Optimal Approximate Quantum Error Correction for Quantum Metrology

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(Dated: November 8, 2021)

For a generic set of Markovian noise models, the estimation precision of a parameter associated with the Hamiltonian is limited by the $1/\sqrt{t}$ scaling where $t$ is the total probing time, in which case the maximal possible quantum improvement in the asymptotic limit of large $t$ is restricted to a constant factor. However, situations arise where the constant factor improvement could be significant, yet no effective quantum strategies are known. Here we propose an optimal approximate quantum error correction (AQEC) strategy asymptotically saturating the precision lower bound in the most general adaptive parameter estimation scheme where arbitrary and frequent quantum controls are allowed. We also provide an efficient numerical algorithm finding the optimal code. Finally, we consider highly-biased noise and show that using the optimal AQEC strategy, strong noises are fully corrected, while the estimation precision depends only on the strength of weak noises in the limiting case.

Introduction.— Quantum metrology is one of the most important state-of-the-art quantum technologies, studying the precision limit of parameter estimation in quantum systems [1–6]. The task involves preparing a suitable initial state of the system, allowing it to evolve under quantum controls for a specific time, performing a suitable measurement, and inferring the value of the unknown system parameter from the measurement outcome. To enhance the estimation precision, a variety of quantum strategies have been proposed, such as squeezing the initial state [7–12], optimizing the probing time [13], monitoring the environment [14–16], exploiting non-Markovian effects [17–19], optimizing the control Hamiltonian [20–22] and quantum error correction [23–34].

Quantum mechanics places a fundamental limit on estimation precision, the Heisenberg limit (HL), where the estimation precision scales like $1/N$ for $N$ probes; or equivalently, $1/t$ for a total probing time $t$. In the noiseless case, the HL is achievable using the maximally entangled state among probes [1, 35]. In practice, decoherence plays an indispensable role. Under many typical noise models, the estimation precision will follow the standard quantum limit (SQL) with scaling $1/\sqrt{N}$ (or $1/\sqrt{t}$) [29–31, 36–41], the same as the central limit theorem scaling using classical strategies. Nevertheless, the superiority of quantum strategies over classical strategies by a constant-factor improvement, as opposed to a scaling improvement, was proven in several cases [11, 37, 40]. There were also situations where the HL is achievable using quantum strategies even in the presence of noise [16, 31].

Due to the difficulty in obtaining the exact precision limits for general noise models using different quantum strategies, several asymptotical lower bounds have been proposed [29–31, 36–42]. For example, the channel simulation method was used to prove the SQL lower bound for programmable channels [38–40]. A necessary and sufficient condition of achieving the HL under Markovian noise was established using the channel extension method [29–31]. Although these bounds have been successful at showing the scaling limit of quantum strategies, only in several special cases, the saturability of these lower bounds was established, e.g. for dephasing and erasure noise [40] and for teleportation-covariant channels as a special type of programmable channels [43, 44]. A saturability statement of the SQL lower bound under general noise models and an efficient algorithm solving the optimal strategy remain missing up to the present day.

We address both of these open questions in this work. Here we consider parameter estimation under general Markovian noise using the most general adaptive sequential strategy (see Fig. 1). We propose an approximate quantum error correction (AQEC) strategy saturating the SQL lower bound of precision (asymptotically) and an efficient numerical algorithm solving the optimal AQEC codes for different noises. The saturability of the precision lower bound we prove here not only answers an important question in quantum metrology theory, but also paves the way for identifying the optimal quantum strategies in future experiments.

Quantum error correction (QEC) [45] was first shown useful in quantum metrology in a typical scenario where the dephasing noise in a qubit probe is corrected by QEC, while the $X$ (the Pauli-X operator) signal remains intact [23–26]. Later on, the result was generalized to arbitrary Markovian noise [30, 31], stating that the HL is achievable using the sequential QEC strategy if and only if the HNLS condition is satisfied, i.e. the signal Hamiltonian is not in the Lindblad span—an operator subspace defined using Lindblad operators [46–48]. In practice, however, HNLS is often violated and the estimation precision is limited by the SQL, for example, sensing any single-qubit signal under depolarizing noise. Standard QEC strategies would be useless in this case, as the signal will be completely eliminated if the noise is fully corrected. However, here we show that by performing the QEC in an approximate fashion, the highest possible precision limit is achievable, marking another triumph of the QEC strategy in quantum metrology.

In this Letter, we first review the SQL precision lower bound under Markovian noise using the sequential strategy when HNLS is violated. Then we describe our AQEC strategy consisting of both a two-dimensional AQEC code and an optimal recovery channel. This allows the original quantum channel to be reduced to an effective channel where a $Z$ (the Pauli-Z operator) signal was sensed under dephasing noise—a special case where the precision lower bound was known to be saturable [11, 37, 40]. Finally, we optimize the achievable...
precision over all possible AQEC codes, which coincides with the precision lower bound, completing the proof.

\[ F(t) \leq 4t \min_{|\alpha|} \||\alpha||, \]  

where \( \|\cdot\| \) is the operator norm of a matrix, \( h \in \mathbb{R}, h \in \mathbb{C}^r, h \in \mathbb{C}^{r \times r} \) is hermitian,

\[ \alpha = (h \mathbb{1} + h \mathbb{L})^\dagger (h \mathbb{1} + h \mathbb{L}), \]  

\[ \beta = H + h \mathbb{1} + h \mathbb{L} + L^\dagger h + L h \mathbb{L}, \]  

where \( L := (L_1, L_2, \ldots, L_r)^T \) and \( h \mathbb{1} := (h_1, \ldots, h_\ell)^T \). Here we introduce an AQEC strategy which (asymptotically) saturates the QFI upper bound up to an arbitrarily small error under arbitrary Markovian noise. That is, for any small \( \delta > 0 \), there exists an AQEC strategy such that

\[ \mathcal{F} := \max_{t>0} \frac{F(t)}{t} = 4 \min_{h,h,\beta} \|\alpha\| - \delta, \]  

where we define the normalized QFI \( \mathcal{F} \) as the objective function we maximize. The upper bound is saturated asymptotically in the sense that \( \lim_{t \to \infty} F(t)/t = \mathcal{F} \).

**Approximate quantum error correction.** Here we propose a set of AQEC codes for quantum metrology and show that the effective channel under fast AQEC is an effective qubit dephasing channel in the logical space. In this way, identifying the optimal recovery channel for quantum metrology is equivalent to minimizing the noise rate of the dephasing channel where a closed-form solution exists, as opposed to generic AQEC scenarios where many known AQEC recovery channels are only suboptimal [55–60].

Let \( P \) be the projection onto the code space \( |0_0\rangle \langle 0_0| + |1_k\rangle \langle 1_k| \), where \( |0_0\rangle \) and \( |1_k\rangle \) are the logical zero and one states. Applying the AQEC quantum operation \( \mathcal{P} + \mathcal{R} \circ \mathcal{P}_L \) infinitely fast, the effective evolution would be (up to the first order of \( dt \) [31, 33])

\[ \frac{d\rho}{dt} = -i[\omega \mathcal{P}(H), \rho] + \sum_{i=1}^r \left( \mathcal{P}(L_i \rho L_i^\dagger) + \mathcal{R}(\mathcal{P}_L(L_i \rho L_i^\dagger)) - \frac{1}{2} \mathcal{P}(L_i L_i^\dagger) \right), \]  

where \( \mathcal{P}_L = 1 - P, \mathcal{P}(\cdot) = \mathcal{P}(\cdot)P, \mathcal{P}_L(\cdot) = \mathcal{P}_L(\cdot)P_L \) and \( \mathcal{R} \) is a CPTP map describing the AQEC recovery channel. We define the following class of AQEC codes

\[ |0_0/1_k\rangle = \sum_{i,j} A_{01,ij} |i\rangle_{H_0} |j,0/1\rangle_{H_A}, \]  

where \( A_{01,ij} \in \mathbb{C}^{d \times d} \) and \( A_{01,ij} = C_{ij} + \varepsilon D_{ij} \) satisfy \( \text{Tr}(A_{01}^\dagger A_{10}) = \text{Tr}(A_1 A_0^\dagger) = 1 \) and \( \text{Tr}(C^D) = 0 \). Here \( C \) describes the part of the code which \( |0_0\rangle \) and \( |1_k\rangle \) have in common and \( D \) describes the part distinguishing \( |0_0\rangle \) from \( |1_k\rangle \) which generates non-zero signal and noise. In the special case where \( \varepsilon = 0 \), the effective signal and noise are zero. Let \( H_A = H_{\mathbb{A}_1} \otimes H_2 \) where \( \dim H_{\mathbb{A}_1} = d \) and \( \dim H_2 = 2 \), the last ancillary qubit in \( H_2 \) makes the signal and noises both diagonal in the code space, i.e. \( \langle 0_0| H |1_k\rangle = \langle 0_0| S |1_k\rangle = 0 \) for all \( S \in \mathbb{S} \). Later on, we will assume \( \varepsilon \) is a small parameter and consider the perturbation expansion of the effective dynamics around \( \varepsilon = 0 \). We consider the recovery channel restricted to
the structure (we will show that this type of recovery channels is sufficient for our purpose)

\[
\mathcal{R}(\cdot) = \sum_m (|0_m\rangle \langle R_m, 0| + |1_m\rangle \langle S_m, 1|) \cdot \\
\frac{1}{2} \left[ \langle R_m, 0| \langle 0_l| + |S_m, 1\rangle \langle 1_l| \right],
\]

where \(|R_m\rangle\}\{S_m\}\subseteq \mathcal{H}_S \otimes \mathcal{H}_A\) are two sets of orthonormal basis and \(\mathcal{R}\) is CPTP. A few lines of calculation shows the effective channel (Eq. (6)) under the AQEC code (Eq. (7)) and the recovery channel (Eq. (8)) is

\[
\frac{d\rho}{dt} = -i \left[ \frac{\omega}{2} (H_{Z_L} + H_S, \rho) + \gamma(\mathcal{R}) \rho Z_L - \rho, \right],
\]

where \(Z_L = |0_L\rangle \langle 0_L| - |1_L\rangle \langle 1_L|\), \(H_S\) is independent of \(\omega\), and

\[
\gamma(\mathcal{R}) = -\text{Re} \left[ \sum_{i=1}^r (|0_L\rangle \langle 0_l|, \mathcal{R}(P_L|0_l\rangle \langle 1_l| L_i^\dagger) + P(L_i|0_l\rangle \langle 1_l| L_i^\dagger) - \frac{1}{2} \{P(L_i^\dagger L_i, |0_L\rangle \langle 1_L|) \} |1_L| \right].
\]

We can remove the term \(H_S\) in Eq. (9) by applying a reverse Hamiltonian constantly [29]. For dephasing channels, the optimal \(\tilde{\gamma}\) is reached using a special type of spin-squeezed state as the input [9–11, 37, 40], where we have

\[
\tilde{\gamma} = \frac{\text{Tr}(H_{Z_L})^2}{2\gamma(\mathcal{R})}.
\]

To simulate the evolution of multipartite spin-squeezed states using the sequential strategy where we have only a single probe, one could first prepare the desired spin-squeezed state in \(\otimes_{i=1}^N \mathcal{H}_i\) by entangling the logical qubit in the effective dephasing channel (\(\mathcal{H}_i = H_S \otimes H_A\)) with a large number of ancillas (\(\otimes_{i=2}^N \mathcal{H}_i\)) where \(\dim \mathcal{H}_i = \dim \mathcal{H}_1\) for \(2 \leq i \leq N\), and then perform swap operations between \(\mathcal{H}_1\) and \(\mathcal{H}_i\) for \(i = 2, \ldots, N\) successively every time \(t/N\). The optimal \(\tilde{\gamma}\) in Eq. (11) is asymptotically attainable at \(N \to \infty\) [11].

For simplicity in further calculation, we perform a two-step gauge transformation on the Lindblad operators \(\{L_i\}_{i=1}^r\) to simplify the dynamics: (1) Let \(L_i \leftarrow L_i - \text{Tr}(C^\dagger L_i C) \cdot 1\), such that \(L_i\) satisfies \(\text{Tr}(C^\dagger L_i C) = 0\) for all \(L_i\). (2) Perform a unitary transformation \(u \in \mathbb{C}^{r \times r}\) on the Lindblad operators \(L_i \leftarrow u L_i\), such that \(\text{Tr}(C^\dagger L_i^\dagger L_i C)\) is a diagonal matrix. Note that above transformations only induce another parameter-independent shift \(\mathcal{H}_1\) in the Hamiltonian which could be eliminated by a reverse Hamiltonian. Now we have a new set of Lindblad operators \(\{J_i\}_{i=1}^r\), satisfying

\[
\text{Tr}(C^\dagger J_i C) = 0, \quad \text{Tr}(C^\dagger J_i^\dagger J_i C) = \lambda_i \delta_{ij},
\]

and we replace \(\{L_i\}_{i=1}^r\) with \(\{J_i\}_{i=1}^r\) in Eq. (10).

First, we maximize \(\tilde{\gamma}\) over the recovery \(\mathcal{R}\), which is equivalent to minimizing \(\gamma(\mathcal{R})\) over \(\mathcal{R}\). We claim that the minimum noise rate \(\gamma = \min_{\mathcal{R}} \gamma(\mathcal{R})\) is

\[
\gamma = -\frac{1}{\max_{\mathcal{U}; \mathcal{U} \in \mathcal{U}} \text{Tr} (MU + M^\dagger U^\dagger) = 2 \parallel M||_1, \text{for arbitrary square matrices} M \text{ and } U, \text{where } ||\cdot||_1 \text{ is the trace norm (see details in Appx. A of } [61]).
\]

Next, we could like to maximize \(\tilde{\gamma}\) over all possible AQEC codes of the form Eq. (7). It is not clear yet how that could be done mathematically with the presence of trace norm in the denominator. To arrive at an expression of \(\gamma\) free of the trace norm, we further sacrifice the generality of our AQEC code and assume \(\varepsilon \ll 1\). We call it the “perturbation” code in the sense that the signal and the noise are both infinitesimally small when \(\varepsilon \to 0\). Under the limit \(\varepsilon \to 0\), we have \(\text{Tr}(H_{Z_L}) = 2\varepsilon \text{Tr}(H_C) + O(\varepsilon^2)\), where

\[
\hat{C} = \hat{C}_D^\dagger + \hat{D}_C^\dagger,
\]

and the noise rate is (ignoring all \(o(\varepsilon^2)\) terms)

\[
\gamma = \varepsilon^2 \left( \sum \frac{2 \text{Tr}(J_i \hat{C})^2}{\lambda_i} + \sum_{ij; \lambda_i + \lambda_j \neq 0} \frac{|\text{Tr}(J_i^\dagger J_j^\dagger \hat{C})|^2}{2(\lambda_i + \lambda_j)} \right).
\]

For a detailed derivation of the noise rate, see Appx. B of [61] and [62]. Finally, we have the following expression of the normalized QFI (up to the lowest order of \(\varepsilon\))

\[
\tilde{\gamma}(C, \hat{C}) \approx \frac{\text{Tr}(H_C^2)}{\sum_i \text{Tr}(J_i \hat{C})^2 + \sum_{ij; \lambda_i + \lambda_j \neq 0} \frac{|\text{Tr}(J_i^\dagger J_j^\dagger \hat{C})|^2}{2(\lambda_i + \lambda_j)}},
\]

as a function of \(\hat{C}\) and \(C\) (implicitly through the choice of \(\{J_i\}_{i=1}^r\)). The effective dynamics of the perturbation code has the feature that both the signal and the noises are equally weak and only the ratio between them matters. Therefore the exact value of \(\varepsilon\) will not influence the normalized QFI \(\tilde{\gamma}\) as long as it is sufficiently small. On the other hand, it does influence how fast \(F(t)/t\) reaches its optimum \(\tilde{\gamma}\), characterized by a coherence time \(O(1/\varepsilon^2)\).

### Saturating the bound.

Now we maximize the normalized QFI (up to the lowest order of \(\varepsilon\)) over \(C\) and \(\hat{C}\) and show that the optimal \(\tilde{\gamma}\) is exactly equal to its upper bound in Eq. (2). The domain of \(C\) is all complex matrices satisfying \(\text{Tr}(C^\dagger C) = 1\). We assume the domain of \(\hat{C}\) is all traceless Hermitian matrices satisfying \(\text{Tr}(J_i^\dagger J_i C) = 0\) for all \(i, j \in n \equiv \{i; \lambda_i = 0\}\). When \(C\) is full-rank, \(n\) is empty and for arbitrary traceless \(\hat{C}\), we could always take \(D^\dagger = \frac{1}{C} C^{-1} C\) such that Eq. (14) is satisfied. When \(C\) is singular, we could replace it with an approximate full-rank version (e.g. \(C \leftarrow C + \delta I\)). In this case, \(\tilde{\gamma}\) will only be decreased by an infinitesimal small amount when \(\varepsilon = o(\delta^2)\) because the numerator in Eq. (16) is only slightly perturbed after the replacement.

Consider the following optimization problem over \(h, h, \hat{h}\) and \(C\),

\[
\max_{\hat{C}} \min_{\hat{h}, h, \hat{h}} 4\text{Tr}(C^\dagger a C), \\
\text{subject to } \beta = 0, \quad \text{Tr}(C^\dagger C) = 1,
\]

Fixing \(C\), we introduce a Hermitian matrix \(\tilde{C}\) as the Lagrange multiplier associated with the constraint \(\beta = 0\) [63]. Strong
duality implies Eq. (17) has the same solution as the following dual program (see Appx. C of [61])
\[
\max_{C,\tilde{C}} \tilde{\xi}(C,\tilde{C}), \text{ subject to } \Tr(C^{\dagger}C) = 1, \Tr(\tilde{C}) = 0, \quad \text{(18)}
\]
and \(\Tr(J_{j}^{\dagger}J_{\tilde{C}}) = 0, \forall i, j \in n.\)
whose optimal value could be achieved using the perturbation code up to an infinitesimal small error according to the discussion above. On the other hand, thanks to Sion’s minimax theorem [64, 65], we can exchange the order of the maximization and minimization in Eq. (17) because we could always confine \((h, h)\) in a convex and compact set (see Appx. D of [61]) such that the solution of Eq. (17) is not altered and the objective function \(4\Tr(C^{\dagger}\alpha C)\) is concave (linear) with respect to \(CC^{\dagger}\) and convex (quadratic) with respect to \((h, h)\). Therefore, the optimal value of Eq. (18) is also equal to \(4\min_{h,h}∥\alpha∥,\) the upper bound of the normalized QFI.

Numerical algorithm.— It is known that the upper bound in Eq. (2) could be calculated via a semidefinite program (SDP) [30, 42],
\[
F(t) \leq 4t \min_{h,h} \min_{β=0} x, \text{ subject to } A' \geq 0, \beta = 0, \quad \text{(19)}
\]
where \(A' = \left(\begin{array}{cc}
\bar{h} & \bar{h}L
\\
\bar{h}L & \bar{h}L + \bar{h}L^t
\end{array}\right)\) and \(\geq 0\) means positive semidefinite. However, the minimax theorem does not guarantee an efficient algorithm to solve Eq. (18) after exchanging the order of the maximization and minimization in Eq. (17). Now we provide an efficient numerical algorithm obtaining an optimal \((C^o, C^{o\dagger})\) in three steps. The validity of this algorithm is proven in Appx. E of [61]. The algorithm runs as follows: (a) Solving \(\min_{h,h} \min_{β=0} ∥α∥\) using the SDP gives us an optimal \(α^o\) (and corresponding \(h^o, h^o, h^o\)) satisfying \(∥α^o∥ = \min_{h,h} ∥α∥\). (b) Suppose \(Π^o\) is the projection onto the subspace spanned by all eigenstates corresponding to the largest eigenvalue of \(α^o\), we find an optimal \(C^{o\dagger}C^o\) satisfying \(Π^oC^oC^{o\dagger}Π^o = C^oC^{o\dagger}\) and
\[
\Re[\Tr(C^oC^{o\dagger}(\Delta h L + \Delta h L^t)(h^o L^t + h^o L))] = 0, \quad \text{(20)}
\]
for all \((\Delta h, \Delta h)\) such that \(\Delta h L + \Delta h L^t + \Delta h L^t + \Delta h L = 0\) for some \(\Delta h\). That is, this step is simply solving a system of linear equations. (c) Find \(\{J_{ij}\}_{i=1}^n\) via the gauge transformation. Let \(S_0 = \text{span}\{I, J_{ij}, \forall i, j \in n\}\). Decompose \(M = J_1\) or \(J_{ij} (:= J_{ij})\), into \(M = M^h + iM^ah + M^b + iM^{ah}\) where \(M^{ah}, M^{ah}_{ij}\) are Hermitian, \(M^{ah}_{ij}, i \in S_0\) and \(M^{ah}_{ij} \perp S_0\) (in terms of the Hilbert-Schmidt norm). Using the vectorization of matrices \(|\rangle\rangle = \sum_{jk} γ(jk) |j⟩⟨k| |j⟩⟨k|,\) let
\[
B = \sum_{i} \sum_{lj=0} J_{ij}^{\dagger} ⟨⟨J_{ij}^h⟩⟩ + J_{ij}^{\dagger} ⟨⟨J_{ij}^h⟩⟩ + \sum_{ij, λ_i+λ_j ≠ 0} 2(J_{ij}^{\dagger} ⟨⟨J_{ij}^h⟩⟩ + J_{ij}^{\dagger} ⟨⟨J_{ij}^h⟩⟩) / (λ_i+λ_j). \quad \text{(21)}
\]
According to the Cauchy-Schwarz inequality,
\[
\max_C \tilde{\xi}(C^o, \tilde{C}) = \max_C \frac{∥⟨⟨H^h/⟨⟨C⟩⟩⟩⟩∥^2}{∥⟨⟨C/⟨⟨B⟩⟩⟩⟩∥} = ∥H^h∥B^{-1}∥H^h∥, \quad \text{(22)}
\]
and the optimal \(∥\tilde{C}^o∥ = B^{-1}⟨⟨H^h⟩⟩\). Here \(-1\) means the Moore-Penrose pseudoinverse.

Highly-biased noise.— We consider a special case where noises are separated into two groups — strong ones and weak ones [31–33]. To be specific, we consider the following quantum master equation
\[
\frac{dρ}{dt} = -i[ωH, ρ] + \sum_{i ∈ s} η(L_{i}\rho L_{i}^t - \frac{1}{2}\{L_{i}^tL_{i}, ρ\}) + \sum_{i ∈ Π} \left(\frac{L_{i}\rho L_{i}^t - \frac{1}{2}\{L_{i}^tL_{i}, ρ\}}{R\bar{s}}\right), \quad \text{(23)}
\]
where the indices of Lindblad operators \(\{L_{i}\}_{i=1}^n\) are separated into \(s\) and \(Π\), representing weak and strong noises respectively. \(η ≪ 1\) is a small parameter characterizing the relative strength of the weak noises. Moreover, we assume that \(H \notin \text{span}\{L_{i}, L_{i}^t, L_{i}^tL_{j}, i, j \in s\}\) so that it is possible to fully correct all strong noises and also preserve a non-trivial signal in the code space. Taking \(η → 0\), it is easy to show that (see Appx. F of [61]), the optimal \(\tilde{\xi}\) in this case is equal to
\[
\tilde{\xi}^o = \frac{1}{η} ∥|\bar{α}|⟩⟨h, h, h|β=0⟩ + O(1), \quad \text{(24)}
\]
where \(α = (h^o L + h^o L^t)Π_{s}(h^o L + h^o L^t)\) and \(Π_{s}\) is a diagonal matrix whose \(i\)-th diagonal element is one when \(i \in s\) and zero when \(i \notin s\). This reduces the running time of the SDP in Eq. (19) by reducing \(A'\) from a \((d(r+1)×d(r+1)\) matrix to a \((d(|s|+1)×d(|s|+1)\) matrix. Using the optimal AQEC strategy, \(\tilde{\xi}\) is boosted by a factor of \(O(1/η)\), compared to the case where no QEC is performed. To find the optimal AQEC code, we can solve the dual program of a modified version of Eq. (17) where \(α\) is replaced with \(α^o\):
\[
\max_{C,\tilde{C}} \tilde{\xi}(C,\tilde{C}), \text{ subject to } \Tr(C^{\dagger}C) = 1, \Tr(\tilde{C}) = 0, \quad \text{(25)}
\]
\[\forall i, j \in s, \text{ Tr}(L_{i}^tL_{j}^t) = \text{Tr}(L_{i}^tL_{j}^t) = 0, \quad \text{(26)}\]
and some other linear constraints on \(\tilde{C}\) when \(\text{Tr}(CL^tLC) - \text{Tr}(CL^tLC)\text{Tr}(CLC)\) is singular. Here \(\tilde{\xi}\) is the dominant part of \(\tilde{\xi}\) such that \(\tilde{\xi} = \tilde{\xi}/η + O(1)\). Detailed calculations including the expression of \(\tilde{\xi}\) and the dual program are provided in Appx. F of [61]. The constraint Eq. (26) on \(\tilde{C}\) is equivalent to the Knill-Laflamme condition for Lindblad operators \(\{L_{i}\}_{i∈s} [66, 67]\). It implies strong noises are fully corrected by the optimal AQEC code and explains why the estimation precision depends only on the strength of weak noises in Eq. (24).

Conclusions and outlook.— In this Letter, we proposed an AQEC strategy such that the optimal SQL in Hamiltonian parameter estimation could be achieved asymptotically. An interesting open question is whether the perturbation code we introduced here could be turned into non-perturbative ones. We provide an example in Appx. G of [61], where by slightly modifying the ancilla-free QEC code (non-perturbation) proposed in Ref. [33], we show that the optimal \(\tilde{\xi}\) could be achieved in the correlated dephasing noise model. However, it is unclear how to generalize the result to generic noise models.
Another two interesting open questions are (1) how to characterize the power of QEC in improving quantum metrology for parameters encoded in generic quantum channels [39, 40], for example when the rate of quantum controls is constant, rather than infinitely fast; (2) how to optimize the QEC strategy when considering a constant probing time, rather than an infinitely long probing time.

Acknowledgements.– We thank Kyungjoo Noh, Rafal Demkowicz-Dobrzański, Zhou Fan, Jing Yang, Yuxiang Yang for helpful discussions. We acknowledge support from the ARL-CDQI (W911NF15-2-0067, W911NF-18-2-0237), ARO (W911NF-18-1-0020, W911NF-18-1-0212), ARO MURI (W911NF-16-1-0349), AFOSR MURI (FA9550-15-1-0015), DOE (DE-SC0019406), NSF (EFMA-1640959), and the Packard Foundation (2013-39273).

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Appendix A: Minimizing the noise rate $\gamma(\mathcal{R})$ over recovery channels $\mathcal{R}$

In this appendix we prove Eq. (13) in the main text. According to Eq. (10),

$$\gamma(\mathcal{R}) = -\Re \left[ \sum_{i=1}^{r} \langle 0_{i} | \left( \mathcal{R} (P_{\perp} (J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} )) + P (J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} ) - \frac{1}{2} \{ P(J_{i}^{\dagger} J_{i}), | 0_{i} \rangle \langle 1_{i} | \} \right) | 1_{i} \rangle \right].$$  \hspace{1cm} (A1)

In order to calculate $\gamma = \min_{\mathcal{R}} \gamma(\mathcal{R})$, we only need to calculate the first term minimized over $\mathcal{R}$:

$$-\max_{\mathcal{R}} \Re \left[ \sum_{i} \langle 0_{i} \mathcal{R} (P_{\perp} (J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} ) ) | 1_{i} \rangle \right]$$

$$= -\max_{|R_{m}, |S_{m}\rangle} \Re \left[ \sum_{i,m} \langle R_{m} | P_{\perp} J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} P_{\perp} | S_{m} \rangle 1 \right]$$

$$= -\frac{1}{2} \max_{|R_{m}, |S_{m}\rangle} \Tr \left( \sum_{m} |R_{m}\rangle \langle S_{m}| \cdot \sum_{i} \langle 0_{i} \langle 1_{i} | J_{i}^{\dagger} P_{\perp} | 1 \rangle \right) + h.c.$$  \hspace{1cm} (A2)

where $h.c.$ means Hermitian conjugate and we have used $\max_{U:U^{\dagger}U=1} \Tr (MU + M^{\dagger}U^{\dagger}) = 2 \| M \|_{1}$ for arbitrary square matrices $M$ and $U$, which could be proven easily using the singular value decomposition of $M$.

Appendix B: Perturbation expansion of the noise rate $\gamma$

In this appendix we expand the minimum noise rate $\gamma$ around $\varepsilon = 0$ using the perturbation code. For simplicity, the equal sign “$=$” in this appendix means approximate equality up to the second order of $\varepsilon$ (ignoring all $o(\varepsilon^{2})$ terms). We also state a useful lemma here:

Lemma 1 (68). $\| X + \varepsilon Y \|_{1} = \| X \|_{1} + O(\varepsilon)$ for arbitrary $X$ and $Y$.

To calculate Eq. (13), we first consider the terms independent of $\mathcal{R}$,

$$-\Re \left[ \sum_{i=1}^{r} \langle 0_{i} \right. \mathcal{R} (P_{\perp} (J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} ) ) | 1_{i} \rangle \left. - \frac{1}{2} \{ P(J_{i}^{\dagger} J_{i}), | 0_{i} \rangle \langle 1_{i} | \} \right) | 1_{i} \rangle \right]$$

$$= -\sum_{i} \Re\left[ \Tr (A_{0} A_{0}^{\dagger} J_{i}) \right] \langle 1_{i} | J_{i}^{\dagger} P_{\perp} \rangle 1 \right] - \frac{1}{2} \left( \Tr (A_{0} A_{0}^{\dagger} J_{i}^{\dagger} J_{i}) + \Tr (A_{1} A_{1}^{\dagger} J_{i}^{\dagger} J_{i}) \right)$$  \hspace{1cm} (B1)

The remaining term is equal to (thanks to Lemma 1) minus

$$\left\| \sum_{i} P_{\perp} J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} P_{\perp} \right\|_{1} = \left\| \left( \sqrt{\Lambda^{-1}} \langle X_{1} + \varepsilon^{2} X_{1}^{\dagger} \rangle + \varepsilon^{2} X_{1}^{\dagger} \right) \right\|_{1},$$  \hspace{1cm} (B2)

where $\Lambda \in \mathbb{R}^{r \times r}$ is a diagonal matrix whose $k$-th diagonal element is $\lambda_{k}$ and $\hat{\Lambda} \in \mathbb{R}^{r \times r}$ is a diagonal matrix whose $k$-th diagonal element is $\lambda_{k}$ if $\lambda_{k} > 0$ and 1 if $\lambda_{k} = 0$. Assume $\{ \lambda_{k} \}_{k=1}^{r}$ is arranged in a non-ascending order and $r_{0}$ is the largest integer such that $\lambda_{r_{0}}$ is positive. $X_{1}, X_{1}^{\dagger} \in \mathbb{C}^{r \times r}$ satisfy

$$\langle \Lambda + \varepsilon X_{1} + \varepsilon^{2} X_{1}^{\dagger} \rangle_{j} = \langle \sqrt{\Lambda_{j,0}} \mathcal{R} (P_{\perp} J_{i} | 0_{i} \rangle \langle 1_{i} | J_{i}^{\dagger} ) P_{\perp} | 0_{i} \rangle \rangle_{j,0} = \langle \mathcal{R} (C^{\dagger} J_{i}^{\dagger} J_{i} A_{0}) - \Tr (C^{\dagger} J_{i}^{\dagger} J_{i} A_{0}) \rangle_{j}$$

$$= \lambda_{i} \delta_{ij} + \varepsilon \Tr (C^{\dagger} J_{i}^{\dagger} J_{i} A_{0}) - \varepsilon^{2} \Tr (C^{\dagger} J_{i}^{\dagger} J_{i} D) \Tr (\hat{C} J_{i})$$  \hspace{1cm} (B3)

for $1 \leq j \leq r_{0}$ and

$$\langle \Lambda + \varepsilon X_{1} + \varepsilon^{2} X_{1}^{\dagger} \rangle_{j} = \langle \varepsilon \Tr (\hat{J}_{i}^{\dagger} J_{i} A_{0}) \rangle_{j}$$

$$= \varepsilon \Tr (\hat{J}_{i}^{\dagger} J_{i} D) - \varepsilon^{2} \Tr (\hat{J}_{i}^{\dagger} D) \Tr (\hat{C} J_{i})$$  \hspace{1cm} (B4)
for $r_0 + 1 \leq j \leq r$. $X_2, X'_2 \in \mathbb{C}^{(d^2-r)\times r}$ satisfy

$$
(A + \varepsilon X_2 + \varepsilon^2 X'_2)_{ji} = \langle \tilde{J}_{j+r,0} | P_{L} J_i | 0_k \rangle = \text{Tr}(\tilde{J}_{j+r}^\dagger J_i A_0) - \text{Tr}(\tilde{J}_{j+r}^\dagger A_0^\dagger J_i A_0) = \varepsilon \text{Tr}(\tilde{J}_{j+r}^\dagger J_i D) - \varepsilon^2 \text{Tr}(\tilde{J}_{j+r}^\dagger D) \text{Tr}(\hat{C} J_i),
$$

(B5)

for $r + 1 \leq j \leq d^2 - 1$ and

$$
(A + \varepsilon X_2 + \varepsilon^2 X'_2)_{di2} = \langle \tilde{J}_{d^2,0} | P_{L} J_i | 0_k \rangle = \text{Tr}(\tilde{J}_{d^2}^\dagger J_i A_0) - \text{Tr}(\tilde{J}_{d^2}^\dagger A_0^\dagger J_i A_0) = \varepsilon \text{Tr}(C^\dagger J_i D) - \varepsilon \text{Tr}(\hat{C} J_i).
$$

(B6)

Here

$$
|\tilde{J}_{j,0/1}\rangle = \begin{cases}
\frac{1}{\sqrt{N}} \sum_{ik} C_{ik} J_j |i\rangle |k, 0/1\rangle, & j \leq r_0, \\
\sum_{ik} (\tilde{J}_{j})_{ik} |j\rangle |k, 0/1\rangle, & r_0 < j \leq d^2, \\
\sum_{ik} \frac{C_{ik}}{\sqrt{\text{Tr}(C^\dagger C)}} |i\rangle |k, 0/1\rangle, & j = d^2,
\end{cases}
$$

(B7)

are two sets of orthonormal basis of $\mathcal{H}_S \otimes \mathcal{H}_A$.

To calculate the first and second order expansion of Eq. (B2), we consider the singular value decompositions

$$
\begin{align*}
\left( \frac{1}{\varepsilon X_2 + \varepsilon^2 X'_2} \right)^{1/2} &= U(\varepsilon) \left( \begin{array}{cc} \Sigma(\varepsilon) & 0 \\ 0 & 0 \end{array} \right) V(\varepsilon)^\dagger, \\
\left( A - \varepsilon X_1 + \varepsilon^2 X'_1 \right)^{1/2} &= V(-\varepsilon) \left( \begin{array}{cc} \Sigma(-\varepsilon) & 0 \\ 0 & 0 \end{array} \right) U(-\varepsilon)^\dagger,
\end{align*}
$$

(B8)

Then

$$
\text{Eq. (B2)} = \left\| U(\varepsilon) \left( \begin{array}{cc} \Sigma(\varepsilon) & 0 \\ 0 & 0 \end{array} \right) V(-\varepsilon) \right\|_1 = \left\| \Sigma(\varepsilon) V(\varepsilon)^\dagger V(-\varepsilon) \Sigma(-\varepsilon) \right\|_1
$$

$$
= \text{Tr} \left( (V(\varepsilon) \Sigma(\varepsilon) V(\varepsilon)^\dagger V(-\varepsilon) \Sigma(-\varepsilon) V(-\varepsilon)^\dagger V(\varepsilon) \Sigma(\varepsilon) V(\varepsilon)^\dagger) \right)^{1/2}
$$

(B9)

with $Y(\varepsilon) = V(\varepsilon) \Sigma(\varepsilon)^2 V(\varepsilon)^\dagger = \Lambda + \varepsilon (X'_1 \Pi \Lambda + \Pi \Lambda X'_1) + \varepsilon^2 (X'_1 \hat{\Lambda}^{-1} X_1 + X'_1 \Pi \Lambda + \Pi \Lambda X'_1^\dagger + X'_2 X_2)$,

(B10)

and $\Pi \Lambda$ is the projector onto the support of $\Lambda$.

Using Theorem 2 in Ref. [62], we have

$$
\text{Tr} \left( \sqrt{Y(\varepsilon)} Y(-\varepsilon) \sqrt{Y(\varepsilon)} \right) =
$$

$$
\text{Tr} (\Lambda + \varepsilon^2 \text{Tr}(X'_1 \hat{\Lambda}^{-1} X_1 + X'_1 \Pi \Lambda + \Pi \Lambda X'_1^\dagger + X'_2 X_2) - \varepsilon^2 \sum_{i,j;i:\lambda_i + \lambda_j \neq 0} |X_{1,ij} + X_{1,ji}|^2 \frac{\lambda_i + \lambda_j}{\lambda_i + \lambda_j}).
$$

(B11)

Note that

$$
\text{Tr}(X'_1 \hat{\Lambda}^{-1} X_1 + X'_2 X_2) = \sum_{i=1}^{r} \text{Tr}(D D^\dagger J_i^\dagger J_i) + \sum_{i=1}^{r} \left| \text{Tr}(\tilde{C} J_i) \right|^2 - \sum_{i=1}^{r} \text{Tr}(D^\dagger J_i C) \text{Tr}(\hat{C} J_i) + \text{Tr}(C^\dagger J_i D) \text{Tr}(\hat{C} J_i),
$$

(B12)

$$
\text{Tr}(X'_1 \Pi \Lambda + \Pi \Lambda X'_1^\dagger) = - \sum_{i=1}^{r} \left( \text{Tr}(C^\dagger J_i D) \text{Tr}(\hat{C} J_i) + \text{Tr}(D^\dagger J_i C) \text{Tr}(\hat{C} J_i) \right),
$$

(B13)

therefore

$$
\gamma = \text{Eq. (B1)} - \text{Eq. (B2)}
$$

$$
= 2 \varepsilon^2 \sum_{i} \left| \text{Tr}(J_i \hat{C}) \right|^2 + \varepsilon^2 \sum_{i,j;i:\lambda_i + \lambda_j \neq 0} \left| \text{Tr}(J_i J_j \hat{C}) \right|^2 (\lambda_i + \lambda_j),
$$

(B14)
where \( \tilde{C} = CD^\dagger + DC^\dagger \). According to Eq. (11), we have

\[
\tilde{\mathcal{R}} = \frac{\text{Tr}(H \tilde{C})^2}{\sum_i |\text{Tr}(J_i \tilde{C})|^2 + \sum_{ij: \lambda_i + \lambda_j \neq 0} \frac{\text{Tr}(J_i J_j \tilde{C})^2}{2(\lambda_i + \lambda_j)}} + O(\varepsilon).
\] (B15)

**Appendix C: Lagrange dual program of Eq. (17)**

Here we show the Lagrange dual program of Eq. (17) is Eq. (18). From the definition of \( \alpha \) (Eq. (3)) and \( \beta \) (Eq. (4)), we see that the upper bound in Eq. (2) is invariant under the transformation \( L \to J \), that is, after the transformation \( L \to J \) there is always another set of \((h, \mathbf{h}, \mathbf{h})\) such that \( \beta = 0 \) and \( \alpha \) is the same. Therefore we let

\[
\alpha = (h\mathbb{1} + bJ)(h\mathbb{1} + bJ), \quad (C1)
\]

\[
\beta = H + h\mathbb{1} + h^\dagger J, \quad (C2)
\]

where \( \mathbf{J} = (J_1, J_2, \ldots, J_r)^T \). To proceed, we simplify the notations by letting

\[
\hat{J}_i = \frac{\text{Tr}(J_i \tilde{C})}{\text{Tr}(HC)}, \quad i_{ij} = \frac{\text{Tr}(J_i J_j \tilde{C})}{\text{Tr}(HC)}, \quad (C3)
\]

Note that the \( r \)-dimensional vector \( j \) is to be distinguished from the index \( j \), then we have

\[
\tilde{\mathcal{R}}(C, \tilde{C}) = \left( j\hat{j} + \sum_{ij: \lambda_i + \lambda_j \neq 0} \frac{|i_{ij}|^2}{2(\lambda_i + \lambda_j)} \right)^{-1}, \quad (C4)
\]

and \( 4\text{Tr}(C^\dagger \alpha C) = 4(h^\dagger h + \text{Tr}(\lambda h^2)) \).

Fixing \( C \), we introduce a Hermitian matrix \( \tilde{C} \) as a Lagrange multiplier of \( \beta = 0 \) [63], the Lagrange function is

\[
L(\tilde{C}, h, h, h) = 4(h^\dagger h + \text{Tr}(\lambda h^2)) + \text{Tr}(\tilde{C}(H + h\mathbb{1} + J^\dagger h + h^\dagger J + J^\dagger \mathbf{h}J)). \quad (C5)
\]

Then the dual program of Eq. (17) is

\[
\max_{\tilde{C}} \min_h \min_{h,h} \left. L(\tilde{C}, h, h, h) \right|_{\tilde{C}:Tr(\tilde{C})=0, Tr(\tilde{C}H) \neq 0}
\]

\[
= \max_{\tilde{C}, \text{Tr}(\tilde{C})=0, \forall i,j \in n, Tr(\tilde{C}J_i^\dagger J_j) = 0} \frac{-1}{4}\text{Tr}(\tilde{C}H)^2 j^\dagger j - \frac{1}{8}\text{Tr}(\tilde{C}H)^2 \sum_{ij: \lambda_i + \lambda_j \neq 0} \frac{|i_{ij}|^2}{\lambda_i + \lambda_j} + \text{Tr}(\tilde{C}H)} \quad (C6)
\]

\[
= \max_{\tilde{C}, \text{Tr}(\tilde{C})=0, \forall i,j \in n, Tr(\tilde{C}J_i^\dagger J_j) = 0} \left( j^\dagger j + \sum_{ij: \lambda_i + \lambda_j \neq 0} \frac{|i_{ij}|^2}{2(\lambda_i + \lambda_j)} \right)^{-1} \quad \tilde{\mathcal{R}}(C, \tilde{C}), \quad (C5)
\]

as in Eq. (18).

**Appendix D: Confining \((h, \mathbf{h})\) in a compact set**

The minimax theorem [65] states that for convex compact sets \( P \subset \mathbb{R}^m \) and \( Q \subset \mathbb{R}^n \) and \( f : P \times Q \to \mathbb{R} \) such that \( f(x, y) \) is a continuous convex (concave) function in \( x \) (y) for every fixed \( y \) (x), then

\[
\max_{y \in Q} \min_{x \in P} f(x, y) = \min_{x \in P} \max_{y \in Q} f(x, y). \quad (D1)
\]
In Eq. (17), the operator \( CC^\dagger \) satisfying \( \text{Tr}(CC^\dagger) = 1 \) is contained in a convex compact set, but the domain of \((h, h, h)\) is not compact. Here we show that we could always confine \((h, h, h)\) in a convex and compact set such that the solution of Eq. (17) is not altered. First we note that \( \min_{h,h,h} \|\alpha\| = a < \infty \) when \( H \in \mathcal{S} \). Note that

\[
\|\alpha\| = \left\| \begin{pmatrix} h_1^\dagger + \sum_{i=1}^r b_1^i L_1 \\ h_2^\dagger + \sum_{i=1}^r b_2^i L_1 \\ \vdots \\ h_r^\dagger + \sum_{i=1}^r b_r^i L_1 \end{pmatrix} \right\|^2.
\]

It is clear that there exists some \( b > 0 \) such that for all \( \|(h, h)\|_2 > b \) (\( \|\cdot\|_2 \) is the Euclidean norm), we have \( \|\alpha\| > a \). Therefore it is easy to find some \( b' > 0 \) such that

\[
\min_{h,b} \max_C 4 \text{Tr}(C^\dagger \alpha C),
\]

subject to \( \exists h, \beta = 0, \quad \text{Tr}(C^\dagger C) = 1, \quad \|(h, h)\|_2 \leq b', \) (D3)

and

\[
\max_{h,b} \min_C 4 \text{Tr}(C^\dagger \alpha C),
\]

subject to \( \exists h, \beta = 0, \quad \text{Tr}(C^\dagger C) = 1, \quad \|(h, h)\|_2 \leq b', \) (D4)

has the same optimal value equal to \( 4a \), and there exists a saddle point \((h^*, h^*, C^*)\) such that

\[
\text{Tr}(C^\dagger \alpha^* C) \leq \text{Tr}(C^{*\dagger} \alpha^* C^*) \leq \text{Tr}(C^{*\dagger} \alpha^* C^*)
\]

for all \((h, h, C)\) satisfying \( \exists h, \beta = 0, \text{Tr}(C^\dagger C) = 1, \) and \( \|(h, h)\|_2 \leq b' \), where \( \alpha^* = (h^* \mathbb{I} + h^* L)^\dagger (h^* \mathbb{I} + h^* L) \). Moreover, based on the above discussion, \((h^*, h^*)\) is not on the boundary, i.e. \( \|(h^*, h^*)\|_2 < b' \). The second inequality in Eq. (D5) is then equivalent to

\[
\text{Re}[\text{Tr}(C^{*\dagger} \Delta h \mathbb{I} + \Delta h L)^\dagger (h^* \mathbb{I} + h^* L) C^*)] = 0,
\]

for all \((\Delta h, \Delta h)\) satisfying

\[
\Delta h \mathbb{I} + \Delta h \dagger L + L^\dagger \Delta h + L^\dagger \Delta h L = 0
\]

for some \( \Delta h \). Therefore, \((h^*, h^*, C^*)\) is also a saddle point of Eq. (17):

\[
\max_{h,b} \min_C 4 \text{Tr}(C^\dagger C),
\]

subject to \( \exists h, \beta = 0, \quad \text{Tr}(C^\dagger C) = 1, \) (D8)

proving that the optimal value of Eq. (17) must also be equal to \( 4a \).

**Appendix E: The validity of the numerical algorithm**

Here we prove the validity of the three-step algorithm introduced in the main text. Let \((h^*, h^*, C^*)\) be the saddle point of Eq. (17). The first inequality in Eq. (D5) implies

\[
\text{Tr}(C^{*\dagger} \alpha^* C^*) = \|\alpha^*\| = \min_{h,h,h} \|\alpha\|,
\]

which means that \( \Pi^* C^* = C^* \) where \( \Pi^* \) is the projection onto the subspace spanned by all eigenstates corresponding to the largest eigenvalue of \( \alpha^* \).

Now assume we have a solution \((h^\circ, h^\circ)\) of Eq. (2) such that \( \alpha^\circ = (h^\circ \mathbb{I} + h^\circ L)^\dagger (h^\circ \mathbb{I} + h^\circ L) \) satisfies

\[
\|\alpha^\circ\| = \min_{h,h,h} \|\alpha\|.
\]

We prove that \((h^\circ, h^\circ, C^*)\) is also a saddle point. Choose \( p \in (0, 1) \) and let

\[
(h, h) = (ph^\circ + (1-p)h^*, ph^\circ + (1-p)h^*).
\]

(3)
Then
\[
\text{Tr}(C^\dagger \alpha C^\ast) = p^2 \text{Tr}(C^\dagger \alpha C^\ast) + (1 - p)^2 \text{Tr}(C^\dagger \alpha C^\ast) + 2p(1 - p) \text{Re}[\text{Tr}(C^\dagger (h^\dagger \mathbb{1} + h^\circ \mathbf{L})(h^\ast \mathbb{1} + h^\circ \mathbf{L})C^\ast)]
\leq p^2 \text{Tr}(C^\dagger \alpha C^\ast) + (1 - p)^2 \text{Tr}(C^\dagger \alpha C^\ast) + 2p(1 - p) \sqrt{\text{Tr}(C^\dagger \alpha C^\ast)\text{Tr}(C^\dagger \alpha C^\ast)} \leq \|\alpha\|^2. \quad (E4)
\]

On the other hand, we know \( \text{Tr}(C^\dagger \alpha C^\ast) \geq \|\alpha\| \). Therefore the equality in Eq. (E4) must hold, which means
\[
\text{Tr}(C^\dagger \alpha C^\ast) = \|\alpha\|^2, \quad (h^\dagger \mathbb{1} + h^\circ \mathbf{L})C^\ast = (h^\circ \mathbb{1} + h^\circ \mathbf{L})C^\ast. \quad (E5)
\]

As a result, we have \( \text{Tr}(C^\dagger \alpha C^\ast) \leq \text{Tr}(C^\dagger \alpha C^\ast) \) for arbitrary \( C \) satisfying \( \text{Tr}(C^\dagger C) = 1 \). Moreover,
\[
\text{Re}[\text{Tr}(C^\dagger ((\Delta h^\dagger L + \Delta h^\circ L)^{(h^\circ \mathbb{1} + h^\circ \mathbf{L})C^\ast})] = \text{Re}[\text{Tr}(C^\dagger (\Delta h^\dagger L + \Delta h^\circ L)^{(h^\dagger \mathbb{1} + h^\circ \mathbf{L})C^\ast})] = 0, \quad (E6)
\]
and \( \text{Tr}(C^\dagger \alpha C^\ast) \leq \text{Tr}(C^\dagger \alpha C^\ast) \), proving \((h^\circ, h^\circ, C^\circ)\) is also a saddle point. Hence, step (b) in our algorithm will at least have one solution \( C^\ast \), and the solution of step (b) \((h^\circ, h^\circ, C^\circ)\) is also a saddle point satisfying
\[
\text{Tr}(C^\circ \alpha C^\circ) \leq \text{Tr}(C^\circ \alpha C^\circ) \leq \text{Tr}(C^\circ \alpha C^\circ), \quad (E7)
\]
for all \((h, h, C)\) satisfying \( \beta = 0 \) and \( \text{Tr}(C^\dagger C) = 1 \). Strong duality [63] implies the optimal value of
\[
\max_{\bar{C}} \mathfrak{F}(C^\circ, \bar{C}), \quad \text{subject to } \text{Tr}(C^\dagger C) = 1, \bar{C}(\bar{C}) = 0 \quad \text{and} \quad \text{Tr}(J^\dagger_i J^\dagger_j \bar{C}) = 0, \forall i, j \in n, \quad (E8)
\]
is equal to that of \( \min_{h, h, \beta} 4\text{Tr}(C^\dagger \alpha C^\circ) = \min_{h, h, \beta} \|\alpha\|, \) proving the optimality of \((C^\circ, \bar{C})\).

Appendix F: Highly-biased noise

We derived the optimal \( \mathfrak{F} \) and the corresponding optimal AQEC code under the highly-biased noise model, taking the limit \( \eta \to 0 \). Using the highly-biased model (Eq. (23)), we need to replace \( L \) by \((\Pi_\bar{s} + \Pi_\bar{s} \sqrt{\bar{\eta}}) L \) in Eq. (2), where \( \Pi_{\bar{s}(or \bar{s})} \) is an \( r \)-by-\( r \) diagonal matrix whose \( i \)-th diagonal element is one when \( i \in \bar{s} \) (or \( \bar{s} \)) and zero when \( i \in s \) (or \( s \)). After the following parameter transformation
\[
\mathbf{h} \leftarrow (\Pi_\bar{s} + \Pi_\bar{s} / \sqrt{\bar{\eta}}) \mathbf{h}, \quad \mathbf{b} \leftarrow (\Pi_\bar{s} + \Pi_\bar{s} / \sqrt{\bar{\eta}}) \mathbf{b}(\Pi_\bar{s} + \Pi_\bar{s} / \sqrt{\bar{\eta}}), \quad (F1)
\]
in Eq. (2), we have the optimal QFI is equal to \( \mathfrak{F}^\circ = \min_{h, h, \beta} \|\alpha\|, \) where
\[
\alpha = (h^\dagger \mathbb{1} + h^\circ \mathbf{L})(\Pi_\bar{s} + \Pi_\bar{s} / \eta)(h^\ast \mathbb{1} + h^\circ \mathbf{L}), \quad (F2)
\]
\[
\beta = H + h^\dagger \mathbf{L}^\dagger h^\ast \mathbf{L} + \mathbf{L}^\dagger h^\ast \mathbf{L} \quad (F3)
\]
Letting \( \bar{\alpha} = (h^\dagger \mathbb{1} + h^\circ \mathbf{L})\Pi_\bar{s}(h^\ast \mathbb{1} + h^\circ \mathbf{L}) \), we have
\[
\mathfrak{F}^\circ = \frac{1}{\eta} \min_{\bar{h}, h, \beta} \|\bar{\alpha}\| + O(1), \quad (F4)
\]
where \( \min_{h, h, \beta} \|\bar{\alpha}\| > 0 \) as long as \( H \notin \text{span}\{\mathbb{1}, L_i, L_i^\dagger, L_j^\dagger \} \).

Now consider the dual program of the modified version of Eq. (17) with \( \alpha \) replaced by \( \bar{\alpha} \). We first simplify the calculation by performing a gauge transformation such that the new set of Lindblad operators \( J \) satisfies \( \text{Tr}(C^\dagger J J^\dagger C) = 0 \), \( J^\dagger_{\bar{s}\bar{s}} \) is diagonal with the \( i \)-th diagonal element equal to \( \lambda_i \) when \( i \in \bar{s} \) and zero when \( i \in s \), and its Schur complement \( J^\dagger_{\bar{s}\bar{s}} = J^\dagger_{\bar{s}\bar{s}} J^\dagger_{\bar{s}\bar{s}} J^\dagger_{\bar{s}\bar{s}} \) is diagonal with the \( i \)-th diagonal element equal to \( \lambda_i \) when \( i \in \bar{s} \) and zero when \( i \in \bar{s} \). Here \( -1 \) means the Moore-Penrose pseudoinverse and we use the notations \((\cdot)^\dagger = \Pi_\bar{s}(\cdot)^\dagger \Pi_\bar{s} \) for \( \Pi_\bar{s} \) and \( \Pi_{\bar{s}} = \Pi_{\bar{s}} \Pi_\bar{s} \). Note that the gauge transformation here is divided into two steps (1) \( L_i \leftarrow L_i - \text{Tr}(C^\dagger L_i^\dagger C) \) and (2) \( L_i \leftarrow (u \Pi_{\bar{s}} + v \Pi_{\bar{s}}) \mathbf{L} \) where \( u = \Pi_{\bar{s}} u \Pi_{\bar{s}} \) and \( v = \Pi_{\bar{s}} v \Pi_{\bar{s}} \) are some unitary operators within the subspaces defined by \( \Pi_{\bar{s}} \) and \( \Pi_{\bar{s}} \). In this way, the solution is invariant. Again, we introduce a Hermitian matrix \( C \) as a Lagrange multiplier, the Lagrange function is
\[
L(\bar{C}, h, \mathbf{h}, \eta) = 4\text{Tr}(C^\dagger (h^\dagger \mathbb{1} + h^\circ \mathbf{J})^\dagger \Pi_\bar{s}(h^\dagger \mathbb{1} + h^\circ \mathbf{J})C) + \text{Tr}(\bar{C}(H + h^\dagger \mathbf{J} + J^\dagger h^\ast \mathbf{J} + J^\dagger J^\ast h^\ast \mathbf{J})). \quad (F5)
\]
Then we have

\[
\begin{align*}
\min_{\mathbf{h}, \mathbf{h}, \mathbf{h}} & \| \hat{\alpha} \| = \max_{\tilde{C}} \min_{\mathbf{h}, \mathbf{h}, \mathbf{h}} \mathcal{L}(\tilde{C}, \mathbf{h}, \mathbf{h}, \mathbf{h}) \\
= & \max_{\mathcal{C}} \min_{\mathbf{h}, \mathbf{h}, \mathbf{h}} 4(\mathbf{h}^\dagger \mathbf{h} + \mathcal{S}(\mathbf{h}^\dagger \mathbf{h} \mathbf{h}^T)) + \mathcal{S}(\mathbf{h}^\dagger \mathbf{h}^\dagger \mathbf{h} + \mathbf{h}^\dagger \mathbf{h} \mathbf{h}^T) \\
= & \max_{\mathcal{C}} \min_{\mathbf{h}, \mathbf{h}, \mathbf{h}} \left( -\frac{1}{4} \mathcal{S}(\mathbf{h}^\dagger \mathbf{h})^2 \mathbf{h} + \mathcal{S}(\mathbf{h}^\dagger \mathbf{h}) + \mathcal{S}(\mathbf{h}^T) \right),
\end{align*}
\]

and

\[
(\ast) = \min_{\mathbf{h}, \mathbf{h}, \mathbf{h}} 4 \mathcal{S}(\mathbf{h}^\dagger \mathbf{h} \mathbf{h}^\dagger \mathbf{h} + \mathbf{h}^\dagger \mathbf{h} \mathbf{h}^\dagger) + \mathcal{S}(\mathbf{h}^\dagger \mathbf{h} \mathbf{h}^\dagger) + \mathcal{S}(\mathbf{h}^\dagger \mathbf{h}^\dagger \mathbf{h}^\dagger) + \mathcal{S}(\mathbf{h}^\dagger \mathbf{h}^\dagger \mathbf{h}^\dagger) - \frac{\mathcal{S}(\mathbf{h}^\dagger \mathbf{h})^2}{4} \mathcal{S}(\mathbf{h}^\dagger \mathbf{h}) \mathcal{S}(\mathbf{h}^\dagger \mathbf{h})
\]

\[
= -\frac{\mathcal{S}(\mathbf{h}^\dagger \mathbf{h})^2}{4} \left( \sum_{i, i^\prime \in \mathbf{s}} \frac{|(\mathbf{h}^\dagger \mathbf{h} + \mathbf{h}^\dagger \mathbf{h}^\dagger)_{ii^\prime}}{2(\lambda_i + \lambda_{i^\prime})} + \sum_{i \in \mathbf{s}, j \in \mathbf{s}, \lambda_j \neq 0} \frac{|i_j|^2}{\lambda_j} \right),
\]

under the constraints that \( \Pi_s \mathbf{h} = 0 \), \( \Pi_s \mathbf{h} = 0 \),

\[
\Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0, \quad \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0,
\]

\[
\Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0, \quad \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0.
\]

where \( \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0 \), \( \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0 \), \( \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0 \), \( \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0 \).

To conclude, the dual program after replacing \( \alpha \) by \( \hat{\alpha} \) is equal to

\[
\max_{\mathcal{C}, \tilde{C}} \bar{\mathcal{F}}(\mathcal{C}, \tilde{C}), \text{ subject to } \mathcal{S}(\mathcal{C} \mathcal{C}) = 1, \mathcal{S}(\mathcal{C} \mathcal{C}) = 0, \mathcal{S}(\mathcal{C} \mathcal{C}) \neq 0,
\]

\[
\Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0, \quad \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0, \quad \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0, \quad \Pi_s \mathbf{J} + \Pi_s \mathbf{J} = 0.
\]

where

\[
\bar{\mathcal{F}}(\mathcal{C}, \tilde{C}) = \left( \mathbf{J} \mathbf{h} + \sum_{i, i' \in \mathbf{s}} \frac{|(\mathbf{h}^\dagger \mathbf{h} + \mathbf{h}^\dagger \mathbf{h}^\dagger)_{ii^\prime}}{2(\lambda_i + \lambda_{i^\prime})} + \sum_{i \in \mathbf{s}, j \in \mathbf{s}, \lambda_j \neq 0} \frac{|i_j|^2}{\lambda_j} \right)^{-1}.
\]

**Appendix G: The optimal AQEC code for correlated-dephasing noise**

In this appendix, we provide a non-perturbation QEC code achieving the optimal \( \bar{\mathcal{F}} \) in a correlated noise-dephasing noise model [32, 33]. We have \( N \geq 3 \) qubits evolving under

\[
\frac{d\rho}{dt} = -i\omega[\mathbf{w}, \mathbf{Z}, \rho] + \frac{1}{2} \sum_{j, k} (\mathbf{G}_{jk} \mathbf{Z}_j \rho \mathbf{Z}_k - \frac{1}{2} \{\mathbf{Z}_k \mathbf{Z}_j, \rho\})
\]

\[
= -i\omega[\mathbf{w}, \mathbf{Z}, \rho] + \sum_{j} \frac{\mu_j}{2} \left( (\mathbf{v}_j \cdot \mathbf{Z}) \rho (\mathbf{v}_j \cdot \mathbf{Z}) - \frac{1}{2} \{\mathbf{v}_j \cdot \mathbf{Z}^2, \rho\} \right),
\]

where \( \mathbf{Z}_i \) is the Pauli-Z operator on the \( i \)-th qubit, \( \mathbf{w} \) and \( \mathbf{v}_i \) are all unit vectors and \( \mathbf{G} = \sum_i \mu_i \mathbf{v}_i \mathbf{v}_i^T \) with \( \mu_i > 0 \) and \( \{\mathbf{v}_i\} \) an orthonormal set of vectors. The HLS condition \( \mathcal{H} \in \mathcal{S} \) is equivalent to \( \mathbf{w} \in \text{span}\{\mathbf{v}_i, \forall i\} \).

We first calculate the optimal normalized QFI

\[
4 \min_{\mathbf{h}, \mathbf{h}, \mathbf{h}, \mathbf{h}} \| \hat{\alpha} \| = 4 \mathbf{h}^\dagger \mathbf{h} = 2 \sum_i \frac{(\mathbf{v}_i \cdot \mathbf{w})^2}{\mu_i} = 2 \mathbf{w}^T \Gamma^{-1} \mathbf{w},
\]
where $-1$ means the Moore-Penrose pseudoinverse, and

$$
\beta = H + h\mathbb{1} + h\dagger h + L\dagger h + L\dagger h L
$$

$$
= w \cdot Z + \left( \mathbb{1} \sqrt{\frac{\mu_1}{2}} v_1 \cdot Z \right) \cdots \left( \mathbb{1} \sqrt{\frac{\mu_N}{2}} v_N \cdot Z \right) \begin{pmatrix}
\mathbb{1} \\
h \\
h \dagger
\end{pmatrix} \begin{pmatrix}
\sqrt{\frac{\mu_1}{2}} v_1 \cdot Z \\
\sqrt{\frac{\mu_2}{2}} v_2 \cdot Z \\
\vdots \\
\sqrt{\frac{\mu_N}{2}} v_N \cdot Z
\end{pmatrix}
$$

$$
= w \cdot Z + \sum_i (h_i + h_i\dagger) \sqrt{\frac{\mu_i}{2}} v_i \cdot Z = 0 \quad \Rightarrow \quad h_i = \frac{v_i \cdot w}{\sqrt{2\mu_i}}, \quad h = 0.
$$

(G3)

Now we introduce a QEC code

$$
|0_L\rangle = \bigotimes_{j=1}^N \left( \cos \theta_j |0_j\rangle + i \sin \theta_j |1_j\rangle \right), \quad |1_L\rangle = X^{\otimes N} |0_L\rangle,
$$

(G4)

where $\theta = \frac{1}{2} \arccos \chi u$, defined element-wise, satisfying

$$
P(v \cdot Z)P = v \cdot \cos(2\theta) Z_L = \chi(v \cdot u) Z_L,
$$

(G5)

and $P(v \cdot Z)(v' \cdot Z)P \propto P$ for any $v$ and $v'$. $\chi$ is a tunable parameter $\in (0, \|u\|^{-1}_\infty)$ where $\|\cdot\|_\infty$ is the infinity norm.

The QEC code is designed to correct every mode $v$ perpendicular to $u$. Using the recovery channel introduced in Appx. F of [33], we would have an effective channel

$$
\frac{d\rho}{dt} = -i[\omega \chi(u^T w) Z_L, \rho] + \frac{(u^T \Gamma u)\chi^2}{2} (Z_L \rho Z_L - \rho).
$$

(G6)

Using spin-squeezed states as input states, we could achieve the optimal $\mathfrak{F}$ because

$$
\mathfrak{F} = 4\chi^2(u^T w)^2 \leq 2w\Gamma^{-1} w,
$$

where the second equality holds when

$$
(u^T \Gamma u)(w^T \Gamma^{-1} w) = (u^T w)^2 \quad \Rightarrow \quad u \propto \Gamma^{-1} w.
$$

Note that $\mathfrak{F} = 2w\Gamma^{-1} w$ could be very large if there exist an $i$ such that $\mu_i \ll (v_i \cdot w)^2$. 