\textbf{L}^{2, \overline{\partial}}\text{-COHOMOLOGY GROUPS OF SOME SINGULAR COMPLEX SPACES}

NILS ØVRELID AND SOPHIA VASSILIADOU

Abstract. Let $X$ be a pure $n$-dimensional (where $n \geq 2$) complex analytic subset in $\mathbb{C}^N$ with an isolated singularity at $0$. In this paper we express the $L^2(0,q)\overline{\partial}$-cohomology groups for all $q$ with $1 \leq q \leq n$ of a sufficiently small deleted neighborhood of the singular point in terms of resolution data. We also obtain identifications of the $L^2(0,q)\overline{\partial}$-cohomology groups of the smooth points of $X$, in terms of resolution data, when $X$ is either compact or an open relatively compact complex analytic subset of a reduced complex space with finitely many isolated singularities.

1. Introduction

Let $X$ be a reduced pure $n$-dimensional complex analytic set in $\mathbb{C}^N$ with an isolated singularity at $0$ and let $X'$ denote the set of smooth points of $X$. Let $(z_1, \ldots, z_N)$ be the coordinates in $\mathbb{C}^N$, and set $|z| := (\sum_{j=1}^{N}|z_j|^2)^{\frac{1}{2}}$. The set of smooth points $X'$ inherits a Kähler metric from its embedding in $\mathbb{C}^N$, which we call the ambient metric. Due to the incompleteness of the metric there are many possible $L^2$-extensions of the $\overline{\partial}$-operator originally acting on smooth forms on $X$. We consider the maximal (distributional) $\overline{\partial}_{\text{max}}$-operator. For positive $r$ we let $B_r := \{z \in \mathbb{C}^N; \|z\| < r\}$, $X_r := X \cap B_r$, and $X'_r := X' \cap B_r$. We shall choose an $R > 0$ small enough, so that $bB_r$ intersects $X$ transversally for all $0 < r < R$. Unless otherwise noted in what follows by $\overline{\partial}$ we shall mean $\overline{\partial}_{\text{max}}$. We define the local (resp. global) $L^2\overline{\partial}$-cohomology groups

$$H^{p,q}_{(2)}(X'_r) := \frac{\ker (\overline{\partial}) \cap L^p_{(2)}(X'_r)}{\Im (\overline{\partial}) \cap L^p_{(2)}(X'_r)},$$

(resp. $H^{p,q}_{(2)}(X') := \frac{\ker (\overline{\partial}) \cap L^p_{(2)}(X')}{\Im (\overline{\partial}) \cap L^p_{(2)}(X')}$.)

In [10] we showed that the above local $L^2\overline{\partial}$-cohomology groups are finite dimensional when $p + q < n$ and $q > 0$ and zero when $p + q > n$. The idea of the proof in the case $p + q < n$, was based on constructing complete Kähler metrics to obtain a weighted $L^2$-solution for square-integrable, $\overline{\partial}$-closed forms on $X'_r$, with compact support on $X_r$ and identifying the obstructions to solving $\overline{\partial}u = f$ on $X'_r$ to certain $L^2\overline{\partial}$-cohomology groups of “spherical shells” around 0. Sharp regularity results for $\overline{\partial}$ (which could yield finite dimensionality results for the above cohomology groups when $p + q \leq n - 2$, $q > 0$) have been obtained by Pardon and Stern for projective varieties with isolated singularities in [31]. We also presented in [10] various sufficient conditions on the complex analytic set to guarantee that the local $L^2\overline{\partial}$-cohomology groups vanish. Our results were most complete when 0 was an isolated singular point in a hypersurface $X$ and when $p + q \leq n - 1$, $1 \leq q \leq n - 2$ ($n \geq 3$). In [27] we proved finite dimensionality of $H^{n-1,1}_{(2)}(X'_r)$ using a global finite dimensionality result of $L^2\overline{\partial}$-cohomology groups on projective varieties with arbitrary singularities.

All of the results in [10] were obtained while working on the original singular space. This paper started as an attempt to provide a short proof of the finite dimensionality of $L^2$-Dolbeault cohomology groups of...
complex spaces with isolated singularities by passing to an appropriate desingularization of $X$. The second author had presented such results in conference talks since 2006. Since then, new techniques have evolved to describe the $L^2$-cohomology groups in some cases (see the work of Ruppenthal $[33,34]$ that deals with cones over smooth projective varieties and his most recent preprint $[35]$). Using results from earlier papers of ours, some classical theorems from algebraic geometry and singularity theory and some key observations from $[35]$ and $[30]$, we were able to obtain a rather complete description of both local and global (the latter result when $X$ is compact or open relatively compact complex analytic set in a reduced complex space with finitely many isolated singularities in $X$) $L^2$-cohomology groups on $X_{\xi}$ or $X'$ in terms of resolution data. Earlier work of Pardon (section 4 in $[29]$) indicated the importance of such descriptions in understanding birational invariants of singular projective varieties.

The first main result in the paper is the following theorem:

**Theorem 1.1.** Let $X$ be a complex analytic subset of $\mathbb{C}^N$ of pure dimension $n \geq 2$ with an isolated singularity at $0$, and let $\pi : \tilde{X} \to X$ be a desingularization. Then, there exists a well-defined, linear mapping $\phi_* : H^q(\tilde{X}, \mathcal{O}) \to H^q_{(2)}(X')$ such that $\phi_*$ is bijective if $1 \leq q \leq n - 2$ and injective if $q = n - 1$. Here $\tilde{X} = \pi^{-1}(X_r)$ and $X_r := X \cap \{z \in \mathbb{C}^N ; ||z|| < r\}$.

The above theorem generalizes results of Ruppenthal in $[34]$. In that paper, he considered affine cones over smooth projective varieties. For these varieties the exceptional locus of a desingularization is a smooth submanifold of $\tilde{X}$. We impose no such restriction on the exceptional locus of the desingularization. Key ingredient in the proof of Theorem 1.1 is a theorem of Stephen Yau and Ulrich Karras ($[12,19]$) that describes the local cohomology along exceptional sets. For complex analytic subsets of $\mathbb{C}^N$ with an isolated singular point, the exceptional locus of a desingularization is an exceptional set in the sense of Grauert (see part $\alpha$) in the Characterization of exceptional sets in section 3.1).

The cokernel of the map $\phi_*$ will play a prominent role in the paper. As we mentioned earlier, due to the incompleteness of the metric, there are many $L^2$-extensions of the $\partial$-operator acting on smooth forms on $X'$. So far we have been considering the maximal (distributional) extension. We can also consider the $L^2$-closure of $\partial$ acting on forms with coefficients in $C^\infty_0(\tilde{X} \setminus \{0\})$. Let us denote this extension by $\partial^1$. We shall see in section 4, that the cokernel of $\phi_*$ (or more precisely the dual of it) measures somehow the obstructions to having $\partial_{\text{max}} = \partial^1$ at the level of holomorphic $(n,0)$-forms.

In January of 2010, we became aware of a recent preprint of Ruppenthal that appeared at the Erwin Schrödinger Institute preprint series. Its purpose was to describe explicitly the $L^2$-$\partial$-cohomology of compact complex spaces in terms of resolution data and thus answer a conjecture by MacPherson on the birational invariance of the $L^2$-Euler characteristic of projective varieties. After having seen his preprint and using lemma 6.2 from $[35]$, we were able to strengthen Theorem 1.1. More precisely we show the following:

**Theorem 1.2.** Let $X$ be a complex analytic subset of $\mathbb{C}^N$ of pure dimension $n \geq 2$ with an isolated singularity at 0. Let $\pi : \tilde{X} \to X$ be a desingularization such that the exceptional locus $E$ of $\pi$ is a simple, normal crossings divisor. Let $Z = \pi^{-1}(\text{Sing}X)$ be the unreduced exceptional divisor of the resolution, let the support of $Z$ be denoted by $|Z| := E$ and let $D := Z - |Z|$. Then, there exists a natural surjective linear map

$$T : H^{n-1}(\tilde{X}, \mathcal{O}(D)) \to H^{0,n-1}_{(2)}(X')$$

whose kernel is naturally isomorphic to $H^{n-1}_{\partial E}(\tilde{X}, \mathcal{O}(D))$. Here $H^{n-1}_{\partial E}(\tilde{X}, \mathcal{O}(D))$ means cohomology with support on $E$.

As a corollary of theorem 1.2, we recover Theorem 7.1 from $[35]$ (for $q = n - 1$). This theorem asserts that when the line bundle associated to the divisor $-D = |Z| - Z$ is locally semi-positive with respect to $X$, then $H^q_{(2)}(X') \cong H^q(\tilde{X}, \mathcal{O}(D))$ for all $0 \leq q \leq n$. Indeed, using Serre duality and Takegoshi’s
twisted vanishing theorem (Torsion freeness of the main theorem in the introduction of [11]), we see that $H^n_c(\tilde{X}_r, \mathcal{O}(D)) = 0$ in this case. A result by Karras will guarantee the isomorphism between $H^{n-1}_c(\tilde{X}_r, \mathcal{O}(D))$ and $H^{n-1}(\tilde{X}_r, \mathcal{O}(D))$, which combined with Theorem 1.2 will yield the desired isomorphism $H^{0,n-1}_{(2)}(X'_r) \cong H^{n-1}(\tilde{X}_r, \mathcal{O}(D))$. We can also recover Ruppenthal’s result for all $q \leq n - 2$ (see Remark 4.5.1 in section 4).

In order to prove Theorem 1.2, we construct a non-degenerate pairing

$$
\frac{H^{0,n-1}_{(2)}(X'_r)}{\ker(\bar{D})^n,0} \times \frac{\ker(\bar{D})^n,0}{\mathbb{C}}
$$

where $\bar{D}$ is as above. In [10] we showed that the map $j_* : H^{0,n-1}_{(2)}(X'_r) \to H^{n-1}(X'_r, \mathcal{O})$ induced by the natural inclusion $j : L^{0,n-1}_{(2)}(X'_r) \to L^{0,n-1}(X'_r)$ is injective. An understanding of the Im $j_*$ will turn out to be instrumental in the construction of the map $T$. We will therefore present some necessary and sufficient conditions to describe elements in Im $j_*$. (using Lemma 6.2 in [35] and [11]). Now, there exists a natural map $\ell_* : H^{n-1}(\tilde{X}_r, \mathcal{O}(D)) \to H^{n-1}(X'_r, \mathcal{O})$. Using a twisted version of an $L^2$-Cauchy problem we will show that Im $j_*$ $\subset$ Im $\ell_*$ and construct a map $S : H^{0,n-1}_{(2)}(X'_r) \to H^{n-1}(\tilde{X}_r, \mathcal{O}(D))$. Then the proof of Theorem 1.2 will be based on the following key observation: the map $\ell_*$ is surjective on the Im $j_*$. The composition $j_*^{-1} \circ \ell_*$ will be the desired map $T$ and $T = S = Id$.

When $q = n$, we can easily show that the map $\phi_* : H^n(\tilde{X}_r, \mathcal{O}) \to H^{0,n}_{(2)}(X'_r)$ described by $\phi_*([g]) = [(\pi^{-1})^*g]$ is surjective. Since $\tilde{X}_r$ contains no compact $n$-dimensional irreducible components, by Siu’s theorem (37) we have $H^n(\tilde{X}_r, \mathcal{O}) = 0$. Hence, $H^{0,n}_{(2)}(X'_r) = 0$.

With a little bit more work, we can obtain global versions of Theorems 1.1 and 1.2. More precisely, let $X$ be a pure $n$-dimensional, relatively compact domain in a reduced complex analytic space $Y$. We give Reg $Y$ a hermitian metric compatible with local embeddings. Assume that $X \cap \text{Sing } Y := \Sigma = \{a_1, a_2, \ldots, a_m\} \subset X$. Let $\pi : \hat{Y} \to Y$ be a desingularization such that $E = \pi^{-1}(\Sigma)$ is a normal crossings reduced divisor in $\hat{X} = \pi^{-1}(X)$. Let $Z := \pi^{-1}(\Sigma)$ be the unreduced exceptional divisor and $D := Z - E$. Give $\hat{Y}$ a non-degenerate hermitian metric. Let $H^{0,q}_{(2)}(\hat{X}, \mathcal{O}(D))$ denote the $L^2$-cohomology of $(0,q)$-forms in $\hat{X}$ with values in $L_D$, the holomorphic line bundle associated to the divisor $D$ (see Remark 2.2.2 in section 2). Then we have

**Theorem 1.3.** The map $\phi_* : H^{0,q}_{(2)}(\hat{X}) \to H^{0,n}_{(2)}(X'_r)$, defined by $\phi_*([f]) = [(\pi^{-1})^*f]$ is an isomorphism, when $1 \leq q \leq n - 2$ and where $X'_r := X \setminus \Sigma$.

**Theorem 1.4.** There exists a natural surjective map $\hat{T} : H^{0,n-1}_{(2)}(\hat{X}, \mathcal{O}(D)) \to H^{0,n-1}_{(2)}(X'_r)$, whose kernel is naturally isomorphic to $H^{n-1}_{(2)}(\hat{X}, \mathcal{O}(D))$ and where $X'_r := X \setminus \Sigma$.

Let us point out that in the most interesting cases, i.e. when $X$ is compact or $\partial X$ is smooth, strongly pseudoconvex submanifold of Reg $Y$, we have $H^{0,q}_{(2)}(\hat{X}, \mathcal{O}(D)) \cong H^q(\hat{X}, \mathcal{O}(D))$ for $q > 0$ and $F$ any holomorphic line bundle, so Theorems 1.1 and 1.2 carry over verbatim. In [35], Ruppenthal proved (Theorem 1.6) that when the line bundle associated to the divisor $-D$ is locally semi-positive with respect to $X$, then $H^{0,n-1}_{(2)}(X'_r) \cong H^{n-1}(\hat{X}, \mathcal{O}(D))$. This follows from Theorem 1.4 taking into account Takegoshi’s or Silva’s relative vanishing theorem and Karras’ results. For projective surfaces with isolated singularities, we can say more:

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1For a proper, generically finite to one holomorphic map $p : X \to Y$ where $X$ is a complex connected manifold and $Y$ is a reduced analytic space, this relative vanishing theorem was already known to A. Silva (see A.2 Lemma in [36]).

2More precisely, Theorem 1.6 in [35], states that when the line bundle associated to the divisor $-D$ is locally semi-positive with respect to $X$, then for all $q$, with $0 \leq q \leq n$ one has $H^{0,q}_{(2)}(X'_r) \cong H^q(\hat{X}, \mathcal{O}(D))$. 

Corollary 1.5. Let $X$ be a projective surface with finitely many isolated singularities. Then the map $T : H^{0,1}_0(\tilde{X}, \mathcal{O}(D)) \to H^{1,0}_0(X')$ is an isomorphism (the right-hand side $L^2$-cohomology is computed with respect to the restriction of the Fubini-Study metric in $X'$).

This Corollary was first conjectured by Pardon in [29], while studying MacPherson’s conjecture. It appeared later as a special case of Theorem B in [30]. A key observation from the Appendix in [30] along with Theorem 1.4 will help us settle Pardon’s conjecture in the case of projective surfaces with isolated singularities and bypass the difficulties that were encountered with the proof of Theorem B in [30]. It would be interesting to determine whether the kernel of $T$ vanishes for higher dimensional projective varieties with an isolated singularity (Professor Kollár offered some insight on when this vanishing could occur; see Remark 5.2.3 in section 5). In that case the global cohomology group $H^{0,n-1}_0(X')$ would be isomorphic to $H^{n-1}(\tilde{X}, \mathcal{O}(D))$.

Correspondingly, this $L^2$-Dolbeault cohomology group would not be a birational invariant.

Now we follow the assumptions and notation as in the paragraph just above Theorems 1.3, 1.4 and consider the case where $X$ is compact or $\partial X$ is smooth strongly pseudoconvex submanifold of Reg $Y$. The map $\phi^n _0 : H^n(X, \mathcal{O}) \to H^{0,n}_0(X')$ defined by $\phi^n ([f]) = [(\pi^{-1})^* f]$ is easily seen to be surjective. Let $i^n _0 : H^n(\tilde{X}, \mathcal{O}) \to H^n(\tilde{X}, \mathcal{O}(D)) \cong H^{0,n}_0(\tilde{X}, L_D)$ be the map on cohomology induced by the sheaf inclusion $i : \mathcal{O} \to \mathcal{O}(D)$. We will show

Corollary 1.6. With $X, \tilde{X}, D, \phi^n _0, i^n _0$ as above we have $\ker (\phi^n _0) = \ker (i^n _0)$ and $H^{0,n}_0(X') \cong H^n(\tilde{X}, \mathcal{O}(D))$.

The kernel of $i^n _0$ can be computed using standard long exact sequences on cohomology and cohomology with support on $E$. Thus one of the benefits of the above Corollary is that it allows us to describe the kernel of $\phi^n _0$ which in some sense measures the difference between the $L^p_{(2), \bar{\partial}_{min}}$-cohomology group on $X'$ (which is isomorphic to $H^n(\tilde{X}, \mathcal{O})$), and the corresponding cohomology group using the $\bar{\partial}_{max}$-operator (i.e. $H^p_{(2)}(X')$).

The organization of the paper is as follows: Apart for some preliminaries, in section 2 we will give a short proof of the finite dimensionality of $L^p_{(2), \bar{\partial}}$-cohomology groups of small deleted neighborhoods of the singular point 0. In section 3 we prove Theorem 1.1. Section 4 contains the proof of Theorem 1.2. Section 5 contains the proofs of the global theorems and section 6, the identification of $H^{0,n}_0(X')$ with $H^n(\tilde{X}, \mathcal{O}(D))$.

In section 7 we discuss the vanishing or not of some local $L^2$-$\bar{\partial}$-cohomology groups of some complex spaces $X$ with isolated singularities.

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2. Preliminaries

2.1. Desingularization and pull-back metrics. Our results in [10] were obtained while working mostly on the original singular space. However, we can desingularize $X$, i.e. consider a proper, holomorphic, surjective map $\pi : \tilde{X} \to X$ such that $\tilde{X}$ is smooth, $\pi : \tilde{X} \backslash E \to X \backslash \text{Sing}X$ is a biholomorphism and $E = \pi^{-1}(\text{Sing}X)$ is a divisor with normal crossings (we only need this extra condition on the exceptional
locus for the proof of Theorems 1.2 and 1.4). Since the singular locus of \( X \) consists of one point we can cover \( E \) by finitely many coordinate charts \((U_i, z)\) with \( i = 1, \ldots, M \) and near each \( x_0 \in E \) we can find local holomorphic coordinates \((z_1, \ldots, z_n)\) in terms of which \( E \cap U_i \) is given by \( h_i(z) = z_1 \cdots z_{n_i} = 0 \), where \( 1 \leq n_i \leq n \).

Let \( \sigma \) be a positive definite metric on \( \tilde{X} \). We can then consider, volume element \( dV_\sigma \) and pointwise norms/norms on \( \Lambda T\tilde{X} \) and \( \Lambda T^*\tilde{X} \). For every open subset \( U \) of \( \tilde{X} \), let \( \mathcal{L}^{p,q}(U) \) be

\[
\mathcal{L}^{p,q}(U) := \{ u \in L^{p,q}_{2,\text{loc}}(U) ; \; \overline{\partial}u \in L^{p,q+1}_{2,\text{loc}}(U) \}
\]

and for each open subset \( V \subset U \), let \( r^U_V : \mathcal{L}^{p,q}(U) \to \mathcal{L}^{p,q}(V) \) be the obvious restriction maps. Here square-integrability is with respect to the metric \( \sigma \). Then the map \( u \to \overline{\partial}u \) defines an \( \mathcal{O}_X \)-homomorphism \( \overline{\partial} : \mathcal{L}^{p,q} \to \mathcal{L}^{p,q+1} \) and the sequence

\[
0 \to \mathcal{O}_X \to \mathcal{L}^{p,0} \to \mathcal{L}^{p,1} \to \cdots \to \mathcal{L}^{p,n} \to 0
\]

is exact by the local Poincaré lemma for \( \mathcal{O}_X \). Since each \( \mathcal{L}^{p,q} \) is closed under multiplication by smooth cut-off functions we have a fine resolution of \( \mathcal{O}_X \).

We introduce some notational convention: For the manifold \( \tilde{X}, \gamma \) will always denote a positive semi-definite hermitian metric on \( \tilde{X} \), which is generically definite. More specifically in this paper we shall let \( \gamma \) denote the pull-back of the ambient metric on \( X' \). It degenerates along a divisor \( D' \), supported on the exceptional divisor \( E \). One is faced with the hard task of understanding how the pull back of the ambient metric looks like on \( \tilde{X} \). This has been done by Hsiang-Pati [18], Nagase [20] for projective surfaces with isolated singularities and recently by Taalman [39] (following an idea of Pardon and Stern [32]) for three-dimensional projective varieties with isolated singularities. Youssin in [13] considered desingularizations \((X, \pi)\) of \( X \) that factor through the Nash blow-up of \( X \) and found a way to describe the pull-back of forms defined on \( X' \) with measurable coefficients and square-integrable with respect to the ambient metric, in terms of data on \( \tilde{X} \). Similar descriptions of such forms for projective surfaces with isolated singularities appeared in the 1997 preprint of Pardon and Stern [32].

2.2. Locally free sheaves and twisted cohomology groups. Let us consider any effective divisor \( D = \sum_{i=1}^m d_i E_i \) where \( d_i \in \mathbb{N} \) and where \( \{ E_i \}_{i=1}^m \) are the irreducible components of \( E = \pi^{-1}(\text{Sing} \ X) \). By \( \mathcal{O}(D) \) we denote the sheaf of germs of meromorphic functions \( f \) such that \( \text{div} (f) + D \geq 0 \). If \( \{ U_a \} \) is a covering of \( \tilde{X} \) and \( u_a \) is a meromorphic function on \( U_a \) such that \( \text{div} (u_a) = D \) on \( U_a \), then \( \mathcal{O}(D)|_{U_a} = u_a^{-1} \mathcal{O} \).

Hence \( \mathcal{O}(D) \) is a locally free sheaf of rank 1. This sheaf can be identified with the sheaf of sections of a line bundle \( \mathcal{L}_D \) over \( \tilde{X} \) defined by the cocycle \( g_{ab} := \frac{u_a}{u_b} \in \mathcal{O}^*(U_a \cap U_b) \). In fact there is a sheaf homomorphism \( \mathcal{O}(D) \to \mathcal{O}(\mathcal{L}_D) \) defined by

\[
\mathcal{O}(D)(W) \ni f \to sf \in \mathcal{O}(\mathcal{L}_D)(W) \quad \text{with} \quad \theta_a(sf) = f u_a \quad \text{on} \quad W \cap U_a
\]

where \( \theta_a \) is the corresponding trivialization of \( L_{D|U_a} \). The constant function \( f = 1 \) induces a meromorphic section \( s \) of \( \mathcal{L}_D \) and its zero set \( s^{-1}(0) \) is the support of \( D \), usually denoted by \( |D| \). Hence, we can identify sections in \( \Gamma(U, \mathcal{O}(D)) \) with sections in \( \Gamma(U, \mathcal{O}(\mathcal{L}_D)) \) via the isomorphism \( f \to f \otimes s \). The inverse of this map is given by taking any section \( \Gamma(U, \mathcal{O}(\mathcal{L}_D)) \ni A \to A \cdot s^{-1} \in \Gamma(U, \mathcal{O}(D)) \).

Locally this map is described by sending \( A = u \otimes e \to s^{-1}(e) u \), where \( e \) is a local holomorphic frame for \( \mathcal{O}(\mathcal{L}_D) \) and \( s^{-1} \) is a meromorphic section of \( L_{-D} \cong \mathcal{L}_D^* \) (the dual of \( \mathcal{L}_D \)) satisfying \( s \cdot s^{-1} := s^{-1}(s) = 1 \).

For any open set \( U \subset \tilde{X} \) we set

\[
L^{p,q}_{2,\text{loc}}(U, \mathcal{O}(D)) := \{ f \in L^{p,q}_{2,\text{loc}}(U \setminus E) \mid \chi f \in L^{p,q}_{2,\text{loc}}(V) \quad \text{for all} \quad V^\text{open} \subset U \quad \text{and} \quad \forall \chi \in \mathcal{O}(-D)(V) \}.
\]

Remark 2.2.1 In principle one could define \( L^{p,q}_{2,\text{loc}}(U, \mathcal{O}(D)) \) to consist of all forms \( f \in L^{p,q}_{2,\text{loc}}(U \setminus |D|) \) such that \( \chi f \in L^{p,q}_{2,\text{loc}}(V) \) for all \( V^\text{open} \subset U \) and \( \forall \chi \in \mathcal{O}(-D)(V) \). But then for points \( x \in E \setminus |D| \)
one sees that \( f \) extends as an \( L^p_{(2)} \) form across these points. Hence, we do not lose any information by defining \( L^p_{(2),\text{loc}}(U, \mathcal{O}(D)) \) the way we did before the Remark.

Similarly, for a sufficiently small, relatively compact open neighborhood \( U \) of \( E \) in \( \tilde{X} \), one can define the following spaces

\[
L^{p,q}_{(2)}(U, \mathcal{O}(D)) := \{ f \in L^p_{2,\text{loc}}(U \setminus E) \mid u_a f \in L^p_{(2)}(U \cap U' \setminus E) \text{ for all } a \}
\]

where \( \{U'_a\} \) is a finite open covering of \( E \) and if \( D = \sum d_j E_j \), then \( u_a := \prod g_{j,a} \). In the above definition, square-integrability is with respect to any non-degenerate metric \( \sigma \) on \( \tilde{X} \). It is clear from the definitions that for such a \( U \) we have: \( L^{p,q}_{(2)}(U, \mathcal{O}(D)) \rightarrow L^{p,q}_{(2),\text{loc}}(U, \mathcal{O}(D)) \). Using a partition of unity \( \{ \rho_a \} \) subordinate to the covering \( \{U'_a\} \), we can define a norm on this space:

\[
\|f\|_{L^{p,q}_{(2)}(U, \mathcal{O}(D))} := \left( \int_U \sum_a \rho_a |u_a f|^2 dV \right)^{\frac{1}{2}}
\]

This definition seems to depend on the covering \( \{U'_a\} \), the partition of unity \( \{ \rho_a \} \), and the choice of the local defining function for the divisor \( D \). Since \( \overline{U} \) is bounded, by passing to a slightly smaller covering of \( U \), we will see that the corresponding norms, if we choose different coverings, defining functions for \( D \) and partitions of unity, would be equivalent.

Now the map \( U \rightarrow L^{p,q}(\mathcal{O}(D))(U) := \{ f \in L^p_{2,\text{loc}}(U, \mathcal{O}(D)) \text{ such that } \overline{f} f \in L^p_{2,\text{loc}}(U, \mathcal{O}(D)) \} \) (here \( \overline{f} \) is with respect to open subsets of \( \tilde{X} \setminus E \)) is a fine sheaf on \( \tilde{X} \) and

\[
0 \to \Omega^p_X \otimes \mathcal{O}(D) \to L^{0,0}(\mathcal{O}(D)) \overset{\overline{\partial}}{\to} L^{0,1}(\mathcal{O}(D)) \overset{\overline{\partial}}{\to} \cdots \overset{\overline{\partial}}{\to} L^{p,q}(\mathcal{O}(D)) \to 0
\]

is a fine resolution of \( \Omega^p_X \otimes \mathcal{O}(D) \). To see this we can argue as follows: For \( x \in U_a \), the maps of germs \( f_x \to (u_a f_x) \otimes u_a^{-1} \) from \( L^{p,q}(\mathcal{O}(D))_x \to L^{p,q}_x \otimes \mathcal{O}_x, \mathcal{O}(D)_x \) are independent of \( a \), where \( L^{p,q} \) are defined in section 2.1. These maps of germs define sheaf isomorphisms \( L^{p,q}(\mathcal{O}(D)) \to L^{p,q} \otimes \mathcal{O}(D) \), commuting with \( \overline{\partial} \) and \( \overline{\partial} \otimes \text{Id} \) respectively. Moreover, the operation \( - \otimes \mathcal{O}(D) \) preserves exact sequences, since \( \mathcal{O}(D) \) is a locally free sheaf over \( \mathcal{O} \). Hence the cohomology of \( \left( \Gamma(\tilde{X}_r, \mathcal{L}^{p,q}(\mathcal{O}(D))), \overline{\partial} \right) \) is \( H^*(\tilde{X}_r, \Omega^p \otimes \mathcal{O}(D)) \) for any \( p \geq 0 \).

### 2.2.1. An alternative characterization of \( L^{p,q}_{(2)}(U, \mathcal{O}(D)) \)

In section 4 of this paper we would need another realization of \( L^{p,q}_{(2)}(U, \mathcal{O}(D)) \) for \( U \) a smoothly bounded strongly pseudoconvex neighborhood of \( E \) in \( \tilde{X} \). We would like to identify this space with the square-integrable sections of \( \wedge^p q T^* \tilde{X} \otimes L_D \) over \( U \), where \( L_D \) is the holomorphic line bundle associated to the divisor \( D \). We would also need in section 4, some general results about differential operators acting on sections of holomorphic line bundles, cohomology groups with coefficients in line bundles etc. In this section we will systematically discuss these notations. Let \( \tilde{X} \) be given a non-degenerate metric \( \sigma \) and let \( F \) be a holomorphic line bundle endowed with a Hermitian metric \( h \). Let \( C^\infty_{p,q}(U, F) := C^\infty(\mathcal{U}, \wedge^p q T^* \tilde{X} \otimes F) \) denote the space of smooth \((p, q)\)-forms in \( U \) with coefficients in \( F \), \( C^\infty_{p,q}(\mathcal{U}, F) \) denote the smooth up-to the boundary of \( U \), \((p, q)\)-forms with coefficients in \( F \) and let \( D^{p,q}(U, F) \) the compactly supported sections with coefficients in \( F \). Using a trivialization \( \theta_U : F_U \rightarrow U \times \mathbb{C}^n \) we can choose a frame \( e(x) := \theta_U^{-1}(x, 1) \) of \( F \). Locally for each \( x \in U \), any element \( A \in C^\infty(U, F) \) can be written as \( A = \phi \otimes e \) in a smaller neighborhood \( W \subset U \) of \( x \) where \( \phi \in C^\infty(W) \) and \( e \in \mathcal{O}(F)(W) \). Let \( \tau : F \rightarrow F^* \) be the conjugate-linear isomorphism of \( F \) onto its dual \( F^* \) defined by

\[3\]
\[ \tau(e)(e') := h(e', e) \] whenever \( e, e' \in F_x \). The dual bundle \( F^* \) is given the metric \( h^* := h^{-1} \) that makes \( \tau \) an isometry. Then we can define the generalized Hodge-star-operator

\[ \overline{\tau}_F : C^\infty_{p,q}(U, F) \to C^\infty_{n-p, n-q}(U, F^*) \]

(2) \[ \overline{\tau}_F(\phi \otimes e) = \overline{\tau} \phi \otimes \tau(e) \]

where \( \phi \in \wedge^{p,q}T^*_x U \) and \( e \in F_x \).

For sections \( A \in C^\infty_{p,q}(U, F) \) we can easily check that the following equality holds: \( \overline{\tau}_F, \overline{\tau}_F A = (-1)^{p+q} A \), where \( \overline{\tau}_F \) is the Hodge-star operator associated to \( F^* \).

We can also define a wedge product \( \wedge : C^\infty_{p,q}(U, F) \times C^\infty_{r,s}(U, F^*) \to C^\infty_{p+r, q+s}(U, C) \) described by

\[ (\phi \otimes e) \wedge (\psi \otimes f) = \phi \wedge \psi f(e) \]

(3) where \( A := \phi \otimes e \) and \( B := \psi \otimes f \) are the local descriptions of two sections \( A \in C^\infty_{p,q}(U, F) \) and \( B \in C^\infty_{r,s}(U, F^*) \) and where \( e, f \) are local frames for \( F, F^* \) respectively.

Using the metric \( \sigma \) on \( X \), the hermitian metric \( h \) on \( F \) and the local description of elements in \( C^\infty_{p,q}(U, F) \) we can define a pointwise inner product for two elements \( A, B \in C^\infty_{p,q}(U, F) \)

\[ \langle A, B \rangle_{F,x} = h(e, e) \quad \langle \phi, \psi \rangle_{\sigma, x} \]

(4) where \( A = \phi \otimes e \) and \( B = \psi \otimes e \) in a small neighborhood \( W \subset U \) of \( x \) and \( , \rangle_{\sigma, x} \) is the standard pointwise inner product on \( X \) arising from the metric \( \sigma \). By integrating with respect to the volume element \( dV_x \) we obtain a global \( L^2 \) inner product on \( U \).

For any two sections \( A, B \in C^\infty_{p,q}(U, F) \) given locally by \( A = \phi \otimes e \) and \( B = \psi \otimes e \) with \( \phi, \psi \) smooth \((p,q)\)-forms in smaller neighborhood of \( x \) we have

\[ A \wedge \overline{\tau}_F B = \phi \wedge \overline{\tau} \psi (\tau(e))(e) = h(e, e) \quad \langle \phi, \psi \rangle_{\sigma, x} dV = \langle A, B \rangle_{F,x} dV \]

As before, we obtain a global inner product on sections in \( C^\infty_{p,q}(U, L_D) \) given by

(5) \[ \langle A, B \rangle_{F,D} = \int_U A \wedge \overline{\tau}_F B. \]

Let \( \overline{\partial}_F = \overline{\partial} \otimes Id : C^\infty_{p,q}(U, F) \to C^\infty_{p,q+1}(U, F) \). Then we can define the formal adjoint

\[ \partial_F : C^\infty_{p,q}(U, F) \to C^\infty_{p,q-1}(U, F) \]

via the identity \( \partial_F := -\overline{\tau}_F, \overline{\partial}_F, \overline{\tau}_F \), where by \( \overline{\tau}_F \) we denote the \( \overline{\partial} \) operator associated to the \( F^* \).

Let \( L^{p,q}_{(2)}(U, F) \) denote the completion of \( D^{p,q}(U, F) \) under the inner product defined above. This completion is independent of the choice of the bundle metric \( h \), with different choices of metrics leading to equivalent inner products. The wedge product, inner product, the generalized Hodge \( \overline{\tau}_F \) operator defined earlier for smooth sections, extend naturally to square-integrable sections. One also obtains various extensions of the operators \( \overline{\partial}_F, \partial_F \) on \( L^{p,q}_{(2)}(U, F) \) just as in the case of complex-valued forms. By abuse of notation we shall denote the weak extension of \( \overline{\partial}_F \) on \( L^{n-q}_{(2)}(U, F) \) by \( \overline{\partial}_F \) (instead of the cumbersome \( (\overline{\partial}_F)_w \)), the minimal extension of \( \overline{\partial}_F \) by \( \overline{\partial}_{F, min} \), the weak extension of \( \partial_F \) on \( L^{n-q}_{(2)}(U, F) \) by \( \partial_F, h \) (instead of \( (\partial_F)_w \)) and finally \( \overline{\partial}_{F, h} \) will denote the Hilbert space adjoint of \( \overline{\partial}_F \). Let
Remark 2.2.2 In sections 4 and 6 of the paper we will be considering forms with coefficients in line bundles \( F \) that arise from various divisors \( D \) on \( \tilde{X} \) (i.e. \( F = L_D \) for various divisors \( D \)). There exists a map

\[
L^{p,q}_{(2)}(U, O(D)) \to L^{p,q}_{(2)}(U, L_D)
\]

(6)

which is easily seen to be a bicontinuous isomorphism between \( L^{p,q}_{(2)}(U, O(D)) \equiv L^{p,q}_{(2)}(U, L_D) \). The inverse to the above map is given by sending an \( L^{p,q}_{(2)}(U, L_D) \) \( \ni A \to A \cdot s^{-1} \), where \( s, s^{-1} \) were defined in the first paragraph of section 2.2. Based on this remark, in subsequent sections we will be tacitly identifying \( H^{p,q}_{(2)}(U, O(D)) \) and \( H^{p,q}_{(2)}(U, L_D) \).

In section 4 of the paper we shall need a generalized density lemma and closed-range property for \( \overline{\mathcal{D}}_\mathcal{D}(\tilde{U}) \) (i.e. the \( \mathcal{D} \) operator associated to the line bundle \( L_D^p \equiv L_D \) for some divisor \( D \)). To simplify notation, we will consider a holomorphic line bundle \( F \) over \( \tilde{X} \) and a hermitian metric \( h \) on it that is smooth up to \( \tilde{U} \). Consider the \( \overline{\mathcal{D}}_F, \overline{\mathcal{D}}_F^* \) operators, defined in an analogous manner as before.

Lemma 2.1. The space \( C^\infty((\tilde{U}, F) \cap \text{Dom}(\overline{\mathcal{D}}_{F,h}) \) is dense in the Dom(\( \overline{\mathcal{D}}_F \) \cap Dom(\( \overline{\mathcal{D}}_{F,h} \) \cap L^{p,q}_{(2)}(U, F) \) for the graph norm \( A \to ||A|| + ||\overline{\mathcal{D}}_F A|| + ||\overline{\mathcal{D}}_{F,h} A|| \).

Proof. By a partition of unity argument, it is enough to consider sections supported by \( \overline{U} \cap V \), where \( V \) is a small coordinate chart over which we have a local holomorphic trivialization \( e \) of \( F \). Writing \( h(e, e) = e^{-\psi} \) on \( V \), we see that \( \overline{\mathcal{D}}_F (u \otimes e) = \overline{\mathcal{D}}u \otimes e \) and \( \vartheta_{F,h} (u \otimes e) = (\vartheta_{\psi}u) \otimes e \), where \( \vartheta_{\psi}u := \vartheta u - \vartheta_{\psi,u} \) is the formal adjoint of \( \vartheta \) with respect to the weighted \( L^2 \)-inner product \( (f, g)_\psi := \int f \overline{g} e^{-\psi} \, dV \). We see that \( u \otimes e \in \text{Dom}(\overline{\mathcal{D}}_{F,h}) \) if and only if \( u \in \text{Dom}(\overline{\mathcal{D}}_F) \), and then the result follows from the ordinary density lemma for scalar-valued forms. Q.E.D.

Let us consider the following complex

\[
L^{p,q-1}_{(2)}(U, F) \overset{\overline{\mathcal{D}}_F}{\longrightarrow} L^{p,q}_{(2)}(U, F) \overset{\overline{\mathcal{D}}_F^*}{\longrightarrow} L^{p,q+1}_{(2)}(U, F).
\]

Recall that \( U \) is a smoothly bounded strongly pseudoconvex domain in \( \tilde{X} \), the Hilbert spaces are taken using the metric \( h \) and \( \overline{\mathcal{D}}_F, \overline{\mathcal{D}}_F^* \) denotes the Hilbert space adjoint of \( \overline{\mathcal{D}}_F \). We want to show that

Lemma 2.2. The Range(\( \overline{\mathcal{D}}_F \)) is closed in \( L^{p,q}_{(2)}(U, F) \), if \( q > 0 \).

Proof. For any element \( A \in \mathcal{D}_F := \text{Dom}(\overline{\mathcal{D}}_F) \cap \text{Dom}(\overline{\mathcal{D}}_{F,h}) \subset L^{p,q}_{(2)}(U, F) \) we set \( |||A|||_2^2 := ||A||^2 + ||\overline{\mathcal{D}}_F A||^2 + ||\overline{\mathcal{D}}_{F,h} A||^2 \) where all the norms are computed with respect to \( h \) and a fixed non-degenerate metric on \( \tilde{X} \). The key observation in order to prove Lemma 2.2 is that if a ball in \( \mathcal{D}_F \) (with respect to \( ||| \cdot |||_2 \)) is relatively compact in \( L^{p,q}_{(2)}(U, F) \), then \( \overline{\mathcal{D}}_F \) has closed image in \( L^{p,q}_{(2)}(U, F) \) and in \( L^{p,q+1}_{(2)}(U, F) \), if \( q > 0 \). We know that when \( \partial \tilde{U} \) is smooth, strongly pseudoconvex and \( F := U \times \mathbb{C} \) (the scalar valued case), this observation is true (combining Theorem 5.3.7 in [17] and Rellich’s lemma). We set \( \mathcal{D} := \text{Dom}(\overline{\mathcal{D}}) \cap \text{Dom}(\overline{\mathcal{D}}^*) \subset L^{p,q}_{(2)}(U) \) and \( |||f|||_2^2 := ||f||^2 + ||\overline{\mathcal{D}}f||^2 + ||\overline{\mathcal{D}}^* f||^2 \) in this case. Then we have the following general result:

Lemma 2.3. Let \( U \) be a relatively compact subdomain in \( \tilde{X} \). Assume that \( \{ f \in \mathcal{D} : |||f|||_2 \leq 1 \} \) is relatively compact in \( L^{p,q}_{(2)}(U) \). Then \( \{ A \in \mathcal{D}_F : |||A|||_F \leq 1 \} \) is relatively compact in \( L^{p,q}_{(2)}(U,F) \), for any holomorphic line bundle in a neighborhood of \( \overline{U} \) and any choice of smooth metric \( h \) on \( F \).
Proof. Cover $\overline{U}$ by relatively compact open sets $V_1, \cdots, V_m$ where we have holomorphic trivializations $e_j$ of $F$ over $V_j$ for each $j$. Choose $\zeta_j \in C^\infty_0(V_j)$; $0 \leq \zeta_j \leq 1$ that form a partition of unity on $\overline{U}$. Given $A \in L^{p,q}_{(2)}(U, F)$, we have $s = f_j \otimes e_j$ on $U \cap V_j$ for all $1 \leq j \leq m$. The linear map

$$\Theta : L^{p,q}_{(2)}(U, F) \rightarrow \left( L^{p,q}_{(2)}(U)^m \right)$$

$$A \rightarrow (\zeta_1 f_1^0, \zeta_2 f_2^0, \cdots, \zeta_m f_m^0)$$

where $k^0$ denotes extension of the form $k$ by zero to $U$, is a bounded map from $L^{p,q}_{(2)}(U, F)$ to $\left( L^{p,q}_{(2)}(U)^m \right)$, and maps $D_F$ into $D^m$.

Let $\chi_j \in C^\infty(V_j)$ such that $\chi_j = 1$ on supp $\zeta_j$ for all $j \leq m$ and let us define a map $K$

$$K : \left( L^{p,q}_{(2)}(U)^m \right) \rightarrow L^{p,q}_{(2)}(U, F)$$

$$(g_1, \cdots g_m) \rightarrow \sum_{j=1}^m (\chi_j g_j \otimes e_j)^0.$$ 

One can easily check that $K$ is a bounded left inverse to $\Theta$.

Now, by elementary estimations we can show that for all $j$ with $1 \leq j \leq m$ we have $||| \zeta_j f_j ||| \leq C ||| A |||_F$ for some positive constant $C$ and for all $A \in D_F$. It follows that when $B$ is a $||| \cdot |||_F$-ball in $D_F$, then $\Theta(B)$ is relatively compact in $L^{p,q}_{(2)}(U)^m$, so $B = K(\Theta(B))$ is relatively compact in $L^{p,q}_{(2)}(U, F)$. Q.E.D.

One can obtain a more direct proof of Lemma 2.2 by suitably modifying Hörmander’s arguments in the proof of Theorem 3.4.1 in [17]. The key observation is that the assertion of the lemma is independent of a “conformal” change of the metric $h$ of $F$. Setting for example $\hat{h} := h \xi$ where $\xi \in C^0(\overline{U})$ and $\xi > 0$ on $\overline{U}$, would only produce equivalent norms on the Hilbert spaces that appear just before Lemma 2.2. Then one can use as $\xi := e^{-\tau} \phi$, where $\phi$ is chosen as in the proof of Theorem 3.4.1 in [17] and follow Hörmander’s argument to show that the range of $\overline{\partial} F$ is closed in $L^{p,q}_{(2),\hat{h}}(U, F)$ for $q > 0$.

2.3. A short proof of the finite dimensionality of $L^2$-Dolbeault cohomology groups. Let $\ell : L^{p,q}_{2,\text{loc}}(\hat{X}_r, \mathcal{O}(D)) \rightarrow L^{p,q}_{2,\text{loc}}(X'_r)$ be the map defined by $\ell(g) = (\pi^{-1})^*(g)$ for $g \in L^{p,q}_{2,\text{loc}}(\hat{X}_r, \mathcal{O}(D))$. Clearly $\ell$ commutes with $\overline{\partial}$ and induces a map on cohomology

$$\ell_* : H^q(\hat{X}_r, \Omega^p \otimes \mathcal{O}(D)) \rightarrow H^q(X'_r, \Omega^p).$$

In section 6 of [10] we compared various $L^2$-cohomology groups with certain sheaf cohomology groups. We considered the natural inclusion $j : L^{p,q}_{(2)}(X'_r) \rightarrow L^{p,q}_{2,\text{loc}}(X'_r)$ and studied the corresponding induced homomorphism $j_* : H^q_{(2)}(X'_r) \rightarrow H^q(X'_r, \Omega^p_{(2)}(X'_r))$.

**Theorem 2.4.** (Corollary 1.6 in [10]) Let $j_* : H^q_{(2)}(X'_r) \rightarrow H^q(X'_r, \Omega^p_{(2)}(X'_r))$ be the obvious homomorphism induced by the inclusion $j : L^{p,q}_{(2)}(X'_r) \rightarrow L^{p,q}_{2,\text{loc}}(X'_r)$. Then the map $j_*$ is injective for $p + q < n$ and $q > 0$ and bijective for $p + q \leq n - 2$ and $q > 0$.

Proof of finite dimensionality of Dolbeault cohomology groups. For a form $(p, q)$ form $f$ defined on $X'_r$ and square-integrable with respect to the ambient metric, its pull-back $\pi^* f$ need not belong to $L^{p,q}_{2,\sigma}(\hat{X}_r \setminus E)$ where $\sigma$ is any non-degenerate metric on $\hat{X}$. However, given $f \in L^{p,q}_{2,\sigma}(X'_r) \cap \text{Dom}(\overline{\partial})$ we can show, using lemma 3.1 in [11] (comparison estimates of weighted $L^2$-norms between forms and their pull-backs under resolution of singularities maps), that $\pi^* f \in \Gamma(\hat{X}_r, L^{p,q}(\mathcal{O}(D)))$ for some divisor $D = \sum_{i=1}^m d_i E_i$ supported on $E$ whenever $d_1, \cdots, d_m$ are chosen large enough ($d_i > 1$ for all $i = 1, \cdots, m$). In addition,
\[ \bar{\partial} \pi^* f = \pi^*(\bar{\partial} f) \] for any \( f \in L^{p,q}_{(2)}(X') \cap \text{Dom}(\bar{\partial}) \) and any \( p, q \geq 0 \). Taking into account all these we obtain a commutative diagram

\[
\begin{array}{ccc}
H^{p,q}_{(2)}(X') & \xrightarrow{\pi^*} & H^q(\hat{X}_r, \Omega^p \otimes \mathcal{O}(D)) \\
\downarrow j_* & & \downarrow j_* \\
H^q(X', \Omega^p) & & 
\end{array}
\]

By Theorem 2.4 we know that the map \( j_* \) is injective for \( p + q \leq n - 1 \), \( q > 0 \). Hence the map \( \pi^* \) is injective for such \( p, q \). As \( \hat{X}_r \) is a smoothly bounded strongly pseudoconvex domain the cohomology groups \( H^q(\hat{X}_r, \Omega^p \otimes \mathcal{O}(D)) \) are finite dimensional. Hence \( H^{p,q}_{(2)}(X') \) are finite dimensional for all \( p+q \leq n-1, \ q > 0 \).

Global identifications of the \( L^2-\overline{\partial} \)-cohomology groups on projective surfaces with isolated singularities with cohomology groups of appropriate sheaves on the desingularized manifolds have been obtained by Pardon (for cones over smooth projective curves) in [29], by Pardon and Stern in [30] for \( L^0_{(2)}-\overline{\partial} \)-cohomology groups of projective varieties with arbitrary singularities and recently by Ruppenthal [35] for a large class of compact pure dimensional Hermitian complex spaces with isolated singularities.

3. Proof of Theorem 1.1

3.1. Exceptional sets. We shall recall the notion of exceptional sets (in the sense of Grauert [13]) and some key results regarding these sets that will be needed in the paper.

Definition. Let \( X \) be a complex space. A compact nowhere discrete, nowhere dense analytic set \( A \subset X \) is exceptional if there exists a proper, surjective map \( \pi : X \to Y \) such that \( \pi(A) \) is discrete, \( \pi : X \setminus A \to Y \setminus \pi(A) \) is biholomorphic and for every open set \( D \subset Y \) the map \( \pi^* : \Gamma(D, \mathcal{O}) \to \Gamma(\pi^{-1}(D), \mathcal{O}) \) is surjective.

We usually say that \( \pi \) collapses or blows down \( A \).

If \( V \) is a Stein neighborhood of \( \pi(A) \) then \( \pi^{-1}(V) \) is a 1-convex space with maximal compact analytic set \( A \) and \( \pi |_{\pi^{-1}(V)} \) is the Remmert reduction.

Characterization of exceptional sets

Below we collect some basic results regarding exceptional sets.

\( \alpha \) (Theorem 4.8, page 57 in [24]) Let \( X \) be an analytic space and \( A \) a compact, nowhere discrete analytic subset. \( A \) is exceptional if and only if there exists a neighborhood \( U \) of \( A \) such that the closure of \( U \) in \( X \) is compact, \( U \) is strictly Levi pseudoconvex and \( A \) is the maximal compact analytic subset of \( U \). Also, \( A \) is exceptional if and only if \( A \) has arbitrarily small strictly pseudoconvex neighborhoods.

\( \beta \) (Lemma 3.1 in [23]) Let \( \pi : U \to Y \) exhibit \( A \) as exceptional in \( U \) with \( Y \) a Stein space. If \( U \supset V \) with \( V \) holomorphically convex neighborhood of \( A \) and \( \mathcal{F} \) is a coherent analytic sheaf on \( U \), then the restriction map \( \rho : H^i(U, \mathcal{F}) \to H^i(V, \mathcal{F}) \) is an isomorphism for \( i \geq 1 \).

3.2. Local cohomology along exceptional sets. In this section we recall Stephen Yau’s and Karras’ results that describe the local cohomology along exceptional sets. Our earlier work on Hartogs’ extension theorems on Stein spaces (see [28]) indicated to us the importance of the local cohomology exact sequences and led us to the discovery of these theorems.

For a sheaf of abelian groups \( \mathcal{F} \) on a paracompact, Hausdorff space \( X \) and for \( K \) a closed subset of \( X \), let \( \Gamma_K(X, \mathcal{F}) := \Gamma(X, \mathcal{F}) \setminus \Gamma(X \setminus K, \mathcal{F}) \). Since each \( \mathcal{F}^\bullet \) is flabby, we have a short exact sequence \( 0 \to \Gamma_K(X, \mathcal{F}^\bullet) \to \Gamma(X, \mathcal{F}^\bullet) \to \Gamma(X \setminus K, \mathcal{F}^\bullet) \to 0 \). This induces a long exact sequence on cohomology.
(7) \[ 0 \to H^0_K(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(X \setminus K, \mathcal{F}) \to H^1_K(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to ... \]

It is a standard fact from sheaf cohomology theory that \( H^i_K(X, \mathcal{F}) \cong H^i(U, \mathcal{F}) \) where \( U \) is an open neighborhood of \( K \) in \( X \). The fact that \( H^i_K(U, \mathcal{F}) \) is independent of the neighborhood \( U \) of \( K \) is referred to as excision.

On the other hand, one could also consider the cohomology with compact support on \( X \) and define \( \Gamma_c(X, \mathcal{F}) \) to be the group of global sections of \( \mathcal{F} \) whose supports are compact subsets of \( X \). Let \( Y \) be a compact subset of \( X \). Letting \( \{C_i\} \) denote the canonical resolution of \( \mathcal{F} \), we have an inclusion of complexes

\[(\Gamma_Y(X, C_i), d_i) \hookrightarrow (\Gamma_c(X, \mathcal{C}), d)\]

which induces natural homomorphisms \( \gamma_i : H^i_c(X, \mathcal{F}) \to H^i_c(M, \mathcal{F}) \) for all \( i \geq 0 \).

In general we do not have enough information on the maps \( \gamma_i \) but in the special case of exceptional sets we can obtain very precise information about them. For the remainder of this section let \( X \) be a reduced complex space and \( Y = E \) be an exceptional subset of \( X \). Then we know that there exists a strongly pseudoconvex neighborhood \( M \) of \( E \) in \( X \) and a non-negative exhaustion function \( \phi \) on \( M \) such that \( \phi \) is strongly plurisubharmonic on \( M \setminus E \) and \( E = \{ x \in M : |\phi(x)| = 0 \} \). By excision, \( H^i_c(X, \mathcal{F}) = H^i_c(M, \mathcal{F}) \) for all \( i \geq 0 \) and therefore we have natural homomorphisms \( \gamma_i : H^i_c(X, \mathcal{F}) \to H^i_c(M, \mathcal{F}) \) for all \( i \geq 0 \). Karras showed that under circumstances these maps \( \gamma_i \) are isomorphisms for some \( i \).

**Theorem 3.1.** (Proposition 2.3 in [19]) Let \( X \) be a reduced complex space and \( E \) an exceptional subset of \( X \). If \( \mathcal{F} \) is a coherent analytic sheaf on \( X \) such that \( \text{depth}_x \mathcal{F} \geq d \) for \( x \in M \setminus E \), then

\[ \gamma_i : H^i_c(X, \mathcal{F}) \to H^i_c(M, \mathcal{F}) \]

is an isomorphism for \( i < d \).

Once we have Theorem 3.1, we can very easily obtain the following corollary:

**Corollary 3.2.** Let \( E \) be an exceptional set of an \( n \)-dimensional complex manifold \( M \). Then

\[ H^i_c(M, \mathcal{O}_M) = 0 \quad \text{for} \quad i < n. \]

**Proof.** Since \( E \) is an exceptional set of \( M \), \( M \) is a strongly pseudoconvex manifold and let \( p : M \to S \) denote the Remmert reduction map. For every coherent analytic sheaf \( \mathcal{F} \), \( H^i(M, \mathcal{F}) \) are finite dimensional for \( i > 0 \). Hence we can apply Serre’s duality theorem for \( \mathcal{F} = \omega_M = \Omega^n \), the sheaf of holomorphic \( n \)-forms on \( M \). Then \( H^{n-i}(M, \omega_M) \cong H^i_c(M, \mathcal{O}) \) for all \( i < n \). Since \( R^i p_* \omega_M = 0 \) for all \( i > 0 \), by Takegoshi’s relative vanishing theorem in [40], we have \( H^{n-i}(M, \omega_M) = H^{n-i}(S, p_*(\omega_M)) \). But the latter cohomology groups vanish since \( S \) is Stein, \( p_*(\omega_M) \) is coherent and \( n - i > 0 \). Therefore \( H^i_c(M, \mathcal{O}_M) = 0 \) for all \( i < n \).

Since \( M \) is a manifold \( \text{depth}_x(\mathcal{O}_M) = n \) for all \( x \in M \setminus E \); hence we can apply Theorem 3.1 to conclude that \( H^i_c(M, \mathcal{O}_M) \cong H^i_c(M, \mathcal{O}_M) = 0 \) for all \( i < n \).

### 3.3. Proof of Theorem 1.1.

Suppose now that \( X \) is a pure \( n \)-dimensional \((n \geq 2)\) complex analytic set in \( \mathbb{C}^N \) with an isolated singularity at 0 and let \( X_r = X \cap B_r \) be a small Stein neighborhood of 0 with smooth boundary. Let \( \pi : \tilde{X} \to X \) be a desingularization of \( X \). Then \( E := \pi^{-1}(0) \) (the exceptional locus of the desingularization) is an exceptional set in the sense of Grauert and let \( \tilde{X}_r := \pi^{-1}(X_r) \). Let \( \sigma \) be a positive definite metric on \( \tilde{X} \). In what follows \( \mathcal{L}^{\bullet, \bullet} \) represents the sheaves of differential forms that were introduced in section 2.1.

Let \( r > 0 \) be a regular value of \( || \circ \pi \) on \( \tilde{X}_r \) with \( 0 < r < R \), \( \tilde{X}_r \) is a relatively compact domain with smooth strongly pseudoconvex boundary in \( \tilde{X}_r \). It is a standard fact that the inclusion of the following complexes

\[ L^0_{(R)}(\tilde{X}_r) \cap \mathcal{D}(\overline{D}) \subseteq L^0_{(R)}(\tilde{X}_r) \]

...
induces isomorphisms on the corresponding cohomology groups $H^{0,q}_{(2)}(\tilde{X}_r) \cong H^q(\tilde{X}_r, \mathcal{O})$ for $q > 0$.

By Theorem 2.4, we know that for $1 \leq q \leq n - 2$ we have $H^{0,q}_{(2)}(X'_r) \cong H^q(X'_r, \mathcal{O})$. The latter sheaf cohomology groups are isomorphic to $H^q(\tilde{X}_r \setminus E, \mathcal{O}_{\tilde{X}_r})$. Consider the long exact local cohomology sequence

\[ \ldots \rightarrow H^i_E(\tilde{X}_r, \mathcal{O}) \rightarrow H^i(\tilde{X}_r, \mathcal{O}) \xrightarrow{j} H^i(\tilde{X}_r \setminus E, \mathcal{O}) \rightarrow H^{i+1}_E(\tilde{X}_r, \mathcal{O}) \rightarrow \ldots \]

If $1 \leq q \leq n - 2$, then by Corollary 3.2 we have $H^q_E(\tilde{X}_r, \mathcal{O}) = H^{q+1}_E(\tilde{X}_r, \mathcal{O}) = 0$; hence from (8) we can conclude that $H^q(\tilde{X}_r, \mathcal{O}) \cong H^q(\tilde{X}_r \setminus E, \mathcal{O}) \cong H^q(X'_r, \mathcal{O}) \cong H^{0,q}_{(2)}(X'_r)$.

We shall construct now the map $\phi_*$ that appears in Theorem 1.1. Recall that for a $(0, q)$ form in $\tilde{X}_r$, we have $\|g\|_{L^2(\tilde{X}_r)} \leq C \|g\|_{L^2(\tilde{X}_r)}$, where $\gamma$ is the "pseudometric" that arises from the pull-back of the Euclidean metric in $X'$, since $|\gamma| \leq C_0 |\sigma|$ and $p = 0$. Moreover $\|\gamma\|_{L^2(\tilde{X}_r)} = \|g\|_{L^2(\tilde{X}_r)}$. Now, for $g \in L^{0,q}_{(2)}(\tilde{X}_r)$ we have $\|g\|_{L^2(\tilde{X}_r)} \leq C \|g\|_{L^2(\tilde{X}_r)}$, since we pass to a smaller norm. Hence for all $q$ with $0 \leq q \leq n$, there exists a bounded linear map:

$$
\phi : L^{0,q}_{(2)}(\tilde{X}_r) \rightarrow L^{0,q}_{(2)}(X'_r)
$$

$$
u \rightarrow (\pi^{-1})^*(\nu).
$$

Then we have a commutative diagram of complexes

\[
\begin{array}{ccc}
L^0(\tilde{X}_r) & \xrightarrow{r} & L^0_{(2)}(\tilde{X}_r) \cap D(\mathcal{D}) \\
\downarrow \phi & & \downarrow \phi \\
L^0_{(2)}(X'_r) \cap D(\mathcal{D}) & \xrightarrow{j} & L^0((X'_r)
\end{array}
\]

which induces the following commutative diagram:

\[
\begin{array}{ccc}
H^q(\tilde{X}_r, \mathcal{O}) & \xrightarrow{r_*} & H^q(\tilde{X}_r \setminus E, \mathcal{O}) \\
\phi_* \downarrow & & \downarrow \phi_* \\
H^q_{(2)}(\tilde{X}_r) & \xrightarrow{j_*} & H^q(X'_r, \mathcal{O}).
\end{array}
\]

Since for $1 \leq q \leq n - 2$ $r_*$ and $j_*$ are isomorphisms, the commutativity of the above diagram will imply that $\phi_*$ is an isomorphism for $1 \leq q \leq n - 2$. On the other hand for $q = n - 1$ the maps $r_*$ and $j_*$ are only injective, hence $\phi_*$ is an injective map.
4. Proof of Theorem 1.2

4.1. Different extensions for ∂-operator. For the proof of Theorem 1.2 another closed extension of the ∂-operator will play a key role. Let ∂∗ denote the graph closure in L² of ∂ acting on forms f with coefficients in C∞₀(X∪ \ {0}). We can also consider the minimal extension of the ∂-operator on X'. More precisely we let ∂min denote the graph closure in L² of ∂ acting on forms with coefficients in C∞₀(X∪ \ {0}) (Dirichlet conditions on both the boundary of X and the singularity 0). It is easy to check that

Lemma 4.1. \( \text{Dom}(∂) = \{ f ∈ \text{Dom}(∂) : χ f ∈ \text{Dom}(∂_{\text{min}}) \text{ for a cut-off function } χ ∈ C∞₀(X) \} \).

Forms of bidegree \((0, n - 1)\) in \( \text{Dom}(∂)(X_r) \) are in the domain of \( ∂¹ \). More precisely, we have:

Lemma 4.2. If \( h ∈ \text{Dom}(∂) ∩ L⁰ⁿ⁻¹(X_r) \), then \( φ(h) ∈ \text{Dom}(∂) \).

Proof. We will distinguish two cases:

Case I. Let \( h ∈ \text{Dom}(∂_{X_r}) ∩ L⁰ⁿ⁻¹(X_r) \) and smooth in \( X_r \) (thus bounded near \( E \)). By a partition of unity argument we can assume that the support of \( h \) is contained in a coordinate domain \( U \), where \( U \supseteq U' ⊂ \mathbb{C}^n \) when \( E ∩ U = \{ z ∈ U : z_1 z_2 · · · z_m = 0 \} \) for some \( m \) with \( 1 ≤ m ≤ n \). We choose a family of cut-off functions \( χ_k \) that satisfy: i) \( χ_k(z) = 1 \) when \( \text{dist}(z, E) ≥ \frac{1}{k} \) and \( χ_k(z) = 0 \) near \( E \), and ii) \( |∂χ_k(z)| ≤ Ck \) for all \( k \). Now \( φ(h) \) has compact support \( π(U) ∩ X_r \) and in order to show that it belongs to \( \text{Dom}(∂) \) it suffices by Lemma 4.1 to show that \( φ(h) ∈ \text{Dom}(∂_{\text{min}}) \). Since \( ∂_{\text{min}} = ∂^*_{\text{max}} \) (the Hilbert space adjoint of \( ∂_{\text{max}} \)) we must show that

\( (∂φ(h), w) = (∂(h), ∂^*_{\text{max}} w) \)

for all \( w ∈ \text{Dom}(∂_{\text{max}}) ∩ L⁰ⁿ(X_r) \). Let us set \( ψ_k := χ_k ∘ π^{-1} \). Then we have

\( (∂φ(h), w) = \lim (∂ψ_k φ(h), ∂^*_{\text{max}} w) = \lim (∂ψ_k φ(h), w + (∂φ(h), w)). \)

But

\[ \int_{X_r} |∂χ_k ∧ h ∧ π^*(w)| ≤ ||h||∞ \left( \int_{X_r} |∂χ_k|^2 dV \right)^{\frac{1}{2}} \left( \int_{\text{supp}∂χ_k} |π^* w|^2 dV \right)^{\frac{1}{2}} = ABC \]

where \( C → 0 \) as \( k → ∞ \), while \( B \) is easily seen to be uniformly bounded. Hence, \( φ(h) ∈ \text{Dom}(∂) \).

Case II. Let \( h ∈ \text{Dom}(∂) ∩ L⁰ⁿ⁻¹(X_r) \). Since the smooth forms in \( X_r \) are dense in \( \text{Dom}(∂) \) in the graph norm, there exist \( h_ν ∈ C∞₀(X_r) \) such that \( h_ν → h \) in the graph norm as \( ν → ∞ \). But then \( φ(h_ν) ∈ \text{Dom}(∂) \) by Case I and converge to \( φ(h) \) in the graph norm in \( L⁰ⁿ⁻¹(X_r) \). Recall that \( ∂¹ \) is a closed operator, hence \( φ(h) ∈ \text{Dom}(∂) \).

4.2. Preliminaries from [10]. In a previous work (lemma 3.4 in [10]), we showed that for \( f ∈ Z^{p, q}_0 := L⁰ⁿ(X_r) \) with \( p + q ≤ n - 1 \) and \( q > 0 \), the equation \( ∂u = f \) is solvable in \( L⁰ⁿ⁻¹(X_r) \) if and only if the equation \( ∇v = f \) is solvable in \( L⁰ⁿ⁻¹(X_r ∪ \overline{B}_r) \) for some \( 0 < r_0 < r \). In addition we showed (Proposition 3.5 in [10]) that the equation \( ∂u = f \) is solvable in \( L⁰ⁿ⁻¹(X_r ∪ \overline{B}_r) \) with \( 0 < r_0 < r \), for \( f \) in a closed subspace of finite codimension in \( Z^{p, q}_0 \) when \( p + q < n, q > 0 \). Let us recall Case II in the proof of Proposition 3.5 in [10]. Let \( f ∈ Z^{0, n⁻¹}(X_r) \) and let \( χ ∈ C∞₀(X_r) \) with \( χ = 1 \) near 0 and \( \text{supp} χ ⊂ X_ρ \) with \( 0 < ρ < r \). It was shown that \( ∇w = π^*(∂χ ∧ f) \) had a solution in \( L⁰ⁿ⁻¹(X_r) \), compactly supported in \( X_r \) if and only if
\[
\int_{X^\prime_r} f \wedge \bar{\nabla}_X \wedge \psi = 0 \quad \text{for all } \psi \in L^{n,0}_2(X^\prime_r) \cap \ker(\overline{D}) := H^{n,0}_2(X^\prime_r).
\]

Condition (9) can be derived from the following weaker condition:

\[
\int_{X^\prime_r} f \wedge \bar{\nabla}_X \wedge \psi = 0 \quad \text{for all } \psi \in L^{n,0}_2(X^\prime_r) \cap \ker(\overline{D}) := H^{n,0}_2(X^\prime_r).
\]

This is a consequence of the following fact:

**Lemma 4.3.** The pair \((X^\prime_r, X^\prime_\rho)\) is an \(L^2\)-Runge pair, i.e. the restriction map \(r : \mathcal{O}L^2(X^\prime_r) \to \mathcal{O}L^2(X^\prime_\rho)\) has dense image.

**Proof.** Let \(h \in \mathcal{O}L^2(X^\prime_\rho)\). We need to show that there exists a sequence of functions \(h_\nu \in \mathcal{O}L^2(X^\prime_r)\) such that \(\|h_\nu - h\|_{L^2(K)} < \epsilon\), where \(\epsilon > 0\) and \(K\) is a compact subset of \(X^\prime_r\). Suppose that \(K \subset A_0 := X \cap \{r_0 < \|z\| < r_1\}\), where \(0 < r_0 < r_0^\ast < r_1 < \rho < r\). Let \(\phi \in C^\infty_0(A)\) with \(\phi = 1\) on \(A_0\) and let us look at the \((0,1)\)-form \(g := \overline{\nabla}(\phi h)\); we can write \(g = g' + g''\) where \(g'\) is supported on \(X \cap \{\|z\| > r_1\}\), \(g''\) is supported in \(X^\prime_r\) and both are \(\overline{D}\)-closed on \(X^\prime_r\). Using Proposition 3.1 from [10], we know that there exists a solution \(u''\) satisfying \(\overline{D}u'' = g''\) on \(X^\prime_r\), compactly supported in \(X^\prime_r\) and in \(L^{0,0}_2(X^\prime_r)\). We consider a convex, increasing function \(\xi \in C^\infty(\mathbb{R})\) with \(\xi(t) = 0\) if \(t \leq r_1\) and \(\xi(t) > 0\) if \(t > r_1\). Let \(\psi(z) = \xi(\|z\|)\). By our choice of \(\xi\), we know that the min \(\{\psi(z) ; z \in \text{supp} g\} = c > 0\). Applying Theorem 1.3 in [10], we obtain a solution \(u'_\nu\) satisfying \(\overline{D}u'_\nu = g'\) in \(X^\prime_r\) and

\[
\int_{X^\prime_r} |u'_\nu|^2 e^{-\nu \psi} dV \leq C \|g'\|^2
\]

for \(\nu \geq r_0\). Hence, \(\int_{A_0} |u'_\nu|^2 dV \leq C e^{-\nu c} \|g'\|^2\).

Let \(h_\nu := \phi h - u'_\nu - u''\). Then \(h_\nu \in L^{0,0}_2(X^\prime_r)\), \(\overline{D}h_\nu = 0\) on \(X^\prime_r\) and

\[
\int_K |h_\nu - h|^2 dV < \epsilon
\]

for \(\nu\) sufficiently large. Q.E.D.

Condition (10) is independent of the choice of the cut-off function \(\chi\). Also, if \(f = \overline{\nabla}u\) near the support of \(\overline{\nabla}_X\) then \(f\) satisfies (10), since

\[
\int_{X^\prime_r} f \wedge \overline{\nabla}_X \wedge \psi = \int_{X^\prime_r} d(u \wedge \overline{\nabla}_X \wedge \psi) = 0
\]

by Stokes’ theorem. Hence condition (10) depends only on the equivalence class \([f] \in H^{0,n-1}_2(X^\prime_r)\).

Let \(\mathcal{M} := \{f \in Z^{0,n-1}_2(X^\prime_r) ; \ f \text{ satisfies } (10)\}\).

If \(f \in \mathcal{M}\), we can write \(f = \phi(w + \pi^*((1-\chi)f)) + (\chi f - \phi(w))\) -where \(w\) is the square-integrable, compactly supported form in \(X_r\) that satisfies \(\overline{\nabla}w = \pi^*(\overline{\nabla}_X f)\). Each term to the right-hand side of the previous equation is \(\overline{\nabla}\)-closed and the second one has compact support in \(X_r\), hence it is \(\overline{\nabla}\)-exact by Proposition 3.1 in [10] (which is an \(L^2\)-solvability result for square-integrable, \(\overline{\nabla}\)-closed forms with compact support in \(X_r\)). Therefore we can write \([f] = \phi_*([w + \pi^*((1-\chi)f)])\).

On the other hand, if \(f = \phi(g)\) for some \(g \in L^{0,n-1}_2(\hat{X}_r) \cap \ker(\overline{D})\), Lemma 4.2 from section 4.1 tell us that \(\phi(g)\) will belong in the domain of \(\overline{\nabla}\) and \(\overline{\nabla} \phi(g) = 0\). Hence there exist \(h_\nu \in C^\infty_0(\hat{X}_r \setminus \{0\})\) such that \(h_\nu \to \phi(g)\) and \(\overline{\nabla}h_\nu \to 0\) in \(L^2\). The latter would imply for \(\psi \in H^{n,0}_2(X^\prime_r)\) that
\begin{equation}
\int_{X'_r} f \wedge \overline{\partial} \chi \wedge \psi = \lim_{\nu \to \infty} \int_{X'_r} h_\nu \wedge \overline{\partial} \chi \wedge \psi = (-1)^{n-1} \lim_{\nu \to \infty} \int_{X'_r} d(h_\nu \wedge \chi \wedge \psi) = 0,
\end{equation}

by Stokes’ theorem on $X'_r$.

We have thus shown the following

**Lemma 4.4.** $\mathcal{M} = \phi(Z_{(2)}^{0,n-1}(\tilde{X}_r)) + \text{Im} \overline{\partial}^{n-2}(X'_r)$.

In [10], we showed that $\int_{X'_r} \overline{\partial} \chi \wedge f \wedge \psi = 0$ is satisfied for all $f \in Z_{(2)}^{0,n-1}(X'_r)$ when $\psi \in \Gamma(X_\rho, \tilde{\omega})$ for some coherent $\mathcal{O}_X$-module $\tilde{\omega}$. The module $\Gamma(X_\rho, \tilde{\omega})$ was shown to have finite codimension in $\Gamma(X_\rho, \omega)$-where $\omega$ was Grothendieck’s dualizing sheaf-and a fortiori in $L_{2,\text{loc}}^0(X_\rho) \cap \ker(\overline{\partial})$. Now, if $a_1, \ldots, a_m$ span the complementary subspace to $\Gamma(X_\rho, \tilde{\omega})$ we see that

$$\mathcal{M} = \{ f \in Z_{(2)}^{0,n-1}(X'_r) ; \int_{X'_r} f \wedge \overline{\partial} \chi \wedge a_j = 0, \text{ for all } j = 1, \ldots, m \}.$$ 

Hence the codimension of $\mathcal{M}$ in $Z_{(2)}^{0,n-1}(X'_r)$ is at most $m$. In what follows we will identify the subspace of $H_{(2)}^{0,0}(X'_r)$ for which $\int_{X'_r} f \wedge \overline{\partial} \chi \wedge \psi = 0$ for all $f \in Z_{(2)}^{0,n-1}(X'_r)$.

**4.3. Construction of a non-degenerate pairing.** The construction in section 4.2 allow us to consider the following pairing: Let $\chi \in C_0^\infty(X_\rho)$ such that $\chi = 1$ in a neighborhood of $0$ and supp $\chi \subset X_\rho$ with $0 < \rho < r$. Take a pair $(f, \psi) \in Z_{(2)}^{0,n-1}(X'_r) \times H_{(2)}^{0,0}(X'_r)$ and assign to it the number

\begin{equation}
< f, \psi > = \int_{X'_r} f \wedge \overline{\partial} \chi \wedge \psi.
\end{equation}

Recall from our discussion above that $< f, \psi > = 0$ for all $\psi \in H_{(2)}^{0,0}(X'_r)$ is equivalent to the fact that $f \in \mathcal{M}$ in which its turn is equivalent to the fact that $f = \phi(g) + \overline{\partial} u$ where $g \in L_{(2)}^{0,n-1}(\tilde{X}_r) \cap \ker(\overline{\partial})$ (hence $\phi(g) \in \text{Dom}(\overline{\partial})$) and $u \in L_{(2)}^{0,n-2}(X'_r)$, compactly supported in $X_r$.

**Proposition 4.5.** We have $< f, \psi > = 0$ for all $f \in Z_{(2)}^{0,n-1}(X'_r)$ if and only if $\psi \in \ker(\overline{\partial})_{X'_r}$.

**Remark:** The first paragraph in section 4.3 and Proposition 4.4 will allow us to say that \([12]\) is a non-degenerate pairing from \[
\frac{Z_{(2)}^{0,n-1}(X'_r)}{\phi(Z_{(2)}^{0,n-1}(\tilde{X}_r)) + \text{Im} \overline{\partial}^{n-2}(X'_r)} \times \frac{\ker(\overline{\partial})_{X'_r}^{n,0}}{\ker(\overline{\partial})_{X'_r}^{1,n,0}} \to \mathbb{C}
\]
or equivalently \[
\frac{H_{(2)}^{0,n-1}(X'_r)}{\phi_*(H_{(2)}^{0,n-1}(X_\rho))} \times \frac{\ker(\overline{\partial})_{X'_r}^{0,n,0}}{\ker(\overline{\partial})_{X'_r}^{n,0}} \to \mathbb{C}
\]

**Remark:** Due to the injectivity of $\phi_*$ one can obtain the following bound on the complex dimension of $H_{(2)}^{0,n-1}(X'_r)$

\begin{equation}
\dim_{\mathbb{C}} H_{(2)}^{0,n-1}(X'_r) = \dim_{\mathbb{C}} H_{(2)}^{0,n-1}(\tilde{X}_r) + \dim_{\mathbb{C}} \frac{\ker(\overline{\partial})_{X'_r}^{n,0}}{\ker(\overline{\partial})_{X'_r}^{1,n,0}}.
\end{equation}
Proof of Proposition 4.5. $\Leftarrow$ If $\psi \in \text{ker} \left( \bar{\partial} \right)_{X_r'}$, then there exist $\psi_\nu \in C^\infty (X_r \setminus \{0\})$ such that $\psi_\nu \to \psi$ in $L^2$ and $\bar{\partial}_\nu \psi_\nu \to 0$ in $L^2$ as $\nu \to \infty$. But then $\int_{X'_r} f \wedge \bar{\partial}_\nu \psi_\nu = \lim_{\nu \to \infty} \int_{X'_r} f \wedge \bar{\partial}_\nu \psi_\nu = \lim_{\nu \to \infty} \left[ \int_{X'_r} f \wedge \bar{\partial}(\psi_\nu) - \int_{X'_r} f \wedge \bar{\partial}_\nu \psi_\nu \right] = (-1)^n \lim_{\nu \to \infty} \int_{X'_r} \bar{\partial} f \wedge \psi_\nu - 0 = 0.$

$\Rightarrow$ Let us assume that $< f, \psi > = 0$ for all $f \in Z^{0,n-1}_{(2)}(X'_r)$. We want to show that $\psi \in \text{ker} \left( \bar{\partial} \right)_{X_r'}$. It suffices to show that $\psi \in \text{Dom}(\bar{\partial})$. By Lemma 4.1, this is equivalent to showing that $\chi \psi \in \text{Dom}(\bar{\partial}_{\text{min}})$. Recall that $(\bar{\partial}_{\text{min}})^* = \partial_{\text{max}}$. Hence to show that $\chi \psi \in \text{Dom}(\bar{\partial}_{\text{min}})$ it would suffice to show that $\chi \psi \in \text{Dom}(\partial_{\text{max}})^*$ or equivalently

\[(14) \quad (\bar{\partial}(\chi \psi), g) = (\chi \psi, \partial_{\text{max}} g)\]

for all $g \in \text{Dom}(\partial_{\text{max}})^{n-1} = \{ g \in L^{n-1}_{(2)}(X'_r) : \partial_{\text{max}} g \in L^2 \text{ (weakly)} \}$. The operator $\bar{\pi} : L^{n-1}_{(2)}(X'_r) \to L^{0,n-1}_{(2)}(X'_r)$ is an isometry mapping from $\text{Dom}(\partial_{\text{max}}) \to \text{Dom}(\bar{\partial})$ (here $\bar{\partial}$ denotes the maximal (weak) extension). Hence (14) is equivalent to

\[(15) \quad \int_{X'_r} \bar{\partial}(\chi \psi) \wedge w = -(-1)^n \int_{X_r'} \chi \psi \wedge \partial w\]

for all $w \in \text{Dom}(\bar{\partial}) \cap L^{0,n-1}_{(2)}(X'_r)$.

Clearly (15) holds for all $w \in Z^{0,n-1}_{(2)}(X'_r)$, by the assumption $< f, \psi > = 0$ for all $f \in Z^{0,n-1}_{(2)}(X'_r)$. Let us consider an arbitrary element $w \in \text{Dom}(\bar{\partial}) \cap L^{0,n-1}_{(2)}(X'_r)$. Then $\bar{\partial} w \in Z^{0,n}_{(2)}(X'_r)$ and $\pi^* (\bar{\partial} w) \in H^{0,n}_{(2)}(\tilde{X}_r)$ (a few words are in order here: a) $\pi^* (\bar{\partial} w) \in L^{0,n}_{(2),\gamma}(\tilde{X}_r)$, where $\gamma$ is a non-degenerate metric on $\tilde{X}_r$ and $\gamma$ is the pull-back of the Euclidean metric under $\pi$. b) $\pi^* (\bar{\partial} w)$ is $\bar{\partial}$-closed in $\tilde{X}_r \setminus E$, but since $\pi^* (\bar{\partial} w) \in L^{0,n}_{(2),\sigma}(\tilde{X}_r)$ it can be extended as a $\bar{\partial}$-closed form in $\tilde{X}_r$. We shall still denote the extended form as $\pi^* (\bar{\partial} w)$. Now, as $\tilde{X}_r$ is a smoothly bounded domain with strongly pseudoconvex boundary we have $H^{0,n}_{(2)}(\tilde{X}_r) \cong H^n(\tilde{X}_r, O) = 0$ (the latter due to work of Siu, [37]). Hence, there exists a solution $g \in L^{0,n-1}_{(2)}(\tilde{X}_r)$ such that $\bar{\partial} g = \pi^* (\bar{\partial} w)$ in $\tilde{X}_r$. But then, $\bar{\partial} \phi(g) = \bar{\partial} w$ on $X'_r$ and

\[
\int_{X'_r} \bar{\partial}(\chi \psi) \wedge w = \int_{X'_r} \bar{\partial}(\chi \psi) \wedge (w - \phi(g)) + \int_{X'_r} \bar{\partial}(\chi \psi) \wedge \phi(g) = F + G
\]

Since $w - \phi(g) \in Z^{0,n-1}_{(2)}(X'_r)$ we know that $F = -(-1)^n \int_{X'_r} \chi \psi \wedge \bar{\partial}(w - \phi(g)) = 0$. To finish the proof of the proposition, we need to show that $G := \int_{X'_r} \bar{\partial}(\chi \psi) \wedge \phi(g) = -(-1)^n \int_{X'_r} \chi \psi \wedge \bar{\partial} \phi(g)$. Using Lemma 4.2, we have that $\phi(g) \in \text{Dom}(\bar{\partial})$. Hence there exist $g_\nu \in C^\infty_{0,(0,n-1)}(\tilde{X}_r \setminus \{0\})$ such that $g_\nu \to \phi(g)$. But then, using Stokes' theorem we obtain

\[
\int_{X'_r} \bar{\partial}(\chi \psi) \wedge \phi(g) = \lim_{\nu \to \infty} \int_{X'_r} \bar{\partial}(\chi \psi) \wedge g_\nu = \lim_{\nu \to \infty} \int_{X'_r} (\chi \psi \wedge g_\nu) - (-1)^n \lim_{\nu \to \infty} \int_{X'_r} \chi \psi \wedge \bar{\partial} g_\nu = -(-1)^n \int_{X'_r} \chi \psi \wedge \bar{\partial} \phi(g).
\]

Hence the Proposition is proven.
4.4. Towards an understanding of $\text{Im } j_\ast$. In section 2 of the paper, we defined two maps $j_\ast, \ell_\ast$ that would be crucial for the proof of Theorem 1.2. The map $j_\ast : H^{0,n-1}_{(2)}(X'_r) \to H^{n-1}(X'_r, \mathcal{O})$ was induced by the inclusion $j : L^{0,n-1}_{(2)}(X'_r) \to L^{0,n-1}_{2,\text{loc}}(X'_r)$ and by Theorem 2.4 we know it is injective. In section 2.2 we showed that there exists a natural map $\ell : L^{0,n-1}_{2,\text{loc}}(X', \mathcal{O}(D)) \to L^{0,n-1}_{2,\text{loc}}(X'_r)$ defined by

$$\ell(g) := (\pi^{-1})^\ast(g).$$

Clearly $\ell$ commutes with $\overline{\partial}$ and induces a map $\ell_\ast : H^{n-1}(X', \mathcal{O}(D)) \to H^{n-1}(X'_r, \mathcal{O})$ in cohomology.

We will begin by obtaining a characterization for forms $f \in L^{0,n-1}_{2,\text{loc}}(X'_r) \cap \ker(\overline{\partial})$ that arise as $j_\ast([h]) = [f]$ for some $h \in Z^{0,n-1}_{(2)}(X'_r)$. The lemma below describes some necessary and sufficient conditions to address this question.

**Lemma 4.6.** Let $f \in L^{0,n-1}_{2,\text{loc}}(X'_r) \cap \ker(\overline{\partial})$. Then,

i) If $[f] \in \text{Im } j_\ast$, then $<f, \psi> := \int_{X'_r} f \wedge \overline{\partial} \wedge \psi = 0$ when $\psi \in \text{Kern}(\overline{\partial})^{n,0}$. Here $\chi$ is a cut-off function as in section 4.3.

ii) On the other hand, if $\int_{X'_r} |f|^2 \|z\|^B dV < \infty$ for some $B > 0$ large enough and $<f, \psi> = 0$ when $\psi \in \text{Ker}(\overline{\partial})^{n,0}$, then $[f] \in \text{Im } j_\ast$.

**Proof.** i) In section 4.3, we constructed a pairing $<, > : Z^{0,n-1}_{(2)}(X'_r) \times H^{n,0}_{(2)}(X'_r) \to \mathbb{C}$ described by: for $(f, \psi) \in Z^{0,n-1}_{(2)}(X'_r) \times H^{n,0}_{(2)}(X'_r)$

$$<f, \psi> := \int_{X'_r} f \wedge \overline{\partial} \wedge \psi$$

where $\chi \in C^\infty_0(X_r)$ such that $\chi = 1$ near 0. Certainly this pairing can be defined also for forms $f \in L^{0,n-1}_{2,\text{loc}}(X'_r) \cap \ker(\overline{\partial})$.

Let $[f] = j_\ast [h]$ for some $h \in Z^{0,n-1}_{(2)}(X'_r)$. Then $f = h + \overline{\partial} u$ for some $u \in L^{0,n-2}_{2,\text{loc}}(X'_r)$. In Proposition 4.4, we showed that whenever $\psi \in \text{Ker}(\overline{\partial})^{n,0}$ we have $<a, \psi> = 0$ for all $a \in Z^{0,n-1}_{(2)}(X'_r)$. Hence, to prove i), it suffices to show that $<\overline{\partial} u, \psi> = 0$ when $\psi \in \text{Ker}(\overline{\partial})^{n,0}$. But this follows easily from Stokes’ theorem as

$$<\overline{\partial} u, \psi> = \int_{X'_r} d(u \wedge \overline{\partial} \chi \wedge \psi) = 0$$

since the integrand is compactly supported in $X'_r$.

ii) As $<f, \psi> = 0$ for all $\psi \in \text{Ker}(\overline{\partial})^{n,0}$, the bounded linear functional

$$\lambda : \text{Ker}(\overline{\partial})^{n,0} \to \mathbb{C}$$

defined by $\lambda(\psi) = <f, \psi> = \int_{X'_r} f \wedge \overline{\partial} \chi \wedge \psi$ factors to a well-defined bounded linear functional (still denoted by $\lambda$ for simplicity)

$$\lambda : \frac{\text{Ker}(\overline{\partial})^{n,0}}{\text{Ker}(\overline{\partial})^{n,0}} \to \mathbb{C}$$

such that $\lambda([\psi]) = <f, \psi>$. In Proposition 4.5, we saw that

$$\left( \frac{\text{Ker}(\overline{\partial})^{n,0}}{\text{Ker}(\overline{\partial})^{n,0}} \right) ^\prime \cong \frac{H^{0,n-1}_{(2)}(X'_r)}{\phi_\ast(H^{0,n-1}(X_r))}.$$
Hence there exists a \( g \in Z^{0,n-1}_2(X'_r) \) such that \( \lambda([\psi]) = \langle g, \psi \rangle \) for all \( \psi \in H^{n,0}_2(X'_r) \), i.e.

\[
(16) \quad \int_{X'_r} f \wedge \partial \chi \wedge \psi = \int_{X'_r} g \wedge \partial \chi \wedge \psi
\]

for all \( \psi \in H^{n,0}_2(X'_r) \).

Arguing now verbatim as in section 4.2, condition (16) will guarantee the existence of a \( w \in L^{0,n-1}_2(\tilde{X}_r) \), compactly supported in \( \tilde{X}_r \), such that \( \overline{\partial}w = \pi^*(\overline{\partial} \chi \wedge (f - g)) \) or equivalently the existence of a \( u = \phi(w) \in \text{Im} \phi \), compactly supported in \( X_r \), satisfying \( \overline{\partial}u = \overline{\partial} \chi \wedge (f - g) \) on \( X'_r \). Then, as in section 4.2, we can split \( f - g = f_1 + f_2 \), where

\[
f_1 = \chi(f - g) - u, \text{ compactly supported in } X_r,
\]

\[
f_2 = (1 - \chi)(f - g) + u \in L^{0,n-1}_2(X'_r)
\]

and \( f_1, f_2 \) are \( \overline{\partial} \)-closed. Moreover, we have \( \int_{X'_r} |f_1|^2 \|z\|^B \, dv < \infty \).

We shall now recall a result about weighted \( L^2 \)-estimates for solutions to \( \overline{\partial} \)-closed forms defined on \( \text{Reg} \Omega \), compactly supported in \( \Omega \), where \( \Omega \) Stein relatively compact subdomain of a Stein space and \( A = \text{Sing } X \):

**Theorem 4.7.** (Theorem 5.3 in [28]) Let \( f \) be a \((p,q)\) form defined on \( \text{Reg} \Omega \) and \( \overline{\partial} \)-closed there with \( 0 < q < n \), compactly supported in \( \Omega \) and such that \( \int_{\text{Reg} \Omega} |f|^2 \, d^N_A \, dv < \infty \) for some \( N_0 \geq 0 \). Then there exists a solution \( u \) to \( \overline{\partial} u = f \) on \( \text{Reg} \Omega \) satisfying \( \text{supp}_X u \subset \Omega \) and such that

\[
\int_{\text{Reg} \Omega} |u|^2 \, d^N_A \, dv \leq C \int_{\text{Reg} \Omega} |f|^2 \, d^N_A \, dv
\]

where \( N \) is a positive integer that depends on \( N_0 \) and \( \Omega \) and \( C \) is a positive constant that depends on \( N_0, \Omega \) and \( \text{supp } f \). Here \( d_A \) denotes the distance function to \( A \).

Using the above theorem, we know there exists a \( v \in L^{0,n-2}_2(X'_r) \) such that \( \overline{\partial}v = f_1 \). Therefore we have

\[
f = ((1 - \chi)f + \chi g + u) + \overline{\partial}v.
\]

Hence \( [f] = j_*[(1 - \chi)f + \chi g + u] \). Q.E.D.

In [35], Ruppenthal identified more or less the \( \ker(\overline{\partial})^{n,0} \) in terms of resolution data. Using the notation of Theorem 1.2 in the introduction we have:

**Lemma 4.8.** (Lemma 6.2 in [35]) \( \ker(\overline{\partial})^{n,0} = (\pi^{-1})^* \left( L^{n,0}_2(\tilde{X}_r) \cap \Gamma(\tilde{X}_r, K_{\tilde{X}_r} \otimes \mathcal{O}(|Z| - Z)) \right) \).

**Remark:** Lemma 6.2 in [35] only states that \( \Gamma(\tilde{X}_r, K_{\tilde{X}_r} \otimes \mathcal{O}(|Z| - Z)) = \ker(\overline{\partial}_{s, \text{loc}}) \cap \Gamma(\tilde{X}_r, F^{n,0}_{\gamma,E}) \), where \( \overline{\partial}_{s, \text{loc}} \) is defined as follows: Let \( f \in L^{p,q}_{\gamma,\text{loc}}(\tilde{X}_r) \), where \( \gamma \) is the “pseudometric” from section 2. We say that \( f \in \text{Dom}(\overline{\partial}_{s, \text{loc}})(\tilde{X}_r) \) if \( \overline{\partial}f \in L^{p,q+1}_{\gamma,\text{loc}}(\tilde{X}_r) \) and there exist a sequence of smooth forms \( f_j \) compactly supported away from \( E \) such that \( f_j \to f \) in the graph norm in \( L^{p,\gamma}_E(K) \) for any \( K \) compact subset of \( \tilde{X}_r \). We write in this case \( \overline{\partial}_{s, \text{loc}}f = \overline{\partial}f \) (where the right hand side is taken in the weak sense). Also \( F^{n,0}_{\gamma,E}(\tilde{X}_r) := L^{n,0}_{\gamma,\text{loc}}(\tilde{X}_r) \cap \text{Dom}(\overline{\partial}_{s, \text{loc}})(\tilde{X}_r) \).

**Proof of Lemma 4.8.** It follows immediately from Lemma 6.2 in [35] that \( \ker(\overline{\partial})^{n,0} \subset (\pi^{-1})^*(L^{n,0}_2(\tilde{X}_r) \cap \Gamma(\tilde{X}_r, K_{\tilde{X}_r} \otimes \mathcal{O}(|Z| - Z))) \).
It remains to prove, assuming that \( L^{n,0}_{(2)}(\tilde{X}_r) \cap \Gamma(\tilde{X}_r, K_{\tilde{X}} \otimes \mathcal{O}([Z] - Z)) \) is non-trivial, we proceed as follows: By Ruppenthal’s result we know that given \( f \in \Gamma(\tilde{X}_r, K_{\tilde{X}} \otimes \mathcal{O}([Z] - Z)) \), there exists a sequence of \( f_j \) smooth compactly supported away from \( E \) such that \( f_j \to f \) in the graph norm in \( L^{n,*}_r(K) \) for every compact subset \( K \) of \( \tilde{X}_r \). Choose a cut-off function \( c \in C^\infty_0(\tilde{X}_r) \) such that \( c = 1 \) near \( E \). Then \( c f_j \to c f \) in graph norm in \( L^{n,*}_r(\tilde{X}_r) \) and the same holds true for their push-forward. The push-forward of \((1 - c) f \) is easily approximated in graph norm by smooth forms supported away from \( 0 \) (as \( f \) is assumed to be now in \( L^{n,0}_r(\tilde{X}_r) = L^{n,0}_{(2)}(\tilde{X}_r) \)). Hence \((\pi^{-1})^* f \in \ker(\overline{\partial}^1)^{n,0} \).

4.4.1. An alternative description of \( \text{Im } j_* \). The second key step in the proof of Theorem 1.2 is the realization that \( \text{Im } j_* = \text{Im } \ell_* \) or equivalently

**Lemma 4.9.** The map \( \ell_* : H^{n-1}(\tilde{X}_r, \mathcal{O}(D)) \to H^{n-1}(X'_r, \mathcal{O}) \) is surjective on \( \text{Im } j_* \).

**Proof.** We need to show that i) \( \text{Im } \ell_* \subset \text{Im } j_* \) and ii) \( \text{Im } j_* \subset \text{Im } \ell_* \). To prove i) it suffices to show for any \( g \in L^{0,1}_{2,\text{loc}}(\tilde{X}_r, \mathcal{O}(D)) \cap \ker(\overline{\partial}) \), that \( \ell(g) \) satisfies the conditions of Lemma 4.6 ii), i.e.

\[
\alpha) \quad \int_{X'_r} |\ell(g)|^2 \|z\|^B dV < \infty,
\]

\[
\beta) \quad <\ell(g), \psi> = 0, \text{ for all } \psi \in \ker(\overline{\partial}^1)^{n,0}
\]

Property \( \alpha) \) follows easily, for some \( B > 0 \) sufficiently large, by the estimates in section 3 of an earlier paper of ours, see Lemma 3.1 in [11]. There, we compared weighted \( L^2 \)-norms between forms and their pull-backs under resolution of singularities maps.

It remains to prove \( \beta) \). When \( \psi \in \ker(\overline{\partial}^1)^{n,0} \) let \( \tilde{\psi} := \pi^* \psi \). Lemma 4.8 yields immediately that \( \tilde{\psi} \in L^{0,0}_{(2)}(\tilde{X}_r) \cap \Gamma(\tilde{X}_r, K \otimes \mathcal{O}(-D)) \). Let also \( \tilde{\chi} := \chi \circ \pi \). Then

\[
<\ell(g), \psi> = -\int_{\tilde{X}_r} \overline{\partial}\tilde{\chi} \wedge g \wedge \tilde{\psi}.
\]

But then \( g \wedge \tilde{\psi} \in L^{0,1}_{2,\text{loc}}(\tilde{X}_r) \) is \( \overline{\partial} \)-closed outside \( E \); thus it extends as a \( \overline{\partial} \)-closed form \( b \) in \( \tilde{X}_r \). Hence

\[
<\ell(g), \psi> = -\int_{\tilde{X}_r} \overline{\partial}\tilde{\chi} \wedge b = -\int_{\tilde{X}_r} d(\tilde{\chi} b) = 0
\]

by Stokes’ theorem, since \( \tilde{\chi} b \) has compact support in \( \tilde{X}_r \).

To prove ii) i.e. that \( \text{Im } j_* \subset \text{Im } \ell_* \) we will use a “twisted” version of arguments that appeared in sections 4.2-4.4. Let \( f \in Z^{0,1}_{(2)}(X'_r) \) and let \( \chi \in C^\infty_0(X_r) \) such that \( \chi = 1 \) near 0 and \( \text{supp } \chi \subset X_\rho \) for some 0 < \( \rho < r \). Let \( \tilde{f} := \pi^* f \) and \( \tilde{\chi} := \chi \circ \pi \). Then for all \( \psi \in \ker(\overline{\partial}^1)^{n,0}_{X'_r} \) we have that

\[
<\tilde{f}, \psi>_{X'_r} = \int_{X'_r} \tilde{f} \wedge \overline{\partial}\tilde{\chi} \wedge \tilde{\psi} = 0
\]

Using Lemma 4.8 this implies that

\[
\int_{X'_r} \tilde{f} \wedge \overline{\partial}\tilde{\chi} \wedge \tilde{\psi} = 0
\]

for all \( \tilde{\psi} \in L^{0,0}(\tilde{X}_r, \mathcal{O}(-D)) \cap \ker(\overline{\partial}) \) and where \( D := Z - |Z| \) is as in Theorem 1.2.

We may consider \( A := \tilde{f} \wedge \overline{\partial}\tilde{\chi} \otimes s \) as an element in \( L^{0,0}_{(2)}(\tilde{X}_r, L_D) \) with \( \text{supp } u \subset \tilde{X}_\rho \) and let \( B := \tilde{\psi} \otimes s^{-1} \in L^{0,0}_{(2)}(U, L_{-D}) \). Then we can rewrite (17) as

\[
\int_U A \wedge B = 0
\]
Now, the generalized moment condition \[13\] will permit us to solve the equation \( \partial_D F = A = \hat{f} \wedge \partial \hat{\chi} \otimes s \) with \( F \in L^{n-1}_{(2)}(\hat{X}_r, L_D) \) and \( \text{supp } F \subset \hat{X}_p \subset \hat{X}_r \). This is a consequence of the following \( L^2 \)-Cauchy problem:

**Proposition 4.10.** Let \( U \subset \subset \hat{X}_r \) be an open neighborhood of \( E \) with smooth strongly pseudoconvex boundary, and let \( h \in L^{0,1}_{(2)}(\hat{X}_r, L_D) \), \( \text{supp } h \subset U \). If

\[
\int_U h \wedge a = 0 \quad \text{for all } a \in L^{n,0}_{(2)}(U, L_D)
\]

then there exists a solution \( v \in L^{0,n-1}_{(2)}(\hat{X}_r, L_D) \) satisfying \( \partial v = h \) with \( \text{supp } v \subset U \subset \subset \hat{X}_r \).

**Proof.** Since \( U \) is a smoothly bounded strongly pseudoconvex domain in a complex manifold we know from Lemma 2.2 of section 2, that the Range(\( \partial_D \)) is closed in \( L^{n,1}_{(2)}(U, L_D) \). Hence we have the following strong decomposition

\[
L^{n,0}_{(2)}(U, L_D) = \text{Rang}(\partial_D^\perp) \oplus H^{n,0}_{(2)}(U, L_D)
\]

where \( H^{n,0}_{(2)}(U, L_D) := \ker(\partial_D) \cap L^{n,0}_{(2)}(U, L_D) \).

Now, \[13\] implies that \( \pi_D h \) is orthogonal to \( H^{n,0}_{(2)}(U, L_D) \), hence it belongs to the range of \( \partial_D^\perp \), i.e. there exists an element \( a \in \text{Dom}(\partial_D) \) such that \( \partial_D a = \pi_D h \). By Proposition 1 in [6], we know that \( \partial_D a \in \text{Dom}(\partial_{D,min}) \) and \( \partial_D = -\pi_D (\partial_{D,min}) \pi_D \). Hence there exist a sequence of compactly supported sections \( a_n \in D^{0,n-1}_{(2)}(U, L_D) \) such that \( a_n \to \partial_D a \) and \( \partial_D a_n \to -(-1)^n \pi_D h \) in \( L^{n,0}_{(2)}(U, L_D) \).

Set \( \hat{a} := (-\pi_D a_n)^0 \), i.e. the trivial extension by zero outside \( U \). Then we claim that \( \partial \hat{a} = h \) in \( \hat{X}_r \). Indeed, take \( \psi \in D^{0,n}(\hat{X}_r, L_D) \) and let us look at

\[
(\hat{a}, \partial_D \psi)_{\hat{X}_r} = (-\partial_D a, \partial_D \psi)_U = -\lim(a_n, \partial_D \psi)_U \\
- \lim(\partial_D a_n, \psi)_U = (h, \psi)_U = (h, \psi)_{\hat{X}_r}.
\]

Here we used the fact that \( a_n \) are compactly supported in \( U \) in order to perform integration by parts in the second line, and that \( h \) is compactly supported in \( U \).

Using the above proposition for \( U = \hat{X}_p \), we obtain a solution \( w \) to \( \partial w = u = \hat{f} \wedge \partial \hat{\chi} \) with \( w \in L^{0,n-1}_{(2)}(\hat{X}_r, \mathcal{O}(D)) \) and \( \text{supp } w \subset \hat{X}_r \).

Then, we can write \( \hat{f} = (\chi \hat{f} - w) + (w + (1 - \chi) \hat{f}) =: g_1 + g \). Let \( h := (\pi^{-1})^* g_1 \). Then \( \text{supp } h \subset \subset X_r \), \( \partial h = 0 \) on \( X'_r \) and \( \int_{X'_r} \|z\|^B |h|^2 dV < \infty \) for some \( B > 0 \) sufficiently large. Then, by Theorem 4.7 we know that there exists a solution \( t \in L^{0,n-2}_{2,loc}(X'_r) \) such that \( \partial t = h \). Hence we can write

\[
f = \ell(g) + \partial t.
\]

Hence we have \( j_*([f]) = \ell_*([g]) \) and thus ii) is proven.

Let \([f_1], \ldots, [f_m] \) be a basis of \( H^{0,n-1}_{(2)}(X'_r) \). Then we can define a map \( S: H^{0,n-1}_{(2)}(X'_r) \to H^{n-1}_{(2)}(\hat{X}_r, \mathcal{O}(D)) \) such that \( S(\sum c_j [f_j]) = \sum c_j [g_j] \), where \([g_j] \) satisfy \( j_*[f_j] = \ell_*[g_j] \).
4.5. **Proof of Theorem 1.2.** The sheaf inclusion $m : \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}}(D)$ induces a commutative diagram between long exact local cohomology sequences

\[
\begin{array}{ccccccc}
H^{n-2}(\tilde{X}, \mathcal{O}) & \xrightarrow{\delta} & H^{n-1}_{E}(\tilde{X}, \mathcal{O}) & \xrightarrow{k_{\ast}} & H^{n-1}(\tilde{X}, \mathcal{O}) & \xrightarrow{r_{\ast}} & H^{n-1}(\tilde{X}, \mathcal{O}) \\
\downarrow{m_{\ast}} & & \downarrow{m_{\ast}} & & \downarrow{m_{\ast}} & & \downarrow{m_{\ast}} \\
H^{n-2}(\tilde{X}, \mathcal{O}(D)) & \xrightarrow{\delta} & H^{n-1}_{E}(\tilde{X}, \mathcal{O}(D)) & \xrightarrow{k_{\ast}} & H^{n-1}(\tilde{X}, \mathcal{O}(D)) & \xrightarrow{r_{\ast}} & H^{n-1}(\tilde{X}, \mathcal{O}(D)).
\end{array}
\]

By Karras’ result we know that $H^{n-1}_{E}(\tilde{X}, \mathcal{O}) = 0$. Taking into account this and the commutativity of the above diagram (in particular of the left square) we obtain the following exact sequence:

\[(20) \quad 0 \to H^{n-1}_{E}(\tilde{X}, \mathcal{O}(D)) \xrightarrow{k_{\ast}} H^{n-1}(\tilde{X}, \mathcal{O}(D)) \xrightarrow{r_{\ast}} H^{n-1}(\tilde{X}, \mathcal{O}). \]

Using this information we construct the following diagram:

\[
\begin{array}{ccccccc}
& & H^{0,n-1}_{(2)}(X_{\ell}') & \xrightarrow{j_{\ast}} & (\text{Im } j_{\ast}) & \xrightarrow{\pi} & H^{n-1}(\tilde{X}, \mathcal{O}) \\
0 & \rightarrow & H^{n-1}_{E}(\tilde{X}, \mathcal{O}(D)) & \xrightarrow{k_{\ast}} & H^{n-1}(\tilde{X}, \mathcal{O}(D)) & \xrightarrow{r_{\ast}} & H^{n-1}(\tilde{X}, \mathcal{O}) \\
& & \downarrow{T} & & \downarrow{\ell_{\ast}} & & \downarrow{m_{\ast}} \\
& & H^{0,n-1}_{(2)}(X_{\ell}') & \xrightarrow{j_{\ast}} & (\text{Im } j_{\ast}) & \xrightarrow{\pi} & H^{n-1}(\tilde{X}, \mathcal{O}).
\end{array}
\]

The top row is the exact sequence from (20). From the right rectangle of the diagram, we observe that $r_{\ast} = m_{\ast} \circ (\pi^{\ast}) \circ \ell_{\ast}$.

We shall show in a moment that

**Lemma 4.11.** (i) The natural map $k_{\ast} : H^{n-1}_{E}(\tilde{X}, \mathcal{O}(D)) \to H^{n-1}(\tilde{X}, \mathcal{O}(D))$ is injective with $\text{Im } k_{\ast} = \text{kern } \ell_{\ast}$.

(ii) The map $j_{\ast}^{-1} \circ \ell_{\ast} : H^{n-1}(\tilde{X}, \mathcal{O}(D)) \to H^{0,n-1}_{(2)}(X_{\ell}')$ is a surjective map $T$ as in Theorem 1.2.

**Proof.** (i) The injectivity of $k_{\ast}$ follows from the exactness of (20). Also, from the exactness of (20) we have that $\text{Im } k_{\ast} = \text{kern } r_{\ast}$. Due to the commutativity of the right rectangle of the above diagram, we see that $\text{kern } r_{\ast} = \text{kern } \ell_{\ast}$.

(ii) As $j_{\ast}$ is an isomorphism between $H^{0,n-1}_{(2)}(X_{\ell}')$ and $\text{Im } j_{\ast}$, we can define the map $T := j_{\ast}^{-1} \circ \ell_{\ast} : H^{n-1}(\tilde{X}, \mathcal{O}(D)) \to H^{0,n-1}_{(2)}(X_{\ell}')$. Clearly $\text{kern } T = \text{kern } \ell_{\ast} = \text{Im } k_{\ast} \cong H^{n-1}_{E}(\tilde{X}, \mathcal{O}(D))$. The surjectivity of $T$ follows from the fact that $T \circ S = \text{Id}$ on $H^{0,n-1}_{(2)}(X_{\ell}')$.

**Remark 4.5.1** When $q < n - 1$ the map $j_{\ast} : H^{0,q}_{(2)}(X_{\ell}') \to H^{q}(X_{\ell}', \mathcal{O})$ is an isomorphism. Arguing in a similar manner as in section 4.5 we obtain the following short exact sequence of sheaves for each $q \leq n - 2$:

\[
0 \to H^{q}_{E}(\tilde{X}, \mathcal{O}(D)) \to H^{q}(\tilde{X}, \mathcal{O}(D)) \to H^{0,q}_{(2)}(X_{\ell}') \to H^{q+1}_{E}(\tilde{X}, \mathcal{O}(D))
\]

where the $H^{0,q}_{(2)}(X_{\ell}')$ entry appears due to the fact that $H^{q}(\tilde{X}, \mathcal{O}) \cong H^{q}(X_{\ell}', \mathcal{O})$.

As a consequence of the above sequence and in the special case where $-D$ is locally semi-positive with respect to $X$ we obtain (via Theorem 3.1 and Takegoshi’s vanishing theorem) for all $q$ with $0 \leq q \leq n - 2$ that $H^{0,q}_{(2)}(X_{\ell}') \cong H^{q}(\tilde{X}, \mathcal{O}(D))$. The isomorphism when $q = n - 1$ in this case has already been observed in the introduction as a consequence of Theorem 1.2. Hence we can recover Ruppenthal’s Theorem 7.1 from [35] for all $q \leq n - 1$.

5. **Proofs of Theorems 1.3 and 1.4**
5.1. Proof of Theorem 1.3. We choose neighborhoods \( \{V_j\}_{j=1}^m \) of \( \{a_j\}_{j=1}^m \) with \( V_j \subset X \) and such that for all \( j = 1, \ldots, m \), \( \tilde{X}_j \subset B(0, R) \subset \mathbb{C}^N \), where \( \tilde{X}_j \) are subvarieties with 0 as an isolated singular point. Assume \( \tilde{V}_i \cap V_j = 0 \), if \( i \neq j \). Set \( \tilde{V} := \bigcup_{j=1}^m V_j \), \( \tilde{V} = \pi^{-1}(V) \). Choose a partition of unity \( \chi_0, \chi_1, \ldots, \chi_m \) with \( \text{supp} \chi_0 \subset X \setminus \Sigma \) and \( \text{supp} \chi_j \subset V_j \) if \( j > 0 \). Thus \( \chi_j = 1 \) near \( a_j \) and \( \chi_0 = 1 \) near \( X \setminus V \). Let \( \phi_* : H^0,q(\tilde{X}, O) \rightarrow H^0,q(X') \) be the map sending \([f]\) to \([(\pi^{-1})_* f] \). We need to show that \( \phi_* \) is bijective for \( 1 \leq q \leq n - 2 \).

We show first surjectivity. Let \([f] \in H^0,q(\tilde{X}) \). By Theorem 1.1, we know that \( f_{|V_i} = \phi(g_i) + \overline{\partial} u_i \) for \( i = 1, \ldots, m \) where \( g_i \in Z^0,q(\tilde{X}_i) \) and \( u_i \in L^0,q-1(\tilde{X}_i) \). Set \( g := \pi^* f \) on \( \tilde{X} \setminus \tilde{V} \) and \( g := g_i + \overline{\partial} \pi^*(\chi_i u_i) \), on \( V_i \). Then \( g \in Z^0,q(\tilde{X}) \) and \( f - \phi(g) = \sum_{i=1}^m \overline{\partial}(\chi_i u_i)^m \) where by \( k^a \) we mean trivial extension of a function \( k \) by zero outside \( V_i \). Then \([f] = \phi_* [g] \).

To see injectivity, we let \( g \in Z^0,q(\tilde{X}) \) and assume that \( \phi(g) = \overline{\partial} u \) for some \( u \in L^0,q-1(\tilde{X}) \). Write \( g = \overline{\partial}(\pi^*(\chi_0 u)) + \sum_{i}^m g_i \), where \( g_i = \chi_i g + \overline{\partial} \chi_i \wedge \pi^* u \). We have \( g_i \in L^0,q(\tilde{V}_i) \) with \( \overline{\partial} g_i = 0 \) and \( \text{supp} g_i \) compact in \( \tilde{V}_i \). Hence \( [g_i] \in H^0,q(\tilde{V}_i, O) \). If \( A_i = \pi^{-1}(a_i) \) is the exceptional set of the desingularization \( \pi : \tilde{V}_i \rightarrow V_i \), then by Karras’ result we have that \( H^0,q(\tilde{V}_i, O) = 0 \). Hence there exists \( v_i \in L^0,q-1(\tilde{V}_i) \), compactly supported in \( \tilde{V}_i \) such that \( \overline{\partial} v_i = g_i \). Then we set \( v := (\pi^*(\chi_0 u)] + \sum_{i}^m v_i \). We can easily check that \( v \in L^0,q-1(\tilde{X}) \) and \( \overline{\partial} v = g \); hence \([g] = 0 \).

5.2. Proof of Theorem 1.4. In the proof of Theorem 1.2, the map \( j_* : H^p,q(\tilde{X}') \rightarrow H^p,q(X' \setminus \Sigma) \) was induced by the inclusion \( j : L^p,q(\tilde{X}') \rightarrow L^p,q(\tilde{X}') \) played a crucial role. For the situation we consider in Theorem 1.4, we need to introduce some auxiliary spaces and a modified map \( j' \). More precisely, let us set

\[
\'L^p,q(\tilde{X}) := \{ f \in L^p,q(\tilde{X}) \mid f \in L^p,q(X \setminus V) \}
\]

Let \( \'L^p,q(\tilde{X}) \) and let \( \'H^p,q(\tilde{X}) \) denote the cohomology of the complex \( \{\'L^p,*\}, \overline{\partial} \) \), where \( \overline{\partial} \) is taken with respect to the open subsets in \( X' \). Let us consider the inclusion map \( j' : L^p,q(\tilde{X}) \rightarrow \'L^p,q(\tilde{X}) \). Then we have:

**Proposition 5.1.** For \( p + q \leq n - 1 \), \( q > 0 \) the map \( j' : H^p,q(\tilde{X}') \rightarrow \'H^p,q(\tilde{X}) \) is injective.

Proof. Let \( f \in Z^0,q(\tilde{X}) \) and assume that \( j'_*([f]) = 0 \), i.e. \( \overline{\partial} u = f \) for some \( u \in \'L^0,q-1(\tilde{X}) \). Using the partition of unity \( \{\chi_i\}_{i=0}^m \), we can rewrite \( f \) as

\[
f = \overline{\partial} u = \overline{\partial}(\sum_{i=0}^m \chi_i u) = \overline{\partial}(\chi_0 u) + \sum_{i=1}^m (\chi_i f + \overline{\partial} \chi_i u).
\]

Now the forms \( g_i := \chi_i f + \overline{\partial} \chi_i \wedge u \in L^p,q(\tilde{V}_i \setminus \{a_i\}) \) are \( \overline{\partial} \)-closed there and \( \text{supp} g_i \subset V_i \); hence by Proposition 3.1 in II, we know that there exists a \( v_i \in L^0,q-1(\tilde{V}_i \setminus \{a_i\}) \), with compact support in \( V_i \) such that \( \overline{\partial} v_i = g_i \) for \( i = 1, \ldots, m \). Setting \( v := \chi_0 u + \sum_{i=1}^m v_i \in L^0,q-1(\tilde{X}) \) we have \( \overline{\partial} v = f \). Q.E.D.

**Remark 5.2.1:** Using a similar argument one can further show that the map \( j'_* : H^p,q(\tilde{X}') \rightarrow \'H^p,q(\tilde{X}) \) is bijective for \( p + q \leq n - 2 \), \( q > 0 \).

Let us consider the map \( \ell' : L^0,q-1(\tilde{X}, O(D)) \rightarrow \'L^0,q-1(\tilde{X}) \) which sends \( g \rightarrow (\pi^{-1})^* g \) and let \( \ell'_* : H^0,q-1(\tilde{X}, O(D)) \rightarrow \'H^0,q-1(\tilde{X}) \) be the corresponding map in cohomology. The first step in the proof of Theorem 1.4 is to show that \( \text{Im}(j'_*) = \text{Im}(\ell'_*) \).
We shall show first that $\text{Im}(\ell') \subset \text{Im}(\ell'')$. Let us consider an element $g \in Z^{0,n-1}_{(2)}(\tilde{X}, O(D)) := L^{0,n-1}_{(2)}(\tilde{X}, O(D)) \cap \ker(\nabla)$. By Lemma 4.9, we know that on $V_i$ we have $\ell''(g) = f_i + \nabla u_i$ where $f_i \in Z^{0,n-1}_{(2)}(V_i \setminus a_i)$ and $u_i \in L^{2,1}_{\text{loc}}(V_i \setminus a_i)$. Set $f := \ell''(g)$ on $X \setminus V$ and $f := f_i + \nabla (\chi_a u_i)$ on $V_i \setminus a_i$. Then $f \in L^{0,n-1}_{(2)}(X')$, is well-defined and $\nabla$-closed and $\ell''(g) - f = \sum_{i=1}^\infty \nabla(\chi_a u_i)^{a_i}$. Hence $\ell''([f]) = \ell''[g]$.

To show the other direction, we consider an element $f \in Z^{0,n-1}_{(2)}(X')$. By Lemma 4.9, we have on each $V_i$; $f = \ell''(g_i) + \nabla u_i$, where $g_i \in Z^{0,n-1}_{(2)}(V_i, O(D))$ and $u_i \in L^{2,1}_{\text{loc}}(V_i \setminus a_i)$. Set $g := \pi^* f$ on $\tilde{X} \setminus \tilde{V}$, and $g := g_i + \nabla (\chi_a u_i)$ on $\tilde{V}_i$ for $i = 1, \ldots, m$. Then $g \in L^{0,n-1}_{(2)}(\tilde{X}, O(D))$, is well-defined with $\nabla g = 0$. Then $f = \ell''(g) + \sum_{i=1}^m \nabla(\chi_a u_i)^{a_i}$; hence $\ell''[f] = \ell''[g]$.

Then we can consider the operator $\tilde{T} : H^{0,n-1}_{(2)}(\tilde{X}, O(D)) \rightarrow H^{0,n-1}_{(2)}(X')$ defined by $\tilde{T} := j'' \circ \ell''$; clearly $\tilde{T}$ is surjective. It remains to show that the kernel of $\tilde{T}$ is naturally isomorphic to $H^{-1}_{(2)}(\tilde{X}, O(D))$. As in the local case, we have the following short exact sequence

$$0 \rightarrow H^{-1}_{(2)}(\tilde{X}, O(D)) \rightarrow H^{-1}_{(2)}(X, O(D)) \rightarrow H^{-1}_{(2)}(X \setminus E, O(D))$$

By Karras' result we know that $H^{-1}_{(2)}(\tilde{X}, O(D)) \cong H^{-1}_{(2)}(\tilde{V}, O(D))$ and we have the following commutative diagram:

$$
\begin{array}{ccc}
0 & \longrightarrow & H^{0,n-1}_{(2)}(\tilde{X}, O(D)) \\
\downarrow \cong & & \downarrow \ell'' \\
H^{0,n-1}_{(2)}(\tilde{V}, O(D)) & \longrightarrow & H^{0,n-1}_{(2)}(X', O(D))
\end{array}
$$

From the commutativity of the left upper triangular part of the diagram we can conclude that the oblique map from $H^{0,n-1}_{(2)}(\tilde{V}, O(D)) \rightarrow H^{0,n-1}_{(2)}(\tilde{X}, O(D))$ is injective. Similarly, from the commutativity of the bottom left triangular part of the diagram we can conclude that $\ell''$ is injective. Now, by the definition of $\tilde{T}$ we know that $\ker \tilde{T} = \ker \ell''$. If we could show that the kernel $\ell'' = \text{Im} \ell''$, then we could finish the proof of Theorem 1.4, since the map $\ell''$ is injective and hence $\ker \tilde{T} \cong H^{-1}_{(2)}(\tilde{X}, O(D))$.

We first observe that $\ell'' \circ \ell'' = 0$ (i.e. $\text{Im} \ell'' \subset \text{Im} \ell''$). Indeed, consider an element $f \in L^{0,n-1}_{\text{comp}}(\tilde{V}, O(D)) \cap \ker(\nabla)$ (where the sub-index comp indicates that $f$ has compact support in $\tilde{V}$). Then $\ell''(f)|_{\tilde{V}_i}$ is $\nabla$-closed, with compact support on $\tilde{V}_i$ and with “polynomial blow-up”. By Theorem 4.7, we know that there exists $u_i \in L^{0,n-1}_{\text{loc}}(V_i \setminus a_i)$, compactly supported in $V_i$ such that $\nabla u_i = \ell''(f)$ on $V_i \setminus a_i$. Hence $[\ell''f] = 0$ in $H^{0,n-1}_{(2)}(X')$.

On the other hand, let $\ell''(g) = \nabla u$ with $u \in L^{0,n-2}_{(2)}(X')$, for some $g \in Z^{0,n-1}_{(2)}(\tilde{X}, O(D))$. Then $A := g - \nabla (\chi_a u)$ has compact support in $\tilde{V}$, so $[g] \in \text{Im} \ell''$. Q.E.D.

**Remark 5.2.2:** In the case of compact varieties $X$, we do not need to introduce the auxiliary spaces $L^{0,n}_{(2)}(X')$, ordinary local $L^2$-cohomology will do and Theorem 1.4 will be valid. Moreover, in the case of projective surfaces we can prove the following corollary:

**Corollary 5.2.** For projective surfaces $X$ with finitely many isolated singularities, the map

$$
\tilde{T} : H^{0,1}_{(2)}(\tilde{X}, O(D)) \rightarrow H^{0,1}_{(2)}(X')
$$

of Theorem 1.4 is an isomorphism (the right-hand side $L^2$-cohomology is computed with respect to the restriction of the Fubini-Study metric in $X'$). Here $\pi : \tilde{X} \rightarrow X$ is a desingularization of $X$ such that $E := \pi^{-1}(\text{Sing} X)$ is a divisor with simple normal crossings, $Z := \pi^{-1}(\text{Sing} X)$ is the unreduced exceptional divisor and $D := Z - E$. 

Proof. We shall show that $H^1_E(\tilde{X}, \mathcal{O}(D)) = 0$ by showing that the map $H^1_E(\tilde{X}, \mathcal{O}(D)) \to H^1_E(\tilde{X}, \mathcal{O}(Z))$ is injective. Now $H^1_E(\tilde{X}, \mathcal{O}(Z)) \cong H^1(\tilde{U}, \mathcal{O}(Z))$ where $\tilde{U}$ is a smooth strongly pseudoconvex neighborhood of $E$. The latter cohomology group is isomorphic to the dual of $H^1(\tilde{U}, \mathcal{K}(-Z))$, which vanishes by Takegoshi’s or Silva’s relative vanishing theorem, since $L \cdot Z$ is locally semi-positive with respect to $X$ (see example 11.22, page 56 in [8], or [35] pages 24-25). Hence, the proof of the corollary will be complete once we prove

**Lemma 5.3.** Under the assumptions of the corollary, the map

$$H^1_E(\tilde{X}, \mathcal{O}(D)) \to H^1_E(\tilde{X}, \mathcal{O}(Z))$$

is injective.

Proof. We introduce some auxiliary 1-cycles supported on $E = \cup_{i=1}^N E_i$ and where $E_i$ are the irreducible components of $E$. For a special ordering of the irreducible components of $E$ (to be determined later on), we set $D_0 := Z = \sum_{k=1}^N m_k E_k$, $D_j := Z - \sum_{k=1}^j E_k$. Then $D_N = Z - E$. Consider the standard short exact sequences of sheaves

$$0 \to \mathcal{O}(D_j) \to \mathcal{O}(D_{j-1}) \to \mathcal{O}_{E_j}(D_{j-1}) \to 0. \quad (21)$$

Taking long exact sequence on cohomology with support on $E$ we obtain for each $j \geq 1$

$$\cdots \to H^0_E(\tilde{X}, \mathcal{O}_{E_j}(D_{j-1})) \to H^1_E(\tilde{X}, \mathcal{O}(D_j)) \to H^1_E(\tilde{X}, \mathcal{O}(D_{j-1})) \to \cdots \quad (22)$$

Suppose were able to show that $E_j \cdot D_{j-1} < 0$ for all $j \geq 1$ for some ordering of the irreducible components. Then $H^0_E(\tilde{X}, \mathcal{O}_{E_j}(D_{j-1})) = H^0(E_j, \mathcal{O}_{E_j}(D_{j-1})) = 0$. This will imply that each map $H^1_E(\tilde{X}, \mathcal{O}(D_{j})) \to H^1_E(\tilde{X}, \mathcal{O}(D_{j-1}))$ is injective for each $j = 1, \ldots, N$. From this we can infer the injectivity of $H^1_E(\tilde{X}, \mathcal{O}(D_N)) \to H^1_E(\tilde{X}, \mathcal{O}(D_0))$ which is precisely what we want in the lemma.

To conclude the proof of the lemma it suffices to show that it is possible to rearrange the irreducible components $\{E_j\}$ of $E$ in such a way as to have $E_j \cdot D_{j-1} < 0$ for all $j \geq 1$. The proof below is a generalization of the proof of property a) in the Appendix of [30] (there they assumed that $E$ is connected, while we do not impose such a restriction).

Let $E_{(1)}, \ldots, E_{(m)}$ denote the connected components of $E$. We can write for each $1 \leq i \leq m$ $E_{(i)} := \cup_{j \in J_i} E_j$ where $J_1, J_2, \ldots, J_m$ partition $\{1, 2, \ldots, N\}$ and let $N_i := |J_i|$. As the set $E$ is exceptional in $\tilde{X}$, let $\Phi : \tilde{X} \to Y$ be the blow-down map. By Proposition 4.6 in [24], since $\tilde{X}$ is normal, $Y$ is normal. But then, using Lemma 4.1 in [24], each connected component $E_{(i)}$ of $E$ ($1 \leq i \leq m$) is mapped to a different point $\{y_i\}$ of $Y$ under $\Phi$. By theorem 4.4 in [24], the intersection matrix for each connected component $E_{(i)}$ of $E$, denoted by $S_{(i)} := (E_{(i)}^j \cdot E_{(i)}^k)$ for any ordering $E_{(i)}^1, \ldots, E_{(i)}^n$ of the irreducible components in $E_{(i)}$, is negative definite.

Set $Z_{(i)} := \sum_{k \in J_i} m_k E_{(i)}^k$. Let us observe that $E_{(i)}^j \cdot Z_{(i)} = E_{(i)}^j \cdot Z$ for $j \in J_i$, since irreducible components of $E$ that belong to different connected components do not intersect. Following an idea of Gonzalez-Sprinberg (Lemma 2.1 in [12], Pardon and Stern observed (in the proof of property a) in the Appendix in [30] as well as in Proposition 3.6 in [32]) that for each irreducible component $E_{(i)}^k$ of $E_{(i)}$ one has $E_{(i)}^k \cdot Z_{(i)} \leq 0$. Hence we have $E_{(i)}^j \cdot Z_{(i)} = E_{(i)}^j \cdot Z \leq 0$ for all $j \in J_i$. We claim now that there exists a $j \in J_i$ such that $E_{(i)}^j \cdot Z_{(i)} < 0$. Indeed, if for all $j \in J_i$ we had $E_{(i)}^j \cdot Z_{(i)} = 0$ this would imply that for all $j \in J_i$, we have $E_{(i)}^j \cdot E_{(i)}^k = 0$, which would contradict the negative definiteness of the matrix $S_{(i)}$. Hence there exists a $j \in J_i$ such that $E_{(i)}^j \cdot Z_{(i)} < 0$. Let us call this $E_{(i)}^j := E_{(i)}^j$. Since $E_{(i)}$ is connected, we can inductively define $E_{(i)}^1, \ldots, E_{(i)}^N$, such that $E_{(i)}^j$ intersects some $E_{(i)}^k$ for some $k < j; j > 1$ and such that
isomorphism when dim $X = 2$ in the 2-dimensional case, and hence the map

$$T : H^{n-1}(X, \mathcal{O}(D)) \to H^{0,n-1}(X'_r)$$

of Theorem 1.2 is an isomorphism when dim $X = n = 2$. It follows in an a similar way that the map $\tilde{T}$ of Theorem 1.4 is always an isomorphism when $n = 2 = \dim X$. 

\[
E_{ij} \cdot (Z - \sum_{k=1}^{j-1} E_{ik}) = E_{ij} \cdot (Z(i) - \sum_{k=1}^{j-1} E_{ik}) < 0.
\]
6. Proof of Corollary 1.6

In what follows, we use the assumptions and notation that were introduced in the paragraph just above Corollary 1.6 (and in the paragraph above Theorems 1.3, 1.4) in section 1.

6.1. A description of the kernel of \(i_*^n\). Recall that \(i_*^n : H^n(\tilde{X}, \mathcal{O}) \to H^n(\tilde{X}, \mathcal{O}(D))\) is the map on cohomology induced by the sheaf inclusion \(i : \mathcal{O} \to \mathcal{O}(D)\). Let us consider the following short exact sequence of sheaves

\[ 0 \to \mathcal{O} \to \mathcal{O}(D) \xrightarrow{i_*} \mathcal{O}_D(D) \to 0. \]

It yields two long exact sequences on cohomology

\[ \ldots H^{n-1}(\tilde{X}, \mathcal{O}) \xrightarrow{\delta} H^{n-1}(\tilde{X}, \mathcal{O}(D)) \xrightarrow{\mu_*} H^n(\tilde{X}, \mathcal{O}(D)) \to 0 \]

\[ \ldots H^{n-1}(\tilde{U}, \mathcal{O}) \xrightarrow{\delta} H^{n-1}(\tilde{U}, \mathcal{O}(D)) \xrightarrow{\mu_*} H^n(\tilde{U}, \mathcal{O}(D)) \to 0 \]

where \(\tilde{U} = \pi^{-1}(U)\) with \(U\) a disjoint union of smoothly bounded strongly pseudoconvex neighborhoods of the singular points \(\{a_j\}_{j=1}^m\). The vanishing of \(H^n(\tilde{U}, \mathcal{O})\) is due to Siu’s theorem in [37]. Also, \(H^n(\tilde{X}, \mathcal{O}_D(D)) = H^n(|D|, \mathcal{O}_D(D)) = 0\), since the support of \(D\), denoted by \(|D|\), is an \((n-1)\)-dimensional variety. From the exactness of the first long exact sequence we know that \(\ker(i_*^n) = \operatorname{Im}(\delta)\), and that

\[ \operatorname{Im}(\mu_*^n) \cong \frac{H^{n-1}(\tilde{X}, \mathcal{O}(D))}{i_*^n(H^{n-1}(\tilde{X}, \mathcal{O}))}. \]

We also have the following short exact sequence that defines \(\ker(i_*^n)\)

\[ 0 \to \operatorname{Im}(\mu_*^n) \xrightarrow{\delta} H^{n-1}(|D|, \mathcal{O}_D(D)) \xrightarrow{\delta} \ker(i_*^n) \to 0, \]

where \(I\) is the inclusion map.

From the exactness of the second long exact sequence above we obtain that

\[ H^{n-1}(\tilde{U}, \mathcal{O}_D(D)) = H^{n-1}(|D|, \mathcal{O}_D(D)) \xrightarrow{\Omega} \frac{H^{n-1}(\tilde{U}, \mathcal{O}(D))}{\ker(i_*^n)} = \frac{H^{n-1}(\tilde{U}, \mathcal{O}(D))}{i_*^n(H^{n-1}(\tilde{U}, \mathcal{O}))}. \]

The short exact sequence that defines \(\ker(i_*^n)\) can be rewritten now as follows:

\[ 0 \to \frac{H^{n-1}(\tilde{X}, \mathcal{O}(D))}{i_*^n(H^{n-1}(\tilde{X}, \mathcal{O}))} \xrightarrow{\nu} \frac{H^{n-1}(\tilde{U}, \mathcal{O}(D))}{i_*^n(H^{n-1}(\tilde{U}, \mathcal{O}))} \xrightarrow{\delta \circ \Omega^{-1}} \ker(i_*^n) \to 0, \]

for some injective map \(\nu := \Omega \circ I \circ \Psi^{-1}\).

Using the commutativity of the right grid in the very first diagram of section 4.5 and Karras’ results, we can conclude that the maps \(i_*^{n-1}, i_*^{n-1}\) (denoted in section 4.5 as \(m_*\)) are injective. By identifying \(i_*^{n-1}(H^{n-1}(\tilde{X}, \mathcal{O}))\) with \(H^{n-1}(\tilde{X}, \mathcal{O})\) and \(i_*^{n-1}(H^{n-1}(\tilde{U}, \mathcal{O}))\) with \(H^{n-1}(\tilde{U}, \mathcal{O})\) from the above short exact sequence we see that if

\[ \dim \mathcal{C} \frac{H^{n-1}(\tilde{U}, \mathcal{O}(D))}{H^{n-1}(\tilde{U}, \mathcal{O})} \neq \dim \mathcal{C} \frac{H^{n-1}(\tilde{X}, \mathcal{O}(D))}{H^{n-1}(\tilde{X}, \mathcal{O})}, \]

then the kernel of \(i_*^n\) would be non-trivial.
6.2. Proof of Corollary 1.6. We need to show that i) $\text{ker}(\phi_n^u) \subset \text{ker}(i_n^u)$ and ii) $\text{ker}(i_n^u) \subset \text{ker}(\phi_n^u)$. In what follows we shall think of $i_n^u : H^n(\tilde{X}, \mathcal{O}) \to H^n(\tilde{X}, \mathcal{O}(D)) \cong H^{0,n}(\tilde{X}, L_D)$ as the map on cohomology induced by the sheaf map $\mathcal{O} \to \mathcal{O}(L_D)$, sending $f \to f \otimes s$, where $s$ is the canonical section of $L_D$ introduced in section 2.2.

To prove i), let $[c] \in H^n(\tilde{X}, \mathcal{O})$ such that $\phi_n^u([c]) = [0]$. Without loss of generality we can assume that $[c]$ can be represented by an element $g \in L_{0,n}^2(\tilde{X})$ with $g = 0$ in $\tilde{U}$ (since we can solve $\partial t = g$ in a neighborhood of $\tilde{U}$ we can replace $g$ by $g - \bar{\partial}(\xi t^o)$, where $\xi$ is a cut-off function with $\xi = 1$ on $\tilde{U}$ and $t^o$ denotes trivial extension by zero outside $\tilde{U}$). Now, $\phi_n^u(g) = \bar{\partial}u$ for some $u \in L_{0,n}^2(\tilde{X}')$. Using a cut-off function $\chi \in C_\infty(\tilde{X})$ with $\chi = 1$ near the singular locus $A$ and $\text{supp} \chi \subset U$, we can rewrite $\phi_n^u(g)$ in $U'$ as

\[(\ast) \quad \phi_n^u(g) = \bar{\partial}u = \bar{\partial}\chi \wedge u + \bar{\partial}((1 - \chi)u).\]

This is possible since $g$ was taken to be 0 on $\tilde{U}$ yielding $u \in Z_{n-1}^0(U')$. Using the surjectivity of the map $\ell_*$ on $\text{Im} \ j_*$ (Lemma 4.9 in our paper), we know that there exists an $A \in Z_{n-1}^0(\tilde{U}, L_D)$ and a $v \in L_{2,\text{loc}}^{0,n}(\tilde{U})$ such that $u|_U = (\pi^{-1})^* (A \cdot s^1) + \bar{\partial}v$. Setting $\tilde{\chi} = \chi \circ \pi$ and applying $\pi^*$ on both sides of $(\ast)$, we obtain on $\tilde{U} \setminus E$

\[\ast \ast \quad g = \bar{\partial}\tilde{\chi} \wedge \pi^* u + \bar{\partial}((1 - \tilde{\chi}) \pi^* u)\]

\[= \bar{\partial}\tilde{\chi} \wedge (A \cdot s^1 + \bar{\partial}\pi^* v) + \bar{\partial}((1 - \tilde{\chi}) \pi^* u).\]

From $(\ast \ast)$ we obtain that $g \otimes s = \bar{\partial}(\tilde{\chi} A^o - \bar{\partial}\tilde{\chi} \wedge \pi^* v^o \otimes s + (1 - \tilde{\chi}) \pi^* u \otimes s) = \bar{\partial}B$, with $B = \tilde{\chi} A^o - \bar{\partial}\tilde{\chi} \wedge \pi^* v^o \otimes s + (1 - \tilde{\chi}) \pi^* u \otimes s \in L_{0,n}^2(\tilde{X}, L_D)$, since $\pi$ is a quasi-isometry from $\{\tilde{\chi} < 1\}$ onto $\{\chi < 1\}$. Hence $i_n^u([g]) = [0]$.

To prove ii) we consider again an element $[c] \in \text{ker}(i_n^u)$. As in the proof of i), we can assume that this class may be represented by an element $g \in L_{0,n}^2(\tilde{X})$ with $g = 0$ in $\tilde{U}$. By assumption we have that there exists an element $A \in L_{0,n}^2(\tilde{X}, L_D)$ such that $g \otimes s = \bar{\partial}A = \bar{\partial}\tilde{\chi} \wedge A + \bar{\partial}((1 - \tilde{\chi}) A)$ and thus we have on $\tilde{U} \setminus E$

\[\ast \ast \ast \quad g = \bar{\partial}\tilde{\chi} \wedge A \cdot s^{-1} + \bar{\partial}((1 - \tilde{\chi}) (A \cdot s^{-1}))\]

since $g = 0$ on $\tilde{U}$ and therefore $A \in Z_{n-1}^0(\tilde{U}, L_D)$. Now by the surjectivity of $\ell_*$ on $\text{Im} j_*$ (Lemma 4.9 in our paper) we know that there exist elements $t \in Z_{n-1}^0(U')$ and $v \in L_{2,\text{loc}}^{0,n}(U')$ such that on $U'$ we have $(\pi^{-1})^*(A \cdot s^{-1}) = t + \bar{\partial}v$. Applying $\phi_n^u$ on $(\ast \ast \ast)$ we can express $\phi_n^u(g)$ on $X'$ as $\phi_n^u(g) = \bar{\partial}\chi \wedge (t^o + \bar{\partial}v^o) + \bar{\partial}((1 - \chi) \phi_n^u(A \cdot s^{-1})) = \bar{\partial}(\chi t^o - \bar{\partial}\chi \wedge v^o - (1 - \chi) \phi_n^u(A \cdot s^{-1})) = \bar{\partial}C$ where $C := \chi t^o - \bar{\partial}\chi \wedge v^o - (1 - \chi) \phi_n^u(A \cdot s^{-1}) \in L_{n-1}^{0,n}(X')$. Hence $\phi_n^u([g]) = [0]$. Q.E.D.

Using Corollary 1.6, we can easily show the existence of the following commutative diagram

\[\begin{array}{ccc}
H^n(\tilde{X}, \mathcal{O}) & \xrightarrow{\phi_n^u} & H^{0,n}_2(X') \\
\downarrow{i_n^u} & & \downarrow{\text{some map}} \\
H^n(\tilde{X}, \mathcal{O}(D)) & & \\
\end{array}\]

As the maps $\phi_n^u$, $i_n^u$ are surjective and $\text{ker}(i_n^u) = \text{ker}(\phi_n^u)$ the dotted map will be an isomorphism. Thus $H^{0,n}_2(X') \cong H^n(\tilde{X}, \mathcal{O}(D))$. Q.E.D.
7. Examples

The purpose of this section is to produce various examples for which we have or not vanishing of $L^2$-$\mathcal{D}$-cohomology groups.

In [10] we showed that whenever 0 was a Cohen-Macaulay point of a pure $n$-dimensional complex analytic variety with $n \geq 3$, then we have $H^q_{(2)}(X') = 0$ for all $q$ with $1 \leq q \leq n - 2$ (this result was obtained using Theorem 2.4 and an extension theorem of cohomology classes by Scheja). Classical examples of Cohen-Macaulay singularities are rational singularities of dimension $n \geq 2$ (Corollary 4.3 in [10]-attributed to Kempf). Recall that in a complex space $X$, a normal point $p \in X$ is called rational if given a resolution of singularities $\pi : \tilde{X} \to X$ we have that $(R^i\pi_*\mathcal{O}_{\tilde{X}})_p = 0$ for all $i > 0$. It follows from Hironaka’s work that the condition on $R^i\pi_*\mathcal{O}_{\tilde{X}}$ is independent of the choice of $\tilde{X}$. There is a plethora of rational singularities as the following examples suggest.

Example 1: Quotient singularities. We have the following theorem proven by Burns:

Theorem 7.1. (Proposition 4.1 in [4]) Let $M$ be a complex manifold and $G$ a properly discontinuous group of automorphisms of $M$. Then $X = M/G$ has only rational singularities.

This implies all the double point singularities in dimension 2 ($A_k, D_k, E_6, E_7, E_8$) are rational singularities.

Example 2: Arnold’s singularities (Example 3.4 in [4]). These are direct generalizations of the rational double points of dimension 2.

Example 3: Some affine cones over smooth projective hypersurfaces (Example 1.2 in [4]). Let $V \subset \mathbb{CP}^n$ be a smooth hypersurface of degree $d$, described by the nonsingular homogeneous polynomial $F(Z)$ in the homogeneous coordinates $Z_1, \cdots, Z_{n+1}$. Let $X$ be defined by $F(Z) = 0$ in $\mathbb{C}^{n+1}$. $X$ is called the affine cone over $V$. Let $\mathcal{O}_V(-1)$ denote the restriction of the universal line bundle on $V$, $p : \mathcal{O}_V(-1) \to V$ the corresponding projection map and let $\tilde{X}$ denote the total space of $\mathcal{O}_V(-1)$. Now, the map $\pi : \tilde{X} \to X$ is the contraction of the zero section of $\mathcal{O}_V(-1)$, hence $\pi$ a resolution of singularities of $X$ with $\pi^{-1}(0) = V$. In the algebraic category, using this, we can see that $0 \in X$ is a rational singularity if and only if $d \leq n$. Indeed, by the Leray spectral sequence of $\pi$ we have that $H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}) = H^0(X, R^j\pi_*\mathcal{O}_{\tilde{X}}) = (R^j\pi_*\mathcal{O}_{\tilde{X}})_0$. Using the Leray spectral sequence for $p$ (the fact that $R^i\pi_*\mathcal{O}_{\tilde{X}} = 0$ for all $i > 0$, hence $H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}) \cong H^j(V, p_*\mathcal{O}_{\tilde{X}})$ and expanding cohomology classes into Taylor series along the fibers of $\mathcal{O}_V(-1)$ we get that $H^j(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \oplus_{k \geq 0} H^j(V, \mathcal{O}_V(k))$. Using the fact that the canonical line bundle of $V$ is given by $K_V = [(d-n-1)H|_V]$ (adjunction formula) where $H$ is the hyperplane bundle in $\mathbb{CP}^n$, Serre’s duality and the dual version of Kodaira’s vanishing theorem for negative line bundles, we see that all these cohomology groups vanish for $j > 0$ if $d \leq n$. In this case 0 is a rational singularity for the affine cone over $V$.

On the other hand, when $d = n+1$ the calculation in Example 3 will yield $H^{n-1}(\tilde{X}, \mathcal{O}) = H^{n-1}(V, \mathcal{O})$. If $\dim_c H^{n-1}(V, \mathcal{O}) \neq 0$ then one produces a non-rational singularity. This happens for example if $V$ is any Riemann surface in $\mathbb{CP}^2$, of genus $g \geq 1$.

Remark: The local $L^2$-$\mathcal{D}$-cohomology groups are completely determined in the case of affine cones over smooth projective varieties. In this case $Z = |Z|$ and Theorem 1.2 in our paper or Theorem 7.1 of Ruppenthal in [35] guarantees that $H^q_{(2)}(X') \cong H^q(\tilde{X}, \mathcal{O})$ for all $q$ with $1 \leq q \leq n - 1$.

7.1. Some remarks on non Cohen-Macaulay spaces. There are many irreducible complex analytic spaces that are not Cohen-Macaulay. In a very interesting paper [36] Stückrad and Vogel constructed a wealth of examples of smooth projective varieties $V$ (Proposition 9 in [36]) whose affine cone over $V$, denoted by $X(V)$ and abbreviated by $X$ when there is no confusion, had the property that its local ring at the vertex of the cone was not Cohen-Macaulay. The precise construction is as follows:

Let $d \geq 3$ and consider the variety $W \subset \mathbb{CP}^{d-1}$ defined by the equation $z_0^d + z_1^d + \cdots + z_{d-1}^d = 0$. Let $V$ be the Segre embedding of $W \times \mathbb{CP}^1$ in $\mathbb{CP}^{d-1}$. In [36] it is shown that the local ring $\mathcal{O}_{X,0}$ (0 is the vertex of

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5 This vanishing is a consequence of Exercises 8.1-8.2 page 252 in [14].
affine cone $X$ over $V$) is a normal non Cohen-Macaulay ring. Andreatta and Silva used this construction to produce in [II] another example of non-rational singularity.

Using a Künneth formula for Segre products one has

$$H^{d-1}(V, \mathcal{O}_V(k)) \cong \oplus_{r+s=d-1} (H^r(W, \mathcal{O}_W(k)) \otimes H^s(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(k)))$$

By the version of Kodaira's vanishing theorem for negative line bundles and using also the fact that $K_W \cong \mathcal{O}_W$ by the adjunction formula, the dual of the cohomology groups $H^r(W, \mathcal{O}_W(k))$ can be computed

$$(H^r(W, \mathcal{O}_W(k)))' \cong H^{d-2-r}(W, K_W \otimes \mathcal{O}_W(-k)) \cong H^{d-2-r}(W, \mathcal{O}_W(-k)) = 0$$

for $r > 0$ and $k > 0$. By the Künneth formula above we have

$$H^{d-1}(V, \mathcal{O}_V(k)) \cong H^{d-2}(W, \mathcal{O}_W(k)) \otimes H^1(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(k)) = 0.$$  

since $H^1(\mathbb{CP}^1, \mathcal{O}_{\mathbb{CP}^1}(k)) = 0$ for all $k \geq -1$. We know that there must exist an $i$ with $1 \leq i \leq d-2$, such that $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0$, where $\tilde{X}$ is a desingularization of the affine cone $X$ over $V$ (otherwise, the vertex of the cone would have been a rational singularity, thus Cohen-Macaulay). As mentioned in the remark above, using Theorem 1.2 we have that $H^{0,d-1}_2(X_i) = 0$ while, for some $i$ with $1 \leq i \leq d-2$ we have that $H^{0,d-1}_2(X_i) \neq 0$ (using Theorem 1.1).

The vertex of the cone in this example is a new type of singularity called weakly rational singularity. Recall that in an $n$-dimensional complex space $(X, \mathcal{O}_X)$, a point $p \in X$ is called a weakly rational singularity of $X$ if $(R^{n-1}p, \pi, \mathcal{O}_X)_p = 0$. As before $\pi : \tilde{X} \to X$ is a resolution of $X$. The above definition is independent of the resolution $\pi$. Andreatta and Silva and Yau studied to what extent Laufer's results on rational singularities from [28] generalize to this category of singularities. It is clear that when $n = 2$, then the definitions of weakly rational and rational coincide. For higher dimensional singularities this is no longer true as the following example shows:

**Example.** Consider a compact Riemann surface $V$ of genus $1$ and $F$ a sufficiently negative vector bundle of rank $r \geq 2$ over $V$. Let $X$ denote the total space of $F$. Let $\pi : X \to V$ be the blow-down of $V$ in $\tilde{X}$ and $x := \pi(V)$. Then, $x$ is weakly rational since $H^{d_{\dim X} - 1}(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ but not rational since $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \neq 0$.

To show this we need the following facts: a) Proposition 26 from Andreotti-Grauert [2]-that discusses filtrations of cohomology groups of vector bundles over complex manifolds and their associated graded complex- and asserts that $\text{Grad} H^i(X, \mathcal{O}_X) \cong \oplus_{k=0}^n H^i(V, (F^*)^k)$, b) the fact that $\dim \mathcal{H}^i(V, (F^*)^k) = 1$ for $k = 0$ and 0 otherwise, and last c) the knowledge that the cohomology groups $H^i(V, (F^*)^k) = 0$ for all $k \geq 0$ and $q \geq 2$.

Moreover $\mathcal{O}_{X,x}$ is not Cohen-Macaulay; recall that if the homological codimension $\text{codim} \mathcal{O}_{X,x} = \dim_x X \geq 3 \geq 2 + 1$ then by Theorem 3.1, page 37 in [3] we should have $H^1_x(X, \mathcal{O}_X) = 0$ for all $i \leq 2$. From the local cohomology exact sequence and taking into account that $X$ is Stein, we see that $H^2_x(X, \mathcal{O}) = H^1(X \setminus \{x\}, \mathcal{O}) \cong H^1(\tilde{X} \setminus V, \mathcal{O})$. Using Karras' results the latter cohomology group is isomorphic to $H^1(\tilde{X}, \mathcal{O})$. The latter space is nonzero from earlier computations. Hence $\text{codim} \mathcal{O}_{X,x} \neq \dim_x X$.

Using Theorem 1.1 and these calculations we obtain $\dim H^{0,d}_2(U \setminus \{x\}) \neq 0$, where $U$ is a small Stein neighborhood of $x$ with smooth boundary and $H^{0,d}_2(U \setminus \{x\}) = 0$ for $2 \leq q \leq \dim_x X - 2$ (if $\dim_x X \geq 4$).

Now by further blowing up $V$ inside $\tilde{X}$ (for example blowing-up the ideal sheaf of $V$) we can obtain a manifold $\tilde{X}$, a map $p : \tilde{X} \to X$ such that $p^{-1}(V) = E$ ($E$ is non-singular as it is locally isomorphic to $V \times \mathbb{P}^{s-1}$, with $s = \text{codim}(V, \tilde{X})$ and locally principal—see for example Theorem 8.24, page 186 in [14]). Then we obtain a resolution map $\tilde{p} : \tilde{X} \to X$, such that $\tilde{p}^{-1}(x) = E$. Using Theorem 1.2 (as $-D$ is locally semi-positive with respect to $\tilde{X}$), we obtain $H^{0,d}_2(U \setminus \{x\}) \cong H^{d_{\dim X} - 1}(\tilde{p}^{-1}(U), \mathcal{O}(D))$ where $D = Z - |Z|$ and $Z = mE$ is the unreduced divisor $\tilde{p}^{-1}(x)$ and $|Z| = E$. If $m = 1$ then $\mathcal{O}(D) \cong \mathcal{O}$.

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Let us recall Proposition 2.1 from [28]: Let $N$ be a vector bundle over a compact Riemann surface $A$. Suppose that $N = \bigoplus_{i=1}^n L_i$, where $L_i$ is a line bundle of Chern class $c_i$ on $A$. Then $A$ is exceptional in $N \iff c_i < 0$, $1 \leq i \leq n - 1$. 

hence $H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}((m - 1) E)) = H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}) = 0$ by the calculation above. Determining the vanishing or not of $H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}((m - 1) E))$ when $m \geq 2$ is slightly more involved.

We know that $\mathcal{O}_X(E)_{1E} = N_{E|\tilde{X}} = \mathcal{O}_{P(N_{V|X})}(-1)$ (see Proposition 12.4 in section 12, Chapter VII of [2]), where $N_{V|X}$ is the normal bundle of $V$ in $\tilde{X}$ and $\mathcal{O}_{P(N_{V|X})}(-1)$ is the tautological line bundle over $E = P(N_{V|X})$. Here the projectivized normal bundle $P(N_{V|X})$ is defined by considering lines in $N_{V|X}$.

Using observation $\beta$ in the Characterization of Exceptional Sets (section 3.1) on page 10, and Proposition 26 from [2], we have

$$ H^{\dim_r X - 1}(\hat{p}^{-1}(U), \mathcal{O}(D)) \cong H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}((m - 1) E)), $$

$$ \text{Grad } H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}((m - 1) E)) = \oplus_{k \geq 0} H^{\dim_r X - 1}(E, \mathcal{O}((m - 1) E)_{1E} \otimes \mathcal{O}_E(1)^k), $$

where $\mathcal{O}_E(1) = \mathcal{O}_E(-1)^*$ (the dual of the tautological line bundle over $E$). Hence,

$$ \text{(25)} \quad \text{Grad } H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}((m - 1) E)) = \oplus_{k \geq 0} H^{\dim_r X - 1}(E, \mathcal{O}_E(k - m + 1)) $$

$$ \text{(26)} \quad = \oplus_{k \geq 0} H^0(E, K_E \otimes \mathcal{O}_E(m - 1 - k)). $$

The first equality follows from the fact that $\mathcal{O}((m - 1) E)_{1E} = \mathcal{O}_E(-(m - 1))$. The second follows from Serre duality. For a fixed rank $r$ of the vector bundle $F$, the vanishing or not of $H^{\dim_r X - 1}(\tilde{X}, \mathcal{O}((m - 1) E))$ depends on the multiplicity $m$ of the divisor $Z$. To determine what happens for $m \geq 2$ we need to recall some facts about projectivized vector bundles. We summarize them in the following Proposition:

**Proposition 7.2.** Let $W$ be a holomorphic vector bundle of rank $r \geq 2$ over a complex manifold $M$, let $P(W) := \mathbb{P}(W)$ be the projectivized bundle and let $p : P(W) \to M$ be the associated projection map. Let $\mathcal{L}_1^{-1}(W) := \{((z, [v]), \lambda v) \mid (z, [v]) \in P(W), \lambda \in \mathbb{C}\}$ be the tautological line bundle associated to $W$ (it is a sub-bundle of $p^* W$), let $\mathcal{L}_{P(W)}$ denote its dual bundle (“hyperplane bundle”) and let $\otimes^k W^*$ denote the $k$-th symmetric product of $W^*$. Then we have:

i) (Proposition 2.2 in [20] or Theorem 3.5 in [5]) The canonical bundle $K_{P(W)}$ is given by

$$ K_{P(W)} = p^*(\det W^* \otimes K_M) \otimes \mathcal{L}_{P(W)}^{-1}, $$

where $\mathcal{L}_{P(W)}^{-1}$ is the dual of the $r$-fold tensor product of the “hyperplane bundle” $\mathcal{L}_{P(W)}$.

ii) (Lemma 3.1 in [13] or Theorem attributed to Grothendieck on page 403 in [5]) For all $m \geq 0$, $i > 0$ and for any coherent analytic sheaf $S$ on $M$, we have

$$ \text{ii.a) } p_* (\mathcal{L}_m^{P(W)} \otimes p^* S) \cong \otimes^m W^* \otimes S, $$

$$ \text{ii.b) } R^i p_* (\mathcal{L}_m^{P(W)} \otimes p^* S) = 0. $$

In what follows we will abbreviate the normal bundle $N_{V|X}$ of $V$ in $\tilde{X}$ by $NV$. Recall that $E = P(NV)$ and $p : P(NV) \to V$ is the associated projection map. Identifying $\mathcal{O}_E(-1) = \mathcal{L}_{P(NV)}^{-1}$ and using Proposition 7.2 i) (taking into account that $K_V = 0$), we can rewrite the right-hand side of (23) as

$$ \oplus_{k \geq 0} H^0(E, p^*(\det NV)^* \otimes \mathcal{L}_{P(NV)}^{-k-m-1}). $$

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7Kobayashi-Ochiai’s $L(W)$ corresponds in our notation to $\mathcal{L}_{P(W)}^{-1}$.

8Hartshorne’s $P(W)$ corresponds in our notation to $P(W^*)$.

9La theorie des classes de Chern, Bull. S. M. F, tome 86, (1958), 137-154.
Now, if \(-r - k + m - 1 = 0\) (this can happen for example for certain values of \(k\) when \(m \geq 3\) and \(r = 2\) or more generally when \(m \geq r + 1\), we obtain from Proposition 7.2 ii.a)

\[
H^0(E, p^*(\det(NV)^*) \otimes L_{P(NV)}^{-r-k+m}) \cong H^0(V, \det(NV)^* \otimes O_V),
\]

since \(\rho^*(\det(NV)^*) = O_V\). Now \(\det(NV)^*\) is a line bundle of positive degree over \(V\) so by Riemann-Roch we know it has a non-zero section. Hence, the right-hand side of (27) (and therefore \(H^{\dim_xX-1}(\hat{X}, O((m-1)E))\)) does not vanish.

On the other hand, if \(-r - k + 1 + m < 0\) (this can happen for all non-negative integers \(k\), when \(m = 2\) and \(r \geq 2\) or more generally when \(m < r + 1\) then taking into account that \(p_*(L_{P(E)}^\nu \otimes p^*S) = 0\) for all \(\nu \geq 1\) and all coherent analytic sheaves \(S\) on \(V\), we obtain that \(H^0(E, p^*(\det(NV)^*) \otimes L_{P(NV)}^{-r-k+1+m}) = 0\).

Hence \(H^{\dim_xX-1}(\hat{X}, O((m-1)E)) = 0\) in this case (i.e. when \(m < r + 1\)).

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Dept. of Mathematics, University of Oslo, P.B 1053 Blindern, Oslo, N-0316 NORWAY

Dept. of Mathematics, Georgetown University, Washington, DC 20057 USA

E-mail address: nilsov@math.uio.no, sv46@georgetown.edu