COHOMOLOGIES OF SASAKIAN GROUPS AND SASAKIAN SOLVMANIFOLDS

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Abstract. We show certain symmetry of the dimensions of cohomologies of the fundamental groups of compact Sasakian manifolds by using the Hodge theory of twisted basic cohomology. As applications, we show that the polycyclic fundamental groups of compact Sasakian manifolds are virtually nilpotent and Sasakian solvmanifolds are finite quotients of Heisenberg nilmanifolds.

1. Introduction

The purpose of this paper is to study the fundamental groups of compact Sasakian manifolds. Sasakian manifolds constitute an odd-dimensional counterpart of the class of Kähler manifolds. The fundamental groups of compact Kähler manifolds satisfy various properties. Since the first cohomology of manifolds and the first group cohomology of their fundamental groups are isomorphic, the first group cohomology of the fundamental group of a compact Kähler manifold was studied precisely by using the Hodge theory. In this paper, we study the first cohomologies of the fundamental groups of compact Sasakian manifolds for 1-dimensional representations by using the Hodge theory of the basic cohomology.

Let $\Gamma$ be a group. We denote by $C(\Gamma)$ the space of characters $\Gamma \to \text{GL}_1(\mathbb{C})$ which can be factored as

$$\Gamma \to H_1(\Gamma, \mathbb{Z})/\text{(torsion)} \to \text{GL}_1(\mathbb{C}).$$

For $\rho \in C(\Gamma)$, we denote by $H^*(\Gamma, C_{\rho})$ the group cohomology of $\Gamma$ with values in the module associated with the 1-dimensional representation $\rho$. Considering the exponential map $\mathbb{C} \to \mathbb{C}^* = \text{GL}_1(\mathbb{C})$, we have the surjective map $E : H^1(\Gamma, C) \to C(\Gamma)$. We define the "real" action of $\mathbb{R}^*$ on $C(\Gamma)$ such that for $f_1 + \sqrt{-1}f_2 \in H^1(\Gamma, \mathbb{C})$ with $f_1, f_2 \in H^1(M, \mathbb{R})$, the action is given by

$$t \cdot E(f_1 + \sqrt{-1}f_2) = E(tf_1 + \sqrt{-1}f_2)$$

for $t \in \mathbb{R}^*$.

In this paper we prove the "$\mathbb{R}^*$-symmetry" of cohomologies of the fundamental group of a compact Sasakian manifold.

Theorem 1.1. Let $(M, g)$ be a compact Sasakian manifold. Then for each $\rho \in C(\pi_1(M))$ and $t \in \mathbb{R}^*$, we have

$$\dim H^*(\pi_1(M), \rho) = \dim H^*(\pi_1(M), t \cdot \rho).$$

For a group $\Gamma$, we consider the set

$$\mathcal{J}_k(\Gamma) = \{ \rho \in C(\Gamma) | H^1(\Gamma, C_{\rho}) \geq k \}.$$
For $C(\pi_1(M)) \ni \rho = \mathcal{E}(f_1 + \sqrt{-1}f_2)$ with $f_1, f_2 \in H^1(M, \mathbb{R})$, $\mathcal{E}(f_1 + \sqrt{-1}f_2)$ is fixed by the $\mathbb{R}^*$-action if and only if $f_1 = 0$ (equivalently $\rho$ is unitary). Hence we have the following corollary.

**Corollary 1.2.** Let $(M, g)$ be a compact Sasakian manifold. If there exists a non-unitary character $\rho : \pi_1(M) \to \text{GL}_1(\mathbb{C})$ satisfying $H^1(\pi_1(M), \mathbb{C}_\rho) \geq k$, then the set $\mathcal{J}_k(\pi_1(M))$ is an infinite set.

By this corollary, we prove the following result which is analogous to the Arapura-Nori’s result in [2].

**Corollary 1.3.** Let $(M, g)$ be a compact Sasakian manifold. Suppose that the fundamental group $\pi_1(M)$ is polycyclic. Then $\pi_1(M)$ is virtually nilpotent.

We consider nilmanifolds and solvmanifolds. Solvmanifolds (resp. nilmanifolds) are compact homogeneous spaces of solvable (resp. nilpotent) Lie groups. It is known that every nilmanifold can be represented by $G/\Gamma$ such that $G$ is a simply connected nilpotent Lie group and $\Gamma$ is a lattice in $G$ (see [13]).

In [5], it is proved that a compact $2n+1$-dimensional nilmanifold admits a Sasakian structure if and only if it is a Heisenberg nilmanifold $H_{2n+1}/\Gamma$ where $H_{2n+1}$ is the $(2n+1)$-dimensional Heisenberg Lie group and $\Gamma$ is its lattice.

By Corollary 1.2, we can easily extend the result in [5] for solvmanifolds as in [8].

**Corollary 1.4.** A compact $2n+1$-dimensional solvmanifold admitting a Sasakian structure is a finite quotient of a Heisenberg nilmanifold.

**Proof.** It is known that the fundamental group of a compact solvmanifold is a torsion-free polycyclic group and solvmanifolds with isomorphic fundamental groups are diffeomorphic (see [13]). Hence by Corollary 1.3, we can easily show that a compact $2n+1$-dimensional solvmanifold admitting a Sasakian structure is a finite quotient of a Sasakian nilmanifold. Thus the Corollary follows from the result in [5].

In particular we have the following result.

**Corollary 1.5.** Let $G$ be a $2n+1$-dimensional simply connected solvable Lie group with a lattice $\Gamma$. We assume that $G$ is completely solvable (i.e. for any $g \in G$, all eigenvalues of the adjoint operator $Ad_g$ are real). Then the compact solvmanifold $G/\Gamma$ admits a Sasakian structure if and only if it is a Heisenberg nilmanifold.

**Proof.** By the Saito’s rigidity theorem in [14], if a simply connected completely solvable Lie group contains a nilpotent lattice, then it is nilpotent. Hence if $G/\Gamma$ admits a Sasakian structure, then by Corollary 1.3 we can easily show that $G$ is nilpotent. Thus the Corollary follows from the result in [5].

\[ \square \]

2. Preliminary

Let $M$ be a compact smooth manifold and $\Lambda^*(M)$ the de Rham complex of $M$. For a $\mathbb{C}$-valued closed 1-form $\phi \in \Lambda^1(M) \otimes \mathbb{C}$, we consider the operator $\phi \wedge : \Lambda^r(M) \otimes \mathbb{C} \to \Lambda^{r+1}(M) \otimes \mathbb{C}$ of left-multiplication. Define $d_\phi = d + \phi \wedge$. Then we have $d_\phi d_\phi = 0$ and hence $(\Lambda^*(M) \otimes \mathbb{C}, d_\phi)$ is a cochain complex. We denote by $H^*(M, \phi)$ the cohomology of $(\Lambda^*(M) \otimes \mathbb{C}, d_\phi)$. The cochain complex $(\Lambda^*(M) \otimes \mathbb{C}, d_\phi)$ is considered as the de Rham complex with values in the topologically trivial flat bundle $M \times \mathbb{C}$ with the connection form $\phi$. Hence the structure of the cochain complex $(\Lambda^*(M) \otimes \mathbb{C}, d_\phi)$ is determined by the character $\rho_\phi : \pi_1(M) \to \text{GL}_1(\mathbb{C})$.
Lemma 3.1. We have an isomorphism $H^1(M, \phi) \cong H^*(\pi_1(M), \rho_\phi)$. The map $H^1(M, \mathbb{C}) \ni [\phi] \mapsto \rho_\phi \in \mathcal{C}^1_1(M)$ is identified with the map $\mathcal{E}$ as in Introduction. Hence the action of $\mathbb{R}^*$ on $\mathcal{C}(\pi_1(M))$ as in Introduction is given by

$$t \cdot \rho_\phi = \rho_{t \text{Re}_\phi + \sqrt{-1} \text{Im}_\phi}$$

for $t \in \mathbb{R}^*$ and $[\phi] \in H^1(M, \mathbb{C})$.

### 3. Proof of Theorem 1.1

3.1. **Basic cohomology.** Let $(M, g)$ be a compact $(2n+1)$-dimensional sasakian manifold and $\eta$ the contact structure associated with the sasakian metric $g$. Take $\xi$ the Reeb vector field. Let $A^*(M)$ be the de Rham complex of $M$. A differential form $\alpha \in A^*(M)$ is basic if $i_\xi \alpha = 0$ and $i_\xi d\alpha = 0$. Denote $A^*_B(M)$ the differential graded algebra of the basic differential forms on $M$ and denote by $H^*_B(M, \mathbb{R})$ (resp. $H^*_B(M, \mathbb{C})$) the cohomology of $A^*_B(M)$ (resp. $A^*_B(M) \otimes \mathbb{C}$). Then it is known that the inclusion $A^*_B(M) \subset A^*(M)$ induces a cohomology isomorphism $H^*_B(M, \mathbb{R}) \cong H^*(M, \mathbb{R})$ and $H^*_B(M, \mathbb{C}) \cong H^*(M, \mathbb{C})$ (see [1]

Consider the Hodge star operator $*: A^*(M) \to A^{2n+1-r}(M)$ for the sasakian metric $g$. We define the transverse Hodge star operator $*_{T}: A^*_B(M) \to A^{2n+1-r}(M)$ as $*_{T}(\alpha) = *_{T}(\eta \wedge \alpha)$, Then we have $*_{T}\alpha = (-1)^r\alpha$. Restricting the scalar product $\langle \cdot, \cdot \rangle_{A^*(M) \times A^*(M)} \to \mathbb{R}$ on the basic forms $A^*_B(M)$ we consider formal adjoint $\delta_{B} : A^*_B(M) \to A^{r-1}_B(M)$ of the differential $d$ on $A^*_B(M)$. Then we have $\delta_{B} = -*_{T}d*_{T}$. Consider the basic Laplacian $\Delta_{B} = dd_{B} + \delta_{B}d_{B}$. A basic form $\alpha \in A^*_B(M)$ is harmonic if $\Delta_{B}\alpha = 0$. Denote $\mathcal{H}^*_B(M) = \ker \Delta_{B,A^*_B(M)}$. Then we have the Hodge decomposition (see [9], [7] and [16])

$$A^*_B(M) = \mathcal{H}^*_B(M) \oplus \text{im} d_{|A_{B}^{r-1}(M)} \oplus \text{im} \delta_{B,A_{B}^{r+1}(M)}.$$

We have the transverse complex structure on $\ker i_\xi \subset \Lambda^* TM$ and we obtain the bi-grading $A^*_B(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} A^{p,q}_B(M)$ with the bi-differential $d = \partial_{B} + \bar{\partial}_{B}$. We consider the formal adjoint $\partial_{B} : A^{p,q}_B(M) \to A^{p-1,q}_B(M)$ and $\bar{\partial}_{B} : A^{p,q}_B(M) \to A^{p,q+1}_B(M)$ of $\partial_{B}$ and $\bar{\partial}_{B}$ respectively for the restricted Hermitian inner product on $A^*_B(M) \otimes \mathbb{C}$. Then we have $\partial_{B} = -*T \partial_{B}*T$ and $\bar{\partial}_{B} = -*T \bar{\partial}_{B}*T$. Denote $\Delta' = \partial_{B}\partial_{B} + \bar{\partial}_{B}\bar{\partial}_{B}$ and $\Delta'' = \partial_{B}\partial_{B} + \bar{\partial}_{B}\bar{\partial}_{B}$.

Let $\omega = d\eta$. Then $\omega$ gives a transverse Kähler structure. We can take a complex coordinate $(z_1, \ldots, z_n)$ which is transvers to $\xi$ such that $\omega$ is a Kähler form on $(z_1, \ldots, z_n)$. Define the operator $L : A^{p,q}_B(M) \to A^{p+1,q+1}_B(M)$ by $L\alpha = \omega \wedge \alpha$ and consider the formal adjoint $\Lambda : A^{p,q}_B(M) \to A^{p-1,q-1}_B(M)$ of $L$. We have $\Lambda = -*T L*_{T}$. By using the transverse Kähler geometry, we obtain the following Kähler identity.

**Lemma 3.1.**

$$\Lambda \partial_{B} - \partial_{B}\Lambda = -\sqrt{-1}\partial_{B} \quad \text{and} \quad \Lambda \bar{\partial}_{B} - \bar{\partial}_{B}\Lambda = -\sqrt{-1}\bar{\partial}_{B}.$$

This implies $\Delta_{B} = 2\Delta'_{B} = 2\Delta'_{B}$ and hence we have the Hodge structure

$$\mathcal{H}^*_B(M) \otimes \mathbb{C} = \bigoplus_{p+q=r} \mathcal{H}^{p,q}_B(M) \quad \text{and} \quad \overline{\mathcal{H}^{p,q}_B(M)} = \mathcal{H}^{p,q}_B(M)$$

where $\mathcal{H}^{p,q}_B(M) = \ker \Delta'_{B,A^*_B(M)} = \ker \Delta''_{B,A^*_B(M)}$.
3.2. Twisted basic cohomology. Let $\phi \in A^1_B(M) \otimes \mathbb{C}$ be a closed basic 1-form. Then $(A^*_B(M) \otimes \mathbb{C}, d_\phi)$ is a cochain complex. Denote by $H^*_B(M, \phi)$ the cohomology of this complex. It is known that there exists a sub-torus $T \subset \text{Isom}(M, g)$ such that $A^*_B(M) \otimes \mathbb{C} \subset (A^*(M) \otimes \mathbb{C})^T$ and we have the exact sequence of complexes

$$0 \longrightarrow A^*_B(M) \otimes \mathbb{C} \longrightarrow (A^*(M) \otimes \mathbb{C})^T \longrightarrow A^*_B^{-1}(M) \otimes \mathbb{C} \longrightarrow 0$$

for the usual differential $d$ (see [3] Section 7.2.1). We can say that this is also exact for twisted differential $d_\phi$. Hence, taking the long exact sequence, we have the exact sequence

$$0 \longrightarrow H^1_B(M, \phi) \longrightarrow H^1(M, \phi) \longrightarrow H^0_B(M, \phi).$$

We can easily check

$$H^0_B(M, \phi) = H^0(M, \phi) = 0$$

and hence we have:

**Lemma 3.2.**

$$H^1_B(M, \phi) \cong H^1(M, \phi).$$

Consider the formal adjoint $(\phi\wedge)_B^* : A^*_B(M) \otimes \mathbb{C} \rightarrow A^{* -1}(M) \otimes \mathbb{C}$ of the operator $\phi\wedge$ for the restricted Hermitian inner product on $A^*_B(M) \otimes \mathbb{C}$. Then we have

$$(\phi\wedge)_B^* = *_T (\phi\wedge)_T$$

(see the proof of [12] Corollary 2.3). Taking the formal adjoint $(\phi\wedge)^* : A^*(M) \otimes \mathbb{C} \rightarrow A^{* -1}(M) \otimes \mathbb{C}$ on the usual de Rham complex $A^*(M) \otimes \mathbb{C}$, we have

$$(\phi\wedge)^*_{A^*(M) \otimes \mathbb{C}} = (\phi\wedge)_B^*.$$ 

Let $\delta_B, \phi = \delta_B + (\phi\wedge)^*$, $\Delta_B, \phi = d_\phi \delta_B, \phi + \delta_B, \phi d_\phi$ and $\ker \Delta_B, \phi|_{A^*(M) \otimes \mathbb{C}} = \mathcal{H}_B^1(M, \phi)$. As in [9], [7] and [10], we have the Hodge decomposition

$$A^*_B(M) \otimes \mathbb{C} = \mathcal{H}_B^*(M, \phi) \oplus \text{im} d_{\phi|_{A^*_B(M) \otimes \mathbb{C}}} \oplus \text{im} \delta_{\phi|_{A^*_B(M) \otimes \mathbb{C}}}.$$

Consider the double complex $(A^*_B, \partial_B, \partial_B^*)$ for the usual differential $d$. For a $(1,0)$-basic form $\theta$, considering the operators $\theta\wedge : A^p_B(M) \rightarrow A^{p+1}_B(M)$ and $\Lambda : A^p_B(M) \rightarrow A^{p-1}_B(M)$, as the local argument for the Kähler identities, see, e.g., [17] Lemma 6.6], we have the following identity.

**Lemma 3.3.**

$$\Lambda(\theta\wedge) - (\theta\wedge)\Lambda = -\sqrt{-1} *_T (\theta\wedge) *_T \quad \text{and} \quad \Lambda(\tilde{\theta}\wedge) - (\tilde{\theta}\wedge)\Lambda = \sqrt{-1} *_T (\tilde{\theta}\wedge) *_T.$$

Let $\phi \in \mathcal{H}_B^1(M) \otimes \mathbb{C}$. By $\mathcal{H}_B^1(M) = H^1_B(M) = H^0_B(M) + H^0_B^1(M)$, we can take unique $\theta_1, \theta_2 \in H^0_B^1(M)$ such that $\phi = \theta_1 + \theta_2 - \bar{\theta}_2$. Define

$$\partial_B, \theta_1, \theta_2 = \partial_B + \theta_2 \wedge + \bar{\theta}_1 \wedge \quad \text{and} \quad \partial_B, \theta_1, \theta_2 = \partial_B - \theta_2 \wedge + \theta_1 \wedge.$$

By $H^p_B(M) = \ker \Delta_B^{p+1}_B = \ker \Delta_B^p|_{A^p_B(M) \otimes \mathbb{C}, \partial_B, \theta_1, \theta_2}$ and $(A^*_B(M) \otimes \mathbb{C}, \partial_B, \theta_1, \theta_2)$ are cochain complexes. Denote by $H^*_B(M, \theta_1, \theta_2)$ the cohomology of $(A^*_B(M) \otimes \mathbb{C}, \partial_B, \theta_1, \theta_2)$. As similar to [11] Lemma 2.1, we obtain the following lemma.

**Lemma 3.4.** For any $t \in \mathbb{R}^*$, we have

$$\dim H^*_B(M, \theta_1, \theta_2) = \dim H^*_B(M, t\theta_1, \theta_2).$$
Consider the formal adjoint
\[ \partial_{B, \theta_1, \theta_2}^* = \partial_{B} + (\theta_2 \wedge)^* + (\bar{\theta}_1 \wedge)^* \quad \text{and} \quad \partial_{B, \theta_1, \theta_2}^\dag = \partial_{B}^\dag - (\theta_2 \wedge)^* + (\theta_1 \wedge)^*. \]

Since we have \((\theta_1 \wedge)^* = \ast_T (\bar{\theta}_1 \wedge)^*\), by the Lemma 3.4 and 3.3, we obtain the following Kähler identity (cf. [I Section 1.2], [13 Page 15]).

**Lemma 3.5.**

\[ \Lambda \partial_{B, \theta_1, \theta_2} - \partial_{B, \theta_1, \theta_2} \Lambda = \sqrt{-1} \partial_{B, \theta_1, \theta_2}^\dag \quad \text{and} \quad \Lambda \partial_{B, \theta_1, \theta_2} - \partial_{B, \theta_1, \theta_2} \Lambda = -\sqrt{-1} \partial_{B, \theta_1, \theta_2}^\dag. \]

Define

\[ \Delta_{B, \theta_1, \theta_2} = \partial_{B, \theta_1, \theta_2} \partial_{B, \theta_1, \theta_2} + \partial_{B, \theta_1, \theta_2} \partial_{B, \theta_1, \theta_2} \quad \text{and} \quad \Delta_{B, \theta_1, \theta_2} = \partial_{B, \theta_1, \theta_2} \partial_{B, \theta_1, \theta_2} + \partial_{B, \theta_1, \theta_2} \partial_{B, \theta_1, \theta_2}. \]

Then Lemma 3.3 implies \( \Delta_{B, \phi} = 2 \Delta_{B, \theta_1, \theta_2} = 2 \Delta_{B, \theta_1, \theta_2} \). Denote \( H^*(M, \theta_1, \theta_2) = A_{B, \theta_1, \theta_2}(M) \otimes \mathbb{C} \).

We have the Hodge decomposition

\[ A_B^*(M) \otimes \mathbb{C} = H^*(M, \theta_1, \theta_2) \oplus \text{im} \partial_{B, \theta_1, \theta_2} A_{B, \theta_1, \theta_2}(M) \otimes \mathbb{C} \oplus \text{im} \partial^*_{B, \theta_1, \theta_2} A_{B, \theta_1, \theta_2}(M) \otimes \mathbb{C}. \]

Hence we obtain an isomorphism

\[ H_B^*(M, \phi) \cong H_B^*(M, \theta_1, \theta_2). \]

By Lemma 3.2 and 3.4 we have the following result.

**Theorem 3.6.** For any \( t \in \mathbb{R}^* \), we have the equation

\[ \dim H^1(M, \phi) = \dim H^1(M, t\theta_1 + \theta_2 - \bar{\theta}_2). \]

Since we have isomorphisms \( H^1(M, \mathbb{C}) \cong H^1_B(M, \mathbb{C}) \cong H^1_B(M) \otimes \mathbb{C} \), we obtain Theorem 1.1

4. **Proof of Theorem 1.3**

Let \( G \) be a simply connected solvable Lie group and \( N \) be the nilradical (i.e. maximal connected nilpotent normal subgroup) of \( G \). Denote by \( \mathfrak{g} \) the Lie algebra of \( G \). Then we can take a simply connected nilpotent subgroup \( C \subset G \) such that \( G = C \cdot N \) (see [6 Proposition 3.3]). Since \( C \) is nilpotent, the map

\[ \Phi : C \ni c \mapsto (\text{Ad}_c)_s \in \text{Aut}(\mathfrak{g}) \]

is a diagonalizable representation where \( (\text{Ad}_c)_s \) is the semi-simple part of the adjoint operator \( \text{Ad}_c \) (see [10]). We take a diagonalization \( \Phi = \text{diag}(\alpha_1, \ldots, \alpha_n) \) where \( \alpha_1, \ldots, \alpha_n \) are \( \mathbb{C} \)-valued characters of \( C \). Since the adjoint representation \( \text{Ad} \) on the nilradical \( N \) is unipotent and we have an isomorphism \( G/N \cong C/C \cap N \), \( \alpha_1, \ldots, \alpha_n \) are considered as characters of \( G \). For each \( g \in G \), \( \alpha_i(g) \) is an eigenvalue of \( \text{Ad}_g \).

Suppose \( G \) admits a lattice \( \Gamma \). We consider the solvmanifold \( G/\Gamma \). \( G/\Gamma \) is an aspherical manifold with the fundamental group \( \Gamma \). In [11], the set \( \mathcal{J}_1(\Gamma) \) was studied. The author proved that \( \mathcal{J}_1(\Gamma) \) is a finite set ([11 Corollary 5.9]) and if one of the characters \( \alpha_1, \ldots, \alpha_n \) is non-unitary, then there exists a non-unitary character \( \rho \in \mathcal{C}(\Gamma) \) of \( \Gamma \) such that \( \rho \in \mathcal{J}_1(\Gamma) \) ([11 Corollary 5.10]). We suppose that \( \Gamma \) can be the fundamental group of a compact Sasakian manifold. Then by Corollary 1.2 characters \( \alpha_1, \ldots, \alpha_n \) are all unitary. Hence, for any \( g \in G \), all eigenvalues of \( \text{Ad}_g \) are unitary. In this case, a lattice \( \Gamma \) of \( G \) is virtually nilpotent (see [3 Chapter IV, 5]). It is known that every polycyclic group contains a lattice of some simply connected solvable Lie group as a finite index normal subgroup (see [13 Theorem 4.28]). Hence Corollary 1.3 follows.
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