ON A PERTURBED COMPOUND POISSON MODEL WITH VARYING PREMIUM RATES

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(Communicated by Hailiang Yang)

Abstract. In this paper, we consider a perturbed compound Poisson model with varying premium rates. The surplus process is observed at a sequence of review times. The effective premium rate is adjusted according to the surplus increment between the inter-review times. We study the Gerber-Shiu functions by Laplace transform method. When the claim size density is a combination of exponentials, the explicit expressions for the Laplace transforms of ruin time are derived.

1. Introduction. In insurance risk theory, the compound Poisson model perturbed by diffusion is defined by

\[ \tilde{U}_t = u + ct - S_t + \sigma B_t, \quad t \geq 0, \]

where \( u \geq 0 \) is the initial surplus, and \( c > 0 \) is the constant premium rate. The aggregate claims process \( S_t = \sum_{j=1}^{N_t} X_j \) is a compound Poisson process, where \( \{N_t\}_{t \geq 0} \) is a homogeneous Poisson process with intensity \( \lambda > 0 \), and \( \{X_j\}_{j \geq 1} \)

\( 2010 \) Mathematics Subject Classification. Primary: 91B30; Secondary: 62P05.

Key words and phrases. Varying premium rates, Gerber-Shiu function, Laplace transform, ruin.

Z.M. Zhang was supported by the National Natural Science Foundation of China [11471058, 11101451, 11301303] and the Natural Science Foundation Project of CSTC of China [cstc2014jcjyA00007]. The research of Y. Yang was supported by National Natural Science Foundation of China (No. 71471090), the Humanities and Social Sciences Foundation of the Ministry of Education of China (No. 14YJCZH182), China Postdoctoral Science Foundation (No. 2014T70449, 2012M520964), Natural Science Foundation of Jiangsu Province of China (No. BK20131339), the Major Research Plan of Natural Science Foundation of the Jiangsu Higher Education Institutions of China (No. 15KJA110001), Qing Lan Project, PAPD, Program of Excellent Science and Technology Innovation Team of the Jiangsu Higher Education Institutions of China, Project of Construction for Superior Subjects of Statistics of Jiangsu Higher Education Institutions, Project of the Key Lab of Financial Engineering of Jiangsu Province. The research of C.L. Liu was supported by the Fundamental Research Funds for the Central Universities (No. 106112015CDJXY100006).

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is a sequence of i.i.d. r.v.’s with common density \( f_X \) (on \((0, \infty))\), mean \( \mu_X = \int_0^\infty x f_X(x)dx \) and Laplace transform \( f_X(s) = \int_0^\infty e^{-sx} f_X(x)dx \). Finally, \( \{B_t\}_{t \geq 0} \) is a standard Brownian motion starting from zero, and \( \sigma > 0 \) is a diffusion parameter.

The perturbed compound Poisson model was first proposed to extend the classical risk model [6]. Since then, a lot of contributions to model [11] have been made in the literature. See e.g. [6, 13, 14, 17, 18, 4, 10], to name a few. We note that it is typically assumed that the premium rate is constant over time. However, the constant premium rate assumption is often justified at the macro level by assuming that the insurance company’s portfolio is stable over time. Although the diffusion component can describe uncertainty of the premium rate, it cannot effectively represent the point view of the company. [19] considered a perturbed risk model, where the premium rates depend on the claim sizes. In their paper, the premium rate may be adjusted at every claim arrival time. [9] studied a risk model with varying premium rates, where the effective premium rate is determined by the surplus increments between successive review times. In [9], it is assumed that the inter-review time has a combination-of-exponentials density. Although the combination-of-exponentials density can approximate any continuous density function on \((0, \infty)\), the constant inter-review time assumption is more appropriate from the point view of practical applications.

In this paper, we modify model [11] by allowing the premium rate to vary according to the recent claim experience. We consider the embedded premium policy proposed by Li et al. [9]. Let \( 0 = Z_0 < Z_1 < Z_2 < \cdots \) be a sequence of successive review times, and for \( j = 1, 2, \cdots \), define \( \eta_j \) \( (\eta_j = c_1 \text{ or } c_2) \) to be the effective premium rate between review times \( Z_{j-1} \) and \( Z_j \), where we assume that \( 0 < c_1 < c_2 < \infty \). For \( j = 1, 2, \cdots \), let \( T_j = Z_j - Z_{j-1} \) be the \( j \)th inter-review time, and conditioning on \( \{\eta_j\}_{j \geq 1} \), it is assumed that the inter-review times \( \{T_j\}_{j \geq 1} \) are i.i.d, as well as independent of other random quantities. The premium rate is adjusted as follows: (1) if the surplus increment between \( Z_{j-1} \) and \( Z_j \) is larger than zero, the effective premium rate \( \eta_{j+1} \) in the next inter-review time interval \((Z_j, Z_{j+1})\) is \( c_1 \); (2) if the surplus increment between \( Z_{j-1} \) and \( Z_j \) is smaller than zero, the effective premium rate \( \eta_{j+1} \) in the next inter-review time interval \((Z_j, Z_{j+1})\) is \( c_2 \). Furthermore, we assume that \( c_1 > \lambda \mu_X \) so that the safety loading condition holds. After the above modification, we denote the surplus process by \( \{U_t\}_{t \geq 0} \).

In this paper, we assume that the inter-review times are Erlang \((\gamma, n)\) distributed with density

\[
\gamma^n t^{n-1} e^{-\gamma t} \quad (n-1)!, \quad t > 0,
\]

where \( \gamma > 0 \) is the scale parameter, and \( n \) is the shape parameter that is a positive integer. The Erlang distribution has been widely used in risk theory. On the one hand, the Erlang assumption can lead to explicit expressions for ruin related functions due to its mathematical tractability. On the other hand, the constant inter-review time can be approximated by the Erlangization techniques proposed by Asmussen et al. [3]. (see also e.g. [11, 12, 11, 16, 10]).

Suppose that the solvency is discretely monitored, and define the ruin time by \( \tau = Z_j \), where \( J = \inf\{j \geq 1 : U_{Z_j} < 0\} \). Let \( \tau = \infty \) if \( U_{Z_j} \geq 0 \) for all \( j = 1, 2, \cdots \). Let \( \delta \geq 0 \) be the interest force, and let \( I(A) \) be the indicator function of the event \( A \). We are interested in the Gerber-Shiu function

\[
\phi_i(u) = E\left[e^{-\delta \tau w (U_{Z_{j-1}}, U_{Z_j})} I(\tau < \infty) | U_0 = u, \eta_1 = c_i\right], \quad i = 1, 2.
\]
where \( w \) defined on \([0, \infty) \times (0, \infty)\) is a penalty function of the surplus prior to ruin and the deficit at ruin.

The remainder of this paper is organized as follows. In Section 2, we derive the discounted density of the surplus increment. The Laplace transforms of the Gerber-Shiu function is given in Section 3. Finally, when the claim size density is a combination of exponentials, we find the explicit expressions for the Laplace transforms of the ruin time.

2. Preliminaries. For \( q > 0 \), let \( e_q \) denote an exponential r.v. with mean \( 1/q \) (independent of other random quantities), and let \( e_{q,1}, e_{q,2}, \ldots \) be independent copies of \( e_q \). For \( i = 1, 2, \ldots \), let \( Y_i(t) = c_i t - S_i + \sigma B_t, \ t \geq 0 \). For \( n = 1, 2, \ldots \), define

\[
F_{q,n,i}(dx) = P (Y_i(e_{q,1} + \cdots + e_{q,n}) \in dx),
\]

and let \( f_{q,n,i}(x) \) be the corresponding density function. Since \( Y_i(t) \) has stationary independent increments property, we have for \( n \geq 2 \)

\[
f_{q,n,i}(x) = \int_{-\infty}^{\infty} f_{q,n-1,i}(y)f_{q,1,i}(x-y)dy,
\]

which gives a recursive approach to compute \( f_{q,n,i} \) with the starting point \( f_{q,1,i} \).

For convenience, we put

\[
f_{q,n,i}(x) = f_{q,n,i}^{-}(x)I(x < 0) + f_{q,n,i}^{+}(x)I(x > 0)
\]

in the remainder of this paper. Note that \( f_{q,n,i}^{+} \) and \( f_{q,n,i}^{-} \) are defined on \((0, \infty)\).

First, let us consider the case \( n = 1 \). For \( i = 1, 2, \ldots \), let

\[
\psi_i(s) = \frac{1}{t} \ln E e^{s Y_i(t)} = \frac{\sigma^2}{2} s^2 + c_i s - \lambda \left(1 - f_X(s)\right), \ s \geq 0,
\]

be the Laplace exponent of \( Y_i \), and for any \( q > 0 \), let \( \rho_{q,i} \) be the positive root of equation \( \psi_i(s) = q \). Note that \( \rho_{q,i} \) is unique and it is the right inverse of \( \psi_i \). It follows from Corollary 8.9 in [8] that

\[
F_{q,1,i}(dx) = \int_0^\infty q e^{-qt} P(Y_i(t) \in dx) dt = \left(\frac{q}{\psi_i'(\rho_{q,i})}\right) e^{-\rho_{q,i} x} - q W_{q,i}^{(q)}(-x) dx,
\]

where \( W_{q,i}^{(q)} \) is the \( q \)-scale function (see [8], Theorem 8.1(i)). The \( q \)-scale function is such that \( W_{q,i}^{(q)}(x) \equiv 0 \) for \( x < 0 \), and for \( x \geq 0 \) it is characterized by the Laplace transform

\[
\int_0^\infty e^{-sx} W_{q,i}^{(q)}(x)dx = \frac{1}{\psi_i(s) - q}, \ s > \rho_{q,i}.
\]

By [5] we immediately obtain

\[
f_{q,1,i}^{+}(x) = \frac{q}{\psi_i'(\rho_{q,i})} e^{-\rho_{q,i} x}, \ f_{q,1,i}^{-}(x) = \frac{q}{\psi_i'(\rho_{q,i})} e^{\rho_{q,i} x} - q W_{q,i}^{(q)}(x), \ x > 0.
\]

Lemma 1. For \( n \geq 1 \), we have

\[
f_{q,n,i}^{+}(x) = \sum_{j=1}^{n} a_{q,n,i,j} \frac{x^{j-1}}{(j-1)!} e^{-\rho_{q,i} x}, \ x > 0,
\]

where \( a_{q,n,i,j} \)'s are some constants defined in the following proof.
Finally, plugging (9)-(11) back into (8) gives (7) with proof.

For $f_{q,n,i}^+(x) = \int_{-\infty}^{0} f_{q,n-1,i}(y)f_{q,1,i}(x-y)dy + \int_{0}^{x} f_{q,n-1,i}(y)f_{q,1,i}(x-y)dy$

$$f_{q,n,i}^+(x) = I + II + III. \tag{8}$$

For $I$, by (3) we have

$$I = \int_{-\infty}^{0} f_{q,n-1,i}(y)f_{q,1,i}(x-y)dy = a_{q,1,i,1} \int_{-\infty}^{0} f_{q,n-1,i}(y)e^{-\rho_{q,i}(y)}dy.$$

For $II$, by (3) and assumption on $f_{q,n-1,i}$, we have

$$II = \int_{0}^{x} f_{q,n-1,i}(y)f_{q,1,i}(x-y)dy = \sum_{j=1}^{n-1} a_{q,1,i,1} a_{q,n-1,i,j} \int_{0}^{x} \frac{y^{j-1}}{(j-1)!} e^{-\rho_{q,i}y} \cdot e^{-\rho_{q,i}(y)}dy.$$

For $III$, it follows from the assumption on $f_{q,i,n-1}^+$ that

$$III = \sum_{j=1}^{n-1} a_{q,n-1,i,j} \int_{x}^{\infty} \frac{y^{j-1}}{(j-1)!} e^{-\rho_{q,i}y} f_{q,1,i}(x-y)dy$$

$$= \sum_{j=1}^{n-1} a_{q,n-1,i,j} \int_{0}^{\infty} \frac{(x+y)^{j-1}}{(j-1)!} e^{-\rho_{q,i}(x+y)} f_{q,1,i}(y)dy$$

$$= \sum_{j=1}^{n-1} a_{q,n-1,i,j} \sum_{k=1}^{j} \frac{x^{k-1}}{(k-1)!} e^{-\rho_{q,i}x} \int_{0}^{\infty} \frac{y^{j-k}}{(j-k)!} e^{-\rho_{q,i}y} f_{q,1,i}(y)dy$$

$$= \sum_{k=1}^{n-1} \sum_{j=k}^{n-1} a_{q,n-1,i,j} \int_{0}^{\infty} \frac{y^{j-k}}{(j-k)!} e^{-\rho_{q,i}y} f_{q,1,i}(y)dy \cdot \frac{x^{k-1}}{(k-1)!} e^{-\rho_{q,i}x}$$

$$= \sum_{j=1}^{n-1} a_{q,n-1,i,j} \int_{0}^{\infty} \frac{y^{j-x}}{(j-1)!} e^{-\rho_{q,i}y} f_{q,1,i}(y)dy \cdot \frac{x^{j-1}}{(j-1)!} e^{-\rho_{q,i}x}, \tag{11}$$

Finally, plugging (9)-(11) back into (8) gives (7) with

$$a_{q,n,i,j} = \begin{cases} a_{q,1,i,1} \int_{0}^{\infty} f_{q,n-1,i}(y)e^{-\rho_{q,i}y}dy \\ + \sum_{k=1}^{n-1} a_{q,n-1,i,k} \int_{0}^{\infty} \frac{y^{k-1}}{(k-1)!} e^{-\rho_{q,i}y} f_{q,1,i}(y)dy, & j = 1, \\
\sum_{k=1}^{n-1} a_{q,n-1,i,k} \int_{0}^{\infty} \frac{y^{k-1}}{(k-1)!} e^{-\rho_{q,i}y} f_{q,1,i}(y)dy, & j = 2, \cdots n-1, \\
a_{q,1,i,1} a_{q,n-1,i,n-1} + \sum_{k=1}^{n-1} a_{q,n-1,i,k} \int_{0}^{\infty} \frac{y^{k-1}}{(k-1)!} e^{-\rho_{q,i}y} f_{q,1,i}(y)dy, & j = n. \end{cases}$$

This completes the proof. \qed
3. Analysis of the Gerber-Shiu functions. First, we derive integral equations for the Gerber-Shiu functions. Suppose that \( \eta_1 = \epsilon_1 \). By conditioning on the time of the first review time \( Z_1 \) and the surplus increment \( U_{Z_1} - U_0 \), we have

\[
\phi_1(u) = \int_0^\infty e^{-st} \left( \int_0^\infty \phi_1(u + x)P(Y_1(t) \in dx)f_T(t)dt \right. \\
+ \int_0^\infty e^{-st} \left. \int_{-u}^{-\infty} \phi_2(u + x)P(Y_1(t) \in dx)f_T(t)dt \right. \\
+ \int_0^\infty e^{-st} \left. \int_{-\infty}^{-u} w(u, -x - u)P(Y_1(t) \in dx)f_T(t)dt \right) \\
= \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_0^\infty (\gamma + \delta)^n e^{-(\gamma + \delta)t} \left( \int_0^\infty \phi_1(u + x)P(Y_1(t) \in dx)dt \right. \\
+ \left. \int_0^\infty \phi_2(u + x)P(Y_1(t) \in dx)dt \right) \\
+ \left. \int_0^\infty w(u, -x - u)P(Y_1(t) \in dx)dt \right) \\
= \left( \frac{\gamma}{\gamma + \delta} \right)^n \phi_1(u + x)f_{\gamma + \delta, n, 1}^+(x)d \omega_1(u), \\
\phi_2(u) = \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_0^\infty \phi_1(u + x)f_{\gamma + \delta, n, 2}^+(x)d \omega_1(u), \\
- \left. \int_0^\infty \phi_2(u - x)f_{\gamma + \delta, n, 2}^-(x)d \omega_1(u), \\
\phi_1(u) = \sum_{j=1}^n a_{n, i, j} \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_0^\infty \phi_1(u + x) \left( x^{j-1} \right)^e^{-\rho_i} dx \\
+ \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_0^\infty \phi_2(u - x)f_{\gamma + \delta, n, 2}^- \left( \frac{\gamma}{\gamma + \delta} \right)^n \omega_1(u) \\
= \sum_{j=1}^n a_{n, i, j} \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_0^\infty \phi_1(u) \left( \frac{x - u}{j - 1} \right)^{j-1} e^{-\rho_i(x-u)} dx \\
+ \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_0^\infty \phi_2(u - x)f_{\gamma + \delta, n, 2}^- \left( \frac{\gamma}{\gamma + \delta} \right)^n \omega_1(u). 
\]
Taking Laplace transforms on both sides of (15) gives

\[
\hat{\phi}_1(s) = \sum_{j=1}^{n} a_{n,1,j} \left( \frac{\gamma}{\gamma + \delta} \right)^n \int_{0}^{\infty} e^{-su} \int_{0}^{\infty} \phi_1(x) \left( \frac{x-u}{j-1} \right)^{j-1} e^{-\rho_1(x-u)} dx du \\
+ \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,1}^{-}(s) \hat{\phi}_2(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{\omega}_1(s). \tag{16}
\]

By changing the order of integrals, we have

\[
\int_{0}^{\infty} e^{-su} \int_{0}^{\infty} \phi_1(x) \left( \frac{x-u}{j-1} \right)^{j-1} e^{-\rho_1(x-u)} dx du \\
= \int_{0}^{\infty} e^{-sx} \phi_1(x) \int_{0}^{x} (x-u)^{j-1} e^{-(\rho_1-s)(x-u)} du dx \\
= \int_{0}^{\infty} e^{-sx} \phi_1(x) \int_{0}^{x} y^{j-1} e^{-(\rho_1-s)y} dy dx \\
= \int_{0}^{\infty} e^{-sx} \phi_1(x) \cdot \frac{1}{(\rho_1-s)^j} \left( 1 - \sum_{k=1}^{j} \frac{((\rho_1-s)x)^k}{(k-1)!} e^{-(\rho_1-s)x} \right) dx \\
= \frac{\hat{\phi}_1(s)}{(\rho_1-s)^j} - \sum_{k=1}^{n} \frac{A_{1,k}}{(\rho_1-s)^{j-k+1}}, \tag{17}
\]

where \( A_{1,k} = \int_{0}^{\infty} \frac{x^{j-1}}{(k-1)!} e^{-\rho_1 s x} \phi_1(x) dx, \ k = 1, \ldots, n. \) Hence, (16) becomes

\[
\hat{\phi}_1(s) = \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,1}^{-}(s) \hat{\phi}_1(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,1}^{-}(s) \hat{\phi}_2(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{\omega}_1(s) \\
- \left( \frac{\gamma}{\gamma + \delta} \right)^n \sum_{j=1}^{n} a_{n,1,j} \sum_{k=1}^{j} \frac{A_{1,k}}{(\rho_1-s)^{j-k+1}} \\
= \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,1}^{-}(s) \hat{\phi}_1(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,1}^{-}(s) \hat{\phi}_2(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{\omega}_1(s) \\
- \left( \frac{\gamma}{\gamma + \delta} \right)^n \sum_{k=1}^{n} A_{1,k} l_{1,k}(s), \tag{18}
\]

where \( l_{1,k}(s) = \sum_{j=k}^{n} \frac{a_{n,j}}{(\rho_1-s)^{j-k+1}} \) for \( k = 1, \ldots, n. \) Similarly, it follows from (14) that

\[
\hat{\phi}_2(s) = \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,2}^{-}(s) \hat{\phi}_1(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{f}_{n,2}^{-}(s) \hat{\phi}_2(s) + \left( \frac{\gamma}{\gamma + \delta} \right)^n \hat{\omega}_2(s) \\
- \left( \frac{\gamma}{\gamma + \delta} \right)^n \sum_{k=1}^{n} A_{2,k} l_{2,k}(s), \tag{19}
\]

where for \( k = 1, \ldots, n,

\[
A_{2,k} = \int_{0}^{\infty} \frac{x^{j-1}}{(k-1)!} e^{-\rho_2 s x} \phi_1(x) dx, \ l_{2,k}(s) = \sum_{j=k}^{n} \frac{a_{n,j}}{(\rho_2-s)^{j-k+1}}.
\]
By (15) and (19) one easily obtains

\[
\hat{\phi}_1(s) = \left( \left( \frac{\gamma + \delta}{\gamma} \right)^n - \hat{f}_{n,2}(s) \right) \left( \tilde{\omega}_1(s) - \sum_{k=1}^{n} A_{1,k}l_{1,k}(s) \right) \\
+ \hat{f}_{n,1}(s) \left( \tilde{\omega}_2(s) - \sum_{k=1}^{n} A_{2,k}l_{2,k}(s) \right) \\
\left( \hat{f}_{n,1}^+(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) \left( \hat{f}_{n,2}^-(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) - \hat{f}_{n,1}(s) \hat{f}_{n,2}(s)
\]

(20)

and

\[
\hat{\phi}_2(s) = \left( \left( \frac{\gamma + \delta}{\gamma} \right)^n - \hat{f}_{n,1}^+(s) \right) \left( \tilde{\omega}_2(s) - \sum_{k=1}^{n} A_{2,k}l_{2,k}(s) \right) \\
+ \hat{f}_{n,2}(s) \left( \tilde{\omega}_1(s) - \sum_{k=1}^{n} A_{1,k}l_{1,k}(s) \right) \\
\left( \hat{f}_{n,1}^+(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) \left( \hat{f}_{n,2}^-(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) - \hat{f}_{n,1}(s) \hat{f}_{n,2}(s).
\]

(21)

The Laplace transforms \(\hat{\phi}_1(s)\) and \(\hat{\phi}_2(s)\) can be determined by (20) and (21), provided that we can determine the unknown constants, \(A_{1,k}, A_{2,k}, k = 1, \ldots, n\). In order to determine \(A_{1,k}\)'s and \(A_{2,k}\)'s, we consider the roots of the following equation,

\[
\left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_1 - s)^j} - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) \left( \hat{f}_{n,2}^-(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) - \hat{f}_{n,1}(s) \sum_{j=1}^{n} \frac{a_{n,2,j}}{(\rho_2 - s)^j} = 0.
\]

(22)

**Theorem 1.** When \(\delta > 0\), equation (22) has exactly \(2n\) roots with positive real parts.

**Proof.** First, it is easily seen that \(\rho_1\) and \(\rho_2\) are not roots of equation (22). Hence, (22) is equivalent to the following equation

\[
(\rho_1 - s)^n(\rho_2 - s)^n \left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_1 - s)^j} - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) \left( \hat{f}_{n,2}^-(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right)
\]

\[
= (\rho_1 - s)^n(\rho_2 - s)^n \hat{f}_{n,1}(s) \sum_{j=1}^{n} \frac{a_{n,2,j}}{(\rho_2 - s)^j}.
\]

(23)

Now we prove that equation (23) has the same number of roots on the right half complex plane as the following equation

\[
(\rho_1 - s)^n(\rho_2 - s)^n \left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_1 - s)^j} - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) \left( \hat{f}_{n,2}^-(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^n \right) = 0.
\]

(24)

To this end, we use Rouche’s theorem. It is easily seen that both sides of (23) are analytic function of \(s\). Let \(r > 0\) be a sufficiently large number, and denote by \(C_r\) the contour containing the imaginary axis running from \(-ir\) to \(ir\) and a semicircle with radius \(r\) running clockwise from \(ir\) to \(-ir\).
For $s$ on the semicircle, we have: as $r \to \infty$, $|\hat{f}_{n,1}(s)| \to 0$, $\left| \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_{i} - s)^{j}} \right| \to 0$, $i = 1, 2$, which result in

$$\left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_{1} - s)^{j}} - \left( \frac{\gamma + \delta}{\gamma} \right)^{n} \right) \left( \hat{f}_{n,2}(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^{n} \right) \to \left( \frac{\gamma + \delta}{\gamma} \right)^{2n},$$

and

$$\hat{f}_{n,1}(s) \sum_{j=1}^{n} \frac{a_{n,2,j}}{(\rho_{2} - s)^{j}} \to 0.$$

Hence, when the radius $r$ is sufficiently large, we know that the module of the left side of (23) is larger than that of the right side.

Let us consider the case when $s$ is a pure imaginary number. By definition, we have

$$\hat{f}_{n,1}(s) = \int_{0}^{\infty} e^{-sx} f_{n,1}^{-}(x)dx = \int_{-\infty}^{0} e^{sx} f_{n,1}^{-}(-x)dx = E \left[ e^{sY_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n})} ; Y_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) < 0 \right],$$

and by Lemma 1

$$\sum_{j=1}^{n} \frac{a_{n,i,j}}{(\rho_{i} - s)^{j}} = \int_{0}^{\infty} e^{sx} f_{n,i}^{-}(x)dx = E \left[ e^{sY_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n})} ; Y_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) > 0 \right]$$

which lead to

$$|\hat{f}_{n,1}(s)| \leq P \left( Y_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) < 0 \right)$$

and

$$\left| \sum_{j=1}^{n} \frac{a_{n,i,j}}{(\rho_{i} - s)^{j}} \right| \leq P \left( Y_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) > 0 \right).$$

Hence, we have

$$\left| \left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_{i} - s)^{j}} - \left( \frac{\gamma + \delta}{\gamma} \right)^{n} \right) \left( \hat{f}_{n,2}(s) - \left( \frac{\gamma + \delta}{\gamma} \right)^{n} \right) \right| \geq \left( \frac{\gamma + \delta}{\gamma} \right)^{n} - \left| \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_{i} - s)^{j}} \right| \left( \frac{\gamma + \delta}{\gamma} \right)^{n} - \left| \hat{f}_{n,2}(s) \right| \geq \left( \frac{\gamma + \delta}{\gamma} \right)^{n} - P \left( Y_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) > 0 \right) \times \left( \frac{\gamma + \delta}{\gamma} \right)^{n} - P \left( Y_{2}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) > 0 \right) \geq P \left( Y_{1}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) < 0 \right) P \left( Y_{2}(e_{\gamma+\delta,1} + \cdots + e_{\gamma+\delta,n}) < 0 \right) \geq \left| \hat{f}_{n,1}(s) \right| \sum_{j=1}^{n} \frac{a_{n,2,j}}{(\rho_{2} - s)^{j}},$$
which implies that the module of the right side of (23) is larger than that of the right side for \( s \) on the imaginary axis. By Rouche’s theorem we know that equation (23) and (24) have the same number of roots on the right half complex plane.

Since for \( s \) with positive real part, 
\[
\left| \hat{f}_{n,2}(s) \right| = E \left[ e^{\sum_{j=1}^{n} (\gamma + \delta \cdot s_j) - \frac{(\gamma + \delta)^n}{\gamma}} \right] 
\]
\[
\leq P (\sum_{j=1}^{n} (\gamma + \delta \cdot s_j) - \frac{(\gamma + \delta)^n}{\gamma}) < 0, 
\]
Equation (24) has the same number of roots with positive real parts as the following equation
\[
(n_1 - s)^n (n_2 - s)^n \left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_1 - s)^j} - \frac{(\gamma + \delta)^n}{\gamma} \right) = 0. 
\]
Again, by Rouche’s theorem, one easily finds that (25) has the same number of roots (on the right half complex plane) as the following equation
\[
(n_1 - s)^n (n_2 - s)^n \left( \frac{\gamma + \delta}{\gamma} \right)^n = 0. 
\]
Obviously, (26) has \( 2n \) positive roots (counting multiciity). Hence, the proof is completed.

\[ \square \]

Remark 1. The proof of Theorem 1 relies on Rouche’s theorem, which requires that there are no roots on the boundary of some domains. However, for \( \delta = 0 \) the suitable domain can not be constructed. As a result, we use the generalized Rouche’s theorem proposed in [7]. See e.g. [15].

Theorem 2. For \( \delta = 0 \), equation (22) has exactly \( 2n - 1 \) roots with positive real parts and a simple root \( s = 0 \).

Proof. When \( \delta = 0 \), (22) reduces to
\[
\left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_1 - s)^j} - 1 \right) (\hat{f}_{n,2}(s) - 1) = \hat{f}_{n,1}(s) \sum_{j=1}^{n} \frac{a_{n,2,j}}{(\rho_2 - s)^j}. 
\]
Since \( (\hat{f}_{n,1}(0) - 1) (\hat{f}_{n,2}(0) - 1) = \hat{f}_{n,1}(0) \hat{f}_{n,2}(0) \), then (27) has a root \( s = 0 \).

For convenience, we put
\[
\theta_1(s) = (\rho_1 - s)^n (\rho_2 - s)^n \left( \sum_{j=1}^{n} \frac{a_{n,1,j}}{(\rho_1 - s)^j} - 1 \right) (\hat{f}_{n,2}(s) - 1), 
\]
\[
\theta_2(s) = (\rho_1 - s)^n (\rho_2 - s)^n \hat{f}_{n,1}(s) \sum_{j=1}^{n} \frac{a_{n,2,j}}{(\rho_2 - s)^j}. 
\]
On the open disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) and its boundary \( \partial D = \{ z \in \mathbb{C} : |z| = 1 \} \), we define 
\[
\Theta_1(z) = \theta_1(R(1 - z)), \quad \Theta_2(z) = \theta_2(R(1 - z)), 
\]
where \( R \) is a sufficiently large positive number. Immediately we have
\[
\Theta_1(1) = \Theta_2(1) = |\rho_1 \rho_2|^n \hat{f}_{n,1}(0) \hat{f}_{n,2}(0) > 0. 
\]
Using the same arguments (for $\delta = 0$) as in the proof of Theorem 1, we can deduce $|\Theta_1(z)| > |\Theta_2(z)|$ for $z \in \partial D$ (excluding $z = 1$). Moreover, we have

$$
\Theta'_1(1) = -R\theta'_1(0) = Rn\rho_1^{-1}\rho_2\left(\hat{f}^+_1(0) - 1\right) + Rn\rho_2^{-1}\rho_1\left(\hat{f}^+_2(0) - 1\right) + R[\rho_1\rho_2]n\left(d\hat{f}^+_{n,1}(s)\right)_{s=0} - R[\rho_1\rho_2]n\left(d\hat{f}^-_{n,2}(s)\right)_{s=0}
$$

and similarly

$$
\Theta'_2(1) = -R\theta'_2(0) = Rn\rho_1^{-1}\rho_2\left(\hat{f}^+_2(0) - 1\right) + Rn\rho_2^{-1}\rho_1\left(\hat{f}^+_1(0) - 1\right) + R[\rho_1\rho_2]n\left(d\hat{f}^+_{n,2}(s)\right)_{s=0} - R[\rho_1\rho_2]n\left(d\hat{f}^-_{n,1}(s)\right)_{s=0}
$$

Hence,

$$
\frac{\Theta'_1(1) - \Theta'_2(1)}{\Theta_1(1)} = \frac{R[\rho_1\rho_2]n\left(d\hat{f}^-_{n,1}(s)\right)_{s=0} - R[\rho_1\rho_2]n\left(d\hat{f}^+_{n,1}(s)\right)_{s=0}}{\rho_1\rho_2\left(\hat{f}^+_1(0)\right) + R[\rho_1\rho_2]n\left(d\hat{f}^+_{n,2}(s)\right)_{s=0}}
$$

$$
= \frac{R\left(d\hat{f}^-_{n,1}(s)\right)_{s=0} - R\left(d\hat{f}^+_{n,1}(s)\right)_{s=0}}{\hat{f}^+_{n,1}(0) - \hat{f}^-_{n,1}(0)}
$$

$$
+ \frac{R\left(d\hat{f}^+_{n,2}(s)\right)_{s=0} - R\left(d\hat{f}^-_{n,2}(s)\right)_{s=0}}{\hat{f}^+_{n,2}(0) - \hat{f}^-_{n,2}(0)}
$$

$$
= R\frac{1}{\hat{f}^-_{n,1}(0)}E[Y_1(e_{\gamma,1} + \cdots e_{\gamma,n})] + R\frac{1}{\hat{f}^+_{n,2}(0)}E[Y_2(e_{\gamma,1} + \cdots e_{\gamma,n})]
$$

$$
= \frac{Rn}{\gamma\hat{f}^-_{n,1}(0)}(c_1 - \lambda\mu_X) + \frac{Rn}{\gamma\hat{f}^+_{n,2}(0)}(c_2 - \lambda\mu_X) > 0
$$

thanks to the safety loading condition $c_2 > c_1 > \lambda\mu_X$. From Theorem 1 in [7], we conclude that the number of roots of $\Theta_1(z) - \Theta_2(z) = 0$ inside $D$ is equal to the number of roots of $\Theta_1(z) = 0$ inside $D$ minus 1. As a result, the number of roots of $\theta_1(s) - \theta_2(s) = 0$ inside the contour $\{s : |s - R| = R\}$ is equal to the number of roots of $\theta_1(s) = 0$ inside the contour $\{s : |s - R| = R\}$ minus 1. Since $R$ can be taken large enough, the number of roots with positive real parts of $\theta_1(s) - \theta_2(s) = 0$
is equal to that of \( \theta_1(s) = 0 \) minus 1. Using the same arguments used in Theorem 1, we can show that equations \( \theta_1(s) = 0 \) and \( (\rho_1 - s)^n (\rho_2 - s)^n = 0 \) have the same number of roots on the right half complex plane. Since the latter has 2n positive roots (counting multiplicity), the former also has 2n roots with positive real parts. Hence, \( \theta_1(s) - \theta_2(s) = 0 \) has 2n - 1 roots with positive real parts. This completes the proof. \( \square \)

In the remainder of this paper, we denote these 2n roots by \( \nu_1, \nu_2, \ldots, \nu_{2n} \), and assume they are distinct, since the other case is rare.

Now we are ready to determine the unknown constant \( A_{1,1}, \ldots, A_{1,n}, A_{2,1}, \ldots, A_{2,n} \). Since \( \hat{\phi}_i(s) \) is finite for \( Re(s) \geq 0 \), it follows from Theorem 1 that \( \nu_1, \ldots, \nu_{2n} \) are also zero points of the numerators of (20) and (21). Hence, we have for \( j = 1, \ldots, 2n \)

\[
\left( \frac{\gamma + i}{\gamma} \right)^n - \hat{f}_{n,2}(-\nu_j) \left( \hat{\omega}_1(\nu_j) - \sum_{k=1}^n A_{1,k} l_{1,k}(\nu_j) \right) + \hat{f}_{n,1}(-\nu_j) \left( \hat{\omega}_2(\nu_j) - \sum_{k=1}^n A_{2,k} l_{2,k}(\nu_j) \right) = 0,
\]

and

\[
\left( \frac{\gamma + i}{\gamma} \right)^n - \hat{f}_{n,2}(\nu_j) \left( \hat{\omega}_1(\nu_j) - \sum_{k=1}^n A_{1,k} l_{1,k}(\nu_j) \right) + \hat{f}_{n,1}(\nu_j) \left( \hat{\omega}_2(\nu_j) - \sum_{k=1}^n A_{2,k} l_{2,k}(\nu_j) \right) = 0.
\]

Note that (30) and (31) are equivalent, since \( \nu_j \)'s are roots of equation (22).

4. Combination-of-exponentials jumps. In this section, we assume that the individual claim size density is a combination of exponentials of the following form

\[
f_X(x) = \sum_{j=1}^m B_j \alpha_j e^{-\alpha_j x}, \quad x > 0,
\]

where \( \sum_{j=1}^m B_j = 1, 0 < \alpha_1 < \cdots < \alpha_m < \infty \).

Instead of using the recursive method given in Section 2, we can apply Laplace inverse to determine the density functions \( f_{n,1} \) and \( f_{n,2} \). First, for any \( 0 \leq s < \rho_i \), we have

\[
\hat{f}_{n,i}(-s) = \int_{-\infty}^{\infty} e^{sx} f_{n,i}(x) dx = \left( \hat{f}_{1,i}(-s) \right)^n = \left( E e^{\psi_i(s)} \right)^n = \left( \int_0^\infty (\gamma + \delta) e^{-y} (e^{\psi_i(s)} y \psi_i(s)) dt \right)^n \] \tag{33}

Using (32) we have

\[
\gamma + \delta - \psi_i(s) = \gamma + \delta - \frac{1}{2} \sigma^2 s^2 - c_i s + \lambda s \sum_{j=1}^m \frac{B_j}{s + \alpha_j} = \frac{-\frac{1}{2} \sigma^2}{\prod_{j=1}^m (s + \alpha_j)} (s - \rho_i) \prod_{k=1}^{m+1} (s + r_{i,k}),
\]

where \(-r_{i,1}, \ldots, -r_{i,m+1}\) are roots (with negative real parts) of equation \( \psi_i(s) = \gamma + \delta \). Plugging (35) back into (33) gives

\[
\hat{f}_{n,i}(-s) = \left( \frac{\psi_i(s)}{(\rho_i - s) \prod_{k=1}^{m+1} (s + r_{i,k})} \right)^n = \frac{\mathcal{H}_n(s)}{\rho_i - s} \prod_{k=1}^{m+1} (s + r_{i,k})^n. \tag{36}
\]
By partial fractions we have

\[ \mathcal{H}_n(s) = \left( \frac{\sigma^2}{\sqrt{2\pi}} \right)^n (\gamma + \delta)^n \prod_{j=1}^{m} (s + \alpha_j)^n. \]

By partial fractions we have

\[ \hat{f}_{n,i}(-s) = \sum_{j=1}^{n} \frac{a_{n,i,j}}{(\rho_i - s)^j} + \sum_{j=1}^{n} \sum_{k=1}^{m+1} \frac{b_{n,i,j,k}}{(s + r_{i,k})^j}, \quad (37) \]

where

\[ a_{n,i,j} = \frac{(-1)^{n-j} d^{n-j} \mathcal{H}_n(s)}{(n-j)! ds^{n-j} \prod_{k=1}^{m+1} (s + r_{i,k})^n} \bigg|_{s=\rho_i}, \quad j = 1, \ldots, n, \]

\[ b_{n,i,j,k} = \frac{1}{(n-j)! ds^{n-j}} \mathcal{H}_n(s) \bigg|_{s=-r_{i,k}}, \quad j = 1, \ldots, n, \]

\[ k = 1, \ldots, m + 1. \]

It follows from (37) that

\[ \hat{f}_{n,i}^+(s) = \sum_{j=1}^{n} \frac{a_{n,i,j}}{(s + \rho_i)^j}, \quad \hat{f}_{n,i}^-(s) = \sum_{j=1}^{n} \sum_{k=1}^{m+1} \frac{b_{n,i,j,k}}{(s + r_{i,k})^j}, \quad (38) \]

and

\[ \hat{f}_{n,i}^+(x) = \sum_{j=1}^{n} \frac{a_{n,i,j} x^{j-1}}{(j-1)!} e^{-\rho_i x}, \quad \hat{f}_{n,i}^-(x) = \sum_{j=1}^{n} \sum_{k=1}^{m+1} b_{n,i,j,k} x^{j-1} \frac{e^{-r_{i,k} x}}{(j-1)!}, \quad x > 0. \quad (39) \]

Let us consider the case when \( w \equiv 1 \). Then the Gerber-Shiu functions reduce to the Laplace transforms of the ruin time. By (39) we have

\[ \omega_i(u) = \int_u^{\infty} f_{n,i}^-(x) dx = \sum_{j=1}^{n} \sum_{k=1}^{m+1} b_{n,i,j,k} \int_u^{\infty} \frac{x^{j-1}}{(j-1)!} e^{-r_{i,k} x} dx \]

\[ = \sum_{j=1}^{n} \sum_{k=1}^{m+1} b_{n,i,j,k} \sum_{l=1}^{j-1} \frac{u^{l-1}}{(l-1)!} e^{-r_{i,k} u} \]

\[ = \sum_{l=1}^{j} \sum_{k=1}^{m+1} \sum_{i=1}^{n} b_{n,i,j,k} \frac{u^{l-1}}{(l-1)!} e^{-r_{i,k} u} \]

\[ = \sum_{j=1}^{n} \sum_{k=1}^{m+1} \sum_{l=1}^{j} \frac{c_{n,i,j,k}}{(j-1)!} u^{l-1} e^{-r_{i,k} u}, \]

where \( c_{n,i,k,l} = \sum_{l=j}^{n} b_{n,i,l,k} r_{i,k}^{j-l-1} \). Hence,

\[ \hat{\omega}_i(s) = \sum_{j=1}^{n} \sum_{k=1}^{m+1} \frac{c_{n,i,j,k}}{(s + r_{i,k})^j}, \quad i = 1, 2. \quad (40) \]

Denote the common denominator of (20) and (21) by \( \text{Den}(s) \). By (39) we have that

\[ \prod_{i=1}^{2} \left( \frac{(\rho_i - s)^n \prod_{k=1}^{m+1} (s + r_{i,k})^n}{\text{Den}(s)} \right) \cdot \text{Den}(s) \]
is a polynomial function with degree $2(m+2)n$ and leading coefficient \((\frac{\gamma+\delta}{\gamma})^{2n}\).

Hence, we have
\[
\prod_{i=1}^{2} \left( (\rho_i - s)^n \prod_{k=1}^{m+1} (s + r_{i,k})^n \right) \cdot \text{Den}(s) = \left( \frac{\gamma+\delta}{\gamma} \right)^{2n} 2n \prod_{i=1}^{2} (s - \nu_i)^{2(m+1)n} \prod_{j=1}^{2} (s + R_j),
\]
where \(-R_j, j = 1, 2, \ldots, 2(m+1)n\), are roots (with negative real parts) of equation \((22)\).

Denote the numerator of \((20)\) by \(Nu_1(s)\). By \((38)\) and \((40)\) we know that
\[
\prod_{i=1}^{2} \left( (\rho_i - s)^n \prod_{k=1}^{m+1} (s + r_{i,k})^n \right) \cdot Nu_1(s)
\]
is a polynomial function with degree \(2(m+2)n - 1\). Since \(-\nu_1, \ldots, -\nu_{2n}\) are also zero points of \(Nu_1(s)\), then
\[
\prod_{i=1}^{2} \left( (\rho_i - s)^n \prod_{k=1}^{m+1} (s + r_{i,k})^n \right) \cdot Nu_1(s) = \prod_{i=1}^{2n} (s - \nu_i) \cdot L_1(s), \quad (42)
\]
where \(L_1(s)\) is a polynomial function with degree \(2(m+1)n - 1\). Denote the numerator of \((21)\) by \(Nu_2(s)\). As above, we have
\[
\prod_{i=1}^{2} \left( (\rho_i - s)^n \prod_{k=1}^{m+1} (s + r_{i,k})^n \right) \cdot Nu_2(s) = \prod_{i=1}^{2n} (s - \nu_i) \cdot L_2(s), \quad (43)
\]
where \(L_2(s)\) is a polynomial function with degree \(2(m+1)n - 1\).

Substituting \((41)-(43)\) into \((20)\) and \((21)\) gives
\[
\hat{\phi}_i(s) = \frac{L_i(s)}{\left( \frac{\gamma+\delta}{\gamma} \right)^{2n} \prod_{j=1}^{2(m+1)n} (s + R_j)}, \quad i = 1, 2.
\]

By partial fractions we obtain
\[
\hat{\phi}_i(s) = \sum_{j=1}^{2(m+1)n} \frac{L_{ij}}{s + R_j}, \quad i = 1, 2,
\]
where
\[
L_{ij} = \frac{L_i(-R_j)}{\left( \frac{\gamma+\delta}{\gamma} \right)^{2n} \prod_{l=1, l\neq j}^{2(m+1)n} (R_l - R_j)}, \quad j = 1, \ldots, 2(m+1)n.
\]

Upon Laplace inversion we get
\[
\phi_i(u) = \sum_{j=1}^{2(m+1)n} L_{ij} e^{-R_j u}, \quad i = 1, 2, \quad (44)
\]
which are general formulas for the Laplace transforms of the ruin time under the
claim size density assumption \((32)\).

We present some numerical examples. Let us consider the following three claim
size distributions

(1) a sum of two exponentials with mean 1/3 and 2/3;
(2) an exponential distribution with mean 1;
(3) a mixture of two exponentials: one exponential with mean 2 (mixing probability 1/3) and one exponential with mean 1/2 (mixing probability 2/3).

The corresponding claim size densities take the form of combination of exponentials:
(1) \( f_X(x) = 3e^{-1.5x} - 3e^{-3x}; \)
(2) \( f_X(x) = e^{-x}; \)
(3) \( f_X(x) = \frac{1}{6}e^{-\frac{1}{2}x} + \frac{4}{3}e^{-2x}. \)

Moreover, we find that the common mean of these three distributions is 1, while the variances are 0.56, 1 and 2, respectively. See [2].

Assume that the inter-review times are Erlang(2) distributed with density function \( f_T(t) = te^{-t}, \ t \geq 0. \) Set \( c_1 = 1.1, c_2 = 2, \sigma^2 = 0.5. \) Since the mean of claim size is 1, the safety loading condition \( c_1 > \lambda\mu_X \) holds true. Furthermore, set \( \delta = 0 \) and \( w \equiv 1, \) then the Gerber-Shiu functions reduce to ruin probability. We can use (44) to compute \( \phi_i(u), \ i = 1, 2. \) We plot the ruin probability curves in Figure 1. It follows that both \( \phi_1(u) \) and \( \phi_2(u) \) are decreasing functions of the initial surplus level. Furthermore, we observe that \( \phi_1(u) > \phi_2(u), \) which is possibly due to that ruin is more likely to occur when the initial effective premium rate \( \eta_1 = c_1 \) because of \( c_1 < c_2. \) In Tables 1-3, we illustrate some exact values of ruin probabilities. Comparing the values of the same cells across these tables, we find that the ruin probabilities appear to increase with the variance of the claim size distribution, which implies that the claim with larger variance has higher risk for the insurer.

![Figure 1. Ruin probabilities for Erlang(2) inter-review times. (a) \( f_X(x) = 3e^{-1.5x} - 3e^{-3x}; \) (b) \( f_X(x) = e^{-x}; \) (c) \( f_X(x) = \frac{1}{6}e^{-\frac{1}{2}x} + \frac{4}{3}e^{-2x}. \)](image)

**Table 1.** Exact values of ruin probabilities when \( f_X(x) = \frac{1}{6}e^{-\frac{1}{2}x} + \frac{4}{3}e^{-2x}. \)

| \( u \) | 0 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( \phi_1(u) \) | 0.64215 | 0.33207 | 0.15448 | 0.06979 | 0.03119 | 0.01388 | 0.00617 | 0.00274 | 0.00121 | 0.00054 |
| \( \phi_2(u) \) | 0.41629 | 0.19545 | 0.08830 | 0.03946 | 0.01757 | 0.00781 | 0.00347 | 0.00154 | 0.00068 | 0.00030 |

**Acknowledgments.** The author would like to thank the Editor and two anonymous referees for providing very helpful comments and suggestions.
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Table 2. Exact values of ruin probabilities when \( f_X(x) = e^{-x} \).

| \( u \) | \( \phi_1(u) \) | \( \phi_2(u) \) |
|------|---------|---------|
| 0    | 0.70195 | 0.48671 |
| 2    | 0.42231 | 0.27859 |
| 4    | 0.23766 | 0.15357 |
| 6    | 0.13018 | 0.08339 |
| 8    | 0.07048 | 0.04500 |
| 10   | 0.03797 | 0.02421 |
| 12   | 0.02041 | 0.01301 |
| 14   | 0.01096 | 0.00699 |
| 16   | 0.00588 | 0.00375 |
| 18   | 0.00316 | 0.00201 |

Table 3. Exact values of ruin probabilities when \( f_X(x) = \frac{1}{6}e^{-\frac{1}{3}x} + \frac{1}{3}e^{-2x} \).

| \( u \) | \( \phi_1(u) \) | \( \phi_2(u) \) |
|------|---------|---------|
| 0    | 0.75332 | 0.59185 |
| 2    | 0.55726 | 0.43080 |
| 4    | 0.40147 | 0.30693 |
| 6    | 0.28352 | 0.21544 |
| 8    | 0.19789 | 0.14989 |
| 10   | 0.13718 | 0.10374 |
| 12   | 0.09472 | 0.07158 |
| 14   | 0.06524 | 0.04929 |
| 16   | 0.04488 | 0.03390 |
| 18   | 0.03084 | 0.02330 |

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Received November 2015; revised January 2016.

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