A CASTELNUOVO-MUMFORD REGULARITY BOUND FOR SCROLLS

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Dedicated to Professor Lawrence Ein on the occasion of his sixtieth birthday.

Abstract. Let $X \subseteq \mathbb{P}^r$ be a scroll of codimension $e$ and degree $d$ over a smooth projective curve of genus $g$. The purpose of this paper is to prove a linear Castelnuovo-Mumford regularity bound that $\text{reg}(X) \leq d - e + 1 + g(e - 1)$. This bound works over an algebraically closed field of arbitrary characteristic.

1. Introduction

Throughout the paper, we work over an algebraically closed field $k$ of arbitrary characteristic. Let $X \subseteq \mathbb{P}^r$ be a projective variety defined by an ideal sheaf $\mathcal{I}_X$. We say that $X$ is $m$-regular if $H^i(\mathbb{P}^r, \mathcal{I}_X(m - i)) = 0$ for all $i > 0$. The minimal such number $m$ is called the Castelnuovo-Mumford regularity of $X$ and is denoted by $\text{reg}(X)$. It has attracted considerable attentions in the past thirty years to bound the regularity in terms of geometric or algebraic invariants. One optimal bound has been conjectured by Eisenbud-Goto [3] that $\text{reg}(X) \leq \deg X - \text{codim } X + 1$.

Very recently, counterexamples involving singular varieties for the regularity conjecture have been found by McCullough-Peeva [12]. However, it is still interesting to study whether this conjecture or a weaker variant holds for important cases, and the conjecture actually has been proven for integral curves by [5], for smooth complex surfaces by [17] and [10], and certain singular surfaces by [13]. Slightly weaker results for lower dimensional smooth varieties in characteristic zero were also obtained in [6] and [7]. As one of the promising cases of the conjecture, nonsingular scrolls of arbitrary dimension were studied in [1, Theorem 3], while the proof there contains a miscalculation (see Remark 3.1 for details). On the other hand, Noma showed that the double point divisor associated to a generic inner projection of a smooth projective variety $X \subseteq \mathbb{P}^r$ is semiample except when $X$ is a scroll, a Roth variety, or the second Veronese surface [15], and he proved a weaker bound for the regularity of Roth varieties [16]. Thus the scroll case is of special interest to us.

Motivated by the work of [5] and [1], we establish a regularity bound for scrolls in this paper. Precisely, let $C$ be a smooth projective curve of genus $g \geq 0$ and let $E$ be a very ample vector bundle on $C$ of rank $n$ and degree $d$. The variety $X = \mathbb{P}(E) \subseteq \mathbb{P}^r = \mathbb{P}(V)$ embedded by a base-point-free subspace $V \subseteq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ is called a scroll over $C$. Note that $X$ is a non-degenerate smooth projective variety of dimension $n$ and degree $d$. The main result is the following theorem.

**Theorem 1.1.** Let $X \subseteq \mathbb{P}^r$ be a scroll of codimension $e$ and degree $d$ over a smooth projective curve $C$ of genus $g \geq 0$. Then one has

$$\text{reg}(X) \leq d - e + 1 + g(e - 1).$$

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When $g = 0$, Theorem 1.1 gives a sharp bound. This was first proved by Bertin since the proof in [1] works in this case and is independent on the characteristic of the base field. Another proof using a different method in characteristic zero was also given in [8, Theorem 5.2]. If $g = 1$, we get an interesting bound $\text{reg}(X) \leq d$. If we further impose extra conditions, the above bound can be established for certain singular scrolls (see Remark 3.2 for details).

One of the essential points in the proof of Theorem 1.1 is to pick a particular line bundle on the curve with certain cohomological properties and then to resolve its pullback on the scroll as an $\mathcal{O}_{\mathbb{P}^r}$-module. This idea goes back to [5] and has been used in [1] and [14]. We point out that the choice of such line bundle is not unique. In [5] and [14], for instance, a different line bundle with even stronger properties has been considered (see Remark 3.3 for this direction in our case).

It is interesting to mention that a sharp regularity bound for the structure sheaf of a smooth projective variety in characteristic zero has been established in [9, Theorem A], where the characteristic zero and the smoothness assumption are crucial. In some sense, the regularity bound would be relatively easy to establish for structure sheaves. We point out some evidence in this direction in Corollary 2.3.

The paper is organized as follows: in Section 2, we collect basic facts and properties; Section 3 is devoted to the proof of Theorem 1.1.

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2. Preliminaries

In this section, we briefly review relevant basic facts for the convenience of the reader. By a variety, we mean a separated reduced and irreducible scheme of finite type over the field $k$.

A coherent sheaf $\mathcal{F}$ on a projective variety $X$ with a very ample line bundle $L$ is said to be $m$-regular with respect to $L$ if

$$H^i(X, \mathcal{F} \otimes L^{m-i}) = 0 \quad \text{for all } i > 0.$$ 

When the line bundle $L$ is clear from the context, we simply say $\mathcal{F}$ is $m$-regular. The least such number, if exists, is denoted by $\text{reg}(\mathcal{F})$. If $X$ is a subvariety of a projective space $\mathbb{P}^r$, the line bundle $L$ is always assumed to be $\mathcal{O}_X(1)$. By Mumford’s regularity theorem ([11, Theorem 1.8.5]), if $\mathcal{F}$ is $m$-regular, then $\mathcal{F}$ is $(m+1)$-regular. For more details, see [11, Section 1.8]. We shall use the following property of regularity.

**Proposition 2.1** ([10, Lemmas 2.5]). Consider an exact sequence of coherent sheaves on $\mathbb{P}^r$

$$\cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow F \longrightarrow 0.$$

If $F_i$ is $(p + i)$-regular for some integer $p$ and for each $i \geq 0$, then $F$ is $p$-regular.

The line bundle $L$ induces an embedding $X \subseteq \mathbb{P}^r$ so that $X$ is defined by an ideal sheaf $\mathcal{I}_X$ on $\mathbb{P}^r$. Then $X$ is said to be $m$-regular if $\mathcal{I}_X$ is $m$-regular. From the definition, that $X$ is $m$-regular is equivalent to the following two conditions:

1. $X$ is $(m-1)$-normal, i.e., the natural restriction map

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(m-1)) \longrightarrow H^0(X, \mathcal{O}_X(m-1))$$

is surjective;

2. The structure sheaf $\mathcal{O}_X$ is $(m-1)$-regular.

The number $\text{reg}(\mathcal{I}_X)$ is called the Castelnuovo-Mumford regularity of $X$ and is denoted by $\text{reg}(X)$. 

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Under suitable conditions, the regularity of $X$ can be read off from the regularity of its hyperplane section. We state this observation here.

**Lemma 2.2.** Let $X \subseteq \mathbb{P}^r$ be a non-degenerate projective variety of dimension $n \geq 2$. Suppose that a general hyperplane section $Y = X \cap \mathbb{P}^{r-1}$ is $m$-regular for some integer $m > 0$. Then $\mathcal{O}_X$ is $(m-1)$-regular. If moreover $X$ is $(m-1)$-normal, then it is $m$-regular.

**Proof.** Consider the short exact sequence

$$0 \rightarrow \mathcal{O}_X(-1) \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0.$$ 

Since $Y \subseteq \mathbb{P}^{r-1}$ is $k$-normal for all $k \geq m-1$, we obtain the surjective map

$$H^0(X, \mathcal{O}_X(k)) \twoheadrightarrow H^0(Y, \mathcal{O}_Y(k)).$$

Furthermore, since $\mathcal{O}_Y$ is $k$-regular for all $k \geq m-1$, we have

$$H^{i-1}(Y, \mathcal{O}_Y(k-i+1)) = H^i(Y, \mathcal{O}_Y(k-i+1)) = 0 \quad \text{for all } i \geq 2.$$ 

It now follows that

$$H^i(X, \mathcal{O}_X(k-i)) = H^i(X, \mathcal{O}_X(k-i+1)) \quad \text{for all } i \geq 1.$$ 

Note that $H^i(X, \mathcal{O}_X(l)) = 0$ for $i > 0$ and a sufficiently large integer $l$. Thus $\mathcal{O}_X$ is $(m-1)$-regular. If further assume that $X$ is $(m-1)$-normal, then it is clear that $X$ is $m$-regular. 

In some cases, it is relatively easy to obtain an optimal regularity bound for the structure sheaf, as showed in the following proposition.

**Proposition 2.3.** Let $X \subseteq \mathbb{P}^r$ be a non-degenerate projective variety of codimension $e$ and degree $d$ over an algebraically closed field $k$. Assume one the following conditions holds:

(i) $\dim X = 2$;

(ii) $\dim X = 3$, $X$ has isolated singularities and $\text{char } k = 0$.

Then $\text{reg}(\mathcal{O}_X) \leq d - e$.

**Proof.** Take a general hyperplane section $Y = X \cap \mathbb{P}^{r-1}$. Since we are working over an infinite field, we can apply Bertini’s theorem to $Y$. Thus $Y$ is an integral curve under the condition (i) and a nonsingular surface under the condition (ii). The embedding $Y \subseteq \mathbb{P}^{r-1}$ is non-degenerate. By [5] and [10], we know that $\text{reg}(Y) \leq d - e + 1$. Now by the lemma above, $\mathcal{O}_X$ is $(d-e)$-regular. 

Next, we briefly recall the definition of Koszul groups associated to a line bundle of a projective variety. For the details, we refer the reader to Green’s paper [4]. Let $V \subseteq H^0(X, L)$ be a basepoint-free subspace. There is a canonical surjective evaluation homomorphism $\epsilon_Y : V \otimes \mathcal{O}_X \rightarrow L$. Define $M_V$ as the kernel of $\epsilon_Y$ to form a short exact sequence

$$0 \rightarrow M_V \rightarrow V \otimes \mathcal{O}_X \xrightarrow{\epsilon_Y} L \rightarrow 0.$$ 

Fix a line bundle $B$ on $X$, and define

$$R = R(X, B, L) = \bigoplus_{m \geq 0} H^0(X, B + mL).$$

Then $R$ is naturally a graded $S = \text{Sym}^*(V)$-module so that it has a minimal free resolution

$$\cdots \rightarrow \bigoplus_q K_{p, q}(X, B, V) \otimes_k S(-p - q) \rightarrow \cdots \rightarrow \bigoplus_q K_{0, q}(X, B, V) \otimes_k S(-q) \rightarrow R \rightarrow 0,$$ 

in which the vector space $K_{p, q}(X, B, V)$ is the Koszul group associated to $B$ with respect to $L$. We shall use the following proposition to compute these groups.
Proposition 2.4 ([2, Proposition 3.2]). Assume that $H^i(X, B + mL) = 0$ for $i > 0$ and $m > 0$. Then for $q \geq 2$, we have

$$K_{p,q}(X, B, V) = H^1(X, \wedge^{p+1} M_V \otimes (B + (q-1)L)).$$

If moreover $H^1(X, B) = 0$, then we allow $q = 1$ in the above formula.

We conclude this section by listing a couple of technical results that we shall use in our proof.

Lemma 2.5 (cf. [5, Lemma 1.7], [1, Lemma 4.1]). Let $C$ be a smooth irreducible projective curve of genus $g$, and $p: C \to \mathbb{P}^{r+1}$ be a morphism of degree $d = \deg p^* \mathcal{O}_{\mathbb{P}^{r+1}}(1)$. Assume that $p(C)$ is non-degenerate, and set $M = p^* \Omega^1_{\mathbb{P}^{r+1}}(1)$. Then for a general line bundle $A$ of degree $d - e + g$ on $C$, one has $h^0(C, A) = d - e + 1$ and $h^1(C, A) = h^1(C, M \otimes A) = h^1(C, \wedge^2 M \otimes A) = 0$.

Proof. As in [5, Proof of Lemma 1.7], we first consider a filtration

$$M = F^1 \supseteq F^2 \supseteq \cdots \supseteq F^{e+1} \supseteq F^{e+2} = 0$$

of $M$ by vector bundles such that each of the quotients $L_i = F^i/F^{i+1}$ is a line bundle of strictly negative degree. Then $h^1(C, L_i \otimes A) = h^1(C, L_i \otimes L_j \otimes A) = 0$ implies $h^1(C, M \otimes A) = h^1(C, \wedge^2 M \otimes A) = 0$. Since $\deg M = -d$ and $\text{rank } M = e + 1$, it follows that $\deg L_i \geq e - d$ and $\deg L_i \otimes L_j \geq -d + e - 1$. Note that a generic line bundle of degree $\geq g - 1$ is non-special. Thus the assertion now follows. \hfill \square

Lemma 2.6 ([11, Theorem B.2.2]). Let $u: E \to F$ be a generically surjective homomorphism of vector bundles of ranks $e$ and $f$ on a smooth variety. Associated to $u$, the Eagon-Northcott complex

$$0 \to \wedge^e E \otimes \text{Sym}^{e-f-1}(F^*) \otimes \det F^* \to \cdots$$

$$\text{EN}_u : \quad \cdots \to \wedge^f E \otimes F^* \otimes \det F^* \to \wedge^{f+1} E \otimes \det F^* \to E \xrightarrow{u} F \to 0.$$

is exact away from the support of $\text{coker}(u)$.

Lemma 2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be coherent sheaves on $\mathbb{P}^r$. Consider the following diagram

$$\begin{align*}
\mathcal{O}_{\mathbb{P}^r}^a & \xrightarrow{u} \mathcal{F} \xrightarrow{\phi} 0 \\
\mathcal{O}_{\mathbb{P}^r}^b & \xrightarrow{v} \mathcal{G} \xrightarrow{\phi'} 0
\end{align*}$$

(2.1)

where $u$ and $v$ are surjective morphisms, $\phi$ is a morphism, and $a \leq b$. Assume that $u$, $v$, and $\phi$ induce injective morphisms on global sections and satisfy the following condition

$$(\phi \circ u)(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}^a)) \subseteq v(H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}^b)) \subseteq H^0(\mathbb{P}^r, \mathcal{G}).$$

Then there exists an injective morphism $\phi' : \mathcal{O}_{\mathbb{P}^r}^b \hookrightarrow \mathcal{O}_{\mathbb{P}^r}^b$ lifting $\phi$ with $\text{Coker } \phi' \simeq \mathcal{O}_{\mathbb{P}^r}^{b-a}$ such that the diagram (2.1) is commutative.

Proof. By assumption, we may assume that $\mathcal{O}_{\mathbb{P}^r}^a \simeq V \otimes \mathcal{O}_{\mathbb{P}^r}$ and $\mathcal{O}_{\mathbb{P}^r}^b \simeq W \otimes \mathcal{O}_{\mathbb{P}^r}$ for vector subspaces $V$ and $W$ of $H^0(\mathbb{P}^r, \mathcal{G})$ such that $V \subseteq W$. Hence $\phi'$ is induced by the inclusion $V \subseteq W$. The commutativity of the diagram (2.1) is checked locally by using the fact that at a closed point $x$, the elements of $V$ and $W$ induce sets of generators for $\mathcal{F}_x$ and $\mathcal{G}_x$, respectively. \hfill \square
3. Proof of the main result

This section is devoted to the proof of Theorems 1.1. Recall that $C$ is a nonsingular irreducible projective curve of genus $g \geq 0$ and $E$ is a very ample vector bundle on $C$ of rank $n$ and degree $d$. The scroll $X = \mathbb{P}(E)$ is embedded into a projective space $\mathbb{P}^r = \mathbb{P}(V)$ by a map $p$ determined by a base-point-free subspace $V \subseteq H^0(X, \mathcal{O}_X(1))$ of dimension $r + 1$. Denote by $t : X \to C$ the canonical projection. Write $e = \text{codim} \ X$ and set $S = \text{Sym}^* (V)$ as the symmetric algebra of $V$. Our setting can be summarized in the following diagram

\[
X = \mathbb{P}(E) \xrightarrow{p} \mathbb{P}^r \xrightarrow{t} C.
\]

Proof of Theorem 1.1. First of all, the vector space $V$ is naturally a base-point-free subspace of $H^0(C, E)$ under the identification $H^0(C, E) = H^0(X, \mathcal{O}_X(1))$. Let $M$ be the kernel of the surjective evaluation map $V \otimes \mathcal{O}_X \to \mathcal{O}_X(1)$. Then there is a short exact sequence

\[
0 \to M \to V \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \to 0.
\]

Since $R^1t_*M = 0$, pushing down this sequence to $C$ by $t$ yields a short exact sequence

\[
0 \to t_*M \to V \otimes \mathcal{O}_C \to E \to 0.
\]

Take a general linear subspace $W \subseteq V$ of dimension $n - 1$ and let $\overline{V} = V/W$ be the quotient space. The linear space $\mathbb{P}(\overline{V})$ is naturally a linear subspace of $\mathbb{P}^r$. As $W$ is general, the following conditions hold.

(3.3) There is a short exact sequence

\[
0 \to W \otimes \mathcal{O}_C \to E \to \det E \to 0.
\]

(3.4) The intersection $C_0 = X \cap \mathbb{P}(\overline{V})$ is a curve such that the restriction map $t|_{C_0} : C_0 \to C$

is an isomorphism.

Snake lemma gives rise to the following diagram

\[
\begin{array}{ccccccc}
0 & & 0 \\
\downarrow & & \downarrow \\
W \otimes \mathcal{O}_C & \longrightarrow & W \otimes \mathcal{O}_C \\
\downarrow & & \downarrow \\
0 \longrightarrow t_*M \longrightarrow V \otimes \mathcal{O}_C \longrightarrow E \longrightarrow 0. \\
\end{array}
\]

\[
\begin{array}{cccccc}
0 \longrightarrow t_*M \longrightarrow \overline{V} \otimes \mathcal{O}_C \longrightarrow \det E \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0
\end{array}
\]

We identify $C$ as $C_0$ so that $\mathcal{O}_{\mathbb{P}^r}(1)|_C = \det E$ and the embedding $C \subseteq \mathbb{P}(\overline{V})$ is determined by the subspace $\overline{V} \subseteq H^0(C, \det E)$. Therefore

\[
t_*M = \Omega^1_{\mathbb{P}(\overline{V})}|_C.
\]
We apply Lemma 2.5 to the curve $C \subseteq \mathbb{P}(V)$. So there exists a general line bundle $A$ on $C$ of degree $d - r + n + g$ satisfying the properties:

\begin{align*}
(3.5) \quad & h^1(C, A) = h^1(C, t_*M \otimes A) = h^1(C, \wedge^2 t_*M \otimes A) = 0, \\
(3.6) \quad & h^0(C, A) = d - r + n + 1, \\
(3.7) \quad & h^0(C, t_*M \otimes A) = (r - n)(d - r + n) + 1.
\end{align*}

Since $d - r + n + g \geq g$, we can further assume $A$ is effective, i.e.,

\[(3.8) \quad A = \mathcal{O}_C(D) \quad \text{where} \quad D = \sum_{i=1}^{d-r+n+g} x_i, \quad \text{with distinct general points} \quad x_i \in C.\]

Consider the graded $S$-module

\[R = \bigoplus_{m \geq 0} H^0(X, t^*A \otimes \mathcal{O}_X(m))\]

Note that $H^0(X, t^*A \otimes \mathcal{O}_X(m)) = 0$, for $m < 0$. We claim that the line bundle $t^*A$ is 1-regular (with respect to $\mathcal{O}_X(1)$). To see this, it is enough to verify $H^1(X, t^*A \otimes \mathcal{O}_X(1-i)) = 0$ for $1 \leq i \leq n$. Since $R^1 t_* \mathcal{O}_X(l) = 0$ for $l \geq 1 - n$ and $j \geq 1$, by the projection formula we have $H^1(X, t^*A \otimes \mathcal{O}_X(1-i)) = H^1(C, A \otimes S^{1-i}E)$. The latter one is automatically zero if $i > 1$ and for $i = 1$, $H^1(C, A) = 0$ by the choice of $A$. Hence $t^*A$ is indeed 1-regular. So $R$ is 1-regular too as a graded $S$-module. Thus a minimal free resolution of $R$ over $S$ should have the following shape

\[
0 \rightarrow S^{\alpha_{r-n}}(-r+n) \oplus S^{\beta_{r-n}}(-r+n-1) \rightarrow \cdots \rightarrow S^{\alpha_2}(-2) \oplus S^{\beta_2}(-3) \rightarrow \rightarrow S^{\alpha_1}(-1) \oplus S^{\beta_1}(-2) \rightarrow S^{\alpha_0} \oplus S^{\beta_0}(-1) \rightarrow R \rightarrow 0.
\]

Using Proposition 2.4 and properties (3.5)-(3.7), we can determine that $\beta_0 = \beta_1 = 0$ and

\[
\alpha_0 = h^0(C, A) = d - r + n + 1,
\]

\[
\alpha_1 = h^0(C, t_*M \otimes A) = (r - n)(d - r + n) + 1.
\]

After sheafification, we then obtain a free resolution of $t^*A$

\[
\cdots \rightarrow \mathcal{O}(r-n)(d-r+n+1)(-1) \rightarrow \mathcal{O}_{pr}^{d-r+n+1} \rightarrow t^*A \rightarrow 0.
\]

Recall that $A = \mathcal{O}_C(D)$ for an effective divisor $D$. Denote by $s_D \in H^0(C, A)$ the corresponding global section. Then there is a short exact sequence

\[0 \rightarrow \mathcal{O}_C \xrightarrow{s_D} A \rightarrow A_D \rightarrow 0.
\]

Let $X_i := t^{-1}(x_i) \simeq \mathbb{P}^{n-1}$. Pulling back this sequence by $t$ gives rise to a short exact sequence

\[0 \rightarrow \mathcal{O}_X \xrightarrow{s_D} t^*A \rightarrow \mathcal{F} = \bigoplus_{i=1}^{d-r+n+g} \mathcal{O}_{X_i} \rightarrow 0,
\]

where $\mathcal{F}$ is a direct sum of $\mathcal{O}_{X_i}$ as indicated. Resolving each $\mathcal{O}_{X_i}$ by the Koszul resolution, we obtain a free resolution of $\mathcal{F}$ as follows:

\[
0 \rightarrow \mathcal{O}_{pr}^{n_{r-n+1}}(-r+n-1) \rightarrow \cdots
\]

\[
\rightarrow \mathcal{O}_{pr}^{n_2}(-2) \xrightarrow{w'} \mathcal{O}_{pr}^{(r-n+1)(d-r+n+g)}(-1) \xrightarrow{w'} \mathcal{O}_{pr}^{d-r+n+g} \rightarrow \mathcal{F} \rightarrow 0.
\]
On the other hand by Snake lemma (see also [5, Proof of Theorem 2.1]), we obtain the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N & \rightarrow & \mathcal{O}^{(r-n)(d-r+n)+1}_{\mathbb{P}^r}(-1) & \rightarrow & \mathcal{O}^{d-r+n+1}_{\mathbb{P}^r} & \rightarrow & t^*A & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K & \rightarrow & \mathcal{O}^{(r-n)(d-r+n)+1}_{\mathbb{P}^r} & \rightarrow & \mathcal{O}^{d-r+n+1}_{\mathbb{P}^r} & \rightarrow & t^*A & \rightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow & & \\
0 & & & & & & & & & \\
\end{array}
\]

(3.12)

in which all horizontal and vertical sequences are exact. A truncation of the free resolution (3.10) induces a free resolution of the sheaf $K$, from which we see that $H^1(\mathbb{P}^r, K(m)) = H^2(\mathbb{P}^r, K(m)) = 0$ for all $m \in \mathbb{Z}$. Therefore from the left vertical sequence in the diagram (3.12), we get

\[
H^1(\mathbb{P}^r, N(m)) = H^1(\mathbb{P}^r, \mathcal{I}_X(m)) \quad \text{for all } m \in \mathbb{Z}. 
\]

In the diagram (3.12), the sheaf $\mathcal{F}$ is also resolved by the exact sequence

\[
0 \rightarrow N \rightarrow \mathcal{O}^{(r-n)(d-r+n)+1}_{\mathbb{P}^r}(-1) \xrightarrow{v'} \mathcal{O}^{d-r+n}_{\mathbb{P}^r} \rightarrow \mathcal{F} \rightarrow 0.
\]

We point out that the kernel $N$ of $v'$ may not be locally free. We shall compare it with the free resolution (3.11) of $\mathcal{F}$. Our goal is to lift the identity morphism $\text{id}_\mathcal{F}$ of $\mathcal{F}$ consecutively to construct two injective morphisms $\phi_0$ and $\phi_1$ to achieve the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N & \rightarrow & \mathcal{O}^{(r-n)(d-r+n)+1}_{\mathbb{P}^r}(-1) & \xrightarrow{v'} & \mathcal{O}^{d-r+n}_{\mathbb{P}^r} & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
\downarrow \phi_2 & & \downarrow \phi_1 & & \downarrow \phi_0 & & \downarrow \text{id}_\mathcal{F} & & \downarrow & & \\
0 & \rightarrow & P & \rightarrow & \mathcal{O}^{(r-n+1)(d-r+n+g)}_{\mathbb{P}^r}(-1) & \xrightarrow{w'} & \mathcal{O}^{d-r+n+g}_{\mathbb{P}^r} & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
\end{array}
\]

(3.14)

where $P = \ker(w')$ and $\phi_2 = \phi_1|_N$. For this, let us construct $\phi_0$ first. Set $\mathcal{N} = \ker v$ and $\mathcal{W} = \ker w$. It is easy to check that $H^0(\mathbb{P}^r, \mathcal{N}) = H^0(\mathbb{P}^r, \mathcal{W}) = H^1(\mathbb{P}^r, \mathcal{W}) = 0$. So we apply Lemma 2.7 to get $\phi_0$ lifting $\text{id}_\mathcal{N}$. Next, we construct $\phi_1$. The restricted morphism $\phi_0|_\mathcal{N} : \mathcal{N} \rightarrow \mathcal{W}$ induces a morphism $\phi'' : \mathcal{N}(1) \rightarrow \mathcal{W}(1)$. Hence, we are in the situation

\[
\begin{array}{ccccccccc}
0 & \rightarrow & N(1) & \rightarrow & \mathcal{O}^{(r-n)(d-r+n)+1}_{\mathbb{P}^r} & \xrightarrow{v'} & \mathcal{N}(1) & \xrightarrow{\phi''} & \mathcal{W}(1) & \rightarrow & 0 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
0 & \rightarrow & P(1) & \rightarrow & \mathcal{O}^{(r-n+1)(d-r+n+g)}_{\mathbb{P}^r} & \xrightarrow{w'} & \mathcal{W}(1) & \rightarrow & \mathcal{F} & \rightarrow & 0 \\
\end{array}
\]
We can check that $H^0(\mathbb{P}^r, N(1)) = 0$ and $H^0(\mathbb{P}^r, P(1)) = 0$ by using (3.12) and (3.11). Then Lemma 2.7 can be applied to lift $\phi^0$ to obtain the desired $\phi_1$. Therefore, we obtain the commutative diagram (3.14) as claimed.

Immediately from the diagram (3.14), we have

$$\text{Coker } \phi_0 = \mathcal{O}_{\mathbb{P}^r}^g$$

and as a consequence of Snake lemma, we obtain a short exact sequence

$$(3.15) \quad 0 \to F \to \mathcal{O}_{\mathbb{P}^r}^{f+g}(-1) \to \mathcal{O}_{\mathbb{P}^r}^g \to 0,$$

where $f = d - r + n - 1 + g(r - n)$ and as a consequence of Snake lemma, we obtain a short exact sequence

$$(3.16) \quad 0 \to \mathcal{O}_{\mathbb{P}^r}^g \to \mathcal{O}_{\mathbb{P}^r}^{f+g}(1) \to F^* \to 0.$$

By Proposition 2.1, $F^*$ is $(-1)$-regular. In addition, for each $i > 0$, the short exact sequence (3.16) induces an exact complex

$$(3.17) \quad \ldots \to \wedge^2(\mathcal{O}_{\mathbb{P}^r}^g) \otimes \text{Sym}^{i-2}(\mathcal{O}_{\mathbb{P}^r}^{f+g}(1)) \to \wedge^1(\mathcal{O}_{\mathbb{P}^r}^g) \otimes \text{Sym}^{i-1}(\mathcal{O}_{\mathbb{P}^r}^{f+g}(1)) \to \text{Sym}^i(\mathcal{O}_{\mathbb{P}^r}^{f+g}(1)) \to \text{Sym}^i(F^*) \to 0,$$

which implies that $\text{Sym}^i(F^*)$ is $(-i)$-regular.

Finally, we form the following commutative diagram

$$\begin{array}{ccccccc}
0 & & & & & & \phi_2 \\
\downarrow & & & & & & \downarrow \\
0 & \to & \ker(w'') & \to & \mathcal{O}_{\mathbb{P}^r}^{n_2}(-2) & \to & P & \to & 0 \\
\downarrow & & & & & & \downarrow & & \downarrow \\
0 & \to & \ker(q) & \to & \mathcal{O}_{\mathbb{P}^r}^{n_2}(-2) & \to & q & \to & 0 \\
\downarrow & & & & & & \downarrow \\
N & \to & 0 & \to & 0 \\
\downarrow & & & & & & \downarrow \\
0 & & & & & & 0
\end{array}$$

where the morphism $q$ is induced by $w''$ composing with the projection from $P$ to $F$. We compute a regularity bound for $N$ to finish the proof. By a truncation of the resolution (3.11), it is easy to check that $H^2(\mathbb{P}^r, \ker(w'')(m)) = 0$ for $m \in \mathbb{Z}$. The Eagon-Northcott complex associated to the morphism $q$ (Lemma 2.6) gives rise to a free resolution of $\ker(q)$

$$(3.18) \quad \ldots \to \wedge^{f+2}(\mathcal{O}_{\mathbb{P}^r}^{n_2}(-2)) \otimes \text{Sym}^{1}(F^*) \otimes \det F^* \to \wedge^{f+1}(\mathcal{O}_{\mathbb{P}^r}^{n_2}(-2)) \otimes \det F^* \to \ker(q) \to 0.$$

Note that $\det F^* = \mathcal{O}_{\mathbb{P}^r}(f + g)$. In addition, for $i \geq 0$, the $i$-th term in the resolution (3.18) is of the form

$$\wedge^{f+i+1}(\mathcal{O}_{\mathbb{P}^r}^{n_2}(-2)) \otimes \text{Sym}^{i}(F^*) \otimes \det F^*$$

which is a direct sum of copies of $\text{Sym}^{i}(F^*) (-f - 2 + g - 2i)$. Since we have seen that $\text{Sym}^{i}(F^*)$ is $(-i)$-regular, we obtain that $\wedge^{f+i+1}(\mathcal{O}_{\mathbb{P}^r}^{n_2}(-2)) \otimes \text{Sym}^{i}(F^*) \otimes \det F^*$ is $(f + 2 - g + i)$-regular. By Proposition 2.1, we see that $\ker(q)$ is $(f + 2 - g)$-regular. In particular, this implies that
$H^1(\mathbb{P}^r, \ker(q)(m)) = 0$ for all $m \geq f + 1 - g$. Thus $H^1(\mathbb{P}^r, N(m)) = 0$ for all $m \geq f + 1 - g$. Hence by (3.13), we deduce that
\[ X \subseteq \mathbb{P}^r \text{ is } m\text{-normal, for } m \geq f + 1 - g. \]
Recall that $f = d - r + n - 1 + g(r - n) = d - e + 1 + ge$. Observe that a general hyperplane section of a scroll is again a scroll. Furthermore, we know the regularity bound for the curve case by [5]. Hence inductively by Lemma 2.2, we obtain
\[ \text{reg}(X) \leq f + 2 - g = d - e + 1 + g(e - 1), \]
which finishes the proof of the theorem. □

**Remark 3.1.** The idea in the proof of Theorem 1.1 follows the one in Bertin’s work [1], which goes back to the work of [5]. We choose the same line bundle $A$ as in [1] to deduce a locally free resolution of $t^*A$. However, in [1, p.180], the sheafification of a minimal free resolution of the cokernel sheaf $\mathcal{F}$ was claim to have the form
\begin{equation}
\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^r}^{(r-n+1)(d+n-r)}(-1) \longrightarrow \mathcal{O}_{\mathbb{P}^r}^{d-r+n} \longrightarrow \mathcal{F} \longrightarrow 0,
\end{equation}
from which the conjectured regularity bound was proved. Unfortunately, the resolution (3.19) only works when genus $g = 0$. For arbitrary genus $g$, since $h^0(\mathbb{P}^r, \mathcal{F}) = d - r + n + g$, the complex (3.19) cannot be deduced from a minimal free resolution. The correct free resolution is (3.11) but the difference between $d - r + n$ and $d - r + n + g$ makes the proof complicated.

**Remark 3.2.** We can use the same argument to generalize Theorem 1.1 to singular case under certain conditions. For the convenience of the reader, we formulate this direction here and leave the details. Assume that $C$ is a smooth projective curve of genus $g \geq 0$ and let $E$ be a vector bundle on $C$ of rank $n$ and degree $d$ (not necessarily ample). Assume that $V \subseteq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ is a base-point-free subspace of dimension $r + 1$ and it induces a birational morphism $p: \mathbb{P}(E) \to \mathbb{P}^r = \mathbb{P}(V)$ to the image. This time we denote by $X$ the image of $p$. So we have the diagram as follows:
\[ \mathbb{P}(E) \xrightarrow{p} X \subseteq \mathbb{P}^r = \mathbb{P}(V) \]
\[ \downarrow t \]
\[ C. \]

We regard $X$ as a generalized scroll. In order to get a regularity bound for $X$, we still use a general line bundle $A = \mathcal{O}_C(D)$ for an effective divisor $D$ consisting of general $d - r + n + g$ points. The global section $s_D \in H^0(C, A)$ gives rise to a short exact sequence
\[ 0 \longrightarrow \mathcal{O}_C \xrightarrow{s_D} A \longrightarrow A_D \longrightarrow 0. \]
Under the following two conditions:
\begin{enumerate}
\item $p_*\mathcal{O}_{\mathbb{P}(E)} = \mathcal{O}_X$,
\item the induced morphism $R^1p_*(t^*\mathcal{O}_C) \to R^1p_*(t^*A)$ is injective,
\end{enumerate}
the proof of Theorem 1.1 works without much change and we can obtain the same regularity bound. Note that the condition (2) is equivalent to
\[ H^1(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m)) = H^1(C, \text{Sym}^m(E)) = 0 \text{ for } m \gg 0, \]
which may not hold in general if $E$ is not ample and $g > 0$.

**Remark 3.3.** [5] and [14] show a different choice of the line bundle $A$ which gives a better regularity bound for curves. It is achieved by the key observation that a minimal free resolution of such line bundle $A$ has a good shape (see [5, Lemma 2.3] and [14, Lemma 4]). However, for the higher dimensional scroll cases, it seems that such choice of $A$ does not work. For instance, consider an elliptic surface scroll $S \subseteq \mathbb{P}^4$ of degree 5. Then the choice of $A$ is nothing but the
structure sheaf $A = \mathcal{O}_S$. It is expected that the sheafification of a minimal free resolution of $\mathcal{O}_S$ is of the form

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^4}^{-n} (-1) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$  

However, the actual sheafification of a minimal free resolution of $\mathcal{O}_S$ is

$$\cdots \longrightarrow \mathcal{O}_{\mathbb{P}^4}^{-n} (-2) \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_S \longrightarrow 0.$$  

References

[1] M.-A. Bertin, *On the regularity of varieties having an extremal secant line*, J. Reine Angew. Math. 545 (2002), 167-181.
[2] L. Ein and R. Lazarsfeld, *Asymptotic syzygies of algebraic varieties*, Invent. Math. 190 (2012), 603-646.
[3] D. Eisenbud and S. Goto, *Linear free resolutions and minimal multiplicity*, J. Algebra 88 (1984), 89-133.
[4] M. Green, *Koszul cohomology and the geometry of projective varieties*, J. Differential Geom. 19 (1984), 125-171.
[5] L. Gruson, R. Lazarsfeld and C. Peskine, *On a theorem of Castelnuovo, and the equations defining space curves*, Invent. Math. 72 (1983), 491-506.
[6] S. Kwak, *Castelnuovo regularity for smooth subvarieties of dimensions 3 and 4*, J. Algebraic Geom. 7 (1998), 195-206.
[7] S. Kwak, *Generic projections, the equations defining projective varieties and Castelnuovo regularity*, Math. Z. 234 (2000), 413-434.
[8] S. Kwak and E. Park, *Some effects of property $N_p$ on the higher normality and defining equations of nonlinear-early normal varieties*, J. Reine Angew. Math. 582 (2005), 87-105.
[9] S. Kwak and J. Park, *A bound for Castelnuovo-Mumford regularity by double point divisors*, preprint, arXiv:1406.7404.
[10] R. Lazarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. 55 (1987), 423-429.
[11] R. Lazarsfeld, *Positivity in algebraic geometry I. Classical Setting: line bundles and linear series*, A Series of Modern Surveys in Math. 48, Springer-Verlag, Berlin, (2004).
[12] J. McCullough and I. Peeva. *Counterexamples to the Eisenbud-Goto regularity conjecture*, to appear.
[13] W. Niu, *Castelnuovo-Mumford regularity bounds for singular surfaces*, Math. Z. 280 (2015), 609-620.
[14] A. Noma, *A bound on the Castelnuovo-Mumford regularity for curves*, Math. Ann. 322 (2002), 69-74.
[15] A. Noma, *Generic inner projections of projective varieties and an application to the positivity of double point divisors*, Trans. Amer. Math. Soc. 366 (2014), 4603-4623.
[16] A. Noma, *Castelnuovo-Mumford regularity of projected Roth varieties*, J. Algebra 466 (2016), 153-168.
[17] H. Pinkham, *A Castelnuovo bound for smooth surfaces*, Invent. Math. 83 (1986), 321-332.