Some Novel Cosine Similarity Measures Based on Complex Hesitant Fuzzy Sets and Their Applications

Ronnason Chinram, Tahir Mahmood, Ubaid Ur Rehman, Zeeshan Ali, and Aiyared Iampan

1. Introduction

The fuzzy set (FS) is the modification of crisp set which was given by Zadeh [1] to manage the vagueness and uncertainty in the information in real-life decisions. In the theory of FS, the positive grade belongs to closed interval [0, 1], where greatest value designated greatest positive grade. FS has numerous applications in various fields [2–4]. Bustince et al. [5] operated on FSs and their models, extensions portrayal, and aggregation. SMs between FSs play an essential role in the theory of FS, which attracted a lot of attention from the authors. SMs have a lot of applications in real-world problems and are extremely useful in numerous fields [6, 7].

Chen [8] interpreted the similarity function to find the similarity degree among FSs. Pedrycz [9] presented fuzzy control and fuzzy systems. FSs in pattern recognition, methodology, and methods are also presented by Pedrycz [10]. Rangel-Valdez et al. [11] described parallel designs for metaheuristics that solve portfolio selection problems using fuzzy outranking relations. Mahmood [12] described a novel approach towards bipolar soft set and their applications.

Numerous authors mentioned the issue, what will be the impact when we alter the range of FS into a unit circle of a complex plane. To deal with such sorts of circumstance, Ramot et al. [13] described the notion of CFS as a modification of FS to handle the complex and tricky data in real-world. The idea of CFS is represented by complex-valued positive grade, which carries two-dimensional data in a particular set. Moreover, Tamir et al. [14] presented the Cartesian form of CFS and the Cartesian complex fuzzy positive grade where both real and imaginary parts carry the fuzzy data. In polar portrayal, the fuzzy data carries the
phase value and absolute value of complex positive grade. The complex fuzzy number is not the same as the CFS. The \( \delta \)-equalities and operation properties of CFS were introduced by Zhang et al. [15].

HFSs are the significant expansions of the theory of FS. Torra [16, 17] described the notion of HFS. An HFS is represented by positive grade which is in the shape of a finite subset of closed interval [0, 1]. Torra and Narukawa [17] characterized some fundamental operations on HFS. Rodriguez et al. [18] built up the idea of hesitant fuzzy linguistic term sets. Farhadinia [19] interpreted the idea of similarity and distance measures for higher order HFS. The notion of hesitant fuzzy data aggregation in decision-making (DM) was described by Xia and Xu [20]. Wei et al. [21] interpreted the idea of hesitant fuzzy Choquet aggregation operators and their applications to multiple attribute decision-making (MADM). Zhang [22] characterized hesitant fuzzy aggregation operators and their application to MADM. Xu and Xia [23] explored the idea of separation and correlation measures of HFS. Zhu et al. [24] gave the idea of hesitant fuzzy geometric Bonferroni means. Herrera et al. [25] described HFSs, an emerging tool in decision-making. A review of HFSs, quantitative and qualitative extensions, was explored by Rodriguez et al. [26]. Li et al. [27] characterized the consistency of hesitant fuzzy linguistic preference relations. Muhiuddin et al. [28] interpreted the generalized hesitant fuzzy ideals in semigroups.

The idea of similarity is a fundamental idea in human cognizance. Similarity has a key role in recognition, taxonomy, and several different fields. There are numerous aspects of the notion of the similarity that have escaped formalization. As per (HFS) detailing of a substantial, broadly useful definition of similarity is a difficult issue. There does not exist a legitimate, universally useful definition of similarity. There exist numerous specific definitions that have been utilized with accomplishment in diagnostics, classification, cluster analysis, and recognition. There are a few comparability measures that are interpreted and utilized for different purposes [29]. The SMs are categorized into 3 classifications: (1) measures based on implicators. (2) Measure based on metric. (3) Measure based on set-theoretic. While managing SMs based on distance, examples have been developed for perceptual similarity where each distance adage is obviously damaged by dissimilarity measures, especially the triangle inequality [17], and thusly the relating SM ignores transitivity. This model hypothesizes that the perceptual distance fulfills the metric adages, the observational legitimacy of which has been tentatively tested by a few authors, especially the triangle inequality (for subtleties see [16] and [17, 30, 31]). Thus, in the event of set-theoretic SMs, it is seen that crisp transitivity is a lot more grounded condition to be put upon SM. Set-theoretic SMs are additionally partitioned in three gatherings: (i) measures dependent on crisp logic; (ii) measures dependent on fuzzy logic; (iii) measures dependent on HFSs.

In this paper, we present complex HFSs. The inspiration is that when characterizing the positive grade of the element, the struggle of establishing the positive grade is not as we have margin of error (as in complex intuitionistic FS [32]), or some chance circulations (as in type 2 CFSs) on the probable values. In the existing theories, numerous scholars have faced several troubles. When a decision-maker provides such types of information for the grade of truth in the form of \( 0.22e^{0.3} \) and \( 0.5e^{0.3} \), this circumstance can emerge in a multicriteria DM. Basically, the theory of complex hesitant fuzzy set contains the grade of truth in the form of complex number, whose real and imaginary parts are in the form of the finite subset of the unit interval. In this unique situation, rather than considering only an aggregation operator [12], it is helpful to manage all the possible values. This circumstance, as we will talk about later, can be demonstrated utilizing multisets. Therefore, the existing theories are not able to cope with such types of troubles. The investigated ideas are more able to cope with it effectively.

Due to this and preserving the advantages of the SMs, in this manuscript, the notion of CHFS is explored, which is the fusion of HFS and CFS to manage the uncertainty and complicated data in real world. The positive membership in CHFS is in the form of a finite subset of unit disc in the complex plane. Moreover, in this manuscript, we interpreted some similarity measures (SMs) and weighted SMs (WSMs). Additionally, we use our explored SMs and weighted SMs of CHFS in the environment of medical diagnosis and pattern recognition to assess the practicality and competence of the described SMs. The comparison between explored measures with some already defined measures and their graphical representations are also discussed in detail.

The structure of this manuscript is given as follows: in Section 2 of this manuscript, we present preliminaries. In Section 3, the notion of the CHFS and its fundamental properties are explored. In Section 4 of this manuscript, we explore some similarity measures (SMs) and weighted SMs (WSMs) of CHFS. In Section 5, we use proposed SMs and weighted SMs in the environment of medical diagnosis and pattern recognition. The comparison between explored measure with some already defined measures and their graphical representations are also discussed in detail in Section 6. In Section 7, we discuss the conclusion of the article.

2. Preliminaries

In this section, we revise fundamental definitions such as FS, CFS, and HFS. Throughout this paper, \( x \) denotes the fix set.

**Definition 1** (see [1]). An FS \( E \) is of the shape,

\[
E = \{ (x, \mu_E (x)) \mid x \in \chi \},
\]

with a condition \( 0 \leq \mu_E (x) \leq 1 \), where \( \mu_E (x) \) stands for the grade of membership. Throughout this paper, the family of all FSs on \( X \) are designated by FS(\( X \)). The pair \( E = (x, \mu_E (x)) \) is said to be fuzzy number (FN).

**Definition 2** (see [13]). A CFS \( E \) is of the shape,

\[
E = \{ (x, \mu_E (x)) \mid x \in \chi \},
\]

where \( \mu_E (x) = g(x)e^{2\pi i \mu(x)} \) stands for the complex-valued membership grade in the shape of polar coordinate, where
the subset of unit disc in complex plane with a condition expressed the complex-valued grade of membership which is

\[
\gamma_E(x), \omega_E(x) \in [0, 1]. \text{ Moreover, the pair } E = (x, \gamma_E(x), \omega_E(x)) \text{ is said to be complex fuzzy number (CFN).}
\]

**Definition 3** (see [16, 17]). An HFS \( E \) is of the shape,

\[
E = \{(x, \mu_E(x)) | x \in \chi \},
\]

where \( \mu_E(x) \) is a finite subset of \([0, 1]\) standing for the grade of membership for every element \( x \in \chi \). Moreover, the pair \( E = (x, \mu_E(x)) \) is said to be hesitant fuzzy number (HFN).

**Definition 4** (see [29]). For any two HFSs \( E \) and \( F \), the SM \( \mathcal{D}(E, F) \) fulfills the following axioms:

1. \( 0 \leq \mathcal{S}(E, F) \leq 1 \);
2. \( \mathcal{S}(E, F) = 1 \iff E = F \);
3. \( \mathcal{S}(E, F) = \mathcal{S}(F, E) \).

From the discussion we did above, we get that the \( \mathcal{S}(E, F) = 1 = \mathcal{D}(E, F) \).

### 3. Complex Hesitant Fuzzy Sets

In this section, we explored the notion of complex hesitant fuzzy sets (CHFSs) and some of its properties.

**Definition 6.** A CHFS \( E \) is of the shape,

\[
E = \{(x, \mu_E(x)) | x \in X \},
\]

where

\[
\mu_E(x) = \begin{cases} \gamma_{E_1}(x) \cdot e^{i2\pi(\omega_{E_1}(x))}, & j = 1, 2, 3, \ldots, n \\ \gamma_{E_n}(x) \cdot e^{i2\pi(\omega_{E_n}(x))}, & \ldots, \ldots, \gamma_{E_2}(x) \cdot e^{i2\pi(\omega_{E_2}(x))} \end{cases}
\]

expressed the complex-valued grade of membership which is the subset of unit disc in complex plane with a condition

\[
\gamma_{E_1}(x), \omega_{E_1}(x) \in [0, 1]. \text{ Further, } E = (x, \gamma_{E_1}(x) \cdot e^{i2\pi(\omega_{E_1}(x))}) \text{ is known as the complex hesitant fuzzy number (CHFN).}
\]

**Definition** \( \gamma_{E_1}(x), \omega_{E_1}(x) \) be two CHFNs. Then,

1. \( c(\gamma_E(x)) = \begin{cases} (x, 1 - \gamma_E(x)) \cdot e^{i2\pi(1 - \omega_E(x))} \end{cases} \)
2. \( E \cup F = \begin{cases} (x, \max(\gamma_E(x), \gamma_F(x))) \cdot e^{i2\pi(\max(\omega_E(x), \omega_F(x)))} \end{cases} \)
3. \( E \cap F = \begin{cases} (x, \min(\gamma_E(x), \gamma_F(x))) \cdot e^{i2\pi(\min(\omega_E(x), \omega_F(x)))} \end{cases} \)

The theory of CHFS is a powerful tool to deal with unsure and complicated real-world issues. The CHFS holds the grade of membership in the shape of a finite subset of the unit disc in the complex plane, whose entities are in the shape of polar coordinates. Essentially, the CHFS holds two-dimensional data in a particular set. The explored CHFS is more general than the existing notions such as FS, CFS, and HFS.

**Example** \( E = \{(x, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0), (x, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1), (x, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2), (x, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3), (x, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4)\} \)

\( F = \{(x, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0), (x, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1), (x, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2), (x, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3), (x, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4)\} \)

be two CHFSs. Then,

**Definition 5** (see [29]). For any two HFSs \( E \) and \( F \), the distance measure \( \mathcal{D}(E, F) \) fulfills the following properties:

1. \( 0 \leq \mathcal{D}(E, F) \leq 1 \);
2. \( \mathcal{D}(E, F) = 1 \iff E = F \);
3. \( \mathcal{D}(E, F) = \mathcal{D}(F, E) \).

4. **Similarity Measures Based on the Cosine Function for CHFSs**

In this section, we interpreted some SMs such as cosine SMs for CHFSs, SMs of CHFSs based on cosine function, and SMs of CHFSs based on cotangent function.

**Definition 8.** Let \( E \) and \( F \) be two CHFSs on set \( X \). Then, SM between \( E \) and \( F \) is represented by \( S_\chi(E, F) \), which fulfills the following postulate:

1. \( 0 \leq S_\chi(E, F) \leq 1 \);
2. \( S_\chi(E, F) = 1 \iff E = F \);
3. \( S_\chi(E, F) = S_\chi(F, E) \).

4.1. **Cosine Similarity Measures for CHFS.** Let \( E \) be a CHFS on a set \( X \). Then, the elements contained in CHFS can be presented as the function of membership degree \( \mu_E(x) \), which is a subset of a unit disc in a complex plane. Consequently, a cosine SM and weighted cosine SM with CHF data are expressed similarly to the cosine SM based on Bhattacharya’s distance [33].
Definition 9. Let $E$ and $F$ be two CHFSs on a set $X$. Then, the cosine SM between $E$ and $F$ can be presented as

$$S^1(E, F) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j}^* (x_k) \cdot y_{F_j} (x_k) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}} (x_k) \cdot \omega_{y_{F_j}} (x_k)}{\sqrt{1/L \sum_{j=1}^{L} y_{E_j}^2 (x_k) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}}^2 (x_k)} \sqrt{1/L \sum_{j=1}^{L} y_{E_j}^2 (x_k) + (1/L) \sum_{j=1}^{L} \omega_{y_{F_j}}^2 (x_k)}} \right).$$

(6)

In Definition 9, if we assume the imaginary parts zero, then the interpreted SM transforms for HFS. Likewise, if we assume the CHFS as a singleton set, then the interpreted SM transforms for CFS. Moreover, if we assume the CHFS as a singleton set and the imaginary part zero, then the interpreted SM transforms for FS. Its structure makes it important and expert to deal with unknown and undependable data in real decision theory.

Theorem 1. The SM $S^1(E, F)$ fulfills the following postulates:

1. $0 \leq S^1(E, F) \leq 1$;
2. $S^1(E, F) = 1$ if $E = F$;

(3) $S^1(E, F) = S^1(F, E)$.

Proof

(1) Since $1/L \sum_{j=1}^{L} y_{E_j} (x_k) \cdot y_{F_j} (x_k) \in [0, 1]$, $1/L \sum_{j=1}^{L} \omega_{y_{E_j}} (x_k) \cdot \omega_{y_{F_j}} (x_k) \in [0, 1]$, $1/L \sum_{j=1}^{L} y_{E_j}^2 (x_k) \in [0, 1]$, and $1/L \sum_{j=1}^{L} \omega_{y_{E_j}}^2 (x_k) \in [0, 1]$, and denominator will always remain greater than the nominator. So, for $k = 1$, we have

$$\frac{1/L \sum_{j=1}^{L} y_{E_j} (x_1) \cdot y_{F_j} (x_1) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}} (x_1) \cdot \omega_{y_{F_j}} (x_1)}{1/L \sum_{j=1}^{L} y_{E_j}^2 (x_1) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}}^2 (x_1)} \in [0, 1].$$

(7)

For $k = 2$, we have

$$\frac{1/L \sum_{j=1}^{L} y_{E_j} (x_2) \cdot y_{F_j} (x_2) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}} (x_2) \cdot \omega_{y_{F_j}} (x_2)}{1/L \sum_{j=1}^{L} y_{E_j}^2 (x_2) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}}^2 (x_2)} \in [0, 1].$$

(8)

By continuing this procedure, we obtain

$$\sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j} (x_k) \cdot y_{F_j} (x_k) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}} (x_k) \cdot \omega_{y_{F_j}} (x_k)}{1/L \sum_{j=1}^{L} y_{E_j}^2 (x_k) + (1/L) \sum_{j=1}^{L} \omega_{y_{E_j}}^2 (x_k)} \right) \in n[0, 1].$$

(9)
This implies that

\[
0 \leq \sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j}(x_k) \cdot y_{F_k}(x_k) + 1/L \sum_{j=1}^{L} \omega_{y_{E_j}}(x_k) \cdot \omega_{y_{F_k}}(x_k)}{1/L \sum_{j=1}^{L} y_{E_j}^2(x_k) + 1/L \sum_{j=1}^{L} \omega_{y_{E_j}}^2(x_k)} \right) \leq n,
\]

(10)

\[
0 \leq \frac{1}{2} \sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j}(x_k) \cdot y_{F_k}(x_k) + 1/L \sum_{j=1}^{L} \omega_{y_{E_j}}(x_k) \cdot \omega_{y_{F_k}}(x_k)}{1/L \sum_{j=1}^{L} y_{E_j}^2(x_k) + 1/L \sum_{j=1}^{L} \omega_{y_{E_j}}^2(x_k)} \right) \leq 1,
\]

which implies that

\[
0 \leq S_{E}^1(x_k) \leq 1.
\]

(11)

We have

\[
S_{E}^1(E, F) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j}(x_k) \cdot y_{F_k}(x_k) + 1/L \sum_{j=1}^{L} \omega_{y_{E_j}}(x_k) \cdot \omega_{y_{F_k}}(x_k)}{1/L \sum_{j=1}^{L} y_{E_j}^2(x_k) + 1/L \sum_{j=1}^{L} \omega_{y_{E_j}}^2(x_k)} \right).
\]

(12)

Now, as \( E = F \Rightarrow \mu_{E_j}(x_k) = \mu_{F_j}(x_k) \) for \( k = 1, 2, \ldots, n \Rightarrow y_{E_j}(x_k)e^{i\omega y_j(x_k)} = y_{F_j}(x_k)e^{i\omega y_j(x_k)} \) for \( k = 1, 2, \ldots, n \). Then,

\[
e^{i2\pi(\omega y_j(x_k))} = \]

\[
S_{E}^1(E, F) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j}(x_k) + \cdots + y_{E_j}(x_k)}{1/L \sum_{j=1}^{L} y_{E_j}^2(x_k) + \cdots + y_{E_j}^2(x_k)} \right).
\]

(13)

\[
S_{E}^1(E, F) = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1/L \sum_{j=1}^{L} y_{E_j}^2(x_k) + \cdots + y_{E_j}^2(x_k)}{1/L \sum_{j=1}^{L} y_{E_j}^2(x_k) + \cdots + y_{E_j}^2(x_k)} \right).
\]

(14)

\[
S_{E}^1(E, F) = 1.
\]
We have

\[
S^1_c(E, F) = \frac{1}{n} \sum_{n, x, \ell = 1}^{n, x, \ell} \left( \frac{1}{\sqrt{\sum_{j=1}^{n} x_{\ell} y_{\ell} x_{\ell}}} \right)^2 \left( \frac{1}{\sqrt{\sum_{j=1}^{n} y_{\ell} y_{\ell} x_{\ell}}} \right)^2
\]

\[
= \frac{1}{n} \sum_{n, x, \ell = 1}^{n, x, \ell} \left( \frac{1}{\sqrt{\sum_{j=1}^{n} x_{\ell} y_{\ell} x_{\ell}}} \right)^2 \left( \frac{1}{\sqrt{\sum_{j=1}^{n} y_{\ell} y_{\ell} x_{\ell}}} \right)^2
\]

We defined distance measure of the angle as

\[
d(E, F) = \arccos(S^1(E, F)).
\]

It holds the following axioms:

1. \(d(E, F) \geq 0\) if \(0 \leq S^1_c(E, F) \leq 1\);
2. \(d(E, F) = \arccos(1) = 0\) if \(S^1_c(E, F) = 1\);

We have

\[
S^1_{cw}(E, F) = \sum_{n, x, \ell = 1}^{n, x, \ell} \frac{1}{\sqrt{\sum_{j=1}^{n} x_{\ell} y_{\ell} x_{\ell}}} \left( \frac{1}{\sqrt{\sum_{j=1}^{n} y_{\ell} y_{\ell} x_{\ell}}} \right)^2
\]

where \(w = (w_1, w_2, \ldots, w_n)^T\) represents the weight vector of every element \(x\) included in CHFS and the weight vector satisfies \(w_x \in [0, 1]\) for every \(x = 1, 2, \ldots, n\). When we suppose the weight vector to be \(w = (1/n, 1/n, \ldots, 1/n)^T\), the weighted cosine SM will transform into cosine SM. Otherwise speaking, when \(w_x = 1/n, x = 1, 2, 3, \ldots, n\), the \(S^1_{cw}(E, F) = S^1_c(E, F)\).

4.2. Similarity Measures of CHFSs Based on Cosine Function.

In this part of the paper, we interpreted SMs of CHFSs based on cosine function and studied their properties.

Definition 11. Let \(E\) and \(F\) be two CHFSs on a set \(X\). Then, the SMs based on the cosine function between \(E\) and \(F\) can be presented as

\[
S^2_c(E, F) = \frac{1}{n} \sum_{n, x, \ell = 1}^{n, x, \ell} \cos \left( \frac{\pi}{2} \left( \max \left( \frac{1}{2} \sum_{j=1}^{n} y_{\ell} (x) - y_{\ell}(x) \right) \right) \right),
\]

where \(S^2_c(E, F)\) means the SM based on the cosine function between \(E\) and \(F\), which considers the maximum distance based on the amplitude and phase terms.

Theorem 2. The SM \(S^2_c(E, F)\) fulfills the following postulates:

1. \(0 \leq S^2_c(E, F) \leq 1\);
(2) $S^2_1(E, F) = 1$ if $E = F$;
(3) $S^2_1(E, F) = S^2_1(F, E)$.

Proof

By continuing this procedure, we obtain

$$\cos\left[\frac{\pi}{2} \left( \max\left( \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |y_{Fj}(x_1) - y_{Fj}(x_1)|, \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |\omega_{\gamma_j}(x_1) - \omega_{\gamma_j}(x_1)| \right) \right]\right] \in [0, 1]. \quad (19)$$

For $\mathcal{K} = 2$, we have

$$\cos\left[\frac{\pi}{2} \left( \max\left( \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |y_{Fj}(x_2) - y_{Fj}(x_2)|, \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |\omega_{\gamma_j}(x_2) - \omega_{\gamma_j}(x_2)| \right) \right]\right] \in [0, 1]. \quad (20)$$

By continuing this procedure, we obtain

$$\sum_{\mathcal{K}=1}^{n} \cos\left[\frac{\pi}{2} \left( \max\left( \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |y_{Fj}(x_{\mathcal{K}}) - y_{Fj}(x_{\mathcal{K}})|, \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |\omega_{\gamma_j}(x_{\mathcal{K}}) - \omega_{\gamma_j}(x_{\mathcal{K}})| \right) \right]\right] \in n[0, 1]. \quad (21)$$

This implies that

$$0 \leq \sum_{\mathcal{K}=1}^{n} \cos\left[\frac{\pi}{2} \left( \max\left( \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |y_{Fj}(x_{\mathcal{K}}) - y_{Fj}(x_{\mathcal{K}})|, \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |\omega_{\gamma_j}(x_{\mathcal{K}}) - \omega_{\gamma_j}(x_{\mathcal{K}})| \right) \right]\right] \leq n, \quad (22)$$

$$0 \leq \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \cos\left[\frac{\pi}{2} \left( \max\left( \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |y_{Fj}(x_{\mathcal{K}}) - y_{Fj}(x_{\mathcal{K}})|, \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |\omega_{\gamma_j}(x_{\mathcal{K}}) - \omega_{\gamma_j}(x_{\mathcal{K}})| \right) \right]\right] \leq 1,$$

which implies that

$$0 \leq S^2_{\mathcal{K}}(x_{\mathcal{K}}) \leq 1. \quad (23)$$

We have

$$S^2_1(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \cos\left[\frac{\pi}{2} \left( \max\left( \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |y_{Fj}(x_{\mathcal{K}}) - y_{Fj}(x_{\mathcal{K}})|, \frac{1}{\tilde{L}} \sum_{j=1}^{\tilde{L}} |\omega_{\gamma_j}(x_{\mathcal{K}}) - \omega_{\gamma_j}(x_{\mathcal{K}})| \right) \right]\right]. \quad (24)$$

Now, as $E = F \implies \mu_E(x_{\mathcal{K}}) = \mu_F(x_{\mathcal{K}})$, for $\mathcal{K} = 1, 2, \ldots, n \implies y_{Ej}(x_{\mathcal{K}}) e^{i2\pi(\omega_{\gamma_j}(x_{\mathcal{K}}))} = y_{Fj}(x_{\mathcal{K}}) e^{i2\pi(\omega_{\gamma_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1, 2, \ldots, n \implies y_{Fj}(x_{\mathcal{K}}) = y_{Ej}(x_{\mathcal{K}})$ and $e^{i2\pi(\omega_{\gamma_j}(x_{\mathcal{K}}))} = e^{i2\pi(\omega_{\gamma_j}(x_{\mathcal{K}}))}$ for $\mathcal{K} = 1, 2, \ldots, n$. Then, $|y_{Ej}(x_{\mathcal{K}}) - y_{Fj}(x_{\mathcal{K}})| = 0$ and $|\omega_{\gamma_j}(x_{\mathcal{K}}) - \omega_{\gamma_j}(x_{\mathcal{K}})| = 0$ for $\mathcal{K} = 1, 2, \ldots, n$. This implies that
Theorem 3. The SM $S(E, F) \equiv 1$.

(3) We have

\[ S(E, F) = 1, \]

(25)

\[ S(E, F) = \frac{1}{n} \sum_{\mathcal{X} = 1}^{n} \cos \left[ \frac{\pi}{2} \left( \max \left( \frac{1}{L} \sum_{j=1}^{L} |y_{E,j}(x_{\mathcal{X}}) - y_{F,j}(x_{\mathcal{X}})|, \frac{1}{L} \sum_{j=1}^{L} |\omega_{x_{\mathcal{X}}} - \omega_{y_{\mathcal{X}}}(x_{\mathcal{X}})| \right) \right) \right], \]

\[ = \frac{1}{n} \sum_{\mathcal{X} = 1}^{n} \cos \left[ \frac{\pi}{2} \left( \max \left( \frac{1}{L} \sum_{j=1}^{L} |y_{E,j}(x_{\mathcal{X}}) - y_{F,j}(x_{\mathcal{X}})|, \frac{1}{L} \sum_{j=1}^{L} |\omega_{x_{\mathcal{X}}} - \omega_{y_{\mathcal{X}}}(x_{\mathcal{X}})| \right) \right) \right], \]

\[ = S(E, F). \]

Proof

(4) $0 \leq S(E, F) \leq 1$;

(5) $S(E, F) = 1$ if $E = F$;

(6) $S(E, F) = S(F, E)$.

For $\mathcal{X} = 2$, we have

\[ \cos \left[ \frac{\pi}{4} \left( \frac{1}{L} \sum_{j=1}^{L} |y_{E,j}(x_{2}) - y_{F,j}(x_{2})| + \frac{1}{L} \sum_{j=1}^{L} |\omega_{x_{2}} - \omega_{y_{2}}(x_{2})| \right) \right] \in [0, 1]. \]

By continuing this procedure, we obtain

\[ \sum_{\mathcal{X} = 1}^{n} \cos \left[ \frac{\pi}{4} \left( \frac{1}{L} \sum_{j=1}^{L} |y_{E,j}(x_{\mathcal{X}}) - y_{F,j}(x_{\mathcal{X}})| + \frac{1}{L} \sum_{j=1}^{L} |\omega_{x_{\mathcal{X}}} - \omega_{y_{\mathcal{X}}}(x_{\mathcal{X}})| \right) \right] \in n[0, 1]. \]

This implies that

\[ 0 \leq \sum_{\mathcal{X} = 1}^{n} \cos \left[ \frac{\pi}{4} \left( \frac{1}{L} \sum_{j=1}^{L} |y_{E,j}(x_{\mathcal{X}}) - y_{F,j}(x_{\mathcal{X}})| + \frac{1}{L} \sum_{j=1}^{L} |\omega_{x_{\mathcal{X}}} - \omega_{y_{\mathcal{X}}}(x_{\mathcal{X}})| \right) \right] \leq n, \]

\[ 0 \leq \frac{1}{n} \sum_{\mathcal{X} = 1}^{n} \cos \left[ \frac{\pi}{4} \left( \frac{1}{L} \sum_{j=1}^{L} |y_{E,j}(x_{\mathcal{X}}) - y_{F,j}(x_{\mathcal{X}})| + \frac{1}{L} \sum_{j=1}^{L} |\omega_{x_{\mathcal{X}}} - \omega_{y_{\mathcal{X}}}(x_{\mathcal{X}})| \right) \right] \leq 1, \]
which implies that
\[ 0 \leq S^3_\xi(x, \mathcal{X}) \leq 1. \]

(2) We have
\[ S^3_\xi(E, F) = \frac{1}{n} \sum_{\mathcal{X}=1}^{n} \cos \left( \frac{\pi}{4} \left( \frac{1}{n} \sum_{j=1}^{n} |\gamma_{E_j}(x, \mathcal{X}) - \gamma_{F_j}(x, \mathcal{X})| + \frac{1}{n} \sum_{j=1}^{n} \omega_{\gamma_{E_j}(x, \mathcal{X})} - \omega_{\gamma_{F_j}(x, \mathcal{X})} \right) \right) \]

Now, as \( E = F \) implies \( \mu_E(x, \mathcal{X}) = \mu_F(x, \mathcal{X}) \), for \( \mathcal{X} = 1, 2, \ldots, n \) we have \( \gamma_{E_j}(x, \mathcal{X}) = \gamma_{F_j}(x, \mathcal{X}) \) and \( \omega_{\gamma_{E_j}(x, \mathcal{X})} = \omega_{\gamma_{F_j}(x, \mathcal{X})} \) for \( \mathcal{X} = 1, 2, \ldots, n \). This implies that
\[ S^3_\xi(E, F) = 1. \]

(3) We have
\[ S^3_\xi(E, F) = \frac{1}{n} \sum_{\mathcal{X}=1}^{n} \cos \left( \frac{\pi}{4} \left( \frac{1}{n} \sum_{j=1}^{n} |\gamma_{E_j}(x, \mathcal{X}) - \gamma_{F_j}(x, \mathcal{X})| + \frac{1}{n} \sum_{j=1}^{n} \omega_{\gamma_{E_j}(x, \mathcal{X})} - \omega_{\gamma_{F_j}(x, \mathcal{X})} \right) \right) \]

\[ = \frac{1}{n} \sum_{\mathcal{X}=1}^{n} \cos \left( \frac{\pi}{4} \left( \frac{1}{n} \sum_{j=1}^{n} |\gamma_{E_j}(x, \mathcal{X}) - \gamma_{F_j}(x, \mathcal{X})| + \frac{1}{n} \sum_{j=1}^{n} \omega_{\gamma_{E_j}(x, \mathcal{X})} - \omega_{\gamma_{F_j}(x, \mathcal{X})} \right) \right) \]

\[ = S^3_\xi(F, E). \]
where $S^4_\kappa(E, F)$ means the cotangent SM between $E$ and $F$, which considers the maximum distance based on the amplitude and phase terms.

\[
S^5_\kappa(E, F) = \frac{1}{n} \sum_{\mathcal{X}=1}^{n} \text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{m} \sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})| \left| \frac{1}{m} \sum_{j=1}^{m} \omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}}) \right| \right) \right],
\]

(37)

where $S^5_\kappa(E, F)$ means the cotangent SM between $E$ and $F$, which considers the sum of distance based on the amplitude and phase terms.

**Theorem 4.** The SM $S^4_\kappa(E, F)$ fulfills the following postulates:

1. $0 \leq S^4_\kappa(E, F) \leq 1$;
2. $S^4_\kappa(E, F) = 1$ if $E = F$;
3. $S^4_\kappa(E, F) = S^2_\kappa(F, E)$.

**Proof**

1. Since $1/\sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})| \in [0, 1]$, $1/\sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})| \in [0, 1]$, this implies that

\[
\max (1/\sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})|, 1/\sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})|) \in [0, 1].
\]

So, for $\mathcal{X} = 1$, we have

\[
\text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \max \left( \frac{1}{m} \sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})|, \frac{1}{m} \sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})| \right) \right) \right] \in [0, 1].
\]

(38)

For $\mathcal{X} = 2$, we have

\[
\text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \max \left( \frac{1}{m} \sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})|, \frac{1}{m} \sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})| \right) \right) \right] \in [0, 1].
\]

(39)

By continuing this procedure, we obtain

\[
\sum_{\mathcal{X}=1}^{n} \text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \max \left( \frac{1}{m} \sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})|, \frac{1}{m} \sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})| \right) \right) \right] \in n[0, 1].
\]

(40)

This implies that

\[
0 \leq \sum_{\mathcal{X}=1}^{n} \text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \max \left( \frac{1}{m} \sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})|, \frac{1}{m} \sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})| \right) \right) \right] \leq n,
\]

(41)

\[
0 \leq \frac{1}{n} \sum_{\mathcal{X}=1}^{n} \text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \max \left( \frac{1}{m} \sum_{j=1}^{m} |\gamma_E(x_{\mathcal{X}}) - \gamma_F(x_{\mathcal{X}})|, \frac{1}{m} \sum_{j=1}^{m} |\omega_{\gamma_{E_j}}(x_{\mathcal{X}}) - \omega_{\gamma_{F_j}}(x_{\mathcal{X}})| \right) \right) \right] \leq 1,
\]
which implies that

\[ 0 \leq \mathcal{S}^4_\epsilon (x, \mathcal{H}) \leq 1. \]  \hspace{1cm} (42)

(2) We have

\[ |\gamma^{E}_j (x, \mathcal{H}) - \gamma^{F}_j (x, \mathcal{H})| = 0 \quad \text{and} \quad |\omega^{E}_j (x, \mathcal{H}) - \omega^{F}_j (x, \mathcal{H})| = 0 \]

for \( j = 1, 2, \ldots, n \). This implies that

\[ \mathcal{S}^4_\epsilon (E, F) = 1. \]  \hspace{1cm} (44)

(3) We have

\[ \mathcal{S}^4_\epsilon (E, F) = \mathcal{S}^5_\epsilon (F, E). \]

**Theorem 5.** The SM \( \mathcal{S}^5_\epsilon (E, F) \) fulfils the following postulates:

(10) \( 0 \leq \mathcal{S}^5_\epsilon (E, F) \leq 1; \)

(11) \( \mathcal{S}^5_\epsilon (E, F) = 1 \) if \( E = F; \)

(12) \( \mathcal{S}^5_\epsilon (E, F) = \mathcal{S}^5_\epsilon (F, E). \)

**Proof**

(1) Since \( \frac{1}{L} \sum_{j=1}^{L} |\gamma^{E}_j (x, \mathcal{H}) - \gamma^{F}_j (x, \mathcal{H})| \in [0, 1], \frac{1}{L} \sum_{j=1}^{L} |\omega^{E}_j (x, \mathcal{H}) - \omega^{F}_j (x, \mathcal{H})| \in [0, 1] \), this implies that,

\[ 1/2 \max (1/\mathcal{H} \sum_{j=1}^{L} |\gamma^{E}_j (x, \mathcal{H}) - \gamma^{F}_j (x, \mathcal{H})|, \sum_{j=1}^{L} |\omega^{E}_j (x, \mathcal{H}) - \omega^{F}_j (x, \mathcal{H})|) \in [0, 1]. \]  

So, for \( \mathcal{H} = 1 \), we have

\[ \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{L} \sum_{j=1}^{L} |\gamma^{E}_j (x_1) - \gamma^{F}_j (x_1)| + \frac{1}{L} \sum_{j=1}^{L} |\omega^{E}_j (x_1) - \omega^{F}_j (x_1)| \right) \right] \in [0, 1]. \]  \hspace{1cm} (46)

For \( \mathcal{H} = 2 \), we have

\[ \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{L} \sum_{j=1}^{L} |\gamma^{E}_j (x_2) - \gamma^{F}_j (x_2)| + \frac{1}{L} \sum_{j=1}^{L} |\omega^{E}_j (x_2) - \omega^{F}_j (x_2)| \right) \right] \in [0, 1]. \]  \hspace{1cm} (47)
By continuing this procedure, we obtain

\[
\sum_{\mathcal{K}=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{2} \sum_{j=1}^{k} \left| y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}}) \right| + \frac{1}{2} \sum_{j=1}^{k} \left| \omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}}) \right| \right) \right] \in [0, 1].
\] (48)

This implies that

\[
0 \leq \sum_{\mathcal{K}=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{2} \sum_{j=1}^{k} \left| y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}}) \right| + \frac{1}{2} \sum_{j=1}^{k} \left| \omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}}) \right| \right) \right] \leq n,
\] (49)

\[
0 \leq \sum_{\mathcal{K}=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{2} \sum_{j=1}^{k} \left| y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}}) \right| + \frac{1}{2} \sum_{j=1}^{k} \left| \omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}}) \right| \right) \right] \leq 1,
\]

which implies that

\[
0 \leq S_5^\mathcal{K}(x_{\mathcal{K}}) \leq 1.
\] (50)

(2) We have

\[
S_5^\mathcal{K}(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{2} \sum_{j=1}^{k} \left| y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}}) \right| + \frac{1}{2} \sum_{j=1}^{k} \left| \omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}}) \right| \right) \right].
\] (51)

Now, as \( E = F \implies \mu_E(x_{\mathcal{K}}) = \mu_F(x_{\mathcal{K}}) \), for \( \mathcal{K} = 1, 2, \ldots, n \implies y_{E_1}(x_{\mathcal{K}}) e^{i\pi(x_{\mathcal{K}})} = y_{F_1}(x_{\mathcal{K}}) \)
eq e^{i\pi(x_{\mathcal{K}})} \) for \( \mathcal{K} = 1, 2, \ldots, n \implies y_{E_1}(x_{\mathcal{K}}) = y_{F_1}(x_{\mathcal{K}}) \) and \( e^{i\pi(x_{\mathcal{K}})} \) for \( \mathcal{K} = 1, 2, \ldots, n \). Then, \( |y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}})| = 0 \) and \( |\omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}})| = 0 \) for \( \mathcal{K} = 1, 2, \ldots, n \). This implies that

\[
S_5^\mathcal{K}(E, F) = 1.
\] (52)

(3) We have

\[
S_5^\mathcal{K}(E, F) = \frac{1}{n} \sum_{\mathcal{K}=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{2} \sum_{j=1}^{k} \left| y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}}) \right| + \frac{1}{2} \sum_{j=1}^{k} \left| \omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}}) \right| \right) \right],
\]

\[
= \frac{n}{n} \sum_{\mathcal{K}=1}^{n} \cot \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{2} \sum_{j=1}^{k} \left| y_{E_1}(x_{\mathcal{K}}) - y_{F_1}(x_{\mathcal{K}}) \right| + \frac{1}{2} \sum_{j=1}^{k} \left| \omega_{y_{E_1}}(x_{\mathcal{K}}) - \omega_{y_{F_1}}(x_{\mathcal{K}}) \right| \right) \right],
\]

\[
= S_5^\mathcal{K}(F, E).
\] □
Definition 14. Let $E$ and $F$ be two CHFs on a set $X$. Then, the weighted cotangent SMs between $E$ and $F$ can be presented as

$$
S_{cw}^4(E, F) = \sum_{x \in X}^{n} w_x \text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \max \left( \frac{1}{n} \sum_{j=1}^{n} |y_{E_i}(x) - y_{F_i}(x)|, \frac{1}{n} \sum_{j=1}^{n} |\omega_{y_{E_i}}(x) - \omega_{y_{F_i}}(x)| \right) \right) \right],
$$

$$
S_{cw}^5(E, F) = \sum_{x \in X}^{n} w_x \text{Cot} \left[ \frac{\pi}{4} + \frac{\pi}{8} \left( \frac{1}{n} \sum_{j=1}^{n} |y_{E_i}(x) - y_{F_i}(x)|, \frac{1}{n} \sum_{j=1}^{n} |\omega_{y_{E_i}}(x) - \omega_{y_{F_i}}(x)| \right) \right],
$$

where $w = (w_1, w_2, \ldots, w_n)^T$ represents the weight vector of every element $x \in X$ ($X = 1, 2, \ldots, n$) carried in CHFS and the weight vector satisfies $w \in [0, 1]$ for every $X = 1, 2, 3, \ldots, n$, $\sum_{x \in X} w_x = 1$. When we assume the weight vector to be $w = (1/n, 1/n, \ldots, 1/n)^T$, the weighted cotangent SMs will transform into cotangent SMs. Otherwise speaking, when $w_x = 1/n, X = 1, 2, \ldots, n$, the $S_{cw}^m(E, F) = S_{cw}^m(E, F)$ for $m = 4, 5$.

5. Applications

In this section, we gave two applications about cosine SM, SMs based on cosine function, and cotangent SM under CHF environment. The interpreted SMs are applied to pattern recognition and medical diagnosis to express the usefulness of these SMs.

5.1. Pattern Recognition

Example 2. Without any hesitancy, the quantity of construction usually relies on the standard of building materials. Accordingly, building material scrutiny is the assumption of good engineering standards. The selection of material must be strictly controlled. Scrutiny authorizes the builders to correctly recognize qualified materials and upgrade the standard of the project. To resolve the abovementioned issues, we choose the building materials $E_j (j = 1, 2, 3, 4, 5)$, which are discussed as follows:

$$
E_1 = \left\{ \begin{array}{l}
(x_1, \{0.6e^{2\pi(0.1)}, 0.5e^{2\pi(0.5)}\}), (x_2, \{0.7e^{2\pi(0.4)}\}), (x_3, \{0.6e^{2\pi(0.8)} , 0.4e^{2\pi(0.7)}\}) \\
(x_4, \{0.8e^{2\pi(0.9)}, 0.2e^{2\pi(0.7)}\}), (x_5, \{0.2e^{2\pi(0.3)}, 0.6e^{2\pi(0.5)}, 0.4e^{2\pi(0.6)}\})
\end{array} \right\},
$$

$$
E_2 = \left\{ \begin{array}{l}
(x_1, \{0.1e^{2\pi(0.4)}\}), (x_2, \{0.5e^{2\pi(0.1)}, 0.1e^{2\pi(0.6)}\}), (x_3, \{0.2e^{2\pi(0.6)}, 0.7e^{2\pi(0.4)}\}) \\
(x_4, \{0.1e^{2\pi(0.4)}, 0.3e^{2\pi(0.1)}\}), (x_5, \{0.5e^{2\pi(0.6)}\})
\end{array} \right\},
$$

$$
E_3 = \left\{ \begin{array}{l}
(x_1, \{1e^{2\pi(0.8)}, 0.6e^{2\pi(0.8)}, 0.5e^{2\pi(0.9)}\}), (x_2, \{0.5e^{2\pi(0.7)}\}), (x_3, \{0.8e^{2\pi(0.1)}, 0.7e^{2\pi(0.9)}\}) \\
(x_4, \{0.9e^{2\pi(0.8)}, 0.7e^{2\pi(0.6)}\}), (x_5, \{0.7e^{2\pi(0.5)}, 0.2e^{2\pi(0.4)}, 0.3e^{2\pi(0.7)}\})
\end{array} \right\},
$$

$$
E_4 = \left\{ \begin{array}{l}
(x_1, \{0.3e^{2\pi(0.9)}, 1e^{2\pi(1)}\}), (x_2, \{0.4e^{2\pi(0.2)}, 0.2e^{2\pi(0.5)}\}), (x_3, \{0.2e^{2\pi(1)}\}) \\
(x_4, \{0.8e^{2\pi(0.6)}\}), (x_5, \{0.5e^{2\pi(0.1)}, 0.6e^{2\pi(0.3)}, 0.8e^{2\pi(0.5)}\})
\end{array} \right\},
$$

$$
E_5 = \left\{ \begin{array}{l}
(x_1, \{0.4e^{2\pi(0.2)}, 0.2e^{2\pi(0.5)}\}), (x_2, \{0.4e^{2\pi(0.2)}, 0.4e^{2\pi(1)}\}), (x_3, \{0.2e^{2\pi(0.4)}, 0.3e^{2\pi(0.1)}\}) \\
(x_4, \{0.6e^{2\pi(0.7)}, 0.3e^{2\pi(0.5)}, 0.6e^{2\pi(0.1)}\}), (x_5, \{0.1e^{2\pi(0.3)}\})
\end{array} \right\}.
$$

To resolve the abovementioned issue, we choose the complex hesitant fuzzy set in the form of unknown materials.

$$
E = \left\{ \begin{array}{l}
(x_1, \{0.9e^{2\pi(0.8)}, 0.7e^{2\pi(0.4)}, 1e^{2\pi(0.8)}\}), (x_2, \{0.4e^{2\pi(0.6)}\}), (x_3, \{0.8e^{2\pi(0.6)}, 0.5e^{2\pi(0.8)}\}) \\
(x_4, \{0.9e^{2\pi(0.6)}, 0.4e^{2\pi(0.3)}\}), (x_5, \{0.5e^{2\pi(0.7)}, 0.3e^{2\pi(0.9)}, 0.2e^{2\pi(0.5)}\})
\end{array} \right\}.
$$
The aim of this issue is to categorize the unspecified building material \( E \) in one of the categories \( E_j \) \((j = 1, 2, 3, 4, 5)\). For it, the cosine SM, SMs based on cosine function, and cotangent SMs which are explored in this paper have been used to determine the similarity from \( E \) to \( E_j \) \((j = 1, \ldots, 5)\) and calculations are introduced in Tables 1 and 2.

As stated by the above-computed calculations given in Table 1, we simply note that the degree of similarity between \( E \) and \( E_3 \) is the greatest one as an extract by all five SMs. This specifies that all five SMs assign the unspecified building material \( E \) to the specified building material \( E_3 \) based on the principle of the maximum degree of similarity. Ranking of the explored cosine and cotangent SMs between \( E \) and \( E_j \) \((j = 1, \ldots, 5)\) is also introduced in Table 1. The graphical representation of the interpreted WSMs between \( E \) and \( E_j \) \((j = 1, \ldots, 5)\) is indicated in Figure 1.

The weight of elements has great significance to suppose in real decision-making problems. If we suppose the weight of elements \( x \in \mathcal{X} = \{1, 2, 3, 4, 5\} \) to be \( w_x = (0.15, 0.1, 0.25, 0.2, 0.3) \), respectively, then the interpreted WSMs (weighted cosine SMs and weighted cotangent SMs) have been used to determine the similarity from \( E \) to \( E_j \) \((j = 1, \ldots, 5)\) and calculations are introduced in Table 2.

As stated by the above-computed calculations given in Table 2, we simply note that the degree of similarity between \( E \) and \( E_3 \) is the greatest one as an extract by all five WSMs. This specifies that all five WSMs assign the unspecified building material \( E \) to the specified building material \( E_3 \) based on the principle of the maximum degree of similarity. Ranking of the explored weighted cosine SMs, weighted SMs based on cosine function, and weighted cotangent SMs between \( E \) and \( E_j \) \((j = 1, \ldots, 5)\) is indicated in Figure 2.

5.2. Medical Diagnosis. Symptoms of every diseases are almost different. To examine that the victim is suffering from what type of diseases, the medical diagnosis relies on the victim’s symptoms. The victim’s symptoms are a set of symptoms and unspecified diseases will be a set of diagnostic diseases. The interpreted SMs are illustrated by a following numerical example of medical diagnosis.

Example 3. Let a set of diagnosis \( D = \{D_1\) (Typhoid), \( D_2\) (Flu), \( D_3\) (Heart problem), \( D_4\) (Pneumonia), \( D_5\) (Coronavirus)\} and set of symptoms \( \mathcal{X} = \{x_1\) (fever), \( x_2\) (cough) \( x_3\) (short of breath)\}. The victim’s symptoms are represented in the form of CHFSs as follows:

\[
P = \left\{ \begin{array}{c}
(x_1, \{0.6e^{2\pi(0.9)}, 0.9e^{2\pi(0.5)}\}), \\
(x_2, \{0.5e^{2\pi(0.7)}, 0.9e^{2\pi(1)}, 1e^{2\pi(0.5)}\}), \\
(x_3, \{0.1e^{2\pi(0.2)}\})
\end{array} \right\}.
\]

The indications of each disease \( D_j \) \((j = 1, 2, 3, 4, 5)\) are represented in the form CHFSs as follows:

\[
D_1 = \left\{ \begin{array}{c}
(x_1, \{0.7e^{2\pi(1)}, 0.9e^{2\pi(0.8)}\}), \\
(x_2, \{1e^{2\pi(0.8)}, 0.5e^{2\pi(0.6)}, 0.6e^{2\pi(0.9)}\}), \\
(x_3, \{0.4e^{2\pi(0.6)}\})
\end{array} \right\},
\]

\[
D_2 = \left\{ \begin{array}{c}
(x_1, \{0.5e^{2\pi(0.6)}, 0.9e^{2\pi(0.8)}\}), \\
(x_2, \{0.8e^{2\pi(1)}, 0.7e^{2\pi(0.8)}\}), \\
(x_3, \{0.1e^{2\pi(0.05)}\})
\end{array} \right\},
\]

\[
D_3 = \left\{ \begin{array}{c}
(x_1, \{0.6e^{2\pi(0.5)}\}), \\
(x_2, \{0.2e^{2\pi(0.2)}, 0.4e^{2\pi(0.1)}\}), \\
(x_3, \{0.8e^{2\pi(1)}, 1e^{2\pi(1)}, 0.7e^{2\pi(0.9)}\})
\end{array} \right\},
\]

\[
D_4 = \left\{ \begin{array}{c}
(x_1, \{0.6e^{2\pi(0.9)}, 0.7e^{2\pi(0.8)}, 0.4e^{2\pi(0.7)}\}), \\
(x_2, \{0.5e^{2\pi(0.7)}, 0.7e^{2\pi(0.3)}, 0.1e^{2\pi(0.6)}\}), \\
(x_3, \{0.1e^{2\pi(0.0)}\}), \\
(x_4, \{0.6e^{2\pi(0.8)}\}), \\
(x_5, \{0.4e^{2\pi(0.1)}, 0.2e^{2\pi(0.4)}\})
\end{array} \right\},
\]

\[
D_5 = \left\{ \begin{array}{c}
(x_1, \{0.8e^{2\pi(0.4)}, 0.5e^{2\pi(0.7)}\}), \\
(x_2, \{0.6e^{2\pi(0.7)}, 0.7e^{2\pi(0.9)}\}), \\
(x_3, \{0.1e^{2\pi(0.4)}, 0.3e^{2\pi(0.2)}\}), \\
(x_4, \{0.3e^{2\pi(0.4)}\}), \\
(x_5, \{0.8e^{2\pi(0.7)}, 0.9e^{2\pi(1)}, 1e^{2\pi(0.7)}\})
\end{array} \right\}.
\]

The aim of this issue is to find the disease of the victim \( P \) in one of the diseases \( D_j \) \((j = 1, 2, 3, 4, 5)\). For it, the cosine SM, SMs based on cosine function, and cotangent SMs which are explored in this paper have been utilized to determine the similarity from \( P \) to \( D_j \) \((j = 1, \ldots, 5)\) and calculations are introduced in Tables 3 and 4.
Table 1: The explored SMs between $E$ and $E_j$ ($j = 1, 2, 3, 4, 5$).

| SMs   | (E, E₁) | (E, E₂) | (E, E₃) | (E, E₄) | (E, E₅) | Ranking |
|-------|---------|---------|---------|---------|---------|---------|
| $S_{1}^{1}$ (E, E₁) | 0.4733  | 0.2829  | 0.517   | 0.3429  | 0.2844  | $E_5 \geq E_1 \geq E_4 \geq E_3 \geq E_2$ |
| $S_{1}^{2}$ (E, E₂) | 0.8674  | 0.6372  | 0.9321  | 0.6047  | 0.7278  | $E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$ |
| $S_{1}^{3}$ (E, E₃) | 0.9022  | 0.796   | 0.959   | 0.7405  | 0.8054  | $E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$ |
| $S_{1}^{4}$ (E, E₄) | 0.6056  | 0.3777  | 0.6988  | 0.3762  | 0.4605  | $E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$ |
| $S_{1}^{5}$ (E, E₅) | 0.6528  | 0.5264  | 0.7563  | 0.4746  | 0.7479  | $E_3 \geq E_1 \geq E_5 \geq E_2 \geq E_4$ |

Table 2: The explored WSMs between $E$ and $E_j$ ($j = 1, 2, 3, 4, 5$).

| SMs   | (E, E₁) | (E, E₂) | (E, E₃) | (E, E₄) | (E, E₅) | Ranking |
|-------|---------|---------|---------|---------|---------|---------|
| $S_{cw}^{1}$ (E, E₁) | 0.4247  | 0.2933  | 0.4557  | 0.3128  | 0.2826  | $E_3 \geq E_1 \geq E_4 \geq E_3 \geq E_2$ |
| $S_{cw}^{2}$ (E, E₂) | 0.8829  | 0.6705  | 0.9219  | 0.672   | 0.7072  | $E_3 \geq E_1 \geq E_4 \geq E_3 \geq E_2$ |
| $S_{cw}^{3}$ (E, E₃) | 0.911   | 0.816   | 0.9551  | 0.7603  | 0.7964  | $E_3 \geq E_1 \geq E_4 \geq E_3 \geq E_2$ |
| $S_{cw}^{4}$ (E, E₄) | 0.628   | 0.4011  | 0.6737  | 0.4207  | 0.4359  | $E_3 \geq E_1 \geq E_4 \geq E_3 \geq E_2$ |
| $S_{cw}^{5}$ (E, E₅) | 0.6664  | 0.5434  | 0.7431  | 0.4933  | 0.7386  | $E_3 \geq E_1 \geq E_4 \geq E_3 \geq E_2$ |

Figure 1: The graphical representation of interpreted SMs.

Figure 2: The graphical representation of interpreted WSMs.

As stated by the above-computed calculations described in Table 3, we simply note that the degree of similarity between $P$ and $D_j$ is the greatest one as an extract by five SMs. This specifies that all five SMs express that the victim $P$ has typhoid based on the principle of the maximum similarity degree. Ranking of the explored cosine and cotangent SMs between $P$ and $D_j$ ($j = 1, \ldots, 5$) is also introduced in Table 3. Next, the graphical representation of the interpreted SMs between $P$ and $D_j$ ($j = 1, \ldots, 5$) is indicated in Figure 3.

The weight of elements has great significance to suppose in real decision-making problems. If we suppose the weight of elements $x_{\mathcal{X}}$ ($\mathcal{X} = 1, 2, 3, 4, 5$) to be $w_{\mathcal{X}} = (0.15, 0.1, 0.25, 0.2, 0.3)$, respectively, then the interpreted WSMs (weighted cosine SMs and weighted cotangent SMs) have been used to determine the similarity from $P$ to $D_j$ ($j = 1, \ldots, 5$) and calculations are introduced in Table 4.

As stated by the above-computed calculations described in Table 3, we simply note that the degree of similarity between $P$ and $D_j$ is the greatest one as an extract by WSMs except $S_{cw}^{1}$. This specifies that WSMs $S_{cw}^{2}$, $S_{cw}^{3}$, $S_{cw}^{4}$, and $S_{cw}^{5}$ show that the victim $P$ has typhoid based on the principle of the maximum similarity degree. We also note that degree of similarity between $P$ and $D_j$ is the highest one as an extract by WSM $S_{cw}^{1}$. This specifies that the WSM $S_{cw}^{1}$ shows that the victim $P$ has coronavirus. Ranking of the explored weighted cosine SMs, weighted SMs based on cosine function, and...
weighted cotangent SMs between $P$ and $D_i$ ($i = 1, \ldots, 5$) is also introduced in Table 4. Next, we have the graphical representation of the interpreted WSMs between $P$ and $D_i$ ($i = 1, \ldots, 5$) in Figure 4.

6. Comparison

In this section of the paper, we expressed the effectiveness and advantages of the interpreted SMs by comparing with some already defined SMs.

Example 4. Without any hesitancy, the quantity of construction usually relies on the standard of building materials. Accordingly, building material scrutiny is the assumption of good engineering standards. The selection of material must be strictly controlled. Scrutiny authorizes the builders to correctly recognize qualified materials and upgrade the standard of the project. Suppose pattern recognition problem about the categorization of building materials. Let five specified building materials $E_i$ ($i = 1, 2, 3, 4, 5$) which are represented in the form of HFSs as follows:

$$
\begin{align*}
E_1 &= \left\{ (x_1, [0.6, 0.5]), (x_2, [0.7]), (x_3, [0.6, 0.4]) \right\}, \\
E_2 &= \left\{ (x_1, [0.1]), (x_2, [0.5, 0.1]), (x_3, [0.2, 0.7]) \right\}, \\
E_3 &= \left\{ (x_1, [1, 0.6, 0.5]), (x_2, [0.5]), (x_3, [0.8, 0.7]) \right\}, \\
E_4 &= \left\{ (x_1, [0.3, 1]), (x_2, [0.4, 0.2]), (x_3, [0.2]) \right\}, \\
E_5 &= \left\{ (x_1, [0.4, 0.2]), (x_2, [0.4, 0.4]), (x_3, [0.2, 0.3]) \right\}.
\end{align*}
$$

Next, let an unspecified building material $E$ in the form of CHFS which needs to be recognized be

$$
E = \left\{ (x_1, [0.9, 0.7, 1]), (x_2, [0.4]), (x_3, [0.8, 0.5]) \right\}.
$$
Table 5: The comparison between interpreted and some already defined SMs of Example 4.

| Method         | Score value                                                                 | Ranking                        |
|----------------|------------------------------------------------------------------------------|--------------------------------|
| Xu and Xia [29] | $S(E, E_t) = 0.7667, S(E, E_e) = 0.5967$                                     | $E_3 \geq E_4 \geq E_2 \geq E_1$ |
|                | $S(E, E_t) = 0.7733, S(E, E_e) = 0.3467$                                     |                                |
|                | $S(E, E_t) = 0.7067, S(E, E_e) = 0.8367, S(E, E_e) = 0.5283$                  |                                |
| Zeng et al. [30]| $S(E, E_t) = 0.845, S(E, E_e) = 0.415, S(E, E_e) = 0.6872$                   | $E_3 \geq E_4 \geq E_2 \geq E_1$ |
|                | $\text{Cos}_\text{HFS}(E, E_t) = 0.963, \text{Cos}_\text{HFS}(E, E_e) = 0.841, \text{Cos}_\text{HFS}(E, E_e) = 0.925$ |
| Jun [31]       | $\text{Cos}_\text{HFS}(E, E_t) = 0.911, \text{Cos}_\text{HFS}(E, E_e) = 0.883, \text{Cos}_\text{HFS}(E, E_e) = 0.925$ |
|                | $S(E, E_t) = 0.5006, S(E, E_e) = 0.3334, S(E, E_e) = 0.5234$                  | $E_3 \geq E_1 \geq E_4 \geq E_2$ |
| Proposed SM    | $S(E, E_t) = 0.453, S(E, E_e) = 0.3147, S(E, E_e) = 0.8914$                   | $E_3 \geq E_1 \geq E_4 \geq E_2$ |
|                | $S(E, E_t) = 0.7565, S(E, E_e) = 0.9719, S(E, E_e) = 0.7922$                  |                                |
|                | $S(E, E_t) = 0.8036, S(E, E_e) = 0.9721, S(E, E_e) = 0.9342$                  |                                |
| Proposed SM    | $S(E, E_t) = 0.9929, S(E, E_e) = 0.9456, S(E, E_e) = 0.9486$                  | $E_3 \geq E_1 \geq E_4 \geq E_2$ |
|                | $S(E, E_t) = 0.6513, S(E, E_e) = 0.5454$                                     |                                |
| Proposed SM    | $S(E, E_t) = 0.7994, S(E, E_e) = 0.5352, S(E, E_e) = 0.5395$                  | $E_3 \geq E_1 \geq E_2 \geq E_3 \geq E_4$ |
|                | $S(E, E_t) = 0.8105, S(E, E_e) = 0.7457, S(E, E_e) = 0.8953$                  |                                |
| Proposed SM    | $S(E, E_t) = 0.7418, S(E, E_e) = 0.8671$                                     | $E_3 \geq E_1 \geq E_2 \geq E_3 \geq E_4$ |

We convert the HFSs in the CHFs by taking $1 = e^0$ as follows:

\[
\begin{align*}
E_1 &= \left\{ \left( x_1, \left\{ 0.6 e^{2\pi (0.0)}, 0.5 e^{2\pi (0.0)} \right\} \right), \left( x_2, \left\{ 0.7 e^{2\pi (0.0)} \right\} \right), \left( x_3, \left\{ 0.6 e^{2\pi (0.0)}, 0.4 e^{2\pi (0.0)} \right\} \right) \right\}, \\
E_2 &= \left\{ \left( x_1, \left\{ 0.1 e^{2\pi (0.0)} \right\} \right), \left( x_2, \left\{ 0.5 e^{2\pi (0.0)}, 0.1 e^{2\pi (0.0)} \right\} \right), \left( x_3, \left\{ 0.2 e^{2\pi (0.0)}, 0.7 e^{2\pi (0.0)} \right\} \right) \right\}, \\
E_3 &= \left\{ \left( x_1, \left\{ 1 e^{2\pi (0.0)}, 0.6 e^{2\pi (0.0)}, 0.5 e^{2\pi (0.0)} \right\} \right), \left( x_2, \left\{ 0.5 e^{2\pi (0.0)} \right\} \right), \left( x_3, \left\{ 0.8 e^{2\pi (0.0)}, 0.7 e^{2\pi (0.0)} \right\} \right) \right\}, \\
E_4 &= \left\{ \left( x_1, \left\{ 0.3 e^{2\pi (0.0)}, 1 e^{2\pi (0.0)} \right\} \right), \left( x_2, \left\{ 0.4 e^{2\pi (0.0)}, 0.2 e^{2\pi (0.0)} \right\} \right), \left( x_3, \left\{ 0.2 e^{2\pi (0.0)} \right\} \right) \right\}, \\
E_5 &= \left\{ \left( x_1, \left\{ 0.8 e^{2\pi (0.0)}, 0.6 e^{2\pi (0.0)}, 0.8 e^{2\pi (0.0)} \right\} \right), \left( x_2, \left\{ 0.5 e^{2\pi (0.0)}, 0.6 e^{2\pi (0.0)}, 0.8 e^{2\pi (0.0)} \right\} \right), \left( x_3, \left\{ 0.2 e^{2\pi (0.0)} \right\} \right) \right\}.
\end{align*}
\]
For Example 4, we need to find that the unknown building material $E$ belongs to which of the specified building material $E_i$ ($i = 1, 2, 3, 4, 5$). In Example 4, the data are in the shape of HFSs. We found similarity between $E$ and $E_i$ ($i = 1, . . . , 5$) through some already defined SMs for HFSs, as shown in Table 5. As $1 = \delta^0$, then the data given in Example 4 are transformed into CHFSs. Then, through interpreted SMs, we found the similarity between $E$ and $E_i$ ($i = 1, 2, 3, 4, 5$) which is also given in Table 5. Our interpreted SMs showed that unspecified building material $E$ belongs to the specified building material $E_3$ because the similarity between $E$ and $E_3$ is the greatest one. Ranking of the interpreted and already defined SMs is also described in Table 5. Next, we have the graphical representation of the

![Comparison of interpreted SMs with some existing SMs](image.png)

**Figure 5:** The graphical representation of interpreted SMs with some existing SMs for Example 4.

### Table 6: The comparison between interpreted and some already defined SMs of Example 2.

| Method          | Score value | Ranking   |
|-----------------|-------------|-----------|
| Xu and Xia [29] | Unsuccessful | Unsuccessful |
| Zeng et al. [30] | Unsuccessful | Unsuccessful |
| Jun [31]        | Unsuccessful | Unsuccessful |

$S_1^x (E, E_i) = 0.4733, S_2^x (E, E_i) = 0.2829$

- $S_1^x (E, E_1) = 0.517$
- $S_1^x (E, E_2) = 0.3429$
- $S_1^x (E, E_3) = 0.2844$
- $S_1^x (E, E_4) = 0.8674$
- $S_1^x (E, E_5) = 0.6372$
- $S_2^x (E, E_1) = 0.9321$
- $S_2^x (E, E_2) = 0.6047$
- $S_2^x (E, E_3) = 0.7278$
- $S_2^x (E, E_4) = 0.9022$
- $S_2^x (E, E_5) = 0.796$
- $S_3^x (E, E_1) = 0.959$
- $S_3^x (E, E_2) = 0.7405$
- $S_3^x (E, E_3) = 0.8054$
- $S_3^x (E, E_4) = 0.6056$
- $S_3^x (E, E_5) = 0.3777$
- $S_4^x (E, E_1) = 0.6988$
- $S_4^x (E, E_2) = 0.3762$
- $S_4^x (E, E_3) = 0.4605$
- $S_4^x (E, E_4) = 0.6528$
- $S_4^x (E, E_5) = 0.5264$
- $S_5^x (E, E_1) = 0.7563$
- $S_5^x (E, E_2) = 0.4746$
- $S_5^x (E, E_3) = 0.7479$

The data $E$ can be presented as

$$E = \left\{ \begin{array}{c}
(x_1, \{0.9 e^{2\pi i (0.0)}, 0.7 e^{2\pi i (0.0)}, 1 e^{2\pi i (0.0)}\}), \\
(x_2, \{0.4 e^{2\pi i (0.5)}, 0.8 e^{2\pi i (0.0)}, 0.5 e^{2\pi i (0.0)}\}), \\
(x_3, \{0.5 e^{2\pi i (0.0)}, 0.3 e^{2\pi i (0.0)}, 0.2 e^{2\pi i (0.0)}\}), \\
(x_4, \{0.9 e^{2\pi i (0.0)}, 0.4 e^{2\pi i (0.0)}\}), \\
(x_5, \{0.5 e^{2\pi i (0.0)}, 0.3 e^{2\pi i (0.0)}, 0.2 e^{2\pi i (0.0)}\}) \end{array} \right\}. \quad (62)$$
comparison of the proposed and already defined SMs which is represented in Figure 5.

Now, we discuss the comparison between interpreted and already defined SMs for Example 2. In Example 2, the data are in the shape of CHFSs. We know that no SM exists in the literature to solve this kind of data. The existing SMs are ineffective to find the similarity between $E$ and $E_j$ ($j = 1, \ldots, 5$) as demonstrated in Table 6. From Table 6, we observe that the data given in Example 2 are solvable by the interpreted SMs. The interpreted SMs get the similarity between $E$ and $E_j$ ($j = 1, \ldots, 5$), as demonstrated in Table 6. Our interpreted SMs showed that unspecified building material $E$ belongs to the specified building material $E_j$ because the similarity between $E$ and $E_j$ is the greatest one. Ranking of the explored SMs is also introduced in Table 6. Next, we have the graphical representation of the comparison of proposed and already defined SMs which is represented in Figure 6.

From the above discussion, our explored SMs can represent extra fuzzy information and put it broadly in circumstances in real-life problems. Based on CHFS, we explored the SMs; our SMs are more satisfactory for real-life problems, and the existing SMs and our SMs are more general than the existing SMs.

7. Conclusion

The CHFS is one of the enlargements of the CFS in which the possibility of the enrollment work is stretched out from the subset of the genuine number to the unit disc which is interpreted. In this article, we explored another type of similarity measure (SM) which relies on the cosine and cotangent functions. At that stage, we use our introduced SMs and weighted SMs (based on the cosine and cotangent functions) between CHFSs to manage pattern recognition and medical diagnosis problems including design acknowledgment and plan choice. Finally, two numerical models are given to represent the logic and effectiveness of the likeness measures for design acknowledgment and conspire choice. The comparison between explored measure with some existing measures and their graphical representations are also discussed in detail.

Consequently, the measures defined in this manuscript can be utilized in a larger range of applications. In future research, we will extend this work to suppose the two facts: (1) similarity measures and aggregation operators [34–42]; (2) methods [43].

Data Availability

The data used in this article are artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This article was supported by “Algebra and Applications Research Unit”.

References

[1] L. A. Zadeh, “Fuzzy sets,” Information and Control, vol. 8, no. 3, pp. 338–353, 1965.
[2] W. Siler and H. Ying, “Fuzzy control theory: the linear case,” Fuzzy Sets and Systems, vol. 33, no. 3, pp. 275–290, 1989.
[3] J. Yen and R. Langari, Fuzzy Logic: Intelligence, Control, and Information, Prentice-Hall, Upper Saddle River, NJ, USA, 1999.
[4] D. Dubois and H. Prade, “Fuzzy sets in approximate reasoning, Part 1: inference with possibility distributions,” Fuzzy Sets and Systems, vol. 40, no. 1, pp. 143–202, 1991.
[5] H. Bustince, F. Herrera, and J. Montero, “Fuzzy sets and their extensions: representation, aggregation and models: intelligent systems from decision making to data mining,” in Web Intelligence and Computer Vision Springer, Berlin, Germany, 2007.
[6] D. Li, W. Zeng, and J. Li, “New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making,” Engineering Applications of Artificial Intelligence, vol. 40, pp. 11-16, 2015.
[7] X. Zhang and Z. Xu, “Novel distance and similarity measures on hesitant fuzzy sets with applications to clustering analysis,” Journal of Intelligent & Fuzzy Systems, vol. 28, no. 5, pp. 2279–2296, 2015.
[8] S.-M. Chen, “A weighted fuzzy reasoning algorithm for medical diagnosis,” Decision Support Systems, vol. 11, no. 1, pp. 37–43, 1994.
[9] W. Pedrycz, Fuzzy Control and Fuzzy Systems, Research Studies Press Ltd, Boston, MA, USA, 2nd edition, 1993.
[10] W. Pedrycz, “Fuzzy sets in pattern recognition: methodology and methods,” Pattern Recognition, vol. 23, no. 1-2, pp. 121–146, 1990.
[11] N. Rangel-Valdez, C. Gómez-Santillán, J. C. Hernández-Marin, M. L. Morales-Rodriguez, L. Cruz-Reyes, and H. J. Fraire-Huacuja, “Parallel designs for metaheuristics that solve portfolio selection problems using fuzzy outranking relations,” International Journal of Fuzzy Systems, vol. 22, no. 8, pp. 2747–2759, 2020.
[12] T. Mahmood, “A novel approach towards bipolar soft sets and their applications,” Journal of Mathematics, vol. 2020, Article ID 4699808, 2020.
[13] D. Ramot, R. Milo, M. Friedman, and A. Kandel, “Complex fuzzy sets,” IEEE Transactions on Fuzzy Systems, vol. 10, no. 2, pp. 171–186, 2002.
[14] D. E. Tamir, L. Jin, and A. Kandel, “A new interpretation of complex membership grade,” *International Journal of Intelligent Systems*, vol. 26, no. 4, pp. 285–312, 2011.

[15] G. Zhang, T. S. Dillon, K.-Y. Cai, J. Ma, and J. Lu, “Operation properties and,” *International Journal of Approximate Reasoning*, vol. 50, no. 8, pp. 1227–1249, 2009.

[16] V. Torra, “Hesitant fuzzy sets,” *International Journal of Intelligent Systems*, vol. 25, no. 6, pp. 529–539, 2010.

[17] V. Torra and Y. Narukawa, “On hesitant fuzzy sets and decision,” in Proceedings of the 2009 IEEE International Conference on Fuzzy Systems, pp. 1378–1382, IEEE, Jeju Island, South Korea, August 2009.

[18] R. M. Rodríguez, L. Martínez, and F. Herrera, “Hesitant fuzzy linguistic term sets for decision making,” *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 4, pp. 109–119, 2011.

[19] B. Farhadinia, “Distance and similarity measures for higher order hesitant fuzzy sets,” *Knowledge-Based Systems*, vol. 55, pp. 43–48, 2014.

[20] M. Xia and Z. Xu, “Hesitant fuzzy information aggregation in decision making,” *International Journal of Approximate Reasoning*, vol. 52, no. 3, pp. 395–407, 2011.

[21] G. Wei, X. Zhao, H. Wang, and R. Lin, “Hesitant fuzzy choquet integral aggregation operators and their applications to multiple attribute decision making,” *International Information Institute (Tokyo)*, vol. 15, no. 2, p. 441, 2012.

[22] Z. Zhang, “Hesitant fuzzy power aggregation operators and their application to multiple attribute group decision making,” *Information Sciences*, vol. 234, pp. 150–181, 2013.

[23] Z. Xu and M. Xia, “On distance and correlation measures of hesitant fuzzy information,” *International Journal of Intelligent Systems*, vol. 26, no. 3, pp. 410–425, 2011.

[24] B. Zhu, Z. Xu, and M. Xia, “Hesitant fuzzy geometric Bonferroni means,” *Information Sciences*, vol. 205, pp. 72–85, 2012.

[25] F. Herrera, L. Martínez-López, V. Torra, and Z. Xu, “Hesitant fuzzy sets: an emerging tool in decision making,” *International Journal of Intelligent Systems*, vol. 29, no. 6, pp. 493–494, 2014.

[26] R. M. Rodríguez, L. Martínez, F. Herrera, and V. Torra, “A review of hesitant fuzzy sets: Quantitative and qualitative extensions,” in *Fuzzy Logic in its 50th Year*, pp. 109–128, Springer, Cham, Switzerland, 2016.

[27] C. C. Li, R. M. Rodríguez, L. Martínez, Y. Dong, and F. Herrera, “Consistency of hesitant fuzzy linguistic preference relations: An interval consistency index,” *Information Sciences*, vol. 432, pp. 347–361, 2018.

[28] G. Muhiddin, A. M. Alanazi, A. Mahboob, and D. Al-Kadi, “Generalized hesitant fuzzy ideals in semigroups,” *Journal of Mathematics*, vol. 2020, Article ID 8856287, 12 pages, 2020.

[29] Z. Xu and M. Xia, “Distance and similarity measures for hesitant fuzzy sets,” *Information Sciences*, vol. 181, no. 11, pp. 2128–2138, 2011.

[30] W. Zeng, D. Li, and Q. Yin, “Distance and similarity measures between hesitant fuzzy sets and their application in pattern recognition,” *Pattern Recognition Letters*, vol. 84, pp. 267–271, 2016.

[31] Y. Jun, “Vector similarity measures of hesitant fuzzy sets and their multiple attribute decision making,” *Economic Computation & Economic Cybernetics Studies & Research*, vol. 48, no. 4, 2014.

[32] V. Torra and Y. Narukawa, *Modeling Decisions: Information Fusion and Aggregation Operators*, Springer Science & Business Media, Berlin, Germany, 2007.

[33] A. Bhattacharya, “On a measure of divergence between two multinomial populations,” *The Indian Journal of Statistics*, vol. 7, no. 4, pp. 401–406, 1946.

[34] K. T. Atanassov, “Intuitionistic fuzzy sets,” in *Intuitionistic Fuzzy Sets*, pp. 1–137, Springer, Berlin, Germany, 1999.

[35] G. Wei, “Some similarity measures for picture fuzzy sets and their applications,” *Iranian Journal of Fuzzy Systems*, vol. 15, no. 1, pp. 77–89, 2018.

[36] F. Xiao and W. Ding, “Divergence measure of Pythagorean fuzzy sets and its application in medical diagnosis,” *Applied Soft Computing*, vol. 79, pp. 254–267, 2019.

[37] P. Liu, T. Mahmood, and Z. Ali, “Complex q-rung orthopair fuzzy aggregation operators and their applications in multi-attribute group decision making,” *Information*, vol. 11, no. 1, p. 5, 2020.

[38] P. Liu, Z. Ali, and T. Mahmood, “A method to multi-attribute group decision-making problem with complex q-rung orthopair linguistic information based on heronian mean operators,” *International Journal of Computational Intelligence Systems*, vol. 12, no. 2, pp. 1465–1496, 2019.

[39] K. Ullah, T. Mahmood, Z. Ali, and N. Jan, “On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition,” *Complex & Intelligent Systems*, vol. 6, no. 1, pp. 15–27, 2019.

[40] K. Ullah, H. Garg, T. Mahmood, N. Jan, and Z. Ali, “Correlation coefficients for T-spherical fuzzy sets and their applications in clustering and multi-attribute decision making,” *Soft Computing*, vol. 24, no. 3, pp. 1647–1659, 2019.

[41] N. Jan, L. Zedam, T. Mahmood, K. Ullah, and Z. Ali, “Multiple attribute decision making method under linguistic cubic information,” *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 1, pp. 253–269, 2019.

[42] K. Ullah, T. Mahmood, N. Jan, and Z. Ali, “A note on geometric aggregation operators in spherical fuzzy environment and its application in multi-attribute decision making,” *Journal of Engineering and Applied Sciences*, vol. 37, no. 2, 2018.

[43] X. Peng and W. Li, “Algorithms for hesitant fuzzy soft decision making based on revised aggregation operators, WDBA and CODAS,” *Journal of Intelligent & Fuzzy Systems*, vol. 36, no. 6, pp. 6307–6323, 2019.