On intertwining operators and finite automorphism groups of vertex operator algebras

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Abstract

Let $V$ be a simple vertex operator algebra and $G$ a finite automorphism group. We give a construction of intertwining operators for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules by using intertwining operators for $V$.

We also determine some fusion rules for a vertex operator algebra as an application.

1 Introduction

Let $V$ be a simple vertex operator algebra (cf. [1], [8] and [9]), and $G$ a finite automorphism group. It is an important problem to understand the module category for the vertex operator algebra $V^G$ of $G$-invariants. In [2], this question was asked and several ideas are proposed.

For a simple vertex operator algebra $V$, it is shown in [6] that every irreducible $V$-module is a completely reducible $V^G$-module as a natural consequence of a duality theorem of Schur-Weyl type. In this paper we give a construction of intertwining operators for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules by using intertwining operators for $V$.

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Let’s state our results more explicitly. Firstly, we need to recall the results of Dong–Yamskulna[6]. For an irreducible $V$-module $(L, Y_L)$ and $a \in G$ we define a new irreducible $V$-module $(L \circ a, Y_{L \circ a})$. Here $L \circ a$ is equal to $L$ and $Y_{L \circ a}(u, z) = Y_L(au, z)$. Let $\mathcal{S}$ be a finite set of inequivalent irreducible $V$-modules which is closed under the right action of $G$. In [3] they define a finite dimensional semisimple associative algebra $\mathcal{A}_a(G, \mathcal{S})$ over $\mathbb{C}$ and show a duality theorem of Schur-Weyl type for the actions of $V^G$ and $\mathcal{A}_a(G, \mathcal{S})$ on the direct sum of $V$-modules in $\mathcal{S}$ which is denoted by $\mathcal{L}$. That is, as a $\mathcal{A}_a(G, \mathcal{S}) \otimes V^G$-module, $\mathcal{L} = \bigoplus_{(j, \lambda) \in \Gamma} W_{(j, \lambda)} \otimes M_{(j, \lambda)}$, where $\{W_{(j, \lambda)}\}_{(j, \lambda) \in \Gamma}$ is the set of all inequivalent irreducible $\mathcal{A}_a(G, \mathcal{S})$-modules and $M_{(j, \lambda)}$ is the multiplicity space of $W_{(j, \lambda)}$ in $\mathcal{L}$. Each $M_{(j, \lambda)}$ is a nonzero irreducible $V^G$-module and the different multiplicity spaces are inequivalent $V^G$-modules.

In this paper we consider intertwining operators for irreducible $V^G$-modules constructed from irreducible $V$-modules in this way.

For each $i = 1, 2, 3$ let $\mathcal{S}_i$ be a finite set of inequivalent irreducible $V$-modules which is closed under the action of $G$ and let $\mathcal{L}_i$ be the direct sum of $V$-modules in $\mathcal{S}_i$. We have the decomposition $\mathcal{L}_i = \bigoplus_{(j_i, \lambda_i) \in \Gamma_i} W_{(j_i, \lambda_i)}^i \otimes M_{(j_i, \lambda_i)}^i$ as a $\mathcal{A}_{a_i}(G, \mathcal{S}_i) \otimes V^G$-module. Set $\mathcal{I} = \bigoplus_{(L_1, L_2, L_3) \in \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3} \mathcal{I}_V(L_1, L_2, L_3)^G \otimes L_1 \otimes L_2$, where $\mathcal{I}_V(L_1, L_2, L_3)^G \otimes L_1 \otimes L_2$ is the set of all intertwining operators of type $(L_1, L_2)$. $\mathcal{I}$ has a natural $\mathcal{A}_{a_3}(G, \mathcal{S}_3)$-module structure. For each $i = 1, 2, 3$, fix $(j_i, \lambda_i) \in \Gamma_i$ and nonzero $v^{10} \in M_{(j_1, \lambda_1)}^1$, $v^{20} \in M_{(j_2, \lambda_2)}^1$. Set $\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)} = \text{Span}_C \{ f \otimes (w^1 \otimes v^{10}) \otimes (w^2 \otimes v^{20}) \in \mathcal{I} \mid w^1 \in W_{(j_1, \lambda_1)}^1, w^2 \in W_{(j_2, \lambda_2)}^2 \}$. $\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}$ is a $\mathcal{A}_{a_3}(G, \mathcal{S}_3)$-submodule of $\mathcal{I}$. We will construct an injective linear map from the multiplicity space of $W_{(j_3, \lambda_3)}^3$ in $\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}$ to the set of all intertwining operators for $V^G$ of type $(M_{(j_3, \lambda_3)}^3, M_{(j_1, \lambda_1)}^1, M_{(j_2, \lambda_2)}^2)$. Therefore, the fusion rule for $V^G$ of type $(M_{(j_3, \lambda_3)}^3, M_{(j_1, \lambda_1)}^1, M_{(j_2, \lambda_2)}^2)$ is greater than or equal to the multiplicity of $W_{(j_3, \lambda_3)}^3$ in $\mathcal{I}_{(j_1, \lambda_1), (j_2, \lambda_2)}$.

This paper is organized as follows. In Sect.2 we first recall a construction of irreducible $V^G$-modules from irreducible $V$-modules in [3]. We also recall the definitions of intertwining operators and fusion rules. In Sect.3 we give a construction of intertwining operators for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules. In Sect.4 we apply the main result to determine some fusion rules for a vertex operator algebra $W$ studied in [3]. In Appendix, we give some singular vectors in $W$-modules used in Sect. 4.
2 Preliminary

We assume that the reader is familiar with the basic knowledge on vertex operator algebras as presented in [1], [8], and [9].

The following notation will be in force throughout the paper: \( V = (V, Y, 1, \omega) \) is a simple vertex operator algebra and \( G \) a finite automorphism group of \( V \). For any \( V \)-module \( L \), we always arrange the grading on \( L = \bigoplus_{n=0}^{\infty} L(n) \) so that \( L(0) \neq 0 \) if \( L \neq 0 \) by using a grading shift.

2.1 Irreducible \( V^G \)-modules constructed from irreducible \( V \)-modules

In this subsection we review the results of Dong–Yamskulna [6]. For a simple vertex operator algebra \( V \), they showed that every irreducible \( V \)-module is a completely reducible \( V^G \)-module as a natural consequence of a duality theorem of Schur-Weyl type.

Let \((L, Y_L)\) be an irreducible \( V \)-module and \( a \in G \). We define a new irreducible \( V \)-module \((L \circ a, Y_L \circ a)\). Here \( L \circ a \) is equal to \( L \) and \( Y_L \circ a(u, z) = Y_L(au, z) \). Note that \( L \circ a \) is also an irreducible \( V \)-module. A set \( S \) of irreducible \( V \)-modules is called stable if for any \( L \in S \) and \( a \in G \) there exists \( M \in S \) such that \( L \circ a \simeq M \).

Now we take a finite \( G \)-stable set \( S \) consisting of inequivalent irreducible \( V \)-modules. Let \( L \in S \) and \( a \in G \). Then there exists \( M \in S \) such that \( M \simeq L \circ a^{-1} \). That is, there is a linear map \( \phi(a, L) : L \to M \) satisfying the condition: \( \phi(a, L)Y_L(v, z) = Y_M(av, z)\phi(a, L) \). By simplicity of \( L \), there exists \( \alpha_L(b, a) \in \mathbb{C}^\times \) such that \( \phi(b, M)\phi(a, L) = \alpha_L(b, a)\phi(ba, L) \). Moreover, for \( a, b, c \in G \) we have

\[
\alpha_L(c, ba)\alpha_L(b, a) \alpha_L(c, b) = \alpha_M(c, b)\alpha_L(cb, a).
\]

For \( L \in S \) and \( a \in G \), we denote \( M \in S \) such that \( L \circ a \simeq M \) by \( L \cdot a \).

Define a vector space \( \mathbb{C}S = \bigoplus_{L \in S} \mathbb{C}e(L) \) with a basis \( e(L) \) for \( L \in S \). The space \( \mathbb{C}S \) is an associative algebra under the product \( e(L)e(M) = \delta_{L,M}e(L) \). Let \( U(\mathbb{C}S) = \{ \sum_{L \in S} \lambda_L e(L) \mid \lambda_L \in \mathbb{C}^\times \} \) be the set of unit elements on \( \mathbb{C}S \). \( U(\mathbb{C}S) \) is a multiplicative right \( G \)-set by the action \( (\sum_{L \in S} \lambda_L e(L)) \cdot a = \sum_{L \in S} \lambda_L e(L \cdot a) \) for \( a \in G \). Set \( \alpha(a, b) = \sum_{L \in S} \alpha_L(a, b)e(L) \). Then \( (\alpha(a, b) \cdot c)\alpha(ab, c) = \alpha(a, bc)\alpha(b, c) \) hold for all \( a, b, c \in G \). So \( \alpha : G \times G \to U(\mathbb{C}S) \) is a 2-cocycle.
Define the vector space $A_\alpha(G,S) = \mathbb{C}[G] \otimes \mathbb{C}S$ with a basis $a \otimes e(L)$ for $a \in G$ and $L \in S$ and a multiplication on it:

$$a \otimes e(L) \cdot b \otimes e(M) = \alpha_M(a,b)ab \otimes e(L \cdot b)e(M).$$

Then $A_\alpha(G,S)$ is an associative algebra with the identity element $\sum_{L \in S} 1 \otimes e(L)$.

We define an action of $A_\alpha(G,S)$ on $L = \bigoplus_{L \in S} L$ as follows: For $L, M \in S$, $w \in M$ and $a \in G$ we set

$$a \otimes e(L) \cdot w = \delta_{L,M} \phi(a, L)w.$$

Note that the actions of $A_\alpha(G,S)$ and $V^G$ on $L$ commute with each other.

For each $L \in S$ set $G_L = \{a \in G \mid L \circ a \simeq L\}$ as $V$-modules. Let $O_L$ be the orbit of $L$ under the action of $G$ and let $G = \sqcup_{j=1}^k G_L g_j$ be a right coset decomposition with $g_1 = 1$. Then $O_L = \{L \cdot g_j \mid j = 1,\ldots,k\}$ and $G_L g_j = g_j^{-1} G_L g_j$. We define several subspaces of $A_\alpha(G,S)$ by:

\begin{align*}
S(L) &= \text{Span}_\mathbb{C}\{a \otimes e(L) \mid a \in G_L\}, \\
D(L) &= \text{Span}_\mathbb{C}\{a \otimes e(L) \mid a \in G\} \quad \text{and} \\
D(O_L) &= \text{Span}_\mathbb{C}\{a \otimes e(L \cdot g_j) \mid j = 1,\ldots,k, a \in G\}.
\end{align*}

Decompose $S$ into a disjoint union of orbits $S = \sqcup_{j \in J} O_j$. Let $L^{(j)}$ be a representative element of $O_j$. Then $O_j = \{L^{(j)} \cdot a \mid a \in G\}$ and $A_\alpha(G,S) = \bigoplus_{j \in J} D(O^{(j)}_L)$. We recall the following properties of $A_\alpha(G,S)$.

**Lemma 1.** (\cite{4}, Lemma 3.4) Let $L \in S$ and $G = \sqcup_{j=1}^k G_L g_j$. Then

1. $S(L)$ is a subalgebra of $A_\alpha(G,S)$ isomorphic to $\mathbb{C}^{\alpha_L}[G_L]$ where $\mathbb{C}^{\alpha_L}[G_L]$ is the twisted group algebra with 2-cocycle $\alpha_L$.

2. $D(O_L) = \oplus_{j=1}^k D(L \cdot g_j)$ is a direct sum of left ideals.

3. Each $D(O_L)$ is a two sided ideal of $A_\alpha(G,S)$ and $A_\alpha(G,S) = \oplus_{j \in J} D(O^{(j)}_L)$. Moreover, $D(O_L)$ has the identity element $\sum_{M \in O_L} 1 \otimes e(M)$.

**Lemma 2.** (\cite{4}, Theorem 3.6)

1. $D(O_L)$ is semisimple for all $L \in S$ and the simple $D(O_L)$-modules are precisely equal to $\text{Ind}_{S(L)}^{D(L)} U = D(L) \otimes_S U$ where $U$ ranges over the simple $\mathbb{C}^{\alpha_L}[G_L]$-modules.
(2) $\mathcal{A}_\alpha(G, S)$ is semisimple and simple $\mathcal{A}_\alpha(G, S)$-modules are precisely $\text{Ind}_{S(L(j))}^{D(L(j))} U$ where $U$ ranges over the simple $C^{\alpha_L}[G_L]$-modules and $j \in J$.

For $L \in S$ let $\Lambda_{G_L, \alpha_L}$ be the set of all irreducible characters $\lambda$ of $C^{\alpha_L}[G_L]$. We denote the corresponding simple module by $U(L, \lambda)$. Note that $L$ is a semisimple $C^{\alpha_L}[G_L]$-module. Let $L^\lambda$ be the sum of simple $C^{\alpha_L}[G_L]$-submodules of $L$ isomorphic to $U(L, \lambda)$. Then $L = \bigoplus_{\lambda \in \Lambda_{G_L, \alpha_L}} L^\lambda$. Moreover $L^\lambda = U(L, \lambda) \otimes L_\lambda$ where $L_\lambda = \text{Hom}_{C^{\alpha_L}[G_L]}(U(L, \lambda), L)$ is the multiplicity space of $U(L, \lambda)$ in $L$. We can realize $L_\lambda$ as a subspace of $L$ in the following way: Let $w \in U(L, \lambda)$ be a fixed nonzero vector. Then we can identify $\text{Hom}_{C^{\alpha_L}[G_L]}(U(L, \lambda), L)$ with the subspace $\{ f(w) \mid f \in \text{Hom}_{C^{\alpha_L}[G_L]}(U(L, \lambda), L) \}$ of $L^\lambda$. Note that the actions of $C^{\alpha_L}[G_L]$ and $V^G$ on $L$ commute with each other. So $L^\lambda$ and $L_\lambda$ are ordinary $V^G$-modules. Furthermore, $L^\lambda$ and $L_\lambda$ are ordinary $V^G$-modules.

For convenience, we set $G_j = G_{L(j)}, \Lambda_j = \Lambda_{L(j), \alpha_{L(j)}}$ and $U(j, \lambda) = U(L(j), \lambda)$ for $j \in J$ and $\lambda \in \Lambda_j$. We denote by $\Gamma$ the set $\{(j, \lambda) \mid j \in J, \lambda \in \Lambda_j\}$. We have a decomposition

$$L^{(j)} = \bigoplus_{\lambda \in \Lambda_j} U_{(j, \lambda)} \otimes M_{(j, \lambda)}$$

as a $C^{\alpha_L}[G_j] \otimes V^{G_j}$-module. We also have

$$\mathcal{L} = \bigoplus_{(j, \lambda) \in \Gamma} \text{Ind}_{S(L^{(j)})}^{D(L^{(j)})} U_{(j, \lambda)} \otimes M_{(j, \lambda)}$$

as a $\mathcal{A}_\alpha(G, S) \otimes V^G$-module.

For $(j, \lambda) \in \Gamma$ we set $W_{(j, \lambda)} = \text{Ind}_{S(L^{(j)})}^{D(L^{(j)})} U_{(j, \lambda)}$. Then $W_{(j, \lambda)}$ forms a complete list of simple $\mathcal{A}_\alpha(G, S)$-modules by Lemma 2.

A duality theorem of Schur–Weyl type holds.

Theorem 1. (6 Theorem 6.14) As a $\mathcal{A}_\alpha(G, S) \otimes V^G$-module,

$$\mathcal{L} = \bigoplus_{(j, \lambda) \in \Gamma} W_{(j, \lambda)} \otimes M_{(j, \lambda)}.$$
(1) Each \( M_{(j,\lambda)} \) is a nonzero irreducible \( V^G \)-module.

(2) \( M_{(j_1,\lambda_1)} \) and \( M_{(j_2,\lambda_2)} \) are isomorphic \( V^G \)-modules if and only if \( (j_1, \lambda_1) = (j_2, \lambda_2) \).

2.2 Intertwining operators and fusion rules

We recall the definition of intertwining operators for \( V \)-modules which is introduced in [8].

**Definition 1.** Let \( L^i \) \((i = 1, 2, 3)\) be \( V \)-modules. An intertwining operator of type \( (L^3_{L^1 L^2}) \) is a linear map

\[
I(\cdot, z) : L^1 \rightarrow \text{Hom}_\mathbb{C}(L^2, L^3)\{z\},
\]

\[
v \mapsto I(v, z) = \sum_{\gamma \in \mathbb{C}} v_{\gamma^{-1}}^{-1},
\]

which satisfies the following conditions: Let \( u \in V, v \in L^1, \) and \( w \in L^2 \).

1. For any fixed \( \gamma \in \mathbb{C}, v_{\gamma+n}w = 0 \) for \( n \in \mathbb{Z} \) sufficiently large.

2. \( I(L(-1)v, z) = \frac{d}{dz}I(v, z) \).

3.

\[
z^{-1}_0\delta(z_1 - z_2)Y_{L^3}(u, z_1)I(v, z_2)w - z^{-1}_0\delta(-z_2 + z_1)I(v, z_2)Y_{L^2}(u, z_1)w
\]

\[
= z^{-1}_2\delta(z_1 - z_0)I(Y_{L^1}(u, z_0)v, z_2)w.
\]

We denote by \( I_V(L^3_{L^1 L^2}) \) the set of all intertwining operators of type \( (L^3_{L^1 L^2}) \). The dimension of \( I_V(L^3_{L^1 L^2}) \) is called the fusion rule of type \( (L^3_{L^1 L^2}) \).

For a \( V \)-module \( L = \oplus_{n=0}^{\infty} L(n) \), it is shown in [8, Theorem 5.2.1.] that the graded vector space \( L' = \oplus_{n=0}^{\infty} L(n)^* \) carries the structure of a \( V \)-module, where \( L(n)^* = \text{Hom}_\mathbb{C}(L(n), \mathbb{C}) \). \( L' \) is called the contragredient module of \( L \).

The fusion rules have some symmetries.

**Lemma 3.** ([8], Proposition 5.4.7 and 5.5.2) Let \( L^i \) \((i = 1, 2, 3)\) be \( V \)-modules. Then

\[
\dim_\mathbb{C} I_V\left( \frac{L^3}{L^1 L^2} \right) = \dim_\mathbb{C} I_V\left( \frac{L^3}{L^2 L^1} \right) = \dim_\mathbb{C} I_V\left( \frac{(L^3)'}{(L^1 L^2)'} \right).
\]
Let $L^1$ and $L^2$ be irreducible $V$-modules. We use a notation $L^1 \times L^2 = \sum_{L^3} \dim_{\mathbb{C}} I_V(L^1 \otimes L^2) L^3$ to represent the fusion rules, where $L^3$ ranges over the irreducible $V$-modules. Note that $L^1 \times L^2 = L^2 \times L^1$ by Lemma 3.

3 Intertwining operators for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules

In this section we give a construction of intertwining operators for irreducible $V^G$-modules which occur as submodules of irreducible $V$-modules by using intertwining operators for $V$.

Let $S_i$ $(i = 1, 2, 3)$ be finite $G$-stable sets consisting of inequivalent irreducible $V$-modules. Set $L_i = \bigoplus_{L \in S_i} L$. For $L^i \in S_i$ and $a \in G$, $\phi_i(a,L^i) : L^i \to L^i \cdot a^{-1}$ denote the fixed $V$-module isomorphisms. For $L^i \in S_i$ and $a, b \in G$, $\alpha_i, \lambda_i \in \mathbb{C}^*$ denote nonzero complex numbers such that $\phi_i(b,L^i) \cdot \alpha_i, \lambda_i$ denote the fixed $1$-dimensional irreducible $G$-submodule $L^i$, and set $\Gamma_i = \{(j_i, \lambda_i) \mid j_i \in J_i, \lambda_i \in \Lambda_i\}$.

For $f \in I_V(L^3 \otimes L^1 \otimes L^2)$ and $a \in G$, we define $af \in I_V(L^3 \otimes L^1 \otimes L^2)$ as follows: For $v \in L^1$ we set

$$af(v,z) = \phi_3(a,L^3) f(\phi_1(a,L^1) v, z) \phi_2(a,L^2)^{-1}.$$ 

Set

$$\mathcal{I} = \bigoplus_{(L^1, L^2, L^3) \in S_1 \times S_2 \times S_3} I_V(L^3 \otimes L^1 \otimes L^2).$$

We define an action of $A_{\alpha_3}(G,S)$ on $\mathcal{I}$ as follows: Let $L^i \in S_i$ $(i = 1, 2, 3)$. For $a \otimes e(M) \in A_{\alpha_3}(G,S)$, $v \in L^1$, $w \in L^2$, and $f \in I_V(L^3 \otimes L^1 \otimes L^2)$, we set

$$(a \otimes e(M)) \cdot (f \otimes v \otimes w) = \delta_{M,L^3} \cdot af \otimes \phi_1(a,L^1) v \otimes \phi_2(a,L^2) w$$

$$\in I_V(L^3 \otimes L^1 \cdot a^{-1} \otimes L^2 \cdot a^{-1}).$$

We define a map $\Psi : \mathcal{I} \to \mathcal{L}_3 \{z\}$ by

$$\Psi(f \otimes v^1 \otimes v^2) = f(v^1, z)v^2$$
for $v^1 \in L^1, v^2 \in L^2$, and $f \in I_V(L^3_{L^1 L^2}),$ where $L^i \in S_i$ ($i = 1, 2, 3$). Note that $\Psi$ is a $A_{\alpha \beta \gamma}(G, S_3)$-module homomorphism.

**Lemma 4.** The map $\Psi : \mathcal{I} \to L_3 \{z\}$ is injective.

**Proof.** We use the same method that was used in the proof of Lemma 3.1. of [5]. Assume false. Then there is a nonzero $X \in \text{Ker} \Psi$. Since $L_3 = \oplus L \in S_3$, we may assume $X = \sum f_{ij} \otimes v^{1i} \otimes v^{2j}$, where $v^{1i} \in L^{1i}(i = 1, \ldots, l_1)$ are linearly independent homogeneous vectors in $L_1$, $v^{2j} \in L^{2j}(j = 1, \ldots, l_2)$ are linearly independent homogeneous vectors in $L_2$, $f_{ij} \in I_V(L^{1i}_{L_1 L^3}), L^i \in S_i$, $L^{2j} \in S_2$, and $L^3 \in S_3$. We may also assume $f_{11} \otimes v^{11} \otimes v^{21}$ is nonzero. Since $\sum f_{ij}(v^{1i}, z)v^{2j} = 0$, for all $u \in V$ we have

$$\sum_{i,j} Y_{L_3}(u, z_1) f_{ij}(v^{1i}, z)v^{2j} = 0.$$  

Using the associativity of intertwining operators [4, Proposition 11.5],

$$\sum_{i,j} f_{ij}(Y_{L_1}(u, z_1)v^{1i}, z)v^{2j} = 0. \quad (1)$$

We denote $Y_{L_1}(u, z_1) = \sum_{n \in \mathbb{Z}} u^{L_1}_{n} z_1^{n-1}$. Fix $N \in \mathbb{Z}$ such that $v^{1i} \in \oplus_{n=0}^{N} L^{1i}(n)$ for all $i$. Since $S_1$ consists of inequivalent irreducible $V$-modules, the linear map $\sigma_N : V \to \oplus_{L \in S_1} \oplus_{m=0}^{N} \text{End}_C L(m)$ defined by $\sigma_N(u) = u^{L_1}_{n} z_1^{n-1}$ for homogeneous $u \in V$ is an epimorphism by Lemma 6.13 of [5]. So there exists $u^1 \in V$ such that $\sigma_N(u^1)v^{1i} = \delta_{1,i} v^{11}$. From formula (1), we have

$$0 = \sum_{i,j} f_{ij}(\sigma_N(u^1)v^{1i}, z)v^{2j} = \sum_{j} f_{1j}(v^{11}, z)v^{2j}. \quad (2)$$

Therefore, for all $u \in V$ we have

$$\sum_{j} Y_{L_3}(u, z_1) f_{1j}(v^{11}, z)v^{2j} = 0.$$  

Using the commutativity of intertwining operators [4, Proposition 11.4],

$$\sum_{j} f_{1j}(v^{11}, z) Y_{L_2}(u, z_1)v^{2j} = 0.$$
Since $S_2$ consists of inequivalent irreducible $V$-modules, we have
\[ f^{11}(v^{11}, z)v^{21} = 0 \]
for the same reason to obtain formula (2). So $f^{11} \otimes v^{11} \otimes v^{21} \in \text{Ker} \Psi$. Since $L^{11}$ and $L^{21}$ are irreducible $V$-modules and $f^{11} \otimes v^{11} \otimes v^{21}$ is nonzero, this contradicts Proposition 11.9 of [4].

We have the decomposition of each $L_i$ as a $A_{\alpha_i}(G, S_i)$-module in Theorem 11.

\[ L_i = \bigoplus_{(j, \lambda_i) \in \Gamma_i} W^i_{(j, \lambda_i)} \otimes M^i_{(j, \lambda_i)}. \]

For $i = 1, 2$ let $(j_1, \lambda_i) \in \Gamma_i$ and let $v^{i0} \in M^i_{(j_1, \lambda_i)}$. Set
\[ I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20}) = \text{Span}_C \{ f \otimes (w^1 \otimes v^{10}) \otimes (w^2 \otimes v^{20}) \in I \mid w^1 \in W^1_{(j_1, \lambda_1)}, w^2 \in W^2_{(j_2, \lambda_2)} \}. \]

$I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20})$ is a $A_{\alpha_3}(G, S_3)$-submodule of $I$. For any nonzero $v^{10}, v^1 \in M^1_{(j_1, \lambda_1)}$ and nonzero $v^{20}, v^2 \in M^2_{(j_2, \lambda_2)}$, $I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20})$ and $I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^1, v^2)$ are isomorphic $A_{\alpha_3}(G, S_3)$-modules.

**Theorem 2.** Fix a nonzero $v^{10} \in M^1_{(j_1, \lambda_1)}$ and a nonzero $v^{20} \in M^2_{(j_2, \lambda_2)}$.

For any $((j_1, \lambda_1), (j_2, \lambda_2), (j_3, \lambda_3)) \in \Gamma_1 \times \Gamma_2 \times \Gamma_3$, there exists an injective linear map from $\text{Hom}_{A_{\alpha_3}(G, S_3)}(W^{3}_{(j_3, \lambda_3)}, I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20}))$ to $I_{V_3}(M^3_{(j_3, \lambda_3)} M^1_{(j_1, \lambda_1)} M^2_{(j_2, \lambda_2)})$.

In particular,
\[ \dim C I_{V_3}(M^3_{(j_3, \lambda_3)} M^1_{(j_1, \lambda_1)} M^2_{(j_2, \lambda_2)}) \geq \dim C \text{Hom}_{A_{\alpha_3}(G, S_3)}(W^{3}_{(j_3, \lambda_3)}, I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20})). \]

**Proof.** For convenience, we set $W^i = W^i_{(j_i, \lambda_i)}, M^i = M^i_{(j_i, \lambda_i)}$ ($i = 1, 2, 3$), $A_3 = A_{\alpha_3}(G, S_3)$, and $I_{12}(v^{10}, v^{20}) = I_{(j_1, \lambda_1), (j_2, \lambda_2)}(v^{10}, v^{20})$. Fix a nonzero $w^{30} \in W^3$. Let $F \in \text{Hom}_{A_3}(W^3, I_{12}(v^{10}, v^{20}))$. We shall define $\Phi(F) \in I_{V_3}(M^3_{12})$.

For $v^1 \in M^1, v^2 \in M^2$, we set
\[ H(v^1, v^2) \in \text{Hom}_{A_3}(W^3, I_{12}(v^1, v^2)) \]
as follows: For $w^3 \in W^3$, let $F(w^3) = \sum f^i \otimes (w^{1i} \otimes v^{10}) \otimes (w^{2i} \otimes v^{20}) \in I_{12}(v^{10}, v^{20})$, where $w^{1i} \otimes v^{10} \in L^{1i}, w^{2i} \otimes v^{20} \in L^{2i}, f^i \in I_{V}(L^{1i}, L^{2i})$ and
\( L^j \in S_j \ (j = 1, 2, 3) \). Note that \( w^j \otimes v^j \in L^j \) by the definition of \( M^3 \) for \( j = 1, 2, 3, \) and \( 3 \). We define

\[
H(v^1, v^2)(w^3) = \sum_i f^i \otimes (w^i \otimes v^1) \otimes (w^3 \otimes v^2)
\]

It is clear that \( H(v^1, v^2) \in \text{Hom}_{A_3}(W^3, \mathcal{I}_1 W^3(v^1, v^2)) \). Since the map \( \Psi : \mathcal{I} \rightarrow \mathcal{L}_3(z) \) is a \( A_3 \)-module homomorphism, the map \( w^3 \mapsto \Psi(H(v^1, v^2)(w^3)) \) is a \( A_3 \)-module homomorphism from \( W^3 \) to \( \mathcal{L}_3(z) \). So \( \Psi(H(v^1, v^2)(w^3)) \) is a \( A_3 \)-submodule of \( \mathcal{L}_3(z) \). Let \( w^{3,1}, w^{3,2}, \ldots, w^{3,\dim W^3} \) be a basis of \( W^3 \). Since \( W^3 \) is an irreducible \( A_3 \)-module, there exists \( a \in A_3 \) such that \( aw^{3,i} = \delta_1, w^{3,1} \). Let \( \Psi(H(v^1, v^2)(w^{3,1})) = \sum_i w^{3,i} \otimes p^i \), where \( p_i \in M^3(z) \).

Then

\[
\Psi(H(v^1, v^2)(w^{3,1})) = a \sum_i w^{3,i} \otimes p^i = a \sum_i (aw^{3,i}) \otimes p^i = w^{3,1} \otimes p^1.
\]

So \( \Psi(H(v^1, v^2)(w^3)) \in (w^3 \otimes M^3, z) \) for all \( w^3 \in W^3 \). We hence have an unique \( \Phi(F)(v^1, z)v^2 \in M^3(z) \) such that

\[
w^{30} \otimes \Phi(F)(v^1, z)v^2 = \Psi(H(v^1, v^2)(w^{30})). \tag{3}
\]

Since \( f^i \) are intertwining operators, we have \( \Phi(F) \in I_{V^0}(M^3_1, M^3_2) \) from formula (3).

We will show that \( \Phi \) is injective. Suppose \( \Phi(F) = 0 \). Then

\[
0 = w^{30} \otimes \Phi(F)(v^{10}, z)v^{20} = \Psi(F(w^{30})).
\]

Since \( \Psi \) is injective by Lemma 3, \( F(w^{30}) = 0 \). Since \( W^3 \) is an irreducible \( A_3 \)-module and \( F \in \text{Hom}_{A_3}(W^3, \mathcal{I}_1 W^3(v^{10}, v^{20})) \), \( F = 0 \).

Let \( CG \) be the group algebra of \( G \) and \( \text{Irr}G \) the set of all irreducible characters of \( G \). We set \( S_i = \{ V \} \ (i = 1, 2, 3) \) in Theorem 2. Then \( A_{\alpha_i}(G, S_i) = CG \) and \( \Gamma_i = \text{Irr}G \). Note that \( \dim CG \ni \dim (V^1) \Gamma = 1 \) since \( V \) is simple. In this case we have the following result:

**Corollary 3.** Let \( \chi_i \in \text{Irr}G \ (i = 1, 2, 3) \). Then

\[
\dim CG \left( \begin{array}{c} V_{\chi_3}^1 \\ V_{\chi_1} \end{array} \right) \geq \dim CG \left( W_{\chi_3}, W_{\chi_1} \otimes CG W_{\chi_2} \right).
\]
In [2, Section 3], it is conjectured that if $V$ is rational then for all $\chi_1, \chi_2 \in \text{Irr} G$,

$$V_{\chi_1} \times V_{\chi_2} = \sum_{\chi_3 \in \text{Irr} G} \dim C \text{Hom}_C(G(W_{\chi_3}, W_{\chi_1} \otimes C W_{\chi_2})) V_{\chi_3}.$$ 

The conjecture implies that if $V$ is rational then the representation algebra of the finite group $G$ is always realized as a subalgebra of the fusion algebra of $V^G$.

4 An application

In [3] we studied a vertex operator algebra $\mathcal{W}$ which is a realization of an algebra denoted by $[\mathbb{Z}_3^{(5)}]$ in [7]. $\mathcal{W}$ is a fixed point subalgebra of a vertex operator algebra $\mathcal{M}_0^k$. It is expected that the $\mathbb{Z}_3$ symmetry of $\mathcal{W}$ affords $3B$ elements of the Monster simple group [11]. For $\mathcal{M}_0^k$, the irreducible modules are classified and the fusion rules are determined in [12]. Let $L^i$ ($i = 1, 2, 3$) be irreducible $\mathcal{W}$-modules such that $L^1$ and $L^2$ occur as submodules of irreducible $\mathcal{M}_0^k$-modules. In this section we determine the fusion rule of type $(L^3 L^1 L^2)$ using Theorem [2].

4.1 Subalgebra $\mathcal{M}_0^k$ of $V_{\sqrt{2}A_2}$

In this subsection we review some properties of $\mathcal{M}_0^k$ in [12]. Let $A_2$ be the ordinary root lattice of type $A_2$ and $V_{\sqrt{2}A_2}$ the lattice vertex operator algebra associated with $\sqrt{2}A_2$. Let $\alpha_1, \alpha_2$ be the simple roots of type $A_2$ and set $\alpha_0 = -\alpha_1 - \alpha_2$.

For basic definitions concerning lattice vertex operator algebras we refer to [1] and [2]. Our notation for the lattice vertex operator algebra is standard [9]. In particular, $\mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} \sqrt{2}A_2$ is an abelian Lie algebra, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ is the corresponding affine Lie algebra, $M(1) = \mathbb{C}[\alpha(n); \alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\hat{\mathfrak{h}}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}, n > 0$, and $c = 1$. As a vector space $V_{\sqrt{2}A_2} = M(1) \otimes \mathbb{C}[\sqrt{2}A_2]$ and for each $v \in V_{\sqrt{2}A_2}$, a vertex operator $Y(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \in \text{End}(V_{\sqrt{2}A_2})[[z, z^{-1}]]$ is defined. The vector $1 = 1 \otimes 1$ is called the vacuum vector. We use the symbol $e^\alpha, \alpha \in \sqrt{2}A_2$ to denote a basis of $\mathbb{C}[\sqrt{2}A_2]$. 

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There exists an isometry \( \tau \) of \( \sqrt{2}A_2 \) such that \( \tau(\sqrt{2}\alpha_1) = \sqrt{2}\alpha_2 \) and \( \tau(\sqrt{2}\alpha_2) = \sqrt{2}\alpha_0 \). The isometry \( \tau \) lifts naturally to an automorphism of \( V_{\sqrt{2}A_2} \):

\[
\alpha^1(-n_1) \cdots \alpha^k(-n_k)e^\beta \mapsto (\tau\alpha^1)(-n_1) \cdots (\tau\alpha^k)(-n_k)e^{\tau\beta}.
\]

By abuse of notation, we denote it by \( \tau \). Let \( G \) be the cyclic group generated by \( \tau \). Set

\[
\omega^3 \equiv \frac{1}{15}(\alpha_1(-1)^2 + \alpha_2(-1)^2 + \alpha_0(-1)^2) + \frac{1}{10}(e^{\sqrt{2}\alpha_1} + e^{-\sqrt{2}\alpha_1} + e^{\sqrt{2}\alpha_2} + e^{-\sqrt{2}\alpha_2} + e^{\sqrt{2}\alpha_0} + e^{-\sqrt{2}\alpha_0})
\]

and \( M^0_k = \{ v \in V_{\sqrt{2}A_2} \mid (\omega^3)_1 v = 0 \} \). Since \( \tau\omega^3 = \omega^3 \), \( M^0_k \) is invariant under the action of \( \tau \). \( \tau \) is an automorphism group of \( M^0_k \) of order 3 by [3, Theorem 2.1]. Set

\[
L_0 = L, \quad L_a = \frac{\sqrt{2}\alpha_2}{2} + L, \quad L_b = \frac{\sqrt{2}\alpha_0}{2} + L, \quad L_c = \frac{\sqrt{2}\alpha_1}{2} + L
\]

and

\[
M^i_k = \{ v \in V_{L_i} \mid (\omega^3)_1 v = 0 \},
W^i_k = \{ v \in V_{L_i} \mid (\omega^3)_1 v = \frac{2}{5} v \}, \quad \text{for } i = 0, a, b, c.
\]

It is shown in [12] that \( \{M^i_k, W^i_k \mid i = 0, a, b, c\} \) is the set of all irreducible \( M^0_k \)-modules and the fusion rules are determined.

### 4.2 Subalgebra \( \mathcal{W} \) in \( M^\tau \)

We denote by \( \mathcal{W} \) the subalgebra \( (M^0_k)^\tau \) of fixed points of \( \tau \) in \( M^0_k \). We recall some properties of \( \mathcal{W} \) in [3]. \( \mathcal{W} \) is generated by the Virasoro element \( \omega \) and an element \( J \) of weight 3. Let \( Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2} \) and \( Y(J, z) = \sum_{n \in \mathbb{Z}} J(n)z^{-n-2} \). 

\[ \sum_{n \in \mathbb{Z}} J(n) z^{n-3} \]. They satisfy the following commutation relations:

\[
[L(m), L(n)] = (m-n)L(m+n) + \frac{m^3-m}{12} \cdot \frac{6}{5} \cdot \delta_{m+n,0},
\]

\[
[L(m), J(n)] = (2m-n)J(m+n),
\]

\[
[J(m), J(n)] = (m-n)(22(m+n+2)(m+n+3) + 35m+2(n+2)L(m+n) - 120(m-n) \left( \sum_{k \leq -2} L(k)L(m+n-k) + \sum_{k \geq -1} L(m+n-k)L(k) \right) - \frac{7}{10}m(m^2-1)(m^2-4)\delta_{m+n,0}.
\]

(4)

\( \mathcal{W} \) has exactly 20 irreducible modules. 8 irreducible \( \mathcal{W} \)-modules occur as submodules of irreducible \( M^0_k \)-modules. We introduce those 8 \( \mathcal{W} \)-modules. \( \tau \) acts on the irreducible \( M^0_k \)-modules as follows:

\[
M^0_k \circ \tau \simeq M^0_k, \quad W^0_k \circ \tau \simeq W^0_k,
\]

\[
M^a_k \circ \tau \simeq M^a_k, \quad M^b_k \circ \tau \simeq M^b_k, \quad M^0_k \circ \tau \simeq M^0_k,
\]

\[
W^a_k \circ \tau \simeq W^a_k, \quad W^b_k \circ \tau \simeq W^b_k, \quad \text{and} \quad W^k \circ \tau \simeq W^a_k.
\]

So \( \{M^0_k\}, \{W^0_k\}, \{M^a_k, M^b_k, M^c_k\}, \) and \( \{W^a_k, W^b_k, W^c_k\} \) are \( G \)-stable sets. The automorphism \( \tau \) of \( V_{2A_2} \) fixes \( \omega^3 \) and so \( W^0_k \) is invariant under \( \tau \). Hence we can take \( \tau \) as \( \phi(\tau, W^0_k) \) in Section 2.1. For these \( G \)-stable sets, we can take the 2-cocycles \( \phi \) in Section 2.1 to be trivial. Let \( \xi = e^{2\pi \sqrt{-1}/3} \). We set \( M^{(i)}_k = \{ v \in M^0_k \mid \tau v = \xi^i v \} \) and \( W^{(i)}_k = \{ v \in W^0_k \mid \tau v = \xi^i v \} \) for \( i \in \mathbb{Z} \).

Note that \( \mathcal{W} = M^{(0)}_k \). By Theorem, we have \( M^{(i)}_k, W^{(i)}_k, (i = 0, 1, 2) M^a_k, \) and \( W^a_k \) are inequivalent irreducible \( \mathcal{W} \)-modules. Moreover, \( M^a_k \simeq M^b_k \simeq M^c_k \) and \( W^a_k \simeq W^b_k \simeq W^c_k \) as \( \mathcal{W} \)-modules. The contragredient modules of these \( \mathcal{W} \)-modules are

\[
(M^{(i)}_k)' \simeq M^{(0)}_k, \quad (W^{(i)}_k)' \simeq W^{(0)}_k, \quad (i = 0, 1, 2),
\]

\[
(M^a_k)' \simeq M^b_k \text{ and } (W^a_k)' \simeq W^b_k.
\]

All the other irreducible \( \mathcal{W} \)-modules occur as submodules of irreducible \( \tau^i \)-twisted \( M^0_k \)-modules for \( i = 1, 2 \).

### 4.3 An upper bound for the fusion rule

We review some notations and formulas for the Zhu algebra \( A(V) \) of an arbitrary vertex operator algebra \( V \) in [14] and the \( A(V) \)-bimodule \( A(L) \).

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of an arbitrary $V$-module $L$ in [10] and [13]. For $u, v \in V$ with $u$ being homogeneous, define two bilinear operations

$$u \ast v = \text{Res}_z (Y(u, z)v \frac{(1 + z)^{	ext{wt}u}}{z}),$$
$$u \circ v = \text{Res}_z (Y(u, z)v \frac{(1 + z)^{	ext{wt}u}}{z^2}).$$

We extend $\ast$ and $\circ$ for arbitrary $u, v \in V$ by linearity. Let $O(V)$ be the subspace of $V$ spanned by all $u \circ v$ for $u \in V, v \in L$. Set $A(V) = V/O(V)$. By [11, Theorem 2.1.1.], $O(V)$ is a two-sided ideal with respect to the operation $\ast$ and $(A(V), \ast)$ is an associative algebra with identity $1 + O(V)$. For every $V$-module $N, N(0)$ is a left $A(V)$-module.

Let $L$ be a $V$-module. For $u \in V, v \in L$ with $u$ being homogeneous, define three bilinear operations

$$u \ast v = \text{Res}_z (Y(u, z)v \frac{(1 + z)^{	ext{wt}u}}{z}),$$
$$v \ast u = \text{Res}_z (Y(u, z)v \frac{(1 + z)^{	ext{wt}u-1}}{z}),$$
$$u \circ v = \text{Res}_z (Y(u, z)v \frac{(1 + z)^{	ext{wt}u}}{z^2}).$$

We extend $\ast$ and $\circ$ for arbitrary $u \in V, v \in L$ by linearity. Let $O(L)$ be the subspace of $L$ spanned by all $u \circ v$ for $u \in V, v \in L$. By [10, Theorem 1.5.1.], $O(L)$ is a two-sided ideal with respect to the operation $\ast$. Thus it induces an operation on $A(L) = L/O(L)$. Denote by $[v]$ the image of $v \in L$ in $A(L)$. $A(L)$ is a $A(V)$-bimodule under the operation $\ast$. Using $A(L)$, we have an upper bound for every fusion rule.

**Lemma 5.** ([13] Proposition 2.10.) Let $L^i = \bigoplus_{n=0}^{\infty} L^i(n)$ ($i = 1, 2, 3$) be irreducible $V$-modules. Then

$$\dim_{\mathbb{C}} I_V \left( \begin{array}{c} L^3 \\ L^1, L^2 \end{array} \right) \leq \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}} \left( L^3(0)^* \otimes A(V) A(L^1) \otimes A(V) L^2(0) \right).$$

**4.4 The fusion rules for irreducible $\mathcal{W}$-modules which occur as submodules of irreducible $M^0_k$-modules**

Let $L^i$ ($i = 1, 2, 3$) be irreducible $\mathcal{W}$-modules such that $L^1$ and $L^2$ occur as submodules of irreducible $M^0_k$-modules. In this subsection we determine the
fusion rule of type $\left(\frac{L^3_{12}}{L^3_{12}}\right)$. As a first step, we give a lower bound for every fusion rule by using Theorem 2.

Lemma 6. (1) The fusion rule of following types is greater than or equal to 1: Let $i, j \in \{0, 1, 2\}$.

\[
(M^0_{k} M^0_{k}), \quad (W^0_{k} M^0_{k}), \quad (M^0_{k} M^0_{k}), \quad (M^0_{k} M^0_{k}), \quad (M^0_{k} W^0_{k}), \quad (M^0_{k} W^0_{k}).
\]

(2) The fusion rule of following types is greater than or equal to 2:

\[
(M^a_{k} M^a_{k}), \quad (W^a_{k} M^a_{k}), \quad (W^a_{k} W^a_{k}).
\]

Proof. We use notations in Section 3. The fusion rules for $M^0_{k}$ are obtained in [12].

(1) We consider the case that $S_1 = S_2 = S_3 = \{W^0_{k}\}$. We have $\dim C I_M^0(W^0_{k}) = 1$. Let $\mathbb{C}x_i (i = 0, 1, 2)$ be the one dimensional $G$-modules such that $\tau \cdot x_i = \xi^i x_i$. Fix nonzero $v^j \in \text{Hom}_G(\mathbb{C}x_i, W^0_{k})$. Then

\[
W^0_{k} = \bigoplus_{i=0}^{2} x_i \otimes \text{Hom}_G(\mathbb{C}x_i, W^0_{k}),
\]

\[
\mathcal{I} = I^0_{M} \begin{pmatrix} W^0_{k} & W^0_{k} \\ W^0_{k} & W^0_{k} \end{pmatrix} \otimes \mathbb{C} W^0_{k} \otimes \mathbb{C} W^0_{k}, \quad \text{and}
\]

\[
\mathcal{I}_{W^0_{k}(i), W^0_{k}(j)}(v^i, v^j) = I^0_{M} \begin{pmatrix} W^0_{k} & W^0_{k} \\ W^0_{k} & W^0_{k} \end{pmatrix} \otimes (x_i \otimes v^i) \otimes (x_j \otimes v^j).
\]

Let $f \in I^0_{M} \begin{pmatrix} W^0_{k} & W^0_{k} \\ W^0_{k} & W^0_{k} \end{pmatrix}$ be a nonzero intertwining operator. Then $\tau f = f$ by the construction of intertwining operators in [12]. So

\[
(\tau \otimes e(W^0_{k})) \cdot f \otimes (x_i \otimes v^i) \otimes (x_j \otimes v^j) = \xi^{i+j} f \otimes (x_i \otimes v^i) \otimes (x_j \otimes v^j)
\]

and $\dim C I_W(W^0_{k}(i), W^0_{k}(j)) \geq 1$. We can compute the other cases in the same way.
(2) We consider the case that $S_1 = S_2 = S_3 = \{M^a_k, M^b_k, M^c_k\}$. For $i_1, i_2, i_3 \in \{a, b, c\}$, we have

$$\dim_{\mathbb{C}} I_{M^0_k} \left( \frac{M^{i_3}_{k}}{M^{i_1}_{k} \otimes M^{i_2}_{k}} \right) = \begin{cases} 1, & \text{if } \{i_1, i_2, i_3\} = \{a, b, c\}, \\ 0, & \text{otherwise}. \end{cases}$$

We define an action of $\tau$ on $\{a, b, c\}$ by $\tau(a) = b, \tau(b) = c$ and $\tau(c) = a$. It is possible to take $\phi(\tau, M^a_k)$ ($i = a, b, c$) such that

$$\phi(\tau, M^a_k)\phi(\tau, M^b_k)\phi(\tau, M^c_k) = \text{id}_{M^0_k}.$$

Fix a nonzero $v^a \in M^a_k$ and set $v^b = \phi(\tau, M^a_k)v^a \in M^b_k$, $v^c = \phi(\tau, M^b_k)v^b \in M^c_k$. Fix nonzero $f_{a,b,c} \in I_{M^0_k}(\frac{M^a_k}{M^b_k \otimes M^c_k}), f_{b,a,c} \in I_{M^0_k}(\frac{M^b_k}{M^c_k \otimes M^a_k}), f_{c,b,a} \in I_{M^0_k}(\frac{M^c_k}{M^a_k \otimes M^b_k})$ and set

$$f_{b,c,a} = \tau f_{a,b,c} \in I_{M^0_k}(\frac{M^a_k}{M^b_k \otimes M^c_k}),$$
$$f_{c,a,b} = \tau f_{b,c,a} \in I_{M^0_k}(\frac{M^b_k}{M^c_k \otimes M^a_k}),$$
$$f_{c,b,a} = \tau f_{b,a,c} \in I_{M^0_k}(\frac{M^c_k}{M^a_k \otimes M^b_k}).$$

We have

$$\mathcal{I} = \bigoplus_{i_1, i_2, i_3 \in \{a, b, c\}} I_{M^0_k} \left( \frac{M^{i_3}_{k}}{M^{i_1}_{k} \otimes M^{i_2}_{k}} \right) \otimes_{\mathbb{C}} M^{i_1}_{k} \otimes_{\mathbb{C}} M^{i_2}_{k}$$

and

$$\mathcal{I}_{M^a_k, M^b_k} = \bigoplus_{i=0,1,2} \mathbb{C} f_{\tau^i(a), \tau^i(b), \tau^i(c)} \otimes v^{\tau^i(a)} \otimes v^{\tau^i(b)} \oplus \bigoplus_{i=0,1,2} \mathbb{C} f_{\tau^i(b), \tau^i(a), \tau^i(c)} \otimes v^{\tau^i(b)} \otimes v^{\tau^i(a)}.$$
We will show that every fusion rule meets the lower bound obtained in Lemma 6. We use Lemma 5. For an irreducible $\mathcal{W}$-module $N = \bigoplus_{n=0}^\infty N(n)$, we fix a nonzero vector in $N(0)$ and denote it by $w_N$. By the same argument as in [3, Lemma 5.2], $N$ is spanned by the vectors of the form

$$L(-m_1) \cdots L(-m_p) J(-n_1) \cdots J(-n_q) w_N$$

with $m_1 \geq \cdots \geq m_p \geq 1, n_1 \geq \cdots \geq n_q \geq 1, p = 0, 1, \ldots, q = 0, 1, \ldots$. By the same argument as in [3, Section 5.4], we have for $n \leq -1$ and $u \in N$,

$$L(n)u = (-1)^{n-1}[\omega] * [u] + (-1)^{n-1}n[u] * [\omega] + (-1)^n \text{wt}_u[u],$$

$$J(n)u = (-1)^n (n[J(-1)u] + (n+1)[J0u] - (n+1)[J] * [u] - \frac{n(n+1)}{2}[u] * [J])$$

in $A(N)$. Using formula (3) and commutation relations (4) repeatedly, it is shown that $A(N)$ is generated by $\{J(-1)^i w_N\}_{i=0}^\infty$ as an $A(\mathcal{W})$-bimodule.

A singular vector $w$ of weight $h$ for $N$ is by definition a vector which satisfies $L(0)w = hw$ and $L(n)w = J(n)w = 0$ for $n \geq 1$. By commutation relations (4), it is easy to show that $w$ is a singular vector of weight $h$ if and only if $L(0)w = hw$ and $L(1)w = L(2)w = J(1)w = 0$. If $h - \text{wt}N(0) > 0$, then the submodule of $N$ generated by $w$ does not contain $N(0)$. We hence have $w = 0$.

**Theorem 4.** The fusion rule $L^1 \times L^2$ is given by the following list, where $\{L^1, L^2\}$ is an arbitrary pair of irreducible $\mathcal{W}$-modules which occur as sub-
modules of irreducible $M^0_k$-modules: Let $i, j \in \{0, 1, 2\}$.

\[
M^0_k \times M^0_k = M^0_{0+i+j}, \\
M^0_k \times W^0_k = W^0_{0+i+j}, \\
W^0_k \times W^0_k = M^0_{0+i+j} + W^0_{0+i+j}, \\
M^0_k \times M^a_k = M^a_k, \\
M^0_k \times W^a_k = W^a_k, \\
W^0_k \times M^a_k = W^a_k, \\
W^0_k \times W^a_k = M^a_k + W^a_k, \\
M^a_k \times M^a_k = \sum_{i=0}^{2} M^0_{0+i} + 2M^a_k, \\
M^a_k \times W^a_k = \sum_{i=0}^{2} W^0_{0+i} + 2W^a_k, \\
W^a_k \times W^a_k = \sum_{i=0}^{2} M^0_{0+i} + \sum_{i=0}^{2} W^0_{0+i} + 2M^a_k + 2W^a_k.
\]

Proof. For an irreducible $W$-module $N$, $h_N$ denotes the eigenvalue for $L(0)$ on $N(0)$ and $k_N$ denotes the eigenvalue for $J(0)$ on $N(0)$. For all irreducible $W$-modules those eigenvalues are computed in [3]. For irreducible $W$-modules $L^i$ ($i = 1, 2, 3$), set

\[
F(L^1, L^2, L^3) = \dim_{\mathbb{C}} \left( L^3(0)^* \otimes_{A(W)} A(L^1) \otimes_{A(W)} L^2(0) \right).
\]

Note that $F(L^1, L^2, L^3)$ is lower than or equal to the number of generators of $A(L^1)$ as a $A(W)$-bimodule since the dimension of top level of every irreducible $W$-module is 1.

The simplicity of $W$ implies that

\[
\dim_{\mathbb{C}} I_W \left( \begin{array}{c}
L^3 \\
W L^2
\end{array} \right) = \begin{cases}
1, & \text{if } L^2 \simeq L^3, \\
0, & \text{otherwise}
\end{cases}
\]

for all irreducible $W$-modules $L^2$ and $L^3$. We consider the case $L^1 \neq W$. We will show that $F(L^1, L^2, L^3)$ is lower than or equal to the lower bound for $\dim_{\mathbb{C}} I_W \left( \begin{array}{c}
L^3 \\
W L^2
\end{array} \right)$ given in Lemma [6]. Then, by Lemma [3] and Lemma [5] we get the desired results. We use a computer algebra system Risa/Asir to find
singular vectors used in the following argument. The explicit forms of those singular vectors are given in Appendix.

(1) We consider the case $L^1 = W_{k}^{0(1)}$. Since $(5\sqrt{-3}L(-1) + J(-1)) w_{W_{k}^{0(1)}}$ is a singular vector and $A(W_{k}^{0(1)})$ is generated by $\{J(-1)^{i} w_{W_{k}^{0(1)}}\}_{i=0}^{\infty}$ as a $A(W)$-bimodule, $A(W_{k}^{0(1)})$ is generated by $w_{W_{k}^{0(1)}}$ as a $A(W)$-bimodule. So $F(W_{k}^{0(1)}, L^2, L^3) \leq 1$. Set

$$w_1 = \left(-30\sqrt{-3}L(-1)J(-1) + 39\sqrt{-3}J(-2) + 5J(-1)^2 + 336L(-2) + 405L(-1)^2\right) w_{W_{k}^{0(1)}}.$$

Since $w_1$ is a singular vector, we have a relation

$$0 = 50\sqrt{-3}([\omega]^2 * [w_{W_{k}^{0(1)}}] + [w_{W_{k}^{0(1)}}] * [\omega]^2) - 20\sqrt{-3}([\omega] * [w_{W_{k}^{0(1)}}] + [w_{W_{k}^{0(1)}}] * [\omega]) + 4\sqrt{-3}[w_{W_{k}^{0(1)}}] - 100\sqrt{-3} [\omega] * [w_{W_{k}^{0(1)}}] * [\omega] - 5[w_{W_{k}^{0(1)}}] * [J] + 5[J] * [w_{W_{k}^{0(1)}}]$$

in $A(W_{k}^{0(1)})$ by using formulas (3). Therefore,

$$0 = (50\sqrt{-3}(h_{L,2}^2 + h_{L,3}^2) - 20\sqrt{-3}(h_{L,2} + h_{L,3}) + 4\sqrt{-3} - 100\sqrt{-3}h_{L,2}h_{L,3} - 5k_{L,2} + 5k_{L,3}) w_{(L^3)^{\otimes} [w_{W_{k}^{0(1)}}] \otimes w_{L^2}}$$

in $L^3(0)^{\otimes} \otimes_{A(W)} A(W_{k}^{0(1)}) \otimes_{A(W)} L^2(0)$. Set

$$\psi(h_{L,2}, k_{L,2}, h_{L,3}, k_{L,3}) = \frac{(50\sqrt{-3}(h_{L,2}^2 + h_{L,3}^2) - 20\sqrt{-3}(h_{L,2} + h_{L,3}) + 4\sqrt{-3} - 100\sqrt{-3}h_{L,2}h_{L,3} - 5k_{L,2} + 5k_{L,3})}{w_{(L^3)^{\otimes} [w_{W_{k}^{0(1)}}] \otimes w_{L^2}} = 0}$$

If $\psi(h_{L,2}, k_{L,2}, h_{L,3}, k_{L,3}) \neq 0$, then $w_{(L^3)^{\otimes} [w_{W_{k}^{0(1)}}] \otimes w_{L^2}} = 0$ and $F(W_{k}^{0(1)}, L^2, L^3) = 0$. By computing $\psi(h_{L,2}, k_{L,2}, h_{L,3}, k_{L,3})$ for all pairs $(L^2, L^3)$ of 20 irreducible $W$-modules, we have $F(W_{k}^{0(1)}, L^2, L^3) \leq 1$ if the pair $(L^2, L^3)$ is one of

$$(M_{k}^{1}, W_{k}^{0(1)+1}), (W_{k}^{1}, M_{k}^{0(1)+1}), (W_{k}^{0(1)}, W_{k}^{0(1)+1}), (i = 0, 1, 2), (M_{k}^{a}, W_{k}^{a}), (W_{k}^{a}, M_{k}^{a}), \text{ and } (W_{k}, W_{k}^{a})$$

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and $F(W_k^{0(1)}, L^2, L^3) = 0$ otherwise. Combining these results and Lemma 6, we have

\[
W_k^{0(1)} \times M_k^{0(i)} = W_k^{0(1+i)},
\]

\[
W_k^{0(1)} \times W_k^{0(i)} = M_k^{0(1+i)} + W_k^{0(1+i)},
\]

\[
W_k^{0(1)} \times M_k^a = W_k^a,
\]

\[
W_k^{0(1)} \times W_k^a = M_k^a + W_k^a.
\]

In the case $L^1 = W_k^{0(2)}$, we can compute the fusion rules in the same way. In the case $L^1 = M_k^{0(i)}$ ($i = 1, 2$), it is shown that $A(M_k^{0(i)})$ is generated by $[w_{M_k^{0(i)}}]$ in the same way. But in these cases we need two singular vectors $v_{31} \in M_k^{0(i)}(3)$ and $v_{41} \in M_k^{0(i)}(4)$. We can compute the fusion rules by using two relations $v_{41} = 0$ and $J(-1)v_{31} = 0$.

(2) We consider the case $L^1 = W_k^{0(0)}$. There are two singular vectors $u_{2i} \in W_k^{0(0)}(2)$ ($i = 1, 2$) and there is one singular vector $u_{41} \in W_k^{0(0)}(4)$. Since $u_{21} = 0$, $A(W_k^{0(0)})$ is generated by $\{w_{W_k^{0(0)}}, J(-1)w_{W_k^{0(0)}}\}$ as a $A(W)$-bimodule and for $X \in W_k^{0(0)}$ we have $a_1(L^2, L^3; X), a_2(L^2, L^3; X) \in \mathbb{C}$ such that

\[
w_{(L^3)^t} \otimes [X] \otimes w_{L^2} = a_1(L^2, L^3; X)w_{(L^3)^t} \otimes [w_{W_k^{0(0)}}] \otimes w_{L^2} + a_2(L^2, L^3; X)w_{(L^3)^t} \otimes [J(-1)w_{W_k^{0(0)}}] \otimes w_{L^2}.
\]

Therefore, $F(W_k^{0(0)}, L^2, L^3) \leq 2$. Set a matrix

\[
A = \begin{pmatrix}
a_1(L^2, L^3; u_{22}) & a_2(L^2, L^3; u_{22}) \\
a_1(L^2, L^3; J(-1)u_{22}) & a_2(L^2, L^3; J(-1)u_{22}) \\
a_1(L^2, L^3; v_{41}) & a_2(L^2, L^3; u_{41}) \\
a_1(L^2, L^3; J(-1)u_{41}) & a_2(L^2, L^3; J(-1)u_{41})
\end{pmatrix}.
\]

We have

\[
A \begin{pmatrix} (w_{(L^3)^t} \otimes [w_{W_k^{0(0)}}] \otimes w_{L^2} \\ w_{(L^3)^t} \otimes [J(-1)w_{W_k^{0(0)}}] \otimes w_{L^2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\]
If \( \text{rank} A = 2 \), then 
\[
\begin{align*}
  w_{(L^3)^\gamma} \otimes [w_{W^0(0)}] \otimes w_{L^2} &= w_{(L^3)^\gamma} \otimes [J(-1)w_{W^0(0)}] \otimes w_{L^2} = 0 \text{ in } L^3(0)^* \otimes A(W) A(W) L^2(0).
\end{align*}
\]
If \( \text{rank} A = 1 \), then 
\[
\begin{align*}
  w_{(L^3)^\gamma} \otimes [w_{W^0(0)}] \otimes w_{L^2} \text{ and } w_{(L^3)^\gamma} \otimes [J(-1)w_{W^0(0)}] \otimes w_{L^2} \text{ are linearly dependent. We hence have}
\end{align*}
\]
\[
F(W^0_k, L^2, L^3) \leq \begin{cases} 
0, & \text{if } \text{rank} A = 2, \\
1, & \text{if } \text{rank} A = 1, \\
2, & \text{if } \text{rank} A = 0.
\end{cases}
\]

By computing the rank of \( A \) for all pairs \((L^2, L^3)\) of 20 irreducible \( W \)-modules, we have for \( i = 0, 1, 2 \)
\[
\begin{align*}
  W^0_k \times M^0_k &= W^0_k, \\
  W^0_k \times W^0_k &= M^0_k + W^0_k, \\
  W^0_k \times M^a_k &= W^a_k, \\
  W^0_k \times W^a_k &= M^a_k + W^a_k.
\end{align*}
\]

In the case that \( L^1 \) is one of \( M^a_k \) and \( W^a_k \), we can compute the fusion rules in the same way. We roughly explain each case. In the case \( L^1 = M^a_k \), there are two singular vectors \( v_{21}, v_{22} \) in \( M^a_k(2) \) and three singular vectors \( v_{61}, v_{62}, v_{63} \) in \( M^a_k(6) \). Set the same matrix for \( v_{22}, v_{61}, v_{62}, v_{63}, J(-1)v_{61}, J(-1)v_{62} \) and \( J(-1)v_{63} \) as in the case of \( W^0_k \). By computing the rank of the matrix, we can determine the fusion rules. In the case \( L^1 = W^a_k \), there is a singular vector \( v_{21} \) in \( W^a_k(2) \) and there are two singular vectors \( v_{41}, v_{42} \) in \( W^a_k(4) \). Set the same matrix for \( v_{41}, v_{42}, J(-1)v_{41} \) and \( J(-1)v_{42} \) as in the case of \( W^0_k \). By computing the rank of the matrix, we can determine the fusion rules.

\[\square\]

5 Appendix

We give some singular vectors in irreducible \( W \)-modules used in Theorem 4. We omit \( w_N \in N(0) \) from the explicit form of every singular vector in a \( W \)-module \( N \).
We obtain singular vectors in $M_k^{0(1)}$ by replacing $J(n)$ with $-J(n)$ in the above vectors.

- $M_k^{0(1)}$.
  
  (1) $9\sqrt{-3}L(-1) + J(-1)$,
  (2) $6255L(-1)J(-2) - 375\sqrt{-3}L(-1)J(-1)^2 + 36960L(-2)J(-1)$
  $\quad + 5175L(-1)^2J(-1) - 26208J(-3) + 1425\sqrt{-3}J(-2)J(-1)$
  $\quad + 25J(-1)^3 + 23040\sqrt{-3}L(-3) + 147600\sqrt{-3}L(-2)L(-1)$
  $\quad - 44625\sqrt{-3}L(-1)^3$,
  (3) $-36720\sqrt{-3}L(-1)J(-3) + 324L(-1)J(-2)J(-1)$
  $\quad - 16\sqrt{-3}L(-1)J(-1)^3 + 9360\sqrt{-3}L(-2)J(-2)$
  $\quad + 12852\sqrt{-3}L(-1)^2J(-2) + 2400L(-2)J(-1)^2$
  $\quad + 462L(-1)^2J(-1)^2 - 5040\sqrt{-3}L(-3)J(-1)$
  $\quad + 8640\sqrt{-3}L(-2)L(-1)J(-1) - 5232\sqrt{-3}L(-1)^3J(-1)$
  $\quad + 35280\sqrt{-3}J(-4) - 819J(-2)^2$
  $\quad + 76\sqrt{-3}J(-2)J(-1)^2 + J(-1)^4$
  $\quad - 751680L(-4) - 1028160L(-3)L(-1)$
  $\quad + 254880L(-2)L(-1)^2 + 16929L(-1)^4$.

We obtain singular vectors in $M_k^{0(2)}$ by replacing $J(n)$ with $-J(n)$ in the above vectors.

- $W_k^{0(0)}$.
  
  (1) $-70\sqrt{-3}L(-1)J(-1) + 91\sqrt{-3}J(-2) - 5J(-1)^2 - 2496L(-2) + 195L(-1)^2$,
  (2) $-70\sqrt{-3}L(-1)J(-1) + 91\sqrt{-3}J(-2) + 5J(-1)^2 + 2496L(-2) - 195L(-1)^2$,
  (3) $-1500L(-1)J(-2)J(-1) + 1200L(-2)J(-1)^2 + 750L(-1)^2J(-1)^2$
  $\quad + 3600J(-3)J(-1) + 825J(-2)^2 + J(-1)^4$
  $\quad - 633600L(-4) + 46800L(-3)L(-1) + 230400L(-2)^2$
  $\quad - 126000L(-2)L(-1)^2 + 50625L(-1)^4$.

- $W_k^{0(1)}$.
  
  (1) $5\sqrt{-3}L(-1) + J(-1)$,
  (2) $-30\sqrt{-3}L(-1)J(-1) + 39\sqrt{-3}J(-2) + 5J(-1)^2 + 336L(-2) + 405L(-1)^2$.

We obtain singular vectors in $W_k^{0(2)}$ by replacing $J(n)$ with $-J(n)$ in the above vectors.
There are another three singular vectors $v_{6i}$ ($i = 1, 2, 3$) in $M^a_k$, 

- $W^a_k$:

   1. $J(-1)^2 - 30L(-2) + 75L(-1)^2,$
   2. $-5040\sqrt{-3}L(-1)J(-3) - 4980L(-1)J(-2)J(-1) + 200\sqrt{-3}L(-1)J(-1)^3 + 7668\sqrt{-3}L(-2)J(-2) + 990\sqrt{-3}L(-1)^2(J(-2) + 9660L(-2)J(-1)^2 + 7350L(-1)^2J(-1) + 35760\sqrt{-3}L(-3)J(-1) - 78960\sqrt{-3}L(-2)L(-1)J(-1) + 34200\sqrt{-3}L(-1)^3J(-1) + 5208\sqrt{-3}J(-4) + 12780J(-3)J(-1) + 2622J(-2)J(-2) - 310\sqrt{-3}J(-2)J(-1)^2 + 25J(-1)^4 - 9000L(-4) - 255780L(-3)L(-1) - 28044L(-2)^2 + 557460L(-2)L(-1)^2 - 78975L(-1)^4;$

   3. $5040\sqrt{-3}L(-1)J(-3) - 4980L(-1)J(-2)J(-1) - 200\sqrt{-3}L(-1)J(-1)^3 - 7668\sqrt{-3}L(-2)J(-2) - 990\sqrt{-3}L(-1)^2J(-2) + 9660L(-2)J(-1)^2 + 7350L(-1)^2J(-1) - 35760\sqrt{-3}L(-3)J(-1) + 78960\sqrt{-3}L(-2)L(-1)J(-1) - 34200\sqrt{-3}L(-1)^3J(-1) - 5208\sqrt{-3}J(-4) + 12780J(-3)J(-1) + 2622J(-2)J(-2) + 310\sqrt{-3}J(-2)J(-1)^2 + 25J(-1)^4 - 9000L(-4) - 255780L(-3)L(-1) - 28044L(-2)^2 + 557460L(-2)L(-1)^2 - 78975L(-1)^4.$

- $M^a_k$:

   1. $8\sqrt{-3}L(-1)J(-1) - 6\sqrt{-3}J(-2) + J(-1)^2 + 90L(-2) + 27L(-1)^2,$
   2. $-8\sqrt{-3}L(-1)J(-1) + 6\sqrt{-3}J(-2) + J(-1)^2 + 90L(-2) + 27L(-1)^2,$

There are another three singular vectors $v_{6i}$ ($i = 1, 2, 3$) in $M^a_k$, 

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\[v_{61} = 63987840\sqrt{-3}L(-1)J(-5) - 1059480L(-1)J(-4)J(-1) + 1587600L(-1)J(-3)J(-2) + 14400\sqrt{-3}L(-1)J(-3)J(-1)^2 - 26208\sqrt{-3}L(-1)J(-2)^2J(-1) + 4260L(-1)J(-2)J(-1)^3 + 32\sqrt{-3}L(-1)J(-1)^5 + 23284800\sqrt{-3}L(-2)J(-4) - 19867680\sqrt{-3}L(-1)^2J(-4) + 145800L(-2)J(-3)J(-1) - 1414260L(-1)^2J(-3)J(-1) - 2471400L(-2)J(-2) - 155358L(-1)^2J(-2)^2 - 15840\sqrt{-3}L(-2)J(-2)J(-1)^2 + 10224\sqrt{-3}L(-1)^2J(-2)J(-1)^2 + 3990L(-2)J(-1)^4 - 447L(-1)^2J(-1)^4 - 25401600\sqrt{-3}L(-3)J(-3) - 5443200\sqrt{-3}L(-2)L(-1)J(-3) - 1995840\sqrt{-3}L(-1)^3J(-3) - 659880L(-3)J(-2)J(-1) + 1666440L(-2)L(-1)J(-2)J(-1) + 341388L(-1)^3J(-2)J(-1) - 41280\sqrt{-3}L(-3)J(-1)^3 + 32640\sqrt{-3}L(-2)L(-1)J(-1)^3 + 7872\sqrt{-3}L(-1)^3J(-1)^3 + 4294080\sqrt{-3}L(-4)J(-2) + 16420320\sqrt{-3}L(-3)L(-1)J(-2) + 4989600\sqrt{-3}L(-2)^2J(-2) - 1360800\sqrt{-3}L(-2)L(-1)^2J(-2) + 122472\sqrt{-3}L(-1)^4J(-2) - 3483720L(-4)J(-1)^2 - 1742220L(-3)L(-1)J(-1)^2 + 38700L(-2)^2J(-1)^2 + 588420L(-2)L(-1)^2J(-1)^2 - 169749L(-1)^4J(-1)^2 - 42370560\sqrt{-3}L(-5)J(-1) - 45861120\sqrt{-3}L(-4)L(-1)J(-1) - 2332800\sqrt{-3}L(-3)L(-2)J(-1) - 31164480\sqrt{-3}L(-3)L(-1)^2J(-1) - 3542400\sqrt{-3}L(-2)^2L(-1)J(-1) + 4429440\sqrt{-3}L(-2)L(-1)^3J(-1) + 23328\sqrt{-3}L(-1)^2J(-1)^3 - 101787840\sqrt{-3}J(-6) - 546480J(-5)J(-1) - 1186920J(-4)J(-2) + 5280\sqrt{-3}J(-4)J(-1)^2 - 2381400J(-3)^2 - 47520\sqrt{-3}J(-3)J(-2)J(-1) - 3420J(-3)J(-1)^3 + 11088\sqrt{-3}J(-2)^3 + 870J(-2)^2J(-1)^2 - 88\sqrt{-3}J(-2)J(-1)^4 + J(-1)^6 + 879076800L(-6) + 990072720L(-5)L(-1) + 65091600L(-4)L(-2) + 323666280L(-4)L(-1)^2 + 102173400L(-3)^2 + 73823400L(-3)L(-2)L(-1) - 3027780L(-3)L(-1)^3 - 152523000L(-2)^3 + 95652900L(-2)^2L(-1)^2 - 15326010L(-2)L(-1)^4 - 505197L(-1)^6.\]
The vector $v_{62}$ is obtained by replacing $J(n)$ with $-J(n)$ in $v_{61}$.

$$v_{63} = -1478136600L(-1)J(-4)J(-1) + 423979920L(-1)J(-3)J(-2)$$
$$- 1273500L(-1)J(-2)J(-1)^3 - 29005560L(-2)J(-3)J(-1)$$
$$+ 58538700L(-1)^2J(-3)J(-1) + 134322300L(-2)J(-2)^2$$
$$- 133840350L(-1)^2J(-2)^2 + 1341750L(-2)J(-1)^4$$
$$+ 680625L(-1)^2J(-1)^4 - 778588200L(-3)J(-2)J(-1)$$
$$+ 143310600L(-2)J(-2)J(-1) + 37975500L(-1)^3J(-2)J(-1)$$
$$- 2232505800L(-4)J(-1)^2 - 311863500L(-3)L(-1)J(-1)^2$$
$$+ 586133100L(-2)^2J(-1)^2 - 239341500L(-2)L(-1)^2J(-1)^2$$
$$+ 127186875L(-1)^4J(-1)^2 - 393666480J(-5)J(-1)$$
$$- 1236672360J(-4)J(-2) - 104786136J(-3)^2$$
$$+ 3968100J(-3)J(-1)^3 + 2139750J(-2)^2J(-1)^2$$
$$+ 625J(-1)^6 + 275225065920L(-6)$$
$$+ 450006898320L(-5)L(-1) + 221829042960L(-4)L(-2)$$
$$- 60223224600L(-4)L(-1)^2 + 121440173400L(-3)^2$$
$$+ 145205865000L(-3)L(-2)L(-1) - 22975690500L(-3)L(-1)^3$$
$$- 5826518800L(-2)^3 + 65516566500L(-2)L(-1)^2$$
$$- 55664516250L(-2)L(-1)^4 + 5974171875L(-1)^6.$$ 

**References**

[1] R. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Nat. Acad. Sci. U.S.A.* 83 (1986), 3068–3071.

[2] R. Dijkgraaf, C. Vafa, E. Verlinde, and H. Verlinde, The operator algebra of orbifold models, *Comm. Math. Phys.* 123 (1989), 485–526.

[3] C. Dong, C.H. Lam, K. Tanabe, H. Yamada, and K. Yokoyama, $\mathbb{Z}_3$ symmetry and $W_3$ algebra in lattice vertex operator algebras, *Pacific J. Math.* 215 (2004), 245–296.

[4] C. Dong and J. Lepowsky, Generalized Vertex Algebras and Relative Vertex Operators, Progress in Math., Vol. 112, Birkhauser, Boston, 1993.

[5] C. Dong and G. Mason, On quantum Galois theory, *Duke Math. J.* 86 (1997), 305–321.
[6] C. Dong and G. Yamskulna, Vertex operator algebras, generalized doubles and dual pairs, Math. Z. 241 (2002), 397–423.

[7] V. A. Fateev and A. B. Zamolodchikov, Conformal quantum field theory models in two dimensions having $\mathbb{Z}_3$ symmetry, Nuclear Physics B280 (1987), 644–660.

[8] I. Frenkel, Y. Huang, and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, Mem. Amer. Math. Soc. 104 (1993).

[9] I. B. Frenkel, J. Lepowsky and A. Meurman, Vertex Operator Algebras and the Monster, Pure and Applied Math., Vol. 134, Academic Press, 1988.

[10] I. Frenkel and Y. Zhu, Vertex operator algebras associated to representations of affine and Virasoro algebras, Duke Math. J. 66 (1992), 123–168.

[11] K. Kitazume, C. Lam and H. Yamada, 3-state Potts model, moonshine vertex operator algebra and $3A$ elements of the monster group, IMRS, 23 (2003), 1269–1303.

[12] C. Lam and H. Yamada, $\mathbb{Z}_2 \times \mathbb{Z}_2$ codes and vertex operator algebras, J. Algebra 224 (2000), 268–291.

[13] H. Li, Determining fusion rules by $A(V)$-modules and bimodules, J. Algebra 212 (1999), 515–556.

[14] Y. Zhu, Modular invariance of characters of vertex operator algebras, J. Amer. Math. Soc. 9 (1996), 237–302.