HIGHER ORDER DERIVATIVES OF APPROXIMATION POLYNOMIALS ON $\mathbb{R}$

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Abstract. D. Leviatan has investigated the behavior of the higher order derivatives of approximation polynomials of the differentiable function $f$ on $[-1,1]$. Especially, when $P_n$ is the best approximation of $f$, he estimates the differences $\|f^{(k)} - P_n^{(k)}\|_{L_\infty([-1,1])}$, $k = 0, 1, 2, \ldots$. In this paper, we give the analogies for them with respect to the differentiable functions on $\mathbb{R}$, and we apply the result to the monotone approximation.

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$. We say that $f : \mathbb{R} \to \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$ for $0 < x < y$. The notation $f(x) \sim g(x)$ means that there are positive constants $C_1, C_2$ such that for the relevant range of $x$, $C_1 \leq f(x)/g(x) \leq C_2$. The similar notation is used for sequences and sequences of functions. Throughout $C, C_1, C_2, \ldots$ denote positive constants independent of $n, x, t$. The same symbol does not necessarily denote the same constant in different occurrences. We denote the class of polynomials with degree $n$ by $P_n$.

First, we introduce some classes of weights. Levin and Lubinsky introduced the class of weights on $\mathbb{R}$ as follows.

Definition 1.1. Let $Q : \mathbb{R} \to [0, \infty)$ be a continuous even function, and satisfy the following properties:

(a) $Q'(x)$ is continuous in $\mathbb{R}$, with $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R}\setminus\{0\}$.
(c) $\lim_{x \to \infty} Q(x) = \infty$.
(d) The function

$$T_w(x) : = \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$, with $T_w(x) \geq \Lambda > 1, x \in \mathbb{R}\setminus\{0\}$.

(e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad a.e. \ x \in \mathbb{R}.$$
Furthermore, if there also exists a compact subinterval \( J(\ni 0) \) of \( \mathbb{R} \), and \( C_2 > 0 \) such that
\[
\frac{Q''(x)}{Q'(x)} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J,
\]
then we write \( w = \exp(-Q) \in \mathcal{F}(C^2+) \).

For convenience, we denote \( T \) instead of \( T_w \), if there is no confusion. Next, we give some typical examples of \( \mathcal{F}(C^2+) \).

**Example 1.2** (\([5]\)). (1) If \( T(x) \) is bounded, then we call the weight \( w = \exp(-Q(x)) \) the Freud-type weight and we write \( w \in \mathcal{F}^* \subset \mathcal{F}(C^2+) \).
(2) When \( T(x) \) is unbounded, then we call the weight \( w = \exp(-Q(x)) \) the Erdös-type weight: (a) For \( \alpha > 1, \, l \geq 1 \) we define
\[
Q(x) := Q_{l,\alpha}(x) = \exp(l(|x|^\alpha) - \exp_1(0)),
\]
where \( \exp_1(x) = \exp(\exp(\exp(\ldots \exp(x) \ldots))(l \text{ times}) ) \). More generally, we define
\[
Q_{l,\alpha,m}(x) = |x|^m \{ \exp_1(|x|^\alpha) - \hat{\alpha} \exp_1(0) \}, \quad \alpha + m > 1, \, m \geq 0, \, \alpha \geq 0,
\]
where \( \hat{\alpha} = 0 \) if \( \alpha = 0 \), and otherwise \( \hat{\alpha} = 1 \). We note that \( Q_{l,0,m} \) gives a Freud-type weight, and \( Q_{l,\alpha,m} \) \( (\alpha > 0) \) gives an Erdös-type weight.
(3) For \( \alpha > 1 \), \( Q_{\alpha}(x) = (1 + |x|)^{|x|\alpha} - 1 \) gives also an Erdös-type weight.

For a continuous function \( f : [-1, 1] \rightarrow \mathbb{R} \), let
\[
E_n(f) = \inf_{P \in \mathcal{P}_n} \|f - P\|_{L^\infty((-1,1))} = \inf_{P \in \mathcal{P}_n} \max_{x \in [-1,1]} |f(x) - P(x)|.
\]

D. Leviatan [7] has investigated the behavior of the higher order derivatives of approximation polynomials for the differentiable function \( f \) on \([-1, 1]\), as follows:

**Theorem** (Leviatan [7]). For \( r \geq 0 \) we let \( f \in C^{(r)}[-1,1] \), and let \( P_n \in \mathcal{P}_n \) denote the polynomial of best approximation of \( f \) on \([-1,1]\). Then for each \( 0 \leq k \leq r \) and every \(-1 \leq x \leq 1\),
\[
|f^{(k)}(x) - P_n^{(k)}(x)| \leq \frac{C_r}{n^k} \Delta_n^{(k)}(x) E_n \left( f^{(k)} \right), \quad n \geq k,
\]
where \( \Delta_n(x) := \sqrt{1 - x^2}/n + 1/n^2 \) and \( C_r \) is an absolute constant which depends only on \( r \).

In this paper, we estimate \( \left| \left( f^{(k)}(x) - P_{n;f}^{(k)}(x) \right) w(x) \right| \), \( x \in \mathbb{R}, \, k = 0, 1, \ldots, r \) for \( f \in C^{(r)}(\mathbb{R}) \) and for some exponential type weight \( w \) in \( L^p(\mathbb{R}) \)-space, \( 1 < p \leq \infty \), where \( P_{n;f} \in \mathcal{P}_n \) is the best approximation of \( f \). Furthermore, we give an application for a monotone approximation with linear differential operators. In Section 2 we write the theorems in the space \( L^\infty(\mathbb{R}) \), then we also denote a certain assumption and some notations which need to state the theorems. In Section 3 we give some lemmas and the proofs of theorems. In Section 4, we consider the similar problem in \( L^p(\mathbb{R}) \)-space, \( 1 < p < \infty \). In Section 5, we give a simple application of the result to the monotone approximation.

2. Theorems and Preliminaries

First, we introduce some well-known notations. If \( f \) is a continuous function on \( \mathbb{R} \), then we define
\[
\|fw\|_{L^\infty(\mathbb{R})} := \sup_{t \in \mathbb{R}} |f(t)w(t)|,
\]
and for $1 \leq p < \infty$ we denote

$$\|f w\|_{L^p(\mathbb{R})} := \left( \int_{\mathbb{R}} |f(t) w(t)|^p \, dt \right)^{1/p}.$$  

Let $1 \leq p \leq \infty$. If $\|w f\|_{L^p(\mathbb{R})} < \infty$, then we write $w f \in L^p(\mathbb{R})$, and here if $p = \infty$, we suppose that $f \in C(\mathbb{R})$ and $\lim_{|x| \to \infty} |w(x) f(x)| = 0$. We denote the rate of approximation of $f$ by

$$E_{\mu,n}(w; f) := \inf_{P \in \mathcal{P}_n} \|f - Pw\|_{L^\infty(\mathbb{R})}.$$  

The Mhaskar-Rakhmanov-Saff numbers $a_x$ is defined as follows:

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{\sqrt{1 - u^2}} \, du, \quad x > 0.$$  

To write our theorems we need some preliminaries. We need further assumptions.

**Definition 2.1.** Let $w = \exp(-Q) \in \mathcal{F}(C^2)$ and $0 < \lambda < (r + 2)/(r + 1)$. Let $r \geq 1$ be an integer. Then we write $w \in \mathcal{F}_\lambda(C^{r+2})$ if $Q \in C(r+2)(\mathbb{R}\setminus\{0\})$ and there exist two constants $C > 1$ and $K \geq 1$ such that for all $|x| \geq K$,

$$\frac{|Q'(x)|}{Q^\lambda(x)} \leq C \quad \text{and} \quad \frac{|Q''(x)|}{|Q'(x)|} \sim C \frac{|Q^{(k+1)}(x)|}{Q^{(k)}(x)}$$

for every $k = 2, \ldots, r$ and also

$$\frac{|Q^{(r+2)}(x)|}{Q^{(r+1)}(x)} \leq C \frac{|Q^{(r+1)}(x)|}{Q^{(r)}(x)}.$$  

In particular, $w \in \mathcal{F}_\lambda(C^{3})$ means that $Q \in C(3)(\mathbb{R}\setminus\{0\})$ and

$$\frac{|Q'(x)|}{Q^\lambda(x)} \leq C \quad \text{and} \quad \frac{|Q''(x)|}{Q'(x)} \leq C \frac{|Q'''(x)|}{Q''(x)}$$

hold for $|x| \geq K$. In addition, let $\mathcal{F}_\lambda(C^2) := \mathcal{F}(C^2)$.

From [5], we know that Example 1.2 (2),(3) satisfy all conditions of Definition 2.1. Under the same condition of Definition 2.1 we obtain an interesting theorem as follows:

**Theorem 2.2 (10, Theorem 4.1, 4.2]).** Let $r$ be a positive integer, $0 < \lambda < (r + 2)/(r + 1)$ and let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+2})$. Then for any $\mu, \nu, \alpha, \beta \in \mathbb{R}$, we can construct a new weight $w_{\mu,\nu,\alpha,\beta} \in \mathcal{F}_\lambda(C^{r+1})$ such that

$$T^n_\alpha(x)(1 + x^2)^\nu(1 + Q(x))^\alpha(1 + |Q'(x)|)^\beta w(x) \sim w_{\mu,\nu,\alpha,\beta}(x)$$

on $\mathbb{R}$, and

$$a_{n/c}(w) \leq a_{n}(w_{\mu,\nu,\alpha,\beta}) \leq a_{cn}(w), \quad c \geq 1,$$

$$T^n_{\mu,\nu,\alpha,\beta}(x) \sim T^n_\alpha(x)$$

hold on $\mathbb{R}$.

For a given $\alpha \in \mathbb{R}$ and $w \in \mathcal{F}(C^2)$, we let $w_\alpha \in \mathcal{F}(C^2)$ satisfy $w_\alpha(x) \sim T^n_\alpha w(x)$, and let $P_n;w,\alpha \in \mathcal{P}_n$ be the best approximation of $f$ with respect to the weight $w_\alpha$, that is,

$$\|(f - P_n;w,\alpha)w_\alpha\|_{L^\infty(\mathbb{R})} = E_n(w_\alpha, f) := \inf_{P \in \mathcal{P}_n} \|(f - P)w_\alpha\|_{L^\infty(\mathbb{R})}.$$  

Then we have the main result as follows:
Theorem 2.3. Let $r \geq 0$ be an integer. Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+2})$ and $0 < \lambda < (r + 2)/(r + 1)$. Suppose that $f \in C^{(r)}(\mathbb{R})$ with

$$
\lim_{|x| \to \infty} T^{1/4}(x)f^{(r)}(x)w(x) = 0.
$$

Then there exists an absolute constant $C_r > 0$ which depends only on $r$ such that for $0 \leq k \leq r$ and $x \in \mathbb{R}$,

$$
\left| \left( f^{(k)}(x) - P^{(k)}_{n;w}(x) \right) w(x) \right| \leq C_r T^{k/2}(x)E_{n-k} \left( w_{1/4}, f^{(k)} \right)
\leq C_r T^{k/2}(x) \left( \frac{a_n}{n} \right)^{r-k} E_{n-r} \left( w_{1/4}, f^{(r)} \right).
$$

When $w \in \mathcal{F}^*$, we can replace $w_{1/4}$ with $w$ in the above.

Applying Theorem 2.3 with $w$ or $w_{-1/4}$, we have the following corollary.

Corollary 2.4. (1) Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+2})$, $0 < \lambda < (r + 2)/(r + 1)$, $r \geq 0$. We suppose that $f \in C^{(r)}(\mathbb{R})$ with

$$
\lim_{|x| \to \infty} T^{1/4}(x)f^{(r)}(x)w(x) = 0,
$$

then for $0 \leq k \leq r$ we have

$$
\left\| f^{(k)} - P^{(k)}_{n;w} \right\|_{L^\infty(\mathbb{R})} \leq C_r E_{n-k} \left( w^1_{4}, f^{(k)} \right)
\leq C_r \left( \frac{a_n}{n} \right)^{r-k} E_{n-r} \left( w_{1/4}, f^{(r)} \right).
$$

(2) Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3})$, $0 < \lambda < (r + 3)/(r + 2)$, $r \geq 0$. We suppose that $f \in C^{(r)}(\mathbb{R})$ with

$$
\lim_{|x| \to \infty} f^{(r)}(x)w(x) = 0,
$$

then for $0 \leq k \leq r$ we have

$$
\left\| f^{(k)} - P^{(k)}_{n;w(1/4)} \right\|_{L^\infty(\mathbb{R})} \leq C_r E_{n-k} \left( w, f^{(k)} \right)
\leq C_r \left( \frac{a_n}{n} \right)^{r-k} E_{n-r} \left( w, f^{(r)} \right).
$$

When $w \in \mathcal{F}^*$, we can replace $w_n$ with $w$ in the above.

Corollary 2.5. Let $r \geq 0$ be an integer. Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3})$, $0 < \lambda < (r + 3)/(r + 2)$, and let $w(2r+1)/4f^{(r)} \in L^\infty(\mathbb{R})$. Then, for each $k(0 \leq k \leq r)$ and the best approximation polynomial $P_{n;w_{k/2}}$;

$$
\left\| f - P_{n;w_{k/2}} \right\|_{L^\infty(\mathbb{R})} = E_n \left( w_{k/2}, f \right),
$$

we have

$$
\left\| f^{(k)} - P^{(k)}_{n;w_{k/2}} \right\|_{L^\infty(\mathbb{R})} \leq C_r E_{n-k} \left( w_{(2k+1)/4}, f^{(k)} \right)
\leq C_r \left( \frac{a_n}{n} \right)^{r-k} E_{n-r} \left( w_{(2k+1)/4}, f^{(r)} \right).
$$

When $w \in \mathcal{F}^*$, we can replace $w_{1/4}$ with $w$ in the above.
3. Proof of Theorems

Throughout this section we suppose \( w \in \mathcal{F}(C^2+) \). We give the proofs of theorems. First, we give some lemmas to prove the theorems. We construct the orthonormal polynomials \( p_n(x) = p_n(w^2, x) \) of degree \( n \) for \( w^2(x) \), that is,

\[
\int_{-\infty}^{\infty} p_n(w^2, x)p_m(w^2, x)w^2(x)dx = \delta_{mn} \text{(Kronecker delta)}.
\]

Let \( fw \in L_2(\mathbb{R}) \). The Fourier-type series of \( f \) is defined by

\[
\hat{f}(x) := \sum_{k=0}^{\infty} a_k(w^2, f)p_k(w^2, x), \quad a_k(w^2, f) := \int_{-\infty}^{\infty} f(t)p_k(w^2, t)w^2(t)dt.
\]

We denote the partial sum of \( \hat{f}(x) \) by

\[
s_n(f, x) := s_n(w^2, f, x) := \sum_{k=0}^{n-1} a_k(w^2, f)p_k(w^2, x).
\]

Moreover, we define the de la Vallée Poussin means by

\[
v_n(f, x) := \frac{1}{n} \sum_{j=n+1}^{2n} s_j(w^2, f, x).
\]

**Proposition 3.1** ([11, Theorem 1.1 (1.5), Corollary 6.2 (6.5)]) Let \( w \in \mathcal{F}_\lambda(C^3+) \), \( 0 < \lambda < 3/2 \), and let \( 1 \leq p \leq \infty \). When \( T^{1/4}wf \in L_p(\mathbb{R}) \), we have

\[
\|v_n(f)w\|_{L_p(\mathbb{R})} \leq C \left\| T^{1/4}wf \right\|_{L_p(\mathbb{R})},
\]

and so

\[
\|(f - v_n(f))w\|_{L_p(\mathbb{R})} \leq CE_{p,n} \left(T^{1/4}w, f \right).
\]

So, equivalently,

\[
\|v_n(f)w\|_{L_p(\mathbb{R})} \leq C \left\| w^{1/4}f \right\|_{L_p(\mathbb{R})},
\]

and so

\[
(3.1) \quad \|(f - v_n(f))w\|_{L_p(\mathbb{R})} \leq CE_{p,n} \left(w^{1/4}, f \right).
\]

When \( w \in \mathcal{F}^* \), we can replace \( w^{1/4} \) with \( w \).

**Lemma 3.2.** (1) ([8 Lemma 3.5, (a)]) Let \( L > 0 \) be fixed. Then, uniformly for \( t > 0 \),

\[
a_{Lt} \sim a_t.
\]

(2) ([8 Lemma 3.4 (3.17)]) For \( x > 1 \), we have

\[
|Q'(ax)| \sim \frac{x}{a_x} T(ax) \quad \text{and} \quad |Q(ax)| \sim \frac{x}{\sqrt{T(ax)}}.
\]

(3) ([8 Lemma 3.2 (3.8)]) Let \( x \in (0, \infty) \). There exists \( 0 < \varepsilon < 1 \) such that

\[
T \left( x \left[ 1 + \frac{\varepsilon}{T(x)} \right] \right) \sim T(x).
\]

(4) ([8 Proposition 3]) If \( T(x) \) is unbounded, then for any \( \eta > 0 \) there exists \( C(\eta) > 0 \) such that for \( t \geq 1 \),

\[
a_t \leq C(\eta)t^\eta.
\]
To prove the results, we need the following notations. We set
\[ \sigma(t) := \inf \left\{ a_n : \frac{a_n}{u} \leq t \right\}, \quad t > 0, \]
and
\[ \Phi_t(x) := \sqrt{1 - \frac{|x|}{\sigma(t)}} + T^{-1/2}(\sigma(t)), \quad x \in \mathbb{R}. \]
Define for \( f, w \in L_p(\mathbb{R}), 0 < p \leq \infty, \)
\[ \omega_p(f, w, t) := \sup_{0 < h \leq t} \left\| w(x) \left\{ f \left( x + \frac{h}{2} \Phi_t(x) \right) - f \left( x - \frac{h}{2} \Phi_t(x) \right) \right\} \right\|_{L_p(|x| \leq \sigma(2t))} \]
and
\[ + \inf_{c \in \mathbb{R}} \| w(x)(f - c)(x) \|_{L_p(|x| \geq \sigma(4t))} \]
(see [2, 3]).

**Proposition 3.3** (cf. [3, Theorem 1.2], [2, Corollary 1.4]). Let \( w \in \mathcal{F}(C^2) \). Let \( 0 < p \leq \infty \). Then for \( f : \mathbb{R} \to \mathbb{R} \) which \( f w \in L_p(\mathbb{R}) \) (and for \( p = \infty \), we require \( f \) to be continuous, and \( f w \) to vanish at \( \pm \infty \)), we have for \( n \geq C_3 \),
\[ E_{p,n}(f, w) \leq C_1 \omega_p(f, w, C_2 \frac{a_n}{n}), \]
where \( C_j, j = 1, 2, 3, \) do not depend on \( f \) and \( n \).

**Proof.** Damelin and Lubinsky [3] or Damelin [2] have treated a certain class \( \mathcal{E}_1 \) of weights containing the conditions (a)-(d) in Definition 1.1 and
\[ \frac{y Q'(y)}{x Q'(x)} \leq C_1 \left( \frac{Q(y)}{Q(x)} \right)^{C_2}, \quad y \geq x \geq C_3, \]
where \( C_i, i = 1, 2, 3 > 0 \) are some constants, and they obtain this Proposition for \( w \in \mathcal{E}_1 \). Therefore, we may show \( \mathcal{F}(C^2) \subset \mathcal{E}_1 \). In fact, from Definition 1.1 (d) and (e), we have for \( y \geq x > 0, \)
\[ \frac{Q'(y)}{Q'(x)} = \exp \left( \int_x^y \frac{Q'(t)}{Q'(x)} dt \right) \leq \exp \left( C_3 \int_x^y \frac{Q'(t)}{Q(t)} dt \right) = \left( \frac{Q(y)}{Q(x)} \right)^{C_3}, \]
and
\[ \frac{1}{x} = \exp \left( \int_x^y \frac{1}{t} dt \right) \leq \exp \left( \frac{1}{x} \int_x^y \frac{Q'(t)}{Q(t)} dt \right) = \left( \frac{Q(y)}{Q(x)} \right)^{\frac{1}{x}}. \]
Therefore, we obtain [3,2] with \( C_2 = C_3 + \frac{1}{x} \), that is, we see \( \mathcal{F}(C^2) \subset \mathcal{E}_1 \). \( \square \)

**Theorem 3.4.** Let \( w \in \mathcal{F}(C^2) \). (1) If \( f \) is a function having bounded variation on any compact interval and if
\[ \int_{-\infty}^{\infty} w(x)|df(x)| < \infty, \]
then there exists a constant \( C > 0 \) such that for every \( t > 0, \)
\[ \omega_1(f, w, t) \leq Ct \int_{-\infty}^{\infty} w(x)|df(x)|, \]
and so
\[ E_{1,n}(f, w) \leq C \frac{a_n}{n} \int_{-\infty}^{\infty} w(x)|df(x)|. \]
(2) Let us suppose that \( f \) is continuous and \( \lim_{|x| \to \infty} |(\sqrt{T}w)(x)| = 0 \), then we have

\[
\lim_{t \to 0} \omega_\infty(f, w, t) = 0.
\]

To prove this theorem we need the following lemma.

**Lemma 3.5** ([9], Lemma 7). (1) For \( t > 0 \) there exists \( a_u \) such that

\[
t = \frac{a_u}{u} \quad \text{and} \quad \sigma(t) = a_u.
\]

(2) If \( t = a_u/u, \ u > 0 \) large enough and

\[
|x - y| \leq t\Phi_t(x),
\]

then there exist \( C_1, C_2 > 0 \) such that

\[
C_1w(x) \leq w(y) \leq C_2w(x).
\]

**Proof of Theorem 3.4** (1) Let \( g(x) := f(x) - f(0) \). For \( t > 0 \) small enough let \( 0 < h \leq t \) and \( |x| \leq \sigma(2t) < \sigma(t) \). Hence we may consider \( \Phi_t(x) \leq 2 \). Then by Lemma 3.5

\[
\int_{-\infty}^{\infty} w(x) \left| g \left( x + \frac{h}{2} \Phi_t(x) \right) - g \left( x - \frac{h}{2} \Phi_t(x) \right) \right| dx
\]

\[
= \int_{-\infty}^{\infty} w(x) \left| \int_{x - \frac{h}{2} \Phi_t(x)}^{x + \frac{h}{2} \Phi_t(x)} df(v) \right| dx \leq C \int_{-\infty}^{\infty} \left| \int_{x - \frac{h}{2} \Phi_t(x)}^{x + \frac{h}{2} \Phi_t(x)} w(v) df(v) \right| dx
\]

\[
\leq \int_{-\infty}^{\infty} \int_{x-h}^{x+h} w(v) |df(v)| dx \leq \int_{-\infty}^{\infty} w(v) \int_{v-h \leq x \leq v+h} |dx| |df(v)|
\]

\[
\leq 2h \int_{-\infty}^{\infty} w(v) |df(v)|.
\]

Hence we have

\[
\int_{-\infty}^{\infty} w(x) \left| g \left( x + \frac{h}{2} \Phi_t(x) \right) - g \left( x - \frac{h}{2} \Phi_t(x) \right) \right| dx \leq 2t \int_{-\infty}^{\infty} w(x) |df(x)|. \tag{3.3}
\]

Moreover, we see

\[
\inf_{c \in \mathbb{R}} \|w(x)(f - c)(x)\|_{L_1(|x| \geq \sigma(4t))} \leq \frac{1}{Q'(\sigma(4t))} \|Q'(x)w(x)g(x)\|_{L_1(|x| \geq \sigma(4t))}. \tag{3.4}
\]

Here we see

\[
\frac{\sqrt{T(\sigma(t))}}{Q'(\sigma(t))} \sim t. \tag{3.5}
\]

In fact, from Lemma 3.2 (2), for \( t = \frac{a_u}{u} \)

\[
Q'(\sigma(t)) = Q'(a_u) \sim \frac{u\sqrt{T(a_u)}}{a_u} \sim \frac{\sqrt{T(\sigma(t))}}{t}.
\]
On the other hand, we have
\[
\int_{0}^{\infty} Q'(x)w(x)|g(x)|dx = \int_{0}^{\infty} Q'(x)w(x) \int_{0}^{x} dg(u) \, dx \\
\leq \int_{0}^{\infty} Q'(x)w(x) \int_{0}^{x} |df(u)| \, dx \\
= -w(x) \int_{0}^{\infty} |df(u)| \, dx + \int_{0}^{\infty} w(x)|df(x)| \\
= \int_{0}^{\infty} w(x)|df(x)|
\]
because \( \lim_{x \to \infty} w(x) = 0 \) from (c) of Definition 1.1. Similarly we have
\[
\int_{-\infty}^{0} |Q'(x)w(x)g(x)| \, dx \leq \int_{-\infty}^{0} w(x)|df(x)|.
\]
Hence we have
\[
(3.6) \quad \|Q'wg\|_{L_1(\mathbb{R})} \leq \int_{-\infty}^{\infty} w(u)|df(u)|.
\]
Therefore, using (3.4), (3.5) and (3.6), we have
\[
(3.7) \quad \inf_{c \in \mathbb{R}} \|w(x)(f - c)(x)\|_{L_1(\{x : |x| \geq \sigma(u)\})} = O(t) \int_{-\infty}^{\infty} w(x)|df(x)|.
\]
Consequently, by (3.3) and (3.7) we have
\[
\omega_1(f, w, t) \leq Ct \int_{-\infty}^{\infty} w(x)|df(x)|.
\]
Hence, setting \( t = C_2 \frac{a_n}{n} \), if we use Proposition 3.3, then
\[
E_{1,n}(f, w) \leq C_3 \frac{a_n}{n} \int_{-\infty}^{\infty} w(x)|df(x)|.
\]
(2) Given \( \varepsilon > 0 \), and let us take \( L = L(\varepsilon) > 0 \) large enough as
\[
\sup_{|x| \geq L} |w(x)f(x)| \leq \frac{1}{\sqrt{T(L)}} \sup_{|x| \geq L} |\sqrt{T(x)}w(x)f(x)| < \varepsilon \quad \text{(by our assumption)}.
\]
Then we have
\[
\inf_{c \in \mathbb{R}} \sup_{|x| \geq L} |w(x)(f - c)(x)| \leq \frac{1}{\sqrt{T(L)}} \sup_{|x| \geq L} |\sqrt{T(x)}w(x)f(x)| < \varepsilon.
\]
Now, there exists \( \varepsilon > 0 \) small enough such that
\[
\frac{h}{2} \Phi_t(x) \leq \varepsilon \frac{1}{T(x)} \quad |x| \leq \sigma(2t),
\]
because if we put \( t = \frac{a_u}{u} \), then we see \( \sigma(t) = a_u \) and \( |x| \leq \sigma(2t) < a_u \). Hence, noting [3] Lemma 3.7, that is, for some \( \varepsilon > 0 \), and for large enough \( t \),
\[
T(a_u) \leq Ct^{2-\varepsilon},
\]
and if \( w \) is the Erdős-type weight, then from Lemma 3.2 (4), we have
\[
(3.8) \quad t\Phi_t(x) \leq \frac{a_u}{u} \leq \varepsilon \frac{1}{T(a_u)} \leq \varepsilon \frac{1}{T(x)}.
\]
If \( w \in \mathcal{F}^* \), we also have \((3.8)\), because for some \( \delta > 0 \) and \( u > 0 \) large enough,
\[
T \Phi_t(x) \leq \frac{a_n}{u} \leq u^{-\delta} \leq \frac{1}{T(x)}.
\]

Therefore, using Lemma \(\text{(3.2)}\), Lemma \(\text{(3.3)}\) and the assumption
\[
\lim_{|x| \to \infty} \sqrt{T(x)} w(x) f(x) = 0,
\]
for \( 2L \leq |x| \leq \sigma(2t), \ h > 0, \)
\[
\left| w(x) \left\{ f \left( x + \frac{h}{2} \Phi_t(x) \right) - f \left( x - \frac{h}{2} \Phi_t(x) \right) \right\} \right|
\leq C \left[ \frac{1}{\sqrt{T(x)}} \left| \sqrt{T \left( x + \frac{h}{2} \Phi_t(x) \right) w \left( x + \frac{h}{2} \Phi_t(x) \right) f \left( x + \frac{h}{2} \Phi_t(x) \right) \right| \right. \\
+ \frac{1}{\sqrt{T(x)}} \left| \sqrt{T \left( x - \frac{h}{2} \Phi_t(x) \right) w \left( x - \frac{h}{2} \Phi_t(x) \right) f \left( x - \frac{h}{2} \Phi_t(x) \right) \right| \right]
\leq 2\varepsilon.
\]

On the other hand,
\[
\lim_{t \to 0} \sup_{0 < h \leq t} \left\| w(x) \left\{ f \left( x + \frac{h}{2} \Phi_t(x) \right) - f \left( x - \frac{h}{2} \Phi_t(x) \right) \right\} \right\|_{L_\infty(|x| \leq 2L)} = 0.
\]
Therefore, we have the result. \(\square\)

**Lemma 3.6 (cf. [2] Lemma 4.4).** Let \( g \) be a real valued function on \( \mathbb{R} \) satisfying \( \|gw\|_{L_\infty(\mathbb{R})} < \infty \) and
\[
(3.9) \quad \int_{-\infty}^{\infty} gPw^2dt = 0 \quad P \in \mathcal{P}_n.
\]
Then we have
\[
(3.10) \quad \left\| w(x) \int_{0}^{x} g(t)dt \right\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} \| gw \|_{L_\infty(\mathbb{R})}.
\]
Especially, if \( w \in \mathcal{F}_\lambda(C^{3+}), 0 < \lambda < 3/2, \) then we have
\[
(3.11) \quad \left\| w(x) \int_{0}^{x} (f'(t) - v_n(f')(t)) dt \right\|_{L_\infty(\mathbb{R})} \leq C \frac{a_n}{n} E_\lambda \left( w_{1/4}, f' \right).
\]
When \( w \in \mathcal{F}^* \), we also have \((3.11)\) replacing \( w_{1/4} \) with \( w \).

**Proof.** We let
\[
(3.12) \quad \phi_x(t) = \begin{cases} \frac{w^{-2}(t)}{x}, & 0 \leq t \leq x; \\ 0, & \text{otherwise}, \end{cases}
\]
then we have for arbitrary \( P_n \in \mathcal{P}_n, \)
\[
(3.13) \quad \left| \int_{0}^{x} g(t)dt \right| = \left| \int_{-\infty}^{\infty} g(t)\phi_x(t)w^2(t)dt \right| = \left| \int_{-\infty}^{\infty} g(t)(\phi_x(t) - P_n(t))w^2(t)dt \right|.
\]
Therefore, we have
\[
\left| \int_{0}^{x} g(t)dt \right| \leq \|gw\|_{L_\infty(\mathbb{R})} \inf_{P_n \in \mathcal{P}_n} \int_{-\infty}^{\infty} |\phi_x(t) - P_n(t)| w(t)dt
\]
\[
= \|gw\|_{L_\infty(\mathbb{R})} E_{1,n}(w : \phi_x).
\]
Here, from Theorem 3.6 we see that
\[ E_{1,n}(w : \phi_x) \leq C_{a_n} \frac{a_n}{n} \int_{-\infty}^{\infty} w(t) |d\phi_x(t)| \leq C_{a_n} \frac{a_n}{n} \int_{0}^{x} w(t) |Q'(t)| w^{-2}(t) dt \]
\[ = C_{a_n} \frac{a_n}{n} \int_{0}^{x} Q'(t) w^{-1}(t) dt \leq C_{a_n} \frac{a_n}{n} w^{-1}(x). \]

So, we have
\[ \left| w(x) \int_{0}^{x} g(t) dt \right| \leq \|gw\|_{L_{\infty}(\mathbb{R})} w(x) E_{1,n}(w : \phi_x) \leq C_{a_n} \frac{a_n}{n} \|gw\|_{L_{\infty}(\mathbb{R})}. \]

Therefore, we have (3.10). Next we show (3.11). Since
\[ v_n(f')(t) = \frac{1}{n} \sum_{j=n+1}^{2n} s_j(f', t), \]
and for any \( P \in \mathcal{P}_n, j \geq n + 1, \)
\[ \int_{-\infty}^{\infty} (f'(t) - s_j(f'; t)) P(t) w^2(t) dt = 0, \]
we have
\[ \int_{-\infty}^{\infty} (f'(t) - v_n(f')(t)) P(t) w^2(t) dt = 0. \]

Using (3.10) and (3.1), we have (3.11). □

**Lemma 3.7.** Let \( w = \exp(-Q) \in F_\lambda(C^3 +), 0 < \lambda < 3/2. \) Let \( \|w_{1/4} f\|_{L_{\infty}(\mathbb{R})} < \infty, \) and let \( q_{n-1} \in \mathcal{P}_n \) be the best approximation of \( f' \) with respect to the weight \( w, \) that is,
\[ \|(f' - q_{n-1}) w\|_{L_{\infty}(\mathbb{R})} = E_{n-1}(w, f'). \]

Now we set
\[ F(x) := f(x) - \int_{0}^{x} q_{n-1}(t) dt, \]
then there exists \( S_{2n} \in \mathcal{P}_{2n} \) such that
\[ \|w (F - S_{2n})\|_{L_{\infty}(\mathbb{R})} \leq C_{a_n} \frac{a_n}{n} E_n(w_{1/4}, f'), \]
and
\[ \|w S_{2n}\|_{L_{\infty}(\mathbb{R})} \leq CE_{n-1}(w_{1/4}, f'). \]
When \( w \in \mathcal{F}^*, \) we also have same results replacing \( w_{1/4} \) with \( w. \)

**Proof.** Let
\[ (3.14) \quad S_{2n}(x) = f(0) + \int_{0}^{x} v_n(f' - q_{n-1}) (t) dt, \]
then by Lemma 3.6 (3.11),
\[ \|w (F - S_{2n})\|_{L_{\infty}(\mathbb{R})} \]
\[ = \|w \left( f - \int_{0}^{x} q_{n-1}(t) dt - f(0) - \int_{0}^{x} v_n(f' - q_{n-1}) (t) dt \right)\|_{L_{\infty}(\mathbb{R})} \]
\[ = \|w \left( \int_{0}^{x} [f'(t) - v_n(f')(t)] dt \right)\|_{L_{\infty}(\mathbb{R})} \leq C_{a_n} \frac{a_n}{n} E_n(w_{1/4}, f'). \]
Now by Proposition \[3.1\] and (\[3.1\]),
\[
\|uw^2\|_{L^\infty(\mathbb{R})} = \|w(v_n(f' - q_{n-1}))\|_{L^\infty(\mathbb{R})}
\leq \|w(v_n(f') - q_{n-1})\|_{L^\infty(\mathbb{R})} + \|v_n(f' - q_{n-1})\|_{L^\infty(\mathbb{R})}
\leq E_n(w_{1/4}, f') + E_{n-1}(w, f') \leq E_{n-1}(w_{1/4}, f') .
\]

To prove Theorem 2.3. we need the following theorems.

**Theorem 3.8** ([9 Corollary 3.4]). Let \( w \in F(C^2) \), and let \( r > 0 \) be an integer. Let \( 1 \leq p \leq \infty \), and let \( w f^{(r)} \in L_p(\mathbb{R}) \). Then we have
\[
E_{p,n}(f, w) \leq C\left(\frac{a_n}{n}\right)^k \|f^{(k)}\|_{L_p(\mathbb{R})}, \quad k = 1, 2, \ldots, r,
\]
and equivalently,
\[
E_{p,n}(f, w) \leq C\left(\frac{a_n}{n}\right)^k E_{p,n-k}(f^{(k)}, w).
\]

**Theorem 3.9** ([10 Corollary 6.2]). Let \( r \geq 1 \) be an integer and \( w \in F_s(C^{r+2}) \), \( 0 < \lambda < (r + 2)/(r + 1) \), and let \( 1 \leq p \leq \infty \). Then there exists a constant \( C > 0 \) such that for any \( 1 \leq k \leq r \), any integer \( n \geq 1 \) and any polynomial \( P \in \mathcal{P}_n \),
\[
\|P^{(k)} w\|_{L_p(\mathbb{R})} \leq C\left(\frac{n}{a_n}\right)^k T^{k/2} P_w \|_{L_p(\mathbb{R})} .
\]

**Proof of Theorem 2.3.** We prove the theorem only in case of unbounded \( T(x) \), in the case of Freud case \( F^* \) we can prove it similarly. We show that for \( k = 0, 1, \ldots, r \),
\[
(3.15) \quad \left|\left(f^{(k)} - P^{(k)}_{n,f,w}\right) w(x)\right| \leq C T^{k/2}(x) E_{n-k}(w_{1/4}, f^{(k)}).
\]

If \( r = 0 \), then (3.15) is trivial. For some \( r \geq 0 \) we suppose that (3.15) holds, and let \( f \in C^{(r+1)}(\mathbb{R}) \). Then \( f' \in C^{(r)}(\mathbb{R}) \). Let \( q_n \in \mathcal{P}_{n-1} \) be the polynomial of best approximation of \( f' \) with respect to the weight \( w \). Then, from our assumption we have for \( 0 \leq k \leq r \),
\[
(3.16) \quad \left|\left(f^{(k+1)} - q_{n-1}^{(k)}\right) w(x)\right| \leq C T^{k/2}(x) E_{n-k}(w_{1/4}, f^{(k+1)}),
\]
that is, for \( 1 \leq k \leq r + 1 \)
\[
(3.17) \quad F(x) := f(x) - \int_0^x q_{n-1}(t) dt = f(x) - Q_n(x),
\]
then
\[
|F'(x) w(x)| \leq CE_{n-1}(w, f') .
\]

As 3.14 we set \( S_{2n} = \int_0^x (v_n f'(t) - q_{n-1}(t)) dt + f(0) \), then from Lemma 3.7
\[
(3.18) \quad \|F - S_{2n}\|_{L^\infty(\mathbb{R})} \leq C\left(\frac{a_n}{n}\right) E_{n-1}(w_{1/4}, f'),
\]
and
\[
\|S_{2n}' w\|_{L^\infty(\mathbb{R})} \leq CE_{n-1}(w_{1/4}, f') .
\]
Here we apply Theorem 3.9 with the weight \( w_{-(k-1)/2} \). In fact, by Theorem 2.2 we have \( w_{-(k-1)/2} \in \mathcal{F}_\lambda(C^{r+2}+) \). Then, noting \( a_{2n} \sim a_n \) from Lemma 3.2 (1), we see

\[
|S_{2n}^{(k)}(x)w_{-(k-1)/2}(x)| \leq C \left( \frac{n}{a_n} \right)^{k-1} \|S_{2n}'w\|_{L_\infty(\mathbb{R})} \\
\leq C \left( \frac{n}{a_n} \right)^{k-1} E_{n-1}(w_{1/4}, f'),
\]

that is,

\[
(3.19) \quad |S_{2n}^{(k)}(x)w(x)| \leq C \left( \frac{n\sqrt{T(x)}}{a_n} \right)^{k-1} E_{n-1}(w_{1/4}, f'), \quad 1 \leq k \leq r + 1.
\]

Let \( R_n \in \mathcal{P}_n \) denote the polynomial of best approximation of \( F \) with \( w \). By Theorem 3.9 with \( w_{-\frac{a}{2}} \) again, for \( 0 \leq k \leq r + 1 \) we have

\[
|R_n^{(k)} - S_{2n}^{(k)}(x)w_{-\frac{a}{2}}(x)| \leq C \left( \frac{n}{a_n} \right)^k \|R_n - S_{2n}w_{-\frac{a}{2}}(x)T^{k/2}(x)\|_{L_\infty(\mathbb{R})} \\
\leq C \left( \frac{n}{a_n} \right)^k \|R_n - S_{2n}w\|_{L_\infty(\mathbb{R})},
\]

and by \( 3.18 \)

\[
\|R_n - S_{2n}w\|_{L_\infty(\mathbb{R})} \leq C \left[ \|F - R_nw\|_{L_\infty(\mathbb{R})} + \|F - S_{2n}w\|_{L_\infty(\mathbb{R})} \right] \\
\leq C \left[ \frac{\alpha_n}{n} E_n(w_{1/4}, f') + \frac{\alpha_n}{n} E_{n-1}(w_{1/4}, f') \right] \\
\leq C \frac{\alpha_n}{n} E_{n-1}(w_{1/4}, f').
\]

Hence, from \( 3.20 \) and \( 3.21 \) we have for \( 0 \leq k \leq r + 1 \)

\[
|(R_n^{(k)} - S_{2n}^{(k)}(x))w(x)| \leq C \left( \frac{n\sqrt{T(x)}}{a_n} \right)^k \alpha_n E_{n-1}(w_{1/4}, f')
\]

\[
(3.22) \quad \leq C \left( \frac{n\sqrt{T(x)}}{a_n} \right)^k \alpha_n E_{n-1}(w_{1/4}, f').
\]

Therefore by \( 3.19, 3.22 \) and Theorem 3.8

\[
|R_n^{(k)}(x)w(x)| \leq C T^{k/2}(x) \left( \frac{n}{a_n} \right)^{k-1} E_{n-1}(w_{1/4}, f') \\
\leq C T^{k/2}(x) E_{n-k}(w_{1/4}, f^{(k)}).
\]

Since \( E_n(F, w) = E_n(w, f) \) and

\[
E_n(F, w) = \|w(F - R_n)\|_{L_\infty(\mathbb{R})} = \|w(f - Q_n - R_n)\|_{L_\infty(\mathbb{R})}
\]

(3.34)
(see (3.17)), we know that \( P_{n;f,w} := Q_n + R_n \) is the polynomial of best approximation of \( f \) with \( w \). Now, from (3.16), (3.17) and (3.23) we have for \( 1 \leq k \leq r + 1, \)
\[
\left| \left( f^{(k)}(x) - P_{n;f,w}^{(k)}(x) \right) w(x) \right| = \left| \left( f^{(k)}(x) - Q_n^{(k)}(x) - R_n^{(k)}(x) \right) w(x) \right| \\
\leq \left| \left( f^{(k)}(x) - q_n^{(k-1)}(x) \right) w(x) \right| + \left| R_n^{(k)}(x) w(x) \right| \\
\leq C T^{k/2} E_{n-k} \left( w_{1/4}, f^{(k)} \right).
\]

For \( k = 0 \) it is trivial. Consequently, we have (3.15) for all \( r \geq 0 \). Moreover, using Theorem 3.8 we conclude Theorem 2.3.

\[ \square \]

**Proof of Corollary 2.7** It follows from Theorem 2.3.

\[ \square \]

**Proof of Corollary 2.8** Applying Theorem 2.3 with \( w_{k/2} \), we have for \( 0 \leq j \leq r \)
\[
\left\| \left( f^{(j)} - P_{n;f,w_{k/2}}^{(j)} \right) w \right\|_{L_\infty(\mathbb{R})} \leq C E_{n-k} \left( w_{(2k+1)/4}, f^{(j)} \right).
\]

Especially, when \( j = k \), we obtain
\[
\left\| \left( f^{(k)} - P_{n;f,w_{k/2}}^{(k)} \right) w \right\|_{L_\infty(\mathbb{R})} \leq C E_{p,n-k} \left( w_{(2k+1)/4}, f^{(k)} \right).
\]

\[ \square \]

4. **Theorems in \( L_p(\mathbb{R}) \) \( (1 \leq p \leq \infty) \)**

In this section we will give an analogy of Theorem 2.3 in \( L_p(\mathbb{R}) \)-space \( (1 \leq p \leq \infty) \) and we will prove it using the same method as the proof of Theorem 2.3.

Let \( 1 \leq p \leq \infty \). Let \( w = \exp(-Q) \in \mathcal{F}_\lambda(C^3+) \), \( 0 < \lambda < 3/2 \), and let \( \beta > 1 \) be fixed. Then we set \( w^\sharp \) and \( w^\flat \) as follows;
\[
\begin{align*}
w(x) & \sim w^\sharp(x) \in \mathcal{F}(C^2+) ; \\
w(x) & \sim w^\flat(x) \in \mathcal{F}(C^2+) \end{align*}
\]

(see Theorem 2.2).

**Theorem 4.1.** Let \( r \geq 0 \) be an integer. Let \( w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+2}+) \), \( 0 < \lambda < (r + 2)/(r + 1) \), and let \( \beta > 1 \) be fixed. Suppose that \( T^{1/4} f^{(r)} w \in L_p(\mathbb{R}) \).

Let \( P_{p,n;f,w} \in P_n \) be the best approximation of \( f \) with respect to the weight \( w \) in \( L_p(\mathbb{R}) \)-space, that is,
\[
E_{p,n} (w, f) := \inf_{P \in P_n} \left\| (f - P) w \right\|_{L_p(\mathbb{R})} = \left\| (f - P_{p,n;f,w}) w \right\|_{L_p(\mathbb{R})}.
\]

Then there exists an absolute constant \( C_r > 0 \) which depends only on \( r \) such that for \( 0 \leq k \leq r \) and \( x \in \mathbb{R} \),
\[
\left\| \left( f^{(k)} - P_{p,n;f,w}^{(k)} \right) w_{-k/2}^\sharp \right\|_{L_p(\mathbb{R})} \leq C_r E_{p,n-k} \left( w_{1/4}, f^{(k)} \right) \\
\leq C_r \left( \frac{a_n}{b_n} \right)^{r-k} E_{p,n-r} \left( w_{1/4}, f^{(r)} \right).
\]

When \( w \in \mathcal{F}^* \), we can replace \( w_{1/4} \) and \( w^\sharp_{-k/2} \) with \( w \) and \( w^\sharp \), respectively in the above.

If we apply Theorem 4.1 with \( w_{-1/4} \), then we have the following.
Corollary 4.2. Let $r \geq 0$ be an integer. Let $w = \exp(-Q) \in F_\lambda(C^{r+3+})$, $0 < \lambda < (r+3)/(r+2)$, and let $\beta > 0$ be fixed. Suppose that $w^r fb^r \in L_p(\mathbb{R})$. Then for $0 \leq k \leq r$ we have

$$\left\| (f^{(k)} - P^{(k)}_{p,n,f,w_{k/2}}) w_r \right\|_{L_p(\mathbb{R})} \leq C_r E_{p,n-k} \left( w, f^{(k)} \right) \leq C_r \left( \frac{\alpha_n}{n} \right)^{r-k} E_{p,n-r} \left( w, f^{(r)} \right).$$

When $w \in F^*$, we can omit $T^{-2(k+1)/4}$ in the above.

Corollary 4.3. Let $r \geq 0$ be an integer. Let $w = \exp(-Q) \in F_\lambda(C^{r+3+})$, $0 < \lambda < (r+3)/(r+2)$, and let $\beta > 0$ be fixed.

1. Let $w_{(2r+1)/4} f^{(r)} \in L_p(\mathbb{R})$. Then, for each $k (0 \leq k \leq r)$ and the best approximation polynomial $P_{p,n,f,w_k/2}$, we have

$$\left\| \left( f^{(k)} - P^{(k)}_{p,n,f,w_k/2} \right) w_r \right\|_{L_p(\mathbb{R})} \leq C_r E_{p,n-k} \left( w_{(2k+1)/4}, f^{(k)} \right) \leq C_r \left( \frac{\alpha_n}{n} \right)^{r-k} E_{p,n-r} \left( w_{(2k+1)/4}, f^{(r)} \right).$$

2. Let $w_{(2r+1)/4} f^{(r)} \in L_p(\mathbb{R})$. Then, for each $k (0 \leq k \leq r)$ and the best approximation polynomial $P_{p,n,f,w_k/2}$, we have

$$\left\| \left( f^{(k)} - P^{(k)}_{p,n,f,w_k/2} \right) w_r \right\|_{L_p(\mathbb{R})} \leq C_r E_{p,n-k} \left( w_{(2k+1)/4}, f^{(k)} \right) \leq C_r \left( \frac{\alpha_n}{n} \right)^{r-k} E_{p,n-r} \left( w_{(2k+1)/4}, f^{(r)} \right).$$

When $w \in F^*$, we can replace $w_{(2k+1)/4}$ and $w_{(2k+1)/4}$ with $w$ and $w^r$, respectively in the above.

Especially when $p = \infty$, we can refer to $w^r$ or $w^r$ as $w$. In this case, we can note that Corollary 4.2 and Corollary 4.3 imply Corollary 2.4 and Corollary 2.5 respectively.

To prove Theorem 4.4 we need to prepare some notations and lemmas.

Lemma 4.4 (R). Let $w \in F(2^r+)$ and let $1 \leq p \leq \infty$. If $g : \mathbb{R} \to \mathbb{R}$ is absolutely continuous, $g(0) = 0$, and $w^f g \in L_p(\mathbb{R})$, then

$$\|Q^f w g\|_{L_p(\mathbb{R})} \leq C \|w^f g\|_{L_p(\mathbb{R})}.$$  

Lemma 4.5 (cf. [9] Theorem 4.1). Let $w \in F(2^r+)$ and let $1 \leq p \leq \infty$. If $w^f \in L_p(\mathbb{R})$, then

$$E_{p,n}(w, f) \leq C \omega_p \left( f, w, \frac{\alpha_n}{n} \right) \leq C \frac{\alpha_n}{n} \|w^f\|_{L_p(\mathbb{R})}.$$  

Proof. The first inequality follows from Proposition 4.3. We show the second inequality. By [9] Lemma 7] we have

$$\left\| w(x) \left\{ f \left( x + \frac{h}{2} \Phi_{t}(x) \right) - f \left( x - \frac{h}{2} \Phi_{t}(x) \right) \right\} \right\|_{L_p(|x| \leq \sigma(2t))}^p \leq h^p \int_{\mathbb{R}} |w(x)\Phi_{t}(x)f'(x)|^p dx \leq Ch^p \int_{\mathbb{R}} |w(x)f'(x)|^p dx.$$
Hence we see
\begin{equation}
(4.1) \quad \sup_{0 < h \leq t} \| w(x) \left\{ f \left( x + \frac{h}{2} \Phi_t(x) \right) - f \left( x - \frac{h}{2} \Phi_t(x) \right) \right\|_{L_p(|x| \leq \sigma(2t))} \leq C t \| w' \|_{L_p(\mathbb{R})}.
\end{equation}

Now, we estimate
\[ \| w(x)(f - c)(x) \|_{L_p(|x| \geq \sigma(4t))}. \]

Let \( g(x) := f(x) - f(0) \).
\[ \inf_{c \in \mathbb{R}} \| w(x)(f - c)(x) \|_{L_p(|x| \geq \sigma(4t))} \leq \frac{1}{Q'(\sigma(4t))} \| Q'w \|_{L_p(\mathbb{R})}. \]

Then we have from \((3.9)\),
\begin{equation}
(4.2) \quad \inf_{c \in \mathbb{R}} \| w(x)(f - c)(x) \|_{L_p(|x| \geq \sigma(4t))} \leq C t \| Q'w \|_{L_p(\mathbb{R})}.
\end{equation}

Here, from Lemma \(4.3\) we have
\begin{equation}
(4.3) \quad \| Q'w(f - f(0)) \|_{L_p(\mathbb{R})} \leq C \| w' \|_{L_p(\mathbb{R})}.
\end{equation}

From \((4.2)\) and \((4.3)\) we have
\begin{equation}
(4.4) \quad \inf_{c \in \mathbb{R}} \| w(x)(f - c)(x) \|_{L_p(|x| \geq \sigma(4t))} \leq C t \| w' \|_{L_p(\mathbb{R})}.
\end{equation}

From \((4.1)\) and \((4.4)\) we have the result. \(\Box\)

**Lemma 4.6** (cf. [1] Lemma 4.4). Let \( 1 \leq p \leq \infty \) and \( \beta > 1 \), and let us define \( w^q \) with \( p, \beta \). Let \( g \) be a real valued function on \( \mathbb{R} \) satisfying \( \| gw \|_{L_p(\mathbb{R})} < \infty \) and \( \| Q'w \|_{L_p(\mathbb{R})} \leq C \| w' \|_{L_p(\mathbb{R})} \), then we have
\begin{equation}
(4.5) \quad \left\| w^q \right\|_{L_p(\mathbb{R})} \leq C \frac{a_n}{n} \| gw \|_{L_p(\mathbb{R})}.
\end{equation}

Especially, if \( w \in F_\lambda(C^3+) \), \( 0 < \lambda < 3/2 \), then we have
\begin{equation}
(4.6) \quad \left\| w^q \right\|_{L_p(\mathbb{R})} \leq C \frac{a_n}{n} E_{p,n} \left( w_{1/4}, f' \right).
\end{equation}

When \( w \in F^\ast \), we also have \((4.6)\) replacing \( w_{1/4} \) with \( w \).

**Proof.** For arbitrary \( P_n \in \mathcal{P}_n \), we have by \((3.13)\) and Hölder inequality
\[ \left\| \int_0^x g(t) dt \right\|_{L_p(\mathbb{R})} \leq \| g(t)w(t) \|_{L_p(\mathbb{R})} E_{q,n} \left( w, \Phi_x \right), \quad 1 \leq p \leq \infty, 1/p + 1/q = 1, \]
where \( \phi \) is defined in \((3.12)\). Then, we obtain by Lemma \(4.6\)
\[ \left\| \int_0^x g(t) dt \right\|_{L_p(\mathbb{R})} \leq \| g(t)w(t) \|_{L_p(\mathbb{R})} \frac{a_n}{n} \left( \int_{\mathbb{R}} |w(t)\phi_x'(t)|q dt \right)^{1/q} \]
\[ \leq \frac{a_n}{n} \| g(t)w(t) \|_{L_p(\mathbb{R})} |Q'(x)|^{1-1/q} \left( \int_0^x Q'(t)w^{-q}(t) dt \right)^{1/q} \]
\[ \leq C \frac{a_n}{n} \| g(t)w(t) \|_{L_p(\mathbb{R})} |Q'(x)|^{1-1/q} w^{-1}(x). \]

Here, for \( p = 1 \) we may consider
\[ \lim_{q \to \infty} \left( \int_0^x Q'(t)w^{-q}(t) dt \right)^{1/q} = \lim_{q \to \infty} \left( w^{-q}(t) \right)^{1/q} = w^{-1}(x). \]
Hence, we have
\[ \left\| \frac{w(x)}{(1 + |Q'(x)|)(1 + |x|^\beta)} \right\|_{L_p(\mathbb{R})} \leq C \left( \frac{a_n}{n} \right) \left\| (1 + |x|)^{-\beta/p} \right\|_{L_p(\mathbb{R})} \| gw \|_{L_p(\mathbb{R})} \]
\[ \leq C \left( \frac{a_n}{n} \right) \| gw \|_{L_p(\mathbb{R})} , \]
Therefore, we have (4.5). From (4.5), (3.9) and Proposition 3.1, we have (4.6). \( \square \)

**Lemma 4.7.** Let \( w = \exp(-Q) \in \mathcal{F}_\lambda(C^3+) \), \( 0 < \lambda < 3/2 \). Let \( 1 \leq p \leq \infty \), \( \| w_{1/4} f' \|_{L_p(\mathbb{R})} < \infty \), and let \( q_{n-1} \in \mathcal{P}_n \) be the best approximation of \( f' \) with respect to the weight \( w \) on \( L_p(\mathbb{R}) \) space, that is,
\[ \| (f' - q_{n-1})w \|_{L_p(\mathbb{R})} = E_{p,n-1}(w,f'). \]
Using \( q_{n-1} \), define \( F(x) \) and \( S_{2n} \) as \( (3.17) \) and \( (3.14) \). Then we have
\[ \| w^2 (F - S_{2n}) \|_{L_p(\mathbb{R})} \leq C \frac{a_n}{n} E_{p,n} \left( w_{1/4}, f' \right) , \]
and
\[ \| w S_{2n}^\prime \|_{L_p(\mathbb{R})} \leq CE_{p,n-1} \left( w_{1/4}, f' \right) . \]
When \( w \in \mathcal{F}^* \), we also have same results replacing \( w_{1/4} \) with \( w \).

**Proof.** By Lemma 4.6, we have the result using the same method as the proof of Lemma 3.7. \( \square \)

**Proof of Theorem 4.1.** We will prove it similarly to the proof of Theorem 2.3. First, let \( q_{n-1} \in \mathcal{P}_n \) be the polynomial of best approximation of \( f' \) with respect to the weight \( w \) on \( L_p(\mathbb{R}) \) space. Then using \( q_{n-1} \), we define \( F(x) \) and \( S_{2n} \) in the same method as \( (3.17) \) and \( (3.14) \). Then we have using Lemma 4.7
\[ \| F' w \|_{L_p(\mathbb{R})} = E_{p,n-1}(w,f') , \]
\[ \| (F - S_{2n}) w^2 \|_{L_p(\mathbb{R})} \leq C \frac{a_n}{n} E_{p,n} \left( w_{1/4}, f' \right) \]
and
\[ (4.7) \quad \| S_{2n}^\prime w \|_{L_p(\mathbb{R})} \leq CE_{p,n-1} \left( w_{1/4}, f' \right) . \]
Then we see from Theorem 3.9 and (4.7),
\[ (4.8) \quad \| S_{2n}^{(k)} w_{-(k-1)/2} \|_{L_p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^{k-1} E_{p,n-1} \left( w_{1/4}, f' \right) \]
and using \( w^2 \leq w \) and Theorem 3.8
\[ (4.9) \quad \| (R_{2n}^{(k)} - S_{2n}^{(k)}) w_{-(k-1)/2} \|_{L_p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^{k-1} E_{p,n-1} \left( w_{1/4}, f' \right) , \]
where \( R_{2n} \in \mathcal{P}_n \) denotes the polynomial of best approximation of \( F \) with \( w \) on \( L_p(\mathbb{R}) \) space (by the similar calculation as \( (3.20) \) and \( (3.21) \)). Then, we see \( w_{-(k-1)/2}^\prime(x) \leq w_{-(k-1)/2}^\prime(x) \). By (4.8) and (4.9) and Theorem 3.8 we have
\[ (4.10) \quad \| R_{2n}^{(k)} w_{-(k-1)/2} \|_{L_p(\mathbb{R})} \leq C \left( \frac{n}{a_n} \right)^{k-1} E_{p,n-1} \left( w_{1/4}, f' \right) \leq CE_{p,n-k} \left( w_{1/4}, f^{(k)} \right) . \]
By the same reason to [3.24], we know that $P_{p,n,f,w} := Q_n + R_n$ is the polynomial of best approximation of $f$ with $w$ on $L_p(\mathbb{R})$ space. Therefore, using $P_{p,n,f,w}$, \cite{10} and the method of mathematical induction, we have for $1 \leq k \leq r + 1$,

$$\left\| (f^{(k)} - P_{p,n,f,w}^{(k)}) w_{r/2}^k \right\|_{L_p(\mathbb{R})} \leq CE_{p,n-k} \left( w_{1/4}, f^{(k)} \right).$$

□

Proof of Corollary 4.2 It follows from Theorem 4.1.

Proof of Corollary 4.3 If we apply Theorem 4.1 with $w_{1/4}$ and $w_{r/2}^k$, then we can obtain the results.

5. Monotone Approximation

Let $r > 0$ be an integer. Let $k$ and $\ell$ be integers with $0 \leq k \leq \ell \leq r$. In this section, we consider a real function $f$ on $\mathbb{R}$ such that $f^{(r)}(x)$ is continuous in $\mathbb{R}$ and we let $a_j(x)$, $j = k, k + 1, \ldots, \ell$ be bounded on $\mathbb{R}$.

Now, we define the linear differential operator (cf. [1])

\begin{equation}
L := L_{k,\ell} := \sum_{j=k}^{\ell} a_j(x)[d^j / dx^j].
\end{equation}

G. A. Anastassiou and O. Shisha \cite{1} consider the operator (5.1) with $a_j(x)$ under some condition on $[-1, 1]$. They showed that if $L(f) \geq 0$ for $f \in C^{(r)}[-1, 1]$, there exist $Q_n \in \mathcal{P}_n$ such that $L(Q_n) \geq 0$ and for some constant $C > 0$,

$$\|f - Q_n\|_{L_\infty([-1, 1])} \leq C n^{\ell - r} \omega \left( f^{(p)}; 1 / n \right),$$

where $\omega (f^{(p)}; t)$ is the modulus of continuity. In this section, we will obtain a similar result with exponential-type weighted $L_\infty$-norm as the above result. Our main theorem is as follows.

Theorem 5.1. Let $k$ and $\ell$ be integers with $0 \leq k \leq \ell \leq r$. Let $w = \exp(-Q) \in \mathcal{F}_\lambda(C^{r+3}+)$, and let $T(x)$ be continuous on $\mathbb{R}$. Suppose that $w(x)f^{(r)}(x) \to 0$ as $|x| \to \infty$. Let $P_{n,f,w_{-1/4}} \in \mathcal{P}_n$ be the best approximation for $f$ with the weight $w_{-1/4}$ on $\mathbb{R}$. Suppose that for a certain $\delta > 0$,

$$L(f; x) \geq \delta, \ x \in \mathbb{R}.$$

Then, for every integer $n \geq 1$ and $j = 0, 1, \ldots, \ell$,

\begin{equation}
\left\| \left( f^{(j)} - P_{n,f,w_{-1/4}}^{(j)} \right) wT^{-2j+1/4} \right\|_{L_\infty(\mathbb{R})} \leq C_j \left( \frac{\alpha n}{\delta} \right)^{r-j} E_n^{r-j} \left( w, f^{(r)} \right),
\end{equation}

where $C_j > 0$, $0 \leq j \leq \ell$, are independent of $n$ or $f$, and for any fixed number $M > 0$ there exists a constant $N(M, \ell, \delta) > 0$ such that

\begin{equation}
L(P_{n,f,w_{-1/4}}; x) \geq \frac{\delta}{2}, \ \ |x| \leq M, \ \ n \geq N(M, \ell, \delta).
\end{equation}
Proof. From Corollary 2.4 we have (5.2). Hence, we also have

\[
\left| (L(f;x) - L(P_{n,f,w_{-1/4}};x)) w(x) T^{-(2\ell+1)/4}(x) \right|
\]

\[
= \sum_{j=k}^{\ell} a_j(x) \left\{ f^{(j)}(x) - P_{n,f,w_{-1/4}}^{(j)}(x) \right\} w(x) T^{-(2\ell+1)/4}(x)
\]

\[
\leq E_{n-r} \left( w, f^{(r)} \right) \sum_{j=k}^{\ell} |a_j(x)| C_j \left( \frac{a_n}{n} \right)^{r-j}
\]

\[
\leq C_{k,\ell} \left( \frac{a_n}{n} \right)^{r-\ell} E_{n-r} \left( w, f^{(r)} \right),
\]

where we set \( C_{k,\ell} := \sum_{j=k}^{\ell} \|a_j\|_{L_\infty(x)} C_j \). Then we have for \(|x| \leq M\)

\[
\left| L(f;x) - L(P_{n,f,w_{-1/4}};x) \right|
\]

\[
\leq C_{k,\ell} \left\| w^{-1}(x) T^{(2\ell+1)/4}(x) \right\|_{L_\infty(|x| \leq M)} \left( \frac{a_n}{n} \right)^{r-\ell} E_{n-r} \left( w, f^{(r)} \right).
\]

Here, for \( \delta > 0 \) there exists \( N(M, \ell, \delta) > 0 \) such that for \( n \geq N(M, \ell, \delta) \)

\[
C_{k,\ell} \left\| w^{-1}(x) T^{(2\ell+1)/4}(x) \right\|_{L_\infty(|x| \leq M)} \left( \frac{a_n}{n} \right)^{r-\ell} E_{n-r} \left( w, f^{(r)} \right) \leq \frac{\delta}{2}.
\]

This follows from \( w(x) f^{(r)}(x) \rightarrow 0 \) as \(|x| \rightarrow \infty\). Therefore we see

\[
\frac{\delta}{2} \leq L(f;x) - \frac{\delta}{2} \leq L(P_{n,f,w_{-1/4}};x).
\]

Consequently, we have (5.3). \( \square \)

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