On stress singularities at plane bi- and tri-material junctions - A way to derive some closed-form analytical solutions

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Abstract

Stress singularities at 2D bi- and tri-material junctions, consisting of dissimilar, homogeneous, isotropic and linear-elastic wedges under a plane strain state are considered. The stresses formed at the vertex of this composite situation are analyzed by the complex variable method, based on an appropriate choice of the Kolosov-potentials which are applicable in the vicinity of the vertex. In doing so, the identification of the singularity exponent is performed. With the help of a novel approach it is demonstrated how to derive some solutions for the orders of the stress singularities at bi- and tri-material combinations in a closed-form analytical manner.

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1. Introduction

In various exact solutions of boundary value problems in linear elasticity the stress field is found to have singularities, e.g. when discontinuities are present in the geometry or mechanical properties of the material. In many technical areas dissimilar materials have to be joined together, and the stress state around points, where several materials meet can be expected to be singular. Although it might in general not be sufficient to consider only the orders of the singularities, the type of singularity affects the structural strength of elastic materials in a decisive manner, because these singularities correspond to locations of high stress from which the initiation of fracture is prone to occur. For an assessment in the scope of linear elasticity the knowledge of the orders of the stress singularities is of particular interest.

There is a multitude of contributions related to stress singularities in linear elastic media and a review article on notch problems and crack stress-singularities has been given by Paggi and Carpinteri [1]. Some
early works related to stress singularities in linear elastic media are from Williams [2], Bogy [3,4], Hein and Erdogan [5], England [6] and Theocaris [7]. Dempsey and Sinclair reported and studied power-logarithmic singularities in a series of papers [8,9,10]. Recently, there is a renewed interest in singularity problems of linear elasticity, because solutions of such problems can e.g. be useful for checking the effectiveness of new numerical approaches for anisotropic multi-material junctions by comparison with limit solutions.

In all works dealing with singularity analyses, to the best knowledge of the authors of this contribution, the results are at last obtained numerically. In this work it is demonstrated with the help of a novel approach how to derive some solutions for the orders of the stress singularities at bi- and tri-material combinations in a closed-form analytical manner. From different techniques available for singularity analyses, the complex variable method has been chosen to analyze the stresses formed at the vertex of a plane multi-material composite situation. Based on an appropriate choice of the Kolosov-potentials (Theocaris [7]) which are applicable in the vicinity of the vertex, the orders of the stress singularities are determined.

2. Theoretical Setting

As shown in Fig. 1a, the general geometry consisting of dissimilar, homogeneous, isotropic and linear elastic (Young's moduli $E_k$ and Poisson's ratios $\nu_k$) material sectors under a plane strain state is considered. The interfaces between the respective sectors are perfectly bonded and the stresses and displacements are expressed in terms of two complex potentials $\Phi_k$ and $\Psi_k$ consistent with the complex potential method given by Muskhelishvili [11].

$$\text{Fig. 1a) } n\text{-material-junction; b) } n\text{-material-junction with notch; c) a special bi-material configuration}$$

It is convenient to formulate stress continuity and stress boundary conditions with the help of stress resultants. Omitting further details, the continuity conditions at a perfectly bonded interface $\Gamma_k$ ($\varphi = \varphi_k$) become

\begin{align}
\left[\Phi_k(z) + z\Phi_k'(z) + \Psi_k(z)\right]_{z=0}^{z=z_k} &= \left[\Phi_{k+1}(z) + z\Phi_{k+1}'(z) + \Psi_{k+1}(z)\right]_{z=0}^{z=z_k}, \quad (1) \\
\frac{1}{\mu_k} \left[k_k \Phi_k(z_k) - z_k \Phi_k'(z_k) - \Psi_k(z_k)\right] &= \frac{1}{\mu_{k+1}} \left[k_{k+1} \Phi_{k+1}(z_k) - z_k \Phi_{k+1}'(z_k) - \Psi_{k+1}(z_k)\right], \quad (2)
\end{align}

where the subscripts $k$ and $k+1$ define quantities or functions referred to the sectors $k$ and $k+1$, respectively. Furthermore $z = re^{i\theta}$ and $z_k = re^{i\theta_k}$ hold, $\mu = E/(2(1+\nu))$ denotes the shear modulus, and if $\nu$ is Poisson’s ratio, then the Kolosov-constant $\kappa$ takes on the value $3-4\nu$ for plane strain and $(3-\nu)/(1+\nu)$ for plane stress. Equation (1) enforces stress continuity along the interface $\Gamma_k$ between the
two materials $k$ and $k+1$ and equation (2) enforces displacement continuity along the same interface. If an $n$-material-junction with notch is considered, as shown in Fig. 1b, the stress-free surfaces $\Gamma_0$ and $\Gamma_n$ require:

$$\left[\Phi_1(z) + z\Phi'_1(z) + \Psi_1(z)\right]_{z=0} = 0 \quad , \quad \left[\Phi_n(z) + z\Phi'_n(z) + \Psi_n(z)\right]_{z=0} = 0 \quad . \quad (3) \text{ and } (4)$$

In case of an $n$-material-junction as in Fig. 1a, the possibility of a branch cut must be taken into account. This can be done without loss of generality by defining $\phi_0 = 0$ for material 1 and $\phi_n = 2\pi$ for material $n$. In order to investigate the asymptotic behaviour of the stress distribution in the vicinity of the vertex of the composite, complex functions of the form

$$\Phi_k(z) = a_{1k}z^\lambda + a_{2k}z^{-\lambda} \quad \text{ and } \quad \Psi_k(z) = h_{1k}z^\lambda + h_{2k}z^{-\lambda} \quad \text{ with } \lambda, a_{1k}, h_{2k} \in \mathbb{C} \quad \quad (5)$$

are assumed, following an idea of Theocaris [7]. Again, the subscript $k$ defines quantities referred to the sector $k$. Introducing the potentials (5) into any of the conditions (1) - (4) leads to equations that must hold for every value of the variable $r$ and thus the coefficients for $r^z$ and $r^{-z}$ must be equal to zero. In this manner two equations can be gained and after conjugating the second equation, two additional homogeneous equations for the unknown constants $a_{1k}, a_{2k}, h_{1k}, h_{2k}$ will result ([7] gives further details). Finally there will be $4n$ homogeneous equations for the unknowns $a_{1k}, a_{2k}, h_{1k}, h_{2k}$, $\{k = 1, 2, 3, ..., n\}$ and for a non-trivial solution of this system of equations $B \cdot \alpha = 0$ the determinant of the coefficient matrix $B$ must be equal to zero. $\det B = 0$ yields the characteristic equation, from which the roots (or "eigenvalues") $\lambda$ can be gained. The general expression of the stress-field is then given by the asymptotic expansion:

$$\sigma_{ij}(r, \varphi) = \sum_{m=1} K_m e^{\text{Re}(\lambda_m) - 1} \left[ \cos(\text{Im}(\lambda_m) \ln r) f_{ij}^\cos(\varphi) + \sin(\text{Im}(\lambda_m) \ln r) f_{ij}^\sin(\varphi) \right] \quad , \quad (6)$$

with the generalized stress-intensity factors $K_m \in \mathbb{R}$. Note, that the orders of the stress singularities are defined by the exponents $\lambda_m$ and an imaginary part of $\lambda_m$ gives an oscillatory term superimposed on the singular term. Singular stresses result, when $\text{Re} \lambda < 1$ and requiring finite strain energy in any region of the body leads to $0 < \text{Re} \lambda < 1$. Another form of singularity that may appear is the power-logarithmic stress singularity, i.e. a singularity of the form $O(r^{\alpha-1} \ln r)$, $\lambda \in \mathbb{R}$, which shall not be taken into account in this study.

3. Examples

As an example the configuration of Fig. 1c is analyzed. This bi-material configuration has already been investigated by Hein and Erdogan [5] using the Mellin transformation technique applied to an Airy stress function formulation. Therefore an in-depth discussion of the configuration can be omitted. All results of [5] were obtained numerically, whereas in the following a novel approach is presented, that allows to derive closed-form analytical solutions in many cases. It is assumed that there are two traction-free boundaries and one perfectly bonded interface. The elastic constants are given by $E_k$ and $\nu_k$ ($k=1,2$) and no branch cut has to be taken into account. The traction-free boundary $\varphi = 0$, the perfectly bonded interface $\varphi = -\alpha$ ($\alpha > 0$) and the traction-free boundary $\varphi = -\pi$ require:

$$a_{11} + \lambda \bar{a}_{21} + \bar{b}_{21} = 0 \quad \text{ and } \quad \lambda a_{11} + b_{11} + \bar{a}_{21} = 0 \quad , \quad (7) \text{ and } (8)$$

$$\frac{1}{\mu_1} \left[ k_1 a_{11} e^{-2ia} - \lambda \bar{a}_{21} e^{-2ia} - \bar{b}_{21} \right] = \frac{1}{\mu_2} \left[ k_2 a_{12} e^{-2ia} - \lambda \bar{a}_{22} e^{-2ia} - \bar{b}_{22} \right] \quad , \quad (9)$$
Equations (9) and (11) are multiplied with $\mu_1 \cdot \mu_2$ and this leads to the corresponding homogeneous system of linear equations $B \cdot a = 0$ with the unknown quantities assembled in $a = [a_{11} \ a_{21} \ b_{11} \ a_{12} \ a_{22} \ b_{12} \ b_{22}]^T$. There will be nontrivial solutions if and only if $\det B = 0$ is fulfilled.

With the modular ratio $\eta := \mu_1 / \mu_2$, the characteristic polynomial $D := 1/(\mu_2^2) \det B$ has been calculated with the help of the computer algebra system Mathematica which took only a few seconds on a standard personal computer. After some simplifications, the following expression results:

$$D = 2\eta^2 \left[ \lambda^2 \left( 2\cos(\pi\alpha)\cos(2\pi\alpha) + 6\sin^2(\alpha) - 2\cos(2\pi\alpha) - 8\lambda^2 \sin^4(\alpha) \right) - 2\sin^2(\alpha) + \kappa_2 \left( \cos(2(\pi - 2\pi)\lambda) + \cos(2\pi\lambda) + 2\cos(2(\pi - \alpha)\lambda)(\lambda^2 - 1) + (2\lambda^2 \cos(2(\pi - 2\pi)\lambda) - 2\sin^2(\alpha))\kappa_2 \right) \right]$$

$$+ 2\eta \left[ 6\lambda^4 \sin^4(\alpha) + 4\lambda^2 \cos(2(\pi - \alpha)\lambda) - 3 + \cos(2\pi\alpha) \right] \sin^2(\alpha) + 1 + \cos(2\pi\lambda) - \cos(2(\pi - \alpha)\lambda) - \cos(2\alpha\lambda) - 2 \sin((2\pi - 3\alpha)\lambda) \sin(\alpha) + 4\lambda^2 \cos(2\pi\alpha) \sin^2(\alpha)$$

$$- \kappa_1 \left[ 4\lambda^2 \cos(2\alpha\lambda) \sin^2(\alpha) + 2\sin(\alpha) \sin((2\pi - 3\alpha)\lambda) + \sin(\alpha) \right] - \left[ 1 - 4\lambda^2 \sin^2(\alpha) + \cos(2\pi\lambda) - \cos(2(\pi - \alpha)\lambda) - \cos(2\alpha\lambda) \right]$$

$$+ 2 \left[ 2\lambda^2 \sin^2(\alpha) - 2\sin^2(\lambda(\pi - \alpha)) \right] \left[ 1 - 4\lambda^2 \sin^2(\alpha) + 2\cos(2\alpha\lambda) \kappa_1 + \kappa_2^2 \right].$$

The studies showed that changes in Poisson’s ratio hardly affect the results, so a variation of $\nu_1$ will not be taken into account. With $\nu_1 = \nu_2 = 2/10$ (hence $\kappa_1 = \kappa_2 = 3 - 8/10$ for plane strain and $\eta = E_1 / E_2$) the characteristic polynomial can be written as $D = D(\alpha, \lambda, \eta) = f_q(\alpha, \lambda) \eta^2 + f_1(\alpha, \lambda) \eta + f_c(\alpha, \lambda)$, with

$$f_q(\alpha, \lambda) = 2^{1/5} \left[ 2\lambda^2 \sin^2(\alpha) - 1 \right] \left[ 1^{1/5} + 2\cos(2(\pi - \alpha)\lambda) + \cos(2(\pi - 2\alpha)\lambda) \cos(2\pi\lambda) + 1^{1/5} \cos(2\alpha\lambda) \right]$$

$$+ \cos(2\alpha\lambda) - 1 + 2\lambda^2 \left[ 3 - 4\lambda^2 \sin^2(\alpha) - 2\cos(2\alpha\lambda) \right] \sin^2(\alpha),$$

$$f_1(\alpha, \lambda) = \sqrt{2} \left[ 200\lambda^4 \sin^4(\alpha) - 4\lambda^2 \left( 98 + 15 \cos(2\alpha\lambda) \right) \sin^2(\alpha) \right.$$

$$\left. - 2\sin(2(\pi - \alpha)\lambda) \sin((2\pi - 3\alpha)\lambda) \right] \left[ 73 + 60\lambda^2 \sin^2(\alpha) \right]$$

$$+ 146 \cos^2(\pi\lambda) - 710 \sin((2\pi - 3\alpha)\lambda) \sin(\alpha) - 110 \sin^2(\lambda(\pi - \alpha)) - 73 \cos(2\alpha\lambda) \right]$$

$$f_c(\alpha, \lambda) = \sqrt{25} \left[ 73 + 55 \cos(2\alpha\lambda) - 50\lambda^2 \sin^2(\alpha) \right] \left[ \lambda^2 \sin^2(\alpha) - \sin^2((\pi - \alpha)\lambda) \right].$$

In the following, some configurations with a constant value $\alpha$ are investigated. It is obvious, that closed-form analytical solutions $\lambda = \lambda(\eta)$ cannot be found, since $D = 0$ is a (highly) non-linear transcendental equation and that is why in all studies known to the authors of this article, the results are at last obtained numerically. However, there is a possibility to derive solutions in a closed-form analytical manner by considering the "inverse" problem and identifying solutions $\eta = \eta(\lambda)$. In this case, solutions can simply be calculated as:

$$\eta_{1/2} = \frac{-f_1 \pm \sqrt{f_1^2 - 4f_q \cdot f_c}}{2f_q}.$$
with \( f_q, f_1, f_2 \) given by (18) - (20). It is needless to say, that only real-valued solutions \( \eta \in \mathbb{R} \) are of interest. There is however a slight restriction since complex roots \( \lambda \) cannot be investigated with this method. In that case all the complex-valued branches were calculated by solving \( D = 0 \) with Newton's method. Calculating the roots turned out to be a bad conditioned problem, so the analysis had to be done very carefully. Summing up, it can be stated, that a well-adjusted Mathematica-implementation enables to do calculations in a robust and highly efficient manner: The pure calculating time for a plot as shown below is only about 4 seconds on a standard PC. The ordinates of all following plots presented are limited to the real part of \( \lambda \) in the range \( 0 < \text{Re}\lambda < 1 \) and corresponding (positive) imaginary parts of the roots are given when the roots are complex. Note, that complex roots always occur as a pair of complex conjugates, but in the plots only the positive imaginary part is displayed. Furthermore, it has to be kept in mind, that \( \sigma_{ij} \sim r^{\text{Re} \lambda - 1} \) holds, thus \( \text{Re} \lambda = 0.5 \) indicates the "classical" square root - stress singularity. In Figs. 2-3 the angle \( \alpha \) of the bi-material configuration takes on the values \( \alpha = 15^\circ, 45^\circ, 60^\circ, 90^\circ \).

Fig. 2) Orders of singularities for depicted configurations

Fig. 3) Orders of singularities for depicted configurations
With the method proposed in this work it is possible to derive closed-form analytical solutions for any other bi-material configuration and in addition for tri-material configurations, too. This possibility to the best of the authors' knowledge is a novelty. It should be noted, that analyzing a tri-material configuration will lead to a quartic equation with the variable $\eta$. It is a well-known fact, that quartic equations can be solved algebraically in terms of a finite number of operations such as addition, subtraction, multiplication, division and root extraction.

4. Summary and Conclusion

In this contribution, stress singularities at two-dimensional multi-material-junctions, consisting of dissimilar, homogeneous, isotropic and linear-elastic wedges under a plane strain state have been considered. The stresses formed at the vertex of this multi-material composite situation were analyzed by the complex variable method, based on an appropriate choice of the Kolosov-potentials which are applicable in the vicinity of the vertex. For the case of a bi-material configuration it has been demonstrated how to derive some closed-form analytical solutions for the orders of the stress singularities $\lambda \in \mathbb{R}$ from the characteristic equation. Complex-valued roots $\lambda$ were calculated by solving the characteristic equation with Newton's method. With the help of a well-adjusted Mathematica-implementation calculations were carried out in a robust and highly efficient manner. With the method proposed in this work it is possible to calculate all real-valued eigenvalues $\lambda$ of any bi-material- and any tri-material configuration in a closed-form analytical manner. This possibility is a valuable novelty, since in all works dealing with singularity analyses, to the best knowledge of the authors of this article, the results are at last obtained numerically.

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