Influence of the interface internal energy on monotone disturbances of a creeping stationary flow with a velocity field of the Hiemenz type

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Abstract. The stability of a creeping stationary flow, which arises as a result of the interface internal energy, is studied. It is shown that the system of amplitude equations allows exact integration. Complex decrement as a solution to the transcendental equation is found. It was found that long-wave perturbations decay with increasing time. Neutral curves for the transformer oil - formic acid system were constructed.

1. Introduction
Questions of hydrodynamic stability of states of rest and stationary unidirectional flows with interfaces under the influence of various physical and chemical factors (surface viscosity, adsorption, evaporation, condensation) have been the subject of research by many authors (see [1] - [4] and the references in them). However, there is a poorly studied important class of problems on the effect of increments of the interface internal energy on the development of convective motions in its vicinity. According to calculations, this effect can be significant if the system consists of liquids with a sufficiently low viscosity [5]. The influence of the internal interphase energy changes on the stability of rest and stationary flows was taken into account in [1], [6]. In particular, in the framework of the classical Benard-Marangoni instability problem of a two-layer system, it was shown that the interface internal energy changes cause an increase of critical temperature gradients. A similar study of the stability of unidirectional stationary flows allows us to describe the available experimental data [6]. However, in these works, the basic stationary flow was independent of the parameter $E$, which is responsible for the interface internal energy change. In the present work, in the framework of the creeping flow model, we study the stability of two-dimensional conjugate flows depending on the parameter $E$.

2. Main flow
A two-layer stationary flow of viscous heat-conducting fluids in a flat channel bounded by solid walls $y = 0, y = h$ in the absence of mass forces is considered. The fluid 1 occupies the region $0 < y < l < h, -\infty < x < \infty$, the fluid 2 occupies the region $l < y < h, -\infty < x < \infty$, and the line $y = l$ is their common interface. The velocity and temperature field as a solution of the system of viscous heat-conducting fluid equations is sought in the form
For the first time, the solution of purely viscous fluid equations with such a velocity field was studied by K. Hiemenz [7]. The surface tension on the interface line \( y = l \) linearly depends on temperature: 
\[
\sigma(\theta) = \sigma_0 - \alpha (\theta - \theta_0) \quad \text{with constants } \sigma_0 > 0, \alpha > 0, \theta_0 > 0, \theta = \theta_1(x, l) = \theta_2(x, l).
\]
In addition, complete energy equality is fulfilled on it
\[
k_2 \frac{\partial \theta_2(x, l)}{\partial y} - k_1 \frac{\partial \theta_1(x, l)}{\partial y} = \alpha \theta(x, l) \text{div}_2 \mathbf{u},
\]
where \( k_j = \) are thermal conductivities of liquids, \( \text{div}_2 \mathbf{u} \) is surface divergence of the velocity vector \( \mathbf{u}(x, l) = u_1(x, l) = u_2(x, l). \) For the Hiemenz velocity field (1) \( \text{div}_2 \mathbf{u} = w_1(l). \)

Notice [1], that the relation order of equation right-hand side (2) to the first term of its left-hand side is estimated by the parameter 
\[
E = \frac{\alpha \theta^*}{\mu_2 k_2} \quad \text{(for the second term it is necessary to assume } \mu_1 k_1, \mu_j = \rho_j v_j \text{ are dynamic viscosities). It determines the influence of interphase energy on the liquid layers flow, } \theta^* \text{ is the characteristic temperature on the interface } y = l. \text{ In normal conditions } [1] E \ll 1. \text{ However, for low-viscosity and cryogenic liquids, it can reach final values. Therefore, for such mediums, it is necessary to consider the right-hand side in (2).}

**Remark 1.** For unidirectional flows, the right-hand side in (2) is identically equal to zero.

Substitution of the solution (1) into the equations of motion and heat transfer leads to the following system of nonlinear equations:
\[
\begin{align*}
\nu_j w_{jy} + \nu_j^2 &= v_j w_{jyy} + f_j, \quad w_j + v_j y = 0, \\
2w_j a_j + v_j a_{yy} &= \chi_j a_{jyy}, \\
v_j b_{jy} &= \chi_j b_{jyy} + 2 \chi_j a_j,
\end{align*}
\]
where \( \nu_j, \chi_j \) are kinematic viscosities and thermal diffusivity (positive constants), \( f_j \) are constants. The pressure in the liquids is distributed according to the laws
\[
\frac{1}{\rho_j} p_j(x, y) = \nu_j v_{jy} - \frac{\nu_j^2}{2} - f_j \frac{x^2}{2} + d_{0j}, \quad d_{0j} = \text{const},
\]
so \( f_j \) can be considered as the pressure gradient along the axis of \( x. \) Suppose the temperature distribution \( \theta_1(x, 0) = a_{10} x^2 + b_{10} \) with constants \( a_{10}, b_{10}, \) is set on the rigid wall \( y = 0. \) At \( a_{10} > 0, \) the temperature has a minimum value at the point \( x = 0, \) and at \( a_{10} < 0 \) the temperature is maximal. The upper wall is thermally insulated, i.e. \( \theta_2 y(x, h) = 0. \) Thus, on the rigid walls, the conditions for the unknowns have form
\[
\begin{align*}
w_1(0) = v_1(0) &= 0, \quad a_1(0) = a_{10}, \quad b_1(0) = b_{10}, \\
w_2(h) = v_2(h) &= 0, \quad a_{2y}(h) = b_{2y}(h) = 0.
\end{align*}
\]

The relations are satisfied on the interface \( y = l \) [1]
\[
\begin{align*}
w_1(l) &= w_2(l), \quad v_1(l) = v_2(l) = 0, \quad a_1(l) = a_2(l), \\
b_1(l) &= b_2(l), \quad \mu_2 w_{2y}(l) - \mu_1 w_{1y}(l) = -2 \alpha a_1(l), \\
k_2 a_{2y}(l) - k_1 a_{1y}(l) &= \alpha a_1(l) w_2(l), \\
k_2 b_{2y}(l) - k_1 b_{1y}(l) &= \alpha b_1(l) w_2(l).
\end{align*}
\]

**Remark 2.** This problem is inverse, because, along with functions \( w_j(y), v_j(y), a_j(y), b_j(y) \) the constants \( f_j \) (the pressure gradients along the layers) are also unknown. The functions \( b_j(y) \) are determined after finding the remaining unknowns and do not affect the velocity fields in liquids.

The following dimensionless variables and parameters are introduced
\[ \xi = \frac{v}{h}, P_j = \frac{v_j}{X_j}, M = \frac{ax_{10}h^3}{X_1\mu_2}, E = \frac{ax_{10}h^2}{\mu_2 k_2}, \]

where \( P_j \) are the Prandtl numbers, \( M \) is the Marangoni number, \( \theta^* = |a_{10}|h^2 \) is the characteristic temperature; the parameters \( M \) and \( E \) can be either positive or negative. Suppose that \( |M| \ll 1 \) and the number \( E \) is finite. We find solution of the nonlinear conjugate boundary value problem (3), (5), (6) by the perturbation method

\[ w_j(y) = \frac{X_j}{h^2} MW_j(\xi) = \frac{X_j}{h^2} M(W_j^0(\xi) + MW_j^1(\xi) + \ldots), \]
\[ v_j(y) = \frac{X_j}{h^2} MV_j(\xi) = \frac{X_j}{h^2} M(V_j^0(\xi) + MV_j^1(\xi) + \ldots), \]
\[ f_j = \frac{X_j}{h^4} MF_j = \frac{X_j}{h^4} M(F_j^0 + MF_j^1 + \ldots), \]
\[ a_j(y) = a_{10}A_j(\xi) = a_{10}(A_j^0(\xi) + MA_j^1(\xi) + \ldots), \]
\[ b_j(y) = a_{10}h^2 B_j(\xi) = a_{10}h^2(B_j^0(\xi) + AB_j^1(\xi) + \ldots). \]

The resulting boundary value problem for \( W_j^0, V_j^0, F_j^0, A_j^0 \) has a simple analytical solution [8]:

\[ W_j^0(\xi) = \frac{v(1-y)^2}{6y^2P_1} F_2^0(-3\xi^2 + 2y\xi), \]
\[ V_j^0(\xi) = \frac{v(1-y)^2}{6y^2P_1} P_2^0(\xi^3 - y\xi^2) \]
\[ A_j^0(\xi) = 1 + D_1 \xi, \quad 0 \leq \xi \leq \gamma; \]
\[ W_j^0(\xi) = \frac{X_j}{h^4} F_2^0(-3\xi^2 + 2(\gamma + 2)\xi - 1 - 2\gamma), \]
\[ V_j^0(\xi) = \frac{X_j}{h^4} P_2^0(\xi^3 - (\gamma + 2)\xi^2 + (1 + 2\gamma)\xi - \gamma), \]
\[ A_j^0(\xi) = 1 + \gamma D_1 \equiv D_2, \quad \gamma \leq \xi \leq 1, \]

at that

\[ F_1^0 = \frac{v(1-y)^2}{y} F_2^0, \quad F_2^0 = \frac{3y P_1 D_2}{\nu(y-1)[\nu+\mu(1-y)]} \]

and \( D_2 \) is a solution of the quadratic equation

\[ \frac{\gamma^2(1-y)\nu}{2k[y+\mu(1-y)]} D_2^2 + D_2 - 1 = 0. \]

The following designation is introduced: \( v = v_1/v_2, \quad \chi = \chi_1/\chi_2, \quad \gamma = l/h < 1, \mu = \mu_1/\mu_2. \) A complete analysis of the arising flows (and there may be two, one, or none) is given in [8]. For example, for \( a_{10} > 0 \) the equation (12) has two real roots, for \( E = E^* = -k[\gamma + \mu(1-\gamma)] \left[ 2y^2(1-\gamma) \right]^{-1} \) there is one root and for \( E < E^* \) there are no real roots. The functions \( B_j^0(\xi) \) are found by the formulas

\[ B_1^0(\xi) = B_{10} + C_1 \xi - \xi^2 - \frac{D_1 \xi^3}{3}, \quad 0 \leq \xi \leq \gamma, \]
\[ B_2^0(\xi) = C_2 + D_2(2\xi - \xi^2), \quad \gamma \leq \xi \leq 1, \]

where

\[ C_1 = \frac{2D_2(1-\gamma) + k(2\gamma - D_1 \gamma^2) + E W_1^0(\gamma) \left( \gamma^2 + \frac{D_1 \gamma^3}{3} - B_{10} \right)}{k + E W_1^0(\gamma)}, \]
\[ C_2 = \gamma C_1 + B_{10} - \gamma^2 - \frac{D_1 \gamma^3}{3} + D_2(\gamma^2 - 2\gamma), \quad B_{10} = \frac{b_{10}}{a_{10}h^2}. \]

**Remark 3.** For \( E = 0 \) there is only one stationary mode:
\[ W_1^0(\xi) = \frac{(1 - \gamma)(3\xi^2 - 2\gamma\xi)}{2\delta\gamma}, \quad V_1^0(\xi) = \frac{(\gamma - 1)(\xi^3 - \gamma\xi^2)}{2\delta\gamma}, \]
\[ A_1^0(\xi) = 1, \quad 0 \leq \xi \leq \gamma; \]
\[ W_2^0(\xi) = -\frac{\gamma[3\xi^2 - 2(\gamma + 2)\xi + 2\gamma + 1]}{2\delta(\gamma - 1)}, \]
\[ V_2^0(\xi) = -\frac{\gamma[\xi^3 - (\gamma + 2)\xi^2 + (2\gamma - 1)\xi - \gamma]}{2\delta(\gamma - 1)}, \quad \delta = \gamma + \mu(1 - \gamma), \]
\[ A_2^0(\xi) = 1, \quad \gamma \leq \xi \leq 1. \]

3. Small perturbation equations

According to (8), the velocity field of the main flow in the layers is proportional to the number \( M \). The same is true for pressures (4); the temperatures in the layers are finite at \( M \to 0 \). If \((U_j, V_j, Q_j, T_j)\) are dimensionless perturbations of the velocity, pressure and temperature fields, then they satisfy the Stokes systems

\[ U_{1\tau} + Q_{1\xi} = P_1(U_{1\xi} + U_{1\xi}), \quad V_{1\tau} + Q_{1\xi} = P_1(V_{1\xi} + V_{1\xi}), \]
\[ U_{1\xi} + V_{1\xi} = 0, \quad T_{1\tau} = T_{1\xi} + T_{1\xi} \]

(14) at \( \tau > 0, -\infty < \xi < \infty, 0 < \xi < \gamma \);

\[ U_{2\tau} + \rho Q_{2\xi} = \frac{P_2}{\chi}(U_{2\xi} + U_{2\xi}), \quad V_{2\tau} + \rho Q_{2\xi} = \frac{P_2}{\chi}(V_{2\xi} + V_{2\xi}), \]
\[ U_{2\xi} + V_{2\xi} = 0, \quad T_{2\tau} = \frac{1}{\chi}T_{2\xi} + T_{2\xi} \]

(15) at \( \tau > 0, -\infty < \xi < \infty, y < \xi < 1 \). In systems (14), (15) \( \tau = \chi t/h^2 \) is dimensionless time, \( \zeta = z/h, \rho = \rho_1/\rho_2 \). The systems of equations (14), (15) are supplemented by the boundary conditions:

on the wall \( \xi = 0 \)

\[ U_1 = 0, \quad V_1 = 0, \quad T_1 = 0; \]

(16)

on the wall \( \xi = 1 \)

\[ U_2 = 0, \quad V_2 = 0, \quad T_2 = 0; \]

(17)

on the interface \( \xi = \gamma \)

\[ U_1 = U_{2\xi}, \quad V_2 = 0, \quad V_2 = 0, \quad T_2 = T_2, \]
\[ \frac{P_2}{\chi}(U_{2\xi} + V_{2\xi}) - P_1(U_{1\xi} + V_{1\xi}) = -T_{1\xi}, \]
\[ T_{2\xi} - kT_{1\xi} = E(\theta_1 U_{1\xi} + W_{1\xi}) \]

(18) where \( E \) is determined from (7), \( k = k_1/k_2 \). According to the equalities (9)–(11)

\[ W_1(\gamma) = \frac{\gamma(1-\gamma)}{2[\gamma+\mu(1-\gamma)]} D_2, \quad \theta_1(\gamma, \zeta) = D_2 \zeta^2 + B_1(\gamma), \]

(19) and \( D_2 \) is a solution of the equation (12).

**Remark 4.** It seems interesting that the information about the main flow is contained only in the energy condition at the interface. This is the last relation in (18).

The normal wave method is not applicable to the problem (14) - (18) due to the presence of the term \( D_2 \zeta^2 \) (see the last relation in (18)). However, if the temperature \( \theta_1(\gamma, \zeta) \) varies little in limits of the perturbation wavelength \( d \)

\[ d \ll \sqrt{B_1^0(\gamma)}, \]

(20)
then $\theta_1(\gamma, \zeta)$ can be approximated simply as $B^\theta_1(\gamma)$ (see (13)) [1]. We differentiate the energy equality in (18) twice by $\zeta$, and then we take into account the limitation (20) to more accurately take into account information about the temperature change along the interface (the value of $D_2$). We get

$$
(T_2\zeta - kT_1\zeta)\zeta = E\left[2D_2U_\zeta + B^\theta_1(\gamma)U_\zeta\zeta\zeta + W_1(\gamma)T_1\zeta\zeta\right], \quad \zeta = \gamma.
$$

(21)

Now the solution of the problem is sought in the normal waves form

$$(U_j, V_j, Q_j, T_j) = \left(U_j(\zeta), V_j(\zeta), Q_j(\zeta), T_j(\zeta)\right) e^{i(\alpha\zeta - \lambda t)}.
$$

After substituting into the equation and boundary conditions, a spectral problem arises for the perturbation amplitudes and the parameter $\lambda' = d/d\zeta$

$$
-\lambda U_1 + i\alpha Q_1 = P_1(U''_1 - \alpha^2U_1), \quad V'_1 = i\alpha U_1 = 0,
$$

$$
-\lambda V_1 + Q'_1 = P_1(V''_1 - \alpha^2V_1),
$$

$$
-\lambda T_1 = T''_1 - \alpha^2T_1, \quad 0 < \zeta < \gamma;
$$

$$
-\lambda U_2 + \rho Q_2 = P_2 \left(\frac{\zeta}{\bar{\chi}}(U''_2 - \alpha^2U_2), \quad V'_2 = i\alpha U_2 = 0,
$$

$$
-\lambda V_2 + \rho Q'_2 = P_2 \left(\frac{\zeta}{\bar{\chi}}(V''_2 - \alpha^2V_2),
$$

$$
-\lambda T_2 = \frac{1}{\bar{\chi}}(T''_2 - \alpha^2T_2), \quad \gamma < \zeta < 1.
$$

The conditions on the walls (16), (17) and the first relations at the interface remain unchanged. And the rest along with (21) will be rewritten like this

$$
\frac{P_2}{\rho\chi} U'_2 - P_1 U'_1 = -i\alpha T_1,
$$

(24)

$$
a(kT_1' - T_2') = E\left[i(2D_2 - \alpha^2 B_1(\gamma))U_1 - \alpha W_1(\gamma)T_1\right], \quad \zeta = \gamma.
$$

(25)

4. The solution of the boundary value problem for amplitude equations

The functions $Q_j(\zeta)$ are not included in the boundary conditions. We exclude them from the equations and introduce vorticities in the layers

$$\Omega_j = i\alpha V_j - U'_j.
$$

(26)

Then $\Omega_j$ are solutions of the equations

$$
\Omega''_1 + \left(\frac{i\chi}{P_1} - \alpha^2\right)\Omega_1 = 0, \quad \Omega''_2 + \left(\frac{i\chi}{P_2} - \alpha^2\right)\Omega_2 = 0
$$

at $0 < \zeta < \gamma$ and $\gamma < \zeta < 1$ respectively. Let

$$
C_1^2 = \frac{i\chi}{P_1} - \alpha^2, \quad C_2^2 = \frac{i\chi}{P_2} - \alpha^2,
$$

(27)

then

$$\Omega_j = a_{1j} \cos C_j\zeta + a_{2j} \sin C_j\zeta
$$

with constants $a_{1j}, a_{2j}, j = 1,2$. Therefore, the functions $U_j, V_j$ satisfy the system of ordinary differential equations

$$
i\alpha V_j - U'_j = a_{1j} \cos C_j\zeta + a_{2j} \sin C_j\zeta, \quad V'_j + i\alpha U_j = 0,
$$

(28)

at that for $j = 1 0 < \zeta < \gamma$ and for $j = 2 \gamma < \zeta < 1$. The system solution (26) has an explicit form. For this, we exclude the functions $V_j$ by differentiation. We get the equations
\[ U_j'' - a^2 U_j = C_j \left( a_{1j} \sin C_j \xi - a_{2j} \cos C_j \xi \right). \]

Assuming \( \lambda \neq 0, C_j^2 \neq -\alpha^2 \), we find
\[
U_j(\xi) = b_{1j} ch a \xi + b_{2j} sh a \xi + \frac{C_j}{\alpha} \int_0^\xi \left( a_{1j} \sin C_j \xi - a_{2j} \cos C_j \xi \right) sh a (\xi - \zeta)d\zeta, \tag{29}
\]
where \( 0 < \xi < \gamma \) at \( j = 1 \) and \( \gamma < \xi < 1 \) at \( j = 2 \); \( b_{1j}, b_{2j} \) are constants.

**Remark 5.** The case \( \lambda = 0 \) corresponds to monotonic perturbations and will be considered separately.

Since \( V_j' = -i\alpha U_j \) and taking into account the sticking conditions \( V_1(0) = 0, V_2(1) = 0 \) we find representations for \( V_j(\xi) \):
\[
V_1(\xi) = -i\alpha \int_0^\xi U_1(\zeta)d\zeta, \quad 0 \leq \xi \leq \gamma, \tag{30}
\]
\[
V_2(\xi) = -i\alpha \int_0^\xi U_2(\zeta)d\zeta, \quad \gamma \leq \xi \leq 1. \tag{31}
\]

**Remark 6.** The integrals on the right-hand sides of the expressions (30), (31) can be calculated exactly.

The equations for \( T_j(\xi) \) have solutions
\[
T_j(\xi) = a_{3j} \cos(C_{2+1j}\xi) + a_{4j} \sin(C_{2+1j}\xi) \tag{32}
\]
with constants \( a_{3j}, a_{4j} \). Also, in addition to (27),
\[
C_2^2 = i\lambda - \alpha^2, \quad C_4^2 = i\lambda \chi - \alpha^2. \tag{33}
\]

Since \( T_1(0) = 0, T_2'(1) = 0 \), then \( a_{31} = 0, \)
\[
T_1(\xi) = a_{41} \sin C_3 \xi, \quad T_2(\xi) = \frac{a_{42} \cos(C_4(1-\xi))}{\sin C_4}. \tag{34}
\]

In deriving the formula for \( T_2(\xi) \), it is assumed that \( C_4 \neq m\pi, m = 0, 1, 2, \ldots \) We obtain a system of linear equations for constants \( a_{1j}, a_{2j}, b_{1j}, b_{2j}, a_{41}, a_{42}, j = 1, 2 \). We have successively: the condition \( U_1(0) = 0 \) implies the equality
\[
b_{11} + \frac{C_j}{\alpha} \int_0^\gamma sh a \xi \sin C_4 \xi d\xi + a_{21} \int_0^\gamma sh a \xi \cos C_4 \xi d\xi = 0; \tag{35}
\]
the sticking condition on another wall \( U_2(1) = 0 \) gives the equality
\[
b_{12} ch a + b_{22} sh a + \frac{C_j}{\alpha} \left[ a_{12} \int_0^1 sh a (1-\xi) \sin C_4 \xi d\xi - a_{22} \int_0^1 sh a (1-\xi) \cos C_4 \xi d\xi \right] = 0; \tag{36}
\]
from the condition \( U_1(\gamma) = U_2(\gamma) \) we deduce the equality
\[
b_{11} ch a \gamma + b_{21} sh a \gamma = b_{12} ch a \gamma + b_{22} sh a \gamma; \tag{37}
\]
from the conditions \( V_1(\gamma) = V_2(\gamma) = 0 \) we get
\[
\int_0^\gamma U_1(\zeta)d\zeta = 0, \quad \int_\gamma^1 U_2(\zeta)d\zeta = 0. \tag{38}
\]

Taking into account (34), the temperature equality \( T_1(\gamma) = T_2(\gamma) \) at the interface gives the relation
\[
a_{41} \sin C_3 \gamma = \frac{a_{42} \cos(C_4(1-\gamma))}{\sin C_4}. \tag{39}
\]
From the condition for a jump of the tangential stresses at the interface \((24)\) we obtain
\[
\frac{P_2}{\rho x} (b_{21} \text{sh} \alpha y + b_{22} \text{ch} \alpha y) - P_1 (b_{11} \text{sh} \alpha y + b_{21} \text{ch} \alpha y) = -i a_{41} \sin C_3 y. \tag{40}
\]

Finally, the energy condition \((25)\) leads to the equality
\[
\alpha \left[ k C_3 a_{41} \cos C_3 y - \frac{c_4 a_{42} \sin \{c_4 (1 - \gamma)\}}{\sin C_4} \right] = E \left[ i \{2 A_1 (y) - \alpha^2 B_1 (y)\} \times \right.
\]
\[
\times (b_{11} \text{ch} \alpha y + b_{21} \text{sh} \alpha y) - a W_1 (y) a_{41} \sin C_3 y \}. \tag{41}
\]

We got eight connections between ten constants \(a_{1j}, a_{2j}, b_{1j}, b_{2j}, a_{41}, a_{42}\). However, to derive second-order equations for the functions \(U_j (\xi)\), we used differentiation by \(\xi\). Therefore, it is necessary to turn to the system \((28)\), from which \(U'_1 (0) = -a_{11}, U'_2 (1) = -a_{12} \cos C_2 - a_{22} \sin C_2\), or, according to the \((29)\),
\[
abla b_{21} + C_1 \left[ a_{11} \int_0^y \text{ch} \alpha \zeta \sin C_3 \zeta \, d\zeta - a_{21} \int_y^0 \text{ch} \alpha \zeta \cos C_1 \zeta \, d\zeta \right] = -a_{11},
\]
\[
abla (b_{12} \text{sh} \alpha + b_{22} \text{ch} \alpha) + C_2 \left[ a_{12} \int_y^1 \text{ch} \alpha (1 - \zeta) \sin C_2 \zeta \, d\zeta - a_{22} \int_1^y \text{ch} \alpha (1 - \zeta) \cos C_2 \zeta \, d\zeta \right] = -a_{12} \cos C_2 - a_{22} \sin C_2. \tag{42}
\]

So, \((35)-(42)\) is a system of ten linear homogeneous equations with respect to constants \(a_{1j}, a_{2j}, b_{1j}, b_{2j}, a_{41}, a_{42}\). It should have a non-trivial solution. Therefore, the determinant of the corresponding matrix is equal to zero. This is the equation on \(\lambda\). If we denote
\[
z = \sqrt{i \lambda},
\]

| Table 1. The first three roots of the equation \((43)\). |
|---|---|---|---|---|---|---|
| \(E = 0, \gamma\) | \(z_01\) | \(z_02\) | \(z_03\) | \(\gamma = 0.5, E\) | \(z_01\) | \(z_02\) | \(z_03\) |
| 0.9 | 1.350 | 4.477 | 7.622 | 0.9 | 1.338 | 5.463 | 8.225 |
| 0.4 | 1.471 | 5.089 | 8.431 | 0.4 | 1.330 | 5.462 | 8.223 |
| 0.1 | 1.639 | 5.137 | 8.791 | 0.1 | 1.326 | 5.4617 | 8.2221 |
| 0.01 | 1.838 | 5.527 | 9.191 | 0.01 | 1.324 | 5.4615 | 8.2218 |

then from \((27), (33)\)
\[
C_1 = \frac{z^2}{P_1} - \alpha^2, C_2 = \sqrt{\frac{\chi z^2}{P_2} - \alpha^2}, C_3 = \sqrt{z^2 - \alpha^2}, C_4 = \sqrt{\chi z^2 - \alpha^2}.
\]

Therefore, instead of \(\lambda\), the unknown value will be \(z\).

**Remark 7.** All integrals in \((29)-(32), (35), (36), (38), (41)\) are easily calculated. The mentioned system of linear equations becomes cumbersome and is not given here.

**5. Long wave approximation**

For such an approximation, the dimensionless wave number \(\alpha \to 0\). We represent unknown quantities in the form
\[
U_j = \alpha U_j^{(0)} + \alpha^2 U_j^{(1)} + \ldots, \quad V_j = \alpha^2 V_j^{(0)} + \alpha^3 V_j^{(1)} + \ldots,
\]
\[
T_j = T_j^{(0)} + \alpha T_j^{(1)} + \ldots, \quad \lambda = \lambda^{(0)} + \alpha \lambda^{(1)} + \ldots
\]

In this case, the mass conservation equations and boundary conditions at the interface\((18)\) contain all of their terms. Turning to the system \((28)\), we must assume that
\[
a_{kj} = \alpha \bar{a}_{kj}, \lim_{\alpha \to 0} \bar{a}_{kj} = \bar{a}_{kj}^0 \equiv \text{const}, k, j = 1, 2.
\]
We also note that for $\alpha \to 0$
\[ C_1^2 \sim \frac{i}{P_1} \lambda^{(0)}, C_2^2 \sim \frac{ix}{P_2} \lambda^{(0)}, C_3^2 \sim -i\lambda^{(0)}, C_4^2 \sim -i\lambda^{(0)} \chi. \]

After calculations, as a similar section 3, we arrive to the equation for $z_0 = \sqrt{i\lambda^{(0)}}$:

\[ \cos[(1-\gamma)\sqrt{\chi}z_0]H_1 + H_2 H_3 \cos(\sqrt{\chi}z_0) = 0 \] (43)

with known $H_j(z_0), j = 1, \ldots, 6$. Equation (43) has a countable number of roots $z_{0n}$ and $\lambda^{(0)} = \lambda^{(0)}n = -iz_{0n}^2, n = 1, 2, \ldots$; in particular, when $E = 0, k = \sqrt{\chi}$ we find $z_{0n} = \frac{n}{2} (1 + 2(n - 1))(\gamma + k(1 - \gamma))^{-1}$. Table 1 shows the values of the first three roots of equation (43) for the transformer oil-formic acid system. The dimensionless parameters of the physical system are $\rho = 0.74, \nu = 15.41, \chi = 0.71, k = 0.41, P_1 = 308.2, P_2 = 14.2$. In general, calculations show that the wave attenuation decrement is a purely imaginary number with a negative imaginary part, i.e. $\lambda^{(0)} = \text{Im}\lambda^{(0)} < 0$. Therefore, the main flow is stable with respect to long-wave disturbances.

6. Monotone disturbances
In this case, in equations (22), (23) $\lambda = 0$, all quantities $C_1^2, C_2^2$ in (27) and $C_3^2, C_4^2$, in (33) have the same value $\alpha^2$.

Therefore $\Omega_j = a_{1j} \text{ch} \alpha \xi + a_{2j} \text{sh} \alpha \xi$ and instead of (29) we get

\[ U_j(\xi) = \left(b_{1j} - \frac{a_{2j} \xi}{2}\right) \text{ch} \alpha \xi + \left(b_{2j} - \frac{a_{1j} \xi}{2}\right) \text{sh} \alpha \xi. \]

The expressions (32) for temperature amplitudes take the form

\[ T_j(\xi) = a_{3j} \text{ch} \alpha \xi + a_{4j} \text{sh} \alpha \xi, \quad j = 1, 2. \]

From the conditions on the lower wall $\xi = 0$ we have

\[ b_{11} = 0, \quad a_{31} = 0. \]

Figure 1. Neutral curves depending on the dimensionless parameter $\gamma$ (ratio of layer thicknesses).
From the remaining conditions we obtain the equation for $E(\alpha)$. A numerical study of the equation is presented in figure 1, which shows the neutral curves for different values of $\gamma$ in the transformer oil-formic acid system. Stability areas are located under the curves. Negative values of the number $E(\alpha)$ correspond to the case of cooling of the lower channel wall. It can be seen that when the thickness of the layers (parameter $\gamma$) changes, the nature of the dependence $E(\alpha)$ does not change.

7. Conclusions
In the framework of the creeping flow model, the stability of two-dimensional conjugate flows in plane layers is investigated. Such flows arise due to changes of the interface internal energy. The intensity of this change is characterized by the parameter $E$. An interesting fact is that information about the main flow is contained only in the energy condition at the interface. It is established that the main flow is stable with respect to long-wave disturbances. For monotonic perturbations, the dependence of the parameter $E$ from the wave number with varying layer thicknesses is studied.

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