System–environment correlations and non-Markovian dynamics

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Abstract
We determine the total state dynamics of a dephasing open quantum system using the standard environment of harmonic oscillators. Of particular interest are random unitary approaches to the same reduced dynamics and system–environment correlations in the full model. Concentrating on a model with an at times negative dephasing rate, the issue of ‘non-Markovianity’ will also be addressed. Crucially, given the quantum environment, the appearance of non-Markovian dynamics turns out to be accompanied by a loss of system–environment correlations. Depending on the initial purity of the qubit state, these system–environment correlations may be purely classical over the whole relevant time scale, or there may be intervals of genuine system–environment entanglement. In the latter case, we see no obvious relation between the build-up or decay of these quantum correlations and ‘non-Markovianity’.

1. Introduction

In open quantum system dynamics one accounts for influences of the external environment [1–3]. Despite the unitary time-evolution of the total state of system plus environment, the dynamics of the system itself will in general be non-unitary. Growing correlations between the system of interest and its surroundings lead to a decay of the initially present coherences. This line of thought is at the heart of decoherence theory and is put forward to explain the appearance of classical properties in quantum systems [4–8]. Decoherence in particular is of relevance for quantum technologies trying to make use of the vast computational potential forecast to applied quantum information processing [9].

In the regime of weak system–environment coupling and short environmental correlation times the dynamics of an open quantum system may be described in terms of the Born–Markov approximation. The corresponding Markov master equation then has a generator in Lindblad form [10, 11]. The time evolution of the system depends on its present state only. Often, however, such an approximation is not justified. Then, memory effects—often incorporated by means of integrals over the past [12, 13]—start to play an essential role. Yet, it is known that for arbitrary bath correlation functions approximate and sometimes even exact time-local master equations can be derived [2, 14]. An important example is given by the exact master equation for a damped harmonic oscillator bilinearly coupled to a bath of harmonic oscillators [14–16]. Here, we focus on a dephasing qubit as an open quantum system using the standard harmonic oscillator environment, whose exact time local master equation is known.

In more recent developments, the question of how to define and distinguish ‘Markovian’ from ‘non-Markovian’ dynamics from a local (system) perspective was addressed. The analysis has been based both on a single snapshot of the dynamics [17] and on the full time evolution of the open quantum system within a certain time interval [18, 19]. In the latter approach, memory effects associated with non-Markovian dynamics are expected to cause temporary increase in the distinguishability of states (in terms of trace distance, e.g. [18]) and the dynamics may no longer be divisible [19]. Under Markovian dynamics, on the other hand, the decay of distinguishability will be monotonic throughout and divisibility will be ensured.

In this context the ‘flow of quantum information’ from system to environment and back is an often employed, and certainly intuitive picture. However, without a proper conceptual and theoretical framework, such a picture should be used with caution.

Care has also to be taken with respect to the need for a proper (quantum) environment. It should be noted that for single-qubit decoherence the dynamics may always...
be described in terms of stochastic fluctuations of external fields, i.e. the dynamics has a random unitary representation [26–28]. The dynamics may thus be modeled without invoking a quantum environment at all. Higher dimensions then two are needed to see decoherence that can only be understood in terms of a proper quantum environment [29].

The role and nature of system–environment correlations in open quantum system dynamics has raised some notable interest lately. In the context of quantum discord [20], for example, total states with no quantum correlation (zero discord) were shown to be the most general class of initial states allowing for completely positive (CP) reduced dynamics [21–23]. Another interesting line are investigations about the relation between decoherence and system–environment entanglement. Here, it is possible that the system essentially decoheres completely without becoming entangled with its environment at all [24, 25]. Cases in point are classical systems with no environment or even those with an environment at all [24, 25]. Such cases show that classical system–environment correlations alone may account for a vast number of phenomena related to ‘open quantum systems’. As we will also see in this paper, often an exchange of quantum information between system and environment cannot be proven.

In order to shed light on the nature of system–environment correlations in open system dynamics in a non-trivial (non-Lindblad) regime, and the possible relation to recent definitions of ‘non-Markovianity’ we here investigate single-qubit dephasing due to the coupling to an oscillator environment [25, 30, 31]. In so doing, we favour to investigate the dynamics and ‘quantumness’ of system–environment correlations thus avoiding the study of the somewhat vague notion of (quantum) ‘information flow’ in open quantum system dynamics.

The paper is structured as follows. In section 2 we will present our model and give an exact, useful expression for the total system–environment state. Section 3 will be concerned with the derivation of a random unitary representation for the reduced qubit state at time t, showing that from the system perspective alone, no quantum environment is necessary to model the dynamics. The two inherently different approaches in sections 2 and 3 are exact and thus describe decoherence in the Markovian as well as the non-Markovian regime. Accounting for non-Markovianity, in section 4 we concentrate on a super-ohmic spectral density which ensures an at times negative dephasing rate. In section 5 we employ a measure for system–environment correlations and relate it to ‘non-Markovianity’ in the sense of [18, 19]. Following earlier work [25], in section 6 we finally investigate the nature of these correlations. We find that for most qubit initial states there is no relation between ‘non-Markovianity’ and the build-up or decay of quantum correlations. Furthermore, even if there are time intervals where the total state of system and environment is entangled, there is no obvious relation to the periods of ‘non-Markovianity’. We will draw our conclusions in section 7.

2. Quantum decoherence model: reduced and full dynamics

Continuing our earlier work on the dynamics of system–environment correlations for a dephasing qubit [25], we start from a typical model [32, 33] with total Hamiltonian

\[ H_{\text{tot}} = H_{\text{sys}} + H_{\text{int}} + H_{\text{env}}, \]

by coupling a qubit non-dissipatively to a bath of harmonic oscillators through the choices [34–38]

\[ H_{\text{sys}} = \frac{\hbar \Omega}{2} \sigma_z, \]

\[ H_{\text{int}} = \sigma_z \otimes \sum_{\lambda=1}^{N} h_{\lambda} a_{\lambda}^+ + h.c., \]

\[ H_{\text{env}} = \sum_{\lambda=1}^{N} \hbar \omega_{\lambda} a_{\lambda}^+ a_{\lambda}. \]

Here \( \Omega \) denotes the energy difference between the qubit states and the coefficients \( g_{\lambda} \) describe the coupling strengths between the qubit and each environmental mode of frequency \( \omega_{\lambda} \) and annihilation and creation operators \( a_{\lambda}, a_{\lambda}^+ \).

As environmental initial state we choose a thermal state \( \rho_{\text{therm}} = (\bar{n}_{\lambda} + 1)^{-1} \exp[-\hbar \omega_{\lambda} a_{\lambda}^+ a_{\lambda}/k_B T] \) for each oscillator with the mean thermal occupation number \( \bar{n}_{\lambda} = (\exp[\hbar \omega_{\lambda}/k_B T] - 1)^{-1} \) at temperature \( T \). Initially, we assume no system–environment correlations such that the total initial state is simply given by the product \( \rho_{\text{tot}}(0) = \rho_{\text{sys}} \otimes \rho_{\text{therm}} \). Accordingly, for the reduced system state \( \rho_{\text{red}}(t) = \text{Tr}_{\text{env}} [\rho_{\text{tot}}(t)] \) the dynamical map \( E(t, 0) : \rho_{\text{tot}}(0) \rightarrow \rho_{\text{red}}(t) \) is completely positive (CP) with \( \rho_{\text{red}}(0) = \rho_{\text{sys}} \).

Already at this stage we emphasise that on the reduced level, this dynamics can equally well be described by random unitary dynamics as will be elaborated upon in section 3.

The quantum dephasing model (1) may be solved without any approximation. A possible approach to the time-local master equation for the system state is provided by the non-Markovian quantum state diffusion approach to open systems [14, 39]. We find for the reduced density operator \( \rho_{\text{red}}(t) = \text{Tr}_{\text{env}} [\rho_{\text{tot}}(t)] \)

\[ \dot{\rho}_{\text{red}} = -i \frac{\Omega}{2} \sum_{\sigma_z} \rho_{\text{red}} \sigma_z \rho_{\text{red}} - \frac{\gamma(t)}{2} (\rho_{\text{red}} \sigma_z - \sigma_z \rho_{\text{red}} \sigma_z). \]

This equation is solved by

\[ \rho_{\text{red}}(t) = \left( D^*(t) \rho_{01} D(t) \rho_{10} \right), \]

with

\[ D(t) = \exp \left[ -i \Omega t - \int_0^t \gamma(s) ds \right]. \]

and where the \( \rho_{ij} \) represent the initial state of the qubit.

Equations (3) and (4) involve the time dependent dephasing rate \( \gamma(t) \) which by means of the spectral density of the environment \( J(\omega) = \sum_{\lambda=0}^{N} |g_{\lambda}|^2 \delta(\omega - \omega_{\lambda}) \) can be written as

\[ \gamma(t) = 4 \int_0^\infty ds \int_0^\infty d\omega J(\omega) \coth(\hbar \omega/2k_B T) \cos(\omega s). \]

Later we will concentrate on environments that lead to periods in time with a negative dephasing rate.
Recently, we investigated system–environment correlations of this model [25] and found the useful representation
\[
\rho_{\text{red}}(t) = \int \frac{d^2 \xi}{\pi} \frac{1}{\pi} e^{-|\xi|^2/\pi} \hat{P}(t; \xi, \xi^*) \otimes |\xi\rangle \langle \xi| \tag{7}
\]
of the total state. It represents a partial P-representation where the environmental degrees of freedom are expanded in terms of coherent states |\xi\rangle. Here, \(\xi = (\xi_1, \xi_2, \ldots)\) is a vector of complex numbers and we consistently make use of the notation \(d^2 \xi/\pi := d\xi_1/\pi d\xi_2/\pi \ldots\) (see also [40]). Furthermore, we symbolically write \(\exp[-|\xi|^2/\pi]/\pi = \prod \exp[-|\xi_i|^2/\pi]\pi_i\), involving the mean thermal occupation number \(\bar{n}_i\) of the \(\lambda\)th environmental mode. The system part of the total state is encoded in a matrix-valued partial P-function \(\hat{P}(t)\) with values in the \(2 \times 2\) dimensional state space of the qubit.

In order to represent a solution of the total Schrödinger–von Neumann equation with initial \(\rho_{\text{red}}(0) = \rho_{\text{sys}} \otimes \rho_{\text{therm}}\), the partial P-function in (7) reads
\[
\hat{P}(t; \xi, \xi^*) = \left( \begin{array}{cc} A^+(t; \xi, \xi^*) \rho_{\text{sys}} B(t; \xi, \xi^*) \rho_{\text{sys}} \\ B^+(t; \xi, \xi^*) \rho_{\text{sys}} A(t; \xi, \xi^*) \rho_{\text{sys}} \end{array} \right). \tag{8}
\]

Here, \(A^\pm = \exp[-A(t) \pm \{a(t)\xi\} + \{\xi^* a(t)\}]\) and \(B = \exp[-i2\tau] \exp[B(t) - \{b(t)\xi\} - \{\xi^* b(t)\}]\), where we have introduced the complex time dependent vectors \(a(t) = (a_1(t), a_2(t), \ldots)\) and \(b(t)\) with scalar product \(a(t)\xi = \sum a_i(t)\xi_i\) and vector components
\[
a_i(t) = \frac{1}{\bar{n}_i} \int_0^t (g_{s} e^{i\omega_3 s}) \, ds \tag{9}
\]
\[
b_i(t) = \frac{2\bar{n}_i + 1}{\bar{n}_i} \int_0^t (g_{s} e^{i\omega_3 s}) \, ds. \tag{10}
\]

Furthermore, we use the abbreviations
\[
A(t) = 2 \Re \int_0^t ds \int_0^t dr \left[ \sum \frac{1}{\bar{n}_i} |g_{s}|^2 e^{-i\omega_2 (s-r)} \right]
\]
\[
B(t) = 2 \Re \int_0^t ds \int_0^t dr \left[ \sum \frac{2\bar{n}_i + 1}{\bar{n}_i} |g_{s}|^2 e^{-i\omega_2 (s-r)} \right].
\]

Initially, \(\hat{P} = \rho_{\text{sys}} = \rho_{\text{red}}(0)\) and note that there are no approximations necessary to achieve the result (8) and thus via (7) to obtain the exact state of the composite system (see also [25]).

Later, we study the dynamics of system–environment correlations. Therefore, a useful representation of the total state as in (7) is of central importance. The local dynamics alone is insufficient for the study of any quantities related to genuine open quantum system dynamics, i.e. involving a proper quantum environment as in equation (1). As we will elaborate upon next, in our case the same reduced dynamics (3) could have been obtained from a stochastic Schrödinger dynamics, not invoking a quantum environment at all.

3. Random unitary representations

With an eye on experimental conditions, decoherence of qubits is random by unitary random dynamics [30, 41]. In terms of a dynamical map, this implies that there exists a relation
\[
\rho_{\text{red}}(t) = \sum_k p_k U_k \rho_{\text{red}}(0) U_k^\dagger \tag{11}
\]
with suitably chosen probabilities \(p_k > 0\) and unitary maps \(U_k\). Indeed, on the level of the reduced state, single-qubit decoherence (and indeed, all single-qubit unital CP maps) can always be modelled in this way [26–28]. Thus, from a reduced point of view no quantum environment as in equation (1) is required. The reduced dynamics can be obtained from a local Schrödinger equation driven by a random Hermitian Hamiltonian. By contrast, genuine quantum decoherence may be found in two-qubit systems [29]. It is also worth noting that random unitary dynamics emerging from an open quantum system with environmental initial pure state can always be ‘undone’ (quantum error correction) [42, 43].

The most straightforward random unitary realization of single-qubit decoherence with state (4) at time \(t\) is provided by the simple quantum operation
\[
\rho_{\text{red}}(t) = \left( \begin{array}{cc} 1 + \frac{|D(t)|^2}{2} & e^{-i\sigma_z} \rho_{\text{red}}(0) e^{i\sigma_z} \\ \frac{1 - |D(t)|^2}{2} e^{i\sigma_z} \rho_{\text{red}}(0) e^{-i\sigma_z} \end{array} \right), \tag{12}
\]
which is obviously of the form (11) employing just two unitaries \(U_1 = \exp[-i\sigma_z/2], U_2 = \exp[i\sigma_z/2]\) and probabilities \(p_{1,2} = (1 \pm |D(t)|^2)/2\). Recall that according to (5), \(|D(t)| = \exp[-\int_0^t \gamma(s) \, ds]\).

It is worth noting that the very same formal relation holds true for the two-time map
\[
\hat{E}(t, t') : \rho_{\text{red}}(t') \mapsto \rho_{\text{red}}(t) \tag{13}
\]
such that
\[
\rho_{\text{red}}(t) = \left( \begin{array}{cc} 1 + \frac{|D(t, t')|^2}{2} U_1(t - t') \rho_{\text{red}}(t') U_1^\dagger(t - t') \\ \frac{1 - |D(t, t')|^2}{2} U_2(t - t') \rho_{\text{red}}(t') U_2^\dagger(t - t') \end{array} \right) \tag{14}
\]
with \(|D(t, t')| = \exp[-\int_{t'}^t \gamma(s) \, ds]\). However, as \(\gamma(s)\) need not be positive for all times (see later), the prefactor of the second contribution, \(\gamma(s)^2/\bar{n}_s\), may turn negative for \(t'\) near times of negative \(\gamma(s)\). Thus, for such times \((t', t)\), the map \(\hat{E}(t, t')\) in the form (14) ceases to take the form of a random unitary map. Indeed, using the Jamiołkowski isomorphism [44] it is straightforward to see that \(|D(t, t')| < 1\) or
\[
\int_{t'}^t \gamma(s) \, ds > 0 \tag{15}
\]
sγ( dynamical map quantum environment is involved. Here, the unitary system dynamics is determined by the random reduced dynamics reflects an ensemble of experiments where in our case of a single qubit there is a single decoherence factor D(t) = Tr[Un(t)ρenv(0)Un†(t)] as in (5). The two propagators are determined by the environment Hamiltonians \( H_t = i[H_0, \hat{\rho}] \) with \( H_t = \hbar \Omega/2 \sum_{\lambda=1}^{N} h_{g_{\lambda}}(\alpha_{\lambda}^\pm) + \sum_{\lambda=1}^{N} \lambda a_{\lambda}^\dagger a_{\lambda}^{} \), and two sign changes for \( H_0 \). We next employ the Wigner representation of the environmental initial state \( \rho_{\text{env}}(0) \) and for the operator \( U_n(t)U_t^\dagger(t) \), accordingly. The latter's Wigner Weyl symbol we denote by \( W_0(\alpha, \alpha^*) = \frac{1}{(2\pi \hbar)^2} \int e^{-ia_{\lambda}^\dagger a_{\lambda}^\prime} \text{Tr}[e^{ia_{\lambda}^\dagger a_{\lambda}^\prime} \rho_{\text{env}}] \). We find

\[
D(t) = \int \frac{d^2\alpha}{\pi} W_0(\alpha, \alpha^*) U_{01}(\alpha, \alpha^t, t).
\]

Due to the harmonic properties of the environment, the corresponding propagators' \( U_{01} \) are known explicitly and lead to the phase factor \( U_{01}(\alpha, \alpha^t, t) = \exp[-i\Phi(\alpha, \alpha^t, t)] \) with

\[
\Phi(\alpha, \alpha^t, t) = \frac{\Omega}{2} t - 2 \sum_{\lambda=1}^{N} g_{\lambda} a_{\lambda}^\dagger \int_0^t e^{-i\omega_{\lambda}s} ds + \text{c.c.}
\]

We see that \( D(t) \) is just an average over a random complex number of unit norm. For a thermal initial state the initial Wigner function \( W_0 = \frac{1}{\pi \hbar} \exp[-|\alpha|^2/\hbar(\hbar + \frac{1}{2})] \) is positive. Thus, expression (16) leads to a random unitary representation for \( \mathcal{E}(t, 0) \):

\[
\rho_{\text{red}}(t) = \int \frac{d^2\alpha}{\pi} W_0(\alpha, \alpha^*) U_n(t) \rho_{\text{red}}(0) U_n^\dagger(t).
\]

This corresponds to a random unitary evolution of the qubit with \( U_n = \exp[-i\int_0^t H_n(s) ds/\hbar] \) and diagonal random Hamiltonian \( H_n(t) = \alpha_{\lambda}^2 \int_0^t e^{-i\omega_{\lambda}s} ds + \alpha_{\lambda}^{*2} \int_0^t e^{i\omega_{\lambda}s} ds \). Note that in this representation the probability of occurrence of a particular unitary evolution is given by the value of the initial Wigner distribution and is thus time independent.

The second random unitary representation (18) of the reduced dynamics reflects an ensemble of experiments where the unitary system dynamics is determined by the random \( H_n(t) \), driven by some (classical) stochastic process. No quantum environment is involved.

Note that both random unitary representations of the dynamical map \( \mathcal{E}(t, 0) \) are exact—no restriction on the sign of \( \gamma(t) \) is necessary.

It may appear tempting to define a two-time map \( \mathcal{F}(t, t') \) through (18) with \( U_n \rightarrow U_n(t, t') = \exp[-i\int_0^t H_n(s) ds/\hbar] \). However, it is clear that \( \mathcal{F}(t, t') \neq \mathcal{E}(t, t') \) unless \( t' = 0 \). Indeed, while \( \mathcal{F}(t, t') \) is a CP map for all \( t, t' \) this ceases to be true for \( \mathcal{E}(t, t') \) (see the next section).

We close this section by pointing out an interesting additional observation: the random unitary representation (18) for the quantum dephasing model is not restricted to single-qubit-dephasing. In fact, for an arbitrary system Hilbert space dimension, the very same construction works for all dephasing factors \( \mathcal{D}_{nm}(t) = \text{Tr}[U_n(t)\rho_{\text{env}}(0)U_{m}^\dagger(t)] \) of a quantum oscillator environment model. So even for larger Hilbert space dimension than two—on a local level—pure dephasing based on a quantum oscillator model like (1) cannot be distinguished from random unitary dynamics. For genuine quantum decoherence, one needs ‘more quantum mechanical’ environments [29].

4. Negative dephasing rate and ‘non-Markovianity’

The physics of the harmonic oscillator environment model is encoded in its spectral density \( J(\omega) \). As we are here interested in instances of negative dephasing rate, we choose a particular super-ohmic spectral density with sharp cutoff at frequency \( \omega_c \)

\[
J(\omega) = \kappa \frac{\omega^3}{\omega_c^4} \Theta(\omega - \omega_c)
\]

with \( \Theta(\omega) \) the Heaviside step function and \( \kappa \) a dimensionless coupling constant. In the high-temperature limit \( k_B T > \hbar \omega_c \), the time dependent dephasing rate (6) can easily be evaluated analytically, we get

\[
h\gamma(t) = 8\kappa k_B T \left( \frac{\sin(\omega_c t)}{(\omega_c t)^2} - \frac{\cos(\omega_c t)}{\omega_c t} \right).
\]

Though many of our results do not rely on any special choice of \( J(\omega) \), in the following, whenever we show figures, we will use the spectral density (19) and account for the high-temperature limit by choosing \( T = 10 \hbar \omega_c/k_B \). Furthermore, we choose \( \kappa = 10^{-2} \) throughout this paper.

Single qubit decoherence with negative dephasing rate is interesting in connection with two recently proposed ‘measures of non-Markovianity’ [18, 19]. In both definitions the notion of divisibility, related to the decomposition of the dynamical map according to \( \mathcal{E}(t, t') = \mathcal{E}(t, t'') \mathcal{E}(t'', t') \) with \( t \geq t'' \geq t' \) is at the heart of ‘non-Markovianity’. As explained around equation (15), our \( \mathcal{E}(t, t') \) ceases to be a CP map for time intervals of negative \( \gamma(t) \). Indeed, as can be confirmed easily, for single qubit dephasing the two measures of non-Markovianity from [18, 19] are non-zero whenever \( \gamma(t') < 0 \) for some \( 0 < t' < t \).

The fact that the map \( \mathcal{E}(t, t') \) is no longer CP at times \( t' > 0 \) is a consequence of growing correlations between system and environment. These correlations may be due to entanglement, but they need not be as will be shown in section 6.

As can be seen in figure 1 our \( \gamma(t) \) turns negative in certain restricted periods, while the integral

\[
\int_0^t \gamma(s) ds = 8\kappa k_B T \ln \left( 1 - \frac{1 - \sin(\omega_c t)}{(\omega_c t)} \right)
\]

stays positive, as expected for the CP map \( \mathcal{E}(t, 0) \).

5. Total state and system–environment correlations

Coupling to an environment leads to the build-up of correlations between system and environment and thus to changes in local entropies. From the point of view of information theory, the reduced dynamics is regarded as a ‘channel’ for quantum information. In this context, several quantities related to von Neumann entropy \( S = -\text{Tr}(\rho \log \rho) \)
are of interest: e.g. $C_S(t) = S_{\text{sys}} + S_{\text{env}} - S_{\text{tot}}$ as a measure for system–environment correlations [47]. These quantities are hard to compute, unless one deals with very small systems or Gaussian states.

Here we choose purity $P = \text{Tr}(\rho^2)$ as an indicator for the mixedness of states, which is related to the ‘linear entropy’ via $S_L = 1 - P$. Clearly, as with entropy, total purity $P_{\text{tot}}$ is preserved under unitary evolution with $H_{\text{sys}}$. The sum of local purities $P_{\text{sys}}(t), P_{\text{env}}(t)$ however, will be smaller as $t > 0$. For the initial product state we have $P_{\text{tot}} = P_{\text{sys}}P_{\text{env}}$ and it appears natural for all $t > 0$ to consider the difference of logarithms of $P$ as a simple measure of correlations

$$C = \log(P_{\text{tot}}) - \log(P_{\text{sys}}) - \log(P_{\text{env}}). \quad (22)$$

$C$ is easier to compute than $C_S$, but still one finds $C = C_S = 0$ for uncorrelated states and $C = C_S = 2 \ln N$ for maximally entangled bipartite pure states of equal dimension $N$. The dynamics of system–environment correlations is given by

$$C(t) = \log\left(\frac{P_{\text{sys}}(0)}{P_{\text{sys}}(t)}\right) + \log\left(\frac{P_{\text{env}}(0)}{P_{\text{env}}(t)}\right)
= C_{\text{sys}}(t) + C_{\text{env}}(t). \quad (23)$$

Here the contributions $C_{\text{sys}}$ and $C_{\text{env}}$ correspond to the amount of correlations created between system and environment as indicated by the increase of the local entropies in the two subsystems.

Having the total state (7) at hand, all these quantities can be determined easily for our dephasing qubit. For instance, the qubit purity is readily determined to give

$$P_{\text{sys}}(t) = \frac{1}{2}(1 + z^2 + (x^2 + y^2)|D(t)|^2). \quad (24)$$

Here and in the following we denote by $r = (x, y, z) = \text{Tr}[\sigma \rho]$ the coordinates of the Bloch vector of the \textit{initial state of the qubit}. Somewhat more involved, yet still easy to determine is the purity of the environment. We find

$$P_{\text{env}}(t) = \frac{1}{2}(1 + z^2 + (1 - z^2)|G(t)|^2)P_{\text{env}}(0) \quad (25)$$

with the initial environmental purity

$$\log P_{\text{env}}(0) = \int_0^\infty d\omega J(\omega) \log(\tanh(\hbar \omega / 2 k_B T)). \quad (26)$$

In (24), the time dependence arises from the decoherence factor $|D(t)| = \exp\left[\int_0^t \gamma(s) \, ds\right]$ of qubit dephasing with the rate $\gamma(t) = 4 \int_0^\infty d\omega J(\omega) \coth(\hbar \omega / 2 k_B T) \cos(\omega s)$ from (6). By contrast, for the environment the time dependence is governed by a factor $|G(t)| = \exp\left[\int_0^t \Gamma(s) \, ds\right]$ with a dual rate $\Gamma(t) = 4 \int_0^\infty d\omega J(\omega) \tanh(\hbar \omega / 2 k_B T) \cos(\omega s)$.

The rate of change of the correlation $C(t)$ from (23) stems from the two contributions $\dot{C}_{\text{sys}}(t) = \dot{C}_{\text{sys}} + \dot{C}_{\text{env}}$ with

$$\dot{C}_{\text{sys}}(t) = \frac{2 y(t)}{a[D(t)]^2 + 1}, \quad (27)$$

and

$$\dot{C}_{\text{env}}(t) = \frac{2 \Gamma(t)}{b[G(t)]^2 + 1}, \quad (28)$$

where the initial state of the qubit determines the factors $a = (1 + z^2)/(1 - z^2)$ and $b = (1 + z^2)/(x^2 + y^2)$.

Equations (27) and (28) reflect a first important result: system and environment become more correlated for $\gamma(t), \Gamma(t) > 0$. More interestingly, system–environment correlations \textit{decrease} for negative dephasing rates. In other words, during ‘non-Markovian’ periods system and environment recover some of their initial independence. As we will elaborate in the next section, these system–environment correlations may well be purely classical without the build-up of entanglement.

In the high temperature limit considered here, by means of (27) and (28) all quantities in (23) can be obtained readily. Reflecting the huge dimension of the environmental Hilbert space, it turns out that its contribution $C_{\text{env}}(t)$ is small compared to $C_{\text{sys}}(t)$. Therefore, from (27) we expect $C(t) \sim \gamma(t)$. In figure 2 we display system–environment correlations $C(t)$ and dephasing rate $\gamma(t)$. Clearly, changes in $C$ correlate with the sign of $\gamma$, and thus $C(t)$ decreases in domains of non-Markovianity. As we will explain in the next section, in this case the total state is not entangled and thus $C(t)$ reflects classical correlations only.

Recall that this connection between system–environment correlations and non-Markovianity can only be established on the basis of the total quantum state (7). For the random unitary representation (18) of the same reduced dynamics, due to the
lack of an environment, the notion of system–environment correlations ceases to make sense.

### 6. Quantum and classical system–environment correlations

Having access to the total state we can also investigate the nature of system–environment correlations. In earlier work we have shown that quantum correlations need not exist in such open system models, in particular in the high-temperature limit. In such a case, the total state may still be separable and thus all correlations could be established using classical communication. Here we argue very much as in [25].

With the time and temperature dependent function

\[ S(T, t) = 4 \int_0^T ds \int_0^t dr \int_0^\infty d\omega \times J(\omega) \exp[i\hbar \omega/kT] \cos[\omega(s - t)] \]  

(29)

we have shown in [25] that the total state is separable, as long as

\[ S(T, t) \leq \ln \sqrt{1 - \frac{z^2}{x^2 + y^2}}. \]  

(30)

In the high temperature limit, with our special choice of \( J(\omega) \) this quantity can be easily evaluated yielding

\[ S(T, t) = 4k \left( \frac{1}{2} - \frac{\sin[\omega_0 t]}{\omega_0 t} \cos[\omega_0 t] + \frac{1}{\omega_0^2 t^2} \right). \]  

(31)

With criterion (30) we can indeed prove that the total state underlying the correlation displayed in figure 2 is separable.

These findings show that the existence of a quantum environment does not imply (growing) entanglement. Moreover, there is no connection between the non-Markovian character of the dynamics and the nature of system–environment correlations.

We can prove system–environment entanglement when the partial transpose \( \rho_{\text{tot}}^{\text{PT}} \) of the total state yields a negative expectation value \( \langle \Psi | \rho_{\text{tot}}^{\text{PT}} | \Psi \rangle \) in some state \( |\Psi\rangle \) of the composite system [48]. By means of the representation (7) we have shown in [25] that with the time and temperature dependent function

\[ E(T, t) = 8 \int_0^T ds \int_0^t dr \int_0^\infty d\omega \times J(\omega) \sinh[\hbar \omega/kT] \cos[\omega(s - t)] \]  

(32)

entanglement is present whenever

\[ E(T, t) > \ln \left[ \frac{r - \frac{z^2}{x^2 + y^2}}{x^2 + y^2} \right]. \]  

(33)

In contrast to the separable case studied in figure 2, choosing a qubit initial state with a purity closer to one, we can indeed prove the existence of system–environment entanglement (see highlighted regions in figure 3). Remarkably, there is a close connection between our ‘entanglement witness’ (33) and the environmental contribution \( C_{\text{env}}(t) \) of the correlations. In the high temperature limit \( kT \gg \hbar \omega_0 \), we find

\[ C_{\text{env}} = \frac{1}{2} \frac{E}{b e^{-\delta E} + 1}. \]  

(34)

### Figure 3. \( C_{\text{env}}(t) \) against time for a qubit with initial purity \( r = 0.997 \). The highlighted domains correspond to time intervals with non-zero system–environment entanglement. Clearly, an increase of \( C_{\text{env}}(t) \) above a critical value indicates these quantum correlations. and for \( E(T, t) \ll 1 \), \( C_{\text{env}}(t) = E(T, t)/2(b + 1) \). Now we are able to reformulate the entanglement criterion (33) in terms of the environmental part of the system–environment correlations

\[ C_{\text{env}}(t) > \ln \left[ \frac{r - \frac{z^2}{x^2 + y^2}}{x^2 + y^2} \right]. \]  

(35)

In figure 3 we choose a larger initial state purity \( r = 0.997 \), leading to time intervals with system–environment entanglement (highlighted areas), according to criterion (33). The appearance of quantum correlations is closely related to the dynamics of \( C_{\text{env}}(t) \) as explained earlier: in figure 3 values of \( C_{\text{env}}(t) \) larger then the threshold given by (35) indicate entanglement. Comparing these findings to the domains of negative \( \gamma \) from figure 2 (which are independent of the qubit initial state), we see no obvious relation between time intervals of quantum correlations and non-Markovianity of the dynamics.

### 7. Conclusions

We have investigated non-Markovian dynamics of a decohering qubit and its environment. Since the reduced dynamics can be modelled by means of random unitary evolution, we have stressed that a genuine ‘open quantum system’ point of view is not required in this case. In particular, we argue that approaches based solely on the reduced description may be misleading with respect to interpretations. A study of ‘information flow to the environment’, e.g., is questionable without the existence of environmental dynamical degrees of freedom.

Considering the full dynamics of system plus quantum environment, we have investigated the measure \( C = C_{\text{sys}} + C_{\text{env}} \) for system–environment correlations that emerge from an increase of the local entropies of the two subsystems. We have found (the time derivative of) this quantity to be closely related to the sign of the dephasing rate \( \gamma(t) \), reflecting the non-Markovian character of the dynamics. Referring to earlier work, we are able to show that the total state underlying the correlations described by \( C \), is separable for a large class of mixed qubit initial states. Therefore, given the quantum
environment, ‘non-Markovianity’ is still unrelated to the build-up or decay of quantum correlations (entanglement) between system and environment.

For qubit initial states with large purity, by contrast, we were able to find periods where the total state is entangled. But again, we see no obvious relation between ‘non-Markovianity’ and the build-up or decay of these quantum correlations. Interestingly though, we are able to relate the environmental part of the correlations C to entanglement.

We are confident that our approach will be helpful for further investigations with respect to system–environment correlations and ‘information flow’ in open system dynamics.

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