Force in Kappa-Deformed Relativistic Dynamics

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Abstract
We consider the physical implications of various choices of the three-momentum basis in the \( \kappa \)-deformed Poincaré algebra. In particular, we find that the energy dependence of the velocity of a \( \kappa \)-particle leads to unexpected features in \( \kappa \)-deformed kinematics. We also discuss the notion of force in \( \kappa \)-deformed dynamics, and as a tool example we investigate the motion of a \( \kappa \)-deformed particle under the action of a constant force.

1 Introduction

The possibility of description of space-time at subatomic distances of order Planck-length \( (l_p \sim 1.6 \cdot 10^{-35} m) \) using the quantum deformed space-time symmetries (i.e. the Hopf algebra extension of the Poincaré symmetry) is investigated since ten years \[1\],\[2\],\[3\]. This quantum deformation of the Poincaré algebra and group leads us to the concept of noncommutative space-time structure. There are some arguments that this noncommutativity of space-time coordinates could be the effect caused by quantum gravity \[4\],\[5\].

Further, we shall discuss a deformed Poincaré symmetry based on quantum deformation with a dimensional (massive) parameter \( \kappa \), so-called \( \kappa \)-deformation (see, e.g.,\[2\],\[3\]). This type of deformation of the relativistic symmetry seems to be very mild because of the following properties:

i) three dimensional nonrelativistic rotations \( O(3) \) are not deformed,

ii) the energy remains the additive quantity (i.e. it has a trivial coproduct, in the language of Hopf algebras), therefore, the energy conservation law is still valid,

iii) the commutativity of space coordinates \( x_i \) is preserved

\[
[x_i, x_j] = 0 \quad i, j = 1, 2, 3. \quad (i.1)
\]

iv) the generators of the four-momentum \( p_\mu \) commute

\[
[p_\mu, p_\nu] = 0 \quad \mu, \nu = 0, ..., 3. \quad (i.2)
\]
v) the relativistic time coordinate $x_0$ appears to be a quantum number and noncommuting space-time takes the form

$$[x_i, x_0] = \frac{i\hbar}{\kappa c} x_i = il_\kappa x_i,$$  \hspace{1cm} (i.3)

where $\hbar$ - Planck constant, $c$ - velocity of light and $l_\kappa$ describes the fundamental length (related to $\kappa$ deformation parameter) at which the time coordinate $x_0 = ct$ has to be considered as noncommutative \cite{3}. In ordinary quantum mechanics (which becomes in the limit $\kappa \to \infty$) the fundamental length $l_\kappa = 0$. It is estimated that $\kappa > 10^{12}$ GeV, therefore $l_\kappa < 10^{-28}m$, in practical considerations of the deformations effects one can assume $l_\kappa \sim l_p$.

The commutation relation (i.3) is a source of recently discussed generalized uncertainty relations in the $\kappa$-deformed framework. It appears that using Wigner’s procedure of measurement of distances and the relation (i.3) one can conclude the existence of a "minimum length" as a minimum uncertainty for the measurement of distances (see also \cite{6,7}). This effect, in principle could be tested experimentally in the gravity-wave interferometers (see discussion in \cite{8}).

The property (iv) of $\kappa$-deformed Poincaré algebra tell us that as in nondeformed relativistic symmetry, the four-momentum generators commute (contrary to our space-time (i.3)). However, taking into account the Hopf algebra structure of three-momentum, because of nonsymmetric coproduct, the addition law of momentum $p_i$ is more complicated (see, e.g.\cite{2,9-11}). One can also consider the basis of three-momentum generators $\tilde{p}_i$ given by a nonlinear transformation respecting $O(3)$ rotational covariance i.e.

$$\tilde{p}_i = f \left( \frac{p_0}{\kappa c} \right) p_i$$  \hspace{1cm} (i.4)

the only physical requirement for the function $f$ is a correct nondeformed limit i.e. $f(p_0/\kappa c) \to 1$ for $\kappa \to \infty$. This condition follows from the Hopf algebra $\kappa$-deformation under assumption that it is an extension of Poincaré algebra. However, the physical consequences strongly depend on a choice of the momentum basis i.e. the function $f$. In particular, $\kappa$-deformed particle kinematics depends on the choice of three-momentum basis. We shall discuss this problem in detail for the standard \cite{2} and bicrossproduct basis \cite{3}.

The problem how the $\kappa$-deformed particle kinematics depends on the choice of three-momentum basis has been partially discussed in \cite{9,10}. In particular, the velocity of the particle has unexpected features from the point of view of the relativistic ideas - the velocity diminishes for large energy. The first attempt to find the correct relativistic behaviour of $\kappa$-deformed velocity can be found in \cite{2} where, by using the Poisson bracket, the monotonic function of the velocity was found, but the generators of the configuration space were changed. However, let us stress that within this approach it is possible to obtain a $\kappa$-particle velocity with physically reasonable properties. This result suggests that one can allow to look for other momentum generators which are more acceptable from the physical point of view.

In our paper we consider only the four-momentum algebra. Thus, we will not
discuss how the choice of the momentum basis changes the momentum coalgebra, boost generators algebra or its coalgebra. However, different momentum bases lead to different forms of Casimir i.e. the mass square operator $M^2$ so, different dispersion relations.

We discuss the particle kinematics with the use the Hamilton's formalism (see [10]). We start from the general $\kappa$-deformed quadratic Casimir and use it to obtain the relations between the momentum, energy and velocity for a particle with an arbitrary mass. We discuss it for the two simple momentum bases (standard and bicrossproduct ones) and show that these bases lead to unexpected properties of the velocity or momentum. Then, using the Newtonian relation between the force $\vec{F}$ and three-momentum, i.e. $\vec{F} = \dot{\vec{p}}$ (also valid in the relativistic case) we derive the general formula for the dependence of the force on the deformed velocity. This relation shows us that the $\kappa$-deformation effects for the moving massive particle are of order $\frac{1}{\kappa}$ so, it is practically impossible to measure experimentally because of the magnitude of $\kappa$. Our considerations suggest that the possible $\kappa$-deformed effects can appear in very sensitive experiments at the energies of order of $\kappa c^2$, therefore recently discussed gamma-ray bursts observations [8] give us a hope for verification or not the $\kappa$-deformed model of space-time.

2 Generalized $\kappa$-deformed mass condition and its consequences for momentum and velocity

In the $\kappa$-Poincaré algebra we can construct a deformed quadratic Casimir $M^2$, describing the deformed mass square operator. In this way the energy of the particle and the quadratic momentum operator are related and this relation can be used as a starting point for the description of the deformed particle kinematics and dynamics. Naturally, its form depends on the particular choice of the momentum basis. The various choices of the three-momentum basis and associated $\kappa$-deformed Casimir can be found in [11], in particular the standard basis [2] and bicrossproduct basis [3]. In all cases the energy generator is assumed to be additive quantity, in fact this leads to the appearance of the term $\sinh(E/\kappa c^2)$ in the dispersion relation. Strictly speaking, this form of the energy dependence is related to $\kappa$-deformation.

If we consider the general three-momentum basis of the type (i.4) then the deformed mass condition takes the form:

$$
\left(2\kappa \sinh\left(\frac{E}{2\kappa c^2}\right)\right)^2 - \frac{1}{c^2} f^2\left(\frac{E}{2\kappa c^2}\right)\vec{p}^2 = M^2.
$$

(1)

where $M^2$ is the quadratic Casimir of the $\kappa$-deformed Poincaré algebra and $f$ is an arbitrary invertible function of energy, satisfying the boundary condition $\kappa \to \infty \Rightarrow f \to 1$ so

$$
\kappa \to \infty \Rightarrow M^2 c^4 = E^2 - c^2 \vec{p}^2.
$$

(2)
In the rest frame \(p = 0\) we get

\[
M^2 = 4\kappa^2 \sinh^2 \left( \frac{m_0}{2\kappa} \right) = 2\kappa^2 \left( \cosh \left( \frac{m_0}{\kappa} \right) - 1 \right). \tag{3}
\]

where \(m_0\) is the rest mass of the particle. As was shown in [12] for \(\kappa \to \infty\) the relation (3) in the case \(f = 1\) is consistent with various definitions of mass for a massive particle which are introduced in the ordinary special relativity. Let us notice, that the final form of (3) is the same for the rest mass definitions considered in [12]. Moreover, taking the Casimir \(M^2\) as given in (3) we obtain in the limit \(\kappa \to \infty\) the standard mass shell condition. Therefore, for optional \(f\), eq. (1) can be rewritten as follows \((E \geq m_0c^2)\)

\[
\cosh \left( \frac{E}{\kappa c^2} \right) - \cosh \left( \frac{m_0}{\kappa} \right) = \frac{1}{2\kappa^2 c^2 f^2} \left( \frac{E}{2\kappa c^2} \right) \hat{p}^2. \tag{4}
\]

Assuming that the Hamilton’s formalism holds for the \(\kappa\)-deformed momentum space the standard definition of the velocity

\[
v_i \equiv \hat{x}_i = \frac{\partial E}{\partial p_i} \tag{5}
\]

can be used to find the relation between the momentum and velocity. From (4) we obtain:

\[
p_i = \frac{\kappa}{f^2} \left\{ \sinh \left( \frac{E}{\kappa c^2} \right) - \frac{f'}{f} \left[ \cosh \left( \frac{E}{\kappa c^2} \right) - \cosh \left( \frac{m_0}{\kappa} \right) \right] \right\} v_i \tag{6}
\]

Using the quadratic momentum \(\hat{p}^2\) which can be easily obtained from (4)

\[
p^2(E) = \frac{2\kappa^2 c^2}{f^2} \left( \cosh \left( \frac{E}{\kappa c^2} \right) - \cosh \left( \frac{m_0}{\kappa} \right) \right), \tag{7}
\]

we get the dependence of the velocity on energy

\[
v^2 = 2c^2 f^2 \left( \frac{E}{2\kappa c^2} \right) \cosh \left( \frac{E}{\kappa c^2} \right) - \cosh \left( \frac{m_0}{\kappa} \right) \left\{ \sinh \left( \frac{E}{\kappa c^2} \right) - \frac{f'}{f} \left[ \cosh \left( \frac{E}{\kappa c^2} \right) - \cosh \left( \frac{m_0}{\kappa} \right) \right] \right\}^2. \tag{8}
\]

These energy dependencies of the momentum and velocity will now be used for two different functions \(f\) associated with the earlier mentioned choices of the momentum basis:

(a) the standard \(\kappa\)-deformed momentum basis \[2\]

it corresponds to the choice \(f \left( \frac{E}{2\kappa c^2} \right) = 1\)

\[
\lim_{E \to \infty} p^2(E) = \lim_{E \to \infty} \left( \kappa^2 c^2 e^{\frac{E}{\kappa c^2}} \right) = \infty. \tag{9}
\]

\[
\lim_{E \to \infty} v^2(E) = 2c^2 \lim_{E \to \infty} \frac{\cosh \left( \frac{E}{\kappa c^2} \right) - \cosh \left( \frac{m_0}{\kappa} \right)}{\sinh^2 \left( \frac{E}{\kappa c^2} \right)} = 0. \tag{10}
\]
The expression (10), which is similar to the one obtained in [2], means that 
\( v(E) \) is not a monotonic energy function, but there exists the maximum of the velocity

\[
v_{\text{max}} = c \exp\left(-\frac{m_0}{\kappa}\right).
\]

(11)
The region of energy for \( v \geq v_{\text{max}} \) is nonrelativistic - the velocity diminishes as energy increases (see also [10]). For small energies \( E \ll \kappa c^2 \) we can derive

\[
p^2(E) \sim \frac{1}{c^2} \left( E^2 - m_0^2 c^4 \right) \left\{ 1 + \frac{m_0^2}{12 \kappa^2} + \frac{1}{12} \left( \frac{E}{\kappa c^2} \right)^2 \right\} + O\left(\frac{1}{\kappa^4}\right),
\]

(12)

\[
v^2(E) \sim c^2 \left( 1 - \frac{m_0^2 c^4}{E^2} \right) \left\{ 1 + \frac{m_0^2}{12 \kappa^2} \right\} \left\{ 1 - \frac{1}{4} \left( \frac{E}{\kappa c^2} \right)^2 \right\} + O\left(\frac{1}{\kappa^4}\right).
\]

(13)

It is easy to estimate the value of \( v_{\text{max}} \) (11) for instance for the electron \( (m_e = 9.1 \cdot 10^{-31} kg, \kappa \sim 1.7 \cdot 10^{-15} kg) \) we get \( v_{\text{max}} \sim c \exp\left(-10^{-16}\right) \) and it corresponds to the electron energy \( E_e \sim 10^4 GeV \). Because \( \kappa c^2 \sim 10^{12} >> E_e \sim 1 GeV \) therefore, we can use expansion (13) to estimate an energy dependent variation in velocity

\[
\frac{\delta v}{v} = \frac{v_0 - v}{v_0} \sim \frac{1}{8 \kappa^2 c^4} \left( E^2 - m_0^2 c^4 \right) \sim 10^{-26}.
\]

(14)

This kind of the variation in velocity is discussed by Ellis at al. [8].

(b) the bicrossproduct \( \kappa \)-deformed momentum basis [3] it corresponds to the choice \( f\left(\frac{E}{\kappa c^2}\right) = \exp\left(\frac{E}{\kappa c^2}\right) \)

\[
\lim_{E \to \infty} p^2(E) = 2 \kappa^2 c^2 \lim_{E \to \infty} \left[ e^{-\frac{E}{\kappa c^2}} \left( \cosh\left( \frac{E}{\kappa c^2} \right) - \cosh\left( \frac{m_0}{\kappa} \right) \right) \right] = \kappa^2 c^2
\]

(15)

\[
\lim_{E \to \infty} v^2(E) = 2 c^2 \lim_{E \to \infty} \left\{ e^{-\frac{E}{\kappa c^2}} \left( \cosh\left( \frac{E}{\kappa c^2} \right) - \cosh\left( \frac{m_0}{\kappa} \right) \right) \right\} = \infty
\]

(16)

And for small energies \( E \ll \kappa c^2 \) we get

\[
p^2(E) \sim \frac{1}{c^2} \left( E^2 - m_0^2 c^4 \right) \left( 1 + \frac{m_0^2}{12 \kappa^2} \right) e^{-\frac{E}{\kappa c^2}} + O\left(\frac{1}{\kappa^4}\right),
\]

(17)

\[
v^2(E) \sim c^2 \left( 1 - \frac{m_0^2 c^4}{E^2} \right) \left( 1 + \frac{m_0^2}{12 \kappa^2} \right) e^{-\frac{E}{\kappa c^2}} + O\left(\frac{1}{\kappa^4}\right).
\]

(18)
Using (18) at low energies $E_e \sim 1GeV$, we find an energy dependent variation in velocity

$$\frac{\delta v}{v} \sim \frac{1}{2\kappa c^2} \left( 3E + \frac{m_0^2 c^2}{12\kappa} \right) \sim 10^{-13}. \quad (19)$$

This form of the energy dependence is discussed also in [8]. We see that in the case (a) we get the quadratic dependence on energy of $\delta v/v$ contrary to the linear dependence in the case (b). Therefore, the choice of the momentum basis, i.e. the function $f$, leads to different physical properties at low energies. Also for large energies we observe some unconventional features in $\kappa$-deformed kinematics. In particular, in the standard basis (a) when energy grows the velocity of the particle tends to zero (10) and in the bicrossproduct basis (b) the velocity goes to $\infty$ (16) and the limit of the momentum is proportional to $\kappa c$ (15).

It appears that the formula (1) allows one to choose the momentum basis which for all energies the momentum and velocity behaviour is similar to standard relativistic one [13]. Obviously, this choice would involve a more complicated form of the $f$ function.

### 3 Force in $\kappa$-deformed dynamics

Using the $\kappa$-deformed mass condition (1) and demanding that the standard relation between the force and momentum vectors should be conserved by any deformation

$$\vec{F} = \dot{\vec{p}}, \quad (20)$$

we obtain

$$\left[ 2\kappa^2 \sinh \left( \frac{E}{\kappa c^2} \right) - \frac{1}{c^2 f f' \tilde{p}^2} \right] \dot{E} = 2\kappa f^2 \tilde{p} \vec{F} \quad (21)$$

or equivalently the same relation as in the case of nondeformed relativistic dynamics

$$\dot{E} = \vec{v} \vec{F}. \quad (22)$$

The force $\vec{F}$ introduced in (18) can be considered as a function $\vec{F} = \vec{F}(E, \vec{v}, \vec{v})$ of the energy, velocity and acceleration. For simplicity, all expressions will be derived for the standard $\kappa$-deformed momentum basis (the (a) choice in our case).

From (18) we get a simple formula

$$p_i = \kappa \sinh \left( \frac{E}{\kappa c^2} \right) v_i, \quad (23)$$

and therefore
\[ \vec{F} = \vec{p} = \kappa \sinh \left( \frac{E}{\kappa c^2} \right) \vec{v} + \frac{1}{c^2} \cosh \left( \frac{E}{\kappa c^2} \right) \dot{E} \vec{v}. \]  

(24)

Multiplying this expression by \( \vec{v} \) and using (20) we obtain

\[ \dot{E} \left[ 1 - \frac{1}{c^2} \cosh \left( \frac{E}{\kappa c^2} \right) v^2 \right] = \kappa \sinh \left( \frac{E}{\kappa c^2} \right) \dot{v} \vec{v}. \]  

(25)

Using (22) we get the following formula

\[ \vec{F} = \kappa \sinh \left( \frac{E}{\kappa c^2} \right) \left\{ \dot{v} + \frac{1}{c^2} \frac{\cosh \left( \frac{E}{\kappa c^2} \right) \left( \dot{v} \vec{v} \right)}{1 - \frac{v^2}{c^2} \cosh \left( \frac{E}{\kappa c^2} \right)} \right\}. \]  

(26)

The relation (24) in the nondeformed case \( \kappa \to \infty \) gives the standard relativistic formula

\[ \vec{F} = \frac{E_{rel}}{c^2} \left\{ \dot{v} + \frac{\dot{v} \vec{v}}{c^2 - v^2} \right\} \quad E_{rel} = \frac{m_0 c^2}{\sqrt{1 - \frac{v^2}{c^2}}}. \]  

(27)

The formula (24) can be rewritten in a more familiar form with the use of the unit direction vector \( \vec{n} \). Then \( \vec{v} = v \vec{n} \) and

\[ \vec{F} = \kappa \sinh \left( \frac{E}{\kappa c^2} \right) \left\{ \frac{\dot{v} \vec{n}}{1 - \frac{v^2}{c^2}} + \dot{v} \vec{n} \right\}. \]  

(28)

This expression in the limit \( \kappa \to \infty \) leads to the well-known relativistic formula

\[ \vec{F} = \frac{E_{rel}}{c^2} \left\{ \frac{\dot{v} \vec{n}}{1 - \frac{v^2}{c^2}} + \dot{v} \vec{n} \right\}. \]  

(29)

4 The motion of a particle under a constant force

The simplest example of dynamics is the motion of a particle with the rest mass \( m_0 \) under the action of a constant force. Fixing the force as a constant quantity \( F = m_0 g \) (where \( g \) is a constant acceleration, for instance the gravity) with the same direction as the initial velocity \( \vec{v} \), we can discuss the consequences of the force \( \kappa \)-deformation for the equations of motion.

It is easy to see that from (24) a straight line motion in the velocity direction follows. Taking this line as the \( x \) coordinate, the equation (20) can be rewritten in a simpler form

\[ \dot{E} = v F = m_0 g v. \]  

(30)

Using again the equation (10), we get
\[ v(E) = c \sqrt{\frac{2 \cosh \left( \frac{E}{\kappa c^2} \right) - 2 \cosh \left( \frac{m_0}{\kappa} \right)}{\sinh \left( \frac{E}{\kappa c^2} \right)}}. \]  
(31)

Therefore

\[ \dot{E} = 2m_0gc \sqrt{\frac{\sinh^2 \left( \frac{E}{\kappa c^2} \right) - \sinh^2 \left( \frac{m_0}{\kappa} \right)}{\sinh \left( \frac{E}{\kappa c^2} \right)}} \]  
(32)

and from the derivative the following relation can be obtained

\[ E(t) = 2\kappa c^2 \arcsinh \sqrt{\left( \frac{m_0g}{2\kappa c^2} \right)^2 t^2 + \sinh^2 \left( \frac{m_0}{2\kappa} \right)}. \]  
(33)

Using the relation (28) and integrating the velocity (naturally \( v = \dot{x} \)) the motion of the particle is derived

\[ \cosh \left[ \frac{m_0g}{\kappa c^2} \left( x + \frac{c^2}{g} \right) \right] = \frac{1}{2} \left( \frac{m_0g}{\kappa c^2} \right)^2 t^2 + \cosh \left( \frac{m_0}{\kappa} \right) \]  
(34)

With the assumptions that the mass of the particle is much smaller than \( \kappa \) \((m_0 \leq \kappa)\) and \( gx \leq c^2 \) we obtain the following expansion of (22)

\[ \left( x + \frac{c^2}{g} \right)^2 = c^2 t^2 + \frac{c^4}{g^2} \left( 1 + \frac{1}{12} \frac{m_0^2}{\kappa^2} \right). \]  
(35)

Therefore, the last relation up to the quadratic term in \( \frac{1}{\kappa} \) has the same form as the standard relativistic hyperbolic motion of a particle under the action of the constant force \([14]\). Because the corrections are too small, the "deformed hyperboloid" of motion fits the standard relativistic curve closely, with only minor departures. We see, that in order to obtain significant differences between the deformed and standard case, the mass of the moving particle should have the value of the order of \( \kappa \).

### 5 Closing remarks

We showed that kinematics of the \( \kappa \)-deformed particle depends on the particular choice of the three-momentum basis. The standard and bicrossproduct momentum bases which are usually used have some unconventional properties from the physical point of view. Therefore, if one advocates the model of \( \kappa \)-deformed Poincaré symmetry then the conventional relativistic notions have to be revisited or one should find such the three-momentum basis i.e the function \( f \) for which the energy dependencies of the momentum and velocity behave similarly to the ordinary special relativity. This problem will be considered elsewhere (see \([13]\)).

The considerations of \( \kappa \)-deformed motion of a particle under the action of a constant force show that the departures of the hyperboloid obtained in this work
from the standard one are too small to be observed in today’s experiments. Therefore, it seems that the dynamical behaviour of $\kappa$-particle can not decide the validity of $\kappa$-deformed relativistic symmetry.

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