Short-wavelength soliton in a fully degenerate quantum plasma

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We present a novel one-dimensional nonlinear evolution equation governing the dynamics short-wavelength longitudinal waves in a nonrelativistic fully degenerate quantum plasma using kinetic equation for the Wigner function. The linear dispersion of the equation has a form of "zero sound" $\omega \sim k \exp(-k^2)$, where $k$ is the wave number, and it strongly differs from previously known nonlinear evolution equations. We numerically find the corresponding soliton solutions and demonstrate that the collisions between three solitons turn out to be elastic resulting only in phase shifts.

I. INTRODUCTION

Nonlinear evolution equations are widely used as models to describe many phenomena in various field of nonlinear science. The classical examples of these equations are well-known universal models in dispersive nonlinear media, such as the Korteweg-de Vries (KdV) and nonlinear Schrodinger (NLS) equations etc. [1, 2]. A common feature of nonlinear evolution equations is the presence of dispersion and nonlinearity which in some cases can effectively balance each other and lead to soliton formation.

In classical plasma, the most known examples of solitons are the ion-acoustic soliton described by the KdV equation which corresponds to the linear dispersion $\omega \sim k^3$, the Langmuir soliton of the NLS equation (in the subsonic regime) and Alfvén soliton of the derivative NLS equation with the dispersion $\omega \sim k^2$. Here, $\omega$ and $k$ are the frequency and wave number respectively in suitable dimensionless variables. At present, a comparatively large number of nonlinear equations, including multidimensional ones, and, accordingly, their soliton solutions in classical plasma are known [3, 4]. Note here that, generally speaking, most nonlinear equations admitting soliton solutions are not completely integrable (unlike, for instance, the KdV, NLS and derivative NLS equations and some others [2]), and, therefore, have no the exact solutions describing the elastic collisions between solitons.

Quantum effects in plasmas are important in the limit of low-plasma temperature and high-particle number density [5, 6]. Such plasmas are ubiquitous in micro-electronic devices, in dense astrophysical plasma, in microplasmas, and in laser plasmas (for example, see the reviews [8, 9] and references therein). There are two well-known models for describing the quantum effects in a plasma. The Wigner and Hartree models are based upon the Wigner-Poisson and Schrödinger-Poisson systems which, respectively, correspond to the statistical and hydrodynamic description of the plasma particles. Kinetic models of quantum plasmas are based on the time evolution equation for one-particle Wigner function in the mean-field approximation, in which the self-consistent electrostatic or electromagnetic fields are described by either Poisson’s equation or Maxwell’s equations, respectively. The quantum hydrodynamic (QHD) model generalizes the fluid model by including of quantum statistical pressure and quantum diffraction (the Bohm potential) terms. Later on, the quantum hydrodynamics model for plasmas was extended to include magnetic fields and spin dynamics. In the framework of kinetic description, explicit nonlinear one-dimensional solutions for the stationary Wigner and Wigner-Poisson equations were presented in Ref. [10]. Based on the kinetic Wigner-Poisson model of quantum plasma, the kinetic quantum Zakharov equations that describe nonlinear coupling of Langmuir waves to low frequency plasma density variations for cases of non-degenerate and degenerate plasma electrons were obtained in Refs. [13, 14]. On the other hand, it is the QHD model that is most often used to study nonlinear phenomena, in particular, nonlinear waves and solitons in quantum plasmas.

In this case, the reductive perturbation technique is usually used to obtain evolution equations that take into account weak dispersion and weak nonlinearity. In the framework of the QHD model, the KdV equation and acoustic solitons in nonrelativistic, unmagnetized cold quantum plasmas were obtained in Ref. [16]. In the same model, nonlinear periodic waves (cnoidal waves), which generalize soliton solutions of the KdV equation to the case of periodic boundary conditions, were investigated in Ref. [17]. In Ref. [18] linear ion-acoustic waves and ion-acoustic solitons of the KdV equation were studied in the QHD model for nonrelativistic, unmagnetized quantum plasma with electrons with an arbitrary degeneracy degree. Under this, the equation of state for electrons follows from a local Fermi-Dirac distribution function and applies equally well both to fully degenerate and classical, nondegenerate limits. In the long-wavelength limit, the results agree with quantum kinetic theory. The KdV equation and the corresponding soliton solutions were obtained also in a magnetized quantum plasma in the form of the magnetoacoustic solitons, and in a plasma with relativistically degenerate electrons [20]. Solitons of the Kadomtsev-Petviashvili equation, which is a multidimensional generalization of the KdV equation for an unmagnetized or weakly magnetized plasma, were obtained...
for quantum plasmas in Ref. [21]. The same equation was derived for describing ion-acoustic nonlinear waves in cold quantum electron-positron-ion plasma [22] and for magnetoacoustic solitons in magnetized quantum plasma [23]. The modified KdV equation (the KdV with cubic nonlinearity, and also completely integrable) in a quantum plasma and its soliton were also obtained in Ref. [24]. Solitons of the Zakharov-Kuznetsov equation [3, 25], which is a multidimensional generalization of the KdV equation for magnetized plasma, were considered for cold quantum plasma in Refs. [26, 27] in the form of the usual ion-acoustic solitons, explosive solitons and nonlinear periodic waves in terms of the Jacobi elliptic functions and in Ref. [28] for magnetized quantum plasma with arbitrary degeneracy of electrons. Recently, a nonlinear string equation (one of the forms of the Boussinesq equation) with the linear dispersion $\omega^2 \sim -k^2 + k^4$ and quadratic nonlinearity for describing quantum two-stream instability has been suggested in Ref. [29]. The corresponding analytical solutions have the form of the blow-up solitons.

As is known, the considered hydrodynamic models are valid only in the long-wavelength case $k \ll 1$ (the wave number is normalized to the Fermi-Debye length) [2]. Then, the appearance of solitons is due to the balance of weak dispersion $k^3 \ll 1$ for the KdV equation, and $k^2 \ll 1$ for the NLS, for example, and weak nonlinearity (quadratic and cubic respectively). The goal of this paper is to derive a novel evolution equation describing short-wavelength ($k \gg 1$) nonlinear waves in a nonrelativistic fully degenerate quantum plasma using the kinetic approach. Despite the specific nature of the dispersion in the short-wavelength limit (the dispersion of the zero sound) $\omega \sim k \exp(-k^2)$, which has no counterpart in classic plasmas, we show that balance between the weak dispersion and weak quadratic nonlinearity lead to the formation of solitons. Moreover, we show that collisions between even three solitons are elastic.

The paper is organized as follows. In Sec. II, we present our model nonlinear equation that governs the dynamics of longitudinal waves in a fully degenerate quantum plasma in the short-wavelength limit. The soliton solutions and elastic collisions between solitons are presented in Sec. III. Finally, Sec. IV concludes the paper.

II. MODEL EQUATION

Dielectric functions and dispersion relations of quantum degenerate plasmas have been calculated using kinetic theory [6, 22, 23]. In the semiclassical limit $\hbar k \ll p_F = h(3\pi^2 n_0)^{1/3}$, the longitudinal dielectric function in a plasma with completely degenerate electrons in the zero-temperature limit reads

$$\varepsilon(\omega, k) = 1 + 3\frac{\omega^2}{k^2 p_F^2} \left[ 1 - \frac{\omega}{2k p_F} \ln \left( \frac{\omega + k p_F}{\omega - k p_F} \right) \right] + \frac{\omega}{2k p_F} \theta(k^2 p_F^2 - \omega^2),$$  \hspace{1cm} (1)

where $\theta(x)$ is the Heaviside step function, $\omega$ and $k$ are the frequency and the wave vector respectively. The goal of this paper is to derive a novel evolution equation describing short-wavelength limit $k \gg 1$ for the KdV, and $k^2 \gg 1$ for the NLS, for example. The goal of this paper is to derive a novel evolution equation describing short-wavelength ($k \gg 1$) nonlinear waves in a fully degenerate quantum plasma using the kinetic approach. Despite the specific nature of the dispersion in the short-wavelength limit (the dispersion of the zero sound) $\omega \sim k \exp(-k^2)$, which has no counterpart in classic plasmas, we show that balance between the weak dispersion and weak quadratic nonlinearity lead to the formation of solitons. Moreover, we show that collisions between even three solitons are elastic.

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for the quantum electron distribution function (Wigner function) $F(x, v, t)$ can be written as

$$
\frac{\partial F}{\partial t} + v \frac{\partial F}{\partial x} = -\frac{i e}{2 m} \hbar \int d\lambda d\lambda^\prime \exp \left[ \frac{m}{\hbar} (v - v^\prime) \lambda \right] \
\times \left[ \varphi \left( x + \frac{\lambda}{2} t \right) - \varphi \left( x - \frac{\lambda}{2} t \right) \right] F(x, v^\prime, t),
$$

(5)

where $\varphi$ is the electrostatic potential. In the momentum space eq. 5 can be written as

$$
(\omega - kv) f_q(v) = \frac{e \hbar}{m} \varphi_q \left[ f^{(0)}(v + \frac{\hbar k}{2m}) - f^{(0)}(v - \frac{\hbar k}{2m}) \right] \
+ \frac{e}{\hbar} \sum_{q=q_1+q_2} \varphi_{q_1} \left[ f_{q_2}(v + \frac{\hbar k_1}{2m}) - f_{q_2}(v - \frac{\hbar k_1}{2m}) \right].
$$

(6)

where $f_q(v)$ is the deviation of the electron distribution function from the equilibrium one $f^{(0)}(v)$, and $\varphi$ is the electrostatic potential. The distribution function $f^{(0)}(v)$ is normalized to the equilibrium plasma density, $\int f^{(0)}(v) dv = n_0$. Integrating over $\lambda$ in eq. (6) yields

$$
\exp \left[ \frac{m}{\hbar} (v - v^\prime) \pm \frac{i k}{2} \right] \lambda \rightarrow 2\pi \delta \left[ \frac{m}{\hbar} (v - v^\prime) \pm \frac{k}{2} \right],
$$

(7)

and then integrating over $v^\prime$ yields

$$
(\omega - kv) f_q(v) = \frac{e}{\hbar} \varphi_q \left[ f^{(0)}(v + \frac{\hbar k}{2m}) - f^{(0)}(v - \frac{\hbar k}{2m}) \right] \
+ \frac{e}{\hbar} \sum_{q=q_1+q_2} \varphi_{q_1} \left[ f_{q_2}(v + \frac{\hbar k_1}{2m}) - f_{q_2}(v - \frac{\hbar k_1}{2m}) \right].
$$

(8)

We present the function $f_q(v)$ as a series in powers of the field strength (i.e. $f_q^{(n)} \sim \varphi^n$)

$$
f_q(v) = \sum_{n=1}^{\infty} f_q^{(n)}(v).
$$

(9)

In the linear approximation from Eqs. (3) and (4) we have

$$
f_q^{(1)} = \frac{e \varphi_q}{\hbar (\omega - kv)} \left[ f^{(0)}(v + \frac{\hbar k}{2m}) - f^{(0)}(v - \frac{\hbar k}{2m}) \right],
$$

(10)

and then one can write the recurrence relation

$$
f_q^{(n)} = \frac{e}{\hbar (\omega - kv)} \sum_{q=q_1+q_2} \varphi_{q_1} \left[ f_{q_2}^{(n-1)}(v + \frac{\hbar k_1}{2m}) - f_{q_2}^{(n-1)}(v - \frac{\hbar k_1}{2m}) \right].
$$

(11)

For the nonlinear terms ($n \geq 2$) we use the quasiclassical approximation and in the limit $\hbar k/(2m) \ll v$ in Eq. (11) one can expand

$$
f_{q_2}^{(n-1)} \left( v \pm \frac{\hbar k_1}{2m} \right) \approx f_{q_2}^{(n-1)}(v) \pm \frac{\partial f_{q_2}^{(n-1)}(v)}{\partial v} \frac{\hbar k_1}{2m}
$$

(12)

whence we get

$$
f_q^{(n)} = \frac{m}{m(\omega - kv)} \sum_{q=q_1+q_2} k_1 \varphi_{q_1} \frac{\partial f_{q_2}^{(n-1)}}{\partial v}.
$$

(13)

Retaining terms in Eq. (9) up to second order in the wave fields and substituting $f_q$ into the Poisson equation

$$
k^2 \varphi_q = -4\pi e \int f_q(v) dv,
$$

(14)

where the ion contribution is omitted, we get

$$
\varepsilon_q \varphi_q = \sum_{q=q_1+q_2} V_{q_1, q_2} \varphi_{q_1} \varphi_{q_2},
$$

(15)

where

$$
\varepsilon_q = 1 + \frac{4\pi e^2}{m} \int \frac{f^{(0)}(v + \frac{\hbar k}{2m}) - f^{(0)}(v - \frac{\hbar k}{2m})}{(\omega - kv)} dv,
$$

(16)

or, after suitable change of variables (17)

$$
\varepsilon_q = 1 - \frac{4\pi e^2}{m} \int \frac{f^{(0)}(v)}{(\omega - kv)^2 - \hbar^2 k^4/(4m^2)} dv,
$$

(17)

is the linear electron dielectric response function from which one can obtain Eq. (1) and, in particular, the linear dispersion law Eq. (2), and the interaction matrix element (the nonlinear dielectric susceptibility) $V_{q_1, q_2}$ is determined by

$$
V_{q_1, q_2} = -\frac{e}{2m n_0 k^2} \int \frac{k_1}{\omega_1 + \omega_2 - (k_1 + k_2)v} \
\times \frac{\partial}{\partial v} \frac{k_2}{\omega_2 - k_2v} \frac{\partial f^{(0)}}{\partial v} dv + (\omega_1, k_1 \rightarrow \omega_2, k_2).
$$

(18)

Note that the expression (18) for the interaction matrix element $V_{q_1, q_2}$ is written in a symmetrized form. Singularities in the denominators in Eqs. (18) and (19) are avoided, as usual, using Landau’s rule by replacing $\omega \rightarrow \omega + i0$ and then

$$
(\omega - kv)^{-1} = \mathcal{P}(\omega - kv)^{-1} - i\pi \delta(\omega - kv),
$$

(19)

$$
[\omega_1 + \omega_2 - (k_1 + k_2)v]^{-1} = \mathcal{P}[\omega_1 + \omega_2 - (k_1 + k_2)v]^{-1} \
- i\pi \delta[\omega_1 + \omega_2 - (k_1 + k_2)v],
$$

(20)

where $\mathcal{P}$ is the principal value of the integrals. Imaginary parts in Eqs. (19) and (20) account for the linear and nonlinear Landau damping respectively. As noted above, the linear Landau damping in a fully degenerate quantum plasma for the wave with dispersion Eq. (3) is absent, but the nonlinear Landau damping due to the interaction of the beat waves with plasma particles is possible. In this work we neglect the nonlinear Landau damping and only the principal value of the integrals is understood, although the corresponding damping term can be easily
obtained from Eq. (20) in the same way as the nonlinear Landau damping is obtained in the kinetic derivation of the NLS equation for Langmuir waves in classic plasma [4]. After two partial integrations in Eq. (15), one can write

\[
V_{q_1,q_2} = -\frac{e}{2m n_0 k^2} \int \left[ \frac{2k^2 k_1 k_2}{\omega - k v^2} f(0) dv + (\omega_1, k_1 \rightarrow \omega_2, k_2), \right]
\]

(21)

where \( \omega = \omega_1 + \omega_2 \) and \( k = k_1 + k_2 \). Expanding \( \epsilon(\omega, k) \) given by Eq. (11) near the \( \omega_k \) determined by Eq. (9) yields

\[
\epsilon(\omega, k) = \epsilon(\omega_k, k) + \epsilon'(\omega_k)(\omega - \omega_k)
\]

(22)

where \( \epsilon'(\omega_k) \equiv \partial \epsilon(\omega)/\partial \omega \big|_{\omega=\omega_k} \) and in the leading order one can get

\[
\epsilon'(\omega_k) = \frac{3\omega_p^2}{4k^2 v_F^2} \exp \left( \frac{2k^2 v_F^2}{3\omega_p^2} + 2 \right).
\]

(23)

After substituting Eq. (22) into Eq. (15) we have

\[
(\omega - \omega_k)\varphi_q = \frac{1}{\epsilon'(\omega_k)} \sum_{q=q_1+q_2} V_{q_1,q_2} \varphi_{q_1} \varphi_{q_2}.
\]

(24)

Note that the direct substitution of the equilibrium distribution function for the one-dimensional fully degenerate electrons \( \partial f(0)/\partial v = -n_0/(2v_F)\delta(v_F - |v|) \) into Eq. (15) after one partial integration is in excess of accuracy, since it takes into account dispersion terms in the nonlinearity which correspond to \( k_1v \) in the denominators. Thus, we neglect these terms in Eq. (21) and in the leading order the nonlinear dielectric susceptibility Eq. (21) becomes

\[
V_{q_1,q_2} = -\frac{e}{2m k^2} \left\{ \frac{2k^2 k_1 k_2}{\omega_1 \omega_2} + \frac{kk_1 k_2}{\omega_2^2} + (\omega_1, k_1 \rightarrow \omega_2, k_2) \right\}.
\]

(25)

Essentially, that the wave dispersion in Eq. (3) has an acoustic type and in the leading term satisfies the three-wave resonance condition

\[
\omega_k = \omega_{k_1} + \omega_{k_2}, \quad k = k_1 + k_2.
\]

(26)

In particular, this means that this condition, together with taking into account only the quadratic nonlinearity, ensures the validity of the successive approximation in Eq. (9), and this is equivalent [35, 40] to the multi-time-scale perturbation expansion, i. e. the secular terms are removed automatically. Taking into account Eqs. (4) and (20) when calculating Eq. (25), then substituting Eqs. (23) and (25) into Eq. (21), and introducing the slow time scale \( \Omega = \omega - k v_F \) which balances the dispersion in Eq. (3) (compare, for example, kinetic derivation of the KdV equation in Ref. [3]), we finally get

\[
\left[ \Omega - 2k v_F \exp \left( -\frac{k^2 v_F^2}{3\omega_p^2} - 2 \right) \right] \varphi_q = -\frac{2e}{mv_F} k \exp \left( -\frac{k^2 v_F^2}{3\omega_p^2} - 2 \right) \sum_{q=q_1+q_2} \varphi_{q_1} \varphi_{q_2}.
\]

(27)

After rescaling

\[
k \rightarrow \frac{v_F}{\omega_p}, \quad \Omega \rightarrow \frac{\exp(2)}{2\omega_p}, \quad \Phi \rightarrow -\frac{4e}{mv_F}\varphi,
\]

(28)

equation (27) can be written in the dimensionless form

\[
\left[ \Omega - k \exp \left( -\frac{2k^2}{3} \right) \right] \Phi_q = k \exp \left( -\frac{2k^2}{3} \right) \sum_{q=q_1+q_2} \Phi_{q_1} \Phi_{q_2}.
\]

(29)

By introducing the operator \( L \) acting in the physical space as

\[
\hat{L}f(x) = \int i k \exp(-2k^2/3)e^{-ikx} \hat{f}(k) \, dk,
\]

(30)

where \( f(x) \) is an arbitrary function and \( \hat{f}(k) \) is its Fourier transform, and using the convolution identity

\[
(fg)_k = \int \hat{f}_{k_1} \hat{g}_{k_2} \delta(k - k_1 - k_2) \, dk_1 \, dk_2.
\]

(31)

one can write Eq. (29) in the physical space as

\[
\partial_t \Phi + \hat{L} \Phi + \hat{L} \Phi^2 = 0,
\]

(32)

so that the nonlinearity has a nonlocal character. It is seen that "motionless" \( (\partial_t = 0) \) solutions is not possible. Note also that in the considered short-wavelength case \( k > 1 \), Eq. (29) can not be simplified by any expansion in \( k \).

### III. SOLITON SOLUTION AND COLLISIONS BETWEEN SOLITONS

We look for stationary traveling solutions of Eq. (32) of the form \( \Phi(x,t) = \Phi(x - vt) \), where \( v \) is the velocity of propagation in the \( x \) direction. In the Fourier space the stationary solution corresponds to \( \Phi_q = \Phi_q \delta(\Omega - k v) \) and from Eq. (29) we have

\[
[\nu - \exp(-2k^2/3)] \Phi_k = \exp(-2k^2/3) \sum_{k=k_1+k_2} \Phi_{k_1} \Phi_{k_2}.
\]

(33)

Finding an analytical solution of Eq. (33) apparently does not seem possible but one can find soliton solutions numerically using the Petviashvili method [41, 42]. Equation (33) are written in the form

\[
G_k \Phi_k = B_k,
\]

(34)
where \( G_k = \left[ v - \exp \left( -2k^2/3 \right) \right] \) and \( B_k \) accounts for the nonlinear term. Then the Petviashvili iteration procedure at the \( n \)-th iteration is

\[
\Phi_k^{(n+1)} = sG_k^{-1}B_k^{(n)},
\]

where \( s \) is the so called stabilizing factor determined by

\[
s = \left( \frac{\int |\Phi_k^{(n)}|^2 dk}{\int \Phi_k^{*(n)}G_k^{-1}B_k^{(n)} dk} \right)^\gamma.
\]

and \( \gamma = 2 \) for the quadratic nonlinearity, the parenthetic superscript denotes the iteration step index. Nonlinear terms at each step were calculated by using Eq. (31). The procedure always converges to the nonlinear ground state, i.e. soliton, regardless of the initial guess. Moreover, the rate of convergence is almost independent of the initial approximation. We used \( \Phi(x) = \exp(-x^2) \) as the initial guess in all runs. Iterations rapidly converge to a soliton solution provided that \( v > 1 \). In physical variables, the soliton velocity should satisfy the condition \( v > 2 \exp(-2)\nu_F \sim 0.27\nu_F \). Note that for the group velocity \( \nu_{gr} = \partial \omega / \partial k \) of linear waves with dispersion Eq. (3) (taking into account \( kv_F/\omega_p \gg 1 \)) is

\[
\nu_{gr} = \nu_F \left[ 1 - \frac{8k^2\nu_F^2}{3\exp(2)\omega_p^2} \exp \left( - \frac{2k^2\nu_F^2}{3\omega_p^2} - 2 \right) \right]
\]

and \( \nu_{gr} < \nu_F \). Simulations show that the soliton amplitude grows linearly with increasing the velocity \( v \), as for the solitons of the KdV equation (but not for the Langmuir solitons of the NLS equation, for which the soliton velocity and amplitude are independent parameters). Examples of the solitons with different velocities (amplitudes) are presented in Fig. 1.

We note that, generally speaking, collisions between solitons in nonintegrable models can be almost elastic under certain conditions, for instance, if the soliton amplitudes and velocities are sufficiently close to each other. To study the time evolution of the solitons under their collisions, we numerically solve the nonlinear equation (32) with the initial conditions given by a superposition of \( N \leq 3 \) soliton solutions

\[
\Phi(x,t) = \sum_{i=1}^{N} \Phi_i(x-x_i,t)
\]

at the time \( t = 0 \), where \( \Phi_i \) correspond numerically found (up to machine accuracy) soliton solutions with essentially different velocities \( v_i \). The time integration is per-
formed by a fourth order Runge-Kutta method with the variable time step and local error control. The periodic boundary conditions are assumed. The linear and nonlinear terms are computed in spectral space. The simulations have been performed for various values of soliton velocities (and, therefore, amplitudes) both for two soliton ($N = 2$) and three ($N = 3$) soliton collisions. An example of the elastic collision between three solitons with the velocities $v_1 = 4.1, v_2 = 2.5$ and $v_2 = 1.5$ is shown in Fig. 2. In particular, it can be seen that at the time $t = 23$ in the inset Fig. 2 all three solitons undergo distortion simultaneously so that two distant solitons feel each other through an intermediate soliton – this is a typical many-soliton effect. Then, the solitons fully reconstruct their initial form without any emitting wakes of radiation ($t = 60$), resulting only in phase shifts. The overall picture closely resembles the elastic soliton collisions in the integrable models \([2, 45, 46]\).

**IV. CONCLUSIONS**

In conclusion, we have derived the nonlinear evolution equation governing dynamics of the short-wavelength longitudinal waves in the fully degenerate quantum plasma with the ”zero-sound” dispersion and numerically found the soliton solutions. By numerical simulation we have shown that soliton collisions are elastic.

The elastic collisions between three solitons might suggest that equation \([42]\) has exact $N$-soliton solutions and is completely integrable just like for KdV equation and others \([2, 45, 46]\), but this is most likely not the case. In the inverse scattering transform approach there exists a relationship between some function $\hat{\omega}(\lambda)$, where $\lambda$ is the spectral parameter, and the dispersion relation $\omega(k)$ of the linearized equation \([45]\). In all known cases $\hat{\omega}(\lambda)$ is the rational function of $\lambda$ though the associated spectral problem may involve meromorphic functions of the spectral parameter $\lambda$ like the elliptic Jacobi functions, as in the case of the Landau-Lifshitz equation \([46]\). In any case, the integrability of equation \([42]\) seems to be an open question.

Note that the dispersion relation Eq. \([3]\) is the same as in classical nondegenerate ultrarelativistic plasmas, i.e. when the plasma particle temperature significantly exceeds the particle rest mass \([6, 34, 47, 48]\), in the short-wavelength limit $k \gg \omega_p/c$ with the replacement $v_F \to c$, where $c$ is the speed of light and $\omega_p \to \sqrt{4\pi e^2\lambda_0/(3T)}$ is the ultrarelativistic electron Langmuir frequency and $T$ is the electron temperature. In reality, ultrarelativistic plasma consists of electrons and positrons and, in most cases, a small impurity of nonrelativistic ions. A preliminary consideration shows that for pure electron-positron plasma in the thermal equilibrium (electron and positron temperatures are equal), quadratic nonlinearity vanishes identically. The situation changes drastically if ions are present, then one can obtain an equation similar to Eq. \([42]\). Under this, the dominant nonlinear term comes from the electrons and positrons while the ion contribution in nonlinearity is negligible. The detailed analysis will be presented elsewhere.

**V. DATA AVAILABILITY**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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