Bethe ansatz matrix elements as non-relativistic limits of form factors of quantum field theory

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Abstract. We show that the matrix elements of integrable models computed by the algebraic Bethe ansatz (BA) can be put in direct correspondence with the form factors of integrable relativistic field theories. This happens when the $S$-matrix of a Bethe ansatz model can be regarded as a suitable non-relativistic limit of the $S$-matrix of a field theory, and when there is a well-defined mapping between the Hilbert spaces and operators of the two theories. This correspondence provides an efficient method to compute matrix elements of Bethe ansatz integrable models, overcoming the technical difficulties of their direct determination. We analyze this correspondence for the simplest example in which it occurs, that is, the quantum nonlinear Schrödinger and the Sinh–Gordon models.

Keywords: form factors, integrable quantum field theory, quantum integrability (Bethe ansatz)

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1. Introduction

The aim of this paper is to show that a direct correspondence exists between the matrix elements\(^4\) computed by the algebraic Bethe ansatz in non-relativistic integrable models (in the following simply referred to as Bethe ansatz models) [1] and the form factors considered in relativistic integrable quantum field theories [2]. As shown below, the relation between the two quantities can be established along the lines of recent studies on the non-relativistic limit of a quantum field theory [3, 4].

The discovery of such a correspondence may greatly help deepen our general knowledge of integrable models and, in particular, shed new light on the calculation of their correlation functions. The reason is the following: while the direct computation of Bethe ansatz matrix elements proves to be quite a difficult task (often carried out successfully only for few operators), the computation of form factors is instead a simpler problem. In the latter case, for instance, one can take advantage of additional constraints coming from the relativistic invariance of the field theory and as a result, explicit expressions of form factors can be usually found not only for a few operators but also for a larger number of them: actually, the classification of all operators of a quantum field theory can be obtained in terms of the different solutions of the form factor equations [5, 6].

At the heart of this correspondence lies the $S$-matrix, both of the relativistic field theory and the Bethe ansatz model: if the $S$-matrix of the latter is obtained by a suitable non-relativistic limit of the $S$-matrix of the former, then the form factors of the quantum field theory go smoothly to the Bethe ansatz matrix elements. Obviously one has to be sure that there is also a one-to-one mapping between the Hilbert spaces and operators of the two theories. But if one can prove that such a mapping exists, it is then easy to understand why the field theory form factors reduce to the Bethe ansatz matrix elements: this happens because the analytic properties of the form factors and the Bethe ansatz matrix elements are dictated by the $S$-matrices of the corresponding theories. In the

\(^4\) Matrix elements of the Bethe ansatz models are also called ‘form factors’ in the literature. Although the main conclusion of this paper is that they are proper limit expressions of the relativistic form factors. To avoid confusion about which quantities we are referring to in various parts of the paper, we will call the Bethe ansatz quantities ‘matrix elements’ while we will reserve the term ‘form factors’ for those relative to quantum field theories (QFT).
following we provide evidence for this correspondence by analyzing the simplest models in which it occurs: the quantum nonlinear Schrödinger (QNLS) model on one side, and the Sinh–Gordon (ShG) model on the other. In particular, our approach provides a universal method to compute matrix elements of any local operator in the QNLS model.

2. Matrix elements in the algebraic Bethe ansatz

In this section we summarize without derivation some basic properties and results of the algebraic Bethe ansatz solution of the QNLS system associated to the Lieb–Liniger model [7]. The interested reader is referred to the book [1] and references therein.

The Hamiltonian of the model in volume $L$ with periodic boundary conditions is given by

$$H_{\text{QNLS}} = \int_0^L dx \left( \partial_x \psi^\dagger \partial_x \psi + c \psi^\dagger \psi^\dagger \psi \psi \right),$$

where $\psi(x, t)$ and $\psi^\dagger(x, t)$ are canonical Bose fields

$$[\psi(x, t), \psi^\dagger(y, t)] = \delta(x - y),$$

and $c$ is the coupling constant. The Fock-vacuum is defined as

$$\psi|0\rangle = 0, \quad \langle 0|\psi^\dagger = 0.$$  

The QNLS model can be solved via the algebraic Bethe ansatz (BA). The monodromy matrix reads

$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

and its entries act in a space consisting of states

$$|\lambda_1, \ldots, \lambda_N\rangle = \prod_{j=1}^N B(\lambda_j)|0\rangle, \quad N = 0, 1, \ldots,$$

where $B(\lambda) = B(\lambda) \exp(-i\lambda L/2)$, $\{\lambda\}$ are arbitrary complex parameters, and the pseudo-vacuum $|0\rangle$ coincides with the Fock-vacuum. Similarly, dual states can be constructed using the operators $C(\lambda) = C(\lambda) \exp(-i\lambda L/2)$.

The $R$-matrix describes the commutation relations of the monodromy matrix entries and it satisfies the Yang–Baxter equations. For the QNLS model it can be written in the form

$$R(\lambda, \mu) = \begin{pmatrix} f(\mu, \lambda) & 0 & 0 & 0 \\ 0 & g(\mu, \lambda) & 1 & 0 \\ 0 & 1 & g(\mu, \lambda) & 0 \\ 0 & 0 & 0 & f(\mu, \lambda) \end{pmatrix}$$

with

$$f(\mu, \lambda) = \frac{\mu - \lambda + ic}{\mu - \lambda}, \quad g(\mu, \lambda) = \frac{ic}{\mu - \lambda}.$$
The transfer matrix $\tau(\lambda) = \text{tr} T(\lambda) = A(\lambda) + D(\lambda)$ generates the complete set of the conservation laws of the model. The eigenstates of the transfer matrix have the form (5), however the parameters $\{\lambda\}$ are not arbitrary but they satisfy the system of Bethe equations
\begin{equation}
\exp[i\lambda_j L] \prod_{k=1, k \neq j}^N \tilde{S}_{\text{QNLS}}(\lambda_j, \lambda_k) = 1,
\end{equation}
where the two-particle $S$-matrix is given by
\begin{equation}
\tilde{S}_{\text{QNLS}}(\lambda_j, \lambda_k) \equiv \frac{f(\lambda_k, \lambda_j)}{f(\lambda_j, \lambda_k)} = \frac{\lambda_j - \lambda_k - ic}{\lambda_j - \lambda_k + ic}.
\end{equation}

The $S$-matrix gives the phase factor by which the state gets multiplied when the particles $i$ and $j$ are interchanged. Hence the Bethe equations say that the total phase-shift acquired when a particle of momentum $\lambda_j$ is taken to a round trip comes from the usual phase which is proportional to the momentum plus the scattering phase shifts picked up when the particle scatters through all the other particles. Taking the logarithm leads us to
\begin{equation}
\tilde{Q}_j = \lambda_j L + \sum_{k=1, k \neq j}^N \frac{1}{4} \log \tilde{S}_{\text{QNLS}}(\lambda_j, \lambda_k) = 2\pi I_j, \quad I_j \in \mathbb{Z}.
\end{equation}

Using the algebra satisfied by the monodromy matrix, the scalar products of the BA states (5) can be worked out explicitly, as well as the action of the operator $\psi$ on these states. However, the calculation of the scalar products proved to be a highly non-trivial combinatorial problem (see [1] and references therein). As a result, the norms of the states with parameters $\lambda$ that satisfy the Bethe equations (8) are
\begin{equation}
\langle \lambda_1, \ldots, \lambda_N | \lambda_1, \ldots, \lambda_N \rangle = c_N \prod_{j,k=1, j \neq k}^N f(\lambda_j, \lambda_k) \tilde{\rho}_N,
\end{equation}
where
\begin{equation}
\tilde{\rho}_N = \det \left( \frac{\partial \tilde{Q}_j}{\partial \lambda_k} \right)
\end{equation}
is the Gaudin determinant associated to the Bethe equations (8). Knowing the action of $\psi$ on the BA states and the scalar products, its unnormalized matrix elements
\begin{equation}
\tilde{F}_N^\psi(\lambda_1, \ldots, \lambda_{N-1} | \lambda_1, \ldots, \lambda_N) = \langle \lambda'_1, \ldots, \lambda'_{N-1} | \psi(0,0) | \lambda_1, \ldots, \lambda_N \rangle
\end{equation}
can be given explicitly. These matrix elements are originally defined for states which solve the Bethe equations. However, one can define a function $F_N$ such that the actual matrix elements will be given by the value of this function taken at the particular set of momenta which satisfy the Bethe equations. Hence, the function $F_N$ itself does not carry any information about the system size $L$: in fact, this only enters the Bethe equations satisfied by the physical momenta. Note that the only non-zero matrix elements of $\psi$ are
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for states where the difference of the particle numbers is one and that the functions $F_N$ are symmetric separately in the momenta $\lambda$ and $\lambda'$.

We will give explicit examples for matrix elements in section 4. However, it is important to note that they satisfy the recursion relation [8]

$$
\tilde{F}_N^{\psi}(\lambda'_1, \ldots, \lambda'_{N-1}|\lambda_1, \ldots, \lambda_N) \xrightarrow{\lambda_1 - \lambda_1' \to i c} \frac{i c}{\lambda_1 - \lambda_1'} 
\times \left( \prod_{j=2}^{N-1} f(\lambda'_1, \lambda'_j) \prod_{j=2}^{N} f(\lambda_j, \lambda_1) - \prod_{j=2}^{N-1} f(\lambda'_j, \lambda'_1) \prod_{j=2}^{N} f(\lambda_1, \lambda_j) \right)
\times \tilde{F}_{N-1}^{\psi}(\lambda'_1, \ldots, \lambda'_{N-2}|\lambda_1, \ldots, \lambda_{N-1}) + \cdots,
$$

(14)

where the dots stand for non-singular parts. We would like to remark that similar recursive equations appear also for the form factors of the infinite XXZ spin chain [9].

Similarly, matrix elements of the current operator $j(x) = \psi^\dagger(x)\psi(x)$ can also be determined. The current has non-zero matrix elements only between states having the same number of particles,

$$
\tilde{F}_N^{\psi}(\lambda'_1, \ldots, \lambda'_{N}|\lambda_1, \ldots, \lambda_N) = \langle \lambda'_1, \ldots, \lambda'_{N}|\psi^\dagger(0,0)\psi(0,0)|\lambda_1, \ldots, \lambda_N \rangle,
$$

(15)

and they also satisfy the recursion relations (14) with the obvious change in the number of particles of the dual vector $(N - 1) \to N$.

3. Form factors in the Sinh–Gordon model

The ShG model is an integrable relativistic field theory in 1 + 1 dimensions defined by the Lagrangian density

$$
\mathcal{L} = \frac{1}{2} \left( \frac{\partial \phi}{c_l \partial t} \right)^2 - \frac{1}{2} \left( \frac{\partial \phi}{\partial x} \right)^2 - \frac{m_0^2 c_l^2}{g^2 \hbar^2} \left( \cosh(g \phi) - 1 \right),
$$

(16)

where $\phi = \phi(x,t)$ is a real scalar field, $m_0$ is a mass scale and $c_l$ is the speed of light. The parameter $m_0$ is related to the physical (renormalized) mass $m$ of the particle by [10]

$$
m_0^2 = m^2 \frac{n \pi}{\sin(n \pi)}.
$$

(17)

The integrability of the ShG model implies the absence of particle production processes and its $n$-particle scattering amplitudes are purely elastic. Moreover, they factorize into $n(n - 1)/2$ two-body $S$-matrices which can be determined exactly via the $S$-matrix bootstrap [11]. The energy $E(\theta)$ and the momentum $P(\theta)$ of a particle can be written as $E(\theta) = M c_l^2 \cosh \theta$, $P(\theta) = M c_l \sinh \theta$, where $\theta$ is the rapidity. In terms of the rapidities the exact two-body $S$-matrix is given by [12]

$$
S_{\text{ShG}}(\theta_1, \theta_2) \equiv S(\theta_1, \theta_2) = \frac{\sinh \theta_{12} - i \sin(\alpha \pi)}{\sinh \theta_{12} + i \sin(\alpha \pi)},
$$

(18)

where $\theta_{12} = \theta_1 - \theta_2$ and $\alpha$ is the renormalized coupling constant

$$
\alpha = \frac{\hbar c_l g^2}{8 \pi + \hbar c_l g^2}.
$$

(19)

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The key observation, made in papers [3,4], is that the QNLS model can be regarded as a suitable non-relativistic limit of the ShG model, under which the two-particle $S$-matrices, the Hamiltonian and the thermodynamic Bethe ansatz equations of the ShG model go into the corresponding quantities of the QNLS model. The connection between the two theories is realized by taking a double limit, in which the speed of light $c_l$ goes to infinity, the coupling $g$ goes to zero, but with their product kept fixed and given by

$$c_l \to \infty, \quad g \to 0, \quad g c_l = \frac{4\sqrt{\epsilon}}{\hbar},$$

(20)

where $c$ is the coupling constant of the QNLS model. Taking such a double limit of the $S$-matrix of the ShG model one arrives at the $S$-matrix (9) of the QNLS model, once we set $m = 1/2$ and $\hbar = 1$. Note that $m_0 \to m$ in the limit.

Such a correspondence between $S$-matrices gives a hint that an exact mapping exists between the two models. Given that in a relativistic integrable model the two-particle $S$-matrix governs its entire dynamics (its thermodynamic properties and the form factors of all its operators), if this mapping exists then it has several interesting and far-reaching consequences. For instance, in the past a mapping between two relativistic $S$-matrices was used to establish the correspondence between the form factors and, based on this, between the operators of the theories [13]. As discussed below, in the present case the situation is more subtle, because the mapping operates between a relativistic theory and a non-relativistic model. The analysis in this case requires additional care about the structure of states and operators of the two Hilbert spaces.

Here we would like to shed light on the direct relation between the Bethe ansatz matrix elements of the QNLS and the form factors of the ShG model. In a relativistic field theory defined in an infinite volume the elementary form factors of a local operator $O$ are the matrix elements of $O(0, 0)$ between the vacuum and a set of $n$-particle asymptotic states:

$$F_n^O(\theta_1, \theta_2, \ldots, \theta_n) = \langle 0 | O(0, 0) | \theta_1, \ldots, \theta_n \rangle_{\text{in}}.$$  

(21)

The knowledge of all form factors of an operator is enough to realize how it acts on any state of the theory. In fact, a generic matrix element of the operator can be expressed in terms of its form factors by using the translation operator and the crossing symmetry which is implemented by an analytic continuation in the rapidity variables$^5$:

$$\langle \theta'_1, \ldots, \theta'_n | O(0, 0) | \theta_1, \ldots, \theta_k \rangle = F_{n+k}^C (\theta'_1 + i\pi, \ldots, \theta'_n + i\pi, \theta_1, \ldots, \theta_k).$$  

(22)

The form factors satisfy a set of functional and recursive equations, which for integrable models makes it possible to find in many cases their explicit expressions (for a review, see [2]). For a scalar operator, unitarity and crossing symmetry dictate the following functional equations

$$F_n(\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n) = S(\theta_k - \theta_{k+1}) F_n(\theta_1, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_n),$$  

(23a)

$$F_n(\theta_1 + 2\pi i, \ldots, \theta_n) = F_n(\theta_2, \ldots, \theta_n, \theta_1).$$  

(23b)

The form factors of integrable theories can have two kinds of simple poles and, except for these singularities, they are analytic in the strip $0 < \text{Im} \theta_{ij} < 2\pi$ (here $\theta_{ij} = \theta_i - \theta_j$). The

$^5$ If $\theta'_i = \theta_j$ for some $i$ and $j$, this formula gets modified by contact terms.
first kind of poles corresponds to kinematical singularities at \( \theta_{ij} = i\pi \) and their residues give rise to a set of recursive equations between the \( n \)-particle and the \((n+2)\)-particle form factors

\[
F_{n+2}(\theta' + i\pi, \theta, \theta_1, \ldots, \theta_n \rightarrow \theta', \theta, \theta_1, \ldots, \theta_n) \rightarrow \frac{i}{\theta' - \theta} \left(1 - \prod_{j=1}^{n} S(\theta - \theta_j)\right) F_n(\theta_1, \ldots, \theta_n) + \cdots,
\]

where the dots stand for regular parts. The second kind of poles is related to the bound states of the theory, but since there are no bound states in the ShG model, there are no such poles in the form factors of this theory and we will not consider them any further.

Note the striking similarity between the recursive equations (24) and (14): we will show below that, indeed, they exactly correspond one to the other in the limit (20). These recursive equations, together with the requirement of the correct asymptotic behavior and of the desired analyticity properties, are the key tools in finding explicit solutions for the form factors. In the ShG model a concise expression is provided by the form factor of the exponential operator [14]

\[
F_n(k) = \langle 0 | e^{k\phi} | \theta_1, \ldots, \theta_n \rangle = \frac{\sin(k\pi\alpha)}{\sin(\pi\alpha)} \left(\frac{4\sin(\pi\alpha)}{F_{\min}(1\pi)}\right)^{n/2} \det M_n(k) \prod_{j<l} F_{\min}(\theta_j - \theta_l) e^{\theta_j} + e^{\theta_l},
\]

Here \( k \) is an arbitrary real number, \( F_{\min}(\theta) \) is the minimal solution of the form factor bootstrap equations and \( M_n \) is a \((n-1) \times (n-1)\) matrix

\[
[M_n(k)]_{j,l} = \sigma_{2j-l}^{(n)} \frac{\sin((j-l+k)\pi\alpha)}{\sin(\pi\alpha)},
\]

where \( \sigma_{j}^{(n)} \) are the elementary symmetric polynomials of the variables \( e^{\theta_j} \):

\[
\sigma_{j}^{(n)} = \sum_{i_1 < \cdots < i_j} e^{\theta_{i_1}} \cdots e^{\theta_{i_j}}.
\]

We will see that the form factors of \( e^{k\phi} \) act as generating functions for the form factors of all the powers of the field \( \phi \).

4. QNLS matrix elements from ShG form factors

Before going on with the analysis, it is worth emphasizing the strong similarities of the QNLS and ShG models, in particular, the similarity of the key equations for the Bethe ansatz matrix elements and the form factors. Both theories are integrable and they contain a single type of massive particle without additional bound states. The pseudo-vacuum, on which the BA states are built, coincides with the Fock-vacuum, i.e. the zero-particle state. The ShG form factors are built on the physical vacuum, but in the non-relativistic limit the zero-point fluctuations disappear and we obtain the Fock-vacuum of the QNLS model. The two-particle \( S \)-matrix, which in the relativistic context governs every aspect of integrable theories, also corresponds between the two models. From this it is clear that the Bethe equations (10) and (29) are also mapped to each other. The striking similarity between the recursive equations (14) and (24) is very important, because these are the key equations that allow for the determination of the form factors and the matrix elements.
Moreover, the building block of the relativistic form factors, $F_{\text{min}}(\theta)$, has the following behavior: in the limit (20)

$$F_{\text{min}}(\theta_{jl}) \to \frac{\lambda_j - \lambda_l}{\lambda_j - \lambda_l + ic} = f(\lambda_j, \lambda_l), \quad F_{\text{min}}(\theta_{jl} + i\pi) \to 1 \quad \text{for } \theta_{jl} \in \mathbb{R},$$

thus it goes into an important building block of the Bethe ansatz quantities. With all these correspondences it should be clear that the ShG form factors will be mapped to the QNLS matrix elements. In what follows we give the details of this mapping and provide explicit examples.

In order to obtain QNLS matrix elements from the ShG form factors using the limit (20), we have to understand both the relation between the operators and the correspondence between the states of the two models. We also have to take into account that the Bethe ansatz matrix elements are defined in finite volume whereas the relativistic ones discussed so far are defined in infinite volume. The latter difference can be cured using the results of [15]. Even in relativistic quantum field theories the states in a finite volume $L$ can be described as multi-particle scattering states, where the rapidities of the particles are solutions of the Bethe quantization conditions:

$$Q_j = mc_l L \sinh \theta_j + \sum_{k \neq j}^{n} \frac{1}{i} \log S(\theta_j - \theta_k) = 2\pi I_j, \quad j = 1, \ldots, n. \quad (29)$$

However, this procedure is to be understood as an ‘asymptotic Bethe ansatz’ in the sense that it gives the correct results to all orders in $1/L$ but it misses residual finite size effects which decay exponentially with the volume. These corrections can be associated to processes involving virtual particles with some of them ‘traveling around the world’. The main idea of [15] is that this picture of ‘asymptotic BA’ applies to the form factors as well: the matrix elements in a finite volume are given by the infinite volume form factors at the particular set of rapidities which solve the corresponding Bethe equations. In addition, one has to introduce normalization factors given by the corresponding Gaudin determinants

$$\rho_n(\theta_1, \ldots, \theta_n) = \det \frac{\partial Q_j}{\partial \theta_k}, \quad (30)$$

which can be interpreted as the density of states (in rapidity-space) in the corresponding sector, or alternatively as the norms of the BA states as in (11). Thus the finite volume form factors are given by the infinite volume ones taken at the rapidities satisfying the Bethe equations (29), divided by the density of states:

$$\langle \theta_1', \ldots, \theta_n'|\mathcal{O}(0,0)|\theta_1, \ldots, \theta_n \rangle_L = \frac{F^{\mathcal{O}}(\theta_1' + i\pi, \ldots, \theta_n' + i\pi, \theta_1, \ldots, \theta_n)}{\sqrt{\rho_1(\theta_1', \ldots, \theta_n') \rho_n(\theta_1, \ldots, \theta_n)}} + \mathcal{O}(e^{-\mu L}), \quad (31)$$

where $\mathcal{O}(e^{-\mu L})$ stands for the above-mentioned exponentially small corrections in $L$. In the non-relativistic limit both the Bethe equations and the norms of the states go over to the QNLS quantities (10) and (12). In particular, the relation between the norms is given by

$$\rho_N \longrightarrow (mc_l)^N \tilde{\rho}_N. \quad (32)$$

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We note that (31) is valid as long as there are no coinciding rapidities in the ‘bra’ and ‘ket’ vectors. In the latter case disconnected terms arise; a proper treatment of such contributions was given in [16]. In this paper we do not elaborate on disconnected pieces, i.e. we assume that the two sets of rapidities are completely distinct.

Let us turn now to the relation between the operators in the two theories which was given in [3]:

\[ \phi(x, t) \sim \sqrt{\frac{\hbar^2}{2m}} (\psi(x, t) e^{-i\frac{mc^2}{\hbar}lt} + \psi^\dagger(x, t) e^{+i\frac{mc^2}{\hbar}lt}) . \] (33)

The exponential terms have to be separated, because the relativistic Hamiltonian contains also the rest energy which is absent in the non-relativistic case. The sign \( \sim \) means that in any functional expression of \( \phi \) the surviving exponential terms should be dropped, because in the non-relativistic limit \( c_l \to \infty \) they are rapidly oscillating and give zero when integrated over any small but finite time interval.

Similarly, we must compensate for the rest energy in the time evolution of the states, so (without the proper normalizations)

\[ |\theta_1, \ldots, \theta_n\rangle \leftrightarrow e^{-inmc^2lt}|\lambda_1, \ldots, \lambda_n\rangle, \quad \langle \theta_1, \ldots, \theta_n| \leftrightarrow e^{+inmc^2lt}\langle \lambda_1, \ldots, \lambda_n| . \] (34)

The relation between the rapidities and the momenta in the non-relativistic limit is \( \lambda_i = \theta_i/mc_l \). Note that the encounter of the oscillating terms in the states (34) and in the operator (33) (as well as in all its powers) will ensure that in the non-relativistic limit a given operator (e.g. \( \psi \)) have non-zero matrix elements only between states in which the number of particles differ by a fixed amount (e.g. by one).

Let us now deal with the question of the normalization of the states. First of all, one should note that in contrast with the BA states the relativistic asymptotic states are not symmetric in the rapidities, they rather obey

\[ |\theta_1, \ldots, \theta_k, \theta_{k+1}, \ldots, \theta_n\rangle = S(\theta_k - \theta_{k+1})|\theta_1, \ldots, \theta_{k+1}, \theta_k, \ldots, \theta_n\rangle . \] (35)

Hence, in order to establish the correspondence with the BA states, these states should be symmetrized in the rapidities, which can be done by multiplying with the appropriate phase factors:

\[ |\theta_1, \ldots, \theta_n\rangle_{\text{symm}} = \prod_{j>k} \sqrt{S(\theta_j - \theta_k)} |\theta_1, \ldots, \theta_n\rangle . \] (36)

The normalization (11) of the BA states should also be taken into account. From (32) it is clear that for the proper normalization we have to include a factor of \( \sqrt{mc_l} \) for every particle.

Collecting everything we finally arrive at the relation

\[ \prod_{j>k} \sqrt{S(\theta_j - \theta_k)} |\theta_1, \ldots, \theta_n\rangle \sim \frac{(mc_l)^{n/2}}{e^{n/2} \prod_{j<k} \sqrt{f(\lambda_j, \lambda_k)}} e^{-inmc^2lt}|\lambda_1, \ldots, \lambda_n\rangle . \] (37)

Taking the \( S \)-matrices to the right hand side and using formula (9) for the \( S \)-matrix recovered in the double limit of the ShG model, this can be written as

\[ |\theta_1, \ldots, \theta_n\rangle \sim \frac{(mc_l)^{n/2}e^{-inmc^2lt}}{e^{n/2} \prod_{j<k} f(\lambda_j, \lambda_k)} |\lambda_1, \ldots, \lambda_n\rangle . \] (38)
The relation between the dual vectors is given by the complex conjugate expression, in particular, the sign of the time-dependent phase (34) is opposite.

Using the $S$-matrix (9) of the QNLS model and the correspondence of the states (38) it is easy to prove, as shown below, that the relativistic recursive equations (24) transform exactly into the recursive equations (14). Note also that due to (28) the $f(\lambda_i, \lambda_j)$ factors in (38) will exactly cancel the limiting forms of $F_{\text{min}}(\theta)$. Now we are in a position to obtain a generic QNLS matrix element of the form

$$\langle \lambda'_1, \ldots, \lambda'_{N+p-q} | \psi^\dagger_p \psi^q | \lambda_1, \ldots, \lambda_N \rangle$$

by performing the following steps:

1. We determine the infinite volume ShG form factor

$$\langle 0 | : \phi^{p+q} : | \theta'_1 + i\pi, \ldots, \theta'_{N+p-q} + i\pi, \theta_1, \ldots, \theta_N \rangle$$

by picking the $O(k^{p+q})$ term in the expansion of (25) and (for $p+q > 2$) by taking into account the normal ordering issues. The pre-factor in (33) is 1 with the choice $m = 1/2, \hbar = 1$, but simple combinatorial factors may arise. As we said, the number of crossed rapidities will automatically select the correct combination $\psi^\dagger_p \psi^q$ out of the binomial expansion of $\phi^{p+q}$.

2. We calculate the finite volume form factors using (31). However, in the double limit the norms of the finite volume states will go to those of the BA states (11) up to a factor of $\sqrt{mc_l}$ for every particle, which needs to be included.

3. We take the double scaling limit (20) using (19) as well as the relation $\theta_i \to \lambda_i/mc_l$ and the limit of the minimal form factor $F_{\text{min}}(\theta)$ (28).

4. We take into account the different normalizations according to (38).

5. To get a proper matching with the Bethe ansatz matrix elements we include a factor of $-i$ for each $\psi$ and a $+i$ for each $\psi^\dagger$.

These steps can be summarized in the formula

$$\langle \lambda'_1, \ldots, \lambda'_{N+p-q} | \psi^\dagger_p \psi^q | \lambda_1, \ldots, \lambda_N \rangle = i^{p-q} \left( \frac{p+q}{p} \right)^{-1} \left( \frac{2m}{\hbar^2} \right)^{(p+q)/2} c^{(2N+p-q)/2} \prod_{j<k}^{N} f(\lambda_j, \lambda_k) \prod_{j<k}^{N+p-q} f(\lambda'_j, \lambda'_k)$$

$$\times \tilde{\lim} \left\{ (mc_l)^{-2N+p-q/2} \langle 0 | : \phi^{p+q} : | \theta'_1 + i\pi, \ldots, \theta'_{N+p-q} + i\pi, \theta_1, \ldots, \theta_N \rangle \right\},$$

where $\tilde{\lim}$ denotes the double scaling limit (20).

- We note that the form factors of normal ordered operators $: \phi^n :$ for $n > 2$ are obtained as linear combinations of terms coming from the expansion of the exponential; we refer the reader to paper [4] for a detailed discussion of this technical issue.

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BA matrix elements as non-relativistic limits of form factors of QFT

As a first check of this procedure let us explicitly show the correspondence of the recursive equations (24) and (14). Starting from the relativistic case we have

\[ F_{2N-1}(\theta'_1 + i\pi, \ldots, \theta'_{N-1} + i\pi, \theta_1, \ldots, \theta_N) \]

\[ = \prod_{j=2}^{N-1} S(\theta'_j + i\pi - \theta_1) F_{2N-1}(\theta'_1 + i\pi, \theta_1, \theta'_2 + i\pi, \ldots, \theta'_{N-1} + i\pi, \theta_2, \ldots, \theta_N) \]

\[ \times \prod_{j=2}^{N-1} S(\theta'_j + i\pi - \theta_1) F_{2N-3}(\theta'_2 + i\pi, \ldots, \theta'_{N-1} + i\pi, \theta_2, \ldots, \theta_N) \]

\[ = \frac{i}{\theta'_1 - \theta_1} \left\{ \prod_{j=2}^{N-1} S(\theta'_1 - \theta'_j) - \prod_{j=2}^{N} S(\theta_1 - \theta_j) \right\} \times F_{2N-3}(\theta'_2 + i\pi, \ldots, \theta'_{N-1} + i\pi, \theta_2, \ldots, \theta_N). \]  

(41)

In the limit using (40) this relation becomes

\[ i \left( \frac{mc}{c} \right)^{(2N-1)/2} \sqrt{\frac{h^2}{2m}} \prod_{j<k}^{N} f(\lambda_j, \lambda_k) \prod_{j<k}^{N-1} f(\lambda'_j, \lambda'_k) \tilde{F}_N(\lambda'_1, \ldots, \lambda'_{N-1}, \lambda_1, \ldots, \lambda_N) \]

\[ \times \frac{i mc}{\lambda'_1 - \lambda_1} \left\{ \prod_{j=2}^{N-1} \tilde{S}_{\text{QNLS}}(\lambda'_1 - \lambda'_j) - \prod_{j=2}^{N} \tilde{S}_{\text{QNLS}}(\lambda_1 - \lambda_j) \right\} \times \tilde{F}_{N-1}(\lambda'_2, \ldots, \lambda'_{N-1}, \lambda_2, \ldots, \lambda_N). \]  

(42)

Simplifying with the common pre-factors and using relation (9) between the \( S \)-matrix and the function \( f(\lambda, \mu) \) we arrive at equation (14).

Let us take now some explicit examples of the limit for the matrix elements of the field operator \( \psi \) and the current operator \( \psi^\dagger \psi \). For the matrix elements of \( \psi \) a nice determinant representation was found in [17]. They can be expressed also as [8]

\[ \tilde{F}_N^\psi(\lambda'_1, \ldots, \lambda'_{N-1}|\lambda_1, \ldots, \lambda_N) = P_N(\lambda'_1, \ldots, \lambda'_{N-1}|\lambda_1, \ldots, \lambda_N) \]

\[ \frac{1}{\prod_{k=1}^{N-1} (\lambda_j - \lambda'_k)}, \]  

(43)

where \( P_N \) are polynomials in \( \{\lambda\} \). The first few examples are given by

\[ P_1 = -i\sqrt{c}, \]  

(44a)

\[ P_2 = -2i\sqrt{c}c^2, \]  

(44b)

\[ P_3 = -4i\sqrt{c}c^4[-c^2 + (\lambda_1 - \lambda'_1)(\lambda_2 - \lambda'_2) + (\lambda_1 - \lambda'_2)(\lambda_3 - \lambda'_1) + (\lambda_2 - \lambda'_1)(\lambda_3 - \lambda'_2)]. \]  

(44c)

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The form factors for $\psi^\dagger \psi$ can be extracted from the matrix elements of the non-local operator
\[ Q_1(x) = \int_0^x dy \, j(y) \]  
by differentiating with respect to $x$. Matrix elements of $Q_1(x)$ are listed for example in [18] and a determinant formula can be found in [19], which yield
\[ \bar{F}_1^j(\lambda'_1 | \lambda_1) = c, \]  
\[ \bar{F}_2^j(\lambda'_1, \lambda'_2 | \lambda_1, \lambda_2) = -2e^3\left(\lambda_2 + \lambda'_1 - \lambda'_2\right)^2 \prod_{j,k=1}^2 (\lambda'_j - \lambda_k), \]
and so on.

On the ShG side the form factors of the various powers of $\phi$ can be obtained by series expanding formula (25) in the auxiliary real variable $k$ (see footnote 6). For example, the form factors of $\phi$ are given by
\[ \langle 0| \phi | \theta_1, \ldots, \theta_n \rangle = \frac{\pi \alpha}{g} \left( \frac{4}{F_{\min}(1\pi)} \right)^{n/2} (\sin(\pi \alpha))^{(n/2)-1} \det M_n(0) \prod_{j<l}^n \frac{F_{\min}(\theta_j - \theta_l)}{e^{\theta_j} + e^{\theta_l}}, \]
or explicitly
\[ \langle 0| \phi | \theta_1 \rangle = \frac{2}{\sqrt{F_{\min}(1\pi)}} \frac{\pi \alpha}{g \sqrt{\sin(\pi \alpha)}}, \]
\[ \langle 0| \phi | \theta_1, \theta_2, \theta_3 \rangle = \frac{8}{F_{\min}(1\pi)^{3/2}} \frac{\pi \alpha}{g} \sqrt{\sin(\pi \alpha)} e^{\theta_1 + \theta_2 + \theta_3} \prod_{j<l}^3 \frac{F_{\min}(\theta_j - \theta_l)}{e^{\theta_j} + e^{\theta_l}}. \]

Now we can apply the rule (40) with $p = 0$, $q = 1$ and $N = 1, 2, \ldots$. It is useful to note that
\[ c_1 \alpha \to \frac{2c}{\pi \hbar}, \quad \frac{\pi \alpha}{g} \to \frac{\sqrt{c}}{2\pi}. \]

From (28) we see that $F_{\min}(1\pi) \to 1$ and that the surviving $F_{\min}$ factors exactly cancel the $f$-functions appearing in (40). Taking care of the pre-factors the double limit yields a final result which coincides exactly with the expressions (44).

Similarly, the first form factors of $\phi^2$ obtained from (25) are
\[ \langle 0| \phi^2 | \theta_1, \theta_2 \rangle = \frac{8}{F_{\min}(1\pi)^2} \frac{\pi^2 \alpha^2}{g^2 \sin(\pi \alpha)} F_{\min}(\theta_1 - \theta_2), \]  
\[ \langle 0| \phi^2 | \theta_1, \theta_2, \theta_3, \theta_4 \rangle = \frac{32}{F_{\min}(1\pi)^2} \frac{\pi^2 \alpha^2}{g^2} \prod_{j<l}^4 \frac{F_{\min}(\theta_j - \theta_l)}{e^{\theta_j} + e^{\theta_l}} \times (e^{\theta_1 + \theta_2 + \theta_3 + \theta_4} (e^{\theta_1} + e^{\theta_2} + e^{\theta_3} + e^{\theta_4})^2 - e^{2(\theta_1 + \theta_2 + \theta_3 + \theta_4)} (e^{-\theta_1} + e^{-\theta_2} + e^{-\theta_3} + e^{-\theta_4})^2). \]

Applying the double limit formula (40) with $p = q = 1$ to (51) one retrieves the matrix elements (46). We have also checked the matrix elements with higher numbers of particles finding a perfect agreement with the corresponding Bethe ansatz matrix elements.

\[ \text{doi:10.1088/1742-5468/2010/05/P05014} \]
5. Conclusions

In this paper we have analyzed the close correspondence between the matrix elements of Bethe ansatz models and the form factors of relativistic integrable field theories. If the former models can be regarded as non-relativistic limits of the latter theories, then the Bethe ansatz matrix elements can be efficiently obtained as the non-relativistic expressions of the corresponding form factors, whose explicit computation is much simpler.

We have discussed in detail this correspondence between the quantum nonlinear Schrödinger model and the Sinh–Gordon model, where we gave a universal method to compute all the matrix elements of every local operator, but there are strong arguments in favor of its validity for other pairs of models as well. Indeed, both in Bethe ansatz models and quantum field theories the main properties of matrix elements are dictated by their $S$-matrices: therefore, if there is a mapping between the Hilbert spaces and operators of the two theories and the expressions of the two $S$-matrices coincide in the non-relativistic limit of the quantum field theory, these two facts induce a mapping between the matrix elements of the two theories. The use of form factors may help in solving most of the technical obstacles that have prevented so far the computation of matrix elements in non-relativistic Bethe ansatz integrable models, thus opening new perspectives in the computation of their correlation functions. Along this direction it would be interesting to investigate multi-component systems, where the BA results are limited due to the nested nature of the Bethe ansatz.

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References

[1] Korepin V E, Bogoliubov N M and Izergin A G, 1993 Quantum Inverse Scattering Method and Correlation Functions (Cambridge: Cambridge University Press)
[2] Smirnov F A, 1992 Form Factors in Completely Integrable Models of Quantum Field Theory (Singapore: World Scientific)
[3] Kornos M, Mussardo G and Trombettoni A, 2009 Phys. Rev. Lett. 103 210404 arXiv:0909.1336
[4] Kornos M, Mussardo G and Trombettoni A, 2010 Phys. Rev. A 81 043606 arXiv:0912.3502
[5] Cardy J L and Mussardo G, 1990 Nucl. Phys. B 340 387
[6] Dellino G, 2009 Nucl. Phys. B 807 455 arXiv:0806.1883
[7] Lieb E H and Liniger W, 1963 Phys. Rev. 130 1605
[8] Izergin A G, Korepin V E and Reshetikhin N Yu, 1987 J. Phys. A: Math. Gen. 20 4799
[9] Pakuliak S, 1994 Int. J. Mod. Phys. A 9 2087 [arXiv:hep-th/9307090]
[10] Babujian H and Karowski M, 2002 J. Phys. A: Math. Gen. 35 9081 [arXiv:hep-th/0204097]
[11] Zamolodchikov A B and Zamolodchikov A B, 1979 Nucl. Phys. B 133 525
[12] Arinshtein A E, Fateev V A and Zamolodchikov A B, 1977 JETP Lett. 25 457
[13] Alt C, Mussardo G and Delfino G, 1993 Phys. Lett. B 317 573 [arXiv:hep-th/9306103]
[14] Koubek A and Mussardo G, 1993 Phys. Lett. B 311 193 [arXiv:hep-th/9306044]
[15] Pozsgay B and Takacs G, 2008 Nucl. Phys. B 788 167 arXiv:0706.1445
[16] Pozsgay B and Takacs G, 2008 Nucl. Phys. B 788 209 arXiv:0706.3605
[17] Kojima T, Korepin V E and Slavnov N A, 1997 Commun. Math. Phys. 188 657 [arXiv:hep-th/9611216]
[18] Izergin A G and Korepin V E, 1984 Commun. Math. Phys. 94 67
[19] Slavnov N A, 1990 Teor. Mat. Fiz. 82 389

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