Spacetime Spin and Chirality Operators for Minimal 4D, $\mathcal{N} = 1$ Supermultiplets From BC$_4$ Adinkra-Tessellation of Riemann Surfaces

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ABSTRACT

We propose an explicit mathematical construction and plausibility arguments for how spacetime chirality and Lorentz generators emerge for minimal, off-shell 4D, $\mathcal{N} = 1$ supermultiplets by use of a 4.4.4.4 tessellation of Riemann surfaces based on plaquettes originating from Coxeter Group BC$_4$ adinkras.

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1 Introduction

Sometimes there occur difficulty in understanding how “spacetime spin” arises. An example of this is the, still not completely solved, problem of the “proton spin crisis.” Before 1987, it was expected the quarks were the principal carriers of spin in the proton. An experiment \[1\] that year revealed this was not supported by observation. This raises the question, “From where does the proton get its spin?” The spin of any relativistic spin-1/2 particle is closely related to the question of the Lorentz transformation properties of the particle.

At a recent meeting held at Brown University, a closely related question about adinkras was raised and addressed in an impromptu lecture by the author. Given the extemporaneous nature of this presentation, it was not a polished one and a main purpose of this note is to mitigate, remediate (any errors in the talk), and provide a formal set of definitions that seem needful in order to address the question, “From where do adinkras \[2\] get their spacetime spin?” As the talk was being given, it was also realized that many of the concepts introduced had not appeared previously in the literature. So a secondary purpose of this work is to make these more widely available to any interested party.

2 The Emergence of Lorentz Symmetry From $BC_4$ Adinkras: Stage I

We begin with the adjoint representation of SO(4) and indicate its generators by $t_{13}$, where this denotes the generator whose infinitesimal effect is to rotate in the $I - J$ plane. Let the symbol $\Lambda_{13}$ denote six parameters so that $\Lambda \cdot t$ is a vector in the algebra of $so(4)$. Let $\ast \Lambda$ be the Hodge dual of $\Lambda$ when it is considered to be a 2-form.

The two quantities $\Lambda_{\pm} \ast \Lambda$ can now be shown to belong to the two commuting $su(2)$ algebras that occur in $so(4)$. The $\alpha$ and $\beta$ matrices in (2.1) represent the generators of these two distinct and commuting $su(2)$ algebras. The results in (2.2) and (2.3) are simply some of their properties. We define

$$\alpha^I = [\sigma^2 \otimes \sigma^1], \quad \alpha^2 = [I_2 \otimes \sigma^2], \quad \alpha^3 = [\sigma^2 \otimes \sigma^3],$$

$$\beta^I = [\sigma^1 \otimes \sigma^2], \quad \beta^2 = [\sigma^2 \otimes I_2], \quad \beta^3 = [\sigma^3 \otimes \sigma^2],$$

where these matrices satisfy the identities

$$\alpha^I \alpha^J = \delta^I J I_4 + i\epsilon^{IJKL} \alpha^L, \quad \beta^I \beta^J = \delta^I J I_4 + i\epsilon^{IJKL} \beta^L, \quad [\alpha^I, \beta^J] = 0,$$

$$\text{Tr}(\alpha^I \alpha^J) = 4 \delta^I J, \quad \text{Tr}(\beta^I \beta^J) = 0,$$

$$\text{Tr}(\alpha^I) = \text{Tr}(\beta^I) = 0.$$

Next we denote four traceless matrices by $\tilde{K}^\mu$ (where $\mu = (0, 1, 2, 3)$) and defined via the equations

$$\tilde{K}^\mu = i a^{\mu \tilde{I}} \alpha^\tilde{I} + i b^{\mu \tilde{I}} \beta^\tilde{I} + c^{\mu \tilde{I} \tilde{J}} \alpha^\tilde{I} \beta^\tilde{J},$$

where $a^{\mu \tilde{I}}$, $b^{\mu \tilde{I}}$, and $c^{\mu \tilde{I} \tilde{J}}$ are 60 real constants. To determine a solution space of values for these constants we impose the conditions

$$\left\{ \tilde{K}^\mu, \tilde{K}^\nu \right\} = 2\eta^{\mu \nu} I_4,$$

This question was raised by J. Lukierski at a meeting in 2015 \[2\].
where \( \eta^{\mu \nu} \) is the Minkowski metric (our conventions are given in the work [3]) for a space of one temporal and three spatial dimensions. As the matrices \( I_4, \ i\alpha^\dagger, \ i\beta^\dagger, \) and \( \alpha^\dagger \beta^\dagger \) constitute a complete basis for constructing any real \( 4 \times 4 \) matrix, this guarantees solutions must exist.

Upon calculating the left hand side, we find

\[
\left\{ \hat{K}^\mu, \hat{K}^\nu \right\} = -a^{\mu\dagger}a^{\nu\dagger} \left\{ \alpha^\dagger, \alpha^\dagger \right\} - b^{\mu\dagger}b^{\nu\dagger} \left\{ \beta^\dagger, \beta^\dagger \right\} \\
+ i \left( b^{\mu\dagger}c^{\nu} \hat{R} \hat{L} + b^{\nu\dagger}c^{\mu} \hat{R} \hat{L} \right) \left\{ \beta^\dagger, \alpha^\dagger \hat{R} \beta^\dagger \right\} \\
+ i \left( a^{\mu\dagger}c^{\nu} \hat{R} \hat{L} + a^{\nu\dagger}c^{\mu} \hat{R} \hat{L} \right) \left\{ \alpha^\dagger, \alpha^\dagger \hat{R} \beta^\dagger \right\} \\
- \left( a^{\mu\dagger}b^{\nu\dagger} + a^{\nu\dagger}b^{\mu\dagger} \right) \left\{ \alpha^\dagger, \beta^\dagger \right\} \\
+ c^{\mu\dagger}c^{\nu} \hat{R} \hat{L} \left\{ \alpha^\dagger \beta^\dagger, \alpha^\dagger \beta^\dagger \right\} ,
\]

\[
\left\{ \hat{K}^\mu, \hat{K}^\nu \right\} = -2 \left( a^{\mu\dagger}a^{\nu\dagger} + b^{\mu\dagger}b^{\nu\dagger} \right) I_4 \\
+ i 2 \left( b^{\mu\dagger}c^{\nu} \hat{J} \hat{I} + b^{\nu\dagger}c^{\mu} \hat{J} \hat{I} \right) \alpha^\dagger \\
+ i 2 \left( a^{\mu\dagger}c^{\nu} \hat{J} \hat{I} + a^{\nu\dagger}c^{\mu} \hat{J} \hat{I} \right) \beta^\dagger \\
- 2 \left( a^{\mu\dagger}b^{\nu\dagger} + a^{\nu\dagger}b^{\mu\dagger} \right) \alpha^\dagger \beta^\dagger \\
+ c^{\mu\dagger}c^{\nu} \hat{R} \hat{L} \left( \alpha^\dagger \alpha^\dagger \hat{R} \beta^\dagger \beta^\dagger + \alpha^\dagger \alpha^\dagger \beta^\dagger \beta^\dagger \right) ,
\]

\[
\left\{ \hat{K}^\mu, \hat{K}^\nu \right\} = -2 \left( a^{\mu\dagger}a^{\nu\dagger} + b^{\mu\dagger}b^{\nu\dagger} - c^{\mu\dagger}c^{\nu} \hat{R} \hat{L} \right) I_4 \\
+ i 2 \left( b^{\mu\dagger}c^{\nu} \hat{J} \hat{I} + b^{\nu\dagger}c^{\mu} \hat{J} \hat{I} \right) \alpha^\dagger \\
+ i 2 \left( a^{\mu\dagger}c^{\nu} \hat{J} \hat{I} + a^{\nu\dagger}c^{\mu} \hat{J} \hat{I} \right) \beta^\dagger \\
- 2 \left( a^{\mu\dagger}b^{\nu\dagger} + a^{\nu\dagger}b^{\mu\dagger} \right) \alpha^\dagger \beta^\dagger \\
- 2 c^{\mu\dagger}c^{\nu} \hat{R} \hat{L} \left( \alpha^\dagger \beta^\dagger \right) ,
\]

The condition in (2.5) is thus seen to be equivalent to 160 quadratic polynomial constraints that define an algebraic variety whose explicit form is given by,

\[
\eta^{\mu \nu} = - \left( a^{\mu\dagger}a^{\nu\dagger} + b^{\mu\dagger}b^{\nu\dagger} \right) + c^{\mu\dagger}c^{\nu} \hat{R} \hat{L} , \quad \text{(\# of constraints : 10)}
\]

\[
0 = a^{\mu\dagger}c^{\nu} \hat{J} \hat{I} + a^{\nu\dagger}c^{\mu} \hat{J} \hat{I} , \quad \text{(\# of constraints : 30)}
\]

\[
0 = b^{\mu\dagger}c^{\nu} \hat{J} \hat{I} + b^{\nu\dagger}c^{\mu} \hat{J} \hat{I} , \quad \text{(\# of constraints : 30)}
\]

\[
0 = c^{\mu\dagger}c^{\nu} \hat{R} \hat{L} \epsilon \hat{I} \hat{R} \epsilon \hat{J} \hat{S} + \left( a^{\mu\dagger}b^{\nu\dagger} + a^{\nu\dagger}b^{\mu\dagger} \right) , \quad \text{(2.9)}
\]

\[
= \delta^{\hat{R} \hat{S}} \left( e^{\mu\dagger}c^{\nu} \hat{J} \hat{I} - c^{\nu\dagger}c^{\mu} \hat{J} \hat{I} \right) \\
+ \left( e^{\hat{S} \hat{J}} \epsilon^{\nu} \hat{R} - e^{\hat{J} \hat{S}} \epsilon^{\nu} \hat{R} \hat{S} \right) \\
+ \left( e^{\nu} \hat{R} \hat{S} - c^{\nu\dagger}c^{\mu} \hat{S} \hat{R} \right) \\
+ \left( a^{\mu\dagger}b^{\nu\dagger} + a^{\nu\dagger}b^{\mu\dagger} \right) , \quad \text{(\# of constraints : 90)}
\]
to impose constraints on the 60 constants represented as $a^\nu I$, $b^\nu I$, and $c^{\nu\bar{\nu}} I$.

So the next question becomes what structures “native and natural” to adinkras associated with BC$_4$ can give rise to the basis seen in (2.4) above?

3 The Emergence of Lorentz Symmetry From BC$_4$ Adinkras: Stage II

To every 4-four color, 4-open node, 4-closed node adinkra, such as shown in the figures below,

![Figure 1: (R) = (CM)](image1)

![Figure 2: (R) = (TM)](image2)

![Figure 3: (R) = (VM)](image3)

there corresponds a set of “L-matrices” and “R-matrices” $L_i^{(R)}$ and $R_i^{(R)}$ that satisfy the “Garden Algebra.”

\[
\begin{align*}
L_i^{(R)} R_j^{(R)} + L_j^{(R)} R_i^{(R)} &= 2 \delta_{IJ} I_{d \times d}, \\
R_i^{(R)} L_j^{(R)} + R_j^{(R)} L_i^{(R)} &= 2 \delta_{IJ} I_{d \times d}, \\
R_i^{(R)} &= [L_i^{(R)}]^{-1}.
\end{align*}
\] (3.1)

and the fermionic holoraumy matrix $\tilde{V}_{ij}^{(R)}$ for each representation is defined via the equation

\[
R_i^{(R)} L_j^{(R)} - R_j^{(R)} L_i^{(R)} = i2 \tilde{V}_{ij}^{(R)}. \tag{3.2}
\]

Due to the definitions in (3.1), it follows the matrices $\tilde{V}_{ij}^{(R)}$ are elements in the so(4) algebra and thus for any two representation (R) and (R') we may write

\[
\tilde{V}_{ij}^{(R)} = \ell_{ij}^{(R)} \alpha^i + \ell_{ij}^{(R)} \beta^i, \quad \tilde{V}_{ij}^{(R')} = \ell_{ij}^{(R')} \alpha^i + \ell_{ij}^{(R')} \beta^i. \tag{3.3}
\]
for some set of coefficients \( \ell_{ij}^{(R)} \), \( \ell_{ij}^{(R')} \), \( \ell_{ij}^{(\tilde{R})} \), and \( \ell_{ij}^{(\tilde{R}')} \).

Next, we consider two adinkra representations \( \mathcal{R} \) and \( \mathcal{R}' \) associated with \( \text{BC}_4 \) and construct a matrix \( k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle \) via the equation

\[
k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle = i A^\mu_{1J} \tilde{V}_{1J}^{(R)} + i B^\mu_{1J} \tilde{V}_{1J}^{(R')} + C^\mu_{KL} \tilde{V}_{KL}^{(R)} \tilde{V}_{KL}^{(R')} ,
\]

which is seen to be a quadratic polynomial in the fermionic holoraumy matrices. By use of the equations in (3.3), \( k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle \) becomes

\[
k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle = i \left[ A^\mu_{1J} \left( \ell_{ij}^{(R)} + \ell_{ij}^{(R')} \right) + C^\mu_{KL} \ell_{ij}^{(R)} \ell_{ij}^{(R')} \right] \alpha^\mathbf{I} + \left[ B^\mu_{1J} \left( \ell_{ij}^{(R)} + \ell_{ij}^{(R')} \right) + C^\mu_{KL} \ell_{ij}^{(R)} \ell_{ij}^{(R')} \right] \beta^\mathbf{I} + \left[ C^\mu_{KL} \ell_{ij}^{(R)} \ell_{ij}^{(R')} \right] \alpha^\mathbf{I} \beta^\mathbf{I} .
\]

This has exactly the form of (2.4) where the \( a, b, c \) coefficients there are related to the the \( \ell \), and \( \tilde{\ell} \) parameters in (3.3) and \( A, B, C \) coefficients in (3.4) as well through the relationships

\[
a^\mu_{iJ} = \left[ A^\mu_{1J} \left( \ell_{ij}^{(R)} + \ell_{ij}^{(R')} \right) + C^\mu_{KL} \ell_{ij}^{(R)} \ell_{ij}^{(R')} \right] , \\
b^\mu_{iJ} = \left[ B^\mu_{1J} \left( \ell_{ij}^{(R)} + \ell_{ij}^{(R')} \right) + C^\mu_{KL} \ell_{ij}^{(R)} \ell_{ij}^{(R')} \right] , \\
c^\mu_{iJ} = \left[ C^\mu_{KL} \ell_{ij}^{(R)} \ell_{ij}^{(R')} \right] \equiv C^\mu_{KL} \Delta^{iJ}_{LJKL} \langle (\mathcal{R})| (\mathcal{R}') \rangle .
\]

Given the equations in (3.6), we can now formulate a conjecture about the construction of Dirac Gamma matrices on the basis of adinkras related to \( \text{BC}_4 \).

**Conjecture One:**
Let \( \mathcal{R} \) and \( \mathcal{R}' \) denote any four color adinkra graphs associated with \( \text{BC}_4 \). To each such graph, there exist six associated “fermionic holoraumy matrices” \( \tilde{V}_{ij}^{(R)} \) and \( \tilde{V}_{ij}^{(R')} \).

If \( \Delta^{iJ}_{LJKL} \langle (\mathcal{R})| (\mathcal{R}') \rangle \equiv 0 \), then the equation

\[
 k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle k^\nu \langle (\mathcal{R})| (\mathcal{R}') \rangle + k^\nu \langle (\mathcal{R})| (\mathcal{R}') \rangle k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle = 2 \eta^{\mu\nu} \mathbf{I}_4 ,
\]

possesses no solutions.

An argument for the validity of this is suggested by the following observations.

We can look back to the first equation in (2.9). This equation makes it clear for the diagonal entries of the Minkowski metric on the left-hand side to be indefinite requires that the \( c^{iJ} \) be non-vanishing. Otherwise the diagonal entries of \( \eta^{\mu\nu} \) must be negative definite. In order to have \( c^{iJ} \) be non-vanishing, it must be the case that \( \Delta^{iJ}_{LJKL} \langle (\mathcal{R})| (\mathcal{R}') \rangle \) must be non-vanishing.

One can calculate the quantity \( \Delta^{iJ}_{LJKL} \langle (\mathcal{R})| (\mathcal{R}') \rangle \) on the \((CM), (TM)\) and \((VM)\) pairs, to find when it is vanishes. When it vanishes, then \( c^{iJ} \) vanishes, by looking at the purely diagonal entries in the first equation of (2.9), one sees the impossibility to satisfy this equation due to the indefinite nature of the Minkowski metric. This brings us to a second conjecture.
Conjecture Two:
Let \((R)\) and \((R')\) denote any four color adinkra graphs associated with \(BC_4\). If the condition \(\Delta_{IJKL}((R)|(R')) \neq 0\), is satisfied, then the equation in (2.5) possesses multiple solutions.

An argument for the validity of this conjecture is seen by simply carrying out the explicit calculations indicated to examine \(\Delta_{IJKL}((R)|(R'))\) among the adinkras in Fig. #1, Fig. #2, and Fig. #3. This will be done shortly. We emphasize these are conjectures as we expect these to hold over all the adinkras associated with \(BC_4\). The work in [5] shows this number is 36,864 adinkras.

As Boolean Factors \(\times\) permutations cycles (BFPC) \(^3\) the three adinkras correspond to,

\[
\begin{align*}
\text{Red} & \quad L^{(CM)}_1 = (10)_b(234), \quad L^{(CM)}_2 = (12)_b(132), \quad L^{(CM)}_3 = (6)_b(143), \quad L^{(CM)}_4 = (0)_b(124), \\
\text{Green} & \quad L^{(TM)}_1 = (14)_b(243), \quad L^{(TM)}_2 = (4)_b(123), \quad L^{(TM)}_3 = (2)_b(134), \quad L^{(TM)}_4 = (6)_b(1243), \\
\text{Blue} & \quad L^{(VM)}_1 = (10)_b(1342), \quad L^{(VM)}_2 = (12)_b(23), \quad L^{(VM)}_3 = (0)_b(14), \quad L^{(VM)}_4 = (6)_b(124),
\end{align*}
\]

(3.8)

that were introduced in the work of [4]. Above, each column of L-matrices is listed vertically under the rainbow color shown in the corresponding adinkra. In more explicit form these become

**CM Adinkra L – Matrices**

\[
\begin{align*}
L^{(CM)}_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad L^{(CM)}_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \\
L^{(CM)}_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad L^{(CM)}_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (3.9)
\end{align*}
\]

**TM Adinkra L – Matrices**

\[
\begin{align*}
L^{(TM)}_1 &= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad L^{(TM)}_2 &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \\
L^{(TM)}_3 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad L^{(TM)}_4 &= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (3.10)
\end{align*}
\]

\(^3\) Here we use the “read down” convention for relating cycle notation to matrix notation of permutations as discussed in the work of [5].
**VM Adinkra L – Matrices**

\[
L_{1}^{(VM)} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}, \quad L_{2}^{(VM)} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

\[
L_{3}^{(VM)} = \begin{bmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}, \quad L_{4}^{(VM)} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0
\end{bmatrix}. \tag{3.11}
\]

Given these sets of L-matrices, we use (3.2) to find the corresponding “tilde-V” matrices with the results given respectively in (3.12), (3.13), and (3.14)

\[
\tilde{V}_{12}^{(CM)} = + \tilde{V}_{34}^{(CM)} = + \alpha^2, \tag{3.12}
\]

\[
\tilde{V}_{13}^{(CM)} = - \tilde{V}_{24}^{(CM)} = + \alpha^3, \tag{3.13}
\]

\[
\tilde{V}_{14}^{(CM)} = + \tilde{V}_{23}^{(CM)} = + \alpha^1, \tag{3.14}
\]

Upon comparing these equations with the forms of the equations that appear in (3.3), one can extract the \(\ell\) and \(\tilde{\ell}\) parameters for each adinkra representation. For the (CM) representation, all of the \(\tilde{\ell}\)'s vanish, but for (TM) and (VM) representations, all of the \(\ell\)'s vanish.

The condition in (3.7) informs us the two adinkras corresponding to the BC\(_4\) adinkra representations \((\mathcal{R})\) and \((\mathcal{R}')\) permit the definition of a set of Lorentzian signature Dirac Gamma matrices by simply making the identification

\[
\gamma^\mu = k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle, \tag{3.15}
\]

from which it follows a 4D Lorentz generator is given by

\[
\Sigma^{\mu\nu} = i^\frac{1}{4} \left[ k^\mu \langle (\mathcal{R})| (\mathcal{R}') \rangle , k^\nu \langle (\mathcal{R})| (\mathcal{R}') \rangle \right]. \tag{3.16}
\]

and we can express this in the form

\[
\Sigma^{\mu\nu} = \frac{1}{2} \left\{ \left( a^{\mu} \tilde{\alpha} \tilde{\gamma} \tilde{\alpha} - c^{\mu} \tilde{\alpha} \tilde{\gamma} \tilde{\alpha} \right) \tilde{\epsilon} \tilde{\alpha} \tilde{\beta} \\
+ \left( b^{\mu} \tilde{\alpha} \tilde{\gamma} \tilde{\beta} - c^{\mu} \tilde{\alpha} \tilde{\gamma} \tilde{\beta} \right) \tilde{\epsilon} \tilde{\alpha} \tilde{\beta} \\
- i \left( a^{\mu} \tilde{\alpha} \tilde{\gamma} \tilde{\alpha} - a^{\nu} \tilde{\alpha} \tilde{\gamma} \tilde{\alpha} \right) \tilde{\epsilon} \tilde{\alpha} \tilde{\beta} \tilde{\alpha} \tilde{\beta} \\
- i \left( b^{\mu} \tilde{\alpha} \tilde{\gamma} \tilde{\beta} - b^{\nu} \tilde{\alpha} \tilde{\gamma} \tilde{\beta} \right) \tilde{\epsilon} \tilde{\alpha} \tilde{\beta} \tilde{\alpha} \tilde{\beta} \right\}, \tag{3.17}
\]
but with the very important understanding that the coefficients \(a^I, b^I,\) and \(c^{I\tilde{J}}\) that appear in (3.17) are here strictly defined in terms of the definitions given in (3.6). This operator (3.17) has the interpretation of being a spacetime spin operator constructed from the adinkra representations \((\mathcal{R})\) and \((\mathcal{R}')\).

On the solution shown in (2.5) for the coefficients \(a^I, b^I,\) and \(c^{I\tilde{J}}\) it is found that

\[
\begin{bmatrix}
\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}
\end{bmatrix} = -\eta^{\mu\rho} \Sigma^{\nu\sigma} + \eta^{\mu\sigma} \Sigma^{\nu\rho} + \eta^{\nu\rho} \Sigma^{\mu\sigma} - \eta^{\nu\sigma} \Sigma^{\mu\rho},
\]

which guarantees the result in (3.17) defines a spacetime spin operator in terms the adinkras \((\mathcal{R})\) and \((\mathcal{R}')\) whose \(\ell\) and \(\tilde{\ell}\) parameters determine the coefficients via (3.6). So we can pick \((\mathcal{R}) = (CM)\) and \((\mathcal{R}') = (TM)\) or \((VM)\), but not \((\mathcal{R}') = (CM)\), in order to guarantee that \(\Delta_{HKL}^{\hat{I}\hat{J}IJKL}\) does not vanish in the expressions given in (3.15) - (3.18).

In a similar manner, a 4D chirality operator constructed from the adinkra representations \((\mathcal{R})\) and \((\mathcal{R}')\) follows from the expression

\[
\gamma^5 = \frac{i}{3!} \epsilon_{\mu\nu\rho\sigma} \Sigma^{\mu\nu} \Sigma^{\rho\sigma}.
\]

Given the realization of the 4D Minkowski space Lorentz Generator on such a pair of adinkra representations, the Lorentz transformation properties on fermions in all adinkra are now well defined. As well, the same can be said of chirality properties. However, we can go even beyond two such adinkras.

Consider two unordered sets of adinkras which we write (with \((\mathcal{R})_1 = (\mathcal{R})\) and \((\mathcal{R}')_1 = (\mathcal{R}')\)) as

\[
\{A\} = \left\{ (\mathcal{R})_1, \ldots, (\mathcal{R})_P \right\},
\]

\[
\{B\} = \left\{ (\mathcal{R}')_1, \ldots, (\mathcal{R}')_Q \right\},
\]

where \(P\) and \(Q\) are some integers. Furthermore, let us assume that any member of \(\{A\}\) possesses the same values of all of their \(\ell\) and \(\tilde{\ell}\) parameters as any other member of the set. We also assume that any member of \(\{B\}\) possesses the same values of all of their \(\ell\) and \(\tilde{\ell}\) parameters as any other member of the set. If the \(a, b, c, A, B,\) and \(C\) coefficients are universally used to construct Dirac Gamma matrices, they will all yield the same set of Dirac Gamma matrices independent of which elements from \(\{A\}\) and \(\{B\}\) are utilized.

So all adinkras with the same values of their \(\ell\) and \(\tilde{\ell}\) parameters belong to an equivalence class with regard to their Lorentz spacetime symmetry properties...a very satisfying result.

### 4 Connecting To Riemann Surfaces

Due to the works of [6,7], we now know adinkras provide a periodic tessellation of Riemann surfaces with spin-structures and integer valued Morse divisors. Due to these observations, the discussion given in this work provides a way in which the regular 4.4.4.4 tessellation of Riemann surfaces...
surfaces constructed from $\text{BC}_4$ minimal adinkras is related to spin-structures of four dimensional Minkowski space supermultiplets with $\mathcal{N} = 1$ SUSY. In other words, the adinkras have provided a “bridge” between the spin-structures on a Riemann surface and spin-structures of a four dimensional Minkowski spacetime.

In preparation to see the connection to Riemann surfaces, we go back to the results presented in (2.1) and take the absolute values of those equations which yields the following

$$
\alpha^1 = [\sigma^1 \otimes \sigma^1], \quad \alpha^2 = [I_2 \otimes \sigma^1], \quad \alpha^3 = [\sigma^1 \otimes I_2], \quad (4.1)
$$

and, though the order is different when considering each set of matrices, $\{|\alpha^1\rangle\}$ and $\{|\beta^1\rangle\}$, their actual elements are the same. We may disregard the ordering and add to these three matrices the $4 \times 4$ identity element (i.e., $I_2 \otimes I_2$) to form a set we denote by $\{\mathcal{V}(4)\}$ that can be written as

$$
\{\mathcal{V}(4)\} = \{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\} = \{I_{2\times 2} \otimes I_{2\times 2}, I_{2\times 2} \otimes \sigma^1, I_{2\times 2} \otimes \sigma^1, I_{2\times 2} \otimes \sigma^1\}, \quad (4.2)
$$

which alternately can also be written in the form of

$$
\{\mathcal{V}(4)\} = \{(1), (12)(34), (13)(24), (14)(23)\}, \quad (4.3)
$$

by use of cycle notation. These matrices form a group, the Klein Vierergruppe, but they do not satisfy the conditions in (3.1). In order to achieve this, L-matrices [4]

$$
L = \mathcal{S} \cdot \mathcal{P} \quad (4.4)
$$

which are signed permutations must be introduced. The factors $\mathcal{P}$ denote special quartets of permutations (as identified in the work of [4]) acting on four objects and the factors $\mathcal{S}$ are $4 \times 4$ matrices given the nomenclature of “Boolean Factors” in this work. These have the forms indicated by

$$
(p_12^0 + p_22^1 + p_32^2 + p_42^3)_b \equiv \begin{pmatrix}
(-1)^{p_1} & 0 & 0 & 0 \\
0 & (-1)^{p_2} & 0 & 0 \\
0 & 0 & (-1)^{p_3} & 0 \\
0 & 0 & 0 & (-1)^{p_4}
\end{pmatrix}, \quad (4.5)
$$

where $p_1, p_2, p_3$, and $p_4$ are bits taking on values of either one or zero. There are sixteen “even” Boolean Factor quartets that may be combined with the elements of $\{\mathcal{V}(4)\}$ to form the L-matrices indicated in (4.4). These are,

$$
\mathcal{S}_{\mathcal{V}(4)}[\alpha] = \{(12)_b, (10)_b, (0)_b, (6)_b\}, \{(6)_b, (12)_b, (0)_b, (10)_b\}, \{(14)_b, (8)_b, (2)_b, (4)_b\}, \{(4)_b, (14)_b, (2)_b, (8)_b\}, \{(8)_b, (14)_b, (4)_b, (2)_b\}, \{(2)_b, (8)_b, (4)_b, (14)_b\}, \{(10)_b, (12)_b, (6)_b, (0)_b\}, \{(0)_b, (10)_b, (6)_b, (12)_b\}, \{(14)_b, (4)_b, (8)_b, (2)_b\}, \{(4)_b, (2)_b, (8)_b, (14)_b\}, \{(12)_b, (6)_b, (10)_b, (0)_b\}, \{(6)_b, (0)_b, (10)_b, (12)_b\}, \{(10)_b, (0)_b, (12)_b, (6)_b\}, \{(0)_b, (6)_b, (12)_b, (10)_b\}, \{(8)_b, (2)_b, (14)_b, (4)_b\}, \{(2)_b, (4)_b, (14)_b, (8)_b\}. \quad (4.6)
$$
Though only even Boolean Factors are listed above, “odd” quartets exist. Given a Boolean Factor quartet of the form \( \{ (p)_b, (q)_b, (r)_b, (s)_b \} \), its “antipodal antonym” \([11]\) \( \{ (P)_b, (Q)_b, (R)_b, (S)_b \} \) via

\[
\{(P)_b, (Q)_b, (R)_b, (S)_b \} = \{(15 - p)_b, (15 - q)_b, (15 - r)_b, (15 - s)_b \} \ .
\]

(4.7)

If \( \{ (p)_b, (q)_b, (r)_b, (s)_b \} \) together with a specific quartet of permutations \( \mathcal{P} \) form a set of L-matrices, this will remain true if any number of the Boolean Factors within the quartet are replaced by their antonyms. An explicit example shows how this works. We can form a set of L-matrices by forming an inner product using

\[
L = \{(12)_b, (10)_b, (0)_b, (6)_b \} \cdot \{(0), (12)(34), (13)(24), (14)(23)\}
\]

(4.8)

which can be shown to satisfy the conditions in (3.1). The antipodal antonym L-matrices are obtained from

\[
antipodal \ antonym \ [L] = \{(3)_b, (5)_b, (15)_b, (9)_b \} \cdot \{(0), (12)(34), (13)(24), (14)(23)\}
\]

(4.9)

Simple calculations show these L-matrices satisfy the conditions in (3.1). The antipodal antonym corresponds to a sign exchange of a single fermionic node. By performing a simultaneous sign change on all the boson nodes or all the fermion nodes, one can eliminate these as independent representations. In terms of the field theory described by the adinkras, this is simply a fermionic field redefinition by a minus sign.

We partition the sixteen Boolean Factors according to

\[
S_{\mathcal{V}(4)}[\alpha] = S_{\mathcal{V}(4)}[\alpha^-] \cup S_{\mathcal{V}(4)}[\alpha^+]
\]

(4.10)

where \( \alpha^- = 1, 3, 5, 7, 10, 12, 14, \) and \( 16 \) while \( \alpha^+ = 2, 4, 6, 8, 9, 11, 13, \) and \( 15 \). The \( \ell \) coefficients vanish for all members of \( S_{\mathcal{V}(4)}[\alpha^-] \) while the \( \ell \) coefficients vanish for all members of \( S_{\mathcal{V}(4)}[\alpha^+] \). From discussion in previous chapters, any choice of spin structure on the Riemann surface that includes one factor from the \( \alpha^- \) set and one from the \( \alpha^+ \) set should lead to a spacetime spin generator among minimal 4D, \( \mathcal{N} = 1 \) supermultiplets.

We are now in position to make contact with the work in prior papers \([6,7]\). For our purposes, the most relevant points from these works can be captured in one formula and one figure. It was noted when adinkras are utilized to reconstruct Riemann surfaces, there arises a relation between the genus of the Riemann surface \( g \), the number of distinct colors \( N \) of the links according to the formula,

\[
g = 1 + d (N - 4)
\]

(4.11)

where the number of closed nodes (also equal to the number of open nodes) is \( d \). Since all adinkras associated with BC4 correspond to \( d = 4 \) and possess exactly four colors, the second factor in the sum in (4.11) vanishes. For the special case of \( N = 4 \), all adinkras, even ones not associated with BC4, “live” on the torus independent of the number of nodes.

The formula in (4.11) informs us the any BC4 adinkra, such as the chiral supermultiplet shown in Fig. # 1, can be used as a tessellation of a torus. In Fig. # 4, the chiral supermultiplet adinkra
has been “cut open,” “laid flat,” “replicated,” “glued together,” and used for a 4.4.4.4 tiling of a torus.

Figure 4: Chiral Supermultiplet Adinkra On Torus

The image in Fig. #4 also includes the a-cycle and b-cycle of the torus. Though we only show the adinkra from Fig. #1, the adinkras from Fig. #2 and Fig. #3, can be treated in the same manner.

Forgetting about the dashing on the adinkras, the monodromy data associated to the Riemann surfaces in [6] leads to a set of matrices of the type shown here in (4.2) and (4.3). The authors of [7] construct a dictionary between odd dashings and certain spin structures on the adinkra Riemann surface. The Boolean factors used above encode odd dashings on the adinkra. By combining these two dictionaries, the spin operators constructed from adinkras for the four dimensional minimal supermultiplets can be built from pairs of (non isomorphic!) spin structures on these Riemann surfaces.

When these are clarified, it will connect Riemann surfaces with adinkra tessellation to the spin operators constructed from adinkras for the four dimensional minimal supermultiplets. The fermionic holoraumy matrices $\tilde{V}^{(R)}_{1j}$ of (3.2) provide “colored and decorated” plaquettes as shown in Fig. #4, equipped with an “isospin R-symmetry,” used for the tessellation of the Reimann surfaces and thus are the origins of the spacetime chirality and Lorentz symmetries for the 4D, $\mathcal{N} = 1$ supermultiplets via a “spin from iso-spin” approach.

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