ASYMPTOTIC BASE LOCI VIA OKOUNKOV BODIES

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Abstract. An Okounkov body is a convex body in Euclidean space associated to a divisor on a smooth projective variety with respect to an admissible flag. In this paper, we recover the asymptotic base loci from the Okounkov bodies by studying various asymptotic invariants such as the asymptotic valuations and the moving Seshadri constants. Consequently, we obtain the nefness and ampleness criteria of divisors in terms of the Okounkov bodies. Furthermore, we compute the divisorial Zariski decomposition by the Okounkov bodies, and find upper and lower bounds for moving Seshadri constants given by the size of simplexes contained in the Okounkov bodies.

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1. Introduction

It is a fundamental problem to understand the geometry of linear series or divisors on a variety in algebraic geometry. Since the introduction and treatment of the Okounkov bodies associated to big divisors by Lazarsfeld-Mustață ([LM]) and Kaveh-Khovanskii ([KK]) motivated by earlier works by Okounkov ([O1], [O2]), there have been considerable attempts to extract various properties of divisors from the Okounkov bodies. Let $D$ be a divisor on a smooth projective variety $X$ of dimension $n$. Then the Okounkov body $\Delta_{\mathcal{Y}}(D)$ is a convex body in the Euclidean space $\mathbb{R}^n$ associated to $D$ with respect to an admissible flag $\mathcal{Y}$. In [CHPW], we defined and studied the valuative Okounkov body $\Delta_{\mathcal{Y}}^{\text{val}}(D)$ and the limiting Okounkov body $\Delta_{\mathcal{Y}}^{\text{lim}}(D)$ of a pseudoeffective divisor $D$ with respect to an admissible flag $\mathcal{Y}$. They are also convex bodies in the Euclidean space $\mathbb{R}^n$ which coincide with the classical Okounkov body $\Delta_{\mathcal{Y}}(D)$ if $D$ is big. For more details of Okounkov bodies, see Section 3.

It was shown that two pseudoeffective divisors are numerically equivalent to each other if and only if the associated limiting Okounkov bodies with respect to all admissible flags coincide ([LM] Proposition 4.1), ([J] Theorem A), ([CHPW] Theorem C]). Thus, in principle, every numerical property of pseudoeffective divisors can be encoded in the associated limiting Okounkov bodies with respect to all admissible flags. On the other hand, the valuative Okounkov bodies are not numerical in nature and the main results of this paper do not hold for such bodies (see [CPW] Remark 4.10).

One of the most important numerical properties of pseudoeffective divisors is the asymptotic base loci. The principal aim of this paper is to study how to extract asymptotic base loci, more precisely, the restricted base locus $B_-(D)$ and the augmented base locus $B_+(D)$, from the limiting Okounkov bodies of a pseudoeffective divisor $D$. See Subsection 2.1 for definitions of asymptotic base loci.

The following is the first main result of this paper on the restricted base loci.

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Theorem A (=[Theorem 4.2]. Let $D$ be a pseudoeffective divisor on a smooth projective variety $X$ of dimension $n$. Then the following are equivalent.

1. $x \in B_-(D)$.
2. For any admissible flag $Y_x$ centered at $x$, the origin of $\mathbb{R}^n$ is not contained in $\Delta^\lim_{\nu}(D)$.
3. For some admissible flag $Y_x$ centered at $x$, the origin of $\mathbb{R}^n$ is not contained in $\Delta^\lim_{\nu}(D)$.

Note that a pseudoeffective divisor $D$ is nef if and only if $B_-(D) = \emptyset$. Thus we immediately obtain the following nefness criterion of divisors.

Corollary B (=[Corollary 4.3]. Let $D$ be a pseudoeffective divisor on a smooth projective variety $X$ of dimension $n$. Then the following are equivalent.

1. $D$ is nef.
2. For any admissible flag $Y_x$, the origin of $\mathbb{R}^n$ is contained in $\Delta^\lim_{\nu}(D)$.
3. For any point $x \in X$ and for some admissible flag $Y_x$ centered at $x$, the origin of $\mathbb{R}^n$ is contained in $\Delta^\lim_{\nu}(D)$.

To prove Theorem A we use the asymptotic valuation at a given pseudoeffective divisor (see Subsection 2.2 for the definition). Furthermore, we recover the divisorial components of $B_-(D)$ from the limiting Okounkov bodies, thereby obtaining the movability criterion of divisors (see Theorem 4.4). We also compute the divisorial Zariski decomposition of a pseudoeffective divisor (see Section 5).

Next we prove the analogous results for the augmented base locus. We define $U_{\geq 0} := U \cap \mathbb{R}_{\geq 0}$ where $U$ is a small open neighborhood of the origin of $\mathbb{R}^n$.

Theorem C (=[Theorem 6.4]. Let $D$ be a pseudoeffective divisor on a smooth projective variety $X$ of dimension $n$. Then the following are equivalent.

1. $x \in B_+(D)$.
2. For any admissible flag $Y_x$ centered at $x$, $U_{\geq 0}$ is not contained in $\Delta^\lim_{\nu}(D)$ for any small open neighborhood $U$ of the origin of $\mathbb{R}^n$.
3. For some admissible flag $Y_x$ centered at $x$, $U_{\geq 0}$ is not contained in $\Delta^\lim_{\nu}(D)$ for any small open neighborhood $U$ of the origin of $\mathbb{R}^n$.

Note that a pseudoeffective divisor $D$ is ample if and only if $B_+(D) = \emptyset$. Thus we immediately obtain the following ampleness criterion of divisors.

Corollary D (=[Proposition 6.2]. Let $D$ be a big divisor on a smooth projective variety $X$ of dimension $n$. Then the following are equivalent.

1. $D$ is ample.
2. For any admissible flag $Y_x$, $U_{\geq 0}$ is contained in $\Delta^\lim_{\nu}(D)$ for some small open neighborhood $U$ of the origin of $\mathbb{R}^n$.
3. For any point $x \in X$ and for some admissible flag $Y_x$ centered at $x$, $U_{\geq 0}$ is contained in $\Delta^\lim_{\nu}(D)$ for some small open neighborhood $U$ of the origin of $\mathbb{R}^n$.

The main ingredient of the proof of Theorem C is the results on the relation between the moving Seshadri constants of big divisors and the Okounkov bodies (see Subsection 2.5 for the definition of moving Seshadri constants).

We can also give both lower and upper bounds for the moving Seshadri constants of pseudoeffective divisors by analyzing the structure of the limiting Okounkov bodies. A simplex of length $\lambda = (\lambda_1, \cdots, \lambda_n)$ is a convex subset of $\mathbb{R}_{\geq 0}^n$ defined as

$$
\Delta_\lambda := \left\{ (x_1, \cdots, x_n) \in \mathbb{R}_{\geq 0}^n \left| \frac{x_1}{\lambda_1} + \cdots + \frac{x_n}{\lambda_n} \leq 1 \right. \right\}
$$

where $\lambda_i \geq 0$ (1 $\leq i \leq n$) are nonnegative real numbers. Let $x_i = 0$ for $i$ such that $\lambda_i = 0$. If $x \notin B_-(D)$, then the origin of $\mathbb{R}^n$ is contained in $\Delta^\lim_{\nu}(D)$ for all admissible flag $Y_x$ by Theorem A. Thus $\Delta_\lambda \subseteq \Delta^\lim_{\nu}(D)$ for some length $\lambda$. For $x \notin B_-(D)$ and an admissible flag $Y_x$ centered at $x$, we consider the maximal sub-simplex $\Delta_{\text{max}}$ of length $(\lambda_1, \cdots, \lambda_n)$ contained in $\Delta^\lim_{\nu}(D)$. In this case, we set $\lambda_i(D; x, Y_x) := \lambda_i$ and $\lambda_{\text{min}}(D; x, Y_x) := \min\{\lambda_i(D; x, Y_x)\}$. If $x \in B_-(D)$ so that the origin is not contained in $\Delta^\lim_{\nu}(D)$, then we define $\Delta_{\text{max}}$ as the origin. Now we can state our result on bounds for moving Seshadri constants.
Theorem E (=Theorem 7.3). Let $D$ be a pseudoeffective divisor on a smooth projective variety $X$ of dimension $n$, and $x$ be a point on $X$. Then we have
\[ \sup_{\lambda_\ast} \{ \lambda_{\min}(D; x, Y_\ast) \} \leq \varepsilon(||D||; x) \leq \inf_{\lambda_\ast} \{ \lambda_n(D; x, Y_\ast) \} \]
where $\sup$ and $\inf$ are taken over the admissible flags $Y_\ast$ centered at $x$.

We prove Theorem E by basically reducing our statement to the case of nef divisors (Theorem 7.3) using a version of the Fujita approximation for the Okounkov bodies.

Note that both inequalities in Theorem E can be strict in general (see Example 7.5). Thus it is still too much to expect to obtain the exact values of the moving Seshadri constants by only considering the limiting Okounkov bodies on $X$. On the other hand, one can obtain the exact values by using the infinitesimal Okounkov bodies (see [J], [KL1], [KL3] Theorem C). However, computing the infinitesimal Okounkov bodies is quite difficult in general. Moreover, it is already very interesting to give some bounds for moving Seshadri constants using the Okounkov bodies only (cf. [J], [KL1]).

Our main results are higher dimensional generalizations of some results in [KL1]. We remark that Küronya and Lozovanu also independently obtained Theorem A and Corollary B in [KL2] when the divisor $D$ is big. They also showed Theorem C and Corollary D under the assumption that $Y_1$ is ample. Our results do not require such strong condition on the admissible flags $Y_\ast$ and extend to the pseudoeffective case as well.

The organization of the paper is as follows. We start in Section 2 by collecting basic facts on the asymptotic base loci, asymptotic valuations, divisorial Zariski decompositions, restricted volumes, and moving Seshadri constants. In Section 3 we review the construction and basic properties of limiting Okounkov bodies. The next two sections concern asymptotic valuations via limiting Okounkov bodies. We give the proofs of Theorem A and Corollary B in Section 4 and we calculate the divisorial Zariski decomposition via the limiting Okounkov bodies in Section 5. We then turn to the augmented base loci and moving Seshadri constants. In Section 6 we show Corollary D first, and then prove Theorem C. Section 7 is devoted to proving Theorem E.

2. Preliminaries

In this section, we recall basic notions and properties which we use later on. By a variety, we mean a smooth projective variety defined over the field $\mathbb{C}$ of complex numbers. Unless otherwise stated, a divisor means an $\mathbb{R}$-Cartier divisor. A divisor $D$ is pseudoeffective if its numerical equivalence class $[D] \in N^1(X)_{\mathbb{R}}$ lies in the pseudoeffective cone $\overline{\text{Eff}}(X)$, the closure of the cone spanned by effective divisor classes. A divisor $D$ on a variety $X$ is big if $[D]$ lies in the interior $\text{Big}(X)$ of $\overline{\text{Eff}}(X)$.

2.1. Asymptotic base loci. We will define the asymptotic base loci of divisors which will be used throughout the paper. Let $D$ be a $\mathbb{Q}$-divisor on a variety $X$. The stable base locus $\text{SB}(D)$ of $D$ is defined as
\[ \text{SB}(D) := \bigcap_{m \geq 0} \text{Bs}([mD]) \]
where the intersection is taken over the positive integers $m$ such that $mD$ are $\mathbb{Z}$-divisors.

We recall that $\text{SB}(D)$ is not a numerical property of $D$ (see [La, Example 10.3.3]). However, the following asymptotic base loci which are defined for $\mathbb{R}$-divisors $D$ depend only on the numerical class $[D] \in N^1(X)_{\mathbb{R}}$.

Definition 2.1. Let $D$ be a divisor on a variety $X$. The restricted base locus $\text{B}_-(D)$ of $D$ is defined as
\[ \text{B}_-(D) := \bigcup_{A} \text{SB}(D + A) \]
where the union is taken over all ample divisors $A$ such that $D + A$ are $\mathbb{Q}$-divisors. The augmented base locus $\text{B}_+(D)$ is defined as
\[ \text{B}_+(D) := \bigcap_{A} \text{SB}(D - A) \]
where the intersection is taken over all ample divisors $A$ such that $D - A$ are $\mathbb{Q}$-divisors.
We recall that $D$ is nef if and only if $\mathcal{B}_-(D) = \emptyset$, and $D$ is ample if and only if $\mathcal{B}_+(D) = \emptyset$. It is also easy to see that $D$ is not pseudoeffective if and only if $\mathcal{B}_-(D) = X$, and $D$ is not big if and only if $\mathcal{B}_+(D) = X$. It is well known that $\mathcal{B}_-(D)$ and $\mathcal{B}_+(D)$ do not contain any isolated points. For more details on the asymptotic base loci, we refer to [La], [ELMNP] and [ELMNP2].

2.2. Asymptotic valuations. Let $\sigma$ be a divisorial valuation of a variety $X$, and $V := \text{Cent}_X \sigma$ be its center on $X$. If $D$ is a big divisor on $X$, we define the asymptotic valuation of $\sigma$ at $x$ as

$$
\text{ord}_V(||D||) := \inf \{ \sigma(D') \mid D \equiv D' \geq 0 \}.
$$

If $D$ is only a pseudoeffective divisor on $X$, we define

$$
\text{ord}_V(||D||) := \lim_{\epsilon \to 0^+} \text{ord}_V(||D + \epsilon A||)
$$

for some ample divisor $A$ on $X$. This definition is independent of the choice of $A$, and the number $\text{ord}_V(||D||)$ depends only on the numerical class $[D] \in N^1(X)_{\mathbb{R}}$. Note that $V \subseteq \mathcal{B}_-(D)$ if and only if $\text{ord}_V(||D||) > 0$ (see [ELMNP] Proposition 2.8, [Ny] V.1.9 Lemma). For more details, we refer to [ELMNP] and [Ny].

2.3. Divisorial Zariski decompositions. Let $D$ be a pseudoeffective divisor on a variety $X$ of dimension $n$.

**Definition 2.2.** The divisorial Zariski decomposition of $D$ is the expression

$$
D = P + N
$$

such that the negative part $N$ of $D$ is defined as

$$
N = \sum_{\text{codim } E = 1} \text{ord}_E(||D||)E
$$

where the summation is over the codimension 1 irreducible subvariety $E$ of $X$ such that $\text{ord}_E(||D||) > 0$ and the positive part $P$ of $D$ is defined as $P := D - N$.

It is well known that the summation for the negative part $N$ is finite and the components of $N$ are linearly independent in $N^1(X)_{\mathbb{R}}$. Furthermore, the positive part is movable, that is, $\mathcal{B}_-(D)$ has no divisorial components. For more details, see [B] and [Ny] Chapter III.

2.4. Restricted volumes. Let $D$ be a $\mathbb{Q}$-divisor on a variety $X$ of dimension $n$, and $V$ be a $v$-dimensional proper subvariety of $X$ such that $V \not\subseteq \mathcal{B}_+(D)$. The restricted volume of $D$ along $V$ is defined as

$$
\text{vol}_{X|V}(D) := \limsup_{m \to \infty} \frac{h^0(X|V, mD)}{m^v/v!}
$$

where $h^0(X|V, mD)$ is the dimension of the image of the natural restriction map $\varphi : H^0(X, \mathcal{O}_X(mD)) \to H^0(V, \mathcal{O}_V(mD))$ ([ELMNP2] Definition 2.1]). As the volume function, the restricted volume $\text{vol}_{X|V}(D)$ depends only on the numerical class of $D$, and it extends uniquely to a continuous function

$$
\text{vol}_{X|V} : \text{Big}^V(X) \to \mathbb{R}
$$

where $\text{Big}^V(X)$ is the set of all $\mathbb{R}$-divisor classes $\xi$ such that $V$ is not properly contained in any irreducible component of $\mathcal{B}_+(\xi)$. By [ELMNP2] Theorem 5.2], if $V$ is an irreducible component of $\mathcal{B}_+(D)$, then $\text{vol}_{X|V}(D) = 0$. When $V = X$, then we have $\text{vol}_{X|X}(D) = \text{vol}_X(D)$ for any divisor $D$, so $\text{vol}_X(D) = 0$ when $D$ is not big. For more details, see [ELMNP2].

2.5. Moving Seshadri constants. We first recall the definition of the Seshadri constant of a nef divisor at a point.

**Definition 2.3.** Let $D$ be a nef divisor on a variety $X$. Then the Seshadri constant $\varepsilon(D; x)$ of $D$ at a point $x$ on $X$ is defined as

$$
\varepsilon(D; x) := \sup \{ s \mid f^* D - sE \text{ is nef} \}
$$

where $f : \tilde{X} \to X$ is the blow-up of $X$ at $x$ with the exceptional divisor $E$. 
We now let
\[
\varepsilon'(D; x) := \inf_{C} \left\{ \frac{D \cdot C}{\text{mult}_x C} \right\}
\]
where \(\inf\) runs over all irreducible curves containing \(x\). It is well known that when \(D\) is nef, \(\varepsilon(D; x) = \varepsilon'(D; x)\) ([La Proposition 5.1.3]). Furthermore, by the Seshadri’s ampleness criterion ([La Theorem 1.4.13]), a divisor \(D\) on a variety \(X\) is ample if and only if \(\inf_{x \in X} \varepsilon'(D; x) > 0\). Thus for a nef divisor \(D\), the Seshadri constant \(\varepsilon(D; x)\) measures the local positivity of \(D\) at \(x\). For more details, we refer to [La Chapter 5].

For pseudoeffective divisors, Nakamaye ([Nm], see also [ELMNP2]) defined the following measurement.

**Definition 2.4.** Let \(D\) be a pseudoeffective divisor on a variety \(X\). If \(x \notin B_+(D)\), then the moving Seshadri constant \(\varepsilon(||D||; x)\) of \(D\) at a point \(x\) on \(X\) is defined as
\[
\varepsilon(||D||; x) := \sup_{f \cdot D = A + E} \varepsilon(A; x)
\]
where the sup runs over all morphisms \(f : \bar{X} \to X\) with \(\bar{X}\) smooth, that are isomorphic over a neighborhood of \(x\), and decompositions \(f^{-1}D = A + E\) with an ample \(\mathbb{Q}\)-divisor \(A\) and an effective divisor \(E\) such that \(f^{-1}(x)\) is not in the support of \(E\). If \(x \in B_+(D)\), then we simply let \(\varepsilon(||D||; x) = 0\).

If \(D\) is nef, then \(\varepsilon(||D||; x) = \varepsilon(D; x)\). Note that \(\varepsilon(||D||; x)\) depends only on the numerical class of \(D\). Furthermore, by [ELMNP2 Theorem 6.2], for every point \(x\) of a variety \(X\), the map \(D \mapsto \varepsilon(||D||; x)\) defines a continuous function on the entire Néron-Severi space \(N^1(X)_\mathbb{R}\). For more details, we refer to [ELMNP2].

### 3. Construction and basic properties of Okounkov bodies

In this section, we first explain the construction of Okounkov bodies in [LM], [KK] and limiting Okounkov bodies in [CHPW] and review some of their basic properties. Throughout this subsection, we fix an admissible flag \(Y_*\) on a smooth projective variety \(X\) of dimension \(n\), which is defined as a sequence of irreducible subvarieties \(Y_i\) of \(X\) such that
\[
Y_* : X = Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_{n-1} \supseteq Y_n = \{x\}
\]
where each \(Y_i\) is of codimension \(i\) in \(X\) and is smooth at \(x\). We denote the \(\mathbb{R}\)-linear system of a divisor \(D\) by
\[
||D||_{\mathbb{R}} := \{D' \mid D \sim_{\mathbb{R}} D' \geq 0\}.
\]

Let us first consider a big divisor \(D\) on \(X\). For a given admissible flag \(Y_*\), we define a valuation-like function
\[
(3.1) \quad \nu_\bullet : ||D||_{\mathbb{R}} \to \mathbb{R}_0^n
\]
as follows. For \(D' \in ||D||_{\mathbb{R}}\), let
\[
\nu_1 = \nu_1(D') := \text{ord}_{Y_1}(D').
\]
Since \(D' - \nu_1(D')Y_1\) is also effective, we can define
\[
\nu_2 = \nu_2(D') := \text{ord}_{Y_2}((D' - \nu_1 Y_1)_{|Y_1}).
\]
Once \(\nu_i = \nu_i(D')\) is defined, we define \(\nu_{i+1} = \nu_{i+1}(D')\) inductively as
\[
\nu_{i+1}(D') := \text{ord}_{Y_{i+1}}(\cdots ((D' - \nu_i Y_1)_{|Y_1} - \nu_2 Y_2)_{|Y_2} - \cdots - \nu_i Y_i)_{|Y_i}).
\]
By collecting the values \(\nu_i(D')\), we can define a function \(\nu_\bullet\) in (3.1) as
\[
\nu(D') = (\nu_1(D'), \nu_2(D'), \cdots, \nu_n(D')).
\]

**Remark 3.1.** By definition, it is easy to see that for any \(D' \in ||D||_{\mathbb{R}}\), we have
\[
\nu_i(D') \leq \text{ord}_{Y_i}(D') \leq \text{ord}_{Y_i}(D'_{|Y_{i-1}}).
\]
\textbf{Definition 3.2.} The Okounkov body $\Delta_{Y_\bullet}(D)$ of a big divisor $D$ with respect to an admissible flag $Y_\bullet$ is a closed convex subset of $\mathbb{R}_{\geq 0}$ defined as follows:

\[(3.2) \quad \Delta_{Y_\bullet}(D) := \text{the closure of the convex hull of } \nu_{Y_\bullet}(D|_R) \subseteq \mathbb{R}_{\geq 0}.\]

The limiting Okounkov body $\Delta_{Y_\bullet}^{\text{lim}}(D)$ of a pseudoeffective divisor $D$ with respect to an admissible flag $Y_\bullet$ is defined as

$$\Delta_{Y_\bullet}^{\text{lim}}(D) := \bigcap_{\epsilon > 0} \Delta_{Y_\bullet}(D + \epsilon A)$$

where $A$ is an ample divisor on $X$. If $D$ is not pseudoeffective, we simply put $\Delta_{Y_\bullet}^{\text{lim}}(D) := \emptyset$.

It is easy to see that the limiting Okounkov body $\Delta_{Y_\bullet}^{\text{lim}}(D)$ is also a closed convex subset of $\mathbb{R}^n$. Note also that if $D$ is big, then $\Delta_{Y_\bullet}(D) = \Delta_{Y_\bullet}^{\text{lim}}(D)$ by the continuity of $\Delta_{Y_\bullet}(D)$ (\cite[Theorem B]{LM}). For this reason, we will simply use the notation $\Delta_{Y_\bullet}(D)$ instead of $\Delta_{Y_\bullet}^{\text{lim}}(D)$ when $D$ is big.

\textbf{Lemma 3.3.} Let $D$ be a pseudoeffective divisor on $X$. Consider a birational morphism $f : \tilde{X} \to X$ with $\tilde{X}$ smooth and an admissible flag

$$\tilde{Y}_\bullet : \tilde{X} = \tilde{Y}_0 \supseteq \tilde{Y}_1 \supseteq \cdots \supseteq \tilde{Y}_{n-1} \supseteq \tilde{Y}_n = \{x'\}.$$ on $\tilde{X}$. Suppose that $f$ is isomorphic over $f(x')$ and

$$Y_\bullet := f(\tilde{Y}_\bullet) : X = f(\tilde{Y}_0) \supseteq f(\tilde{Y}_1) \supseteq \cdots \supseteq f(\tilde{Y}_{n-1}) \supseteq f(\tilde{Y}_n) = \{f(x')\}.$$

is an admissible flag on $X$. Then we have $\Delta_{Y_\bullet}^{\text{lim}}(f^*D) = \Delta_{Y_\bullet}^{\text{lim}}(D)$.

\begin{proof}
It is enough to consider for the case where $D$ is big. In this case, the assertion follows from the construction of Okounkov bodies of big divisors and the fact that $H^0(X, \mathcal{O}_X(D')) = H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(f^*D'))$ for any $\mathbb{Z}$-divisor $D'$ on $X$.
\end{proof}

By the following lemma, we can assume that every subvariety $Y_i$ from the admissible flag $Y_\bullet$ is smooth.

\textbf{Lemma 3.4.} Let $D$ be a pseudoeffective divisor on $X$ and $Y_\bullet$ be an admissible flag on $X$. Then we can take a birational morphism $f : \tilde{X} \to X$ with $\tilde{X}$ smooth and an admissible flag $\tilde{Y}_\bullet$ on $\tilde{X}$ such that each $\tilde{Y}_i$ is smooth and $\Delta_{\tilde{Y}_\bullet}^{\text{lim}}(f^*D) = \Delta_{\tilde{Y}_\bullet}^{\text{lim}}(D)$.

\begin{proof}
Recall that each subvariety $Y_i$ from the admissible flag $Y_\bullet$ is smooth at $x$. By successively taking embedded resolutions of singularities of $Y_{n-1}, \ldots, Y_1$ in $X$, we can take a birational morphism $f : \tilde{X} \to X$ with $\tilde{X}$ smooth such that $f$ is isomorphic over $x$. For $1 \leq i \leq n - 1$, let $\tilde{Y}_i$ be the strict transform of $Y_i$. Then we obtain an admissible flag on $\tilde{X}$ as follows:

$$\tilde{Y}_\bullet : \tilde{X} = \tilde{Y}_0 \supseteq \tilde{Y}_1 \supseteq \cdots \supseteq \tilde{Y}_{n-1} \supseteq \tilde{Y}_n = \{f^{-1}(x')\}.$$ Now the assertion follows from Lemma [3.3] \hfill \Box
\end{proof}

The following lemma will be helpful to compute the limiting Okounkov bodies using the divisorial Zariski decompositions.

\textbf{Lemma 3.5.} Let $D$ be a pseudoeffective divisor on $X$, and $D = P + N$ be the divisorial Zariski decomposition. Fix an admissible flag $Y_\bullet$ on $X$. Then we have $\Delta_{Y_\bullet}^{\text{lim}}(D) = \Delta_{Y_\bullet}^{\text{lim}}(P) + \Delta_{Y_\bullet}^{\text{lim}}(N)$. In particular, if $Y_n = \{x\} \not\subseteq \text{Supp}(N)$, then $\Delta_{Y_\bullet}^{\text{lim}}(D) = \Delta_{Y_\bullet}^{\text{lim}}(P)$.

\begin{proof}
When $D$ is big, the assertion is exactly the same as \cite[Theorem C (3)]{KL}. If $D$ is only pseudoeffective, then the assertion follows from the big case and the definition of limiting Okounkov bodies. \hfill \Box
\end{proof}

It is sometimes useful to work in the following restricted situations. For $1 \leq k \leq n$, we define the $k$-th partial flag $Y_{k\bullet}$ of $Y_\bullet$ as

$$Y_{k\bullet} := Y_k \supseteq \cdots \supseteq Y_n.$$ Suppose that $D$ is a big divisor such that $Y_k \not\subseteq B_+(D)$. We define $\nu_{k\bullet} : |D|_R \to \mathbb{R}_{\geq 0}$ as the function $\nu_{k\bullet}$ defined above (3.1) where we let $\nu_{i}(D') = 0$ for all $D' \in |D|_R$ if $i \leq k$.
Definition 3.6. The $k$-th restricted Okounkov body $\Delta_{Y_k}(D)$ of $D$ with respect to the partial flag $Y_k$ is defined as the following subset of $\mathbb{R}^{n-k}_0$:

$$\Delta_{Y_k}(D) := \text{the closure of the convex hull of } \nu_k([D|_R]).$$

By convention, the 0-th restricted Okounkov body is the usual Okounkov body, i.e., $\Delta_{Y_k}(D) = \Delta_{Y_k}(D)$. We can easily check that $\Delta_{Y_k}(D)$ is a closed convex subset of $\mathbb{R}^n$. We note that the $k$-th restricted Okounkov body $\Delta_{Y_k}(D)$ is nothing but the Okounkov body of a graded linear series $W_*$ in [LM] p.804, where $W_{m} = \text{Im} (H^0(X, O_X(mD)) \rightarrow H^0(Y_k, O_{Y_k}(mD|_Y)))$.

The following is one of the most important properties of Okounkov bodies.

Theorem 3.7 ([LM] (2,7) p.804). Let $D$ be a big divisor on $X$, and $Y_k$ be an admissible flag on $X$. Assume that $Y_k \not\subseteq B_+(D)$. Then we have

$$\text{vol}_{\mathbb{R}^{n-k}}(\Delta_{Y_k}(D)) = \frac{1}{(n-k)!} \text{vol}_X(Y_k(D)).$$

Remark 3.8. Assume that $Y_k$ is smooth. We can regard the $k$-th partial flag $Y_k$ as an admissible flag on $Y_k$ so that $\Delta_{Y_k}(D|_{Y_k})$ is a subset of $\mathbb{R}^{n-k}$. If $D$ is a pseudoeffective Cartier divisor, then

$$\{0\}^k \times \Delta_{Y_k}(D|_{Y_k}) \supseteq \Delta_{Y_k}(D)$$

holds in general. If $D$ is nef and big and $Y_k \not\subseteq B_+(D)$, then the equality holds and $\text{vol}_{\mathbb{R}^{n-1}}(\Delta_{Y_k}(D|_{Y_k})) = \frac{1}{(n-k)!} \text{vol}_X(Y_k(D)).$

Definition 3.9. Let $x \in \Delta_{Y_k}(D)$ be a point. If $x$ is of the form $x = \nu_k(D')$ for some $D' \in |D|_R$, then $x$ is called a $(k,\epsilon)$ valuative point of $\Delta_{Y_k}(D)$. We denote by

$$\Gamma_k := \{\nu_k(D') \mid D' \in |D|_R\} \subseteq \{0\}^k \times \mathbb{R}^{n-k}$$

the set of $k$-valuative points.

It is known that $\Gamma_k$ forms a dense subset in $\Delta_{Y_k}(D)$ ([KMS Lemma 2.6]). Thus it is enough to take the closure of the image $\nu_k([D|_R])$ in Definition 3.2 and 3.3 to obtain $\Delta_{Y_k}(D)$.

Lemma 3.10. Let $D, D'$ be pseudoeffective divisors on $X$. Then we have $\Delta_{Y_k}(D)+\Delta_{Y_k}(D') \subseteq \Delta_{Y_k}(D+D')$.

Proof. For an ample divisor $A$ and any $\epsilon, \epsilon' > 0$, we have

$$\Delta_{Y_k}(D+\epsilon A) + \Delta_{Y_k}(D'+\epsilon' A) \subseteq \Delta_{Y_k}(D+D'+\epsilon+\epsilon')A.$$ 

It follows from the convexity of $\Delta_{Y_k}(D)$ (cf. [LM Proof of Corollary 4.12]). By taking the limit, we obtain the statement. $\square$

For any subset $\Delta \subseteq \mathbb{R}^n$, we denote $\Delta_{x_1=\ldots=x_k=0} := \Delta \cap \{0\}^k \times \mathbb{R}^{n-k}$. It is easy to check that the condition $Y_k \not\subseteq B_+(D)$ implies that

$$\Delta_{Y_k(D)}|_{x_1=\ldots=x_k=0} \neq \emptyset.$$

Lemma 3.11 (cf. [LM] Theorem 4.24)). Let $D$ be a big divisor and $Y_k$ be an admissible flag on $X$ such that $Y_k \not\subseteq B_+(D)$ for some $k \geq 1$. Then

$$\left(\Delta_{Y_{k-1}}(D)\right)_{x_1=\ldots=x_k=0} = \Delta_{Y_k}(D).$$

Furthermore, $\Delta_{Y_k}(D)|_{x_1=\ldots=x_k=0} = \Delta_{Y_k}(D)$.

Proof. Since $\Delta_{Y_k}(D)|_{x_1=\ldots=x_k=0} \neq \emptyset$ by the assumption $Y_k \not\subseteq B_+(D)$, there is a $k$-th valuative point $\nu_k(D') \neq 0$ of $\Delta_{Y_k}(D)$ for some $D' \in |D|_R$. Note that $\nu_k(D') = 0$. Thus $(\Gamma_{k-1})_{x_1=0}$ coincides with the image of $\Gamma_k \subseteq \{0\}^k \times \mathbb{R}^{n-k}$ under the injective map

$$\{0\}^k \times \mathbb{R}^{n-k} \rightarrow \{0\}^{k-1} \times \mathbb{R}^{n-(k-1)}$$

given by the identity. It is easy to see that $(\Gamma_{k-1})_{x_1=0} = (\Gamma_{k-1})_{x_k=0}$ and this implies the desired equality of the sets. By applying the equality successively, we obtain $\Delta_{Y_k}(D)|_{x_1=\ldots=x_k=0} = (\Delta_{Y_k}(D)|_{x_1=0})_{x_2=\ldots=x_k=0} = \cdots = \Delta_{Y_k}(D)$. $\square$
4. Restricted base loci via Okounkov bodies

In this section, we prove Theorem A and Corollary B. More specifically, we extract the restricted base locus $B_+(D)$ of a pseudoeffective divisor $D$ from its associated limiting Okounkov bodies. We also recover the divisorial components of $B_-(D)$ from the limiting Okounkov bodies, and consequently obtain the movability criterion of divisors in terms of limiting Okounkov bodies (Theorem 4.3). Throughout this section, $X$ is a smooth projective variety of dimension $n$.

We show the following lemma first.

Lemma 4.1. Let $D$ be a big divisor on $X$, and fix an admissible flag $Y_\bullet$ on $X$.

(1) If $Y_1 \subseteq B_-(D)$, then the origin of $\mathbb{R}^n$ is not contained in $\Delta_{Y_\bullet}(D)$.

(2) If $Y_1 \not\subseteq B_-(D)$, then for any integer $k \geq 1$ with $Y_k \not\subseteq B_-(D)$, we have $\Delta_{Y_\bullet}(D)_{x_1=\cdots=x_k=0} \neq \emptyset$.

Proof. (1) In this case, $Y_1$ is an irreducible component of $B_-(D)$ and $\text{ord}_{Y_1}(||D||) > 0$ by Proposition 2.8. Thus for any $D' \in |D|_{\mathbb{R}}$, we have $\nu_1(D') = \text{ord}_{Y_1}D' \geq \text{ord}_{Y_1}(||D||) > 0$. It follows that

$$\inf\{x_1 \mid (x_1, \cdots, x_n) \in \Delta_{Y_\bullet}(D)\} > 0.$$ 

In particular, the origin of $\mathbb{R}^n$ is not contained in $\Delta_{Y_\bullet}(D)$.

(2) Let $k \geq 1$ be an integer such that $Y_k \not\subseteq B_-(D)$. For any $\varepsilon > 0$, there exists an effective divisor $D' \sim_{\mathbb{R}} D$ such that $\text{ord}_{Y_k}(D') < \varepsilon$. Since $Y_i \subseteq Y_k$ for $i \leq k$, we have $\text{ord}_{Y_1}(D') \leq \text{ord}_{Y_k}(D')$, and hence, $\text{ord}_{Y_i}(D') < \varepsilon$ for all $i \leq k$. By Remark 3.3, we see that $\nu_i(D') \leq \text{ord}_{Y_i}(D')$. Thus

$$\nu_i(D') < \varepsilon$$

for all $i \leq k$.

This implies that for any $\varepsilon > 0$, there exists a valuative point $(x_1, \cdots, x_n)$ in $\Delta_{Y_\bullet}(D)$ such that $x_i \in [0, \varepsilon)$ for all integers $i$ with $1 \leq i \leq k$. Thus we obtain $\Delta_{Y_\bullet}(D)_{x_1=\cdots=x_k=0} \neq \emptyset$. \qed

We now prove Theorem A as Theorem 4.2.

Theorem 4.2. Let $D$ be a pseudoeffective divisor on $X$. Then the following are equivalent.

(1) $x \in B_-(D)$.

(2) For any admissible flag $Y_\bullet$ centered at $x$, the origin of $\mathbb{R}^n$ is not contained in $\Delta_{Y_\bullet}^\text{lim}(D)$.

(3) For some admissible flag $Y_\bullet$ centered at $x$, the origin of $\mathbb{R}^n$ is not contained in $\Delta_{Y_\bullet}^\text{lim}(D)$.

Proof. We first treat the case where $D$ is big. In this case, $\Delta_{Y_\bullet}^\text{lim}(D) = \Delta_{Y_\bullet}(D)$.

$\Rightarrow$ (2): Assume that $x \in B_-(D)$ and fix an admissible flag $Y_\bullet$ centered at $x$. If $Y_1 \subseteq B_-(D)$, then by Lemma 4.1 (1), the origin of $\mathbb{R}^n$ is not contained in $\Delta_{Y_\bullet}(D)$. Thus assume that $Y_1 \not\subseteq B_-(D)$. If $k \geq 1$ is the largest integer among $i$ such that $Y_i \not\subseteq B_-(D)$, then $Y_{k+1} \subseteq B_-(D)$ and by Lemma 4.1 (2),

$$S_k := \Delta_{Y_\bullet}(D)_{x_1=\cdots=x_k=0} \neq \emptyset.$$

We claim that $\inf\{x_{k+1} \mid (0, \cdots, 0, x_{k+1}, \cdots, x_n) \in S_k\} > 0$, and hence, the origin of $\mathbb{R}^n$ is not contained in $\Delta_{Y_\bullet}(D)$. Since $S_k \neq \emptyset$ and the valuative points are dense in $\Delta_{Y_\bullet}(D)$, it follows that for any small $\varepsilon > 0$, there exists an effective divisor $D' \in |D|_{\mathbb{R}}$ which defines a valuative point $\nu(D') = (\nu_1, \cdots, \nu_n) \in \Delta_{Y_\bullet}(D)$ such that $0 \leq \nu_i < \varepsilon$ for all integers $i$ satisfying $1 \leq i \leq k$. Let $D'_1 := D'$, $D'_2 := (D'_1 - \nu_1 Y_1)|_{Y_1}$, and define $D'_i$ inductively as

$$D'_i := (D'_{i-1} - \nu_{i-1} Y_{i-1})|_{Y_{i-1}}.$$

Then we get

$$\nu_{k+1} = \nu_{k+1}(D') = \text{ord}_{Y_{k+1}}((D'_k - \nu_k Y_k)|_{Y_k})$$

$$= \text{ord}_{Y_{k+1}}(D'_k|_{Y_k}) - \nu_k$$

$$= \text{ord}_{Y_{k+1}}((D'_{k-1} - \nu_{k-1} Y_{k-1})|_{Y_k}) - \nu_k$$

$$= \text{ord}_{Y_{k+1}}(D'_{k-1}|_{Y_k}) - \nu_k - \nu_{k-1} - \nu_{k-2} - \cdots - \nu_1.$$ 

We have $\text{ord}_{Y_{k+1}}(||D||) > 0$ since $Y_{k+1} \subseteq B_-(D)$. Suppose that $0 < \varepsilon < \frac{1}{2k} \text{ord}_{Y_{k+1}}(||D||)$. Since $0 \leq \nu_i < \varepsilon$ for $1 \leq i \leq k$, we obtain

$$\nu_{k+1} = \nu_{k+1}(D'|_{Y_k}) = \text{ord}_{Y_{k+1}}(D'_k|_{Y_k}) - (\nu_1 + \cdots + \nu_k)$$

$$> \text{ord}_{Y_{k+1}}(D'_k|_{Y_k}) - k \cdot \frac{1}{2k} \text{ord}_{Y_{k+1}}(||D||)$$

$$\geq \text{ord}_{Y_{k+1}}(||D||) - \frac{1}{2} \text{ord}_{Y_{k+1}}(||D||)$$

$$= \frac{1}{2} \text{ord}_{Y_{k+1}}(||D||)$$.
Therefore, we get
\[
\inf\{x_{k+1} \mid (0, \ldots, 0, x_{k+1}, \ldots, x_n) \in S_k\} \geq \frac{1}{2}\ord_{y_{k+1}}(||D||) > 0.
\]
We have shown the claim so that the origin of \(\mathbb{R}^n\) is not contained in \(\Delta_{\nu_{k+1}}(D)\).

(2) \(\Rightarrow\) (3): Obvious.

(3) \(\Rightarrow\) (1): Assume that there exists an admissible flag \(Y_{\bullet}\) centered at \(x\) such that the origin of \(\mathbb{R}^n\) is not contained in \(\Delta_{\nu_{k+1}}(D)\). To derive a contradiction, suppose that \(x \notin B_{-}(D)\). By letting \(k = n\) in Lemma 4.1 (2), we obtain a contradiction. Thus we have proved this theorem for big divisors.

We now turn to the proof for the case where \(D\) is pseudoeffective. Fix an ample divisor \(A\) on \(X\). First, observe that
\[
B_{-}(D) = \bigcup_{\epsilon > 0} B_{-}(D + \epsilon A).
\]
Thus \(x \in B_{-}(D)\) if and only if \(x \in B_{-}(D + \epsilon A)\) for all sufficiently small \(\epsilon > 0\). Since \(\Delta_{\nu_{k+1}}(D) = \bigcap_{\epsilon > 0} \Delta_{\nu_{k+1}}(D + \epsilon A)\), the assertion easily follows from the case where \(D\) is big. \(\square\)

Note that a pseudoeffective divisor \(D\) is nef if and only if \(B_{-}(D) = \emptyset\). Thus we immediately obtain the following nefness criterion.

**Corollary 4.3.** Let \(D\) be a pseudoeffective divisor on \(X\). Then the following are equivalent.

1. \(D\) is nef.
2. For any admissible flag \(Y_{\bullet}\), the origin of \(\mathbb{R}^n\) is contained in \(\Delta_{\nu_{k+1}}(D)\).
3. For any point \(x \in X\) and for some admissible flag \(Y_{\bullet}\) centered at \(x\), the origin of \(\mathbb{R}^n\) is contained in \(\Delta_{\nu_{k+1}}(D)\).

Next we prove the movability criterion of divisor. Recall that a divisor \(D\) is movable if \(B_{-}(D)\) has no irreducible components of dimension \(n - 1\).

**Theorem 4.4.** Let \(D\) be a pseudoeffective divisor on \(X\). The following are equivalent:

1. \(D\) is movable.
2. For any admissible flag \(Y_{\bullet}\) on \(X\), we have \(\Delta_{\nu_{k+1}}(D)_{x_1=0} \neq \emptyset\).

**Proof.** As in the proof of Theorem 4.2 we prove the case when \(D\) is big. The pseudoeffective case is easy and left to the readers.

(1) \(\Rightarrow\) (2): Fix an arbitrary admissible flag \(Y_{\bullet}\) of \(X\). If \(D\) is movable, then \(Y_1 \subseteq B_{-}(D)\). By Lemma 4.1 (2), we have \(\Delta_{\nu_{k+1}}(D)_{x_1=0} \neq \emptyset\).

(2) \(\Rightarrow\) (1): Suppose that \(D\) is not movable. Then there exists a prime divisor \(E \subseteq B_{-}(D)\). Take an admissible flag \(Y_{\bullet}\) such that \(Y_1 = E\). Then for any effective divisor \(D'\) such that \(D' \sim_D D\), we have \(\nu_1(D') = \ord_{Y_1}(D') \geq \ord_{Y_1}(||D||) > 0\).

Thus \(\Delta_{\nu_{k+1}}(D)_{x_1=0} = \emptyset\), which is a contradiction. \(\square\)

**Corollary 4.5.** If \(X\) is a surface, then in addition to the conditions (1),(2), and (3) in Corollary 4.3 the following condition is also equivalent:

4. The Okounkov body \(\Delta_{\nu_{k+1}}(D)\) with respect to any admissible flag \(Y_{\bullet}\) intersects the \(x_2\)-axis of the plane \(\mathbb{R}^2\).

**Proof.** The condition (4) is the movability condition for divisors on a surface. On a surface, a divisor is movable if and only if it is nef. \(\square\)

5. **DIVISORIAL ZARISKI DECOMPOSITIONS VIA OKOUNKOV BODIES**

In this section, we compute the divisorial Zariski decomposition of a big divisor using the limiting Okounkov bodies. The Zariski decomposition plays a crucial role in computing the Okounkov body of a big divisor in the surface case (see [LM, Theorem 6.4]). As before, \(X\) is a smooth projective variety of dimension \(n\).

We first define the following set for a pseudoeffective divisor \(D\) on \(X\):
\[
\text{div}(\Delta_{\nu_{k+1}}(D)) := \left\{ E \mid \begin{array}{l} Y_{\bullet} \text{ is an admissible flag with } Y_1 = E \text{ such that } \Delta_{\nu_{k+1}}(D)_{x_1=0} = \emptyset. \end{array} \right\}.
\]
Lemma 5.1. The set \( \text{div}\Delta^\lim(D) \) is finite, and \(#(\text{div}\Delta^\lim(D)) = 0 \) if and only if \( D \) is movable.

Proof. Note that for an admissible flag \( Y = P \) on \( X \), we have

\[
\sigma_Y(||D||) = \min \{x_1, x_2, \ldots, x_n \in \Delta^\lim_Y(D)\}.
\]

Thus \(#(\text{div}\Delta^\lim(D)) \) is the number of divisorial components of \( B_-(D) \), so it is finite by [Ny]. The second statement follows from Theorem 4.4. \( \Box \)

We now explain how to obtain the divisorial Zariski decomposition of a pseudoeffective divisor \( D \) using the limiting Okounkov bodies of \( D \). If \( D \) is not movable, then by Theorem 4.4 there exists an admissible flag \( Y \) with \( Y = P \) such that \( \Delta^\lim_Y(D) \) does not intersect the hyperplane \( H \) defined by \( x_1 = 0 \). Define a positive number

\[
a_1 := \inf \{x_1 \geq 0 \mid (x_1, \ldots, x_d) \in \Delta^\lim_Y(D)\} > 0
\]

and consider the divisor \( D - a_1E_1 \). Note that \( a_1 = \text{ord}_{E_1}(||D||) \).

Lemma 5.2. Let \( D \) be a pseudoeffective divisor on \( X \). If \( Y \) an admissible flag on \( X \) such that \( Y = P \) is a divisorial component of \( B_-(D) \), then

\[
\Delta^\lim_Y(D - a_1E_1) = \Delta^\lim_Y(D) - (a_1, 0, \ldots, 0).
\]

Proof. First, we assume that \( D \) is big. It is easy to see that

\[
||D - a_1E_1||_R = ||D||_R - a_1E_1.
\]

By applying the function \( \nu \), we obtain

\[
\nu(D - a_1E_1) = \nu(D) - (a_1, 0, \ldots, 0).
\]

This implies the required statement. Now we consider the case that \( D \) is only pseudoeffective. For an ample divisor \( A \) and a positive number \( \epsilon \), the divisor \( D_\epsilon := D + \epsilon A \) is big. Thus we have

\[
\Delta^\lim_{Y}(D_\epsilon - \text{ord}_{E_1}(||D_\epsilon||)E_1) = \Delta^\lim_Y(D_\epsilon) - (\text{ord}_{E_1}(||D_\epsilon||), 0, \ldots, 0).
\]

By taking \( \epsilon \to 0 \), we obtain the required statement by the definitions of \( \text{ord}_{E_1}(||D||) \) and \( \Delta^\lim_Y(D) \). \( \Box \)

By Lemma 5.2 we have \( \Delta^\lim_Y(D - a_1E_1) = \Delta^\lim_Y(D) - (a_1, 0, \ldots, 0) \). The Okounkov body \( \Delta^\lim_Y(D - a_1E_1) \) touches the hyperplane \( H \) since

\[
\inf \{x_1 \geq 0 \mid (x_1, \ldots, x_d) \in \Delta^\lim_Y(D - a_1E_1)\} = \text{ord}_{E_1}(||D - a_1E_1||) = 0.
\]

We also have

\[
#(\text{div}\Delta^\lim(Y - a_1E_1)) = #(\text{div}\Delta^\lim(D)) - 1.
\]

We can continue this process by replacing \( D \) by \( D - a_1E_1 \). Thus after \( n = #(\text{div}\Delta^\lim(D)) \) steps, we arrive at a situation where

\[
\Delta^\lim_Y(D - a_1E_1 - \cdots - a_nE_n)_{x_1 = 0} \neq \emptyset
\]

for all admissible flags \( Y \) on \( X \). Since \( #(\text{div}\Delta^\lim(Y - a_1E_1 - \cdots - a_nE_n)) = 0 \), Lemma 5.1 implies that \( P = D - a_1E_1 - \cdots - a_nE_n \) is movable. Thus we obtain the divisorial Zariski decomposition \( D = P + N \) where \( N = a_1E_1 + \cdots + a_nE_n \).

6. Augmented Base Loci via Okounkov Bodies

In this section, we prove Theorem \( C \) and Corollary \( D \). More precisely, we extract the augmented base locus \( B_+(D) \) of a pseudoeffective divisor \( D \) from its associated limiting Okounkov bodies. Throughout this section, \( X \) is a smooth projective variety of dimension \( n \).

Although Corollary \( D \) can be proved as a consequence of Theorem \( C \), we first show Corollary \( D \) as Proposition 6.2 in order to clarify the ideas and to make the proofs more transparent. For this purpose, we need the following easy lemma.

Lemma 6.1. Let \( D \) be a pseudoeffective divisor on \( X \), and fix an admissible flag \( Y \) centered at a point \( x \) on \( X \). If there is an irreducible curve \( C \) passing through \( x \) such that \( \frac{D - a_1E_1}{\text{mult}_x C} < \epsilon \) for some \( \epsilon > 0 \) and \( C \not\subseteq Y \), then \( C \subseteq B_-(D - \epsilon Y_1) \). In particular, \( (\epsilon, 0, \ldots, 0) \not\in \Delta^\lim_Y(D) \).
Proof. Note that \( \text{mult}_x C \leq Y_1 \cdot C \). Thus we have

\[
(D - \varepsilon Y_1) \cdot C \leq D \cdot C - \varepsilon \text{mult}_x C < 0,
\]

and hence, \( C \subseteq B_-(D - \varepsilon Y_1) \). In particular, we have \( x \in B_-(D - \varepsilon Y_1) \). Therefore Theorem A implies that the origin of \( \mathbb{R}^n \) is not contained in \( \Delta_{Y\lambda}^{\text{lim}}(D - \varepsilon Y_1) \). Thus we get \( (\varepsilon, 0, \ldots, 0) \notin \Delta_{Y\lambda}^{\text{lim}}(D) \). \( \square \)

The following gives the ampleness criterion of divisors via limiting Okounkov bodies.

**Proposition 6.2.** Let \( D \) be a big divisor on \( X \) of dimension \( n \). Then the following are equivalent.

1. \( D \) is ample.
2. For any admissible flag \( Y_\bullet, U_{\geq 0} \) is contained in \( \Delta_{Y_\bullet}(D) \) for some small open neighborhood \( U \) of the origin of \( \mathbb{R}^n \).
3. For any point \( x \in X \) and for some admissible flag \( Y_\bullet \) centered at \( x, U_{\geq 0} \) is contained in \( \Delta_{Y_\bullet}(D) \) for some small open neighborhood \( U \) of the origin of \( \mathbb{R}^n \).

**Proof.** (1) \( \Rightarrow \) (2): We use the induction on dimension \( n \). The case \( n = 1 \) is clear since we have \( \Delta_{Y_\bullet}(D) = [0, \deg D] \) ([LM] Example 1.14). We now assume that \( n \geq 2 \). Since \( D - \varepsilon Y_1 \) is ample for all sufficiently small \( \varepsilon > 0 \), it follows from Corollary [B] and [LM] Theorem 4.26] that

\[
0 \neq \Delta_{Y_\lambda}(D - \varepsilon Y_1) \cap \{ x_1 = \varepsilon \} \subseteq \Delta_{Y_\lambda}(D) \cap \{ x_1 = \varepsilon \}.
\]

In particular, \( (\varepsilon', 0, \ldots, 0) \in \Delta_{Y_\lambda}(D) \) for some \( \varepsilon' > 0 \).

Let \( f : \tilde{X} \to X \) be a birational morphism as in Lemma 3.4 so that \( \Delta_{Y_\lambda}(D) = \Delta_{Y_\lambda}(f^*D) \). Let \( E_1, \ldots, E_r \) be all the exceptional prime divisors of \( f \) and write \( E := \sum_{i=1}^r E_i \). Then \( (f^*D - \varepsilon E)|_{Y_\lambda} \) is ample on \( \tilde{Y}_\lambda \) for a sufficiently small \( \varepsilon > 0 \). By the induction hypothesis, there exists a simplex \( \Delta_{Y_\lambda} \subseteq \mathbb{R}^{n-1} \) of length \( \lambda' = (\lambda_2, \ldots, \lambda_n) \) with all \( \lambda_i > 0 \) such that \( \Delta_{Y_\lambda} \subseteq \Delta_{Y_\lambda}(\{(f^*D - \varepsilon E)|_{Y_\lambda}\}) \). Thus by Remark 14.8, we have

\[
\{0\} \times \Delta_{Y_\lambda} \subseteq \Delta_{Y_\lambda}(f^*(D - \varepsilon E)|_{Y_\lambda}) = \Delta_{Y_\lambda}(f^*D - \varepsilon E) = \Delta_{Y_\lambda}(f^*D - \varepsilon E)|_{x_1=0}.
\]

We can easily show that \( \Delta_{Y_\lambda}(\varepsilon E) = \{0\} \). Thus we obtain

\[
\{0\} \times \Delta_{Y_\lambda} \subseteq \Delta_{Y_\lambda}(f^*(D - \varepsilon E) + \Delta_{Y_\lambda}(\varepsilon E) \subseteq \Delta_{Y_\lambda}(f^*D) = \Delta_{Y_\lambda}(D).
\]

By the convexity of \( \Delta_{Y_\lambda}(D) \), the simplex \( \Delta_{Y_\lambda} \subseteq \mathbb{R}^n \) of length \( \lambda = (\varepsilon', \lambda_2, \ldots, \lambda_n) \) is contained in \( \Delta_{Y_\lambda}(D) \).

(2) \( \Rightarrow \) (3): Obvious.

(3) \( \Rightarrow \) (1): Suppose that \( D \) is not ample. Then by the Seshadri’s ampleness criterion, there exist a point \( x \) on \( X \) and a sequence \( \{C_i\} \) of irreducible curves passing through \( x \) such that \( \lim_{i \to \infty} \frac{D \cdot C_i}{\text{mult}_x C_i} = 0 \). To derive a contradiction, we further assume that for some admissible flag \( Y_\bullet \) centered at \( x \), the simplex \( \Delta_{Y_\lambda} \subseteq \mathbb{R}^n \) of length \( \lambda = (\varepsilon', \varepsilon, \varepsilon) \) is contained in \( \Delta_{Y_\lambda}(D) \) for a sufficiently small \( \varepsilon > 0 \). We may assume that \( \frac{D \cdot C_i}{\text{mult}_x C_i} < \varepsilon \) for all \( i \). Take a birational morphism \( f : \tilde{X} \to X \) isomorphic over \( x \) as in Lemma 3.4. Then we obtain an admissible flag \( \tilde{Y}_\bullet \) on \( \tilde{X} \) such that all subvarieties \( \tilde{Y}_i \) are obtained by the strict transforms of \( Y_i \), and we have \( \Delta_{Y_\lambda}(D) = \Delta_{\tilde{Y}_\lambda}(f^*D) \). If we let \( x' := f^{-1}(x) \) and denote by \( C_i \) the strict transforms of \( C_i \), then we also have \( \frac{D \cdot C_i}{\text{mult}_x C_i} = \frac{f^*D \cdot \tilde{C}_i}{\text{mult}_{x'} \tilde{C}_i} \).

Suppose that \( \tilde{C}_j \neq \tilde{Y}_{n-1} \) for some \( j \). Then there exists an integer \( k \geq 0 \) such that \( \tilde{C}_j \subseteq \tilde{Y}_k \) and \( \tilde{C} \not\subseteq \tilde{Y}_{k+1} \). For such \( k \), the following holds

\[
\frac{f^*D \cdot \tilde{C}_j}{\text{mult}_{x'} \tilde{C}_j} < \varepsilon.
\]

Thus Lemma 6.4 implies that \( (\varepsilon, 0, \ldots, 0) \notin \Delta_{\tilde{Y}_\lambda}(f^*D|_{\tilde{Y}_\lambda}) \). However, this is a contradiction since

\[
\Delta_{\tilde{Y}_\lambda}(f^*D)|_{x_1 = \ldots = x_k = 0} \subseteq \Delta_{\tilde{Y}_\lambda}(f^*D|_{\tilde{Y}_\lambda}).
\]

It remains to consider the case where \( \{C_i\} \) is a constant sequence; \( \tilde{C}_i = \tilde{Y}_{n-1} \) for all \( i \). In this case, \( f^*D \cdot \tilde{C}_i \) is constant and we have \( f^*D \cdot \tilde{C}_i = f^*D|_{\tilde{Y}_2} \cdot \tilde{C}_i = 0 \). Let \( f^*D|_{\tilde{Y}_2} = P + N \) be the Zariski decomposition on the surface \( \tilde{Y}_2 \). Then we have \( \tilde{P} \cdot \tilde{C}_i = 0 \). Now [LM] Theorem 6.4 implies \( (0, \varepsilon) \notin \Delta_{\tilde{Y}_2}(f^*D|_{\tilde{Y}_2}) \), which however is again a contradiction. \( \square \)
The following is the main ingredient of the proof of Theorem C.

**Theorem 6.3.** Let \( x \in \mathbf{B}_+(D) \) and \( x \not\in \mathbf{B}_-(D) \). Then for any \( \varepsilon > 0 \), there exist a birational morphism \( f : Y \to X \) which is isomorphic over a neighborhood of \( x \) and a curve \( C \) on \( Y \) passing through \( x' := f^{-1}(x) \) such that

\[
0 \leq \frac{P \cdot C}{\mult_x C} < \varepsilon
\]

where \( P \) is the positive part of the divisorial Zariski decomposition of \( f^*D \).

**Proof.** Fix a real number \( \varepsilon > 0 \). Let \( A_i \) be a sequence of ample divisors on \( X \) such that \( D + A_i \) is a \( \mathbb{Q} \)-divisor for each \( i \) and \( \lim_{i \to \infty} A_i = 0 \). Note that \( x \not\in \mathbf{B}_+(D + A_i) \) for each \( i \). Since \( \varepsilon(|| \cdot ||; x) \) is continuous and \( \lim_{i \to \infty} \varepsilon(||D + A_i||; x) = \varepsilon(||D||, x) = 0 \), we may assume that

\[
\varepsilon(||D + A||, x) < \varepsilon
\]

holds for all \( i \). For each \( i \), as in [Le] Proposition 3.7, there exist an integer \( m \) and a birational morphism \( f_{i,m} : X_{i,m} \to X \) centered at \( \mathbf{B}_+(D + A_i) \) which resolves the base locus of the linear system \( |m(D + A_i)| \).

To simplify the notation, we denote \( x' = f_{i,m}^*(x) \). Let \( f_{i,m}^*(m(D + A_i)) = M_{i,m} + F_{i,m} \) be a decomposition into a base point free divisor \( M_{i,m} \) and the fixed part \( F_{i,m} \). Since \( \varepsilon(||D + A||, x) = \lim_{m \to \infty} \varepsilon(\frac{1}{m} M_{i,m} ; x') \), we have

\[
\varepsilon \left( \frac{1}{m} M_{i,m}, x' \right) < \varepsilon
\]

for all sufficiently large and divisible \( m \). This implies that on \( X_{i,m} \) for a sufficiently large and divisible \( m \), there exists an irreducible curve \( C \) on \( X_{i,m} \) passing through \( x' \) such that

\[
\frac{1}{m} M_{i,m} \cdot C < \varepsilon.
\]

Consider the divisorial Zariski decompositions:

\[
f_{i,m}^*(D + A_i) = P_{i,m}' + N_{i,m}' \quad \text{and} \quad f_{i,m}^*(D) = P_{i,m} + N_{i,m}.
\]

By [Le] Proposition 3.7, we have \( \lim_{m \to \infty} \frac{1}{m} M_{i,m} = P_{i,m}' \), and we also have \( \lim_{i \to \infty} P_{i,m}' = P_{i,m} \). Thus we finally obtain

\[
0 \leq \frac{P_{i,m} \cdot C}{\mult_x C} < \varepsilon.
\]

for some large \( m \).

Now we prove Theorem C as Theorem 6.4.

**Theorem 6.4.** Let \( D \) be a pseudoeffective divisor on \( X \). Then the following are equivalent.

1. \( x \in \mathbf{B}_+(D) \).
2. For any admissible flag \( Y_x \) centered at \( x \), \( U_{>0} \) is not contained in \( \Delta_{Y_x}^\lim(D) \) for any small open neighborhood \( U \) of the origin of \( \mathbb{R}^n \).
3. For some admissible flag \( Y_x \) centered at \( x \), \( U_{>0} \) is not contained in \( \Delta_{Y_x}^\lim(D) \) for any small open neighborhood \( U \) of the origin of \( \mathbb{R}^n \).

**Proof.** Suppose first that \( D \) is not big. Then \( \mathbf{B}_+(D) = X \) and \( \vol_{\mathbb{R}^n}(\Delta_{Y_x}^\lim(D)) = 0 \) for any admissible flag \( Y_x \). Thus \( \Delta_{Y_x}^\lim(D) \) cannot contain any \( n \)-dimensional convex set. In this case, there is nothing to prove. Thus, from now on, we assume that \( D \) is big and we write \( \Delta_{Y_x}^\lim(D) = \Delta_{Y_x}(D) \).

(1) \( \Rightarrow \) (2): Fix an admissible flag \( Y_x \) centered at \( x \). If \( x \in \mathbf{B}_-(D) \), then this implication follows from Theorem A. Thus we assume that \( x \in \mathbf{B}_+(D) \) \( \setminus \mathbf{B}_-(D) \) so that the origin of \( \mathbb{R}^n \) is contained in \( \Delta_{Y_x}(D) \).

First, we assume that \( D|_{Y_x} \) is not big for some \( 1 \leq k \leq n - 1 \). Then arguing as above, we see that \( \Delta_{Y_x}^\lim(D|_{Y_x}) \) does not contain \( U_{>0} \cap \{0\}^k \times \mathbb{R}^{n-k} \) for any small neighborhood \( U \) of the origin of \( \mathbb{R}^n \).

Since \( \Delta_{Y_x}(D)_{x_1=\ldots=x_k = 0} \subseteq \Delta_{Y_x}^\lim(D|_{Y_x}) \), we have

\[
U_{>0} \cap \{0\}^k \times \mathbb{R}^{n-k} \not\subseteq \Delta_{Y_x}(D)_{x_1=\ldots=x_k = 0},
\]

hence \( U_{>0} \not\subseteq \Delta_{Y_x}(D) \).

Thus we only have to consider the case where \( D|_{Y_x} \) is big for all \( i \) such that \( 1 \leq i \leq n - 1 \). To derive a contradiction, suppose that a simplex \( \Delta_\lambda \) of length \( \lambda = (\varepsilon, \ldots, \varepsilon) \) with \( \varepsilon > 0 \) is contained in
so that ▲

\[\Delta_{\mathcal{Y}_i}(D).\]

Since \(x \in \mathcal{B}_+(D) \setminus \mathcal{B}_-(D),\) by Theorem \[6.3\] there exist a birational morphism \(f : \widetilde{X} \to X\) and an irreducible curve \(C\) on \(\widetilde{X}\) passing through \(x' := f^{-1}(x)\) such that

\[P \cdot C < \varepsilon\]

where \(P\) is the positive part of the divisorial Zariski decomposition of \(f^*D.\) By Lemma \[3.3\], we may assume that the admissible flag \(\mathcal{Y}_i\) on \(\widetilde{X}\) obtained by taking strict transforms of \(\mathcal{Y}_i\) consists of smooth subvarieties. By Lemma \[3.5\] we have

\[\Delta_{\mathcal{Y}_i}(D) = \Delta_{\mathcal{Y}_i}(f^*D) = \Delta_{\mathcal{Y}_i}(P)\]

so that \(\mathbf{\Delta}_\lambda \subseteq \Delta_{\mathcal{Y}_i}(P).\) In particular, \(P|_{\mathcal{Y}_k}\) is big for any \(1 \leq k \leq n-1,\) and \((\varepsilon, 0, \cdots, 0) \in \Delta_{\mathcal{Y}_i}(P)|_{x_1=\cdots=x_k=0}\) holds in \(\mathbb{R}^{n-k}\) for any \(k \geq 0.\)

We claim that \(C = \mathcal{Y}_{n-1}.\) Otherwise, there exists an integer \(k \geq 0\) such that \(C \subseteq \mathcal{Y}_k\) and \(C \not\subseteq \mathcal{Y}_{k+1}.\) In this case, we have

\[P \cdot C = P|_{\mathcal{Y}_k} \cdot C < \varepsilon.\]

Therefore, by Lemma \[6.1\] \((\varepsilon, 0, 0, \cdots, 0) \notin \Delta_{\mathcal{Y}_{n-k}}(P|_{\mathcal{Y}_k})\) holds in \(\mathbb{R}^{n-k}\). However, this is a contradiction since

\[\Delta_{\mathcal{Y}_{n-k}}(P)|_{\mathcal{Y}_{n-k}} \triangleq \Delta_{\mathcal{Y}_{n-k}}(P)|_{\mathcal{Y}_{n-k}}\]

holds for any \(k \geq 0.\) Thus \(C = \mathcal{Y}_{n-1}.\) In this case, \(C = \mathcal{Y}_{n-1}\) is a big divisor on a variety \(X\) and for some sufficiently small \(\varepsilon > 0,\) we have \(\mathcal{B}_+(D) \cap \mathcal{B}_-(D - \varepsilon A)\). By Theorem \[1.2\] for any admissible flag \(\mathcal{Y}_i\) centered at \(x,\) the origin \(O\) of \(\mathbb{R}^n\) is contained in \(\Delta_{\mathcal{Y}_i}(D - \varepsilon A).\) By Proposition \[6.2\] the Okounkov body \(\Delta_{\mathcal{Y}_i}(\varepsilon A)\) contains \(U_{\geq 0}\) for some open neighborhood \(U\) of the origin of \(\mathbb{R}^n.\) By Lemma \[3.10\] we have

\[\Delta_{\mathcal{Y}_i}(D - \varepsilon A) + \Delta_{\mathcal{Y}_i}(\varepsilon A) \subseteq \Delta_{\mathcal{Y}_i}(D).\]

Therefore, the set \(U_{\geq 0} = O + U_{\geq 0}\) is also contained in \(\Delta_{\mathcal{Y}_i}(D).\)

\[\mathbf{7. \text{Bounds for moving Seshadri constants via Okounkov bodies}}\]

In this section, we prove Theorem \[6.2\] Throughout this section, \(X\) is a smooth projective variety of dimension \(n.\)

For a convex subset \(\Delta \subseteq \mathbb{R}^n\) containing the origin of \(\mathbb{R}^n,\) we define the \textit{maximal sub-simplex} of \(\Delta\) as the simplex \(\mathbf{\Delta}_\lambda\) of length \(\lambda = (\lambda_1, \cdots, \lambda_n)\) where \(\lambda_i = \max\{x_i|[0, \cdots, 0, x_i, 0, \cdots, 0] \in \Delta\}\) for each \(i.\)

Note that we may have \(\lambda_i = 0\) for some \(i.\) If the origin of \(\mathbb{R}^n\) is not contained in \(\Delta,\) then we define the origin as its maximal sub-simplex. If \(\Delta = \Delta_{\mathcal{Y}_i}(D)\) where \(D\) is a big divisor on a variety \(X\) of dimension \(n\) and \(\mathcal{Y}_i\) is an admissible flag centered at \(x \in X,\) then the \(i\)-th maximal length \(\lambda_i\) depends on \(D, x, \) and \(\mathcal{Y}_i.\) Thus we can write \(\lambda_i = \lambda_i(D; x, \mathcal{Y}_i).\)

We first compute the bounds for the Seshadri constant of nef and big divisors.

\[\mathbf{Theorem 7.1.}\] Let \(D\) be a nef and big divisor on \(X,\) and \(x\) be a point on \(X\) with \(x \notin \mathcal{B}_+(D).\) Fix an admissible flag \(\mathcal{Y}_i\) centered at \(x.\) Let \(\mathbf{\Delta}_\lambda\) be the maximal sub-simplex of \(\Delta_{\mathcal{Y}_i}(D)\) of length \(\lambda = (\lambda_1, \cdots, \lambda_n)\) where \(\lambda_i = \lambda_i(D; x, \mathcal{Y}_i).\) Then we have

\[\lambda_{\min} \leq \varepsilon(D; x) \leq \lambda_n\]

where \(\lambda_{\min} := \min_{1 \leq i \leq n} \{\lambda_i\}.\)
Theorem 7.3. Let $D$ be a pseudoeffective divisor on $X$, and $x$ be a point on $X$. Then we have
\[ \sup_{Y_\bullet} \{ \lambda_{\min}(D; x, Y_\bullet) \} \leq \varepsilon(||D||; x) \leq \inf_{Y_\bullet} \{ \lambda_n(D; x, Y_\bullet) \} \]
where sup and inf are taken over the admissible flags $Y_\bullet$ centered at $x$. 

Proof. Note first that by Theorem [C] $\Delta_{Y_\bullet}(D)$ contains $U_{\geq 0}$ for some open neighborhood $U$ of the origin in $\mathbb{R}^n$. Thus $\lambda_i > 0$ for all $i$. First, we show the upper bound. Recall that
\[ \varepsilon(D; x) = \inf \left\{ \frac{D \cdot C}{\mult_x C} \right\} \]
where inf runs over all irreducible curves $C$ passing through $x$. Since $Y_n = \{ x \} \not\subseteq B_+(D)$, we have $Y_{n-1} \not\subseteq B_+(D)$. Then it follows from Theorem [3.7] and [LM] Theorem 6.4 that
\[ \Delta_{Y_n}(D)_{x_1=\ldots=x_{n-1}=0} = \{ (0, \ldots, 0, x_n) \mid 0 \leq x_n \leq \text{vol}_{X|Y_{n-1}}(D) \} \]
Note that $\text{vol}_{X|Y_{n-1}}(D) = D \cdot Y_{n-1} = \lambda_n$ and $\mult_x Y_{n-1} = 1$. Thus we have
\[ \varepsilon(D; x) = \inf \left\{ \frac{D \cdot C}{\mult_x C} \right\} \leq \frac{D \cdot Y_{n-1}}{\mult_x Y_{n-1}} = \lambda_n. \]

For the inequality concerning the lower bound, we only have to prove that
\[ \lambda_{\min} \leq \frac{D \cdot C}{\mult_x C} \]
for any irreducible curve $C$ passing through $x$. Note that $(\lambda_{\min}, 0, \ldots, 0) \in \Delta_{Y_\bullet}(D)$. If $C \not\subseteq Y_1$, then by Lemma [6.1] we get $(D - \lambda_{\min}Y_1) \cdot C \geq 0$. Thus we have
\[ \lambda_{\min} \leq \frac{D \cdot C}{\mult_x Y_1} \leq \frac{D \cdot C}{\mult_x C}. \]

When $C \subseteq Y_1$, we use the induction on the dimension $n$ of $X$. By Lemma [5.3], we can assume that all subvarieties from $Y_\bullet$ are smooth. Suppose that $n = 2$. In this case, $C = Y_1$. Since $\mult_x C = 1$, it follows that
\[ \lambda_{\min} \leq \lambda_2 = \text{vol}_{X|C}(D) = D \cdot C = \frac{D \cdot C}{\mult_x C}. \]
This completes the proof for the case $n = 2$. Now we suppose that $n \geq 3$. In this case, by induction, we obtain
\[ \lambda_{\min} \leq \frac{D|_{Y_1} \cdot C}{\mult_x C} = \frac{D \cdot C}{\mult_x C}. \]

Hence, in any case we have $\lambda_{\min} \leq \frac{D \cdot C}{\mult_x C}$. \hfill $\Box$

To prove Theorem [E] we need the following lemma.

Lemma 7.2. Let $D$ be a big divisor on $X$. Let $Y_\bullet$ be an admissible flag on $X$ centered at a point $x$ such that $x \not\in B_+(D)$. Then for any $\varepsilon > 0$, there exists a birational morphism $f : \overline{X} \to X$ isomorphic over a neighborhood of $x$ and a decomposition $f^*D = A + E$ into an ample divisor $A$ and an effective divisor $E$ such that for each $i$, we have
\[ \lambda_i(D; x, Y_\bullet) - \varepsilon < \lambda_i(A; x'; \overline{Y}_\bullet) \leq \lambda_i(D; x, Y_\bullet) \]
where $\overline{Y}_\bullet$ is an admissible flag on $\overline{X}$ whose subvarieties are obtained by the strict transforms of subvarieties of $Y_\bullet$ and $x' = f^{-1}(x)$. In particular, we also have
\[ \lambda_{\min}(D; x, Y_\bullet) - \varepsilon < \lambda_{\min}(A; x'; \overline{Y}_\bullet) \leq \lambda_{\min}(D; x, Y_\bullet). \]

Proof. By the Fujita approximation for Okounkov bodies (cf. [P] Theorem 2.3), we can take a birational morphism $f : \overline{X} \to X$ in the statement such that
\[ \Delta_{\overline{Y}_\bullet}(A) \subseteq \Delta_{\overline{Y}_\bullet}(f^*D) = \Delta_{Y_\bullet}(D) \subseteq \Delta_{Y_\bullet}((1 + \delta A) \leq \Delta_{Y_\bullet}(f^*D)) \]
for some $\delta > 0$, and $\text{vol}_{\overline{Y}_\bullet}(\Delta_{\overline{Y}_\bullet}(f^*D)) - \text{vol}_{\overline{Y}_\bullet}(\Delta_{\overline{Y}_\bullet}(A))$ and $\text{vol}_{\overline{Y}_\bullet}(\Delta_{Y_\bullet}((1 + \delta A)) - \text{vol}_{\overline{Y}_\bullet}(\Delta_{\overline{Y}_\bullet}(f^*D))$ are arbitrarily small. The assertion immediately follows from this observation. \hfill $\Box$

We finally prove Theorem [E] as Theorem [7.3].

Theorem 7.3. Let $D$ be a pseudoeffective divisor on $X$, and $x$ be a point on $X$. Then we have
\[ \sup_{Y_\bullet} \{ \lambda_{\min}(D; x, Y_\bullet) \} \leq \varepsilon(||D||; x) \leq \inf_{Y_\bullet} \{ \lambda_n(D; x, Y_\bullet) \} \]
where sup and inf are taken over the admissible flags $Y_\bullet$ centered at $x$. 

Proof. If \( x \in B_+(D) \), then \( \varepsilon(||D||; x) = 0 \). Since \( \lambda_{\min}(D; x, Y_\bullet) = 0 \) for any admissible flag \( Y_\bullet \) centered at \( x \), the assertion follows from Theorem \( \text{C} \). Thus we assume that \( x \notin B_+(D) \) in the remaining.

For the upper bound, it is enough to prove that for any admissible flag \( Y_\bullet \) centered at \( x \), \( \varepsilon(A; x' = f^{-1}(x)) \leq \lambda_n(D; x, Y_\bullet) \) holds for a birational morphism \( f : Y \rightarrow X \) isomorphic over a neighborhood of \( x \) and where \( A \) is an ample divisor such that \( f^*D - A \) is effective. Let \( \tilde{Y}_\bullet \) be the induced admissible flag on \( Y \) where each \( \tilde{Y}_i \) is the proper transform of \( Y_i \) on \( Y \). Then by Theorem \( \text{L.1} \) we have \( \varepsilon(A; x') \leq \lambda_n(A; x', \tilde{Y}_\bullet) \).

Since \( \lambda_n(A; x', \tilde{Y}_\bullet) \leq \lambda_n(D; x, Y_\bullet) \), we have

\[ \varepsilon(A; x') \leq \lambda_n(D; x, Y_\bullet). \]

For the lower bound, we need to prove that \( \lambda_{\min}(D; x, Y_\bullet) \leq \varepsilon(||D||; x) \) for any admissible flag \( Y_\bullet \) centered at \( x \). By the definition of \( \varepsilon(||D||; x) \), it is enough to prove that for any \( \varepsilon > 0 \) there exists a birational morphism \( f : Y \rightarrow X \) with an ample divisor \( A \) on \( Y \) as in the above paragraph such that

\[ \lambda_{\min}(D; x, Y_\bullet) < \varepsilon < \varepsilon(A; x'). \]

This follows from Lemma \( \text{L.2} \) since \( \lambda_{\min}(A; x', \tilde{Y}_\bullet) \leq \varepsilon(A; x') \) by Theorem \( \text{L.1} \).

\[ \square \]

Example 7.4. Let \( m \) be a positive integer. Let \( X = \mathbb{P}^2 \) and \( D \sim L \) where \( L \) is a line on \( \mathbb{P}^2 \). Consider an admissible flag \( Y_\bullet \) where \( Y_1 \) is a general member of \( |mL| \). Then

\[ \Delta_{Y_\bullet}(D) = \{ (x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid m^2x_1 + x_2 \leq m \} \]

Thus \( \lambda_{\min}(D; x, Y_\bullet) = \frac{1}{m} \) and \( \lambda_2(D; x, Y_\bullet) = m \). Since \( \varepsilon(D; x) = 1 \), the inequalities in Theorem \( \text{L.1} \) are strict if \( m > 1 \). However, if \( m = 1 \), then the equalities in Theorems \( \text{L.1} \) and \( \text{L.2} \) hold.

Example 7.5. As in [KL1] Remark 4.9, we also consider a fake projective plane \( S \) such that \( K_S = 3H \) where \( \text{Pic}(S) = \mathbb{Z} \cdot [H] \) and \( H^2 = 1 \) (see [PS] 10.4 for the existence of such a surface). Fix an admissible flag \( Y_\bullet : Y_0 = S \supseteq Y_1 = C \supseteq Y_0 = \{ x \} \) where \( x \) is a very general point on \( S \). Note that \( H^0(S, K_S) = 0 \) and so \( H^0(S, H) = 0 \). Thus \( C \in |KH| \) for some integer \( k > 1 \). As in the previous example, we have

\[ \Delta_{Y_\bullet}(H) = \{ (x_1, x_2) \in \mathbb{R}^2_{\geq 0} \mid k^2x_1 + x_2 \leq k \} \]

Note that \( \varepsilon(H; x) = 1 \). However, since we always have \( k > 1 \), both inequalities in Theorem \( \text{L.3} \) are strict.

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