Some Existence Theorems for Semilinear Neumann Problems with Landesman-Lazer Condition Revisited

Sheng Ma\(^a\), Zhihua Hu\(^a\), Jing Jin\(^a\), Qin Jiang\(^a\)

\(^a\)Department of Mathematics, Huanggang Normal University, Hubei 438000, China

Abstract. In this paper, existence theorems are established for Neumann problems for semilinear elliptic equations at resonance together with Landesman-Lazer condition revisited. Our existence results follow as an application of the Saddle point Theorem together with a standard eigenspace decomposition.

1. Introduction and main results

In the paper, we are concerned with the following Neumann boundary value problems

\[
\begin{aligned}
-\Delta u &= \mu_k u + g(u) - h(x) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \(\Delta\) is the Laplacian operator, \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is a bounded domain with smooth boundary and outer normal vector \(n = n(x)\), \(\frac{\partial u}{\partial n} = n(x) \cdot \nabla u\), the function \(g : \mathbb{R} \to \mathbb{R}\) is a bounded continuous function with \(G(u) = \int_0^u g(s)ds\) as its primitive, \(h \in L^2(\Omega)\) and \(\mu_k, k \geq 1\), is the \(k\)-th eigenvalue of the eigenvalue problem

\[
\begin{aligned}
-\Delta u &= \mu u \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Let \(m \geq 1\) be a multiplicity of \(\mu_k\). Then we set the eigenvalues of (2) be the increasing sequence:

\[\mu_1 < \mu_2 \leq \cdots \leq \mu_{k-1} < \mu_k = \cdots = \mu_{k+m-1} < \mu_{k+m} \leq \mu_{k+m+1} \leq \cdots \to \infty.\]

Define the functional \(\varphi\) on \(H^1(\Omega)\) by

\[
\varphi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |u|^2 dx - \int_{\Omega} G(u) dx + \int_{\Omega} h u dx, \quad u \in H^1(\Omega),
\]

2010 Mathematics Subject Classification. 35J20, 35J25, 35B34, 35B38

Keywords. elliptic equations; Neumann problems; critical point; Landesman-Lazer condition; Saddle point Theorem

Received: 02 March 2019; Accepted: 17 May 2019

Communicated by Miodrag Spalević

Corresponding author: Qin Jiang

Supported by the Natural Science Foundation of Hubei Provincial(No.2018CFC825).

Email address: jiangqin9990126.com (Qin Jiang)
where the Sobolev space $H^1(\Omega)$ is the usual space of $L^2(\Omega)$ functions with weak derivative in $L^2(\Omega)$, endowed with the norm defined by

$$
\|u\| = \left( \int_\Omega |u(x)|^2 \, dx + \int_\Omega |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}}
$$

for all $u \in H^1(\Omega)$. It’s well known that finding solutions of problem (1) is equivalent to finding critical points of $I$ in $H^1(\Omega)$.

There exists a lot of published literatures related to the solvability conditions for Neumann boundary value problems, see [2][17][21][22][31][32] and there references. For problem (1), the common solvability conditions were the periodicity condition, see [26], the monotonicity condition, see [23][24], the sign condition, see [14][15], the Landesman-Lazer type condition, see [16][18][29][30][33].

We focus on the so-called Landesman-Lazer condition, introduced by Lazer and Leach [20] in 1969 in the case

$$
g(t, x) = \lambda_N x + h(x) - e(t),
$$

where $\lambda_N = \left( \frac{2N}{\pi^2} \right)^{\frac{1}{2}}$ and $h$ is bounded. In the settings of [20], this condition ensures existence of one periodic solution for the following problem

$$
\begin{aligned}
&u'' + g(t, x) = 0, \\
u(0) = u(T), \ u'(0) = u'(T),
\end{aligned}
$$

In the case when $g(t, x) = \lambda_N x + h(t, x)$, it can be written as follows:

$$
\int_{x=0}^{\pm\infty} \lim\inf_{t \to \pm\infty} h(t, x)v(t) \, dt + \int_{x=0}^{\pm\infty} \lim\sup_{t \to \pm\infty} h(t, x)v(t) \, dt > 0,
$$

for every $v$ solving the homogeneous equation

$$
x'' + \lambda_N x = 0.
$$

Just as an intuitive idea, one can qualitatively think that a suitable shape for $h(t, x)$ to satisfy such a condition requires that $h$ is positive for $x \to +\infty$ and negative for $x \to -\infty$.

This paper [20] opened the way towards what today is usually called the Landesman-Lazer condition, introduced one year later to a semilinear elliptic problem by Landesman and Lazer [19], and read as follows:

(LL) for any nontrivial $\phi$ in the eigenspace associated with $\mu_k$,

$$
g(\mp\infty) \int_\Omega \phi^+ \, dx - g(\pm\infty) \int_\Omega \phi^- \, dx < \int_\Omega h \phi \, dx < g(\pm\infty) \int_\Omega \phi^+ \, dx - g(\mp\infty) \int_\Omega \phi^- \, dx.
$$

After the pioneering works [19][20], this type of conditions has inspired several authors in the attempt of finding the right abstract formulation and providing different generalizations. Contributions in this direction were given, among others, by [1][3][4][5][7][8][9][11][12][13] for a quite rich bibliography about the subject see [10]. In particular, in [28], Tang defined the function $F(t) = 2G(t)/t - g(t)$ and the constants $F(\pm\infty) = \lim\inf_{t \to \pm\infty} F(t)$, $\bar{F}(\pm\infty) = \lim\sup_{t \to \pm\infty} F(t)$ to prove that a resonance problem about the first eigenvalue of a linear operator

$$
\begin{aligned}
u''(x) + m^2 u + g(x, u) = h(x), \quad x \in (0, \pi) \\
u(0) = u(\pi) = 0,
\end{aligned}
$$

is solvable under the Landesman-Lazer type condition:

$$
\int_0^\pi \left[ \bar{F}(\pm\infty)(\sin x)^+ - F(\pm\infty)(\sin x)^- \right] \, dx
$$

$$
< \int_0^\pi h \sin x \, dx < \int_0^\pi \left[ F(\pm\infty)(\sin x)^+ - \bar{F}(\pm\infty)(\sin x)^- \right] \, dx.
$$
Later in 2001, Tomiczek [33] studied two-point boundary value problems (3) and introduced a rather general sufficient condition so called potential Landesman-Lazer type:

\[
(p-\text{LL})_\pm \text{ for any nontrivial } \phi \text{ in the eigenspace associated with } \mu_k,
\]

\[
G^\pm \int_\Omega \phi^+ dx - G^- \int_\Omega \phi^- dx < \int_\Omega \bar{h} \phi dx < G^\pm \int_\Omega \phi^+ dx - G^- \int_\Omega \phi^- dx,
\]

as a generalization to conditions (4) and (\text{LL})_\pm, where \( G^\pm = \lim_{s \to \pm \infty} G(s) \) and in [33], \( \mu_k = m^2, \phi = \sin x \).

In addition, in 2001, Tang [30] considered the Neumann boundary value problem (1) under the condition similar to (4) and obtained the following results:

**Theorem A** [30] Suppose that \( g \in C(R, R) \) such that

\[
0 \leq \lim \inf \frac{g(t)}{t} \leq \lim \sup \frac{g(t)}{t} < \mu_2.
\]

Assume that \( h \in L^q(\Omega) \) satisfying

\[
\bar{F}(-\infty) < \frac{1}{|\Omega|} \int_\Omega h(x)dx < \bar{F}(+\infty),
\]

where \( q > \frac{2N}{N+2} \) if \( N \geq 3 \) (\( q > 1 \), if \( N = 1, 2 \)), \( |\Omega| \) is the volume of \( \Omega \),

\[
\bar{F}(+\infty) = \lim_{t \to +\infty} F(t), \quad \bar{F}(-\infty) = \lim_{t \to -\infty} F(t),
\]

and

\[
F(t) = 2G(t)/t - g(t), \text{ for } t \neq 0, \ F(0) = g(0).
\]

Then the problem (1), where \( k = 1 \), has at least one solution in the Sobolev space \( H^1(\Omega) \).

**Theorem B** [30] Suppose that \( g \in C(R, R) \) such that

\[
\lim_{|t| \to \infty} \frac{g(t)}{t} = 0.
\]

Assume that \( h \in L^q(\Omega) \) satisfying either

\[
\left| \int_\Omega h \phi dx \right| < \frac{1}{2} (\bar{F}(-\infty) - \bar{F}(+\infty)),
\]

or

\[
\left| \int_\Omega h \phi dx \right| < \frac{1}{2} (\bar{F}(+\infty) - \bar{F}(-\infty)),
\]

for any nontrivial \( \phi \) in the eigenspace associated with \( \mu_k \), with \( \|\phi\|_1 = 1 \), where \( q > \frac{2N}{N+2} \) if \( N \geq 3 \) (\( q > 1 \), if \( N = 1, 2 \)), Then the problem (1), where \( k > 1 \), has at least one solution in the Sobolev space \( H^1(\Omega) \).

The purpose of this paper is to introduce a rather generalization of (\text{LL})_\pm and (p-\text{LL})_\pm for the existence of a solution of problem (1). For readers’ convenience, we first give the following statements.

The corresponding eigenfunctions, \( (\phi_n) \), form an orthogonal basis for both \( L^2(\Omega) \) and \( H^1(\Omega) \). Assume that every \( \phi_n \) with respect to the \( L^2 \) norm \( \|\phi_n\|_2 = 1, n = 1, 2, \cdots \). We split the space \( H^1(\Omega) \) into the following three subspaces spanned by the eigenfunctions of (2) as follows:

\[
\mathcal{H} = \text{span}\{\phi_1, \cdots, \phi_{k-1}\},
\]
\[ \tilde{H} = \text{span}\{\phi_2, \cdots, \phi_{k+m-1}\}, \]
\[ \overline{H} = \text{span}\{\phi_k, \phi_{k+1}, \cdots\}. \]

Then
\[ H^1(\Omega) = H \oplus \tilde{H} \oplus \overline{H} \]
with \( \dim\tilde{H} = k - 1 \), \( \dim\overline{H} = m \), \( \dim H = \infty \). Of course, if \( k = 1 \) then \( m = 1 \) (\( \mu_1 \) is a simple eigenvalue) and \( \tilde{H} = \emptyset \). We also split an element \( u \in H^1(\Omega) \) as \( u = \tilde{u} + \bar{u} + \hat{u} \), and split a function \( h \in L^2(\Omega) \) as \( h = h^+ + h^- \), where \( \tilde{u} \in \tilde{H}, \bar{u} \in \overline{H}, \hat{u} \in H \) and
\[ \int_{\Omega} h^+ v \, dx = 0, \quad \text{for any } v \in \tilde{H}. \]

The generalization of (LL)\( _{k} \) and (p-LL)\( _{k} \) for the existence of a solution of problem (1), reads as follows:

\[(GLL)_{\pm} \quad \text{If } \{u_n\} \subset H^1(\Omega) \text{ is a sequence such that } \|u_n\|_2 \to \infty \text{ and there exists } \phi_0 \in \tilde{H}, \frac{u_n}{\|u_n\|_2} \to \phi_0 \text{ in } L^2(\Omega) \text{ as } n \to \infty, \text{ then}\]
\[ \lim_{n \to \infty} \left( \int_{\Omega} G(u_n) \, dx - \int_{\Omega} h u_n \, dx \right) = \pm \infty. \]

Suppose \( \|u_n\|_2 \to \infty \) and \( \frac{u_n}{\|u_n\|_2} \to \phi_0 \) for some eigenfunction \( \phi_0 \). Then an easy computation yields, by l'Hospital's rule,
\[
\lim_{n \to \infty} \frac{1}{\|u_n\|_2} \left( \int_{\Omega} G(u_n) \, dx - \int_{\Omega} h u_n \, dx \right) = \lim_{n \to \infty} \int_{\Omega} \left( \frac{G(u_n)}{u_n} - h \right) \frac{u_n}{\|u_n\|_2^2} \, dx = \int_{\Omega} (g(+\infty) + \bar{h})\phi_0^+ \, dx - \int_{\Omega} (g(-\infty) + \bar{h})\phi_0^- \, dx,
\]
and directly
\[
\lim_{n \to \infty} \frac{1}{\|u_n\|_2} \left( \int_{\Omega} G(u_n) \, dx - \int_{\Omega} h u_n \, dx \right) = \lim_{n \to \infty} \int_{\Omega} \left( \frac{G(u_n)}{u_n} - \bar{h} \right) \frac{u_n}{\|u_n\|_2^2} \, dx = \int_{\Omega} (G^+ + \bar{h})\phi_0^+ \, dx - \int_{\Omega} (G^- + \bar{h})\phi_0^- \, dx,
\]
where \( G^\pm = \lim_{s \to \infty} \frac{G(s)}{s} \). Due to the last two expressions above, either (LL)\( _{k} \) or (p-LL)\( _{k} \) imply (GLL)\( _{k} \). In addition, from [33], we know (p-LL)\( _{k} \) is more general than the condition (4). That is, (GLL)\( _{k} \) are more general than conditions (LL)\( _{k} \), (p-LL)\( _{k} \) and (4).

In this paper, we consider Neumann boundary value problems (1) under the Landesman-Lazer type condition (GLL)\( _{k} \), and obtain the existence theorems by saddle point theorem together with a standard eigenspace decomposition. The main results in the paper are next summarized.

**Theorem 1.** Under the hypothesis (GLL)\( _{-} \), the problem (1) has at least one solution in the Sobolev space \( H^1(\Omega) \).

**Theorem 2.** Under the hypothesis (GLL)\( _{+} \), the problem (1) has at least one solution in the Sobolev space \( H^1(\Omega) \).

**Remark 3.** Compared with conditions (LL)\( _{k} \), (p-LL)\( _{k} \) and (5)-(7), the advantages of (GLL)\( _{k} \) are illustrated by some examples.
imply the conditions (LL) and the condition (p-LL) for which implies that conditions (5), (6) and (7) are empty. That is, they do not apply. Moreover, (GLL) obviously it holds
\[ \lim_{|s| \to \infty} G(s) = \lim_{|s| \to \infty} \left( s \arctan s - \frac{1}{2} \ln(1 + s^2) + \pi s \right) = \infty, \]
which means (GLL) holds for \( h \in L^2(\Omega)^{+} \), where
\[ L^2(\Omega)^{+} = \left\{ h \in L^2(\Omega) : \int_{\Omega} h \phi dx = 0 \text{ for all } \phi \in H \right\} \subseteq L^2(\Omega). \]
However, the limits \( g(\pm \infty) \) do not exist. That is, the condition (LL) do not apply.

(ii) (GLL) hold for \( h \in L^2(\Omega)^{+} \). However, both (LL) and (p-LL) do not apply even if the limits \( g(\pm \infty) \) exist. Set \( g(s) = \frac{s \text{sgn}(s)}{\ln(1 + |s|)} \). Then we easily obtain
\[ \lim_{|s| \to \infty} G(s) = \lim_{|s| \to \infty} \ln(\ln(e + |s|)) = +\infty, \]
which means (GLL) holds for \( h \in L^2(\Omega)^{+} \). However, we also get \( g(\pm \infty) = 0 \) and \( G^+ = 0 \) which, respectively, imply the conditions (LL) and (p-LL) are empty.

(iii) (GLL) hold for \( h \in L^2(\Omega)^{+} \). However, all of the conditions (5)-(7), (LL) and (p-LL) do not apply. Set \( g(s) = \frac{\ln(1 + s^2) + 2 \sin s}{s} \). Then we easily obtain
\[ \lim_{|s| \to \infty} G(s) = \lim_{|s| \to \infty} \frac{\ln(1 + s^2) + 2 \sin s}{s} = 0, \]
and
\[ F(s) = \frac{2G(s)}{s} - g(s) = \frac{\ln(1 + s^2) + 2 \sin s}{s} - \frac{2s}{1 + s^2} - 2 \cos s. \]
Obviously it holds
\[ F(-\infty) = F(+\infty) = -2, \quad F(+\infty) = F(-\infty) = 2, \]
which implies that conditions (5), (6) and (7) are empty. That is, they do not apply. Moreover, (GLL) holds for \( h \in L^2(\Omega)^{+} \). However, the conditions (LL) and (p-LL) do not apply since the limits \( g(\pm \infty) \) do not exist and the condition (p-LL) is empty by \( G^+ = 0 \).

The functions \( g(s) \) and \( h(x) \) satisfy our Theorems but not satisfying the corresponding results published in the literature so far, such as Theorems A and B.

2. Proof of Theorems

The methods to prove the theorems are variational basically based upon minmax methods together with a standard eigenspace decomposition. To make the statements precise, let us introduce some notation.

It is well known that, by Sobolev’s inequality, there exists a constant \( M > 0 \) such that
\[ ||u||_{L^2(\Omega)} \leq M||u||. \]
Since the function \( g \) is a bounded continuous, we can easy prove that \( \varphi \) is continuously differentiable in \( H^1(\Omega) \), in a way similar to Theorem 1.4 in [25]. To prove Theorems 1 and 2, we recall an abstract critical point theorem, i.e., the Saddle point Theorem under the (PS) condition, the readers can refer to [27].

Lemma 1 Let \( H \) be a Banach space with a decomposition \( H = H^{-} + H^{+} \), where \( H^{-} \) and \( H^{+} \) are two
subspaces of $H$ with $\dim H^- < +\infty$. Assume that $\varphi : X \to R$ is a $C^1$-function, satisfying (PS) condition and

(a) there exist constants $\rho > 0$ and $\alpha$ such that $\varphi|_{B_R} \leq \alpha$,

(b) there exist a constant $\beta > \alpha$ such that $\varphi|_{B_R} \geq \beta$,

Then the functional $\varphi$ possesses a critical point in $H$.

In addition, we need the following lemmas.

**Lemma 2** There exist $C_1 > 0, C_2 > 0$ such that for any $u \in H$ we have

\[
\begin{align*}
\int_\Omega |\nabla \bar{u}|^2 dx - \mu_k \int_\Omega |\bar{u}|^2 dx & \leq -C_1 ||\bar{u}||^2, \\
\int_\Omega |\nabla \bar{u}|^2 dx - \mu_k \int_\Omega |\bar{u}|^2 dx & \geq C_2 ||\bar{u}||^2. 
\end{align*}
\] (9) (10)

**Proof** The inequalities (9) and (10) follow from the variational characterization of $\mu_k$.

**Lemma 3** There exist $C_3 > 0, C_4 > 0, C_5 > 0$ such that for any $u \in H$ we have

\[
\begin{align*}
\left| \int_\Omega g(u) \bar{u} dx - \int_\Omega h \bar{u} dx \right| & \leq C_3 ||\bar{u}||, \\
\left| \int_\Omega g(u) \bar{u} dx - \int_\Omega h \bar{u} dx \right| & \leq C_4 ||\bar{u}||, \\
\left| \int_\Omega G(u) dx - \int_\Omega h u dx \right| & \leq C_5 ||u||_2. 
\end{align*}
\] (11) (12) (13)

**Proof** The inequalities (11), (12) and (13) follow from the Hölder inequality, the boundedness of $g$ and the fact $h \in L^2(\Omega)$.

**Lemma 4** Under the assumption (GLL)$_k$, the functional $\varphi$ satisfies (PS) condition. That is, $\{u_n\}$ possesses a convergent subsequence if $\{u_n\}$ is a sequence of $H$ such that $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \to 0$ as $n \to \infty$.

**Proof** Step 1. We claim that $\{u_n\}$ is bounded in $L^2(\Omega)$. We argue by contradiction. So, suppose that $||u_n||_2 \to \infty$ as $n \to \infty$. Put $v_n = \frac{u_n}{||u_n||_2}$. Then $||v_n||_2 = 1$. So, by boundedness of $\{\varphi(u_n)\}$ and $||u_n||_2 \to \infty$, it holds

\[
\frac{\varphi(u_n)}{||u_n||_2^2} = \frac{1}{2} \int_\Omega |\nabla v_n|^2 dx - \frac{1}{2} \mu_k \int_\Omega |v_n|^2 dx - \int_\Omega \frac{G(u_n)}{||u_n||_2^2} dx + \frac{1}{||u_n||_2^2} \int_\Omega h u_n dx
\]

\[
= \frac{1}{2} ||v_n||^2 - \frac{\mu_k + 1}{2} - \int_\Omega \frac{G(u_n)}{||u_n||_2^2} dx + \frac{1}{||u_n||_2^2} \int_\Omega h u_n dx \to 0. 
\] (14)

Due to (13), we easily obtain

\[
\int_\Omega \frac{G(u_n)}{||u_n||_2^2} dx + \frac{1}{||u_n||_2^2} \int_\Omega h u_n dx \to 0.
\]

It follows from (14) that

\[
||v_n||^2 \to \mu_k + 1,
\]

which means $\{v_n\}$ is bounded in $H$. Passing to a subsequence, if necessary, we may assume that there exists $v \in H$ such that

\[
v_n \to v \text{ in } H \text{ and } v_n \to v \text{ in } L^2(\Omega).
\]
For arbitrary \( w \in H \), then we obtain
\[
\int_{\Omega} \nabla v_n \nabla w dx \rightarrow \int_{\Omega} \nabla v \nabla w dx \text{ by } v_n \rightarrow v \text{ in } H,
\]
\[
\int_{\Omega} v_n w dx \rightarrow \int_{\Omega} v w dx \text{ by } v_n \rightarrow v \text{ in } L^2(\Omega),
\]
\[
\frac{1}{\| u_n \|^2} \int_{\Omega} g(u_n) w dx \rightarrow 0 \text{ and } \frac{1}{\| u_n \|^2} \int_{\Omega} h w dx \rightarrow 0,
\]
by the boundedness of \( g, h \in L^2(\Omega) \) and the hypothesis \( \| u_n \|_2 \rightarrow \infty \). Moreover, by \( \varphi'(u_n) \rightarrow 0 \) and \( \| u_n \|_2 \rightarrow \infty \), one has
\[
0 \leftarrow \frac{(\varphi'(u_n), w)}{\| u_n \|^2} = \int_{\Omega} \nabla v_n \nabla w dx - \mu_k \int_{\Omega} v_n w dx
- \frac{1}{\| u_n \|^2} \int_{\Omega} g(u_n) w dx + \frac{1}{\| u_n \|^2} \int_{\Omega} h w dx.
\]
Thus by (15), for arbitrary \( w \in H \), we have
\[
\int_{\Omega} \nabla v \nabla w dx - \mu_k \int_{\Omega} v w dx = 0,
\]
which means \( v = \varphi_0 \in H \) is an eigenfunction corresponding to \( \mu_0 \). Obviously,
\[
v_n = \frac{u_n}{\| u_n \|^2} \rightarrow \varphi_0 \text{ in } L^2(\Omega).
\]
An easy computation yields, by (9) and (11),
\[
(\varphi'(u_n), \hat{u}_n) = \int_{\Omega} |\nabla \hat{u}_n|^2 dx - \mu_k \int_{\Omega} |\hat{u}_n|^2 dx - \int_{\Omega} g(u_n) \hat{u}_n dx + \int_{\Omega} h \hat{u}_n dx
\leq -C_1|\hat{u}_n|^2 + C_3|\hat{u}_n|.
\]
(16)
Due to (16) and \( \varphi'(u_n) \rightarrow 0 \), we obtain \( \| \hat{u}_n \| \) is bounded. Similarly, it holds
\[
(\varphi'(u_n), \hat{u}_n) = \int_{\Omega} |\nabla \hat{u}_n|^2 dx - \mu_k \int_{\Omega} |\hat{u}_n|^2 dx - \int_{\Omega} g(u_n) \hat{u}_n dx + \int_{\Omega} h \hat{u}_n dx
\geq C_2|\hat{u}_n|^2 - C_4|\hat{u}_n|,
\]
which implies \( \| \hat{u}_n \| \) is bounded by \( \varphi'(u_n) \rightarrow 0 \).

Now we rewrite \( \varphi(u_n) \) as follows:
\[
\varphi(u_n) = \int_{\Omega} |\nabla \hat{u}_n|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |\hat{u}_n|^2 dx + \frac{1}{2} \mu_k \int_{\Omega} |\hat{u}_n|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |\hat{u}_n|^2 dx
\]
\[
- \int_{\Omega} G(u_n) dx + \int_{\Omega} h \hat{u}_n dx + \int_{\Omega} h \hat{u}_n dx + \int_{\Omega} h \hat{u}_n dx + \int_{\Omega} h \hat{u}_n dx.
\]
(17)
Since \( \| \hat{u}_n \| \) and \( \| \hat{u}_n \| \) are bounded, \( A, B \) and \( D \) are bounded. Moreover, since \( \| u_n \|_2 \rightarrow \infty \), \( \frac{u_n}{\| u_n \|^2} \rightarrow \varphi_0 \), and (GLL) holds, we have
\[
- \int_{\Omega} G(u_n) dx + \int_{\Omega} h \hat{u}_n dx \rightarrow -\infty \text{ and } + \infty,
\]
Proof of Theorem 1. H  

We first prove that the functional $\varphi(\mu_{n})\rightarrow\pm\infty$. Thus by (17) it holds

$$
\varphi(\mu_{n}) \rightarrow \pm\infty.
$$

Obviously it contradicts the assumption of the boundedness of $\varphi(\mu_{n})$. So $\{\mu_{n}\}$ is bounded in $L^{2}(\Omega)$.

Step 2. We claim that $\{\mu_{n}\}$ is bounded in $H$. In fact, we again use the following equation:

$$
\varphi(\mu_{n}) = \frac{1}{2} \int_{\Omega} |\nabla \mu_{n}|^{2} dx - \frac{1}{2} \mu_{k} \int_{\Omega} |\mu_{n}|^{2} dx - \int_{\Omega} G(\mu_{n}) dx + \int_{\Omega} h \mu_{n} dx
$$

$$
= \frac{1}{2} |\mu_{n}|^{2} - \frac{\mu_{k} + 1}{2} \int_{\Omega} |\mu_{n}|^{2} dx - \int_{\Omega} G(\mu_{n}) dx + \int_{\Omega} h \mu_{n} dx.
$$

(18)

Since $\{\mu_{n}\}$ is bounded in $L^{2}(\Omega)$, $\frac{1}{2} \int_{\Omega} |\nabla \mu_{n}|^{2} dx$, $\int_{\Omega} G(\mu_{n}) dx$ and $\int_{\Omega} h \mu_{n} dx$ are bounded. Moreover $\varphi(\mu_{n})$ is bounded, thus by (18), we have $||\mu_{n}||$ must be also bounded.

Step 3. We claim $\{\mu_{n}\}$ has a strongly convergent subsequence in $H$. In fact, since $||\mu_{n}||$ is bounded in $H$, $\{\mu_{n}\}$ has a subsequence, still denoted by $\{\mu_{n}\}$ for the convenience, such that

$$
u_{n} \rightarrow u \quad \text{in} \quad H \quad \text{and} \quad u_{n} \rightarrow u \quad \text{in} \quad L^{2}(\Omega).
$$

Then one has

$$
-\mu_{k} \int_{\Omega} \nabla \mu_{n} (\nabla u_{n} - \nabla u) dx - \int_{\Omega} g(\mu_{n}) (\mu_{n} - u) dx + \int_{\Omega} h (\mu_{n} - u) dx \rightarrow 0.
$$

Moreover, it holds

$$
0 \leftarrow (\varphi'(\mu_{n}), \mu_{n} - u) = \int_{\Omega} \nabla \mu_{n} \nabla (\mu_{n} - u) dx - \mu_{k} \int_{\Omega} \mu_{n} (\mu_{n} - u) dx - \int_{\Omega} g(\mu_{n}) (\mu_{n} - u) dx + \int_{\Omega} h(\mu_{n} - u) dx.
$$

So we deduce that

$$
\int_{\Omega} \nabla \mu_{n} \nabla (\mu_{n} - u) dx \rightarrow 0.
$$

That is,

$$
\int_{\Omega} |\nabla \mu_{n}|^{2} dx - \int_{\Omega} \nabla \mu_{n} \nabla u dx \rightarrow 0.
$$

Due to the weak convergence $u_{n} \rightarrow u$ in $H$, it holds

$$
\int_{\Omega} \nabla \mu_{n} \nabla u dx - \int_{\Omega} \nabla u \nabla u dx \rightarrow 0.
$$

Thus we get

$$
\int_{\Omega} |\nabla \mu_{n}|^{2} dx - \int_{\Omega} |\nabla u|^{2} dx \rightarrow 0,
$$

which, together with $u_{n} \rightarrow u$ in $L^{2}(\Omega)$, implies

$$
\int_{\Omega} |\nabla \mu_{n}|^{2} dx + \int_{\Omega} |\mu_{n}|^{2} dx = ||\mu_{n}||^{2} \rightarrow ||u||^{2} = \int_{\Omega} |\nabla u|^{2} dx + \int_{\Omega} |u|^{2} dx.
$$

The uniform convexity of $H$ then implies that $u_{n} \rightarrow u$ in $H$. Hence the functional $\varphi$ satisfies (PS) condition.

**Proof of Theorem 1.** Under the assumption (GLL)$_{-}$, we set $H = H^{1}(\Omega) = H^{-} \oplus H^{+}$, where $H^{-} = \hat{H}$ is a finite dimension subspace and $H^{+} = \hat{H}$. 

On the one hand, we claim that there is a constant $\beta$ such that
\[
\inf_{w \in H^*} \varphi(u) \geq \beta.
\]
If not, there exists a sequence $\{u_n\} \subset H^*$ such that
\[
\lim_{n \to \infty} \varphi(u_n) = -\infty.
\]
Then $\|u_n\|_2 \to \infty$, and for $\nu_n = \frac{u_n}{\|u_n\|_2} \in H^+$, by (19) we obtain
\[
0 \geq \frac{\varphi(u_n)}{\|u_n\|^2_2} = \frac{1}{2} \int_{\Omega} |\nabla \nu_n|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |\nu_n|^2 dx - \int_{\Omega} \frac{G(u_n)}{\|u_n\|^2_2} dx + \int_{\Omega} h \frac{u_n}{\|u_n\|^2_2} dx
\]
\[
= \frac{1}{2} \|\nu_n\|^2 - \frac{\mu_k}{2} - \int_{\Omega} \frac{G(u_n)}{\|u_n\|^2_2} dx + \int_{\Omega} h \frac{u_n}{\|u_n\|^2_2} dx.
\]
However, by (13), we know
\[
- \int_{\Omega} \frac{G(u_n)}{\|u_n\|^2_2} dx + \int_{\Omega} h \frac{u_n}{\|u_n\|^2_2} dx \to 0.
\]
So by (20) and (21), we get
\[
\|\nu_n\|^2 \to \mu_k + 1,
\]
which implies $\|\nu_n\|$ is bounded. Passing to a subsequence, if necessary, we may assume that there is $v \in H^+$ such that
\[
v_n \to v \text{ in } H \text{ and } v_n \to v \text{ in } L^2(\Omega).
\]
Due to the weak lower semicontinuity of the norm in $H$, we know
\[
\liminf_{n \to \infty} \int_{\Omega} |\nabla \nu_n|^2 dx \geq \int_{\Omega} |\nabla v|^2 dx.
\]
Thus by (20), (21) and (22), we have
\[
\int_{\Omega} |\nabla v|^2 dx - \mu_k \int_{\Omega} |v|^2 dx \leq 0,
\]
which, together with (10), implies that $v = \phi_0 \in H$ is an eigenfunction associated with $\mu_k$. Clearly,
\[
v_n = \frac{u_n}{\|u_n\|_2} \to \phi_0 \text{ in } L^2(\Omega).
\]
For all $u_n = \bar{u}_n + \tilde{u}_n \in H^+$, one has
\[
\varphi(u_n) = \frac{1}{2} \int_{\Omega} |\nabla \bar{u}_n|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |\bar{u}_n|^2 dx - \int_{\Omega} \frac{G(u_n)}{\|u_n\|^2_2} dx + \int_{\Omega} h u_n dx + \int_{\Omega} h^+ \tilde{u}_n dx
\]
\[
\geq C_2 \|\bar{u}\|^2 - \|h^+\|_2 \|\bar{u}\|_2 - \int_{\Omega} G(u_n) dx + \int_{\Omega} h u_n dx + \int_{\Omega} G(u_n) dx + \int_{\Omega} h u_n dx,
\]
which, together with (GLL)-, yields
\[
\varphi(u_n) \to +\infty \text{ as } n \to +\infty.
\]
Obviously it contradicts with (9). That is, the conclusion is verified.

On the other hand, for $\bar{u} \in H^-$, we have
\[
\varphi(\bar{u}) = \frac{1}{2} \int_{\Omega} |\nabla \bar{u}|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |\bar{u}|^2 dx - \int_{\Omega} \frac{G(\bar{u})}{\|\bar{u}\|^2_2} \bar{u} dx + \int_{\Omega} h \bar{u} dx
\]
\[
\leq -C_1 \|\bar{u}\|^2 - \int_{\Omega} G(\bar{u}) dx + \int_{\Omega} h \bar{u} dx,
\]
which implies \( \varphi(\hat{u}) \rightarrow -\infty \) as \( \| \hat{u} \| \rightarrow +\infty \).

Hence there exist constants \( \alpha \) and \( R > 0 \) such that

\[
\sup_{u \in D} \varphi(u) < \alpha < \beta,
\]

where \( D = \{ u \in H^- \mid \| u \| \leq R \} \).

Therefore, the hypotheses (a) and (b) in Lemma 1 are satisfied. Recall that the functional \( \varphi \) satisfies (PS) condition in Lemma 4, the proof of Theorem 1 is finished via Lemma 1.

**Proof of Theorem 2.** Under the assumption \((\text{GLL})_+\), we put \( H = H^1(\Omega) = H^- \oplus H^+ \), where \( H^- = \hat{H} \oplus \bar{H} \) and \( H^+ = \tilde{H} \).

On the one hand, we claim that

\[
\lim_{\| u \| \rightarrow \infty} \varphi(u) = -\infty, \quad u \in H^-.
\]

If not, there exist a sequence \( \{ u_n \} \) in \( H^- \) and a constant \( C_6 \) such that \( \| u_n \| \rightarrow \infty \) and

\[
\varphi(u_n) \geq C_6.
\]  \hspace{1cm} (24)

Since \( H^- \) is a finite dimension space, the two norms \( \| \cdot \| \) and \( \| \cdot \|_2 \) are equivalent on \( H^- \). In fact, for all \( u \in H^- \), one has

\[
\int_{\Omega} |\nabla u|^2 dx - \mu_k \int_{\Omega} |u|^2 dx \leq 0.
\]  \hspace{1cm} (25)

Thus it holds

\[
\| u \|^2 = \int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |u|^2 dx \leq (1 + \mu_k) \int_{\Omega} |u|^2 dx \leq (1 + \mu_k) \| u \|_2^2.
\]  \hspace{1cm} (26)

Obviously, by the definition of the two norms, one has

\[
\| u \|_2^2 = \int_{\Omega} |u|^2 dx \leq \| u \|^2.
\]  \hspace{1cm} (27)

Due to (25) and (26), the two norms \( \| \cdot \| \) and \( \| \cdot \|_2 \) are equivalent on \( H^- \). Then it holds

\[
\| u_n \|_2 \rightarrow \infty.
\]

Put \( v_n = \frac{u_n}{\| u_n \|_2} \in H^- \). Since \( H^- \) is a finite dimension space, there exists \( v \in H^- \) satisfying

\[
v_n \rightharpoonup v \quad \text{both in } H \text{ and } L^2(\Omega).
\]  \hspace{1cm} (28)

Moreover, by (13), we know

\[
- \int_{\Omega} G(u_n) dx + \int_{\Omega} \frac{h u_n}{\| u_n \|_2^2} dx \rightarrow 0.
\]  \hspace{1cm} (29)

Then via (27) and (28) we obtain

\[
0 \leq \liminf_{n \rightarrow \infty} \frac{\varphi(u_n)}{\| u_n \|_2^2} = \liminf_{n \rightarrow \infty} \left[ \frac{1}{2} \int_{\Omega} |\nabla v_n|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |v_n|^2 dx - \int_{\Omega} G(u_n) dx + \int_{\Omega} \frac{h u_n}{\| u_n \|_2^2} dx \right]
\]

\[
= \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \frac{1}{2} \mu_k \int_{\Omega} |v|^2 dx.
\]
However, we all know
\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx - \frac{1}{2} \mu_k \int_{\Omega} |v|^2 \, dx \leq 0.
\]
Thus it holds
\[
\frac{1}{2} \int_{\Omega} |\nabla v|^2 \, dx = \frac{1}{2} \mu_k \int_{\Omega} |v|^2 \, dx,
\]
which implies that \( v = \phi_0 \in \bar{H} \) is an eigenfunction associated with \( \mu_k \). Clearly,
\[
v_n = \frac{u_n}{\|u_n\|_2} \to \phi_0 \text{ in } L^2(\Omega).
\]
For all \( u_n = \hat{u}_n + \bar{u}_n \in H^- \), one has
\[
\phi(u_n) = \frac{1}{2} \int_{\Omega} |\nabla \hat{u}_n|^2 \, dx - \frac{1}{2} \mu_k \int_{\Omega} |\bar{u}_n|^2 \, dx - \int_{\Omega} G(u_n) \, dx + \int_{\Omega} \hat{h} u_n \, dx + \int_{\Omega} h^+ \bar{u}_n \, dx
\leq -C_1||\hat{u}_n||^2 + ||h^+||_2 ||\bar{u}_n||_2 - \int_{\Omega} G(u_n) \, dx + \int_{\Omega} \hat{h} u_n \, dx,
\]
which, together with (GLL), implies
\[
\phi(u_n) \to -\infty \text{ as } n \to +\infty.
\]
This contradicts (24). The conclusion is verified.

On the other hand, by (8), (10) and (13), for \( u \in H^+ \), we have
\[
\phi(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{2} \mu_k \int_{\Omega} |u|^2 \, dx - \int_{\Omega} G(u) \, dx + \int_{\Omega} hu \, dx
\geq C_3||u||^2 - C_5||u||_2 \geq C_3||u||^2 - C_7||u||,
\]
which implies
\[
\phi(u) \to +\infty \text{ as } ||u|| \to +\infty.
\]
Hence there exist constants \( \alpha \) and \( R > 0 \) such that
\[
\sup_{u \in \partial D} \phi(u) < \alpha < \beta,
\]
where \( D = \{ u \in H^- \mid ||u|| \leq R \} \).

Therefore, the hypotheses (a) and (b) in Lemma 1 are satisfied. Recall that the functional \( \phi \) satisfies (PS) condition in Lemma 4, Theorem 2 is proved via Lemma 1.

References

[1] H. Brezis, L. Nirenberg, Characteristics of the ranges of some nonlinear operators and applications to boundary value problems, Ann. Scuola Norm. Sup. Pisa 5 (1978), 225-326.
[2] D. G. Costa; An invitation to variational methods in differential equations, Birkhauser, 2007.
[3] D. de Figueiredo, The Dirichlet problem for nonlinear elliptic equations: a Hilbert space approach, Lecture Notes in Math., vol. 446, Springer, Berlin, 1975.
[4] P. Drábek, Landesman-Lazer type condition and nonlinearities with linear growth, Czechoslovak Math. J. 40 (1990), 70-86.
[5] P. Drábek, Landesman-Lazer condition for nonlinear problems with jumping nonlinearities, J. Differential Equations. 85 (1990), 186-199.
[6] P. Drábek, Landesman-Lazer condition revisited: the influence of vanishing and oscillating nonlinearities, Electronic Journal of Qualitative Theory of Differential Equations. 2015, No. 68, 1-11.
[7] C. Fabry, Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities, J. Differential Equations. 116 (1995), 405-418.
[8] C. Fabry and A. Fonda, Periodic solutions of nonlinear differential equations with double resonance, Ann. Mat. Pura Appl. 157 (1990), 99-116.
[9] C. Fabry and A. Fonda, Nonlinear equations at resonance and generalized eigenvalue problems, Nonlinear Anal. 18 (1992), 427-444.
[10] A. Fonda, M. Garrione, Double resonance with Landesman-Lazer conditions for planar systems of ordinary differential equations, J. Differential Equations. 250 (2011), 1052-1082. doi:10.1016/j.jde.2010.08.006.

[11] S. Fučík, Solvability of Nonlinear Equations and Boundary Value Problems, Reidel, Boston, 1980.

[12] A. M. Krasnosel’skiî, On bifurcation points of equations with Landesman-Lazer type nonlinearities, Nonlinear Anal. 18 (1992), 1187-1199.

[13] J. Mawhin, Landesman-Lazer’s type problems for nonlinear equations, Conf. Sem. Mat. Univ. Bari 147 (1977), 1-22.

[14] C.P. Gupta, Perturbations of second order linear elliptic problems by unbounded nonlinearities, Nonlinear Anal. 6 (1982), 919-933.

[15] R. Iannacci, M.N. Nkashama, Nonlinear two point boundary value problem at resonance without Landesman-Lazer condition, Proc. Amer. Math. Soc. 10 (1989), 943-952.

[16] R. Iannacci, M.N. Nkashama, Nonlinear boundary value problems at resonance, Nonlinear Anal. 11 (1987), 455-473.

[17] Q. Jiang, S. Ma, Existence and multiplicity of solutions for semilinear elliptic equations with Neumann boundary conditions, Electronic Journal of Differential Equations. 2015, No. 200, pp. 1-8.

[18] C.C. Kuo, On the solvability of a nonlinear second-order elliptic equations at resonance, Proc. Amer. Math. Soc. 124 (1996), 83-87.

[19] E. M. Landesman, A. C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19(1970), 609-623.

[20] A. C. Lazer, D. E. Leach, Bounded perturbations of forced harmonic oscillators at resonance, Ann. Mat. Pura Appl. 82 (1969), 49-68.

[21] C.Y. Li, Q. Zhang, F.F. Chen, Pairs of sign-changing solutions for sublinear elliptic equations with Neumann boundary conditions, Electronic Journal of Differential Equations. 2014, No. 112, 1-9.

[22] C. Li, S. J. Li, Multiple solutions and sign-changing solutions of a class of nonlinear elliptic equations with Neumann boundary condition, J. Math. Anal. Appl. 298(2004), 14-32.

[23] J. Mawhin, Necessary and sufficient conditions for the solvability of nonlinear equations through the dual least action principle, in: X. Pu (Ed.), Workshop on Applied Differential Equations, Beijing, 1985, World Scientific, Singapore, 1986, 91-108.

[24] J. Mawhin, Semi-coercive monotone variational problems, Acad. Roy. Belg. Bull. Cl. Sci. 73 (1987), 118-130.

[25] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Springer-Verlag, New York, 1989.

[26] P.H. Rabinowitz, On a class of functionals invariant under a Zn action, Trans. Amer. Math. Soc. 310 (1) (1988) 303-311.

[27] P.H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, Amer. Math. Soc., Providence, RI, 1986.

[28] C.L. Tang, Solvability for two boundary value problems, J. Math. Anal. Appl. 216(1997), 368-374.

[29] C.L. Tang, Solvability of Neumann problem for elliptic equation at resonance, Nonlinear Anal. 44 (2001), 323-335.

[30] C.L. Tang, Some existence theorems for sublinear Neumann boundary value problem, Nonlinear Anal. 48 (2002), 1003-1011.

[31] C.L. Tang, X.P. Wu, Existence and multiplicity for solutions of Neumann problem for semilinear elliptic equations, J. Math. Anal. Appl. 288 (2003), 660-670.

[32] C.L. Tang, X.P. Wu; Multiple solutions of a class of Neumann problem for semilinear elliptic equations, Nonlinear Analysis: TMA. 62(2005), 455-465.

[33] P. Tomiczek, A generalization of the Landesman-Lazer condition, Electronic Journal of Differential Equations. 2001, No. 4, 1-11.