ON THE GROWTH OF THE OPTIMAL CONSTANTS OF THE MULTILINEAR BOHNEBLUST–HILLE INEQUALITY

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Abstract. Let \((K_n)_{n=1}^\infty\) be the optimal constants satisfying the multilinear (real or complex) Bohnenblust–Hille inequality. The exact values of the constants \(K_n\) are still waiting to be discovered since eighty years ago; recently, it was proved that \((K_n)_{n=1}^\infty\) has a subexponential growth. In this note we go a step further and address the following question: Is it true that

\[
\lim_{n \to \infty} (K_n - K_{n-1}) = 0?
\]

Our main result is a Dichotomy Theorem for the constants satisfying the Bohnenblust–Hille inequality; in particular we show that the answer to the above problem is essentially positive in a sense that will be clear along the note. Another consequence of the dichotomy proved in this note is that \((K_n)_{n=1}^\infty\) has a kind of subpolynomial growth: if \(p(n)\) is any non-constant polynomial, then \(K_n\) is not asymptotically equal to \(p(n)\). Moreover, if

\[
q > \log_2 \left( \frac{e^{1-\frac{1}{q^2}}}{\sqrt{2}} \right) \approx 0.526,
\]

then

\[
K_n \approx n^q.
\]

1. Introduction

Let \(\mathbb{K}\) be the real or complex scalar field. The multilinear Bohnenblust–Hille inequality (see, for example, [1, 4, 5, 9] and also [3] for a polynomial version) asserts that for every positive integer \(n \geq 1\) there exists a constant \(c_{\mathbb{K},n}\) such that

\[
\left( \sum_{i_1, \ldots, i_n=1}^N |T(e_{i_1}, \ldots, e_{i_n})| \right)^{\frac{n+1}{2n}} \leq c_{\mathbb{K},n} \sup_{z_1, \ldots, z_n \in \mathbb{D}^N} |T(z_1, \ldots, z_n)|
\]

for all \(n\)-linear form \(T : \mathbb{K}^N \times \cdots \times \mathbb{K}^N \to \mathbb{K}\) and every positive integer \(N\), where \((e_i)_{i=1}^N\) denotes the canonical basis of \(\mathbb{K}^N\) and \(\mathbb{D}^N\) represents the open unit polydisk in \(\mathbb{K}^N\). It is well-known that \(c_{\mathbb{K},n} \in [1, \infty)\) for all \(n\) and that the power \(\frac{2n}{n+1}\) is sharp but, on the other hand, the optimal values for \(c_{\mathbb{K},n}\) remain a mystery. To the best of our knowledge the unique precise
The original constants obtained by Bohnenblust and Hille (for the complex case) are
\[c_{\mathbb{C},n} = n^{\frac{n+1}{2n}} 2^{\frac{n-1}{2n}}.\]
Later, these results were improved to
\[c_{\mathbb{C},n} = 2^{\frac{n-1}{2}}\] (Davie, Kaiser, 1973 ([2, 8])),
and
\[c_{\mathbb{C},n} = \left(\frac{2}{\sqrt{\pi}}\right)^{n-1}\] (Quèffelec, 1995 ([11])).

In 2012 ([6]) it was proved that the best constants satisfying the Bohnenblust–Hille inequality have a subexponential growth (for both real and complex scalars). A step further would be to verify if these optimal constants have an even better asymptotic behavior. More precisely:

**Problem 1.** Let \((K_{\mathbb{K},n})_{n=1}^{\infty}\) be the sequence of optimal constants satisfying the Bohnenblust–Hille inequality. Is it true that
\[\lim_{n \to \infty} (K_{\mathbb{K},n} - K_{\mathbb{K},n-1}) = 0?\]

In this note, among other results, we essentially show that the answer to this problem is positive. More precisely, as a consequence of our main result (Dichotomy Theorem) we show that if there exist \(L_1, L_2 \in [0, \infty]\) so that
\[L_1 = \lim_{n \to \infty} (K_{\mathbb{K},n} - K_{\mathbb{K},n-1}) \quad \text{and} \quad L_2 = \lim_{n \to \infty} \frac{K_{\mathbb{K},2n}}{K_{\mathbb{K},n}},\]
then
\[L_1 = 0\]
and
\[L_2 \in [1, \frac{e^{1-\frac{1}{4}\gamma}}{\sqrt{2}}],\]
where \(\gamma\) denotes the Euler constant (see ([3, 11])). The non-existence of the above limits would be an extremely odd event since there is no reason for a pathological behavior for the optimal constants \((K_{\mathbb{K},n})_{n=1}^{\infty}\) satisfying the Bohnenblust–Hille inequality.

Another corollary of the Dichotomy Theorem is that the sequence \((K_{\mathbb{K},n})_{n=1}^{\infty}\) of optimal constants satisfying the Bohnenblust–Hille inequality can not have any kind of polynomial growth. Also, if
\[q > \log_2 \left(\frac{e^{1-\frac{1}{4}\gamma}}{\sqrt{2}}\right) \approx 0.526,\]
then
\[K_{\mathbb{K},n} \sim n^q.\]
2. The Dichotomy Theorem

From now on our arguments will hold for both real and complex scalars, so we will use the same notation for both cases. In all this note \((K_n)_{n=1}^\infty\) denotes the sequence of the optimal constants satisfying the Bohnenblust–Hille inequality.

From now on we say that a sequence of positive real numbers \((R_n)_{n=1}^\infty\) is well-behaved if there are \(L_1, L_2 \in [0, \infty)\) such that

\[
\lim_{n \to \infty} \frac{R_{2n}}{R_n} = L_1
\]

and

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = L_2.
\]

Note that the above requirements are quite weak (observe that \(L_1, L_2\) may be infinity). So, any sequence of the form

\[
R_n = b \cdot c^n \quad \text{for} \quad (a, b, c) \in [0, \infty)^2 \times (-\infty, \infty), \quad \text{or}
\]

\[
R_n = b \cdot a^n \quad \text{for} \quad (a, b, c) \in [0, \infty)^2 \times (-\infty, \infty), \quad \text{or}
\]

\[
R_n = b \cdot n^a \quad \text{for} \quad (a, b) \in (-\infty, \infty) \times [0, \infty), \quad \text{or}
\]

\[
R_n = b \log n, \quad \text{for} \quad b \in (0, \infty) \quad \text{or}
\]

\[
R_n = \sum_{j=0}^{k} a_j n^j \quad \text{with} \quad a_k > 0
\]

is well-behaved. Since the elements of \((K_n)_{n=1}^\infty\) belong to \([1, \infty)\), we will restrict our attention to well-behaved sequences in \([1, \infty)\). We also remark that, even restricted to sequences in \([1, \infty)\), the limits (2.1) and (2.2) are, in fact, independent. For example

\[
R_n := \begin{cases} 
\sqrt{n}, & \text{if } n = 2^k \text{ for some } k, \\
2\sqrt{n}, & \text{otherwise}
\end{cases}
\]

satisfies (2.1) with \(L_1 = \sqrt{2}\) but does not fulfil (2.2). On the other hand let, for all positive integers \(k > 1\),

\[
B_k := \{2^k - 1, ..., 2^{k+1} - 2\}.
\]

The sequence

\[
R_n := \begin{cases} 
\sqrt{n}, & \text{if } n \in B_k \text{ for some } k \text{ odd,} \\
(\min B_k)^k + kn, & \text{if } n \in B_k \text{ for some } k \text{ even,}
\end{cases}
\]

satisfies (2.2) but does not satisfy (2.1).

Henceforth the subexponential sequence of constants satisfying the multi-linear Bohnenblust–Hille inequality constructed in [6] is denoted by \((C_n)_{n=1}^\infty\). Since we are interested in the growth of \((K_n)_{n=1}^\infty\), we will restrict our attention to sequences \((R_n)_{n=1}^\infty\) so that \(1 \leq R_n \leq C_n\) for all \(n\).

Our main result is the following dichotomy:
**Dichotomy Theorem.** If \( 1 \leq R_n \leq C_n \) for all \( n \), then exactly one of the following assertions is true:

(i) \((R_n)_{n=1}^{\infty}\) is subexponential and not well-behaved.

(ii) \((R_n)_{n=1}^{\infty}\) is well-behaved with

\[
\lim_{n \to \infty} \frac{R_{2n}}{R_n} \in \left[1, \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \right]
\]

and

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0.
\]

As a corollary we extract the following information on the optimal constants \((K_n)_{n=1}^{\infty}\):

**Corollary 1.** The optimal constants \((K_n)_{n=1}^{\infty}\) satisfying the Bohnenblust–Hille inequality is

(i) subexponential and not well-behaved or

(ii) well-behaved with

\[
\lim_{n \to \infty} \frac{K_{2n}}{K_n} \in \left[1, \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \right]
\]

and

\[
\lim_{n \to \infty} (K_n - K_{n-1}) = 0.
\]

If (i) is true, then we will have a completely surprising result: the bad behavior of \((K_n)_{n=1}^{\infty}\). On the other hand, if (ii) is true we will have an almost ultimate and surprising information on the growth of \((K_n)_{n=1}^{\infty}\).

We remark that there exist well-behaved sequences \((R_n)_{n=1}^{\infty}\) such that

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0
\]

but

\[
\lim_{n \to \infty} \frac{R_{2n}}{R_n} \notin \left[1, \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \right].
\]

In fact, since \(\frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \approx 1.44\), for

\[R_n = n^{\frac{3}{5}}\]

we have

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0
\]

and

\[
\lim_{n \to \infty} \frac{R_{2n}}{R_n} = 2^{\frac{3}{5}} \notin \left[1, \frac{e^{1-\frac{1}{2}\gamma}}{\sqrt{2}} \right].
\]
On the other hand, as a consequence of our results we observe the simple (but useful) fact: if \((R_n)_{n=1}^\infty\) is well-behaved and
\[
\lim_{n \to \infty} \frac{R_{2n}}{R_n} \in [1, 2),
\]
then necessarily
\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0.
\]
So, \textit{a fortiori}, condition (2.5) is superfluous.

3. The proofs

Henceforth the letter \(\gamma\) denotes the Euler constant

\[
\gamma = \lim_{m \to \infty} \left( - \log m + \sum_{k=1}^{m} \frac{1}{k} \right) \approx 0.577.
\]

Lemma 1. If \(1 \leq R_n \leq C_n\) for all \(n\) and there exists \(L \in [0, \infty)\) so that
\[
L = \lim_{n \to \infty} \frac{R_{2n}}{R_n},
\]
then
\[
L \in [1, e^{1-\frac{1}{2}\gamma}].
\]

Proof. Suppose that \(L < 1\). For any \(0 < \varepsilon < 1\), there is a \(N_0\) so that
\[
n \geq N_0 \Rightarrow \frac{R_{2n}}{R_n} < 1 - \varepsilon.
\]
Arguing by induction we have
\[
R_{2^l N_0} < R_{N_0}(1 - \varepsilon)^l
\]
for all positive integer \(l\) and we conclude that
\[
\lim_{l \to \infty} R_{2^l N_0} = 0,
\]
which is impossible, since \(R_n \geq 1\) for all \(n\). To simplify the notation, we will write
\[
\alpha := e^{1-\frac{1}{2}\gamma}.
\]
Now let us show that \(L > \alpha\) is also not possible. From [6] we know that
\[
\lim_{n \to \infty} \frac{C_{2n}}{C_n} = \alpha.
\]
We will show that there is a sufficiently large \(N\) so that \(R_N > C_N\) (which is a contradiction).

Given a small \(0 < \varepsilon < L - \alpha\), there is a \(n_0\) so that
\[
n \geq n_0 \Rightarrow \frac{C_{2n}}{C_n} < \alpha + \frac{\varepsilon}{2} := A \text{ and } \frac{R_{2n}}{R_n} > \alpha + \varepsilon := B.
\]
Using induction we have
\[ C_{2^n_0} < A^l C_{n_0} \]
\[ R_{2^n_0} > B^l R_{n_0} \]
for all positive integer \( l \). Hence
\[ \frac{R_{2^n_0}}{C_{2^n_0}} > \frac{B^l R_{n_0}}{A^l C_{n_0}} = \left( \frac{B}{A} \right)^l \frac{R_{n_0}}{C_{n_0}}. \]
Since \( \frac{B}{A} > 1 \) we conclude that
\[ \lim_{l \to \infty} \left( \frac{B}{A} \right)^l \frac{R_{n_0}}{C_{n_0}} = \infty \]
and thus there is a positive integer \( N_1 \) so that
\[ \frac{R_{2^{N_1}n_0}}{C_{2^{N_1}n_0}} > 1, \]
which is a contradiction. \( \square \)

**Lemma 2.** If \( 1 \leq R_n \leq C_n \) for all \( n \), and the limit (3.3) exists, then there is a positive integer \( N_0 \) so that
\[ \frac{2^l N_0}{R_{2^l N_0}} > \frac{N_0}{R_{N_0}} \left( \frac{4}{3} \right)^l \]
for all positive integers \( l \).

**Proof.** From the previous lemma we have
\[ \lim_{n \to \infty} \frac{R_{2n}}{R_n} = L \in [1, e^{1-\frac{1}{2}\gamma} \sqrt{2}]. \]
So, since
\[ \frac{e^{1-\frac{1}{2}\gamma} \sqrt{2}}{2} < \frac{3}{2}, \]
let us fix \( N_0 \) so that
\[ \frac{R_{2n}}{R_n} < \frac{3}{2} \]
for all \( n \geq N_0 \). Hence, by induction,
\[ R_{2^l N_0} < \left( \frac{3}{2} \right)^l R_{N_0} \]
for all positive integer \( l \). We conclude that
\[ \frac{2^l N_0}{R_{2^l N_0}} > \frac{2^l N_0}{\left( \frac{3}{2} \right)^l R_{N_0}} = \frac{N_0}{R_{N_0}} \left( \frac{4}{3} \right)^l \]
for all \( l \). \( \square \)
Lemma 3. If \((R_n)_{n=1}^{\infty}\) is well-behaved and \(1 \leq R_n \leq C_n\) for all \(n\), then

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0
\]

Proof. Let \(M := \lim_{n \to \infty} (R_n - R_{n-1})\). The first (and main) step is to show that \(M \notin (0, \infty)\).

Suppose that \(M \in (0, \infty)\). In this case, from (3.3) there is a positive integer \(N_1\) such that

\[
n \geq N_1 \Rightarrow R_n - R_{n-1} > \frac{M}{2}.
\]

So,

\[
n \geq N_1 \Rightarrow R_{2n} - R_n > \frac{nM}{2}
\]

Hence

\[
n \geq N_1 \Rightarrow \frac{R_{2n}}{R_n} - 1 > \left(\frac{nM}{2}\right) \frac{1}{R_n}.
\]

From Lemma \(\Box\) we have

\[
1 \leq \lim_{n \to \infty} \frac{R_{2n}}{R_n} \leq e^{1 - \frac{1}{\gamma}}\frac{\sqrt{2}}{4}.
\]

So, there is a positive integer \(N_2\) so that

\[
n \geq N_2 \Rightarrow \frac{R_{2n}}{R_n} > 3.
\]

Hence if \(N_3 = \max\{N_1, N_2\}\) we have

\[
n \geq N_3 \Rightarrow \frac{1}{2} > \frac{R_{2n}}{R_n} - 1 > \left(\frac{nM}{2}\right) \frac{1}{R_n}.
\]

From the previous lemma we know that there is a \(N_0\) so that

\[
\frac{2^l N_0}{R_{2^l N_0}} > \frac{N_0}{R_{N_0}} \left(\frac{4}{3}\right)^l
\]

for all positive integers \(l\).

Now we choose a positive integer \(l_0\) such that

\[
l > l_0 \Rightarrow N_l := 2^l N_0 > N_3.
\]

Hence, from (3.4), we know that

\[
l \geq l_0 \Rightarrow N_l > \frac{1}{2} > \left(\frac{N_l M}{2}\right) \frac{1}{R_{N_l}} = \left(\frac{M}{2}\right) \frac{2^l N_0}{R_{2^l N_0}}.
\]

But, since

\[
\lim_{l \to \infty} \frac{N_0}{R_{N_0}} \left(\frac{4}{3}\right)^l = \infty,
\]

from (3.5) and (3.6) we have a contradiction.

The argument to show that \(\lim_{n \to \infty} (R_n - R_{n-1})\) can not be infinity is an immediate consequence of the previous case. \(\Box\)
Note that a simple adaptation of the proof of the above lemmata provides the following simple but apparently useful general result:

**Proposition 1.** If \((R_n)_{n=1}^\infty\) is well-behaved and

\[
1 \leq \lim_{n \to \infty} \frac{R_{2n}}{R_n} < 2,
\]

then

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0.
\]

In particular, this result reinforces that the information \((2.4)\) implies \((2.5)\).

Our main result is a straightforward consequence of the previous lemmata:

**Theorem 1 (Dichotomy).** If \(1 \leq R_n \leq C_n\) for all \(n\), exactly one of the following assertions is true:

(i) \((R_n)_{n=1}^\infty\) is subexponential and not well-behaved.

(ii) \((R_n)_{n=1}^\infty\) is well-behaved with

\[
\lim_{n \to \infty} \frac{R_{2n}}{R_n} \in [1, e^{1-\frac{1}{2}\gamma} \sqrt{2}]
\]

and

\[
\lim_{n \to \infty} (R_n - R_{n-1}) = 0.
\]

As we have just mentioned (it is a consequence of Proposition 1), the information \(\lim_{n \to \infty} (R_n - R_{n-1}) = 0\) in (ii) is in fact a consequence of the fact that \((R_n)_{n=1}^\infty\) is well-behaved with \(\lim_{n \to \infty} \frac{R_{2n}}{R_n} \in [1, e^{1-\frac{1}{2}\gamma} \sqrt{2}]\).

In the real case it is known that \((K_n)_{n=1}^\infty\) satisfies

\[
2^{1-\frac{1}{n}} \leq K_n \leq C_n
\]

for all positive integer \(n\) (see [7]). It is not difficult to obtain an example of subexponential and not well-behaved sequence satisfying the above inequality. For example,

\[
R_n = \begin{cases} 
2^{1-\frac{1}{n}}, & \text{if } n = 2^k \text{ for some } k, \\
C_n, & \text{otherwise}.
\end{cases}
\]

### 4. Final Remarks

It is well-known that the powers \(\frac{R_n}{n+1}\) in the Bohnenblust–Hille inequality are sharp; so, it is a common feeling that the optimal constants from the Bohnenblust–Hille inequality must have an uniform behavior, without strange fluctuations on their growth. The fact that (i) is fulfilled would, indeed, be a strongly unexpected result. On the other hand, if (ii) is true (and we conjecture that this is the case) we would also have a noteworthy information on the growth of these constants:

\[
\lim_{n \to \infty} (K_n - K_{n-1}) = 0,
\]
and this is also a surprising result in view of the previous known estimates for the growth of these constants (see [11, 15, 22]).

As we mentioned in the previous section, for the case of real scalars we know that

\begin{equation}
K_n \geq 2^{1 - \frac{1}{n}}
\end{equation}

and $K_2 = \sqrt{2}$ (and also that $K_3 > K_2$). On the one hand all known estimates for the constants in the Bohnenblust–Hille inequality indicate that we “probably” have $\lim_{n \to \infty} K_n = \infty$; but, as a matter of fact, we do not know any proof that the sequence $(K_n)_{n=1}^{\infty}$ tends to infinity. Also, the sequence $(2^{1 - \frac{1}{n}})_{n=1}^{\infty}$ is obviously well-behaved and it is not completely impossible that the above estimates (4.1) are sharp.

As a final remark mention that a consequence of the Dichotomy Theorem asserts that if

\begin{equation}
q \in \mathbb{R} - [0, \beta]
\end{equation}

with

$$\beta := \log_2 \left( \frac{e^{1 - \frac{1}{2} \gamma}}{\sqrt{2}} \right) \approx 0.526$$

and $c \in (0, \infty)$, then the sequence $(K_n)_{n=1}^{\infty}$ can not be of the form $K_n \sim cn^q$.

In fact, denoting $B_n^{(q)} = cn^q$ we have

$$\lim_{n \to \infty} \frac{K_{2n}}{K_n} = \lim_{n \to \infty} \left( \frac{B_n^{(q)}}{B_{2n}^{(q)}} \frac{K_{2n}}{B_{2n}^{(q)}} \frac{B_{2n}^{(q)}}{K_n} \right)$$

$$= \lim_{n \to \infty} \left( \frac{B_n^{(q)}}{B_n^{(p)}} \right) \lim_{n \to \infty} \left( \frac{K_{2n}}{B_{2n}^{(q)}} \right) \lim_{n \to \infty} \left( \frac{B_{2n}^{(q)}}{K_n} \right)$$

$$= 2^q.$$

Since $q \in \mathbb{R} - \left[ 0, \log_2 \left( \frac{e^{1 - \frac{1}{2} \gamma}}{\sqrt{2}} \right) \right]$, we have

$$2^q \notin [1, \frac{e^{1 - \frac{1}{2} \gamma}}{\sqrt{2}}]$$

and it contradicts the Dichotomy Theorem. The case $q < 0$ is in fact impossible since we know that $K_n$ belongs to $[1, \infty)$.

A similar reasoning shows that if $p(n)$ is any non-constant polynomial, then

$$K_n \sim p(n).$$
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