Symmetries: From Proofs To Algorithms And Back

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Abstract. We call an objective function or algorithm symmetric with respect to an input if after swapping two parts of the input in any algorithm, the solution of the algorithm and the output remain the same. More formally, for a permutation \( \pi \) of an indexed input, and another permutation \( \pi' \) of the same input, such that swapping two items converts \( \pi \) to \( \pi' \), \( f(\pi) = f(\pi') \), where \( f \) is the objective function.

After reviewing samples of the algorithms that exploit symmetry, we give several new ones, for finding lower-bounds, beating adversaries in online algorithms, designing parallel algorithms and data summarization. We show how to use the symmetry between the sampled points to get a lower/upper bound on the solution. This mostly depends on the equivalence class of the parts of the input that when swapped, do not change the solution or its cost.

Keywords: Parallel Algorithms · Nash Equilibrium · Online Algorithms · Lower and upper bounds · Clustering · Gray codes.

1 Introduction

By symmetry, we mean after swapping the points in any algorithm, the solution of the algorithm and the output remain the same. More formally, for a permutation \( \pi \) of an indexed input, and another permutation \( \pi' \) of the same input, such that swapping two items converts \( \pi \) to \( \pi' \), \( f(\pi) = f(\pi') \), where \( f \) is the objective function.

Each of the following sections discusses an example of the use of symmetry in known problems or algorithms.

2 Lower Bounds Based on Symmetries

Several lower bounds based on the swapping method exist, for example [24], in which by setting the distances to the optimal solution equal for specific subsets of the input points, the symmetry is preserved. The lower bounds set all distances equally, so this type of lower bounds work for any objective function that only depends on the pairwise distances of the points.

In [10], the authors state there is no optimal clustering, where a clustering is a partitioning \( \Gamma \) of the input set \( S \) into a set of clusters using a function \( f \) that takes a distance function \( d \) on \( S \) and returns \( \Gamma \). Also, \( d \) is not required to be a metric. The authors claim no clustering satisfies the following three conditions:

- Scale-invariance: for any distance function \( d \) and any \( \alpha > 0 \), \( f(d) = f(\alpha \cdot d) \) holds.
- Richness: the range of \( f \) is the set of all possible partitions of \( S \).
Consistency: for two distance functions \( d \) and \( d' \), if \( f(d) = \Gamma \) and \( d' \) is a \( \Gamma \)-transformation of \( d \), then \( f(d') = \Gamma \). A \( \Gamma \)-transformation of \( d \) is a transformation that for each pair \( i, j \in S \) that are in the same cluster of \( \Gamma \), \( d'(i, j) \leq d(i, j) \) and for each pair \( i, j \in S \) in different clusters of \( \Gamma \), \( d'(i, j) \geq d(i, j) \).

Based on the symmetry, consider a partitioning of the input that represents the optimal clustering \( O \). The possible solutions to a clustering are the ones that are symmetric based on the 3 conditions of [10] with respect to \( O \). Trivial examples of optimal clusterings are empty clustering, where no cluster center is selected, and all clusters, where each element is a single cluster. A non-trivial example is the random clustering, where \( f \) is a random function that selects the centers uniformly at random and assigns the points uniformly at random to the clusters. Since random clustering depends only on the internal randomness of \( f \), it satisfies all the distance conditions, and it satisfies richness because there is a non-zero probability that any partitioning is selected.

A reasonable alternative is to use multiple clustering criterias, for example multiple distance functions at the same time.

The consistency condition in the paper contradicts the triangle inequality. Unless \( d' \) is shrunk while preserving the triangle inequality. The examples for the contradiction between the triangle inequality and the consistency condition are as follows:

- Example 1: Assume \( a, b, c \) are three points in the same cluster. \( ab + bc > ac \). The transformation is to shrink \( ab \) and \( bc \), until \( ab + bc < ac \).

- Example 2: Assume \( a, b, c \) are three points in a different cluster. We increase their distances such that the triangle inequality no longer holds.

Based on these examples, adding the triangle inequality restriction is not as trivial as the paper [10] suggests.

One useful example of symmetry is the one between clusterings that choose a center and the shapes of their clusters are disks. In [3], it is shown that sampling the points with probability \( p \) gives a \((\delta p)\)-approximation with the failure probability \( 1/\delta \) and \( pn \) centers, for any \( \delta > 0 \), if the number of centers \( k = \Omega(1) \).

While the proof of [3] uses disks of equal size to prove a constant factor approximation for a random sampling for k-center, it is possible to use the same proof on the disk graph of disks with different radii, such as the one in k-means. Note that the proof does not depend on the objective function. The only difference is the centroid is chosen as the center instead of one of the points.

**Lemma 1.** [1] Using the nearest point to the centroid instead of the centroid multiplies the approximation factor of Euclidean k-means by 2.

**Proof.** Let \( p \) be the closest input point to the centroid \( c \) in its cluster, and let \( p_i, i = 1, \cdots, t \) be all the other points. So the optimal clustering has cost \( \sqrt{\sum_{i=1}^t d(p_i, c)^2} \). Using the law of cosines, we bound \( d(p, p_i) \) by \( d(p, c) \) and \( d(p, c) \):

\[
\sum_{i=1}^t d(p_i, p)^2 \leq \sum_{i=1}^t [d(p_i, c)^2 + d(p, c)^2 + 2d(p_i, c)d(p, c)] \\
\leq \sum_{i=1}^t [d(p_i, c)^2 + d(p_i, c)^2 + 2d(p_i, c)d(p_i, c)] = 4 \sum_{i=1}^t d(p_i, c)^2.
\]

[1] The result of this lemma is well-known.
Taking the root of each side gives \( \sqrt{\sum_{i=1}^{t} d(p_i, c)^2} \leq 2 \sqrt{\sum_{i=1}^{t} d(p_i, c)^2} \).

Since the proof of [8] for \( k \)-center does not depend on the cost of the objective function, and Lemma [1] proves using the cost of choosing a centroid instead of an input point can at most multiply the approximation factor by a constant factor, then for \( p = \frac{k}{n} \), the probability of finding a \( c \)-approximation at an iteration is \( 1 - \frac{k}{nc} \). So, in \( \Theta\left(\frac{1}{n}\right) \) iterations, Lloyd’s algorithm finds a constant factor approximation:

**Theorem 1.** Lloyd’s algorithm \([12]\) is a constant factor approximation for \( k \)-means if it is run for \( \Omega\left(\frac{1}{n}\right) \) iterations and \( k = \Omega(1) \).

### 3 A Summary of Equilibriums To Defeat All Adversaries

The Nash equilibrium \([13]\) is a concept in game theory, where a set of selfish players in a game, each player with a set of actions and its own objective function, cannot improve their objective function by changing their actions if other players do not change their actions.

Each algorithm can be modeled by a zero-sum two-player game, where the adversary is one of the players and the algorithm is the other one. Since the Nash equilibrium always exists, there is always an algorithm that is not affected by the adversary. The Nash equilibrium does not necessarily give the optimal solution, so this does not give any information on the performance of the algorithm. The use of algorithms versus inputs as a zero-sum game has already been discussed in Yao’s minimax principle \([16]\).

For example, in the secretary problem, a sequence of people are interviewed for a single job position, where after each interview the value of each person for that job is determined and the choices are to hire the last person or reject him. A Deterministic algorithm \([8]\) is to choose the first value that exceeds \( \max_{i=1,\ldots,i} v_j \), where \( i \) satisfies \( \sum_{t=i+1}^{n} \frac{1}{t} \leq 1 < \sum_{t=i}^{n} \frac{1}{t} \) after visiting the first \( i - 1 \) candidates. For \( n \to \infty \), the competitive ratio is \( 1/e \). An approximation algorithm is based on the prophet inequality \([11]\), and hires the first candidate with value at least as much as half of the maximum seen so far. Formally if \( v_i \geq \max_{i=1,\ldots,j} v_i/2 \), the algorithm hires \( i \)-th person.

Any algorithm not dependent on the choices of the adversary works. Consider an algorithm that removes \( t \) random values of the input stream and runs the \( 1/e \)-competitive algorithm.

**Theorem 2.** This algorithm is \( (1 - \frac{t}{n}) \)-competitive with probability \( 1 - t/n \) and fails to choose anyone, otherwise.

A trade-off between the competitive ratio and the success probability can be useful in some applications.

In \([9]\), the algorithm used for solving the problem creates a set of \( n \) random numbers \( u_1, \ldots, u_n \), and chooses the threshold based on them, i.e. \( \max_{i=1,\ldots,j} u_i/2 \), and compares \( v_i \) with that value. This is an example of combining a equilibrium algorithm with the competitive algorithm. It uses the extra information about the distribution of the input. Their algorithm as well as their lower bound allows the cases where no-one is hired. The summary is \( \max_{j=1,\ldots,i} u_j \) whose size is constant, and therefore sublinear.
4 Equivalence Classes Based on Critical Paths in Parallelization

Consider the dependency graph of an algorithm. The same swapping symmetry exists in these graphs, where swapping the order of the execution of two commands does not change the output of the algorithm, assuming everything else remains unchanged.

Existing algorithms for the minimum spanning tree (MST) problem in the massively parallel computations (MPC) [6] either use $\Omega(\log n)$ rounds, $\omega(n + m)$ space, or solve special cases of the problem, even for computing a $(1 + \epsilon)$-approximation of the problem [5,17].

We call the variation of the Prim’s algorithm that prioritizes choosing the vertex adjacent to a previous vertex to a new vertex, the prioritized Prim’s algorithm.

**Theorem 3.** A prioritized version of Prim’s algorithm outperforms Kruskal’s algorithm for MST in MPC.

**Proof.** Consider an expander graph of degree $m$ with unit weights. Kruskal’s algorithm for MST chooses the edges in an arbitrary order, which in the worst case requires $\theta(mn)$ checks in a graph with $n$ vertices. Prim’s algorithm chooses each neighboring vertex equally likely, so it takes $O(n)$ checks. The prioritized Prim’s algorithm, solves the problem in $O(\log_m n)$ rounds, since at each step it adds every neighbor of the previous vertices, if it does not already exist in the tree. \(\square\)

The equivalence classes for a MST are the vertices at the same depth in a breadth-first search of the tree. Given such an ordering of the vertices, the prioritized Prim’s algorithm solves MST in $O(\log_m n)$ rounds in MPC on a partitioning of data based on the ordering.

In [4], an $O(1)$-round algorithm for indexing the vertices based on a grid in a balanced manner is given. Using the $z$-order curve in the indexing algorithm, compute an ordering of the vertices. This gives a partitioning of MST based on distances. Using Prim’s algorithm for MST and visiting the vertices in the order of their indices computes MST in $O(1)$ rounds.

5 A Trick for Solving The Recurrence Relations of Monotone Functions

Assume a given recurrence relation uses only monotone functions with the same monotonicity in its formula. We try to make them symmetric by lower-bounding and upper-bounding the literals with the minimum and maximum literals, respectively.

For example, consider the linear recurrence relation $T(n) = T(n/2) + T(3n/4) + O(n), T(1) = 1, \forall c = O(1)$. We can bound $T(n)$ with the following recurrences:

\[
T_1(n) = 2T_1(3n/4) + O(n), T_1(1) = 1, \forall c = O(1) \\
T_2(n) = 2T_2(n/2) + O(n), T_2(c) = 1, \forall c = O(1) \\
\min(T_1(n), T_2(n)) \leq T(n) \leq \max(T_1(n), T_2(n))
\]

This will help us use the master theorem [7], instead of the recursion tree method for solving this problem. Using the master theorem, $T_1(n) = \theta(\max(n^{\log_{4/3} 2}, n)) = \theta(n^{\log_{4/3} 2})$, since $\log_{4/3} 2 \approx 2.41$.

Similarly, $T_2(n) = \theta(\max(n^{\log_{2} 2}, n)) = O(n \log n)$.

Both bounds are superlinear, so $T'(an) \geq \alpha T'(n)$, for $\alpha \geq 1$ and $T'(\alpha n) \leq \alpha T'(n)$ for $\alpha \leq 1$. Using a better upper bound $T_1'(n) = \frac{5}{3} T_1'(3n/4) + O(n), T_1'(c) = 1, \forall c = O(1)$.

\[
T_1'(n) = \frac{5}{3} T_1'(3n/4) + O(n), T_1'(1) = 1, \forall c = O(1).
\]
Which using the master’s theorem solves to \( T_1'(n) = \theta(\max(n^{\log_{4/3} 5/3}, n)) = \theta(n^{\log_{4/3} 5/3}) \), since \( \log_{4/3} 5/3 \approx 1.776 \). So far, \( T(n) = \theta(n^\beta), \beta \in [1, 1.776] \). Similarly for \( T_2'(n) = \frac{3}{4} T_2'\left(\frac{n}{2}\right) + O(n) \), \( T_2'(c) = 1, \forall c = O(1) \), the solution is \( T_2'(n) = \theta(n^{\log_2 5/2}) \). Since \( \log_2 5/2 \approx 1.32 \), the bound on \( \beta \) improves to \([1.321, 1.776]\). This process can be repeated to get a more exact solution, by applying the new powers to the factors of \( n/2 \) and \( 3n/4 \). This is also a proof, so it does not the substitute method for verification.

Using the substitute method for \( n^\beta \) instead, and finding the minimum \( \beta \) that satisfies \( 1 \leq \left(\frac{3}{4}\right)^\beta + \left(\frac{1}{4}\right)^\beta \) gives \( \beta \approx 1.51 \).

6 Gray Codes on Sets of Data Blocks

In [14,15], a binary locally repairable code (BLRC) is a binary parity replication method that allows the recovery of data after a set of errors by accessing a few number of replicas and the errors can be fixed by taking the XOR of the replicas.

The algorithm can be described as a Gray code with 2-bit groupings. For example, consider the case with 5 digits and at most 2 errors. The parity blocks in a BLRC for data blocks \( B_1, \cdots, B_{10} \) are as follows:

\[
\begin{align*}
P_1 &= B_1 + B_2 + B_3 + B_4 + B_5 + B_6 \\
P_2 &= B_1 + B_2 + B_3 + B_7 + B_8 + B_9 \\
P_3 &= B_3 + B_4 + B_5 + B_7 + B_8 + B_{10} \\
P_4 &= B_2 + B_4 + B_6 + B_7 + B_9 + B_{10} \\
P_5 &= B_1 + B_5 + B_6 + B_8 + B_9 + B_{10}
\end{align*}
\]

There are \( \binom{5}{3} = 10 \) possible cases, where each row is a data block and each column is a parity block:

\[
\begin{array}{cccc}
1 & 1 & - & 1 \\
2 & 2 & - & - \\
3 & 3 & 3 & - \\
4 & - & 4 & 4 \\
5 & - & 5 & 5 \\
6 & - & 6 & 6 \\
7 & 7 & 7 & - \\
8 & 8 & 8 & 8 \\
9 & - & 9 & 9 \\
10 & 10 & 10 & 10 \\
\end{array}
\]

\[
\begin{array}{c}
11001 \\
11010 \\
11100 \\
10110 \\
10101 \\
10011 \\
01110 \\
01101 \\
01011 \\
00111 \\
\end{array}
\]

\[
\begin{array}{c}
11001 \\
11010 \\
11100 \\
10110 \\
10101 \\
10011 \\
01110 \\
01101 \\
01011 \\
00111 \\
\end{array}
\]

We suggest the following steps for creating parities in general:

- For \( n \) data blocks and \( d \) errors, the number of required parity blocks is \( \binom{n}{d} \), since we need to choose the places of \( d \) errors among \( n \) positions.
- Building a Gray code with \( d \)-bit distances can be done by computing the Cartesian product of \( d \) Gray codes, and concatenating them.
- Minimizing the number of XORs requires the placement of \( n \)-bit strings such that each column has the maximum intersection with the previous column. Since sorting is based on the bit significance, it is sufficient to sort the strings.
Theorem 4. A Gray code with \(d\)-bit distance between its consecutive strings exists for any length \(n\), where \(n > d\).

Proof. We use induction on the recursive algorithm we discussed before. Based on the induction hypothesis the Gray code of length \(n'\) with distance \(d'\) exist, for all \(n' < n, d' < d\). Assuming \(G(n, 0)\) is the Gray code of length \(n\) and \(G(n, d)\) is the Gray code of length \(n\) with distance \(d\) between its consecutive strings, the following recursion solves the problem:

\[
G(n, d) = G(n - i, d - 1) + G(i, 0) \quad \forall i < n, 
\]

and the base cases are \(G(i, 0)\). Since the time required for constructing the Gray code of length \(n\) is \(O(n)\), setting \(i = \lceil n/2 \rceil\) minimizes the construction time, as long as \(n - i > d - 1\). Otherwise, we use \(i = 1\). The construction time is:

\[
G(n, d) = G(\frac{n}{2^d}, 0) + \sum_{j=1}^{d} \frac{n}{2^j} = O(n). 
\]

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