Abstract: We establish the regularity in 2 dimensions of $L^2$ solutions to critical elliptic system in divergence form involving involution operator of finite $W^{1,2}$-energy.

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I Introduction

In [24] the second author discovered a compensation phenomenon for the linear elliptic systems of the form

$$-\Delta u = \Omega \cdot \nabla u \quad \text{in} \quad D^2,$$

where $u \in W^{1,2}(D^2, \mathbb{R}^n)$ and $\Omega$ is an $L^2$ map into the antisymmetric matrices of $\mathbb{R}^2$ vectors. That is to say there exists a matrix $(\Omega^i_j)_{i,j=1,\ldots,n}$ of $L^2$ functions into $\mathbb{R}^2$ such that

$$\forall \ i = 1, \ldots, n \quad -\Delta u_i = \sum_{j=1}^{n} \Omega^i_j \cdot \nabla u_j \quad \text{and} \quad \Omega^i_j = -\Omega^j_i \quad \forall \ i, j = 1, \ldots, n.$$

A-priori the system (I.1) is critical for the chosen norms, with a right hand side in $L^1$. Without antisymmetry of $\Omega$ no improved regularity has to be expected in general while $W^{1,2}$ solutions to (I.1) for $\Omega \in L^2(D^2, \mathbb{R}^2 \otimes so(n))$ are known to be in $\bigcap_{p < 2} W^{2,p}_{\text{loc}}(D^2)$.

One of the main strategy introduced in [24] was to use the antisymmetry of $\Omega$ in order to construct a “gauge” $A \in L^\infty \cap W^{1,2}(D^2, Gl(n)(\mathbb{R}))$ satisfying

$$\text{div}(\nabla \Omega A) := \text{div}(\nabla A - A \Omega) = 0$$

Taking a “primitive” $\nabla B = (-\partial_x B, \partial_y B) := \nabla \Omega A \in L^2(D^2, M_n(\mathbb{R}) \otimes \mathbb{R}^2)$ the system (I.1) becomes equivalent to the conservation law

$$\text{div}(A \nabla u) = \nabla \cdot (\nabla A - A \Omega) = 0 \quad (I.2)$$

The Jacobian form of the right-hand-side of (I.2) permits to use now classical integrability by compensation phenomena originally discovered by H.Wente [35] and related to the ones by R.Coifman, Rochberg and Weiss [4] (see [5]).

Following the main ideas of [24], extensions of this compensation phenomenon were obtained in [25] for critical systems of the form (for $m > 2$)

$$\Delta v = \Omega v \quad \text{in} \quad B^m \quad (I.3)$$

where $\Omega \in L^{m/2}(B^m, so(n))$ and $v \in L^{m/(m-2)}(B^m, \mathbb{R}^n)$ as well as for systems of the form

$$\Delta^{1/2} v = \Omega v \quad \text{in} \quad \mathbb{R} \quad (I.4)$$

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where this time \( v \in L^2(\mathbb{R}, \mathbb{R}^n) \) and \( \Omega \in L^2(\mathbb{R}, so(n)) \) (see [12]). More recently the two authors are extending their results to non local right-hand-side

\[
\Delta^{1/4} v = \int_{\mathbb{R}} H(x, y) v(y) \, dy
\]

(1.5)

where pointwise antisymmetry has to be replaced by the more general notion of anti-self-duality of the underlying non-local operator \( K(x, y) \) where \( K(x, y) := H(x, y) - \omega(x) \delta_{x=y} \in L^1_{\text{loc}}(\mathbb{R}^2) \) (see [13]).

In the present work we are exhibiting a new compensation phenomenon which does not enter in none of the previous existing ones. Our main result is the following

**Theorem I.1.** Let \( S \in W^{1,2}(D^2, \text{Sym}(n)) \), where \( \text{Sym}(n) \) denotes the vector space of symmetric \( n \times n \) matrices, such that \( S^2 = \text{id}_n \) and let \( u \in L^2(D^2, \mathbb{R}^n) \) be a solution of the following linear elliptic system in divergence form

\[
\text{div}(S \nabla u) = \sum_{j=1}^n \text{div}(S_{ij} \nabla u^j) = 0.
\]

Then \( u \in \bigcap_{p>2} W^{1,p}_{\text{loc}}(D^2, \mathbb{R}^n) \). \( \square \)

**Remark I.1.** The system (1.6) is elliptic with Principal Symbol \( |\xi|^2 S \). It is however not strongly elliptic in the sense of Legendre Hadamard since obviously \( \langle \lambda, SA \rangle \) can change sign as \( \lambda \) varies. \( \square \)

**Remark I.2.** Structural conditions on \( S \) for the regularity are necessary in the following sense. In [22] an \( L^2 \) solution to

\[
\text{div}(A \nabla u) = 0
\]

is produced where \( A \in W^{1,2}(D^2, \text{Sym}(2)) \) and \( A \) is satisfying the strong ellipticity condition\(^1\)

\[
\langle A(x)\xi, \xi \rangle \geq |\xi|^2
\]

uniformly on \( D^2 \) but \( u \notin W^{1,p}_{\text{loc}}(D^2, \mathbb{R}) \) for any \( p > 1 \). \( \square \)

**Remark I.3.** Contrary to the case of the systems (1.7) in [23], we have not found yet striking applications in geometry or physics of systems (1.6) while nevertheless they look very “natural” and enjoy numerous formulations that we are going to present in this work. The system (1.7) is nothing but the harmonic Map equation into a pseudo-riemannian manifold (see remark 1.4). The formulation using Dirac operator below (see [14]) moreover corresponds to the Weierstrass representation of Lagrangian surfaces in four-dimensional space by Hélein and Romon (21) Theorem 1). The assumption \( u \in L^2 \) is also faithful to the Hilbert Space framework in mathematical physics\(^2\) and makes this function space natural in that sense. \( \square \)

Behind the proof of theorem 1.1 there is an \( \varepsilon \)-regularity type of estimate which implies the following concentration-compactness result

**Theorem I.2.** Let \( S_k \to S_\infty \) weakly in \( W^{1,2}(D^2, \text{Sym}(n)) \) where \( S_k^2 = \text{id}_n \) and let \( u_k \to u_\infty \) weakly in \( L^2(D^2, \mathbb{R}^n) \), and satisfying

\[
\text{div}(S_k \nabla u_k) = 0
\]

\(^1\)The matrix \( A \) is acting on the different vertical components of \( \nabla u \), \( u \) is in fact scalar in this case while in theorem 1.1 the matrix \( S \) acts on the horizontal components of \( \nabla u \) that is \( \nabla u_1, \ldots, \nabla u_n \).

\(^2\)Original works in mathematical physics which have nourish the growth of analysis with problems from quantum mechanics, such as the study of Schrödinger semigroups for instance [31] etc, take the \( L^2 \) space and not the “energy space” \( W^{1,2} \) as the “configuration space”.

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then, modulo extraction of a subsequence, there exists finitely many points $a_1 \ldots a_Q \in D^2$ s.t.
\[
u_k \rightharpoonup u_\infty \quad \text{strongly in } \bigcap_{p<2} W^{1,p}_{\text{loc}}\left(D^2 \setminus \{a_1 \ldots a_Q\}\right).
\]
Moreover $u_\infty$ satisfies $\text{div}(S_\infty \nabla u_\infty) = 0$ in $D'(D^2)$.

We shall call $S \in W^{1,2}(D^2, \text{Sym}(n))$ where $S^2 = \text{id}_n$ a chirality operator. Ethimologically, in old greek $\chi\epsilon\iota\rho$ (kheir) means “hand”. The word chirality refers to an intrinsic disymmetry of the space where a left and a right directions are given. More precisely almost everywhere on $D^2$ we have the existence of two orthogonal projections, $P_R$ and $P_L$ complementary to each other ($P_R = \text{id}_n - P_L$), the left and the right, such that $S = P_R - P_L$.

Remark I.4. The system (I.6) is then the Euler-Lagrange equation of the Dirichlet energy into the pseudo-riemannian manifold $(\mathbb{R}^n, g)$ where $g(X,Y) := \langle X, P_R Y \rangle - \langle X, P_L Y \rangle$. In other words (I.6) is the harmonic map equation from $D^2$ into $(\mathbb{R}^n, g)$, it correspond to critical points of
\[
E_g(u) := \int_{D^2} |P_R \nabla u|^2 - |P_L \nabla u|^2 \, dx^2.
\]

As we will see theorem I.1 can be rephrased as follows.

Theorem I.3. Let $P_R \in W^{1,2}(D^2, \text{Sym}(n))$ such that $P_R \circ P_R = P_R$ and denote $P_L := \text{id}_n - P_R$ and let $f \in L^2(D^2, \mathbb{C}^n)$ satisfying
\[
\begin{align*}
&\quad P_R \frac{\partial f}{\partial z} = 0 \\
&\quad P_L \frac{\partial f}{\partial \bar{z}} = 0
\end{align*}
\]
then $f \in \bigcap_{p<2} W^{1,p}_{\text{loc}}(D^2, \mathbb{C}^n)$.

In the course of the paper we will give a third formulation of our main result. For $n = 2$ it takes a simpler following form.

Theorem I.4. Let $\Omega \in L^2(D^2, \text{so}(2) \otimes \mathbb{C})$ and let $f \in L^2(D^2, \mathbb{C})$ such that
\[
\frac{\partial f}{\partial z} = \Omega \bar{f}.
\]
Assume $\Im(\partial_z \Omega) = 0$ then $f \in \bigcap_{p<+\infty} W^{1,p}_{\text{loc}}(D^2, \mathbb{C}^2)$.

The system
\[
\frac{\partial f}{\partial z} = \Omega f.
\]
where $\Omega \in L^2(D^2, \text{so}(2) \otimes \mathbb{C})$ and $\Im(\partial_z \Omega) = 0$ enjoys the same compensation property as (1.8) for $f \in L^2(D^2, \mathbb{C})$ but this last fact is a consequence of the theory in [24] while the theorem I.4 is new. It can be recasted in the following way. Recall the definition of the Dirac Operator in $\mathbb{C}^2$
\[
\mathcal{D} := \begin{pmatrix} 0 & \frac{\partial}{\partial z} \\ -\frac{\partial}{\partial \bar{z}} & 0 \end{pmatrix}
\]
Then we have the following corollary

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Corollary I.1. Let \( U \in L^2(D^2, \mathbb{C}) \) such that \( \Im(\partial_z U) = 0 \). Let \( \Psi \in L^2(D^2, \mathbb{C}^2) \) be a solution of
\[
D\Psi = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \Psi ,
\]
then \( \Psi \in \bigcap_{p<+\infty} W^{1,p}_{loc}(D^2, \mathbb{C}^2) \).
\(
\square
\)

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II Preliminaries

We will denote by \( S(\mathbb{R}^n) \) the space of Schwarz functions.

II.1 Bourgain-Brezis Inequalities

In \cite{2} Bourgain and Brezis proved the following striking result:

Theorem II.5 (Lemma 1 in \cite{2}). Let \( u \) be a \( 2\pi \)-periodic function in \( \mathbb{R}^n \) such that \( \int_{\mathbb{R}^n} u = 0 \), and let \( \nabla u = f + g \), where \( f \in W^{-1, \infty} \) and \( g \in L^1 \) are \( 2\pi \)-periodic vector valued functions. Then
\[
\| u \|_{L^{\infty}} \leq c \| f \|_{W^{-1, \infty}} \| g \|_{L^1}.
\]
(II.11)

As a consequence of Theorem II.5 they get the following

Corollary II.2 (Theorem 1 in \cite{2}). For every \( 2\pi \)-periodic function \( h \in L^\infty(\mathbb{R}^n) \) with \( \int_{\mathbb{R}^n} h = 0 \) there exists a \( 2\pi \)-periodic \( v \in W^{1,\infty} \cap L^\infty \) satisfying
\[
\text{div } v = h,
\]
and
\[
\| v \|_{L^\infty} + \| v \|_{W^{1,\infty}} \leq C(n) \| h \|_{L^\infty}.
\]

II.2 Bourgain-Brezis inequality in 2 dimension revisited

For the convenience of the reader we provide a proof of (II.11) in 2-dimension which has the advantage of not assuming the periodicity. The proof is related to some compensation phenomena observed first in \cite{15} in the analysis of 2-dimensional perfect incompressible fluids. This observation has also been used by the second author in the analysis of isothermic surfaces \cite{26} (see also \cite{17, 23, 24}).

Lemma II.1. Let \( u \in D'(\mathbb{C}) \) be such that
\[
\nabla u = f + g \in (W^{-1,2} + L^1)(\mathbb{C}) \text{ in } D'(\mathbb{C}).
\]
(II.12)

Then \( u \in L^2(\mathbb{C}) \) and
\[
\| u \|_{L^2(\mathbb{C})} \leq C(\| f \|_{W^{-1,2}(\mathbb{C})} + \| g \|_{L^1(\mathbb{C})}).
\]
(II.13)

Proof of Lemma II.1

Classical estimates give first that \( u \in L^{2,\infty}(\mathbb{C}) \)
The Hodge decomposition of $f$ in $L^2(\mathbb{C})$ gives the existence of $w, v \in \dot{H}^1(\mathbb{C})$ such that

$$f = \nabla w + \nabla^\perp v$$  \hspace{1cm} (II.14)

with

$$\|w\|_{L^2(\mathbb{C})} + \|v\|_{L^2(\mathbb{C})} \leq C \|f\|_{W^{-1,2}(\mathbb{C})}.$$  

We rewrite (II.12) in the form

$$\nabla(u - w) = \nabla^\perp v + g.$$  \hspace{1cm} (II.15)

Let $h = u - w - iv \in L^{2,\infty}(\mathbb{C})$, we have

$$\partial_x h = 2(1 + i)g \in L^1(\mathbb{C}).$$  \hspace{1cm} (II.16)

Let $\psi$ be the solution of

$$\partial\bar{z}\psi = h$$  \hspace{1cm} (II.17)

given explicitly by $\psi = 4\pi z^* h$.  We also have

$$\partial^2_{zz} \psi = 2(1 + i)g.$$  \hspace{1cm} (II.18)

**Claim:** $\psi \in L^\infty(\mathbb{C})$

**Proof of the Claim** We take the Fourier Transform of both sides in (II.18) and get

$$\hat{\psi} = \frac{1}{\xi^2} \hat{g}.$$  \hspace{1cm} (II.19)

Hence

$$\psi = \mathcal{F}^{-1} \left[ \frac{1}{\xi^2} \right] * 2(1 + i)g.$$  \hspace{1cm} (II.20)

We set $\xi = \xi_1 + i\xi_2$. Observe that

$$\mathcal{F}^{-1} \left[ \frac{1}{\xi^2} \right] = \mathcal{F}^{-1} \left[ \frac{\xi^2}{|\xi|^4} \right] = \mathcal{F}^{-1} \left[ \frac{\xi^2 - \xi_2^2 + 2i\xi_1\xi_2}{|\xi|^4} \right].$$

Since $\xi_1^2 - \xi_2^2 + 2i\xi_1\xi_2$ is homogeneous harmonic polynomial, we can apply Theorem 5 in 3.3 of [32] and deduce the existence of an universal constant $c_0$ such that

$$\mathcal{F}^{-1} \left[ \frac{\xi^2 - \xi_2^2 + 2i\xi_1\xi_2}{|\xi|^4} \right] = c_0 \frac{x_1^2 - x_2^2 + 2ix_1x_2}{|x|^2}.$$  

It follows then from (II.20)

$$\|\psi\|_{L^\infty} \leq 2c_0 \|g\|_{L^1} \left\| \frac{x_1^2 - x_2^2 + 2ix_1x_2}{|x|^2} \right\|_{L^\infty} = 2c_0 \|g\|_{L^1}. $$  \hspace{1cm} (II.21)

This proves the claim.

It follows from (II.21)

$$\left| \int_{\mathbb{C}} h^2 dx_1 dx_2 \right| = \left| \int_{\mathbb{C}} h \partial_z \psi dx_1 dx_2 \right|$$

$$= \left| \int_{\mathbb{C}} \partial_z (h\psi) dx_1 dx_2 - \int_{\mathbb{C}} \partial_z h \psi dx_1 dx_2 \right| \leq \|g\|_{L^1} \|\psi\|_{L^\infty} \leq C \|g\|_{L^1}^2. $$  \hspace{1cm} (II.22)

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\(3\) We recall that $\partial_t (4\pi z^* \frac{1}{z}) = \delta_0$. 

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We conclude the proof. ✷

We observe that if in the Lemma II.1 Remark II.5.

From (II.22) and (II.23) it follows that

simply get the estimate

\[ \frac{1}{2} \left( \left( \int_C |u - w|^2 \, dx \right)^2 + \left( \int_C v(u - w) \, dx \right)^2 \right)^{1/2} \] (II.23)

From (II.22) and (II.23) it follows that

\[ \int_C |u - w|^2 \, dx \leq \int_C |v|^2 \, dx + C \| g \|_{L^1}^2 \] (II.24)

and

\[ \int_C |u|^2 \, dx \leq 2 \| u \|_{L^2}^2 + 2 \| v \|_{L^2}^2 + C \| g \|_{L^1}^2 \]
\[ \leq C(\| f \|_{W^{-1,2}} + \| g \|_{L^1}^2) \] (II.25)

We conclude the proof. □

Remark II.5. We observe that if in the Lemma II.1 \( \nabla u = \nabla^1 v + g \) with \( g \in L^1 \) and \( v \in L^2 \) then we simply get the estimate

\[ \int_C |u|^2 \, dx \leq \| v \|_{L^2}^2 + C \| g \|_{L^1}^2 \] (II.26)

namely the constant in front of \( \| v \|_{L^2}^2 \) is 1.

Remark II.6. Lemma II.1 still holds if \( u \in \mathcal{D}'(B(0,1)) \) satisfies

\[ \nabla u = f + g \in (W^{-1,2} + L^1)(B(0,1)) \text{ in } \mathcal{D}'(B(0,1)). \] (II.27)

In this case we have to estimate the solution \( \psi \) of (II.18) in \( B(0,1) \). Let \( \tilde{\psi} \) be the solution of

\[ \partial^2_{zz} \tilde{\psi} = 2(1 + i)g \mathbf{1}_{B(0,1)} \text{ in } \mathcal{D}'(\mathbb{C}). \] (II.28)

Then \( w = \tilde{\psi} - \psi \) satisfies \( \partial^2_{zz} w = 0 \) in \( \mathcal{D}'(B(0,1)) \). This implies that \( w = z\phi_1(z) + \phi_2(z) \) where \( \phi_1 \) and \( \phi_2 \) are holomorphic. Since \( \psi \in L^\infty(\mathbb{C}) \) it follows that \( \psi \in L^\infty(B(0,1)) \). □

III Regularity of solutions to \( \text{div}(S \nabla u) = 0 : \) Proof of theorem I.1.

In this section we are going to investigate the regularity of \( L^2 \) solutions to the following system

\[ \text{div}(S \nabla u) = 0 \text{ in } \mathcal{D}'(\mathbb{C}) \] (III.29)

where \( S \in W^{1,2}(\mathbb{C}, O(n)) \) with \( S^2 = 1 \).

It has been shown in [22] that there exists solutions \( u \in W^{1,1}_{\text{loc}}(B(0,1)) \) of \( \text{div}(A \nabla u) = 0 \) in \( \mathcal{D}'(B(0,1)) \) where \( A \) is a uniformly elliptic and continuous matrix which is in none of the spaces \( W^{1,p}_{\text{loc}}(B(0,1)) \) for any \( p > 2 \).

\(^4\) \( O(n) \) denotes the group of orthogonal \( n \times n \) matrices, \( U(n) \) is the group of unitary \( n \times n \) matrices and \( SU(n) \) is the Lie group of \( n \times n \) unitary matrices with determinant 1. The Lie algebra of \( U(n) \) consists of \( n \times n \) skew-Hermitian matrices, with the Lie bracket given by the commutator.
Actually they construct a counter-example of a matrix $A$ which turns to be also in $W^{1,2}(B(0,1))$. The matrix $A(x) = (a_{ij}(x))_{1 \leq i \leq n}$ is defined as follows

$$a_{ij}(x) = \delta_{ij} + \alpha(|x|) \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right)$$

where

$$\alpha(r) = \frac{-\beta n}{(n-1) \left( \log \frac{r_0}{r} \right)} + \frac{\beta(n+1)}{(n-1) \left( \log \frac{r_0}{r} \right)^2}. \quad (\text{III.30})$$

where $r_0$ is large enough so that $\alpha \geq -\frac{1}{2}$ and $\beta > 1$.

Clearly $a_{ij} \in L^2(B(0,1))$. A direct computation for any $i,j,k$ gives

$$\frac{\partial a_{ij}}{\partial x_k} = \alpha'(|x|) \frac{x_k}{|x|} \left( \delta_{ij} - \frac{x_i x_j}{|x|^2} \right) - \alpha(|x|) \left( \frac{\delta_{ik} x_j + \delta_{jk} x_i}{|x|^2} - \frac{2 x_i x_j}{|x|^4} \right). \quad (\text{III.31})$$

Therefore

$$\left| \frac{\partial a_{ij}}{\partial x_k}(x) \right| \leq C \frac{1}{r} \frac{1}{\log \left( \frac{r_0}{r} \right)}.$$

Since $\frac{1}{r} \frac{1}{\log \left( \frac{r_0}{r} \right)} \in L^2(B(0,1))$ then $\nabla a_{ij} \in L^2(B(0,1))$ as well. It is proved in [22] that

$$u(x) = x_1 \frac{1}{r^2 \log \left( \frac{r_0}{r} \right)^\beta} \in L^2(B(0,1)) \quad \text{solves} \quad \sum_{i,j=1}^2 \partial_{x_i}(a_{ij} \partial_{x_j} u) = 0.$$

We are now proving the following result

**Theorem III.6.** There is an $\varepsilon_0 > 0$ such that if $S \in \dot{W}^{1,2}(\mathbb{C}, O(n))$ with $S^2 = I_n$ and $\|\nabla S\|_{L^2(\mathbb{C})} \leq \varepsilon_0$ then there is $Q \in \dot{W}^{1,2}(\mathbb{C}, SO(n))$ such that

$$S = Q S^0 Q^{-1}$$

where

$$S^0 = \left( \begin{array}{c|c} I_{m \times m} & 0_{m \times n-m} \\ \hline 0_{n-m \times m} & I_{n-m \times n-m} \end{array} \right) \quad (\text{III.32})$$

with $m \leq n$ and

$$\|\nabla Q\|_{L^2} \leq C \|\nabla S\|_{L^2}. \quad (\text{III.33})$$

where $C > 0$ only depends on $n$. \hfill $\Box$

**Proof of Theorem III.6.** Let $S \in \dot{W}^{1,2}(\mathbb{C}, O(n))$ be with $S^2 = I_n$.

We have det $S$, Trace($S$) $\in W^{1,2}(\mathbb{R}^2, \mathbb{Z})$. Precisely

$$\det S = (-1)^{n-m}, \quad \text{and} \quad \text{Trace}(S) = 2m - n \quad (\text{III.34})$$

where $m = \#$ positive eigenvalues and $n - m = \#$ negative eigenvalues (we recall that the eigenvalues of $S$ can be either 1 or $-1$. Since det $S$, Trace($S$) $\in W^{1,2}(\mathbb{R}^2)$ it follows that det $S$ and Trace($S$) are both constant a.e. in $\mathbb{R}^2$.

We set

$$P_R := \frac{I - S}{2} \quad \text{and} \quad P_L = \frac{I + S}{2} \quad (\text{III.35})$$

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$P_L, P_R$ are idempotent since $(I - S)^2 = S^2 - 2S + I = 2(I - S)$ and $(I + S)^2 = 2(I + S)$ and the ranks of $P_L$ and $P_R$ are constant.

We can see $P_L$ (resp. $P_R$) as $W^{1,2}$ maps with values into the Grassmanian $Gr_m(R^n)$ (resp. $Gr_{n-m}(R^n)$) of un-oriented $m$--planes (resp. $n - m$--planes) in $R^n$.

By applying Lemma 5.1.4 in Hélein book [20] one can find 2 $W^{1,2}$ frames forming orthonormal basis $e = e_1, \ldots, e_m$ and $f_1, \ldots, f_{n-m}$ of $Im(P_L)$ and $Im(P_R)$ respectively such that

$$\|\nabla e_i\|_{L^2} \leq C\|\nabla P_L\|_{L^2} \quad \text{and} \quad \|\nabla f_j\|_{L^2} \leq C\|\nabla P_R\|_{L^2}$$

(III.36)

for $i = 1, \ldots, m$ and $j = 1, \ldots, n - m$.

Let $(\epsilon_k)_{k=1,\ldots,n}$ be the canonical basis of $R^n$ and let $Q_L \in SO(m)$, $Q_R \in SO(n - m)$ be such that

$$Q_L \epsilon_k = e_i \quad \text{for} \quad k = 1, \ldots, m \quad \text{and} \quad i = 1, \ldots, m$$

$$Q_R \epsilon_k = f_j \quad \text{for} \quad k = m + n, \ldots, n \quad \text{and} \quad j = 1, \ldots, n - m$$

and

$$P^0_L = Q^{-1}_L P_L Q_L \quad \text{and} \quad P^0_R = Q^{-1}_R P_R Q_R$$

where

$$P^0_L = \begin{pmatrix} I_{m \times m} & 0_{m \times n-m} \\ 0_{n-m \times m} & 0_{n-m \times n-m} \end{pmatrix}$$

(III.37)

and

$$P^0_R = \begin{pmatrix} 0_{m \times m} & 0_{m \times n-m} \\ 0_{n-m \times m} & I_{n-m \times n-m} \end{pmatrix}$$

(III.38)

We define

$$Q = \begin{pmatrix} Q_L & 0_{n \times n-n} \\ 0_{m \times m} & -Q_R \end{pmatrix}$$

(III.39)

By construction we have $Q^t Q = Id$, $S^0 = Q^{-1} S Q$ and

$$\|\nabla Q\|_{L^2} \leq C\|\nabla S\|_{L^2}.$$ 

(III.40)

This concludes the proof of Theorem [III.6]

Next we show how theorem [I.1] implies theorem [I.3]. More precisely we establish that (III.29) is equivalent to (I.7) for a suitable choice of $f$.

**Proposition III.1.** Let $S \in W^{1,2}(C, O(n))$ with $S^2 = I_n$ and let $u \in L^2(C, R^n)$ be a solution of

$$\text{div}(S \nabla u) = 0 \quad \text{in} \quad D'(C).$$

(III.41)

Then there exists $v \in L^2(C, R^n)$ such that $\nabla^t v = S \nabla u$ in $D'(C)$. Moreover the function $f = u + iv$ satisfies

$$\begin{cases}
P_R \frac{\partial f}{\partial z} = 0 \quad \text{in} \quad D'(C) \\
P_L \frac{\partial f}{\partial \bar{z}} = 0 \quad \text{in} \quad D'(C).
\end{cases}$$

(III.42)

$\square$
Proof of Proposition III.1. Let \( v \in D'(\mathbb{C}) \) be such that \( \nabla^\perp v = S \nabla u \) in \( D'(\mathbb{C}) \). It holds \( \nabla^\perp v = \nabla(Su) - \nabla Su \in W^{-1,2} + L^1 \). Lemma III.1 gives that \( v \in L^2(\mathbb{C}) \). We have

\[
\begin{cases}
S \partial_{x_1} u = -\partial_{x_2} v \\
S \partial_{x_2} u = \partial_{x_1} v
\end{cases}
\]  

(Ill.43)

Therefore

\[
S \partial_{x_1} (u + iv) = i \partial_{x_2} (u + iv)
\]  

(Ill.44)

Let us introduce \( f: \mathbb{C} \to \mathbb{C}^n \) given by \( f = u + iv \). Obviously \( f \) satisfies:

\[
\begin{cases}
S \partial_{x_1} f - i \partial_{x_2} f = 0 \\
i \partial_{x_1} f - \partial_{x_2} f = 0
\end{cases}
\]  

(Ill.45)

In other words

\[
(S - I) \partial_z f = 0
\]  

(Ill.46)

From (Ill.45) it follows

\[
S(\partial_z f + \partial_{\bar{z}} f) = (\partial_z f - \partial_{\bar{z}} f)
\]

\[
(I - S) \partial_{\bar{z}} f = (S + I) \partial_z f.
\]

Therefore \( f \) satisfies (Ill.42) and we conclude the proof. \( \square \)

III.1 Proof of theorem I.1: the case \( n = 2 \)

In this section we focus the case the function \( u \) takes values into \( \mathbb{R}^2 \), since as we will see the formulation will become simpler and maybe more enlightening.

Let \( Q \in W^{1,2}(\mathbb{C}, \text{SO}(2)) \) then a classical result by Carbou gives the existence of \( \alpha \in W^{1,2}(\mathbb{C}, \mathbb{R}) \) such that

\[
Q(x) = \begin{pmatrix}
\cos(\alpha(x)) & -\sin(\alpha(x)) \\
\sin(\alpha(x)) & \cos(\alpha(x))
\end{pmatrix}
\]  

(Ill.47)

We also set

\[
S^0 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}
\]  

(Ill.48)

Next we re-formulate the system (I.6) in the \( n = 2 \) case. Precisely we have

Proposition III.2. Let \( S \in W^{1,2}(\mathbb{C}, \text{O}(2)) \) with \( S^2 = 1 \) and \( \|\nabla S\|_{L^2(\mathbb{C})} \leq \varepsilon_0 \) (with \( \varepsilon_0 > 0 \) as in Theorem III.6). Let \( Q \in \text{SO}(2) \) as in III.1 such that \( S = Q^{-1} S^0 Q \) and let \( u, v \) be as in the statement of Proposition III.1. Then function \( f: \mathbb{C} \to (\mathbb{C}^2) \)

\[
f := S^0 Qu + iQv
\]  

(Ill.49)

satisfies the following equation

\[
\partial_z f = 2 \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \partial_{z\alpha} \bar{f}.
\]  

(Ill.50)

\( \square \)
Proof of proposition III.2 Let \( u \in L^2(\mathbb{R}^2) \) be a solution of (III.41) and \( v \in L^2(\mathbb{C}) \) be such that

\[ \nabla^\perp v = S \nabla u. \]  

(III.51)

Since \( S = Q^{-1} S^0 Q \) we can write (III.51) as

\[ Q \nabla^\perp v = S^0 Q \nabla u. \]  

(III.52)

We set \( f_R := S^0 Qu \) and \( f_\Im := Qv \). From the fact that \( S^0 Q \nabla u - Q \nabla^\perp v = 0 \) it follows

\[ \nabla(f_R) - \nabla^\perp(f_\Im) = \nabla(S^0 Q)u - \nabla^\perp Q^\perp S^0 f_R - \nabla^\perp Q^{-1} f_\Im. \]  

(III.53)

Therefore

\[ \begin{cases} \partial_{x_1} f_R + \partial_{x_2} f_\Im = S^0 (\partial_{x_1} Q)Q^{-1} S^0 f_R + \partial_{x_2} Q Q^{-1} f_\Im, \\ \partial_{x_2} f_R - \partial_{x_1} f_\Im = S^0 (\partial_{x_2} Q)Q^{-1} S^0 f_R + \partial_{x_1} Q Q^{-1} f_\Im. \end{cases} \]  

(III.54)

We have

\[ S^0 \nabla Q Q^{-1} S^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla \alpha. \]

We have

\[ \begin{cases} \partial_{x_1} f_R + \partial_{x_2} f_\Im = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_{x_1} \alpha f_R + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_{x_2} \alpha f_\Im, \\ -\partial_{x_2} f_R + \partial_{x_1} f_\Im = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \partial_{x_2} \alpha f_R + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_{x_1} \alpha f_\Im. \end{cases} \]  

(III.54)

From (III.54) it follows

\[ \partial_{x_1} f_R + \partial_{x_2} f_\Im - i(\partial_{x_2} f_R - \partial_{x_1} f_\Im) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} (\partial_{x_1} \alpha - i \partial_{x_2} \alpha)(f_R - if_\Im). \]

Hence

\[ \partial_z f = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \partial_z \alpha \bar{f}. \]  

(III.55)

This concludes the proof of proposition III.2 ∎

Now we present the regularity of the equation (III.50) and therefore of (III.55). We would like first to explain the reasons why the equation (III.55) does not fall within the classical theory of systems with a \( L^2 \) potential.

Let us represent a function \( f = u + iv \) with \( u = (u_1, u_2), v = (v_1, v_2) \) as

\[ f = \begin{pmatrix} u_1 + iu_2 \\ v_1 + iv_2 \end{pmatrix}. \]

We observe that the equation (III.55) can be written as

\[ \begin{cases} \partial_z (u_1 + iu_2) = \partial_{x_2} \alpha (v_1 - iv_2), \\ \partial_z (v_1 + iv_2) = -\partial_{x_2} \alpha (u_1 - iu_2). \end{cases} \]  

(III.56)
The system (III.56) is of the form
\[
\begin{cases}
\partial_z \phi = \omega \bar{\psi} \\
\partial_z \psi = -\omega \bar{\phi}
\end{cases}
\]
(III.57)
where \(\omega = \partial_z \alpha \in L^2(C, C)\). The difficulty is that in the right hand side of (III.57) there are the conjugate of the unknows \((\phi, \psi)\). Suppose we would have instead a system of the form
\[
\begin{cases}
\partial_z \phi = \partial_z \alpha \psi \\
\partial_z \psi = -\partial_z \alpha \phi
\end{cases}
\]
(III.58)
Then the function \(\Phi := (\phi_1, \phi_2)\) solves
\[
\partial_z \Phi = \Omega \Phi \quad \text{(III.59)}
\]
where
\[
\Omega = \begin{pmatrix} 0 & \partial_z \alpha \\ -\partial_z \alpha & 0 \end{pmatrix} = \partial_z Q^{-1} Q
\]
Hence we would deduce \(\partial_z (Q \Phi) = 0\) which would imply that \(\Phi \in W^{1,2}_{\text{loc}}\). Unfortunately the multiplication of \(\Phi\) solving (III.57) by a matrix in \(SO(2)\) does not permit to absorb the potential \(\Omega\) which is the case of interest in the present work. Therefore we have to find a different Lie group that permit us to absorb the potential.

Recall standard notations regarding the algebra of Quaternions that we denote by \(\mathbb{H}\):
\[
\mathbb{H} := \{a + bi + cj + dk, \quad (a, b, c, d) \in \mathbb{R}^4\},
\]
where \(i, j\) and \(k\) are the fundamental quaternion units satisfying \(i^2 = j^2 = k^2 = -1\) and \(ij = -ji = k, \quad jk = -kj = i \quad \text{and} \quad ki = -ik = j\). The set \(\mathbb{H}\) of all quaternions is a vector space over the real numbers with dimension 4. The conjugate of \(q \in \mathbb{H}\) is the quaternion \(q^{-1} = \frac{q^*}{|q|^2}\), where \(|q| = \sqrt{qq^*}\) is the norm of \(q\).

Given \(q = q_1 + q_2 i + q_3 j + q_4 k\) we set
\[
\Pi_i(q) = q_2 i \quad \text{and} \quad \Pi_j(q) = q_3 j + q_4 k.
\]
We also denote by \(\mathbb{H}_p\) the quaternion of the form \(q = q_2 i + q_3 j + q_4 k\) (the pure quaternions) and \(U(\mathbb{H}) := \{q \in \mathbb{H} : \quad |q| = 1\}\). \(\mathbb{H}_p\) is the Lie Algebra of the Lie Group \(U(\mathbb{H})\).

Finally given \(f: \mathbb{C} \rightarrow \mathbb{H}\) we introduce the following differential operators (Cauchy-Riemann-Fueter operators):
\[
\partial_L f := 2^{-1}(\partial_{x_1} f - i \partial_{x_2} f) \
\partial_R f := 2^{-1}(\partial_{x_1} f - \partial_{x_2} f i). 
\]
(III.60)
and
\[
\bar{\partial}_L f := 2^{-1}(\partial_{x_1} f + i \partial_{x_2} f) \
\bar{\partial}_R f := 2^{-1}(\partial_{x_1} f + \partial_{x_2} f i).
\]
(III.62)
(III.63)
We observe that if \(f\) takes values in \(\mathbb{C}\) then
\[
\partial_L f = \partial_R \bar{f} = \partial_L \bar{f} \quad \text{and} \quad \bar{\partial}_L f = \bar{\partial}_R \bar{f} = \bar{\partial}_L \bar{f}.
\]
We are going to rewrite the equation (III.56) using the quaternions.
Lemma III.2. Let
\[ f = \begin{pmatrix} u_1 + iu_2 \\ v_1 + iv_2 \end{pmatrix} \]
be a solution of (III.50) then the quaternion
\[ \mathbf{f} = u_1 + u_2i + v_1j + v_4k \]
satisfies
\[ \partial_L \mathbf{f} = -\partial_z \alpha j \mathbf{f} . \quad (III.64) \]

Proof of lemma III.2. Equation (III.50) can also be written using \( \partial_L \) operator (which coincides with \( \partial_R \) at this stage since the variables \( u_1 + iu_2 \) and \( v_1 + iv_2 \) are \( \mathbb{C} \)-valued).

\[
\begin{cases}
\partial_L(u_1 + iu_2) = \partial_z \alpha (v_1 - iv_2) \\
\partial_L(v_1 + iv_2) = -\partial_z \alpha (u_1 - iu_2)
\end{cases} \tag{III.65}
\]

We multiply from the right the second equation in (III.65) by \( j \) and we get (recall that \( ij = k = -ji \))
\[ \partial_L(v_1j + v_2k) = -\partial_z \alpha j(u_1 + iu_2) . \tag{III.66} \]

We can write the equation (III.66)
\[ \partial_L(u_1 + iu_2) = -\partial_z \alpha j^2 (v_1 - iv_2) = -\partial_z \alpha j(v_1j + v_2k) . \tag{III.67} \]

By summing (III.66) and (III.67) we find
\[ \partial_L(u_1 + u_2i + v_1j + v_4k) = -\partial_z \alpha j(u_1 + u_2i + v_1j + v_2k) . \tag{III.68} \]

Hence we get (III.64) and we can conclude. \[ \square \]

III.2 Bootstrap test for \( \partial_L \mathbf{f} = \partial_z \alpha j \mathbf{f} \)

In the sequel up to exchange \( \alpha \) and \( -\alpha \) we study the following equation
\[ \partial_L \mathbf{f} = \partial_z \alpha j \mathbf{f} \tag{III.69} \]

The first main goal of this section is to show that the operator
\[ \mathbf{f} \in L^2(\mathbb{C}, \mathbb{H}) \mapsto \partial_L \mathbf{f} - \partial_z \alpha j \mathbf{f} \]
is injective if the \( L^2 \) norm of \( \partial_z \alpha \) is sufficiently small. This is what we call the “bootstrap test”.

Theorem III.7. There exists \( \varepsilon_0 > 0 \) such that for any \( \alpha \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{R}) \) satisfying
\[ \| \nabla \alpha \|_{L^2} \leq \varepsilon_0 , \]
and any \( \mathbf{f} \in L^2(\mathbb{C}, \mathbb{H}) \) solving
\[ \partial_L \mathbf{f} = \partial_z \alpha j \mathbf{f} , \tag{III.70} \]
then \( \mathbf{f} \equiv 0. \) \[ \square \]
Before going to the proof of theorem III.7, we will introduce a non-linear operator \( N \).

Let \( q \in U(\mathbb{H}) \) We multiply the equation (III.71) on the left by \( q \):

\[
q[\partial_{x_1} f - i \partial_{x_2} f] = q[\partial_{x_1} \alpha - \partial_{x_2} \alpha \ i] \dot{f}.
\]

Observe that

\[
q[\partial_{x_1} f - i \partial_{x_2} f] = \partial_{x_1} [q f] - \partial_{x_2} [q i f] - \partial_{x_1} q \ f + \partial_{x_2} q \ i f.
\]

By combining (III.71) and (III.72) we get

\[
\dot{q} \partial_{x_1} [q f] - \partial_{x_2} [q i f] = q[\partial_{x_1} \alpha - \partial_{x_2} \alpha \ i] \dot{f} + \partial_{x_2} q \ i f.
\]

We introduce the following operator

\[
N: W^{1,2}(\mathbb{C}, U(\mathbb{H})) \rightarrow W^{-1,2}(\mathbb{C}, \text{Span}\{i\}) \times L^2(\mathbb{C}, \text{Span}\{j, k\})
\]

\[
q \mapsto (\Pi_i(\partial_{x_1} (q^{-1} \partial_{x_1} q) + \partial_{x_2} (q^{-1} \partial_{x_2} q)), \Pi_{j,k}(q^{-1} \partial_{x_1} q - q^{-1} \partial_{x_2} q i))
\]

We shall prove the following result.

**Lemma III.3.** There is \( \varepsilon_0 > 0 \) and \( C > 0 \) such that for any choice of \( \omega \in W^{-1,2}(\mathbb{C}, \mathbb{R}) \) and \( g \in L^2(\mathbb{C}, \text{Span}\{j, k\}) \) satisfying

\[
\|\omega\|_{W^{-1,2}} \leq \varepsilon_0, \quad \|g\|_{L^2} \leq \varepsilon_0
\]

then there is \( q \in W^{1,2}(\mathbb{C}, U(\mathbb{H})) \) such that

\[
N(q) = (i \omega, g)
\]

and

\[
\|\nabla q\|_{L^2} \leq C(\|\omega\|_{W^{-1,2}} + \|g\|_{L^2})
\]

In order to prove lemma III.3 we shall need to introduce some notations and establish some intermediate results. As in [12] Proof of Theorem 1.2, Step 4, by an approximation argument it suffices to prove Lemma III.3 assuming that \( \omega \) and \( g \) are slightly more regular. We fix \( 1 < p < +\infty \) and for \( \varepsilon > 0 \) we introduce let

\[
U_\varepsilon := \left\{ \omega \in W^{-1,p} \cap W^{-1,p'}(\mathbb{C}, \text{Span}\{i\}) \times \alpha \in L^p \cap L^{p'}(\mathbb{C}, \text{Span}\{j,k\}) \mid \|\omega\|_{W^{-1,2}} + \|g\|_{L^2} \leq \varepsilon \right\}
\]

For constants \( \varepsilon, \Theta > 0 \) let \( V_{\varepsilon, \Theta} \subseteq U_\varepsilon \) be the set where we have the decomposition (III.76) with the estimates

\[
\|\nabla q\|_2 \leq \Theta(\|\omega\|_{W^{-1,2}} + \|g\|_{L^2}) \quad ,
\]

\[
\|\nabla q\|_p \leq \Theta(\|\omega\|_{W^{-1,p}} + \|g\|_{L^p}) \quad ,
\]
\[ \| \nabla q \|_{p'} \leq \Theta(\| \omega \|_{W^{-1,p'}} + \| g \|_{L^{p'}}). \]  

That is
\[ V_{\varepsilon, \Theta} := \left\{ \omega, g \in U_{\varepsilon} : \begin{array}{c} \text{there exists } q \in (\hat{W}^{1,1} \cap W^{1,1})(\mathbb{R}^2), \text{ so that } q - 1 \in L^{\frac{p}{2}} \end{array} \right\} \]

(III.76) and (III.79), (III.80), (III.81) hold.

The strategy in order to prove lemma III.3 follows the one K. Uhlenbeck introduced in [34] to construct Coulomb gauges in critical dimensions. Precisely lemma III.3 is going to be a consequence of the following proposition.

**Proposition III.3.** There exist \( \Theta > 0 \) and \( \varepsilon > 0 \) so that \( V_{\varepsilon, \Theta} = U_{\varepsilon} \).

**Proof of Proposition III.3.**

Proposition III.3 follows, once we show the following four properties

(i) \( V_{\varepsilon} \) is connected.

(ii) \( V_{\varepsilon, \Theta} \) is nonempty.

(iii) For any \( \varepsilon, \Theta > 0 \), \( V_{\varepsilon, \Theta} \) is a relatively closed subset of \( U_{\varepsilon} \).

(iv) There exist \( \Theta > 0 \) and \( \varepsilon > 0 \) so that \( V_{\varepsilon, \Theta} \) is a relatively open subset of \( U_{\varepsilon} \).

Property (i) is clear, since \( U_{\varepsilon} \) is obviously starshaped with center 0.

Property (ii) is also obvious since : \( (0, 0) \in V_{\varepsilon, \Theta} \).

The closeness property (iii) follows almost verbatim from [12] Proof of Theorem 1.2, Step 1, p.1315: there one replaces \((-\Delta)^{1/4}\) by \( \nabla \). Observe that a uniform bound of the \( L^p \)-norm as in (III.81) implies by Sobolev embedding in particular a uniform bound of \( q - 1 \) in \( L^{\frac{p}{2}}(\mathbb{R}^2) \).

We show now the openness property (iv). For this let \( \omega_0, g_0 \) be arbitrary in \( V_{\varepsilon, \Theta} \), for some \( \varepsilon, \Theta > 0 \) chosen below.

Let \( q_0 \in \hat{W}^{1,1} \cap \hat{W}^{1,p'}(\mathbb{C}, \mathbb{H}) \), \( q_0 - 1 \in L^{\frac{2}{p}}(\mathbb{C}) \) so that the decomposition (III.76) as well as the estimates (III.79), (III.80) and (III.81) are satisfied for \( \omega_0 \) and \( g_0 \).

We consider perturbations of \( q_0 \) of the form \( q = q_0 e^u \) where \( u \in (\hat{W}^{1,1} \cap \hat{W}^{1,p'} \cap L^{\frac{2}{p}})(\mathbb{C}, \mathbb{H}_p) \). Observe that the exponent \( p > 2 \) has been chosen in particular to ensure \( u \in C^0 \cap L^{\infty}(\mathbb{C}) \) and \( q_0 e^u - 1 \in L^{\frac{p}{2}} \).

We set
\[ \hat{N}_0 : (\hat{W}^{1,1} \cap \hat{W}^{1,p'} \cap L^{\frac{2}{p}})(\mathbb{C}, \mathbb{H}_p) \rightarrow \left( \hat{i}(W^{-1,1} \cap W^{-1,1})(\mathbb{C}), L^p \cap L^{p'}(\mathbb{C}, \text{span}\{j, k\}) \right) \]
\[ u \mapsto \hat{N}(q_0 e^u). \]  

(III.82)

We will write \( u = u_1 i + u_2 j + u_3 k \).

---

5Indeed for a Schwartz function one has
\[ u(x) = \frac{1}{2\pi} \int_{\mathbb{C}} \nabla_x \log |x - y| : \nabla u(y) \, dy \Rightarrow \| u \|_{L^\infty} \leq (2\pi)^{-1} \| \nabla_x \log |x - y| \|_{L^{2,\infty}} \| \nabla u \|_{L^{2,1}} \]

Generalized Hölder inequality (see [19]) gives moreover
\[ \| \nabla u \|_{L^{2,1}}^2 \leq C \| \nabla u \|_{L^p} \| \nabla u \|_{L^{p'}}. \]

The statement \( u \in L^\infty \) follows by the density of Schwartz functions in \( W^{1,1} \cap W^{1,p'} \cap L^{\frac{2}{p}} \).
We have $\tilde{N}_{q_0} \in C^1$ and we can compute $D\tilde{N}_{q_0}(0)$ as

$$D\tilde{N}_{q_0}(0) = \left. \frac{d}{dt} \tilde{N}_{q_0}(tu) \right|_{t=0} = : L_{q_0}(u),$$

where for $u \in L^{2p_2} \cap \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{C}, H_p)$

$$L_{q_0}(u) := (\Pi_i(\Delta u) + \partial_{x_1}[q_0^{-1}\partial_{x_1}q_0 u - uq_0^{-1}\partial_{x_1}q_0] + \partial_{x_2}[q_0^{-1}\partial_{x_2}q_0 u - uq_0^{-1}\partial_{x_2}q_0]),$$

(III.83)

Indeed we have

$$\Pi_{jk}(\partial_{x_1}u - \partial_{x_2}u + [q_0^{-1}\partial_{x_1}q_0 u - uq_0^{-1}\partial_{x_1}q_0] - [q_0^{-1}\partial_{x_2}q_0 u - uq_0^{-1}\partial_{x_2}q_0]i) \quad .$$

In order to use a fixed-point argument for $\tilde{N}_{q_0}$, we will show that $L_{q_0}$ is an isomorphism. Precisely we prove the following lemma.

**Lemma III.4.** For any $\Theta > 0$ there exists $\varepsilon > 0$ so that the following holds for any $\omega_0, \alpha_0$ and $q_0$ as above.

For any $\omega \in i(W^{-1,p} \cap W^{-1,p'})(\mathbb{C}, \mathbb{R})$, $g \in (L^p \cap L^{p'})'(\mathbb{C}, \text{span}\{j, k\})$ there exists a unique $u \in L^{2p_2} \cap \dot{W}^{1,p} \cap \dot{W}^{1,p'}(\mathbb{C}, H_p)$ so that

$$(\omega, g) = L_{q_0}(u)$$

and for some constant $C = C(\omega_0, \alpha_0, \Theta) > 0$ it holds

$$\|u\|_{L^{2p_2}} + \|\nabla u\|_{L^p(\mathbb{C})} + \|\nabla u\|_{L^{p'}(\mathbb{C})} \leq C \left( \|\omega\|_{W^{-1,p}(\mathbb{C})} + \|\omega\|_{W^{-1,p'}(\mathbb{C})} \right) + (\|g\|_{L^p(\mathbb{C})} + \|g\|_{L^{p'}(\mathbb{C})}).$$

(III.84)

**Proof of lemma III.4.**

**Claim 1.** $L_1(u)$ is invertible $(q_0 = 1)$ as a map

$$L_1 : \dot{W}^{1,p'} \cap L^{2p_2}(\mathbb{C}, H_p) \to \left(iW^{-1,p'}(\mathbb{C}), L^p \cap L^{p'}(\mathbb{C}, \text{span}\{j, k\})\right)$$

Indeed we have

$$L_1(u) = \frac{d}{dt} N(e^{tu})_{t=0} = (\Pi_i(\Delta u), \Pi_{jk} (\partial_{x_1}u - \partial_{x_2}u))$$

(III.85)

Indeed

$$\Pi_{jk}(\partial_{x_1}u - \partial_{x_2}u) = (\Delta u_1, i, (\partial_{x_1}u_2 - \partial_{x_2}u_3)) + (\partial_{x_1}u_3 + \partial_{x_2}u_2)k \quad .$$

Given $f \in W^{-1,p}(\mathbb{C})$, $a, b \in L^p \cap L^{p'}(\mathbb{C}, \mathbb{R})$ there is a unique triple $u_1, u_2, u_3 \in \dot{W}^{1,p}(\mathbb{C}) \cap \dot{W}^{1,p'}(\mathbb{C}) \cap L^{2p_2}$ such that

$$L_1(u) = (f, aj + bk) \quad .$$

Indeed

$$\begin{cases}
\Delta u_1 = f \\
\partial_x(u_2 - u_3 i) = a - bi
\end{cases}$$

(III.86)

$$u_1(x) = \frac{1}{2\pi} \log(|x|) \ast f(x), \quad u_2 - u_3 i = \frac{1}{4\pi} (a - bi) \ast \frac{x}{|x|} \quad .$$

Classical estimates give

$$\|\nabla u_1\|_{L^{p'}} \lesssim \|f\|_{W^{-1,p'}}, \quad \|u_2 - u_3 i\|_{L^{2p_2}} + \|\nabla(u_2 - u_3 i)\|_{L^{p'}} \lesssim \|a - bi\|_{L^{p'}} \quad .$$

The claim is proved.
Observe that
\[ L_{q_0}(u) - L_1(u) = (\partial_x [q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0] + \partial_x [q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0], \]
\[ \Pi_{jk} \left( \frac{q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0}{2} \right). \]
We have
\[ \| \partial_x \left( q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0 \right) \|_{W^{-1,p'}} \leq \| q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0 \|_{L^{p'}} \]
\[ \leq \| \nabla q_0 \|_{L^{2}} \| u \|_{L^{\frac{2p}{2p-1}}} \leq \varepsilon \| u \|_{L^{\frac{2p}{2p-1}}} \] (III.87)
Choosing \( \varepsilon > 0 \) small enough (depending on \( \Theta \)) we obtain that \( L_{q_0} \) is an invertible map from \( \hat{W}_1^{-1,p'}(\mathbb{C},\mathbb{R}) \) to \( (i(W^{-1,p} \cap W^{-1,p'})(\mathbb{C},\mathbb{R}), g \in (L^{p} \cap L^{p'})(\mathbb{C},\text{span}\{j,k\}) \) we prove that the unique solution \( u \) of \( L_{q_0}(u) \) is in \( W_1^{1,p} \).

From the fact that \( (w,i,\beta) = \hat{N}_{q_0}(u) \) it follows:
\[ \Delta u_1 = \omega + i \Pi \left( \sum_{k=1}^{2} \partial_x \left( q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0 \right) \right) \]
\[ \partial_x (u_2 - u_3) = -j \Theta - j \Pi \Pi \left( \sum_{k=1}^{2} \partial_x \left( q_0^{-1} \partial_x q_0 u - u q_0^{-1} \partial_x q_0 \right) \right). \] (III.88)
Let \( p' < r < 2 \), since \( \nabla q_0 \in L^{p} \) we have for \( \ell = 1,2 \)
\[ \| q_0^{-1} \partial_x q_0 u \|_{L^{p'}} \lesssim \| \nabla q_0 \|_{L^{p}} \| u \|_{L^{\frac{2p}{2p-1}}} \] (III.89)
for \( \frac{1}{p} = \frac{1}{p'} + \frac{2r}{2p-1} \). Observe that \( p > t > 2 \), since \( r > p' \).

From (III.83) and (III.84) it follows \( \nabla u \in L^{t} \). We have also \( \nabla u \in L^{p'} \). This implies that \( u \in L^{\infty} \) (see previous footnote). Therefore
\[ \| q_0^{-1} \partial_x q_0 u \|_{L^{p'}} \lesssim \| \nabla q_0 \|_{L^{p}} \| u \|_{L^{\infty}} \] (III.90)
From (III.88) it follows that \( \nabla u \in L^{p} \) and the Claim 2 is proved. This concludes the proof of lemma III.3.

\[ \square \]

Proof of proposition III.3 continued.

For \( \varepsilon = \varepsilon(\Theta) > 0 \) chosen small enough, and for any \( (\omega_0, g_0) \in \mathcal{V}_{\varepsilon,\Theta} \) the local inversion theorem applied to \( \mathbb{N} \) some \( \delta > 0 \) (that might depend on \( (\omega_0, g_0) \)) such that, for any \( (\omega, g) \in \mathcal{U}_{\varepsilon} \) with
\[ \| \omega - \omega_0 \|_{W^{-1,p}(\mathbb{C})} + \| \omega - \omega_0 \|_{W^{-1,p'}(\mathbb{C})} < \delta \] (III.91)
\[ \| g - g_0 \|_{L^{p}(\mathbb{C})} + \| g - g_0 \|_{L^{p'}(\mathbb{C})} < \delta, \] (III.92)
we find \( q = q_0 c^2 u \in \hat{W}_1^{1,p} \cap \hat{W}_1^{1,p'}(\mathbb{C}, S(\mathbb{H})) \), so that \( q - 1 \in L_{\frac{2p}{2p-1}}(\mathbb{C}, \mathbb{H}) \) and (III.76) is satisfied.

It remains to prove (III.79), (III.80) and (III.81). This will be implied by the following lemma.

**Lemma III.5.** There exists a \( \Theta > 0 \) and a \( \sigma > 0 \) so that whenever \( q \in \hat{W}_1^{1,p} \cap \hat{W}_1^{1,p'}(\mathbb{C}, \mathcal{U}(\mathbb{H})) \) and \( q - 1 \in L_{\frac{2p}{2p-1}}(\mathbb{C}, \mathbb{H}) \) so that (III.76) is satisfied and it holds
\[ \| \nabla q \|_{L^{2}(\mathbb{C})} \leq \sigma, \] (III.93)
then (III.79), (III.80) and (III.81) hold. \[ \square \]
Choosing \( \Theta = \frac{C}{\sigma} \) we have
\[
\| \nabla q \|_{L^q} \leq \Theta (\| \omega \|_{W^{-1,q}} + \| g \|_{L^q}).
\]
This concludes the proof of lemma [III.5].

End of the proof of Proposition [III.3].

Thanks to lemma [III.5] the openness property (iv) is proven, Proposition [III.3] is then established.

Proof of Theorem [III.7]. Let \( q \in N^{-1}(0, -\partial_x \alpha_j + \partial_x \alpha_k) \) and \( \| \nabla q \|_{L^2} \leq \Theta \| \nabla \omega \|_{L^2}. \) By definition \( q \) satisfies
\[
\Pi_i (\partial_{x_i} (q^{-1} \partial_{x_i} q) + \partial_{x_j} (q^{-1} \partial_{x_j} q)) = 0
\]
\[
\Pi_{jk} (q^{-1} \partial_{x_j} q - q^{-1} \partial_{x_k} q) i = -\partial_{x_i} \alpha_j + \partial_{x_j} \alpha_k.
\]
We analyze the first equation in (III.101).

We have
\[
\Pi_i (\partial_{x_i} (q^{-1} \partial_{x_i} q) + \partial_{x_j} (q^{-1} \partial_{x_j} q)) = 0
\]
\[
\Pi_i (\partial_{x_i} (\Pi_j (q^{-1} \partial_{x_j} q) + \partial_{x_j} (q^{-1} \partial_{x_j} q))) = 0.
\]

---

6We use the fact that if \( \nabla \omega \in L^{2, \infty}, \nabla b \in L^q \), with \( q \in [p', \bar{p}] \) and if \( -\Delta \phi = \nabla \alpha \cdot \nabla b \) in \( C \) then \( \nabla \phi \in L^q \) with \( \| \nabla \phi \|_{L^q} \leq C_q \| \nabla b \|_{L^q} \| \nabla \alpha \|_{L^{2, \infty}}. \) The constant \( C_q \) is uniformly bounded if \( q \in [p', \bar{p}] \) (see [22]).
Therefore there exists \( \zeta \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{R}) \) such that

\[
\begin{aligned}
\Pi_i(q^{-1}\partial x_i q) &= -\partial x_j \zeta \\
\Pi_i(q^{-1}\partial x_q q) &= \partial x_j \zeta .
\end{aligned}
\]  

(III.103)

From (III.103) it follows in particular that

\[
-\Delta \zeta = \partial x_2 (\Pi_i(q^{-1}\partial x_i q)) - \partial x_1 (\Pi_i(q^{-1}\partial x_q q)) = \Pi_i (\partial x_2 (q^{-1}\partial x_1 q) - \partial x_1 (q^{-1}\partial x_2 q)) .
\]  

(III.104)

The right hand side of (III.105) is a sum of jacobians, hence it is in the Hardy space \( \mathcal{H}^1(\mathbb{C}) \). It follows in particular that \( \nabla \zeta \in L^{2,1}(\mathbb{C}) \), with

\[
\| \nabla \zeta \|_{L^{2,1}} \lesssim \| \nabla q \|_{L^2} .
\]

We have

\[
q^{-1}\partial x_i q - q^{-1}\partial x_q q = \Pi_i (q^{-1}\partial x_i q) + \Pi_{jk} (q^{-1}\partial x_i q) - \Pi_i (q^{-1}\partial x_q q) - \Pi_{jk} (q^{-1}\partial x_q q) = -\partial x_2 \zeta - \partial x_1 \zeta i + \Pi_{jk} (q^{-1}\partial x_1 q - q^{-1}\partial x_2 q) .
\]  

(III.105)

\[\text{In (III.105) we use the fact that } \Pi_{jk} (ai) = \Pi_{jk} (a) i \text{ for } a \in \mathbb{R}. \]

By combining (III.73), (III.101) and (III.105) we get

\[
\partial x_1 [qf] - \partial x_2 [qi f] = -2 q i \{ \partial x_2 [q \zeta] f \} .
\]  

(III.106)

We set

\[
-\Delta A = 2 q i \{ \partial x_2 [q \zeta] f \} .
\]  

(III.107)

Observe that

\[
\| \nabla A \|_{L^{2,2}} \lesssim \| qf \nabla \zeta \|_{L^1} \lesssim \| \nabla \zeta \|_{L^{2,1}} \| qf \|_{L^2} \lesssim \| qf \|_{L^{2,2}} \| \nabla \zeta \|_{L^{2,2}} \lesssim \varepsilon_0^2 \| qf \|_{L^{2,2}} .
\]  

(III.108)

Since

\[
\partial x_1 (qf - \partial x_1 A) - \partial x_2 (q i f + \partial x_2 A) = 0 ,
\]

there exists \( B \in \dot{W}^{1,(2,\infty)} \) such that

\[
\begin{aligned}
qf - \partial x_1 A &= -\partial x_2 B \\
-qi f - \partial x_2 A &= \partial x_1 B .
\end{aligned}
\]  

(III.109)

Therefore

\[
\begin{aligned}
f &= q^{-1}(\partial x_1 A - \partial x_2 B) \\
f &= i q^{-1}(\partial x_2 A + \partial x_1 B) .
\end{aligned}
\]  

(III.110)
From (III.110) it follows
\[
\begin{align*}
\partial_{x_1} A - \partial_{x_2} B &= q i q^{-1}(\partial_{x_2} A + \partial_{x_1} B) \\
-\partial_{x_2} A - \partial_{x_1} B &= q i q^{-1}(\partial_{x_1} A - \partial_{x_2} B) \\
-\partial_{x_2} B &= q i q^{-1}(\partial_{x_2} A + \partial_{x_1} B) - \partial_{x_1} A \\
-\partial_{x_1} B &= q i q^{-1}(\partial_{x_1} A - \partial_{x_2} B) + \partial_{x_2} A \\
-\Delta B &= \partial_{x_1}(q i q^{-1}(\partial_{x_1} A - \partial_{x_2} B)) + \partial_{x_2}(q i q^{-1}(\partial_{x_2} A + \partial_{x_1} B))
\end{align*}
\]

We observe that \(-\partial_{x_1}[q i q^{-1}\partial_{x_2} B] + \partial_{x_2}[q i q^{-1}\partial_{x_1} B]\) is sum of Jacobians and therefore we can apply Wente’s Lemma (case \(L^2 - L^{2,\infty}\)):
\[
\|\nabla B\|_{L^{2,\infty}} \lesssim \|\nabla A\|_{L^{2,\infty}} + \|\nabla q\|_{L^2} \|\nabla B\|_{L^{2,\infty}}
\]
\[
\lesssim \|\nabla A\|_{L^{2,\infty}} + \epsilon_0 \|\nabla B\|_{L^{2,\infty}}.
\]

(III.112)

Estimate (III.112) implies that
\[
\|\nabla B\|_{L^{2,\infty}} \lesssim \|\nabla A\|_{L^{2,\infty}}
\]
\[
\|q f\|_{L^{2,\infty}} \lesssim \|\nabla A\|_{L^{2,\infty}} + \|\nabla B\|_{L^{2,\infty}}
\]
\[
\lesssim \epsilon_0 \|f\|_{L^{2,\infty}}.
\]

(III.113)

If \(\epsilon_0\) is small enough then \(f \equiv 0\). This concludes the proof of Theorem (III.7) \(\Box\)

III.3 Morrey-Type Estimates

In this section we prove Morrey-type estimates for solutions to (III.29) in the case \(n = 2\).

Proposition III.4. Let \(S \in W^{1,2}(C, O(2))\) with \(S^2 = 1\) and \(u \in L^2(C)\) be a solution of (III.29). Then \(u \in W^{1,p}_{loc}\) for every \(p \in [1, 2)\). \(\Box\)

Proof of Proposition III.4. Step 1. Assume that \(\|\nabla S\|_{L^1(B(0, 1))} \leq \epsilon_0\). By arguing as in the previous section we can find \(q \in W^{1,2}(B(0,1), \mathcal{D}'(\mathbb{R}^2))\) such that \(\|\nabla q\|_{L^1(B(0, 1))} \leq C \|\nabla S\|_{L^2(B(0, 1))}\) and

\[
\partial_{x_1} [q f] - \partial_{x_2} [q i f] = -2 q i [\partial_{x_2} \zeta] f \quad \text{in} \quad \mathcal{D}'(B(0, 1))
\]

(III.114)

with
\[
\|\partial_{x_2} \zeta\|_{L^1(B(0, 1))} \leq \epsilon_0.
\]

We set
\[
-\Delta A = 2 q i [\partial_{x_2} \zeta] f.
\]

(III.115)

We have
\[
\|\nabla A\|_{L^{2,\infty}(B(0, r))} \lesssim \epsilon_0^2 \|f\|_{L^{2,\infty}(B(0, r))}.
\]

(III.116)
Since
\[ \partial_{x_1}(q \partial_x - \partial_{x_1}A) - \partial_{x_2}(q \partial_x + \partial_{x_2}A) = 0, \]
there exists \( B \in W^{1,2}(\cdot, \infty) \) such that
\[
\begin{cases}
q \partial_x - \partial_{x_1}A = -\partial_{x_2}B \\
-q \partial_x - \partial_{x_2}A = \partial_{x_1}B.
\end{cases}
\]  
We have
\[
-\Delta B = \partial_{x_1} (q \partial_x^{-1}(\partial_{x_1}A - \partial_{x_2}B)) + \partial_{x_2} (q \partial_x^{-1}(\partial_{x_2}A + \partial_{x_1}B)) \quad \text{in} \quad D'(B(0,1))
\]  
Let \( x \in B(0,1/2) \) and \( 0 < r < 1/4 \). We decompose \( B = \beta_1 + \beta_2 \) in \( B(x,r) \) where \( \beta_1 \) and \( \beta_2 \) satisfy respectively
\[
\begin{cases}
\Delta \beta_1 = 0 & \text{in } B(0,r) \\
\beta_1 = B & \text{on } \partial B(0,r)
\end{cases} \quad \text{and} \quad \begin{cases}
\Delta \beta_2 = \Delta B & \text{in } B(0,r) \\
\beta_2 = 0 & \text{on } \partial B(0,r)
\end{cases}
\]  
The following estimates hold:

**Estimate of \( \beta_2 \):**

Wente inequality \((L^2, \infty - L^2 \text{ case})\) combined with classical Calderon Zygmund inequalities give
\[
\| \nabla \beta_2 \|_{L^2, \infty(B(x,r))} \leq \| \nabla A \|_{L^2, \infty(B(x,r))} + \varepsilon^2_0 \| \nabla B \|_{L^2, \infty(B(x,r))}. 
\]  

**Estimate of \( \beta_1 \):**

Since \( \beta_1 \) is harmonic, for every \( \delta > 0 \) we have
\[
\| \beta_1 \|_{L^2, \infty(B(x,\delta r))} \leq \| \beta_2 \|_{L^2(B(x,\delta r))} \leq \left( \frac{4\delta}{3} \right)^2 \| \beta_1 \|_{L^2(B(x,\delta/4r))} \leq C \left( \frac{4\delta}{3} \right)^2 \| \beta_1 \|_{L^2, \infty(B(x,r))},
\]  
where \( C \) is a constant independent of \( r \).

**Estimate of \( B \):**

Combining the previous estimates we obtain
\[
\| \nabla B \|_{L^2, \infty(B(x,\delta r))} \leq \| \beta_1 \|_{L^2, \infty(B(x,\delta r))} + \| \beta_2 \|_{L^2, \infty(B(x,\delta r))} \leq \left( \frac{4\delta}{3} \right)^2 \| \beta_1 \|_{L^2, \infty(B(x,r))} + \| \nabla A \|_{L^2, \infty(B(x,r))} + \varepsilon^2_0 \| \nabla B \|_{L^2, \infty(B(x,r))} \leq \left( \frac{4\delta}{3} \right)^2 \left[ \| \nabla A \|_{L^2, \infty(B(x,r))} + \varepsilon^2_0 \| \nabla B \|_{L^2, \infty(B(x,r))} \right] + \| \nabla A \|_{L^2, \infty(B(x,r))} + \varepsilon^2_0 \| \nabla B \|_{L^2, \infty(B(x,r))}.
\]  

(III.122)
From (III.122) it follows that
\[
\|f\|_{L^{2,\infty}(B(x,\delta r))} \lesssim \|\nabla A\|_{L^{2,\infty}(B(x,\delta r))} + \|\nabla B\|_{L^{2,\infty}(B(x,\delta r))} \\
\leq \varepsilon_0^2 \|f\|_{L^{2,\infty}(B(0,r))} + \left(\frac{\varepsilon_0^2}{3}\right) \left[\|\nabla A\|_{L^{2,\infty}(B(x,\delta r))} + \varepsilon_0^2 \|\nabla B\|_{L^{2,\infty}(B(x,\delta r))}\right] \\
+ \|\nabla A\|_{L^{2,\infty}(B(x,r))} + \varepsilon_0^2 \|\nabla B\|_{L^{2,\infty}(B(x,r))} \\
\lesssim \gamma \|f\|_{L^{2,\infty}(B(0,r))}.
\] (III.123)
where \(\gamma = \gamma(\delta, \varepsilon_0) < 1\). By iterating (III.123) we get the existence of a constant \(0 < \alpha < 1\) such that
\[
\sup_{x \in B(0,1/2), 0 < r < 1/4} r^{-\alpha} \|f\|_{L^{2,\infty}(B(x,r))} < +\infty.
\] (III.124)

Now we plug the estimate (III.124) into (III.115) and we get
\[
\sup_{x \in B(0,1/2), 0 < r < 1/4} r^{-\alpha} \|\Delta A\|_{L^1(B(x,r))} < +\infty.
\] (III.125)
and therefore using the main result of [1]
\[
\sup_{x \in B(0,1/2), 0 < r < 1/4} r^{-\alpha} \|\nabla A\|_{L^{2,\infty}(B(x,r))} < +\infty.
\] (III.126)

From (III.125) it follows in particular that \(\nabla A \in L^q(B(0,1/4))\) for all \(q < \frac{2-\alpha}{4-\alpha}\) (See again Adams [1], Remark after Proposition 3.2).

From (III.117), (III.124), (III.126) it follows that
\[
\sup_{x \in B(0,1/2), 0 < r < 1/4} r^{-\alpha} \|\nabla B\|_{L^{2,\infty}(B(x,r))} < +\infty.
\] (III.127)

By plugging (III.127) into (III.118) and (III.119) one gets that
\[
\sup_{x \in B(0,1/2), 0 < r < 1/4} r^{-\alpha} \|\Delta B\|_{L^{2,\infty}(B(x,r))} < +\infty.
\] (III.128)
which implies that \(\nabla B \in L^q(B(0,1/4))\) for all \(q < \frac{2-\alpha}{4-\alpha}\) as well. Therefore \(f \in L^q(B(0,1/4))\) for all \(q < \frac{2-\alpha}{4-\alpha}\) as well. Actually one can show by bootstrap arguments that \(f \in L^q_{\text{loc}}\) for all \(q < +\infty\).

**Step 2.** From Step 1 it follows that \(Su \in L^q_{\text{loc}}(\mathbb{C})\) for all \(q < +\infty\). Since \(u\) solves (III.29) we have
\[
\Delta(Su) = \text{div}(\nabla(Su)) = \text{div}(\nabla SSu) \text{ in } D'(\mathbb{C})
\] (III.129)
From (III.129) one gets that \(\nabla(Su) \in L^q_{\text{loc}}\) for all \(q < +\infty\) and therefore \(\nabla u = \nabla(Su) - S\nabla(Su) \in L^p_{\text{loc}}\) for all \(p < 2\). This concludes the proof of proposition (III.4) which itself implies theorem (I.1) in the \(n = 2\) case. 

**IV Proof of theorem I.1: the general case \(n \geq 2\).**

We are going to present here another approach to study the regularity of the equation (III.29) which works for every \(n \geq 2\). We start by showing the bootstrap test:
Theorem IV.8. Let $S \in \dot{W}^{1,2}(\mathbb{C}, O(n))$ with $S^2 = 1$ and $u \in L^2(\mathbb{C}, \mathbb{R}^n)$ be a solution to the equation (III.29). There is $\varepsilon_0 > 0$ such that if $\|\nabla S\|_{L^2(\mathbb{C})} \leq \varepsilon_0$ then $u \equiv 0$.

Proof of theorem IV.8. From Lemma II.1 we can find $v \in L^2(\mathbb{C}, \mathbb{R}^n)$ such that $\nabla v = S \nabla u$. Assume that $\|\nabla S\|_{L^2(\mathbb{C})} \leq \varepsilon_0$ where $\varepsilon_0$ is the constant appearing in Theorem III.6. Then there is $Q \in \dot{W}^{1,2}(\mathbb{C}, SO(n))$ such that

$$S = Q^{-1} S^0 Q$$

where $S^0$ is the matrix (III.32).

We set

$$f = f_R + i f_\Im = S^0 Q u + i Q v.$$ 

Equation (III.50) is equivalent to the system:

$$
\begin{align*}
\partial_{x_1} f_R + \partial_{x_2} f_\Im &= S^0 \partial_{x_1} Q Q^{-1} S^0 f_R + \partial_{x_2} Q Q^{-1} f_\Im \\
- \partial_{x_2} f_R + \partial_{x_1} f_\Im &= - S^0 \partial_{x_2} Q Q^{-1} S^0 f_R + \partial_{x_1} Q Q^{-1} f_\Im
\end{align*}
$$

(IV.130)

We can write

$$S^0 = \left( (-1)^{\min(2m+1,2i)} \delta_{ij} \right)_{1 \leq i,j \leq n}.$$

Let $\Omega = (\omega_{ij})_{1 \leq i,j \leq n}$ be an anti-symmetric real matrix (i.e. $\omega_{ij} = -\omega_{ji}$), then

$$\tilde{\Omega} = S^0 \Omega S^0 = \left( \omega_{ij} (-1)^{\min(2m+1,2i)+\min(2m+1,2j)} \right).$$

Therefore

$$\begin{align*}
\tilde{\omega}_{ij} = \omega_{ij} &\iff i,j \leq m \text{ and } i,j > m \\
\omega_{ij} = -\omega_{ji} &\iff \text{ otherwise .}
\end{align*}
$$

(IV.131)

Observe that the matrix $\tilde{\Omega}$ is still anti-symmetric. We set $\Omega^t := \partial_{x_i} Q Q^{-1}$ and $\tilde{\Omega}^t = S^0 \partial_{x_i} Q Q^{-1} S^0$.

$$
\begin{align*}
\partial_{x_1} f_R + \partial_{x_2} f_\Im &= \tilde{\Omega}^1 f_R + \Omega^2 f_\Im \\
\partial_{x_2} f_R - \partial_{x_1} f_\Im &= \tilde{\Omega}^2 f_R - \Omega^1 f_\Im
\end{align*}
$$

(IV.132)

Then we get

$$\left( \partial_{x_1} - i \partial_{x_2} \right) (f_R + i f_\Im) = \tilde{\Omega}^1 f_R + \Omega^2 f_\Im - i (\tilde{\Omega}^2 f_R - \Omega^1 f_\Im)$$

$$= \left( \tilde{\Omega}^1 - i \tilde{\Omega}^2 \right) f_R + i \left( \Omega^1 - i \Omega^2 \right) f_\Im$$

$$= \left( \frac{\tilde{\Omega}^1 - i \tilde{\Omega}^2}{2} \right) \left( (f_R + i f_\Im) + (f_R - i f_\Im) \right)$$

$$+ \left( \frac{\Omega^1 - i \Omega^2}{2} \right) \left( (f_R + i f_\Im) - (f_R - i f_\Im) \right).$$

(IV.133)
Which gives
\[
(\partial_{x_1} - i \partial_{x_2})(f_\mathcal{R} + if_\mathcal{I}) = \frac{1}{2} \left[ (\tilde{\Omega}^1 + \Omega_1) - i(\tilde{\Omega}^2 + \Omega_2) \right] (f_\mathcal{R} + if_\mathcal{I}) + \frac{1}{2} \left[ (\tilde{\Omega}^1 - \Omega_1) - i(\tilde{\Omega}^2 - \Omega_2) \right] (f_\mathcal{R} - if_\mathcal{I}) .
\] (IV.134)

From (IV.131) it follows for \( \ell = 1, 2 \)
\[
\frac{\tilde{\Omega}^\ell + \Omega^\ell}{2} = \begin{pmatrix}
\omega_{ij}^\ell & 0_{m \times n - m} \\
0_{n - m \times m} & \omega_{ij}^\ell
\end{pmatrix}
\] (IV.135)
and
\[
\frac{\tilde{\Omega}^\ell - \Omega^\ell}{2} = \begin{pmatrix}
0_{m \times m} & -\omega_{ij}^\ell \\
-\omega_{ij}^\ell & 0_{n - m \times n - m}
\end{pmatrix} .
\] (IV.136)

We can write the system (IV.134) as
\[
\partial z f = \frac{1}{2} \Omega^+ f + \frac{1}{2} \Omega^- \bar{f} ,
\] (IV.137)
where
\[
\Omega^+ = \frac{(\tilde{\Omega}^1 + \Omega_1) - i(\tilde{\Omega}^2 + \Omega_2)}{2} \quad \text{and} \quad \Omega^- = \frac{(\tilde{\Omega}^1 - \Omega_1) - i(\tilde{\Omega}^2 - \Omega_2)}{2} .
\] (IV.138) (IV.139)

We observe that by construction for every \( i, j \) we have
\[
\partial_{x_2} (\Omega_{ij}^+) - \partial_{x_1} (\Omega_{ij}^+) \in \mathcal{H}^1(\mathbb{R}^2) \quad \text{and} \quad \partial_{x_2} (\Omega_{ij}^-) - \partial_{x_1} (\Omega_{ij}^-) \in \mathcal{H}^1(\mathbb{R}^2)
\] (IV.140)
since these quantities are linear combinations of Jacobians of functions with gradient in \( L^2 \).

Let \( Q \) be defined as follows:
\[
Q = \begin{pmatrix}
Q_C & 0_{m \times n - m} \\
0_{n - m \times m} & Q_L
\end{pmatrix}
\] (IV.141)
where \( Q_C \in \dot{W}^{1,2}(\mathbb{R}^2, SO(m)) \) and \( Q_L \in \dot{W}^{1,2}(\mathbb{R}^2, SO(n - m)) \). The following identity hold
\[
\partial z (Qf) = \partial z Qf + Q \partial z f = (\partial z QQ^{-1}) (Qf) + \frac{1}{2} Q (\Omega^+ f + \Omega^- \bar{f})
\]
\[
= [\partial z QQ^{-1} + \frac{1}{2} Q \Omega^+ Q^{-1}] Q f + \frac{1}{2} (Q \Omega^- Q^{-1}) \bar{Q} \overline{f} .
\] (IV.142)

Claim 1: There are two constants \( \varepsilon(n) > 0 \) and \( C(n) > 0 \) depending only on \( n \) such that if \( ||\nabla \Omega^+||_{L^2} < \varepsilon(n) \) there exist a matrix \( Q \) of the form (IV.141) and \( \eta \in \dot{W}^{1,2,1}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) such that
\[
\partial z QQ^{-1} + \frac{1}{2} Q \Omega^+ Q^{-1} = -i \partial z \eta .
\]
Proof of Claim 1. By the same arguments in Lemma A.3 in [24] we can find $Q \in \tilde{W}^{1,2}(\mathbb{C}, SO(n))$ of the form (IV.141) and $\eta \in \tilde{W}^{1,2}(\mathbb{C}, so(n))$ such that
\[
\begin{align*}
-\partial_{x_2}\eta &= \partial_{x_1}QQ^{-1} + Q[\Omega_1 + \Omega_1]Q^{-1} \\
\partial_{x_1}\eta &= \partial_{x_2}QQ^{-1} + Q[\Omega_2 + \Omega_2]Q^{-1}.
\end{align*}
\] (IV.143)

It follows that
\[
-\Delta\eta = \frac{\partial_{x_1}(\partial_{x_1}QQ^{-1}) - \partial_{x_1}(\partial_{x_2}QQ^{-1})}{(1)} + \frac{\partial_{x_2}[Q[\Omega_1 + \Omega_1]Q^{-1}] - \partial_{x_1}[Q[\Omega_2 + \Omega_2]Q^{-1}]}{(2)}.
\] (IV.144) (IV.145)

The first term (1) on the right hand side of (IV.144) is in the Hardy Space $H^1(\mathbb{R}^2)$ since it is a linear combination of Jacobians of functions with derivative in $L^2$.

Regarding the second term in (2) we now show that it is in $W^{-1,2}(\mathbb{R}^2)$. Indeed we observe that each component of (2) can be written in the form
\[
\partial_{x_1}(a\omega_1) - \partial_{x_1}(a\omega_2)
\] (IV.146)

where $a \in (\tilde{W}^{1,2} \cap L^\infty)(\mathbb{R}^2)$ and $\omega_1, \omega_2 \in L^2(\mathbb{R}^2)$ satisfy
\[
\partial_{x_2}\omega_1 - \partial_{x_1}\omega_2 \in H^1(\mathbb{R}^2).
\] (IV.147)

Let $c, b \in W^{1,2}(\mathbb{R}^2)$ be such that
\[
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix} = \nabla c + \nabla b.
\]

We can deduce from (IV.147) that $\Delta c \in H^1(\mathbb{R}^2)$ hence $c \in \tilde{W}^{1,2}(\mathbb{R}^2)$. We can now rewrite (IV.146) as follows
\[
\partial_{x_2}[a(\partial_{x_1}b - \partial_{x_2}c)] - \partial_{x_1}[a(\partial_{x_2}b - \partial_{x_1}c)] = \partial_{x_2}a\partial_{x_1}b - \partial_{x_1}a\partial_{x_2}b + \partial_{x_1}[a\partial_{x_1}c] - \partial_{x_2}[a\partial_{x_2}c].
\] (IV.148) (IV.149)

We observe that $\partial_{x_2}a\partial_{x_1}b - \partial_{x_1}a\partial_{x_2}b \in H^1(\mathbb{R}^2)$ and $\partial_{x_1}[a\partial_{x_1}c] - \partial_{x_2}[a\partial_{x_2}c] \in W^{-1,2}(\mathbb{R}^2)$. This gives that (2) is in $W^{-1,2}(\mathbb{R}^2)$, and this concludes the proof of claim 1.

The system (IV.142) can then be written as
\[
\partial_z(Qf) = A(Qf) + B\overline{f}
\] (IV.150)

where $A \in L^{2,1}(\mathbb{R}^2, \mathcal{M}_{n \times n}(\mathbb{C}))$ and $B \in L^2(\mathbb{R}^2, \mathcal{M}_{n \times n}(\mathbb{C}))$ satisfying for every $i, j$ $B_{ij} = -B_{ji}$ and
\[
\partial_{x_2}\left(B^\mathbb{R}_{ij}\right) - \partial_{x_1}\left(B^\mathbb{C}_{ij}\right) \in W^{-1,2}(\mathbb{R}^2).
\] (IV.151)

In the sequel we are going to study a system of the type:
\[
\partial_z g = Ag + B\overline{g}
\] (IV.152)
where \( A \in L^{2,1}(\mathbb{R}^2, \mathcal{M}_{n\times n}(\mathbb{C})) \) and \( B \in L^2(\mathbb{R}^2, \mathcal{M}_{n\times n}(\mathbb{C})) \) satisfying \( B_{ij} = -B_{ji} \) and \((IV.151)\).

**Step 1.** We first observe that
\[
\partial_z g = Ag - B_j g_j
\]
(IV.153)
where \( j \) is the quaternion number satisfying \( j^2 = -1 \) and \( ij = -ji \).

**Step 2.** The function \( g_j \) satisfies the system
\[
\partial_z g_j = Ag_j + B_j g.
\]
(IV.154)

**Step 3.** We set
\[
G = \begin{pmatrix} g^1 \\ \vdots \\ g^n \\ g^1 j \\ \vdots \\ g^n j \end{pmatrix}
\]
(IV.155)
\[
G \text{ satisfies } \partial_z G = \Omega G + \Omega_1 G
\]
(IV.156)
where
\[
\Omega_1 = \Omega_1^R - i\Omega_1^3 = \begin{pmatrix} A & 0_{n\times n} \\ 0_{n\times n} & A \end{pmatrix},
\]
(IV.157)
and
\[
\Omega = \Omega^R - i\Omega^3 = \begin{pmatrix} 0_{n\times n} & -B_j \\ B_j & 0_{n\times n} \end{pmatrix},
\]
(IV.158)
where we have set
\[
\Omega^R := \begin{pmatrix} 0_{n\times n} & (-B_j)^R \\ B_j^R & 0_{n\times n} \end{pmatrix} \text{ and } \quad \Omega^3 := \begin{pmatrix} 0_{n\times n} & (-B_j)^3 \\ B_j^3 & 0_{n\times n} \end{pmatrix}.
\]
Observe that
\[
\overline{\Omega^R} := \begin{pmatrix} 0_{n\times n} & B_j^R \\ -B_j^R & 0_{n\times n} \end{pmatrix}
\]
and then
\[
(\overline{\Omega^R})^t = \begin{pmatrix} 0_{n\times n} & -(B_j)^R \\ (B_j)^R & 0_{n\times n} \end{pmatrix} = -\Omega^R
\]
Therefore
\[
(\overline{\Omega^R})^t + \Omega^R = 0.
\]
(IV.159)
Similarly \((\Omega^3)^t + \Omega^3 = 0\). Recall the general rule for the conjugacy operation in \(\mathbb{H}\): \(i\Omega^3 = -\Omega^3 i\). Since we have that the coefficients of \(\Omega^3\) are in \(j\mathbb{R}\), we obtain \(i\Omega^3 = -\Omega^3 i = i\Omega^3\). Hence finally we have established

\[
(i\Omega^3)^t = i(\Omega^3)^t = -i\Omega^3
\]

The matrix \(\Omega = \Omega^R - i\Omega^I\) satisfies then \((\Omega)^t + \Omega = 0\) which means that it belongs to the Lie algebra of the hyper-unitary group: the compact Lie group \(U(n, \mathbb{H})\) of invertible \(n \times n\) quaternions matrices \(D\) satisfying \(D^* D = D D^* = \text{Id}_n\).

**Lemma IV.6.** Let \(n \in \mathbb{N}\). There are two constants \(\varepsilon(n) > 0\) and \(C(n) > 0\) depending only on \(n\) such that for each vector \(\Omega \in L^2(\mathbb{R}^2, u(n, \mathbb{H}))\) with \(\|\nabla \Omega\|_{L^2} < \varepsilon_0\) there exist \(P \in U(n, \mathbb{H})\) and \(\xi \in W^{1, 2}(\mathbb{R}^2, u(n, \mathbb{H}))\) such that

\[
\begin{align*}
2\Omega^R &= -P^{-1}\partial_{x_1}P - P^{-1}\partial_{x_2}\xi P \\
2\Omega^I &= -P^{-1}\partial_{x_1}\xi P + P^{-1}\partial_{x_2}P
\end{align*}
\]

and

\[
\|\nabla P\|_{L^2} + \|\nabla \xi\|_{L^2} \leq C\|\nabla \Omega\|_{L^2}.
\]

\(\square\)

For the proof of Lemma IV.6 we refer either to Lemma A.3 in [24] or to Lemmas 2.2 and 2.4 in [30]. We just point out that in this case one replace \(so(n)\) and \(SO(n)\) respectively with \(u(n, \mathbb{H})\) and \(U(n, \mathbb{K})\).

We observe that in the case \(\Omega\) is given by (IV.158) then \(\xi \in W^{1, 2}(\mathbb{R}^2, u(n, \mathbb{H}))\) be as in Lemma IV.6. Then the following estimate holds

\[
\begin{align*}
\partial_{x_1}(PG) &= (\partial_{x_1}PP^{-1} + 2P\Omega^R P^{-1})PG + (P\Omega^I P^{-1})PG \\
\partial_{x_2}(PG) &= (\partial_{x_2}PP^{-1} + 2P\Omega^I P^{-1})PG + (P\Omega^I P^{-1})PG.
\end{align*}
\]

(IV.160) - (IV.161)

From (IV.160) - (IV.161) we deduce that

\[
\begin{align*}
\partial_{x_1}(PG) &= (-\partial_{x_2}\xi + P\Omega^I P^{-1})PG \\
\partial_{x_2}(PG) &= (\partial_{x_1}\xi + P\Omega^I P^{-1})PG
\end{align*}
\]

(IV.164) - (IV.165)

or in other words

\[
\partial_{x_1}(PG) = (i\partial_{x_2}\xi + P\Omega^I P^{-1})PG
\]

(IV.166)

From (IV.166) it follows that

\[
\|PG\|_{L^2, \infty} \lesssim (\|\partial_{x_2}\xi\|_{L^2, 1} + \|\Omega^I\|_{L^2, 1})\|PG\|_{L^2, \infty}
\]

\[
\lesssim \varepsilon \|PG\|_{L^2, \infty}.
\]

(IV.167)

If \(\varepsilon\) is small enough then \(G \equiv 0\). This concludes the proof of theorem IV.8
IV.1 Dirichlet growth estimates

We are now proving the following $\epsilon$-regularity theorem.

**Theorem IV.9.** Let $1 < p < 2$. There is $\epsilon_0 > 0$ and $C$ such that for any $S \in W^{1,2}(B(0,1), O(\mathbb{N}))$ with $S^2 = 1$ and $u \in L^2(B(0,1), \mathbb{R}^n)$ solution to the equation (IV.139), if $\|\nabla S\|_{L^2(B(0,1))} \leq \epsilon_0$ then

$$\|\nabla u\|_{W^{1,\infty}(B(0,1/2))} \leq C \|u\|_{L^2(B(0,1))}. \tag{IV.168}$$

\[\square\]

**Proof of theorem IV.9**

Suppose that $\|\nabla S\|_{L^2(B(0,1))} \leq \epsilon_0$. We can repeat the arguments in the previous sections and obtain:

$$\partial_z(PG) = (i\partial_z \xi + (P\Omega_1 P^{-1}))PG, \quad \text{in } \mathcal{D}'(B(0,1)) \tag{IV.169}$$

where $\|\partial_z \xi\|_{L^2(B(0,1))} \leq \epsilon_0$ and $\|\Omega_1\|_{L^{2,1}(B(0,1))} \leq \epsilon_0$.

Let $w \in W^{1,2}(B(0,1))$ be such that $PG = \partial_z w$ and such that

$$\|\nabla w\|_{L^2(B(0,1))} \leq C \|f\|_{L^2(B(0,1))} \tag{IV.170}$$

Recall that $G$ is constructed out of $g := Q f$ following (IV.155). $w$ solves

$$\Delta w = 4(i\partial_z \xi + [P\Omega_1 P^{-1}])\partial_z w, \quad \text{in } \mathcal{D}'(B(0,1)) \tag{IV.171}$$

Let $x_0 \in B(0,1/2)$ and $0 < r < 1/4$. We decompose $w$ in $B(x_0, r)$ as follows

$$\begin{cases}
\Delta \sigma_1 = 0 & \text{in } B(x_0, r) \\
\sigma_1 = w & \text{on } \partial B(x_0, r)
\end{cases}
\quad \begin{cases}
\Delta \sigma_2 = 4(i\partial_z \xi + [P\Omega_1 P^{-1}])\partial_z w & \text{in } B(x_0, r) \\
\sigma_2 = 0 & \text{on } \partial B(x_0, r)
\end{cases} \tag{IV.172}$$

**Estimate of $\nabla \sigma_1$:**

Since $\sigma_1$ is harmonic, for every $\delta > 0$ we have

$$\|\nabla \sigma_1\|_{L^2(B(x_0,\delta r))}^2 \leq \|\nabla \sigma_1\|_{L^2(B(x_0,4\delta r))}^2 \leq C \left(\frac{4\delta}{3}\right)^2 \|\nabla \sigma_1\|_{L^2(B(x_0,3/4r))}^2 \tag{IV.173}$$

where $C$ is a constant independent of $r$.

**Estimate of $\nabla \sigma_2$:**

A classical result on Riez potentials implies that there exists a constant $C_0$ independent of $r$ such that

$$\|\nabla \sigma_2\|_{L^2(B(x_0, r))} \leq C \|\Delta \sigma_2\|_{L^1(B(x_0, r))} \leq \epsilon_0 \|\nabla u\|_{L^2,\infty(B(x_0, r))} \leq \epsilon_0 \|\nabla u\|_{L^2,\infty(B(0,1))} \tag{IV.174}$$

\[\text{We recall that if } u \in W^{1,2(2,\infty)}_0(B(0,1)) \text{ then } \|\nabla u\|_{L^2,\infty(B(0,1))} \leq C \|\Delta u\|_{L^1(B(0,1))}.\]  

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Now we combine (IV.173) and (IV.174).

\[ \| \nabla w \|_{L^2(B(x_0, \delta)}} \lesssim \| \nabla \sigma_1 \|_{L^2, \infty(B(p, 2\delta r))} + \| \nabla \sigma_2 \|_{L^2, \infty(B(x_0, \delta r))} \]

\[ \lesssim (\frac{4\delta}{3}) \| \nabla w \|_{L^2, \infty(B(x_0, r))} + \| \nabla \sigma_2 \|_{L^2, \infty(B(x_0, r))} + \varepsilon_0 \| \nabla w \|_{L^2, \infty(B(x_0, r))} \]

\[ \lesssim \left( \frac{4\delta}{3} + \left( \frac{4\delta}{3} \right) \varepsilon_0 + \varepsilon_0 \right) \| \nabla w \|_{L^2, \infty(B(x_0, r))} \quad \text{(IV.175)} \]

If we choose \( \delta \) and \( \varepsilon_0 \) small enough we get

\[ \| \nabla w \|_{L^2, \infty(B(x_0, \delta r))} \leq \eta \| \nabla w \|_{L^2, \infty(B(x_0, r))}. \quad \text{(IV.176)} \]

By iterating the estimate (IV.176) which is valid for every \( p \in B(0, 1/2) \) and \( 0 < r < 1/4 \) we get

\[ \sup_{x_0 \in B(0, 1/2), 0 < r < 1/4} r^{-\alpha} \| \nabla w \|_{L^2, \infty(B(x_0, r))} < C \| f \|_{L^2(B(0, 1))} \quad \text{(IV.177)} \]

Plugging the estimate (IV.177) into the equation (IV.171) we obtain that

\[ \sup_{p \in B(0, 1/2), 0 < r < 1/4} r^{-\alpha} \| \nabla w \|_{L^2, \infty(B(x_0, r))} < C \| f \|_{L^2(B(0, 1))} \quad \text{(IV.178)} \]

The estimate (IV.178) implies that for \( q > \frac{2 - \alpha}{1 - \alpha} > 2 \) (see Remark to Proposition 3.2 in Adams \[1\]) one has

\[ \| \nabla w \|_{L^q(B(0, 1/2))} \leq C \| f \|_{L^2(B(0, 1))} \]

The PDE (III.29) is becoming subcritical and a standard bootstrapping argument yields (IV.168). This concludes the proof of theorem (IV.9).

\[ \square \]

V Proof of theorem I.2

A standard covering argument gives that, modulo extraction of a subsequence, there exists finitely many points \( a_1 \cdots a_Q \) such that, for any \( \delta > 0 \)

\[ \lim_{k \to +\infty} \inf \left\{ \rho > 0 ; \int_{B_p(x)} | \nabla S_k |^2 (y) \, dy = \frac{\varepsilon_0}{2} \text{ where } x \in D^2 \backslash \cup_{i=1}^Q B_{\delta_k}(a_i) \right\} > 0 \]

Where \( \varepsilon_0 > 0 \) is given by the epsilon-regularity theorem (IV.2) for some \( 2 > p > 1 \). This theorem implies then that \( u_k \to u_\infty \) strongly in \( L^2_{\text{loc}}(D^2 \backslash \{ a_1 \cdots a_Q \}) \) hence we can pass in the limit in the equation away from the points and one gets

\[ \text{div} (S_\infty \nabla u_\infty) = 0 \quad \text{in } D'(D^2 \backslash \{ a_1 \cdots a_Q \}) \]

It remains to establish the point removability. Since \( S_\infty \nabla u_\infty = \nabla (S_\infty u_\infty) - \nabla S_\infty u_\infty \in H^{-1} + L^1(D^2) \) a classical result on distributions supported by points gives the existence of \( \alpha_1 \cdots \alpha_Q \in \mathbb{R}^n \) such that

\[ \text{div} (S_\infty \nabla u_\infty) = \sum_{i=1}^Q \alpha_i \delta_{a_i} \quad \text{in } D'(D^2) \]

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We pick a point $a_{i_0}$ arbitrary and we consider an axially symmetric function $\chi$ centred at $a_{i_0}$ such that $\chi \equiv 1$ in a neighborhood of $a_{i_0}$ and $\text{Supp} \chi \subset B_r(a_{i_0})$ where $0 < r < \inf_{i \neq j} |a_i - a_j|$. We have
\[ 0 = \int_{B_r(a_{i_0})} \nabla \chi \cdot S_k \nabla u_k \, dx \]
Because of the weak convergence of $\nabla u_k$ towards $\nabla u_\infty$ in $L^q$ for any $q < 2$ away from the points $a_1 \cdots a_Q$ and the strong convergence of $S_k$ towards $S_\infty$ in any $L^p$ for $p < +\infty$ we have
\[ 0 = \int_{B_r(a_{i_0})} \nabla \chi \cdot S_\infty \nabla u_\infty \, dx \]
which gives $\alpha_{i_0} = 0$. This concludes the proof of theorem I.2.

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