Transformation of the spin and orbital angular momentum of Hermite-Gaussian beams of a complex and real argument propagating through a uniaxial crystal is considered. We revealed that the spin and orbital angular momentum of the complex argument HG beam experience a sharp splashes whereas such an intense conversion process is not inherent in a real argument HG beams. The reasons and conditions of the resonance effect are brought to light.

**OCIS Codes**: 260.0260, 260.1180, 260.6042

It is well known that Hermite-Gaussian beams (HG) propagating through free space or a homogeneous isotropic medium cannot carry over an orbital angular momentum (OAM) [1] while their spin angular momentum (SAM) is defined by the initial polarization states [2]. On the other hand, a traditional lens converter [3] enables us to transform HG beam into a singular one bearing optical vortices that possess the OAM. However, presence of optical vortices in the beam is not the exclusive requirement. For example, even a fundamental Gaussian beam subjected to an astigmatic transformation gains also the OAM [4]. Transmitting through a homogeneous isotropic medium, light beam changes neither the OAM nor SAM. Absolutely other situation appears in a non-homogeneous isotropic medium, for example, in an optical fiber [5, 6]. Polarization transformations and change of a vortex structure in the optical fiber are controlled by a spin-orbital coupling, a total angular momentum flux along the fiber axis being conserved. A sharp gradient of a refractive index causes an intense spin-orbit coupling resulting in essential field transformations such as, for example, a transverse shift of the reflected and refracted beams on the boundary face of two homogeneous isotropic media [7].

A circularly polarized Gaussian beam propagating along the optical axis of a homogeneous uniaxial crystal gets also the OAM owning to nucleation of a double-charged optical vortex in the orthogonally polarized beam component [8,9]. Ciattoni et.al.[10] showed that such a vortex nucleation in an anisotropic medium is a result of a spin-orbit coupling on the stipulation that a total angular momentum flux along the crystal optical axis is conserved. A gradual beam depolarization in the Gaussian beam is accompanied by a slow growth of the OAM so that the OAM can reach a maximum value only at infinitely large crystal length. By analyzing this situation we supposed that an intense spin-orbit coupling at a relatively small crystal length can be in line with sharp intensity variations over the beam cross-section. Such characteristic features have a whole family of high-order beams: Hermite-Gaussian, Laguerre-Gaussian, Bessel-Gaussian at alias.

The aim of this Letter is to consider intrinsic features of a spin-orbital coupling in Hermite-Gaussian beams propagating along the optical axis of a homogeneous uniaxial crystal and to bring to light the conditions for the resonance transformations of the spin and orbital AM.

Let us consider the propagation of a circularly polarized HG beam along the optical axis of a uniaxial crystal with a permittivity tensor in diagonal form: $\hat{\varepsilon} = \text{diag}(\varepsilon_o, \varepsilon_o, \varepsilon_3)$, where $n_o = \sqrt{\varepsilon_o}$ and $n_3 = \sqrt{\varepsilon_3}$ being the refractive indices along a major crystallographic axes (see Fig.1). The basic equation for the transverse components of the complex amplitudes $\hat{E}_\perp = (e_+ \hat{E}_+ + e_- \hat{E}_-) \perp$ (where $e_+$ and $e_-$ being the unit vectors of a circularly polarized basis) of the paraxial beam field $E_{\perp}(x, y, z) \exp(-ik_0z)$ (where $k_0 = k_0n_o$, $k_0$ stands for a wavenumber in free space) is presented in the paper [9]. Generally speaking, there are two types of HG beams [11]: the HG beams of a real argument $\psi_{m,n}$ (or standard beams) and the HG beams $\psi_G(z=0)$ propagating through the crystal and polarization filters. P - a polarizer, $B$ - a quarter-wave retarder, A - a transmittance axis of the polarizer, C - a unit vector of the crystal optical axis, e and o - crystallographic axes of the birefringent phase retarder and the intensity distributions of the circularly polarized components of the HG beam with $m=n=4$, $w_0 = 8 \mu m$; $z=4.7 \text{ mm}$

**Fig. 1 (on-line)** Sketch of the beam propagation through the crystal and polarization filters.: $P$ - a polarizer, $B$ - a quarter-wave retarder, $A$ - a transmittance axis of the polarizer, $C$ - a unit vector of the crystal optical axis, e and o - crystallographic axes of the birefringent phase retarder and the intensity distributions of the circularly polarized components of the HG beam with $m=n=4$, $w_0 = 8 \mu m$; $z=4.7 \text{ mm}$

Transformed: the unit vectors of a circularly

# Equation

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Transferred: the unit vectors of a circularly
beams of a complex argument $\theta_{m,n}^j$ (or elegant HG beams):

$$
\theta_{m,n}^{(i,e)}(x,y,z) = H_m(\sqrt{2X/L}) \times
$$

$$
\times H_n(\sqrt{2Y/L}) \sigma_{o,e} e^{i(m+n)\Gamma_{o,e} w_0^2} \Psi_0^{(i,e)},
$$

where $H_m(x)$ stands for a Hermite polynomial, $\Psi_0^{(i,e)}$ is a Hermite polynomial, $\sigma_{o,e} = 1 - i z / z_0$, $z_0 = k_0 w_0^2 / 2$, $\sigma_e = 1 - i z / z_e$, $z_e = k_0 w_0^2 / 2$, $k_0$ is the refractive index of the extraordinary beam, $w_0$ is a beam waist radius at the $z=0$ plane, indices $o$- and $e$- designate the ordinary or extraordinary beam in the crystal, respectively. We suppose that the electric field $\vec{E}_{\perp}$ of the HG beam is right hand polarized at the $z=0$ plane:

$$
\vec{E}_{\perp}^{(m,n)}(x,y,z=0) = e_{e} e_{\theta_{m,n}^{(o,e)}}(x,y,z=0) \quad \text{(or)} \quad \vec{E}_{\perp}^{(m,n)}(x,y,z=0) = e_{o} e_{\theta_{m,n}^{(o,e)}}(x,y,z=0).
$$

A particular solution to the paraxial wave equation (1) in terms of $\theta_{m,n}^{(i,e)}$ mode beams of a complex argument can be presented as

$$
\vec{E}_{+} = e_{\theta_{m,n}^{(i)}} = \left[ e_{\theta_{m,n}^{(o)}} + e_{\theta_{m,n}^{(e)}} \right] / \sqrt{2},
$$

$$
\vec{E}_{-} = e_{\theta_{m,n}^{(i)}} = \left[ -w_0^{2m+n} / \sqrt{2} \right] \sigma_{o,e} \partial_x^m \partial_y^n \theta_{m,n}^{(i,e)}, \quad (4)
$$

where $\ell = e^{i/2} \mathbf{\Psi}_0 - \mathbf{\Psi}_e / R^2 + \mathbf{\Psi}_o - \mathbf{\Psi}_e$,

$$
n_e = n_3 / n_o, \quad R^2 = X^2 + Y^2, \quad \tan \varphi = y / x.
$$

The field equations for the standard $\mathcal{H}$-beams can be rewritten in terms of the complex amplitude $\theta_{m,n}^{(i,e)}$-beams by means of the expression

$$
\mathcal{H}_{2n+s}(x) = a \sum_{k=0}^{n} \frac{2^k}{[(n-k)!]!(2k+s)!]} \mathbf{e}^{(2n+s)}(x), \quad (5)
$$

and $s = 0, 1$, $a = 2^{s/2} (2n+s)!$. The major difference between the HG beams of the real and complex argument is that the standard $\mathcal{H}$-beams do not change their structure when propagating in free space up to the scale. Structure of the elegant $\mathcal{H}$-beam is radically transformed over all length of propagation. A little displacement of the beam cross-section from the initial plane $z=0$ results in vanishing edge dislocations in the $\mathcal{H}$-beam while the edge dislocations in the $\mathcal{H}$-beam are slightly shifted (see Fig. 2). As the $\mathcal{H}$-beam propagates along the crystal, the beam field is concentrated around four symmetric spots (see the intensity distributions in the dotted frame in Fig.2) provided that $m=n$ regardless of the magnitude of the indices. However, the sizes of the spots decrease when enlarging the index value. At a large crystal length, the spots are modulated by a set of interference fringes. Naturally, such a beam reconstruction cannot but have an impact on the spin-orbit coupling.

In the paraxial approximation, the conservation law of the AM flux along the crystal optical axis can be presented as [10]

$$
\ell_{z} = s_{z} + s_{z}(z=0) = const, \quad (6)
$$

where the OAM $\ell_{z}$ and the SAM $s_{z}$ are

$$
\ell_{z} = -i K \mathbf{\vec{E}} \mathbf{\vec{z}}, \quad s_{z} = i K \mathbf{\vec{E}} \mathbf{\vec{z}}, \quad \mathbf{\vec{z}} = \mathbf{\vec{x}} / y - \mathbf{\vec{y}} / x. \quad \text{Taking into account eqs (4), the forth integration of eqs (7) gives the expression for SAM and OAM in the \theta_{m,n}^{(i,e)}}-	ext{beams:}
$$

$$
\ell_{z} = A \cos N \gamma, \quad s_{z} = 1 - s_{z}, \quad (9)
$$

$$
A = (1 + Z^2)^{(m+n+1)/2} N, \quad \gamma = Z, \quad Z = 1 / z_0 - 1 / z_e / 2. \quad A \text{ similar expression for the real argument $\mathcal{H}$-beams has a cumbersome form and is not presented in the Letter. The above expressions are of a generalization of the expression (52) in the paper [10] obtained for a circularly polarized fundamental Gaussian beam. Notice that the SAM and OAM fluxes vanish for the linearly polarized initial beam. A set of curves shown in Fig.3 describes evolution of the SAM $s_{z}$ and OAM $\ell_{z}$ along the crystal length. In contrast to a smooth

Fig. 2 Transformation of the circularly polarized components in the $\mathcal{H}_{10,10}^{(i,e)}$ and $\mathcal{H}_{10,10}^{(i,e)}$ beams along the LiNbO$_3$ crystal length, $w_0 = 10 \mu m$.}
as a function of the index m: m=n, and

\[ Z_{\text{opt}} = \frac{L_{\text{opt}}}{K} \text{ as functions of the length } Z \]

curves for a Gaussian beam (m=n=0), the SAM and OAM of the complex argument \( \mathcal{R} \)-beams experience a sharp spike near an optimal crystal length \( Z = Z_{\text{opt}} \) where the AM reaches the extreme value \( s_z = s_{\text{extr}}, \ell_z = \ell_{\text{extr}} \). A magnitude of the AM splashes depends on the beam indices m and n: the more the index value, the more the more modulus of the spin and orbital AM. However, a total AM is conserved. At the same time, the SAM and OAM for the real argument \( \mathcal{R} \)-beams have more smooth behavior. For example, the \( \mathcal{R} \)-beam with indices \( m=n=50 \) has the extreme OAM less than \( \ell_{\text{extr}} < 1.2 \) while the \( \mathcal{R} \)-beam even for the indices \( m=n=10 \) gains the extreme OAM \( \ell_{\text{extr}} = 1.8 \).

The equation for the optimal \( Z_{\text{opt}} \) can be derived from the requirement: \( ds_z/\ell_z = 0 \). After a simple algebra we come to the characteristic equation: \( \tan[(m+n+1)\arctan Z] = -Z \). The second root of this equation corresponds to the optimal crystal length \( Z = Z_{\text{opt}} \). When enlarging the indices m and n, the optimal length \( Z_{\text{opt}} \) tends to zero: \( Z_{\text{opt}} \ll 1 \) and we can write approximate solution to the above equation: \( (m+n+1)Z_{\text{opt}} = -\pi = -Z_{\text{opt}} \), \( m+n \gg 1 \) or \( Z_{\text{opt}} = \pi/(m+n+2) \). Fig.4,a illustrates the extreme magnitudes of the AM as a function of the indices \( m=n \). Noteworthy that the extreme OAM reaches \( \ell_{\text{extr}} = 2 \) for \( m > 40 \) and then changes very slowly.

There are three physical processes in the crystal that underline the extreme spikes of OAM. First of all, when transmitting, high-order \( \mathcal{R} \)-beams are transformed in such a way that light forms only four symmetric maxima regardless of the beam indices. The second, a uniaxial crystal forms a non-uniformly polarized field distribution in the on-axis propagating beam consisting of interlaced rings of right- and left-hand circular polarization. Typical map of the field distribution in \( \mathcal{R} \) beam illustrates Fig.4,b. The left hand circular polarization

Fig. 4 (a) Extreme values of the SAM \( s_z^{(\text{extr})} \) and OAM \( \ell_z^{(\text{extr})} \) as a function of the index m: m=n, and (b) The map of the polarization state distribution on the background of the total intensity distribution in the \( \mathcal{E}K_{m,n} \) beam with \( m=n=10 \), \( Z=0.0785 \).

positioned at the central parts of all four intensity maxima of the beam results in arising a black strip in the \( \mathcal{E}K_{m,n} \) component in Fig.2 (the framed pictures). The left hand circular polarized component dominates in the beam i.e. the SAM changes a sign. The resonance condition corresponds to coinciding the radius of the intensity maxima of the beam and the radius of the ring with a left hand circular polarization. A conservation law of the total AM flux causes a slash of the OAM at the expense of a spin-orbit coupling in a homogeneous anisotropic medium of a uniaxial crystal.

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