THE WEINSTEIN CONJECTURE AND THE THEOREMS OF
NEARBY AND ALMOST EXISTENCE

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Dedicated to Alan Weinstein on the occasion of his sixtieth birthday

Abstract. The Weinstein conjecture, as the general existence problem for periodic orbits of Hamiltonian or Reeb flows, has been among the central questions in symplectic topology for over two decades and its investigation has led to understanding of some fundamental properties of Hamiltonian flows.

In this paper we survey some recently developed and well-known methods of proving various particular cases of this conjecture and the closely related almost existence theorem. We also examine differentiability and continuity properties of the Hofer–Zehnder capacity function and relate these properties to the features of the underlying Hamiltonian dynamics, e.g., to the period growth.

1. Introduction

Without exaggeration, one can say that Arnold’s conjecture and the Weinstein conjecture have been the two problems determining the development of symplectic topology over the past twenty years. The Weinstein conjecture, [We4], the problem we are interested in here, concerns the existence of closed characteristics on a compact hypersurface of contact type. To be more precise, consider a regular compact contact type level of a Hamiltonian on a symplectic manifold. Then, the Weinstein conjecture as we understand it now asserts that the level must carry at least one periodic orbit of the Hamiltonian flow.

This conjecture was motivated by numerous results establishing the existence of periodic orbits under various, often rather restrictive, conditions on the level (e.g., convex or bounding a star-shaped domain); see, e.g., [Mo, Ra, We1, We2, We3]. The feature distinguishing the Weinstein conjecture from these results is that the requirement that the level has contact type is invariant under symplectomorphisms (as is the assertion) while the hypotheses of the earlier theorems are not. In its original form for Hamiltonians on a linear symplectic space, the Weinstein conjecture was proved by Viterbo, [Vi1]. Since then, the conjecture has been established for many other symplectic manifolds (sometimes under additional requirements on the level); see, e.g., [FHV, HV1, HV2, LT, Lu1, Lu2, Lu4, Vi4], to mention just a few results. Among these manifolds are products of complex projective spaces, manifolds of the form $P \times \mathbb{C}^n$ (where $P$ is compact symplectic and the product
is given a split symplectic structure), and sub-critical Stein manifolds. However, in general, the conjecture is still neither proved nor disproved. For example, the Weinstein conjecture in its full generality is open for cotangent bundles and $\mathbb{T}^4$.

Starting from the Weinstein conjecture, one can go in two different directions. One direction is to dispose of the Hamiltonian and the ambient symplectic manifold and focus exclusively on the level of contact type. This naturally leads to the question of whether a Reeb flow on a compact contact manifold necessarily has a periodic orbit; see, e.g., [Ho3]. Along these lines, Hofer, [Ho2], proved the existence of periodic orbits for the Reeb flow of a contact form on $S^3$ or on a closed three-dimensional contact manifold $M$ with $\pi_2(M) \neq 0$ and also for the Reeb flow of an overtwisted contact form. This approach interprets the conjecture as a question about the dynamics of Reeb flows and leads to the notions of contact homology and symplectic field theory, [EGH, BEHWZ].

Another direction is to view the conjecture as a question about the dynamics of Hamiltonian flows on the ambient symplectic manifold. This is the perspective with which we are concerned here. More specifically, we focus on such problems as whether and how the assumption that the level has contact type can be relaxed, whether the existence of periodic orbits is typical, etc. The contact type requirement cannot be dropped entirely: there exists a proper function on $\mathbb{R}^{2n}$ ($C^\infty$-smooth if $2n \geq 6$ and $C^2$-smooth if $2n = 4$) with a regular level carrying no periodic orbits; see [Gi1, Gi3, GG1, GG2, He1, He2, Ke2]. (Constructions of such functions are known as counterexamples to the Hamiltonian Seifert conjecture.) Nevertheless, the existence of periodic orbits on the level sets of a fixed Hamiltonian is a generic phenomenon: almost all (in the sense of measure theory) regular levels of a $C^2$-smooth proper function on $\mathbb{R}^{2n}$ carry periodic orbits of the Hamiltonian flow. This result, due to Hofer and Zehnder and to Struwe, is known as the almost existence theorem; [HZ3, St]. The almost existence theorem holds for many, but not all, symplectic manifolds (sometimes under some additional restrictions on the Hamiltonian, cf. [GG3]). For example, the theorem holds for $\mathbb{CP}^n$, products $P \times C^n$ with split symplectic structure (where $P$ is geometrically bounded), [HV1, Lu1, MDSl, Schl], sub-critical Stein manifolds, [Ci, Ke3, Lu1], and small neighborhoods of certain non-Lagrangian submanifolds, [Ke3, Sc3, Schl]. However, as has been observed by Zehnder, [Ze], the almost existence theorem fails for certain Hamiltonians on $T^{2n}$ equipped with an irrational symplectic structure (Zehnder’s torus). In fact, for this Hamiltonian system, there is an interval of energy values without periodic orbits. Moreover, this phenomenon is stable under $C^k$-small ($k > 2n$) perturbations of the Hamiltonian, [He3, He4]. A sibling of the almost existence theorem is the theorem of dense or nearby existence which guarantees the existence of periodic orbits for a dense set of energy levels or, equivalently, near a fixed level; see, e.g., [CGK, FH, FS, HZ2, HZ3]. Both of these theorems imply the Weinstein conjecture and below we will discuss the relation between these theorems.

Another reason to consider the almost existence and nearby existence theorems is that a broad class of Hamiltonian systems for which energy levels fail to have contact type naturally arises in classical mechanics. In this class are, for example, the systems describing the motion of a charge in a magnetic field, which we will discuss shortly; see also [Gi2]. For such systems one can expect the almost existence theorem to hold and periodic orbits to exist for all low energy values. However,
periodic orbits need not exist on all energy levels even when the symplectic form is exact near a level (the horocycle flow, see Example 4.7). Furthermore, as has been pointed out by Kerman, [Ke1], the analysis of such systems is closely related to a generalization of the Weinstein–Moser theorem, [Mo, We2].

The Weinstein–Moser theorem asserts that a smooth function $H$ on $\mathbb{R}^{2n}$ attaining a non-degenerate minimum at the origin must have at least $n$ distinct periodic orbits on every level near the origin, [Mo, We2]. Let us now replace $\mathbb{R}^{2n}$ by an arbitrary symplectic manifold $W$ and the non-degenerate minimum at one point by a Morse–Bott non-degenerate minimum along a closed symplectic submanifold $M \subset W$. Then, conjecturally, every level of $H$ near $M$ carries at least one periodic orbit or even a number of periodic orbits. We will refer to this conjecture as the generalized Weinstein–Moser conjecture. As a particular example motivating our interest in this question, consider the standard kinetic energy $H$ on the cotangent bundle $W = T^* M \xrightarrow{\pi} M$, equipped with a twisted symplectic structure $\omega_0 + \pi^* \Omega$, where $\omega_0$ is the standard symplectic structure $dp \wedge dq$ and $\Omega$ is a closed two-form on $M$. This system describes the motion of a charge on $M$ in the magnetic field $\Omega$; see [Gi2]. When $\Omega$ is non-degenerate $M$ turns into a symplectic submanifold of $W$.

The generalized Weinstein–Moser conjecture has been proved in a number of particular cases, [GK1, Ke1], but in general the question is still open. Recently, however, some progress has been made along the lines of the almost existence theorem. Namely, it has been shown that almost all levels close to $M$ of a function $H$ attaining a minimum along $M$ carry contractible periodic orbits, [CGK, GG3, Ma2] – this is the relative (with respect to $M$) almost existence theorem for small energy values. (Note that unless $M$ is a Morse–Bott non-degenerate minimum of $H$, we cannot expect such periodic orbits to exist on all levels near $M$, [GG3].) This implies the almost existence theorem for low energy periodic orbits of a charge in a non-degenerate magnetic field. Moreover, under suitable additional hypothesis, the genuine, non-relative, almost existence theorem holds near $M$, [Ke3]. In the setting of magnetic fields, these results can be further refined: periodic orbits must exist whenever $\Omega \neq 0$, [Schl]. (We will briefly discuss the proof of this result in Section 3.5.) We also refer the reader to [GK2, Ma1, Pol2] for related results.

The nearby existence theorem is weaker and often easier to prove than the almost existence theorem. The pattern has been that, in many cases, the nearby existence theorem was proved first and then followed by the almost existence theorem; cf., e.g., [HZ1] and [HZ3, St], [CGK] and [GG3]. As we have pointed out above, both theorems imply the Weinstein conjecture. Conversely, essentially every proof of a particular case of the Weinstein conjecture in the Hamiltonian setting translates into a proof of either the nearby existence or almost existence theorem, although it is not always easy to establish which of these theorems is proved. As of today, almost existence is verified in virtually all the cases where the nearby existence has been proved. Probable exceptions are some of the results from [Vi4] on periodic orbits in cotangent bundles and, perhaps, the results of [LT] and hence of [Lu2, Lu3, Lu4].

In the present paper we focus on the aspects of the Weinstein conjecture related to global Hamiltonian dynamics, and hence our treatment of the conjecture in the large is by no means comprehensive. For example, we do not even touch upon the Weinstein conjecture for contact manifolds. (The reader interested in this conjecture is referred to the surveys [Ho3, Ho4, Ho5] in addition to the references given above.) Furthermore, we do not mention the fruitful connection between the
Weinstein conjecture and Gromov–Witten invariants, although we do discuss the holomorphic curve approach to the proof of the conjecture.

In Sections 2 and 3 we outline methods of proving the nearby existence and almost existence theorems and, hence, the Weinstein conjecture. These sections can be viewed as a brief introduction to certain concepts of symplectic topology (action selectors, constructions of symplectic capacities, Hofer’s metric, symplectic homology, etc.), albeit strictly focused on a specific task and not even mentioning many aspects of the subject. However, these sections should not be taken as an introduction to symplectic topology in general. For example, we assume the reader’s familiarity with Floer homology. Section 4 concerns some simple, but apparently not present in the literature, properties of the Hofer–Zehnder capacity function. Although the paper is complemented by an extensive bibliography, the list of references contains only the papers immediately relevant to our discussion. Inevitably, this list is incomplete and omits many important contributions to the subject, and our exposition emphasizes the publications that have most influenced the author’s thinking.

Conventions. In this paper, all manifolds are assumed to be without boundary. A symplectic manifold \( W \) will be called convex if \( W \) is either closed (i.e., compact) or open and convex at infinity. Here \( W \) is said to be convex at infinity if there exists: a hypersurface \( \Sigma \subset W \) which separates \( W \) into one set with compact closure and another, \( U \), with non-compact closure; and a flow \( \varphi_t \) (for \( t \geq 0 \)) of symplectic dilations on \( U \), which is transversal to \( \partial U = \Sigma \). Recall also that \( (W, \omega) \) is symplectically aspherical if \( c_1|_{\pi_2(W)} = \omega|_{\pi_2(W)} = 0 \). (In some instances, this condition can be replaced by \( \omega|_{\pi_2(W)} = 0 \).) We refer the reader to, e.g., [AL, CGK] for a discussion of geometrically bounded symplectic manifolds and to [HZ3, MDSa1, MDSa2, Pol3] for a general introduction to symplectic topology.

Let us now fix the sign conventions in the definition of the action functional. Let \( H \in C^\infty(S^1 \times W) \) and let \( x : S^1 \to W \) be a contractible loop in \( W \). Here we will use the action functional \( A_H \) defined as

\[
A_H(x) = -\int_{D^2} \bar{x}^* \omega + \int_{S^1} H(t,x) dt,
\]

where \( \bar{x} : D^2 \to W \) is a map of a disk, bounded by \( x \).

By the least action principle, contractible one-periodic orbits of \( H \) are precisely the critical points of \( A_H \). The action spectrum \( S(H) \) of \( H \) is the set of critical values of \( A_H \), i.e., the collection of action values of \( A_H \) on the contractible periodic orbits. (Since we are assuming that \( \omega|_{\pi_2(W)} = 0 \), the action \( A_H \) is single valued. Otherwise, one has to pass to a suitable covering of the loop space and deal with numerous other technical difficulties.) We set \( H_t = H(t, \cdot) \), where \( t \in S^1 \), and denote by \( X_H \) the Hamiltonian vector field of \( H \).

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2. The nearby existence theorems

Nearby existence theorems are usually proved by Floer homological methods. Below we discuss some of these methods, focusing on the key ideas and omitting many (often quite non-trivial) technical details.
Throughout this section, all ambient symplectic manifolds \((W,\omega)\) are assumed to be convex and symplectically aspherical, unless explicitly stated otherwise as is, for example, in Section 2.4.

The argument is particularly transparent for open manifolds which are convex at infinity. Hence we consider this case first.

2.1. The action selector method: convex at infinity open manifolds. In this section we focus on convex at infinity open manifolds and outline the approach to the proof of the nearby existence theorem utilizing the notion of an action selector. In our proof of the existence of an action selector and its properties, we mainly follow [FS, Sc3]. However, many elements of our argument are already contained in [HZ2, HZ3], where the action selector is defined differently; see also, e.g., [Oh2, Vi3].

2.1.1. Action selectors and the nearby existence theorem. Let \(W\) be a convex at infinity open symplectic manifold. We denote by \(H_U\), where \(U\) is an open subset of \(W\), the space of compactly supported smooth Hamiltonians \(H: S^1 \times U \rightarrow \mathbb{R}\).

For our purposes, it is convenient to adopt the following definition. An action selector is a \(C^0\)-continuous function \(\sigma: H_W \rightarrow \mathbb{R}\) such that

1. \(\sigma(H) \in S(H)\),
2. \(\sigma\) is monotone, i.e., \(H_1 \leq H_2\) implies that \(\sigma(H_1) \leq \sigma(H_2)\), and
3. \(\sigma(H) = \max H\), when \(H\) is a \(C^2\)-small function on \(W\) with a unique maximum.

Let us point out some consequences of (AS1)–(AS3) which are essential for what follows. First recall that the action spectrum \(S(H)\) is compact and nowhere dense; see, e.g., [HZ3, Sc3]. (This elementary, but not entirely trivial, fact can be thought of as a version of Sard’s theorem for \(A_H\), cf. [Zv].) Furthermore, as is easy to see, \(S(H)\) depends only on the time-one flow \(\varphi_H\) of \(H\) viewed as an element of the universal cover \(D\) of the group of Hamiltonian symplectomorphisms; the homotopy to identity is given by \(H\). (In fact, when \(W\) is symplectically aspherical, \(S(H)\) depends only on the time-one flow of \(H\) regarded as an element of the group of symplectomorphisms.) Then it follows from (AS1) and the continuity of \(\sigma\) that \(\sigma(H)\) also depends only on \(\varphi_H \in D\). Furthermore,

\[\sigma(H) > 0\] when \(H\) is non-negative and non-constant, (2.1)

by (AS2) and (AS3) and, by (AS1) and (AS2), \(\sigma(H) = 0\) when \(H \leq 0\).

The nearby existence theorem for functions on \(U \subset W\) can be (and often is) easily derived from the existence of an action selector \(\sigma\) which meets one additional requirement that \(\sigma\) is \textit{a priori} bounded on \(H_U\). To state the result, it is convenient to introduce the notion of a shell or thickening of a hypersurface. A shell \(\{\Sigma_s\}\) in \(U\) is an embedding \(\Sigma \times (-\epsilon, \epsilon) \hookrightarrow U\), where \(\Sigma\) is a closed manifold such that \(\dim \Sigma = \dim U - 1\). We identify \(\Sigma \times (-\epsilon, \epsilon)\) with its image, set \(\Sigma_s = \Sigma \times s\) and assume, in addition, that \(\Sigma = \Sigma_0\) divides \(U\) into two components: one bounded (i.e., such that its closure in \(U\) is compact) and the other one, bounded or unbounded. (In many instances, this assumption can be omitted.)

1The properties (AS1)–(AS3) should not be taken as an attempt to axiomatize the notion of an action selector (cf. [FGS]) — such axioms would almost certainly be different from (AS1)–(AS3) and would include a version of the inequality (2.4) below. Our objective is to list the properties of the action selector that are most essential for the proof of the nearby existence theorem.
The following, nearly trivial, proposition sums up a number of versions of the nearby existence theorem:

**Proposition 2.1.** Assume that there exists an action selector \( \sigma: \mathcal{H}_W \to \mathbb{R} \) such that \( \sigma(K) \leq C_U \) for every function \( K \in \mathcal{H}_U \) and some constant \( C_U \) independent of \( K \). Then the nearby existence theorem holds for proper functions \( H \) on \( U \): contractible in \( W \) periodic orbits exist for a dense set of values of \( H \). Equivalently, for every shell \( \{ \Sigma_s \} \) there exists a value \( s_0 \) such that \( \Sigma_{s_0} \) carries a contractible in \( W \) closed characteristic.\(^2\)

As we will soon see, there exists an action selector on \( \mathbb{R}^{2n} \) which is *a priori* bounded on bounded domains, [HZ3] and hence the nearby existence theorem holds for \( \mathbb{R}^{2n} \).

**Remark 2.2.** In this proposition and in Proposition 3.1, the assumption that the shell divides \( W \) is not essential; see [MS].

**Proof of Proposition 2.1.** Given a shell \( \{ \Sigma_s \} \), consider a smooth function \( F \) on \( W \) such that
- \( F \) is equal to some positive constant \( C > C_U \) on the bounded component of the complement to the shell and is identically zero on the unbounded component,
- within the shell, \( F \) depends only on \( s \) and is a monotone decreasing function with range \([0, C]\) and such that 0 and \( C \) are the only critical values of \( F \).

Note that the only critical values of \( F \) on \( W \) are again 0 and \( C \). Furthermore, \( 0 < \sigma(F) < C \) by (2.1) and since \( \sigma(F) \leq C_U < C \). Hence, any contractible one-periodic orbit of \( F \) with action \( \sigma(F) \) must be non-trivial and lie on one of the regular levels of \( F \), i.e., on some hypersurface \( \Sigma_{s_0} \). Hence, at least one of these hypersurfaces carries a contractible in \( W \) closed characteristic. \( \square \)

2.1.2. **The existence of an action selector.** Let us turn now to the problem of existence of an action selector. There are numerous constructions of action selectors for different classes of manifolds \( W \): in [HZ2, HZ3] an action selector has been constructed for \( W = \mathbb{R}^{2n} \) using a direct variational method on the space of \( H^{1/2} \)-loops; in [Vi3] an action selector has been defined for cotangent bundles using generating functions; this construction has been extended to Lagrangian submanifolds in [Oh1, Oh2] using Floer homology; in [Sc3, Oh2] an action selector has been introduced for symplectically aspherical closed manifolds \( W \), again by utilizing the Floer homology methods; these results have been extended to convex at infinity manifolds in [FS]. Finally, in [Oh3], an action selector was defined for closed symplectic manifolds which are not necessarily convex or symplectically aspherical.

Let us now outline the construction of an action selector, following mainly [FS, Sc3], for open symplectically aspherical manifolds which are convex at infinity.

Let \( W \) be such a manifold. Recall that the Floer homology \( HF(H) \) for \( H \in C_c^\infty(S^1 \times W) \) is defined and independent of \( H \), i.e., for any two functions these groups are canonically isomorphic, [Fl1, Fl2], see also [PSS, Sa, Sc1]. Since \( W \) is not closed but convex at infinity, one has to extend \( H \) to a function with a suitable growth at infinity, without creating new periodic orbits, and then work with the

\(^2\)Depending on whether this result is thought of in terms of \( H \) or a shell, it is referred to as a dense existence or nearby existence theorem.
Floer homology of the resulting function; see [FS, Oh1, Vi4]. Moreover, \(HF(H)\) is isomorphic to \(H_*(W)\) up to a shift of degrees, but this fact and the specific nature of the isomorphism are not very essential for us until later stages of the argument. At the moment, one may interpret \(H_*(W)\) as a common notation for the groups \(HF(H)\) identified with each other for different functions \(H\). Furthermore, the filtered Floer homology \(HF^{[a, b]}(H)\) is defined for any interval \([a, b]\) with end-points outside of \(\mathcal{S}(H)\). This homology is constant under deformations of \(a, b\) and \(H\) as long as \(a\) and \(b\) are outside of the action spectrum of \(H\), see, e.g., [Vi4].

There is a natural map

\[ j^a: H_*(W) = HF(H) \to HF^{[a, \infty]}(H) \]

induced by the quotient of complexes. Fix a non-zero element \(u \in H_*(W)\) and set

\[ \sigma_u(H) = \inf\{a \mid j^a u = 0\}. \]

It is easy to see by using the invariance and monotonicity properties of Floer homology that \(\sigma_u\) meets the requirements (AS1) and (AS2). Let \([\max] \in H_*(W)\) be the class of the maximum of a \(C^2\)-small bump function. By calculating the Floer homology of such a bump function, one can show that \([\max] \neq 0\). The selector \(\sigma = \sigma_{[\max]}\) obviously satisfies (AS1)-(AS3). However, the \(C^0\)-continuity of \(\sigma\), or in general of \(\sigma_u\), requires a proof (see, e.g., [Oh2, Sc3]) and this proof relies on the explicit construction of Floer’s continuation map.

The key to the proof is the following observation. Let \(H_0 \geq H_1\) be two Hamiltonians whose one-periodic orbits are non-degenerate. Consider periodic orbits \(x_0\) of \(H_0\) and \(x_1\) of \(H_1\) such that there exists a connecting homotopy trajectory joining \(x_0\) and \(x_1\) for a certain “linear” monotone homotopy from \(H_0\) to \(H_1\). Then

\[ A_{H_0}(x_0) - A_{H_1}(x_1) \leq \int_0^1 \max_{p \in W} (H_0(t, p) - H_1(t, p)) \, dt. \tag{2.2} \]

This inequality can be obtained by a direct calculation and the \(C^0\)-continuity of \(\sigma\) readily follows from (2.2); see [Sc3] for details.

In a similar vein, (2.2) implies the inequality

\[ \sigma(H) \leq \int_0^1 \max_{p \in W} H(t, p) \, dt, \tag{2.3} \]

which can be established by setting \(H = H_0\) in (2.2), taking a \(C^2\)-small function as \(H_1\) and then letting \(H_1\) go to zero.

2.1.3. **An a priori bound for \(\sigma\).** One class of domains \(U\) for which \(\sigma\) is a priori bounded is the class of displaceable domains, i.e., \(U\) such that there exists a Hamiltonian symplectomorphism \(\varphi_H\) with \(H \in \mathcal{H}_W\) moving \(U\) away from itself: \(\varphi_H(U) \cap U = \emptyset\). For example, bounded open subsets of \(\mathbb{R}^{2n}\) are displaceable.

To show that \(\sigma\) is a priori bounded on a displaceable domain \(U\), one argues as follows.

Let \(\varphi_H^t\) and \(\varphi_K^t\) be time-dependent Hamiltonian flows generated by \(H\) and \(K\) in \(\mathcal{H}_W\). Denote by \(H \# K \in \mathcal{H}_W\) the Hamiltonian generating \(\varphi_H^t \circ \varphi_K^t\), i.e.,

\[ H \# K(t, p) = H(t, p) + K(t, (\varphi_H^t)^{-1}(p)). \]

The crucial feature of the selector \(\sigma\) defined in Section 2.1.2 is that \(\sigma\) is sub-additive:

\[ \sigma(H \# K) \leq \sigma(H) + \sigma(K) \tag{2.4} \]
or more generally $\sigma_{u \cap v}(H \# K) \leq \sigma_u(H) + \sigma_v(K)$, where $u \cap v$ is the intersection of homology classes $u$ and $v$.

The sub-additivity of $\sigma$ is proved in [Oh2, Sc3] by making use of the pair-of-pants product introduced in [Sc2]; see also [PSS]. This is a product

$$HF^{[a, \infty)}(H) \otimes HF^{[b, \infty)}(K) \to HF^{[a+b, \infty)}(H \# K)$$

which is intersection of cycles on the full Floer homology $H_*(W)$ and such that the corresponding diagram commutes. (At this stage one specifically utilizes the isomorphism $HF(H) \to H_*(W)$ defined in [PSS], for this isomorphism sends the pair-of-pants product to the intersection of cycles. However, it might also be possible to prove the sub-additivity of $\sigma_{\text{max}}$ by using the Floer continuation map.) Once the existence of such a product is established, (2.4) follows from the definition of $\sigma$; see [Sc3, Section 4].

The sub-additivity of $\sigma$ implies an a priori bound for $\sigma$. Let $K$ be a Hamiltonian whose time-one flow displaces $\text{supp} H$, where, by definition, $\text{supp} H = \bigcup_{t \in S^1} \text{supp} H_t$. Denote by $K^-$ the Hamiltonian generating the flow $(\varphi^K_t)^{-1}$ so that $K \# K^- = 0$. Then

$$\sigma(H) \leq \sigma(K) + \sigma(K^-). \quad (2.5)$$

To prove (2.5), we argue as in [FS]. First observe that since $\varphi^K$ displaces $\text{supp} H$, one-periodic orbits of $H \# K$ are exactly one-periodic orbits of $K$, and, as a consequence, $S(H \# K) = S(K)$. The same is, of course, true when $H$ is replaced by the Hamiltonian $sH$ with $s \in \mathbb{R}$, i.e., $S((sH) \# K) = S(K)$. By continuity of $\sigma$ and “discontinuity” of $S(K)$, we conclude that $\sigma((sH) \# K) \in S((sH) \# K) = S(K)$ is independent of $s$. Setting $s = 0$ and $s = 1$ yields $\sigma(K) = \sigma(H \# K)$. Therefore,

$$\sigma(H) = \sigma(H \# K \# K^-) \leq \sigma(H \# K) + \sigma(K^-) = \sigma(K) + \sigma(K^-),$$

which proves (2.5).

Among other important examples of displaceable domains $U$ are small neighborhoods of closed non-Lagrangian submanifolds of middle dimension whose normal bundles have non-vanishing sections, [LS, Pol1]. This (combined with Macarini’s stabilization trick, [Ma1]), leads to a local version of the nearby existence theorem for twisted cotangent bundles over surfaces (other than the torus) or, in higher dimensions, for exact magnetic fields; see [FS].

2.2. The action selector method: closed manifolds. In this section we will briefly point out changes required in the action selector proof of the nearby existence theorem when $W$ is a closed manifold.

First, let us observe that the construction of an action selector outlined in Section 2.1.2 carries over to Hamiltonians on closed manifolds $W$. (Here, as everywhere in this section, $W$ is assumed to be symplectically aspherical.) Such an action selector has the property $\sigma(H + \text{const}) = \sigma(H) + \text{const}$ and hence is never a priori bounded.

To circumvent this problem, one chooses a suitable normalization of Hamiltonians. (This is also necessary to ensure that $\sigma(H)$ depends only on $\varphi_H$.) However, in contrast with open manifolds where compactly supported Hamiltonians are automatically normalized, there appears to be no natural choice of normalization for closed manifolds.

One possible choice is to restrict the action selector to Hamiltonians vanishing on a neighborhood of a fixed point. This should allow one to extend word-for-word the results and constructions of Section 2.1 to closed manifolds. However, certain
details of this approach are still to be worked out and we leave its discussion to a later occasion. Here, we use the traditional normalization requiring the mean value of Hamiltonians to be zero. More specifically, let \( \mathcal{H}_W \) be the class of smooth functions on \( S^1 \times W \) normalized so that \( \int_W H_x u^n = 0 \) and let \( \mathcal{H}_U \) be formed by Hamiltonians \( H \in \mathcal{H}_W \) such that \( \text{supp} \ X_{H_t} \subset U \) for all \( t \in S^1 \).

Proposition 2.1 still holds when the class \( \mathcal{H}_U \) is defined in such a fashion. However, when \( W \) is closed, obtaining an \( \text{a priori} \) bound for a selector on \( \mathcal{H}_U \) is more difficult than in the case of an open manifold, even though this still might be possible. We refer the reader to [FGS] for a detailed analysis of this question. (The proof of the \( \text{a priori} \) bound from Section 2.1.3 does not go through because inequality (2.5) need not hold. The reason is that, even when \( W \) is connected, \( S(H \# K) \) will differ from \( S(K) \) by the constant equal to \( H_{|W \setminus U|} \).

Here, following [Sc3], we choose a somewhat different approach. Let, as in Section 2.1.2, \( \sigma_{[\max]} \) be the action selector associated with the Floer homology class \( [\max] \in H_s(W) \) corresponding to the maximum of a \( C^2 \)-small bump function (shifted to have zero mean). Set

\[
\gamma(H) = \sigma_{[\max]}(H) + \sigma_{[\max]}(H^-).
\]

Equivalently, \( \gamma(H) = \sigma_{[\max]}(H) - \sigma_{[\min]}(H) \), where \( \sigma_{[\min]} \) is the action selector associated with the Floer homology class \( [\min] \in H_s(W) \) corresponding to the minimum of a negative \( C^2 \)-small shifted bump function. Similarly to action selectors for normalized functions, \( \gamma(H) \) depends only on \( \varphi_H \in \mathcal{D} \).

When \( H \) is \( C^2 \)-small, we have \( \gamma(H) = \int_0^1 (\max H - \min H) \, dt \). Furthermore, \( \gamma \) is still \( C^0 \)-continuous in \( H \), sub-additive, but not necessarily monotone. Then, since \( H_t(p) = -H_t((\varphi_H)^{-1}(p)) \), inequality (2.3) translates to

\[
\gamma(H) \leq \int_0^1 \left( \max_{p \in W} H(t, p) - \min_{p \in W} H(t, p) \right) \, dt.
\]

Note also that \( \gamma(K^-) = \gamma(K) \). As we have pointed out above, the proof of (2.5) given in Section 2.1.3 does not go through for the normalization we use here. However, a similar but more involved argument proves the upper bound

\[
\gamma(H) \leq 2 \gamma(K),
\]

where the time-one flow of \( K \) displaces \( U \supset \text{supp} \ (X_H) \). (Here, by definition, \( \text{supp} \ X_{H_t} = \bigcup_{s \in S^1} \text{supp} \ X_{H_t,s} \).) We refer the reader to [Sc3] for detailed proofs of these facts.

Observe now that Proposition 2.1 holds when an \( \text{a priori} \) bounded selector \( \sigma \) is replaced by \( \gamma \) which satisfies (2.6) and is \( \text{a priori} \) bounded on \( U \). (To see this, only minor modifications in the proof are needed.) As a consequence of (2.7), we immediately obtain the nearby existence theorem for displaceable domains in compact symplectically aspherical manifolds.

### 2.3. Limitations of the action selector method.

The class of manifolds \( W \) and domains \( U \) to which the action selector method applies is rather limited. For example, a bounded selector or a bounded function \( \gamma \) do not exist for \( U = W = \mathbb{T}^2 \) (nor for \( \mathbb{T}^4 \) or \( \mathbb{T}^2 \times S^2 \) with split symplectic structure). Indeed, it is easy to see that there is a function \( H \in \mathcal{H}_{\mathbb{T}^2} \) with \( S(H) = \{ \min H, \max H \} \) and such that \( \sigma_{[\max]}(H) = \max H \) and \( \sigma_{[\min]}(H) = \max H \) are both arbitrarily large. The same is true for tubular neighborhoods of the zero section in cotangent bundles of some compact manifolds (e.g., of surfaces with genus \( g \geq 1 \)). As a consequence,
Proposition 2.1 cannot be applied to prove nearby existence on these manifolds as long as the standard Floer homology (even accounting for non-contractible orbits) is employed. This is the case already for $W = \mathbb{T}^2$ even though every regular level of $H$ is comprised of (not-necessarily contractible) periodic orbits. The problem is that the periodic orbit can “migrate” from one homotopy class to another depending on the function in question.

Another class of manifolds to which this method does not apply is formed by geometrically bounded, but not convex manifolds. Among such manifolds are many twisted cotangent bundles and also universal coverings of some manifolds, which justifies the interest in this class.

The problem is that there is no known satisfactory definition of an action selector for general geometrically bounded manifolds. For example, the obstacle in the homological approach is that $HF^{[a, b)}(H)$ for $H \in \mathcal{H}_W$ has been defined only for intervals $[a, b]$ which do not contain zero, while the homological definition of $\sigma$ requires $HF^{[a, b)}(H)$ to be defined for all intervals. One possible solution is to consider an action selector only on the class of non-negative (autonomous or time-dependent) Hamiltonians or a yet more narrow class of functions (containing the functions $F$ from the proof of Proposition 2.1) and to extend the homological definition to this class. The proof of the \textit{a priori} bound outlined in this section will not carry over to such a narrow class, but a different argument (e.g., akin to the symplectic homology method from Section 2.4) may.

Furthermore, when a geometrically bounded manifold is “asymptotically convex”, as are for instance twisted cotangent bundles, one should be able to define Floer homology for all intervals of action by suitably extending the function at infinity as in the convex case. When this is done, many of the results that hold for convex manifolds should remain valid, although certain technical difficulties have to be dealt with.

However, as of today, no work in either of these directions has been carried out and it is not clear whether or not this would lead to new results.

### 2.4. Symplectic homology

A slightly different approach to proving the nearby existence theorem, which also utilizes Floer homology, relies on symplectic homology introduced and investigated in [CFH, CFHW, FH, FHW].

Let, as above, $U$ be a domain in a symplectic manifold $W$. The symplectic homology of $U$ is defined as

$$\text{SH}^{[a, b)}(U) = \lim_{\leftarrow H} \text{HF}^{[a, b)}(H),$$

where the inverse limit is taken over all $H \in C^\infty_c(S^1 \times U)$.

The symplectic homology of $U$ detects closed characteristics in an arbitrarily narrow thickening $\{\Sigma_s\}$ of $\Sigma = \partial U$, provided that $\Sigma$ is smooth. In particular, when $\text{SH}^{[a, b)}(U) \neq 0$, in any thickening $\{\Sigma_s\}$ there exists a hypersurface $\Sigma_{s_0}$ carrying a closed characteristic. (This fact is an essentially immediate consequence of the definition; see [FH].) Therefore, the nearby existence theorem holds for a function $H$ whenever the symplectic homology of the sublevels $\{H < c\}$ is non-trivial. The problem is thus reduced to verifying that symplectic homology does not vanish.

Let us show how this approach works, for example, when $W = \mathbb{R}^{2n}$. Let $U$ be a bounded domain in $\mathbb{R}^{2n}$ which we assume to contain the origin. Thus, there exist two open balls $B_r$ and $B_R$ centered at the origin and such that $B_r \subset U \subset B_R$. The
symplectic homology has a natural monotonicity property: an inclusion \( U \subset V \) induces a map \( \text{SH}^{[a, b)}(V) \to \text{SH}^{[a, b)}(U) \). The map \( \Psi : \text{SH}^{[a, b)}(B_r) \to \text{SH}^{[a, b)}(B_r) \) factors as

\[
\text{SH}^{[a, b)}(B_r) \to \text{SH}^{[a, b)}(U) \to \text{SH}^{[a, b)}(B_r),
\]

and hence \( \text{SH}^{[a, b)}(U) \neq 0 \) when \( \Psi \neq 0 \). The symplectic homology of a ball and the map \( \Psi \) can be calculated explicitly (see [FHW, Her1]) and, indeed, it turns out that \( \Psi \neq 0 \) for some \( b > a > 0 \) and the nearby existence in \( \mathbb{R}^{2n} \) follows. Today, this calculation can be carried out particularly easily (cf. [BPS, CGK]) if one makes use of Poźniak’s theorem giving Floer homology in the Morse–Bott non-degenerate case, [Poz].

This method also applies in the setting of the generalized Weinstein–Moser theorem discussed in the introduction. Let \( U \) be a neighborhood of a closed symplectic submanifold \( M \) of a geometrically bounded symplectically aspherical manifold \( W \). Then, by taking suitably defined symplectic tubular neighborhoods of \( M \) as \( B_r \) and \( B_R \), one can show that \( \text{SH}^{[a, b)}(U) \neq 0 \), [CGK]. This leads to a proof of a nearby existence theorem for narrow shells enclosing \( M \) or small values of functions having a minimum (say, equal to zero) along \( M \). Note that, since here we utilize Floer homology only for intervals that do not contain zero, the method can be used for geometrically bounded manifolds which are not necessarily convex.

Furthermore, this method can also be cast in the framework of energy selectors defined on a suitable class of functions (cf. [Her1]). However, the benefits of this approach are unclear and we omit the details here.

**Remark 2.3 (Stability of the area spectrum, [CFHW]).** As a side remark, let us mention one more application of symplectic homology and Poźniak’s theorem. Here, again, we focus on the main idea of the argument rather than on a complete proof. Let \( \Sigma \) be a smooth hypersurface in \( W \). The area (or action) spectrum \( \mathcal{A}(\Sigma) \) is the collection of symplectic areas bounded by contractible closed characteristics on \( \Sigma \), including iterated closed characteristics. It is not hard to see using Poźniak’s theorem that, when \( \Sigma \) is the boundary of an open domain \( U \),

\[
\mathcal{A}(\Sigma) = \{ a \in \mathbb{R} \mid \text{SH}^{[a-\epsilon, a+\epsilon)}(U) \neq 0 \text{ for any small } \epsilon > 0 \},
\]

provided that \( \Sigma \) has contact type and all (iterated) closed characteristics on \( \Sigma \) are non-degenerate. (An additional, more subtle, argument is needed here when different closed characteristics bound equal areas.) Observing that the right hand side of this equality depends solely on \( U \), we conclude that under these hypotheses \( \mathcal{A}(\Sigma) \) is determined by \( U \). This is the stability of the area spectrum theorem, [CFHW]. More specifically, let \( \Sigma \) and \( \Sigma' \) be smooth contact type hypersurfaces in \( W \) bounding open domains \( U \) and \( U' \), respectively. Assume that all (iterated) closed characteristics on \( \Sigma \) and \( \Sigma' \) are non-degenerate and that there exists a symplectomorphism of \( W \) sending \( U \) to \( U' \). Then, \( \mathcal{A}(\Sigma) = \mathcal{A}(\Sigma') \).

**2.5. Other applications.** In this section we briefly discuss some other applications of the action selector, which are important for what follows.

2.5.1. **Homological capacity.** An action selector \( \sigma \) can be used to introduce an invariant, the homological capacity \( c_{\text{hom}}(U) \), of a domain \( U \) in a symplectic manifold \( W \). Assume first that \( W \) is open and convex at infinity. To an action selector \( \sigma \) on
We associate a function $c_{\text{hom}}$ on open subsets of $W$ by setting
\[ c_{\text{hom}}(U) = \sup\{ \sigma(H) \mid H \in \mathcal{H}_U \} \in (0, \infty]. \]
By definition, $c_{\text{hom}}(U) < \infty$ if and only if $\sigma$ is a priori bounded on $U$, and hence the nearby existence theorem holds for $U$.

When $W$ is closed, we set
\[ c_{\text{hom}}(U) = \sup\{ \gamma(H) \mid H \in \mathcal{H}_U \} \in (0, \infty]. \]

The homological capacity is invariant under symplectomorphisms of $W$, monotone with respect to inclusions of domains, and homogeneous of degree one with respect to scaling of $\omega$, i.e.,
\[ c_{\text{hom}}(U, \lambda \omega) = |\lambda| c_{\text{hom}}(U, \omega) \text{ for any non-zero } \lambda \in \mathbb{R}. \]

Here we do not touch upon the general definition and properties of symplectic capacities and refer the reader to, e.g., [Ha, Her1, Her2, Ho1, HZ2, HZ3] for a detailed discussion of this subject.

In what follows we will always assume that $c_{\text{hom}}$ is associated to $\sigma = \sigma_{\text{max}}$ or $\gamma = \sigma_{\text{max}} - \sigma_{\text{min}}$. Then, it is not hard to show that $c_{\text{hom}}(B_R) = \pi R^2$ by calculating the Floer homology of bump functions on $B_R$ (see [FHW] and Section 2.4). Similarly, $c_{\text{hom}}(S^2) = \text{area}(S^2)$. However, $c_{\text{hom}}(W) = \infty$, when $W$ is a closed orientable surface other than $S^2$ or $W = \mathbb{T}^2 \times S^2$ or $W = \mathbb{T}^4$.

In Section 3.3 we will utilize the homological capacity $c_{\text{hom}}$ as an upper bound for the Hofer–Zehnder capacity.

2.5.2. Hofer’s geometry. Denote by $\mathcal{D}$ the universal cover of the group of Hamiltonian symplectomorphisms of $W$. To be more precise, $\mathcal{D}$ is formed by time-one flows $\varphi_K$ of Hamiltonians $K \in \mathcal{H}_W$, where each $\varphi_K$ comes together with the homotopy class (with fixed end points) of the path $\varphi^t_H$ for $t \in [0, 1]$. Hofer’s norm $\| \cdot \|_H$ is defined as
\[ \| K \|_H = \int_0^1 (\max K_t - \min K_t) \, dt \]
for $K \in \mathcal{H}_W$. For $\psi \in \mathcal{D}$, set
\[ \rho(\psi) = \inf\{ \| K \|_H \mid K \text{ generates } \psi, \text{i.e., } \psi = \varphi_K \}. \]
It is easy to see that $\rho(\psi, \varphi) := \rho(\psi \varphi^{-1})$ is a bi-invariant metric on $\mathcal{D}$, provided that $\rho$ is non-degenerate, i.e.,
\[ \rho(\psi) > 0 \text{ iff } \psi \neq \text{id}. \]

It turns out that $\rho$, known as Hofer’s metric, is indeed non-degenerate for any symplectic manifold $W$. Non-degeneracy of $\rho$ has been established through a series of more and more general results starting with $W = \mathbb{R}^{2n}$ (see e.g., [HZ3]) and ending with the proof for an arbitrary $W$ in [LMcD1]. We refer the reader to [Po3] for an introduction to Hofer’s geometry and further references. Here we only outline the proof of non-degeneracy for symplectically aspherical convex manifolds.

Assume first that $W$ is open and convex at infinity. Let $\varphi_K \neq \text{id}$. Pick an open set $U$ displaced by $\varphi_K$, i.e., such that $\varphi_K(U) \cap U = \emptyset$, and let $H \in \mathcal{H}_U$. The idea is to again utilize (2.5), but this time to obtain a lower bound for the right hand side. Namely, (2.3) yields the inequality $\sigma(K) \leq \int_0^1 \max K_t \, dt$ and also
\( \sigma(K^-) \leq -\int_0^1 \min K_t \, dt \), as can be easily seen from the definition of \( K^- \). Therefore, by (2.5),
\[
\sigma(H) \leq \| K \|_H,
\]
and hence, once \( \text{supp} \, H \subset U \) and \( \sigma(H) > 0 \),
\[
0 < \sigma(H) \leq \rho(\varphi_K). \tag{2.8}
\]
This proves non-degeneracy of \( \rho \) for convex at infinity symplectically aspherical open manifolds. Furthermore, let us define the displacement energy of \( U \) as \( e(U) = \inf \rho(\psi) \), where \( \psi \in \mathcal{D} \) displaces \( U \). Then (2.8) translates into the upper bound
\[
c_{\text{hom}}(U) \leq e(U), \tag{2.9}
\]
which is a minor improvement over the upper bound \( c_{\text{hom}}(U) \leq 2e(U) \) established in [FS].

When \( W \) is closed and symplectically aspherical, we argue in a similar fashion. Let, as above, \( \varphi_K \) displace \( U \). Note that \( \gamma(K) \leq \rho(\varphi_K) \), by (2.6). Hence, we conclude from (2.7) that
\[
\gamma(H) \leq 2\rho(\varphi_K)
\]
for any \( H \in \mathcal{H}_U \). When \( H \) is a \( C^2 \)-small nonzero function, \( \gamma(H) = \int_0^1 (\max H_t - \min H_t) \, dt > 0 \). Therefore, \( \rho(\varphi_K) > 0 \), which proves non-degeneracy of \( \rho \). Furthermore, we also obtain the upper bound, [Sc3]:
\[
c_{\text{hom}}(U) \leq 2e(U). \tag{2.10}
\]
In fact, the more accurate upper bound (2.9) still holds in this case; see [FGS].

Non-degeneracy of \( \rho \) for an arbitrary symplectic manifold has been established in [LMcD1] by showing, using entirely different methods, that
\[
c_{\text{Gr}}(U) \leq 2\rho(\varphi_K),
\]
where \( c_{\text{Gr}}(U) \) is the Gromov capacity of \( U \), i.e.,
\[
c_{\text{Gr}}(U) = \sup \{ \pi R^2 \mid B_R \text{ symplectically embeds into } U \}.
\]
Note also that since \( c_{\text{Gr}}(U) \leq c_{\text{hom}}(U) \), (2.8) yields that in fact \( c_{\text{Gr}}(U) \leq \rho(\varphi_K) \), whenever \( W \) is symplectically aspherical, open, and convex at infinity.

3. The almost existence theorems

Virtually all known proofs of the almost existence theorems are based on the notion of the Hofer–Zehnder capacity, [HZ3], with the exception of [St] which precedes this notion.

3.1. The Hofer–Zehnder capacity and almost existence. Let \( V \) be a symplectic manifold without boundary. Denote by \( \mathcal{H}_{HZ}(V) \) the class of smooth non-negative functions \( K \) on \( V \) such that
\begin{itemize}
  \item \( K \) is compactly supported if \( V \) is not closed or \( K \) vanishes on some open set if \( V \) is compact,
  \item \( K \) is constant near its maximum.
\end{itemize}
We will refer to such functions as Hofer–Zehnder functions. Also, let us call a non-trivial periodic orbit of \( K \) with period \( T \leq 1 \) fast. Otherwise, an orbit will be called slow. A Hofer–Zehnder function without non-trivial fast periodic orbits
will be said to be admissible. Following [HZ2, HZ3], recall that the Hofer–Zehnder capacity of $V$ is defined as

$$c_{HZ}(V) = \sup \{ \max K \mid K \in H_{HZ}(V) \text{ and } K \text{ is admissible} \} \in (0, \infty].$$

The capacity $c_{HZ}(V)$ does not change when the assumptions that $K$ is non-negative and/or that $K$ is constant near its maximum are dropped, [GG3]. The Hofer–Zehnder capacity has the same general properties as the homological capacity. One should think of $c_{HZ}$ as a higher dimensional analogue of the area. We will elaborate on this point in Section 4 and here we only mention that by using the area–period relation (see Section 4) one can show that for closed orientable surfaces $c_{HZ}$ is exactly equal to the area, [Sib], in contrast with $c_{hom}$.

The following result asserts that to prove almost existence in $V$, it suffices to establish that $c_{HZ}(V)$ is finite.

**Proposition 3.1** ([HZ3]). Assume that $c_{HZ}(V) < \infty$. Then the almost existence theorem holds for proper $C^2$-functions $H$ on $V$: periodic orbits of $H$ exist on almost all, in the sense of measure theory, regular levels of $H$. Equivalently, for every shell $\{ \Sigma_s \}$ which bounds a domain in $V$, the hypersurfaces $\Sigma_s$ carry closed characteristics for almost all $s$.

This proposition is a rather simple consequence of the definition of $c_{HZ}$ and the Arzela–Ascoli theorem; see [HZ3]. The assumption that $\{ \Sigma_s \}$ bounds a domain is superfluous, [MS]. In Section 4 we will prove a more precise version of Proposition 3.1.

One can also incorporate the homotopy class of an orbit in the almost existence theorem and in the definition of $c_{HZ}$, [Sc3]. The simplest way to do this is as follows. Fix an ambient symplectic manifold $W$. Let us now modify the definition of the Hofer–Zehnder capacity by requiring $V$ to be an open subset of $W$ and requiring $K \in H_{HZ}(V)$ to have no non-trivial contractible in $W$ fast periodic orbits. We denote the resulting capacity by $c_{HZ}^o(V)$. Then Proposition 3.1 holds for contractible in $W$ periodic orbits provided that $c_{HZ}^o(V) < \infty$. Clearly, $c_{HZ}(V) = c_{HZ}^o(V)$, when $W$ is simply connected, and

$$c_{HZ}(V) \leq c_{HZ}^o(V) \quad (3.1)$$

in general. The strict inequality is possible: for example, $c_{HZ}(S^1 \times (0, 1)) = \text{area}(S^1 \times (0, 1)) < c_{HZ}^o(S^1 \times (0, 1)) = \infty$, where we view the annulus $S^1 \times (0, 1)$ as a subset of itself. This also shows that the choice of the ambient manifold, even though it is not included in the notation, effects the value of $c_{HZ}^o$. (Replace $W = S^1 \times (0, 1)$ by $W = \mathbb{R}^2$.) In this connection, we also note that $c_{HZ}(U) < \infty$ for any bounded open subset $U$ of $T^*\mathbb{T}^n$, [Ji], while $c_{HZ}^o(U) = \infty$ when $U$ contains the zero section. Indeed, $U$ can be symplectically embedded into a bounded subset of $\mathbb{R}^{2n}$ since $\mathbb{T}^n$ admits a Lagrangian embedding in $\mathbb{R}^{2n}$. In particular, the almost existence theorem holds in $T^*\mathbb{T}^n$.

Beyond dimension two, little is known about the capacity $c_{HZ}$ in contrast with the capacity $c_{HZ}^o$. For example, all results discussed in what follows deal with contractible periodic orbits and thus concern the capacity $c_{HZ}^o$. Of course, by (3.1), an upper bound for $c_{HZ}^o$ implies an upper bound for $c_{HZ}$.

3.2. **Finiteness of the Hofer–Zehnder capacity.** In view of Proposition 3.1, to prove the almost existence theorem in $V$ it suffices to show that $c_{HZ}(V) < \infty$. Let
us start by proving that the Hofer–Zehnder capacity of a bounded domain in $\mathbb{R}^{2n}$ is finite.

**Theorem 3.2 ([HZ2, HZ3]).** Let $U$ be a bounded domain in $\mathbb{R}^{2n}$. Then $c_{HZ}(U) < \infty$. Moreover, $c_{HZ}(B_R) = \pi R^2$, where $B_R$ is the ball of radius $R$.

**Proof.** By monotonicity, the first assertion is an immediate consequence of the inequality $c_{HZ}(B_R) < \infty$. To show that $c_{HZ}(B_R) = \pi R^2$, it suffices to prove that $c_{HZ}(B_R) \leq \pi R^2$; the opposite inequality is easy. (It is usually the case that establishing an upper bound for the capacity is hard while a lower bound can be easily obtained by definition.) The key is the following result:

**Proposition 3.3 ([HZ3]).** Let $H$ be a Hofer–Zehnder function on $B_R$ such that $\max H > \pi R^2$. Then the Hamiltonian flow of $H$ has a non-trivial one-periodic orbit.

Following [GG3], let us outline a proof of the proposition which relies on the calculation of the Floer homology and generalizes to some other situations.

Let $H$ be as in Proposition 3.3. There exist non-negative functions $K^- \leq H \leq K^+$, supported in $B_R$ and depending only on the distance to the origin, and such that the action of $K^\pm$ on the nearest to the origin non-trivial one-periodic orbits is greater than $\max K^+$. (These orbits are Hopf circles on a sphere enclosing the origin.) To find such functions, we make use of the assumption that $H$ is constant near its maximum and take, as $K^\pm$, functions that squeeze $H$ from above and below as tightly as possible and that depend only on the distance to the origin. Now, applying Poźniak’s theorem [Poz], one can show that

$$HF^{[a, \infty)}_{m}(K^\pm) = \mathbb{Z}_2$$

for some $m$ and $a > \max K^+ \geq \max H$. This homology group is “generated” by the non-trivial one-periodic orbits closest to the origin. Moreover, by examining the natural homotopy from $K^+$ to $K^-$, one can prove that the monotonicity map, which factors as

$$\mathbb{Z}_2 = HF^{[a, \infty)}_{m}(K^+) \rightarrow HF^{[a, \infty)}_{m}(H) \rightarrow HF^{[a, \infty)}_{m}(K^-) = \mathbb{Z}_2,$$

is an isomorphism. Hence, $HF^{[a, \infty)}_{m}(H) \neq 0$. Every trivial periodic orbit of $H$ has action in the interval $[0, \max H]$. Thus, since $a > \max H$, the flow of $H$ must have a non-trivial one-periodic orbit. This concludes the proof of Proposition 3.3 and Theorem 3.2.

Observe that Proposition 3.3 is stronger than what is needed: the proposition guarantees the existence of a non-trivial orbit with period $T = 1$ while $T \leq 1$ is sufficient to prove Theorem 3.2. Yet, this fact, somewhat surprisingly, appears to have no interesting applications.

Also note that there are compact aspherical symplectic manifolds with infinite Hofer–Zehnder capacity. The basic example of such a manifold is Zehnder’s torus, [Ze], i.e., a torus $\mathbb{T}^{2n}$ with an irrational symplectic structure. All other examples of such manifolds known to the author are derivatives of Zehnder’s torus. As has been pointed out in the introduction, the nearby/almost existence theorem also fails for Zehnder’s torus.

Let us now elaborate on some general principles concerning the almost existence or nearby existence theorem and the Weinstein conjecture. The existence problem
for periodic orbits on a given energy level is dual in a certain, rather vague, sense to the existence problem for periodic orbits of a fixed period. The orbits of a fixed period are known to exist (Arnold’s conjecture) and a variety of methods (e.g., Floer homology) of proving this fact have been developed during the last two decades. In contrast with this, unless the function is assumed to be convex (and then the duality acquires a precise meaning), the problem for a fixed energy level may fail to have a solution, as counterexamples to the Hamiltonian Seifert conjecture show. Moreover, there seems to be no direct method to tackle the problem of existence of periodic orbits for a sufficiently large set of energy values (e.g., dense or full measure). All known methods, including those outlined above, first reduce the problem to proving that the flow of a function $H$ with sufficiently large variation $\max H - \min H$ possesses (non-trivial, fast, or period one) periodic orbits. Thus, all proofs of these theorems hinge on a general principle that a compactly supported function with sufficiently large variation must have fast non-trivial periodic orbits or even one-periodic orbits if the function is constant near its maximum. This principle, which is often true but fails in general (Zehnder’s torus!), can already be proved for some manifolds by utilizing, for example, Floer homology.

Coming back to the discussion of the Hofer–Zehnder capacity, let us mention just one more of its properties. Namely, $c_{HZ}(Z_R) = \pi R^2$, where $Z_R = D^2_R \times \mathbb{R}^{2n-2} \subset \mathbb{R}^{2n}$ is a symplectic cylinder. [HZ2, HZ3]. (Thus, the Hofer–Zehnder capacity of $Z_R$ is finite even though its volume is infinite.) This can be easily proved as follows, [GG3]. First note that our proof of Proposition 3.3 readily extends to ellipsoids and shows that the capacity of an ellipsoid is equal to its minimal cross-section area (by a plane through the origin). Then exhausting $Z_R$ by more and more elongated ellipsoids we see that $c_{HZ}(Z_R) = \pi R^2$.

The next application of our method concerns the generalized Weinstein–Moser conjecture, the motion of a charge in a magnetic field, and the relative almost existence theorem. Let $U$ be a neighborhood of a closed symplectic submanifold $M$ of a geometrically bounded symplectically aspherical manifold $W$. Then, we can emulate the proof of Proposition 3.3 by taking, as $K^{\pm}$, functions that depend only on the distance to $M$ in a suitable metric and that are supported near $M$. As a consequence, we obtain the relative almost existence theorem: near $M$, almost all levels of a function $H$ attaining a minimum along $M$ carry contractible periodic orbits, [GG3]. (Note that this, in turn, requires replacing the ordinary Hofer–Zehnder capacity in Theorem 3 by its relative counterpart. The relative Hofer–Zehnder capacity $c_{HZ}(W, M)$, where $M \subset W$ is a compact subset, is defined just as the ordinary capacity, but the function $K$ is required to attain its maximum along $M$, see, e.g., [GG3, HZ2, La] for more details.)

Furthermore, a more refined version of the Floer homology calculation outlined here can be used to show that $c_{HZ}^0(U) < \infty$ when $U$ is a small neighborhood of a closed symplectic submanifold $M$ with homologically trivial normal bundle in a closed symplectically aspherical manifold, [Ke3]. This implies the almost existence theorem in $U$.

3.3. Comparison of $c_{HZ}^0$ with $c_{hom}$. In this section we will outline a different approach to obtaining an upper bound for $c_{HZ}^0$. To be more specific, we prove that $c_{HZ}^0(U) \leq c_{hom}(U)$. In our proof of this inequality we draw heavily on [HZ3, FS, Sc3], but some details appear to be new.
Let $W$ be an open manifold (with $\omega|_{\pi_2(W)} = 0$) on which a continuous selector $\sigma$ satisfying (AS1) and (AS2) is defined. Assume also that

$$\sigma(H) = \max H, \text{ when } H \text{ is } C^2\text{-small and independent of time}. \quad (3.2)$$

This requirement strengthens (AS3) and holds for the energy selector defined in Section 2.1.2 because the Floer complex of a $C^2$-small autonomous Hamiltonian is equal to its Morse complex; see [FHS]. Let $c_{\text{hom}}$ be the associated homological capacity.

**Proposition 3.4** ([FS, Sc3]). $c_{\text{HZ}}(U) \leq c_{\text{hom}}(U)$ for any $U \subset W$.

This, by (3.1), implies that $c_{\text{HZ}}(U) \leq c_{\text{hom}}(U)$.

**Proof.** The proposition is an immediate consequence of the following

**Lemma 3.5.** Let $H \in \mathcal{H}_W$ be an autonomous Hamiltonian whose flow has no fast non-trivial contractible periodic orbits. Then $\sigma(H) = \max H$.

To derive the proposition from the lemma, we just use the definition of the capacities:

$$c_{\text{hom}}(U) = \sup \{ \sigma(H) \mid H \in \mathcal{H}_U \} \geq \sup \{ \max H \mid H \text{ as in the lemma} \} = c_{\text{HZ}}(U).$$

The assertion of the lemma is very plausible. When $W$ is convex at infinity and $\sigma = \sigma_{\text{max}}$, one can argue as follows. By continuity, it suffices to prove the lemma for a generic $H$. Then, since $H$ does not have non-trivial contractible periodic orbits, the Floer complex of $H$ is generated by the critical points of $H$ and hence $\sigma(H)$ is a critical value of $H$. (Moreover, without loss of generality we may assume that the eigenvalues of $d^2H$ at the critical points of $H$ are small. Indeed, this can be achieved by replacing $H$ by $f \circ H$, where $f : \mathbb{R} \to \mathbb{R}$ is a suitable diffeomorphism which is $C^0$-close to identity, see [GG3, Schl]. Then $\sigma(H)$ must be the value of $H$ at one of its local maxima, for maxima are the only critical points of $H$ with correct index. Hence, $\max H = \sigma(H)$ if $H$ has a unique maximum.) However, to show that $\sigma(H)$ is delivered by the global maximum, one has to analyze the Floer differential of $H$ and this indeed can be done; see, e.g., [FS, Sc3]. At this point one considers the homotopy $\lambda H$ with $\lambda \in (0, 1]$ and fully uses the assumptions on $H$. Note also that it seems to be unknown whether in the hypotheses of the lemma fast orbits can be replaced by one-periodic orbits.

One can also prove the lemma in a formal set-theoretic way, using only the continuity of $\sigma$, (AS1), (AS2) and (3.2). Denote by $\mathcal{S}$ the action spectrum $\mathcal{S}(H)$. Observe that by the hypothesis of the lemma, every contractible one-periodic orbit of $\lambda H$ is trivial for all $\lambda \in (0, 1]$, and hence the action spectrum $\mathcal{S}(\lambda H)$ is again comprised entirely of the critical values of $\lambda H$. Therefore, $\mathcal{S}(\lambda H) = \lambda \mathcal{S}$. Set

$$\sigma(\lambda) = \sigma(\lambda H) \in \lambda \mathcal{S}.$$ 

Then $\sigma(\lambda)/\lambda$ is a continuous function of $\lambda$ with values in $\mathcal{S}$. Since the action spectrum $\mathcal{S}$ is a nowhere dense, $\sigma(\lambda)/\lambda = \text{const.}$ For a small $\lambda > 0$, the function $\lambda H$ is $C^2$-small and hence $\sigma(\lambda H) = \max(\lambda H) = \lambda \max H$. It follows that $\sigma(\lambda) = \lambda \max H$ for all $\lambda \in (0, 1]$ which concludes the proof of the lemma and of the proposition.

Proposition 3.4 also holds (see [Sc3]) when $W$ is closed and symplectically aspherical and $c_{\text{hom}}$ is the homological capacity defined in Section 2.5.1 for $\gamma =$
\[ \sigma_{\text{max}} - \sigma_{\text{min}}. \] This can be proved essentially in the same way as the result for open manifolds.

Combining Proposition 3.4 with (2.9) and (2.10), we infer that a displaceable open set has a finite Hofer–Zehnder capacity:

\[
\begin{align*}
c_{\text{HZ}}^a(U) &\leq e(U) \text{ if } W \text{ is open,} \\
c_{\text{HZ}}^a(U) &\leq 2e(U) \text{ if } W \text{ is closed,}
\end{align*}
\]
whenever \( W \) is convex and symplectically aspherical. These are versions of the so-called “capacity–displacement energy inequality”. In particular, the almost existence theorem holds for displaceable sets. For example, let \( L \subset W \) be a closed non-Lagrangian submanifold of middle dimension whose normal bundle has a non-vanishing section. Assume also that on \( W \) there exists a selector \( \sigma \) satisfying (3.2). Then, by [LS, Pol1], a small neighborhood \( U \) of \( L \) is displaceable and hence \( c_{\text{HZ}}^a(U) < \infty \). As in Section 2.1.3, this fact leads to a local version of the almost existence theorem for twisted cotangent bundles over surfaces (other than the torus) or, in higher dimensions, for exact magnetic fields; see [FS, Ma1].

Proposition 3.4 explains why the almost existence theorem can often be proved in the same setting as that of the weaker nearby existence theorem. Indeed, most of the proofs of the former, with some notable exceptions, rely on first establishing that a version of the homological capacity is finite. However, in this case the Hofer–Zehnder capacity is also finite, which implies almost existence theorem.

**Remark 3.6.** Proposition 3.3 suggests that in the definition of the Hofer–Zehnder capacity one may replace the condition that \( K \) has no fast periodic orbits by a weaker requirement that it has no one-periodic orbits (cf. [Ke3]). The resulting invariant is still a capacity. For this capacity the upper bound from Proposition 3.4 may fail to hold, unless an additional requirement on the critical set of \( K \) is imposed, cf. [FGS]. However, this capacity is \textit{a priori} bounded on \( \mathbb{R}^{2n} \) and sufficient for the proof of the almost existence theorem. Furthermore, the inequality \( T \leq C_{c_H}(h) \) in Proposition 4.2 is then replaced by the equality. (In fact, when \( c_{\text{HZ}} \) is modified in this way, for every converging sequence \( (c_H(h) - c_H(h_i))/h_i \to A \) there exists an \( A \)-periodic orbit (not necessarily simple) on the level \( H = h \).) As a consequence, this capacity is represented on hypersurfaces of restricted contact type in \( \mathbb{R}^{2n} \) (cf. Proposition 4.4).

### 3.4. Holomorphic curves and almost existence

The holomorphic curve approach to the nearby existence and almost existence theorems goes back to [FHV, HV2]. Here we outline only the main idea of the method, following essentially [HV2] and omitting technical details which are in some instances quite involved.

Let \((W, \omega)\) be a closed symplectic manifold and let \( L_- \) and \( L_+ \) be two disjoint closed submanifolds of \( W \). Fix a generic almost complex structure \( J \) compatible with the symplectic structure and consider the space of \( J \)-holomorphic spheres \( u: S^2 \to W \) in a given free homotopy class \( A \in [S^2, W] \) and with the North pole on \( L_+ \) and the South pole on \( L_- \). The resulting metric space of \( J \)-holomorphic curves is not compact: the non-compact group of conformal transformations of \( S^2 \) fixing the poles acts properly on it. To make this space compact, we require that

\[
\int_{D_-} u^* \omega = \langle \omega, A \rangle / 2,
\]
where $D_-$ is the Southern hemisphere in $S^2$. This condition eliminates the conformal flow from one pole to another, parametrized by $\mathbb{R}$, and forces the resulting space $\mathcal{M}_0$ to be compact. This space depends on $L_\pm$, $A$, and $J$.

For a generic $J$, the space $\mathcal{M}_0$ is a smooth manifold and its free $S^1$-equivariant cobordism class $[\mathcal{M}_0]$ is independent of $J$. (Here the action of $S^1$ on $\mathcal{M}_0$ arises from the $S^1$-action on $S^2$ by rotations about the vertical axis.) Note that in reality the manifold $W$ and the class $A$ must satisfy some additional conditions, e.g., $A$ must be minimal, i.e., $\langle \omega, A \rangle = m(W, \omega)$; the definition of $m(W, \omega)$ is recalled below.

Let now $H$ be a smooth non-negative function on $W$ such that $H \equiv 0$ near $L_-$ and $H \equiv \max H$ near $L_+$. Consider the perturbed Cauchy–Riemann equation for the maps $u : S^2 \to W$:

$$\bar{\partial}Ju + \lambda \nabla H = 0,$$

where $\lambda \geq 0$ is real parameter. Note that since $\nabla H = 0$ in some neighborhoods of $L_\pm$, this equation simply means that $u$ is holomorphic near the poles. To make sense of this equation away from the poles, we view $S^2$ as the cylinder $S^1 \times \mathbb{R}$ with the two poles attached. Then the coordinates on the cylinder are used to define $\bar{\partial}Ju$ as a vector field along $u$.

Let $\mathcal{M}_\lambda$ be the space of solutions of (3.6) which are in the class $A$, send the North and South poles to $L_\pm$, and satisfy (3.5). When $\lambda = 0$, we obtain the space $\mathcal{M}_0$ introduced above.

In general, the solutions of (3.6) can be thought of as gradient trajectories for $A_\lambda H$ connecting trivial periodic orbits on $L_+$ with those on $L_-$. Then, a calculation shows that the difference of actions on these periodic orbits, which is $\lambda \max H$, is bounded from above by the symplectic area of $u$, i.e.,

$$\lambda \max H \leq \langle \omega, A \rangle.$$

As a consequence, $\mathcal{M}_\lambda = \emptyset$ when $\lambda$ is large.

Let us examine now the disjoint union of the spaces $\mathcal{M}_\lambda$. Under suitable genericity hypotheses, this is a smooth manifold. If this manifold is compact, it gives a cobordism from $\mathcal{M}_0$ to the empty set, i.e., $[\mathcal{M}_0] = 0$. Just as in Floer’s theory, compactness can fail only when a family of solutions $u_\lambda \in \mathcal{M}_\lambda$ converges as $\lambda \to \lambda_0 \leq \langle \omega, A \rangle / \max H$ to a broken solution which “hangs up” on a contractible one-periodic orbit of $\lambda_0 H$, i.e., a contractible $1/\lambda_0$-periodic orbit of $H$. We conclude that $H$ must have contractible one-periodic orbits, provided that $\max H > \langle \omega, A \rangle$ and $[\mathcal{M}_0] \neq 0$. (Note that at this point one still has to show that the orbits found are non-trivial.)

This approach leads to a few versions of the almost existence theorem, all relying on the same basic requirement that the space $\mathcal{M}_0$ is not cobordant to zero. In other words, the space of holomorphic curves “connecting” $L_+$ and $L_-$ should be sufficiently large for the method to apply. In particular, it may be helpful to start with larger submanifolds $L_\pm$ or with a function $H$ such that $H \equiv 0$ and $H \equiv \max H$ on large subsets. (The trade-off is that this may result in the almost existence theorem for a restricted class of functions $H$.) Let us illustrate these considerations by some examples.

When $L_\pm$ are points and the method is applicable to $W$ and $A$, we conclude that $c_{u2}(W) \leq \langle \omega, A \rangle$ and hence prove the almost existence theorem in $W$. There are, however, rather few manifolds $W$ for which $[\mathcal{M}_0] \neq 0$ when $L_\pm$ are points. One example is $\mathbb{C}P^n$ with the standard symplectic form normalized as the reduction of
the unit sphere in $\mathbb{C}^{n+1}$. In this case the standard $J$ is already generic, $M_0 = S^1$ with $S^1$ acting by translations, and we see that $c_{HZ}(\mathbb{C}P^n) = \pi$, [HV2]. It is also possible that this reasoning can be utilized to show that other coadjoint orbits of compact semi-simple Lie groups have finite Hofer–Zehnder capacity.

Let now $W = P \times S^2$, where $(P, \beta)$ is a compact symplectic manifold, as in [HV2]. Then taking the class of the fiber $S^2$ as $A$ and applying, with some modifications, this method to $L_− = P \times \{0\}$ and $L_+ = P \times \{\infty\}$, one can show that the relative capacity $c_{HZ}(P \times D_R^2, P \times \{0\}) = \pi R^2$ as long as area($D_R^2$) = $\pi R^2$ < $m(P, \beta, J)$, where

$$m(P, \beta, J) := \inf \left\{ \int_{S^2} u^* \beta > 0 \mid u \text{ is a non-constant } J\text{-holomorphic sphere} \right\}.$$ 

In particular, $m(P, \beta, J) = \infty$ when $\pi_2(P) = 0$, [FHV]. Note also that $m(P, \beta, J)$ is always positive by Gromov’s compactness theorem. Furthermore, assume that

$$m(P, \beta) := \inf \left\{ \int_{S^2} u^* \beta > 0 \mid u : S^2 \to P \right\} \geq 0.$$ 

In contrast with $m(P, \beta, J) \geq m(P, \beta)$, this constant can be zero. (For instance, this is the case for $P = S^2 \times S^2$ where the area of the first component is 1 and the second component has an irrational area.) By taking a point in $P \times \{0\}$ as $L_−$ and $L_+$ as before, one can show that $c_{HZ}(P \times S^2, L_+) = \text{area}(S^2)$ as long as area($S^2$) < $m(P, \beta)$, [HV2]. As a consequence, $c_{HZ}(P \times D_R^2) = \pi R^2$ if $\pi R^2 < m(P, \beta)$. (In fact, there are strong indications that $c_{HZ}(P \times D_R^2) = \pi R^2$ for any geometrically bounded symplectic manifold $P$ and any $R > 0$; see [MDSI, Schl].) Symplectic manifolds $P$ for which $m(P, \beta) > 0$ are called rational.

The method also extends to the setting where $P$ is not closed but is geometrically bounded, [Lu1]. (These results have further applications to the existence problem for periodic orbits of a charge in a magnetic field, see [Ma1, Ma2].) For some other incarnations and applications of the holomorphic curve method, we refer the reader to [GL, Ke3, LT, Lu2, Lu3, Lu4]. In particular, for uniruled symplectic manifolds (such as $\mathbb{T}^2 \times S^2$) and symplectic toric manifolds the Weinstein conjecture was established in [Lu2, Lu4]. In the context of the generalized Weinstein–Moser conjecture, it was proved in [Ke3] that a small neighborhood of a rational symplectic submanifold $M \subset W$ has finite Hofer–Zehnder capacity $c_{HZ}$.

Finally, one may replace holomorphic spheres by holomorphic curves of higher genus to obtain a sufficiently large space $M_0$ as in [LT, Lu3]. Extra care in interpreting (3.6) is then needed. For instance, imposing some additional conditions on $H$ may be necessary, e.g., requiring $H$ to be locally constant outside of a shell separating $L_−$ from $L_+$. This, depending on the details of the approach, leads to a version of either the nearby existence or almost existence theorem, cf. [LT, Lu2, Lu3, Lu4].

The holomorphic curve approach in the form outlined above does not apply to compact manifolds that generically have too few holomorphic curves. Among such manifolds is, for example, $\mathbb{T}^4$ with the standard symplectic structure. It is not known whether or not the Hofer–Zehnder capacity of $\mathbb{T}^4$ is finite.

3.5. **Hofer’s geometry and almost existence.** Applications of Hofer’s geometry to nearby and almost existence theorems are based on a principle relating minimizing properties of geodesics in Hofer’s metric and fast periodic orbits. Namely, consider the Hamiltonian flow $\varphi^t_H$ which is a one-parameter subgroup in $D$ and
hence can be viewed as a geodesic in $D$. Then, conjecturally, $\varphi_t^H$ is length minimizing for $t \in [0, 1)$, provided that $H$ has no fast periodic orbits; see [MDSI] and references therein. Various particular cases of this conjecture have been proved. We refer the reader to [Pol3] for a detailed discussion and additional references and to [MDSI], following [LMcD2], for more recent results relevant to our discussion.

The application of Hofer’s geometry to the circle of questions considered in this paper was pioneered in [Pol2]. Below we will just briefly indicate the logic of the argument following closely the most recent work [Schl] and in fact suppressing the connection with minimizing properties of geodesics in Hofer’s metric.

Let $W$ be a geometrically bounded symplectic manifold, which we do not require to be symplectically aspherical or convex, and let $H$ be an autonomous Hamiltonian supported in $U \subset W$. The cornerstone of the method is the inequality

$$c^{o}_{HZ}(U) \leq 4e(U),$$

which implies almost existence in $U$, provided that $e(U) < \infty$. (Note that stronger inequalities (3.3) and (3.4) hold when $W$ is symplectically aspherical and convex. Hence, the emphasis here is on extending the class of manifolds for which a version of the capacity–displacement energy inequality is proved.)

The proof of (3.7) is based on the following two results:

(i) Assume that $supp H \subset U \subset W$ and $\| H \|_H > 4e(U)$. Then $\rho(\varphi_H) < \| H \|_H$.

(ii) $H$ has contractible fast periodic orbits, provided that $\rho(\varphi_H) < \| H \|_H$.

The first assertion (i) is proved by the curve shortening method, ubiquitous in Hofer’s geometry and going back to [Sik]; see the references above for other incarnations of this method. As stated, the second assertion (ii) is obtained in [Schl] as a consequence of the results of [MDSI]. (This is the point where length minimizing geodesics enter the picture.)

The inequality (3.7) is sufficient to establish the almost existence theorem for only a very limited class of domains $U$. However, (3.7) can be combined with Macarini’s stabilization trick, [Ma1]. This leads to an upper bound similar to (3.7), but with $e(U)$ replaced by the displacement energy of $U \times S^1$ in $W \times T^* S^1$, [Schl], which has a somewhat broader range of applications. In particular, by modifying (3.7) in this way, one can prove the almost existence theorem for low energy periodic orbits of a charge in a non-vanishing magnetic field and a conservative force field, [Schl].

We conclude this section by pointing out that it may be possible to prove (3.7) directly, without invoking minimizing properties of geodesics in Hofer’s geometry, by adapting the argument from [MDSI].

4. The Hofer–Zehnder capacity function

What we see as a central open problem concerning the almost existence theorem is the question whether or not this theorem is sharp. Consider, for example, a smooth proper function $H : \mathbb{R}^{2n} \to \mathbb{R}$ which we assume to be bounded from below. By the almost existence theorem, almost all regular levels in $[\min H, \infty)$ carry periodic orbits. The counterexamples to the Hamiltonian Seifert conjecture show that $H$ may have a discrete collection of aperiodic levels, i.e., regular levels without periodic orbits, [G1, G3, G4, GG1, GG2, He1, He2, Ke2]. Moreover, such levels can accumulate to a degenerate critical level of $H$, [GG3]. However, it is still unknown if aperiodic levels can accumulate to a regular level either with or without
periodic orbits. In particular, it is not known whether the set of aperiodic energy values can be dense or be a Cantor set.

4.1. The definition of the capacity function. Throughout this section, we will focus on functions on \( \mathbb{R}^{2n} \) although most of our discussion carries over to functions on any manifold of bounded capacity. To concentrate on the essential part of the problem, let us assume that the function has only one critical point, the minimum. The key to the proof of the almost existence theorem is the Hofer–Zehnder capacity function associated with \( H \):

\[
c_H(h) = c_{HZ}(\{H < h\}) < \infty, \text{ where } h > \min H.
\]

This is a monotone increasing function on \((\min H, \infty)\) and the levels \( H = h \), where \( c_H \) is Lipschitz, carry periodic orbits, [HZ3]. (This is a simple consequence of the Arzela–Ascoli theorem and shortly we will recall the argument.) Since \( c_H \) is monotone increasing, it is differentiable almost everywhere and the almost existence theorem follows. (Note that there is no reason to expect every non-Lipschitz value to be aperiodic. In particular, aperiodic points are not entirely visible from the properties of \( c_H \).)

Any zero measure set is the set of non-Lipschitz points of some monotone increasing function, [Na, p. 214]. Hence, one possible approach to the problem (but not the only one) is to investigate additional properties of the capacity function which distinguish it from an arbitrary monotone increasing function. We will soon see that some of these readily arise from the Arzela–Ascoli theorem.

The following elementary observation illustrates our point, [Gi5]. Denote by \( Z \) the collection of aperiodic values of \( H \). Furthermore, let \( Z_T \) be the collection of levels where all periodic orbits have period greater than \( T \). It is clear that

\[
Z = \bigcap_{T \in \mathbb{Z}^+} Z_T
\]

and that, by the Arzela–Ascoli theorem, \( Z_T \) is open. Hence, \( Z \) is a \( G_\delta \) set. Furthermore, if \( Z \) is dense, every set \( Z_T \) is also dense. From this we infer that \( Z \) must be a residual set when \( Z \) is dense. In particular, the set \( Z \) of aperiodic values cannot be a countable dense set.

4.2. The area–period relation. The role of the capacity function in the proof of the almost existence theorem can be best understood in terms of the classical area–period relation; see, e.g., [Ar, p. 282]. Let us recall this result.

Let \( H \) be a function on \( \mathbb{R}^2 \). Denote by \( \text{area}(h) \) the area bounded by a regular level \( H = h \) and by \( T(h) \) the period of the periodic orbit on this level. (Here we are assuming that the levels are connected.)

**Proposition 4.1** (Area–period relation), \( \frac{d\text{area}(h)}{dh} = T(h) \).

Note that the same is true for any proper function on a symplectic surface, provided that the level \( H = h \) is regular. When this level is comprised of more than one connected component, the right hand side is the sum of their periods.

To prove Proposition 4.1, denote by \( \omega \) the area form (i.e., the symplectic form) on \( \mathbb{R}^2 \). Dividing \( \omega \) by \( dH \) near the level, we can write \( \omega = dH \wedge \alpha \), where \( \alpha \) is a one-form such that \( \alpha(X_H) = 1 \). Let \( \Pi_\epsilon \) be the annulus \( h \leq H \leq h + \epsilon \). The
following calculation concludes the proof:

\[
\frac{d \text{area}(h)}{dh} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} \omega = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\Omega} dH \wedge \alpha = \int_{\{H = h\}} \alpha = T(h).
\]

This result generalizes to higher dimensions when \( H \) is convex. Namely, in this case, on any level \( H = h \) there exist periodic orbits \( \gamma_l \) and \( \gamma_r \) such that the periods of these orbits are equal to the left and, respectively, right derivatives of \( c_H \) at \( h \) and the symplectic areas bounded by \( \gamma_l \) and \( \gamma_r \) are equal to \( c_H(h) \), [Ne]. When the convexity assumption is dropped, the derivative of the capacity function gives only an upper bound on the period:

**Proposition 4.2.** Assume that the lower derivative

\[
\underline{c}_H(h) = \liminf_{\delta \to 0} \frac{c_{H}(h + \delta) - c_{H}(h)}{\delta}
\]

is finite. Then the level \( H = h \) carries a periodic orbit with period \( T \leq \underline{c}_H(h) \) and, as a consequence, \( \underline{c}_H(h) > 0 \).

The proposition shows, in particular, that the Hofer–Zehnder capacity is strictly increasing. More specifically, we have

**Corollary 4.3.**

(i) The capacity function \( c_H \) is strictly increasing.

(ii) Let \( U \) and \( V \) be open bounded subsets of \( \mathbb{R}^{2n} \) such that \( \overline{U} \subset V \). Then \( c_{HZ}(U) < c_{HZ}(V) \).

It is easy to see that the assumption that the sets are bounded is essential and the requirement \( \overline{U} \subset V \) cannot be replaced by \( U \subseteq V \).

**Proof of Proposition 4.2.** The proof of the proposition is a modification of an argument from [HZ3]. Let \( h_i \to h \) be a sequence such that

\[
A_i := \frac{c_H(h_i) - c_H(h)}{h_i - h} \to A,
\]

where \( 0 \leq A < \infty \). By passing if necessary to a subsequence, we may assume that \( h_i \) converges to \( h \) either from the right or from the left. In what follows we will assume that \( h_i > h \). The other case can be handled in a similar fashion. Observe that either \( A_i > 0 \) for all (sufficiently large) indexes \( i \) or \( c_H \) is constant on some interval \([h, h + \delta]\), with \( \delta > 0 \), since \( c_H \) is monotone.

Assume first that \( A_i > 0 \) and pick a sequence \( b_i > 1 \) converging to one and a sequence

\[
0 < \epsilon_i < (b_i - 1)(h_i - h)A_i.
\]

By the definition of the Hofer–Zehnder capacity, on the domain \( \{H < h\} \) there exists an admissible function \( K_i \) without fast non-trivial periodic orbits and such that \( \max K_i = c_H(h) - \epsilon_i \). Let \( f_i \) be a monotone decreasing function on the interval \([h, h_i]\) identically equal to \( b_i(h_i - h)A_i \) near \( h \) and zero near \( h_i \) and such that \(|f'_i| \leq a_i b_i A_i \) for some sequence \( a_i \to 1 \). Set \( F_i = f_i \circ H \) on the shell \( h \leq H \leq h_i \) and smoothly extend this function to \( \mathbb{R}^{2n} \) by requiring it to be constant inside and outside of the shell. Then

\[
\max (K_i + F_i) = c_H(h_i) + [(b_i - 1)(h_i - h)A_i - \epsilon_i] > c_H(h_i).
\]

Hence, \( K_i + F_i \) must have a non-trivial fast periodic orbit which can only be a periodic orbit of \( F_i \). Therefore, \( H \) has a periodic orbit with period less than or
equal to \(a_i b_i A_i\) in the shell \(h \leq H \leq h_i\). By passing to the limit and applying the Arzela–Ascoli theorem, we conclude that \(H\) has a periodic orbit of period less than or equal to \(A = \lim a_i b_i A_i\) on the level \(H = h\).

To finish the proof it suffices to show that \(c_H\) cannot be constant on any interval \([h, h + \delta]\). (This will also be a direct proof of the corollary.) Assume the contrary. Then \(c_H(h_i) = c_H(h)\) for all large enough \(i\). In this case we again choose a sequence of decreasing functions \(f_i\) on \([h, h_i]\) such that \(f_i\) is zero near \(h_i\) and positive constant near \(h\). (Thus \(f_i\) is equal to \(\max f_i > 0\) near \(h_i\).) Furthermore, we can make the slope of \(f_i\) so small that the function \(F_i\) has no fast non-trivial periodic orbits. Let \(0 < \epsilon_i < \max f_i\) and let \(K_i\) be as above. It is clear that the function \(K_i + F_i\) has no fast non-trivial periodic orbits. On the other hand, we have

\[
\max(K_i + F_i) = c_H(h) - \epsilon_i + \max f_i > c_H(h) = c_H(h_i),
\]

and hence \(K_i + F_i\) must have fast non-trivial periodic orbits. This contradiction completes the proof. \(\square\)

A question closely related to the area–period relation is that of representability of a capacity. Let \(U\) be the domain bounded by a compact smooth hypersurface \(\Sigma\) in \(\mathbb{R}^{2n}\). We say that a capacity \(c\) is represented on \(\Sigma\) if there is a closed characteristic on \(\Sigma\) that bounds a disc of symplectic area \(c(U)\). For example, as is easy to see from the definition, \(c_{\text{hom}}\) is represented on smooth hypersurfaces of contact type. Since there are hypersurfaces without closed characteristics, no capacity is represented on every hypersurface. The Hofer–Zehnder capacity is represented on convex hypersurfaces, [HZ2]. In fact, when \(\Sigma\) is convex, \(c_{\text{H}}(U)\) is the minimal symplectic area of a closed characteristic on \(\Sigma\), [HZ2]. However, it is not known if \(c_{\text{H}}\) is represented on every contact type, or even restricted contact type, hypersurface. Arguing as in the proof of Proposition 4.2 and utilizing Lemma 3.5, one can prove the following

**Proposition 4.4.** Let \(U\) be the domain bounded by a smooth hypersurface \(\Sigma\) of restricted contact type. Then, \(c_{\text{H}}\) is sub-represented on \(\Sigma\), i.e., there exists a closed characteristic on \(\Sigma\) with area less than or equal to \(c_{\text{H}}(U)\).

Recall in this connection that even when \(U\) is star-shaped there can be a closed characteristic on \(\Sigma\) with area strictly less than \(c_{\text{H}}(U)\): the “Bordeaux bottle” (see [HZ3, p. 99]) is an example. We refer the reader to, e.g., [FS, Her2] for other representability results.

### 4.3. Period Growth and the Hofer–Zehnder Capacity Function

As is immediately clear from the definition, the Hofer–Zehnder capacity function is necessarily continuous from the left. Beyond this trivial observation, little is known about continuity or differentiability properties of the capacity function. For example, it is not known if this function can be discontinuous at regular values. One, admittedly very naive, approach to this question is based on the estimates of the period growth.

To make this more precise, fix a regular level of \(H\), say \(H = 0\). Let \(\tau(h)\) be the infimum of all periods of periodic orbits in the open shell \(0 < H < h\) or \(h < H < 0\), depending on whether \(h\) is positive or negative. Then the growth of the function \(\tau(h)\) is related to continuity and smoothness of \(c_H\) at zero as the next proposition shows.

**Proposition 4.5.**

(i) \(\tau(h) = O(1/h)\) as \(h \to 0^+\) and \(\tau(h) = o(1/|h|)\) as \(h \to 0^-\).
(ii) Assume that $\tau(h_k) \geq C/h_k$ for some sequence $h_k \to 0^+$. Then $c_H(0^+) - c_H(0) \geq C$.

(iii) Assume that $\tau(h_k) \geq C/|h_k|^\alpha$ for some sequence $h_k \to 0$ and $0 \leq \alpha \leq 1$. Then $|c_H(h) - c_H(0)| \geq C|h|^{1-\alpha}$.

The proofs of these facts follow the same line as the proof of the almost existence theorem in [HZ3], cf. the proof of Proposition 4.2. Omitting a detailed argument here, we only mention that (i) is a consequence of the fact that the capacity is bounded. In fact, $c_{HZ}(\{H < 0\}) = \infty$ if (i) fails for the left limit and $c_{HZ}(V) = \infty$ for any open set $V \supset \{H \leq 0\}$ if (i) fails for the right limit.

Remark 4.6. The analogues of (ii) and (iii) hold when the difference of capacities is replaced by the relative capacity.

Proposition 4.5 is difficult to apply to examine discontinuity or non-smoothness of the capacity function. Indeed, let, for example, $H$ be a Hamiltonian on $\mathbb{R}^{2n}$ constructed in [Gi1, Gi3, GG1, GG2, He1, He2, Ke2] such that $H = 0$ carries no periodic orbits. Then $c_H$ is not Lipschitz at zero as is clear already from the results of [HZ3]. To utilize Proposition 4.5, we would need to bound from below minimal periods on the nearby levels $H \approx 0$. However, the constructions of $H$ afford little insight into the dynamics on these levels and obtaining such lower bounds appears to be an extremely difficult problem. Preliminary estimates indicate that $c_H$ can probably be Hölder with any $\alpha \in (0, 1)$ at $H = 0$ or even fail to be Hölder. However, there is no convincing evidence that $c_H$ can be discontinuous. (Note in this connection that the property of $c_H$ to be Hölder with a specific $\alpha$ or to be continuous, just as to be Lipschitz (see [HZ3]), is determined by the level $H = 0$ and is independent of the choice of $H$.)

Consider, however, the following example.

Example 4.7 (The horocycle flow). Let $M$ be a closed surface equipped with a metric of constant negative curvature $-1$ and let $\Omega$ be the area form on $M$. Consider the twisted symplectic structure $\omega = \omega_0 + \pi^*\Omega$ on $W = T^*M$, where $\pi$ is the natural projection $T^*M \to M$ and $\omega_0$ is the standard symplectic structure. Set $H = \|p\|^2 - 1$. The Hamiltonian flow on the level $H = 0$ is the horocycle flow and hence has no periodic orbits; see, e.g., [Gi2, Gi4] for details. Assume that the Hofer–Zehnder capacity of the sets $\{H < h\}$ for $h$ near zero is finite. (This is not known, although is likely to be true, even for $h < 0$ close to zero. For small neighborhoods of the zero section (i.e., $h \approx -1$), finiteness has been recently proved in [FS]; however, for $h > 0$, finiteness of $c_{HZ}$ appears to be beyond reach of the methods considered in this paper.) Then, we claim that

$$c_H(h) - c_H(0) \geq l_{\min}\sqrt{h} \text{ for } h > 0, \quad (4.1)$$

where $l_{\min}$ is the minimal length of a closed geodesic on $M$. Therefore, $c_H$ is not smoother at $h = 0$ than $1/2$-Hölder on the right.

To prove this, let us first note that

$$\tau(h) = l_{\min}/\sqrt{h} \text{ for } h > 0, \quad (4.2)$$

and hence, by (iii), $c_H(h) - c_H(0) \geq l_{\min}\sqrt{h}$. Establishing (4.2) directly appears to be rather difficult. Instead, let us observe that there exists a symplectomorphism $(T^*M \setminus \{H \leq 0\}, \omega) \to (T^*M \setminus M, \omega_0)$ which sends $\|p\|^2$ to $\|p\|^2 - 1$. (Such a symplectomorphism can be constructed, for instance, by judiciously applying
Moser’s method.) This allows one to translate the period growth on \( (T^*M, \omega_0) \) as \( \| p \| \to 0 \), which is \( l_{\text{min}} / \| p \| \) for the Hamiltonian \( \| p \|^2 / 2 \), to the period growth on \( (T^*M, \omega) \) for \( H > 0 \).

Note also that by applying the results of [BPS] and using the above symplectomorphism, one can prove that \( c_{\text{HZ}}(\{ H < h \}, \{ H \leq 0 \}) \leq l_{\text{min}} \sqrt{|h|} \). This, together with the analogue of (4.1) for the relative capacity, proves that \( c_{\text{HZ}}(\{ H < h \}, \{ H \leq 0 \}) = l_{\text{min}} \sqrt{|h|} \).

One can also estimate the period growth on the left. All orbits on the level \( H = h < 0 \) are closed and project to geodesic circles on \( M \) with geodesic curvature \( k_g > 1 \). A straightforward calculation shows that \( \tau(h) = 2\pi / \sqrt{|h|} \) for \( h < 0 \) and hence, by (iii), \( c_H(0) - c_H(h) \geq 2\pi \sqrt{|h|} \). Thus, on the left, \( c_H \) is again not smoother than \( 1/2 \)-Hölder.

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