Dissipation and Vortex Creation in Bose-Einstein Condensed Gases

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We solve the Gross-Pitaevskii equation to study energy transfer from an oscillating ‘object’ to a trapped Bose-Einstein condensate. Two regimes are found: for object velocities below a critical value, energy is transferred by excitation of phonons at the motion extrema; while above the critical velocity, energy transfer is via vortex formation. The second regime corresponds to significantly enhanced heating, in agreement with a recent experiment.

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The concept was first introduced by Landau in his famous criterion \( \text{[1]} \), where elementary excitations are produced above a velocity \( v_L \). In liquid \(^4\)He this process refers to the excitation of rotons, with \( v_L \approx 58 \text{ ms}^{-1} \). However, much smaller critical values are observed experimentally, which prompted Feynman to propose that quantized vortices may be responsible \( \text{[2]} \).

Vortex nucleation in superfluid \(^4\)He is difficult to explain quantitatively. Strong interactions within the liquid, plus thermal and quantum fluctuations, impede formulation of a satisfactory microscopic theory. In contrast, Bose-Einstein condensation (BEC) in trapped alkali gases \( \text{[3]} \) provides a relatively simple system for exploring superfluidity. Weakly-interacting condensates can be produced with a negligibly small non-condensed component. This allows an accurate description by a nonlinear Schrödinger equation, often known as the Gross-Pitaevskii (GP) equation. The system also offers excellent control over the temperature, number of atoms and interaction strength, as well as allowing manipulation of the condensate using magnetic and optical forces \( \text{[3]} \).

Recent experiments have produced vortices by coherent excitation \( \text{[4]} \) and cooling of a rotating cloud \( \text{[5]} \). The existence of vortices was also inferred by Raman et al. \( \text{[6]} \), where a condensate was probed by an oscillating laser beam blue-detuned far from atomic resonance. The optical dipole force expels atoms from the region of highest intensity, resulting in a repulsive potential. Although vortices were not directly imaged, significant heating of the cloud was observed only above a critical velocity, indicating a transition to a dissipative regime. This heating was found to depend upon the existence of a condensate, indicating that it must be due to the production of elementary excitations, that subsequently populate the non-condensed fraction.

Critical velocities for vortex formation in superflow past an obstacle have been studied numerically by solution of the GP equation in a homogeneous condensate \( \text{[6]} \). Simulations have also confirmed that vortices are nucleated when a laser beam is translated inside a trapped Bose condensed gas \( \text{[7]} \). In this paper, we attempt to clarify the role of vortices in the MIT experiment \( \text{[8]} \) by presenting 2D and 3D simulations of an oscillating repulsive potential in a condensate. The motion transfers energy to the condensate, and we observe that the transfer rate increases significantly above the critical velocity for vortex formation.

Our simulations employ the GP equation for the condensate wavefunction \( \Psi(r,t) \) in a harmonic trap \( V_{\text{trap}}(r) = \frac{1}{2} \sum_{j=1}^{3} \omega_j^2 \dot{r}_j^2 \); \( j = x, y, z \). For convenience, we scale in harmonic oscillator units (h.o.u.), where the units of length, time, and energy are \( (\hbar/2m\omega_x)^{1/2}, \omega_x^{-1}, \) and \( \hbar\omega_x \), respectively. The scaled GP equation is then

\[
\imath \partial_t \Psi = (-\nabla^2 + V + C|\Psi|^2)\Psi, \tag{1}
\]

where \( V \) represents a time-dependent ‘object’ potential superimposed upon a stationary trap: \( V = \frac{1}{4}(x^2 + cy^2 + \eta z^2) + V_{\text{ob}}(r,t) \). The atomic interactions are parameterized by \( C = (NU_0/h\omega_x)(2m\omega_x/h)^{1/2} \); \( N \) atoms of mass \( m \) interact with a s-wave scattering length \( a \), such that \( U_0 = 4\pi\hbar^2a/m \). The number of dimensions is \( \gamma \). For most of the simulations here, \( \gamma = 2 \), corresponding to the limit \( \eta \rightarrow 0 \). In this case, \( N \) represents the number of atoms per unit length along \( z \). For the general 3D situation, a moving laser beam focused to a waist \( \tilde{w}_0 \) (in h.o.u.) at \( (0,y(t),0) \), is simulated using

\[
V_{\text{ob}}(r,t) = \frac{U_{\text{ob}}}{\sigma} \exp \left[ -\frac{2(x^2 + (y - y'(t))^2)}{\sigma w_0^2} \right], \tag{2}
\]

where \( \sigma = 1 + (z/z_0)^2 \). The Rayleigh range is \( z_0 = \pi w_0^2/\lambda \), where \( \lambda \) is the laser wavelength \( \text{[13]} \).

Our numerical methods are discussed elsewhere \( \text{[10–12]} \). Briefly, initial states are found by propagating \( \text{[4]} \) in imaginary time with \( V_{\text{ob}}(r,0) \), using a spectral method. Then, real-time simulations are performed subject to motion of the object potential. To recover the essential physics behind the MIT experiment \( \text{[8]} \), we describe the oscillatory motion by \( y'(T) = \alpha - vT \) \((T < 1/2f)\) and

\[
y'(T) = vT - 3\alpha \left( 1/2f < T < 1/f \right), \tag{3}
\]

where \( T = t/s \) and \( s \) is the number of completed oscillations. The velocity between the motion extrema is constant, \( v = \pm 4\alpha/f \), where \( \alpha \) is the amplitude and \( f \) is the frequency. The condensate is anisotropic, with its long axis along \( y (\epsilon < 1) \).
As a consequence, for small $\alpha$, the beam moves through regions of near-constant density. Initially, the object creates a density minimum at $y = \alpha$, which follows closely behind the moving object. For $v > v_c$, where $v_c \propto c_s$ and $c_s = \sqrt{2C/|\Psi|^2}$ is the sound velocity, the density inside the beam evolves to zero. This is accompanied by a $\pi$ phase slip [11], at which point the density minimum splits into a pair of vortex lines of equal but opposite circulation [12]. The vortex pair separates, and the process begins again.

The creation of phonons or vortices increases the energy of the condensate, which was calculated numerically using the functional $E = \int (|\nabla \Psi|^2 + V|\Psi|^2 + \frac{C}{2}|\Psi|^4) d^3r$. The time-independent ground state of the wavefunction represents the minimum of this functional. The energy is related to the drag force on the object $F_{ob}$ by $dE/dt = F_{ob} \cdot v$. The drag can be calculated independently over the whole condensate using $F_{ob} = -\int |\Psi|^2 \nabla V_{ob} d^3r$, allowing a numerical check. Superfluidity corresponds to the situation where $E$ remains constant when $V_{ob}$ is time-dependent; i.e. when there is no drag on the object.

Fig. 1 shows the energy and drag as a function of time, as calculated for two different frequencies in 2D simulations. At low frequency, the energy transfer is relatively small and characterized by ‘jumps’ at the motion extrema, whereas at higher frequency the energy transfer is two orders of magnitude larger and more continuous. Further insight can be gained by considering the drag. At low $f$, there is little drag except at the motion extrema (Fig. 1(c)), while at high $f$ appreciable drag is observed at all times (Fig. 1(d)).

To measure the average rate of energy transfer, a linear regression analysis is performed on the energy-time data. The gradients are plotted against $v$ in Fig. 2. It can be seen that the curves are characterized by two different regimes. Small energy transfer at low $v$, gives way to enhanced heating above the critical velocity, $v_c$. At high $v$, the three plots follow a single linear curve.

![FIG. 1. Time-dependent 2D simulations of laser beam oscillation, with grid spacing of 0.156 (512 × 128 points) and parameters $C = 1000$, $\epsilon = 0.0625$, $\alpha = 4$, $V_{ob} = 20$ and $\tilde{\alpha}_0 = 1.0$. Condensate energy as a function of time are plotted for (a) $f = 0.05$ and (b) $f = 0.2$. The drag $F_{ob}$ is also plotted for both frequencies in (c) and (d) respectively.](image)

![FIG. 2. Mean rate of energy change as a function of velocity, for $\alpha = 3$ (triangles), $\alpha = 4$ (squares) and $\alpha = 5$ (bullets). Otherwise, parameters are the same as Fig. 1. The dashed line shows the speed of sound in the condensate center, $c_s = \sqrt{\langle \Psi^2 \rangle} \simeq 3.55$. The plot shows a sharp transition between phonon heating (low $v$) and vortex heating at $v_c \simeq 0.4c_s$.](image)
where $\xi$ is the healing length and $d$ is the distance between the vortices. Equation (3) is valid for the inhomogeneous condensate when $\xi \ll d \ll R$, where $R$ is the radial extent of the condensate. Eq. (3) with $d = 2w_0$ is plotted in Fig. 3 (inset), and is found to agree with the numerical data. Recall that the vortex pair separates immediately after formation, when the pair still resides within the density minimum created by the object. The pair also moves in the direction of the object motion: however, it is slower, and is eventually left behind. At this point, it has an energy approximately equal to $E_{\text{pair}}$, and the formation process is complete. The heating rate can be expressed as $\frac{dE}{dt} = E_{\text{pair}}f_s$ [7], where $f_s$ is the shedding frequency, which is found to be proportional to $v$. This accounts for the linear dependence of the energy transfer rate.

\[ E_{\text{pair}} = \frac{2\pi \hbar^2}{m} \ln \left( \frac{d}{\xi} \right), \]

FIG. 3. Number of vortex pairs created up to $t = 10$ against rate of energy transfer. Simulation parameters are the same as Fig. 1, with $\alpha = 4$ and $C = 2000$ (plotted with squares, fit with a solid linear regression line); $C = 3000$ (circles, dotted line); $C = 4000$ (diamonds, dashed); and $C = 5000$ (triangles, long-dashed). The data points closely follow the regression lines, suggesting a constant energy for each vortex pair. Inset: the average pair energy against $C$, where the dashed line shows the pair energy predicted by (3).

The subsequent vortex dynamics involve an interplay between velocity fields induced by other vortices, and effects arising from the condensate inhomogeneity. In the absence of the object, an isolated pair follows a trajectory similar in character to that of a vortex ring [16], culminating in self-annihilation. However, the object moves back through its wake, interacting with the original pairs and creating more vortices. The circulation of a pair depends upon the direction of the object motion when it is created. So, vortex pairs of opposite circulation are formed and interact when sufficiently close. This leads to situations where vortices annihilate or move towards the edge. The number of vortices remaining within the condensate bulk is found to reach an equilibrium value.

The critical velocity for vortex formation, $v_c$, as a function of potential height and nonlinear coefficient is shown in Fig. 4. The critical velocity is not as well defined as in the homogeneous case [8–10] for a number of reasons. First, a density inhomogeneity along the direction of motion leads to a variation in $c_s$, and therefore $v_c$. However, this is less than $\sim 3\%$ in the simulations considered here. The oscillatory nature of the object motion is important. The time taken for a vortex pair to form diverges to infinity as $v$ approaches $v_c$ from above. So, the measured value of $v_c$ increases from its true value as $\alpha$ decreases. In addition, the object travels through its own low-density wake, where $c_s$ is lower. Vortices can therefore be formed after the first half-oscillation, when $v$ is slightly below $v_c$. Nevertheless, we can obtain a good estimate for $v_c$ by choosing intermediate values of the amplitude (e.g. $\alpha = 4$) and considering only vortex formation during the first half-cycle.

FIG. 4. Critical velocity for vortex formation $v_c$ at $C = 2000$ as a function of potential height $U_{\text{ob}}$, expressed as a fraction of the speed of sound at the condensate center, $c_s$. Inset: critical velocity plotted against $C$, with $U_{\text{ob}} = 20$. The other parameters in both plots are $\alpha = 4$ and $w_0 = 1$. The critical velocity for vortex formation, $v_c$, decreases as a function of increasing object potential height, $U_{\text{ob}}$, allowing an experimental diagnostic for vortex formation at varying beam intensities. This behavior agrees with simulations of 1D soliton creation [16] and vortex ring formation in 3D [17]. We have also studied the case of $U_{\text{ob}} < 0$, which corresponds to a red-detuned laser. Atoms are attracted to the potential minimum, creating a density peak which moves with the beam. Vortex pairs are created from a density minimum which develops ahead of the beam. Fig. 3 (inset) shows $v_c$ as a function of $C$. The critical velocity tends to a constant value as $C$ increases.
the experiment \[7,18\], where a relatively low critical ve-
cordingly, vortex lines first appear in these regions and
condensate edge where the speed of sound is lower. Ac-
critical velocities: a result of the beam intersecting the
Fig. 5. Similar behaviour to 2D is observed, with smaller
energy transfer rate as a function of velocity is presented in
(triangles) it is
\[ v \times 3D \] simulations with grid spacing 0.234 (256 \times 64 \times 64 \text{ points}).

Parameters are \( C = 1000, \epsilon = 0.0625, \eta = 1, \alpha = 3.0, \bar{\omega}_0 = 1, \)
and \( \lambda \approx 0.281. \) The speed of sound at the condensate center
\( c_s \approx 2.54 \) is represented by the dashed line. For \( U_{ob} = 40 \)
(bullets) the critical velocity is \( v_c \approx 0.13c_s \), while for \( U_{ob} = 20 \)
(triangles) it is \( v_c \approx 0.20c_s \).

Simulations in 3D were performed, and the mean en-
ergy transfer rate as a function of velocity is presented in
Fig. 5. Similar behaviour to 2D is observed, with smaller
critical velocities: a result of the beam intersecting the
condensate edge where the speed of sound is lower. Ac-
cordingly, vortex lines first appear in these regions and
penetrate into the center. This conclusion agrees with the experiment \[1,18\], where a relatively low critical ve-
cy (\( v_c \approx 0.26c_s \)) was measured. The dependence of \( v_c \)
on \( U_{ob} \) and \( C \) was found to be similar to 2D, where e.g.
v\( c \approx 0.29c_s \) for \( C = 4000 \) and \( U_{ob} = 35 \). Enhanced heat-
ing is also observed for \( v_c > c_s \), due to phonon emission
between the extrema.

In this paper, we have studied the role of vortex for-
tation in the breakdown of superfluidity, by an oscillat-
ing object in a trapped Bose-Einstein condensate. We
find that at low object velocities, energy is transferred by phonon emission at the motion extrema, while a much
larger energy is transferred above the critical velocity for vortex formation. To generalize these conclusions to re-
alistic experimental situations, the model should include
the non-condensed thermal cloud. Energy would then be transferred from the condensed to the thermal cloud by phonon damping \[1\], or vortex decay \[19\].

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