Actions of compact groups, $C^*$-index theorem, and families

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Abstract

We prove the index theorem for elliptic operators acting on sections of bundles where fiber is equal to a projective module over a $C^*$-algebra, in the situation of action of a compact Lie group on this algebra as well as on the total space commuting with symbol. As an application the equivariant index theorem for a direct product of base by the space of parameters is obtained.

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1 Introduction

The rational version of index theorem for $C^*$-elliptic operators [29, 31] has numerous applications in differential topology (to proof the Novikov conjecture), differential geometry (curvature of spin manifolds), theory of elliptic operators with random and periodic coefficients, etc. After that the equivariant generalization of this theorem [11], as well as equivariant generalization taking the torsion into account [42] was obtained. They also have found their applications (see, e.g. [37, 39]). In these theorems a compact Lie group acts on the total space of a bundle of $A$-modules, but not on the algebra $A$ itself. For some applications, the most classic of which is calculation of equivariant index of a family of elliptic operators, it is necessary for the group to act on $A$ compatible with the action on the total space. The present paper is devoted to the decision of this problem.

We use intensively the foundations of theory of $C^*$-Hilbert modules. One can find it in [27, 45, 28]. The modern state of index theory for families and its applications can be found e.g. in [44, 10].

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2 An averaging theorem

Let $G$ be a compact Lie group acting continuously on $C^*$-algebra $A$ by involutive automorphisms. If $A$ is unital, then the unity has to be invariant. Then $g \in G$ takes self-adjoint elements to self-adjoint and positive ones to positive.

Remark 2.1 If $a \geq b \geq 0$, then $ga \geq gb$. Indeed, $a - b = c^*c \geq 0$. Hence, $ga - gb = g(c)^*g(c) \geq 0$.

An $A$-module $M$ is called $GGA$-module if it is equipped with a $C^*$-linear action of $G$, such that

$$g(m \cdot a) = g(m) \cdot g(a), \quad m \in M, \quad a \in A.$$

Definition 2.2 An inner $A$-valued product on Hilbert module is called invariant or $GGA$-product if

$$\langle gx, gy \rangle = g(\langle x, y \rangle).$$

Let us remark that the averaging theorem of [11] (see also [39]) does not define an invariant product in the $GGA$-case. Moreover, the formula

$$\langle u, v \rangle = \int_G \langle T_xu, T_xv \rangle' \, dx,$$

does not define an A-inner product:

Example 2.3 Let $\mathcal{M} = A = C(G)$. Then the new “product” is valued in constants only. Some effects of such originality are studied in [13].

Definition 2.4 Let us define an action of $G$ on the module $l_2(A)$ over a unital $G$-algebra $A$ by the formula $g(u_1, u_2, \ldots) = (gu_1, gu_2, \ldots)$. 
Lemma 2.5 This is a continuous action, and the initial $A$-valued product is invariant with the respect to it.

Proof: First of all

$$g(ua) = g((u_1, u_2, \ldots) a) = (g(u_1) g(a), g(u_2) g(a), \ldots) = (g(u_1), g(u_2), \ldots) g(a) = g(u) g(a).$$

Further,

$$\langle gu, gv \rangle = \sum_{i=1}^{\infty} (g(u_i))^* g(v_i) = \sum_{i=1}^{\infty} (u_i^* g(v_i)) = \sum_{i=1}^{\infty} (u_i^* v_i) = g(\langle u, v \rangle).$$

Let us demonstrate the continuity. Let $\|g - h\| < \varepsilon$ as automorphisms of the algebra $A$. Then

$$\left\|\langle (g - h) u, (g - h) u \rangle\right\| = \left\| \sum_{i=1}^{\infty} ((g - h)(u_i))^* (g - h)(u_i) \right\| = \left\| \sum_{i=1}^{\infty} (g - h)(u_i^* u_i) \right\| = \left\| (g - h) \sum_{i=1}^{\infty} u_i^* u_i \right\| \leq \varepsilon \|u\|^2. \quad \square$$

Let us recall some facts about the integration of operator-valued functions \cite{25}. Let $X$ be a compact space, $A$ be a $C^*$-algebra, $\varphi : C(X) \to A$ be an involutive homomorphism of algebras with unity, and $F : X \to A$ be a continuous map, such that for every $x \in X$ the element $F(x)$ commutes with the image of $\varphi$. In this case the integral

$$\int_X F(x) \, d\varphi \in A$$

can be defined in the following way. Let $X = \bigcup_{i=1}^{n} U_i$ be an open covering and

$$\sum_{i=1}^{n} \alpha_i(x) = 1$$

be a corresponding partition of unity. Let us choose the points $\xi_i \in U_i$ and compose the integral sum

$$\sum (F, \{U_i\}, \{\alpha_i\}, \{\xi_i\}) = \sum_{i=1}^{n} F(\xi_i) \varphi(\alpha_i).$$

If there is a limit of such integral sums then it is called the corresponding integral.

If $X$ is a Lie group $G$ then it is natural to take $\varphi$ equal to the Haar measure

$$\varphi : C(X) \to C, \quad \varphi(\alpha) = \int_G \alpha(g) \, dg$$

(though this is only a positive linear map, not a $*$-homomorphism) and to define for a norm-continuous $Q : G \to L(H)$ ($A$ is realized as a subalgebra in algebra $L(H)$ of all bounded operators on Hilbert space $H$)

$$\int_G Q(g) \, dg = \lim \sum_i Q(\xi_i) \int_G \alpha_i(g) \, dg.$$
If $Q : G \to P^+(A) \subset L(H)$, then, since $\int_{G} \alpha_i(g) \, dg = 0$, we get
\[
\sum_i Q(\xi_i) \cdot \int_{G} \alpha_i(g) \, dg \in P^+(A) \quad \text{and} \quad \int_{G} Q(g) \, dg \in P^+(A)
\]
(the positive cone $P^+(A)$ is convex and closed). So we have proved the following Lemma.

**Lemma 2.6** Let $Q : G \to P^+(A)$ be a continuous function. Then for the integral in the sense of [25] we have
\[
\int_{G} Q(g) \, dg \geq 0. \quad \square
\]

**Theorem 2.7** Let $GL = GL(A)$ be the full general linear group, i.e., the group of all bounded $A$-linear automorphisms of $l_2(A)$. Let $g \mapsto T_g$ ($g \in G, T_g \in GL$) be a representation of $G$ such that the map
\[
G \times l_2(A) \to l_2(A), \quad (g, u) \mapsto T_g u
\]
is continuous. Then on $l_2(A)$ there is a GGA-product equivalent to the original one.

**Proof:** Let $\langle \cdot, \cdot \rangle'$ be the original product. We have a continuous map $G \to A, \quad x \mapsto x^{-1}(\langle T_x u, T_x v \rangle')$. for every $u$ and $v$ from $l_2(A)$. We define the new product by
\[
\langle u, v \rangle = \int_{G} x^{-1}(\langle T_x u, T_x v \rangle') \, dx,
\]
where the integral can be defined in the sense of either of the two definitions from [25, p. 810] because the map is continuous with the respect to the norm of the $C^*$-algebra. This product is an $A$-Hermitian map $l_2(A) \times l_2(A) \to A$. Indeed,
\[
\langle u \cdot a, v \cdot b \rangle = \int_{G} x^{-1}(\langle T_x (u \cdot a), T_x (v \cdot b) \rangle') \, dx =
\]
\[
= \int_{G} x^{-1}(\langle T_x (u) \cdot (xa), T_x (v) \cdot (xb) \rangle') \, dx =
\]
\[
= \int_{G} x^{-1}(\langle xa^*(T_x (u)), T_x (v) \rangle')(xb) \, dx =
\]
\[
= \int_{G} x^{-1}(\langle a^*(T_x (u)), T_x (v) \rangle')(xb) \, dx =
\]
\[
= \int_{G} (a^*)x^{-1}(\langle T_x (u), T_x (v) \rangle')b \, dx = a^* \langle u, v \rangle b.
\]
Since $x^{-1} : A \to A$ is an involutive mapping, it takes positive elements to positive ones and by Lemma [24]
\[
\langle u, u \rangle = \int_{G} x^{-1}(\langle T_x u, T_x u \rangle') \, dx, \geq 0.
\]
For $T_x u = (a_1(x), a_2(x), \ldots) \in l_2(A)$ the equality $\langle u, u \rangle = 0$ takes the form
\[
\int_{G} \sum_{i=1}^{\infty} x^{-1}(a_i(x)a_i^*(x)) \, dx = 0.
\]
Since \( x^{-1}(a_i(x)a_i^*(x)) \geq 0 \), we have \( a_i(x) = 0 \) almost everywhere. Hence, \( a_i(x) = 0 \) for all \( x \) by the continuity, and \( T_xu = 0 \). In particular, \( u = 0 \).

Let us demonstrate the continuity of this new product. Fix \( u \in l_2(A) \). Then by Lemma 2.5, \( x \mapsto T_x(u) \), \( G \to l_2(A) \) is a continuous mapping of a compact space. Hence, the set \( \{T_x(u) \mid x \in G\} \) is bounded. So, by the principle of uniform boundness \([4, \text{vol. 2, p. 309}]\)

\[
\lim_{v \to 0} T_x(v) = 0
\]

uniformly with respect to \( x \in G \). If \( u \) is fixed then

\[
\|T_x(u)\| \leq M_u = \text{const}
\]

and by (1)

\[
\|\langle u, v \rangle\| = \| \int_G x^{-1}(\langle T_xu, T_xv \rangle') \, dx \| \leq M_u \cdot \text{vol} \, G \cdot \sup_{x \in G} \|T_x(v)\| \to 0 \quad (v \to 0).
\]

This gives the continuity at 0 and hence on the whole space \( l_2(A) \times l_2(A) \). Since each operator \( T_y \) is an automorphism, we get

\[
\langle T_yu, T_yv \rangle = \int_G x^{-1}(\langle T_xT_yu, T_xT_yv \rangle') \, dx = \int_G y(xy)^{-1}(\langle T_xyu, T_xyv \rangle') \, dx = y(\langle u, v \rangle),
\]

i. e. this product is invariant.

Now we will show the equivalence of the two norms and, in particular, the continuity of the representation. There is a number \( N > 0 \) such that \( \|T_x\|' \leq N \) for every \( x \in G \). So by \([23]\) (for the simplicity \( \text{vol} \, G = 1 \))

\[
\|u\|^2 = \|\langle u, u \rangle\|_A = \| \int_G x^{-1}(\langle T_xu, T_xu \rangle') \, dx \|_A \leq \left( \sup_{x \in G} \|T_xu\|' \right)^2 \leq N^2(\|u\|')^2.
\]

On the other hand, let \( \langle u, u \rangle' = 1 \). Then applying Lemma \([2,6]\) and Remark \([2,1]\), we obtain

\[
\langle u, u \rangle' = \int_G g^{-1}(\langle u, u \rangle) \, dg = \int_G g^{-1}(\langle g^{-1}T_gu, g^{-1}T_gu \rangle') \, dg \leq \int_G g^{-1}(\|T_g^{-1}\|^2 \langle g^{-1}T_gu, g^{-1}T_gu \rangle') \, dg \leq \int_G g^{-1}(N^2 \langle T_gug, T_gug \rangle') \, dg = N^2 \int_G g^{-1}(\langle T_gu, T_gu \rangle') \, dg = N^2 \langle u, u \rangle.
\]

Then \( (\|u\|')^2 = \|\langle u, u \rangle'\|_A \leq N^2(\|u, u\|)_A = N^2\|u\|^2 \). By linearity we obtain a similar estimate for \( u \) with invertible \( \langle u, u \rangle' \), while the elements of such a form are dense (see \([13]\)) in \( l_2(A) \). \( \square \)

Remark 2.8 \( l_2(P) \) is a direct summand in \( l_2(A) \), so the previous theorem holds for \( l_2(P) \).
3 \textit{K-theory of GGA-bundles}

Let us recall some general constructions of K-theory, contained in [24], and also [22].

\begin{definition}[24, I.6.7] An additive category $\mathcal{C}$ is called pseudo-Abelian, if for each object $E$ from $\mathcal{C}$ and each morphism $p : E \to E$, satisfying to a condition $p^2 = p$ (i.e. an idempotent) there exists the kernel $\text{Ker} p$. For an arbitrary additive category $\mathcal{C}$ there exists associated pseudo-Abelian category $\tilde{\mathcal{C}}$ which is a solution of the appropriate universal problem and is defined as follows [24, I.6.10]. Objects of $\mathcal{C}$ are pairs $(E,p)$, where $E \in \text{Ob}(\mathcal{C})$ and $p$ is a projector in $E$. A morphism from $(E,p)$ to $(F,q)$ is such a morphism $f : E \to F$ of the category $\mathcal{C}$, that $f \circ p = q \circ f = f$.
\end{definition}

\begin{definition}[24, § II.1] We call \textit{symmetrization} of an Abelian monoid $M$ the following Abelian group $S(M)$. Consider the product $M \times M$ and its quotient monoid with respect to the equivalence relation

$$(m, n) \sim (m', n') \iff \exists p, q : (m, n) + (p, p) = (m', n') + (q, q).$$

This quotient monoid is a group denoted $S(M)$. If we consider now an additive category $\mathcal{C}$ and denote through $\tilde{E}$ the isomorphism class of an object $E$ from $\mathcal{C}$, then the set $\Phi(\mathcal{C})$ of these classes is equipped with a structure of an Abelian monoid with respect to operation $\tilde{E} \oplus \tilde{F} = (E \oplus F)^\ast$. In this case the group $S(\Phi(\mathcal{C}))$ is denoted through $K(\mathcal{C})$ and is called Grothendieck group of the category $\mathcal{C}$.

\begin{definition}[24, § II.2] Banach structure on an additive category $\mathcal{C}$ is defined by the introducing of the structure of a Banach space on all groups $\mathcal{C}(E, F)$, where $E$ and $F$ are arbitrary objects from $\mathcal{C}$. It is assumed, that applications of composition of morphisms $\mathcal{C}(E, F) \times \mathcal{C}(F, G) \to \mathcal{C}(E, G)$ are bilinear and continuous. In this case we call $\mathcal{C}$ Banach category.
\end{definition}

\begin{definition}[24, § II.2] Let $\mathcal{C}$ and $\mathcal{C}'$ be additive categories. An additive functor $\varphi : \mathcal{C} \to \mathcal{C}'$ is called \textit{quasi-surjective} if each object of $\mathcal{C}'$ is a direct summand of an object of type $\varphi(E)$. A functor $\varphi$ called \textit{full} if for any $E, F \in \text{Ob}(\mathcal{C})$ the map $\varphi(E, F) : \mathcal{C}(E, F) \to \mathcal{C}'(\varphi(E), \varphi(F))$ is surjective. For Banach categories $\varphi$ is called \textit{Banach} if this map $\varphi(E, F)$ linear and continuous.
\end{definition}

\begin{definition}[24, II.2.13] Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a quasi-surjective Banach functor. We shall denote by $\Gamma(\varphi)$ the set consisting of triples of the form $(E, F, \alpha)$, where $E$ and $F$ are objects of the category $\mathcal{C}$ and $\alpha : \varphi(E) \to \varphi(F)$ is an isomorphism. The triples $(E, F, \alpha)$ and $(E', F', \alpha')$ are named \textit{isomorphic}, if there are such isomorphisms $f : E \to E'$ and $g : F \to F'$, that the diagram

$$\begin{array}{ccc}
\varphi(E) & \xrightarrow{\alpha} & \varphi(F) \\
\varphi(f) \downarrow & & \downarrow \varphi(g) \\
\varphi(E') & \xrightarrow{\alpha'} & \varphi(F')
\end{array}$$

commutes. A triple $(E, F, \alpha)$ is \textit{elementary} if $E = F$ and isomorphism $\alpha$ is homotopic in the set of automorphisms of $\varphi(E)$ to the identical isomorphism $\text{Id}_{\varphi(E)}$. We define \textit{sum} of two triples $(E, F, \alpha)$ and $(E', F', \alpha')$ as

$$(E \oplus E', F \oplus F', \alpha \oplus \alpha').$$
The Grothendieck group $K(\varphi)$ of a functor $\varphi$ is defined as quotient set of the monoid $\Gamma(\varphi)$ with respect to the following equivalence relation: $\sigma \sim \sigma'$ if and only if there exist such elementary triples $\tau$ and $\tau'$, that the triple $\sigma + \tau$ is isomorphic to the triple $\sigma' + \tau'$. The operation of addition introduces on $K(\varphi)$ a structure of Abelian group. The class of a triple we shall denote by $d(E,F,\alpha)$.

**Definition 3.6** [24, II.3.3] Consider the set of pairs of the form $(E,\alpha)$, where $E$ is an object of the category $\mathcal{C}$ and $\alpha$ is an automorphism of $E$. Two pairs $(E,\alpha)$ and $(E',\alpha')$ are called *isomorphic*, if there is such isomorphism $h : E \to E'$ in category $\mathcal{C}$, that the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{h} & E' \\
\alpha & \downarrow & \alpha' \\
E & \xrightarrow{h} & E'
\end{array}
\]

commutes. The direct sum defines the operation of *addition* of pairs. A pair $(E,\alpha)$ is called *elementary*, if the automorphism $\alpha$ is homotopic to $\text{Id}_E$ in the set of automorphisms of $E$. Abelian group $[24, II.3.4]$ $K^{-1}(\mathcal{C})$ is defined as a quotient set (with operation of addition) of the set of pairs $\{(E,\alpha)\}$ with respect to the following equivalence relation: $\sigma \sim \sigma'$ if and only if there are such elementary pairs $\tau$ and $\tau'$, that $\sigma + \tau$ is isomorphic to $\sigma' + \tau'$.

**Definition 3.7** [24, II.4.1] Let $\mathcal{C}$ be a Banach category and $C^{p,q}$ be the Clifford algebra. We shall denote by $\mathcal{C}^{p,q}$ the category, objects of which are pairs $(E,\rho)$, where $E \in \text{Ob}(\mathcal{C})$ and $\rho : C^{p,q} \to \text{End}(E)$ is a homomorphism of algebras. A morphism from a pair $(E,\rho)$ to a pair $(E',\rho')$ is such a $\mathcal{C}$-morphism $f : E \to E'$, that $f \circ \rho(\lambda) = \rho(\lambda) \circ f$ for each element $\lambda \in C^{p,q}$.

**Definition 3.8** [24, III.4.11] Let $\mathcal{C}$ be a pseudo-Abelian Banach category. The group $K^{p,q}(\mathcal{C})$ is defined as the Grothendieck group of the functor $C^{p,q+1} \to C^{p,q}$ in the sense of the Definition 3.5.

The following statement can be easily obtained by the properties of Clifford algebras.

**Theorem 3.9** [24, III.4.6, III.4.12] The groups $K^{p,q}(\mathcal{C})$ depend only on the difference $p-q$. Besides, the groups $K^{0,0}(\mathcal{C})$ and $K^{0,1}(\mathcal{C})$ are canonically isomorphic to groups $K(\mathcal{C})$ and $K^{-1}(\mathcal{C})$.

**Definition 3.10** Now we can define $K^{p,q}(\mathcal{C}) = K^{p,q}(\mathcal{C})$ and similarly for K-groups of functors.

We need also another description of K-groups, which is equivalent [24, §§ III.4, III.5] to the initial.

**Definition 3.11** [24, III.4.11, III.5.1] Let $\mathcal{C}$ be a pseudo-Abelian Banach category and let $E$ be a $C^{p,q}$-module (an object of the category $\mathcal{C}^{p,q}$). Gradation of $E$ is such endomorphism $\eta$ of object $E$ (considering as an object from $\mathcal{C}$), that

1. $\eta^2 = 1$,

2. $\eta\rho(e_i) = -\rho(e_i)\eta$, where $e_i$ are the generators of Clifford algebra and $\rho : C^{p,q} \to \text{End}(E)$ is the homomorphism, determining the $C^{p,q}$-structure on $E$. 

In other words, a gradation of $E$ is a $C^{p,q+1}$-structure on $E$, extending the initial $C^{p,q}$-structure (if we put $\rho(\epsilon_{p+q+1}) = \eta$).

The term “gradation” arises from the following fact. Morphism $\eta$ determines the decomposition into a direct sum

$$E = E_0 \oplus E_1, \quad E_0 = \text{Ker} \left( \frac{1 - \eta}{2} \right), \quad E_1 = \text{Ker} \left( \frac{1 + \eta}{2} \right),$$

while $\rho : C^{p,q} \to \text{End}(E_0 \oplus E_1)$ is a morphism of $\mathbb{Z}/2\mathbb{Z}$-graded algebras.

Let us define group $K^{p,q}(C)$ as a quotient group of the free Abelian group, generated by triples $(E, \eta_1, \eta_2)$, where $E$ is a $C^{p,q}$-module and $\eta_1, \eta_2$ is a gradation of $E$, with respect to the subgroup, generated by relations

1. $(E, \eta_1, \eta_2) + (F, \xi_1, \xi_2) = (E \oplus F, \eta_1 \oplus \xi_1, \eta_2 \oplus \xi_2),$
2. $(E, \eta_1, \eta_2) = 0$, if $\eta_1$ is homotopic to $\eta_2$ in the set of gradations of $E$.

As usual, by $d(E, \eta_1, \eta_2) \in K^{p,q}(C)$ we shall denote the class of triple $(E, \eta_1, \eta_2)$.

We pass to the necessary specification of these constructions.

If $X$ is a paracompact topological space, let $\text{Vect}(X; A)$ be the category of locally trivial bundles $p : E \to X$ with fiber $M = p^{-1}(x) \in \mathcal{P}(A)$ and structure group equal to $\text{Aut}_AM$. Such bundles are called $A$-bundles. This category is Banach in the sense of Definition 3.3 (see also [22, 31, 23]). Let $G$ be a compact Lie group, acting on $X$ and algebra $A$ continuously.

An $A$-bundle $p : E \to X$ is called $GGA$-bundle if a $G$-space structure is given on $E$, and for any elements $g \in G, e \in E, x \in X$

1. $gp(e) = pg(e)$, and
2. $g : p^{-1}(x) \to p^{-1}(gx)$ is an $GGA$-linear mapping (i.e. $g(e \cdot a) = g(e) \cdot g(a)$).

We form the Banach category $\text{Vect}_{G}(X; A)$ whose objects are $GGA$-bundles and whose morphisms are the morphisms of $\text{Vect}(X; A)$ that commute with the action of $G$.

The set of all continuous sections $s : X \to E$ of a $GGA$-bundle $E$ over a compact space $X$ forms a Banach $A$-module $\Gamma(E)$ (in the topology of the maximum of the norm). The group $G$ strongly continuously acts on $\Gamma(E)$ according to the rule $(Gs)(x) = gs(g^{-1}x)$, where $g \in G, s \in \Gamma(E), x \in X$. Let us note that this module is a $GGA$-module over the algebra of all continuous $A$-valued functions on $X$:

$$g(s \cdot a)(x) = g((s \cdot a)(g^{-1}x)) = g(s(g^{-1}x) \cdot a(g^{-1}x)) = g(s(g^{-1}x)) \cdot g(a(g^{-1}x)) = g(s)(x) \cdot g(a)(x) = (gs \cdot g(a))(x).$$

The averaging mapping $s \mapsto \int_G gs$ determines a projection $\mu : \Gamma(E) \to \Gamma^G(E)$, where $\Gamma^G(E)$ is the space of $G$-invariant sections.

**Lemma 3.12** Let $s'$ be a $G$-invariant cross-section of a $GGA$-bundle $E \to X$ over a closed $G$-stable subset $Y$ of the compact $G$-space $X$. Then $s'$ can be extended to a $G$-invariant cross-section over $X$.\footnote{That is $GY \subseteq Y$.}
Proof: Similarly to [1, 2.1.1].

Consider two GGA-bundles $E$ and $F$ over the compact base $X$. The $G$-bundle $\text{Hom}(E, F)$ is introduced in the standard way. It is not an $A$-bundle, but Lemma 3.12 remains valid for it. As usual, we can identify $\Gamma^G(\text{Hom}(E, F))$ with the set of GGA-morphisms $\varphi : E \to F$. Using Lemma 3.12, we obtain the following assertion.

**Lemma 3.13** Let $\varphi' : E|_Y \to F|_Y$ be a morphism of the restrictions of GGA-bundles $E$ and $F$ over a compact space $X$ to a closed $G$-stable subset $Y$. Then $\varphi'$ can be extended to a morphism of $G$-bundles $\varphi : E \to F$ over $X$. If $\varphi'$ is an isomorphism, then there exists a $G$-stable open neighborhood $U$ of $Y$ such that $\varphi|_U : E|_U \to F|_U$ is an isomorphism. Any two such extensions $\varphi_0$ and $\varphi_1$ are homotopic to each other over some $G$-neighborhood $U' \supset Y$ in the class of isomorphisms.

If $E$ is a $G$-space and $I$ is the unit interval, then we define the action of $G$ on $Y \times I$ by the formula $g(y, t) = (gy, t)$. From Lemma 3.13, as well as in a classical case, we obtain the following fact.

**Lemma 3.14** Let $Y$ be a compact $G$-space, $f_t : Y \to X$ a homotopy of $G$-mappings $(0 \leq t \leq 1)$, and $E$ a $G$-$A$-bundle over $X$. Then $f_0^*E \cong f_1^*E$.

By Lemma 3.13 we can define GGA-bundle

$$E_1 \cup \varphi E_2 \to X$$

in the usual way, where $X = X_1 \cup X_2$, $Y = X_1 \cap X_2$, $X_1$ and $X_2$ are closed $G$-subspaces of the compact space $X$, $E_1 \to X_1$ and $E_2 \to X_2$ are GGA-bundles, and $\varphi : E_1|_Y \to E_2|_Y$ is an isomorphism. Further, as follows from Lemma 3.14, $E_1 \cup \varphi E_2$ up to isomorphism depends only on the $G$-homotopy class of $\varphi$.

**Theorem 3.15** (see, e.g. [39]) Let $E$ and $F$ be $A$-bundles over $X$, and $\alpha : E \to F$ a morphism such that $\alpha_x : E_x \to F_x$ is an epimorphism for all points $x \in X$. Then there exists a morphism $\beta : F \to E$, such that $\alpha \beta = \text{Id}_F$.

**Definition 3.16** Consider a strongly continuous action of $G$ on an arbitrary Banach space $\Gamma$. A vector $s \in \Gamma$ is called periodic, if the orbit $Gs$ lies in a finite-dimensional subspace of space $\Gamma$.

According to a lemma of Mostow (see [32]) periodic vectors form a dense subset of $\Gamma$.

**Theorem 3.17** Let $X$ be a compact $G$-space, and $E \to X$ a GGA-bundle. Then there exists a trivial GGA-bundle $M = X \times M$ and GGA-bundle $E'$ such that $M \cong E \oplus E'$, where $M$ is a projective GGA-module.
Proof: By a lemma of Mostow [32] and Lemma [31] there exist sections \( v_1, \ldots, v_n \) periodic in \( \Gamma(E) \), as in complex Banach space, with \( \{ v_j(x) \} \) generating \( E|_x \) for any \( x \). Taking their orbits, we consider finite-dimensional \( G \)-C-module \( W \subset \Gamma(E) \) spanned by them: \( W \) is the \( \mathbb{C} \)-linear span of \( \{ Gv_j \}_{j=1, \ldots, n} \). Let \( s_1, \ldots, s_N \) be a basis for \( W \). We have the trivial \( G \)-bundle
\[
M = X \times M = X \times (A \otimes \mathbb{C} W)
\]
with the diagonal action of \( G \). Define a morphism \( \theta : M \to E \) on the generators by the formula \( \theta(x,a \otimes s_i) = s_i(x) \cdot a \). This is an epimorphism of \( A \)-bundles. Since
\[
\theta(g(x,a \otimes s_i)) = \theta(gx, ga \otimes g(s_i)) = (g(s_i))(gx) \cdot ga = \]
\[
gs_i(g^{-1}gx) \cdot ga = g(s_i(x)) \cdot ga = g(s_i(x) \cdot a) = g(\theta(x,a \otimes s_i)),
\]
it follows that \( \theta \) is \( G \)-mapping. We introduce in \( M \) the following fiberwise \( A \)-Hermitian product. If \( (, ) \) is a \( G \)-invariant inner product in \( W \), then let
\[
\langle (x,a \otimes s_i), (x,b \otimes s_j) \rangle = a^*b(s_i,s_j).
\]
Then
\[
\langle g(x,a \otimes s_i), g(x,b \otimes s_j) \rangle = g(a^*b)(g(s_i), g(s_j)) = g(a^*b)(s_i,s_j) =
\]
\[
g(\langle (x,a \otimes s_i), (x,b \otimes s_j) \rangle).
\]
Hence, our \( A \)-product is \( G \)-invariant (in the sense of Def. [2.2]). Such a product gives, in particular, the structure of a Hilbert module in any fiber. Let \( E' = \text{Ker } \theta \). With the help of [30] it is easy to obtain that \( M \cong E' \oplus E \). \( \square \)

Corollary 3.18 A fiberwise \( GGA \)-product can be introduced on every \( GGA \)-bundle over a compact base.

Let \( E \to X \) be an \( A \)-bundle. We consider the bundle \( \text{Hom}(E,E) \) with fiber \( \text{Hom}(E_x,E_x), x \in X \). An element \( p_x = p_x^2 \) is called a projection in the fiber. Denote by \( \text{Proj}(E) \subset \text{Hom}(E,E) \) the set of projections. Let \( Q(E) \subset \text{Hom}(E,E) \) consist of all \( T \) such that \( z \cdot 1_{E_x} - T_x \) is an isomorphism for any \( x \in X, z \in \mathbb{C} \) if \( \text{Re } z \neq 1/2 \). It is clear, that \( \text{Proj}(E) \subset Q(E) \). The usual method of Cauchy integrals (cf. [22, p. 184]) can be used to prove

Lemma 3.19 There exists a retraction \( \alpha : Q(E) \to \text{Proj}(E) \). \( \square \)

Construction 3.20 If \( E' \) is a \( GGA \)-bundle over a closed \( G \)-invariant subspace \( Y \) of a compact \( G \)-space \( X \), then \( E' \oplus F' = Y \times M \) for some \( F' \), by Theorem 3.17. We define \( p' : Y \times M \to Y \times M \) by the formula \( p'(e',f') = (e',0) \), bearing in mind the identifications indicated above. Then \( p' \) is a \( G \)-projection with \( \text{Im } p' = E' \). We extend \( p' \) as a cross-section in \( G \text{Hom}(E,F) \) over \( Y \) to a \( G \)-section \( p \) over \( X \), where \( F = X \times M \). By Lemma on generators from [31], there exists a neighborhood \( U \) of \( Y \) such that \( p|_U \in Q(F|_U) \). Taking \( GU \), we may assume that \( U \) is \( G \)-stable. It is easy to see that \( E = \text{Im } \alpha(p|_U) \) is a \( GGA \)-bundle and \( E|_Y = E' \).
Definition 3.21 Following the general scheme presented above, we define the $K$-groups for a compact $G$-space $X$ by setting

$$K_{p,q}^G(X,A) = K_{p,q}(\text{Vect}_G(X,A)),$$
$$K_{0}^G(X,A) = K_{0}(\text{Vect}_G(X,A)),$$
$$K_{G}(X,A) = K_{0,0}(\text{Vect}_G(X,A)).$$

Let $\rho^{X,Y} : \text{Vect}_G(X,A) \to \text{Vect}_G(Y,A)$ be the restriction functor, where $Y$ is a closed $G$-subspace of $X$. The following assertion is a consequence of Theorem 3.17 and Lemma 3.13.

Lemma 3.22 $\rho^{X,Y}$ is a full quasi surjective Banach functor in the sense of 3.4. $\square$

Therefore, setting $K_{n}^G(X,Y,A) := K_{n}(\rho^{X,Y})$ in the sense of Definition 3.10, we get (by [24, II.3.22] and [22, 2.3.1]) the exact sequence of a pair

$$\ldots \to K_{n-1}^G(X,Y,A) \to K_{n-1}^G(X,A) \to K_{n}^G(Y,A) \to K_{n}^G(X,Y,A) \to \ldots$$

Consider the trivial $G$-pair $(B^n, S^{n-1})$, where $B^n$ is the $n$-dimensional closed ball, and $S^{n-1}$ its boundary. Then, according to a construction in [24, 22] for an arbitrary Banach category, the categories $\text{Vect}_G(X,A)(B^n)$, $\text{Vect}_G(X,A)(S^{n-1})$ are defined along with the group $K(B^n, S^{n-1}; \text{Vect}_G(X,A)) = K(\varphi)$, where

$$\varphi : \text{Vect}_G(X,A)(B^n) \to \text{Vect}_G(X,A)(S^{n-1})$$

is the restriction functor.

We now give the definitions and show, that $\varphi$ coincides with

$$\rho^{B^n \times X, S^{n-1} \times X} : \text{Vect}_G(B^n \times X, A) \to \text{Vect}_G(S^{n-1} \times X, A).$$

Since both functors are induced by restrictions, it is suffices to show, that for a $G$-trivial compact space $Z$ the categories $\text{Vect}_G(Z \times X, A)$ and $\text{Vect}_G(X,A)(Z)$ are naturally isomorphic to each other. By definition, $\text{Vect}_G(X,A)(Z)$ is associated in the sense of 3.4 with the category $\text{Vect}_G(X,A)_{T}(Z)$, defined as follows. The objects of $\text{Vect}_G(X,A)_{T}(Z)$ coincide with those of $\text{Vect}_G(X,A)$, and the morphisms are continuous mappings

$$Z \to \text{Mor}_{\text{Vect}_G(X,A)}(E,F).$$

Thus, $\text{Vect}_G(X,A)_{T}(Z)$ is identified with a full subcategory in $\text{Vect}_G(X \times Z, A)$. The category

$$\text{Vect}_G^{T}(X \times Z, A)$$

of trivial $GGA$-bundles over $X \times Z$, with which $\text{Vect}_G(X \times Z, A)$ is associated according to Theorem 3.17 is a full subcategory of $\text{Vect}_G(X,A)_{T}(Z)$. Passing to the associated categories, we get the required result. Comparison of it with constructions in [24, 22], yields two important corollaries.
**Theorem 3.23** (Bott-Clifford periodicity) We have a natural isomorphism
\[ K^n_G(X, Y, A) \cong K^{n-2}_G(X, Y, A). \]

**Theorem 3.24** (Bott periodicity) We have a natural isomorphism
\[ K^1_G(X, A) \cong K_G(X \times B^1, X \times S^0, A). \]

We get the next result from Construction 3.20.

**Lemma 3.25** Let \( Y \) be a closed \( G \)-subspace of a compact space \( X \). Then
\[ K^*_G(Y, A) = \lim_{Y \subset U \to X} K^*_G(U, A). \]

**Definition 3.26** Let \( X \) be a locally compact paracompact (Hausdorff) \( G \)-space. Let
\[ K^*_G(X, A) := \text{Ker} \{ K^*_G(\tilde{X}, A) \to K^*_G(pt, A) \}, \]
where \( \tilde{X} = X \cup pt \) is the one-point compactification.

It is easy to get another description of groups \( K^*_G \) from a consideration of the sequence (2) for the pair \( (\tilde{X}, pt) \):

**Lemma 3.27**
\[ K^*_G(X, A) \cong K^*_G(\tilde{X}, pt, A). \]

**Definition 3.28** Let
\[ K^{-1,n}_G(X, A) = \text{Ker} \{ K^{-1}_G(\tilde{X}, A) \to K^{-1}_G(pt, A) \}, \]
\[ K^n_G(X, A) = K^n_G((X \setminus Y) \times R^n, A). \]

It is necessary to verify compatibility:
\[ K^{-1,n}_G(Y, A) \cong K^{-1,n}_G(Y \times R, A). \]

This is done in the same way as for Theorem [24, II.4.8]. If \( X \) and \( Y \) are compact, then \( K^n_G(X, Y, A) \cong K^n_G(X, Y, A) \); therefore, we omit the sign \( c \).

We finish this Section with another description of our K-groups.

**Definition 3.29** Let \( X \) be a locally compact paracompact \( G \)-space. A complex of GGA-bundles over \( X \) is defined to be a sequence
\[ (E, d) = \left( \ldots \xrightarrow{d_i} E_i \xrightarrow{d_{i+1}} E_{i+1} \xrightarrow{d_{i+2}} \ldots \right), \quad i \in \mathbb{Z}, \]
where \( i \in \mathbb{Z} \), the \( E_i \) are GGA-bundles over \( X \), and \( d_i \) are morphisms with \( d_{i+1}d_i = 0 \) for every \( i \); also, \( E^i = 0 \) for all but perhaps finitely many \( i \). A morphism of complexes \( f : (E, d) \to (F, h) \) is defined to be a sequence of morphisms \( f_i : E^i \to F^i \) connected by the condition \( f_{i+1}d_{i+1} = h_{i+1}f_i \). Isomorphism in this category will be denoted by \( (E, d) \cong (F, h) \). A point \( x \in X \) is called a point of acyclicity \((E, d)\) if the restriction of \((E, d)\) to \( x \), i.e., the sequence of \( A \)-modules
\[ (E, d)_x = \left( \ldots \xrightarrow{(d_i)_x} E^i_x \xrightarrow{(d_{i+1})_x} E^i_{i+1} \xrightarrow{(d_{i+2})_x} \ldots \right), \]
is exact. The support \( \text{supp} (E, d) \) is the complement in \( X \) of the set of points of acyclicity.
Proposition 3.30  supp \((E, d)\) is a closed \(G\)-subset of \(X\).

Proof: The \(G\)-invariance is obvious. The proof of the fact that it is closed can be found, e. g., in [39, 1.3.34].

Remark 3.31  The remaining assertions of this section can be proved by the classical scheme (cf. [4, 14]) with the specific nature of \(C^*\)-algebras taken into account, as it was demonstrated in two previous lemmas; therefore, the proofs are omitted.

Definition 3.32  Let \(Y\) be a closed \(G\)-subspace of \(X\). Denote by \(\text{L}_G(X, Y; A)\) the semigroup (with respect to the direct sum) of classes of isomorphic \(GGA\)-complexes \((E, d)\) on \(X\) such that \(\text{supp} \,(E, d)\) is a compact subset of \(X \setminus Y\). Two elements in \(\text{L}_G(X, Y; A)\) are said to be \(\text{homotopic}\) if for representatives \((E_0, d_0)\) and \((E_1, d_1)\) of them there is a complex \((E, d)\) representing an element of \(\text{L}_G(X \times I, Y \times I; A)\) such that \((E_0, d_0) = (E, d)|_{X \times \{0\}}\) and \((E_1, d_1) = (E, d)|_{X \times \{1\}}\); in this case we write \((E_0, d_0) \simeq (E_1, d_1)\). We introduce the following equivalence relation: \((E_0, d_0) \sim (E_1, d_1)\) iff there are acyclic \((F_0, f_0)\) and \((F_1, f_1)\) such that

\[(E_0, d_0) \oplus (F_0, f_0) \simeq (E_1, d_1) \oplus (F_1, f_1).\]

By acyclicity we understand the condition \(\text{supp} \,(E, d) = \emptyset\). Let \(\text{M}_G(X, Y; A) = \text{L}_G(X, Y; A)/\sim\). Denote by \(\text{L}^n_G(X, Y; A)\) and \(\text{M}^n_G(X, Y; A)\) the corresponding semigroups constructed from the complexes of length \(n\). We have the natural injective semigroup homomorphisms (addition of the zero term):

\[\text{L}^n_G(X, Y; A) \to \text{L}^{n+1}_G(X, Y; A), \quad \text{L}_G(X, Y; A) := \lim\rightarrow \text{L}^n_G(X, Y; A).\]

The equivalence relation \(\sim\) commutes with imbeddings; therefore, the indicated morphisms induce morphisms \(\text{M}^n_G(X, Y; A) \to \text{M}^{n+1}_G(X, Y; A)\).

Lemma 3.33  Suppose that \(E \to X\) and \(F \to X\) are \(GGA\)-bundles, \(\alpha : E|_Y \to F|_Y\) and \(\beta : E \to F\) are monomorphisms, and in the class of monomorphisms \(E|_Y \to F|_Y\) there exists a \(G\)-homotopy joining \(\alpha\) and \(\beta|_Y\). Then there exists a monomorphism \(\bar{\alpha} : E \to F\) such that \(\bar{\alpha}|_Y = \alpha\).

Lemma 3.34  \(\text{M}^n_G(X, Y; A) \to \text{M}^{n+1}_G(X, Y; A)\) is an isomorphism.

Remark 3.35  Suppose that \(X\) is compact and \(Y = \emptyset\). Then we have a natural isomorphism \(\chi_1 : \text{M}^1_G(X, \emptyset; A) \to \text{K}_G(X; A)\), assigning to a class of the complex \((0 \to E^0 \to E^1 \to 0)\) the element \([E^1] - [E^0]\).

Lemma 3.36  There exists and unique natural equivalence of functors

\[\chi_1 : \text{M}^1_G(X, Y; A) \to \text{K}_G(X, Y; A),\]

on the category of compact \(G\)-pairs, with the equivalence of the form indicated in 3.35 for \((X, \emptyset)\).
Definition 3.37  We return to the case of (locally compact, paracompact) noncompact, in general, $G$-pairs $(X, Y)$. Two complexes $(E, d)$ and $(F, b)$ in $L_G(X, Y; A)$ are said to be \emph{compactly isomorphic} if there exists a compact $G$-subset $C$ with
\[
\text{supp } (E, d) \cup \text{supp } (F, b) \subset \text{int } C \subset C \subset X \setminus Y,
\]
and $G$-isomorphisms $\psi^i : E^i \to F^i$ over $X$, where the following diagram commutes over $X \setminus \text{int } C$:
\[
\begin{array}{ccccccc}
0 & \to & E^0 & \overset{d^0}{\to} & E^1 & \overset{d^1}{\to} & \cdots & \overset{d^n}{\to} & E^n & \to & 0 \\
0 & \downarrow{\psi^0} & \downarrow{\psi^1} & & \downarrow{\psi^n} & & & & \downarrow{\psi^0} & & \\
0 & \to & F^0 & \overset{b^0}{\to} & F^1 & \overset{b^1}{\to} & \cdots & \overset{b^n}{\to} & F^n & \to & 0.
\end{array}
\]

Complex of the form $0 \to \ldots \to 0 \to E \overset{\text{id}}{\to} E \to 0 \to \ldots \to 0$ is said to be \emph{elementary}. Two complexes $(E, d)$ and $(F, b)$ are said to be \emph{compact equivalent}, if there exist elementary $Q_1, \ldots, Q_r$ and $P_1, \ldots, P_s$ such that $(E, d) \oplus Q_1 \oplus \ldots \oplus Q_r$ is compact isomorphic to $(F, b) \oplus P_1 \oplus \ldots \oplus P_s$. This equivalence is denoted by $\approx$.

Lemma 3.38 $(E, d) \approx (F, b)$ implies $(E, d) \sim (F, b)$.

Lemma 3.39 Suppose that
\[
(E, d) = \left(0 \to E^0 \overset{d}{\to} E^1 \to 0 \right), \quad (F, b) = \left(0 \to F^0 \overset{b}{\to} F^1 \to 0 \right)
\]
and there exist compact $G$-sets $C_1$ and $C_2$ such that
\[
\text{supp } (E, d) \cup \text{supp } (F, b) \subset C_1 \subset \text{int } C_2 \subset C_2 \subset X \setminus Y,
\]
and a GGA-bundle $L$ over $C_2$ with an isomorphism
\[
\theta : (E^0 \oplus F^1)|_{C_2} \oplus L \to (E^1 \oplus F^0)|_{C_2} \oplus L,
\]
for which $\theta|_{C_2 \setminus C_1} = d \oplus b^{-1} \oplus 1$. Then $(E, d) \approx (F, b)$.

Lemma 3.40 $M^1_G(X \setminus Y; A) \cong M^1_G(X, Y; A)$.

Corollary 3.41

1) $\quad M^1_G(\hat{X}, pt; A) \cong M^1_G(X; A)$
\[
\downarrow \cong \\
K_G(\hat{X}, pt; A) \cong K_G(X; A),
\]

2) $\quad M^1_G(X, Y; A) \cong M^1_G(X \setminus Y; A)$
\[
\downarrow \cong \\
K_G(X, Y; A) \cong K_G(X \setminus Y; A)
\]

Theorem 3.42 There is a natural isomorphism
\[
M_G(X, Y; A) \cong M^1_G(X, Y; A) \cong K_G(X, Y; A).
\]
4 *-Fredholm operators

Let us remind the definition of C*-Fredholm operator \([31, 41]\).

**Definition 4.1** A bounded \(A\)-operator \(F : H_A \to H_A\), is called Fredholm, if

1) The operator \(F\) supposes adjoint, and

2) There exist decompositions of the domain \(H_A = M_1 \oplus N_1\) and the values space \(H_A = M_2 \oplus N_2\) of \(F\) (where \(M_1, M_2, N_1, N_2\) are closed \(A\)-submodules, \(N_1\) and \(N_2\) have a finite number of generators) such that the operator \(F\) has with respect to these decompositions the matrix \(F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}\), and \(F_1 : M_1 \to M_2\) is an isomorphism.

In the equivariant case we change the definition as follows. Let \(l_2(P)\) be equipped with an invariant \(A\)-inner product (it is possible to do this by Remark 2.8). We can apply the Stabilization Theorem of Kasparov \([26]\) to \(l_2(P)\): \(l_2(P) \oplus \mathcal{H}_A \cong \mathcal{H}_A\), where \(\mathcal{H}_A = \sum_{i=1}^{\infty} (A \otimes \mathbb{C} V_i)\), and \(\{V_i\}\) is the countable collection of finite-dimensional spaces, in which all (up to isomorphism) irreducible unitary representations of \(G\) are realized, and each representation is repeated an infinite number of times. Isomorphism (3) is a \(GGA\)-isomorphism of Hilbert modules and the sum on the left in (3) is orthogonal. We introduce the following notation

\[ R_m = \sum_{i=m+1}^{\infty} (A \otimes \mathbb{C} V_i), \quad R_m^\perp = \sum_{i=1}^{m} (A \otimes \mathbb{C} V_i). \]

For any bounded \(G\)-\(A\)-operator \(F : l_2(P_1) \to l_2(P_2)\) let \(S(F) : \mathcal{H}_A \to \mathcal{H}_A\) (\(S\) due to the word “stabilization”) denote the operator

\[ \begin{pmatrix} 1 & 0 \\ 0 & F \end{pmatrix} : \mathcal{H}_A \cong \mathcal{H}_A \oplus l_2(P_1) \to \mathcal{H}_A \oplus l_2(P_2) \cong \mathcal{H}_A. \]

Of course, everything is determined up to a \(GGA\)-isomorphism.

**Theorem 4.2** cf. \([31, 41]\) Let \(\mathcal{H}_A \cong \mathcal{M} \oplus \mathcal{N}\), where \(\mathcal{M}\) and \(\mathcal{N}\) are closed \(GGA\)-modules, and \(\mathcal{N}\) has a finitely many generators \(a_1, \ldots, a_s\). Then \(\mathcal{N}\) is a projective \(GGA\)-module of finite type.

**Proof:** Just as in \([31, 41]\). \(\square\)

**Definition 4.3** A bounded \(GGA\)-operator

\[ F : l_2(P_1) \to l_2(P_2), \]

is called Fredholm operator \((GGA\)-Fredholm\), if

1) \(F\) admits an adjoint;

2) for \(S(F)\) there exist an inverse image decomposition \(\mathcal{H}_A = \mathcal{M}_1 \oplus \mathcal{N}_1\) and an image decomposition \(\mathcal{H}_A = \mathcal{M}_2 \oplus \mathcal{N}_2\), where \(\mathcal{M}_1, \mathcal{M}_2, \mathcal{N}_1, \mathcal{N}_2\) are closed \(GGA\)-modules, \(\mathcal{N}_1, \mathcal{N}_2\) have finitely many generators, and the operator \(S(F)\) has the matrix form \(S(F) = \)
\[
\begin{pmatrix} F_1 & 0 \\
0 & F_2 \end{pmatrix}
\]
in these decompositions, with \(F_1 : M_1 \to M_2\) being an isomorphism of GGA-modules. By Theorem 4.2, \(N_1\) and \(N_2\) are projective GGA-modules; we can form the index element
\[
\text{index } F = [N_1] - [N_2] \in K^G(A).
\]

**Theorem 4.4** (see, e. g. [39]) In the decomposition in the definition of \(A\)-Fredholm operator (see 4.1) we can always assume \(M_0\) and \(M_1\) admitting an orthogonal complement. More precisely, there exists a decomposition for
\[
\begin{pmatrix} F_3 & 0 \\
0 & F_4 \end{pmatrix} : H_A = V_0 \oplus W_0 \to V_1 \oplus W_1 = H_A,
\]
such that \(V_0 \perp \hat{V}_0 = H_A\), \(V_1 \perp \hat{V}_1 = H_A\), or (what is just the same) such that the projections
\[
p_0 : V_0 \oplus W_0 \to V_1, \quad p_1 : V_1 \oplus W_1 \to V_1
\]
admit conjugates.

**Lemma 4.5** Let \(\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1\) be a decomposition into an orthogonal sum and \(\mathcal{M}\) and \(\mathcal{M}_0\) are GGA-modules with an invariant inner product. Then \(\mathcal{M}_1\) is invariant.

**Proof:** Let \(x \in \mathcal{M}_0\), \(y \in \mathcal{M}_1\) and \(g \in G\) be arbitrary. Then
\[
\langle x, gy \rangle = g(\langle g^{-1}x, y \rangle) = g(0) = 0. \quad \Box
\]

**Corollary 4.6** The previous Theorem is valid in the GGA-case as well.

**Proof:** While proving the mentioned theorem we took \(V_0 = (N_0)^\perp\) and \(V_1 = F(V_0)\). Hence, by Lemma all modules are invariant. \(\Box\)

The remaining statements of the present section can be proved similarly to the GA-case (see [31, 33]).

**Theorem 4.7** The index is well-defined.

**Lemma 4.8** Let \(F : l_2(A) \to l_2(A)\) be a GGA-Fredholm operator, then there exists an \(\varepsilon > 0\) such that any bounded GGA-operator \(D\), restricted to satisfy \(\|F - D\| < \varepsilon\) and admitting an adjoint, is a GGA-Fredholm operator and \(\text{index } D = \text{index } F\).

**Lemma 4.9** Let \(F\) and \(D\) be GGA-Fredholm operators,
\[
F : l_2(P_1) \to l_2(P_2), \quad D : l_2(P_2) \to l_2(P_3).
\]
Then \(DF : l_2(P_1) \to l_2(P_3)\) is a GGA-Fredholm operator and \(\text{index } DF = \text{index } D + \text{index } F\).
Lemma 4.10 Let $K : l_2(\mathcal{P}) \to l_2(\mathcal{P})$ be a compact GGA-operator. Then $1 + K$ is a GGA-Fredholm operator and index $(1 + K) = 0$.

Lemma 4.11 Consider a GGA-Fredholm operator $F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2)$. Let an operator $K \in K(l_2(\mathcal{P}_1), l_2(\mathcal{P}_2))$ be $G$-equivariant. Then the operator $F + K$ is a GGA-Fredholm operator and index $(F + K) = \text{index } F$.

Lemma 4.12 Let $F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2)$ be a bounded GGA-operator admitting an adjoint, $D = S(F) + K \in \text{End}^* \mathcal{H}_A$ and $K \in K(\mathcal{H}_A)$ be GGA-operators. Let $D$ have a decomposition of $\mathcal{H}_A$ from the definition of $G$-$A$-Fredholm operator. Then $F$ is a GGA-Fredholm operator.

Theorem 4.13 Let

$$F : l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2), \quad D : l_2(\mathcal{P}_2) \to l_2(\mathcal{P}_1), \quad D' : l_2(\mathcal{P}_2) \to l_2(\mathcal{P}_1)$$

be bounded GGA-operators admitting an adjoint and

$$S(DF) = 1_{\mathcal{H}_A} + K_1, \quad S(D'F) = 1_{\mathcal{H}_A} + K_2, \quad K_1, K_2 \in K(\mathcal{H}_A).$$

Then $F$ is a GGA-Fredholm operator.

Lemma 4.14 Let $D$, $D'$, and $F$ be bounded GGA-operators admitting adjoint. Let $FD$ and $D'F$ be GGA-Fredholm operators. Then $F$ is a GGA-Fredholm operator.

5 The Thom isomorphism

In this section we discuss the theorem on the Thom isomorphism in $K_G(\ , A)$-theory. As in other cases (see e.g. [4]) it plays an important role.

Let $X$ be a $G$-space, $p : F \to X$ a complex bundle over $X$, and $s : X \to F$ an invariant section. We denote by $\Lambda^i(F)$ the complex $G$-bundle of $i$-vectors over $X$. Let us define the complex $\Lambda(F, s)$ over $X$ of lengths $n = \dim F$:

$$\Lambda(F, s) := (0 \to \Lambda^0(F) \xrightarrow{\alpha^0} \Lambda^1(F) \xrightarrow{\alpha^1} \ldots \xrightarrow{\alpha^{n-1}} \Lambda^n(F) \to 0),$$

where $\alpha^k(v_x) = s(x) \wedge v_x$ for $v_x \in \Lambda^k(F)_x$.

Lemma 5.1 (see [7])

1. $(\Lambda(F, s), \alpha)$ is really a complex.
2. $\text{supp } (\Lambda(F, s)) = \{ x \in X | s(x) = 0 \}$ and if this set is compact, then the element $[\Lambda(F, s)] \in K_G(X)$ is defined. □
Let $\pi : p^*F \to F$ be the bundle with total space

$$p^*F = \{(f_1, f_2) \in F \times F \mid p(f_1) = p(f_2)\}, \quad \pi(f_1, f_2) = f_1.$$ 

The vector bundle $(p^*F, \pi, F)$ has the canonical section

$$s_F : F \to p^*F, \quad s_F(f) = (f, f).$$

The support of $s_F$ is equal to $X$. Hence, if $X$ is a compact set, the element $[\Lambda(p^*F, s_F)] = \lambda_F \in K_G(X)$ is defined. Let us define by $a \cdot b$ the element $p^*(a) \otimes b \in K_G(F; A)$.

**Definition 5.2** If the base of a vector bundle is compact, then the mapping

$$\varphi : K_G(X; A) \to K_G(F; A), \quad \varphi(a) = a \cdot \lambda_F.$$

is called the *Thom homomorphism.*

The following statement is obvious.

**Proposition 5.3** The Thom homomorphism $\varphi$ is a morphism of $R(G)$-modules. □

Let $i : X \hookrightarrow F$ be the enclosure of the zero section of $F$. It induces the homomorphisms

$$i^* : K_G(F; A) \to K_G(X; A), \quad i^*\varphi : K_G(X; A) \to K_G(X; A).$$

**Proposition 5.4 1.** If $X$ is compact, then the following sequence is defined

$$0 \longrightarrow \Lambda^0F \overset{0}{\longrightarrow} \Lambda^1F \overset{0}{\longrightarrow} \Lambda^2F \longrightarrow \cdots \overset{0}{\longrightarrow} \Lambda^nF \longrightarrow 0.$$

It defines an element of $K_G(X)$. For any element $a \in K_G(X; A)$

$$i^*\varphi(a) = a \cdot [0 \longrightarrow \Lambda^0F \overset{0}{\longrightarrow} \cdots \overset{0}{\longrightarrow} \Lambda^nF \longrightarrow 0].$$

2. For $a \in K_G(X; A) \cong M_G(X; A)$ (see 3.42)

$$i^*\varphi(a) = a \cdot \sum_{i=0}^{n} (-1)^i \Lambda^i F.$$

**Proof:** By the construction of the natural isomorphism between $K_G$ and $M_G$, these assertions are equivalent to each other. Let us prove the item 1. Let $a \in K_G(X; A)$ be represented by the complex

$$a = [(\mathcal{E}, \alpha)] = [0 \longrightarrow E^k \overset{\alpha^k}{\longrightarrow} E^\alpha \longrightarrow E^0 \longrightarrow 0].$$

Then

$$\varphi(a) = a \cdot \lambda_F = [(p^*\mathcal{E}, p^*\alpha) \otimes \Lambda(p^*F, s_F)],$$

hence

$$i^*\varphi(a) = [(i^*p^*\mathcal{E}, i^*p^*\alpha)] \otimes [i^*\Lambda(p^*F, s_F)].$$
Since \( p^i = 1 \), we have \((i^*p^*E, i^*p^*\alpha) = (E, \alpha) = a\), while the restriction of complex \( \Lambda(p^*F, s_F) \) on \( X \) is equal to
\[
[0 \rightarrow \Lambda^0 F \rightarrow \cdots \rightarrow \Lambda^n F \rightarrow 0]. \tag*{\Box}
\]

Let us pass to the case of a locally compact space \( X \). The complex \( \Lambda(p^*F, s_F) \) has no compact support now and does not determine the element \( \lambda_F \in K_G(F) \). However, if \( a = [(E, \alpha)] \in K_G(X; A) \), then
\[
\text{supp } \{(p^*E, p^*\alpha) \otimes \Lambda(p^*F, s_F)\} \subset
\subset \text{supp } (p^*E, p^*\alpha) \cap \text{supp } \Lambda(p^*F, s_F) \subset
\subset \text{supp } (p^*E, p^*\alpha) \cap X = \text{supp } (E, \alpha).
\]
Thus, the complex \( (p^*E, p^*\alpha) \otimes \Lambda(p^*F, s_F) \) has the compact support. We obtain a homomorphism of \( R(G) \)-modules
\[
\varphi : K_G (X; A) \rightarrow K_G (F; A), \quad \varphi (a) = [(p^*E, p^*\alpha) \otimes \Lambda(p^*F, s_F)].
\]
As well as in the compact case,
\[
i^*\varphi (a) = a \)[0 \rightarrow \Lambda^0 F \rightarrow \Lambda^1 F \rightarrow \cdots \rightarrow \Lambda^n F \rightarrow 0].
\]
Passing by the Bott periodicity to \( K^1 \) (see Theorem 3.23), we define the Thom homomorphism in the general case:
\[
\varphi = \varphi^F_A : K_G(X; A) \rightarrow K_G^*(F; A).
\]
Let now \( E \) and \( F \) be two complex \( G \)-bundles over \( X \). Then a product is defined by the formula
\[
\mu : K_G (E) \otimes K_G (F) \rightarrow K_G (E \times F).
\]
Since the enclosure \( E \oplus F \rightarrow E \times F \) induces a homomorphism
\[
K_G (E \times F) \rightarrow K_G (E \oplus F),
\]
we obtain the multiplication \( \cdot : K_G (E) \otimes K_G (F) \rightarrow K_G (E \oplus F) \).

**Theorem 5.5 1.** \cite{16} If \( X \) is compact, then
\[
\lambda_E \cdot \lambda_F = \lambda_{E \oplus F}.
\]

2. In the general case \( \varphi^F_A : \varphi^F_C = \varphi^F_C \cdot \varphi^F_A = \varphi^{E \oplus F}. \) The equalities hold in the following sense. Let, for example, \( ab \in K_G (X; A) \), where \( a \in K_G (X) \), \( b \in K_G (X; A) \). Then
\[
\varphi^F_C (a) \cdot \varphi^F_A (b) = \varphi^{E \oplus F} (ab).
\]
3. \cite{16} Let \( F_1 \) and \( F_2 \) be two complex bundles over \( X \), and \( s_1, s_2 \) their section. Then
\[
\Lambda (F_1 \oplus F_2, s_1 \oplus s_2) = \Lambda (F_1, s_1) \otimes \Lambda (F_2, s_2).
Proof: The item 2 immediately follows from the item 1 and the definition of the Thom homomorphism. The items 1 and 3 are proved in [16]. □

Proposition 5.6 Consider $E \oplus F$ as a bundle over $X$ as well as a bundle over $E$. Then the diagram
\[
\begin{array}{ccc}
K_\ast_G(X; A) & \xrightarrow{\varphi_E} & K_\ast_G(E; A) \\
\downarrow \varphi_{E \oplus F}^1 & & \downarrow \varphi_{E \oplus F}^2 \\
K_\ast_G(E \oplus F; A) & = & K_\ast_G(E \oplus F; A)
\end{array}
\]
is commutative.

Proof: Consider the projections
\[ p : E \to X, \quad q : F \to X, \quad r : E \oplus F \to X, \quad t : E \oplus F \to E. \]
Let $x \in K_G(X; A)$. Then $\varphi^E(x) = p^*(x) \Lambda(p^*E, s_E)$,
\[
\varphi_{E \oplus F}^2 \varphi^E(x) = t^*(p^*(x) \Lambda(p^*E, s_E)) \Lambda(t^*(E \oplus F), s_{E \oplus F}),
\]
and $\Lambda(t^*(E \oplus F), s_{E \oplus F}) = \Lambda(r^*F, t^*s_F)$. Since $t^*s_E + t^*s_F = s_{E \oplus F}$, by the item 3 of Theorem 5.3 (we denote the elements of $K$-groups and their representatives by the same symbols),
\[
\varphi_{E \oplus F}^2 \varphi^E(x) = r^*(x) \Lambda(r^*E, t^*s_F) = r^*(x) \Lambda(r^*(E \oplus F), s_{E \oplus F}) = \varphi_{E \oplus F}^1(x). \quad \square
\]

Let starting from this moment $X$ be separable. Then $C_0(X) \rtimes G$ is of the class of algebras for which the results of [38] are valid.

Theorem 5.7 [12] Let $X$ be separable and metrizable, and let action of $G$ on $A$ be trivial. Then $\varphi_A$ is an isomorphism.

Proof: Let us form the geometrical resolution [38]
\[
0 \to A \otimes K \otimes C_0(\mathbb{R}) \xrightarrow{\iota} C \xrightarrow{\nu} 0,
\]
where $C \neq C$ and $F$ are $C^*$-algebras, and $K$ is the algebra of compact operators in Hilbert space. Let us prove the following Lemma.

Lemma 5.8 The diagram
\[
\begin{array}{ccc}
K_\ast_G(X; A \otimes K \otimes C_0(\mathbb{R})) & \xrightarrow{\iota_*} & K_\ast_G(X; C) \\
\downarrow \varphi_{A \otimes K \otimes C_0(\mathbb{R})} & & \downarrow \varphi_C \\
K_\ast_G(V; A \otimes K \otimes C_0(\mathbb{R})) & \xrightarrow{\iota_*} & K_\ast_G(V; C)
\end{array}
\]
is commutative. Here $V \to X$ is a $C$-vector bundle,
\[ \iota : A \otimes K \otimes C_0(\mathbb{R}) = \text{Ker } \nu \hookrightarrow C, \]
$\iota_*$ is an extension of scalars.
5 THE THOM ISOMORPHISM

Proof: Let \( a \in K^0_G(X; A \otimes K \otimes C_0(R)) \) be defined by the following complex:
\[
(\mathcal{E}, \alpha) = (0 \to E^1 \xrightarrow{\alpha} E^0 \to 0).
\]
Then (by \( \otimes \) we define the tensor product over the cartesian product of bases)
\[
\varphi_{A \otimes K \otimes C_0(R)}(a) = [(p^* \mathcal{E}, \psi^* \alpha) \otimes \Lambda(p^* F, s_F)] = \\
= \begin{bmatrix}
0 & p^* E^1 \otimes p^* \Lambda^0 F & (p^* \alpha \otimes 1) \otimes (1 \otimes s) \\
& (p^* E^1 \otimes p^* \Lambda^0 F) \oplus (p^* E^1 \otimes p^* \Lambda^1 F) & \ldots
\end{bmatrix}_{\text{diag}},
\]
and
\[
\iota_* \varphi_{A \otimes K \otimes C_0(R)}(a) = \begin{bmatrix}
0 & (p^* E^1 \otimes_A \otimes_{K \otimes C_0(R)} C) \otimes p^* \Lambda^0 F & (p^* \alpha \otimes 1 \otimes 1) \otimes (1 \otimes 1 \otimes s) \\
& ((p^* E^1 \otimes_A \otimes_{K \otimes C_0(R)} C) \otimes p^* \Lambda^1 F) & \ldots
\end{bmatrix}_{\text{diag}} = \\
= \varphi_{C \otimes A}(a).
\]

By definition \( \varphi \) commutes with the Bott isomorphism, which is natural \[19\], in particular, it commutes with \( \iota_* \). Hence, Lemma is true for \( K^1 \) too. \( \square \)

If \( G \) is a compact group, \( X \) is a paracompact Hausdorff \( G \)-space, then \( X/G \) is a paracompact \( G \)-space (see, e.g. \[16, p. 7\]). The stabilizer \( G_x \) is closed \[11\]. Hence, it is compact and has the type I. Therefore, by the following theorem, \( C^* \)-crossed product \( C_0(X) \rtimes G \) has the type I in our case.

**Theorem 5.9** \[18\] \( C_0(X) \rtimes G \) has the type I iff \( X/G \) is a \( \tau_0 \)-space and all isotropy groups \( G_x \) has the type I. \( \square \)

Let \( \gamma : K^*_G(X; B) \xrightarrow{\cong} K^*((C_0(X) \otimes B) \rtimes G) \) be the natural isomorphism \[13, 21\]. Let \( B := A \otimes K \otimes C_0(R) \) and \( C^X_G := C_0(X) \rtimes G \) (and similarly for \( V \)). Consider the following diagram
\[
\begin{array}{c c c c}
0 & \xrightarrow{\cong} & \text{Tor}(K_*(C^X_G), K_*(A)) & \xrightarrow{\gamma \varphi C^Y \gamma^{-1} \otimes 1} & K_*(C^X_G) \otimes K_*(C) & \xrightarrow{\gamma \varphi C^Y \gamma^{-1}} & K_*(C^Y_G) \otimes K_*(C) \\
\xrightarrow{\cong} & 1 & 2 & \xrightarrow{\gamma \varphi C^Y \gamma^{-1}} & \text{Tor}(K_*(C^V_G), K_*(A)) & \xrightarrow{\gamma \varphi C^Y \gamma^{-1}} & K_*(C^V_G) \otimes K_*(C) \xrightarrow{\gamma \varphi C^Y \gamma^{-1}} & K_*(C^V_G) \otimes K_*(C),
\end{array}
\]
where the exact rows represent a part of (4.5) from \[38\].
Consider also the diagram:

\[
\begin{align*}
\bigoplus_{q \in \mathbb{Z}_2} K_q(C_G^X \otimes B) \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbb{Z}_2} K_q(C_G^X \otimes C) \\
\cong & \oplus_{q \in \mathbb{Z}_2} K_q(C_G^X \otimes B) \\
0 \to K_*(C_G^X) \otimes K_*(A) \xrightarrow{\alpha_X} & K_*(C_G^X) \otimes A \xrightarrow{\beta_X} \text{Tor}(K_*(C_G^X), K_*(A)) \to 0 \\
\cong & K_*(((C_0(X) \otimes A) \rtimes G)) \\
0 \to K_G^*(X) \otimes K_*(A) \xrightarrow{\varphi_A} & K_G^*(X; A) \\
\cong & K_*(((C_0(V) \otimes A) \rtimes G)) \\
0 \to K_G^*(V) \otimes K_*(A) \xrightarrow{\sigma_V} & K_G^*(V; A) \\
\cong & K_*(C_G^V) \otimes K_*(A) \xrightarrow{\alpha_V} K_*(C_G^V) \otimes A \xrightarrow{\beta_V} \text{Tor}(K_*(C_G^V), K_*(A)) \to 0 \\
\cong & K_*(C_G^V) \otimes K_*(A) \\
\bigoplus_{q \in \mathbb{Z}_2} K_q(C_G^V \otimes B) \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbb{Z}_2} K_q(C_G^V \otimes C) \\
\cong & \oplus_{q \in \mathbb{Z}_2} K_q(C_G^V \otimes B) \\
\end{align*}
\]

where the rows are exact: they present the main result of \([8]\). Let us consider the following diagram:

\[
\begin{align*}
\bigoplus_{q \in \mathbb{Z}_2} K_q(C_G^X \otimes B) \xrightarrow{\iota_*} & \bigoplus_{q \in \mathbb{Z}_2} K_q(C_G^X \otimes C) \xrightarrow{\gamma} \text{Tor}(K_*(C_G^X), K_*(A)) \to 0 \\
K_G^*(X; A) \xrightarrow{\varphi_B \gamma^{-1}} & 10 \xrightarrow{\gamma \varphi_C \gamma^{-1}} 7 \xrightarrow{\gamma \varphi_C \gamma^{-1} \ast 1} \text{Tor}(K_*(C_G^V), K_*(A)) \to 0. \\
\end{align*}
\]

Let us suppose that all squares \([1], \ldots, 10\) are commutative. We obtain the diagram

\[
\begin{align*}
0 \to K_G^*(X) \otimes K_*(A) \xrightarrow{\varphi_A \gamma^{-1}} & K_G^*(X; A) \xrightarrow{\gamma \varphi_C \gamma^{-1}} \text{Tor}(K_*(C_G^X), K_*(A)) \to 0 \\
0 \to K_G^*(V) \otimes K_*(A) \xrightarrow{\varphi_A \gamma^{-1}} & K_G^*(V; A) \xrightarrow{\gamma \varphi_C \gamma^{-1} \ast 1} \text{Tor}(K_*(C_G^V), K_*(A)) \to 0.
\end{align*}
\]

Its rows are exact by the commutativity of \([3]\) and \([4]\), and \([11] = [5]\). If \(a \in K_G^*(X; A), \)
then
\[
(\gamma \varphi C \gamma^{-1} \ast 1) (\beta_X \circ \sigma_X \circ \gamma(A)) = \begin{pmatrix} 7 & 8 \\ 6 & 10 \end{pmatrix} (\beta \circ \sigma_X \circ \gamma(a)) = \begin{pmatrix} 7 & 9 \end{pmatrix} (\beta \circ \sigma_V \circ \gamma \circ \varphi_A(a)).
\]

Hence, (4) is a commutative diagram with exact rows. Let us show that the squares 1, \ldots, 10 are commutative.

- **1** commutes by the standard algebraic argument;
- **5** commutes by the associativity of \( \otimes \): rows and columns are induced by \( \otimes \);
- **2** commutes if we can prove the commutativity of 3 or 4 for any unital \( A \). Indeed, \( C \) is not unital, but we can consider the unitalization \( C^+ \) of it and the diagram:

\[
\begin{array}{ccc}
K_*(C^X_G) \otimes K_*(C) & \longrightarrow & K_*(C^X_G \otimes C) \\
K_*(C^X_G) \otimes K_*(C^+) & \longrightarrow & K_*(C^X_G \otimes C^+) \\
K^C_*(C_0(X)) \otimes K_*(C) & \longrightarrow & K^C_*(C_0(X) \otimes C) \\
K^C_*(C_0(X)) \otimes K_*(C^+) & \longrightarrow & K^C_*(C_0(X) \otimes C^+)
\end{array}
\]

Then the forward square is commutative by 3. The commutativity of the others is evident. Hence, the back square is commutative and we get 2.

- **8** and **9** commute by the construction of the exact sequence from 38.
- **6** commutes since \( \gamma \) for \( K_1 \) is defined in 19, 21 via the Bott isomorphism.
- **7** = 1 + 2.

So, we have to verify the commutativity of 3 (for any unital algebra) and 10. Let \( P \) be a projective \( G \)-\( C_0(X) \)-module, \( M \) a projective \( A \)-module. Then

\[
\alpha(\gamma \otimes 1)([P] \otimes [M]) = \alpha([P \underbrace{L^1(G,C_0(X)) \otimes}_{\begin{pmatrix} 7 \end{pmatrix}} (C_0 \otimes G)] \otimes [M]) = ([P \underbrace{L^1(G,C_0(X)) \otimes}_{\begin{pmatrix} 6 \end{pmatrix}} (C_0 \otimes G)] \otimes [M]),
\]

and

\[
\sigma \gamma \alpha([P] \otimes [M]) = \sigma([P \otimes [M]) = \sigma([P \otimes M] \underbrace{L^1(G,C_0(X)) \otimes}_{\begin{pmatrix} 7 \end{pmatrix}} ((C_0(X) \otimes A) \otimes G]) = \begin{pmatrix} 7 \end{pmatrix} \begin{pmatrix} 6 \end{pmatrix} \begin{pmatrix} 9 \end{pmatrix} ((C_0(X) \otimes A) \otimes_{\text{max}} A)),
\]

\[
= [P \otimes M] \underbrace{L^1(G,C_0(X)) \otimes}_{\begin{pmatrix} 6 \end{pmatrix}} ((C_0(X) \otimes G) \otimes_{\text{max}} A)).
\]
Let us define the following mapping of the obtained modules
\[ r : p \otimes m \otimes (f \otimes a) \mapsto (p \otimes f) \otimes ma. \]

To prove that it is well defined, we have to verify the following equality

\[ r((p \otimes m)h \otimes (f \otimes a)) = r((p \otimes m) \otimes h(f \otimes a)), \]

where \( h \in L^1(G, C_0(X)) \otimes A \). Without loss of generality it can be assumed that

\[ h(g) = \sum_{i=1}^{k} h^i(g) \otimes a^i, \]

where \( h^i \in L^1(G, C_0(X)), a_i \in A \). Then

\[ r((p \otimes m)h \otimes (f \otimes a)) = \sum_{i=1}^{k} \int_G g^{-1}(p \cdot h^i(g)) \, dg \otimes f \otimes m \cdot a^i. \]

On the other hand,

\[ r((p \otimes m) \otimes h(f \otimes a)) = \sum_{i=1}^{k} p \otimes h^i f \otimes m \cdot a^i a = \sum_{i=1}^{k} \int_G g^{-1}(p \cdot h^i(g)) \, dg \otimes f \otimes m \cdot a^i a. \]

Hence, the mapping is well defined for the algebraic tensor product. Since \( A \) is a unital algebra, \( r \) is an isomorphism of the algebraic tensor products. Consider

\[ t : (P \otimes_{L^1(G, C_0(X))} C_0(X) \rtimes G) \otimes M \longrightarrow (P \otimes_{L^1(G, C_0(X))} C_0(X) \rtimes G) \otimes A, \]

\[ t(p \otimes f \otimes m) := (p \otimes m) \otimes (f \otimes 1_A). \]

Then

\[ t \circ r(p \otimes m \otimes (f \otimes a)) = (p \otimes ma) \otimes (f \otimes 1_A) = \left( \frac{1}{l} p \otimes m \right) \otimes (l \otimes a) \otimes (f \otimes 1_A) = \]

\[ = \int_G g^{-1} \left( \frac{1}{l} p \otimes m \right) (l \otimes a)(g) \, dg \otimes (f \otimes 1_A) = \left( \frac{1}{l} p \otimes m \right) \otimes (lf \otimes a) = (p \otimes m) \otimes (f \otimes a), \]

i.e., \( t \circ r = \text{Id} \). We obtained an algebraic isomorphism of modules over \((C_0(X) \rtimes G) \otimes A\). Let us consider its behavior with the respect to norms. Considering a dense subset, we can assume that the element has the form

\[ x = \sum_{i=1}^{k} (p_i \otimes m_i) \otimes f_i \otimes a_i, \]

where

\[ f_i \otimes a_i \in L^1(G, C_0(X)) \otimes_{\text{max}} A \subset (C_0(X) \rtimes G) \otimes_{\text{max}} A. \]

On this dense subset \( r \) is the following isometry:

\[ P \otimes M \otimes \{1\} \longrightarrow (P \otimes \{1\}) \otimes M. \]
Hence, $r$ is an isomorphism of $(C_0(X) \rtimes G) \otimes A$-modules and
\[ \sigma \gamma \alpha = \alpha(\gamma \otimes 1). \]

Let us verify the commutativity of $[10]$. By Lemma $[5,8]$ it is sufficient to obtain the equality $\gamma t_* = t_*\gamma$. We have:
\[ \gamma t_* P = \left( P_{A \otimes K \otimes C_0(\mathbb{R})} \otimes C \right)_{L^1(G, C_0(X) \otimes C)} ((C_0(X) \otimes C) \rtimes G) = \]
\[ \left( P_{A \otimes K \otimes C_0(\mathbb{R})} \otimes C \right)_{L^1(G, C_0(X)) \otimes_{\max} C} ((C_0(X) \rtimes G) \otimes C) = \]
\[ \left( P_{A \otimes K \otimes C_0(\mathbb{R})} \otimes C \right)_{L^1(G, C_0(X))} (C_0(X) \rtimes G), \]
and
\[ t_* \gamma P = \left( P_{L^1(G, C_0(X) \otimes B)} \otimes (C_0(X) \otimes B) \rtimes G \right)_{C_0(X) \otimes B} C_0(X) \otimes C = \]
\[ \left( P_{L^1(G, C_0(X)) \otimes B} \otimes (C_0(X) \rtimes G) \otimes B \right)_B C = \]
\[ \left( P_{L^1(G, C_0(X))} \otimes (C_0(X) \rtimes G) \right)_{A \otimes K \otimes C_0(\mathbb{R})} C, \]
where as before $B := A \otimes K \otimes C_0(\mathbb{R})$. Let us define
\[ \bar{r} : \gamma t_* P \rightarrow t_* \gamma P, \quad \bar{r}(p \otimes c \otimes h) = p \otimes h \otimes c. \]
Then $\bar{r}$ is well defined. Indeed, if $\alpha \in A \otimes K \otimes C_0(\mathbb{R})$ then
\[ \bar{r}(p \alpha \otimes c \otimes h) = p \alpha \otimes h \otimes c, \]
\[ \bar{r}(p \otimes \alpha c \otimes h) = p \otimes h \otimes \alpha c = p \alpha \otimes h \otimes c, \]
and if $f \in L^1(G, C_0(X))$ then
\[ \bar{r}((p \otimes c) f \otimes h) = \bar{r}(pf \otimes c \otimes h) = pf \otimes h \otimes c, \]
\[ \bar{r}(p \otimes c \otimes fh) = p \otimes fh \otimes c = pf \otimes h \otimes c. \]
Let $f \otimes d \in (C_0(X) \rtimes G) \otimes C$. Then
\[ \bar{r}((p \otimes c \otimes h)(f \otimes d)) = \bar{r}(p \otimes cd \otimes hf) = p \otimes hf \otimes cd, \]
\[ \bar{r}(p \otimes c \otimes h)(f \otimes d) = (p \otimes c \otimes h)(f \otimes d) = p \otimes hf \otimes cd. \]
Hence, $\bar{r}$ defines on the algebraic tensor product a well-defined homomorphism of modules over $(C_0(X) \rtimes G) \otimes C$. On the dense subset
\[ \left( P_{A \otimes K \otimes C_0(\mathbb{R})} \otimes C \right)_{L^1(G, C_0(X))} \subset \left( P_{A \otimes K \otimes C_0(\mathbb{R})} \otimes C \right)_{L^1(G, C_0(X))} C_0(X) \rtimes G \]
the homomorphism $\bar{r}$ is the following isometry
\[ \left( P_{A \otimes K \otimes C_0(\mathbb{R})} \otimes C \right) \otimes \{1\} \rightarrow \left( P \otimes \{1\} \right)_{A \otimes K \otimes C_0(\mathbb{R})} C. \]
Hence, $\bar{r}$ is an isomorphism of modules over the algebra $(C_0(X) \rtimes G) \otimes C$ completed with respect to $\otimes_{\max}$; and $\gamma t_* = t_* \gamma$.

Now we apply Five-lemma to the diagram $[4]$. The proof of Theorem is completed. $\square$
Theorem 5.10 If $X$ is a separable metrizable trivial $G$-space, then $\varphi_A$ is an isomorphism.

Proof: Let us consider the diagram:

$$K^*_G(X; A) \cong K^*_G(C(X) \otimes A) \cong K^*_G((C(X) \otimes A) \times G) \cong K^*_G((C_0(V) \otimes A) \times G) \cong K^*_G(C(V) \otimes A) \cong K^*(X; A \times G) \cong K^*(V; A \times G).$$

To prove the commutativity of this diagram (which implies Theorem) it is sufficient to demonstrate the following. Let $M$ be a projective $G, C_0(V) \otimes A$-module of finite type and $E$ is a projective $G, C(V)$-module, i.e. $C(V)$-module, because the action of $G$ on $C(V)$ is trivial. Then

$$\gamma(M \otimes C(V) E) = \gamma(M) \otimes C(V) E.$$

Since these are “the same modules” by the definition of $\gamma$ in [21] (where it was denoted by $\Psi$), it is necessary to verify first, that the module structure survives, and second, that the constructions behaves properly with the respect to morphisms $f : E \to F$. Indeed,

$$(m \otimes e) \cdot \gamma(M \otimes C(V) E) \left( \int_G a(g) U_g dg \otimes \varphi \right) = \left( \int_G g^{-1}((m \otimes e) \cdot (a(g) \otimes \varphi)) dg \right) =
= \left( \int_G g^{-1}(m \cdot a(g)) \otimes (e \cdot \varphi) dg \right) =
= \left( \int_G g^{-1}(m \cdot a(g)) dg \right) \otimes (e \cdot \varphi) = (m \otimes e) \cdot \gamma(M) \otimes C(V) E \left( \int_G a(g) U_g dg \otimes \varphi \right).$$

The statement about morphisms is immediate consequence of the triviality of the action of $G$ on $C(V)$.

Now, after the particular cases, we are able to prove the general theorem about the Thom isomorphism.

Theorem 5.11 Let $X$ be a manifold, then $\varphi_A$ is an isomorphism.

Proof: First, let us prove the theorem for the trivial bundle $X \times V$, where $V$ is a complex finite-dimensional $G$-space.

Let us denote by $1$ the trivial 1-dimensional $G$-module, and the projective space $P(V \oplus 1)$ is a compactification of $V$ and we have the following natural homomorphism

$$j : K_G(V; A) \to K_G(P(V \oplus 1); A), \quad j : K_G(V \times X; A) \to K_G(P(V \oplus 1) \times X; A).$$

Let $X$ be compact. Let us consider an arbitrary element $x \in K_G(P(V \oplus 1) \times X; A)$. Let us consider the analytical index of the correspondent family of Dolbeault operators over $P(V \oplus 1)$ with coefficients in $x$ (cf. [4]), i.e. an operator over $C^*$-algebra $C(X) \otimes A$ (see Section [4]). This is an element of $K_G(X; A)$. Taking the composition with $j$ (cf. [4], p. 123)) we get a family of mappings $\alpha = \alpha_{x,A} : K_G(V \times X; A) \to K_G(X; A)$, having the following properties:
(a1) \( \alpha \) is functorial with the respect to \( G \)-morphisms of \( X \) and \( A \);

(a2) the following diagram commutes:

\[
\begin{align*}
K_G(V \times X; B) \otimes K_G(X; A) & \longrightarrow K_G(V \times X; B \otimes A) \\
\alpha_{X,B \otimes 1} & \\
K_G(X; B) \otimes K_G(X; A) & \longrightarrow K_G(X; B \otimes A),
\end{align*}
\]

\( \text{in particular, } \alpha \text{ is a morphism of } K_G(X)\)-modules;

(a3) \( \alpha_{pt,C}(\lambda_V^*) = 1 \in R(G) \).

In fact, transferring the space of parameters into coefficients:

\[
K_G(P(V \oplus 1) \times X; B) \cong K_G(P(V \oplus 1); C(X) \otimes B),
\]

we reduce the situation to the case \( X = pt \), and (a2) takes form

\[
\begin{align*}
K_G(V; B) \otimes K_G(pt; D) & \longrightarrow K_G(V; B \otimes D) \\
\alpha_{pt,B \otimes 1} & \\
K_G(pt; B) \otimes K_G(pt; D) & \longrightarrow K_G(pt; B \otimes D),
\end{align*}
\]

and the functoriality of (a1) means functoriality with the respect to the algebra of coefficients. In a usual way with the help of commutative diagram

\[
\begin{align*}
0 \rightarrow K_G(V; A) & \longrightarrow K_G(V; A^+) \longrightarrow K_G(V; C) \\
\alpha & \\
0 \rightarrow K_G(pt; A) & \longrightarrow K_G(pt; A^+) \longrightarrow K_G(pt; C)
\end{align*}
\]

\( \alpha \) can be extended to non-unital (in particular, non-compact) case. The diagram (5) is still commutative. We will need the following two particular cases of it. For \( B = C_0(X) \otimes A, D = C_0(V) \) we get for \( x \in K_G(V \times X; A), \ y \in K_G(V) \)

\[
\alpha_{X \times V,A}(x \otimes y) = \alpha_{X,A}(x) \otimes y \in K_G(X \times V; A).
\]

For \( D = C_0(X) \otimes A, B = C_0(W) \) we get for \( y' \in K_G(W \times X; A), \ x' \in K_G(V) \)

\[
\alpha_{W \times X,A}(x' \otimes y') = \alpha_{pt,C}(x') \otimes y' \in K_G(W \times X; A).
\]

Let \( x \in K_G(X; A) \), then by (6) with \( W = 0 \) and (a2)

\[
\alpha(\lambda_V^* x) = \alpha(\lambda_V^* x) = x.
\]

Let \( y \in K_G(V \times X; A) = K_G(V; C_0(X) \otimes A) \), then by (5) and (7) with \( W = V \)

\[
\alpha(y) \otimes \lambda_V^* = \alpha(y \otimes \lambda_V^*) = \alpha(\lambda_V^* \otimes y) = \alpha(\lambda_V^*) y = y \in K_G(X \times V; A),
\]
where by \( \tilde{y} \in K_G(X \times V; A) \) we denote the element, obtained from \( y \) under the mapping \( X \times V \to V \times X, (x, v) \mapsto (-v, x) \) (such that \( V \times X \times V \to V \times X \times V, (u, x, v) \mapsto (-v, x, u) \) is homotopic to the identity). Let us apply to the both parts of (9) the isomorphism \( K_G(X \times V; A) \to K_G(V \times X; A) \). We obtain that \( \varphi \alpha \) is an isomorphism. But by (8) \( \alpha \varphi = \text{Id} \), hence, \( \alpha \) is the two-sides inverse. And the unknown automorphism is the identity.

The pass to the case of general complex \( H \)-bundle \( E \to Y \) we make similarly to [2, p. 124]. Namely, let us take \( G = U(n) \times H \) and the principal \( G \)-bundle \( X \), with which \( E \) is associated, \( Y = X/U(n) \), in such a way that

\[
K_G(X; A) \cong K_H(Y; A), \quad K_G(V \times X; A) \cong K_H((V \times X)/U(n); A) \cong K_H(E; A).
\]

Since the identifications commute with multiplication by \( \lambda_1^* \), we get the desired result. (Other way is to enclose the bundle into a trivial one and use the transitivity of the Thom homomorphism). \( \square \)

We shall pass now to some further constructions, connected with the Thom homomorphism. They will be necessary for the definition of topological index.

Let \( X \) and \( Y \) be smooth \( G \)-manifolds, \( i : X \to Y \) the equivariant enclosure, \( Y \) is equipped with a \( G \)-invariant Riemannian metric, \( (TX, p_T, X) \) is the tangent bundle of \( X \), \( (N, p_N, X) \) is the normal bundle for \( i \). Let us choose a function \( \varepsilon : X \to (0, \infty) \) such that the map of \( N \) to itself

\[
n \mapsto \varepsilon \frac{n}{1 + |n|}
\]

is \( G \)-equivariant and determines a \( G \)-diffeomorphism \( \Phi : N \to W \) on an open tubular neighborhood \( W \supset X \) in \( Y \). The enclosure \( i : X \to Y \) is decomposed in a composition of two enclosures \( i_1 : X \to W \) and \( i_2 : W \to Y \). Passing to differentials we obtain

\[
TX \xrightarrow{d_1} TW \xrightarrow{d_2} TY, \quad d\Phi : TN \to TW.
\]

**Lemma 5.12** [16, p. 112] The manifold \( TN \) can be identified with \( p_T^*(N \oplus N) \) with the help of a \( G \)-diffeomorphism \( \psi \) such that the following diagram is commutative

\[
\begin{array}{ccc}
p_T^*(N \oplus N) & \to & TN \\
\downarrow \psi & & \downarrow \\
TX & \xrightarrow{p_T} & N \\
\downarrow \ & & \downarrow \ & & \downarrow p_N \\
X.
\end{array}
\]

**Proof:** The manifold \( TN \) as the vector bundle over \( N \) can be identified with \( p_N^*(TX) \oplus p_N^*(N) \). A point of the total space \( TN \) is a pair of the form \( (n_1, t + n_2) \), where both vectors are from the fiber over the point \( x \in X \). Similarly, we represent elements \( p_T^*(N \oplus N) \) as pairs of the form \( (t, n_1 + n_2) \). Let us define \( \psi \) by the equality \( \psi(n_1, t + n_2) = (t, n_1 + n_2) \). \( \square \)
With the help of the relation \( i \cdot (n_1, n_2) = (-n_2, n_1) \), we can equip \( p_T^*(N \oplus N) = p_T^*(N) \oplus p_T^*(N) \) with a structure of a complex manifold. Then we can consider the Thom homomorphism

\[ \varphi : K_G(TX; A) \to K_G(p_T^*(N \oplus N), A). \]

Since \( TW \) is an open \( G \)-stable subset of \( TY \) and \( di_2 : TW \to TY \) is an enclosure, by the construction \[\text{[3.20]}\], there is the homomorphism \( (di_2)_* : K_G(TW; A) \to K_G(TY; A) \).

**Definition 5.13** Let \( i : X \to Y \) be an enclosure. The Gysin homomorphism is the mapping

\[ i_! : K_G(TX; A) \to K_G(TY; A), \quad i_! = (di_2)_*(d\Phi^{-1})^* \psi^* \varphi. \]

In other words, it is obtained by passage to \( K \)-groups in the upper part of the diagram

\[
\begin{array}{ccc}
\psi & \leftarrow & p_T^*(N \oplus N) \\
\downarrow p_T & & \downarrow q_T \\
TX & \leftarrow & TN \\
\downarrow & & \downarrow \Phi \\
N & \leftarrow & TW \\
\downarrow & & \downarrow di_2 \\
X & \leftarrow & TY \\
\downarrow i_1 & & \downarrow i_2 \\
W & \leftarrow & Y.
\end{array}
\]

Another choice of metric and neighborhood \( W \) induces the homotopic map and (by the item 3 of Theorem \[\text{[5.14]}\] below) the same homomorphism.

**Theorem 5.14** ( )

1. \( i_! \) is a homomorphism of \( R(G) \)-modules.
2. Let \( i : X \to Y \) and \( j : Y \to Z \) be two \( G \)-inclusions, then \((j \circ i)_! = j_! \circ i_! \).
3. Let enclosures \( i_1 : X \to Y \) and \( i_2 : X \to Y \) are \( G \)-homotopic in the class of enclosures. Then \((i_1)_! = (i_2)_! \).
4. Let \( i_1 : X \to Y \) be a \( G \)-diffeomorphism, then \( i_! = (di_1^{-1})^* \).
5. An enclosure \( i : X \to Y \) can be represented as a compositions of enclosures \( X \) in \( N \) (as the zero section \( s_0 : x \to N \) and \( N \to Y \) by \( i_2 \circ \Phi : N \to Y \). Then \( i_! = (i_2 \circ \Phi)_!(s_0)_! \).
6. Consider the complex bundle \( p_T^*(N \otimes C) \) over \( TX \). Let us form the complex \( \Lambda(p_T^*(N \otimes C), 0) : \)

\[
0 \to \Lambda^0(p_T^*(N \otimes C)) \to \ldots \to \Lambda^k(p_T^*(N \otimes C)) \to 0
\]

with the noncompact support. If \( a \in K_G(TX; A) \), then the complex \( a \otimes \Lambda(p_T^*(N \otimes C), 0) \) has the compact support and determines an element of \( K_G(TX; A) \). Then \((di)^*i_!(a) = a \cdot \Lambda(p_T^*(N \otimes C), 0) \).
7. For \( x \in K_G(TX; A) \) and \( y \in K_G(TY) \) \( i_!(x(di)^*y) = i_!(x) \cdot y \).

**Proof:** 1. By the definition of \( i_! \).
2. To simplify the argument, let us identify the tubular neighborhood with the normal bundle. Then \((j^*i)_! \) is the composition

\[
K_G(TX; A) \xrightarrow{\varphi} K_G(TN \oplus TN'_1; A) \to K_G(TZ; A),
\]
where $N'$ is the normal bundle of $Y$ in $Z$, $N'_1 = N'|_X$, and $TN \oplus TN'_1$ is considered in the same way as at p. 29 as a complex bundle over $TX$. On the other hand, $j^*_i i_!$ represents the composition

$$K_G(TX; A) \xrightarrow{\varphi} K_G(TN; A) \to K_G(TY; A) \xrightarrow{\varphi} K_G(TN'; A) \to K_G(TZ; A).$$

By properties of $\varphi$, the following diagram is commutative

$$K_G(TX; A) \xrightarrow{\varphi} K_G(TN; A) \to K_G(TY; A) \quad \text{with} \quad K_G(TN \oplus TN'_1; A) \xrightarrow{\varphi} K_G(TN'; A) \to K_G(TZ; A).$$

This completes the proof of item 2.

3. The homotopy of enclosures does not influence on $q_T$, but only on the further maps in the definition of the Gysin homomorphism. But for these maps the assertion follows from the homotopy invariance of $K$-theory.

4. In this case $N = X$, $W = Y$, $\Phi = i$, $i_2 = \text{Id}_Y$, and the formula is obvious.

5. By 2.

6. By definition,

$$(di)^* i_! = (di_1)^*(di_2)^*(di_2)_* (d\varphi^{-1})^* \psi^* \varphi^* = (\psi^* d\Phi^{-1} \circ di_1)^* \varphi,$$

where $i_1 : X \to W$, $i_2 : W \to Y$. Let $(n_1, t + n_2) \in TN = p_N^*(TX) \oplus p_N^*(N)$, where $n_1$ is the shift under the exponential mapping, $t + n_2$ a tangent vector to $W$. If $d\Phi(n_1, t + n_2)$ is in $TX$, then $n_1 = n_2 = 0$. Hence,

$$d\Phi^{-1} di_1(t) = (0, t + 0), \quad \psi^* d\Phi^{-1} di_1(t) = (t, 0 + 0).$$

Therefore, $\psi^* d\Phi^{-1} di_1 : TX \to p_T^*(N \oplus N)$ is the enclosure of the zero section. Since $\varphi(a) = a \cdot \Lambda(q_T^* p_T^*(N \otimes C), s_{p_T^*(N \otimes C)})$, it follows that $(di)^* i_!(a) = a \cdot \Lambda(p_T^*(N \otimes C), 0)$.

7. The mapping $di_1^* q_T^* \psi^* d\Phi^{-1} : TW \to TW$ is homotopic to the identical mapping. Hence,

$$i_!(x \cdot (di)^* y) = (di_2)_* (d\Phi^{-1})^* \psi^* \varphi (x \cdot (di)^* y) =
= (di_2)_* (d\Phi^{-1})^* \psi^* [(q_T^* (x) \lambda p_T^*(N \otimes C))(q_T^* (di)^* y)]
= (di_2)_* [ (d\Phi^{-1})^* \psi^* (q_T^* (x) \lambda p_T^*(N \otimes C)) (d\Phi^{-1})^* \psi^* (q_T^* (di_1)^* (di_2)^* y)]
= [(di_2)_* (d\Phi^{-1})^* \psi^* (q_T^* (x) \lambda p_T^*(N \otimes C))] [(di_2)_* (di_2)^* y] = i_!(x) \cdot y. \quad \square$$

**Theorem 5.15 1.** Let $V$ be a $G \cdot R$-vector space and $X = pt$ a trivial $G$-manifold. Hence, $TX = pt$ and $TV = V \otimes C$. Consider the enclosure of zero $i : X \to V$. Then the mapping

$$i_! : K_G^*(A) = K_G(TX; A) \to K_G(TV; A) = K_G(V \otimes C; A)$$
coincides with the Thom homomorphism \( \varphi^V \otimes C \).

2. Let \( V_1 \) and \( V_2 \) be \( G \)-\( R \)-spaces, \( i : X \to V_1 \) an enclosure. Let us define the enclosure \( k : X \to V_1 \oplus V_2 \) by the formula \( k(x) = i(x) + 0 \). Then the following diagram is commutative

\[
\begin{array}{ccc}
K_G(TV_1; A) & \xrightarrow{i_!} & K_G(TX; A) \\
\downarrow \varphi & & \downarrow \varphi \\
K_G(T(V_1 \oplus V_2), A), & \xrightarrow{k_!} & K_G(T(V_1 \oplus V_2), A),
\end{array}
\]

where \( \varphi \) the Thom homomorphism of the complex bundle

\[
T(V_1 \oplus V_2) = V_1 \otimes C \oplus V_2 \otimes C \to TV_1 = V_1 \otimes C.
\]

**Proof:**

1. The assertion follows from the definition of \( i_! \). More precisely, \( X = 0 \in V, N = V \). Also, \( W \) can be chosen equal to the interior \( D_1 \) of the ball of radius 1 in \( V \) with respect to an invariant metric. In this case the diagram from the definition of the Gysin homomorphism \[5.13\] takes the following form

\[
\begin{array}{ccc}
V \otimes C & \xrightarrow{\psi} & V \otimes C \\
\downarrow & & \downarrow \\
TX = 0 & \xrightarrow{\Phi} & D \oplus V \oplus V^2
\end{array}
\]

In our case \( \Psi = \text{Id} \) and \( di_2 \cdot d\Phi \) is homotopic to \( \text{Id} \), since it has the form \( v \otimes z \mapsto (v \otimes z)/(1 + |v \otimes z|) \). Hence, \( i_! = \varphi \).

2. Let the enclosure \( i : X \to V_1 \) has the normal bundle \( N \) and the tubular neighborhood \( W \). Then \( N \oplus V_2 \) is a normal bundle for the enclosure \( k \) with the tubular neighborhood \( W \oplus D(V_2) \), where \( D(V_2) \) is a ball. If \( a \in K_G(TX; A) \), then

\[
k_!(a) = (di_2 \oplus 1)_* (d\Phi^{-1} \oplus 1)^*(\psi \oplus 1)^* \varphi^N \oplus N \oplus V_2 \oplus V_2(a).
\]

By item 2 of theorem \[5.5\], \( \varphi_A^{N \oplus N \oplus V_2 \oplus V_2} = \varphi_A^{N \oplus N} \cdot \varphi_{V_2 \oplus V_2}. \) Since \( a = a \cdot C \), where \( C \) is the trivial line bundle,

\[
k_!(a) = (di_2)_* (d\Phi^{-1})^* \varphi^N_A(a) \cdot \varphi_{V_2 \oplus V_2}(C) =
= i_!(a) \cdot \lambda_{T(V_1 \oplus V_2)} = \varphi(i_!(a)). \quad \square
\]
6 Analytical index

Let us remind how a pseudo differential operator \( \chi(\sigma) \) over \( A \) can be defined starting from a symbol \( \sigma \) in the case \( G = e \) (see, e.g. [39]).

Suppose \( X \) is a compact closed smooth manifold, \( \pi : T^*X \to X \) is the projection of the cotangent bundle, \( E_1 \) and \( E_2 \) are smooth \( A \)-bundles over \( X \). Let \( \{ U_j \} \) be a trivializing cover of \( X \) for \( E_1 \) and \( E_2 \), \( \{ \phi_j \} \) be the subordinate partition of unity, \( \{ \psi_j \} \) a collection of functions such that \( \psi_j|_{\text{supp} \phi_j} = 1 \). A symbol of order \( m \) is a morphism of \( A \)-bundles \( \sigma : \pi^*E_1 \to \pi^*E_2. \)

Let \( u_j \in \Gamma^\infty(U_j, E_1) \) be a smooth section tending to 0 at infinity. Let us assume

\[
(\chi_j \sigma)(u_j)(x) = \frac{1}{(2\pi)^n} \int e^{i\langle x, \xi \rangle} \sigma(x, \xi) \int e^{-i\langle y, \xi \rangle} u_j(y) \, dy \, d\xi.
\]

For \( u \in \Gamma^\infty(X, E_1) \) let us define \( \chi \sigma \) by the formula

\[
(\chi \sigma)(u)(x) = \sum_j \psi_j(x)(\chi_j \sigma)(\phi_j u)(x).
\]

This is the section of the symbol-map (see, e.g. [39, §2.1]). We obtain a bounded \( A \)-homomorphism admitting an adjoint in Sobolev spaces:

\[
\chi \sigma : H^s(E_1) \cong l_2(\mathcal{P}_1) \to l_2(\mathcal{P}_2) \cong H^{s-m}(E_2).
\]

Suppose \( G \) is an arbitrary compact Lie group, \( X \) is a compact \( G \)-manifold, \( E_1 \) and \( E_2 \) are \( GGA \)-bundles over \( X \), the symbol \( \sigma : \pi^*E_1 \to \pi^*E_2 \) is a \( GGA \)-morphism. The action of \( G \) on Sobolev spaces is naturally defined by the formula \( g(u)(x) = g(u(g^{-1}x)) \). Without loss of generality, we can assume the Sobolev inner products on \( H^s(E_i) \cong l_2(\mathcal{P}_i) \) to be \( GGA \)-products (see Remark 2.8).

More precisely, let us choose a \( C^* \)-Hermitian metrics in bundles to be \( G \)-invariant, we obtain that under the action of \( G \) on Sobolev spaces we get each time admissible products (see, e.g. [39, §2.1]). Hence, this \( \Psi DO \) admits an adjoint with the respect to the averaged Sobolev product.

Suppose \( \sigma \) is an elliptic symbol, then (see, e.g. [39, §2.1])

\[
(\chi \sigma)(\chi \tau) = 1 + K_1, \quad (\chi \tau)(\chi \sigma) = 1 + K_2
\]

for some symbol (parametrix) \( \tau \), where \( K_1 \) and \( K_2 \) are \( A \)-compact operators. The action of group \( G \) on \( \Psi DO \) is given by \( g(P)u = g(P(g^{-1}(u))) \).

Let \( g \in G \) be fixed. Let us denote by \( g\{ \} \) the action of \( g \) on sections and by \( g \cdot \) the action on the elements of total space of a bundle. By [39, §2.1] \( \text{OP}_{r-1} \) and \( \text{CZ}_r \), hence, \( \text{Int}_r \) are invariant under diffeomorphisms, in particular, under \( g \). Hence \( \text{Int}_r \) has this invariance property. Moreover (see, e.g. [39, §2.1]),

\[
\sigma_m(g(\chi \sigma))(x, \xi) = \sigma_m(g \cdot \chi \sigma \cdot g)(x, \xi) = \\
= \sigma_m(\chi \sigma)(gx, g\xi) = \sigma(gx, g\xi) = \sigma(x, \xi),
\]
since $g$ acts on covectors by
$$g\xi = g_*\xi = (D(g^{-1}))^*\xi,$$
where the tangential mapping (derivative) of an arbitrary smooth map $\varphi$ we denote by $D(\varphi)$. Hence
$$g(\chi\sigma) = g\{\chi\sigma\}g^{-1} = \chi'\sigma,$$
i. e., $g(\chi\sigma)$ is another $\Psi$DO with the same symbol $\sigma$, where $\chi'$ is some other section of the symbol-map.

**Lemma 6.1** Let us fix $D = \chi\sigma$. Then the map $G \to \text{Int}_m(E,F)$, given by the formula
$$g \mapsto g(D), \quad g(D)(u) = g\{Dg^{-1}\{u\}\},$$
is continuous, i. e., for any $s$ this map is continuous as a map from the group $G$ to the space of operators $H^s \to H^{s-m}$ equipped with the uniform topology.

**Proof:** (cf. [3]) Let $a \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of group $G$. The action of $a$ defines a vector field $\vec{a}_x$. The corresponding differential operator $a_E$ acts on sections of the bundle $E$:
$$a_E(u) = \left(\vec{a}_x, \frac{\partial}{\partial x_i}\right)(u).$$
The symbol of this operator is equal to $\sigma_{a,E} = \langle \vec{a}, \xi \rangle \text{Id}_E$. Similarly for $F$. Then $\sigma\sigma_{a,E} = \sigma_{a,F}\sigma$ (see, e. g. [39, §2.1]),
$$D(\chi\sigma_{a,E}) - (\chi\sigma_{a,F}) D \in \text{Int}_m(E,F),$$
$$\|D(\chi\sigma_{a,E}) - (\chi\sigma_{a,F}) D\|_m^m \leq \|D\|\|a_E\| + \|a_F\|,$$
where $a_E = \chi\sigma_{a,E}$. If $a$ is in a bounded neighborhood of $0 \in \mathfrak{g}$, there exist numbers $c_s$ such that
$$\|D a_E - a_F D\|_m^m < c_s,$$
where $\|\cdot\|_m^m$ is the norm in $\text{Int}_m$, i. e., the uniform norm in the space of operators $H^s \to H^{s-m}$.

For $g_t = \exp ta$ and a smooth section $u$ let us assume $f(t) := g_t(D)u$. Then
$$f_t(t) = \exp(ta_F)D\exp(-ta_E)u.$$  
Indeed, it is sufficient to prove this equality in one chart for analytical functions, because the analytical basis was constructed in an explicit form in [31, p. 854]. For an analytical section $u(x)$ we have to prove that
$$e^{ta_E}u(x) = u(\varphi_x(t)), \quad (11)$$
where $\varphi_x(t)$ is the orbit of the action of the one-parameter subgroup, generated by $a$. For this purpose let us consider the function $F(t) = u(\varphi_x(t))$. It is analytical for $|t| \leq 1$. Then for $|t| \leq 1$ the series
$$F(t) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!}t^n$$
is convergent. Since \( \varphi_x(t) \) is an integral curve of the vector field \( \tilde{a}_x \),
\[
F'(t) = \frac{du(\varphi_x(t))}{dt} = \frac{d\varphi_x(t)}{dt} u = \tilde{a}_{\varphi_x(t)} u = (a_E u)(\varphi_x(t)),
\]
i. e., \( F'(t) \) plays the role of \( F(t) \) for the section \( a_E u \). By induction
\[
F^{(n)}(t) = (a^n_E u)(\varphi_x(t)).
\]
Therefore
\[
F^{(n)}(0) = (a^n_E u)(\varphi_x(0)) = (a^n_E u)(x).
\]
Hence
\[
F(t) = \sum_{n=0}^{\infty} \frac{(a^n_E u)(x)}{n!} t^n = \sum_{n=0}^{\infty} \frac{(ta_E)^n u}{n!}(x) = (e^{ta_E} u)(x),
\]
and we have (11). Further, since the action of \( G \) on \( H^s \) is an isometry,
\[
\left\| \frac{df_t}{dt} \right\|_{m-s} = \| \exp(ta_F (D a_E - a_F D) \exp(-ta_E) u) \| \leq c_s \| u \|_s.
\]
This implies continuity. \( \square \)

Let \( \chi^1 = D_1 \) and \( \chi^2 = D_2 \) be two \( \Psi DO \) with the same symbol \( \sigma \). Then \( D_1 - D_2 \in \mathcal{K}(l_2(P_1, \mathcal{P}_2)) \). If \( \sigma \) is a \( G \)-equivariant symbol, then we have shown that \( g(D) = D' = \chi' \sigma \). Hence the orbit \( G(D) \) is in the closed linear manifold
\[
D + \mathcal{K}(l_2(P_1), l_2(P_2)).
\]
By the previous lemma there exists the averaged operator \( Av D \) and it lies in this linear manifold. Thus for an elliptic \( G \)-symbol the operator \( Av D \) is an \( A \)-Fredholm one. Really, from (10) it follows
\[
Av(D) Av(\chi \tau) = (\chi \sigma + K'_1)(\chi \tau + K''_1) =
\]
\[
= (\chi \sigma) (\chi \tau) + K'_1 (\chi \tau) + (\chi \sigma) K''_1 + K'_1 K''_1 = 1 + \tilde{K}_1,
\]
\[
Av(\chi \tau) Av(D) = (\chi \tau + K''_1)(\chi \sigma + K'_1) =
\]
\[
= (\chi \tau) (\chi \sigma) + K''_1 (\chi \sigma) + (\chi \tau) K'_1 + K''_1 K'_1 = 1 + \tilde{K}_2,
\]
where \( K'_1, K''_1, \tilde{K}_1, \tilde{K}_2 \) are \( A \)-compact operators. It was proved that each \( \Psi DO \) over unital \( C^* \)-algebra admits an adjoint. Hence each element of the orbit \( G(D) \) has an adjoint. Since the operators admitting an adjoint form a Banach space (algebra), the averaged operator \( Av D \) has an adjoint. It remains to apply our theory of Fredholm operators. Thus the equivariant analytical index
\[
a \text{-ind} \sigma = \text{index} Av D \in K^G(A) = K^0_G(pt; A).
\]
is defined. It is clear, that all argument about homotopies (see, e. g. [12]) is valid in the equivariant case. Hence in this case the analytical index defines a homomorphism of Abelian groups
\[
\text{a-ind}_G^X : K_G(TX; A) \to K^G(A) = K^0_G(pt; A).
\]
7 The axiomatic approach

Let us start from the definition of the topological index. Let $G$ be a compact Lie group and $X$ be a compact $G$-manifold. From [33] it follows that there exists a representation of $G$ in orthogonal group $O(V)$ of some real finite-dimensional space $V$ and $G$-enclosure $i : X \to V$. Thus, the Gysin homomorphism (see 5):

$$i_! : K_G(TX; A) \to K_G(TV; A) = K_G(V \otimes C; A)$$

is defined. Since $TV = V \otimes C$ is a complex vector space, we have the following Thom isomorphism (see 5):

$$\varphi : K^G_0(A) = K^0_G(pt; A) \to K_G(TV; A).$$

**Definition 7.1** The topological index is the following map:

$$t \text{-} \text{ind}_G^X : K_G(TX; A) \to K^G(A), \quad t \text{-} \text{ind}_G^X := \varphi^{-1} \circ i_!.$$

**Theorem 7.2**

1. The index $t \text{-} \text{ind}_G^X$ does not depend on the choice of $V$, enclosure $i : X \to V$, and representation $G \to O(V)$.

2. The index $t \text{-} \text{ind}_G^X$ is a $R(G)$-homomorphism.

3. If $X = pt$, then the map

$$t \text{-} \text{ind}_G^X : K^G(A) = K_G(TX; A) \to K^G(A)$$

coincides with $\text{Id}_{K^G(A)}$.

4. Suppose $X$ and $Y$ are compact $G$-manifolds, $i : X \to Y$ is a $G$-enclosure. Then the diagram

$$\begin{array}{ccc}
K_G(TX; A) & \xrightarrow{i} & K_G(TY; A) \\
\downarrow^{t \text{-} \text{ind}_G^X} & & \downarrow^{t \text{-} \text{ind}_G^Y} \\
K^G(A) & & \\
\end{array}$$

commutes.

**Proof:** 1). Let us consider the enclosures

$$i_1 : X \to V_1, \quad i_2 : X \to V_2.$$

Denote by $j = i_1 + i_2$ the induced enclosure $j : X \to V_1 \oplus V_2$. It is sufficient to show that the topological index, which comes from $i_1$, coincides with the index, which comes from $j$. Let us define a homotopy of $G$-enclosures by the formula

$$j_s(x) = i_1(x) + s \cdot i_2(x) : X \to V_1 \oplus V_2, \quad 0 \leq s \leq 1.$$
Then by Theorems 5.14.3 and 5.15.1 the indexes for \( j \) and \( j_0 \) coincide. Let us show now that \( j_0 = i_1 + 0 \) and \( i_1 \) define the same topological indexes. For this purpose consider the diagram

\[
\begin{array}{ccc}
K_G(TX; A) & \rightarrow & K_G(T(V_1 \oplus V_2); A) \\
\downarrow (i_1)_! & & \downarrow \varphi_2 \\
K_G(TV_1; A) & \rightarrow & K^G(A) \\
\varphi_1 \uparrow \uparrow \uparrow \uparrow \uparrow \uparrow & & \varphi_3 \\
\end{array}
\]

where \( \varphi_i \) are the corresponding Thom homomorphisms. The upper triangle is commutative by Theorem 5.15.2, and the lower is commutative by Proposition 5.6. Hence \( \varphi_1^{-1} \circ (i_1)_! = \varphi_3^{-1} \circ (j_0)_! \) as desired.

2). This statement follows from 5.3 and 5.14.1.
3). This follows immediately from the definition of the index and from 5.15.1
4). Let us consider the diagram

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow j_0 & & \downarrow j \\
V & \rightarrow & \\
\end{array}
\]

Let us apply 5.14.2. We have the following commutative diagram

\[
\begin{array}{ccc}
K_G(TX; A) & \rightarrow & K_G(TY; A) \\
\downarrow (j_0)_! & & \downarrow j_! \\
K_G(TV; A) & \rightarrow & K^G(A) \\
\end{array}
\]

or

\[
\begin{array}{ccc}
K_G(TX; A) & \rightarrow & K_G(TY; A) \\
\downarrow (j_0)_! & & \downarrow j_! \\
K_G(TV; A) & \rightarrow & K^G(A) \\
\downarrow t-\text{ind}_G^X & & \downarrow t-\text{ind}_G^Y \\
K^G(A). & & \\
\end{array}
\]

Definition 7.3 An index function is a family of \( R(G) \)-homomorphisms \( \{ \text{ind}_G^X \} \)

\[\text{ind}_G^X : K_G(TX; A) \rightarrow K^G(A),\]

where \( G \) runs through the set of compact Lie groups and \( X \) is a smooth compact \( G \)-manifold. This family is restricted to satisfy the following two conditions:
1. If $f : X \to Y$ is a $G$-diffeomorphism, then the diagram

$$
\begin{array}{ccc}
K_G(TX; A) & \xrightarrow{(df^{-1})^*} & K_G(TY; A) \\
\downarrow \text{ind}_G^X & & \downarrow \text{ind}_G^Y \\
K^G(A) & & K^G(A)
\end{array}
$$

is commutative.

2. If $\psi : H \to G$ is a homomorphism of groups, then the diagram

$$
\begin{array}{ccc}
K_G(TX; A) & \xrightarrow{\psi^*} & K_H(TX; A) \\
\downarrow \text{ind}_G^X & & \downarrow \text{ind}_H^X \\
K^G(A) & \xrightarrow{\psi^*} & K^H(A)
\end{array}
$$

is commutative.

**Assertion 7.4** The topological index $t\text{-ind}_G^X$ is an index function.

**Proof:** 1). Suppose $i : Y \hookrightarrow V$, $j = i \circ f : X \hookrightarrow V$. By 5.14.2, the following diagram is commutative

$$
\begin{array}{ccc}
K_G(TY; A) & \xrightarrow{f_{\text{ind}}} & K_G(TV; A) \\
\downarrow f_! & & \downarrow f_! \\
K_G(TX; A) & \xrightarrow{\text{ind}_G^X} & K^G(A)
\end{array}
$$

By 5.14.4, we have in this case $f_! = (df^{-1})^*$ and it remains to use the definition of $t\text{-ind}$.

2). Immediately follows from the definition. \qed

Let us consider the following two axioms.

**Axiom A1.** If $X = \text{onepoint}$, then $\text{ind}_G^X : K_G(TX; A) \to K^G(A)$ coincides with $\text{Id}_{K^G(A)}$.

**Axiom A2.** Suppose $i : X \to Y$ is a $G$-enclosure, then the diagram

$$
\begin{array}{ccc}
K_G(TX; A) & \xrightarrow{i_!} & K_G(TY; A) \\
\downarrow \text{ind}_G^X & & \downarrow \text{ind}_G^Y \\
K^G(A) & & K^G(A)
\end{array}
$$

is commutative.

**Corollary 7.5** (from Theorem 7.2) The topological index $t\text{-ind}_G^X$ satisfies axioms A1 and A2. \qed
Theorem 7.6 Let \( \text{ind}^X_G \) be an index function satisfying axioms A1 and A2. Then \( \text{ind}^X_G = t\text{-ind}^X_G \).

Proof: Consider a \( G \)-enclosure \( i : X \to V \) of a manifold \( X \) in a real vector \( G \)-space \( V \). The one-point compactification \( V^+ \) (i.e., a sphere) is a \( G \)-manifold with the canonical \( G \)-inclusion \( \varepsilon^+ : V \to V^+ \). Suppose \( i^+ = \varepsilon^+ \circ i : X \to V^+ \). If \( P = 0 \in V \) and \( j : P \to V \) is the inclusion, then we obtain the diagram

\[
\begin{array}{c}
\text{K}_G(TX; A) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\text{K}_G(TV; A) \quad \quad \text{K}_G(TV^+; A) \quad \quad \text{K}_G(A) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\text{K}_G(TP; A) = K^G(A),
\end{array}
\]

where \( j^+ = \varepsilon^+ \circ j : P \to V^+ \). By 5.14.2 (resp., by axiom A2), the left (resp., right) triangles commute. By A1, \( \text{ind}^P_G \) is the identity mapping. Hence

\[
\text{ind}^X_G = \text{ind}_G^{V^+} i^+ = \text{ind}_G^{V^+} (\varepsilon^+) i \ast = \text{ind}_G^{V^+} (j^+) j^{-1} i \ast = \text{ind}_G^{V^+} j^{-1} i \ast = t\text{-ind}^X_G.
\]

Since \( j : K^G(A) \to K_G(TV; A) = K_G(V \otimes \mathbb{C}; A) \) coincides with the Thom homomorphism, the theorem is proved.

Axiom B1 (excision). Let \( U \) be a (noncompact) \( G \)-manifold and

\[
j_1 : U \to X_1, \quad j_2 : U \to X_2
\]

be \( G \)-enclosures of the manifold \( U \) on open subsets of compact \( G \)-manifolds \( X_1 \) and \( X_2 \). Then the diagram

\[
\begin{array}{c}
\text{K}_G(TX_1; A) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
\text{K}_G(TU; A) \quad \quad \text{K}_G(A) \\
\downarrow \quad \quad \downarrow \\
\text{K}_G(TX_2; A)
\end{array}
\]

is commutative.

Suppose there exists though one of indicated enclosures. Then by the axiom, the index

\[
\text{ind}^U_G : K_G(TU; A) \to K^G(A)
\]

can be well defined.

Let us denote by \( \text{c-ind}_H^Y \) the classical (complex) index

\[
\text{c-ind}_H^Y : K_H(TY) \to R(H).
\]
We have the following statement (see [3]). Suppose \( j : * \to \mathbb{R}^n \) is the enclosure of \( \overline{0} \), hence \( j_! : R(O(n)) \to K_{O(n)}(TR^n) \). Then \( \text{c-ind}_{O(n)}^R j_!(1) = 1 \).

Let \( \pi : P \to X \) be a compact differentiable principal bundle for a group \( H \) (compact Lie group). Then we have (right) free action of \( H \) on \( P \) and \( X = P/H \). Suppose we have left action of \( G \) on \( P \) and these two actions commute. Let \( F \) be a compact left \((G \times H)\)-manifold. We can form the associated bundle \( \pi_1 : Y = P \times_H F \to X \) with the natural action of \( G \). Consider the tangent bundle along the fibers of \( \pi_1 \). Let us denote it by \( T_F Y \). Then \( T_F Y \) is a \( G \)-invariant real subbundle of \( TY \) and \( T_F Y = P \times_H TF \).

Using the metric it is possible to decompose \( TY \) into a direct sum \( TY = T_F Y \oplus \pi_1^*(TX) \). Therefore the multiplication

\[
\mathbf{K}_G(TX; A) \otimes \mathbf{K}_G(T_F Y) \\
\downarrow \\
\mathbf{K}_G(\pi_1^*TX; A) \otimes \mathbf{K}_G(T_F Y) \to \mathbf{K}_G(TY; A)
\]

is defined. There exists the map

\[
\gamma : \mathbf{K}_G(TX; A) \otimes \mathbf{K}_{G \times H}(TF) \to \mathbf{K}_G(TY; A).
\]

Let us denote \( \gamma(a \otimes b) \) by \( a \cdot b \).

If \( V \) is a complex vector \((G \times H)\)-space, then \( P \times_H V \) is a complex vector \( G \)-bundle over \( X \). We obtain the following ring homomorphism being a homomorphism of \( R(G) \)-modules:

\[
\mu_P : R(G \times H) \to K_G(X), \quad [V] \mapsto [P \times_H V].
\]

Since \( \mathbf{K}_G(TX; A) \) has a \( K_G(X) \)-module structure, we can formulate the following axiom.

**Axiom B2.** If \( a \in \mathbf{K}_G(TX; A), \ b \in \mathbf{K}_{G \times H}(TF) \), then

\[
\text{ind}_G^Y (a \cdot b) = \text{ind}_G^X (a \cdot \mu_P (\text{c-ind}_{G \times H}^F (b)) ),
\]

i.e., the diagram

\[
\begin{array}{ccc}
\mathbf{K}_G(TX; A) \otimes \mathbf{K}_{G \times H}(TF) & \overset{1 \otimes \text{C-ind}_{G \times H}^F}{\longrightarrow} & \mathbf{K}_G(TX; A) \otimes R(G \times H) \\
\downarrow \gamma & & \downarrow 1 \otimes \mu_P \\
\mathbf{K}_G(TY; A) & & \mathbf{K}_G(TX; A) \otimes \mathbf{K}_G(X) \\
\downarrow \text{ind}_G^Y & & \downarrow \text{ind}_G^X \\
\mathbf{K}_G^{G}(A) & & \mathbf{K}_G(TX; A)
\end{array}
\]

is commutative

**Theorem 7.7** Let \( \pi : P \to X \) be a principal right \( H \)-bundle with a left action of \( G \) commuting with \( H \). Suppose \( F \) is a \((G \times H)\)-space. Let us denote by \( Y \) the space
Let $j : X_1 \to X$ and $k : F_1 \to F$ be $G$- and $(G \times H)$-enclosures, respectively; let $\pi^1 : P_1 \to X_1$ be the principal $H$-bundle induced by $j$ on $X_1$; assume $Y_1 := P_1 \times_H F_1$. The enclosures $j$ and $k$ induce $G$-enclosure $j \ast k : Y_1 \to Y$. In this situation the diagram

$$
\begin{array}{ccc}
K_G(TX; A) \otimes_{R(G)} K_{G \times H}(TF) & \xrightarrow{\gamma} & K_G(TY; A) \\
\downarrow j \otimes k_1 & & \downarrow (j \ast k)_1 \\
K_G(TX_1; A) \otimes_{R(G)} K_{G \times H}(TF_1) & \xrightarrow{\gamma} & K_G(TY_1; A)
\end{array}
$$

is commutative.

**Proof:** Let us describe $\gamma$ explicitly:

$$
\begin{align*}
\cong K_G(TX; A) & \otimes K_{G \times H}(P \times_H TF) \\
\xrightarrow{\beta} K_G(TX; A) & \otimes K_{G \times H}(P \times TF) \\
\xrightarrow{\alpha} K_G(\pi^*TX; A) & \otimes K_{G \times H}(P \times_H TF_1) \\
\xrightarrow{\gamma} K_G((\pi^1)^*TX_1; A) & \otimes K_{G \times H}(P_1 \times_H TF_1) \\
\xrightarrow{\gamma} K_G((\pi^1)^*TX \oplus (P \times_H TF); A) & = K_G(TY; A) \\
\xrightarrow{(j \ast k)_1} K_G((\pi^1)^*TX_1 \oplus (P_1 \times_H TF_1); A) & = K_G(TY_1; A).
\end{align*}
$$

Let us remind the diagram, which was used for the definition of the Gysin homomorphism of an enclosure $j : X_1 \to X$:

From the similar diagrams for $k_1$ and $(j \ast k)_1$ and the explicit form of the maps it follows that the square $[4]$ in $[(12)]$ iff $\alpha$ has the following form:

$$
\alpha(\sigma \otimes \rho) = (\pi^1)_! \left( \left( dj_2 \right)_* \left( d\Phi_{X_1}^{-1} \right)^* \psi_{X_1}^* \right) \circ \varphi_X^T(\sigma) \otimes \left( \pi^* j_2 \times_H dk_2 \right)_* \left( \left( \pi^* \Phi_{X_1} \times_H d\Phi_{F_1} \right)^{-1} \right)^* (1 \times_H \psi_{F_1})^* \varphi_C^T(\rho),
$$
where $S$ and $T$ are bundles of the form
\[
\pi_1^* \left( (p_T^X)\{N_{X_1} \oplus N_{X_1}\} \right) \quad \pi^*N_{X_1} \times_H (p_T^{F_1})^* (N_{F_1} \oplus N_{F_1})
\]
\[
S: \quad \downarrow (\pi_1)^* q_{T}^X \quad \quad \quad T: \quad \downarrow (\pi_1)^* (p_{N_{X_1}} \times_H q_{T}^{F_1})
\]
\[
(\pi_1)^*_T (TX_1), \quad \quad \quad \quad \quad \quad \pi^*X_1 \times_H TF_1 = P_1 \times_H TF_1.
\]

Hence the square $[3]$ in (12) is commutative iff the homomorphism $\beta$ has the form
\[
\beta(\tau \otimes \rho) = j_1(\tau) \otimes \otimes(\pi^*J_2 \times_H dk_2)_* \left( (\pi^*\Phi_{X_1} \times_H d\Phi_{F_1})^{-1} \right)^* (1 \times_H \psi_{F_1})^* \phi_T^G(\rho),
\]
\[
\tau \in K_G(TX_1; A), \quad \rho \in K_G(P_1 \times_H TF_1).
\]

In turn, the square $[2]$ in (12) is commutative iff the homomorphism $\varepsilon$ has the form
\[
\varepsilon(\tau \otimes \delta) = j_1(\tau) \otimes \otimes(\pi^*J_2 \times_H dk_2)_* \left( (\pi^*\Phi_{X_1} \times_H d\Phi_{F_1})^{-1} \right)^* (1 \times_H \psi_{F_1})^* \phi_T^G(\delta),
\]
\[
\tau \in K_G(TX_1; A), \quad \delta \in K_{G \times H}(P_1 \times_H TF_1),
\]
where $\bar{T}$ is the following bundle:
\[
\bar{T}: \quad \downarrow (\pi_1)^* (p_N \times_H q_{T}^{F_1})
\]
\[
P_1 \times TF_1.
\]

Suppose $\delta = [C] \otimes \omega$, where $[C] \in K_{G \times H}(P_1)$, $C$ is the one-dimensional trivial bundle and $\omega \in K_{G \times H}(TF_1)$. Then
\[
\varepsilon(\tau \otimes \delta) = j_1(\tau) \otimes \left\{ \pi^*((j_2)_*(\Phi_{X_1}^{-1})*[C] \otimes k_1(\omega)) \right\} = j_1(\tau) \otimes \left\{ [C] \otimes k_1(\omega) \right\}.
\]

Since the map $K_{G \times H}(TF) \rightarrow K_{G \times H}(P \times TF)$ (as well as the lower line in (12)) has the form $\omega \mapsto [C] \otimes \omega$, we have proved the commutativity of $[1]$ in (12).

Let in the situation of axiom B2
\[
\text{c-ind}^F_{G \times H}(G) \in R(G) \subset R(G \times H).
\]

Since $\mu_F$ and $\text{ind}^X_G$ are $R(G)$-homomorphisms, the following property holds.

Axiom B2’. (corollary of B2) If $\text{c-ind}^F_{G \times H}(G) \in R(G) \subset R(G \times H)$, then
\[
\text{ind}^Y_G(a \cdot b) = \text{ind}^X_G(a) \cdot \text{c-ind}^F_G(b).
\]

Assume in B2 $X = P$, $H = 1$. We obtain the following axiom.

Axiom B2’’. (corollary of B2) If $X$ and $F$ are $G$-manifolds, then
\[
\text{ind}^{X \times F}_G(a \cdot b) = \text{ind}^X_G(a) \cdot \text{c-ind}^F_G(b).
\]
Theorem 7.8 Suppose an index function \( \text{ind} \widetilde{\chi} \) satisfies A1, B1, B2, then

\[
\text{ind} \widetilde{\chi}_G = \text{t-ind} \chi_G.
\]

Proof: Suppose in the axiom B2' \( F \) is equal to an open \((G \times H)\)-subset of the compact manifold \( \tilde{F} \). Let \( j : F \hookrightarrow \tilde{F} \). Then

\[
\text{ind} Y_G(a \cdot b) = \text{ind} \widetilde{Y}_G(dJ_*)(a \cdot b) = \text{ind} \widetilde{Y}_G(a \cdot ((dj)_*,b)) =
\]

\[
= \text{ind} \chi_G(a) \cdot \text{c-ind} \widetilde{F}_G(H \times H)((dj)_*,b) =
\]

\[
= \text{ind} \chi_G(a) \cdot \text{c-ind} \widetilde{F}_G(H \times H)(b),
\]

(13)

where \( J \) is the enclosure

\[
Y = P \times_H F \xrightarrow{\text{Id} \times_H j} P \times_H \tilde{F} = \tilde{Y}.
\]

Indeed, let us consider the diagram

\[
\begin{align*}
K_G(TX; A) \otimes K_{G \times H}(TF) & \rightarrow K_G(TX; A) \otimes K_{G \times H}(P \times TF) \\
\downarrow 1 \otimes (dj)_* & \downarrow 1 \otimes (\text{Id} \times dj)_* \\
K_G(TX; A) \otimes K_{G \times H}(TF) & \rightarrow K_G(TX; A) \otimes K_{G \times H}(P \times TF) \\
\end{align*}
\]

\[
\cong K_G(TX; A) \otimes K_G(P \times_H TF) = K_G(TX; A) \otimes K_G(T_F Y) \rightarrow
\]

\[
\downarrow 1 \otimes (\text{Id} \times_H dj)_* & \downarrow 1 \otimes \alpha_* \\
\cong K_G(TX; A) \otimes K_G(P \times_H TF) = K_G(TX; A) \otimes K_G(T_F Y) \rightarrow
\]

\[
\rightarrow K_G(\pi^*_1 TX; A) \otimes K_G \otimes K_G(T_F Y) \rightarrow K_G(TY; A) \\
\] \( \downarrow 1 \otimes \alpha_* \) \( \downarrow (dj)_* \)

\[
\rightarrow K_G(\pi^*_1 TX; A) \otimes K_G(T_F Y) \rightarrow K_G(TY; A).
\]

This diagram is commutative. In fact, we have

\[
\begin{align*}
TY & = T_F Y \oplus \pi^*_1(TX) \\
\downarrow dJ & \downarrow \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \\
T\tilde{Y} & = T_{\tilde{F}} Y \oplus \pi^*_1(TX)
\end{align*}
\]

and \( \alpha = \text{Id} \times_H dj \) under the identification \( T_F Y = P \times_H TF \). We have proved the second equality in (13), the remaining are obvious.

Let us now take, in particular,

\[
F = \mathbb{R}^n, \quad \tilde{F} = (\mathbb{R}^n)^+ = S^n, \quad H = O(n), \quad b = j(1), \quad 1 = [\mathbb{C}],
\]

where \( j : \emptyset \hookrightarrow \mathbb{R}^n \) is the natural enclosure. Then \( P \) is a principal \( O(n) \)-bundle over \( X \), the group \( G \) acts on \( P \) commuting with \( O(n) \). Suppose \( G \) acts on \( \mathbb{R}^n \) in a trivial way. We form the associated real \( G \)-bundle

\[
P \times_{O(n)} \mathbb{R}^n = Y \rightarrow X.
\]
Let us denote by $i : X \to Y$, $i = 1_X \ast j$, the enclosure of $X$ as of the zero section. Assume in Theorem 7.7 $F = \mathbb{R}^n$, $X_1 = X$, $F_1 = pt$. Then we obtain the commutative diagram

$$
\begin{array}{ccc}
K_G(TX; A) \otimes_{R(G)} K_{G \times O(n)}(T\mathbb{R}^n) & \xrightarrow{\gamma} & K_G(TY; A) \\
\uparrow (1_X)! \otimes j! & & \uparrow (1_X \ast j)! = i! \\
K_G(TX; A) \otimes_{R(G)} K_{G \times O(n)}(pt) & \xrightarrow{\gamma} & K_G(TX; A).
\end{array}
$$

Since $\gamma (a \otimes 1) = a$,

$$
i_t(a) = \gamma ((1_X)! \otimes j!)(a \otimes 1) = \gamma (a \otimes j!(1)) = a \cdot j!(1) = a \cdot b.
$$

By the property of $c$-ind indicated above,

$$
c\text{-ind}_{\mathbb{R}^n_{O(n)}} j!(1) = 1, \quad c\text{-ind}_{G \times O(n)} j!(1) = 1,
$$

where $G$ acts on $\mathbb{R}^n$ in a trivial way. Now by the axiom B2,

$$
\begin{align*}
\text{ind}^X_G & = \text{ind}^X_G(a \cdot 1) = \text{ind}^X_G(a \cdot \mu_P(1)) = \\
& = \text{ind}^X_G(a \cdot \mu_P(c\text{-ind}_{\mathbb{R}^n_{O(n)}}(b))) = \\
& = \text{ind}^Y_G(a \cdot b) = \text{ind}^Y_G i_t(a).
\end{align*}
$$

Let $k : X \to Z$ be an enclosure of $X$ in a compact $G$-manifold $Z$ with the normal bundle $N$ and a tubular neighborhood $\Phi : N \to W$. By the definition of the Gysin homomorphism, $k_t = (d_{i_2} \circ d\Phi)_* j_t$, where $d_{i_2} : TW \to TZ$ is an enclosure of tangent bundle and $j : X \to N$ is the enclosure of $X$ as of the zero section in the normal bundle. In the diagram

$$
\begin{array}{ccc}
K_G(TX; A) & \xrightarrow{j!} & K_G(TN; A) \xrightarrow{(d_{i_2} \circ d\Phi)_*} K_G(TZ; A) \\
\downarrow \text{ind}^X_G & & \downarrow \text{ind}^N_G & \downarrow \text{ind}^{\mathbb{Z}}_G \\
K^G(A) & & &
\end{array}
$$

the left triangle is commutative by (14). The map $i_2 \cdot \Phi$ is an open enclosure. Hence by B2, the right triangle is commutative too. Therefore, $\text{ind}^X_G = \text{ind}^{\mathbb{Z}}_G \circ k_t$. Hence A2 is satisfied. To complete the proof it remains to apply Theorem 7.6.

\section{Proof of the index theorem}

First of all obviously, the analytical index is an index function.

\begin{lemma}
The analytical index $a$-ind satisfies the axiom A1.
\end{lemma}
**Proof:** Elliptic operator over a point is a GGA-mapping \( P : V \to W \) of projective GGA-modules and \( [\sigma(P)] = [V] - [W] = \text{index } P \in K^G(A). \)

\[ \square \]

**Theorem 8.2** The index a-ind satisfies the axiom B1.

**Proof:** Suppose \( a \in K_G(TU; A) \); \( j_1 : U \to X_1 \) and \( j_2 : U \to X_2 \) are \( G \)-enclosures; \( \pi : TU \to U \) is the natural projection. Let the similar equalities hold for \( \tilde{F} \) to construction of operators \( \tilde{D} \).

\[ 0 \to \pi^*E \xrightarrow{\rho} \pi^*F \to 0 \]

be exact for \( x \in U \setminus L, |\xi| > c \) (point and (co)vector). Suppose

\[ \alpha : E|_{U \setminus L} \cong (U \setminus L) \times N, \quad \beta : F|_{U \setminus L} \cong (U \setminus L) \times N, \]

\[ \rho = (\pi^*\beta)^{-1}(\pi^*\alpha), \]

\( L \) is some \( G \)-invariant compact set. Then it is possible to assume the symbols \( \sigma_1 \in \text{Smbl}_0(X_1, E_1, F_1) \), \( \sigma_2 \in \text{Smbl}_0(X_2, E_2, F_2) \) be as follows. Suppose

\[ E_1 = E \cup_{j_1 \alpha} (X_1 \setminus j_1 L) \times N, \quad E_2 = E \cup_{j_2 \alpha} (X_1 \setminus j_2 L) \times N. \]

Let the similar equalities hold for \( F_1 \) and \( F_2 \), and \( \sigma_1 = \rho \cup_{j_1} \text{Id} , \sigma_2 = \rho \cup_{j_2} \text{Id} . \) Let us pass to construction of operators \( \tilde{D}_1 \) and \( \tilde{D}_2 \), which represent these symbols in \( \text{Int}_0(X_1; E_1, F_1) \) \( \text{Int}_0(X_2; E_2, F_2) \), respectively. Let us take a trivializing cover, a partition of unity and smoothing functions on \( U \). Pull them back on \( j_1 U \) and \( j_2 U \), and then complete these collections of open sets (to obtain covers) by some open sets not intersecting with \( j_1 L \) and \( j_2 L \), respectively. By our symbols and with the help of this data let us construct in the usual way (non-invariant) operators \( D_1, D_2 \in CZ_0 \), and then

\[ \tilde{D}_1 = \text{Av}_G D_1 \in \text{Int}_0(X_1), \quad \tilde{D}_2 = \text{Av}_G D_2 \in \text{Int}_0(X_2). \]

It is necessary to check up the equality

\[ \text{index } \tilde{D}_1 = \text{index } \tilde{D}_2 \in K^G(A). \]

Since \( L \) is invariant, the averaging over this set is the same for both operators. Since the operators have the order 0, we compute index in \( L_2 \)-spaces. For these spaces

\[ L_2(X_1, E_1) \cong L_2(j_1 L, E_1|_{j_1 L}) \oplus L_2(X_1 \setminus j_1 L, E_1|_{X_1 \setminus j_1 L}) \]

and

\[ \tilde{D}_1 : L_2(X_1 \setminus j_1 L, E_1|_{X_1 \setminus j_1 L}) \cong L_2(X_1 \setminus j_1 L, E_1|_{X_1 \setminus j_1 L}) \]

(this is the identity operator). Similar relations hold for \( \tilde{D}_2 \). On the second summand of the decomposition of \( L_2 \) we have the commutative diagram

\[ \begin{array}{ccc}
\Gamma(E_1|_{j_1 L}) & \xrightarrow{\tilde{D}_1} & \Gamma(F_1|_{j_1 L}) \\
(j_2 j_1^{-1}) \cong & \text{and} & (j_2 j_1^{-1}) \\
\Gamma(E_2|_{j_2 L}) & \xrightarrow{\tilde{D}_2} & \Gamma(F_2|_{j_2 L}).
\end{array} \]

This diagram demonstrates the coincidence of indices. \( \square \)
\textbf{Theorem 8.3} \textit{The analytical index} \textbf{a-ind} \textit{satisfies the axiom B2.}

\textbf{Proof:} Consider the manifold \( Y = P \times_H F \) over a compact manifold \( X \), where \( P \to X \) is the principal bundle with compact Lie group \( H \), and \( F \) is a compact left \( H \)-manifold. The compact Lie group \( G \) acts on \( X = P/H \) and \( Y = P \times_H F \). Hereinafter the metrics on \( P, X, F, Y \) and on \( A \)-vector bundles are supposed to be invariant. Let us recall that we considered in the axiom B2 two elements and their product of special form:

\[
a \in K_G(TX; A), \quad b \in K_{G \times H}(TF), \quad a \cdot b \in K_G(TY; A).
\]

Let us represent these elements by symbols of elliptic operators. Let \( a \) be represented by a smooth \( G \)-symbol of order 1. Let \( A_1 \in C_{Z1} \) (not necessary invariant) have the symbol \( \alpha \). Let us choose a trivializing cover \( \{ U_j \} \) of the manifold \( X \) for \( P \) and \( Y \) with a subordinate partition of unity \( \{ \varphi_j^2 \} \). Let us consider the operator \( \widehat{A}_1 = \varphi_j A_1 \varphi_j \) on \( U_j \).

Suppose \( p : Y \to X \) is the projection, and \( Y_j = p^{-1}(U_j) \cong U_j \times F \). The lifting \( \widehat{A}_1 \) of \( A_1 \) to \( Y_j \) is a tensor product. Hence it belongs to \( \text{Int}_1(Y_j) \). Extend this lifting by zero and get the element of \( \text{Int}_1(Y) \). Let us average over \( G \):

\[
\tilde{A} = \text{Av}_G(\sum \widehat{A}_1) \in \text{Int}_1(Y).
\]

Then \( \sigma_1(\tilde{A}) = \tilde{\alpha} \), where the lifting \( \tilde{\alpha} \) of the symbol \( \alpha \) is defined globally. This follows from the invariance of the metric: for the decomposition of cotangent space \( Y \) into the vertical component \( \eta \) horizontal component \( \xi \) we assume \( \tilde{\alpha}(\xi, \eta) = \alpha(\xi) \).

Let us restrict \( \tilde{A}_1 \) on sections, which are constants along the fibers of \( Y \), then we receive \( A_1 \). Therefore the restriction \( A \) on the space of these sections is a \( G \)-invariant operator \( A = \text{Av}_G(\sum A_1) \in C_{Z1}(X) \) with symbol \( \alpha \).

Let \( b = [\beta] \in K_{G \times H}(TF) \), \( \beta = \sigma(B) \), \( \tilde{B} \in C_{Z1}(F) \) is a \((G \times H)\)-operator. Let \( \tilde{B}_1 \) be the operator over \( \tilde{P} \times \hat{F} \), obtained by a lifting of \( B \). Since this operator is \((G \times H)\)-invariant, it induces a \( G \)-invariant operator \( \tilde{B} \) over \( Y = P \times_H F \) by the restriction of \( \tilde{B}_1 \) on the constant sections along fibers of \( \tilde{P} \times \hat{F} \to Y \). Since \( \tilde{P} \) is locally trivial, the restriction \( \tilde{B}_j = \tilde{B}|_{U_j} \) over \( Y_j = p^{-1}(U_j) \cong U_j \times F \) is the lifting of \( B \). Then \( \tilde{B}_j \in \text{Int}_1(Y_j), \tilde{B} \in \text{Int}_1(Y) \).

Suppose \( \tilde{\beta} = \sigma_1(\tilde{B}) \), \( \tilde{\beta}(\xi, \eta) = \beta(\eta) \), where \( \xi \) is the horizontal component and \( \eta \) is the vertical one. There is a \( G \)-invariant operator

\[
D = \begin{pmatrix} \tilde{A} & -\tilde{B}^* \\ \tilde{B} & \tilde{\alpha}^* \end{pmatrix} \in \text{Int}_1(Y, E^0 \oplus G^0 \oplus E^1 \oplus G^1, E^0 \oplus G^1 \oplus E^1 \oplus G^0),
\]

\[
\sigma(D) = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta}^* \\ \beta & \tilde{\alpha}^* \end{pmatrix},
\]

and \( [\sigma(D)] = [\alpha][\beta] = a \cdot b \) by the definition of the multiplication \( \gamma \). Let us calculate now index \( D \). Since \( \tilde{B} \) is “of complex origins”, its kernel and cokernel are invariant modules from \( \mathcal{P}(A) \) and index \( [\ker \tilde{B}] = [\text{Coker} \tilde{B}] \in K_G(A) \). The operator \( \tilde{B} \) is the extension of \( B \) to fibers; \( \ker \tilde{B} \) consists of those smooth sections, which lay in \( \ker B_x \) for each fiber \( Y_x \). Here \( B_x \) is the operator over \( Y_x \) acting as \( B \) over the standard fiber \( F \). Hence \( \ker \tilde{B} \) is the space of smooth sections of the vector bundle \( K_B = P \times_H \ker B \) over \( X \). Since \( \tilde{A} \)
and $\widetilde{B}$ commute, $\widetilde{A}$ induces an operator $C^K$ on the sections $K_B$. From the definition of $\widetilde{A}$ it follows that

$$C^K = Av_G \left( \sum \varphi_j C^K_j \varphi_j \right),$$

where $C^K_j$ is the operator induced by $\widetilde{A}^j_1$ on $K_B|_{U_j}$. By the definition of $\widetilde{A}^j_1$, this means that $C^K_j = \widetilde{A}^j_1 \otimes \text{Id}_{K_B}$. Therefore

$$C^K_j \in \text{Int}_1, \quad C^K \in \text{Int}_1, \quad \sigma(C^K) = \alpha \otimes \text{Id}_{K_B}.$$

Thus $C^K$ is an elliptic $G$-invariant operator on $X$, $[\sigma(C^K)] = a[K_B] \in K_G(TX; A)$, and

$$a \text{-ind}^X a[K_B] = \text{index}^X C^K \in K^G(A).$$

Similarly, if $L_B = P \times_H \text{Coker } B$, then $\sigma(C^L) = \alpha \otimes \text{Id}_{L_B}$,

$$[\sigma(C^L)] = a[L_B] \in K_G(TX; A),$$

$$a \text{-ind}^X a[L_B] = \text{index}^X C^L \in K^G(A),$$

and

$$\text{index}^X C^K - \text{index}^X C^L = a \text{-ind}^X \left( a([K_B] - [L_B]) \right) = a \text{-ind}^X \left( a \mu_p (c \text{-ind}_{G \times H}^F b) \right) \in K^G(A).$$

It remains to show that

$$\text{index}^Y D = \text{index}^X C^K - \text{index}^X C^L.$$

By the definition of the index of an $A$-Fredholm operator,

$$\text{index}^X C^K = [N^K_0] - [N^K_1], \quad C^K : M^K_0 \cong M^K_1.$$

Let us remark the following. If

$$F = \begin{pmatrix} F_1 & 0 \\ 0 & F_2 \end{pmatrix}, \quad F_1 : M_0 \cong M_1, \quad F_2 : N_0 \to N_1,$$

is a decomposition for $GGA$-Fredholm operator, index $F = [N_0] - [N_1]$, then index $(F^*) = - \text{index} F$ and

$$- \text{index}^X C^L = \text{index}^X (C^L)^*,$$

where $(C^L)^*$ is constructed in the same way as $C^L$, but instead of the operator $\widetilde{A}$ we take $\widetilde{A}^*$. Suppose,

$$\text{index}^X (C^L)^* = [N^{L*}_0] - [N^{L*}_1], \quad (C^L)^* : M^{L*}_0 \cong M^{L*}_1.$$

Then by the definition of $K_B$ and $L_B$ as the kernels and cokernels, and $C^K$ and $(C^L)^*$, respectively, we have

$$\widetilde{A} : M^K_0 \cong M^K_1, \quad \widetilde{B}(M^K_0) = 0, \quad \widetilde{A}^* : M^{L*}_0 \cong M^{L*}_1, \quad \widetilde{B}^*(M^{L*}_0) = 0.$$
Further,
\[
D \left( \begin{array}{c} M_0^K \\ M_{0x}^L \\ M_0^{Lx} \end{array} \right) = \left( \begin{array}{c} \tilde{A}(M_0^K) \\ \tilde{A}^*(M_0^{Lx}) \end{array} \right) = \left( \begin{array}{c} M_1^K \\ M_1^{Lx} \end{array} \right),
\]
(15)
\[
D \left( \begin{array}{c} N_0^K \\ N_{0x}^L \\ N_0^{Lx} \end{array} \right) = \left( \begin{array}{c} \tilde{A}(N_0^K) \\ \tilde{A}^*(N_0^{Lx}) \end{array} \right) \subset \left( \begin{array}{c} N_1^K \\ N_1^{Lx} \end{array} \right).
\]
(16)

This is the description of the action of \( D \) on \( \text{Ker} \tilde{B} \oplus \text{Ker} \tilde{B}^* \).

Let now \( x \in \text{Ker} \tilde{B}^\perp \), i.e., \( (x, y) = 0 \) for any \( y \) such that \( \tilde{B}y = 0 \). For these \( x \) and \( y \) we have
\[
\tilde{B}(\tilde{A}^*y) = \tilde{A}^*(\tilde{B}y) = 0, \quad (\tilde{A}x, y) = (x, \tilde{A}^*y) = 0.
\]

Therefore \( \mathcal{A}\left((\text{Ker} \tilde{B})^\perp \right) \subset \left( \text{Ker} \tilde{B} \right)^\perp \). Similarly,
\[
\mathcal{A}^*\left((\text{Coker} \tilde{B})^\perp \right) \subset \left( \text{Coker} \tilde{B} \right)^\perp.
\]

Hence,
\[
D \left( \begin{array}{c} (\text{Ker} \tilde{B})^\perp \\ (\text{Coker} \tilde{B})^\perp \end{array} \right) \subset \left( \begin{array}{c} (\text{Ker} \tilde{B})^\perp \\ (\text{Coker} \tilde{B})^\perp \end{array} \right).
\]
(17)

From
\[
D^*D = \left( \begin{array}{cc} \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}\tilde{A}^* + \tilde{B}\tilde{B}^* \end{array} \right) \geq \left( \begin{array}{cc} \tilde{B}^\ast \tilde{B} & 0 \\ 0 & \tilde{B}\tilde{B}^\ast \end{array} \right)
\]

it follows that the operator \( D \) is an isomorphism on \( (\text{Ker} \tilde{B})^\perp \oplus (\text{Coker} \tilde{B})^\perp \). Taking into account the inclusion (17), (15) and (16), we obtain the following formula for the inverse image:
\[
D^{-1} \left( (\text{Ker} \tilde{B})^\perp \oplus (\text{Coker} \tilde{B})^\perp \right) \subset \left( \text{Ker} \tilde{B} \right)^\perp \oplus (\text{Coker} \tilde{B})^\perp.
\]
(18)

To prove that
\[
D : (\text{Ker} \tilde{B})^\perp \oplus (\text{Coker} \tilde{B})^\perp \rightarrow (\text{Ker} \tilde{B})^\perp \oplus (\text{Coker} \tilde{B})^\perp
\]
is an isomorphism, it remains to check that it is an epimorphism. By calculations as above, we obtain that \( (\text{Ker} \tilde{B})^\perp \) is an \( \tilde{A}^* \)-invariant submodule and \( (\text{Coker} \tilde{B})^\perp \) is an \( \tilde{A} \)-invariant submodule. The composition \( DD^* \) defines the isomorphism
\[
DD^* : (\text{Ker} \tilde{B})^\perp \oplus (\text{Coker} \tilde{B})^\perp \cong (\text{Ker} \tilde{B})^\perp \oplus (\text{Coker} \tilde{B})^\perp.
\]

By (13), \( D \) is an epimorphism.

Now we can calculate the index:
\[
\text{index}^{\times} D = [N_0^K \oplus N_0^{Lx}] - [N_1^K \oplus N_1^{Lx}] = \\
= ([N_0^K] - [N_1^K]) + ([N_0^{Lx}] - [N_1^{Lx}]) = \\
= \text{index}^{\times} C^K + \text{index}^{\times} C^{Lx} = \\
= \text{index}^{\times} C^K - \text{index}^{\times} C^L = \\
= a \cdot \text{ind}^{\times} \left( a \cdot \mu_p \left( \text{C-ind} \_{F \times H} b \right) \right).
\]
Theorem 8.4 Index functions a-ind and t-ind coincide.

Proof: From the statements proved above, it follows that we can apply Theorem 7.8.

9 Families and algebras

Let us remind the connection between $C^*$-elliptic operators and families of elliptic operators for non-equivariant case (cf. [30]). The equivariant case and the case of families of $C^*$-operators can be obtained just similarly.

Let $M$ be a smooth closed manifold, $X$ be a compact Hausdorff space, $A = C(X)$ is a unital $C^*$-algebra. Let us consider a family of operators elliptic along the fibers of $M \times X \to X$. More precisely, we have complex bundles $E \to M \times X$, $F \to M \times X$ and an operator $D_x : H^s(E_x) \to H^{s-m}(F_x)$. By Theorem of Jänich [20] the index

\[ \text{index}_a \{ D_x \} = [\text{Ker} D_x] - [\text{Coker} D_x] \in R^0(X) = K_0(A) \]

is defined (we get bundles after a compact perturbation). On the other hand, $E$ can be considered as an $A$-bundle $E_A$ over $M$ (similarly, $F_M$), and we can define an operator $D_A$ by commutativity of the following diagram (we consider the continuous cross-sections of subbundle of product bundle of $X$ by Sobolev space of stabilized fiber)

\[
\begin{array}{ccc}
\Gamma(X, \{ H^s(E_x) \}) & \xrightarrow{(D_x)_*} & \Gamma(X, \{ H^{s-m}(F_x) \}) \\
\psi \cong & \psi \cong & \\
H^s(E_A) & \xrightarrow{D_A} & H^{s-m}(F_A).
\end{array}
\]

So \( D_x \cdot (h)(x) := D_x(h(x)), \) where \( h : X \to \{ H^s(E_x) \} \), and locally (when $E_A = E_x \otimes A$)

\[
\psi(f \otimes a)(x) = f \cdot a(x), \quad \psi(D_A(f \otimes a))(x) = D_x(f \cdot a(x)) = D_x(f) \cdot a(x).
\]

Lemma 9.1 The operator $D_A$ is an $A$-elliptic operator and

\[ \text{a-ind } D_A = \text{index}_a \{ D_x \} . \]

Proof: As it was demonstrated, $D_A$ is an $A$-homomorphism. By calculation in local coordinates we obtain immediately, that $D_A$ is an $A$-$\Psi$DO with the correspondent symbol. The compactness of $X$ implies its invertibility over the complement of a compact neighborhood in the cotangent bundle, i.e. ellipticity. The coincidence of (co)kernels follows from the definition of $D_A$ after a compact perturbation, which has no influence for indices.

\[ \square \]
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