On the hyper-singular boundary integral equation methods for dynamic poroelasticity: three dimensional case

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Abstract

In our previous work [SIAM J. Sci. Comput. 43(3) (2021) B784-B810], an accurate hyper-singular boundary integral equation method for dynamic poroelasticity in two dimensions has been developed. This work is devoted to studying the more complex and difficult three-dimensional problems with Neumann boundary condition and both the direct and indirect methods are adopted to construct combined boundary integral equations. The strongly-singular and hyper-singular integral operators are reformulated into compositions of weakly-singular integral operators and tangential-derivative operators, which allow us to prove the jump relations associated with the poroelastic layer potentials and boundary integral operators in a simple manner. Relying on both the investigated spectral properties of the strongly-singular operators, which indicate that the corresponding eigenvalues accumulate at three points whose values are only dependent on two Lamé constants, and the spectral properties of the Calderón relations of the poroelasticity, we propose low-GMRES-iteration regularized integral equations. Numerical examples are presented to demonstrate the accuracy and efficiency of the proposed methodology by means of a Chebyshev-based rectangular-polar solver.

Keywords: Poroelasticity, hyper-singular operator, Calderón relation, regularized integral equation

1 Introduction

The dynamic poroelastic problems describing the physical behavior of the wave propagation in the elastic solid and the interstitial fluid can be found in many fields of applications such as petroleum industry, materials science, soil mechanics and biomechanics, etc. In accordance to Biot’s theory [6, 10, 19, 22], the dynamic poroelastic problems can be modeled by the coupled equations of the pore pressure and the solid displacement field, and the targeted degrees of freedom can be changed [39]. For the numerical solutions of such kind of wave scattering problems, it is known that the boundary integral equation (BIE) methods [21, 27] take advantages over the volumetric discretization methods [23, 25, 31, 42] in the sense of dimensions reduction, discretization of boundary and natural satisfactory of radiation condition, while in particular, the volumetric methods requires introducing appropriate artificial boundary conditions, such as absorbing boundary conditions or perfectly matched layers for the treatment of problems on unbounded domains. As a continuation of our previous work [44] for the two-dimensional poroelastic scattering problems, this work is devoted to proposing efficient BIE methods for solving the three-dimensional problems with Neumann boundary condition and it requires more complex technical investigations of the poroelastic boundary integral operators (BIOs).

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In the classical BIE theory, both the direct methods based on Green’s formula and the indirect methods
based on potential theory have been extensively discussed. In practice, the combined boundary integral
equations (CBIEs) resulting from a combination of single-layer and double-layer BIOs (for Dirichlet case)
or a combination of double-layer and hyper-singular BIOs (for Neumann case) are generally employed to
avoid the influence of possible eigenfrequencies. In this work, we employ both the direct method and the
indirect method to construct two types of CBIEs for solving the three-dimensional poroelastic problems
with Neumann boundary condition. As mentioned in [44], it still remains open to prove the unique
solvability of the CBIEs for the poroelastic scattering problem, but these BIEs still can provide efficient
numerical tools for the solutions of the problems imposed on unbounded domains. Then analogous to
the two-dimensional case [44], the following three issues should be addressed:

(i). The jump relations between the layer-potentials and the BIOs in dynamic poroelasticity case are not
easy to be observed.

(ii). The double-layer operators $K, K'$ (strongly-singular) and hyper-singular operator $N$ are well defined
in the sense of Cauchy principle value and Hadamard finite part [27], respectively. Then it requires
appropriate solvers for the accurate evaluation of these operators.

(iii). It is known that the eigenvalues of the hyper-singular operator accumulate at infinity and as a
result, solving the CBIEs by means of Krylov-subspace iterative solvers, such as GMRES, generally
requires a relatively large number of iterations for the convergence of numerical solution. Then
low-GMRES-iteration integral formulations are highly desirable.

To resolve the first and second issues, it is necessary to take a comprehensive study on the single kernels
of the poroelastic BIOs. To reduce/transform the singularities, some methodologies, for instance, adding-
and-subtracting appropriate terms and regularization using integration-by-parts, have been discussed in
open literatures. Inspired by the idea of reformulating the acoustic/Laplace hyper-singular integral operator
into a combination of weakly-singular integral operators and tangential derivatives [27,36], a novel
regularization technique using Günter derivative and Stokes formulas has been developed for the elastic
and thermoelastic problems [3,4,32,43]. In two-dimensions, the Günter derivative can be simplified
as the classical tangential derivative multiplied by a constant matrix, and the regularized formulations
for two-dimensional poroelastic BIOs have been investigated in [44]. But the three-dimensional Günter
derivative is more complex. Although the thermoelastic problem [30] takes a similar Biot’s model as the
poroelastic problem, the results presented in [4] can not be extended to the three-dimensional poroelastic
case trivially according to the more complicated coupled boundary operator and its adjoint, see Section 3.
It is proved in Theorems 4.1-4.6 that the three-dimensional strongly-singular and hyper-singular poroe-
lastic integral operators can be re-expressed in terms of multiple weakly-singular integral operators and
tangential-derivative operators. Compared with the formulations given in [34,35], the derived regularized
expressions in this work take simpler forms, and as a consequence, the jump relations between the layer-
potentials and the BIOs in dynamic poroelasticity case can be proved in an extremely simple manner,
see Theorem 3.1 and its proof in Section 4.3. In addition, owing to the new regularized expressions, the
numerical evaluation of the poroelastic BIOs amounts to the evaluation of weakly-singular type integrals,
for which the so-called Chebyshev-based rectangular-polar method proposed in [12] is applicable, and the
evaluation of three-dimensional tangential derivatives can be implemented via FFT [15].

The third issue is related to the spectral regularization or preconditioning. In addition to algebraic
preconditioning approaches, such as sparse approximate inverse [5,17] and multigrid methods [28], the
analytical preconditioning approach based on the Calderón relation, which in fact utilizes the compositions
$NS, SN$ of single-layer operator $S$ and hyper-singular operator $N$, has been discussed for solving wave
scattering problems by closed surfaces [11,15,20] or open surfaces [2,13,15]. Due to the weakly-singularity
of double-layer operator, it follows easily that the acoustic Calderón relation can be viewed as a compact
perturbation of an identity operator for the smooth closed-surface case. But this does not hold trivially
in elastic case, and also in poroelastic case (which can be understood naturally since the elastic single
kernels are involved in the poroelastic BIOs), on account of the fact that the classical elastic double-
layer operators are not compact. It has been proved in [14] for two-dimensional case and in [15] for

2
three-dimensional case that the elastic double-layer operators $K, K'$ are polynomially compact and as a result, the values of the finite accumulation points of the eigenvalues of $NS, SN$ only depend on the Lamé parameters of the elastic medium. The two-dimensional poroelastic case has been discussed in [44], whereas the result does not directly fit for the three-dimensional context. It is shown in this work (see Theorem 3.3) that the three-dimensional poroelastic double-layer operators are compact in the sense of a third-order polynomial and interestingly, the corresponding accumulation points of the eigenvalues are independent of the poroelastic parameters (see Table 1) except the two Lamé parameters. On a basis of the spectral properties of the poroelastic BIOs and analogous to the two-dimensional approach [44], we propose two regularized CBIEs for which the eigenvalues of the combined integral operators are bounded away from zero and infinity and then it leads to significant reductions in the number of GMRES iterations required for convergence to a given residual tolerance over the original CBIEs.

This paper is organized as follows. The dynamic poroelastic scattering problem is introduced in Section 2 and then in Sections 3.1-3.2 we present both the direct and indirect methods to derive the classical CBIEs, respectively. Section 3.3 is arranged to give a theoretical investigation of the spectral properties of the poroelastic integral operators and the corresponding Calderón relation. In Section 4, regularized expressions of the strongly-singular and hyper-singular operators are presented and then the jump relations between the layer-potentials and the BIOs are proved. Section 5.1 proposes two new RBIEs based on the Calderón relation and Section 5.2 briefly describes the numerical discretization method for poroelastic BIOs. Some numerical examples are presented in Section 6 to demonstrate the accuracy and efficiency of the proposed method.

2 Poroelastic problem

Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\Gamma := \partial \Omega$, and its exterior complement is denoted by $\Omega^c = \mathbb{R}^3 \setminus \Omega$. This work is devoted to studying the numerical solutions of the three-dimensional time-harmonic problems of wave propagation in the domain $\Omega^c$ which is occupied by a linear isotropic poroelastic medium characterized by the physical parameters listed in Table 1.

| Notation | Physical meaning |
|----------|------------------|
| $\lambda, \mu (\mu > 0, 3\lambda + 2\mu > 0)$ | Lamé parameters |
| $\nu_p$ | Poisson ratio |
| $\nu_u$ | undrained Poisson ratio |
| $B$ | Skempton porepressure coefficient |
| $\rho_s$ | solid density |
| $\rho_f$ | fluid density |
| $\rho_a = C \phi \rho_f$ | apparent mass density |
| $\phi$ | porosity |
| $\kappa$ | permeability coefficient |
| $\rho = (1 - \phi) \rho_s + \phi \rho_f$ | bulk density |
| $\alpha = \frac{2B(1-2\nu_f)(1+\nu_f)}{3(\nu_f(1-2\nu_f)(1+\nu_f))^2}$ | compressibility |
| $R = \frac{2\nu_f B^2(1-2\nu_f)(1+\nu_f)^2}{9(\nu_f(1-2\nu_f)(1+\nu_f))^2}$ | constitutive coefficient |

Following the Biot’s theory [6, 8, 9], the solid displacements $u = (u_1, u_2, u_3)^\top \in H^1_{loc}(\Omega^c)^3$ and the pore pressure $p \in H^1_{loc}(\Omega^c)$ characterizing the wave propagation in poroelastic medium can be modeled by the following coupled partial differential equations

$$
\begin{align*}
\Delta^* u + (\rho - \beta \rho_f) \omega^2 u - (\alpha - \beta) \nabla p &= 0 \\
\Delta p + qp + i\omega \gamma \nabla \cdot u &= 0
\end{align*}
$$

in $\Omega^c$, (2.1)
The wave numbers or equivalently, in a matrix form

\[ LU = 0, \quad L = \begin{bmatrix} \Delta^* + (\rho - \beta \rho_f)\omega^2 I & -(\alpha - \beta)\nabla \\ i\omega \gamma \nabla \end{bmatrix}, \quad U = (u_1, u_2, u_3, p)^T, \]

where

\[
\beta = \frac{\omega \rho_f^2 \rho_f}{\bar{\rho}^2 + \omega \rho_f (\rho_a + \delta \rho_f)}, \quad q = \frac{\omega^2 \rho_f}{\beta R}, \quad \gamma = -\frac{i \omega \rho_f (\alpha - \beta)}{\beta},
\]

are abbreviations defined to simplify the representation of the problem, \( \omega \) denotes the frequency, \( I \) is the identity operator and \( \Delta^* \) denotes the Lamé operator given by

\[ \Delta^* := \nabla \cdot \tilde{\sigma}(u), \]

with

\[ \tilde{\sigma}(u) = \lambda (\nabla \cdot u) I + 2\mu \varepsilon(u) \quad \text{and} \quad \varepsilon(u) = \frac{1}{2} (\nabla u + (\nabla u)^\top). \]

Given some data \( F \in H^{-1/2}(\Gamma) \), the Neumann boundary condition

\[ T(\partial, \nu) U := \begin{bmatrix} T_f(\partial, \nu) & -\alpha \nu \\ -\rho_f \omega^2 \nu^\top & \partial \nu \end{bmatrix} U = F \quad \text{on} \quad \Gamma \]

(2.2)
is imposed for the poroelastic problem in which the traction operator \( T(\partial, \nu) \) is defined as

\[ T(\partial, \nu) u := 2 \mu \partial \nu u + \lambda \nu \nabla \cdot u + \mu \nu \times \nabla \times u, \quad \nu = (\nu_1, \nu_2, \nu_3)^\top, \]

where \( \nu \) denotes the outward unit normal to the boundary \( \Gamma \) and \( \partial \nu := \nu \cdot \nabla \) is the normal derivative.

It follows [4,26,38] that the solution \( U \) of (2.1) admits a representation of the form

\[ U = (u, p)^\top = (u^1, p^1)^\top + (u^2, p^2)^\top + (u^s, p^s)^\top \]

where \( (u^k, p^k), k = 1, 2, s, \) satisfy

\[
\begin{align*}
(\Delta + k_1^2) u^1 &= 0, & (\Delta + k_2^2) u^2 &= 0, & (\Delta + k_s^2) u^s &= 0, \\
\text{curl} u^1 &= 0, & \text{curl} u^2 &= 0, & \text{div} u^s &= 0, \\
(\Delta + k_1^2) p^1 &= 0, & (\Delta + k_2^2) p^2 &= 0, & p^s &= 0.
\end{align*}
\]

(2.3)

Here, we denote by \( k_p \) and \( k_s \) the compressional and shear wave numbers, respectively, and they are given by

\[ k_p := \omega \sqrt{\frac{\rho - \beta \rho_f}{\lambda + 2\mu}}, \quad k_s := \omega \sqrt{\frac{\rho - \beta \rho_f}{\mu}}. \]

The wave numbers \( k_1, k_2 \) in (2.3), which represent the wave numbers of the fast compressional wave and slow compressional wave in poroelastic medium, respectively, are determined through

\[ k_1^2 + k_2^2 = q(1 + \varepsilon) + k_p^2, \quad k_1^2 k_2^2 = q k_p^2, \quad \text{Im}(k_i) \geq 0, i = 1, 2, \]

with \( \varepsilon = \frac{i \omega (\alpha - \beta)}{q(\lambda + 2\mu)} \). In particular,

\[
\begin{align*}
k_1 &= \sqrt{\frac{1}{2} \left\{ k_p^2 + q(1 + \varepsilon) + \sqrt{[k_p^2 + q(1 + \varepsilon)] - 4q k_p^2} \right\}}, \\
k_2 &= \sqrt{\frac{1}{2} \left\{ k_p^2 + q(1 + \varepsilon) - \sqrt{[k_p^2 + q(1 + \varepsilon)] - 4q k_p^2} \right\}}.
\end{align*}
\]
To complete the statement of the poroelastic problem, we assume that the solution \( U \) satisfies the following Kupradze radiation conditions as \( r = |x| \to \infty \) for \( l = 1, 2, 3 \) and \( j = 1, 2, 3 \),

\[
\begin{align*}
u^j &= o(r^{-1}), \quad \partial_{x^j} u^j = O(r^{-2}), \\
p^j &= o(r^{-1}), \quad \partial_{x^j} p^j = O(r^{-2}), \\
u^s &= o(r^{-1}), \quad r(\partial_r u^s - ik_s u^s) = O(r^{-1}).
\end{align*}
\]

**Remark 2.1.** For the poroelastic problem, the degrees of freedom can be determined in different ways [39]. Compared with the formulation in terms of the solid displacement and the fluid displacement, and the formulation in terms of the solid displacement and the seepage displacement, the above model enjoys the lowest number of unknowns. For the uniqueness analysis of the dynamic poroelastic problem, we refer to [24].

### 3 Boundary integral equations

In this section, we introduce the hyper-singular BIEs for solving the poroelastic problem together with some theoretical study of the properties of BIEs. Based on the Green’s identities and potential theory, direct and indirect boundary integral formulations are derived, respectively. We begin with the first and second Green’s identities for the poroelastic problems in \( \Omega \) (analogous to the problem in \( \Omega^c \)).

For \( U = (u^\top, p)^\top \) and \( V = (v^\top, \theta)^\top \), the first Green’s identity reads

\[
\int_\Omega \nabla u^\top \cdot \nabla v + A_\Omega(U, V) = \int_\Omega \tilde{T}(\partial, \nu) U^\top \cdot \nabla v ds, \tag{3.1}
\]

while the second Green’s identity admits

\[
\int_\Omega (\nabla u^\top \cdot \nabla v - U^\top \cdot \nabla^* V) dx = \int_\Gamma \left( \tilde{T}(\partial, \nu) U^\top \cdot \nabla v - U^\top \cdot \tilde{T}^*(\partial, \nu) V \right) ds. \tag{3.2}
\]

Here, \( A_\Omega \) denotes a bilinear form defined by

\[
A_\Omega(U, V) := \int_\Omega \left( \tilde{\sigma}(u) : \tilde{\varepsilon}(v) + (\rho - \beta \rho_f) \omega^2 u \cdot v - \alpha \nabla \cdot v \\
+ \beta \nabla p \cdot v - \rho_f \omega^2 u \cdot \nabla \theta + \frac{\rho_f \omega^2 \alpha}{\beta} \nabla \cdot u \theta + \nabla p \cdot \nabla \theta + q p \theta \right) dx,
\]

where \( L^* \) denotes the adjoint operator of \( L \) defined by

\[
L^* = \begin{bmatrix}
\Delta^* + (\rho - \beta \rho_f) \omega^2 I & -i \omega \gamma \nabla \\
(\alpha - \beta) \nabla & \Delta + q
\end{bmatrix},
\]

and \( \tilde{T}^*(\partial, \nu) \) is the boundary operator given by

\[
\tilde{T}^*(\partial, \nu) = \begin{bmatrix}
T(\partial, \nu) & -\frac{\rho_f \omega^2 \alpha}{\beta} \\
-\beta \nu^\top & \partial_{\nu}
\end{bmatrix}. \tag{3.3}
\]

It is known [18,38] that the fundamental solution of the operator \( L^* \) in \( \mathbb{R}^3 \) is given by

\[
E(x, y) = \begin{bmatrix}
E_{11}(x, y) & E_{12}(x, y) \\
E_{21}(x, y) & E_{22}(x, y)
\end{bmatrix}, \quad x \neq y,
\]
with
\[
E_{11}(x,y) = \frac{1}{\mu} \gamma_{k_t}(x,y) I + \frac{1}{(\rho - \beta \rho_f) \omega^2} \nabla_x \nabla_y^\top \left( \frac{\gamma_{k_t}(x,y) - \frac{k_1^2 - k_p^2}{k_1^2 - k_p^2} \gamma_{k_1}(x,y)}{+ \frac{k_2^2 - k_1^2}{k_2^2 - k_1^2} \gamma_{k_2}(x,y)} \right),
\]
\[
E_{12}(x,y) = \frac{i \omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_p^2)} \nabla_x [\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)],
\]
\[
E_{21}(x,y) = -\frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_p^2)} \nabla_x [\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)],
\]
\[
E_{22}(x,y) = -\frac{1}{(k_1^2 - k_p^2)} [(k_1^2 - k_p^2) \gamma_{k_1}(x,y) - (k_2^2 - k_p^2) \gamma_{k_2}(x,y)],
\]
in which
\[
\gamma_{k_t}(x,y) = \frac{\exp(i k_t |x - y|)}{4\pi |x - y|}, \quad x \neq y, \quad t = s, p, 1, 2,
\]
is the fundamental solution of the Helmholtz equation in \( \mathbb{R}^3 \) with wave number \( k_t \).

### 3.1 Direct method

It follows from the Green’s formulas (3.2) that the solution of (2.1) can be represented in the form
\[
U(x) = \mathcal{D}(U)(x) - \mathcal{S}(\mathcal{T}(\partial, \nu) U)(x), \quad x \in \Omega^c,
\]
where \( \mathcal{S} \) and \( \mathcal{D} \) are the single-layer and double-layer potentials given by
\[
\mathcal{S}(\varphi)(x) := \int_{\Gamma} (E(x,y))^\top \varphi(y) ds_y, \quad x \notin \Gamma,
\]
\[
\mathcal{D}(\varphi)(x) := \int_{\Gamma} (\mathcal{T}^*(\partial_y, \nu_y) E(x,y))^\top \varphi(y) ds_y, \quad x \notin \Gamma,
\]
respectively. Introduce the BIOs for the poroelasticity in the sense of principle value or Hadamard finite part as follows
\[
\mathcal{S}(\varphi)(x) := \int_{\Gamma} (E(x,y))^\top \varphi(y) ds_y, \quad x \in \Gamma,
\]
\[
\mathcal{K}(\varphi)(x) := \int_{\Gamma} (\mathcal{T}^*(\partial_y, \nu_y) E(x,y))^\top \varphi(y) ds_y, \quad x \in \Gamma,
\]
\[
\mathcal{K}^*(\varphi)(x) := \mathcal{T}(\partial_y, \nu_z) \int_{\Gamma} (E(x,y))^\top \varphi(y) ds_y, \quad x \in \Gamma,
\]
\[
\mathcal{N}(\varphi)(x) := \mathcal{T}(\partial_z, \nu_x) \int_{\Gamma} (\mathcal{T}^*(\partial_y, \nu_y) E(x,y))^\top \varphi(y) ds_y, \quad x \in \Gamma,
\]
where \( \mathcal{S}, \mathcal{K}, \mathcal{K}^* \) and \( \mathcal{N} \) are called, respectively, the single-layer, double-layer, transpose of double-layer, and hyper-singular BIOs. Then we conclude the jump relation results associated with the poroelastic layer potentials and BIOs in the following theorem.

**Theorem 3.1.** For \( x \in \Gamma \), the following jump relations hold:
\[
\lim_{h \to 0^+, z = x \pm hv_x} \mathcal{S}(\varphi)(z) = \mathcal{S}(\varphi)(x),
\]
\[
\lim_{h \to 0^+, z = x \pm hv_x} \mathcal{D}(\varphi)(z) = \left( \pm \frac{1}{2} I + \mathcal{K} \right)(\varphi)(x),
\]
\[
\lim_{h \to 0^+, z = x \pm hv_x} \mathcal{T}(\partial_z, \nu_x) \mathcal{S}(\varphi)(z) = \left( \mp \frac{1}{2} I + \mathcal{K}^* \right)(\varphi)(x),
\]
\[
\lim_{h \to 0^+, z = x \pm hv_x} \mathcal{T}(\partial_z, \nu_x) \mathcal{D}(\varphi)(z) = \mathcal{N}(\varphi)(x).
\]
Remark 3.2. The proof of this theorem relies on the study of the regularized expressions of the integral operators that will be presented in Section 4 and thus, will be reported after that.

Now applying the jump conditions, we are led to the BIEs on $\Gamma$

$$U(x) = \left(\frac{1}{2} I + K\right)(U)(x) - S(\tilde{T}(\partial, \nu)U)(x), \quad x \in \Gamma,$$

and

$$\tilde{T}(\partial_x, \nu_x)(U)(x) = N(U)(x) + \left(\frac{1}{2} I - K'\right)(\tilde{T}(\partial, \nu)U)(x), \quad x \in \Gamma. \quad (3.12)$$

Combining the BIEs (3.11)-(3.12) results into the so-called Burton-Miller formulation [16] on $\Gamma$

$$i\eta \left(\frac{1}{2} I - K\right) - N \right)(U)(x) + \left[\frac{1}{2} I + K' + i\eta S\right](\tilde{T}(\partial, \nu)U)(x) = 0, \quad (3.13)$$

where $\eta \neq 0$ is a combination coefficient. Using the boundary condition (2.2), we obtain the direct combined boundary integral equation (DCBIE)

$$\left[i\eta \left(\frac{1}{2} I - K\right) - N\right](U)(x) = -\left[\frac{1}{2} I + K' + i\eta S\right](F)(x), \quad x \in \Gamma. \quad (3.14)$$

3.2 Indirect method

The indirect boundary integral formulations can also be used for solving the poroelastic problems, which also allow for a suitable tool to test each operator separately. From the potential theory, the unknown function $U$ of (2.1) can be represented by a combination of the single-layer and double-layer potentials

$$U(x) = (D - i\eta S)(\varphi)(x), \quad x \in \Omega^c, \quad \eta \neq 0. \quad (3.15)$$

Operating with the boundary operator $\tilde{T}(\partial, \nu)$ on (3.15), taking the limit as in Theorem 3.1 and applying the boundary condition (2.2), we can obtain the indirect combined boundary integral equation (ICBIE)

$$\left[i\eta \left(\frac{1}{2} I - K\right) - N\right](\varphi)(x) = F, \quad x \in \Gamma. \quad (3.16)$$

3.3 Operator properties

Assuming that the boundary $\Gamma$ is sufficiently smooth, the BIOs are continuous mappings between the following spaces [27,43]

$$S : (H^{-1/2}(\Gamma))^4 \rightarrow (H^{1/2}(\Gamma))^4, \quad (3.17)$$

$$K, K' : (H^{\pm 1/2}(\Gamma))^4 \rightarrow (H^{\pm 1/2}(\Gamma))^4, \quad (3.18)$$

$$N : (H^{1/2}(\Gamma))^4 \rightarrow (H^{-1/2}(\Gamma))^4. \quad (3.19)$$

and the following Calderón relations hold:

$$SN = K^2 - \frac{1}{4} I, \quad NS = K'^2 - \frac{1}{4} I, \quad (3.20)$$

$$KS = SK', \quad NK = K'N. \quad (3.21)$$

As analytical preconditioning techniques, the Calderón relations have been investigated and utilized in regularized BIE methods [11,15,44], which require the spectral study of the BIOs, to construct BIE
systems possessing highly favorable spectral properties. The main reason is that the eigenvalues of the hyper-singular integral operator $N$ accumulate at infinity. As a result, obtaining the solutions of some integral equations, for example (3.14) and (3.16) in this work, by means of Krylov-subspace iterative solvers such as GMRES generally requires large numbers of iterations. To overcome this difficulty, the spectral properties of the integral operators $K, K'$ and the associated Calderón relations $NS$ for two-dimensional poroelasticity are investigated in [44] and then a regularized BIE method is proposed. However, as proved in the following theorem, the three-dimensional integral operators $K$ and $K'$ enjoy spectral properties different from those in the two-dimension case.

**Theorem 3.3.** Let $\tilde{I}_{\lambda,\mu}$ be a matrixized operator given by

$$\tilde{I}_{\lambda,\mu} = \begin{bmatrix} C_{\lambda,\mu}^2 I & 0 \\ 0 & 0 \end{bmatrix},$$

where $C_{\lambda,\mu}$ is a constant satisfying

$$0 < C_{\lambda,\mu} = \frac{\mu}{2(\lambda + 2\mu)} < \frac{3}{8}.$$

Then $K'(K'^2 - \tilde{I}_{\lambda,\mu}):(H^{1/2}(\Gamma))^4 \to (H^{1/2}(\Gamma))^4$ is compact. Furthermore, the spectrum of $K'$ consists of three non-empty sequences of eigenvalues which converge to $0$, $C_{\lambda,\mu}$ and $-C_{\lambda,\mu}$ respectively.

**Proof.** Analogous to the proof of [44, Theorem 3.1], it is sufficient to consider the static case ($\omega = 0$) BIO corresponding to $K'$ which can be formulated as

$$K'_0(U)(x) = \tilde{T}_0(\partial_x, \nu_x) \int_{\Gamma} (E_0(x, y))^\top U(y) ds_y = \begin{bmatrix} K'_{1,0} & K'_{2,0} \\ K'_{3,0} & K'_{4,0} \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}(x),$$

where

$$\tilde{T}_0(\partial_x, \nu_x) = \begin{bmatrix} T(\partial_x, \nu_x) & -\alpha \nu_x \\ 0 & \partial_{\nu_x} \end{bmatrix},$$

and

$$E_0(x, y) = \begin{bmatrix} E_{0,11} & E_{0,12} \\ E_{0,21}^\top & E_{0,22} \end{bmatrix} = \begin{bmatrix} \frac{E_{c,0} \alpha(x-y)^\top}{8\pi(\lambda + 2\mu)|x-y|} & 0 \\ 0 & \frac{1}{4\pi|x-y|} \end{bmatrix}$$

is the fundamental solution of static poroelastic problem

$$E_{c,0}(x, y) = \frac{\lambda + 3\mu}{8\pi \mu(\lambda + 2\mu)} \begin{bmatrix} \frac{1}{|x-y|} I + \frac{\lambda + \mu}{(\lambda + 3\mu)} \frac{(x-y)(x-y)^\top}{|x-y|^3} \end{bmatrix}$$

being the fundamental solution of Lamé equation. It can be verified that $K'_{1,0} = 0$ and the kernels of $K'_{j,0}, j = 2, 4$ admit weak singularity implying that $K'_{j,0}, j = 2, 4$ are compact. From [1], we know that $K'_{1,0}(K'^2_{2,0} - C_{\lambda,\mu}^2 I):(H^{1/2}(\Gamma))^4 \to (H^{1/2}(\Gamma))^4$ is compact. Therefore,

$$K'_0(K'^{2}_{0} - \tilde{I}_{\lambda,\mu}) = \begin{bmatrix} K'_{1,0}(K'^{2}_{2,0} - C_{\lambda,\mu}^2 I) & K'^{2}_{1,0}K'_{2,0}K'_{3,0} + K'_{1,0}K'^{2}_{4,0} + K'_{2,0}K'^{2}_{4,0} \\ 0 & K'^{2}_{3,0} \end{bmatrix}$$

is compact. A direct calculation yields

$$K'(K'^{2} - \tilde{I}_{\lambda,\mu}) = K'_0(K'^{2} - \tilde{I}_{\lambda,\mu}) + K_c,$n

$$K_c = K'_0(K' - K'_0)K' + K'^{2}_{2,0}(K' - K'_0) + (K' - K'_0)(K'^{2} - \tilde{I}_{\lambda,\mu}).$$

The compactness of $K' - K'_0$ indicates the compactness of $K_c$ and then further implies that $K'(K'^{2} - \tilde{I}_{\lambda,\mu})$ is compact. The proof is completed. \qed
The spectral property of the operator \( K \) is similar to \( K' \). Relying on the Calderón relations \([27]\)

\[
NS = K'^2 - \frac{1}{4} I, \quad SN = K^2 - \frac{1}{4} I,
\]

we can conclude that the spectrum of both the composite operators \( NS \) and \( SN \) consist of two non-empty sequences of eigenvalues which converge to \(-\frac{1}{4}\) and \(-\frac{1}{4} + C_{\lambda,\mu}^2\), respectively.

To verify the above results numerically, we consider the problem of poroelastic scattering by a unit ball, and we choose the same values of parameters as in Section 6. Consequently, the constant \( C_{\lambda,\mu} = 0.1875 \).

Utilizing the following re-expressions of the BIOs together with the discretization method presented in Sections 4-5, the eigenvalue distributions of the integral operators \( K', K, NS \) and \( SN \) are displayed in Figure 1 showing an agreement with our theoretical results. Based on these results, the corresponding regularized BIE method is proposed in Section 5 for solving the three-dimensional poroelastic problem.

Figure 1: Eigenvalue distributions of the operators \( K' \) (a), \( K \) (b), \( NS \) (c) and \( SN \) (d) for a unit ball scatterer.

### 4 Strong-singularity and hyper-singularity regularization

As aforementioned, the integral operators \( K \) (as well as \( K' \)) and \( N \) are strongly-singular and hyper-singular, respectively. In this section, we will present new (regularized) expressions for these operators. More precisely, with the help of the tangential Günter derivative \([4]\), these operators will be expressed in terms of compositions of weakly-singular integral operators (\( K_i^j \), \( K'_i^j \), \( N_i^j \) appearing in the following Theorems (4.1)-(4.6)) and tangential-derivative operators (\textit{The formulations to derive these regularized expressions will be shown in the supplemented material}). Employing these formulations together with the rectangular-polar quadrature method proposed in \([12,15]\) and the iteration solver GMRES then leads to our boundary integral solver for the poroelastic problem.

We begin with the Günter derivative operator \( M(\partial, \nu) \) defined as

\[
M(\partial, \nu)u(x) = \partial_\nu u - \nu(\nabla \cdot u) + \nu \times \text{curl } u.
\]
From [15], we know that

\[ M(\partial, \nu) = \begin{pmatrix} 0 & -\bar{\partial}_2^S & -\bar{\partial}_3^S \\ \bar{\partial}_2^S & 0 & -\bar{\partial}_3^S \\ -\bar{\partial}_3^S & -\bar{\partial}_2^S & 0 \end{pmatrix}, \]

where \( \bar{\partial}_i^S, i = 1, 2, 3 \) are the components of \( \nu \times \nabla^S = (\bar{\partial}_2^S, \bar{\partial}_3^S, \bar{\partial}_1^S)^\top \), in which \( \nabla^S \) is the surface gradient defined as \( \nabla^S = \nabla u - \nu \partial \nu u \). Then the traction operator \( T(\partial, \nu) \) can be rewritten as

\[ T(\partial, \nu) u(x) = (\lambda + 2\mu) \nu (\nabla \cdot u) + \mu \partial \nu u + \mu M(\partial, \nu) u. \]

### 4.1 Strong-singularity regularization

We first consider the operators \( K \) and \( K' \) in the forms of

\[ K(U)(x) = \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}(x), \quad K'(U)(x) = \begin{bmatrix} K'_1 & K'_2 \\ K'_3 & K'_4 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}(x), \quad x \in \Gamma. \]

Then the following regularized formulations can be obtained.

**Theorem 4.1.** The operators \( K_j, j = 1, \ldots, 4 \) can be expressed as

\[ K_j = K_j^3 + K_j^2 M(\partial, \nu) + \{K_j^3 M(\partial, \nu)\}^\top, \quad j = 1, \ldots, 4, \tag{4.1} \]

where \( K_1^3 = K_2^3 = K_3^3 = K_4^3 = 0 \), and

\[ K_1^1(u)(x) = \int_{\Gamma} \left[ \partial_{y_1} \gamma_k(x, y) I - \frac{\rho I \omega^2}{\beta} E_{21}(x, y) \nu_y^\top \right] u(y) ds_y \]

\[ - \int_{\Gamma} \nabla_y \left[ (\gamma_k(x, y) - \gamma_{k_1}(x, y)) - \frac{k_2^2 - q}{k_1^2 - k_2^2}(\gamma_{k_2}(x, y) - \gamma_{k_2}(x, y)) \right] \nu_y^\top u(y) ds_y, \]

\[ K_1^2(u)(x) = \int_{\Gamma} [2\mu E_{11}(x, y) - \gamma_{k_1}(x, y) I] u(y) ds_y, \]

\[ K_1^3(p)(x) = - \int_{\Gamma} \left[ \frac{\alpha - \beta}{(k_1^2 - k_2^2)(\lambda + 2\mu)} (k_2^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) + \beta E_{11}(x, y) \right] \nu_y p(y) ds_y, \]

\[ K_1^3(u)(x) = \int_{\Gamma} \left[ \frac{i \omega \gamma}{(k_1^2 - k_2^2)} (k_2^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) - \frac{\rho I \omega^2}{\beta} E_{22} \nu_y^\top u(y) ds_y, \]

\[ K_3^2(u)(x) = - \frac{2i \mu \omega \gamma}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int_{\Gamma} \nabla_y^\top [\gamma_k(x, y) - \gamma_{k_2}(x, y)] u(y) ds_y, \]

\[ K_3^3(p)(x) = \frac{\omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_{\Gamma} \partial_{y_1} [\gamma_k(x, y) - \gamma_{k_2}(x, y)] p(y) ds_y \]

**Theorem 4.2.** The operators \( K'_j, j = 1, \ldots, 4 \) can be expressed as

\[ K'_j = K_j^3 + M(\partial_x, \nu_x) K_j^2 + M(\partial_x, \nu_x) : K_j^3, \quad j = 1, \ldots, 4, \tag{4.2} \]
where $K_1^3 = K_2^3 = K_3^2 = K_4^2 = K_5^0 = 0$, and

\[
K_1^4(u)(x) = \int_\Gamma \left[ \nu_x \gamma_k(x, y) \right] (\gamma_k(x, y) - \gamma_k(x, y) - k_3^2 - \frac{q}{k_1^2 - k_2^2} \gamma_k(x, y) - \gamma_k(x, y)) u(y) ds_y
\]

\[
K_2^4(u)(x) = \int_\Gamma \left[ 2\mu E_{11}(x, y) - \gamma_k(x, y) I \right] u(y) ds_y,
\]

\[
K_3^4(p)(x) = \frac{\alpha - \beta}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int_\Gamma \nu_x [\gamma_k(x, y) - \gamma_k(x, y)] p(y) ds_y,
\]

\[
K_4^4(u)(x) = -\rho f \omega^2 \int_\Gamma \nu_x E_{11}(x, y) u(y) ds_y
\]

\[
- \frac{i\omega \gamma}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int_\Gamma \nu_x [\gamma_k(x, y) - \gamma_k(x, y)] p(y) ds_y.
\]

\[
K_5^4(u)(x) = \frac{i\omega \gamma}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int_\Gamma u(y) \nu_x [\gamma_k(x, y) - \gamma_k(x, y)] ds_y.
\]

4.2 Hyper-singularity regularization

In this subsection, we investigate the hyper-singular operator $N$ with

\[
N(\psi)(x) = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} (x), \quad x \in \Gamma.
\]

The regularized formulations for the operators $N_j$, $j = 1, \ldots, 4$ are given in the following theorems.

**Theorem 4.3.** The hyper-singular operator $N_1$ can be expressed as

\[
N_1 = N_1^1 + M(\partial, \nu) N_1^2 M(\partial, \nu) + \tau_2 N_1^3 \tau_1 + M(\partial, \nu) N_1^4 + N_1^5 M(\partial, \nu),
\]

where

\[
N_1^1(u)(x) = -(\rho - \beta \rho f) \omega^2 \int_\Gamma \gamma_k(x, y) \left( \nu_x \nu_y - \nu_x \nu_y I \right) u(y) ds_y
\]

\[
+ \int_\Gamma \left[ C_1 \gamma_k(x, y) - C_2 \gamma_k(x, y) \right] \nu_x \nu_y u(y) ds_y,
\]

\[
N_1^2(u)(x) = \int_\Gamma \left[ 4\mu^2 E_{11}(x, y) - 3\mu \gamma_k(x, y) I \right] u(y) ds_y,
\]

\[
N_1^3(u)(x) = \mu \int_\Gamma \gamma_k(x, y) u(y) ds_y.
\]
Theorem 4.4. The hyper-singular operator $N_2$ can be expressed as

$$N_2 = N_2^1 + M(\partial, \nu)N_2^2 M(\partial, \nu) + M(\partial, \nu)N_2^3,$$

where

$$N_2^1(p)(x) = \alpha - \beta \frac{k_1^2 - q}{k_1^2 - k_2^2} \int \partial_{v_\nu} \gamma k_1(x,y) \nu_x p(y) ds_y$$

$$+ \beta \int \partial_{v_\nu} \nu_x \gamma_k(x,y) - \gamma_k(x,y)) - \partial_{v_\nu} \gamma_k(x,y) I \nu_y p(y) ds_y$$

$$+ \frac{\alpha}{k_1^2 - k_2^2} \int \partial_{v_\nu} [k_1^2 - k_2^2 \gamma_k(x,y) - (k_1^2 - k_2^2) \gamma_k(x,y)] \nu_x p(y) ds_y,$$

$$N_2^2(p)(x) = \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int \{ \nabla_y \gamma_k(x,y) - \gamma_k(x,y) \} p(y) \nu_y ds_y,$$

$$N_2^3(p)(x) = -\beta \int [2 \mu E_{11}(x,y) - \gamma_k(x,y) I] \nu_y p(y) ds_y$$

$$- \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int [k_1^2 \gamma_k(x,y) - k_2^2 \gamma_k(x,y)] \nu_y p(y) ds_y.$$

Theorem 4.5. The hyper-singular operator $N_3$ can be expressed as

$$N_3 = N_3^1 + N_3^2 M(\partial, \nu) + M(\partial, \nu) : N_3^3 M(\partial, \nu).$$
where

\[ N_4^1(u)(x) = \frac{i\omega \gamma}{k_1^2 - k_2^2} \int_\Gamma \partial_{\nu_x} \left[ k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y) \right] \nu_y^T u(y) ds_y \]

\[-\rho_f \omega^2 \int_\Gamma \left[ \partial_{\nu_x} \left( \gamma_{k_1}(x, y) - \gamma_{k_1}(x, y) \right) \nu_y^T + \partial_{\nu_y} \gamma_{k_1}(x, y) \nu_x^T \right] u(y) ds_y \]

\[ + \left( \frac{\rho_f \omega^2 (k_2^2 - q)}{k_1^2 - k_2^2} - \frac{\rho_f^2 \omega^4 (\alpha - \beta)}{\beta (\lambda + 2\mu)(k_1^2 - k_2^2)} \right) \int_\Gamma \partial_{\nu_x} \left[ \gamma_{k_1}(x, y) - \gamma_{k_2}(x, y) \right] \nu_y^T u(y) ds_y \]

\[ + \frac{\rho_f \omega^2 \alpha}{\beta (k_1^2 - k_2^2)} \int_\Gamma \partial_{\nu_x} \left[ (k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_1^2 - k_2^2) \gamma_{k_2}(x, y) \right] \nu_y^T u(y) ds_y, \]

\[ N_4^2(u)(x) = -\rho_f \omega^2 \int_\Gamma \nu_x^T \left[ 2\mu E_{11}(x, y) - \gamma_{k_1}(x, y) I \right] u(y) ds_y \]

\[ - \frac{2i\mu \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] \nu_x^T u(y) ds_y, \]

\[ N_4^3(u)(x) = \frac{2i\mu \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma u(y) \nabla_x^T \left[ \gamma_{k_1}(x, y) - \gamma_{k_2}(x, y) \right] ds_y. \]

**Theorem 4.6.** The hyper-singular operator \( N_4 \) can be expressed as

\[ N_4 = N_4^1 + \tau_2 N_4^2 \tau_1, \]

where

\[ N_4^1(p)(x) = \rho_f \omega^2 \beta \int_\Gamma \nu_x^T E_{11}(x, y) \nu_y p(y) ds_y \]

\[ + \frac{i\omega \gamma \beta + \rho_f \omega^2 (\alpha - \beta)}{\lambda + 2\mu} \int_\Gamma [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] \nu_x^T \nu_y p(y) ds_y \]

\[ - \frac{1}{k_1^2 - k_2^2} \int_\Gamma \left[ (k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_1^2 - k_2^2) \gamma_{k_2}(x, y) \right] \nu_x^T \nu_y p(y) ds_y, \]

\[ N_4^2(p)(x) = \rho_f \omega^2 (\alpha - \beta) + i\omega \beta \gamma \int_\Gamma \left[ \gamma_{k_1}(x, y) - \gamma_{k_2}(x, y) \right] p(y) ds_y \]

\[ - \frac{1}{k_1^2 - k_2^2} \int_\Gamma \left[ (k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_1^2 - k_2^2) \gamma_{k_2}(x, y) \right] p(y) ds_y. \]

### 4.3 Proof of Theorem 3.1

Thanks to the derived regularized formulations of the integral operators, now we can prove the jump conditions stated in Theorem 3.1.

The jump condition for the single-layer potential follows trivially. Relating to the proof of Theorem 4.1, it can be concluded that \( D(\varphi)(z), x \in \Gamma \) takes a form similar to (4.1) for \( z = x \pm h \nu_x \notin \Gamma \), i.e.,

\[ D = \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}, \quad D_j = D_1 + D_2^2 M(\partial, \nu) + \{ D_2^3 M(\partial, \nu) \}^\top, \quad j = 1, \ldots, 4. \]

Letting \( h \to 0^+ \) (i.e. \( z \to x \in \Gamma \)), and applying the classical jump relations for acoustic and elastic problems [27], it is only necessary to study the jumps of \( D_1, D_2, D_3 \) and \( D_4 \) in the forms associated with \( K_1, K_2, K_3 \) and \( K_4 \), respectively. Note that for \( x \in \Gamma \),

\[ \lim_{h \to 0^+, z = x \pm h \nu_x} \int_\Gamma \nabla_y \gamma_{k_1}(z, y) \varphi(y) ds_y = \pm \frac{\nu_x}{2} \varphi(x) + \int_\Gamma \nabla_y \gamma_{k_1}(x, y) \varphi(y) ds_y, \]

13
which implies
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} \int_{\Gamma} \partial_{\nu_x} \gamma_{k_1}(z, y) u(y) ds_y = \frac{1}{2} u(x) + \int_{\Gamma} \partial_{\nu_x} \gamma_{k_2}(x, y) u(y) ds_y,
\]
\[- \lim_{h \to 0^+, z = x \pm h\nu_x} \int_{\Gamma} \partial_{\nu_x} \left[ \frac{k_p^2 - k_1^2}{k_1^2 - k_2^2} \gamma_{k_1}(z, y) - \frac{k_p^2 - k_2^2}{k_1^2 - k_2^2} \gamma_{k_2}(z, y) \right] p(y) ds_y,
\]
\[= \frac{1}{2} p(x) + \int_{\Gamma} \partial_{\nu_x} \left[ \frac{k_p^2 - k_1^2}{k_1^2 - k_2^2} \gamma_{k_1}(x, y) - \frac{k_p^2 - k_2^2}{k_1^2 - k_2^2} \gamma_{k_2}(x, y) \right] p(y) ds_y.
\]
It follows that
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} D_1^1(u)(z) = \frac{1}{2} u(x) + K_1^1(u)(x), \quad x \in \Gamma,
\]
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} D_2^1(p)(z) = K_2^1(p)(x), \quad x \in \Gamma,
\]
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} D_3^1(u)(z) = K_3^1(u)(x), \quad x \in \Gamma,
\]
and
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} D_4^1(p)(z) = \frac{1}{2} p(x) + K_4^1(p)(x), \quad x \in \Gamma.
\]
Therefore, we have
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} D(\varphi)(z) = \frac{1}{2} \varphi(x) + K(\varphi)(x), \quad x \in \Gamma.
\]
Analogously, it can be derived that \( \overline{T}(\partial_{\nu}, \nu_x)S(\varphi)(z) \), \( x \in \Gamma \) takes a form similar to (4.2) for \( z = x \pm h\nu_x \notin \Gamma \). Due to the fact that for \( x \in \Gamma \),
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} \int_{\Gamma} \nu_x \cdot \nabla_{z} \gamma_{k_1}(z, y) u(y) ds_y = \frac{1}{2} u(x) + \int_{\Gamma} \partial_{\nu_x} \gamma_{k_2}(x, y) u(y) ds_y,
\]
\[- \lim_{h \to 0^+, z = x \pm h\nu_x} \int_{\Gamma} \nu_x \cdot \nabla_{z} \left[ \frac{k_p^2 - k_1^2}{k_1^2 - k_2^2} \gamma_{k_1}(z, y) - \frac{k_p^2 - k_2^2}{k_1^2 - k_2^2} \gamma_{k_2}(z, y) \right] p(y) ds_y,
\]
\[= \frac{1}{2} p(x) + \int_{\Gamma} \partial_{\nu_x} \left[ \frac{k_p^2 - k_1^2}{k_1^2 - k_2^2} \gamma_{k_1}(x, y) - \frac{k_p^2 - k_2^2}{k_1^2 - k_2^2} \gamma_{k_2}(x, y) \right] p(y) ds_y,
\]
we arrive at the jump conditions
\[
\lim_{h \to 0^+, z = x \pm h\nu_x} \overline{T}(\partial_{\nu}, \nu_x)S(\varphi)(z) = \pm \frac{1}{2} \varphi(x) + K'(\varphi)(x), \quad x \in \Gamma.
\]
It remains to prove the jump conditions for \( \overline{T}(\partial_{\nu}, \nu_x)D(\varphi)(z) \) as \( h \to 0^+ \). We can write it as
\[
\overline{T}(\partial_{\nu}, \nu_x)D(\varphi)(z) = \begin{bmatrix} N_1 & N_2 \\ N_3 & N_4 \end{bmatrix}(\varphi)(z),
\]
and it can be proved that \( N_j, j = 1, \cdots, 4 \) take forms similar to \( N_j, j = 1, \cdots, 4 \), respectively. We first study \( N_1 \) in the form
\[
N_1(u)(z) = N_1^1(u)(z) + M(\partial_{\nu}, \nu_x)N_1^2(M(\partial_{\nu}, \nu_x)u)(z) + \tau_1^z z^x N_1^3(\tau_1 u)(z) + M(\partial_{\nu}, \nu_x)N_1^4(u)(z) + N_1^5(M(\partial_{\nu}, \nu_x)u)(z),
\]
}\]
where \( \tau_2^{2-x} u(z) = (\nu_x \times \nabla_S) \cdot u(z) \). Obviously,

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} N_j^2(u)(z) = N_j^2(u)(x), \quad x \in \Gamma, \quad j = 1, 2, 3,
\]

and thus,

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} M(\partial_x, \nu_x)N_1^2(M(\partial, \nu)u)(z) = M(\partial_x, \nu_x)N_1^2(M(\partial, \nu)u)(x), \quad x \in \Gamma,
\]

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} \tau_2^{3-x} N_1^3(\tau_1 u)(z) = \tau_2 N_1^3(\tau_1 u)(x), \quad x \in \Gamma.
\]

In addition, note that

\[
N_1^4(u)(z) = \int_\Gamma \nabla_y [ -2\mu(\gamma k_z(z, y) - \gamma k_1(z, y)) + C_3(\gamma k_1(z, y) - \gamma k_2(z, y))] \nu_y^T u(y) ds_y + \mu \int_\Gamma \partial_y \gamma k_z(z, y) u(y) ds_y,
\]

\[
N_1^5(u)(z) = \int_\Gamma \nu_x \nabla_x [ -2\mu(\gamma k_z(z, y) - \gamma k_1(z, y)) + C_4(\gamma k_1(z, y) - \gamma k_2(z, y))] u(y) ds_y + \mu \int_\Gamma \nu_x \nabla_x \gamma k_z(z, y) u(y) ds_y,
\]

we have

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} N_1^4(u)(z) = \frac{\mu}{2} u(x) + N_1^4(u)(x), \quad x \in \Gamma,
\]

and

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} N_1^5(u)(z) = \frac{\mu}{2} u(x) + N_1^5(u)(x), \quad x \in \Gamma.
\]

Therefore,

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} M(\partial_z, \nu_z)N_1^4(u)(z) + N_1^5(M(\partial, \nu)u)(z) = \frac{\mu}{2} M(\partial_x, \nu_x) u(x) + M(\partial_x, \nu_x) N_1^4(u)(x)
\]

\[
\mp \frac{\mu}{2} M(\partial_x, \nu_x) u(x) + N_1^5(M(\partial, \nu)u)(x) = M(\partial_x, \nu_x) N_1^4(u)(x) + N_1^5(M(\partial, \nu)u)(x), \quad x \in \Gamma,
\]

which gives that

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} N_1^4(u)(z) = N_1(u)(x), \quad x \in \Gamma.
\]

The proof of the other three jump conditions

\[
\lim_{h \to 0^+, z = x \pm h\nu_x} N_j^4(u)(z) = N_j^4(u)(x), \quad x \in \Gamma, j = 2, 3, 4,
\]

is analogous and hence is omitted. This completes the proof of Theorem 3.1.
5 Regularized boundary integral equation solver

5.1 Regularized boundary integral equations

Making use of the spectral properties of the poroelastic integral operators presented in Section 3.3, we are able to construct regularized boundary integral equations (RBIEs) with favorable features of better spectral properties. According to the regularized integral equation method discussed in [44] for two-dimensional poroelastic problems, we can choose the static single-layer operator $\mathcal{R} = S_0$ given by

$$S_0(\varphi)(x) := \int_{\Gamma} (E_0(x, y))^\top \varphi(y) ds_y, \quad x \in \Gamma,$$

as the regularized operator.

Figure 2: Eigenvalue distributions of the operators $i\eta \left( \frac{1}{2} I - K \right) - \mathcal{R}N$ (a) and $i\eta \left( \frac{1}{2} I - K' \right) + N\mathcal{R}$ (b) for a unit ball scatterer.

Therefore, for the direct method, the DCBIE (3.14) can be regularized as (called DRBIE)

$$\left[i\eta \left( \frac{1}{2} I - K \right) - \mathcal{R}N\right] U(x) = -\left[\mathcal{R}\left(\frac{1}{2} I + K'\right) + i\eta S\right] (F(x)), \quad x \in \Gamma.$$  

(5.2)

For the indirect method, replacing the solution representation (3.15) by

$$U(x) = (DR - i\eta S)(\varphi)(x), \quad x \in \Omega^c, \quad \eta \neq 0,$$

(5.3)

we can obtain the regularized form of ICBIE (called IRBIE) as follows

$$\left[i\eta \left( \frac{1}{2} I - K' \right) + N\mathcal{R}\right] (\varphi) = F \quad \text{on} \quad \Gamma,$$

(5.4)

instead of the classical ICBIE (3.16). In view of the spectra results in Section 3.3, we can conclude that the spectrum of the regularized integral operator on the left-hand side of DRBIE (5.2) consists of three nonempty sequences of eigenvalues which converge to $1/4 - i\eta/2$, $1/4 - C_{\lambda,\mu}^2 + i\eta(1/2 + C_{\lambda,\mu})$ and $1/4 - C_{\lambda,\mu}^2 + i\eta(1/2 - C_{\lambda,\mu})$, respectively. Similarly, we can also observe that the eigenvalues of the integral operator on the left-hand side of IRBIE (5.4) accumulated at $-1/4 + i\eta/2$, $-1/4 + C_{\lambda,\mu}^2 + i\eta(1/2 + C_{\lambda,\mu})$ and $-1/4 + C_{\lambda,\mu}^2 + i\eta(1/2 - C_{\lambda,\mu})$, see for example Figure 2 a numerical verification.

5.2 Numerical discretization

In this section, we briefly introduce the application of the Chebyshev-based rectangular-polar solver discussed in [12,15] to the numerical discretization the poroelastic BIOs. Based on a partition of the
boundary using non-overlapping parametric curvilinear patches, this approach interpolates the unknowns on a Chebyshev grid on each patch in terms of Chebyshev polynomials. For the corresponding acceleration of this method, we refer to [29].

Let \( \Gamma \) be partitioned into a set of \( M \) non-overlapping parametrized (logically-rectangular) patches \( \Gamma_q, q = 1, \ldots, M \) as

\[
\Gamma = \bigcup_{q=1}^{M} \Gamma_q, \quad \Gamma_q = \left\{ r^q(u, v) = (x^q(u, v), y^q(u, v), z^q(u, v))^\top : [-1, 1]^2 \to \mathbb{R}^3 \right\}.
\]

Introducing the tangential covariant basis vectors and surface normal on \( \Gamma_q \):

\[
a^q_u = \frac{\partial r^q(u, v)}{\partial u}, \quad a^q_v = \frac{\partial r^q(u, v)}{\partial v}, \quad \nu^q = \frac{a^q_u \times a^q_v}{|a^q_u \times a^q_v|},
\]

we can obtain the metric tensor as

\[
G^q = \begin{bmatrix} g^q_{uu} & g^q_{uv} \\ g^q_{vu} & g^q_{vv} \end{bmatrix},
\]

where \( g^q_{ij}, i, j = 1, 2 \) denote the components of the inverse of the matrix \( G^q \).

Here, \( |G^q| \) is the determinant of \( G^q \). As a result, the surface gradient of a given density \( \varphi = \varphi(r^q(u, v)) \) can be expressed as

\[
\nabla^S \varphi = \sum_{i,j=1}^{2} g^{ij} \partial_i \varphi \partial_j r^q(u_i, v_j), \quad \partial_1 = \frac{d}{du}, \partial_2 = \frac{d}{dv},
\]

where \( g^{ij}, i, j = 1, 2 \) denote the components of the inverse of the matrix \( G^q \).

Given a density \( \varphi \), it can be approximated on \( \Gamma_q \) by the Chebyshev polynomials as

\[
\varphi(x) = \sum_{i,j=0}^{N-1} \varphi^q_{ij} a_{ij}(u, v), \quad x \in \Gamma_q,
\]

where

\[
a_{ij}(u, v) = \frac{1}{N^2} \sum_{m,n=0}^{N-1} \alpha_n \alpha_m T_n(u) T_m(v), \quad \alpha_n = \begin{cases} 1, & n = 0, \\ 2, & n \neq 0, \end{cases}
\]

and the coefficients \( \varphi^q_{ij} = \varphi(x^q_{ij}) \) denote the values of the continuous density \( \varphi \) at the discretization points \( x^q_{ij} = r^q(u_i, v_j) \) with

\[
u_i = \cos \left( \frac{2i + 1}{2N} \pi \right), \quad v_j = \cos \left( \frac{2j + 1}{2N} \pi \right), \quad i, j = 0, \ldots, N - 1.
\]

Therefore, we can obtain the approximation of the surface gradient as

\[
\left. (\nabla^S \varphi) \right|_{x=x^q_{ij}} = \sum_{n,m=0}^{N-1} B^q_{ij,nm} \varphi^q_{nm}, \quad B^q_{ij,nm} = \left( \sum_{i,j=1}^{2} g^{ij} \partial_i a_{nm} \partial_j r^q \right) \bigg|_{u=u_i, v=v_j}.
\]
We now discuss the discretizations of the proelastic BIoS. On a basis of the regularized formulations of integral operators given in Sections 4, the numerical implementations can be converted into evaluating multiple operators of two types, (i) Integral operators with weakly-singular kernels $H(x, y)$

$$\mathcal{H}\varphi(x) = \int_{\Gamma} H(x, y)\varphi(y)ds_y,$$

and (ii) surface-differentiation operators $M(\partial, \nu)$, $\gamma_1$, $\gamma_2$ appearing in Section 4 which can be extracted from the approximation of surface gradient $\nabla_x^\tau$. Clearly, the integrals $\mathcal{H}\varphi(x)$ over $\Gamma$ can be split into the sum of integrals over each of the $M$ patches,

$$\mathcal{H}\varphi(x) = \sum_{q=1}^{M} \mathcal{H}_q(x), \quad \mathcal{H}_q(x) := \int_{\Gamma_q} H(x, y)\varphi(y)ds_y, \quad x \in \Gamma.$$

In the “non-adjacent” integration case, in which the target point $x_{ij}^q$ is far from the integration patch, the integral $\mathcal{H}_q(x_{ij}^q)$ is non-singular. Then the classical Fejér’s first quadrature rule can be utilized to obtain the approximation

$$\mathcal{H}_q(x_{ij}^q) \approx \sum_{m,n=0}^{N-1} A_{ij,nm}^q \varphi_{nm}^q,$$

where

$$A_{ij,nm}^q = H(x_{ij}^q, r^q(u_n, v_m)) J^q(u_n, v_m) w_n w_m,$$

with the quadrature weights

$$w_j = \frac{2}{N} \left( 1 - 2 \frac{[N/2]}{4l^2 - 1} \cos(lu_j) \right), \quad j = 0, ..., N - 1.$$

In the “adjacent” integration case, in which the point $x_{ij}^q$ either lies within the integration patch or is located very close to it, the integral $\mathcal{H}_q(x_{ij}^q)$ becomes weakly-singular and nearly-singular, respectively. It is suggested in [12] to constructed a new graded mesh, relying on a smoothing change of variables $\xi_u(s)$ (for more details, see [12,15]), around the point which lies closest to $x_{ij}^q$ with parameters

$$\bar{\xi}^q = \arg\min_{(u,v) \in [-1,1]^2} \left\{ \left| x_{ij}^q - r^q(u,v) \right| \right\}.$$

Then the single integral can be approximated by

$$\mathcal{H}_q(x_{ij}^q) \approx \sum_{n,m=0}^{N-1} \varphi_{nm}^q \int_{-1}^{1} \int_{-1}^{1} H(x_{ij}^q, r(u,v)) J^q(u,v) a_{nm}(u,v) du dv$$

$$= \sum_{n,m=0}^{N-1} \varphi_{nm}^q \int_{-1}^{1} \int_{-1}^{1} \tilde{H}(x_{ij}^q, s,t) \tilde{J}^q(s,t) \tilde{a}_{nm}(s,t) \xi_u^q(s) \xi_v^q(t) ds dt$$

$$\approx \sum_{n,m=0}^{N-1} C_{ij,nm}^q \varphi_{nm}^q,$$

where

$$C_{ij,nm}^q = \sum_{n,m=0}^{N-1} H(x_{ij}^q, \xi_u^q(s), \xi_v^q(t)) J^q(\xi_u^q(s), \xi_v^q(t)) \times a_{nm}(\xi_u^q(s), \xi_v^q(t)) \xi_u^q(\tilde{t}_1) \xi_v^q(\tilde{t}_2) \tilde{w}_1 \tilde{w}_2.$$
6 Numerical experiments

In this section, several numerical examples, involving three bounded obstacles depicted in Fig. 3, are presented to demonstrate the accuracy and efficiency of the proposed methods for solving three-dimensional poroelastic problems. Utilizing the dimensionless technique discussed in [18,40], we set \( \mu = 2, \nu_p = 0.2, \nu_u = 0.33, B = 0.62, C = 0.66, \phi = 0.333, \rho_s = 1, \rho_f = 0.5, \kappa = 1 \). We additionally choose \( \eta = 1 \). Here, we use the fully complex version of the iterative solver GMRES to produce the solutions of the integral equations and the maximum errors defined by

\[
\epsilon_\infty := \max_{x \in \Omega_c} \frac{|U_e^{num}(x) - U^{exa}(x)|}{\max_{x \in \Omega_c} |U^{exa}(x)|},
\]

will be displayed. Here, \( S \) is the square \([-2,2] \times [-2,2] \times \{2\} \subset \Omega_c \), \( U^{exa} \) is the exact solution of the poroelastic problem (2.1), and \( U^{num} \) is the numerical solution generated from the DCBIE (3.14), ICBIE (3.16), DRBIE (5.2) or IRBIE (5.4), respectively. The particular implementation for the numerical experiments is programmed in Fortran and is parallelized using OpenMP.

We first test the accuracy of the proposed methods. The exact solution \( U^{exa} = (u^{exa}, p^{exa})^T \) is given by

\[
u^{exa}(x) = E_{21}(x, z), \quad p^{exa}(x) = E_{22}(x, z), \quad x \in \Omega_c,
\]

with \( z = (0,0.5,0.3) \) for the obstacle Fig. 3(a) and \( z = (0,0,0) \) for the obstacles Fig. 3(b,c), which gives the boundary data \( F = \hat{T}(\partial, \nu)U^{exa} \) on \( \Gamma \). We first consider \( \omega = \pi \) and the Chebyshev grid with \( M = 6, N = 16 \) and \( N_3 = 100 \). Fig. 4 shows the errors \( |U^{num}(x) - U^{exa}(x)| \) between the numerical solution \( U^{num} \) resulting from solving DRBIE (5.2) and the exact solution \( U^{exa} \) for the obstacle Fig. 3(a) and \( x \in \{x \in \mathbb{R}^3 : |x| = 2\} \) with a maximum value \( 1.814 \times 10^{-5} \). Next, we consider the poroelastic problem of scattering by obstacle Fig. 3(b) on a basis of six \( 2 \times 2 \) patches \( (M = 24) \) with \( \omega = 20, N = 16, N_3 = 200 \) and employ IRBIE (5.4), the point-wise values of the numerical and exact solutions on the line segment \( \{x \in \mathbb{R}^3 : x_1 = 2, x_2 \in [-2,2], x_3 = 2\} \) are displayed in Fig. 5. The relative error for this case is \( \epsilon_\infty = 6.67 \times 10^{-6} \). In Fig. 6, the numerical errors \( \epsilon_\infty \) with respect to \( N \) using DRBIE and IRBIE for the obstacles Fig. 3 are presented while choosing \( \omega = 2\pi \) and Chebyshev grid with \( M = 6 \). Higher accuracy can be achieved by increasing the parameter \( N^3 \) and treating the evaluation of weakly-singular kernels for small \( |x-y| \) with cares.

Next, we verify the efficiency of the regularized integral equation methods. Choosing \( \omega = 2\pi \) and the Chebyshev grid with \( M = 6, N = 32 \) and \( N_3 = 200 \), Fig. 7 displays the history of GMRES residuals as functions of the number of iterations for the method of using DCBIE (3.14), ICBIE (3.16), DRBIE (5.2) and IRBIE (5.4), respectively. The rapid convergence results of regularized methods demonstrate that use of the regularized integral equations is highly beneficial compared to the un-regularized ones. With \( \omega = 20 \) and \( M = 24 \), Table 6 lists the precomputation time, time per iteration and number of iterations required by the regularized integral equation methods. For the DRBIE method, an accuracy of \( 7.6 \times 10^{-3} \) (resp. \( 2.5 \times 10^{-6} \)) can be achieved by setting \( N = 8 \) (resp. \( N = 16 \)), while an accuracy of \( 2.1 \times 10^{-2} \) (resp. \( 1.7 \times 10^{-6} \)) can be obtained for the IRBIE method.

![Figure 3: Obstacles considered in the numerical tests.](image)

(a) Ball  (b) Ellipsoid  (c) Bean
Finally, we consider the scattering of an incident point source $U^{inc}$ in the form

$$U^{inc} = (u^{inc}^\top, p^{inc}^\top)^\top, \quad u^{inc}(x) = E_{12}(x, z), \quad p^{inc}(x) = E_{22}(x, z)$$

by the obstacle Fig. 3(b) where $z = (3, 2, 0)$ denotes the location of the point source. The numerical solutions in $\Omega^c$ with $\omega = 20$ are presented in Figs. 8 and 9 based on the DRBIE and IRBIE, respectively. A total of 41 (resp. 32) iterations sufficed for the DRBIE (resp. IRBIE) method to reach the GMRES residual tolerance value $\epsilon_r = 1 \times 10^{-4}$. The numerical results demonstrate the accuracy and efficiency of the proposed regularized boundary integral equation methods.

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Appendix. Regularized expressions of the strongly-singular and hyper-singular operators and proofs.

This appendix presents the main approach for deriving the regularized formulations of the strongly-singular and hyper-singular operators. Analogous technique has been shown in [4] for the three-dimensional elastic and thermoelastic problems. But the derivations for three-dimensional poroelastic problem are more complex and thus, we present the full proof to make this appendix individually readable.

Given the Günter derivative operator

\[ M(\partial, \nu)u(x) = \partial_\nu u - \nu(\nabla \cdot u) + \nu \times \text{curl} u, \]

we can rewrite the traction operator \( T(\partial, \nu) \) as

\[ T(\partial, \nu)u(x) = (\lambda + \mu)\nu(\nabla \cdot u) + \mu \partial_\nu u + \mu M(\partial, \nu)u. \]

Then

\[
T(\partial, \nu)\nabla = (\lambda + \mu)\nu\Delta + \mu \partial_\nu \nabla + \mu M(\partial, \nu)\nabla \\
= (\lambda + \mu)\nu\Delta + \mu \partial_\nu \nabla - \mu M(\partial, \nu)\nabla + 2\mu M(\partial, \nu)\nabla,
\]

Figure 6: Numerical errors \( \epsilon_\infty \) for the problem of scattering by the obstacles Fig. 3.

Figure 7: GMRES residual \( \epsilon_r \) for the problem of scattering by the obstacles Fig. 3.
Table 2: Computation times and number of iterations required by the DRBIE method.

| $N$ | $N_\beta$ | $N_{DOF}$ | DRBIE | IRBIE |
|-----|------------|-----------|-------|-------|
|     |            |           | Time(prec.) | Time(liter.) | Niter($\epsilon_r$) | $\epsilon_\infty$ | Time(prec.) | Time(liter.) | Niter($\epsilon_r$) | $\epsilon_\infty$ |
| 8   | 100        | $4 \times 1536$ | 4.7 s | 11.7 s | 67 ($9.9 \times 10^{-6}$) | 9.1 $\times 10^{-3}$ | 3.6 s | 25.5 s | 31 ($8.5 \times 10^{-6}$) | 2.1 $\times 10^{-2}$ |
| 8   | 200        | $4 \times 1536$ | 18.3 s | 11.6 s | 49 ($9.8 \times 10^{-6}$) | 7.6 $\times 10^{-3}$ | 13.8 s | 24.7 s | 30 ($8.6 \times 10^{-6}$) | 2.1 $\times 10^{-2}$ |
| 16  | 100        | $4 \times 6144$ | 24.4 s | 2.18 min | 54 ($9.4 \times 10^{-6}$) | 5.9 $\times 10^{-6}$ | 18.1 s | 2.14 min | 22 ($9.6 \times 10^{-6}$) | 2.4 $\times 10^{-6}$ |
| 16  | 200        | $4 \times 6144$ | 1.46 min | 2.18 min | 33 ($9.8 \times 10^{-9}$) | 2.5 $\times 10^{-6}$ | 1.09 min | 2.65 min | 20 ($8.5 \times 10^{-9}$) | 1.7 $\times 10^{-6}$ |

Together with

$$\partial_x \nabla - M(\partial, \nu) \nabla = \nu \Delta,$$  \hspace{2cm} (A.2)

imply that

$$T(\partial, \nu) \nabla = (\lambda + 2\mu) \nu \Delta + 2\mu M(\partial, \nu) \nabla.$$  \hspace{2cm} (A.3)

Letting $M(\partial_x, \nu_x) = [m_{ij}^{ij}]_{i,j=1}^3$, it can be known that

$$m_{ij}^{ij} = \partial_x \nu_x^j - \partial_x \nu_x^i = -m_{ij}^{ji} \text{ for } i,j = 1, 2, 3.$$

Therefore, we have the following properties of the operator $M(\partial_x, \nu_x)$ [27]. For any scalar fields $p, q$, vector fields $u, v$ and tensor field $\Pi$, there hold the Stokes formulas

$$\int_{\Gamma} (m_{ij}^{ij}) q ds = -\int_{\Gamma} p m_{ij}^{ij} q ds,$$  \hspace{2cm} (A.4)

$$\int_{\Gamma} (Mu) \cdot v ds = \int_{\Gamma} u \cdot (Mv) ds,$$  \hspace{2cm} (A.5)

$$\int_{\Gamma} (Mq) v ds = -\int_{\Gamma} q (Mv) ds,$$  \hspace{2cm} (A.6)

$$\int_{\Gamma} (M\Pi)^T v ds = \int_{\Gamma} \Pi^T (Mv) ds.$$  \hspace{2cm} (A.7)

We first have the following result [4] Lemma 4.2. For $x \neq y$, it follows that

$$T(\partial_x, \nu_x) E_{11}(x, y)$$

$$= -\nu_x \nabla_x^T [\gamma_k(x, y) - \gamma_k(x, y)] + \frac{k_2^2 - q}{k_1^2 - k_2^2} \nu_x \nabla_x^T [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)]$$

$$+ \partial_x \gamma_{k_1}(x, y) I + M(\partial_x, \nu_x) [2\mu E_{11}(x, y) - \gamma_k I],$$  \hspace{2cm} (A.8)

and

$$T(\partial_y, \nu_y) E_{11}(x, y)$$

$$= -\nu_y \nabla_y^T [\gamma_k(x, y) - \gamma_k(x, y)] + \frac{k_2^2 - q}{k_1^2 - k_2^2} \nu_y \nabla_y^T [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)]$$

$$+ \partial_y \gamma_{k_1}(x, y) I + M(\partial_y, \nu_y) [2\mu E_{11}(x, y) - \gamma_k I].$$  \hspace{2cm} (A.9)

With these identities, the proofs of Theorems 4.1-4.6 can be established.
Proof. The operator $K$ can be written as

$$K(U)(x) = \int_\Gamma (\hat{T}^*(\partial_y, \nu_y)E(x, y))\top U(y)ds_y$$

$$= \begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}(x),$$

where

$$K_1(u)(x) = \int_\Gamma ((T(\partial_y, \nu_y)E_{11}(x, y))\top - \frac{\beta f \omega^2 \alpha}{\beta} E_{21}(x, y)\nu_y\top) u(y)ds_y,$$

$$K_2(p)(x) = \int_\Gamma (-\beta E_{11}(x, y)\nu_y + \partial_{\nu_y} E_{21}(x, y)) p(y)ds_y,$$

$$K_3(u)(x) = \int_\Gamma ((T(\partial_y, \nu_y)E_{12}(x, y))\top - \frac{\beta f \omega^2 \alpha}{\beta} E_{22}(x, y)\nu_y\top) u(y)ds_y,$$

$$K_4(p)(x) = \int_\Gamma (-\beta E_{12}(x, y)\nu_y + \partial_{\nu_y} E_{22}(x, y)) p(y)ds_y.$$

Using (A.7) and (A.9), we have

$$\int_\Gamma (T(\partial_y, \nu_y)E_{11}(x, y))\top u(y)ds_y$$

$$= -\int_\Gamma \nabla_y \left[ (\gamma_k(x, y) - \gamma_k(y, y)) - \frac{k_1^2 - q}{k_1^2 - k_2^2} (\gamma_k(x, y) - \gamma_k(y, y)) \right] \nu_y\top u(y)ds_y$$

$$+ \int_\Gamma \partial_{\nu_y} \gamma_k(x, y) I u(y)ds_y + \int_\Gamma [2\mu E_{11}(x, y) - \gamma_k(x, y) I] M(\partial_y, \nu_y)u(y)ds_y.$$  (A.10)
Then we can obtain that

\[
K_1(u)(x) = -\int_{\Gamma} \nabla_y \left[ (\gamma_{k_z}(x, y) - \gamma_{k_1}(x, y)) - \frac{k_2^2 - q}{k_1^2 - k_2^2} (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \right] \nu_y^T u(y) ds_y \\
\quad + \int_{\Gamma} \left[ \partial_{x_y} \gamma_{k_z}(x, y) I - \frac{\rho f \omega^2 \alpha}{\beta} E_{21}(x, y) \nu_y^T \right] u(y) ds_y \\
\quad + \int_{\Gamma} \partial_{y_y} \gamma_{k_z}(x, y) I u(y) ds_y + \int_{\Gamma} [2\mu E_{11}(x, y) - \gamma_{k_z}(x, y) I] M(\partial_y, \nu_y) u(y) ds_y.
\]
It follows from (A.2) and (A.6) that

\[ \int_\Gamma \partial_{\nu_y} E_{21}(x, y) \, p(y) \, ds_y \]

\[ = \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma \partial_{\nu_y} \nabla_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \, p(y) \, ds_y \]

\[ = \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma \partial_{\nu_y} \nabla_y (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \, p(y) \, ds_y \]

\[ = \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma (M(\partial_y, \nu_y) \nabla_y + \nu_y \Delta_y) (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \, p(y) \, ds_y \]

\[ = \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) \nu_y \, p(y) \, ds_y \]

\[ = \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma M(\partial_y, \nu_y) \nabla_y (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \, ds_y \]

\[ = \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) \nu_y \, p(y) \, ds_y \]

\[ + \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \left\{ \int_\Gamma \nabla_y^\top (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) M(\partial_y, \nu_y) \, p(y) \, ds_y \right\}^\top, \quad (A.11) \]

which implies that

\[ K_2(p)(x) \]

\[ = - \int_\Gamma \left[ \frac{\alpha - \beta}{(k_1^2 - k_2^2)(\lambda + 2\mu)} (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) + \beta E_{11}(x, y) \right] \nu_y \, p(y) \, ds_y \]

\[ + \frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \left\{ \int_\Gamma \nabla_y^\top (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) M(\partial_y, \nu_y) \, p(y) \, ds_y \right\}^\top. \]

From (A.3) and (A.5), we can obtain that

\[ \int_\Gamma (T(\partial_y, \nu_y) E_{12}(x, y))^\top u(y) \, ds_y \]

\[ = \frac{i \omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma (T(\partial_y, \nu_y) \nabla_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)))^\top u(y) \, ds_y \]

\[ = - \frac{i \omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma (T(\partial_y, \nu_y) \nabla_y (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)))^\top u(y) \, ds_y \]

\[ = - \frac{i \omega \gamma}{k_1^2 - k_2^2} \int_\Gamma \Delta_y (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \nu_y^\top u(y) \, ds_y \]

\[ - \frac{2iY \omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma (M(\partial_y, \nu_y) \nabla_y (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)))^\top u(y) \, ds_y \]

\[ = \frac{i \omega \gamma}{k_1^2 - k_2^2} \int_\Gamma (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) \nu_y^\top u(y) \, ds_y \]

\[ - \frac{2iY \omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma \nabla_y^\top (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) M(\partial_y, \nu_y) \, u(y) \, ds_y, \quad (A.12) \]

which yields

\[ K_3(u)(x) \]

\[ = \int_\Gamma \left[ \frac{i \omega \gamma}{(k_1^2 - k_2^2)} (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) - \frac{\beta_0 \omega^2 \alpha}{\beta} E_{22} \right] \nu_y^\top u(y) \, ds_y \]

\[ - \frac{2iY \omega \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma \nabla_y^\top (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) M(\partial_y, \nu_y) \, u(y) \, ds_y. \]
Theorem 4.1.

From (A.8), it can be obtained that

\[ K'(U)(x) = \begin{bmatrix} K'_1 & K'_2 \\ K'_3 & K'_4 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}(x), \quad x \in \Gamma, \]

where the operators \( K'_j \), \( j = 1, \cdots, 4 \) are denoted as

\[ K'_1(u)(x) = \int_{\Gamma} (T(\partial_x, \nu_x)E_{11} - \alpha \nu_x E_{12}^T) u(y) ds_y, \]
\[ K'_2(p)(x) = \int_{\Gamma} (T(\partial_x, \nu_x)E_{21} - \alpha \nu_x E_{22}) p(y) ds_y, \]
\[ K'_3(u)(x) = \int_{\Gamma} (-\rho_f \omega^2 \nu_x^T E_{11} + \partial_x \nu_x^T) u(y) ds_y, \]
\[ K'_4(p)(x) = \int_{\Gamma} (-\rho_f \omega^2 \nu_x^T E_{21} + \partial_x \nu_x) p(y) ds_y. \]

From (A.8), it can be obtained that

\[ K'_1(u)(x) = -\int_{\Gamma} \nu_x \nabla_x^T \left[ (\gamma_{k_1}(x, y) - \gamma_{k_1}(x, y)) - \frac{k_2^2 - q}{k_1^2 - k_2^2} (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \right] u(y) ds_y \]
\[ \quad + \int_{\Gamma} \left[ \partial_x \gamma_{k_1}(x, y) I - \alpha \nu_x E_{12}^T(x, y) + M(\partial_x, \nu_x)(2\mu E_{11}(x, y) - \gamma_{k_2}(x, y) I) \right] u(y) ds_y. \]

It follows from (A.3) that

\[ \int_{\Gamma} T(\partial_x, \nu_x)E_{21}(x, y)p(y) ds_y \]
\[ = -\frac{\alpha - \beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_{\Gamma} T(\partial_x, \nu_x) \nabla_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y \]
\[ = -\frac{\alpha - \beta}{(k_1^2 - k_2^2)} \int_{\Gamma} \nu_x \Delta_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y \]
\[ - \frac{2\mu(\alpha - \beta)}{\lambda + 2\mu(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, \nu_x) \nabla_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y \]
\[ = \frac{\alpha - \beta}{(k_1^2 - k_2^2)} \int_{\Gamma} (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) \nu_x p(y) ds_y \]
\[ - \frac{2\mu(\alpha - \beta)}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, \nu_x) \nabla_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y. \]

Therefore,

\[ K'_2(p)(x) \]
\[ = \int_{\Gamma} \left( \frac{\alpha - \beta}{k_1^2 - k_2^2} (k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)) - \alpha E_{22}(x, y) \right) \nu_x p(y) ds_y \]
\[ - \frac{2\mu(\alpha - \beta)}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, \nu_x) \nabla_x (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y. \]
On the other hand, we obtain from (A.2) that
\[ \int_{\Gamma} \partial_{\nu_{x}} E_{12}^T(x,y)u(y)ds_{y} \]
\[ = \frac{i\omega_{\gamma}}{(\lambda + 2\mu)(k_{1}^2 - k_{2}^2)} \int_{\Gamma} \partial_{\nu_{x}} \nabla_{x}^T (\gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y)) u(y)ds_{y} \]
\[ = \frac{i\omega_{\gamma}}{(\lambda + 2\mu)(k_{1}^2 - k_{2}^2)} \int_{\Gamma} \nu_{x}^T \Delta_{x} (\gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y)) u(y)ds_{y} \]
\[ + \frac{i\omega_{\gamma}}{(\lambda + 2\mu)(k_{1}^2 - k_{2}^2)} \int_{\Gamma} \{ M(\partial_{x}, \nu_{x}) \nabla_{x} (\gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y)) \}^T u(y)ds_{y} \]
\[ = -\frac{i\omega_{\gamma}}{(\lambda + 2\mu)(k_{1}^2 - k_{2}^2)} \int_{\Gamma} \{ k_{1}^2 \gamma_{k_{1}}(x,y) - k_{2}^2 \gamma_{k_{2}}(x,y) \} \nu_{x}^T u(y)ds_{y} \]
\[ + \frac{i\omega_{\gamma}}{(\lambda + 2\mu)(k_{1}^2 - k_{2}^2)} \int_{\Gamma} \{ M(\partial_{x}, \nu_{x}) \nabla_{x} (\gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y)) \}^T u(y)ds_{y}. \] (A.14)

Letting \( R_{1}(x,y) = (\gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y)) \), note that
\[ \{ M(\partial_{x}, \nu_{x}) \nabla_{x} R_{1}(x,y) \}^T u(y) \]
\[ = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} \partial_{x_{1}} R_{1}(x,y) \\ \partial_{x_{2}} R_{1}(x,y) \\ \partial_{x_{3}} R_{1}(x,y) \end{pmatrix}^T \begin{pmatrix} u_{1}(y) \\ u_{2}(y) \\ u_{3}(y) \end{pmatrix} \]
\[ = \sum_{i,j=1}^{3} m_{ij} \partial_{x_{j}} \left( \gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y) \right) u_{i}(y) \]
\[ = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}: \begin{pmatrix} \partial_{x_{1}} R_{1}(x,y)u_{1}(y) \\ \partial_{x_{2}} R_{1}(x,y)u_{2}(y) \\ \partial_{x_{3}} R_{1}(x,y)u_{3}(y) \end{pmatrix} \]
\[ = M(\partial_{x}, \nu_{x}): \begin{pmatrix} u(y) \nabla_{x}^T R_{1}(x,y) \end{pmatrix}. \]

Then we have
\[ K'_{3}(u)(x) \]
\[ = -\int_{\Gamma} \left[ \rho_{J} \omega^{2} \nu_{x}^T E_{11}(x,y) + \frac{i\omega_{\gamma}}{(\lambda + 2\mu)(k_{1}^2 - k_{2}^2)}(k_{1}^2 \gamma_{k_{1}}(x,y) - k_{2}^2 \gamma_{k_{2}}(x,y)) \nu_{x}^T \right] u(y)ds_{y} \]
\[ + \frac{i\omega_{\gamma}}{(k_{1}^2 - k_{2}^2)(\lambda + 2\mu)} M(\partial_{x}, \nu_{x}): \int_{\Gamma} u(y) \nabla_{x}^T \left[ \gamma_{k_{1}}(x,y) - \gamma_{k_{2}}(x,y) \right] ds_{y}. \]

Using the definition, the formula for \( K'_{3} \) can be obtained directly. Then the Theorem (4.2) can be proved.

Now we investigate the hyper-singular operator \( N \). Note that the hyper-singular operator of \( N \) can be written as
\[ N(\psi)(x) = \begin{bmatrix} N_{1} & N_{2} \\ N_{3} & N_{4} \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix}(x), \quad x \in \Gamma, \]
Thus, we can obtain from (A.2) and (A.3) that

\[ N_1(u)(x) = \int_{\Gamma} \left[ T(\partial_x, \nu_x)(T(\partial_y, \nu_y)E_{11}(x,y))^\top - \frac{\rho f \omega^2 \alpha}{\beta} T(\partial_x, \nu_x)E_{21}(x,y) \nu_y^\top \right] u(y) ds_y \]

\[ - \int_{\Gamma} \left[ \alpha \nu_x(T(\partial_y, \nu_y)E_{12}(x,y))^\top - \frac{\rho f \omega^2 \alpha^2}{\beta} E_{22}(x,y) \nu_y^\top \right] u(y) ds_y. \]

\[ N_2(p)(x) = - \int_{\Gamma} [\beta T(\partial_x, \nu_x)E_{11}(x,y) \nu_y - \partial_x T(\partial_y, \nu_x)E_{21}(x,y)] p(y) ds_y \]

\[ + \int_{\Gamma} \left[ \alpha \beta \nu_x E_{11}^\top \nu_y - \alpha \partial_x \nu_y E_{22}(x,y) \nu_y \right] p(y) ds_y. \]

\[ N_3(u)(x) = - \int_{\Gamma} \left[ \rho f \omega^2 \nu_x^\top (T(\partial_y, \nu_y)E_{11}(x,y))^\top - \frac{\rho f \omega^2 \alpha}{\beta} \nu_x^\top E_{21} \nu_y^\top \right] u(y) ds_y \]

\[ + \int_{\Gamma} \left[ \partial_x (T(\partial_y, \nu_y)E_{12}(x,y))^\top - \frac{\rho f \omega^2 \alpha}{\beta} \partial_x E_{22}(x,y) \nu_y^\top \right] u(y) ds_y. \]

\[ N_4(p)(x) = \int_{\Gamma} \left[ \rho f \omega^2 \beta \nu_x^\top E_{11}(x,y) \nu_y - \rho f \omega^2 \nu_x^\top \partial_x \nu_y E_{21}(x,y) \right] p(y) ds_y \]

\[ - \int_{\Gamma} \left[ \beta \partial_x E_{12}^\top (x,y) \nu_y - \partial_x (\partial_x \nu_y E_{22}(x,y)) \right] p(y) ds_y. \]

Considering the following term

\[ T(\partial_x, \nu_x) \int_{\Gamma} (T(\partial_y, \nu_y)E_{11}(x,y))^\top u(y) ds_y, \]

we first set that

\[ f_1(x) = \int_{\Gamma} \nabla_y (\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)) \nu_y^\top u(y) ds_y, \]

\[ f_2(x) = \int_{\Gamma} \nabla_y (\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)) \nu_\nu^\top u(y) ds_y, \]

\[ f_3(x) = \int_{\Gamma} \partial_x (\gamma_{k_1}(x,y)) u(y) ds_y, \]

\[ f_4(x) = \int_{\Gamma} (2\mu E_{11}(x,y) - \gamma_{k_1}(x,y) I) \partial_x \nu_y u(y) ds_y, \]

and

\[ g_i(x) = \mu \partial_x f_i(x) + (\lambda + \mu) \nu_x \nabla_x \cdot f_i(x) + \mu M(\partial_x, \nu_x) f_i(x). \]

Thus, we can obtain from (A.2) and (A.3) that

\[ g_1(x) = (\lambda + 2\mu) \int_{\Gamma} \left[ k_1^2 \gamma_{k_1}(x,y) - k_2^2 \gamma_{k_2}(x,y) \right] \nu_x \nu_y^\top u(y) ds_y \]

\[ + 2\mu \int_{\Gamma} M(\partial_x, \nu_x) \nabla_y (\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)) \nu_y^\top u(y) ds_y, \]

(A.15)

and

\[ g_2(x) = (\lambda + 2\mu) \int_{\Gamma} \left[ k_1^2 \gamma_{k_1}(x,y) - k_2^2 \gamma_{k_2}(x,y) \right] \nu_x \nu_y^\top u(y) ds_y \]

\[ + 2\mu \int_{\Gamma} M(\partial_x, \nu_x) \nabla_y (\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)) \nu_y^\top u(y) ds_y. \]

(A.16)
Relying on the results of the Helmholtz equation, we have

\[
\int_\Gamma \frac{\partial_{\nu_x}}{\partial_{\nu_y}} \gamma_{k_s}(x, y) u(y) ds_y = \int_\Gamma (\nu_x \times \nabla_{x} \gamma_{k_s}(x, y)) \cdot (\nu_y \times \nabla_{y} u(x)) ds_y + k^2_s \int_\Gamma \gamma_{k_s}(x, y) \nu_{x}^T \nu_{y} u(y) ds_y \quad (A.17)
\]

Therefore, \( g_3(x) \) can be expressed as

\[
g_3(x) = \mu \int_\Gamma \frac{\partial_{\nu_x}}{\partial_{\nu_y}} \gamma_{k_s}(x, y) u(y) ds_y + (\lambda + \mu) \int_\Gamma \nu_{x}^T \frac{\partial_{\nu_x}}{\partial_{\nu_y}} \gamma_{k_s}(x, y) u(y) ds_y + \mu \int_\Gamma M(\partial_{x}, \nu_{x}) \frac{\partial_{\nu_x}}{\partial_{\nu_y}} \gamma_{k_s}(x, y) u(y) ds_y \]

\[
= \mu \int_\Gamma (\nu_x \times \nabla_{x} \gamma_{k_s}(x, y)) \cdot (\nu_y \times \nabla_{y} u(y)) ds_y + \mu k^2_s \int_\Gamma \gamma_{k_s}(x, y) \nu_{x}^T \nu_{y} u(y) ds_y + (\lambda + \mu) \int_\Gamma \nu_{x}^T \frac{\partial_{\nu_x}}{\partial_{\nu_y}} \gamma_{k_s}(x, y) u(y) ds_y + \mu \int_\Gamma M(\partial_{x}, \nu_{x}) \frac{\partial_{\nu_x}}{\partial_{\nu_y}} \gamma_{k_s}(x, y) u(y) ds_y. \quad (A.18)
\]

For \( g_4(x) \), we know from (A.8) that

\[
g_4(x) = \mu \int_\Gamma \nu_{x}^T \nabla_{x} \gamma_{k_s}(x, y) M(\partial_{y}, \nu_{y}) ds_y
\]

\[
-2\mu \int_\Gamma \nu_{x}^T [\gamma_{k_1}(x, y) - \gamma_{k_s}(x, y)] M(\partial_{y}, \nu_{y}) u(y) ds_y
\]

\[
+ \frac{2\mu(k^2_s - q)}{k^2_1 - k^2_2} \int_\Gamma \nu_{x}^T [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] M(\partial_{y}, \nu_{y}) u(y) ds_y
\]

\[
+ 4\mu^2 \int_\Gamma M(\partial_{x}, \nu_{x}) E_{11}(x, y) M(\partial_{y}, \nu_{y}) u(y) ds_y
\]

\[
- 3\mu \int_\Gamma M(\partial_{x}, \nu_{x}) \gamma_{k_s}(x, y) M(\partial_{y}, \nu_{y}) u(y) ds_y
\]

\[
- (\lambda + \mu) \int_\Gamma \nu_{x}^T \gamma_{k_s}(x, y) M(\partial_{y}, \nu_{y}) u(y) ds_y. \quad (A.19)
\]
Therefore, \( A.15 \) - \( A.19 \) yields

\[
T(\partial_x, \nu_x) \int_\Gamma (T(\partial_y, \nu_y) E_{11}(x, y))^\top u(y) ds_y
= -g_1(x) + \frac{k_2^2 - q}{k_1^2 - k_2^2} g_2(x) + g_3(x) + g_4(x)
\]

\[
= -(\rho - \beta \rho_f)\omega^2 \int_\Gamma \gamma_k(x, y) (\nu_x \nu_{y}^\top - \nu_{x}^\top \nu_y I) u(y) ds_y
+ \int_\Gamma \left[ D_1 \gamma_{k_1}(x, y) \right] \nu_x \nu_{y}^\top u(y) ds_y
+ \mu \int_\Gamma (\nu_x \times \nabla_x \gamma_{k_1}(x, y)) \cdot (\nu_y \times \nabla_y u(y)) ds_y
+ 4\mu^2 \int_\Gamma M(\partial_x, \nu_x) E_{11}(x, y) M(\partial_y, \nu_y) u(y) ds_y
- 2\mu \int_\Gamma \nu_x \nabla_x^\top [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] M(\partial_y, \nu_y) u(y) ds_y
- 2\mu \int_\Gamma M(\partial_x, \nu_x) \nabla_y [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] \nu_{y}^\top u(y) ds_y
+ \frac{2\mu(k_2^2 - q)}{k_1^2 - k_2^2} \int_\Gamma \nu_x \nabla_x^\top [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] M(\partial_y, \nu_y) u(y) ds_y
\]

\[
+ \frac{2\mu(k_2^2 - q)}{k_1^2 - k_2^2} \int_\Gamma M(\partial_x, \nu_x) \nabla_y [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] \nu_{y}^\top u(y) ds_y
+ \mu \int_\Gamma M(\partial_x, \nu_x) \partial_{\nu_y} \gamma_{k_2}(x, y) u(y) ds_y + \mu \int_\Gamma \partial_{\nu_y} \gamma_{k_2}(x, y) M(\partial_y, \nu_y) ds_y,
\]

with

\[
D_1 = \frac{k_2^2(\lambda + 2\mu)(k_2^2 - q)}{k_1^2 - k_2^2}, \quad D_2 = \frac{k_2^2(\lambda + 2\mu)(k_2^2 - q)}{k_1^2 - k_2^2}.
\]

On the other hand, we obtain from \( A.3 \) that

\[
\int_\Gamma T(\partial_x, \nu_x) E_{21}(x, y) \nu_{y}^\top u(y) ds_y
= \frac{\alpha - \beta}{k_1^2 - k_2^2} \int_\Gamma [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] \nu_x \nu_{y}^\top u(y) ds_y
+ \frac{2\mu(\alpha - \beta)}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma M(\partial_x, \nu_x) \nabla_y [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] \nu_{y}^\top u(y) ds_y,
\]

and

\[
\int_\Gamma \nu_x (T(\partial_y, \nu_y) E_{12}(x, y))^\top u(y) ds_y
= \frac{i\omega \gamma}{k_1^2 - k_2^2} \int_\Gamma [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] \nu_x \nu_{y}^\top u(y) ds_y
+ \frac{2i\omega \mu \gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int_\Gamma \nu_x \nabla_x^\top (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) M(\partial_y, \nu_y) ds_y.
\]
Combining (A.20)–(A.22), we have
\[
N_1(u)(x) = -(\rho - \beta \rho_f) \omega^2 \int_{\Gamma} \gamma_{k_1}(x, y) (v_x v_y^\top - v_x^\top v_y) u(y) ds_y
\]
\[
+ \int_{\Gamma} [C_1 \gamma_{k_1}(x, y) - C_2 \gamma_{k_2}(x, y)] v_x v_y^\top u(y) ds_y,
\]
\[
+ \int_{\Gamma} M(\partial_x, v_x) \left[4 \mu^2 E_{11}(x, y) - 3 \mu \gamma_{k_2}(x, y) I\right] M(\partial_y, v_y) u(y) ds_y
\]
\[
+ \mu \int_{\Gamma} \gamma_{k_1}(x, y) \tau_1 u(y) ds_y
\]
\[
+ \int_{\Gamma} M(\partial_x, v_x) \nabla_y \left[-2 \mu (\gamma_{k_1}(x, y) - \gamma_{k_1}(x, y)) + C_3 (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y))\right] v_y^\top u(y) ds_y
\]
\[
+ \mu \int_{\Gamma} M(\partial_x, v_x) \partial_{v_x} \gamma_{k_1}(x, y) u(y) ds_y
\]
\[
+ \int_{\Gamma} v_x \nabla_y^\top \left[-2 \mu (\gamma_{k_1}(x, y) - \gamma_{k_1}(x, y)) + C_4 (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y))\right] M(\partial_y, v_y) u(y) ds_y
\]
\[
+ \mu \int_{\Gamma} \partial_{v_x} \gamma_{k_1}(x, y) M(\partial_y, v_y) u(y) ds_y,
\]
For $N_2$, it follows from (A.2) and (A.3) that

\[
\int_{\Gamma} \partial_{v_y} T(\partial_x, v_x) E_{21}(x, y) p(y) ds_y
\]
\[
= - \frac{\alpha - \beta}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int_{\Gamma} \partial_{v_y} T(\partial_x, v_x) \nabla_x \left[\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)\right] p(y) ds_y
\]
\[
= \frac{\alpha - \beta}{k_1^2 - k_2^2} \int_{\Gamma} \partial_{v_y} [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] v_x p(y) ds_y
\]
\[
+ \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, v_x) \partial_{v_y} \nabla_y \left[\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)\right] p(y) ds_y
\]
\[
= \frac{\alpha - \beta}{k_1^2 - k_2^2} \int_{\Gamma} \partial_{v_y} [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] v_x p(y) ds_y
\]
\[
- \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, v_x) \partial_{v_y} \left[\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)\right] v_y p(y) ds_y
\]
\[
+ \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, v_x) M(\partial_y, v_y) \nabla_y \left[\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)\right] p(y) ds_y
\]
\[
= \frac{\alpha - \beta}{k_1^2 - k_2^2} \int_{\Gamma} \partial_{v_y} [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] v_x p(y) ds_y
\]
\[
- \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, v_x) \partial_{v_y} \left[\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)\right] v_y p(y) ds_y
\]
\[
+ \frac{2 \mu (\alpha - \beta)}{(\lambda + 2 \mu)(k_1^2 - k_2^2)} \int_{\Gamma} M(\partial_x, v_x) \left\{ \nabla_y^\top \left[\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)\right] M(\partial_y, v_y) p(y) \right\}^\top ds_y. \]
which, in corporation with (A.8), yields that

\[
N_2(p(x) = \beta \int \nu_x \nabla_x^T (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) - \partial_{\nu_x} \gamma_{k_x}(x, y) I \nu_y p(y) dy \\
+ \frac{i \omega \gamma \alpha \beta}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int \nu_x \nabla_x^T (\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)) \nu_y p(y) dy \\
- \frac{\alpha - \beta}{k_1^2 - k_2^2} \int \partial_{\nu_x} [((k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_2^2 - k_1^2) \gamma_{k_2}(x, y))] \nu_x p(y) dy \\
+ \frac{\alpha}{k_1^2 - k_2^2} \int \partial_{\nu_x} [((k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_2^2 - k_1^2) \gamma_{k_2}(x, y))] \nu_x p(y) dy \\
+ \frac{2\mu (\alpha - \beta)}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int \{ \nabla_y^T [\gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] M(\partial_y, \nu_y) p(y) \}^T dy \\
- \frac{\beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int M(\partial_x, \nu_x) [((k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_2^2 - k_1^2) \gamma_{k_2}(x, y)) \nu_y p(y) dy \\
- \frac{\beta}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int M(\partial_x, \nu_x) [((k_1^2 - k_2^2) \gamma_{k_1}(x, y) - (k_2^2 - k_1^2) \gamma_{k_2}(x, y))] \nu_y p(y) dy.
\]

For \(N_3\), we mainly need to consider the following term

\[
\int \partial_{\nu_x} \{ T(\partial_y, \nu_y) E_{12}(x, y) \}^T u(y) dy \\
= \frac{i \omega \gamma}{k_1^2 - k_2^2} \int \partial_{\nu_x} [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] \nu_y^T u(y) dy \\
+ \frac{(2i \mu \omega \gamma)}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int [k_1^2 \gamma_{k_1}(x, y) - k_2^2 \gamma_{k_2}(x, y)] \nu_x^T M(\partial_y, \nu_y) u(y) dy \\
+ \frac{(2i \mu \omega \gamma)}{(k_1^2 - k_2^2)(\lambda + 2\mu)} \int [\nabla_x \gamma_{k_1}(x, y) - \gamma_{k_2}(x, y)] M(\partial_y, \nu_y) u(y) dy.
\]
By a combination of (A.10) and (A.12), we can obtain that

\[ N_3(u)(x) = -\rho j\omega^2 \int \Gamma \left[ \partial_{x_u} (\gamma_k(x, y) - \gamma_{k_2}(x, y)) \nu_y^T + \partial_{x_u} \gamma_k(x, y) \nu_x^T \right] u(y) ds_y \]

\[ + \left( \frac{\rho j\omega^2(k_2^2 - q)}{k_1^2 - k_2^2} - \frac{\rho j\omega^2(\alpha - \beta)}{\beta(\lambda + 2\mu)(k_1^2 - k_2^2)} \right) \int \Gamma \partial_{x_u} \left[ \gamma_k(x, y) - \gamma_{k_2}(x, y) \right] \nu_y^T u(y) ds_y \]

\[ + \frac{i\omega\gamma}{k_1^2 - k_2^2} \int \Gamma \partial_{x_u} \left[ k_1^2 \gamma_k(x, y) - k_2^2 \gamma_{k_2}(x, y) \right] \nu_y^T u(y) ds_y \]

\[ + \frac{\rho f j\omega^2}{\beta(k_1^2 - k_2^2)} \int \Gamma \partial_{x_u} \left[ (k_2^2 - k_2^2) \gamma_k(x, y) - (k_2^2 - k_2^2) \gamma_{k_2}(x, y) \right] \nu_y^T u(y) ds_y \]

\[ - \rho j\omega^2 \int \Gamma \nu_x^T \left[ 2\mu E_{11}(x, y) - \gamma_k(x, y) \right] u(y) ds_y \]

\[ - \frac{2i\mu\omega\gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int \Gamma \left[ k_1^2 \gamma_k(x, y) - k_2^2 \gamma_{k_2}(x, y) \right] \nu_x^T u(y) ds_y \]

\[ + \frac{2i\mu\omega\gamma}{(k_1^2 - k_2^2)(\lambda + 2\mu)} M(\partial_{x_u}, \nu_u) : \int \Gamma M(\partial_{x_y}, \nu_y) u(y) \nabla_x^T [\gamma_k(x, y) - \gamma_{k_2}(x, y)] ds_y. \]

Using (A.11), we have

\[ -\rho j\omega^2 \int \Gamma \nu_x^T \partial_{x_y} E_{21}(x, y) p(y) ds_y \]

\[ = \frac{\rho f j\omega^2(\alpha - \beta)}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int \Gamma \partial_{x_u} \partial_{x_y} (\gamma_k(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y \]

\[ + \frac{\rho f j\omega^2(\alpha - \beta)}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int \Gamma \left[ k_1^2 \gamma_k(x, y) - k_2^2 \gamma_{k_2}(x, y) \right] \nu_x^T \nu_y p(y) ds_y \]

Due to (A.14), we can obtain that

\[ -\beta \int \partial_{x_u} E_{11}^T \nu_y p(y) ds_y \]

\[ = \frac{i\omega\beta\gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int \Gamma \partial_{x_u} \partial_{x_y} (\gamma_k(x, y) - \gamma_{k_2}(x, y)) p(y) ds_y \]

\[ + \frac{i\omega\beta\gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)} \int \Gamma \left[ k_1^2 \gamma_k(x, y) - k_2^2 \gamma_{k_2}(x, y) \right] \nu_x^T \nu_y p(y) ds_y \]
Following the result (A.17), we have

\[
\int_\Gamma \partial_{\nu_x} \partial_{\nu_y} E_{22}(x,y)p(y)ds_y = - \frac{1}{k_1^2 - k_2^2} \int_\Gamma \partial_{\nu_x} \partial_{\nu_y} \left[ (k_p^2 - k_1^2) \gamma_{k_1}(x,y) - (k_1^2 - k_2^2) \gamma_{k_2}(x,y) \right] p(y)ds_y
\]

\[
= - \frac{1}{k_1^2 - k_2^2} \int_\Gamma \left[ (k_p^2 - k_1^2)k_1^2 \gamma_{k_1}(x,y) - (k_1^2 - k_2^2)k_2^2 \gamma_{k_2}(x,y) \right] \nu_x^\top \nu_y p(y)ds_y
\]

\[
= - \frac{1}{k_1^2 - k_2^2} \int_\Gamma \left[ (k_p^2 - k_1^2)k_1^2 \gamma_{k_1}(x,y) - (k_1^2 - k_2^2)k_2^2 \gamma_{k_2}(x,y) \right] \nu_x^\top \nu_y p(y)ds_y
\]

\[
- \frac{1}{k_1^2 - k_2^2} \int_\Gamma \tau_2 \left[ (k_p^2 - k_1^2) \gamma_{k_1}(x,y) - (k_1^2 - k_2^2) \gamma_{k_2}(x,y) \right] \tau_1 p(y)ds_y.
\]

Hence,

\[
N_4(p)(x) = \rho f \omega^2 \beta \int_\Gamma \nu_x^\top E_{11}(x,y) \nu_y p(y)ds_y
\]

\[
+ i \omega \gamma \beta + \rho f \omega^2 (\alpha - \beta) \frac{\lambda + 2\mu}{k_1^2 - k_2^2} \int_\Gamma \left[ k_1^2 \gamma_{k_1}(x,y) - k_2^2 \gamma_{k_2}(x,y) \right] \nu_x^\top \nu_y p(y)ds_y
\]

\[
- \frac{1}{k_1^2 - k_2^2} \int_\Gamma \left[ (k_p^2 - k_1^2)k_1^2 \gamma_{k_1}(x,y) - (k_1^2 - k_2^2)k_2^2 \gamma_{k_2}(x,y) \right] \nu_x^\top \nu_y p(y)ds_y
\]

\[
+ \rho f \omega^2 (\alpha - \beta) + i \omega \beta \gamma \frac{\lambda + 2\mu}{k_1^2 - k_2^2} \int_\Gamma \tau_2 [\gamma_{k_1}(x,y) - \gamma_{k_2}(x,y)] \tau_1 p(y)ds_y
\]

\[
- \frac{1}{k_1^2 - k_2^2} \int_\Gamma \tau_2 \left[ (k_p^2 - k_1^2) \gamma_{k_1}(x,y) - (k_1^2 - k_2^2) \gamma_{k_2}(x,y) \right] \tau_1 p(y)ds_y.
\]

\[
\square
\]

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