The mean flow, velocity dispersion, energy transfer and evolution of rotating and growing dark matter halos

Zhijie (Jay) Xu,1★
1Physical and Computational Sciences Directorate, Pacific Northwest National Laboratory; Richland, WA 99352, USA

ABSTRACT

By decomposing velocity dispersion into non-spin and spin-induced, mean flow and dispersion are analytically solved for axisymmetric rotating and growing halos. The polar flow can be neglected and azimuthal flow is directly related to dispersion. The fictitious ("Reynolds") stress acts on mean flow to enable energy transfer from mean flow to random motion and maximize system entropy. For large halos (high peak height $v$ at early stage of halo life) with constant concentration, there exists a self-similar radial flow (outward in core and inward in outer region). Halo mass, size and specific angular momentum increase linearly with time via fast mass accretion. Halo core spins faster than outer region. Large halos rotate with an angular velocity proportional to Hubble parameter and spin-induced dispersion is dominant. All specific energies (radial/rotational/kinetic/potential) are time-invariant. Both halo spin ($\sim 0.031$) and anisotropic parameters can be analytically derived. For "small" halos with stable core and slow mass accretion (low peak height $v$ at late stage of halo life), radial flow vanishes. Small halos rotate with constant angular velocity and non-spin axial dispersion is dominant. Small halos are more spherical in shape, incompressible, and isotropic. Radial and azimuthal dispersion are comparable and greater than polar dispersion. Due to finite spin, kinetic energy is not equipartitioned with the greatest energy along azimuthal direction. Different from normal matter, small halos are hotter with faster spin. Halo relaxation (evolution) from early to late stage involves continuous variation of shape, density, mean flow, momentum, and energy. During relaxation, halo isotopically "stretches" with conserved specific rotational kinetic energy, increasing concentration and momentum of inertial. Halo "stretching" leads to decreasing angular velocity, increasing angular momentum and spin parameter.

Key words: Dark matter halo; N-body simulations; Theoretical models

1 INTRODUCTION

The large-scale structure formation and evolution can be rigorously studied based on the self-gravitating collisionless fluid dynamics (SG-CFD) that deals with the motion of collisionless dark matter under its own gravity. While SG-CFD and hydrodynamic turbulence are different in many aspects, both contain the same essential ingredients (randomness, nonlinearity, and multiscale nature) and share many similarities with each other.

Turbulence is ubiquitous in nature and might be the last unresolved problems in classical physics. More specifically, homogeneous isotropic incompressible turbulence has been well-studied for many decades and of important relevance to SG-CFD. The classical picture of turbulence is a eddy-mediated cascade process, where large eddies feed smaller eddies, which feed even smaller eddies, and so on to the smallest scale when viscous dissipation becomes dominant, i.e. a direct (kinetic) energy cascade (Richardson 1922). A key question for turbulence is "how the kinetic energy is transferred from the mean flow to turbulence, cascaded through scales, and destroyed by viscosity?" Or equivalently, how the turbulence initiates, propagates, and dies out.

The energy cascade in turbulence starts with the kinetic energy obtained from mean flow by the largest eddies through Reynolds stress (arising from velocity fluctuation) acting on the mean flow. This kinetic energy is further cascaded successively to smaller and smaller...
eddy dissipation rate proportional to the rate of mass transfer (Xu 2021f). The energy transfer between mean flow and random motion in halos is discussed in Section 5, along with the halo evolution from early to late stage in Section 6. A halo stretching mechanism (counterpart of vortex stretching) is proposed and studied extensively along with the energy and momentum evolution.

2 N-BODY SIMULATIONS AND NUMERICAL DATA
The numerical data for this work is publicly available and generated from the N-body simulations carried out by the Virgo consortium, an international collaboration that aims to perform large N-body simulations of the formation of large-scale structures. A comprehensive description of the simulation data can be found in (Frenk et al. 2000; Jenkins et al. 1998). The same set of simulation data has been widely used in a number of different studies from clustering statistics (Jenkins et al. 1998) to the formation of halo clusters in large scale environments (Colberg et al. 1999), and testing models for halo abundance and mass functions (Sheth et al. 2001). Some key parameters of N-body simulations are listed in Table 1.

Two relevant datasets from this N-body simulation, i.e. halo-based and correlation-based statistics of dark matter flow, can be found at Zenodo.org (Xu 2022a,b), along with the accompanying presentation slides, “A comparative study of dark matter flow & hydrodynamic turbulence and its applications” (Xu 2022c). All data files are also available on GitHub (Xu 2022d).

3 SOLUTIONS FOR ROTATING AND GROWING HALOS
3.1 Continuity and momentum equations and azimuthal flow
Jeans’ equation and solutions for spherical, stationary, and non-rotating halos can be found in many literature (Hoef et al. 2004; Binney & Tremaine 1987). Solutions for spherical, growing, and non-rotating halos were also studied, where the effect of nonzero radial flow on halo density is formulated (Xu 2021b). While vortex volume/mass conserved for incompressible flow, halos are much more complex and dynamic objects that are constantly growing, spinning, shape-changing, with a nonuniform density profile, and usually not volume- or mass-conserved. The purpose of this paper is to explore relevant solutions and evolution of rotating and growing halos and the role of halos in energy transfer and cascade in SG-CFD.

The rest of paper is organized as follows: Section 2 introduces the simulation and numerical data used for this work. Section 3 presents solutions for the mean flow and velocity dispersions of an axisymmetric rotating and growing halo (the building block of SG-CFD) at their early and late stage of life. The momentum and energy solutions of rotating and growing halos are presented in Section 4. The energy transfer between mean flow and random motion in halos is discussed in Section 5, along with the halo evolution from early to late stage in Section 6. A halo stretching mechanism (counterpart of vortex stretching) is proposed and studied extensively along with the energy and momentum evolution.

Table 1. Numerical parameters of N-body simulation

| Run | $Q_0$ | $\Lambda$ | $h$ | $\Gamma$ | $\sigma_8 (\text{Mpc}/h)$ | $L (\text{Mpc}/h)$ | $m_p / M_{\odot}/h$ | $L_{\text{soft}} (\text{Kpc}/h)$ |
|-----|-------|-----------|-----|---------|-----------------|-----------------|----------------|------------------|
| SCDM1 | 1.0 | 0.0 | 0.5 | 0.5 | 0.51 | 239.5 | 256 | $2.27 \times 10^{11}$ | 36 |

Tremaine 1987). Solutions for non-rotating growing halos with a nonzero radial flow were recently studied (Xu 2021b). While vortex volume/mass conserved for incompressible flow, halos are much more complex and dynamic objects that are constantly growing, spinning, shape-changing, with a nonuniform density profile, and usually not volume- or mass-conserved. The purpose of this paper is to explore relevant solutions and evolution of rotating and growing halos and the role of halos in energy transfer and cascade in SG-CFD.
the axisymmetry about axis of rotation, the mean azimuthal flow is the polar angle between the radial vector $r$ and axis of rotation, and $\phi$ is the azimuthal angle in plane perpendicular to that axis. Distance to that axis is $r_\phi = r \sin \theta$.

The starting point of our formulation is the continuity equation in spherical coordinates,

$$\frac{\partial \rho_h}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho_h u_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \rho_h u_\theta \right) = 0,$$

where $\rho_h \equiv \rho_h (r, \theta, \varphi)$ is the halo density. Mean flow along three coordinates are introduced as the radial flow $u_r$, polar flow (meridional flow) $u_\theta$, and azimuthal flow (zonal flow) $u_\varphi$. By considering the axisymmetry about axis of rotation, the mean azimuthal flow $u_\varphi = u_\varphi (r, \theta, \varphi)$ should be independent of the azimuthal angle $\varphi$. The polar flow $u_\theta = u_\theta (r, \theta, \varphi)$ is also independent of $\varphi$ with symmetry $u_\theta (r, \pi - \theta, \varphi)$ such that $u_\theta (r, \pi/2, \varphi) = 0$.

Observations of flow on rotating sphere strongly suggest that as the rotation rate increases, the azimuthal flow (zonal flow) will become dominant and the polar flow (meridional flow) $u_\theta$ may be neglected ($u_\theta \approx 0$) (also discussed in Fig. 2). The original continuity Eq. (1) reduces to

$$\frac{\partial \rho_h}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho_h u_r \right) = 0,$$

where the density $\rho_h = \rho_h (r, \theta, \varphi)$ and the radial flow $u_r = u_r (r, \theta, \varphi)$ are functions of $r$ and $\theta$ only. Equation (2) was extensively studied in our previous work (Xu 2021a) and used to solve for the mean radial flow $u_r = u_r (r, \theta, \varphi)$ for a given halo density $\rho_h$. In current model, (in-plane) flow in concentric spherical shells is incompressible (term 1 in Eq. (1) vanishes). However, radial flow (out-of-plane) is not incompressible with $u_r \neq 0$. The special case is an isothermal density profile where $u_\varphi = 0$ such that the mean flow of entire halo is incompressible everywhere.

The full momentum equations (Jeans’s equation) along three spherical coordinates read

$$\begin{align*}
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_r}{\partial \theta} + u_\varphi \frac{\partial u_r}{\partial \varphi} - u_\varphi^2 - \frac{u_\varphi^2}{r} & = -\frac{\partial \phi_r}{\partial r} + \frac{\sigma_\theta^2 + \sigma_\varphi^2}{r}, \\
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho_h \sigma_r^2 \right) & = \frac{\partial}{\partial \theta} \left( \rho_h \sigma_\theta^2 \sin \theta \right) + \frac{\partial}{\partial \varphi} \left( \rho_h \sigma_\varphi^2 \right),
\end{align*}$$

and

$$\begin{align*}
\frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + u_\theta \frac{\partial u_\varphi}{\partial \theta} + u_\varphi \frac{\partial u_\varphi}{\partial \varphi} + u_r u_\varphi \frac{\partial \phi_r}{\partial \theta} + \frac{u_\varphi^2 \cot \theta}{r} & = -\frac{1}{r} \frac{\partial \phi_r}{\partial \theta} + \frac{\sigma_\theta^2 \cot \theta - \sigma_\varphi^2 + \sigma_\varphi^2 \cot \theta}{r}, \\
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho_h \sigma_{\varphi r}^2 \right) & = \frac{\partial}{\partial \theta} \left( \rho_h \sigma_{\varphi \theta}^2 \sin \theta \right) + \frac{\partial}{\partial \varphi} \left( \rho_h \sigma_{\varphi \varphi}^2 \right),
\end{align*}$$

where the gravitational potential $\phi_r$ is related to halo density via the halo mass $m_r = m_r (r, \theta)$ within a shell of radius $r$,

$$\frac{\partial \phi_r}{\partial r} = \frac{G m_r (r, \theta)}{r^2} \quad \text{and} \quad \rho_h = \frac{1}{4\pi r^2} \frac{\partial m_r (r, \theta)}{\partial r}.$$

By assuming vanishing off-diagonal velocity dispersions and the fact that all variables should be independent of the azimuthal angle $\varphi$ due to axisymmetry, i.e.,

$$\sigma_{rr}^2 = \sigma_{\theta \theta}^2 = \sigma_{\varphi \varphi}^2 = \sigma_{\theta \varphi}^2 = \sigma_{\varphi \theta}^2 = 0,$$

and

$$\sigma_\theta^2 = 0, \quad \sigma_\varphi^2 = 0, \quad \sigma_\varphi^2 = 0,$$

momentum equations (Eq. (3)-(5)) can be significantly reduced to

$$\begin{align*}
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + u_\theta \frac{\partial u_r}{\partial \theta} + \frac{1}{\rho_h} \frac{\partial}{\partial \varphi} \left( \rho_h \sigma_{\varphi r}^2 \right) + \frac{2}{r \sigma_{rr}^2} \left( 1 - \frac{\sigma_{\theta \theta}^2 + \sigma_{\varphi \varphi}^2 + u_\varphi^2}{2\sigma_{rr}^2} \right) \frac{\partial \phi_r}{\partial r} & = 0, \\
\sigma_{rr}^2 & = \sigma_{\theta \theta}^2 = \sigma_{\varphi \varphi}^2 = \sigma_{\theta \varphi}^2 = \sigma_{\varphi \theta}^2 = 0.
\end{align*}$$
\[ \frac{\partial u_\varphi}{\partial t} + u_r \frac{\partial u_\varphi}{\partial r} + \frac{u_r u_\varphi}{r} = 0. \quad (11) \]

The mean azimuthal flow \( u_\varphi^2 \) is directly related to in-plane velocity dispersions \( \sigma_{\theta \varphi}^2 \) and \( \sigma_{\varphi \varphi}^2 \) in Eq. (10). The azimuthal flow \( u_\varphi \) can be solved from Eq. (11) if \( u_r \) is known. Note that an exact definition of the halo anisotropic parameter \( \beta_{h1} \) should be (term 1 in Eq. (9))

\[ \beta_{h1} = 1 - \frac{\sigma_{\theta \varphi}^2 + \sigma_{\varphi \varphi}^2 + u_\varphi^2}{2\sigma_{\varphi \varphi}^2}, \quad (12) \]

where the effect of azimuthal flow due to halo spin should be included. However, \( u_\varphi^2 \) might be relatively small compared to in-plane velocity dispersions \( \sigma_{\theta \varphi}^2 \) and \( \sigma_{\varphi \varphi}^2 \) for massive halos with large velocity dispersion such that \( u_\varphi^2 \) can be neglected. If the azimuthal flow \( u_\varphi \) can be neglected, the anisotropic parameter \( \beta_{h1} \) reduces to the standard definition in literature,

\[ \beta_{h} = 1 - \frac{\sigma_{\theta \varphi}^2 + \sigma_{\varphi \varphi}^2}{2\sigma_{\varphi \varphi}^2}. \quad (13) \]

Clearly, the two definitions are only consistent with each other for massive or large halos, where azimuthal flow \( u_\varphi^2 \) can be neglected when compared to in-plane velocity dispersions. However, small halos spin much faster than large halos at the same redshift (see Xu 2021f, Fig. 15) and the effect of \( u_\varphi^2 \) can be strong. Two definitions are different with \( \beta_{h1} = 0 \) and \( \beta_{h} = 0 \) for small and fast spinning halos. We will discuss and compare two definitions in Fig. 9.

We will close this section by presenting the mean flow from N-body simulations. For every halo identified in the system, the axis of rotation can be determined first by calculating the halo angular momentum vector \( \mathbf{H}_h \) (see Xu 2021f, Eq. (56)). All halos are positioned and aligned by the axis of rotation as shown in Fig. 1 such that \( u_\varphi > 0 \) is always true. The mean flow of every particle in halo can be obtained by projecting its peculiar velocity along three spherical coordinates. The statistics is then taken over all particles in the same spherical shell (spherical averaging) and for all halos in the same group (group averaging) to increase signal noise ratio. Groups of small halos have enough halos for reliable statistics, while groups of large halos may not have sufficient number of halos, where the average can be taken over multiple halo groups of similar sizes of a given range.

Figure 2 plots the variation of the mean (peculiar) radial (\( u_{rP} \), 'square'), azimuthal (\( u_{\varphi} \), 'circles'), and polar (\( u_{\theta} \), 'diamond') flow (unit: km/s) with radius \( r \) for halo groups of size \( n_p = 2, 3, 4, 10 \) and 20 at \( z = 0 \). For \( n_p = 2 \), planar motion leads to \( u_{\theta} = 0 \). The azimuthal flow is predicted to be \( u_{\varphi} \sim r^{-1/2} \) for \( n_p = 2 \) and gradually shifts to \( u_{\varphi} \sim r^{-1/2} \) for inner region and approaching \( u_{\varphi} \sim r \) for outer region. For all size of halos in figure, the peculiar radial flow \( u_{rP} \sim -H_r \) from stable cluster hypothesis (Xu 2021d). The mean polar flow is negligible, i.e. \( u_{\theta} \approx 0 \) almost everywhere.

![Figure 2](image2.png)

Figure 3 plots the variation of angular velocity \( \omega_{r} \) (\( r_z \) with \( r_z \) (distance from axis of rotation) for halo groups of size \( n_p = 2, 3, 4, 10, 20 \) and 40. For \( n_p = 2 \), the angular velocity is predicted to be \( \omega_{r} \approx 13r^{-3/2} \) (see Xu 2021d, Eq. (103)). Angular velocity \( \omega_{r} \) decreases with halo size. For a given size, \( \omega_{r} \) decreases with distance \( r_z \) and approaches a constant \( \omega_{r} \) in outer region of halos. Halo core spins faster than outer region.

3.2 Evolution of halo momentum and energy

The evolution of halo momentum and energy can be studied exactly by the continuity and momentum equations. The first example is to multiply the continuity equation (Eq. (2)) and momentum Eq. (9) with \( u_r \) and \( \beta_h \) respectively and add them together that leads to an
The integration of Eq. (14) over the entire halo by applying

\[
\int_0^{r_h} 2\pi r^2 \int_0^\pi (\bullet) \sin \theta d\theta dr \quad \text{to both sides of Eq. (14) leads to}
\]

\[
\frac{\partial \tilde{L}_h}{\partial t} + 4\pi r^2 \tilde{\rho}_h (r_h) u_r (r_h) \left[ u_r (r_h) - \frac{\partial \tilde{r}_h}{\partial t} \right] +
\]

\[
2\pi r^2 \tilde{\rho}_h (r_h) \int_0^\pi \sigma^2_{rr} (r_h, \theta) \sin \theta d\theta + \int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \frac{\partial \tilde{r}_h}{\partial t} dr +
\]

\[
- \frac{1}{2} \int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \int_0^\pi \left( \sigma^2_{\theta\theta} + \sigma^2_{\varphi\varphi} + u^2_r \right) \sin \theta d\theta dr = 0,
\]

where the (zeroth order) halo radial momentum is defined as

\[
\tilde{L}_h (a) = \int_0^{r_h} u_r (r, a) 4\pi r^2 \tilde{\rho}_h (r, a) dr.
\]

The integration of Eq. (14) over the entire halo by applying \( \int_0^{r_h} 2\pi r^2 \int_0^\pi (\bullet) r \sin \theta d\theta dr \) leads to a complete virial theorem for rotating and growing halos,

\[
\frac{\partial \tilde{G}_h}{\partial t} + 4\pi r^2 \tilde{\rho}_h (r_h) u_r (r_h) \left[ u_r (r_h) - \frac{\partial \tilde{r}_h}{\partial t} \right] +
\]

\[
2\pi r^2 \tilde{\rho}_h (r_h) \int_0^\pi \sigma^2_{rr} (r_h, \theta) \sin \theta d\theta + \int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \frac{\partial \tilde{r}_h}{\partial t} dr +
\]

\[
- \frac{1}{2} \int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \int_0^\pi \left( \sigma^2_{\theta\theta} + \sigma^2_{\varphi\varphi} + u^2_r \right) \sin \theta d\theta dr = 0,
\]

where halo virial quantity (first order radial momentum) is defined as

\[
\tilde{G}_h (a) = \int_0^{r_h} u_r (r, a) 4\pi r^2 \tilde{\rho}_h (r, a) dr.
\]

Term 2 is the surface energy due to radial flow and mass accretion at halo surface and term 3 is the surface energy due to radial velocity dispersion. Term 4 is for halo radial kinetic energy and term 5 is for the halo rotational kinetic energy, both of which are from mean flow of halo (coherent motion). Term 6 is for the kinetic energy due to the random motion. Term 7 is for the halo potential energy. The similar equation has been extensively studied (see Xu 2021a, Eq. (75)) for an isotropic, growing, and non-rotating halo, where term 5 is not present. For virialized, non-rotating, and non-growing halos, \( u_r = 0 \) and \( \partial \tilde{r}_h / \partial t = 0 \) such that terms 2, 4, and 5 are not present.

The second example is for the radial kinetic energy. Multiplying Eqs. (9) and (14) with \( \tilde{\rho}_h u_r \), \( u_r \), respectively and adding them together leads to the evolution of radial kinetic energy \( \left( \tilde{\rho}_h u_r^2 \right) \),

\[
\frac{\partial \left( \tilde{\rho}_h u_r^2 \right)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \tilde{\rho}_h u_r^2 \right) = 0,
\]

where the first term is the time derivative of radial kinetic energy. The second term is the advection in radial direction. The last three terms are the production of radial kinetic energy including two contributions, i.e. \( P_1 \) and \( P_3 \) from velocity radial dispersion \( \sigma^2_{rr} \) and \( P_2 \) from the gravitational interaction. With \( P_2 + P_3 \approx 0 \) (gravitational force balances the pressure gradient), there is no net energy transfer between the radial mean flow \( \rho_h u_r^2 \) and random motion \( \rho_h \sigma^2_{rr} \) (term \( P_1 \)). The direction of transfer depends on the sign of \( u_r \). Using the radial momentum equation Eq. (9), we have the identity

\[
\frac{\partial \left( \tilde{\rho}_h u_r^2 \right)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \tilde{\rho}_h u_r^2 \right) = 0,
\]

which will be further used to study two contributions (\( P_1 \) and \( P_2 \)) for the production of radial kinetic energy (see Eq. (122)).

Integrating \( \left( \tilde{\rho}_h u_r^2 \right) \) in Eq. (19) leads to the total halo radial kinetic energy

\[
\tilde{K}_r (a) = \frac{1}{2} \int_0^{r_h} u_r^2 (r, a) 4\pi r^2 \tilde{\rho}_h (r, a) dr.
\]

Integrating Eq. (19) over the entire halo by applying \( \int_0^{r_h} 2\pi r^2 \int_0^\pi (\bullet) r \sin \theta d\theta dr \) leads to the evolution of halo radial kinetic energy,

\[
\frac{\partial \tilde{K}_r}{\partial t} + 2\pi r^2 \tilde{\rho}_h (r_h) u_r (r_h) \left[ u_r (r_h) - \frac{\partial \tilde{r}_h}{\partial t} \right] +
\]

\[
\int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \frac{\partial \tilde{r}_h}{\partial t} dr + \int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \frac{\partial \tilde{r}_h}{\partial t} dr +
\]

\[
- \frac{1}{2} \int_0^{r_h} 4\pi r^2 \tilde{\rho}_h \int_0^\pi \left( \sigma^2_{\theta\theta} + \sigma^2_{\varphi\varphi} + u^2_r \right) \sin \theta d\theta dr = 0,
\]

where halo angular momentum is defined as

\[
\tilde{M}_\theta (r_h) = \int_0^{r_h} u_r (r, a) 4\pi r^2 \tilde{\rho}_h (r, a) dr.
\]

The third example is for the halo angular momentum. Multiplying the continuity equation (Eq. (2)) and Eq. (11) with \( u_\varphi \) and \( \tilde{\rho}_h \), respectively and adding them together leads to the evolution of \( \rho_h u_\varphi \) that is relevant to the angular momentum,

\[
\frac{\partial \rho_h u_\varphi}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho_h u_\varphi r^2 \right) + \frac{u_r}{r} \left( \rho_h u_\varphi \right) = 0.
\]

Multiplying all terms with \( r_\varphi = r \sin \theta \) and integrating Eq. (23) over the entire halo, i.e. applying the integration

\[
\int_0^{r_h} \int_0^\pi \int_0^{2\pi} (\bullet) r \sin \theta d\theta d\phi d\theta dr = \int_0^{r_h} \int_0^\pi (\bullet) r \sin^2 \theta d\theta dr,
\]

leads to the time variation of halo angular momentum \( \tilde{H}_h \)

\[
\frac{\partial \tilde{H}_h}{\partial t} = 2\pi r^2 \rho_h (r_h) \int_0^\pi u_\varphi (r_h, \theta) \sin^2 \theta d\theta \left( \frac{\partial \tilde{r}_h}{\partial t} - u_r (r_h) \right),
\]
where angular momentum $\tilde{H}_h$ is defined as
\[
\tilde{H}_h = \int_0^{r_h} 2\pi r^3 \rho_h (r) \left( \int_0^\pi u_\varphi r^2 \sin^2 \vartheta d\vartheta \right) dr.
\] (26)

Note that integration of the first term in Eq. (23) can be separated into two contributions using the Leibniz’s rule (the integration limit $r_h = r_h (t)$ is a function of $t$),
\[
\int_0^{r_h (t)} 2\pi r^2 \int_0^\pi \rho_h (r) u_\varphi (r, \vartheta) \sin^2 \vartheta d\vartheta dr
d\vartheta
\frac{\partial \tilde{H}_h}{\partial t} = -2\pi r_h^3 \rho_h (r_h) \frac{\partial r_h}{\partial t} \int_0^\pi u_\varphi (r_h, \vartheta) \sin^2 \vartheta d\vartheta.
\] (27)

Here we demonstrate that the change of halo momentum comes only from the halo growth and radial flow at halo surface (infall of matter) (Eq. (25)). Mean radial and azimuthal flow in halos do not contribute to the change of halo angular momentum. Since $\partial r_h/\partial t > 0$ and $u_r (r_h) < 0$ for a growing halo, the angular momentum $\tilde{H}_h$ should be always increasing with time for growing halos. The halo angular momentum is conserved only if $\partial r_h/\partial t = u_r (r_h) = 0$.

The Tidal Torque Theory relates the origin and evolution of angular momentum to the gravitational tidal forces from the environment in which halos form (Peebles 1969; White 1984). The Tidal Torque Theory (TTT) predicts a linear increase of $\tilde{H}_h$ with time $t$ for a halo with a fixed mass. Most of the halo angular momentum is obtained from the misalignment between the tidal shear field and halo shape. However, a growing halo may obtain its momentum through continuous mass acquisition (see Eq. (25)). Similar ideas were also discussed before (Vitvitska et al. 2002). Mass accretion leads to a linear increase of the specific angular momentum $\tilde{H}_h \sim t$ (or total angular momentum $\tilde{H}_h \sim t^2$) at the early stage of halos (Table 3).

The final example is the halo rotational kinetic energy. Multiplying Eqs. (11) and (23) with $\rho_h u_\varphi$ and $u_\varphi$ respectively and adding them together leads to the evolution for term $\rho_h u_\varphi^2$,
\[
\frac{\partial \left( \rho_h u_\varphi^2 \right)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( \rho_h u_\varphi^2 u_r r^2 \right) + 2 \rho_h u_\varphi u_r = 0.
\] (28)

Since $u_r > 0$ in the halo core region and $u_r < 0$ in the halo outer region for fast growing halos (see Xu 2021b, Fig. 2), the rotational kinetic energy is consumed in the halo core region and generated in outer region.

In hydrodynamic turbulence, Reynolds stress arising from velocity fluctuation continuously transfers kinetic energy from mean flow to turbulence and sustain the energy cascade. Note that $u_\varphi^2$ is closely related to the in-plane velocity dispersion (Eq. (10)), the production term in Eqs. (19) and (28) describe the energy transfer between the mean flow and random motion (turbulence) in halos. The fictitious stresses $\rho_h \sigma_{rr}$ and $\rho_h \sigma_{\varphi \varphi}$ (equivalent to the "Reynolds stress") acts on the gradient of mean flow ($u_r / r$) to facilitate the energy transfer between mean flow and random motion.

While the energy transfer in turbulence is always one-way from mean flow to random motion, the energy transfer is two-way in halos of dark matter flow, where energy can be drawn from random motion to mean flow in outer region ($u_r < 0$ in Eq. (28)) or from mean flow to random motion in core region ($u_r > 0$), depending on the local sign of $u_r$. However, for entire halo, there is a net transfer from mean flow to random flow (see Table 4).

Just like the radial kinetic energy in Eq. (21), halo rotational kinetic energy is defined as,
\[
\tilde{K}_a = \frac{1}{2} \int_0^{r_h} 2\pi r^2 \int_0^\pi \left( \rho_h u_\varphi^2 \right) \sin \theta d\theta dr.
\] (29)

Integrating Eq. (28) with $\int_0^{r_h} 2\pi r^2 \int_0^\pi 1/2 \left( \rho_h u_\varphi^2 \right) \sin \theta d\theta dr$ leads to the evolution of the total rotational kinetic energy for entire halo,
\[
\frac{\partial \tilde{K}_a}{\partial t} = \pi r_h^2 \rho_h (r_h) \int_0^\pi u_\varphi^2 (r_h, \vartheta) \sin \vartheta d\vartheta \left( \frac{\partial r_h}{\partial t} - u_r (r_h) \right)
- \int_0^{r_h} 2\pi r^2 u_r \rho_h \left( \int_0^\pi u_\varphi^2 (r, \vartheta) \sin \vartheta d\vartheta \right) dr.
\] (30)

where the rotational kinetic energy can be changed due to halo growth and radial flow (term 1 in Eq. (30)) or from mean flow and random motion in bulk of halo (term 2 in Eq. (30)). By contrast, angular momentum can only be changed due to the surface term (see Eq. (25)).

A complete understanding of the evolution and transfer of radial and rotational kinetic energies will require solutions of mean flow and velocity dispersions. Obviously Eqs. (2), (9), (10), and (11) is not a closed system. Additional assumptions are required to obtain complete solutions of the mean flow and velocity dispersions, which will be discussed in the next section.

### 3.3 General solutions for axisymmetric rotating&growing halos

We now turn to the axisymmetric solutions of a rotating and growing spherical halo with a non-zero angular velocity. In principle, such halos can be characterized by four time-varying parameters, i.e. the halo mass $m_h (t)$, the angular velocity $\omega_h (t)$, concentration parameter $c (t)$ and scale radius $r_s (t)$. The halo size (virial radius) is $r_h (t) = c (t) r_s (t)$. A reduced spatial-temporal variable $x$ is introduced (see Xu 2021b, Eq. (60)),
\[
x (r, t) = \frac{r}{r_s (t)} = \frac{c (t) r}{r_h (t)}.
\] (31)

The time and spatial derivatives with respect to $t$ and $r$ can be derived in terms of the reduced variable $x$ using the chain rule,
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial x} \frac{\partial x}{\partial t} = -x \frac{\partial \ln r_s}{\partial t} \frac{\partial}{\partial x} \frac{\partial r_s}{\partial x} \quad \text{and} \quad \frac{\partial}{\partial r} = \frac{\partial}{\partial x} \frac{\partial x}{\partial r} = \frac{1}{x} \frac{\partial}{\partial x}.
\] (32)

A unknown function $F (x)$ is introduced such that halo density $\rho_h$ and the mass $m_r$ enclosed in the radius $r$ can all be expressed in terms of function $F (x)$,
\[
\rho_h (r, t) = m_h (t) F' (x) \quad \text{and} \quad m_r (r, t) = m_h (t) F (x).
\] (33)

The total mass of a virialized halo is expected to be proportional to the background density $\rho_0$ at present epoch,
\[
m_h (t) = \frac{4}{3} \pi r_h^3 \Delta_c \rho_0 a^{-3},
\] (34)

where the critical ratio $\Delta_c = 18\pi^2$ can be obtained from a spherical collapse model or a two-body collapse model (see Xu 2021d, Eq. (89)) for a matter dominant universe. The circular velocity at the
surface of a halo and at any given radius \( r \) can be defined as,
\[
v^2_{\text{cir}} (a) = \frac{Gm_h (a)}{r_h (a)} = \frac{4\pi^2 r_h^2}{t_2^2} = (3\pi Hr_h)^2
\]
and
\[
v^2_c (r, a) = \frac{Gm_r (r, a)}{r} = \frac{cF (x)}{F (c)} r^2 v^2_{\text{cir}}.
\]
A relation between \( c (t), r_s (t), \) and \( m_h (t) \) is found from Eq. (34),
\[
\frac{\partial \ln r_s}{\partial \ln t} + \frac{\partial \ln m_h}{\partial \ln t} = \frac{1}{3} \frac{\partial \ln m_h}{\partial \ln r} + \frac{2}{3}.
\]

We will focus on the solutions for two limiting situations in terms of a reduced amplitude parameter (peak height) of density fluctuation

\[
v = \delta_{\text{cr}} / \sigma (m_h, z),
\]
where \( \delta_{\text{cr}} = 1.68 \) is the critical overdensity from spherical collapse model and \( \sigma (m_h, z) \) is the rms (root mean square) fluctuation of the smoothed density field. Halos at their early stage with fast mass accretion have their angular momentum increasing with time. The mass accretion and increase of angular momentum will gradually slow down with halos evolving toward the late stage of their life.

At the same redshift, large halos tend to have a higher \( \nu \) and small halos have a lower \( \nu \). From this point on, "large" halos refer to the halos at early stage of its life with fast mass accretion (high \( \nu \)) and a growing core such that the concentration \( c (t) \) is relatively time-invariant and the halo mass \( m_h (t) \sim t \) from inverse mass cascade (see Xu 2021a, Fig. 7). From Eq. (36), we should have
\[
r_h (t) \sim t \quad \text{and} \quad r_s (t) \sim t.
\]

"Small" halos refer to low \( \nu \) halos at the late stage of halo life with slow mass accretion and a stable core, where the radius \( r_s (t) \) and the halo core mass (mass enclosed within \( r_s \)) \( m_r (r_s, t) \) are all relatively time-invariant such that (from Eqs. (33) and (36))
\[
m_r (r_s, t) / m_h (t) = F (1) / F (c) = C_F \quad \text{and} \quad c^3 \sim F (c) / F (1)^2 = t^2 / C_F.
\]
Here \( C_F \) is the ratio of core mass to halo mass and concentration \( c \sim t^{2/3} \sim a \) for small halos with halo mass increases slowly with \( m_h (t) \propto F (c) \). This simple relation is consistent with concentration models in (Butlack et al. 2001b; Wechsler et al. 2002).

The complete solution of the mean radial flow \( u_r \) can be obtained by solving the continuity equation (Eq. (2)) for a given unknown function \( F (x) \) (see Xu 2021b, Eq. (23)),
\[
u_r (r) = u_r r_s / t
\]
and the normalized radial flow
\[
\nu_h (x) = \frac{\partial \ln r_s}{\partial \ln t} + \left( \frac{\partial \ln F (c)}{\partial \ln t} - \frac{\partial \ln m_h}{\partial \ln t} \right) F (x) / F (c).
\]
Obviously, \( \nu_h (x) = 0 \) for small halos with a stable core (using Eq. (39) with constant \( r_s \), halo mass \( m_h \propto F (c) \)). While for large halos (using Eq. (38) with a constant concentration \( c \)),
\[
\nu_h (x) = x - F (x) / F (c).
\]

To derive full solutions for mean flow and velocity dispersions, the first assumption we made here is to use the separation of variables to express the mean azimuthal flow \( u_\varphi \) as
\[
u_\varphi (r, \theta, t) = \omega_h (t) r_s (t) F_\varphi (x) K_\varphi (\theta),
\]
where \( F_\varphi (x) \) and \( K_\varphi (\theta) \) are the radial and angular functions for \( u_\varphi \), respectively. The azimuthal flow \( u_\varphi \) is expected to be proportional to the effective halo angular velocity \( \omega_h \). The exact solution of \( F_\varphi (x) \) can be derived from the momentum equation for \( u_\varphi \) (Eq. (11)) with help of chain rule from Eq. (32),
\[
\frac{\partial \ln F_\varphi}{\partial \ln x} = \frac{u_h (x) + x \left( \frac{\partial \ln \omega_h}{\partial \ln t} + \frac{\partial \ln r_s}{\partial \ln t} \right)}{x \frac{\partial \ln r_s}{\partial \ln t} - \alpha_h (x)}.
\]

Velocity dispersions are expected to be isotropic for non-rotating halos with a spherical symmetry. The halo spin \( (\omega_h \neq 0) \) breaks the spherical symmetry and leads to the anisotropy in velocity dispersion. For spherical halos with a finite angular velocity \( \omega_h \), velocity dispersions are only isotropic along the axis of rotation \( (r_z = 0 \text{ or } \theta = 0 \text{ such that } u_\varphi = 0 \text{ on that axis}) \),
\[
\sigma^2_{\varphi r} (r, \theta = 0, t) = \sigma^2_{\theta \varphi} (r, \theta = 0, t) = \sigma^2_{r_0} (r, t),
\]
where \( \sigma^2_{r_0} (r, t) \) is the axial velocity dispersion along the axis of rotation. With spin causing the velocity dispersion anisotropy, velocity dispersions can be a function of azimuthal flow \( u^2_\varphi \).

The second assumption is to express velocity dispersions as functions of the azimuthal flow \( u^2_\varphi \). The first order approximation for three dispersions should read
\[
\sigma^2_{\theta \varphi} (r, \theta, t) = \sigma^2_{r_0} (r, t) + \alpha_{\varphi} (r, t) u^2_\varphi (r, \theta, t),
\]
\[
\sigma^2_{\varphi r} (r, \theta, t) = \sigma^2_{r_0} (r, t) + \beta_{\varphi} (r, t) u^2_\varphi (r, \theta, t),
\]
\[
\sigma^2_{r r} (r, \theta, t) = \sigma^2_{r_0} (r, t) + \gamma_{\varphi} (r, t) u^2_\varphi (r, \theta, t),
\]
where expansion coefficients \( \alpha_{\varphi}, \beta_{\varphi} \) and \( \gamma_{\varphi} \) will be determined later. This approximation decomposes the velocity dispersions into a non-spin induced axial dispersion (term 1) and a spin-induced dispersion (term 2). Substitution of Eqs. (45) and (46) into the momentum equation in polar direction (Eq. (10)) leads to the solution for angular function \( K_{\varphi} (\theta) \),
\[
\frac{\partial \ln u_\varphi}{\partial \ln \sin \theta} = \frac{\partial \ln K_{\varphi}}{\partial \ln \sin \theta} = \frac{1 + \beta_{\varphi} - \alpha_{\varphi}}{2 \alpha_{\varphi}}.
\]

With expression of \( u_\varphi \) in Eq. (42), the angular function \( K_{\varphi} (\theta) \) is
\[
K_{\varphi} (\theta) = (\sin \theta)^{\alpha_{\varphi}} \quad \text{and} \quad \alpha_{\varphi} = \frac{1 + \beta_{\varphi} - \alpha_{\varphi}}{2 \alpha_{\varphi}}.
\]

Next, substitution of velocity dispersions (Eqs. (45)-(47)) into the momentum equation in radial direction (Eq. (9)) leads to two separate equations, i.e. an equation for the isotropic velocity dispersion \( \sigma^2_{r_0} \) (term 1 in Eq. (45)),
\[
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{1}{\rho_h} \frac{\partial \left( \rho_h u_r^2 \right)}{\partial r} + \frac{\partial \phi_h}{\partial r} + F_a (r, t) = 0,
\]
and an equation for anisotropic velocity dispersions via coefficients \( \alpha_{\varphi}, \beta_{\varphi} \) and \( \gamma_{\varphi} \) (term 2 in Eqs. (45)-(47)),
\[
\frac{\partial \ln \gamma_{\varphi}}{\partial \ln x} + \frac{2}{\gamma_{\varphi}} \frac{\partial \ln u_\varphi}{\partial \ln x} + \frac{\partial \ln \rho_h}{\partial \ln x} + \frac{1}{\gamma_{\varphi}} u_\varphi = \frac{r F_a (r, t)}{\gamma_{\varphi} u_\varphi^2}.
\]
Here \( \alpha_{\varphi} \) is a dimensionless coefficient for the effect of anisotropy on the radial velocity dispersion through functions \( \alpha_{\varphi}, \beta_{\varphi}, \) and \( \gamma_{\varphi} \),
\[
\alpha_{\varphi} = (\alpha_{\varphi} + \beta_{\varphi} + 1) / 2 \gamma_{\varphi}.
\]
where $\alpha_a$ can be related to the anisotropic parameter $\beta_{h1}$. The new and the old (standard) anisotropy parameters defined in Eqs. (12) and (13) can be expressed in terms of the coefficients $\alpha_\varphi$, $\beta_\varphi$ and $\gamma_\varphi$ as,

$$
\beta_{h1} = \frac{1 - (1 + \alpha_\varphi + \beta_\varphi) / (2\gamma_\varphi)}{1 + \alpha_\varphi^2 / (\gamma_\varphi u_\varphi^2)} = \frac{1 - \alpha_a}{1 + \alpha_\varphi^2 / (\gamma_\varphi u_\varphi^2)}
$$

(53)

and

$$
\beta_h = \frac{1 - (\alpha_\varphi + \beta_\varphi) / (2\gamma_\varphi)}{1 + \alpha_\varphi^2 / (\gamma_\varphi u_\varphi^2)}.
$$

(54)

The coupling function $F_\alpha (r, t)$ (with a unit of acceleration) reflects the coupling between term 1 and term 2 in Eq. (45), i.e., how velocity dispersion $\gamma_\varphi u_\varphi^2$ due to halo spin and the axial dispersion $\sigma_{rr}^2$ are coupled. Two terms are decoupled if and only if $F_\alpha (r, t) = 0$.

The radial velocity dispersion $\sigma_{rr}^2 (r, t)$ for a non-rotating isotropic spherical growing halo ($\omega_h = 0$ and $\beta_{h1} = 0$ in Eq. (9)) has been extensively studied previously (Xu 2021b), where

$$
\frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} + \frac{1}{\rho_h} \frac{\partial (\rho_h u_r^2)}{\partial r} = 0.
$$

(55)

The logarithmic slope of pressure can be obtained from Eq. (55) (see Xu 2021b, Eq. (73)),

$$
\frac{\partial \ln \left( \frac{\rho_h u_r^2}{\sigma_r^2} \right)}{\partial \ln x} = \frac{\sigma_r^2}{\sigma_r^2} \left( \frac{v_r^2}{4\pi^2 c^2} - \frac{v_c^2}{c^2} \right).
$$

(56)

Obviously, $\sigma_{rr}^2 = \sigma_r^2$ and Eq. (50) reduces to Eq. (55) only if the coupling term $F_\alpha (r, t) = 0$. Comparison of Eq. (55) with (50) leads to a relation between two dispersions

$$
\frac{\partial \ln \left( \frac{\rho_h (\sigma_r^2 - \sigma_{rr}^2)}{\sigma_r^2 - \sigma_{rr}^2} \right)}{\partial \ln r} \equiv \frac{r F_\alpha (r, t)}{\sigma_r^2 - \sigma_{rr}^2}.
$$

(57)

where the coupling term $F_\alpha (r, t)$ contributes to the difference between $\sigma_r^2$ of an isotropic non-rotating halo and the axial dispersion $\sigma_{rr}^2$ of a rotating halo. The relation between the other two radial dispersions is obtained by subtracting Eq. (55) from Eq. (9),

$$
\frac{\partial \ln \left( \frac{\rho_h (\sigma_r^2 - \sigma_{rr}^2)}{\sigma_r^2 - \sigma_{rr}^2} \right)}{\partial \ln r} \equiv -\frac{2\beta_{h1}\sigma_{rr}^2}{\sigma_r^2 - \sigma_{rr}^2},
$$

(58)

where $\beta_{h1}$ is the new anisotropic parameter defined in Eq. (12). However, $\sigma_{rr}^2$ does not necessarily equal $\sigma_r^2$ even for $\beta_{h1} = 0$ because of the additional dependence of $\sigma_{rr}^2$ on $u_\varphi^2$ in Eq. (47).

Finally, the difference between radial velocity dispersion $\sigma_{rr}^2$ and axial dispersion $\sigma_{rr}^2$ reads

$$
\frac{\partial \ln \left( \frac{\rho_h (\sigma_r^2 - \sigma_{rr}^2)}{\sigma_r^2 - \sigma_{rr}^2} \right)}{\partial \ln r} \equiv \frac{r F_\alpha (r, t) - 2\beta_{h1}\sigma_{rr}^2}{\sigma_r^2 - \sigma_{rr}^2},
$$

(59)

which is consistent with Eq. (51) and includes two contributions from $F_\alpha (r, t)$ and $\beta_{h1}$, respectively.

### 3.4 Solutions for small halos at late stage (low peak height $\nu$)

We first focus on small halos with a stable core and slow mass accretion rate. Figure 4 plots the variation of (spherical and group averaged) velocity dispersions and mean azimuthal flow $u_\varphi^2$ with the radius $r$ for all halos with a size $n_p$ between [20 40]. For velocity dispersions (Eqs. (45) to (47)), the contribution from $\sigma_{rr}^2$ (term 1) is dominant at small $r$, while the contribution from $u_\varphi^2$ (term 2) can be dominant at large $r$. We also found a good agreement of $u_\varphi^2 = \sigma_{rr}^2 - \sigma_r^2$ for large $x$ (Eq. (63)), i.e., a surprisingly simple result that directly connects the mean flow and random motion (turbulence) at halo scale.

For small halos with a stable core, coupling term $F_\alpha (r, t) = 0$ is expected such that $\sigma_{rr}^2 = \sigma_r^2$ (Eqs. (50) and (57)). For core region with a small $r$, $\sigma_{rr}^2 \approx \sigma_{rr}^2 = \sigma_r^2$, while $\sigma_{rr}^2 \gg \sigma_{rr}^2 = \sigma_r^2$ for outer region due to a significant contribution from azimuthal flow $u_\varphi^2$ at large $r$ (see Fig. 4). In addition, the radial flow vanishes with $u_r (r) = 0$ (see Eq. (40) small halos are well bound and virialized structure). Small halos are incompressible in (proper) velocity field with $u_r = \theta_\varphi = u_\varphi = 0$ at large $x$, i.e. $\nabla \cdot \mathbf{u} = 0$. While in comoving system, the peculiar velocity $\mathbf{v}$ field has constant divergence with

$$
\nabla \cdot \mathbf{v} = \frac{1}{r^2} \frac{\partial (r^2 v_r)}{\partial r} = -3Ha,
$$

(60)

where peculiar radial flow $v_r = u_r - Har = -Har$ if $u_r = 0$ (also from stable clustering hypothesis demonstrated by a two-body collapse model (Xu 2021d)). The constant divergence flow in small halos was also supported by the correlation-based statistical analysis, where dark matter flow is shown to be constant divergence on small scale and rotational on large scale (Xu 2022e).

The angular velocity $\omega_h$ of small halos is relatively time-invariant (small halos grow slowly with a constant $r_s$ and vanishing radial flow $u_r = 0$). Small halos with a stable core are expected to be relatively isotropic with the anisotropic parameter $\beta_{h1} = 0$ (Eq. (54)) (however, the old definition of anisotropic parameter in Eq. (13) $\beta_h \neq 0$ for small halos), i.e.

$$
\alpha_a = \frac{\alpha_\varphi + \beta_\varphi + 1}{2\gamma_\varphi} = 1, \quad 2\sigma_{rr}^2 = \sigma_\varphi^2 + \sigma_{r\theta}^2 + u_\varphi^2.
$$

(61)

With $\beta_{h1} = 0$ (or equivalently $\alpha_a = 1$) and $F_\alpha (r, t) = 0$, Eq. (51)
Table 2. Dispersions and mean flow for rotating and non-rotating halos

|                  | Radial (r)          | Azimuthal (θ) | Polar (φ)          |
|------------------|---------------------|---------------|-------------------|
| Rotating         | $\sigma_{rr} = \sigma_{rr0} + \sigma_{\theta\phi} = \sigma_{\theta\phi0} + \sigma_{\phi\phi} = \sigma_{\phi\phi0}$ | $2u_{q0}$     | $u_{q0}$          |
| Non-rotating     | $\sigma_{rr} = \sigma_{rr0} + \sigma_{\theta\phi} = \sigma_{\theta\phi0} + \sigma_{\phi\phi} = \sigma_{\phi\phi0}$ | $\frac{u_{p}}{\sqrt{2}}$ | $0$               |

Mean flow

- Random
  - Rotating: $\sigma_{rr0} = \sigma_{\theta\phi0} = \sigma_{\phi\phi0}$
  - Non-rotating: $\sigma_{rr0} = \sigma_{\theta\phi0} = \sigma_{\phi\phi0}$

For $\gamma_{\phi}$ reduces to,

$$\frac{\partial \ln \gamma_{\phi}}{\partial \ln r} \approx \frac{\partial \ln u_{\phi}}{\partial \ln r} + \frac{\partial \ln \rho_{h}}{\partial \ln r} = 0.$$  \hspace{1cm} (62)

For outer region (large $x$) of small halos with an isothermal density profile (the logarithmic slope of density is -2) and $u_{\phi} \sim \omega_{h} r_{x} \sim \omega_{h} r_{x} \sin \theta$ (as shown in Figs. 2 and 3), Eq. (62) predicts $\gamma_{\phi}/\partial x = 0$, i.e. $\gamma_{\phi} (r, t)$ is almost a constant of location $r$. If we also expect $\sigma_{rr}^{2} = \sigma_{\theta\phi}^{2}$, i.e. $\gamma_{\phi} = \beta_{\phi}$ (as shown in Figs. 4 and 8) for large $r$, Eq. (61) requires $1 + \alpha_{\phi} = \beta_{\phi}$ such that

$$\sigma_{rr}^{2} = \sigma_{\phi\phi}^{2} = \sigma_{\theta\theta}^{2} + u_{r}^{2} \text{ and } u_{r} = u_{\phi} = 0,$$  \hspace{1cm} (63)

as shown in both Fig. 4 and Fig. 8 for small halos. Equation (63) can be considered as how energy is partitioned along each direction for isotropic ($\rho_{h} = 0$), incompressible, fully virialized ($u_{r} = 0$), and rotating halos with extremely slow mass accretion. As shown in Table 2, the total kinetic energy (both random motion and mean flow) is partitioned along each coordinate:

- Radial: $\sigma_{rr}^{2} = \sigma_{\phi\phi}^{2}$
- Azimuthal: $\sigma_{\theta\theta}^{2} + u_{\phi}^{2}$
- Polar: $\sigma_{\phi\phi}^{2} + u_{\phi}^{2}$

Energy is not equipartitioned along each direction, with the largest kinetic energy in azimuthal direction and the smallest kinetic energy in polar direction. The exponent $\alpha_{\phi} = 1/\alpha_{\phi}$ for angular function $K_{\phi} (\theta)$ can be obtained from Eq. (49). For $K_{\phi} (\theta) \sim \sin \theta$ such that $\alpha_{\phi} = 1$, we should have $\alpha_{\phi} = 1$ and $\beta_{\phi} = \gamma_{\phi} = 2$ for small halos. This can be confirmed by simulation data in Fig. 8.

Now let’s compare the energy of an initially virialized non-rotating halo that has an isotropic velocity dispersion $\sigma_{rr}$ with the energy of a rotating halo of the same size. The density profile and the potential energy should be the same for both halos. The axial dispersions of rotating halo is always $\sigma_{rr0} = \sigma_{\phi\phi0}$ (Eq. (57) with $F_{A}(r, t) = 0$). For a rotating halo with a rotational kinetic energy of $K_{\theta}$ (see Eq. (29) for definition), there will be around $5K_{\theta}$ extra kinetic energy in the form of random motion with $\beta_{\phi} = \gamma_{\phi} = 2\alpha_{\phi} = 2$ when compared with non-rotating halo (see Table 2). In addition, small halos with a finite spin will have an additional spin-induced pressure $\rho_{h} u_{\phi}^{2}$ when compared to a non-rotating halo. The spin-induced pressure is independent of $r$ for an isothermal density profile ($\rho_{h} \propto r^{-2}$ and $u_{\phi} \propto r$) such that the gradient of spin-induced pressure vanishes. Total pressure gradient of rotating halos is the same as the non-rotating halo to balance the gravitational force (Eq. (9)).

In contrast to normal object whose temperature is independent of the speed of spin, faster rotating halos (with fixed mass) are expected to be hotter with greater entropy due to the random motion associated with velocity dispersion. Figure 5 plots the variation of two parameters $\alpha_{\phi}$ (Eq. (52)) and $\alpha_{\phi}$ (Eq. (49)) with radius $r$ for halo groups of different sizes $n_{p}$. For small halos, $\alpha_{\phi} = 1$ and halo is isotropic with $\beta_{h1} = 0$ almost everywhere. Large halos tend to have an anisotropic outer region with $\alpha_{\phi} < 1$ and an isotropic core with $\alpha_{\phi} = 1$. In addition, the azimuthal flow $u_{\phi}$ tends to strongly depend on the polar angle $\theta$ for small halos, while $\alpha_{\phi} \ll 1$ for large halos such that $u_{\phi}$ is less dependent on angle $\theta$.

3.5 Solutions for large halos at early stage (high peak height $\nu$)

We now turn to solutions for the other limiting situation, i.e. large halos (high peak height $\nu$) with an expanding core, fast mass accretion, and constant halo concentration $c$. We first focus on the solution for azimuthal flow $u_{\phi}$. For large halos with fast mass accretion, there exists a non-zero radial flow $u_{r}$. (Eq. (41)), where the normalized radial flow $u_{h}$ is

$$u_{h}(x) = \frac{u_{r}(r)}{r_{x}(t)} = x - \frac{F(x)}{F'(x)} \text{ and } u_{h}(c) = c \left(1 - \frac{1}{a_{h}} \right).$$  \hspace{1cm} (65)

A halo deformation parameter is introduced here as

$$\alpha_{h} = cF'(c)/F(c)$$  \hspace{1cm} (66)

to quantify the radial deformation at halo surface (no deformation if $\alpha_{h} = 1$ for isothermal density profile). The (normalized) peculiar radial flow that excludes the Hubble flow is

$$u_{p}(x) = \frac{u_{r}(-r)}{r_{x}(r)} = \left|u_{r}(r) - Hr\right| \frac{t}{r_{x}(r)} = u_{h}(x) - 2x = \frac{x}{3} - \frac{F(x)}{F'(x)}$$  \hspace{1cm} (67)

and

$$u_{p}(x = c) = c \left(1 - \frac{1}{a_{h}} \right).$$  \hspace{1cm} (68)

With radial flow from Eq. (65), the logarithmic slope of density at halo center can be related to a halo deformation parameter $\gamma_{h}$.
The peculiar radial velocity at halo virial radius $r = r_h$ is proportional to circular velocity with a proportional constant $1/3\pi$ (using Eq. (68))

$$ u_{rp}(r_h) = u_r(r_h) - H r_h $$

$$ = u_p(c) r_s(t) t = u_r(c) r_h \left( \frac{1}{3} - \frac{1}{\alpha_h} \right) = -\frac{v_{cir}}{3\pi} $$

This is true for an isothermal density profile with $u_r = 0$ and $\alpha_h = 1$, where $v_{cir}$ is the circular velocity at the virial radius. The proportional constant $1/3\pi$ is essentially related to the angle of incidence (see Xu 2021b, Section 3.4), i.e., the angle for single merger merging with halos in mass cascade (see Xu 2021a, Eq. 8). It is also required to interpret the critical MOND (modified Newtonian dynamics) acceleration $a_0$ by the mass and energy cascade in dark matter flow (see Xu 2022j, Eq. (12) and Fig. 8).

Specifically, for large halos with an isothermal profile, $F(x) = x/c$ and $\alpha_f = 1$, we have $u_f = 2c^2/3$ and the mean azimuthal flow

$$ u_\varphi (r, \theta, t) = \frac{2}{3} \omega_h(t) r_h(t) \sin \theta $$

that is independent of the radius $r$.

Here if we assume the mean azimuthal flow $u_\varphi$ on halo surface with a polar angle of $\pi/2$ (halo equator) is equal to the peculiar radial flow (two velocities are equal on the halo equator), from Eq. (78),

$$ u_\varphi \left(r_h, \frac{\pi}{2}, t\right) = -u_{rp} \left(r_h, t\right) = -u_p (x = c) \frac{r_s}{t} = -\frac{v_{cir}}{3\pi} $$

Substitution of expression for $u_\varphi$ from Eq. (72) and $u_{rp}$ from Eq. (68) into Eq. (80) leads to the expression of halo angular velocity,

$$ \omega_h = \left( \frac{3}{2a_h} - \frac{1}{2} \right) \frac{c^2}{F(c)} \alpha H $$

where the angular velocity of large halos $\omega_h \sim H \sim t^{-1} - a^{-3/2}$.

Now we can compare our solution of the mean azimuthal flow $u_\varphi$ with $N$-body simulation. The spherical averaged azimuthal flow $u_{avg}$ (normalized by the Hubble flow) can be defined as (with solutions of $u_\varphi$ and $\omega_h$ from Eqs. (72) and (81)),

$$ u_{avg} = 1/2 \int_0^{\pi} u_\varphi (r, \theta, t) \sin \theta \, d\theta $$

Figure 6 presents the variation of normalized (spherical and group averaged) azimuthal flow $u_{avg}$ with radius $r$ for different size of halos. Function $F(x)$ for a NFW density profile

$$ F(x) = \ln (1 + x) - \frac{x}{1 + x} $$

is used for comparison along with other parameters $c = 3.5$, $r_s = 0.34 Mpc/h$, and $\alpha_\varphi = 1/2$.

Halos of the same size $n_g$ are first aligned by the axis of rotation and assembled into a composite halo containing all particles from the same halo group. The average is taken over the normalized azimuthal flow $u_{avg}$ of all particles in the same spherical shell of radius $r$ of composite halos. Next, average is also taken over all halo groups with size $n_p$ in the given range as indicated in Fig. 6. The azimuthal flow $u_\varphi$ approaches 10 times of Hubble flow $Hr$ at the halo core region and is comparable to Hubble flow at halo outer region. This solution also suggests a faster spinning core and slower spinning outer region of halos with $\omega_r \sim H$ (Fig. 3).

Next let us turn to solutions for velocity dispersions of large halos.
Figure 7 plots velocity dispersions and azimuthal flow $u_{\varphi}$ with radius $r$ for halos of size $n_P$ between [500 1000] at $z=0$. The spin-induced dispersion from azimuthal flow $u_{\varphi}$ is dominant in large halos over the axial dispersion $\sigma^2_{r0}$.

The coupling function $F_\theta (r, t) < 0$ in Eqs. (50) and (51) such that (from Eqs. (56) and (57)),

$$r F_a (r, t) / \sigma^2_T \approx \frac{\partial \ln [\rho_h \sigma^2_T]}{\partial \ln r} \approx \frac{\gamma u_{\varphi}^2}{\gamma u_{\varphi}^2} \left( \frac{x^2 - xu h \partial u h}{4\pi^2 c^2} \frac{dx}{\partial x} \right) \frac{v^2_{\text{c ir}}}{v^2_{\text{c ir}}},$$

(84)

This can be further reduced to (with $u_h$ from Eq. (65))

$$r F_a (r, t) / \gamma u_{\varphi}^2 \approx \left[ \frac{x^2}{4\pi^2 c^2} \frac{\partial \ln F}{\partial \ln x} \right]^{-1/2} \frac{\partial \ln F}{\partial \ln x} - \frac{cF (x)}{xF (c)} \frac{v^2_{\text{c ir}}}{\gamma u_{\varphi}^2},$$

(85)

that is in terms of the unknown function $F (x)$. Term 1 in Eq. (85) is the contribution from mean radial flow and is expected to be much smaller when compared to term 2 from the gravitational potential.

The approximation of coupling function $F_a (r, t)$ from Eq. (85))

$$F_a (r, t) = - \frac{F (x) r h^2}{F (c)} \frac{v^2_{\text{c ir}}}{\gamma u_{\varphi}^2} = - \frac{\partial \sigma}{\partial r},$$

(86)

can be obtained and used in Eq. (50) for large halos.

With $\alpha_{\varphi}$ and $\beta_{\varphi}$ are comparable and both are much greater than 1, we will have $\alpha_{\varphi}$ (exponent of sin $\theta$ in Eq. (72) for $u_{\varphi}$),

$$\alpha_{\varphi} = \frac{1 + \beta_{\varphi} - \alpha_{\varphi}}{2\alpha_{\varphi}} \ll 1 \quad \text{with} \quad \alpha_{\varphi} \gg 1 \quad \text{and} \quad \beta_{\varphi} \gg 1,$$

(87)

such that the dependence on the coordinate variable $\theta$ can be eliminated, i.e. all variables are only weakly dependent on $\varphi$. This is also clearly shown in the plot of $\alpha_{\varphi}$ in Fig. 5, where azimuthal flow $u_{\varphi}$ is weakly dependent on $\theta$ for large halos.

With approximation of coupling function $F_a (r, t)$ in Eq. (86), Eq. (50) for axial velocity dispersion $\sigma^2_{r0}$ reduces to

$$\frac{\partial \sigma^2_{r0}}{\partial t} + u_r \frac{\partial \sigma^2_{r0}}{\partial r} + \frac{1}{\rho_h} \frac{\partial \left( \rho_h \sigma^2_{r0} \right)}{\partial r} = 0,$$

(88)

where $\sigma^2_{r0}$ is entirely determined by the mean radial flow $u_r$. Using solution of $u_{\varphi}$ in Eq. (65), the solution of $\sigma^2_{r0}$ reads

$$\sigma^2_{r0} (x) = \frac{\gamma u_{\varphi}^2}{4\pi^2 c^2 F (x)} \left[ \int F^2 (x) dx - \frac{2}{x^2 F (c) \gamma u_{\varphi}^2} \right],$$

(89)

which is the first term in the solution for radial dispersion $\sigma^2_r$ of isotropic and non-rotating halos (see Xu 2021b, Eq. (68)).

Next, Eqs. (51) and (85) are now used to solve for the in-plane and radial velocity dispersions. The equation for $\gamma_{\varphi}$ now reads,

$$\frac{\partial \ln \gamma_{\varphi}}{\partial \ln x} + \frac{2}{2} \frac{\partial \ln \gamma_{\varphi}}{\partial \ln x} = - \frac{c F (x)}{xF (c) \gamma u_{\varphi}^2} v^2_{\text{c ir}},$$

(90)

Substitution of the solution of $u_{\varphi}$ (Eq. (72)) into Eq. (90) leads to

$$\frac{\partial \ln \gamma_{\varphi}}{\partial \ln x} + \frac{2}{2} \frac{\partial \ln \gamma_{\varphi}}{\partial \ln x} = \frac{\lambda_{\varphi} x}{2\alpha_{\varphi}} \gamma_{\varphi}.$$
Figure 8. The variation of $C_2 = \beta_\varphi - \alpha_\varphi$ (solid line) and $C_1 = \gamma_\varphi - \alpha_\varphi$ (dash line), i.e. $\sigma_{\varphi\varphi} - \sigma_{\theta\theta} = C_2 u_\varphi^2$ and $\sigma_{\varphi\varphi}^2 - \sigma_{\theta\theta}^2 = C_1 u_\varphi^2$, with radius $r$ for halo groups of different sizes at $z=0$. Small halos are entirely isotropic with $C_1 = C_2 = 1$, i.e. $\sigma_{\varphi\varphi} = \sigma_{\theta\theta} = u_\varphi^2$ and $\sigma_{\varphi\varphi}^2 = \sigma_{\theta\theta}^2$ such that the anisotropic parameter $\beta_{h1} = 0$. For large halos, $C_1$ and $C_2$ are more likely to be dependent on $r$ with $C_1 \gg C_2$. At halo surface, $C_1 = \gamma_\varphi - \alpha_\varphi \approx 10$ and $C_2 = \beta_\varphi - \alpha_\varphi \approx 1$.

With $v_{\text{cir}}^2$ from Eq. (35) and $\omega_h$ from Eq. (81), the dimensionless constant $\lambda_f$ is defined as

$$
\lambda_f = \frac{c v_{\text{cir}}^2}{a_f^2 \omega_h^2 F(c)} = \frac{9\pi^2 F(c)}{(3/2\omega_h - 1/2)^2}. 
$$

(92)

To obtain a solution of $\gamma_\varphi$ and hence the solution of velocity dispersions, we need to introduce some additional constraints between three expansion coefficients,

$$
\beta_\varphi = \alpha_\varphi + C_2(x) \quad \text{and} \quad \gamma_\varphi = \alpha_\varphi + C_1(x),
$$

(93)

where $C_1$ and $C_2$ are two functions of $x$ that can be determined from simulation. This requires

$$
C_2 u_\varphi^2 = \sigma_{\varphi\varphi} - \sigma_{\theta\theta}^2 \quad \text{and} \quad C_1 u_\varphi^2 = \sigma_{\varphi\varphi}^2 - \sigma_{\theta\theta}^2,
$$

(94)

i.e. the difference between velocity dispersions is always proportional to $u_\varphi^2$. Figure 8 presents the variation of $C_1$ and $C_2$ with radius $r$ for halo groups of different sizes. Clearly, $C_1 = C_2 = 1$ for small halos, as predicted in the previous section since small halos are relatively isotropic with anisotropic parameter $\beta_{h1} = 0$. However, large halos are anisotropic with $\beta_{h1} > 0$, where $C_1$ and $C_2$ are $r$-dependent with $C_1 \gg C_2$. At halo surface, $C_1 = \gamma_\varphi - \alpha_\varphi \approx 10$ and $C_2 = \beta_\varphi - \alpha_\varphi \approx 1$.

We first look at a special case: large halos with extremely fast mass accretion and infinitesimal halo lifespan, where the radial flow $u_r$ vanishes (see Xu 2021b, Fig. 3) and axial velocity dispersion $\sigma_{\varphi\varphi}^2 = 0$ from Eq. (88)). These halos should have an isothermal density profile with $F(x) = x/c$ (see Xu 2021d, Section 3.7). Therefore, from Eq. (91), the expansion coefficients for large halos with isothermal density profile should be,

$$
\alpha_\varphi = \frac{9\pi^2 - C_2 - 1}{2}, \quad \beta_\varphi = \frac{9\pi^2 + C_2 - 1}{2}, \quad \gamma_\varphi = \frac{9\pi^2 - C_2 - 1 + 2C_1}{2}.
$$

(95)

For a general density profile, with these relations, the final equation for the expansion coefficient $\gamma_\varphi$ reads (from Eq. (91))

$$
\frac{\partial \gamma_\varphi}{\partial x} + \frac{\gamma_\varphi}{x} \left[ \frac{\partial \ln \left( F^2 F'/x^4 \right)}{\partial \ln x} + \frac{\lambda_f}{F(x)} \right] = \frac{C_2}{x} + \frac{1 - 2C_1}{x}. 
$$

(96)

Exact solution of $\gamma_\varphi$ will depend on the model of $C_1$ and $C_2$. One reasonable simplification is to neglect term 1 in Eq. (96) because of $C_1 \gg C_2 \approx 1$ and assume a constant $C_1(x) = C_1 = 10$. The corresponding solution for $\gamma_\varphi$ can be obtained in terms of $F(x)$,

$$
\gamma_\varphi(x) = \frac{x^4}{F^2(x) F'(x)} \left( \frac{2C_1 - 1 - C_2}{\int_x^\infty F^2(y) F'(y) dy} \right).
$$

(97)

With $F(x) \sim x^2$ for small $x$ (NFW profile), we should expect $\gamma_\varphi \sim x^{-1}$ from Eq. (96). For any given density profile (or function $F(x)$), the velocity dispersions (Eqs. (45) to (47)) can be eventually obtained with solution of $\sigma_{\varphi\varphi}^2$ from Eq. (89) and solutions of $u_\varphi^2$ and $\gamma_\varphi$ from Eqs. (72) and (97), respectively. For NFW profile, the two terms in Eq. (97) can be obtained analytically,

$$
term1 = \frac{2 + 83x + 147x^2 + 68x^3}{6x(1+x)^3} - \frac{35}{12} \ln x \left[ -35 + 8 \ln (1 + x) (5 + 6 \ln (1 + x)) \right]
$$

$$
+ \frac{\ln (1 + x)}{3x^3 (1 + x)^2} \left[ -2x + 6x^2 + 45x^3 + 34x^4 \right]
$$

$$
+ \frac{\ln (1 + x)}{3x^3 (1 + x)} \left[ -1 + 2x + x^2 (x - 2) (3 + 5x) \right]
$$

$$
+ \frac{2}{3} [5 + 12 \ln (1 + x)] \text{poly log} (2, 1 + x)
$$

$$
- 8 \text{poly log} (3, 1 + x) + \frac{5\pi (21i - 8\pi)}{36} - \frac{4}{3} \ln^3 (1 + x)
$$

and

$$
term2 = \frac{\ln (1 + x)}{2x^2 (1 + x)^2} + \frac{1}{2x (1 + x)^2} \left[ -1 - 9x - 7x^2 \right]
$$

$$
+ \frac{1}{2x - 8x - 4x^2 + x^3} \ln (1 + x)
$$

$$
+ \left( \pi^2 + 6 \text{poly log} (2, -x) - \ln x + 3 (\ln (1 + x)^2) x (1 + x)^2 \right).
$$

(98)

For large halos with $\sigma_{\varphi\varphi}^2 \ll \gamma_\varphi u_\varphi^2$, the anisotropy parameters $\beta_{h1}$ and $\beta_h$ (Eq. (54)) are equal and reduced to the same expression in terms of $\gamma_\varphi$,

$$
\beta_{h1} \approx 1 - \frac{1 + \sigma_\varphi + \beta_\varphi}{2\gamma_\varphi} \approx \frac{2C_1 - 1 - C_2}{2\gamma_\varphi} = \beta_h, 
$$

(99)

which is inversely proportional to coefficient $\gamma_\varphi$ in Eq. (97).

Figure 9 plots the variation of anisotropy parameters $\beta_{h1}$ (dash lines) and $\beta_h$ (solid lines) with radius $r$ for halo groups of different sizes $n_p$. For small halos that are isotropic, $\beta_{h1} \approx 0$ while $\beta_h \neq 0$ since $\beta_h$ does not include the effect of azimuthal flow $u_\varphi$. However,
The virial quantity (excluding Hubble flow) is (Eq. (67)),

\[
G_{hp} = \frac{1}{m_h} \int_0^{r_h} 4\pi r^3 \rho(r) u_r dr
\]

\[
= \frac{1}{2} \left[ 1 - \frac{5}{c^2 F(c)} \int_0^c x F(x) dx \right] Hr_h^2.
\]

(103)

For any density profiles, the specific halo angular momentum reads

\[
H_h = \left( \frac{1}{a_h} - \frac{1}{3} \right) \frac{c^2}{F(c)} \frac{G_h - G_{hp}}{\alpha_f} = \left( \frac{1}{a_h} - \frac{1}{3} \right) \frac{c^2}{F(c)} \alpha_f Hr_h^2
\]

(104)

from Eqs. (75), (102), (103), and (81).

With Eq. (73) for relations between \( r_0^2 \) and \( r_1^2 \), the halo angular momentum from Eq. (104) can be finally written in terms of \( r_h \),

\[
H_h = \gamma_H Hr_h^2 = \frac{1}{8} \left( \frac{3}{a_h} - 1 \right) \sqrt{\Gamma(3/2 + \alpha_f/2)} \Gamma(2 + \alpha_f/2) Hr_h^2
\]

(105)

where the coefficient \( \gamma_H \) for angular momentum is

\[
\gamma_H = \frac{1}{8} \left( \frac{3}{a_h} - 1 \right) \Gamma(3/2 + \alpha_f/2) \Gamma(2 + \alpha_f/2)
\]

(106)

The specific momentum tensor of a spherical halo reads from Eqs. (76) and (103),

\[
\frac{1}{m_h} \int_V \mathbf{x} \otimes \mathbf{u}_p \rho h dV = \begin{bmatrix} G_{hp}/3 & -Hh/2 & 0 \\ Hh/2 & G_{hp}/3 & 0 \\ 0 & 0 & G_{hp}/3 \end{bmatrix}
\]

(107)

It can be found the diagonal terms of halo momentum tensor are the virial quantity in Eq. (103), while the off-diagonal terms are the angular momentum in Eq. (105). The evolution of momentum tensor on both halo and large scales is extensively studied in a separate paper (see Xu 2022g, Section 5).

Finally, the halo specific radial kinetic energy is derived as (with Eq. (65) for \( u_r \)) (also see Xu 2021b, Eq. (54)),

\[
K_r = \frac{1}{2 m_h} \int_0^{r_h} u_r^2(r,a) 4 \pi r^2 \rho_h(r,a) dr
\]

\[
= \frac{9}{8} \left( 1 - \frac{4}{c^2 F(c)} \int_0^c x F(x) dx + \frac{1}{c^2 F(c)} \int_0^c \frac{F^2(x)}{F'(x)} dx \right) Hr_h^2.
\]

(108)

The halo (specific) peculiar radial kinetic energy (excluding Hubble flow) can be obtained as (with Eq. (68) for \( u_{r,p} \)),

\[
K_{rp} = \frac{1}{2 m_h} \int_0^{r_h} u_{r,p}^2 4 \pi r^2 \rho_h(r,a) dr
\]

\[
= \frac{1}{8} - \frac{1}{c^2 F(c)} \int_0^c x F(x) dx + \frac{9}{8 c^2 F(c)} \int_0^c \frac{F^2(x)}{F'(x)} dx \right) Hr_h^2.
\]

(109)

The halo (specific) rotational kinetic energy is derived as (with Eq. (72) for \( u_\phi \)),

\[
K_\alpha = \frac{1}{m_h} \int_0^{r_h} 2 \pi r^3 \rho_h(r) \left( \int_0^\pi \frac{1}{2} \sin \theta d\theta \right) d\phi
\]

\[
= \frac{1}{4} \left( \frac{3}{2 a_h} - \frac{1}{c^2 F(c)} \frac{\sqrt{\Gamma(1 + \alpha_f/2)}}{\Gamma(3/2 + \alpha_f/2)} \right) \int_0^c \frac{F^2(x)}{x^2} dx Hr_h^2
\]

(110)

All these momentum and energy quantities are derived in terms of function \( F(x) \) (Eq. (33)) that is dependent on halo density profile (Xu 2021b) and summarized in Table 3 for isothermal and NFW profiles.
4.2 Calculation of halo spin parameter

The halo spin parameter $\lambda_p$ is commonly used to characterize the importance of angular momentum to the random motion. The energy solutions obtained can be used to estimate the value of $\lambda_p$ for large halos with fast mass accretion. With angular momentum explicitly derived in Eq. (105), the two usual definitions of dimensionless spin parameter can be defined as (Peebles 1969; Bullock et al. 2001a),

$$
\lambda_p = \frac{H_p |E_h|^{1/2}}{G m_h} \quad \text{and} \quad \lambda_p' = \frac{H_p}{\sqrt{2} v_{circ} r_h},
$$

(111)

where $E_h = \Phi_h + K_h$ is the total specific energy. The halo specific potential energy

$$
\Phi_h = -\gamma \Phi \frac{G m_h}{r_h}
$$

(112)

where the coefficient $\gamma \Phi$ for potential energy is

$$
\gamma \Phi = \left( \frac{c}{F''(c)} \right) \left( \int_0^c \frac{F(x) F'(x)}{x} \, dx \right) \approx 1.
$$

(113)

The critical density ratio $\Delta_c = 18 \pi^2$ can be obtained from spherical collapse model or two-body collapse model (Xu 2021d). The halo specific kinetic energy $K_h = 3/2 \sigma_v^2 = (n_c / 2) \Phi_h$, with $n_c \approx -1.3$ for large halos is the effective potential exponent for virial theorem that considers surface energy due to non-zero radial flow and velocity dispersion (see Xu 2021b, Eq. (96)).

It should be noted that Eq. (112) can be used to derive the relation for virial kinetic energy $\sigma_v^2$. Halo size $r_h = \gamma_g \sigma_v r_h$ can be written as (see Xu 2021f, Eq. (61)),

$$
r_g = \gamma_g \sigma^2 \left( \frac{G m_h}{\Delta_c H_0} \right)^{1/3}
$$

(114)

such that (with Eq. (112) for $\Phi_h$)

$$
\sigma_v^2 = -\Phi_h \frac{\gamma_v}{3} = \frac{1}{3} \gamma \Phi \frac{\Delta_c}{2} \left( \frac{G m_h H_0}{h} \right)^{2/3} a^{-3/4},
$$

(115)

where $\gamma_v / \approx -n_c$ is the virial ratio and $\Delta_c$ is the critical density ratio. Here $\gamma_v / \approx 1.3$ for NFW profile and $\gamma_v / \approx 1.5$ for isothermal profile, (Xu 2021b, Eq. (96)). Combining (115) with the model of $\sigma_v^2$ from N-body simulation (Xu 2021f, Eq. (19)) leads to a good equation for velocity dispersion $\sigma_h^2$ of entire N-body system (see Xu 2022g, Fig. 1a),

$$
\sigma_h^2 = \gamma \Phi \gamma_v \left( \frac{G H_0 5.8 \times 10^{-12} M_\odot}{h} \right)^{2/3} \frac{t}{t_0}.
$$

(116)

Rotational kinetic energy $K_\alpha$ can be approximated as (Eq. (75))

$$
K_\alpha \approx \frac{1}{2} |H_h| \omega_h = \frac{3}{4} \left( |H_h| / r_h \right)^2.
$$

(117)

With Eq. (73) for root mean square radius $r_g$ and (111), the two halo spin parameters read

$$
\lambda_p = \gamma \Phi \gamma_v \frac{\sqrt{4 / 3} (1 + n_c / 2) K_\alpha}{|\Phi_h|} \frac{2}{3} \gamma \Phi \gamma_v \sqrt{1 - \frac{\gamma_v}{2}} \frac{K_a}{\sigma_v^2}
$$

and

$$
\lambda_p' = \gamma \frac{2 \gamma \Phi K_\alpha}{3 |\Phi_h|} = \frac{1}{3} \gamma \Phi \sqrt{2 \gamma \Phi \gamma_v} \frac{K_a}{\sigma_v^2},
$$

(118)

where both definitions reflect the ratio of rotational kinetic energy $K_\alpha$ to virial kinetic energy $\sigma_v^2$.

With Eq. (105) for $H_h$, circular velocity $v_{circ} = \sqrt{\Delta_c / 2 H_h} = 3 \pi H r_h (\Delta_c$ is the critical density ratio), and Eq. (111), spin parameters $\lambda_p$ and $\lambda_p'$ finally read (for NFW profile in Table 3)

$$
\lambda_p \approx \frac{\gamma H}{3 \pi v} \left( \frac{1 + n_c}{2} \right) \approx 0.03, \quad \lambda_p' \approx \frac{\gamma H}{3 \pi v \sqrt{2}} \approx 0.038.
$$

(119)

Results for halo spin parameter agrees well with other simulations (Hetznecker & Burkert 2006). In addition, the halo mass dependence of spin parameter that decreases with halo size is discussed in a separate paper (Xu 2022g). All relevant parameters are summarized in Table 3 for two density profiles.

| Symbol         | Physical meaning          | Equation | Isothermal profile with $\alpha_p = 0$ and $c = 3.5$ | NFW profile with $\alpha_p = 0$ and $c = 3.5$ |
|----------------|---------------------------|----------|-----------------------------------------------|-----------------------------------------------|
| $F(x)$         | Function for density $\rho_h$ | Eq. (33) | $x/c$                                          | $\ln (1 + x) - x / (1 + x)$                     |
| $\alpha_p$     | Deformation parameter     | Eq. (66) | 1.0                                            | 0.833                                          |
| $\gamma_h$     | Deformation rate parameter| Eq. (69) | 0                                              | 1/2                                           |
| $\alpha_f$     | Constant for function $F(x)$ | Eq. (77) | $2c^2 / 3$                                      | 9.20                                          |
| $\Lambda_p$    | Constant for equation for $\gamma_p$ | Eq. (92) | $9 \pi^2 / c$                                   | 10.895                                        |
| $\gamma_H$     | Coefficient for $H_p$      | Eq. (106) | 1/3                                            | 0.511                                          |
| $\gamma_v$     | Coefficient for potential $\Phi_h$ | Eq. (113) | 1                                              | 0.936                                          |
| $\gamma_v$     | Virial ratio               | Eq. (115) | 1.5                                            | 1.3                                           |
| $\gamma_v'$    | Ratio of two halo sizes    | Eq. (73) | 1/3                                            | 0.3214                                        |
| $L_h$          | Specific radial momentum   | Eq. (100) | 0                                              | 0                                             |
| $L_{h,p}$      | Peculiar radial momentum   | Eq. (101) | $-H r_h / 2$                                   | $-0.501 H r_h$                                 |
| $G_h$          | Specific virial quantity   | Eq. (102) | 0                                              | $-0.027 H r_h^2$                               |
| $G_{h,p}$      | Peculiar virial quantity   | Eq. (103) | $-H r_h^2 / 3$                                 | $-0.348 H r_h^2$                               |
| $H_h$          | Specific angular momentum  | Eq. (105) | $H r_h^2 / 3$                                | 0.511 $H r_h^2$                                |
| $\omega_h$     | Angular velocity           | Eq. (81) | 1.5 $H$                                       | 2.38 $H$                                      |
| $K_r$          | Radial kinetic energy      | Eq. (108) | 0                                              | 0.0062 $H^2 r_h^2$                              |
| $K_{r,P}$      | Peculiar radial kinetic energy | Eq. (109) | $H^2 r_h^2 / 6$                                | 0.1937 $H^2 r_h^2$                              |
| $K_{\alpha}$   | Rotational kinetic energy  | Eq. (110) | $H^2 r_h^2 / 3$                                | 0.7658 $H^2 r_h^2$                              |
| $\Phi_h$       | Halo potential energy      | Eq. (112) | $-9 \pi^2 H^2 r_h^2$                          | $-8.424 \pi^2 H^2 r_h^2$              |
| $\Lambda_p$    | First halo spin parameter  | Eq. (119) | 0.018                                          | 0.031                                          |
| $\Lambda_p'$   | Second halo spin parameter | Eq. (119) | 0.025                                          | 0.038                                          |

(118)

Vol. 000, 1–22 (2022)
5 ENERGY TRANSFER BETWEEN MEAN FLOW AND RANDOM MOTION

The energy exchange between mean flow and random motion is the key to understand how the turbulence initiates, propagates and evolves in dark matter flow to maximize system entropy.

5.1 General formulation for energy transfer

First, we present a generalized formulation for the evolution of an arbitrary scalar quantity $S(r, \theta, t)$. Using continuity Eq. (2), it is easy to write down the evolution of an arbitrary quantities $\rho S^n (n = 1$ for momentum and $n = 2$ for kinetic energy if $S$ is velocity),

$$\frac{\partial (\rho S^n)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ (\rho S^n) u_r r^2 \right] - \nu \rho S^{n-1} \frac{\partial S}{\partial t} + u_r \frac{\partial S}{\partial r} = 0,$$

where the term $P_S$ stands for the production or consumption of scalar $S$. Integrating Eq. (120) with $\int_0^{r_h} 2\pi r^2 k \int_0^\pi (\cdot) \sin \theta d\theta dr$ leads to the time evolution of the $k$th moment of $S^n$ in entire halo,

$$\frac{\partial}{\partial t} \left[ \int_0^{r_h} 2\pi r^2 k \int_0^\pi (\cdot) \sin \theta d\theta dr \right] = 2\pi \rho \left[ \frac{\partial S}{\partial t} + u_r \frac{\partial S}{\partial r} \right] \sin \theta d\theta dr .$$

In general, the rate of change of scalar $S$ includes two parts: the surface contribution from mass accretion ($S_2$) and the bulk contribution from exchange between mean flow and random motion ($S_1$). By replacing $S$ with the mean flow $u_r$ and $u_{\varphi}$, we can choose appropriate values for $k$, $m$ and $n$ to recover the equations for radial and rotational momentum and energy evolution (Eqs. (14) to (30)).

The evolution of radial & peculiar radial kinetic energy, and rotational kinetic energy can be easily obtained by applying Eq. (120) with $S$ replaced by $u_r$, $u_{\varphi}$, respectively,

$$\frac{\partial (\rho h u_r^2)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ (\rho h u_r^2) u_r r^2 \right] - 2\rho h u_r u_r = 0,$$

$$\frac{\partial (\rho h u_{\varphi}^2)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ (\rho h u_{\varphi}^2) u_r r^2 \right] - 2\rho h u_r u_{\varphi} = 0,$$

These equations can be used to illustrate the energy transfer between mean flow and the random motion.

5.2 Energy transfer between mean flow and random motion

We first substitute Eq. (88) for large halos into Eq. (122) to show that

$$\frac{\partial (\rho h u_r^2)}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left[ (\rho h u_r^2) u_r r^2 \right] + 2\rho h u_r u_r = 0,$$

which is essentially the same as Eq. (28) that has been directly derived from the continuity and momentum equations. Here $u_{\varphi}$ in production term $P_{u_{\varphi}}$ is actually related to in-plane velocity dispersions as shown in Eq. (10). Equ. (126) describes the energy transfer between azimuthal flow and random motion (the in-plane velocity dispersions) of SG-CFD via a fictitious stress $\rho h u_{\varphi}$ (similar to Reynolds stress) acting on the mean flow gradient $u_r/r$.

The production terms in Eq. (122)-(124) can be explicitly written in terms of function $F(x)$ (using Eqs. (33), (65), (68) and (72)),

$$P_{ur} = \frac{2\rho h u_{cir}^2}{t} (x - u_h) \frac{\partial u_h}{\partial x},$$

$$P_{u_{\varphi}} = \frac{2\rho h u_{cir}^2}{t} \frac{\partial u_{\varphi}}{\partial x},$$

where $\rho h$ is the mean halo density. The $P_2$ contribution of $P_{ur}$ in Eq. (19) reads,

$$P_2 = \frac{2\rho h c^2 F(x)}{3} u_h,$$

while the contributions $P_1$ and $P_3$ (from radial velocity dispersions in Eq. (19)) of $P_{ur}$ can be obtained as $P_1 + P_3 = P_{u_{\varphi}} - P_2$.

Figure 10 presents the variation of function $F(x)$, normalized...
radial flow $u_h(x)$, peculiar radial flow $u_p(x)$, and production terms $P_{ur}(x)$, $P_{urp}(x)$ and $P_{urp}$ (Eqs. (127)-(129)) with a reduced coordinate $x = r/r_s(t)$ for NFW profile. The mean radial flow $u_h(x)$ is positive (out-flow) at halo core region, reaching its maximum at $x = 1$, and become negative (in-flow) at around $x = x_0$ for outer region with $x_0 \approx 2.15$, as indicated by the ‘+’ and ‘-’ signs in figure. The peculiar radial flow $u_p(x)$ is always negative. A positive production term means the energy transfer from mean flow to random motion, and vice versa. The radial flow loses its energy to random motion in halo core region $x < 1$ and gains energy for $x = [1, 2]$, and loses it energy again for outer region with $x > x_0$. The production term $P_{urp}(x) > 0$ for $x < x_0$ means the rotational flow loses its energy to random motion in core region, while gains energy from random motion in outer region.

The net rate of change of quantity $\rho_h S\theta^2$ for the entire halo has two contributions (as shown in the general Eq. (121)), i.e. term $S_1$ due to the energy transfer with the random motion inside halo, and the term $S_2$ from the halo surface due to the halo mass accretion (growth) and mass cascade. One example is for radial momentum $L_h$ by replacing $S$ with $u_r$ in Eq. (121) and $n = 1, m = 1, k = 0$,

$$\frac{\partial L_h}{\partial t} = 4\pi \rho_h (r_h) r_h^2 u_r (r_h) \left( \frac{\partial r_h}{\partial t} - u_r (r_h) \right)$$

\[ + \int_0^r 4\pi \rho_r r^2 \left( \frac{\partial u_r}{\partial t} + u_r \frac{\partial u_r}{\partial r} \right) dr. \tag{131} \]

With Eq. (65) for $u_r$ and Eq. (32) for derivatives, the final expression can be expressed as,

$$\frac{\partial L_h}{\partial t} = \frac{m_h r_h}{t^2} \left[ \frac{1}{\alpha_h} - \frac{1}{\alpha_r} + \frac{2}{cF(c)} \int_0^c F(x) \, dx \right], \tag{132}$$

with two contributions, i.e. $S_2$ from halo surface and $S_1$ from bulk respectively. For different kinetic energy, i.e. $K_r$ (radial), $K_{rp}$ (peculiar radial), and $K_{az}$ (rotational), two terms $S_1$ and $S_2$ can all be computed with given density profiles and listed in Table 4.

Finally, this section describes the evolution of momentum and energies and the energy transfer between coherent (mean) flow and random motion for large halos with fast mass accretion. For radial momentum $L_h$, $S_1 = -S_2 > 0$ and the total $L_h = 0$ is time-invariant for large halos (see Xu 2021b, Eq. (51)). For angular momentum $H_h$, $S_1 = 0$ and $S_2 > 0$ such that the total angular momentum $H_h$ increases as $H_h \propto t^2$ (specific angular momentum $H_h \propto t$ and angular velocity $\omega_h \propto t^{-1}$) with all contributions from $S_2$ due to the mass accretion ($m_h(t) \sim t$ and $r_h(t) \sim t$ in Eq. (38)). The radial and rotational kinetic energies of mean flow ($K_r$ and $K_{az}$) increase proportional to $t$ with $S_1 < 0$ and $S_2 > 0$, i.e. the mean flow kinetic energy of entire halo is increasing mainly due to the mass accretion ($S_2 > 0$). The energy transfer between mean flow and random motion is described by Eqs. (122) to (129) and Fig. 10. The local energy transfer can be two-way between coherent and random motion. For entire halo, a net kinetic energy is transferred from mean flow to random motion in SG-CFD (the bulk contribution is always negative $S_1 < 0$).

### Table 4. The rate of change of halo momentum and energy

| Symbol | Physical meaning | Isothermal profile with $\alpha_\theta = 0$ | NFW profile with $\alpha_\theta = 0$ and $c = 3.5$ |
|--------|------------------|------------------------------------------|--------------------------------------------|
| $\frac{\partial L_h}{\partial t}$ | Radial momentum | $0$ | $0$ |
| $S_1$ | Bulk contribution | $0$ | $0.2\rho_h \frac{m_h}{t^2}$ |
| $S_2$ | Surface contribution | $0$ | $-0.2\rho_h \frac{m_h}{t^2}$ |
| $\frac{\partial H_{kr}}{\partial t}$ | Angular momentum | $\frac{\pi m_h r_h^2}{t} \left( \frac{3}{\alpha_r} - \frac{1}{2} \right)$ | $\frac{\pi m_h r_h^2}{t} \left( \frac{3}{\alpha_r} - \frac{1}{2} \right)$ |
| $S_1$ | Bulk contribution | $0$ | $0$ |
| $S_2$ | Surface contribution | $0$ | $0.0062H^2 r_h^2 m_h t$ |
| $\frac{\partial H_{kr}}{\partial t}$ | Rotational kinetic energy | $0$ | $0.0453H^2 r_h^2 m_h t$ |
| $S_1$ | Bulk contribution | $0$ | $0$ |
| $S_2$ | Surface contribution | $0$ | $0.0801H^2 r_h^2 m_h t$ |

6 HALO RELAXATION FROM EARLY TO LATE STAGE

Previous sections provide the mean flow and velocity dispersion solutions for large halos (high $v$ at the early stage of halo life with fast mass accretion and constant concentration) with a non-zero radial
flow (Eq. (41)). The other limiting situation consists of halos with a stable core, low mass accretion and vanishing radial flow (low peak height v at the late stage of halo life with a constant core mass, scale radius and a time-varying concentration). This section focuses on the transition (relaxation) of halos from their early to late stages.

Let’s assume a typical large halo of mass $m_h^L(t)$ that is constantly growing with the waiting time exactly to be $\tau_g \sim a m_h^{-2/3}$ for every single merging event during its entire mass accretion history (see Xu 2021a, Eq. (45)). With $m_h^L(t) \sim a^{3/2}$, the time span of that typical halo $\tau_g^L \equiv \tau_g \sim a^{0}$ should be time-invariant. The actual halo lifespan $\tau_g$ can be random in nature and either less or greater than $\tau_g$. If for any merging event, the random waiting time $\tau_g > \tau_g$ such that the actual halo mass $m_h(t) < m_h^L(t)$ after that merging. A positive feedback process is established since the waiting time $\tau_g \propto a m_h^{-2/3}$ in the propagation range such that $m_h(t)$ will increase slower and slower with longer and longer waiting time or lifespan $\tau_g$. On the other hand, if the random waiting time $\tau_g < \tau_g$ for a merging event such that halo mass $m_h(t) > m_h^L(t)$ in the deposition range after that merging, where the average waiting time for a single merging is significantly longer. A negative feedback will be established to self-limit and slower down the further growth of $m_h(t)$ such that rare halos can have mass much greater than $m_h^L(t)$.

The feedback process leads to the transition (relaxation) from high $v$ to low $v$ halos with slower mass accretion. During halo relaxation, there is a continuous variation of halo shape, density profile, mean flow, momentum, and energies. We will start from the general solution for mean radial flow, which facilitates the mass and momentum exchange between different spherical shells and the energy transfer between random motion and mean flow (Eqs. (19) and (28)).

### 6.1 Evolution of mean radial flow from early to late stage

To discuss the halo relaxation, the starting point is to extend the key function $F \equiv F(x, a)$ (Eq. (33)) to a more general form of $F \equiv F(x, a, \alpha)$, where an additional shape parameter $\alpha \equiv \alpha(t)$ is introduced. A good example is the function $F(x, a)$ of an Einasto profile in Eq. (146). During halo relaxation, we assume a continuous variation of function $F(x, a)$ with time-dependent shape parameter $\alpha$ and concentration $c$. Like Eq. (33), the halo density and mass $m_r$ within radius $r$ is

$$\rho_h(r, t) = \frac{1}{4\pi r^2} \frac{\partial m_r(r, a)}{\partial r} = \frac{m_h(t)}{4\pi r^3} \frac{F'(x, a)}{x^2 F(x, a)}$$

and

$$m_r(r, t) = m_h(t) \frac{F(x, a)}{F'(x, a)}.$$  \hspace{1cm} (133)

The time derivative of halo density is obtained from Eq. (133),

$$\frac{\partial \rho_h(r, a)}{\partial t} = \frac{1}{4\pi r^2} \frac{\partial^2 m_r(r, a)}{\partial r \partial t}.$$ \hspace{1cm} (135)

Using the continuity Eq. (2) and Eq. (135), the time derivative of $m_r(r, a)$ reads

$$\frac{\partial m_r(r, a)}{\partial t} = -4\pi r^2 u_r(r, a) \rho_h(r, a).$$ \hspace{1cm} (136)

A general expression of the mean radial flow reads,

$$u_r = \frac{1}{4\pi r^2} \frac{\partial}{\partial t} \ln m_r \frac{m_r}{\rho_h(r, a)} = \frac{r_s \partial \ln m_r}{t} \frac{F(x, a)}{F'(x, a)}.$$ \hspace{1cm} (137)

From the definition of $m_r(r, a)$ in Eq. (134), the logarithmic derivative of $m_r(r, a)$ reads,

$$\frac{\partial \ln m_r}{\partial t} = \frac{\partial \ln m_h}{\partial t} - \frac{\partial \ln F(x, a)}{\partial t} \frac{\partial \ln r_s}{\partial t}$$

$$= \frac{\partial \ln L F(c, a)}{\partial c} \frac{\partial \ln c }{\partial t} + \frac{\partial \ln F(x, a)}{\partial F(c, a)} \frac{\partial \ln \alpha}{\partial t}.$$ \hspace{1cm} (138)

Substitution of Eq. (138) into Eq. (137) leads to the dimensionless radial flow $u_h = u_r t/r_s$,

$$u_h = x \frac{\partial \ln r_s}{\partial t} - \frac{F(x, a)}{F'(x, a)} \frac{\partial \ln m_h}{\partial t}$$

$$+ \frac{F(x, a)}{F'(x, a)} \left[ \frac{\partial \ln F(c, a)}{\partial c} \frac{\partial \ln c}{\partial t} + \frac{\partial \ln F(x, a)}{\partial F(c, a)} \frac{\partial \ln \alpha}{\partial t} \right].$$ \hspace{1cm} (139)

where terms 1, 2 and 3 represent the contributions from mass accretion, change of concentration, and change of the shape of halo density profile, respectively. Here $F'(x, a)$ stands for the derivative with respect to x, not a. For constant $\alpha$ and $c$, Eq. (139) reduces to Eq. (41) for large halos (high v) with fast mass accretion. The mean radial flow is given by $u_h = u_{hm} + u_{hc} + u_{ha}$ from Eq. (139) with contributions from mass accretion ($u_{hm}$), concentration ($u_{hc}$) and shape parameter ($u_{ha}$).

$$u_{hm}(x, t) = \frac{\partial \ln r_s}{\partial t} - \frac{F(x, a)}{F'(x, a)} \frac{\partial \ln m_h}{\partial t}.$$ \hspace{1cm} (140)

$$u_{hc}(x, t) = \frac{\partial \ln c}{\partial t} \frac{F'(x, a)}{F(x, a)} = \frac{\partial \ln F(c, a)}{\partial c}.$$ \hspace{1cm} (141)

$$u_{ha}(x, t) = \frac{\partial \ln \alpha}{\partial t} \frac{F(x, a)}{F'(x, a)} = \frac{\partial \ln F(c, a)}{\partial \ln \alpha} - \frac{\partial \ln F(x, a)}{\partial \ln r_s}.$$ \hspace{1cm} (142)

and the relevant boundary conditions are

$$u_h(0, t) = 0, \quad \frac{\partial u_{hm}}{\partial x} = 0,$$

$$u_{hc}(c, t) = c \frac{\partial \ln c}{\partial t}, \quad \text{and} \quad u_{ha}(c, t) = 0.$$ \hspace{1cm} (143)

It can be easily verified that

$$u_{hc}(c, t) = c \frac{\partial \ln c}{\partial t} = \frac{\partial \ln r_h}{\partial \ln r_s}$$. \hspace{1cm} (144)

For large halos with high peak height $v$ and constant concentration $c$, $u_{hc} = u_{ha} = 0$ and only radial flow $u_{hm} \sim 0$ is dominant such that halo angular momentum increases with time (Eq. (25)). However, halo relaxation involves an increasing concentration with a fixed scale radius $r_s$, i.e. an isotropic "halo stretching" along all directions with increasing concentration and halo size $r_h$. The radial flow $u_{hc}$ on halo surface is (Eq. (144))

$$u_{hc}(c, t) = \frac{r_s}{t} = \frac{\partial r_h}{\partial t}.$$ \hspace{1cm} (145)

From Eq. (25), the concentration flow $u_{hc}$ by itself does not change the angular momentum of halos. Since $u_{hc} > 0$ for all $r$ and using Eq. (145), the radial flow $u_{hc}$ leads to decreasing rotational kinetic energy (Eq. (30)). This can be understood as the increase of moment of inertial from halo stretching (Eq. (154)). The shape induced radial
flow $u_{hk}$ vanishes on halo surface ($u_{hk}(c, t) = 0$) and can be neglected for small change in $a$. Hence, both radial flows $u_{hc}$ and $u_{hb}$ does not lead to the change of halo angular momentum (Eq. (25)). This is important as the halo angular momentum should be conserved if no mass accretion ($u_{hm} = 0$) if the halo mass $m_h$ is also fixed with no mass accretion (see Eq. (140)).

However, we do expect a slow but nonzero mass accretion during halo relaxation. Halo angular momentum slowly increases with time and should not be conserved. Instead, during halo stretching, a vanishing total radial flow $u_h = 0$ is expected in Eq. (139) that requires the radial flow $u_{hm}$ from mass accretion to cancel the concentration flow $u_{hc}$, i.e. $m_h \propto F(c, a)$ from Eqs. (140) and (141). It turns out a conserved rotational kinetic energy during halo relaxation (Section 6.4).

For Einasto profile, $F(x, a)$ reads,
\[
F(x, a) = \Gamma(3/a) - \Gamma(3/a, 2x^a/a),
\]
where $u_{hc}$ can be explicitly obtained from Eq. (141)
\[
u_{hc} = \frac{\partial \ln \nu \Gamma(3/a) - \Gamma(3/a, 2x^a/a) \frac{c^3}{\Gamma(3/a)} \exp \left[-2 \frac{c^a - x^a}{x^a} \right]}.
\]
\[
(147)
\]

6.2 Path of evolution in $(c, a)$ space from early to late stage

To better describe the halo evolution from early stage (high $v$) to late stage (low $v$), a relation between shape parameter $a$ and concentration $c$ can be identified from Eq. (134),
\[
\frac{F(1, a)}{F(c, a)} = \frac{m_r(x)}{m_h} = C_F(t),
\]
where $C_F \equiv 1$ is the ratio of core mass to total mass of halo. The ratio $C_F$ should approach a constant for small halos ($v \to 0$) with extremely slow mass accretion, where the scale radius $r_s$, core mass $m_r(r_s)$ and halo mass $m_h$ are all relatively time-invariant.

Shape parameter $a$ and concentration $c$ for halos of different sizes at different redshifts can be conveniently expressed in terms of the peak height $v = \delta_c/\sigma(m_h, z)$ of density fluctuation (Klypin et al. 2016). The relevant expressions read,
\[
a = 0.115 + 0.0165v^2 \quad \text{and} \quad c = 6.5v^{-1.6},
\]
where $\delta_c = 1.68$ is the critical overdensity from spherical collapse model and $\sigma(m_h, z)$ is the root mean square fluctuation of the smoothed density field. This equation gives minimum values of $\min(a) = 0.115$ and $\min(c) = 3.08$ for arbitrary peak height $v$.

Figure 11 plots different paths of halo evolution in the space of shape parameter $a$ and concentration $c$. The thick red line gives a path of evolution in $(c, a)$ space that follows a constant ratio $C_F = 0.27$. Other solid lines plot different paths along different ratio $C_F$ using Eq. (148). All paths end with a limiting shape parameter $a$ when concentration $c \to \infty$ and $\partial a/\partial c \to 0$. The corresponding evolution path in $(c, a)$ space (for halos in N-body simulations from Eq. (149)) is presented as the green dash line with peak height $v$ between [0.5, 5.0]. Halos with fast mass accretion and vanishing radial momentum should have a constant $a = 0.2$, a limiting concentration $c = 3.5$ (see Xu 2021b, Eq. (53)) and $C_F = 0.27$ (Eq. (148)) that is denoted by the blue dot in Fig. 11. Halos at their early stage of life (high $v$) will gradually evolving to the low $v$ (late stage) along the green dash line in N-body simulation. Both the shape parameter $a$ and ratio $C_F$ are decreasing along that path, while $c$ is increasing along that path. With $v \to 0$ along the green line, we have limiting $a = 0.115$ and $C_F \approx 0.03$ for halos reaching their final stage. For blue dash line with constant $a = 0.2$, the limiting ratio $C_F \approx 0.083$.

6.3 Evolution of density profile and moment of inertia

Now let us look at the density profile variation during halo relaxation. Halo density profile reads (from Eq. (33))
\[
\rho_h \equiv \frac{m_h F(1, a)}{(4/3) \pi r_s^3 F(c, a)} \frac{F'(x, a)}{3F(1, a) x^2} = \rho_c \frac{F'(x, a)}{3F(1, a) x^2},
\]
where $\rho_c$ is the mean density of core region with $r < r_s$.

Figure 12 plots the variation of normalized density profile of $\rho_h^* = \rho_h/\rho_c$ along the path $3$ (blue dash line in Fig. 11) using an Einasto (red lines) and a NFW model (blue lines). The first segment for high $v$ halos with constant $c$ (before blue dot) is also almost along a constant $C_F$ path (see the red line in Fig. 11). The change of density profile is from black solid line to green, and to red solid lines in Fig. 12. With decreasing $a$ and constant $c$, fast mass accretion leads to an increasing core mass that is proportional to the total halo mass $m_h$.

During fast mass accretion stage (high $v$), the mass accretion induced radial flow ($u_{hm}$) is dominant and core structure is changing significantly. The NFW and Einasto profiles are different in inner region during this stage. It was shown that Einasto profile is a better choice for massive (high $v$) halos (Klypin et al. 2016). The reason is that NFW is a single parameter profile and cannot reflect the change in shape parameter $a$ during this stage of evolution. In addition, the mean core density $\rho_c \sim r^{-2} \sim a^{-3}$ with $m_h \sim t$ and $r_s \sim t$. Note that for NFW profile with only one parameter $c$, the first segment simply
reduces to the blue dot for high $v$ halos (full solutions are discussed in Sections 3.5 and 4.1).

The second segment for evolving toward low $v$ halos with a constant $\alpha$ (after blue dot) should have decreasing $C_F$ with time. During this slower mass accretion stage (low $v$), the radial flow is negligible with contributions from both $u_{hm}$ and $u_{hc}$ canceling each other such that $m_h = F(c, \alpha)$ (Eqs. (140) and (141)). This means a constant core mass $m_r (r_s)$ (Eq. (148)) and core density $\rho_c$ during this stage. The density profile during this stage simply stretches to larger $c$ with inner density fixed ($\rho_c$ is fixed in Eq. (150)). A NFW profile (Blue) is also plotted for comparison that is quite different from Einasto profile for high $v$ halos.

To better understand the halo relaxation ("stretching"), the variation of normalized density $\rho_h^*(x)$ with time along the relaxation path 3 in Fig. 11. An Einasto profile is used for the calculation. For high $v$ halos with fast mass accretion, the evolution is along a constant $c$ and $C_F$ path in Fig. 11. The decreasing shape parameter $\alpha$ leads to a steeper density of inner region and core mass increases proportional to the total halo mass (constant $C_F$). For low $v$ halos with slower mass accretion, the evolution is along a constant $\alpha$ path. The density profile is simply stretching to larger $c$ with inner density fixed ($\rho_c$ is fixed in Eq. (150)). A NFW profile (Blue) is also plotted for comparison that is quite different from Einasto profile for high $v$ halos.

\[
\begin{align*}
\rho_h^*(x) &= \frac{\rho_c}{x^\alpha} \\
\rho_c &= \frac{m_h}{4\pi r_s^3} \\
\rho_c &= \frac{m_h}{4\pi r_s^3} \\
\rho_c &= \frac{m_h}{4\pi r_s^3} \quad \text{for} \quad c \to 0
\end{align*}
\]
proportional to time $t$ ($r_g \sim t$ and $|\mathbf{H}_h| \sim t$ in Table 3) such that the specific rotational kinetic energy $K_\alpha$ is always conserved. During halo “stretching” (second segment of blue line in Fig. 11), the root mean square radius $r_g(c) = r_s \sqrt{F_{\omega}(c)}$ (Eq. (153)) that can be different from scaling of $r_g \sim t$ for high $v$ halos. However, a reasonable estimate is that the scaling $|\mathbf{H}_h| \sim r_g$ continuously extends beyond early stage during halo stretching such that rotational kinetic energy $K_\alpha$ is still conserved and the angular velocity $\omega_h \sim r_g^{-1}$ (Eq. (156)). At least, the scaling $|\mathbf{H}_h| \sim r_g$ should be a good approximation at the beginning of halo stretching.

To summarize, along path 3) in Fig. 11 with constant $r_x$ and core mass, the increasing concentration $c$ leads to a decreasing core mass ratio $C_F$. The halo stretching with inner density fixed (Fig. 12) leads to the increasing moment of inertia (Eq. (154)) and angular momentum $\mathbf{H}_h$, while halo angular velocity $\omega_h$ and azimuthal flow $u_\varphi$ decreases along that path. With the coupling term $F_\alpha$ (Eqs. (50) and (51)) approaching zero for low $v$ halos, there is a net transfer of spin-induced velocity dispersion to axial dispersion ($\sigma_\varphi^2$ dispersion due to gravity) (from part 2 to part 1 in Eq. (45)), i.e. an increasing in $\sigma_\varphi^2$. Coefficients $\alpha_\varphi$, $\beta_\varphi$, and $\gamma_\varphi$ also decreases with time (Fig. 8) such that halos become more isotropic with $\beta_{h1,2} \rightarrow 0$ (Fig. 9).

The halo specific potential energy (see Xu 2021b, Eq. (90)) reads

$$
\Phi_h \frac{Gm_h}{r_h} = -\frac{1}{m_h} \int_0^{r_h} 4\pi r^2 \gamma_h \frac{Gm_r(r)}{r} dr = -\frac{Gm_h F(1)}{r_s F(c)} \Phi_h^*,
$$

(157)

where the dimensionless number $\Phi_h^*$ reads (due to constant $r_s$ and core mass $m_h F(1)/F(c)$),

$$
\Phi_h^* = \frac{1}{F(1) F(c)} \int_0^c F(x) F'(x) dx.
$$

(158)

With $m_h \propto F(c)$, constant scale radius $r_x$, and conserved rotational kinetic energy $K_\alpha$ along path 3) in Fig. 11, the variation of all relevant quantities can be summarized in Fig. 14 for an Einasto profile. The mass ratio $C_F$ decreases from 0.27 to 0.08. Other quantities are normalized by their initial values at $c = 3.5$, i.e. the values for halos in their early stage (blue dot in Fig. 11 and shown in Table 3). Halo spin parameter $\lambda_p$ increases from 0.031 when $c = 3.5$ and increases with time during halo stretching due to the faster increase in angular momentum than halo mass (Eq. (111) and Fig. 14). This is consistent with simulation results (Ahn et al. 2014), where $\lambda_p$ increases with time. In addition, $\lambda_p$ for halos of different size should converge to a limiting value of $\lambda_p$ of low $v$ halos (late stage) with $c \rightarrow \infty$.

7 CONCLUSIONS

By revisiting fundamental ideas of energy transfer and cascade in hydrodynamic turbulence, self-gravitating collisionless dark matter flow (SG-CFD) shares many similarities, but also exhibits some unique features. In hydrodynamic turbulence, Reynolds stress arising from velocity fluctuations acts as a conduit to continuously transfer energy from mean flow to turbulence and sustain the continuous energy cascade. To quantitatively describe the energy transfer between mean flow and random motion in SG-CFD, general solutions of mean flow and velocity dispersions are derived for axisymmetric, growing, and rotating halos in spherical coordinate. The polar flow can be neglected (Fig. 2). The azimuthal flow is directly related to in-plane velocity dispersions (Eq. (10)). The radial flow facilitates the exchange of momentum and energy across different spherical shells (Eqs. (19), (23) and (28)).

Evolution of halo momentum and kinetic energy are extensively studied (Eqs. (14) to (30)) based on the continuity and momentum equations (Eqs. (9) to (11)). A growing halo may obtain its momentum through a continuous mass acquisition as quantitatively described by Eq. (25). For large halo at the early stage of its life (Table 3), the specific angular momentum $H_h$ increases linearly with time $t$,
while the specific halo angular kinetic energy $\mathcal{K}_h$ is a constant. Halo angular momentum can only be changed from mass accretion and radial flow at halo surface (Eq. (25)). Halo rotational kinetic energy can be generated from both mass accretion and the energy transfer with random motion (Eq. (30)). The fictitious stress $\rho_h u'_r$ (equivalent to “Reynolds stress”) acts on the gradient of mean flow $\langle u_r/r \rangle$ to facilitate the energy transfer between mean flow and random motion (Eq. (28)). While the energy transfer in turbulence is always one-way from mean flow to random motion, the local energy transfer can be two-way in SG-CFD depending on the sign of radial flow $u_r$.

By assuming that velocity anisotropy is due to finite halo spin, velocity dispersions can be decomposed into a gravity induced non-spin axial dispersion ($\sigma^2_{r_0}$) and a spin-induced dispersion that is dependent on the azimuthal flow $u'_\phi$ (Eqs. (45) to (47)). A new definition of halo anisotropic parameter $\beta_{h1}$ is proposed to include the effect of azimuthal flow $u'_\phi$ on anisotropy (Eq. (12)). Parameter $\beta_{h1}$ reduces to the usual definition $\beta_h$ (Eq. (13)) if $u'_\phi$ can be neglected. General solutions of mean flow and velocity dispersion are obtained in Section 3.3 (Eqs. (42), (43), (49), (50) and (51)) and subsequently applied to two limiting situations in Sections 3.4 and 3.5.

For "large" halos (high peak height $h$ at the early stage of halo life) with fast mass accretion and constant concentration, there exists a non-zero self-similar radial flow induced by fast halo growth (Eq. (65)). The radial flow drives outward mass flow in the core region and inward mass flow in the outer region (the gravitational infall). The halo surface energy can be significant due to the non-zero radial flow and low halo concentration such that the halo virial ratio $\gamma_v \approx 1.3 > 1$ (see Xu 2021b, Fig. 9). Angular momentum and rotational kinetic energy are transported by the radial flow (Eqs. (23) and (28)). The random motion draws kinetic energy from mean flow in core region, and vice versa in the outer region (Eq. (28) and Fig. 10). There is a net transfer from mean flow to random motion for the entire halo to maximize system entropy (negative $S_1$ in Table 4). A growing halo (the early stage of halo life) obtains its angular momentum through continuous mass acquisition (Eq. (25)) that predicts a linear increase of specific angular momentum $H_h$ with time $t$ (Eq. (105) and Table 3). The self-similar azimuthal flow is only dependent on radius $r$ and not significantly dependent on the polar angle $\theta$ with $\sigma_{r\theta} \ll 1$ (Eqs. (72), (87) and Fig. 5). The effective halo angular velocity $\omega_h$ is proportional to the Hubble parameter $H$ and decreases with time (Eq. (81)). Large halos rotate with a faster spinning core and slower outer region. For large halos, spin-induced dispersions are dominant ($\sigma^2_{r_0} \ll \gamma_v u'^2_\phi$) and two anisotropy parameters are equal, i.e. $\beta_{h1} \approx \beta_h$ (Fig. 9). The radial velocity momentum vanishes for large halos leads to a limiting concentration $c = 3.5$ (see Xu 2021b, Eq. (53)). Halo mass $m_h$, size $r_h$, and specific angular momentum $H_h$ all increase linearly with time $t$. All specific energies (radial/rotational/kinetic/potential) are time invariant for large halos (Table 3). The halo spin parameter $\lambda_p = 0.031$ and the variation of anisotropic parameter $\beta_{h1}$ in halo can be obtained analytically (Eq. (119), Eqs. (97) to (99) and Fig. 9).

The other limiting situation consists of "small" halos with a stable core (well bound and virialized) and low mass accretion (low peak height $h$ and the late stage of halo life with an almost constant halo mass, core mass, scale radius and an increasing halo concentration). The radial flow vanishes for small halos (Eq. (40)) without mass, momentum, and energy exchange between different spherical shells. Halo surface energy can be negligible due to the vanishing radial flow and high halo concentration (extremely low density at halo surface). Small halos rotate more like a rigid body. The halo angular velocity $\omega_h$ is relatively time-invariant. For small halos, non-spin axial dispersion is dominant ($\sigma^2_{r_0} \gg \gamma_v u'^2_\phi$) and the anisotropy parameters $\beta_{h1} \approx 0$ (Fig. 9). Small halos are more spherical in shape, incompressible for proper velocity, and isotropic ($\beta_{h1} = 0$). The radial and azimuthal dispersions are comparable for small halos and greater than the polar dispersion, i.e. $\sigma^2_{r_0} = \sigma^2_{\theta_0} = \sigma^2_{\phi_0}$ (Eq. (63)) that reflects a direct connection between mean flow and random motion in SG-CFD. The total kinetic energy including both random motion and mean flow is not equipartitioned along each direction with the greatest kinetic energy along azimuthal direction and the smallest along polar direction, i.e. $\sigma^2_{r_0} > \sigma^2_{\theta_0} > \sigma^2_{\phi_0}$. In small, short halos are isotropic ($\beta_{h1} = 0$), incompressible ($u_r = u_\theta = 0$), well bound and virialized structures.

Finally, the halo relaxation from high $\nu$ (early stage) to low $\nu$ (late stage) is studied with a continuous variation of halo shape, density profile, mean flow, momentum, and energy (dash lines in Fig. 11). Overall, shape parameter $\alpha$ decreases and concentration $c$ increases during relaxation (Eq. (149) and Fig. 11). The “vortex stretching” plays an important role for the energy cascade from large to small scales in turbulence. Due to the conservation of angular momentum, the stretching of vortex along the axis of rotation decreases the moment of inertia and increases the rotational kinetic energy. In SG-CFD, A isotropic "halo stretching" is proposed with increasing concentration and constant inner density (Fig. 12) and core mass. Halo stretching leads to increasing halo mass, moment of inertia (Eq. (154) and Fig. 13). In contrast to "vortex stretching", the halo angular momentum is not conserved and increasing with time (Fig. 14). The specific rotational kinetic energy is relatively conserved during halo stretching such that angular velocity $\omega_h$ decreases with time (Eq. (156)). With the coupling term $F_h$ (Eqs. (50) and (51)) approaching zero for low $\nu$ halos, there is a net transfer of spin-induced velocity dispersion to the non-spin axial dispersion ($\sigma^2_{r_10}$) (from part 2 to part 1 in Eq. (45)), i.e. an increasing in $\sigma^2_{r_10}$ and decreasing in $u'^2_\phi$, and coefficients $\alpha_{\phi}$, $\beta_{\phi}$ and $\gamma_{\phi}$. Halo becomes more isotropic with $\beta_{h1} \rightarrow 0$ during relaxation. The halo spin parameter increases with time due to faster increasing angular momentum than halo mass.

DATA AVAILABILITY

Two datasets underlying this article, i.e. a halo-based and correlation-based statistics of dark matter flow, are available on Zenodo (Xu 2022a, b), along with the accompanying presentation slides "A comparative study of dark matter flow & hydrodynamic turbulence and its applications” (Xu 2022c). All data files are also available on GitHub (Xu 2022d).

REFERENCES

Ahn J., Kim J., Shin J., Kim S. S., Choi Y. Y., 2014, Journal of the Korean Astronomical Society, 47, 77
Andersson B., Andersson R., 2012, Computational Fluid Dynamics for Engineers. Cambridge University Press, New York
Binney J., Tremaine S., 1987, Galactic Dynamics. Princeton University Press, Princeton
Bullock J. S., Kolatt T. S., Sigad Y., Somerville R. S., Kravtsov A. V., Klypin A. A., Primack J. R., Dekel A., 2001a, Monthly Notices of the Royal Astronomical Society, 321, 559
Bullock J. S., Dekel A., Kolatt T. S., Kravtsov A. V., Klypin A. A., Porciani C., Primack J. R., 2001b, Astrophysical Journal, 555, 240
Colberg J. M., White S. D. M., Jenkins A., Pearce F. R., 1999, Monthly Notices of the Royal Astronomical Society, 308, 593

Vol. 000, 1–22 (2022)
Cooray A., Sheth R., 2002, Physics Reports-Review Section of Physics Letters, 372, 1
Despali G., Giocoli C., Tormen G., 2014, Monthly Notices of the Royal Astronomical Society, 443, 3208
Frenk C. S., et al., 2000, arXiv:astro-ph/0007362v1
Hetznecker H., Burkert A., 2006, Monthly Notices of the Royal Astronomical Society, 370, 1905
Hoefl M., Macket J. P., Gottlober S., 2004, Astrophysical Journal, 602, 162
Jenkins A., et al., 1998, Astrophysical Journal, 499, 20
Klypin A., Yepes G., Gottlober S., Prada F., Hess S., 2016, Monthly Notices of the Royal Astronomical Society, 457, 4340
Kolmogoroff A., 1941a, Comptes Rendus De L Academie Des Sciences De L Urss, 30, 301
Kolmogoroff A. N., 1941b, Comptes Rendus De L Academie Des Sciences De L Urss, 32, 16
Kraichnan R. H., 1967, Physics of Fluids, 10, 1417
Neyman J., Scott E. L., 1952, Astrophysical Journal, 116, 144
Peebles P. J. E., 1969, Astrophysical Journal, 155, 393
Richardson L. F., 1922, Weather Prediction by Numerical Process. Cambridge University Press, Cambridge, UK
Sheth R. K., Mo H. J., Tormen G., 2001, Monthly Notices of the Royal Astronomical Society, 323, 1
Taylor G. I., 1932, Proceedings of the Royal Society of London Series a-Containing Papers of a Mathematical and Physical Character, 135, 685
Taylor G. I., 1938, Proceedings of the Royal Society of London Series a-Mathematical and Physical Sciences, 164, 0015
Vitvitska M., Klypin A. A., Kravtsov A. V., Wechsler R. H., Primack J. R., Bullock J. S., 2002, Astrophysical Journal, 581, 799
Weihsler R. H., Bullock J. S., Primack J. R., Kravtsov A. V., Dekel A., 2002, Astrophysical Journal, 568, 52
White S. D. M., 1984, Astrophysical Journal, 286, 38
Xu Z., 2021a, arXiv e-prints, p. arXiv:2109.09985
Xu Z., 2021b, arXiv e-prints, p. arXiv:2109.12244
Xu Z., 2021c, arXiv e-prints, p. arXiv:2110.03126
Xu Z., 2021d, arXiv e-prints, p. arXiv:2110.05784
Xu Z., 2021e, arXiv e-prints, p. arXiv:2110.09676
Xu Z., 2021f, arXiv e-prints, p. arXiv:2110.13885
Xu Z., 2022c, A comparative study of dark matter flow & hydrodynamic turbulence and its applications, doi:10.5281/zenodo.6569901, http://dx.doi.org/10.5281/zenodo.6569901
Xu Z., 2022d, Dark matter flow dataset, doi:10.5281/zenodo.6586212, https://github.com/ZhijieXu2022/dark_matter_flow_dataset
Xu Z., 2022a, Dark matter flow dataset Part I: Halo-based statistics from cosmological N-body simulation, doi:10.5281/zenodo.6541230, http://dx.doi.org/10.5281/zenodo.6541230
Xu Z., 2022b, Dark matter flow dataset Part II: Correlation-based statistics from cosmological N-body simulation, doi:10.5281/zenodo.6569898, http://dx.doi.org/10.5281/zenodo.6569898
Xu Z., 2022c, arXiv e-prints, p. arXiv:2202.00910
Xu Z., 2022d, arXiv e-prints, p. arXiv:2202.02991
Xu Z., 2022e, arXiv e-prints, p. arXiv:2202.04054
Xu Z., 2022f, arXiv e-prints, p. arXiv:2202.06515
Xu Z., 2022g, arXiv e-prints, p. arXiv:2202.07240
Xu Z., 2022h, arXiv e-prints, p. arXiv:2203.05606
Xu Z., 2022i, arXiv e-prints, p. arXiv:2203.06899
Zhao D. H., Jing Y. P., Mo H. J., Bower G., 2009, Astrophysical Journal, 707, 354