BRST Quantisation and the Product Formula for the Ray-Singer Torsion

by

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Abstract: We give a quantum field theoretic derivation of the formula obeyed by the Ray-Singer torsion on product manifolds. Such a derivation has proved elusive up to now. We use a BRST formalism which introduces the idea of an infinite dimensional Universal Gauge Fermion, and is of independent interest being applicable to situations other than the ones considered here. We are led to a new class of Fermionic topological field theories. Our methods are also applicable to combinatorially defined manifolds and methods of discrete approximation such as the use of a simplicial lattice or finite elements. The topological field theories discussed provide a natural link between the combinatorial and analytic torsion.

§ 1. Introduction

Quantum field theories have, since their inception, received a considerable onslaught from a variety of mathematical techniques, particularly those drawn from analysis. However in the last decade there has been ample evidence that the way forward is considerably illuminated if techniques from topology are used. It has been learned, particularly in the case of gauge theories, that quantum and statistical fluctuations are sensitive to the global properties of the manifold on which they occur, topological field theories being, perhaps, the ultimate example.

In this paper we discuss the field theoretic realizations of the Ray-Singer torsion, which arises as a power of the partition function of an appropriate topological field theory. This subject has of course been considered before however one obstacle remained, and that was a field theoretic demonstration of the product formula (cf. Birmingham et al. [1]) which states that the torsion on a product manifold, \( M = M_1 \times M_2 \), is the torsion of one factor raised to the the Euler character of the the other factor. The principal purpose of this paper is therefore to give a field theoretic demonstration of this formula.

The techniques we employ are a mixture of harmonic analysis and field theory. A detailed understanding of the harmonic analysis is necessary in order to establish certain isomorphisms, and to construct a detailed decomposition of the harmonic forms appropriate for a product manifold. The principal results of the paper are
(i) The decomposition of the kernels of $d$ and $d^*$ on a product manifold in terms of appropriate data on the factors.

(ii) The construction of the gauge fixing sector for an $n$-form field in arbitrary dimensions, together with the decomposition of this on a product manifold.

(iii) The introduction of a new class of Fermionic topological field theories, and a larger graded class of topological field theories whose partition function $Z$ is the quarter power of the torsion $T (Z = T^{1/4})$, for all $M$. This latter graded class exists on all odd dimensional manifolds $M$; and if $\dim M = 2n + 1$, then the corresponding topological field theories are Bosonic for $n$ odd and Fermionic for $n$ even.

(iv) A proof of the product formula in topological field theory. This field theory approach makes manifest the connection between the analytic and combinatorial torsions since the topological BRST action that arises is easily considered on a simplicial lattice. This amounts to one of many possible regularisation procedures in field theory. The convergence of the eigenvalues of the combinatorial Laplacian to those of the continuum Laplacian (see Müller [2]) then establishes the equivalence of the combinatorial and analytic torsions in the field theory context.

The layout of the paper is as follows. In section 2 we introduce the Ray-Singer torsion, and analyse of the Laplacians on $r$-forms, establish some isomorphisms between the kernel of $d$ and the kernel of $d^*$ for these spaces and the associated “shifted Poincaré duality”. We establish the corresponding decomposition for product manifolds. The section ends with a simple derivation of the product formula for the torsion. The starting point for this analysis is the definition of the analytic Ray-Singer torsion in terms of zeta functions.

Section 3 contains a description of the BRST method for constructing the path integral measure for the quantisation of an $n$-form gauge field where the action for this field is invariant under the addition to this field of a closed $(n - 1)$-form field. The method is the gauge Fermion construction of Batalin and Vilkovisky [7,8]. We introduce a powerful notation for handling this construction, whereby the dimension of the manifold plays a minor rôle. The gauge Fermion triangle is formally infinite, the gauge fixing necessary is carried out by a certain finite sub-triangle determined by the particular field to be found at its apex. We finish the section by considering the gauge Fermion on product manifolds. The procedures used are purely algebraic and do not rely on the manifold being a differentiable one. The results are readily translatable to a simplicial manifold.

In section 4 we introduce our topological field theories. By choosing the field which forms the lower apex of the gauge Fermion triangle to be Fermionic we find an interesting class of topological field theories, which alternates between Fermionic and Bosonic as the dimension changes. Each theory has the torsion to the quarter power as its partition function. We also consider briefly the wider class of topological actions where two fields are used and show that these have the torsion to an appropriate power as partition function. We finish the section by considering these theories on product manifolds. An analysis of the resulting partition function readily gives a derivation of the product formula. This procedure is readily implementable on simplicial manifolds and can be interpreted as establishing the equivalence of the combinatorial and analytic torsions.

§ 2. The Ray-Singer torsion

The Ray-Singer torsion, which we denote by $T(M, E)$, is a positive real number which is
defined when one has a flat connection \( A \) on a bundle \( E \) over a compact manifold \( M \) together with a certain representation of the fundamental group \( \pi_1(M) \).

In brief the definition of \( T(M, E) \) arises as follows: For a flat connection \( A \) on \( M \) to give rise to a non-trivial situation one requires \( A \) to have non-zero holonomy and thus \( M \) must have a non-trivial \( \pi_1(M) \). The holonomy of \( A \) round a based loop \( C \) is then realised in the usual way by the path ordered integral

\[
h(C) = P \exp \left[ \int_C A \right]
\]

These \( h(C) \) then provide a representation of \( \pi_1(M) \), the one referred to above, in the gauge group \( G \) of \( A \). For the torsion \( T(M, E) \) one must consider the volume elements \( V^r \) associated with each cohomology group \( H^r(M; E) \). Determinants give a natural definition for volume elements and, in terms of de Rham cohomology, one can realise these \( V^r \) as determinants of Laplacians

\[
Det \Delta^E_r
\]

where

\[
\Delta^E_r = (d_A d^*_A + d^*_A d_A)
\]

is the Laplacian on \( r \)-forms with coefficients in \( E \) and \( d_A^2 = 0 \) since \( A \) is flat*. Zeta functions can be used to define these determinants provided \( \Delta^E_r \) is positive which is assumed. The torsion is then defined by

\[
\ln T(M, E) = \sum_{r=0}^{n} (-1)^r r \ln Det \Delta^E_r, \quad n = \dim M
\]

\[
= - \sum_{r=0}^{n} (-1)^r r \zeta'_{\Delta^E_r}(0)
\]

where we have used the standard definition of a determinant via the zeta function: i.e. if \( O \) is an elliptic differential or pseudo-differential operator with positive spectrum \( \{\lambda_n\} \) then \( Det O \) is defined via

\[
\ln Det O = -\zeta'_O(0), \quad \text{where } \zeta_O(s) = \sum_{\lambda_n} \frac{\Gamma_n}{\lambda_n^s}
\]

and \( \Gamma_n \) denotes the degeneracies of the \( \lambda_n \). For convenience we also write

\[
\ln T(M, E) = \frac{d\tau(s)}{ds} \bigg|_{s=0}
\]

\[
\text{where } \tau(s) = - \sum_{r=0}^{n} (-1)^r r \zeta_{\Delta^E_r}(s)
\]

* Because of this fact, from now on, we shall abbreviate \( d_A \) to simply \( d \).
Some standard properties [2,3,4] of $T(M,E)$ are:

(i) $T(M,E) > 0$

(ii) $T(M,E) = 1$ if dim $M$ is even

(iii) $T(M,E)$ is independent of the metric used to construct the $\Delta^E_r$.

Property (ii) above means that the non-trivial cases occur, in the main, when $M$ is odd-dimensional; although one can sometimes make non-trivial deductions in which $T(M,E)$ plays a role when $M$ is even dimensional e.g. if $M$ is a complex manifold cf. Ray-Singer [5] and Witten [6]. Property (iii) asserts that $T(M,E)$ is a metric independent quantity and suggests, what is in fact true, that $T(M,E)$ may be a topological invariant of $M$.

We now need to establish some notation and also properties of the torsion relevant for us here. Central to the discussion are the spectra of the Laplacians $\Delta^E_r$; so we define $\mathcal{E}_r(M,\lambda)$ to be the eigenspace of $\Delta^E_r$ corresponding to the eigenvalue $\lambda$.

$$\mathcal{E}_r(M,\lambda) = \{ \omega \in \Gamma(M, \Lambda^r T^* M \otimes E) : \Delta^E_r \omega = \lambda \omega \}$$

(2.7)

where $\Gamma(M, \Lambda^r T^* M \otimes E)$ is the space of sections, over $M$, of the $r$-form bundle $\Lambda^r T^* M \otimes E$. Now this eigenspace $\mathcal{E}_r(M,\lambda)$ has a decomposition [2] which is particularly useful for the subsequent quantum field theory. Recall that $d$ and $d^*$ act as shown below

$$\cdots \xrightarrow{d} \mathcal{E}_r(M,\lambda) \xrightarrow{d} \mathcal{E}_{r+1}(M,\lambda) \xrightarrow{d} \cdots$$

(2.8)

$$\cdots \xleftarrow{d^*} \mathcal{E}_r(M,\lambda) \xleftarrow{d^*} \mathcal{E}_{r+1}(M,\lambda) \xleftarrow{d^*} \cdots$$

and so

$$\mathcal{E}_r(M,\lambda) \simeq \text{Im} \, d \oplus \ker d^* \simeq \text{Im} \, d^* \oplus \ker d$$

(2.9)

But our positivity requirement means that

$$\ker d \cap \ker d^* = \phi$$

and it is then easy to check that if we further define

$$\mathcal{E}'_r(M,\lambda) = \{ \omega \in \mathcal{E}_r(M,\lambda) : d\omega = 0 \} \quad \mathcal{E}''_r(M,\lambda) = \{ \omega \in \mathcal{E}_r(M,\lambda) : d^*\omega = 0 \}$$

then $\mathcal{E}_r(M,\lambda)$ has the decomposition

$$\mathcal{E}_r(M,\lambda) = \mathcal{E}'_r(M,\lambda) \oplus \mathcal{E}''_r(M,\lambda)$$

(2.10)

allowing one to verify that

$$\mathcal{E}''_r(M,\lambda) \xrightarrow{d^*/\sqrt{\lambda}} \mathcal{E}'_{r+1}(M,\lambda)$$

(2.11)

is an isomorphism with inverse $d^*/\sqrt{\lambda}$ so that

$$\mathcal{E}''_r(M,\lambda) \simeq \mathcal{E}'_{r+1}(M,\lambda)$$

(2.12)

$\dagger$ $M$ is always a compact closed Riemannian manifold
For the computation of the torsion it is useful to display the information encapsulated in 2.10 and 2.12 above in a table. Doing this we have

\[ \mathcal{E}_n(M, \lambda) = \mathcal{E}'_n \oplus \phi \]
\[ \mathcal{E}_{n-1}(M, \lambda) = \mathcal{E}'_{n-1} \oplus \mathcal{E}''_{n-1} \]
\[ \vdots \]
\[ \mathcal{E}_r(M, \lambda) = \mathcal{E}'_r \oplus \mathcal{E}''_r \]
\[ \mathcal{E}_{r-1}(M, \lambda) = \mathcal{E}'_{r-1} \oplus \mathcal{E}''_{r-1} \]
\[ \vdots \]
\[ \mathcal{E}_1(M, \lambda) = \mathcal{E}'_1 \oplus \mathcal{E}''_1 \]
\[ \mathcal{E}_0(M, \lambda) = \phi \oplus \mathcal{E}''_0 \]

Note that the arrow in 2.13 simply connects pairs of spaces which are isomorphic.

A final stage in making use of the refinement of the spectral data embodied in the decomposition table above is to restrict the Laplacians \( \Delta^E_r \) to the spaces \( \mathcal{E}'_r(M, \lambda) \) and \( \mathcal{E}''_r(M, \lambda) \). These two restrictions correspond precisely to the operators \( dd^* \) (acting on \( \mathcal{E}'_r(M, \lambda) \)) and \( d^*d \) (acting on \( \mathcal{E}''_r(M, \lambda) \)) respectively. In this way we construct the associated zeta functions for these operators giving

\[ \zeta_{dd^*}(s) = \sum_\lambda \frac{\Gamma'_r(\lambda)}{\lambda^s}, \quad \zeta_{d^*d}(s) = \sum_\lambda \frac{\Gamma''_r(\lambda)}{\lambda^s} \]

and

\[ \zeta_{\Delta^E}(s) = \zeta_{dd^*}(s) + \zeta_{d^*d}(s) \]

in an obvious notation, and clearly \( \Gamma_r(\lambda) = \Gamma'_r(\lambda) + \Gamma''_r(\lambda) \). However if we now return to the definition 2.4 of the torsion \( T(M, E) \) which was

\[ \ln T(M, E) = -\sum_0^n (-1)^r r \zeta'_{(dd^*+d^*d),r}(0) \]

and use the information contained in table 2.13, we find that

\[ \ln T(M, E) = -\sum_0^{n-1} (-1)^r \zeta'_{d^*d}(0) \]

Of course an analogous formula exists for \( \tau(s) \).

It will also be useful later to take account of isomorphisms that arise because of Poincaré duality. In the first instance Poincaré duality asserts that

\[ \mathcal{E}_r(M, \lambda) \simeq \mathcal{E}_{n-r}(M, \lambda) \]
Refining this with respect to the decomposition 2.10 gives the further statement that

\[
\mathcal{E}_r'(M, \lambda) \simeq \mathcal{E}_{n-r}'(M, \lambda)
\]
\[
\mathcal{E}_r''(M, \lambda) \simeq \mathcal{E}_{n-r}'(M, \lambda)
\]

Finally if we use our table above we find that a ‘shifted Poincaré duality’ exists for the spaces \(\mathcal{E}_r''(M, \lambda)\) alone, namely

\[
\mathcal{E}_r''(M, \lambda) \simeq \mathcal{E}_{n-r-1}'(M, \lambda)
\]

This gives a further refinement to the formula for the torsion since we can now easily verify that, for odd \(n, n = 2m + 1\), one has

\[
\ln T(M, E) = -2 \sum_{r=0}^{m-1} (-1)^r \zeta_{d^r d^r}(0) - (-1)^m \zeta_{d^r d^r}(0), \quad \text{where } n = 2m + 1
\]

and for even \(n, n = 2m\) we have a cancellation of terms rather than a doubling and hence

\[
\ln T(M, E) = 0
\]

This completes our brief account of the properties of the torsion that we need for this paper. We are particularly interested in \(T(M, E)\) for the case where

\[
M = M_1 \times M_2
\]

and we turn to this matter in the next section.

\section*{§§ 2.1 The Decomposition of \(\mathcal{E}_r'\) and \(\mathcal{E}_r''\) on Product Manifolds}

We shall now assume that

\[
M = M_1 \times M_2, \quad \text{with } \dim M \text{ odd}
\]

we also take the convention that \(\dim M_1\) is odd and \(\dim M_2\) is even. In this situation if we take \(M_2\) to be simply connected, so that it supports no non-trivial flat connections, then as we shall demonstrate below \(T(M, E)\) has the property that

\[
T(M, E) = \{T(M, E_1)\}^{\chi(M_2)}
\]

where \(E_1\) denotes the flat bundle over \(M_1\) and \(\chi(M_2)\) is the usual Euler-Poincaré characteristic of \(M_2\).

Now using the notation defined above on \(M_1 \times M_2\) we have

\[
\mathcal{E}_r(M_1 \times M_2, \nu) = \mathcal{E}_r'(M_1 \times M_2, \nu) \oplus \mathcal{E}_r''(M_1 \times M_2, \nu)
\]
\[
= \bigoplus_{p+q=r} \mathcal{E}_p(M_1, \lambda) \wedge \mathcal{E}_q(M_2, \mu), \quad \text{with } \nu = \lambda + \mu
\]
Vital for us is the relation implicit in eq. 2.24 above between the spaces on its RHS. In particular we need an explicit construction of the space $\mathcal{E}_{r''}(M_1 \times M_2, \nu)$. This we now provide. The appropriate projection operator is $d^*d$ and so, formally, we have

$$\mathcal{E}_{r''}(M_1 \times M_2, \nu) = \frac{dd^*}{\nu} \mathcal{E}_r(M_1 \times M_2, \nu)$$

(2.25)

Hence we write

$$\mathcal{E}_{r''}(M_1 \times M_2, \nu) = \bigoplus_{p+q=r} \frac{d^*d}{\nu} (\mathcal{E}_p(M_1, \lambda) \wedge \mathcal{E}_q(M_2, \mu))$$

$$= \bigoplus_{p+q=r} \frac{d^*}{\nu} \{d\mathcal{E}_p(M_1, \lambda) \wedge \mathcal{E}_q(M_2, \mu) \oplus (-1)^p \mathcal{E}_p(M_1, \lambda) \wedge d\mathcal{E}_q(M_2, \mu)\}$$

$$= \bigoplus_{p+q=r} \frac{d^*}{\nu} \{d\mathcal{E}_{p'}(M_1, \lambda) \wedge \mathcal{E}_q(M_2, \mu) \oplus (-1)^p d\mathcal{E}_p(M_1, \lambda) \wedge d\mathcal{E}_q''(M_2, \mu)\}$$

$$\oplus (-1)^p d^*\mathcal{E}_p(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu) \oplus (-1)^2 p \mathcal{E}_p(M_1, \lambda) \wedge d^*\mathcal{E}_q''(M_2, \mu)\}$$

(2.26)

Next we must use the information from table 2.13 and, doing this, we find that

$$\mathcal{E}_{r''}(M_1 \times M_2, \nu) = \bigoplus_{p+q=r} \frac{1}{\nu} \{\lambda \mathcal{E}_{p''}(M_1, \lambda) \wedge \mathcal{E}_q(M_2, \mu)$$

$$\oplus (-1)^{p+1} \sqrt{\lambda \nu} \mathcal{E}_{p+1}(M_1, \lambda) \wedge \mathcal{E}_{q-1}(M_2, \mu)$$

$$\oplus (\nu) \mathcal{E}'_p(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu) \oplus \mu \mathcal{E}_p(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu)\}$$

(2.27)

Expanding now the terms $\mathcal{E}_q(M_2, \mu)$ and $\mathcal{E}_p(M_1, \lambda)$, these being the only terms in the above without a prime or a double-prime, we get the (temporarily) longer expression

$$\mathcal{E}_{r''}(M_1 \times M_2, \nu) = \bigoplus_{p+q=r} \frac{1}{\nu} \{\lambda \mathcal{E}_{p''}(M_1, \lambda) \wedge \mathcal{E}_q'(M_2, \mu) \oplus \lambda \mathcal{E}'_p(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu)$$

$$\oplus (-1)^{p+1} \sqrt{\lambda \nu} \mathcal{E}_{p+1}(M_1, \lambda) \wedge \mathcal{E}_{q-1}(M_2, \mu)$$

$$\oplus (-1)^p \sqrt{\lambda \nu} \mathcal{E}'_{p-1}(M_1, \lambda) \wedge \mathcal{E}_{q+1}(M_2, \mu)$$

$$\oplus \mu \mathcal{E}'_p(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu) \oplus \mu \mathcal{E}'_p(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu)\}$$

(2.28)

This now simplifies to

$$\mathcal{E}_{r''}(M_1 \times M_2, \nu) = \bigoplus_{p=0}^{r-1} \frac{1}{\nu} \mathcal{E}_p''(M_1, \lambda) \wedge \mathcal{E}_p'(M_2, \mu) \bigoplus \mathcal{E}_p''(M_1, \lambda) \wedge \mathcal{E}_p'(M_2, \mu)$$

$$\bigoplus (-1)^p \sqrt{\lambda \nu} \mathcal{E}_{p+1}(M_1, \lambda) \wedge \mathcal{E}_{q-1}(M_2, \mu)$$

$$\bigoplus (-1)^p \sqrt{\lambda \nu} \mathcal{E}'_{p-1}(M_1, \lambda) \wedge \mathcal{E}_{q+1}(M_2, \mu)$$

$$\bigoplus \mu \mathcal{E}_p'(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu) \bigoplus \mu \mathcal{E}_p'(M_1, \lambda) \wedge \mathcal{E}_q''(M_2, \mu)\}$$

(2.29)
Some summands are now pairable off and after further minor algebraic adjustments we find that

\[ E_r''(M_1 \times M_2, \nu) = \bigoplus_{p=0}^{r-1} \left\{ C_p(\lambda, \mu) E_p''(M_1, \lambda) \wedge E_{r-p}'(M_2, \mu) \right\} + C_p(\mu, \lambda) E_{p+1}'(M_1, \lambda) \wedge E_{r-p-1}''(M_2, \mu) \bigoplus_{p=0}^{r} E_p''(M_1, \lambda) \wedge E_{r-p}''(M_2, \mu) \]

where \( C_p(\lambda, \mu) = \frac{(\lambda + (-1)^p \sqrt{\lambda \mu})}{\nu} \)

(2.30)

We can now proceed to give a much more compact formula for \( E_r''(M_1 \times M_2, \nu) \) by first defining the spaces \( V_r''(\nu), V_{p,r}'(\nu), \) and \( W_r''(\nu), W_{p,r}'(\nu) \). We define

\[ V_{p,r}''(\nu) = C_p(\lambda, \mu) E_p''(M_1, \lambda) \wedge E_{r-p}'(M_2, \mu) \]

with \( V_r''(\nu) = \bigoplus_{p=0}^{r} V_{p,r}''(\nu) \)

and \( W_{p,r}''(\nu) = E_p''(M_1, \lambda) \wedge E_{r-p}''(M_2, \mu) \)

with \( W_p''(\nu) = \bigoplus_{p=0}^{r} W_{p,r}''(\nu) \)

Our final formula for \( E_r''(M_1 \times M_2, \nu) \) is then

\[ E_r''(M_1 \times M_2, \nu) = V_r''(\nu) \oplus W_r''(\nu), \quad \nu = \lambda + \mu, \lambda > 0, \mu > 0 \]

(2.32)

Of course an exactly analogous formula exists for \( E_r'(M_1 \times M_2, \nu) \) and to give it we now make the parallel definitions

\[ V_{p,r}'(\nu) = C_p(\lambda, \mu) E_p'(M_1, \lambda) \wedge E_{r-p}''(M_2, \mu) \]

with \( V_r'(\nu) = \bigoplus_{p=1}^{r} V_{p,r}'(\nu) \)

and \( W_{p,r}'(\nu) = E_p'(M_1, \lambda) \wedge E_{r-p}''(M_2, \mu) \)

with \( W_p'(\nu) = \bigoplus_{p=0}^{r-1} W_{p,r}'(\nu) \)

and so we have

\[ E_r'(M_1 \times M_2, \nu) = V_r'(\nu) \oplus W_r'(\nu), \quad \nu = \lambda + \mu, \lambda > 0, \mu > 0 \]

(2.34)

A particularly important simplification happens to the decompositions 2.32 and 2.34 above when, as happens in the present paper, one can have \( \mu = 0 \). When \( \mu = 0 \) the spaces \( E_r'(M_2, 0) \) and \( E''(M_2, 0) \) coincide and contain harmonic forms. We shall denote these
harmonic spaces by $\mathcal{E}_r^H(M_2)$. It is then easy to check that the decompositions above are replaced by

$$
\mathcal{E}'_r(M_1 \times M_2, \nu) = \bigoplus_{p=1}^r \mathcal{E}'_p(M_1, \lambda) \wedge \mathcal{E}^H_{r-p}(M_2), \quad \nu = \lambda + \mu, \ \mu = 0
$$

$$
\mathcal{E}''_r(M_1 \times M_2, \nu) = \bigoplus_{p=0}^r \mathcal{E}''_p(M_1, \lambda) \wedge \mathcal{E}^H_{r-p}(M_2), \quad \nu = \lambda + \mu, \ \mu = 0
$$

These decompositions are essential for sections 3 and 4.

§§ 2.2 The Product Formula

Recalling from (2.6) that the logarithm of the torsion is given by the derivative at $s = 0$ of $\tau(s)$ which from the above discussion takes the form

$$
\tau(s) = - \sum_{r=0}^{m-1} \sum_{\nu} (-1)^r \frac{\Gamma''_r(\nu)}{\nu^s}
$$

(2.36)

Now from our discussion above we have with $\nu = \lambda + \mu$ and $\lambda, \mu > 0$, and recalling that $\Gamma''_r = \text{dim} \mathcal{E}_r''$ we have

$$
\Gamma''_r(\nu) = \sum_{p=0}^r \Gamma''_p(\lambda) \Gamma''_{r-p}(\mu) + \sum_{p=0}^{r-1} \Gamma''_p(\lambda) \Gamma'(r-p)(\mu)
$$

(2.37)

which by the isomorphisms identifying $\mathcal{E}'_q \simeq \mathcal{E}'_{q-1}$ gives

$$
\Gamma''_r(\nu) = \sum_{p=0}^r \Gamma''_p(\lambda) \Gamma''_{r-p}(\mu) + \sum_{p=0}^{r-1} \Gamma''_p(\lambda) \Gamma''_{r-p-1}(\mu)
$$

(2.38)

Noticing that in (2.36) if we consider a fixed eigenvalue $\nu$ this latter decomposition implies when summed over $r$

$$
\sum_{r=0}^{d-1} (-1)^r \sum_{p=0}^r \Gamma''_p(\lambda) \Gamma''_{r-p}(\mu) + \sum_{r=0}^{d-1} (-1)^r \sum_{p=0}^{r-1} \Gamma''_p(\lambda) \Gamma''_{r-p-1}(\mu)
$$

(2.39)

Now letting $r' = r - 1$ in the second sum and setting $\Gamma''_{-1} = 0$ we get on some cancellation of terms

$$
(-1)^{d-1} \sum_{p=0}^{d-1} \Gamma''_p(\lambda) \Gamma''_{d-p-1}(\mu)
$$

Now letting $\text{dim} M_1 = d_1$ and $\text{dim} M_2 = d_2$, so that $d = d_1 + d_2$; then since $\Gamma''_p(\lambda) = 0$ for $p > (d_1 - 1)$, and $\Gamma''_q = 0$ for $q > (d_2 - 1)$, there is no non-zero contribution to this sum. We conclude that the sum over degeneracies contributing to $\tau(s)$ vanishes for any non-zero $\mu$ and $\nu$. 
Thus the only contribution to (2.36) comes from \( \nu \) with \( \mu = 0 \) since the other terms cancel. It is important to observe that the terms with \( \mu \) and \( \nu \) non-zero cancel eigenvalue by eigenvalue, due to the alternation in signs from different degree forms and the isomorphisms described above. This will be useful to our discussion of the field theory also. A little rearrangement of this sum then shows that the sum \( \sum_{q=0}^{d_2} (-1)^q \tau_q(0) \) can be factored out; but \( \tau_q(0) = b_q \) the \( q \)th Betti number, this means that the preceding sum is the Euler character of \( M_2 \). Thus

\[
\tau(s) = \chi(M_2)\tau_1(s)
\] (2.40)

Taking the derivative of this at \( s = 0 \) establishes the product formula for the Ray-Singer torsion.

### § 2.3 Action Terms and Product Manifolds

We employ an important and useful notation in the action to denote the degree of the forms appearing therein:

- \( f_r \) denotes a form on \( M \) of degree \( r \)
- \( \bar{f}_r \) denotes a form on \( M \) of degree \((n - r)\)

where \( n = \dim M \) (2.41)

This allows us to develop a much clearer notation for the BRST field theory below; in particular the dimension \( n \) of \( M \) does not explicitly appear and expressions such as \( g_r \wedge f_r \) are immediately recognisable as volume forms.

To actually compute the functional integral for the torsion we have only three types of contribution to the action to consider, namely

\[
\int_M f''_{r-1} \wedge d\bar{f}''_r, \quad \int_M \bar{f}'_{r-1} \wedge d^* f'_r, \quad \text{and} \quad \int_M \bar{f}'_r \wedge d^* d f''_r
\] (2.42)

However these are expressible using natural inner products induced by the metric on \( M \).

In general for two \( r \)-forms \( \omega \) and \( \nu \) on \( M \) we define their inner product by

\[
< \omega, \nu > = \int_M tr(\omega \wedge *\nu)
\] (2.43)

In what follows we shall relieve the notation by omitting the trace on the RHS and the wedge between the forms.

Turning now to the first of our three contributions above we have*

\[
\int_M f''_{r-1} d\bar{f}''_r = (-1)^{(n+1)(r-1)} \int_M f''_{r-1} *^2 d\bar{f}''_r
\]

\[
= (-1)^{(n+1)(r-1)} < f''_{r-1}, *d\bar{f}''_r >
\] (2.44)

* For keeping abreast of the signs in some of these formulae it is useful to note that the operators \( * : \Omega_r(M) \rightarrow \Omega_{n-r}(M) \) and \( d^* : \Omega_r(M) \rightarrow \Omega_{r-1}(M) \) satisfy \( *^2 = (-1)^{r(n-r)} \) and \( d^* = (-1)^{nr+n+1} * d^* \).
But we know from 2.12 that \(*d*\ gives rise to an isomorphism between the spaces \(E''_\nu(M, \nu)\) and \(E'_{\nu+1}(M, \nu)\) for each \(\nu\). To make use of that here we use the decomposition

\[
E''_\nu(M) = \bigoplus \nu E''_{\nu}(M, \nu) \quad (2.45)
\]

Now we let \(\{e''_\nu(M, \nu, i)\}\) denote an orthonormal basis for \(E''_{\nu}(M, \nu)\) \((i\) labels the degeneracy\) and write, remembering that \(\bar{f}\) denotes an \((n-r)\)-form,

\[
\begin{align*}
\bar{f''}_r &= \sum_{\nu, i} \bar{c''}_r(\nu, i)e''_{n-r}(M, \nu, i) \\
\bar{f'_{r-1}} &= \sum_{\nu, j} \bar{c'_{r-1}}(\nu, j)e'_{r-1}(M, \nu, j)
\end{align*} \quad (2.46)
\]

where \(\bar{c''}_r(\nu, i)\) and \(\bar{c'_{r-1}}(\nu, j)\) are constants. But using the isomorphisms 2.12 and 2.19 we calculate that

\[
\int_M \bar{f''}_{r-1}d\bar{f''}_r = (-1)^{(n+1)(r-1)} \sum_{\nu_1, \nu_2, j} \bar{c'_{r-1}}(\nu_1, j) \bar{c''}_r(\nu_2, i) \sqrt{\nu} < e''_{r-1}(M, \nu_1, j), e''_{r}(M, \nu_2, i) > \\
= (-1)^{(n+1)(r-1)} \sum_{\nu, i} \bar{c''}_r(\nu, i) \sqrt{\nu} \quad (2.47)
\]

Similarly, and in an analogous notation, we calculate that

\[
\begin{align*}
\int_M \bar{f'_{r-1}}d^*f'_{r} &= (-1)^{(n+1)(r-1)} \sum_{\nu, i} \bar{c'_{r-1}}(\nu, i) \sqrt{\nu} \\
\text{and} \int_M \bar{f'_{r}}d^*df''_{r} &= (-1)^{(n+1)(r-1)} \sum_{\nu, i} \bar{c'_{r}}(\nu, i) \sqrt{\nu} \quad (2.48)
\end{align*}
\]

Hence 2.48 and 2.49 give our evaluations of the three generic action terms of 2.42 above.

Let us now consider what happens on a product manifold, \(M = M_1 \times M_2\). In this case the spaces \(E'_r(M, \nu)\) and \(E''_r(M, \nu)\) decompose further as described above. It is convenient for our purposes to separate off the harmonic contribution to \(M_2\) from the remainder. Thus, denoting the orthonormal basis for the space \(E^H_j(M_2)\) of harmonic \(j\)-forms by \(\{\varepsilon^a_j\}\), we have

\[
f_r = \tilde{f}_r + \sum_{a=1}^{b_2} \sum_{j=0}^{\min(r, d_2)} f_{r-j}^a \varepsilon^a_j, \quad d_2 = \dim M_2
\]

The term \(\tilde{f}_r\) which denotes the field remaining after separating off the harmonic contribution, further decompose into \(f'_r + f''_r\) of the form (2.46) where \(\mu > 0\). But, we have shown that
a basis for such forms on a product manifold is provided by 2.31 and 2.33, thus \( \tilde{f}''' \) further decomposes as

\[
\tilde{f}'''_r = \sum_{p=0}^{r-1} f^{V''}_{r,p} + \sum_{p=0}^{r} f^{W''}_{r,p}
\]

Similarly \( \tilde{f}_r \) decomposes as

\[
\tilde{f}_r = \tilde{f}_r + \sum_{a=1}^{\min(r,d_2)} \sum_{j=0}^{b_j} \tilde{f}_r^{a} f^{a}_{r-j} \epsilon_j^a, \quad d_2 = \dim M_2
\]

where this time the sum over \( a \) is a sum over an orthonormal basis for the space \( E^H_{d_2-j}(M_2) \) whose dimension is \( (d_2 - j) = b_j; \ b_j \) being the \( (d_2 - j) \)-th Betti number. For an orientable manifold \( \tilde{b}_j = b_j \) and we will assume this to be the case though little modification is necessary to treat the more general case. Were both factors to have harmonic contributions the above formulae generalize in a symmetrical fashion, this case is not of interest here.

Now, retaining only terms which do not integrate to zero, our three fiducial terms have the decomposition

\[
\tilde{f}_{r-1} d^* f_r = \tilde{f}_{r-1} d^* \tilde{f}_r + \sum_{j=0}^{r} \sum_{a=1}^{b_j} \tilde{f}^{a}_{r-j-1} d^* f^{a}_{r-j} \epsilon^{a}_{j}
\]

where \( \epsilon^a = \tilde{\epsilon}^a_j \epsilon_j^a \) and so is a volume form on \( M_2 \).

Similarly

\[
f_{r-1} d \tilde{f}_r = f_{r-1} d \tilde{f}_r + \sum_{j=0}^{r} \sum_{a=1}^{b_j} f^{a}_{r-j-1} d f^{a}_{r-j} \epsilon^{a}_{j}
\]

and

\[
\tilde{f}_r d^* df_r = \tilde{f}_r d^* \tilde{f}_r + \sum_{j=0}^{r} \sum_{a=1}^{b_j} \tilde{f}^{a}_{r-j} d^* f^{a}_{r-j} \epsilon^{a}_{j}
\]

We will have occasion to use these formulae in the next sections. Now we turn to the matter of the BRST quantization which we will need for the computation of the torsion.

\section*{§ 3. BRST Quantisation: The Gauge Fermion Construction}

We wish to calculate the partition function associated with the functional integral over the action \( S[f_n] \)

\[
Z = \int \mu[f_n] e^{-S[f_n]} \tag{3.1}
\]

where the action \( S[f_n] \) is invariant under the local gauge transformation

\[
f_n \rightarrow f_n + d\omega_{n-1} \tag{3.2}
\]

The functional measure \( \mu[f_n] \) must contain gauge fixing delta functionals to ensure that the integration is performed only over one of the set of \( f_n \) which are equivalent under such
a gauge transformation. An action that will be of special concern to us in the context of topological field theory and the Ray-Singer torsion is

\[ S[f_n] = i \int_M f_n df_n \]  

(3.3)

where \( f_n \) is a matrix valued \( n \)-form. However since the methods of this section are not particular to the action \( S[f_n] \) chosen we will not restrict ourselves to any particular action for the present. Our focus is on the construction of the measure \( \mu[f_n] \).

This measure is most simply constructed by extending the set of fields being integrated over, in such a way that on integrating out the additional fields the result yields the desired measure \( \mu[f_n] \). The necessary extended set of fields and resulting action are obtained by replacing the original transformation (3.2) by a BRST transformation

\[ \delta f_n = df_{n-1} \]  

(3.4)

The transformation \( \delta \) is of Fermionic character in that the field \( f_{n-1} \) has the opposite statistics to those of the field \( f_n \). Notice that the original action is invariant under this BRST transformation despite its Fermionic nature, so that

\[ \delta S[f_n] = 0 \]  

(3.5)

A set of transformation laws must similarly be given for the additional fields. The ‘ghost’ field \( f_{n-1} \) is the first member of this extended set of additional fields. The method we will follow for the construction of the auxiliary fields and action is that of Batalin and Vilkovisky [7,8]. In this method the gauge fixing is performed at the level of a “gauge Fermion” \( \Psi \) whose BRST variation \( \delta \Psi \) gives the necessary addition to the original action 3.3. The BRST action is then

\[ \hat{S} = S[f_n] + \int_M \delta \Psi \]  

(3.6)

If we ensure that \( \delta^2 = 0 \) then the original gauge invariance of the action (3.3) has been replaced by a BRST invariance of the action (3.6).

We now proceed with the construction of the gauge Fermion \( \Psi \) for the gauge fixing of the field \( f_n \). It is convenient to construct the ghost Fermion as a triangle and to start from its bottom which we declare to be of level zero. For this purpose it is convenient to introduce some important notation: We will label our fields by \( f_{(i,j)} \) and \( \bar{f}_{(i,j)} \) where

\[ f_{(i,j)} \]  

denotes a form of degree \( i \) and level \( j \)

\[ \bar{f}_{(i,j)} \]  

denotes a form of degree \( (n - i) \) and level \( j \).

(3.7)

Summarising: the first suffix indicates the degree of the form and the second indicates the ghost level (as distinct from ghost number*). Our notation will ensure that details of the

* Actually the ghost number of \( f_{(i,j)} \) is \( (n - j) \) and that of the anti-ghost \( \bar{f}_{(i,j)} \) is \( -(n - j) \). This can be deduced from the requirement that the action should have ghost number zero and the fact that \( \delta \) increases ghost number by one.
underlying manifold play a minimal rôle. This is convenient since the construction is local and algebraic.

§§ 3.4 The Gauge Fermion and its Variation

The pattern that emerges for the gauge Fermion may be compactly written down for the general term, at level \( k \) for the gauge Fermion. It is

\[
\Psi_k = \sum_{j=0}^{\lfloor k/2 \rfloor} \Psi_{(k-2j,k)}, \quad \text{with } \lfloor k/2 \rfloor = \text{the integer part of } k \tag{3.8}
\]

where

\[
\Psi_{(k-2j,k)} = \bar{f}_{(k-2j-1,k-1)} d^* f_{(k-2j,k)} + f_{(k-2j-2,k)} d \bar{f}_{(k-2j-1,k-1)} \tag{3.9}
\]

Note that the expression 3.9 for \( \Psi_{(k-2j,k)} \) has two terms: the first term contains a gauge fixing for the field \( f_{(k-2j,k)} \) but in the process introduces a new field \( \bar{f}_{(k-2j-1,k-1)} \), this new field also requires gauge fixing which, one notes, is done by the second term; however this second term contains a further new field \( f_{(k-2j-2,k)} \) which needs dealing with. But this latest field is of the same form of the original one except that its first suffix is smaller by two. Hence we can repeat the process just described until the first suffix reaches zero.

We use the subscripts \((k-2j,k)\) for \( \Psi_{(k-2j,k)} \) since they are also those of the leading field being gauge fixed by this structure, and all information is recoverable from these subscripts.

The pattern of the gauge Fermion can be summarized in the following triangular diagram

![Gauge Fermion Triangle](image)

Fig 1: The Gauge Fermion Triangle

We now come to the BRST variation of this whole structure. The BRST variations of
the corresponding fields are
\[
\begin{align*}
\delta f_{(k,k)} &= df_{(k-1,k-1)}, \\
\delta f_{(k-2j,k)} &= f_{(k-2j,k-1)}, \quad \delta f_{(k-2j,k-1)} = 0, \quad j \neq 0 \\
\delta \bar{f}_{(k-2j,k-1)} &= \bar{f}_{(k-2j-1,k)}, \quad \delta \bar{f}_{(k-2j-1,k)} = 0, \quad \forall j
\end{align*}
\]  
(3.10)
We note that \(\delta\) commutes with \(d\) and \(d^*\) and \(\delta^2 = 0\).

The variation of \(\Psi_k\) itself comprises two sets of terms: one set of ghost level \(k\) the other of ghost level \(k - 1\). This splitting into two levels arises from the Leibnitz rule applied to the product of a level \(k\) field with a level \(k - 1\) anti-field; one can see how it operates by following the downward and upward arrows of the variation triangle of Fig. 2 below.

It is convenient to organize the resulting fields into groups of a given level on this basis. The resulting variation is
\[
\delta \Psi_k = \sum_{j=0}^{[k/2]} F_{(k-2j,k)} + \bar{f}_{(k-1,k-1)} d^* df_{(k-1,k-1)} + \sum_{j=0}^{[k/2]} \bar{F}_{(k-2j-1,k-1)}
\]  
(3.11)
where the basic structures that appear are
\[
\begin{align*}
F_{(k-2j,k)} &= f_{(k-2j-1,k)} d^* f_{(k-2j,k)} + (-1)^F f_{(k-2j-2,k)} d\bar{f}_{(k-2j-1,k)} \\
\bar{F}_{(k-2j,k)} &= f_{(k-2j-1,k)} d\bar{f}_{(k-2j,k)} + (-1)^F \bar{f}_{(k-2j-2,k)} d^* f_{(k-2j-1,k)}
\end{align*}
\]  
(3.12) and
(3.13)
where \(F\) is the Fermion number of the field appearing immediately to its right. These are of similar form to \(\Psi_{(k-2j,k)}\), however \(F_{(k-2j,k)}\) and \(\bar{F}_{(k-2j,k)}\) both consist entirely of fields at the same level \(k\).

Note that in the expression 3.11 for \(\delta \Psi_k\) we have separated out, as special, the diagonal term, \(f_{(k-1,k-1)}\). This field will acquire its gauge fixing from the level \(k - 1\) ghost Fermion, \(\Psi_{k-1}\), just as the field \(f_{(k,k)}\) is gauge fixed by the first sum of (3.11).

The structure which emerges as a result of the BRST transformation is a new triangle which can be described as follows: one starts with the original triangle of Fig. 1. and, after BRST variation, two triangles appear: one displaced upwards to the right the other displaced downwards to the right. These two triangles partially overlap to form a new triangle containing a distinguished diagonal. This structure, where we include \(\Psi\) and \(\delta \Psi\) in the same triangle, is illustrated in Fig. 2 below.

\[\text{Fig 2: The gauge Fermion triangle and its variation}\]
The south east arrow represents the variation of an un-barred field, the north east arrow that of the barred field. The variation of a barred field *increases* its second index by one while the variation of an unbarred field *decreases* its second index by one. We see the variation of diagonal elements is special as the triangle is upper diagonal. Similarly if the gauge Fermion triangle terminates the top terminal row is special having no associated downward partner.

We are now in a position to complete the gauge fixing for the field \( f(n,n) \) in the action \( S[f(n,n)] \). The gauge Fermion required is simply

\[
\Psi = \sum_{k=1}^{n} \Psi_k
\]  

(3.14)

and is a sum of all rows of the triangle up to level \( n \). The resulting BRST invariant gauge fixed Lagrangian in terms of our compact notation is just

\[
\hat{L} = \mathcal{L}_n + \mathcal{L}
\]

where

\[
\mathcal{L} = \sum_{j=0}^{[n/2]} F(n-2j,n) + \sum_{k=0}^{n-1} \mathcal{L}_k
\]  

(3.15)

and

\[
\mathcal{L}_k = \bar{f}'(k,k) d^* df(k,k) + \sum_{j=0}^{[k/2]} \left( \bar{F}(k-2j,k) + F(k-2j,k) \right)
\]  

(3.16)

and \( \mathcal{L}_n \) is the the original Lagrangian 3.3 for the field \( f(n,n) \).

The boundary contribution, i.e. level \( n \) contribution to \( \mathcal{L} \) takes the form

\[
\sum_{j=0}^{[n/2]} F(n-2j,n)
\]  

(3.17)

and does not have corresponding \( \bar{F} \) contributions.

Our next task is to integrate out all the fields \( f(i,j) \) and \( \bar{f}(i,j) \) to obtain the measure \( \mu[f(n,n)] \). The first observation is that none the Lagrangians \( \mathcal{L}_k \) ‘interact’ with each other and we can therefore integrate them out separately. Next we observe that if we use the decomposition described earlier into primed and double primed forms \( (d\phi' = 0 \text{ and } d^* \phi'' = 0) \) then we find each term in the Lagrangian involves a distinct field, i.e. we can write

\[
\bar{f}'(k,k) d^* df(k,k) = \bar{f}''(k,k) dd^* f'(k,k)
\]  

(3.18)

\[
F(k-2j,k) = \bar{f}''(k-2j-1,k) d^* f'(k-2j,k) + (-1)^F f''(k-2j-2,k) d f'(k-2j-1,k)
\]  

(3.19)

and

\[
\bar{F}(k-2j,k) = f''(k-2j-1,k) d f''(k-2j,k) + (-1)^F \bar{f}'(k-2j-2,k) d^* f'(k-2j-1,k)
\]  

(3.20)

Up to this point we have been careful not to specify the statistics of the fields, other than to remark that the statistics alternate with the level; in addition all the terms of a given
level have the same statistics. It is then clear that the statistics of all fields are determined by the choice made for the apex field $f_{(0,0)}$. We make such a choice and assign the apex statistics first (though conventionally one would start by specifying the statistics of the field $f_{(n,n)}$)

We will consider in turn both options for the statistics of these fields, but begin with the Fermionic case; this being the one which occurs when these fields arise in the gauge fixing of a 1-form field such as in QED.

Integrating out just these fields at the apex then gives $\text{Det} (d^* d_0)$ which we simply denote by $X_0$, the subscript denoting the fact that this object is associated with an operator acting on zero forms.

Let us now consider the level $k$ Lagrangian, $\mathcal{L}_k$. The leading term in this Lagrangian is

$$\tilde{f}''_{(k,k)} d^* df'_{(k,k)}$$

and it will involve Fermionic fields if $k$ is even and Bosonic fields if $k$ is odd. Integrating out this term then gives

$$X^{p_k}_k,$$

where $X_k = \text{Det} (d^* d_k)$, and $p_k = (-1)^k$  

Now if we integrate out the term

$$\mathcal{F}_{(k-2j,k)} = f'_{(k-2j-1,k)} d^* f'_{(k-2j,k)} + (-1)^F f''_{(k-2j-2,k)} d f''_{(k-2j-1,k)}$$

we obtain

$$\text{Det} (d^*_{k-2j})^{p_k} \text{Det} (d_{k-2j-2})^{p_k}$$

By the isomorphisms discussed previously, we obtain

$$\text{Det} (d^*_{k-2j}) = X^{1/2}_{k-2j-1} \quad \text{Det} (d_{k-2j-2}) = X^{1/2}_{k-2j-2}$$

Therefore the contribution from integrating out $\mathcal{F}_{(k-2j,k)}$ is

$$X^{p_k/2}_{k-2j-1} X^{p_k/2}_{k-2j-2}$$

Similarly integrating out the fields in $\tilde{\mathcal{F}}_{(k-2j-1,k)}$ yields

$$\text{Det} (d^*_{k-2j-1})^{p_k} \text{Det} (d_{k-2j-1})^{p_k} = X^{p_k/2}_{k-2j-2} X^{p_k/2}_{k-2j-1}$$

We see that both $\mathcal{F}$ and $\tilde{\mathcal{F}}$ terms give similar contributions when integrated out. In effect as a consequence of the field theory Poincaré duality is automatically implemented.

The result therefore of integrating out all fields contributing to $\mathcal{L}_k$ is

$$X_0 X_1 X_2 \ldots X_k \quad \text{for } k \text{ even}$$

$$(X_0 X_1 X_2 \ldots X_k)^{-1} \quad \text{for } k \text{ odd}$$

with the statistics of $f_{(0,0)}$ Fermionic.
Now integrating out all fields in the Lagrangians $\sum_{i=0}^{k} L_i$ yields

\[ X_0 X_2 X_4 \ldots X_k \text{ for } k \text{ even} \]
\[ (X_1 X_3 \ldots X_k)^{-1} \text{ for } k \text{ odd} \]  

(3.29)

There remains only to consider the final boundary term 3.17. These fields when integrated out give the contribution

\[ X_0^{1/2} X_1^{1/2} X_2^{1/2} \ldots X_1^{1/2} \]  
\[ (X_0^{1/2} X_1^{1/2} X_2^{1/2} \ldots X_1^{1/2})^{-1} \]  

(3.30)

Integrating out all of the fields associated with the gauge fixing of $f_{(n,n)}$ yields the complete measure

\[ \mu[f_{(n,n)}] = \mu(n,n) \delta[d^* f_{(n,n)}] \]  

(3.31)

with

\[ \mu(n,n) = \frac{X_0^{1/2} X_1^{1/2} X_2^{1/2} \ldots X_1^{1/2}}{X_1^{1/2} X_3^{1/2} \ldots X_1^{1/2} \ldots} \]  

(3.32)

where the dots terminate with the last term $X_{n-1}$ and the field $f_{(n,n)}$ is Fermionic for $n$ even and Bosonic for $n$ odd, since we have chosen $f_{(0,0)}$ to be Fermionic. If the statistics of the field $f_{(n,n)}$ are opposite to those above then the corresponding result can be read off by replacing $X_k$ by $X_k^{-1}$ in all expressions above. We emphasize that the above considerations leading to the result (3.32) are quite general and do not rely on the form of the action to be gauge fixed; they rely only on the fact that the field to be gauge fixed is an $n$-form, and that the action is invariant under the addition of a closed form. The gauge Fermion itself is formally infinite and can be considered to be a Universal Gauge Fermion $\Psi$, the finite dimensionality of the manifold chosen and the field to be gauge fixed determine the relevant portion of the gauge Fermion for a particular problem.

§§ 3.5 The Gauge Fermion on Product Manifolds

Our next task is to analyse this structure on a product manifold $M_1 \times M_2$ where $M_2$ admits harmonic forms. As mentioned above there are three separate types of term to analyse. If we focus on the gauge Fermion only two of these enter. However, we have to be a little careful to ensure that our notation does not become degenerate. Such a problem arises now since there are new forms at level $k$ which, if we track them using the labels of the product, will acquire the same labels. What saves us is that fields acquiring the same subscripts will have different harmonic labels. No confusion should then arise once this is kept in mind.

Let us now consider the level $k$ gauge Fermion given by (3.8) on a product manifold where we separate off the harmonic contributions from $M_2$. The decomposition (2.50) and (2.51) are the relevant ones and give

\[ \Psi(r,k) = \tilde{\Psi}(r,k) + \sum_{j=0}^{r} \sum_{a=1}^{b_j} \Psi^a_{(r-j,k)} \varepsilon^a_j \varepsilon^a_j \]  

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The term $\Psi_k$ further decomposes using 2.50 and the orthogonality of such fields. We will not, however, pursue this here. Now noting that when the first index becomes zero $\Psi_{(r-j,k)}$ vanishes we can regroup the level $k$ gauge Fermion into a triangle graded by Betti number, thus

$$\Psi_k = \tilde{\Psi}_k + \sum_{j=0}^k \sum_{a=1}^{b_j} \Psi_{k-j}^a(k) \bar{\epsilon}_j \epsilon_j$$

where we include the bracketed $k$ to indicate that these terms are arising from the level $k$ gauge Fermion, and to avoid a degeneracy of notation.

We further simplify our notation by using a repeated index convention for the sums over harmonic forms thus

$$\Psi = \tilde{\Psi} + \Psi^a \epsilon^a$$

The sum is understood to be over all harmonic forms and $\epsilon^a = \bar{\epsilon}_j \epsilon_j$ for $j$-forms. The gauge Fermions $\Psi^a$ are understood to terminate when the fields comprising them do not exist. Thus in the gauge fixing of an $n$-form on $M$ the gauge Fermion associated with $j$-forms on $M_2$ will terminate at the level $k$ for which $(n - j - k)$ reaches 1.

The variation of the gauge Fermion can be similarly organized as before to give

$$\mathcal{L} = \tilde{\mathcal{L}} + \mathcal{L}^a \epsilon^a$$

where $\mathcal{L}^a$ are the results of varying the gauge Fermions $\Psi^a$, the superscript $a$ runs over all allowable Betti numbers; these will depend on the field being gauged fixed, and the manifold dimensions $d_1$ and $d_2$.

The combined action remaining once the harmonic contribution on $M_2$ has been extracted is then

$$\tilde{\mathcal{L}} = \sum_{j=0}^{[n/2]} \tilde{\mathcal{F}}_{(n-2j,n)} + \sum_{k=0}^{n-1} \tilde{\mathcal{L}}_k$$

It can then be integrated as before to give

$$\tilde{\mu}_{(n,n)} = \prod_{j=0}^{\lfloor n/2 \rfloor} X_0^{1/2} X_2^{1/2} \cdots X_{2j}^{1/2} \cdots$$

for all $n$ (3.34)

where $\tilde{X}_i = \text{Det} (d^*d_i)$ denotes the determinant of $d^*d_i$ on the orthogonal complement to the harmonic forms on $M_2$.

Finally examining the harmonic sector we find on integrating out the fields in $\mathcal{L}^e \epsilon^a$ the contribution

$$\mu^H_{(n,n)} = \prod_{k=0}^{n-1} X_k(M_1)^{p_k \sigma_k}$$

where $\sigma_k = \frac{1}{2} \sum_{j=0}^{n-k-1} (-)^j b_j$ (3.35)

and $X_k(M_1)$ is (3.22) restricted $M_1$.

§ 4. Topological Field Theories

Let us now consider a topological field theory, by choosing the starting Lagrangian to be $f_{(n,n)} df_{(n,n)}$. Note that Stokes’ theorem shows that this Lagrangian will integrate to zero
if the field $f_{(n,n)}$ is Bosonic for $n$ even or Fermionic for $n$ odd. Hence we are naturally led to consider $f_{(n,n)}$ to have statistics consistent with the descendant BRST field $f_{(0,0)}$ being Fermionic.

The field term $\overline{f}_{(n,n)}d f_{(n,n)} = f'_{(n,n)}d f'_{(n,n)}$ integrates to give

$$X_n^{1/4} \quad \text{for } n \text{ even}$$

$$X_n^{-1/4} \quad \text{for } n \text{ odd}$$

When included with the contribution from BRST gauge fixing the partition function is

$$Z_n = \frac{X_0^{1/2}X_2^{1/2} \cdots X_n^{1/4}}{X_1^{1/2}X_3^{1/2} \cdots}$$

$$= \frac{X_0^{1/2}X_2^{1/2} \cdots X_{n-1/4}}{X_1^{1/2}X_3^{1/2} \cdots} \quad \text{for } n \text{ odd}$$

$$= \prod_{k=0}^{n-1} X_k^{p_k/2} X_{n/4} \quad \text{where } p_k = (-1)^k$$

The logarithm of this can be expressed in terms of zeta functions by recalling that

$$\ln X_k = -\zeta'_d(a_k)(0)$$

thus

$$\ln Z_n = -\frac{1}{2} \sum_{k=0}^{n-1} (-1)^k \zeta'_d(a) + \frac{1}{4} (-1)^{n+1} \zeta'_d(a_n)(0)$$

$$= \frac{1}{4} \ln T \quad \text{from our previous discussion.}$$

We note that the partition function of our alternating sequence of Fermionic and Bosonic actions always yields the torsion to the same power $1/4$.

This latter identification of the partition function with the quarter power of the torsion relies on regulating the expressions $X_k$. We have adopted the zeta function procedure of the proceeding discussion. We emphasize however that we are not restricted to this regularisation procedure. In fact, by noting the close relationship of the zeta function and the heat kernel, it is possible to use any covariant regularisation at the lever of the heat kernel. A natural one to consider from the point of view of physics is a lattice regulator. This could be implemented by working on a simplicial lattice. The preceding analysis will go through without modification, provided the equivalent objects are given their natural lattice interpretation. This is then equivalent to Müllers’s proof of the equivalence of the combinatorial and lattice torsions.

Let us mention some further points regarding the partition function in general before proceeding to a derivation of the product formula. The first observation is that if we wish to consider a topological action in even dimensions we need to begin with an alternative to

$$\int_M f_{(n,n)} d f_{(n,n)}$$

(4.5)
since this action only exists for odd dimensional manifolds.

The simplest possibility is to consider an action made from two distinct fields say \( f_{(k,k)} \) and \( g_{(l,l)} \). The topological action we consider is \( g_{(l,l)} df_{(k,k)} \) and \( \text{dim} M = (k + l + 1) \). Of course this action can be considered in arbitrary dimensions and for both statistics for the respective fields provided they have the same statistics. From our work above these fields have separate gauge Fermions for their gauge fixing, which do not mix. The statistics of \( f_{(0,0)} \) and \( g_{(0,0)} \) are not independent but are determined by the dimension of the manifold since to be able to form an action from the two fields \( f_{(k,k)} \) and \( g_{(l,l)} \) dictates that both of these fields have the same statistics; but since the two gauge Fermion triangles meet, there is a constraint.

For simplicity let us consider the case of an even dimensional manifold of dimension \( 2n \) and pick the starting fields to be \( f_{(n,n)} \) and \( g_{(n-1,n-1)} \). Since the total height of the \( f \) triangle is one more than that of the \( g \) triangle it is clear that the fields \( f_{(0,0)} \) and \( g_{(0,0)} \) must have opposite statistics. If we take the case when \( f_{(0,0)} \) has Fermionic statistics we obtain the contribution from the \( f \) triangle as in (3.32) and that from the \( g \) triangle is the inverse of this but terminating one term sooner. For definiteness let us take \( n = 2m + 1 \) in which case \( f_{(n,n)} \) has the opposite statistics to \( f_{(0,0)} \), and is now a Boson. Thus we have a contribution to the partition function of

\[
X_0^{1/2}X_2^{1/2} \ldots X_{2m}^{1/2} \quad \text{for the } f \text{ triangle} \quad (4.6)
\]

\[
X_1^{1/2}X_3^{1/2} \ldots X_{2m-1}^{1/2} \quad \text{for the } g \text{ triangle} \quad (4.7)
\]

Integrating out the original action gives \( X_{2m+1}^{-1/2} \) which by shifted Poincaré duality is equal to \( X_{2m}^{-1/2} \). Taking the product of all this gives the partition function \( Z_n \) so that we have

\[
Z_n = X_0^{1/2}X_2^{1/2} \ldots X_{2m}^{1/2} X_1^{1/2}X_3^{1/2} \ldots X_{2m-1}^{1/2} X_{2m}^{-1/2} = 1 \quad (4.8)
\]

We see that the partition function is automatically unity. This agrees, as it must, with the torsion which is also unity in even dimensions.

In the case of an odd dimensional manifold of dimension \( 2n + 1 \) with starting action chosen to be

\[
\int_M g_{(n,n)} df_{(n,n)} \quad (4.9)
\]

the fields \( f_{(0,0)} \) and \( g_{(0,0)} \) will now have the same statistics and the contribution from each BRST triangle will be identical. The contribution from the original action will then be

\[
X_n^{1/2} \quad (4.10)
\]

plus for Fermionic fields and minus for Bosonic ones. This is also the square of the contribution we had previously, modulo statistics. Thus we see that the the partition function in this case is

\[
Z_n = T^{\pm 1/2} \quad (4.11)
\]
where the plus and minus depend on whether the fields were chosen Bosonic or Fermionic. In the case in which the fields \( f_{(n,n)} \) and \( g_{(n,n)} \) are always chosen Bosonic irrespective of dimension, the partition function is

\[
T_1^\frac{1}{2} \text{ for odd } n \quad T_1^{1/2} \text{ for even } n. \tag{4.12}
\]

Finally choosing a different starting field \( f_{(k,k)} \) would, in a similar fashion, yield a partition function which is \( T_1^{\pm 1/2} \).

Now let us consider the case of a product manifold. We have described what happens when we consider the gauge fixing of a particular field \( f_{(n,n)} \) on the product manifold \( M_1 \times M_2 \) in section 3. The further contribution that enters the partition function is the contribution from the original action. We take for definiteness the topological action \( f_{(n,n)} df_{(n,n)} \) with \( n \) even. With our gauge fixing measure 3.31 the only field to be integrated over is \( f''_{(n,n)} \).

The action becomes on extraction of the harmonic contributions from \( M_2 \)

\[
f_{(n,n)} df_{(n,n)} = \tilde{f}_{(n,n)} d\tilde{f}_{(n,n)} + \sum_{j=0}^{n} \sum_{a=1}^{b_j} f_{(n-j,n-j)}^a df_{(n-j,n-j)}^a \tag{4.13}
\]

Integrating these fields out gives

\[
\tilde{X}^\frac{1}{4}_n \prod_{j=0}^{n} X_j(M_1)^\frac{1}{4} b_{n-j} \tag{4.14}
\]

Including the contribution \( \tilde{X}^{1/4}_n \) with the measure \( \tilde{\mu}_{(n,n)} \) of 3.34 we have

\[
\tilde{Z}(M_1 \times M_2) = \tilde{\mu}_{(n,n)} \tilde{X}_n = 1 \tag{4.15}
\]

from our analysis of degeneracies in section 2. The relevant cancellation occurs mode by mode. In fact this does not rely on the form of regulator used to regulate the determinants as long as it is implemented in a uniform manner. Combining the remaining term from the measure 3.35 with the contribution from 4.14, together with the fact that \( \tilde{Z}(M_1 \times M_2) = 1 \), gives

\[
Z(M_1 \times M_2) = \prod_{k=0}^{n-1} X_k(M_1)^{p_k \sigma_k} \prod_{k=0}^{n} X_k(M_1)^{b_k/4} \tag{4.16}
\]

Noting the range of the sums concerned and using Poincaré duality we see that

\[
4\sigma_k + (-1)^k b_k = \sum_{j=0}^{d_2} (-1)^j b_j \quad \text{ for each } k \tag{4.17}
\]

\[
= \chi(M_2) \quad \text{the Euler character of } M_2.
\]

The final result is that

\[
Z(M_1 \times M_2) = Z(M_1) \chi(M_2) \tag{4.18}
\]

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which establishes the product formula. The argument given here in fact can also be used to demonstrate the same formula for the combinatorial torsion.

§ 5. Conclusion

In this paper we have followed two primary themes, the first being a detailed examination of the kernels of $d$ and $d^*$, the second being the BRST quantization procedure using the gauge Fermion construction of Batalin and Vilkovisky. We have examined both topics on an arbitrary manifold $M$ as well as specifically on product manifolds. We then combined these two pieces of work to give a field theoretic derivation of the formula for the Ray-Singer torsion on product manifolds.

This derivation can be seen to encompass both the Ray-Singer and combinatorial torsions, depending on which regularisation procedure is adopted for the field theory: Considering the procedure on a simplicial manifold yields the combinatorial torsion, whereas regulating the continuum theory by defining the determinants using zeta functions yielded the analytic torsion. Müller’s proof [2] that the torsion defined on a simplicial mesh, and invariant under mesh refinement, is equal to the analytic torsion provides us here with concrete examples of field theories which exists in the combinatorial sense and whose continuum limit is also well defined. The equality of these two field theories under lattice subdivision is an example of strong renormalization group invariance.

We obtained several useful results which should be of use in a wider context also. We have provided a detailed analysis of the spectral decomposition of the eigenspaces of Laplacians on $r$-forms, establishing the decomposition of the kernels of $d$ and $d^*$. We took apart this structure for product manifolds and obtained compact expressions showing in detail how $\ker d$ and $\ker d^*$ are constructed from the spectral differential form data for the factors $M_1$ and $M_2$. This information was essential for a proper understanding of the decomposition of the gauge Fermion on product manifolds as well as for the proof of the product formula for the torsion.

We constructed the gauge fixing for an $n$-form gauge field on $M$ using the gauge Fermion method of Batalin and Vilkovisky. By concentrating on this structure, and developing a compact notation, we were able, very simply, to construct the necessary measure for the quantisation of such a field. Our procedure constructed a formalism in which the details of the manifold $M$ play very much a subordinate rôle, much as is the case in the theory of universal characteristic classes. This allowed us to give a completely general gauge Fermion $\Psi$ which can be viewed as a ‘Universal Gauge Fermion’ $\Psi$—it is an infinite dimensional upper triangular matrix, but, just as in the universal characteristic class formalism, the finite dimensionality of the particular $M$ chosen and the field being gauge fixed determine which finite dimensional subset of $\Psi$ to use.

We also observed that by fixing the statistics of one of the fields at the lower apex of $\Psi$ the statistics of all other fields are fixed. We considered both possibilities for the statistics of the apex field—the choice, of a Fermionic apex corresponds, for example, to gauge fixing a Bosonic connection. Our notation makes manifest the nested structure of the gauge Fermion so that the only modification necessary to gauge fixing an $n$-form field, rather than an $(n-1)$-form field, is the inclusion of another row of the gauge Fermion triangle. This is an example of how the universality referred to in the previous paragraph operates.
This nested structure suggested a new class of topological actions of the form $f_{(n,n)} df_{(n,n)}$ which exist only for odd dimensional manifolds; where the fields $f_{(n,n)}$ are Bosonic for odd $n$ and Fermionic for even $n$. Each of these actions gives a partition function which is the quarter power of the torsion.

Further work that is worth considering along these lines is an examination of the BRST procedure on a manifold where the metric is Wick rotated to give it a Lorentzian signature. On the initial Riemannian manifold the gauge fixing is complete, however after a Wick rotation the gauge fixing develops singularities and no longer appears to be complete. We have given sufficient spectral information and a sufficiently simple method of constructing the gauge Fermion that these and similar questions should be within easy grasp. A further and more detailed analysis of the combinatorial torsion would be worthy of attention. It would be nice to examine what additional information can be gleaned from this procedure on complex manifolds where complex torsion can be defined.

Interesting examples of field theories defined discretely (in addition to the usual lattice QCD and Regge calculus) are the simplicial theory of Sorkin [11] and the finite element theories constructed by Bender et al. [9,10]. Much of the present work is applicable to these latter cases. This general area deserves further attention.

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