What spatial geometry does the (2+1)-dimensional QFT vacuum prefer?

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We consider relativistic (2+1)-QFTs on a product of time with a two-space and study the vacuum free energy as a functional of the temperature and spatial geometry. We focus on free scalar and Dirac fields on arbitrary perturbations of flat space, finding that the free energy difference from flat space is finite and always negative to leading order in the perturbation. Thus free (2+1)-QFTs appear to always energetically favor a crumpled space on all scales; this is true both as a purely quantum effect at zero temperature and as a purely thermal effect at high temperature. Importantly, we show that this quantum effect is non-negligible for the relativistic Dirac degrees of freedom on monolayer graphene even at room temperature, so we argue that this vacuum energy effect should be included for a proper analysis of the equilibrium configuration of graphene or similar materials.

I. INTRODUCTION

The presence of matter gives a surface embedded in an ambient space an energy. This matter may be external to the surface – like the pressure of air on a soap bubble – or may comprise the material nature of the surface itself – like a membrane with surface tension and bending energy. These energies determine the equilibrium (i.e. static) configuration of such a surface: for instance, the presence of surface tension tends to make membranes favor (smooth) minimal-area configurations, while finite-temperature thermodynamic effects may render membranes unstable to crumpling or rippling.

In this Letter we initiate a study of the free energy contribution to the equilibrium configuration of a surface due to free relativistic quantized matter fields living on it. In particular, we include zero-temperature (Casimir) effects. Such relativistic quantum fields occur in various physical settings: for example, in graphene and related materials, the electronic structure gives rise to an effective description in terms of relativistic Dirac fermions propagating on the two-dimensional crystal. In cosmology domain wall defects may exist and could carry upon them relativistic degrees of freedom. More exotically, in braneworld models our universe is itself a surface on which the Standard Model fields live.

The setting is then (2+1)-d QFT on a product of time with a two-space. By studying both the free non-minimally coupled scalar and the free Dirac fermion we will see that such a field lowers the free energy of the surface on which it lives when the surface is deformed away from being intrinsically flat. This energy difference is UV finite (and thus well-defined) and present at any temperature including $T = 0$ (in which case it can be interpreted as a Casimir effect) both for massless and massive fields, and any scalar non-minimal coupling. It is then natural to wonder whether a classical membrane action is able to counteract this quantum tendency to crumple. We will perform a naïve analysis of this question for monolayer graphene, which is indeed seen to ripple on short scales. We show that at room temperature the quantum vacuum energy of the Dirac fermions give a scale at which one would expect crumpling effects on the order of the lattice spacing. The effective membrane description which would validate our analysis breaks down at this scale, so our results make no definitive statement about the rippling of graphene. However, they do indicate that a careful consideration of these quantum effects is important for a proper treatment of equilibrium configurations of graphene and similar materials even at room temperature.

II. FREE ENERGY DIFFERENCE

We consider a spacetime which is a product of time with a two-space $\Sigma$ (for now taken to be general). Since we are interested in QFT at finite temperature $T$ we work in Euclidean time, so the metric is

$$ds^2 = d\tau^2 + ds^2_{\Sigma}$$

with $\tau$ periodic with period $\beta = 1/T$. We will consider a free scalar $\phi$ and Dirac spinor $\psi$ with equations of motion

$$(-\nabla^2 + \xi R + M^2) \phi = 0, \quad (\slashed{D} + M) \psi = 0$$

respectively, where $R$ is the Ricci scalar and $\slashed{D}$ is understood as being defined by the spin connection (our conventions can be found in the Supplemental Material).

\footnote{Note that free massless vector fields are equivalent to free minimally coupled massless scalar fields by duality in (2+1)-d, so our analysis indirectly includes massless vector fields as well.}

\footnote{Our analysis was motivated by holographic considerations: for (2+1)-d conformal field theories (CFTs), flat space is energetically disfavored at zero temperature. This was shown globally in holographic CFTs and perturbatively in general.}

\footnote{Unless otherwise stated we use natural units $\hbar = c = k_B = 1$ with $c$ the “effective” speed of light of the relativistic fields (not necessarily equal to the actual speed of light).}
The free energy $F[\Sigma]$ is a functional of the geometry $\Sigma$ (and temperature $T$) and is given in terms of the partition function $Z[\Sigma]$ as $F = -T \ln Z$. We are specifically interested in the difference $\Delta F$ between the free energy on $\Sigma$ and some reference background space $\Sigma^*$ at the same temperature, which satisfies

$$e^{-\beta \Delta F} = \frac{Z[\Sigma]}{Z[\Sigma^*]} = \frac{\int D\Phi e^{-S_\Sigma[\Phi]}}{\int D\Phi e^{-S_{\Sigma^*}[\Phi]}} = (e^{-\Delta S})_{\Sigma^*},$$

where $\Phi$ stands for the matter field (scalar or fermion) being integrated over in the path integral, $\Delta S = S_\Sigma - S_{\Sigma^*}$ is the difference between the action on $S^1 \times \Sigma$ and $S^1 \times \Sigma^*$, and the expectation value is defined by the path integral on the background geometry $S^1 \times \Sigma$. To evaluate $\Delta F$, recall that for free fields the path integrals in (3) yield functional determinants, giving

$$Z = (\det L)^q \text{ with } L = -\partial_x^2 + \mathcal{O} + M^2,$$

where $q = -1/2$ (+1) for the scalar (fermion), $\mathcal{O}$ is an elliptic self-adjoint scalar operator on $\Sigma$ given explicitly in (11), and the determinant is evaluated over Matsubara frequencies on the thermal circle (with appropriate periodicity or antiperiodicity in the scalar and fermion cases respectively). For the scalar, (11) is obtained straightforwardly. The fermion case is more subtle, and we leave full details to the Supplemental Material. In short, a direct path integral yields $Z = \det(i\partial - iM)$. However, by exploiting the direct product structure of the metric (1) along with the fact that the two-dimensional rotation group only has a single generator, we may eliminate the spinor structure and reduce the determinant to that of an elliptic operator of the form (1) with the determinant taken over the space of complex functions with antiperiodicity on the thermal circle.

The free energy can then be evaluated via heat kernel methods [14]: defining the heat kernel as $K_L(t)$, one obtains

$$\beta \Delta F = q \int_0^\infty \frac{dt}{t} \Delta K_L(t),$$

where $\Delta K_L(t) = K_L(t) - K_{\Sigma^*}(t)$. This expression is UV divergent unless $\Delta K_L(t)$ vanishes at $t = 0$; this condition can be ensured by an appropriate choice of background $\Sigma^*$. Specifically, the heat kernel expansion gives [14]

$$K_L(t) = \beta \left[ \frac{c_1 V_\Sigma}{t^{3/2}} + \frac{c_2 \chi_\Sigma + c_3 V_\Sigma M^2}{t^{1/2}} + O(t^{1/2}) \right],$$

where $V_\Sigma$ and $\chi_\Sigma$ are the volume and Euler number of $\Sigma$, respectively, and $c_1$, $c_2$, and $c_3$ are dimensionless constants independent of the geometry (though they depend on the choice of matter field). Thus requiring that $\Delta F$ be UV-finite imposes that we choose a background geometry $\Sigma$ with the same volume and topology as $\Sigma$. It is worth emphasizing that although the undifferenced functional determinant $\det L$ is UV-divergent, we do not need to invoke any regularization to evaluate the differentiated free energy [4].

Now we specialize to our case of interest. Ultimately we wish to take $\Sigma$ to be a deformation of flat space, $\Sigma$. Since these are two-dimensional we introduce conformally flat coordinates $x^A$, in terms of which the metrics on $\Sigma$ and $\Sigma^*$ take the form

$$ds^2_\Sigma = e^{2f(x)} \delta_{AB} dx^A dx^B, \quad ds^2_{\Sigma^*} = \delta_{AB} dx^A dx^B.$$

In order to have good control over the spectrum of $L$ (which is essential for computing the heat kernel), we compactify these to tori $\Sigma_L$, $\Sigma_{\Sigma^*}$ via the identifications $x^A \sim x^A + L_A$ with $L_1 = L$ and $L_2 = rL$. We consider a family of deformations $f_L(x)$ so that as $L \to \infty$ (with $r$ fixed) we recover (7) with the $x^A$ uncompactified. Moreover, at any finite $L$, we may choose $f_L$ such that $V_{\Sigma_L} = V_{\Sigma_{\Sigma^*}}$. By the arguments above, this condition will ensure that for every $L$, the energy difference between the deformed and flat torus will be UV finite [4].

Our object of interest is the free energy difference with this IR regulator removed:

$$\beta \Delta F_{\infty}[f] \equiv q \lim_{L \to \infty} \int_0^\infty \frac{dt}{t} \Delta K_L[f_L; L](t),$$

with $\Delta K_L[f_L; L](t) \equiv K_L[f_L; L](t) - K_L[0; L](t)$. For notational convenience we will henceforth forego writing the arguments of these functionals explicitly, and we will drop the subscripts $L$ on $f_L$ and $\infty$ on $\Delta F_{\infty}$. Using (4), we finally obtain

$$\beta \Delta F[f] = q \lim_{L \to \infty} \int_0^\infty \frac{dt}{t} e^{-M^2 t} \Theta(T^2 t) \Delta K_O(t),$$

where the sum over Matsubara frequencies yields

$$\Theta(\zeta) = \sum_{n = -\infty}^{\infty} e^{-(2\pi)^2 (n-q+1/2)^2 \zeta},$$

and the operators $O$ are given explicitly in terms of $f$ as

$$O_s = -e^{-2f} \left( \nabla^2 + 2\xi \left( \nabla^2 f \right) \right),$$

$$O_t = O_s |_{\xi = 1/4} = -e^{-2f} \left( i \tilde{\star} (df \wedge d) - \frac{(\nabla_A f)^2}{4} \right),$$

Note that since $\det L$ is UV-divergent, $\Delta F$ is not necessarily the same as a difference of separately renormalized free energies on $\Sigma$ and $\Sigma^*$, which could be scheme-dependent and therefore unphysical.

For finite $L$ one may choose between periodic and antiperiodic boundary conditions for the fermion on the torus cycles; since our torus is only an IR regulator, and this distinction vanishes in the limit $L \to \infty$, we take the periodic case for simplicity.
with $\nabla_A$ and $\ast$ the covariant derivative and Hodge dual on the flat background $f = 0$, and the subscripts $s$ and $f$ denoting the scalar and fermion.

### III. PERTURBATION THEORY

We introduce a perturbation parameter $\epsilon$ to expand our deformation $\Sigma$ of the flat $\Sigma$ as

$$f = \epsilon f^{(1)} + \epsilon^2 f^{(2)} + O(\epsilon^3).$$

(12)

Preservation of the volume requires its eigenvalues and eigenfunctions as

$$\lambda_i \equiv \bar{\lambda}_i + \epsilon \lambda_i^{(1)} + \epsilon^2 \lambda_i^{(2)} + O(\epsilon^3),$$

(15a)

$$h_i = \bar{h}_i + \epsilon \sum_j b^{(1)}_{ij} \bar{h}_j + \epsilon^2 \sum_j b^{(2)}_{ij} \bar{h}_j + O(\epsilon^3),$$

(15b)

so the $\bar{h}_i$ are eigenfunctions of the flat space Laplacian with corresponding eigenvalues $\bar{\lambda}_i$. We choose these eigenfunctions to be normalized as

$$\langle \bar{h}_i | \bar{h}_j \rangle \equiv \int d^2x \bar{h}_i \bar{h}_j = \delta_{ij}.$$  

(16)

Then defining

$$P^{(1)}_{ij} = \langle \bar{h}_i | \mathcal{O}^{(1)} | \bar{h}_j \rangle,$$

(17a)

$$P^{(2)}_{ij} = \langle \bar{h}_i | \mathcal{O}^{(2)} | \bar{h}_j \rangle + \sum_{k; \lambda_k \neq \lambda_i} \frac{P^{(1)}_{ik} P^{(1)}_{kj}}{\lambda_i - \lambda_k},$$

(17b)

standard perturbation theory yields the eigenvalue shifts

$$\lambda_i^{(1)} = P^{(1)}_{ii}, \quad \lambda_i^{(2)} = P^{(2)}_{ii} \quad \text{(no sum).}$$

(18)

Note that we have glossed over a subtlety: recall from QM perturbation theory that the presence of degenerate subspaces imposes additional constraints on the unperturbed eigenfunctions $\bar{h}_i$ for the expansion (15b) to be consistent. The first order eigenvalue problem requires we arrange our basis $\bar{h}_i$ such that $P^{(1)}_{ij}$ is diagonal within such subspaces (i.e. if $\bar{\lambda}_i = \bar{\lambda}_j$ but $i \neq j$, then $P^{(1)}_{ij} = 0$).

If any degeneracies remain at first order, we must further ensure at second order that $P^{(2)}_{ij}$ be diagonal in the remaining degenerate subspaces. We discuss this issue explicitly in the Supplemental Material.

Finally, we expand the heat kernel as

$$\Delta K_O(t) = \epsilon K^{(1)}(t) + \epsilon^2 K^{(2)}(t) + O(\epsilon^3),$$

(19)

with

$$K^{(1)}(t) = t \sum_i e^{-\lambda_i t} P^{(1)}_{ii},$$

(20a)

$$K^{(2)}(t) = \frac{t}{2} \sum_i e^{-\lambda_i t} \left( P^{(1)}_{ii} - P^{(2)}_{ii} \right)$$

(20b)

### IV. RESULTS

In order to perform our computations we Fourier decompose the perturbation

$$f^{(1)}(x) = \frac{(2\pi)^2}{rL^2} \sum_N \int_\mathbb{R} [f^{(1)}(\kappa x) e^{i\kappa x}] / L, \hspace{1cm} (21a)$$

$$\to \int d^2k \tilde{f}^{(1)}(\kappa) e^{i\kappa \cdot x} \text{ as } L \to \infty,$$  

(21b)

where the sum runs over all pairs of integers $N = \{n_1, n_2\}$, and the second line defines $k_A = \lim_{L \to \infty} 2\pi n_A / L_A$.

An explicit calculation on the torus for fixed $L$ reveals that (for both the scalar and fermion) while the eigenvalues are indeed shifted at first order in $\epsilon$, their contribution to the heat kernel vanishes: $K^{(1)} = 0$. The leading order perturbation to the heat kernel is then the second order term $K^{(2)}$. A lengthy but straightforward computation yields the finite-$L$ expressions presented in the Supplemental Material; in the limit $L \to \infty$ they become

$$K^{(2)}(t) = t \int d^2k k^4 \left| \tilde{f}^{(1)}(\kappa) \right|^2 I(k^2 t)$$

(22)

with $k = |\kappa|$,

$$I_s(\zeta) = -\frac{\pi}{4k^2} \left[ 6 + \zeta(1 - 8\xi) - \left( 6 + 2\zeta(1 - 4\xi) + \frac{\zeta^2}{2}(1 - 4\xi)^2 \right) F\left( \frac{\sqrt{\zeta}}{2} \right) \right],$$

(23a)

$$I_t(\zeta) = \frac{\pi}{4k^2} \left[ 6 + \zeta F\left( \frac{\sqrt{\zeta}}{2} \right) - 6 \right],$$

(23b)

and $F(\zeta) = \zeta^{-1} e^{-\zeta^2} \int_0^\infty d\zeta' e(\zeta'^2)$. Thus using (9) we find

$$\Delta F = -\epsilon^2 \int d^2k a(k) \left| \tilde{f}^{(1)}(\kappa) \right|^2,$$  

(24)
TABLE I. The scaling of $\Delta F$ for the minimally coupled free scalar field and Dirac fermion for different relative magnitudes of $\ell$, $\ell_M$, and $\ell_T$. Note that for the non-minimally coupled scalar (i.e. $\xi \neq 0$), factors of $\ln(\ell_M/\ell)$ appear in the last two rows.

| $\ell_T \gg \ell \gg \ell_M$ | $\ell_M/\ell$ |
| $\ell_T \gg \ell_M \gg \ell$ | $1$ |
| $\ell \gg \ell_T \gg \ell_M$ | $\ell_T/\ell_M$ |
| $\ell \gg \ell_M \gg \ell_T$ | $\ell_M/\ell_T$ |
| $\ell_M \gg \ell_T \gg \ell$ | $1$ |
| $\ell_M \gg \ell \gg \ell_T$ | $\ell_T/\ell$ |

A few comments are in order. Firstly we see the leading variation in $\Delta F$ is quadratic in $\epsilon$. Next, we have that $I(\zeta)$ is finite and $\Theta(\zeta)$ is $O(\zeta^{-1/2})$ at small $\zeta$, and thus $\Delta F$ is UV-finite. Likewise, since $I(\zeta)$ and $\Theta(\zeta)$ are $O(1)$ at large $\zeta$, $\Delta F$ is also IR-finite for $M > 0$; in fact, the large-$\zeta$ decay of $I(\zeta)$ also implies IR finiteness in the massless case $M = 0$ for both the fermion and minimally-coupled scalar ($\xi = 0$). Finally, a key physical point is that, as can be seen by explicitly plotting $\pi I(\zeta)$, the effect is a purely quantum one: $\Delta E \sim -\epsilon^2 \hbar c/\ell$ for $\ell \ll \ell_M$. On the other hand, at high temperatures the energy decrease is a thermal effect: e.g. $\Delta F_I \sim -\epsilon^2 k_B T$ for $\ell \ll \ell_M$.

As a final note, the small-temperature limit $\ell_T \gg \max[\ell, \ell_M]$ is in fact analytically tractable: Poisson resummation gives $\Theta(T^2 t) = \beta / \sqrt{4 \pi t}$ up to terms that are exponentially suppressed in $\beta^2 / t$, which allows us to compute $a(k)$ explicitly as

\begin{align}
  a_s^{(T=0)}(k) &= \frac{\pi k^3}{128} \left[ \frac{2(3-32\xi)M}{k^3} - \frac{24M^3}{k^3} + \left( 3 - 32\xi + 128\xi^2 - 8(1 - 16\xi) \frac{M^2}{k^2} + \frac{48M^4}{k^4} \right) \arccot \left( \frac{2M}{k} \right) \right], \quad (26a) \\
  a_t^{(T=0)}(k) &= \frac{\pi k^3}{64} \left[ \frac{2M}{k} + \frac{24M^3}{k^3} + \left( 1 - \frac{8M^2}{k^2} - \frac{48M^4}{k^4} \right) \arccot \left( \frac{2M}{k} \right) \right]. \quad (26b)
\end{align}

V. MEMBRANE CRUMPLING

We have seen that at any temperature, free relativistic $(2+1)$-dimensional degrees of freedom energetically prefer deformations of flat space to flat space itself. Let us now consider how this effect competes with a membrane’s bending energy, which at zero temperature favors a flat geometry.

Consider three-dimensional flat space with Cartesian coordinates $\{X^A, Z\}$ and parametrize a surface in it by $X^A = x^A + \epsilon v^A(x^B)$, $Z = \sqrt{\epsilon} h(x^A)$. Then for small $\epsilon$ and suitable $v^A$, the intrinsic metric on the membrane in

\[ \epsilon^2 \hbar c/\ell \]

the coordinates $x^A$ is as in equation (7) with $-\nabla^2 f = \epsilon \det(\partial_A \partial_B h)$. The bending energy due to extrinsic curvature is

\[ H = \epsilon \kappa \int d^2 x (\nabla^2 h)^2, \quad (27) \]

where $\kappa$ is the bending rigidity. If the membrane is deformed from flat over a region of characteristic size $\ell \ll \ell_M$, then the (positive) bending energy $E_B$ and (negative) vacuum energy $E_Q$ (at zero temperature) for $N$ free relativistic quantum fields are parametrically given as

\[ E_B \sim \epsilon \kappa, \quad E_Q \sim -\epsilon^2 N \frac{\hbar c}{\ell}. \quad (28) \]

The ground state equilibrium configuration of the membrane should minimize $E = E_B + E_Q$. One might expect that because $E_B$ is lower order in $\epsilon$ than $E_Q$, a perturbative analysis guarantees that $E > 0$ for any deformation of flat space. However, note that $E_B$ and $E_Q$ have different scale dependence, with $E_Q$ dominating at sufficiently small scales. Thus if $\ell \lesssim \ell_{\text{crumple}} \equiv N \hbar c/\kappa$, $E_Q$ can be comparable to and even dominate $E_B$ while still in
the perturbative regime \( \epsilon \lesssim 1 \). Whether or not \( E \) actually decreases for (sufficiently large) deformations of flat space – therefore implying that the membrane’s equilibrium configuration is crumpled at a scale \( \ell \lesssim \ell_{\text{crumple}} \) – then depends on nonlinear and higher-derivative contributions to its bending action and whether or not these are relevant at the scale \( \ell_{\text{crumple}} \) at amplitudes \( O(\epsilon^2) \).

It is instructive to consider the case of a graphene monolayer, for which the bending rigidity is \( \kappa \sim 1 \) eV, the unit cell has size \( \ell_{\text{cell}} \sim 1 \) Å, and the relativistic fields are two Dirac fermions with effective speed \( c \sim c_{\text{light}}/100 \), with \( c_{\text{light}} \) the actual speed of light \( [5, 13] \). Our effective membrane description is valid for \( \ell \gg \ell_{\text{cell}} \), while from Table I the scaling properties \( [28] \) are valid at room temperature for \( \ell \ll \ell_T=300\kappa \sim 10^4\ell_{\text{cell}} \). Computing the crumpling scale, we find \( \ell_{\text{crumple}} \sim 10\ell_{\text{cell}} \), which is sufficient close to \( \ell_{\text{cell}} \) to make our effective membrane description suspect. Thus while this naïve analysis is insufficient to imply the existence of a crumpled equilibrium configuration for graphene, it does indicate that long range quantum properties of the conduction electrons (which give rise to the effective Dirac fermions) are important for understanding the energetics of equilibrium monolayer graphene even at room temperature; such effects are presumably highly challenging to correctly incorporate into Monte-Carlo simulations. Indeed, it is intriguing to note that for freely suspended graphene at room temperature, one does see low amplitude ripples on short scales \( \sim 50 \) Å, close our \( \ell_{\text{crumple}} \) \([12] \).

We emphasize that in future two-dimensional crystal materials whose electronic structures similarly give rise to Dirac fermions (or perhaps scalars or vectors), the situation may be different. In particular, if one wishes to have such a monolayer material that is flat on scales above the unit cell scale \( \ell_{\text{cell}} \), one presumably requires \( N\hbar c/\kappa \lesssim \ell_{\text{cell}} \), which may be regarded as a bound on the speed or number of relativistic species, given the bending mechanics of the crystal.

SUPPLEMENTAL MATERIAL

1. Fermion Partition Function

We will follow the Clifford algebra conventions of \([13]\): in Euclidean signature, the Clifford algebra is

\[ \{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}, \]

which allows us to take the \( \gamma^\mu \) to be Hermitian. With such conventions, a natural choice of representation of the gamma matrices in three dimensions is \( \gamma^\mu = \sigma^\mu \) with \( \sigma^\mu \) the Pauli matrices, though we note that none of our statements will depend on such a choice. Scalars are formed from spinors \( \chi, \psi \) as the bilinears \( \bar{\chi}\psi \) with \( \bar{\chi} = \chi^\dagger \), and the massive Euclidean Dirac action on a curved space with metric \( g_{ab} \) is

\[ S_E[\bar{\psi}, \psi] = \int d^3x \sqrt{g} \bar{\psi}(i\partial - iM)\psi, \]

where \( \partial = \gamma^\mu(e_\mu)^aD_a \) with \( \{e_\mu\} \) for \( \mu = 1, 2, 3 \) a vielbein,

\[ D_a = \nabla_a + \frac{1}{2} \omega_{a\mu\nu}S^{\mu\nu}, \]

\( \nabla_a \) the usual Riemann connection compatible with \( g_{ab} \), \( S^{\mu\nu} = [\gamma^\mu, \gamma^\nu]/4 \) the generators of the Lorentz group, and \( \omega_{a\mu\nu} = (e_\mu)^b\nabla_a(e_\nu)_b \) the spin connection. Note that the operator \( i\partial \) is self-adjoint, but the \( i \) in the mass term renders the Euclidean action non-Hermitian. This factor of \( i \) is necessary to ensure that the action obeys the Osterwalder-Schrader positivity conditions; see e.g. \([10]\) for a discussion of such subtleties associated with spinors in Euclidean space.

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Performing the path integral yields

\[ Z = \int D\bar{\psi} D\psi e^{-S_{\mathcal{E}}[\bar{\psi}, \psi]} = \det(i\hat{D} - iM). \tag{32} \]

Because \( i\hat{D} \) is self-adjoint, its eigenvalues are real. Moreover, in the direct product geometry \( \Pi \), eigenspinors of \( i\hat{D} \) can be decomposed into Fourier modes \( \psi = e^{-i\Omega_n \tau} \psi_{\Sigma} \), with \( \psi_{\Sigma} \) a spinor on \( \Sigma \) and \( \Omega_n = (2n + 1)\pi/\beta \) a Matsubara frequency (with \( n \in \mathbb{Z} \)). It is then straightforward to show that if \( e^{-i\Omega_n \tau} \psi_{\Sigma} \) is an eigenspinor of \( i\hat{D} \) with eigenvalue \( \lambda \), then \( e^{i\Omega_n \tau} \psi_{\Sigma} \) is an eigenspinor with eigenvalue \(-\lambda\). Thus the spectrum of \( i\hat{D} \) on the background \( \Pi \) is symmetric about zero \(^{10}\) so we have

\[ Z^2 = \det(i\hat{D} - iM) \det(-i\hat{D} - iM) = \det(\hat{D}^2 - M^2). \tag{33} \]

(See e.g. \(^{11}\) for more on this trick in \( d = 4 \).) Now, writing the metric on \( \Sigma \) in the conformally flat form \( \Pi \), one can evaluate \( \hat{D}^2 - M^2 \). Noting that there is only one generator \( S^2 = (i/2)\gamma^\tau \) of rotations in two dimensions, we obtain

\[ \hat{D}^2 - M^2 = -\mathcal{L} P_L - \mathcal{L}^* P_R, \tag{34} \]

where \( P_{L,R} = (1 \pm \gamma^\tau)/2 \) are left and right Weyl projectors on \( \Sigma \) and \( \mathcal{L} \) is as given in \(^{12}\). Decomposing \( \psi = e^{-i\Omega_n \tau} \psi_{\Sigma} \), we see that \( \mathcal{L} \) and \( \mathcal{L}^* \) act only on left- and right-helicity Weyl spinors \( P_L \psi_{\Sigma}, P_R \psi_{\Sigma} \), respectively. Since these spinors only have one component each, we may just interpret \( \mathcal{L} \) and \( \mathcal{L}^* \) as acting only on complex functions (albeit with antiperiodic boundary conditions on the thermal circle). We therefore have

\[ \det(-\mathcal{L} P_L - \mathcal{L}^* P_R) = \det(-\mathcal{L}) \det(-\mathcal{L}^*) = (\det \mathcal{L})^2, \tag{35} \]

where in the second expression we take the determinants only over the space of functions on which \( \mathcal{L} \) and \( \mathcal{L}^* \) act, and in the last equality we noted that because \( \mathcal{L} \) is self-adjoint (with respect to the usual \( L_2 \) norm), \( \mathcal{L} \) and \( \mathcal{L}^* \) have the same spectrum (and thus determinant). Thus the partition function for the fermion can be evaluated by just taking the functional determinant of a scalar differential operator acting on complex functions.

### 2. Finite-\( L \) Heat Kernels

Here we provide more details on the computation of the heat kernel at finite \( L \). First, in order to deal with the issue of eigenfunction degeneracy, it is convenient to take \( r^2 \) irrational (so that no eigenvalue \( \lambda_\xi \) has degeneracy greater than four) and choose the label \( i \) to consist of \( \{\bar{N}^+, \bar{S}\} \), where \( \bar{N}^+ = \{n_1^+, n_2^+\} \) is a pair of nonnegative integers and \( \bar{S} = \{s_1, s_2\} \) is a pair of signs, with \( s_A = \pm 1 \) if \( n_A^+ \not= 0 \) and \( s_A = 0 \) if \( n_A^+ = 0 \). The values of \( \bar{S} \) index the degenerate subspaces; for a given \( \bar{N}^+ \), there are \( d_{\bar{N}^+} = (2 - \delta_{n_1^-,0})(2 - \delta_{n_2^-,0}) \) possible such values. The eigenvalues of \(-\sqrt{-\hat{V}}^2\) are then given by \( \bar{N}^+ \) as \( \lambda_{\bar{N}^+, \bar{S}} = (2\pi/\beta)^2((n_1^+)^2 + (n_2^+)^2) \) and have degeneracies \( d_{\bar{N}^+} \), while we write the eigenfunctions as

\[ \bar{h}_{\bar{N}^+, \bar{S}}(x) = \frac{1}{\sqrt{\mathcal{F} L}} e^{2\pi i \sum_{\lambda S}^\mathcal{F} s_A n_{\lambda S}^+ x^A / L}, \tag{36} \]

where the sum runs over all \( d_{\bar{N}^+} \) possible choices of \( \bar{S} \) and for fixed \( \bar{N}^+ \), \( \sum_{\mathcal{F} S}^\mathcal{F} \) is an arbitrary \( d_{\bar{N}^+} \times d_{\bar{N}^+} \) unitary matrix. In other words, for given \( \bar{N}^+ \) the \( \bar{h}_{\bar{N}^+, \bar{S}} \) form an arbitrary orthonormal basis of the degeneracy subspace of \(-\sqrt{-\hat{V}}^2\) with eigenvalue \( \lambda_{\bar{N}^+} \); the freedom to choose this basis is what allows us to satisfy the perturbation theory constraints on \( \bar{h}_{\bar{N}^+, \bar{S}} \).

Using this formalism, we may compute the second-order correction to the heat kernel at finite \( L \) using \(^{20b}\). After some rearrangement, we find

\[ K_{\mathcal{E}}^{(2)}(t) = \frac{4(2\pi)^4}{(L_1 L_2)^2} \left[ \frac{t}{2} \sum_{\bar{N}^+} e^{-\lambda_{\bar{N}^+} t} \sum_{S, \bar{S}, S \neq \bar{S}} \left| \hat{f}^{(1)}_{\bar{S} \bar{S} \bar{N} \bar{N}} \right|^2 \left( \bar{\lambda}_{\bar{N}^+} - \xi \bar{\lambda}_{\bar{N}^+} \right)^2 \right. \]

\[ + \sum_{\bar{N}, \bar{N}'} \left| \hat{f}^{(1)}_{\bar{N} \bar{N}'} \right|^2 e^{-\lambda_{\bar{N}'} t} \left( \bar{\lambda}_{\bar{N}'} - \xi \bar{\lambda}_{\bar{N}'} \right) \left( -\delta_{\bar{N} \bar{S}, \bar{S}} \bar{\lambda}_{\bar{N} \bar{S}} + \delta_{\bar{N} \bar{S}', \bar{S}} \bar{\lambda}_{\bar{N} \bar{S}}^+ \right) \left( \frac{\lambda_{\bar{N}^+} - \xi \lambda_{\bar{N}^+}}{\lambda_{\bar{N}^+} - \lambda_{\bar{N}^+}^+} \right) \left. \right], \tag{37a} \]

\(^{10}\) Note that the direct product structure of \( \Pi \) was crucial; in a general odd-dimensional geometry the spectrum of \( i\hat{D} \) need not be symmetric \(^{17}\).
\begin{equation}
K_4^{(2)}(t) = t \frac{4(2\pi)^4}{(L_1 L_2)^2} \left[ \frac{1}{2} \sum_{\vec{s}, \vec{s}' \neq \vec{s}} e^{-\lambda_{\vec{s} \vec{s}' + t} \Delta \vec{s} \vec{s}^+} \left| \hat{f}_{\vec{s}}^{(1)} \right|^2 \left( \left( \lambda_{\vec{s}^+} - \frac{1}{4} \bar{\lambda} \Delta \vec{s} \vec{s}^+ \right)^2 - D^2_{\vec{s} \vec{s}' + \vec{s}' \vec{s}^+} \right) \right]
+ \sum_{\vec{s}, \vec{s}'} \left| \hat{f}_{\vec{s}}^{(1)} \right|^2 e^{-\lambda_{\vec{s} \vec{s}' + t} \Delta \vec{s} \vec{s}^+} \left( \frac{3}{16} \lambda_{\vec{s}^+} - \bar{\lambda}_{\vec{s}'} + \delta_{\vec{s}, \vec{s}'} \left( \frac{\lambda_{\vec{s}^+} - \bar{\lambda}_{\vec{s}'}}{4 + D_{\vec{s} \vec{s}' + \vec{s}' \vec{s}^+}} \right) \lambda_{\vec{s}^+ - \vec{s}'} - \bar{\lambda}_{\vec{s}'} \right),
\end{equation}

where we defined $\vec{s} \vec{s}^+ \equiv \{ s_1 n_1^+, s_2 n_2^+ \}$, $\Delta \vec{s} \vec{s}^+ \equiv (\vec{s} - \vec{s}') \vec{s}^+$, $D_{\vec{s} \vec{s}' + \vec{s}' \vec{s}^+} = i(2\pi)^2 (n_1 n_2' - n_2 n_1')/(2L_1 L_2)$, and $\delta_{\vec{s}, \vec{s}'} \neq \lambda_{\vec{s}^+ - \vec{s}'} = 1$ if $\lambda_{\vec{s}'} = \bar{\lambda}_{\vec{s}^+}$, and 0 otherwise.

Note that the precise form of the matrices $c_{\vec{s}, \vec{s}'}^{(\vec{s}^+)}$ does not matter since they cancel out in traces, but the presence of the sums over degenerate subspaces in the first terms in the above expressions is an artifact of needing to treat the degenerate subspaces properly.

We may now take the limit $L \to \infty$. The first term in each expression above vanishes in this limit (essentially because $L^{-4} \sum_{\vec{s}} \to L^{-2} \int d^2k \to 0$); for the same reason, the terms containing $\delta_{\vec{s}, \vec{s}'} \neq \lambda_{\vec{s}^+ - \vec{s}'}$ also vanish. We then obtain equation (22) with

\begin{equation}
I_s(k^2 t) = \frac{4}{k^4} P \int d^2q e^{-q^2 t} \left( -\frac{1}{16} k^2 \right. \\
\left. + \frac{(q^2 - k^2/4)^2 + (\vec{k} \times \vec{q})^2/4}{k^2 - 2q \cdot \vec{k}} \right),
\end{equation}

where $q = |\vec{q}|$, $k = |\vec{k}|$, $\vec{k} \times \vec{q} = k_1 q_2 - k_2 q_1$, and $P$ denotes a Cauchy principal value (which comes about since terms in which the denominator vanishes are excluded in the discrete sums). After integration, we obtain (23a) and (23b).