Topology, Quasiperiodic functions and the Transport phenomena.

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Abstract

In this article we give the basic concept of the “Topological Numbers” in theory of quasiperiodic functions. The main attention is paid to appearance of such values in transport phenomena including Galvanomagnetic phenomena in normal metals (Chapter 1) and the modulations of 2D electron gas (Chapter 2). We give just the main introduction to both of these areas and explain in a simple way the appearance of the “integral characteristics” in both of these problems. The paper can not be considered as the detailed survey article in the area but explains the main basic features of the corresponding phenomena.

Introduction.

Galvanomagnetic phenomena in normal metals, Transport in 2D electron gas and Topology of Quasiperiodic functions.

We are going to consider the transport phenomena connected with the geometry of quasiclassical electron trajectories in the magnetic field \( B \).
Let us start with the most fundamental case where this kind of phenomena appears in the conductivity of normal metals having complicated Fermi surfaces in the presence of the rather strong magnetic field. This classical part of solid state physics was started by Kharkov school of I.M. Lifshitz (I.M. Lifshitz, M.Ya. Azbel, M.I. Kaganov, V.G. Peschansky) in 1950’s and has become the essential part of conductivity theory in normal metals. Let us give here some small excurse in this area. We will start with the classical work of I.M. Lifshitz, M.Ya. Azbel and M.I. Kaganov ([1]) where the importance of topology of the Fermi surface for the conductivity was established. Namely, there was shown the difference between the "simple" Fermi surface (topological "sphere") (Fig. 1a) and more complicated surfaces where the non-closed quasiclassical electron trajectories can arise. In particular, the detailed consideration for the "simple" Fermi surface and the surfaces like "warped cylinder" (Fig. 1b) for the different directions of B was made.

Figure 1: The "simple" Fermi surface having the form of the sphere in the Brillouen zone and the periodic "warped cylinder" extending through the infinite number of Brillouen zones. The quasiclassical electron orbits in p-space are also shown for a given direction of B.
Both the pictures on Fig. represent the forms of the Fermi surfaces in \( p \)-space and we should remember that only one Brillouen zone should be taken in the account to get the right phase space volume for the electron states. The values of \( p \) different by any reciprocal lattice vector \( n_1a_1 + n_2a_2 + n_3a_3 \) (where \( n_i \) are integers) are then physically equivalent to each other and represent the same electron state. The Brillouen zone can then be considered as the parallelogram in the \( p \)-space with the identified opposite sides on the boundary.

Also the Fermi surfaces \( S_F \) will then be periodic in \( p \)-space with periods \( a_1, a_2, a_3 \).

**Remark.** From topological point of view we can consider the Brillouen zone as the compact 3-dimensional torus \( T^3 \). The corresponding Fermi-surfaces will then be also compact surfaces of finite size embedded in \( T^3 \).

The presence of the homogeneous magnetic field \( B \) generates the evolution of electron states in the \( p \)-space which can be described by the dynamical system

\[
\dot{p} = \frac{e}{c} [v_{gr}(p) \times B] = \frac{e}{c} [\nabla \epsilon(p) \times B]
\]  

(1)
where $\epsilon(p)$ is the dependence of energy on the quasimomentum (dispersion relation) and $v_{gr}(p) = \nabla \epsilon(p)$ is the group velocity at the state $p$. Both the functions $\epsilon(p)$ and $v_{gr}(p)$ are also the periodic functions in $p$-space and can be considered as the one-valued functions in $T^3$.

The system (11) has two conservative integrals which are the electron energy and the component of $p$ along the magnetic field. The electron trajectories can then be represented as the intersections of the constant energy surfaces $\epsilon(p) = const$ with the planes orthogonal to $B$ and only the Fermi level $\epsilon(p) = \epsilon_F$ is actually important for the conductivity. Easy to see then that global geometry of the ”essential” electron trajectories will depend strongly on the form of Fermi surface in $p$-space.

Coming back to the Fig. 1 we can see that the form of electron trajectories can be quite different for the Fermi surfaces shown at Fig. 1a and Fig. 1b. Such, for the Fermi surface shown at Fig. 1b we can have periodic non-closed electron trajectories (if $B$ is orthogonal to vertical axis) while for the surface on Fig. 1a all the trajectories are just closed curves lying in one Brillouen zone for all directions of $B$.

Let us say now that this global geometry plays the main role in the electron motion in the coordinate space also (despite the factorization in $p$-space). Thus the electron wave-packet motion in $x$-space ($x = (x, y, z)$) can be found from the additional system

$$\dot{x} = v_{gr}(p(t)) = \nabla \epsilon(p(t))$$

for any trajectory in $p$-space after the integration of system (11). The structure of system (11) permits to claim for example that the $xy$-projection of ”electron motion” in $x$-space has the same form as the trajectory in $p$-space just rotated by $\pi/2$. We can see then that the electron drift in $x$-space in magnetic field is also very different for the trajectories shown at Fig. 3a and 3b due to the action of the crystal lattice.

The effect of this ”geometrical drift” can be measured experimentally in the rather pure metallic monocrystals if the mean free electron motion time is big enough (such that electron packet ”feels” the geometric features of trajectory between the two scattering acts). The geometric picture requires then that the time between the two scatterings is much longer than the ”passing time” through one Brillouen zone for the periodic trajectory and much longer than the ”inverse cyclotron frequency” for closed trajectories.\(^1\)

\(^1\)This criterium can be actually more complicated for trajectories of more complicated

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Figure 3: Electron trajectories in $p$-space given by the intersections of planes orthogonal to $B$ for the Fermi surfaces shown at Fig. 1a and Fig. 1b for $B$ orthogonal to vertical axis.

For the approximation of effective mass $m^*$ in crystal this condition can be roughly expressed as $\omega_B \tau \gg 1$, where $\omega_B = eB/m^*c$ is the formal cyclotron frequency and $\tau$ is the mean free electron motion time. Let us note that this requirement is satisfied better for the big values of $B$ and we will consider the formal limit $B \to \infty$ in our paper. We will call this situation "geometric strong magnetic field limit" and consider the asymptotic of conductivity tensor for this case.\footnote{Formally also another condition $\hbar \omega_B \ll \epsilon_F$ should also be imposed on the magnetic field $B$. However, this condition is always satisfied for the real metals and all experimentally available magnetic fields (the upper limit is $B \sim 10^3 - 10^4 T$). So we will not pay special attention to this second restriction and assume that the limit $B \to \infty$ is considered in the "experimental sense" where the second condition is satisfied.}

Let us give here the asymptotic form of conductivity tensor obtained in \cite{1} for the case of trajectories shown at Fig. 3a and Fig. 3b. Let us take the $z$-axis in the $x$-space along the magnetic field $B$. The axes $x$ and $y$ can be chosen arbitrarily for the case of Fig. 3a and we take the $y$-axis along the mean electron drift direction in $x$-space for the case Fig. 3b. (Easy to see form.
that the $x$-axis will then be directed along the mean electron drift in $p$-space in this situation). The asymptotic forms of the conductivity tensor can then be written as:

**Case 1** (Closed trajectories, Fig. 3a):

$$
\sigma_{ik} \approx \frac{n e^2 \tau}{m^*} \left( \begin{array}{llll}
(\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & * \\
(\omega_B \tau)^{-1} & (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & * \\
(\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & *
\end{array} \right), \omega_B \tau \gg 1
$$

**Case 2** (open periodic trajectories, Fig. 3b):

$$
\sigma_{ik} \approx \frac{n e^2 \tau}{m^*} \left( \begin{array}{llll}
(\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & * \\
(\omega_B \tau)^{-1} & (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & * \\
(\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & *
\end{array} \right), \omega_B \tau \gg 1
$$

where $*$ mean some dimensionless constants of order of 1.

We can see that the conductivity reveals the strong anisotropy in the plane orthogonal to $B$ in the second case and the mean direction of the electron trajectory in $p$-space (not in $x$) can be measured experimentally as the zero eigen-direction of $\sigma_{ik}$ for $B \to \infty$.

More general types of open electron trajectories were considered in [2, 3]. For example, the open trajectories which are not periodic were found in [2] for the "thin spatial net" (Fig. 4, a).

The open trajectories exist here only for the directions of $B$ close to main crystallographic axes $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ (Fig. 4, b). It was shown in [2] that the open trajectories lie in this case in the straight strip of the finite width in the plane orthogonal to $B$ and pass through them. The mean direction of open trajectories are given here by the intersections of plane orthogonal to $B$ with the main crystallographic planes $(xy)$, $(yz)$ and $(xz)$.

The form of conductivity tensor for this kind of trajectories used in [2] coincides with (3).

Some analytical dispersion relations were also considered in [3]. Let us mention here also the works [1, 5, 6, 7, 8, 10, 11, 12] where different experimental (and theoretical) investigations for some real metals were made.

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3Actually this work contains some conceptual mistakes but gives also some correct features concerning the existence of open trajectories for these dispersion relations.
Figure 4: The picture from [2] representing the "thin spatial net" and the corresponding directions of $\mathbf{B}$ on the unit sphere where the non-closed electron trajectories exist.

The detailed consideration of these results can be found also in the survey articles [9, 13] and the book [14] (see also [15]).

Let us say now about the topological approach to the problem of general classification of all possible electron trajectories regardless the concrete features of the dispersion relation $\epsilon(p)$ started by S.P. Novikov ([16]) (see also [17, 18, 19]). Let us formulate here the Novikov problem:

**Novikov problem.** Let any smooth 3-periodic function $\epsilon(p)$ be given in the 3-dimensional space $\mathbb{R}^3$ (with arbitrary lattice of periods). Fix any non-degenerate energy level $\epsilon(p) = \text{const}$ (i.e. $\nabla \epsilon(p) \neq 0$ on this level) and consider the intersections of corresponding smooth 3-periodic surface by any set of parallel planes in $\mathbb{R}^3$. Describe the global geometry of all possible non-singular (open) trajectories which can arise in the intersections.

The words "the global geometry" mean here first of all the asymptotic behavior of the trajectory when $t \to \pm \infty$ in sense of dynamical systems. Let us formulate here also Novikov conjecture about the generic non-singular trajectories which was proved later by his pupils:
Novikov conjecture. The generic non-singular open trajectories lie in the straight strips of finite width (in the plane orthogonal to $B$) and pass through them.

Let us emphasize also that Novikov conjecture is connected with the generic open trajectories and can be not valid in the special degenerate cases (S.P. Tsarev, I.A. Dynnikov) as we will see later.

The topological problem of S.P. Novikov was considered later in his school (A.V. Zorich, I.A. Dynnikov, S.P. Tsarev) where the basic theorems about the non-closed trajectories were obtained. Let us say here about the main breakthroughs in this problem made in [20] (A.V. Zorich) and [23] (I.A. Dynnikov).

We note first that even for the rather complicated periodic Fermi surface the electron trajectories will be quite simple if the direction of $B$ is purely rational (with respect to reciprocal lattice), i.e. if the plane $\Pi(B)$ orthogonal to $B$ contains two linearly independent reciprocal lattice vectors. This property can also be formulated in the form that the magnetic fluxes through the faces of elementary cell in the $x$-space are proportional to each other with rational coefficients. In this situation the picture arising in $\Pi(B)$ is purely periodic and all open electron trajectories can be also just the periodic curves corresponding precisely to the case (3). However, the condition of rationality is completely unstable with respect to any small rotations of $B$ such that the rational directions give just a set of measure zero among all the directions of $B$.

The remarkable fact proved by A.V. Zorich is that the open trajectories reveal the “topologically regular” properties even after the small rotations purely rational direction. Namely, they lie in the straight strips of the finite width in accordance with Novikov conjecture (but are not periodic anymore) and pass through them. Let us formulate this in more precise form.

**Theorem 1.** (A.V. Zorich, [20]) Consider arbitrary smooth Fermi surface and the rational direction of magnetic field $B_0$ such that no singular trajectory connects two different (not equivalent modulo the reciprocal lattice) singular (stagnation) points of the system (4). Then there exists small open region $\Omega$ on the unit sphere around direction $B_0$ such that all open trajectories (if they exist) lie in the straight strips of finite width in the plane orthogonal to $B$ if $B/B \in \Omega$, (Fig. 5).
Let us mention also that the additional topological condition in Theorem 1 has a generic form and generically does not impose anything on the direction $B_0$.

Theorem of Zorich claims actually that all the rational directions of $B$ can be extended to some ”small open spots” on the unit sphere (parameterizing directions of $B$ where we can not have situation more complicated than represented at Fig. 5. This set already has the finite measure on the unit sphere and moreover we can conclude that any stable open trajectory can have only the form shown at Fig. 5 since the rational directions are everywhere dense on the unit sphere. Zorich theorem, however, does not permit to state that this situation is the only possible one since the sizes of the ”spots” become smaller and smaller for big rational numbers and we can not claim that they cover all the unit sphere in general situation.

The next important result was obtained by I.A. Dynnikov ([23]) who proved that the trajectories shown at Fig. 5 can be the only stable trajectories with respect to the small variation of the Fermi energy $\epsilon_F$ for a given dispersion relation $\epsilon(p)$. Let us formulate the exact form Dynnikov theorem in Chapter 1 where we will consider this picture in more details. We will just
say here that the methods developed in [23] permitted to prove later that all the cases of open trajectories different from shown at Fig. 5 can appear only "with probability zero" (i.e. for the directions of $\mathbf{B}$ from the set of measure zero on the unit sphere) for generic Fermi surfaces $S_F : \epsilon(p) = \epsilon_F$ ([27, 32]) which gave the final proof of Novikov conjecture for generic open trajectories.

The methods of proofs of Zorich and Dynnikov theorems gave the basis for the invention of the "Topological Quantum Numbers" introduced in [24] by present authors (see also the survey articles [30, 35, 36]) for the conductivity in normal metals. Let us say also that another important property, called later the "Topological Resonance" played the crucial role for physical phenomena in [24]. The main point of this property can be formulated as follows: all the trajectories having the form shown at Fig. 4 have the same mean direction in all the planes orthogonal to $\mathbf{B}$ for the generic directions of $\mathbf{B}$ (actually for any not purely rational direction of $\mathbf{B}$) and give the same form (3) of contribution to conductivity tensor in the same coordinate system. This important fact makes experimentally observable the integer-valued topological characteristics of the Fermi surface having the form of the integral planes of reciprocal lattice and corresponding "stability zones" on the unit sphere. We are going to describe in details these quantities in the Chapter 1 of our paper. Our goal is to give here the main features of the corresponding picture and we don’t give all the details of the classification of all open trajectories for general Fermi surfaces. However, the picture we are going to describe serves as the "basic description" of conductivity phenomena and all the other possibilities can be considered as the special additional features for the non-generic directions of $\mathbf{B}$. Let us also say here that the final classification of open trajectories for generic Fermi surfaces was finished in general by I.A. Dynnikov in [32] which solves in main the Novikov problem. The physical phenomena connected with different types of open trajectories can be found in details in the survey articles [35, 36].

Let us say now some words about the general Novikov problem connected with the quasiperiodic functions on the plane with $N$ quasiperiods. According to the standard definition the quasiperiodic function in $\mathbb{R}^m$ with $N$ quasiperiods ($N \geq m$) is a restriction of a periodic function in $\mathbb{R}^N$ (with $N$ periods) to any plane $\mathbb{R}^m \subset \mathbb{R}^N$ of dimension $m$ linearly embedded in $\mathbb{R}^N$. In our situation we will always have $m = 2$ and the quasiperiodic functions on the plane will be the restrictions of the periodic functions in $\mathbb{R}^N$ to some
2D plane.

**General Novikov problem.** Describe the global geometry of open level curves of quasiperiodic function $f(r)$ on the plane with $N$ quasiperiods.

Easy to see that the general Novikov problem gives the Novikov problem for the electron trajectories if we put $N = 3$. Indeed, all the trajectories in the planes orthogonal to $B$ can be considered as the level curves of quasiperiodic functions $\epsilon(p)|_{\Pi(B)}$ with 3 quasiperiods. According to the said above we can say that the general Novikov problem is solved in main for $N = 3$. However, the case $N > 3$ becomes very complicated from topological point of view and no general classification in this case exists at the moment. Let us say that the only topological result existing now for general Novikov problem is the analog of Zorich theorem (Theorem 1) for the case $N = 4$ (S.P. Novikov, [38]) and the general situation is still under investigation by now.

In Chapter 2 of our paper we consider another application of general Novikov problem connected with the "superlattice potentials" for the two-dimensional electron gas in the presence of orthogonal magnetic field. This kind of potentials is connected with modern techniques of "handmade" modulations of 2D electron gas such as the holographic illumination, "gate modulation", piezoelectric effect etc. ... All such modulations are usually periodic in the plane and in many situations the level curves play the important role for the transport phenomena in such systems. The most important thing for us will be the conductivity phenomena in these 2D structures in the presence of orthogonal magnetic field $B$. According to the quasiclassical approach the cyclotron electron orbits drift along the level curves of modulation potential in the magnetic field which gives the "drift contribution" to conductivity in the plane. Among the papers devoted to this approach we would like to mention here the paper [40] (C. Beenakker) where this approach was introduced for the explanation of "commensurability oscillations" of conductivity in potential modulated just in one direction and [41] (D.E. Grant, A.R. Long, J.H. Davies) where the same approach was used for explanation of suppression of these oscillations by the second orthogonal modulation in the periodic case. Let us add that all these phenomena correspond to the long free electron motion time which will now play the role of the "geometric limit" (not $B \to \infty$) in this second situation.

We are going to show that the general Novikov problem can also arise naturally in these structures if we consider the independent superposition of
different periodic modulations. It can be proved that in this case we always obtain the quasiperiodic functions where the number of quasiperiods depends on the complexity of total modulation. The results in Novikov problem can then help to predict the form of the ”drift conductivity” in the limit of long free electron motion time. In Chapter 2 we give the main features of the situation of superposition of several ”1D modulations” where the potentials with small number of quasiperiods can arise. The detailed consideration of this situation can be found in [42]. However, the Novikov problem arise also in much more general case of arbitrary superpositions of more complicated (but periodic) structures.

At last we would like to say that the quasiperiodic functions with big number of quasiperiods can be a model for the random potentials on the plane. The corresponding Novikov problem arise in the percolation theory for such potentials. We will also say some words about this situation at the end of Chapter 2.

1 The classification of Fermi surfaces and the ”Topological Quantum Numbers”.

Let us start with the definitions of genus and Topological Rank of the Fermi surface.

Definition 1.

Let us consider the phase space $T^3 = \mathbb{R}^3/L$ introduce above. After the identification every component of the Fermi surface becomes the smooth orientable 2-dimensional surface embedded in $T^3$. We can then introduce the standard genus of every component of the Fermi surface $g = 0, 1, 2, ...$ according to standard topological classification depending on if this component is topological sphere, torus, sphere with two holes, etc ... (Fig. 6)

Definition 2.

Let us introduce the Topological Rank $r$ as the characteristic of the embedding of the Fermi surface in $T^3$. It’s much more convenient in this case to come back to the total $p$-space and consider the connected components of the three-periodic surface in $\mathbb{R}^3$. 
1) The Fermi surface has Rank 0 if every its connected component can be bounded by a sphere of finite radius.

2) The Fermi surface has Rank 1 if every its connected component can be bounded by the periodic cylinder of finite radius and there are components which can not be bounded by the sphere.

3) The Fermi surface has Rank 2 if every its connected component can be bounded by two parallel (integral) planes in $\mathbb{R}^3$ and there are components which can not be bounded by cylinder.

4) The Fermi surface has Rank 3 if it contains components which can not be bounded by two parallel planes in $\mathbb{R}^3$.

The pictures on Fig. 6 a-d represent the pieces of the Fermi surfaces in $\mathbb{R}^3$ with the Topological Ranks 0, 1, 2 and 3 respectively.

As can be seen the genuses of the surfaces represented on the Fig. 6 a-d are also equal to 0, 1, 2 and 3 respectively. However, the genus and the Topological Rank are not necessary equal to each other in the general situation.

Let us discuss briefly the connection between the genus and the Topological Rank since this will play the crucial role in further consideration.

It is easy to see that the Topological Rank of the sphere can be only zero and the Fermi surface consists in this case of the infinite set of the periodically repeated spheres $S^2$ in $\mathbb{R}^3$.

The Topological Rank of the torus $T^2$ can take three values $r = 0$, $r = 1$ and $r = 2$. Indeed, it is easy to see that all the three cases of periodically
Figure 7: The Fermi surfaces with Topological Ranks 0, 1, 2 and 3 respectively.
repeated tori $T^2$ in $\mathbb{R}^3$ (Rank 0), periodically repeated "warped" integral cylinders (Rank 1) and the periodically repeated "warped" integral planes (Rank 2) give the topological 2-dimensional tori $T^2$ in $T^3$ after the factorization (see Fig. 8).

It’s not difficult to prove that these are the only possibilities which we can have for embedding of the 2-dimensional torus $T^2$ in $T^3$. We just note here that the mean direction of the "warped periodic cylinder" (embedding of Rank 1) can coincide with any reciprocal lattice vector $n_1a_1 + n_2a_2 + n_3a_3$ in $\mathbb{R}^3$. Also the "directions" of the corresponding "warped planes" (embedding of Rank 2) are always generated by two (linearly independent) reciprocal lattice vectors $m_1^{(1)}a_1 + m_2^{(1)}a_2 + m_3^{(1)}a_3$ and $m_1^{(2)}a_1 + m_2^{(2)}a_2 + m_3^{(2)}a_3$. We can see so that both the embeddings of Rank 1 and Rank 2 of $T^2$ in $T^3$ are characterized by some integer numbers connected with the reciprocal lattice.

Let us make also one more remark about the surfaces of Ranks 0, 1 and 2 in this case. Namely the case $r = 2$ has actually one difference from the cases $r = 0$ and $r = 1$. The matter is that the plane in $\mathbb{R}^3$ is not homological to zero in $T^3$ (i.e. does not restrict any domain of "lower energies") after the factorization. We can conclude so that if these planes appear as the connected
components of the physical Fermi surface (which is always homological to zero) they should always come in pairs, $\Pi_+$ and $\Pi_-$, which are parallel to each other in $\mathbb{R}^3$. The factorization of $\Pi_+$ and $\Pi_-$ gives then the two tori $T^2_+$, $T^2_-$ with the opposite homological classes in $T^3$.

It can be shown that the Topological Rank of any Fermi surface of genus 2 can not exceed 2 also. The example of the corresponding immersion of such component with maximal Rank is shown at Fig. 7 c and represents the two parallel planes connected by cylinders. We will not give the proof of this theorem here and just say that this fact plays important role in the classification of non-closed electron trajectories on the Fermi surface of genus 2. Namely, it can be proved that the open trajectories on the Fermi surface of genus 2 can not be actually more complicated than the trajectories on the surface of genus 1. In particular they always have the ”topologically regular form” in the same way as on the Fermi surface of genus 1 (see later). Also the same integral characteristics in the cases when this surface has Rank 1 or 2 as in the case of genus 1 can be introduced for genus 2 (actually for any genus if Rank is equal to 1 or 2).

At last we say that the Topological Rank of the components with genus $g \geq 3$ can take any value $r = 0, 1, 2, 3$.

**Definition 3.**

We call the open trajectory topologically regular (corresponding to ”topologically integrable” case) if it lies within the straight line of finite width in $\Pi(B)$ and passes through it from $-\infty$ to $\infty$. All other open trajectories we will call chaotic.

Let us discuss now the connection between the geometry of the non-singular electron orbits and the topological properties of the Fermi surface. We will briefly consider here the simple cases of Fermi surfaces of Rank 0, 1 and 2 and come then to our basic case of general Fermi surfaces having the maximal rank $r = 3$. We have then the following situations:

1) The Fermi surface has Topological Rank 0.

Easy to see that in this simplest case all the components of the Fermi surface are compact (Fig. 7a, 8a) in $\mathbb{R}^3$ and there is no open trajectories at all.
2) The Fermi surface has Topological Rank 1.
In this case we can have both open and compact electron trajectories. However the open trajectories (if they exist) should be quite simple in this case. They can arise only if the magnetic field is orthogonal to the mean direction of one of the components of Rank 1 (periodic cylinder) and are periodic with the same integer mean direction (Fig. 7b, 8b). The corresponding sets of the directions $B/B$ are just the one-dimensional curves and there can not be the open regions on the unit sphere for which we can find the open trajectories on the Fermi surface.

3) The Fermi surface has Topological Rank 2.
It can be easily seen that this case gives much more possibilities for the existence of open orbits for different directions of the magnetic field. In particular, this is the first case where the open orbits can exist for the generic direction of $B$. So, in this case we can have the whole regions on the unit sphere such that the open orbits present for any direction of $B$ belonging to the corresponding region. It is easy to see, however, that the open orbits have also a quite simple description in this case. Namely, any open orbit (if they exist) lies in this case in the straight strip of the finite width for any direction of $B$ not orthogonal to the integral planes given by the components of Rank 2. The boundaries of the corresponding strips in the planes $\Pi(B)$ (orthogonal to $B$) will be given by the intersection of $\Pi(B)$ with the pairs of integral planes bounding the corresponding components of Rank 2. It can be also shown ([21], [22]) that every open orbit passes through the strip from $-\infty$ to $+\infty$ and can not turn back. We can see then that all the trajectories are "topologically regular in this case also.

According to the remarks above the contribution to the conductivity given by every family of orbits with the same mean direction reveals the strong anisotropy when $\omega B \tau \rightarrow \infty$ and coincides in the main order with the formula (3) for the open periodic trajectories.

Let us say that the trajectories of this type have already all the features of the general topologically integrable situation.

Let us start now with the most general and complicated case of arbitrary Fermi surface of Topological rank 3.
We describe first the convenient procedure ([27], [32]) of reconstruction of the constant energy surface when the direction of $B$ is fixed.
We will assume that the system has generically only the non-degenerate
Figure 9: The cylinder of compact trajectories bounded by the singular orbits. (The simplest case of just one critical point on the singular trajectory.)

singularities having the form of the non-degenerate poles or non-degenerate saddle points. The singular trajectories passing through the critical points (and the critical points themselves) divide the set of trajectories into different parts corresponding to different types of trajectories on the Fermi surface. We will not be interested here in the geometry of compact electron orbits in the "geometric limit" $\omega_B \tau \to \infty$. It’s not difficult to show that the pieces of the Fermi surface carrying the compact orbits can be either infinite or finite cylinders in $\mathbb{R}^3$ bounded by the singular trajectories (some of them maybe just points of minimum or maximum) at the bottom and at the top (see Fig. 9).

Let us remove now all the parts containing the non-singular compact trajectories from the Fermi surface. The remaining part

$$S_F/(\text{Compact Nonsingular Trajectories}) = \bigcup_j S_j$$

is a union of the 2-manifolds $S_j$ with boundaries $\partial S_j$ who are the compact singular trajectories. The generic type in this case is a separatrix orbit with just one critical point like on the Fig. 9.

Easy to see that the open orbit will not be affected at all by the construc-
Figure 10: The reconstructed constant energy surface with removed compact trajectories and the two-dimensional discs attached to the singular trajectories in the generic case of just one critical point on every singular trajectory.

Definition 4.
We call every piece \( S_j \) the "Carrier of open trajectories".

Let us fill in the holes by topological 2D discs lying in the planes orthogonal to \( B \) and get the closed surfaces

\[
\bar{S}_j = S_j \cup (2\text{D discs})
\]

(see Fig. 10).

This procedure gives again the periodic surface \( \bar{S}_e \) after the reconstruction and we can define the "compactified carriers of open trajectories" both in \( \mathbb{R}^3 \) and \( \mathbb{T}^3 \).

Easy to see then that the reconstructed surface can be used instead of the original Fermi surface for the determination of open trajectories. Let us ask the question: can the reconstructed surface be simpler than the original
one?

The answer is positive and moreover it can be proved that "generically" the reconstructed surface consists of components of genus 1 only. This remarkable fact gives the very powerful instrument for the consideration of open trajectories on the arbitrary Fermi surface.

In fact, the proof of the Theorem 1 was based on the statement that genus of every compactified carrier of open orbits $\bar{S}_j$ is equal to 1 in this case.

Let us formulate now the Theorem of I.A. Dynnikov ([23]) which made the second main breakthrough in the Novikov problem.

**Theorem 2.** (I.A. Dynnikov, [23]).

*Let a generic dispersion relation $\epsilon(p) : \mathbb{T}^3 \to \mathbb{R}$ be given such that for level $\epsilon(p) = \epsilon_0$ the genus $g$ of some carrier of open trajectories $\bar{S}_i$ is greater than 1. Then there exists an open interval $(\epsilon_1, \epsilon_2)$ containing $\epsilon_0$ such that for all $\epsilon \neq \epsilon_0$ in this interval the genus of carrier of open trajectories is less than $g$.*

The Theorem 2 claims then that only the "Topologically Integrable case" can be stable with respect to the small variations of energy level also.

The formulated theorems permit us to reduce the consideration of open orbits in any stable situation to the case of the surfaces of genus 1 where the Fermi surface can have Topological Rank 0, 1 or 2 only. Easy to see that the Rank 0 can not appear just by definition in the reconstructed surface $\bar{S}$ since it can contain only the compact trajectories. The Rank 1 is possible in $\bar{S}$ only for special directions of $\mathbf{B}$. Indeed, the component of Rank 1 has the mean integral direction in $\mathbb{R}^3$ and can contain the open (periodic) trajectories only if $\mathbf{B}$ is orthogonal this integral vector in $p$-space. The corresponding open trajectories is then not absolutely stable with respect to the small rotations of $\mathbf{B}$ and can not exist for the open region on the unit sphere.

We can claim then that the only generic situation for $\bar{S}_i$ is a set of components of Rank 2 which are the periodic warped planes in this case. The corresponding electron trajectories can then belong just to "Topologically integrable" case being the intersections of planes orthogonal to $\mathbf{B}$ with the periodically deformed planes in the $p$-space.

The important property of the compactified components of genus 1 aris-
ing for the generic directions of $\mathbf{B}$ is following: they are all parallel in average in $\mathbb{R}^3$ and do not intersect each other. This property mentioned in [24] and called later the "Topological resonance" plays the important role in the physical phenomena connected with geometry of open trajectories. Such, in particular, all the stable topologically regular open trajectories in all planes orthogonal to $\mathbf{B}$ have then the same mean direction and give the same form (3) of contribution to conductivity in the appropriate coordinate system common for all of them. This fact gives the experimental possibility to measure the mean direction of non-compact topologically regular orbits both in $\mathbf{x}$ and $\mathbf{p}$ spaces from the anisotropy of conductivity tensor $\sigma^{ik}$.

Let us say again that the surface $\bar{S}_\epsilon$ is the abstract construction depending on the direction of $\mathbf{B}$ and do not exist apriori in the Fermi surface $S_{\epsilon_F}$. The important fact, however, is the stability of the surface $\bar{S}_\epsilon$ with respect to the small rotations of $\mathbf{B}$. This means in particular that the common direction the components of Rank 2 is locally stable with respect to the small rotations of $\mathbf{B}$ which can be then discovered in the conductivity experiments. From the physical point of view, all the regions on the unit sphere where the stable open orbits exist can be represented as the "stability zones" $\Omega_\alpha$ such that each zone corresponds to some integral plane $\Gamma_\alpha$ common to all the points of stability zone $\Omega_\alpha$. The plane $\Gamma_\alpha$ is then the integral plane in reciprocal lattice which defines the mean directions of open orbits in $\mathbf{p}$-space for any direction of $\mathbf{B}$ belonging to $\Omega_\alpha$ just as the intersection with the plane orthogonal to $\mathbf{B}$. As can be easily seen from the form of (3) this direction always coincides with the unique direction in $\mathbb{R}^3$ corresponding to the decreasing of conductivity as $\omega_B \tau \to \infty$.

The corresponding integral planes $\Gamma_\alpha$ can then be given by three integer numbers $(n_1^\alpha, n_2^\alpha, n_3^\alpha)$ (up to the common multiplier) from the equation

$$n_1^\alpha [\mathbf{x}]_1 + n_2^\alpha [\mathbf{x}]_2 + n_3^\alpha [\mathbf{x}]_3 = 0$$

where $[\mathbf{x}]_j$ are the coordinates in the basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of the reciprocal lattice, or equivalently

$$n_1^\alpha (\mathbf{x}, \mathbf{l}_1) + n_2^\alpha (\mathbf{x}, \mathbf{l}_2) + n_3^\alpha (\mathbf{x}, \mathbf{l}_3) = 0$$

where $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\}$ is the basis of the initial lattice in the coordinate space.

We see then that the direction of conductivity decreasing $\hat{\eta} = (\eta_1, \eta_2, \eta_3)$ satisfies to relation

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\[ n_1^a(\hat{\eta}, \hat{l}_1) + n_2^a(\hat{\eta}, \hat{l}_2) + n_3^a(\hat{\eta}, \hat{l}_3) = 0 \]

for all the points of stability zone \( \Omega_\alpha \) which makes possible the experimental observation of numbers \((n_1^a, n_2^a, n_3^a)\).

The numbers \((n_1^a, n_2^a, n_3^a)\) were called in [24] the "Topological Quantum numbers" of a dispersion relation in metal.

Let us note, that we can consider now the result of [2] about the "thin spatial net” as the particular case of this general theorems where the integer planes take the simplest possibility being the main planes \( xy, yz, xz \). If we introduce now the "Topological Quantum numbers" for this situation we will have only the triples \((\pm 1, 0, 0), (0, \pm 1, 0)\) and \((0, 0, \pm 1)\) for this Fermi surface.

In general, we can state that the unit sphere should be divided into the (open) parts where the open orbits are absent at all on the Fermi level for given directions of \( \mathbf{B} \) and the "stability zones" \( \Omega_\alpha \) where the open orbits exist on the Fermi level and have "topologically regular" form. Every stability zone corresponds to the triple of "Topological quantum numbers" giving the integral direction of periodically deformed two-dimensional planes in \( \bar{S}_{\epsilon_F}(\mathbf{B}) \) which are swept by the zero eigen-vector of \( \sigma_{ik} \) for \( \mathbf{B} \in \Omega_\alpha \).

Let us say now that the "Topologically regular" trajectories are the generic open trajectories but nonetheless they are not the only possible for rather complicated Fermi surfaces. Namely, for rather complicated Fermi surfaces and the special directions of \( \mathbf{B} \) the chaotic cases can also arise (S.P. Tsarev, I.A. Dynnikov).

It was first shown by S.P. Tsarev ([33]) that the more complicated chaotic open orbits can still exist on rather complicated Fermi surfaces \( S_F \). Such, the example of open trajectory which does not lie in any finite strip of finite width was constructed. However, the trajectory had in this case the asymptotic direction even not being restricted by any straight strip of finite width in the plane orthogonal to \( \mathbf{B} \). The corresponding asymptotic behavior of conductivity should reveal also the strong anisotropy properties in the plane orthogonal to \( \mathbf{B} \) although the exact form of \( \sigma_{ik} \) will be slightly different from (3) for this type of trajectories. By the same reason, the asymptotic direction of orbit can be measured experimentally in this case.

The more complicated examples of chaotic open orbits were constructed in [27] for the Fermi surface having genus 3. These types of the open orbits do not have any asymptotic direction in the planes orthogonal to \( \mathbf{B} \) and have
rather complicated form “walking everywhere” in these planes.

The corresponding contribution to $\sigma^{ik}$ is also very different for this kind of trajectories (29). In particular, it appears that this contribution becomes zero in all the directions including direction of $B$ for $B \to \infty$. The total conductivity tensor $\sigma^{ik}$ has then only the contribution of compact electron trajectories in the conductivity along $B$ which does not disappear when $B \to \infty$. The corresponding effect can be observed experimentally as the local minima of the longitudinal (i.e. parallel to $B$) conductivity for the points of the unit sphere where this kind of trajectories can appear. The more detailed description of $\sigma^{ik}$ in this case can be found in [29].

Let us add that it was proved recently by I.A. Dynnikov that the measure of chaotic cases on the unit sphere is zero for generic Fermi surface [27, 32]. The systematic investigation of the open orbits was completed in general after the works [20, 23, 24, 27] in [32]. In particular the total picture of different types of the open orbits for generic dispersion relations was presented. Let us just formulate here the main results of [32] in the form of Theorem.

**Theorem 3** (I.A. Dynnikov, [32]).

Let us fix the dispersion relation $\epsilon = \epsilon(p)$ and the direction of $B$ of irrationality $3$ and consider all the energy levels for $\epsilon_{\min} \leq \epsilon \leq \epsilon_{\max}$. Then:

1) The open electron trajectories exist for all the energy values $\epsilon$ belonging to the closed connected energy interval $\epsilon_1(B) \leq \epsilon \leq \epsilon_2(B)$ which can degenerate to just one energy level $\epsilon_1(B) = \epsilon_2(B) = \epsilon_0(B)$.

2) For the case of the nontrivial energy interval the set of compactified carriers of open trajectories $\tilde{S}_\epsilon$ is always a disjoint union of two-dimensional tori $\mathbb{T}^2$ in $\mathbb{T}^3$ for all $\epsilon_1(B) \leq \epsilon \leq \epsilon_2(B)$. All the tori $\mathbb{T}^2$ for all the energy levels do not intersect each other and have the same (up to the sign) indivisible homology class $c \in H_2(\mathbb{T}^3, \mathbb{Z})$, $c \neq 0$. The number of tori $\mathbb{T}^2$ is even for every fixed energy level and the corresponding covering $\tilde{S}_\epsilon$ in $\mathbb{R}^3$ is a locally stable family of parallel (“warped”) integral planes $\Pi^2_i \subset \mathbb{R}^3$ with common direction given by $c$. The form of $\tilde{S}_\epsilon$ described above is locally stable with the same homology class $c \in H_2(\mathbb{T}^3)$ under small rotations of $B$. All the open electron trajectories at all the energy levels lie in the strips of finite width with the same direction and pass through them. The mean direction of the trajectories is given by the intersections of planes $\Pi(B)$ with the integral family $\Pi^2_i$ for the corresponding ”stability zone” on the unit sphere.

3) The functions $\epsilon_1(B)$, $\epsilon_2(B)$ defined for the directions of $B$ of irrationality $3$ can be continuated on the unit sphere $S^2$ as the piecewise smooth
functions such that \( \epsilon_1(B) \geq \epsilon_2(B) \) everywhere on the unit sphere.

4) For the case of trivial energy interval \( \epsilon_1 = \epsilon_2 = \epsilon_0 \) the corresponding open trajectories may be chaotic. Carrier of the chaotic open trajectory is homologous to zero in \( H_2(T^3, \mathbb{Z}) \) and has genus \( \geq 3 \). For the generic energy level \( \epsilon = \epsilon_0 \) the corresponding directions of magnetic fields belong to the countable union of the codimension 1 subsets. Therefore a measure of this set is equal to zero on \( S^2 \).

Let us say that we give here the results connected with generic directions of \( B \) and do not consider the special cases when \( B \) is purely or "partly" rational. The corresponding effects are actually simpler then formulated above and can be easily added to this general picture. Let us give here the references to the survey articles \[30, 32, 35, 36\] where all the details (both from mathematical and physical point of view) can be found.

\section{Quasiperiodic modulations of 2D electron gas and the general Novikov problem.}

In this chapter we will give the general impression about the quasiperiodic modulations of 2D electron gas and describe the main topological aspects for the special class of such structures. Let us say first some words about different modern modulation techniques and the quasiclassical electron behavior in such systems.

We first point here the holographic illumination of high-mobility 2D electron structures (AlGaAS – GaAs hetero-junctions) at the temperatures \( T \leq 4.2K \) (see, for example \[39\]). In these experiments the expanded laser beam was splitted into two parts which gave an interference picture with the period \( a \) on the 2D sample. The illumination caused the additional ionization of atoms near the 2D junction which remains for a rather long time after the illumination. During this relaxation time the additional periodic potential \( V(r) = V(x), V(x) = V(x + a) \) arised in the plane and the electron behavior was determined by the orthogonal magnetic field \( B \) and the potential \( V(x) \).

The quasiclassical consideration for the case \( |V(x)| \ll \epsilon_F \) was first considered by C.W.J. Beenakker (\[40\]) for the explanation of "commensurability oscillations" in such structures found in \[39\]. According to this approach the quasiclassical electrons near Fermi level move around the cyclotron orbits in
Figure 11: The averaging of the the potential $V(x)$ over the cyclotron orbit with radius $r_B$ centered at the point $r$.

the magnetic field and drift due to potential $V(x)$ in the plane. Since only the electrons near Fermi level $\epsilon_F$ play the main role in the conductivity we can introduce the characteristic cyclotron radius $r_B = m^*v_F/eB$ for the Fermi velocity $v_F$. The corresponding drift of the electron orbits near Fermi will then be determined by the averaged effective potential $V_{B_{eff}}(x)$ given by the averaging of $V(r) = V(x)$ over the cyclotron orbit with radius $r_B$ centered at the point $r$ (Fig 11).

The potential $V_{B_{eff}}(x)$ is different from $V(x)$ but has the same symmetry and also depends only on $x$. The drift of the cyclotron orbits is going along the level curves of $V_{B_{eff}}(x)$ which are very simple in this case (just the straight lines along the $y$-axis) and the corresponding velocity $v_{drift}$ is proportional to the absolute value of gradient $|V_{B_{eff}}(x)|$ at each level curve. The analytic dependence of $|V_{B_{eff}}(x)|$ on the value of $B$ (based on the commensurability of $2r_B$ with the (integer number) $\times a$) was used in [40] for the explanation of the oscillations of conductivity along the fringes with the value of $B$.

In the paper [41] the situation with the double-modulated potentials made by the superposition of two interference pictures was also considered. The corresponding potential $V(r)$ is double-periodic in $\mathbb{R}^2$ in this case and the
same is true for potentials $V_{\text{eff}}^B(r)$. The consideration used the same quasiclassical approach for the potential $V_{\text{eff}}^B(r)$ based on the analysis of its level curves. It was shown then in [41] that the second modulation should suppress the commensurability oscillations in this case which disappear at all for the equal intensities of two (orthogonal) interference pictures.

Easy to see also that all the open drift trajectories can be only periodic in the case of periodic $V_{\text{eff}}^B(r)$.

It seems that the situation with the quasiperiodic modulations of 2D electron gas did not appear in experiments. However, we think that this situation is also very natural for the technique described above and can be considered from the point of view of general Novikov problem. The corresponding approach was developed in [42] for the special cases of superpositions of several (3 and 4) interference pictures on the plane. Nonetheless, as we already mentioned, Novikov problem arise actually also for any picture given by superposition of several periodic pictures in the plane. The corresponding potentials can have many quasiperiods in this case and the Novikov problem can reveal then much more complicated properties (chaotic) than described in [42].

We are going however to describe here just the main points of ”topologically regular” behavior in the case of the superpositions of 3 and 4 interference pictures which give the quasiperiodic potentials $V(r)$ and $V_{\text{eff}}^B(r)$ with 3 and 4 quasiperiods on the plane. Unlike the previous papers we don’t pay here much attention on the analytic dependence on $B$ and investigate in main the geometric properties of conductivity in this case.

Before we start the geometric consideration we want to say also that the holographic illumination is not the unique way to produce the superlattice potentials for the two-dimensional electron gas. Let us mention here the works [43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53] where the different techniques using the biasing of the specially made metallic gates and the piezoelectric effect were considered. Both 1D and 2D modulated potentials as well as more general periodic potentials with square and hexagonal geometry appeared in this situation. Actually these techniques give much more possibilities to produce the potentials of different types with the quasiperiodic properties.

Let us have now three independent interference pictures on the plane with three different generic directions of fringes $\eta_1, \eta_2, \eta_3$ and periods $a_1, a_2, a_3$ (see Fig. 12).
The total intensity $I(\mathbf{r})$ will be the sum of intensities

\[ I(\mathbf{r}) = I_1(\mathbf{r}) + I_2(\mathbf{r}) + I_3(\mathbf{r}) \]

of the independent interference pictures.

We assume that there are at least two non-coinciding directions (say $\eta_1, \eta_2$) among the set $(\eta_1, \eta_2, \eta_3)$.

It can be shown that the potentials $V(\mathbf{r})$ and $V_{\text{eff}}^B(\mathbf{r})$ can be represented in this situation as the quasiperiodic functions with 3 quasiperiods in the plane.

Let us introduce now the important definition of the "quasiperiodic group" acting on the potentials described above.

**Definition 5.**

Let us fix the directions $\eta_1, \eta_2, \eta_3$ and periods $a_1, a_2, a_3$ of the interference fringes on picture 12 and consider all independent parallel shifts of positions of different interference pictures in $\mathbb{R}^2$. We will say that all the potentials $V'(\mathbf{r})$ (and the corresponding $V_{\text{eff}}^B'(\mathbf{r})$) made in this way are related by the transformations of a quasiperiodic group.
According to the definition the quasiperiodic group is a three-parametric Abelian group isomorphic to the 3-dimensional torus $\mathbb{T}^3$ due to the periodicity of every interference picture.\(^4\)

We will say that potential $V(r)$ is generic if it has no periods in $\mathbb{R}^2$. We say that potential $V(r)$ is periodic if it has two linearly independent periods in $\mathbb{R}^2$ and that $V(r)$ is "partly periodic" if it has just one (up to the integer multiplier) period in $\mathbb{R}^2$.

It can be also shown that the quasiperiodic group does not change the "periodicity" of potentials $V(r)$, $V_{\text{eff}}^B(r)$.

Let us say now that the results for Novikov problem can be applied also in this situation. We will formulate here the main results for the generic potentials $V(r)$ (the special additional features can be found in [42]). Let us formulate here the theorem from [42] about the drift trajectories for the generic potentials of this kind based on the topological theorems for Novikov problem in 3-dimensional case (formulated above).

**Theorem 4. ([42])**

Let us fix the value of $B$ and consider the generic quasiperiodic potential $V_{\text{eff}}^B(r)$ made by three interference pictures and taking the values in some interval $\epsilon_{\text{min}}(B) \leq V_{\text{eff}}^B(r) \leq \epsilon_{\text{max}}(B)$. Then:

1) Open quasiclassical trajectories $V_{\text{eff}}^B(r) = c$ always exist either in the connected energy interval

$$\epsilon_1(B) \leq c \leq \epsilon_2(B)$$

($\epsilon_{\text{min}}(B) < \epsilon_1(B) < \epsilon_2(B) < \epsilon_{\text{max}}(B))$ or just at one energy value $c = \epsilon_0(B)$.

2) For the case of the finite interval ($\epsilon_1(B) < \epsilon_2(B)$) all the non-singular open trajectories correspond to topologically regular case, i.e. lie in the straight strips of the finite width and pass through them. All the strips have the same mean directions for all the energy levels $c \in [\epsilon_1(B), \epsilon_2(B)]$ such that all the open trajectories are in average parallel to each other for all values of $c$.

3) The values $\epsilon_1(B)$, $\epsilon_2(B)$ or $\epsilon_0(B)$ are the same for all the generic potentials connected by the "quasiperiodic group".

—

\(^4\)Easy to see that the quasiperiodic group contains the ordinary translations as the algebraic subgroup.
4) For the case of the finite energy interval ($\epsilon_1(B) < \epsilon_2(B)$) all the non-singular open trajectories also have the same mean direction for all the generic potentials connected by the "quasiperiodic group" transformations.

We see again that the "topologically regular" open trajectories are also generic for this situation as previously.

Let us consider now the asymptotic behavior of conductivity tensor when $\tau \to \infty$ (mean free electron motion time). We will consider here only the "topologically regular" case. Let us point out that the full conductivity tensor can be represented as the sum of two terms

$$\sigma_{0}^{ik}(B) = \sigma_{0}^{ik}(B) + \Delta\sigma^{ik}(B)$$

In the approximation of the drifting cyclotron orbits the parts $\sigma_{0}^{ik}(B)$ and $\Delta\sigma^{ik}(B)$ can be interpreted as caused respectively by the (infinitesimally small) difference in the electron distribution function on the same cyclotron orbit (weak angular dependence) and the (infinitesimally small) difference in the occupation of different trajectories by the centers of cyclotron orbits at different points of $\mathbb{R}^2$ (on the same energy level) as the linear response to the (infinitesimally) small external field $E$.

The first part $\sigma_{0}^{ik}(B)$ has the standard asymptotic form:

$$\sigma_{0}^{ik}(B) \sim \frac{ne^2}{m\varepsilon_f} \left( \frac{\omega_B \tau}{\omega_B \tau - 1} \right) - \frac{\omega_B \tau}{\omega_B \tau - 1} \left( \frac{\omega_B \tau}{\omega_B \tau - 1} \right)^{-1}$$

for $\omega_B \tau \gg 1$ due to the weak angular dependence ($\sim 1/\omega_B \tau$) of the distribution function on the same cyclotron orbit. We have then that the corresponding longitudinal conductivity decreases for $\tau \to \infty$ in all the directions in $\mathbb{R}^2$ and the corresponding condition is just $\omega_B \tau \gg 1$ in this case.

For the part $\Delta\sigma^{ik}(B)$ the limit $\tau \to \infty$ should, however, be considered as the condition that every trajectory is passed for rather long time by the drifting cyclotron orbits to reveal its global geometry. Thus another parameter $\tau/\tau_0$ where $\tau_0$ is the characteristic time of completion of close trajectories should be used in this case and we should put the condition $\tau/\tau_0 \gg 1$ to have the asymptotic regime for $\Delta\sigma^{ik}(B)$. In this situation the difference between the open and closed trajectories plays the main role and the asymptotic behavior of conductivity can be calculated in the form analogous to that used in [1, 2, 3] for the case of normal metals. Namely:
\[
\Delta \sigma^{ik}(B) \sim \frac{ne^2\tau}{m^{eff}} \left( \begin{array}{cc}
\frac{(\tau_0/\tau)^2}{\tau_0/\tau} & \frac{\tau_0/\tau}{(\tau_0/\tau)^2} \\
\end{array} \right)
\]
in the case of closed trajectories and

\[
\Delta \sigma^{ik}(B) \sim \frac{ne^2\tau}{m^{eff}} \left( \begin{array}{cc} *
\frac{\tau_0/\tau}{\tau_0/\tau} & \frac{\tau_0/\tau}{(\tau_0/\tau)^2} \\
\end{array} \right)
\]
(* \sim 1) for the case of open topologically regular trajectories if the x-axis coincides with the mean direction of trajectories.

The condition \(\tau/\tau_0 \gg 1\) is much stronger then \(\omega_B \tau \gg 1\) in the situation described above just according to the definition of the slow drift of the cyclotron orbits. We can keep then just this condition in our further considerations and assume that the main part of conductivity is given by \(\Delta \sigma^{ik}(B)\) in this limit. Easy to see also that the magnetic field \(B\) should not be "very strong" in this case.

According to the remarks above we can write now the main part of the conductivity tensor \(\sigma^{ik}(B)\) in the limit \(\tau \to \infty\) for the case of topologically regular open orbits. Let us take the x axis along the mean direction of open orbits and take the y-axis orthogonal to \(x\). The asymptotic form of \(\sigma^{ik}\), \(i,k = 1,2\) can then be written as:

\[
\sigma^{ik} \sim \frac{ne^2\tau}{m^{eff}} \left( \begin{array}{cc} *
\frac{\tau_0/\tau}{\tau_0/\tau} & \frac{\tau_0/\tau}{(\tau_0/\tau)^2} \\
\end{array} \right), \quad \tau_0/\tau \to 0
\]

where * is some value of order of 1 (constant as \(\tau_0/\tau \to 0\)).

The asymptotic form of \(\sigma^{ik}\) makes possible the experimental observation of the mean direction of topologically regular open trajectories if the value \(\tau/\tau_0\) is rather big.

Let us introduce now the "topological numbers" characterizing the regular open trajectories analogous to introduced in [24] for the case of normal metals. We will give first the topological definition of these numbers using the action of the "quasiperiodic group" on the quasiperiodic potentials ([23]).

We assume that we have the "topologically integrable" situation where the topologically regular open trajectories exist in some finite energy interval \(\epsilon_1(B) \leq c \leq \epsilon_2(B)\). According to Theorem 4 the values \(\epsilon_1(B), \epsilon_2(B)\) and the mean directions of open trajectories are the same for all the potentials constructed from our potential with the aid of the "quasiperiodic group". It
follows also from the topological picture that all the topologically regular trajectories are absolutely stable under the action of the "quasiperiodic group" for the generic $V_{B}^{\text{eff}}(r)$ and can just "crawl" in the plane for the continuous action of such transformations.

We take the first interference picture $((\eta_1, a_1))$ and shift continuously the interference fringes in the direction of $\text{grad} X(r)$ (orthogonal to $\eta_1$) to the distance $a_1$ keeping two other pictures unchanged. Easy to see that we will have at the end the same potentials $V(x, y)$ and $V_{B}^{\text{eff}}(x, y)$ due to the periodicity of the first interference picture with period $a_1$. Let us fix now some energy level $c \in (\epsilon_1(B), \epsilon_2(B))$ and look at the evolution of non-singular open trajectories (for $V_{B}^{\text{eff}}(x, y)$) while making our transformation. We know that we should have the parallel open trajectories in the plane at every time and the initial picture should coincide with the final according to the construction. The form of trajectories can change during the process but their mean direction will be the same according to Theorem 4 ("Topological resonance").

We can claim then that every open trajectory will be "shifted" to another open trajectory of the same picture by our continuous transformation. It's not difficult to prove that all the trajectories will then be shifted by the same number of positions $n_1$ (positive or negative) which depends on the potential $V_{B}^{\text{eff}}(x, y)$ (Fig. 13).

The number $n_1$ is always even since all the trajectories appear by pairs with the opposite drift directions.

Let us now do the same with the second and the third sets of the interference fringes and get an integer triple $(n_1, n_2, n_3)$ which is a topological characteristic of potential $V_{B}^{\text{eff}}(x, y)$ (the "positive" direction of the numeration of trajectories should be the same for all these transformations).

The triple $(n_1, n_2, n_3)$ (defined up to the common sign) can be represented as:

$$(n_1, n_2, n_3) = M (m_1, m_2, m_3)$$

where $M \in \mathbb{Z}$ and $(m_1, m_2, m_3)$ is the indivisible integer triple.

The numbers $(m_1, m_2, m_3)$ play now the role of "Topological numbers" for this situation. Let us say that for direct experimental observation of these numbers the connection between these numbers and the mean direction of the "Topologically regular" trajectories can play important role. Let us describe here this connection:
Let us draw three straight lines $q_1, q_2, q_3$ with the directions $\eta_1, \eta_2, \eta_3$ (Fig. 12) and choose the "positive" and "negative" half-planes for every line $q_i$ on the plane. Let us consider now three linear functions $X(r), Y(r), Z(r)$ on the plane which are the distances from the point $r$ to the lines $q_1, q_2, q_3$ with the signs "+" or "−" depending on the half-plane for the corresponding line $q_i$ (Fig. 14). Let us choose here the signs "+" or "−" such that the gradients of $X(r), Y(r), Z(r)$ coincide with directions of shifts of the corresponding interference pictures in the definition of $(m_1, m_2, m_3)$.

**Theorem 6** ([12])

Consider the functions

$$X'(r) = X(r)/a_1, \quad Y'(r) = Y(r)/a_2, \quad Z'(r) = Z(r)/a_3$$

in $\mathbb{R}^2$. The mean direction of the regular open trajectories is given by the linear equation:

$$m_1X'(x, y) + m_2Y'(x, y) + m_3Z'(x, y) = 0 \quad (5)$$

where $(m_1, m_2, m_3)$ is the indivisible integer triple introduced above.
Let us say now about the situation with 4 independent sets of interference fringes in the plane (see also [42]). In general we get here the quasiperiodic potentials $V(r), V_{eff}^B(r)$ with 4 quasiperiods. The situation in this case is more complicated than in the case $N = 3$ and no general classification of open trajectories exists at the time. At the moment just the theorem analogous to Zorich result can be formulated in this situation (S.P. Novikov, [38]). According to Novikov theorem we can claim just that the ”small perturbations” of purely periodic potentials having 4 quasiperiods have the ”topologically regular” level curves like in the previous case.

Let us say that the purely periodic potentials $V(r)$ give the everywhere dense set in the space of parameters $\eta_1, \eta_2, \eta_3, \eta_4, a_1, a_2, a_3, a_4$ and can be found in any small open region of this space. Novikov theorem claims then that every potential of this kind can be surrounded by the ”small open ball” in the space of parameters $\eta_1, \eta_2, \eta_3, \eta_4, a_1, a_2, a_3, a_4$ where the open level curves will always demonstrate the ”topologically regular” behavior. The set of potentials thus obtained has the finite measure among all potentials and the ”topologically regular” open trajectories can be found with finite probability also in this case. However, we don’t claim here that the chaotic behavior has measure zero for 4 quasiperiods and moreover we expect the nonzero probability also for the chaotic trajectories in this more complicated
The topologically regular cases demonstrate here the same "regularity properties" as in the previous case including the "Topological numbers". Thus, we can introduce in the same way the action of the quasiperiodic group on the space of potentials with 4 quasiperiods and define in the same way the 4-tuples \((m_1, m_2, m_3, m_4)\) of integer numbers characterizing the topologically regular cases in this situation.

Also the analogous theorem about mean directions of the regular trajectories can be formulated in this case. Namely, if we introduce the functions \(X(r), Y(r), Z(r), W(r)\) in the same way as for the case of 3 quasiperiods (above) and the corresponding functions

\[ X'(r) = X(r)/a_1, \ldots, W'(r) = W(r)/a_4 \]

we can write the equation for the mean direction of open trajectories on the plane in the form:

\[ m_1X'(r) + m_2Y'(r) + m_3Z'(r) + m_4W'(r) = 0 \]

The numbers \((m_1, m_2, m_3, m_4)\) are stable with respect to the small variations of \(\eta_1, \eta_2, \eta_3, \eta_4, a_1, a_2, a_3, a_4\) (and the intensities of the interference pictures \(I_1, I_2, I_3, I_4\) and correspond again to some "stability zone' in this space of parameters.

Let us say now some words about the limit of Novikov problem for the large values of \(N\). Namely, the following problem can be formulated:

Give a description of global geometry of the open level curves of quasiperiodic function \(V(r)\) in the limit of large numbers of quasiperiods.

We can claim that the open level curves should exist here also in the connected energy interval \([\epsilon_1, \epsilon_2]\) on the energy scale which can degenerate just to one point \(\epsilon_0\). We expect that the "topologically regular" open trajectories can exist also in this case. However the probability of "chaotic behavior" should increase for the cases of large \(N\) which is closer now to random potential situation. The corresponding behavior can be considered then as the "percolation problem" in special model of random potentials given by quasiperiodic approximations. Certainly, this model can be quite different from the others. Nevertheless, we expect the similar behavior of the

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\(^5\)The proof given in [26] for the case of 3 quasiperiod works actually for any \(N\).
chaotic trajectories for rather big \( N \) also in this rather special model. This area, however, is still under investigation by now.

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