Gauge Symmetry Enhancement and $N = 2$ Supersymmetric Quantum Black Holes in Heterotic String Vacua

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Abstract

$N = 2$ supersymmetric quantum black holes in the heterotic $S$-$T$-$U$ model are presented. In particular three classes of axion-free quantum black holes with half the $N = 2$, $D = 4$ supersymmetries unbroken are considered. First, these quantum black holes are investigated at generic points in moduli space. Then “linearized” non-abelian black holes are investigated representing a subset of non-abelian black hole solutions at critical points of perturbative gauge symmetry enhancement in moduli space. It is shown that the entropy of “linearized” non-abelian black holes can be obtained, starting at non-critical points in moduli space, by continuous variation of the moduli and a proper identification of the non-abelian charges.

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Four-dimensional string models including non-perturbative excitations provide a possibly consistent description of all interactions. At low-energies it is convenient to describe these models using an effective supergravity action for the light string modes where heavy string modes have been integrated out. At the level of the effective supergravity action, neglecting non-perturbative effects, the vacuum expectation value of the so-called moduli, i.e. massless scalar fields with flat potentials, parametrize different string vacua. The corresponding moduli spaces of four dimensional effective string models have a very rich structure. First, they give rise to duality symmetries. Second, there exist certain critical points/lines in moduli space giving rise, for instance, to the stringy version of the Higgs effect \[1,2\]. At these critical points in moduli space a finite number of additional massless states may appear in the string spectrum giving rise to gauge symmetry enhancement. In this context the role of the Higgs field is taken by the moduli. The corresponding one-loop running coupling constant encounters a logarithmic singularity parametrized by the Higgs field \[3\]. This logarithmic singularity takes the threshold effects of massive modes, becoming massless at the critical points, into account \[2–6\].

In the context of string theory it has been shown in \[7\] that four-dimensional non-rotating black hole solutions in the BPS limit depend classically only on the bare quantized charges on the horizon. Thus, the black hole solutions in the BPS limit are independent of the values of the moduli at spatial infinity. In \[8\] it has been shown how one can understand this result from a supersymmetric point of view: On the horizon the central charge of the extended supersymmetry algebra acquires a minimal value and thus the extremization of the central charge provides the specific moduli values on the horizon \[8,9\].

Although the BPS limit of black hole solutions in four dimensions with \(N \geq 4\) is by now well understood \[10\], new features of black hole physics arise in four-dimensional \(N = 2\) string theory. In particular there exists a large number of different \(N = 2\) string vacua so that the extreme black hole solutions depend on the specific details of the particular \(N = 2\) string model.

If one considers, for example, four-dimensional \(N = 2\) heterotic string compactifications on \(K3 \times T_2\) with \(N_V + 1\) vector multiplets (including the graviphoton), the classical pre-potential is completely universal and corresponds to a scalar non-linear \(\sigma\)-model based on
the coset space $\frac{SU(1,1)}{U(1)} \otimes \frac{SO(2,N_V-1)}{SO(2) \times SO(N_V-1)}$. However, since in heterotic $N = 2$ string compactifications the dilaton can be described by a vector multiplet, the heterotic prepotential receives perturbative quantum corrections at the one-loop level \[4,5\]; in addition there are non-perturbative contributions.

In \[11,12\] it has been shown that string loops affect the entropy of $N = 2$ heterotic quantum black holes only through a perturbative modification of the string coupling. Thus, near critical points of perturbative gauge symmetry enhancement in moduli space the entropy of $N = 2$ supersymmetric quantum black holes receives logarithmic quantum corrections \[13,14\]. In this paper we will also be concerned with black holes at these critical points themselves. More results on black hole solutions in $N = 2$ supersymmetric vacua are given e.g. in \[15,16\].

The paper is organized as follows: In section two we will briefly introduce $N = 2$ supergravity, special geometry and the Bekenstein-Hawking entropy in terms of the $N = 2$ prepotential. In section three we introduce the heterotic $S-T-U$ model. Then, in section four, we discuss abelian axion-free quantum black holes in the $S-T-U$ model. In section five we investigate non-abelian quantum black holes associated with critical points of perturbative gauge symmetry enhancement in moduli space. Finally we summarize the results and in the appendix we review, for the sake of completeness, the classical gauge symmetry enhancement in the $S-T-U$ model.

II. $N = 2$ SUPERGRAVITY AND SPECIAL GEOMETRY

The vector couplings of local $N = 2$ supersymmetric Yang-Mills theory are encoded in the holomorphic function $F(X)$, where the $X^I$ ($I = 0 \ldots N_V$) denote the complex scalar fields of the vector supermultiplets. Here $N_V$ counts the number of physical scalars, and $I$ counts the number of physical vectors. The special geometry \[17\] of this theory can be defined in terms of a symplectic section $V$. This is a $(2N_V + 2)$-dimensional complex symplectic vector given by $V^T = (X^I, F_I)$ with periods $F_I = \partial F/\partial X^I$. The $N_V$ physical scalars parametrize a $N_V$ dimensional complex hypersurface, defined by the condition that the section satisfies a symplectic constraint:

$$i \left( \bar{X}^I F_I - F_I X^I \right) = 1 \quad (\text{II.1})$$
This hypersurface can be described in terms of a complex projective space with coordinates $z^A (A = 1, \ldots, N_V)$, if the complex coordinates are proportional to some holomorphic sections $X^I(z)$ of the complex projective space: $X^I = e^{K(z, \bar{z})/2} X^I(z)$ with

$$K(z, \bar{z}) = -\log \left( i\bar{x}^I(\bar{z}) F_I(X^I(z)) - iX^I(z) \bar{F}_I(\bar{x}^I(z)) \right).$$ (II.2)

Moreover one can introduce special coordinates $X^0(z) = 1$ and $X^A(z) = z^A$. In this special coordinates the Kähler potential is

$$K(z, \bar{z}) = -\log \left( 2(F + \bar{F}) - (z^A - \bar{z}^A)(F_A + \bar{F}_A) \right)$$ (II.3)

with $F(z) = i(X^0)^{-2} F(X)$.

The mass of any physical state is given by the central charge in the BPS limit:

$$M_{BPS}^2 = |Z|^2 = e^{K(z, \bar{z})} |q_I X^I - p^I F_I|^2$$ (II.4)

In the BPS limit the entropy of a black hole is also given by the central charge, if the central charge has been extremized with respect to the moduli ($\partial_A |Z| = 0$) [8]. Thus, the moduli take their fixed values at the horizon of the BPS saturated black hole. This extremization problem is equivalent to the algebraic solution of the following $2N_V + 2$ “stabilization equations”

$$\bar{Z}V - Z\bar{V} = iQ$$ (II.5)

with the symplectic magnetic/electric charge vector $Q^T = (p^I, q_I)$. To solve these equations it is convenient to go to the so-called Y-basis [11]. The corresponding Y-coordinates are defined as $Y^I = \bar{Z} X^I$. Hence the $2N_V + 2$ stabilization equations in the Y-basis are given by

$$Y^I - \bar{Y}^I = ip^I, \quad F_I - \bar{F}_I = iq_I$$ (II.6)

and the Bekenstein-Hawking entropy in the Y-basis reads

$$S_{BH} = i\pi \left( \bar{Y}^I F_I(Y^I) - Y^I \bar{F}_I(\bar{Y}^I) \right)_{\text{fix}} = \pi |Z|^2_{\text{fix}}.$$ (II.7)

Note that the special projective coordinates are invariant under this change of basis.
III. THE PERTURBATIVE HETEROTIC S-T-U MODEL

Compactifying the $D = 10$ effective heterotic string theory on $K3 \times T_2$ one can construct the $D = 4$, $N = 2$ $S$-$T$-$U$ model \cite{22,24}. This model has 244 hypermultiplets, which we will ignore in the following. Moreover it contains three vector moduli $S$, $T$ and $U$, where $S$ denotes the heterotic dilaton and $T, U$ the $T_2$-moduli. This model exhibits a non-perturbative symmetry (exchange symmetry) which exchanges the dilaton $S$ with one of the two vector moduli $T$ or $U$ \cite{22,24}. Moreover the model is invariant under mirror symmetry which exchanges $T$ and $U$.

In special projective coordinates the prepotential reads

$$F(S,T,U) = -STU + h(T,U) \quad (\text{III.1})$$

with

$$S = -iz^1, \quad T = -iz^2, \quad U = -iz^3. \quad (\text{III.2})$$

Here $h(T,U)$ denotes the perturbative quantum corrections. Then the prepotential in the $Y$-basis is $F(Y) = -i(Y^0)^2 F(S,T,U)$ with periods

$$F_0 = iY^0 [-STU - 2h + Th_T + Uh_U],$$
$$F_1 = Y^0 T U,$$
$$F_2 = Y^0 [SU - h_T],$$
$$F_3 = Y^0 [ST - h_U]. \quad (\text{III.3})$$

The Kähler potential in special coordinates reads

$$K(S,T,U) = -\log(S + \bar{S} + V_{GS}) - \log(T + \bar{T}) - \log(U + \bar{U}) \quad (\text{III.4})$$

with

$$V_{GS}(T,U) = \frac{2(h + \bar{h}) - (T + \bar{T})(h_T + \bar{h}_T) - (U + \bar{U})(h_U + \bar{h}_U)}{(T + \bar{T})(U + \bar{U})}, \quad (\text{III.5})$$

Here $V_{GS}(T,U)$ denotes the Green-Schwarz term \cite{1}, which yields the true perturbative target-space duality invariant string coupling

$$\frac{8\pi}{g^2_{\text{pert}}} = S + \bar{S} + V_{GS}(T,U). \quad (\text{III.6})$$
The corresponding semiclassical quantum corrections have been determined in [21,23] and read in the fundamental Weyl chamber $\text{Re } T > \text{Re } U$

$$h(T, U) = -\frac{1}{3} U^3 - c - \frac{1}{4\pi^3} \sum_{k,l \geq 0} c_n(4kl) \, \text{Li}_3 \left( e^{-2\pi(kT+\rho U)} \right)$$

(III.7)

with $c = \frac{\chi \zeta(3)}{2(2\pi)^3}$ where $\chi = -480$ denotes the Euler number. In the semiclassical limit the exchange symmetry is broken, but the mirror symmetry is still valid. The singularities of the semiclassical prepotential at $T = U \neq 1$, $T = U = 1$ and $T = U = e^{i\pi/6}$ reflect the perturbative gauge symmetry enhancement of $U(1)^2$ to $SU(2) \times U(1)$, $SU(2)^2$ and $SU(3)$ respectively (see appendix). Near these critical points in moduli space the quantum corrections take the specific form [3,4]

$$h(T, U) = \frac{1}{\pi} (T - U)^2 \log(T - U) + \Delta(T, U)$$

$$h(T, U) = \frac{1}{\pi} (T - 1) \log(T - 1)^2 + \Delta'(T, U)$$

$$h(T, U) = \frac{1}{\pi} (T - \rho) \log(T - \rho)^3 + \Delta''(T, U).$$

(III.8)

Here, the functions $\Delta(T, U)$ are finite and single valued at the critical points. In the large moduli limit $S, T, U \rightarrow \infty$ ($\text{Re } S > \text{Re } T > \text{Re } U$) the semiclassical quantum corrections reduce to

$$h(T, U) = -\frac{1}{3} U^3 - c.$$ 

(III.9)

In this particular limit in moduli space the exchange symmetry is restored perturbatively but mirror symmetry is broken. Moreover, the quantum corrected $S-T-U$ model is dual to the type II model described by the elliptically fibered CY space $WP_{1,1,2,8,12}(24)$. In the large moduli limit this duality becomes manifest if one takes for the three Kähler class moduli [24]

$$t_1 = U, \quad t_2 = S - T, \quad t_3 = T - U.$$ 

(III.10)

Ignoring constant contributions the dual prepotential on the type II side reads ($t_2 > t_3 > 0$)

$$\mathcal{F}^0(t_1, t_2, t_3) = -\frac{4}{3} t_1^3 - t_1^2(t_2 + 2t_3) - t_1 t_2 t_3 - t_1 t_3^2.$$ 

(III.11)

In the classical limit ($h = 0$) the exchange and mirror symmetry are both restored and combine together to the classical triality symmetry [18,19].
IV. ABELIAN QUANTUM BLACK HOLES IN THE S-T-U MODEL

In this section we will consider axion-free quantum black holes in the $S$-$T$-$U$ model, only. This restriction implies

$$z^A(2Y^0 - ip^0) = ip^A. \hspace{1cm} (IV.1)$$

Moreover we will work in a definite Weyl chamber. Thus, the perturbative quantum corrections $h(T, U)$ are real. This implies that our results are not manifest target-space duality invariant $[13, 11, 14]$. On the other hand, all the results we will obtain can be given in a manifest target-space duality invariant form following $[14]$.

A. Axion-free Quantum Black Holes in the S-T-U Model

In particular we will discuss three different dyonic axion-free black hole configurations (i)-(iii):

1. First class of dyonic axion-free quantum black holes

In this case we take $Y^0 + \bar{Y}^0 = \lambda \neq 0$. Then the first set of stabilization equations yields

$$S = \frac{p^1}{\lambda}, \hspace{1cm} T = \frac{p^2}{\lambda}, \hspace{1cm} U = \frac{p^3}{\lambda}, \hspace{1cm} (IV.2)$$

and from the second set we obtain

$$h = \frac{1}{2p^0\lambda}(p^1q_1 - p^2q_2 - p^3q_3 - p^0q_0), \hspace{1cm} \lambda = \pm \sqrt{\frac{p^0p^2p^3}{q_1}},$$

$$h_T = \frac{p^1q_1}{p^0p^2} - \frac{q_2}{p^0}, \hspace{1cm} h_U = \frac{p^1q_1}{p^0p^3} - \frac{q_3}{p^0}. \hspace{1cm} (IV.3)$$

In the classical limit this yields $STU = -q_0/\lambda$ and the classical charge constraints

$$p^1q_1 = p^2q_2 = p^3q_3 = -p^0q_0. \hspace{1cm} (IV.4)$$

The classical entropy is given by

$$S_{BH}^{class} = 2\pi \frac{\lambda^2 + (p^0)^2}{p^0\lambda} p^1q_1. \hspace{1cm} (IV.5)$$
The restrictions from the stabilization equations are such that the quantum corrections of the entropy are of a restricted form, i.e. the charges have to obey constraints. In the case of perturbative quantum corrections the charges obey the general semiclassical constraint

\[ p^1 q_1 = p^2 q_2 + p^0 p^2 h_T = p^3 q_3 + p^0 p^3 h_U = p^2 q_2 + p^3 q_3 + p^0 q_0 + 2p^0 \lambda h \]  

(IV.6)

and the entropy reads

\[ S_{BH} = \frac{\pi}{2} \frac{\lambda^2 + (p^0)^2}{p^0 \lambda} \left( p^1 q_1 + p^2 q_2 + p^3 q_3 - p^0 q_0 \right) \]  

(IV.7)

This entropy includes all perturbative quantum corrections. The parameter \( \lambda \) can be determined because the perturbative quantum corrections are independent of the dilaton [4,5]. On the other hand, including non-perturbative quantum corrections one finds for this black hole configuration the same entropy (IV.7), but the parameter \( \lambda \) remains an undetermined parameter.

2. Second class of dyonic axion-free quantum black holes

For this configuration we take \( Y^0 + \bar{Y}^0 = 0 \). Then we obtain from the first set of stabilization equations \( Y^0 = \frac{i}{2} p^0 \) and \( p^A = 0 \). The second set yields

\[ q_0 = 0, \quad TU = \frac{q_1}{p^0}, \quad h_T = SU - \frac{q_2}{p^0}, \quad h_U = ST - \frac{q_3}{p^0}. \]  

(IV.8)

In the classical limit one finds for the fixed values of the moduli on the horizon

\[ S = \sqrt{\frac{q_2 q_3}{q_1 p^0}}, \quad T = \sqrt{\frac{q_1 q_3}{q_2 p^0}}, \quad U = \sqrt{\frac{q_1 q_2}{q_3 p^0}} \]  

(IV.9)

and the classical entropy reads

\[ S_{BH}^{\text{class}} = 2\pi \sqrt{p^0 q_1 q_2 q_3}. \]  

(IV.10)

In the case of quantum corrections, on the other hand, the entropy is in general

\[ S_{BH} = \pi p^0 \left( q_2 T + q_3 U + p^0 h \right)_{\text{fix}}. \]  

(IV.11)

Here, \( T \) and \( U \) take their fixed values on the horizon. To find these fixed values without further restrictions on the charges or limits in moduli space seems to be difficult at this point.
3. Third class of dyonic axion-free quantum black holes

For this configuration we take $Y^0 - \overline{Y}^0 = 0$. Thus, the first set of stabilization equations yields $Y^A = \frac{i}{2}p^A$ and hence

$$S = \frac{p^1}{2Y^0}, \quad T = \frac{p^2}{2Y^0}, \quad U = \frac{p^3}{2Y^0}. \quad (IV.12)$$

From the second set one obtains

$$q_A = 0, \quad q_0 = 2Y^0[-STU - 2h + Th_T + Uh_U]. \quad (IV.13)$$

In the classical limit, with $q_0 < 0$, the fixed values of the moduli on the horizon are

$$Y^0 = \frac{1}{2} \sqrt{\frac{p^1p^2p^3}{|q_0|}}, \quad S = \sqrt{\frac{p^1|q_0|}{p^2p^3}}, \quad T = \sqrt{\frac{p^2|q_0|}{p^1p^3}}, \quad U = \sqrt{\frac{p^3|q_0|}{p^1p^2}}. \quad (IV.14)$$

and the corresponding classical entropy reads

$$S^{\text{class}}_{BH} = 2\pi \sqrt{|q_0|p^1p^2p^3}. \quad (IV.15)$$

Moreover, in the case of quantum corrections the entropy is in general

$$S_{BH} = 4\pi (Y^0)^2 (2STU + h - h_T T - h_U U)_{\text{fix}} \quad (IV.16)$$

with

$$Y^0 = -\frac{q_0}{2} (STU + 2h - h_T T - h_U U)^{-1}_{\text{fix}}. \quad (IV.17)$$

Near the critical point of perturbative gauge symmetry enhancement $T = U \neq 1$ one finds that $Y^0$ is single valued

$$Y^0 = -\frac{q_0}{2} \left( STU - \frac{1}{\pi} (T - U)^2 + 2\Delta - \Delta_T T - \Delta_U U \right)^{-1}_{\text{fix}}. \quad (IV.18)$$

The corresponding quantum corrected entropy reads

$$S_{BH} = 4\pi (Y^0)^2 \left( 2STU - \frac{1}{\pi} (T - U)^2 [\log(T - U) + 1] + \Delta - \Delta_T T - \Delta_U U \right)_{\text{fix}}. \quad (IV.19)$$

Thus, the entropy of the quantum black hole receives logarithmic and polynomial quantum corrections due to one-loop effects in the effective action of the $S$-$T$-$U$ model.
Moreover, at the microscopic level the logarithmic quantum corrections are a possible origin of subleading terms in the degeneracy of an underlying (unknown) quantum theory \[13\]. If we consider, for example, the dyonic case at hand and omit polynomial quantum corrections encoded in $\Delta(T, U)$ we find

$$Y^0 = \frac{1}{4\pi} \frac{(p^2 - p^3)^2}{q_0} + \sqrt{\left(\frac{1}{4\pi} \frac{(p^2 - p^3)^2}{q_0}\right)^2 - \frac{p^1 p^2 p^3}{4q_0}}. \quad (IV.20)$$

Now we consider the large $q_0$ limit ($q_0 < 0, p^A > 0$) and find that $Y^0$ is given by its classical value. Thus, the quantum corrected entropy in this limit reads

$$S_{BH} = 2\pi \sqrt{|q_0|^2} - \frac{1}{2} \log \left(\frac{(p^2 - p^3)^2 |q_0|}{p^1 p^2 p^3}\right) + \cdots \quad (IV.21)$$

Here the dots stand for polynomial quantum corrections encoded in $\Delta(T, U)$. Eq. (IV.21) is the perturbative corrected black hole entropy in a special region of moduli space and represents the entropy of an abelian black hole. However, if we consider the limit $T \to U$ ($p^2 \to p^3$) in moduli space, the $U(1)$ gauge symmetry becomes enhanced to $SU(2)$ and the effective action changes. This is a pure stringy effect and reflects the fact that at the line $T = U$ additional states become massless (see appendix) giving rise to a perturbative gauge symmetry enhancement. The quantum corrected entropy approaches this line in moduli space smooth, because the perturbative quantum corrections already take this light states, becoming massless at $T = U$, into account \[3\]. Thus, approaching the line of perturbative gauge symmetry enhancement, the logarithmic quantum corrections in (IV.21) vanish. However, on the line of perturbative gauge symmetry enhancement (IV.21) is, first of all, not the correct entropy, since the effective action that has been used to calculate the entropy, is not correct anymore. Instead the effective action has a non-abelian $SU(2)$ sector on the line $T = U$. Of course, if the black hole solution breaks this non-abelian gauge group explicitly down to $U(1)$, we can take the limit $T = U$ explicitly.

**B. Dual Quantum Black Hole Pairs in the S-T-U Model**

In this subsection we will consider the large moduli limit of the $S-T-U$ model. In this limit the $S-T-U$ model is dual to certain CY compactifications of type II string models. Here we consider, as an example, the CY space described by $WP_{1,1,2,8,12}(24)$. An analogous discussion for other CY spaces using, for instance, the results of \[21\] is straightforward.
Classical $N = 2$ supersymmetric black holes in the context of Calabi-Yau compactifications have been already extensively discussed in [18,11,20,23]. Nevertheless it is instructive, at this point of our discussion, to consider these heterotic quantum black holes explicitly.

Let us recall that in the large moduli limit the mirror symmetry is broken, the exchange symmetry, on the other hand, is valid. Thus, the corresponding black hole solutions are in the type II and the heterotic description identical and have, in addition, a *perturbative* exchange symmetry. In the following we will consider again the three different dyonic classes (i) - (iii) in the large moduli limit:

1. **First class of dual quantum black hole pairs**

In this particular configuration the fields on the heterotic side are already determined in terms of the charges. The charges obey the constraints

$$p^1 q_1 = p^2 q_2 = p^3 q_3 - \left(\frac{p^3}{p^2}\right)^2 = p^2 q_2 + p^3 q_3 + p^0 q_0 - \frac{2}{3} \left(\frac{p^3}{p^2}\right)^2 q_1$$

and the quantum corrected entropy is (IV.7) obeying (IV.22). In addition the exchange symmetry exchanges $p^1 \leftrightarrow p^2$. The corresponding quantum corrected entropy reads

$$S_{BH} = \pi p^0 q_1 q_2 q_3 \sqrt{\frac{2}{\beta q_3 p^0}} + \pi \sqrt{\frac{p^0 \beta q_3^2}{2}} + \pi (p^0)^2 h(T, U)_{fix}$$

If we consider the limit of small electric charges ($q_A \ll 1$) and large magnetic charge $p^0 \gg 1$, then the leading contribution to the entropy has its origin in the constant part of the perturbative quantum corrections. In this particular limit we find
Thus, we recover the result of [14] that in a particular limit in moduli space the leading contribution to the entropy is proportional to $\zeta(3)$. Note that in this particular context the leading contribution is finite.

3. Third class of dual quantum black hole pairs

For this configuration we obtain, first of all,

$$Y^0 = \frac{1}{2} \sqrt{\frac{p_1^2 p_3 + (p_3)^3}{3|q_0|}}.$$  \hspace{1cm} (IV.26)

Thus, the fixed values of the heterotic moduli are given by

$$S = \sqrt{\frac{(p_1)^2 |q_0|}{p_1^2 p_2^3 + (p_3)^3/3}}, \quad T = \sqrt{\frac{(p_2)^2 |q_0|}{p_1^2 p_2^3 + (p_3)^3/3}}, \quad U = \sqrt{\frac{p_3 |q_0|}{p_1^2 p_2^3 + (p_3)^3/3}}.$$  \hspace{1cm} (IV.27)

Again the exchange symmetry exchanges $p_1 \leftrightarrow p_2$ and the corresponding quantum corrected entropy reads

$$S_{BH} = 2\pi \sqrt{|q_0| p_1^2 p_2^3 + \frac{|q_0| (p_3)^3}{3}}.$$  \hspace{1cm} (IV.28)

V. NON-ABELIAN QUANTUM BLACK HOLES IN THE S-T-U MODEL

In the previous section we have considered quantum black holes at generic points in moduli space, i.e. the effective four-dimensional supergravity action had only abelian gauge groups. In this section we will discuss critical points in moduli space, where the effective action also has a non-abelian gauge sector. Thus, the corresponding black hole solutions can be non-abelian.

There are two classes of non-abelian black holes: i) linearized black hole solutions [27] and ii) non-linear black hole solutions [28].

The linearized non-abelian black hole solutions i) are related to abelian black hole solutions by construction. In the following we will discuss these linearized non-abelian black holes.
hole solutions, only. Clearly these solutions represent only a certain subset of solutions of the equations of motion. Moreover, the linearized non-abelian black hole solutions of [27] have been studied in a pure Yang-Mills context. In order to study linearized non-abelian black holes in string vacua, one must reformulate the theorem given in [27].

A. Linearized solution theorem in string theory

The “linearized solution theorem” of [27] in the context of string theory can be formulated as follows

**Theorem:** Let $\mathcal{G}$ be a $N$-parameter Lie-group with an invariant metric $\gamma_{ab}$ ($a, b = 1 \ldots N$). Then for every solution of the field-dependent (source-free) coupled Einstein-Maxwell equations there is a $(N - 1)$ parameter set of solutions of the field-dependent coupled Einstein-massless-Yang-Mills equations for the gauge group $\mathcal{G}$.

**Proof:** The field-dependent coupled Einstein-Maxwell Lagrangian is in general of the form

$$e^{-1} \mathcal{L}_{EM} = R + f(\phi)F^2 + g(\phi)F\tilde{F}$$

in the Einstein frame. Here $f(\phi), g(\phi)$ are arbitrary field-dependent functions with $\phi_i$ ($i = 1, \ldots$), At tree level in heterotic string vacua, for instance, $f(\phi)$ is given by the dilaton and $g(\phi)$ by the model-independent axion. The corresponding solution of the field-dependent coupled Einstein-Maxwell equations of motion of (V.1) are given by $g_{\mu\nu}^0, A_{\mu}^0, \phi_i^0$. Then the solution of the field-dependent coupled Einstein-Yang-Mills equation for gauge group $\mathcal{G}$ with metric $\gamma_{ab}$ are given by $g_{\mu\nu}, A_{\mu}^a, \phi_i$ with

$$g_{\mu\nu} \equiv g_{\mu\nu}^0, \quad A_{\mu}^a \equiv \beta^a A_{\mu}^0, \quad \phi_i \equiv \phi_i^0.$$  (V.2)

The $N$-parameters $\beta^a$ are subject to the constraint

$$\langle \beta | \beta \rangle = 1.$$  (V.3)

Here, we have defined a scalar product $\langle X | X \rangle = X^a \gamma_{ab} X^b$. The Lagrangian of the field-dependent coupled Einstein-Yang-Mills model is given by

$$e^{-1} \mathcal{L}_{EYM} = R + f(\phi)\text{tr} F^2 + g(\phi)\text{tr} F\tilde{F}$$

(V.4)
with \( \text{tr}F^2 = F_{\mu\nu}^a \gamma_{ab} F^{\mu\nu b} \) and an analogous expression for \( \text{tr}\tilde{F}^2 \). Note that the field-dependent coupling functions \( f(\phi), g(\phi) \) have to be the same as in the pure abelian case. Using now (V.2) one obtains \( F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a = \beta^a F_{\mu\nu}^0 \) and therefore \( \mathcal{L}_{EYM} = \mathcal{L}_{EM} \). Since the same action has the same solutions of the equations of motion, the proof is complete.

Note that the spacetime symmetry of the abelian and the non-abelian solution are the same. The non-abelian solution, however, depends on \( N-1 \) independent parameters. These parameters are related to the non-abelian charges: The electric and magnetic charges of the non-abelian solution, given by \( F_{0r}^a \sim \frac{Q^a}{r} \) and \( \tilde{F}_{0r}^a \sim \frac{P^a}{r} \) for large \( r \), are

\[
Q^a = \beta^a Q^0, \quad P^a = \beta^a P^0. \tag{V.5}
\]

Here \((Q^0, P^0)\) denote the electric and magnetic charge of the corresponding abelian solution respectively. Using (V.3) we find that the charges have to obey the charge constraints

\[
\langle Q | Q \rangle = (Q^0)^2, \quad \langle P | P \rangle = (P^0)^2 \tag{V.6}
\]

and Dirac’s quantization condition

\[
\langle P | Q \rangle = Q^0 P^0 = 2\pi n. \tag{V.7}
\]

The difference between the abelian and the non-abelian solution is that in the non-abelian case the solution depends on \( N-1 \) independent non-abelian charges. The particular dependence is constrained by (V.6) and (V.7). These linearized non-abelian solutions contain the case of a pure abelian solution, which is reached if \( \beta^1 = 1 \) and \( \beta^n = 0 \) with \( n = 2, \ldots, N-1 \).

**B. Application to quantum black holes**

From the “linearized solution theorem” follows: The entropy of a given abelian and non-abelian black hole solution is identical, if they satisfy the “linearized solution conditions” (V.2) and (V.3). Thus, for the case at hand, if we approach lines/points of gauge symmetry enhancement in moduli space, the entropy, derived in the abelian effective field theory, is still

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Note that the generators \( T_a \in \mathcal{G} \) obey \([T_a, T_b] = f_{abc} T_c\) with antisymmetric structure constants \( f_{abc} \).
VI SUMMARY AND CONCLUSION

valid at the critical points themselves, if the non-abelian black hole solution satisfies (V.2) and (V.3). One must only identify the non-abelian charges in terms of the abelian ones. To be more concrete, let us consider the classical dyonic solutions (i)-(iii) at the critical line $T = U$. For the first class of dyonic axion-free black hole solutions we find $p^2 = p^3$ and $q_2 = q_3$. Thus, the entropy of the non-abelian black hole solution reads

$$S^{\text{class}}_{BH} = 2\pi \frac{\lambda^2 + (p^0)^2}{p^0 \lambda} p^1 q_1, \quad \lambda = \pm \sqrt{\frac{q^0}{q_1} \langle P|P \rangle}. \quad (V.8)$$

Here $P^a (a = 1, 2, 3)$ denote the non-abelian magnetic charges corresponding to the $SU(2)$ group. Analogous one finds for the second class (ii) $(q_2)^2 = (q_3)^2 = \langle Q|Q \rangle$ and for the third class (iii) $(p)^2 = (p^3)^2 = \langle P|P \rangle$. Finally, since we can approach the lines/points of perturbative gauge symmetry enhancement explicitly, if the corresponding non-abelian black hole solution satisfies the “linearized solution theorem”, we find that quantum corrections yield polynomial corrections to the entropy at these critical points, only. Note that these points of perturbative gauge symmetry enhancement are non-critical in the sense that no phase transitions as discussed in [29] occur. Moreover, the non-abelian black holes satisfying the “linearized solution theorem” do not really create a so-called “gauge charge hair” [27,28,30]. This can be seen already via the charge constraints (V.6) and (V.7). The linearized black hole solution does not depend on particular data with respect to the gauge group other than the charges.$^2$

VI. SUMMARY AND CONCLUSION

$N = 2$ supersymmetric quantum black holes in the heterotic $S$-$T$-$U$ model have been studied. First, three classes of axion-free quantum black holes with half the $N = 2$, $D = 4$ supersymmetries unbroken have been investigated. Remarkably, the entropy of one class of solutions is valid everywhere at generic points in moduli space, while the entropy of the other two classes depend on the specific perturbative quantum corrections in moduli space. However, these results were, first of all, only valid at generic points in moduli space, where the corresponding low-energy effective action contains only abelian gauge groups. In the

$^2$ Note that non-extreme black holes in string theory depend on the values of the moduli at infinity [31].
second part we considered “linearized” non-abelian black holes. It is shown that the entropy of linearized non-abelian black holes at critical points of perturbative gauge symmetry enhancement can be reached, starting at non-critical points, by continuous variation of the moduli and a proper identification of the non-abelian charges. These linearized non-abelian black hole solutions contain the pure abelian black hole solutions valid already at generic points in moduli space. On the other hand, the linearized non-abelian black hole solutions represent only a subset of solutions in the general non-abelian $S$-$T$-$U$ model. Thus, it would be very interesting to consider also non-linear non-abelian black hole solutions (see e.g. [28,32]) in the context of the $S$-$T$-$U$ model at critical points of perturbative gauge symmetry enhancement.

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VII. APPENDIX

The heterotic string on $K3 \times T_2$ has two moduli $T$ and $U$ corresponding to the torus $T_2$. These two scalars are members of two $U(1) N = 2$ vector multiplets at generic points in moduli space. In addition to the corresponding $U(1)_L \times U(1)_R$ symmetry of $T_2$ the model contains the heterotic dilaton and the graviphoton. Thus, at generic points in moduli space there is a $U(1)_L^2 \times U(1)_R^2$ abelian symmetry. At special lines/points in the classical $(T, U)$ moduli space additional gauge bosons become massless and the $U(1)_L^2$ becomes enlarged to a non-abelian gauge symmetry. First of all there are four inequivalent lines in the classical moduli space where two charged gauge bosons become massless [2,6]: Using the mass formula for $N = 2$ BPS states one finds

$$M_{BPS}^2 = e^{K(z, \bar{z})} |\mathcal{M}(S, T, U)|^2 \sim |\mathcal{M}(T, U)|^2 = |m_2 - im_1 U + in_1 T - n_2 T U|^2. \quad (VII.9)$$

Here, $\mathcal{M}$ denotes the holomorphic mass [26] and $m_{1,2} [n_{1,2}]$ the momentum [winding] quantum numbers in the 1, 2 direction of $T_2$. The four critical lines of perturbative gauge symmetry enhancement can be read off straightforward.

| critical lines | quantum numbers | gauge symmetry |
|----------------|-----------------|----------------|
| $T = U$        | $m_1 = n_1 = \pm 1, m_2 = n_2 = 0$ | $SU(2)_L \times U(1)_L$ |
| $T = 1/U$      | $m_1 = n_1 = 0, m_2 = n_2 = \pm 1$ | $SU(2)_L \times U(1)_L$ |
| $T = U + i$    | $m_1 = n_1 = m_2 = \pm 1, n_2 = 0$ | $SU(2)_L \times U(1)_L$ |
| $T = \frac{U}{U+1}$ | $m_1 = n_1 = -n_2 = \pm 1, m_2 = 0$ | $SU(2)_L \times U(1)_L$ |

If two [three] critical lines intersect with each other four [six] additional states, corresponding to gauge bosons, become massless at the intersection point. At this points the gauge group is enlarged to $SU(2)_L^2 [SU(3)_L].$

| critical points | quantum numbers | gauge symmetry |
|-----------------|-----------------|----------------|
| $T = U = 1$     | $m_1 = n_1 = \pm 1, m_2 = n_2 = \pm 1$ | $SU(2)_L \times SU(2)_L$ |
| $T = U = e^{i\pi/6}$ | $m_2 = n_2 = 0, m_1 = n_1 = \pm 1$ | $SU(3)_L$ |
|                 | $m_2 = n_2 = 1, m_1 = -1, n_1 = 0$ |             |
|                 | $m_2 = n_2 = 1, m_1 = 0, n_1 = 1$ |             |
|                 | $m_2 = n_2 = -1, m_1 = 0, n_1 = -1$ |             |
|                 | $m_2 = n_2 = -1, m_1 = 1, n_1 = 0$ |             |
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