Black hole entropy in loop quantum gravity: the role of internal symmetries

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Abstract. I will discuss here the role of the internal symmetry group in the computations of black hole entropy in loop quantum gravity according to the standard prescription given by Domagala and Lewandowski [1]. In particular I will show how it is possible to take into account the possible choice of either $SO(3)$ or $SU(2)$ as the internal symmetry groups of general relativity in Loop Quantum Gravity and how this choice changes the combinatorial problem of counting the black hole degrees of freedom.

The problem of understanding the microphysics behind black hole thermodynamics is a good testing ground for candidate quantum gravity theories. In this respect both string theories and LQG can claim a reasonable success because both of them can explain the Bekenstein-Hawking law. In the first case this is done for extremal or almost extremal black holes whereas the computations in LQG can deal with the more physical Schwarzschild or Kerr spacetimes.

The Bekenstein-Hawking area law is arguably the most firmly rooted result obtained to date concerning the semiclassical behavior of black holes. However this is not the only prediction that has been put forward for the behavior of black hole entropy. Another interesting idea is the quantization of entropy and area suggested in the late seventies by Bekenstein and Mukhanov [2]. In their proposal the area spectrum of black holes is an equally space one (only quantized areas are possible) and, consequently, the entropy is also effectively quantized.

A remarkable achievement of LQG has been the construction of well defined operators in a suitable Hilbert space that represent geometrical quantities such as areas and volumes (see the nice review [3] and references therein). The quantization of the areas is of particular interest for us in the study of black hole entropy because they are predicted to be quantized as suggested by Bekenstein and Mukhanov. There are however some problems with this picture. The most obvious one is due to the fact that the area spectrum in LQG is not equally spaced.

A surprising and unexpected way to reconcile these two alternates views (the equally spaced
spectrum of Bekenstein-Mukhanov and the LQG one) was suggested by the authors of [4]. In this paper they show that the actual computations for the entropy of small black holes performed by a direct brute force counting of the relevant microstates display an intriguing behavior: when the black hole entropy is plotted as a function of area the expected linear growth is modulated by an oscillating term that gives rise to a staircase structure for the entropy that becomes effectively quantized and jumps at equally spaced values for the area. This means that, at least for microscopic black holes, the area and the entropy are essentially quantized in the way suggested by Bekenstein and Mukhanov. This is a highly non-trivial result because of the non-evenness of the spectrum of the area operator in LQG. Subsequently a rather detailed description of this phenomenon has been obtained by using some techniques borrowed from number theory and combinatorics [5]. I will briefly discuss them here and, in particular, concentrate on how a proposal [6] to study and understand the effects of changing the internal gauge group of LQG from SU(2) to SO(3) can be incorporated in our framework.

The computation of the entropy of a black hole modeled as an isolated horizon was reduced to a concrete counting problem in [7]. This was subsequently simplified in [1] and reduced to a much simpler combinatorial problem\(^1\). This will be my starting point here so I will use the following prescription to obtain the black hole entropy [1].

**Definition 1** The entropy \( S_{\text{bh}}(a) \) of a quantum horizon of the classical area \( a \), according to Quantum Geometry and the Ashtekar-Baez-Corichi-Krasnov framework [7], is

\[
S(a) = \log n(a),
\]

where \( n(a) \) is 1 plus the number of all the finite, arbitrarily long, sequences \( \vec{m} = (m_1, \ldots, m_n) \) of non-zero half integers, such that the following equality and inequality are satisfied:

\[
\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a}{8\pi\gamma\ell_P^2}. \tag{1}
\]

Here \( \gamma \) is the Barbero-Immirzi parameter of Quantum Geometry and \( \ell_P \) the Planck length. The extra term 1 above comes from the trivial sequence.

The counting that must be performed to obtain the black hole entropy according to the previous prescription can be carried out in different ways. In fact, a proposal [8] that appeared simultaneously with [1] gives a solution that leads to the Bekenstein-Hawking area law. However, as pointed out in [9], this result is not useful to see the effective entropy quantization described in [4] (it does not easily lend itself to numerical computations). Also a full understanding of the asymptotic behavior of the entropy for large areas is difficult (though maybe not impossible) to get.

An alternative way of studying the entropy was proposed in [5, 10]. The main idea was to start from a precise characterization of the area spectrum in LQG, in particular the degeneracies of the different eigenvalues, and then use this to obtain a concrete algorithm to compute the black hole entropy. In fact it is not only possible to build this algorithm but also to “implement” it in the form of a black hole generating function [10] that can be used to check some of the results obtained to date on this issue and also as the starting point of an asymptotic analysis for large black hole areas.

In the Ashtekar-Baez-Corichi-Krasnov approach [7] to black hole entropy one works in a spacetime with an inner boundary that models the black hole horizon (technically it will become an isolated horizon [13]). The gravitational degrees of freedom can be classified either as bulk

\(^1\) Although there are some subtle differences between both approaches for small black holes, it is expected that the values for the entropy obtained with either definition are the same, at least in the macroscopic regime.
degrees of freedom or horizon degrees of freedom. The latter are described by a \( U(1) \) Chern-Simons theory on the sphere \( S^2 \) whose level \( \kappa \in \mathbb{N} \) is related to the classical area of the horizon \( a_0 \) according to \( a_0 = 4\pi\gamma\ell_P^2\kappa \). In this sense the entropy definition appearing above is restricted only to such values of the area. However, it is possible to argue that the horizon area should be given by an eigenvalue in the spectrum of the area operator in \( \text{LQG}^2 \). My point of view here will be then to extend the definition above to arbitrary values of the area \( a \), actually, by taking units such that \( 4\pi\gamma\ell_P^2 = 1 \) the inequality appearing in (1) will become

\[
\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} \leq \frac{a}{2},
\]

and hence the difference between the two different points of view boils down to choosing \( a \) as an integer or a real number.

The counting of the allowed sequences mentioned in the previous definition can be conveniently performed in four steps

(i) Start by fixing a value for the area \( a \) and obtain all the possible choices for the half integers \( |m_i| \) compatible with the area, i.e. satisfying

\[
\sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} = \frac{a}{2}.
\]

Obviously this number will be zero if \( a \) does not belong to the spectrum of the area operator. At this point we do not worry about the ordering of the labels but, rather, we just try to find out how many times the different values of each \( |m| \in \mathbb{N}/2 \) actually appear. In this step we count then the number of different finite multisets of \( |m_i| \) compatible with the area \( a \).

(ii) Count the different ways in which the previous multisets can be reordered.

(iii) Count all the different ways to introduce signs in the sequences of positive half-integers obtained in the previous step in such a way that the projection constraint \( \sum_{i=1}^{n} m_i = 0 \) is satisfied.

(iv) Repeat the same procedure for all the area eigenvalues smaller than \( a \) and add the number of sequences obtained for each value of the area.

The first step can be thought of as a characterization of the part of the spectrum of the area operator relevant in the computation of black hole entropy. This is so because the area eigenvalues for the isolated horizon that represents the black hole have the form

\[
a(\vec{j}) := 8\pi\gamma\ell_P^2 \sum_{i=1}^{n} \sqrt{j_i(j_i+1)}
\]

where the \( j_i \) “spin” labels are associated to the edges that pierce the horizon of the graph labeling the state vector in the \( \text{LQG} \) Hilbert space. For convenience I introduce here the integer variables \( k_i := 2|m_i| \) and write

\[
\sum_{i=1}^{n} \sqrt{|m_i|(|m_i|+1)} = \frac{a}{2} \Rightarrow \sum_{i=1}^{n} \sqrt{(k_i+1)^2 - 1} = a \Rightarrow
\]

\[
\sum_{i=1}^{k_{\text{max}}(a)} n_k \sqrt{(k+1)^2 - 1} = a \tag{2}
\]

It is possible to justify, also, that in the case of isolated horizons one can work with an area operator different from the standard one in \( \text{LQG} \) [14].
where the $n_k$ in the last sum tell us the number of times that the label $k \in \mathbb{N}$ appears. Also I denote as $k_{\text{max}}(a)$ the maximum value of the positive integer $k$ compatible with the area $a$. At this point we want to find out all the sets of pairs $\{(k, n_k) : k \in \mathbb{N}, n_k \in \mathbb{N}\}$ such that the previous equation is satisfied\(^3\). It is obvious that we can always write $\sqrt{(k+1)^2 - 1}$ as the product of an integer times the square root of a square-free positive number\(^4\) $p_I$ by using its prime factor decomposition. This means that the area eigenvalue $a$ must be an integer linear combination of square roots of squarefree numbers of the form

$$a = \sum_{l=1}^{r} q_l \sqrt{p_I},$$

or else the condition given by equation (2) cannot be satisfied. This leads then to the following equation

$$\sum_{k=1}^{k_{\text{max}}(a)} n_k \sqrt{(k+1)^2 - 1} = \sum_{l=1}^{r} q_l \sqrt{p_I}. \quad (3)$$

This can be solved by first considering each of the square-free numbers $p_I$ separately and computing the possible values of $k$ such that $\sqrt{(k+1)^2 - 1}$ is an integer multiple of $\sqrt{p_I}$. This amounts to solving the equations

$$\sqrt{(k_I + 1)^2 - 1} = y_I \sqrt{p_I}$$

in the two non-negative, integer unknowns $k_I$ and $y_I$ (here the label $I$ refers to the square-free number $p_I$). These are equivalent to

$$x_I^2 - p_I y_I^2 = 1 \quad (4)$$

where $x_I := k_I + 1$. Equation (4) is the well known Pell equation \cite{15}. Its general solution is known and can be found in the following way. Start by obtaining the so called fundamental solution \cite{15} $(x_I^I, y_I^I)$ corresponding to the smallest positive value for $x$. Once this is known the remaining solutions are given by the sequence $\{ (x_{\alpha}^I, y_{\alpha}^I) : \alpha \in \mathbb{N} \}$

$$x_{\alpha}^I = \frac{1}{2} \left[ (x_I^I + y_I^I \sqrt{p_I})^\alpha + (x_I^I - y_I^I \sqrt{p_I})^\alpha \right]$$

$$y_{\alpha}^I = \frac{1}{2 \sqrt{p_I}} \left[ (x_I^I + y_I^I \sqrt{p_I})^\alpha - (x_I^I - y_I^I \sqrt{p_I})^\alpha \right].$$

From these we get the solutions $(k_I^I, y_I^I)$ to the original equation. For instance for $p_I = 2$ the previous sequence starts as $(2, 2), (16, 12), (98, 70), (576, 408), \ldots$ It is interesting to note at this point that its terms grow exponentially fast.

Once the values of the $k_I$ are known we have to obtain the corresponding $n_{k_I}$. This is easily accomplished by solving a system of uncoupled, linear, diophantine equations (with non-negative, integer, unknowns). Indeed, by writing

$$\sum_{k=1}^{k_{\text{max}}} n_k \sqrt{(k+1)^2 - 1} = \sum_{l=1}^{r} \sum_{\alpha=1}^{\infty} n_{k_{\alpha}} y_{\alpha}^I \sqrt{p_I} = \sum_{l=1}^{r} q_l \sqrt{p_I},$$

\(^3\) Notice that by requiring that $n_k \neq 0$ we restrict ourselves to find only the values of $k$ that actually do appear.

\(^4\) We will enumerate these numbers as $p_1 = 2, p_2 = 3, p_3 = 5$, and so on.
and using the fact that the square roots of the square-free numbers are linearly independent over the rationals (and hence also over the integers) the previous equation is equivalent to the following system of linear, uncoupled, diophantine equations

$$\sum_{\alpha=1}^{\infty} y_{\alpha} n_{\alpha}^{I} = q_{I}, \quad I = 1, \ldots, r.$$  

Several comments are in order now. The first is that for a fixed value of the area $a$ (necessarily an integer linear combination with a finite number of coefficients $q_{I}$ corresponding to the square-free integers $p_{I}$) only a finite number of labels $k_{\alpha}^{I}$ come into play in these equations. Second, it may happen that some of these equations admit no solutions, in that case $a$ does not belong to the spectrum of the area operator. Finally when they can be solved the solution tell us exactly what the allowed values for the spin labels.

In the entropy definition introduced above the $m_{i}$ labels are half-integers. This is a consequence of the fact that the internal symmetry group in the Ashtekar formulation of general relativity is $SU(2)$. As long as one does not want to couple fermionic matter\(^5\) to gravity the Ashtekar formulation can also be obtained by using $SO(3)$ as the internal symmetry group. When considering quantization basically all the results corresponding to the standard $SU(2)$ formulation can be translated. The main difference lies in the fact that the edges in the spin network states are labeled only by integers. As far as the problem that I am considering here is concerned this has two main consequences. On one hand the spectrum of the area operator is modified, on the other we can only use integer values for the $|m_{i}|$ (and consequently only even values for the $k_{i}$).

Up to this point we have found all the possible choices of $|m_{i}|$ compatible with a given value of the area $a$, together with their multiplicities, that can be represented in the following schematic form\(^6\) as the multiset

$$\left(\frac{1}{n_{1}}, \ldots, \frac{1}{n_{2}}, \ldots, \frac{1}{n_{k_{\max}(a)}}, \ldots, \frac{k_{\max}(a)}{2}, \ldots, \frac{k_{\max}(a)}{2} \right).$$

The number of different sequences obtained from each of these multisets by reordering its elements is immediately given by the multinomial coefficient

$$\frac{(\sum_{k=1}^{k_{\max}(a)} n_{k})!}{\prod_{k=1}^{k_{\max}(a)} n_{k}!}.$$  

The degeneracy originating in this type of relabeling is responsible for many of the features of the black hole degeneracy spectrum and, in particular, the staircase structure of the black hole entropy in LQG discovered in [4]. This completes the second step mentioned above.

Once we have identified all the possible sequences of positive half-integers satisfying the area condition in (1) step three consists of finding out how many of them satisfy the so-called projection constraint

$$\sum_{i=1}^{n} m_{i} = 0$$  

for each of the sequences of positive half-integers $\{|m_{1}|, \ldots, |m_{n}|\}$ obtained in the previous two steps. In fact we will consider the slightly more general problem of solving

$$\sum_{i=1}^{n} m_{i} = p, \quad p \in \mathbb{Z}/2 \iff \sum_{i=1}^{n} k_{i} = 2p, \quad k_{i} := 2m_{i}.$$  

\(^{5}\) At least according to the most simple choices for its dynamics.

\(^{6}\) If some $n_{k}$ is zero then there are no terms with the number $k/2$.  

There are several approaches to solve this problem that can be found in the literature. Some of them are interesting because they suggest a connection of the present problem with conformal field theories [11] and problems arising in quantum computing [12]. Here I will give a simple solution based on the use of generating functions. Let us consider then that we have a multiset of positive half integers $k_1/2, k_2/2, \ldots, k_n/2$ ($k_i \in \mathbb{N}$) with multiplicities given by $n_i \in \mathbb{N}$ and take the following function of the variable $z$ (a Laurent polynomial)

$$
\prod_{i=1}^{n}(z^{k_i} + z^{-k_i})^{n_i}.
$$

By expanding it in powers of $z$ it is easy to see that the coefficient of the power $z^{2p}$ is, precisely, the number of different ways to distribute signs among the elements of the multiset of $k_i$ in such a way that

$$
\sum_i k_i = 2p.
$$

In particular if we look for the constant term in (6) we get the number of solutions for the projection constraint that can be built from a given multiset. A convenient way to extract this type of information is by using Cauchy’s theorem. This allows us to extract these coefficients (for a given $p$) by computing the contour integral

$$
\frac{1}{2\pi i} \oint_{\gamma} \frac{dz}{z^{p+1}} \prod_{i=1}^{n}(z^{k_i} + z^{-k_i})^{n_i},
$$

where $\gamma$ is an index 1 curve surrounding the origin. By choosing for it a unit circumference the previous integral can be written in the following useful alternative form

$$
\frac{2^{N-1}}{\pi} \int_{-\pi}^{\pi} d\theta \cos p\theta \prod_{i=1}^{n} \cos n_i k_i \theta
$$

where $N$ denotes the number of elements in the multiset (considering multiplicities).

Up to this point we have given a procedure to compute what we call the black hole degeneracy spectrum. This is defined, for every value of the area spectrum, as the number of sequences of non-zero half-integers satisfying the two conditions

$$
\sum_{i=1}^{n} m_i = 0, \quad \sum_{i=1}^{n} \sqrt{|m_i|(|m_i| + 1)} = \frac{a}{2}.
$$

The addition of the degeneracies corresponding to all the possible values of the area spectrum smaller or equal than the one for which the entropy is computed is the last remaining step to carry our program to completion. This can be done in principle by repeating the previous procedure for each of the relevant area eigenvalues and adding up the results. For an equally spaced spectrum this task can be easily accomplished by using generating functions (as has been done, for example, in [16] for a simplified, equally spaced, area spectrum). However, for the general case that we are considering here, one has to resort to more complicated methods based on the use of functional equations or a combination of generating functions and integral transforms [8, 9]. Here we only quote the result for completeness.

$$
\exp S_{SU(2)}(a) = \frac{1}{8\pi^2} \int_{0}^{4\pi} e^{\pi i \omega} \int_{0}^{\infty} s^{-1} \left(1 - 2 \sum_{k=1}^{\infty} e^{-s\sqrt{k(k+2)}} \cos \frac{\omega k}{2} \right)^{-1} e^{as} ds d\omega, \quad a \geq 0
$$

* Due precisely to this fact this solution is especially suited to obtain the black hole generating functions of [10].
where \( x_0 \) is a real number larger than the real part of all the singularities of the integrand in the previous expression. From the last formula it is especially easy to find the expression corresponding to the choice of \( SO(3) \) [6] as the internal symmetry group: it suffices to restrict the values of the label \( k \) appearing in the sum in the denominator of the integrand to even values.

\[
\exp S_{SO(3)}(a) = \frac{1}{8\pi^2i} \int_0^{4\pi} \int_{x_0-i\infty}^{x_0+i\infty} s^{-1} \left( 1 - 2 \sum_{k=1}^{\infty} e^{-s\sqrt{k(k+1)}} \cos \omega k \right)^{-1} e^{as} \, ds \, d\omega , \quad a \geq 0
\]

This result can also be derived either by the functional methods originally used by Meissner to study the entropy or by the use of generating functions of the type developed in [10]. Though the value of the Immirzi parameter changes [6], the staircase structure of the entropy survives (with a periodicity different from the one corresponding to the \( SU(2) \) case [17]).

Finally the black hole generating function corresponding to the choice of either \( SU(2) \) or \( SO(3) \) as the internal group is

\[
G(z, x_1, x_2, \ldots) = \left( 1 - \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} (z^{k^l_m} + z^{-k^l_m}) y^l_m \right)^{-1}, \quad (9)
\]

where the variables \( x_1, x_2, \ldots \) are associated to the different square-free numbers \( p_l \), \( z \) is an additional variable, and the \( (k^l_m, y^l_m) \) are the solutions to the Pell equation. In the case of working with \( SU(2) \) all the solutions should be taken, whereas for \( SO(3) \) only even values of \( k^l_m \) (for each \( p_l \)) have to be considered. The coefficient of the term \( z^{0} x_1^{q_1} \cdots x_q^{r} \) tells us the value of the exponential of the entropy \( e^{S(a)} \) for an area \( a \) given by \( a = \sum_l q_l \sqrt{p_l} \).

In conclusion the results and techniques developed to study in detail black hole entropy in loop quantum gravity can be easily adapted to work with internal gauge groups different from the standard \( SU(2) \) choice. This allows to study the kind of microstructure found in the black hole spectrum for this alternative choices, in particular it is straightforward to write down an integral expression for the entropy in the form of an inverse Laplace transform and also to derive the corresponding generating functions along the lines of [10].

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