Abstract

Yang-Mills theories on a 1+1 dimensional cylinder are considered. It is shown that canonical quantization can proceed following different routes, leading to inequivalent quantizations.

The problem of the non-free action of the gauge group on the configuration space is also discussed. In particular we re-examine the relationship between “θ-states” and the fundamental group of the configuration space. It is shown that this relationship does or does not hold depending on whether or not the gauge transformations not connected to the identity act freely on the space of connections modulo connected gauge transformations.
To the present day, the understanding of the canonical quantization of non-abelian
gauge theories in 3+1 dimensions and the knowledge of their physical degrees of freedom
are lacking the necessary rigor. Recent papers \cite{1,2} have dealt with the somewhat simpler
task of quantizing $SU(N)$ or $U(N)$ Yang-Mills theory on a 1+1 dimensional cylinder.

These cases are interesting not only because they may cast some light on the features
of the 3+1 dimensional case, but also because they deal with the quantization of a gauge
theory on a compact spatial manifold. Indeed, it is known \cite{3} that on compact spaces, the
action of the gauge group on the space of connections may not be free, causing the phase
space to be no longer a manifold but only an orbifold. In relation to both Yang-Mills
theories and gravity, some simplified finite-dimensional examples have been worked out
in \cite{5} and \cite{6}.

In this paper, we describe some novel features that arise in the quantization of pure
Yang-Mills theory on a cylinder. It will be shown that this model admits inequivalent
quantizations, the ambiguities arising because of two different reasons:

1. Let $G$ denote the gauge group, i.e. the group of all gauge transformations that act
on the phase space, and $G_0$ the subgroup of $G$ connected to the identity, which is
generated by Gauss law. As in any gauge theory, $G_0$ should leave a physical state
invariant. Towards this end, one will have to go to a reduced phase space and/or
impose Gauss law on the physical states. In the problem at hand it will be shown
that there are inequivalent ways of carrying out this procedure.

2. In our problem, the action of the gauge group is not free and the reduced phase
space is an orbifold. After suitably treating the singular points \cite{5}, it will be shown
that there are different self-adjoint extensions of the relevant operators, and hence
different quantum theories.

The fact that the action of the gauge group is not free has also another consequence.
In non-abelian gauge theories in 3+1 dimensions, the possibility of inequivalent quantizations and θ-states arises because of the existence of gauge transformations that are not connected to the identity (i.e. elements of $G^\infty$ which are not elements of its identity component $G_0^\infty$, the superscript ∞ indicating that the gauge transformations have to vanish at spatial infinity). It is known [4] that this is closely related to the multiple connectivity of the reduced configuration space. But the usual arguments relating them require a free action of the group $G^\infty$ on the space $\mathcal{A}$ of connections. Indeed, if this action is free one can show that the fundamental group $\Pi_1(Q)$ of the reduced configuration space $Q = \mathcal{A}/G^\infty$ equals the group of disconnected components of $G^\infty$, given by $G^\infty/G_0^\infty$. That this action is free can be proven whenever the gauge transformations in question are pointed gauge transformations, i.e. they reduce to identity at a point of the base space (as it happens at the point at $\infty$ in the case of $\mathbb{R}^4$). Now in the context of gravity, [6] gives an example in which this relationship does not hold because the action of the group of gauge transformations (diffeomorphisms in this case) is not free. In particular it is proven that the configuration space is simply connected despite the existence of gauge transformations not connected to the identity.

Let us make a few initial remarks about this problem in our context.

We will consider here pure YM theories when the group $G$ is $U(N)$ or $SU(N)$ and the spatial manifold is $S^1$. In all these cases the action of the gauge group $\mathcal{G}$ on the space of connections $\mathcal{A}$ is never free.

For $G = SU(N)$, $\mathcal{G}/\mathcal{G}_0$ is trivial and therefore there exist no θ-states. Neither is the reduced configuration space multiply connected, so that there is nothing much to say about the relation between θ-states and the fundamental group of the reduced configuration space in this case.

For $G = U(N)$, we do have gauge transformations not connected to the identity and $\mathcal{G}/\mathcal{G}_0 = \mathbb{Z}$. As already mentioned above, the action of the gauge group $\mathcal{G}$ on the space
A of connections is not free. However the group \( G/G_0 \) of transformations not connected to the identity still acts freely on the space \( A/G_0 \) of connections modulo connected gauge transformations. This is because of the following reason. [Below we denote the Lie algebra of \( G \) by \( \mathcal{G} \):]

If \( g(x) \) is an element of \( G \) which is not connected to the identity, then its action on a gauge potential \( A(x) \) is

\[
A(x) \longrightarrow g(x) \cdot A(x) := g(x)A(x)g(x)^{-1} + g(x)dg(x)^{-1}.
\]

(1)

In particular \( A_{U(1)} \), the component of \( A \) along the \( U(1) \) central subalgebra of \( U(N) \), transforms according to

\[
A_{U(1)} \longrightarrow (g \cdot A)_{U(1)} = A_{U(1)} + (gdg^{-1})_{U(1)},
\]

(2)

where \( (gdg)^{-1} \) is a closed, non-exact form because the non-trivial map \( g(x) \) is homotopic to a map which necessarily winds around the \( U(1) \) center of \( U(N) \). This shows that \( G/G_0 \) acts freely on \( A \). That its action is free also on \( A/G_0 \) follows because otherwise we would have

\[
g \cdot A = g_0 \cdot A \Leftrightarrow (g_0^{-1}g) \cdot A = A \quad \text{for } g \in G/G_0 \quad \text{and } g_0 \in G_0,
\]

(3)

which cannot be since \( g_0^{-1}g \) is also an element in \( G/G_0 \).

Thus the fundamental group \( \Pi_1(Q) \) of the reduced configuration space \( Q = A/\mathcal{G} = (A/G_0)/(G/G_0) \) is once again given by \( G/G_0 = \mathbb{Z} \). For such a theory, then, the usual relation between \( \theta \)-states and the multiple connectivity of the configuration space continues to hold.

Since in this paper we are looking at pure YM theories (without fermions), when \( G = SU(N) \), the relevant group can be very well said to be not \( SU(N) \) but \( SU(N)/\mathcal{C}(SU(N)) \) where \( \mathcal{C}(SU(N)) \) is the center of \( SU(N) \). This is because the center of \( SU(N) \) is a discrete subgroup and therefore the phase space variables which are Lie algebra valued, as well
as the gauge transformations of such variables, live only on the quotient of $SU(N)$ by its discrete center. [This would not have been the case if $G$ were $U(N)$ because the center of the latter is $U(1)$ and hence not discrete.]

Thus, for example, if $G = SU(2)$, we can alternatively think of a theory with $G = SU(2)/\mathbb{Z}_2 = SO(3)$. In this case it turns out that $G/G_0 = \mathbb{Z}_2$ and $\theta$-states associated to gauge transformations not connected to the identity do exist in this theory. [By “$\theta$-states” we mean those associated with irreducible representations of $G/G_0$, even though, as here, they are not labelled by an angle $\theta$.] However, unlike in the $U(N)$ case, here the reduced configuration space $Q = \mathcal{A}/G$ is simply connected: since $\mathbb{Z}_2$ does not act freely on the space of connections modulo connected gauge transformations, the relationship between $\theta$-states and non-simple connectivity of the configuration space is spoiled.

We now turn to the demonstration of the results 1 and 2.

Consider pure YM theory for a semisimple compact group $G$ on a cylinder. Let us also assume that space is a circle running from $x = 0$ to $x = 2\pi$.

The action is

$$S = -\frac{1}{4} \int Tr(F_{\mu\nu}F^{\mu\nu}),$$

(4)

where the curvature tensor $F$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

(5)

If $T^a$ are the generators of the Lie algebra $G$ of $G$, then a Lie algebra valued field $X$ is defined in terms of its components $X^a$ by

$$X = X^a T^a,$$

(6)

where summation over $a$ is assumed.

The Hamiltonian and Poisson Brackets (PB) obtained from (4) are

$$H = \frac{1}{2} \int Tr(E^2) \, dx,$$

(7)
\{ A^a_1(x), A^b_1(y) \} = 0 = \{ E^a(x), E^b(y) \} \\
\{ A^a_1(x), E^b_1(y) \} = \delta^{ab} \delta(x - y). \quad (8)

The Hamiltonian (7) is to be complemented by the Gauss law

$$\frac{\partial E}{\partial x} + [A_1, E] \approx 0 ,$$

which is the generator of gauge transformations

$$(A_1 , E) \rightarrow (gA_1g^{-1} + g\frac{\partial g^{-1}}{\partial x} , gEg^{-1}) \quad (10)$$

where $g(x)$ is valued in $G$.

Following Rajeev \[2\], in order to isolate the gauge invariant degrees of freedom, we define variables $S(x)$ and $\tilde{E}(x)$:

$$S(x) = \mathcal{P} \exp[- \int_0^x A_1(y) \, dy] ,$$
$$\tilde{E}(x) = S^{-1}(x)E(x)S(x). \quad (11)$$

$\mathcal{P}$ denotes path ordering.

Notice that since $S(2\pi)$ need not to be equal to $S(0) = 1$, this redefinition imposes a twisted boundary condition on $\tilde{E}(x)$:

$$S(2\pi)\tilde{E}(2\pi)S^{-1}(2\pi) = \tilde{E}(0). \quad (12)$$

The Hamiltonian and PB in these variables are

$$H = \frac{1}{2} \int Tr(\tilde{E}^2) \, dx ,$$
$$\{ S(x), S(y) \} = 0 ,$$
$$\{ S(x), \tilde{E}^a(y) \} = -\theta(x - y)S(x)T^a ,$$
$$\{ \tilde{E}^a(x), \tilde{E}^b(y) \} = f^{abc}\tilde{E}^c(y) , \quad y < x . \quad (14)$$

where $f^{abc}$ refer to the structure constants of the Lie Algebra $G$. 

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The gauge transformations now assume the simpler form

\[(S(x) , \tilde{E}(x)) \rightarrow (g(x)S(x)g^{-1}(0) , g(0)\tilde{E}(x)g^{-1}(0)) .\]  

(15)

Furthermore, using

\[\frac{dS}{dx} = -A_1 S ,\]  

(16)

Gauss law becomes

\[\frac{\partial \tilde{E}}{\partial x} \approx 0\]  

(17)

and can be easily solved:

\[\tilde{E}(x) = \tilde{E}(0) .\]  

(18)

Using the equivalence under gauge transformations (15) and Gauss law (18), it is easy to see [2] that the reduced phase space consists of

\[q = S(2\pi) \in G,\]  

\[p = \tilde{E} \in G,\]  

(19)

along with the constraint arising from (12) and the gauge equivalence (15):

\[qpq^{-1} = p ,\]  

(20)

\[(q , p) \sim (gqg^{-1} , gpg^{-1}) .\]  

(21)

Here \(g = g(0) , g(x)\) being the gauge transformation.

The Hamiltonian (13) and PB’s (14) can be restricted to the \((q , p)\) space:

\[H = \pi Tr(p^2) ,\]  

(22)

\[\{q,q\} = 0 ,\]  

\[\{q,p^a\} = -qT^a ,\]  

\[\{p^a , p^b\} = f^{abc}p^c ,\]  

(23)
\( p^a \) is defined here using rule (6).

In addition, it can be checked, using (23), that the function \( qpq^{-1} - p \) involved in the constraint (20) generates exactly the gauge transformations (21).

The following point about (20) is to be noted. In quantum theory, \( qpq^{-1} \) is not well defined unless an ordering of the operators is also defined. However, using the PB’s (23) between \( q \) and \( p^a \) and the fact that classical Poisson Brackets are replaced by Lie Brackets in quantum theory, it can be checked that the difference between the quantum operators \( qp^a T^a q^{-1} \) and (say) \( p^a q T^a q^{-1} \), both of which correspond classically to \( qpq^{-1} \), is only a constant. Due to this lucky fact, \( qpq^{-1} - p \) is well defined (without having to define an ordering) even in quantum theory up to a constant. Thus the gauge transformations generated by this function are independent of the ordering.

Having got the canonical structure of the phase space, we can proceed to quantize the system. We will argue that one can follow three different routes.

Following Rajeev [2], the first approach to quantization can be to declare wave functions as complex functions of \( q \) which are invariant under \( g \rightarrow g q g^{-1} \). Then the Hamiltonian, being just the quadratic Casimir, has as eigenstates the characters of the irreducible representations. We have nothing new to add to this approach.

The remaining two alternatives, developed for the first time in this paper, insist on carrying through the reduction of the phase space in stages. In these approaches, we show how quantization ambiguities arise because of the two reasons mentioned at the beginning.

The first step in this reduction is the observation that the constraint (20) allows us to choose a representative \(( \tilde{q} , \tilde{p} )\) for each gauge equivalence class of the following type [1]:

\[
\tilde{q} \in \text{Cartan subgroup of } G, \text{ denoted } C_G,
\]

\[
\tilde{p} \in \text{Cartan subalgebra of } G, \text{ denoted } C_G. \tag{24}
\]

The consequence is that (22) and (23) can now be restricted to the (partially) reduced
phase space $C_G \times C_G$, this restriction making sense because of the gauge invariance of the Hamiltonian and the gauge covariance of the PB relations. [The meaning of the latter is that (23) is invariant in form under a gauge transformation $q \rightarrow g q g^{-1}, p \rightarrow g p g^{-1}$.] We thus have

$$H = \pi Tr(\tilde{p}^2),$$

$$\{\tilde{q}, \tilde{q}\} = 0,$$

$$\{\tilde{p}^i, \tilde{p}^j\} = 0,$$

$$\{\tilde{q}, \tilde{p}^i\} = -q C^i,$$

(26)

(where $i$ runs over the Cartan Subalgebra generators and $C^i$ are the corresponding generators) along with a residual gauge equivalence left over from (21):

$$(\tilde{q}, \tilde{p}) \sim (W\tilde{q}W^{-1}, W\tilde{p}W^{-1}) = (\tilde{q}', \tilde{p}').$$

(27)

$W$ here belongs to the Weyl subgroup of $G$. [For the case where $G$ is $U(N)$ or $SU(N)$, this action corresponds to a permutation of the diagonal entries of the matrices corresponding to $\tilde{q}$ and $\tilde{p}$ in a representation where the basis is chosen so that the $C^i$, the generators of $C_G$, are diagonal.]

(25) and (26) represent the free Hamiltonian for $n$ non-interacting particles on an $n$-torus with some identifications ($n$ is the dimension of $C_G$ which equals $N-1$ for $SU(N)$ and $N$ for $U(N)$). That this is so, can be seen by defining variables $\alpha_i$:

$$e^{-\alpha_i C^i} = \tilde{q}.$$  

(28)

If $C^i$ are chosen such that

$$e^{-L_i C^i} = 1 \text{ and } e^{-\alpha_i C^i} \neq 1 \text{ for } 0 < \alpha_i < L,$$

(29)

then the $\alpha_i$ are coordinates on an $n$-torus taking values in the range:

$$0 \leq \alpha_i \leq L_i.$$

(30)
Now (27) imposes further identifications on points of this torus.

In terms of $\alpha_i$ and $\tilde{p}_i$, (25) and (26) are

$$H = \pi \sum_i \tilde{p}_i^2,$$

$$\{\alpha_i, \alpha_j\} = 0 = \{\tilde{p}_i, \tilde{p}_j\},$$

$$\{\alpha_i, \tilde{p}_j\} = \delta_{ij}.$$  \hspace{1cm} (31)

(32)

whereas (27) can be rewritten as

$$(\alpha_i, \tilde{p}_i) \sim ((W \circ \alpha)_i, (W \circ \tilde{p})_i),$$

where $\circ$ is the action induced on the variables $\alpha_i$ and $\tilde{p}_i$ by conjugation in the right hand side of (27). (33) is a discrete identification of the elements of the phase space arising because of permutations of diagonal entries in the representation of $C^i$ mentioned above.

The second method of quantizing pure YM on a cylinder now follows by taking wavefunctions as complex square integrable functions of $\{\alpha_i\}$, thought of as coordinates of the $n$-torus which are however invariant under (33). More generally, the wavefunctions can be functions defined on the universal cover of the $n$-torus which transform according to an irreducible unitary representation of the fundamental group of the configuration space, compatibly with the condition (33) lifted to the universal cover. In this more general case, the wave functions can change by a phase when transported around a non-trivial loop on the torus.

In the case of $SU(N)$ or $SU(N)/C(SU(N))$, this more general case trivializes because the equivalence (33) forces the wave function to remain unchanged when transported around one of the $N - 1$ loops.

For $U(N)$, this happens for all the loops except the one corresponding to its $U(1)$ center. Thus, in this case, wavefunctions can change by a phase when they go around this particular loop while they are single-valued with respect to the other $N - 1$ loops.
The Hamiltonian (31) is just the Laplacian in this representation and so finding the
eigenstates reduces to the easy problem of “particle in a box”, where along some directions
[exactly one for $U(N)$ and none for $SU(N)$ or $SU(N)/\mathcal{C}(SU(N))$] wave functions are
allowed to pick up a phase around a closed loop. Similar results, but using a somewhat
different approach have also been obtained in reference [8].

The third method of quantization is to go to the completely reduced phase space which
is $(T^n \times \mathbb{R}^n)/\sim$ where $\sim$ is the equivalence under (33). This quotienting however has fixed
points and so the reduced phase space becomes an orbifold. Wave functions as before will
be complex functions of the reduced configuration space in the non-singular regions of
the phase space while the Hamiltonian will, as before, be the Laplacian in these regions.
The Laplacian will in general admit several self-adjoint extensions to the singular regions
(of necessarily lower dimension) and therefore will give rise to a family of inequivalent
quantum theories.

As an illustration of the above general statements three specific examples will be
considered below.

$G = SU(2)$.

This is a prototype of the $SU(N)$ case in which there are no $\theta$-states and the reduced
configuration space is also simply connected.

First, we have Rajeev’s quantization [2] according to which wavefunctions are char-
acters of irreducible representations. Since the Hamiltonian is $\pi \text{Tr}(p^2)$, the eigenvalues
here are $2\pi j(j+1)$ where $j = 0, \frac{1}{2}, 1, \ldots$.

In the second way of quantizing the system, we first observe that $\mathbb{C}G$ is 1-dimensional,
consisting of the the only element $\frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

Eq. (31) is now

$$ H = \pi p_1^2, \quad (34) $$
while (32) (in its quantum version) is

\[
\begin{align*}
[\alpha_1, \alpha_1] &= 0 = [\tilde{p}_1, \tilde{p}_1] \\
[\alpha_1, \tilde{p}_1] &= i.
\end{align*}
\]  

(35)

Here $\alpha_1$ is the angular coordinate on a circle of length $L = 2\pi \sqrt{2}$.

(33) leads to

\[
(\alpha_1, \tilde{p}_1) \sim (L - \alpha_1, -\tilde{p}_1).
\]  

(36)
as can be seen in the following way. Since $\tilde{q} = \begin{pmatrix} e^{-i\frac{\alpha_1}{\sqrt{2}}} & 0 \\ 0 & e^{i\frac{\alpha_1}{\sqrt{2}}} \end{pmatrix}$ while $\tilde{p} = \frac{1}{\sqrt{2}} \begin{pmatrix} i\tilde{p}_1 & 0 \\ 0 & -i\tilde{p}_1 \end{pmatrix}$, the permutation that $W$ causes, takes $\tilde{q}$ to $\tilde{q}' = \begin{pmatrix} e^{i\frac{\alpha_1}{\sqrt{2}}} & 0 \\ 0 & e^{-i\frac{\alpha_1}{\sqrt{2}}} \end{pmatrix}$ and $\tilde{p}$ to $\tilde{p}' = \frac{1}{\sqrt{2}} \begin{pmatrix} -i\tilde{p}_1 & 0 \\ 0 & i\tilde{p}_1 \end{pmatrix}$. Hence (36) follows.

Thus the second method of quantization would correspond to considering wave functions $\Psi(\alpha_1)$ in the Hilbert space of square integrable functions on $S^1$ which are invariant under:

\[
\alpha_1 \rightarrow L - \alpha_1.
\]  

(37)

[The more general wavefunction which may transform by a phase under $\alpha_1 \rightarrow L + \alpha_1$ is automatically excluded because of the requirement that $\Psi(0) = \Psi(L)$ arising from (37).]

A complete set of functions on a circle which are also eigenfunctions of the Hamiltonian (34) (which in this representation is just given by the operator $-\pi \frac{d^2}{d\alpha_1^2}$) are $\Psi(\alpha_1) = \sqrt{\frac{L}{\pi}} e^{i\frac{2\pi}{L}n\alpha_1}$, $n \in \mathbb{Z}$. However, with restriction (37), the set of functions allowed are only

\[
\cos \frac{2\pi}{L} n\alpha_1, \quad n \in \mathbb{Z}.
\]

Thus the eigenfunctions are

\[
\Psi_n(\alpha_1) = \sqrt{\frac{2}{L}} \cos \frac{2\pi}{L} n\alpha_1,
\]  

(38)

while the corresponding eigenvalues are

\[
E_n = \frac{\pi n^2}{2}.
\]  

(39)
The third way of quantizing $SU(2)$ YM on a cylinder is by doing the quotient by the equivalence relation (36). It yields the segment $[0, L_2]$ as the configuration space. The corresponding phase space is given by $(\tilde{q}, \tilde{p}) \in [0, \frac{L}{2}] \times \mathbb{R}$ with the identifications $(0, \tilde{p}_1) \equiv (0, -\tilde{p}_1)$ and $(\frac{L}{2}, \tilde{p}_1) \equiv (\frac{L}{2}, -\tilde{p}_1)$. Thus the interior of the phase space is $(0, \frac{L}{2}) \times \mathbb{R}$ and the interior of the coordinate space is the open interval $(0, \frac{L}{2})$. Here the wavefunctions can therefore be considered as functions of $x \in [0, \frac{L}{2}]$ while the Hamiltonian $H$ is any self-adjoint extension of $-\pi \frac{d^2}{d\alpha_1^2}$ from the interior $(0, \frac{L}{2})$ to the closed segment, obtained by imposing suitable boundary conditions.

Let us recall that an operator $A$ is self-adjoint on a domain $\mathcal{D}$ if

$$< \Psi | A \Phi > = < A \Psi | \Phi > \quad \forall \Phi \in \mathcal{D} \iff \Psi \in \mathcal{D},$$

$< \cdot | \cdot >$ indicating the scalar product. For $A = H = -\frac{d^2}{d\alpha_1^2}$ on $[0, \frac{L}{2}]$, one can easily show that the above condition is equivalent to find a domain $\mathcal{D} \subset L^2([0, \frac{L}{2}])$ which satisfies:

$$\frac{d}{d\alpha_1} \Psi(\alpha_1) \Phi(\alpha_1) - \Psi(\alpha_1) \frac{d}{d\alpha_1} \Phi(\alpha_1) \big|_{0}^{L/2} = 0 \quad \forall \Phi \in \mathcal{D} \iff \Phi \in \mathcal{D}.$$

There is an infinite number of such domains [9], corresponding to different boundary conditions for the wavefunction $\Psi(\alpha_1)$ at $\alpha_1 = 0, \frac{L}{2}$.

The spectrum and eigenfunctions of $H$ depend on the extension chosen.

For example, for the self-adjoint extension corresponding to the boundary condition $\frac{d}{d\alpha_1} \Psi(0) = \frac{d}{d\alpha_1} \Psi(\frac{L}{2}) = 0$, one recovers the set of eigenfunctions and eigenvalues (38) and (39).

If the boundary condition reads $\Psi(0) = \Psi(\frac{L}{2}) = 0$ instead, the eigenfunctions are

$$\Psi_n(\alpha_1) = \sqrt{\frac{4}{L}} \sin \frac{2\pi}{L} n\alpha_1,$$

whereas the eigenvalues are still given by

$$E_n = \frac{\pi n^2}{2}.$$
On the contrary, the eigenvalue do change for other boundary conditions, such as
\[ \Psi(L/2) = e^{i\theta} \Psi(0) \]
\[ \frac{d}{d\alpha_1} \Psi(L/2) = e^{i\theta} \frac{d}{d\alpha_1} \Psi(0) . \]
In this case we have:
\[ \Psi_n(\alpha_1) = \sqrt{\frac{2}{L}} e^{i\frac{2\pi}{L}(2n+\frac{\theta}{\pi})\alpha_1} , \]
\[ E_n = \frac{\pi(2n + \frac{\theta}{\pi})^2}{2} . \]
(39) and (43) are not the same, nor do they agree with the spectrum found by Rajeev.

\( G = SO(3) \).

This is the prototype of the \( SU(N)/C(SU(N)) \) case, where there are \( \theta \)-states but the reduced configuration space is simply connected so that the usual relation between them is violated. This example is very similar to the previous one except that the group is now doubly connected. This means that the analogue of \( q \) here when written in terms of an \( SU(2) \) group element is the equivalence class \( \{ \pm q \} \) \((-1 \) is the element of \( SU(2) \) which also maps to the identity of \( SO(3) \)).

Firstly, note that we have the relation
\[ (q, p) \sim (gqg^{-1}, gp\tilde{g}^{-1}) \] (18)
where \( q, g \in SU(2) \) and \( p \in SU(2) \).

(21) gives the gauge transformations that are generated by Gauss law and therefore wave functions have to be invariant under these transformations.

There are also gauge transformations not connected to identity, which are not generated by Gauss law. Wave functions can transform under unitary representations of this group of transformations, this less stringent requirement being enough to guarantee invariance of the matrix elements of observables. In this example \( g(x) \) such that \( g(0) = 1 \) and \( g(2\pi) = -1 \) is a gauge transformations that is not connected to identity. The equivalence that it causes is:
\[ (q, p) \sim (-q, p) . \]
(44)
In terms of the variables defined in (24), this means that
\[(\tilde{q}, \tilde{p}) \sim (-\tilde{q}, \tilde{p}), \tag{45}\]
or in terms of the \(\alpha_1\) and \(p_1\) defined in (28) and (8) respectively,
\[(\alpha_1, \tilde{p}_1) \sim (\alpha_1 + \frac{L}{2}, \tilde{p}_1). \tag{46}\]
The corresponding group of transformations here is \(\mathbb{Z}_2\). Thus wave functions transform under unitary representations of \(\mathbb{Z}_2\):
\[\Psi(\alpha_1 + \frac{L}{2}) = \pm \Psi(\alpha_1). \tag{47}\]
This is just the usual argument given for the existence of \(\theta\)-states.

In Rajeev’s method of quantization, since wave functions are taken as characters of irreducible representations of \(SU(2)\), the representations corresponding to \(\frac{1}{2}\)-integers satisfy \(\Psi(-q) = -\Psi(q)\) while the integral representations satisfy \(\Psi(-q) = \Psi(q)\). They correspond to the two choices in (17). Clearly we cannot allow wavefunctions of both types to occur in a single quantum theory because then a superposition of these wave functions would not satisfy (47). Thus two quantum theories are super-selected, one for each sign in (17). The energies for the quantum theories corresponding to the + sign and – sign are \(E_j = 2\pi j(j + 1)\) with \(j\) being integer or half-integer respectively.

In the second method of quantization, eigenfunctions are as in (38) above. With condition (17) there is again a super-selection into two quantum theories with eigenfunctions
\[\Psi^I_n(\alpha_1) = \sqrt{\frac{4}{L}} \cos \frac{2\pi}{L}(2m)\alpha_1 \quad \text{for} \quad \Psi(\alpha_1 + \frac{L}{2}) = \Psi(\alpha_1), \tag{48}\]
\[\Psi^\#_n(\alpha_1) = \sqrt{\frac{4}{L}} \cos \frac{2\pi}{L}(2m + 1)\alpha_1 \quad \text{for} \quad \Psi(\alpha_1 + \frac{L}{2}) = -\Psi(\alpha_1)\]
and the corresponding energy eigenvalues
\[E^I_n = \frac{\pi(2m)^2}{2}, \]
\[E^\#_n = \frac{\pi(2m + 1)^2}{2}. \tag{49}\]
In the third way of quantization, we have to quotient first by (33) to get the completely reduced phase space as in the SU(2) example. In addition we have relation (46) and the corresponding rule for wave functions (47) to contend with. In the reduced phase space, (46) and (47) will read

\[
(\alpha_1, \tilde{p}_1) \sim \left( \frac{L}{2} - \alpha_1, -\tilde{p}_1 \right),
\]

\[
\psi\left(\frac{L}{2} - \alpha_1\right) = \pm \psi(\alpha_1).
\]

In this case, while the boundary conditions at \(\alpha_1 = 0\) are arbitrary, those at \(\alpha_1 = \frac{L}{4}\) are not. This is so because the reduced configuration space is (as before) the interval \([0, \frac{L}{2}]\), but now condition (51) forces \(\frac{d}{d\alpha_1} \psi\left(\frac{L}{4}\right)\) or \(\psi\left(\frac{L}{4}\right)\) to be zero depending on the + or − choices of (51). Thus the problem reduces to finding those self-adjoint extensions of \(\frac{d^2}{d\alpha_1^2}\) on the interval \([0, \frac{L}{2}]\) such that either \(\frac{d}{d\alpha_1} \psi\left(\frac{L}{4}\right) = 0\) or \(\psi\left(\frac{L}{4}\right) = 0\). Here, the most general boundary condition at \(\alpha_1 = 0\) for \(\frac{d^2}{d\alpha_1^2}\) to be self-adjoint is

\[
\frac{d}{d\alpha_1} \psi(0) = k\psi(0),
\]

where \(k\) is an arbitrary real constant.

The energy levels are dependent on the parameter \(k\) and whether the BC at \(\alpha_1 = \frac{L}{4}\) is the one corresponding to the + or − sign of (51). For example, when \(k = 0\), the eigenstates and eigenvalues reduce to those obtained in (48) and (49) above.

Incidentally, (50) also shows that the completely reduced configuration space is just a segment (the closed interval \([0, \frac{L}{2}]\)) and hence is simply connected. So, the argument which relates \(\theta\)-states to the multiple connectivity of the configuration space clearly breaks down here since \(\theta\)-states do exist despite the configuration space being simply connected.

One further point to be noted here is the following. If instead of sticking to condition (51), we treat (50) as another exact equivalence and quotient the phase space by this equivalence, we arrive at a completely reduced phase space identical in structure to that of the SU(2) example, save for the fact that the coordinate variable \(\alpha_1\) here takes values
in $[0, \frac{L}{4}]$ instead of $[0, \frac{L}{2}]$. So, the eigenfunctions and eigenvalues obtained in this approach match those obtained in the third way of quantizing $SU(2)$ YM [Equations (37-41)] with the following replacements:

$$\begin{align*}
\alpha_1 & \rightarrow 2\alpha_1 , \\
E_n & \rightarrow 4E_n .
\end{align*}$$

(53)

In this approach the BC at $\alpha_1 = \frac{L}{4}$ has been relaxed and a two-fold ambiguity $[\frac{d}{d\alpha_1}\Psi(\frac{L}{4}) = 0 \text{ or } \Psi(\frac{L}{4}) = 0]$ has been allowed to become an $\infty$-fold ambiguity. This infinity of BC’s now also contains those corresponding to $\theta$-states as special cases.

$$G = U(2).$$

This is a prototype of the $U(N)$ case in which $\theta$-states exist and are also related to the fundamental group of the reduced configuration space in the standard way. In this example, there are gauge transformations not connected to the identity with the added feature that now they act freely on $A/G_0$, so that $G/G_0$ acts freely on $A/G_0$. Thus the reduced configuration space is

$$Q = A/G = (A/G_0)/(G/G_0)$$

(54)

and therefore

$$\pi_1(Q) \equiv \pi_1(A/G) = \pi_0(G/G_0) = G/G_0 = \mathbb{Z} .$$

(55)

Therefore $\theta$-states here are related to the first homotopy group $\pi_1(Q)$ of the reduced configuration space and the former can be thought of as arising because of quantization ambiguities corresponding to different unitary representations of $\pi_1(Q) = \mathbb{Z}$.

If $U(2)$ is parametrized using an $U(1)$ element $u$ and an $SU(2)$ element $q$, then an element of $U(2)$ corresponds to the pair $(u, q)$ provided the following identification is made:

$$(u, q) \sim (-u, -q) .$$

(56)
Thus wave functions in Rajeev’s approach are complex functions of the above pair which are such that

\[ \Psi[(u, gqq^{-1})] = \Psi[(u, q)] , \]
\[ \Psi[(-u, -q)] = \Psi[(u, q)] e^{i\theta} . \]

The first condition simply restates the similar condition imposed on wave functions even in the $SU(2)$ case. The second condition is a consequence of (56) and the multiple connectivity of $U(2)$ which allows wave functions to transform under a unitary representation of the fundamental group (here $\mathbb{Z}$) when the coordinate variable goes around a non-trivial loop.

The Hamiltonian in Rajeev’s method is the quadratic Casimir of $U(2)$ and the eigenstates are the $SU(2)$ characters multiplied by functions of one coordinate (parametrizing the $U(1)$ center) which are of the form (42). More specifically, if $\psi_n(\alpha) = e^{i(n + \frac{\theta}{2\pi})\alpha}$ and $\chi$ is one of the characters of $SU(2)$, then the eigenstates are

\[ \Psi[(u, q)] = \psi_n(\alpha)\chi(q) , \]

where $u = e^{i\alpha}$ and $n = 2m + 1$ or $n = 2m$ depending on $\chi$ being a character of a half-integer or integer representation respectively. The energy eigenvalues are

\[ E_{j,n} = 2\pi j(j + 1) + \pi(n + \frac{\theta}{2\pi})^2 , \]

where $j$ takes values $0, \frac{1}{2}, 1, \ldots$ and $n$ takes values $0, 1, 2, \ldots$.

The inequivalent quantizations here correspond to different values of the parameter $\theta$ and they can be thought of as arising due to the existence of gauge transformations not connected to the identity ($\theta$-states approach) or equivalently be thought of as arising due the existence of a non-trivial fundamental group for the reduced configuration space.

Similar analysis (not repeated here) goes through for the other approaches to quantization. In each of the other approaches too, an extra term of the form (12) multiplies
the already existing $SU(2)$ eigenstates while a corresponding extra term of the form (43) adds to the energy eigenvalues of the $SU(2)$ example.

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