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CANCELLATION IN ADDITIVELY TWISTED SUMS ON $GL(n)$

By Stephen D. Miller

Abstract. In a previous paper with Schmid we considered the regularity of automorphic distributions for $GL(2, \mathbb{R})$, and its connections to other topics in number theory and analysis. In this paper we turn to the higher rank setting, establishing the nontrivial bound

$$\sum_{n \leq T} a_n e^{2\pi i n \alpha} = O(T^{3/4+\varepsilon}),$$

uniformly in $\alpha \in \mathbb{R}$, for $a_n$ the coefficients of the $L$-function of a cusp form on $GL(3, \mathbb{Z}) \setminus GL(3, \mathbb{R})$. We also derive an equivalence (Theorem 7.1) between analogous cancellation statements for cusp forms on $GL(n, \mathbb{R})$, and the sizes of certain period integrals. These in turn imply estimates for the second moment of cusp form $L$-functions.

1. Introduction. Consider a sequence $a_n$ of arithmetic quantities of order 1, and the sums of their twists by additive characters

$$S(T, \alpha) = \sum_{n=1}^{T} a_n e^{2\pi i n \alpha}, \quad \alpha \in \mathbb{R}.$$ (1.1)

In this paper we shall be concerned with obtaining estimates for $S(T, \alpha)$ which are uniform in $\alpha$. This problem was considered already by Hardy and Littlewood in 1914 [15], and is well understood when the $a_n$ are the normalized Fourier coefficients of a modular or Maass form on the upper half plane, i.e., automorphic forms on $GL(2, \mathbb{R})$ (see [29]). In the case of cusp forms (which is simpler to state), one has the estimate

$$S(T, \alpha) = O_{\varepsilon}(T^{1/2+\varepsilon}), \quad \text{for any } \varepsilon > 0,$$ (1.2)

uniformly in $\alpha$. This can be seen in a variety of ways, perhaps most naturally in terms of the boundedness of cusp forms (e.g. (1.7)). The exponent of $T^{1/2}$ is best-possible, as can be seen by estimating the $L^2$-norm of the trigonometric polynomial $S(T, \alpha)$

$$\int_{0}^{1} |S(T, \alpha)|^2 \, d\alpha = \sum_{n \leq T} |a_n|^2,$$ (1.3)
which should be of order $T$ if the $a_n$ are of order 1. See, for example, [5], [6], [10], [14], [15], [29], [30], [37] for background on techniques used to bound $S(T, \alpha)$. A folklore conjecture asserts that the estimate (1.2) holds for the Fourier coefficients of any cusp form, on any group. The purpose of this paper is to provide a nontrivial, uniform estimate for $S(T, \alpha)$ beyond the classical case of $GL(2, \mathbb{R})$ (Theorem 1.1 below).

Such sums have long been connected to important questions in analytic number theory. Most notably, Titchmarsh’s method [36, p. 165] derives from (1.2) the correct order of magnitude

$$\int_{-T}^{T} |L(1/2 + it)|^2 \, dt = O(\varepsilon(T^{1+\varepsilon})), \quad \varepsilon > 0$$

for the second moment of the $L$-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ formed from the cusp form’s coefficients $a_n$. We include a proof of this in Theorem 1.3 below. A slight variant for the coefficients of arbitrary automorphic forms on $GL(m)$, $m \geq 2$ implies estimates for the higher moments of $L(s)$ as well:

$$\int_{-T}^{T} |L(1/2 + it)|^{2k} \, dt = O(\varepsilon(T^{1+\varepsilon})), \quad \varepsilon > 0, \quad \text{for all } k \geq 1.$$

The latter is equivalent to the generalized Lindelöf conjecture in the $t$-aspect, which states that

$$L(1/2 + it) = O(\varepsilon(t^\varepsilon)), \quad \varepsilon > 0.$$

It is widely believed that it is just as difficult to obtain, for example, the correct order of magnitude (1.5) of the second moment of the standard $L$-function of an automorphic form on $GL(3)$, as it is to obtain the correct order of magnitude for the sixth power moment of the Riemann $\zeta$-function. This has long been a major challenge in analytic number theory.

Thus the estimate (1.2), not for modular forms but for automorphic cusp forms of higher rank, is evidently a very difficult one to obtain. It is not surprising that good bounds for $S(T, \alpha)$, $\alpha \in \mathbb{Q}$, can be obtained; a classical result of Landau (see [36, Chapter 12]) gives bounds of the form $O(T^\Theta)$, where $\Theta < 1$. This cancellation is closely related to the analytic continuation and functional equation of the multiplicatively twisted $L$-functions $L_\chi(s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$, where $\chi$ is a Dirichlet character. The main challenge is to also provide estimates when $\alpha$ is irrational, and uniform ones at that.

In this paper we will deduce such bounds by approximating $\alpha$ by rational numbers. As an illustration, consider a holomorphic cusp form $f(z) = \sum_{n \geq 1} a_n n^{(k-1)/2} e(nz)$ of weight $k$ for $SL(2, \mathbb{Z})$. (The $a_n$ are the coefficients of the standard $L$-function of $f$.) Such a form satisfies the uniform bound $|f(z)| = O((\text{Im } z)^{-k/2})$ for all $z = \alpha + i/T$ in the upper half plane, and so in particular for
any $\alpha \in \mathbb{R}$

$$\sum_{n=1}^{\infty} |a_n e(n\alpha)| n^{(k-1)/2} e^{-2\pi n/T} = O(T^{k/2}), \quad \text{as } t \to 0. \quad (1.7)$$

This is essentially a smoothed form of (1.2). The bound for $f(z)$, in turn, comes from the modularity of $f$ under a suitably chosen matrix $\gamma \in SL(2, \mathbb{Z})$ which maps $z$ to a point $\gamma z$ in a fixed fundamental domain, on which $f(z)$ is bounded. Finding such a $\gamma$ is a diophantine problem. In particular, if $\alpha \in \mathbb{Q}$, $\gamma$ can be chosen so that $\gamma z$ is very close to the cusp, where $f(z)$ in fact decays rapidly; this partly explains the remark of the previous paragraph. In order to generalize this argument to non-holomorphic cusp forms such as Maass forms, or to automorphic forms on $GL(3)$, we will use a Voronoi-style summation formula (Section 2) to give bounds on smoothed sums analogous to (1.7).

Our main result (Theorem 1.1 below) is the nontrivial uniform estimate for $S(T, \alpha)$ of $O(\varepsilon T^{3/4+\varepsilon})$ when the $a_n$ are the Fourier coefficients of an automorphic form $\Phi$ on $GL(3, \mathbb{Z}) \setminus GL(3, \mathbb{R})$. (The techniques of this paper and [28] apply to the general congruence subgroup of $GL(3, \mathbb{Z})$, but the coefficients $a_n$ are no longer uniquely determined by the $L$-function data.) These coefficients, which we shall denote $a_{q,n}$, are naturally indexed by two integral parameters, the first of which we will hold fixed and not attempt to measure the dependence of (though it is of course possible to do so). The coefficients $a_{n,q}$ are actually the Fourier coefficients of the form $\tilde{\Phi}$ contragredient to $\Phi$, so there is no loss of generality in fixing the first index instead of the second. Thanks to the Rankin-Selberg theory ([23]; see also [3, §2] and [17, §5]), we know the $a_{q,n}$ obey the Ramanujan conjecture on average. More precisely, the Rankin-Selberg $L$-function

$$L(s, \Phi \otimes \tilde{\Phi}) = \sum_{n,q \geq 1} |a_{q,n}|^2 (nq^2)^{-s},$$

initially convergent for $\Re s$ large, has a meromorphic continuation to $\mathbb{C}$ with only a simple pole at $s = 1$; this translates into the estimates

$$\sum_{nq^2 \leq T} |a_{q,n}|^2 = O(T), \quad (1.8)$$

$$\sum_{n \leq T} |a_{q,n}|^2 = O(q^2 T),$$

and

$$\sum_{n \leq T} |a_{q,n}| = O(q T) \quad (1.9)$$

by Cauchy-Schwartz. Thus the trivial estimate for $S(T, \alpha)$—obtained by taking
the absolute value of each term in (1.1)—is $O(T)$ for any fixed $q$. Our main result is the following improvement, which goes halfway between the trivial bound and the best-possible bound of $O(T^{1/2})$.

**Theorem 1.1.** Let $a_{q,n}$ denote the Fourier coefficients of a cusp form on $\text{GL}(3, \mathbb{Z}) \setminus \text{GL}(3, \mathbb{R})$. Then for any $\varepsilon > 0$

$$
\sum_{n=1}^{T} a_{q,n} e^{2\pi i n \alpha} = O(\varepsilon T^{3/4+\varepsilon}), \quad \text{uniformly in } \alpha \in \mathbb{R},
$$

with the implied constant depending on $q$, $\varepsilon$, and the cusp form.

Through partial summation, this statement implies a bound for the analogous smoothed sums (actually Theorem 1.1 is derived from a similar statement—see (5.18)):

**Corollary 1.2.** Let $a_{q,n}$ denote the Fourier coefficients of a cusp form on $\text{GL}(3, \mathbb{Z}) \setminus \text{GL}(3, \mathbb{R})$, and $\phi$ be any Schwartz function. Then for any $\varepsilon > 0$

$$
\sum_{n \neq 0} a_{q,n} e^{2\pi i n \alpha} \phi \left( \frac{n}{T} \right) = O(\varepsilon T^{3/4+\varepsilon}),
$$

where the implied constant depends on $\varepsilon$, $q$, $\phi$, and the cusp form.

As we mentioned before, the folklore conjecture that $S(T, \alpha) = O(\varepsilon T^{1/2+\varepsilon})$ implies the correct order of magnitude for the second moment (1.4). Weaker estimates on $S(T, \alpha)$ still give cancellation bounds via the classical method of Titchmarsh alluded to above. Though the following theorem appears to be well known to experts, we have been unable to locate a suitable statement in the literature, and so have chosen to include a proof in Section 6.

**Theorem 1.3.** Let $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be the $L$-function of a cusp form on $\text{GL}(m) \setminus \mathbb{Q}$, other than the Riemann $\zeta$-function. Suppose that

$$
S(T, \alpha) = \sum_{n=1}^{T} a_n e(n \alpha) = O(\varepsilon T^{\beta+\varepsilon}), \quad \text{uniformly in } \alpha,
$$

for some $\beta \geq \frac{1}{2}$ and any $\varepsilon > 0$. Then the second moment of $L(s)$ satisfies the bound

$$
\int_{-T}^{T} |L(\frac{1}{2} + it)|^2 dt = O(\varepsilon T^{1+(2\beta-1)m+\varepsilon}), \quad \text{for any } \varepsilon > 0.
$$

Some brief remarks are in order. First, the omission of $\zeta(s)$ is made for a technical reason; besides the fact that the precise asymptotics of the second moment of $\zeta(\frac{1}{2} + it)$ have long been known (see [36, §7]), Theorem 1.3 requires
some modification for $L$-functions which have poles. Such an adjustment can be used to study the $2k$-th moment (1.5), though we shall not pursue this here. Second, we have chosen to state Theorem 1.3 for cusp form $L$-functions on $GL(m)$ over $\mathbb{Q}$ because they and their products are believed to account for the totality of $L$-functions. Titchmarsh’s method could equally be used to derive results for other classes, such as the Selberg class [34]. Thirdly, Theorem 1.3 is of interest only for $\beta \leq \frac{3}{4} - \frac{1}{2m}$ (and hence $m > 2$), because the second moment (1.13) can always be bounded by $O_{\varepsilon}(T^{m/2+\varepsilon})$ using the approximate functional equation (see [4, p. 31]).

Though Theorem 1.1 is the first nontrivial bound for $S(T, \alpha)$ on $GL(m)$, $m > 2$, it still falls far short of improving any estimates on the critical values of a $GL(3)$ $L$-function. Our obstacle to sharpening the estimate of Theorem 1.1 is the appearance of Kloosterman sums in formula (2.4), which we bound in Section 5 only by their absolute value (Weil’s bound). Future improvements would necessarily obtain cancellation in sums of products of the $a_n$ with Kloosterman sums. We are unable to prove any interesting statements for $GL(m)$, $m > 3$, but there obtaining cancellation in sums of $a_n$ times hyper-Kloosterman sums could in principle be used to attack the second moment. Though this appears no easier, it is perhaps of interest that the moment problem is connected to exponential sums in this fashion.

Sections 2–5 of this paper contain the proof of Theorem 1.1; in Section 6 we turn to the proof of Theorem 1.3. Finally, in Section 7 we give an equivalence between bounds on $S(T, \alpha)$ for cusp forms, and the sizes of certain period integrals studied by Jacquet, Piatetski-Shapiro, and Shalika in their construction of the standard $L$-function on $GL(m)$. In particular, the equivalence given by Theorem 7.1b for the optimal case of $S(T, \alpha) = O_{\varepsilon}(T^{1/2+\varepsilon})$ can be viewed as a condition on an individual cusp form which, together with Theorem 1.3, implies the correct order of magnitude for the second moment of its standard $L$-function. (Again, a modification for noncusp form $L$-functions can be used to discuss higher moments and the full Lindelöf conjecture in the $t$-aspect.)

Our interest in this problem originated in joint work with Wilfried Schmid on questions regarding the Hölder regularity of the boundary distributions associated to cusp forms on $GL(3, \mathbb{R})$ (see [29] for a survey on the case of $GL(2, \mathbb{R})$). Theorem 1.1 can be used to give the following nontrivial estimate:

**Corollary 1.4.** Let

$$\tau_{n,q}(x) = \sum_{n \neq 0} c_{n,q} e(nx)$$

denote the abelian Fourier components of the of the boundary value distribution of an automorphic cusp form on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$, where

$$c_{n,q} = a_{n,q} |n|^{-\lambda_1} |q|^\lambda_3 \operatorname{sgn}(n)^{\delta_1} \operatorname{sgn}(q)^{\delta_3}$$
(the $\lambda_j$ and $\delta_j$ are representation-theoretic parameters as in the next section—see [28, §7] for details). Then $\tau_{x,q}$ lies in the Hölder class $C^{< \Re \lambda - 1/2}$. Interestingly, the techniques from partial differential equations and representation theory used in [33]—which obtain an essentially sharp estimate for $GL(2, \mathbb{R})$—seem to only recover a very weak bound for $GL(m, \mathbb{R})$, $m \geq 3$. This is consistent with the expected overall difficulty of (1.4), which is a consequence of $S(T, \alpha) = O_{\varepsilon}(T^{1/2+\varepsilon})$.

Finally, a remark is in order about the coefficients of noncuspidal automorphic forms. For example, the early papers of [6, 10, 15, 37] studied the Fourier coefficients of Eisenstein series on the upper half plane, notably the divisor function $d(n)$. There is an Eisenstein series on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$, whose standard $L$-function is $\zeta(s)^3$, with Fourier coefficients $a_{1,n}$ equal to the triple divisor function

$$d_3(n) := \# \{a b c = n \mid a, b, c \in \mathbb{N}\}. \tag{1.14}$$

The method used in this paper can be extended to study additively twisted sums of $d_3(n)$ as well, although there is of course no nontrivial bound for even $S(T, 0)$ here because $d_3(n) > 0$. One must settle for almost-everywhere bounds, which could not possibly be uniform (or else by continuity they would extend everywhere). Strong nonuniform results, however, can be obtained via Carleson’s theorem on Fourier series (see [30]) by using only the fact that $d_3(n) = O_{\varepsilon}(n^\varepsilon)$, and nothing about automorphy. In any event, for many applications—such as in studying moments—a more useful form of (1.2) would uniformly bound the difference between $S(T, \alpha)$ and a main term.

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2. Voronoi Summation for $GL(3)$. Our main tool is the $GL(3)$ analog of the Voronoi summation formula, recently proven in [28] (see also [26] for a less technical exposition). We will now give a brief summary of the formula, referring the reader to [28] for definitions not fully explained here, and for its connections to the functional equations of twisted $L$-functions. Denote the embedding parameters of a cusp form $\Phi$ on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$ by $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{C}^3$ and $(\delta_1, \delta_2, \delta_3) \in (\mathbb{Z}/2\mathbb{Z})^3$. These are representation-theoretic parameters connected to $\Phi$, and will be related to the functional equation for the $L$-function $L(s, \Phi)$ below in (2.11).
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We may and shall make the normalizing assumption that

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \quad \text{and} \quad \delta_1 + \delta_2 + \delta_3 \equiv 0 \pmod{2} \quad (2.1)$$

The summation formula which we are about to state involves doubly-indexed Fourier coefficients $a_{n,m}$ of a cusp form $\Phi$ on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$. These are perhaps simplest described in terms of the standard and contragredient $L$-functions of $\Phi$,

$$L(s, \Phi) = \sum_{n=1}^{\infty} a_{n,1} n^{-s} \quad \text{and} \quad L(s, \tilde{\Phi}) = \sum_{n=1}^{\infty} a_{n,1} n^{-s}, \quad (2.2)$$

respectively. If $\Phi$ is a Hecke eigenform—an assumption we do not make, yet one which entails no loss of generality—the coefficients $a_{n,m}$ are eigenvalues of the Hecke operators $T_{n,m}$, and accordingly satisfy certain recursion identities (for a full description see [2, §9]). In particular, when $\Phi$ is a Hecke eigenform, the $a_{n,m}$ can be derived from the $a_{n,1}$ and $a_{1,m}$ via the identity

$$a_{n,m} = \sum_{d | (n,m)} \mu(d) a_{n/d,1} a_{1,m/d}, \quad (2.3)$$

where $\mu(d)$ denotes the Möbius $\mu$-function.

The following is the Voronoi-style summation formula for automorphic forms on $GL(3, \mathbb{Z}) \backslash GL(3, \mathbb{R})$.

**Theorem 2.1.** ([28]) Suppose that $a_{n,m}$ are the Fourier coefficients of a cuspidal $GL(3, \mathbb{Z})$-automorphic representation of $GL(3, \mathbb{R})$, with embedding parameters $(\lambda_1, \lambda_2, \lambda_3)$ and $(\delta_1, \delta_2, \delta_3)$ as in (2.1). Let $f$ be a Schwartz function which vanishes to infinite order at the origin, or more generally, a function on $\mathbb{R} \setminus \{0\}$ such that $(\text{sgn} \cdot x)^{\delta_3} |x|^{-\lambda_3} f(x)$ is a Schwartz function. Then for $T > 0$, $(a,c) = 1$, $c \not= 0$, $a \bar{a} \equiv 1 \pmod{c}$, and $q > 0$, 

$$\sum_{n \not= 0} a_{q,n} e(-na/c) f \left( \frac{n}{T} \right) = \sum_{d | cq} \left| \frac{c}{d} \right| \sum_{n \not= 0} a_{n,d} S(q \bar{a}, n; qc/d) F \left( \frac{nd^2}{c^2q^3} T \right), \quad (2.4)$$

where $S(n, m; c) = \sum_{x \in (\mathbb{Z}/c\mathbb{Z})^*} e \left( \frac{nx + mx^2}{c} \right)$ denotes the Kloosterman sum and, in symbolic notation,

$$F(t) = \int_{\mathbb{R}^3} f \left( \frac{x_1 x_2 x_3}{t} \right) \prod_{j=1}^{3} \left( (\text{sgn} x_j)^{\delta_j} |x_j|^{-\lambda_j} e(-x_j) \right) dx_3 dx_2 dx_1. \quad (2.5)$$

This integral expression for $F$ converges when performed as repeated integral in the indicated order—i.e., with $x_3$ first, then $x_2$, then $x_1$—and provided $\text{Re} \lambda_1 >$
Re $\lambda_2 > \text{Re} \lambda_3$; it has meaning for arbitrary values of $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ by analytic continuation. (An alternative description of the relation (2.5) is given in (2.9) below.)

The embedding parameters $(\lambda_1, \lambda_2, \lambda_3)$ obviously play an important role in Theorem 2.1, so it is worthwhile to describe them in more detail. The parameter $\lambda_3$ may always be chosen to have the maximal real part among $\{\lambda_1, \lambda_2, \lambda_3\}$. We can assume

(2.6) $\text{Re} \lambda_1, \text{Re} \lambda_2 \leq \text{Re} \lambda_3$, $\text{Re} \lambda_1, \text{Re} \lambda_2 < \frac{1}{2}$, and $\text{Re} \lambda_3 \geq 0$.

The second inequality requires some explanation. There are essentially two types of representations of $GL(3, \mathbb{R})$ corresponding to cusp forms. The first possibility is that $\Phi$ comes from a fully induced principal series representation, in which case $|\text{Re} \lambda_j| < 1/2$ and the $\lambda_j$’s may be freely permuted; otherwise $\Phi$ is connected to a induced representation of $GL(3, \mathbb{R})$ constructed from the discrete series $D_k$ of $GL(2, \mathbb{R})$ (corresponding to weight $k \geq 2$ modular forms). In this latter situation we may and do chose the $\lambda_j$’s to be written as

(2.7) $\lambda_1 = -2it$, $\lambda_2 = -\frac{k - 1}{2} + it$, $\lambda_3 = -\frac{k - 1}{2} + it$, $t \in \mathbb{R}$,

with $\delta_1 \equiv \delta_2 + \delta_3 \equiv k \pmod{2}$. In either case (2.6) certainly holds. This bound of $1/2$ comes from knowledge of the unitary dual of $GL(3, \mathbb{R})$ and has a generalization to $GL(n, \mathbb{R})$. For automorphic representations one can in fact do better, though this is not necessary for our purposes (see [25, Appendix 2] for the most recent improvements).

A more useful characterization of the relation between the functions $f$ and $F$ in (2.5) is provided by the (signed) Mellin transforms. For this we must split the functions $f$ and $F$ into odd and even parts; the relation (2.5) clearly preserves parity. If a function $g$ has parity $\eta \in \mathbb{Z}/2\mathbb{Z}$ (i.e., $g(-x) = (-1)^\eta g(x)$), then we define the signed Mellin transform of $g$ to be

(2.8) $M_\eta g(s) = \int_{\mathbb{R}} g(x)|x|^{s-1} \text{sgn}(x)^\eta \, dx$

for values of $s$ where the integral is absolutely convergent, and elsewhere by analytic continuation. When both $f$ and $F$ have parity $\eta$, the relation (2.5) can be succinctly described by the formula

(2.9) $M_\eta F(s-1) = G_{\delta_1+\eta}(s-\lambda_1) G_{\delta_2+\eta}(s-\lambda_2) G_{\delta_3+\eta}(s-\lambda_3) M_\eta f(1-s)$
Here the function

\[ G_\delta(s) = (2\pi)^{-s} \Gamma(s) \left[ e \left( \frac{s}{4} \right) + (-1)^\delta e \left( -\frac{s}{4} \right) \right] \]

\[ = \begin{cases} 
2 (2\pi)^{-s} \Gamma(s) \cos(\pi s/2), & \delta \equiv 0 \pmod{2} \\
2i (2\pi)^{-s} \Gamma(s) \sin(\pi s/2), & \delta \equiv 1 \pmod{2}
\end{cases} \tag{2.10} \]

has only simple poles and simple zeroes, at the points \( s \in (2\mathbb{Z} + \delta) \cap \mathbb{Z}_{<0} \) and \( s \in (2\mathbb{Z} + \delta + 1) \cap \mathbb{Z}_{>0} \), respectively. The functional equation relating \( L(s, \Phi) \) and \( L(s, \tilde{\Phi}) \) can be cleanly stated in terms of the \( G_\delta \) as

\[ L(1-s, \tilde{\Phi}) = G_{\delta_1}(s + \lambda_1) G_{\delta_2}(s + \lambda_2) G_{\delta_3}(s + \lambda_3) L(s, \Phi); \tag{2.11} \]

from this it is also possible to relate the \( \lambda_j \) and \( \delta_j \) to the \( \Gamma \)-factors appearing in the usual form of the functional equation (see [28, §6]).

The functions \( G_\delta \) also arise in relating the Mellin and Fourier transforms. Suppose that \( g \) is a Schwartz function of parity \( \eta \); then

\[ \hat{g}(r) = \int_{\mathbb{R}} g(x) e(-xr) \, dx \tag{2.12} \]

is also, and

\[ M_\eta \hat{g}(s) = (-1)^\eta G_\eta(s) M_\eta g(1-s) \tag{2.13} \]

([27, (4.58)]). The Fourier inversion formula is then equivalent to the identity

\[ G_\eta(s) G_\eta(1-s) = (-1)^\eta \tag{2.14} \]

([27, (4.11)]).

We end this section with a remark about the product of \( G_\delta \)'s occurring in (2.9), namely that

\[ G_{\delta_1+\eta}(s - \lambda_1) G_{\delta_2+\eta}(s - \lambda_2) G_{\delta_3+\eta}(s - \lambda_3) \]

is holomorphic for \( \Re s \geq \frac{1}{2} \).

In light of our assumptions and discussion around (2.6), the only possible poles must come from the third factor, and even then only when \( \Re \lambda_3 \geq \frac{1}{2} \), in which case we also assume (2.7). In this case the product of the last two functions in (2.15) in fact equals

\[ i^k (2\pi)^{1-2s-2it} \frac{\Gamma(s + \frac{k-1}{2} + it)}{\Gamma(1 - s + \frac{k-1}{2} - it)} \]

([28, (6.12)]), which has no poles even in the larger region \( \Re (s + \frac{k-1}{2}) > 0 \).
In this section we will describe the test functions \( f \) that will be inserted into (2.4) in order to obtain our eventual results. Our goal now is to collect some estimates on \( F(x) \) for the analysis of the right-hand side of (2.4) in Proposition 5.1. At this stage it is probably helpful to list which of our variables are considered fixed, and which we will make estimates in terms of. The parameters \( q, \lambda_1, \lambda_2, \) and \( \lambda_3 \) are all considered fixed. At times we will need to introduce some finite parameters indexed by \( \sigma, K, M, \) or \( N \), for example to shift contour integrals or integrate by parts; this amount will always be bounded in terms of the fixed parameters \( q \) and \( \{ \lambda_j \} \). The dependence on these latter parameters—ultimately traceable back—will not be explicitly mentioned, though it is possible and obviously cumbersome to do so. The estimates on \( F(x) \) in this section and the next mainly involve the quantities \( x \) and \( Y \) (a non-negative parameter); the most important aspect of the bounds on \( F(x) \) is their dependence in \( Y \) for \( Y \geq 1 \).

In order to use (2.9) it is necessary that \( f \) (and hence \( F \)) be of parity \( \eta \in \mathbb{Z}/2\mathbb{Z} \); we shall accordingly describe choices of \( f \) for both parities. To make the notation uniform and convenient, we will from now on regard the parameter \( \delta_3 \) as an element of \( \{0, 1\} \), not just of \( \mathbb{Z}/2\mathbb{Z} \). Let \( \omega \in \mathbb{Z}/2\mathbb{Z} \) be an arbitrary parity parameter, and fix a smooth function \( \phi_0 \) of parity \( \eta + \omega \in \mathbb{Z}/2\mathbb{Z} \) with support in the interval \((-1, 1)\). From \( \phi_0 \) we will define a number of auxiliary functions in terms of the non-negative parameter \( Y \). First will be

\[
\phi(x) = \begin{cases} 
\phi_0(x - Y) + (-1)^\omega \phi_0(x + Y), & \delta_3 \equiv 0 \pmod{2} \\
\frac{1}{2\pi i} \left[ \phi_0'(x - Y) + (-1)^\omega \phi_0'(x + Y) \right], & \delta_3 \equiv 1 \pmod{2}.
\end{cases}
\]

We let \( f(x) = |x|^\lambda_3 \text{sgn}(x)^{\delta_3} \hat{\phi}(x) \), so that

\[
f(x) = \begin{cases} 
2 |x|^{\lambda_3 + \delta_3} \cos(2\pi Yx) \hat{\phi}_0(x), & \omega \equiv 0 \pmod{2} \\
-2i |x|^{\lambda_3 + \delta_3} \sin(2\pi Yx) \hat{\phi}_0(x), & \omega \equiv 1 \pmod{2}.
\end{cases}
\]

Clearly \( \phi \) and \( \hat{\phi} \) are Schwartz functions, so \( f \) is admissible in (2.4) and has parity \( \eta \). We have now

\[
M_\eta f(s) = M_{\delta_3 + \eta} \hat{\phi}(s + \lambda_3) = (-1)^{\delta_3 + \eta} G_{\delta_3 + \eta}(s + \lambda_3) M_{\delta_3 + \eta} \hat{\phi}(1 - s - \lambda_3)
\]

by (2.13), and

\[
M_\eta F(s - 1) = G_{\delta_1 + \eta}(s - \lambda_1) G_{\delta_2 + \eta}(s - \lambda_2) M_{\delta_3 + \eta} \hat{\phi}(s - \lambda_3),
\]

by (2.9) and (2.14). This last expression is holomorphic in \( \text{Re} \ s \geq \frac{1}{2} \) by (2.15), because the signed Mellin transform \( M_{\delta_3 + \eta} \hat{\phi}(s) \) of a Schwartz function can only have poles where \( G_{\delta_3 + \eta}(s) \) does (see [27, (3.31)]). Moreover, \( M_\eta F(s) \) decays rapidly in vertical strips. We may therefore calculate \( F(x) \) using the Mellin inversion
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formula, e.g.,

\[(3.5)\]
\[
\frac{4\pi i F(x)}{|x| \text{ sgn } (x)^n} = \int_{\text{Re } s = \sigma \geq \frac{1}{2}} M_{\eta} F(s - 1) |x|^{-s} ds \\
= \int_{\text{Re } s = \sigma \geq \frac{1}{2}} G_{\delta_1 + \eta}(s - \lambda_1) G_{\delta_2 + \eta}(s - \lambda_2) M_{\delta_3 + \eta} \phi(s - \lambda_3) |x|^{-s} ds,
\]
or

\[(3.6)\]
\[
\frac{4\pi i F(xY)}{|xY| \text{ sgn } (x)^n} Y^{\lambda_3} = \int_{\text{Re } s = \sigma \geq \frac{1}{2}} G_{\delta_1 + \eta}(s - \lambda_1) G_{\delta_2 + \eta}(s - \lambda_2) M_{\delta_3 + \eta} \phi_1(s - \lambda_3) |x|^{-s} ds.
\]

In this last expression, which will be useful for $Y$ large, we have introduced the function $\phi_1(x) = \phi(Yx)$, which has $M_{\delta_3 + \eta} \phi(s) = Y^s M_{\delta_3 + \eta} \phi_1(s)$.

**Lemma 3.1.** For $Y \leq 1$ we have the uniform estimate

\[(3.7)\]
\[
F(x) \ll |x|^{-N} \text{ for any real number } N \geq -\frac{1}{2},
\]

where the implied constant depends continuously on $N$.

**Proof.** We have just remarked above that the integrand in (3.5) is holomorphic for $\text{Re } s = \sigma \geq \frac{1}{2}$. For $\sigma$ in this range

\[(3.8)\]
\[
F(x) \ll |x|^{-\sigma} \int_{\mathbb{R}} |G_{\delta_1 + \eta}(\sigma + it - \lambda_1) G_{\delta_2 + \eta}(\sigma + it - \lambda_2) M_{\delta_3 + \eta} \phi(\sigma + it - \lambda_3)| dt.
\]

The lemma will follow with $\sigma = N + 1$ once we show the integral in (3.8) is bounded independently of $Y \leq 1$. To estimate the function $G$ along vertical lines, we use the asymptotic

\[(3.9)\]
\[
|G(\sigma + it)| \sim \left( \frac{\sigma}{2\pi} \right)^{\sigma - 1/2}, \quad t \to \infty,
\]

which is a direct consequence of Stirling’s formula applied to definition (2.10). Bounds on $M_{\delta_3 + \eta} \phi(\sigma + it)$ can be obtained from

\[
M_{\delta_3 + \eta} \phi(\sigma + it) = \int_{\mathbb{R}} \phi(x) |x|^{|\sigma + it - 1} \text{ sgn } (x)^{\delta_3 + \eta} dx \leq \int_{\mathbb{R}} |\phi(x)| |x|^{|\sigma - 1} dx.
\]
Because $\phi$ is supported in $(-2, 2)$ and is bounded independently of $Y \leq 1$, this last integral is uniformly bounded in $Y \leq 1$ with a continuous dependence on $\sigma \geq \frac{1}{2}$. The same holds true when $\phi$ is replaced by any of its derivatives $\phi^{(K)}$, so $M_{\delta_{3}+\eta+K}^{(K)}(\sigma + it + K)$ is uniformly bounded in $t$ and $Y$ for $Y \leq 1$. Integration by parts $K$ times then shows

$$M_{\delta_{3}+\eta+K}^{(K)}(\sigma + it + K) \ll |t|^{-K},$$

again uniformly for $Y \leq 1$. Consequently the integral in (3.8) converges rapidly and is bounded independently of $Y \leq 1$, with a continuous dependence on $\sigma$.

The situation for $Y \geq 1$ is more complicated. A helpful difference is that $\phi$ vanishes in a neighborhood of the origin when $Y \geq 1$, making its Mellin transform entire. We first state a lemma about the Mellin transform’s dependence on $Y$:

**Lemma 3.2.** For any real numbers $Y \geq 1$, $N \geq 0$, and $\sigma$,

$$M_{\delta_{3}+\eta+K}^{\phi}(\sigma + it - \lambda_{3}) \ll |t|^{-N} M_{\delta_{3}+\eta+K}^{\phi}(\sigma + it + N) \ll \frac{1}{Y} \left( \frac{Y}{|t|} \right)^{N}.\tag{3.10}$$

Here the implied constants are independent of $Y$ and $t$, and depend continuously on $\sigma$ and $N$.

**Proof.** The first inequality comes directly from integration by parts. We shall prove the second when $N \geq 0$ is an integer; it extends to reals by interpolation. We have

$$M_{\delta_{3}+\eta+K}^{\phi}(\sigma + it) = 2 \int_{0}^{\infty} \phi_{1}^{(N)}(x) x^{\sigma + it} dx \ll \frac{1}{Y} \int_{0}^{\infty} \phi_{0}^{(N+\delta_{3})}(Y(x - 1)) |x|^{|\sigma-1|} dx.\tag{3.11}$$

The derivatives of $\phi_{0}$ are bounded by an absolute constant, and furthermore the integrand above is supported in the interval $(1 - c Y^{-1}, 1 + c Y^{-1})$ for some absolute constant $0 < c < 1$. So (3.11) is bounded by $O\left( Y^{N} \int_{1-c}^{1+c} Y^{-1} x^{\sigma-1} dx \right) = O(Y^{N-1})$ for $Y \geq 1$, where the implied constant depends continuously on $\sigma$. This establishes the second inequality in (3.10).

The arguments used earlier to bound $F(x)$ when $Y \leq 1$ generalize to the case $Y \geq 1$ as well, but with an inadequate $Y$-dependence. To improve upon Lemma 3.1 for $|x| \leq Y$, we shift the contour further to the left and estimate the contribution from the poles, rather than merely a contour integral positioned just
to their right. (The estimates for $|x| \geq Y \geq 1$ will be given in Proposition 4.2 at the end of the next section.)

LEMMA 3.3. For $Y \geq 1$ and $x \leq Y$ we have the bound

$$F(x) \ll |x|^{1/2} Y^{-1/2 - \Re \lambda_3}. \tag{3.12}$$

Proof. Suppose momentarily that the poles of $G_{\delta_1 + \eta}(s - \lambda_1)$ and $G_{\delta_2 + \eta}(s - \lambda_2)$ do not overlap. Shifting the contour in (3.6) to sufficiently negative and avoiding the poles, we obtain the expression

$$4 \pi i \frac{F(x Y)}{|x| \sgn (x)^{\eta}} Y^{\lambda_3 - 1} = \sum_{j=1,2} \sum_{0 \leq k < \Re \lambda_j - \sigma} c_{k,j} |x|^{k - \lambda_j} M_{\delta_j + \eta} \phi_1 (\lambda_j - \lambda_3 - k) + R, \tag{3.13}$$

where

$$R \ll |x|^{-\sigma} \int_{\Re s = \sigma \leq 0} |G_{\delta_1 + \eta}(s + it - \lambda_1) G_{\delta_2 + \eta}(s + it - \lambda_2) M_{\delta_3 + \eta} \phi_1 (s + it - \lambda_3)| \, dt \ll \frac{|x|^{-\sigma}}{Y}. \tag{3.14}$$

Here we have used the fact $M_{\delta_j + \eta} \phi_1 (s)$ is entire, (2.1), (3.9), and (3.10) with $N = 0$; also the implied constants in (3.14) depend continuously on $\sigma$. Finally the $c_{k,j}$ are constants coming from the residues of $G_{\delta_1 + \eta}(s - \lambda_1) G_{\delta_2 + \eta}(s - \lambda_2)$ at the points $s = \lambda_j - k$. Another application of (3.10) bounds the $M_{\delta_1 + \eta} \phi_1$ factor in the sum on the right-hand side of (3.13) by $O(1/Y)$ as well.

If the poles of $G_{\delta_1 + \eta}(s - \lambda_1)$ and $G_{\delta_2 + \eta}(s - \lambda_2)$ in fact do overlap, then (3.13) remains correct provided an additional factor of $(\log |x|)^{j-1}$ is included. Bounding the right-hand side of (3.13) therefore gives the estimate

$$F(x Y) Y^{\lambda_3} \ll \varepsilon |x|^{1 - \Re \lambda_1 - \varepsilon} + |x|^{1 - \Re \lambda_2 - \varepsilon}, \quad |x| \leq 1 \tag{3.15}$$

for any $\varepsilon > 0$. These exponents are both greater than $\frac{1}{2}$ for $\varepsilon$ small, thanks to (2.6), and so the right-hand side of (3.15) is bounded by $O(|x|^{1/2})$, for $x \leq 1$, proving (3.12).

4. A substitute for stationary phase. To bound $F(x)$ from the integral (2.5) one could attempt to use stationary phase. We instead find it more convenient to apply a device of [4, p. 33] to the Mellin transform $MF$ of $F$ instead. This allows us to express the transformed $F$ in terms of an asymptotic series of Fourier
transforms, which in practice are often simpler to estimate. Let us use the notation $f(s) \approx g(s)$ if, for any integer $M \geq 0, f(s)/g(s)$ has an asymptotic expansion of the form $1+c_1 s^{-1} + \cdots + c_M s^{-M} + O(|s|^{-M-1})$ for large values of $s$ in any vertical strip of finite width. In this notation Stirling’s formula reads

$$\Gamma(s) \approx \sqrt{2\pi} e^{-s} s^{s-1/2}.$$  

Consequently,

$$\frac{\Gamma(s + a)}{\Gamma(s + b)} \approx s^{a-b}. \hspace{2cm} (4.1)$$

Though we only need a special case, we will state the following lemma in enough generality that it can be applied to arbitrary $L$-functions. Indeed, the ratio of $\Gamma$-factors in the functional equation of any $L$-function can always be written in the form of the left-hand side of (4.2) below (see [28, §6], for example). The method here can often be applied instead of stationary phase on $\mathbb{R}^n$ to give asymptotic expansions of the transformed functions in general Voronoi-style summation formulas (e.g. [26]) and approximate functional equations, in terms of the ordinary, one-variable Fourier transform.

**Lemma 4.1.** For any $(\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ with mean $\bar{\mu}$, and $(\varepsilon_1, \ldots, \varepsilon_n) \in (\mathbb{Z}/2\mathbb{Z})^n$, we have

$$\prod_{j=1}^n G_{\varepsilon_j}(s - \mu_j) \approx \sum_{\gamma=0}^1 C_\gamma n^{-n s} G_\gamma \left( ns - n \bar{\mu} + \frac{1-n}{2} \right) \hspace{2cm} (4.2)$$

for explicitly computable constants $C_0$ and $C_1$.

**Proof.** To simplify the notation, add $\bar{\mu}$ to $s$ and denote the sum $\sum_{j=1}^n \varepsilon_j$ as simply $\varepsilon$. We can thereby assume that $\bar{\mu} = 0$. The left-hand side of (4.2) can be expressed using (2.10) as

$$\prod_{j=1}^n (2\pi)^{-s-\mu_j} \Gamma(s - \mu_j) \left[ e \left( \frac{s - \mu_j}{4} \right) + (-1)^{\varepsilon_j} e \left( \frac{\mu_j - s}{4} \right) \right] \approx (2\pi)^{-n s} \left[ e \left( \frac{ns}{4} \right) + (-1)^{\varepsilon} e \left( -\frac{ns}{4} \right) \right] \prod_{j=1}^n \Gamma(s - \mu_j). \hspace{2cm} (4.3)$$

We now use (4.1) and the identity

$$\prod_{j=0}^{n-1} \Gamma \left( \frac{s + j}{n} \right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - ns} \Gamma(ns) \hspace{2cm} (4.4)$$
to rewrite (4.3) as

\[ (4.5) \]
\[
(2\pi)^{-ns} \left[ e^{\left(\frac{ns}{4}\right)} + (-1)^{\varepsilon} e^{\left(-\frac{ns}{4}\right)} \right] \prod_{j=1}^{n} \Gamma\left(s + \frac{j-1}{n}\right) s^{-\mu_{j} - (j-1)/n} \\
\approx (2\pi)^{-ns} \left[ e^{\left(\frac{ns}{4}\right)} + (-1)^{\varepsilon} e^{\left(-\frac{ns}{4}\right)} \right] s^{(1-n)/2} n^{1/2-ns} \Gamma(ns) \\
\approx (2\pi)^{-ns} \left[ e^{\left(\frac{ns}{4}\right)} + (-1)^{\varepsilon} e^{\left(-\frac{ns}{4}\right)} \right] n^{n/2-ns} \Gamma\left( n s + \frac{1-n}{2} \right). 
\]

It is clear from (2.10) that \( G_{0}(s) \pm G_{1}(s) \) span all linear combinations of \((2\pi)^{-s} \Gamma(s) e^{(\pm \frac{1}{2})}\), so this last expression is indeed a linear combination of \( n^{-ns} G_{\gamma}(ns + \frac{1-n}{2}) \), \( \gamma = 0, 1 \), as (4.2) asserts.

We next remark that the same argument used in the last step of (4.5) shows that

\[ (4.6) \]
\[
\frac{G_{\delta}(s+1)}{s} = (2\pi)^{-s-1} \Gamma(s) \left[ e^{\left(s+\frac{1}{4}\right)} + (-1)^{\varepsilon} e^{\left(-s+\frac{1}{4}\right)} \right] \]

is also a linear combination of \( G_{0}(s) \) and \( G_{1}(s) \). That means that the higher terms in the asymptotic expansion in (4.2) can also be written in terms of linear combinations of the \( G_{\delta} \)'s, with shifted arguments. We shall now apply this specifically to the product of \( G_{\delta} \)'s in (3.4):

\[ (4.7) \]
\[
G_{\delta_{1}+\eta}(s - \lambda_{1}) G_{\delta_{2}+\eta}(s - \lambda_{2}) \approx \sum_{\gamma=0}^{1} C_{\gamma} 2^{-2s} \Gamma_{\gamma}(2s - \lambda_{1} - \lambda_{2} - \frac{1}{2}), 
\]

and return to bounding \( F(x) \) in the regime \(|x| \geq Y \geq 1\). We can use (2.1) and (4.7) to restate (3.4) as

\[ (4.8) M_{\eta}F(s-1) = \sum_{\gamma=0}^{1} \left( \sum_{j=0}^{M-1} C_{\gamma,j} 2^{-2s} \Gamma_{\gamma}\left(2s + \lambda_{3} - \frac{1}{2} - j\right) \right) + O \left( 2^{-2s} \Gamma_{\gamma}\left(2s + \lambda_{3} - \frac{1}{2} - M\right) \right) M_{\delta_{1}+\eta} \phi(s - \lambda_{3}). 
\]

The error term represented by the \( O \)-notation here comes from the asymptotic expansion; the implied constants of course depend only on \( \lambda_{1} \) and \( \lambda_{2} \), which we consider fixed. We will take \( M \) to be a large positive integer, and evaluate \( F(x) \) using (4.8) in the contour integral representation (3.6) along \( \text{Re } s = \sigma \), where \( 2\sigma + \text{Re } \lambda_{3} - M = \varepsilon \), an arbitrarily small positive real number. Recall the remark at the beginning of Section 3 that \( M \) will be bounded in terms of \( q \) and the \( \{ \lambda_{j} \} \).
Changing the value of the constants $C_{\gamma,j}$, we may write

\[ Y^{\lambda_3} \text{sgn}(x)^\eta \frac{F(xY)}{|x|} Y \]

\[ = \sum_{\gamma=0}^{1} \sum_{j=0}^{M-1} C_{\gamma,j} \int_{\text{Re } s = \sigma} |4x|^{-s} G_{\gamma}(2s + \lambda_3 - \frac{1}{2} - j) M_{\delta_3 + \eta} \phi_1(s - \lambda_3) \, ds \]

\[ + O \left( |4x|^{-\sigma} \int_{\text{Re } s = \sigma} |G_{\gamma}(2\sigma + 2it + \lambda_3 - \frac{1}{2} - M) M_{\delta_3 + \eta} \phi_1(\sigma + it - \lambda_3)| \, dt \right) \]

The sum of these error terms, which we denote $R$, can be bounded by (3.9) and Lemma 3.2 as

\[ R \ll |x|^{-\sigma} \left( \int_0^1 dt + \int_1^t \frac{Y^2}{Y} \, dt + \int_{t+}^\infty \frac{Y^{N-1} \sigma \text{Re } \lambda_3 - M - 1 - N \, dt }{Y^{N-1} \sigma \text{Re } \lambda_3 - M - 1} \right) \]

\[ = O \left( |x|^{-\sigma} \right) = O \left( \frac{x}{Y^2} \right) \]

for any large $N \geq 0$ (recall $2\sigma + \text{Re } \lambda_3 - M = \varepsilon > 0$). Changing variables in (4.9) gives an expression—again with different constants $C_{\gamma,j}$—of the form

\[ Y^{\lambda_3 - 1} \text{sgn}(x)^\eta \frac{F(xY)}{|x|} - R = \sum_{\gamma=0}^{1} \sum_{j=0}^{M-1} C_{\gamma,j} \left| x \right|^{(\lambda_3 - j - 1)/2} \tilde{\psi}_{\gamma,j} \left( 2|x|^{1/2} \right), \]

where

\[ M_{\gamma} \psi_{\gamma,j}(s) = M_{\delta_3 + \eta} \phi_1 \left( \frac{1 - s - 3\lambda_3 + \frac{1}{2} + j}{2} \right), \]

or in other words,

\[ \psi_{\gamma,j}(x) = 2 \text{sgn}(x)^{\delta_3 + \eta + \gamma} \left| x \right|^{3\lambda_3 - j - 2 - \frac{3}{2}} \phi_1 \left( \frac{1}{x^2} \right). \]

Recall from (3.1) that each $\psi_{\gamma,j}$ is smooth, and supported in a neighborhoods of width $O(Y^{-1})$ about $\pm 1$. In addition, a straightforward calculation writing $\psi_{\gamma,j}$ in the form $\text{sgn}(x)^{K|x|^A} \phi(\frac{2}{x})$ shows that the $K$-th derivative of $\psi_{\gamma,j}$ is bounded by $O(Y^K)$. We conclude that

\[ \hat{\psi}(r) \ll r^{-K} \|\psi^{(K)}\|_1 \ll r^{-K} Y^{K-1} \]

for each of our functions $\psi = \psi_{\gamma,j}$, and any integer $K \geq 0$. 

Inserting (4.10) and (4.12) into (4.11), we obtain the following bound for $F(x)$ when $|x| \geq 1$:

\begin{equation}
\frac{Y^{\lambda_3-1}}{|x|} F(xY) \ll Y^{K-1} |x|^{-K + \frac{\Re \lambda_3}{2} - \frac{1}{2}} + \left| \frac{x}{Y^2} \right|^{-\sigma} Y^{\Re \lambda_3 - M - 1},
\end{equation}

\begin{equation}
F(xY) \ll Y^{K - \Re \lambda_3} |x|^{\frac{3}{2} - \frac{K + \Re \lambda_3}{2}} + Y^{2\sigma - M} |x|^{1 - \sigma},
\end{equation}

or

\begin{equation}
F(x) \ll Y^{3K - \frac{3}{2} - \frac{3\Re \lambda_3}{2}} |x|^{\frac{3}{2} - \frac{K + \Re \lambda_3}{2}} + Y^{3\sigma - M - 1} |x|^{1 - \sigma}.
\end{equation}

Recalling our choice that $2\sigma + \Re \lambda_3 - M = \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, we may deduce our final estimates on $F(x)$:

**Proposition 4.2.** Let $Y \geq 1$.

(a) If $Y \leq |x| \leq Y^3$ then for any $\varepsilon > 0$

\begin{equation}
F(x) \ll_{\varepsilon} Y^{\varepsilon - \Re \lambda_3} \left| \frac{x}{Y} \right|^{\frac{3}{2} - \frac{K + \Re \lambda_3}{2}}.
\end{equation}

(b) If $|x| \geq Y^3$ then for any $N > 0$

\begin{equation}
F(x) \ll_{N} Y^{3/2} \left| \frac{x}{Y^3} \right|^{-N}.
\end{equation}

**Proof.** These both follow directly from (4.15). For part (a), set $K = 0$ to handle the first term, and note that $\sigma$ is large in the second. To settle part (b) it suffices to prove the prove (4.17) for $N$ large, which is straightforward because $\sigma$, $M$, and $K$ may be taken to be large.

**Remark.** The method used here can be used to obtain more precise information about the asymptotic behavior of $F(x)$. In particular, the fact that $\psi_{\gamma,j}(x)$ is concentrated near $x = \pm 1$ allows one to understand the oscillatory behavior of $F(x)$ as well (see [5], where this is explored in much more detail for summation formulas connected to Dedekind zeta functions).

**5. Proof of Theorem 1.1.** In this section we prove Theorem 1.1 by inserting our choice of $f$ into the summation formula (2.4). First we need to specify some of the parameters in that formula. Let $T \geq 1$, $a = -p_k$, and $c = q_k$, where $p_j/q_j$ are the continued fraction approximants to $\alpha$, and $k$ is chosen such that

\begin{equation}
q_k^2 \leq T \leq q_{k+1}^2.
\end{equation}
We set \( Y = T |\alpha + \frac{a}{c}| \geq 0 \) so that \( \alpha = \pm \frac{Y}{T} - \frac{a}{c} \), and hence

\[
\frac{1}{q_k q_{k+1}} \ll \frac{Y}{T} = \left| \frac{\alpha - p_k}{q_k} \right| \ll \frac{1}{q_k q_{k+1}}
\]

by the standard properties of continued fractions (see, for example, [1, p. 47]).

The following proposition applies our bounds on \( F \) to the right-hand side of (2.4) in Theorem 2.1; afterwards we will conclude Theorem 1.1 by a standard analysis of the left-hand side.

**Proposition 5.1.** With the choice of \( f \) given in (3.1-3.2), the right-hand side of (2.4) is \( O_\varepsilon (T^{3/4+\varepsilon}) \), independent of \( \alpha \).

**Proof.** First, note that the GCD of the parameters of the Kloosterman sum in (2.4) is bounded by \( \gcd(q\bar{a},qc/d) \leq q \), which we consider to be fixed. This Kloosterman sum is therefore bounded by \( O_\varepsilon ((qc/d)^{1/2+\varepsilon}) = O_\varepsilon ((c/d)^{1/2+\varepsilon}) \), for any \( \varepsilon > 0 \), according to Weil’s bound (the implied constant here of course depends on \( \varepsilon \)). That means the right-hand side of (2.4) is bounded by

\[
\sum_{d|c} \left| \frac{c}{d} \right|^{3/2+\varepsilon} \sum_{n \neq 0, |n| \leq X} \left| \frac{a_{n,d}}{|n|} \right| \left| F \left( \frac{nd^2}{c^3q} T \right) \right|.
\]

Now we will use our bounds on \( F \) to bound this expression. First, let us settle the case of \( Y \leq 1 \), which is simpler. Here we break up (5.3) as the sum of

\[
\sum_{d|c} \left| \frac{c}{d} \right|^{3/2+\varepsilon} \sum_{n \neq 0, |n| \leq X} \left| \frac{a_{n,d}}{|n|} \right| \left| F \left( \frac{n}{X} \right) \right|
\]

and

\[
\sum_{d|c} \left| \frac{c}{d} \right|^{3/2+\varepsilon} \sum_{|n| > X} \left| \frac{a_{n,d}}{|n|} \right| \left| F \left( \frac{n}{X} \right) \right|,
\]

where for convenience we have set \( X = \frac{c^3q}{d^2T} \). We will use the bound of \( F(x) \ll x^{-N} \) from Lemma 3.1 here: for (5.4) we take \( N = 0 \), and for (5.5) we take \( N \) arbitrarily large. Then the sums over \( n \) are therefore bounded by

\[
\sum_{0 \neq |n| \leq X} |a_{n,d}| |n|^{-1} \text{ and } \sum_{|n| > X} |a_{n,d}| |X|^N |n|^{-1-N},
\]

respectively, with the remaining value of \( N \) a large positive integer. Let us assume
momentarily that \( X \geq 1 \). The partial summation identity

\[
\sum_{n=1}^{K} a_n b_n = \sum_{n=1}^{K-1} A_n (b_n - b_{n+1}) + A_K b_K, \quad A_n = \sum_{k=1}^{n} a_n
\]

(5.7)

can now be used to replace the coefficients \(|a_{n,d}|\) in (5.6) by their average size of \( O(d) \) from (1.9). One then bounds the two sums by \( O(d \log (X + 1)) = O_\varepsilon (d X^\varepsilon) \) and

\[
\ll d \int_{X}^{\infty} \frac{dn}{n} \left( \frac{n}{X} \right)^{-N} = d \int_{1}^{\infty} n^{-N-1} dn = O(d),
\]

(5.8)

respectively. Thus the right-hand side of (2.4) is

\[
\ll \varepsilon |c|^{3/2+\varepsilon} \sum_{d \mid c} d^{-1/2-\varepsilon} X^\varepsilon \ll \varepsilon |c|^{3/2+\varepsilon} X^\varepsilon = O_\varepsilon (T^{3/4+\varepsilon})
\]

(5.9)

by (5.1), and the fact that \( \# \{d \mid c\} = O_\varepsilon (c^\varepsilon) \). This has been done subject to the assumption that \( X \geq 1 \), but actually the argument simplifies if \( X < 1 \) because the first sum in (5.6) has no terms and (5.8) is taken over a shorter range.

Now we turn to the case where \( Y \geq 1 \), which is more involved. We now break the sum over \( n \) in (5.3) into three ranges:

\[
\sum_{n \neq 0} |a_{n,d}| \left| F \left( \frac{n}{X} \right) \right|
\]

(5.10)

\[
\leq \sum_{\substack{0 \neq |n| < XY \ 0 \neq |n| = XY \ 0 \neq |n| > XY^3}} |a_{n,d}| \left| F \left( \frac{n}{X} \right) \right|
\]

(5.11)

\[
+ \sum_{XY \leq |n| \leq XY^3} |a_{n,d}| \left| F \left( \frac{n}{X} \right) \right|
\]

(5.12)

\[
+ \sum_{|n| > XY^3} |a_{n,d}| \left| F \left( \frac{n}{X} \right) \right|
\]

(5.13)

with again \( X = \frac{c^3 q}{2T} \). We will again make the assumption that \( XY \geq 1 \); otherwise the analysis is simpler as it was just above in the argument for \( Y \leq 1 \). For (5.11) we use the estimate \( F(n/X) \ll Y^{-\text{Re}\lambda_3} \) from Lemma 3.3. After again using partial summation to replace \(|a_{n,d}|\) by \( O(d)\), this results in the bound

\[
\ll \varepsilon d Y^{-\text{Re}\lambda_3} \sum_{|n| \leq XY} \frac{1}{|n|} \ll d Y^{-\text{Re}\lambda_3} \log (XY + 1) \ll \varepsilon d (XY)^\varepsilon.
\]

(5.14)

for (5.11) (recall \( \text{Re}\lambda_3 \geq 0 \)).
For the remaining pieces (5.12–5.13), we turn to Proposition 4.2. The bound (4.16) allows us to bound (5.12) by

\[ \ll \varepsilon X^{-D} Y^{e-\Re \lambda_3-\Re \lambda_3} \sum_{|n| \leq XY^3} |a_{n,d}| \frac{1}{n-D} \varepsilon, \]

where \( D = \frac{3}{4} + \Re \lambda_3 > 0 \). We then estimate (5.15) again by partial summation, and find it is

\[ \ll \varepsilon dX^{-D} Y^{e-\Re \lambda_3-\Re \lambda_3} \left( XY^3 \right)^D = dY^{\Re \lambda_3+2D} = dY^{3/2+\varepsilon}. \]

Finally, for (5.13) we use the bound of (4.17), namely

\[ F \left( \frac{n}{X} \right) \ll Y^{3/2} \left| \frac{n}{XY^3} \right| - N \]

for any \( N \) large. Again after removing the \(|a_{n,d}| \) by partial summation, (5.13) is bounded by

\[ \ll dY^{3/2} |XY^3|^N \sum_{n \geq XY^3} |n|^{-N-1} \ll dY^{3/2}. \]

All told (5.10) is \( O(dY^{3/2+\varepsilon}) \), plus the negligible term (5.14), which is \( O(\varepsilon dT^{\varepsilon Y}) \). The final contribution of the sum over \( d \) in (5.3) is again \( O(\varepsilon^c) \) just as immediately after (5.9), so the right-hand side of (2.4) is bounded by

\[ \varepsilon (c^3 q k^{1/2} Y^{3/2+\varepsilon} + q^{3/2+\varepsilon}) = \varepsilon (c^{3/2+\varepsilon} + q^{3/2+\varepsilon}), \]

because of (5.1) and (5.2).

We have just bounded the right-hand side of (2.4). By taking linear combinations of the left-hand side for the functions (3.2) for \( \eta, \omega \in \mathbb{Z}/2\mathbb{Z} \), one obtains the result

\[ \sum_{n \neq 0} a_{n,d} \left| \left( \frac{n}{XY^3} \right)^{\lambda_3+\delta_3} e(na) \phi_0 \left( \frac{n}{T} \right) \right| = O_{\varepsilon} \left( T^{\lambda_3+\delta_3+3/4+\varepsilon} \right), \]

uniformly in \( \alpha \in \mathbb{R} \), for any smooth function \( \phi_0 \) with support in \((-1, 1)\). Here in (5.18) the implied constant depends also on \( \phi_0 \). By rescaling \( \phi_0 \) with the parameter \( T \), (5.18) remains valid for any smooth function \( \phi_0 \) of compact support.

**Lemma 5.2.** For any \( g \in C^1(\mathbb{R}) \), the \( L^1 \) norm of the kernel

\[ D_{k,N}(x) = \sum_{n=1}^{N} e(nx) g \left( \frac{n}{N} \right) \]

is bounded by \( O_{\varepsilon} \left( \frac{1}{N^{1/2+\varepsilon}} \right) \).
taken over $\mathbb{R}/\mathbb{Z}$ satisfies
\[ \|D_{g,N}\|_1 = O(\log N), \]
where the implied constant depends on $g$.

**Proof.** By the partial summation formula (5.7),
\[ D_{g,N}(x) = \sum_{n=1}^{N-1} D_n(x) \left( g \left( \frac{n}{N} \right) - g \left( \frac{n+1}{N} \right) \right) + D_N(x) g \left( \frac{N}{N} \right), \]
where $D_n(x) = \sum_{k \leq n} e(kx) = \frac{e((n+1)x)-e(x)}{e(x)-1}$ is essentially the classical Dirichlet kernel, and has $\|D_n\|_1 = O(\log n)$. The $L^1$ norm of $D_{g,N}$ is thus bounded by
\[ \|D_{g,N}\|_1 \leq \sum_{n=0}^{N-1} \|D_n\|_1 \left| g \left( \frac{n}{N} \right) - g \left( \frac{n+1}{N} \right) \right| + \|D_N\|_1 \left| g \left( \frac{N}{N} \right) \right|. \]

As $g \left( \frac{n}{n} \right) - g \left( \frac{n+1}{N} \right) = O(1/N)$, (5.21) is bounded by $O(N\log N) + O(\log N) = O(\log N)$.

**Proof of Theorem 1.1.** We have shown (5.18) holds for any smooth function $\varphi_0$ with compact support. Choose $\varphi_0$ so that $\hat{\varphi}_0$ is nonzero on $[0, 1]$ (that such a function exists can be seen simply by rescaling). Letting $T = N$ and $g = 1/\hat{\varphi}_0(x)$, convolve the left-hand side of (5.18) over $\mathbb{R}/\mathbb{Z}$ against the kernel $D_{g,N}$ from (5.19). The uniform upper bound in (5.18) and $L^1$-norm estimate from Lemma 5.2 provides us the following estimate:
\[ \sum_{n \leq T} a_{n,q} n^{\lambda_1+\lambda_3} e(n \alpha) = O_{\varepsilon} \left( T^{\lambda_1+\lambda_3+3/4+\varepsilon} \right). \]
The power of $n$ may be then removed using the partial summation formula (5.7), proving Theorem 1.1.

**5. Proof of Theorem 1.3.** In this section we prove Theorem 1.3, which gives a bound on the second moment of an $L$-function based on the amount of cancellation present in additive twists of its coefficients. Our assumption in Theorem 1.3 is that the $L$-function $L(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, assumed to not be the Riemann $\zeta$-function, is the standard $L$-function of a cusp form on $GL(m)$ over $\mathbb{Q}$. We now quickly review their basic analytic properties (see for example [13] and [3], [11], [12], [17], [32]). The coefficients $a_n$ are of polynomial growth and $L(s)$
satisfies a functional equation of the form

\[ L(s) = \omega N^{1/2-s} G(1-s) \tilde{L}(1-s), \tag{6.1} \]

where \( N \geq 1 \) is the “conductor,” \( \omega \) is a complex number of modulus 1, and \( G(s) \) is a ratio of \( \Gamma \)-factors. The latter is customarily written in a variety of styles, though this choice is inessential here; for a later stage in this argument it will be convenient to write

\[ G(s) = \prod_{j=1}^{m} G_{\delta_j}(s + \lambda_j), \tag{6.2} \]

where \( \lambda_j \in \mathbb{C}, \delta_j \in \mathbb{Z}/2\mathbb{Z} \), and \( G_{\delta}(s) \) is the function defined in (2.10). The parameters \( \{(\lambda_j, \delta_j)\} \) can be viewed as principal series embedding parameters for the representation of \( \text{GL}(m, \mathbb{R}) \) associated to the cusp form. In fact they have already made an appearance in Theorem 2.1 (for a fuller discussion of the case \( m = 3 \), see [28]). Finally the dual \( L \)-function in (6.1) is defined by

\[ \tilde{L}(s) = \sum_{n=1}^{\infty} a_n \frac{\text{sgn}(n)}{|n|^{-s}}, \]

and both \( L(s) \) and \( \tilde{L}(s) \) are entire and of finite order, except for the excluded case of \( m = 1 \) and \( L(s) = \zeta(s) \). Furthermore, \( L(s) \) vanishes at certain points where the \( \Gamma \)-factors have poles; all that we will utilize is that there exists an index \( 1 \leq k \leq m \) with \( \text{Re} \lambda_k \leq 0 \) such that \( G_{\delta_k}(s - \lambda_k)L(s) \) is also entire. For shorthand we denote the pair \( (\delta_k, \lambda_k) \) as \( (\delta, \nu) \).

At the heart of the connection between cancellation bounds and the second moment is the classical method of Titchmarsh [36, p. 165], which uses Parseval’s identity for the Mellin transform (our conventions here are carried over from Section 2). Let \( \phi(x) \in C^\infty_c(\mathbb{R}) \) have parity \( \delta \in \mathbb{Z}/2\mathbb{Z} \). The function

\[ f(x) = \sum_{n \neq 0} a_{|n|} \text{sgn}(n)^\delta \hat{\phi}(nx) |nx|^{-\nu} \tag{6.3} \]

also has parity \( \delta \), and its signed Mellin transform is

\[ M_{\delta}f(s) = \sum_{n \neq 0} a_{|n|} \text{sgn}(n)^\delta \int_{\mathbb{R}} \hat{\phi}(nx) |nx|^{-\nu} |x|^{s-1} \text{sgn}(x)^\delta \, dx \]

\[ = 2L(s)M_{\delta} \hat{\phi}(s - \nu) \]

\[ = 2(-1)^\delta L(s) G_{\delta}(s - \nu) M_{\delta}(1 - s + \nu) \quad \text{(cf. (2.13)).} \]

Parseval’s identity

\[ \int_{\mathbb{R}} |f(x)|^2 \, dx = \frac{1}{4\pi} \int_{\mathbb{R}} |M_{\delta}f(\frac{1}{2} + it)|^2 \, dt \tag{6.5} \]

relates the second moment of \( L(\frac{1}{2} + it) \) to the \( L^2 \) norm of \( f \) as follows. Because
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of (3.9), one has

$$\left| M_{\delta f}(\frac{1}{2} + it) \right| = 2 \left| L(\frac{1}{2} + it) \right| \left| G_{\delta}(\frac{1}{2} + it - \nu) \right| \left| M_{\delta \phi}(\frac{1}{2} - it + \nu) \right|$$

$$\sim 2 \left( \frac{\eta}{\pi} \right)^{-\Re \nu} \left| L(\frac{1}{2} + it) \right| \left| M_{\delta \phi}(\frac{1}{2} - it + \nu) \right|.$$ 

We shall now pick $\phi$ more specifically in order to bound the second moment through (6.5). The main idea is to ensure that $M_{\delta \phi}(\frac{1}{2} - it + \nu)$, $t \in \mathbb{R}$, approximates the characteristic function of $[-T, T]$, for $T$ is a large real parameter. Let $\phi_0 \geq 0$ be an even smooth function supported in the interval $(-\frac{1}{2}, \frac{1}{2})$, and let

$$\phi(x) = \phi_0(T(x - 1)) + (-1)^\delta \phi_0(T(x + 1)),$$

so that

$$\hat{\phi}(r) = \left[ e(-r) + (-1)^\delta e(r) \right] T^{-1} \hat{\phi}_0 \left( \frac{r}{T} \right)$$

and

$$f(x) = \sum_{n \neq 0} a_{|n|} \operatorname{sgn}(n)^\delta |nx|^{\nu} [e(-nx) + (-1)^\delta e(nx)] T^{-1} \hat{\phi}_0 \left( \frac{nx}{T} \right).$$

For $T$ large, the function $\phi_0(T(x - 1))$ is concentrated near $x = 1$ and has mass on the order of $T^{-1}$. It is straightforward to choose $\phi_0$ such that $M_{\delta \phi}(\frac{1}{2} - it + \nu)$ is nonzero in the range $t \in [-cT, cT]$ for some $c > 0$. By rescaling $\phi_0$ if necessary, one may further ensure $|M_{\delta \phi}(\frac{1}{2} - it + \nu)| \gg T^{-1}$ for $t \in [-T, T]$; as a result of (6.5),

$$T^{-2} \int_{-T}^{T} \left| |n|^{-\nu} |x|^{-\nu} \left| L(\frac{1}{2} + it) \right|^2 \right| \, dt \ll \int_{|x| \leq T^{1-m-\epsilon}} |f(x)|^2 \, dx$$

$$+ \int_{|x| \geq T^{1+\epsilon}} |f(x)|^2 \, dx$$

$$+ \int_{T^{1-m-\epsilon} < |x| < T^{1+\epsilon}} |f(x)|^2 \, dx.$$ 

Since $\phi$ is supported away from the origin for $T$ large, its Mellin transform is entire. Therefore the last expression for $M_{\delta f}(s)$ in (6.4) is also entire because of our assumption on $\nu$. A standard contour shift and application of Stirling’s formula (or alternatively the asymptotic analysis developed for $F$ in Section 4), produces rapid decay of $f$ near 0 and $\infty$—enough to make the first two terms on the right-hand side of (6.10) decay rapidly in $T$ because of the ranges so chosen. The expression for $f(x)$ in (6.9) is a linear combination of smoothed variants of $|x|^{-\nu} T^{-1} S(T, x)$, except of course for the added presence of the $|n|^{-\nu}$
term. These differences can be removed by partial summation as at the end of the last section, and so our assumption that $S(T, x) = O_s(T^{3+\varepsilon})$ gives the bound $f(x) = O_s(|x|^{-\text{Re } \nu} T^{-1} |L^{\beta-\text{Re } \nu + \varepsilon}|)$. We conclude that the right-hand side of (6.10) is $O_s(T^{-2 \text{Re } \nu - \beta m + 2 m + \varepsilon})$. Recalling that $\text{Re } \nu \leq 0$, this implies that $\int_{-T}^{T} |L(1/2 + it)|^2 dt = O_s(T^{2 \beta m - m + 1 + \varepsilon})$.

7. Period bounds: an analytic analog of the Ramanujan conjecture. In this section we establish an equivalence of the folklore cancellation conjecture (1.1–1.2) for the $L$-function coefficients of a cusp form $\phi$ on $\Gamma_1(N) \backslash GL(m, \mathbb{R})$, where $\Gamma_1(N)$ denotes the subgroup of $GL(m, \mathbb{Z})$ consisting of matrices whose last row equals $[0 \cdots 0 \ 1]$ (mod $N$). The methods and results in this section are not truly particular to the subgroups $\Gamma_1(N)$ themselves, but this family is a very canonical one to study because it captures every cusp form on $GL(m)$: namely, every adelic automorphic representation has a vector which is (left-)invariant under $\Gamma_1(N)$ [21]. So there is essentially no loss of generality entailed by this restriction.

Let us first introduce the period alluded to above, which first originated in the construction of the standard $L$-function on $GL(m)$ by Jacquet, Piatetski-Shapiro, and Shalika [19, 20] (see also [3], [7], [17] as references for this section). Let $P$ denote the standard $(2, 1, 1, \ldots, 1)$ parabolic of $GL(m)$, so that its unipotent subgroup $N$ consists of all upper triangular unit matrices which have zero as their second entry in the first row (blank entries are zero):

\[
P = \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ \vdots & * & * & * & * \\ 1 & 0 & * & * & * \end{pmatrix} \in GL(m) \right\},
\]

\[
N = \left\{ \begin{pmatrix} 1 & 0 & * & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \\ \vdots & * & * & * & * \\ 1 & \end{pmatrix} \in GL(m) \right\}.
\]

Let $\psi$ denote the standard additive character of unit upper triangular matrices, which maps a matrix $n$ to $e^{2\pi i s(n)}$, $s(n)$ being the sum of the entries of $n$ lying one position above the diagonal. Clearly $\psi$ is invariant under $N(\mathbb{Z})$. The period
under consideration is

\begin{equation}
V(g) = \int_{N(\mathbb{Z}) \backslash N(\mathbb{R})} \phi(ng) \overline{\psi(n)} \, dn.
\end{equation}

Of course our notation suppresses the implicit dependence of $V$ on $\phi$.

We can now state the main result of this section. For $m = 2$ this result is well
known, though the history (especially of the implication (b) $\Rightarrow$ (a)) is somewhat
muddled—the first complete proof we are aware of is in [14].

**Theorem 7.1.** Let $0 \neq \phi \in C^\infty(\Gamma_1(N) \backslash GL(m, \mathbb{R}))$ be a cusp form and $\frac{1}{2} \leq \beta < 1$. Then the following two statements are equivalent:

(a) The $L$-function coefficients $a_n$, $n \geq 1$, satisfy the cancellation bound

\begin{equation}
\sum_{n=1}^{T} a_n e(nx) = O_{\varepsilon}(T^{\beta+\varepsilon})
\end{equation}

uniformly in $x$ for any $\varepsilon > 0$;

(b) The period $V$ satisfies the bound

\begin{equation}
V\begin{pmatrix} y & x & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\
0 & & \ddots & \ddots \\
0 & & \ddots & \ddots \\
0 & & \ddots & \ddots \\
\end{pmatrix} = O_{\varepsilon}\left(y^{\frac{m-1}{2} - \beta - \varepsilon}\right), \quad y > 0,
\end{equation}

uniformly in $x$ for any $\varepsilon > 0$.

**Remarks.** (1) The reason we have termed this an “analytic analog of the
Ramanujan conjecture” is that the conjectured optimal bound in (7.4) with $\beta = \frac{1}{2}$
is reminiscent of the following classical situation. Let $\phi(z) = \sum_{n \geq 1} c_n e(nz)$ be
the Fourier expansion of a classical holomorphic form of weight $k$. The trivial
bound on the $n$th-coefficient

\begin{equation}
c_n = e^{2\pi ny} \int_{x=0}^{1} \phi(x+iy) e(-nx) \, dx \leq e^{2\pi ny} \int_{x=0}^{1} |\phi(x+iy)| \, dx
\end{equation}

is obtained by invoking the bound $\phi(x+iy) = O(y^{-k/2})$ and taking $y$ to be of
order $1/n$: $c_n = O(n^{k/2})$. This estimate is on the order of $\sqrt{n}$ short of the truth of
$c_n = O(n^{(k-1)/2})$ predicted by the Ramanujan conjecture (in this case a theorem of
Deligne [8]). The reason for this loss of $\sqrt{n}$ in (7.5) is that we have used absolute
values and forfeited any cancellation from the oscillation of this period integral.

A similar phenomenon likely happens in (7.4), for bounding (7.2) trivially via
absolute values presumably gives an estimate which is off by some power of $n$.
The analogy with the Ramanujan conjecture is only meant in this analytic sense
and is not meant to have any algebraic connotation.
(2) Note that the period in part (b), like $\phi$, decays rapidly as $y \to \infty$. So the (7.4) is only an issue for $y$ small.

(3) Since different vectors in the same representation space share common $L$-function coefficients, assertion (b) is either true for all or none of the nonzero smooth vectors in the irreducible subrepresentation of $L^2(\Gamma_1(N) \backslash GL(m, \mathbb{R}))$ generated by right translates of $\phi$.

(4) Finally, the reason we have focused on the range $\frac{1}{2} \leq \beta < 1$ is because in practice this is only interesting situation ($\beta = 1$ being trivial, and $\beta = \frac{1}{2}$ conjectured to be optimal).

One of the advantages of taking $\phi$ to be invariant under $\Gamma_1(N)$ is the Fourier expansion (see [31], [35] and [3, (2.1.6)])

$$\phi(g) = \sum_{n_1, \ldots, n_{m-1} \geq 1} \sum_{\gamma \in \Gamma_\infty \backslash GL(m-1, \mathbb{Z})} a_{n_1, \ldots, n_{m-1}} \frac{1}{\prod_{j=1}^{m-1} j^{(m-j)/2}} \sum_{\|n\|} W \left( \begin{pmatrix} n_1 \\ n_2 \\ \vdots \\ n_{m-1} \\ 1 \end{pmatrix} (\gamma \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix})^T g \right),$$

where $\Gamma_\infty$ refers to the subgroup of unit upper triangular matrices in $GL(m-1, \mathbb{Z})$. Here $W(g)$ is the (archimedean) Whittaker function, formed from the same type of integral as (7.2), but with $N$ replaced by the maximal unipotent subgroup $N_0 = \{ \text{all unit upper triangular matrices} \}$. The coefficients of the standard $L$-function in this notation are $a_n = a_{n,1,1,\ldots,1}$. The identification of these coefficients with those of the standard $L$-function (as opposed to the contragredient $L$-function) is somewhat arbitrary and not completely universal—the difference is of no essential consequence here, and was introduced purely as a matter of convenience. Automorphic representations always satisfy the transformation law $\phi(gh) = (-1)^{\delta} \phi(g)$ for some $
 \delta \in \mathbb{Z}/2\mathbb{Z}$, where $h = \begin{pmatrix} -1 & 1 & \cdots \\ 1 & 1 & \cdots \\ \cdots & \cdots & \cdots \\ 1 & 1 & \cdots \end{pmatrix}$; the Whittaker functions naturally inherit this right-transformation property as well. Formulas (7.2) and (7.6) then give the following expression for $V(g)$ in terms of the $L$-function coefficients:

$$V(g) = \sum_{n \neq 0} \frac{a_n}{|n|^{(m-1)/2}} W \left( \begin{pmatrix} n \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & \cdots \\ 1 & 1 & \cdots \\ \cdots & \cdots & \cdots \\ 1 & 1 & \cdots \end{pmatrix}^T g \right).$$
The integral representation of the standard $L$-function on $GL(m)$ by Jacquet, Piatetski-Shapiro, and Shalika (see [3, 7, 17, 20, 22, 24]) uses the signed Mellin transform

$$I(s, \phi) = \int_{\mathbb{R}} V \left( \begin{array}{c} y \\ 1 \\ \vdots \\ 1 \end{array} \right) |y|^{s-\frac{m+1}{2}-1} \text{sgn}(y)^s dy$$

$$= 2J(s, W) \sum_{n=1}^{\infty} a_n n^{-s} = 2J(s, W) L(s, \phi),$$

where

$$J(s, W) = \int_{\mathbb{R}} W \left( \begin{array}{c} y \\ 1 \\ \vdots \\ 1 \end{array} \right) |y|^{s-\frac{m-1}{2}-1} \text{sgn}(y)^s dy.$$
and—thanks to (7.10)—have only mild growth near the origin. Let

$$w(y) = W \left( \begin{array}{cccc} y & 1 \\ 1 & 1 \\ & \ddots & \ddots \\ & & & 1 \end{array} \right),$$

(7.11)

so that the formula for $V \left( \begin{array}{cccc} y & x \\ 1 & 1 \\ & \ddots & \ddots \\ & & & 1 \end{array} \right)$ (denoted more succinctly as just $V \left( \begin{array}{cccc} y & x \\ 1 & 1 \end{array} \right)$) in (7.7) reads

$$V \left( \begin{array}{cccc} y & x \\ 1 & 1 \end{array} \right) = y^{(m-1)/2} \sum_{n \neq 0} a_n \text{sgn}(n)^\delta e(nx) |ny|^{(1-m)/2} w(|ny|), \quad y > 0.$$  

(7.12)

Letting $f(t) = |t|^{(1-m)/2}w(|t|)$ and keeping in mind that $y$ may be assumed to be small (Remark 2), we bound (7.12) by partial summation:

$$V \left( \begin{array}{cccc} y & x \\ 1 & 1 \end{array} \right) \ll \varepsilon y^{(m-1)/2} \sum_{n=1}^{\infty} n^{\beta+\varepsilon} y |f'(ny)|, \quad y > 0.$$  

(7.13)

The Whittaker function $w(t)$ (or, more accurately, the restriction of the Whittaker function to the one parameter subgroup in (7.11)) has rapid decay as $t \to \infty$, which means that in order to achieve (7.4) we need just to establish the bound

$$\sum_{i=1}^{T} n^{\beta+\varepsilon} |f'(ny)| = O_\varepsilon(y^{-1-\beta-\varepsilon}),$$

(7.14)

where $T$ is on the order of $y^{-1-\varepsilon'}$ for $\varepsilon'$ very close to 0, say $\varepsilon' = \varepsilon^2$. This itself follows from knowing that

$$f'(t) = \frac{1-m}{2} t^{-(m+1)/2} w(t) + t^{(1-m)/2} w'(t)$$

$$= O(t^A), \quad t > 0, \quad \text{for some } A \geq -\beta - 1,$$

(7.15)

for then the sum in (7.14) is bounded by $O_\varepsilon(y^A \sum_{n=1}^{T} n^{\beta+\varepsilon}) = O_\varepsilon(y^{-1-\beta-\varepsilon})$ (for $\varepsilon'$ sufficiently small). The condition (7.15) in turn follows from

$$w(t), \quad tw'(t) = O(t^{m/2-1}),$$

(7.16)
where we have used the assumption that $1/2 \leq \beta < 1$. The last assertion is literally (7.10) for $w(t)$ itself, and in fact also for $tw'(t)$ too, which is the Whittaker function of the derivative $Ad(e_1)\phi$ of $\phi$, $e_1$ being the matrix which is zero everywhere except for the entry 1 in its first position. This completes the proof of (a) $\implies$ (b).

The reverse implication (b) $\implies$ (a) can be proven using the technique of Hafner [14]. Namely, the rapid decay as $y \to \infty$ and the bound as $y \to 0$ of (7.4) gives an analytic continuation of the Mellin transform of

\[
(7.17) \quad \int_\mathbb{R} V \left( \begin{array}{cc} y & x \\ 1 & 1 \end{array} \right) |y|^{s-\frac{m-1}{2}} \sum_{n \neq 0} a_n e(nx) |n|^{-s} \delta \, dy = J(s, W) \sum_{n \neq 0} a_n e(nx) |n|^{-s}
\]

(cf. (7.8)) to $\text{Re} \, s > \beta$. In this range $J(s, W)$ is holomorphic by the remark above (7.10). For real $s > \beta$ such that $J(s, W) \neq 0$, (7.17) gives the bound

\[
(7.18) \quad \sum_{n \neq 0} a_n e(nx) |n|^{-s} = O(1),
\]

uniformly in $x$, and with the implied constant depending on $s$. In fact the sum on the left-hand side is a continuous function of $x$ because of dominated convergence applied to the integral in (7.17). This in turn implies (a) by [29, Prop. 3.7].

The implication (b) $\implies$ (a) can alternatively be proven using the techniques in [33], or instead by the method of Voronoi summation in [28].
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