Fermion in the Nonabelian Gauge Field Theory in 2+1 Dimensions

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Abstract

The massive $SU(2)$ gauge field theory coupled with fermions is considered in $2 + 1$ dimensions. Quark energy spectrum and radiative shift in constant external nonabelian field, being exact solution of the gauge field equations with the Chern-Simons term, are calculated. Under the condition $m = \theta/4$ the quark state is shown to be supersymmetric.

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Gauge theory models in (2+1)-space-time are useful in developing ideas for four-dimensional models, such as their high temperature behavior, boundaries of the spatial region occupied by gauge field etc. In this paper, we discuss topologically massive SU(2) gauge theory coupled with fermions and compute one-loop correction to quark energy.

The total Lagrangian of three-dimensional topologically massive SU(2)-gluodynamics of potentials $A_\mu \equiv \tau^a A^{a\mu}/2$, where $\tau^a$ are Pauli matrices in color space, and quarks is described as follows

$$\mathcal{L} = \frac{1}{4} \mathcal{F}_{\mu \nu}^a \mathcal{F}^{a\mu \nu} - \frac{\theta}{4} \varepsilon^{\mu \nu \alpha} \left( \mathcal{F}_{\mu \nu}^a A^{a\alpha}_{\mu} - \frac{g}{3} \varepsilon^{abc} A^{a\mu}_{\mu} A^{b}_{\nu} A^{c}_{\alpha} \right) - \frac{1}{2\xi} \left( \nabla_{\mu} a^{a \mu} \right)^2 + \bar{\psi}(D_{\mu} \gamma^\mu - m)\psi, \quad (1)$$

where $\mu, \nu = 0, 1, 2$, $a = 1, 2, 3$, the color field tensor is given by $\mathcal{F}_{\mu \nu}^a = \partial_\mu A_{\nu}^a - \partial_\nu A_{\mu}^a + g\varepsilon^{abc} A_{\mu}^b A_{\nu}^c$. The total gauge field $A^{a\mu}$ is represented as the sum of the classical background field and quantum fluctuations $a^{a\mu}$, i.e. $A^{a\mu} = A^{a\mu} + a^{a\mu}$, so that $\nabla_{\mu} a^{a \mu} = \delta^{ab} \partial_\mu + g\varepsilon^{abc} A^{c}_{\mu}$ is the background covariant derivative and $D_{\mu} a^{a \mu} = \delta^{ab} \partial_\mu + g\varepsilon^{abc} A^{c}_{\mu}$ is the total covariant derivative. The third term in (1) is the gauge-fixing term. The coefficient $\theta$ in front of the second term in (1) (Chern-Simons term) is the Chern-Simons (CS) mass of the gauge field. We use the following representation for the $\gamma$-matrices in $2 + 1$ dimensions: $\gamma^0 = \sigma^z$, $\gamma^{1,2} = i\sigma^{1,2}$, with $\sigma^i$ as Pauli matrices. The $\gamma$-matrices obey the following relation:

$$\gamma^\mu \gamma^\nu = g^{\mu \nu} - i\varepsilon^{\mu \nu \alpha} \gamma_{\alpha}.$$

It was shown in [2] that the classical constant nonabelian potentials

$$A^{a\mu} = \frac{\theta}{2g} \delta^{a\mu} \chi^{(a)}_{\lambda \omega}, \quad (2)$$

with normalized constant vector $\chi^{(a)}_{\lambda \omega} = (\lambda_i, \omega \lambda_i, \omega)$ satisfy the field equations with the Chern-Simons term without external currents. In [2] $\lambda = \pm 1$ and $\omega = \pm 1$ take its values independently. The Kronecker delta $\delta^{a\mu}$ in [2] implies that directions 1, 2, 3 in the color space correspond to directions 1, 2, 0 in the Minkowski $2 + 1$ space-time, respectively. In what follows, like in [3], where corrections to the gluon energy were considered, these solutions are chosen as the background.

Considering the one-loop corrections, it is sufficient to retain only the terms in the Lagrangian (1) quadratic in the quantum fields. They determine the quark energy spectrum in the gauge field (2)

$$\varepsilon_1^2 = p^2 + m_{\text{eff}1}^2, \quad \varepsilon_2^2 = p^2 + m_{\text{eff}2}^2,$$

where

$$m_{\text{eff}1}^2 = (m - \bar{\theta})^2, \quad m_{\text{eff}2}^2 = (m - \bar{\theta})(m + 3\bar{\theta}) \quad (3)$$

and $\bar{\theta} = \theta/4$. These branches of the energy spectrum correspond to the plane-wave solutions $\psi_s(x) = \exp(-i\varepsilon_s t + i\vec{p}\vec{x})\Psi_s$, with $s = 1, 2$, and $\Psi_s$ as constant spinors, of the Dirac equation

$$[\gamma^\mu (p_\mu + g A_{\mu}) - m] \psi = 0,$$

and are related to two opposite projections of the particle color spin operator [2]

$$\mathbf{J} = J^a \tau^a/2 = \bar{\theta}^{-1} g A_{\mu} p_\mu - \omega \gamma^0 \tau^3 (p_\mu \gamma^\mu + \bar{\theta} - m), \quad (4)$$

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defined as eigenvalues of the equation
\[ J \Psi_s = (-1)^s(\bar{\theta} - m)\Psi_s. \]

We note, that the r.h.s. of the last equation vanishes, when \( m = \bar{\theta} \). In this special case, the quark effective masses in this gauge field (3) are also equal to zero, and the two branches of the energy spectrum coincide. For values of the mass lying in the interval \(-\bar{\theta} - 2|\bar{\theta}| < m < -\bar{\theta} + 2|\bar{\theta}|\), the energy squared, \( \varepsilon^2 \), becomes negative for certain values of the quark momentum \( p \), and tachyonic modes arise in the quark spectrum. Moreover, it appears that \( \Psi_1 = \Psi_2 \) under this condition.

We now consider the one-loop radiative shift of the quark energy. According to [3], the quark energy radiative correction \( \Delta \varepsilon \) is obtained by averaging the mass operator over the quark state in the external field \( A^\mu \), specified by equation (2). The one-loop radiative correction to the quark energy is given by formula
\[ \Delta \varepsilon = -i \frac{T}{\pi} \int d^3x \int d^3x' \bar{\psi}_k(x) M_{kl}(x, x') \psi_l(x'), \]
where
\[ iM_{kl}(x, y) = -ig^2\gamma^\mu(\tau^a/2)_{kn}S_{nm}(x, y)\gamma^\nu(\tau^b/2)_{ml}D^{ab}_\mu(x, y) \]
is the one-loop mass operator. The expression for \( \Delta \varepsilon \) has to be applied with due regard to the fact that the quark Hamiltonian in the background (2) becomes non-Hermitian. In the momentum representation we have
\[ \Delta \varepsilon = \bar{\Psi}_k(p) iM_{kl}(p) \Psi_l(p), \]
\[ iM_{kl}(p) = ig^2\gamma^\mu(\tau^a/2)_{kn} \int d^3k S_{nm}(p - k)\gamma^\nu(\tau^b/2)_{ml}D^{ab}_{\mu}(k). \]

In order to calculate the quark mass operator in the external field the quark and gluon propagators have to be found. The quark Green’s function satisfies the equation
\[ [\gamma^\mu(i\partial_\mu + gA_\mu) - m]S(x, y) = \delta(x - y). \]

In the background gauge field \( A^\mu \) considered above, the quark Green’s function in the momentum representation has the form
\[ S(p) = [(p^2 - m_{\text{eff}1}^2)(p^2 - m_{\text{eff}2}^2)]^{-1} \left\{ (p^2 - (m - \bar{\theta})^2)[\gamma^\mu(p_\mu + gA_\mu) + m] - 2(\gamma^\mu p_\mu + m - \bar{\theta})(gA^\nu p_\nu + \bar{\theta}(m - \bar{\theta})) \right\}, \]
with the quark effective masses \( m_{\text{eff}1} \) and \( m_{\text{eff}2} \) defined above.

The gluon propagator in the external field in the gauge \( \xi = 1 \) takes the form
\[ D^{ab}_{\mu\nu} = \delta^{ab} g_{\mu\nu} \left[ \frac{1}{2\mathcal{E}} + \frac{\mathcal{E}}{2\alpha} \right] - \frac{4g^2}{\theta^2} A^a_\mu A^b_\nu \left[ \frac{1}{3\mathcal{E}} - \frac{\mathcal{E}}{3\beta} \right] + \frac{4g^2}{\theta^2} A^a_\nu A^b_\mu \left[ \frac{1}{2\mathcal{E}} - \frac{\mathcal{E}}{2\alpha} \right] + \frac{16g^2}{\theta^2\alpha\beta} F^{a\mu}_{\nu\alpha} F^{b\nu}_{\beta\delta} F^{\delta\nu}_{\alpha\beta}, \]
\[ + \frac{8ig^2}{\theta^2\beta} (F^{a\mu}_{\nu\alpha} A^b_\nu - F^{b\nu}_{\nu\alpha} A^a_\mu) p^\alpha + \frac{16ig^2}{\theta^2\alpha} \varepsilon^{\alpha\beta\gamma} F^{a\mu}_{\nu\alpha} F^{b\nu}_{\beta\gamma} p^\gamma, \]
where we used the notations \( E = p^2 + \frac{1}{2} \theta^2 \), \( \alpha = \mathcal{E}^2 - 4 \theta^2 p^2 \) and \( \beta = \mathcal{E}^2 - 6 \theta^2 p^2 \).

In the case of an arbitrary quark mass value \( m \neq \bar{\theta} \), the quark has two different color states. It is interesting to consider the special case, when \( m = \bar{\theta} \), and, as it has been mentioned above, the quark effective masses vanish. We remind that under this condition, \( m_{\text{eff}1} = m_{\text{eff}2} = 0 \), only one quark state survives. The corresponding plane wave solution of the Dirac equation \( \psi(x) = (2\pi)^{-1} \exp(-i\varepsilon t + ip\cdot x) \Psi(p) \) is determined by the constant spinor

\[
\Psi(p) = \frac{1}{8} \begin{pmatrix}
[\lambda(\kappa - 1) + (\kappa + 1)][(\omega + 1)e^{i\phi} + (\omega - 1)] \\
- [\lambda(\kappa - 1) - (\kappa + 1)][(\omega - 1)e^{-i\phi} + (\omega + 1)] \\
[\lambda(\kappa + 1) - (\kappa - 1)][(\omega - 1)e^{i\phi} + (\omega + 1)] \\
[\lambda(\kappa + 1) + (\kappa - 1)][(\omega + 1)e^{-i\phi} + (\omega - 1)]
\end{pmatrix}.
\]

It should be emphasized that in this formula the quark momentum \( p \) is assumed to be nonzero. Here \( \kappa = \pm 1 \) is the sign of the energy \( (\varepsilon \equiv p_0 = \kappa |p|) \) and the phase \( \phi \) is defined by the relation \( p^2 \pm ip^1 = |p|e^{\pm i\phi} \). We used representation for the Dirac operator and the Hamiltonian, where \( \gamma \)-matrices are inserted into isospin color Pauli matrices \( \tau^a \). It should be also emphasized that, since the fermion Hamiltonian is non-Hermitian, it provides only one solution of the Dirac equation for one value of the energy sign.

After integration over intermediate momentum \( k \) the mass operator is expressed in the form

\[
i M(p) = ig^2 \frac{\pi^2 \sqrt{2}}{|\theta|} \left[ - b_1 \theta + \frac{5}{9} \left( \frac{2}{\theta} g A^\mu + \frac{1}{2} \gamma^\mu \right)p^\mu - g A^\mu \gamma^\nu (b_2 g_{\mu\nu} - b_3 \theta^{-2} p^\mu p^\nu) \right],
\]

where

\[
b_1 = \frac{5}{6} - \frac{i}{12} (3\sqrt{2} + 4\sqrt{3}),
\]
\[
b_2 = \frac{10}{9} + \frac{i}{9} (8\sqrt{3} + 9\sqrt{2}),
\]
\[
b_3 = \frac{4}{45} - \frac{4\sqrt{3}i}{15}.
\]

Now we can find the one-loop radiative correction to the quark energy in the external field, which turns out to be vanishing in this particular state:

\[
\Delta \varepsilon = \frac{5\pi^2 \sqrt{2}}{18} ig^2 |\theta|^{-1} \left( p_0 - \kappa \sqrt{(p^1)^2 + (p^2)^2} \right) \equiv 0.
\]

It should be emphasized that this result is valid only for finite values of the quark momenta \( p = (p^1, p^2) \). For the case of vanishing \( p \), a special consideration is needed.

The fact that a fermion remains effectively massless even in the one-loop approximation signals the presence of a certain symmetry of the problem in question. In fact, the results obtained demonstrate the supersymmetry property of the state considered.

As is well known (see, e.g., [4]), the minimal representation of SUSYQ (SUSY Quantum Mechanics) is provided with the supercharges \( Q_1, Q_2 \) and the Hamiltonian \( H_S \):

\[
H_S = Q_1^2 = Q_2^2, \quad [H_S, Q_i] = 0, \quad \{Q_i, Q_j\} = 2\delta_{ij} H_S, \quad i, j = 1, 2.
\]
The quark Hamiltonian in the external field (2) can be written in the form

$$H = ip^1\gamma^2 - ip^2\gamma^1 + \bar{\theta}(\gamma^0 - \omega\tau^3) - \lambda\bar{\theta}(\tau^1\gamma^2 - \omega\tau^2\gamma^1).$$

The SUSYQ Hamiltonian $H_S = H^2$ is found to be

$$H_S = p^2 + 2\bar{\theta}[i\lambda(p^1\tau^1 + \omega p^2\tau^2) - i\omega\tau^3(p^1\gamma^2 - p^2\gamma^1)].$$

It is interesting to note that $H_S$ can be written in the form

$$H_S = p^2 - 2\bar{\theta}J,$$

where $J$ is the operator defined in (4). Under the condition $m = \bar{\theta}$ we have $J^2 = 0$. The supercharges, corresponding to the above Hamiltonian $H_S$, can be found:

$$Q_1 = \left[\sqrt{2}\gamma^0 - \frac{i\lambda\omega\bar{\theta}}{|p|}\right](p^1\gamma^1 + p^2\gamma^2) - \frac{\omega\bar{\theta}}{|p|}\gamma^0\tau^3(\omega p^1\tau^1 + p^2\tau^2) + \frac{1}{|p|}(p^1\gamma^1 + p^2\gamma^2)(\omega p^1\tau^1 + p^2\tau^2),$$

$$Q_2 = \left[i\omega\sqrt{2}\frac{\lambda\bar{\theta}}{|p|^2}\gamma^0\right](p^1\gamma^1 + p^2\gamma^2)(\omega p^1\tau^1 + p^2\tau^2) + i\bar{\theta}\tau^3 + i\omega\gamma^0(p^1\gamma^1 + p^2\gamma^2).$$

It is interesting to consider the quark ground state, when its momentum is equal to zero. Solution of the Dirac equation is easy to find in this case. It has the form

$$\Psi(p = 0) = \frac{c_1}{\sqrt{2}} \begin{pmatrix} \omega + 1 \\ \omega - 1 \\ 0 \\ 0 \end{pmatrix} + \frac{c_2}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ \omega - 1 \\ \omega + 1 \end{pmatrix} + \frac{c_3}{\sqrt{8}} \begin{pmatrix} \omega - 1 \\ \omega + 1 \\ \lambda(\omega + 1) \\ \lambda(\omega - 1) \end{pmatrix},$$

(5)

where constants $c_i$ are arbitrary up to a normalization condition.

In order to demonstrate the supersymmetry property of the Dirac equation $D\psi = 0$, where

$$D = \gamma^\mu(p_\mu + gA_\mu) - m$$

is the Dirac operator, we consider its zero-mode solutions with $p^0 = 0$ under the condition $m = \theta/4$ and with vanishing fermion mass in virtue of the condition $gA^0\gamma^0\psi = m\psi$, which means $\sigma^3\tau^3\psi = \omega\psi$. In this case the Dirac equation takes the form

$$-(\gamma^1\nabla^1 + \gamma^2\nabla^2)\psi_0 = 0.$$ 

Then we introduce combinations

$$b^\pm = (2i)^{-1}(\gamma^1 \pm i\gamma^2),$$

which play the role of fermionic anticommuting operators (creation and annihilation operators). The Dirac operator has the form

$$D = Q_+ + Q_-.$$
where \( Q_+ = 2b^+ \nabla_u \), \( Q_- = 2b^- \nabla_u \) while \( \nabla_u = \partial_u - igA_u \), \( \bar{\nabla}_u = \bar{\partial}_u - ig\bar{A}_u \), and here \( \partial_u = \frac{1}{2}(\partial_1 - i\partial_2) \), \( \bar{\partial}_u = \frac{1}{2}(\partial_1 + i\partial_2) \), etc. The standard SUSY hamiltonian has the form

\[
H_S = (D^2) = \{Q_+, Q_-\}, \quad Q^2_\pm = 0, \quad [H_S, Q_\pm] = 0.
\]

We set \( \psi_0 = \psi_\tau \psi_s \), where \( \psi_\tau \) describes the color (boson) state and \( \psi_s \) specifies the spin (fermion) state. For the ground (zero-mode) state we have: \( H_S \psi = 0 \). This means

\[
Q_+ \psi_0 = 0, \quad Q_- \psi_0 = 0.
\]

Consider \( Q_+ \psi_0 = 0 \), or \( b^+ D_u \psi_0 = 0 \). Let \( \psi_s = |0\rangle_s \), then \( b^+ \psi_s = |1\rangle_s \) and then \( D_u \psi_\tau = 0 \) (|0\rangle and |1\rangle are fermionic states). For \( \partial_1, \partial_2 \psi_0 = 0 \) we have

\[
(\tau^1 - i\tau^2) \psi_\tau = 0.
\]

Then if \( \omega = +1 \), we recall that \( \tau^3 \sigma^3 = \omega = +1 \), and for \( \sigma^3 \psi_s = -\psi_s \), we have \( \tau^3 \psi_\tau = -\psi_\tau \) and \( (\tau_1 - i\tau_2) \psi_\tau = 0 \). Hence \( \psi_\tau = |0\rangle_\tau \). Excited solutions are constructed in a standard way. Now it is clear that the first two terms in (5) describe the states with the SUSY property.

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