Antiwindup Design Approach to Constrained Primal-Dual Dynamics

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Abstract: An antiwindup control framework is developed for primal-dual dynamics in convex optimization. The proposed architecture provides for straightforward implementation of the ensuing primal-dual dynamics and for establishing global asymptotic stability using the notion of shifted-passivity with Lyapunov function commonly encountered in antiwindup literature.

Keywords: Primal-Dual Dynamics, Constrained Control, Shifted Passivity, Antiwindup Compensator, Constrained Optimization

1. INTRODUCTION

Primal-dual dynamics belong to the class of continuous-time gradient based algorithms for real-time solution of convex optimization problems (Arrow et al., 1958). The need for fast, efficient and architecturally simple computational methods in resource constrained and in critical infrastructure systems has made the primal-dual dynamics approach a method of choice (Ma and Elia, 2015; Stegink et al., 2015). As a result, the primal-dual dynamics approach has attracted significant research interest seeking to establish provable stability and robustness guarantees (Feijer and Paganini, 2010; Cherukuri et al., 2016; Nguyen et al., 2018; Qu and Li, 2019), and to develop mechanisms for hardware circuit realization (Kose, 1956; Vichik et al., 2015). However, for problems with inequality constraints, additional nonlinearity must be introduced at the input of the gradient system to account for the violation of the inequality constraint. To address this, previous works have focused on reformulating the primal-dual dynamics has a projected dynamical system with convex properties (Jayawardhana et al., 2007; Stegink et al., 2015). However, for problems with inequality constraints, additional nonlinearity must be introduced at the input of the gradient system to account for the violation of the inequality constraint. To address this, previous works have focused on reformulating the primal-dual dynamics has a projected dynamical system with convex properties (Jayawardhana et al., 2007; Stegink et al., 2015) or as a switched system (Feijer and Paganini, 2010; Kosaraju et al., 2018) and invoking tools from hybrid control to establish stability. Others have employed a proximal-like Lagrangian reformulation of the problem, and then establishing stronger stability results using non-diagonal Lyapunov functions (Qu and Li, 2019; Ding and Jovanović, 2019). In Adegbege (2020), an artificial switching element is introduced at the input of the controller to counter the effect of the input nonlinearity and to enforce the optimality of the equilibrium condition of the modified primal-dual dynamics.

In this work, we adopt a different approach. We interpret the primal-dual dynamics within the context of antiwindup control systems (Mulder et al., 2001; Gomes da Silva Jr et al., 2014). We consider the interconnection without input nonlinearity as the nominal control loop for which a suitable linear controller has already been designed. The controller ensures that the equilibrium point of the primal-dual dynamics is the optimal solution of a corresponding equality constrained convex optimization problem. Then an antiwindup augmentation is incorporated into the closed loop to account for the effect of the input nonlinearity. This additional augmentation is constructed such that the input-output passivity of the nominal interconnection is preserved while the equilibrium point solves the original inequality constrained convex optimization problem. By proper characterization of the input nonlinearity, asymptotic stability and convergence of the augmented system can be established using the notion of shifted passivity (Van der Schaft, 2000; Monshizadeh et al., 2019) and employing Lyapunov function similar to those in standard antiwindup control literature (Mulder et al., 2001).

The paper is structured as follows. In section 2, we provide the mathematical background necessary for setting up the proposed modified primal-dual dynamics and we introduce the notion of shifted passivity for constrained gradient dynamics. In section 3, we state our main result and contribution. We show that with antiwindup augmentation, a corrective signal is injected into the loop such that the modified interconnected system recovers the shifted-passivity property of the nominal closed loop. Global asymptotic stability of the equilibrium then follows from applying classical invariance principle. Finally in section 4, we provide a computational example to illustrate the effectiveness of the proposed approach to solving nonlinear optimization problems arising in model predictive control.

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2. MATHEMATICAL PRELIMINARIES AND PROBLEM SETUP

2.1 Constrained Gradient Dynamics

We consider the gradient system
\[ \dot{x} = -\nabla f(x) + g^T v \]  
\[ y = gx \]  
(1a)
(1b)
where \( f(x) \) is a continuously differentiable strongly convex function that maps from \( \mathbb{R}^n \) to \( \mathbb{R} \) and \( \nabla f(x) \) denotes the gradient vector defined as \( [\nabla f(x)]_i = (\partial f)(\partial x_i) \). The constant matrix \( g \in \mathbb{R}^m \rightarrow \mathbb{R}^m \) is assumed to be full row-ranked. The vectors \( v \in \mathbb{R}^m_+ \) (nonnegative orthant) \( \subset \mathbb{R}^m \) and \( y \in \mathbb{R}^m \) are respectively the system input and the output.

The task is to construct an appropriate controller for (1) such that the equilibrium point \( x^* \) is globally asymptotically stable and corresponds to the optimum solution of the convex optimization problem:
\[ \min_x f(x) \]  
subject to \( gx \geq w \)  
(2a)
(2b)
where \( w \in \mathbb{R}^m \) may be time dependent. The classical gradient method for solving (2) involves the construction of the Lagrangian function:
\[ L(x, \lambda) = f(x) + \lambda^T (w - gx) \]  
(3)
where \( \lambda \in \mathbb{R}^m_+ \) is the vector of Lagrangian multipliers associated with the inequality constraint (2b), and then followed by the saddle-point dynamics (Feijer and Paganini, 2010; Cherukuri et al., 2016):
\[ \eta_x \dot{x} = -\nabla_x L(x, \lambda) = -\nabla f(x) + g^T \lambda \]  
\[ \eta_\lambda \dot{\lambda} = [\nabla_\lambda L(x, \lambda)]_i^\lambda = [w - gx]_i^\lambda \]  
(4a)
(4b)
where \( \eta_x \) and \( \eta_\lambda \) are time constants for the \( x \) and \( \lambda \) trajectories respectively, and \( [P_i^\lambda]^\lambda \) denote the positive projection of \( P \) to ensure that \( \lambda \) remains in \( \mathbb{R}^m_+ \). It is typically defined elementwise as:
\[ [P_i^\lambda]^\lambda = \begin{cases} P_i & ; \lambda_i > 0 \text{ or } P_i > 0 \vspace{1mm} \\ 0 & ; \text{otherwise.} \end{cases} \]  
(5)

The presence of projection such as (5) in the primal-dual dynamics (4) renders the gradient flow discontinuous with associated implementation issues (Qu and Li, 2019). By proper characterization of the input nonlinearity, we avoid the use of projection while ensuring feasibility at equilibrium.

2.2 Input Nonlinearity Characterization

We define the input nonlinearity \( v = \phi(u) \) with
\[ \phi(u) = \begin{cases} \phi_1(u_1) & ; u_1 \geq 0 \\ \phi_m(u_m) & ; u_1 < 0 \end{cases} \]  
(6)

to capture the satisfaction or the violation of the constraint. We also define a complementary nonlinearity
\[ \psi(u) = \begin{cases} \psi_1(u_1) & ; u_i \geq 0 \\ \psi_m(u_m) & ; u_i < 0 \end{cases} \]  
(7)
such that \( \psi(u) + \phi(u) = u \). As constructed, the nonlinearity \( \phi(\cdot) \) satisfies \( \phi(0) = 0 \), and the incremental sector condition (Desoer and Vidyasagar, 1975):
\[ \langle v - v^* \rangle^T ((u - u^*) - (v - v^*)) \geq 0 \]  
(8)
for all \( u, u^*, v, v^* \) satisfying \( v = \phi(u) \) and \( v^* = \phi(u^*) \).

When \( \phi_i(u_i) = u_i \) for all \( i \), the gradient system (1) is unconstrained and we say the nonlinearity is inactive. We will subsequently take advantage of this characterization to establish global asymptotic stability for the primal-dual dynamics.

2.3 Linear Controller

It is well known that the problem of regulating nonlinear problems of the form (1) to a non-zero equilibrium \( x^* \) requires integral control (Jayawardhana et al., 2007). Here we consider the proportional-integral controller:
\[ \dot{\xi} = K_I (w - y) \]  
\[ u = \xi + (w - y) \]  
(9a)
(9b)
where \( K_I \) is diagonal positive definite matrix. The integral part of the controller ensures that at equilibrium, the equality constraints \( y^* = gx^* = w \) holds. To handle inequality constraints, the integral controller must be modified to guarantee \( gx^* \geq w \) at equilibrium. We will explore this in section 3.

2.4 Shifted Passivity As Input-Output Property

As we are interested in the forced equilibrium of the constrained gradient dynamic, we rely on the concept of shifted passivity (Van der Schaft, 2000; Monshizadeh et al., 2019). Suppose we define the equilibrium condition of the gradient system (1) as
\[ \mathcal{E} = \{ (x^*, v^*) \in \mathbb{R}^n \times \mathbb{R}^m_+ \mid \nabla f(x^*) - g^T v^* = 0 \} \]  
(10)
with the corresponding output as \( y^* = gx^* \), shifted passivity is defined in terms of the input \( v - v^* \) and the output \( y - y^* \) of the incremental model
\[ \dot{x} = - (\nabla f(x) - \nabla f(x^*)) + g^T (v - v^*) \]  
\[ y - y^* = g(x - x^*) \]  
(11a)
(11b)
for a fixed \( (x^*, v^*) \).

Definition 1. The gradient system (1) is exponentially shifted passive (resp. shifted passive) if for a fixed \( (x^*, v^*) \in \mathcal{E} \) and a corresponding \( y^* = gx^* \), there exists a positive semidefinite function \( V(z, x - x^*) \) such that:
\[ \dot{V}_z = (\nabla V_z^T) \dot{x} \geq -\varepsilon V_z + (v - v^*)^T (y - y^*) \]  
(12)
with a scalar \( \varepsilon > 0 \) (resp. \( \varepsilon = 0 \)) for all \( (x, v) \in \mathbb{R}^n \times \mathbb{R}^m_+ \).

Lemma 2. Suppose \( (x^*, v^*) \in \mathcal{E} \) is fixed for the gradient system (1) with a strongly convex \( f(x) \) and a full row-ranked \( g \). Then the gradient system is exponentially shifted passive with the storage function
\[ V_z (x - x^*) = \frac{1}{2} (x - x^*)^T (x - x^*) \]  
(13)

Proof. By strong convexity of \( f(x) \), the following monotonicity property (Boyd and Vandenberghe, 2004)
\[ (x - x^*)^T (\nabla f(x) - \nabla f(x^*)) \geq \delta (x - x^*)^T (x - x^*) \]  
(14)
holds for a positive constant \( \delta \). Since \((x^*, v^*) \in \mathcal{E}\) and \(g\) is full-row ranked, it follows that (cf. Simpson-Porco, 2018, corollary 3.6), (Ishizaki et al., 2016, lemma 3)):
\[
v^* = (gg^T)^{-1} g \nabla f(x^*) \quad \text{and} \quad x^* = \nabla f^{-1}(g^Tv^*)
\]
(15) and thus, \(x^*\) and \(v^*\) are uniquely defined. Evaluating the derivative of \(V_x\) along (1) gives:
\[
\dot{V}_x = -(x - x^*)^T(\nabla f(x) - \nabla f(x^*)) + (x - x^*)^T g^Tv - v^* \\
\leq -\delta (x - x^*)^T (x - x^*) + (y - y^*)^T (v - v^*)
\]
(16)
\[
= -\varepsilon V_x + (y - y^*)^T (v - v^*)
\]
(17)
with \(\varepsilon = 2\delta\). Condition (17) is exactly the definition of exponentially shifted passivity.

It is well known that for linear systems, passivity implies shifted passivity for any forced equilibrium (Jayawardhana et al., 2007). For the proportional-integral controller (9), the equilibrium condition for a fixed \(w\) is \((y^w = w, u^w = \xi^w)\) and hence the incremental form is given by
\[
\xi = -K_f (y - y^*), \quad u = u^w - (\xi - \xi^w) - (y - y^*).
\]
(18a, 18b)
It is straightforward to show that with the storage function
\[
V_k(\xi - \xi^*) = \frac{1}{2} (\xi - \xi^*)K_f^{-1}(\xi - \xi^*),
\]
(19)
the controller satisfies the following shifted passivity condition with respect to \(-(y - y^*)\) and \((u - u^w)\):
\[
\dot{V}_k = -(y - y^*)^T (u - u^w) - (y - y^*)^T (y - y^*)
\]
(20a, 20b)
For static nonlinearity (6), we define the following notions of shifted passivity (Monshizadeh et al., 2019).

Definition 3. The nonlinear map \(v = \phi(u)\) with \(0 = \phi(0)\) and satisfying \(v^* = \phi(u^*)\) is output shifted passive (resp. shifted passive) if the following inequality holds
\[
0 \leq (u - u^*)^T (v - v^*) - \rho(u - v^*)^T (v - v^*)
\]
(21)
for scalar \(\rho > 0\) (resp. \(\rho = 0\)) for all \((u, v) \in \mathbb{R}^m \times \mathbb{R}^m\).

3. CONSTRAINED PRIMAL-DUAL DYNAMICS AS FEEDBACK INTERCONNECTION

We follow the two step antiwindup design approach (Milder et al., 2001; Gomes da Silva Jr et al., 2014) where a nominal controller is first constructed ignoring the input nonlinearity i.e. setting \(\phi(u) = u\) and then an antiwindup augmentation is introduced to account for the effect of the input nonlinearity in closed loop.

3.1 Ignoring Input Nonlinearity-Nominal Interconnection

For the nominal interconnection, we ignore the input nonlinearity such that the output of the linear controller (9) is coupled directly to the gradient system (1) with \(v = u\). The resulting interconnection becomes:
\[
\dot{x} = -\nabla f(x) + g^T u, \quad y = gx
\]
(22a)
\[
\dot{\xi} = K_f (w - y)
\]
(22b)
\[
u = \xi + (w - y)
\]
(22c)
where \(w\) can be considered an exogenous input. This is illustrated in Fig. 1. At equilibrium, we have
\[
0 = -\nabla f(x^*) + g^T u^*,
\]
(23a)
\[
0 = w - gx^*
\]
(23b)

Fig. 1. Closed-Loop Interconnection with Linear Control

This is exactly the Karush Kuhn Tucker (KKT) optimality condition for the equality constrained optimization problem (Bertsekas, 2014):
\[
\min f(x),
\]
subject to \(gx = w\).
(24a, 24b)

To study the stability of the equilibrium point \((x^*, \xi^*)\), we express (22) in incremental form relative to \((x^*, \xi^*, u^*, y^*)\) where \(u^*\) and \(y^* = gx^*\) are the corresponding points satisfying the equilibrium condition (23), and then employ the concept of shifted-passivity (Van der Schaft, 2000).

Defining the displacement variables \(\hat{x} = x - x^*, \hat{\xi} = \xi - \xi^*\), \(\hat{u} = u - u^*\) and \(\hat{y} = y - y^*\), the incremental form of (22) is obtained as:
\[
\dot{\hat{x}} = - (\nabla f(x) - \nabla f(x^*)) + g^T \hat{u}, \quad \hat{y} = g\hat{x}
\]
(25a)
\[
\dot{\hat{\xi}} = -K_f \hat{y}, \quad \hat{u} = \hat{\xi} - \hat{y}
\]
(25b)
We state the following result.

Proposition 4. Consider the closed-loop system comprising the feedback interconnection of the gradient dynamics (1) and the proportional-integral controller (9) with \(v = u\). Suppose that \(f(x)\) is strongly convex and that \(g\) is full rank. Then the equilibrium point \((x^*, \xi^*)\) is globally asymptotically stable. Moreover \(x^*\) is the optimal solution of (24).

Proof. We consider the Lyapunov function candidate
\[
V = V_x + V_k = \frac{1}{2} x^T x + \frac{1}{2} \xi^T K_f^{-1} \xi.
\]
(26)
By construction \(V > 0\) for all \(x \neq x^*\) and \(\xi \neq \xi^*\). Evaluating the derivative of \(V\) along the trajectories of the incremental model (25) yields:
\[
\dot{V} = -\hat{x}^T (\nabla f(x) - \nabla f(x^*)) + g^T \hat{u} - \hat{u}^T \hat{y} - \hat{y}^T \hat{y}
\]
(27a)
\[
\leq -\epsilon \hat{y}^T \hat{y} \leq 0.
\]
(27b)
Note the first inequality follows from the exponential shifted passivity of the gradient dynamics and the shifted passivity of the controller. Since \(V\) is radially unbounded, all solution trajectories must be bounded (Wassim and Chellaboina, 2008). Define the invariant set \(\mathcal{M} = \{(\hat{x}, \hat{\xi}) \in \mathbb{R}^n \times \mathbb{R}^m | V = 0\}\). Observe that the first term of (27b) vanishes only when \(x = x^*\) and by continuity argument \(\hat{x} \equiv 0 \rightarrow \hat{y} \equiv 0\). It then follows from (25b) that \(\xi\) is uniformly continuous and hence \(\lim_{t \to \infty} \hat{\xi} = 0\).

So the largest invariant set contained in \(\mathcal{M}\) is the equilibrium point \((x^*, \xi^*)\). We therefore conclude asymptotic
**Proposition 5.** Consider the closed-loop system (28) comprising the feedback interconnection of the gradient dynamics (1) with $v = \phi(u)$ and the augmented nonlinear controller (28b)-(28c) controller. Suppose that $f(x)$ is strongly convex and that $q$ is full row ranked. Then the equilibrium point $(x^*, \xi^*)$ is globally asymptotically stable. Moreover $x^*$ is the optimal solution of (2).

**Proof.** We employ the candidate Lyapunov function

$$ V = V_x + V_y + V_\psi $$

where $V_x$ and $V_y$ are as defined in (13) and (19) respectively, and $V_\psi$ is introduced to account for incremental sector boundedness of the complementary nonlinearity $\psi(\cdot)$ and it is specified as (cf. Mulder et al. (2001)):

$$ V_\psi = \sum_{i=1}^m \int_0^t (q_i(\tau) - q_i^*) ((u(\tau) - u_i^*) - (q_i(\tau) - q_i^*) \, d\tau. $$

Note that $V_\psi$ is nonnegative, $V_x > 0$ for all $\tilde{x} \neq 0$ and $V_y > 0$ for all $\xi \neq 0$, so we have $V > 0$ for all $(\tilde{x}, \tilde{\xi}) \neq (0, 0)$. Since $\tilde{x} \equiv 0 \Rightarrow \tilde{y} \equiv 0$ and $(\tilde{\xi} \equiv 0 \Rightarrow \tilde{u} \equiv 0 \Rightarrow \tilde{q} \equiv 0$, we also have $V(0, 0) = 0$.

Evaluating the derivative of $V$ along the incremental model (30) gives:

$$ \dot{V} = -\tilde{x}^T (\nabla f(x) - \nabla f(x^*)) + \tilde{y}^T \tilde{u} - \tilde{q}^T \tilde{q} $$

$$ = -\tilde{x}^T (\nabla f(x) - \nabla f(x^*)) - (\tilde{y} + \tilde{q})^T (\tilde{y} + \tilde{q}) $$

$$ \leq -\varepsilon V_x - (\tilde{y} + \tilde{q})^T (\tilde{y} + \tilde{q}) \leq 0. $$

Observe that the first inequality is due to the exponential shifted passivity of the map $\tilde{u} \to (\tilde{y} + \tilde{q})$ and the shifted passivity of the map $\tilde{y} \to \tilde{y}$.

To establish asymptotic stability we follow similar route as in the proof of Proposition 4. Let $\mathcal{M} = \{ (\tilde{x}, \tilde{\xi}) \in \mathbb{R}^n \times \mathbb{R}^m | V = 0 \}$. Observe that $V_x$ vanishes only at $\tilde{x} = 0$. Also, the second term in (32c) vanishes when $\tilde{y} = -\tilde{q}$. But as $\tilde{x} \equiv 0 \Rightarrow \tilde{y} \equiv 0$ and $(\tilde{\xi} \equiv 0 \Rightarrow \tilde{u} \equiv 0 \Rightarrow \tilde{q} \equiv 0$, the largest invariant set in $\mathcal{M}$ in the equilibrium point $(x^*, \xi^*)$. Asymptotic stability follows from the LaSalle invariance principle. Since $V$ is a global Lyapunov function, we conclude global asymptotic stability of the equilibrium point.

### 4. NUMERICAL EXAMPLE

To illustrate the utility of the proposed approach to optimization problem arising in model predictive control (MPC) (Maeder et al., 2009), we consider a continuous-time system with the following dynamics:

$$ x_p(t) = (a - 1)x_p(t) + x_a(t) + d(t), \quad a < 1 $$

$$ y_p(t) = x_p(t) $$

where $x_p \in \mathbb{R}$ is the plant state, $x_a \in \mathbb{R}$ is the control input which must satisfy the constraint $x_a \geq 0$ and $d \in \mathbb{R}$ is an input disturbance. Suppose the output $y_p$ is to track a zero reference with zero offset at steady state. To achieve this, we employ linear offset-free model predictive control (Maeder et al., 2009). First, we discretized the plant using forward difference approximation and a sampling time of 1 second to obtain

$$ x_p[k + 1] = ax_p[k] + x_a[k] + d[k], \quad a < 1 $$

$$ y_p[k] = x_p[k] $$
and we adopt the following unity prediction horizon model predictive control:

$$\min_{x_u[0]} (x_u[0] - x_u^*)^2 + (x_p[1] - x_p^*)^2$$  \hspace{1cm} (37a)$$

subject to $x_u[0] \geq 0$,

$$x_p[1] = ax_p[0] + x_u[0] + d[0],$$  \hspace{1cm} (37b)$$

$$a - 1 \begin{bmatrix} x_p' \\ x_u' \end{bmatrix} = \begin{bmatrix} -d[0] \\ 0 \end{bmatrix},$$  \hspace{1cm} (37c)$$

$$x_p[0] = \hat{x}_p[k], \ d[0] = \hat{d}[k],$$  \hspace{1cm} (37d)$$

where $x_p'$ and $x_u'$ are the steady-state targets enforced by (37d), and $\hat{x}_p[k]$ and $\hat{d}[k]$ are the observed values at the current time instant. The MPC can be reduced to the following quadratic program (QP) problem that has to be solved at every time instant:

$$\min_{x_u[0]} 2x_u[0]^2 + 4 \left( d[0] + \frac{1}{2} ax_p[0] \right) x_u[0] + \text{constant terms}$$  \hspace{1cm} (38a)$$

subject to $x_u[0] \geq 0$  \hspace{1cm} (38b)$$

The QP has a closed-form solution:

$$x_u'[k] = \begin{cases} -\hat{d}[k] - \frac{1}{2} ax_p'[k] & -\hat{d}[k] - \frac{1}{2} ax_p'[k] \geq 0 \\ 0 & -\hat{d}[k] - \frac{1}{2} ax_p'[k] < 0 \end{cases}$$  \hspace{1cm} (39)$$

Using the proposed continuous-time solver, we obtain the controller dynamics without antiwindup augmentation as:

$$\begin{bmatrix} \dot{x}_u[1] \\ \dot{x}_u[0] \end{bmatrix} = \begin{cases} -4 \begin{bmatrix} x_u[1] \\ \xi[0] \end{bmatrix} - 4 \begin{bmatrix} \hat{d}(t) + \frac{1}{2} a \hat{x}_p(t) \end{bmatrix} & \text{for } \sigma(\xi - x_u) = \xi - x_u, \\ -9 \begin{bmatrix} x_u[1] \\ \xi[0] \end{bmatrix} - 4 \begin{bmatrix} \hat{d}(t) + \frac{1}{2} a \hat{x}_p(t) \end{bmatrix} & \text{for } \sigma(\xi - x_u) = 0, \end{cases}$$  \hspace{1cm} (40)$$

and with antiwindup augmentation as:

$$\begin{bmatrix} \dot{x}_u[1] \\ \dot{x}_u[0] \end{bmatrix} = \begin{cases} -4 \begin{bmatrix} x_u[1] \\ \xi[0] \end{bmatrix} - 4 \begin{bmatrix} \hat{d}(t) + \frac{1}{2} a \hat{x}_p(t) \end{bmatrix} & \text{for } \sigma(\xi - x_u) = \xi - x_u, \\ -9 \begin{bmatrix} x_u[1] \\ \xi[0] \end{bmatrix} - 4 \begin{bmatrix} \hat{d}(t) + \frac{1}{2} a \hat{x}_p(t) \end{bmatrix} & \text{for } \sigma(\xi - x_u) = 0. \end{cases}$$  \hspace{1cm} (41)$$

Note that (40) and (41) correspond to (22) and (28) respectively and that the equilibrium conditions of (41) solve the QP problem (38) for fixed plant and disturbance estimates. As seen in (41), the effect of the antiwindup augmentation is to decouple the $x_u$ and the $\xi$ dynamics whenever $\sigma(\xi - x_u) = 0$ holds.

For zero offset tracking, we construct a continuous-time observer of the form:

$$\dot{\hat{x}}_p(t) = (a - 1) \hat{x}_p(t) + x_u(t) + \hat{d}(t) + L_z (y_p(t) - \hat{x}_p(t))$$

where we have incorporated a constant input disturbance model. The observer gains $L_z$ and $L_d$ are chosen as:

$$L_z = \frac{a - 1}{2} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$  \hspace{1cm} (43)$$

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Fig. 4. Input \( [x_u(t)]: \) solver with antiwindup (solid-black), solver without antiwindup (Dashed-blue), and solver with closed-form solution (Dotted-red)

Fig. 5. Lagrange \([\xi(t)]: \) solver with antiwindup (solid-black) and solver without antiwindup (Dashed-blue)

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5603