Non-Symmetrized Hyperspherical Harmonics Method for Non-Equal Mass Three-Body Systems

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The non-symmetrized hyperspherical harmonics method for a three-body system, composed by two particles having equal masses, but different from the mass of the third particle, is reviewed and applied to the $^3$H, $^3$He nuclei and $^3ΛH$ hyper-nucleus, seen respectively as nnp, ppm and NNA three-body systems. The convergence of the method is first tested in order to estimate its accuracy. Then, the difference of binding energy between $^3$H and $^3$He due to the difference of the proton and the neutron masses is studied using several central spin-independent and spin-dependent potentials. Finally, the $^3ΛH$ hypernucleus binding energy is calculated using different NN and AN potential models. The results have been compared with those present in the literature, finding a very nice agreement.

I. INTRODUCTION

The hyperspherical harmonics (HH) method has been widely applied in the study of the bound states of few-body systems, starting from $A=3$ nuclei [1, 2]. Usually, the use of the HH basis is preceded by a symmetrization procedure that takes into account the fact that protons and neutrons are fermions, and the wave function has to be antisymmetric under exchange of any pair of these particles. For instance, for $A=3$, antisymmetry is guaranteed by writing the wave function as

$$\Psi = \sum_p \Psi_p,$$

$p = 1, 2, 3$ corresponding to the three different particle permutations [1]. However, it was shown in Refs. [3–7] that this preliminary step is in fact not strictly necessary, since, after the diagonalization of the Hamiltonian, the eigenvectors turn out to have a well-defined symmetry under particle permutation. In this second version, the method is known as non-symmetrized hyperspherical harmonics (NSHH) method. As we will also show below, the prize to pay for the non-antisymmetrization is that a quite larger number of the expansion elements are necessary with respect to the “standard” HH method. However, the NSHH method has the advantage to reduce the computational effort due to the symmetrization procedure, and, moreover, the same expansion can be easily re-arranged for systems of different particles with different masses. In fact, the steps to be done within the NSHH method from the case of equal-mass to the case of non-equal mass particles are quite straightforward and will be illustrated below. In this work, we apply the NSHH method to study the $^3$H, $^3$He and $^3ΛH$ systems, seen as nnp, ppm, NNA respectively (we used the standard notation of N for nucleon and Y for hyperon). In order to test our method, we study the first two systems listed above with five different potential models, and the hypernucleus with three potential models. We start with simple central spin-independent NN and YN interactions, and then we move to central spin-dependent potentials. To be noticed that none of the interactions considered is realistic. Furthermore, we do not include three-body forces. Therefore, the comparison of our results with the experimental data is meaningless. However, the considered interactions are useful to test step by step our method and to compare with results obtained in the literature.

The paper is organized as follows: in Section II we describe the NSHH method, in Section III we discuss the results obtained for the considered nuclear systems. Some concluding remarks and an outlook are presented in Section IV.

II. THEORETICAL FORMALISM

We briefly review the formalism of the present calculation. We start by introducing the Jacobi coordinates for a system of $A=3$ particles, with mass $m_i$, position $r_i$, and momentum $p_i$. By defining $x_i = \sqrt{m_i} r_i$, [8], they are taken as a linear combination of $x_i$, i.e.

$$y_i = \sum_{j=1}^A c_{ij} x_j,$$

(2)
where the coefficients $c_{ij}$ need to satisfy the following conditions [8]

\[
\sum_{i=1}^{3} c_{ji}c_{ji} = \frac{1}{M} \quad (j = 1, 2) ,
\]

\[
\sum_{i=1}^{3} c_{ji}c_{ki} = 0 \quad (j \neq k = 1, 2) .
\]

Here $M$ is a reference mass. The advantage of using Eqs. (2)–(4) is that the kinetic energy operator can be cast in the form

\[
T = -\frac{\hbar^2}{2m_{\text{tot}}} \nabla^2 y_3 - \frac{\hbar^2}{2M} \left( \nabla^2 y_1 + \nabla^2 y_2 \right) ,
\]

where $y_3$ is the center-of-mass coordinate. For a three-body system, there are three possible permutations of the particles. Therefore, the Jacobi coordinates depend on this permutations. For $p = 3$, i.e. $i, j, k = 1, 2, 3$, the Jacobi coordinates are explicitly given by

\[
y^{(3)}_2 = -\sqrt{\frac{m_2}{M(m_1 + m_2)}} x_1 + \sqrt{\frac{m_1}{M(m_1 + m_2)}} x_2 ,
\]

\[
y^{(3)}_1 = -\sqrt{\frac{m_1 m_3}{M m_{\text{tot}}(m_1 + m_2)}} x_1 - \sqrt{\frac{m_2 m_3}{M m_{\text{tot}}(m_1 + m_2)}} x_2 + \sqrt{\frac{m_1 + m_2}{M m_{\text{tot}}}} x_3 .
\]

They reduce to the familiar expressions for equal-mass particles when $m_1 = m_2 = m_3 = M$ (see for instance Ref. [1]). We then introduce the hyperspherical coordinates, by replacing, in a standard way, the moduli of $y^{(3)}_1, y^{(3)}_2$ by the hyperradius and one hyperangle, given by

\[
\rho^2 = y^{(p)}_1 + y^{(p)}_2 ,
\]

\[
\tan \phi^{(p)} = \frac{y^{(p)}_1}{y^{(p)}_2} .
\]

To be noticed that the hyperangle $\phi^{(p)}$ depends on the permutation $p$, while the hyperradius $\rho$ does not. The well-known advantage of using the hyperspherical coordinates is that the Laplace operator can be cast in the form [8]

\[
\nabla^2 = \nabla^2 y_1 + \nabla^2 y_2 + \frac{\partial^2}{\partial \rho^2} + \frac{5}{\rho} \frac{\partial}{\partial \rho} + \frac{\Lambda^2(\Omega^{(p)})}{\rho^2} ,
\]

where $\Lambda^2(\Omega^{(p)})$ is called the grand-angular momentum operator, and is explicitly written as

\[
\Lambda^2(\Omega^{(p)}) = \frac{\partial^2}{\partial \phi^{(p)}^2} - \hat{\ell}^2_1(\phi^{(p)}) \sin^2 \phi^{(p)} - \hat{\ell}^2_2(\phi^{(p)}) \cos^2 \phi^{(p)} + 2 \left[ \cot \phi^{(p)} - \tan \phi^{(p)} \right] \frac{\partial \phi^{(p)}}{\partial \phi^{(p)}} .
\]

Here $\hat{\ell}^1_1$ and $\hat{\ell}^2_2$ are the (ordinary) angular momentum operators associated with the Jacobi vectors $y^{(p)}_1$ and $y^{(p)}_2$ respectively, and $\Omega^{(p)} \equiv (y^{(p)}_1, y^{(p)}_2, \phi^{(p)})$. The HH functions are the eigenfunctions of the grand-angular momentum operator $\Lambda^2(\Omega^{(p)})$, with eigenvalue $-G(G + 4)$, i.e.

\[
\Lambda^2(\Omega^{(p)}) Y_G(\Omega^{(p)}) = -G(G + 4) Y_G(\Omega^{(p)}) .
\]

Here the HH function $Y_G(\Omega^{(p)})$ is defined as

\[
Y_G(\Omega^{(p)}) = N^{\ell_1, \ell_2}_n (\cos \phi^{(p)})^{\ell_1} (\sin \phi^{(p)})^{\ell_2} Y_{\ell_1 m_1}(\hat{y}^{(p)}_1) Y_{\ell_2 m_2}(\hat{y}^{(p)}_2) \times P^{\ell_1 + \frac{1}{2}, \ell_2 + \frac{1}{2}}_n (\cos 2\phi^{(p)}) ,
\]

with $N^{\ell_1, \ell_2}_n$ a normalization factor [8] and

\[
G = 2n + \ell_1 + \ell_2 , \quad n = 0, 1, \ldots ,
\]
We now consider our system made of three particles, two with equal masses, different from the mass of the third.

is the so-called grand-angular momentum. We remark that the HH functions depend on the considered permutation via $\Omega^{(p)}$. It is useful to combine the HH functions in order to assign them a well defined total orbital angular momentum $\Lambda$. Using the Clebsch-Gordan coefficients, we introduce the functions $H_G(\Omega^{(p)})$ as

$$H_G(\Omega^{(p)}) = \sum_{m_1,m_2} Y_G(\Omega^{(p)})(\ell_1 m_1 \ell_2 m_2 | \Lambda \Lambda_z)$$

$$= \left| Y_{\ell_1}(\hat{y}_1^{(p)}) Y_{\ell_2}(\hat{y}_2^{(p)}) \right| \Lambda \Lambda_z P_{n}^{\ell_1,\ell_2}(\phi^{(p)}),$$

(14)

where $[G]$ stands for $[\ell_1, \ell_2, \Lambda, n]$, and

$$P_{n}^{\ell_1,\ell_2}(\phi^{(p)}) = N_{e}^{\ell_1,\ell_2}(\cos \phi^{(p)})^{\ell_2} (\sin \phi^{(p)})^{\ell_1} P_{n}^{\ell_1+\ell_2,\ell_2}(\cos 2\phi^{(p)}).$$

(15)

We now consider our system made of three particles, two with equal masses, different from the mass of the third particle. We choose to fix the two equal mass particles in position 1 and 2, and we set the third particle with different mass as particle 3. Therefore, we will work with the Jacobi and hyperspherical coordinates with fixed permutation $p = 3$.

The wave function that describes our system can now be cast in the form

$$\Psi = \sum_{\{G\}} BH_{\{G\}}^{J}(\Omega^{(3)}) \ u_{\{G\}}(\rho),$$

(16)

where $u_{\{G\}}(\rho)$ is a function of only the hyperradius $\rho$, and $BH_{\{G\}}^{J}(\Omega^{(3)})$ is given by Eq. (14) multiplied by the spin part, i.e.

$$BH_{\{G\}}^{J}(\Omega^{(3)}) = \sum_{\Lambda,\Sigma_s} H_{\{G\}}(\Omega^{(3)}) \times \left[ \frac{1}{2} \otimes \frac{1}{2} \right]_{S_s} \otimes \frac{1}{2} \right]_{\Sigma_s} \times (\Lambda \Lambda_z, \Sigma \Sigma_s | JJ_z).$$

(17)

Here $S$ is the spin of the first couple with third component $s$, $\Sigma$ is the total spin of the system and $\Sigma_z$ its third component, and $\{G\}$ now stands for $\{\ell_1, \ell_2, n, \Lambda, S, \Sigma\}$. To be noticed that the LS-coupling scheme is used, so that the total spin of the system is combined, using the Clebsch-Gordan coefficient ($\Lambda \Lambda_z, \Sigma \Sigma_s | JJ_z$), with the total orbital angular momentum to give the total spin $J$. Furthermore, (i) $\ell_1, \ell_2$ and $n$ are taken such that Eq. (13) is satisfied for $G$ that runs from $G^{min} = \ell_1 + \ell_2$ to a given $G^{max}$, to be chosen in order to reach the desired accuracy, and (ii) we have imposed $\ell_1 + \ell_2 = \text{even}$, since the systems under consideration have positive parity. The possible values for $\Lambda, \Sigma, \Sigma_z$ and $G^{min}$, which together with $G^{max}$ identify a channel, are listed in Table I for a system with $J^\pi = 1/2^+$. Note that, since we are using central potentials, only the first channel of Table I will be in fact necessary.

In the present work, the hyperradial function is itself expanded on a suitable basis, i.e. a set of generalized Laguerre polynomials $[\mathbf{1}]$. Therefore, we can write

$$u_{\{G\}}(\rho) = \sum_{l} c_{\{G\},l} f_l(\rho),$$

(18)

where $c_{\{G\},l}$ are unknown coefficients, and

$$f_l(\rho) = \sqrt{\frac{l!}{(l+5)!}} \gamma^{3} (5) L_l(\gamma \rho) e^{-\frac{\rho}{2}}.$$

(19)

Here $^{(5)}L_l(\gamma \rho)$ are generalized Laguerre polynomials, and the numerical factor in front of them is chosen so that $f_l(\rho)$ are normalized to unit. Furthermore, $\gamma$ is a non-linear parameter, whose typical values are in the range $(2-5)$ fm$^{-1}$.

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### Table I: List of the channels for a $J^\pi = 1/2^+$ system. $\Lambda$ and $\Sigma$ are the total orbital angular momentum and the total spin of the nuclei. See text for more details.

| ch | A | $\Sigma$ | $G^{min}$ |
|----|---|---------|----------|
| 1  | 0 | 1/2    | 0        |
| 2  | 1 | 1/2    | 2        |
| 3  | 1 | 3/2    | 2        |
| 4  | 2 | 3/2    | 2        |

The possible values for $\Lambda$ are:

- $\min \{ \ell_1, \ell_2 \} = \ell_1 = \ell_2$ (even, since the systems under consideration have positive parity). The possible values for $\Lambda$ and $\Sigma$ are:

- $(\Lambda, \Sigma, \Sigma_z) = (0, 0, 0)$,
- $(2, 1, 0)$,
- $(3, 2, 0)$,
- $(4, 3, 0)$.

The possible values for $\Sigma z$ are:

- $(\Lambda, \Sigma, \Sigma z) = (0, 0, 0)$,
- $(2, 1, 0)$,
- $(3, 2, 0)$,
- $(4, 3, 0)$.

The possible values for $J$ are:

- $J = 1/2$,
- $J = 2$. 

The possible values for $\pi$ are:

- $\pi = 1/2^+$. 

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The results have to be stable against $\gamma$, as we will show in Section III. With these assumptions, the functions $f_i(\rho)$ go to zero for $\rho \to \infty$, and constitute an orthonormal basis.

By using Eq. (18), the wave function can now be cast in the form

$$\Psi = \sum_{\{G\}} \sum_{i=1}^{N_{\text{max}}} c_{\{G\},i} B H_{\{G\}}^i(\Omega) f_i(\rho) ,$$

(20)

where we have dropped the superscript (3) in $\Omega^{(3)}$ to simplify the notation, and we have indicated with $N_{\text{max}}$ the maximum number of Laguerre polynomials in Eq. (18).

In an even more compact notation, we can write

$$\Psi = \sum_{\xi} c_{\xi} \Psi_{\xi} ,$$

(21)

where $\Psi_{\xi}$ is a complete set of states, and $\xi$ is the index that labels all the quantum numbers defining the basis elements. The expansion coefficients $c_{\xi}$ can be determined using the Rayleigh-Ritz variational principle [1], which states that

$$\langle \delta \xi | H - E | \xi \rangle = 0 ,$$

(22)

where $\delta \xi \Psi$ denotes the variation of the wave function with respect to the coefficients $c_{\xi}$. By doing the differentiation, the problem is then reduced to a generalized eigenvalue-eigenvector problem of the form

$$\sum_{\xi'} \langle \Psi_{\xi'} | H - E | \Psi_{\xi} \rangle c_{\xi'} = 0 ,$$

(23)

that is solved using the Lanczos diagonalization algorithm [9]. The use of the Lanczos algorithm is dictated by the large size ($\sim 50000 \times 50000$) of the involved matrices (see below).

All the computational problem is now shifted in having to calculate the norm, kinetic energy and potential energy matrix elements. One of the advantage of using a fixed permutation is that the norm and kinetic energy matrix elements are or analytical, or involve just a one-dimensional integration. In fact, they are written as

$$N_{\{G'\},k;\{G\},l} \equiv \langle \Psi_{\xi'} | \Psi_{\xi} \rangle = J \delta_{\xi,\xi'} ,$$

(24)

$$T_{\{G'\},k;\{G\},l} \equiv \langle \Psi_{\xi'} | T | \Psi_{\xi} \rangle = -\frac{\hbar^2}{2M} J \delta_{\{G\},\{G'\}} \int d\rho \rho^5 f_k(\rho)$$

$$\times \left[ -G(G+4) \frac{f_i(\rho)}{\rho^2} + 5 \frac{f'_i(\rho)}{\rho} + f''_i(\rho) \right] ,$$

(25)

where $J$ is the total Jacobian of the transformation, given by

$$J = \left( M \sqrt{\frac{m_{\text{total}}}{m_1 m_2 m_3}} \right)^3 ,$$

(26)

and $f'_i(\rho)$ and $f''_i(\rho)$ are, respectively, the first and the second derivatives of the functions $f_i(\rho)$ defined in Eq. (19).

The potential matrix elements in Eq. (23) can be written as

$$V_{\{G'\},k;\{G\},l} \equiv \langle \Psi_{\xi'} | V_{12} + V_{23} + V_{13} | \Psi_{\xi} \rangle ,$$

(27)

with $V_{ij}$ indicating the two-body interaction between particle $i$ and particle $j$. Note that in the present work we do not consider three-body forces. Since it is easier to evaluate the matrix elements of $V_{ij}$ when the Jacobi coordinate $y_2$ is proportional to $r_i - r_j$, we proceed as follows. We make use of the fact that the hyperradius is permutation-independent, and we use the fact that the HH function written in terms of $\Omega^{(p)}$ can be expressed as function of the HH written using $\Omega'^{(p')}$, with $p' \neq p$. Basically it can be shown that [1]

$$H_{\{G\}}(\Omega^{(p)}) = \sum_{\{G'\}} \delta^{(p-p')}_{\{G\},\{G'\}} G,\Lambda \ H_{\{G'\}}(\Omega'^{(p')}) ,$$

(28)

where the grandangular momentum $G$ and the total angular momentum $\Lambda$ remain constant, i.e. $G = G'$ and $\Lambda = \Lambda'$, but we have $[G] \neq [G']$, since all possible combinations of $\ell_1, \ell_2, n$ are allowed. The spin-part written in terms of
permutation $p$ can be easily expressed in terms of permutation $p'$ via the standard $6j$ Wigner coefficients $[10]$. The transformation coefficients $a_{[G],[G']}^{(p 	o p'),G,A}$ can be calculated, for $A = 3$, through the Raynal-Revai recurrence relations $[11]$. Alternatively, we can use the orthonormality of the HH basis $[1]$, i.e.

$$a_{[G],[G']}^{(p 	o p'),G,A} = \int d\Omega^{(p')} [H_{[G']}[(\Omega^{(p')})]^\dagger H_{[G]}[(\Omega^{(p)})].$$

Their explicit expression can be found for instance in Ref. $[1]$ as is reported in the Appendix for completeness. The final expression for the potential matrix elements is given by

$$\langle \Psi_{\xi'} | V_{12} + V_{23} + V_{13} | \Psi_{\xi} \rangle = J \int d\rho \rho^5 f_k(\rho)f_l(\rho) \times \left\{ \int d\Omega^{(3)} BH_{\xi'}^{(3)}(\Omega^{(3)}) V_{12} BH_{\xi}(\Omega^{(3)}) + \sum_{\xi''} \left[ a_{\xi' \rightarrow \xi''}^{(3\rightarrow1),G',A'} a_{\xi \rightarrow \xi''}^{(3\rightarrow1),G,A} \right] \times \int d\Omega^{(1)} BH_{\xi''}(\Omega^{(1)}) V_{23} BH_{\xi'}(\Omega^{(1)}) + a_{\xi \rightarrow \xi''}^{(3\rightarrow2),G',A'} a_{\xi \rightarrow \xi''}^{(3\rightarrow2),G,A} \times \int d\Omega^{(2)} BH_{\xi''}(\Omega^{(2)}) V_{13} BH_{\xi'}(\Omega^{(2)}) \right\}. \quad (30)$$

It is then clear the advantage of using the NSHH method also for the calculation of the potential matrix elements, as in fact all what is needed is the calculation of one integral of the type

$$I(\rho) = \int d\rho \rho^5 f_k(\rho)f_l(\rho) \int d\Omega^{(p)} BH_{\xi'}^{(p)}(\Omega^{(p)}) V_{ij} BH_{\xi''}(\Omega^{(p)}) \, , \quad (31)$$

with $p$ the permutation corresponding to the order $i, j, k$.

### III. RESULTS

We present in this section the results obtained with the NSHH method described above. In particular, we present in Section III A the study of the convergence of the method, in the case of the triton binding energy, calculated with $m_p = m_n$. We then present in Section III B the results for the triton and $^3$He binding energy, when $m_p \neq m_n$. In Section III C we present the results of the hypertriton.

The potential models used in our study are central spin-independent and spin-dependent. In particular, the $^3$H and $^3$He systems have been investigated using the spin-independent Volkov $[12]$, Afnan-Tang $[13]$ and Malfliet-Tjon $[14]$ potential models, and the two spin-dependent Minnesota $[15]$ and Argonne AV4$'$ $[16]$ potential models. Note that the AV4$'$ potential is a reprojection of the much more realistic Argonne AV18 $[17]$ potential model. In the case of the hypernucleus $^4_\Lambda$H, we have used the Gaussian spin-independent central potential of Ref. $[18]$, and two spin-dependent potentials: the first one, labeled MN9 $[19]$, combines a Minnesota $[12]$ potential for the $NN$ interaction with the $S = 1$ component of the same Minnesota potential multiplied by a factor 0.9 for the $\Lambda N$ interaction. The second one, labeled AU, uses the Argonne AV4$'$ of Ref. $[16]$ for the $NN$ interaction, and the Usmani potential of Ref. $[20]$ for the $\Lambda N$ interaction (see also Ref. $[21]$).

#### A. Convergence study

We recall that the wave function is written as in Eq. $[24]$, and that, since we are using central potentials, only the first channel of Table II is considered, as for instance in Ref. $[1]$. Therefore we need to study the convergence of our results on $G^{\text{max}}$ and $N^{\text{max}}$. Furthermore, we introduce the value of $j$ as $j = ^3\ell_2 + ^2S$, $\ell_2$ and $S$ being the orbital angular momentum and the spin of the pair $ij$ on which the potential acts. This allows to set up the theoretical framework also in the case of projecting potentials. Therefore, we will study the convergence of our results also on the maximum value of $j$, called $j^{\text{max}}$. Finally, the radial function written as in Eq. $[19]$, presents a non-linear parameter $\gamma$, for which
we need to find a range of values such that the binding energy is stable. Note that in these convergence studies we have used $m_n = m_p$.

We start by considering the parameter $\gamma$. The behaviour of the binding energy as a function of $\gamma$ is shown for the Volkov potential in the top panel of Fig. 1. We mention here that for all the other potential models we have considered, the results are similar. The other parameters were kept constant, i.e. $G_{\text{max}} = 20$, $N_{\text{max}} = 16$ and $j_{\text{max}} = 6$. The particular dependence on $\gamma$ of the binding energy, that increases for low values of $\gamma$, is constant for some central values, and decreases again for large values of $\gamma$, allows to determine a so-called plateau, and the optimal value for $\gamma$ has to be chosen on this plateau. Alternatively, we can chose $\gamma$ such that for a given $N_{\text{max}}$ the binding energy is maximum. A choice of $\gamma$ outside the plateau would require just a larger value of $N_{\text{max}}$. To be noticed that this particular choice of $\gamma$ is not universal. As an example, in the “standard” HH method, $\gamma = 2.5 - 4.5$ fm$^{-1}$ for the AV18 potential, but much larger ($\approx 7$ fm$^{-1}$) for the chiral non-local potentials [1]. In our case, different values of $\gamma$ for different potentials might improve the convergence on $N_{\text{max}}$, but not that on $j_{\text{max}}$ and $G_{\text{max}}$, determined by the structure of the HH functions. Since, as shown below, the convergence on $N_{\text{max}}$ is not difficult to be achieved, we have chosen to keep $\gamma$ at a fixed value, i.e. $\gamma = 4$ fm$^{-1}$ for all the potentials.

In the bottom panel of Fig. 1 we fix $j_{\text{max}} = 8$, $\gamma = 4$ fm$^{-1}$ and $G_{\text{max}} = 20$, and we show the pattern of convergence for the binding energy $B$ with respect to $N_{\text{max}}$, in the case of the Argonne AV4′ model. Here convergence is reached for $N_{\text{max}} = 24$, i.e. we have verified that, for higher $N_{\text{max}}$ value, $B$ changes by less than 1 keV. To be noticed that for the other potentials, convergence is already reached for $N_{\text{max}} = 16 - 20$.

The variation of the binding energy as a function of $j_{\text{max}}$ and $G_{\text{max}}$ depends significantly on the adopted potential model. Therefore, we need to analyze every single case. As we can see from the data of Tables II and III, the convergence on $j_{\text{max}}$ and $G_{\text{max}}$ for the Volkov and the Minnesota potentials is really quick, and we can reach an accuracy better than 2 keV for $j_{\text{max}} = 10$ and $G_{\text{max}} = 40$. On the other hand, in the case of the Afnan-Tang potential, we need to go up to $j_{\text{max}} = 14$ and $G_{\text{max}} = 50$, in order to get a total accuracy of our results of about 2 keV (1 keV is due to the dependence on $N_{\text{max}}$). This can be seen by inspection of Table IV. The Malfliet-Tjon potential model implies a convergence even slower of the expansion, and we have to go up to $G_{\text{max}} = 90$ and $j_{\text{max}} = 22$, to get an uncertainty of about 3 keV, as shown in Table V. In fact, being a sum of Yukawa functions, the Malfliet-Tjon potential model is quite difficult to be treated also with the “standard” symmetrized HH method [1].

In Tables VI and VII we show the convergence study for the AV4′, which is the most realistic potential model used

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**TABLE II:** The $^3\text{H}$ binding energy $B$ (in MeV) calculated with the Volkov potential model [12], using $m_n = m_p$, $N_{\text{max}} = 16$ and $\gamma = 4$ fm$^{-1}$, as function of $j_{\text{max}}$ and $G_{\text{max}}$.

| $j_{\text{max}}$ | 6   | 8   | 10  |
|------------------|-----|-----|-----|
| $G_{\text{max}}$ | B   | B   | B   |
| 20   | 8.460 | 8.462 | 8.462 |
| 30   | 8.461 | 8.464 | 8.464 |
| 40   | 8.461 | 8.464 | 8.465 |

**TABLE III:** Same as Table II but for the Minnesota potential model [15].

| $j_{\text{max}}$ | 6   | 8   | 10  |
|------------------|-----|-----|-----|
| $G_{\text{max}}$ | B   | B   | B   |
| 20   | 8.376 | 8.381 | 8.381 |
| 30   | 8.378 | 8.383 | 8.385 |
| 40   | 8.378 | 8.383 | 8.385 |

**TABLE IV:** Same as Table II but for the Afnan-Tang potential model [13].

| $j_{\text{max}}$ | 6   | 8   | 10  |
|------------------|-----|-----|-----|
| $G_{\text{max}}$ | B   | B   | B   |
| 20   | 6.608 | 6.608 | 6.608 |
| 30   | 6.605 | 6.604 | 6.604 |
| 40   | 6.608 | 6.608 | 6.608 |
| 50   | 6.608 | 6.608 | 6.608 |
FIG. 1: Top panel: The binding energy $B$ (in MeV) as function of the parameter $\gamma$ (in fm$^{-1}$) for the Volkov potential model [12], with $G_{max} = 20$, $j_{max} = 6$ and $N_{max} = 16$, and using $m_n = m_p$. Bottom panel: The binding energy $B$ (in MeV) as function of the parameter $N_{max}$ for the AV4’ potential model [16], with $G_{max} = 20$, $j_{max} = 8$ and $\gamma = 4$ fm$^{-1}$, and using $m_n = m_p$.

here for the $A = 3$ nuclear systems. As we can see by inspection of the tables, in order to reach an accuracy of about 3 keV, we have to push the calculation up to $G_{max} = 80$, $N_{max} = 24$ and $j_{max} = 20$. Our final result of $B = 8.991$ MeV, though, agrees well with the one of Ref. [22], obtained with the “standard” symmetrized HH method, for which $B = 8.992$ MeV.

The results for the binding energy of $^3$H and $^3$He with the different potentials will be summarized in the next Subsection.
Having verified that our method can be pushed up to convergence, we present in the third column of Table VIII the results for the $^3$H binding energy with all the different potential models, obtained still keeping $m_p = m_n$. The results are compared with those present in the literature, finding an overall nice agreement.

We now turn our attention to the $^3$H and $^3$He nuclei, considering them as made of different mass particles. Therefore, we impose $m_p \neq m_n$ and we calculate the $^3$H and $^3$He binding energy and the difference of these binding energies, i.e.

$$\Delta B = B_{^3H} - B_{^3He}.$$  \hspace{2cm} (32)

To be noticed that we have not yet included the effect of the (point) Coulomb interaction. The results are listed in Table VII. By inspection of the table, we can see that $\Delta B$ is not the same for all the potential models. In fact, while for the spin-independent Afnan-Tang and Malfliet-Tjon central potentials, and for the spin-dependent AV4’ potential, $\Delta B = 14$ keV, for the Volkov and the Minnesota potential we find a smaller value. In all cases, though, we have

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
\hline
$j_{\text{max}}$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$
\hline
10 & 7.943 & 20 & 7.943 & 20 & 7.943 & 20 & 7.943 & 20 \\
12 & 8.155 & 30 & 8.179 & 30 & 8.179 & 30 & 8.179 & 30 \\
14 & 8.182 & 40 & 8.229 & 40 & 8.229 & 40 & 8.229 & 40 \\
16 & 8.190 & 50 & 8.243 & 50 & 8.243 & 50 & 8.243 & 50 \\
18 & 8.192 & 60 & 8.246 & 60 & 8.246 & 60 & 8.246 & 60 \\
20 & 8.193 & 70 & 8.248 & 70 & 8.248 & 70 & 8.248 & 70 \\
22 & 8.194 & 80 & 8.249 & 80 & 8.249 & 80 & 8.249 & 80 \\
24 & 8.194 & 90 & 8.250 & 90 & 8.250 & 90 & 8.250 & 90 \\
\hline
\end{tabular}
\caption{Same as Table II but for the Malfliet-Tjon potential model [14].}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
\hline
$N_{\text{max}}$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$
\hline
16 & 8.682 & 20 & 8.682 & 20 & 8.682 & 20 & 8.682 & 20 \\
40 & 8.927 & 60 & 8.960 & 60 & 8.981 & 60 & 8.981 & 60 \\
80 & 8.927 & 80 & 8.960 & 80 & 8.981 & 80 & 8.981 & 80 \\
\hline
20 & 8.686 & 20 & 8.686 & 20 & 8.686 & 20 & 8.686 & 20 \\
40 & 8.927 & 40 & 8.961 & 40 & 8.974 & 40 & 8.974 & 40 \\
60 & 8.931 & 60 & 8.964 & 60 & 8.977 & 60 & 8.977 & 60 \\
80 & 8.931 & 80 & 8.964 & 80 & 8.977 & 80 & 8.977 & 80 \\
\hline
24 & 8.687 & 20 & 8.687 & 20 & 8.687 & 20 & 8.687 & 20 \\
40 & 8.928 & 40 & 8.962 & 40 & 8.975 & 40 & 8.975 & 40 \\
60 & 8.932 & 60 & 8.965 & 60 & 8.980 & 60 & 8.980 & 60 \\
80 & 8.933 & 80 & 8.966 & 80 & 8.981 & 80 & 8.981 & 80 \\
\hline
\end{tabular}
\caption{The $^3$H binding energy $B$ (in MeV) calculated with the AV4’ potential model [16], using $m_n = m_p$, $G_{\text{max}} = 60$, $N_{\text{max}} = 24$ and $\gamma = 4$ fm$^{-1}$.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{cccccccc}
\hline
$j_{\text{max}}$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$ & $G_{\text{max}}$ & $B$
\hline
10 & 8.932 & 8.965 & 8.980 & 8.986 & 8.989 & 8.991 \\
12 & 8.932 & 8.965 & 8.980 & 8.986 & 8.989 & 8.991 \\
14 & 8.932 & 8.965 & 8.980 & 8.986 & 8.989 & 8.991 \\
16 & 8.932 & 8.965 & 8.980 & 8.986 & 8.989 & 8.991 \\
18 & 8.932 & 8.965 & 8.980 & 8.986 & 8.989 & 8.991 \\
20 & 8.932 & 8.965 & 8.980 & 8.986 & 8.989 & 8.991 \\
\hline
\end{tabular}
\caption{The $^3$H binding energy $B$ (in MeV) calculated with the AV4’ potential model [16], using $m_n = m_p$, $G_{\text{max}} = 60$, $N_{\text{max}} = 24$ and $\gamma = 4$ fm$^{-1}$.}
\end{table}
where we have indicated

| Potential model literature | $B(m_n = m_p)$ | $B_{3^1H}$ | $B_{3^2He}$ | $\Delta B$ | $BC_{3^3He}$ |
|---------------------------|----------------|------------|-------------|-------------|---------------|
| Volkov                    | 8.465 [1]      | 8.465      | 8.470       | 8.459       | 0.011         | 7.754         |
| Afnan-Tang                | 6.698 [22]     | 6.697      | 6.704       | 6.690       | 0.014         | 5.990         |
| Malfliet-Tjon             | 8.253 [1]      | 8.250      | 8.257       | 8.243       | 0.014         | 7.516         |
| Minnesota                 | 8.386 [1]      | 8.385      | 8.389       | 8.381       | 0.008         | 7.706         |
| AV4'                      | 8.992 [22]     | 8.991      | 8.998       | 8.984       | 0.014         | 8.272         |

TABLE VIII: The $^3$H binding energy obtained using $m_n = m_p$ ($B(m_p = m_n)$), the $^3$H and $^3$He binding energies calculated taking into account the difference of masses but no Coulomb interaction in $^3$He ($B_{3^1H}$ and $B_{3^2He}$), the difference $\Delta B = B_{3^1H} - B_{3^2He}$; and the $^3$He binding energy calculated including also the (point) Coulomb interaction ($BC_{3^3He}$). All the values are given in MeV. The results present in the literature for $B(m_p = m_n)$ are also listed with the corresponding references.

verified that $\Delta B$ is equally distributed, i.e. we have verified that

$$B_{m_n = m_p} = B_{3^1H} - \frac{\Delta B}{2} = B_{3^2He} + \Delta B - \frac{\Delta B}{2},$$

as can be seen from Table VIII. We would like to remark that in the NSH H method, the inclusion of the difference of masses is quite straightforward, and $\Delta B$ can be calculated “exactly”. This is not so trivial within the symmetrized HH method. Furthermore, we compare our results with those of Ref. [23], where $\Delta B$ was calculated within the Faddeev equation method using realistic Argonne AV18 [17] potential, and it was found $\Delta B = 14$ keV, in perfect agreement with our AV4’ result.

In order to test our results for $\Delta B$, we try to get a perturbative rough estimate of $\Delta B$, proceeding as follows: since the neutron-proton difference of mass $\Delta m = m_n - m_p = 1.2934$ MeV is about three orders of magnitude smaller than their average mass $m = (m_n + m_p)/2 = 938.9187$ MeV, we can assume also $\Delta B$ to be small. Furthermore, we suppose the potential to be insensitive to $\Delta m$, and we consider only the kinetic energy. In the center of mass frame, the kinetic energy operator can be cast in the form

$$T = \sum_{i=1}^{3} \frac{\mathbf{p}_i^2}{2m_i} = \frac{\mathbf{p}_e^2 + \mathbf{p}_n^2 + \mathbf{p}_d^2}{2m_e},$$

where $m_e$ stands for the mass of the two equal particles, i.e. $m_n$ for $^3$H and $m_p$ for $^3$He, and $m_d$ is the mass of the third particle, different from the previous ones. By defining $E = \langle H \rangle = \langle T + V \rangle$, where $\langle H \rangle$ is the average value of the Hamiltonian $H$, we obtain

$$\frac{\partial E}{\partial m_e} = \langle \frac{\partial H}{\partial m_e} \rangle = \langle \frac{\partial T}{\partial m_e} \rangle = -\langle T_e \rangle \frac{m_e}{m_e},$$

$$\frac{\partial E}{\partial m_d} = \langle \frac{\partial H}{\partial m_d} \rangle = \langle \frac{\partial T}{\partial m_d} \rangle = -\langle T_d \rangle \frac{m_d}{m_d},$$

where we have indicated $\langle T_{e/d} \rangle \approx \mathbf{p}_{e/d}^2/(2m_{e/d})$. Moreover, we define the proton and neutron mass difference $\Delta m_{p/n}$ as

$$\Delta m_{p} = m_p - m = \frac{\Delta m}{2},$$

$$\Delta m_{n} = m_n - m = \frac{\Delta m}{2},$$

and the $^3$He and $^3$H binding energy difference $\Delta B_{3^2He/3^1H}$ as

$$\Delta B_{3^2He} = B_{m_n = m_p} - B_{3^2He},$$

$$\Delta B_{3^1H} = B_{m_n = m_p} - B_{3^1H}.$$
Then using Eqs. (35)–(38), we obtain

$$\Delta B_{3\text{He}} \approx \frac{\partial E}{\partial m_e} \Delta m_p + \frac{\partial E}{\partial m_o} \Delta m_n = \frac{\langle 2T_e \rangle \Delta m}{m_p} - \frac{\langle T_d \rangle \Delta m}{m_o} \quad (41)$$

$$\Delta B_{3\text{H}} \approx \frac{\partial E}{\partial m_e} \Delta m_n + \frac{\partial E}{\partial m_o} \Delta m_p = -\frac{\langle 2T_e \rangle \Delta m}{m_e} + \frac{\langle T_d \rangle \Delta m}{m_o} \quad (42)$$

In conclusion

$$\Delta B_{PT} \equiv B_{3\text{He}} - B_{3\text{He}} \approx \langle 2T_e - T_d \rangle \frac{\Delta m}{2m} \approx \langle T \rangle \frac{\Delta m}{3m} \quad (43)$$

where the last equality holds assuming that $\langle T_e \rangle = \langle T_d \rangle = \langle T \rangle / 3$, since the $^3\text{He}$ and $^3\text{H}$ have a large S-wave component (about 90%). The results of $\langle T \rangle$ and $\Delta B_{PT}$ are listed in Table IX and are compared with the values for $\Delta B$ calculated within the NSHH and already listed in Table VIII. By inspection of the table we can see an overall nice agreement between this rough estimate and the exact calculation for all the potential models. Only in the case of the Minnesota and AV4 potentials, $\Delta B_{PT}$ is 4 and 3 keV larger than $\Delta B$, respectively. This can be understood by noticing that these potentials are spin-dependent, giving rise to mixed-symmetry components in the wave functions. These components are responsible for a reduction in $\Delta B_{PT}$ [24], related to the fact that the nuclear force for the $^3S_1$ np pair is stronger than for the $^1S_0$ nn (or pp) pair. Therefore, the kinetic energy for equal particles $\langle T_e \rangle$ is less than the kinetic energy for different particles $\langle T_d \rangle$.

### C. The $^3\Lambda$ hypernucleus

The hypernucleus $^3\Lambda$ is a bound system composed by a neutron, a proton and the $\Lambda$ hyperon. In order to study this system, we have considered the proton and the neutron as reference-pair, with equal mass $m_n = m_p = m$, while the $\Lambda$ particle has been taken as the third particle with different mass. The $\Lambda$ hyperon mass has been chosen depending on the considered potential. We remind that we have used three different potential models: a central spin-independent Gaussian model [18], and two spin-dependent central potentials, labelled MN9 [19] and AU [21] potentials. Therefore, when the $^3\Lambda$ hypernucleus has been studied using the Gaussian potential of Ref. [18], we have set $M_\Lambda = 6/5 \, m_N$, accordingly. In the other two cases, we have used $M_\Lambda = 1115.683$ MeV. We first study the convergence pattern of our method, which in the case of the Gaussian potential of Ref. [18] is really fast, with a reached accuracy of 1 keV on the binding energy already with $N_{\text{max}} = 20$, $j_{\text{max}} = 10$ and $G_{\text{max}} = 50$. This can be seen directly by inspection of Tables [X] and [XI].

The convergence pattern in the case of the spin-dependent central MN9 and AU potentials has been found quite slower. This is shown in Tables [XII] and [XIII] respectively. By inspection of Table [XII] we can conclude that $B = 2.280$ MeV, with an accuracy of about 3 keV, obtained with $G_{\text{max}} = 100$, $N_{\text{max}} = 34$, and $j_{\text{max}} = 14$. By inspection of Table [XIII] $B = 2.532$ MeV, with an accuracy of about 4 keV, going up to $G_{\text{max}} = 140$, $N_{\text{max}} = 24$ and $j_{\text{max}} = 16$.

The results obtained with our method for the three potential models considered in this work are compared with those present in the literature [18, 19, 21] in Table XIV finding a very nice agreement, within the reached accuracy.

| Potential model | $\langle T \rangle$ (MeV) | $\Delta B_{PT}$ (MeV) | $\Delta B_{NSHH}$ (MeV) |
|-----------------|--------------------------|-----------------------|--------------------------|
| Volkov          | 23.798                   | 0.011                 | 0.011                    |
| Afnan-Tang      | 30.410                   | 0.014                 | 0.014                    |
| Malfliet-Tjon   | 30.973                   | 0.014                 | 0.014                    |
| Minnesota       | 27.216                   | 0.012                 | 0.008                    |
| AV4             | 37.599                   | 0.017                 | 0.014                    |

TABLE IX: Mean value for the kinetic energy operator $\langle T \rangle$, $\Delta B$ estimated with the perturbative theory ($PT$), and $\Delta B$ calculated with the NSHH for the different potential models considered in this work. See text for more details.
| $j_{\text{max}}$ = 6 | $j_{\text{max}}$ = 8 | $j_{\text{max}}$ = 10 |
|-----------------|-----------------|-----------------|
| $G_{\text{max}}^B$ | $G_{\text{max}}^B$ | $G_{\text{max}}^B$ |
| 0 | 0.510 | 0 | 0.510 | 0 | 0.510 |
| 2 | 1.070 | 2 | 1.070 | 2 | 1.070 |
| 4 | 1.776 | 4 | 1.776 | 4 | 1.776 |
| 6 | 2.211 | 6 | 2.211 | 6 | 2.211 |
| 8 | 2.371 | 8 | 2.371 | 8 | 2.371 |
| 10 | 2.476 | 10 | 2.476 | 10 | 2.476 |
| 12 | 2.551 | 12 | 2.551 | 12 | 2.551 |
| 20 | 2.659 | 20 | 2.660 | 20 | 2.660 |
| 30 | 2.692 | 30 | 2.693 | 30 | 2.693 |
| 40 | 2.700 | 40 | 2.701 | 40 | 2.701 |
| 50 | 2.702 | 50 | 2.703 | 50 | 2.703 |

TABLE X: The $^3\Lambda$H binding energy $B$ (in MeV) as function of $j_{\text{max}}$ and $G_{\text{max}}^B$, calculated with the Gaussian potential model of Ref. [18], using $N_{\text{max}} = 20$ and $\gamma = 4$ fm$^{-1}$.

| $N_{\text{max}}$ | 8 | 12 | 16 | 20 |
|-----------------|---|---|---|---|
| $B$             | 2.552 | 2.651 | 2.660 | 2.660 |

TABLE XI: The $^3\Lambda$H binding energy $B$ (in MeV) as function of $N_{\text{max}}$, calculated with the the Gaussian potential model of Ref. [18], using $G_{\text{max}}^B = 20$, $j_{\text{max}} = 8$ and $\gamma = 4$ fm$^{-1}$.

### IV. CONCLUSIONS AND OUTLOOK

In this work we present a study of the bound state of a three-body system, composed of different particles, by means of the NSHH method. The method has been reviewed in Section II. In order to verify its validity, we have started by considering a system of three equal-mass nucleons interacting via different central potential models, three spin-independent and two spin-dependent. We have studied the convergence pattern, and we have compared our results at convergence with those present in the literature, finding an overall nice agreement. Then, we have switched on the difference of mass between protons and neutrons and we have calculated the difference of binding energy $\Delta B$ due to the difference between the neutron and proton masses. We have found that $\Delta B$ depends on the considered potential model, but is always symmetrically distributed (see Eq. (33)).

Finally we have implemented our method for the $^3\Lambda$H hypernucleus, studied with three different potentials, i.e. the Gaussian potential of Ref. [18], for which we have found a fast convergence of the NSHH method, the MN9 and the AU potentials of Ref. [19], for which the convergence is much slower. In these last two cases, in particular, we had found necessary to include a large number of the HH basis (46104 for the MN9 and 52704 for the AU potentials), but the agreement with the results in the literature has been found quite nice. To be noticed that we have included only two-body interactions, and therefore a comparison with the experimental data is meaningless.

In conclusion, we believe that we have proven the NSHH method to be a good choice for studying three-body systems composed of two equal mass particles, different from the mass of the third particle. Besides $^3\Lambda$H, $^9$He, and $^3\Lambda$H, several other nuclear systems can be viewed as three-body systems of different masses. This applies in all cases where a strong clusterization is present, as in the case of $^6$He and $^6$Li nuclei, seen as $NN\alpha$, or the $^9$Be and $^9$B, seen as a $\alpha\alpha N$ three-body systems. Furthermore, taking advantage of the versatility of the HH method also for scattering systems, the NSHH approach could be extended as well to scattering problems. Work along these lines are currently underway.

**Acknowledgments**

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V. APPENDIX: THE TRANSFORMATION COEFFICIENTS

Let us start by writing Eq. (29) as

$$a^{(p-p')G,L}_{\ell_1,\ell_2,n,\ell_1',\ell_2',n'} = \int d\Omega^{(p')} [H_{[\ell_1',\ell_2',n',\Lambda_{\Lambda_1}]}(\Omega^{(p')})]^\dagger H_{[\ell_1,\ell_2,n,\Lambda_{\Lambda_1}]}(\Omega^{(p)}) \, ,$$

(44)

where $\ell_i$ ($\ell'_i$) is the orbital angular momentum associated with the Jacobi coordinate $y_i^{(p)}$ ($y_i^{(p')}$). It can be demonstrated by direct calculation and exploiting the spherical harmonics properties that

$$a^{(p-p')G,L}_{\ell_1,\ell_2,n,\ell_1',\ell_2',n'} = \frac{N^\ell_n N^\ell_n}{2} \frac{1}{2} \int_0^1 d\phi \int_{-1}^1 d\mu (\cos \phi)^{2+\ell'_2}(\sin \phi)^{2+\ell'_1} \times P_{\ell_1+1/2,\ell_2+1/2}(\cos 2\phi^{(p')})P_{\ell_1+1/2,\ell_2+1/2}(\cos 2\phi^{(p)}) \times \sum_{\lambda_1,\lambda_2} C^{(p),\ell_1}_{\ell_2,\lambda_1,\lambda_2}(\sin \phi^{(p')}, \cos \phi^{(p')})P_{\lambda}(\mu) \times (-)^{\Lambda_1+\Lambda_2+\ell'_2}(2\Lambda+1) \hat{\ell}_1 \hat{\ell}_2 \hat{\lambda}_1 \hat{\lambda}_2 \times \left\{ \begin{array}{c} \ell'_1 \ \ell'_2 \ \Lambda \\ \ell_2 \ \lambda_1 \ \lambda \\ 0 \ \ 0 \ \ 0 \end{array} \right\} \left\{ \begin{array}{c} \ell'_1 \ \ell'_2 \ \Lambda \\ \ell_2 \ \lambda_1 \ \lambda \\ 0 \ \ 0 \ \ 0 \end{array} \right\} \, .$$

(45)

Here the curly brackets indicate the $6j$ Wigner coefficients, and the coefficients $C^{(p),\ell_1}_{\ell_2,\lambda_1,\lambda_2}(\sin \phi^{(p')}, \cos \phi^{(p')})$ are defined as

$$C^{(p),\ell_1}_{\ell_2,\lambda_1,\lambda_2}(\sin \phi^{(p')}, \cos \phi^{(p')}) = \sum_{\lambda_1+\lambda_2=\ell_1} \sum_{\lambda'_1+\lambda'_2=\ell_2} (\sin \phi^{(p')})^{\lambda_1+\lambda'_1}(\cos \phi^{(p')})^{\lambda_2+\lambda'_2} \times (a_{11(p')})_{\lambda_1}(a_{12(p')})_{\lambda_2}(a_{21(p')})_{\lambda'_1}(a_{22(p')})_{\lambda'_2} \times (-)^{\lambda_1+\lambda_2+\lambda'_1+\lambda'_2} D_{\ell_1,\lambda_1,\lambda_2} D_{\ell_2,\lambda_1,\lambda_2} \times \hat{\ell}_1 \hat{\ell}_2 \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}'_1 \hat{\lambda}'_2 \left( \begin{array}{c} \lambda_1 \ \lambda'_1 \ \ell_1' \\ 0 \ \ 0 \ \ 0 \end{array} \right) \times \left( \begin{array}{c} \lambda_2 \ \lambda'_2 \ \ell_2' \\ 0 \ \ 0 \ \ 0 \end{array} \right) \left\{ \begin{array}{c} \lambda_1 \ \lambda_2 \ \ell_1 \\ \lambda'_1 \ \lambda'_2 \ \ell_2' \\ \ell_1' \ \ell_2' \ \Lambda \end{array} \right\} \, .$$

(46)

In Eq. (46) $\ell \equiv \sqrt{2\ell+1}$, and the round (curly) brackets denote $3j$ ($9j$) Wigner coefficients. The coefficients $a_{ij(p')}$, with $ij = 1,2$ are given by

$$y_i^{(p)} = \sum_{j=1}^2 a_{ij(p')} y_j^{(p')} \, ,$$

(47)

and depend on the (different) masses of the three particles. and $D_{\ell,\ell_6,\ell_6}$ is defined as

$$D_{\ell,\ell_6,\ell_6} = \sqrt{\frac{(2\ell+1)!}{(2\ell_6+1)!(2\ell_6+1)!}} \, .$$

(48)

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| \( N_{\text{max}} \) | \( G_{\text{max}} \) | \( B \) | \( G_{\text{max}} \) | \( B \) | \( G_{\text{max}} \) | \( B \) | \( G_{\text{max}} \) | \( B \) | \( G_{\text{max}} \) | \( B \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 50 | 2.174 | 50 | 2.201 | 50 | 2.205 | 50 | 2.207 |
| 50 | 2.178 | 50 | 2.205 | 50 | 2.209 | 60 | 2.211 |
| 70 | 2.181 | 70 | 2.207 | 70 | 2.211 | 70 | 2.213 |
| 80 | 2.181 | 80 | 2.208 | 80 | 2.212 | 80 | 2.214 |
| 90 | 2.182 | 90 | 2.208 | 90 | 2.212 | 90 | 2.215 |
| 100 | 2.182 | 100 | 2.208 | 100 | 2.212 | 100 | 2.215 |
| 20 | 50 | 2.206 | 50 | 2.232 | 50 | 2.236 | 50 | 2.239 |
| 50 | 2.213 | 60 | 2.238 | 60 | 2.242 | 60 | 2.245 |
| 70 | 2.216 | 70 | 2.241 | 70 | 2.246 | 70 | 2.248 |
| 80 | 2.219 | 80 | 2.243 | 80 | 2.248 | 80 | 2.250 |
| 90 | 2.220 | 90 | 2.244 | 90 | 2.249 | 90 | 2.251 |
| 100 | 2.220 | 100 | 2.244 | 100 | 2.249 | 100 | 2.252 |
| 24 | 50 | 2.219 | 50 | 2.243 | 50 | 2.247 | 50 | 2.250 |
| 50 | 2.227 | 60 | 2.251 | 60 | 2.255 | 60 | 2.257 |
| 70 | 2.232 | 70 | 2.255 | 70 | 2.259 | 70 | 2.261 |
| 80 | 2.235 | 80 | 2.257 | 80 | 2.261 | 80 | 2.264 |
| 90 | 2.236 | 90 | 2.259 | 90 | 2.263 | 90 | 2.266 |
| 100 | 2.236 | 100 | 2.260 | 100 | 2.264 | 100 | 2.267 |
| 28 | 50 | 2.222 | 50 | 2.248 | 50 | 2.252 | 50 | 2.255 |
| 50 | 2.233 | 60 | 2.257 | 60 | 2.261 | 60 | 2.264 |
| 70 | 2.240 | 70 | 2.263 | 70 | 2.267 | 70 | 2.269 |
| 80 | 2.244 | 80 | 2.266 | 80 | 2.270 | 80 | 2.272 |
| 90 | 2.246 | 90 | 2.268 | 90 | 2.272 | 90 | 2.274 |
| 100 | 2.248 | 100 | 2.269 | 100 | 2.273 | 100 | 2.276 |
| 32 | 50 | 2.225 | 50 | 2.249 | 50 | 2.253 | 50 | 2.256 |
| 50 | 2.236 | 60 | 2.258 | 60 | 2.262 | 60 | 2.265 |
| 70 | 2.243 | 70 | 2.264 | 70 | 2.268 | 70 | 2.272 |
| 80 | 2.247 | 80 | 2.268 | 80 | 2.273 | 80 | 2.275 |
| 90 | 2.250 | 90 | 2.272 | 90 | 2.275 | 90 | 2.277 |
| 100 | 2.252 | 100 | 2.273 | 100 | 2.276 | 100 | 2.279 |
| 34 | 50 | 2.225 | 50 | 2.249 | 50 | 2.253 | 50 | 2.256 |
| 50 | 2.237 | 60 | 2.259 | 60 | 2.263 | 60 | 2.266 |
| 70 | 2.244 | 70 | 2.265 | 70 | 2.269 | 70 | 2.272 |
| 80 | 2.248 | 80 | 2.269 | 80 | 2.274 | 80 | 2.276 |
| 90 | 2.251 | 90 | 2.273 | 90 | 2.276 | 90 | 2.278 |
| 100 | 2.253 | 100 | 2.274 | 100 | 2.277 | 100 | 2.280 |

TABLE XII: The \(^{3}\text{H}\) binding energy \( B \) (in MeV) as function of \( G_{\text{max}} \), \( j_{\text{max}} \) and \( N_{\text{max}} \), calculated with the MN9 potential model of Ref. [19], using \( \gamma = 4 \text{ fm}^{-1} \).
\[ \text{TABLE XIII: Same as Table XII but using the AU potential model of Ref. [21] for } N_{\text{max}} = 16, 20, 24. \]

\begin{tabular}{cccccccc}
\hline
 & \( j_{\text{max}} = 6 \) & \( j_{\text{max}} = 10 \) & \( j_{\text{max}} = 12 \) & \( j_{\text{max}} = 14 \) & \( j_{\text{max}} = 16 \) \\
\hline
\( N_{\text{max}} = 16 \) & & & & & & \\
20 & 1.436 & 20 & 1.508 & 20 & 1.521 & 20 & 1.521 \\
30 & 1.924 & 30 & 2.037 & 30 & 2.051 & 30 & 2.056 \\
40 & 2.137 & 40 & 2.243 & 40 & 2.258 & 40 & 2.270 \\
50 & 2.250 & 50 & 2.349 & 50 & 2.363 & 50 & 2.372 \\
60 & 2.314 & 60 & 2.408 & 60 & 2.421 & 60 & 2.430 \\
70 & 2.355 & 70 & 2.445 & 70 & 2.455 & 70 & 2.461 \\
80 & 2.379 & 80 & 2.466 & 80 & 2.476 & 80 & 2.484 \\
90 & 2.394 & 90 & 2.479 & 90 & 2.489 & 90 & 2.497 \\
100 & 2.402 & 100 & 2.486 & 100 & 2.498 & 100 & 2.503 \\
110 & 2.406 & 110 & 2.489 & 110 & 2.504 & 110 & 2.510 \\
120 & 2.408 & 120 & 2.491 & 120 & 2.507 & 120 & 2.513 \\
130 & 2.409 & 130 & 2.492 & 130 & 2.509 & 130 & 2.515 \\
140 & 2.409 & 140 & 2.492 & 140 & 2.510 & 140 & 2.516 \\
150 & & & & & & & \\
\hline
\end{tabular}

\begin{tabular}{cccccccc}
\hline
 & \( N_{\text{max}} = 20 \) & & & & & & \\
20 & 1.438 & 20 & 1.522 & 20 & 1.522 & 20 & 1.522 \\
30 & 1.925 & 30 & 2.038 & 30 & 2.053 & 30 & 2.057 \\
40 & 2.139 & 40 & 2.245 & 40 & 2.259 & 40 & 2.268 \\
50 & 2.252 & 50 & 2.351 & 50 & 2.364 & 50 & 2.373 \\
60 & 2.317 & 60 & 2.409 & 60 & 2.423 & 60 & 2.431 \\
70 & 2.357 & 70 & 2.446 & 70 & 2.458 & 70 & 2.466 \\
80 & 2.384 & 80 & 2.470 & 80 & 2.480 & 80 & 2.488 \\
90 & 2.402 & 90 & 2.485 & 90 & 2.495 & 90 & 2.501 \\
100 & 2.413 & 100 & 2.494 & 100 & 2.504 & 100 & 2.512 \\
110 & 2.421 & 110 & 2.501 & 110 & 2.510 & 110 & 2.518 \\
120 & 2.426 & 120 & 2.505 & 120 & 2.514 & 120 & 2.521 \\
130 & 2.429 & 130 & 2.507 & 130 & 2.516 & 130 & 2.524 \\
140 & 2.430 & 140 & 2.508 & 140 & 2.517 & 140 & 2.526 \\
150 & & & & & & & \\
\hline
\end{tabular}

\begin{tabular}{cccccccc}
\hline
 & \( N_{\text{max}} = 24 \) & & & & & & \\
20 & 1.438 & 20 & 1.523 & 20 & 1.523 & 20 & 1.524 \\
30 & 1.926 & 30 & 2.039 & 30 & 2.053 & 30 & 2.057 \\
40 & 2.139 & 40 & 2.245 & 40 & 2.259 & 40 & 2.268 \\
50 & 2.252 & 50 & 2.351 & 50 & 2.364 & 50 & 2.373 \\
60 & 2.317 & 60 & 2.410 & 60 & 2.423 & 60 & 2.432 \\
70 & 2.357 & 70 & 2.446 & 70 & 2.458 & 70 & 2.467 \\
80 & 2.385 & 80 & 2.471 & 80 & 2.481 & 80 & 2.488 \\
90 & 2.402 & 90 & 2.485 & 90 & 2.496 & 90 & 2.502 \\
100 & 2.414 & 100 & 2.494 & 100 & 2.505 & 100 & 2.513 \\
110 & 2.423 & 110 & 2.502 & 110 & 2.511 & 110 & 2.519 \\
120 & 2.429 & 120 & 2.507 & 120 & 2.516 & 120 & 2.523 \\
130 & 2.433 & 130 & 2.510 & 130 & 2.519 & 130 & 2.527 \\
140 & 2.437 & 140 & 2.511 & 140 & 2.521 & 140 & 2.530 \\
150 & & & & & & & \\
\hline
\end{tabular}

\section*{TABLE XIV: The \( ^3 \text{H} \) binding energy \( B \) (in MeV) obtained in the present work is compared with the results present in the literature.}

\begin{tabular}{|c|c|c|}
\hline
Potential model & \( B \) & literature \\
\hline
Gaussian & 2.703 & 2.71 [18] \\
MN9 & 2.280 & 2.27 [19] \\
AU & 2.532 & 2.530 [21] \\
\hline
\end{tabular}