Robust and efficient transport of two-qubit entanglement via disordered spin chains

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Abstract We investigate how robust is the modified XX spin-1/2 chain of [R. Vieira and G. Rigolin, Phys. Lett. A 382, 2586 (2018)] in transmitting entanglement when several types of disorder and noise are present. First, we consider how deviations about the optimal settings that lead to almost perfect transmission of a maximally entangled two-qubit state affect the entanglement reaching the other side of the chain. Those deviations are modeled by static, dynamic, and fluctuating disorder. We then study how spurious or undesired interactions and external magnetic fields diminish the entanglement transmitted through the chain. For chains of the order of hundreds of qubits, we show for all types of disorder and noise here studied that the system is not appreciably affected when we have weak disorder (deviations of less than 1% about the optimal settings) and that for moderate disorder it still beats the standard and ordered XX model when deployed to accomplish the same task.

1 Introduction

An important tool in the practical implementation of quantum computation and communication tasks is the reliable transmission of quantum states from one location to another [1]. Quantum information, or equivalently a quantum state, can be sent from one party (Alice) to another (Bob) in at least three ways, namely, via direct transmission, via the quantum teleportation protocol [2], and via spin chains [3]. This last strategy, where Alice and Bob are connected by a spin chain through which they can send quantum states to each other.

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other is the main focus of this work. This task is accomplished by properly tuning the interaction among the qubits in the chain such that after the system evolves a certain time $t > 0$, a state prepared by one party at $t = 0$ reaches the other one with high fidelity. Note that here as well as in the quantum teleportation protocol no physical system is transmitted from Alice to Bob.

One important characteristic of a spin chain when employed to transmit quantum states is that the interaction strength among the qubits, once set up to give a high fidelity transmission, is fixed along the time evolution of the system. Also, by using spin chains as the mean through which quantum information is sent back and forth within a silicon-based quantum chip, we will deal with the same physical system in order to process and transmit quantum information. In this way we avoid the technologically difficult task of integrating different physical platforms, one for processing and other for transmitting information [3].

The several works investigating the usefulness of spin chains to perform quantum communication tasks have centered their focus on one of the three following scenarios, namely, the transmission of a single qubit [3,4,5,6,7,11,12,15,17,18,19,20,21,23,24,25,27,28,30,31,32,34], the generation of entanglement between the two ends or two specific sites of the chain [3,11,14,17,20,25,33,35,36], and the transmission of quantum states composed of two or more qubits [5,6,11,12,25,26,29,53]. We should also mention important contributions on the usefulness of chains of continuous variable systems in transmitting quantum states [8,9,10,17,33]. Most of the aforementioned works studied strictly one dimensional chains with open or periodic boundary conditions and extensive investigations on the performance of those systems in the presence of disorder and noise can be found in [5,37,38,39,40,41,42,43,44,45,46,47,48,49,50].

In this work our main goal is to investigate the performance of the quantum state transfer protocol presented in [53] when subjected to several types of noise and disorder. The protocol of [53] was specifically built to transmit maximally entangled two-qubit states from Alice to Bob and to be simple in its construction and operation, namely, we avoided any modulation in the coupling constants between the qubits along the chain [4,5,16] and we excluded the use of external magnetic fields to drive the transmission of the quantum state from Alice to Bob [12,22,25,29]. With such stringent requirements, we could achieve almost perfect entanglement transmission by going slightly beyond a one dimensional spin chain. In Fig. [1] we show the optimal configurations for the model proposed in [53] and for the standard strictly one dimensional system usually seen in the literature. We also point out a related geometry given in the work of Chen et al. [32] aiming at sending a single qubit from 1 to $N$. 
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Fig. 1 Upper panel: Alice’s qubits A and 1 are set at $t = 0$ as the maximally entangled Bell state $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$ while the remaining qubits are set in the state $|0\rangle$. The system then evolves according to the Hamiltonian $H$, Eq. (1), and at a specific time $t > 0$ Bob’s qubits $N$ and $B$ become highly entangled. For the ordered and noiseless case Bob’s qubits become an almost perfect replica of Alice’s qubits if $J$ and $J_m$ are properly adjusted [53]. Lower panel: Alice’s qubits 1 and 2 are initially $|\Psi^+\rangle$ and the remaining ones are in the state $|0\rangle$. The system then evolves according to the standard XX Hamiltonian and at a specific time $t > 0$ Bob’s qubits $N - 1$ and $B$ become entangled. However, the performance for the standard model decreases considerably when we add more and more spins to the chain [53]. This is not the case with the proposed model, where one can always find for any size of the chain a pair of values $J$ and $J_m$ such that the entanglement reaching Bob is for all practical purposes the maximum value possible, i.e., Bob always gets the Bell state $|\Psi^+\rangle$ [53].

2 The model

The ordered system is described by the following slightly modified XX Hamiltonian,

$$H = H_A + H_M + H_B,$$

where

$$H_A = J(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + J(\sigma_A^x \sigma_2^x + \sigma_A^y \sigma_2^y),$$

$$H_M = \sum_{j=2}^{N-2} J_m(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y),$$

$$H_B = J(\sigma_{N-1}^x \sigma_N^x + \sigma_{N-1}^y \sigma_N^y) + J(\sigma_{N-1}^x \sigma_B^x + \sigma_{N-1}^y \sigma_B^y).$$

(2)

The notation $\sigma_\alpha^\beta \sigma_\beta^\gamma$ means $\sigma_\alpha^\beta \otimes \sigma_\beta^\gamma$, the superscript indicates a specific Pauli matrix, and the subscript denotes the qubit acted by it. Here $\sigma^x|0\rangle = |0\rangle, \sigma^x|1\rangle = -|1\rangle, \sigma^y|0\rangle = |1\rangle, \sigma^y|1\rangle = |0\rangle, \sigma^y|0\rangle = i|1\rangle, \sigma^y|1\rangle = -i|0\rangle$, and $i = \sqrt{-1}$. We can get the standard XX model setting $J = 0$ and letting the summation for $H_M$ run from 1 to $N - 1$. In Ref. [53] it was shown that for a system of $N+2$ qubits, with $N = 100$, Bob gets about 99% of the entanglement created by Alice if $J_m/J = 49.98$ in the proposed model. And for small values
of \( J_m/J \), namely, \( J_m/J \leq 5 \), we get 82\% of the entanglement created by Alice arriving at Bob if \( J_m/J = 2.86 \). The strictly linear chain (standard model with \( N = 100 \) qubits) transmits only about 40\% of the entanglement with Alice in the best scenario (when we set \( J_m/J = 2.32 \)).

In order to study how disorder and noise affect the proposed model, we work with the following Hamiltonian,

\[
\hat{H} = \hat{H}_A + \hat{H}_M + \hat{H}_B + \hat{H}_z + \hat{H}_{zz},
\]

where

\[
\begin{align*}
\hat{H}_A &= J_{1,2}(t)(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + J_{A,2}(t)(\sigma_A^x \sigma_2^x + \sigma_A^y \sigma_2^y), \\
\hat{H}_M &= \sum_{j=2}^{N-1} J_{j,j+1}(t)(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y), \\
\hat{H}_B &= J_{N-1,N}(t)(\sigma_N \sigma_1^x + \sigma_N \sigma_1^y) + J_{N-1,B}(t)(\sigma_N \sigma_B^x + \sigma_N \sigma_B^y). \\
\hat{H}_z &= h_A(t)(1 - \sigma_A^z) + \sum_{j=1}^N h_j(t)(1 - \sigma_j^z) + h_B(t)(1 - \sigma_B^z), \\
\hat{H}_{zz} &= \Delta_{A,2}(t)\sigma_A^z \sigma_2^z + \sum_{j=1}^N \Delta_{j,j+1}(t)\sigma_j^z \sigma_{j+1}^z + \Delta_{N-1,B}(t)\sigma_N^z \sigma_B^z.
\end{align*}
\]

Here \( \hat{H}_A, \hat{H}_M, \) and \( \hat{H}_B \) extend the proposed model such that the coupling constants \( J_{A,J}(t) \) may now change with time and with position (qubit’s lattice location), allowing us to model several types of disorder as later explained. \( \hat{H}_z \) and \( \hat{H}_{zz} \) represent, respectively, external magnetic fields and the \( \sigma_j^z \sigma_{j+1}^z \) interaction, which are two types of noise that may act on the proposed model. When \( J_{A,2}(t) = J_{1,2}(t) = J_{N-1,N}(t) = J_{N-1,B}(t) = J, \) \( J_{j+1}(t) = J_m, \) for \( j = 2, \ldots, N - 2, \) and all \( h_j(t) \) and \( \Delta_{i,j}(t) \) are zero, we recover the proposed model of Ref. [53].

Hamiltonian [53] is such that \([H, Z] = 0\), i.e., it commutes with the operator \( Z = \sigma_A^+ + \sum_{j=1}^N \sigma_j^+ + \sigma_B^+ \). This implies that the number of spins up and down (excitations) does not change in time, which restricts considerably the size of the Hilbert space needed to describe the system. In this scenario, the Schrödinger equation can be efficiently solved when the system has just a few excitations [53].

Following Ref. [53], we aim at sending from Alice to Bob entangled states containing just one excitation. In this case, any system of \( N + 2 \) qubits, as depicted in Fig. 1, is described by the linear combination of \( N + 2 \) single-excitation states,

\[
|\psi(t)\rangle = \sum_j c_j(t)|1_j\rangle,
\]

where \( j = A, 1, 2, \ldots, N - 1, N, B, \) and

\[
|1_j\rangle = \sigma_j^z|00\cdots0\cdots0\rangle = \underbrace{|00\cdots1\cdots0\rangle}_{j\text{-th qubit}}.
\]
We assume that when \( t = 0 \) Alice’s qubits 1 and \( A \) are the maximally entangled Bell state \( |\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2} \), leading to the following initial state for the whole system, \( |\Psi(0)\rangle = (|010\cdots0\rangle + |10\cdots0\rangle)/\sqrt{2} \). In the notation of Eq. (5) we have at \( t = 0 \)

\[
c_j(0) = c_1(0) = 1/\sqrt{2} \text{ and } c_j(0) = 0, \text{ for } j \neq A, 1.
\]

After substituting Eq. (5) into the Schrödinger equation

\[
\frac{d}{dt} |\Psi(t)\rangle = \tilde{H}(t)|\Psi(t)\rangle
\]

and left multiplying it by the bra \( \langle 1_k | \) we get

\[
\frac{d}{dt} c(t) = \sum_j c_j(t) \langle 1_k | \tilde{H}(t) | 1_j \rangle.
\]

Using \( \tilde{H}(t) \), Eq. (3), we can compute \( \langle 1_k | \tilde{H}(t) | 1_j \rangle \) via the same strategy presented in Ref. [53]. This allows us to rewrite the Schrödinger equation as

\[
\frac{d c(t)}{dt} = \tilde{M}(t) \ c(t),
\]

where

\[
c(t) = (c_A(t), c_1(t), c_2(t), \ldots, c_{N-1}(t), c_N(t), c_B(t))^T
\]

is a column vector (\( T \) denotes transposition) and \( \tilde{M}(t) \) is the matrix of dimension \((N + 2) \times (N + 2)\) shown in Eq. (11):

\[
\tilde{M}(t) = \frac{i2}{\hbar}
\begin{pmatrix}
D_A(t) & 0 & J_{A,2}(t) & 0 & 0 & 0 & \cdots \\
0 & D_1(t) & J_{1,2}(t) & 0 & 0 & 0 & \cdots \\
J_{A,2}(t) & J_{1,2}(t) & D_2(t) & J_{2,3}(t) & 0 & 0 & \cdots \\
0 & 0 & J_{2,3}(t) & D_3(t) & J_{3,4}(t) & 0 & \cdots \\
0 & 0 & 0 & J_{3,4}(t) & D_4(t) & J_{4,5}(t) & \cdots \\
0 & 0 & 0 & 0 & J_{4,5}(t) & D_5(t) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

The diagonal elements of Eq. (11) are

\[
D_A(t) = h_A(t) + \Delta(t)/2 - \Delta_{A,2}(t), \quad (12)
\]
\[
D_1(t) = h_1(t) + \Delta(t)/2 - \Delta_{1,2}(t), \quad (13)
\]
\[
D_2(t) = h_2(t) + \Delta(t)/2 - \Delta_{A,2}(t) - \Delta_{1,2}(t) - \Delta_{2,3}(t), \quad (14)
\]
\[
D_3(t) = h_3(t) + \Delta(t)/2 - \Delta_{j-1,j}(t) - \Delta_{j,j+1}(t), \text{ for } 3 \leq j \leq N-2, \quad (15)
\]
\[
D_{N-1}(t) = h_{N-1}(t) + \Delta(t)/2 - \Delta_{N-2,N-1}(t) - \Delta_{N-1,N}(t) - \Delta_{N-1,B}(t), \quad (16)
\]
\[
D_N(t) = h_N(t) + \Delta(t)/2 - \Delta_{N-1,N}(t), \quad (17)
\]
\[
D_B(t) = h_B(t) + \Delta(t)/2 - \Delta_{N-1,B}(t), \quad (18)
\]
with

\[ \Delta(t) = \Delta_{A,2}(t) + \sum_{j=1}^{N-1} \Delta_{j,j+1}(t) + \Delta_{N-1,B}(t). \tag{19} \]

The solution to Eq. (9) can be formally written as a time-ordered matrix exponential with the aid of the time-ordering operator \( \hat{T} \),

\[
c(t) = \hat{T} \exp \left( \int_0^t \tilde{M}(t')dt' \right) c(0), \tag{20}\]

where \( c(0) \) is the column vector containing the initial conditions listed in Eq. (7).

3 Quantifying entanglement

We use the Entanglement of Formation (EoF) \([51,52]\) to quantify the entanglement shared between two qubits. Specifically, we want to determine as a function of time the EoF between qubits \( N \) and \( B \) with Bob. Tracing out the other \( N \) qubits from \( \rho(t) = |\Psi(t)\rangle \langle \Psi(t)| \), the density matrix describing the whole system, we get \([53]\)

\[
\rho_{N,B}(t) = \begin{pmatrix}
1 - |c_N(t)|^2 & -|c_N(t)|^2 & 0 & 0 & 0 \\
-|c_N(t)|^2 & |c_N(t)|^2 & c_N(t)c_B^*(t) & 0 & 0 \\
0 & c_N(t)c_B^*(t) & |c_B(t)|^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \tag{21}\]

where \( * \) means complex conjugation. Eq. (21) is the density matrix describing the two qubits \( N \) and \( B \) with Bob written in the basis \( \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \).

With the aid of \( \rho_{N,B}(t) \) the EoF can be computed following the recipe in \([52,53]\),

\[
\text{EoF}(\rho_{N,B}) = -f(C) \log_2 f(C) - [1 - f(C)] \log_2 [1 - f(C)], \tag{22}\]

with \( f(C) = (1 + \sqrt{1 - C^2})/2 \) and concurrence \( C \) given by \( C(t) = 2|c_N(t)c_B(t)| \).

Note that a maximally entangled two-qubit state has EoF = 1 and when the pair of qubits shares no entanglement EoF = 0.

4 Modeling disorder and noise

We deal with the three major types of disorder that can affect the system’s coupling constants \( J_{i,j} \), namely, static, dynamic, and fluctuating disorders \([54,55]\), and assume that any fluctuation in those coupling constants is about the optimal settings for the ordered case with \( N = 100 + 2 \) qubits. Specifically, we either employ \( J_m/J = 2.86, \) the optimal setting for \( J_m/J \leq 5.0 \) and whose transmitted entanglement is EoF = 0.816, or \( J_m/J = 49.98, \) the best case
when $J_m/J \leq 50.0$ and with transmitted entanglement given by EoF = 0.986 [53] (see the Appendix for calculations related to chains of $N = 1000 + 2$ qubits). In the rest of this work we set $J = 1.0$ and $\hbar = 1.0$, which defines $J$ as our unit of energy, and work with the two values of $J_m$ given above. Following Ref. [54,55], we also assume that when dealing with time dependent disorder, the Hamiltonian changes with time only a finite number of times and after the same period $\tau$.

In this scenario, we can model disorder if we set

$$J_{i,j}(t_k) = J_{i,j}(t_{k-1}) [1 + \delta J_{i,j}(k)],$$

with $J_{A,2}(0) = J_{1,2}(0) = J_{N-1,N}(0) = J_{N-1,B}(0) = J, \sum_{j=2}^{N-2} J_{j,j+1}(0) = J_m$, and $J_{i,j}(t) = 0$, any $t$, for other values of $i,j$. Here

$$(k-1)\tau < t_k \leq k\tau,$$

where $k$ is an integer such that $1 \leq k \leq n$. We also have $t_0 = 0$ and we define $T_{\text{max}} = \max\{t_n\} = n\tau$. Sometimes we call $T_{\text{max}}$ by $Jt/\hbar$, both notations meaning the time when in the optimal ordered model the transmitted entanglement from Alice to Bob is maximal. The quantity $\delta J_{i,j}(k)$ is responsible for introducing disorder in our system. Its behavior depends on which type of disorder we have but all cases boil down to random continuous uniform distributions centered in zero and whose ranges lie between $-p$ and $p$, with $p$ the maximal percentage fluctuation of $J_{i,j}$ about its optimal value. Note that $\delta J_{i,j}(k)$ is a function of the index $k$, i.e., whenever $k$ changes we have a new uniform distribution from which the random values of $\delta J_{i,j}$ are drawn.

In a similar way we can introduce noise. Noting that the optimal settings have no external magnetic field ($h_j(t) = 0$) and no $\sigma_i^z \sigma_j^z$ interaction ($\Delta_{i,j}(t) = 0$), the following prescription allows us to insert noise in our system,

$$h_j(t_k) = h_j(t_{k-1}) + \delta h_j(k),$$

$$\Delta_{i,j}(t_k) = \Delta_{i,j}(t_{k-1}) + \delta \Delta_{i,j}(k),$$

where $t_k$ and $k$ were defined above, $h_j(0) = 0$ for $j = A,1,2,\ldots,N-1,N,B$, $\Delta_{A,2}(0) = \Delta_{1,2}(0) = \Delta_{N-1,N}(0) = \Delta_{N-1,B}(0) = 0$, $\sum_{j=2}^{N-2} \Delta_{j,j+1}(0) = 0$, and $\Delta_{i,j}(t) = 0$, any $t$, for other values of $i,j$. Analogously to $\delta J_{i,j}(k)$, $\delta h_j(k)$ or $\delta \Delta_{i,j}(k)$ depends on the kind of disorder we have and will be explained shortly. But as before these uniform distributions are centered in zero, ranging between $-p$ and $p$, where $p$ now means the greatest possible value of $\delta h_j(k)$ and $\delta \Delta_{i,j}(k)$. Since our unit of energy is $J = 1.0$, $p$ can be read as the percentage of $J$ defining the upper bound for $\delta h_j(k)$ or $\delta \Delta_{i,j}(k)$ in a given uniform distribution.
4.1 Static disorder

In this scenario we have noise or fluctuations in the coupling constants that are time independent but site (position) dependent,

\[ \delta J_{i,j}(1) = \delta J_{i,j} \quad \delta h_j(1) = \delta h_j \quad \delta \Delta_{i,j}(1) = \delta \Delta_{i,j}, \quad (27) \]

\[ \delta J_{i,j}(k) = \delta h_j(k) = \delta \Delta_{i,j}(k) = 0, \quad k \neq 1. \quad (28) \]

This means that \( J_{i,j}(t) = J_{i,j}(0)(1 + \delta J_{i,j}) \), \( h_j(t) = \delta h_j \), \( \Delta_{i,j}(t) = \delta \Delta_{i,j} \), where \( 0 \leq t \leq n\tau \). In other words, each quantity above is chosen from independent continuous uniform distributions at \( t = 0 \) and fixed during the time evolution.

In this case Eq. (20) can be integrated and we get

\[ c(t) = \exp(\tilde{M} t) c(0). \quad (29) \]

By numerically computing the matrix exponential we obtain \( c(t) \) and then, using Eq. (22), the entanglement at any \( t \) between the qubits \( N \) and \( B \) with Bob.

4.2 Dynamic disorder

In contrast to the case of static disorder, we have time dependent but position independent noise or fluctuations in the coupling constants,

\[ \delta J_{i,j}(k) = \delta J(k), \quad \delta h_j(k) = \delta h(k), \quad \delta \Delta_{i,j}(k) = \delta \Delta(k). \quad (30) \]

Now, each coupling constant changes equally by \( \delta J(k) \) at the time \( (k - 1)\tau \), remaining fixed from \( (k - 1)\tau \) to \( k\tau \). After each period of time \( \tau \) a new random number is drawn from a continuous uniform distribution as previously explained and a new value of the coupling constant is obtained: \( J_{i,j}(t_k) = J_{i,j}(t_{k-1})[1 + \delta J(k)] \). Similarly, all qubits are subjected to the same external field from \( (k - 1)\tau \) to \( k\tau \) and the \( \sigma_z^i \sigma_z^j \) interaction strength is the same for any pair of nearest neighbors during this time span. After a period of time \( \tau \) new values for the field and for the \( \sigma_z^i \sigma_z^j \) interaction strengths are obtained from two independent continuous uniform distributions \( \delta h(k) \) and \( \delta \Delta(k) \) according to the prescription given in Eq. (23) and (26). We should note that it is convenient sometimes to express \( \tau \) as a percentage of \( T_{max} = Jt/\hbar \). Thus \( \tau = (Jt/\hbar)/n \), with \( 1/n \) denoting the percentage of the total time of evolution \( T_{max} \) giving the period of changes.

Noting that between time \( (k - 1)\tau \) and \( k\tau \), where \( k \) is an integer between 1 and \( n \), the matrix \( \tilde{M}(t) \) is time independent, the time evolution of the system all the way down to \( T_{max} = n\tau \) can be written as

\[ c(n\tau) = \exp(\tilde{M}_n \tau) \exp(\tilde{M}_{n-1} \tau) \ldots \exp(\tilde{M}_1 \tau) c(0), \quad (31) \]

where \( \tilde{M}_k \) is a time independent matrix given by Eq. (11) such that \( \tilde{M}_k = \tilde{M}_{k-1} + \delta \tilde{M}_k \). Here \( \tilde{M}_0 \) is associated to the optimal ordered Hamiltonian, Eq. (1), and \( \delta \tilde{M}_k \) is the modification to \( \tilde{M}_{k-1} \) coming from the presence of the dynamical disorder given by Eqs. (23)–(26) and (30).
4.3 Fluctuating disorder

This is the case where both aspects of static and dynamic disorders are combined, namely, where we have time and position dependent noise or time and position dependent fluctuations in the coupling constants. The coupling constants between qubits as well as the strength of the noise are given by Eqs. (23)-(26), with $\delta J_{i,j}(k)$, $\delta h_j(k)$, and $\delta \Delta_{i,j}(k)$ depending on the position of the qubits and on time in the same way already described, respectively, for the cases of static and dynamic disorders. The time evolution of the system is given by Eq. (31) and $\tilde{M}_k$ is a time independent matrix with position dependent coefficients given by Eq. (11).

5 Results

We first study static disorder and dynamic and fluctuating disorders for $\tau = 10\% \tau_{\text{max}} = 10\% J t/\hbar$ (Figs. 2 and 3).

**Fig. 2** Proposed model with $N = 100 + 2$ qubits with 1000 static (left) and dynamic (right) disorder realizations for each value of the x-axis (percentage deviation $p$ as explained in the text), starting at 0.1% and going up to $p = 10\%$, in increments of 0.1%. All solid curves and the dashed ones refer to the model here reported (proposed model). The only curves referring to the standard model (strictly one dimensional chain) are the dotted ones. Upper panels: The solid curves give the average of the entanglement transmitted after 1000 realizations for each value of $p$ and at $J t/\hbar = 9.54$, the time where the optimal transmission of entanglement in the ordered system occurs if $J_m/J = 2.86$. This value for $J_m/J$ is that giving the best performance when $J_m/J$ ranges from 0 to 5. The circle-black curves denote the noiseless case, where only the coupling constants are subjected to disorder. The diamond-red curves represent the case where the noise is given by external magnetic fields and the star-green ones when noise is given by the $\sigma_z \sigma_z$ interaction. The square-blue curves depict the case where both noises are present as well as disorder in the coupling constants. The dashed curves are the optimal entanglement transmitted for the clean system and the dotted ones show the greatest entanglement transmitted for the standard $N = 100$ qubits ordered model. Lower panels: Again we have the proposed model but now $J t/\hbar = 20.53$, the time where the optimal transmission of entanglement for the clean system occurs when $J_m/J = 49.98$. This value for $J_m/J$ is the one giving the best performance for the proposed model when $J_m/J$ ranges from 0 to 50. See text for more details.
This value of $\tau$ means that the Hamiltonian changes 10 times during the time evolution of the system from $t = 0$ to $t = T_{\text{max}}$ according to the prescription given above.

Looking at the upper panels of Figs. 2 and 3, we note the following trends for the entanglement transmitted to Bob. For small values of $J_m/J$ the presence of noise barely affects the efficiency of the protocol, i.e., the presence of random external magnetic fields and $\sigma_i^z\sigma_j^z$ interactions do not affect considerably the system’s performance up to a deviation of 10% about the optimal settings for the ordered case. On the other hand, the presence of disorder in the coupling constants affects the system performance much more strongly than the presence of noise. However, for deviations of 1% about the optimal settings, the efficiency is barely reduced.

Moreover, among the three types of disorder in the coupling constants, static disorder is the least severe. In this scenario, even for a deviation of 10% about the optimal settings, we still beat the entanglement transmitted using...
the optimal ordered standard model (dotted lines). Note that dynamic and fluctuating disorders in the coupling constants affect the system nearly in the same way. When noise and disorder in the coupling constants act simultaneously, the reduction in efficiency is almost the same as if the noise was absent, another confirmation of the fact that the types of noise here studied barely affect the system’s efficiency for small values of $J_m/J$.

![Fluctuating Disorder for Several $\tau$'s](image)

**Fig. 4** Proposed model with noise and coupling constants simultaneously modeled by fluctuating disorder. Here $J_m/J = 49.98$ and we study how the efficiency of the entanglement transmission is affected as we decrease $\tau$, the period of changes in the Hamiltonian. The meaning of all curves above and how they were computed are similar to those given in Figs. 2 and 3. It is clear that the smaller the period of changes, the lower the amount of entanglement reaching Bob. Note that all curves refer to the proposed model, with the exception of the dotted one, which is the optimal entanglement transmitted for the ordered standard model (strictly one dimensional chain).

Moving to the lower panels of Figs. 2 and 3, we note different trends for the entanglement transmitted to Bob. Indeed, for high values of $J_m/J$ the presence of noise becomes relevant in almost all cases, reducing the efficiency of the protocol, and the presence of disorder in the coupling constants no longer dominates the reduction of the system’s performance. Of all three types of disorder, dynamic disorder is now the least severe. In this case, even for a deviation of 10% about the optimal settings, the entanglement transmitted to Bob is very high, close to the optimal value for the clean system and way above the entanglement reaching Bob if we employ the standard model (dotted lines). Dynamic and fluctuating disorders, on the other hand, no longer affect the system in the same way, with the latter being much more severe. When noise and disorder in the coupling constants act simultaneously, we get almost always the worst scenario, where the reduction in efficiency is most severe (square-blue curves). However, even for high values of $J_m/J$, deviations of 1%
about the optimal settings do not affect much the amount of entanglement reaching Bob.

In Fig. 4 we study how the rate of change of the Hamiltonian during the time evolution affects the system’s performance. We work with the worst of the six scenarios shown in Figs. 2 and 3, namely, fluctuating disorder with noise and disorder in the coupling constants simultaneously affecting the system. Looking at Fig. 4 it is not difficult to convince ourselves that the higher the frequency of changes in the Hamiltonian or, equivalently, the smaller the period \( \tau \) of changes, the lower the entanglement transmitted to Bob.

The reason why a small period in the fluctuating disorder leads to a poor transmission of entanglement is the fact that the smaller the period of changes, the higher the frequency of changes in the Hamiltonian. This higher number of changes in the Hamiltonian is cumulative and ultimately leads to a greater deviation from the optimal values of the coupling constants. For longer periods, nevertheless, the Hamiltonian changes only once or twice during the whole time evolution, leading to almost no change in the optimal settings when we work with small disorder. This fact reflects in a better performance for systems with longer periods of changes in the Hamiltonian when compared with the shorter ones.

\[ \text{Disorder and Noise Simultaneously} \]

\[ \tau = 10\% \text{ J/h} \]

\[ N = 100 + 2 \text{ Qubits} \]

\[ J = 1.0, \ J_m = 2.86, \ J/t = 9.54 \]

\[ J = 1.0, \ J_m = 49.98, \ J/t = 20.53 \]

1000 simulations for each value of the x axis in increments of 0.1%

**Fig. 5** We show for two values of \( J_m/J \) how different types of disorder affect the proposed model’s performance. The meaning of all quantities shown here is the same as explained in Figs. 2, 3, and 4. Note that all curves refer to the proposed model, with the exception of the dotted ones, which are the optimal entanglement transmitted for the ordered standard model (strictly one dimensional chain).

In order to make it clearer to analyze how different types of disorder affect the efficiency of the protocol, we show in Fig. 5 and in the same graphic the
entanglement transmitted for the three different types of disorder studied here. We fix the value of \( \tau \) such that it is 10% of \( J_t/\hbar \), when dealing with dynamic and fluctuating disorders, and work with the scenario where noise and disorder in the coupling constants are simultaneously affecting the system.

For \( J_m/J = 2.86 \), upper panel of Fig. 5, we note that for small fluctuations about the optimal settings (< 5%), dynamic disorder is the most severe type of disorder. For greater values of deviations about the optimal settings, fluctuating disorder becomes the worst case. Also, for small values of \( J_m/J \), static disorder is the least severe type of disorder. Looking at the lower panel of Fig. 5, when \( J_m/J = 49.98 \), we note that dynamic disorder becomes very benign, little affecting the system’s performance. We also note that fluctuating disorder is the most severe type of disorder for all range of fluctuations about the optimal settings of the clean system.

6 Conclusion

Summing up, we have extensively studied how the protocol presented in Ref. [53], specially devised to transmit maximally entangled Bell states, responds to disorder and noise for systems of size of the order of a hundred qubits. We have studied three types of disorder, namely, static, dynamic, and fluctuating disorders [54,55] and two types of noise that may affect the system. In one type of noise we have random external magnetic fields acting on the qubits of the system and in the other one we investigated how spurious \( \sigma^z_i \sigma^z_j \) interactions reduce the system’s capacity to transmit entanglement.

For all the several cases of noise and disorder presented here, we showed that for deviations of up to 1% about the optimal settings of the clean system, we still have excellent entanglement transmission, with little or no reduction in the efficiency of the protocol. Moreover, up to deviations about the optimal settings of the order of 3%, the proposed model (upper panel of Fig. 1) still beats the optimal entanglement transmission efficiency of the standard model (lower panel of Fig. 1). For a system’s size of the order of a thousand qubits, we obtain excellent transmission of entanglement for deviations in the optimal settings of up to 0.1% and acceptable ones for deviations of up to 0.5% (see the Appendix).

All those results show that the proposed model is very robust to noise and disorder and we believe it might prove fruitful to test in future works its robustness when the number of excitations is no longer conserved and when we have residual Dzyaloshinskii-Moriya interactions affecting the system [56,57].

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A The 1000 + 2 qubits case

Whenever we have dynamic or fluctuating disorders, specially for small values of \( \tau \), the computational resources needed to handle systems of the order of thousands of qubits are very demanding. For that reason, we report here only two calculations for the \( N = 1000 + 2 \) qubits system by implementing one tenth of the realizations made for the \( N = 100 + 2 \) qubits case, namely, instead of 1000 realizations for each disorder percentage, we work with 100 realizations. We have also worked with the worst possible scenario, in which the system is most severely affected, i.e., fluctuating disorder affecting the coupling constants as well as fluctuating random external fields and spurious \( \sigma_z \sigma_z \) interactions. Note that for less stringent disorder and noise scenarios, the \( N = 1000 + 2 \) qubits case gives much better results.

![Graph showing average entanglement versus range of relative fluctuation about the optimal ordered settings for \( N = 100 + 2 \) and \( N = 1000 + 2 \) qubits under fluctuating disorder and noise.](image)

**Fig. 6** For \( N = 100 + 2 \) qubits we have \( J_m/J = 2.86 \), the optimal setting for the clean system when \( J_m/J \leq 5.0 \), giving a transmitted entanglement of \( E_{oF} = 0.81 \) at the time \( Jt/\hbar = 9.54 \) (square-black curve). For \( N = 1000 + 2 \) qubits we have \( J_m/J = 4.25 \), the optimal setting for the clean system when \( J_m/J \leq 5.0 \), with transmitted entanglement given by \( E_{oF} = 0.71 \) at the time \( Jt/\hbar = 9.54 \) (circle-red curve). Solid curves are related to the proposed model when affected by fluctuating disorder and noise about those optimal settings while the dotted curves refer to the optimal entanglement transmitted if we employ the clean standard model (strictly linear chain). The period \( \tau \) of changes in the Hamiltonian is shown in the figure.

As expected, the results depicted in Figs. 6 and 7 show that the longer the chain the more the system is affected by disorder and noise. However, we still have very good entanglement transmission for disorder of less than 0.1% about the optimal settings of the clean system and an acceptable one for less than 0.5%. Moreover, for disorder of less than 1%, we still beat the optimal transmission of the clean standard model (strictly linear chain).
Robust and efficient transport of entanglement via disordered spin chains

For $N = 100 + 2$ qubits we have $J_m/J = 49.98$, the optimal setting for the clean system when $J_m/J \leq 50$, giving a transmitted entanglement of $E_{oF} = 0.98$ at the time $Jt/\hbar = 20.53$ (square-black curve). For $N = 1000 + 2$ qubits we have $J_m/J = 298.27$, the optimal setting for the clean system when $J_m/J \leq 300$, with transmitted entanglement given by $E_{oF} = 0.99$ at the time $Jt/\hbar = 118.55$ (circle-red curve). Solid curves are related to the proposed model when affected by fluctuating disorder and noise about those optimal settings while the dotted curves refer to the optimal entanglement transmitted if we employ the clean standard model (strictly linear chain). The period $\tau$ of changes in the Hamiltonian is shown in the figure.

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