Simultaneous measurement of coordinate and momentum on a von Neumann lattice

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Abstract – It is shown that on a finite phase plane the kq-coordinates and the sites of a von Neumann lattice are conjugate to one another. This elementary result holds when the number M defining the size of the phase plane can be expressed as a product, \( M = M_1 M_2 \), with \( M_1 \) and \( M_2 \) being relatively prime. As a consequence of this result a hitherto unknown wave function is defined giving the probability of simultaneously measuring the momentum and coordinate on the von Neumann lattice.

Following the pioneering work of Schwinger [1] in the finite phase plane, there has been much interest and activity in a great variety of physical problems, including quantum measurements [2], quantum computing [3], quantum maps [4], Landau levels in a magnetic field and von Neumann lattices [5], quantum teleportation [6], and others (see review article by Vourdas [7]). In general, when limiting the motion to a finite phase plane, one might obtain some surprising results that do not exist in the infinite phase plane. An example of such a result are the mutually unbiased bases [8] which are characteristic for a finite phase plane. Another example is the Wigner function which requires different definitions for even and odd dimensionality \( M \) of the phase plane [9]. Yet another example is the \( kq \)-representation in finite phase plane [5,10]. In this example, when \( M = M_1 M_2 \), with \( M_1 \) and \( M_2 \) being relatively prime, one can construct a \( KQ \)-representation, which is conjugate to the \( kq \)-representation. This is not possible in an infinite phase plane.

In this letter we show that in a finite phase plane with lengths \( Mc \) in the \( x \)-direction and \( \hbar \frac{2\pi}{c} \) in the \( p \)-direction (\( c \) is a constant), the \( kq \)-coordinates [11] and the sites of the corresponding von Neumann lattice are mutually conjugate. This result holds when \( M = M_1 M_2 \), with \( M_1 \) and \( M_2 \) being relatively prime. The \( kq \)-coordinates are defined by the eigenvalues of the commuting operators

\[
T(a) = \exp \left( \frac{i}{\hbar} pa \right) \quad \text{and} \quad \tau \left( \frac{2\pi}{a} \right) = \exp \left( i x \frac{2\pi}{a} \right), \quad \text{where} \quad a = M_1 c.
\]

The unit cell of the corresponding von Neumann lattice has the dimensions \( a \) and \( \hbar \frac{2\pi}{a} \) in the \( x \) and \( p \)-directions, respectively. We call it the \( a \)-von Neumann lattice. An additional set of \( KQ \)-coordinates is built which are eigenvalues of the commuting operators

\[
T(b) = \exp \left( \frac{i}{\hbar} pb \right) \quad \text{and} \quad \tau \left( \frac{2\pi}{b} \right) = \exp \left( i x \frac{2\pi}{b} \right), \quad \text{where} \quad b = M_2 c \quad [10].
\]

Similarly, the \( KQ \)-coordinates are conjugate to the sites of the \( b \)-von Neumann lattice, having as the unit cell \( b \) and \( \hbar \frac{2\pi}{b} \), respectively. Then a very surprising and unexpected result is proven, namely, that the eigenvalues of the operators \( T(a) \) and \( \tau \left( \frac{2\pi}{a} \right) \) are also the sites \( u'b \) and \( v'h \frac{2\pi}{a} \) of the \( b \)-von Neumann lattice, where \( u' \) and \( v' \) are integers modulo \( M_1 \) and \( M_2 \), respectively. A similar proof is given for the \( KQ \)-coordinates and the sites of the \( a \)-von Neumann lattice. Having proven this we show how to define a new wave function which gives the probability of a simultaneous measurement of the momentum and coordinate on the von Neumann lattice.

As is known [5,10] a finite phase plane can be achieved by using boundary conditions on the wave function \( \psi(x) \) and its Fourier transform \( F(p) \)

\[
\psi(x + Mc) = \psi(x); \quad F \left( p + \hbar \frac{2\pi}{c} \right) = F(p), \quad (1)
\]

where \( M \) is an integer and \( c \) is a constant which determines the discreteness of \( x \) and \( p \) in the phase plane. Following the boundary conditions in eq. (1), the coordinate \( x \) and

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the momentum $p$ turn out to be discrete and assume the values

$$x = sc, \ s = 0, 1, \ldots, M - 1; \quad p = \hbar \frac{2\pi}{Mc} t, \ t = 0, 1, \ldots, M - 1.$$  

(2)

In the finite phase plane the operators $x$ and $p$ are then replaced by the following exponential operators:

$$\tau \left( \frac{2\pi}{Mc} \right) = \exp \left( i \frac{2\pi}{Mc} x \right), \quad T(c) = \exp \left( \frac{i}{\hbar} pc \right).$$  

(3)

Let us assume that

$$M = M_1 M_2.$$  

(4)

One then can define a $kq$-representation, based on the constant $a = M_1 c$, with the commuting operators [11]

$$T(a) = \exp \left( i \frac{2\pi}{Mc} pa \right)$$

and

$$\tau \left( \frac{2\pi}{a} \right) = \exp \left( i \frac{2\pi}{a} q \right).$$  

(5)

The eigenvalues of these operators are the $kq$-coordinates in the finite phase plane $k_n$ and $q_m$

$$T(a) : \ \exp(ik_n a) \quad \text{and} \quad \tau \left( \frac{2\pi}{a} \right) : \ \exp \left( i \frac{2\pi}{a} q_m \right).$$  

(6)

where

$$k_n = \frac{2\pi}{Mc} n, \quad n = 0, 1, \ldots, M_2 - 1;$$

$$q_m = mc, \quad m = 0, 1, \ldots, M_1 - 1.$$  

(7)

For the case where $M = 15$, $M_1 = 3$ and $M_2 = 5$, these eigenvalues are shown in fig. 1 by open circles. According to the definition of the eigenvalues in eqs. (6) and (7), $k_n$ is given modulo $\frac{2\pi}{a}$, and $q_m$ is given modulo $a$. This means that any of the unit cells of the von Neumann lattice in fig. 1 could be used for plotting the eigenvalues in eqs. (6) and (7). We would like to draw to the attention of the reader that the area of the unit cell of the von Neumann lattice is $\hbar$, the Planck constant, which makes the von Neumann lattice very attractive.

Similarly, we can define another von Neumann lattice with the constant $b = M_2 c$, and, respectively, another $KQ$-representation (here we use capital $K$ and $Q$) based on the commuting operators [10]

$$T(b) = \exp \left( i \frac{2\pi}{b} pb \right)$$

and

$$\tau \left( \frac{2\pi}{b} \right) = \exp \left( i \frac{2\pi}{b} q \right).$$  

(8)

The eigenvalues of these commuting operators will be labelled by $K_{n'}$ and $Q_{m'}$ which will assume the values (cf. eq. (7))

$$K_{n'} = \frac{2\pi}{Mc} n', \quad n' = 0, 1, \ldots, M_1 - 1;$$

$$Q_{m'} = m' c, \quad m' = 0, 1, \ldots, M_2 - 1.$$  

(9)

It is clear that for $M = 15$ one can draw a figure similar to fig. 1, but with $a = 3c$ replaced by $b = 5c$. Correspondingly, we will have 3 unit cells in the $x$-direction and 5 unit cells in the $p$-direction, see fig. 2. Up to now there was no restriction on the factors $M_1$ and $M_2$ in the partition of $M$ in eq. (4). However, as was shown in ref. [10], a very basic new result is obtained when $M_1$ and $M_2$ are relatively prime. Then the operators in eqs. (5) and (8) are
mutually conjugate. In operator language this conjugacy is expressed in the following way [10]:

\[ T(a) \tau (\frac{2\pi}{b}) = \tau (\frac{2\pi}{b}) T(a) \exp \left( 2\pi i \frac{M_1}{M_2} \right), \]

\[ T(b) \tau (\frac{2\pi}{a}) = \tau (\frac{2\pi}{a}) T(b) \exp \left( 2\pi i \frac{M_2}{M_1} \right). \]  

(10)

On the other hand, in the language of the eigenstates of the operators in eqs. (5) and (8) this conjugacy assumes the following form [10]:

\[ \langle k, q | K, Q \rangle^2 = \frac{1}{M}. \]

(11)

where we have omitted the subscripts on \( k, q, K \) and \( Q \) (see eqs. (7) and (9)). In eq. (11) \( |k, q \rangle \) are the eigenvectors of the commuting operators in eq. (5), while \( |K, Q \rangle \) are the eigenvectors of the commuting operators in eq. (8).

We are now ready to prove that the \( kq \)-coordinates and the sites of the \( a\)-von Neumann lattice are mutually conjugate. Here the \( a\)-von Neumann lattice is the lattice that corresponds to the \( kq\)-representation (see eq. (5)) defined by means of the constant \( a \). In the example for \( M = 15 \), this is given in fig. 1. Let us denote the sites of the \( a\)-von Neumann lattice by \( \alpha_{st} \)

\[ \alpha_{st} = sa + t \frac{2\pi}{a}, \]  

where \( s \) runs from 0 till \( M_2 - 1 \), and \( t \) from 0 till \( M_1 - 1 \). In our example of \( M = 15 \), \( M_1 = 3 \), and \( M_2 = 5 \), \( s \) runs from 0 to 4 and \( t \) runs from 0 to 2 (see fig. 1). At this juncture we have that the couples \( kq \) and \( KQ \) are mutually conjugate coordinates. In order to prove that \( kq \) and \( \alpha_{st} \) are mutually conjugate, we have to show that there is a one-to-one correspondence between the coordinates \( KQ \) and \( \alpha_{st} \). Let us consider the \( K_{n'} \) and \( Q_{m'} \) coordinates in eq. (9) and write the following equations:

\[ n' \frac{2\pi}{M_C} + v' \frac{2\pi}{b} = \frac{2\pi}{a}, \quad m'c + u'b = sa, \]  

(13)

where \( s, t, u' \) and \( v' \) are integers. In what follows we shall use primed letters to relate to the \( b\)-von Neumann lattice (unprimed are for the \( a\)-lattice). The meaning of eqs. (13) is very simple. On the left-hand sides we have a \( K_{n'}, Q_{m'} \)-point in the unit cell of the \( b\)-von Neumann lattice modulo the \( (u', v') \)-lattice site (see eq. (9) and fig. 2). On the right-hand side, we have a \( (t, s) \)-site of the \( a\)-von Neumann lattice. We can now convince ourselves that the \( K_{n'}, Q_{m'} \)-point determines uniquely the \( (t, s) \)-site. For this let us rewrite the first equation of eqs. (13) by dividing both sides of it by \( \frac{2\pi}{M_C} \). We get the following Diophantine equation:

\[ n' + M_1 v' = M_2 t. \]  

(14)

The particular feature of this equation is that for any \( n' = 0, 1, \ldots, M_1 - 1 \) there is a single well determined pair \( (v', t) \) solving it [10]. This means that there is a well-defined integer \( t \) (mod \( M_1 \)) for any \( n' \). Similarly, the second equation in eqs. (13), after dividing by \( c \), turns into

\[ m' + M_2 v' = M_1 s. \]  

(15)

Again, this equation gives a well-defined pair \( (u', s) \) for every \( m' \). From eqs. (13)–(15) it follows that the \( (K_{n'}, Q_{m'}) \)-points in the unit cell of the \( b\)-von Neumann lattice determines uniquely the site \( (\frac{2\pi}{M_C} t, \frac{a}{c} s) \) in the \( a\)-von Neumann lattice. This leads to a one-to-one correspondence between the \( M \) \( (K_{n'}, Q_{m'}) \)-points in the unit cell of the \( b\)-von Neumann lattice (see eq. (9)) and the \( M \) sites \( (\frac{2\pi}{M_C} t, \frac{a}{c} s) \) (with \( t = 0, 1, \ldots, M_1 - 1; s = 0, 1, \ldots, M_2 - 1 \)) of the \( a\)-von Neumann lattice. A similar analysis shows the one-to-one correspondence between the \( K_{n'}, q_{m'} \)-points in the unit cell of the \( a\)-von Neumann lattice and the sites of the \( b\)-von Neumann lattice. From this we deduce the result stated in the abstract, namely, that the \( kq \)-coordinates (we drop the subscripts) in a unit cell of an \( a\)-von Neumann lattice are conjugate to the sites of the same lattice. This we deduce from the fact that the \( kq \) and \( KQ \) are conjugate coordinates [10], and, as we have shown, the \( KQ \)-coordinates determine uniquely the sites of the \( a\)-von Neumann lattice. In fig. 1 the \( kq \)-coordinates in the unit cell are denoted by open circles, while the sites of the \( a\)-von Neumann lattice are denoted by \( \times \)’s. Visually, the conjugacy of the \( kq \) coordinates and the von Neumann lattice can be expressed by saying that the open circles and the \( \times \)’s are conjugate. A similar statement holds for fig. 2.

Let us now restate the results in the language of operators and their eigenfunctions. In the finite phase plane, the eigenfunctions of the \( b \)-operators (eq. (8)) are [5,10]

\[ \langle x | K_{n'}, Q_{m'} \rangle = \frac{1}{\sqrt{M_1}} \sum_{\ell=0}^{M_1-1} \exp(iK_{n'}\ell/b) \Delta(x - Q_{m'} - \ell b), \]  

(16)

where \( \Delta(x) \) is 1 when \( x \) is a multiple of \( M_C \) and zero otherwise. These functions satisfy the eigenvalue equations

\[ T(b) \langle x | K_{n'}, Q_{m'} \rangle = e^{iK_{n'}b} \langle x | K_{n'}, Q_{m'} \rangle, \]

\[ \tau \left( \frac{2\pi}{b} \right) \langle x | K_{n'}, Q_{m'} \rangle = e^{iQ_{m'} \frac{2\pi}{a}} \langle x | K_{n'}, Q_{m'} \rangle, \]  

(17)

where \( e^{iK_{n'}b} \) and \( e^{iQ_{m'} \frac{2\pi}{a}} \) are the eigenvalues of the operators in eq. (8) (see eq. (9)). These eigenvalues can, respectively, be replaced by

\[ e^{iK_{n'}b} = e^{it \frac{2\pi}{M_C}}; \quad e^{iQ_{m'} \frac{2\pi}{a}} = e^{isa \frac{2\pi}{M_C}} \]

(18)

This follows from eq. (13) because the term \( \frac{2\pi}{M_C} v' \) in the first equation and the term \( u'b \) in the second equation do not contribute to eq. (18). Equations (17) and (18) show that the operators \( T(b) \) and \( \tau \left( \frac{2\pi}{b} \right) \) (eq. (8)) have as their eigenvalues not only the coordinates \( K_{n'} \) and \( Q_{m'} \) but also
the sites $\alpha_{st}$ in eq. (12) of the a-von Neumann lattice. A similar analysis can be carried out for the eigenfunctions $(x|k_n, q_m)$ of the operators $T(a)$ and $T^\pi$ in eq. (5) (see eqs. (6) and (7)). As in eq. (13), we can connect $k_n$ and $q_m$ in eq. (7) to the sites of the a-von Neumann lattice (see eq. (7))

$$k_n + \frac{2\pi}{a} = v' \frac{2\pi}{b}, \quad q_m + sa = u'b,$$

(19)

where again $s, t, u'$ and $v'$ are integers, with $s, t$ relating to the a-von Neumann lattice, and $u', v'$ to the b-von Neumann lattice. As before in eq. (13), $k_n$ and $q_m$ determine uniquely these integers. We can therefore write equations, similar to eqs. (18),

$$e^{ik_n a} = e^{iu'2\pi a}, \quad e^{iq_m 2\pi} = e^{iu'b 2\pi}.$$  

(20)

This follows from eq. (19). We obtain the result that not only $k_n$ and $q_m$ but also the sites

$$\beta_{u'v'} = u'b + v'\hbar \frac{2\pi}{b}$$

(21)

of the b-von Neumann lattice are eigenvalues of the operators in eq. (5). The results in eqs. (18) and (20) are surprising and clearly hold only when $M_1$ and $M_2$ in eq. (4) are relatively prime.

We are now in a position to write down a new quantum-mechanical wave function, which has as its variables the sites of a von Neumann lattice. Consider the wave function $C^{(a)} (k_n, q_m)$ [10,11] in the $kq$ representation. By using eq. (19), that connects $k_n$ and $q_m$ in eqs. (6) and (7) to the sites of the a- and b-von Neumann lattices, the wave function $C^{(a)} (k_n, q_m)$ will become

$$C^{(a)} (k_n, q_m) = \exp \left(2\pi i s u' \frac{M_1}{M_2}\right) C^{(a)} \left(v' \frac{2\pi}{b}, u'b\right),$$

(22)

where we have used the periodic boundary conditions satisfied by a $kq$-function [11]. Equation (22) shows that the $kq$-function $C^{(a)} (k_n, q_m)$, on the a-von Neumann lattice, determines a wave function $C^{(a)} (v' \frac{2\pi}{b}, u'b)$ which has as its arguments the sites of the b-von Neumann lattice. This is a hitherto unknown function in quantum mechanics. The square of its absolute value

$$|C^{(a)} \left(v' \frac{2\pi}{b}, u'b\right)|^2$$

(23)

gives the probability for a simultaneous measurement of the momentum $v'\hbar \frac{2\pi}{b}$ and the coordinate $u'b$ on the b-von Neumann lattice. Similarly, we can use the $C^{(b)} (k'_n, q'_m)$ for defining the wave function $C^{(b)} \left(\frac{2\pi}{a}, sa\right)$. The square of its absolute value

$$|C^{(b)} \left(\frac{2\pi}{a}, sa\right)|^2$$

(24)

gives the probability for a simultaneous measurement of the momentum $\theta \hbar \frac{2\pi}{a}$ and the coordinate $sa$ on the a-von Neumann lattice.

In summary, it has been shown that on a finite phase plane of length $M\pi$ in the $x$-direction and $h\frac{2\pi}{b}$ in the $p$-direction, the $kq$-coordinates and the sites of the corresponding von Neumann lattice form mutually conjugate coordinates. This result holds when $M$ is a product of two relatively prime numbers $M_1$ and $M_2$. By using this conjugacy, a new wave function was introduced into elementary quantum mechanics, which has as its arguments the sites of the von Neumann lattice. The square of the absolute value of this wave function gives the probability of simultaneously measuring the momentum and coordinate on a von Neumann lattice. This result brings us back to the work of von Neumann in the very beginning of quantum mechanics [12], where he was looking for a complete and orthogonal set of eigenfunctions on a von Neumann lattice. Much work has been done and published on this subject [13–15], attempting to achieve the von Neumann hypothesis [15] in one or another approximation or modification. We refer, in particular, to a recently published paper on the subject with an original approach to the problem, and which gives also a good summary of the literature [16]. It is worth noting that the results of this work and of ref. [10] establish properties of mutually unbiased bases (MUB) in a space of dimension $d$ which is not a power of a prime. This may be of interest in the attempts to obtain a set of $d + 1$ MUB for any dimension.

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