The Shapovalov Determinant for the Poisson Superalgebras

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Abstract

Among simple \( Z \)-graded Lie superalgebras of polynomial growth, there are several which have no Cartan matrix but, nevertheless, have a quadratic even Casimir element \( C_2 \): these are the Lie superalgebra \( K'(1|\beta) \) of vector fields on the \((1|\beta)\)-dimensional supercircle preserving the contact form, and the series: the \( n \)-dimensional Lie superalgebra \( sh(0|2k) \) of special Hamiltonian fields in \( 2k \) odd indeterminates, and the \( Kac(M) \) odd version of \( sh(0|2k) \). Using \( C_2 \) we compute N. Shapovalov determinant for \( K'(1|\beta) \) and \( sh(0|2k) \), and for the Poisson superalgebras \( po(0|2k) \) associated with \( sh(0|2k) \). A. Shapovalov described irreducible finite dimensional representations of \( po(0|n) \) and \( sh(0|n) \); we generalize his result for Verma modules: give criteria for irreducibility of the Verma modules over \( po(0|2k) \) and \( sh(0|2k) \).

Introduction

Every simple finite dimensional Lie algebra has a symmetrizable Cartan matrix. Moreover, if the simple \( Z \)-graded Lie algebra \( g \) of polynomial growth (SZGLAPG, for short) has a Cartan matrix, i.e., \( g = g(\mathcal{A}) \), then \( \mathcal{A} \) is symmetrizable. These Cartan matrices correspond to Dynkin diagrams and extended Dynkin diagrams. More exactly, the algebras corresponding to extended diagrams are not simple, they are certain "relatives", called Kac(M) odd algebras, of central extensions of simple ones; in applications Kac(M odd algebras are even more interesting than simple ones, cf. [14].

For finite dimensional simple Lie algebras \( g(\mathcal{A}) \) N. Shapovalov [22] suggested a powerful method for description of irreducible highest weight \( g(\mathcal{A}) \)-modules (Verma modules and their quotients). The method (a development of an idea due to Gelfand and Kirillov cf. [1]) with [22]) was to consider what is now called the Shapovalov determinant.

The Cartan matrices \( \mathcal{A} \) corresponding to finite dimensional Lie algebras and to those of class SZGLAPG are very special. Kac and Kazhdan [11] extended Shapovalov’s result to the Lie algebras with any symmetrizable Cartan matrix \( \mathcal{A} \). Namely, if \( \mathcal{A} \) is symmetrizable, then \( g(\mathcal{A}) \) possesses a nondegenerate invariant bilinear form \( B \) and with the help of the associated with \( B \) quadratic Casimir element \( C_2 \) they computed the Shapovalov determinant for all such Lie algebras.

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Using absence of zero divisors in the enveloping algebra of any Lie algebra, they further obtained a description of irreducible modules that occur in the Jordan (Hölder series of an arbitrary Vem a module over these algebras.

Kac later conjectured a formula for the Shapovalov determinant for the Lie superalgebras with symmetric Cartan matrix \( \frac{1}{2}j^2 \); the formula had an obvious mistake and a correction was offered in [11], for the proof of the corrected formula see [21]. As shown in [11], the formula for the determinant is a corollary of an explicit form of the quadratic Casimir element. Moreover, thanks to the existence of the Casimir element, the Shapovalov determinant turns out to be equal to the product of linear functions. For Lie superalgebras with symmetric Cartan matrix this element is even, so the arguments of [11] are applicable literally. Still, the direct analogy soon stops: (a) to describe irreducible modules that occur in the Jordan (Hölder series of an arbitrary Vem a module over Lie superalgebras we need new ideas due to the presence of zero divisors (cf. [21]), (b) the form of the product of linear factors depends on properties of odd roots involved and is more complicated, see Th.2.4 [21]. Still, one obtains a criterion for irreducibility of Vem a modules.

Our result: calculation of the even quadratic Casimir element for simple finite dimensional Lie superalgebras without symmetric Cartan matrix. Such are only the Lie superalgebras \( \text{sh}(0\mathfrak{2n}) \) of special hamiltonian vector fields. We also consider a "relative" of \( \text{sh}(0\mathfrak{2n}) \), the Poisson superalgebra \( \text{po}(0\mathfrak{2n}) \). Actually we derive the result for \( \text{sh}(0\mathfrak{2n}) \) from that for \( \text{po}(0\mathfrak{2n}) \).

Corollary: a criterion for irreducibility of Vem a modules over all these algebras and calculation of the Shapovalov determinant: it is the product of the linear terms of the above criterion.

Quantization sends \( \text{sh}(0\mathfrak{2k}) \) and \( \text{po}(0\mathfrak{2k}) \) into Lie superalgebras with Cartan matrix \( \frac{1}{2}j^2 \); so our result can be read as a "dequantization" of the Shapovalov determinant for \( \text{psl}(2^k, 1^{p^k}) \) and \( \text{gl}(2^k, 1^{p^k}) \).

Another corollary related with existence of the quadratic Casimir element is an explicit solution to the classical Yang-Baxter equation with values in the above Lie superalgebras; for finite dimensional simple Lie superalgebras these solutions are given in [12].

Since it is natural to describe Poisson Lie superalgebras as subalgebras of the Lie superalgebra of contact vector fields, we describe them and recall our earlier result on the Shapovalov determinant for \( k(\mathfrak{g}) \).

Related open problems: 1) Extension of our result to loop algebras is straightforward, elsewhere we will consider their \( \text{Kac}(M\text{ oody}) \) versions.

2) Even in the absence of the even quadratic Casimir element one can define the Shapovalov determinant provided the algebra possesses an involutive anti-automorphism which sends positive roots into negative ones. Among Lie superalgebras of type \( \text{SZGAPG} \) only stringy or \"superconf\" all Lie superalgebras (for their complete list see [12]) possess this property together with relatives of the \"queer\" series and \( \text{sh}(0\mathfrak{2n} - 1) \) as well as its relative, \( \text{po}(0\mathfrak{2n} - 1) \), together with loop algebras and \( \text{Kac}(M\text{ oody}) \) versions thereof. For the queer Lie superalgebras, even finite dimensional ones, nobody had yet computed Shapovalov determinant. For some (but not all!) of the distinguished, see [12], stringy superalgebras the Shapovalov determinant is computed. In these cases it turned out to be the product of indecomposable polynomial ideals, cf. [3, 15,16] and refs. therein.

3) A graded algebras it is natural to consider the following ones, prime examples being the Lie algebras of (a) differential operators with polynomial coefficients and of (b) \"co-
plex size matrices"; for a description of a large class of them see [5]; most of them have nondegenerate invariant symmetric bilinear forms. Though even for the simplest of these algebras the Cashimirelement is not calculated yet, Shokhet [22] calculated the Shapovalov determinant for some modules, conjecturally it is possible to calculate Cashimirelement on a wider class of modules, cf. [20].

1 A description of $k^1(1j\!\!6)$

Supercircle. A supercircle or (for a physicist) a closed superstring of dimension 1j6 is the real supermanifold $S^{1j\!\!6}$ associated with the rank $m$ trivial vector bundle over the circle. Let $t = e^{i\theta}$, where $\theta$ is the angle parameter on the circle, be the even indeterminate of the Fourier transform; let $\Lambda = (1; \ldots; m)$, be the odd coordinates on the supercircle formed by a basis of the number of the trivial bundle over the circle. Then $(t; \Lambda)$ are the coordinates on $(C^1)^{1j\!\!6}$, the complexification of $S^{1j\!\!6}$.

Let $m = 2k$ and the contact form be

$$X = dt\sum_{i \in \Lambda} (d_i + d_i);$$

For the classical case of simple stringy" Lie superalgebras of vector elds and their non-trivial central extensions see [11]. Among the "mah" series are: vect$(1j\!\!6) = \text{derC}[t, t^\theta; \Lambda]$, of all vector elds and $k^1(1j\!\!6)$ that preserves the Pfaff equation $= 0$. The superscript $L$ indicates that we consider vector elds with Laurent coe cients, not polynomial ones.

The modules of tensor elds. To advance further, we recall the description of the modules of tensor elds over the general vectorial Lie superalgebra vect$(1j\!\!6)$ and its subalgebras, see [2]. Let $\mathfrak{g} = \text{vect}(1j\!\!6)$ realized by vector elds on the $1j\!\!6$-dimensional linear supermanifold $C^{1j\!\!6}$ with coordinates $x = (u; \Lambda)$ with the standard grading (deg $x_i = 1$ for any $i = 1, \ldots, n + m$) and $g_0 = \mathfrak{gl}(1j\!\!6)$. Let $V$ be the $\mathfrak{gl}(1j\!\!6)$-module with the lowest weight $\mathfrak{w}(V)$. Make $V$ into a $g_0$-module by setting $g \cdot V = 0$ for $g \in \mathfrak{g}_i$. The superspace $T(V) = \text{Hom}_{(\mathfrak{g}_0)}(U(\mathfrak{g}); V)$ is isomorphic to due to the Poincare-Birkho (Witt theorem, to $\mathbb{C}[x]$) $V$. Its elements have a natural interpretation as formal tensor elds of type $V$ or $(\mathfrak{g}_+)$, we will simply write $T(\mathfrak{g})$ instead of $T(\mathfrak{g})$. In what follows we consider irreducible $g_0$-modules; for any other $Z$-graded vectorial Lie superalgebra construction of $g_0$-modules with lowest weight is identical.

Examples. $T(\mathfrak{g})$ is the superspace of functions; $\text{Vol}(1j\!\!6) = T(1; \ldots; 1; 1; \ldots; 1)$ (the semi colon separates the $m$ coordinates of the weight with respect to the matrix units $E_{ij}$ of $\mathfrak{gl}(1j\!\!6)$) is the superspace of densities or volume forms. We denote the generator of $\text{Vol}(1j\!\!6)$ corresponding to the ordered set of indeterminates $x$ by $\text{vol}(x)$. The space of densities is $\text{Vol}(1j\!\!6) = T(1; \ldots; 1; \ldots; 1)$. In particular, $\text{Vol}(1j\!\!6) = T(1)$, while $\text{Vol}(0j\!\!6) = T(\Lambda)$.

Modules of tensor elds over stringy superalgebras. Denote by $T^L(V) = C[t, t^\theta] V$ the vect$(1j\!\!6)$-module that differs from $T(V)$ by allowing the Laurent polynomial coe cients of its elements instead of just polynomials. Clearly, $T^L(V)$ is a vect$(1j\!\!6)$-module. Define the twisted with weight version of $T^L(V)$ by setting:

$$T^X(V) = C[t, t^\theta] V:
The simplest modules. These are analogs of the standard or identity representation of the matrix algebras. The simplest modules over the Lie superalgebras of series are, clearly, the submodules Vol. These modules are characterized by the fact that over \( F \), the algebra of functions, they are of rank 1, i.e., have only one generator. Over stringy superalgebras, we can as well twist these modules and consider Vol. Observe that for \( \mathbb{Z} \) this module has only one submodule, the image of the exterior differential, see [1]; for \( 2 \mathbb{Z} \) this submodule coincides with the kernel of the residue:

\[
\text{Res : } \text{Vol}^L \to C ;
\]

\( f_{\text{vol}}(t) \) the coe. of \( \frac{1 \cdots n}{t} \) in the power series expansion of \( f \):

Over contact superalgebras \( k(2n + 1) \mathbb{N} \), it is more natural to express the simplest modules not in terms of -densities but in terms of powers of:

\[
F = \begin{cases} 
F & \text{for } n = m = 0; \\
F^{-2} & \text{otherwise};
\end{cases}
\]

Observe that, as \( k(2n + 1) \mathbb{N} \) modules, \( \text{Vol} = F \ (2n+2) \) and \( F = F_0 \). In particular, the Lie superalgebra of series \( k \) does not distinguish between \( \frac{\partial}{\partial t} \) and \( 1 \); their transformation rules are identical. Hence, \( k(2n + 1) \mathbb{N} \) - functions usually set \( \text{deg } f = \frac{1}{2} \) and \( \text{deg } t = 1 \), whereas we prefer, as is custom among mathematicians, the integer values of the highest weight with respect to the Cartan subalgebra of \( k(1.\mathbb{N})_0 = o(2n) \) \( \mathbb{C} \), so we use doubled physicists weights.)

Convenient formulas. A laconic way to describe the Lie superalgebras of series \( k \) is via generating functions. For \( 2 \mathbb{C} \left[ t ; \right] \), where \( = ( ; ) \), set:

\[
K_f = \left( 2 \ E \right) (f) \frac{\partial}{\partial t} H_f + \frac{\partial f}{\partial t} E ; \quad \text{where } \ E = \ X \frac{\partial}{\partial t} \frac{\partial}{\partial t} \frac{\partial}{\partial t} + \frac{\partial f}{\partial t} \frac{\partial}{\partial t} \frac{\partial}{\partial t} ;
\]

and where \( H_f \) is the Hamiltonian \( \text{en} \) with Hamiltonian \( f \) that preserves \( d \) :

\[
H_f = (1)^{\text{deg}(f)} \ \ X \frac{\partial f}{\partial t} \frac{\partial}{\partial t} + \frac{\partial f}{\partial t} \frac{\partial}{\partial t} ;
\]

Since

\[
L_{K_f} ( ) = K_1 (f) ;
\]

it follows that \( K_f \ 2 \ k(2n + 1) \mathbb{N} \).

To the (super)commutator \( [K_f ; K_g] \) there corresponds the contact bracket of the generating functions:

\[
[K_f ; K_g] = \ K_{[f,g]};
\]

An explicit formula for the contact brackets is as follows. Let us first define the brackets on functions that do not depend on \( t \). The Poisson bracket \( f ; g \) is given by the formula

\[
2 \ f ; g \ : = \ (1)^{\text{deg}(f)} \ \ X \frac{\partial f}{\partial t} \frac{\partial g}{\partial t} + \frac{\partial f}{\partial t} \frac{\partial g}{\partial t} .
\]
Now, the contact bracket is

$$ff;gg_{|z_\mathfrak{z}_A} = (2 \mathcal{E}(f) \otimes g) \frac{\partial g}{\partial t} \mathcal{E}(g) ff;gg_{|z_\mathfrak{z}_B}.$$ 

It is not difficult to prove that $k(1|\mathfrak{p}k) = \text{Span } (K_\mathcal{E} : f 2 C [\xi])$ as superspaces.

The Poisson superalgebra is $\text{po}(0|\mathfrak{m}n) = \text{Span } (K_\mathcal{E} : f 2 C [\xi])$. Its quotient modulo the center, $z = C K_\mathcal{E}$, is called the Hamiltonian Lie superalgebra $h(0|\mathfrak{m}n)$; clearly, $h(0|\mathfrak{m}n) = \text{Span } (H_\mathcal{E} : f 2 C [\xi]; g; q; g)$. On $\text{po}(0|\mathfrak{m}n)$ and $h(0|\mathfrak{m}n)$, there are supertraces:

$$Z = K_\mathcal{E}; H_\mathcal{E} \int f \text{ vol } (\cdot ).$$

The traceless elements span ideals $\text{spo}(0|\mathfrak{m}n)$, special Poisson superalgebra, and $\text{sh}(0|\mathfrak{m}n)$, special Hamiltonian superalgebra. For $m = 2k$ quantization sends them into $\mathfrak{gl} (2k; 1|\mathfrak{p}k; 1)$ and $\mathfrak{psl} (2k; 1|\mathfrak{p}k; 1)$, respectively, and the integral becomes the usual supertrace [17].

**Roots of $k^L (1|\mathfrak{p})$.** The Cartan subalgebra of $g = k^L (1|\mathfrak{p})$ is the span of

$$H_1 = K_{1 1}; \quad H_2 = K_{2 2}; \quad H_3 = K_{3 3}; \quad H_4 = K_{4 k}; \quad H_5 = K_{t 2 3 3 2}; \quad H_6 = K_{t 1 3 3 1}; \quad H_7 = K_{t 1 2 2 0}; \quad H_8 = K_{t 1 2 3 3 3 2 1};$$

The weight of the elements is taken, however, only with respect to the diagonalizing part of the Cartan subalgebra, namely, with respect to $H_4$ and, after semicolon, $H_1, H_2, H_3$:

$$\text{wht } K_{t 1 2 3 3 3 2 1} = (2(a 1) + (j + j) 1 1; 2 2; 3 3);$$

where $j = 0, 1, 2$. The root vectors are ordered lexicographically.

**Roots of $\text{po}(0|\mathfrak{p}k)$ and $\text{sh}(0|\mathfrak{p}k)$.** The Cartan subalgebra of $\text{po}(0|\mathfrak{p}k)$ is the span of

$H_1 = K_{1 1}, H_2 = K_{1 1}, H_{1 i} = K_{1 1 1 1}, \ldots, H_{1; 1; \ldots; n} = K_{1 1 1; \ldots; k 1 1; \ldots; k}$. In what follows we denote: $I = (1; \ldots; n)$ and let $P (I)$ be the set of all the subsets of $I$.

Observe that the diagonalizing subalgebra $d$ of the Cartan subalgebra of $\text{po}(0|\mathfrak{p}k)$ is the Cartan subalgebra of $\mathfrak{p}(2|k)$ spanned by the $H_i$. We will call it the $\text{sm all }^\dagger$ Cartan subalgebra. The roots are given with respect to $d$ completed with (to enable a n grading) the exterior derivation $E = \frac{\partial}{\partial \eta}$, where $\eta = (i)$. The weight with respect to $E$, separated by a semicolon, is given first (shifted by 2).

The root vectors and their weights are:

$$\text{wht } K_{1 1 1; \ldots; k 1 1; \ldots; k} = (j + j) 2; 1 1; \ldots; k k);$$

and the root vectors are ordered lexicographically. A shorthand for the vector

$K_{1 1 1; \ldots; k 1 1; \ldots; k}$ is $K_\mathcal{E}$.

The roots of $\text{sh}(0|\mathfrak{p}k)$ are similar to those of $\text{po}(0|\mathfrak{p}k)$: just delete $K_1$ and $K_{1 1; \ldots; k k}$ and replace all $K_\mathcal{E}$ with $H_\mathcal{E}$.

The root vectors corresponding to simple roots are

$$X_{1 1} = K_{1 1} \quad \text{and} \quad X_{1 1} = K_{1 1} \quad \text{for any subset } I \text{ of } I \text{ except } I = \text{itself}.$$
Kac-Moody superalgebras associated with \( \text{po}(0|2k) \) and \( \text{sh}(0|2k) \). Recall that for each nine dimensional Lie superalgebra \( g \), the loop algebra associated with it is denoted by \( g^{(1)} = C[t^1; t] \). Let \( B \) be the nondegenerate supersymmetric bilinear form on \( g \). A nontrivial central extension \( g^{(1)} = g^{(1)} \cdot Cz \) of \( g^{(1)} \) is given by means of the cocycle
\[
c(f; g) = \text{Res} B(f; dg) \quad \text{for any} \quad f; g \in g^{(1)}:
\]
On \( g^{(1)} \), the form \( B \) induces the following nondegenerate invariant supersymmetric form \( B^{(1)} \):
\[
B^{(1)}((f; a); (g; b)) = B(f; g)(0) + ab \quad \text{for any} \quad f; g \in g^{(1)} \text{ and } a; b \in C:
\]
The algebra \( g^{(1)} \cdot Ct_{\frac{d}{dt}} \) is called (a ne) Kac-Moody Lie superalgebra, cf. [14] for the list of simple Kac-Moody Lie superalgebras see [2].
The Cartan subalgebra of \( \text{po}(0|2k)^{(1)} \) (resp. \( \text{sh}(0|2k)^{(1)} \)) is the span of the Cartan subalgebra of \( \text{po}(0|2k) \) (resp. \( \text{sh}(0|2k) \)) and the central element \( z \); hence, the roots are given with respect to \( t_{\frac{d}{dt}} \) and \( E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), and, after semicolon, the \( \text{sm all" Cartan subalgebra, spanned by the } H_i. \)
The weights of the root vectors are
\[
\text{wh}(e^K) = (n; j^2; j + j 2; 1, 1; ; k; k)
\]
and the root vectors are ordered lexicographically.

2 The bilinear forms and related Casimir elements

In what follows the root elements are normalized so that \( B(e; e) = 1 \) for the series \( \text{po} \) and \( \text{sh} \) as well as for \( k^2(1|j) \). The vector \( e \) is called the right dual of \( e \), cf. [12].

In the realization of \( k^2(1|j) \) by means of generating functions the invariant nondegenerate even supersymmetric bilinear form \( B \) is given by the formula
\[
B(K;K) = \text{Res}\ f g; \quad \text{where}\ \text{Res}(f) = \text{the coefficient of} \quad \begin{pmatrix} 1 \cdots & 3 \cdots & 3 \\ t \end{pmatrix} \quad (2)
\]
It is easy to verify directly that
\[
H_1 = H_5; \quad H_2 = H_6; \quad H_3 = H_7; \quad H_4 = H_8;
\]
Lem m a 1. The following Casimir element corresponds to the form \( B \) and, therefore, belongs to (the completion of) the center of the enveloping algebra of \( k^2(1|j) \):
\[
C_2 = e \cdot e + \sum_{i=1}^{4} X_i H_i H_1 + 2H_3 + 4H_6 + 4H_8; \quad (4)
\]
Observe, that if in formula (4) we replace the right dual elements with the left dual ones, i.e., such that \( B(e; e) = 1 \), we obtain an element which does not belong to the center.
In the realization of \( p^\ell(0;\mathfrak{g}) \) by means of generating functions, the invariant nondegenerate bilinear form \( B \) is given by the formula

\[
B(\mathfrak{k}^\ell;\mathfrak{k}_g) = f_g \text{vol}(\mathfrak{g}): \]

Clearly, this form induces an invariant nondegenerate form \( B(\mathfrak{h}_i;\mathfrak{h}_g) = B(\mathfrak{k}^\ell;\mathfrak{k}_g) \) on \( \text{Sh}(0;\mathfrak{g}) \). Obviously, up to a sign, \( \mathfrak{h}_i = H_{\mathfrak{inf}_g}, \mathfrak{h}_{ij} = H_{\mathfrak{inf}_g}, \) etc.

Lemm a 2. The following Casimir element corresponds to the form \( B \) and, therefore, belongs to the center of the enveloping algebra of \( p^\ell(0;\mathfrak{g}) \):

\[
C_2 = \sum_{J>0} X_j X_j \mathcal{H}_j \mathcal{H}_j + \left( 2^k \right)^{1/2} H_{\mathfrak{inf}_g}; \quad (5)
\]

The Casimir element for \( \text{Sh}(0;\mathfrak{g}) \) is obtained from the above one by deleting term \( s \) with \( \mathfrak{h}_i \) and \( \mathfrak{h}_i \).

Observe that the third summand in (5) is precisely the element of the small Cartan subalgebra corresponding to the weight \( \lambda/2 \), which is defined for any finite-dimensional Lie superalgebra as the halfsum of positive even roots minus the halfsum of positive odd roots. It is remark able that its form is so simple.

In the realization of \( p^\ell(0;\mathfrak{g}) \) with the help of generating functions the invariant nondegenerate bilinear form \( B \) is given by the formula

\[
B(\mathfrak{t}^m;\mathfrak{k}_g;ab) = f_g \text{vol}(\mathfrak{g}) + ab:
\]

Clearly, this form induces an invariant nondegenerate form on \( \text{Sh}(0;\mathfrak{g}) \).

The Casimir element corresponding to the invariant form \( B \) belongs to the center of the (completed) enveloping algebra of \( p^\ell(0;\mathfrak{g}) \).

On proofs. The proof of Lemmas 1 and 2 is a direct verification: it suffices to apply root vectors corresponding to simple negative roots. This is a routine done by hand, but tests are much easier to perform with the help of Grozman's SuperLie package, see \([6]\).

3 The Shapovalov determinant. Irreducible Verma modules over \( k^2(1;\mathfrak{j}) \), and \( p^\ell(0;\mathfrak{g}) \) with its relatives

Let \( a = (a_1;\ldots;a_n) \) be the highest weight of the Verma module \( \mathbb{M}^a \) over \( g \) and \( b = (b_1;\ldots;b_n) \) (here \( m = 8 \) for \( k^2(1;\mathfrak{j}) \), and \( m = 2^k \mathbb{P} \) for \( p^\ell(0;\mathfrak{g}) \) and \( m = 2^k \mathbb{P}^2 \) for \( \text{Sh}(0;\mathfrak{g}) \)) be one of the weights of \( \mathbb{M}^a \), i.e., \( b = n_1 \mathbf{r}_1, \) where the \( r_1 \) are positive roots and \( n_1 \geq 2 \) if the weight \( r_1 \) is even, \( n_1 = 0 \) or 1 if the weight \( r_1 \) is odd, i.e., \( n_1 \mathbf{r}_1 \) is a quasimodule. By applying \( C_2 \) to the highest weight vector of the Verma module we get, as in \([11]\), the following theorems.

Theorem 1. Let \( g = k^2(1;\mathfrak{j}) \). The module \( \mathbb{M}^a \) is irreducible if and only if for every quasimodule

\[
a_1 a_1 + a_2 a_4 + a_3 a_7 + a_4 a_8 \neq a_1 b_5 + a_2 b_6 + a_3 b_7 + a_4 b_8 + b_1 a_5 + b_2 a_6
\]

\[
+ b_3 a_7 + b_4 a_8 + b_2 b_5 + b_2 b_6 + b_3 b_7 + b_4 b_8 + 4b_5 + 2b_6 + 4b_8;
\]
or, in other words, if and only if $2(a + ;) \notin (;)$ for every quasiroot and $= 2H_5 + H_6 = 2H_8$.

**Theorem 2.** Let $g = po(0; k)$. The module $M^a$ is irreducible if and only if

$$2(a + ;) \notin (;) \text{ for every quasiroot and } = (2)^k 2H_{\text{inf} k}.$$

More explicitly, the above formula can be written as

$$X \sum a_{j} a_{j} \notin X \sum a_{j} b_{0} + X \sum b_{0} a_{j} + X \sum b_{0} b_{0} = (2)^k 2H_{\text{inf} k}:$$

The description of the irreducible Verma modules $M^a$ over $sh (0; k)$ is similar: in the above formula delete the terms with subscripts $j$ and $I$.

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