Generalized Deformed Oscillators and Algebras

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ABSTRACT

The generalized deformed oscillator schemes introduced as unified frameworks of various deformed oscillators are proved to be equivalent, their unified representation leading to a correspondence between the deformed oscillator and the N=2 supersymmetric quantum mechanics (SUSY-QM) scheme. In addition, several physical systems (two identical particles in two dimensions, isotropic oscillator and Kepler system in a 2-dim curved space) and mathematical structures (quadratic algebra QH(3), finite W algebra W_0) are shown to possess the structure of a generalized deformed su(2) algebra, the representation theory of which is known. Furthermore, the generalized deformed parafermionic oscillator is identified with the algebra of several physical systems (isotropic oscillator and Kepler system in 2-dim curved space, Fokas–Lagerstrom, Smorodinsky–Winternitz and Holt potentials) and mathematical constructions (generalized deformed su(2) algebra, finite W algebras W_0 and W_3^{(2)}). The fact that the Holt potential is characterized by the W_3^{(2)} symmetry is obtained as a by-product.

1 Introduction

Quantum algebras (quantum groups) [1, 2], are nonlinear generalizations of the usual Lie algebras to which they reduce for appropriate values of the deformation parameter(s). From the mathematical point of view they are Hopf algebras [3]. Their use in physics became popular with the introduction of the q-deformed harmonic oscillator [4, 5, 6] as a tool for providing a boson realization of the quantum algebra su_q(2), although similar mathematical structures had already been known [3]. Initially used for solving the quantum Yang–Baxter equation [4], quantum algebras have subsequently found applications in several branches of physics, as, for example, in the description of spin chains, squeezed states, rotational and vibrational nuclear and molecular spectra, and in conformal field theories. By now several kinds of generalized deformed oscillators have been introduced [10, 11, 12] and unification schemes for them have been suggested [13].

Furthermore, generalized deformed su(2) algebras have been introduced [11, 15] in a way that their representation theory remains as close as possible to the usual su(2) one. It will be shown here that several physical systems (two identical particles in two dimensions [1], isotropic oscillator and Kepler system in a 2-dim curved space [17], as well as the quadratic algebra QH(3) [18] and the finite W algebra W_0 [14] can be accommodated within this scheme. The advantage this unification offers is that the representation theory of the generalized deformed su(2) algebras is known [14].

In addition, generalized parafermionic oscillators have been introduced [20], in analogy to generalized deformed oscillators. It will be shown here that several physical systems (isotropic oscillator and Kepler system in a 2-dim curved space [17], Fokas–Lagerstrom potential [21], Smorodinsky–Winternitz potential [22], Holt potential [23]) and mathematical constructions (generalized deformed su(2) algebra [14], finite W algebras W_0 [19] and W_3^{(2)} [24, 25, 26, 27]) can be accommodated within this framework. As a by-product the fact that the Holt potential is characterized by the W_3^{(2)} symmetry occurs [28].

In section 2 the various deformed oscillators will be put in a common mathematical framework, while in section 3 various unification schemes will be proved to be equivalent. In section 4 the relation of the deformed oscillator to the N=2 supersymmetric quantum mechanics (SUSY-QM) scheme will be pointed out. In section 5 the cases related to the generalized deformed su(2) algebra will be studied, while in section 6 the systems related to generalized deformed parafermionic oscillators will be considered. Section 7 will contain discussion of the present results and plans for further work.
2 The generalized deformed oscillator

By now many kinds of deformed oscillators have been introduced in the literature. All of them can be accommodated within the common mathematical framework of the generalized deformed oscillator [10, 11, 12], which is defined as the algebra generated by the operators \( \{1, a, a^\dagger, N\} \) and the structure function \( \Phi(x) \), satisfying the relations

\[
[\{a, N\} = a, \quad [a^\dagger, N] = -a^\dagger, \quad a^\dagger a = \Phi(N) = [N], \quad a a^\dagger = \Phi(N + 1) = [N + 1],
\]

where \( \Phi(x) \) is a positive analytic function with \( \Phi(0) = 0 \) and \( N \) is the number operator. From eq. (2.2) we conclude that

\[
N = \Phi^{-1}(a^\dagger a),
\]

and that the following commutation and anticommutation relations are obviously satisfied:

\[
[a, a^\dagger] = [N + 1] - [N], \quad \{a, a^\dagger\} = [N + 1] + [N].
\]

The structure function \( \Phi(x) \) is characteristic to the deformation scheme. In table 1 the structure functions corresponding to different deformed oscillators are given. They will be further discussed at the end of this section.

It can be proved that the generalized deformed algebras possess a Fock space of eigenvectors \( |0\rangle, |1\rangle, \ldots, |n\rangle, \ldots \) of the number operator \( N \)

\[
N|n\rangle = n|n\rangle, \quad \langle n|m\rangle = \delta_{nm},
\]

if the vacuum state \( |0\rangle \) satisfies the following relation:

\[
a|0\rangle = 0.
\]

These eigenvectors are generated by the formula:

\[
|n\rangle = \frac{1}{\sqrt{|n|!}} (a^\dagger)^n |0\rangle,
\]

where

\[
|n|! = \prod_{k=1}^{n} [k] = \prod_{k=1}^{n} \Phi(k).
\]

The generators \( a^\dagger \) and \( a \) are the creation and annihilation operators of this deformed oscillator algebra:

\[
a|n\rangle = \sqrt{|n|} a|n - 1\rangle, \quad a^\dagger|n\rangle = \sqrt{|n + 1|} a|n + 1\rangle.
\]

These eigenvectors are also eigenvectors of the energy operator

\[
H = \frac{\hbar \omega}{2} (a a^\dagger + a^\dagger a),
\]

corresponding to the eigenvalues

\[
E(n) = \frac{\hbar \omega}{2} (\Phi(n) + \Phi(n + 1)) = \frac{\hbar \omega}{2} ([n] + [n + 1]).
\]

For

\[
\Phi(n) = n
\]

one obtains the results for the ordinary harmonic oscillator. For

\[
\Phi(n) = \frac{q^n - q^{-n}}{q - q^{-1}} = [n]_q
\]

one has the results for the \( q \)-deformed harmonic oscillator [4, 5, 6], while the choice

\[
\Phi(n) = \frac{Q^n - 1}{Q - 1} = [n]_Q
\]
Table 1: Structure functions of special deformation schemes

| Φ(x) | Reference |
|------|-----------|
| i    | x         | harmonic oscillator, bosonic algebra |
| ii   | \( \frac{q^x - q^{-x}}{q - q^{-1}} \) | q-deformed harmonic oscillator [4, 5, 6] |
| iii  | \( \frac{q^x - 1}{q - 1} \) | Arik–Coon, Kuryshkin, or Q-deformed oscillator [7, 8] |
| iv   | \( \frac{q^x - p^{-x}}{q - p^{-1}} \) | 2-parameter deformed oscillator [29, 30, 31] |
| v    | \( x(p + 1 - x) \) | parafermionic oscillator [32] |
| vi   | \( \frac{\sinh(\tau x) \sinh(\tau(p + 1 - x))}{\sinh^2(\tau)} \) | \( q \)-deformed parafermionic oscillator [33, 34] |
| vii  | \( x \cos^2(\pi x/2) + (x + p - 1) \sin^2(\pi x/2) \) | parabosonic oscillator [32] |
| viii | \( \frac{\sinh(\tau x) \cosh(\tau(x + 2N_0 - 1))}{\sinh(\tau)} \cos^2(\pi x/2) + \frac{\sinh(\tau(x + 2N_0 - 1)) \cosh(\tau x)}{\sinh(\tau)} \sin^2(\pi x/2) \) | \( q \)-deformed parabosonic oscillator [33, 34] |
| ix   | \( \sin^2 \pi x/2 \) | fermionic algebra [35] |
| x    | \( q^x - 1 \sin^2 \pi x/2 \) | \( q \)-deformed fermionic algebra [36, 37, 38, 39, 40, 41] |
| xi   | \( \frac{1 - (-q)^x}{1 + q} \) | generalized \( q \)-deformed fermionic algebra [42] |
| xii  | \( x^n \) | [10] |
| xiii | \( \frac{\text{sn}(x \tau)}{\text{sn}(\tau)} \) | [10] |

leads to the results of the \( Q \)-deformed harmonic oscillator [6, 8]. Many more cases are shown in table 1, on which the following comments apply:

i) Two-parameter deformed oscillators have been introduced [29, 30, 31], in analogy to the one-parameter deformed oscillators.

ii) Parafermionic oscillators [32] of order \( p \) represent particles of which the maximum number which can occupy the same state is \( p \). Parabosonic oscillators [32] can also be introduced.

iii) \( q \)-deformed versions of the parafermionic and parabosonic oscillators have also been introduced [33, 34].

iv) \( q \)-deformed versions of the fermionic algebra [35] have also been introduced [36, 37, 38, 39, 40, 41], as well as \( q \)-deformed versions of generalized \( q \)-deformed fermionic algebras [42]. It has been proved, however, that \( q \)-deformed fermions are fully equivalent to the ordinary fermions [43, 44, 45].

3 Equivalence of unification schemes of deformed oscillators

Using the definition (eqs (2.1), (2.2)) of the deformed algebra we can find the relation between the present formalism and the various deformation frameworks introduced recently. For each deformation framework the structure function \( \Phi(x) \) can be determined. Thus each deformation scheme is equivalent to the unified treatment expressed by eqs (2.1) and (2.2).

3.1 The Beckers–Debergh unification scheme

Beckers and Debergh proposed recently [46] a unification scheme based on the following set of relations:

\[
\begin{align*}
aa^\dagger + g(q)a^\dagger a &= |N + 1| + g(q)||N||, \\
c^\dagger |n> &= \sqrt{|n + 1||n + 1>},
\end{align*}
\]
\[ c|n> = \sqrt{[n]}|n-1>, \]

where \( g(q) \) is an ordinary function of the parameter \( q \) and the bracket \([n]\) is a function of \( n \). Obviously the correspondence between the Beckers-Debergh formalism and the algebra defined by eqs (2.1), (2.2) is given by defining the structure function \( \Phi(x) \) as follows:

\[ \Phi(x) = [|x|]. \]

In [10] eq. (2.2) is not used explicitly, eq. (3.1) being used instead. If one considers eq. (2.2) as the fundamental one, then eqs (3.1)-(3.3) are satisfied, simplifying considerably the demonstrations of the theorems.

### 3.2 The Odaka–Kishi–Kamefuchi unification scheme

Odaka–Kishi–Kamefuchi [34] proposed an algebra generated by the following relations:

\[
[a, \mathcal{N}] = a, \quad [a^\dagger, \mathcal{N}] = -a^\dagger, \tag{3.5}
\]

\[
[a^\dagger, a]_\alpha = a^\dagger a + \alpha a a^\dagger = G(\mathcal{N}), \tag{3.6}
\]

where the spectrum of the operator \( \mathcal{N} \) is given by:

\[ \mathcal{N}|n> = (n+N_0)|n>. \tag{3.7} \]

After a little algebra Odaka–Kishi–Kamefuchi [34] define the numbers \( I_n \):

\[ I_n = (\langle n|a|n+1>)^2 = \sum_{m=0}^{n} (-1)^m \alpha^{-(m+1)} G(n-m), \tag{3.8} \]

where \( G_n = \langle n|G(\mathcal{N})|n> \). Without difficulty we can find the correspondence between the general definition (eqs (2.1), (2.2)) and the formalism of the Odaka–Kishi–Kamefuchi (eqs (3.5)-(3.8)). The number operator \( N \) in eq. (2.2) is related to the operator \( \mathcal{N} \) by the relation:

\[ N = \mathcal{N} - N_0 = \mathcal{N} - \langle 0|\mathcal{N}|0>, \tag{3.9} \]

while the structure function \( \Phi(x) \) is the solution of the equation:

\[ \Phi(x) + \alpha \Phi(x+1) = G(x+N_0). \tag{3.10} \]

The structure function \( \Phi(x) \) is also related to the numbers \( I_n \) in eq. (3.8) by:

\[ \Phi(n+1) = I_n. \tag{3.11} \]

In many concrete cases eq. (3.11) can be used directly for finding the correspondence between the various formalisms. An example is the case of the parabosonic oscillator [32]. In eq. (6) of [34] the numbers \( I_n \) are determined by:

\[ I_n = \begin{cases} n + 2N_0 & \text{for } n=\text{even} \\ n + 1 & \text{for } n=\text{odd} \end{cases} \tag{3.12} \]

From eq. (3.11) we find that

\[ \Phi(x) = x \cos^2(\pi x/2) + (x + 2N_0 - 1) \sin^2(\pi x/2), \tag{3.13} \]

while in the case of the \( q \)-deformed version of the parabosons (eq. (10) of [34]) the structure function is given by

\[ \Phi(x) = \frac{\sinh(\tau)}{\sinh(\tau/2)} \frac{\cosh(\tau(x + 2N_0 - 1))}{\cosh(\tau/2)} \cos^2(\pi x/2) + \frac{\sinh(\tau(x + 2N_0 - 1))}{\sinh(\tau)} \cosh(\tau) \sin^2(\pi x/2). \tag{3.14} \]

### 3.3 The generalized deformed oscillator

The method of the generalized deformed oscillator proposed by Daskaloyannis [10, 11, 12] is essentially the same with the algebra generated by eqs (2.1) and (2.2). In [10] the initial assumption was that there is a function \( g(x) \) such that:

\[ aa^\dagger = g(a^\dagger a). \tag{3.15} \]

This assumption is equivalent to the statement given by eq. (2.2), the structure function \( \Phi(x) \) being given by the solution of the functional equation:

\[ \Phi(1 + \Phi^{-1}(x)) = g(x). \tag{3.16} \]
3.4 The bosonization scheme
This method was initially proposed by Jannussis et al. [35]. The algebra is defined by:
\[ a = F(N + 1)b, \quad a^\dagger = b^\dagger F(N + 1), \]
where \( \{ 1, b, b^\dagger, N \} \) is the usual oscillator algebra
\[ b^\dagger b = N, \quad bb^\dagger = N + 1, \quad [b, N] = b. \]
This theory is formally equivalent to the one proposed in this paper by defining the structure function to be:
\[ \Phi(x) = xF(x). \]

3.5 The generalized Q-deformed oscillator
This algebra was proposed by Brodimas et al. [47]. It is based on the relation:
\[ aa^\dagger - Qa^\dagger a = f(N). \]
The corresponding structure function is:
\[ \Phi(x + 1) - Q\Phi(x) = f(x). \]

4 Relation between the generalized deformed oscillator and the SUSY-QM
The generation of the unified deformed oscillator schemes from eq. (2.2) shows the relation of this kind of theories with the \( N = 2 \) supersymmetric quantum mechanics (SUSY-QM) [48, 49, 50].

Using the vocabulary of the SUSY-QM, we can identify the operators \( a^\dagger a \) and \( aa^\dagger \) as being the partner hamiltonians \( H_+ \) and \( H_- \) of the SUSY-QM. These partner hamiltonians have common eigenstates \( |n> \) corresponding to the eigenvalues \( \Phi(n) \) and \( \Phi(n + 1) \),
\[ H_+|n> = E_+^n|n>, \quad E_+^n = \Phi(n), \]
\[ H_-|n> = E_-^n|n>, \quad E_-^n = \Phi(n + 1). \]
It is clear that \( E_-^n = E_-^{n+1} \), while \( \Phi(0) = 0 \) is an eigenvalue only of the Hamiltonian \( H_+ \) and not of \( H_- \). Thus for \( H_+ \) the state \( |0> \) represents the ground state with eigenvalue \( E_0^+ = 0 \), while no eigenstate with zero eigenvalue exists for \( H_- \). The ground state eigenvalue of \( H_- \) is non-zero.
The operators \( a^\dagger \) and \( a \) correspond to the raising and lowering operators of the SUSY-QM. The supersymmetric charges can be defined as:
\[ Q = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & a^\dagger \\ 0 & 0 \end{pmatrix}. \]
The SUSY hamiltonian is defined by:
\[ H_S = \begin{pmatrix} Q, Q^+ \end{pmatrix} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}. \]
It follows then that
\[ [Q, H_S] = [Q^+, H_S] = 0, \quad Q^2 = (Q^+)^2 = 0. \]
Eqs (4.3)-(4.5) suggest that the unified schemes for deformed oscillators are essentially similar to the SUSY-QM.

Starting from this point of view one can generate a SUSY-QM using, for example, the \( q \)-deformed oscillator of Biedenharn [44] and Macfarlane [45]. The present construction is an example of usual SUSY-QM, different from the \( q \)-deformed \( N=2 \) SUSY-QM recently constructed by Spiridonov [51].

We have so far proved that the various unified deformation schemes existing in the literature are equivalent methods of creating a deformed oscillator. Their common root is based on the fact that all these theories can be deduced from an operator algebra \( \{ 1, a, a^\dagger, N \} \) with the properties:
\[ [a, N] = a, \quad [a^\dagger, N] = -a^\dagger, \]
\[ a^\dagger a = \Phi(N) = [N], \quad aa^\dagger = \Phi(N + 1) = [N + 1], \]
where \( \Phi(x) \) is a positive analytic structure function with \( \Phi(0) = 0 \). The above formulation shows the relation between the deformed oscillator formalism and the usual \( N=2 \) SUSY-QM. To every deformed oscillator corresponds an \( N=2 \) SUSY example.
An interesting problem is to examine which kind of multidimensional deformed oscillator algebra could correspond to N=2 SUSY-QM. Another interesting question is to study if the correspondence is invertible, i.e. if every N=2 SUSY algebra corresponds to a deformed oscillator scheme. It is also well known that one can find SUSY algebras generated by superpotentials. The relation between superpotentials and deformed oscillators should be examined. From the physical point of view, the implications of the present findings on the recent effort of describing isospectral superdeformed bands in neighbouring nuclei by SUSY-QM techniques should also be studied.

5 Generalized deformed su(2) algebras

Generalized deformed su(2) algebras having representation theory similar to that of the usual su(2) have been constructed in [14]. It has been proved that it is possible to construct an algebra

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = \Phi(J_0(J_0 + 1)) - \Phi(J_0(J_0 - 1)), \]

where \( J_0, J_+, J_- \) are the generators of the algebra and \( \Phi(x) \) is any increasing entire function defined for \( x \geq -1/4 \). Since this algebra is characterized by the function \( \Phi \), we use for it the symbol \( \text{su}_\Phi(2) \). The appropriate basis \( |l, m> \) has the properties

\[ J_0|l, m> = m|l, m>, \]
\[ J_+|l, m> = \sqrt{\Phi(l(l + 1)) - \Phi(m(m + 1))}|l, m + 1>, \]
\[ J_-|l, m> = \sqrt{\Phi(l(l + 1)) - \Phi(m(m - 1))}|l, m - 1>, \]

where

\[ l = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots, \]

and

\[ m = -l, -l - 1, -l + 1, \ldots, l - 2, l, l + 1. \]

The Casimir operator is

\[ C = J_- J_+ + \Phi(J_0(J_0 + 1)) = J_+ J_- + \Phi(J_0(J_0 - 1)), \]

its eigenvalues indicated by

\[ C|l, m> = \Phi(l(l + 1))|l, m>. \]

The usual su(2) algebra is recovered for

\[ \Phi(x(x + 1)) = x(x + 1), \]

while the quantum algebra \( \text{su}_q(2) \)

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_0]_q, \]

occurs for

\[ \Phi(x(x + 1)) = [x]_q[x + 1]_q, \]

with \( q \)-numbers defined as

\[ [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \]

The \( \text{su}_\Phi(2) \) algebra occurs in several cases, in which the rhs of the last equation in (5.1) is an odd function of \( J_0 \).

5.1 Two identical particles in two dimensions

Let us consider the system of two identical particles in two dimensions. For identical particles observables of the system have to be invariant under exchange of particle indices. A set of appropriate observables in this case is

\[ u = (x_1)^2 + (x_2)^2, \quad v = (x_1)^2 - (x_2)^2, \quad w = 2x_1x_2, \]
\[ U = (p_1)^2 + (p_2)^2, \quad V = (p_1)^2 - (p_2)^2, \quad W = 2p_1p_2, \]
\[ C_1 = \frac{1}{4}(x_1p_1 + p_1x_1), \quad C_2 = \frac{1}{4}(x_2p_2 + p_2x_2), \quad M = x_1p_2 + x_2p_1, \]
where the indices 1 and 2 indicate the two particles. These observables are known to close an sp(4,R) algebra. A
representation of this algebra can be constructed \[16, 53\] using one arbitrary constant $\eta$
and three matrices $Q$, $R$, and $S$ satisfying the commutation relations
\[
[S, Q] = -2iR, \quad [S, R] = 2iQ, \quad [Q, R] = -8iS(\eta - 2S^2). \quad (5.16)
\]
The explicit expressions of the generators of sp(4,R) in terms of $\eta$, $S$, $Q$, $R$ are given in \[16\] and need not be repeated
here. Defining the operators
\[
X = Q - iR, \quad Y = Q + iR, \quad S_0 = S/2, \quad (5.17)
\]
one can see that the commutators of eq. (5.16) take the form
\[
[S_0, X] = X, \quad [S_0, Y] = -Y, \quad [X, Y] = 32S_0(\eta - 8(S_0)^2), \quad (5.18)
\]
which is a deformed version of su(2). It is clear that the algebra of eq. (5.18) is a special case of an su$_\Phi$(2) algebra
with structure function
\[
\Phi(J_0(J_0 + 1)) = 16\eta J_0(J_0 + 1) - 64(J_0(J_0 + 1))^2. \quad (5.19)
\]
The condition that $\Phi(x)$ has to be an increasing function of $x$ implies the restriction $x < \eta/8$.

5.2 Kepler problem in 2-dim curved space

Studying the Kepler problem in a two-dimensional curved space with constant curvature $\lambda$ one finds the algebra \[17\]
\[
[L, R_\pm] = \pm R_\pm, \quad [R_-, R_+] = F\left(L + \frac{1}{2}\right) - F\left(L - \frac{1}{2}\right), \quad (5.20)
\]
where
\[
F(L) = \mu^2 + 2\lambda L^2 - \lambda L^2\left(L^2 - \frac{1}{4}\right). \quad (5.21)
\]
It is then easy to see that
\[
[R_+, R_-] = 2L\left(-2H + \frac{\lambda}{4}\right) + 4\lambda L^3, \quad (5.22)
\]
which corresponds to an su$_\Phi$(2) algebra with
\[
\Phi(J_0(J_0 + 1)) = \left(-2H + \frac{\lambda}{4}\right)J_0(J_0 + 1) + \lambda(J_0(J_0 + 1))^2. \quad (5.23)
\]
For $\Phi(x)$ to be an increasing function, the condition
\[
\lambda x > H - \frac{\lambda}{8} \quad (5.24)
\]
has to be obeyed.

5.3 Isotropic oscillator in 2-dim curved space

In the case of the isotropic oscillator in a two-dimensional curved space with constant curvature $\lambda$ one finds the algebra \[17\]
\[
[L, S_\pm] = \pm 2S_\pm, \quad [S_-, S_+] = G(L + 1) - G(L - 1), \quad (5.25)
\]
with
\[
G(L) = H^2 - \left(\omega^2 + \frac{\lambda^2}{4} + \lambda H\right)L^2 + \frac{1}{4}\lambda^2 L^4. \quad (5.26)
\]
Using $\tilde{L} = L/2$ one easily sees that
\[
[L, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 8\tilde{L}\left(\omega^2 - \frac{\lambda^2}{4} + \lambda H\right) - 16\lambda^2 \tilde{L}^3, \quad (5.27)
\]
which corresponds to an su$_\Phi$(2) algebra with
\[
\Phi(J_0(J_0 + 1)) = 4\left(\omega^2 - \frac{\lambda^2}{4} + \lambda H\right)J_0(J_0 + 1) - 4\lambda^2(J_0(J_0 + 1))^2. \quad (5.28)
\]
For $\Phi(x)$ to be an increasing function, the condition
\[
x < \frac{1}{2\lambda^2} \left( \omega^2 - \frac{\lambda^2}{4} + \lambda H \right)
\] (5.29)
has to be satisfied.

### 5.4 The quadratic Hahn algebra QH(3)

The quadratic Hahn algebra QH(3) \[18\]
\[
[K_1, K_2] = K_3, \quad (5.30)
\]
\[
[K_2, K_3] = A_2 K_2^2 + C_1 K_1 + DK_2 + G_1, \quad (5.31)
\]
\[
[K_3, K_1] = A_2 (K_1 K_2 + K_2 K_1) + C_2 K_2 + DK_1 + G_2, \quad (5.32)
\]
can be put in correspondence to an $su_\Phi(2)$ algebra in the special case in which $C_1 = -1$ and $D = G_2 = 0$. The equivalence can be seen \[74\] by defining the operators
\[
J_1 = \frac{J_+ + J_-}{2}, \quad J_2 = \frac{J_+ - J_-}{2i}. \quad (5.33)
\]
Then the $su_\Phi(2)$ commutation relations can be written as
\[
[J_0, J_1] = iJ_2, \quad [J_0, J_2] = -iJ_1, \quad (5.34)
\]
\[
[J_1, J_2] = \frac{i}{2} (\Phi(J_0(J_0 + 1)) - \Phi(J_0(J_0 - 1))). \quad (5.35)
\]
Subsequently one can see that the two algebras are equivalent for
\[
K_1 = J_1 + A_2 J_0^2 + G_1, \quad K_2 = J_0, \quad K_3 = -iJ_2, \quad (5.36)
\]
and
\[
\Phi(J_0(J_0 + 1)) = -(2A_2 G_1 + C_2) J_0(J_0 + 1) - A_2^2 (J_0(J_0 + 1))^2. \quad (5.37)
\]
For $\Phi(x)$ to be an increasing function, the condition
\[
x < -\frac{2A_2 G_1 + C_2}{2A_2^2} \quad (5.38)
\]
has to be obeyed.

### 5.5 The finite W algebra $\bar{W}_0$

It is worth remarking that the finite W algebra $\bar{W}_0$ \[12\]
\[
[L_0^+, L_0^-] = \pm L_0^+, \quad (5.39)
\]
\[
[L_0^+, L_0^-] = (-k(k-1) - 2(k+1)) U_0 + 2(U_0)^3, \quad (5.40)
\]
is also an $su_\Phi(2)$ algebra with
\[
\Phi(J_0(J_0 + 1)) = \left( -\frac{k(k-1)}{2} - (k + 1)h \right) J_0(J_0 + 1) + \frac{1}{2} (J_0(J_0 + 1))^2. \quad (5.41)
\]
For $\Phi(x)$ to be an increasing function, the condition
\[
x > \frac{k(k-1)}{2} + (k + 1)h \quad (5.42)
\]
has to be satisfied.

In all of the above cases the representation theory of the $su_\Phi(2)$ algebra immediately follows from eqs. (5.2)–(5.4). In each case the range of values of the free parameters is limited by the condition that $\Phi(x)$ has to be an increasing entire function defined for $x \geq -1/4$. The results of this section are summarized in table 2.
Table 2: Structure functions of generalized deformed su(2) algebras. For conditions of validity for each of them see the corresponding subsection of the text.

| $\Phi(J_0(J_0 + 1))$ | Reference |
|-----------------------|-----------|
| i $J_0(J_0 + 1)$      | usual su(2) |
| ii $[J_0,J_0+1]_q$   | su$_q$(2) [1, 2] |
| iii $16\eta J_0(J_0 + 1) - 64(J_0(J_0 + 1))^2$ | 2 identical particles in 2-dim [10] |
| iv $(-2H + \frac{\lambda}{4}) J_0(J_0 + 1) + \lambda(J_0(J_0 + 1))^2$ | Kepler system in 2-dim curved space [17] |
| v $\frac{4(\omega^2 \frac{\lambda^2}{4} + \lambda H)}{J_0(J_0 + 1) - 4\lambda^2(J_0(J_0 + 1))^2}$ | isotropic oscillator in 2-dim curved space [17] |
| vi $-(2A_2 G_1 + C_2) J_0(J_0 + 1) - A_2^2(J_0(J_0 + 1))^2$ | quadratic Hahn algebra QH(3) [18] |
| vii $\left( -\frac{k(k-1)}{2} - (k+1\hbar) \right) J_0(J_0 + 1) + \frac{1}{2}(J_0(J_0 + 1))^2 + \frac{1}{2}(J_0(J_0 + 1))^2 + \frac{1}{2}(J_0(J_0 + 1))^2$ | finite W algebra W$_0$ [19] |

6 Generalized deformed parafermionic oscillators

The relation of the above mentioned algebras, and of additional ones, to generalized deformed parafermions is also worth studying.

It has been proved [20] that any generalized deformed parafermionic algebra of order $p$ can be written as a generalized oscillator with structure function

$$F(x) = x(p + 1 - x)(\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots),$$

where $\lambda, \mu, \nu, \rho, \sigma, \ldots$ are real constants satisfying the conditions

$$\lambda + \mu x + \nu x^2 + \rho x^3 + \sigma x^4 + \ldots > 0, \quad x \in \{1, 2, \ldots, p\}.$$  \hfill (6.2)

6.1 The su$_\Phi$(2) algebra

Considering an su$_\Phi$(2) algebra [4] with structure function

$$\Phi(J_0(J_0 + 1)) = AJ_0(J_0 + 1) + B(J_0(J_0 + 1))^2 + C(J_0(J_0 + 1))^3,$$  \hfill (6.3)

and making the correspondence

$$J_+ \rightarrow A^\dagger, \quad J_- \rightarrow A, \quad J_0 \rightarrow N,$$ \hfill (6.4)

one finds by equating the rhs of the first of eq. (2.4) and the last of eq. (5.1) that the su$_\Phi$(2) algebra is equivalent to a generalized deformed parafermionic oscillator of the form

$$F(N) = N(p + 1 - N)[-\eta^2(p + 1)C + pB] + (p^3 C + (p - 1)B)N + ((p^2 - p + 1)C + B)N^2 + (p - 2)CN^3 + CN^4],$$ \hfill (6.5)

if the condition

$$A + p(p + 1)B + p^2(p + 1)^2C = 0$$ \hfill (6.6)

holds. The condition of eq. (6.2) is always satisfied for $B > 0$ and $C > 0$.

In the special case of $C = 0$ one finds that the su$_\Phi$(2) algebra with structure function

$$\Phi(J_0(J_0 + 1)) = AJ_0(J_0 + 1) + B(J_0(J_0 + 1))^2$$ \hfill (6.7)

is equivalent to a generalized deformed parafermionic oscillator characterized by

$$F(N) = BN(p + 1 - N)(-p + (p - 1)N + N^2),$$ \hfill (6.8)

if the condition

$$A + p(p + 1)B = 0$$ \hfill (6.9)
is satisfied. The condition of eq. (6.2) is satisfied for $B > 0$.

Including higher powers of $J_0(J_0 + 1)$ in eq. (6.3) results in higher powers of $N$ in eq. (6.5) and higher powers of $p(p+1)$ in eq. (6.6). If, however, one sets $B = 0$ in eq. (6.7), then eq. (6.8) vanishes, indicating that no parafermionic oscillator equivalent to the usual $su(2)$ rotator can be constructed.

6.2 The finite W algebra $\bar{W}_0$

The $\bar{W}_0$ algebra \cite{19} of eqs (5.39)-(5.40) is equivalent to a generalized deformed parafermionic algebra with

$$F(N) = N(p + 1 - N) \frac{1}{2} (-p + (p - 1)N + N^2),$$

(6.10)

provided that the condition

$$k(k-1) + 2(k+1)h = p(p+1)$$

(6.11)

holds. One can easily check that the condition of eq. (6.2) is satisfied without any further restriction.

6.3 Isotropic harmonic oscillator in a 2-dim curved space

The algebra of the isotropic harmonic oscillator in a 2-dim curved space with constant curvature $\lambda$ for finite representations can be put in the form \cite{55}

$$F(N) = 4N(p + 1 - N) \left(\lambda(p + 1 - N) + \sqrt{\omega^2 + \lambda^2/4}\right) \left(\lambda N + \sqrt{\omega^2 + \lambda^2/4}\right),$$

(6.12)

the relevant energy eigenvalues being

$$E_p = \sqrt{\omega^2 + \lambda^2/4}(p + 1) + \frac{\lambda}{2}(p + 1)^2,$$

(6.13)

where $\omega$ is the angular frequency of the oscillator. It is clear that the condition of eq. (6.2) is satisfied without any further restrictions.

6.4 The Kepler problem in a 2-dim curved space

The algebra of the Kepler problem in a 2-dim curved space with constant curvature $\lambda$ for finite representations can be put in the form \cite{21}

$$F(N) = N(p + 1 - N) \left(\frac{4\mu^2}{(p + 1)^2} + \lambda \frac{(p + 1 - 2N)^2}{4}\right),$$

(6.14)

the corresponding energy eigenvalues being

$$E_p = -\frac{2\mu^2}{(p + 1)^2} + \frac{\lambda p(p + 2)}{8},$$

(6.15)

where $\mu$ is the coefficient of the $-1/r$ term in the Hamiltonian. It is clear that the restrictions of eq. (6.2) are satisfied automatically.

6.5 The Fokas–Lagerstrom potential

The Fokas–Lagerstrom potential \cite{21} is described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{x^2}{2} + \frac{y^2}{18}.$$  

(6.16)

It is therefore an anisotropic oscillator with ratio of frequencies 3:1. For finite representations it can be seen \cite{55} that the relevant algebra can be put in the form

$$F(N) = 16N(p + 1 - N) \left(p + \frac{2}{3} - N\right) \left(p + \frac{4}{3} - N\right)$$

(6.17)

for energy eigenvalues $E_p = p + 1$, or in the form

$$F(N) = 16N(p + 1 - N) \left(p + \frac{2}{3} - N\right) \left(p + \frac{1}{3} - N\right)$$

(6.18)
for eigenvalues $E_p = p + 2/3$, or in the form

$$F(N) = 16N(p + 1 - N) \left( p + \frac{5}{3} - N \right) \left( p + \frac{4}{3} - N \right)$$ (6.19)

for energies $E_p = p + 4/3$. In all cases it is clear that the restrictions of eq. (6.2) are satisfied.

### 6.6 The Smorodinsky–Winternitz potential

The Smorodinsky–Winternitz potential [22] is described by the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2) + k(x^2 + y^2) + \frac{c}{x^2},$$ (6.20)

i.e. it is a generalization of the isotropic harmonic oscillator in two dimensions. For finite representations it can be seen [55] that the relevant algebra takes the form

$$F(N) = 1024k^2N(p + 1 - N) \left( N + \frac{1}{2} \right) \left( p + 1 + \frac{\sqrt{1 + 8c}}{2} - N \right)$$ (6.21)

for $c \geq -1/8$ and energy eigenvalues

$$E_p = \sqrt{8k} \left( p + \frac{5}{4} + \frac{\sqrt{1 + 8c}}{4} \right), \quad p = 1, 2, \ldots.$$ (6.22)

In the special case of $-1/8 \leq c \leq 3/8$ and energy eigenvalues

$$E_p = \sqrt{8k} \left( p + \frac{5}{4} - \frac{\sqrt{1 + 8c}}{4} \right), \quad p = 1, 2, \ldots$$ (6.23)

the relevant algebra is

$$F(N) = 1024k^2N(p + 1 - N) \left( N + \frac{1}{2} \right) \left( p + 1 - \frac{\sqrt{1 + 8c}}{2} - N \right).$$ (6.24)

In both cases the restrictions of eq. (6.2) are satisfied.

### 6.7 Two identical particles in two dimensions

Using the same procedure as above, the algebra of eq. (5.18) can be put in correspondence with a parafermionic oscillator characterized by

$$F(N) = N(p + 1 - N)64(p + (1 - p)N - N^2),$$ (6.25)

if the condition

$$\eta = 4p(p + 1)$$ (6.26)

holds. However, the condition of eq. (6.2) is violated in this case.

### 6.8 The quadratic Hahn algebra QH(3)

For the quadratic Hahn algebra QH(3) of eqs (5.30)-(5.32) one obtains the parafermionic oscillator with

$$F(N) = N(p + 1 - N)A_2^2(p + (1 - p)N - N^2),$$ (6.27)

if the condition

$$p(p + 1)A_2^2 + 2A_2G_1 + C_2 = 0$$ (6.28)

holds. Again, eq. (6.2) is violated in this case.
6.9 The finite W algebra $W_3^{(2)}$

The finite W algebra $W_3^{(2)}$ [24, 25, 26, 27] is characterized by the commutation relations

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H^2 + C,$$

$$[C, E] = [C, F] = [C, H] = 0. \quad (6.29)$$

Defining $\hat{H} = H/2$ these can be put in the form

$$[\hat{H}, E] = E, \quad [\hat{H}, F] = -F, \quad [E, F] = 4\hat{H}^2 + C,$$

$$[C, E] = [C, F] = [C, \hat{H}] = 0. \quad (6.30)$$

This algebra is equivalent to a parafermionic oscillator with

$$F(N) = \frac{2}{3} N(p + 1 - N)(2p - 1 + 2N), \quad (6.31)$$

provided that the condition

$$C = -\frac{2}{3} p(2p + 1) \quad (6.32)$$

holds. One can easily see that the condition of eq. (6.2) is satisfied without any further restriction.

6.10 The Holt potential

The Holt potential [23]

$$H = \frac{1}{2} (p_x^2 + p_y^2) + (x^2 + 4y^2) + \frac{\delta}{x^2} \quad (6.33)$$

is a generalization of the harmonic oscillator potential with a ratio of frequencies 2:1. The relevant algebra can be put in the form of an oscillator with

$$F(N) = \frac{2^{3/2}}{3} N(p + 1 - N) \left( p + 1 + \frac{\sqrt{1 + 8\delta}}{2} - N \right), \quad (6.34)$$

where $(1 + \delta) \geq 0$, the relevant energies being given by

$$E_p = \sqrt{8} \left( p + 1 + \frac{\sqrt{1 + 8\delta}}{4} \right). \quad (6.35)$$

In this case it is clear that the condition of eq. (6.2) is always satisfied without any further restrictions.

In the special case $-\frac{1}{8} \leq \delta \leq \frac{3}{8}$ one obtains

$$F(N) = \frac{2^{3/2}}{3} N(p + 1 - N) \left( p + 1 - \frac{\sqrt{1 + 8\delta}}{2} - N \right), \quad (6.36)$$

the relevant energies being

$$E_p = \sqrt{8} \left( p + 1 - \frac{\sqrt{1 + 8\delta}}{4} \right). \quad (6.37)$$

The condition of eq. (6.2) is again satisfied without any further restrictions within the given range of $\delta$ values.

The deformed oscillator commutation relations in these cases take the form

$$[N, A^\dagger] = A^\dagger, \quad [N, A] = -A, \quad (6.38)$$

$$[A, A^\dagger] = 2^{3/2} \left( 3N^2 - N \left( 4p + 1 \pm \sqrt{1 + 8\delta} \right) + p^2 \pm \frac{1}{2} p\sqrt{1 + 8\delta} \right). \quad (6.39)$$

It can easily be seen that they are the same as the $W_3^{(2)}$ commutation relations [24, 25, 26, 27] with the identifications

$$F = \sigma A^\dagger, \quad E = \rho A, \quad C = f(p), \quad H = -2N + k(p), \quad (6.40)$$

where

$$\rho \sigma = 2^{-19/2}/3, \quad k(p) = \frac{1}{3} \left( 4p + 1 \pm \sqrt{1 + 8\delta} \right), \quad (6.41)$$

$$f(p) = \frac{2}{9} \left( 14p^2 + 4p \pm (7p + 3)\sqrt{1 + 8\delta} + 1 + 4\delta \right). \quad (6.42)$$

It is thus shown that the Holt potential possesses the $W_3^{(2)}$ symmetry.

The results of this section are summarized in table 3.
In summary, we have proved that the deformation schemes introduced in the literature as unifying frameworks for various deformed oscillators (summarized in Table 1) are equivalent, their unified representation leading to a correspondence between the deformed oscillator and the N=2 supersymmetric quantum mechanics (SUSY-QM) scheme. In addition, we have shown that several physical systems (two identical particles in two dimensions, isotropic oscillator and Kepler system in a 2-dim curved space) and mathematical structures (quadratic Hahn algebra QH(3), finite W algebra \( \mathcal{W} \)) are identified with a generalized deformed su(2) algebra, the representation theory of which is known. The results are summarized in Table 2. Furthermore, the generalized deformed parafermionic oscillator is found to describe several physical systems (isotropic oscillator and Kepler system in a curved space, Fokas–Lagerstrom, Smorodinsky–Winternitz and Holt potentials) and mathematical constructions (generalized deformed su(2) algebras, finite W algebras \( \mathcal{W}_0 \) and \( \mathcal{W}_3^{(2)} \)). The results are summarized in Table 3. The framework of the generalized deformed parafermionic oscillator is more general than the generalized deformed su(2) algebra, since in the rhs of the relevant basic commutation relation in the former case (first equation in eq. (2.4)) both odd and even powers are allowed, while in the latter case (eq. (5.1)) only odd powers are allowed.

The relevance of deformed oscillator algebras, finite W algebras and quadratic algebras in the study of the symmetries of the anisotropic quantum harmonic oscillator in two \([50]\) and three \([57]\) dimensions is receiving attention.

### Table 3: Structure functions of deformed oscillators.

For conditions of validity and further explanations in the case of the various generalized deformed parafermionic oscillators see the corresponding subsection in the text.

| \( F(N) \) | Reference |
|---|---|
| \( \frac{q^N - q^{-N}}{q - q^{-1}} = [N]_q \) | \( q \)-deformed harmonic oscillator \([4, 6, 16]\) |
| \( N(p + 1 - N) \) | parafermionic oscillator \([32]\) |
| \( [N]_q[p + 1 - N]_q \) | \( q \)-deformed parafermionic oscillator \([33, 14]\) |
| \( N(p + 1 - N)(\lambda + \mu N + \nu N^2 + \rho N^3 + \sigma N^4 + ...) \) | generalized deformed parafermionic oscillator \([24]\) |
| \( N(p + 1 - N)[- (p^2 (p + 1)C + pB) + (p^3 C + (p - 1)B)N + (p^2 - p + 1)C + B]N^2 + (p - 2)CN^3 + CN^4 \) | 3-term su\(_{\phi}(2)\) algebra (eq. 6.3) |
| \( BN(p + 1 - N)( -p + (p - 1)N + N^2) \) | 2-term su\(_{\phi}(2)\) algebra (eq. 6.7) |
| \( N(p + 1 - N)\frac{1}{2}(-p + (p - 1)N + N^2) \) | finite W algebra \( \mathcal{W}_0 \) \([19]\) |
| \( 4N(p + 1 - N)(\lambda(p + 1 - N) + \sqrt{\omega^2 + \lambda^2}/4) \) | isotropic oscillator in 2-dim curved space \([17, 23]\) |
| \( N(p + 1 - N) \frac{4\mu^2}{(p+1)^2} + \lambda(p+1)^2(1-2N)^2/4 \) | Kepler system in 2-dim curved space \([17, 23]\) |
| \( 16N(p + 1 - N)(p + \frac{5}{2} - N)(p + \frac{3}{2} - N) \) or \( 16N(p + 1 - N)(p + \frac{1}{2} - N)(p + \frac{1}{2} - N) \) or \( 16N(p + 1 - N)(p + \frac{1}{2} - N)(p + \frac{1}{2} - N) \) | Fokas–Lagerstrom potential \([21, 55]\) |
| \( 1024k^2N(p + 1 - N)(N + \frac{1}{2})(p + 1 \pm \frac{\sqrt{1 + 26k 2}}{2} - N) \) | Smorodinsky-Winternitz potential \([24, 55]\) |
| \( \frac{2}{3}N(p + 1 - N)(2p - 1 + 2N) \) | finite W algebra \( \mathcal{W}_3^{(2)} \) \([24, 25, 26, 27]\) |
| \( 2^{23/2}N(p + 1 - N)(p + 1 \pm \frac{\sqrt{1 + 26k 2}}{2} - N) \) | Holt potential \([23, 55]\) |
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