Abstract

New exact completely closed homogeneous Generalized Master Equations (GMEs), governing the evolution in time of equilibrium two-time correlation functions for dynamic variables of a subsystem of \( s \) particles \((s < N)\) selected from \( N >> 1 \) particles of a classical many-body system, are obtained. These time-convolution and time-convolutionless GMEs differ from the known GMEs (e.g. Nakajima-Zwanzig GME) by absence of inhomogeneous terms containing correlations between all \( N \) particles at the initial moment of time and preventing the closed description of \( s \)-particles subsystem evolution. Closed homogeneous GMEs describing the subdynamics of fluctuations are obtained by applying a special projection operator to the Liouville equation governing the dynamics of \( N \)-particle system. In the linear approximation in the particles’ density, the linear Generalized Boltzmann equation accounting for initial correlations and valid at all timescales is obtained. This equation for a weak inter-particle interaction converts into the generalized linear Landau equation in which the initial correlations are also accounted for. Connection of these equations to the nonlinear Boltzmann and Landau equations are discussed.
Subdynamics of fluctuations in an equilibrium classical many-particle system and generalized linear Boltzmann and Landau equations

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1 Introduction

One of the long-standing problems of statistical physics of \( N \)-particle (\( N >> 1 \)) system remains the derivation of the closed kinetic equations for \( s \)-particle (\( s < N \)) distribution functions sufficient for calculation of the measurable values characterizing a non-equilibrium state of that many-particle system. The natural starting point is the Liouville (classical system) or von-Neumann (quantum system) linear equation for an \( N \)-particle distribution function or statistical operator. In the reduced description method leading to the BBGKY chain, the closed Boltzmann kinetic equation can be obtained by employing the Boltzmann "molecular chaos" approximation at any time moment (beginning from the initial state) or more sophisticated Bogoliubov principle of weakening of initial correlations [1]. In the latter case, in the first approximation in particles' density, the nonlinear Boltzmann equation follows from the BBGKY chain on a large (kinetic) timescale. Note, that nonlinearity of the Boltzmann equation, obtained from the linear Liouville (or von-Neumann) equation, is a consequence of the above mentioned approximations. Lanford’s derivation of the Boltzmann equation (however only on a small timescale) [2] seems to be the most relevant result in the mathematical foundation of the kinetic theory.

For the case of small inter-particle interaction, the nonlinear Landau equation follows from the Boltzmann equation (see, e.g., [3]). But, strictly speaking, the Landau equation should be derived from the particles system dynamics (the Liouville equation) in the weak-coupling limit. The partial result (for the short timescale) was obtained in [4].

In the projection operators approach leading to the Generalized Master Equations (GMEs) (see, e.g., [5]), in order to obtain the completely closed
(homogeneous) linear equation for the reduced $s$-particle ($s < N$) distribution function (statistical operator), the undesired inhomogeneous term (a source) containing all $N$-particle initial correlations should be disregarded (which is incorrect in principle [6]).

These procedures allowing for obtaining the desired closed kinetic equations are not completely satisfactory: They implies, e.g., the "propagation of chaos" in time hypothesis for the factorized initial state, the general proof of which is still lacking (see, e.g. [7]), and do not allow for considering the evolution process on any timescale for arbitrary initial state. The natural desire then arises to abandon the "molecular chaos" (or other mentioned assumption) and include initial correlations into consideration. This can be effectively done, e.g., by deriving from the Liouville (von Neumann) equation the completely closed (homogeneous with no source) evolution equations valid on any timescale. For arbitrary initial conditions it has been attempted in [8] [9] [10] [11].

On the other hand, the equilibrium state for the full system provides the natural initial state for the system under consideration [6]. Moreover, this case provides an additional opportunity for including initial correlations into consideration. For quantum system, the closed homogeneous linear evolution equations accounting for initial correlations was obtained in [12] [13] with application to polaron mobility problem.

It turns out, that the initial equilibrium state in the classical physics case is even more favorable, than the case of quantum physics, for realization of the program with no "molecular chaos" (factorized initial state) assumption. In this paper we show that there is a special projection operator selecting the relevant part of the equilibrium two-time $s$-particle correlation function for the $N$-particle ($N > s$) system of classical particles which obeys the exact time-convolution (TC) or time-convolutionless (TCL) homogeneous GMEs. Thus, it is shown that there is a subdynamics in the subspace of $s$ particles, and, therefore, the evolution of the correlation function (thermal fluctuations) is completely closed (no undesirable terms defined on the full phase space of $N$ particles). The initial correlations are "hidden" in the projection operator which can be expanded in the particles density $n$ or in a small inter-particle interaction. In the first case, the linear generalized Boltzmann equation accounting for initial correlations and valid at all timescales follows from the obtained homogeneous TC GME in the first in $n$ order expansion of the kernel governing the evolution of a one-particle correlation function. We show how this equation leads to the known linear Boltzmann equation with additional term related to initial correlations. In the case of small inter-particle interaction, we obtain from this equation the linear equation for one-particle correlation function and discuss its connection to the Landau equation.
Projection operator formalism for s-particle equilibrium correlation function

We consider an \( N \)-particle (\( N >> 1 \)) system of interacting classical particles. Let us select the subsystem \( s \), i.e., the complex of \( s \) (\( s < N \)) particles (s-complex), which interacts with the environment \( \Sigma \) of remaining \( N - s \) particles. Note, that the particles, making up a subsystem, can be of different from the environment particles sort. Then, we assume that the Hamilton function of the full system can be presented as

\[
H = H_s + H_\Sigma + \bar{H}_{s\Sigma},
\]

(1)

where \( H_s \), \( H_\Sigma \) and \( \bar{H}_{s\Sigma} \) are the Hamilton functions of the subsystem \( s \), environment \( \Sigma \) and the subsystem-environment interaction \( \bar{H}_{s\Sigma} \), respectively. More specific form of these functions will be considered later.

We consider a two-time equilibrium correlation function for subsystem's dynamic functions \( A_s \) and \( B_s \), which depend on the set of variables characterizing the subsystem, i.e on \( x_i = (r_i, p_i), \) i = 1, 2, ..., \( s \), where \( x_i \) is the coordinate of the \( i \)-th particle in the phase space. The time dependence of dynamic functions is given by

\[
A_s(t) = \exp(Lt)A_s(0), \quad B_s(t) = \exp(Lt)B_s(0),
\]

where \( L \) is the Liouville operator

\[
L = L_s + L_\Sigma + \bar{L}_{s\Sigma}
\]

related to the Hamilton function (1) and defined by the Poisson bracket. Thus, we consider the correlation function

\[
\varphi_{AB}(t) = < A_s(t)B_s(0) > = \int \ldots \int dx^s A_s(0)[\int \ldots \int dx^\Sigma G_N(t, \beta)]B_s(0),
\]

(2)

Here,

\[
\rho(\beta) = Z^{-1}\exp(-\beta H), \quad Z = \int \ldots \int dx^N \exp(-\beta H),
\]

\[
LC_N = \{H, C_N\}_p = \sum_{i=1}^N \left[ \frac{\partial C_N}{\partial r_i} \frac{\partial H}{\partial p_i} - \frac{\partial C_N}{\partial p_i} \frac{\partial H}{\partial r_i} \right],
\]

\[
dx^N = dx^s dx^\Sigma,
\]

(3)

\( \{H, C_N\}_p \) is the Poisson bracket, \( C_N \) is some dynamic function defined on the full phase space of the \( N \)-particle system under consideration. We see, that the dynamics of correlation function (2) is defined by function \( G_N(t, \beta) \) which depends on the whole set of variables \( x_1, ..., x_N \) and obeys the equation

\[
\frac{\partial}{\partial t} G_N(t, \beta) = -LG_N(t, \beta).
\]

(4)

The formal solution to Eq. (4) is

\[
G_N(t, \beta) = U(t, 0)G_N(0, \beta), \quad U(t, 0) = \exp(-Lt).
\]

(5)
However, it is seen from the definition (2), that dynamics of the subsystem fluctuations is governed by function dependent on much smaller number of variables $x_1, \ldots, x_s$ than the whole set of $N$ variables $x_1, \ldots, x_N$, i.e., by function $G_N(t, \beta)$ integrated over the environment variables $x_{s+1}, \ldots, x_N$

$$F_s(t, \beta) = \int \ldots \int dx^\Sigma G_N(t, \beta).$$

In order to obtain the equation for the reduced function $F_s(t, \beta)$ (6), it is convenient to employ the projection operator technique [14], [15], [16] and to break $G_N(t, \beta)$ by some projection operators $P$ and $Q = 1 - P$ (with the properties $P^2 = P$, $Q^2 = Q$, $P + Q = 1$, $PQ = 0$) into the relevant $R_N(t, \beta)$ and irrelevant $I_N(t, \beta)$ parts

$$G_N(t, \beta) = R_N(t, \beta) + I_N(t, \beta),
R_N(t, \beta) = PG_N(t, \beta), I_N(t, \beta) = QG_N(t, \beta) = G_N(t, \beta) - R_N(t, \beta).$$

We note, that the relevant and irrelevant parts generally depend on coordinates and momenta of all $N$ particles in contrast to the reduced function $F_s(t, \beta)$. The relevant part $R_N(t, \beta)$ is conveniently defined in such a way that it comprises the reduced function of interest $F_s(t, \beta)$ as a multiplier. Thus, we consider the projection operators of the form

$$P = \Phi_{s\Sigma} \int \ldots \int dx^\Sigma,$$

where the function $\Phi_{s\Sigma}$ generally depends on the coordinates of a subsystem and an environment and normalized as

$$\int \ldots \int \Phi_{s\Sigma} dx^\Sigma = 1.$$

Then, it is easily seen that for projectors given by (8) and (9), the correlation function (2) is completely defined by the relevant part of $G_N(t, \beta)$

$$\varphi_{AB}(t) = \int \ldots \int dx^s \int \ldots \int dx^\Sigma A_s(0)R_N(t, \beta)B_s(0).$$

If, e.g.,

$$\Phi_{s\Sigma} = \rho_{\Sigma} = Z^{-1}_\Sigma \exp(-\beta H_\Sigma), Z_\Sigma = \int \ldots \int dx^\Sigma \exp(-\beta H_\Sigma),$$

then we have the "standard" projectors (see, e.g., [5])

$$P = P_\Sigma = \rho_\Sigma \int \ldots \int dx^\Sigma, Q_\Sigma = 1 - P_\Sigma$$

conventionally used for such types of problems (interaction of a subsystem with a reservoir).
By application of operators (12) to equation (4), we obtain the equations for the relevant and irrelevant parts of $G_N(t, \beta)$

\[
\frac{\partial}{\partial t} R_N(t, \beta) = -P_S L[R_N(t, \beta) + I_N(t, \beta)],
\]

\[
\frac{\partial}{\partial t} I_N(t, \beta) = -Q_S L[R_N(t, \beta) + I_N(t, \beta)],
\] (13)

where now

\[
R_N(t, \beta) = P_S G_N(t, \beta) = \rho_s F_s(t, \beta),
\]

\[
I_N(t, \beta) = G_N(t, \beta) - \rho_s F_s(t, \beta) = G^i_N(t, \beta).
\] (14)

Finding $G^i_N(t, \beta)$ from the second equation (13) as a function of $G^r_N(\tau, \beta)$ and $G^i_N(0, \beta)$ and inserting it in the first equation (13), we arrive at the conventional exact time-convolution generalized master equation (TC-GME) known as the Nakajima-Zwanzig equation for the relevant part of function $G^r_N(t, \beta)$ [14, 15]

\[
\frac{\partial}{\partial t} G^r_N(t, \beta) = -P_S L G^r_N(t, \beta) + \int_0^t P_S L U Q_S(t, \tau) Q_S L G^r_N(\tau, \beta) d\tau
\]

\[
- P_S L U Q_S(t, 0) G^i_N(0, \beta),
\]

\[
U Q_S(t, \tau) = \exp[-Q_S L Q_S(t - \tau)].
\] (15)

Equation (15), which, in fact, gives the equation for the reduced function $F_s(t, \beta)$, is quite general and formally closed. Serving as a basis for many applications [5], this equation, nevertheless, contains the undesirable and non-negligible inhomogeneous initial condition term (the last term in the right hand side of (15))

\[
G^i_N(0, \beta) = G_N(0, \beta) - P_S G_N(0, \beta) = \rho(\beta) - \rho_s \rho_S,
\]

\[
\rho_s = \int \cdots \int dx \Sigma \rho(\beta)
\] (16)

(see (2) and (12)). This term is not equal to zero due to initial (at $t = 0$) correlations ($\rho(\beta) \neq \rho_s \rho_S$). Therefore, Eq. (15) does not provide for a complete reduced description of a multiparticle system in terms of the relevant (reduced) function $F_s(t, \beta)$. Applying Bogoliubov’s principle of weakening of initial correlations, allowing to eliminate the influence of initial correlations on the large enough time scale $t \gg t_{cor}$ ($t_{cor}$ is the time for damping of initial correlations) or using a factorized initial condition, when $\rho(\beta) = \rho_s \rho_S$, one can achieve the desirable goal and obtain the homogeneous GME for $F_s(t, \beta)$, i.e. Eq. (15) with no initial condition term. However, obtained in such a way homogeneous GME is either approximate and valid only on a large enough time scale (when all initial correlations vanish) or applicable only for a rather artificial (actually unreal, as pointed in [6]) initial conditions (no correlations at an initial instant of time). In addition, Eq. (15) poses the problem to deal with due to its time-nonlocality. However, it is possible to obtain the time-local equation for the
relevant part $G_N(t, \beta)$ [5] [17] [18] which also contains the inhomogeneous source term.

3 Completely closed (homogeneous) GMEs for s-particle correlation function

Let us now introduce the following projection operators $P_s$ and $Q_s$

\[ P = P_s = \rho_s^\Sigma \int \ldots \int dx^\Sigma, Q = Q_s = 1 - P_s, \]

\[ \rho_s^\Sigma = \frac{1}{Z_s^\Sigma} \exp[-\beta(H_\Sigma + \tilde{H}_s^\Sigma)], \]

\[ Z_s^\Sigma = \int \ldots \int dx^\Sigma \exp[-\beta(H_\Sigma + \tilde{H}_s^\Sigma)]. \]  

(17)

It is not difficult to see that $P_s^2 = P_s, Q_s^2 = Q_s, P_sQ_s = 0$. Then, we can divide $G_N(t, \beta)$ into the relevant $g_r^N(t, \beta)$ and irrelevant $g_i^N(t, \beta)$ components as

\[ G_N(t, \beta) = g_r^N(t, \beta) + g_i^N(t, \beta), \]

\[ g_r^N(t, \beta) = P_sG_N(t, \beta) = \rho_s^\Sigma F_S(t, \beta), \]

\[ g_i^N(t, \beta) = Q_sG_N(t, \beta) = G_N(t, \beta) - \rho_s^\Sigma F_S(t, \beta). \]  

(18)

It is not difficult to see that the dynamics of the correlation function (2) is completely defined by the relevant part $g_r^N(t, \beta)$ of $G_N(t, \beta)$, i.e.,

\[ \varphi_{AB}(t) = \int \ldots \int dx^s A_s(0)F_s(t, \beta)B_s(0) = \int \ldots \int dx^N A_s(0)g_r^N(t, \beta)B_s(0). \]  

(19)

The projection operator $P_s$ [17] has an interesting property, namely,

\[ P_s \rho(\beta) = \rho(\beta), Q_sG_N(0, \beta) = 0. \]  

(20)

Thus, by applying the introduced projection operators $P_s$ and $Q_s$ [17] to equation (4), we arrive at the following exact homogeneous time-convolution GME (compare with (15))

\[ \frac{\partial}{\partial t} g_r^N(t, \beta) = -P_sLg_r^N(t, \beta) + \int_0^t P_sLU_Q_s(t, \tau)Q_sLg_r^N(\tau, \beta)d\tau, \]

\[ U_Q_s(t, \tau) = \exp[-Q_sL(t - \tau)]. \]  

(21)

Equation (21) is the completely closed equation for the relevant part of the correlation function that we are looking for. It shows, that in the considered case, the dynamics of correlation function [2] can be exactly projected.
on the dynamics within its relevant subspace. It follows, that the correlation of fluctuations of the selected complex of \( s \) particles can be described by the linear equation in the subspace of the corresponding coordinates \( x_i = (r_i, p_i) \) \( (i = 1, ..., s) \). To make it more clear, we rewrite Eq. (21) as the equation for an \( s \)-particle function \( F_s(t, \beta) \) governing the subsystem’s fluctuations in time (see also (2) and (18))

\[
\frac{\partial}{\partial t} F_s(t, \beta) = -\left[ \int \ldots \int dx^\Sigma L^s \right] F_s(t, \beta) + \left[ \int \ldots \int dx^\Sigma L \int_0^t d\tau U_{QR}(t, \tau) Q_s L^s \right] F_s(\tau, \beta).
\]

(22)

Generally, the evolution equation (21) poses some problem to deal with due to its time-nonlocality. It is possible, however, to obtain the exact homogeneous time-local equation for the relevant part of the correlation function. The idea is to take advantage of the evolution of \( G_N(t, \beta) \), defined by (5), which leads to the relation

\[
G_N(\tau, \beta) = U^{-1}(t, \tau) G_N(t, \beta),
\]

\[
U^{-1}(t, \tau) = \exp[L(t - \tau)].
\]

(23)

Using (23) and the conventional projection operator (12), the well known time-convolutionless equation for the relevant part \( G^r_N(t, \beta) \) of \( G_N(t, \beta) \), which contains the undesirable inhomogeneous term (16) comprising the initial correlations, can be obtained. (see [5, 17, 18]).

We will show now, that the use of the projector (17) instead of (12) leads to the completely closed homogeneous time-convolutionless GME for the relevant part of the correlation function. We will briefly conduct the derivation which is rather a standard one. First, we apply the projector (17) to (23) and obtain additional equation connecting the relevant and irrelevant parts of \( G_N(t, \beta) \)

\[
g^r_N(\tau, \beta) = P_s U^{-1}(t, \tau) [g^r_N(t, \beta) + g^i_N(t, \beta)].
\]

(24)

We also have the equation for the irrelevant part \( g^i_N(t, \beta) \) which follows from the solution of the second equation (13) with the projection operator (17)

\[
g^i_N(t, \beta) = -\int_0^t U_{QR}(t, \tau) Q_s L g^r_N(\tau, \beta) d\tau,
\]

(25)

where \( U_{QR}(t, \tau) \) is given by (21) and the property (20) was used. From two equations (24) and (25) one finds that

\[
g^r_N(t, \beta) = [1 - \alpha(t)]^{-1} \alpha(t) g^r_N(t, \beta),
\]

\[
\alpha(t) = -\int_0^t U_{QR}(t, \tau) Q_s L P_s U^{-1}(t, \tau) d\tau.
\]

(26)
Substituting $g_r^\beta(t, \beta)$ into the projected by $P_s$ equation (4) into equation (26)

$$\frac{\partial}{\partial t} g_r^\beta(t, \beta) = -P_s L [g_r^\beta(t, \beta) + g_N^\beta(t, \beta)],$$

we finally obtain

$$\frac{\partial}{\partial t} g_r^\beta(t, \beta) = -P_s L [1 - \alpha(t)]^{-1} g_r^\beta(t, \beta).$$

If it is possible to expand the operator $[1 - \alpha(t)]^{-1}$ into the series in $\alpha(t)$, then the first two terms of this expansion results in the following time-local equation (compare with (21))

$$\frac{\partial}{\partial t} g_r^\beta(t, \beta) = -P_s L g_r^\beta(t, \beta) + P_s L \int_0^t d\tau U_{Q_s}(t, \tau) Q_s L P_s U_{Q_s}^{-1}(t, \tau) g_r^\beta(t, \beta).$$

Equations (21) and (28) present the main results of this section. They show that the projector (17) allows for selecting the relevant part $g_r^\beta(t, \beta)$ of the multiparticle function $G_N(t, \beta)$ governing the dynamics of correlation function function (2) which satisfies the completely closed linear time-convolution and time-convolutionless equations. They, in fact, describe the evolution of the $s$-particles marginals (6) on the arbitrary timescale. Thus, one remains in the scope of the linear evolution given by the Liouville equation (4) but should pay for this simplification by accounting for initial correlations, which are conveniently ignored. It is also worth noting that the developed formalism only works in the framework of classical physics (when the terms of the Hamilton function (1) commutes with each other). For quantum physics a different approach is needed (see [12, 13]).

4 Equations for a more specific case

Let us specify the Hamilton function $H$ for the case of the identical particles with the two-body interparticle interaction $V_{ij}$ as

$$H = H_s + H_\Sigma + \bar{H}_s,\quad H_s = \sum_{i=1}^s \frac{p_i^2}{2m} + \sum_{1 \leq i < j \leq s} V_{ij}(|r_i - r_j|) + < H_{s\Sigma} >_\Sigma,$$

$$H_\Sigma = \sum_{i=s+1}^N \frac{p_i^2}{2m} + \sum_{s+1 \leq i < j \leq N} V_{ij}(|r_i - r_j|),$$

$$\bar{H}_s = H_s - < H_{s\Sigma} >_\Sigma, H_{s\Sigma} = \sum_{i=1}^s \sum_{j=s+1}^N V_{ij}(|r_i - r_j|).$$

(30)
Here, for convenience, we introduce the energy of the mean field $<H_{s\Sigma}>_\Sigma$ acting on the $s$-complex by the ”equilibrium” environment

$$<H_{s\Sigma}>_\Sigma = \int \ldots \int dx^\Sigma \rho_{s\Sigma} H_{s\Sigma},$$

(31)

where $\rho_{s\Sigma}$ is given by (11). Note, that $<H_{s\Sigma}>_\Sigma$ depends only on the coordinates of $s$ selected particles $r_i \ (i = 1, \ldots, s)$. For a space-homogeneous case, this mean field does not depend on $r_i \ (i = 1, \ldots, s)$.

The corresponding to (30) Liouville operator $L$ is

$$L = L_s + L_{\Sigma} + \tilde{L}_{s\Sigma},$$

$$L_s = \sum_{i=1}^s \left[ v_i \nabla_i - (\nabla_i <H_{s\Sigma}>_\Sigma) \frac{\partial}{\partial p_i} \right] - \sum_{1 \leq i < j \leq s} (\nabla_i V_{ij}) \cdot \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \right),$$

$$L_{\Sigma} = \sum_{i=s+1}^N v_i \nabla_i - \sum_{s+1 \leq i < j \leq N} (\nabla_i V_{ij}) \cdot \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \right),$$

$$\tilde{L}_{s\Sigma} = - \sum_{i=1}^s \sum_{j=s+1}^N (\nabla_i V_{ij}) \cdot \left( \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \right) + \sum_{i=1}^s (\nabla_i <H_{s\Sigma}>_\Sigma) \frac{\partial}{\partial p_i},$$

$$v_i = p_i / m, \ \nabla_i = \frac{\partial}{\partial x^i}, \ V_{ij} = V_{ij}(|r_i - r_j|).$$

(32)

Equations (21) and (28) can be rewritten in the simplified form if we take into account the following operators properties

$$P_s L_s = P_s L_s P_s = \rho_{s\Sigma}^2 L_s \int \ldots \int dx^\Sigma, P_s L_s Q_s = 0,$$

$$Q_s L_s P_s = L_s P_s - \rho_{s\Sigma}^2 L_s \int \ldots \int dx^\Sigma,$$

$$P_s L_{\Sigma} = 0, P_s L_{\Sigma} Q_s = 0, Q_s L_{\Sigma} P_s = L_{\Sigma} P_s,$$

$$P_s \tilde{L}_{s\Sigma} P_s = \sum_{i=1}^s [ < F_i >_{s\Sigma}^\Sigma - < F_i >_{s\Sigma} ] \frac{\partial}{\partial p_i} P_s,$$

(33)

where

$$F_i = - \sum_{j=s+1}^N (\nabla_i V_{ij}), < \ldots >_{s\Sigma} = \int \ldots \int dx^\Sigma (\ldots \rho_{s\Sigma}^\Sigma),$$

(34)

i.e., $F_i$ is the force acting on the $i$-particle ($i = 1, \ldots, s$) from the the ”environment” of $N-s$ particles. Here and further on, we use, as usual, that all functions $\Phi(x_1, \ldots, x_N; t)$, defined on the phase space, and their derivatives vanish at the boundaries of the configurational space and at $p_i = \pm \infty$.

Now Eq. (21) can be presented as

$$\frac{\partial}{\partial t} F_s (t, \beta) = - \left[ L_s + \sum_{i=1}^s [ < F_i >_{s\Sigma}^\Sigma - < F_i >_{s\Sigma} ] \frac{\partial}{\partial p_i} \right] F_s (t, \beta) + C(t, \beta),$$

(35)
where the collision term $C(t, \beta)$ is defined by

$$C(t, \beta) = \int_0^t d\tau \cdots \int dx^\Sigma \tilde{L}_s \Sigma U_Q_s(\tau, 0)\tilde{L}_s \Sigma \rho_s^\Sigma + (L_s \rho_s^\Sigma) + L_s \rho_s^\Sigma$$

$$- \sum_{i=1}^s (\langle F_i >_\Sigma^s - \langle F_i >_\Sigma^s) \rho_s^\Sigma \frac{\partial}{\partial p_i} F_s(t - \tau, \beta).$$

(36)

and $(L_s \rho_s^\Sigma)$ means that $L_s$ acts only on $\rho_s^\Sigma$. Note that if we use in (35) and (36) the standard projector (12), i.e., substitute $\rho_s^\Sigma$ with $\rho^\Sigma$, then Eq. (35) will acquire the "standard" form

$$\frac{\partial}{\partial t} F_s(t, \beta) = -L_s F_s(t, \beta) + \int_0^t d\tau \cdots \int dx^\Sigma \tilde{L}_s \Sigma U_Q_s(\tau, 0)\tilde{L}_s \Sigma \rho_s^\Sigma F_s(t - \tau, \beta)$$

if we neglect in (15) the inhomogeneous source term. Thus, we see that the extra terms in Eq. (35) are due to initial correlations which are "hidden" in the projector (17).

5 Evolution equation for one-particle correlation function in the first approximation in the particles density

For what follows, we consider the equation for $F_1(t, \beta) = F_1(r_1, p_1; t, \beta)$. One can see that for $s = 1$ the Hamilton function (30) and the Liouville operator (32) have more simple form. In order to expand the kernel of Eq. (35) in the density of particles $n = N/V$ ($V$ is the system's volume), we need the expansion for the distribution function $\rho_1^\Sigma$. In order to do that, it is convenient to express $\exp(-\beta H_\Sigma)$ and $\exp(-\beta H_1\Sigma)$ in terms of the Mayer functions $f_{ij}$ [19]. Then,

$$\rho_1^\Sigma = \frac{\exp(-\beta H_\Sigma) \exp(-\beta H_1\Sigma)}{\int \cdots \int d\Sigma \exp(-\beta H_\Sigma) \exp(-\beta H_1\Sigma)}$$

$$= \rho^\Sigma(p) \prod_{2 \leq i < j \leq N}^N \left(1 + f_{ij}\right) \prod_{j=2}^N (1 + f_{ij}), d\Sigma = dr_2...dr_N$$

(38)
where
\[ f_{ij} = e^{-\beta V_{ij}} - 1, \quad V_{ij} = V(|r_i - r_j|), \]
\[ \rho_{\Sigma}(p) = \exp[-\beta H_{\Sigma}(p)]/\int \cdots \int dp^{\Sigma} \exp[-\beta H_{\Sigma}(p)], \]
\[ H_{\Sigma}(p) = \sum_{j=2}^{N} p_j^2/2m, \quad dp^{\Sigma} = dp_2 \cdots dp_N, \quad (39) \]

and we used that \( \exp(\beta \langle H_{\Sigma} \rangle_{\Sigma}) \) does not depend on the "environment" \( \Sigma \) variables and is cancelled out of \( \rho_{\Sigma} \).

Let us consider the denominator of \( \rho_{\Sigma} \)
\[ \int \cdots \int dr^{\Sigma} \prod_{2 \leq i < j \leq N} (1 + f_{ij}) \prod_{j=2}^{N} (1 + f_{1j}) \]
\[ = \int dr_2 \cdots \int dr_N (1 + f_{23} + f_{24} + f_{34} + \ldots + f_{N-1,N} + f_{23}f_{24} + \ldots) \times (1 + f_{12} + f_{13} + \ldots + f_{12}f_{13} + \ldots) \]
\[ = V^{N-1} + (N - 1)V^{N-2} \int f_{12} dr_2 + N(N - 1)V^{N-3} \int dr_2 \int dr_3 (f_{23} + f_{12}f_{13}) + \ldots \]
\[ = V^{N-1}[1 + n \int dr_2 f_{12} + n^2 \int dr_2 \int dr_3 (f_{23} + f_{12}f_{13}) + \ldots], \quad N \gg 1. \quad (40) \]

One can see that the terms with one integration over the particle coordinate is proportional to \( n \), while the terms with integration over coordinates of two, three and more particles are proportional to \( n^2 \), \( n^3 \) and higher powers of \( n \), respectively. In what follows, we will restrict ourselves to the linear in \( n \) approximation to the kernel of Eq. (35), and, therefore, only the terms with one integration over the particle coordinate should be taken into consideration. The products of terms with one integration over the particle coordinate will also be (naturally) disregarded. Thus, in the linear in \( n \) approximation
\[ \rho_{\Sigma} = \frac{1}{V^{N-1}} \rho_{\Sigma}(p)(1 + \sum_{j=2}^{N} f_{1j})(1 - n \int dr_2 f_{12}) \quad (41) \]

where we approximated \( \prod_{j=2}^{N} (1 + f_{1j}) \) by \( 1 + \sum_{j=2}^{N} f_{1j} \) for the abovementioned reasons (in view of the further integration over the environment \( N - s \) particles coordinates). In the same way we can consider the difference between \( \rho_{\Sigma} \) and \( \rho_{\Sigma} \). In the linear in \( n \) approximation
\[ \rho_{\Sigma} - \rho_{\Sigma} = \frac{1}{V^{N-1}} \rho_{\Sigma}(p)[(\sum_{j=2}^{N} f_{1j})(1 - n \int dr_2 f_{12}) - n \int dr_2 f_{12}], \quad (42) \]
where we used that in this approximation $\rho_\Sigma = \frac{1}{V_N} \rho_\Sigma(p)$. Then, applying \( \int \ldots \int \ ) to (12), we see that in the adopted approximation
\[
< F_1 >_\Sigma - < F_1 >_\Sigma = -n \int d\mathbf{r}_2 f_{12} (\nabla_1 V_{12}). 
\] (43)

Thus, Eq. (35) for \( s = 1 \) in the linear in \( n \) approximation acquires the form
\[
\frac{\partial}{\partial t} F_1(t, \beta) = -[v_1 \nabla_1 - (\nabla_1 < H_{1\Sigma} >_\Sigma) \frac{\partial}{\partial p_1} - n \int d\mathbf{r}_2 f_{12} (\nabla_1 V_{12}) \frac{\partial}{\partial p_1}] F_1(t, \beta) 
+ C_{\text{col}}(t, \beta) + C_{\text{ic}}(t, \beta), 
\] (44)
where \( C_{\text{col}}(t, \beta) \) is the collision term
\[
C_{\text{col}}(t, \beta) = n \int_0^t d\tau \int d\mathbf{r}_2 \int d\mathbf{p}_2 L_{12} \exp[-(L_{12}\tau)] \rho^0(\mathbf{p}_2) e^{-\beta V_{12}} F_1(t - \tau, \beta) 
\times F_1(t - \tau, \beta) 
L_{12} = L_1^0 + L_2^0 + L'_{12}, L_1^0 = v_1 \nabla_1, L_2^0 = v_2 \nabla_2, 
L'_1 = F_{12} \partial_{12}, F_{12} = - (\nabla_1 V_{12}), \partial_{12} = \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}, 
\rho^0(\mathbf{p}_2) = \exp(-\beta p_2^2/2m) \int d\mathbf{p}_2 \exp(-\beta p_2^2/2m), 
\] (45)
and \( C_{\text{ic}}(t, \beta) \) is the additional term due to initial correlations
\[
C_{\text{ic}}(t, \beta) = n \int_0^t d\tau \int d\mathbf{r}_2 \int d\mathbf{p}_2 L'_{12} \exp[-(L_{12}\tau)] g_{12} (\nabla_1 f_{12}) \rho^0(\mathbf{p}_2) F_1(t - \tau, \beta) 
= n \beta \int_0^t d\tau \int d\mathbf{r}_2 \int d\mathbf{p}_2 \partial_{12} F_{12} \exp[-(L_{12}\tau)] \rho^0(\mathbf{p}_2) e^{-\beta V_{12}} F_1(t - \tau, \beta) 
\times F_{12} \partial_{12}, \partial_{12} = \frac{\partial}{\partial p_1} - \frac{\partial}{\partial p_2}, 
g_{12} = v_1 - v_2. 
\] (46)
Here we used Eqs. (30), (32), (41) and that in the linear in \( n \) approximation
\[
< H_{1\Sigma} >_\Sigma = n \int d\mathbf{r}_2 V_{12}, 
\] (47)
and that \( \nabla_1 f_{12} = -\nabla_2 f_{12} \).

Obtained Eq. (44) with (45) and (46) is the main result of this section and can be considered as a generalized linear Boltzmann equation accounting
for initial correlations. The term (45) is the generalized linear version of the Boltzmann collision term. In the space-homogeneous case when $F_1(t, \beta) = F_1(p_1; t, \beta)$, and $\nabla_1 < H_{1\Sigma} >_{\Sigma} = 0$ and $\int d\mathbf{r}_2 f_{12}(\nabla_1 V_{12}) = 0$, Eq. (44) reduces to

$$\frac{\partial}{\partial t} F_1(p_1; t, \beta) = C_{col}(t, \beta) + C_{ic}(t, \beta),$$

where $C_{col}(t, \beta)$ and $C_{ic}(t, \beta)$ are given by (45) and (46) but with $F_1(t - \tau, \beta) = F_1(p_1; t - \tau, \beta)$.

We see, that the evolution in time in Eqs. (45) and (46) is defined by the exact two-particle propagator which satisfies the integral equation

$$\exp[-(L_{12} \tau)] = U_{12}(\tau) = U_{12}^0(\tau) + \int_0^\tau d\tau_1 U_{12}^0(\tau - \tau_1)L_{12} U_{12}(\tau_1),$$

where $U_{12}^0(\tau) = \exp[-(L_1^0 + L_2^0)\tau]$ is a "free" two-particle propagator.

6 Connection to the Boltzmann equation

Note, that Eqs. (44) and (48) are the reversible in time ones. They become irreversible if it is possible to extend in them the upper limit of integration over $\tau$ to infinity (and this limit exists). It can be the case of the short time of interparticle interaction $\tau_{cor}$, when

$$\tau_{cor} << t \sim \tau_{rel},$$

where $\tau_{rel}$ is the timescale on which the $F_1(t, \beta)$ changes (see also below). Then, in the obtained equations we can approximate the one-particle function as $F_1(t - \tau, \beta) \approx F_1(t, \beta)$ and simultaneously extend the upper limit of integration to infinity. The latter can be done, e.g., if the time-dependent force-force correlation function in Eqs. (45) and (46)

$$\int d\mathbf{r}_2 F_{12} \exp[-(L_{12} \tau)] F_{12}$$

$\pi$ quickly vanishes on the timescale $\tau_{cor} \sim r_{cor} / v << \tau_{rel}$ ($r_{cor}$ is a radius of the inter-particle interaction $V_{ij}$ and $v$ is the average particle velocity), i.e. when the interaction is rather a short-range one. Thus, in this case, Eqs. (44), and (48) on the timescale $t \sim \tau_{rel}$ become Markovian (time local) and irreversible.

Let us consider (following the approach of [3]) in this approximation and in the space-homogeneous case the collision term (45), which in new convenient
variables \( v_i = p_i/m, \) \( r = r_2 - r_1 \) and \( g = v_1 - v_2 \) can be written as

\[
C_{col}(v_1; t, \beta) = n \int dV_2 J(v_1, v_2),
\]

\[
J(v_1, v_2) = \int dr \int_0^\infty d\tau L'U(\tau)L' \varphi(v_1, v_2, r; t, \beta),
\]

\[
L' = [\nabla V(r)] * \partial, \nabla = \frac{\partial}{\partial r}, \partial = \frac{2}{m} \frac{\partial}{\partial g},
\]

\[
\varphi(v_1, v_2, r; t, \beta) = \rho^0(v_2)e^{-\beta V(r)}F_1(v_1; t, \beta),
\]

where we dropped for brevity the indexes \( 12 \) in \((48)\), which cannot lead to misunderstanding because we deal in the adopted first order in \( n \) approximation only with a pair of particles. It can be shown \[3\], that the integral over \( \tau \) in \((52)\) can be presented as (see \((49)\))

\[
Z = GL'Z, G = \int_0^\infty d\tau U^0(\tau), Z = \int_0^\infty d\tau U(\tau).
\]

In the matrix form, \( Z(r, g; r', g') \) satisfies the equation

\[
\{g\nabla - [\nabla V(r)] * \partial\} Z(r, g; r', g') = \delta(r - r')\delta(g - g'),
\]

i.e., it is the Green function of the two-particle Liouville equation (see \[22\]), whereas the matrix \( G(r, g; r', g') \) is diagonal with respect to velocity indexes \( G(r, g; r', g') = G^0(r - r')\delta(g - g') \) and \( G^0(r, g; r', g') \) is the Green function of the unperturbed Liouville equation, i.e.,

\[
g\nabla G^0(r - r') = \delta(r - r').
\]

If we introduce the function

\[
f(r, g; t, \beta) = \varphi(v_1, v_2, r; t, \beta) + \int dr' \int dg' Z(r, g; r', g') [\nabla' V(r')] \partial' \varphi(v_1', v_2', r'; t, \beta),
\]

then it is not difficult to show, using \((54)\), that

\[
\lim_{V(r) \to 0} f(r, g; t, \beta) = \rho^0(v_2)F_1(v_1; t, \beta).
\]

Thus, it is easy to verify that function \( f(r, g; t, \beta) \), presented as

\[
f(r, g; t, \beta) = \varphi(v_1, v_2, r; t, \beta) + \int dr' G^0(r - r') [\nabla' V(r')] \partial f(r', g; t, \beta),
\]

satisfies the equation \((57)\).
Now we can write down the function \( J(v_1, v_2) \) defining the collision term as
\[
J(v_1, v_2) = \int dr [\nabla V(r)] \partial \{f(r, g; t, \beta) - \varphi(v_1, v_2, r; t, \beta)\}
\]
\[
= \int drg \nabla V(r) f(r, g; t, \beta). \tag{59}
\]
Here we used (56), (57) and that \( \varphi(v_1, v_2, r; t, \beta) \) depends on the relative distance \( r \) as \( \exp[-\beta V(r)] \), and, therefore, \( \int dr [\nabla V(r)] \exp[-\beta V(r)] = -\beta^{-1} \int dr \{\exp[-\beta V(r)]\} = 0 \) in the space-homogeneous case.

Let us select the coordinate system in which axis \( z \) is directed along vector \( g \). Then the Green function \( G^0(r - r') \) has the form
\[
G^0(r - r') = g^{-1} \delta(x - x') \delta(y - y') \theta(z - z'),
\]
\[
\theta(x) = 1, x > 0,
\]
\[
\theta(x) = 0, x < 0,
\]
\[\tag{60}\]
which is in agreement with Eq. (55). Inserting (60) in (58), we obtain
\[
f(r, g; t, \beta) = \varphi(v_1, v_2, r; t, \beta) + \int_{-\infty}^{z} dz' g^{-1} [\nabla' V(x, y, z')] \partial f(x, y, z', g; t, \beta)
\]
\[
= \varphi(v_1, v_2, r; t, \beta) + \int_{-\infty}^{z} dz' \left( \frac{\partial}{\partial z'} \right) \{f(x, y, z', g; t, \beta) - \varphi(v_1, v_2, z'; t, \beta)\}
\]
\[
= \varphi(v_1, v_2, r; t, \beta) + \int_{-\infty}^{z} dz' \left( \frac{\partial}{\partial z'} \right) f(x, y, z', g; t, \beta)
\]
\[
= \varphi(v_1, v_2, r; t, \beta) + \Phi(x, y, z; g; t, \beta),
\]
\[
\Phi(x, y, z, g; \beta) = \int_{-\infty}^{z} dz' \left( \frac{\partial}{\partial z'} \right) f(x, y, z', g; t, \beta). \tag{61}\]

Finally, introducing (61) in (59), we obtain
\[
J(v_1, v_2) = \int drg \nabla V(r) f(r, g; t, \beta)
\]
\[
= \int_{-\infty}^{\infty} dzg \left( \frac{\partial}{\partial z} \right) \Phi(x, y, z; g) = g [\Phi(x, y, +\infty, g; t, \beta) - \Phi(x, y, -\infty, g; t, \beta)]
\]
\[
= g \int_{-\infty}^{\infty} dz' \left( \frac{\partial}{\partial z} \right) f(x, y, z', g; t, \beta)
\]
\[
= g [f(x, y, +\infty, g; t, \beta) - f(x, y, -\infty, g; t, \beta)]. \tag{62}\]
It is then evident from (62), that function \( f(x, y, -\infty, gt, \beta) \) can be identified with the distribution function (62) \( \varphi(v_1, v_2, -\infty; t, \beta) \) prior to collision, i.e.

\[
\varphi(v_1, v_2, -\infty; t, \beta) = \rho_0(v_2)F_1(v_1; t, \beta),
\]

(63)

where we used that \( \exp[-\beta V(\pm \infty)] = 1 \) \( (V(\pm \infty) = 0) \). Function \( f(x, y, +\infty, gt, \beta) \) can be considered as the distribution function of particles after collision with the relative velocity \( g \). But according to the Liouville theorem, this distribution function is equal to the distribution function before collision with velocities \( v'_1, v'_2 \) which correspond to the velocities \( v_1, v_2 \), i.e.,

\[
f(x, y, +\infty, gt, \beta) = \rho_0(v'_2)F_1(v'_1; t, \beta),
\]

\[
\begin{align*}
v_1 + v_2 &= v'_1 + v'_2, \\
v_1^2 + v_2^2 &= v'_1^2 + v'_2^2.
\end{align*}
\]

(64)

Taking into account that in the adopted coordinate system with \( g \) directed along the \( z \)-axis

\[
dx dy = bdbd\varphi,
\]

(65)

where \( b \) is the impact parameter and \( \varphi \) is the azimuth angle. As a result, we arrive at the linear Boltzmann collision term in Eq. (48)

\[
C_{\text{col}}(v_1; t, \beta) = n \int dv_2 \int d\varphi dbb \rho_0(v_2)[\rho_0(v'_2)F_1(v'_1; t, \beta) - \rho_0(v_2)F_1(v_1; t, \beta)].
\]

(66)

Making in (66) the substitutions

\[
F_1(v_1; t, \beta) = \rho_0(v_1)W(v_1, t),
\]

(67)

the collision term for the function \( W \) can be rewritten as (see, e.g. [20])

\[
C_{\text{col}}(v_1; t, \beta) = \rho_0(v_1)n \int dv_2 \int d\varphi dbb \rho_0(v_2)[W(v'_1, t) - W(v_1, t)],
\]

(68)

where we used that according to definition (45) for \( \rho_0(v_2) \) and the conservation of energy (64)

\[
\rho_0(v_1)\rho_0(v_2) = \rho_0(v'_1)\rho_0(v'_2)
\]

(69)

Note, that \( \rho_0(v_1) \) can be cancelled out in Eq. (68) written for function \( W(v_1, t) \).

7 Connection to the Landau equation

It is interesting to consider Eq. (44) for the case of a weak interparticle interaction when

\[
< p_i^2 / 2m > \sim k_B T >> V_{ij},
\]

(70)
i.e., the interparticle interaction is small as compared to the average particle’s kinetic energy. Then, in the second order in the small parameter, defined by inequality (70), the collision term (45) can be rewritten as

\[ C_{\text{col}}(t, \beta) = n \int_0^t d\tau \int d\mathbf{r}_2 \int d\mathbf{p}_{12} \partial_{12} \exp[-(v_1 \nabla_1 + v_2 \nabla_2)\tau] \mathbf{F}_{12} \partial_{12} \rho^0(\mathbf{p}_2) e^{v_1 \nabla_1 \tau} F_1(t, \beta). \]  

(71)

Here, in order to remain within adopted accuracy, we took \( F_1(t - \tau, \beta) \) in the zero in the interaction approximation (see (44))

\[ F_1(t - \tau, \beta) = \exp[-v_1 \nabla_1 (t - \tau)] F_1(0, \beta) = e^{v_1 \nabla_1 \tau} F_1(t, \beta). \]  

(72)

In the same approximation

\[ C_{\text{ic}}(t, \beta) = n \int_0^t d\tau \int d\mathbf{r}_2 \int d\mathbf{p}_{12} \partial_{12} \exp[-(v_1 \nabla_1 + v_2 \nabla_2)\tau] \mathbf{F}_{12} \partial_{12} \rho_0(\mathbf{p}_2) e^{v_1 \nabla_1 \tau} F_1(t, \beta). \]  

(73)

Then, we have for any function of the particles coordinates \( \Phi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) \)

\[
\exp[-(v_1 \nabla_1 + v_2 \nabla_2)\tau] \Phi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N) = \Phi(\mathbf{r}_1 - v_1 \tau, \mathbf{r}_2 - v_2 \tau, ..., \mathbf{r}_N). \]  

(74)

If we also take into account the commutation rule

\[ [\exp[-(v_1 \nabla_1 + v_2 \nabla_2)\tau], \partial_{12}] = \exp[-(v_1 \nabla_1 + v_2 \nabla_2)\tau] \frac{\tau}{m} (\nabla_1 - \nabla_2), \]  

(75)

the collision term acquires the final form

\[ C_{\text{col}}(t, \beta) = n \int_0^t d\tau \int d\mathbf{p}_{12} \partial_{12} G_L(x_1, \mathbf{g}_{12}; \tau)(\partial_{12} + \frac{\tau}{m} \nabla_1) \rho^0(\mathbf{p}_2) F_1(x_1; t, \beta), \]

\[ G_L(x_1, \mathbf{g}_{12}; \tau) = \int d\mathbf{r}_2 \mathbf{F}_{12}(0) \mathbf{F}_{12}(\tau), \mathbf{F}_{12}(\tau) = -\nabla_1 V(\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{g}_{12} \tau). \]  

(76)

Using (74), the initial correlation term (73) can be rewritten as

\[ C_{\text{ic}}(t, \beta) = n\beta \int_0^t d\tau \int d\mathbf{p}_{12} \partial_{12} G_L(x_1, \mathbf{g}_{12}; \tau) \mathbf{g}_{12} \rho^0(\mathbf{p}_2) F_1(x_1; t, \beta). \]  

(77)

The collision integral (76) coincides with the corresponding collision integral in the nonlinear equation for inhomogeneous system of weakly interacting classical particles (see [3]) if in the latter, the distribution function for the particle, with which the tagged particle collides, is replaced by the equilibrium distribution function for this particle \( \rho^0(\mathbf{p}_2) \). In addition, for such a coincidence, the integral over \( d\tau \) should be extended to infinity. It can be done, if the interaction
is rather a short-range one and if for the timescale (50) the force acting on the particle vanishes \((F_{12}(t) = 0)\).

In the space-homogeneous case equation (48) for a small interparticle interaction reads

\[
\frac{\partial}{\partial t} F_1(p_1; t, \beta) = n \int_0^t \int d\tau \int d\vec{p}_2 \partial_{12} G_L(x_1, g_{12}; \tau) (\partial_{12} + \beta g_{12}) \rho^0(\vec{p}_2) F_1(p_1; t, \beta).
\]

The first (collision) term in the r.h.s. of Eq. (78) coincides with the Landau collision integral if it is possible to extend integral over \(\tau\) to infinity (short-range interparticle interaction) and to replace the distribution function for the second tagged particle with \(\rho^0(\vec{p}_2)\) (which seems natural for considered second order in interaction approximation for the kernel governing evolution of \(F_1(p_1; t, \beta)\)). The second term in the r.h.s. of (78) is caused by initial correlations.

\section{Conclusion}

We have rigorously derived the exact completely closed (homogeneous) Generalized Master Equations governing the evolution in time of an equilibrium two-time correlation function for dynamic variables of a selected group of \(s\) \((s < N)\) particles of an \(N\)-particle \((N \gg 1)\) system of classical particles. These time-convolution and time-convolutionless GMEs differ from known equations (such as Nakajima-Zwanzig equation) by absence of the undesirable inhomogeneous terms containing the correlations of all \(N\) particles in the initial moment of time. Such reduced description has become possible due to employing a special projection operator.

This projection operator (comprising initial correlations) can be expanded into series in the density of particles or in the weak interparticle interaction. In the linear in \(n\) approximation for the kernel governing the evolution of a one-particle correlation function, the generalized linear Boltzmann equation accounting for initial correlations and valid at any timescale has been rigorously obtained. At the timescale \(t \sim \tau_{rel} >> \tau_{cor}\) and for a short-range interaction this equation becomes irreversible with the collision integral of the known linear Boltzmann equation but with additional term due to initial correlations.

If in addition the interparticle interaction is weak, the generalized linear Boltzmann equation converts into the generalized linear Landau equation accounting for initial correlations and valid on all timescale. Again, at the timescale \(t \sim \tau_{rel} >> \tau_{cor}\) and for a short-range interaction this equation becomes irreversible. For the space-homogeneous case, the collision integral coincides with the Landau collision integral in which the distribution function of the second tagged particle is replaced with the equilibrium Maxwell distribution function. But there is also an additional term in the kernel governing the evolution of a one-particle correlation function caused by initial correlations.
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