On the summability of truncated double Fourier series

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Abstract

We estimate the truncated double trigonometric series
\[ \sum_{n=0}^{N} \sum_{m=0}^{M} a_{mn} e^{2\pi i (mx + ny)}, \quad a_{mn} \in \mathbb{C}, \]
in Lebesgue spaces with mixed norms in terms of the \( p^{th} - q^{th} \) power finite double sums of its coefficients. We obtain these estimates for all possible values of the exponents involved then we provide examples of matrices in \( \mathbb{C}^{M \times N} \) that maximize some of them up to a constant independent of \( M \) and \( N \).

Keywords: double trigonometric sums, integrability, \( L^p \) spaces with mixed norms

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1. The problem

Let \( 1 \leq p, q, r, s \leq \infty \). Consider the Banach space \( l^{p,q}(M, N) \) of all complex matrices \( A \in \mathbb{C}^{M \times N} \) with the norm

\[
\| A \|_{l^{p,q}(M, N)} =:\begin{cases} 
\left( \sum_{n=1}^{N} \left( \sum_{m=1}^{M} |a_{mn}|^p \right)^{q/p} \right)^{1/q}, & 1 \leq p, q < \infty; \\
\left( \sum_{n=1}^{N} (\max_{1 \leq m \leq M} |a_{mn}|)^q \right)^{1/q}, & p = \infty, 1 \leq q < \infty; \\
\max_{1 \leq n \leq N} \left( \sum_{m=1}^{M} |a_{mn}|^p \right)^{1/p}, & 1 \leq p < \infty, q = \infty; \\
\max_{1 \leq m \leq M, 1 \leq n \leq N} |a_{mn}|, & p = \infty, q = \infty.
\end{cases}
\]

Consider in addition the Banach space \( L^{r,s}([a, b]; [c, d]) =: L^q([c, d]; L^p([a, b])) \) of all functions \( f : [a, b] \times [c, d] \to \mathbb{C} \) that are Lebesgue measurable on \( [a, b] \times [c, d] \) and satisfy that \( \| f \|_{L^{r,s}([a, b]; [c, d])} = \| f(x, y) \|_{L^q([c, d])} \| L^p([a, b]) \|_{L^s([c, d])} < \infty \).

Let \( T_{M,N} : l^{p,q}(M, N) \to L^{r,s}([0, 1]; [0, 1]) \) be the linear operator that assigns
to each matrix $A \in \ell^{p,q}(M,N)$ the double trigonometric sum $S_{M,N} = T_{M,N}A$ defined by

$$S_{M,N}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} a_{mn} e^{2\pi i ((m-1)x+(n-1)y)}.$$  \hspace{1cm} (1)

The function $S_{M,N}$ is smooth and 1-periodic in each variable. If the complex entries $a_{mn}$ in (1) are the Fourier coefficients of some function in $L^1([0,1] \times [0,1])$ then $S_{M,N}$ is a rectangular partial sum of the double Fourier series of that function. This truncated sum proved useful in many applications (cf. [2, 10]). Let $Q$ be the hypercube $[0,1]^2$ and, for simplicity, let $\ell^{p,q}$, $L^{r,s}$ and $C_{M,N}(p,q,r,s)$ denote the spaces $\ell^{p,q}(M,N)$ and $L^{r,s}([0,1];[0,1])$ and the operator norm $\| T_{M,N} \|_{\ell^{p,q} \rightarrow L^{r,s}} = \sup_{A \in \ell^{p,q}, A \neq 0} (\| T_{M,N}A \|_{L^{r,s}} / \| A \|_{\ell^{p,q}})$ respectively. We are interested in estimating $S_{M,N}$ in the mixed $L^{r,s}$ norm in terms of the $\ell^{p,q}$ norm of its coefficients matrix $A$. That is, we would like to prove estimates of the form

$$\| S_{M,N} \|_{L^{r,s}} \leq c_{M,N}(p,q,r,s) \| A \|_{\ell^{p,q}} \text{ for all points } \left(\frac{1}{p},\frac{1}{q},\frac{1}{r},\frac{1}{s}\right) \in Q.$$  \hspace{1cm} (2)

Since the linear space $\ell^{p,q}$ is finite dimensional then we guarantee not only the boundedness of $T_{M,N}$ but also the existence of a maximizing matrix $A_{p,q,r,s} \in \mathbb{C}^{M \times N}$ for which

$$\| T_{M,N}A_{p,q,r,s} \|_{L^{r,s}} = C_{M,N}(p,q,r,s) \| A_{p,q,r,s} \|_{\ell^{p,q}}.$$ 

Foschi [3] studied this kind of boundedness for the one dimensional trigonometric sum $\sum_{n=0}^{N-1} a_n e^{inx}$. In [8], Vukolova and Dyachenko considered the sums of double trigonometric series in sines and cosines with multiply monotonous coefficients (see [9] by the same authors) and proved some estimates of these sums in $L^p$ spaces with a mixed norm.

2. $\ell^{p,q} - L^{r,s}$ estimates

If we take absolute values of both sides of (1) then apply the triangle inequality we easily get the estimate

$$\| S_{M,N} \|_{L^{\infty,\infty}} \leq \| A \|_{\ell^{1,1}}.$$  \hspace{1cm} (3)
The set \( O = \left\{ e^{2\pi i (m x + n y)}, (m, n) \in \mathbb{Z} \times \mathbb{Z} \right\} \) is an orthonormal system in \( L^{2,2} \). Thus

\[
\| S_{M,N} \|_{L^{2,2}}^2 = \int_0^1 \int_0^1 |S_{M,N}(x,y)|^2 \, dx \, dy = \int_0^1 \int_0^1 S_{M,N}(x,y)\overline{S_{M,N}(x,y)} \, dx \, dy
\]

\[
= \sum_{k=1}^N \sum_{j=1}^M \sum_{n=1}^N \sum_{m=1}^M a_{jk} \overline{a_{mn}} \int_0^1 e^{2\pi i (j-m)x} \, dx \int_0^1 e^{2\pi i (k-n)y} \, dy
\]

\[
= \sum_{n=1}^N \sum_{k=1}^M \sum_{m=1}^M \sum_{j=1}^M a_{jk} \overline{a_{mn}} \delta_{jm} \delta_{kn} = \sum_{k=1}^N \sum_{j=1}^M |a_{jk}|^2 = \| A \|_{l^{2,2}}^2.
\]

Hence we have

\[
\| S_{M,N} \|_{L^{2,2}} = \| A \|_{l^{2,2}}.
\]  

(4)

To this end we can obtain the estimates (2) on \( Q \) without further looking at the properties of the operator \( T_{M,N} \). First observe that, by Hölder’s inequality, we have

\[
\| f \|_{L^{r,s}} \leq \| f \|_{L^{r',s'}} \quad \text{if} \quad 1 \leq r' \leq r \leq \infty,
\]

\[
\| f \|_{L^r} \leq \| f \|_{L^s} \quad \text{if} \quad 1 \leq s \leq r \leq \infty,
\]  

(5)

for any function \( f \in L^{r,s} \). It also follows from Hölder’s inequality that

\[
\| A \|_{l^{p,q}} \leq M^{\frac{p}{r'} - \frac{q}{s'}} \| A \|_{l^{q,p}} \quad \text{if} \quad 1 \leq p, q \leq \infty,
\]

\[
\| A \|_{l^{p,q}} \leq N^{\frac{1}{r'} - \frac{1}{s'}} \| A \|_{l^{q,p}} \quad \text{if} \quad 1 \leq q, \bar{q} \leq \infty,
\]  

(6)

for any \( A \in C^{M \times N} \). Using (5) we deduce from the estimate (3) that \( \| S_{M,N} \|_{L^{r,s}} \leq \| A \|_{l^{1,1}} \), \( 1 \leq r, s \leq \infty \), which tells us that all \( L^{r,s} \) norms of \( S_{M,N} \) are controlled by the sum \( \sum_{n=1}^N \sum_{m=1}^M |a_{mn}| \). Of course we could furthermore apply (6) to the latter estimate and get that \( \| S_{M,N} \|_{L^{r,s}} \leq M^{1-\frac{1}{p}} N^{1-\frac{1}{q}} \| A \|_{l^{p,q}} \), \( 1 \leq p, q, r, s \leq \infty \). But we will momentarily find stronger estimates everywhere in \( Q - \{(1,1,0,0)\} \). For instance, if we apply (5) to the equality (4) we obtain the estimate

\[
\| S_{M,N} \|_{L^{r,s}} \leq \| A \|_{l^{2,2}} \quad \text{if} \quad \frac{1}{2} \leq \frac{1}{r} \leq 1, \quad \frac{1}{2} \leq \frac{1}{s} \leq 1,
\]  

(7)
and since, by (6), \( \| A \|_{L^2} \leq M^{\frac{1}{p} - \frac{1}{q}} N^{\frac{1}{p} - \frac{1}{q}} \| A \|_{l^p,q} \) for all \( 2 \leq p, q \leq \infty \) then it follows from (7) that

\[
\| S_{M,N} \|_{L^r,s} \leq M^{\frac{1}{p} - \frac{1}{q}} N^{\frac{1}{p} - \frac{1}{q}} \| A \|_{l^p,q}, \quad 0 \leq \frac{1}{p}, \frac{1}{q} \leq \frac{1}{2}, \quad \frac{1}{2} \leq \frac{1}{r}, \frac{1}{s} \leq 1. \tag{8}
\]

By standard \( L^p \) interpolation (cf. [1]) between the \( l^{1,1} - L^{\infty,\infty} \) estimate (3) and the \( l^{2,2} - L^{2,2} \) estimate (4) we obtain

\[
\| S_{M,N} \|_{L^r,s} \leq \| A \|_{l^p,q}, \tag{9}
\]

for all points \( (\frac{1}{p}, \frac{1}{q}, \frac{1}{r}, \frac{1}{s}) \) on the line segment joining the two points \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) and \( (1, 1, 0, 0) \) in \( Q \) given by \( \frac{1}{2} \leq \frac{1}{p} = \frac{1}{q} \leq 1, \quad 0 \leq \frac{1}{r} = \frac{1}{s} \leq \frac{1}{2}, \quad \frac{1}{2} \leq \frac{1}{p} \leq 1 \).

Applying (5) to the estimate (9) we get

\[
\| S_{M,N} \|_{L^r,s} \leq \| A \|_{l^p,q}, \quad 1 - \frac{1}{p} \leq \frac{1}{r} \leq 1, \quad 1 - \frac{1}{q} \leq \frac{1}{s} \leq 1, \quad \frac{1}{2} \leq \frac{1}{p} = \frac{1}{q} \leq 1. \tag{10}
\]

Moreover, using the first part of (6), it follows from (10) that

\[
\| S_{M,N} \|_{L^r,s} \leq M^{\frac{1}{p} - \frac{1}{q}} \| A \|_{l^p,q}, \quad 1 - \frac{1}{p} \leq \frac{1}{r} \leq 1, \quad 1 - \frac{1}{q} \leq \frac{1}{s} \leq 1, \quad 0 \leq \frac{1}{p} \leq \frac{1}{q}, \quad \frac{1}{2} \leq \frac{1}{q} \leq 1. \tag{11}
\]

While applying the second part of (6) to (10) we obtain

\[
\| S_{M,N} \|_{L^r,s} \leq N^{\frac{1}{p} - \frac{1}{q}} \| A \|_{l^p,q}, \quad 1 - \frac{1}{p} \leq \frac{1}{r} \leq 1, \quad 1 - \frac{1}{p} \leq \frac{1}{s} \leq 1, \quad 0 \leq \frac{1}{q} \leq \frac{1}{p}, \quad \frac{1}{2} \leq \frac{1}{p} \leq 1. \tag{12}
\]

Observe here that (8) follows either from (11) with \( q = 2 \) after reapplying (6) to the norm \( \| A \|_{l^p,2} \) or from (12) with \( p = 2 \) after reapplying (6) to the norm \( \| A \|_{l^2,q} \).
Interestingly, if we reverse the order in which we apply the consequences of Hölder’s inequality, (5) and (6), to the estimate (9) we recover the estimate (2) for another range of the exponents $p, q, r, s$. Indeed, applying (6) first to the estimate (9) yields

\[
\| S_{M,N} \|_{L^{r,s}} \leq M^{1 - \frac{1}{r} - \frac{1}{p}} N^{1 - \frac{1}{s} - \frac{1}{q}} \| A \|_{l^{p,q}},
\]

\[
0 \leq \frac{1}{p} \leq 1 - \frac{1}{r}, \quad 0 \leq \frac{1}{q} \leq 1 - \frac{1}{s}, \quad 0 \leq \frac{1}{r} = \frac{1}{s} \leq \frac{1}{2},
\]

(13)

Moreover, if we carefully use the inequalities (5) in (13) we are led to the estimates

\[
\| S_{M,N} \|_{L^{r,s}} \leq M^{1 - \frac{1}{r} - \frac{1}{p}} N^{1 - \frac{1}{s} - \frac{1}{q}} \| A \|_{l^{p,q}},
\]

\[
0 \leq \frac{1}{p} \leq 1 - \frac{1}{r}, \quad 0 \leq \frac{1}{q} \leq 1 - \frac{1}{r}, \quad \frac{1}{r} \leq \frac{1}{s} \leq 1, \quad 0 \leq \frac{1}{r} \leq \frac{1}{2},
\]

(14)

\[
\| S_{M,N} \|_{L^{r,s}} \leq M^{1 - \frac{1}{r} - \frac{1}{p}} N^{1 - \frac{1}{s} - \frac{1}{q}} \| A \|_{l^{p,q}},
\]

\[
0 \leq \frac{1}{p} \leq 1 - \frac{1}{s}, \quad 0 \leq \frac{1}{q} \leq 1 - \frac{1}{s}, \quad \frac{1}{s} \leq \frac{1}{r} \leq 1, \quad 0 \leq \frac{1}{s} \leq \frac{1}{2}
\]

(15)

Again, using (5), the estimate (8) results from (14) with $r = 2$ and from (15) with $s = 2$. The estimate (14) coincides with (11) only in the region $1/q + 1/r = 1$ and coincides with (12) only in the region $1/p + 1/r = 1$. Similarly, the estimate (15) coincides with (11) and (12) exclusively in the regions $1/q + 1/s = 1$ and $1/p + 1/s = 1$, respectively.

The relation between the exponents $p, q, r, s$ for which the estimates (11), (12), (14) and (15) hold can be demonstrated by the following respective four sets of figures.
One way to summarize the estimates obtained above is the following theorem.

**Theorem 1.** Let $\Theta : Q \to \left[\frac{1}{2}, 1\right]$ be the continuous surjection defined by
\[ \Theta(\alpha, \beta, \gamma, \delta) := \begin{cases} \frac{1}{2}, & 0 \leq \alpha \leq \frac{1}{2}, \ 0 \leq \beta \leq \frac{1}{2}, \ \frac{1}{2} \leq \gamma \geq 1, \ \frac{1}{2} \leq \delta \geq 1; \\ \alpha, & \frac{1}{2} \leq \alpha \leq 1, \ \alpha \geq \beta, \ \alpha + \gamma \geq 1, \ \alpha + \delta \geq 1; \\ \beta, & \frac{1}{2} \leq \beta \leq 1, \ \beta \geq \alpha, \ \beta + \gamma \geq 1, \ \beta + \delta \geq 1; \\ 1 - \gamma, & 0 \leq \gamma \leq \frac{1}{2}, \ \gamma \leq \delta, \ \alpha + \gamma \leq 1, \ \beta + \gamma \leq 1; \\ 1 - \delta, & 0 \leq \delta \leq \frac{1}{2}, \ \delta \leq \gamma, \ \alpha + \delta \leq 1, \ \beta + \delta \leq 1. \end{cases} \]

Then
\[ \| S_{M,N} \|_{L^{p,q}} \leq (MN)^{\Theta(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})} \| A \|_{l^{p,q}}. \quad (16) \]

Next, we try to find maximizers \( A_{p,q,r,s} \in C_{M,N} \) for the estimate (16).

3. Search for the maximizers

We begin with discussing a potential maximizer for the estimate (8). We anticipate, for this purpose, the estimate (19) in Lemma 4 which is an implication of Lemma 2 below when \( g \) is a constant function. Lemma 2 is due to Van der Corput. It provides an approximation for exponential sums with certain phases by oscillatory integrals.

Lemma 2. (\cite{7}, Lemma 4.10). Let \( f \) be a smooth function such that \( f'(x) \) is decreasing with \( f'(a) = \beta, \ \ f'(b) = \alpha. \) Let \( g(x) \) be a real positive function with a continuous derivative and \( |g'(x)| \) decreasing. Let \( \theta \in (0,1) \) be a constant. Then
\[ \sum_{a<n<b} g(n) e^{2\pi i f(n)} = \sum_{a-\theta < \lambda < b + \theta} \int_{a}^{b} g(t) e^{2\pi i (f(z) - \lambda z)} \, dz + O \left( g(a) \log (\beta - \alpha + 2) + O \left( |g'(a)| \right) \right). \]

Corollary 3. If \( f \) is a smooth real-valued function such that \( f' \) is monotone and \( |f'(x)| \leq \theta < 1 \) then the exponential sum
\[ \sum_{a<n<b} e^{2\pi i f(n)} = \int_{a}^{b} e^{2\pi i f(z)} \, dz + O(1). \quad (17) \]
Exploiting the assertion of Corollary 3 we get to prove the following lemma.

**Lemma 4.** Let $M >> 1$ and let $x \in [\eta, 1-\eta]$ for some fixed $0 < \eta < 1$. Then

$$
\sum_{m=0}^{M-1} e^{2\pi i m \left(x - \frac{x}{4M}\right)} = \sqrt{\frac{2}{\eta}} e^{-\frac{\pi i}{4}} e^{\frac{2\pi \eta M}{4}} x^2 \sqrt{M} + O(1) \quad (18)
$$

so that

$$
\left| \sum_{m=0}^{M-1} e^{2\pi i m \left(x - \frac{x}{4M}\right)} \right| \gtrsim \sqrt{M}. \quad (19)
$$

**Proof 5.** Since $z \mapsto z \left(x - \frac{x}{4M}\right)$ has a monotonically decreasing first derivative bounded by $1 - \frac{\eta}{2}$ when $0 \leq z \leq M$ and $0 \leq x \leq 1 - \eta$, then, by (17) of Corollary 3 we have

$$
\sum_{m=0}^{M-1} e^{2\pi i m \left(x - \frac{x}{4M}\right)} = I(M) + O(1), \quad I(M) = \int_0^{M-1} e^{2\pi i z \left(x - \frac{x}{4M}\right)} dz.
$$

Next, we manipulate the oscillatory integral $I(M)$ as follows

(i) Rescale $z \to Mz$ then complete the square for the quadratic polynomial in $z$ in the exponent

(ii) Rewrite the resulting integral $\int_0^{1-1/M} = \int_{-\infty}^{+\infty} - \int_{-\infty}^0 - \int_{1-1/M}^{+\infty}$.

(iii) For the integral $\int_{-\infty}^{+\infty}$, translate $z \to z + \frac{2}{\eta} x$ then apply the formula (cf. [6], Section 2.2) $\int_{-\infty}^{+\infty} e^{-iaz^2} \, dz = \sqrt{\frac{\pi}{a}} e^{-\frac{\pi a}{4}}$, $a > 0$, that can be justified by a contour integral.

(iv) Show that both integrals $\int_{-\infty}^0$, $\int_{1-1/M}^{+\infty}$ are at most $O(\eta^{-1}M^{-1})$ via integration by parts.
Following the steps (i) - (iv) we see that

$$I(M) = M \int_{0}^{1} e^{2\pi M z (x - \frac{4}{7} z)} \, dz = Me^{\frac{2\pi M}{\eta} x^2} \int_{0}^{1} e^{-\frac{\pi}{2} \eta M (z - \frac{4}{7} x)^2} \, dz. \quad (20)$$

Now, we have

$$\int_{-\infty}^{+\infty} e^{-\frac{\pi}{2} \eta M (z - \frac{4}{7} x)^2} \, dz = \int_{-\infty}^{+\infty} e^{-\frac{\pi}{2} \eta M x^2} \, dz = \sqrt{2} e^{-\frac{\pi}{4} \eta M}. \quad (21)$$

Also

$$-\pi \eta M \int_{-\infty}^{0} e^{-\frac{\pi}{2} \eta M (z - \frac{4}{7} x)^2} \, dz = \int_{-\infty}^{0} \frac{\partial}{\partial z} e^{-\frac{\pi}{2} \eta M (z - \frac{2}{7} x)^2} \, dz = O(1), \quad (22)$$

because when $\eta \leq x \leq 1 - \eta$

$$\left| \int_{-\infty}^{0} e^{-\frac{\pi}{2} \eta M (z - \frac{2}{7} x)^2} \, dz \right| \leq \int_{-\infty}^{0} \left( z - \frac{2}{7} x \right)^{-2} \, dz = \frac{-1}{z - \frac{2}{7} x} \bigg|_{-\infty}^{0} \in [\eta/(2(1 - \eta)), 1/2].$$

Similarly, one can verify that for $\eta \leq x \leq 1 - \eta$

$$\eta M \int_{1-1/M}^{+\infty} e^{-\frac{\pi}{2} \eta M (z - \frac{2}{7} x)^2} \, dz = O(1). \quad (23)$$

The identity (20) together with the estimates (21) - (23) yield (18).

Now, let $B \in \mathbb{C}^{M \times N}$ be such that $b_{jk} = e^{-\frac{\pi}{2} \eta \sqrt{(j-1)^2/M+(k-1)^2/N}}$ so that

$$T_{M,N}B(x, y) = \sum_{k=1}^{N} \sum_{j=1}^{M} e^{-\frac{\pi}{2} \eta \sqrt{(j-1)^2/M+(k-1)^2/N}} e^{2\pi i((j-1)x+(k-1)y)} \left( \sum_{j=0}^{M-1} e^{2\pi i j \left( x - \frac{4}{7} x \right)} \right) \left( \sum_{k=0}^{N-1} e^{2\pi i k \left( y - \frac{4}{7} y \right)} \right). \quad (24)$$
For \((x, y) \in [\eta, 1 - \eta]^2\), we can use (19) to estimate the exponential sums in (24) and obtain \(|T_{M,N}B(x, y)| \gtrsim M^{\frac{1}{2}} N^{\frac{1}{2}}\). This implies that \(\|T_{M,N}B\|_{L^{r,s}} \gtrsim M^{\frac{1}{2}} N^{\frac{1}{2}}\). And evidently \(\|B\|_{l^{p,q}} = M^{\frac{1}{p}} N^{\frac{1}{q}}\).

Thus

\[
\|T_{M,N}B\|_{L^{r,s}} \gtrsim \frac{1}{M^{\frac{1}{r} - \frac{1}{p}}} N^{\frac{1}{2} - \frac{1}{q}} \|B\|_{l^{p,q}}.
\] (25)

Estimate (25) stands behind our intuition that the matrix \(B\) is a candidate maximizer for (8). Let \(C \in \mathbb{C}^{M \times N}\) be a column matrix of ones so that \(c_{jk} = 1\) for some fixed \(1 \leq k \leq N\), \(c_{j\bar{k}} = 0\), \(\bar{k} \neq k\). Then

\[
T_{M,N}C(x, y) = e^{2\pi i (k-1) y} \sum_{m=1}^{M} e^{2\pi i (m-1) x} = e^{2\pi i (k-1) y} e^{\pi i (M-1) x} \frac{\sin (\pi M x)}{\sin (\pi x)}.
\] (26)

Since \(\sin (1) \leq \frac{\sin (\theta)}{\theta} \leq 1\) whenever \(0 \leq \theta \leq 1\). Then

\[
\frac{\sin (\pi M x)}{\pi M x} \geq \sin (1), \quad \frac{\sin (\pi x)}{\pi x} \leq 1, \quad \text{whenever} \quad 0 \leq x \leq \frac{1}{\pi M}.
\]

Hence \(\frac{\sin (\pi M x)}{\sin (\pi x)} \geq \sin (1) M\) when \(0 \leq x \leq \frac{1}{\pi M}\). Applying this inequality to (26), taking into account that \(e^{\pi i ((M-1)x + 2(k-1)y)}\) is a unit vector in \(\mathbb{C}\), we get

\[
|T_{M,N}C(x, y)| \geq \sin (1) M, \quad x \in [0, 1/\pi M].
\]

Turning to the \(L^{r,s}\) norm, the latter estimate implies

\[
\|T_{M,N}C\|_{L^{r,s}} \geq \|T_{M,N}C\|_{L^{r,s}([0,1/\pi M]; [0,1])} \geq \frac{\sin (1)}{\pi^{\frac{1}{r}}} M^{1 - \frac{1}{r}}.
\]

But \(\|C\|_{l^{p,q}} = M^{\frac{1}{p}}\). Therefore

\[
\|T_{M,N}C\|_{L^{r,s}} \geq \frac{\sin (1)}{\pi^{\frac{1}{r}}} M^{1 - \frac{1}{r} - \frac{1}{p}} \|C\|_{l^{p,q}}.
\] (27)

We deduce from (27) that, up to the constant \(\sin (1)/\pi^{\frac{1}{r}}\), the vector \(C\) maximizes both estimates (11) and (14) in their region of coincidence \(1/q + \ldots\)
Analogously, if $R \in \mathbb{C}^{M \times N}$ is a row matrix of ones then $\| R \|_{l^{p,q}} = N^{\frac{1}{q}}$ and we have

$$\| T_{M,N} R \|_{l^{r,s}} \geq \| T_{M,N} R \|_{L^{r,s}([0,1],[0,1/\pi N])} \geq \frac{\sin (1)}{\pi} N^{1-\frac{1}{r}-\frac{1}{q}} \| R \|_{l^{p,q}}.$$  \hspace{1cm} (28)

From (28) we see that, up to the constant $\sin (1)/\pi$, the row matrix $R$ maximizes both estimates (12) and (15) in the region $1/p + 1/s = 1$.

Furthermore, if $D \in \mathbb{C}^{M \times N}$ is a matrix of ones then it is easy to verify in the same spirit that

$$\| T_{M,N} D \|_{l^{r,s}} \geq \frac{\sin^2 (1)}{\pi^{r+s}} M^{1-\frac{1}{r}-\frac{1}{p}} N^{1-\frac{1}{r}-\frac{1}{q}} \| D \|_{l^{p,q}}.$$  \hspace{1cm} (29)

The inequality (29) shows that the matrix $D$ maximizes the estimate (13) up to the constant $\sin^2 (1)/\pi^{r+s}$.

Notice that we can achieve (27) applying the same argument if the nonzero entries, the ones, in $C$ are replaced by an arbitrary complex constant. The same claim holds for the matrices $R$ and $D$.

Finally, let $E \in \mathbb{C}^{M \times N}$ be such that $e_{mn} = 1$ for some $1 \leq m \leq M$, $1 \leq n \leq N$ and $e_{jk} = 0$ for $j \neq m$, $k \neq n$. Obviously $\| E \|_{l^{p,q}} = 1$ and since $|T_{M,N} E(x,y)| = 1$ then we have $\| T_{M,N} E \|_{L^{r,s}} = 1$ as well for all values of Lebesgue exponents $p, q, r, s$. So

$$\| T_{M,N} E \|_{L^{r,s}} = \| E \|_{l^{p,q}}.$$  \hspace{1cm} (30)

Observe here that not only does the nonzero entry $e_{mn}$ enjoy an arbitrary position but it can also be taken to be an arbitrary complex constant. We would always have the equality (30). From (30) we realize that the estimate (10) is sharp and that the matrix $E$ is a maximizer for it.

4. Asymptotic behaviour of $\| T_{M,N} \|_{l^{p,q} \to L^{r,s}}$ as $M, N \to \infty$.

On one hand the inequalities (8), (11), (12), (14) and (15) provide upper bounds for the positive constants $C_{M,N}(p,q,r,s)$ in the hypercube $Q$. While on the other hand each of the estimates (25), (27) - (30) gives a lower bound for them in a certain range of exponents values as pointed out in Section 3. Putting these bounds together, we can describe the asymptotic behaviour of $C_{M,N}(p,q,r,s)$ as $M, N \to \infty$ in some regions in $Q$. 

Theorem 6. Let $\Phi : Q \rightarrow \left[ \frac{1}{2}, 1 \right]$ be the restriction of the continuous surjection $\Theta$ in Theorem 1 defined by

$$\Phi(\alpha, \beta, \gamma, \delta) := \begin{cases} \frac{1}{2}, & 0 \leq \alpha \leq \frac{1}{2}, 0 \leq \beta \leq \frac{1}{2}, \frac{1}{2} \leq \gamma \geq 1, \frac{1}{2} \leq \delta \geq 1; \\ \alpha, & \frac{1}{2} \leq \alpha \leq 1, \alpha \geq \beta, \alpha + \gamma \geq 1, \alpha + \delta = 1; \\ \sqrt{\alpha \beta}, & \frac{1}{2} \leq \alpha \leq 1, \alpha = \beta, \alpha + \gamma > 1, \alpha + \delta > 1; \\ \beta, & \frac{1}{2} \leq \beta \leq 1, \beta \geq \alpha, \beta + \gamma = 1, \beta + \delta \geq 1; \\ \sqrt{(1-\gamma)(1-\delta)}, & 0 \leq \gamma \leq \frac{1}{2}, \gamma = \delta, \alpha + \gamma \leq 1, \beta + \delta \leq 1. \end{cases}$$

Then

$$C_{M,N}(p, q, r, s) \sim \frac{(MN)^\Phi(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})}{M^{\frac{1}{p}} N^{\frac{1}{q}}}. \tag{31}$$

5. The estimates when the sum terms are not orthogonal in $L^{2,2}$

We have seen in Section 2 how we obtained all $l^{p,q} - L^{r,s}$ estimates from the $l^{1,1} - L^{\infty,\infty}$ and $l^{2,2} - L^{2,2}$ estimates with the help of the powerful tool of interpolation and manipulation of Hölder’s inequality. In order to get the equality (4) we relied on the orthogonality of the terms of the double trigonometric sum in (1) in $L^{2,2}([0, 1]; [0, 1])$.

Nevertheless, we can still prove estimates of the form $\| S_{M,N} \|_{L^{2,2}} \lesssim \| A \|_{l^{2,2}}$ when the oscillatory terms in $S_{M,N}(x, y)$ are not orthogonal.

To illustrate the idea, assume that $V_{M,N}(x, y) = \sum_{n=1}^{N} \sum_{m=1}^{M} a_{mn} e^{i((m-1)x + (n-1)y)}$. By Urysohn’s lemma [3], let $\chi$ be a nonnegative cutoff function supported in $]-1/4, 5/4[$ such that $\chi(x) = 1$ on $[0, 1]$. Then

$$\| V_{M,N} \|_{L^{2,2}} \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \chi(x) \chi(y) |V_{M,N}(x, y)|^2 dx dy = \sum_{n_2=1}^{N} \sum_{n_1=1}^{N} \sum_{m_2=1}^{M} \sum_{m_1=1}^{M} a_{m_1 n_1} a_{m_2 n_2} \int_{\mathbb{R}} \chi(x) e^{i(m_1-m_2)x} dx \int_{\mathbb{R}} \chi(y) e^{i(n_1-n_2)y} dy. \tag{31}$$
By the localisation principle for oscillatory integrals \([5]\) we have

\[
\left| \int \chi(x) e^{i\nu x} \, dx \right| \lesssim \frac{1}{1 + \nu^2}, \quad \nu \in \mathbb{R}.
\]

Using this in (31) we get

\[
\| V_{M,N} \|^2_{L^2,2} \lesssim \sum_{n_2=1}^N \sum_{n_1=1}^N \frac{1}{1 + (n_1 - n_2)^2} \sum_{m_2=1}^M \sum_{m_1=1}^M \frac{|a_{m_1n_1}| |a_{m_2n_2}|}{1 + (m_1 - m_2)^2}. \tag{32}
\]

Using Young’s inequality for the convolution of sequences we find

\[
\sum_{m_2=1}^M \sum_{m_1=1}^M \frac{|a_{m_1n_1}| |a_{m_2n_2}|}{1 + (m_1 - m_2)^2} \lesssim \left( \sum_{m_1=1}^M a_{m_1n_1}^2 \right)^{\frac{1}{2}} \left( \sum_{m_2=1}^M a_{m_2n_2}^2 \right)^{\frac{1}{2}}.
\]

Plugging this estimate into (32) then reapplying Young’s inequality gives the estimate

\[
\| V_{M,N} \|^2_{L^2,2} \lesssim \| A \|^2_{L^2,2}.
\]

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