Abstract

We show that non-chiral $N = 2$ supergravity in ten-dimensions admits a family of dual actions where the one-form, two-form or three-form is replaced by the seven-form, six-form or five-form respectively. The dual actions and supersymmetry transformations are given.
1. Introduction

In the last year there has been a resurging interest in supergravity theories, especially in eleven-dimensions [1] and ten-dimensions [2] in connection with duality properties of superstring theories [3]. There are, however, supergravity theories which do not correspond to superstring theories, notably eleven-dimensional supergravity and ten-dimensional $N = 1$ supergravity formulated with a six-form [4] rather than a two-form. The two formulations in ten-dimensions are dual to each other, even in the presence of super Yang-Mills multiplet or higher curvature terms [4,5]. The formulation of supergravity with the six-form has been conjectured to be the low-energy limit of the four-brane [6], but since the quantization of membranes gives continuous spectra [7], the link at present is not clearly established. Further, eleven dimensional supergravity, when compactified on a circle, is thought to be equivalent to the type II superstring [3], and this in turn have some compactifications identical to those of the heterotic string when solitonic modes are taken into account. On the other hand eleven dimensional supergravity does not admit a dual formulation and the three-form could not be replaced with a six-form [9], although the two-brane theory is conjectured to admit a dual five-brane [6]. The trivial dimensional reduction of eleven-dimensional supergravity to ten-dimensions gives type IIA non-chiral $N = 2$ supergravity [11], and when truncated to $N = 1$ supergravity is known to have a dual formulation [2,4]. What prevents the eleven-dimensional theory from admitting a dual form is the presence of a Chern-Simons term $\int A_3 \wedge dA_3 \wedge dA_3$ where $A_3$ is a three form. As $A_3$ appears explicitly in the action a duality transformation is not possible, because the action cannot be expressed solely in terms of the field strength.

The purpose of this note is to show that type IIA $N = 2$ supergravity in ten-dimensions, although obtained from the eleven dimensional theory by trivial dimensional reduction, admits more than one dualisation. As the three form in eleven dimensions is reduced to a three-form and a two-form in ten-dimensions, the Chern-Simons term can always be rewritten in terms of either the field strength of the two-form or three-form. Further, a one-form which originates from the eleven dimensional metric, mixes with the two and three forms, in such a way that one of the one-form, two-form, or three-form appears only through its field strength. This will allow us to pass to the dual versions in ten-dimensions, provided one shows that the supersymmetry invariance continues to hold for the dual action. The plan of this
paper is as follows. In section two we start from the type IIA supergravity in ten dimensions, and show the modifications needed to allow for the duality transformations to be performed. In sections three, four and five we perform these transformations to obtain three other dual versions of the theory. In section 6 we indicate how to obtain another dual formulation and the connection between this family of theories.

2. $N = 2$ supergravity in ten-dimensions: (1,2,3)

Non-chiral $N = 2$ supergravity in ten dimensions was obtained [9] by trivially reducing the eleven dimensional theory [1]. The action is expressed in terms of the bosonic fields $A_{\mu\nu}$ (or $A_2$), $A_{\mu\nu\rho}$ (or $A_3$), $B_\mu$ (or $B$), $\phi$ and the vielbein $e_\mu^a$. Because of the presence of the one-form, two-form and three-form we will denote this formulation by (1,2,3). The fermionic fields are the gravitino $\psi_\mu$ and the spinor $\lambda$ both of which are Majorana spinors. The action is given by [9]

$$I = \int d^{10}x e \left( -\frac{1}{4\kappa^2} R(\omega(e)) - \frac{i}{2} \psi_\mu \Gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{48} \kappa \Gamma^{\mu\nu\rho\sigma} F^\prime_{\mu\nu\rho\sigma} ight. $$
$$+ \frac{1}{12} e^{-2\kappa\phi} F^\prime_{\mu\nu\rho} F^{\mu\nu\rho} - \frac{1}{4} e^{3\kappa\phi} G_{\mu\nu} G^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi $$
$$+ \frac{i}{2} \lambda \Gamma^\mu D_\mu \lambda - \frac{i\kappa}{\sqrt{2}} \lambda \Gamma^{11} \Gamma^\mu \psi_\mu \partial_\nu \phi $$
$$+ \frac{\kappa}{8(12)^2} e^{-1} e^{\mu_1 \cdots \mu_{10}} F_{\mu_1 \cdots \mu_4} F_{\mu_5 \cdots \mu_8} A_{\mu_9 \mu_{10}} $$
$$+ \frac{\kappa}{96} e^{\frac{2}{3} \phi} \left( \bar{\psi}_\mu \Gamma^{\mu\nu\alpha\beta\gamma\delta} \psi_\nu + 12 \bar{\psi} \Gamma^{\alpha\beta\gamma\delta} \psi \right) + \frac{1}{\sqrt{2}} \lambda \Gamma^{\mu} \Gamma^{\alpha\beta\gamma\delta} \psi_\mu + \frac{3}{4} \lambda \Gamma^{\alpha\beta\gamma\delta} \lambda \right) F^\prime_{\alpha\beta\gamma\delta} $$
$$- \frac{\kappa}{24} e^{\frac{2}{3} \phi} \left( \bar{\psi}_\mu \Gamma^{11} \Gamma^{\mu\nu\alpha\beta\gamma} \psi_\nu - 6 \bar{\psi} \Gamma^{11} \Gamma^{\beta\gamma} \psi \right) F_{\alpha\beta\gamma} $$
$$- \frac{i\kappa}{8} e^{\frac{3}{3} \phi} \left( \bar{\psi}_\mu \Gamma^{11} \Gamma^{\mu\nu\alpha\beta} \psi_\nu + 2 \bar{\psi} \Gamma^{11} \psi \right) + \frac{3}{2} \lambda \Gamma^{\mu} \Gamma^{\alpha\beta} \psi_\mu + \frac{5}{4} \lambda \Gamma^{11} \Gamma^{\alpha\beta} \lambda \right) G_{\alpha\beta} $$
$$+ \text{quartic fermionic terms} \right) \tag{2.1}$$

where $G_{\mu\nu}$, $F_{\mu\nu\rho}$ and $F^\prime_{\mu\nu\rho\sigma}$ are field strengths of $B_\mu$, $A_{\mu\nu}$ and $A_{\mu\nu\rho}$ respectively. Because of the eleven dimensional origin of this theory one has the modified field strength $F^\prime_{\mu\nu\rho\sigma}$ where

$$G_{\mu\nu} = 2 \partial_{[\mu} B_{\nu]} $$
$$F_{\mu\nu\rho} = 3 \partial_{[\mu} A_{\nu\rho]} \tag{2.2} $$
$$F^\prime_{\mu\nu\rho\sigma} = 4 (\partial_{[\mu} A_{\nu\rho\sigma]} + 2 B_{[\mu} F_{\nu\rho\sigma]})$$

As can be seen by compactifying the eleven-dimensional theory working in a flat frame [4,11], we can write the field strength $F^\prime$ in terms of a modified potential $A_3'$,
where
\[
A'_{\mu\nu\rho} = A_{\mu\nu\rho} - 6B_{[\mu}A_{\nu\rho]} \tag{2.3}
\]
\[
F'_{\mu\nu\rho\sigma} = 4(\partial_{[\mu}A'_{\nu\rho\sigma]} + 3G_{[\mu\nu}A_{\rho\sigma]})
\]
These identities will play a vital role in allowing for duality transformations. The supersymmetry transformations are given by
\[
\delta e^a_{\mu} = -i\epsilon\Gamma^a\psi_{\mu}
\]
\[
\delta\psi_{\mu} = D_{\mu}(\omega) - \frac{1}{32}e^{\frac{3\phi}{2}}(\Gamma_{\mu}^{\nu\rho} - 14\delta_{\mu}^{\nu}\Gamma^{\rho})\Gamma^{11}\epsilon G_{\nu\rho}
\]
\[
\delta\psi_{\mu} = D_{\mu}(\omega) - \frac{i}{48}e^{-\kappa\phi}(\Gamma_{\mu}^{\nu\rho\sigma} - 9\delta_{\mu}^{\nu}\Gamma^{\rho\sigma})\Gamma^{11}\epsilon F_{\nu\rho\sigma}
\]
\[
\delta B_{\mu} = \frac{i}{2}e^{-\frac{3\phi}{2}}(\overline{\psi}_{[\mu}\Gamma_{\nu]}\Gamma^{11}\epsilon - \frac{\sqrt{2}}{4}\epsilon\Gamma_{\mu}\epsilon)
\]
\[
\delta A_{\mu\nu} = e^{\kappa\phi}(\overline{\psi}_{[\mu}\Gamma_{\nu]}\Gamma^{11}\epsilon - \frac{1}{2\sqrt{2}}\epsilon\Gamma_{\mu\nu}\epsilon)
\]
\[
\delta A_{\mu\nu\rho} = -\frac{3}{2}e^{-\frac{3\phi}{2}}(\overline{\psi}_{\nu}\Gamma_{\rho]}\Gamma^{11}\epsilon - \frac{1}{6\sqrt{2}}\epsilon\Gamma^{11}\Gamma_{\mu\nu}\epsilon)
\]
\[
\delta A_{\mu\nu\rho} = -\frac{3}{2}e^{-\frac{3\phi}{2}}(\overline{\psi}_{\nu}\Gamma_{\rho]}\Gamma^{11}\epsilon - \frac{1}{6\sqrt{2}}\epsilon\Gamma^{11}\Gamma_{\mu\nu}\epsilon)
\]
\[
\delta\lambda = \frac{1}{\sqrt{2}}D_{\mu}\phi(\Gamma_{\mu}\Gamma^{11}\epsilon) + \frac{3}{8\sqrt{2}}e^{\frac{3\phi}{2}}\Gamma^{\mu\nu}\epsilon G_{\mu\nu}
\]
\[
\delta\phi = \frac{i}{\sqrt{2}}\epsilon\Gamma^{11}\epsilon
\]

Another important piece is the Chern-Simons term which can be written in terms of differential forms as \( \int A_2 \wedge dA_3 \wedge dA_3 \) where \( A_2 \) and \( A_3 \) stand for the two- and three-forms: \( A_2 = A_{\mu\nu}dx^{\mu} \wedge dx^{\nu} \), and \( A_3 = A_{\mu\nu\rho}dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \). This can be reexpressed in such a way that \( A_{\mu\nu} \) appears only through its field strength \( F_{\mu\nu} \). We derive this by using
\[
A_2 \wedge dA_3 \wedge dA_3 = d(A_2 \wedge A_3 \wedge dA_3) - dA_2 \wedge A_3 \wedge dA_3 \tag{2.5}
\]
and discarding the surface term after integration. Next, although the field \( B_{\mu} \) does not appear in the Chern-Simons term, it appears explicitly in the field strength \( F'_{\mu\nu\rho\sigma} \) in eq (2.2). If equation (2.3) is used instead of (2.2), then \( B_{\mu} \) appears only through its field strength \( G_{\mu\nu} \) but then the Chern-Simons term must be expressed in
terms of $A'_{\mu\nu\rho}$. It is not difficult to show that

$$
\begin{align*}
A_2 \wedge dA_3 \wedge dA_3 &= A_2 \wedge dA'_3 \wedge dA'_3 + 6A_2 \wedge A_2 \wedge dB \wedge dA'_3 \\
&\quad + 12A_2 \wedge A_2 \wedge A_2 \wedge dB \wedge dB \\
&\quad + 6d(A_2 \wedge A_2 \wedge B \wedge (dA'_3 + 4A_2 \wedge dB)) \\
\end{align*}
$$

(2.6)

Discarding the surface term, we see that the action (2.1) is expressible in terms of $A_2$, $dA'_3$ and $dB$. From all of these considerations it is very suggestive that we can apply duality transformations to the following fields $(A_6, A_2)$, or $(A_3, A_2)$ or $(B, A_7)$. We now consider these transformations one at a time.

3. A dual theory with a six-form $A_6$: $(1,6,3)$

To obtain the dual theory where the two-form is replaced with a six-form, we add to the action (2.1) the term

$$
\frac{1}{3!6!} \int A_6 \wedge dF_3
$$

(3.1)

where $A_6 = A_{\mu_1\cdots\mu_6} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_6}$ is a six-form and $F_3$ is a three-form, $F_3 = F_{\mu\nu\rho} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho}$, which in (2.1) is not assumed now to be a field strength. The equation of motion of $A_6$ forces $F_3$, locally, to be $dA_2$. Integrating by parts and discarding the surface term, eq (3.1) can be rewritten in the form

$$
\frac{1}{3!6!} \int F_3 \wedge dA_6
$$

(3.2)

Since $F_3$ appears in the action (2.1) and (3.2) at most quadratically, we can perform the $F_3$ gaussian integration to obtain the dual version as a function of $A_6$. Therefore, the action in the form (2.1) plus (3.1) can give either one of the two dual actions, depending on what is integrated first, $A_6$ or $F_3$. The supersymmetry transformations of the combined action can be found as follows [10]. The supersymmetry transformations of $F_3$ are taken to be identical to those of $d\delta A_2$ as given in eq (2.2) (without identifying $F_3$ with $dA_2$ ), then the action (2.1) will be invariant except for one term proportional to $dF_3$ which does not vanish now because the Bianchi identity is no longer available. The non-invariant term will be cancelled by the transformation of the new term (3.1) which is also proportional to $\int \delta A_6 \wedge dF_3$. This determines $\delta A_6$ to be given by

$$
\delta A_{\mu_1\cdots\mu_6} = -3ie^{-\kappa\phi} (\overline{\psi} \gamma_{\mu_1\cdots\mu_5} \psi_{\mu_6}) - \frac{i}{6\sqrt{2}} \overline{\psi} \gamma_{\mu_1\cdots\mu_6} \Gamma^{11} \lambda
$$

(3.3)

and explicitly shows that the action (2.1) plus (3.1) admits a duality transformation between the two-form and the six-form. The duality transformation is at the level
of the action and not only the equations of motion. As the field $F_{\mu\nu\rho}$ appears at most quadratically, doing the gaussian integration for $F_{\mu\nu\rho}$, or solving its equation of motion and substituting back into the action, are equivalent. The equation of motion gives

$$M_{\alpha\beta\gamma}^{\mu\nu\rho} F_{\mu\nu\rho} = X_{\alpha\beta\gamma}$$  \hspace{1cm} (3.4)$$

where the tensors $M_{\alpha\beta\gamma}^{\mu\nu\rho}$ and $X_{\alpha\beta\gamma}$ are given by

$$M_{\alpha\beta\gamma}^{\mu\nu\rho} = \left( \frac{1}{3!} e^{-2\kappa \phi} (1 - e^{3\kappa \phi} B_\sigma B^\sigma) \delta_{\alpha\beta\gamma}^{\mu\nu\rho} + \frac{1}{4} e^{\kappa \phi} B_\mu \delta_{\alpha\beta\gamma}^{[\mu\nu\rho] B} \right)$$  \hspace{1cm} (3.5)$$

and

$$X_{\alpha\beta\gamma} = \frac{1}{216} \epsilon_{\alpha\beta\gamma}^{\mu_1 \cdots \mu_7} \left( \frac{1}{7!} F_{\mu_1 \cdots \mu_7} + A_{\mu_1 \mu_2 \mu_3} \partial_{\mu_4} A_{\mu_5 \mu_6 \mu_7} \right) + \frac{\kappa}{24} e^{-\frac{1}{2} \kappa \phi} \left( \bar{\psi}_\mu \Gamma_{\alpha\beta\gamma}^{\mu\nu\rho} \psi_\nu - 6 \bar{\psi}_\alpha \Gamma_{\beta\gamma}^{11} \Gamma_{\mu\nu\rho} \psi_\mu \right) + \frac{\kappa}{12} e^{\frac{1}{2} \kappa \phi} \left( \bar{\psi}_\mu \Gamma_{\alpha\beta\gamma}^{\mu\nu\rho} \psi_\nu + 12 \bar{\psi}_\alpha \Gamma_{\beta\gamma}^{11} \Gamma_{\mu\nu\rho} \psi_\mu \right) + \frac{1}{\sqrt{2}} \bar{\lambda} \Gamma_{\alpha\beta\gamma}^{\mu\nu\rho} \psi_\mu + \frac{3}{4} \bar{\lambda} \Gamma_{\alpha\beta\gamma}^{\mu\nu\rho} \lambda B_\rho$$  \hspace{1cm} (3.6)$$

and we have denoted $F_{\mu_1 \cdots \mu_7} = 7 \partial_{[\mu_1} A_{\mu_2 \cdots \mu_7]}$.

Solving equation (3.4) for $F_{\mu\nu\rho}$ gives

$$F_{\mu\nu\rho} = M^{-1}_{\mu\nu\rho} X_{\alpha\beta\gamma}$$  \hspace{1cm} (3.7)$$

where the tensor $M^{-1}_{\mu\nu\rho}$ is the inverse of $M_{\alpha\beta\gamma}^{\mu\nu\rho}$:

$$M^{-1}_{\mu\nu\rho} M_{\alpha\beta\gamma}^{\mu\nu\rho} = \frac{1}{3!} \delta_{\alpha\beta\gamma}^{\kappa\lambda\sigma}$$  \hspace{1cm} (3.8)$$

The explicit form of $M^{-1}$ is

$$M^{-1}_{\mu\nu\rho} = \frac{6 e^{2k\phi}}{1 - e^{3k\phi} B_\sigma B^\sigma} \left( \frac{1}{3!} \delta_{\alpha\beta\gamma}^{\mu\nu\rho} - \frac{3}{2} e^{k\phi} B_\mu \delta_{[\alpha\beta\gamma]}^{[\mu\nu\rho] B} \right)$$  \hspace{1cm} (3.9)$$

Therefore to obtain the dual action from (2.1) plus (3.1), we discard all the $F_{\mu\nu\rho}$ contributions and replace them with

$$\frac{1}{2} X_{\alpha\beta\gamma} M^{-1}_{\mu\nu\rho} X_{\mu\nu\rho}$$  \hspace{1cm} (3.11)$$

4. The dual action with a five-form: (1,2,5)

To find the $N = 2$ supergravity action where the three-form is replaced with a five-form we proceed as before. First, we write the action (2.1) in such a way that the
three-form appears only through its field strength. We use eq (2.3) for $F'_{\mu\nu\rho\sigma}$, and write it as $F'_{\mu\nu\rho\sigma} + 12G_{[\mu\nu} A_{\rho\sigma]}$. Then we assume that $F'_{\mu\nu\rho\sigma}$ is an independent field and not the field strength of $A'_{\mu\nu\rho}$, and add the following term to the action:

$$\frac{1}{4!5!} \int A_5 \wedge dF_4$$

(4.1)

where $A_5 = A_{\mu_1...\mu_5} dx^{\mu_1} \wedge ... \wedge dx^{\mu_5}$. The $A_5$ equation implies, locally, that $F_4 = dA'_3$ and this gives again the action (2.1). If, however, we integrate eq (4.1) by parts, and then do the gaussian integration of $F_{\mu\nu\rho\sigma}$ we will be left with an action in terms of the dual field strength $F_{\mu_1...\mu_6}$. To restore the supersymmetry invariance after adding (4.1) to the action (2.1) we assume that $\delta F_{\mu\nu\rho\sigma} = 4 \partial^{[\mu} \delta A_{\nu\rho\sigma]}$, then the extra terms that spoil the invariance of the action (2.1) are cancelled by those arising from the non-invariance of the term (4.1). This is achieved by taking

$$\delta A_{\mu_1...\mu_5} = \frac{5}{2} i e^{\frac{i}{2}\kappa\phi} \epsilon^{\Gamma_111} \Gamma_{[\mu_1...\mu_4} \psi_{\mu_5]}$$

(4.2)

The sum of the actions (2.1) and (4.1) gives both dual actions depending on the order of integration and is invariant under the new supersymmetry transformations.

The gaussian integration of $F_{\mu\nu\rho\sigma}$ gives

$$\frac{1}{2} X_{\mu\nu\rho\sigma} M^{-1}_{\alpha\beta\gamma\delta} X^{\alpha\beta\gamma\delta}$$

(4.3)

where $X_{\mu\nu\rho\sigma}$ is defined by

$$X_{\mu\nu\rho\sigma} = \frac{\kappa}{4!} \epsilon_{\mu\nu\rho\sigma} \mu_1...\mu_6 \left( \frac{1}{6!} F_{\mu_1...\mu_6} + \frac{1}{16} A_{\mu_1\mu_2} A_{\mu_3\mu_4} G_{\mu_5\mu_6} \right)$$

$$+ \frac{\kappa}{96} e^{\frac{i}{2}\kappa\phi} (\overline{\psi}_\rho \Gamma^{\alpha\beta}_{\mu\rho\sigma} \psi_\beta + 12 \overline{\psi}_\rho [\mu \Gamma_{\nu\rho} \psi_{\sigma}] + \frac{1}{\sqrt{2}} \overline{\lambda} \Gamma^{\alpha\beta}_{\mu\nu\rho\sigma} \psi_\alpha + \frac{3}{4} \overline{\lambda} \Gamma_{\mu\nu\rho\sigma} \lambda)$$

$$- \frac{1}{2} e^{\kappa\phi} G_{[\mu\nu} A_{\rho\sigma]}$$

(4.4)

and the matrix $M^{-1}$ is the inverse of

$$M^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} = \left( \frac{1}{4!} \right)^2 \left( e^{\kappa\phi} \delta^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma} - \kappa \epsilon_{\mu\nu\rho\sigma}^{\alpha\beta\gamma\delta} A_{\lambda\tau} \right)$$

(4.5)

defined by

$$M^{-1}_{\alpha\beta\gamma\delta} M_{\tau\mu\nu\rho\sigma} = \frac{1}{4!} \delta_{\alpha\beta\gamma\delta}$$

(4.6)

The explicit expression of $M^{-1}$ is too long to give here. The field strength $F_6$ is given by

$$F_{\mu_1...\mu_6} = 6 \partial_{[\mu_1} A_{\mu_2...\mu_6]}$$

(4.7)
Therefore, to obtain the dual action we discard all the terms containing $F_{\mu\nu\rho\sigma}$ and replace them with (4.3). This completes the derivation of the dual action where the three-form is replaced by the five-form.

5. The dual action with a seven-form: $(7,2,3)$

As we have seen in section 2, there exists the possibility of writing the $N = 2$ supergravity action IIA in such a way that the one-form $B$ appears only through its field strength. This required a redefinition of the three-form. The procedure of obtaining the action where the one-form is replaced with the seven-form is the same as before. We first manipulate the action (2.1) so that the field $B_\mu$ appears only through its field strength $G_{\mu\nu}$ then we assume that $G_{\mu\nu}$ is an independent field and add a term to the action (2.1) of the form:

$$\frac{1}{2!7!} \int A_7 \wedge dG$$

(5.1)

where we have defined the seven-form $A_7 = A_{\mu_1 \cdots \mu_7} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_7}$. Integrating the $A_7$ field out implies the constraint $dG = 0$ whose solution, locally, is $G_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}$ and this takes us back to the action (2.1). Integrating the action (5.1) by parts and discarding the surface term we obtain

$$\frac{1}{2!7!} \int dA_7 \wedge G$$

(5.2)

The field $F'_{\mu\nu\rho\sigma}$ in the action (2.1) is taken to be of the form (2.3) and the Chern-Simons term is rearranged to be given by (2.5). Then the full action is at most quadratic in $G_{\mu\nu}$ and the gaussian integration can be performed. This will give the dual action expressed in terms of the field strength of $A_7$. The non-invariance of (2.1) under the supersymmetry transformations due to the removal of the identification $G = dB$ is cancelled by the variation of (5.1) provided one identifies the variation of $G$ with

$$\delta G_{\mu\nu} = 2\partial_{[\mu} \delta B_{\nu]}$$

(5.3)

and the variation of $A_7$ with

$$\delta A_{\mu_1 \cdots \mu_7} = e^{\frac{3}{2}\kappa\phi} \left( -\frac{7}{2} \bar{\epsilon} \Gamma_{[\mu_1 \cdots \mu_6} \psi_{\mu_7]} + \frac{\sqrt{2}}{8} \bar{\epsilon} \Gamma_{\mu_1 \cdots \mu_7} \Gamma^{11} \lambda \right)$$

(5.4)

The gaussian integration of $G_{\mu\nu}$ gives

$$-\frac{1}{4} X_{\mu\nu} M^{-1}{}_{\alpha\beta} X^{\alpha\beta}$$

(5.5)
where $X_{\mu \nu}$ is defined by

$$X_{\mu \nu} = -\frac{1}{8} e^{\kappa \phi} A^\rho_\sigma (4 \partial_{[\mu} A_{\nu \rho \sigma]}')$$

$$+ \frac{3\kappa}{16} (e^{\alpha \beta \mu \nu \rho \sigma} \psi_\beta + 12 e^{\alpha \beta \mu \nu \rho \sigma} \psi_\beta + 1 \sqrt{2} e^{\alpha \beta \mu \nu \rho \sigma} \psi_\alpha + \frac{3}{4} e^{\alpha \beta \mu \nu \rho \sigma} \lambda) A^\rho_\sigma$$

$$- i \kappa \frac{3}{8} e^{3 \kappa \phi} (e^{\alpha \beta \mu \nu \rho \sigma} \psi_\beta + 2 e^{\alpha \beta \mu \nu \rho \sigma} \psi_\beta + \frac{3}{2} e^{\alpha \beta \mu \nu \rho \sigma} \psi_\alpha + \frac{5}{4} e^{\alpha \beta \mu \nu \rho \sigma} \lambda)$$

$$+ \epsilon_{\mu \nu \rho \sigma \tau} \left( \frac{1}{2!7!} \partial_{\mu_1} A_{\mu_2 \cdots \mu_8} - \frac{\kappa}{192} A_{\mu_1 \mu_2} \cdots A_{\mu_7 \mu_8} \right)$$

(5.6)

and the tensor $M^{-1\alpha \beta}_{\mu \nu}$ is the inverse of

$$M^{\alpha \beta}_{\mu \nu} = \frac{1}{4} e^{3 \kappa \phi} (1 + \frac{3}{2} A_{\rho \sigma} A^\rho_\sigma) \delta^{\alpha \beta}_{\mu \nu} + A_{\mu \nu} A^{\alpha \beta}$$

$$- 4 \delta^{[\alpha}_{\mu_1} A^{\beta]}_{\nu} A^{\rho \sigma}_{\mu_2} - \frac{\kappa}{96} e^{\alpha \beta}_{\mu \nu \mu_1 \cdots \mu_6} A_{\mu_1 \mu_2} A_{\mu_3 \mu_4} A_{\mu_5 \mu_6}$$

(5.7)

The inverse of $M$ is defined by:

$$M^{-1\alpha \beta}_{\mu \nu} M^{\rho \sigma}_{\alpha \beta} = \frac{1}{2!} \delta^{\rho \sigma}_{\mu \nu}$$

(5.8)

but again the explicit expression is too long to give here. Finally, $G_{\mu \nu}$ is related to its dual by the relation

$$G_{\mu \nu} = M^{-1\alpha \beta}_{\mu \nu} X_{\alpha \beta}$$

(5.8)

The dual action is obtained by discarding all the $G_{\mu \nu}$ contributions in (2.1) plus (5.2) and replacing them with (5.5). This completes the derivation of the dual action where the one-form is replaced with a seven-form.

6. Conclusion: connections between the different formulations

In this letter we have shown that the original formulation of $N = 2$ supergravity type IIA given in terms of a one-form, a two-form and a three-form (we denote this by (1,2,3)), admits three other dual formulations. In the first, the two-form is replaced with a six-form giving rise to a formulation in terms of a one-form, a six-form and a three-form (denoted by (1,6,3)). In the second the three-form is replaced with a five-form giving rise to a formulation in terms of (1,2,5) forms. Finally, in the third the one-form is replaced with a seven-form giving rise to the (7,2,3) formulation. It is easy to see that the (1,2,5) formulation depends on the three-form through its field strength suggesting that it is possible to find a duality transformation that takes the one-form to a seven-form. This will give the (7,2,5) formulation. This can also be reached by performing a duality transformation on the three-form in the (7,2,3)
formulation as it appears only through its field strength. This also implies that the 
(7,2,5) formulation can be reached by applying a double duality transformation to the 
one-form and three-form simultaneously. If we arrange the (1,2,3), (7,2,3), (7,2,5) and 
(1,25) formulations at the corners of a square in a clockwise fashion, then all adjacent 
vertices could be transformed to each other by a simple duality transformation, and 
the opposite edges by a double duality transformation. But it seems that the (1,6,3) 
formulation can only be connected to the (1,2,3) formulation as it depends on the 
one-form and three-form explicitly. It will be interesting to understand the relation 
of these field theories and their duality properties to those of extended objects.

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