ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO
A HIGHER-ORDER KDV-TYPE EQUATION
WITH CRITICAL NONLINEARITY

MAMORU OKAMOTO∗

Division of Mathematics and Physics, Faculty of Engineering
Shinshu University, 4-17-1 Wakasato
Nagano City 380-8553, Japan

(Communicated by Thierry Cazenave)

Abstract. We consider the Cauchy problem of the higher-order KdV-type equation:

\[ \partial_t u + \frac{1}{m} |\partial_x|^{m-1} \partial_x u = \partial_x (u^m) \]

where \( m \geq 4 \). The nonlinearity is critical in the sense of long-time behavior. Using the method of testing by wave packets, we prove that there exists a unique global solution of the Cauchy problem satisfying the same time decay estimate as that of linear solutions. Moreover, we divide the long-time behavior of the solution into three distinct regions.

1. Introduction. We consider the Cauchy problem for the higher-order Korteweg-de Vries (KdV) type equation

\[ \partial_t u + \frac{1}{m} |\partial_x|^{m-1} \partial_x u = \partial_x (u^m), \quad (1) \]

where \( u \) is a real valued function, \( |\partial_x| = (\partial_x^2)^{\frac{1}{2}} \), and \( m \in \mathbb{Z}_{\geq 3} \). This equation describes the propagation of nonlinear waves in a dispersive medium. In particular, (1) with \( m = 3, 5 \) are called the modified KdV (mKdV) and the modified Kawahara equations, respectively.

If \( u \) is a solution to (1), then the total mass, the momentum, and the energy are conserved:

\[
\begin{align*}
\int_{\mathbb{R}} u(t, x) dx, & \quad \int_{\mathbb{R}} u(t, x)^2 dx, \\
\int_{\mathbb{R}} \left\{ \frac{1}{2m} |\partial_x|^{m-1} u(t, x)^2 - \frac{1}{m+1} u(t, x)^{m+1} \right\} dx.
\end{align*}
\]

Local-in-time well-posedness of the Cauchy problem for (1) follows from the same argument as in [19] (see also [17, 18, 4] and references therein). Owing to the conserved quantities, the local-in-time solution can be extended to the global-in-time one if the values of the initial data are small. Thus, it is of interest to obtain the asymptotic behavior of solutions to (1).

2000 Mathematics Subject Classification. Primary: 35Q53; Secondary: 35B40.

Key words and phrases. Higher-order KdV-type equation, asymptotic behavior, critical nonlinearity, self-similar solution.

∗ Corresponding author: Mamoru Okamoto.
Sidi et al. [26] studied the long-time behavior of solutions to the generalized KdV equations

\[ \partial_t u + \frac{1}{\alpha} |\partial_x|^{\alpha-1} \partial_x u = \partial_x (u^p) \]  

for \( \alpha \in \mathbb{R}, \alpha \geq 1, \) and \( p \in \mathbb{Z}_{\geq 2}. \) More precisely, they proved that when the initial data values are small, the global-in-time solution scatters to a linear solution if \( \alpha \geq 1 \) and \( p > \frac{\alpha+\sqrt{\alpha^2+4\alpha}}{2} + 1. \) Kenig et al. [18] improved the results in [26], that is the scattering for small initial data values holds true if \( \alpha \geq 1 \) and \( p > \max(\alpha+1, \frac{5}{2} + 3). \) Because there exists a blow up solution in some cases when initial data values are large ([21, 23]), the assumption of small initial data values is essential.

The asymptotic behavior for (2) with \( \alpha = 3 \) has been studied by several researchers (see [1, 7, 8, 9, 28, 22, 13, 5, 3, 2] and references therein). In particular, \( p = 3 \) is critical in the sense of long-time behavior. In other words, while the solutions scatter for \( p > 3, \) asymptotic behavior of the solution differs from that of \( p = 3. \) Moreover, Hayashi and Naumkin [10, 12] showed the criticality of the quartic derivative fourth-order nonlinear Schrödinger equation (see also [11, 14]), which is related to (1) with \( m = 4. \)

From these results, we expect that the nonlinearity of (1) is critical in the sense of long-time behavior. However, there is a gap between the exponent of nonlinearity \( m \) in previous results and that to be critical in general. In this paper, we study the asymptotic behavior of solutions to (1) for \( m \geq 4. \) Even though we used \( u^m \) in (1) in our study, the same asymptotic behavior is obtained for (1) with short-range perturbations (see Remark 1.3).

To explain the critical phenomenon, we roughly derive the asymptotic behavior of linear solutions for Schwartz initial data \( u_0 \in S(\mathbb{R}). \) Let \( U(t) \) denote the linear propagator, i.e., \( U(t) := e^{-\frac{i}{\hbar} t [\partial_x]^m \partial_x}. \) We note that, for \( t > 0, \) the linear solution is written as follows:

\[ U(t) u_0(x) = t^{-\frac{m}{2}} \int_{\mathbb{R}} Q_0 \left( t^{-\frac{1}{m}} (x-y) \right) u_0(y) dy, \]

where \( Q_0 \) is defined by \( Q_0(y) := \frac{1}{2\pi} \int_{\mathbb{R}} e^{i(y \xi - \frac{1}{m} |\xi|^{m-1})} d\xi, \) that is, \( Q_0 \) satisfies the ordinary differential equation

\[ \frac{d}{dy} \right|^{m-1} Q_0 - y Q_0 = 0. \]

Sidi et al. [26] proved the following estimate for \( Q_0: \)

\[ \left| \frac{d^k Q_0}{dy^k} (y) \right| \lesssim \langle y \rangle \frac{k}{m-2}, \]

where \( \langle y \rangle := (1+|y|^2)^{\frac{1}{2}}. \)

The Hamiltonian flow corresponding to the linear equation is given by

\[ (x, \xi) \mapsto (x + t|\xi|^{m-1}, \xi). \]

We expect that, for \( v > 0, \) solutions initially localized spatially near zero and at frequencies near \( \pm \xi_v, \) where \( \xi_v := \frac{1}{v^{\frac{1}{m-1}}}, \) travel along the ray \( \Gamma_v := \{ x = vt \}. \) Hence, the linear solution \( U(t) u_0(x) \) decays rapidly as \( t^{-\frac{1}{m}} x \to -\infty \) and oscillates as \( t^{-\frac{1}{m}} x \to +\infty. \) In particular, the stationary phase method shows that, as \( t^{-\frac{1}{m}} x \to +\infty, \) there exists a constant \( c_0 \) such that

\[ U(t) u_0(x) = c_0 t^{-\frac{1}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{m-2}{m-1}} \Re \left\{ \tilde{u}_0 \left( t^{-\frac{1}{m}} x \right) \right\} e^{i\phi(t, x)} \] + error,
where the phase function is given by
\[
\phi(t, x) := \frac{m - 1}{m} t^{-\frac{1}{m}} |x|^\frac{m}{m - 1} - \frac{\pi}{4},
\]

Moreover, in the self-similar region $|t^{-\frac{1}{m}} x| \lesssim 1$,
\[
\mathcal{U}(t)u_0(x) = t^{-\frac{k}{m}} Q_0(t^{-\frac{1}{m}} x) \int_\mathbb{R} u_0(y) dy + \text{error}.
\]

This observation implies that if $\|u_0\|_{L^2} + \|xu_0\|_{L^2} \leq \varepsilon$ then we have
\[
|\partial_x^k \mathcal{U}(t) u_0(x)| \lesssim \varepsilon t^{-\frac{k+1}{m}} (t^{-\frac{1}{m}} x)^{\frac{k}{m} - \frac{m+2}{m+1}}
\]
for $k = 0, 1, \ldots, m - 2$. We expect that solutions to (1) satisfy the same pointwise estimates as linear solutions above when the initial data values are small. Then,
\[
\|u(t)\|_{L^2} \lesssim \|u(0)\|_{L^2} + \varepsilon^{m-1} \int_0^t \|u(t')\|_{L^2} dt'.
\]
Because the integral is not bounded by $\sup_{t \geq 1} \|u(t)\|_{L^2}$, we only expect that the solution behaves like a linear solution up to $t \sim \exp(\varepsilon^{-m+1})$. Especially, the asymptotic behavior of the solution differs from that of linear solutions.

1.1. **Main result.** To state our results precisely, we introduce notation and function spaces. We denote the set of positive and negative real numbers by $\mathbb{R}_+$ and $\mathbb{R}_-$, respectively. We denote the usual Sobolev space by $H^s(\mathbb{R})$. For $s \in \mathbb{R}$, we denote the weighted Sobolev spaces by $\Sigma^s(\mathbb{R})$ equipped with the norm $\|f\|_{\Sigma^s} := \|f\|_{H^s} + \|xf\|_{L^2}$.

**Theorem 1.1.** Let $m \geq 4$. Assume that the initial datum $u_0$ at time 0 satisfies
\[
\|u_0\|_{\Sigma^\frac{m-1}{m}} \leq \varepsilon \ll 1.
\]
Then, there exists a unique global solution $u$ to (1) with $\mathcal{U}(-t) u \in C(\mathbb{R}; \Sigma^\frac{m-1}{m}(\mathbb{R}))$ satisfying the estimates
\[
\left\| \left( t^{-\frac{1}{m}} x \right)^{-\frac{k+1}{m} + \frac{m-2}{2(m-1)}} \partial_x^k u(t) \right\|_{L^\infty} \lesssim \varepsilon t^{-\frac{k+1}{m}}
\]
for $t \geq 1$ and $k = 0, 1, \ldots, m - 2$. Moreover, we have the following asymptotic behavior as $t \to +\infty$.

In the decaying region $\mathcal{X}^- (t) := \{ x \in \mathbb{R}_- : t^{-\frac{1}{m}} |x| \geq t^{(m-1)\rho} \}$, where $\rho := \frac{1}{m} (\frac{1}{2m} - \varepsilon)$, we have
\[
\left\| t^{\frac{1}{m}} (t^{-\frac{1}{m}} x)^{\frac{m-3}{2(m-1)}} u \right\|_{L^\infty(\mathcal{X}^- (t))} + \left\| t^{\frac{1}{m}} (t^{-\frac{1}{m}} x) u \right\|_{L^2(\mathcal{X}^- (t))} \lesssim \varepsilon.
\]

In the self-similar region $\mathcal{X}^0 (t) := \{ x \in \mathbb{R}_- : t^{-\frac{1}{m}} |x| \leq t^{(m-1)\rho} \}$, there exists a solution $Q = Q(y)$ to the nonlinear ordinary differential equation
\[
\left| \frac{d}{dy} \right|^{m-1} Q - y Q - mQ^m = 0,
\]
satisfying $\|Q\|_{L^\infty} \lesssim \varepsilon$ and
\[
\left\| u(t) - t^{-\frac{k}{m}} Q(t^{-\frac{1}{m}} x) \right\|_{L^\infty(\mathcal{X}^0 (t))} \lesssim \varepsilon t^{-\frac{k}{m} - \frac{m-1}{m}},
\]
\[
\left\| u(t) - t^{-\frac{k}{m}} Q(t^{-\frac{1}{m}} x) \right\|_{L^2(\mathcal{X}^0 (t))} \lesssim \varepsilon t^{-\frac{k}{m} - \frac{m-1}{m}}.
\]
In the oscillatory region \( \mathcal{X}^+(t) := \{ x \in \mathbb{R}_+: t^{-\frac{1}{m}}|x| \gtrsim t^{(m-1)p}\} \), there exists a unique complex-valued function \( W \) satisfying \( W(\xi) = \overline{W(-\xi)} \) and \( \|W\|_{L^\infty \cap L^2} \lesssim \varepsilon \) such that
\[
u(t) = \frac{2}{\sqrt{m-1}} t^{-\frac{1}{m}} x^{-\frac{m-1}{2m-1}} \Re \left\{ W \left( t^{-\frac{1}{m}} |x|^{\frac{1}{m}} \right) e^{i\phi(t,x)} \right\} + \text{err}_x,
\]
where the error, \( \text{err}_x \), satisfies the estimates
\[
\left\| t^{-\frac{1}{m}} x^{-\frac{m-1}{2m-1}} \text{err}_x \right\|_{L^\infty(\mathcal{X}^+(t))} + \left\| t^{-\frac{1}{m}} x^{-\frac{m-1}{2m-1}} \text{err}_x \right\|_{L^2(\mathcal{X}^+(t))} \lesssim \varepsilon.
\]
In the corresponding frequency region \( \widehat{\mathcal{X}}^+(t) := \{ \xi \in \mathbb{R}_+: t^\frac{1}{m} |\xi| \gtrsim t^{(\frac{1}{m}-\varepsilon)} \} \), we have
\[
\widehat{u}(t,\xi) = W(\xi) e^{\frac{i}{m} \xi \cdot t} + \text{err}_\xi,
\]
where the error, \( \text{err}_\xi \), satisfies
\[
\left\| \left( t^\frac{1}{m} \xi \right)^{-\frac{m-2}{m-1}} \text{err}_\xi \right\|_{L^\infty(\widehat{\mathcal{X}}^+(t))} + \left\| t^{-\frac{1}{m}} x^{-\frac{m-1}{2m-1}} \text{err}_\xi \right\|_{L^2(\widehat{\mathcal{X}}^+(t))} \lesssim \varepsilon.
\]

As (1) has time reversal symmetry given by \( u(t,x) \mapsto u(-t,-x) \), we get the corresponding asymptotic behavior as \( t \to -\infty \). In addition, we can obtain the same result as in Theorem 1.1 for (1) of which nonlinearity is replaced by \(-\partial_x(u^m)\), because the sign of the nonlinearity is not important in our proof.

Theorem 1.1 presents the leading asymptotic term not only in \( L^\infty(\mathbb{R}) \), but also in \( L^2(\mathbb{R}) \). In addition, as with linear solutions, we divide the long-time behavior of the solution to (1) into three distinct regions. Moreover, Theorem (1.1) says that there is a difference between \( u \) and linear solutions in \( \mathcal{X}^0(t) \), while the leading term of \( u \) in \( \mathcal{X}^+(t) \) behaves like a linear solution.

The regularity assumption \( u_0 \in H^{\frac{m-1}{m}}(\mathbb{R}) \) needs to show the existence of a local-in-time solution \( u \) with \( \mathcal{U}(-t)u \in C([-T,T];\Sigma^0(\mathbb{R})) \) (see Remark 1.5 below). In other words, regularity is no longer required in the proof of the global existence and asymptotic behavior.

**Remark 1.2.** Because \( Q \) satisfies (5), \( u(t,x) := t^{-\frac{1}{m}} Q(t^{-\frac{1}{m}} x) \) is a solution to (1) with the initial datum \( u(0) = \int_{\mathbb{R}} u_0(x) dx \delta_0 \), where \( \delta_0 \) denotes the Dirac delta measure concentrated at the origin. Moreover, by (50) below, we can roughly state that
\[
\|u(t) - u(0)\|_{L^\infty(\mathbb{R})} \lesssim te^{-\frac{1}{m} - \varepsilon}.
\]

If \( \int_{\mathbb{R}} u_0(x) dx = 0 \), then the self-similar solution vanishes, and the solution \( u \) to (1) decays faster than \( t^{-\frac{1}{m}} \). Accordingly, the nonlinearity of (1) is not critical in the long-time behavior in this case.

**Remark 1.3.** Theorem 1.1 is also true for short-range perturbations of the form
\[
\partial_t u + \frac{1}{m} |\partial_x|^m \partial_x u = \partial_x (u^m + F(u)),
\]
where there exists a real number \( p > m \) such that \( F \in C^3(\mathbb{R}) \) and
\[
\left| \frac{d^j}{du^j} F(u) \right| \lesssim |u|^{p-j}
\]
for \( j = 0,1,2,3 \). In fact, if \( u_0 \in \Sigma^{\frac{p}{m-1}}(\mathbb{R}) \) with \( \|u_0\|_{\Sigma^{\frac{p}{m-1}}} \leq \varepsilon \), then there exists a unique global solution \( u \) to (6) with \( \mathcal{U}(-t)u \in C^3(\mathbb{R};\Sigma^{\frac{p}{m-1}}(\mathbb{R})) \) satisfying the estimate (4). Moreover, the global solution has the same asymptotic behavior as
in Theorem 1.1, as long as \( \rho = \frac{1}{m} (\frac{1}{2m} - \varepsilon) \) is replaced with \( \bar{\rho} := \frac{1}{m} (\frac{1}{2m} - a - \varepsilon) \), where \( a := \max (\frac{2m+1-2p}{2m}, 0) < \frac{1}{2m} \). For completeness, we briefly outline these modifications in Appendix B.

1.2. Outline of the proof. We give an outline of the proof of Theorem 1.1. Let \( \mathcal{L} \) denote the linear operator of (1):

\[
\mathcal{L} := \partial_t + \frac{1}{m} |\partial_x|^{m-1} \partial_x.
\]

To obtain pointwise estimates of solutions, we use the “vector field”

\[
\mathcal{J} := \mathcal{U}(t) \partial_t \mathcal{U}(-t) = x - t |\partial_x|^{m-1}.
\]

However, \( \mathcal{J} \) does not behave well with respect to the nonlinearity, so as in \([7, 8, 5]\) we instead work with

\[
\Lambda := \partial_x^{-1} (mt \partial_t + x \partial_x + 1).
\]

Here, \( \mathcal{S} := mt \partial_t + x \partial_x + 1 \) is the generator of the scaling transformation for (1):

\[
u(t, x) \mapsto \lambda \nu(\lambda^m t, \lambda x)
\]

for any \( \lambda > 0 \). Moreover, \( \mathcal{S} \) is related to \( \mathcal{L} \) and \( \mathcal{J} \) as follows:

\[
\mathcal{S} = mt \mathcal{L} + \mathcal{J} \partial_x + 1.
\]

We introduce the norm with respect to the spatial variable as follows:

\[
\|u(t)\|_X := \left( \|u(t)\|^2_{H^\frac{m-1}{2m}} + \|\Lambda u(t)\|_{L^2}^2 \right)^{\frac{1}{2}}.
\]

We note that

\[
\|u_0\|_X \sim \|u_0\|_{H^\frac{m-1}{2m}} + \|xu_0\|_{L^2} \sim \|u_0\|_{\Sigma^\frac{m-1}{2m}}.
\]

Well-posedness in \( X \) follows from a similar argument as in \([19]\).

**Proposition 1.4.** Let \( m \geq 4 \). If \( u_0 \in \Sigma^\frac{m-1}{2m} (\mathbb{R}) \), then there exists a time \( T = T(||u_0||_{\Sigma^\frac{m-1}{2m}}) > 0 \) and a unique solution \( u \in C([-T, T]; X) \) to (1) satisfying

\[
\sup_{t \in [-T, T]} \|u(t)\|_X \lesssim \|u_0\|_{\Sigma^\frac{m-1}{2m}}.
\]

Moreover, the flow map \( u_0 \in \Sigma^\frac{m-1}{2m} (\mathbb{R}) \mapsto u \in C([-T, T]; X) \) is locally Lipschitz continuous.

We give a proof of this well-posedness result in Appendix A.

**Remark 1.5.** Because (8) implies

\[
\mathcal{J} u = \Lambda u - mt \partial_x^{-1} \mathcal{L} u = \Lambda u - mt u^m,
\]

the regularity \( u_0 \in H^\frac{m-1}{2m} (\mathbb{R}) \) ensures \( \mathcal{J} u \in C \left( [-T, T]; L^2 (\mathbb{R}) \right) \), which is equivalent to \( \mathcal{U}(-t) u \in C \left( [-T, T]; \Sigma^0 (\mathbb{R}) \right) \).

For initial data \( u_0 \) with \( ||u_0||_{\Sigma^\frac{m-1}{2m}} \ll 1 \), we can find an existence time \( T > 1 \) and a unique solution \( u \in C([-T, T]; X) \) to (1). Moreover, Proposition 1.4 and Lemma 2.1 below say that the existence of a global solution \( u \in C(\mathbb{R}; X) \) follows from the decay estimate (4). In addition, Proposition 1.4 implies that the solution \( u \in C([-T, T]; X) \) is approximated by smooth solutions. Hence, it suffices to show the estimate (4) for smooth solutions.
Because (1) is time reversal invariant, it suffices to consider the case \( t \geq 0 \). We then make the bootstrap assumption that \( u \) satisfies the linear pointwise estimates: there exists a constant \( D \) with \( 1 \ll D \ll \varepsilon^{-\frac{3}{2}} \) such that
\[
\left\| (t^{-\frac{1}{2}} x)^{-\frac{m-2}{m}} \partial_x^k u(t) \right\|_{L^\infty} \leq D \varepsilon t^{-\frac{k+1}{2m}}
\] (10)
for \( t \in [1, T] \) and \( k = 0, 1, \ldots, m - 2 \).

In §2, under this assumption, for \( \varepsilon > 0 \) sufficiently small, we have the energy estimate
\[
\sup_{0 \leq t \leq T} \| u(t) \|_X \leq C \varepsilon(T) \varepsilon,
\]
where \( C \) is a constant independent of \( D \) and \( T \). To complete the proof of global existence, we need to close the bootstrap estimate (10).

In §3, we prove decay estimates in \( L^\infty(\mathbb{R}) \) and \( L^2(\mathbb{R}) \) that allow us to reduce closing the bootstrap argument to considering the behavior of \( u \) along the ray \( \Gamma_u \).

We also observe that (10) holds true at \( t = 1 \).

To close the bootstrap argument, we use the method of testing by wave packets as in [5, 6, 15, 25]. Here, a wave packet is an approximate solution localized in both space and frequency on the scale of the uncertainty principle. Our main task in §4 is to construct a wave packet \( \Psi_v(t, x) \) to the corresponding linear equation and observe its properties. Here, to show that \( \Psi \) is a natural number. Because \( \Psi_v(t, x) \) is essentially frequency localized near \( \xi_v = v \frac{x}{m} \) (see Lemma 4.1), the linear operator \( L \) acts on \( \Psi_v \) as \( \partial_t + i(-i\partial_x)^m \). Hence, we can avoid applying the nonlocal operator \( |\partial_x| \) directly in the calculation of \( L\Psi \) (see (39)).

To observe decay of \( u \) along the ray \( \Gamma_u \), we use the function
\[
\gamma(t, v) := \int_\mathbb{R} u(t, x) \overline{\Psi_v(t, x)} dx.
\] (11)
In §4, we also prove that \( \gamma \) is a reasonable approximation of \( u \). We then reduce closing the bootstrap estimate (10) to proving global bounds for \( \gamma \).

In §5, we show that \( \gamma \) is the leading asymptotic term of \( u \) in \( X^+(t) \). Moreover, we prove existence of a solution \( Q \) to (5).

1.3. Notation. We summarize the notation used throughout this paper. We set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). We denote the space of all rapidly decaying functions on \( \mathbb{R} \) by \( S(\mathbb{R}) \). We define the Fourier transform of \( f \) by \( \mathcal{F}[f] \) or \( \hat{f} \).

In estimates, we use \( C \) to denote a positive constant that can change from line to line. If \( C \) is absolute or depends only on parameters that are considered fixed, then we often write \( X \lesssim Y \), which means \( X \leq CY \). When an implicit constant depends on a parameter \( a \), we sometimes write \( X \lesssim_a Y \). We define \( X \ll Y \) to mean \( X \leq C^{-1}Y \) and \( X \sim Y \) to mean \( C^{-1}Y \leq X \leq CY \). We write \( X = Y + O(Z) \) when \( |X - Y| \lesssim Z \).

Let \( \sigma \) be a smooth even function with \( 0 \leq \sigma \leq 1 \) and \( \sigma(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq 1, \\ 0, & \text{if } |\xi| \geq 2. \end{cases} \) For any \( R, R_1, R_2 > 0 \) with \( R_1 < R_2 \), we set
\[
\sigma_{\leq R}(\xi) := \sigma \left( \frac{\xi}{R} \right), \quad \sigma_{> R}(\xi) := 1 - \sigma_{\leq R}(\xi),
\]
\[
\sigma_{< R}(\xi) := \sigma_{\leq \frac{R}{2}}(\xi), \quad \sigma_{\geq R}(\xi) := 1 - \sigma_{< R}(\xi), \quad \sigma_R(\xi) := \sigma_{\leq R}(\xi) - \sigma_{< R}(\xi),
\]
ASYMPTOTIC BEHAVIOR FOR A KDV-TYPE EQ 573

\[ \sigma_{R_1 \leq \leq R_2}(\xi) := \sigma_{R_2}(\xi) - \sigma_{R_1}(\xi), \quad \sigma_{R_1 < \leq R_2}(\xi) := \sigma_{R_2}(\xi) - \sigma_{R_1}(\xi). \]

For any \( N, N_1, N_2 \in 2^\mathbb{Z} \) with \( N_1 < N_2 \), we define
\[
P_N f := \mathcal{F}^{-1}[\sigma_N \hat{f}], \quad P_{N_1 \leq \leq N_2} f := \mathcal{F}^{-1}[\sigma_{N_1 \leq \leq N_2} \hat{f}].
\]

We denote the characteristic function of an interval \( I \) by \( 1_I \). For \( N \in 2^\mathbb{Z} \), we define
\[
P^\pm f := \mathcal{F}^{-1}[1_{\mathbb{R}^+ \pm \hat{f}}], \quad P_N^\pm := P^\pm P_N.
\]

2. Energy estimates. We show energy estimates for solutions \( u \) to (1) under (10).

**Lemma 2.1.** Assume \( m \geq 4 \). Let \( u \) be a solution to (1) in a time interval \([0, T] \) satisfying
\[
\|u_0\|_{\Sigma^{\frac{m-1}{m}}} \leq \varepsilon \ll 1
\]
and (10). Then,
\[
\|u(t)\|_X \lesssim \varepsilon(t)^\varepsilon,
\]
where the implicit constant is independent of \( T, \varepsilon \).

To treat the fractional derivatives, we use the Kato-Ponce commutator estimate (see [16, 19]):

**Lemma 2.2.** For \( 0 < s < 1 \), we have
\[
\|\partial_x^s(fg) - f\partial_x^s g\|_{L^2} \lesssim \|\partial_x^s f\|_{L^2} \|g\|_{L^\infty}.
\]

**Proof of Lemma 2.1.** Because the desired bound for \( 0 \leq t \leq 1 \) follows from Proposition 1.4, we consider the case \( t \geq 1 \).

Integration by parts yields
\[
\frac{1}{2} \partial_t \|u(t)\|_{L^2}^2 = \int_{\mathbb{R}} \partial_x (u^m) u dx = 0.
\]

By Lemma 2.2 and (10), we have
\[
\frac{1}{2} \partial_t \left\| \partial_x^{\frac{m-1}{m}} u(t) \right\|_{L^2}^2
\]
\[
= m \int_{\mathbb{R}} \partial_x^{\frac{m-1}{m}} u \cdot \partial_x^{\frac{m-1}{m}} (u^{m-1} \partial_x u) dx
\]
\[
= m \int_{\mathbb{R}} \partial_x^{\frac{m-1}{m}} u \cdot u^{m-1} |\partial_x^{\frac{m-1}{m}} \partial_x u| dx + O \left( \|u(t)\|_{H^{\frac{m-1}{m}}}^2 \|u(t)\|_{L^\infty} \|\partial_x u(t)\|_{L^\infty} \right)
\]
\[
= -\frac{m(m-1)}{2} \int_{\mathbb{R}} u^{m-2} \partial_x (|\partial_x^{\frac{m-1}{m}} u|^2) dx + O \left( (D\varepsilon)^{m-1} t^{-1} \|u(t)\|_{H^{\frac{m-1}{m}}}^2 \right)
\]
\[
\lesssim (D\varepsilon)^{m-1} t^{-1} \|u(t)\|_{H^{\frac{m-1}{m}}}^2.
\]

From \( \|\partial_x^{m-1} \partial_x, x\| = m |\partial_x|^{m-1} \), a simple calculation yields
\[
[\mathcal{L}, S] = m\mathcal{L}, \quad [S, \partial_x] = -\partial_x,
\]
which imply that for solutions \( u \) to (1)
\[
\mathcal{L}u = \partial_x^{\frac{1}{m}} (S + m) \mathcal{L}u = mu^{m-1} \partial_x Au.
\]

From (12), integration by parts, and (10), we obtain
\[
\frac{1}{2} \partial_t \|Au(t)\|_{L^2}^2 = -\frac{m(m-1)}{2} \int_{\mathbb{R}} u^{m-2} \partial_x (Au)^2 dx \lesssim (D\varepsilon)^{m-1} t^{-1} \|Au(t)\|_{L^2}^2.
\]

From \( (D\varepsilon)^{m-1} \ll \varepsilon \), Gronwall’s inequality with the above estimates implies
\[
\|u(t)\|_X \leq 10 \|u(1)\|_X t^{\varepsilon} \lesssim \varepsilon t^{\varepsilon}.
\]

\[ \square \]
We define the auxiliary space
\[ \|u(t)\|_{\tilde{X}} := \|Ju(t)\|_{L^2} + t^\frac{1}{m} \left\| \left( t^{\frac{1}{m}} \partial_x \right)^{-1} u(t) \right\|_{L^2} . \]

**Lemma 2.3.** Let \( u \) be a solution to (1) which satisfies \( \|u_0\|_{Y^m} \leq \varepsilon < 1 \) and (10). Then, for \( t \geq 1 \), we have
\[ \|u(t)\|_{\tilde{X}} \lesssim \varepsilon t^\frac{1}{m}, \]
where the implicit constant is independent of \( D, T, \) and \( \varepsilon \).

**Proof.** By (10), we see that
\[ \|u_m\|_{L^2} \leq (D\varepsilon)^{m-1} \left( \int_{|x| \leq t^\frac{1}{m}} dx + \int_{|x| \geq t^\frac{1}{m}} \left( t^{-\frac{1}{m}} |x| \right)^{-m \frac{m-2}{m-1}} dx \right)^{\frac{1}{2}} \lesssim \varepsilon t^{-1} + \frac{1}{t^\frac{1}{m}}. \]
From (9) and Lemma 2.1, we therefore have
\[ \|Ju(t)\|_{L^2} \lesssim \|\Lambda u(t)\|_{L^2} + t \|u_m\|_{L^2} \lesssim \varepsilon t^\frac{1}{m}. \]

We use a self-similar change of variables by defining
\[ U(t, y) := t^\frac{1}{m} u(t, t^\frac{1}{m} y). \]
A direct calculation shows
\[ \partial_t U(t, y) = \frac{1}{m} t^{-1+\frac{1}{m}} (Su)(t, t^\frac{1}{m} y) = \frac{1}{m} t^{-1} \partial_y \left( (\Lambda u)(t, t^\frac{1}{m} y) \right). \]
Hence, we have
\[ \partial_t \left\| (\partial_y)^{-1} U(t) \right\|_{L^2_y} \lesssim t^{-1} \|\Lambda u(t)\|_{L^2_x} \lesssim \varepsilon t^{-1} + \frac{1}{t^\frac{1}{m}}. \]
By integrating this with respect to \( t \), we have
\[ \left\| (\partial_y)^{-1} U(t) \right\|_{L^2_y} \lesssim \varepsilon. \]
From \( \left\| (\partial_y)^{-1} U(t) \right\|_{L^2_y} = t^\frac{1}{m} \left\| \left( t^{\frac{1}{m}} \partial_x \right)^{-1} u(t) \right\|_{L^2_x} \), we obtain the desired bound. \( \square \)

**Remark 2.4.** The estimate \( \|u(t)\|_{\tilde{X}} \lesssim \varepsilon \) for \( 0 < t < 1 \) holds true. Indeed, by Proposition 1.4, Remark 1.5, and a sufficiently small value of \( \varepsilon > 0 \), we have
\[ \sup_{0 \leq t \leq 1} \|u(t)\|_{\tilde{X}} \lesssim \sup_{0 \leq t \leq 1} \left( \|\Lambda u(t)\|_{L^2} + \|u(t)^m\|_{L^2} + \|u(t)\|_{L^2} \right) \]
\[ \lesssim \sup_{0 \leq t \leq 1} \left( \|u(t)\|_{X} + \|u(t)\|_{Y} \right) \]
\[ \lesssim \varepsilon. \]

3. **Decay estimates.** In this section, we prove a number of estimates for \( u \) without assuming (10).

We divide \( u \) into two parts on which \( J \) acts hyperbolically and elliptically. For simplicity, we use the following notation:
\[ u_N := P_N u, \quad u^\pm := P^\pm u, \quad u^+_N := P^+_N u. \]
Since $\mathcal{J} u_N = P_N(\mathcal{J} u) + i F^{-1} [\partial_x \sigma_N \tilde{u}]$ and $u$ is real valued, we have

$$\|u(t)\|_{X} \sim \left( \|u_{t}^{-\frac{1}{m}} \|_{X}^2 + \sum_{N \in 2^2} \|u_N^+(t)\|_{X}^2 \right)^{\frac{1}{2}}, \quad u_{t}^{-\frac{1}{m}} := \sum_{N \in 2^2} u_N.$$  \hspace{1cm} (16)

For $t \geq 1$ and $N > t^{-\frac{1}{m}}$, we define the hyperbolic and elliptic parts of $u_N^+$ as follows:

$$u_N^{hyp,+} := \sigma_N^{hyp} u_N^+, \quad u_N^{ell,+} := u_N^+ - u_N^{hyp,+},$$

where $\sigma_N^{hyp}(t, x) := \sigma_{\frac{1}{|t|N^{m-1}} \leq \kappa t N^{m-1}}(x) |_{R^+}(x)$ and $\kappa := 2^{2m+3}$. This large constant $\kappa$ is needed to show (21) in Lemma 3.3 below. Moreover, we define

$$u_{hyp,+} = \sum_{N \in 2^2} u_N^{hyp,+}, \quad u_{ell} = u - 2 \mathcal{R} u_{hyp,+}.$$  

We note that $u^{hyp,+}$ is supported in $\{ x \in \mathbb{R}^2 : t^{-\frac{1}{m}} x \geq \frac{1}{2^N} \}$. For $(t, x) \in \mathbb{R}^2$ with $t^{-\frac{1}{m}} |x| \geq \frac{1}{2^N}$, the number of dyadic numbers $N \in 2^2$ satisfying $\frac{1}{2^N} |x| \leq 2^{2m+3}$ is less than 10. Hence, $u^{hyp,+}(t, x)$ is a finite sum of $u^{hyp,+}(t, x)$’s.

The functions $u^{hyp,+}$ and $u^{ell,+}$ are essentially localized at frequency $N$ in the following sense: For any $a \geq 0$, $b \in \mathbb{R}$, and $c \geq 0$,

$$\left\| \left( 1 - P_{\frac{1}{2} \leq |x| \leq 2N} \right) \left| \partial_x \right|^a \left| x \right|^b u_N^{hyp,+} \right\|_{L^2} \lesssim_{a, b, c} t^{-\frac{1}{m} a} \left( \frac{a}{2N} \right)^{-c} \left\| u_N \right\|_{L^2},$$  \hspace{1cm} (17)

$$\left\| \left( 1 - P_{\frac{1}{2} \leq |x| \leq 2N} \right) \left| \partial_x \right|^a u_N^{ell,+} \right\|_{L^2} \lesssim_{a, b, c} t^{-\frac{1}{m} a} \left( \frac{a}{2N} \right)^{-c} \left\| u_N \right\|_{L^2},$$  \hspace{1cm} (18)

$$\left\| \left( 1 - P_{\frac{1}{2} \leq |x| \leq 2N} \right) \left( x \right)^b \sigma_{\frac{1}{|t|N^{m-1}} \leq \kappa t N^{m-1}}(x) u_N^{ell,+} \right\|_{L^2} \lesssim_{a, b, c} t^{-\frac{1}{m} a} \left( \frac{a}{2N} \right)^{-c} \left\| u_N \right\|_{L^2},$$  \hspace{1cm} (19)

These estimates are consequences of the following lemma (see, for example, [24, 25]):

**Lemma 3.1.** For $2 \leq p \leq \infty$, any $a$, $b$, $c \in \mathbb{R}$ with $a \geq 0$ and $a + c \geq 0$, and any $R > 0$, we have

$$\left\| \left( 1 - P_{\frac{1}{2} \leq |x| \leq 2N} \right) \left| \partial_x \right|^a (|x|^b \sigma_R P_N^+ f) \right\|_{L^p} \lesssim_{a, b, c} N^{-c+\frac{1}{2} - \frac{1}{p} + b} R^{-a+b-c} \left\| P_N^+ f \right\|_{L^2}.$$  

Moreover, we may replace $\sigma_R$ on the left hand side by $\sigma_{>R}$ if $a + c > b + 1$ and by $\sigma_{<R}$ if $a + c \geq 0$ and $b = 0$.

**3.1. Frequency localized estimates.** Since

$$\mathcal{J} u^+ = (x - t(-i \partial_x)^{m-1}) u^+,$$

by factorizing out the term $x - t \xi^{m-1}$, we define

$$\mathcal{J}_+ := |x|^{\frac{1}{m-1}} + it \frac{1}{m-1} \partial_x, \quad \mathcal{J}_- := \sum_{j=0}^{m-2} |x|^{\frac{m-2-j}{m-1}} (-it \frac{1}{m-1} \partial_x)^j.$$  

These operators are useful in our analysis.

We begin with the following preliminary observation.
Lemma 3.2. Let $a$ be a real number and let $g$ be a (C-valued) smooth function supported in $\mathbb{R}_+$ or $\mathbb{R}_-$. For any integer $k > 0$, the following equations hold:

\[
\Re \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^{k-1} g(x)} dx = \sum_{l=1}^k C_{2k-1,l}^{a,\pm} \int_{\mathbb{R}_\pm} \left| x)^{a-1-l} \partial_x^{k-l} g(x) \right|^2 dx,
\]

\[
\Re \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^2 g(x)} dx = \sum_{l=0}^k C_{2k,l}^{a,\pm} \int_{\mathbb{R}_\pm} \left| x)^{a-1} \partial_x^{k-l} g(x) \right|^2 dx,
\]

\[
\Im \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^{k-1} g(x)} dx = \sum_{l=0}^{k-1} D_{2k-1,l}^{a,\pm} \int_{\mathbb{R}} \xi \left| \mathcal{F} \left[ \cdot | \cdot \right] \partial_x^{k-l-1} g(x) \right| (\xi)^2 d\xi,
\]

\[
\Im \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^2 g(x)} dx = \sum_{l=1}^k D_{2k,l}^{a,\pm} \int_{\mathbb{R}} \xi \left| \mathcal{F} \left[ \cdot | \cdot \right] \partial_x^{k-l-1} g(x) \right| (\xi)^2 d\xi,
\]

where $C_{k,l}^{a,\pm}$ and $D_{k,l}^{a,\pm}$ are real constants depending on $a$, $k$, and $l$. In particular, $C_{2k,0}^{a,\pm} = D_{2k-1,0}^{a,\pm} = (-1)^k$.

Proof. For $k = 1$, integration by parts yields

\[
\Re \int_{\mathbb{R}_\pm} |x|^a g(x) \partial_x g(x) dx = \frac{a}{2} \int_{\mathbb{R}_\pm} \left| x)^{a-1} g(x) \right|^2 dx,
\]

\[
\Re \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^2 g(x)} dx = -\Re \int_{\mathbb{R}_\pm} \left( |x|^a \partial_x g(x) \pm a |x|^{a-1} g(x) \right) \overline{\partial_x g(x)} dx
\]

\[
= - \int_{\mathbb{R}_\pm} \left| x)^{a} \partial_x g(x) \right|^2 dx + \frac{1}{2} a(a - 1) \int_{\mathbb{R}_\pm} \left| x)^{a-1} g(x) \right|^2 dx.
\]

Similarly, we have

\[
\Im \int_{\mathbb{R}_\pm} |x|^a g(x) \partial_x g(x) dx = \Im \int_{\mathbb{R}_\pm} \left| x)^{a} \partial_x g(x) \right|^2 dx
\]

\[
= - \int_{\mathbb{R}} \xi \left| \mathcal{F} \left[ \cdot | \cdot \right] g(x) \right| (\xi)^2 d\xi,
\]

\[
\Im \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^2 g(x)} dx = -\Im \int_{\mathbb{R}_\pm} \left( |x|^a \partial_x g(x) \pm a |x|^{a-1} g(x) \right) \overline{\partial_x g(x)} dx
\]

\[
= \pm a \int_{\mathbb{R}} \xi \left| \mathcal{F} \left[ \cdot | \cdot \right] g(x) \right| (\xi)^2 d\xi.
\]

Hence, the equations hold when $k = 1$ with $C_{1,1}^{a,\pm} = \frac{a}{2}$, $C_{2,0}^{a,\pm} = -1$, $C_{2,1}^{a,\pm} = \frac{1}{2} a(a - 1)$, $D_{1,0}^{a,\pm} = -1$, and $D_{2,1}^{a,\pm} = \pm a$.

Next, we assume that these equalities hold up to $k - 1$. Integration by parts yields

\[
\Re \int_{\mathbb{R}_\pm} |x|^a g(x) \overline{\partial_x^{k-1} g(x)} dx
\]

\[
= -\Re \int_{\mathbb{R}_\pm} |x|^a \partial_x g(x) \overline{\partial_x^{k-2} g(x)} dx + a \Re \int_{\mathbb{R}_\pm} |x|^{a-1} g(x) \overline{\partial_x^{k-1} g(x)} dx
\]
\[
\begin{align*}
&= -\sum_{l=1}^{k-1} C_{2k-3,l}^a \int_{\mathbb{R}_\pm} \left| \left| x \right|^\frac{a+1}{2} \partial_x^{-l} g(x) \right|^2 dx \\
&\quad + a \sum_{l=1}^{k-1} C_{2k-2,l}^a \int_{\mathbb{R}_\pm} \left| \left| x \right|^\frac{a+1}{2} \partial_x^{-l} g(x) \right|^2 dx \\
&= \sum_{l=1}^{k-1} \left( -C_{2k-3,l}^a + aC_{2k-2,l-1}^a \right) \int_{\mathbb{R}_\pm} \left| \left| x \right|^\frac{a+1}{2} \partial_x^{-l} g(x) \right|^2 dx \\
&\quad + a \sum_{l=1}^{k-1} C_{2k-2,l-1}^a \int_{\mathbb{R}_\pm} \left| \left| x \right|^\frac{a+1}{2} \right|^2 dx \\
&\quad \quad + \sum_{l=1}^{k-1} \left( -C_{2k-3,l}^a + aC_{2k-2,l-1}^a \right) \int_{\mathbb{R}_\pm} \left| \left| x \right|^\frac{a+1}{2} \partial_x^{-l} g(x) \right|^2 dx.
\end{align*}
\]

Hence, by setting

\[
C_{2k-1,l}^\pm := \begin{cases} 
-C_{2k-3,l}^a + aC_{2k-2,l-1}^a, & \text{if } l = 1, 2, \ldots, k - 1, \\
+ aC_{2k-2,k-1}^a, & \text{if } l = k,
\end{cases}
\]

\[
C_{2k,l}^\pm := \begin{cases} 
-C_{2k-2,0}^a + aC_{2k-1,l}^a, & \text{if } l = 1, 2, \ldots, k - 1, \\
+ aC_{2k-1,k}^a, & \text{if } l = k,
\end{cases}
\]

we obtain the equations for the real part. Similarly, by setting

\[
D_{2k-1,l}^\pm := \begin{cases} 
-D_{2k-3,0}^a, & \text{if } l = 0, \\
-D_{2k-3,l}^a + aD_{2k-2,l}^a, & \text{if } l = 1, 2, \ldots, k - 2, \\
+ aD_{2k-2,k-1}^a, & \text{if } l = k - 1,
\end{cases}
\]

\[
D_{2k,l}^\pm := \begin{cases} 
-D_{2k-2,0}^a + aD_{2k-1,l}^a, & \text{if } l = 1, 2, \ldots, k - 1, \\
+ aD_{2k-1,k-1}^a, & \text{if } l = k,
\end{cases}
\]

we obtain the equations for the imaginary part. From the recurrence relations with \( C_{2,0}^a = D_{1,0}^a = -1 \), we have \( C_{2k,0}^a = D_{2k-1,0}^a = (-1)^k \).

We show the frequency localized estimates.
Lemma 3.3. For \( t \geq 1 \) and \( N > t^{-\frac{1}{2}} \), we have
\[
\left\| \left( |x|^{\frac{m}{2} + \frac{1}{2}} + t^\frac{m}{2} N^{m-2} \right) \mathcal{J}_+ u_N^{\text{hyp}^+}(t) \right\|_{L^2} \lesssim \| u_N(t) \|_{X},
\]
(20)
\[
\left\| \left( |x| + t N^{m-1} \right) u_N^{\text{ell}^+}(t) \right\|_{L^2} \lesssim \| u_N(t) \|_{X},
\]
(21)

Proof. Set \( f := \mathcal{J}_+ u_N^{\text{hyp}^+} \). We apply Lemma 3.2 to obtain
\[
\| \mathcal{J}_- f \|_{L^2}^2 = \sum_{j=0}^{m-2} \left\| \frac{1}{t^{\frac{m}{2} - \frac{j}{2}}} |x|^{-\frac{m-2-j}{2}} \partial_x^j f \right\|_{L^2}^2
\]
\[
= 2 \sum_{j=0}^{m-3} \sum_{k=1}^{m-2-j} \mathcal{R} \int_{\mathbb{R}} (-1)^k t^{\frac{2j+k}{m-1}} |x|^{2 - \frac{2j+k+2}{m-1}} \partial_x^j f(x) \overline{\partial_x^k f(x)} \, dx
\]
\[
= 2 \sum_{j=0}^{m-3} \sum_{k=1}^{m-2-j} \mathcal{R} \int_{\mathbb{R}} (-1)^k t^{\frac{2j+k}{m-1}} |x|^{2 - \frac{2j+k+2}{m-1}} \partial_x^j f(x) \overline{\partial_x^k f(x)} \, dx
\]
\[
+ 2 \sum_{j=0}^{m-3} \sum_{k=1}^{m-2-j} \sum_{l=0}^{k-1} \mathcal{R} \int_{\mathbb{R}} (-1)^{\frac{k+l}{2}} t^{\frac{2j+k}{m-1}} |x|^{2 - \frac{2j+k+2}{m-1}} \partial_x^j f(x) \overline{\partial_x^k f(x)} \, dx
\]
\[
= 2 \sum_{j=0}^{m-3} \sum_{k=1}^{m-2-j} \sum_{l=0}^{k-1} \mathcal{R} \int_{\mathbb{R}} (-1)^{\frac{k+l}{2}} t^{\frac{2j+k}{m-1}} |x|^{2 - \frac{2j+k+2}{m-1}} \partial_x^j f(x) \overline{\partial_x^k f(x)} \, dx
\]
\[
+ 2 \sum_{j=0}^{m-3} \sum_{k=1}^{m-2-j} \sum_{l=0}^{k-1} \mathcal{R} \int_{\mathbb{R}} (-1)^{\frac{k+l}{2}} t^{\frac{2j+k}{m-1}} |x|^{2 - \frac{2j+k+2}{m-1}} \partial_x^j f(x) \overline{\partial_x^k f(x)} \, dx
\]
Here, (17) yields that for \( l \geq 1 \)
\[
t^{\frac{2j+k}{m-1}} \int_{\mathbb{R}} \left| |x|^{1 - \frac{2j+k+2}{2(m-1)} - l} \partial_x^{j + \frac{k-1}{2} - l} f(x) \right|^2 \, dx
\]
\[
\lesssim t^{\frac{2j+k}{m-1}} \left( t N^{m-1} \right)^{2 - \frac{2j+k+2}{2(m-1)} - 2l} \| \partial_x^{j + \frac{k-1}{2} - l} \mathcal{J}_+ u_N^{\text{hyp}^+} \|_{L^2}^2
\]
\[
\lesssim (t N^m)^{2(1-l)-\frac{m}{m-1}} N^{-2j-k+2l-2+\frac{m}{m-1}} \left( N^{2j+k-2l} \right) \mathcal{P}_{\frac{1}{2} - 2l} \lesssim 2 \mathcal{J}_+ u_N^{\text{hyp}^+} \|_{L^2}^2
\]
We observe that the equation

\[ \langle f \rangle \langle g \rangle \leq (tN^m)^{2(1-t)} N^{-2} \| u_N \|_{L^2}^2 + N^{-2} \| u_N \|_{L^2}^2 \]

\[ \lesssim N^{-2} \| u_N \|_{L^2}^2 , \]

\[ t \frac{2 + k}{m-1} \int_{\mathbb{R}} |\xi| \left| \mathcal{F} \left[ \cdot \cdot 1 - \frac{2 + k + 2}{2(m-1)} \partial_x^{j+1} f(x) \right] \right| dx \]

\[ \lesssim \frac{2 + k}{m-1} N \left| \mathcal{P}_{N} \right| \lesssim (tN^m)^{2(1-t)} \| u_N \|_{L^2}^2 \]

\[ \lesssim (tN^m)^{2(1-t)} N^{-2} \| u_N \|_{L^2}^2 + N^{-2} \| u_N \|_{L^2}^2 \]

\[ \lesssim N^{-2} \| u_N \|_{L^2}^2 . \]

Similarly, we get

\[ t \frac{2 + k}{m-1} \int_{\mathbb{R}} |\xi| \left| \mathcal{F} \left[ \cdot \cdot 1 - \frac{2 + k + 2}{2(m-1)} \partial_x^{j+1} f(x) \right] \right| dx \]

\[ \lesssim \frac{2 + k}{m-1} N \left| \mathcal{P}_{N} \right| \lesssim (tN^m)^{2(1-t)} \| u_N \|_{L^2}^2 \]

\[ \lesssim N^{-2} \| u_N \|_{L^2}^2 . \]

Hence, we have

\[ \| J_f \|_{L^2}^2 \lesssim \sum_{j=0}^{m-2} \| t^{\frac{m-j}{m-1}} |x|^{\frac{m-j}{m-1}} \partial_x^{j} f \|_{L^2}^2 - N^{-2} \| u_N \|_{L^2}^2 . \]

On the other hand, because

\[ (J_- J_+ u_N^{hyp,+})(x) = (x - t(-i \partial_x)^{m-1}) u_N^{hyp,+} (x) \]

\[ + \sum_{j=1}^{m-2} (-it^{\frac{m-j}{m-1}})^j |x|^{\frac{m-j}{m-1}} \sum_{k=0}^{j-1} \left( \begin{array}{c} j \\ k \end{array} \right) \partial_x^{j-k} (x^{\frac{m-j}{m-1}}) \partial_x^k u_N^{hyp,+} , \]

\[ (x - t(-i \partial_x)^{m-1}) u_N^{hyp,+} = J u_N^{hyp,+} - t ((-i \partial_x)^{m-1} - \partial_x^{m-1}) u_N^{hyp,+} , \]

(17) and \( tN^m > 1 \) yield

\[ \| J_- f \|_{L^2} \lesssim \| (x - t(-i \partial_x)^{m-1}) u_N^{hyp,+} \|_{L^2} + \sum_{j=1}^{m-2} \sum_{k=0}^{j-1} \| t^{\frac{m-j}{m-1}} \|_{L^2} \| \partial_x^k u_N^{hyp,+} \|_{L^2}^2 \]

\[ \lesssim \| J u_N^{hyp,+} \|_{L^2} + \sum_{j=1}^{m-2} \sum_{k=0}^{j-1} (tN^m)^{1-j+k} N^{-1} \| u_N \|_{L^2} + N^{-1} \| u_N \|_{L^2} \]

\[ \lesssim \| u_N (t) \|_{N^c} . \]

Combining this with (22), we obtain (20).

For the elliptic bound, we decompose \( u_N^{ell,+} \) into three parts

\[ u_N^{ell,+} = u_{\leq k}^{ell,+} + u_{k}^{ell,+} + u_{\geq k}^{ell,+} . \]

We observe that the equation

\[ \| x f \|_{L^2}^2 + \| t \partial_x f \|_{L^2}^2 = \| J f \|_{L^2}^2 + 2 \int_{\mathbb{R}} |\partial_x^{m-1} f(x)|^2 dx . \]
holds for any smooth function \( f \) and odd \( m \). Similarly, for even \( m \),

\[
\| x|\partial_x|^{\frac{m}{2}} f \|_{L^2}^2 + \| t|\partial_t|^{m+\frac{1}{2}} f \|_{L^2}^2 = \| \mathcal{J}|\partial_x|^\frac{m}{2} f \|_{L^2}^2 + 2 \int_R tx |\partial_x|^m f(x) dx \tag{24}
\]

holds for any smooth function \( f \).

In what follows, we only consider the case when \( m \) is even, because the case when \( m \) is odd can be similarly handled.

Since \( |x, \partial_x|^m \partial_x|^{-\frac{1}{2}}| \leq -\left( \frac{m}{2} + 1 \right) \partial_x^\frac{m}{2} |\partial_x|^{-\frac{1}{2}} - \frac{3}{2} \partial_x^m|\partial_x|^{-\frac{1}{2}} \) and \( \kappa = 2^{2m+3} \), (19) implies

\[
\left| \int_R tx \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} \right) (t, x) \right| dx \right| 
\leq \frac{2}{\kappa} N^{-m+1} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
\leq \frac{4}{\kappa} N^{-m+1} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
+ C N^{-m+1} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
\leq \frac{2^{2m+3}}{\kappa} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
+ C N \| u_N(t) \|_{L^2}^2 
\leq \frac{1}{4} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 + C N \| u_N(t) \|_{L^2}^2.
\]

Taking \( f = \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} \) in (24), we have

\[
\left| \int_R tx \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right| dx \right| 
\leq \left| \mathcal{J}|\partial_x|^{\frac{m}{2}} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 + C N \| u_N(t) \|_{L^2}^2,
\]

which shows

\[
\left| \int_R tx \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right| dx \right| 
\leq \frac{1}{8} \left| \int_R tx \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right| dx \right|.
\]

By (18), we have

\[
\left| \int_R tx \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t, x) \right) \right| dx \right| 
\leq \frac{2}{\kappa} N^{-m+1} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
\leq \frac{2^{2m}}{\kappa} \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
+ C N \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right|_{L^2}^2 
\leq \frac{1}{8} \left| \int_R tx \left| \partial_x^\frac{m}{2} \left( \sigma_{\geq \kappa tN^{m-1}u_N^{\text{ell}+,+}} (t) \right) \right| dx \right|.
\]
Taking $f = \sigma_{\leq \frac{1}{2}tN^{-m-1}}u_N^{\text{ell},+}$ in (24), we have
\[
\left\| t|\partial_x|^{m+\frac{1}{2}} \left( \sigma_{\leq \frac{1}{2}tN^{-m-1}}u_N^{\text{ell},+}(t) \right) \right\|_{L^2} \lesssim \left\| \mathcal{F}|\partial_x|^\frac{3}{2} \left( \sigma_{\leq \frac{1}{2}tN^{-m-1}}u_N^{\text{ell},+}(t) \right) \right\|_{L^2} + N^{\frac{1}{2}} \| u_N(t) \|_{L^2},
\]
which shows that
\[
\left\| t|\partial_x|^{m-1} \left( \sigma_{\leq \frac{1}{2}tN^{-m-1}}u_N^{\text{ell},+}(t) \right) \right\|_{L^2} \lesssim \| u_N(t) \|_{\tilde{X}}.
\]

From
\[
\int_{\mathbb{R}} tx \left| \partial_x^{\frac{m}{2}+1} \left( \sigma_{\leq \frac{1}{2}tN^{-m-1}<\kappa tN^{-m-1}}u_N^{\text{ell},+}(t,x) \right) \right|^2 dx < 0,
\]
taking $f = \sigma_{\leq \frac{1}{2}tN^{-m-1}<\kappa tN^{-m-1}}u_N^{\text{ell},+}$ in (24), we have
\[
tN^{-m-1} \left\| \sigma_{\leq \frac{1}{2}tN^{-m-1}<\kappa tN^{-m-1}}u_N^{\text{ell},+}(t) \right\|_{L^2} \lesssim \| u_N(t) \|_{\tilde{X}}.
\]

3.2. Decay estimates in $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$. First, by summing up the frequency localized estimates, we show the $L^2$-estimates.

**Corollary 3.4.** For $t \geq 1$, we have
\[
\sum_{k=0}^{m-2} \sum_{l=0}^{k} \left\| t^{\frac{k+1}{m+1}} |x|^{-\frac{mk+1}{m+1}+l} \partial_x^l \partial_y^\text{hyp} \right\|_{L^2} \lesssim \| u(t) \|_{\tilde{X}},
\]
\[
\sum_{k=0}^{m-2} \left\| t^{\frac{k}{m+1}} |x|^{-\frac{k+1}{m+1}+\nu} \mathcal{F} \partial_x^k \partial_y^\text{hyp} \right\|_{L^2} \lesssim \| u(t) \|_{\tilde{X}},
\]
\[
\sum_{k=0}^{m-2} \left\| t^{\frac{k+1}{m+1}} \partial_x^k u_N^{\text{ell},+} \right\|_{L^2} \lesssim \| u(t) \|_{\tilde{X}}.
\]

**Proof.** By (16) and (17), we have
\[
\sum_{k=0}^{m-2} \sum_{l=0}^{k} \left( \sum_{N \in 2^\mathbb{Z}} \left\| t^{\frac{k+1}{m+1}} |x|^{-\frac{mk+1}{m+1}+l} \partial_x^l \partial_y^\text{hyp} \right\|_{L^2} \right)^{\frac{1}{2}}
\]
\[
\lesssim \sum_{k=0}^{m-2} \sum_{l=0}^{k} \left( \sum_{N \in 2^\mathbb{Z}} \left\| t^{\frac{k+1}{m+1}} |x|^{-\frac{mk+1}{m+1}+l} \partial_x^l \partial_y^\text{hyp} \right\|_{L^2} \right)^{\frac{1}{2}}
\]
\[
+ \sum_{k=0}^{m-2} \sum_{l=0}^{k} \left( 1 - P_{N^{-m+k} \leq \kappa tN^{-m-1}} \right) \left\| t^{\frac{k+1}{m+1}} |x|^{-\frac{mk+1}{m+1}+l} \partial_x^l \partial_y^\text{hyp} \right\|_{L^2}
\]
\[
\lesssim \sum_{k=0}^{m-2} \sum_{l=0}^{k} \left( \sum_{N \in 2^\mathbb{Z}} \left\| t^{-k-l} N^{-m-k+(m-1)l-1} \partial_x^l \partial_y^\text{hyp} \right\|_{L^2} \right)^{\frac{1}{2}} + \| u(t) \|_{\tilde{X}}
\]
\[
\lesssim \| u(t) \|_{\tilde{X}}.
\]
We use (16), (17), and (20) to obtain
\[
\sum_{k=0}^{m-2} \left\| t^{\frac{k}{m-1}} |x|^{-\frac{k-m+2}{m-1}} \partial_x^k \mathcal{J}_+ u_{\text{hyp}^+} \right\|_{L^2}^2
\lesssim \sum_{k=0}^{m-2} \left( \sum_{N \in 2^k N > t^{-\frac{1}{m}}} \left\| t^{\frac{k}{m-1}} |x|^{-\frac{k-m+2}{m-1}} \partial_x^k \mathcal{J}_+ u_{\text{hyp}^+}^N (t) \right\|_{L^2}^2 \right)^{\frac{1}{2}}
+ \sum_{k=0}^{m-2} \sum_{N \in 2^k N > t^{-\frac{1}{m}}} \left\| \left( 1 - P^N_{\frac{N}{2} \leq t \leq \frac{N}{2}} \right) t^{\frac{k}{m-1}} |x|^{-\frac{k-m+2}{m-1}} \partial_x^k \mathcal{J}_+ u_{\text{hyp}^+}^N (t) \right\|_{L^2}
\lesssim \|u(t)\|_{\tilde{H}}.
\]
This gives (26) with \( k = 0 \). For \( k \geq 1 \), because
\[
\mathcal{J}_+ \partial_x^k u_{\text{hyp}^+} = \partial_x^k (\mathcal{J}_+ u_{\text{hyp}^+}) + t^{-\frac{k}{m-1}} |x|^{-\frac{k-m+2}{m-1}} \sum_{l=0}^{k-1} C_{k,l} t^{\frac{(k-1)+l}{m-1}} |x|^{-\frac{m(k-1)+l}{m-1}} + t \partial_x u_{\text{hyp}^+},
\]
the estimate (26) follows from (20) and (25).

For the elliptic bound, we note that
\[
\begin{align*}
\begin{aligned}
u^{\text{ell}} &= u_{\leq t^{-\frac{1}{m}}} + 2\mathbb{N} \sum_{N \in 2^k N > t^{-\frac{1}{m}}} u_{\text{hyp}^+}. \tag{28}
\end{aligned}
\end{align*}
\]

We use (16), (18), and (21) to obtain
\[
\sum_{k=0}^{m-2} \left\| t^{\frac{k}{m-1}} \partial_x^k \nu^{\text{ell}} \right\|_{L^2} \lesssim t^{\frac{k}{m}} \|u_{\leq t^{-\frac{1}{m}}}\|_{L^2} + \sum_{k=0}^{m-2} \left( \sum_{N \in 2^k N > t^{-\frac{1}{m}}} \left( t^{\frac{k+1}{m}} N^k \| u_{\text{hyp}^+}^N \|_{L^2} \right)^2 \right)^{\frac{1}{2}}
+ \sum_{k=0}^{m-2} \sum_{N \in 2^k N > t^{-\frac{1}{m}}} \left\| \left( 1 - P^N_{\frac{N}{2} \leq t \leq \frac{N}{2}} \right) \partial_x^k u_{\text{hyp}^+}^N \right\|_{L^2}
\lesssim \|u(t)\|_{\tilde{H}}.
\]
From \( t^\frac{k}{m-1} |x|^{-\frac{k-m+1}{m-1}} N^k \lesssim |x| + t N^{m-1} \) and (19), we have
\[
t^\frac{k}{m-1} \left\| |x|^{-\frac{k-m+1}{m-1}} \partial_x^k u_{\text{hyp}^+}^N \right\|_{L^2(|x| \geq t^{\frac{1}{m}})} \lesssim N^{-k} \left\| (|x| + t N^{m-1}) \sigma_{t^{\frac{1}{m}}} (x) \partial_x^k u_{\text{hyp}^+}^N \right\|_{L^2}
\lesssim \|(|x| + t N^{m-1}) u_{\text{hyp}^+}^N \|_{L^2} + t^{\frac{1}{m}} N^{-2} \|u_N\|_{L^2}.
\]
Because
\[
\begin{align*}
\begin{aligned}
t^\frac{k}{m-1} \left\| |x|^{-\frac{k-m+1}{m-1}} \partial_x^k u_{\leq t^{-\frac{1}{m}}} \right\|_{L^2(|x| \geq t^{\frac{1}{m}})} &\lesssim t^{\frac{k}{m}} \|x \partial_x^k u_{\leq t^{-\frac{1}{m}}} \|_{L^2(|x| \geq t^{\frac{1}{m}})} \\
&\lesssim \|\mathcal{J} u_{\leq t^{-\frac{1}{m}}} \|_{L^2} + t^{\frac{k}{m}} \|u_{\leq t^{-\frac{1}{m}}} \|_{L^2} \\
&\lesssim \|u(t)\|_{\tilde{H}},
\end{aligned}
\end{align*}
\]
by (16), (19), and (21), we obtain
\[
\left\| \frac{\mathrm{ii}}{\sqrt{m+1}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \partial_x^k u_{N+1}^{\text{hyp}}(t,x) \right\|_{L^2(|x| \geq t^{-\frac{1}{3}})} \leq \|u(t)\|_{\tilde{X}} + \left( \sum_{N \in \mathbb{Z}} \left\| \langle |x| + tN^{m-1} \rangle u_N^{\text{hyp}}(t,x) \right\|_{L^2}^2 \right)^{\frac{1}{2}} + \sum_{N \in \mathbb{Z}} t^{-\frac{3}{2}} N^{-2} \|u_N\|_{L^2} \]
\[
+ \sum_{N \in \mathbb{Z}} \left\| \left( P_{\leq 2N} - P_{\leq 2N} \right) t^{\frac{m+1}{m}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \sigma > t^{\frac{1}{3}} (x) \partial_x^k u_N^{\text{hyp}}(t,x) \right\|_{L^2} \leq \|u(t)\|_{\tilde{X}}.
\]
Second, we show the pointwise decay estimates.

**Proposition 3.5.** For \( t \geq 1 \) and \( k = 0, 1, \ldots, m - 2 \), we have
\[
\left\| \frac{\mathrm{ii}}{\sqrt{m+1}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \partial_x^k u_{N+1}^{\text{hyp}}(t,x) \right\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|u(t)\|_{\tilde{X}},
\]
\[
\left\| \frac{\mathrm{ii}}{\sqrt{m+1}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \partial_x^k u_N^{\text{hyp}}(t,x) \right\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|u(t)\|_{\tilde{X}}.
\]

**Proof.** The Gagliardo-Nirenberg inequality
\[
|f| \lesssim \|f\|_{L^2}^{\frac{1}{2}} \|\partial_x f\|_{L^2}^{\frac{1}{2}}
\]
with \( f = e^{-i\phi} u_N^{\text{hyp}} \), \( \partial_x (e^{-i\phi} u^{\text{hyp}}) = -it^{-\frac{k}{m+1}} e^{-i\phi} \mathcal{J}_x u^{\text{hyp}} \), Lemma 3.3, and (17) imply
\[
\left\| \frac{\mathrm{ii}}{\sqrt{m+1}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \partial_x^k u_{N+1}^{\text{hyp}}(t,x) \right\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|\partial_x^k u_{N+1}^{\text{hyp}}(t)\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|\partial_x^k u_N^{\text{hyp}}(t)\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|u(t)\|_{\tilde{X}}.
\]
Because \( u_N^{\text{hyp}}(t,x) \) is a finite sum of \( u_N^{\text{hyp}}(t,x) \)'s, this yields the desired hyperbolic bound (29).

Next, we show the elliptic bound. First, we consider the low frequency part. For \( |x| \leq t^{-\frac{1}{3}} \), Bernstein's inequality implies
\[
\left\| \frac{\mathrm{ii}}{\sqrt{m+1}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \partial_x^k u \right\|_{L^2} \leq \|u\|_{L^2} \leq \frac{1}{t^{\frac{1}{3}}} \|u(t)\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|u(t)\|_{\tilde{X}}.
\]
Similarly, for \( |x| \geq t^{-\frac{1}{3}} \), we have
\[
\left\| \frac{\mathrm{ii}}{\sqrt{m+1}} \left( t^{-\frac{1}{3}} |x| \right)^{- \frac{k}{m+1} + \frac{m-3}{2m-4} \partial_x^k u \right\|_{L^2} \leq \|u\|_{L^2} \leq \frac{1}{t^{\frac{1}{3}}} \|u(t)\|_{L^2} \leq t^{-\frac{3}{2m+1}} \|u(t)\|_{\tilde{X}}.
\]
Second, we consider the high frequency part. For \(|x| \leq t^{\frac{1}{a}}\), the Gagliardo-Nirenberg inequality and (18) yield
\[
\left| t^{\frac{k+1}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{k}{m-a} + \frac{2m-3}{2(m-a)}} \partial_x^k u_N^{\text{ell}+}(t, x) \right|
\lesssim t^{\frac{k+1}{m}} N^k \left\| u_N^{\text{ell}+}(t, x) \right\|_{L^2} + t^{\frac{1}{2m}} (t^{\frac{1}{m}} N)^{-2} \| u_N(t) \|_{L^2}
= t^{\frac{k-m+1}{m}} N^{k-m+1} \left\| tN^{m-1} u_N^{\text{ell}+}(t, x) \right\|_{L^2} + t^{\frac{1}{2m}} (t^{\frac{1}{m}} N)^{-2} \| u_N(t) \|_{L^2}.
\]

Thus, by (21), we have
\[
\sum_{N \in 2^\mathbb{Z}, \ N > t^{-\frac{1}{a}}} \left| t^{\frac{k+1}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{k}{m-a} + \frac{2m-3}{2(m-a)}} \partial_x^k u_N^{\text{ell}+}(t, x) \right| \lesssim t^{-\frac{1}{2m}} \| u(t) \|_{\widetilde{X}}.
\]

For \(|x| > t^{\frac{1}{a}}\), there exists \(M \in 2^\mathbb{Z}\) such that
\[
u_N^{\text{ell}+}(t, x) = \sigma_{tM^{-1}}(x) u_N^{\text{ell}+}(t, x).
\]

Then, the Gagliardo-Nirenberg inequality and Lemma 3.1 lead to
\[
\left| t^{\frac{k+1}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{k}{m-a} + \frac{2m-3}{2(m-a)}} \partial_x^k u_N^{\text{ell}+}(t, x) \right|
\lesssim t^{\frac{2m-1}{2m}} M^{2m-3-k} \left\| \sigma_{tM^{-1}}(x) \partial_x^k u_N^{\text{ell}+}(t, x) \right\|_{L^2}
= t^{\frac{2m-1}{2m}} M^{2m-3-k} N^{k+\frac{1}{2}} \left\| \sigma_{tM^{-1}}(x) u_N^{\text{ell}+}(t) \right\|_{L^2}
+ t^{-k} M^{2m-\frac{3}{2} - mk} \max \left(1, tN^{m-1}\right)^{-2} \| u_N(t) \|_{L^2}.
\]

Hence, we have
\[
\sum_{N \in 2^\mathbb{Z}, \ N > t^{-\frac{1}{a}}} \left| t^{\frac{k+1}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{k}{m-a} + \frac{2m-3}{2(m-a)}} \partial_x^k u_N^{\text{ell}+}(t, x) \right|
\lesssim \sum_{N \in 2^\mathbb{Z}, \ t^{-\frac{1}{a}} < N \leq M} t^{-\frac{1}{2m}} M^{-k-\frac{1}{2}} N^{k+\frac{1}{2}} \left\| x u_N^{\text{ell}+}(t) \right\|_{L^2}
+ \sum_{N \in 2^\mathbb{Z}, \ N > M} t^{-\frac{1}{2m}} M^{2m-3-k} N^{-2m+3+k} \left\| tN^{m-1} u_N^{\text{ell}+}(t) \right\|_{L^2} + t^{-\frac{1}{2m}} \| u(t) \|_{\widetilde{X}}
\lesssim t^{-\frac{1}{2m}} \| u(t) \|_{\widetilde{X}}.
\]

**Remark 3.6.** For \(t \geq 1\), the estimate
\[
\left| t^{\frac{k+3}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{k}{m-a} + \frac{m-2}{m-a}} \partial_x^k u_N^{\text{hyp}+}(t, x) \right| \lesssim \| u(t) \|_{L^2} + t^{-\frac{1}{2m}} \| u(t) \|_{\widetilde{X}}
\]
holds true. Indeed, the Gagliardo-Nirenberg inequality, (17), and (20) yield
\[
\left| t^{\frac{k+3}{m}} \left( t^{-\frac{1}{m}} x \right)^{-\frac{k}{m-a} + \frac{m-2}{m-a}} \partial_x^k u_N^{\text{hyp}+}(t, x) \right|
\lesssim t^{\frac{2m-1}{2m}} N^{-\frac{m-2}{2}} \left\| \partial_x^k u_N^{\text{hyp}+}(t) \right\|_{L^\infty}
\lesssim t^{\frac{2m-3}{2m-1} - k} \left\| \partial_x^k u_N^{\text{hyp}+}(t) \right\|_{L^2}^{\frac{1}{2}} \left\| \mathcal{T}_x \partial_x^k u_N^{\text{hyp}+}(t) \right\|_{L^2}^{\frac{1}{2}}.
\]
We note that Lemma 4.1.

Following sense:

Proof.

We show that \( \Psi \) is essentially localized at frequency \( \xi_\alpha := v^{\frac{1}{m-1}} \) in the following sense:

**Lemma 4.1.** For \( t \geq 1 \) and \( v \geq t^{-\frac{m-1}{m}} \), we have

\[
\mathcal{F}[\Psi_v](t, \xi) = \frac{1}{\sqrt{m-1}} \lambda^{-1} \chi_1(\lambda^{-1}(\xi - \xi_\alpha), \lambda^{-1} \xi_\alpha) e^{-\frac{1}{m}t \xi^m},
\]

where \( \chi_1(\cdot, \alpha) \in \mathcal{S}(\mathbb{R}) \) satisfies

\[
\sup_{\alpha \geq 1} \sup_{\zeta \in \mathbb{R}} |\zeta|^k \partial^l_\zeta \chi_1(\zeta, \alpha) | \lesssim_{k,l} 1
\]

for any \( k, l \in \mathbb{N}_0 \). Moreover, there exists a constant \( C_1 > 0 \) such that for any \( \alpha \geq 1 \),

\[
\left| \int_{\mathbb{R}} \chi_1(\zeta, \alpha) d\zeta - 1 \right| \leq \frac{C_1}{\alpha}.
\]

**Proof.** From Taylor’s theorem, we can write for \( x > 0 \)

\[
\phi(t, x) = \phi(t, vt) + \partial_x \phi(t, vt)(x - vt) + \frac{1}{2} \partial^2_x \phi(t, vt)(x - vt)^2 + \int_{vt}^x \frac{(x - y)^2}{2} \partial^2_y \phi(t, y) dy
\]

\[
= -\frac{\pi}{4} + \frac{m - 1}{m} t \xi^m_v + \xi_v(x - vt) + \frac{1}{2(m-1)} \lambda^2(x - vt)^2 + R(\lambda(x - vt), \lambda^{-1} \xi_v),
\]

where

\[
R(z, \alpha) := -\frac{m - 2}{2(m-1)^2} \frac{\zeta^3}{\alpha} \int_0^1 (1 - \theta)^2 \left( \theta \frac{z}{\alpha} + 1 \right)^{-\frac{2(m-3)}{m-1}} d\theta.
\]

We note that \( R(z, \alpha) \) is well-defined provided that \( z > -\alpha \). By a change of variables using \( z = \lambda(x - vt) \), we have

\[
\mathcal{F}[\Psi_v](t, \xi)
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i x \xi} \chi(\lambda(x - vt)) e^{i \phi(t, x)} dx
\]

\[
= e^{-i \frac{\pi}{4} \lambda^{-1} e^{\frac{1}{m}t ((m-1) \xi^m_v) - i R(z, \lambda^{-1} \xi_v) \chi(z)} dx
\]

\[
= \frac{1}{\sqrt{m-1}} \lambda^{-1} \chi_1(\lambda^{-1}(\xi - \xi_\alpha), \lambda^{-1} \xi_\alpha) e^{-\frac{1}{m}t \xi^m},
\]
Finally, (32) follows from (34) and (35).

By definition, $\chi_1(\cdot, \alpha) \in \mathcal{S}(\mathbb{R})$ for $\alpha \geq 1$. From the Fresnel integrals,

$$(1 - i) \sqrt{\frac{m - 1}{2}} \mathcal{F}[\chi]\left(\frac{z}{\sqrt{2}}(\zeta, \alpha)\right) = \int_{\mathbb{R}} e^{-i(\eta + \sqrt{\frac{m-1}{2}}(\zeta - \frac{x}{m-1}))^2} d\eta$$

holds for any $z, \zeta \in \mathbb{R}$. Accordingly, we have

$$\mathcal{F}[e^{\frac{m-1}{2}z^2+iR(z,\alpha)}\chi]\left(\frac{\zeta}{\sqrt{2}}\right) = e^{-i\frac{m-1}{2}z^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\frac{m-1}{2}(\zeta - \frac{x}{m-1})^2+iR(z,\alpha)}\chi dz$$

$$= \frac{1}{1 - i} \sqrt{\frac{2}{\pi}} e^{-\frac{m-1}{2}z^2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-i(\eta^2 + \sqrt{2(m-1)}(\zeta - \frac{x}{m-1}))^2} e^{iR(z,\alpha)}\chi(z) d\eta dz$$

$$= \frac{1}{1 - i} \sqrt{\frac{2}{\pi}} e^{-\frac{m-1}{2}z^2} \int_{\mathbb{R}} e^{-i\eta^2} e^{-i\sqrt{2(m-1)}\eta\zeta} \hat{\chi}_2\left(-\sqrt{\frac{2}{m-1}}\eta, \alpha\right) d\eta,$$

where

$$\chi_2(\cdot, \alpha) := \chi e^{iR(\cdot, \alpha)} \in \mathcal{S}(\mathbb{R})$$

for $\alpha \geq 1$. Hence, we can write

$$\chi_1(\zeta, \alpha) = \frac{m - 1}{2} \sqrt{\frac{2}{\pi}} e^{\frac{m-3}{m-1}(\zeta, m-1)} \int_{\mathbb{R}} e^{-i\frac{m-1}{2}\eta^2} e^{i\eta(\zeta, \alpha)} \hat{\chi}_2(\eta, \alpha) d\eta. \quad (34)$$

As $\chi_2(\cdot, \alpha) \in \mathcal{S}(\mathbb{R}), e^{\frac{m-3}{m-1}(\zeta, m-1)} = 1 + O\left(\frac{|\zeta|^3}{\alpha}\right)$, and

$$\sup_{\alpha \geq 1} \sup_{\eta \in \mathbb{R}} |\langle \eta \rangle^k \partial^l_{\eta} \hat{\chi}_2(\eta, \alpha)| \lesssim_{k, l} 1, \quad (35)$$

we obtain

$$\int_{\mathbb{R}} \chi_1(\zeta, \alpha) d\zeta = \sqrt{2\pi} \hat{\chi}_2(0, \alpha) + O\left(\frac{1}{\alpha}\right).$$

Because $e^{iR(z,\alpha)} = 1 + O\left(\frac{1}{\alpha}\right)$ for $|z| < 1$ and $\alpha \geq 1$, we have

$$\sqrt{2\pi} \hat{\chi}_2(0, \alpha) = \int_{\mathbb{R}} \chi(z) e^{iR(z,\alpha)} dz = 1 + O\left(\frac{1}{\alpha}\right).$$

Finally, (32) follows from (34) and (35). \hspace{1cm} \Box

For $v \geq t^{-\frac{m-1}{2}}$, we define the nearest dyadic number to $\xi_v$ by $N_v \in 2^\mathbb{Z}$. Then, $\frac{\xi_v}{2} < N_v < 2 \xi_v$ holds.

Integration by parts with (32) yields

$$\left|\left(1 - P_{\frac{m-1}{2} \leq \xi_v, \xi_v \leq 2N_v}^+ \right) |\partial_x|^a \Psi_v(t, x)\right| \lesssim_{a,l} t^{-\frac{m-1}{2}} \left(t^{\frac{m-1}{m}} v\right)^{-l} \min(1, |x|^{-1}t)^2$$

for any $a, l \geq 0$, which implies

$$\left\|\left(1 - P_{\frac{m-1}{2} \leq \xi_v, 2N_v}^+ \right) |\partial_x|^a \Psi_v(t)\right\|_{L^1} \lesssim_{a,c} t^{\frac{m-1}{m}} \left(t^{\frac{m-1}{m}} v\right)^{-c} \quad (36)$$

for $v \geq t^{-\frac{m-1}{2}}$ and any $a, c \geq 0$.

Next, we show that $\Psi_v$ is a good approximate solution for the linear equation.
We begin with the following preliminary observation. Set
\[ S_j := \{ k = (k_1, \ldots, k_j) \in \mathbb{N}_0 : 0 \leq k_1 \leq \cdots \leq k_j \leq j, k_1 + \cdots + k_j = j \} \]
for \( j \in \mathbb{N} \).

**Lemma 4.2.** Let \( f \) be a smooth function. For any \( j \in \mathbb{N} \), we have
\[
\frac{d^j}{dx^j} e^{f(x)} = \sum_{k \in S_j} C_k^{(j)} e^{f(x)} \prod_{l=1}^{j} f^{(k_l)}(x),
\]
(37)
where
\[
f^{(l)} := \begin{cases} 1, & \text{if } l = 0, \\ \frac{d^l f}{dx^l}, & \text{otherwise}, \end{cases}
\]
and \( C_k^{(j)} \) is a constant depending on \( j \in \mathbb{N}, k \in S_j \). In particular,
\[
C_{(1, \ldots, 1)}^{(j)} = 1, \quad C_{(0,1,\ldots,1,2)}^{(j)} = \binom{j}{2}.
\]

**Proof.** A direct calculation shows
\[
\frac{d}{dx} e^{f(x)} = e^{f(x)} f'(x), \quad \frac{d^2}{dx^2} e^{f(x)} = e^{f(x)} \left( f'(x)^2 + f''(x) \right),
\]
\[
\frac{d^3}{dx^3} e^{f(x)} = e^{f(x)} \left( f'(x)^3 + 3 f'(x) f''(x) + f'''(x) \right).
\]
Hence, we have
\[
C_{(1, \ldots, 1)}^{(1)} = C_{(1, \ldots, 1)}^{(2)} = C_{(0,2)}^{(2)} = C_{(1,1,1)}^{(3)} = C_{(0,0,3)}^{(3)} = 1, \quad C_{(0,1,2)}^{(3)} = 3.
\]
(38)

We assume that (37) holds up to \( j - 1 \). Because
\[
\frac{d^j}{dx^j} e^{f(x)}
\]
\[
= \frac{d}{dx} \left( \sum_{k \in S_{j-1}} C_k^{(j-1)} e^{f(x)} \prod_{l=1}^{j-1} f^{(k_l)}(x) \right)(x)
\]
\[
= \sum_{k \in S_{j-1}} C_k^{(j-1)} e^{f(x)} \left( f'(x) \prod_{l=1}^{j-1} f^{(k_l)}(x) + \sum_{n \in \{1, \ldots, j-1\}} f^{(k_n+1)}(x) \prod_{l \neq n} f^{(k_l)}(x) \right),
\]
the constants \( C_k^{(j)} \) are determined by \( C_k^{(j-1)} \), which shows (37).

In particular, the following recurrence equations hold true:
\[
C_{(1, \ldots, 1)}^{(j)} = C_{(1, \ldots, 1)}^{(j-1)}, \quad C_{(0,1,\ldots,1,2)}^{(j)} = C_{(0,1,\ldots,1,2)}^{(j-1)} + (j - 1) C_{(1, \ldots, 1)}^{(j-1)}.
\]
By (38), we obtain \( C_{(1, \ldots, 1)}^{(j)} = 1 \) and \( C_{(0,1,\ldots,1,2)}^{(j)} = \binom{j}{2} \).
\]

Set
\[
S'_j := S_j \setminus \{(1, \ldots, 1)\}, \quad S''_j := S'_j \setminus \{(0,1,\ldots,1,2)\}.
\]
Lemma 4.2 and \( \partial_t \phi = -\frac{i}{m} (\partial_x \phi)^m \) yield
\[
\left( \partial_t + \frac{i}{m} (-i \partial_x)^m \right) \Psi_v(t, x)
\]
\begin{align*}
&= -\frac{x + vt}{2t} \lambda' e^{i\phi} + i\partial_t \phi e^{i\phi} \\
&\quad + \frac{i}{m} (-i)^m \left\{ \chi \partial_x^m (e^{i\phi}) + m \lambda' \partial_x^{m-1} (e^{i\phi}) + \frac{m(m-1)}{2} \lambda^2 \partial_x^{m-2} (e^{i\phi}) \\
&\quad + m(m-1)(m-2) \frac{6}{m} \lambda^3 \partial_x^{m-3} (e^{i\phi}) + \sum_{j=0}^{m-4} \left( \frac{m}{j} \right) \lambda^{m-j} \chi^j \partial_x^{j}(e^{i\phi}) \right\} e^{i\phi} \\
&- \frac{x + vt}{2t} \lambda' e^{i\phi} + i\partial_t \phi e^{i\phi} \\
&\quad + \frac{i}{m} \left\{ \lambda' (\partial_x \phi)^m - \frac{m}{2} \right\} \left( \partial_x \phi \right)^m - i \left( \frac{m-1}{2} \right) \left( \partial_x \phi \right)^m - 2 \partial_x^2 \phi \\
&\quad + (-i)^m \sum_{k \in S'_m} \lambda^k \left( \partial_x \phi \right)^m - i \left( \frac{m-1}{2} \right) \left( \partial_x \phi \right)^m - 2 \partial_x^2 \phi \\
&\quad + (-i)^m \sum_{k \in S'_m} \lambda^k \left( \partial_x \phi \right)^m - i \left( \frac{m-1}{2} \right) \left( \partial_x \phi \right)^m - 2 \partial_x^2 \phi \\
&\quad + \left\{ \lambda^m \chi^m + \sum_{k \in S'_m} \frac{m-1}{2} \lambda^k \partial_x^2 \phi \right\} e^{i\phi} \\
&\quad + \frac{x - vt}{2t} \chi^m \partial_x (e^{i\phi}) \\
&\quad - \frac{(m-1)(m-2)}{6} \frac{m}{m-1} \partial_x \left( \partial_x \phi \right)^m - \frac{(m-1)(m-3)}{12} \partial_x \left( \partial_x \phi \right)^m - \frac{(m-1)(m-4)}{24} \partial_x \left( \partial_x \phi \right)^m \\
&\quad + \frac{i}{m} \sum_{k \in S'_m} \lambda^k \left( \partial_x \phi \right)^m - i \left( \frac{m-1}{2} \right) \left( \partial_x \phi \right)^m - 2 \partial_x^2 \phi \\
&\quad + \frac{x - vt}{2t} \chi^m \partial_x (e^{i\phi}) \\
&\quad + \left\{ \lambda^m \chi^m + \sum_{k \in S'_m} \frac{m-1}{2} \lambda^k \partial_x^2 \phi \right\} e^{i\phi} \\
&= \frac{\partial_x \chi}{t} + O \left( t^{-1} \left( \frac{m+1}{m} v \right)^{\frac{m}{m-1}} X(\lambda(x - vt)) \right),
\end{align*}
where
\[ \widetilde{\chi}(t, x) := \chi \left( \frac{x - vt}{2} \right) \chi(\lambda(x - vt)) - i \frac{m - 1}{2} \lambda^2 t^{\frac{1}{m - 1}} |x|^{\frac{m - 2}{m - 1}} \chi'(\lambda(x - vt)) \]
\[ - \frac{(m - 1)(m - 2)}{6} \lambda^3 t^{\frac{2}{m - 1}} |x|^{\frac{m - 3}{m - 1}} \chi''(\lambda(x - vt)) \]
and \( X \) is a nonnegative continuous function supported in \([ - \frac{1}{2}, \frac{1}{2} ] \). Therefore, we obtain the following:
\[ (L \Psi_v)(t, x) = e^{i\phi} t^\lambda \partial_x \tilde{\chi} + \frac{1}{m} \left( (\partial_x |m - 1 \partial_x - i(-i\partial_x)^m) P^\nu \Psi_v(t, x) \right) + O \left( t^{-1} (t^{\frac{m - 1}{2}})^{-\frac{m - 1}{2}} \right). \] (39)

Because \( \tilde{\chi} \) has the same localization as \( \chi \), the first term on the right hand side of (39) is essentially localized at frequency \( \xi_v \). For the sake of completeness, we give a proof here, although the proof is similar to that of Lemma 4.1.

**Lemma 4.3.** For \( t \geq 1 \) and \( v \geq t^{-\frac{m - 1}{2}} \) and any \( a, c \geq 0 \), we have
\[ \left\| \left( 1 - P^\nu_{\frac{m - 1}{2} \lambda v} \right) \left| \partial_x \right|^{a} (e^{i\phi} \tilde{\chi}) \right\|_{L^2_x} \lesssim_{a, c} t^{-\frac{m - 1}{2}} \left( t^{\frac{m - 1}{2}} v \right)^{-c}. \] (40)

**Proof.** We write
\[ \tilde{\chi}(t, x) := \chi_0(\lambda(x - vt), \lambda vt), \]
where
\[ \chi_0(z, \alpha) := \frac{z}{2} \chi(z) - i \frac{m - 1}{2} \alpha^{-\frac{m - 2}{m - 1}} |z + \alpha|^{\frac{m - 2}{m - 1}} \chi'(z) \]
\[ - \frac{(m - 1)(m - 2)}{6} \alpha^{-\frac{2(m - 2)}{m - 1}} |z + \alpha|^{\frac{m - 3}{m - 1}} \chi''(z). \]

The same calculation as in the proof of Lemma 4.1 yields
\[ \mathcal{F}[e^{i\phi} \tilde{\chi}](t, \xi) := \frac{1}{\sqrt{m - 1}} \lambda^{-1} \chi_1(\lambda^{-1}(\xi - \xi_v), \lambda^{-1} \xi_v) e^{-\frac{1}{2} \lambda r \xi^m}, \]
where
\[ \chi_1(\zeta, \alpha) := (1 - i) \sqrt{\frac{m - 1}{2}} \sum_{l=0}^{m - 2} \left( \begin{array}{c} m \vspace{1mm} \\
\end{array} l \right) \zeta^{m - l} \alpha^{-m + l - 2} \mathcal{F}[e^{i\phi} \chi_0(\cdot, \alpha)](\xi). \]

Since \( \chi_0(\cdot, \alpha) \in S(\mathbb{R}) \) for \( \alpha \geq 1 \), we have
\[ \sup_{\alpha \geq 1} \sup_{\zeta \in \mathbb{R}} \left| \langle \zeta \rangle^k \partial_\zeta^l \chi_1(\zeta, \alpha) \right| \lesssim_{k, l} 1. \]

From \( |\xi - \xi_v| \geq \frac{\xi_v}{2} \) provided that \( \xi \notin [\frac{N_v}{2}, 4N_v] \), integration by parts yields
\[ \left\| \left( 1 - P^\nu_{\frac{m - 1}{2} \lambda v} \right) \left| \partial_x \right|^{a} (e^{i\phi} \chi)(x) \right\|_{L^2_x} \lesssim_t t^{-\frac{m - 1}{2}} \min(1, |x|^{-1}tv)^2, \]
which implies the desired bound. \( \square \)
4.2. Testing by wave packets. Let $C_2 > 0$ be the constant appearing in (32) with $k = 2$ and $l = 0$, that is,

$$\sup_{\alpha \geq 1} \sup_{\xi \in \mathbb{R}} \| \langle \xi \rangle^2 \chi_1(\xi, \alpha) \| \leq C_2.$$ 

For $t \geq 1$, we define

$$\Omega(t) := \left\{ v \in \mathbb{R}_+ : v \geq C_* t^{-\frac{m-1}{m}} \right\},$$

where

$$C_* := (2(C_1 + C_2 + 1))^{\frac{2(m-1)}{m}}.$$ (41)

Here, $C_1$ is the constant appearing in (33). The large constant $C_*$ is needed to show the pointwise estimate (43) in the frequency space.

We observe that the output $\gamma(t, v)$ defined by (11) is a “good” approximation of $u$ for $v \in \Omega(t)$.

**Proposition 4.4.** For $t \geq 1$ and $k = 0, 1, \ldots, m - 2$, we have the bounds

$$\partial^k_x u^+(t, vt) = i^k v^\frac{k}{m-2} e^{i\theta(t, vt)} \gamma(t, v) + R_k(t, v),$$ (42)

where $R_k$ is a function satisfying

$$\begin{align*}
\left\| \frac{k+1}{m-1} (t^{-\frac{m-1}{m}} v)^{-\frac{k}{m-1}} + \frac{1}{2} (t^{-\frac{m-1}{m}} v)^{-\frac{k}{m-1}} R_k(t, v) \right\|_{L^\infty_V(\Omega(t))} &\lesssim t^{-\frac{k}{m}} \| u(t) \|_{\tilde{X}}, \\
\left\| \frac{k+1}{m-1} (t^{-\frac{m-1}{m}} v)^{-\frac{k}{m-1}} + \frac{1}{2} (t^{-\frac{m-1}{m}} v)^{-\frac{k}{m-1}} R_k(t, v) \right\|_{L^2_V(\Omega(t))} &\lesssim t^{-\frac{k}{m}} \| u(t) \|_{\tilde{X}}.
\end{align*}$$

Moreover, in the frequency space, we have

$$\tilde{u}(t, \xi_v) = \sqrt{\frac{m-1}{m}} t e^{-\frac{1}{m} \xi v^m} \gamma(t, v) + R_\xi(t, v),$$ (43)

where $R_\xi$ is a function satisfying

$$\begin{align*}
\left\| \frac{m-1}{m} (t^{-\frac{m-1}{m}} v)^{-\frac{m}{m-1}} R_\xi(t, v) \right\|_{L^\infty_V(\Omega(t))} &\lesssim t^{-\frac{1}{m}} \| u(t) \|_{\tilde{X}}, \\
\left\| \frac{m-1}{m} R_\xi(t, v) \right\|_{L^2_V(\Omega(t))} &\lesssim t^{-\frac{1}{m}} \| u(t) \|_{\tilde{X}}.
\end{align*}$$

**Proof.** First, we show that

$$\left\| v^{-\frac{m}{m-1}} \int_{\mathbb{R}} f(t, x) \chi(\lambda(x - vt)) dx \right\|_{L^2_V(\Omega(t))} \lesssim \| f(t, \cdot) \|_{L^2_V(\Omega(t))},$$ (44)

holds true. By a change of variables using $z = \lambda(x - vt)$,

$$\text{L.H.S. of (44)} = t^\frac{1}{2} \left\| \int_{\mathbb{R}} f(t, \frac{1}{t^\frac{1}{2}} v^\frac{m-2}{2(m-1)} z + vt) \chi(z) dz \right\|_{L^2_V(\Omega(t))}.$$

Setting $\tilde{v} = t^\frac{1}{2} v^\frac{m-2}{2(m-1)} z + vt$, we note that

$$t^{-\frac{1}{2}} \tilde{v} = t^{\frac{m-1}{m}} v \left\{ 1 + \left( t^{\frac{m-1}{m}} v \right)^{-\frac{m}{m-1}} z \right\} \geq 1,$$

$$\frac{d\tilde{v}}{dv} = t \left\{ 1 + \frac{m-2}{2(m-1)} \left( t^{\frac{m-1}{m}} v \right)^{-\frac{m}{m-1}} z \right\} \geq \frac{t}{2},$$

for $v \in \Omega(t)$ and $|z| \leq \frac{1}{2}$. Then, we have

$$\begin{align*}
\text{L.H.S. of (44)} &\lesssim t^\frac{1}{2} \int_{\mathbb{R}} \left\| f(t, \frac{1}{t^\frac{1}{2}} v^\frac{m-2}{2(m-1)} z + vt) \chi(z) dz \right\|_{L^2_V(\Omega(t))} \\
&\lesssim \| f(t, \cdot) \|_{L^2_V(\Omega(t))}.
\end{align*}$$
Second, we show that $u$ in the definition of $\gamma$ is replaced with $u^{\text{hyp.}+}$ up to error terms;

$$
i^k \lambda v \frac{\partial^{k \ell}}{\partial \gamma} \gamma(t, v) = i^k \lambda v \frac{\partial^{k \ell}}{\partial \gamma} \int_R u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)} dx + R_k(t, v). \tag{45}
$$

In fact, Proposition 3.5 and (36) imply

$$
\left| \int_R u^+(t, x) \overline{\Psi_v(t, x)} dx \right| \leq \|u\|_{L^\infty} \|P^{-}\Psi_v(t)\|_{L^1} \lesssim (t^{-\frac{m-1}{m}} v)^{-1} \cdot t^{-\frac{1}{2\pi}} \|u(t)\|_{\tilde{X}}. \tag{46}
$$

Moreover, (30) yields

$$
\left| \int_R u^{\text{ell}}(t, x) \overline{\Psi_v(t, x)} dx \right| \lesssim \lambda^{-1} (t^{-\frac{m-1}{m}} v)^{-\frac{2m-3}{4m-4}} \|\langle t^{-\frac{1}{2\pi}} x \rangle \frac{2m-3}{4m-4} u^{\text{ell}}(t)\|_{L^\infty} \lesssim (t^{-\frac{m-1}{m}} v)^{-\frac{1}{2}} \cdot t^{-\frac{1}{2\pi}} \|u(t)\|_{\tilde{X}}.
$$

In addition, by (44) and (27), we have

$$
\left\| \int_R u^{\text{ell}}(t, x) \overline{\Psi_v(t, x)} dx \right\|_{L^2((0, \infty), L^2)} \lesssim t^{-\frac{m-2}{2(4m-4)}} \|\langle x \rangle \frac{m-2}{2(4m-4)} u^{\text{ell}}(t)\|_{L^2((0, \infty), L^2)} \lesssim t^{-\frac{m-2}{2m-2}} \|\langle t^{-\frac{1}{2\pi}} x \rangle \frac{m-2}{2(4m-4)} u^{\text{ell}}(t)\|_{L^2} \lesssim t^{-\frac{m-2}{2m-2}} \cdot t^{-\frac{1}{2\pi}} \|u(t)\|_{\tilde{X}}.
$$

Hence, from $\lambda v \frac{\partial^{k \ell}}{\partial \gamma} = t^{-\frac{k-1}{2m}} (t^{-\frac{m-1}{m}} v)^{\frac{k-1}{m}} \frac{\partial^{k \ell}}{\partial \gamma}$, we obtain (45).

Third, we observe that the equation

$$
i^k \lambda v \frac{\partial^{k \ell}}{\partial \gamma} \gamma(t, v) = \lambda \int_R \partial_x k u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)} dx + R_k(t, v). \tag{47}
$$

holds true. We note that

$$u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)} = -i v^{-\frac{1}{m-1}} \partial_x u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)}$$

$$- it^{\frac{k-1}{2m}} \left(x^{-\frac{1}{m-1}} - (vt)^{-\frac{1}{m-1}}\right) \partial_x u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)}$$

$$+ it^{\frac{1}{2m}} x^{-\frac{1}{m-1}} \partial_x (u^{\text{hyp.}+} e^{-i\phi})(t, x) (\lambda(x - vt)).$$

Here, (29) yields

$$v^{-\frac{k-1}{2m}} \left| \int_R t^{\frac{k-1}{2m}} \left(x^{-\frac{1}{m-1}} - (vt)^{-\frac{1}{m-1}}\right) \partial_x k u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)} dx \right|$$

$$\lesssim \frac{t^{k}}{(t^{-\frac{1}{2\pi}} v)^{-\frac{k-1}{2m-1}}} \int_R \left| \partial_x k u^{\text{hyp.}+}(t, x) (\lambda(x - vt)) \right| dx$$

$$\lesssim (t^{-\frac{m-1}{m}} v)^{-\frac{1}{2}} \cdot t^{-\frac{1}{2\pi}} \|u(t)\|_{\tilde{X}}.$$

By (44) and (25), we have

$$\left\| v^{-\frac{k-1}{2m}} \int_R t^{\frac{k-1}{2m}} \left(x^{-\frac{1}{m-1}} - (vt)^{-\frac{1}{m-1}}\right) \partial_x k u^{\text{hyp.}+}(t, x) \overline{\Psi_v(t, x)} dx \right\|_{L^2((0, \infty), L^2)}$$

$$\lesssim t^{-\frac{1}{2}} \left\| v^{-\frac{k-1}{2m}} \int_R x^{-\frac{k+1}{2m}} \partial_x k u^{\text{hyp.}+}(t, x) (\lambda(x - vt)) dx \right\|_{L^2((0, \infty), L^2)}.$$
By (27), (30), and (47), we have

\[ L, M \]

These estimates and (45) show (47).

With a change of variables using \( z \), we have

\[ w_k(t, x) := e^{-i\phi(t,x)} \partial_x^k u_{\text{bvp}+}(t, x) \]

By (27), (30), and (47), we have

\[ \partial_x^k u_{\text{bvp}}(t, v) - i^k \lambda v^{\frac{1}{m-1}} e^{i\phi(t,v)} \gamma(t, v) \]

\[ = \partial_x^k u_{\text{bvp}+}(t, v) - \lambda e^{i\phi(t,v)} \int \partial_x^k u_{\text{bvp}+}(t, x) \Psi_v(t, v) dx + R_k(t, v) \]

With a change of variables using \( z = \lambda(x - vt) \) and (26), we see that

\[ \int_{\mathbb{R}} \left| (w_k(t, v) - w_k(t, x)) \chi(\lambda(x - vt)) \right| dx \]

\[ \leq \lambda^{-1} \int_{\mathbb{R}} \left| (w_k(t, v) - w_k(t, \lambda^{-1}z + vt)) \chi(z) \right| dz \]

\[ = \lambda^{-2} \int_{\mathbb{R}} \left| \varphi w_k(t, vt + (1 - \theta)\lambda^{-1}z) d\theta \cdot z \chi(z) \right| dz \]

\[ \lesssim t^{-\frac{m-1}{2}} \lambda^{-\frac{1}{2}} (tv)^{\frac{k+m+2}{m-1}} \left|x\right|^{-\frac{k+m+2}{m-1}} \mathcal{J}_+ \partial_x^k u_{\text{bvp}+}(t) \right|_{L^2} \]

\[ \lesssim t^{-\frac{m-1}{2}} (tv)^{-\frac{k}{m-1}} + e^{\frac{1}{m-1}} \cdot t^{-\frac{1}{m-1}} \|u(t)\|_{\overline{X}}. \]

These estimates and (45) show (47).

We are now in position to prove (42). We set \( w_k(t, x) := e^{-i\phi(t,x)} \partial_x^k u_{\text{bvp}+}(t, x) \).
which shows the $L^2$-estimate in (42).

Next, we consider the estimates in the frequency spaces. Because
\[\left|\int_{-\infty}^{0} \lambda^{-1} \chi_1(\lambda^{-1}(\xi - \xi_v), \lambda^{-1} \xi_v) d\xi\right| = \left|\int_{-\infty}^{-\lambda^{-1} \xi_v} \chi_1(\zeta, \lambda^{-1} \xi_v) d\zeta\right| \leq C_2(t^{\frac{m-1}{m}} v)^{-\frac{m}{m-1}},\]

Proposition 3.5, Lemma 4.1, (36), and (46) yield
\[
\left|\hat{u}(t, \xi_v) - \sqrt{m-1} e^{-\frac{1}{m} \xi^m} \gamma(t, v)\right| \leq \left|\int_{\mathbb{R}^+} \left(\hat{u}(t, \xi_v)e^{\frac{1}{m} \xi^m} - \hat{u}(t, \xi)e^{\frac{1}{m} \xi^m}\right) \lambda^{-1} \chi_1(\lambda^{-1}(\xi - \xi_v), \lambda^{-1} \xi_v) d\xi\right| \\
+ (C_1 + C_2)(t^{\frac{m-1}{m}} v)^{-\frac{m}{m-1}} \left|\hat{u}(t, \xi_v)\right| + C(t^{\frac{m-1}{m}} v)^{-1} \cdot t^{-\frac{1}{2m}} \|u(t)\|_{\tilde{X}}.
\]

With a change of variables using $\zeta = \lambda^{-1}(\xi - \xi_v)$, we have
\[
\left|\int_{\mathbb{R}^+} \left(\hat{u}(t, \xi_v)e^{\frac{1}{m} \xi^m} - \hat{u}(t, \xi)e^{\frac{1}{m} \xi^m}\right) \lambda^{-1} \chi_1(\lambda^{-1}(\xi - \xi_v), \lambda^{-1} \xi_v) d\xi\right| \\
\leq \left|\int_{\mathbb{R}} |\xi - \xi_v| \int_{0}^{1} \left|\hat{F}u(t, \theta(\xi_v - \xi) + \xi)\right| d\theta \lambda^{-1} |\chi_1(\lambda^{-1}(\xi - \xi_v), \lambda^{-1} \xi_v)| d\xi\right| \\
= \lambda \int_{\mathbb{R}} \int_{0}^{1} \left|\hat{F}u(t, \xi_v + \lambda \zeta(1 - \theta))\right| d\theta |\chi_1(\zeta, \lambda^{-1} \xi_v)| d\zeta.
\]

Because
\[
|\hat{u}(t, \xi_v)| \leq \left|\hat{u}(t, \xi_v) - \sqrt{m-1} e^{-\frac{1}{m} \xi^m} \gamma(t, v)\right| + \sqrt{m-1} |\gamma(t, v)|
\]
and $\chi_1(\cdot, \alpha) \in \mathcal{S}(\mathbb{R})$ for $\alpha \geq 1$, by Proposition 3.5, (31), and (41), we have
\[
\left|\hat{u}(t, \xi_v) - \sqrt{m-1} e^{-\frac{1}{m} \xi^m} \gamma(t, v)\right| \\
\lesssim \lambda^\frac{1}{2} \|\hat{F}u(t)\|_{L^2} + (t^{\frac{m-1}{m}} v)^{-\frac{m}{m-1}} |\gamma(t, v)| + (t^{\frac{m-1}{m}} v)^{-1} \cdot t^{-\frac{1}{2m}} \|u(t)\|_{\tilde{X}}.
\]
\[
\lesssim (t^{\frac{m-1}{m}} v)^{-\frac{m-2}{m-1}} \cdot t^{-\frac{1}{2m}} \|u(t)\|_{\tilde{X}},
\]
which shows the $L^\infty$-estimate in (43).

For the $L^2$-estimate in the frequency space, we change variables using $\bar{v} = \xi_v + \lambda \zeta(1 - \theta)$. Because
\[
\frac{d\bar{v}}{dv} = \frac{1}{m-1} v^{-\frac{m}{m-1}} \left\{1 - \frac{m - 2}{2} \zeta(1 - \theta)t^{-\frac{1}{2}} v^{-\frac{m}{m-1}}\right\},
\]
(48) yields
\[
\left\|\hat{u}(t, \xi_v) - \sqrt{m-1} e^{-\frac{1}{m} \xi^m} \gamma(t, v)\right\|_{L^2_t(\Omega(t))} \\
\lesssim \int_{0}^{1} \lambda \int_{\mathbb{R}} \left|\hat{F}u(t, \xi_v + \lambda \zeta(1 - \theta))\right| |\zeta \chi_1(\zeta, \lambda^{-1} \xi_v)| d\zeta \left\|_{L^2_t(\Omega(t))} d\theta.
\]
Moreover, from (39), (36), Lemma 4.3, and Proposition 3.5, we have
\[
\|\nabla \gamma(t, v)\|_{L^2 (\Omega(t))} + t^{-\frac{m-1}{2m}} \cdot t^{-\frac{1}{2m}} \|u(t)\|_{\tilde{X}} \leq t^{-\frac{1}{2}} \|\mathcal{J}u(t, \tilde{v})\|_{L^2 (\Omega(t))} + t^{-\frac{m-1}{2m}} \cdot t^{-\frac{1}{2m}} \|u(t)\|_{\tilde{X}}
\]
which concludes the $L^2$-estimate in (43) \qed

5. Proof of the main theorem. We show the following estimate for $\dot{\gamma}$.

**Proposition 5.1.** Let $u$ be a solution to (1) that satisfies (10). Then, for $t \geq 1$, we have
\[
\|t^{\frac{m-1}{m}} v\|_{L^2 (\Omega(t))} + \|t^{\frac{m-1}{m}} \dot{\gamma}(t)\|_{L^2 (\Omega(t))} \lesssim \varepsilon,
\]
where the implicit constant is independent of $D$, $T$, and $\varepsilon$.

**Proof.** A direct calculation yields
\[
\dot{\gamma}(t, v) = \int e^{i\phi} (Lu - \overline{\nabla} u) \cdot \nabla \chi(t, x) \, dx.
\]
The bootstrap assumption (10) yields
\[
\int |u(\partial_x \overline{\nabla} u)(t, x)| \, dx \lesssim t^{-1} \|t^{\frac{m-1}{m}} v\|_{L^2 (\Omega(t))} \lesssim t^{-1} \|\gamma(t)\|_{L^2 (\Omega(t))} \lesssim \varepsilon.
\]
Moreover, from (39), (36), Lemma 4.3, and Proposition 3.5, we have
\[
\int |u(\partial_x \overline{\nabla} u)(t, x)| \, dx \lesssim t^{-1} \|t^{\frac{m-1}{m}} v\|_{L^2 (\Omega(t))} \lesssim t^{-1} \|\gamma(t)\|_{L^2 (\Omega(t))} \lesssim \varepsilon.
\]
Here, from (26), we obtain
\[
t^{\frac{m-1}{m}} |u^{hyp}(t, x)| \lesssim t^{-\frac{m-1}{m}} \|u(t)\|_{L^2 (\Omega(t))} \lesssim \varepsilon.
\]
In addition, we use (26) and (44) to obtain
\[
\|t^{\frac{m-1}{m}} \mathcal{J} u^{hyp}(t, x) \|_{L^2 (\Omega(t))} \lesssim \varepsilon.
\]
First, we prove global existence of the solution to (1). From Proposition 1.4 and Lemma 2.1, this is equivalent to showing (4), that is to say, to closing the bootstrap estimate (10). When \( t^{-\frac{1}{m}}|x| \lesssim 1 \), Lemma 2.3 and Proposition 3.5 yield
\[
\left\| \left( t^{-\frac{1}{m}}x \right)^{-\frac{k+2}{m-1}} \partial_x^k u(t) \right\|_{L^\infty(t^{-\frac{1}{m}}|x| \lesssim 1)} \lesssim \varepsilon t^{-\frac{k+1}{m}}.
\]
When \( t^{-\frac{1}{m}}|x| \gtrsim 1 \), owing to (42), we only need to show that
\[
\| \gamma(t) \|_{L^\infty(\Omega(t))} \lesssim \varepsilon,
\]
where the implicit constant is independent of \( D \), \( T \), and \( \varepsilon \).

For \( v \geq C_* \), where \( C_* \) is defined by (41), the Gagliardo-Nirenberg inequality, Proposition 1.4 and Lemma 4.1 lead to
\[
|\gamma(1,v)| \lesssim \| \hat{u}(1) \|_{L^\infty} = \left\| e^{\frac{i}{2}|k|^{m-1}x} \hat{u}(1) \right\|_{L^\infty} \lesssim \| u(1) \|_{L^\infty}^{\frac{1}{2}} \| \mathcal{J} u(1) \|_{L^2}^{\frac{1}{2}} \lesssim \varepsilon.
\]
The fundamental theorem of calculus and Proposition 5.1 yield
\[
\gamma(t,v) = \gamma(1,v) + O \left( \varepsilon (t^{-\frac{m-1}{m}}v)^{-\frac{m-2}{2(m-1)}} \right),
\]
which implies
\[
|\gamma(t,v)| \lesssim \varepsilon
\]
for \( t \in [1,T] \).

When \( 0 < v < C_* \), set \( t_0 := (C_*v^{-1})^{\frac{m}{m-1}} > 1 \). Note that \( v \in \Omega(t) \) for \( t \geq t_0 \). Then, Bernstein’s inequality, (36), Proposition 3.5, and Lemma 2.3 yield
\[
|\gamma(t_0,v)| \lesssim t_0^{\frac{1}{m}} \sum_{N \in 2^k, N \sim t_0^{-\frac{1}{m}}} \| u_N(t_0) \|_{L^2} + \varepsilon \lesssim \varepsilon.
\]
The fundamental theorem of calculus and Proposition 5.1 lead to
\[
\gamma(t,v) = \gamma(t_0,v) + O (\varepsilon),
\]
which implies
\[
|\gamma(t,v)| \lesssim \varepsilon
\]
for \( t \in [t_0,T] \). Accordingly, we conclude that (4) holds for any \( t \in [1,T] \).

Second, we show the existence of a self-similar solution. We use the self-similar change of variables (13). Let \( \rho > 0 \) be a constant specified later. We set \( \mathcal{Q}(t) := \{ y \in \mathbb{R} : |y| \leq C_* t^{(m-1)/\rho} \} \) and \( \mathfrak{C} := 4\alpha C_*^{(m-1)/\rho} \). For \( k = 0, 1, \ldots, m-2 \), estimates (15) and (21) and Lemmas 2.1 and 2.3 imply that
\[
\left\| \partial_t \left( P_{\leq \rho^\mu} \partial_y^k U \right) \right\|_{L^\infty(\mathcal{Q}(t))} \lesssim t^{(k+\frac{1}{2})\rho} \left\| P_{\leq \rho^\mu} \partial_Y U \right\|_{L^2} + t^{-1} \left\| P_{\rho^\mu \rho^{\frac{1}{2}}} \partial_y^k U \right\|_{L^\infty(\mathcal{Q}(t))} \lesssim t^{(k+\frac{1}{2})\rho - \frac{m-1}{m}} \| u \|_{L^2} + t^{(k+\frac{1}{2})\rho + \frac{m-1}{m}} \sum_{N \in 2^k, N \sim \rho^{-\frac{1}{m}}} \| u_N \|_{L^2} \lesssim \varepsilon t^{-1+(k+\frac{1}{2})\rho - \min\{\frac{m}{m-1},1\},\mu},
\]
Furthermore, (18), (21), and Lemma 2.3 yield
\[
\| P_{>\epsilon^l} \partial_y \varrho \|_{L_y^\infty(\Omega(t))} \lesssim t^{\frac{1}{2\gamma - 1}} \sum_{N \in 2^N} \frac{N^{k+\frac{1}{2}}}{N^2} \| u_N^{\text{ell},+} \|_{L^2}
\]
\[
+ t^{\frac{1}{2\gamma}} \sum_{N \in 2^N} \| \left(1 - P_{\frac{N}{2} \leq \cdot \leq 2N} \right) |\partial_x|^{k+\frac{1}{2}} u_N^{\text{ell},+} \|_{L^2}
\]
\[
\lesssim t^{(k-m+\frac{1}{2})\rho}\epsilon,
\]
\[
\| P_{>\epsilon^l} \partial_x \varrho \|_{L_x^2(\Omega(t))} \lesssim t^{\frac{1}{2\gamma}} + t^{\frac{1}{2\gamma - 1}} \sum_{N \in 2^N} \frac{N^{k}}{N^2} \| u_N^{\text{ell},+} \|_{L^2}
\]
\[
+ t^{\frac{1}{2\gamma}} \sum_{N \in 2^N} \| \left(1 - P_{\frac{N}{2} \leq \cdot \leq 2N} \right) |\partial_x|^{k+\frac{1}{2}} u_N^{\text{ell},+} \|_{L^2}
\]
\[
\lesssim t^{(k-m+1)\rho}\epsilon.
\]

By setting \( \rho := \frac{1}{m} (\frac{1}{2\gamma} - \epsilon) \), there exists \( Q = Q(y) \in L_y^\infty(\mathbb{R}) \) such that
\[
\| \partial_y^k U(t) - \partial_y^k Q \|_{L_y^\infty(\Omega(t))} \lesssim \epsilon t^{(k-m+\frac{1}{2})\rho},
\]
\[
\| \partial_x^k U(t) - \partial_x^k Q \|_{L_x^2(\Omega(t))} \lesssim \epsilon t^{(k-m+1)\rho}.
\]

By (4) and the first estimate in (49), we see that
\[
\left\| \langle \cdot \rangle^{\frac{m-2}{2(m-1)}} Q \right\|_{L_y^\infty(\mathbb{R})} \leq \lim_{t \to \infty} \left( t^{\frac{m-2}{2(m-1)}} \| Q - U(t) \|_{L_y^\infty(\Omega(t))} + \left\| \langle \cdot \rangle^{\frac{m-2}{2(m-1)}} U(t) \right\|_{L_y^\infty(\mathbb{R})} \right)
\]
\[
\lesssim \epsilon.
\]

Because Lemma 2.1 implies that
\[
\left\| \partial_y^m - U - yU - mU^m \right\|_{L_y^2} = \left\| (Au)(t, \frac{1}{t\gamma} y) \right\|_{L_y^2} \lesssim t^{-\frac{1}{m} + \epsilon},
\]
by taking the limit as \( t \to \infty \), we have that \( Q \) is a solution to (5). Moreover, (49) and the mass conservation law yield
\[
\int_{\mathbb{R}} Q(y)dy = \lim_{t \to \infty} \int_{\mathbb{R}} U(t, y)dy = \int_{\mathbb{R}} u_0(x)dx.
\]

Therefore, \( u(t, x) := t^{-\frac{1}{m}} Q(t^{-\frac{1}{m}} x) \) is a solution to (1) with \( u(0) = \int_{\mathbb{R}} u_0(x)dx \delta_0 \), where \( \delta_0 \) denotes the Dirac delta measure concentrated at the origin.

Finally, we prove the asymptotic behavior of the global solution. The estimates in the self-similar region \( X^0(t) \) follow from (49). Moreover, the estimates in the decaying region \( X^{-}(t) \) are consequences of Lemma 2.3, (27), and (30). Hence, we only need to show the estimates in the oscillatory region \( X^+(t) \). Proposition 5.1 yields
\[
\gamma(t, v) = \gamma(1, v) + \tilde{R}(t, v),
\]
where
\[
\left\| t^{\frac{m-1}{2}} \langle \cdot \rangle^{\frac{m-2}{2(m-1)}} \tilde{R}(t, v) \right\|_{L_y^\infty(\Omega(t))} + \left\| t^{\frac{m-1}{2}} \tilde{R}(t, v) \right\|_{L_x^2(\Omega(t))} \lesssim \epsilon.
\]

Here, we set
\[
W(\xi) := \sqrt{m - 1} \gamma(1, \xi^{m-1}),
\]
and extend \( W \) to \( \mathbb{R} \) by defining
\[
W(-\xi) = \overline{W(\xi)}, \quad W(0) = \int_{\mathbb{R}} u_0(x)dx.
\]
Then, from (4) and (44), we see that
\[
\|W\|_{L^\infty \cap L^2} \lesssim \|u(1)\|_{L^\infty \cap L^2} \lesssim \epsilon.
\]
Proposition 4.4 and (52) show the estimates in \( \mathcal{X}^+(t) \).

**Appendix A. Well-posedness.** In this appendix, we show the local-in-time well-posedness of the Cauchy problem of (6) as well as (1). Here, we assume \( p \geq m \geq 3 \).

To show well-posedness for (1), we can apply the Fourier restriction norm method. In fact, Grünrock [4] proved well-posedness for (1) in \( H^s(\mathbb{R}) \) for odd values of \( m \geq 5 \) and \( s > -\frac{1}{2} \). On the other hand, because \( F \) may not be a polynomial with respect to \( u \), some regularity is needed to be well-posed for (6).

We need to introduce some notation. Let \( 1 \leq p, q \leq \infty \) and \( T > 0 \). Define
\[
\|f\|_{L^p_T L^q_x} := \left( \int_{-\infty}^{\infty} \left( \int_{-T}^{T} |f(t,x)|^p dt \right)^{\frac{q}{p}} dx \right)^{\frac{1}{q}},
\]
\[
\|f\|_{L_{x}^m L_{t}^p} := \left( \frac{1}{\int_{-\infty}^{\infty} \left( \int_{-T}^{T} |f(t,x)|^p dx \right)^{\frac{q}{p}} dt} \right)^{\frac{1}{q}},
\]
with \( T = t \) to indicate the case when \( T = \infty \).

The maximal function estimate and the local smoothing estimate are the main tools in the proof (see Theorems 2.5 and 4.1 in [18] respectively).

**Theorem A.1.**
\[
\|U(t)u_0\|_{L^p_T L^q_x} \lesssim \left\| \partial_x A_{1/2} u_0 \right\|_{L^2}, \quad \left\| \partial_x A^{\frac{p-1}{2}} U(t)u_0 \right\|_{L^p_T L^q_x} \lesssim \|u_0\|_{L^2}.
\]

Stein’s interpolation [27] and Theorem A.1 yield the following.

**Lemma A.2.** For any value of \( \alpha \) where \( -\frac{1}{2} \leq \alpha \leq \frac{m-1}{2} \), we have
\[
\left\| \partial_x A^{\alpha} U(t)u_0 \right\|_{L^p T^q (L^1 T^1)} \lesssim \|u_0\|_{L^2}.
\]
In particular, by setting \( \alpha = 0 \) or \( \alpha = 1 \), it follows that
\[
\|U(t)u_0\|_{L^p T^q (L^1 T^1)} \lesssim \|u_0\|_{L^2}.
\]

Moreover, setting \( \alpha = \frac{p+m-3}{2(p-1)} \), we have
\[
\left\| \partial_x A(t)u_0 \right\|_{L^{p_0 T^q} L^1_T} \lesssim \left\| \partial_x A^{\frac{p+m-1}{2(p-1)}} u_0 \right\|_{L^2},
\]
where \( (p_0, q_0) := \left( \frac{2(2m-1)(p-1)}{(m-2)(p-2)}, \frac{2(2m-1)(p-1)}{3p + 2m - 7} \right) \).

Our next result is a Sobolev type of estimate (the proof is the same as that of Lemma 3.15 in [19]).

**Lemma A.3.** For any \( q \in \mathbb{R} \) with \( \max \left( \frac{-p + 2m - 3}{2(2m - 1)(p - 1)}, 0 \right) < \frac{1}{q} < \frac{m-2}{2m(p-1)} \), we have
\[
\|g\|_{L^p T^q (L^1_T)} \lesssim \left\| \partial_x A^{\alpha(q)} g \right\|_{L^{p_0 T^q} L^1_T},
\]
where \( \frac{1}{r(q)} := -\frac{m}{q} + \frac{m-2}{2(p-1)} \) and \( \alpha(q) := \frac{2m-1}{2q} - \frac{m-2}{2(p-1)} + \frac{1}{q} \).
Proof. The assumption implies that $0 < \alpha(q) < 1$. Fix $t$ now and use the fractional integration in $x$ to obtain the representation

$$g(t, x) = c_q \int_{-\infty}^{\infty} \frac{1}{|x - y|^{1 - \alpha(q)}} \left(|\partial_x|^{\alpha(q)}g\right)(y) dy.$$  

By Minkowski’s inequality, we have

$$\|g(x)\|_{L^q_t} \lesssim \int_{-\infty}^{\infty} \frac{1}{|x - y|^{1 - \alpha(q)}} \left\|\left(|\partial_x|^{\alpha(q)}g\right)(y)\right\|_{L^q_t} dy.$$  

From $\alpha(q) = \frac{q - 2}{4q} - \frac{1}{r(q)}$, Hardy-Littlewood-Sobolev’s inequality yields the desired bound.  

Lemma A.2 with $\alpha = \frac{2m - 1}{2q} - \frac{1}{q}$ implies

$$\left\|\left|\partial_x|^{\alpha(q)}U(t)u_0\right\|_{L^2_t} \lesssim \left\|\left|\partial_x|^{\frac{2m - 1}{2q}}u_0\right\|_{L^2} \right.\tag{55}$$

provided that $q \in [2, \infty]$.

Setting $q_1 := \frac{(2m - 1)(p - 1)}{m - 3}$, $q_2 := \frac{2(2m - 1)(p - 1)(p - 2)}{2m - 3(p - 4m + 9)}$, and $q_3 := \frac{2mp(p - 1)}{2m - 3p + 1}$, we define

$$\|u\|_{Y_p} := \|u\|_{L^q_t L^p_x} + \|u\|_{L^q_t L^p_x} + \|\partial_x u\|_{L^q_t L^p_x},$$  

$$\|u\|_{\bar{Y}_p} := \|\partial_x u\|_{L^q_t L^p_x} + \sum_{j=1}^{3} \left\|\left|\partial_x|^{\alpha(j)}u\right|\|_{L^q_t L^p_x} \right.\tag{56}$$

for $p \geq m \geq 3$, $s \in \mathbb{R}$, and $T > 0$.

The linear estimates (53), (54), and (55) lead to

$$\|U(t)u_0\|_{Y_p} \lesssim \|u_0\|_{L^2}, \quad \|U(t)u_0\|_{\bar{Y}_p} \lesssim \|u_0\|_{H^{\frac{2m - 1}{2mp - 1}}}.\tag{56}$$

We note that if $u$ is a solution to (6), then

$$\mathcal{L}u = (mu^{m - 1} + F'(u)) \partial_x \Delta u + mF(u) - F'(u)u.\tag{57}$$

**Proposition A.4.** Let $p \geq m \geq 3$ and $\frac{2m - 1}{2(p - 1)} \leq s < 1$. If $u_0 \in \Sigma^s(\mathbb{R})$, then there exists a value $T > 0$ that depends on $\|u_0\|_{\Sigma^s}$ and a unique solution $u \in Z^s_T$ to (6) satisfying

$$\sup_{t \in [-T, T]} (\|u(t)\|_{H^s} + \|\Delta u(t)\|_{L^2}) \lesssim \|u_0\|_{\Sigma^s}.$$  

Moreover, the flow map $u_0 \in \Sigma^s(\mathbb{R}) \mapsto u \in Z^s_T$ is locally Lipschitz continuous.
Proof. The fractional Leibnitz and chain rules (see Appendix in [19]) and Lemma A.3 yield
\[
\|\partial_x^s (F'(u) \partial_x u)\|_{L_T^{p'}} \leq \|\partial_x^s u\|_{L_T^{p''}} \|F'(u)\|_{L_T^{p''}} + \|\partial_x u\|_{L_T^{p''}} \|u\|^{p-2}_{L_T^{p''}} \leq \|\partial_x^s u\|_{Y_T^m} + \|\partial_x u\|_{L_t^{p''} L_T^{p''}} \|u\|^{p-2}_{L_T^{p''}} \leq \|\partial_x^s u\|_{Y_T^m} \|u\|^{p-1}_{Y_T^m}.
\]
(58)

In addition, Hölder’s inequality and Lemma A.3 imply
\[
\|u\|^p_{L_T^{p'}} = \|u\|^p_{L_T^{p'}} \lesssim T^\frac{1}{p'} \|u\|^p_{L_T^{p''} L_T^{p''}} \lesssim T^\frac{1}{p'} \|u\|^p_{Y_T^m}.
\]
(59)

We define the operator \(K_u_0(u)\) by
\[
K_u_0(u) = U(t)u_0 + \int_0^t U(t-t')\partial_x (u^m + F(u))(t')dt'.
\]

By (57), the estimates (56), (58), and (59) show that
\[
\|K_u_0(t)\|_{Z_T^m} \lesssim \|U(t)u_0\|_{Z_T^m} + \int_{-T}^T \left( \|u^{m-1}\partial_x u\|_{H_T^m} + \|u^{m-1}\partial_x u\|_{L_T^2} \right) dt' \lesssim \|u_0\|_{H_T^m} + \|x u_0\|_{L_T^2} + T^{\frac{1}{2}} \left( \|\partial_x^s (u^{m-1}\partial_x u)\|_{L_T^{p'}} + \|u^{m-1}\partial_x u\|_{L_T^{2}} \right) \lesssim C\|u_0\|_{Z_T^m} + C T^{\frac{1}{2}} \|u\|_{Z_T^m} \left( \|u\|^{m-1}_{Z_T^m} + \|u\|^{p-1}_{Y_T^m} \right).
\]

A similar calculation as above yields
\[
\|K_u_0(u_1) - K_u_0(u_2)\|_{Z_T^m} \lesssim C T^{\frac{1}{2}} \left( \|u_1\|^{m-1}_{Y_T^m} + \|u_2\|^{m-1}_{Y_T^m} + \|u_1\|^{p-1}_{Y_T^m} + \|u_2\|^{p-1}_{Y_T^m} \right) \|u_1 - u_2\|_{Z_T^m}.
\]

Hence, taking \(T > 0\) with
\[
10 C T^{\frac{1}{2}} \left( \left(2C\|u_0\|_{Z_T^m} \right)^{m-1} + \left(2C\|u_0\|_{Y_T^m} \right)^{p-1} \right) \leq 1,
\]
we find that the mapping \(K_u_0\) is a contraction on the ball
\[
B(2C\|u_0\|_{Z_T^m}) := \{u \in Z_T^m : \|u\|_{Z_T^m} \leq 2C\|u_0\|_{Z_T^m} \}.
\]

Accordingly, there exists a unique solution \(u \in B(2C\|u_0\|_{Z_T^m})\) to (6).  \(\square\)
Appendix B. Short-range perturbations. In this appendix, we outline some modifications to Theorem 1.1 in the case of short-range perturbations (6). The main differences appear in the energy estimate.

Let $p$ be a real number with $p > m$. The existence of a local-in-time solution $u$ with $U(-t)u(t) \in \Sigma^{\frac{2m}{2m-1}}(\mathbb{R})$ follows from Proposition A.4 and $H^{\frac{2m}{2m-1}}(\mathbb{R}) \hookrightarrow L^p(\mathbb{R})$.

We need some modifications in the energy estimate for $\Lambda u$ when $p \in (m, m + \frac{1}{2})$, because the second and third terms on the right hand side of (57) do not have enough decay. Set $a := \max(\frac{2m+1-2p}{2m}, 0)$. Because (10) yields

$$\|mF(u) - F'(u)u\|_{L^2} \lesssim \|u^p\|_{L^2},$$

we have

$$\frac{1}{2} \partial_t (t^{-a}\|\Lambda u(t)\|_{L^2})^2$$

$$= -at^{-2a-1}\|\Lambda u(t)\|_{L^2}^2 - \frac{m(m-1)}{2}t^{-2a} \int_{\mathbb{R}} u^{m-2}\partial_x u(\Lambda u)^2 dx$$

$$- \frac{1}{2}t^{-2a} \int_{\mathbb{R}} F''(u)\partial_x u(\Lambda u)^2 dx + O((D\varepsilon)^p t^{-2a-\frac{2p-1}{2m}}\|\Lambda u(t)\|_{L^2})$$

$$\lesssim (D\varepsilon)^{m-1}t^{-1}(t^{-a}\|\Lambda u(t)\|_{L^2})^2 + (D\varepsilon)^{2p-m+1}t^{-2a+1-\frac{2p-1}{2m}}.$$  

Hence, Gronwall’s inequality shows

$$\|u(t)\|_{X} \lesssim \varepsilon(t)^{a+\varepsilon}.$$  

From $a < \frac{1}{2m}$, Lemma 2.3 holds true as long as $\|u_0\|_{\Sigma^{\frac{m}{m-1}}} \lesssim \varepsilon \ll 1$.

Moreover, (51) is replaced with

$$\|\partial_y\|^{m-1}U - yU - muU\|_{L^p_y} \lesssim \|\Lambda u(t, \frac{t}{\varepsilon} y) + tF(u(t, \frac{t}{\varepsilon} y))\|_{L^p_y} \lesssim t^{-\frac{1}{2m}+a+\varepsilon}.$$

The remaining arguments in §5 are unchanged as long as $\rho = \frac{1}{m}(\frac{1}{2m} - a - \varepsilon)$ is replaced with $\rho := \frac{1}{m}(\frac{1}{2m} - a - \varepsilon)$.

Acknowledgments. This work was supported by JSPS KAKENHI Grant number JP16K17624 and Alumni Association “Wakasatokai” of Faculty of Engineering, Shinslu University.

REFERENCES

[1] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems. Asymptotics for the MKdV equation, Ann. of Math. (2), 137 (1993), 295–368.

[2] B. Dodson, Global well-posedness and scattering for the defocusing, mass-critical generalized KdV equation, Ann. PDE, 3 (2017), Art. 5, 35 pp.

[3] P. Germain, F. Pusateri and F. Rousset, Asymptotic stability of solitons for mKdV, Adv. Math., 299 (2016), 272–330.

[4] A. Grünrock, On the hierarchies of higher order mKdV and KdV equations, Cent. Eur. J. Math., 8 (2010), 500–536.

[5] B. Harrop-Griffiths, Long time behavior of solutions to the mKdV, Comm. Partial Differential Equations, 41 (2016), 282–317.

[6] B. Harrop-Griffiths, M. Ifrim, and D. Tataru, The lifespan of small data solutions to the KP-I, Int. Math. Res. Not. IMRN, 2017, 1–28.
ASYMPTOTIC BEHAVIOR FOR A KDV-TYPE EQ

[7] N. Hayashi and P. I. Naumkin, Large time asymptotics of solutions to the generalized Korteweg-de Vries equation, J. Funct. Anal., 159 (1998), 110–136.
[8] N. Hayashi and P. I. Naumkin, Large time behavior of solutions for the modified Korteweg-de Vries equation, Internat. Math. Res. Notices, 1999, 395–418.
[9] N. Hayashi and P. I. Naumkin, On the modified Korteweg-de Vries equation, Math. Phys. Anal. Geom., 4 (2001), 197–227.
[10] N. Hayashi and P. I. Naumkin, Large time asymptotics for the fourth-order nonlinear Schrödinger equation, J. Differential Equations, 258 (2015), 880–905.
[11] N. Hayashi and P. I. Naumkin, Factorization technique for the fourth-order nonlinear Schrödinger equation, Z. Angew. Math. Phys., 66 (2015), 2343–2377.
[12] N. Hayashi and P. I. Naumkin, Global existence and asymptotic behavior of solutions to the fourth-order nonlinear Schrödinger equation in the critical case, Nonlinear Anal., 116 (2015), 112–131.
[13] N. Hayashi and P. I. Naumkin, Factorization technique for the modified Korteweg e Vries equation, SUT J. Math., 52 (2016), 49–95.
[14] H. Hirayama and M. Okamoto, Well-posedness and scattering for fourth order nonlinear Schrödinger type equations at the scaling critical regularity, Commun. Pure Appl. Anal., 15 (2016), 831–851.
[15] M. Iftim and D. Tataru, Global bounds for the cubic nonlinear Schrödinger equation (NLS) in one space dimension, Nonlinearity, 28 (2015), 2661–2675.
[16] T. Kato and G. Ponce, Commutator estimates and the Euler and Navier-Stokes equations, Comm. Pure Appl. Math., 41 (1988), 891–907.
[17] C. E. Kenig, G. Ponce and L. Vega, On the (generalized) Korteweg-de Vries equation, Duke Math. J., 59 (1999), 585–610.
[18] C. E. Kenig, G. Ponce and L. Vega, Oscillatory integrals and regularity of dispersive equations, Indiana Univ. Math. J., 40 (1991), 33–69.
[19] C. E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle, Commun. Pure Appl. Math., 46 (1993), 527–620.
[20] C. E. Kenig, G. Ponce and L. Vega, On the hierarchy of the generalized KdV equations, Singular Limits of Dispersive Waves (Lyon, 1991), 347–356, NATO Adv. Sci. Inst. Ser. B Phys., 320, Plenum, New York, 1994.
[21] C. E. Kenig, G. Ponce, and L. Vega, On the concentration of blow up solutions for the generalized KdV equation critical in $L^2$, Nonlinear Wave Equations (Providence, RI, 1998), 131–156, Contemp. Math., 263, Amer. Math. Soc., Providence, RI, 2000.
[22] H. Koch and J. Marzuola, Small data scattering and soliton stability in $\dot{H}^{-1/6}$ for the quartic KdV equation, Anal. PDE, 5 (2012), 145–198.
[23] F. Merle, Existence of blow-up solutions in the energy space for the critical generalized KdV equation, J. Amer. Math. Soc., 14 (2001), 555–578.
[24] M. Okamoto, Large time asymptotics of solutions to the short-pulse equation, NoDEA Nonlinear Differential Equations Appl., 24 (2017), Art. 42, 24 pp.
[25] M. Okamoto, Long-time behavior of solutions to the fifth-order modified KdV-type equation, Adv. Differential Equations, 23 (2018), 751–792.
[26] A. Sidi, C. Sulem and P. L. Sulem, On the long time behaviour of a generalized KdV equation, Acta Appl. Math., 7 (1986), 35–47.
[27] E. M. Stein, Interpolation of linear operators, Trans. Amer. Math. Soc., 83 (1956), 482–492.
[28] T. Tao, Scattering for the quartic generalised Korteweg-de Vries equation J. Differential Equations, 232 (2007), 623–651.

Received June 2018; revised November 2018.
E-mail address: m.okamoto@shinshu-u.ac.jp