Black ring formation in particle systems

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(Dated: March 24, 2022)

Abstract

It is known that the formation of apparent horizons with non-spherical topology is possible in higher-dimensional spacetimes. One of these is the black ring horizon with $S^1 \times S^{D-3}$ topology where $D$ is the spacetime dimension number. In this paper, we investigate the black ring horizon formation in systems with $n$-particles. We analyze two kinds of system: the high-energy $n$-particle system and the momentarily-static $n$-black-hole initial data. In the high-energy particle system, we prove that the black ring horizon does not exist at the instant of collision for any $n$. But there remains a possibility that the black ring forms after the collision and this result is not sufficient. Because calculating the metric of this system after the collision is difficult, we consider the momentarily-static $n$-black-hole initial data that can be regarded as a simplified $n$-particle model and numerically solve the black ring horizon that surrounds all the particles. Our results show that there is the minimum particle number that is necessary for the black ring formation and this number depends on $D$. Although many particle number is required in five-dimensions, $n = 4$ is sufficient for the black ring formation in the $D \geq 7$ cases. The black ring formation becomes easier for larger $D$. We provide a plausible physical interpretation of our results and discuss the validity of Ida and Nakao’s conjecture for the horizon formation in higher-dimensions. Finally we briefly discuss the probable methods of producing the black rings in accelerators.

PACS numbers: 04.50.+h, 04.70.Bw, 04.20.Ex, 11.10.Kk

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I. INTRODUCTION

There have been two recent proposals known as the brane world scenarios (the ADD scenario and the RS scenario), which attempt to solve the hierarchy problem. In these scenarios, our three-dimensional space is the 3-brane embedded in extra-dimensions. In the ADD scenario, the extra-dimensions are flat and far larger compared to the Planck length. Accordingly the Planck energy $M_P$ determined by the higher-dimensional gravitational constant $G_D$ becomes smaller than the four-dimensional Planck energy $M_4 = 10^{18}\text{GeV}$, and it could be as low as $O(\text{TeV})$. In the RS scenario, there are two branes with positive and negative tension respectively in a five-dimensional spacetime, and the bulk is filled with a negative cosmological constant $\Lambda$. Due to this negative $\Lambda$, the extra-dimension has a warped configuration whose effective volume for observers in the negative tension brane becomes large and this also leads to the small Planck energy.

If the Planck energy is at $O(\text{TeV})$, the properties of the black holes smaller than the scale of the extra-dimensions are totally altered. A black hole is centered on the brane but is extending out into the large extra-dimensions. Hence it becomes a higher-dimensional black hole. Moreover, its gravitational radius is far larger than the four-dimensional gravitational radius with the same mass. It was pointed out that the black hole production in the future-planned accelerators would be possible in this case. In the particle collision with the energy above the Planck scale, the gravity becomes the dominant interaction and the semi-classical treatment is expected to become valid, although recently a counter-statement which is based on the generalized uncertainty principle has been proposed. If we assume that the maximal impact parameter for the black hole production has the same order as the gravitational radius corresponding to the central energy, the cross section becomes sufficiently large to produce one black hole per one second at the Large Hadron Collider (LHC) in CERN. Because such black holes rapidly evaporate due to the Hawking radiation and they radiate mainly on the brane, we are able to detect the signals of the produced black holes at accelerators. The behavior of the produced black holes and the expected signals are first discussed by Dimopoulos and Landsberg and Thomas and Giddings, and subsequently a great amount of papers concerning the black holes at accelerators appeared.

One of the necessary investigations to improve the experimental estimates would be to analyze the black hole formation in the high-energy particle collisions in higher-dimensions.
As a first step, the approximation neglecting the tension of the brane and the structure of the extra-dimensions would be appropriate if we consider the case where the central energy of the particle system is far above the Planck scale and the gravitational radius of the system is smaller compared to the characteristic scale of the extra-dimensions. Eardley and Giddings analytically investigated the apparent horizon formation in this process in the four-dimensional spacetime [8]. Subsequently we investigated the head-on collision in higher-dimensions analytically [9] and the grazing collision in higher-dimensions with numerical calculations [10]. Our results have shown that the cross section for the black hole production is somewhat larger compared to the previous estimates.

The above investigations were concerning the apparent horizons with spherical topology. However it is known that the black hole horizon topology in higher-dimensions are not restricted only to spherical topology. Emparan and Reall generalized the Weyl solution and one of the solution they derived was a five-dimensional, asymptotically-flat, static black hole with horizon topology \( S^1 \times S^2 \) [11], although this spacetime has a conical singularity that supports the ring horizon and prevents it from collapsing. Subsequently they found the rotating version of this black hole solution without the conical singularity, which was named the black ring [12]. Hence in the five-dimensional spacetime, the black holes are not only the generalized Kerr black holes found by Myers and Perry [13], and there is no uniqueness theorem for stationary black hole solutions analogous to the four-dimensional case (see [14] for static cases). Although other solutions with non-spherical topology horizons have not been found, many such solutions are expected to exist. Ida and Nakao [15] discussed the possible horizon topology using the correlation between the power of the Newtonian potential and the apparent horizon formation. According to their discussion, the apparent horizon with torus topology \( T^{D-2} = S^1 \times \cdots \times S^1 \) would be prohibited for all the spacetime dimension numbers \( D \). However various topology horizons such as \( S^1 \times S^{D-3} \) would be allowed to form. Because these black holes with non-spherical horizon topology have features that are characteristic to the higher-dimensional gravity, whether we can produce such black holes would be an interesting theme. In this paper, we would like to analyze the formation of the black ring horizon (i.e. the horizon with topology \( S^1 \times S^{D-3} \)) in particle systems.

Whether the black ring can become the final state depends on its stability. There are many evidences that the black rings are unstable. First, if the angular momentum of the black ring is sufficiently large (i.e. the parameter \( \nu \) in [12] is close to zero), the Gregory-
Laflamme instability occurs because the radius of the ring around $S^2$ becomes small while the radius around $S^1$ becomes large. Next, due to the fact that the sequence of the black ring solution has the minimum value of the angular momentum normalized by an appropriate power of the mass, the black ring would be unstable around this configuration because if one put some mass into the black ring, the mass would increase and the angular momentum would not change and thus there is no black ring solution that corresponds to this situation. Finally, the rotating matter with ring configuration is not stable in the higher-dimensional Newtonian gravity. If one write the effective potential as a function of the ring radius, there is only one unstable point and is no stable point for $D \geq 6$, and there is only a marginally stable configuration for $D = 5$. All these situations support the instability of the black rings for all the parameter range, although the explicit analysis of the black ring perturbation has not been done. However, even if we admit the instability of the black ring, the black ring might form and exist until it changes to the other final state. Ida, Oda, and Park discussed how the black ring evolves if it forms in a high-energy two-particle system and if it traps almost all the energy and the angular momentum of the system using the Newtonian approximation. According to their discussion, such a black ring increases its radius around $S^1$ and expands until the Gregory-Laflamme instability occurs. If it really happens, we might be able to observe some signals from the black rings in accelerators.

In this paper, we investigate the black ring formation in $n$-particle systems for $n \geq 2$, because intuitively the black rings would be easier to form in the system with more particles. We would like to derive the condition for the black ring formation in terms of the particle number $n$ and the location of particles. We analyze two kinds of system: the high-energy particle system which we previously used and the $n$-body black hole initial data. First we will begin with the analysis of the high-energy particle systems. The metric of the spacetime with one high-energy particle has been written down by Aichelburg and Sexl by boosting the Schwarzschild black hole to the speed of light fixing the energy. The resulting spacetime is a massless point particle accompanied by a gravitational shock wave which distributes in the transverse direction of motion. The high-energy two-particle system can be introduced by combining two shock waves. In our previous paper, we analyzed the apparent horizon on the union of two incoming shock waves (i.e. at the instant of collision) and showed that the black hole horizon (i.e. the horizon with topology $S^{D-2}$)
forms. In this paper, we will introduce the equation for the black ring horizon on this slice. Unfortunately, we can easily show that there is no solution of the black ring horizon on this spacetime slice. Intuitively, it is natural to expect that the black ring would form in a high-energy $n$-particle system if $n$ is sufficiently large. But also in the $n$-particle system, we can show that there is no black ring horizon on the union of $n$ incoming shock waves. These results don’t mean that the black ring cannot form in high-energy particle systems: they just mean that there is no black ring horizon at the instant of collision and there remains the possibility of its formation after the collision of the shocks. To obtain the rigorous answer, we should analyze the spacetime slice after the collision of the shocks. However, this would be extremely difficult because the shocks nonlinearly interact after the collision and no one has succeeded to calculate the metric after the collision with the impact parameter. Instead, we next analyze the more simplified situation, the conformally-flat momentarily-static (i.e. time-symmetric) $n$-black-hole initial data originally introduced by Brill and Lindquist in the four-dimensional case [21]. This initial data also can be regarded as a $n$-particle system because it is the solution of the Hamiltonian constraint of the Einstein equation with $\delta$-function sources. Although this setting might be too simplified especially because each particle is not moving, this initial data provides the $n$ particles that are interacting each other and we would be able to obtain a lot of implications about the black ring formation in higher-dimensions. We numerically solve the apparent horizon with $S^1 \times S^{D-3}$ topology that surrounds all the particles, and derive the minimum number $n_{\text{min}}$ that is necessary for the black ring formation for each spacetime dimension number $D$. Using these results, we would like to discuss the feature of the black ring formation in higher-dimensions.

This paper is organized as follows. In Sec. II, we investigate the high-energy particle system and show that the black ring horizon does not exist at the instant of collision. In Sec. III, we analyze the momentarily-static $n$-black-hole system. We explain the numerical methods of finding apparent horizons with $S^1 \times S^{D-3}$ and $S^{D-2}$ topology, and show the numerical results. We also provide a plausible physical interpretation for our results and discuss the validity of Ida and Nakao’s conjecture [15] for the apparent horizon formation in higher-dimensions. Sec. IV is devoted to summary and discussion. The brief discussion concerning the probable methods of black ring production in accelerators is also included.
II. HIGH-ENERGY PARTICLE SYSTEM

In this section, we investigate the existence of the black ring horizon in the high-energy $n$-particle system. First we analyze the two-particle system, in which we previously investigated the black hole formation [10]. The set up of the system is as follows. To obtain the metric of a high-energy one-particle system, we boost the $D$-dimensional Schwarzschild solution to the speed of light. The Schwarzschild metric is given by

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}r^{D-3}}\right)dt^2 + \left(1 - \frac{16\pi G_D M}{(D-2)\Omega_{D-2}r^{D-3}}\right)^{-1}dr^2 + r^2d\Omega_{D-2}^2, \\
&\text{(1)}
\end{aligned}$$

where $d\Omega_{D-2}$ and $\Omega_{D-2}$ are the line element and the $(D-2)$-area of the $(D-2)$-dimensional unit sphere respectively, and the horizon is located at $r = r_h(M)$ where

$$r_h(M) \equiv \left(\frac{16\pi G_D M}{(D-2)\Omega_{D-2}}\right)^{1/(D-3)}. \quad (2)$$

By boosting this metric in $z$-direction and taking a lightlike limit fixing the energy $\mu = M\gamma$, we obtain the Aichelburg-Sexl metric [19]

$$\begin{aligned}
ds^2 &= -d\bar{u}d\bar{v} + \sum_{i=1}^{D-2}d\bar{x}_i^2 + \Phi(\bar{x}_i)d\bar{u}^2, \\
&\text{(3)}
\end{aligned}$$

where $\bar{u} = \bar{t} - \bar{z}$ and $\bar{v} = \bar{t} + \bar{z}$. $\Phi$ depends only on the transverse radius $\bar{\rho} = \sqrt{\sum_{i=1}^{D-2} \bar{x}_i^2}$ and takes the form

$$\Phi(\bar{x}_i) = \begin{cases} 
-8G_4\mu \log \bar{\rho}, & \text{for } D = 4, \\
16\pi\mu G_D/\Omega_{D-3}(D-4)\bar{\rho}^{D-4}, & \text{for } D > 4.
\end{cases} \quad (4)$$

The metric (3) is flat except $\bar{u} = 0$ and the $\delta$ function in eq. (3) indicates that there is a gravitational shock wave propagating in the flat spacetime at the speed of light. Mathematically, the origin of this $\delta$ function is due to the fact that the two flat-spacetime coordinate systems $(\bar{u}, \bar{v}, \bar{x}_i)$ are discontinuously connected at $\bar{u} = 0$. The continuous coordinate system $(u, v, x_i)$ can be introduced by

$$\begin{aligned}
\bar{u} &= u, \\
\bar{v} &= v + \Phi(x_i)\theta(u) + \frac{u}{4}\theta(u)\sum_k \left(\frac{\partial \Phi(x_i)}{\partial x_k}\right)^2, \\
\bar{x}_j &= x_j + \frac{u}{2}\theta(u) \left(\frac{\partial \Phi(x_i)}{\partial x_j}\right),
\end{aligned} \quad (5-7)$$
where $\theta(u)$ is the Heaviside step function. In this coordinate system, the metric is represented without the $\delta$ function:

$$ds^2 = -dudv + H_{ik}H_{jk}dx_idx_j,$$

where

$$H_{ij} = \delta_{ij} + \frac{u}{2}\theta(u)\frac{\partial^2 \Phi}{\partial x_i \partial x_j}.$$

We can superpose two solutions to obtain the exact geometry outside the future light cone of the collision of the shocks:

$$ds^2 = -dudv + \left( H_{ik}^{(+)} H_{jk}^{(+)} + H_{ik}^{(-)} H_{jk}^{(-)} - \delta_{ij} \right) dx_idx_j,$$

where

$$H_{ij}^{(+)} = \delta_{ij} + \frac{u}{2}\theta(u)\frac{\partial^2 \Phi^{(+)}(x)}{\partial x_i \partial x_j},$$

$$H_{ij}^{(-)} = \delta_{ij} + \frac{v}{2}\theta(v)\frac{\partial^2 \Phi^{(-)}(x)}{\partial x_i \partial x_j}.$$

Here $x$ is the point in the $(D-2)$-space $(x_i)$ and $\Phi^{(\pm)}(x)$ are defined as $\Phi^{(\pm)}(x) \equiv \Phi(x - x_{\pm})$ where $x_{\pm}$ denote the locations of the incoming particles in this subspace and we set

$$x_{\pm} = (\pm b/2, 0, ..., 0),$$

where $b$ is the impact parameter. This system was originally introduced by D’Eath and Payne in the four-dimensional case \cite{20}.

Eardley and Giddings \cite{8} derived the equation of the apparent horizon on the union of the incoming shock waves, i.e., $v \leq 0, u = 0$ and $u \leq 0, v = 0$. We briefly review this method because it is useful also for the black ring horizon case. They assumed that the horizon is given by the union of $(D-2)$-surfaces $S_+$ in $v \leq 0, u = 0$ and $S_-$ in $u \leq 0, v = 0$ that are given by $v = -\Psi^{(+)}(x_i)$ and $u = -\Psi^{(-)}(x_i)$ respectively. The surfaces $S_{\pm}$ are combined at $(D-3)$-surface $C$ in $u = v = 0$. Because there is a symmetry in this system, we have only to consider the $S_+$ surface. Although the function $\Psi^{(+)}$ diverges at $x = x_+$, the surface is continuous at that point if one see in the $(\bar{u}, \bar{v}, \bar{x}_i)$ coordinate. In this coordinate, the surface $S_+$ is located at

$$\bar{v} = \Phi^{(+)}(x_i) - \Psi^{(+)}(x_i) \equiv h(x_i),$$

where $x_i = \bar{x}_i$ is satisfied on $u = 0$. The outgoing null normal for this surface is given by

$$k^a = \left( 1, \frac{1}{4} \sum_k \left( \frac{\partial h}{\partial \bar{x}_k} \right)^2, \frac{1}{2} \left( \frac{\partial h}{\partial \bar{x}_i} \right) \right),$$

where

$$ \theta(u) \equiv \begin{cases} 1, & u > 0 \\ 0, & u \leq 0 \end{cases}$$

and

$$\delta_{ij} \equiv \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$
and the condition for zero expansion becomes
\[ \nabla_j^2 h = 0, \]  
where \( \nabla_j^2 \) denotes the flat space Laplacian in the \( (\tilde{x}_i) \) coordinate. To find the black hole horizon, one should calculate the continuous and smooth solution of eq. (16) with two boundary conditions imposed at \( \mathcal{C} \) (see [8] for details).

Now we consider whether the black ring horizon with \( S^1 \times S^{D-4} \) exists on the same slice. Due to this horizon topology, the location of the horizon \( S_+ \) becomes a double-valued function of \( (\tilde{x}_i) \). We divide \( S_+ \) into two parts denoted by
\[ S_1 : \bar{v} = h_1(\tilde{x}_i), \text{ and } S_2 : \bar{v} = h_2(\tilde{x}_i), \]  
where we impose \( h_2 \geq h_1 \). Using the symmetries of this system, \( h_1 \) and \( h_2 \) depends only on \( \tilde{x}_1 \) and \( \tilde{X} \equiv \sqrt{\sum_{k=2}^{D-2} \tilde{x}_k^2} \). The shape of the ring horizon that we expect to exist is shown in Fig. 1 \( S_1 \) and \( S_2 \) are combined at a \( (D-3) \)-surface \( l \), where the gradient of \( h_1 \) and \( h_2 \) diverges. There are two boundaries \( C_+ \) and \( C_- \), where we should impose boundary conditions. The null normal of the surface \( S_2 \) becomes
\[ k^a = (0, 1, 0, ..., 0), \]
and clearly \( S_2 \) has zero expansion for all \( h_2 \). The null normal of \( S_1 \) and the equation for zero expansion coincide with eq. (15) and eq. (16) respectively if we put \( h \equiv h_1 \). However, we can easily show the non-existence of the solution \( h \) that satisfies the above conditions. Assuming the existence of the solution, the location of \( l \) is expressed as

\[ \bar{X} = f(\bar{x}_1), \quad \bar{v} = g(\bar{x}_1). \quad (19) \]

Because the surface \( S_+ \) should be smooth, the surface \( S_1 \) is well-approximated by

\[ \bar{X} \simeq f(\bar{x}_1) - a(\bar{x}_1) \left( h(\bar{x}_1, \bar{X}) - g(\bar{x}_1) \right)^2, \quad (20) \]

in the neighborhood of the \((D-3)\)-surface \( l \) with some function \( a(\bar{x}_1) \). Calculating \( \bar{\nabla}^2 h \), we obtain

\[ \bar{\nabla}^2 h \simeq -\frac{1 + (\partial f/\partial \bar{x}_1)^2}{4a^2} (h - g(\bar{x}_1))^{-3}. \quad (21) \]

If we take a limit \( h \to g(\bar{x}_1) \), the value of \( \bar{\nabla}^2 h \) diverges for any choice of \( a(\bar{x}_1), f(\bar{x}_1), \) and \( g(\bar{x}_1) \), and this shows that the surface like Fig. 4 cannot satisfy the Laplace equation. This result is due to the fact that if there is a \((D-3)\)-dimensional singularity of the \((D-2)\)-dimensional Laplace equation, both values of the solution and its gradient should be finite in the neighborhood of the singularity. Although we have not considered the boundary conditions in this discussion, this is sufficient for the proof of the non-existence of the solution because satisfying eq. (16) is a necessary condition and eq. (21) shows that the necessary condition cannot be satisfied. Hence the black ring horizon does not exist on the union of two incoming shocks, although the black hole horizon can be calculated [10].

Intuitively, it would be obvious that the black ring forms in a \( n \)-particle system for large \( n \). We can write down the metric of the high-energy \( n \)-particle system outside the future light cone of the shock collisions similarly to the two-particle case. However, one can easily see that the black ring horizon does not exist on the union of \( n \) incoming shocks, simply because the horizon equation is the \((D-2)\)-Laplace equation also in this system and it does not allow the divergence of the gradient at the \((D-3)\)-singularity. Hence we cannot find the ring horizon even in the \( n \)-particle system. However, these results just show that the ring horizon does not exist at the instant of collision and it might form after the collision. To obtain the answer, we should investigate the spacetime slice after the collision of the shocks. This analysis is rather difficult because the shocks nonlinearly interact each other after the collision and no one has calculated the metric after the collision. Hence at this
point, we don’t understand anything about the black ring formation in particle systems. To obtain some implications, we consider a rather simplified situation in the next section: the momentarily-static $n$-black-hole initial data.

III. $n$ BLACK HOLE SYSTEM

As we mentioned in previous sections, we consider the momentarily-static $n$-black-hole initial data as a simplified model that we can analyze. This initial data provides $n$-particles that are interacting each other, although each particle is not moving. By analyzing the formation of the black ring horizon in this system, we are able to understand the feature of the black ring formation in $n$-particle systems in higher-dimensions.

A. The Brill-Lindquist initial data

Let $(\mathcal{M}, g_{ij}, K_{ij})$ be the $(D-1)$-dimensional space with the metric $g_{ij}$ and the extrinsic curvature $K_{ij}$. We assume the conformally-flat initial data of which metric is

$$ds^2 = \Psi^{4/(D-3)}(x) \left( dx^2 + dy^2 + dz^2 + z^2 d\Omega_{D-4}^2 \right),$$

where $x$ denotes the location in the flat space and each $(D-4)$-sphere is spanned by a $(\chi_1, ..., \chi_{D-4})$ coordinate. We consider the momentarily static case, where the extrinsic curvature becomes $K_{ij} = 0$. The Hamiltonian constraint becomes

$$\nabla_i^2 \Psi = 0,$$

where $\nabla_i^2$ denotes the Laplacian of the $(D-1)$-dimensional flat space. We select the solution of eq. (23) with $n$ point sources:

$$\Psi = 1 + \sum_{k=1}^{n} \frac{4\pi G D M}{{(D-2)!} \Omega_{D-2} r_k^{D-3}},$$

where $M$ is the ADM mass and $r_k \equiv |x - x_k|$ is the distance between the point $x$ and the location of the point source $x_k$ in the flat space. In the case of $n = 1$ and $D = 4$, this solution provides the Einstein-Rosen bridge. The solution of $n \geq 2$ is the $n$ Einstein-Rosen bridges originally introduced by Brill and Lindquist [21]. Each $x_k$ corresponds to the asymptotically flat region beyond the Einstein-Rosen bridge. We locate the $n$ point sources at $n$ vertexes
of a regular polygon that is inscribed in a circle with the radius $R$. More precisely, we set each $x_k$ as

$$x_k = \left( R \cos \frac{2\pi}{n} (k - 1), R \sin \frac{2\pi}{n} (k - 1), 0 \right),$$

in the $(x, y, z)$ coordinate. As we see from the fact that eq. (24) is a solution of the Hamiltonian constraint (23) with $n$ point sources, this initial data can be regarded as an $n$-point-particle system. Note that the parameters that specify the initial data are the spacetime dimension number $D$, the particle number $n$, and the parameter $R$ to locate the particles in eq. (25).

### B. Methods of finding apparent horizons

Now we explain the methods of finding apparent horizons. Because this space is time-symmetric, the equation for the apparent horizon is

$$\theta_+ = \nabla_i s^i = 0,$$

where $\theta_+$ stands for the expansion of outgoing null geodesic congruence, $s^i$ is the unit normal of the apparent horizon and $\nabla_i$ denotes the covariant derivative with respect to $g_{ij}$. We would like to find the black ring horizon with $S^1 \times S^{D-3}$ topology. Because the apparent horizon is defined as the outermost marginally trapped surface, we also should solve the black hole horizon with $S^{D-2}$ topology that surrounds all the particles to judge whether the black ring horizon is inside the black hole horizon or not.

We introduce a new coordinate $(\rho, \xi, \phi, \chi_1, \ldots, \chi_{D-4})$ by

$$x = (R_0 + \rho \cos \xi) \cos \phi,$$

$$y = (R_0 + \rho \cos \xi) \sin \phi,$$

$$z = \rho \sin \xi.$$

The metric becomes

$$ds^2 = \Psi^{4/(D-3)} \left( d\rho^2 + \rho^2 d\xi^2 + (R_0 + \rho \cos \xi)^2 d\phi^2 + \rho^2 \sin^2 \xi d\Omega_{D-4}^2 \right).$$

This coordinate system is useful for solving the black ring horizon with $S^1 \times S^{D-3}$ topology, because we can specify the location of the black ring horizon by $\rho = h(\xi, \phi)$ if we choose the
value of $R_0$ appropriately. For this purpose, the line $\rho = 0$ should be inside the black ring horizon. The unit normal $s^i$ for this surface is given by
\[ s^i = \frac{\Psi^{-2/(D-3)}}{\sqrt{1 + \frac{h_{\phi}^2}{\rho^2} + \frac{h_{\phi}^2}{(R_0 + \rho \cos \xi)^2}}} \left( 1, \frac{-h_{,\xi}}{\rho^2}, \frac{-h_{,\phi}}{(R_0 + \rho \cos \xi)^2}, 0, \ldots, 0 \right). \]

Calculating $\theta_+ = \nabla_i s^i = 0$, we derive the equation for the black ring horizon as follows:
\[ \left(1 + \frac{h_{\phi}^2}{(R_0 + h \cos \xi)^2}\right) h_{,\xi} + \left( \frac{h_{,\xi} + h_{,\phi}}{(R_0 + h \cos \xi)^2}\right) h_{,\phi} - \frac{2h_{,\xi}h_{,\phi}h_{,\phi}}{(R_0 + h \cos \xi)^2} - \frac{h_{,\xi}^2}{h} \]
\[ - \frac{(h \cos \phi + h_{,\xi} \sin \xi)}{(R_0 + h \cos \xi)^3}h_{,\phi} - \left( \frac{h_{,\xi} + h_{,\phi}}{(R_0 + h \cos \xi)^2}\right) \left[ \frac{D - 3}{h} + \frac{(h \cos \xi + h_{,\xi} \sin \xi)}{h(R_0 + h \cos \xi)} \right] \]
\[ - \frac{(D - 4) \cot \xi}{h^2} h_{,\xi} + \frac{2(D - 2)}{(D - 3)\Psi} \left( \Psi_{,\phi} - \frac{h_{,\xi}h_{,\xi}}{h^2} - \frac{h_{,\phi}h_{,\phi}}{(R_0 + h \cos \xi)^2} \right) \] = 0. \hspace{1cm} \text{(32)}

Because this space has a mirror symmetry about the equatorial plane and a discrete symmetry for the rotation about the z-axis, it is sufficient to solve in the range $0 \leq \xi \leq \pi$ and $0 \leq \phi \leq \pi/n$. We solved eq. (32) under the boundary conditions
\[ h_{,\xi} = 0 \quad \text{at} \quad \xi = 0, \pi, \]
\[ h_{,\phi} = 0 \quad \text{at} \quad \phi = 0, \pi/n, \]
using the SOR method. We selected the value of $R_0$ to be $R_0 = R$ for $n \geq 6$, $R_0 = 0.9R$ for $n = 5$, $R_0 = 0.8R$ for $n = 4$, and $R_0 = 0.7R$ for $n = 3$. The black ring horizon has the $(D - 2)$-dimensional area $A_{D-2}^{(BR)}$, which is given by
\[ A_{D-2}^{(BR)} = 2n\Omega_{D-4} \int_0^{\pi/n} d\phi \int_0^\pi d\xi \Psi^{2(D-2)/(D-3)}(h \sin \xi)^{D-4} \sqrt{(R_0 + h \cos \xi)^2(h_{,\xi}^2 + h_{,\phi}^2) + h_{,\phi}^2}. \]

The black ring horizon is the minimal surface of this area $A_{D-2}^{(BR)}$.

Next we explain the method of finding the black hole horizon with $S^{D-2}$ topology. For this purpose, it is useful to choose the coordinate $(r, \theta, \phi, \chi_1, \ldots, \chi_{D-4})$ where
\[ x = r \sin \theta \sin \phi, \]
\[ y = r \sin \theta \cos \phi, \]
\[ z = r \cos \theta. \]

We give the location of the black hole horizon by $r = \tilde{h}(\theta, \phi)$. The unit normal of this surface becomes
\[ s^i = \frac{\Psi^{-2/(D-3)}}{\sqrt{1 + \frac{\tilde{h}_{,\theta}^2}{r^2} + \frac{\tilde{h}_{,\phi}^2}{r^2 \sin^2 \theta}}} \left( 1, \frac{-\tilde{h}_{,\theta} / r^2}{\tilde{h}_{,\phi} / r^2 \sin^2 \theta}, \frac{-\tilde{h}_{,\phi} / r^2 \sin^2 \theta}{0, \ldots, 0} \right). \]
The equation \( \theta = 0 \) for the black hole horizon becomes

\[
\left( 1 + \frac{\tilde{h}^2_{\phi}}{h^2 \sin^2 \theta^2} \right) \tilde{h}_{\theta \theta} + \left( 1 + \frac{\tilde{h}^2_{\theta}}{h^2} \right) \frac{\tilde{h}_{\phi \phi}}{\sin^2 \theta} - \frac{2 \tilde{h}_{\theta \phi} \tilde{h}_{\theta \phi}}{\tilde{h}^2 \sin^2 \theta} - \left( \frac{\tilde{h}^2_{\phi}}{h^2 \sin^2 \theta} + \frac{\tilde{h}^2_{\theta}}{h^2} \right) \frac{1}{h} \frac{\tilde{h}_{\phi}}{h^2 \sin^3 \theta} - \left( 1 + \frac{\tilde{h}^2_{\phi}}{h^2 \sin^2 \theta} \right) \left[ (D - 2) \tilde{h} ight. \\
+ \frac{2(D - 2)}{(D - 3) \Psi} \left( \Psi,_{r} \tilde{h}^2 - \Psi,_{\theta} \tilde{h}_{\theta} - \frac{\Psi,_{\phi} \tilde{h}_{\phi}}{\sin^2 \theta} \right) - (\cot \theta - (D - 4) \tan \theta) \tilde{h},_{\theta} \right] = 0. \quad (40)
\]

We solved this equation under the boundary conditions

\[
\tilde{h},_{\theta} = 0 \quad \text{at} \quad \theta = 0, \pi/2, \quad (41)
\]
\[
\tilde{h},_{\phi} = 0 \quad \text{at} \quad \phi = 0, \pi/n. \quad (42)
\]

Because there is a coordinate singularity at \( \theta = 0 \), a careful treatment is required for the numerical calculation. We imposed the boundary condition at \( \theta = 0 \) as follows. First, expressing the location of the apparent horizon as \( z = f(x, y) \), we calculated the equation for the apparent horizon in terms of \( f(x, y) \). Next we wrote down the differential equation of \( f(x, y) \) at \( (x, y) = (0, 0) \). Finally we rewrote this differential equation in terms of \( \tilde{h} \). This differential equation at \( \theta = 0 \) is somewhat complicated especially in the cases where \( n \) is a odd number. The \((D - 2)\)-dimensional area of the black hole horizon is given as follows:

\[
A_{D-2}^{(BH)} = 2n \Omega_{D-4} \int_{0}^{\pi/n} \int_{0}^{\pi/2} d\phi d\theta \Psi^{2(D-2)/(D-3)} \tilde{h}^{D-3} \cos^{D-4} \theta \sqrt{(\tilde{h}^2 + \tilde{h}^2_{\theta}) \sin^2 \theta + \tilde{h}^2_{\phi}.} \quad (43)
\]

The black hole horizon is the minimal surface of this area \( A_{D-2}^{(BH)} \).

We solved with \( 40 \times 40 \) grids for both horizons. The error is estimated as follows. In \( n = 2 \) cases, the spherical horizon can be solved using the Runge-Kutta method with a fair good accuracy because the space is axi-symmetric. Comparing the two results, we evaluate the error to be about 1% in \( n = 2 \) cases. The error is expected to be smaller than this value for \( n \geq 3 \) because the parameter range of \( \theta \) becomes smaller while we use the same grid number. To evaluate the error of the ring horizon, we solved with \( 80 \times 80 \) grids in some cases. The error is also about 1% or smaller than this value for the ring horizon. Hence the possible changes of the values in Tables II, III, IV and V which we will show later are expected to be within 0.01.
TABLE I: The values of $R_{\text{min}}^{(BR)}$, $R_{\text{max}}^{(BR)}$, and $R_{\text{max}}^{(BH)}$ for $n = 4, \ldots, 11$ in the $D = 5$ case. The ring horizon does not form for $n \leq 6$. Because $R_{\text{max}}^{(BR)} < R_{\text{max}}^{(BH)}$ for $n = 7$ and 8, the ring horizon is inside the black hole apparent horizon. For the ring apparent horizon formation, $n \geq n_{\text{min}} = 9$ is necessary.

| $n$ | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $R_{\text{min}}^{(BR)}$ | –   | –   | –   | 0.79| 0.79| 0.79| 0.79| 0.79|
| $R_{\text{max}}^{(BR)}$ | –   | –   | –   | 0.82| 0.86| 0.91| 0.95| 0.99|
| $R_{\text{max}}^{(BH)}$ | 0.84| 0.87| 0.88| 0.89| 0.90| 0.90| 0.90| 0.90|

FIG. 2: The black ring horizon for $n = 7$ ($R = 0.8\tilde{r}_h$) and $n = 9$ ($R = 0.91\tilde{r}_h$) in five-dimensions. Although the black ring horizon forms in the $n = 7$ case, it is inside the black hole horizon and hence the system is actually a black hole. For $n \geq 9$, the black ring horizon that is not enclosed by the black hole horizon appears.

C. Numerical results

Now we show the numerical results. In this subsection, we use

$$\tilde{r}_h \equiv \left( \frac{4\pi G D M}{(D - 2) \Omega_{D-2}} \right)^{1/(D-3)}$$

as the unit of the length. In this unit, the location of the apparent horizon in the $R = 0$ cases becomes $r = 1$. We calculated the horizons for each $R$ with 0.01 intervals.

We begin with the $D = 5$ case. In this case, we could find the black ring horizon for $n \geq 7$. For each $n \geq 7$, there are two characteristic values $R_{\text{min}}^{(BR)}$ and $R_{\text{max}}^{(BR)}$ such that the black ring
TABLE II: The values of $R_{\text{min}}^{(BR)}$, $R_{\text{max}}^{(BR)}$, and $R_{\text{max}}^{(BH)}$ for $n = 3, \ldots, 7$ in the $D = 6$ case. The ring horizon does not form for $n = 3$ and the ring horizon is inside the black hole apparent horizon for $n = 4$. For the ring apparent horizon formation, $n \geq n_{\text{min}} = 5$ is necessary.

| $n$ | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|
| $R_{\text{min}}^{(BR)}$ | – | 0.86 | 0.87 | 0.88 | 0.88 |
| $R_{\text{max}}^{(BR)}$ | – | 0.86 | 0.96 | 1.07 | 1.17 |
| $R_{\text{max}}^{(BH)}$ | 0.83 | 0.86 | 0.88 | 0.89 | 0.89 |

**D=6, n=4**

**D=6, n=5**

FIG. 3: The horizons in a four-particle system and a five-particle system in six-dimensions ($R = 0.86\tilde{r}_h$ and $R = 0.9\tilde{r}_h$ respectively). In a four-particle system, the ring horizon is enclosed by a black hole horizon. In a five-particle system, the ring horizon becomes the apparent horizon.

The ring horizon does not exist for $R < R_{\text{min}}^{(BR)}$ and $R > R_{\text{max}}^{(BR)}$, and it exists for $R_{\text{min}}^{(BR)} \leq R \leq R_{\text{max}}^{(BR)}$. In the $R > R_{\text{max}}^{(BR)}$ cases, the throat of the surface between neighboring two particles becomes arbitrarily narrow (i.e. $h(\xi, \pi/n)$ becomes zero) during the iteration. This result shows that the distance between two particles neighboring each other should be sufficiently small for the ring horizon formation. In the $R < R_{\text{min}}^{(BR)}$ cases, the hole of the surface becomes arbitrarily small during the iteration (i.e. $h(\pi, \phi)$ approaches to $R_0$). Hence the space between all the particles at the center should be sufficiently large. Table II shows the values of $R_{\text{min}}^{(BR)}$, $R_{\text{max}}^{(BR)}$, and $R_{\text{max}}^{(BH)}$, where $R_{\text{max}}^{(BH)}$ is the maximal value of $R$ for the black hole formation. According to this Table, the result that the black ring horizon does not form in the $n \leq 6$ cases is realistic, because $R_{\text{min}}^{(BR)}$ is almost constant for all $n \geq 7$ and $R_{\text{max}}^{(BR)}$ linearly decreases with the decrease
TABLE III: The values of $R_{\text{min}}^{(\text{BR})}$, $R_{\text{max}}^{(\text{BR})}$, and $R_{\text{max}}^{(\text{BH})}$ for $n = 3, \ldots, 7$ in the cases of $D = 7$ and $8$. In these dimensions, the ring horizon does not form for $n = 3$ and the ring apparent horizon forms for $n \geq n_{\text{min}} = 4$. The parameter range of $R$ for the black ring formation for $D = 8$ is somewhat larger than the $D = 7$ case.

| $n$ | $D = 7$ | | | $D = 8$ | | | |
|---|---|---|---|---|---|---|---|
| $R_{\text{min}}^{(\text{BR})}$ | – | 0.89 | 0.90 | 0.91 | 0.91 | – | 0.91 | 0.92 | 0.92 | 0.92 |
| $R_{\text{max}}^{(\text{BR})}$ | – | 0.96 | 1.10 | 1.24 | 1.38 | – | 1.03 | 1.19 | 1.35 | 1.51 |
| $R_{\text{max}}^{(\text{BH})}$ | 0.85 | 0.88 | 0.90 | 0.90 | 0.90 | 0.87 | 0.90 | 0.91 | 0.91 | 0.91 |

$D = 7$, $n = 4$

FIG. 4: The black ring horizon in a four-particle system in seven-dimensions ($R = 0.9\tilde{r}_h$).

in $n$ and thus $R_{\text{max}}^{(\text{BR})}$ becomes smaller than $R_{\text{min}}^{(\text{BR})}$ for $n \leq 6$ if we extrapolate them. Because the value of $R_{\text{max}}^{(\text{BH})}$ is larger than $R_{\text{max}}^{(\text{BR})}$ in the cases of $n = 7$ and $8$, the ring horizon is not an apparent horizon because it is inside the black hole horizon. The minimum particle number $n_{\text{min}}$ for the ring apparent horizon formation is nine, although the parameter range of $R$ is small in this case. Hence in the $D = 5$ case, to produce black rings using particle systems would be extremely difficult. Figure 2 shows the shape of the horizons in the cases of $n = 7$ and $n = 9$.

We turn to the $D = 6$ case. Table III shows the values of $R_{\text{min}}^{(\text{BR})}$, $R_{\text{max}}^{(\text{BR})}$, and $R_{\text{max}}^{(\text{BH})}$ for $n = 3, \ldots, 7$. In the $D = 6$ case, the ring horizon forms for $n \geq 4$. Because it is enclosed by the black hole horizon in the $n = 4$ case, the ring apparent horizon appears only for
The ring apparent horizon forms even in the three-particle system in these dimensions.

| n | $D=9$ | $D=10$ |
|---|---|---|
| $R_{\text{min}}^{(BR)}$ | 0.90 | 0.92 |
| $R_{\text{max}}^{(BR)}$ | 0.92 | 0.94 |
| $R_{\text{max}}^{(BH)}$ | 0.89 | 0.91 |

$D=9, \ n=3$

$D=9, \ n=3$

![Diagram](image1.png)

FIG. 5: The black ring horizon in a three-particle system in nine-dimensions ($R = 0.91\tilde{r}_h$).

$n \geq n_{\text{min}} = 5$. This value is far smaller compared to the $D = 5$ case. At the same time, the parameter range of $R$ that the black ring horizon appears is far larger than the $D = 5$ case if we compare fixing $n$. Hence we find that the ring horizon formation becomes easier for larger $D$. The shape of the horizons in the $n = 4$ and $n = 5$ cases are shown in Fig. 8.

Now we show the results of $D = 7$ and 8, which are summarized in Table III. In the $D = 7$ case, the ring horizon does not form in a three-particle system. For $n \geq n_{\text{min}} = 4$, the black ring horizon that is not enclosed by the black hole horizon appears. The result for $D = 8$ is similar to the $D = 7$ case, except that the parameter range of $R$ of the black ring formation becomes somewhat larger than the $D = 7$ case if we compare fixing $n$. We find again that the black ring becomes easier to form for larger $D$. The ring horizon in a four-particle system in the $D = 7$ case is shown in Fig. 4.

Finally we look at the $D = 9$ and $D = 10$ cases. The results are summarized in Table IV.
TABLE V: The minimum particle number $n_{\text{min}}$ that is necessary for the black ring formation for each $D$.

| $D$ | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|-----|----|----|----|----|----|----|----|----|
| $n$ | –  | 9  | 5  | 4  | 4  | 3  | 3  | 3  |

FIG. 6: The behavior of Hawking’s quasi-local mass $M_H^{(BH)}$ (●) and $M_H^{(BR)}$ (○) on the horizons as functions of $R$ in the ten-particle system in five-dimensions. Both $M_H^{(BH)}$ and $M_H^{(BR)}$ are the monotonically decreasing functions of $R$. The value of $M_H^{(BR)}$ is greater than the ADM mass at $R = \tilde{r}_{\text{min}}$, but it rapidly decreases and it is smaller than $M_H^{(BH)}$ at $R = \tilde{r}_{\text{max}}$. The value of the apparent horizon mass $M_{AH}$ at $R = \tilde{r}_{\text{max}}$ is about 90% of the ADM mass.

In these cases, the black ring formation is possible even in the three-particle system, although the parameter range of $R$ is quite small. The shape of the black ring horizon in a three-particle system in nine-dimensions are shown in Fig. 5. We summarized the minimum particle number $n_{\text{min}}$ that is required for the ring apparent horizon formation for each $D$ in Table V.

Now we calculate Hawking’s quasi-local mass \[\text{22}\] on the horizons to understand the amount of trapped energy. It is given by

$$M_H \equiv \frac{(D-2)\Omega_{D-2}}{16\pi G_D} \left(\frac{A_{D-2}}{\Omega_{D-2}}\right)^{(D-3)/(D-2)},$$

(45)

on a surface with zero expansion and becomes an indicator for the energy trapped by the surface. If the surface is an apparent horizon, Hawking’s quasi-local mass becomes the apparent horizon mass $M_{AH}$, that gives the lower bound of the mass of the final state as we
FIG. 7: The behavior of Hawking’s quasi-local mass $M_{H}^{(BH)}$ for $n = 4$ (■), 5 (▲), 6 (♦) and $M_{H}^{(BR)}$ for $n = 4$ (□), 5 (△), 6 (◇) as functions of $R$ in the $D = 7$ case. The values of $R_{min}^{(BR)}$ and $R_{max}^{(BH)}$ are similar and $M_{H}^{(BR)}$ begins at $R = R_{min}^{(BR)}$ with a similar value to $M_{H}^{(BH)}$. Similarly to the $D = 5$ case, $M_{AH}$ is a monotonically decreasing function that is discontinuous at $R = R_{max}^{(BH)}$.

see from the area theorem. Figure 6 shows the values of Hawking’s quasi-local mass on the black hole horizon $M_{H}^{(BH)}$ and the black ring horizon $M_{H}^{(BR)}$ of the ten-particle system in the five-dimensional case. The value of $M_{H}^{(BH)}$ is $M$ at $R = 0$ because it is the Schwarzschild spacetime, and it monotonically decreases as $R$ increases. $M_{H}^{(BR)}$ has its value in the range $R_{min}^{(BR)} \leq R \leq R_{max}^{(BR)}$. Although the value of $M_{H}^{(BR)}$ is greater than $M$ at $R = R_{min}^{(BR)}$, it rapidly decreases with the increase in $R$ and becomes smaller than $M_{H}^{(BH)}$ at $R = R_{max}^{(BH)}$. Hence the apparent horizon mass $M_{AH}$ is always smaller than the ADM mass, and it is a monotonically decreasing function of $R$ that is defined in $0 \leq R \leq R_{max}^{(BR)}$ and is discontinuous at $R = R_{max}^{(BH)}$. The shapes of $M_{H}^{(BH)}$ of different $n$ values almost coincide. The shapes of $M_{H}^{(BR)}$ of different $n$ values are almost similar in the range their values exist: the functions $M_{H}^{(BR)}$ start at almost the same points and show the similar dependence on $R$, although their end points $R = R_{max}^{(BR)}$ are different each other. The minimal value of $M_{AH}$ (i.e. $M_{H}^{(BR)}$ at $R = R_{max}^{(BR)}$) decreases as $n$ increases. In the limit $n \rightarrow \infty$, the value of $R_{max}^{(BR)}$ becomes arbitrarily large while the minimal value of $M_{AH}$ asymptotes to zero, because this system approaches to the ring configuration analyzed in [15]. Figure 7 shows the behavior of Hawking’s quasi-local mass on the horizons in the $D = 7$ case. For $D \geq 6$, the parameter range of the overlapping region $R_{min}^{(BR)} \leq R \leq R_{max}^{(BH)}$ that both the black hole horizon and the black ring horizon exist becomes small and this tendency is enhanced for larger $D$. In the $D \geq 6$ cases, the
value of $M_{H}^{(BR)}$ at $R = R_{\text{min}}^{(BH)}$ is slightly less than $M_{H}^{(BH)}$ for all $n$. Similarly to the $D = 5$ case, the apparent horizon mass $M_{AH}$ is a monotonically decreasing function of $R$ that is discontinuous at $R = R_{\text{max}}^{(BH)}$. If we compare the same $n$ cases, the minimal values of $M_{AH}$ are similar for all $D \geq 7$. It is about 85%, 82%, 79%, and 77% of the ADM mass for $n = 4, 5, 6,$ and 7, respectively. In the $D = 6$ cases, their values are somewhat larger than these values. In the three-particle systems in $D = 9$ and 10, the value of $M_{H}^{(BR)}$ is about 87% of the ADM mass. In summary, the amount of energy that is trapped by the ring horizon is larger than 75% of the ADM mass for all $n \leq 7$ cases, which can be regarded as a fairly large value.

D. Physical interpretation

Now we discuss the physical interpretation for the minimum particle number $n_{\text{min}}$ that is required for the ring apparent horizon formation. One plausible interpretation is as follows. Suppose that $n$ point sources with mass $M/n$ are put at $n$ vertexes of a regular polygon that is inscribed in a circle with radius $R$. Each point source has a trapped region with radius $r_{h}(M/n)$, where we used the expression of eq. (2). The black ring can be expected to form when the trapped regions of neighboring particles overlap each other. This condition leads to

$$R \sin(\pi/n) < r_{h}(M/n).$$

(46)

On the other hand, this $n$-particle system can be expected to become a black hole when all the particles are within a sphere with the gravitational radius of the system mass. Hence for the ring apparent horizon formation,

$$r_{h}(M) < R$$  

(47)

is also necessary. For the existence of $R$ that satisfies these two conditions (46) and (47), $n$ should satisfy

$$\sin(\pi/n) < (1/n)^{1/(D-3)}.$$  

(48)

The minimum number $n \equiv n_{1}$ that satisfies this inequality is summarized in Table VI. Although these values are slightly larger than the values derived by the numerical calculations, this interpretation gives the similar values. The reason why $n_{1}$ depends on $D$ is that the ratio of $r_{h}(M/n)$ and $r_{h}(M)$ has a characteristic value for each $D$. This is in turn because the each dimension number $D$ has the characteristic power of the gravitational potential.
TABLE VI: The minimum particle numbers $n_1$ and $n_2$ that are necessary for the black ring formation for each $D$ estimated by our plausible interpretation and Ida and Nakao’s conjecture \cite{15} respectively.

| $D$ | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|-----|----|----|----|----|----|----|----|----|
| $n_1$ | –  | 10 | 6  | 5  | 4  | 4  | 4  | 3  |
| $n_2$ | –  | 10 | 9  | 8  | 9  | 9  | 9  | 10 |

This leads to the dependence of $n_1$ (and hence the minimum number $n_{\text{min}}$) on the dimension number $D$. We also would like to mention that the boundary surface of the union of particles’ trapped regions does not have $S^1 \times S^{D-3}$ topology for $R < r_h(M/n)$. In this situation, the black ring horizon is expected to disappear. The overlapping region where the black ring horizon and the black hole horizon both exist is given by $r_h(M/n) \leq R \leq r_h(M)$. This parameter range becomes smaller for larger $D$ if we compare fixing $n$. All these pictures correspond to the qualitative behavior of the horizons of our numerical results.

Here we would like to discuss Ida and Nakao’s conjecture \cite{15} which is the generalization of the hoop conjecture \cite{23} in four-dimensional spacetimes. The hoop conjecture gives the condition for the apparent horizon formation in the form $C \leq 2\pi r_h(M)$ where $C$ is the characteristic one-dimensional length scale of the system. This conjecture essentially states that the mass concentration in all directions is necessary for the apparent horizon formation, and is well-confirmed in various systems (see \cite{24} and the references therein). Ida and Nakao investigated various four-dimensional conformally-flat momentarily-static spaces in five-dimensions and found that arbitrarily long black holes can form. This shows that one-dimensional length does not provide the condition for the horizon formation in five-dimensions. They discussed that the $(D-3)$-dimensional volume $V_{D-3}$ characteristic to the system would provide the condition for the horizon formation in $D$-dimensional spacetimes in the form $V_{D-3} \lesssim \Omega_{D-3} r_h^{D-3}(M)$. In our previous paper \cite{10}, we discussed the validity of this conjecture in the high-energy two-particle system and found that this conjecture provides a good condition for the black hole production.

Using our results in this paper, we discuss whether this conjecture gives the condition for the black ring formation in $n$-particle systems. In our opinion, this conjecture has a possibility to provide the condition for the existence of horizons with various topology if we
evaluate the \((D-3)\)-volume appropriately. Here, we assume that the condition of the black hole formation is given by \((D-3)\)-volume \(V_{D-3}^{(BH)}\) of a surface with \(S^{D-3}\) topology, and that the condition of the black ring formation is given by \((D-3)\)-volume \(V_{D-3}^{(BR)}\) of a surface with \(S^1 \times S^{D-4}\) topology. The value of \(V_{D-3}^{(BR)}\) in the \(n\)-particle system is estimated as follows. Because the characteristic radius around \(S^{D-4}\) is given by \(r_h(M/n)\) and that around \(S^1\) becomes \(R\), the characteristic \((D-3)\)-volume becomes \(V_{D-3}^{(BR)} \sim 2\pi R \times \Omega_{D-4} r_{h}^{D-4}(M/n)\). The conjecture gives one condition

\[
2\pi \Omega_{D-4} r_{h}^{D-4}(M/n) R \lesssim \Omega_{D-3} r_{h}^{D-3}(M).
\]

(49)

Because the characteristic \((D-3)\)-volume of a surface with \(S^{D-3}\) topology is \(V_{D-3}^{(BH)} \simeq \Omega_{D-3} R^{D-3}\), the system would become a black hole for \(R \lesssim r_h(M)\) according to the conjecture. Hence we should assume one more condition

\[
r_{h}(M) \lesssim R,
\]

(50)

for the black ring apparent horizon formation. For the existence of \(R\) that satisfies these two conditions (49) and (50), the particle number should satisfy

\[
n \gtrsim \left( \frac{2\pi \Omega_{D-4}}{\Omega_{D-3}} \right)^{(D-3)/(D-4)}.
\]

(51)

The minimum number \(n_2\) satisfying this condition (51) for each \(D\) is shown in Table VI. These values don’t coincide with the values in Table V especially in the point that \(n_2\) is not a monotonically decreasing function of \(D\) and cannot explain the fact that \(n_{\text{min}}\) approaches to three for large \(D\). This result might be because our evaluation of the \((D-3)\)-volume was not appropriate, or might show that the coefficient \(\Omega_{D-3}\) in the inequality \(V_{D-3} \lesssim \Omega_{D-3} r_{h}^{D-3}(M)\) should be taken another appropriate value to judge the existence of the ring horizon. However, this conjecture can explain the existence of the minimum particle number \(n_{\text{min}}\) within the difference by a factor four for \(5 \leq D \leq 11\) even in such a rough estimation. Because the aim of this conjecture would be to provide the phenomenological intuition and would not be to give the exact theorem for the horizon formation, this result can be regarded as successful. Furthermore, because the surface of which we evaluated the \(V_{D-3}^{(BR)}\) value does not have \(S^1 \times S^{D-4}\) topology for \(R \leq r_h(M/n)\), the black ring is expected to disappear in this configuration. Hence the conjecture naturally explains the non-existence of the black ring horizon for small \(R\). In these meanings, Ida and Nakao’s conjecture provides the condition for the horizon formation even if its topology is not spherical.
IV. SUMMARY AND DISCUSSION

In this paper, we investigated the black ring formation in particle systems. In Sec. II, we analyzed the black ring horizon in the high-energy particle collision. In the high-energy two-particle system, we explicitly proved that there is no black ring horizon on the union of two incoming shock waves. Even in the high-energy $n$-particle system, we have seen that there is no ring horizon on the union of $n$ incoming shock waves. These results indicate that the black ring horizon does not exist at the instant of collision, but they don’t imply that the black ring does not form in high-energy particle systems: there remains the possibility of the black ring formation after the collision of incoming shocks. To understand whether the black ring forms in the high-energy particle system or not, we should investigate the spacetime slice after the collision where the particles interact each other. However, to obtain the metric in this phase is difficult.

As a simplified model of $n$-particles that we can calculate, we considered the momentarily-static $n$-black-hole system originally developed by Brill and Lindquist to obtain some indications about the feature of the black ring formation in particle systems. We numerically solved the black ring horizon and the black hole horizon that surround $n$ particles. We located the $n$ particles at $n$ vertexes of a regular polygon that is inscribed in a circle with the radius $R$. There is the minimal value $R_{\text{min}}^{(\text{BR})}$ and the maximal value $R_{\text{max}}^{(\text{BR})}$ such that the black ring horizon exists only in the range $R_{\text{min}}^{(\text{BR})} \leq R \leq R_{\text{max}}^{(\text{BR})}$. For the black ring formation, the distance between the neighboring particles should be small, while the space at the center between all the particles should be large. These lead to the existence of $R_{\text{min}}^{(\text{BR})}$ and $R_{\text{max}}^{(\text{BR})}$. For each $D$, there is the minimum particle number $n_{\text{min}}$ that is necessary for the ring apparent horizon formation. The results are summarized in Table V. This minimum number $n_{\text{min}}$ becomes small and approaches to three for large $D$. If we compare fixing $n$, the parameter range $R$ of the black ring formation becomes larger as $D$ increases. These results show that the black ring formation becomes easier for larger dimension number $D$. Because there is a plausible interpretation for our results, we expect that these features would be valid also in the high-energy particle system.

We calculated Hawking’s quasi-local mass on the horizons to understand the energy trapped by the horizon. Hawking’s quasi-local mass on the apparent horizon $M_{\text{AH}}$ (the apparent horizon mass) is a monotonically decreasing function of $R$ that is discontinuous.
at $R = R_{\text{max}}^{(\text{BH})}$ where the black hole horizon disappears. The amount of the energy trapped inside the ring horizon is always greater than 75% for the $n \leq 7$ cases, which is rather a large value. We don’t consider that so much energy can be trapped in the case of the black ring horizon in the high-energy particle system. In our previous paper [10], we calculated the apparent horizon mass of the black hole horizon that is formed in the high-energy two-particle grazing collision. The energy trapped by the horizon is quite small if $D$ is large and the impact parameter is close to its maximal value, and we discussed that many black holes with small mass would be produced in high-energy particle systems, although further investigations of the gravitational radiation after the collision would be necessary to obtain the rigorous answer. Hence we consider that the trapped energy of the black ring might be small also in the high-energy $n$-particle system. However we expect that the energy trapped by the black ring would be always smaller than that trapped by the black hole if we compare fixing $n$ and $D$ also in the high-energy $n$-particle system.

From these results, we expect that the production of the black rings using systems with many particles would be possible, although it is difficult for the $D = 5$ case. $n = 4$ would be sufficient for this purpose if $D$ is equal to or larger than seven. We should mention that the motion of each particle is important to observe the signals from the black rings in colliders. For example, if each particle has a velocity such that the ring radius decreases, the system would rapidly collapses to a black hole and we would not be able to detect the signals of the black ring even if the ring horizon momentarily forms. To produce the black ring with a long lifetime, each particle should have a velocity in the $\phi$ direction in the coordinate system used in Sec. III. If we take it into account, the ordinary colliders are not appropriate for the black ring formation. A linear collider with four arms, for example, would be able to produce the black rings with a small probability.

Our remaining problem is as follows. Although we expect that the basic feature of the results in the Brill-Lindquist initial data would not be changed in the high-energy particle system, there is a possibility that the momentums of particles have a large effect on the existence of the horizon as we have shown in our previous paper in two-black-hole systems in four-dimensions [24]. To clarify this point, we should investigate the case where each particle has a momentum in the $\phi$ direction. For this purpose, Bowen and York’s method [25] is useful. Using this method, one can calculate the conformally-flat initial data of multi-black holes with momentums. The extrinsic curvature is given analytically, and one should solve the
conformal factor numerically. Although this method was proposed in the four-dimensional case, the extension to the higher-dimensional cases has been done by Shibata [26]. Using this method, we are able to calculate the initial data of moving particles and understand the effect of motion on the ring horizon formation. Next, we would like to investigate the remaining possibility to produce the black rings in two-particle systems. Although it seems that the black ring would not form in the two-point-particle system from the results in this paper, it might be possible if we consider the deformed particles. For example, if we regard the incoming particle as the fundamental string, the classical model for the gravitational field would be provided by a segmental source. Hence in the high-energy collision of two-strings, there is a possibility of the black ring formation. Another example uses the instability of the higher-dimensional Kerr black holes. Recently Emparan and Myers have shown that the ultra-spinning Kerr black holes in higher-dimensions are unstable [27]. Hence, if these ultra-spinning Kerr black holes can be produced in the high-energy two-particle collisions, they would deform due to this instability. Subsequently two deformed black holes might collide each other and form the black ring. To confirm this scenario, we should investigate whether the production of the ultra-spinning black holes is possible in the high-energy two-particle system. This requires the investigation of the balding phase (i.e. the evolution after the particle collision) and the determination of the amount of energy and angular momentum that is radiated away.

Acknowledgments

We would like to thank Masaru Shibata, Ken-ichi Nakao and Daisuke Ida for helpful discussions. We are also in thank of Tetsuya Shiromizu’s helpful comments on the black ring formation in the system of two deformed particles. The work of H. Y. is supported in part by a grant-in-aid from Nagoya University 21st Century COE Program (ORIUM).

[1] N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 429, 263 (1998); I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. R. Dvali, Phys. Lett. B 436, 257 (1998).
[2] L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999).
[3] T. Banks and W. Fischler, hep-th/9906038
[4] S. Dimopoulos and G. Landsberg, Phys. Rev. Lett. 87, 161602 (2001).
[5] S. B. Giddings and S. Thomas, Phys. Rev. D 65, 056010 (2002).
[6] M. Cavaglia, S. Das, and R. Maartens, Class. Quantum Grav. 20, L205 (2003).
[7] R. Emparan, G. T. Horowitz, and R. C. Myers, Phys. Rev. Lett 85, 499 (2000).
[8] D. M. Eardley and S. B. Giddings, Phys. Rev. D 66, 044011 (2002).
[9] H. Yoshino and Y. Nambu, Phys. Rev. D 66, 065004 (2002).
[10] H. Yoshino and Y. Nambu, Phys. Rev. D 67, 024009 (2003).
[11] R. Emparan and H. S. Reall, Phys. Rev. D 65, 084025 (2002).
[12] R. Emparan and H. S. Reall, Phys. Rev. Lett. 88, 101101 (2002).
[13] R. C. Myers and M. J. Perry, Ann. Phys. (N.Y.) 172, 304 (1986).
[14] G. W. Gibbons, D. Ida, and T. Shiromizu, Phys. Rev. D 66, 044010 (2002).
[15] D. Ida and K.-i. Nakao, Phys. Rev. D 66, 064026 (2002).
[16] R. Gregory and R. Laflamme, Phys. Rev. Lett. 70, 2837 (1993).
[17] R. Emparan, private communication.
[18] D. Ida, K.-y. Oda, and S. C. Park, Phys. Rev. D 67, 064025 (2003); ibid. D 69, 049901 (2004).
[19] P. C. Aichelburg and R. U. Sexl, Gen. Relativ. Gravit. 2, 303 (1971).
[20] P. D. D’Eath and P. N. Payne, Phys. Rev. D 46, 658 (1992); D 46, 675 (1992); D 46, 694 (1992).
[21] D. R. Brill and R. W. Lindquist, Phys. Rev. 131, 471 (1963).
[22] S. Hawking, J. Math. Phys. 9, 598 (1968).
[23] K. S. Thorne, in Magic without Magic: John Archbald Wheeler, edited by J.Klauder (Freeman, San Francisco, 1972).
[24] H. Yoshino, Y. Nambu, and A. Tomimatsu, Phys. Rev. D 65, 064034 (2002).
[25] J. M. Bowen and J. W. York, Jr., Phys. Rev. D 21, 2047 (1980).
[26] M. Shibata, unpublished (2002).
[27] R. Emparan and R. C. Myers, J. High Energy Phys. 309, 025 (2003).