Darboux’s Theorem and Quantisation

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Abstract

It has been established that endowing classical phase space with a Riemannian metric is sufficient for describing quantum mechanics. In this letter we argue that, while sufficient, the above condition is certainly not necessary in passing from classical to quantum mechanics. Instead, our approach to quantum mechanics is modelled on a statement that closely resembles Darboux’s theorem for symplectic manifolds.

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1 Introduction

Quantisation may be understood as a prescription to construct a quantum theory from a given classical theory. As such, it is far from being unique. Beyond
canonical quantisation and Feynman’s path–integral, a number of different, often complementary approaches to quantisation are known, each one of them exploiting different aspects of the underlying classical theory. For example, geometric quantisation relies on the geometry of classical phase space. Berezin’s quantisation can be applied to classical systems whose phase space is a homogeneous Kähler manifold [1, 2].

The deep link existing between classical and quantum mechanics has been known for long. Perhaps its simplest manifestation is that of coherent states [3]. More recent is the notion that not all quantum theories arise from quantising a classical system. Furthermore, a given quantum model may possess more than just one classical limit. These ideas find strong evidence in string duality and M–theory [4].

It therefore seems natural to try an approach to quantum mechanics that is not based, at least primarily, on the the metric quantisation of a given classical dynamics. In such an approach one would not take a classical theory as a starting point. Rather, quantum mechanics itself would be more fundamental, in that its classical limit or limits (possibly more than one) would follow from a parent quantum theory.

2 Berezin’s metric quantisation

Below we briefly review the construction of the Hilbert space of states from the metric on complex homogeneous Kähler manifolds [1]. Let \( z^j, \overline{z}^k, j, k = 1, \ldots, n \), be local coordinates on a complex homogeneous Kähler manifold \( M \), and let \( K_M(z^j, \overline{z}^k) \) be a Kähler potential for the metric \( d\bar{s}^2 = g_{j \overline{k}} dz^j d\overline{z}^k \). The Kähler form \( \omega = g_{j \overline{k}} dz^j \wedge d\overline{z}^k \) gives rise to an integration measure \( d\mu(z, \overline{z}) = \omega^n = \det (g_{j \overline{k}}) \prod_{l=1}^{n} dz^l \wedge d\overline{z}^l \). (1)

The Hilbert space of states is the space \( \mathcal{F}_{\hbar}(M) \) of analytic functions on \( M \) with finite norm, the scalar product being

\[
\langle \psi_1 | \psi_2 \rangle = c(\hbar) \int_M d\mu(z, \overline{z}) \exp(-\hbar^{-1}K_M(z, \overline{z})) \overline{\psi_1}(z)\psi_2(z),
\]

and \( c(\hbar) \) a normalisation factor. Let \( G \) denote the Lie group of motions of \( M \), and assume \( K_M(z, \overline{z}) \) is invariant under \( G \). Setting \( \hbar = k^{-1} \), the family of Hilbert spaces \( \mathcal{F}_{\hbar}(M) \) provides a discrete series of projectively unitary representations of \( G \). The homogeneity of \( M \) is used to prove that the correspondence principle is satisfied in the limit \( k \to \infty \). Furthermore, let \( G' \subset G \) be a maximal isotropy subgroup of the vacuum state \( |0\rangle \). Then coherent states \( |\zeta\rangle \) are parametrised by points \( \zeta \) in the coset space \( G/G' \).
3 Quantum mechanics from Darboux’s theorem

Darboux’s theorem locally trivialises any symplectic manifold: every point of a $2n$–dimensional symplectic manifold possesses a local coordinate neighbourhood with coordinates $(p_l, q^l)$, $l = 1, 2, \ldots, n$, in which the symplectic 2–form $\omega$ is expressed as $\omega = dp_l \wedge dq^l$.

Let us now make the statement that

$\textit{Given any quantum system, there always exists a coordinate transformation that transforms the system into the semiclassical regime, i.e., into a system that can be studied by means of a perturbation series in powers of } \hbar \text{ around a certain local vacuum.}$

As with Darboux’s theorem and the Hamilton–Jacobi method\[5\], one can see the use of coordinate transformations in order to trivialise a given system. In our context, however, trivialisation does not mean cancellation of the interaction term, as in the Hamilton–Jacobi technique. Rather, it refers to the choice of a vacuum around which to perform a perturbative expansion in powers of $\hbar$. As we will see presently, this is equivalent to eliminating the metric, thus rendering quantum mechanics metrically trivial. In this sense, Darboux’s theorem for symplectic manifolds falls just short (by Planck’s constant $\hbar$) of being a quantisation, as it otherwise provides the right starting point in the passage from classical to quantum mechanics. (In the strict sense of geometric quantisation\[6\], only those symplectic manifolds that satisfy the integrality conditions can be quantised). Related geometric approaches to quantum mechanics have been presented in refs.\[7, 8, 9, 10\]. More recently, ref.\[11\] has brought to light the relevance for quantisation of infinite–dimensional, complex projective space. Darboux’s theorem has also been used in a very interesting approach to the quantisation of constrained dynamics in ref.\[12\]; this latter approach is an effective alternative to Dirac’s brackets.\[13\] The classical equivalence between Dirac’s method and that of Faddeev–Jackiw is established in ref.\[13\].

4 Discussion

4.1 The choice of a vacuum

The statement above instructs us to choose a local vacuum. Under the choice of a vacuum we understand a specific set of coordinates around which to perform an expansion in powers of $\hbar$. This choice of a vacuum is local in nature, in that it is linked to a specific choice of coordinates. It breaks the group of allowed coordinate transformations to a (possibly discrete) subgroup, leaving behind a (possibly discrete) duality symmetry of the quantum theory. Call $q$ the local coordinate corresponding to the vacuum in question, and $Q$ its quantum operator. The corresponding local momentum $P$ satisfies the usual Heisenberg algebra with $Q$. This fact reflects, at the quantum level, the property that the Darboux coordinates $p, q$ render the symplectic form $\omega$ canonical, $\omega = dp \wedge dq$.

\[1\] I wish to thank R. Jackiw for drawing my attention to this point.
However, as our starting point we have no classical phase space at all, and no Poisson brackets to quantise into commutators. This may be regarded as a manifestly non-perturbative formulation of quantum mechanics.

4.2 Quantum numbers vs. a topological quantum mechanics

Berezin’s quantisation relied heavily on the metric properties of classical phase space. The semiclassical limit could be defined as the regime of large quantum numbers. The very existence of quantum numbers was a consequence of the metric structure. If quantum mechanics is not to be formulated as a quantisation of a given classical mechanics, then we had better do away with global quantum numbers, i.e., with the metric. Metric-free theories usually go by the name of topological theories. Hence our quantum mechanics will be a topological quantum mechanics, i.e., free of global quantum numbers. Locally, of course, quantum numbers do appear, but only after the choice of a local vacuum.

4.3 Classical vs. quantum

After the choice of a local vacuum to expand around, the local quantum numbers one obtains describe excitations around the local vacuum chosen. Hence what appears to be a semiclassical excitation to a local observer need not appear so to another observer. In fact may well turn out to be a highly quantum phenomenon, when described from the viewpoint of a different local vacuum. This point has been illustrated in ref. [10], where coherent states that are local but cannot be extended globally have been analysed. The model of ref. [10] provides an explicit example of the general procedure presented above.

The logic could be summarised as follows: 1) the fact that this quantum mechanics is topological implies the absence of a metric; 2) the absence of a metric implies the absence of global quantum numbers; 3) the absence of global quantum numbers implies the impossibility of globally defining a semiclassical regime. The latter exists only locally.

5 Summary

In this paper we have analysed some general properties that quantum mechanics must satisfy, if it is not to be formulated as a metric quantisation of a given classical mechanics. We have formulated a statement, close in spirit to Darboux’s theorem of symplectic geometry, that provides a starting point for a formulation of quantum mechanics that is explicitly metric-free. Our formalism may be understood as a certain limit of Berezin’s quantisation. The latter relies on the metric properties of classical phase space $\mathcal{M}$, whenever $\mathcal{M}$ is a homogeneous Kähler manifold. In Berezin’s method, quantum numbers arise naturally from the metric on $\mathcal{M}$. The semiclassical regime is then identified with the regime of large quantum numbers. Our method may be regarded as the topological
limit of Berezin’s quantisation, topological meaning that the metric dependence has been removed. As a consequence of this topological nature our quantum mechanics exhibits the added feature that quantum numbers are not originally present. They appear only after a vacuum has been chosen and, contrary to Berezin’s quantisation, they are local in nature, instead of global.

It has been proved in ref. [14] that endowing classical phase space with a Riemannian metric is sufficient for describing quantum mechanics. In this letter we have argued that, while certainly sufficient, the above condition is not necessary in passing from classical to quantum mechanics.

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