Strategy Proof Mechanisms for Facility Location with Capacity Limits

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Abstract

An important feature of many real world facility location problems are capacity limits on the facilities. We show here how capacity constraints make it harder to design strategy proof mechanisms for facility location, but counter-intuitively can improve the guarantees on how well we can approximate the optimal solution. As a baseline, we begin by surveying what is known about strategy proof mechanisms without capacity limits. For example, we identify those strategy proof mechanisms for locating one or two uncapacitated facilities with bounded (in fact optimal) approximation ratios for the total and maximum distance. In addition, we prove that, for two or more uncapacitated facilities, the only deterministic anonymous and strategy proof mechanism with a bounded approximation ratio of the optimal maximum cost solution is the ENDPOINT mechanism for two facilities. We then consider the impact of adding capacity constraints to facilities. With two identical capacitated facilities and no spare capacity, we prove a strong characterization result: a mechanism is anonymous, Pareto optimal and strategy proof iff it is the INNERPOINT mechanism. Spare capacity and facilities of different size interfere with our ability to achieve Pareto optimality and strategy proofness. In particular, with two facilities of different capacity, or two identical facilities with spare capacity, we prove that no mechanism is anonymous, Pareto optimal and strategy proof. By contrast, if we drop capacity limits completely, there are multiple anonymous, Pareto optimal and strategy proof mechanisms for locating two uncapacitated facilities. Additional facilities also interfere with our ability to achieve Pareto optimal and strategy proofness. With three or more capacitated facilities, even if the facilities are identical and there is no spare capacity, we prove that no mechanism is anonymous, Pareto optimal and strategy proof. Again by contrast, if we drop capacity limits, there are multiple anonymous, Pareto optimal and strategy proof mechanisms for locating three facilities. Finally, for the all settings described so far in which no mechanisms exist, we argue that there are multiple mechanisms which are anonymous and strategy proof but not Pareto optimal, or Pareto optimal and strategy proof but not anonymous, or anonymous and Pareto optimal but not strategy proof. Hence, anonymity, Pareto optimality and strategy proofness are a minimal combination of incompatible axioms in all of these settings.

Introduction

The facility location problem is a classic problem in social choice that has attracted attention from researchers in several fields including AI, Operations Research, and Game Theory (e.g. (Drezner and Hamacher 2002; Lu et al. 2010; Fotakis and Tzamos 2010; Escoffier et al. 2011; Procaccia and Tennenholtz 2013; Serafino and Vento 2015; Feigenbaum and Sethuraman 2015; Golowich, Narasimhan, and Parkes 2018; Jagtenberg and Mason 2020)). Our goal here is to design mechanisms to locate one or more facilities fairly and optimally to serve a set of agents. The locations of the agents are privately known to the agents themselves. We therefore wish to identify strategy proof mechanisms that are likely to elicit the true locations of the agents and minimize the total or maximum distance of the agents from the facilities serving them. Facility location problems model many real world problems including locating schools, hospitals, warehouses, and libraries. In many of these problems, facilities have capacity limits (Brimberg et al. 2001; Aziz et al. 2019, 2020). A school has only places for a certain number of students, a warehouse can only serve a given number of shops, a hospital has only a limited number of beds, etc. Our contribution is to demonstrate that such capacity constraints can make it harder to design strategy proof mechanisms, but paradoxically can increase the welfare of the solutions that we are guaranteed to return.

As in much previous work on mechanism design for facility location (e.g. (Procaccia and Tennenholtz 2013)), we consider the one-dimensional setting. This models a number of real world problems such as locating wastewater plants along a river, or warehouses along a highway. There are also various non-geographical settings that can be viewed as one-dimensional facility location problems (e.g. choosing the temperature for a classroom, or selecting a committee to represent people with different political views). In addition, there are settings where we can use the one-dimensional problem to solve more complex problems (e.g. decomposing the 2-d rectilinear problem into a pair of 1-d problems). Finally, the one-dimensional problem is the starting point to consider more complex metrics (e.g., trees and networks) and provides bounds on solutions for these more complex settings (e.g. lower bounds for the 1-d problem provide lower bounds for the 2-d problem).
We have \( n \) agents located on the real line, and wish to locate \( m \) facilities on the real line to serve all the agents. Each agent \( i \) is at location \( x_i \). We suppose agents are ordered so that \( x_1 \leq \ldots \leq x_n \). In the uncapacitated setting, an agent is served by the nearest facility, and a solution is a location \( y_j \) for each facility \( j \). In the capacitated setting, the \( j \)th facility can serve up to \( c_j \) agents. We assume that \( \sum_{j=1}^{m} c_j \geq n \) so that every agent can be served. One special setting we consider is when there is no spare capacity (i.e. \( \sum_{j=1}^{m} c_j = n \)).

Another special setting we consider is when facilities are identical (i.e. \( c_i = c_j \) for all \( i < j \)). A solution in the capacitated setting is both a location \( y_j \) for each facility \( j \), and an assignment of agents to facilities such that the capacity limit \( c_j \) for each facility is not exceeded. Let \( a_i \in \{1, m\} \) denote the facility serving agent \( i \), and \( N_j \) denote the set of agents assigned to facility \( j \), i.e., \( N_j = \{i | a_i = j\} \). Then the capacity constraints in the capacitated setting ensure \( |N_j| \leq c_j \) for all \( j \in \{1, m\} \). We consider an utilitarian objective of the total distance, \( \sum_{i=1}^{n} |x_i - y_{a_i}| \) and an egalitarian objective of the maximum distance, \( \max_{i=1}^{n} |x_i - y_{a_i}| \). Our goal is to minimize one of these two welfare objectives.

We consider a number of mechanisms for the uncapacitated facility location problem. With parameters \( p_1 \) to \( p_m \) with \( 0 \leq p_1 \leq \ldots \leq p_m \leq 1 \), the \textsc{Percentile} mechanism locates facility \( j \) at \( x_j + |p_j(n-1)| \) for \( j \in \{1, m\} \). For example, the \textsc{Leftmost} mechanism has \( m = 1 \) and \( p_1 = 0 \), while the \textsc{Median} mechanism has \( m = 1 \) and \( p_1 = \frac{1}{2} \). The \textsc{EndPoint} mechanism which locates facilities at the left and rightmost agents has \( m = 2 \), \( p_1 = 0 \) and \( p_2 = 1 \). The \textsc{LeftRight} mechanism locates facilities at the leftmost \( j \) distinct locations of the agents, and \( k \) facilities at the rightmost \( k \) distinct locations. If agents declare less than \( j + k \) distinct locations, the mechanism puts multiple facilities at a location. The \textsc{EndPoint} mechanism is the \textsc{LeftRight} mechanism with \( j = k = 1 \), the \textsc{TwoPeaks} mechanism has \( j = 2 \) and \( k = 0 \), the \textsc{TwoPeaks} mechanism has \( j = 0 \) and \( k = 2 \), the \textsc{ThreePeaks} mechanism has \( j = 3 \) and \( k = 0 \), and the \textsc{ThreePeaks} mechanism has \( j = 0 \) and \( k = 3 \).

We also consider a number of mechanisms for the \textit{capacitated} problem. For two capacitated facilities, the \textsc{InnerPoint} mechanism locates one facility at \( x_{c_1} \) serving the leftmost \( c_1 \) agents, and the other facility at \( x_{1+c_1} \) serving the remaining agents. The \textsc{ExtendedEndPoint} mechanism is an extension of the \textsc{EndPoint} mechanism to deal with capacity limits that retains strategy proofness. It was first proposed in (Aziz et al. 2020). It can deal with facilities of different capacity, as well as with spare capacity. It is defined formally as follows.

Let \( f_1 \) and \( f_2 \) be the output locations of the two facilities. Define \( X_1 = \{i | x_i - x_1 \leq \frac{1}{2}(x_n - x_1) \} \) and \( X_2 = \{i | x_n - x_i < \frac{1}{2}(x_n - x_1) \} \). If \( |X_1| \geq |X_2| \), execute one of the following three cases.

**Case 1.** If \( |X_1| \leq c_1 \) and \( |X_2| \leq c_2 \), \( f_1 = x_1 \) and \( f_2 = x_n \). Agents in \( X_1 \) are allocated to \( f_1 \) and the others are allocated to \( f_2 \) as in the \textsc{EndPoint} mechanism.

**Case 2.** If \( |X_1| > c_1 \) and \( |X_2| \leq c_2 \), \( f_1 = 2x_{c_1+1} - x_n \) and \( f_2 = x_n \). Agents \( \{1, \ldots, c_1\} \) are allocated to \( f_1 \) and the others are allocated to \( f_2 \).

**Case 3.** If \( |X_1| \leq c_1 \) and \( |X_2| > c_2 \), \( f_1 = x_1 \) and \( f_2 = 2x_{n-c_2} - x_1 \). Agents in \( \{1, \ldots, n - c_2\} \) are allocated to \( f_1 \) and the others are allocated to \( f_2 \).

We extend the \textsc{Percentile} and \textsc{LeftRight} mechanisms to the capacitated setting by allocating agents left to right to facilities ordered left to right up by their capacity.

We consider three desirable properties of mechanisms for facility location problems: anonymity, Pareto optimality and strategy proofness. Anonymity is a fundamental fairness property that requires all agents to be treated alike. Pareto optimality is one of the most fundamental normative properties in economics. It demands that we cannot improve the solution so one agent is better off without other agents being worse off. Finally, strategy proofness is a fundamental game theoretic property that ensures agents have no incentive to act strategically and try to manipulate the mechanisms.

More formally, a mechanism is \textit{anonymous} iff permuting the agents does not change the solution. For instance, the \textsc{Median} mechanism is anonymous. A mechanism is \textit{Pareto optimal} iff it returns solutions that are always Pareto optimal. A solution is \textit{Pareto optimal} iff there is no other solution in which one agent travels a strictly shorter distance to the facility serving them, and all other agents travel no greater distance. For example, the \textsc{Median} mechanism is Pareto optimal. A mechanism is \textit{strategy proof} iff no agent can mis-report and reduce their distance to travel to the facility serving them. For example, the \textsc{Median} mechanism is strategy proof.

Finally, we will consider strategy proof mechanisms that may not return the optimal solution. In particular, we will consider how well a mechanism approximates the optimal possible welfare. A mechanism achieves an approximation ratio \( \rho \) of the total (maximum) distance iff the total (maximum) distance in any solution it returns is at most \( \rho \) times the optimal. In this case, we say that the mechanism \( \rho \)-approximates the optimal total (maximum) distance. For example, the \textsc{Least} mechanism \( 2 \)-approximates the optimal maximum distance (Theorem 2.1 in (Procaccia and Tennenholz 2013)).

**Uncapacitated Facilities**

We begin by summarizing some existing results about strategy proof mechanisms for the uncapacitated facility location problem, and by filling in a few small gaps in the literature. This summary of uncapacitated results will serve as a baseline to compare against when we add capacity constraints.

**Strategy Proofness and Anonymity**

With any number of facilities, every \textsc{Percentile} mechanisms is \textit{strategy proof} (Theorem 1 in (Sui, Boutilier, and Sandholm 2013)) and anonymous. Similarly, with any number of facilities \( m \), any \textsc{LeftRight} mechanism with \( j + k = m \) is strategy proof and anonymous.
Pareto Optimality

Pareto optimality is somewhat more difficult to achieve than strategy proofness, especially as the number of facilities increases. With a single facility, any PERCENTILE mechanism is Pareto optimal. With two facilities, the only PERCENTILE mechanism that is Pareto optimal is the ENDPOINT mechanism. With three or more facilities, no PERCENTILE mechanism is Pareto optimal. On the other hand, with any number of facilities \( m (m \geq 1) \), every \( j \text{LEFT}k\text{RIGHT} \) mechanism with \( j + k = m \) is Pareto optimal.

Total Distance

With a single facility, the MEDIAN mechanism returns the solution with optimal total distance. With two facilities, no deterministic, anonymous and strategy proof mechanism has a bounded approximation ratio for the total distance except for the ENDPOINT mechanism which has a bounded approximation ratio of \( n - 2 \) (Theorem 3.1 in Fotakis and Tzamos [2013]). With three or more facilities, no deterministic and strategy proof mechanism at all has a bounded approximation ratio for the total distance (Theorem 7.1 in Fotakis and Tzamos [2013]).

Maximum Distance

With a single facility, any PERCENTILE mechanism 2-approximates the optimal maximum distance. Indeed no deterministic and strategy proof mechanism can have a better approximation ratio (Theorem 3.2 of Procaccia and Tennenholtz [2013]). With two facilities, the ENDPOINT mechanism also provides a 2-approximation of the maximum distance. Again, no deterministic and strategy proof mechanism can have a better approximation ratio (Corollary 4.4 of Procaccia and Tennenholtz [2013]). In fact, the ENDPOINT mechanism is the only PERCENTILE mechanism for locating two uncapacitated facilities with a bounded approximation ratio for the maximum distance.

**Theorem 1** With two uncapacitated facilities, the only PERCENTILE mechanism with a bounded approximation ratio for the maximum distance is the ENDPOINT mechanism.

**Proof:** There are three cases. In the first case, \( p_1 = p_2 \). Consider a problem with agents at 0 and 1. The optimal maximum distance is 0 yet the PERCENTILE mechanism returns an allocation with maximum distance of 1. In the second case, \( 0 < p_1 < p_2 \). Consider a problem with \( n = \lceil \frac{2}{p_2 - p_1} \rceil \) agents. Suppose one agent is at 0 and the rest are at 1. The optimal maximum distance is 0 yet the PERCENTILE mechanism returns an allocation with both facilities at 1 giving a maximum distance of 1. In the third case, \( 0 = p_1 < p_2 < 1 \). This case is symmetric to the second.

With three or more facilities, no PERCENTILE mechanism has a bounded approximation ratio for the maximum distance irrespective of the parameters chosen.

**Theorem 2** With three or more uncapacitated facilities, no PERCENTILE mechanism has a bounded approximation ratio for the maximum distance.

**Proof:** Consider three facilities. There are five cases. In the first case, \( p_1 = p_2 = p_3 \). Consider a problem with agents at 0 and 1. The optimal maximum distance is 0 yet the PERCENTILE mechanism returns an solution with maximum distance of 1. In the second case, \( p_1 = p_2 < p_3 \). Consider a problem with \( n = \lceil \frac{3}{p_3 - p_2} \rceil \) agents. By definition, \( n \geq 3 \). Suppose \( 1 + \lfloor p_2 (n - 1) \rfloor \) agents are at 0, one is at \( \frac{1}{2} \) and the rest are at 1. The optimal maximum distance is 0 yet the PERCENTILE mechanism returns an solution with two facilities at 0, and one at 1 giving a maximum distance of \( \frac{1}{2} \). In the third case, \( p_1 < p_2 = p_3 \). This case is symmetric to the second case. In the fourth case, \( p_1 < p_2 < p_3 \) and \( p_3 - p_2 > p_2 - p_1 \). Consider a problem with \( n = \lceil \frac{2}{p_3 - p_2} \rceil \) agents. Suppose \( 1 + \lfloor p_1 (n - 1) \rfloor \) agents are at 0, one is at \( \frac{1}{2} \) and the rest are at 1. The optimal maximum distance is 0 yet the PERCENTILE mechanism returns a solution with one facility at 0, and two at 1 giving a maximum distance of \( \frac{1}{2} \). In the fifth case, \( p_1 < p_2 < p_3 \) and \( p_3 - p_2 \leq p_2 - p_1 \). This case is symmetric to the fourth case. The proof easily extends to four or more facilities.

Actually, the last two results can be strengthened from any PERCENTILE mechanism to any deterministic and strategy proof mechanism and to fewer agents. We need the following simple proposition.

**Proposition 1** A class of mechanisms has a bounded approximation ratio for the maximum distance iff the same class has a bounded approximation ratio for the total distance.

**Proof:** We exploit the following relation between the number of agents \( n \), the total distance \( d_{\text{total}} \), and the maximum distance \( d_{\text{max}} \): \( d_{\text{max}} \leq n d_{\text{max}} \). Therefore if \( d_{\text{max}} \) is bounded then so is \( n d_{\text{max}} \), and by transitivity, \( d_{\text{total}} \) is bounded. For the reverse, we exploit the simple relation: \( d_{\text{max}} \leq d_{\text{total}} \). Therefore if \( d_{\text{total}} \) is bounded then so is \( d_{\text{max}} \). Hence, the maximum distance is bounded iff the total distance is too.

By mapping across the corresponding results about total cost from Fotakis and Tzamos [2013] using this simple proposition, we provide the following characterization result for strategy proof mechanisms with a bounded approximation for the maximum cost.

**Theorem 3** With two or more uncapacitated facilities, the only deterministic, anonymous and strategy proof mechanism with a bounded approximation ratio for the maximum distance is the ENDPOINT mechanism (in the case of two uncapacitated facilities).

**Proof:** This follows immediately from Proposition 1. Theorem 3.1 in Fotakis and Tzamos [2013] which identifies ENDPOINT as the only deterministic, anonymous and strategy mechanism with bounded approximation for the total cost with two uncapacitated facilities and five or more agents, and Theorem 7.1 in Fotakis and Tzamos [2013] which proves that no such mechanism exists with \( m \) uncapacitated facilities (\( m \geq 3 \)) and \( m + 1 \) or more agents.

To summarize, with a single uncapacitated facility, any parameter for the PERCENTILE mechanism provides a 2-approximation of the optimal maximum distance (and no deterministic and strategy proof mechanism can do better); with two facilities, only the parameters 0 and 1 provide a...
Theorem 4

solution set puts the smaller facility at \( x_1 \) serving the leftmost agent and the larger facility somewhere in the interval \([x_2, x_3]\) serving the other two agents. Now move the agent at \( x_3 \) to \( x_2 \). The unique Pareto optimal solution puts the smaller facility at \( x_1 \) serving the leftmost agent and the larger facility at \( x_2 \) serving the two agents there. We next move the rightmost agent from \( x_2 \) back towards its original position at \( x_3 \). Let \( x \) be the distance of the rightmost agent from \( x_2 \) so that \( x \) varies from 0 to \( x_3 - x_2 \), and \( f(x) \) be the distance of the rightmost agent from the facility serving them. It is not hard to show that since the mechanism is strategy proof, \( f(x) \) must be a continuous function of \( x \). Any discontinuity would give the rightmost agent an opportunity to mis-report their location strategically and travel less distance. The location of the rightmost facility therefore tracks continuously to the right, staying within the interval \([x_2, x_2 + x]\) to ensure Pareto optimality. There are four scenarios for where the mechanism locates the rightmost facility as we vary \( x \):

(a) the larger facility remains at \( x_2 \) as with the \textsc{innerPoint} mechanism, (b) the larger facility remains at \( x_2 + x \), (c) the larger facility tracks \( x_2 + x \) until some \( x' \) with \( x_2 + x' < x_3 \) after which the location of the larger facility remains static or (d) the larger facility at some point tracks behind and is strictly between \( x_2 \) and \( x_2 + x \). Note that the larger facility cannot track in front of \( x_2 + x \) as this would not be Pareto optimal. In case (b), consider \( x_1 = \frac{x}{5} \) and \( x = \frac{3}{5} \). Then the middle agent at \( x_2 \) can profitably misreport their location as 0. The leftmost facility will then be located at 0 serving the middle agent. The distance the middle agent travels to be served thereby decreases from \( \frac{3}{5} \) to \( \frac{2}{5} \) violating the assumption that the mechanism is strategy proof. In case (c), suppose \( x_1 = \frac{x}{5} \) and \( x = x_2 \) then the agent at \( x_2 \) can profitably misreport their location as \( \frac{3}{5} \), contradicting the assumption that the mechanism is strategy proof. In case (d), the larger facility tracks strictly behind \( x_2 + x \). By continuity arguments, we can identify two values, \( x = a \) and \( x = b \) with \( a < b \) such that when the rightmost agent is at \( x_2 + b \), the rightmost facility is located at \( x_2 + a \), and when the rightmost agent is at \( x_2 + a \), the rightmost facility is located at \( x_2 + c \) where \( c < a \). Then if agents are at \( x_1 \), \( x_2 \) and \( x_2 + a \), the rightmost agent at \( x_2 + a \) can profitably misreport their location as \( x_2 + b \). This violates the assumption that the mechanism is strategy proof. Therefore the only case that does not lead to a contradiction is case (a). That is, the mechanism acts like the \textsc{innerPoint} mechanism.

In the step case, we suppose that the only anonymous, Pareto optimal and strategy proof mechanism for \( 2k \) agents and two facilities of capacity \( k \), or for \( 2k + 1 \) agents and a facility of capacity \( k \) on the left and a facility of capacity \( k + 1 \) on the right is the \textsc{innerPoint} mechanism. We need to prove two subcases: the first subcase of \( 2k + 2 \) agents and two facilities of capacity \( k + 1 \) and the second subcase of \( 2k + 3 \) agents and a facility of capacity \( k + 1 \) on the left and a facility of capacity \( k + 2 \) on the right. We consider the first subcase. Suppose the \( 2k + 2 \) agents are at \( x_1 \leq x_2 \leq \ldots \leq x_{2k} \leq x_{2k+1} \leq x_{2k+2} \). We move the leftmost agent at \( x_1 \) to 0 and suppose it is fixed and served by the leftmost facility. We now have a facility location prob-

Two Capacitated Facilities

Since we suppose facilities have enough capacity to serve all agents, the first non-trivial capacitated case to consider is two facilities. Capacity constraints can make strategy proofness harder to achieve even in this simplest of settings. For instance, the \textsc{endpoint} mechanism stops being strategy proof when we add capacity limits. In fact, the only \textsc{percentile} mechanism remains strategy proof with the addition of capacity constraints is the \textsc{innerPoint} mechanism (Theorem 9 of [Aziz et al. 2020]).

Supposing we have two identical facilities with no spare capacity, we now prove that the \textsc{innerPoint} mechanism is in fact the only mechanism that is anonymous, Pareto optimal and strategy proof. We contrast this strong characterization result with the uncapacitated problem where there are multiple mechanisms for locating two uncapacitated facilities that are anonymous, Pareto optimal and strategy proof (e.g. the \textsc{twoLeftPeaks} or \textsc{endpoint} mechanisms).

Theorem 4

With \( 2k \) agents and two facilities of capacity \( k \), a mechanism is anonymous, Pareto optimal and strategy proof iff it is the \textsc{innerPoint} mechanism.

Proof: It is easy to see that the \textsc{innerPoint} mechanism is anonymous, Pareto and strategy proof. Therefore it remains to show that if a mechanism is anonymous, Pareto optimal and strategy proof then it is the \textsc{innerPoint} mechanism. We actually prove a stronger statement: if a mechanism is anonymous, Pareto optimal and strategy proof for \( 2k \) agents and two facilities of capacity \( k \), or for \( 2k + 1 \) agents and a facility of capacity \( k \) on the left and another facility of capacity \( k + 1 \) on the right then that mechanism must be the \textsc{innerPoint} mechanism locating one facility at the \( k \)th agent serving the leftmost \( k \) agents, and the other facility at the \( k + 1 \)th agent serving the remaining agents. The proof uses induction on \( k \), the capacity of the facilities. We suppose agents report locations on \([0, 1]\) and scale the problem to ensure this if necessary.

In the base case, \( k = 1 \). There are two subcases. In the first subcase, we have just two agents and two facilities of capacity 1. The unique Pareto optimal solution locates a facility at the location of each agent. A mechanism that does this is a strategy proof and anonymous. This is also the solution return by the \textsc{innerPoint} mechanism. In the second subcase, we have three agents, a facility of capacity 1 on the left and a facility of capacity 2 on the right. Suppose the three agents are at \( x_1 \leq x_2 \leq x_3 \). The Pareto optimal
lem with $2k + 1$ agents, and a facility of capacity $k$ on left and $k + 1$ on the right. By the induction hypothesis, the only anonymous, Pareto optimal and strategy proof mechanism is the \textsc{InnerPoint} mechanism that locates one facility at $x_{k+1}$ serving agents located in the interval $[x_2, x_{k+1}]$, and the other facility at $x_{k+2}$ serving the remaining agents. We next move the leftmost agent from 0 back to $x_1$. By similar continuity arguments used in the base case, the leftmost facility must remain at $x_{k+1}$. Continuity arguments also prevent the rightmost facility moving away from $x_{k+2}$ or for agents switching facility when $x_{k+2} > x_{k+1}$ as such a switch changes the distances traveled. If $x_{k+2} = x_{k+1}$ we don’t care which facility serves which agent as the two facilities are co-located and switching is irrelevant. Hence, with the leftmost agent back at $x_1$, the solution is that returned by the \textsc{InnerPoint} mechanism.

This leaves the final subcase of $2k + 3$ agents and a facility of capacity $k + 1$ on the left and a facility of capacity $k + 2$ on the right. Suppose the agents are at $x_1 \leq x_2 \leq \ldots \leq x_{2k} \leq x_{2k+1} \leq x_{2k+2} \leq x_{2k+3}$. We move the rightmost agent at $x_{2k+3}$ to 1 and suppose it is fixed and served by the rightmost facility. We now have a facility location problem with $2k + 2$ agents, and two facilities of capacity $k + 1$. By the previous case, the only anonymous, Pareto optimal and strategy proof mechanism for such a setting is the \textsc{InnerPoint} mechanism that locates one facility at $x_{k+1}$ serving agents located in the interval $[x_1, x_{k+1}]$, and the other facility at $x_{k+2}$ serving the remaining agents. We next move the rightmost agent from 1 back to $x_{2k+3}$. By similar continuity arguments, the rightmost facility must remain at $x_{k+2}$. Continuity arguments also prevent the leftmost facility moving away from $x_{k+1}$ or for agents to switch facility when $x_{k+2} > x_{k+1}$ and such a switch changes the distances traveled. Hence, with the rightmost agent back at $x_{2k+3}$, the solution is that returned by the \textsc{InnerPoint} mechanism. \hát

It follows immediately that the \textsc{InnerPoint} mechanism is the unique \textsc{Percentile} mechanism that is Pareto optimal. We next consider dropping in turn one of anonymity, Pareto optimality and strategy proofness. If we drop anonymity, then there are multiple other mechanisms besides the \textsc{InnerPoint} mechanism that are Pareto optimal and strategy proof but not anonymous. For example, the \textsc{CapSD} mechanism is a modified serial dictator mechanism for locating capacitated facilities that is Pareto optimal and strategy proof but not anonymous.

\textbf{Example 1 (CapSD mechanism)} Consider the following capacitated version of a serial dictatorship. We order the agents in some arbitrary way. We then locate the first facility at the position of the first agent in this order. If the next agent in the order is at the same location and there is capacity remaining in the first facility, we allocate this next agent to the first facility and repeat. Otherwise we locate the second facility at the position of the next agent in the order. To finish the solution, we must allocate any remaining agents to one of the two facilities to ensure Pareto optimality. We consider the remaining agents in order, allocating each to the nearest facility with remaining capacity. If an agent is equidistant from both facilities and both facilities have spare capacity, we skip allocating them to a facility till the next phase. This prevents spare capacity in a facility being taken up by an agent that doesn’t care to which facility it is allocated, when that spare capacity could be better used by a later agent. Finally, we are left with unallocated agents that are equidistant from the two facilities. We can allocate these agents in an arbitrary way respecting capacities of the facilities. The resulting mechanism is Pareto optimal and strategy proof but not anonymous. The mechanism can be easily extended to allocate three or more facilities, respecting capacity constraints yet ensuring Pareto optimality.

If we drop Pareto optimality, there are multiple mechanisms besides the \textsc{InnerPoint} mechanism that are anonymous and strategy proof but not Pareto optimal (e.g. any \textsc{Percentile} mechanism with $p_1 = p_2$). And if we drop strategy proofness, there are multiple mechanisms besides the \textsc{InnerPoint} mechanism that are anonymous and Pareto optimal but not strategy proof (e.g. the EndPoint and TwoRightPeaks mechanisms). Hence, anonymity, Pareto optimality and strategy proofness are the minimal combination of axioms characterizing the \textsc{InnerPoint} mechanism.

\section*{Three Capacitated Facilities}

If we increase the number of facilities, strategy proofness and Pareto optimality become harder to achieve simultaneously. In fact, even in the restricted setting of just three capacitated facilities of equal size but no spare capacity, no mechanism can be simultaneously anonymous, Pareto optimal and strategy proof. We again contrast this with the uncapacitated problem where there are multiple mechanisms for locating three uncapacitated facilities that are anonymous, Pareto optimal and strategy proof (e.g. the \textsc{ThreeLeftPeaks} and \textsc{ThreeRightPeaks} mechanisms). The following strong impossibility theorem demonstrates that capacity limits make it impossible to achieve anonymity, Pareto optimality and strategy proofness with three or more facilities.

\textbf{Theorem 5} With $km$ agents and $m$ facilities of capacity $k$ ($m \geq 3$ and $k \geq 2$), no mechanism is anonymous, Pareto optimal and strategy proof.

\textbf{Proof:} We demonstrate the absence of an anonymous, strategy proof and optimal mechanism for 6 agents and 3 facilities of capacity 2. The argument easily generalizes to more agents and to more facilities. Suppose there is such a mechanism acting on 6 agents: two at position 0, two at 10, and two at 20. There is an unique Pareto optimal solution, with one facility at 0, another at 10 and a third at 20. Consider moving one of the agents, which we call $a$ from its position at 10. Let $f(x)$ be the function representing the distance of the facility serving $a$ from their reported location $x$. Thus $f(10) = 0$. As the mechanism locating facilities is strategy proof, $f(x)$ needs to be a continuous function of $x$. Indeed, it is possible to show that if $x$ changes by some $\delta$ then $f(x)$ cannot change by more than $2\delta$ or strategic manipulation of the mechanism would be possible. Now suppose we move agent $a$ from position 10 to 11. Then the facility serving $a$ cannot be too far from $a$. Indeed, to ensure Pareto optimality,
it must be the case that one facility remains at 0, the other at 20, and the third lies somewhere in the interval $[10, 11]$.

Suppose now the agents at 0 are fixed, but the other four agents are allowed to vary. Then we effectively have a two facility, four agent mechanism acting on the agents at 10, 11 and two at 20 which locates the two rightmost facilities, while the leftmost facility remains at 0. By Theorem 4, the only anonymous, Pareto optimal and strategy proof mechanism for locating two facilities is the INNERPOINT mechanism. This locates the two rightmost facilities at 11 and 20. Suppose now that the agents at 20 are fixed, but the other four agents are allowed to vary. Then we again effectively have a two facility, four agent mechanism acting on agents at 10, 11 and two at 0 which locates the two leftmost facilities, while the rightmost facility remains at 20. By Theorem 4, this must be the INNERPOINT mechanism which locates the two leftmost facilities at 0 and 10. This is a contradiction as the middle facility cannot simultaneously be at position 10 and 11. Therefore our assumption that there is an anonymous, Pareto optimal and strategy proof mechanism for three capacitated facilities must be incorrect. ♦

It follows immediately that no PERCENTILE mechanism for locating three or more capacitated facilities is Pareto optimal. If we drop anonymity, there are multiple mechanisms that are Pareto optimal and strategy proof (e.g. any CAPASD mechanism). Similarly, if we drop Pareto optimality, there are multiple mechanisms that are anonymous and strategy proof (e.g. any PERCENTILE mechanism with $p_1 = p_2 = p_3$). And if we drop strategy proofness, there are multiple mechanisms that are anonymous and Pareto optimal (e.g. THREELEFTPEAKS, and THREEIGHTPEAKS mechanisms). Hence, anonymity, Pareto optimality and strategy proofness are a minimal combination of incompatible axioms for locating three or more capacitated facilities.

**Welfare Bounds**

We next consider the impact of capacity constraints on the ability of strategy proof mechanisms to approximate the optimal solution. With no spare capacity, we can give a tight lower bound on the approximation ratio for the total distance. No deterministic and strategy proof mechanism can do better than this bound.

**Theorem 6** For $2k$ agents, any deterministic and strategy proof mechanism to locate two facilities of capacity $k$ has an approximation ratio for the total distance of at least $k - 1$.

**Proof:** Suppose there exists a mechanism with a smaller approximation ratio than $k - 1$. A mechanism is partially group strategy proof iff no group of agents at the same location can individually benefit if they misreport simultaneously. As in the uncapacitated setting (Lemma 2.4 in (Fotakis and Tzamos 2013)), a simple transitivity proof demonstrates that in the two capacitated facility game, a strategy proof mechanism is also partially group strategy proof (Lu et al. 2010). Since our mechanism is strategy proof by assumption, it is also partially group strategy proof. And since it has a bounded approximation ratio, it satisfies unanimity. Consider $2k$ agents at 0. By unanimity, the mechanism returns the solution with both facilities at 0. Consider $k + 1$ agents simultaneously moving from 0 to location $x$ for $0 \leq x \leq 1$. Since the mechanism is partial group strategy proof and unanimous, one of three cases must occur. In the first case, both facilities remain at 0. In the second case, both facilities track $x$ to location 1. In the third case, both facilities track $x$ to location $a$ and then remain stationary. The total distance is lowest in the second case and is $k - 1$. However, even this best case contradicts the assumption that there exists a mechanism with a smaller approximation ratio than $k - 1$ since the optimal solution has facilities at 0 and 1 with a total distance of just 1. Note that this lower bound on the approximation ratio would apply to any metric space, not just the line. ♦

With two identical facilities of capacity $k$ and no spare capacity, the INNERPOINT mechanism has an approximation ratio for the total cost of at most $k - 1$ (Theorem 5 in (Aziz et al. 2020)). Hence, the INNERPOINT mechanism is optimal. No other deterministic and strategy proof mechanism can have a better approximation ratio. We contrast this with the uncapacitated setting where the lower bound for any deterministic and strategy proof mechanism is $n - 2$ which is nearly twice as big (Corollary 3.2 in (Fotakis and Tzamos 2013)). Counter-intuitively adding capacity constraints improves the guarantee on the approximation of the optimal solution.

We also prove a tight lower bound on the approximation ratio for the maximum distance. No deterministic and strategy proof mechanism can do better than this bound. Interestingly, this is the same lower bound as in the uncapacitated setting.

**Theorem 7** With two or more facilities of capacity $k$, and no spare capacity, any deterministic and strategy proof mechanism has an approximation ratio for the maximum distance of at least $2$.

**Proof:** Suppose we have an approximation ratio that is at most $a$ with $a \geq 1$. Consider two agents at $4a$, and the other two agents in $[0, 1]$. The optimal allocation has a maximum distance of at most $\frac{a}{2}$. Hence, to meet the approximation ratio, the mechanism will return a solution with a maximum distance of $\frac{a}{2}$ or less. Therefore, the two agents in $[0, 1]$ will be allocated a facility at or to the left of position $1 + \frac{a}{2}$, and the two agents at $4a$ to a facility in $\left[\frac{7a}{2}, \frac{9a}{2}\right]$. Now we can view the allocation of the two agents in $[0, 1]$ as a problem of locating a single facility of unbounded capacity using a deterministic and strategy proof mechanism with bounded approximation ratio for the maximum distance. And by Theorem 3.2 of (Procaccia and Tennenholtz 2013), this leads to a contradiction for $a < 2$. Hence, the approximation ratio $a$ is at least 2. The proof extends to additional agents and facilities easily. ♦

As the INNERPOINT mechanism achieves an approximation ratio of at most 2 (Aziz et al. 2020), it is optimal. No deterministic and strategy proof mechanism is guaranteed to return better quality solutions. It is perhaps surprising that, with two capacitated facilities, we can approximate the maximum distance so well. For the more relaxed problem of approximating the maximum distance with a single uncapsitated facility, the best possible deterministic and strategy
proof mechanism already has an approximation ratio of 2. Despite complicating the problem with an additional facility, and with capacity limits, we can still do just as well.

**Spare Capacity**

So far, we have supposed that there is no spare capacity in the problem. We now relax this assumption. We observe that this can make Pareto optimality and strategy proofness harder to achieve simultaneously even with just two facilities.

**Theorem 8** No mechanism is anonymous, Pareto optimal and strategy proof when we are locating \( m \) facilities of equal capacity \( c \) and there is a spare capacity of size \( k \) where \( m \geq 2, c > 2 \) and \( m(c - 2) > k \geq 1 \).

**Proof:** Consider \( m = 2, c = 3 \) and \( k = 1 \). Suppose we have three agents at 0 and two at 1. Then the unique Pareto optimal solution puts one facility at 0 and the other at 1. Now suppose one agent at 0 is fixed. This modified problem is equivalent to a problem with four agents and two facilities of capacity 2. The only anonymous, Pareto optimal and strategy proof mechanism for such a four agent problem is the INNERPOINT mechanism. Suppose we move one of the agents at 0 to 1. The INNERPOINT mechanism now locates both facilities at 1. This is not Pareto optimal. Our modified problem has two agents at 0 and three at 1. The unique Pareto optimal solution of this modified problem has one facility at 0 and the other at 1. The construction easily extends to larger \( c, m \) and \( k \). For example, for larger \( c \), we put additional agents at both 0 and 1 while for larger \( m \), we put additional agents at 2.

Note that if the spare capacity is sufficiently large then there are mechanisms that are anonymous, Pareto optimal and strategy proof. For example, when the spare capacity is \( m(c - 1) \), we have only \( m \) agents for the \( m \) facilities. We can then simply locate a facility at each agent. Another edge case is three agents and two facilities of capacity 2. Consider the mechanism that locates one facility at the leftmost agent, the other at the rightmost agent, and allocates the middle agent to whichever facility is nearest. This mechanism is anonymous, Pareto optimal and strategy proof.

Note also that if we drop anonymity, there are *multiple* mechanisms that are Pareto optimal and strategy proof despite the presence of spare capacity (e.g., any CAPSD mechanism). Similarly, if we drop Pareto optimality, there are *multiple* mechanisms that are anonymous and strategy proof despite the presence of spare capacity (e.g., any PERCENTILE mechanism with \( p_1 = p_2 = p_3 \). And if we drop strategy proofness, there are *multiple* mechanisms that are anonymous and Pareto optimal despite the presence of spare capacity (e.g., TWOLEFTPEAKS, and TWOIGHTPEAKS mechanisms). Hence, anonymity, Pareto optimality and strategy proofness are a minimal combination of incompatible axioms in the presence of spare capacity.

Counter-intuitively, when we introduce spare capacity, the lower bound on approximating the optimal total distance increases. More capacity means we may have a less good approximation of the optimal solution. More precisely, increasing the spare capacity increases the lower bound on the optimal total distance linearly from \( \frac{n}{2} - 1 \) to \( n - 2 \). Eventually and unsurprisingly it matches the lower bound of \( n - 2 \) seen in the uncapacitated setting (Corollary 3.2 in [Fotakis and Tzamos 2013]). Note that we cannot directly apply lower bounds from the uncapacitated setting to our capacitated problem as the space of mechanisms in the capacitated setting is imposing and thus different to that in the uncapacitated setting [Fotakis and Tzamos 2010]. In the capacitated setting, we force an agent to be served by a particular facility (an “imposing” mechanism) while in the uncapacitated setting agents can visit whichever facility is nearest their true location.

**Theorem 9** Any deterministic and strategy proof mechanism for \( n \) agents (\( n > 2 \)) to locate two facilities both of capacity \( n - k \) or larger for \( \frac{n}{2} \geq k \geq 1 \) has an approximation ratio for the total distance of at least \( n - k - 1 \).

**Proof:** Consider \( k = 1 \). Suppose there exists a deterministic and strategy proof mechanism with an approximation ratio of less than \( n - 2 \). As the approximation ratio is bounded, the mechanism satisfies a generalized form of unanimity in which any zero distance solution is returned. Consider \( n - 1 \) agents at 0, and one agent at \(-n^2\). The optimal total distance is 0 so any mechanism with an approximation ratio of less than \( n - 2 \) puts one facility at 0 and the other at \(-n^2\), both facilities serving the agents at their immediate location. Suppose we now move one agent at 0 rightwards to location \( x \) for \( x \geq 0 \). We then have one agent at \(-n^2\), \( n - 2 \) agents at 0 and one agent at \( x \). To ensure strategy proofness and unanimity, one of three cases occurs. In the first, the facility originally at 0 tracks \( x \) and continues to serve this agent. In the second, the facility at 0 stays at 0 and continues to serve the agent at \( x \). In the third, the facility at 0 tracks \( x \) until some location \( a > 0 \), and then remains fixed. In the second case, the lower bound on the approximation ratio is again surely violated. Hence, we can only be in the first case, with the facility tracking \( x \). Suppose \( x = 1 \). To meet the lower bound, the agents at 0 must continue to be served by the same facility. By similar arguments, starting from \( n - 1 \) agents at 1 and one agent at \(-n^2\) and moving \( n - 2 \) agents at 1 back to 0, we can conclude that the \( n - 2 \) agents at 0 must be served by the facility at 1. But the optimal solution puts one facility at \(-n^2\) and the other at 0. The solution constructed by our mechanism with one facility at 1 violates the lower bound wherever the second facility is located. Hence we have a contradiction that a deterministic and strategy proof mechanism exists which beats the lower bound. For larger \( k \), we construct a similar argument around the possible location of facilities with \( k \) agents at \(-n^2\), \( n - k - 1 \) agents at 0, and one agent at \( x \) for \( x \geq 0 \). Note that for \( n \) even and \( k = \frac{n}{2} \), the lower bound derived here of \( n - k - 1 \) equals the lower bounds of \( \frac{n}{2} - 1 \) derived in Theorem 1.

For two facilities of capacity \( n - 1 \) or greater, the ENDPOINT mechanism (which locates a facility at the leftmost and rightmost locations serving whichever agents are nearest) achieves this lower bound and so is optimal.
other deterministic and strategy proof mechanism can have a better approximation ratio for the total distance. We see therefore that increasing the spare capacity turns the problem inside out – the optimal mechanism goes from the INNERPOINT mechanism that locates facilities on the inside around the median agents when there is no spare capacity to the ENDPOINT mechanism that locates facilities at the end points of the reported locations when there is a large amount of spare capacity. It follows immediately that the ENDPOINT mechanism, which has an approximation ratio for the total distance of $\frac{3m}{2k}$ (Theorem 11 in (Aziz et al. 2020)), is asymptotically optimal. It is an interesting open question to close the gap between $n - k - 1$ and $\frac{3m}{2}$.

For the maximum distance, we can easily adjust the proof of Theorem 7 to show that, even with spare capacity, the approximation ratio for the maximum distance is again at least 2. It immediately follows that the ENDPOINT mechanism, which achieves an approximation ratio of 4 (Theorem 12 in (Aziz et al. 2020)), is asymptotically optimal. It is also an interesting open question to close the gap between 2 and 4.

### Unequal Capacity

So far, we have supposed that all facilities have the same capacity. What happens if facilities can have different capacity to each other? We will show that unsurprisingly this can make it harder to achieve Pareto optimality and strategy proofness simultaneously. For instance, with two facilities of different capacities, there is no anonymous, Pareto optimal and strategy proof mechanism.

**Theorem 10** With two (or more) facilities of two (or more) different capacities, no mechanism is anonymous, Pareto optimal and strategy proof, with or without space capacity, provided the capacity of the smallest facility is at least 2.

**Proof:** Consider no spare capacity, one facility of capacity 3 and the other of capacity 2. Suppose we have three agents at 0 and two at 1. Then the unique Pareto optimal solution puts the facility of capacity 3 at 0 and the facility of capacity 2 at 1. Now suppose we fix one agent at 0. This modified problem is equivalent to a problem with four agents and two facilities of capacity 2. The only anonymous, Pareto optimal and strategy proof mechanism for such a four agent problem is the INNERPOINT mechanism. Suppose we move one of the agents at 0 to 1. The INNERPOINT mechanism locates both facilities at 1. This is not Pareto optimal. The unique Pareto optimal solution of our modified problem has the facility of capacity 2 at 0 and the facility of capacity 3 at 1. The construction easily extends to larger capacities, to more than two facilities, and to problems with spare capacity.

An edge case is when we have two facilities and one has just capacity for a single agent. More generally, suppose we have $m + 1$ agents, one facility of capacity $m$ and the other of capacity 1 where $m > 1$. Consider the mechanism that locates the largest facility at the leftmost agent, and the other at the rightmost agent when the $m$th agent from the left is nearer the leftmost agent, and otherwise locates the largest facility at the rightmost agent and the other facility at the leftmost agent. Agents are allocated to the nearest facility. This mechanism is anonymous, strategy proof and Pareto optimal.

Note also that if we drop anonymity, there are multiple mechanisms that are Pareto optimal and strategy proof despite the presence of facilities of unequal capacity (e.g. any CAPSD mechanism). Similarly, if we drop Pareto optimality, there are multiple mechanisms that are anonymous and strategy proof despite the presence of facilities of unequal capacity (e.g. any PERCENTILE mechanism with $p_1 = p_2 = p_3$). And if we drop strategy proofness, there are multiple mechanisms that are anonymous and Pareto optimal despite the presence of facilities of unequal capacity (e.g. TWOLEFTPEAKS, and TWORIGHTPEAKS mechanisms). Hence, anonymity, Pareto optimality and strategy proofness are a minimal combination of incompatible axioms when facilities can have different capacity.

As might be expected, the more equal the capacity of the different facilities, the better the guarantee on the approximability of the optimal solution.

**Theorem 11** Any deterministic and strategy proof mechanism for $n$ agents ($n > 2$) to locate two facilities of capacity $k$ and $n - k$ with $\frac{k}{2} \geq k \geq 1$ has an approximation ratio for the total cost of at least $n - k - 1$.

**Proof:** The argument is essentially the same as in the proof of Theorem 7.

It again immediately follows from this result that the ENDPOINT mechanism, which achieves an approximation ratio of $\frac{3n}{2k}$ (Theorem 11 in (Aziz et al. 2020)) with facilities of different capacities, is asymptotically optimal. It is an interesting open problem to close the gap between the upper and lower bounds.

For the maximum distance, we can easily adjust the proof of Theorem 7 to show that, even with facilities of different capacities, the approximation ratio for the maximum distance is at least 2. It follows that the ENDPOINT mechanism, which achieves an approximation ratio of 4 (Theorem 12 in (Aziz et al. 2020)), is asymptotically optimal. It is again an interesting open problem to close the gap between the upper and lower bounds.

### Conclusions

We have studied the impact of capacity constraints on mechanisms for facility location considering three important axiomatic properties: anonymity which is a fundamental fairness property, Pareto optimality which is a fundamental efficiency property, and strategy proofness which is a fundamental property about incentives to report sincerely. As a baseline, we began by surveying what is known about strategy proof mechanisms without capacity limits. For example, we identified those mechanisms for locating one or two uncapacitated facilities with bounded (in fact optimal) approximation ratios for the total and maximum distance. In addition, we filled a small gap in the literature by proving that, with three or more uncapacitated facilities, the only deterministic, anonymous and strategy proof mechanism with a bounded approximation ratio for the maximum distance is the ENDPOINT mechanism.
Our main contribution, however, is in the design of strategy proof mechanism for locating facilities with capacity limits on the number of agents that can be served. We provided a comprehensive characterization of the anonymous, Pareto optimal and strategy proof mechanisms for locating capacitated facilities. We also identified tight bounds on how well such mechanisms can approximate the optimal total or maximum distance. In general, and as might be expected, capacity constraints make it more difficult to design strategy proof mechanisms for facility locations. However, a little unexpectedly, capacity constraints can result in better guarantees on the quality of the approximate solutions returned.

There are many directions for future work. For example, randomization is a useful tool to enable mechanisms to ensure desirable axiomatic properties like anonymity and strategy proofness. It would therefore be interesting to extend our analysis from purely deterministic to randomized mechanisms for capacitated facility location. As a second example, many problems are not one dimensional. Can we extend these results to trees, networks, and two-dimensional rectilinear or Euclidean metrics? As a third example, there is a dual class of obnoxious facility location problems where agents wish to be as far as possible from the facility such as a rubbish dump, nuclear power station, or prison. There are also mixed facility location problems where some agents wish to be close to the facility and others far away such as a playground or cell phone tower. It would be interesting to consider the impact of capacity constraints on such problems.

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