Some Properties of the Essential Fuzzy and Closed Fuzzy Submodules

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Abstract
In this paper, we introduce and study the essential and closed fuzzy submodules of a fuzzy module X as a generalization of the notions of essential and closed submodules. We prove many basic properties of both concepts.

Keywords: Fuzzy Module, Closed Fuzzy, essential fuzzy submodule.

Introduction
The notion of a fuzzy subset of a nonempty set S as a function from S into [0,1] was first developed by Zadeh [1]. The concept of fuzzy modules was introduced by Zahedi [2], whereas that of fuzzy submodules was introduced by Martines [3].

A non-zero proper submodule A of a module M is called an essential if \( A \cap B \neq (0) \), for any non-zero submodule B of M [4, 5].

Rabi [6] fuzzified this concept to the essential fuzzy submodule of a fuzzy module X.

Goodeal [4,7] introduced and studied the concept of closed submodules, where a submodule A of an R-module M is said to be a closed submodule of M (A \( \leq_c \) M), if it has no proper essential extension.

In this paper, we shall give some properties of the essential fuzzy submodules. Also, we introduce the concept of the closed fuzzy submodules. We establish many properties and characterizations of these concepts.

Throughout this paper, R is a commutative ring with unity, M is an R-module, and X is a fuzzy module of an R-module M.

1.Preliminaries
In this section, we shall provide the concepts of fuzzy sets and operations on fuzzy sets, with some of their important properties which are used in the next sections of the paper.

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Definition 1.1:[1]
Let $M$ be non-empty set and let $I$ be the closed interval $[0,1]$ of the real line (real number). A fuzzy set $X$ in $M$ (a fuzzy subset $X$ of $M$) is characterized by a membership function $X:M \rightarrow I$, which associates with each point $x \in M$ its degree of membership $X(x) \in [0,1]$.

Definition 1.2: [2]
Let $x_t : M \rightarrow I$ be a fuzzy set in $M$, where $x \in M$, $t \in [0,1]$, defined by:
$$x_t = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases} \text{ for all } y \in M$$
Then $x_t$ is called a fuzzy singleton.

If $x = 0$ and $t = 1$, then :
$$0_1(y) = \begin{cases} 1 & \text{if } y = 0 \\ 0 & \text{if } y \neq 0 \end{cases}$$
We shall call such fuzzy singleton the fuzzy zero singleton.

Definition 1.3: [2]
Let $A$ and $B$ be two fuzzy sets in $M$, then:
1. $A = B$ if and only if $A(x) = B(x)$ for all $x \in M$.
2. $A \subseteq B$ if and only if $A(x) \leq B(x)$, for all $x \in M$. If $A \subseteq B$ and there exists $x \in M$ such that $A(x) < B(x)$, then we write $A \subset B$ and $A$ is called a proper fuzzy subset of $B$.
3. $x_t \subseteq A$ if and only if $X_t(y) \leq A(y)$, for all $y \in M$, and if $t > 0$ then $A(x) \geq t$. Thus, $x_t \subseteq A(x \in A_t)$, (that is, $x_t \in A$ and only if $x_t \subseteq A$).
Next, we give some operations on fuzzy sets:

Definition 1.4: [2]
Let $A$ and $B$ be two fuzzy sets in $M$, then:
1. $(A \cup B)(x) = \max\{A(x), B(x)\}$, for all $x \in M$.
2. $(A \cap B)(x) = \min\{A(x), B(x)\}$, for all $x \in M$.
$A \cup B$ and $A \cap B$ are fuzzy sets in $M$.
In general, if $\{A_\alpha, \alpha \in \Lambda\}$ is a family of fuzzy sets in $M$, then:
$$\bigcap_{\alpha \in \Lambda} A_\alpha(x) = \inf\{A_\alpha(x), \alpha \in \Lambda\}, \text{ for all } x \in S.$$ $$\bigcup_{\alpha \in \Lambda} A_\alpha(x) = \sup\{A_\alpha(x), \alpha \in \Lambda\}, \text{ for all } x \in S.$$ Now, we give the definition of the level subset, which is a set between a fuzzy set and an ordinary set.

Definition 1.5: [3]
Let $A$ be a fuzzy in $M$, for all $t \in [0,1]$, then the set $A_t = \{x \in M, A(x) \geq t\}$ is called level subset of $X$.

Note that, $A_t$ is a subset of $M$ in the ordinary sense.
The following are some properties of the level subset.

Remark 1.6: [1]
Let $A$ and $B$ be two fuzzy subsets of a set $M$ and let $t \in [0,1]$, then:
1. $(A \cap B)_t = A_t \cap B_t$.
2. $(A \cup B)_t = A_t \cup B_t$.
3. $A = B$ if and only if $A_t = B_t$.

Definition 1.7: [1]
Let $A$ be a fuzzy set in $M$, then $A$ is called empty fuzzy set, denoted by $\emptyset$, if and only if $A(x) = 0$, for all $x \in M$.

Definition 1.8: [2]
Let $M$ be an $R$-module. A fuzzy set $X$ of $M$ is called fuzzy module of $M$ if:
1. $X(x-y) \geq \min\{X(x), X(y)\}$, for all $x, y \in M$.
2. $X(rx) \geq X(x)$, for all $x \in M$, $r \in R$.
3. $X(0) = 1$ (0 is the zero element of $M$).
Definition 1.9: [3]

Let X and A be two fuzzy modules of an R-module M. A is called a fuzzy submodule of X if A \subseteq X.

Proposition 1.10:[8]

Let A be a fuzzy set of an R-module M. Then the level subset \( A_t \), \( t \in [0,1] \) is a submodule of M if and only if A is a fuzzy submodule of X, where X is a fuzzy module of an R-module M.

Now, we shall give some properties of the fuzzy submodule, which are used in the next section.

Definition 1.11:[2]

Let A and B be two fuzzy subsets of an R-module M. Then (A \( \oplus \) B)(x) = sup \{\min\{A(a), B(b), x = a + b\} \} \ a, b \in M, for all x \in M. A \( \oplus \) B, is a fuzzy subset of M.

Proposition 1.12:[2]

Let A and B be two fuzzy submodule of a fuzzy module X, then A+B is a fuzzy submodule of X.

Remark 1.13:[9]

If X is a fuzzy module of an R-module M and X, then for all fuzzy singleton \( r_k \) of R , \( r_kX = r_k \), \( \ell = \min\{k, t\} \).

Definition 1.14:[6]

Let X and Y be two fuzzy modules of an R-module M, respectively. Define X \( \oplus \) Y : M \( \oplus \) M \( \rightarrow \) [0,1] by:

\[ (X \oplus Y)(a, b) = \min\{X(a), Y(b), \text{for all} \ (a, b) \in M \oplus M\}. \]

X \( \oplus \) Y is called a fuzzy external direct sum of X and Y.

If A and B are fuzzy submodules of X, Y respectively, then

\[ A \oplus B : M_1 \oplus M_2 \rightarrow [0,1] \] is defined by:

\[ (A \oplus B)(a, b) = \min\{A(a), B(b), \text{for all} \ (a, b) \in M_1 \oplus M_2\} \]

Note that, if X = A \( \oplus \) B and A \( \cap \) B = \( \emptyset \), then X is called internal direct sum of A and B which is denoted by A \( \oplus \) B. Moreover, A and B are called direct summand of X.

Definition 1.15:[9]

Let A be a fuzzy module in M, then we define:

1. \( A^* = \{x \in M: A(x) > 0\} \) is called support of A, also
   \[ A^* = \cup A_t, \ t \in (0,1]. \]
2. \( A_\ast = A_{A(0_M)} = \{x \in M: A(x) = 1 = A(0_M)\}. \)

Definition 1.16:[10]

A fuzzy module X of an R-module M is called simple if X = 0.

Definition 1.17:[6]

Let A be a fuzzy submodule of a fuzzy module X of an R-module M, then A is called an essential fuzzy submodule (briefly A \( \leq_e \) X), if A \( \cap \) B \( \neq \) 0, for any non-trivial fuzzy submodule B of X. Equivalently, a fuzzy submodule A of X is called essential if A \( \cap \) U = 0, implies that U = 0, for all fuzzy submodule U of X.

2. Properties of Essential Fuzzy Submodules

In this section we shall give some properties of essential fuzzy submodule which were introduced in a previous study [6].

Remark 2.1:

If X is a fuzzy module of an R-module M and A is an essential fuzzy submodule of X, then A \( \ell \) is not necessarily an essential of X. For example:

Example:

Let M = \( \mathbb{Z}_6 \) be a \( \mathbb{Z} \)-module. Define X: M \( \rightarrow \) [0,1], A: M \( \rightarrow \) [0,1] by:

\[ X(a) = \begin{cases} 
1 & \text{if } a = 0 \\
\frac{1}{2} & \text{if } a = 2,4 \\
0 & \text{otherwise}
\end{cases}, \quad A(a) = \begin{cases} 
1 & \text{if } a = 0 \\
\frac{1}{3} & \text{if } a = 2,4 \\
0 & \text{otherwise}
\end{cases} \]

It is clear that X is fuzzy module of \( \mathbb{Z}_6 \). A is fuzzy submodule of X and A \( \neq \) 0. Suppose that there exists a fuzzy submodule of X such that A \( \cap \) B = \( \emptyset \). Hence, for all \( a \in \mathbb{Z}_6 \), a \( \neq \) 0 (A\( \cap \)B)(a) = 0. But if \( a \in (2) \) - (0)

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(A\cap B)(a) = \min \{A(a), B(a)\} = \min \left\{ \frac{1}{\frac{1}{a}}, B(a) \right\}, \text{ which implies that } B(a) = 0.

If \( a \notin (2) \), then \( B(a) = 0 \), since \( B(a) \leq X(a) \), for all \( a \in Z_8 \) and \( X(a) \neq 0 \ \forall \ a \in (2) \).

If \( a = 0 \), then \( B(0) = 1 = X(0) \). Thus \( B = 0_1 \), then \( A \) is essential fuzzy submodule of \( X \). On the other hand, \( A_{\frac{1}{2}} = 0, X_{\frac{1}{2}} = (2) \), hence \( A_{\frac{1}{2}} \) is not essential in \( X_{\frac{1}{2}} \).

**Proposition 2.2**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \) and let \( A \) be a fuzzy submodule of \( X \). If \( A_t \) is an essential submodule of \( X_t \ \forall \ t \in [0,1], \) then \( A \) is an essential fuzzy submodule of \( X \).

**Proof:**

Suppose that there exists a fuzzy submodule \( B \) of \( X \) such that \( A \cap B = 0_1 \). Hence, \((A \cap B)_t = (0_1)_t, \ \forall \ t \in [0,1].\) It follows that \( A_t \cap B_t = (0), \ \forall \ t \in (0,1].\) But \( A_t \) is an essential submodule of \( X_t, \) hence \( B_t = (0), \ \forall \ t \in (0,1].\) Thus, \( B = 0_1; \) that is \( A \) is an essential fuzzy submodule of \( X.\)

The following condition remark will be needed in some properties in our work.

Remark: Let \( X \) be a fuzzy module of an \( R \)-module \( M \) and \( A, B \) are non-trivial fuzzy submodules of \( X, \) if \( A \subseteq B, \) implies that \( A \subseteq B.\)

**Proposition 2.3:**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \) such that \( X \) satisfies the previous remark and \( A \) be a fuzzy submodule of \( X, \) then \( A \) is an essential fuzzy submodule of \( X \) if and only if \( A_s \) is an essential submodule in \( X_s.\)

**Proof:**

(\( \Rightarrow \)) Let \( A_s \) be an essential submodule of \( A_s. \) To prove that \( A \) is essential in \( X.\)

Suppose that \( B \) is a fuzzy submodule of \( X, \) \( B \neq 0_1 \) and \( A \cap B = 0_1, \) then \((A \cap B)_t = (0),\) by remark (1.6)(3). So \( (A \cap B)_s = (0), \) implies that \( A_s \cap B_s = (0), \) so \( B_s = (0), \) since \( A_s \) is an essential in \( X_s. \) Then \( B_s = (0), \) \( \forall \ s \in (0,1].\) So by the previous remark \( A_s \subseteq B_s \) implies that \( A \subseteq B. \) Therefore, \( 0_1 = B \cap A = B. \) Thus, \( B = 0_1 \) and \( A \) is essential in \( X.\)

(\( \Leftarrow \)) If \( A \) is essential in \( X, \) then to show that \( A_s \) is essential in \( X_s, \) let \( N \) be a submodule of \( X_s \) and \( A_s \cap N = (0). \) Let \( B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases} \)

It is clear that \( B \) is a fuzzy submodule of \( X \) and \( B_s = N, \) then \( A_s \cap N = A_s \cap B_s = (A \cap B)_s. \) Thus, \((A \cap B)_s = (0) = (0_1)_s. \) Then by the previous remark \( A \cap B = 0_1. \) Since \( A \) is an essential fuzzy submodule of \( X, \) then \((A \cap B)_t = (0), \) so \((A \cap B)_s = (0), \) implies that \( A_s \cap B_s = (0), \) by remark (1.6)(1) so \( B_s = (0), \) since \( A_s \) is an essential in \( X_s. \) Thus \( B_s = (0_1), \) and so by the previous remark \( B = 0_1 \) and so \( A \) is an essential in \( X.\)

**Remark 2.4:**

Every non-trivial fuzzy submodule of \( X \) is essential fuzzy itself.

**Proof:**

Let \( A \) be non-trivial fuzzy submodule of \( X \) and \( B \) be fuzzy submodule of \( A, \) where \( B \neq 0_1, \) then \( A \cap B = B \neq 0_1. \) Therefore, \( A \) is an essential fuzzy submodule of \( A.\)

**Proposition 2.5:**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \) and \( A_1, A_2, B_1, B_2 \) be fuzzy submodules of \( X, \) where \( A_1 \) is an essential fuzzy in \( B_1 \) and \( A_2 \) is an essential in \( B_2, \) then \( A_1 \cap A_2 \) is essential in \( B_1 \cap B_2.\)

**Proof:**

Let \( C \) be any fuzzy submodule of \( X \) such that \( 0_1 \neq C \subseteq B_1 \cap B_2, \) Since \( A_2 \) is essential in \( B_2, \) then \( A_2 \cap C \neq 0_1. \) Therefore, \( A_2 \cap C \) is non-trivial fuzzy submodule of \( X \) such that \( A_2 \cap C \subseteq B_1. \) As \( A_1 \) is essential in \( B_1, \) hence \( A_1 \cap (A_2 \cap C) \neq 0_1, \) then \((A_1 \cap A_2) \cap C \neq 0_1. \) Therefore, \( A_1 \cap A_2 \) is essential in \( B_1 \cap B_2.\)

**Proposition 2.6:**

Let \( X \) be a fuzzy module of an \( R \)-module \( M \) and \( A_1, A_2 \) be fuzzy submodules of \( X. \) If \( A_1 \) is essential in \( X \) and \( A_2 \) is essential in \( X, \) then \( A_1 \cap A_2 \) is essential in \( X.\)

**Proof:**

It is clear by remark(2.4) and proposition(2.5).
Proposition 2.7:
Let $X$ be a fuzzy module of an $R$-module $M$ and $A \leq B \leq X$. If $A$ is an essential fuzzy in $B$ and $B$ is an essential fuzzy in $X$, then $A$ is an essential fuzzy submodule in $X$.

Proof:
Let $A$ be an essential in $B$, $B$ be an essential in $X$, and $C$ be any non-trivial fuzzy submodule of $X$. Since $B$ is an essential fuzzy submodule in $X$, then we have $C \cap B \neq 0_1$, and then since $A$ is essential in $B$, we have $(C \cap B) \cap A \neq 0_1$; that is $C \cap A \neq 0_1$. Thus, $A$ is an essential fuzzy submodule in $X$.

Recall that a submodule $A$ of an $R$-module $C$. A relative complement for $A$ in $C$ is any submodule $B$ of $C$ which is maximal with respect to the property $A \cap B = 0$, [4].

We fuzzify this concept as follows:

Definition 2.8:
Let $X$ be a fuzzy module of an $R$-module $M$ and $A$ be a fuzzy submodule of $X$. A relative complement for $A$ in $X$ is any fuzzy submodule $B$ of $X$ which is maximal with respect to the property $A \cap B = 0$.

Proposition 2.9:
Let $A$ be a fuzzy submodule of $X$ of an $R$-module $M$, such that $X$ satisfies the previous remark. If $B$ is a relative complement for $A$, where $B$ is any fuzzy submodule of $X$, then $B \oplus A$ is essential in $X$.

Proof:
Since $B$ is a relative complement of $A$, we can prove that $B_*$ is a relative complement of $A_*$. As $B \cap A = 0_1$, then $(B \cap A)_* = (0_1)_*$, so $B_* \cap A_* = (0)$. Suppose that $N$ is a submodule of $X_*$ and $B_*$ is a submodule of $N$ such that $N \cap A_* = (0)$. Let

$$C : M \to [0,1],$$

$$C(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that $C$ is a fuzzy submodule of $X$ and $C_* = N$. Thus $C_* \cap A_* = (0_1)_*$. By the previous remark, $C \cap A = 0_1$. But $B_*$ is a submodule of $C_*$, implies that $B_*$ is a fuzzy submodule of $C$ (by the previous remark). Thus $C = B$, since $B$ is a relative complement of $A$. It follows that $C_* = B_*$ (see remark (1.6)(3)); that is $N = B_*$ and $B_*$ is a relative complement of $A_*$. Hence, $A \oplus B_*$ is an essential submodule in $X_*$ [4, proposition(1.3)]. So $(A \oplus B)_*$ is an essential submodule in $X_*$. Therefore, $A \oplus B_*$ is an essential fuzzy submodule of $X$ by proposition (2.4).

3. Properties of Closed Fuzzy Submodules
In this section, we introduce the notion of the closed fuzzy submodule of a fuzzy module as a generalization of (ordinary) notion closed submodule, where a submodule $A$ of an $R$-module $M$ is said to be closed submodule of $M$ (briefly $A \leq_{ce} M$), if $A$ has no proper essential extension ; that is if $A \leq_{ce} B \leq M$, then $A = B$ [4], [7]. We shall give some properties of this concept.

Definition 3.1:
Let $A$ be a fuzzy submodule of $X$ of an $R$-module $M$, then $A$ is called closed fuzzy submodule of $X$ (shortly $A \leq_{ce} X$), if $A$ has no proper essential extension; that is if $A \leq_{ce} B \leq X$, then $A = B$.

Proposition 3.2:
Let $X$ be a fuzzy module of an $R$-module $M$ which satisfies the previous remark, and $A$ is a fuzzy submodule of $X$, then $A$ is closed fuzzy submodule of $X$ if and only if $A_*$ is closed submodule in $X_*$. Proofs:

($\Rightarrow$) Suppose that $A_* \leq_{ce} N \leq X_*$. We have to show that $A_* = N$. Let

$$B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}.$$

It is clear that $B$ is a fuzzy submodule of $X$ and $B_* = N$, $A_* \leq_{ce} B_* = N$. So by the previous remark, $A \leq_{ce} B$. But $A_* \leq_{ce} N = B$, then by proposition(2.4). $A \leq_{ce} B$. Therefore, $A = B$, since $A$ is a closed fuzzy submodule in $X$, so $A_* = B_* = N$, then $A_* = N$.

($\Leftarrow$) If $A_*$ is a closed submodule in $X_*$. To show that $A$ is a closed fuzzy submodule in $X$, assume that $A \leq_{ce} B \leq X$. We must prove that $A = B$. Since $A \leq_{ce} B$, then $A_* \leq_{ce} B_*$. Therefore, $A_* = B_*$, since $A$ is a closed fuzzy submodule in $X$, so $A_* = B_* = N$, then $A_* = N$.

Remarks and Examples 3.3:
1. For every fuzzy module $X$. $0_1$ is closed fuzzy submodule of $X$ and $X$ is closed fuzzy submodule of $X$.
2. Every direct summand of a fuzzy module X is a closed fuzzy submodule of X.

**Proof:**
Let A be a direct summand of X, then there exists fuzzy submodule of X such that $X = A \oplus B$, then $A + B = X$ and $A \cap B = 0$. Suppose that A is an essential fuzzy submodule of C, where C is a fuzzy submodule of X. To show that A = C, we claim that $B \cap C = 0$. Since $(A \cap B) \cap C = 0$, then $A \cap (B \cap C) = 0$. If $B \cap C \neq 0$, then $A \cap (B \cap C) = 0$, since $B \cap C \subseteq C$ and A is essential in C, which is a contradiction. Therefore, $B + C = B \oplus C$.

Now, $X = A \oplus B$, since $A \subseteq C$. Then $X = A \oplus B \leq C + B = C \oplus B$, so $A \oplus B = C \oplus B$, since $(A \oplus B) = (C \oplus B)$.

3. Let $M = \mathbb{Z}_n$ be a Z-module. Let $X : M \to [0,1]$, define $A$ by $A(x) = 1$, and let $B : M \to [0,1]$, define by:

$$A(x) = \begin{cases} 1 & \text{if } x \in (2) \\ 0 & \text{otherwise} \end{cases}, \quad B(x) = \begin{cases} 1 & \text{if } x \in (3) \\ 0 & \text{otherwise} \end{cases}$$

It is clear that A and B are fuzzy submodules of X and $A \oplus B = X$, hence A and B are closed fuzzy submodules by remark and example (3.3)(2).

4. Every fuzzy submodule of semi-simple fuzzy module is closed fuzzy module, where a fuzzy module X of an R-module M is called semi-simple if X is a sum of simple fuzzy submodules of X [10, p.66].

**Proof:**
It is clear.

5. If A is closed fuzzy submodule of X of an R-module M and $A \subseteq B \subseteq X$, then A is a closed fuzzy in B.

**Proof:**
Assume that A is essential in D, where D is a fuzzy submodule of B. It is clear that D is a fuzzy submodule of X. Hence, A = D, since A is a closed fuzzy in X. Thus A is a closed fuzzy in B.

**Theorem 3.4:**
Let $\{A_\alpha\}$ and $\{X_\alpha\}$ be collections of fuzzy modules of an R-module M, such that $A_\alpha$ is a closed fuzzy submodule of $X_\alpha$, for each $\alpha$. Then $\bigoplus A_\alpha$ is closed fuzzy in $\bigoplus X_\alpha$, $\alpha \in \Lambda$ is any index set.

**Proof:**
Suppose that $\bigoplus A_\alpha$ is essential in B, where B is a fuzzy submodule of $\bigoplus X_\alpha$ for any $\alpha \in \Lambda$. $X_\alpha$ is essential in $X_\alpha$. Hence by proposition (2.5), $A_\alpha \oplus X_\alpha = A_\alpha \oplus X_\alpha$ is an essential $B \cap X_\alpha \subseteq X_\alpha$. Hence, $A_\alpha = B \cap X_\alpha$, since $A_\alpha$ is closed fuzzy in $X_\alpha$ by the assumption. But $\bigoplus A_\alpha \subseteq B$, hence $B \cap A_\alpha = A_\alpha \forall \alpha \in \Lambda$. Since $B \cap (A_\alpha \cap X_\alpha) = B \cap X_\alpha = A_\alpha$. It follows that $B \subseteq \bigoplus A_\alpha A_\alpha$. Because $A_\alpha \subseteq B \cap X_\alpha$, it follows that $A_\alpha \subseteq B \cap X_\alpha$, which implies that $X_\alpha \subseteq X_\alpha \cap A_\alpha$ for any $\alpha \in \Lambda$. Thus $B \subseteq \bigoplus A_\alpha A_\alpha$; that is B = $\bigoplus A_\alpha$ and $A_\alpha$ is closed fuzzy in $\bigoplus X_\alpha$.

**Theorem 3.5:**
Let B be a fuzzy submodule of X of an R-module M which satisfies the previous remark, such that $(X/A)_\alpha = X_\alpha/A_\alpha$. Then the following statements are equivalent.

1. B is a closed fuzzy submodule in X.
2. B is a relative complement, for some fuzzy $A \subseteq X$.
3. If A is any relative complement fuzzy of B in X, then B is relative complement fuzzy of A in X.
4. If $B \leq K \leq X$, then $K/B \leq X/B$.

**Proof:**
(1)$\Rightarrow$(2) : If B is a closed fuzzy submodule of X. By proposition (3.2), B is closed submodule in $X_\alpha$. Hence, $B_\alpha$ is a relative complement of some $N$, where N is a submodule of $X_\alpha$. Let $A : M \to [0,1]$, define by:

$$A(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$$
It is clear that $A$ is a fuzzy submodule of $X$ and $A_\ast = N$, so $B_\ast$ is a relative complement of $A_\ast$. Hence, $A_\ast \cap B_\ast = (0)$ and so $B \cap A = 0_1$. Suppose that $C$ is a fuzzy submodule of $X$ and $B$ is a fuzzy submodule of $C$ such that $C_\ast \cap A = 0_1$, so $C_\ast \cap A_\ast = (0)$. But $B_\ast$ is a submodule of $C_\ast$ and $B_\ast$ is a relative complement of $A_\ast$, so $B_\ast = C_\ast$. Thus $B = C$ (by the previous remark) and $B$ is a relative complement of $A$.

(2)$\Rightarrow$(1): If $B$ is a relative complement of $A$. To prove that $B$ is a closed fuzzy submodule in $X$, suppose that $B \leq_C C \leq X$. Then $B \cap A \leq_C C \cap A$, by proposition (2.6). Hence, $0_1 \leq_C C \cap A$ and so $C \cap A = 0_1$. But $B$ is a fuzzy submodule of $C$ and $B$ is a relative complement of $A$, hence $C = B$. Thus, $B$ is closed fuzzy submodule in $X$.

(2)$\Rightarrow$(3): If $B$ is a relative complement of $A$, then $B_\ast$ is a relative complement of $A_\ast$ by proposition (2.10). Then $A_\ast$ is a relative complement of $B_\ast$ [4,.proposition(1.4)] and so $A$ is a relative complement of $B$.

(3)$\Rightarrow$(2): It is clear.

(1)$\Rightarrow$(4): Let $B \leq X \leq K \leq X$. Then $B \leq K \leq K_\ast$, by proposition (1.10) and $K_\ast \leq X_\ast$, by proposition (2.4). Therefore, $B_\ast \leq K_\ast \leq X_\ast$. But $B$ is a closed fuzzy submodule in $X$, implies that $B_\ast$ is a closed submodule in $X_\ast$, by proposition (3.2). Hence $K_\ast / B_\ast \leq X_\ast / B_\ast$, by proposition (1.10), that is $(K / B)_\ast \leq e (X / B)$, and $K / B \leq (X / B)$ by proposition (2.4).

(4)$\Rightarrow$(3): Given that $B \leq X$, $A$ is relative complement of $B$ in $X$, and $B \leq K \leq X$, then $K / B \leq X / B$. We have to show that $B$ is a relative complement of $A$. Since $A \cap B = 0_1$, then $B$ can be enlarged to complement $B_\ast$ of $A$. By the modular law:

$$B_\ast \cap (A \oplus B_\ast) = (B \cap A_\ast) + B_\ast, = (0) + B_\ast = B_\ast.$$ 

This implies that $\left(\frac{B_\ast}{B}\right) \cap (A \oplus B_\ast)_\ast = (B_\ast)_\ast$. That is $\left(\frac{B}{B}\right) \cap (A \oplus B)_\ast = (B_\ast)_\ast$. Also $A \oplus B \leq X$, so $B \leq A \oplus B \leq X_\ast$, therefore by the assumption $A \oplus B / B \leq e X / B$. This implies that $(A \oplus B / B)_\ast \leq e (X / B)_\ast$, by proposition(2.4). So that $\left(\frac{B}{B}\right)_\ast \cap (A \oplus B)_\ast = (B_\ast)_\ast$, implies that $\left(\frac{B}{B}\right)_\ast = (B_\ast)_\ast$.

Let $x \notin B_\ast$, then $\left(\frac{B}{B}\right)_\ast (x + B_\ast) = 0$, implies that sup$[B \cap (x + y) \mid y \in B_\ast] = 0$ Then $B \ast (x + 0) = B (x) = 0$, hence $x \notin B_\ast$, implies that $B_\ast \leq B_\ast$. Hence, $B_\ast = B_\ast$. Then, by the previous remark, $B = B$.

**Proposition 3.6:**

Let $X$ be a fuzzy module of an $R$-module $M$ and let $A \leq B \leq X$. If $A$ is closed fuzzy in $B$ and is $B$ closed fuzzy in $X$, then $A$ is closed fuzzy in $X$.

**Proof:**

Since $A$ is closed fuzzy in $B$ and $B$ is closed fuzzy in $X$, therefore $A_\ast$ is closed in $B_\ast$ and $B_\ast$ is closed in $X_\ast$, by proposition (3.2). This implies that $A_\ast$ is closed in $X_\ast$ [4, proposition(1.5)]. Thus, $A$ is closed fuzzy in $X$ by proposition (3.2).

**Proposition 3.7:**

Let $X$ be a fuzzy module of an $R$-module $M$ and let $A$ be a fuzzy submodule of $X$, then $A$ is a direct summand of $X$ if and only if $A_\ast$ is a direct summand of $X_\ast \forall \ast \in \{0,1\}$.

**Proof:**

($\Rightarrow$) If $A$ is a direct summand of $X$, then $A \oplus B = X$ for some fuzzy submodule of $X$, hence $(A \oplus B)_\ast = X_\ast$, implies that $A_\ast \oplus B_\ast = X_\ast$ [6,.lemma 2.3.3]. Then $A_\ast$ is a direct summand of $X_\ast$.

($\Leftarrow$) If $A_\ast \oplus N = X_\ast$ for some $N$ submodule of $X_\ast$. Define

$$B(x) = \begin{cases} 1 & \text{if } x \in N \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $B$ is a fuzzy submodule of $X$ and $B_\ast = N$, then $B_\ast$ is a submodule of $X_\ast$ by proposition (1.10). But $A_\ast$ is a direct summand of $X_\ast$, then $A_\ast \oplus B_\ast = X_\ast$, hence $(A \oplus B)_\ast = X_\ast$; that is $A \oplus B = X$ by remark(1.6)(3).

**Remark 3.8:**

The intersection of two closed fuzzy submodules needs not be a closed fuzzy submodule in general, as the following example shows:

**Example**
Let $M$ be the $\mathbb{Z}$-module $\mathbb{Z} \oplus \mathbb{Z}_2$. Let $X : M \to [0,1]$, defined by:

$$X(a,b) = 1, \text{ for all } (a, b) \in \mathbb{Z} \oplus \mathbb{Z}_2$$

Let $A : M \to [0,1]$, $B : M \to [0,1]$, defined by:

$$A(a,b) = \begin{cases} 1 & \text{if } (a,b) \in (1,0)Z \cong Z \\ 0 & \text{otherwise} \end{cases} \quad B(a,b) = \begin{cases} 1 & \text{if } (a,b) \in (1,1)Z \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $A$ and $B$ are fuzzy submodules of $X$ and that $A_\ast = (1,0)Z \cong Z$, $B_\ast = (1,1)Z$. Since $A_\ast$ is a direct summand of $X_\ast = M$, then $A$ is direct summand of $X$, by proposition (3.7). Also $B_\ast$ is direct summand of $X_\ast$, then $B$ is direct summand of $X$. Therefore, $A$ and $B$ are closed fuzzy submodules of $X$, by remark and example (3.3)(2). But

$$(A \cap B)(a,b) = \min \{A(a,b), B(b)\} \quad \forall (a, b) \in \mathbb{Z} \oplus \mathbb{Z}_2$$

$$(A \cap B)(a,b) = \begin{cases} 1 & \text{if } (a,b) \in (2,0)Z \\ 0 & \text{otherwise} \end{cases}$$

$$(A \cap B)_\ast = (2,0)Z$$, which is not a closed submodule in $X_\ast$, since $(2,0)Z \leq_e (1,0)Z \leq X$, but $(2,0) \neq (1,0)Z$. Therefore, $A \cap B$ is not a closed fuzzy submodule of $X$, by proposition (3.2).

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