TELEPORTATION AND ENTANGLEMENT DISTILLATION
IN THE PRESENCE OF CORRELATION
AMONG BIPARTITE MIXED STATES

MITSURU HAMADA
Quantum Computation and Information Project (ERATO)
Japan Science and Technology Corporation
5-28-3, Hongo, Bunkyo-ku, Tokyo 113-0033, Japan

The teleportation channel associated with an arbitrary bipartite state denotes the map that represents the change suffered by a teleported state when the bipartite state is used instead of the ideal maximally entangled state for teleportation. This work presents and proves an explicit expression of the teleportation channel for the teleportation using Weyl’s projective unitary representation of \((\mathbb{Z}/d\mathbb{Z})^2\) for integers \(d \geq 2, n \geq 1\), which has been known for \(n = 1\). This formula allows any correlation among the \(n\) bipartite mixed states, and an application shows the existence of reliable schemes for distillation of entanglement from a sequence of mixed states with correlation.

Keywords: Teleportation channels, entanglement distillation, quantum codes

1 Introduction

Relationships between entanglement distillation and quantum error correction have been discussed by Bennett et al. [1]. Especially, they argued that achievable information rates for quantum error correction, i.e., those at which quantum error-correcting codes (quantum codes) reliably, are also achievable as rates for one-way entanglement distillation. More precisely, they associate with an arbitrary bipartite mixed state a map called a teleportation channel, which represents the change suffered by a teleported state when the bipartite mixed state is used for teleportation [2] in place of the ideal maximally entangled state. Then, they argued that an achievable rate for quantum codes on the teleportation channel is also achievable as the asymptotic yield of distillation schemes for the bipartite state. A concrete expression for the teleportation channel using \((\mathbb{Z}/d\mathbb{Z})^2\) was given afterwards [3], which complements the above argument on transformation of achievable rates. Recently, this author gave exponential lower bounds on the highest fidelity and those on the largest information rates that can be attained by standard algebraic quantum codes [4, 5] not only on discrete memoryless quantum channels but also on channels with certain correlation [6, 7, 8, 9]. This work was motivated by interest in exploring implications of these results [6, 7, 8, 9] on entanglement distillation, especially in the presence of correlation among the bipartite states, along the lines of [1].

In what follows, we will do the next three things. (i) To deal with correlated states, the formula for the teleportation channel using \((\mathbb{Z}/d\mathbb{Z})^2\) [3] is generalized to that for teleportation using Weyl’s projective unitary representation of \((\mathbb{Z}/d\mathbb{Z})^{2n}\), which allows any correlation among the \(n\) mixed states shared by two parties, and proved in such a way that the role of (characters of) the underlying group \((\mathbb{Z}/d\mathbb{Z})^{2n}\) becomes clear. (ii) We refine Bennett et al.’s
observation [1]. Namely, while they have discussed only asymptotically achievable rates, we will directly work with fidelity, and show that trade-offs between the fidelity and rates of quantum codes can be transformed into those between the fidelity and rates of one-way distillation protocols. (iii) We apply these arguments to the known results on quantum codes [6, 7, 8, 9]. Namely, we present exponential lower bounds on the largest fidelity that can be attained by one-way distillation protocols using the generalized formula in (i) and transformations in (ii). For example, reliable distillation with a positive asymptotic rate and exponential decay of unity minus fidelity is shown to be possible of a sequence of Bell states $|00\rangle \pm |11\rangle, |01\rangle \pm |10\rangle$ which occur according to the probability measure of a Markov chain.

2 Notation and Basic Notions

2.1 Entanglement Distillation

The set of all linear operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$. As usual, states are represented as unit-trace positive semidefinite element of $\mathcal{L}(\mathcal{H})$ or sometimes, when they are pure, normal vectors in $\mathcal{H}$. For a trace-preserving completely positive (TPCP) linear map $\mathcal{M} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$, we write $\mathcal{M} = \{ M_i \}_{i \in Y}$ if $\mathcal{M}(\sigma) = \sum_{i \in Y} M_i \sigma M_i^\dagger$ (an operator-sum representation). The identity in $\mathcal{L}(\mathcal{H})$ is denoted by $I$ while the identity map from $\mathcal{L}(\mathcal{H})$ onto $\mathcal{L}(\mathcal{H})$ is denoted by $\mathbb{I}$. We distinguish $I$ (or $\mathbb{I}$) for different Hilbert spaces, say $\mathcal{H}_A, \mathcal{H}_{n,R}$, $\mathcal{H}^{\otimes n}_A \otimes \mathcal{H}^{\otimes n}_B$, etc., (by a rather loose notation) using subscripts, say, $A, R, TA, \text{etc.}$, as in $I_{TA}$.

Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces of finite dimensions. Without loss of generality, we assume $\dim \mathcal{H}_A = \dim \mathcal{H}_B =: d$. Our purpose is to distill a bipartite mixed state $\rho_n$ in $\mathcal{L}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n})$ into a maximally entangled state

$$|\Phi_{B_A,B_B}\rangle = \frac{1}{K} \sum_{0 \leq j < K} |j\rangle_A \otimes |j\rangle_B,$$

with some Hilbert spaces $\mathcal{H}_{n,A}, \mathcal{H}_{n,B}$, and orthonormal systems $B_A = \{|j\rangle_A\} \subseteq \mathcal{H}_{n,A}, B_B = \{|j\rangle\} \subseteq \mathcal{H}_{n,B}$ by a TPCP linear map $D_n : \mathcal{L}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n}) \rightarrow \mathcal{L}(\mathcal{H}_{n,A} \otimes \mathcal{H}_{n,B})$, where $D_n$ is in some presupposed class $C$ of distillation protocols. For any such pair $(B_A, B_B)$, we say $|\Phi_{B_A,B_B}\rangle$ is distillable or $C$-distillable from $\rho_n$ with fidelity

$$\langle \Phi_{B_A,B_B} | D_n(\rho_n) | \Phi_{B_A,B_B} \rangle.$$

We want both fidelity and $|B_A| = |B_B|$ to be as large as possible, but there is a trade-off between them, which have been investigated from an information theoretic interest [10, 11]. Thus, we will estimate

$$F^*(\rho_n, K) = F_C^*(\rho_n, K) = \sup \left\{ F \mid \text{a state } |\Phi_{B_A,B_B}\rangle \text{ with } |B_A| = |B_B| \geq K \text{ is } \text{C-distillable from } \rho_n \text{ with fidelity } F \right\}$$

for a state $\rho_n \in \mathcal{L}(\mathcal{H}_A^{\otimes n} \otimes \mathcal{H}_B^{\otimes n})$, where $0 < K \leq d^n$ is a real number.

Our protocol using quantum codes to be discussed below is a simple one-way distillation protocol, which consists of one measurement on the (possibly locally enlarged) site $A$, and a
local quantum operation (TPCP linear map) on the site B that may be chosen according to the measurement result on the site A. We may suppose C denotes the class of such protocols, but we assume C is the slightly more general class $C_1$ of 1-local operations following Rains’ lucid classification [11]. Namely, unless otherwise mentioned, $C = C_1$ is assumed throughout and the subscript $C$ in $F^c_C (\rho_n, K)$ or similar quantities will be suppressed.

It is remarked that requiring $\mathcal{H}_{n,A} = \text{span} \mathcal{B}_A$ and $\mathcal{H}_{n,B} = \text{span} \mathcal{B}_B$ in defining ‘$C$-distillable’, as in [11], does not change the above quantity of distillation protocols, but only mention that $C = C_1$ represents the class of what we call one-way distillation protocols, $C$ is contained in (but significantly differs in capability [12, 1]) from the class of two-way protocols, they are separable and hence positive partial transpose (PPT) ones [10, 11].

### 2.2 Quantum Error Correction

Hereafter throughout the paper, $\mathcal{H}$ is a Hilbert space with dimension $d \geq 2$. We will consider the situation where quantum states are transmitted through a quantum channel with input-output Hilbert space $\mathcal{H}$, which is a sequence of TPCP linear maps $\{A_n\}$ with

$$A_n : \mathcal{L}(\mathcal{H}^\otimes n) \to \mathcal{L}(\mathcal{H}^\otimes n).$$

The map $A_n$ with fixed $n$ is sometimes called a channel also. A quantum code is a pair $(\mathcal{C}_n, \mathcal{R}_n)$ that consists of a subspace $\mathcal{C}_n \subseteq \mathcal{H}^\otimes n$ and a decoder, which is a TPCP linear map,

$$\mathcal{R}_n : \mathcal{L}(\mathcal{H}^\otimes n) \to \mathcal{L}(\mathcal{H}^\otimes n).$$

The subspace $\mathcal{C}_n$ alone is also called a quantum code. While there are two fidelity measures often used for evaluating quantum codes, i.e., entanglement fidelity and minimum pure-state fidelity (minimum fidelity), we will mostly work with the entanglement fidelity since, as its name and definition suggest, it is directly related to entanglement distillation. Entanglement fidelity is defined as follows [14]. Let $\Pi_\mathcal{C}$ and $\tilde{\Pi}_\mathcal{C}$ denote the projection onto $\mathcal{C}$ and its normalization $(\dim \mathcal{C})^{-1} \Pi_\mathcal{C}$, respectively. Then, given a quantum code as above with $\dim \mathcal{C}_n = K$, we prepare a $K$-dimensional Hilbert space $\mathcal{H}_{n,R}$, and a maximally entangled state

$$|\Phi_{\mathcal{B}_R, \mathcal{B}}\rangle = \frac{1}{K} \sum_{0 \leq j < K} |j\rangle_R \otimes |j\rangle,$$

where $\mathcal{B}_R = \{|j\rangle_R\}_{0 \leq j < K} \subseteq \mathcal{H}_{n,R}$ and $\mathcal{B} = \{|j\rangle\}_{0 \leq j < K} \subseteq \mathcal{H}^\otimes n$ are systems of orthonormal vectors, and define the entanglement fidelity $F_e(\Pi_\mathcal{C}, \mathcal{M})$ for the state $\Pi_\mathcal{C}$ and a general TPCP map $\mathcal{M}$ by

$$F_e(\Pi_\mathcal{C}, \mathcal{M}) = \langle \Phi_{\mathcal{B}_R, \mathcal{B}} | \mathcal{M} | \langle \Phi_{\mathcal{B}_R, \mathcal{B}} \rangle | \Phi_{\mathcal{B}_R, \mathcal{B}} \rangle.$$
2.3 Capacity, Reliability Function and Their Analogues for Distillation

Given a channel \( \{A_n\} \), a number \( R \geq 0 \) is said to be an achievable rate on \( \{A_n\} \) if it satisfies

\[
\lim_{n \to \infty} F^*_n(A_n, d^{Rn}) = 1.
\]

The supremum of achievable rates on \( \{A_n\} \) is called the quantum capacity of \( \{A_n\} \) and denoted by \( Q(\{A_n\}) \). A function \( E(R) \geq 0 \) is said to be an attainable (or achievable) error exponent if it satisfies

\[
1 - F^*_n(A_n, d^{Rn}) \leq \exp \left[ -nE(R) + o(n) \right]
\]

and the pointwise supremum of attainable exponents on \( \{A_n\} \) is called the reliability function (the optimum error exponent) of \( \{A_n\} \) and denoted by \( E(R, \{A_n\}) \).

Turning to entanglement distillation, for a sequence of states \( \{\rho_n\} \), we can define the counterparts of \( Q(\{A_n\}) \) and \( E(R, \{A_n\}) \), which this paper is concerned with. For example, the capacity analogue \( D(\{\rho_n\}) \) is defined as the supremum of \( R \) such that \( F^*(\rho_n, d^{nR}) = 1 \). In the literature, \( D(\{\rho^{\otimes n}\}) \) is sometimes called the distillable entanglement of \( \rho \).

2.4 Algebraic Structure Underlying Teleportation and Symplectic Codes

Algebraic structures underlie quantum mechanical phenomena [15] as well as schemes for quantum information processing such as teleportation protocols and symplectic (stabilizer) codes [4, 5]. In such quantum operations, the structure of \( \mathbb{Z}_d = \mathbb{Z}/d\mathbb{Z} \) is exploited, and its link to familiar Hilbert spaces is given by a projective (ray) representation \( N : u \mapsto N_u \) of \( \mathbb{Z}_d \times \mathbb{Z}_d \), which is defined as follows [15]. Fix an orthonormal basis \( \{|0\rangle, \ldots, |d-1\rangle\} \) of \( \mathcal{H} \), put \( \mathcal{X} = \mathbb{Z}_d \times \mathbb{Z}_d \), and

\[
N(i,j) = X^iZ^j, \quad (i,j) \in \mathcal{X},
\]

where \( X, Z \in \mathcal{L}(\mathcal{H}) \) are defined by

\[
X|j\rangle = |j-1\rangle, \quad Z|j\rangle = \omega^j|j\rangle, \quad j \in \mathbb{Z}_d
\]

(4)

with \( \omega \) being a primitive \( d \)-th root of unity. When \( d = 2 \), this representation is essentially the same as the system of Pauli matrices with identity.

To deal with the larger system of \( \mathbb{H}^{\otimes n} \), we put

\[
N_y = N_{y_1} \otimes \ldots \otimes N_{y_n},
\]

(5)

for \( y = (y_1, \ldots, y_n) \in \mathbb{X}^n \). We identify \( \mathbb{X}^n \) with \( \mathbb{Z}_d^{2n} \) via the trivial correspondence

\[
((x_1, z_1), \ldots, (x_n, z_n)) \mapsto (x_1, z_1, \ldots, x_n, z_n).
\]

Observe the commutation relation

\[
N_y N_{y'} = \omega^{(y,y')_{sp}} N_{y'} N_y,
\]

(6)

where

\[
(y, y')_{sp} = \sum_{i=1}^n x_i' z_i - z_i x_i'
\]

(7)

for \( y = (x_1, z_1, \ldots, x_n, z_n) \) and \( y' = (x'_1, z'_1, \ldots, x'_n, z'_n) \in \mathbb{Z}_d^{2n} \).
Fig. 1. Entanglement distillation using a quantum code \((C, R)\). The state \(\varphi\) is a purification of \(\sigma = \hat{\Pi}_C\) to be teleported, and \(\rho_n\) is the shared bipartite state for quantum teleportation. The unitary \(N_x\) is chosen according to the result of the measurement \(\{|\Psi'_x\rangle\langle \Psi'_x|\}_{x \in \mathbb{Z}_{2^n}}\) on site A, which is sent to B by classical communication. After the recovery operation \(R\) of the code, an entangled state is shared between the positions indicated by \(\star\).

3 Teleportation Channel

It is known that for every unitary basis of \(L(H)\) which is orthonormal with respect to the normalized Hilbert-Schmidt inner product, there exists a teleportation protocol using it [16]. Since Weyl’s basis \(\{N_y\}_{y \in X^n}\) is such a unitary basis [17], we have a teleportation protocol using it that teleports states in \(H \otimes n\). Note that

\[
|\Psi_y\rangle = \frac{1}{d^n/2} \sum_{l \in \mathbb{Z}_d^n} |l\rangle \otimes N_y |l\rangle, \quad y \in \mathbb{Z}_{2^n}^d,
\]

and

\[
|\Psi'_x\rangle = \frac{1}{d^n/2} \sum_{l \in \mathbb{Z}_d^n} N_x |l\rangle \otimes |l\rangle, \quad x \in \mathbb{Z}_{2^n}^d,
\]

where \(|(l_1, \ldots, l_n)\rangle = |l_1\rangle \otimes \ldots \otimes |l_n\rangle\), form orthonormal bases of \(H \otimes n \otimes H \otimes n\) [16].

We will present a concrete expression of the teleportation channel for Weyl’s basis \(\{N_y\}_{y \in X^n}\), which allows us to treat correlated states. We will prove this in a simple manner using a property of the underlying additive group \(\mathbb{Z}_{2^n}^d\) for arbitrary integers \(d \geq 2, n \geq 1\).

To describe the protocol, we prepare \(H \otimes n_A, H \otimes n_B\), and \(H \otimes n_T\), where the Hilbert spaces \(H_A, H_B\), and \(H_T\) have the same dimension \(d\) (see Fig. 1 ignoring \(H\) and \(R\)). We will often identify \(H_A, H_B\), and \(H_T\) with \(H\) having a basis \(\{|j\rangle\}_{j \in \mathbb{Z}_d^n}\). Given a bipartite state \(\rho_n\) in \(L(H \otimes n_A \otimes H_B)\), sender A teleports a state in \(H \otimes n_T\) to receiver B. The entire process of the teleportation on the systems of A, B and T can be represented by an operator-sum representation

\[
\mathcal{T}(\sigma \otimes \rho_n) = \sum_{x \in \mathbb{Z}_{2^n}^d} T_x (\sigma \otimes \rho_n) T_x^d
\]

where \(\sigma \in L(H \otimes n_T)\) is the state to be teleported, and

\[
T_x = (I_{TA} \otimes N_x)(|\Psi'_x\rangle\langle \Psi'_x| \otimes I_B) = |\Psi'_x\rangle\langle \Psi'_x| \otimes N_x.
\]

It is natural to ask how \(\sigma\) changes during the teleportation for arbitrary bipartite mixed states \(\rho_n\). This state change, which is of course a TPCP linear map, will be denoted by \(\chi_{\rho_n}\) and called a teleportation channel. The next lemma answers the question.
Lemma 1. The teleportation channel $\chi_{\rho_n}$ for the above protocol is given by

$$\chi_{\rho_n}(\sigma) = \sum_{x \in \mathbb{Z}_{2d}^n} \langle \Psi_x | \rho_n | \Psi_x \rangle N_x \sigma N_x^\dagger.$$ 

Proof. It is enough to calculate (8) for $\sigma = |u\rangle \langle v|$, $u, v \in \mathbb{Z}_d^n$. Write $\rho_n$ as

$$\rho_n = \sum_{y, z \in \mathbb{Z}_{2d}^n} \alpha_{y, z} | \Psi_y \rangle \langle \Psi_z |,$$

and note that by (4),

$$N_x | l \rangle = \omega^{b \cdot l} | l - a \rangle$$

for $x = (a_1, b_1, \ldots, a_n, b_n)$, $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n)$, $l = (l_1, \ldots, l_n)$, and $b \cdot l = \sum_i b_i l_i$. Then, (9) can be rewritten as

$$T_x = \frac{1}{d^n} \sum_{l, m} \omega^{b \cdot (l - m)} | l - a \rangle \langle m - a | \otimes | l \rangle \langle m | \otimes N_x,$$

with which the summand in (8) can be calculated as

$$T_x (| u \rangle \langle v | \otimes \rho_n) T_x^\dagger = \frac{1}{d^n} \sum_{l, l'} \omega^{b \cdot (l - u - l' + v)} | l - a \rangle \langle l' - a |$$

$$\otimes | l \rangle \langle l' | \otimes \sum_{y, z} \alpha_{y, z} N_x N_y | u + a \rangle \langle v + a | N_z^\dagger N_x^\dagger.$$

Using (6) and (11), we then have

$$T_x (| u \rangle \langle v | \otimes \rho_n) T_x^\dagger = \frac{1}{d^n} \left( | \Psi_x' \rangle \langle \Psi_x' | \otimes \sum_{y, z} \alpha_{y, z} \omega^{(x, y - z)_{\text{sp}}} N_y | u \rangle \langle v | N_z^\dagger \right).$$

(13)

We take partial trace of the both sides of (13) over the systems of $T$ and $A$, and sum them over $x \in \mathbb{Z}_{2d}^n$ noting that

$$\sum_{x \in \mathbb{Z}_{2d}^n} \omega^{(x, y - z)_{\text{sp}}} = 0$$

whenever $y \neq z$, 

(14)

which holds because $f_{y - z} : x \mapsto \omega^{(x, y - z)_{\text{sp}}}$ is a character of $\mathbb{Z}_{2d}^n$ (see the next paragraph). Thus, we find the teleportation channel $\chi_{\rho_n}$ is

$$\chi_{\rho_n}(\sigma) = \sum_{y \in \mathbb{Z}_{2d}^n} \alpha_{y, y} N_y \sigma N_y^\dagger,$$

as claimed. □

We include a short proof of (14), though it is merely a property of characters of groups. Put $f(x) = f_a(x) = \omega^{(x, a)_{\text{sp}}}$, $a \in \mathbb{Z}_{2d}^n$. Then, for any $x'$,

$$f(x') \sum_{x \in \mathbb{Z}_{2d}^n} f(x) = \sum_{x \in \mathbb{Z}_{2d}^n} f(x' + x) = \sum_{x \in \mathbb{Z}_{2d}^n} f(x).$$
Hence, if \( f(x') \neq 1 \) for some \( x' \in \mathbb{Z}_d^{2n} \), which is true for \( a \neq 0 \), we have \( \sum_{x \in \mathbb{Z}_d^{2n}} f(x) = 0 \), as desired.

It should be mentioned that there is an inverse process that associates a bipartite mixed state with a channel (TPCP map) \([1]\). The map is

\[
M(\mathcal{A}) = [I \otimes \mathcal{A}](|\Psi_0\rangle\langle \Psi_0|).
\]

Its matrix is Choi’s matrix divided by \( d^n \) \([18]\). Note that when \( \rho_n \) is written in the form \((10)\), we have \( M(\chi_{\rho_n}) = \rho_n \) if and only if \( \alpha_{y,z} \) is diagonal.

It may be worth mentioning that another more straightforward calculation using \((14)\) gives a formula for a discrete ‘twirling’, which is effectively known \([1, \text{Appendix A}], [19, \text{Sec. II.B}]\):

\[
\frac{1}{d^n} \sum_{x \in \mathbb{Z}_d^{2n}} (N_x \otimes N_x)\rho_n (N_x \otimes N_x)^\dagger = \sum_{y \in \mathbb{Z}_d^{2n}} \alpha_{y,y}|\Psi_y\rangle\langle \Psi_y|
\]

for \( \rho_n \) in \((10)\), where \( \langle l|N_x|m \rangle \) is the complex conjugate of \( \langle l|N_x|m \rangle \), \( l, m \in \mathbb{Z}_d^2 \). In general, the resultant states are less disordered (i.e., have less entropy) than those obtained with the continuous or full twirling \([1]\).

### 4 Using Quantum Codes for Entanglement Distillation

Bennett et al. \([1, \text{p.3840, Sec. V.C}]\) argued that an achievable rate for a quantum channel by quantum codes is also achievable as a distillation rate for the corresponding bipartite states by one-way distillation protocols. In this section, we refine this argument working directly with fidelity rather than achievable rates.

Bennett et al.’s distillation process \([1, \text{Fig.14}]\) using a quantum code with no encoding map can be visualized as in Figure 1. Suppose we are given a quantum code \((\mathcal{C} \subseteq \mathcal{K}^{\otimes n}, \mathcal{R})\) with \( \dim \mathcal{C} = K \), which works on the teleportation channel \( \chi_{\rho_n} \), a \( K \)-dimensional Hilbert space \( \mathcal{H}_{n,R} \) and \( n \) copies of \( \mathcal{H}_T \cong \mathcal{H} \) on site A, and a purification of \( \Pi_\mathcal{C} \), i.e., a maximally entangled state \( |\Phi_{\mathcal{B}_R,\mathcal{B}_T}\rangle \) where \( \mathcal{B}_R = \{|j\rangle|0 \leq j \leq K \subseteq \mathcal{H}_{n,R}\text{ and } \mathcal{B}_T = \{|j\rangle|0 \leq j \leq K \subseteq \mathcal{C} \subseteq \mathcal{H}^{\otimes n}_T\text{ are orthonormal systems. Here we identify } \mathcal{H}^{\otimes n}_T \text{ and } \mathcal{C}^{\otimes n}_\mathcal{B} \text{ with } \mathcal{H}^{\otimes n} \text{ via isomorphisms. We perform the teleportation protocol in Section 3 with the bipartite state } \rho_n \in \mathcal{L}(\mathcal{H}^{\otimes n}_A \otimes \mathcal{H}^{\otimes n}_B) \text{ leaving the system of } \mathcal{H}_{n,R} \text{ untouched, and then the recovery operation } \mathcal{R} \text{ on the site B. The role of the code } \mathcal{C} \subseteq \mathcal{K}^{\otimes n} \text{ is to transmit entanglement shared with the system of } \mathcal{H}_{n,R} \text{ from } \mathcal{H}^{\otimes n}_T \text{ to } \mathcal{H}^{\otimes n}_B, \text{ and we see by the definition of entanglement fidelity, the fidelity of this distillation protocol is given by that of the code } (\mathcal{C}, \mathcal{R}):\)

\[
\langle \Phi_{\mathcal{B}_R,\mathcal{B}_T}|D_n(\rho_n)|\Phi_{\mathcal{B}_R,\mathcal{B}_T}\rangle = F_e(\Pi_\mathcal{C}, \mathcal{R} \circ \chi_{\rho_n}), \tag{17}
\]

where \( D_n(\rho_n) = \text{Tr}_\mathcal{R}[(\mathcal{J}_\mathcal{RTA} \otimes \mathcal{R}) \circ (\mathcal{J}_\mathcal{RT} \otimes \mathcal{T})]|(\Phi_{\mathcal{B}_R,\mathcal{B}_T})\langle \Phi_{\mathcal{B}_R,\mathcal{B}_T}| \otimes \rho_n), \mathcal{K} = \mathcal{K}^{\otimes n}_T \otimes \mathcal{K}^{\otimes n}_A, \text{ and } \mathcal{B}_B \text{ is the image of } \mathcal{B}_T \text{ under the isomorphism. We summarize the above argument in the following statement.}

**Theorem 1** Let \( \mathcal{H} \) be a finite-dimensional Hilbert space. Whenever a quantum code \((\mathcal{C} \subseteq \mathcal{H}^{\otimes n}, \mathcal{R} : \mathcal{L}(\mathcal{H}^{\otimes n}) \to \mathcal{L}(\mathcal{H}^{\otimes n}))\) with \( K = \dim \mathcal{C} \) exists, we have

\[
F^*(\rho_n, K) \geq F_e(\Pi_\mathcal{C}, \mathcal{R} \circ \chi_{\rho_n})
\]
Lemma 2
However, we have the following lemma.

where $\hat{\text{vector}}$ or on the state $\rho$
finally claimed in terms of minimum (pure-state) fidelity in place of entanglement fidelity $F_e\rho$. Also that widely investigated symplectic codes \cite{4, 5} have enough flexibility to cope with such teleportation scheme \cite{16} or any other operation that defines a map $\chi_{\rho_n}$ similarly and is allowed in the presupposed class $C$ of distillation protocols, though concrete expressions for $\chi_{\rho_n}$ seem unknown except for the teleportation with \{N_y\} $_y \in Z^2_n$.

The bounds \cite{6, 8, 9} on $1 - F_e(\Pi e, R \circ A_n)$ to be used in the next section were originally claimed in terms of minimum (pure-state) fidelity in place of entanglement fidelity $F_e$. However, we have the following lemma.

**Lemma 2** [13, Theorem 2] For any TPCP linear map $M$,

$$1 - \min |\varphi| M(|\varphi| |\varphi|) |\varphi| \leq G,$$

where $|\varphi|$ is normalized, implies

$$1 - F_e(\Pi e, M) \leq \frac{3}{2} G.$$

In most asymptotic settings, the factor $\frac{3}{2}$ is negligible. Using Lemma 1 and 2, we obtain the following corollary to Theorem 1 where $N_J = \{N_x \ | \ x \in J\}$ for $J \subseteq X^n$ and the term ‘$N_J$-correcting’ is in the sense of Knill and Laflamme \cite{20}.

**Corollary 1** Let $(e, R)$ be an $N_J$-correcting code with $K = \dim e$. Then,

$$1 - F^*(\rho_n, K) \leq \frac{3}{2} \sum_{x \in Z^2_n : x \in J} P_n(x)$$

for a bipartite state $\rho_n \in L(\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n)$, where $P_n(x)$ is given by

$$P_n(x) = \langle \Psi_x | \rho_n | \Psi_x \rangle, \quad x \in Z^2_n.$$

Note that this corollary allows any correlation in $\rho_n$ among the $2n$ factor systems. Note also that widely investigated symplectic codes \cite{4, 5} have enough flexibility to cope with such general states $\rho_n$ in principle. To see this, recall that a symplectic code is obtained from a subspace $L \subseteq Z^2_n$ that are contained in the symplectic dual $L^\perp$ of $L$. If we choose a vector $\tilde{x}(s)$ from each coset $s$ of $L^\perp$ in $Z^2_n$, and denote the set of coset representatives $\tilde{x}(s)$ by $J_0$, we have $N_{J_0}$-correcting symplectic codes $(e, R)$ where $J = J_0 + L$ (see \cite{9} for a self-contained exposition). A natural choice for $\{\tilde{x}(s)\}$ is one that, at least nearly, maximizes $P_n(J) = \sum_{x \in J} P_n(x)$. In fact, the bounds on $F^*_e(A_n, d^{2n})$ in Section 5 below were proved with such choices. Observe that nothing was assumed here on the probability distribution $P_n$ or on the state $\rho_n$ to be distilled. It is remarked that if $e$ is a symplectic code, the factor $\frac{3}{2}$ in the bound in Corollary 1 can be removed since $F_e(\Pi e, R \circ \chi_{\rho_n}) = P_n(J)$ for the above set $J$ and the corresponding codes $(e, R)$. 

**for any state** $\rho_n \in L(\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n)$. In other words,

$$F^*(\rho_n, K) \geq F^*_e(\chi_{\rho_n}, K)$$

for any $0 < K \leq d^n$.

This implies for any $\{\rho_n\}$,

$$Q(\{\chi_{\rho_n}\}) \leq D(\{\rho_n\}),$$

which was known for $\rho_n = \rho^\otimes n$ \cite{1}. Clearly, Theorem 1 as well as (18) is true for an arbitrary state $\rho_n \in L(\mathcal{H}^\otimes n \otimes \mathcal{H}^\otimes n)$ and any other operation that defines a map $\chi_{\rho_n}$ similarly and is allowed in the presupposed class $C$ of distillation protocols, though concrete expressions for $\chi_{\rho_n}$ seem unknown except for the teleportation with \{N_y\} $_y \in Z^2_n$.
5 Known Results on Quantum Codes and Their Implications

In this section, we give more concrete exponential lower bounds on the fidelity of distillation and those on $D(\{\rho_n\})$ assuming $d$ is a prime.

5.1 Distillation of Independent Mixed States

We begin with the easy case in which $\rho_n$ are the tensor power $\rho^\otimes n$ of a state $\rho \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$. By Lemma 1, the teleportation channel $\chi_{\rho_n}$ can be written as $\chi_{\rho}^\otimes n$ with $\chi_{\rho} \sim \{\sqrt{P(u)}X_u\}_{u \in X}$, where $P = P_\rho$ is the probability distribution on $X$ given by

$$P_\rho(u) = \langle \Psi_u | \rho | \Psi_u \rangle, \quad u \in X = \mathbb{Z}_d^2.$$

A known exponential bound for the channel $\{\chi_{\rho_n}\}$ has the form

$$1 - F^*_ch(\chi_{\rho}^\otimes n, d^{Rn}) \leq \exp_d[-n \sup_L \hat{E}_L(R, P_\rho) + o(n)],$$

where $\exp_d x = d^x$ and the supremum is taken over all self-orthogonal even-length subspaces $L$ of $\mathbb{Z}_d^2$ with $d$ prime [9]. We will not give the specifications of $\hat{E}_L(R, P)$ here but only recall that when $L$ consists solely of the zero vector, $\hat{E}_L(R, P)$ is the same as the simple one $E(R, P) = \min_Q\{D(Q||P) + 1 - H(Q) - R^+\}$ (see [6, 7] for the details), which still gives the best numerical lower bound known for most channels.

These bounds are applicable to entanglement distillation through Theorem 1:

$$1 - F^*(\rho^\otimes n, d^{Rn}) \leq \exp_d[-n \sup_L \hat{E}_L(R, P_\rho) + o(n)] \leq \exp_d[-nE(R, P_\rho) + o(n)].$$

The lower bounds on $D(\{\rho^\otimes n\})$ that directly follow from these bounds are

$$D(\{\rho^\otimes n\}) \geq \sup_{m > 0} \sup_{c: \text{symplectic code}} \hat{I}_C(\hat{\Pi}_C, \chi_{\rho}^\otimes m) / m$$

$$\geq I_C(\hat{\Pi}_C, \chi_{\rho}) = 1 - H(P_\rho),$$

where $I_C$ and $H$ denote coherent information and entropy, respectively, the bound in (23) follows from (21) and that $E(R, P_\rho) > 0$ whenever $R < 1 - H(P_\rho)$ [6], and (22) follows from (20) similarly [9]. In (22), the supremum is taken over all symplectic codes designed with the same basis $\{N_x\}_{x \in X}$ as we use for the teleportation protocol.

5.2 Distillation of Mixed States with Classical Markovian Correlation

In this subsection, we consider a sequence of states $\{\rho_n\}$ with certain correlation. For a motivation, imagine, by means of a Gedankenexperiment, we were given a sequence of states $\{\rho_n\}$ and could perform the bipartite measurement

$$\{|\Psi_x\rangle \langle \Psi_x|\}_{x \in \mathbb{Z}_d^2},$$

i.e., $n$ trials of the bipartite measurement $\{|\Psi_u\rangle \langle \Psi_u|\}_{u \in \mathbb{Z}_d^2}$. Then, we would obtain, as measurement results, random variables $X_1, \ldots, X_n$ taking values in $\mathbb{Z}_d^2$ such that Pr$\{\langle X_1, \ldots, X_n \rangle = x\} = \langle \Psi_x | \rho_n | \Psi_x \rangle = P_n(x)$. This distribution is exactly the same as that appearing in the
expression for \( \chi_{\rho_n} \) in Lemma 1. Those \( \{\rho_n\} \) for which \( X_1, \ldots, X_n \) are independent identically distributed random variables, viz., \( P_n(x_1, \ldots, x_2) = P(x_1) \ldots P(x_n) \), have been treated in the previous subsection. We will consider a more general case, where \( X_1, X_2, \ldots \), form a homogeneous (first-order) Markov chain, viz.,

\[
P_n(x_1, \ldots, x_n) = p(x_1) \prod_{j=1}^{n-1} P(x_{j+1}|x_j)
\]  

(24)

for some transition probabilities \( P(v|u), u, v \in \mathcal{X} \), and some initial distribution \( p \). If the Markov chain is irreducible, then by Theorem 1 and the known bound on \( Q(\{\chi_{\rho_n}\}) \) [8], we have

\[
D(\{\rho_n\}) \geq 1 - H(P|q)
\]  

(25)

where \( H(P|q) \) is the entropy of \( P(\cdot|\cdot) \) conditional on the unique stationary distribution \( q \) of the Markov chain. For an attainable error exponent, see [8].

Example 1. Let us assume \( d = 2 \) and define \( P(v|u) \) by

\[
P((0, 0)|u) = 1 - \varepsilon_u \quad \text{and} \quad P(v|u) = \varepsilon_u/3 \quad \text{for} \quad v \neq (0, 0),
\]

\[0 < \varepsilon_u < 1, u \in \mathcal{X} \]. Then, (25) becomes

\[
D(\{\rho_n\}) \geq 1 - \sum_{u \in \mathcal{X}} q(u)[h(\varepsilon_u) + \varepsilon_u \log_2 3],
\]

where \( h(z) = -z \log_2 z - (1 - z) \log_2 (1 - z) \). Note that when \( \varepsilon_u \) is independent of \( u \), \( \{\rho_n\} \) is a sequence of identical isotropic states, and the bound becomes the known one [1].

The bound in (25) is also true when the support of the initial distribution \( p \) is contained in an equivalence class \( \mathcal{X}' \subseteq \mathcal{X} \), where the equivalence relation holds between \( u \) and \( v \) if \( u \) and \( v \) lead to each other, i.e., if \( P^{(n)}(v|u) > 0 \) and \( P^{(m)}(u|v) > 0 \) for some positive integers \( n \) and \( m \) with \( |P^{(n)}(v|u)| \) being the \( n \)-th power of the matrix \( |P(v|u)| \) [21]; in this case, \( q \) is to be understood as the unique stationary distribution \( q \) whose support is \( \mathcal{X}' \). To see this, it is enough to notice that the probability of occurrence of any \( x \notin \mathcal{X}' \) vanishes, so that we can safely replace \( \mathcal{X} \) by \( \mathcal{X}' \) in deriving the bound [8].

Example 2. Let us assume \( \rho_n \) can be written in the form

\[
\rho_n = \sum_{x \in \mathcal{X}^n} P_n(x)\langle \Psi_x | \Psi_x \rangle
\]

with (24), there exists an equivalence class \( \mathcal{X}' \) which is contained in \( \{(0, v) \mid v \in \mathbb{Z}_d\} \), and the initial distribution \( p \) is the stationary one \( q \) whose support is \( \mathcal{X}' \). (The corresponding channel \( \chi_{\rho_n} \) is a probabilistic mixture of conjugation actions of tensor products of several \( Z \) and \( I \).) Then, the bound (25) is tight for this sequence of states (even in the wider class of PPT distillation protocols), which has been known for the special case where the Markov chain is a sequence of independent, identically distributed random variables [22, 11, 19]. This can be shown by examining the derivation of Rains’ upper bound [10]. As a byproduct, \( Q(\{\chi_{\rho_n}\}) = 1 - H(P|q) \) is concluded by (18) for this example. \( \square \)
6 Concluding Remarks

We have presented an explicit formula of the teleportation channel $\chi_{n}$ using Weyl’s projective representation of $(\mathbb{Z}/d\mathbb{Z})^{2n}$, and have seen that the optimum fidelity $F^*(\rho_n, K)$ of a one-way distillation scheme is lower-bounded by that of quantum codes $F^*_{ch}(\chi_{n}, K)$. An application of these results has shown that reliable distillation from correlated mixed states is possible.

Symplectic codes can be used for entanglement distillation in another way, as argued by Shor and Preskill in a security proof for quantum key distribution [23]. Using this scheme and (16), we can reproduce the lower bounds on $F^*(\rho_n, K)$ in Section 5, though the derivation applies only to a restricted class of codes and hence does not imply Theorem 1 or (18).

The problem of determining the optimum achievable rates is not yet settled even for distillation from identical copies of bipartite states, whether the allowed protocols are one-way or two-way (or seemingly more tractable PPT ones), and further investigation is awaited (e.g., [11, 24]). Recently, P. Shor announced that the known upper bound on the quantum capacity written with coherent information [13] is actually the quantum capacity (MSRI Workshop, MSRI, Berkeley, California, Nov. 2002). This implies $D(\{\rho \otimes n\}) \geq \sup_{m>0} \sup_{\sigma \in C(2 \otimes m)} \frac{\operatorname{Ic}(\sigma, \chi_{\rho \otimes m})}{m} = \sup_{m>0} \sup_{\text{subspace of } 2 \otimes m} \frac{\operatorname{Ic}(\hat{\Pi}_C, \chi_{\rho \otimes m})}{m}$, where the equality is due to Lemma 1 of [25] (also [26]). It is not known if this bound is strictly greater than that in (22).

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