MOTIVIC MODEL CATEGORIES AND MOTIVIC DERIVED ALGEBRAIC GEOMETRY

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Abstract. In this paper, we develop an enhancement of derived algebraic geometry to apply to $A^1$-homotopy theory introduced by Morel and Voevodsky. We call the enhancement “motivic derived algebraic geometry”. We shall actually formulate “motivic” versions of $\infty$-categories, $\infty$-topoi, spectral schemes and spectral Deligne–Mumford stacks established by Joyal, Lurie, Toën and Vezzosi.

By using the language of motivic derived algebraic geometry, we construct the Grassmannian and the algebraic $K$-theory. Furthermore we formulate the Thom spaces for vector bundles on (motivic) stacks, and we obtain the algebraic cobordism for (motivic) stacks. As the main result, we prove that the algebraic cobordism corepresents the motivic $\infty$-category which has the universal property of oriented (motivic) $\infty$-categories.

1. Introduction

In the viewpoint of the theory of Grothendieck topoi, the original algebraic geometry consists of the theory of $Sets$-valued sheaves. Here $Sets$ denotes the (large) category of small sets. Derived algebraic geometry, due to Lurie [22], Toën and Vezzosi [36], consists of the theory of sheaves which value in subcategories of the model category $Set_\Delta$ of simplicial sets. Here the model structure of $Set_\Delta$ is defined by Quillen [30], and we will call the canonical model structure “Kan–Quillen model structure”. A fibrant object of this model category $Set_\Delta$ is called Kan complexes which is a derived version of a groupoid. In fact, Kan complexes are said to be $\infty$-groupoids in [19, p.35, Proposition 1.2.5.1]. In the articles [22] and [23], Lurie established the derived versions of schemes and Deligne–Mumford stacks, and they were respectively called spectral schemes and spectral Deligne–Mumford stacks.

The work of this paper is motivic derived algebraic geometry which consists of $A^1$-homotopy spaces-valued sheaves. Namely, motivic derived algebraic geometry is one of the analogies of derived algebraic geometry by replacing the values of sheaves from “Kan complexes” to “$A^1$-homotopy spaces”. We define the motivic versions of groupoids, categories and bicategories.
on the $\mathbb{A}^1$-homotopy category [28]: Let $\mathbf{M}$ be a simplicial model category with some mild conditions (see [20, Definition 1.5.1]). Then the category $\text{Fun}(\Delta^{op}, \mathbf{M})$ has the canonical model structure which is the Bousfield localization of the injective model structure, and the Bousfield localization is called the complete Segal model structure [20, p.33 Proposition 1.5.4]. Fibrant objects $\text{Fun}(\Delta^{op}, \mathbf{M})$ with respect to the complete Segal model structure are said to be complete Segal fibrant. In the view of the complete Segal model structure, we can define a category as a complete Segal fibrant object of the trivial model category $\text{Sets}$. In derived algebraic geometry, an $\infty$-category is regarded as a categorical object of the model category $\text{Set}_\Delta$, and an $\infty$-bicategory is defined to be a complete Segal fibrant object of the model category $\text{Set}_\Delta^*$ of marked simplicial sets. Here the model category $\text{Set}_\Delta^*$ gives another model of $\infty$-categories which is introduced by Lurie [19]. In this paper, we consider the complete Segal model structures of the model categories of $\mathbb{A}^1$-homotopy category, and define the motivic versions of $\infty$-categories and $\infty$-bicategories. Furthermore, we give the motivic versions $\infty$-topoi, classifying $\infty$-topoi, Zariski $\infty$-topoi, étale $\infty$-topoi, spectral schemes and Deligne–Mumford stacks.

The first purpose of the development of motivic derived algebraic geometry is to study moduli problems in the $\mathbb{A}^1$-homotopy space valued functors by (motivic) $\infty$-categorical language. In this paper, we construct motivic version of moduli functors: the Grassmannian, the algebraic $K$-theory and the algebraic cobordism.

The goal of this paper is the construction of the algebraic cobordism for motivic stacks. The algebraic cobordism $\text{MGL}$ is first introduced Voevodsky [38] as the algebraic analogy of the spectrum of the complex cobordism $\text{MU}$. It is expected and proved that $\text{MGL}$ has some properties which are similar to the one of $\text{MU}$. It is known that the complex cobordism $\text{MU}$ is the universal oriented cohomology theory. Panin, Pimenov and Röndigs [29, Theorem 2.7] proved that the set of monoidal maps $\text{MGL} \to R$ between (motivic) $E_\infty$-rings is isomorphic to the set of orientations on the motivic $E_\infty$-ring $R$.

In the final part of this paper, we shall formulate of the algebraic cobordism $\text{MGL}$ by using the motivic version (Definition 6.5) of the following: In [1, p.73, Appendix B.2], Ando, Blumberg, Gepner, Hopkins and Rezk give us a beautiful formulation of the Thom space of line bundles: Let $R$ be an $A_\infty$-ring and $\text{BG}_1(R)$ denote the $\infty$-groupoid of $R$-line objects. Given a map $E : X \to \text{BG}_1(R)$ of $\infty$-groupoids, the Thom module $\text{Thom}_1(E)$ is defined to be the homotopy colimit of $j \circ E$:

$$X \xrightarrow{E} \text{BG}_1(R) \xrightarrow{j} \text{Mod}_R$$

where $\text{Mod}_R$ denote the presentable stable $\infty$-category of modules and $j$ is the canonical embedding. By using language of motivic derived algebraic geometry, the motivic moduli stack $\mathcal{M}_{\text{MGL}}$ can be represented by the universal oriented motivic $\infty$-category $\mathcal{M}_{\text{MGL}}$ which is a variant
of the stable ∞-category realizing the universality of the $K$-theory due to Blumberg, Gepner and Tabuada [8]. The main theorem is Theorem 6.6: The motivic ∞-category $\mathcal{M}_{\text{MGL}}$ is again corepresented by the motivic stack $\mathbb{MGL}$. Furthermore this result gives the motivic ∞-categorical version Corollary 6.7 of the universality theorem [29, Theorem 2.7] due to Panin, Pimenov and Röndigs.

This paper is organized as follows: In section 2 we recall the definition and some properties of left proper combinatorial simplicial model categories. The definition of combinatorial model categories was introduced by J. Smith [19], [11] and [3]. By Dugger’s result [11], it is known that any combinatorial model category has a small presentation which is a Bousfield localization of a model category of a simplicial presheaves on some small category. This result is useful for our formulation of motivic model categories.

In Section 3 we define motivic model categories which are Bousfield localization of simplicial model categories on a Grothendieck site $\mathcal{X}$ with an interval object $I$. In addition, we define the motivic versions of presentable ∞-categories and ∞-topoi as presentable ∞-category valued $I$-local sheaves and ∞-topoi valued $\mathbb{A}^1$-local sheaves, and we will call them motivic presentable ∞-categories and motivic ∞-topoi, respectively.

In Section 4, we define motivic ∞-bicategories and motivic classifying ∞-topoi. Using the motivic version of scaled straightening and unstraightening theorem [20, p.128, Theorem 3.8.1], we can formulate the motivic versions of spectral schemes and spectral Deligne–Mumford stacks (Theorem 4.5).

In Section 5, we apply motivic derived algebraic geometry to $\mathbb{A}^1$-homotopy theory. Our main interest is the case that the Grothendieck site $\mathcal{X}$ is the Nisnevich site of the category $\text{Sm}_S$ of smooth schemes over a regular Noetherian scheme $S$ of finite dimension and with an interval the affine line $\mathbb{A}^1 = \mathbb{A}^1_S$. Then motivic spaces are the same of $\mathbb{A}^1$-homotopy spaces due to Morel and Voevodsky [28], and motivic ∞-categories are ∞-category valued $\mathbb{A}^1$-local presheaves on the Nisnevich site $\text{Sm}_S$. In the theory of stable $\mathbb{A}^1$-homotopy theory [33], an $\mathbb{E}_\infty$-ring object of stable $\mathbb{A}^1$-homotopy category is called a motivic $\mathbb{E}_\infty$-ring (e.g. [12]). In motivic derived algebraic geometry, motivic $\mathbb{E}_\infty$-rings are the coordinate rings of affine (motivic) schemes.

In Section 6 we first construct the Grassmannian $\mathcal{BGL}$ which is the moduli stack of the motivic ∞-groupoid of free modules of an arbitrary motivic $\mathbb{E}_\infty$-ring. Further the algebraic $K$-theory $K = \mathcal{BGL}^+$ is defined to be the group completion of the monoidal structure of the ∞-groupoid of free modules. In section 6.3, we will reformulate the motivic $\mathbb{E}_\infty$-ring $\mathbb{MGL}$ as the initial object of the ∞-category of oriented motivic $\mathbb{E}_\infty$-rings: we obtain the motivic $\mathbb{E}_\infty$-ring $\mathbb{MGL}$ as the oriented completion of the motivic sphere spectrum $\mathbb{S}$ (Theorem 6.4). Finally, we introduce the motivic ∞-categorical version of the Thom modules and the algebraic cobordism $\mathbb{MGL}$, and we prove the main theorem.
In this section, we recall the definition of combinatorial model categories and \( \infty \)-categories.

2.1. Left proper combinatorial simplicial model categories and their Bousfield localization.

**Definition 2.1.** A *model category* is a category \( M \) with three classes of morphisms \( W_M, C_M \) and \( F_M \) such that the following properties are hold:

- **MC1** The category \( M \) admits all small limits and colimits.
- **MC2** The class \( W_M \) has the 2-out-of-3 property.
- **MC3** The three classes \( W_M, C_M \) and \( F_M \) contain all isomorphisms and are closed under all retracts.
- **MC4** The class \( F_M \) has the right lifting property with respect to all morphisms in the class \( C_M \cap W_M \), and the class \( F_M \cap W_M \) has the right lifting property with respect to all morphisms in the class \( C_M \).
- **MC5** The couples \( (C_M \cap W_M, F_M) \) and \( (C_M, F_M \cap W_M) \) are weak functorial factorization systems.

We say that a morphism in \( W_M, C_M \) and \( F_M \) is respectively called a *weak equivalence*, a *cofibration* and a *fibration*. In addition, we say that a morphism in the class \( C_M \cap W_M \) and \( F_M \cap W_M \) is respectively called a *trivial cofibration* and a *trivial fibration*.

**Definition 2.2.** An adjunction \( F : M \rightleftarrows N : G \) between model categories is called a *Quillen adjunction* if \( F \) and \( G \) preserve the factorization systems in the axiom MC5 in Definition 2.1. Moreover, if the Quillen adjunction \( F : M \rightleftarrows N : G \) induces categorical equivalences between their homotopy categories (See [30, Chapter 1]), then it is said to be a *Quillen equivalence* of model categories. Then \( F \) is called a *left Quillen equivalence* and \( G \) a *right Quillen equivalence*.

**Definition 2.3.** A model category is *left proper* if the class of weak equivalences is closed under cobase change by cofibrations. Dually, we say that a model category is *right proper* if the class of weak equivalences is closed under base change by fibrations.

**Example 2.4.** Let \( \text{Set}_A \) denote the category of simplicial sets. Then the category \( \text{Set}_A \) has a model structure described as follows:

- **(C)** A cofibration is a monomorphism of simplicial sets.
- **(F)** A fibration is a Kan fibration of simplicial sets.
- **(W)** A morphism \( f : X \to Y \) of simplicial sets is a weak equivalence if the induced map \( [f] : |X| \to |Y| \) of the geometric realizations is a homotopy equivalence of topological spaces.
This model structure of $\text{Set}_\Delta$ is called Kan–Quillen model structure. Furthermore, Kan–Quillen model structure of $\text{Set}_\Delta$ is both left and right proper. For example, we can refer to a proof of the propernesses in [13, Chapter II.9].

2.2. The definition of monoidal model categories.

**Definition 2.5.** Let $(L, M, N)$ be a triple of model categories. A Quillen bifunctor $F : L \times M \to N$ is a bifunctor which satisfies the following conditions:

- Let $i : A \to B$ be a cofibration in $L$ and $j : A' \to B'$ a cofibration in $M$. Then the induced map

$$F(i \wedge j) : F(A', B) \coprod_{F(A, B)} F(A, B') \to F(A', B')$$

is a cofibration in $N$. Furthermore, if $i$ and $j$ are both trivial cofibrations, then $F(i \wedge j)$ is also a trivial cofibration.

- The bifunctor $F$ preserves all small colimits separately in each valuable.

**Definition 2.6.** Let $M$ be a model category equipped with a monoidal structure. The model category $M$ is said to be a monoidal model category if the monoidal structure satisfies the following conditions:

- The monoidal structure of $M$ is closed.
- The tensor product $- \otimes - : M \times M \to M$ of the monoidal structure of $M$ is a left Quillen bifunctor.
- The unit object of $M$ is a cofibrant object of $M$.

If the monoidal structure of $M$ is symmetric, then we say that $M$ is a symmetric monoidal model category.

**Example 2.7.** The model category $\text{Set}_\Delta$ is a symmetric monoidal model category whose monoidal structure is determined by the Cartesian products of simplicial sets.

**Definition 2.8.** Let $M$ be a model category such that the underlying category is a $\text{Set}_\Delta$-enriched category. If the model category $M$ is tensored and cotensored, and the tensor product $- \otimes - : \text{Set}_\Delta \times M \to M$ is a left Quillen bifunctor, then we say that $M$ is a simplicial model category.

2.2.1. The definition of combinatorial model categories.

**Definition 2.9.** Let $T$ be a collection of morphisms in a locally presentable category $M$. Let $^aT$ denote the set of morphisms in $M$ that it has the right lifting property with respect to all morphisms of $T$. Similarly, we let $T^a$ denote the set of morphisms in $M$ that it has the left lifting property with respect to all morphisms in $T$. We say that the set $(^aT)^a$ is the weakly saturated class of morphisms generated by $T$. 5
Definition 2.10. Let $\mathcal{M}$ be a locally presentable model category. Let $\mathcal{W}_\mathcal{M}$ be the class of weak equivalences in $\mathcal{M}$ and $\mathcal{C}_\mathcal{M}$ the class of cofibrations in $\mathcal{M}$. We say that $\mathcal{M}$ is combinatorial if there exist two small sets $I$ and $J$ such that the classes $\mathcal{C}_\mathcal{M}$ and $\mathcal{C}_\mathcal{M} \cap \mathcal{W}_\mathcal{M}$ are weakly saturated classes of morphisms generated by $I$ and $J$, respectively.

Example 2.11. The model category $\text{Set}_\Delta$ is combinatorial. The collection of cofibrations is generated by morphisms which form $\partial \Delta^n \hookrightarrow \Delta^n$ ($n \geq 0$) and the collection of trivial cofibrations is generated by morphisms which form $\Lambda_i^n \hookrightarrow \Delta^n$ ($0 \leq i \leq n, n \leq 0$). (See e.g. [13, Chapter II.9].)

It is known that any left proper combinatorial simplicial model category has Bousfield localization which are described as the followings:

Definition 2.12. Let $\mathcal{M}$ be a left proper combinatorial simplicial model category. Let $T$ be a collection of cofibrations. We say that a fibrant object of $X \in \mathcal{M}$ is $T$-local if for any morphism $f : Y \rightarrow Y'$ in $T$, the induced map

$$f^* : \text{Map}_\mathcal{M}(Y', X) \rightarrow \text{Map}_\mathcal{M}(Y, X)$$

is a trivial Kan fibration of simplicial sets. A morphism of $f : Y \rightarrow Y'$ is a $T$-weak equivalence if for any $T$-local object $X$, the induced map

$$f^* : \text{Map}_\mathcal{M}(Y', X) \rightarrow \text{Map}_\mathcal{M}(Y, X)$$

is a trivial Kan fibration.

Proposition 2.13. Let $\mathcal{M}$ be a left proper combinatorial simplicial model category with a collection of cofibrations $T$. Let $T^{-1}\mathcal{M}$ denote the category whose underlying category is the same of $\mathcal{M}$. The model structure of $T^{-1}\mathcal{M}$ is defined as follows:

(C) The collection of cofibrations of $\mathcal{M}$ is the same of $\mathcal{M}$.
(W) The collection of weak equivalences of $\mathcal{M}$ is the collection of $T$-weak equivalences.
(F) The collection of fibrations is the collection of morphisms which have the right lifting property with respect to all morphisms which are both cofibrations and $T$-weak equivalences.

Then $T^{-1}\mathcal{M}$ is a left proper combinatorial simplicial model category. Furthermore the functor $L_T : \mathcal{M} \rightarrow T^{-1}\mathcal{M}$ induced by the identity functor on the underlying category is a left Quillen functor of simplicial model categories.

proof. See [19] p.904, Section Appendix A.3.7]. □

The model category $T^{-1}\mathcal{M}$ is said to be the Bousfield localization of $\mathcal{M}$ by $T$. 
Lemma 2.14 ([3], p.21 Definition 2.11 and Lemma 2.12). Let $\mathbf{M}$ be a combinatorial model category and $\mathbf{C}$ a locally presentable model category. Given an adjunction

$$E : \mathbf{M} \rightleftarrows \mathbf{C} : F,$$

we will define a model structure on $\mathbf{C}$ by the following:

1. **(F)** A morphism $f : X \to Y$ in $\mathbf{C}$ is a fibration if $F(f) : F(X) \to F(Y)$ is a fibration in the model category $\mathbf{M}$.
2. **(W)** A morphism $f : X \to Y$ in $\mathbf{C}$ is a weak equivalence if $F(f) : F(X) \to F(Y)$ is a weak equivalence in the model category $\mathbf{M}$.
3. **(WF)** A morphism $f : X \to Y$ in $\mathbf{C}$ is a trivial fibration if $F(f) : F(X) \to F(Y)$ is a trivial fibration in the model category $\mathbf{M}$.
4. **(C)** A morphism $f : X \to Y$ in $\mathbf{C}$ is a cofibration if it has the right lifting property with respect to all trivial fibrations.

Assume that in $\mathbf{C}$, transfinite compositions and pushouts of trivial cofibrations of $\mathbf{C}$ are weak equivalences. Then the locally presentable category $\mathbf{C}$ is a combinatorial model category and it is a tractable model category if $\mathbf{M}$ is. Furthermore the above adjunction is a Quillen adjunction. □

We say that the model structure of $\mathbf{C}$ is the *projective model structure* induced by $E$.

Example 2.15. Let $\mathcal{C}$ be a category and $\mathbf{M}$ a model category. The diagonal functor $D_\mathcal{C} : \mathbf{M} \to \mathbf{M}^{\mathcal{C}}$ has a right adjoint $\prod_\mathcal{C}$. The model structure of $\mathbf{M}^{\mathcal{C}}$ is said to be induced by $\mathbf{M}$.

Now we recall the Dugger’s presentation theorem of combinatorial categories:

Theorem 2.16 ([11] Theorem 1.1). Let $\mathbf{M}$ be a combinatorial model category. Then there exists a small category $\mathcal{C}$ and a left Quillen functor $R : \text{Set}_\Delta^{\text{op}} \to \mathbf{M}$ such that $\mathbf{M}$ is a Quillen equivalent to a Bousfield localization of $\text{Set}_\Delta^{\text{op}}$, where the model structure of $\text{Set}_\Delta^{\text{op}}$ is the projective model structure. □

2.3. The definition of $\infty$-categories. For $n \geq 0$, we let $\Delta^n \in \text{Set}_\Delta$ denote the standard $n$-simplex and $\Lambda^n_i \subset \Delta^n$ be the sub-simplicial set obtained by deleting the interior and the face opposite for the $i$-th vertex.

An *inner fibration* $f : X \to S$ of simplicial sets is a morphism of simplicial sets which has the right lifting property with respect to all inclusion $\Lambda^n_i \to \Delta^n$ for any $n \geq 0$ and $0 < i < n$. Joyal [17] defined $\infty$-categories as follows:

Definition 2.17. A simplicial set $\mathcal{C}$ is an *$\infty$-category* if the canonical map $p : \mathcal{C} \to \ast$ is an inner fibration.
Let $\text{Cat}_{\Lambda}$ denote the large category of simplicial categories. Bergner [4] introduced a model structure which is another model of $\infty$-categories. The model structure is called the Dwyer–Kan model structure. Then $\text{Set}_{\Lambda}$ has a left proper combinatorial model structure that fibrant objects are just $\infty$-categories, and there exists a Quillen equivalence $\mathcal{E} : \text{Set}_{\Lambda} \rightleftarrows \text{Cat}_{\Lambda} : N_{\Lambda}$ between left proper combinatorial model categories (See [19, p.89, Theorem 2.2.5.1]). This model structure of $\text{Set}_{\Lambda}$ is called the Joyal model structure.

Let $\mathbf{M}$ be a simplicial model category and $\mathbf{M}^*$ denote the full subcategory spanned fibrant-cofibrant objects. Then $\mathbf{M}^*$ is a fibrant object of the model category $\text{Cat}_{\Lambda}$. Therefore the simplicial model category $\mathbf{M}$ determines an $\infty$-category $N_{\Lambda}(\mathbf{M}^*)$. We call the $\infty$-category $N_{\Lambda}(\mathbf{M}^*)$ the underlying $\infty$-category of $\mathbf{M}$.

**Definition 2.18.** Consider the Kan–Quillen model structure of $\text{Set}_{\Lambda}$. Write $\mathcal{S}_{\infty} = N_{\Lambda}(\text{Set}_{\Lambda}^*)$. Then we say that $\mathcal{S}_{\infty}$ is the $\infty$-category of spaces. We will refer to a fibrant object (i.e. a Kan complex) as an $\infty$-groupoid.

Lurie [19, Chapter 3] introduced a model category $\text{Set}_{\Lambda}^*$ of marked simplicial sets which gives another model of $\infty$-categories. The model structure is left proper combinatorial simplicial [19, Proposition 3.1.3.7]. A marked simplicial set is a pair $(X, E)$ where $X$ is a simplicial set and $E$ a subset of edges of $X$ which contains the set of all degenerate edges $s_0(X_0)$. Here $s_0 : X_0 \to X_1$ is the degeneracy map. The set $E$ is called marked edges of $X$. A morphism $f : (X, E_X) \to (Y, E_Y)$ of marked simplicial sets is a map $f : X \to Y$ of simplicial sets satisfying $f(E_X) \subset E_Y$. Let $\text{Set}_{\Lambda}^*$ denote the category of marked simplicial sets. We write $X^e = (X, X_1)$ and $X^1 = (X, s_0(X_0))$.

**Definition 2.19** ([19] Definition 2.4.1.3). Let $p : X \to S$ be an inner fibration of simplicial sets. An edge $f : x \to y$ of $X$ is $p$-Cartesian if the induced map $X_{/f} \to X_{/y} \times_{S_{/p(y)}} S_{/p(f)}$ is a trivial Kan fibration.

**Definition 2.20** ([21] Definition 2.4.2.1). A map $p : X \to S$ of simplicial sets is a Cartesian fibration if $p$ is an inner fibration and for every edge $f : x \to y$ in $S$ and every vertex $\tilde{y}$ of $X$ with $p(\tilde{y}) = y$, there exists a $p$-Cartesian edge $\tilde{f} : \tilde{x} \to \tilde{y}$ such that $p(\tilde{f}) = f$. We say that $p$ is a coCartesian fibration if the opposite $p^{op} : X^{op} \to S^{op}$ is a Cartesian fibration.

Let $p : X \to S$ be a Cartesian fibration of simplicial sets. Then $X^e$ denotes the marked simplicial set $(X, E)$, where $E$ is the set of $p$-Cartesian edges of $X$.

Let $X$ and $Y$ be marked simplicial sets. Then the product marked simplicial set $X \times Y$ is defined by the following:

- The underlying simplicial set of $X \times Y$ is the product of the underlying simplicial sets of $X$ and $Y$.
- All marked edges of $X \times Y$ are given by composition $\Delta^1 \xrightarrow{D} \Delta^1 \times \Delta^1 \to X \times Y$. 

\[8\]
where \( D \) denotes the diagonal map and the second map is the product of marked edges of \( X \) and \( Y \).

Let \((X, E_X)\) and \((Y, E_Y)\) be marked simplicial sets. Then the exponential marked simplicial set \( Y^X \) is defined by the following:

- The underlying simplicial set of \( Y^X \) is the functor
  \[
  \text{Set}_\Delta \ni K \mapsto \text{Hom}_{\text{Set}_\Delta}(K \times X, Y) \in \text{Sets}.
  \]

- A marked edge of \( Y^X \) is a morphism determined by a morphism \( \alpha : \Delta^1 \times X \to Y \) such that for any marked edge \( e \) of \( X \), composition
  \[
  \Delta^1 \xrightarrow{D} \Delta^1 \times \Delta^1 \xrightarrow{\text{Id}_{\Delta^1} \times e} \Delta^1 \times X \xrightarrow{\alpha} Y
  \]
is a marked edge of \( Y \).

The marked simplicial set \( Y^X \) has the evaluation map \( Y^X \times X \to Y \) which induces a bijection
\[
\text{Hom}_{\text{Set}_\Delta}(K \times X, Y) \to \text{Hom}_{\text{Set}_\Delta}(K, Y^X)
\]
for each \( K \in \text{Set}_\Delta^+ \).

The exponential determines the following two mapping simplicial sets: Let \( \text{Map}^\flat(X, Y) \) denote the underlying simplicial set of \( Y^X \) and \( \text{Map}^\flat(X, Y) \) the subsimplicial set of \( \text{Map}^\flat(X, Y) \) consisting of all simplices whose all edges are marked edges of \( Y^X \). Let \( S \) be a simplicial set. For marked simplicial sets \( X \) and \( Z \) over \( S \), set
\[
\text{Map}^\flat_S(X, Y) = \text{Map}^\flat(X, Y) \cap \text{Map}_S(X, Y)
\]
and
\[
\text{Map}^\sharp_S(X, Y) = \text{Map}^\sharp(X, Y) \cap \text{Map}_S(X, Y).
\]

Then \( \text{Map}^\sharp_S(X, Y) \) is a Kan complex if \( Y \to S \) is a Cartesian fibration (See [19, 3.1.3.1]).

**Definition 2.21** ([19] p.155). Let \( S \) be a simplicial set and \( p : X \to Y \) a morphism in \( \text{Set}_\Delta^+ / S \).

Then \( p \) is called a *Cartesian equivalence* if for every Cartesian fibration \( Z \to S \), the induced map
\[
\text{Map}^\sharp_S(Y, Z^\natural) \to \text{Map}^\sharp_S(X, Z^\natural)
\]
is a homotopy equivalence of Kan complexes.

**Theorem 2.22** ([19] p.157, Proposition 3.1.3.7). Let \( S \) be a simplicial set. There exists a left proper combinatorial model structure of \( \text{Set}_\Delta^+ / S \) which described as follows:

- **(W)** Weak equivalences in \( \text{Set}_\Delta^+ / S \) are Cartesian equivalences.
- **(C)** Cofibrations are monomorphisms.
- **(F)** Fibrations are those morphisms which have the right lifting property with respect to every morphism satisfying both (W) and (C).
Furthermore the model structure of $(\text{Set}_\Lambda^\ast)_S$ is simplicial and symmetric monoidal whose monoidal model structure is given by the products of marked simplicial sets.

This model structure of $(\text{Set}_\Lambda^\ast)_S$ is called Cartesian model structure. Consider the Joyal model structure of $\text{Set}_\Lambda$. Lurie proved that the functor $(-)^\Lambda : \text{Set}_\Lambda \to (\text{Set}_\Lambda^\ast)/_{A^0}$ is a left Quillen equivalence [19, p.164, Theorem 3.1.5.1]. Therefore a fibration of the simplicial model category $(\text{Set}_\Lambda^\ast)/_{A^0}$ has the underlying simplicial set which is an $\infty$-category. We can refer to fibrant objects of $(\text{Set}_\Lambda^\ast)/_{A^0}$ as $\infty$-categories. Let $\text{Cat}_{\infty}$ denote the simplicial nerve of the full subcategory of $(\text{Set}_\Lambda^\ast)/_{A^0}$ spanned by fibrant objects. We call $\text{Cat}_{\infty}$ the (large) $\infty$-category of small $\infty$-categories.

3. Motivic model categories.

In this section, we define motivic model categories for a Grothendieck site $X$ with an interval object. An interval object $I$ of $X$ is described by a triple $(\mu : I \times I \to I, i_0, i_1 : * \to I)$ satisfying $\mu \circ (i_0 \times \text{id}) = \mu \circ (\text{id} \times i_0) = i_0 \circ 1$ and $\mu \circ (i_1 \times \text{id}) = \mu \circ (\text{id} \times i_1) = \text{id}_I$ where $1 : I \to *$ is a canonical map.

3.1. Definition of motivic model structure of a left proper combinatorial simplicial model category. Let $X$ be a Grothendieck site with an interval object $I$ of $X$. We assume that $X$ has enough points: That is, a morphism $f : X \to Y$ in $X$ is an isomorphism if $f_* : X_* \to Y_*$ is an isomorphism of sets for any point $x : * \to X$ where the functor $(-)_* : X \to \text{Sets}$ denotes the right adjoint of the induced functor $x_* : \text{Sets} \to X$. A simplicial object $U_* : \Delta^\text{op} \to X$ with an augmentation $\pi : U_* \to X \in X$ is a hypercover of $X$ if the following conditions are hold:

- For any $n \geq 0$, $U_*([n])$ is a coproduct of compact objects represented by small objects of $X$.
- The augmentation $\pi : U_* \to X$ is a stalk-wise trivial Kan fibration: That is, $\pi_* : U_* \to *$ is a trivial Kan fibration for any point $x : * \to X$.

Let $M$ be a left proper combinatorial simplicial model category and $M^{\Delta^{op}}$ denote the model category of $M$-valued presheaves on $X$. Here the model structure of $M^{\Delta^{op}}$ is the projective model structure. Then the model structure of $M^{\Delta^{op}}$ is also left proper combinatorial and simplicial. Let $\text{Mot}^I(M)$ denote the Bousfield localization of $M^{\Delta^{op}}$ defined as follows: Cofibrations of $\text{Mot}^I(M)$ are the same of $M^{\Delta^{op}}$. An $M$-valued presheaves $F$ on $X$ is $X$-local if $F$ is a fibrant object of $M^{\Delta^{op}}$ and the induced map $F(f) : F(X) \to F(U_\bullet)$ is a weak equivalence in $M$ for any hypercover $\pi : U_\bullet \to X$ of $x \in X$. Here the functor $| - | : \text{Fun}(\Delta^{op}, X) \to X$ denotes the geometric realization of simplicial objects. Furthermore $F$ is $I$-local if the canonical map $1 : I \to *$ induces a weak equivalence $F(U) \to F(U \times I)$ in $M$ for any $U \in X$. We say that $F$ is motivic $M$-local if $F$ is $X$-local and $I$-local. A map $f : F \to G$ of $M$-valued presheaves on $X$
is a motivic $M$-equivalence if the induced map

$$f^* : \text{Hom}(G, Z) \to \text{Hom}(F, Z)$$

is a weak homotopy equivalence of simplicial sets for each motivic local $M$-valued presheaf $Z$. We call the modal structure of $\text{Mot}_X^I(M)$ the motivic $X$-model structure of $M$.

By [3, p.56, Corollary 4.55], the projective model structure of $M^{\text{op}}$ is left proper combinatorial and symmetric monoidal. Therefore the composition of two Bousfield localization $\text{Mot}_X^I(M)$ of $M^{\text{op}}$ is also left proper. Moreover, by [3, p.54 Proposition 4.47], the Bousfield localization $\text{Mot}_X^I(M)$ is symmetric monoidal localization of $M^X$. Hence we obtain that $\text{Mot}_X^I(M)$ is also a symmetric monoidal model category:

**Theorem 3.1.** Let $M$ be a left proper combinatorial simplicial model category. Let $X$ be a Grothendieck site with an interval $I$. Assume that $X$ has enough points. Then there is a left proper combinatorial simplicial model structure of $\text{Mot}_X^I(M)$ determined by the following:

(C) Cofibrations are point-wise cofibrations.

(W) Weak equivalences are motivic $M$-weak equivalences.

(F) Fibrations are morphisms which has a left lifting property with respect to all morphisms which are both cofibrations and motivic $M$-weak equivalences.

Furthermore, if $M$ is a symmetric monoidal model category, then $\text{Mot}_X^I(M)$ is also symmetric monoidal category. □

Write $\text{MS}_{\infty}^I = \text{Mot}_X^I(\text{Set}_{\Delta})_{\infty}$ and $\text{MCat}_{\infty}^I = \text{Mot}_X^I(\text{Set}_{\Delta})_{\infty}$. We refer to $\text{MS}_{\infty}^I = \text{Mot}_X^I(\text{Set}_{\Delta})_{\infty}$ as the $\infty$-category of motivic spaces and $\text{MCat}_{\infty}^I$ as the $\infty$-category of motivic $\infty$-category, respectively. We will give more explicit view of motivic spaces and motivic $\infty$-categories: By the straightening and unstraightening theorem [19, p.74, Theorem 2.2.1.2], we have a Quillen adjunction

$$\text{St}_X : (\text{Set}_{\Delta})_{/N(X)} \rightleftarrows \text{Set}_{\Delta}^{\text{op}} : \text{Un}_X$$

where the model structure of the left-hand-side is the contravariant model structure [19, Remark 2.1.4.12] and the one of the right-hand-side the projective model structure of the Kan–Quillen model structure of $\text{Set}_{\Delta}$. Let $X$ be a motivic space. Then there exists a right fibration $p_X : \overline{X} \to N(X)$ such that for any $U \in X$, the fiber $X \times_{N(X)} U$ is homotopy equivalent to the Kan complex $X(U)$ satisfying $X(U \times I) \simeq X(U)$.

Similarly, a motivic $\infty$-category $\mathcal{C}$ is an $\text{Set}_{\Delta}^+$-valued presheaf on $X$ satisfying $\mathcal{C}(U \times I) \simeq \mathcal{C}(U)$ for any $U \in X$. In addition, the $\infty$-category $\text{MCat}_{\infty}^I$ is a full subcategory of the $\infty$-category $\text{Fun}(X^{\text{op}}, \text{Cat}_{\infty})$ spanned by $I$-local objects. By the straightening and unstraightening theorem for the model category of marked simplicial sets with the Cartesian model structure [19, p.169, Theorem 3.2.0.1], we have a Quillen adjunction

$$\text{St}_{\Delta}^I : (\overline{\text{Set}_{\Delta}^+}_{/N(X)}) \rightleftarrows (\overline{\text{Set}_{\Delta}^+})^{\text{op}}_{/X} : \text{Un}_{\Delta}^+$$
between left proper combinatorial simplicial model categories. Therefore there exists a Cartesian fibration \( p_c : \mathcal{C} \rightarrow N(\mathcal{X}) \) such that for any \( U \in \mathcal{C} \), the fiber \( \mathcal{C} \times_{N(\mathcal{X})} U \) is weakly equivalent to the \( \infty \)-category \( \mathcal{C}(U) \) satisfying the condition : \( \mathcal{C}(U \times I) \simeq \mathcal{C}(U) \).

Let \( I \) denote the final object of \( \text{MCat}_{\infty} \). Then we have a canonical equivalence:

\[
\text{Fun}_{\text{Cat}_{\infty}}(\Delta^0, \text{Fun}^f(N(\mathcal{X}^{\text{op}}), \mathcal{C})) \simeq \text{Fun}_{\text{MCat}_{\infty}}(I, \mathcal{C}),
\]

where \( \text{Fun}^f(N(\mathcal{X}^{\text{op}}), \mathcal{C}) \) denotes the subcategory of \( \text{Fun}(N(\mathcal{X}^{\text{op}}), \mathcal{C}) \) spanned by those functors satisfying \( F(U \times I) \simeq F(U) \) in \( \mathcal{C} \) for any \( U \in \mathcal{X} \). Note that the model category \( (\text{Set}_{\mathcal{X}})_{/N(\mathcal{X})} \) has a symmetric monoidal model structure with the unit object \( N(\mathcal{X}) \). Hence an object of the motivic \( \infty \)-category \( \mathcal{C} \) is given by a map from the unit \( N(\mathcal{X}) \) to the marked simplicial set \( \mathcal{C} \): an object \( f \) of \( \mathcal{C} \) is described by the following diagram

\[
\begin{array}{ccc}
\Delta^0 \times \{U\} & \xrightarrow{f(U)} & \mathcal{C}(U) \\
\downarrow & & \downarrow \\downarrow \mathcal{C} \\
\Delta^0 \times N(\mathcal{X}) & \xrightarrow{f(U, U)_{U \in \mathcal{X}}} & \mathcal{C} \times_{N(\mathcal{X})} U
\end{array}
\]

satisfying \( f(U \times I) \simeq f(U) \), where the squares are homotopy pull back squares for any \( U \in \mathcal{X} \).

3.2. The large motivic \( \infty \)-category of motivic spaces and the large motivic \( \infty \)-category of small motivic \( \infty \)-categories. Let \( \mathbf{M} \) be a left proper combinatorial simplicial monoidal model category. Assume that \( \mathbf{M} \) has a Cartesian closed symmetric monoidal model structure. Then the model category \( \text{Mot}_{\mathcal{X}}(\mathbf{M}) \) is an \( \mathbf{M} \)-enriched model category. Therefore we have a left Quillen bifunctor \(- \otimes - : \mathbf{M} \otimes \text{Mot}_{\mathcal{X}}(\mathbf{M}) \rightarrow \text{Mot}_{\mathcal{X}}(\mathbf{M})\). Let \( 1 \) be the unit element of \( \text{Mot}_{\mathcal{X}}(\mathbf{M}) \). Then the left Quillen bifunctor \(- \otimes - \) induces a Quillen adjunction

\[
- \otimes 1 : \mathbf{M} \rightleftarrows \text{Mot}_{\mathcal{X}}(\mathbf{M}) : \text{Hom}_{\text{Mot}_{\mathcal{X}}(\mathbf{M})}(1, -).
\]

Consider the case \( \mathbf{M} = (\text{Set}_{\mathcal{X}})_{/\Delta^0} \). Let \( \text{MCat}_{\infty} \) denote the underlying \( \infty \)-category of \( \text{Mot}_{\mathcal{X}}(\mathbf{M}) \). Since the Cartesian model structure of \( (\text{Set}_{\mathcal{X}})_{/\Delta^0} \) is symmetric monoidal, \( \text{MCat}_{\infty} \) is a presentable symmetric monoidal \( \infty \)-category. We have the following adjunction

\[
- \otimes 1 : \text{MCat}_{\infty} \rightleftarrows \text{Map}_{\text{MCat}_{\infty}}(1, -)
\]

between vary large \( \infty \)-categories. The unit object \( 1 \) is described by the single-value point \( \Delta^0 \) constant presheaves on \( \mathcal{X} \). Write \( \mathcal{M} := \mathcal{S}_{\infty} \otimes 1 \) and \( \mathcal{MCat}_{\infty} = \text{Cat}_{\infty} \otimes 1 \). We say that \( \mathcal{M} \) the motivic \( \infty \)-category of motivic spaces and \( \text{MCat}_{\infty} \) the motivic \( \infty \)-category of motivic \( \infty \)-categories.

Let \( \mathcal{D} \) be motivic \( \infty \)-category. Then the product \(- \otimes \mathcal{D} : \text{MCat}_{\infty} \rightarrow \text{MCat}_{\infty} \) has a right adjoint \( \text{Fun}_{\text{MCat}_{\infty}}(-, \mathcal{D}) : \text{MCat}_{\infty} \rightarrow \text{MCat}_{\infty} \). Let \( \mathcal{C} \) be a motivic \( \infty \)-category. Then we say that \( \text{Fun}_{\text{MCat}_{\infty}}(\mathcal{C}, \mathcal{D}) \) is the functor motivic \( \infty \)-category from \( \mathcal{C} \) to \( \mathcal{D} \).
Proposition 3.2. Let \( \mathcal{C} \) be a small motivic \( \infty \)-category. Then the motivic Yoneda functor
\[
y : \mathcal{C} \ni x \mapsto \text{Map}_{\mathcal{C}}(\cdot, x) \in \text{Fun}_{\widehat{\text{MCat}}_{\infty}}(\mathcal{C}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty})
\]
is a fully faithful functor between motivic \( \infty \)-categories.

**proof.** For any object \( U \in X \), we have a weak equivalence
\[
y(U) : \mathcal{C}(U) \to \text{Fun}_{\widehat{\text{Cat}}_{\infty}}(\mathcal{C}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty}(U)) \simeq \text{Fun}_{\widehat{\text{Cat}}_{\infty}}(\mathcal{C}^\text{op}(U), \mathcal{S}_{\infty})
\]
of \( \infty \)-categories by the \( \infty \)-categorical Yoneda lemma [19, p.317, Proposition 5.1.3.1]. Hence the motivic Yoneda functor is also fully faithful. \( \square \)

Lemma 3.3. Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor of small motivic \( \infty \)-categories. Then the restriction functor
\[
F^{-1} : \text{Fun}(\mathcal{D}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty}) \to \text{Fun}(\mathcal{C}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty})
\]
ads a left adjoint.

**proof.** Let \( f : U \to V \) be a morphism in \( \mathcal{X} \). Then we have the commutative diagram
\[
\begin{array}{ccc}
\text{Set}_\Delta^{\mathcal{E}(U)^\text{op}} & \xrightarrow{F(U)_*} & \text{Set}_\Delta^{\mathcal{E}(V)^\text{op}} \\
\mathcal{E}(U)_* \downarrow & & \downarrow \mathcal{E}(V)_* \\
\text{Set}_\Delta^{\mathcal{D}(U)^\text{op}} & \xrightarrow{F(V)_*} & \text{Set}_\Delta^{\mathcal{D}(V)^\text{op}}
\end{array}
\]
of left Quillen functors of simplicial model categories. Therefore we obtain that the functor \( F_* : \text{Fun}(\mathcal{C}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty}) \to \text{Fun}(\mathcal{D}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty}) \) is left adjoint to \( F^{-1} \). \( \square \)

3.3. **Definitions of motivic presentable \( \infty \)-categories and motivic \( \infty \)-topoi.** Let \( \text{Cat}_{\infty} \) denote the very large \( \infty \)-category of large \( \infty \)-categories and \( \text{MCat}_{\infty} \) the very large \( \infty \)-category of large motivic \( \infty \)-categories. Let \( \text{LPr} \) denote the subcategory of \( \text{Cat}_{\infty} \) that those objects are presentable \( \infty \)-categories and those functors are colimit preserving functors. We will define the very large \( \infty \)-category \( \text{LMPr} \) as the pullback of very large \( \infty \)-categories:

\[
\begin{array}{ccc}
\text{LMPr} & \xrightarrow{\text{Fun}(\mathcal{X}^\text{op}, \text{LPr})} & \text{Fun}(\mathcal{X}^\text{op}, \text{Cat}_{\infty}) \\
\text{MCat}_{\infty} \downarrow & & \downarrow \text{Fun}(\mathcal{X}^\text{op}, \text{Cat}_{\infty}) \\
\text{Fun}(\mathcal{X}^\text{op}, \text{MCat}_{\infty}) & \xrightarrow{\text{Fun}(\mathcal{X}^\text{op}, \text{MCat}_{\infty})} & \text{Fun}(\mathcal{X}^\text{op}, \text{MCat}_{\infty})
\end{array}
\]

We say that \( \text{LMPr} \) is the vary large \( \infty \)-category of presentable motivic \( \infty \)-categories.

Proposition 3.4. Let \( F : \mathcal{C} \to \mathcal{D} \) be a functor of motivic \( \infty \)-categories. Assume that \( \mathcal{C} \) is small and \( \mathcal{D} \) is presentable. Let \( y : \mathcal{C} \to \text{Fun}(\mathcal{C}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty}) \) denote the motivic Yoneda embedding. Then the induced functor
\[
y_* : \mathcal{D} \ni d \mapsto \text{Map}_{\mathcal{D}}(\cdot, d) \in \text{Fun}(\mathcal{C}^\text{op}, \mathcal{M}\mathcal{Z}_{\infty})
\]
ads a left adjoint.
**Proof.** For any $U \in \mathcal{X}$, $y_*(F(U)) : \mathcal{D}(U) \to \text{Fun}(\mathcal{C}(U)^{op}, \mathcal{S}_\infty)$ admits a left adjoint $F_*(y)(U) : \text{Fun}(\mathcal{C}(U)^{op}, \mathcal{S}_\infty) \to \mathcal{D}(U)$. We prove that the left adjoints $F_*(y)(U)$ are functorial for morphisms in $\mathcal{X}$: Let $\mathcal{M} \mathcal{T}_\infty$ denote the very large motivic $\infty$-category of motivic spaces. By Lemma 3.3 we have an adjunction:

$$\hat{F}_* : \text{Fun}(\mathcal{C}^{op}, \mathcal{M} \mathcal{T}_\infty) \rightleftarrows \text{Fun}(\mathcal{D}^{op}, \mathcal{M} \mathcal{T}_\infty) : \hat{F}^{-1}.$$ 

Consider the homotopically commutative diagram

$$\begin{array}{ccc}
\text{Fun}(\mathcal{C}^{op}, \mathcal{M} \mathcal{T}_\infty) & \overset{\hat{F}_*}{\longrightarrow} & \text{Fun}(\mathcal{D}^{op}, \mathcal{M} \mathcal{T}_\infty) \\
\text{Fun}(\mathcal{C}^{op}, \mathcal{M} \mathcal{T}_\infty) & \underset{\gamma_*(F)}{\longrightarrow} & \mathcal{D}
\end{array}$$

of motivic $\infty$-categories. Let $f : U \to V$ be a morphism in $\mathcal{X}$. Then we have a diagram

$$\begin{array}{ccc}
\text{Fun}(\mathcal{C}(U)^{op}, \mathcal{S}_\infty) & \overset{\gamma_*(F)_!(U)}{\longrightarrow} & \mathcal{D}(U) \\
\text{Fun}(\mathcal{C}(V)^{op}, \mathcal{S}_\infty) & \underset{\gamma_*(F)_!(V)}{\longrightarrow} & \mathcal{D}(V)
\end{array}$$

of $\infty$-categories and a natural transformation $\alpha : \mathcal{D}(f) \circ F_*(y)(U) \to F_*(y)(V) \circ \gamma(f)$. Since $\hat{\gamma_2}$ is fully faithful and the natural transformation $\hat{\gamma_2}(\alpha) : \hat{\gamma_2}(\mathcal{D}(f) \circ F_*(y)(U)) \to \hat{\gamma_2}(\mathcal{D}(f) \circ F_*(y)(V))$ is weak equivalence, $\alpha$ is also a weak equivalence. □

By the similar way of the definition of presentable motivic $\infty$-categories, we define the motivic version of $\infty$-topoi: Let $^{1}\text{Top}$ denote the very large $\infty$-category of $\infty$-topoi. Those functors of $^{1}\text{Top}$ are left exact colimit preserving functors. Then the very large $\infty$-category $^{1}\mathcal{M}\text{Top}$ is also defined as the pullback

$$\begin{array}{ccc}
^{1}\mathcal{M}\text{Top} & \longrightarrow & \text{Fun}(\mathcal{X}^{op}, ^{1}\text{Top}) \\
\downarrow & & \downarrow \\
\mathcal{M}\mathcal{C}\mathcal{A}_{\infty} & \longrightarrow & \text{Fun}(\mathcal{X}^{op}, \mathcal{C}\mathcal{A}_{\infty}).
\end{array}$$

We say that an object of $^{1}\mathcal{M}\text{Top}$ is a **motivic $\infty$-topos**.

**Proposition 3.5.** Let $\mathcal{C}$ be a small motivic $\infty$-category. Let $\mathcal{M}\mathcal{C}\mathcal{A}_{\infty}$ denote the very large $\infty$-category of motivic $\infty$-categories. Assume that for any morphism $f : U \to V$ in $\mathcal{X}$, the induced functor $\gamma(f) : \mathcal{C}(V) \to \mathcal{C}(U)$ is left exact. Then the motivic $\infty$-category $\text{Fun}_{\mathcal{M}\mathcal{C}\mathcal{A}_{\infty}}(\mathcal{C}^{op}, \mathcal{M} \mathcal{T}_\infty)$ is a motivic $\infty$-topos.

**Proof.** For any object $U \in \mathcal{X}$, the functor $\infty$-category

$$\text{Fun}_{\mathcal{C}\mathcal{A}_{\infty}}(\mathcal{C}(U)^{op}, \mathcal{M} \mathcal{T}_\infty(U)) \simeq \text{Fun}_{\mathcal{C}\mathcal{A}_{\infty}}(\mathcal{C}(U)^{op}, \mathcal{S}_\infty)$$

\[14\]
is an ∞-topos. For any map \( f : U \to V \) in \( \mathcal{X} \), the induced functor \( \mathcal{C}(f) : \mathcal{C}(V) \to \mathcal{C}(U) \) between ∞-categories induces a functor
\[
y(\mathcal{C})(f) : \text{Fun}_{\mathsf{Cat}_{\infty}}(\mathcal{C}(V)^{op}, S_{\infty}) \to \text{Fun}_{\mathsf{Cat}_{\infty}}(\mathcal{C}(U)^{op}, S_{\infty})
\]
which is a left exact colimit preserving functor by [19] p.324, Theorem 5.1.5.6 and p.559, Proposition 6.1.5.2. \( \square \)

4. Motivic derived algebraic geometry.

4.1. Definition of motivic ∞-bicategories. We define the motivic version of ∞-bicategories to formulate the theory of motivic derived algebraic geometry.

First, we recall short review of ∞-bicategories following [20]. Let \( \text{Set}_{\Lambda}^{sc} \) be the category of scaled simplicial sets. The category of scaled simplicial sets \( \text{Set}_{\Lambda}^{sc} \) has a left proper combinatorial model structure [20] p.143, Theorem 4.2.7]. An ∞-bicategory is defined to be a fibrant scaled simplicial set with respect to the model structure on \( \text{Set}_{\Lambda}^{sc} \). However, the model category \( \text{Set}_{\Lambda}^{sc} \) is not simplicial. In order to formulate the motivic model category of \( \text{Set}_{\Lambda}^{sc} \), we use the model category \( (\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}} \) which is left proper combinatorial simplicial symmetric monoidal category. The model structure of \( (\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}} \) is the Bousfield localization of the coCartesian model structure induced by the complete Segal model structure of \( (\text{Set}_{\Lambda}^{sc})^{\Delta^{op}} \) ([20] p.34, Proposition 1.5.7]). There is a left Quillen functor \( \text{sd}^{+} : \text{Set}_{\Lambda}^{sc} \to (\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}} \) which is called a subdivision functor [20] p.145, Definition 4.3.1]. By [20] p.150, Theorem 4.3.1.13], the subdivision functor \( \text{sd}^{+} : \text{Set}_{\Lambda}^{sc} \to (\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}} \) is a left Quillen equivalence.

**Proposition 4.1.** Let \( \mathcal{X} \) be a Grothendieck site with an interval \( I \). Let \( \text{Mot}_{\mathcal{X}}^{I}(\text{Set}_{\Lambda}^{sc}) \) denote the functor category \( (\text{Set}_{\Lambda}^{sc})^{\mathcal{X}^{op}} \) and \( \text{Mot}_{\mathcal{X}}^{I}(\text{sd}^{+}) : \text{Mot}_{\mathcal{X}}^{I}(\text{Set}_{\Lambda}^{sc}) \rightleftarrows \text{Mot}_{\mathcal{X}}^{I}((\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}}) \) the adjunction induced by the Quillen equivalence \( \text{sd}^{+} : \text{Set}_{\Lambda}^{sc} \rightleftarrows (\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}} : F \). We define a model structure on \( \text{Mot}_{\mathcal{X}}^{I}(\text{Set}_{\Lambda}^{sc}) \) as follows:

(C) A morphism \( f : \overline{X} \to \overline{Y} \) is a cofibration if and only if it is a pointwise cofibration.

(W) A morphism \( f : \overline{X} \to \overline{Y} \) is a weak equivalence if and only if \( \text{Mot}_{\mathcal{X}}^{I}(\text{sd}^{+})(f) : \text{sd}^{+}(\overline{X}) \to \text{sd}^{+}(\overline{Y}) \) is a motivic \( (\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}} \)-equivalence.

(F) A morphism \( f : \overline{X} \to \overline{Y} \) is a fibration if and only if it has the right lifting property with respect to all morphisms which satisfies the condition (C) and (W).

Then the Quillen adjunction \( \text{Mot}_{\mathcal{X}}^{I}(\text{sd}^{+}) : \text{Mot}_{\mathcal{X}}^{I}(\text{Set}_{\Lambda}^{sc}) \rightleftarrows \text{Mot}_{\mathcal{X}}^{I}((\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}}) \) : \( \text{Mot}_{\mathcal{X}}^{I}(F) \) is a Quillen equivalence.

**proof.** By the definition of the model structure of \( \text{Mot}_{\mathcal{X}}^{I}(\text{Set}_{\Lambda}^{sc}) \), the induced functor \( \text{Mot}_{\mathcal{X}}^{I}(\text{sd}^{+}) \) is a left Quillen functor. Note that if \( F : \mathcal{M} \to \mathcal{N} \) is a left Quillen equivalence of left proper combinatorial model categories, the induced functor \( F^{\Delta^{op}} : \mathcal{M}^{\Delta^{op}} \to \mathcal{N}^{\Delta^{op}} \) is also a left Quillen equivalence. Let \( \overline{X} \) be a cofibrant object of \( \text{Mot}_{\mathcal{X}}^{I}(\text{Set}_{\Lambda}^{sc}) \) and \( \overline{Y} \) a motivic object of \( \text{Mot}_{\mathcal{X}}^{I}((\text{Set}_{\Lambda}^{sc})_{/\Delta^{op}}) \).
We prove that \( f : \overline{X} \to \text{Mot}_\lambda^t(F)(\overline{Y}) \) is a weak equivalence \( \text{Mot}_\lambda^t(\text{Set}_\lambda^c) \) if and only if the adjoint \( f^* : \text{Mot}_\lambda^t(\text{sd}^+)(\overline{X}) \to \overline{Y} \) is a motivic equivalence. Since \( \overline{Y} \) is motivic local, therefore \( \overline{Y} \) is projective fibrant and the counit map \( \text{Mot}_\lambda^t(\text{sd}^+ \circ F)(\overline{Y}) \to \overline{Y} \) is a projective weak equivalence. Hence the counit \( \text{Mot}_\lambda^t(\text{sd}^+ \circ F)(\overline{Y}) \to \overline{Y} \) is motivic weak equivalence with respect to motivic complete Segal model structure. Write \( C = \text{Mot}_\lambda^t(\text{Set}_\lambda^c) \) and \( D = \text{Mot}_\lambda^t((\text{Set}_\lambda^c)/(\Delta)^{op}) \). Then we have a chain of weak equivalences of simplicial sets

\[
\text{Hom}_C(\text{Mot}_\lambda^t(\text{sd}^+)(\overline{X}), \overline{Z}) \cong \text{Hom}_D(\overline{X}, \text{Mot}_\lambda^t(F)(\overline{Z})),
\]

\[
\text{Hom}_C(\overline{Y}, \overline{Z}) \cong \text{Hom}_D(\text{Mot}_\lambda^t(\text{sd}^+ \circ F)(\overline{Z}), \text{Mot}_\lambda^t(F)(\overline{Z}))
\]

\[
\cong \text{Hom}_D(\text{Mot}_\lambda^t(F)(\overline{Y}), \text{Mot}_\lambda^t(F)(\overline{Z}))
\]

for any motivic local object \( \overline{Z} \) of \( D \). We obtain that \( f^* : \text{Hom}_C(\overline{Y}, \overline{Z}) \to \text{Hom}_C(\text{Mot}_\lambda^t(\text{sd}^+)(\overline{X}), \overline{Z}) \) is a weak equivalence if and only if the induced map \( f^{**} : \text{Hom}_D(\text{Mot}_\lambda^t(F)(\overline{Y}), \text{Mot}_\lambda^t(F)(\overline{Z})) \to \text{Hom}_D(\overline{X}, \text{Mot}_\lambda^t(F)(\overline{Z})) \) is a weak equivalence. \( \square \)

**Definition 4.2.** A motivic \( \infty \)-bicategory is a fibrant object of the model category \( \text{Mot}_\lambda^t(\text{Set}_\lambda^c) \).

A motivic \( \infty \)-bicategory \( \mathcal{F} \) is a \( \text{Set}_\lambda^c \)-valued presheaf on \( \mathcal{X} \) such that each scaled simplicial set \( \mathcal{F}(X) \) is an \( \infty \)-bicategory for \( X \in \mathcal{X} \), and the induced maps \( \mathcal{F}(X \times I) \to \mathcal{F}(X) \) and \( \mathcal{F}(X) \to \mathcal{F}([U_\bullet]) \) are both bicategorical equivalences of scaled simplicial sets for any hypercover \( U_\bullet \to X \).

### 4.2. The motivic \( \infty \)-bicategory of motivic \( \infty \)-categories

By [20] Notation 3.1.9], the model category \( \text{Set}_\lambda^c \) has a Cartesian closed symmetric monoidal structure determined by the Cartesian product \(- \times -\) of scaled simplicial sets. Using [20] Lemma 4.2.6, we obtain that \( \text{Set}_\lambda^c \) is a symmetric monoidal model category. Hence we have a left Quillen functor:

\[- \otimes 1 : \text{Set}_\lambda^c \to \text{Mot}_\lambda^t(\text{Set}_\lambda^c) .\]

Let \( \text{Cat}_\infty \) denote the \( \infty \)-bicategory of \( \infty \)-categories. Write \( \text{MCat}_\infty = \text{Cat}_\infty \otimes 1 \). We say that \( \text{MCat}_\infty \) is the motivic \( \infty \)-bicategory of motivic \( \infty \)-categories. Similarly, let \( ^1\text{Top} \) denote the \( \infty \)-subbicategory of \( \text{Cat}_\infty \) spanned by \( \infty \)-topoi, and those functors are left exact colimit preserving functors. Then we set \( ^1\text{MTop} = ^1\text{Top} \otimes 1 \) and call the \( \infty \)-bicategory of motivic \( \infty \)-topoi.

### 4.3. Motivic \( \infty \)-topoi and motivic classifying \( \infty \)-topoi

Let \( \text{MCat}_\infty \) denote the motivic \( \infty \)-bicategory of (not necessary small) motivic \( \infty \)-categories. Let \( ^1\text{MTop} \) be a subcategory of \( \text{MCat}_\infty \) whose objects are motivic \( \infty \)-topoi and morphisms left exact colimit preserving functors. We call morphisms in \( ^1\text{MTop} \) geometric morphisms.

**Definition 4.3** (cf. [19] p.369, Definition 5.2.8.8(Joyal)). Let \( \mathcal{C} \) be a motivic \( \infty \)-category. A factorization system \((S_L, S_R)\) is a pair of collections of morphisms of \( \mathcal{C} \) which satisfy the following axioms:

1. The collections \( S_L \) and \( S_R \) are closed under retracts.
(2) The collection $S_L$ is left orthogonal to $S_R$.

(3) For any morphism $h : X \to Z$ in $\mathcal{C}$, there exists an object $Y$ of $\mathcal{C}$, morphisms $f : X \to Y$ and $g : Y \to Z$ such that $h = g \circ f$, $f \in S_L$ and $g \in S_R$.

Let $\mathcal{T}$ and $\mathcal{T}'$ be motivic $\infty$-topoi. Let $\text{Fun}(\mathcal{T}, \mathcal{T}')$ denote the full subcategory of $\text{Fun}(\mathcal{T}, \mathcal{T'})$ spanned by those functors $f : \mathcal{T} \to \mathcal{T}'$ which admit geometric left adjoints. Equivalently, a motivic classifying $\infty$-topos $\mathcal{K}$ is a motivic $\infty$-topos which satisfies $\mathcal{K}(U)$ is a classifying $\infty$-topos for any $U \in \mathcal{X}$ and the morphisms $\mathcal{K}(U) \to \mathcal{K}(V)$ compatible with the geometric structures for any morphism $V \to U$ in $\mathcal{X}$.

**Definition 4.4** ([22] p.27, Definition 1.4.3). Let $\mathcal{K}$ be a motivic $\infty$-topos. A geometric structure on $\mathcal{K}$ is a factorization system $(S_L^\mathcal{K}, S_R^\mathcal{K})$ on $\text{Fun}^*(\mathcal{K}, \mathcal{T})$, which depends functorially on $\mathcal{T}$. We say that $\mathcal{K}$ is a classifying motivic $\infty$-topos and a morphism in $S_R^\mathcal{K}$ is a local morphism. For any classifying motivic $\infty$-topos $\mathcal{K}$ and motivic $\infty$-topos $\mathcal{T}$, we let $\text{Str}\^{\mathcal{K}}_\mathcal{K}(\mathcal{T})$ denote the subcategory of $\text{Fun}^*(\mathcal{K}, \mathcal{T})$ spanned by all the objects of $\text{Fun}^*(\mathcal{K}, \mathcal{T})$, and all morphisms of $\text{Str}\^{\mathcal{K}}_\mathcal{K}(\mathcal{T})$ are local. We say that an object of $\text{Str}\^{\mathcal{K}}_\mathcal{K}(\mathcal{T})$ is a $\mathcal{K}$-structured sheaf on $\mathcal{T}$. If a geometric morphism $f : \mathcal{K} \to \mathcal{K}'$ of classifying motivic $\infty$-topoi carries all local morphisms on $\text{Fun}^*(\mathcal{K}, \mathcal{T})$ to local morphisms on $\text{Fun}^*(\mathcal{K}', \mathcal{T})$ for any motivic $\infty$-topos, we say that $f$ is compatible with the geometric structures.

The scaled straightening and unstraightening

$$\text{St}^{\mathcal{K}} : (\text{Set}^+_L)_{\mathcal{T}\text{-Top}} \rightleftarrows (\text{Set}^+_L)^{\infty\text{-Top}} : \text{Un}^{\mathcal{K}}$$

induces a Quillen equivalence

$$\text{Mot}^{\mathcal{K}}_\mathcal{K}(\text{St}^{\mathcal{K}}) : \text{Mot}^{\mathcal{K}}_\mathcal{K}((\text{Set}^+_L)_{\mathcal{T}\text{-Top}}) \rightleftarrows \text{Mot}^{\mathcal{K}}_\mathcal{K}((\text{Set}^+_L)^{\infty\text{-Top}}) : \text{Mot}^{\mathcal{K}}_\mathcal{K}(\text{Un}^{\mathcal{K}})$$

of left proper combinatorial simplicial model categories. We say that a fibrant object $p : X \to \text{L}M\text{Top}$ is a motivic locally coCartesian fibration. That is, for any $U \in \mathcal{X}$, the induced map $p(U) : X(U) \to \text{L}M\text{Top}(U)$ is a locally coCartesian fibration which is functorially morphisms in $\mathcal{X}$. A motivic locally $p$-coCartesian edge of $X$ is an edge which functorially induces a locally $p(U)$-coCartesian edge for any $U \in \mathcal{X}$.

The $\text{Cat}_{\mathcal{K}}$-valued presheaf

$$\text{Str}^{\mathcal{K}}_{\mathcal{K}} : \mathcal{K}^{\mathcal{K}} \ni U \mapsto \text{Str}^{\mathcal{K}}(U)(-) \in \text{Fun}_{\text{Cat}(\mathcal{K})} \left(\text{L}M\text{Top}, \text{Cat}_{\mathcal{K}}\right)$$

determines the $\text{Cat}_{\mathcal{K}}$-valued presheaf

$$\text{L}M\text{Top}(\mathcal{K}) : \mathcal{X} \ni U \mapsto \text{L}M\text{Top}(\mathcal{K}(U)) \in \text{N}(\text{Set}^+_L)^{\mathcal{K}}_{\text{Top}}$$

with a locally coCartesian fibration ([19] p.123, Definition 2.4.2.6] $p(U) : \text{L}M\text{Top}(\mathcal{K}(U)) \to \text{L}\text{Top}^{\mathcal{K}}$ for any $U \in \mathcal{X}$, and the induced maps $f_{UV} : \text{L}M\text{Top}(\mathcal{K}(U)) \to \text{L}M\text{Top}(\mathcal{K}(V))$ carry $p(U)$-locally coCartesian edges to $p(V)$-locally coCartesian edges for any $V \to U$ in $\mathcal{X}$. We
say that \( p : L\mathcal{M}\operatorname{Top}(\mathcal{K}) \to L\mathcal{M}\text{Top}^{op} \) is a motivic \( \infty \)-category of \( \mathcal{K} \)-structured motivic \( \infty \)-topoi. Furthermore we have that the motivic Yoneda functor \( \operatorname{Fun}^\ast(\mathcal{K}, -) : L\mathcal{M}\text{Top} \to \mathcal{M}\text{Cat}_\infty \) classifies an \( \infty \)-category \( L\mathcal{M}\operatorname{Top}_{\mathcal{K}/} \) and a locally motivic coCartesian fibration \( q : L\mathcal{M}\operatorname{Top}_{\mathcal{K}/} \to L\mathcal{M}\text{Top} \). By the similar argument of the proof of [19, p.610, Proposition 6.3.4.6], we have that the \( \infty \)-category \( R\mathcal{M}\operatorname{Top} \) admits pullbacks. In other words, for any geometric morphism \( f : \mathcal{K} \to \mathcal{K}' \), the forgetful functor \( f_* : R\mathcal{M}\operatorname{Top}_{\mathcal{K}/} \to R\mathcal{M}\operatorname{Top}_{\mathcal{K}'/} \) admits a right adjoint. Note that the opposite category of \( L\mathcal{M}\operatorname{Top}_{\mathcal{K}/} \) is weakly equivalent to \( (R\mathcal{M}\operatorname{Top}_{\mathcal{K}/})^{op} \) as motivic \( \infty \)-categories. Hence we have a (homotopically) commutative diagram of \( \infty \)-categories:

\[
\begin{array}{ccc}
L\mathcal{M}\operatorname{Top}_{\mathcal{K}/} & \xrightarrow{f_*} & L\mathcal{M}\operatorname{Top}_{\mathcal{K}'/} \\
\downarrow & & \downarrow \\
L\mathcal{M}\operatorname{Top}(\mathcal{K}) & \xrightarrow{f_*} & L\mathcal{M}\operatorname{Top}(\mathcal{K}') \\
\end{array}
\]

We prove that the lower horizontal functor has a left adjoint:

**Theorem 4.5.** Let \( f : \mathcal{K} \to \mathcal{K}' \) be a geometric morphism of motivic classifying \( \infty \)-topoi such that \( f \) is compatible with geometric structures. Given the commutative diagram

\[
\begin{array}{ccc}
L\mathcal{M}\operatorname{Top}(\mathcal{K}) & \xrightarrow{f_*} & L\mathcal{M}\operatorname{Top}(\mathcal{K}') \\
\downarrow & & \downarrow \\
L\mathcal{M}\text{Top} & & \\
\end{array}
\]

where \( f_* \) is the induced functor by \( f \), and \( p \) and \( q \) are motivic locally coCartesian fibrations. Then \( f_* : L\mathcal{M}\operatorname{Top}(\mathcal{K}') \to L\mathcal{M}\operatorname{Top}(\mathcal{K}) \) admits a left adjoint relative to \( L\mathcal{M}\text{Top} \).

**proof.** Since the unstraightening functor

\[
\operatorname{Mot}_X^!(\operatorname{Un}^{op}): \operatorname{Mot}_X^!(\operatorname{Set}_{\Delta}^{op}[\mathcal{C}]^{op}) \to \operatorname{Mot}_X^!(\operatorname{Set}_{\Delta}^{op}[\mathcal{C}]^{op})
\]

carries fibrant objects to fibrant objects, the induced morphism \( f_* : L\mathcal{M}\operatorname{Top}(\mathcal{K}') \to L\mathcal{M}\operatorname{Top}(\mathcal{K}) \) between fibrant objects over \( L\mathcal{M}\text{Top} \) carries fibrant objects to fibrant objects, the induced morphism \( f_* : L\mathcal{M}\operatorname{Top}(\mathcal{K}') \to L\mathcal{M}\operatorname{Top}(\mathcal{K}) \) between fibrant objects over \( L\mathcal{M}\text{Top} \) and \( f_* \). In particular, \( f_* \) carries motivic locally \( q \)-coCartesian edges to motivic locally \( p \)-coCartesian edges. By [21, Proposition 7.3.2.6], it is sufficient to prove that the functor

\[
f^{-1}_*: \operatorname{Str}_{\mathcal{K}}^{\operatorname{loc}}(\mathcal{T}) \to \operatorname{Str}_{\mathcal{K}}^{\operatorname{loc}}(\mathcal{T})
\]

admits a left adjoint for any motivic \( \infty \)-topos \( \mathcal{T} \). Let \( \mathcal{O}_\mathcal{T} \) be a object of \( \operatorname{Fun}^\ast(\mathcal{K}, \mathcal{T}) \) and \( \mathcal{O}'_\mathcal{T} \) an object of \( \operatorname{Fun}^\ast(\mathcal{K}', \mathcal{T}) \). Let \( \phi : \mathcal{O}_\mathcal{T} \to \mathcal{O}'_\mathcal{T} \) be a local morphism in \( \operatorname{Fun}^\ast(\mathcal{K}, \mathcal{T}) \). We can obtain a left Kan extension \( f_*\mathcal{O}_\mathcal{T} : \mathcal{K}' \to \mathcal{K} \) along \( f \). The transformation \( \phi_* : f_*(\mathcal{O}_\mathcal{T}) \to \mathcal{O}'_\mathcal{T} \) induced by \( \phi \) gives a functorial factorization

\[
f_*(\mathcal{O}_\mathcal{T}) \to \operatorname{MSpc}_{\mathcal{K}}^\mathcal{K}(\mathcal{O}_\mathcal{T}) \xrightarrow{\operatorname{MSpc}(\mathcal{O}_\mathcal{T})} \mathcal{O}'_\mathcal{T}
\]
where MSpc(α) is local. Hence we can get a functor MSpc\(X, T\) : \(f_\ast(O_T) \to O_T'\) which is a left Kan extension of \(f^{-1}_T\).

\[\square\]

**Remark 4.6.** In this paper, the motivic \(\infty\)-category \(\mathcal{MS}_{\text{Top}}(\mathcal{X})\) is constructed by following in \([22]\) Remark 1.4.17.

5. **The application of motivic derived algebraic geometry to \(A^1\)-homotopy theory.**

Fix a regular Noetherian separated scheme \(S\) of finite dimension. We consider the case that the Grothendieck site \(\mathcal{X}\) is the Nisnevich site on the category \(\textbf{Sm}_{S}\) of smooth schemes over \(S\), and the interval object is the affine line \(A^1\). For an arbitrary left proper combinatorial simplicial model category \(\mathcal{M}\), we let \(\text{Mot}_{\mathcal{A}}^\infty(\mathcal{M})\) denote the motivic model category of \(\mathcal{M}\) on the Grothendieck site \(\text{Sm}_S\) with the interval object \(A^1\). For simplify, we write \(\text{MS}_{\infty} = \text{Mot}_{\infty}^\mathcal{A}(\text{Set}_\infty)\) and \(\text{MCat}_{\infty} = \text{Mot}_{\mathcal{A}}^\infty(\text{Set}_{\infty}^\mathcal{A})\). Then the homotopy category of the \(\infty\)-category of \(\text{MS}_{\infty}\) is just \(A^1\)-homotopy category of motivic spectra due to Morel and Voevodsky \([28]\).

5.1. **The monoidal structure of the stable \(\infty\)-category of motivic spectra.** The smash products on the category of pointed simplicial sets \(\text{Set}_\infty\) induces a monoidal structure of the \(\infty\)-category of pointed motivic spaces \(\text{MS}_{\infty}^\ast = \text{Mot}_{\infty}^\mathcal{A}(\text{Set}_{\infty}^\mathcal{A})\). We recall the definition of motivic spectra:

**Definition 5.1.** Let \(i_\infty : \mathbb{P}^0 \to \mathbb{P}^1\) denote the embedding which is given by the point \(\infty \in \mathbb{P}^1\). Then \(\mathbb{P}^1_\infty\) denotes the pointed projective line with the base point \(\infty\). For any motivic space \(X\), write \(\Omega^1_{\mathbb{P}^1_\infty} X = \text{Map}(\mathbb{P}^1 / \mathbb{P}^0, X)\). We say that \(\Omega^1_{\mathbb{P}^1_\infty} X\) is the \(\mathbb{P}^1_\infty\)-loop space of \(X\).

We define the \(\infty\)-category of **motivic spectra** as follows:

**Definition 5.2.** The \(\infty\)-category \(\text{MSp}_{\infty}\) of motivic spectra is defined to be the (homotopy) limit

\[
\text{MSp}_{\infty} = \lim_{\Omega^1_{\mathbb{P}^1_\infty}} \text{MS}_{\infty}^\ast = \lim_{\Omega^1_{\mathbb{P}^1_\infty}} \left( \cdots \rightarrow \text{MS}_{\infty}^\ast \rightarrow \text{MS}_{\infty}^\ast \right).
\]

An object of \(\text{MSp}_{\infty}\) is called **motivic spectra**. Let \(\Omega^\circ_\infty : \text{MSp}_{\infty} \to \text{MS}_{\infty}^\ast\) denote the first projection and \(\Sigma^\circ_\infty : \text{MS}_{\infty}^\ast \to \text{MSp}_{\infty}\) the left adjoint of \(\Omega^\circ_\infty\). Then we call the left adjoint \(\Sigma^\circ_\infty : \text{MS}_{\infty}^\ast \to \text{MSp}_{\infty}\) the \(\mathbb{P}^1_\infty\)-stabilization.

The \(\infty\)-category \(\text{MSp}_{\infty}\) of motivic spectra is a stable \(\infty\)-category in the sense of \([21]\). This follows from by the fact \([28]\) Corollary 2.18 that the map \(S^1 \wedge \mathbb{G}_m \to \mathbb{P}^1_\infty\) is motivic equivalence of pointed motivic spaces, where \(\mathbb{G}_m\) denotes the multiplicative group scheme and \(S^1 = \Delta^1 / \partial \Delta^1\).

**Definition 5.3.** A **motivic \(\mathbb{E}_{\infty}\)-ring** is a commutative algebra object of the symmetric monoidal \(\infty\)-category \(\text{MSp}_{\infty}\) of motivic spectra, and \(\text{CAlg}(\text{MSp}_{\infty})\) denotes the \(\infty\)-category of motivic \(\mathbb{E}_{\infty}\)-rings.
5.2. **The motivic stable ∞-category of modules over a motivic $\mathbb{E}_\infty$-ring.** We introduce the motivic ∞-category of modules over a motivic $\mathbb{E}_\infty$-ring.

Let $R$ be a motivic $\mathbb{E}_\infty$-ring and $\text{Mod}_R$ denote the full subcategory of $\text{MSp}_{\infty}$ spanned by $R$-module objects of motivic spectra. Simply, we call an object of $\text{Mod}_R$ an $R$-module. Let $\text{Mod}_R^\otimes$ denote the symmetric monoidal ∞-category of $R$-modules whose monoidal structure is induced by smash products over $R$. Let $\text{PMod}_R$ denote the full subcategory of $\text{Mod}_R$ generated by compact $R$-modules.

For any scheme $X$ over $S$, $R(X)$ is an $\mathbb{E}_\infty$-ring and the ∞-category of $R(X)$-modules $\text{Mod}_{R(X)}$ is a stable ∞-category. Then we have a functor $\text{Mod}_{R(-)} : N(\text{Sm}_S^\omega) \to \text{Cat}_{\omega}^\infty$, and $\text{Mod}_R$ denotes the motivic ∞-category determined by $\text{Mod}_{R(-)}$. We say that $\text{Mod}_R$ is a **motivic stable ∞-category of $R$-modules.** By the similar way, we get motivic ∞-categories $\mathbb{P}\text{Mod}_R$ and motivic symmetric monoidal ∞-categories $\mathbb{P}\text{Mod}_R^\otimes$ and $\mathbb{P}\text{Mod}_R^\omega$. We say that $\mathbb{P}\text{Mod}_R$ is the motivic ∞-category of perfect $R$-modules.

Furthermore, we consider the motivic ∞-category $\text{CAlg}(\mathbb{P}\text{Mod}_R)$ determined by the functor $\text{CAlg}(\text{Mod}_R) : N(\text{Sm}_S^\omega) \ni X \mapsto \text{CAlg}(\text{Mod}_{R(X)}) \in \text{Cat}_{\infty}$ and call $\text{CAlg}(\text{Mod}_R)$ the motivic ∞-category of commutative motivic $R$-algebras.

**Example 5.4.** Let $S$ denote the motivic $\mathbb{E}_\infty$-ring of the motivic sphere spectrum. Then $\mathbb{M}\text{Sp}_{\infty} = \text{Mod}_S$ is called the motivic ∞-category of motivic spectra. Similarly, we write $\mathbb{M}\text{Sp}_{\omega} = \text{Mod}_S^\omega$, $\mathbb{M}\text{Sp}_{\infty}^\omega = \mathbb{P}\text{Mod}_S$ and $\mathbb{M}\text{Sp}_{\omega}^\omega = \mathbb{P}\text{Mod}_S^\omega$. We call the motivic ∞-category $\text{CAlg}(\mathbb{M}\text{Sp}_{\infty})$ the motivic ∞-category of motivic $\mathbb{E}_\infty$-rings.

**Definition 5.5.** Let $\hat{\text{MCat}}_{\omega}^\infty$ denote the symmetric monoidal ∞-category of (not necessary small) motivic ∞-categories. A **motivic stable ∞-category** is an $\mathbb{M}\text{Sp}_{\infty}$-module object of $\hat{\text{MCat}}_{\infty}$. A small motivic ∞-category $\mathcal{C}$ is stable if $\mathcal{C}$ is an $\mathbb{M}\text{Sp}_{\infty}$-module object of $\hat{\text{MCat}}_{\omega}^\infty$.

**Proposition 5.6.** Let $R$ be a motivic $\mathbb{E}_\infty$-ring. Then $\mathbb{P}\text{Mod}_R$ is a motivic stable ∞-category and $\mathbb{P}\text{Mod}_R$ a small motivic stable ∞-category.

**proof.** For any smooth $S$-scheme $X$, $\text{Mod}_{R(X)}$ is the full subcategory of $\text{Mod}_{S(X)}$ spanned by $R(X)$-module object. Therefore $\text{Mod}_{R(X)}$ is a stable ∞-category. Since the motivic sphere $S$ is an initial object of the ∞-category of motivic $\mathbb{E}_\infty$-rings, $\text{Mod}_{R(X)}$ has a canonical $\text{Mod}_{S(X)}$-module structure in $\text{Cat}_{\omega}^\infty$. Hence $\mathbb{P}\text{Mod}_R$ is a motivic stable ∞-category. By the similar argument, we obtain that $\mathbb{P}\text{Mod}_R$ is a small motivic stable ∞-category. □

**Proposition 5.7.** Let $R$ be a motivic $\mathbb{E}_\infty$-ring. Then a left exact localization of the motivic ∞-category

$$\text{Fun}_{\hat{\text{MCat}}_{\omega}^\infty}(\text{CAlg}(\mathbb{P}\text{Mod}_R)^{\text{op}}, \mathbb{M}\text{Sp}_{\infty})$$

is a motivic ∞-topos. Furthermore a left exact localization of the very large motivic ∞-category

$$\text{Fun}_{\hat{\text{MCat}}_{\omega}^\infty}(\text{CAlg}(\mathbb{P}\text{Mod}_R)^{\text{op}}, \mathbb{M}\text{Sp}_{\infty})$$

is a very large motivic ∞-topos.
is a very large motivic ∞-topos.

**proof.** Let \( f : X \to Y \) be a morphism of smooth \( S \)-schemes. Then \( f \) induces exact functors 
\[-\otimes_{R(Y)} R(X) : \text{PMod}^\otimes_{R(Y)} \to \text{PMod}^\otimes_{R(X)} \text{and} -\otimes_{R(Y)} R(X) : \text{Mod}^\otimes_{R(Y)} \to \text{Mod}^\otimes_{R(X)} \] of stable symmetric monoidal ∞-categories. Since the forgetful functor \( \text{CAlg}(\text{Mod}_A) \to \text{Mod}^\otimes_A \) is conservative for any \( \mathbb{E}_\infty \)-ring \( A \), the both induced functors
\[-\otimes_{R(Y)} R(X) : \text{CAlg}(\text{PMod}^\otimes_{R(Y)}) \to \text{CAlg}(\text{PMod}^\otimes_{R(X)}) \] \[-\otimes_{R(Y)} R(X) : \text{CAlg}(\text{Mod}^\otimes_{R(Y)}) \to \text{CAlg}(\text{Mod}^\otimes_{R(X)}) \] are left exact. By Proposition 3.5, we obtain the conclusion. \( \square \)

Let \( \text{MPr}^L_{\text{St}_\infty} \) denote the subcategory of \( \widehat{\text{MCat}}_\infty \) spanned by motivic presentable stable ∞-categories, and those functors are \( \mathbb{P}^1 \)-stable exact and admit right adjoints. Then \( \text{MPr}^L_{\text{St}_\infty} \) has a symmetric monoidal structure determined by the tensor products of presentable stable ∞-categories.

**Proposition 5.8.** Let \( \text{MPr}^L_{\text{St}_\infty} \) denote the very large ∞-category of motivic stable presentable ∞-categories, and those functors are colimit preserving functors. Then
\[ \mathcal{M} \mathcal{O}d^\otimes : \text{CAlg}(\text{MSp}_\infty) \ni R \mapsto \mathcal{M} \mathcal{O}d^\otimes_R \in \text{CAlg}(\text{MPr}^L_{\text{St}_\infty}) \] is a fully faithful functor between ∞-categories.

**proof.** In fact, for any smooth scheme \( X \) over \( S \), the induced map
\[ \text{Map}_{\text{CAlg}}(R(X), R'(X)) \to \text{Map}_{\text{CAlg}(\text{Pr}^L_{\text{St}_\infty})}(\text{Mod}^\otimes_{R(X)}, \text{Mod}^\otimes_{R'(X)}) \] is a homotopy equivalence of Kan complexes by [21, p.890, Proposition 7.1.2.7] and functorial for morphisms of smooth \( S \)-schemes. \( \square \)

Let \( R \) be a motivic \( \mathbb{E}_\infty \)-ring. By Proposition 5.8. Then \( R \) induces the map
\[ R : \Delta^0 \overset{R}{\to} \text{Map}_{\text{CAlg}(\text{MSp}_\infty)}(\mathbb{S}, R) \simeq \text{Map}_{\text{CAlg}(\text{MPr}^L_{\text{St}_\infty})}(\mathcal{M} \mathcal{O}d^\otimes_\infty, \mathcal{M} \mathcal{O}d^\otimes_R). \] Let \( -\otimes R : \mathcal{M} \mathcal{O}d^\otimes_\infty \to \mathcal{M} \mathcal{O}d^\otimes_R \) denote the functor induced by the map \( R \). Then \( -\otimes R \) induces the map \( -\otimes R : \text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_\infty) \to \text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_R) \). We define the functor \( \text{MSpc} R : \text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_\infty)^{\text{op}} \to \mathcal{M} \mathcal{O}d^\otimes_\infty \) by the following:

**Definition 5.9.** Let \( R \) be a motivic \( \mathbb{E}_\infty \)-ring and \( -\otimes R : \text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_\infty) \to \text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_R) \) the induced functor between motivic ∞-categories. Then the functor \( -\otimes R \) induces the following adjunction
\[ (-\otimes R) : \text{Fun}(\text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_\infty)^{\text{op}}, \mathcal{M} \mathcal{O}d^\otimes_\infty) \rightleftarrows \text{Fun}(\text{CAlg}(\mathcal{M} \mathcal{O}d^\otimes_R)^{\text{op}}, \mathcal{M} \mathcal{O}d^\otimes_\infty) : U_R \]
between motivic $\infty$-categories. Here $U_R$ denote the right adjoint of the left Kan extension $(- \otimes R)_*$. Let $1_R$ denote the final object of $\text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}_R \right)^{\text{op}}, \mathcal{M} \mathcal{Z}_\infty \right)$. Then $\text{MSpc} R$ denotes the functor $U_R(1_R) : \text{CAlg} \left( \mathcal{M} \mathcal{Z}_{p_\infty} \right)^{\text{op}} \to \mathcal{M} \mathcal{Z}_\infty$.

**Proposition 5.10.** Let $R$ and $R'$ be motivic $E_\infty$-rings. Then we have an equivalence

$$\text{Map}_{\text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{P}_{\mathcal{M} \mathcal{S}_\infty} \right)^{\text{op}}, \mathcal{M} \mathcal{S}_\infty \right)}(\text{MSpc} R', \text{MSpc} R) \simeq \text{Map}_{\text{CAlg} \left( \mathcal{M} \mathcal{P}_{\mathcal{M} \mathcal{S}_\infty} \right)}(R, R')$$

of motivic spaces.

**proof.** Let $X$ be a smooth scheme over $S$. Then we have the adjunction of presentable $\infty$-categories:

$$(- \otimes R(X))_* : \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right) \rightleftarrows \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(R(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right) : U_{R(X)}.$$

Let $y : \text{CAlg} \left( \mathcal{M} \mathcal{D}(R(X)) \right) \to \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(R(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right)$ denote the $\infty$-categorical Yoneda embedding. Then the final object of $\text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(R(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right)$ is just $y(R(X))$. Consider the homotopically commutative diagram of very large $\infty$-categories:

$$\text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right) \leftarrow \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(R(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right) \leftarrow \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right)$$

where the vertical arrows are very large Yoneda embeddings. Then $\tilde{y}(R(X)) : \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}} \to \mathcal{S}_\infty$ is the left Kan extension of the functor $\text{MSpc} R(X) : \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}} \to \mathcal{S}_\infty$ along the fully faithful embedding

$$i(X) : \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right) \to \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}(S(X)) \right)^{\text{op}}, \mathcal{S}_\infty \right).$$

Hence we obtain a weak equivalence $\text{Map} \left( \text{MSpc} R'(X), \text{MSpc} R(X) \right) \simeq \text{Map}_{\text{CAlg}}(R, R'(X))$ of Kan complexes. \hfill $\square$

**Corollary 5.11.** Let $R$ and $R'$ be motivic $E_\infty$-rings. Then we have an equivalence

$$\text{Map}_{\text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{D}_{p_\infty} \right)^{\text{op}}, \mathcal{M} \mathcal{Z}_{p_\infty} \right)}(\text{MSpc} R', \text{MSpc} R) \simeq \text{Map}_{\text{CAlg} \left( \mathcal{M} \mathcal{P}_{Z_{p_\infty}} \right)}(\mathcal{M} \mathcal{D}_{p_\infty}, \mathcal{M} \mathcal{D}_{p_\infty}).$$

\hfill $\square$

5.3. **The functor $\text{MSpc}$ for a motivic stable $\infty$-category $\mathcal{C}$.** Let $\mathbf{1}$ denote the unit object of the monoidal structure of $\text{MCat}_\infty$. By Proposition 5.9 the functor

$$\text{MSpc} : \text{CAlg} \left( \mathcal{M} \mathcal{P}_{\mathcal{C}} \right)^{\text{op}} \ni R \mapsto \text{MSpc} R \in \text{Fun}_{\text{MCat}_\infty} \left( \mathbf{1}, \text{Fun} \left( \text{CAlg} \left( \mathcal{M} \mathcal{Z}_{p_\infty}, \mathcal{M} \mathcal{Z}_{p_\infty} \right) \right) \right).$$
induces a fully faithful functor $\text{MSpc}_\ast : \text{CAlg}(\text{MSp}^e_\infty)^{\text{op}} \otimes 1 \to \text{Fun}(\text{CAlg}(\mathcal{M} \circ p_\infty^e), \mathcal{M} \circ p_\infty)$. Since the $\infty$-category $\text{MCat}^e_\infty$ is presentable by [8, Theorem 4.22], we get a homotopically commutative diagram

$$
\begin{array}{ccc}
\text{CAlg}(\text{MSp}^e_\infty)^{\text{op}} \otimes 1 & \xrightarrow{\Psi \circ \text{Mod}^e \circ 1} & \text{Fun}(\text{CAlg}(\mathcal{M} \circ p_\infty^e), \mathcal{M} \circ p_\infty) \\
\text{MCat}^e_\infty, \otimes^{\text{op}} & \xrightarrow{\Psi \circ \text{Mod}^e} & \\
\end{array}
$$

where $\Psi \circ \text{Mod}^e$ denotes the left Kan extension of $\Psi \circ \text{Mod}^e \otimes 1$ along $\text{MSpc}_\ast$. Note that any left Kan extension preserves all small colimit, $\Psi \circ \text{Mod}^e$ has a right adjoint. Then we obtain the adjunction

$$
\Psi \circ \text{Mod}^e : \text{Fun}(\text{CAlg}(\mathcal{M} \circ p_\infty^e), \mathcal{M} \circ p_\infty) \rightleftarrows \text{MCat}^e_\infty, \otimes^{\text{op}} \otimes 1 : \Psi \circ \text{Mod}^e, \ast.
$$

Let $\mathcal{C}$ be a motivic symmetric monoidal stable $\infty$-category. Then $\mathcal{C}$ is regarded as the functor $\hat{\mathcal{C}} : 1 \to \text{MCat}^e_\infty, \otimes^{\text{op}} 1$ between motivic $\infty$-categories. We shall set $\text{MSpc}_\mathcal{C} = \Psi \circ \text{Mod}^e, \ast(\hat{\mathcal{C}}(1)))$.

We shall consider the large version of $\text{MSpc}_\mathcal{C}$: Let $\mathcal{M} \circ \mathcal{M}_\infty$ denote the $\infty$-category of motivic spaces that the carnality of the universe is greater than very large. Let $R$ be a motivic $E_\infty$-ring. Replacing $\Psi \circ \text{Mod}^e$ and $\text{Mod}^e$ in Definition 5.9, the functor $\hat{\text{MSpc}}_R : 1 \to \text{Fun}(\text{CAlg}(\mathcal{M} \circ p_\infty), \mathcal{M} \circ p_\infty)$ induces a functor

$$
\hat{\text{MSpc}}_\ast : \text{CAlg}(\text{MSp}^e_\infty)^{\text{op}} \otimes 1 \to \text{Fun}(\text{CAlg}(\mathcal{M} \circ p_\infty), \mathcal{M} \circ p_\infty)
$$

between motivic $\infty$-categories.

**Example 5.12.** Let $\hat{y} : \text{MP}_{\text{St}}^+ \to \text{Fun}(\text{MP}_{\text{St}}^+, \mathcal{M} \circ \mathcal{M}_\infty)$ denote the Yoneda embedding. By Proposition 3.4, we have a homotopically commutative diagram:

$$
\begin{array}{ccc}
\text{Fun}(\text{CAlg}(\mathcal{M} \circ p_\infty), \mathcal{M} \circ p_\infty) & \xrightarrow{\hat{y} \circ \text{Mod}} & \text{Fun}(\text{MP}_{\text{St}}^+, \mathcal{M} \circ p_\infty) \\
\text{CAlg}(\text{MSp}^e_\infty)^{\text{op}} \otimes 1 & \xrightarrow{\hat{y} \circ \text{Mod}} & \text{MP}_{\text{St}}^+, \otimes^{\text{op}} \otimes 1 \\
\end{array}
$$

where $\text{Mod}$ denote the left Kan extension of $\hat{y} \circ \text{Mod}$ along $\hat{\text{MSpc}}_\ast$. Set $\text{Mod} = \text{Mod}^e(\hat{y} \circ (\mathcal{M} \circ p_\infty))$. Then the functor $\text{Mod}$ is the very large moduli functor of modules: For any motivic $E_\infty$-ring $R$,
one has

$$\text{Map}_{\text{Fun}}(\text{CAlg}(\mathcal{MSp}_\infty, \mathcal{MS}_\infty), \mathcal{MSpR}, \text{Mod}) = \text{Map}_{\text{Fun}}(\mathcal{MSp}_\infty, \mathcal{MS}_\infty, (\mathcal{MSp}_\infty, \mathcal{Mod}) = \text{Map}_{\text{Fun}}(\mathcal{MSp}_\infty, \mathcal{MS}_\infty, (\mathcal{MSp}_\infty, \mathcal{Mod}) = \text{Map}_{\text{Fun}}(\mathcal{MPr}_\infty, \mathcal{MS}_\infty, \mathcal{MSp}_\infty, \mathcal{Mod}) = \text{Map}_{\text{Fun}}(\mathcal{MPr}_\infty, \mathcal{MS}_\infty, \mathcal{MSp}_\infty, \mathcal{Mod}) = \text{Gpd}_\infty(\mathcal{Mod}_R).$$

Let $\text{Sym}_\times : \text{MPr}_\infty \to \text{CAlg}(\text{MPr}_\infty)$ denote the free algebra functor and $U$ its right adjoint. Then we have an adjunction:

$$\text{Mod} \otimes^\ast : \text{Fun}(\text{CAlg}(\mathcal{MSp}_\infty, \mathcal{MS}_\infty), \mathcal{MSpR}, \text{Mod}) \rightleftarrows \text{Fun}(\text{CAlg}(\text{MPr}_\infty, \mathcal{MS}_\infty), \mathcal{MSpR}, \text{Mod}) : U \otimes^\ast,$$

and a weak equivalence $\text{Mod} \simeq U \circ \text{Mod}_\times$. Note that one has a weak equivalences:

$$\text{Fun}_{\text{MPr}_\infty}(\mathcal{MSp}_\infty, U(\text{Mod}_R)) \simeq \text{Fun}_{\text{CAlg}(\text{MPr}_\infty)}(\text{Sym}_\times(\mathcal{MSp}_\infty), \text{Mod}_R)$$

Therefore we obtain that the symmetric monoidal presentable $\infty$-category $\text{Sym}_\times(\mathcal{MSp}_\infty)$ represents the functor $\text{Mod}$ i.e. $\text{Mod} \simeq \text{Mod}_\times(\text{Sym}_\times(\mathcal{MSp}_\infty))$.

For an arbitrary motivic stable presentable symmetric monoidal $\infty$-category $\mathcal{C}$, we will write $\mathcal{MSpC} = \text{Mod}_{\times}(\mathcal{C}).$ Then we have an equivalence : $\text{Mod} \simeq \mathcal{MSpC}(\text{Sym}_\times(\mathcal{MSp}_\infty))$.

5.4. The definition of motivic spectral schemes and motivic (Deligne–Mumford) stacks.

Recall that Zariski topos is classified by the factorization system of local morphisms between local rings, and étale topos is classified by the factorization system of local morphisms between strict Henselian local rings. We introduce the definition of local ring objects, local morphisms and strict Henselian local ring objects of a motivic $\infty$-topos.

5.4.1. Motivic $\infty$-Zariski topos. Let $\mathcal{T}$ be a topos with a final object $1$ and $\mathcal{O}$ a commutative algebra object of $\mathcal{T}$. We say that $\mathcal{O}$ is local [23, p.13, Definition 2.4] if the following conditions are satisfied:

1. Let $0, 1 : 1 \to \mathcal{O}$ denote the additive identity and multiplicative identity in $\mathcal{O}$. Then $1 \times_0 1$ is an initial object of $\mathcal{T}$.
2. Let $\mathcal{O}^\times$ be the multiplicative group of $\mathcal{O}$ which is given by the pullback square

\[
\begin{array}{ccc}
\mathcal{O}^\times & \to & \mathcal{O} \\
\downarrow & & \downarrow \\
1 & \to & \mathcal{O},
\end{array}
\]
where \( m : \emptyset \times \emptyset \to \emptyset \) is the multiplication equipped with \( \emptyset \). Then the map

\[
(1 - e) \coprod e : \emptyset^\times \coprod \emptyset^\times \to \emptyset
\]

is an effective epimorphism (See [19, p.531, Remark 6.1.1.5]).

Let \( \alpha : \emptyset \to \emptyset' \) be a morphism of commutative local ring objects of \( T \). We say that \( \alpha : \emptyset \to \emptyset' \) is a local morphism if the diagram

\[
\begin{array}{ccc}
\emptyset^\times & \xrightarrow{\alpha} & \emptyset'^\times \\
\downarrow & & \downarrow \\
\emptyset & \xrightarrow{\alpha} & \emptyset'
\end{array}
\]

is a pullback square.

**Definition 5.13** (c.f. [23] p.13, Definition 2.5). Let \( \emptyset \) be a commutative ring object of a motivic \( \infty \)-topos on \( T \). We say that \( \emptyset \) is a local if \( \pi_0 \emptyset(U) \) is a local ring object of \( hT(U) \) for any \( U \in T \).

Let \( \mathcal{K}^M_{\text{disc}} \) denote the motivic \( \infty \)-topos \( \text{Fun}(\text{CAlg}(\mathfrak{M} \boxtimes p_{\infty}^{op}, \mathfrak{M} \boxtimes \infty), \mathfrak{M} \boxtimes \infty)) \). We introduce the classifying Zariski topos on \( \mathcal{K}^M_{\text{disc}} \). The motivic Yoneda functor \( y : \mathfrak{M} \boxtimes p^{op}_{\infty} \to \text{Fun}(\mathfrak{M} \boxtimes p_{\infty}^{op}, \mathfrak{M} \boxtimes \infty) \) induces \( \text{CAlg}(y) : \text{CAlg}(\mathfrak{M} \boxtimes p_{\infty}^{op}, \mathfrak{M} \boxtimes \infty) \to \text{CAlg}(\text{Fun}(\mathfrak{M} \boxtimes p_{\infty}^{op}, \mathfrak{M} \boxtimes \infty)) \). Hence we obtain the canonical functor: \( \text{Fun}(\text{CAlg}(\mathfrak{M} \boxtimes p_{\infty}^{op}, \mathfrak{M} \boxtimes \infty), \mathfrak{M} \boxtimes \infty) \to \text{CAlg}(\text{Fun}(\mathfrak{M} \boxtimes p_{\infty}^{op}, \mathfrak{M} \boxtimes \infty)) \). It is well-known that this canonical functor induces weak equivalences on each fiber on \( X \in \mathcal{K} \). Therefore the canonical functor is a weak equivalence of motivic \( \infty \)-categories. We let \( \mathcal{K}^M_{\text{Zar}} \) denote the subcategory of \( \mathcal{K}^M_{\text{disc}} \) whose morphisms are local morphisms. We say that \( \mathcal{K}^M_{\text{Zar}} \) is the motivic \( \infty \)-Zariski topos. Write \( \text{MSch}^M = \text{LAlg}(\mathcal{K}^M_{\text{Zar}})^{op} \). A motivic scheme is an object of the motivic \( \infty \)-category \( \text{MSch} \). Equivalently a motivic scheme is an \( \mathbb{A}^1 \)-homotopy invariant spectral scheme-valued Nisnevich-local sheaf on \( \text{Sm} \).

Write \( \text{MAlgSp}^M = \text{LAlg}(\mathcal{K}^M_{\text{disc}})^{op} \). We refer to \( \text{MAlgSp} \) as the motivic \( \infty \)-category of motivic algebraic spaces. Then the geometric morphism \( \mathcal{K}^M_{\text{disc}} \to \mathcal{K}^M_{\text{Zar}} \) induces the functor

\[
(\dash)^{\text{Zar}} : \text{MAlgSp} \to \text{MSk}
\]

which admits a left adjoint by Theorem 4.5.

**5.4.2. Motivic \( \infty \)-étale topos.**

**Definition 5.14** ([23] p.68, Definition 8.1). Let \( T \) be a topos and \( \mathcal{O}_T \) a commutative ring object of \( T \). For every finitely generated algebra \( R \), let \( \text{Sol}_R(\mathcal{O}_T) \) be an object of \( T \) defined by

\[
\text{Sol}_R(\mathcal{O}_T) : T \ni U \mapsto \text{Hom}_{\text{Ring}}(R, \text{Hom}_T(U, \mathcal{O}_T)) \in \text{Sets}.
\]

We say that \( \mathcal{O}_T \) is strictly Henselian, if for every finite collection of étale maps \( R \to R_\alpha \) which induce a faithfully flat map \( R \to \prod_\alpha R_\alpha \), the induced map

\[
\coprod_\alpha \text{Sol}_{R_\alpha}(\mathcal{O}_T) \to \text{Sol}_R(\mathcal{O}_T)
\]
is an effective epimorphism.

**Definition 5.15** (c.f. [23] p.68, Definition 8.3). Let $\mathcal{T}$ be a motivic $\infty$-topos and $\mathcal{O}_\mathcal{T}$ a commutative algebra object of $\mathcal{T}$. Then we say that $\mathcal{O}_\mathcal{T}$ is **strictly Henselian** if $\pi_0 \mathcal{O}_\mathcal{T}(U)$ is a strictly Henselian commutative ring object of the category $h\mathcal{T}(U)$ for each $U \in \mathcal{T}$. Let $\alpha : \mathcal{O}_\mathcal{T} \to \mathcal{O}'_{\mathcal{T}}$ be a local morphism of local commutative algebra objects of $\mathcal{T}$. We say that $\alpha$ is a **strict Henselian local** if $\mathcal{O}_\mathcal{T}$ and $\mathcal{O}'_{\mathcal{T}}$ are strict Henselian.

Let $\mathcal{K}_M^M$ denote the subcategory of $\mathcal{K}_{\mathcal{disc}}^M$ whose morphisms are strict Henselian local. We say that the motivic $\infty$-topos $\mathcal{K}_M^M$ is the **motivic $\infty$-étale topos**. Write $MStk = \mathcal{K}_M^M$. Then we say that the motivic $\infty$-topos $K^M_{\mathcal{disc}}$ is the motivic $\infty$-étale topos. By the similar argument of the case of motivic schemes, a motivic stack is a $A^1$-homotopy invariant spectral Deligne–Mumford stack-valued Nisnevich-local sheaf. Similarly, the geometric morphism $\mathcal{K}^M_{\mathcal{disc}} \to \mathcal{K}^M_{\mathcal{ét}}$ induces the functor $(-)^{\mathcal{ét}} : MAlgSp \to MStk$ which admits a left adjoint by Theorem 4.5. Let $X$ be a motivic algebraic space. Then we say that $X^{\mathcal{ét}}$ is the motivic stack associated to $X$.

### 6. The algebraic cobordism for motivic stacks.

In this section, we shall reformulate of the motivic $E_\infty$-ring of the algebraic cobordism $MGL$ as the initial object of the $\infty$-category of oriented motivic $E_\infty$-rings. Using this $\infty$-categorical formulation, we construct of the motivic stack $MGL$ of the algebraic cobordism for motivic stable $\infty$-categories.

#### 6.1. The Grassmannian and the $K$-theory for motivic stacks.

**Definition 6.1.** Let $R$ be a motivic $E_\infty$-ring and $n$ a non-negative integer. An $R$-module $M$ is free of rank $n$ if there exists a weak equivalence $f : M \to R^n$ of $R$-modules. Let $\text{Vect}^n_R$ denote the motivic $\infty$-subgroupoid of $\text{Mod}_R$ spanned by free $R$-modules of rank $n$. Let $\text{BGL}_n = \text{Vect}^n_{\mathcal{ét}}(R)$ denote the motivic stack of the $n$-dimensional Grassmannian. Let $\text{BGL}$ denote the colimit $\text{colim}_n \text{BGL}_n$. Then we refer $\text{BGL}$
to the infinite Grassmannian. The algebraic $K$-theory is defined to be the group completion of the Grassmannian $\mathbb{B}GL$: For any motivic $\mathbb{E}_\infty$-ring $R$ and motivic space $X$, the symmetric monoidal $\infty$-groupoid $\mathcal{V}ect_R(X)$ is a monoidal $\infty$-category with respect to the coproduct $\oplus$. In fact $M \times N \simeq M \coprod N$ for any $R$-modules $M$ and $N$. It is known that the group completion is exists [14]. Let $\mathbb{B}GL(R)^+$ the motivic stack defined to be the group completion of $\mathbb{B}GL$:

$\mathbb{B}GL^+(R) = \mathcal{V}ect^+_R$

for any compact motivic $\mathbb{E}_\infty$-ring $R$.

6.2. The motivic $\mathbb{E}_\infty$-ring $MGL$. The algebraic cobordism $MGL$ is a motivic spectrum which is determined by the Thom space of the universal vector bundles $V_n$ over $\mathbb{B}GL_n$ for every $n \geq 0$. It is known that the motivic spectrum $MGL$ has the canonical $\mathbb{E}_\infty$-ring structure [15]. The motivic $\mathbb{E}_\infty$-ring $MGL$ has the following universal property:

**Theorem 6.2** (Panin, Pimenov and Röndigs [29] Theorem 2.7). Let $R$ be a motivic $\mathbb{E}_\infty$-ring. Then the set of monoidal maps $MGL \to R$ is naturally isomorphic to the set of the orientations on $R$.

We shall reformulate the algebraic cobordism by using this universal property. Let $X$ be a compact motivic space and $f : V \to X$ a vector bundle on $X$ of dimension $n$. The Thom space of $V$ is the quotient $V/(V - X)$ which is defined to be $V(\text{Spec}R)/(V(\text{Spec}R) - 0_R)$ for any affine scheme $\text{Spec}R$ over $X$. Here $0_R$ denotes the image of the zero-section $0_R \in V(\text{Spec}R) = \text{Map}(\text{Spec}R, V \times_X \text{Spec}R)$ of $\text{Spec}R$. We refer to the pointed motivic space $V/(V - X)$ as the Thom space of $V$. Let $\text{Thom}_n : (\mathcal{M}sp^\omega_\infty)_{/\mathbb{B}GL_n} \to \mathcal{M}sp^\omega_\infty$ denote the $n$-dimensional Thom space functor defined by the sending $(X \to \mathbb{B}GL_n) \mapsto V/(V - X)$. Since the Thom space of the trivial vector bundle $O^n$ is motivic equivalent to the pointed projective space $\mathbb{P}^n$ by [28 p.112, Proposition 2.17], one has the homotopically commutative diagram:

\[
\begin{array}{cccccccc}
\cdots & \to & (\mathcal{M}sp^\omega_\infty)_{/\mathbb{B}GL_n} & \xrightarrow{-\Theta^n} & (\mathcal{M}sp^\omega_\infty)_{/\mathbb{B}GL_{n+1}} & \xrightarrow{-\Theta^n} & (\mathcal{M}sp^\omega_\infty)_{/\mathbb{B}GL_{n+2}} & \cdots \\
\text{Thom}_n & & \Sigma^1 U & & \Sigma^1 U & & \Sigma^1 U & \\
\cdots & \to & \mathcal{M}sp^\omega_\infty & \xrightarrow{-\Theta^n} & \mathcal{M}sp^\omega_\infty & \xrightarrow{-\Theta^n} & \mathcal{M}sp^\omega_\infty & \cdots \\
\end{array}
\]

where $U$ denotes the forgetful functor. We have the map $\text{Thom}_n = \text{colim}_n \text{Thom}_n : (\mathcal{M}sp^\omega_\infty)_{/\mathbb{B}GL} \to \mathcal{M}sp^\omega_\infty$ and the following coequalizer diagram:

\[
\begin{array}{c}
(\mathcal{M}sp^\omega_\infty)_{/\mathbb{B}GL} \\
\xrightarrow{\text{Thom}_n} \\
\Sigma^1 U \\
\xrightarrow{\text{Thom}_n} \\
\mathcal{M}sp^\omega_\infty \\
\xrightarrow{\text{Th}} \\
\mathcal{M}sp^\omega_\infty \\
\end{array}
\]

where $\text{Th}$ denotes the Thom space.
of presentable $\infty$-categories. Furthermore all functors in this sequence admit left adjoints. Tensoring with the presentable $\infty$-category $\text{CAlg}(\text{MSp}_\infty)$, we obtain a coequalizer sequence

$$\text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty_{\text{BGL}}^{op}} \xrightarrow{\text{Thom},} \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}} \xrightarrow{\text{Th}_1} \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}}$$

of presentable $\infty$-categories. Dually, we obtain an equalizer sequence:

$$\text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty_{\text{BGL}}^{op}} \xleftarrow{\text{Thom},} \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}} \xleftarrow{\text{Th}_1^{-1}} \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}}$$

where all of these functors are respective right adjoints of another coequalizer sequence.

By definition of $\text{th}$ and $\mathcal{M}$, the right adjoint $\text{Th}^{-1} : \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}} \to \text{CAlg}(\text{MSp}_\infty)^{\lambda^{op}}$ is conservative and preserves geometric realizations. Therefore the monad functor

$$\mathbf{T} = \text{Th}_1 \circ \text{Th}^{-1} : \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}} \to \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}}$$

induces an equivalence $\text{Alg}_\mathbf{T}(\text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}}) \to \text{CAlg(\text{MSp}_\infty)^{\lambda^{op}}}$ of presentable $\infty$-categories. We refer to the monad functor $\mathbf{T}$ as the \textit{oriented completion}.

\textbf{Proposition 6.3.} \textit{The monad functor $\mathbf{T} : \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}} \to \text{Alg}_\mathbf{T}(\text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}})$ is equivalent to the localization of presentable $\infty$-category of motivic $\mathbb{E}_\infty$-rings to the full subcategory spanned by oriented motivic $\mathbb{E}_\infty$-rings. }\hfill \Box

\textbf{proof.} Let $R$ be a motivic $\mathbb{E}_\infty$-ring with an orientation $\theta(V/X) : R(\Sigma^n\text{Thom}_n(V)) \to R(X)$ for a vector bundle $V : X \to \text{BGL}_n$ of rank $n$ on a compact motivic space $X$. Then there is an equivalence $R \circ \text{Thom} \simeq R \circ U$ of motivic $\mathbb{E}_\infty$-rings. Hence there exists a unique motivic $\mathbb{E}_\infty$-ring $R'$ such that $\theta^* : R \to \text{Th}^{-1}(R')$ is an equivalence of motivic $\mathbb{E}_\infty$-rings. Since the functor $\text{Th}_1$ is a left adjoint to $\text{Th}^{-1} : \text{CAlg}(\text{MSp}_\infty)^{\text{MSp}_\infty^{op}} \to \text{CAlg}(\text{MSp}_\infty)^{\lambda^{op}}$, an equivalence $\theta^*$ induces a morphism $\theta : \text{Th}_1(R) \to R'$ of motivic $\mathbb{E}_\infty$-rings. Let $u : R \to \text{Th}_1(\text{Th}^{-1}(R))$ denote the unit morphism. Then the homotopy $\theta^* \simeq \text{Th}^{-1}(\theta) \circ u$ induces a homotopy id $\simeq (\theta^*)^{-1} \circ \text{Th}^{-1}(\theta^*) \circ u : R \to \text{Th}_1(\text{Th}^{-1}(R)) \to R$. Note that both $\text{Th}^{-1}$ and $\text{Th}_1$ are colimit preserving functors. Therefore composition $\mathbf{T}$ preserve all geometric realizations. Then $u : R \to \mathbf{T}(R)$ induces a chain of weak equivalences $\mathbf{T}(R) \simeq \mathbf{T}(\mathbf{T}(R)) \simeq \mathbf{T}(\mathbf{T}(\mathbf{T}(R))) \simeq \mathbf{T}(\mathbf{T}(\mathbf{T}(\mathbf{T}(R)))) \simeq R$. Hence $u$ is a weak equivalence between motivic $\mathbb{E}_\infty$-rings. By the definition of $\text{Th}$, the functor $\text{Th}^{-1}$ is conservative. hence $\text{Th}_1(\theta) : \text{Th}_1(R) \to R'$ is an equivalence of motivic $\mathbb{E}_\infty$-ring. Thus, any oriented motivic $\mathbb{E}_\infty$-ring is a $\mathbf{T}$-algebra object of $\text{CAlg}(\text{MSp}_\infty)$ and the localization functor $|\mathbf{T}^*(-)|$ is equivalent to the monad functor $\mathbf{T}$.

\textbf{Theorem 6.4.} \textit{Let MGL denote the motivic $\mathbb{E}_\infty$-ring of the algebraic cobordism and $\mathbb{S}$ the motivic sphere spectrum. Then there is a canonical weak equivalence $\phi : \mathbf{T}(\mathbb{S}) \to \text{MGL}$ of oriented motivic $\mathbb{E}_\infty$-rings.}
Therefore the identity:

\[ \text{Thom} \]

\[ T \]

\[ \varphi \]

Since \( T \) is an initial object of the \( \infty \)-categorical \( \text{Alg}_T(\text{CAlg}(\text{MSp}_\infty)) \), there exists a canonical map \( \phi: T(\mathcal{S}) \to \text{MGL} \) between oriented motivic \( \mathbb{E}_\infty \)-rings. Since \( T(\mathcal{S}) \) is also oriented, the orientation of \( T(\mathcal{S}) \) induces a morphism \( \theta: \text{MGL} \to T(\mathcal{S}) \) of motivic \( \mathbb{E}_\infty \)-rings by Theorem 6.2. Since \( T(\mathcal{S}) \) is an initial object, the composition \( \theta \circ \phi: T(\mathcal{S}) \to \text{MGL} \to T(\mathcal{S}) \) is homotopic to the identity. The initial object \( T(\mathcal{S}) \) is a homotopy limit of all oriented motivic \( \mathbb{E}_\infty \)-rings. Therefore the identity: \( \text{MGL} \to \text{MGL} \) factors as \( \phi \circ \theta: \text{MGL} \to T(\mathcal{S}) \to \text{MGL} \). Hence \( \phi: T(\mathcal{S}) \to \text{MGL} \) is a weak equivalence of motivic \( \mathbb{E}_\infty \)-rings. \( \square \)

6.3. The algebraic cobordism \( \text{MGL} \) for motivic stacks. We shall extend the definition of algebraic cobordism from motivic spaces to motivic stacks.

Definition 6.5. Let \( \mathcal{X} \) be a motivic stack and \( E: \mathcal{X} \to \mathbb{BGL}_n \) a vector bundle of rank \( n \). Then we refer to the Thom module \( \text{Thom}_n(E) \) as the homotopy colimit of \( \text{Mod}^\otimes(\mathcal{X}) \rightarrow \text{Mod}^\otimes(\mathbb{BGL}_n) \):

\[
\begin{align*}
\text{Mod}^\otimes(\mathcal{X}) & \longrightarrow \text{Mod}^\otimes(\mathbb{BGL}_n) \\
\downarrow & \downarrow \\
\text{Thom}(E) & \longrightarrow \text{MGL}_n
\end{align*}
\]

Let \( \iota^n_0: \text{MSp}_\mathbb{S} \to \mathbb{BGL}_n \) denote the embedding corresponding to the trivial vector bundles of rank \( n \). Then we have a natural transformation \( \theta_n: \text{Thom}_n(E) \to \text{Thom}_n(\iota^n_0) \) of Kan extensions. We have the following homotopy colimits:

\[
\begin{align*}
\text{MStk}_{\mathbb{BGL}} & \rightarrow \text{MStk}_{\mathbb{BGL}_{n+1}} \\
\downarrow & \downarrow \\
\text{Thom}(\iota^n_0) & \rightarrow \text{Thom}_{n+1}(\iota^{n+1}_0) \\
\downarrow & \downarrow \\
\text{Mod}^\otimes(\text{MGL}) & \rightarrow \text{Mod}^\otimes(\text{MGL}_{n+1}) \\
\downarrow & \downarrow \\
\text{Th} & \rightarrow \text{Th}
\end{align*}
\]

Since \( \text{Mod}^\otimes(\text{MGL}) \) is presentable, the induced functor \( \text{Th}_n: \text{Mod}^\otimes(\mathbb{BGL}_n) \rightarrow \text{MGL}_n \) admits a right adjoint \( \text{Th}_n^{-1} \).

Let \( \mathcal{C} \) be a motivic stable presentable \( \infty \)-category. A functor symmetric monoidal exact functor \( F: \text{Mod}^\otimes(\text{MGL}) \to \mathcal{C} \) is oriented if the restrictions \( F \circ \text{Thom}_n \) and \( F \circ \text{Thom}(\iota^n_0) \) are...
stable symmetric monoidal \(\infty\)-universality of

Therefore the unit \(u\) of motivic spaces, the functor \(\Theta\) from the motivic stable \(\infty\)-category and \(\hat{\text{Mod}}\) is oriented and there exists a canonical functor

\[
\Theta : \text{Mod}_\ast(MGL) \to \hat{\text{Mod}}(MGL)
\]

such that \(\Theta \simeq \psi \circ \text{Th}\). It is immediately checked that \(\psi\) is a homotopy inverse of \(\phi_{MGL}\) from the universality of \(\hat{MGL}\) and \(\hat{\text{Mod}}(MGL)\).

**Corollary 6.7.** The motivic stack \(\hat{\text{MGL}}\) has the following universal property: Let \(\mathcal{C}\) be a motivic stable symmetric monoidal \(\infty\)-category and \(\hat{\text{Mod}}(\mathcal{C}) = \hat{\text{Mod}}^\ast(\mathcal{C})\) determined by the adjunction:

\[
\hat{\text{Mod}}^\ast: \text{Fun}(\text{CAlg}(\mathbb{M}_{\text{Sp}}^\infty), \mathbb{M}_{\text{Sp}}\infty) \rightleftarrows \text{Fun}(\text{CAlg}(\mathbb{M}_{\text{Sp}}^\infty), \mathbb{M}_{\text{Sp}}\infty) : \hat{\text{Mod}}\]

Then we have a weak equivalence:

\[
\text{Map}(\hat{\text{MSp}}^\ast(\mathcal{C}), \text{MGL}) \simeq \text{Map}_{\text{PMcat}^\infty}(\hat{\text{MGL}}, \mathcal{C}).
\]

**proof.** In fact one has a chain of weak equivalences:

\[
\text{Map}_{\text{Fun}(\text{CAlg}(\mathbb{M}_{\text{Sp}}^\infty), \mathbb{M}_{\text{Sp}}\infty)}(\hat{\text{MGL}}, \mathcal{C}) \simeq \text{Map}_{\text{Fun}(\text{CAlg}(\mathbb{M}_{\text{Sp}}^\infty), \mathbb{M}_{\text{Sp}}\infty)}(\text{Mod}^\ast(\text{MGL}), \mathcal{C})
\]

\[
\simeq \text{Map}_{\text{Fun}(\text{CAlg}(\mathbb{M}_{\text{Sp}}^\infty), \mathbb{M}_{\text{Sp}}\infty)}(\mathcal{C}, \text{Mod}^\ast(\text{MGL})) \simeq \text{Map}_{\text{Fun}(\text{CAlg}(\mathbb{M}_{\text{Sp}}^\infty), \mathbb{M}_{\text{Sp}}\infty)}(\hat{\text{MSp}}^\ast(\mathcal{C}), \text{MGL})
\]
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