Multivariate limits of multilinear polynomial-form processes with long memory

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Multivariate limits of multilinear polynomial-form processes with long memory

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Abstract

We consider the multilinear polynomial-form process

\[ X(n) = \sum_{1 \leq i_1 < \ldots < i_k < \infty} a_{i_1} \ldots a_{i_k} \epsilon_{n-i_1} \ldots \epsilon_{n-i_k}, \]

obtained by applying a multilinear polynomial-form filter to i.i.d. sequence \( \{\epsilon_i\} \) where \( \{a_i\} \) is regularly varying. The resulting sequence \( \{X(n)\} \) will then display either short or long memory. Now consider a vector of such \( X(n) \), whose components are defined through different \( \{a_i\} \)'s, that is, through different multilinear polynomial-form filters, but using the same \( \{\epsilon_i\} \). What is the limit of the normalized partial sums of the vector? We show that the resulting limit is either a) a multivariate Gaussian process with Brownian motion as marginals, or b) a multivariate Hermite process, or c) a mixture of the two. We also identify the independent components of the limit vectors.

1 Introduction

A linear process is generated by applying a linear time-invariant filter to i.i.d. random variables. A common model for stationary long-range dependent (LRD) (or long-memory) time series is a causal linear process with regularly varying coefficients as the lag tends to infinity, namely, \( X(n) = \sum_{i=1}^{\infty} a_i \epsilon_{n-i} \), where the \( \epsilon_i \)'s are i.i.d. with mean 0 and finite variance, and the coefficients satisfy \( a_i = i^{d-1}L(i) \) with \( 0 < d < 1/2 \) and \( L \) is a slowly varying function at infinity (i.e., \( L(x) > 0 \) when \( x \) is large enough and \( \lim_{x \to \infty} L(\lambda x)/L(x) = 1 \) \( \forall \lambda > 0 \)). Note that \( 0 < d < 1/2 \) implies \( \sum_{i=1}^{\infty} |a_i| = \infty \) but \( \sum_{i=1}^{\infty} a_i^2 < \infty \), so \( X(n) \) is well-defined in \( L^2 \) sense. It is well-known that the autocovariance \( \gamma(n) \) of \( X(n) \) is regularly varying with power \( 2d - 1 \), and that the partial sum of \( X(n) \) when suitably normalized converges to fractional Brownian motion with Hurst index \( H = d + 1/2 \). See for example Chapter 4.4 of [Giraitis et al., 2012].

A family of processes related to multilinear processes are the so-called multilinear polynomial-form processes (or discrete-chaos processes), which are defined as

\[ X(n) = \sum_{1 \leq i_1 < \ldots < i_k < \infty} a_{i_1} \ldots a_{i_k} \epsilon_{n-i_1} \ldots \epsilon_{n-i_k}, \]

where \( \sum_{i=1}^{\infty} a_i^2 < \infty \) and \( \epsilon_i \)'s are i.i.d., and the \( k > 0 \) is the order. \( X(n) \) is also said to belong to a discrete chaos of order \( k \). The multilinear polynomial-form process \( X(n) \) can be viewed as generated by nonlinear filters applied to i.i.d. random variables when \( k > 1 \). We call such a nonlinear filter defined in (1) a multilinear polynomial-form filter. Such a process often arises from considering a polynomial of a linear process (see, e.g., [Surgailis, 1982]).
If \( a_i = i^{d-1}L(i) \) with \( 0 < d < 1/2 \), when \( k > 1 \), that is, except for linear processes, the partial sum of \( X(n) \) when suitably normalized no longer converges to a fractional Brownian motion, but depending on \( d \) and \( k \), it either converges to a Hermite process if \( X(n) \) is still LRD, or it converges to a Brownian motion if \( X(n) \) is short-range dependent (SRD), that is, when the autocovariance of \( X(n) \) is absolutely summable. See Giraitis et al. [2012] for more details.

In Statistics, however, one often needs convergence when \( X(n) \) is a vector rather than a scalar. This leads us to the following question: if one applies different multilinear polynomial-form filters to the same \( X(n) \), what is the joint limit behavior of the \( J \)-vector of the partial sums? More specifically, assume that \( \{\epsilon_i\} \) are i.i.d with mean 0 and variance 1. Consider the multilinear polynomial-form processes:

\[
X_j(n) := \sum_{1 \leq i_1 < \ldots < i_k < \infty} a_{i_1,j} \ldots a_{i_k,j} \epsilon_n - i_1 \ldots \epsilon_n - i_k, \quad j = 1, \ldots, J,
\]

where \( k_1, \ldots, k_J \) are orders for \( X_1(n), \ldots, X_J(n) \) respectively, \( \{a_{i,j}\} \) are regularly varying coefficients. Let

\[
Y_{j,N}(t) = \frac{1}{A_j(N)} \sum_{n=1}^{[Nt]} X_j(n), \quad t \geq 0,
\]

where \( A_j(N) \) is a normalization factor such that \( \lim_{N \to \infty} \text{Var}[Y_{j,N}(1)] = 1, \quad j = 1, \ldots, J \). We want to study the limit of the following vector process as \( N \to \infty \):

\[
Y_N(t) := (Y_{1,N}(t), \ldots, Y_{J,N}(t)).
\]

Depending on \( \{a_{i,j}\} \) and \( k_i \), the components of \( Y_N(t) \) can be either purely SRD, or purely LRD, or a mixture of SRD and LRD. In Bai and Taqqu [2012], a similar type of problem is considered for nonlinear functions of a LRD Gaussian process. We show here that the results for multilinear polynomial-form processes are similar to those in Bai and Taqqu [2012]. But in the present context, we are able to provide a complete answer to the problem, in contrast to what happens in Bai and Taqqu [2012], where the mixed SRD and LRD case is stated as a conjecture in some cases.

In addition, we distinguish here between two types of SRD sequences, one involving a linear process \( (k = 1) \) and one involving higher-order multilinear polynomial-form process \( (k \geq 2) \). For the first type of process, we get dependence with the LRD limit component, while for the second type, we get independence.

The paper is organized as follows. In Section 2, some properties of multilinear polynomial-form processes are given and the univariate limit theorems under SRD and LRD are reviewed. In Section 3, we state the multivariate convergence results in three cases: a) pure SRD case, b) pure LRD case and c) mixed SRD and LRD case. The result of the general mixed case is stated in Theorem 3.3. In Section 4, we give the proofs of the results in Section 3.

### 2 Preliminaries

In this section, we introduce some facts about multilinear polynomial-form processes as well as the univariate limit theorems for the partial sums.

Suppose that \( X(n) \) is the multilinear polynomial-form process in (1). Note first, the condition \( \sum_{i=1}^{\infty} a_i^2 < \infty \) guarantees that \( X(n) \) is well-defined in \( L^2 \), since

\[
E[X(n)^2] = \sum_{1 \leq i_1 < \ldots < i_k < \infty} a_{i_1}^2 \ldots a_{i_k}^2 < \infty.
\]

We use throughout a convention \( a_i = 0 \) for \( i \leq 0 \). One can compute the autocovariance of \( X(n) \) as:

\[
\gamma(n) = \sum_{1 \leq i_1 < \ldots < i_k < \infty} a_{n+i_1} a_{i_1} \ldots a_{n+i_k} a_{i_k}, \quad n \in \mathbb{Z}.
\]

The following proposition describes the asymptotic behavior of \( \gamma(n) \) under the assumption: \( a_i = i^{d-1}L(i), \quad i \geq 1, \quad 0 < d < 1/2 \).
Proposition 2.1. Suppose $\gamma(n)$ is defined in [4], $a_i = i^{d-1}L(i)$, $i \geq 1$ with $0 < d < 1/2$ where $L$ is slowly varying at infinity. Then $\gamma(n) = L^*(n)n^{2d-1}$ for some slowly varying function $L^*$ and

$$d_X = \frac{1}{2} - k(\frac{1}{2} - d).$$

Proof. First we claim that as $n \to \infty$,

$$\sum_{i=1}^{\infty} a_{n+i}a_i \sim n^{2d-1}B(d, 1 - 2d)L(n)^2,$$

where $B(\cdot, \cdot)$ is the beta function. Indeed, one can check by Potter’s bound for slowly varying functions (Theorem 1.5.6 in [Bingham et al., 1989]) and the Dominated Convergence Theorem that as $n \to \infty$

$$\frac{1}{L(n)^2n^{2d-1}}\sum_{i=1}^{\infty} a_{n+i}a_i = \sum_{i=1}^{\infty} \frac{i}{n} i^{d-1} (1 + \frac{i}{n})^{d-1} \frac{L(i)}{L(n)} \frac{L(n+i)}{L(n)} \frac{1}{n}$$

$$\to \int_0^\infty u^{d-1}(1 + u)^{d-1}du = B(d, 1 - 2d).$$

Then note that as $n \to \infty$, $\gamma(n) \sim (k!)^{-1}(\sum_{i=1}^{\infty} a_{n+i}a_i)^k$ (the diagonal terms with $i_p = i_q$ are negligible as $n \to \infty$. See also [Giraitis et al., 2012] p.109). Now we can deduce that

$$\gamma(n) = n^{k(2d-1)}L^*(n) = n^{2d-1}L^*(n),$$

where $L^*(n) = (k!)^{-1}B(d, 1 - 2d)kL(n)^{2k}$.

Remark 2.2. According to Proposition 2.1 when $d < \frac{1}{2}(1 - \frac{1}{k})$ (or $k(2d-1) < -1$), we have $\sum |\gamma(n)| < \infty$, and when $d > \frac{1}{2}(1 - \frac{1}{k})$, we have $\sum |\gamma(n)| = \infty$. So if we assume $a_i = i^{d-1}L(i)$, $0 < d < 1/2$, the quantity $\frac{1}{2}(1 - \frac{1}{k})$ is the boundary between SRD and LRD.

We now define precisely what SRD and LRD mean for a multilinear polynomial-form process $X(n)$, and from then on we use this definition whenever we talk about SRD or LRD.

Definition 2.3. Let $X(n)$ be a multilinear polynomial-form process given in [1] with coefficient $\{a_i\}$, autocovariance $\gamma(n)$ and order $k$. We say that $X(n)$ is

(a) SRD, if for some $d \in (-\infty, \frac{1}{2}(1 - \frac{1}{k}))$ and some constant $c > 0$,

$$|a_i| \leq ci^{d-1}, \ i \geq 1, \ \sum_{n=-\infty}^{\infty} \gamma(n) > 0;$$

(b) LRD, if for some $d \in (\frac{1}{2}(1 - \frac{1}{k}), \frac{1}{2})$ and some $L$ slowly varying at infinity,

$$a_i = i^{d-1}L(i), \ i \geq 1, \ \frac{1}{2}(1 - \frac{1}{k}) < d < 1/2.$$

Remark 2.4. The $d$ in (7) and (8) are different. In the SRD case, $\{a_i\}$ is only assumed to decay faster than a power function, which implies $\sum_n |\gamma(n)| \leq \sum_n (\sum_{i=1}^{\infty} |a_{n+i}a_i|)^k < \infty$ by (4), and the particular $d$ chosen will not matter in the limit. While in the LRD case, the regularly varying assumption on $\{a_i\}$ yields a memory parameter $d_X = \frac{1}{2} - k(\frac{1}{2} - d)$ given by (5), and thus $d$ plays an important role.

Next we consider the cross-covariance between of two multilinear polynomial-form processes obtained by applying two multilinear polynomial-form filters to the same $\{\epsilon_i\}$. In particular, set

$$X_1(n) = \sum_{1 \leq i_1 < \ldots < i_p < \infty} a_{i_1} \ldots a_{i_p} \epsilon_{n-i_1} \ldots \epsilon_{n-i_p};$$

$$X_2(n) = \sum_{1 \leq i_1 < \ldots < i_q < \infty} b_{i_1} \ldots b_{i_q} \epsilon_{n-i_1} \ldots \epsilon_{n-i_q}. $$
X_1(n) and X_2(n) share the same \( \{ \epsilon_i \} \) but the sequences \( \{ a_i \} \) and \( \{ b_i \} \) can be different. Then the cross-covariance is

\[
\gamma_{1,2}(n) = \text{Cov}(X_1(n), X_2(0)) = \begin{cases} 
0 & \text{if } p \neq q; \\
\sum_{1 \leq i_1 < \ldots < i_k < \infty} a_{i_1} b_{n+i_1} \ldots a_{i_k} b_{n+i_k} & \text{if } p = q = k
\end{cases}
\]

for any \( n \in \mathbb{Z} \).

The following result will be used to obtain the asymptotic cross-covariance structure between the SRD components of \( Y_N(t) \) in \( (3) \).

**Proposition 2.5.** Let \( X_1(n) \) and \( X_2(n) \) be given as in \( (7) \) and \( (10) \) with \( p = q = k \), and are both SRD in the sense of Definition 2.3. Then the cross-covariance \( \gamma_{1,2}(n) = \text{Cov}(X_1(n), X_2(0)) \) is absolutely summable:

\[
\sum_{n=\infty}^{\infty} |\gamma_{1,2}(n)| < \infty.
\]

Moreover, \( (13) \) implies that as \( N \to \infty \),

\[
\text{Cov} \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt_1]} X_1(n), \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt_2]} X_2(n) \right) \to (t_1 \wedge t_2) \sum_{n=-\infty}^{\infty} \gamma_{1,2}(n).
\]

In addition, if \( k = 1 \), then

\[
\sum_{n=-\infty}^{\infty} \gamma_{1,2}(n) = \sigma_1 \sigma_2
\]

where \( \sigma_j^2 = \sum_n \text{Cov}(X_j(n), X_j(0)) = \lim_{N \to \infty} \text{Var} \left( \frac{1}{\sqrt{N}} \sum_{n=1}^{[N]} X_j(n) \right), j = 1, 2. \)

**Proof.** Suppose that \( \{ a_i \} \) and \( \{ b_i \} \) satisfy the bound in \( (7) \) with \( d = d_1 \) and \( d = d_2 \) respectively. Using a similar argument as in the proof of Proposition 2.1, one can show that

\[
|\gamma_{1,2}(n)| \leq n^{k(d_1 + d_2 - 1)} L^*(n)
\]

for some function \( L^*(n) \) slowly varying at \( \pm \infty \). Since by assumption \( d_1, d_2 < \frac{1}{2} \left( 1 - \frac{1}{k} \right) \), which implies that \( k(d_1 + d_2 - 1) < -1 \), so we have \( \sum_n |\gamma_{1,2}(n)| < \infty \).

The proof of \( (13) \) follows from the argument of Lemma 4.1 in \( \text{Bai and Taqqu, 2012} \), after noting that

\[
\text{Cov} \left( \sum_{n=1}^{[Nt_1]} X_1(n), \sum_{n=1}^{[Nt_2]} X_2(n) \right) = \sum_{n_1=1}^{[Nt_1]} \sum_{n_2=1}^{[Nt_2]} \gamma_{1,2}(n_1 - n_2).
\]

Now let’s prove \( (14) \). When \( k = 1 \), \( X_1(n) = \sum_{i=1}^{\infty} a_i \epsilon_{n-i}, X_2(n) = \sum_{i=1}^{\infty} b_i \epsilon_{n-i} \). Note that by \( (7) \) with \( k = 1 \), we have \( \sum_i |a_i| < \infty \) and \( \sum_i |b_i| < \infty \). The cross-covariance is \( \gamma_{1,2}(n) = \text{Cov}(X_1(n), X_2(0)) = \sum_{i=1}^{\infty} a_i b_{n+i} \). By Fubini,

\[
\sum_{n=-\infty}^{\infty} \gamma_{1,2}(n) = \sum_{n=-\infty}^{\infty} \sum_{i=1}^{\infty} a_i b_{n+i} = \left( \sum_{i=1}^{\infty} a_i \right) \left( \sum_{n=1}^{\infty} b_n \right).
\]

Since \( \left( \sum_{i=1}^{\infty} a_i \right)^2 = \sum_n \gamma_1(n) = \sigma_1^2 \), and \( \left( \sum_{i=1}^{\infty} b_i \right)^2 = \sum_n \gamma_2(n) = \sigma_2^2 \), we get relation \( (14) \). \( \square \)

Let’s now review the limit theorems for partial sum of a single multilinear polynomial-form process \( X(n) \). Let the notation \( \overset{f.d.d.}{\to} \) denote convergence in finite-dimensional distributions.
Theorem 2.6. Suppose that $X(n)$ defined in (1) is SRD. Then
\[
\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \overset{f.d.d.}{\longrightarrow} B(t),
\]
where $A(N)$ is a normalization factor to guarantee unit asymptotic variance at $t = 1$, and $B(t)$ is the standard Brownian motion. In fact, $A(N) \sim \sigma \sqrt{N}$ as $N \to \infty$ with $\sigma^2 = \sum_n \gamma(n)$.

Theorem 2.7. Suppose that $X(n)$ defined in (1) is LRD. Then
\[
\frac{1}{A(N)} \sum_{n=1}^{[Nt]} X(n) \overset{f.d.d.}{\longrightarrow} Z(k)(d)(t),
\]
where $A(N)$ is a normalization factor to guarantee unit asymptotic variance at $t = 1$, and $Z(k)(d)(t)$ is the so-called Hermite process defined with the aid of the $k$-tuple Wiener-Itô stochastic integral denoted by $I_k(\cdot)$.

In fact, $A(N) \sim cN^{1+(d-1/2)/k}L(N)^{k/2}$ as $N \to \infty$ for some $c > 0$.

For the proofs of Theorem 2.6 and Theorem 2.7, we refer the reader to Chapter 4.8 in [Giraitis et al., 2012], respectively Theorem 4.8.1 and Theorem 4.8.2. One may also compare Theorem 2.6 and Theorem 2.7 to their counterparts in the context of nonlinear functions of a LRD Gaussian process, stated as Theorem 2.1 and Theorem 2.2 in [Bai and Taqqu, 2012].

3 Multivariate convergence results

In this section, we state the multivariate joint convergence results for the vector process $Y_N(t)$ in (3). Recall that $Y_N$ is normalized so that the asymptotic variance of every component at $t = 1$ equals 1.

Theorem 3.1. Pure SRD Case. If all the components in $Y_N$ defined in (3) are SRD in the sense of (7), then
\[
Y_N(t) \overset{f.d.d.}{\longrightarrow} B(t) = (B_1(t), \ldots, B_J(t)),
\]
where $B(t)$ is a multivariate Gaussian process with $B_1(t), \ldots, B_J(t)$ being standard Brownian motions with
\[
\text{Cov}(B_p(s), B_q(t)) = (s \wedge t) \frac{\sigma_{p,q}}{\sigma_p \sigma_q}.
\]

(See [Pipiras and Taqqu, 2010].) In fact, $A(N) \sim c N^{1+(d-1/2)/k}L(N)^{k/2}$ as $N \to \infty$ for some $c > 0$.

The results of Chapter 4.8 in [Giraitis et al., 2012] do not include a slowly varying function, nor convergence of finite-dimensional distributions in the case of Theorem 2.6. But they can be easily extended.
Theorem 3.8. Weak convergence in $D$.

The independence between distributions, but one can show that in some cases they extend to weak convergence in the framework of [Bai and Taqqu, 2012], is only a conjecture.

Remark 3.2. $\sigma_{p,q}$ is well-defined by Proposition 2.3.

Remark 3.3. In view of (11) and (17), if all the components of the $Y_N(t)$ have different order, then the limit components $B_j(t)$ uncorrelated and hence independent. Otherwise, they are in general dependent and their covariance is given by (17).

Theorem 3.4. Pure LRD Case. If all the components in $Y_N$ defined in (3) are LRD in the sense of (8) with $d = d_1, \ldots, d_J$ respectively, then

\[
Y_N(t) \xrightarrow{f.d.d.} Z^L_{dl}(t) = (Z^{(k_1)}_{d_1}(t), \ldots, Z^{(k_J)}_{d_J}(t)),
\]

where $Z^{(k_j)}_{d_j}(t)$ are Hermite processes sharing the same random measure $W(.)$ in their Wiener-Itô integral representations. The normalization $A_j(N)$ in (3) satisfies $A_j(N) \sim c_j N^{1+(d_j-1/2)L}(N)^{k_j/2}$ as $N \to \infty$ for some $c_j > 0$. The processes $Z^{(k_j)}_{d_j}$, $j = 1, \ldots, J$ are dependent.

We now consider the mixed SRD and LRD case.

Theorem 3.5. Mixed SRD and LRD Case. Break $Y_N$ in (3) into 3 parts:

\[
Y_N = (Y_{N,S_1}, Y_{N,S_2}, Y_{N,L}),
\]

where within $Y_{N,S_1}$ ($J_{S_1}$-dimensional) every component is SRD and has order $k_{j,S_1} = 1$, within $Y_{N,S_2}$ ($J_{S_2}$-dimensional) every component is SRD and has order $k_{j,S_2} \geq 2$, and within $Y_{N,L}$ ($J_L$-dimensional) every component is LRD. Then

\[
Y_N(t) = (Y_{N,S_1}(t), Y_{N,S_2}(t), Y_{N,L}(t)) \xrightarrow{f.d.d.} (W(t), B(t), Z^{k_L}_{dl}(t)),
\]

where $B(t) := (B_1(t), \ldots, B_{J_{S_2}}(t))$ is the multivariate Gaussian process appearing in Theorem 3.4, $Z^{k_L}_{dl}(t)$ is the multivariate Hermite process appearing in Theorem 3.4,

\[
W(t) = (W(t), \ldots, W(t)),
\]

where $W(t)$ is the Brownian motion integrator for defining $Z^{k_L}_{dl}(t)$ (see (13)), and $B(t)$ is independent of $(W(t), Z^{k_L}_{dl}(t))$.

Remark 3.6. To understand heuristically why $B(t)$ and $(W(t), Z^{k_L}_{dl}(t))$ are independent, note that $Y_{N,S_1}(t)$ belongs to chaos of order $\geq 2$, and is thus uncorrelated with $Y_{N,S_1}(t)$ which belongs to first-order chaos, and also uncorrelated with the random noise $\{c_i\}$ which also belongs to the first-order chaos, and which after summing becomes asymptotically the Brownian measure $W(.)$ defining $Z^{k_L}_{dl}(t)$.

Remark 3.7. The independence between $B(t)$ and $Z^{k_L}_{dl}(t)$ for $k_{j,L} \geq 3$ (the order in LRD component) in the framework of [Bai and Taqqu, 2012], is only a conjecture.

The convergence results in the above theorems are stated in terms of convergence in finite-dimensional distributions, but one can show that in some cases they extend to weak convergence in $D[0,1]^J$ ($J$-dimensional product space where $D[0,1]$ is the space of càdlàg functions on $[0,1]$ with uniform metric).

Theorem 3.8. Weak convergence in $D[0,1]^J$.

1. Theorem 3.4 holds with “$\xrightarrow{f.d.d.}$” replaced by weak convergence in $D[0,1]^J$;
2. If the SRD component in Theorem 3.1 (or Theorem 3.5) satisfies either of the following conditions:
   a. There exists $m \geq 0$, such that the coefficients $a_i$ in $\{j\}$ are zero for all $i > m$;
   b. $\{\epsilon_i\}$ are i.i.d. Gaussian.
   c. The order $k = 1$ and $E(\epsilon_i^2 + \delta) < \infty$ for some $\delta > 0$;
   d. The order $k \geq 2$, $\sum_{i=1}^{\infty} |a_i| < \infty$ and $E(|\epsilon_i|^5) < \infty$;

then Theorem 3.1 (or Theorem 3.5) holds with "f.d.d. replaced by weak convergence in $D[0,1]^J$.

Note that tightness in the SRD case results from an interplay between the dependence structure and the finiteness of the moments.

4 Proofs for the multivariate convergence results

4.1 Pure SRD case

Proof of Theorem 3.1. Following the idea of Giraitis et al., 2012 p.108., we define the truncated multilinear polynomial-form processes:

$$X_j^{(m)}(n) = \sum_{1 \leq i_1 < \ldots < i_k \leq m} a_{i_1,j} \ldots a_{i_k,j} \epsilon_{n-i_1} \ldots \epsilon_{n-i_k}, \quad j = 1, \ldots, J,$$

(20)

where $m > \max_j \{k_j\}$. Note that $X_j^{(m)}(n)$ is $m$-dependent. Set $(\sigma_j^{(m)})^2 = \sum_n \text{Cov} \left( X_j^{(m)}(n), X_j^{(m)}(0) \right)$ (assume $m$ is large enough so that $\sigma_j^{(m)} > 0$), and $\sigma_{p,q}^{(m)} = \sum_n \text{Cov} \left( X_p^{(m)}(n), X_q^{(m)}(0) \right)$ which is well-defined due to Proposition 2.4.

Set

$$Y_{N,j}(t) := \frac{1}{\sigma_j \sqrt{N}} \sum_{n=1}^{\lfloor N \rfloor} X_j(n), \quad Y_{N,j}^{(m)}(t) := \frac{1}{\sigma_j^{(m)} \sqrt{N}} \sum_{n=1}^{\lfloor N \rfloor} X_j^{(m)}(n).$$

Theorem 3.1 follows if one shows that as $N \to \infty$,

$$Y_N^{(m)}(t) = \left( Y_{N,1}^{(m)}(t), \ldots, Y_{N,J}^{(m)}(t) \right) \xrightarrow{f.d.d.} B^{(m)}(t) := \left( B_1^{(m)}(t), \ldots, B_J^{(m)}(t) \right)$$

(21)

where $B_j^{(m)}(t)$'s are Brownian motions with cross-covariance structure:

$$\text{Cov} \left( B_p^{(m)}(t_1), B_q^{(m)}(t_2) \right) = (t_1 \wedge t_2) \frac{\sigma_p^{(m)}}{\sigma_q^{(m)}} \frac{\sigma_{p,q}^{(m)}}{\sigma_{p,q}^{(m)}} , \quad p, q = 1, \ldots, J,$$

(22)

and as $m \to \infty$,

$$\sigma_j^{(m)} \to \sigma_j, \quad \sigma_{p,q}^{(m)} \to \sigma_{p,q}$$

(23)

as well as for any $j = 1, \ldots, J$ and $t \geq 0$, as $m \to \infty$,

$$\text{Var} \left[ Y_{N,j}^{(m)}(t) - Y_{N,j}(t) \right] \to 0$$

(24)

uniformly in $N$. Indeed, combining (21), (22) and (23), one obtains the desired convergence:

$$Y_N(t) = \left( Y_{N,1}(t), \ldots, Y_{N,J}(t) \right) \xrightarrow{f.d.d.} B(t) := \left( B_1(t), \ldots, B_J(t) \right)$$
Relations (23) and (24) can be shown using the same type of arguments in [Giraitis et al., 2012] p.108. We thus only need to show (21) and (22). By the Cramér-Wold device, it suffices to show that for any \((c_1, \ldots, c_J) \in \mathbb{R}^J\),

\[
\sum_j c_j Y_{N,j}^{(m)}(t) = \frac{1}{\sigma_j^{(m)} \sqrt{N}} \left[ \sum_{n=1}^{[Nt]} \sum_j c_j X_j^{(m)}(n) \right] \xrightarrow{f.d.d.} \sum_j c_j B_j^{(m)}(t) =: G(t)
\]

where \(G(t)\) is a non-standardized Brownian motion. This follows from the fact that the sequence \(\{\sum_j c_j X_j^{(m)}(n)\}_n\) is m-dependent and is thus subject to functional central limit theorem ([Billingsley, 1956] Theorem 5.2), which includes convergence in finite-dimensional distributions. The asymptotic cross-covariance structure (22) follows from Proposition 2.5.

4.2 Pure LRD case

**Proof of Theorem 3.4**. The joint convergence is proved by combining Theorem 4.8.2. and Proposition 14.3.3 of [Giraitis et al., 2012] and the arguments leading to them. The dependence between the limit Hermite processes with different orders is shown in Proposition 3.1 in [Bai and Taqqu, 2012].

4.3 Mixed SRD and LRD case

We prove Theorem 3.5 through a number of lemmas, one lemma implying the next.

**Lemma 4.1.** Follow the notations and assumptions in Theorem 3.3. Let \(X_j^{(m)}(n)\) be the m-truncated multilinear polynomial-form process (see (20)) corresponding to the components of \(Y_{N,S_i}(i = 1, 2)\) in Theorem 3.3, where the orders satisfy \(k_{j,S_1} = 1\) and \(k_{j,S_2} \geq 2\). Let

\[
Y_{N,j,S_i}^{(m)}(t) := \frac{1}{\sigma_j^{(m)} \sqrt{N}} \sum_{n=1}^{[Nt]} X_{j,S_i}^{(m)}(n), \quad j = 1, \ldots, J, i = 1, 2,
\]

where (assuming that m is large enough) \(0 < \sigma_j^{(m)} \leq \sigma_j^{(m)}(0) < \infty\), \(i = 1, 2\). Let \(W(t) := N^{-1/2} \sum_{n=1}^{[Nt]} \epsilon_n\), and \(Y_{N,S_i}^{(m)}(t) = (Y_{N,j,S_i}^{(m)}(t), \ldots, Y_{N,J,s_i}^{(m)}(t))\), \(i = 1, 2\). Then

\[
\left( Y_{N,S_1}^{(m)}(t), Y_{N,S_2}^{(m)}(t), W_N(t) \right) \xrightarrow{f.d.d.} \left( W(t), B^{(m)}(t), W(t) \right),
\]

where \(W(t)\) is a standard Brownian motion, \(W(t) = (W(t), \ldots, W(t)) \) (\(J_{S_2}\)-dimensional), \(B^{(m)}(t)\) is as given in (21), namely, its components are standard Brownian motions with cross-covariance (22), and \(B^{(m)}(t)\) is independent of \((W(t), W(t))\).

**Proof.** Fix any \(w = (a_1, \ldots, a_{J_{S_1}}, b_1, \ldots, b_{J_{S_2}}, c) \in \mathbb{R}^{J_{S_1} + J_{S_2} + 1}\). By Cramér-Wold, we want to show that

\[
R_N(t; w) := \sum_j a_j Y_{N,j,1}^{(m)}(t) + \sum_j b_j Y_{N,j,2}^{(m)}(t) + c W_N(t)
\]

\[
\xrightarrow{f.d.d.} \sum_j a_j W(t) + \sum_j b_j B_j^{(m)}(t) + c W(t) =: G(t),
\]

where \(G(t)\) is a non-standardized Brownian motion whose marginal variance is the limit of the marginal variance of \(R_N(t; w)\). Note that one can write

\[
R_N(t; w) = \frac{1}{\sqrt{N}} \sum_{n=1}^{[Nt]} U_n^{(m)}(t),
\]
where
\[
U^{(m)}_w(n) = \sum_{j=1}^{J_{S_1}} a_j^{(m)} \chi_{j,S_1}^{(m)}(n) + \sum_{j=1}^{J_{S_2}} b_j^{(m)} \chi_{j,S_2}^{(m)}(n) + \varepsilon_n^{(m)}
\]
with
\[
\varepsilon_n^{(m)} = \sum_{i=(m-1)n+1}^{mn} \epsilon_i.
\]
Since \(\{U^{(m)}_w(n)\}_n\) is m-dependent, the classical functional central limit theorem applies (Billingsley, 1956), yielding in the limit a Brownian motion \(G(t)\) for \(R_N(t;w)\). Now that the joint normality is shown, we only need to identify the asymptotic covariance structure as \(N \to \infty\) of the left-hand side of (20) to the covariance structure of the right-hand side of (20).

The independence between \(B^{(m)}(t)\) and \((W(t), W(t))\) follows from the uncorrelatedness between \(Y_{N,S_2}^{(m)}(t)\) (involving chaos of order \(\geq 2\)) and \((Y_{N,S_1}^{(m)}(t), W_N(t))\) (involving chaos of order 1 only). The asymptotic covariance structure within \(Y_{N,S_2}^{(m)}(t)\) is given in (22) (apply Theorem 3.1 to \(Y_{N,S_2}^{(m)}(t)\)). Hence we are left to show that the asymptotic covariance structure of \((Y_{N,S_1}^{(m)}(t), W_N(t))\) is that of \((W(t), W(t))\). Note that in \((Y_{N,S_1}^{(m)}(t), W_N(t))\), both \(\{X_{1,S}^{(m)}(n)\}\) and \(\{\epsilon_n\}\) are SRD linear processes. So applying (13) and (14) in Proposition 2.3 with \(\sigma_1 = \sigma_2 = 1\), the desired asymptotic covariance structure is obtained.

**Remark 4.2.** Lemma 4.1 can be rephrased as follows: we define an empirical random measure on a finite interval \(\Delta: W_N(\Delta) := \frac{1}{\sqrt{N}} \sum_{n/N \in \Delta} \epsilon_n\). Then the joint convergence in Lemma 4.1 still holds with \(W(t)\) replaced by \((W_N(\Delta_1), \ldots, W_N(\Delta_I))\) where \(\Delta_i, i = 1, \ldots, I\) are disjoint intervals, and \(W(t)\) in the limit replaced by \((W(\Delta_1), \ldots, W(\Delta_I))\) where \(W(.)\) is the Brownian random measure. Observe that while (20) involves convergence in distribution, the limit components \(W(t)\) and \(W(t)\) both involve the same Brownian motion \(W(t)\).

Now we adopt some notations from Giraitis et al. (2012) Chapter 14.3. Let \(S_M(\mathbb{R}^k)\) be the class of simple functions defined on \(\mathbb{R}^k\) supported on a finite number of \(1/M\)-cubes and vanishing on the diagonals. Suppose that \(h\) is a function defined on \(\mathbb{Z}^k\) which vanishes on diagonals. Let the polynomial form (or discrete multiple integral) with respect to \(h\) be
\[
Q_k(h) = \sum_{i_1, \ldots, i_k \in \mathbb{Z}} h(i_1, \ldots, i_k) \epsilon_{i_1} \cdot \epsilon_{i_k}, \tag{27}
\]
where \(\sum_{i_1, \ldots, i_k} h(i_1, \ldots, i_k)^2 < \infty\). The following lemma plays a key role in the proof of Theorem 3.3.

**Lemma 4.3.** Replace \((Y_{N,S_1}^{(m)}(t), Y_{N,S_2}^{(m)}(t), W_N(t))\) in Lemma 4.1 by \((Y_{N,S_1}^{(m)}(t), Y_{N,S_2}^{(m)}(t), Q_N), \) where \(Q_N = (Q_{k_1}(h_{1,N}), \ldots, Q_{k_{J_{L}}}(h_{J_{L},N}))\) and each \(Q_{k_p}(h_{p,N}), p = 1, \ldots, J_L,\) is a polynomial-form defined in (27) with the same \(\{\epsilon_i\}\) as those defining \(Y_{N,S_1}^{(m)}(t)\) and \(Y_{N,S_2}^{(m)}(t)\). Assume that the “normalized continuous extension” of \(h_{p,N}\), that is,
\[
\hat{h}_{p,N}(x_1, \ldots, x_{k_p}) := N^{k_p/2} h_{p,N}([Nx_1], \ldots, [Nx_{k_p}]), \tag{28}
\]
satisfy that there exists \(f_p \in L^2(\mathbb{R}^{k_p})\) for each \(p = 1, \ldots, J_L,
\[
\lim_{N \to \infty} \|\hat{h}_{p,N} - f_p\|_{L^2(\mathbb{R}^{k_p})} \to 0. \tag{29}
\]

Now define the limit vector \((W(t), B^{(m)}(t), I)\) as follows: \(W(t)\) and \(B^{(m)}(t)\) are as in (27), independent, and \(I = (I_{k_p}(f_p))_{p=1,\ldots,J_L},\) where each Wiener-Itô integral \(I_{k_p}(.)\) has Brownian motion integrator \(W(.)\) the same as the Brownian motion \(W(t)\) defining \(W(t)\). Then as \(N \to \infty,\)
\[
\left(Y_{N,S_1}^{(m)}(t), Y_{N,S_2}^{(m)}(t), Q_N\right) \overset{f.d.d.}{\longrightarrow} \left(W(t), B^{(m)}(t), I\right). \tag{30}
\]
Remark 4.4. Observe that $B^{(m)}$ is independent of $(W, I)$.

Proof. The lemma is proved by combining Lemma 4.1 with the proof of Proposition 14.3.2 of [Giraitis et al., 2012]. By Cramér-Wold, we need to show that for any $a \in \mathbb{R}^{J_1}$, $b \in \mathbb{R}^{J_2}$ and $c \in \mathbb{R}^{J_L}$, as $N \to \infty$,

$$< a, Y^{(m)}_{N,S_1}(t) > + < b, Y^{(m)}_{N,S_2}(t) > + < c, Q_N > \xrightarrow{f.d.d.} < a, W(t) > + < b, B^{(m)}(t) > + < c, I > ,$$

where $< \cdot, \cdot >$ denotes the Euclidean inner product.

Next following the approximation argument that leads to (14.3.14), (14.3.15) and (14.3.16) in [Giraitis et al., 2012], one can show that for any $\epsilon > 0$, there exists $M > 0$ and simple functions $f_{p,\epsilon} \in S_M(\mathbb{R}^p)$, $p = 1, \ldots, J_L$, such that for all $N \geq N_0(\epsilon)$ where $N_0(\epsilon)$ is large enough,

$$\|Q_{k_p}(h_{p,N}) - Q_{k_p}(h_{p,\epsilon,N})\|_{L^2(\Omega)} \leq \epsilon,$$

$$Q_{k_p}(h_{p,\epsilon,N}) \overset{d}{\to} I_{k_p}(f_{p,\epsilon}) \quad \text{as} \quad N \to \infty,$$

$$\|I_{k_p}(f_{p,\epsilon}) - I_{k_p}(f_p)\|_{L^2(\Omega)} \leq \epsilon,$$

where $\|\cdot\|_{L^2(\Omega)}$ denotes the $L^2(\Omega)$ norm.

Set

$$Q_{\epsilon,N} := \left( Q_{k_p}(h_{p,\epsilon,N}) \right)_{p=1,\ldots,J_L}$$

and

$$I_\epsilon := \left( I_{k_p}(f_{p,\epsilon}) \right)_{p=1,\ldots,J_L}.$$

Now note that $Q_{k_p}(h_{p,\epsilon,N})$ is a multivariate polynomial (thus is a continuous function) of random variables of the form $W_N(\Delta_i)$ where $\Delta_i$’s are disjoint finite intervals and $W_N(.)$ is the empirical random measure as given in Remark 4.4. So by Lemma 4.1 (with Remark 4.2) and the Continuous Mapping Theorem, we have that as $N \to \infty$,

$$< a, S_{N,1}^{(m)}(t) > + < b, S_{N,2}^{(m)}(t) > + < c, Q_{\epsilon,N} > \xrightarrow{f.d.d.} < a, W(t) > + < b, B^{(m)}(t) > + < c, I_\epsilon > .$$

By (32) and the Cauchy-Schwartz inequality, we infer that

$$\| (Q_N - Q_{\epsilon,N}) \|_{L^2(\Omega)} \leq \| c \| \| Q_N - Q_{\epsilon,N} \|_{L^2(\Omega)} \leq \| c \| \sqrt{J_L} \epsilon,$$

where $\| \cdot \|$ denotes the Euclidean norm. Similarly using (34),

$$\| (c, I - I_\epsilon) \|_{L^2(\Omega)} \leq \| c \| \| I - I_\epsilon \|_{L^2(\Omega)} \leq \| c \| \sqrt{J_L} \epsilon.$$

We now apply a usual triangular approximation argument (e.g., Lemma 4.2.1 of [Giraitis et al., 2012]). Let

$$U^{(m)}_N(t) = < a, Y^{(m)}_{N,S_1}(t) > + < b, Y^{(m)}_{N,S_2}(t) > + < c, Q_N > ,$$

$$U^{(m)}_{\epsilon,N}(t) = < a, Y^{(m)}_{N,S_1}(t) > + < b, Y^{(m)}_{N,S_2}(t) > + < c, Q_{\epsilon,N} > ,$$

$$U^{(m)}_{\epsilon}(t) = < a, W(t) > + < b, B^{(m)}(t) > + < c, I > ,$$

$$U^{(m)}_\epsilon(t) = < a, W(t) > + < b, B^{(m)}(t) > + < c, I > .$$

By (35), (37) and (36), we have that

$$U^{(m)}_N(t) \xrightarrow{f.d.d.} X^{(m)}_\epsilon(t) \quad \text{as} \quad N \to \infty,$$

$$U^{(m)}_{\epsilon,N}(t) \xrightarrow{f.d.d.} X^{(m)}(t) \quad \text{as} \quad \epsilon \to 0,$$

$$\lim_{\epsilon \to 0} \sup \| U^{(m)}_{\epsilon,N}(t) - U^{(m)}_\epsilon(t) \|_{L^2(\Omega)} = 0, \quad \forall \ t \geq 0,$$

which implies $U^{(m)}_N(t) \xrightarrow{f.d.d.} U^{(m)}(t)$, proving (31).
The next lemma gets rid of the \( m \)-truncation.

**Lemma 4.5.** Lemma 4.3 holds with the \( m \)-truncated normalized partial sums \( Y_{N,S_i}^{(m)}(t), i = 1, 2 \) replaced with the non-truncated ones:

\[
Y_{N,S_i}(t) = \left( \frac{1}{\sigma_{j,S_i} \sqrt{N}} \sum_{n=1}^{[Nt]} X_{j,S_i}(n) \right)_{j=1,\ldots,J_i}, i = 1, 2,
\]

where \( X_{j,S_i}(n) \) is the non-truncated multilinear polynomial-form process corresponding to the component of \( Y_{N,S_i} \) in Theorem 3.5, \( \sigma_{j,S_i} := \sum_n \text{Cov}(X_{j,S_i}(n), X_{j,S_i}(0)) \) and the limit \( B^{(m)}(t) \) is replaced by \( B(t) \), that is, as \( N \rightarrow \infty \),

\[
\left( Y_{N,S_1}(t), Y_{N,S_2}(t), Q_N \right) \xrightarrow{f.d.d.} \left( W(t), B(t), I \right),
\]

where \( W(t) = (W(t), \ldots, W(t)) \), \( B(t) = (B_1(t), \ldots, B_{J_2}(t)) \) are as given in Theorem 3.3.

**Proof.** We apply again the triangular argument at the end of the proof of Lemma 4.3 above, but now with \( m \rightarrow \infty \), namely, to show \( U_N(t) \xrightarrow{f.d.d.} U(t) \), we show

\[
U_N^{(m)}(t) \xrightarrow{f.d.d.} U^{(m)}(t) \text{ as } N \rightarrow \infty,
\]

\[
U^{(m)}(t) \xrightarrow{f.d.d.} U(t) \text{ as } m \rightarrow \infty,
\]

\[
\lim_{m \rightarrow \infty} \limsup_{N \rightarrow \infty} \|U_N^{(m)}(t) - U_N(t)\|_{L^2(\Omega)} = 0, \forall t \geq 0,
\]

The first step follows from Lemma 4.3. The second follows from (23) since that relation implies that the Gaussian vector \( (W, B^{(m)}(t)) \) converges to \( (W, B(t)) \). For the last step, apply the argument leading to (4.8.7) of [Giraitis et al., 2012] and hence for any \( t \geq 0 \) as \( N \rightarrow \infty \),

\[
\|Y_{N,j,i}^{(m)}(t) - Y_{N,j,i}(t)\|_{L^2(\Omega)} \rightarrow 0, j = 1, \ldots, J_i, i = 1, 2.
\]

Now we prove Theorem 3.5.

**Proof of Theorem 3.5.** In view of Lemma 4.5, it is only necessary to verify that the assumption on \( Q_N \) are satisfied, that is, we now focus on the LRD component:

\[
Y_{N,L}(t) = \left( \frac{1}{A_{p,L}(N)} \sum_{n=1}^{[Nt]} X_{p,L}(n) \right)_{p=1,\ldots,J_L}
\]

in Theorem 3.5. Choose as kernels \( h_{p,N} \) in (28) those obtained from \( Y_{N,L} \), that is,

\[
h_{p,N}^{(t)}(s_1, \ldots, s_{k_{p,L}}) = c(p, N) N^{-1+k_{p,L}(1/2-d_{p,L})} \sum_{n=1}^{[Nt]} \prod_{i=1}^{k_{p,L}} a_{n-s_{i,p}},
\]

where \( c(p, N) > 0 \) is some normalization constant. By Theorem 4.8.2 of [Giraitis et al., 2012], (29) holds and so therefore does Lemma 4.3. This concludes the proof of Theorem 3.5.
4.4 Weak convergence in $D[0, 1]^J$

We first state a lemma which will be used to prove case 2a.

**Lemma 4.6.** Let $Q_k(h)$ be a polynomial form defined in (27). If

$$\sum_{i_1, \ldots, i_k} |h(i_1, \ldots, i_k)| < \infty,$$  

(40)

and $E(|\epsilon_i|^5) < \infty$, then we have the following hypercontractivity inequality:

$$E(Q_k(h)^4) \leq c E(Q_k(h)^2)^2,$$  

(41)

where $c = (3 + 2E(\epsilon_i^4))^{2k}$.

**Proof.** Let $h_M$ be the truncated version of $h$, that is,

$$h_M(i_1, \ldots, i_k) = h(i_1, \ldots, i_k) 1_{\{i_1 \leq M, \ldots, i_k \leq M\}}(i_1, \ldots, i_k).$$

By the absolute summability of $h$, we have $E(|Q_k(h_M) - Q_k(h)|) \leq (E|\epsilon_i|^k)^k \sum_{i_1 \geq M, \ldots, i_k \geq M} |h(i_1, \ldots, i_k)| \to 0$ as $M \to \infty$, and thus

$$Q_k(h) \overset{d}{\to} Q_k(h).$$  

(42)

By (11.4.1) of [Nourdin and Peccati, 2012], we have for $M \geq k$,

$$E(Q_k(h_M)^4) \leq (3 + 2E(\epsilon_i^4))^{2k} E(Q_k(h_M)^2)^2$$  

(43)

In addition,

$$E(|Q_k(h_M)|^5) \leq A \left( \sum_{i_1, \ldots, i_k} |h(i_1, \ldots, i_k)| \right)^5 < \infty,$$  

(44)

where $A > 0$ is a constant accounting for the product of absolute moments of $\{\epsilon_i\}$. Note that since $h$ vanishes on the diagonals $i_p = i_q$ when $p \neq q$, there is no moment-order higher than 5 involved there.

Finally, (44) implies that $\{Q_k(h_M)^4, M \geq 1\}$ and $\{Q_k(h_M)^2, M \geq 1\}$ are uniformly integrable, and this combined with (42) and (43) yields (41). \qed

**Proof of Theorem 3.3.** Convergence in finite-dimensional distributions follows from Theorem 3.1, Theorem 3.4 and Theorem 3.5, so we are left to show tightness in $D[0, 1]^J$. Since univariate tightness implies the multivariate tightness in the product space (Lemma 3.10 of [Bai and Taqqu, 2012]), we only need to show that each $\{Y_{j,N}(t), N \geq 1\}$ in (36) is tight with respect to the uniform metric. If $X_j(n)$ is LRD, the tightness is shown in Theorem 4.8.2 of [Giraitis et al., 2012]. We only need to treat the SRD case.

Suppose that $X(n)$ is a process defined in (11) which is SRD.

In case 2a of Theorem 3.3, note that $X_n$ is now a stationary $m$-dependent sequence, so the weak convergence of $S_N(t)$ to Brownian motion, which includes tightness, is classical ([Billingsley, 1956, Theorem 5.2]).

Consider next case 2b. Because $\epsilon_i$ are i.i.d. Gaussian, $X(n)$ belongs to the $k$-th Wiener chaos, or say, can be written as a multiple Wiener-Itô integral of order $k$ (see, e.g., [Nourdin and Peccati, 2012] Chapter 2.2 and Chapter 2.7). Since the $k$-th Wiener chaos is a linear space, $Y_N(t) := \sum_{n=1}^{[Nt]} X(n)$ also belongs to the $k$-th Wiener chaos, and so does $Y_N(t) - Y_N(s)$ for any $0 \leq s < t$. By the hypercontractivity inequality (Theorem 2.7.2 in [Nourdin and Peccati, 2012]), we have

$$E[|Y_N(t) - Y_N(s)|^4] \leq cE[|Y_N(t) - Y_N(s)|^2]^2,$$  

(45)
where \( c \) is some constant which doesn’t depend on \( s, t \) or \( N \). Note that \( \sum_n |\gamma(n)| < \infty \) due to SRD assumption, we have

\[
E[|Y_N(t) - Y_N(s)|^2] = \frac{1}{N} E\left[ \sum_{n=1}^{[Nt]-[Ns]} X(n)^2 \right]
= \frac{[Nt]-[Ns]}{N} \sum_{n=-([Nt]-[Ns])}^{[Nt]-[Ns]} \left(1 - \frac{|n|}{[Nt] - [Ns]}\right) \gamma(n) \leq \frac{[Nt]-[Ns]}{N} \sum_{n=-\infty}^{\infty} |\gamma(n)|.
\] (46)

Combining (45) and (46), we have for some constant \( C > 0 \) that

\[
E[|Y_N(t) - Y_N(s)|^4] \leq c E[|Y_N(t) - Y_N(s)|^2]^2 \leq C |F_N(t) - F_N(s)|^2,
\]

where \( F_N(t) = [Nt]/N \). Now by applying Lemma 4.4.1 and Theorem 4.4.1 of Giraitis et al. 2012, we conclude that tightness holds.

Case 2c is shown by Proposition 4.4.4 of Giraitis et al. 2012 with \( H = 1/2 \).

For case 2d, for \( s < t \),

\[
\frac{1}{A(N)} \sum_{n=1}^{[Nt]-[Ns]} X(n) = \sum_{1 \leq i_1 < \ldots < i_k < \infty} \left( \frac{1}{A(N)} \sum_{n=1}^{[Nt]-[Ns]} a_{n-i_1} \ldots a_{n-i_k} \right) \epsilon_{i_1} \ldots \epsilon_{i_k},
\]

Thus Lemma 4.9 applies with \( h(i_1, \ldots, i_k) = \frac{1}{A(N)} \sum_{n=1}^{[Nt]-[Ns]} a_{n-i_1} \ldots a_{n-i_k} \) since (40) holds due to the assumption \( \sum_{i \geq 1} |a_i| < \infty \). Tightness then follows by applying the same argument as in case 2b.

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References

S. Bai and M.S. Taqqu. Multivariate limit theorems in the context of long-range dependence. *arXiv preprint arXiv:1211.0576v2*, 2012.

Patrick Billingsley. The invariance principle for dependent random variables. *Transactions of the American Mathematical Society*, 83(1):250–268, 1956.

N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation*. Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 1989.

L. Giraitis, H.L. Koul, and D. Surgailis. *Large Sample Inference for Long Memory Processes*. World Scientific Publishing Company Incorporated, 2012.

P. Major. *Multiple Wiener-Itô Integrals*. Citeseer, 1981.

I. Nourdin and G. Peccati. *Normal Approximations With Malliavin Calculus: From Stein’s Method to Universality*. Cambridge Tracts in Mathematics. Cambridge University Press, 2012.

V. Pipiras and M.S. Taqqu. Regularization and integral representations of Hermite processes. *Statistics and probability letters*, 80(23):2014–2023, 2010.

D Surgailis. Zones of attraction of self-similar multiple integrals. *Lithuanian Mathematical Journal*, 22(3):327–340, 1982.

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