New examples of higher-dimensional minimal hypersurfaces

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ABSTRACT. The main results of the paper are Proposition 3 and 4 which provide an effective way to construct minimal hypersurfaces in a Euclidean space. We demonstrate our technique by several new examples.

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1. Introduction

It is well known that any classical solution $u(x) = u(x_1, \ldots, x_n)$ of the following equation

\[(1 + |Du|^2)\Delta u - \sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_i}x_{x_j} = 0,\]

where $u_{x_i} = D_{x_i}u$, gives rise to a minimal (zero mean curvature) hypersurface

$x_{n+1} = u(x), \quad x \in \Omega \subset \mathbb{R}^n$.

1.1. Minimal surfaces with ‘harmonic level sets’. In \[3\], \[4\] and \[5\], the authors studied and classified all zero mean curvature surfaces in the Euclidean and Minkowski spaces given implicitly by

\[M = \{x \in \mathbb{R}^3 : \text{Re} \ h(z) = F(x_3)\}, \quad z = x_1 + x_2\sqrt{-1},\]

where $h(z)$ is a holomorphic function of one complex variable. These surfaces were referred to as minimal surfaces with ‘harmonic level sets’ (\[5\]). More precisely, one has the following result in the Euclidean case.

**Theorem A** (\[3\]). The surface $M$ defined by (2) is minimal if and only if $h'(z) = 1/g(z)$, where $g(z)$ satisfies

\[g''(z)g(z) - g'^2(z) = c \in \mathbb{R}.

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the catenoid \( g(z) = z \) \( h(z) = \ln z \) \( x_1^2 + x_2^2 = \cosh^2 x_3 \)

the helicoid \( g(z) = iz \) \( h(z) = -i \ln z \) \( \frac{x_2}{x_1} = \tan x_3 \)

a Scherk type surface \( g(z) = e^z \) \( h(z) = -e^{-z} \) \( \exp x_3 = \frac{\cos x_2}{\cos x_1} \)

a doubly periodic surface \( g(z) = \sin z \) \( h(w) = -\ln \tanh \frac{w}{2} \) \( \operatorname{cn}\left(\frac{kx_2}{k'}, k\right) = \frac{\sin x_2}{\sinh x_1} \)

\[ \text{Table 1. Some partial solutions} \]

In this case, the function \( F \) is found by \( F''(t) + Y(F(t)) = 0 \), where \( Y(t) \) is well-defined by virtue of

\[ \frac{\operatorname{Re} g'}{|g|^2} = -Y(\operatorname{Re} h(z)). \]

**Remark.** Notice that the resulting surface \( M \), if non-empty, is automatically embedded because

\[ |\nabla f(x)|^2 = F'^2(x_1) + \frac{1}{|g(z)|^2} > 0, \]

where \( f(x) = F(x_1) - \operatorname{Re} h(z) \).

A simple analysis reveals that the only possible solutions of (3) are

(a) \( g(z) = az + b \),
(b) \( g(z) = ae^{bz} \), and
(c) \( g(z) = a \sin(bz + c) \), \( a^2, b^2 \in \mathbb{R} \) and \( c \in \mathbb{C} \)

which, in particular, yields:

2. The higher dimensional case

Let \( f = f(\xi) \) be of class \( C^2 \) in an open set of \( \mathbb{R}^N \). It is well known (see, for instance, [1]) that the non-singular zero-locus

\[ M_f := f^{-1}(0) \cap \{ \xi : |Df(\xi)| \neq 0 \} \]

is a minimal hypersurface in \( \mathbb{R}^N \) if and only if

\[ \Delta_1 f(\xi) = 0 \mod f(\xi). \]
Here
\[ \Delta_1 f := |Df|^2 - \sum_{i,j=1}^{N} f_{\xi_i \xi_j} f_{\xi_i \xi_j} \]
is the mean curvature operator (in fact, the 1-Laplace operator) and the congruence
\[ a(\xi) \equiv b(\xi) \mod c(\xi) \]
is understood in the sense that \( c(\xi) = 0 \) implies \( a(\xi) - b(\xi) = 0 \). Notice that the hypersurface is automatically embedded as a level set.

Below, we study a generalization of (2), i.e. the solutions \( f(\xi) \) of (4) having the form
\[ f(\xi) = \text{Re} \ h(z) - F(t), \quad \xi = (z, t) \in \mathbb{C}^m \times \mathbb{R}^k = \mathbb{R}^{2m+k}, \]
where \( h(z) \) is a holomorphic function and \( F(t) \) is a function of class \( C^2 \). More precisely, recall that a function \( h(z) = h(z_1, \ldots, z_k) : \Omega \to \mathbb{C} \) is called holomorphic in \( \Omega \subset \mathbb{C}^n \), or \( h \in \mathcal{O}(\Omega) \), if \( D_{z_k} h(z) = 0 \) in \( \Omega \) for any \( k = 1, \ldots, m \), where
\[ D_{z_k} = \frac{1}{2}(D_{x_k} - \sqrt{-1}D_{y_k}), \quad D_{\bar{z}_k} = \frac{1}{2}(D_{x_k} + \sqrt{-1}D_{y_k}) \]
and \( z_k = x_k + \sqrt{-1}y_k \in \mathbb{C}, k = 1, \ldots, m \). The reader easily verifies that the following fact holds true.

**Proposition 1.** Let \( h = h(z) \in \mathcal{O}(\Omega) \). Then
\[ D_{x_k} \text{Re} \ h = D_{y_k} \text{Im} \ h = \text{Re} h'_{z_k}, \]
\[ D_{x_k} \text{Im} \ h = -D_{y_k} \text{Re} \ h = \text{Im} h'_{z_k} \]
and
\[ D^2_{x_i x_j} \text{Re} \ h = D^2_{x_i y_j} \text{Im} \ h = -D^2_{y_i y_j} \text{Im} \ h = \text{Re} h''_{z_i z_j}, \]
\[ D^2_{x_i x_j} \text{Im} \ h = -D^2_{x_i y_j} \text{Re} \ h = -D^2_{y_i y_j} \text{Re} \ h = \text{Im} h''_{z_i z_j} \]

It follows from (6) that
\[ |D_z h|^2 := \sum_{i=1}^{m} |h'_{z_i}|^2 = |D_{(x,y)} \text{Re} h|^2 = |D_{(x,y)} \text{Im} h|^2. \]

**Proposition 2.** Let \( f \) is defined by (5) and \( \mathcal{M}_f \neq \emptyset \). The surface \( \mathcal{M}_f \) is an (embedded) minimal hypersurface if and only if
\[ \text{Re} \sum_{i,j=1}^{m} h'_{z_i z_j} h'_{z_i z_j} h'_{z_i z_j} \equiv -|D_z h|^2 \Delta F - \Delta_1 F \mod (F - \text{Re} h) \]
holds whenever
\[ |D_{\xi} f|^2 = \sum_{i=1}^{m} |h'_{z_k}|^2 + |D_{\xi} F|^2 \neq 0. \]

**Proof.** Under the non-singularity assumption (9), it suffices to show that (4) is equivalent to (8). To this end, we notice that by Proposition 1
\[ f'_{x_i} = \text{Re} h'_{z_i}, \quad f'_{y_j} = -\text{Im} h'_{z_i}, \quad f'_{t_k} = -F_{t_k}. \]
Further, \( f_{x_i t_k}'' = f_{y_j t_k}'' = 0, \ f_{t_i t_j}'' = -F_{t_i t_j}'' \) and
\[
\begin{align*}
f_{x_i x_j}'' &= -f_{y_j y_i}'' = \text{Re} \ h_{z_i z_j}'', \\
f_{y_j y_i}' &= -f_{x_i x_j}'' = \text{Im} \ h_{z_i z_j}''.
\end{align*}
\]
This readily yields
\[
2m+k \sum_{i,j=1}^{m} f_{x_i x_j} f_{x_i x_j}'' = \text{Re} \sum_{i,j=1}^{m} h_{z_i z_j}' h_{z_i z_j}' - \sum_{i,j=1}^{k} F_{t_i t_j} F_{t_i t_j}''
\]
hence
\[
-\Delta_1 f = \sum_{k=1}^{m} |h_{z_k}'|^2 + |D F|^2 \Delta F + \text{Re} \sum_{i,j=1}^{m} h_{z_i z_j}' h_{z_i z_j}' - \sum_{i,j=1}^{k} F_{t_i t_j} F_{t_i t_j}''
\]
and the desired property follows. \( \square \)

3. Applications

3.1. The class \( \mathcal{T}^m \). Here we show some examples how Proposition 2 can be used to construct minimal hypersurfaces in \( \mathbb{R}^{2m} \).

Definition 1. We say that a holomorphic function \( h(z_1, \ldots, z_m) \) is \( \mathbb{R} \)-holomorphic equivalently write \( h \in \mathcal{T}^m \), if there exists a real valued function \( \mu : \mathbb{C}^m \to \mathbb{R} \) such that the relation
\[
(10) \sum_{i,j=1}^{m} h_{z_i z_j}' h_{z_i z_j}' = \mu(z) h(z)
\]
holds everywhere in the domain of holomorphy of \( h \).

It follows from the standard theory that if a function \( h \) is \( \mathbb{R} \)-holomorphic in some open set then it is \( \mathbb{R} \)-holomorphic everywhere in the domain of holomorphy. The importance of the introduced class follows from the proposition below.

Proposition 3. Let \( h(z) \in \mathcal{T}^m \). Then
\[
\mathcal{M}_{\text{Re} h} = \{ z \in \mathbb{C}^m = \mathbb{R}^{2m} : \text{Re} h(z) = 0, |D h(z)| \neq 0 \}
\]
is an embedded minimal hypersurface.

Proof. It follows from (10) that
\[
\text{Re} \sum_{i,j=1}^{m} h_{z_i z_j}' h_{z_i z_j}' = \mu(z) \text{Re} h(z),
\]
hence applying Proposition 2 to \( h(z) \) and \( F(z) \equiv 0 \) yields by (8) the required conclusion. \( \square \)

Though the case \( m = 1 \) is trivial (to get a non-trivial minimal hypersurface one need to have at least \( m = 2 \)), it still is very useful for the further constructions. We have the following complete classification of \( \mathcal{T}^1 \).
Proposition 4. Any element of $\mathcal{F}^1$ is either a binomial $h(z) = (az + b)^p$ or the exponential $h(z) = e^{pz}$, where $a, b \in \mathbb{C}$ and $p \in \mathbb{R}$.

Proof. Indeed, let $\Omega$ be the domain of holomorphy of $h(z)$. Then (10) yields
\[
\frac{|h''(z)|^2}{\mu(z)} = \frac{h''(z)h(z)}{h'^2(z)},
\]
where the right hand side is a meromorphic function in $\Omega$, while the left hand side is real-valued in $\Omega$. Thus, the both sides are constant in $\Omega$, say equal to $c \in \mathbb{R}$. This yields $ch'^2(z) = h''(z)h(z)$, or $h'(z) = ch(z)^b$ for some real $b$. This yields the required conclusions. $\square$

The following proposition shows that $\mathbb{R}$-holomorphic functions have a nice multiplicative structure.

Proposition 5. Let $h(z) \in \mathcal{F}^m$ and $g(w) \in \mathcal{F}^n$. Then

(i) $ch(z)^r \in \mathcal{F}^n$ for any $c \in \mathbb{C}$ and $r \in \mathbb{R}$;

(ii) $h(z)g(w) \in \mathcal{F}^{m+n}$.

(iii) $h(z)/g(w) \in \mathcal{F}^{m-n}$.

Proof. Setting $H(z) := h(z)^r$ one easily verifies that
\[
\sum_{i,j=1}^{m} \overline{H'_{z_i}H'_{z_j}} = r^2 \sum_{i,j=1}^{m} (|h|^{2r-4}|D_z h|^4 + \mu |h|^{2r-2})h^r = \mu_1 H,
\]
with $\mu_1 = r^2 \sum_{i,j=1}^{m} (|h|^{2r-4}|D_z h|^4 + \mu |h|^{2r-2})$, obviously a real-valued function, thus yielding $h^r \in \mathcal{F}^m$. Similarly one justifies $ch \in \mathcal{F}^m$ which yields (i).

Further, we have
\[
\sum_{i,j=1}^{m} \overline{H'_{z_i}H'_{z_j}} = \mu h, \quad \sum_{\alpha,\beta=1}^{n} \overline{g'_{w_\alpha}g'_{w_\beta}} = \nu g,
\]
where $\mu(z)$ and $\nu(w)$ are real-valued functions. Therefore, setting $H(z, w) := h(z)g(w)$ one obtains
\[
\sum_{k,l=1}^{m+n} \overline{H'_{z_k}H'_{z_l}} = |g|^2 g \sum_{i,j=1}^{m} \overline{H'_{z_i}H'_{z_j}} + |h|^2 h \sum_{\alpha,\beta=1}^{n} \overline{g'_{w_\alpha}g'_{w_\beta}} + 2|D_z h|^2 |D_w g|^2 h g
\]
\[
= (\mu |g|^2 + \nu |h|^2 + 2|D_z h|^2 |D_w g|^2) H,
\]
yielding (i). Finally, setting $r = -1$ and $c = 1$ in (i) implies that $1/h(z) \in \mathcal{F}^n$, thus together with (ii) implies (iii). $\square$

3.2. Examples.

Example 1. A trivial example of a $\mathbb{R}$-holomorphic function is a linear function $h(z_1, \ldots, z_m)$, where $\mu \equiv 0$. The corresponding minimal hypersurface is a hyperplane in $\mathbb{R}^{2m}$. Another simple example of a $\mathbb{R}$-holomorphic function is (by Proposition 4) the function $h(z_1) = e^{z_1}$. It can be used to construct a highly non-trivial examples as Corollary 1 below shows.
Example 2. A less trivial example is and the quadratic form
\[ h(z_1, \ldots, z_m) = z_1^2 + \ldots + z_m^2 \]
which satisfies (10) with \( \mu = 8 \). The corresponding minimal hypersurface is the Clifford cone
\[ \text{Re} \, h(z) = x_1^2 + \ldots + x_m^2 - y_1^2 - \ldots - y_m^2 = 0. \]
Observe that the non-singularity condition
\[ |Dz h|^2 = 4(|z_1|^2 + \ldots + |z_m|^2) \neq 0 \]
holds everywhere outside the origin of \( \mathbb{R}^{2m} \).

Example 3. Another interesting example is the cubic form
\[ h(z) = \det \begin{pmatrix} z_1 & z_2 & z_3 \\ z_4 & z_5 & z_6 \\ z_7 & z_8 & z_9 \end{pmatrix} = z_1z_5z_9 - z_1z_8z_6 - z_2z_4z_9 + z_2z_7z_6 + z_3z_8z_4 - z_3z_5z_7 \]
in which case \( \text{Re} \, h(z) \) is an irreducible cubic form in \( \mathbb{R}^{18} \). It is straightforward to verify that \( \mu = 2 \sum_{i=1}^{9} |z_i|^2 \) in (10). The equation \( \text{Re} \, h(z) = 0 \) yields a known example of a cubic minimal hypercone in \( \mathbb{R}^{18} \) (see also [?], [?]). The corresponding Hsiang algebra (REC-algebra in terminology of [?]) is isomorphic to the Jordan algebra of Hermitian 3 \( \times \) 3-matrices over \( \mathbb{C} \).

Below we briefly demonstrate how to use Examples 1-5 and Proposition 5 to constructing new examples. For instance, we have the following construction of minimal hypersurfaces in odd-dimensional ambient spaces.

Corollary 1. Let \( h(z) \in \mathcal{I}^m \). Then
\[ x_{2m+1} = \text{arg} \, h(z) \]
is a minimal hypersurface in \( \mathbb{R}^{2m+1} \).

Proof. Indeed, the function \( g(z_1, \ldots, z_m, z_{m+1}) := ie^{z_{m+1}}h(z_1, \ldots, z_m) \) is \( \mathbb{R} \)-holomorphic by Proposition 5 and Proposition 4. Then
\[ \text{Re} \, g = -e^{\text{Re} \, z_{m+1}}(\text{Re} \, h \sin \text{Im} \, z_{m+1} + \text{Im} \, h \cos \text{Im} \, z_{m+1}) \]
yields that \( \text{Re} \, g = 0 \) is equivalently defined by
\[ \text{Im} \, z_{m+1} = -\arctan \frac{\text{Im} \, h}{\text{Re} \, h} = -\text{arg} \, h + C \]
for some real constant \( C \). It is easily seen that the latter equation is equivalent to (11) up to an orthogonal transformation (a reflection) of \( \mathbb{R}^{2m+1} \). \( \square \)

Example 4. Setting \( h(z_1) = z_1 = x_1 + ix_2 \), (11) becomes \( x_3 = \arctan x_2/x_1 \), i.e. the classical helicoid. More generally, one has the following minimal hypersurface:
\[ x_{2m+1} = \text{arg}(z_1^{k_1} \ldots z_m^{k_m}), \quad k_i \in \mathbb{Z}. \]
Combining Example 2 and Proposition 5 yields.
Corollary 2. Let natural numbers $p_i$, $1 \leq i \leq m$, be subject to the GCD-condition $(p_1, \ldots, p_m) = 1$ and let $c \in \mathbb{C}^x$. Then the hypersurface
\[ \text{Re}(cz_1^{p_1} \cdots z_m^{p_m}) = 0, \]
is minimal (in general singularly) immersed cone in $\mathbb{R}^{2n} \cong \mathbb{C}^m$.

Example 5. For $c = 1$, Corollary 2 yields exactly the observation made earlier by H.B. Lawson [2, p. 352]. For instance, when $m = 2$ one obtains the well-known infinite family of immersed algebraic minimal Lawson’s hypercones $\text{Re}(z_1^p z_2^q) = 0$, $(p, q) = 1$, in $\mathbb{R}^4$. The intersection of such a cone with the unit sphere $S^3$ is an immersed minimal surface of Euler characteristic zero of $S^3$ [2].

Example 6. For $c = \sqrt{-1}$, Corollary 2 one obtains minimal hypersurfaces in $\mathbb{R}^{2m}$ of the following kind:
\[ \sum_{i=1}^m p_i \arctan \frac{y_k}{x_k} = 0 \]
which obviously is an algebraic minimal cone in $\mathbb{R}^{2m}$.

3.3. Open questions. Concluding this short paper, we emphasize again, that many more examples can be constructed by combining Examples 1-5 and Proposition 5. An interesting and important question in this direction is how to classify all $\mathbb{R}$-holomorphic functions?

Even some particular results could be interesting. For instance, all the above examples are obtained from simplest elementary blocks $h = z$ and $h = e^z$ by products (ratios) and exponentiations. Is it true that $\mathcal{F}$ is finitely generated?

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