On the Symmetry Algebra of the Discrete States
in $d < 2$ Closed String Theory

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**Abstract**

The symmetry charges associated with the Lian-Zuckerman states for $d < 2$ closed string theory are constructed. Unlike in the open string case, it is shown here that the symmetry charges commute among themselves and act trivially on all the physical states.
I. Introduction

In a beautiful paper, Lian and Zuckerman studied the basic building blocks of the physical states or the BRST cohomology for the \((p,q)\) minimal models coupled to two dimensional gravity \([1]\). Their stimulating result showed that apart from the usual physical states at ghost number zero \([2,3]\) (with the convention that the \(SL(2,C)\) invariant vacuum has ghost number \(-1\)), there are infinite number of physical states at different ghost numbers corresponding to the null vectors present in each conformal block. Some of these physical states at low ghost numbers have been constructed explicitly. For higher ghost number states, the explicit construction is more involved since one has to deal with null vectors at higher levels. Among the low lying ghost number states, the form of the physical states (Lian-Zuckerman states) at ghost number \((-1)\) (or physical operators at ghost number zero) are given in \([4,5]\). The short distance behaviour of the operator product expansions of these physical operators defines an interesting ring structure, the so called ‘ground ring’\([6]\). The importance of this algebraic structure, in the context of \(d=2\) string theory has been emphasized in \([7,8]\). In ref. \([9]\) it is observed that for \(d < 2\) string theory, there exists another generator \(w\) with ghost number \((-1)\) and its inverse \(w^{-1}\) with ghost number one. Furthermore, they proved that any power of \(w\) gives a non-trivial cohomology and thus the ghost number \(n\) sector is described by \(R_n = w^{-n} R_0\) where \(R_0\) is the ring of ghost number zero physical operators. The consequence of this is that for minimal models coupled to gravity, the full ring structure is \(R = R_0 \otimes \mathbb{C}[w, w^{-1}]\). This is found to be non-commutative since the generator \(w\) anticommutes with the generators of \(R_0\), which are usually denoted by \(x\) and \(y\). In a different approach, the physical states with non-zero ghost numbers are shown to be related to the ghost number zero states (Dotsenko-Kitazawa states \([10]\)) by making use of the descent equations of double-cohomology (i.e. the usual string BRST cohomology and Felder’s BRST cohomology) \([11]\). However, these latter type of physical states do not lie inside the primary conformal grid.

In the context of \(d = 2\) string theory, it has been pointed out by Witten \([7]\) that some special physical operators which belong to the zero eigenvalue space of the operator \(b_0\) give rise to the conserved currents modulo BRST commutator. In fact, it is shown that the conserved charges corresponding to the ghost number \((1,0)\) and \((0,1)\) operators generate
an area preserving diffeomorphism in the $x - y$ plane. This result is the reminiscent of the $W_\infty$ symmetry one encounters in $c = 1$ matrix models [12–15], but the clear relationship between these two $W_\infty$ symmetries is still lacking. For $(p, q)$ minimal models coupled to gravity, as we have mentioned the picture is quite different. However, it is probably reasonable to expect a $W^{(q)}$-type symmetry among the conserved charges in analogy with the $(q - 1)$ matrix model.

In the $d < 2$ open string case, the vector fields associated with the physical states have been constructed in ref.[9] and it is shown that their algebra contains a Virasoro algebra as the subalgebra. In this paper, we construct the vector fields for $d < 2$ closed string physical states, at arbitrary ghost numbers, belonging to both relative (annihilated by $b_0$) and semi-relative (annihilated by $b_0 - \bar{b}_0$) cohomology. We show that the vector fields commute among themselves unlike in the open string case. This in turn implies that the conserved currents act trivially on all the physical states. Hence it seems there is no new symmetry associated with these discrete states in the free closed string in $d < 2$.

The organization of our paper is as follows. In sec.II, we briefly review the physical state spectrum for $d < 2$ string theory and fix our notations and conventions. The vector fields of the conserved charges associated with the physical states are constructed in sec.III. In sec.IV, we present our conclusions.

**II. Physical State Spectrum for $d < 2$ Closed String Theory:**

The $(p, q)$ minimal models (where $p > q$ and $gcd(p, q) = 1$) coupled to two dimensional gravity can be described in terms of Coulomb gas representation where the energy momentum tensors for the matter and Liouville sector are given as

\[
T_M(z) = -\frac{1}{2} : \partial X \partial X : + iQ_M \partial^2 X \\
T_L(z) = -\frac{1}{2} : \partial \phi \partial \phi : + iQ_L \partial^2 \phi
\]

(2.1)

Here $X$ and $\phi$ represent the matter and Liouville fields respectively, whereas $2Q_M$ and $2Q_L$ denote the background charges. Since, we are working in the free theory with zero cosmological constant, we can concentrate only on the chiral sector and then finally combine the left and right moving part in order to obtain the full closed string spectrum. The
Virasoro central charges for the matter and Liouville part have the form

\[ c_M = 1 - 12Q_M^2 \]  
\[ c_L = 1 - 12Q_L^2 \]  

The vertex operators \( e^{ipMX} \) and \( e^{ipL\phi} \) have conformal weights \( \Delta(p_M) = 1 + \frac{1}{2} p_M (p_M - 2Q_M) \) and \( \Delta(p_L) = 1 + \frac{1}{2} p_L (p_L - 2Q_L) \) respectively. The screening charges for the matter and Liouville sectors are given by

\[ \alpha_\pm \equiv p_M^\pm = Q_M \pm \sqrt{Q_M^2 + 2} \]  
\[ \beta_\pm \equiv p_L^\pm = Q_L \pm \sqrt{Q_L^2 + 2} \]

From the above equations, we have \( \alpha_+ + \alpha_- = 2Q_M \equiv \alpha_0 \); \( \alpha_+ \alpha_- = -2 \) and similarly \( \beta_+ + \beta_- = 2Q_L \equiv \beta_0 \) and \( \beta_+ \beta_- = -2 \). Also, since the matter sector is the \((p, q)\) minimal models which are characterized by the Virasoro central charge \( 1 - \frac{6(p-q)^2}{pq} \), it, therefore, follows from (2.2a) that \( 2Q_M = \sqrt{\frac{2p}{q}} - \sqrt{\frac{2q}{p}} \) and the screening charges take the values

\[ \alpha_+ = \sqrt{\frac{2p}{q}} \]  
\[ \alpha_- = -\sqrt{\frac{2q}{p}} \]  

Since the central charge of the combined matter and Liouville system should add up to 26, we also have the relation from (2.2) that

\[ Q_M^2 + Q_L^2 = -2 \]  

From this, in terms of \( p, q \), the background charge for the Liouville sector is given by

\[ 2Q_L = i(\sqrt{\frac{2p}{q}} + \sqrt{\frac{2q}{p}}) = i(\alpha_+ - \alpha_-) \equiv (\beta_+ + \beta_-) = \beta_0. \]  
Therefore, the screening charges for the Liouville sector take the values

\[ \beta_+ = i\alpha_+ = i\sqrt{\frac{2p}{q}} \]  
\[ \beta_- = -i\alpha_- = i\sqrt{\frac{2q}{p}} \]
Let us also recall that the primary fields of the \((p, q)\) minimal models are represented by the vertex operators \(e^{i\alpha_{m,m'}X}\) having the conformal weights

\[
\Delta(\alpha_{m,m'}) = \frac{1}{2} \alpha_{m,m'}(\alpha_{m,m'} - 2Q_M) = \frac{1}{4pq} [(pm - qm')^2 - (p - q)^2] \tag{2.7}
\]

where \(1 \leq m \leq q - 1, \ 1 \leq m' \leq p - 1\) and choosing the negative root for \(\alpha_{m,m'}\) we have

\[
\alpha_{m,m'} = \frac{1}{2} [(1 - m)\alpha_+ + (1 - m')\alpha_-] \tag{2.8}
\]

The irreducible highest weight module of the matter sector is obtained by quotienting the Verma module over each primary by its maximum proper submodule. The submodule is generated by a pair of null vectors associated with the primary field. The irreducible Virasoro module can be represented by the following embedding diagram, \(E_{m,m'}:\)

\[
\begin{array}{cccccccc}
  & e_0 & \rightarrow & a_{-1} & \rightarrow & e_{-1} & \rightarrow & a_{-2} & \rightarrow & e_{-2} & \rightarrow & \cdots \\
  &  &  & \times &  & \times &  & \times &  & \times &  & \cdots \\
  & a_0 & \rightarrow & e_1 & \rightarrow & a_1 & \rightarrow & e_2 & \rightarrow & \cdots \\
\end{array}
\]

where

\[
\begin{align*}
  a_t &= \frac{1}{4pq} [(2pq + pm + qm')^2 - (p - q)^2] \\
  e_t &= \frac{1}{4pq} [(2pq + pm - qm')^2 - (p - q)^2] \tag{2.9}
\end{align*}
\]

Each node of the above diagram represents a Verma module with \(a_t\) or \(e_t\) as the dimension of its highest weight state and are null vectors over \(e_0\). An arrow connecting two spaces \(E \rightarrow F\) means that the module \(F\) is contained in the module \(E\). It was proved by Lian and Zuckerman [1] and also by Bouwknegt, McCarthy and Pilch [5] that there exists a unique physical state of the combined matter, Liouville and ghost system at each of these values of \(a_t\) and \(e_t\) if and only if the Liouville momenta satisfy the relation:

\[
\Delta(p_L) = \begin{cases} 1 - a_t \\ 1 - e_t \end{cases} \tag{2.10}
\]

and the ghost number of the state is given as

\[
n_{gh} = \pi(p_L) d(p_L) \tag{2.11}
\]

where \(\pi(p_L) = \text{sign} \ [i(p_L - Q_L)]\) and \(d(p_L)\) is the number of arrows from the top node \(e_0 \equiv \Delta(m,m')\) to that particular node \(a_t\) or \(e_t\). This proof is based on the BRST
quantization scheme where the states are taken to be in the space of conformal fields such that a physical state is BRST closed and it belongs to the zero eigenvalue space of $L^{tot}_0$. The BRST charge is defined as

$$Q = \sum_{n \in \mathbb{Z}} c_{-n} L_n^{(L+M)} - \frac{1}{2} \sum_{m,n \in \mathbb{Z}} (m - n) : c_{-m} c_{-n} b_{m+n} :$$  \hspace{1cm} (2.12)

where $b$, $c$ are the usual reparametrization antighost and ghost with conformal weight 2 and $-1$; and ghost number $-1$ and 1 respectively. The Virasoro generators are written in terms of the oscillators as

$$L_n^{(L)} = \frac{1}{2} \sum_{m \in \mathbb{Z}} : \phi_m \phi_{n-m} : - (n+1) Q L \phi_n$$  \hspace{1cm} (2.13a)

$$L_n^{(M)} = \frac{1}{2} \sum_{m \in \mathbb{Z}} : \alpha_m \alpha_{n-m} : - (n+1) Q M \alpha_n$$  \hspace{1cm} (2.13b)

$$L^{tot}_0 = L_0^{(L+M)} + \sum_{m \in \mathbb{Z}} m : c_{-m} b_m :$$  \hspace{1cm} (2.14)

In the space of conformal fields, a general physical operator takes the form

$$\mathcal{O} = \mathcal{P}(\partial X, \partial \phi, b, c) \ e^{i \alpha_{m,m'} X} \ e^{i \beta_{n,n'} \phi}$$  \hspace{1cm} (2.15)

where, because of the zero eigenvalue of $L^{tot}_0$, $\mathcal{P}$ is a differential polynomial of conformal weight $-\frac{1}{2} \alpha_{m,m'} (\alpha_{m,m'} - 2 Q M) - \frac{1}{2} \beta_{n,n'} (\beta_{n,n'} - 2 Q L)$. Here $\alpha_{m,m'}$ is as given in (2.8) and

$$\beta_{n,n'} = \frac{1}{2} [(1-n) \beta_+ + (1-n') \beta_-]$$  \hspace{1cm} (2.16)

are the Liouville momenta which are the solutions of (2.10). We note that $\alpha_{m,m'}$ has the symmetry $\alpha_{m\pm q,m'\pm p} = \alpha_{m,m'}$ whereas, $\beta_{n,n'}$ has different symmetry property namely, $\beta_{n\pm q,n'\mp p} = \beta_{n,n'}$. Also we have $\alpha_{m,m'} + \alpha_{n,n'} = \alpha_{m+n-1,m'+n'-1}$ and $\beta_{m,m'} + \beta_{n,n'} = \beta_{m+n-1,m'+n'-1}$.

The allowed Liouville momenta of the physical operators in (2.15), obtained by solving
(2.10) can be summarized as follows:

\[ p_L^\pm(a_{\pm t}) = \beta_{\mp qt \pm m, \mp pt \pm m'} \]
\[ p_L^\pm(a_{t-1}) = \beta_{\mp q(t-1) \pm m, \mp p(t-1) \pm m'} \]
\[ p_L^\pm(e_t) = \beta_{\mp qt \pm m, \mp pt \pm m'} \]
\[ p_L^\pm(e_{t-1}) = \beta_{\mp qt \pm m, \mp pt \pm m'} \]
\[ p_L^\pm(e_0) = \beta_{\mp m, \pm m'} \]

where \( t > 0 \), and \( p_L^\pm \) refer to the Liouville momentum greater than \( Q_L \) or less than \( Q_L \). The presence of the generator \( w \) can be seen by noting that all the allowed Liouville momenta can be split into three parts, each part associated with one generator, as follows:

\[ p_L^+(a_{\pm t}) = -2t\beta_w + (m' - 1)\beta_x + (m - 1)\beta_y \quad (2.18a) \]
\[ p_L^+(a_{t-1}) = -2t\beta_w + (p - m' - 1)\beta_x + (q - m - 1)\beta_y \quad (2.18b) \]
\[ p_L^+(e_{t-1}) = -(2t + 1)\beta_w + (p - m' - 1)\beta_x + (m - 1)\beta_y \quad (2.18c) \]
\[ p_L^+(e_t) = -(2t + 1)\beta_w + (m' - 1)\beta_x + (q - m - 1)\beta_y \quad (2.18d) \]
\[ p_L^+(e_0) = -\beta_w + (m' - 1)\beta_x + (q - m - 1)\beta_y \quad (2.18e) \]
\[ p_L^-(a_{\pm t}) = (2t + 1)\beta_w + (p - m' - 1)\beta_x + (m - 1)\beta_y \quad (2.18f) \]
\[ p_L^-(a_{t-1}) = (2t + 1)\beta_w + (m' - 1)\beta_x + (m - 1)\beta_y \quad (2.18g) \]
\[ p_L^-(e_{t-1}) = (2t - 1)\beta_w + (m' - 1)\beta_x + (q - m - 1)\beta_y \quad (2.18h) \]
\[ p_L^-(e_t) = (2t - 1)\beta_w + (p - m' - 1)\beta_x + (m - 1)\beta_y \quad (2.18i) \]
\[ p_L^-(e_0) = -\beta_w + (p - m' - 1)\beta_x + (m - 1)\beta_y \quad (2.18j) \]

where again we have taken \( t > 0 \) and we have defined \( \beta_x = \beta_{1,2} = -\frac{1}{2}\beta_- = -\frac{i}{2}\sqrt{\frac{2q}{p}}; \)
\( \beta_y = \beta_{2,1} = -\frac{1}{2}\beta_+ = -\frac{i}{2}\sqrt{\frac{2p}{q}} \) and \( \beta_w = \beta_{q+1,1} = \beta_{1,p+1} = -\frac{i}{2}\sqrt{2pq} \). Few comments are in order now. First of all, we note that as we fix the matter sector \((m, m')\) with \(1 \leq m \leq q - 1\) and \(1 \leq m' \leq p - 1\) in (2.15), the Liouville sector \((n, n')\) is no longer arbitrary but is getting fixed by (2.10). Secondly, the coefficient of \( \beta_w \) in (2.18) is precisely the ghost number of the operators appearing at different nodes \((a_t \text{ or } e_t)\) of the embedding diagram as can be obtained from (2.11). Finally, it is clear from (2.18) that the whole
spectrum of physical states is generated by the operators \( x, y \) and \( w \) (or \( w^{-1} \)). Since \( \beta_x \) and \( \beta_y \) are not associated with the ghost number, the physical operators \( x \) and \( y \) are spin zero, ghost number zero operators and are the generators of the ground ring \( R_0 \). In our conventions, their explicit forms are

\[
x = [bc + \frac{3}{4} \sqrt{\frac{2q}{p}} (i\partial X + \partial \phi)] e^{i\alpha_{1,2} X + i\beta_{1,2} \phi} \tag{2.19}
\]

\[
y = [bc - \frac{3}{4} \sqrt{\frac{2p}{q}} (i\partial X - \partial \phi)] e^{i\alpha_{2,1} X + i\beta_{2,1} \phi} \tag{2.20}
\]

The other generators \( w, w^{-1} \) in general would have the form,

\[
w = \mathcal{P}(\partial X, \partial \phi, b, c) e^{i\alpha_{q-1,1} X + i\beta_{1,p+1} \phi} \tag{2.21a}
\]

\[
w^{-1} = ce^{i\alpha_{q-1,1} X + i\beta_{1,p+1} \phi}
\]

or,

\[
w = \mathcal{P}(\partial X, \partial \phi, b, c) e^{i\alpha_{1,p-1} X + i\beta_{q+1,1} \phi} \tag{2.21b}
\]

\[
w^{-1} = ce^{i\alpha_{1,p-1} X + i\beta_{1,-p+1} \phi}
\]

where \( \mathcal{P} \) is a differential polynomial of conformal weight \( p + q - 1 \) and ghost number \( (-1) \).

It is clear from (2.18g) that the operators \( x \) and \( y \) belong to the matter sector \((1,2)\) and \((2,1)\) respectively and appear at the position of \( a_0 \), for \( p_L < Q_L \), in the embedding diagram.

Also from (2.18h) and (2.18i), we find that \( w \) belongs to either in \((q-1,1)\) or in \((1,p-1)\) matter sector as given in (2.21a) and (2.21b) and appears at \( e_-1 \) or \( e_1 \), accordingly, for \( p_L < Q_L \). Similarly, from (2.18j) and (2.18c), we see that \( w^{-1} \) belongs to the matter sector \((1,p-1)\) or \((q-1,1)\) for \( p_L < Q_L \) or \( p_L > Q_L \) respectively and appears at the node \( e_0 \) of the embedding diagram. We note that since \( w \cdot w^{-1} \sim I \), so their multiplication is well defined if we take \( w \) from (2.21a) and \( w^{-1} \) from (2.21b) or vice-versa. Notice from (2.18g, f) and (2.18a, b) that all even powers of \( w \) and \( w^{-1} \) belong to the matter sector. Similarly from the other equations in (2.18), it follows that odd powers of \( w \) and \( w^{-1} \) belong to either \((1,p-1)\) or \((q-1,1)\) matter sector. Since the explicit form of \( w \) will involve constructing the operator \( \mathcal{P} \) at level \((p + q - 1)\), so in general it will be quite complicated. But for small values of \( p \) and \( q \), it can indeed be constructed; for example, for \((3,2)\) model (pure Liouville gravity), it has the form \( [16, 17, 9] \)

\[
w = (\partial^2 b - 3\partial b \partial c - \frac{\sqrt{3}}{2} \partial b \partial \phi + \frac{\sqrt{3}}{2} b \partial^2 \phi) e^{\sqrt{3} \phi} \tag{2.22}
\]
For our purpose, the explicit forms are not so important. But as we have seen, in general, the Lian-Zuckerman operators can be written as

\[ O_{n,i,j} = w^n x^i y^j \]

(2.23)

where \( n \in \mathbb{Z} \) and \(-n\) refers to the ghost number of the operator. \( i, j \) are also integers with the restriction \( 0 \leq i \leq p - 2; 0 \leq j \leq q - 2 \). This restriction comes from the fact that the matter momentum of a physical operator should lie inside the Kac table. Note that the Liouville momentum does not necessarily remain inside the Kac table. The other fact to be noted is that the operators \( w, x \) and \( y \) satisfy the commutation relations among themselves as \( wx = -xw, wy = -yw \) but \( xy = yx \). Till now we discussed only the chiral operators. In order to obtain the closed string states we have to combine the holomorphic and anti-holomorphic operators in such a way that their momenta match. Obviously, the solution is to take the Lian-Zuckerman operators for \( d < 2 \) closed string theory as

\[ w^n x^i y^j \bar{w}^n \bar{x}^i \bar{y}^j \]

(2.24)

Since the corresponding states are annihilated by both \( b_0 \) and \( \bar{b}_0 \) separately, these states belong to the so-called relative cohomology of the closed string BRST operator. In the next section, we discuss the other physical states for the closed string theory and the symmetry algebra associated with them.

**III. Symmetry Algebra of \( d < 2 \) Closed String States:**

It is well known in the operator formalism that the definition of a physical state being annihilated by both \( b_0 \) and \( \bar{b}_0 \) is a stronger condition than necessary [18]. On the contrary, what is necessary is that a physical state should be annihilated by \( b_0 - \bar{b}_0 \). As a consequence, besides the operators in (2.24) which satisfy this condition trivially, there are more states (or operators) which are also annihilated by \( b_0 - \bar{b}_0 \) and hence are physical. This can be easily seen, as explained in [7], by the existence of an operator

\[ a = [Q, \phi] = (c \partial \phi + i Q_L \partial c) \]

(3.1)

and its antiholomorphic counterpart

\[ \bar{a} = [\bar{Q}, \phi] = (\bar{c} \partial \phi + i \bar{Q}_L \partial \bar{c}) \]

(3.2)
where $\tilde{Q}$ is the antiholomorphic part of the BRST operator $Q$ as defined in (2.12). Note that $aw = -wa$ and $a$ commutes with $x$ and $y$. Besides, $a$ and $\bar{a}$ have ghost number $(1,0)$ and $(0,1)$ respectively and both of them have conformal weight zero. Thus, we have the new operators

$$\Omega^{(0)} = (a + \bar{a}) w^n x^i y^j \bar{w}^n \bar{x}^i \bar{y}^j$$

which are also annihilated by $b_0 - \bar{b}_0$ and hence are physical operators. Note that they are not independently annihilated by either $b_0$ or $\bar{b}_0$ i.e. they do not belong to the relative cohomology but belong to the semi-relative cohomology of the full BRST charge. Following [8], one can construct the conserved currents or the one-form $\Omega^{(1)}$ from the operators in (3.3) by making use of the descent equation

$$d\Omega^{(0)} = \{Q + \tilde{Q}, \Omega^{(1)}\}$$

In other words, $\Omega^{(1)}$ is simply obtained up to BRST commutator by multiplying (3.3) with $b_{-1}$ and $\bar{b}_{-1}$ separately. Since, $\Omega^{(1)} = \Omega^{(1)}_z dz + \Omega^{(1)}_{\bar{z}} d\bar{z}$ where

$$\Omega^{(1)}_z = \frac{1}{2\pi i} \oint_z dz' b(z')\Omega^{(0)}(z)$$

$$\Omega^{(1)}_{\bar{z}} = \frac{-1}{2\pi i} \oint_{\bar{z}} d\bar{z}' \bar{b}(\bar{z}')\Omega^{(0)}(\bar{z})$$

the symmetry charges would be given by

$$\Omega = \oint \Omega^{(1)}$$

In the above, we have chosen the convention that

$$\frac{1}{2\pi i} \oint \frac{dz}{z} = -\frac{1}{2\pi i} \oint \frac{d\bar{z}}{\bar{z}} = 1$$

Now, one can find the action of the currents or the charges on various physical operators of the type (3.3) and we will give the explicit example later. It is interesting to note that the conserved charge associated with the physical operator $a + \bar{a}$ (i.e. operator of the type (3.3) with $n = i = j = 0$) measures the Liouville momentum of a physical state. Since the one-form associated with $(a + \bar{a})$ is

$$\Omega^{(1)} = \partial \phi dz + \bar{\partial} \phi d\bar{z}$$
so, the vector field of the corresponding conserved charge can be identified as

\[ \sqrt{-2pq}(w_\partial w + \frac{1}{p}x_\partial x + \frac{1}{q}y_\partial y - \bar{w}_\partial \bar{w} - \frac{1}{p}\bar{x}_\partial \bar{x} - \frac{1}{q}\bar{y}_\partial \bar{y}) \]  

(3.9)

It is clear that (3.9) will act trivially on all the physical states, since they have matching left and right Liouville momenta. By working out a few cases explicitly, it is easy to convince oneself that the vector field associated with the conserved charges of the operators \( w^n x^i y^j \) is

\[ (-1)^n \sqrt{-2pq}(n + \frac{i}{p} + \frac{j}{q})w^n x^i y^j \partial_a \]  

(3.10)

Using (3.9) and (3.10), it is now straightforward to write down the vector fields associated with the charges corresponding to both the operators of the type (2.24) and (3.3). Throwing away some overall factors they are respectively given by

\[ G_{n,i,j} = w^n x^i y^j \bar{w}^n \bar{x}^i \bar{y}^j (\partial_a - \partial_{\bar{a}}) \]  

(3.11)

and

\[ K_{n,i,j} = w^n x^i y^j \bar{w}^n \bar{x}^i \bar{y}^j \left[ w_\partial w + \frac{1}{p}x_\partial x + \frac{1}{q}y_\partial y - (n + \frac{i}{p} + \frac{j}{q})a_\partial a - (n + \frac{i}{p} + \frac{j}{q})\bar{a}_\partial \bar{a} \right. \]

\[ \left. - \bar{w}_\partial \bar{w} - \frac{1}{p}\bar{x}_\partial \bar{x} - \frac{1}{q}\bar{y}_\partial \bar{y} + (n + \frac{i}{p} + \frac{j}{q})\bar{a}_\partial \bar{a} \right] \]  

(3.12)

It is now a simple exercise to compute the symmetry algebra of the conserved charges and we find after introducing appropriate cocycle factors [9] that

\[ [G_{n,i,j}, G_{m,k,l}] = 0 \]

\[ [G_{n,i,j}, K_{m,k,l}] = 0 \]  

(3.13)

\[ [K_{n,i,j}, K_{m,k,l}] = 0 \]

Thus, the symmetry charges for the closed string physical states commute among themselves. This is significantly different from the open string case [9], where it is observed that the conserved charges \( K_{n,i,j} \) satisfy a Virasoro algebra when \( i = j = 0 \). Also, the algebra (3.13) does not agree with the algebra conjectured in ref.[19]. The implication of the result (3.13) is that, the currents corresponding to physical operators of the type (2.24) and (3.3)
annihilate the physical states. In general, it is very difficult to have an explicit verification of this statement. But let us consider the (3,2) model as a simpler example and see that this is indeed true. In this case, the matter sector is trivial since the only matter field is the identity field. Consider an operator of the type (3.3) with \( i = j = 0 \) and \( n = -1 \) in this model. Using the explicit form of \( w^{-1} \) from (2.21b) and \( a, \bar{a} \) in (3.1), (3.2), the physical operator \((a + \bar{a})w^{-1}\bar{w}^{-1}\) takes the form

\[
(a + \bar{a})w^{-1}\bar{w}^{-1} = \frac{1}{2\sqrt{3}}(\partial c\bar{c} + \bar{\partial} c\bar{c})e^{-\sqrt{3}\phi}
\]  
(3.14)

The conserved charge associated with this operator is found, using (3.5) and (3.6), to be

\[
\Omega = \frac{1}{2\pi i} \oint dz(\partial c\bar{c} + \bar{\partial} c\bar{c})e^{-\sqrt{3}\phi} - \frac{1}{2\pi i} \oint d\bar{z}(\partial c\bar{c} + \bar{\partial} c\bar{c})e^{-\sqrt{3}\phi}
\]  
(3.15)

where an overall factor of \( \frac{1}{2\sqrt{3}} \) is thrown out. Working out the relevant OPEs, it is easy to check that \( \Omega \) acts trivially on \( W = w\bar{w}, X = x\bar{x} \) and \( (a + \bar{a}) \). The generator \( Y = y\bar{y} \) is absent in this model. Thus, we conclude that the symmetry charges of the \( d < 2 \) free closed string states act trivially on all the physical states.

**IV. Conclusion**

The algebra of the symmetry charges (3.13) is not surprising in the sense that one cannot construct a non-trivial correlator with the free closed string physical states (or operators) given in (2.24) and (3.3). In order to have a non-vanishing correlator, one should be able to construct an operator with ghost number six and matter and Liouville charge equal to the background charges \( \alpha_0 \) and \( \beta_0 \). From the ghost number counting we note that the above condition can be satisfied if one includes the operators of the type \( a\bar{a}w^n x^i y^j \bar{w}^n\bar{x}^i\bar{y}^j \). But, these operators are not present in the closed string cohomology as they are not annihilated by \((b_0 - \bar{b}_0)\).

To conclude we have shown here that the symmetry charges associated with the discrete states in \( d < 2 \) free closed string theory commute among themselves. This in turn implies that they act trivially on all the physical states. It would be interesting to examine how this conclusion is modified when the cosmological constant is taken to be non-zero and whether the symmetry charges in that case reveal any interesting algebraic structure in analogy with the matrix model results.
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