Semiclassical pair production rate for time-dependent electrical fields with more than one component: -WKB-approach and world-line instantons

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We present an analytic calculation of the semiclassical electron-positron pair creation rate by time-dependent electrical fields. We use two methods, first the imaginary time method in the WKB-approximation and second the world-line instanton approach. The analytic tools for both methods are generalized to time-dependent electric fields with more than one component. For the WKB method an expansion of the momentum spectrum of produced pairs around the canonical momentum $\vec{P} = 0$ is presented which simplifies the computation of the pair creation rate. We argue that the world-line instanton method of [1] implicitly performs this expansion of the momentum spectrum around $\vec{P} = 0$. Accordingly the generalization to more than one component is shown to agree with the WKB result obtained via this expansion. However the expansion is only a good approximation for the cases where the momentum spectrum is peaked around $\vec{P} = 0$. Thus the expanded WKB result and the world-line instanton method of [1] as well as the generalized method presented here are only applicable in these cases. We study the two component case of a rotating electric field and find a new analytic closed form for the momentum spectrum using the generalized WKB method. The momentum spectrum for this field is not peaked around $\vec{P} = 0$.

Keywords: Strong electric field, Electron-positron pair production, Semiclassical methods

Introduction

Since Sauter in 1931 [2] and Heisenberg and Euler [3] four years later gave a first description of the vacuum properties of QED, there have been a lot of investigations of the pair creation rate in strong electric fields. In particular, Schwinger [4-6] reformulated their result in an elegant way using quantum-field theoretic methods (see also [7, 8]).

The formulation was extended to space-time-dependent fields using different methods, e.g. the imaginary time method [9-14] and a tunneling picture [15, 16], both using WKB-approximations or the world-line instanton method [17]. By comparing numerical with analytic results it was found that for more complicated field configurations, i.e. for those which have more than one distinct pair of semiclassical turning points, interference effects arise. This was already discussed as a resonance effect for oscillating fields in [18]. Interference effects were recently studied in [19, 20] for the WKB-method and in [21] for the world-line instanton approach. In this paper we consider only fields with one dominant pair of turning points where interference effects are negligible. This enables us to use scalar quantum-electro dynamics, since it is known to give the same results as spinor quantum electrodynamics at the leading non-perturbative order if there are no interference effects [20].

All the analytic methods mentioned above give the same results for electric fields with only one component depending either on space or time. A more general case, namely electric fields with two or three components depending on space was discussed in [22] in the world-line instanton approach.

A special case namely a (two component) rotating electrical field was discussed in [12]. Recently pair production in rotating fields has been studied numerically in [23] using the Wigner formalism. These results can be used to calculate the pair creation rate of a plane wave in a plasma as shown in [24].

So far, electron-positron pair production has not been directly observed in experiments due to the necessity of high field strengths which are out of the range reached by nowadays laser systems. However recent theoretical investigations have shown that less strong fields are needed if one uses carefully-shaped multi-component laser pulses [25-32].

For this reason, we generalize the above mentioned analytic methods to compute the pair creation rate for a general time-dependent periodic electrical field which is characterized by the potential

$$A_\mu(t) = [0, A_1(t), A_2(t), A_3(t)] = \frac{1}{e c}[0, V_1(t), V_2(t), V_3(t)]. \tag{1}$$
To do so we use the WKB-approximation as well as the world-line instanton method of [1]. As is well known, the WKB approach the pair creation rate per volume $V$ takes the general form (see, e.g., [15, 16])

$$\frac{\Gamma_{\text{WKB}}}{V} \sim \int \frac{d^3P}{(2\pi\hbar)^3} \exp \left(-\frac{\pi E_c}{E_0} G(\vec{P})\right).$$

where the integral over $\vec{P}$ is over the momentum modes of the produced pairs. We introduce the critical electrical field

$$E_c = \frac{m^2c^3}{e\hbar}. \quad (3)$$

In Eq. (2) $E_0$ is a characteristic electric field strength and $G(\vec{P})$ is a function depending on the explicit form of the electric field, which is straightforwardly generalized to more than one component.

We find that if the momentum spectrum $\exp(-\pi E_c/E_0 G(\vec{P}))$ is peaked around zero canonical momentum $\vec{P} = 0$ it can be approximated by expanding around this point and it is possible to simplify the result via Gaussian integration. In the world-line instanton framework of [1] the momentum, arising as an integration constant, was implicitly taken to vanish with a Gaussian momentum integration producing the prefactors, as discussed in [21]. We argue that this is a de facto expansion around $\vec{P} = 0$. We generalize this method to the case of electric fields with more than one component and show that the result agrees with the WKB result expanded around $\vec{P} = 0$.

Looking at examples of electric fields with two components we find that the momentum spectrum is not necessarily peaked around zero momentum. In these cases the expanded WKB result as well as the equivalent world-line instanton method based on the one of [1] cannot be applied, because they do not represent a good approximation. In general it becomes obvious that to compute the pair creation rate one needs to have knowledge about the momentum spectrum. This paper is arranged as follows. In Section I we compute the pair creation rate in the WKB approximation for fields with one to three components. We repeat the same for the world-line instanton approach in Section II. In Section III we compare the two methods. We study some examples of interest in Section IV. Section V contains our conclusions and remarks. In order to make the text and ideas more transparent, we relegate some of the technical calculations to A. In B we study the value of the Morse index which is important for the calculations in the world-line instanton approach.

I. PAIR PRODUCTION RATE FOR ELECTRIC FIELDS DEPENDING ON TIME IN THE WKB APPROXIMATION

Here we briefly review the computation of the pair creation rate for time-dependent fields in the WKB-approximation [9–14, 20]. In this case the Klein-Gordon equation reduces to an effective Schrödinger equation. The pair creation rate can thus be connected to the reflection coefficient of a scattering problem.

In Section I A we recall the calculation of the WKB momentum spectrum. It depends on integrals between conjugated pairs of complex turning points in analogy to [19, 20]. If there is more than one pair of turning points interference effects can occur. These are governed by integrals between these different pairs. For the scope of this paper we will however concentrate on the case of one dominant pair of turning points for which interference is negligible. This also enables one to use scalar quantum electro dynamics. Since as shown in [20] for the case of no interference effects the results obtained in this way are equivalent to the ones of spinor quantum electrodynamics at the leading non perturbative order.

In Section I B we show how to simplify the integration between the turning points for analytic purposes. This involves a generalization of a well known variable substitution of the one component case. The pair production rate can be calculated from the transmission probability via an integration over the momentum spectrum as discussed in Section I C. For the comparison of the WKB results with the world-line instanton method we expand the momentum spectrum around $\vec{P} = 0$. By doing so it is possible to perform a Gaussian integration in the momentum space.

A. Momentum spectrum in the WKB approximation

In this Section we shortly recall the computation of the momentum spectrum within in the WKB method [9–14, 20]. We start from the Klein-Gordon equation

$$\left(i\hbar \partial_\mu + eA_\mu(t)\right)^2 - m^2c^2 \phi(x, t) = 0. \quad (4)$$
where the electromagnetical potential takes the form $\hat{s}$. Now the scalar field operator can be decomposed as

$$\hat{\phi}(x,t) = \int \frac{d^3P}{(2\pi\hbar)^3} e^{i\vec{P}\cdot\vec{x}} \left( \phi_\rho(t) \hat{a}_\rho + \phi^*_\rho(t) \hat{b}^\dagger_{-\rho} \right),$$

(5)

where $\hat{a}_\rho$ and $\hat{b}^\dagger_{-\rho}$ are bosonic creation and annihilation operators. The Klein-Gordon equation (4) for the modes becomes

$$\left( -\hbar^2 \partial_t^2 - (\mathcal{E}(t))^2 \right) \phi_\rho(t) = 0,$$

(6)

where we define

$$\mathcal{E}(t)^2 = [cP_j - V_j(t)]^2 + m^2 c^4.$$

(7)

One can now perform a Bogoliubov transformation to time-dependent creation and annihilation operators

$$\hat{c}_\rho(t) = \alpha_\rho(t) \hat{a}_\rho + \beta^*_\rho(t) \hat{b}^\dagger_{-\rho}, \quad \hat{d}^\dagger_{-\rho}(t) = \beta_\rho(t) \hat{a}_\rho + \alpha^*_\rho(t) \hat{b}^\dagger_{-\rho}.$$

(8)

The number of produced pairs for each canonical momentum $\vec{P}$ is now given by the transmission probability

$$W_{\text{WKB}}(\vec{P}) := \lim_{t \to \infty} |\beta_\rho(t)|^2 = \lim_{t \to \infty} \frac{|R_\rho(t)|^2}{1 - |R_\rho(t)|^2} \approx \lim_{t \to \infty} |R_\rho(t)|^2,$$

(9)

which can be connected to the reflection amplitude $R_\rho$ becomes the Riccati equation (see, e.g., [20])

$$\dot{R}_\rho(t) = \frac{\dot{\mathcal{E}}(t)}{2\mathcal{E}(t)} \left[ \exp \left( -2i \frac{\hbar}{\epsilon} \int^t_0 \mathcal{E}(t') dt' \right) - (R_\rho(t))^2 \exp \left( 2i \frac{\hbar}{\epsilon} \int^t_0 \mathcal{E}(t') dt' \right) \right].$$

(10)

For small $R_\rho(t)$ one can ignore the second non-linear term and approximately solve the Riccati equation (10) by integrating

$$\lim_{t \to \infty} R_\rho(t) = \int^{\infty}_{-\infty} \frac{\dot{\mathcal{E}}(t)}{2\mathcal{E}(t)} \exp \left( -2i \frac{\hbar}{\epsilon} \int^t_0 \mathcal{E}(t') dt' \right) dt.$$

(11)

This integral is dominated by the neighborhoods of the turning points $t^\pm_\rho$ defined by

$$\mathcal{E}(t^\pm_\rho) = 0.$$

(12)

It is obvious from Eqs. (7) and (12) that these turning points are momentum dependent and do not take real values. As was discussed in [23] tunneling paths for time-dependent potentials can be described with the help of imaginary times. This “imaginary time method” was applied to the case of pair production in [10–13]. From the definition of the turning points in Eq. (12) we however find that $t^\pm_\rho$ are not necessarily purely imaginary, but are found in conjugated pairs in the complex plane. The turning points are purely imaginary only for potentials which are odd functions of the time. As was already discussed in [21] this is true for the cases which are normally treated in the “imaginary time method” namely $V_1(t) = E_0 t$, $V_1(t) = E_0 / \omega \sin(\omega t)$ and $V_1(t) = E_0 / \omega \tanh(\omega t)$. In general it is however necessary to allow complex values for the turning points.

The reflection coefficient can be evaluated as the sum over turning points. However the approximation of ignoring the second non-linear term in the Riccati equation (10) can lead to a wrong prefactor. By also considering this term one finds (see [20] for details)

$$\lim_{t \to \infty} R_\rho(t) \approx \sum_{t^\pm_\rho} \exp \left( -2i \frac{\hbar}{\epsilon} \int^{t^\pm_\rho}_0 \mathcal{E}(t') dt' \right).$$

(13)

We can now split the integral in real parts along the imaginary axis and imaginary parts along the real axis. In order to do so we define the real part of the turning points $s_\rho$ and the phase integral $\theta(s, s')$ as

$$s_\rho = \text{Re}(t^\pm_\rho), \quad \theta(s, s') = \frac{1}{\hbar} \int^s_{s'} \mathcal{E}(t') dt'.$$

(14)
This allows us to introduce a global phase connected to the first turning point \( t_i^\pm \)

\[
\lim_{t \to \infty} R_{\vec{P}}(t) \approx C \cdot e^{-2i\theta(-\infty, s_i)} \sum_{t_{p_i}^\pm} e^{-2i\theta(s_1, s_p)} \exp \left( -\frac{2}{\hbar} \int_{s_p}^{t_{p_i}^\pm} \kappa(t') dt' \right),
\]

where we introduce

\[
\kappa(t) = \sqrt{-\mathcal{E}(t)^2}.
\]

Now the momentum spectrum of the pair creation rate takes the form [20]

\[
W_{\text{WKB}}(\vec{P}) \approx \lim_{t \to \infty} |R_{\vec{P}}(t)|^2 = \sum_{t_{p_i}^\pm} e^{-2K(t_{p_i}^\pm)} + \sum_{t_{p_i}^\pm \neq t_{p_i}^\pm} 2 \cos(2\theta(s_p, s_{p'})) e^{-K(t_{p_i}^\pm)} e^{-K(t_{p_i}^\pm)},
\]

where we introduce the integral

\[
K(t_{p_i}^\pm) = \frac{1}{\hbar} \int_{t_{p_i}^\pm}^{t_{p_i}^\pm} \kappa(t') dt'.
\]

As described in [20] the first term is related to the pair production for every distinct pair of turning points whereas the second term is related to the interference between the respective turning points \( t_{p_i}^\pm \) and \( t_{p_i}^\pm \). In the following we will concentrate how to best calculate the integral \( K(t_{p_i}^-) \) for the special case that there is one dominant pair of turning points.

**B. Calculation of the integral \( K(t_p) \)**

In this Section we want to introduce analytic tools to calculate the integral \( K(t_p) \) which is defined in Eq. [18]. To enhance the compatibility to existing literature (especially the “imaginary time” and the world-line instanton method of [10, 11, 14] and [1, 17] respectively) we choose to work in natural units in which energies are measured in units of \( mc^2 \) and introduce the adiabatic parameter

\[
\gamma := \frac{m \omega c}{E_0},
\]

as well as the frequency \( \omega \). We can now write the potentials as

\[
V_j(t) = \frac{f_j(\omega t)}{\gamma}
\]

for \( j = 1, 2, 3 \). The WKB transmission probability for one pair of turning points is given by

\[
W_{\text{WKB}}(\vec{P}) = \exp \left( -\frac{\pi E_0}{E_0} G(\vec{P}, \gamma) \right),
\]

where we define the integral

\[
G(\vec{P}, \gamma) = \frac{2}{\pi} \frac{E_0}{E_c} K(t_{p_i}^\pm).
\]

For the calculation of the integral we change the integration variable by analogy with a change of variable in the one-component case (see, e.g., [15]). In order to do so we define the function

\[
F(\vec{P}, t) = \sqrt{[\gamma c P_1 - f_1(t)]^2 + [\gamma c P_2 - f_2(t)]^2 + [\gamma c P_3 - f_3(t)]^2}
\]

and with its help change the integration variable to

\[
\tau = \pm \frac{F(\vec{P}, \omega t)}{\gamma},
\]
such that \( \tau(\vec{P}, t^\pm) = 1 \). Observe that unlike in the one-dimensional case the value of \( \tau \) is 1 at both turning points, which would result in a vanishing integral for an integration between these two points. To resolve this problem we have to choose the sign of \( \tau \) carefully. For the integration from \( t_p^- \) to \( s_p \) we choose the negative sign whereas for the integration from \( t = s_p \) to \( t_p^+ \) we choose the plus sign. Here \( s_p \) is the real part of the pair of turning points \( t_p \) defined in Eq. (14). Because of the symmetry of the problem these two integrals have the same value and we can summarize them in a single one from \( \tau_0 = \tau(s_p) \) to \( \tau = 1 \). We find

\[
G(\vec{P}, \gamma) = i \frac{\omega}{\gamma^2} \frac{2}{\pi} \int_{\omega t_p^-}^{\omega t_p^+} dt \sqrt{(\gamma c P_j - f_j(t))^2 + \gamma^2}
\]

\[
= \frac{4}{\pi} \int_{\tau_0}^{1} d\tau \frac{\sqrt{1 - \tau^2}}{F(\vec{P}, -i \gamma \tau)},
\]

where

\[
F(\vec{P}, z) := \frac{\partial}{\partial t} F(\vec{P}, t) \bigg|_{t = F^{-1}(\vec{P}, z)}
\]

is the derivative of the function \( F(\vec{P}, t) \) defined in Eq. (23) re-expressed as a function of \( \tau \). This function is only uniquely defined for one distinct pair of turning points \( t_p^\pm \). If there is more than one of these pairs we would find a function \( F_{t_p^\pm}(\vec{P}, z) \) for each pair \( t_p^\pm \).

**C. Pair production rate for time-dependent electric fields**

The pair production rate per volume \( V \) can be found by integrating the momentum spectrum defined in Eq. (21) over all the possible momenta with respect to energy momentum conservation for multiphoton absorption. For \( \gamma \ll 1 \) the photon energy spectrum becomes virtually continuous [14, 18]. This leads to

\[
\Gamma_{WKB} V \approx D_s \hbar \omega \int \frac{d^3 P}{(2\pi \hbar)^3} W_{WKB}(\vec{P}) = D_s \hbar \omega \int \frac{d^3 P}{(2\pi \hbar)^3} \exp \left( -\pi \frac{E_c}{E_0} G(\vec{P}, \gamma) \right).
\]

(28)

Here \( D_s \) is a factor connected to the spin of the particles. For electrons with two spin orientations it is equal to 2 [15, 18]. For comparison with the world-line instanton method of Section II it is useful to expand Eq. (22) around \( \vec{P} = 0 \). The explicit calculations for this expansion are performed in A. We find

\[
G(\vec{P}, \gamma) = G(\vec{0}, \gamma) + \frac{1}{2} c P_j G_{jk}(\gamma)c P_k + \cdots,
\]

(29)

where the linear contributions

\[
\frac{\partial G(\vec{P}, \gamma)}{\partial c P_j} \bigg|_{\vec{P} = 0} = 0,
\]

vanish for \( j = 1, 2, 3 \) following from Eq. (A1). We also define

\[
G_{jk}(\gamma) := \frac{\partial^2 G(\vec{P}, \gamma)}{\partial (c P_j) \partial (c P_k)} \bigg|_{\vec{P} = 0}
\]

\[
= \delta_{jk} \frac{4}{\pi} \int_{\tau_0}^{1} \frac{1}{\sqrt{1 - \tau^2}} \frac{1}{F(\vec{0}, -i \gamma \tau)} d\tau + \frac{1}{\gamma^2} \frac{4}{\pi} \int_{\tau_0}^{1} \frac{1}{\sqrt{1 - \tau^2}} \frac{\partial}{\partial \tau} \left( -i \gamma \tau \right) d\tau
\]

(31)

for \( j, k = 1, 2, 3 \) following from Eq. (A5). We also define

\[
F_j(z) := f_j(F^{-1}(\vec{0}, z)).
\]

(32)
After a Gaussian integration the pair creation rate (28) takes the form
\[ \frac{\Gamma_{WKB}}{V} \approx \frac{\Gamma_{WKB}^{\gamma}}{V} := D_s \hbar \omega \left( \frac{mc}{2\pi \hbar} \right)^3 \left( \frac{E_0}{E_c} \right)^{3/2} \exp \left( -\pi \frac{E_0}{E_c} G(\bar{0}, \gamma) \right) \sqrt{\det \left[ \frac{1}{2} G_{ij}(\gamma) \right]}, \]
which is only true if \( G_{ij}(\gamma) \) is a positive definite matrix.

Observe that the expansion discussed here is not always physically justified as can be seen for the example of rotating electrical fields which will be discussed in Sections IV A and IV B.

II. WORLD-LINE INSTANTON PAIR CREATION RATE FOR ELECTRIC FIELDS DEPENDING ON TIME

Following the ideas presented in [1, 17] we start from the Euclidean effective action in the world-line path integral formulation [34, 35]
\[ \Gamma_{\text{Eucl}}[A] = -\int_0^\infty \frac{dT}{T} e^{-T/\hbar} \int_{x(0)}^{x(T)} Dx \exp \left[ -\frac{1}{\hbar} \int_0^T d\tau \left( \frac{\dot{x}^2}{4} + ieA \cdot \dot{x} \right) \right], \]
where the path integral \( \int Dx \) is over all closed Euclidean space-time paths \( x^\mu(\tau) \) with period \( T \) in the proper time \( \tau \). As is well known the pair production rate is connected to the imaginary part of the Minkowski action which can be connected to the Euclidean action (35) for time-dependent fields as [1, 35]
\[ \Gamma = 1 - e^{-2\text{Im}(\Gamma_{\text{Mink}})} \approx \text{Im}(\Gamma_{\text{Mink}}) = \text{Re}(\Gamma_{\text{Eucl}}). \]

The classical Euler-Lagrange equations take the form
\[ m \ddot{x}_\mu = 2ieF_{\mu\nu}(x)\dot{x}_\nu, \]
where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the electromagnetic field strength tensor. For classical solutions
\[ (\dot{x}^\text{cl})^2 = a^2 = \text{const.} \]
follows directly from the antisymmetry of \( F_{\mu\nu} \) together with Eq. (37) by multiplying with \( x_\mu \). Periodic solutions of Eq. (37) are called world-line instantons.

As described in [21] in general these world-line instantons are complex and start and end their trajectories at the semiclassical turning points defined in Eq. (12). If there is more than one distinct pair of turning points the closed trajectories of these instantons may also include interference segments between these pairs. As in the WKB approach we concentrate on potentials with one dominant pair of turning points in this work. For these cases interference effects are negligible.

To sum over all closed loops one can choose to fix a point \( x(0) \) on the loop and allow the loop to fluctuate everywhere but at this point. One now expands
\[ x_\mu(\tau) = \dot{x}^\text{cl}_\mu(\tau) + \eta_\mu(\tau), \]
where the fluctuations \( \eta_\mu \) vanish at \( x(0) \)
\[ \eta_\mu(0) = \eta_\mu(T) = 0. \]
and follow the Jacobi equations [35, 36]
\[ \Lambda_{\mu\nu}\eta_\nu = 0, \]
where the fluctuation operator \( \Lambda_{\mu\nu} \) is defined by
\[ \Lambda_{\mu\nu} = -\frac{1}{2} \delta_{\mu\nu} \frac{d^2}{d\tau^2} - \frac{d}{d\tau} Q_{\mu\nu} + Q_{\mu\nu} \frac{d}{d\tau} + R_{\mu\nu}, \]
where
\[ Q_{\mu\nu} = c^2 \frac{\partial^2 L}{\partial x_\mu \partial \dot{x}_\nu}, \quad R_{\mu\nu} = c^2 \frac{\partial^2 L}{\partial x_\mu \partial x_\nu}, \] (43)

After integrating over \( x^{(0)} \) and using the Gelfand-Yaglom method following \[35\] the semiclassical approximation of the path integral Eq. (35) can be written as \[1, 35, 37\]
\[ \Gamma_{\text{Eucl}}[A] = -\int_0^\infty \frac{dT}{T} \int \frac{d^4 x^{(0)}}{(\hbar c)^4} e^{-T/h} \left( \frac{\hbar}{2\pi T} \right)^2 e^{i\theta} e^{-S[x^{(4)}(T)]/\hbar} \sqrt{\text{det} \left( \eta^{(\nu)}_{\mu, \text{free}}(T) \right) / \text{det} \left( \eta^{(\nu)}_{\mu}(T) \right)}, \] (44)

where \( \eta^{(\nu)}_{\mu}(\tau) \) is the solution to the Jacobi equation \[41\] with the initial conditions
\[ \eta^{(\nu)}_{\mu}(0) = 0, \quad \dot{\eta^{(\nu)}_{\mu}}(0) = \delta_{\mu\nu}. \] (45)
The free operator is defined by
\[ \Lambda_{\mu\nu} = -\frac{1}{2} \delta_{\mu\nu} \frac{d}{d\tau^2} \] (46)
such that
\[ \text{det} \left( \eta^{(\nu)}_{\mu, \text{free}}(T) \right) = T^4 \] (47)
and the phase factor \( e^{i\theta} \) is determined by the Morse index of the operator \( \Lambda \) \[1, 35–37\].

This framework was used in \[1\] to calculate the pair creation rate for one component fields depending either on space or on time and generalized to two and three component fields depending on space in \[22\]. In the following we will study the generalization to three component fields depending on time.

With this method one cannot obtain the momentum spectrum of the pair creation rate. This would however be possible if one starts from the world-line path integral
\[ \Gamma_{\text{Eucl}}[A] = -\int_0^\infty \frac{dT}{T} \int_{x(T)=x(0)} \mathcal{D}x \int \mathcal{D}p \exp \left[ -\frac{1}{\hbar} \int_0^T d\tau \left( \dot{x} \cdot p - \frac{1}{2} (cp + ceA)^2 \right) \right], \] (48)
instead of Eq. (35). This has been done for the one component case in \[21\]. As argued there the version of \[1\] following from the world-line path integral \[35\] takes the momenta, arising as integration constants, to be zero. The prefactor is produced by the Gaussian integration performed in Eq. (44). This can be seen as an implicit expansion of the momentum spectrum around \( \vec{P} = 0 \). We leave the investigation of the world-line instanton momentum spectrum in the general three component case for future work.

**A. Classical solutions for three-dimensional electrical fields depending on time**

We start from the four potential \[1\] in Euclidean form and by analogy with Eq. (20) use
\[ V_j(x_0) = -\frac{1}{\gamma} \int_j \left( \frac{\omega}{c} x_0 \right), \] (49)
where \( j = 1, 2, 3 \). The classical Euler-Lagrange equations \[37\] can be written explicitly as
\[ m\ddot{x}_0 = 2eE_0 \frac{\gamma}{c} \int_j \left( \frac{\omega}{c} x_0 \right) \dot{x}_j, \] (50)
\[ m\ddot{x}_j = -2eE_0 \frac{\gamma}{c} \int_j \left( \frac{\omega}{c} x_0 \right) \dot{x}_0. \] (51)
The last three equations can be directly integrated
\[ \dot{x}_j^{(\text{cl})} = -\frac{2eE_0 c^2}{\omega} \int_j \left( \frac{\omega}{c} x_0^{(\text{cl})} \right). \] (52)
Whereas with the help of Eq. [38] the first one can be solved as

$$\dot{x}_0^{(1)} = \pm a \sqrt{1 - \left( \frac{\dot{f}_j \left( \frac{x_0^{(1)}}{\gamma} \right)}{\gamma} \right)^2},$$

(53)

where like in [1] we define

$$\gamma = \frac{a \omega}{2eE_0 c^2} = \frac{a}{2c} \gamma.$$  

(54)

B. The fluctuation determinant

The fluctuation operator [42] takes the form

$$\Lambda_{\mu \nu} = -\frac{1}{2} \left( \frac{d^2}{dr^2} - \frac{d}{dr} \left( \frac{x_j^{(1)}}{\tau} \right) \right) \left( \frac{d}{dr} \right)^2 - \frac{d}{dr} \left( \frac{x_j^{(1)}}{\tau} \right) \left( \frac{d}{dr} \right)^2 \delta_{lm} \frac{d^2}{dr^2}.$$  

(55)

We obtain the 8 independent solutions to the Jacobi equation [41]

$$\phi^{(0)}(\tau) = \left( \frac{x_j^{(1)}(T)}{\tau} \right),$$  

(56)

$$\phi^{(j)}(\tau) = \left( \frac{x_j^{(1)}(T)}{\tau} \right),$$  

(57)

$$\phi^{(3+j)}(\tau) = \left( 0 \right),$$  

(58)

$$\phi^{(7)}(\tau) = \left( \frac{x_j^{(1)}(T)}{\tau} \right),$$  

(59)

where we define the integrals

$$I(\tau) = \int_0^\tau \frac{dt}{\left[ \dot{x}_0^{(1)}(t) \right]^2},$$  

$$I_j(\tau) = \int_0^\tau \frac{dt}{\dot{x}_j^{(1)}(t)}$$  

$$I_{jk}(\tau) = \int_0^\tau \frac{dt}{\dot{x}_j^{(1)}(t)} I_{jk}(\tau) = \int_0^\tau \frac{dt}{\dot{x}_j^{(1)}(t)} \dot{x}_k^{(1)}(t)$$  

(60)

We can now construct the solutions which fulfill the initial conditions [45] as

$$\eta_{\mu}^{(0)}(\tau) = \phi_{\mu}^{(0)}(\tau) \dot{x}_0^{(1)}(0),$$  

$$\eta_{\mu}^{(j)}(\tau) = \phi_{\mu}^{(0)}(\tau) \dot{x}_j^{(1)}(0) - \phi_{\mu}^{(j)}(\tau).$$  

(61)

Now we want to compute \( \text{det}(\eta_{\mu}^{(0)}(T)) \). To simplify the result one however has to be careful about the integrals defined in Eq. [60]. The reason for this is that the integrals diverge if \( \dot{x}_0(\tau) \) becomes zero in the interval from \( \tau = 0 \) to \( \tau = T \). If one however performs the limit \( \lim_{\tau \to T} \eta_{\mu}^{(0)}(\tau) \) these divergences are canceled. It is possible to separate the divergences into boundary terms with the help of an integration by parts and thus rewrite \( \text{det}(\eta_{\mu}^{(0)}(T)) \) in terms of converging integrals

$$\lim_{\tau \to T} \dot{x}_0^{(1)}(0) \dot{x}_0^{(1)}(T) I(\tau) = \dot{x}_0^{(1)}(0) \dot{x}_0^{(1)}(T) I(T),$$  

(62)

$$\lim_{\tau \to T} \left( \dot{x}_0^{(1)}(T) I(\tau) - I_0(T) \right) = \dot{x}_0^{(1)}(T) I(T) - I_0(T),$$  

(63)

$$\lim_{\tau \to T} \left( I_{jk}(\tau) - \dot{x}_j^{(1)}(T) \dot{x}_k^{(1)}(T) I_0(T) - I_{jk}(T) \right) = I_{jk}(T) - \dot{x}_j^{(1)}(T) \dot{x}_k^{(1)}(T),$$  

(64)

where we define the converging integrals

$$I(\tau) = \int_0^\tau \frac{dt}{\dot{x}_0^{(1)}(t)} \frac{\partial}{\partial t} \left( \frac{1}{\dot{x}_0^{(1)}(t)} \right),$$  

$$I_j(\tau) = \int_0^\tau \frac{dt}{\dot{x}_0^{(1)}(t)} \frac{\partial}{\partial t} \left( \frac{\dot{x}_j^{(1)}(t)}{\dot{x}_0^{(1)}(t)} \right),$$  

$$I_{jk}(\tau) = \int_0^\tau \frac{dt}{\dot{x}_0^{(1)}(t)} \frac{\partial}{\partial t} \left( \frac{\dot{x}_j^{(1)}(t) \dot{x}_k^{(1)}(t)}{\dot{x}_0^{(1)}(t)} \right).$$  

(65)
Using the periodicity of the classical world-line instantons namely \( \dot{x}_j^c(T) = \dot{x}_j^c(0) \), for which \( I_j(T) = 0 \) follows, we find

\[
\eta^{(\nu)}_\mu(T) = \dot{x}_\mu^c(0) \dot{x}_\nu^c(0) I(T) + I_{ij}(T) + T \delta_{ij},
\]

where \( \mu = (0, i) \) and \( \nu = (0, j) \). So that we can compute the fluctuation determinant

\[
\det \left( \eta^{(\nu)}_\mu(T) \right) = \left( \dot{x}_0^c(0) \right)^2 I(T) \det \left( I_{ij}(T) + T \delta_{ij} \right).
\]

For the case of the one component electric field depending on time (\( \dot{x}_3^c(\tau) = \dot{x}_3^c(\tau) = 0 \)) studied in \cite{1} one finds

\[
\tau + I_{11}(\tau) = \int_0^\tau d\tau + I_{11}(\tau) = \int_0^\tau d\tau \frac{1}{\dot{x}_0^c(\tau)} \frac{\partial}{\partial t} \left( \left( \dot{x}_1^c(\tau) \right)^2 + \left( \dot{x}_2^c(\tau) \right)^2 \right) = a^2 I(\tau),
\]

following from Eq. \( \ref{38} \) and thus we recover

\[
\det \left( \eta^{(\nu)}_\mu(T) \right) = \left( \dot{x}_0^c(0) I(T) T a \right)^2.
\]

A factor of \( T^2 \), with respect to Eq. \( \ref{322} \) of \cite{1}, stems from the fact that there only the two dimensional (0,1) part of \( \eta^{(\nu)}_\mu \) is taken into account. This is possible since the (2,3) part is equal to \( T \delta_{ij} \).

Now we need to calculate the Morse index to determine the phase factor in Eq. \( \ref{44} \). It can be derived either as the number of times the determinant \( \det(\eta^{(\nu)}_\mu(T)) \) is zero in between 0 and \( \tau \) or as the number of negative eigenvalues of the operator \( \Lambda \). In \cite{1} it was stated that for the examples studied there \( (A_1(t) \sim \sin(t) \text{ and } A_1(t) \sim \tanh(t)) \) this index is 2, leading to a phase factor of \(-1\). In \cite{3} we show that this is true for all electric fields with one component depending on time. However we have not been able to prove it for the general three-component case.

We now use

\[
\int d^4x(0) = \int dx_0(0) dx_1(0) dx_2(0) dx_3(0) = V \int d\tau_0 \dot{x}_0^c(0) = V \frac{T}{2} \dot{x}_0^c(0),
\]

where \( V \) is the 3-space volume. Using \( \ref{44} \) one obtains the semiclassical Euclidean action

\[
\Gamma_{\text{semi}}^{\text{Eucl}} \approx -\frac{V}{2e^c(2\pi\hbar)^2} e^{i\theta} \int_0^\infty dT \frac{e^{-[T+S(\dot{x}^c(T))/\hbar]}}{\sqrt{I(T) \det \left( I_{ij}(T) + T \delta_{ij} \right)}}.
\]

C. The exponent

We now study the exponent in Eq. \( \ref{71} \) which is proportional to

\[
\Delta(T) := S[\dot{x}^c](T) + T.
\]

Using the classical equations of motion \( \ref{51} \) and \( \ref{50} \) we find

\[
S[\dot{x}^c](T) = \int_0^T d\tau \left( \frac{(\dot{x}^c)^2}{4} + \frac{e^c}{c} \cdot \dot{x}^c \right)
\]

Introducing the function

\[
\tilde{F}(t) = \pm \sqrt{\left( \dot{f}_3(t) \right)^2},
\]

we change the variable of the integral to

\[
y = \tilde{F} \left( \frac{\omega}{c} x_0^c \right) / \gamma.
\]

As result we obtain

\[
\Delta(T) = T \left( 1 - \frac{a(T)^2}{4e^2} \right) + \frac{a(T)^2}{4eE_0 e^2} \pi g(\gamma(T))
\]
with
\[ g(z) = \frac{2}{\pi} \int_{-1}^{1} dy \frac{\sqrt{1 - y^2}}{F(zy)} \operatorname{sgn}(y - y_0) = \frac{4}{\pi} \int_{y_0}^{1} dy \frac{\sqrt{1 - y^2}}{F(zy)}, \]
(77)

where
\[ F(z) := \left| \tilde{F}'(\tilde{F}^{-1}(z)) \right| \]
(78)
is the derivative of \( F(z) \) re-expressed as a function of \( y \). As discussed in \textsuperscript{21}, world-line instantons are closed curves which end and start at the classical turning points which correspond to \( y = \pm 1 \). This can be motivated since \( \tilde{x}_0^1(y) = \pm a \sqrt{1 - y^2} \) following from Eq. \textsuperscript{53} becomes 0 at these points such that the interval in between covers half a period. As in the WKB-case when we perform the substitution \textsuperscript{75} we have to choose the sign in Eq. \textsuperscript{74} carefully.

We introduce \( y_0 = y(s_p) \) by analogy with the one component case of \textsuperscript{1}. We also define
\[ I \]
where
\[ x \]
which end and start at the classical turning points which correspond to \( y \).

Using Eq. \textsuperscript{36} we find that the pair creation rate \( \Gamma \) can be approximated by the imaginary part of the Minkowski action \( \Gamma_{\text{Mink}} \) and that this in turn is approximately equal to the semiclassical world-line instanton pair creation rate \( \Gamma_{\text{WLI}} \)
\[ \Gamma_{\text{WLI}} \approx - e^{i\theta} \frac{V}{2e^4(2\pi\hbar)^2} \sqrt{\frac{\pi\hbar}{2\Delta''(T_c)}} \frac{e^{-\frac{i}{2} \Delta(T_c)}}{\sqrt{I(T_c) \det (I_{ij}(T_c) + T_c \delta_{ij})}} = - e^{i\theta} \frac{V}{(2\sqrt{\pi\hbar} c)^3} \frac{e^{-\pi \frac{E_0}{\hbar} \theta}}{\sqrt{\det (P_{ij}(\gamma) + P(\gamma) \delta_{ij})}}, \]
(81)

Using Eq. \textsuperscript{36} we find that the pair creation rate \( \Gamma \) can be approximated by the imaginary part of the Minkowski action \( \Gamma_{\text{Mink}} \) and that this in turn is approximately equal to the semiclassical world-line instanton pair creation rate \( \Gamma_{\text{WLI}} \)
\[ \Gamma \approx \Gamma_{\text{WLI}} := - e^{i\theta} \frac{V}{(2\pi\hbar c)^3} \left( \frac{E_0}{E_c} \right)^{3/2} \frac{e^{-\pi \frac{E_0}{\hbar} \theta}}{\sqrt{\det (P_{ij}(\gamma) + P(\gamma) \delta_{ij})}}, \]
(82)
where we define
\[ P(\gamma(T)) := \frac{2}{\pi} \int_{y_0}^{1} dy \frac{1}{\sqrt{1 - y^2}} \frac{1}{F(\gamma(T)y)} = \frac{ceE_0}{\pi} T \]
(83)
by analogy with the one component case of \textsuperscript{1}. We also define
\[ P_{jk}(\gamma) := \frac{ceE_0}{\pi} I_{jk}(T_c) = - \frac{1}{\gamma^2 \pi} \int_{y_0}^{1} dy \frac{1}{\sqrt{1 - y^2}} \frac{\partial}{\partial y} \left( \frac{\tilde{F}_j(\gamma y) \tilde{F}_k(\gamma y)}{y F(\gamma y)} \right) \]
(84)
by using definition of the integrals, the substitution \textsuperscript{75} and
\[ \tilde{x}_0^1(t) = -2eEca \sqrt{1 - y^2} F(\gamma(T)y), \]
(85)
which can be derived from Eqs. \textsuperscript{53}, \textsuperscript{74} and \textsuperscript{78}. In \textsuperscript{84} we use
\[ \tilde{F}_j(z) := \tilde{f}_j(\tilde{F}^{-1}(z)). \]
(86)
III. COMPARISON BETWEEN THE WKB AND WORLD-LINE INSTANTON RESULTS

We can now compare the results Eq. (34) of the WKB method discussed in Section I and Eq. (82) of the world-line instanton approach of Section II. Observe that Eq. (34) shows the leading order contribution of the pair production rate (28) where the momentum spectrum was expanded around \( \vec{P} = 0 \) and that Eq. (82) is the counterpart in the world-line instanton approach. We will show that these two results agree with each other.

From the definitions of \( f_j \) and \( \tilde{f}_j \) in Eqs. (20) and (49), and the definitions of \( F \) and \( \tilde{F} \) in Eqs. (23) and (74) respectively we can find

\[
F(\vec{0}, t) = -i \tilde{F}(-it),
\]

We thus find

\[
F(\vec{0}, -iz) = F(z),
\]

which follows from the respective definitions in Eqs. (27), (33), (78) and (86).

Inserting Eq. (88) in Eqs. (26) and (32) and comparing to Eqs. (77), (83) and (84) we find

\[
g(\gamma) = G(\vec{0}, \gamma),
\]

\[
P_{jk}(\gamma) + \delta_{jk} P(\gamma) = \frac{1}{2} G_{jk}(\gamma).
\]

Thus from Eqs. (34) and (82) we find

\[
\Gamma_{WKB}^{\vec{P} \to 0} = \Gamma_{WL} D_s \hbar \omega e^{-i \theta}.
\]

This means that the world-line instanton result agrees with the expansion of the WKB rate around \( \vec{P} = 0 \) except for a factor of \( D_s \hbar \omega \) provided that \( e^{i \theta} = -1 \). The factor of \( D_s \hbar \omega \) stems from the sum over the virtually continuous energy spectrum performed for the WKB-result in Section I C and can be added to the world-line instanton result with the same argumentation (see also [14, 18]). In B we show that \( e^{i \theta} = -1 \) holds for the one-component case but we have not been able to show it generally. This implies that the world-line instanton result agrees with the WKB result where the momentum spectrum was expanded around \( \vec{P} = 0 \).

In the cases where the momentum spectrum is not peaked around \( \vec{P} = 0 \) the expansion (29) around \( \vec{P} = 0 \) is not a good approximation of \( G(\vec{P}, \gamma) \). Accordingly the leading order of the WKB result is not given by Eq. (34) but can be derived from the general form in Eq. (28). Furthermore Eq. (82) derived from the world-line path integral in Eq. (35) does not apply. As discussed in [21] this is due to the fact that the momentum, arising as an integration constant in this framework, was taken to vanish with a Gaussian momentum integration producing the prefactors. As argued before this can be seen as an implicit expansion of the momentum spectrum around \( \vec{P} = 0 \). It would however be possible to get information about the momentum dependence also in the world-line instanton approach by making the more general ansatz of Eq. (48).

Note that it is possible to shift the canonical momentum spectrum by adding constant contributions \( \vec{A}^0 \) to the vector potential \( \vec{A}(t) \). The new potential

\[
\vec{A}'(t) = \vec{A}(t) + \vec{A}^0
\]

gives rise to the same electric field. We define the new momentum modes

\[
\vec{P}' = \vec{P} + e \vec{A}^0.
\]

The momentum spectrum of \( P'_\mu \) is exactly the same as if we had used the original potential \( A_\mu \). Thus the entire momentum spectrum has been shifted by \( -e A_\mu \).

Note that adding constant contributions to the potential does not change the electric field and thus does not change the pair creation rate. This is clear from Eq. (28) since the rate arises from an integration over the whole momentum spectrum and thus is unaffected by a shift. Also the physical spectrum of the kinetic momentum

\[
\vec{p} = \vec{P} - \vec{A}(t) = \vec{P}' - \vec{A}'(t)
\]

remains unaffected.

It would be tempting to try to use this property to shift the peak in the momentum spectrum to \( \vec{P} = 0 \) in order
to use the approximation Eq. [34] to avoid the computation of the momentum spectrum [26]. This would however
require a priori knowledge of the position of the peak and thus the momentum spectrum i.e. one would have to
compute [26]. Additionally the spectrum is not necessarily peaked around a point in the momentum space as is
shown by the example of the constant rotating field in Section [IV A] where it is peaked around a circle defined by
\((\gamma cP_1)^2 + (\gamma cP_2)^2 = 1\). Such that shifting the momentum spectrum does not simplify the problem.

IV. APPLICATIONS

In this Section we use the techniques developed in the previous sections to calculate the pair production rate for
rotating field configurations. In Section [IV A] we study the two-component case of a constant rotating electric field.
Using the WKB-techniques of Section [I] we are able to find an analytic expression for the momentum spectrum. In
Section [IV B] we study the two-component problem of a non-constant rotating field. There because of the higher
complexity we in general cannot obtain the momentum spectrum analytically.

The analytic solutions in the following sections can be formulated with help of the elliptical integrals
\( F(k, \phi) \) and \( E(k, \phi) \) as well as their complete forms \( K(k) \) and \( E(k) \) given by (see [38] Eq. 8.111.2-3)

\[
F(k, \phi) := \int_{0}^{\phi} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}, \quad K(k) := F \left( k, \frac{\pi}{2} \right),
\]

\[
E(k, \phi) := \int_{0}^{\phi} \sqrt{1 - k^2 \sin^2(\theta)} d\theta, \quad E(k) := E \left( k, \frac{\pi}{2} \right).
\]

A. Constant rotating electric field

As an example for a two-component electric field we compute the pair creation rate of a rotating electric field. This
is not a purely academic example although it is one of the simplest cases. As shown in [24] a circularly-polarized laser
wave in a plasma takes exactly this form. The rotating electric field is described by

\[
f_1(t) = \sin(t), \quad f_2(t) = g \cos(t),
\]

where \( g \) defines the sense of the rotation.

Instead of the substitution for the general case discussed in Section [I B] it is more convenient to define

\[
F(\vec{P}, t) = \frac{\sqrt{\gamma cP_1 - f_1(t)}^2 + (\gamma cP_2 - f_2(t))^2}}{\sqrt{(cP_3)^2 + 1}}
\]

instead of [23]. Changing the integration variable according to Eq. [24] we find

\[
G(\vec{P}, \gamma) = \sqrt{1 + (cP_3)^2} \frac{\pi}{4} \int_{\tau_0}^{1} d\tau \frac{\sqrt{1 - \tau^2}}{\mathcal{F}(\vec{P}, -\gamma \tau)},
\]

where \( \mathcal{F}(\vec{P}, z) \) is defined in Eq. [27].

To find \( \mathcal{F}(\vec{P}, t) \) we have to solve the equation \( z = F(\vec{P}, t) \). Using the definition of \( F(\vec{P}, t) \) in Eq. [98] we first find the
following relationship between sine and cosine

\[
-z^2(1 + (cP_3)^2) + (\gamma cP_1)^2 + 1 = 2\gamma cP_1 \sin(t) + 2\gamma g cP_2 \cos(t),
\]

where

\[
P_\parallel := \sqrt{P_1^2 + P_2^2}.
\]

Then by using \( \sin^2(x) + \cos^2(x) = 1 \) we find the following quadratic equation for \( \cos(t) \)

\[
\cos^2(t) - 2h(z) g \frac{P_2}{P_\parallel} \cos(t) + h(z)^2 - \frac{P_2^2}{P_\parallel^2} = 0
\]
with
\[
\begin{align*}
    h(z) &= \frac{(\gamma cP_\parallel)^2 + 1 - z^2(1 + (cP_3)^2)}{2\gamma cP_\parallel},
\end{align*}
\]
which can be solved as
\[
\begin{align*}
    \cos(t) &= h(z)g\frac{P_2}{P_\parallel} \pm \sqrt{1 - h(z)^2}\frac{|P_1|}{P_\parallel}.
\end{align*}
\]
We are interested in
\[
\begin{align*}
    \partial F(\dot{P}, t) = -\frac{[\gamma cP_1 - f_1(t)]f'_1(t) + [\gamma cP_1 - f_2(t)]f'_2(t)}{\sqrt{[\gamma cP_1 - f_1(t)]^2 + [\gamma cP_2 - f_2(t)]^2(cP_3)^2 + 1}} = \mp \frac{|P_1|}{P_1} \frac{cP_\parallel}{z 1 + (cP_3)^2} \sqrt{1 - h(z)^2}.
\end{align*}
\]
Using Eq. (27) we find
\[
\begin{align*}
    F(\dot{P}, z) &= \frac{1}{2} \left| \sqrt{-z^2 - 2\gamma^2 C_-} (z^2 - \gamma^2 C_+) \right|
\end{align*}
\]
with
\[
\begin{align*}
    C_\pm &= \frac{1}{\gamma^2} \frac{(\gamma cP_\parallel \pm 1)^2}{1 + (cP_3)^2}.
\end{align*}
\]
We find a unique solution (107) for Eq. (27) such that there are no interference effects.
The integral (99) takes the form
\[
\begin{align*}
    G(\dot{P}, \gamma) &= \frac{8}{\pi} \frac{\sqrt{1 + (cP_3)^2}}{\sqrt{1 + (cP_3)^2 + C_-}} \left[ \arcsin \left( \frac{\sqrt{1 - \tau_\parallel^2}}{1 + C_-} \right) - \arcsin \left( \frac{\sqrt{1 + \tau_\parallel^2}}{1 + C_+} \right) \right],
\end{align*}
\]
where we change the integration variable to \( x = 1 - 2\gamma^2 C_- \). This integral can be solved and takes real values for \( \tau_\parallel^2 \geq -C_- \) (see [38] Eq. 3.141.5)
\[
\begin{align*}
    G(\dot{P}, \gamma) &= \frac{8}{\pi} \frac{\sqrt{1 + (cP_3)^2}}{\sqrt{1 + (cP_3)^2 + C_-}} \left[ \arcsin \left( \frac{\sqrt{1 - \tau_\parallel^2}}{1 + C_-} \right) - \arcsin \left( \frac{\sqrt{1 + \tau_\parallel^2}}{1 + C_+} \right) \right],
\end{align*}
\]
where we use the elliptic integrals (95) and (96).
Now we have to compute \( \tau_\parallel \). Therefore we need to find the real part of the turning points defined by Eq. (12) or equivalently by \( \tau = \pm 1 \). We find from Eq. (104)
\[
\begin{align*}
    t_p^\pm &= \arccos \left( h(\pm \gamma g)\frac{P_2}{P_\parallel} \pm \sqrt{1 - h(\pm \gamma g)^2}\frac{|P_1|}{P_\parallel} \right),
\end{align*}
\]
which has the real part
\[
\begin{align*}
    s_p &= \text{Re}(t_p^\pm) = \arccos \left( \frac{P_1}{P_\parallel} \right) = \arccos \left( \frac{P_2}{P_\parallel} \right).
\end{align*}
\]
This leads to
\[
\begin{align*}
    \frac{1}{\gamma^2} \left[ \gamma cP_1 - f_1(s_p) \right]^2 + \left[ \gamma cP_2 - f_2(s_p) \right]^2 = \frac{(\gamma cP_\parallel)^2 - 2\gamma cP_\parallel + 1}{(cP_3)^2 + 1} = -C_-,
\end{align*}
\]
\[
1 \text{ The other assumption for the integral, namely } C_- < C_+ \text{, is satisfied for } P_\parallel > 0.
\]
which is used to simplify Eq. (109) to

\[ G(\vec{P}, \gamma) = \frac{8}{\pi \gamma} e^{i \frac{mc^2}{\gamma}} \sqrt{P_x^2 + P_y^2} \left[ K \left( \frac{P_x^2 + P_y^2}{P_x^2 + P_y^2} \right) - E \left( \frac{P_x^2 + P_y^2}{P_x^2 + P_y^2} \right) \right], \]  

(113)

where

\[ \gamma c P_{\pm} := \sqrt{\left( \gamma c P_{||} \pm 1 \right)^2 + \gamma^2}. \]  

(114)

We compare the result Eq. (113) to Eq. (28) of [24]. Using the relations between our variables and theirs which are \( p_x = P_3, p_z = P_1 \) we find

\[ G(\vec{P}, \gamma) \bigg|_{P_3=0, P_1=0, P_5=p_x} = g(\gamma) + c_x(\gamma) \frac{P_x^2}{m^2} + O(p_2^2, p_4^2), \]  

(115)

this means the result is the same for \( P_2 = 0 \). This is due to the fact that they use the purely "imaginary time" picture in contrast to the "complex time" we use. Because of \( \sin(ix) = i \sinh(x) \) and \( \cos(ix) = \cosh(x) \) they have to set the momentum in the direction in which the potential is given by a sine equal to zero to get a real result. As discussed in Section I A and in [21] using a purely imaginary time is only feasible for potentials which are odd functions of the time \( t \).

If we try to expand in \( P_1, P_2 \) around 0 as described in Section I C we run into problems. For \( P_1 = P_2 = 0 \) we find \( P_+ = P_- \) and thus \( G(\vec{0}, \gamma) \) diverges and the pair production rate becomes zero for \( \vec{P} = 0 \). This means in a constant rotating field no pairs are produced with momentum \( \vec{P} = 0 \).

As discussed in Section III the world-line instanton approach does not apply here since the spectrum is not peaked around \( \vec{P} = 0 \). Observe that if we want to construct the world-line instanton for \( \hat{x}_0 \) by solving the equation of motion we find

\[ x_0^c(\tau) = \pm a \sqrt{1 - \frac{1}{\gamma^2} \tau} + C, \]  

(116)

which is not periodic and thus we are not able to construct a world-line instanton for this particular problem. We can however perform the expansion

\[ G(\vec{P}, \gamma) = G(\vec{P}, \gamma) \bigg|_{P_3=0} + G_3(P_1, P_2, \gamma)(cP_3)^2 + \cdots, \]  

(117)
Performing the Gaussian integrals over $P_+$ in $P_\gamma$ we find that
\[ G(\vec{P}, \gamma) \bigg|_{P_\gamma = 0} = \frac{8}{\pi} c P_+ \left[ K \left( \frac{P_+}{P_\gamma} \right) - E \left( \frac{P_-}{P_\gamma} \right) \right], \tag{118} \]
\[ G_3(P_1, P_2, \gamma) = \frac{4}{\pi} \frac{1}{c P_+} K \left( \frac{P_+}{P_\gamma} \right). \tag{119} \]

This expansion is senseful since $E_\gamma$ is proportional to $1/\hbar$ and therefore the exponential in Eq. (21) restricts the perpendicular momentum $P_3$ to be of the order $\sqrt{\hbar}$ as was discussed for the one-component space-dependent case in [14].

We find that $G(\vec{P}, \gamma)$ is peaked around $\gamma c P_\parallel = 1$. This is in accordance with the results of [23] where the momentum spectrum of a rotating pulse is studied numerically. For a high number of rotation cycles per pulse their momentum spectrum is also peaked around a circle of fixed $P_\parallel$.

For $\gamma \ll 1$ an expansion around $\gamma c P_\parallel = 1$ is equal to an expansion for small $P_\parallel$ which follows from the definition in Eq. (114). So that Eqs. (118) and (119) can be expanded for $P_\parallel$ around 0 leading to
\[ G(\vec{P}, \gamma) \bigg|_{P_\parallel = 0} = 1 + \frac{1}{\gamma^3} (\gamma c P_\parallel - 1)^2 + O(P_\parallel^3) \tag{120} \]
\[ G_3(P_1, P_2, \gamma) = 1 + O(P_\parallel^2). \tag{121} \]

Performing the Gaussian integrals over $P_\parallel$ and $P_3$ in Eq. (28) for this approximation we find
\[ \frac{\Gamma_{\text{WKB}}}{V} \approx D_s \frac{mc}{2\pi \hbar} \left[ \frac{E_0}{E_\gamma} \right]^{3/2} \exp \left( -\frac{\pi E_\gamma}{E_0} \right) \frac{1}{\gamma^2} \left[ 1 + \text{Erf} \left( \sqrt{\frac{E_\gamma}{E_0}} \right) \right] \tag{122} \]
\[ \gamma \rightarrow 0 \]
\[ D_s \frac{mc}{2\pi \hbar} \left[ \frac{E_0}{E_\gamma} \right]^{3/2} \frac{1}{\gamma^2} \exp \left( -\frac{\pi E_\gamma}{E_0} \right). \]

For $\gamma \rightarrow 0$ the pair production rate is equal to the one of the constant field. As already mentioned in [24] this is due to the fact that this limit is equivalent to the limit $\omega \rightarrow 0$ in which the electric field becomes the constant one.

We can compare the result obtained by numerically integrating Eq. (28) for $G(\vec{P}, \gamma)$ given by Eq. (113) with the one of the constant field given by Eq. (122) for the same amplitude $E_0$. We find that the ratio between the two can increase several orders of magnitude with increasing frequency $\omega$ (see Fig. 4). This is in accordance with the results found for rotating pulses in Ref. [23]. We additionally find that this increase is significantly higher if the amplitude of $E_0$ of the fields is smaller. As discussed in [23] this can be qualitatively understood by the fact the parameter $\gamma$ is proportional to $\omega$. With increasing $\gamma$ the pair creation rate increases due to the onset of multiphoton pair production. Since $\gamma$ is inversely proportional to $E_0$ this effect is weakened for higher amplitude field strengths (see also [16] for a similar discussion).

### B. Non-constant rotating field

To underline the peculiarities of the expansion (29) we study the case of the potential $A(t) = k(\omega t)/(\gamma \epsilon)$ which rotates with a frequency $\Omega$, described by
\[ f_1(\omega t) = k(\omega t) \sin(\Omega t), \quad f_2(\omega t) = k(\omega t) \cos(\Omega t). \tag{123} \]

It is in general not easy to calculate $G(\vec{P}, \gamma)$ following Eq. (99) since it is non-trivial to invert $F(\vec{P}, t)$. However looking at the expansion around $\vec{P} = 0$ described in Section I C we find
\[ F(\vec{0}, t) = k(t) \tag{124} \]
and thus $G(\vec{0}, \gamma)$ for the rotating electric field is the same as for the non-rotating with potential $A_1(t) = k(\omega t)/(\gamma \epsilon)$. However as we have seen in Section I VA the momentum spectrum is not necessarily peaked around $\vec{P} = 0$ such that the approximation made for the pair creation rate (34) is generally not correct. This point will be further illustrated by the example
\[ k(t) = \sin(t). \tag{125} \]
For this case $G(\vec{0}, \gamma)$ and $G_{33}(\gamma)$ are given by
\begin{align}
G(\vec{0}, \gamma) &= \frac{4}{\pi} \frac{\sqrt{\gamma^2 + 1}}{\gamma^2} \left[ K\left(\sqrt{\frac{\gamma^2}{1 + \gamma^2}}\right) - E\left(\sqrt{\frac{\gamma^2}{1 + \gamma^2}}\right) \right], \\
G_{33}(\gamma) &= \frac{4}{\pi} \frac{1}{\sqrt{1 + \gamma^2}} K\left(\sqrt{\frac{\gamma^2}{1 + \gamma^2}}\right). 
\end{align}

Note that the necessary calculations are analogous to those for an oscillating one-component electric field because of Eq. (124). Additionally $G_{13} = G_{23} = 0$ follow from $\mathcal{F}(t) = 0$.

To calculate the rest of the integrals $G_{jk}(\gamma)$ we need the functions $F_j(z)$ given by
\begin{align}
F_1(z) &= z \sin(\sigma \arcsin(z)), \\
F_2(z) &= z \cos(\sigma \arcsin(z)),
\end{align}
where the ratio of the two frequencies $\sigma := \Omega/\omega$. If the ratio is an integer $\sigma = n$ one can calculate $\mathcal{F}(\vec{P}, z)$ and the integrals $G_{ij}$ defined in Eqs. (27) and (32) respectively analytically by using the following identities for sine and cosine functions with multiple angles (see [38] Eqs. 1.321.1 and 1.331.3)
\begin{align}
\sin(nx) &= \sum_{j=0}^{[n-1]} (-1)^j \binom{n}{2j+1} \sin^{2j+1}(x) \cos^{n-2j-1}(x), \\
\cos(nx) &= \sum_{j=0}^{[n]} (-1)^j \binom{n}{2j} \sin^{2j}(x) \cos^{n-2j}(x).
\end{align}

If one performs this tedious but simple computations one finds $G_{12}(\gamma) = 0$ and $G_{11}(\gamma)$ and $G_{22}(\gamma)$ being combinations of elliptic integrals. However $G_{11}(\gamma) > 0$ and $G_{22}(\gamma) < 0$. This would lead to det$(G_{ij}(\gamma)) < 0$ and thus Eq. (34) would give an imaginary result, which is clearly unphysical. However the Gaussian integral performed to get Eq. (34) is only correct for $G_{ij}$ being a positive definite matrix. So like in Section IV A where the terms of the expansion (29) diverge we cannot use it to simplify the calculation of the pair creation rate in this case.

For $\sigma = 1$ we can show that this is connected to the fact that the momentum spectrum is not centered around $\vec{P} = 0$.

In this case we find (see [38] Eqs. 1.321.1 and 1.333.1)
\begin{align}
f_1(t) &= \sin^2(t) = \frac{1}{2} (1 - \cos(2t)), \\
f_2 &= \sin(t) \cos(t) = \frac{1}{2} \sin(2t).
\end{align}
Since adding a constant part to the vector potential has no influence on the electric field, this is analogous to the case of the constant rotating electric field discussed in Section IV A with twice the frequency. By shifting the momentum spectrum \( P_1 \rightarrow P'_1 = P_1 - 1/(2\gamma c) \) and defining \( P'_\parallel = \sqrt{(|P'_1|^2 + P_2^2)} \), the situation is the same as in Section IV A and the spectrum is peaked around \( \gamma c P'_\parallel = 1 \). As discussed in Section II, this shift has no influence on the physical spectrum of the kinetic momentum \( \langle 94 \rangle \).

V. CONCLUSIONS AND REMARKS

In this article we generalize the analytic methods of the semiclassical WKB-approach and the world-line instanton approach of [1] to calculate the pair production rate of time-dependent electric fields to the case of general three-component fields. For the WKB-approach we obtain the momentum spectrum of the produced pairs. We show that if this spectrum is expanded around \( \vec{P} = 0 \) the results of the two methods are the same.

The momentum spectrum is usually peaked around \( \vec{P} = 0 \) for the examples of one-component fields studied in the literature (see, e.g., [1]). Thus the expansion around \( \vec{P} = 0 \) presents a good approximation. However, this situation changes if one goes to the case of two-component fields. By looking at rotating electric fields we find that their momentum spectra are not peaked around \( \vec{P} = 0 \).

If the momentum spectrum is not peaked around \( \vec{P} = 0 \) one can not use the expanded WKB result since it does not present a good approximation. Also the world-line instanton method of [1] and the generalized form presented here implicitly require the momentum spectrum to be peaked around \( \vec{P} = 0 \). This implies that it is not appropriate to calculate the pair production rate for cases where the momentum spectrum is not peaked around \( \vec{P} = 0 \) in the form discussed here.

However this can possibly be solved in the framework of the modified world-line instanton approach of [21].

In this first investigation we ignored the effects of interference which can play an important role, if there is more than one pair of semiclassical turning points. It has been shown, in [21], that the interference effect is the same in the WKB-approach and the world-line instanton method for the case of electric fields with one component. The investigation of this in the general three-component case is left for future work.

Rotating field configurations such as the one studied here are of interest since they are related to circularly-polarized laser waves. A circularly-polarized wave in medium can be described by a rotating electric field, since it is possible to make a transformation into the co-moving Lorentz frame (see, e.g., [24]).

Recently it has become obvious that the pair production rate of lasers depends sensitively on the pulse shape [25–32]. For the design of feasible experiments to directly measure pair production it is therefore of interest to find a pulse profile which enhances this process. Obviously for complicated laser pulse profiles the calculation has to be done numerically. The development of semiclassical analytical methods discussed in this article certainly helps to provide some physical intuition for these numerical simulations.

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Appendix A: Computation of the Taylor series terms of \( G(\vec{P}, \gamma) \)

In this Appendix we will summarize the calculation of the terms of the Taylor series presented in Section I C. The result (30) can be achieved from the form \( \langle 25 \rangle \) of \( G(\vec{P}, \gamma) \) with the help of

\[
\frac{\partial G(\vec{P}, \gamma)}{\partial c F_j} \bigg|_{\vec{P} = 0} = -\frac{2}{\gamma \pi} \left( \int_{\tau_0}^{1} - \int_{-1}^{\tau_0} \right) \frac{F_j(-i\gamma \tau)}{\sqrt{1 - \tau^2 F(0, -i\gamma \tau)}} d\tau = 0, \tag{A1}
\]
Observe that the boundaries of the integral in Eq. (25), i.e. \( t^+_p \) and \( t^-_p \) defined in Eq. (12), are functions of \( \bar{P} \) but that the term related to them is proportional to
\[
\kappa(t^+_p) \frac{\partial t^+_p(\bar{P})}{\partial P_1} - \kappa(t^-_p) \frac{\partial t^-_p(\bar{P})}{\partial P_1} = 0
\]
and thus vanishes because of Eqs. (12) and (16). Eq. (32) can be found with the help of

\[
G_{jk}(\gamma) = \frac{\partial^2 G(\bar{P}, \gamma)}{\partial (cP_j) \partial (cP_k)} \bigg|_{\bar{P}=0}
\]

\[
= \delta_{jk} \frac{4}{\pi} \int_{\gamma_0}^{1} \frac{1}{\sqrt{1 - \tau^2}} F(0, -i \gamma \tau) d\tau - \frac{1}{\gamma^2} \frac{4}{\pi} \int_{\gamma_0}^{1} \frac{F_j(-i \gamma \tau) F_k(-i \gamma \tau)}{(1 - \tau^2)^{3/2}} F(0, -i \gamma \tau) d\tau
\]

for \( j, k = 1, 2, 3 \). The corresponding boundary term is proportional to

\[
f_j(\omega t^+_p) \frac{\partial t^+_p}{\partial P_k} - f_j(\omega t^-_p) \frac{\partial t^-_p}{\partial P_k}
\]

and does not vanish but reduces to the last term in Eq. (A4) by using

\[
\frac{\partial cP_j}{\partial t} \bigg|_{t=F^{-1}(\sigma, z)} = -\frac{1}{\gamma} F_j(0, z)
\]

which can be deduced by solving Eq. (23) for \( P_j \). The term (A6) is important since it cancels the divergence in the second integral in Eq. (A4).

**Appendix B: Morse Index**

The Morse index can be determined either by the number of negative eigenvalues of the fluctuation operator \( \Lambda \) (42) or the number of times \( \det(\eta^{(\omega)}(\tau)) \) vanishes for \( \tau \) in the interval between 0 and \( T \) where \( \eta^{(\omega)}(\tau) \) are the solutions to the initial value problem (45) of the Jacobi equation (41) [1, 35–37]. We choose the latter method to determine the index because we can readily compute the determinant from the solutions (61) obtained in Section II B.

\[
\det \left( \eta^{(\omega)}(\tau) \right) = \dot{x}^{cl}_0(0) \dot{x}^{cl}_0(\tau) \ddot{I}(\tau) \det \left( \ddot{I}_{kl}(\tau) + \tau \delta_{kl} - \ddot{I}_k(\tau) \ddot{I}_l(\tau) \right) 
\]

Following from the classical solution (53) and using the substitution (75) we find

\[
\dot{x}^{cl}_0(y) = \pm a \sqrt{1 - y^2}
\]

Since the interval for \( \tau \) from 0 to \( T \) is equivalent to twice the one for \( y \) from -1 to 1 we find that \( \dot{x}^{cl}_0(\tau) \) becomes zero twice namely for \( \tau(y = \pm 1) \). This means that the Morse index is at least two.

For the case of the one-component electric fields with \( I_2(\tau) = I_3(\tau) = I_{23}(\tau) = I_{32}(\tau) = 0 \) we show that the Morse index is exactly two. In this case (B1) takes the form

\[
\det \left( \eta^{(\omega)}(\tau) \right) = \dot{x}^{cl}_0(0) \dot{x}^{cl}_0(\tau) \tau^2 \left( \ddot{I}(\tau)[\ddot{I}_{11}(\tau) + \tau(\ddot{I}_1(\tau))^2 - (\ddot{I}_1(\tau))^2] \right)
\]

\[
= \dot{x}^{cl}_0(0) \dot{x}^{cl}_0(\tau) \tau^2 \left[ a\ddot{I}(\tau) - \ddot{I}_1(\tau) \right] \left[ a\ddot{I}(\tau) + \ddot{I}_1(\tau) \right],
\]

where we use Eq. (68). Substituting (75) into the integrals (60) we find

\[
a\ddot{I}(\tau) \pm \ddot{I}_1(\tau) = \frac{1}{2\varepsilon E_0 a} \int_{y(0)}^{y(\tau)} dy \left( \frac{1}{(1 - y^2)^{3/2}} \frac{1}{F(\gamma(T)y)} \right),
\]
where Eq. (52) is used to find $\dot{\mathbf{x}}_1(\tau) = -ay$. Since $-1 < y(\tau) < 1$, the integrand is always positive. This means that the integral $\mathcal{B}_4$ is only zero for $\tau = 0$. This implies the zero points of $\mathcal{B}_4$ are located at $\tau (y = \pm 1)$ and $\tau = 0$. Since these points are the same, the determinant becomes zero twice for $0 < \tau < T$, i.e. the Morse index $\theta = 2$ for the case of one-component electric fields depending on time.

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