The generalization of shift operator related Bessel differential operator

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Abstract. We introduce the generalized shift operator related to the Bessel differential operator and the Hankel integral transform. It generalizes the operator introduced by J Delsarte and studied by J. Delsarte, B.M. Levitan and Ya.I. Zhitomirskii. The considered operator is not the generalized shift operator of the Levitan’s type. The basic properties of the shift operator are proved. The visualization of shift action is given.

1. Introduction
Shift operators $T^\tau$ are a family of linear operators which play an important role in harmonic analysis, for example, it appears in the definitions of almost periodic functions, positive definite functions, convolutions, etc. The term “generalized shift operator” (also the terms “generalized translation operator”, or “generalized displacement operator” are used) are due to J. Delsarte [1–2]. Important ideas and a number of original results in this field are also due to him. The systematic construction of the theory of generalized shift operators was mainly given in the work of B.M. Levitan (see, for example, [3–4]).

In this investigation we consider shift operators corresponding to the Bessel differential operator

$$B = \frac{d^2}{dt^2} + \frac{2\nu + 1}{t} \frac{dt}{d\tau}.$$  

(1)

We deal throughout the paper with weighted Lebesgue spaces $L_p(\mathbb{R}_+; \omega(t) dt)$, $1 \leq p \leq \infty$ with respect to a positive measure $\omega(t) dt$ equipped with the norm for which

$$\|f\|_{L_p(\mathbb{R}_+, \omega(t) dt)} = \left( \int_0^\infty |f(t)|^p \omega(t) dt \right)^{1/p} < \infty.$$ 

Let $T^\tau$ be a shift operators depending on the parameter $\tau \in \mathbb{R}_+$. Thus, a function $T^\tau f(t)$ of two variables corresponds to every function $f(t) \in L_p(\mathbb{R}_+; \omega(t) dt)$.

In 1938 J. Delsarte introduced and studied the generalized shift operator [2]

$$T^\tau g(t) = \int_0^{\infty} g(t) \frac{t^\nu}{\Gamma(\nu+1/2)} \int_0^\pi g\left(\sqrt{t^2 + \tau^2 - 2\tau t \cos \varphi}\right) \sin^{2\nu} \varphi d\varphi,$$

generated by the Bessel differential operator (1).
The operator (2) is the operator of the Levitan’s type. This means that it satisfy four axioms which
generalized the properties of ordinary shift [3]. Note that the operator has a number of additional
properties, thanks to which it finds numerous applications. Detailed studies can be found in [5–7].

The shift (2) defines the B-convolution

\[
\left( f * g \right)(t) = \int_{0}^{\infty} f(\tau) g(\tau^{2v+1}) d\tau
\]

which is well studied and was introduced in the papers [6-7].

The convolution (3) is the classical convolution for the Hankel transform defined by [7]

\[
h_{\nu}[f](s) = \tilde{f}_{\nu}(s) = \int_{0}^{\infty} f(t)j_{\nu}(st)t^{2v+1}dt, \ \nu \geq -1/2.
\]

Here the function

\[
j_{\nu}(st) = \frac{2^{\nu}\Gamma(\nu + 1)}{(st)^{\nu}}j_{\nu}(st) = \sum_{m=0}^{\infty} \frac{(-1)^m\Gamma(\nu + 1)(st)^{2m}}{2^{2m}m!\Gamma(m + \nu + 1)}
\]

is associated with the Bessel function \(j_{\nu}\) of the first kind of order \(\nu\).

The function \(j_{\nu}(st)\) is the solution of the equation

\[
\frac{d^2y}{dt^2} + \frac{2\nu + 1}{t} \frac{dy}{dt} + s^2y = 0
\]

under conditions \(y(0) = 1\) and \(y'(0) = 0\).

![Figure 1](image1.png)

**Figure 1.** The graphs of the function \(y = j_{\nu}(\xi)\) for the four most commonly used values of parameter \(\nu\).

Note that if \(\nu = -1/2\) then the Hankel transform (4) turns into the cosine Fourier transform so we
will consider the case then \(\nu > -1/2\).

The following asymptotic estimate for \(j_{\nu}(\xi)\) at \(\xi \to +\infty\) holds.

\[
j_{\nu}(\xi) = \text{const} \cdot \xi^{-\nu - 1/2}\cos\left(\xi - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) + O(\xi^{-\nu - 3/2}).
\]

In our case \(\nu > -1/2\) so \(j_{\nu}(\xi) \to 0\) at \(\xi \to +\infty\).

Using this asymptotic expansion and the equality \(j_{\nu}(0) = 1\) we can easily show that a positive
number \(C_\beta\) which is independent of \(\xi \in (0, \infty)\) such that

\[
|\xi^\beta j_{\nu}(\xi)| < C_\beta, \ \forall \xi \in (0, \infty) \text{ and } 0 \leq \beta \leq \nu + 1/2
\]

exist. Moreover, the function \(j_{\nu}(\xi) \in L_1(\mathbb{R}_+; \xi^\mu d\xi)\) for \(-1 < \mu < \nu - 1/2\).
The inversion formula for (4) is given by
\[
h_{\nu}^{-1}\{\tilde{f}_\nu\}(t) = f(t) = [2^{2\nu}\Gamma^2(\nu + 1)]^{-1} \int_0^\infty \tilde{f}_\nu(s)j_\nu(st)s^{2\nu + 1}ds.
\] (5)

In this work we study the generalization of shift operator (2) namely the operator defined by
\[
n T^\tau_\nu g = \frac{\hat{c}_{\nu,n}}{(\tau)^n} \int_0^\infty g(\sqrt{t^2 + \tau^2 - 2\tau \cos \phi})C^\nu_n(\cos \phi)\sin^{2\nu}\phi d\phi,
\] (6)
\[
\tau^n n T^\tau_\nu g|_{\tau=0} = 0, \quad t^n n T^\tau_\nu g|_{t=0} = 0, \quad \forall n > 0.
\]

Here $C^\nu_n(z)$ is the $n$-th degree Gegenbauer polynomial with parameter $\nu$.

The operator (6) determines one of generalized convolutions for the Hankel transform (4) which depends on two parameters $\nu \in \mathbb{R}$, $\nu > 0$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and is defined by
\[
\left(f \ast g\right)(t) = \frac{1}{c_{\nu,n}} \int_0^\infty f(\tau) n T^\tau_\nu g(\tau t)\tau^{2\nu+1}d\tau.
\] (7)

The factorization equality for the generalized convolution (7) is written as
\[
h_{\nu+n}\left[f \ast g\right](s) = h_{\nu+n}[f](s)h_\nu[g](s).
\]

The constants $c_{\nu,n}$ and $\hat{c}_{\nu,n}$ are independent of $t \in (0, \infty)$ and are determined by
\[
c_{\nu,n} = \frac{2^{2\nu}\Gamma^2(\nu + n + 1)}{\Gamma^2(\nu + 1)},
\]
\[
\hat{c}_{\nu,n} = c_{\nu,n} \cdot \frac{n! \Gamma(\nu+1)\Gamma(2\nu)}{\Gamma(\nu+1/2)\Gamma(1/2)\Gamma(2\nu+n)}.
\]

If $n = 0$ then we get the corresponding shift operator (2) and the classical convolution (3) defined for $\nu > -1/2$.

The generalized convolution (7) and the corresponding shift operator (6) were obtained using a method for constructing convolutions which is proposed by V.A. Kakichev in 1967 [8]. This method is based on factorization equality. Later V.A. Kakichev generalized his approach and introduced the concept of polyconvolution or generalized convolution [9]. Using this concept the generalized convolutions generated by various linear operators can be constructed [10–19].

2. Shift operator properties

We assume that the functions $f(t)$ and $g(t)$ are piecewise continuous and bounded on $\mathbb{R}_+$, i.e. $\sup_{t \in \mathbb{R}_+} |f(t)| < \infty$ and $\sup_{t \in \mathbb{R}_+} |g(t)| < \infty$. In this case the functions of two variables $n T^\tau_\nu f(t)$ and $n T^\tau_\nu g(t)$ exist as we will see below. Most of the presented properties are proved similarly to the corresponding properties of the operator $g T^\tau_\nu$ (see [5, 6]).

- Linearity and homogeneity: $n T^\tau_\nu\{a \cdot f(t) + b \cdot g(t)\} = a \cdot n T^\tau_\nu f(t) + b \cdot n T^\tau_\nu g(t)$ for all $a, b \in \mathbb{R}$.
- Symmetry: $n T^\tau_\nu f(t) = n T^\tau_\nu f(t)$.
- $(\tau^n) n T^\tau_\nu \{1\} = \delta_{n,0}$ where $\delta_{n,m}$ is the Kronecker delta.

**Corollary 1.** $n T^\tau_\nu \{f(t) + b\} = n T^\tau_\nu f(t) + b \cdot \delta_{n,0}$, $\forall b \in \mathbb{R}$. Therefore, for $n > 0$ we get $n T^\tau_\nu \{f(t) + b\} = n T^\tau_\nu f(t)$, $\forall b \in \mathbb{R}$.

- If $f(t) \equiv 0$ for $t \geq a \geq 0$ then $n T^\tau_\nu f(t) \equiv 0$ for $|t - \tau| \geq a$.
- Continuity of operators: if a sequence of continuous functions $f_k(t)$ converges uniformly to a function $f(t)$ in each finite interval, then the sequence of functions of two variables $n T^\tau_\nu f_k(t)$ converges uniformly to the function $n T^\tau_\nu f(t)$ in each finite region.

\[
\left|\tau^n \cdot n T^\tau_\nu g(t)\right| \leq c_{\nu,n} \cdot \sup_{t \in \mathbb{R}_+} |g(t)|.
\]
Proof. Using the property of Gegenbauer polynomials we get

\[
| (tr)^n \cdot n^T \varphi(t) | \leq \hat{c}_{v,n} \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) | C_n^\nu(\cos \varphi) \sin^{2\nu} \varphi d\varphi
\]

\[
\leq \hat{c}_{v,n} | C_n^\nu(1) | \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) \sin^{2\nu} \varphi d\varphi
\]

\[
= \frac{\Gamma(2\nu + n)}{n! \Gamma(2\nu)} \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) \sin^{2\nu} \varphi d\varphi
\]

\[
= \frac{2^{2n}n! \Gamma^2(\nu + n + 1)}{(tr)^{\nu+n}} \cdot j_{\nu+n}(t) j_{\nu+n}(\tau)
\]

The property is proved.

- \( n^T \varphi(t) = j_{\nu+n}(t) j_{\nu+n}(\tau) \).

Proof. Using the formula (17) from Chapter XI, Section 11.41 of book [20] we obtain

\[
\frac{\Gamma(2\nu + n)}{n! \Gamma(2\nu)} \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) \sin^{2\nu} \varphi d\varphi
\]

\[
= \frac{2^{2(\nu+n)}n! \Gamma^2(\nu + n + 1) \Gamma(\nu)}{\Gamma(2\nu + n)(tr)^n} \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) \sin^{2\nu} \varphi d\varphi
\]

\[
= \frac{2^{2(\nu+n)}n! \Gamma^2(\nu + n + 1)}{(tr)^{\nu+n}} \cdot j_{\nu+n}(t) j_{\nu+n}(\tau)
\]

The equality is proved.

- If \( f(t) \in L_1(\mathbb{R}^2; t^{2\nu+1}dt) \) or \( f(t) \in L_2(\mathbb{R}^2; t^{2\nu+1}dt) \) then the generalized shift (6) of the Hankel transform (4) exists and the equality \( n^T \varphi(t) = h_{\nu+n}[f(t)j_{\nu+n}(ut)](s) \) holds.

Proof. The Hankel transform (4) of the function belongs to the weighted Lebesgue space \( L_1(\mathbb{R}^2; t^{2\nu+1/2}dt) \) or \( L_2(\mathbb{R}^2; t^{2\nu+1}dt) \) exists and

\[
n^T \varphi(t) = \int_0^{\infty} f(t) j_{\nu+n}(st) t^{2\nu+1}dt
\]

\[
= \int_0^{\infty} f(t) j_{\nu+n}(st) t^{2\nu+1}dt = h_{\nu+n}[f(t)j_{\nu+n}(ut)](s)
\]

The property is proved.

- Similarly, using the formula (5) we obtain

\[
\frac{\Gamma(2\nu + n)}{n! \Gamma(2\nu)} \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) \sin^{2\nu} \varphi d\varphi
\]

\[
= \frac{2^{2(\nu+n)}n! \Gamma^2(\nu + n + 1) \Gamma(\nu)}{\Gamma(2\nu + n)(tr)^n} \int_0^{\pi} g \left( \sqrt{t^2 + \tau^2 - 2t\tau \cos \varphi} \right) \sin^{2\nu} \varphi d\varphi
\]

\[
= \frac{2^{2(\nu+n)}n! \Gamma^2(\nu + n + 1)}{(tr)^{\nu+n}} \cdot j_{\nu+n}(t) j_{\nu+n}(\tau)
\]

for the function \( f(t) \in L_2(\mathbb{R}^2; t^{2\nu+1}dt) \).

Corollary 2. If \( f(t) \in L_2(\mathbb{R}^2; t^{2\nu+1}dt) \) then \( n^T \varphi(t) \) belongs to \( L_2(\mathbb{R}^2; t^{2\nu+1}dt) \) for all \( \tau \in \mathbb{R}^2 \) and

\[
\left\| n^T \varphi \right\|^2_{L_2(\mathbb{R}^2; t^{2\nu+1}dt)} \leq \frac{\text{Const}}{t^{2\nu-n}} \left\| f \right\|^2_{L_2(\mathbb{R}^2; t^{2\nu+1}dt)}.
\]

Proof. If \( f(t) \in L_2(\mathbb{R}^2; t^{2\nu+1}dt) \) then \( \tilde{f}_s(s) \in L_2(\mathbb{R}^2; t^{2\nu+1}ds) \) and the Parseval’s identity.
holds. Then we have the following estimation for the norm of the shift of the function \( f(t) \in L_2(\mathbb{R}_+; t^{2v+1} \, dt) \):

\[
\| nT^f \|_{L_2(\mathbb{R}_+; t^{2v+1} \, ds)}^2 \leq 2^{2v} \Gamma^2 (v + 1) \cdot \| f \|_{L_2(\mathbb{R}_+; t^{2v+1} \, dt)}^2
\]

The Corollary 2 is proved.

**Corollary 3.** If \( f(t) \in L_2(\mathbb{R}_+; t^{2v+1} \, dt) \) then the Hankel transform of the shift \( nT^f \) can be presented through the Hankel transform of the function \( f(t) \):

\[
h_{v+n} \left[ nT^f(t) \right] (s) = c_{v,n} \cdot \tilde{f}_v(s)_{v+n}(s t).
\]

The operator (6) is not the generalized shift operators of the Levitan’s type [21]. In particular, the property of associativity and the property of the existence of a single element are not satisfied for this operator.

### 3. Demonstration of the shift action

This section of the article is devoted to demonstrating the action of the shift operator (6) on a number of functions. To do this, we compare its action with the action of the classical operator (2).

We restrict ourselves considering in the case \( v = 1 \) for \( n = 1 \) and \( n = 0 \).

**Example 1.** At first, we consider the action of the shift operator on the function

\[
f(t) = \begin{cases} \sin t, & t \in (0; 4\pi] \\ 0, & t \notin [0; 4\pi) \end{cases}.
\]

The demonstration was executed in the GNU Octave [22]. The result is shown in figure 2. The syntax example of the corresponding function for plotting is located in Appendix.

**Example 2.** Now we present the shift action on the function

\[
f(t) = e^{-t^2}, \quad t > 0.
\]

The result of the action can be seen in figure 3.

**Example 3.** This example demonstrates the action of the shift operator (6) on the function

\[
f(t) = j_1(t).
\]

In this case, we do not consider the action of the classical operator \( qT^f \). Instead, we compare the result of the action of the operator \( nT^f \) obtained numerically and the product of functions \( j_{v+n}(t)j_{v+n}(\tau) \) in the case \( v = 1 \) for \( n = 1 \). The result is shown in figure 4. We see in these graphs that the shift \( nT^f j_1(t) \) coincides with the product of functions \( j_2(t)j_2(\tau) \) what was proved earlier.
Figure 2. Shift action on the function (8) in the case $\nu = 1$ for $n = 1$ and $n = 0$. 
Figure 3. Shift action on the function (9) in the case $\nu = 1$ for $n = 1$ and $n = 0$. 
Figure 4. Shift action on the function (10) in the case $\nu = 1$, $n = 1$: the numerical integration and the using of the property.
4. Conclusion
The studied shift operator $T^\tau$ allows one to solve various equations containing the Bessel differential operator $B$ and (or) the shift operator itself. It can also be used in the construction of generalized wavelet transforms and in the formulation of Tauberian theorems for the Hankel transform. The generalized convolution (7) generated by the operator (6) and the consideration of the convolution applications are of particular interest.

Appendix
function PlottingFunction
    clear;
    figure(1,"position",[0,0, 800, 900])
    st=1/100;
    m=40;
    x=0:st:12;
    f1=fu(x);
    count=1;
    #nu=1
    for tau=[0.01, 1.0, 2.0, 3.0, 4.0, 5.0];
      t=0.01;
      for k=1:length(x)
        phi=0:pi/1000:pi;
        tt=sqrt(x^2+t^2-2*x*t*cos(phi));
        g1=2.*fu(tt).*cos(phi).*sin(phi).^2;#n=1
        g2=fu(tt).*sin(phi).^2;#n=0
        g(k)=(16/pi/t/tau)*trapz(phi,g1);#n=1
        r(k)=(2/pi)*trapz(phi,g2);#n=0
        t+=st;
      endfor;
      str1=\tau=\';
      str2=sprintf('%d',tau);
      str=strcat(str1,str2);
      subplot(3,2,count,"align")
      plot (x,f1,'k', x, g, '-k',"linewidth", 2, x, r, ':k', "linewidth",2)
      axis([0, 12, -1.5, 1.5]);
      grid on;
      title(str, "fontsize", 18);
      xlabel('\{\fontsize{12} t}\}', 'FontSize', 12)
      pause (0.00001);
      count++;
    endfor;
endfunction
function f=fu (x)
    f=sin(x).*(x>0).*((x<=4*pi));
endfunction

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