Courant-like brackets
and loop spaces

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Abstract
We study the algebra of local functionals equipped with a Poisson bracket. We discuss the underlying algebraic structures related to a version of the Courant-Dorfman algebra. As a main illustration, we consider the functionals over the cotangent bundle of the superloop space over a smooth manifold. We present a number of examples of the Courant-like brackets arising from this analysis.
1 Introduction

The higher algebraic structures have been a subject of recent interest in physics and geometry. In this work, we study the algebraic structures underlying the Poisson brackets of local functionals in classical field theory.

Our work is inspired by two different observations. The first interesting observation was made in 1997, in [3] (also see [13, 1]), about the appearance of an $L_\infty$-structure on the space of local functionals in classical field theory. This work is a formalization of the simple idea that a local functional is an integral over some function (integrand), and all operations with the local functionals involve throwing away total derivatives. Thus, the Poisson brackets of local functionals induce a binary operation on the integrands which satisfy the properties of Poisson brackets up to total derivatives. We thus end up with the appropriate $L_\infty$-structure on the space of local functionals in classical field theory. Later, another curious observation was made in [2]. Namely, the authors found an intriguing way to "derive" the Courant and Dorfman brackets on $TM \oplus T^*M$ by calculating the classical Poisson brackets between local functionals of a special form defined over the cotangent bundle of a loop space. The Courant bracket by itself is an example of an $L_\infty$-structure [14]. It would be natural that these two observations are related and that they are just different manifestations of the same algebraic structures.

In the present work we argue that the appropriate formalization of the Courant-Dorfman structure naturally appears when one considers the algebra of local functionals. For the sake of concreteness, we consider the case of the string phase space (i.e. the cotangent bundle of the loop space) and its supersymmetric generalization. However, many results can easily be generalized beyond this concrete framework. Beside arguing about the general structure, we are able to generate infinity many examples of Leibniz algebras for appropriate geometric objects over a smooth manifold $M$. We present four concrete examples.

The paper is organized as follows. In section 2 we review the observation made by Alekseev and Strobl from [2]. We also discuss its supersymmetric generalization and we set the conventions for the rest of the paper. Section 3 presents the main observation: the algebra of local functionals naturally is equipped with a Leibniz bracket and there is a structure of a weak Courant-

\footnote{Similar observations were made in [4] (see [9] for relevant comments) and more elaborated in [11].}
Dorfman algebra. In section 4 we go through different examples of this structure. Section 5 contains a summary of the paper and the concluding remarks. For the reader’s convenience, in the appendix we review and summarize the key properties of the standard Dorfman and Courant brackets on $T M \oplus T^* M$ and we also give the definitions of (weak) Courant-Dorfman algebras.

2 The Alekseev-Strobl observation

In this section we introduce the relevant notation and review the observation by Alekseev and Strobl from [2]. Moreover, we present its supersymmetric generalization.

The bosonic string phase space can be constructed as follows. Let us consider the loop space

$$LM = \{ X : S^1 \longrightarrow M \} ,$$

which is the space of continuously differentiable maps from the circle $S^1$ to a smooth manifold $M$. The cotangent bundle of $LM$ can be defined as the space of continuous vector bundle morphisms $(X, p) : TS^1 \to T^* M$ with differentiable base maps $X : S^1 \to M$. The symplectic structure on $T^*LM$ is of the standard form and it can be written in local coordinates as follows:

$$\omega = \int_{S^1} d\sigma \, \delta X^\mu \wedge \delta p_\mu , \quad (2.1)$$

where $\delta$ is the de Rham differential on $T^*LM$. Thus, we have a Poisson algebra on the space of functionals, $C^\infty(T^*LM)$. For a section $(v + \omega)$ of $TM + T^* M$, we define a local functional of the special form

$$J_\epsilon(v + \omega) = \int_{S^1} d\sigma \, \epsilon(\nu^\mu p_\mu + \omega_\mu \partial X^\mu) , \quad (2.2)$$

where $\partial$ is a derivative along a loop and $\epsilon : S^1 \to \mathbb{R}$ is a test function. It has been observed in [2] that the Poisson bracket of these special local functionals can be written as

$$\{ J_{\epsilon_1}(A), J_{\epsilon_2}(B) \} = -J_{\epsilon_1 \epsilon_2}(A * B) - \int_{S^1} d\sigma \, (\epsilon_2 \partial \epsilon_1)(A, B) , \quad (2.3)$$

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or alternatively as
\[
\{ J_{\epsilon_1}(A), J_{\epsilon_2}(B) \} = -J_{\epsilon_1\epsilon_2}([A, B]_C) + \int_{S^1} d\sigma \ (\epsilon_1 \partial \epsilon_2 - \epsilon_2 \partial \epsilon_1) \langle A, B \rangle .
\] (2.4)

In these expressions \( A, B \in \Gamma(TM + T^*M) \), * stands for the Dorfman bracket, \([\ , \ ]_C\) for the Courant bracket and \(\langle \ , \ \rangle\) is the natural pairing on \(TM + T^*M\) (see the appendix for a review).

This setup can easily be generalized to the supersymmetric case. Let us define the superloop space as the space of the following maps:
\[
\mathcal{L}M = \{ \Phi : T[1]S^1 \longrightarrow M \},
\]
where \(T[1]S^1\) is a superloop parametrized by the coordinates \((\sigma, \theta)\), with \(\sigma\) being a coordinate along \(S^1\) as before and \(\theta\) is its partner which is a section of the tangent bundle to the circle with reversed parity. In local coordinates \(\Phi\) is
\[
\Phi^{\mu}(\sigma, \theta) = X^{\mu}(\sigma) + \theta \lambda^{\mu}(\sigma) .
\]

Therefore, the superloop space can alternatively be defined as
\[
\mathcal{L}M = \{ S^1 \longrightarrow T[1]M \} .
\]
The corresponding phase space is \(T^*\mathcal{L}M\) which can be defined as the space of bundle morphisms \((\Phi, S) : T[1]S^1 \rightarrow T^*[1]M\). \(T^*\mathcal{L}M\) is equipped with a canonical symplectic structure
\[
\omega = i \int_{T[1]S^1} d\sigma d\theta \delta S^\mu \wedge \delta \Phi^\mu ,
\] (2.5)
where \(S\) is a coordinate along the fiber of the bundle:
\[
S^\mu(\sigma, \theta) = \rho^\mu(\sigma) + i\theta p^\mu(\sigma) .
\] (2.6)

Upon the integration over \(\theta\) the bosonic part of (2.5) coincides with (2.1). The symplectic structure (2.5) makes \(C^\infty(T^*\mathcal{L}M)\) into a super-Poisson algebra. Defining the left and right functional derivatives of a functional \(F(\Phi, S)\) as follows:
\[
\delta F = \int_{T[1]S^1} d\sigma d\theta \left( \frac{\overleftarrow{\partial}}{\partial S^\mu} \delta S^\mu + \frac{\overleftarrow{\partial}}{\partial \Phi^\mu} \delta \Phi^\mu \right)
\]
\[
= \int_{T[1]S^1} d\sigma d\theta \left( \delta S^\mu \frac{\overleftarrow{\partial}}{\partial S^\mu} + \delta \Phi^\mu \frac{\overleftarrow{\partial}}{\partial \Phi^\mu} \right) ,
\] (2.7)
we end up with the corresponding Poisson bracket:
\[
\{F, G\} = i \int_{T[1]S^1} d\sigma d\theta \left( \frac{F \delta \delta G}{\delta S_\mu \delta \Phi^\mu} - \frac{F \delta \delta G}{\delta \Phi^\mu \delta S_\mu} \right).
\] (2.8)

As in the bosonic case, we can define a local functional associated to a section of \( TM + T^*M \) as follows:
\[
J_\epsilon(v + \omega) = \int_{T[1]S^1} d\sigma d\theta \epsilon(v^\mu S_\mu + \omega_\mu D\Phi^\mu),
\] (2.9)

with \( \epsilon(\sigma, \theta) \) being an even test function and the derivative \( D \) is defined as
\[
D = \frac{\partial}{\partial \theta} + i\theta \partial, \quad D^2 = i\partial.
\] (2.10)

With respect to the bracket (2.8) the local functionals (2.9) satisfy
\[
\{J_{\epsilon_1}(A), J_{\epsilon_2}(B)\} = iJ_{\epsilon_1\epsilon_2}([A, B]_C) + \frac{i}{2} \int_{T[1]S^1} d\sigma d\theta (D\epsilon_1\epsilon_2 - \epsilon_1 D\epsilon_2) \langle A, B \rangle.
\] (2.11)

This is the supersymmetric generalization of the bosonic bracket (2.4) which has been discussed previously in [11].

The space \( T^*L^*M \) can be equipped with a more general symplectic structure than (2.5). These symplectic structures are labelled by a closed three form on \( M \). All results and observations can easily be generalized to this situation. However, for clarity, we avoid the case of the most general symplectic structure on \( T^*L^*M \). Further details about \( T^*L^*M \) can be found in [5, 16].

### 3 Leibniz algebra on local functionals

For the sake of clarity, we first discuss the bosonic case. The supersymmetric generalization would be straightforward and we will comment on it later on. The space of smooth functionals \( C^\infty(T^*LM) \) is a Poisson algebra. However, in physics the local functionals play a special role. A local functional is defined as follows:
\[
J_\epsilon(A) = \int_{S^1} d\sigma \epsilon(\sigma)A(X, \partial X, \ldots, \partial^k X, p, \partial p, \ldots, \partial^lp),
\] (3.12)
where \( \epsilon \) is a test function and \( A \) is a function of fields and their derivatives (only finite number of derivatives are allowed). The local functionals \( A_{\text{loc}} \) form a subalgebra with respect to the Poisson bracket. This subalgebra is not a Poisson subalgebra though, since we cannot define a product of two local functionals as a local functional. Next, we can define a binary operation on the integrands as follows:

\[
\{ J_1(A), J_\epsilon(B) \} = J_\epsilon(A * B) . \tag{3.13}
\]

Combining the Jacobi identity

\[
\{ J_1(A), \{ J_1(B), J_\epsilon(C) \} \} + \\
\{ J_\epsilon(C), \{ J_1(A), J_1(B) \} \} + \\
\{ J_1(B), \{ J_\epsilon(C), J_1(A) \} \} = 0 \tag{3.14}
\]

and the antisymmetry of the Poisson bracket with the definition (3.13) we arrive at the relation

\[
J_\epsilon (A * (B * C) − B * (A * C) − (A * B) * C) = 0 . \tag{3.15}
\]

Since (3.15) should be true for any test function \( \epsilon \) we get

\[
A * (B * C) = (A * B) * C + B * (A * C) . \tag{3.16}
\]

The binary product \(*\) gives rise to a Leibniz algebra. A (left) Leibniz algebra (sometimes called a Loday algebra) is a module over a commutative ring or field (in our case \( \mathbb{R} \)) with a bilinear product \(*\) such that (3.16) is satisfied. In other words, left multiplication by any element \( A \) is a derivation. Using the definition (3.13) we write the Poisson bracket between two local functionals in the following form:

\[
\{ J_{\epsilon_1}(A), J_{\epsilon_2}(B) \} = J_{\epsilon_1 \epsilon_2}(A * B) + \Lambda(\epsilon_1, \epsilon_2; A, B) , \tag{3.17}
\]

where the last term \( \Lambda \) will be referred to as the anomalous term. The general form of the anomalous term is

\[
\Lambda(\epsilon_1, \epsilon_2; A, B) = \sum_{i=1}^{\infty} \int_{S^1} d\sigma \left( \epsilon_2 \partial^{(i)} \epsilon_1 \right) f_i(A, B) , \tag{3.18}
\]

where \( f_i(A, B) \) are expressions constructed out of \( A \) and \( B \). For concrete \( A \) and \( B \), the sum in (3.18) would only have a finite number of terms. We
conclude that the Poisson bracket between any two local functionals can be represented in the form \((3.17)-(3.18)\), thus inducing a Leibniz algebra structure on the integrands. It is important to stress that the decomposition on the right hand of \((3.17)\) into a \(J\)-term and an anomalous term a priori is not unique. We remove this ambiguity by requiring that \(\Lambda(1, \epsilon_2; A, B) = 0\), in agreement with our definition \((3.13)\). The Alekseev-Strobl formula \((2.3)\) is just a particular example of this general calculation \((3.17)-(3.18)\).

Alternatively, we could choose the test functions to be \(\delta\)-functions and rewrite the formulas \((3.17)-(3.18)\) as

\[
\{A(\sigma), B(\sigma')\} = (A \ast B)(\sigma')\delta(\sigma - \sigma') + \sum_{i=1}^{\infty} f_i(A, B)(\sigma') \partial_i^{(\sigma)}\delta(\sigma' - \sigma), \quad (3.19)
\]

which is a more familiar form in the context of quantum field theory.

Coming back to the general test functions, the antisymmetry of the Poisson bracket implies

\[
J_{\epsilon_1 \epsilon_2}(A \ast B + B \ast A) = -\Lambda(\epsilon_1, \epsilon_2; A, B) - \Lambda(\epsilon_2, \epsilon_1; B, A). \quad (3.20)
\]

Using the prescription \(\Lambda(1, \epsilon_2; A, B) = 0\) together with the property that \((3.20)\) is true for any choice of the test functions we arrive at

\[
A \ast B + B \ast A = \sum_{i=1}^{\infty} (-1)^{i-1} \partial_i f_i(A, B) = \sum_{i=1}^{\infty} (-1)^{i-1} \partial_i f_i(B, A). \quad (3.21)
\]

Let us define a symmetric bilinear form by

\[
\langle A, B \rangle \equiv \frac{1}{2} \sum_{i=1}^{\infty} (-1)^{i-1} \partial^{(i-1)} f_i(A, B) + f_i(B, A), \quad (3.22)
\]

which allows us to rewrite equation \((3.21)\) as

\[
A \ast B + B \ast A = \partial \langle A, B \rangle. \quad (3.23)
\]

Next, we can write the Jacobi identity with two test functions:

\[
\{J_1(A), \{J_{\epsilon_1}(B), J_{\epsilon_2}(C)\}\} + \\
\{J_{\epsilon_1}(B), \{J_{\epsilon_2}(C), J_1(A)\}\} + \\
\{J_{\epsilon_2}(C), \{J_1(A), J_{\epsilon_1}(B)\}\} = 0, \quad (3.24)
\]
which implies, together with the definition (3.17) and the property (3.16), the following relation:

$$\{J_1(A), \Lambda(\epsilon_1, \epsilon_2; B, C)\} = \Lambda(\epsilon_1, \epsilon_2; B, A \ast C) + \Lambda(\epsilon_1, \epsilon_2; A \ast B, C).$$  \hspace{1cm} (3.25)

Combining (3.18), (3.22) and (3.25), we arrive at the identity

$$\partial(\langle A, \partial(B, C) \rangle - \langle A \ast B, C \rangle - \langle B, A \ast C \rangle) = 0. \hspace{1cm} (3.26)$$

Moreover, using the definition (3.13) and the property $J_\epsilon(\partial B) = - \partial J_\epsilon(B)$ we obtain the following relations between $\partial$ and $\ast$:

$$\partial A \ast B = 0 \hspace{1cm} \text{and} \hspace{1cm} \partial(\langle A \ast B \rangle) = A \ast \partial B.$$

Using $\{J_\epsilon(\partial A), J_1(\partial B)\} = 0$ we get $\langle \partial A, \partial B \rangle = 0$.

Let us now discuss possible interpretations of these structures. At this stage, it is useful to formalize our manipulation with the local functionals using the variational bi-complex (see [7] for a review). On the space of expressions $A(X, \partial X, \ldots, \partial^k X, p, \partial p, \ldots, \partial^l p)$ we have two natural operations: variation and taking the full derivative along the loop. These two operations can be defined as two anticommuting differentials with the underlying bi-grading $A^{(p,q)}$. For further details, the reader may consult [7]. Here we only need part of this structure. Let us define $A^{(0,0)}$ to be the zero-forms on the loop and $A^{(1,0)}$ to be one-forms on the loop. There is a differential $d_h$

$$\mathbb{R} \to A^{(0,0)} \xrightarrow{d_h} A^{(1,0)}$$  \hspace{1cm} (3.27)

where $d_h f = (\partial f) d\sigma$, with $\partial$ being a full derivative along the loop. Thus, the space of local functionals is $A_{loc} = A^{(1,0)}/d_h(A^{(0,0)})$ and in the definition of a local functional (3.12) we use an element from $A^{(1,0)}$. Therefore, the previous discussion can be formalized in the following way. The space $A^{(1,0)}$ is equipped with a bracket $\ast : A^{(1,0)} \times A^{(1,0)} \to A^{(1,0)}$ and an inner product $\langle \cdot, \cdot \rangle : A^{(1,0)} \times A^{(1,0)} \to A^{(0,0)}$ such that the following properties are satisfied:

\begin{align*}
A \ast (B \ast C) &= B \ast (A \ast C) + (A \ast B) \ast C, \hspace{1cm} (3.28a) \\
A \ast B + B \ast A &= d_h \langle A, B \rangle, \hspace{1cm} (3.28b) \\
d_h \langle A, d_h \langle B, C \rangle \rangle &= d_h \langle A \ast B, C \rangle + d_h \langle B, A \ast C \rangle, \hspace{1cm} (3.28c) \\
(d_h f) \ast A &= 0, \hspace{1cm} (3.28d) \\
(d_h f, d_h g) &= 0, \hspace{1cm} (3.28e)
\end{align*}

where $A, B, C \in A^{(1,0)}$ and $f, g \in A^{(0,0)}$. Thus, the space of local functionals $A^{(1,0)}$ together with $A^{(0,0)}$ form a weak Courant-Dorfman algebra as defined
in the appendix. $A_{\text{loc}} = A^{(1,0)}/d_h(A^{(0,0)})$ is equipped with a Lie bracket which is the Poisson bracket on the local functionals. Moreover, there is a simple $L_\infty$-structure which has been discussed in [3] (see also [13] and see [8] for specific examples) and this is related to the weak Courant-Dorfman algebra discussed above.

Also, one can study functionals which form a closed Leibniz subalgebra, i.e. functionals of a special form. Our arguments are also applicable to the case when we only look on this Leibniz subalgebra of the integrands $A^{(1,0)}$ with some specific anomalous terms. In the next section we present a few interesting examples of Leibniz subalgebras arising from this sort of calculations. These examples naturally give rise to weak Courant-Dorfman algebras associated to the smooth manifold $M$.

However, in the above discussion of local functionals we have ignored the test functions in all their glory. We have used them only as a technical tool. Keeping the test functions allows us to talk about the Fourier modes of a local functional. With the test functions, we are dealing with the space $A_{\text{loc}}[[t, t^{-1}]]$, where we have Laurent series with respect to the parameter $t = e^{i\sigma}$. In general, imposing the Jacobi identity on the general expressions (3.17)-(3.18) gives infinity many relations between the operations $\ast$ and $f_i$. This is very reminiscent of the Poisson vertex algebra structure. Thus, in all generality, we should have been discussing a sheaf of Poisson vertex algebras associated to $M$. The weak Courant-Dorfman algebra is only the tip of the iceberg if we adopt this general point of view.

All previous discussion has straightforward generalization to the supersymmetric case $T^*\mathcal{L}M$. The formulas (3.17)-(3.18) remain to be true upon using the odd derivative $D$ instead of $\partial$. Then we have a $\mathbb{Z}_2$-graded Leibniz algebra and a $\mathbb{Z}_2$-graded weak Dorfman-Courant algebra.

Moreover, most of the formulas can be extended to the case of higher dimensional field theories. In particular, the argument around (3.13)-(3.16) will remain true and the structure of a Leibniz algebra would be a generic feature of classical field theories.

4 New brackets from local functionals

This section provides an illustration for the general considerations presented in the previous section. We consider local functionals on $T^*\mathcal{L}M$ of special forms which are parametrized by geometrical data on $M$. These local func-
tionsals give rise to interesting Leibniz subalgebras of $\mathcal{A}^{(1,0)}$ which we discuss here in geometrical terms. We recover a well-known generalization of the Courant bracket as well as new generalizations.

4.1 Schouten bracket

Let us start with a very simple example. Consider a local functional on $T^*\mathcal{L}M$ of the following form:

$$J_\epsilon(v) = \int_{T[1]S^1} d\sigma d\theta \epsilon(v^{\mu_1...\mu_p}(\Phi)S_{\mu_1}...S_{\mu_p}),$$

(4.29)

where $v$ is an antisymmetric $p$-multivector field, $v \in \Gamma(\Lambda^pT^*M)$, and $\epsilon$ is a test function with parity $|\epsilon| = (-1)^{p+1}$. Then the Poisson bracket (2.8) between two such local functionals is given by

$$\{J_{\epsilon_1}(v), J_{\epsilon_2}(u)\} = \int_{T[1]S^1} d\sigma d\theta \epsilon_1\epsilon_2 \left( p v^{\mu_1...\mu_{p+1}}\partial_{\rho}u^{\mu_p...\mu_{p+q-1}} - 
-(1)^{(p+1)(q+1)} q u^{\mu_1...\mu_{q-1}}\partial_{\rho}v^{\mu_p...\mu_{p+q-1}} \right)S_{\mu_1}...S_{\mu_{p+q-1}}$$

$$= iJ_{\epsilon_1\epsilon_2}([v, u]_s),$$

(4.30)

where $u \in \Gamma(\Lambda^qT^*M)$ and $[v, u]_s$ is the Schouten bracket. This calculation can be extended to the local functionals parametrized by any sections of $\Lambda^*T^*M$.

4.2 Courant bracket on $TM \oplus \Lambda^*T^*M$

Consider the following functional:

$$J_\epsilon(v + \beta) = \int_{T[1]S^1} d\sigma d\theta \epsilon \left( v^\mu S_\mu + e \frac{1}{p!} \beta_{\nu_1\nu_2...\nu_p} D(D\Phi^{\nu_1}D\Phi^{\nu_2}...D\Phi^{\nu_p}) \right),$$

(4.31)

where $\epsilon$ is an even test function, $e$ is a constant with parity $|\epsilon| = (-1)^p$, $v$ is a vector field and $\beta$ is a $p$-form. Thus, the local functional (4.31) is even. The Poisson bracket between two such local functionals is

$$\{J_{\epsilon_1}(A), J_{\epsilon_2}(B)\} = iJ_{\epsilon_1\epsilon_2}(A * B) + \Lambda(\epsilon_1, \epsilon_2; A, B),$$

(4.32)
where $A \ast B$ is the Dorfman bracket (A.2) generalized to forms of arbitrary degree. The $\Lambda$-term is given by the following expression

$$
\Lambda(\epsilon_1, \epsilon_2; A, B) = i \int_{T[1]S^1} d\sigma d\theta \left( eD^2\epsilon_1\epsilon_2 \frac{1}{(p-1)!} \langle A, B \rangle_{\nu_1...\nu_{p-1}} D\Phi^{\nu_1} \cdots D\Phi^{\nu_{p-1}} - eD\epsilon_1\epsilon_2 \frac{1}{(p-1)!} \langle A, B \rangle_{\nu_1...\nu_{p-1}} D(D\Phi^{\nu_1} \cdots D\Phi^{\nu_{p-1}}) \right),
$$

(4.33)

where $\langle , \rangle$ stands for the pairing (A.1) extended to $\Gamma(TM \oplus \wedge^pT^*M)$.

Furthermore, if we take the test functions $\epsilon_i$ to be purely bosonic, they obey $\int d\theta \epsilon_i = 0$ and as a result there is the relation $\int d\sigma d\theta \ D\epsilon_1 D\epsilon_2 \text{(anything)} = 0$. Using this, we can rewrite the $\Lambda$-term as follows:

$$
\Lambda(\epsilon_1, \epsilon_2; A, B) = i \int_{T[1]S^1} d\sigma d\theta eD\epsilon_1\epsilon_2 \frac{1}{p!} \langle d\langle A, B \rangle \rangle_{\nu_1...\nu_p} D\Phi^{\nu_1} \cdots D\Phi^{\nu_p}.
$$

(4.34)

Anti-symmetrizing the expression (4.32) and assuming bosonic test functions we arrive at the expression

$$
\{J_{\epsilon_1}(A), J_{\epsilon_2}(B)\} = iJ_{\epsilon_1\epsilon_2}([A, B]_C)
$$

$$
+ \frac{i}{2} \int_{T[1]S^1} d\sigma d\theta \ e(D\epsilon_1\epsilon_2 - \epsilon_1 D\epsilon_2) \frac{1}{p!} \langle d\langle A, B \rangle \rangle_{\nu_1...\nu_p} D\Phi^{\nu_1} \cdots D\Phi^{\nu_p},
$$

(4.35)

where $[ , ]_C$ is the Courant bracket generalized to forms of any degree.

Geometrically we can interpret this results as follows. If we choose $E = \Gamma(TM \oplus \wedge^pT^*M)$ equipped with the Dorfman bracket $\ast$ and $R = \Gamma(\wedge^{p-1}T^*M)$. The symmetric bilinear form $\langle , \rangle : E \otimes E \rightarrow R$ is defined by the formula (A.1) and the exterior derivative is understood as $d : R \rightarrow E$. It can easily be checked that this structure $(E, R, d, \langle , \rangle, \ast)$ satisfies the definition of a weak Courant-Dorfman algebra (see the appendix for details).

With this structure we can naturally introduce the notion of a Dirac structure $D$ as a subbundle of $TM \oplus \wedge^pT^*M$ such that $\Gamma(D)$ is closed under $\ast$ and $d\langle A, B \rangle = 0$ for any $A, B \in \Gamma(D)$. For $A \in \Gamma(D)$ the local functionals (4.31) form a closed algebra under the Poisson bracket since $\Gamma(D)$ is a Lie algebra with respect to $\ast$.

### 4.3 Courant-like bracket on $TM \oplus \wedge^pT^*M \oplus \wedge^pT^*M$

Next, we consider a generalization of the functionals (2.9) and (4.31). For a section $A = v + \beta + \gamma \in \Gamma(TM \oplus \wedge^pT^*M \oplus \wedge^{p+1}T^*M)$ we associate a local
functional of the following form:

$$J_\epsilon(A) = \int_{T[1]S^1} d\sigma d\theta \, \epsilon \left( v^\mu S_\mu + e \frac{1}{p!} \beta_{\nu_1 \ldots \nu_p} D (D\Phi^{\nu_1} \ldots D\Phi^{\nu_p}) + e \frac{(-1)^p}{(p+1)!} \gamma_{\nu_1 \ldots \nu_{p+1}} D\Phi^{\nu_1} \ldots D\Phi^{\nu_{p+1}} \right),$$  \hspace{1cm} (4.36)

where $\epsilon$ is a constant with parity $|\epsilon| = (-1)^p$. The Poisson bracket between two such local functionals has the form

$$\{ J_\epsilon_1(A_1), J_\epsilon_2(A_2) \} = i J_\epsilon_1 \epsilon_2 (A_1 * A_2) + \Lambda(\epsilon_1, \epsilon_2; A_1, A_2),$$  \hspace{1cm} (4.37)

where

$$(v_1 + \beta_1 + \gamma_1) * (v_2 + \beta_2 + \gamma_2) \equiv \{v_1, v_2\} + \mathcal{L}_{v_1}(\beta_2 + \gamma_2) - \iota_{v_2} d(\beta_1 + \gamma_1) + (-1)^p \iota_{v_2} \gamma_1$$  \hspace{1cm} (4.38)

is a new Dorfman bracket which differs from the standard by the last term. One can easily check that it satisfies the Leibniz identity. The $\Lambda$-term in (4.37) is

$$\Lambda = i \int_{T[1]S^1} d\sigma d\theta \left[ e D^2 \epsilon_1 \epsilon_2 \frac{1}{(p-1)!} (\iota_{v_1} \beta_2 + \iota_{v_2} \beta_1) \nu_1 \ldots \nu_{p-1} D\Phi^{\nu_1} \ldots D\Phi^{\nu_{p-1}} ight. \right.$$

$$- e D\epsilon_1 \epsilon_2 \frac{1}{(p-1)!} (\iota_{v_1} \beta_2 + \iota_{v_2} \beta_1) \nu_1 \ldots \nu_{p-1} D (D\Phi^{\nu_1} \ldots D\Phi^{\nu_{p-1}})$$

$$+ e D\epsilon_1 \epsilon_2 \frac{(-1)^p}{p!} (\iota_{v_1} \gamma_2 + \iota_{v_2} \gamma_1) \nu_1 \ldots \nu_p D\Phi^{\nu_1} \ldots D\Phi^{\nu_p}. \hspace{1cm} (4.39)$$

If we choose the test functions to be purely bosonic then the $\Lambda$-term can be simplified to

$$\Lambda = i \int_{T[1]S^1} d\sigma d\theta \, e D\epsilon_1 \epsilon_2 \frac{1}{p!} \left( (-1)^p (\iota_{v_1} \gamma_2 + \iota_{v_2} \gamma_1) \right.$$

$$+ d(\iota_{v_1} \beta_2 + \iota_{v_2} \beta_1)) \nu_1 \ldots \nu_p D\Phi^{\nu_1} \ldots D\Phi^{\nu_p}. \hspace{1cm} (4.40)$$

The anti-symmetrized version of expression (4.37) with bosonic test functions is

$$\{ J_\epsilon_1(A_1), J_\epsilon_2(A_2) \} = i J_\epsilon_1 \epsilon_2 ([A_1, A_2]_C) + i \int_{T[1]S^1} d\sigma d\theta \, e (D\epsilon_1 \epsilon_2 - \epsilon_1 D\epsilon_2) \times$$

$$\times \frac{1}{p!} \left( (-1)^p (\iota_{v_1} \gamma_2 + \iota_{v_2} \gamma_1) + d(\iota_{v_1} \beta_2 + \iota_{v_2} \beta_1) \right) \nu_1 \ldots \nu_p D\Phi^{\nu_1} \ldots D\Phi^{\nu_p}, \hspace{1cm} (4.41)$$
where \([ , ]_C\) is the anti-symmetrization of the new Dorfman bracket (4.38).

Now, let us discuss the geometrical meaning of these structures. \(\mathcal{E} = \Gamma(TM \oplus \Lambda^p T^* M \oplus \Lambda^{p+1} T^* M)\) is equipped with the Dorfman bracket \(*\) given by (4.38). Define \(\mathcal{R} = \Gamma(\Lambda^{p-1} T^* M \oplus \Lambda^p T^* M)\). The symmetric bilinear form \(\mathcal{E} \otimes \mathcal{E} \to \mathcal{R}\) is given by the expression

\[
\langle v_1 + \beta_1 + \gamma_1, v_2 + \beta_2 + \gamma_2 \rangle = \iota_{v_1} (\beta_2 + \gamma_2) + \iota_{v_2} (\beta_1 + \gamma_1) .
\]

(4.42)

Define a linear map \(d_h : \mathcal{R} \to \mathcal{E}\) by

\[
d_h(b_{p-1} + a_p) = db_{p-1} + da_p + (-1)^p a_p ,
\]

(4.43)

where \(b_{p-1}\) is \((p-1)\)-form and \(a_p\) is \(p\)-form. If we consider the operation \(d_h\) defined for all values values of \(p\) by this formula then \(d^2_h = 0\). The symmetrization of the bracket (4.38) is given by

\[
A_1 * A_2 + A_2 * A_1 = (-1)^p (\iota_{v_1} \gamma_2 + \iota_{v_2} \gamma_1) + d (\iota_{v_1} (\beta_2 + \gamma_2) + \iota_{v_2} (\beta_1 + \gamma_1)) ,
\]

(4.44)

which can be written as

\[
A_1 * A_2 + A_2 * A_1 = d_h (A_1, A_2)
\]

(4.45)

if we use the above definitions. Indeed, it is easy to check that \((\mathcal{E}, \mathcal{R}, d_h, \langle , , \rangle, \ast)\) is a weak Courant-Dorfman algebra. It is interesting to mention that the bracket (4.38) can be understood as a derived bracket for the differential \(d_h\) (see [12] for a review of derived brackets).

The Dirac structure \(\mathcal{D}\) would be defined as a subbundle of \(TM \oplus \Lambda^p T^* M \oplus \Lambda^{p+1} T^* M\) such that \(\Gamma(\mathcal{D})\) is closed under \(*\) and \(d_h (A_1, A_2) = 0\) for every \(A_1, A_2 \in \Gamma(\mathcal{D})\). Thus \(\Gamma(\mathcal{D})\) is a Lie algebra with respect to \(*\). Decomposing \(d_h (A_1, A_2) = 0\) in form degrees give us two conditions:

\[
(\iota_{v_1} \gamma_2 + \iota_{v_2} \gamma_1) = (-1)^{p+1} d (\iota_{v_1} \beta_2 + \iota_{v_2} \beta_1) ,
\]

(4.46)

\[
d (\iota_{v_1} \gamma_2 + \iota_{v_2} \gamma_1) = 0 ,
\]

(4.47)

where (4.46) implies (4.47). Therefore it is enough that (4.46) is satisfied. This is exactly the same as requiring that the \(\Lambda\)-term in (4.40) vanishes. Thus, the local functionals (4.36) for \(A \in \Gamma(\mathcal{D})\) form a Lie algebra with respect to the Poisson bracket.
4.4 Bracket associated with a symmetric tensor

Next, we consider a different example of a local functional, which is parametrized by a vector field $v$, a symmetric tensor of second rank $\gamma$ and a one form $\rho$:

$$J_\epsilon(A) = \int_{T[S^1]} d\sigma d\theta \, \epsilon \left( v^\mu S_{\mu} + e_{\gamma_{\mu\nu}} \partial \Phi^\mu \partial \Phi^\nu + e_{\rho_{\mu}} \nabla \partial \Phi^\mu \right) , \quad (4.48)$$

where $A = v + \gamma + \rho$ and

$$\nabla \partial \Phi^\mu = \partial^2 \Phi^\mu + \Gamma_{\nu\rho}^\mu \partial \Phi^\nu \partial \Phi^\rho , \quad (4.49)$$

with $\Gamma_{\nu\rho}^\mu$ being a torsionless connection. We need to introduce a connection in order to make the local functional (4.48) invariant under diffeomorphisms of $M$. In (4.48), $e$ is an odd constant, and $J_\epsilon(A)$ is thus even.

The Poisson bracket between two such local functionals yields

$$\{ J_\epsilon_1(A_1), J_\epsilon_2(A_2) \} = i J_{\epsilon_1 \epsilon_2}(A_1 \ast A_2) + \Lambda(\epsilon_1, \epsilon_2, A_1, A_2) . \quad (4.50)$$

The bracket $A_1 \ast A_2$ is given by

$$\left( A_1 \ast A_2 \right)^\mu_{(T)} = \{v_1, v_2\}^\mu , \quad (4.51a)$$

$$\left( A_1 \ast A_2 \right)^{\mu \nu}_{(T \odot T^*)} = (L_{v_1} \gamma_2 - L_{v_2} \gamma_1)_{\mu \nu} + \nabla_{(\mu} (L_{v_1} \rho_2 + 2 t_{v_2} \gamma_1)_{\nu)} , \quad (4.51b)$$

$$\left( A_1 \ast A_2 \right)^{\mu}_{(T^*)} = (L_{v_1} \rho_2 + 2 t_{v_2} \gamma_1)_{\mu} , \quad (4.51c)$$

where we have defined $(\gamma_i)_{\mu \nu} \equiv (\gamma_i)_{\mu \nu} - \nabla_{(\mu} \rho_{\nu)}$. The symmetrization is defined with a normalization factor, e.g. $\gamma(\mu \nu) \equiv \frac{1}{2}(\gamma_{\mu \nu} + \gamma_{\nu \mu})$. This operation * satisfies the Leibniz identity! In (4.50) the anomalous term is given by

$$\Lambda(\epsilon_1, \epsilon_2; A_1, A_2) = i \int_{T[S^1]} d\sigma d\theta \, \partial \epsilon_1 \epsilon_2 \, 2 \left( (t_{v_1} \gamma_2 + t_{v_2} \gamma_1)_{\mu} + \nabla \nu v^\rho_1 \rho_2 - v^\rho_2 \nabla \rho_1 \right) \partial \Phi^\mu \quad (4.52)$$

$$+ i \int_{T[S^1]} d\sigma d\theta \, \partial^2 \epsilon_1 \epsilon_2 \left( t_{v_1} \rho_2 - t_{v_2} \rho_1 \right) .$$

Let us show how this calculation gives rise to another example of a weak Dorfman-Courant algebra. Let $\mathcal{E} = \Gamma(TM \oplus S^2T^*M \oplus T^*M)$ be the space of vector fields plus symmetric tensors of second rank plus one forms on $M$. 

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Define $\mathcal{R} = \Gamma(T^*M)$. The symmetric bilinear form $\langle , \rangle : \mathcal{E} \otimes \mathcal{E} \rightarrow \mathcal{R}$ is given by the expression

$$\langle v_1 + \gamma_1 + \rho_1, v_2 + \gamma_2 + \rho_2 \rangle = (\mathcal{L}_{v_1} \rho_2 + \mathcal{L}_{v_2} \rho_1 + 2\nu_1 \gamma_1 + 2\nu_2 \gamma_2). \quad (4.53)$$

Next, define a map $d_h : \mathcal{R} \rightarrow \mathcal{E}$, which sends a one-form to a symmetric tensor of second rank and a one form, as follows:

$$d_h \alpha = \frac{1}{2} \nabla_\mu \alpha_\nu (dx^\mu \otimes dx^\nu + dx^\nu \otimes dx^\mu) + \alpha_\mu dx^\mu, \quad (4.54)$$

where $\alpha = \alpha_\mu dx^\mu$ is a one form. With these definitions, the symmetrization of the bracket (4.51) is given by

$$A_1 \ast A_2 + A_2 \ast A_1 = d_h \langle A_1, A_2 \rangle. \quad (4.55)$$

Moreover, one can check the property

$$\langle A, d_h \langle B, C \rangle \rangle = \langle A \ast B, C \rangle + \langle B, A \ast C \rangle, \quad (4.56)$$

where $A, B, C \in \mathcal{E}$. Thus, we conclude that with the present definitions, $(\mathcal{E}, \mathcal{R}, d_h, \langle , \rangle, \ast)$ is a weak Courant-Dorfman algebra. This structure depends on the choice of a torsionless connection on $TM$.

5 Summary

In this work we have investigated some of the algebraic properties of the algebra of Poisson brackets between local functionals. We were considering the cotangent bundle of the (super) loop space $T^*LM$ as basis for our consideration. We have argued that there exist a natural structure of weak Courant-Dorfman algebras. We have considered four different examples which gave interesting specific cases of weak Courant-Dorfman algebras, realized on the geometric objects of $M$. One can generate infinity many more examples along those lines. For example, we can consider local functionals of the form

$$J_\epsilon(A) = \int_{T[1]S^1} d\sigma d\theta \epsilon \left( v^\mu S_\mu + \sum_{k_1+\cdots+k_p=\text{fixed}} A(\Phi)_{\nu_1...\nu_p} D^{k_1} \Phi^{\nu_1} \cdots D^{k_p} \Phi^{\nu_p} \right), \quad (5.57)$$

which will give rise to Poisson subalgebras. In (5.57), the components of $A$ are not tensors in general. Everything can be covariantized by introducing
a connection, as in the fourth example in section 4. For these functionals, one can repeat the analysis we have presented and get new weak Courant-Dorfman structures.

There is also another interesting aspect of the current work. The closed subalgebras under the Poisson bracket of local functionals can be interpreted as first class constrains and thus related to gauge symmetries of a theory. This is exactly the case in our analysis when the anomalous term vanishes. In [2] some examples of theories corresponding to first class constraints of the form (2.2) were presented. It could be interesting to study the gauge symmetries which would arise from the closed algebras considered here (the generalization of a Dirac structure when the anomalous term vanishes).

Let us comment on the wider context for our results. The observations presented in this work are closely related to similar statements which have appeared previously in the literature. Historically, the first reference goes back to 1980 when Gel’fand and Dorfman [10] developed the framework for the variational calculus with application to the integrable systems. From these considerations the Dorfman brackets initially appeared. At the same time, the Dorfman bracket appeared in the context of the chiral de Rahm complex (a sheaf of vertex algebras) in [6]. In this respect the Alekseev-Strobl result is a classical version of this observation. We believe that these different observations are closely related. We think that the natural mathematical framework for a better understanding of our observations would be the sheaf of (super)Poisson vertex algebras over $M$. We hope to come back to this problem elsewhere.

Another point we would like to stress is that the considerations presented in section 3 easily can be generalized for a higher dimensional classical field theory. Indeed, a specific example has been discussed previously in [4], [11]. Thus, the appearance of a Leibniz algebra and other related structures is a generic feature of field theory. Hopefully, it may help to understand the quantization in a more algebraic way, in analogy with vertex algebras.

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A Courant-Dorfman algebra

In this appendix we collect the standard properties of the Dorfman and Courant brackets for the reader’s convenience.

On the section of tangent plus cotangent bundle \((T^*M)\) we can define the canonical pairing \(\Gamma(T^*M) \times \Gamma(T^*M) \to \mathbb{C}^\infty(M)\)
\[
\langle v_1 + \beta_1, v_2 + \beta_2 \rangle = (\iota_{v_1} \beta_2 + \iota_{v_2} \beta_1) ,
\]
and the bilinear operation \(\Gamma(T^*M) \times \Gamma(T^*M) \to \Gamma(T^*M)\)
\[
(v_1 + \beta_1) \ast (v_2 + \beta_2) = \{v_1, v_2\} + \mathcal{L}_{v_1} \beta_2 - \iota_{v_2} d \beta_1 ,
\]
which is called the Dorfman bracket. In the above formulas \(\{,\}\) is a Lie bracket of the vector fields, \(\mathcal{L}_v\) is a Lie derivative, \(\iota_v\) is a contraction and \(d\) is the standard exterior derivative on the differential forms. By a direct calculation one can check the Leibniz identity for the Dorfman bracket,
\[
A \ast (f B) = f (A \ast B) + \langle A, df \rangle B ,
\]
\(A, B, C \in \Gamma(T^*M)\), which makes \(\Gamma(T^*M)\) into a Leibniz algebra. Moreover, the following additional properties are satisfied:
\[
A, B, C \in \Gamma(T^*M) \quad \text{and} \quad f, g \in C^\infty(M) .
\]
The Courant bracket is defined as the antisymmetrization of the Dorfman bracket:
\[
[A, B] = \frac{1}{2} (A \ast B - B \ast A) .
\]
As follows from \((A.4b)\) that the Courant bracket is related to the Dorfman bracket as
\[
[A, B] = A \ast B - \frac{1}{2} d \langle A, B \rangle .
\]
If one understands the exterior derivative \(d\) as a map \(d : C^\infty(M) \to \Gamma(TM \oplus T^*M)\) then one can formalize the present structure through the notion of a Courant-Dorfman algebra.

Inspired by the example of \((T^*M)\) and following [15] a Courant-Dorfman algebra \((\mathcal{E}, \mathcal{R}, \partial, \langle, \rangle, \ast)\) consists of the following data:
(a) a commutative $K$-algebra $\mathcal{R}$

(b) an $\mathcal{R}$-module $\mathcal{E}$

(c) a symmetric bilinear form $\langle \ , \ \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{R}$

(d) a derivation $\partial : \mathcal{R} \to \mathcal{E}$

(e) a Dorfman bracket $\ast : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$

which satisfy the following axioms:

1. $A \ast (fB) = f(A \ast B) + \langle A, \partial f \rangle B$
2. $\langle A, \partial (B, C) \rangle = \langle A \ast B, C \rangle + \langle B, A \ast C \rangle$
3. $A \ast B + B \ast A = \partial \langle A, B \rangle$
4. $A \ast (B \ast C) = (A \ast B) \ast C + B \ast (A \ast C)$
5. $(\partial f) \ast A = 0$
6. $\langle \partial f, \partial g \rangle = 0$

where $A, B, C \in \mathcal{E}$ and $f, g \in \mathcal{R}$.

If the symmetric bilinear form $\langle \ , \ \rangle$ is non-degenerate in an appropriate sense [15] then in the above definition the conditions (1), (5) and (6) are redundant.

In the present work, we need a weaker notion of the Courant-Dorfman algebra where $\mathcal{R}$ would not be a commutative algebra and $\mathcal{E}$ would not be an $\mathcal{R}$-module. A weak Courant-Dorfman algebra $(\mathcal{E}, \mathcal{R}, \partial, \langle \ , \ \rangle, \ast)$ is defined by the following data:

(a) a vector space $\mathcal{R}$

(b) a vector space $\mathcal{E}$

(c) a symmetric bilinear form $\langle \ , \ \rangle : \mathcal{E} \otimes \mathcal{E} \to \mathcal{R}$

(d) a map $\partial : \mathcal{R} \to \mathcal{E}$

(e) a Dorfman bracket $\ast : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$

which satisfy the following axioms:

2. $\partial (\langle A, \partial (B, C) \rangle - \langle A \ast B, C \rangle - \langle B, A \ast C \rangle) = 0$
3. $A \ast B + B \ast A = \partial \langle A, B \rangle$
(4) \( A \ast (B \ast C) = (A \ast B) \ast C + B \ast (A \ast C) \)

(5) \((\partial f) \ast A = 0\)

(6) \(\langle \partial f, \partial g \rangle = 0\)

where \(A, B, C \in \mathcal{E}\) and \(f, g \in \mathcal{R}\).

Comparing the two definitions, we modify the property (2) besides changing the properties related to the algebraic structures of \(\mathcal{R}\) and \(\mathcal{E}\).

An example of a weak Courant-Dorfman algebra is given by \(\mathcal{E} = \Gamma(TM \oplus \wedge^p T^*M)\) and \(\mathcal{R} = \Gamma(\wedge^{p-1} T^*M)\), where the symmetric bilinear form \(\langle \, , \, \rangle\) and the Dorfman bracket \(\ast\) are defined formally by the same formulas (A.1) and (A.2) but now \((v_1 + \beta_1), (v_2 + \beta_2) \in \Gamma(TM \oplus \wedge^p T^*M)\). The map \(\partial\) is the exterior derivative acting on \((p - 1)\)-forms. All the properties can easily be checked explicitly.

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