How fast does a random walk cover a torus?

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(Dated: June 20, 2017)

We present high statistics simulation data for the average time $\langle T_{\text{cover}}(L) \rangle$ that a random walk needs to cover completely a 2-dimensional torus of size $L \times L$. They confirm the mathematical prediction that $\langle T_{\text{cover}}(L) \rangle \sim (L \ln L)^2$ for large $L$, but the prefactor seems to deviate significantly from the supposedly exact result $4/\pi$ derived by A. Dembo et al., Ann. Math. 160, 433 (2004), if the most straightforward extrapolation is used. On the other hand, we find that this scaling does hold for the time $T_{N(t)=1}(L)$ at which the average number of yet uncovered sites is 1, as also predicted previously. This might suggest (wrongly) that $\langle T_{\text{cover}}(L) \rangle$ and $T_{N(t)=1}(L)$ scale differently, although the distribution of rescaled cover times becomes sharp in the limit $L \to \infty$. But our results can be reconciled with those of Dembo et al. by a very slow and non-monotonic convergence of $\langle T_{\text{cover}}(L) \rangle / (L \ln L)^2$, as had been indeed proven by Belius et al. [Prob. Theory & Related Fields 167, 1 (2014)] for Brownian walks, and was conjectured by them to hold also for lattice walks.

The problem of how fast a random walk covers a 2-dimensional torus was introduced in the mathematical literature by Wilf [1], who called it the “white screen problem”. But it is also of considerable interest for other sciences, as it relates e.g. to how fast a grazing animal can collect as much food as possible [2–4], or how fast information can be spread on or collected from a network (such as a mobile ad hoc network) whose topology is not known [5–7]. For that reason, it has also been discussed extensively in the statistical physics literature [8–13].

Let us denote by $\langle T_{\text{cover}}(L) \rangle$ the average time needed to cover a torus of $L \times L$ sites completely, and by $T_{N(t)=1}(L)$ the time at which the average number of yet uncovered sites is 1. Naively one would expect that both diverge in the same way with $L$, at least if the distribution of cover times is not too broad.

Aldous [14,15] proved that

$$\langle T_{\text{cover}}(L) \rangle \lesssim \frac{4}{\pi} L^2 \ln^2 L, \quad (1)$$

and proved that the re-scaled time, $T_{\text{cover}}(L)/(L \ln L)^2$, is indeed $\delta$-distributed in the limit $L \to \infty$. He furthermore conjectured that Eq.(1) becomes sharp in this limit.

This conjecture was supported by heuristic arguments in [9,10], where the main quantity of interest was not $\langle T_{\text{cover}}(L) \rangle$ but $T_{N(t)=1}(L)$. These authors argued convincingly that

$$T_{N(t)=1}(L)/(L \ln L)^2 \to \frac{4}{\pi} \quad \text{for} \quad L \to \infty, \quad (2)$$

and then conjectured that the same is true also for the cover times, because mean cover times and times at which the average number of uncovered sites is 1 should scale in the same way.

The story was seemingly closed when Dembo et al. [16] proved rigorously that

$$\lim_{L \to \infty} \frac{T_{\text{cover}}(L)}{(L \ln L)^2} = \frac{4}{\pi} \quad \text{in probability}, \quad (3)$$

i.e. Aldous’ inequality Eq.(1) is saturated and the limit distribution is indeed sharp.

When I re-considered this problem, I was primarily interested in the way how “true self avoiding” walks (or “self-repelling walks”) [17] cover the torus or any other finite lattice [7,18], and wanted just to document the dramatic difference between self-repelling and ordinary random walks. However, soon after I started to simulate ordinary random walks on the 2-torus, it became clear that the data agreed with Eqs.(1) and (2), but not easily with Eq.(3).

The results presented in the following come from simulations that altogether took about 1 year of CPU time on modern workstations. Lattice sizes ranged from $L = 16$ to $L = 65536$ in steps of powers of 2. The number of walks simulated varied between $\approx 4 \times 10^7$ for $L = 16$ and $1350$ for $L = 65536$. For easier coding and faster codes, boundary conditions (b.c.) were not strictly periodic but helical [19]. For large $L$ the difference is negligible. In particular, also for helical b.c. the lattice is a torus, and the difference with periodic b.c. is just that one of the coordinate axes is slightly tilted. We verified that the results obtained with periodic b.c. were identical within statistical errors for $L \geq 16$. We also tested two different random number generators (Ziff’s four-tap generator [20] and the UNIX generator rand48), again with no significant differences.

Results for $\langle T_{\text{cover}}(L)/(L \ln L)^2 \rangle$ against $L$ are shown in Fig. 1. Whenever error bars are not visible on the data points, they are smaller than the line thickness. Also shown is the prediction of Dembo et al. [16] (horizontal line) and a fit for large $L$. This fit is a least square fit (with all three constants fitted) to all data with $L \geq 128$, but the quoted error in the first term is much bigger that the purely statistical error, in order to include plausible further correction terms – where we assume that “plausible” correction terms do not ruin the monotonicity. Our
first conclusion is thus that
\[
\lim_{L \to \infty} \frac{\langle T_{\text{cover}}(L) \rangle}{(L \ln L)^2} = 1.2473 \pm 0.0012. \tag{4}
\]

The right hand side disagrees with the supposedly exact value \(4/\pi = 1.2732 \ldots\) by about 22 standard deviations (similar results have been obtained in [12], albeit with less statistics). This discrepancy can hardly be blamed on statistical fluctuations (the likelihood being about \(10^{-100}\)). It cannot be blamed on the used random number generators, both of which have been proven to be reliable even in problems involving much higher statistics. In view of the extreme simplicity of the code (about one page), also a programming error is very unlikely.

A next problem that could cause a wrong asymptotic estimate could be a very skewed and broad distribution of cover times. But the distribution of normalized cover times is expected [21] to be a (randomly shifted) Gumbel distribution in the limit \(L \to \infty\). This gives a roughly exponential tail, which could not significantly bias any estimates of average cover times.

In any case, in Figs. 2 and 3 we show such distributions. They seem to be indeed exponentially cut off at large times, and definitely do not suggest that estimates of the averages could be influenced significantly by large \(T\) tails.

To add to the last point, we show in Fig. 4 our estimates of the relative fluctuations of \(T_{\text{cover}}\), defined as \(\langle \text{Var}[T_{\text{cover}}] \rangle^{1/2}/\langle T_{\text{cover}} \rangle\). We see that they decrease with \(L\) as predicted by Aldous, although our data are not precise enough to distinguish between a power-law decay with a very small exponent (\(\approx 0.11\)) and a logarithmic behavior.

To shed more light on this problem, we considered next the average number \(N(t)\) of uncovered sites at time \(t\). For \(t \ll L^2\), the number of covered sites is independent of \(L\), and given asymptotically by [23]

\[
s(t) \equiv L^2 - N(t) = \frac{\pi t}{\ln t} \left[ 1 + O\left(\frac{\ln \ln t}{\ln t}\right) \right]. \tag{5}
\]

The finiteness of the lattice becomes relevant for \(t \approx L^2\), and for \(t \gg L^2\) the decay of \(N(t)\) is a pure exponential. The cross-over between these two regimes is shown in Fig. 5. There we show on the y-axis not \(N(t)/L^2\) itself, but we multiplied it with \(\exp(t/\tau(L))\), where the characteristic time \(\tau(L)\) (the inverse decay rate) was estimated
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![FIG. 4](image1) (color online) Relative fluctuations of cover times, plotted against $L$.

![FIG. 5](image2) (color online) The average number $N(t)$ of uncovered sites at time, plotted for different values of $L$ in the regime $t < L^2$. For clarity, we show on the y-axis not $N(t)$ itself but $N(t)/L^2 \exp(t/\tau(L))$, where $\tau(L)$ is the numerically found inverse decay rate of $N(t)$ for $t \gg L^2$. The uppermost curve is for $L = 65536$, the lowest is for $L = 64$.

![FIG. 6](image3) (color online) The average number $N(t)$ of uncovered sites at time, plotted against $t$, for $L = 8192$. The horizontal and vertical straight lines indicate the values $N(t) = 1$ and $t = \langle T_{\text{cover}} \rangle$.

![FIG. 7](image4) (color online) Direct estimates of $T_{N(t)=1}/(L \ln L)^2$ plotted against $L$ on a log-linear plot. The fit just demonstrates that the data are compatible with Eq. (2).

from fits in the regime $L^2 < t < (L \ln L)^2$. The quality of the exponential decay in this regime is illustrated in Fig. 6 for $L = 8192$ (but similarly nice exponentials were also found for all other lattice sizes). In Fig. 6 we plotted $N(t)$ itself, and we verified that the exponential decay continued also for $t \gg (L \ln L)^2$, although statistical errors increase rapidly for large $t$.

This purely exponential decay can be used to determine $\tau(L)$ either by a fit in the regime $L^2 < t < (L \ln L)^2$ or by just finding the value of $t$ where $N(t) = 1$. In the second method we of course have to take into account that the exponential decay holds only for $t > L^2$, but this correction becomes negligible for $L \to \infty$, i.e.

$$T_{N(t)=1} = 2\tau(L) \ln L \times [1 + O(1/\ln^2 L)]. \quad (6)$$

Direct numerical estimates of $T_{N(t)=1}/(L \ln L)^2$ are shown in Fig. 7. We see a much slower (probably logarithmic) convergence than for average cover times, but the data are completely compatible with Eq. (2).

A last reason for a wrong asymptotic estimate would be a very slow (and non-monotonic!) convergence with $L$. We found no indication for this in our data, but it is conjectured in [22] that the behavior for walks on the square lattice is as for off-lattice Brownian walks, which...
analytic curves representing Eq.(7): One with $D = 0$ and the other with $D = 2$. In Fig. 8 we would suggest [22]

$$
\frac{\pi (T_{\text{cover}}(L))}{4(L \ln L)^2} = 1 - \frac{1}{2} \ln \ln L / \ln L + D / \ln L + o(1/\ln L)
$$

(7)

with an unknown constant $D$ (indeed, the conjecture in [22] for lattice walks was slightly weaker). In Fig. 8 we show the data shown already in Fig. 1 together with two analytic curves representing Eq.(7): One with $D = 0$, and the other with $D = 2$. We see that the latter gives a very good fit, from which we conclude that the mathematical predictions are presumably all correct, and $D = 2.02(2)$. We should warn, however, that we could also give decent fits with different coefficients of the $\ln \ln L / \ln L$ term (and, of course, different $D$).

Finally, we show in Fig. 9 the ratios $T_{N(t)=1}/\langle T_{\text{cover}} \rangle$. For very small $L$ they are < 1, because the large-$T$ tails contribute more to $\langle T_{\text{cover}} \rangle$ than to $T_{N(t)=1}$. For larger $L$ this effect is outweighed by the fact that $N(\langle T_{\text{cover}} \rangle) > 1$ because walks that do not yet cover at $t = \langle T_{\text{cover}} \rangle$ might have $\gg 1$ uncovered sites. Finally, at very large $L$, the ratio seems to decrease again, although this is not significant in view of the large error bars. Yet it suggests that the ratio converges to 1 for $L \to \infty$, which would completely reconcile our data with the mathematical proofs. This is supported by the fact (O. Zeitouni, private communication) that the $\ln \ln L / \ln L$ term is absent in $T_{N(t)=1}$.

In summary, our numerical data suggest at face value that $T_{N(t)=1}$ and $\langle T_{\text{cover}} \rangle$ do not scale in the same way with $L$, in contrast to rigorous proofs. But they can be reconciled with the proofs, if the (predicted) corrections to scaling are taken into account. As a result, the convergence towards the asymptotic behavior should be extremely slow (and non-monotonic!). Thus, without knowing the subleading terms, attempts to verify the leading behavior numerically would be futile.

The present paper can be seen as a warning that supposedly rigorous proofs can be wrong (and should thus be checked numerically), but more so as a warning that extrapolations of numerical data can be very subtle and misleading, even if they look completely benign and harmless. The vast number of wrong critical exponent estimates found in the literature bears ample witness to that. Combining rigorous mathematics and numerics can be useful if, as in the present case, the mathematics exclude too naive parametrizations, and the numerics can suggest the value(s) of constants that remain undetermined by the mathematical arguments.

I thank Pradeep K. Mohanty, Bob Ziff, and Ofer Zeitouni for carefully reading the manuscript, and to Ricardo Mendonça for pointing out Ref. [12]. To all of them and also to David Belius, I am indebted for extremely helpful discussions.

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