Conservation laws protect dynamic spin correlations from decay:
Limited role of integrability in the central spin model

Götz S. Uhrig,1 Johannes Hackmann,2 Daniel Stanek,1 Joachim Stolze,1 and Frithjof B. Anders2

1Lehrstuhl für Theoretische Physik I, Technische Universität Dortmund,
Otto-Hahn Straße 4, 44221 Dortmund, Germany
2Lehrstuhl für Theoretische Physik II, Technische Universität Dortmund,
Otto-Hahn Straße 4, 44221 Dortmund, Germany

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Mazur’s inequality renders statements about persistent correlations possible. We generalize it in
a convenient form applicable to any set of linearly independent constants of motion. This approach
is used to show rigorously that a fraction of the initial spin correlations persists indefinitely in the
isotropic central spin model unless the average coupling vanishes. The central spin model describes
a major mechanism of decoherence in a large class of potential realizations of quantum bits. Thus
the derived results contribute significantly to the understanding of the preservation of coherence.
We will show that persisting quantum correlations are not linked to the integrability of the model,
but caused by a finite operator overlap with a finite set of constants of motion.

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a. Introduction. The two-time correlation function of two observables reveals important information about
the dynamics of a system in and out of equilibrium: The noise spectra are obtained from symmetric combinations
of correlation functions, while the causal, antisymmetric combination determines the susceptibilities required for
the theory of linear response.

The two-time correlation function only depends on the time difference if at \( t = 0 \) the system of interest is prepared
in a stationary state whose density operator commutes with the time-independent Hamiltonian. This is
what will be considered in this work. Since correlations generically decay for \( t \to \infty \), important information about
the system dynamics is gained if a non-decaying fraction of correlations prevails at infinite times. Such
non-decaying correlations are clearly connected to a limited dynamics in certain subspaces of the Hilbert space.
The question arises if such a restricted dynamics is always linked to the integrability of the Hamiltonian. Here
integrability means that the Hamiltonian can be diagonalized by Bethe ansatz which implies that there is an
extensive number of constants of motion. Identifying and understanding those non-decaying correlations can be po-
tentially exploited in applications for persistent storage of (quantum) information.

In this Letter we first prove that persisting correlations are not restricted to integrable systems by using
a generalized form of Mazur’s inequality\(^{12}\). This is in contrast to the behavior of the Drude weight in the
frequency-dependent conductivity of one-dimensional systems which appears to vanish abruptly once the in-
tegrability is lost, even if only by including an arbitrarily small perturbation. So far, the Drude weight has been
the most common application of Mazur’s inequality, see for instance Refs.\(^{2,6,9}\) and references therein. Second, we
apply this approach to the central spin model (CSM)\(^7\) describing the interaction of a single spin, e.g., an
electronic spin in a quantum dot\(^8,9\), an effective two-level

b. General Derivation. To establish the key idea
and to fix the notation we present the following mod-
ifed derivation related to Suzuki’s derivation in Ref.\(^2\).
We consider the time-independent Hamiltonian \( \hat{H} \) and the operator \( \hat{A} \) with a vanishing expectation value
\( \langle \hat{A} \rangle = 0 \) with respect to a stationary density operator \( \rho \), i.e., \( [\rho, \hat{H}] = 0 \) so that two-time correlation functions
only depend on the time difference. Note that \( \rho \) does not need to be the equilibrium density operator. Then, \( \rho \) and
\( H \) have a complete common eigenbasis \( \{|j\}\) in a finite-
dimensional Hilbert space, and their spectra are \( \{ \rho_j > 0 \} \) and \( \{ E_j \} \), respectively. We define the correlation function of \( A \) as
\[
S(t) := \langle A(t)A(0) \rangle = \text{Tr} \left[ \rho A(t)A(0) \right] = \sum_{j,m} \rho_j |A_{jm}|^2 \exp(i(E_j - E_m)t),
\]
so that Eq. (1b) is its Lehmann representation, and \( A_{jm} := \langle j|A|m \rangle \) denotes the matrix element of \( A \). Physically, \( S(t) \) stands for a measurement of \( A^\dagger \) at time \( t \) after the evolution from the initial state prepared by applying \( A \) at \( t = 0 \). Especially, for \( A = S^2 \) of a spin \( S = 1/2 \) in a disordered environment, \( S(t) \) is proportional to \( \langle S^z(t) \rangle \) if \( \langle S^z(0) \rangle = 1/2 \), see Supplement A for details. If \( \lim_{t \to \infty} S(t) \) exists, it is given by
\[
S_\infty := \sum_{j,m} \rho_j |A_{jm}|^2 \delta_{E_j,E_m} \geq 0.
\]
If \( S(t \to \infty) \) does not exist, and \( |S(t)| < \infty \), the long-time average \( \lim_{T \to \infty} T^{-1} \int_0^T S(t) dt = S_\infty \) is projecting out the time-independent part \( S_\infty \) and uniquely defines the non-decaying fraction of the correlation.

In practice, the Lehmann representation requires the complete diagonalization of \( H \) which is not feasible for large systems. Hence one resorts to constants of motion. To this end, we define the scalar product for two operators \( X \) and \( Y \) as
\[
(X|Y) := \langle X^\dagger Y \rangle = \text{Tr} \left[ \rho X^\dagger Y \right]
\]
in the super-Hilbert space of the operators. If a set of \( M \) conserved linearly independent operators \( X_i \) with \( [X_i, H] = 0 \) is known, one may assume their orthonormality \( (X_i|X_m) = \delta_{im} \) provided by a Gram-Schmidt process. Then, we expand the operator of interest \( A \)
\[
A = \sum_{i=1}^M a_i X_i + R
\]
in this incomplete operator basis where \( a_i := (X_i|A) \) and \( R \) is the remaining rest with \( (X_i|R) = 0 \) \( \forall i \in \{ 1, \ldots, M \} \). Substituting (1) into the definition (1a) yields
\[
S(t) = \sum_{i=1}^M |a_i|^2 + S^{(R)}(t)
\]
with \( S^{(R)}(t) := \langle R(t)R(0) \rangle \). This relies on the constancy of (i) \( (X_i^\dagger(t)X_m(0)) = \delta_{im} \), of (ii) \( (X_i^\dagger(t)R(0)) = 0 \), and of (iii) \( (R^\dagger(t)X_m(0)) = 0 \) all stemming from \( [X_j, H] = 0 \). For the last relation we have used the cyclic invariance of the trace and \( [\rho, H] = 0 \).

If we knew \( \lim_{t \to \infty} S^{(R)}(t) = 0 \), we would deduce \( S_\infty = \sum_{i=1}^M |a_i|^2 \). But in general this does not hold because \( R \) may still contain a non-decaying part. But (5) implies Mazur’s inequality
\[
S_\infty \geq S_{\text{low}} := \sum_{i=1}^M |a_i|^2.
\]
For a given \( H \), the complete set of conserved eigenstates \( \Gamma \) is spanned by all pairs of energy-degenerate eigenstates
\[
\Gamma := \{ |j\rangle |m\rangle/\sqrt{\rho_m} \text{ with } E_j = E_m \}.
\]
The elements of \( \Gamma \) are orthonormal with respect to the scalar product (3). The coefficient \( a_{jm} := \langle j|m \rangle/\sqrt{\rho_m} \) takes the value \( \sqrt{\rho_m}A_{jm} \) so that the right hand side of (6) equals \( S_\infty \) as given by the Lehmann representation (2). Thus, the inequality (6) is tight because it becomes exact for the complete set \( \Gamma \) of conserved operators. The physical interpretation of Eq. (6) is straightforward in the Heisenberg picture if we view the time-dependent observable \( A^\dagger \) as super vector. Its components parallel to conserved quantities (super vector directions) are constant in time because these quantities commute with the Hamiltonian. But all other components, which are perpendicular to the conserved super subspace, finally decay.

If not all conserved operators are considered, the r.h.s. of (6) decreases and only the inequality holds. Generally, if any subspace of the space spanned by \( \Gamma \) is considered Mazur’s inequality (6) holds. One does not need to know the complete set of eigenstates of \( H \) in order to calculate a lower bound: Any finite (sub)set of conserved operators is sufficient.

Now we proceed to generalize Mazur’s inequality for easy-to-use application. Usually, some conserved operators \( C_i \) are known but they are not necessarily orthonormal in general. Rather their overlaps yield a Hermitian, positive norm matrix \( N \) with matrix elements \( N_{im} := \langle C_i|C_m \rangle \). Each operator \( C_i \) can be represented as a linear superposition of the complete set of orthonormal \( X_i \). These superpositions can be summarized in a matrix \( M \) so that \( c = M^\dagger x \) where the vectors \( x \) and \( c \) contain the operators \( X_i \) and \( C_i \) as coefficients; \( M^\dagger \) is the complex (not Hermitian!) conjugate of \( M \). A short calculation shows that \( N = MM^\dagger \).

If we define the vector \( a_X \) with complex components \( a_i \), the bound \( S_{\text{low}} \) can be expressed by \( S_{\text{low}} = a_X^\dagger a_X \). In analogy, we compute \( a_C \) with complex components \( \langle C_i|A \rangle \). Obviously, \( a_X = M^{-1}a_C \) holds and the lower bound is computed by
\[
S_{\text{low}} = a_C^\dagger (M^{-1})^\dagger M^{-1}a_C = a_C^\dagger N^{-1}a_C
\]
without resorting to orthonormalized operators, relying only on the scalar products of \( C_i \) and \( A \). We have successfully eliminated the construction of a subset of orthogonal operators \( X_i \) and related the lower bound to some known set of linear independent unnormalized conserved operators \( C_i \). The general lower bound (8) is our first key result. A possible route to generalizations to various initial states is sketched in the Supplement.

c. **Central spin model.** The Hamiltonian of the CSM reads
\[
H_0 = \vec{S}_0 \cdot \sum_{k=1}^N J_k \vec{S}_k
\]
where we assume all spins to be $S = 1/2$ for simplicity. It is a generic model to study the interaction between a two-level system and a bath of spins or more generally a set of subsystems with finite number of levels. Currently, it is intensively investigated for understanding the decoherence and dephasing in possible realizations of quantum bit.\footnote{8,9,22,23} Theoretical tools comprise Chebyshev polynomial techniques\footnote{17,24}, perturbative approaches\footnote{15,25,26}, various cluster expansions\footnote{31–34}, Bethe ansatz\footnote{27,28,35,36}, density-matrix renormalization\footnote{37,38}, and studies of the classical analogue.\footnote{12,13,37,38}

By focusing on $A = S_0^z$, the correlation function defined in \textcolor[rgb]{0.5,0.5,0.5}{(10a)} reveals important information on the decay of the central spin. Due to isotropy no other components of the central spin need to be considered. Given the smallness of the hyperfine couplings ($J_k$ is in the range of $\mu$eV corresponding to percents of a Kelvin)\footnote{8,9,12,22,23}, the experimentally relevant temperature can be considered as infinite, and we take the spin system to be completely disordered, i.e., $\rho \propto 1$, prior to the preparation of an initial state of the central spin, cf. Supplement.

For classical spins $S_k^\mu$, there are strong analytical arguments that a fraction of central spin correlations persists unless there is a diverging number of arbitrarily weakly coupled spins in the bath.\footnote{17,18} In the quantum case smaller systems have been studied and evidence for a non-decaying fraction of spin polarization\footnote{17,18} has only been compiled in fairly small ($N < 50$) systems or up to fairly short times.\footnote{16}

Based on the generalized Mazur’s inequality\footnote{39}, we are able to address the nature and the lower bound of these non-decaying correlations for arbitrary system sizes. The total spin $\vec{I} := \sum_{k=0}^N \vec{S}_k$ could serve as a first guess for a useful conserved quantity. Only the $z$-component $C_1 := I^z$ has an overlap $a = (I^z | S_0^z) = 1/4$ (we omit the subscript $C$ for brevity). The norm $N_{11} = (I^z | I^z)$ takes the value $(N + 1)/4$ so that \textcolor[rgb]{0.5,0.5,0.5}{(8)} provides $S_{\text{low}} = 1/(4(N + 1))$. Irrespective of the considered distribution of the couplings $J_k$, using only $I^z$ as single conserved operator does not provide a meaningful lower bound for thermodynamically large, or infinite baths.

The next important conserved quantity is the energy $H_0$ itself. But, of course, $(H_0 | S_0^z) = 0$ because $H_0$ is a scalar and $S_0^z$ a vector component. The $z$-component of the product $\hat{H}_0$, $H_0^z := I^z H_0$, clearly fulfills $[H_0, H_0^z] = 0$ and defines a conserved composite vector operator. We find

$$
(S_0^z | H_0^z) = J_S/16 \quad \text{(10a)}
$$

$$
(H_0^z | H_0^z) = (2 J_S^2 + 3(N - 1) J_Q^2)/64 \quad \text{(10b)}
$$

where $J_S := \sum_{k=1}^N J_k$ and $J_Q^2 := \sum_{k=1}^N J_k^2$. With this input Eq. \textcolor[rgb]{0.5,0.5,0.5}{(8)} yields

$$
S_{\text{low}} = \frac{1}{4} \frac{J_Q^2}{2 J_S^2 + 3(N - 1) J_Q^2}. \quad \text{(11)}
$$

This bound remains finite for $N \to \infty$ if the $J_k$ are drawn from a probability distribution $p(J)$ with average $\overline{J}$ and variance $\overline{J^2}$. For large $N$ one has $J_S = N \overline{J}$ and $J_Q^2 = (\overline{J^2}/20 \overline{J^2} + 12 \overline{J^2})$ so that $S_{\text{low}} = J^2/[20 J^2 + 12 \overline{J^2}]$ ensues for $N \to \infty$. This is a finite lower bound unless the average values $J^2$ vanishes. This rigorous bound is our second key result.

For any finite system with non-vanishing sum $J_S$, Eq. \textcolor[rgb]{0.5,0.5,0.5}{(11)} provides a rigorous finite lower bound which is very easy to compute for any given set of couplings. It can serve to check the validity of numerical results such as provided in Refs.\footnote{12,13,37,38}. Generally, distributions of the $J_k$ have finite values $\overline{J}$ and $\overline{J^2}$. This is the case for nuclear spins in molecules\footnote{41}, or NV centers in diamond\footnote{42} because the spin baths are finite. In quantum dots, the convergence and existence $J_S$ and $J_Q$ is ensured even for arbitrary number of spins because the couplings are bounded from above, but become arbitrarily small due to exponential tails of the electron wave function.\footnote{8,9,12,22,23}

This leads to vanishing $\overline{J}$ implying complete decay for infinite times.

For large, but finite times, however, our results include the possibility of slow decays $S(t) \propto \ln(t)^{-\alpha}$ previously advocated for infinitely large spin baths.\footnote{13,37,38}. Assuming exponential scaling for the couplings $J_k \propto \exp(-\beta k)$, where $\beta$ is inversely proportional to the number of relevant bath spins,\footnote{43} it is clear that $J_Q$ and $J_Q^2$ converge quickly for $N \to \infty$ so that Eq. \textcolor[rgb]{0.5,0.5,0.5}{(11)} implies $S_{\text{low}} \propto 1/N$. Chen et al.\footnote{12} have argued that at any given finite time $t$, only those spins $S_k$ with couplings $t J_k \geq 1$ significantly influence the real-time dynamics of the central

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{(Color online) Spin correlation $S(t)$ for $N = 20$ bath spins with $J_k \approx k$, but normalized such that $J_Q = 1$ is the unit of energy, and various $J_S$ defined in \textcolor[rgb]{0.5,0.5,0.5}{(12)}. The inset compares $S_{\text{low}}$ from the average of the numerical data with $t \in [150/J_Q, 200/J_Q]$ to $S_{\text{low}}$ obtained from \textcolor[rgb]{0.5,0.5,0.5}{(8)} for the 3 quantities ($I^z, I_Q^z, H_0^z$) or for all quantities ($I^z, H_0^z$ with $t \in \{1, 2, \ldots, N\}$). The estimates from the Overhauser correlations $S^{(B)}$ are also shown.}
\end{figure}
spin. Hence, only an effective number \(N_{\text{eff}}(t) \propto \ln(t)\) of spins contribute to the correlation function implying \(S(t) \propto 1/\ln(t)\) for such a distribution function.

The lower bound \([1]\) can be improved by considering the three conserved observables \(I^z, H_0^z\), and \(I_Q^z := I^z \sum_{i<j} \vec{S}_i \cdot \vec{S}_j\). The required vector and matrix elements are given in the supplemental material. Still the bound does not exhaust the numerically found value as depicted in the inset of Fig. 1 for \(J_{\text{ex}} = 0\) (\(J_{\text{ex}}\) makes the system non-integrable, it will be defined in \([12]\)). Even resorting to the integrability of the CSM\(^z\) which implies \(0 = [H_l, H_0]\) with \(H_l := \sum_{k=0}^{N_0} \varepsilon l - \varepsilon k)^{-1} \vec{S}_l \cdot \vec{S}_k\) and \(\varepsilon_0 = 0, \varepsilon_k = -1/J_k\) does not account for the full non-decaying fraction obtained in finite size calculations\(^{15,17}\), see circle in the inset of Fig. 1. The bound has been computed considering \(I^z, H_l^z := I^z H_l\) for \(l \in \{1, 2, \ldots N\}\) (for matrix elements see supplement).

The above results suggest that the integrability is not the key ingredient for a finite non-decaying fraction. To support this claim we extend the Hamiltonian \([9]\) by adding one extra coupling \(H_0 \rightarrow H\)

\[H := H_0 + J_{\text{ex}} \vec{S}_1 \cdot \vec{S}_N\]

(12)
between the most weakly and the most strongly coupled bath spin, defined to be at \(k = 1\) and \(N\), respectively. Its value \(J_{\text{ex}}\) is chosen to be \(O(J_Q)\) so that it constitutes a sizable perturbation even for large spin baths.

The modified time-dependence of \(S(t)\) is depicted for various \(J_{\text{ex}}\) in Fig. 1. A finite \(J_{\text{ex}}\) spoils the integrability completely\(^2\), but leaves the quantities \(I^z, I_Q^z, H^z\) conserved. These three constants of motion generic for isotropic spin models are used to obtain the lower bound (red curve) in the inset of Fig. 1. Obviously, \(S_{\text{low}}\) is decreased smoothly and only moderately upon increasing \(J_{\text{ex}}\) in line with the numerically determined \(S_{\infty}\). There is no abrupt jump to zero, in contrast to what is known for the Drude weight. The conclusion that integrability is only secondary for the non-decaying spin correlation is our third key result.

At present it remains an open question which conserved quantities one has to include to yield a tight lower bound. We presume that higher powers of \(H\), for instance \(I^z H^2\), have to be considered. Such studies are more tedious and left for future research. Instead, we take a mathematically less rigorous route based on the estimate by Merkulov et al.\(^{12}\)

\[S_{\infty} = S_{\text{low}}(B)/(12S_{(B)}(0))\]

(13)
where \(S_{(B)}(t)\) is the correlation of the Overhauser field operator \(\vec{B}_N := \sum_{j=0}^{N} J_k \vec{S}_k\). Note that an arbitrary \(J_0\) can be included because \(\vec{S}_0 \cdot \vec{B}_0\) differs from \(H_0\) in \([9]\) only by an irrelevant constant for spin 1/2. This estimate was derived for a classical, large Overhauser field\(^{12}\) and prevails in the thermodynamic limit of the quantum case: The Overhauser field becomes a classical variable upon \(N \rightarrow \infty\) as shown in Ref.\(^{14}\).

Thus we now apply the general approach \([8]\) to \(A = B_N^z\). Considering only \(C_1 = I^z\) as conserved operator already yields a meaningful lower bound for the Overhauser field correlation function for \(N \rightarrow \infty\)

\[
\frac{S_{\text{low}}^{(B)}}{S^{(B)}(0)} = \frac{(J_S + J_0)^2}{(N + 1)(J_Q^z + J_0^z)}.
\]

Recall \(J_S \propto N\) and \(J_Q^z \propto N\) if the couplings are drawn from a normalized distribution function \(p(J)\). This lower bound can be optimized by choosing the arbitrary value \(J_0\) such that the bound becomes maximal. With the matrix elements given in the supplement \(S_{\text{low}}^{(B)}\) can be improved considering the three constants \(I^z, I_Q^z, H^z\) or all integrals \(I^z\) and \(H_l^z, 1 \leq l \leq N\). The results are also included in Fig. 1 (triangle and square symbols). They hold only for \(J_{\text{ex}} = 0\) because the estimate \([13]\) applies only in this case. Remarkably, the resulting estimates for \(S_{\infty}\) seem to be tight. In particular, the easily evaluated estimate based on all integrals reproduces the numerically found \(S_{\infty}\) to its accuracy. We applied the same estimate to the case \(J_k \propto \exp(-\beta k)\) studied by stochastically evaluating the Bethe ansatz equations and found excellent agreement with the published data with \(N \leq 48\) in Ref.\(^{16}\) as well. Thus we conjecture that the non-decaying fraction \(S_{\infty}\) in the central spin model is quantitatively described by \(S_{\text{low}}^{(B)}/(12S_{(B)}(0))\) if \(S_{\text{low}}^{(B)}\) is determined from the \(N + 1\) integrals \(I^z\) and \(H_l^z\). This constitutes our fourth key result. The small difference, however, between triangle (from three constants of motion) and square (from \(N + 1\) constants of motion) in Fig. 1 indicates again that the significance of the integrability is limited.

In summary, four key results are obtained: (i) An easy-to-use version of Mazur’s inequality to prove persisting correlations; (ii) A rigorous finite lower bound for the infinite-time spin correlation in the CSM, valid for the infinite system if the average coupling is finite; (iii) Only a small part of the persisting correlation is due to the integrability; (iv) A quantitative estimate for the persisting correlation is conjectured, based on the Overhauser field. Clearly, the generalized inequality calls for application to other problems\(^{15}\). The approach is easy to evaluate and can be used for very large systems and large numbers of constants of motion. Thus it can prove fruitful in the intensely studied field of integrable systems, for instance in estimating Drude weights. In the context of coherence in particular, various extensions of the CSM, e.g., by magnetic fields, anisotropies, or more intra-bath couplings suggest themselves to be investigated in the presented manner.

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I. SUPPLEMENTAL MATERIAL

A. Time-Dependent Expectation Values

One may wonder whether the two-time correlations $S(t) = \langle A(t)A(0) \rangle$ reflect time-dependent measurements after the preparation of some initial state. We show that this is the case for the simple, but important example of a spin correlation for $S = 1/2$, i.e., for $A = S_z$. Then we write $S_z(t) = \frac{1}{2}(P_+ - P_-)$ where $P_\sigma$ projects onto the states with $S_\sigma = \sigma/2$. If $\rho$ denotes the density matrix of the total system before any state preparation we calculate

\[ S(t) = \langle \hat{S}_z(t)\hat{S}_z(0) \rangle \]

\[ = \frac{1}{2}\langle \hat{S}_z(t)(P_+ - P_-) \rangle \]

\[ = \frac{1}{2}\langle \hat{S}_z(t)P_+ \rangle \]

\[ = \frac{1}{2}\langle \hat{S}_z(t)P_- \rangle \]

\[ = \text{Tr} (\hat{S}_z(t)P_\sigma \rho) \]

\[ = \frac{1}{2}\langle \hat{S}_z(t) \rangle_{\text{initial}} \]

\[ = \frac{1}{2}\langle \hat{S}_z(t) \rangle_{\text{initial}} \]

where we assumed that the Hamiltonian $H$ and the density matrix $\rho$ are invariant under total inversion $S^z \rightarrow -S^z$ so that the second term in equals the first
one. Finally, in [15] we define the initial density matrix \( \rho_{\text{initial}} := (1/2) P_+ \rho \) which results from \( \rho \) by projecting it to the states with \( S_0^z = 1/2 \) and its proper normalization. This clearly shows that in the studied case \( S(t) \) equals the time-dependent expectation value for a suitably prepared initial state.

The above procedure can be modified to other observables. Generally, we can consider \( \langle A(t) D(0) \rangle \) to focus on the time-dependent expectation value \( \langle A(t) \rangle_D \) starting from the initial density matrix \( \rho_{\text{initial}} := D \rho \). However, do not claim that a suitable operator \( D \) is easy to find. This route remains to be explored in future work.

B. Rigorous Bound for Non-Decaying Spin Correlation

For completeness, we recall the following definitions of conserved quantities. The total angular momentum \( \vec{I} \) and the combination \( \vec{I}_Q \) derived from it read

\[
\vec{I} := \sum_{j=0}^{N} \vec{S}_j \quad \quad \quad (16a)
\]
\[
\vec{I}_Q := \sum_{j=0}^{N} \vec{S}_j \sum_{0 \leq l < p \leq N} (\vec{S}_l \cdot \vec{S}_p), \quad \quad \quad (16b)
\]

Below we only need the corresponding z-components. Furthermore, we consider

\[
H_l^Z := \sum_{j=0}^{N} S_j^z \sum_{j=0, \neq l}^{N} J_j^{(l)}(\vec{S}_l \cdot \vec{S}_j) \quad \quad (16c)
\]

based on the constants of motion \( H_l := \sum_{j=0}^{N} J_j^{(l)}(\vec{S}_l \cdot \vec{S}_j) \) of the integrable CSM, where we use the shorthand \( J_j^{(l)} \) and introduce some further shorthands for future use

\[
J_j^{(l)} := (\varepsilon_l - \varepsilon_j)^{-1} \quad \quad (17a)
\]
\[
S^{(l)} := \sum_{j=0, \neq l}^{N} J_j^{(l)} \quad \quad (17b)
\]
\[
Q^{(l)} := \sum_{j=0, \neq l}^{N} (J_j^{(l)})^2 \quad \quad (17c)
\]

where \( \varepsilon_0 = 0 \) and \( \varepsilon_j = -J_j^{-1} \). Note that \( J_j = J_j^{(0)}, J_S = S^{(0)}, \) and \( J_Q = Q^{(0)} \).

For the disordered spin system with density operator \( \rho \) proportional to the identity the following diagonal scalar products can be determined straightforwardly

\[
\langle I^2 \rangle = (N + 1)/4 \quad \quad (18a)
\]
\[
\langle I_0^z I_0^z \rangle = (N + 1) N (7N - 5)/128 \quad \quad (18b)
\]
\[
\langle H_l^z H_l^z \rangle = (2 S^{(l)})^2 + 3 (N - 1) Q^{(l)}/64. \quad \quad (18c)
\]

We also need the non-diagonal matrix elements

\[
\langle I^2 | H_l^z \rangle = (N + 1) N / 16 \quad \quad (19a)
\]
\[
\langle I_0^z | H_0^z \rangle = (N + 1) N (7N - 5)/128 \quad \quad (18b)
\]
\[
\langle H_l^z | H_p^z \rangle = (2 S^{(l)})^2 + 3 (N - 1) Q^{(l)}/64. \quad \quad (18c)
\]

For the observable \( S_0^z \) we obtain the vector elements

\[
\langle S_0^z | I^2 \rangle = 1/4 \quad \quad (20a)
\]
\[
\langle S_0^z | I_0^z \rangle = N/16 \quad \quad (20b)
\]
\[
\langle S_0^z | H_0^z \rangle = J_S/16 \quad \quad (20c)
\]
\[
\langle S_0^z | H_l^z \rangle = -J_l/16 \quad \quad (20d)
\]

With these matrix and vector elements we can compute \( S_{\text{low}} \) for various sets of conserved quantities. Note that \( H_0^z \) is linearly dependent on the \( N \) quantities \( H_l^z \) with \( 0 < l \leq N \) due to

\[
\sum_{l=0}^{N} H_l^z = 0. \quad \quad (21)
\]

Similarly, \( I_0^z \) depends linearly on them due to

\[
I_0^z = \sum_{l=1}^{N} \varepsilon_l H_l^z. \quad \quad (22)
\]

Hence, one may either consider \( I^2 \) together with the \( N \) quantities \( H_l^z \) with \( 0 < l \leq N \) or the three quantities \( I^2, I_0^z, H_0^z \). The first choice exploits all the known conserved quantities on the considered level of at most trilinear spin combinations. This is what is called ‘all quantities’ in Fig. 1 in the Letter. No explicit formula can be given, but the required matrix inversion is easily performed for up to \( N = O(1000) \) spins with any computer algebra program and up to \( N \approx 10^6 \) spins by any subroutine package for linear algebra.

The second choice of \( I^2, I_0^z, H_0^z \) yields \( 3 \times 3 \) matrices and can be analysed analytically. Inserting the elements in \[15\] and in \[19\] and those in \[20\] into \[8\] yields

\[
S_{\text{low}} = \frac{1}{4(N + 1)^3} (3 J_Q^2 + J_S^3) (N(N + 1) - 10 J_S^2). \quad \quad (23)
\]

Furthermore, these three quantities are conserved for any isotropic spin model so that we may also consider the system with the additional bond \( H = H_0 + J_{\text{ex}} \vec{S}_1 \cdot \vec{S}_N \), see Fig. 1. Thus we extend the above formulæ by passing from \( H_0 \) to \( H \) and hence from \( H_0^z \) to \( H^z = I^2 H \). The modified scalar products are

\[
\langle H^z | H^z \rangle = (H_0^z | H_0^z) + (J_1 + J_N) J_{\text{ex}}/16 + (3 N - 1) J_{\text{ex}}^2/64 \quad \quad (24a)
\]
\[
\langle I^2 | H^z \rangle = (I^2 | H_0^z) + J_{\text{ex}}/8 \quad \quad (24b)
\]
\[
\langle I_0^z | H^z \rangle = (I_0^z | H_0^z) + (7 N - 5) J_{\text{ex}}/64 \quad \quad (24c)
\]
\[
\langle S_0^z | H^z \rangle = (S_0^z | H_0^z). \quad \quad (24d)
\]
They lead to a bound $S_{\text{low}}(J_{\text{ex}})$ as depicted in Fig. 1. The explicit formula is similar to the one in (23), but lengthy so that we do not present it here. It can be easily computed by computer algebra programs.

C. Estimate via Bound for the Overhauser Field

Eq. (13) relates the non-decaying fraction $S_{\infty}$ to the relative bound for the Overhauser field

$$\vec{B} = \sum_{j=0}^{N} J_j \vec{S}_j$$

(25)

where $J_0$ is arbitrary if the central spin has $S = 1/2$. We stress, however, that the derivation yielding (13) in Ref. 12 only holds for the CSM so that we do not consider extensions to finite $J_{\text{ex}}$ in this case.

We use the freedom to choose $J_0$ to maximize the resulting lower bound for $A = B^z$. We reuse all matrix elements of the norm matrix $N$ in (18) and in (19). Since (13) uses the relative correlation we have to compute

$$S^{(B)}(t = 0) = (B^z|B^z) = (J_0^2 + J_0^2)/4$$

(26)

as well. Furthermore, the vector elements of $\mathbf{a}$ must be determined anew

$$(B^z|I^z) = (J_S + J_0)/4$$

(27a)

$$(B^z|I_Q^z) = (J_S + J_0)N/16$$

(27b)

$$(B^z|H_Q^z) = (J^2_Q + J_0J_S)/16$$

(27c)

$$(B^z|H_l^z) = J_l(S^{(l)} + J_0)/8 - J_0(J_S + J_0)/16\text{ for }l > 0.$$  

(27d)

These elements allow us to determine the ratio $S_{\text{low}}^{(B)}(B^z)(0)$ for the three quantities $I^z, I_Q^z, H_Q^z$ or for all quantities, i.e., $\vec{I}$ and $\vec{H}^z_l$ with $1 \leq l \leq N$. The ensuing lower bounds can be optimized by varying $J_0$ in such a way that the ratios become maximum yielding the best bounds. The latter step is easy to perform since the non-linear equation in $J_0$ to be solved to determine the maximum is just a quadratic one. In this way, the triangle and square symbols in Fig. 1 are computed.

The comparison to the Bethe ansatz data for up to $N = 48$ spins in Ref. 18 yields an excellent agreement within the accuracy with which we can read off $S_{\infty}$ from the numerically evaluated Bethe ansatz correlation $S(t)$. This concludes the section on the required input of matrix and vector elements.