On loop spaces with marking

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1 Introduction

An analogous construction of simplicial homotopy group for Kan complex can be applied to saturated complicial sets\footnote{In [2], the analogous construction is applied to non-saturated ones as well.} to give monoids. In this paper, we investigate how the construction of loop spaces of Kan complexes lifts to the complicial setting and relates to such monoids.

Letting $\text{msSet}_*$ denote the category of pointed simplicial sets with marking, we have the following.

**Theorem 1.1.** Loop space functor $\Omega : \text{msSet}_* \to \text{msSet}_*$ is a right Quillen functor from the model category of pointed $(\infty, n+1)$-categories to that of pointed $(\infty, n)$-categories.

Here, the loop space functor is defined to be the right adjoint of reduced suspension functor, namely it is a straightforward generalization of the loop space functor of simplicial sets. We will study this in section 3.

Ozornova-Rovelli, Riehl and Verity ([4], [6], [7]) constructed model structures on the category $\text{msSet}$ of simplicial sets with marking, which are considered as simplicial models of weak higher categories. More precisely, for $n \in \mathbb{N}$, $\text{msSet}$ admits a model structure called the model of $(\infty, n)$-categories. In particular, for $n = 0, 1$, they are equivalent to the models of Kan complexes and quasicategories respectively (see [7] for more detail). The category $\text{msSet}$ also admits a model structure considered to give the model of weak $\omega$-categories.

By the proof of the theorem above, it follows that for any pointed saturated complicial set $X$, $\Omega(X)$ is also a pointed saturated complicial set since they are fibrant with respect to the model structure of weak $\omega$-categories. So its homotopy monoids make sense.

Our main theorem is the following.

**Theorem 1.2.** For any $n \in \mathbb{N}$ and any pointed saturated complicial set $X$, there is a monoid isomorphism $\tau_{n+1}(X) \cong \tau_n(\Omega(X))$.

As the usual simplicial homotopy theory can be understood as the geometry of $(\infty, 0)$-categories, it may worth trying to study simplicial homotopy theory of $(\infty, n)$-categories for general $n$ to understand the geometry of spaces with non-reversible constituents. Since both of the construction of homotopy monoids and
that of loop spaces are straightforward generalizations of the classical ones, this
theorem may be considered to be a natural generalization of the fundamental
fact in the usual simplicial homotopy theory to the higher one.

In the present paper we heavily use the results in [5] especially those about
the suspension functor.

2 Preliminaries

Assuming the reader is familiar with simplicial sets, we recall some notations
about simplicial sets with marking from [4], [5] and [7].

Definition 2.1 ([7]). A pair \((X, mX)\) is a simplicial set with marking\(^2\) if

- \(X\) is a simplicial set,
- \(mX\) is a set of simplices in \(X\) such that \(dX \subset mX\) and \(X_0 \cap mX = \emptyset\),

where \(dX\) denotes the set of degenerate simplices in \(X\).

A map of simplicial sets with marking \(f : (X, mX) \to (Y, mY)\) is a simplicial
map \(f : X \to Y\) such that \(f(x) \in mY\) for all \(x \in mX\).

We say a simplex of \((X, mX)\) is marked when it is an element of \(mX\), and
let msSet denote the category of simplicial sets with marking and their maps.

For simplicial sets with marking \((X, mX)\) and \((Y, mY)\), we say that \((X, mX)\)
is a regular simplicial subset with marking of \((Y, mY)\) if \(X \subset Y\) as simplicial
sets and \(mX = X \cap mY\) (cf. [7]).

For short, we often write \(X\) for a simplicial set with marking \((X, mX)\) omit-
ting \(mX\).

Definition 2.2 ([5], [7]). Let \(n\) be a natural number and \(k \in [n]\).

- The standard thin \(n\)-simplex \(\Delta[n]_t\) is the simplicial set with marking whose
  underlying simplicial set is the standard simplicial set \(\Delta[n]\) and
  \[
  m\Delta[n]_t = \begin{cases} \{d\Delta[n] \cup \{\text{Id}_{[n]}\}\} & (n \neq 0) \\ d\Delta[n] & (n = 0). \end{cases}
  \]

- The \(k\)-complicial \(n\)-simplex \(\Delta^k[n]\) is the simplicial set with marking whose
  underlying simplicial set is the standard simplicial set \(\Delta[n]\) and
  \[
  m\Delta^k[n] = d\Delta[n] \cup \{\alpha \in \Delta[n] | \{k - 1, k, k + 1\} \cap [n] \subset \text{Im}(\alpha)\}.
  \]

- The \(n - 1\)-dimensional \(k\)-complicial horn \(\Lambda^k[n]\) is the regular simplicial
  subset with marking of \(\Delta^k[n]\) whose underlying simplicial set is the usual
  simplicial \(k\)-th horn.

\(^2\)In [7], simplicial sets with marking are called stratified simplicial sets
• $\Delta^k[n]''$ (respectively $\Lambda^k[n]'')$ is the simplicial set with marking whose underlying simplicial set is the same as that of $\Delta^k[n]$ (respectively $\Lambda^k[n]$) and its marked simplices are $m\Delta^k[n]$ (respectively $m\Lambda^k[n]$) with all its $n-1$-simplices.

• $\Delta^k[n]':=\Delta^k[n] \cup \Lambda^k[n]'$.

• $\Delta[3]_{eq}$ is the simplicial set with marking whose underlying simplicial set is the standard 3-simplicial set and the empty one $\Delta[-1]$ is its unit.

\begin{equation}
m\Delta[3]_{eq} = d\Delta[3] \cup \Delta[3][2] \cup \Delta[3][3] \cup \{(01), (13)\},
\end{equation}

where $[01]$ (respectively $[13]$) is the 1-simplex whose image is $\{0, 2\}$ (respectively $\{1, 3\}$).

• $\Delta[3]^2$ is the simplicial set with marking whose underlying simplicial set is the standard 3-simplicial set with

\begin{equation}
m\Delta[3]^2 = \bigcup_{n \geq 1} \Delta[3][n].
\end{equation}

For simplicial sets with marking $X$ and $Y$, $X \star Y$ denotes the join of $X$ and $Y$ (see for example [3, Definition 2.4]). More precisely, the underlying simplicial set is the join of the underlying simplicial sets. So the set of its $n$-simplices is given by

\begin{equation}
(X \star Y)_n = \coprod_{k,l \geq -1 \atop k+l=n-1} X_k \times Y_l,
\end{equation}

where $X_{-1}$ and $Y_{-1}$ are one point sets and a simplex $(x, y) \in X \star Y$ is marked if and only if $x \in X$ and $y \in Y$ are marked.

For a simplex $(x, y) \in X_k \times Y_l \subset (X \star Y)_{k+l+1}$, its face simplices are given by

\begin{equation}
d_i(x, y) = \begin{cases}
(d^X_i(x), y) & 0 \leq i \leq k \\
(x, d^Y_{l-k-1}(y)) & k + 1 \leq i \leq k + 1 + l,
\end{cases}
\end{equation}

and its degeneracy simplices are given by

\begin{equation}
s_i(x, y) = \begin{cases}
(s^X_i(x), y) & 0 \leq i \leq k \\
(x, s^Y_{l-k-1}(y)) & k + 1 \leq i \leq k + 1 + l.
\end{cases}
\end{equation}

The join operation is defined for augmented simplicial sets with marking and the empty one $\Delta[-1]$ is its unit.

\textbf{Definition 2.3} ([7], [4]). Let $n$ be a natural number. We call the following obvious maps $(\infty, n)$-elementary anodyne extensions:

• the complicial horn extension map

\begin{equation}
\Lambda^k[m] \rightarrow \Delta^k[m]
\end{equation}

for $m \geq 1$ and $k \in [m], \ldots
• the thinness extension map
  \[ \Delta^k[m]' \to \Delta^k[m]'' \]
  for \( m \geq 2 \) and \( k \in [m] \),
• the triviality extension map
  \[ \Delta[m] \to \Delta[m]_t \]
  for \( m \geq n + 1 \),
• the saturation extension map
  \[ \Delta[m] \star \Delta[3]_{eq} \to \Delta[m] \star \Delta[3]^2 \]
  for \( m \geq -1 \).

Note that only the triviality extension maps depend on \( n \). We call complicial horn extension maps, thinness extension maps and saturated extension maps from Definition 2.3 elementary anodyne extensions.

Definition 2.4 ([7], [4]). Let \( n \) be a natural number. A simplicial set with marking \( X \) is an \( n \)-trivial saturated complicial set (respectively a saturated complicial set) if it has the right lifting property with respect to \((\infty, n)\)-elementary anodyne extensions (respectively elementary anodyne extensions).

A map \( f : X \to Y \) of simplicial sets with marking is an \((\infty, n)\)-weak equivalence (respectively a saturated complicial weak equivalence) if for any \( n \)-trivial saturated complicial set (respectively any saturated complicial set) \( Z \) the map
  \[ f^* : Z^Y \to Z^X \]
on internal homs is a homotopy equivalence in the sense of [7].

Let \( \text{msSet}_* \) denote the category of pointed simplicial sets with marking and pointed maps.

Theorem 2.5 ([5]). For any \( n \in \mathbb{N} \), \( \text{msSet}_* \) admits the following model structure, where

• the cofibrations are precisely monomorphisms and
• weak equivalences are precisely those maps whose underlying maps of simplicial sets with marking are \((\infty, n)\)-weak equivalences (respectively saturated complicial weak equivalences).

The fibrant objects are precisely the pointed simplicial sets with marking whose underlying simplicial sets with marking are \( n \)-trivial saturated complicial sets (respectively saturated complicial sets).

For \( n \in \mathbb{N} \), we call this model structure the model structure of pointed \((\infty, n)\)-categories. Note that \((\infty, n)\)-elementary anodyne extensions are weak equivalences in this model structure.
3 Results

Using the results and arguments in [5] about the suspension functor of simplicial sets with marking, we consider the reduced one in the following.

For a pointed simplicial set with marking \((X, x_\top)\), \(\Delta[0] \star X\) has two 0-cells \(x_\bot\) and \(x_\top\). The former is the one corresponding to the 0-simplex of \(\Delta[0]\) and the latter is that corresponding to the base point of \(X\). We take \(x_\bot\) as the base point of \(\Delta[0] \star X\).

Also \(\Delta[0] \star X\) has the 1-simplex \(x_\bot x_\top\) whose source is \(x_\bot\) and target is \(x_\top\). Note that this 1-simplex is not marked. We let \(\langle x_\bot x_\top \rangle\) denote the pointed simplicial subset with marking of \(\Delta[0] \star X\) generated by \(x_\bot x_\top\).

**Definition 3.1** (reduced suspension). We define the reduced suspension functor \(\Sigma_+ : \text{msSet}_* \to \text{msSet}_*\) by the following pushout

\[
\begin{array}{c}
\Delta[-1] \star X \cup \langle x_\bot x_\top \rangle \\
\downarrow \\
\Delta[0] \star X \\
\downarrow \\
\Sigma_+ X,
\end{array}
\]

for an object \(X \in \text{msSet}_*\), where the base point of \(\Sigma_+ X\) is induced by \(x_\bot\). For a map \(f\) of pointed simplicial sets with marking, \(\Sigma_+(f)\) is the induced one.

As shown in [5] the suspension functor is left adjoint. The reduced one also has the right adjoint.

**Lemma 3.2.** \(\Sigma_+\) is a left adjoint functor.

**Proof.** We construct the right adjoint functor of \(\Sigma_+\) by using the right adjoint functor \(P^\circ\) of \(\Delta[0] \star (-)\) studied in [7].

Let \((Z, a)\) be a pointed simplicial set with marking. Then we have a simplicial set with marking \(\Omega(Z, a)\) defined by the following pullback

\[
\begin{array}{c}
\Omega(Z, a) \\
\downarrow \\
\Delta[0] \\
\downarrow \\
Z,
\end{array}
\]

where the right-hand side vertical map is induced by the natural inclusions \(\Delta[n] \hookrightarrow \Delta[0] \star \Delta[n]\). We give \(\Omega(Z, a)\) the base point, which we write \(a\) again, by the constant map on \(a \Delta[0] \to P^\circ_a(Z)\) and identity map \(\Delta[0] \to \Delta[0]\).

To show that this gives a right adjoint of \(\Sigma_+\), assume that we have a pointed map \(\Sigma_+(X) \to Z\) with respect to \(a \in Z\). Then we have the following diagram

---

\[\text{In [5], to define the suspension, } X \star \Delta[0] \text{ is used instead of } \Delta[0] \star X. \text{ It may be more reasonable to call } \Sigma_+ \text{ the reduced left suspension functor. If we take the right one, the bijection in Theorem 3.5 may not be a homomorphism of monoids.}\]
due to the adjunction $\Delta[0] \ast (\_ ) \dashv P^\alpha$

This diagram gives the desired map $X \to \Omega(Z, a)$, which defines a right adjoint functor $\Omega$ of $\Sigma_+$.

We may write $\Omega(Z)$ instead of $\Omega(Z, a)$ for short. By definition, an $n$-simplex $\Delta[n] \to \Omega(Z)$ gives rise to a map $\Delta[n] \to \Delta[0] \overset{\alpha}{\to} Z$. Thus the 0-th face of the corresponding $n + 1$-simplex in $Z$ is degenerated on $a$.

This functor is compatible with the model structures in the following sense.

**Lemma 3.3.** The reduced suspension functor $\Sigma_+$ is left Quillen from the model structure of pointed $(\infty, n)$-categories to that of pointed $(\infty, n + 1)$-categories.

**Proof.** It is clear that $\Sigma_+$ preserves cofibrations. Let $f : X \to Y$ be a pointed $(\infty, n + 1)$-weak equivalence. Since $\Delta[-1]$ is the unit for join, we have a pointed $(\infty, n + 1)$-weak equivalence.

$$f_* : \Delta[-1] \ast X \cup \langle x \perp x^\top \rangle \to \Delta[-1] \ast Y \cup \langle y \perp y^\top \rangle,$$

which fits into the following commutative diagram

$$
\begin{array}{ccc}
\Delta[0] & \xleftarrow{=} & \Delta[-1] \ast X \cup \langle x \perp x^\top \rangle \\
\downarrow & & \downarrow \\
\Delta[0] & \xrightarrow{f_*} & \Delta[0] \ast Y
\end{array}
$$

By [5, Proposition 2.5], the right-hand side vertical map is a pointed $(\infty, n + 1)$-weak equivalence since every object is cofibrant. The right-hand side horizontal map is a cofibration. Thus, we obtain the pointed $(\infty, n + 1)$-weak equivalence $\Sigma_+ f : \Sigma_+ X \to \Sigma_+ Y$ as the push out of the diagram. This shows that $\Sigma_+$ sends all $(\infty, n)$-elementary anodyne extensions except for the triviality extension maps to $(\infty, n + 1)$-weak equivalences.

By [5, Lemma 1.8], it is enough to show that the map

$$\Sigma_+ \Delta[n + 1] \to \Sigma_+ \Delta[n + 1]^t$$

is an $(\infty, n + 1)$-weak equivalence. By construction, the underlying simplicial sets of $\Sigma_+ \Delta[n + 1]$ and $\Sigma_+ \Delta[n + 1]^t$ are the same, and the underlying map
of simplicial sets is the identity map. The only difference is that the non-degenerate \((n + 2)\)-simplex of \(\Sigma \Delta [n + 1]_t\) is marked. Therefore we have the following pushout

\[
\begin{array}{ccc}
\Delta [n + 2] & \to & \Delta [n + 2]_t \\
\downarrow & & \downarrow \\
\Sigma_+ \Delta [n + 1] & \to & \Sigma_+ \Delta [n + 1]_t.
\end{array}
\]

Since the upper horizontal map is an acyclic cofibration, the lower map is an \((\infty, n + 1)\)-weak equivalence as desired.

By Theorem 2.5, \(k\)-trivial complicial sets are fibrant with respect to the model structure of \((\infty, k)\)-categories. We get the following.

**Corollary 3.4.** The loop space functor \(\Omega\) is right Quillen from the model structure of pointed \((\infty, n + 1)\)-categories to that of pointed \((\infty, n)\)-categories.

In particular for any pointed \(n + 1\)-trivial saturated complicial set \(X\), \(\Omega(X)\) is a pointed \(n\)-trivial saturated complicial set.

By the same arguments, \(\Omega\) is right Quillen with respect to the model structure for pointed saturated complicial sets. Therefore for any pointed saturated complicial set \(X\), so is \(\Omega(X)\).

As is well known, for a pointed Kan complex \(A\) and a natural number \(n\), there is an isomorphism \(\pi_{n+1}(A) \cong \pi_n(\Omega(A))\) between their homotopy groups, where \(\Omega\) denotes the usual loop space functor. In the following, we show that the analogous result for saturated complicial sets holds. As is shown in [2], the construction of simplicial homotopy groups (see for example [1] or [3]) lifts to the complicial setting in a straightforward way.

For a pointed simplicial set with marking \((X, x)\), we let \(X^x_n\) denote the set of \(n\)-simplices in \(X\) whose boundaries are the degenerate simplices on \(x\). Then there is an evident map \(\psi : X^x_{n+1} \to \Omega(X)^x_n, \alpha \mapsto \alpha\). By definition, for any \(\alpha \in X^x_{n+1}\), its 0-th face is the degenerated simplex on \(x\). So \(\psi(\alpha) = \alpha\) is indeed an element of \(\Omega(X)^x_n\). Note that this map sends marked simplices to marked simplices.

**Theorem 3.5.** For any pointed saturated complicial set \(X\) and \(n \geq 1\), there is a monoid isomorphism \(\tau_{n+1}(X) \cong \tau_n(\Omega(X))\).

**Proof.** The map induces a bijection \(\Psi : \tau_{n+1}(X) \to \tau_n(\Omega(X))\), which is a homomorphism of monoids. Indeed, by construction of homotopy monoids, for \(\alpha, \beta \in X^x_{n+1}\), the multiplication \([\alpha][\beta]\) in \(\tau_{n+1}(X)\) of their homotopy classes can be written as a homotopy class \([d^X_{n+1}(\theta)]\), where \(\theta\) is an \(n + 2\)-simplex obtained by \(\alpha\) and \(\beta\). By the definition of the simplicial structure of the join, the \(n + 1\)-st face of an \(n + 2\)-simplex in \(X\) coincides with the \(n\)-th face of the corresponding \(n + 1\)-simplex in \(\Omega(X)\). So \(\Psi([\alpha][\beta]) = \Psi([d^X_{n+1}(\theta)]) = [d^\Omega(X)_n(\theta)] = [\alpha][\beta] = \Psi([\alpha])\Psi([\beta])\). By definition, \(\Psi\) takes the unit to the unit. \(\square\)
By the same argument, there is a bijection \( \tau_1(X) \cong \tau_0(\Omega(X)) \) for any pointed saturated complicial set \( X \). Thus we can give \( \tau_0(\Omega(X)) \) the monoid structure via this bijection.

Considering this theorem, we may define the spheres in the complicial setting as follows. First the 0-dimensional sphere \( S^0 \) is the simplicial set with marking consisting of two points. For any positive natural number \( n \), the \( n \)-dimensional sphere \( S^n \) is defined to be \( \Sigma_+(S^{n-1}) \). Note that \( S^n \) is not isomorphic to the \( n \)-times smash of \( S^1 \) in general.

By the construction of \( \tau_* \) and the definition of triviality, for any \( n \geq 0 \), \( k \geq 1 \) and any pointed \( n \)-trivial saturated complicial set \( X \), \( \tau_{n+k}(X) \) is a group. This observation is compatible with the theorem above. In other words, for such an \( X \), \( \Omega^n(X) \) is a 0-trivial saturated complicial set. By construction again, \( \tau_k(\Omega^n(X)) \) is isomorphic to the \( k \)-th simplicial homotopy group of the Kan complex associated to \( \Omega^n(X) \).

As a classical fact, for a pointed Kan complex \( A \), \( \pi_{k+1}(A) \) is an abelian group. Hence by the same argument above, for a pointed \( n \)-trivial saturated complicial set \( X \), the monoid \( \tau_{n+k+1}(X) \) is an abelian group.

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