Network Inference and Influence Maximization from Samples*,**,

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Abstract

Influence maximization is the task of selecting a small number of seed nodes in a social network to maximize the influence spread from these seeds. It has been widely investigated in the past two decades. In the canonical setting, the social network and its diffusion parameters are given as input. In this paper, we consider the more realistic sampling setting where the network is unknown and we only have a set of passively observed cascades that record the sets of activated nodes at each diffusion step. We study the task of influence maximization from these cascade samples (IMS) and present constant approximation algorithms for it under mild conditions on the seed set distribution. To achieve the optimization goal, we also provide a novel solution to the network inference problem, that is, learning diffusion parameters and the network structure from the cascade data. Compared with prior solutions, our network inference algorithms require weaker assumptions and do not rely on maximum-likelihood estimation and convex programming. Our IMS algorithms enhance the learning-and-then-optimization approach by allowing a constant approximation ratio even when the diffusion parameters are hard to learn, and we do not need any assumption related to the network structure or diffusion parameters.

Keywords: influence maximization, network inference, optimization from samples, data-driven optimization, end-to-end optimization

1. Introduction

Maximizing the spread of influence through a social network has been widely studied in the past two decades. It models the phenomenon in which a small set of initially \textit{active}
nodes called *seeds* takes some piece of information (news, ideas or opinions, etc.), and the information spreads over the network to *activate* the remaining nodes. The expected number of final active nodes is called the *influence spread* of the seed set. The *influence maximization* problem asks to pick at most $k$ seeds in order to maximize the influence spread. Under many *diffusion models* such as the discrete-time *independent cascade* (IC) model and *linear threshold* (LT) model [Kempe et al. 2003], the problem enjoys a $(1 - 1/e - \varepsilon)$-approximation (with small $\varepsilon > 0$), which is tight assuming P $\neq$ NP [Feige, 1998]. It has found applications in many scenarios.

Traditional influence maximization problem requires as input the whole social network (as well as its parameters), based on which one can compute or estimate the influence spread function. In many scenarios, however, this might be too demanding, especially for those who do not have free access to the network. In this work, we consider influence maximization in the sampling setting where one only has access to a set of passively observed cascades spreading over an implicit social network. Each cascade records the sets of activated nodes at each time step. Such sample data is available in many scenarios, especially on the Internet where the timestamps can be recorded in principle. We are interested in whether we can maximize the influence from such sample data. We model this problem as *influence maximization from samples* below:

**Influence maximization from samples (IMS).** For an unknown social network $G$ with diffusion parameters, given $t$ cascade samples where each seed is independently sampled with an unknown probability, can we find a seed set of size at most $k$ such that its influence spread is a constant approximation of the optimal seed set, when $t$ is polynomial to the size of $G$?

En route to solving the above problem, a natural and reasonable approach is to first learn the network structure as well as its parameters, and then maximize the influence over the learned network. This leads to the well-studied *network inference* problem below:

**Network inference.** For an unknown social network $G$, given polynomial number of cascade samples where each seed is sampled independently with an unknown probability, estimate all diffusion parameters such that with probability at least $1 - \delta$, every parameter is estimated within an additive error $\varepsilon$.

Our contributions in this work are mainly two-fold. First, we revisit the network inference problem and design brand new algorithms for it under both IC and LT models. For the IC model, while all previous algorithms are based on the maximum likelihood estimation and convex programming [Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015; Pouget-Abadie and Horel, 2015], our algorithm builds on a closed-form expression for each edge probability in terms of quantities which can be efficiently estimated. As a result, our algorithm enjoys faster implementation, lower sample complexity and weaker assumptions comparing to previous algorithms. Our assumptions are also easy to verify from cascade samples. We will discuss these differences further in the end of Section 3.1. For the LT
model, to the best of our knowledge, there is no network inference algorithm with theoretical guarantees previously. To resolve the problem, we also build a closed-form expression for each edge weight, which makes the estimation possible.

Second, we provide several end-to-end IMS algorithms with constant approximation guarantee under both IC and LT models. For the IC model, following the canonical learning-and-then-optimization framework, we first present an IMS algorithm by directly invoking our network inference algorithm. The algorithm thus needs the assumptions used for learning. Next, we present alternative algorithms which only need two simple assumptions on the seed distribution and impose no requirements for the underlying network. In contrast, all the known algorithms for network inference (including ours) impose some restrictions on the network. This result is highly non-trivial since it is impossible to resolve network inference problem on arbitrary graphs and hence the learning-and-then-optimization framework fails in this case. For instance, consider a complete graph and another graph with one edge removed from the complete graph, where all edge probabilities are 1. If each node is picked as a seed independently with probability 1/2, one cannot distinguish them within polynomially many cascade samples. For the LT model, an interesting feature of the network inference algorithm is that the learning assumption already has no requirements for the network. Thus, the learning-and-then-optimization framework directly leads to an IMS algorithm that works for arbitrary networks under the LT model. Our IMS follows the general optimization-from-samples framework (Balkanski et al., 2017b), and generalizes the recent result on optimization from structured samples for coverage functions (Chen et al., 2020), see Section 1.1 for details. Finally, we remark that our results not only apply to influence maximization, but also to other learning and optimization settings such as probabilistic maximum cover with application in online advertising (Chen et al., 2016).

1.1. Related Work

Influence maximization from samples follows the framework of optimization from samples (OPS) originally proposed by Balkanski et al. (2017b): given a set of polynomial number of samples \( \{S_i, f(S_i)\}_{i=1}^{t} \) and constraint \( M \), can we find a set \( S \in M \) such that \( f(S) \geq c \cdot \max_{T \in M} f(T) \) for some constant \( c \)? The OPS framework is very important for the data-driven integration of learning and optimization where the underlying model (function \( f \) above) is not readily known. Surprisingly, Balkanski et al. (2017b) showed that even for the maximum coverage problem, there is no constant approximation algorithm under the OPS model, despite prior results that a coverage function \( f \) is learnable from samples (Badanidiyuru et al., 2012) and constant optimization is available when \( f \) is known (Nemhauser et al., 1978). Subsequently, several attempts (Balkanski et al., 2016, 2017a; Rosenfeld et al., 2018; Chen et al., 2020) have been made to circumvent the impossibility result of Balkanski et al. (2017b). Among them the most related one is the optimization from structured samples (OPSS) model for coverage functions (Chen et al., 2020), where the samples carry additional structural information in the form of \( \{S_i, N(S_i)\}_{i=1}^{t} \), where \( N(S_i) \) contains the nodes covered by \( S_i \). It was shown that if the samples are generated from a “negatively correlated” distribution, the maximum coverage problem enjoys constant approximation in the OPSS model. Recall that coverage functions can be regarded as IC
influence spread functions defined over a bipartite graph with edge probabilities in \(\{0, 1\}\). Thus, our result on IMS greatly generalizes OPSS to allow general graphs and stochastic cascades over edges.

End-to-end influence maximization from data has been explored by [Goyal et al., 2011], but they only used a heuristic method to learn influence spread functions and then used the greedy method for influence maximization, so there was no end-to-end guarantee on IMS. A recent work [Balkanski et al., 2017a] revisited IMS problem under the OPS model, and provided a constant approximation algorithm when the underlying network is generated from the stochastic block model. Our study is the first to provide IMS algorithms with theoretical guarantees that work on arbitrary networks.

Network inference has been extensively studied over the past decade [Gomez-Rodriguez et al., 2010; Myers and Leskovec, 2010; Gomez-Rodriguez et al., 2011; Du et al., 2012; Netrapalli and Sanghavi, 2012; Abrahaa et al., 2013; Daneshmand et al., 2014; Du et al., 2013, 2014; Narasimhan et al., 2015; Pouget-Abadie and Horel, 2015]. While most of them focused on the continuous time diffusion model, there are several results under the discrete time IC model [Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015; Pouget-Abadie and Horel, 2015], all of which build on the maximum likelihood estimation. We will compare these results with ours after we present our approach in Section 3.1.

1.2. Organization

In Section 2, we describe the model, some concepts and notations as well as two Chernoff-type lemmas used in the analysis. In Section 3, we present network inference and IMS algorithms under the IC model. In Section 4 we present network inference and IMS algorithms under the LT model. Finally, we conclude the paper in Section 5.

2. Preliminaries

Social network, diffusion model and influence maximization. A social network is modeled as a weighted directed graph \(G = (V, E, p)\), where \(V\) is the set of \(|V| = n\) nodes and \(E\) is the set of directed edges. Each edge \((u, v) \in E\) is associated with a weight or probability \(p_{uv} \in [0, 1]\). For convenience, we assume that \(p_{uv} = 0\) if \((u, v) \notin E\) and \(p_{uv} > 0\) otherwise. We also use \(N(v) = N_{in}(v)\) to denote the in-neighbors of node \(v \in V\).

The information or influence propagates through the network in discrete time steps. Each node \(v \in V\) is either active or inactive, indicating whether it receives the information. Denote by \(S_\tau \subseteq V\) the set of active nodes at time step \(\tau\). The nodes in the set \(S_0\) at time step 0 are called seeds. The diffusion is assumed to be progressive, which means a node will remain active once it is activated. Thus, for all \(\tau \geq 1\), \(S_{\tau - 1} \subseteq S_\tau\).

Given a set of seeds \(S_0\), the diffusion model describes how the information propagates and \(S_\tau\) is generated for each \(\tau \geq 1\). In the literature, the two most well-known diffusion models are the independent cascade (IC) model and the linear threshold (LT) model [Kempe et al., 2003].

In the IC model, at time step \(\tau\), initially \(S_\tau = S_{\tau - 1}\). Next, for each node \(v \notin S_{\tau - 1}\), each node \(u \in N(v) \cap (S_{\tau - 1} \setminus S_{\tau - 2})\) will try to activate \(v\) independently with probability \(p_{uv}\).
(denote $S_{-1} = \emptyset$). Thus, $v$ becomes active with probability $1 - \prod_{u \in \mathcal{N}(v) \cap (S_{t-1} \setminus S_{t-2})} (1 - p_{uv})$ at this step. Once activated, $v$ will be added into $S_{\tau}$. The propagation terminates when at the end of some time step $\tau$, $S_{\tau} = S_{t-1}$. Clearly, the process proceeds in at most $n - 1$ time steps and we use $(S_0, S_1, \cdots, S_{n-1})$ to denote the random sequence of the active nodes.

In the LT model, following the convention, we use $w$ instead of $p$ to denote the edge weight vector. In this model, each node $v \in V$ needs to satisfies the normalization condition that $\sum_{u \in \mathcal{N}(v)} w_{uv} \leq 1$. Besides, each $v \in V$ is associated with a threshold $r_v$, which is sampled independently and uniformly from $[0, 1]$ before the diffusion starts. During the diffusion process, at time step $\tau$, initially $S_{\tau} = S_{t-1}$. Then, for each node $v \notin S_{t-1}$, it will be added into $S_{\tau}$ if $\sum_{u \in S_{t-1} \cap \mathcal{N}(v)} w_{uv} \geq r_v$. The diffusion also terminates if $S_{\tau} = S_{t-1}$ for some time step $\tau$ and we use $(S_0, S_1, \cdots, S_{n-1})$ to denote the random sequence of the active nodes.

Let $\Phi(S_0) = S_{n-1}$ be the final active set given the seed set $S_0$. Its expected size is denoted by $\mathbb{E}[\Phi(S_0)]$ and is commonly called the influence spread of $S_0$. In general, the influence spread function $\sigma : 2^V \to \mathbb{R}_{\geq 0}$ is defined as $\sigma(S) = \mathbb{E}[\Phi(S)]$ for any $S \subseteq V$. Sometimes, we use $\sigma^u(\cdot)$ or $\sigma^v(\cdot)$ to specify the graph parameters explicitly. Influence maximization (IM) asks to find a set of at most $k$ seeds so as to maximize the influence spread of the chosen seed set. Formally, under a specific diffusion model (such as IC or LT models), given a positive integer $k \leq n$, influence maximization corresponds to the following problem: $\arg\max_{S \subseteq V, |S| \leq k} \sigma(S)$.

The sampling setting. Standard influence maximization problem takes as input the social network $G = (V, E, p)$, based on which one can compute or estimate the influence spread function $\sigma$. In this paper, we consider the problem in the sampling setting where $G$ is not given explicitly.

A cascade refers to a realization of the sequence of the active nodes $(S_0, S_1, \cdots, S_{n-1})$. By slightly abusing the notation, we still denote the cascade by $(S_0, S_1, \cdots, S_{n-1})$. In the sampling setting, a set of $t$ independent cascades $(S_{i,0}, S_{i,1}, \cdots, S_{i,n-1})_{i=1}^t$ is given as input, where the seed set $S_{i,0}$ in cascade $i$ is generated independently from a seed set distribution $\mathcal{D}$ over the node sets, and given $S_{i,0}$, the sequence $(S_{i,1}, \cdots, S_{i,n-1})$ is generated according to the specified diffusion rules. Throughout this work, we assume that $\mathcal{D}$ is a product distribution; in other words, each node $u \in V$ is drawn as a seed independently. We aim to solve the following two problems.

1. **Network inference**. Given a set of $t$ samples $(S_{i,0}, S_{i,1}, \cdots, S_{i,n-1})_{i=1}^t$ defined as above, estimate the values of $p_{uv}$ within an additive error. More formally, for some $\varepsilon \in (0, 1)$, compute a vector $\hat{p}$ such that $|\hat{p}_{uv} - p_{uv}| \leq \varepsilon$ for all $u, v \in V$.

2. **Influence maximization from samples (IMS)**. Given a set of $t$ samples $(S_{i,0}, S_{i,1}, \cdots, S_{i,n-1})_{i=1}^t$ defined as above, find a set $S^A$ of at most $k$ seeds such that $\sigma(S^A) \geq \kappa \cdot \sigma(S^*)$ for some constant $\kappa \in (0, 1)$, where $S^*$ denotes the optimal solution.

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In the literature, network inference often means to recover network structure, namely the edge set $E$. Here we slightly abuse the terminology to also mean learning edge parameters.
Notations. Our algorithms actually only use $S_{i,0}$ and $S_{i,1}$ in those cascades to infer information about the graph, and we find it convenient to define some corresponding concepts and notations in advance. These concepts are indeed crucial to our algorithm design. For node $v ∈ V$, we use $q_v = Pr_D[v ∈ S_0]$ to denote the probability that $v$ is drawn as a seed. We denote by $ap_{G,D}(v)$ the activation probability of node $v$ in one time step during a cascade ($S_0, S_1, \cdots, S_{n-1}$) over network $G$ when $S_0$ is drawn from the distribution $D$. Thus, $ap_{G,D}(v) = Pr_{G,D}[v ∈ S_1]$. Note that it contains the possibility that $v$ itself is a seed, namely $v ∈ S_0 ⊆ S_1$. For $u, v ∈ V$, we define $ap_{G,D}(v|u) = Pr_{G,D}[v ∈ S_1|u ∈ S_0]$ and $ap_{G,D}(v|\bar{u}) = Pr_{G,D}[v ∈ S_1|u /∈ S_0]$, respectively, which are the corresponding probabilities conditioned on whether $u$ is selected as a seed. When the context is clear, we often omit the subscripts $G$ and $D$ in the notation.

Chernoff-type bounds. Following are Chernoff-type bounds we will use in our analysis.

Lemma 1 (Multiplicative Chernoff bound, Mitzenmacher and Upfal 2005). Let $X_1, X_2, \cdots, X_n$ be independent random variables in $\{0, 1\}$ with $Pr[X_i = 1] = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \sum_{i=1}^n p_i$. Then, for $0 < a < 1$,

$$Pr[X \geq (1 + a)\mu] \leq e^{-\mu a^2/3},$$

and

$$Pr[X \leq (1 - a)\mu] \leq e^{-\mu a^2/2}.$$

Lemma 2 (Additive Chernoff bound, Alon and Spencer 2008). Let $X_1, \cdots, X_n$ be independent random variables in $\{0, 1\}$ with $Pr[X_i = 1] = p_i$. Let $X = \sum_{i=1}^n X_i$ and $\mu = \sum_{i=1}^n p_i$. Then for any $a > 0$, we have

$$Pr[X - \mu \geq a] \leq \exp(-a \min(1/5, a/4\mu)).$$

Moreover, for any $a > 0$, we have

$$Pr[X - \mu \leq -a] \leq \exp(-a^2/2\mu).$$

3. Algorithms under the IC Model

In this section, we solve both network inference and IMS under the IC model. In Section 3.1 we present network inference algorithms to estimate each edge probability within a small additive error. In Section 3.2 we present several IMS algorithms.

3.1. Network Inference

In this section, we present a novel algorithm under the IC model for estimating the edge probabilities of the underlying graph $G$, namely we need to find an estimate $\hat{p}$ of $p$ such that $|\hat{p}_{uv} - p_{uv}| \leq \varepsilon$ for all $u, v ∈ V$. While all previous studies rely on the maximum likelihood estimation to estimate $\hat{p}$ (Netrapalli and Sanghavi 2012, Narasimhan et al. 2015, Pouget-Abadie and Horel 2015), our algorithm is based on the following key observation on the connection between $p_{uv}$ and the one-step activation probabilities $ap(v)$ and $ap(v|\bar{u})$. We remark that our algorithm does not rely on the knowledge of edges in graph $G$, and in fact it can be used to also reconstruct the edges of the graph.
Lemma 3. Under the IC model, for any \( u, v \in V \) with \( u \neq v \),
\[
p_{uv} = \frac{ap(v) - ap(v \mid \bar{u})}{q_u(1 - ap(v \mid \bar{u}))}.
\]

**Proof.** To avoid confusion, we write the underlying graph \( G \) and the seed distribution \( D \) explicitly in notation \( ap(\cdot) \), namely \( ap(v) = ap_{G,D}(v) \). Consider the subgraph \( G' = G \setminus \{u\} \) by removing node \( u \). Node \( v \) has two chances to be activated in one time step: either by nodes in \( G' \) (including the case where \( v \) itself is a seed) or by node \( u \). Since \( D \) is a product distribution, we have
\[
ap_{G,D}(v) = ap_{G',D}(v) + (1 - ap_{G',D}(v))q_up_{uv}.
\]
Besides, \( ap_{G',D}(v) = ap_{G,D}(v \mid \bar{u}) \) since when considering one-step activation of \( v \), node \( u \) not being the seed is equivalent to removing it from the graph. Plugging the equality into the last one, we obtain
\[
p_{uv} = \frac{ap_{G,D}(v) - ap_{G,D}(v \mid \bar{u})}{q_u(1 - ap_{G,D}(v \mid \bar{u}))},
\]
which proves the lemma.

Equipped with Lemma 3, we are able to estimate \( p_{uv} \) by estimating \( q_u, ap(v) \) and \( ap(v \mid \bar{u}) \) respectively from cascade samples. Let \( t_u = |\{i \in [t] \mid u \in S_t\}| \) be the number of cascades where \( u \) is a seed, \( t_{\bar{u}} = |\{i \in [t] \mid u \notin S_t\}| \) the number of cascades where \( u \) is not a seed, \( t_v = |\{i \in [t] \mid v \in S_t\}| \) the number of cascades where \( v \) is activated in one time step and \( t_{\bar{v}} = |\{i \in [t] \mid u \notin S_t, v \in S_t\}| \) the number of cascades where \( u \) is not a seed and \( v \) is activated in one time step. Then, \( q_u = t_u/t, \hat{ap}(v) = t_v/t \) and \( \hat{ap}(v \mid \bar{u}) = t_{\bar{v}}/t \) are good estimates of \( q_u, ap(v) \) and \( ap(v \mid \bar{u}) \), respectively. The formal procedure is formulated as Algorithm 1.

Algorithm 1 needs to work under Assumption 1 below, which ensures that all quantities are well estimated. Assumption 1 consists of two conditions. The first means that node \( v \in V \) has a non-negligible probability of not being activated in one time step. The second means that the probability of a node \( u \in V \) being selected as a seed is neither too low nor too high.

**Assumption 1 (Edge probabilities estimation under the IC model).** For some parameters \( \alpha \in (0, 1], \gamma \in (0, 1/2], \)
1. \( ap(v) \leq 1 - \alpha \) for all \( v \in V \).
2. \( \gamma \leq q_u \leq 1 - \gamma \) for all \( u \in V \).

We now give an analysis of Algorithm 1. Lemma 4 below gives the number of samples we need to estimate \( q_u, ap(v) \) and \( ap(v \mid \bar{u}) \) within a small accuracy.

**Lemma 4.** Under Assumption 1, for any \( \eta \in (0, 4/5), \delta \in (0, 1) \), for \( \hat{q}_u, \hat{ap}(v) \), and \( \hat{ap}(v \mid \bar{u}) \) defined in Algorithm 1, if the number of samples \( t \geq \frac{16}{\eta \delta} \ln \frac{2n}{\delta} \), with probability at least \( 1 - \delta \), we have
Proof. For a node \( u \in V \), let \( X_i \) be a 0-1 random variable such that \( X_i = 1 \) if \( u \in S_{i,0} \). Thus \( \hat{q}_u = \sum_{i=1}^{t} X_i/t \). By Lemma \( \square \)
\[
\Pr[|\hat{q}_u - q_u| \geq \eta q_u] = \Pr[\sum_{i=1}^{t} X_i - t q_u \geq \eta t q_u] \leq 2 \exp(-t q_u \eta^2/3) \leq 2 \exp(-t \gamma^2 /3) \leq \delta/(3n).
\]
The last inequality requires that \( t \geq \frac{2}{\gamma^2} \ln \frac{6n}{\delta} \).

For a node \( v \in V \), for \( i \in [t] \), let \( Y_i \) be a 0-1 random variable such that \( Y_i = 1 \) if \( v \in S_{i,1} \). Thus \( \hat{ap}(v) = \sum_{i=1}^{t} Y_i/t \). By Lemma \( \square \) \( (a = \eta t, \mu = t \cdot ap(v)) \),
\[
\Pr[|\hat{ap}(v) - ap(v)| \geq \eta] \leq \exp(-\eta t \min(1/5, \eta/(4 \cdot ap(v)))) + \exp(-\eta^2 t / (2 \cdot ap(v))) \\
\leq \exp(-\eta t \min(1/5, \eta/4)) + \exp(-\eta^2 t / 2) \\
\leq \delta/(6n) + \delta/(6n) = \delta/(3n).
\]
The second inequality holds since \( ap(v) \leq 1 \) by definition. The last inequality requires that \( t \geq \frac{\eta}{\eta^2} \ln \frac{6n}{\delta} \).

For \( u \in V \), let \( \tilde{t}_u = |\{i \leq t \mid u / \notin S_{i,0}\} | \) be the number of cascades where \( u \) is not a seed. Since \( q_u \leq 1 - \gamma, 8 \ln(12n/\delta) \gamma^2 \leq t(1 - q_u)/2 \). By Lemma \( \square \) \( (a = 1/2, \mu = t(1 - q_u)) \),
\[
\Pr[\tilde{t}_u \leq 8 \ln(12n/\delta) / \gamma^2] \leq \Pr[\tilde{t}_u \leq t(1 - q_u)/2] \leq \exp(-t(1 - q_u)/8) \leq \exp(-t \gamma/8) \leq \delta/(6n^2).
\]
The last inequality holds as long as \( t \geq 16 \ln(6n/\delta)/\gamma \).

Given a fixed \( \ell \), assume that there are \( t_\ell = \ell \) cascades where \( u \) is not a seed. For \( i \in [\ell] \), let \( Z_i \) be a 0-1 random variable such that \( Z_i = 1 \) iff \( v \in S_{i,1} \) in the \( i \)-th cascade, among the \( \ell \) cascades where \( u \) is not a seed. Then, \( \hat{ap}(v | \bar{u}) = \sum_{i=1}^\ell Z_i/\ell \). By Lemma 2 (\( a = \eta \ell, \mu = \ell \cdot \ap(v | \bar{u}) \)),

\[
\Pr[|\hat{ap}(v | \bar{u}) - \ap(v | \bar{u})| \geq \eta | t_u = \ell]
\leq \exp(-\eta \ell \min(1/5, \eta/(4 \cdot \ap(v | \bar{u})))) + \exp(-\eta^2 \ell/(2 \cdot \ap(v | \bar{u})))
\leq \exp(-\eta \ell \min(1/5, \eta/4)) + \exp(-\eta^2 \ell/2).
\]

The last inequality holds since \( \ap(v | u) \leq 1 \) by definition. If \( \ell \geq 8 \ln(12n/\delta)/\eta^2 \), then

\[
\Pr[|\hat{ap}(v | \bar{u}) - \ap(v | \bar{u})| \geq \eta | t_u = \ell] \leq \delta/(6n^2).
\]

By law of total probability,

\[
\Pr[|\hat{ap}(v | \bar{u}) - \ap(v | \bar{u})| \geq \eta]
\leq \left( \sum_{\ell \leq 8 \ln(12n/\delta)/\eta^2} + \sum_{\ell \geq 8 \ln(12n/\delta)/\eta^2} \right) \Pr[t_u = \ell] \Pr[|\hat{ap}(v | \bar{u}) - \ap(v | \bar{u})| \geq \eta | t_u = \ell]
\leq \Pr[t_\ell < 8 \ln(12n/\delta)/\eta^2] + \Pr[t_\ell \geq 8 \ln(12n/\delta)/\eta^2] \cdot \delta/(6n^2)
\leq \delta/(3n^2).
\]

The lemma follows immediately by union bound.

As stated below, Theorem 1 derives a theoretical guarantee for Algorithm 1.

**Theorem 1.** Under Assumption 1 for any \( \varepsilon, \delta \in (0,1) \), let \( \eta = \varepsilon \alpha \gamma /4 < 1/8 \), and \( \hat{p}_{uv} \) be the set of edge probabilities returned by Algorithm 1. If the number of cascades \( t \geq \frac{16}{\gamma^2} \ln \frac{12n}{\alpha} = \frac{256}{\gamma^2} \ln \frac{12n}{\delta} \), with probability at least \( 1 - \delta \), for any \( u, v \in V \), \( |\hat{p}_{uv} - p_{uv}| \leq \varepsilon \).

**Proof.** With probability at least \( 1 - \delta \), all the events in Lemma 4 occur. We assume that this is exactly the case in the following. Since \( \ap(v | \bar{u}) \leq \ap(v) \leq 1 - \alpha \), we have \( 1 - \ap(v | \bar{u}) \geq \alpha \). By the value of \( \eta \) and the assumption that \( q_u \leq \gamma \), we have

\[
\eta \leq \frac{\varepsilon \gamma}{4}(1 - \ap(v | \bar{u})) \leq \frac{\eta}{4} q_u (1 - \ap(v | \bar{u})).
\]

To prove \( \hat{p}_{uv} \leq p_{uv} + \varepsilon \), we have

\[
\hat{p}_{uv} = \frac{\hat{ap}(v) - \hat{ap}(v | \bar{u})}{q_u (1 - \hat{ap}(v | \bar{u}))}
\leq \frac{\ap(v) - \ap(v | \bar{u}) + 2\eta}{(1 - \eta) q_u (1 - \ap(v | \bar{u}) - \eta)}
\]
The first inequality holds due to Lemma 4. The second inequality holds by applying the first inequality in Eq. (1). The third inequality holds due to Lemma 3 and the second inequality in Eq. (1). To see the correctness of the last inequality, first observe that
\[
(p_{uv} + \varepsilon)(1 - \eta)(1 - \varepsilon \gamma/4) \\
\geq (p_{uv} + \varepsilon)(1 - \eta - \varepsilon \gamma/4) \\
\geq (p_{uv} + \varepsilon) - (1 + \varepsilon)(\eta + \varepsilon \gamma/4).
\]

Next, note that
\[
(1 + \varepsilon)(\eta + \varepsilon \gamma/4) = (1 + \varepsilon)(1 + \alpha)\varepsilon \gamma/4 \leq (1 + \varepsilon)\varepsilon/4 \leq \varepsilon/2.
\]
The equality is due to the definition of \(\eta\). The two inequalities hold since \(\alpha \in (0, 1], \gamma \in (0, 1/2]\) and \(\varepsilon \in (0, 1)\), respectively. Combining the above two observations, we have the desired inequality
\[
(p_{uv} + \varepsilon)(1 - \eta)(1 - \varepsilon \gamma/4) \geq (p_{uv} + \varepsilon) - \varepsilon/2 = p_{uv} + \varepsilon/2.
\]

On the other hand, to prove \(\hat{p}_{uv} \geq p_{uv} - \varepsilon\), first assume that \(p_{uv} \geq \varepsilon\), since otherwise the claim would be trivial for \(\hat{p}_{uv} \geq 0\). We now have
\[
\hat{p}_{uv} = \frac{\hat{a}p(v) - \hat{a}p(v|\bar{u})}{q_{a}(1 - \hat{a}p(v|\bar{u}))} \\
\geq \frac{ap(v) - ap(v|\bar{u}) - 2\eta}{(1 + \eta)q_{a}(1 - ap(v|\bar{u}) + \eta)} \\
\geq \frac{ap_G(v) - ap(v|\bar{u}) - 2\eta}{(1 + \eta)(1 + \varepsilon \gamma/4)q_{a}(1 - ap(v|\bar{u}))} \\
\geq \frac{p_{uv} - \varepsilon/2}{(1 + \eta)(1 + \varepsilon \gamma/4)} \geq p_{uv} - \varepsilon.
\]
The first inequality holds due to Lemma 4. The second inequality holds by applying the first inequality in Eq. (1). The third inequality holds due to Lemma 3 and the second inequality in Eq. (1). The last inequality follows from a similar argument as the one for the last inequality in Eq. (2), and we omit it for conciseness.

With the ability of estimating edge probabilities, we further show that we can recover the graph structure by a standard threshold approach (Netrapalli and Sanghavi, 2012; Pouget-Abadie and Horel, 2015). The formal procedure is depicted as Algorithm 2, which estimates the edge probabilities to a prescribed accuracy and returns the edges whose estimated probabilities are above a prescribed threshold. Its guarantee is shown in Theorem 2, which shows that no “zero-probability edge” is incorrectly recognized as an edge. Besides, only small-probability edges are omitted, which is reasonable for practical use.
$v \in \text{samples}$. However, it is impossible to verify the assumptions in (Netra palli and Sanghavi, 2012; Narasimhan et al., 2015). Besides, Assumption 1 enjoys the advantage that it is verifiable, our assumptions are the weakest compared with those in (Netrapalli and Sanghavi, 2012; Pouget-Abadie and Horel, 2015), it is instead assumed that 1

Theorem 2. Under Assumption 1, if the number of cascades $t \geq \frac{1024}{\alpha^2 \beta^2 \gamma^2} \ln \frac{4n}{\delta}$, with probability at least $1 - \delta$, the edge set $\hat{E}$ returned by Algorithm 2 satisfies (1) $\hat{E} \subseteq E$, and (2) if $p_{uv} > \beta$, then $(u, v) \in \hat{E}$. As a corollary, if $p_{uv} > \beta$ for all $(u, v) \in E$, then $\hat{E} = E$.

Proof. By Theorem 1, $|\hat{p}_{uv} - p_{uv}| \leq \epsilon = \beta/2$ for all $u, v \in V$ with probability at least $1 - \delta$. If $(u, v) \notin E$, then $p_{uv} = 0$ and hence $\hat{p}_{uv} \leq \beta/2$, which implies $(u, v) \notin \hat{E}$. Thus, $\hat{E} \subseteq E$. If $(u, v) \in E$ and $p_{uv} > \beta$. Then, $\hat{p}_{uv} \geq p_{uv} - \beta/2 > \beta/2$ and hence $(u, v) \in \hat{E}$. Finally, if $p_{uv} > \beta$ for all $(u, v) \in E$, then $E \subseteq \hat{E}$ and hence $\hat{E} = E$, which concludes the proof.

Discussion. It is worth comparing the result in (Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015; Pouget-Abadie and Horel, 2015) with ours, since all of them studied network inference under the IC model. Specifically, Netrapalli and Sanghavi (2012) initiated the study of recovering network structure and did not consider the estimation of edge parameters. Narasimhan et al. (2015) and Pouget-Abadie and Horel (2015) studied how to estimate edge parameters. Both of them used the Euclidean norm of the edge probability vector as the measurement of accuracy, while we use the infinite norm. Besides, in (Narasimhan et al., 2015), it was additionally assumed that the network structure is known in advance. In (Pouget-Abadie and Horel, 2015), totally different assumptions were used, which seems incomparable to ours, and thus we will not further compare against it below.

There are several important differences besides the above. First, the approaches taken are different. All the algorithms in the previous works build on the maximum likelihood estimation (MLE) and require to solve a convex program, while we directly find a closed-form expression for the edge probability $p_{uv}$, thus rendering fast implementation.

Second, the assumptions required are different. The assumptions $p_{uv} > \beta$ for all $u, v \in V$ and $\gamma \leq q_u \leq 1 - \gamma$ for all $u \in V$ are also required in the previous works (though may be presented in different forms). The key difference is the condition $a_p(v) \leq 1 - \alpha$ for all $v \in V$. In (Netrapalli and Sanghavi, 2012), its role is replaced by the correlation decay condition, which requires that $\sum_{u \in N(v)} p_{uv} \leq 1 - \alpha$ for all $v \in V$. In (Narasimhan et al., 2015), it is instead assumed that $1 - \prod_{u \in N(v)} (1 - p_{uv}) \leq 1 - \alpha$ for all $v \in V$. By observing that $a_p(v) \leq 1 - \prod_{u \in N(v)} (1 - p_{uv}) \leq \sum_{u \in N(v)} p_{uv}$ (see the appendix), it is easy to see that our assumptions are the weakest compared with those in (Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015). Besides, Assumption 1 enjoys the advantage that it is verifiable, since one can find suitable values for $\alpha$ and $\gamma$ by estimating $a_p(v)$ and $q_u$ from cascade samples. However, it is impossible to verify the assumptions in (Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015) based only on cascade samples. We remark that our network

Algorithm 2: Recover Network Structure

Require: A set of cascades $(S_{i,0}, S_{i,1}, \ldots, S_{i,n-1})_{i=1}^t$, parameter $\beta \in (0, 1)$.

Ensure: An estimated edge set $\hat{E}$.

1: $\{\hat{p}_{uv}\}_{u,v \in V} = \text{Estimate-Edge-Probabilities}$

2: $(S_{t,0}, S_{t,1}, \ldots, S_{t,n-1})_{t=1}^t$. {With estimation accuracy $\beta/2$.}

2: return $\hat{E} = \{(u, v) \mid \hat{p}_{uv} > \beta/2\}$.

\begin{algorithm}

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Algorithm 3 IMS-IC under Assumption 1

Require: A set of cascades \((S_i,0, S_i,1, \ldots, S_i,n-1)_{i=1}^t\) and \(k \in \mathbb{N}_+\).

1: \(\{\hat{p}_{uv}\}_{u,v \in V} = \text{Estimate-Edge-Probabilities}((S_i,0, S_i,1, \ldots, S_i,n-1)_{i=1}^t).\) \(\{\text{With estimation accuracy } \varepsilon k/(2n^3)\}.

2: Let \(\hat{G} = (V, E, \hat{p})\).

3: Let \(S^A = A(\hat{G}, k)\), where \(A\) is a \(\kappa\)-approximation IM algorithm.

4: return \(S^A\).

inference algorithm replies on the assumption that each seed node is independently sampled. This assumption is also made in (Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015) for the MLE method, but conceptually it might be easier to relax this assumption with MLE. We leave the relaxation of the independence sampling assumption of our method as a future work.

Finally, our algorithm has lower sample complexity compared with those in (Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015). Assume that \(ap(v) \leq 1 - \prod_{u \in N(v)} (1 - p_{uv}) \leq \sum_{u \in N(v)} p_{uv} \leq 1 - \alpha\). Then, Netrapalli and Sanghavi (2012) needs \(\tilde{O}\left(\frac{1}{\alpha^2} \beta^2 \gamma D^2 \log \frac{n}{\delta}\right)\) samples to recover network structure, where \(D\) is the maximum in-degree of the network, while we only need \(O\left(\frac{1}{\varepsilon^2} \alpha^2 \beta^2 \gamma \ln \frac{n}{\delta}\right)\) samples by Theorem 2. On the other hand, assume that the network structure is known and \(m = |E|\). Narasimhan et al. (2015) needs \(\tilde{O}\left(\frac{1}{\varepsilon^2} \alpha^2 \beta^2 \gamma (1 - \gamma) mn \ln \frac{n}{\delta}\right)\) samples to achieve \(\|\hat{p} - p\|^2 \leq \varepsilon\), while we only need \(O\left(\frac{1}{\varepsilon^2} \alpha^2 \beta^2 \gamma m \ln \frac{n}{\delta}\right)\) samples by achieving \(|\hat{p}_{uv} - p_{uv}| \leq \sqrt{\varepsilon} m\).

3.2. Influence Maximization from Samples

In this section, we present several IMS algorithms under the IC model. In Section 3.2.1, we present an approximation-preserving algorithm under Assumption 1. In Section 3.2.2, we show that under an alternative assumption (Assumption 2), there is a constant approximation algorithm for the problem. An attractive feature of Assumption 2 (compared to Assumption 1) is that it has no requirement on the network. We also show that by slightly strengthening Assumption 2, we again obtain an approximation-preserving algorithm.

3.2.1. IMS under Assumption 1

Our first IMS algorithm is presented as Algorithm 3. It follows the canonical learning-and-then-optimization approach by first learning a surrogate graph \(\hat{G} = (V, E, \hat{p})\) from the cascades and then executing any \(\kappa\)-approximation algorithm \(A\) for standard influence maximization on \(\hat{G}\) to obtain a solution as output. The construction of \(\hat{G}\) builds on Algorithm 1 and is obtained by estimating all the edge probabilities to a sufficiently small additive error. Algorithm 3 works under Assumption 1 since Algorithm 1 does.

The correctness of Algorithm 3 relies on Lemma 5 which translates the estimation error in edge probabilities into the learning error in the influence spread function. We use it in Theorem 3 to prove that with high probability, Algorithm 3 almost preserves the approximation ratio of any standard influence maximization algorithm \(A\).
Lemma 5 (Narasimhan et al. 2015). Fix $S \subseteq V$. Under the IC model, for any two edge probability vectors $p, \hat{p}$ with $\|p - \hat{p}\|_1 \leq \varepsilon/n$, we have $|\sigma^p(S) - \sigma^{\hat{p}}(S)| \leq \varepsilon$.

Theorem 3. Under Assumption 1, for any $\varepsilon \in (0, 1)$ and $k \in \mathbb{N}_+$, suppose that the number of cascades $t \geq \frac{1000^8}{\varepsilon^2 \gamma^6} \ln \frac{12}{\delta}$ Let $A$ be a $\kappa$-approximation algorithm for influence maximization. Let $S^A$ be the set returned by Algorithm 3 and $S^*$ be the optimal solution on the original graph. We have

$$\Pr[\sigma(S^A) \geq (\kappa - \varepsilon)\sigma(S^*)] \geq 1 - \delta.$$ 

Proof. By Theorem 1, with probability at least $1 - \delta$, for any $u, v \in V$, $|\hat{p}_{uv} - p_{uv}| \leq \varepsilon k/(2n^3)$. Hence, $\|p - \hat{p}\|_1 = \sum_{u,v \in V} |p_{uv} - \hat{p}_{uv}| \leq \varepsilon k/(2n)$. Applying this condition to Lemma 5, we have that $|\sigma^p(S) - \sigma^{\hat{p}}(S)| \leq \varepsilon k/2$ for every seed set $S$. We thus have

$$\sigma(S^A) \geq \sigma^{\hat{p}}(S^A) - \varepsilon k/2 \geq \kappa \cdot \sigma^{\hat{p}}(S^*) - \varepsilon k/2 \geq \kappa \cdot (\sigma(S^*) - \varepsilon k/2) - \varepsilon k/2 = \kappa \cdot \sigma(S^*) - (1 + \kappa)\varepsilon k/2 \geq (\kappa - \varepsilon)\sigma(S^*).$$

The second inequality holds since $S^A$ is a $\kappa$-approximation on $\hat{G}$. The last inequality holds since $\sigma(S^*) \geq k \geq (1 + \kappa)k/2$.

Compared with our learning algorithms for network inference, Algorithm 3 has an additional overhead of $n^6/k^2$ in the number of cascades. This is because it needs to estimate edge probabilities within an additive error of at most $\varepsilon k/(2n^3)$. One can also invoke known network inference algorithms other than ours in Algorithm 3 to obtain a similar approximate solution, but as discussed above, this only incurs higher sample complexity. We are not aware of any approach to reduce the sample complexity and leave it as an interesting open problem.

3.2.2. IMS under Assumptions Independent of the Network

Condition 1 of Assumption 1 depends on the diffusion network, and hence our Algorithm 3 may not be applicable to all networks. In this section, we show that under an alternative assumption (Assumption 2), which is entirely independent of the diffusion network, there still exists a constant approximation IMS algorithm (Algorithm 4).

Assumption 2 (IMS under the IC model, independent of the network). For some constant $c > 0$ and parameter $\gamma \in (0, 1/2]$,

1. $\sum_{u \in V} q_u \leq ck$.
2. $\gamma \leq q_u \leq 1 - \gamma$ for all $u \in V$.

Assumption 2 consists of two conditions. The condition $\sum_{u \in V} q_u \leq ck$ replaces the condition $\text{ap}(v) \leq 1 - \alpha$ in Assumption 1. It means that a random seed set drawn from the product distribution $D$ has an expected size at most linear in $k$ (but not necessarily
bounded above by \( k \)). Assumption 2 puts forward two plausible requirements for the seed distribution \( \mathcal{D} \) and has no requirement for the underlying network. Thus, in principle, one can handle any social networks, as long as the seed set sampling is reasonable according to Assumption 2.

We now describe the high-level idea of Algorithm 4. It might be surprising at first glance that one can remove the condition \( \text{ap}(v) \leq 1 - \alpha \) for all \( v \in V \). After all, it is very hard to learn information about incoming edges of \( v \) if \( \text{ap}(v) \) is very close to 1. To handle this difficulty, recall that \( \text{ap}(v) \) is defined as the activation probability of \( v \) in one time step. Hence, if \( \text{ap}(v) \) is close to 1, \( v \) will be active with high probability starting from a random seed set. The observation suggests that one can divide nodes into two parts according to their \( \text{ap}(\cdot) \). For the nodes with small \( \text{ap}(\cdot) \), Assumption 1 is satisfied and one can find a good approximation for them via a similar approach as Algorithm 3. For the nodes with large \( \text{ap}(\cdot) \), a random seed set is already a good approximation for them. So there is no need to learn their incoming edges. A technical issue here is that a random seed set may not be a feasible solution for the maximization task. This is why we introduce Condition 1 of Assumption 2, by which the expected size of the seed set is at most linear in \( k \). With the condition, we can replace the random seed set by its random subset of size \( k \) while keeping a constant approximation. To summarize, we find two candidate solutions whose union must be a good approximation over the whole network. If we choose one of them randomly, we will finally obtain a feasible solution with constant approximation.

Following the guidance of the above idea, Algorithm 4 first computes an estimate \( \hat{\text{ap}}(v) \) of \( \text{ap}(v) \) for all \( v \in V \) and partitions \( V \) into two disjoint subsets \( V_1 = \{ v \in V \mid \hat{\text{ap}}(v) < 1 - \delta / (4n) \} \) and \( V_2 = V \setminus V_1 \). It then estimates the probabilities of incoming edges of \( V_1 \) using Algorithm 1 and sets the probabilities of incoming edges of \( V_2 \) to 1 directly for technical reasons. The constructed graph is denoted by \( \hat{G} \). Let \( T_1 \) be a \( \kappa \)-approximation on \( \hat{G} \) and \( T_2 = S_{1,0} \) be the first random seed set. Finally, Algorithm 4 selects \( T_1 \) or \( T_2 \) with equal probability, and if it selects \( T_2 \) while \( |T_2| > k \), it further selects a random subset of \( T_2 \) with size \( k \) as the final output seed set \( S_A \).

We now give an analysis of Algorithm 4. Our analysis requires a technical lemma (Lemma 6). Informally, Lemma 6 means that for any set \( R \subseteq V \), the influence of the seed set \( S \) when setting the probabilities of all incoming edges of \( R \) to 1 is no larger than the influence of \( S \cup R \).

**Lemma 6.** Let \( G = (V, E, p) \) be a directed graph and \( R \subseteq V \). Let \( G' = (V, E, p') \) be a directed graph obtained from \( G \) as follows: \( p'_{uv} = 1 \) if \( v \in R \) and \( p'_{uv} = p_{uv} \) otherwise. Then, for any \( S \subseteq V \), we have \( \sigma^p(S \cup R) \geq \sigma^{p'}(S) \).

**Proof.** To prove Lemma 6 we will use live-edge graphs to interpret the IC model and help understand the influence spread. Formally, a live-edge graph corresponding to the IC model is a random subgraph of \( G \) such that each edge \((u, v)\) is picked independently with probability \( p_{uv} \). Let \( \sigma_u(S) \) be the probability that \( u \) is reachable from \( S \) in the live-edge graph. Then, \( \sigma_u(S) \) is also the probability that \( u \) is activated by \( S \) and hence the influence spread function \( \sigma(S) = \sum_{u \in V} \sigma_u(S) \). For a node \( u \in V \), a fixed edge set \( A \) and a seed set \( S \),
The expectation in the first equality is taken over the randomness of edges \( B \) holds since for each edge \( (a, b) \) consider its activation probability. First, we have
\[
\sigma_1 = \sum_{\text{edges } (a, b) \in A} p_{ab} \prod_{(a, b) \in E \setminus A} (1 - p_{ab}) \mathbb{1}_u(A, S).
\]

Let \( B = \{(u, v) \mid u \in V, v \in R\} \) be the set of incoming edges of \( R \). By definition, \( p'_{ab} = 1 \) for \((a, b) \in B\) and \( p'_{ab} = p_{ab} \) otherwise. Let \( \mathcal{B} \subseteq B \) be a random subset of \( B \) such that for each edge \((a, b) \in B\), \((a, b) \in \mathcal{B}\) independently with probability \( p_{ab} \). For a fixed \( u \in V \), we consider its activation probability. First, we have
\[
\sigma_u^p(S \cup R) = \sum_{A \subseteq E \setminus B} \prod_{(a, b) \in A} p_{ab} \prod_{(a, b) \in E \setminus (A \cup B)} (1 - p_{ab}) \mathbb{1}_u(A \cup \mathcal{B}, S \cup R)
\]
\[
= \sum_{A \subseteq E \setminus B} \prod_{(a, b) \in A} p_{ab} \prod_{(a, b) \in E \setminus (A \cup B)} (1 - p_{ab}) \mathbb{1}_u(A \cup B, S \cup R).
\]

The expectation in the first equality is taken over the randomness of \( \mathcal{B} \). The second equality holds since \( R \) itself is part of the seed set, and it makes no difference whether its incoming edges \( B \) are picked into the live-edge graph. Next, by the definition of \( p' \),
\[
\sigma_u^{p'}(S) = \sum_{A \subseteq E \setminus B} \prod_{(a, b) \in A} p_{ab} \prod_{(a, b) \in E \setminus (A \cup B)} (1 - p_{ab}) \mathbb{1}_u(A \cup B, S).
\]
Clearly, $1_u(A \cup B, S \cup R) \geq 1_u(A \cup B, S)$. It follows that $\sigma^p_u(S \cup R) \geq \sigma^p_u(S)$ and therefore $\sigma^p(S \cup R) \geq \sigma^p(S)$.

**Theorem 4.** Under Assumption[3], suppose that the number of cascades $t \geq \frac{36864}{\epsilon^2 \sigma^2 \gamma^2} n^8 \ln \frac{36n}{\delta} + \frac{72n^2}{\epsilon^2} \ln \frac{12n}{\delta}$, and the number of samples used to estimate $\text{ap}(v)$’s is $t' = \frac{72n^2}{\epsilon^2} \ln \frac{12n}{\delta}$. Let $A$ be a $\kappa$-approximation algorithm for influence maximization. Assume that $k \geq \frac{4}{c} \ln \frac{1}{\delta}$. Let $S^A$ be the set returned by Algorithm[4] and $S^*$ be the optimal solution on the original graph. We have that $S^A$ is a feasible solution, and

$$\Pr[\mathbb{E}[\sigma(S^A)] \geq \min \left\{ \frac{1}{2c}, 1 \right\} \frac{\kappa - \epsilon}{2} \sigma(S^*)] \geq 1 - \delta,$$

where the probability is taken over the randomness of $(S_{i,0}, S_{i,1}, \ldots, S_{i,n-1})_{i=1}^T$ and the expectation is taken over the randomness from line 13 of Algorithm[4].

**Proof.** Let $G = (V, E, p)$ be the original graph. Let $V_1 = \{v \in V \mid \hat{\text{ap}}(v) \leq 1 - \frac{\delta}{12n}\}$ and $V_2 = V \setminus V_1$, defined as in Algorithm[1]. Let $B = \{(u, v) \mid u \in V, v \in V_2\}$ be the set of all edges pointing to some node in $V_2$. Let $G' = (V, E, p')$ be a directed graph obtained from $G$ as follows: $p'_{uv} = 1$ if $(u, v) \in B$ and $p'_{uv} = p_{uv}$ otherwise. Let $G = (V, E, \hat{p})$ be a directed graph obtained from $G'$ by replacing $p_{uv}$ with $\hat{p}_{uv}$ for any $(u, v) \notin B$. Clearly, $G$ is exactly the same graph we constructed in Algorithm[4].

For any node $v \in V$, by Lemma[2] when $t' = \frac{72n^2}{\epsilon^2} \ln \frac{12n}{\delta}$,

$$\Pr[|\hat{\text{ap}}(v) - \text{ap}(v)| \geq \delta/(12n)]$$

$$\leq \exp(-t'(\delta/(12n))) \min(1/5, (\delta/(12n)) \cdot (1/\text{ap}(v)))$$

$$+ \exp(-t'(\delta/(12n))^2/2\text{ap}(v))$$

$$\leq \delta/(6n) + \delta/(6n) = \delta/(3n).$$

By union bound, with probability $1 - \delta/3$, for all nodes $v \in V$, $|\hat{\text{ap}}(v) - \text{ap}(v)| \leq \delta/(12n)$. Specifically, for a node $v \in V_1$ with $\hat{\text{ap}}(v) \leq 1 - \delta/(4n)$, we have $\text{ap}(v) \leq \hat{\text{ap}}(v) + \delta/(12n) \leq 1 - \delta/(6n)$. For a node $v \in V_2$ with $\hat{\text{ap}}(v) > 1 - \delta/(4n)$, we have $\text{ap}(v) \geq \hat{\text{ap}}(v) - \delta/(12n) \geq 1 - \delta/(3n)$.

Since for any $v \in V_2$, $\text{ap}(v) \geq 1 - \delta/(3n)$, by union bound, it means that with probability at least $1 - \delta/3$ in one time step all nodes in $V_2$ are activated. Assume that indeed all nodes in $V_2$ are activated in one time step. Then, we have $\sigma^p(T_1 \cup T_2) = \sigma^p(T_1 \cup S_{i,0}) \geq \sigma^p(T_1 \cup V_2)$.

By plugging $S = T_1, R = V_2$ into Lemma[3], we obtain $\sigma^p(T_1 \cup V_2) \geq \sigma^p(T_1)$. Therefore, by submodularity of $\sigma$, $\sigma^p(T_1) + \sigma^p(T_2) \geq \sigma^p(T_1 \cup T_2) \geq \sigma^p(T_1)$.

Since $\{\hat{p}\}_{uv}$ is obtained by running Estimate-Edge-Probability with parameters $\epsilon k/(2n^3), \alpha = \delta/(6n), \gamma$, when the number of cascades $t - t' \geq \frac{36864}{\epsilon^2 \sigma^2 \gamma^2} n^8 \ln \frac{36n}{\delta}$, with probability $1 - \delta/3$, we have $|\hat{p}_{uv} - p_{uv}| \leq \epsilon k/(2n^3)$ for any $(u, v) \notin B$. Since $\hat{p}_{uv} = p'_{uv} = 1$ for $(u, v) \in B$, we have $|\hat{p} - p'\|_1 \leq \epsilon k/(2n)$. Therefore, by Lemma[5] for any $S \subseteq V$,

$$|\sigma^p(S) - \sigma^p(S)| \leq \epsilon k/2.$$
\[ \geq \kappa \cdot (\sigma'(S^*) - \varepsilon k/2) - \varepsilon k/2 \]
\[ \geq \kappa \cdot (\sigma^p(S^*) - \varepsilon k/2) - \varepsilon k/2 \geq (\kappa - \varepsilon)\sigma^p(S^*). \]

The second inequality holds since \( T_1 \) is a \( \kappa \) approximation of \( S^* \) on \( \hat{G} \). The forth inequality holds since \( \sigma'(S) \geq \sigma^p(S) \) for any \( S \subseteq V \), due to \( p'_{uv} \geq p_{uv} \) for any \((u,v) \in E\). The last inequality holds as long as \((1 + \kappa)k/2 \leq \kappa \leq \sigma^p(S^*)\), which holds trivially since \( \kappa \leq 1 \) and \( \sigma^p(S^*) \geq k \).

Combining the previous inequalities, we have

\[ \sigma^p(T_1) + \sigma^p(T_2) \geq \sigma'(T_1) \geq (\kappa - \varepsilon)\sigma^p(S^*), \]

which implies that \( \mathbb{E}[\sigma^p(T)] = \frac{1}{2}(\sigma^p(T_1) + \sigma^p(T_2)) \geq \frac{1}{2}(\kappa - \varepsilon)\sigma^p(S^*). \)

Finally, since \( \sum_{u \in V} q_u \leq \varepsilon k \), \( \Pr[|S_{1,0}| \geq 2ck] \leq e^{-\varepsilon k/3} \leq \delta/3 \) when \( k \geq \frac{3}{\varepsilon} \ln \frac{3}{\delta} \). Assume that \( |T_2| = |S_{1,0}| \leq 2ck \). If \( T = T_1 \) or \( T = T_2 \) but \( |T_2| \leq k \), then \( S^A = T \). If \( T = T_2 \) and \( |T_2| > k \), then \( S^A \) is a uniform subset of \( T \) with size \( k \). Since \( \sigma(\cdot) \) is submodular, we have \( \mathbb{E}[\sigma^p(S^A)] \geq \min \{ \frac{1}{1 - 2\varepsilon}, 1 \} \mathbb{E}[\sigma^p(T)] \geq \min \{ \frac{1}{1 - 2\varepsilon}, 1 \} \frac{\kappa - \varepsilon}{2} \sigma^p(S^*). \)

To conclude, by union bound, with probability at least \( 1 - \delta \), \( S^A \) is a feasible solution and \( \mathbb{E}[\sigma^p(S^A)] \geq \min \{ \frac{1}{1 - 2\varepsilon}, 1 \} \frac{\kappa - \varepsilon}{2} \sigma^p(S^*). \)

**Improving the approximation ratio.** Compared with Algorithm 3 Algorithm 4 has a worse (though still constant) approximation ratio. We show that if the constant \( c \) in Assumption 2 equals to some prescribed small \( \varepsilon \in (0,1/3) \), we can modify Algorithm 4 to be an approximation-preserving algorithm as follows: let \( T_1 = A(\hat{G}, (1 - 2\varepsilon)k) \) and returns \( T_1 \cup T_2 \) directly. It is easy to see that the modified algorithm works since \( T_1 \) loses little in the approximation ratio and \( T_1 \cup T_2 \) is feasible with high probability. The formal procedure is presented in Algorithm 5 and its guarantee is presented below.

**Theorem 5.** Under Assumption 2 with \( c = \varepsilon \in (0,1/3) \), suppose that the number of cascades \( t \geq \frac{202484}{27\varepsilon^2} \ln \frac{202484}{\varepsilon^2} + \frac{22056}{2\varepsilon^2} \ln \frac{22056}{\varepsilon^2} \) and the number of samples used to estimate \( p(\cdot) \)'s is \( t' = \frac{22056}{2\varepsilon^2} \ln \frac{22056}{\varepsilon^2} \). Let \( A \) be an \( \kappa \)-approximation algorithm for influence maximization. Assume that \( k \geq \frac{3}{\varepsilon} \ln \frac{3}{\delta} \). Let \( S^A \) be the set returned by Algorithm 4 and \( S^* \) be the optimal solution on the original graph. We have

\[ \Pr[|S^A| \leq k \wedge \sigma(S^A) \geq (\kappa - 3\varepsilon)\sigma(S^*)] \geq 1 - \delta. \]

**Proof.** By a similar analysis for Algorithm 4 \( \sigma^p(T_1 \cup T_2) \geq \sigma'(T_1) \) with probability at least \( 1 - \delta/3 \), and for any \( S \subseteq V \), \( |\sigma^p(S) - \sigma'(S)| \leq \varepsilon k/2 \) with probability at least \( 1 - \delta/3 \). We thus have

\[ \sigma'(T_1) \geq \sigma^p(T_1) - \varepsilon k/2 \]
\[ \geq \kappa(1 - 2\varepsilon) \cdot \sigma^p(S^*) - \varepsilon k/2 \]
\[ \geq \kappa(1 - 2\varepsilon) \cdot (\sigma^p(S^*) - \varepsilon k/2) - \varepsilon k/2 \]
\[ \geq \kappa(1 - 2\varepsilon) \cdot (\sigma^p(S^*) - \varepsilon k/2) - \varepsilon k/2 \]
Algorithm 5 IMS-IC under Assumption 2 with $c = \varepsilon$

Require: A set of cascades $(S_{i,0}, S_{i,1}, \cdots, S_{i,n-1})_{i=1}^T$ and $k \in \mathbb{N}_+$, parameter $\varepsilon \in (0, 1/3)$, error probability $\delta > 0$, number of samples $t' \in [t]$ used to estimate $\hat{ap}(v)$’s.

1: Set $V_1 = V$ and $V_2 = \emptyset$.
2: for each $v \in V$ do
3: Use the first $t'$ samples $(S_{i,0}, S_{i,1}, \cdots, S_{i,n-1})'_{i=1}$ to compute $\hat{ap}(v) = tv/t'$, where $tv = |\{i \in [t'] \mid v \in S_{i,1}\}|$.
4: if $\hat{ap}(v) \geq 1 - \delta/(4n)$ then
5: Set $\hat{p}_{uv} = 1$ for all $u \in V$.
6: $V_1 = V_1 \setminus \{v\}$ and $V_2 = V_2 \cup \{v\}$.
7: end if
8: end for
9: $\{\hat{p}_{uv}\}_{u \in V, v \in V_1} = \text{Estimate-Edge-Probabilities}$
   $(S_{i,0}, S_{i,1}, \cdots, S_{i,n-1})_{i=t'+1}$ on $V_1$. [With accuracy $\varepsilon k/(2n^3)$, $\alpha = \delta/(6n)$ in Assumption 4]
10: Let $\hat{G} = (V, E, \hat{p})$.
11: $T_1 = A(\hat{G}, (1 - 2\varepsilon)k)$, where $A$ is a $\kappa$-approximation algorithm for influence maximization.
12: $T_2 = S_{1,0}$.
13: return $S^A = T_1 \cup T_2$.

\[
\begin{align*}
&\geq (\kappa - 2\varepsilon) \cdot \sigma^p(S^*) - (1 + \kappa)\varepsilon k/2 \\
&\geq (\kappa - 3\varepsilon) \cdot \sigma^p(S^*).
\end{align*}
\]

The second inequality holds since $T_1$ is a $\kappa(1 - 2\varepsilon)$ approximation of $S^*$ on $\hat{G}$. The forth inequality holds since $\sigma^p(S) \geq \sigma^p(S)$ for any $S \subseteq V$, due to $\hat{p}_{uv} \geq p_{uv}$ for any $(u, v) \in E$. The last inequality holds as long as $(1 + \kappa)k/2 \leq k \leq \sigma^p(S^*)$, which holds trivially since $\kappa \leq 1$ and $\sigma^p(S^*) \geq k$. Therefore, we have $\sigma^p(S^A) = \sigma^p(T_1 \cup T_2) \geq (\kappa - 3\varepsilon)\sigma^p(S^*)$.

Finally, since $\sum_{u \in V} q_u \leq \varepsilon k$, by Lemma 4.1 $\Pr[|S_{1,0}| \geq 2\varepsilon k] \leq e^{-\varepsilon k/3} \leq \delta/3$ when $k \geq \frac{\delta}{2} \ln \frac{3}{\delta}$. If $|S_{1,0}| \leq 2\varepsilon k$, $|S^A| = |T_1 \cup T_2| \leq (1 - 2\varepsilon)k + 2\varepsilon k = k$.

To conclude, by union bound, with probability at least $1 - \delta$, $|S^A| \leq k$ and $\sigma(S^A) \geq (\kappa - 3\varepsilon)\sigma(S^*)$.

4. Algorithms under the LT Model

In this section, we solve both network inference and IMS problems under the LT model. In Section 4.1, we present a network inference algorithm. In Section 4.2, we present an IMS algorithm.

4.1. Network Inference

In this section, we present a network inference algorithm under the LT model which finds an estimate $\hat{w}$ of $w$ such that $|\hat{w}_{uv} - w_{uv}| \leq \varepsilon$ for all $u, v \in V$. The algorithm is similar to
the one under the IC model and based on the key observation on the connection between \(w_{uv}\) and the one-step activation probabilities \(ap(v)\) and \(ap(v | \bar{u})\). We first give an explicit expression of \(ap(v)\) under the LT model.

**Lemma 7.** Under the LT model, \(ap(v) = q_v + (1 - q_v) \sum_{u \in N(v)} q_u w_{uv}\) for any \(v \in V\).

**Proof.** Fix \(v \in V\). For \(v\) to be active in one time step, \(v\) is either picked as a seed or activated by its in-neighbors which are picked as seeds. By the fact that \(D\) is a product distribution, in our notations, we have

\[
ap(v) = q_v + (1 - q_v)ap(v | \bar{v}).
\]

Let \(S_0 \sim D\) and \(R = S_0 \cap N(v)\). For \(u \in N(v)\), let \(X_u \in \{0, 1\}\) indicate whether \(u \in S_0\). Let \(X = \sum_{u \in N(v)} w_{uv} X_u\). By the linearity of expectation, \(E[X] = \sum_{u \in N(v)} q_u w_{uv}\).

On the other hand, by law of total probability, the facts that \(D\) is a product distribution and \(D, \theta_v\) are independent,

\[
ap(v | \bar{v}) = \sum_{R \subseteq N(v)} \Pr[R] \Pr[v \in S_1 | v \notin S_0, R] = \sum_{R \subseteq N(v)} \Pr[R] \sum_{u \in R} w_{uv} = E[X].
\]

Therefore, we have \(ap(v | \bar{v}) = \sum_{u \in N(v)} q_u w_{uv}\).

By plugging it back, we finally obtain \(ap(v) = q_v + (1 - q_v) \sum_{u \in N(v)} q_u w_{uv}\).

Next, we derive a closed-form expression for the edge weight \(w_{uv}\) from Lemma 7.

**Lemma 8.** Under the LT model, for any \(u, v \in V\) with \(u \neq v\),

\[
w_{uv} = \frac{ap(v) - ap(v | \bar{u})}{q_u (1 - q_v)}.
\]

**Proof.** To avoid confusion, we write the underlying graph \(G\) and the seed distribution \(D\) explicitly in notation \(ap(\cdot)\), namely \(ap(v) = ap_{G,D}(v)\). Consider the subgraph \(G' = G \setminus \{u\}\) by removing node \(u\). Since when considering one-step activation of \(v\), node \(u\) not being the seed is equivalent to removing it from the graph, we have

\[
ap_{G',D}(v | \bar{u}) = ap_{G',D}(v).
\]

Next, by Lemma 7, we have

\[
ap_{G',D}(v) = q_v + (1 - q_v) \sum_{u' \in N(v) \setminus \{u\}} q_{u'} w_{u'v},
\]

\[
ap_{G,D}(v) = q_v + (1 - q_v) \sum_{u' \in N(v)} q_{u'} w_{u'v}.
\]
We therefore obtain that
\[
ap_G(v) - ap_G(v | \bar{u}) = (1 - q_v)q_u w_{uv}.
\]

By rearranging the equality, we obtain
\[
w_{uv} = \frac{ap_G(v) - ap_G(v | \bar{u})}{q_u(1 - q_v)}.
\]

Equipped with the lemma, we are able to estimate \( w_{uv} \) by estimating \( q_u, ap(v) \) and \( ap(v | \bar{u}) \) respectively from cascade samples. Let \( t_u = |\{i \in [t] | u \in S_{i,0}\}| \) be the number of cascades where \( u \) is a seed, \( t_\bar{u} = |\{i \in [t] | u \notin S_{i,0}\}| \) the number of cascades where \( u \) is not a seed, \( t^v = |\{i \in [t] | v \in S_{i,1}\}| \) the number of cascades where \( v \) is activated in one time step and \( t^v_\bar{u} = |\{i \in [t] | u \notin S_{i,0}, v \in S_{i,1}\}| \) the number of cascades where \( u \) is not a seed and \( v \) is activated in one time step. Then, \( \hat{q}_u = t_u / t, \hat{ap}(v) = t^v / t \) and \( \hat{ap}(v | \bar{u}) = t^v_\bar{u} / t_\bar{u} \) are good estimates of \( q_u, ap(v) \) and \( ap(v | \bar{u}) \), respectively. The formal procedure is formulated as Algorithm 6.

Algorithm 6 needs to work under Assumption 3 below, which means that the probability of a node \( u \in V \) being selected as a seed is neither too low nor too high. This assumption ensures that the above quantities are well estimated. Compared with Assumption 1, Assumption 3 does not need the condition \( ap(v) \leq 1 - \alpha \) and hence imposes no requirement on the network. This is because the condition \( ap(v) \leq 1 - \alpha \) gives a lower bound for \( 1 - ap(v | \bar{u}) \) and leads to a tighter estimate of it, while in the closed-form expression of \( w_{uv} \), \( 1 - ap(v | \bar{u}) \) does not appear in the denominator, and hence a loose estimate in the lack of the condition still suffices.

**Assumption 3 (Edge weights estimation under the LT model).** For some parameter \( \gamma \in (0, 1/2) \),
\[
\gamma \leq q_u \leq 1 - \gamma, \forall u \in V.
\]

We now give an analysis of Algorithm 6. Lemma 9 gives the number of samples we need to estimate \( q_u, ap(v) \) and \( ap(v | \bar{u}) \) within a small accuracy. Its proof is exactly the same as that of Lemma 4 and therefore omitted.

**Lemma 9.** Under Assumption 3 for any \( \eta \in (0, 4/5) \), \( \delta \in (0, 1) \), for \( \hat{q}_u, \hat{ap}(v), \) and \( \hat{ap}(v | \bar{u}) \) defined in Algorithm 6, if the number of samples \( t \geq \frac{16}{\gamma^2} \ln \frac{12}{\delta} \), with probability at least \( 1 - \delta \), we have
1. \( |\hat{q}_u - q_u| \leq \eta q_u \) for all \( u \in V \),
2. \( |\hat{ap}(v) - ap(v)| \leq \eta \) for all \( v \in V \),
3. \( |\hat{ap}(v | \bar{u}) - ap(v | \bar{u})| \leq \eta \) for all \( u, v \in V \).

**Theorem 6.** Under Assumption 3 for any \( \epsilon, \delta \in (0, 1) \), let \( \eta = \epsilon \gamma^2 / 4 < 1/8 \). Let \( \{\hat{w}_{uv}\}_{u,v \in V} \) be the set of edge weights returned by Algorithm 6. If the number of cascades \( t \geq \frac{16}{\gamma^2} \ln \frac{12}{\delta} = \frac{256}{\epsilon^2 \gamma^2} \ln \frac{12}{\delta} \), with probability at least \( 1 - \delta \), for any \( u, v \in V \), \( |\hat{w}_{uv} - w_{uv}| \leq \epsilon \).
Algorithm 6 Estimate Edge Weights

Require: A set of cascades \((S_{i,0}, S_{i,1}, \ldots, S_{i,n-1})\) \(i=1\).
Ensure: \(\{\hat{w}_{uv}\}_{u,v \in V}\) such that \(|\hat{w}_{uv} - w_{uv}| \leq \varepsilon\) for all \(u,v \in V\).

1: \textbf{for each} \(u \in V\) \textbf{do}
2: \hspace{1em} Compute \(\hat{q}_u = t_u / t\), where \(t_u = |\{i \in [t] \mid u \in S_{i,0}\}|\).
3: \textbf{end for}
4: \textbf{for each} \(v \in V\) \textbf{do}
5: \hspace{1em} Compute \(\hat{ap}(v) = t^v / t\), where \(t^v = |\{i \in [t] \mid v \in S_{i,1}\}|\).
6: \textbf{end for}
7: \textbf{for each} \(v \in V\) \textbf{do}
8: \hspace{1em} \textbf{for each} \(u \in V\) \textbf{do}
9: \hspace{2em} Compute \(\hat{ap}(v \mid \bar{u}) = t^v_{\bar{u}} / t_{\bar{u}}\), where \(t_{\bar{u}} = |\{i \in [t] \mid u \not\in S_{i,0}\}|\) and \(t^v_{\bar{u}} = |\{i \in [t] \mid u \not\in S_{i,0}, v \in S_{i,1}\}|\).
10: \hspace{2em} Let \(\hat{w}_{uv} = \frac{\hat{ap}(v) - \hat{ap}(v \mid \bar{u})}{q_u(1 - q_v)}\).
11: \hspace{1em} \textbf{end for}
12: \textbf{end for}
13: \textbf{return} \(\{\hat{w}_{uv}\}_{u,v \in V}\).

Proof. With probability at least \(1 - \delta\), all the events in Lemma 9 occur. We assume that this is exactly the case in the following. By the choice of \(\eta\) and the assumption that \(\gamma \leq q_u \leq 1 - \gamma\), we have

\[
\eta \leq \frac{\varepsilon \gamma}{4} (1 - q_v) \leq \frac{\varepsilon}{4} q_u (1 - q_v). \tag{3}
\]

To prove \(\hat{w}_{uv} \leq w_{uv} + \varepsilon\), we have

\[
\hat{w}_{uv} = \frac{\hat{ap}(v) - \hat{ap}(v \mid u)}{q_u(1 - q_v)} \\
\leq \frac{ap(v) - ap(v \mid u) + 2\eta}{(1 - \eta)q_u(1 - q_v - \eta q_v)} \\
\leq \frac{ap(v) - ap(v \mid u) + 2\eta}{(1 - \eta)(1 - \varepsilon \gamma / 4)q_u(1 - q_v)} \\
\leq \frac{w_{uv} + \varepsilon / 2}{q_u(1 - q_v)} \\
\leq w_{uv} + \varepsilon.
\]

The first inequality holds due to Lemma 9. The second inequality holds by applying the first inequality in Eq. (3) and the fact that \(q_v \leq 1\). The third inequality holds due to Lemma 8 and the second inequality in Eq. (3). The correctness of the last inequality follows the same argument as Theorem 1.

On the other hand, to prove \(\hat{w}_{uv} \geq w_{uv} - \varepsilon\), first assume that \(w_{uv} \geq \varepsilon\), since otherwise
the claim would be trivial for \( \hat{w}_{uv} \geq 0 \). We now have
\[
\hat{w}_{uv} = \frac{\hat{ap}(v) - \hat{ap}(v | u)}{q_u(1 - \hat{q}_v)} \\
\geq \frac{ap(v) - ap(v | u) - 2\eta}{(1 + \eta)q_u(1 - q_v + \eta q_v)} \\
\geq \frac{ap(v) - ap(v | u) - 2\eta}{(1 + \eta)(1 + \varepsilon\gamma/4)q_u(1 - q_v)} \\
\geq \frac{w_{uv} - \varepsilon/2}{(1 + \eta)(1 + \varepsilon\gamma/4)} \\
\geq w_{uv} - \varepsilon.
\]

The first inequality holds due to Lemma 9. The second inequality holds by applying the first inequality in Eq. (3) and the fact that \( q_v \leq 1 \). The third inequality holds due to Lemma 8 and the second inequality in Eq. (3). The correctness of the last inequality follows the same argument as Theorem 1.

As in the IC model, Algorithm 6 can be adapted to recover the graph structure. For compactness, we omit the adapted algorithm.

In general, the \( \hat{w} \) returned by Algorithm 6 does not necessarily satisfies the normalization condition that \( \sum_{u \in N(v)} \hat{w}_{uv} \leq 1 \) for all \( v \in V \). The condition is crucial in defining the live-edge graph under the LT model, which helps the design of fast IM algorithm such as RR-set (Borgs et al. 2014) and our IMS algorithm in the next subsection. For this reason, we achieve the normalization condition by rescaling \( \hat{w} \), as Corollary 1 below shows.

**Corollary 1.** For any \( \varepsilon, \delta \in (0, 1) \), let \( \{\hat{w}_{uv}\}_{u,v \in V} \) be the edge weights returned by Algorithm 6 under Assumption 3 with \( t \geq \frac{1024}{\varepsilon^2}D^2 \ln \frac{12n}{\delta} \) cascade samples, where \( D \) is the maximum in-degree of the underlying graph \( G \). Let \( w'_{uv} = \frac{\hat{w}_{uv}}{1 + \varepsilon/2} \). Then, with probability at least \( 1 - \delta \),
1. \( \sum_{u \in N(v)} w'_{uv} \leq 1 \) for all \( v \in V \), and
2. \( |w'_{uv} - w_{uv}| \leq \varepsilon \) for any \( u,v \in V \).

**Proof.** By Theorem 6, with probability at least \( 1 - \delta \), for any \( u,v \in V \), \( |\hat{w}_{uv} - w_{uv}| \leq \varepsilon/(2D) \). We assume that this is exactly the case in the following.

Fix \( v \in V \), we have
\[
\sum_{u \in N(v)} w'_{uv} = \sum_{u \in N(v)} \frac{\hat{w}_{uv}}{1 + \varepsilon/2} \leq \frac{1}{1 + \varepsilon/2} \sum_{u \in N(v)} (w_{uv} + \varepsilon/(2D)) \leq \frac{1}{1 + \varepsilon/2} \frac{1}{1 + \varepsilon/2} (1 + \varepsilon/2) = 1.
\]

The last inequality holds since the original \( w \) satisfies \( \sum_{u \in N(v)} w_{uv} \leq 1 \).

Next, we have
\[
|w'_{uv} - w_{uv}| = \frac{1}{1 + \varepsilon/2} |\hat{w}_{uv} - (1 + \varepsilon/2)w_{uv}| \leq \frac{1}{1 + \varepsilon/2} |\hat{w}_{uv} - w_{uv}| + \frac{\varepsilon/2}{1 + \varepsilon/2} |w_{uv}|
\leq \frac{\varepsilon/(2D)}{1 + \varepsilon/2} + \frac{\varepsilon/2}{1 + \varepsilon/2} \leq \frac{\varepsilon}{2}(1 + \frac{1}{D}) \leq \varepsilon.
\]
Algorithm 7 IMS-LT under Assumption 3

Require: A set of cascades \((S_i, 0, S_i, 1, \cdots, S_{i,n-1})_{i=1}^t\) and \(k \in \mathbb{N}_+\).

1: \(\{\hat{w}_{uv}\}_{u,v \in V} = \text{Estimate-Edge-Weights}((S_i, 0, S_i, 1, \cdots, S_{i,n-1})_{i=1}^t)\). \{ With estimation accuracy \(\varepsilon k/(2Dn^3)\). \}
2: Let \(w'_{uv} = \frac{\hat{w}_{uv}}{1+\varepsilon/2}\) for all \(u, v \in V\).
3: Let \(G' = (V, E, w')\).
4: Let \(S_A = A(G', k)\), where \(A\) is a \(\kappa\)-approximation IM algorithm.
5: return \(S_A\).

4.2. Influence Maximization from Samples

In this section, we present an IMS algorithm (Algorithm 7) under the LT model. Our algorithm is approximation-preserving and imposes no requirement on the network. It follows the canonical learning-and-then-optimization approach by first learning a surrogate graph \(G' = (V, E, w')\) from the cascades and then executing any \(\kappa\)-approximation algorithm \(A\) for standard influence maximization on \(G'\) to obtain a solution as output. The construction of \(G'\) builds on Algorithm 6 and is obtained by first estimating all the edge weights to a sufficiently small additive error and then rescale them as in Corollary 1 to meet the normalization condition. Algorithm 7 works under Assumption 3, since Algorithm 6 does. Consequently, Algorithm 7 can handle arbitrary social networks, since Assumption 3 imposes no requirement for the network.

As Lemma 5 for the IC model, to prove the correctness of Algorithm 7, we need to bound the difference between two LT influence functions with different edge parameters by the difference of the parameters. We show this in Lemma 10 below. Note that to apply Lemma 10, the normalization condition must hold. This explains why Algorithm 7 rescales the estimated edge weights before it runs a standard IM algorithm.

We first present Lemma 10 and its proof.

Lemma 10. Under the LT model, for any two edge weight vectors \(w, w'\) such that (1) \(\|w - w'\|_1 \leq \varepsilon/n\), and (2) both \(w\) and \(w'\) satisfy the normalization condition, we have \(|\sigma^w(S) - \sigma^{w'}(S)| \leq \varepsilon\) for all \(S\).

Proof. To prove Lemma 10, we will use live-edge graphs to interpret the LT model and help understand the influence spread. For node \(v \in V\), its (final) activation probability is denoted by \(\sigma^w_v\). Clearly, \(\sigma^w(S) = \sum_{v \in V} \sigma^w_v(S)\) for any seed set \(S \subseteq V\). For node \(b \in V\), let \(E_b\) be the set of its incoming edges. The live-edge graph under the LT model is generated as follows: For each node \(b \in V\), among all of its incoming edges, \((u, v) \in E_b\) is selected exclusively as the single live edge with probability \(w_{uv}\), and no edge is selected with probability \(1 - \sum_{a \in N(b)} w_{ab}\). The selection is independent among all nodes \(b \in V\). Let \(A \subseteq E\) be the edge set of some realization of the live-edge graph. Then, \(A\) satisfies that \(|A \cap E_b| \leq 1\) for all \(b \in V\). For convenience, let \(\mathcal{A} = \{A \subseteq E \mid |A \cap E_b| \leq 1, \forall b \in V\}\) and \(A \cap E_b = \{e(b)\}\) if \(A \cap E_b \neq \emptyset\). Finally, let \(1_v(A, S)\) be the indicator variable such that \(1_v(A, S) = 1\) if and only if \(v\) is reachable from \(S\) via edges in \(A\). It is proved that \(\sigma^w_v(S)\)
can be written as the summation of \( \mathbf{1}_v(A, S) \) over all realizations of the live-edge graph (see Chen et al., 2013):

\[
\sigma^w_v(S) = \sum_{A \in A} \prod_{b \in \mathcal{B} : A \cap E_b \neq \emptyset} w_{v(b)} \prod_{b \notin \mathcal{B} : A \cap E_b = \emptyset} \left(1 - \sum_{a \in N(b)} w_{ab}\right) \mathbf{1}_v(A, S).
\]

With the above interpretation, we now bound the \( L_\infty \) norm of the gradient of \( \sigma_v^w \).

\[
\frac{\partial \sigma^w_v(S)}{\partial w_{cd}} \leq \left| \sum_{A \in A} \prod_{b \notin \mathcal{B} : A \cap E_b \neq \emptyset} w_{v(b)} \prod_{b \notin \mathcal{B} : A \cap E_b = \emptyset} \left(1 - \sum_{a \in N(b)} w_{ab}\right) \mathbf{1}_v(A \cup \{(c, d)\}, S) \right| + \left(1 - \sum_{a \in N(d)} w_{ad}\right) \sum_{A \in A} \prod_{b \notin \mathcal{B} : A \cap E_b \neq \emptyset} w_{v(b)} \prod_{b \notin \mathcal{B} : A \cap E_b = \emptyset} \left(1 - \sum_{a \in N(b)} w_{ab}\right) \mathbf{1}_v(A \cup \{(c, d)\}, S)
\]

\[
\leq \sum_{A \in A} \prod_{b \notin \mathcal{B} : A \cap E_b \neq \emptyset} w_{v(b)} \prod_{b \notin \mathcal{B} : A \cap E_b = \emptyset} \left(1 - \sum_{a \in N(b)} w_{ab}\right) \mathbf{1}_v(A, S)
\]

\[
= 1 - \sum_{a \in N(d)} w_{ad} \leq 1.
\]

The first equality holds since when computing the partial derivative at \( w_{cd} \), we only need to concern the terms where \( w_{cd} \) appears, and \( w_{cd} \) appears when (1) \((c, d)\) is selected as the single live edge of \( d \), or (2) no incoming edges of \( d \) are selected. The inequality follows from the fact that \( 0 \leq |\mathbf{1}_v(A \cup \{(c, d)\}) - \mathbf{1}_v(A, S)| \leq 1 \). The last equality holds since the summation is over \( A \cap E_d = \emptyset \), which equals to the probability that no incoming edges of \( d \) are selected. Clearly, the above inequality means that \( \|\nabla_w \sigma^w_v(S)\|_\infty \leq 1 \).

By the mean-value theorem, there is a \( \bar{w} = sw + (1 - s)w' \) for some \( s \in [0, 1] \) such that

\[
|\sigma^w_v(S) - \sigma^w_v(S')| = \|\nabla_{\bar{w}} \sigma^w_v(S)\|_\infty \|w - w'\|_1 \leq \varepsilon/n.
\]

Therefore, \( |\sigma^w(S) - \sigma^w(S')| \leq \sum_{v \in V} |\sigma^w_v(S) - \sigma^w_v(S')| \leq \varepsilon \), which completes the proof.

Finally, the performance of Algorithm 7 is presented in the following theorem.

**Theorem 7.** Under Assumption 3, for any \( \varepsilon \in (0, 1) \) and \( k \in \mathbb{N}_+ \), suppose that the number of cascades \( t \geq \frac{2\varepsilon^2}{\varepsilon^2 - \frac{k}{2}} \ln \frac{12n}{\delta} \). Let \( A \) be a \( k \)-approximation algorithm for influence maximization. Let \( S^A \) be the set returned by Algorithm 7 and \( S^* \) be the optimal solution on the original graph. Under Assumption 3, we have

\[
\Pr[\sigma(S^A) \geq (k - \varepsilon)\sigma(S^*)] \geq 1 - \delta.
\]
Proof. By Corollary 1, with probability at least $1 - \delta$, we have 1) \( \sum_{u \in N(v)} w'_{uv} \leq 1 \) for all \( v \in V \), and 2) for any \( u, v \in V \), \( |w'_{uv} - w_{uv}| \leq \varepsilon k / (2n^3) \). Hence, \( \|w - w'\|_1 = \sum_{u, v \in V} |w_{uv} - w'_{uv}| \leq \varepsilon k / (2n) \). Applying this condition to Lemma 10, we have that \( |\sigma^w(S) - \sigma^{w'}(S)| \leq \varepsilon k / 2 \) for every seed set \( S \). We thus have
\[
\sigma(S') \geq \sigma^\beta(S') - \varepsilon k / 2 \geq \kappa \cdot \sigma^\beta(S^*) - \varepsilon k / 2 \\
\geq \kappa \cdot (\sigma(S^*) - \varepsilon k / 2) - \varepsilon k / 2 \\
= \kappa \cdot \sigma(S^*) - (1 + \kappa)\varepsilon k / 2 \geq (\kappa - \varepsilon)\sigma(S^*).
\]

The second inequality holds since \( S' \) is a \( \kappa \)-approximation on \( \hat{G} \). The last inequality holds since \( \sigma(S^*) \geq k \geq (1 + \kappa)k / 2 \).

5. Conclusion and Future Work

In this paper, we conduct a rigorous theoretical treatment to the influence maximization from samples (IMS) problem under both IC and LT models, and provide several end-to-end IMS algorithms with constant approximation guarantee. We also provide novel and efficient algorithms for network inference with weaker assumptions.

There are many future directions to extend and improve this work. First, our IMS algorithms require a large number of samples (though polynomial) since we have to estimate edge probabilities to a very high accuracy. It is very interesting to investigate how to improve the sample complexity by leveraging sparsity and different importance of edges in the networks. Second, our samples contain activation sets at every step. One can further study how to do IMS when we only observe the final activation set. Other directions include studying IMS for other stochastic diffusion models (for example, the cumulative activation model in Shan et al., 2019), relaxing the independent seed node sampling assumption, and going beyond influence maximization to study other optimization tasks directly from data samples.

Appendix A. Comparing Assumptions

We summarize the assumptions used in (Netrapalli and Sanghavi, 2012; Narasimhan et al., 2015) below and show that they are strictly stronger than our assumptions.

Assumption 4 (Assumptions in Netrapalli and Sanghavi 2012). For some parameters \( \alpha, \beta \in (0, 1) \),
1. \( p_{uv} \geq \beta \) for all \( (u, v) \in E \).
2. (Correlation decay) \( \sum_{u \in N(v)} p_{uv} < 1 - \alpha \) for all \( v \in V \).
3. \( q_{uv} d_v < 1 / 2 \) for all \( u, v \in V \).

Assumption 5 (Assumptions in Narasimhan et al. 2015). For some parameters \( \beta \geq \alpha \in (0, 1 / 2) \) and \( \gamma \in (0, 1) \),
1. $p_{uv} \geq \beta$ for all $(u,v) \in E$.
2. $1 - \prod_{u \in N^{in}(v)} (1 - p_{uv}) \leq 1 - \alpha$ for all $v \in V$.
3. $\gamma \leq q_{u} \leq 1 - \gamma$ for all $u \in V$.

Lemma 11. \[ ap(v) \leq 1 - \prod_{u \in N^{in}(v)} (1 - p_{uv}) \leq \sum_{u \in N^{in}(v)} p_{uv}. \]

Proof. The first inequality follows from $ap(v) = 1 - \prod_{u \in N^{in}(v)} (1 - q_{u}p_{uv}) \leq 1 - \prod_{u \in N^{in}(v)} (1 - p_{uv})$, since $q_{u} \leq 1$ for all $u \in V$. The second inequality follows from the claim below.

Claim 1. For any $x_1, \ldots, x_n \in [0,1]$, $\prod_{i=1}^{n} (1 - x_i) \geq 1 - \sum_{i=1}^{n} x_i$.

Proof of the Claim. The claim holds trivially when $n = 1$. Assume that the claim holds for any $k < n$. Then,

$$\prod_{i=1}^{n} (1 - x_i) = \prod_{i=1}^{n-1} (1 - x_i) (1 - x_n) \geq (1 - \sum_{i=1}^{n-1} x_i)(1 - x_n) \geq 1 - \sum_{i=1}^{n} x_i.$$

The two inequalities both hold by induction.

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