A note on maximum size of Berge-$C_4$-free hypergraphs

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Abstract

In this paper, we consider maximum possible value for the sum of cardinalities of hyperedges of a hypergraph without a Berge 4-cycle. We significantly improve the previous upper bound provided by Gerbner and Palmer. Furthermore, we provide a construction that slightly improves the previous lower bound.

1 Introduction

A Berge cycle of length $k$, denoted by Berge-$C_k$, is an alternating sequence of distinct vertices and distinct hyperedges of the form $v_1, h_1, v_2, h_2, \ldots v_k, h_k$ where $v_i, v_{i+1} \in h_i$ for each $i \in \{1, 2, \ldots, k-1\}$ and $v_kv_1 \in h_k$.

Throughout the paper we allow hypergraphs to include multiple copies of the same hyperedge (multi-hyperedges).

Let $H$ be a Berge-$C_4$-free hypergraph on $n$ vertices, Győri and Lemons [3] showed that $\sum_{h \in H}(|h| - 3) \leq (1 + o(1))12\sqrt{2}n^{3/2}$. Notice that it is natural to take $|h| - 3$ in the sum, otherwise we could have arbitrarily many copies of a 3-vertex hyperedge. In [2] Gerbner and Palmer improved the upper bound proving that $\sum_{h \in H}(|h| - 3) \leq \frac{\sqrt{6}}{2}n^{3/2} + O(n)$, furthermore they showed that there exists a Berge-$C_4$-free hypergraph $\mathcal{H}$ such that $\sum_{h \in \mathcal{H}}(|h| - 3) \geq (1 + o(1))\frac{1}{3\sqrt{3}}n^{3/2}$.

In this paper we improve their bounds.

Theorem 1. Let $\mathcal{H}$ be a Berge $C_4$-free hypergraph on $n$ vertices, then

$$\sum_{h \in \mathcal{H}}(|h| - 3) \leq (1 + o(1))\frac{1}{2}n^{3/2}.$$

Furthermore, there exists a $C_4$-free hypergraph $\mathcal{H}$ such that

$$(1 + o(1))\frac{1}{2\sqrt{6}}n^{3/2} \leq \sum_{h \in \mathcal{H}}(|h| - 3)$$

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This improves the upper-bound by factor of $\sqrt{6}$ and slightly increases the lower-bound. We introduce couple of important notations and definitions used throughout the paper. Length of a path is the number of edges in the path. For convenience, an edge or a pair of vertices \{a, b\} is sometimes referred to as $ab$.

For a graph (or a hypergraph) $H$, for convenience, we sometimes use $H$ to denote the edge set of the graph (hypergraph) $H$. Thus the number of edges (hyperedges) in $H$ is $|H|$.

## 2 Proof of Theorem 1

We will now construct a graph, existence of which is proved in [2] (page 10). Let us take a graph $H$ on a ground set of $H$ by embedding edges into each hyperedge of $H$. More specifically, for each $h \in H$ we embed $|h| - 3$ edges on the vertices of $h$, such that collection of edges that were embedded in $h$ consists of pairwise vertex-disjoint triangles and edges. We say that $e \in H$ has color $h$ if $e$ was embedded in the hyperedge $h$ of the hypergraph $H$. We will upper bound the number of edges in $H$, which directly gives us an upper bound on $\sum_{h \in H} (|h| - 3)$.

**Observation 2.** For each vertex $x$ of the graph $H$, at most 2 adjacent edges to $x$ have the same color. Moreover, if $xy$ and $xz$ have the same color $h$, then $yz \in H$ and the color of $yz$ is $h$ as well.

The following lemma is stated and proved in [2] (claim 16, page 10).

**Lemma 3.** $H$ is $K_{2,7}$-free.

Now we will upper bound the number of edges in $H$. It should be noted, that the only properties of $H$ that we use during the proof, are Observation 2 and Lemma 3.

For any vertex $v \in V(H)$, let $d(v)$ denote the degree of $v$ in the graph $H$ and let $d$ be the average degree of the graph $H$.

**Claim 1.** We may assume that a maximum degree in $H$ is less than $18\sqrt{n}$.

**Proof.** First, using the standard argument, we will show, that we may assume minimum degree in $H$ is more than $d/3$. Let $u \in V(H)$ be a vertex with degree at most $d/3$. Let us delete the vertex $u$ from $H$, moreover if two distinct edges $ux, uy$ have the same color in $H$, then delete an edge $xy$ as well (by Observation 2 at most $d/6$ edges will be deleted this way). Let the obtained graph be $H'$. Clearly $|H \setminus H'| \leq d/3 + d/6$, i.e. $|H'| \geq \frac{nd}{2} - \frac{d}{2} = \frac{(n-1)d}{2}$ and since $H'$ has $n' = n - 1$ vertices, it means that the average degree of $H'$ is at least $\frac{d}{3}$, and it is easy to see that Observation 2 and Lemma 3 still holds for $H$. So we could upper bound $H'$ in terms of $n'$ and get the same upper bound on $H$ in terms of $n$. We can repeatedly apply this procedure before we will obtain a graph, with increased (or the same) average degree, and for which Observation 2 and Lemma 3 still holds. So we may assume, that the minimum degree in $H$ is more than $d/3$.

Let us assume there is a vertex $u$ with degree at least $18\sqrt{n}$. It is easy to see, that there are at least $18\sqrt{n} \cdot (d/3 - 1)$ paths of length 2 starting at $u$, moreover each vertex of $H$ is
the endpoint of at most 6 of these 2-paths, otherwise there would be a $K_{2,7}$, contradicting Lemma 3. So $n > 18\sqrt{n} \cdot (d/3 - 1)/6$, therefore $d < \sqrt{n} + 3$, i.e. $|H| < n^{3/2}/2 + 1.5n$ and we are done. Therefore, we may assume that degree of each vertex of $H$ is less than $18 \sqrt{n}$. □

Let $N_1(v) = \{x \mid vx \in E(H)\}$ and $N_2(v) = \{y \notin N_1(v) \cup \{v\} \mid \exists x \in N_1(v) \text{ s.t. } yx \in E(H)\}$ denote the first and the second neighborhood of $v$ in $H$, respectively.

Let us fix an arbitrary vertex $v$ and let $G = H[N_1(v)]$ be a subgraph of $H$ induced by the set $N_1(v)$. Clearly, the maximum degree in $G$ is at most 6, otherwise there is a $K_{2,7}$ in the graph $H$, which contradicts Lemma 3. So

$$|G| \leq 3|N_1(v)| = 3d(v).$$  \hspace{1cm} (1)

Let $G_{aux}$ be an auxiliary graph with the vertex set $N_1(v)$ such that $xy \in E(G_{aux})$ if and only if there exists a $w \in N_2(v)$ with $wx, wy \in E(H)$. Let $G'_{aux}$ be the graph with an edge st $E(G_{aux}) \setminus E(G)$, clearly

$$|G_{aux}| \leq |G'_{aux}| + |G| \leq |G'_{aux}| + 3d(v).$$  \hspace{1cm} (2)

Lemma 4. $|G'_{aux}| < d(v)^{9/5}$.

Proof. If we show, that $G'_{aux}$ is $K_{5,5}$-free, then by Kovári-Sos-Turan Theorem $|G'_{aux}| \leq 41/2 \cdot d(v)^{9/5} < d(v)^{9/5}$. So it suffices to prove that $G'_{aux}$ is $K_{5,5}$-free.

First, let us prove the following claim.

Claim 2. Let $xy$ be an edge of $G'_{aux}$ and let $h_x$ and $h_y$ be the colors of $vx$ and $vy$ in $H$, respectively. Then either $x \in h_y$ or $y \in h_x$.

Proof. First note that $h_x \neq h_y$ otherwise, by observation 2, $xy$ would be an edge of $G$ and therefore not an edge of $G_{aux}$. By definition of $G'_{aux}$ there exists $w \in N_2(v)$ such that $wx, wy \in H$. Let $h_1$ and $h_2$ be colors of $wx$ and $wy$ respectively. $h_1 \neq h_2$, otherwise $xy \in G$, a contradiction. If $h_1 = h_x$ or $h_2 = h_y$ then by observation 2 $wv \in E(H)$, therefore $w \in N_1(v)$, a contradiction. Clearly $h_1, h_2, h_x, h_y$ are not all distinct, otherwise they would form a Berge-$C_4$. So either $h_1 = h_y$ or $h_2 = h_x$, therefore $x \in h_y$ or $y \in h_x$. □

Now let us assume for a contradiction, that there is a $K_{5,5}$ in $G'_{aux}$ with parts $A$ and $B$. By the pigeon-hole principle, there exists $v_1, v_2, v_3 \in A$ such that colors of $vv_1, vv_2$ and $vv_3$ are all different. Similarly, there exists $v_4, v_5, v_6 \in B$ such that colors of $vv_4, vv_5$ and $vv_6$ are distinct. For each $1 \leq i \leq 6$ let $h_i \in E(H)$ be the color of $vv_i$. If $v_i \in A, v_j \in B$ and $h_i = h_j$, then $v_i v_j \in G$ therefore $v_i v_j \notin G_{aux}$, a contradiction. So $h_i$ is different for each $i \in \{1, 2, 3, 4, 5, 6\}$. So we have a $K_{3,3}$ in $G'_{aux}$ with parts $v_1, v_2, v_3$ and $v_4, v_5, v_6$ such that color $h_i$ of each $vv_i$ is distinct for each $1 \leq i \leq 6$.

Let $D$ be a bipartite directed graph with parts $v_1, v_2, v_3$ and $v_4, v_5, v_6$, such that $v_i \mapsto v_j \in D$ if and only if $v_i \in h_j$ and $v_i$ and $v_j$ are in different parts. By Claim 2 for each $1 \leq i \leq 3$ and $4 \leq j \leq 6$, either $v_i v_j \in D$ or $v_j v_i \in D$. 

Claim 3. Let $F_1$ and $F_2$ be directed graphs with the edge sets $E(F_1) = \{\vec{y}x, \vec{z}x, \vec{w}z\}$ and $E(F_2) = \{\vec{y}x, \vec{z}x, \vec{z}w, \vec{u}w\}$, where $x, y, z, w, u$ are distinct vertices. Then $D$ is $F_1$-free and $F_2$-free.

Proof. Let us assume, that $D$ contains $F_1$. Then without loss of generality we may assume, that $v_4\vec{v}_1, v_5\vec{v}_1, v_2\vec{v}_5 \in D$. So by definition of $D$, $v_4, v_5 \in h_1$ and $v_2 \in h_5$. Then we have, $vv_4 \subset h_4$, $v_4v_5 \subset h_1$, $v_5v_2 \subset h_5$ and $v_2v \subset h_2$, therefore the hyperedges $h_1, h_1, h_5, h_2$ form a berge $C_4$ in $H$, a contradiction.

If $D$ contains $F_2$, without loss of generality we may assume that $v_4\vec{v}_1, v_5\vec{v}_1, v_5\vec{v}_2, v_6\vec{v}_2 \in D$. So by definition of $D$, we have $v_4, v_5 \in h_1$ and $v_5, v_6 \in h_2$, so $v, h_4, v_4, h_1, v_5, h_2, v_6, h_6$ is a Berge-$C_4$, a contradiction. □

Now since each vertex of $D$ has at least 3 incident edges, there is a vertex in $D$ with at least 2 incoming edges, without loss of generality let this vertex be $v_1$ and let the incoming edges be $v_4\vec{v}_1$ and $v_5\vec{v}_1$. By Claim 3 $D$ does not contain $F_1$, therefore for each $2 \leq i \leq 3$, $v_i\vec{v}_4, v_i\vec{v}_5 \notin D$ i.e. $v_4\vec{v}_i, v_5\vec{v}_i \in D$ for every $1 \leq i \leq 3$. If $v_6\vec{v}_1 \in D$, then $v_4\vec{v}_2, v_5\vec{v}_2, v_5\vec{v}_1, v_6\vec{v}_1$ would form $F_2$ which contradicts Claim 3, therefore $v_6\vec{v}_1 \notin D$. Similarly $v_2\vec{v}_6 \in D$ (and $v_3\vec{v}_6 \in D$), but now $v_1\vec{v}_6, v_2\vec{v}_6$ and $v_3\vec{v}_6$ form $F_1$, a contradiction.

This completes the proof of the lemma. □

Using the information above, we will complete the proof of the upper bound. By the Blackley-Roy inequality there exists a vertex $v \in V(H)$ such that there are at least $d^2$ ordered 2-walks starting at vertex $v$. We now fix this vertex $v$ and define $G$, $G_{aux}$ and $G'_{aux}$ for $v$ similarly as before. Clearly at most $2d(v)$ of these 2-walks may not be a path, so there are at least $d^2 - 2d(v)$ 2-paths starting at $v$.

Let $B$ be a bipartite graph with parts $N_1(v)$ and $N_2(v)$ such that $xy \in B$ if and only if $vx$ is a 2-path of $H$ and $y \in N_2(v)$ (clearly $x \in N_1(v)$). The number of 2 paths $vx$ such that $xy \notin B$ is exactly $2 |G| \leq 6d(v)$ (here we used (1)), therefore we have

$$|B| \geq d^2 - 2d(v) - 6d(v) = d^2 - 8d(v).$$

Let $B'$ be a subgraph of $B$ with the edge set $E(B') = \{xy \in E(B) \mid \exists z \in N_1(v) \setminus \{x\} \text{ such that } yz \in E(B)\}$. Clearly, $xy, yz \in E(B')$ means that $xz \in E(G_{aux})$, moreover, by Lemma 3 for each $xz \in G_{aux}$ there is at most 6 choices of $y \in N_2(v)$ such that $xy, yz \in E(B')$, therefore it is easy to see that the number of 2-paths in $B'$ with terminal vertices in $N_1(v)$ is at most $6|G_{aux}|$. So $|B'| \leq 12|G_{aux}|$, therefore by Lemma 4 and 2 we have $|B'| \leq 12(d(v)^{9/5} + 3d(v))$, so

$$|B \setminus B'| \geq d^2 - 12d(v)^{9/5} - 44d(v) \quad (3)$$
On the other hand, by definition of $B'$ each vertex of $N_2(v)$ is incident to at most 1 edge of $B \setminus B'$, so $|N_2(v)| \geq |B \setminus B'|$, therefore by (3) we have $n > |N_2(v)| \geq d^2 - 12d(v)^{9/5} - 44d(v)$. Using Claim [4] we have $d^2 < n + 12 \cdot (18\sqrt{n})^{9/5} + 44 \cdot 18\sqrt{n}$ i.e. $d^2 < n + 2184n^{0.9} + 792\sqrt{n}$. So for large enough $n$ we have

$$d < \sqrt{n} + 1100n^{0.4}.$$ 

Therefore

$$|H| < \frac{1}{2}n^{2.5} + 550n^{1.4} = \frac{1}{2}n^{1.5}(1 + o(1))$$

$$\sum_{h \in \mathcal{H}}(|h| - 3) = |H| \leq (1 + o(1))\frac{1}{2}n^{3/2}.$$ 

Now it remains to prove the lower bound. Let $G$ be a bipartite $C_4$-free graph on $n/3$-vertices, with $|E(G)| = \left(\frac{n}{6}\right)^{3/2} + o(n^{3/2})$ edges. Let us replace each vertex of $G$ by 3 identical copies of itself, this will transform each edge to a 6-set. Let the resulting 6-uniform hypergraph be $\mathcal{H}$. Clearly

$$\sum_{h \in \mathcal{H}}(|h| - 3) = 3|\mathcal{H}| = \frac{1}{2\sqrt{6}}n^{3/2} + o(n^{3/2}).$$

Now let us show that $\mathcal{H}$ is Berge-$C_4$-free. Let us assume for a contradiction that there is a Berge 4-cycle in $\mathcal{H}$, and let this Berge cycle be $a, h_{ab}, b, h_{bc}, c, h_{cd}, d, h_{da}$. If $a, b, c, d$ are copies of 4 or 3 distinct vertices of $G$, then there would be a $C_4$ or $C_3$ in $G$ respectively, a contradiction. So $a, b, c, d$ are copies of only two vertices of $G$, say $x$ and $y$, so at least two of the pairs $ab, bc, cd, da$ correspond to $xy$ in $G$, therefore it is easy to see, that at least two of the hyperedges $h_{ab}, h_{bc}, h_{cd}, h_{da}$ should be the same, a contradiction.

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