SPHERICAL VARIETIES WITH THE $A_2$-PROPERTY

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Abstract. We show that a spherical variety has the $A_2$-property, i.e., any two points are contained inside an affine open neighbourhood, if and only if the relative interiors of any two cones of its colored fan do not intersect.

Introduction

Throughout the paper, we work with algebraic varieties and algebraic groups over the complex numbers $\mathbb{C}$.

Definition. A normal variety $Y$ is said to have the $A_2$-property if any two points $y_1, y_2 \in Y$ are contained inside an affine open neighbourhood in $Y$.

It has been shown in [Wło93] that a normal variety $Y$ has the $A_2$-property if and only if there exists a closed embedding $Y \hookrightarrow X$ into a toric variety $X$. Every normal quasiprojective variety has the $A_2$-property.

Let $G$ be a connected reductive group and $B \subseteq G$ a Borel subgroup. A closed subgroup $H \subseteq G$ is called spherical if $G/H$ contains an open $B$-orbit. In this case $G/H$ is called a spherical homogeneous space. A $G$-equivariant open embedding $G/H \hookrightarrow Y$ into a normal $G$-variety $Y$ is called a spherical embedding, and $Y$ is called a spherical variety.

According to the Luna-Vust theory (cf. [LV83], [Kno91]), any spherical embedding of $G/H$ can be described by some combinatorial data, which we now recall.

We denote by $\mathcal{M}$ the weight lattice of $B$-semi-invariants in the function field $\mathcal{C}(G/H)$ and by $\mathcal{N} := \text{Hom}(\mathcal{M}, \mathbb{Z})$ the dual lattice. We set $M_\mathbb{Q} := \mathcal{M} \otimes \mathbb{Q}$ and $N_\mathbb{Q} := \mathcal{N} \otimes \mathbb{Q}$. We denote by $D := \{D_1, \ldots, D_r\}$ the set of $B$-invariant prime divisors in $G/H$. The elements of $D$ are called colors.

We denote by $\rho : D \to \mathcal{N}$ the map defined by $\langle \rho(D), \chi \rangle := \nu_D(f_\chi)$ where $f_\chi \in \mathcal{C}(G/H)$ is $B$-semi-invariant of weight $\chi \in \mathcal{M}$. This is possible since $f_\chi$ is unique up to a constant factor. In the same way, we define a map $V \to N$ from the set $V$ of $G$-invariant discrete valuations on $\mathcal{C}(G/H)$. This last map is actually injective, so we may consider $V$ as a subset of $N_\mathbb{Q}$. It is known that $V$ is a cosimplicial cone (cf. [Br90]), called the valuation cone of $G/H$.

Definition. A colored cone is a pair $\sigma := (\mathcal{C}(\sigma), F(\sigma))$ such that $F(\sigma) \subseteq D$ and $\mathcal{C}(\sigma) \subseteq N_\mathbb{Q}$ is a cone generated by $\rho(F(\sigma))$ and finitely many elements of $\mathcal{C}(\sigma)$ such that $\mathcal{C}(\sigma) \cap V \neq \emptyset$. A colored cone is called pointed if $\mathcal{C}(\sigma)$ is pointed and $0 \notin \rho(F(\sigma))$.

A spherical embedding $G/H \hookrightarrow Y$ is called simple if $Y$ contains exactly one closed $G$-orbit. In that case, we denote by $F(Y) \subseteq D$ the set of colors whose closure contains the closed $G$-orbit in $Y$ and by $\mathcal{C}(Y)$ the cone generated by $\rho(F(Y))$ and the elements $\nu_D \in V$ for all $G$-invariant prime divisors $D$ in $Y$.

The map $Y \mapsto (\mathcal{C}(Y), F(Y))$ is a bijection between isomorphism classes of simple spherical embeddings of $G/H$ and pointed colored cones in $N_\mathbb{Q}$.

Definition. A face of a colored cone $\sigma$ is a colored cone $\sigma'$ such that $\mathcal{C}(\sigma')$ is a face of $\mathcal{C}(\sigma)$ and $F(\sigma') = F(\sigma) \cap \rho^{-1}(\mathcal{C}(\sigma'))$.
Definition. A colored fan is a nonempty collection $\Sigma$ of colored cones such that every face of a colored cone in $\Sigma$ also belongs to $\Sigma$ and for every $v \in V$ there is at most one $\sigma \in \Sigma$ with $v \in C(\sigma)^o$. A colored fan $\Sigma$ is called pointed if every colored cone $\sigma \in \Sigma$ is pointed, or equivalently, if $(0, \emptyset) \in \Sigma$.

Let $G/H \hookrightarrow Y$ now be an arbitrary spherical embedding. For any $G$-orbit $Z \subseteq Y$ we can define an open subvariety $Y_Z \subseteq Y$ by removing all $G$-orbits which do not contain $Z$ in their closure. Then $G/H \hookrightarrow Y_Z$ is a simple embedding with closed orbit $Z$.

The map

$$Y \mapsto \{(\mathcal{C}(Y_Z), \mathcal{F}(Y_Z)) : Z \subseteq Y \text{ is a } G\text{-orbit}\}$$

is a bijection between isomorphism classes of spherical embeddings of $G/H$ and pointed colored fans in $N_G$.

Note that the relative interiors of colored cones belonging to a colored fan $\Sigma$ are allowed to intersect outside the valuation cone $V$, which would not be allowed for a fan in the usual sense.

Definition. A colored fan $\Sigma$ is said to have the $A_2$-property if for every $v \in N_{Q}$ there is at most one $\sigma \in \Sigma$ with $v \in C(\sigma)^o$, i.e. if $\{C(\sigma) : \sigma \in \Sigma\}$ can be extended to a fan in the usual sense.

We can now state our Main Theorem.

Main Theorem. Let $\Sigma$ be a colored fan and $G/H \hookrightarrow Y$ be the associated spherical embedding. Then $Y$ has the $A_2$-property if and only if $\Sigma$ has the $A_2$-property.

1. Gale duality for spherical embeddings

The aim of this section is to generalize [ADHL10 Theorem II.2.1.14] to spherical varieties.

Toric varieties can be considered as a special case of spherical varieties. In that case, we have $G = B = T$ where $T$ is a torus and $H = \{e\}$. The character lattice $M := X(T)$ coincides with $M$, and the lattice of one parameter subgroups $N$ of $T$ coincides with $N$. As there are no $T$-invariant prime divisors, we have $D = \emptyset$. Moreover, we have $V = N_{Q} = N_{Q} := N \otimes \mathbb{Q}$, and a colored fan is just a fan in the usual sense.

If we fix primitive elements $u_1, \ldots, u_n \in N$ which span $N$ as convex cone, there are finitely many complete toric varieties $X$, whose fan contains exactly the one-dimensional cones generated by the $u_1, \ldots, u_n \in N_{Q}$. The corresponding Gale dual vector configuration in $Cl(X)_{\mathbb{Q}}$ can be used to describe these toric varieties $X$, which vary only in codimension two. The details can be found in [ADHL10 II.2.1].

We now fix a spherical homogeneous space $G/H$ and continue using the notation of the introduction. We say that a colored cone $\sigma$ is uncolored if $F(\sigma) = \emptyset$. Let $\Sigma_0$ be a colored fan consisting only of one-dimensional uncolored cones with primitive lattice generators $u_1, \ldots, u_n \in V \cap N$ such that the spherical embedding $G/H \hookrightarrow Y$ associated to $\Sigma_0$ satisfies $\Gamma(Y, O_Y) = C$. Note that in this case every non-open $G$-orbit in $Y$ is closed and of codimension one.

The set $D$ of $B$-invariant prime divisors in $Y$ which are not $G$-invariant is in natural bijection with the set $D$. We will identify these two sets. We denote by $\mathcal{N} := \{Y_1, \ldots, Y_n\}$ the set of $G$-invariant prime divisors in $Y$ where $Y_j$ corresponds to $u_j$. We obtain the following exact sequences (cf. [Br107 Proposition 4.1.1]).

$$
\begin{array}{cccccc}
0 & \longrightarrow & L & \longrightarrow & \mathbb{Z}^{D+\mathcal{N}} & \psi \longrightarrow & N \\
0 & \longleftarrow & Cl(Y) & \xleftarrow{\phi} & \mathbb{Z}^{D+\mathcal{N}} & \longleftarrow & M & \longleftarrow & 0
\end{array}
$$
The maps $\psi$ sends $D_i \mapsto d_i := \rho(D_i)$ and $Y_j \mapsto y_j$, while the map $\phi$ sends $D_i \mapsto [D_i]$ and $Y_j \mapsto [Y_j]$. Tensoring everything with $\mathbb{Q}$, we obtain the following pair of mutually dual exact sequences of vector spaces.

\[
\begin{array}{c}
0 \longrightarrow L_Q \longrightarrow \mathbb{Q}^{\mathcal{D} \setminus \mathcal{R}} \overset{\psi}{\longrightarrow} N_Q \longrightarrow 0 \\
0 \longrightarrow C(Y)_Q \overset{\phi}{\longrightarrow} \mathbb{Q}^{\mathcal{D} \setminus \mathcal{R}} \longrightarrow M_Q \longrightarrow 0
\end{array}
\]

For $I \subseteq \mathcal{D}$ or $J \subseteq \mathcal{R}$ we denote by $I^c$ resp. $J^c$ the complement $\mathcal{D} \setminus I$ resp. $\mathcal{R} \setminus J$.

We now give a series of definitions, which are modeled after [ADHL10, II.2.1] before we can state our generalization of [ADHL10, Theorem II.2.1.14].

**Definition 1.1.** A ψ-cone is a pair $(\text{cone}(\psi(I \cup J)), I)$ where $I \subseteq \mathcal{D}$ and $J \subseteq \mathcal{R}$. A ψ-cone is a cone generated by a subset of $\phi(\mathcal{D} \cup \mathcal{R})$. For any collection $\Sigma$ of ψ-cones and any collection $\Theta$ of ψ-cones we define

\[
\Sigma^\sharp := \{ \text{cone}(\psi(I^c \cup J^c)) : I \subseteq \mathcal{D}, J \subseteq \mathcal{R}, \text{cone}(\psi(I \cup J)), I \in \Sigma \},
\]

\[
\Theta^\sharp := \{ \text{cone}(\psi(I^c \cup J^c)) : I \subseteq \mathcal{D}, J \subseteq \mathcal{R}, \text{cone}(\psi(I \cup J)), I \in \Theta \}.
\]

We call a ψ-cone $\sigma$ supported if $\mathcal{C}(\sigma)^\circ \cap V \neq \emptyset$. We call a ψ-cone $\tau$ supported if $\{\tau\}^\sharp$ contains a supported ψ-cone.

We denote by an overline the operation of removing those cones from a collection of ψ-cones which are not supported. For example, a ψ-cone $\tau$ is supported if and only if $\{\tau\}^\sharp$ is not empty. Faces of ψ-cones are defined as for colored cones.

**Definition 1.2.** We call a ψ-cone

(i) pointed if $\mathcal{C}(\sigma)$ is pointed and $0 \notin \psi(\mathcal{F}(\sigma))$,

(ii) simplicial if $\mathcal{C}(\sigma)$ is spanned by part of a $\mathbb{Q}$-basis of $\mathcal{N}_Q$ which contains $\psi(\mathcal{F}(\sigma))$ and $\psi(\sigma) \mapsto \psi(\mathcal{F}(\sigma))$ is injective.

**Definition 1.3.** A ψ-quasifan is a nonempty collection $\Sigma$ of supported ψ-cones such that

(i) for each $\sigma \in \Sigma$ every supported face of $\sigma$ is also in $\Sigma$,

(ii) for each $v \in \mathcal{N}_Q$ there is at most one $\sigma \in \Sigma$ with $v \in \mathcal{C}(\sigma)^\circ$.

A ψ-fan is a ψ-quasifan consisting of pointed ψ-cones. In particular, it is a colored fan with the $A_2$-property. A colored ψ-(quasi)fan is called maximal if it cannot be extended by adding supported ψ-cones. It is called true if it contains the ψ-cone $(0,0)$ for $\mathcal{D} \neq \emptyset$ and the ψ-cones $(cone(u_j),0)$ for $1 \leq j \leq n$.

**Definition 1.4.** A ψ-bunch is a nonempty collection $\Theta$ of supported ψ-cones such that

(i) for any $\tau_1, \tau_2 \in \Theta$ we have $\tau_1^\circ \cap \tau_2^\circ \neq \emptyset$,

(ii) for any $\tau_1 \in \Theta$ every supported ψ-cone $\tau$ with $\tau_1^\circ \subseteq \tau^\circ$ belongs to $\Theta$.

A ψ-bunch $\Theta$ is called maximal if it cannot be extended by adding supported ψ-cones. It is called true if it contains the ψ-cone cone($\phi(\mathcal{D} \cup \mathcal{R})$) for $\mathcal{D} \neq \emptyset$ and the ψ-cones cone($\phi(\mathcal{D} \cup (\mathcal{R} \setminus \{Y_j\}))$) for $1 \leq j \leq n$.

We can now state our generalization of [ADHL10, Theorem II.2.1.14].

**Theorem 1.5.** The maps $\Theta \mapsto \Theta^\sharp$ and $\Sigma \mapsto \Sigma^\sharp$ define mutually inverse order reversing bijections

\[
\{\text{true maximal } \phi\text{-bunches}\} \longleftrightarrow \{\text{true maximal } \psi\text{-fans}\}.
\]

Under these bijections, the true maximal ψ-fans consisting of simplicial cones correspond to the true maximal ϕ-bunches consisting of full-dimensional cones.
Theorem 2.1. \(C\) where omit one ray generated by an element of \(\mathcal{R}\) and the cone \(0\) for \(\mathcal{D} \neq 0\). In Definition 2.3.4, we require all facets which omit one ray generated by an element of \(\mathcal{R}\) and \(\text{cone}(\mathcal{D} \cup \mathcal{R})\) for \(\mathcal{D} \neq 0\). In Definition 2.3.6, we require \(\text{cone}((\phi(\mathcal{D} \cup (\mathcal{R} \setminus \{Y_j\})))\) for \(1 \leq j \leq n\) and \(\text{cone}(\phi(\mathcal{D} \cup \mathcal{R}))\) for \(\mathcal{D} \neq 0\). Finally, in Definition 2.3.9, we require \(\text{cone}(u_j, \emptyset)\) for \(1 \leq j \leq n\) and \((0, \emptyset)\) for \(\mathcal{D} \neq 0\).

The map \(P\) has to be replaced by a map which for \(I \subseteq \mathcal{D}\) and \(J \subseteq \mathcal{R}\) sends \(\text{cone}(I \cup J)\) to \(\text{cone}(\psi(I \cup J), I)\). We have to keep in mind the definition of a face of a colored cone, which also applies to \(\psi\)-cones. Further, the images of elements of \(\mathcal{R}\) generate pairwise different rays in \(N_\mathbb{Q}\), but this does not necessarily hold for the images of elements of \(\mathcal{D} \cup \mathcal{R}\).

We finish by turning our attention to some points in the proof of Proposition II.2.3.11. We explain why \(P\) still preserves separatedness and saturatedness. If \(\delta_1, \delta_2\) admit an \(L_\mathbb{Q}\)-invariant separating linear form \(e\), then \(\mathcal{C}(P(\delta_1))\) and \(\mathcal{C}(P(\delta_2))\) intersect in a common face. Let \(\sigma_1\) resp. \(\sigma_2\) be the corresponding colored face of \(P(\delta_1)\) resp. \(P(\delta_2)\). We have to show \(\sigma_1 = \sigma_2\), i.e. \(\mathcal{F}(\sigma_1) = \mathcal{F}(\sigma_2)\), but this follows from the fact that if \(D \in \mathcal{F}(\sigma_1)\), then \(e\) is zero on \(D \in \mathcal{D}\), hence \(D \in \delta_2\), so separatedness is preserved. Similarly, we see that saturatedness is preserved. \(\square\)

2. Bunched rings

In Section 1 we have obtained a bijection between true maximal \(\phi\)-bunches and true maximal \(\psi\)-fans. The true maximal \(\psi\)-fans can be interpreted as colored fans, to each of which a spherical embedding of \(G/H\) can be associated according to the Luna-Vust theory.

In this section, we will show that to each true maximal \(\phi\)-bunch we can also associate a spherical embedding of \(G/H\) in a natural way. This is done using the theory of bunched rings, which first appeared in \([BH07, Hau08]\). An introduction can be found in \([ADHL10, \text{Chapter III}]\). We assume that the reader is familiar with this theory.

As in Section 1, we consider the colored fan \(\Sigma_0\) consisting of one-dimensional uncolored cones with primitive lattice generators \(u_1, \ldots, u_n \in V \cap N\), but denote the corresponding spherical embedding by \(G/H \hookrightarrow Y_0\). We continue to assume \(\Gamma(Y_0, C_{Y_0}) = \mathbb{C}\). The variety \(Y_0\) has the \(A_2\)-property, as it can be embedded into a toric variety \(X_0\) (cf. \([Gag12, \text{Proposition 2.17}]\)). Now let

\[\mathfrak{g} := (S_{11}, \ldots, S_{1k_1}, \ldots, S_{r1}, \ldots, S_{rs_r}, W_1, \ldots, W_n)\]

be the list of generators of the Cox ring \(\mathcal{R}(Y_0)\) as given in \([Gag12, \text{Theorem 3.6}]\). This is a list of pairwise non-associated \(\text{Cl}(Y_0)\)-prime generators of \(\mathcal{R}(Y_0)\) such that the \(\text{Cl}(Y_0)\)-grading is almost free.

**Theorem 2.1.** The projected \(\mathfrak{g}\)-faces are exactly the supported \(\phi\)-cones, the true \(\mathfrak{g}\)-faces are exactly the true \(\phi\)-bunches, and for a true maximal \(\phi\)-bunch \(\Theta\) the variety associated to the bunched ring \((\mathcal{R}(Y_0), \mathfrak{g}, \Theta)\) is the spherical embedding associated to the colored fan \(\mathcal{G}\).

**Proof.** Let \(\Theta\) be a maximal true \(\phi\)-bunch. For now, we allow it to contain non-supported cones. We denote by \(X\) the toric variety which arises from \((\mathbb{C}[\mathfrak{g}], \mathfrak{g}, \Theta)\) where \(\mathbb{C}[\mathfrak{g}]\) is the polynomial ring with generators \(\mathfrak{g}\) having the same \(\text{Cl}(Y_0)\)-grading as in \(\mathcal{R}(Y_0)\). We denote by \(N\) the lattice of one-parameter subgroups of the acting
We observe that $\Sigma$ has the $A_2$-property if and only if every cone from $\Phi(\sigma)$ is contained in $\Theta^\uparrow$ if and only if every cone from $\Psi(\sigma)$ is contained in $\Theta^\uparrow$.

Any simple embedding of $G/H$ with colored cone $\sigma$ can be embedded into a projective variety, which therefore has the $A_2$-property and arises from a bunched ring. The colored cone $\sigma$ determines exactly in which $B$-invariant divisors (i.e. $G$-invariant divisors and colors) the closed $G$-orbit is contained. Taking into account $[\text{Gag12}]$ Proposition 2.8] for the colors, it follows that the cones in $\Psi(\sigma)$ are required in the fan of $X$ for the orbit corresponding to $\sigma$ to be included in $Y$. But they are also clearly sufficient. It follows that the colored fan of $Y$ is $\Theta^\uparrow$.

Finally, if a projected $\mathfrak{F}$-face were not a supported $\phi$-cone, it would give rise to a non-existing spherical embedding of $G/H$.

It remains to show that the true $\mathfrak{F}$-bunches are exactly the true $\phi$-bunches. This follows from the fact that for the number $s_i$ of generators in $\mathfrak{F}$ corresponding to the color $D_i$ (i.e. $S_{i1}, \ldots, S_{is_i}$) we always have $s_i \geq 2$. □

3. Proof of the Main Theorem

**Proposition 3.1.** Let $\Sigma$ be a colored fan, $G/H \hookrightarrow Y$ the associated spherical embedding, and assume $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}$. Then $Y$ has the $A_2$-property if and only if $\Sigma$ has the $A_2$-property.

**Proof.** If $Y$ has the $A_2$-property, it can be embedded into a variety arising from a bunched ring (cf. $[\text{ADHL10}]$ Theorem 2.1.9]), so it follows from Theorem 2.1] that $\Sigma$ has the $A_2$-property.

On the other hand, if $\Sigma$ has the $A_2$-property, it can be extended to a true maximal $\psi$-fan. It then follows from Theorem 1.5 and Theorem 2.1 that $Y$ is an
open subvariety of a variety arising from a bunched ring. Therefore $Y$ has the $A_2$-property.

It remains to show that the assumption $\Gamma(Y, \mathcal{O}_Y) = \mathbb{C}$ can be removed.

**Proposition 3.2.** Let $\Sigma$ be a colored fan and $G/H \hookrightarrow Y$ the associated spherical embedding. Let $\Sigma'$ be a colored fan obtained from $\Sigma$ by possibly adding some uncolored rays such that the associated spherical embedding $G/H \hookrightarrow Y'$ satisfies $\Gamma(Y', \mathcal{O}_{Y'}) = \mathbb{C}$. Then $Y$ has the $A_2$-property if and only if $Y'$ has the $A_2$-property.

**Proof.** Assume that $Y'$ does not have the $A_2$ property. Then there are two points in $Y'$ which do not admit a common affine neighbourhood. But neither point can lie inside a $G$-orbit of codimension one in $Y'$.

Otherwise, assume that one point lies inside a $G$-orbit of codimension one. Then it is possible to remove orbits of codimension at least two from $Y'$ such that $Y'$ has the $A_2$-property (by Proposition 3.1) and both points are still be contained in $Y'$.

This proves one implication, the other direction is clear. □

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