Private Wireless Federated Learning with Anonymous Over-the-Air Computation

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Abstract

We study the problem of differentially private wireless federated learning (FL). In conventional FL, differential privacy (DP) guarantees can be obtained by injecting additional noise to local model updates before transmitting to the parameter server (PS). In the wireless FL scenario, we show that the privacy of the system can be boosted by exploiting over-the-air computation (OAC) and anonymizing the transmitting devices. In OAC, devices transmit their model updates simultaneously and in an uncoded fashion, resulting in a much more efficient use of the available spectrum, but also a natural anonymity for the transmitting devices. The proposed approach further improves the performance of private wireless FL by reducing the amount of noise that must be injected.

I. INTRODUCTION

Federated learning (FL) [1] allows multiple devices, each with its own local dataset, to train a model collaboratively with the help of a parameter server (PS) without sharing their datasets. At each iteration of FL, the PS shares the most recent global model with the devices, which then use their local datasets to update the global model. Local updates are aggregated and averaged at the PS to update the global model. The fact that data never leaves the devices is considered to protect its privacy. However, recent works [2], [3], [4] show that model updates or gradients also leak a lot of information about the dataset. This calls for additional mechanisms to guarantee privacy. Differential privacy (DP) is a widely-adopted rigorous notion of privacy [5]. Given a mechanism whose output is some statistics about a dataset, DP measures the change in the output probability distribution when the input database is changed with a very close neighbouring dataset. If the change is tolerably small, the algorithm is differentially private. Many recent works exploit DP-based algorithms to provide rigorous privacy guarantees in machine learning [6], [7], [8], [9], [10], [11].

We consider co-located devices communicating with the PS over a wireless multiple access channel (MAC). Recent work has shown that rather than the conventional digital communication, the devices can transmit their local updates simultaneously in an uncoded fashion to enable over-the-air computation (OAC) [12], [13], [14]. This results in a much more efficient use of the available resources, and significantly improves the learning performance. It is also remarked in [13] that the analog computation in wireless domain makes privacy more easily attainable. A common method to provide DP guarantees is adding noise to the output with a variance proportional to the maximum change in the output under neighboring datasets [5]. In the digital implementation of FL, where each device separately communicates with the PS, noise is added to the gradients after each local iteration [9] in order to ensure DP; whereas in analog computation, the PS only sees the sum of the updates together with the channel noise, which effectively protects all the updates. Thus, the same amount of protection can be achieved with less perturbation, improving also the final accuracy.

Privacy in wireless FL through OAC has been recently studied in several works [15], [16], [17], [18]. In [15], distributed stochastic gradient descent (SGD) is studied with quantized gradients and privacy constraints. In [16], if the channel noise is not enough, a subset of the devices add some additional
noise to their updates, benefiting all the devices. In [17] and [18], transmit power is adjusted for the same privacy guarantee. We note that, in [16], [18], [17], channel state information (CSI) at the devices is crucial not only to align their computations at the PS, but also for the privacy guarantee. However, in practice, devices acquire CSI from pilots transmitted by the PS. Hence, as an adversary, the PS can adjust the pilots to degrade the privacy level. Additionally, in [16], devices depend on others for privacy, which introduces additional point-of-failure. In FL [1] only a subset of the devices participate in each iteration, either to reduce the communication load or to save energy. We provide privacy in wireless FL by exploiting the anonymity of the transmitters in OAC. Our contributions are: (1) We study the effects of randomly sampling the devices participating in each iteration, and batching in local datasets on the privacy. By forcing a constant receive power at the PS across iterations, we provide anonymity to the transmitting devices, and employ it for privacy; (2) By distributing the noise generation process across the workers, we make the privacy guarantee resilient against the failure of transmitting nodes; (3) Our protocol is robust against pilot attacks.

II. PRELIMINARIES

Before presenting the system model and the proposed solution, we give some preliminaries on DP and the threat model.

A. Differential Privacy (DP)

**Definition 1.** \( (\epsilon, \delta) \)-DP [5]: A randomized mechanism \( \mathcal{M} : \mathcal{D} \rightarrow \mathcal{R} \) is \( (\epsilon, \delta) \)-differentially private (DP) if

\[
\Pr(\mathcal{M}(D) \in S) \leq e^\epsilon \Pr(\mathcal{M}(D') \in S) + \delta,
\]

\( \forall S \subseteq \mathcal{R} \) and \( \forall D, D' \in \mathcal{D} \) such that \( \|D - D'\|_1 \leq 1 \). This is a shorthand notation for the neighboring datasets, and its definition may differ depending on the context. In this work, we define the neighboring datasets as the two very close datasets with the same cardinality, which differ only in one element.

**Definition 2.** \( (\alpha, \epsilon) \)-Rényi DP [19]: Rényi divergence between two probability distributions is defined as

\[
D_\alpha(P \| Q) = \frac{1}{\alpha - 1} \log \mathbb{E}_{x \sim Q}(P(x)/Q(x))^\alpha.
\]

A randomized mechanism \( \mathcal{M} : \mathcal{D} \rightarrow \mathcal{R} \) satisfies \( (\alpha, \epsilon) \)-Rényi DP (RDP) if \( D_\alpha(\mathcal{M}(D) \| \mathcal{M}(D')) \leq \epsilon \), where \( \alpha \in [1, \infty) \), \( \forall D, D' \in \mathcal{D} \) such that \( \|D - D'\|_1 \leq 1 \). \( \epsilon \) values for \( \alpha = 1 \) and \( \alpha = \infty \) are defined by continuity.

**Lemma 1. Gaussian Mechanism (GM) [5], [19]:** Given \( f : \mathcal{D} \rightarrow \mathcal{R} \), define \( \Delta f \triangleq \max_{D, D'} \|f(D) - f(D')\|_2 \), \( \forall D, D' \in \mathcal{D} \), then the mechanism defined as \( \mathcal{M}(D) \triangleq f(D) + \mathcal{N}(0, \sigma^2) \) satisfies \( (\epsilon, \delta) \)-DP where

\[
\epsilon = \sqrt{2 \log(1.25/\delta)} \Delta f / \sigma
\]

and also satisfies \( (\alpha, \epsilon') \)-RDP where \( \epsilon' = \alpha (\Delta f)^2 / (2\sigma^2) \).

**Lemma 2. RDP to DP [19]:** A mechanism satisfying \( (\alpha, \epsilon) \)-RDP also satisfies \( (\epsilon + \log(1/\delta)/(\alpha - 1), \delta) \)-DP.

**Lemma 3. Composition of GM under RDP [19]:** Composition of \( T \) RDP mechanisms operating on the same dataset with common \( \alpha \) and with \( (\epsilon_1, \epsilon_2, \ldots, \epsilon_T) \) gives an \( (\alpha, \sum_{i=1}^T \epsilon_i) \)-RDP mechanism.

As seen in Lemma 3 composition under Rényi DP is very simple to compute for GM. Moreover, it gives a much tighter bound on the final \( \epsilon \) value compared to the advanced composition theorem [5], which is used in [16], [18].
B. Threat Model

We assume that the PS is semi-honest and curious. It is honest in the sense that it wants to acquire the final trained model, and thus, follows the model averaging step accurately, but it also would like to acquire all possible sensitive information about single individual data points. Therefore, it can attack the CSI estimation process to encourage the devices to increase their transmit power or to add less noise to their transmissions by suggesting that their channel quality is worse than it is in reality. We note that, if a common pilot signal is used to learn the CSI by all the devices, the CSI values can only be scaled by the same parameter across the devices.

We assume that the devices are honest and they do not intentionally attack the learning process; however, we also do not depend on a few devices for privacy as the devices may fail to transmit unintentionally. We further assume that the PS has a single receive antenna, thus it is not capable of determining the identities of the transmitting devices through transmission directions.

III. System Model

We consider $N$ wireless devices each hosting their local dataset $D_i$. A PS orchestrates these devices to learn a global model $w$ by minimizing a global loss function $L = \frac{1}{N} \sum_{i \in [N]} \frac{1}{|D_i|} \sum_{d \in D_i} l(w, d)$, where \( l(w,d) \) is an application-specific empirical loss function. We assume that the dataset is distributed among the devices with the identical probability and independent of each other. We employ distributed SGD to iteratively minimize $L$ across the devices. At the beginning of iteration $t$, the PS broadcasts the global model $w_t$ to all the devices. Then, the devices participating in the learning process in that iteration transmit their gradient estimates over a Gaussian MAC. The PS receives the superposition of the signals transmitted by the participating devices as

$$y[t] = \sum_{i \in A_t} c_{i,t}x_i[t] + z[t],$$

where $z[t] \sim \mathcal{N}(0, N_0)$ is the channel noise and $c_{i,t}$ is the channel coefficient of transmitter $i$ in round $t$. We assume that the transmitters perfectly know and correct the phase shift in their channels. Therefore, for simplicity, we assume real channel gains in the rest of the paper, i.e., $c_{i,t} \in \mathbb{R}^+$. We assume that $c_{i,t}$ is independent across users. We further assume only the agents having a sufficiently good channel coefficients participate in each iteration to increase power efficiency. We assume this happens with probability $p \in (0, 1)$ for each device, independently of other devices. We leave the analysis of this probability for different fading distributions as a future work.

If the device $i$ participates in an iteration, then it samples uniformly from its own local dataset such that every data point is sampled independently from the other samples with probability $q \in (0, 1)$. Let $A_t$ denote the set of participating devices at iteration $t$, and $B_{i,t}$ denote the set of samples in the batch used by the $i$th device if it is participating in iteration $t$. Let $a(t) \triangleq |A(t)|$ and $b_{i,t} \triangleq |B_{i,t}|$. Both $a(t)$ and $b_{i,t}$ are binomial random variables with parameters $p$ and $q$.

Each participating device computes a gradient estimate based on the random samples in its batch. We bound $L_2$-norm of every per-sample gradient by $L$. If this is not the case originally, then the gradient is downscaled by a factor of $\frac{L}{\|\nabla l(w_t, d)\|_2}$; that is, for a sample $d$, we define $g_t(d) \triangleq \nabla l(w_t, d) \times \max\{1, \frac{L}{\|\nabla l(w_t, d)\|_2}\}$, as done in [9].

Device $i$ transmits

$$x_{i}[t] = h_{i,t} \left( \xi_{i,t} \sum_{j \in B_{i,t}} g_t(j) + \beta_{i,t} n_{i,t} \right),$$

where $n_{i,t} \sim \mathcal{N}(0, \sigma_{n_i}^2)$, $\xi_{i,t}$ and $\beta_{i,t}$ are scaling factors, and $h_{i,t} \triangleq (\bar{c}_{i,t})^{-1}$. $\bar{c}_{i,t}$ is the CSI which can be manipulated by the PS by a multiplicative factor of $k \in (0, 1]$, i.e., $\bar{c}_{i,t} = k \cdot c_{i,t}$. For simplicity, we assume that the local datasets are of the same size, although our analysis can be extended easily by adding another scaling factor to the gradient, if this is not the case.
The PS updates the model as \( w_{t+1} = w_t - \eta y[t] \) where \( \eta \) is the learning rate. We assume that the number of participating devices \( a(t) \) and the batch sizes \( b_{i,t} \) are known to the devices but secret to the PS. This can be achieved by keeping a common random number generator state across the devices, or alternatively by encrypted communication between the devices. Since this is only sharing two real numbers, the communication overhead is small.

IV. MAIN RESULTS

As we discussed in the previous section, we sample both the participating devices and the local datasets to be used in each iteration. It is well known that the privacy of randomized mechanisms can be amplified by subsampling [20], [21], [22]; however, these results are for the centralized setting. To directly apply them to the distributed setting and take the advantage of the sampling in the privacy calculation, we make sure that all of the data samples are chosen uniformly with probability \( p \cdot q \) and independently. However, the challenge is that since the local datasets of different devices are distinct, the conditional probability of a data point being chosen given some other data point is already sampled may not be the same as the marginal probability of the same event. For example, if these two data points are hosted by the same device, then the conditional probability of the second data point being chosen is \( q \), instead of \( pq \). One way of overcoming this problem is shuffling the whole dataset across the devices after each iteration to cancel the effect of locality. However, this would incur a large communication cost. A better way is to exploit the wireless nature of the protocol as we explain next.

A. Ensuring the anonymity of devices

Since the PS can only see the aggregated signal, it cannot know which devices have transmitted. Still, if the PS knows the number of participating devices, then it can collect some information across the iterations and infer some dependency between the samples, such as two samples being hosted by the same device. In such cases, the i.i.d. assumption on the sampling probability of data points does not hold. Thus, we also need to mask the number of participating devices. To achieve this, we scale the local gradients by \( b_t \triangleq \sum_{i \in A_t} b_{i,t} \) at the devices so that the received power level at the PS is independent of the number or identity of the transmitting devices, and the PS can only see the average gradient at every iteration. Thus, we set \( \xi_{i,t} = 1/b_t, \forall i \in A_t \).

**Corollary 1.** When \( \xi_{i,t} = 1/b_t, \forall i \in A_t \), the PS cannot infer which devices are actively transmitting in an iteration. Thus, to the PS, learning any set of sampled data points does not convey any information about the remaining sampled data points unless it learns identities of all the chosen samples. Therefore, batching local samples independently with probability \( q \) and similarly sampling the transmitting devices independently with probability \( p \) is equivalent to subsampling in the centralized setting with probability \( p \cdot q \).

B. Robustness to transmission failures

As we have stated, we want our scheme to be robust against the transmission failure of devices that are scheduled to transmit at a certain iteration, but failed to do so for any reason. This is achieved by distributing the noise generation across the devices. Thus, we choose \( \beta_{i,t} = 1/\sqrt{a(t)} \) and \( \sigma_{i,t} = \sigma, \forall i \in A_t \).

Then the received signal at the PS becomes

\[
y[t] = \sum_{i \in A_t} \frac{1}{b_t} \sum_{j \in B_{i,t}} g_t(j) + n[t] + z[t],
\]

where \( n[t] \sim \mathcal{N}(0, \sigma^2) \) is the effective noise added by the devices.
C. Robustness to manipulated CSI values

In practice, given a channel noise variance \( N_0 \), we need to tune \( \sigma \) so that the total noise variance \( N_0 + \sigma^2 \) is sufficient to meet the \((\epsilon, \delta)\) values of the desired privacy level. However, if we consider the effect of channel noise and if our channel is better than we think, then the total noise variance may be less than it should be. Thus, the \( \epsilon \) value we get becomes larger than what we expect. This may result in a privacy breach. To avoid this, in the privacy computations, we simply ignore the channel noise since CSI values are prone to attacks by the PS. Then, the \( \epsilon \) value we get is the upper bound of the real \( \epsilon \) value in the presence of the channel noise.

D. Privacy analysis

In the next theorem, we present a result which shows the benefit of sampling for the overall privacy.

**Theorem 1.** Every iteration \( t \) in our FL scheme with OAC is \((\alpha, \frac{2pq^2\alpha}{\sigma^2})\)-RDP, where \( \sigma' \triangleq \frac{ab_k}{2L} \), if \( pq \leq 1/5 \), \( \sigma' \geq 4 \) and the following inequalities are satisfied

\[
1 < \alpha \leq \frac{1}{2}\sigma'^2 \log \left(1 + \frac{1}{pq(\alpha - 1)}\right) - 2\log(\sigma'),
\]

and

\[
\alpha \leq \frac{\frac{1}{2}\sigma'^2 \log^2\left(1 + \frac{1}{pq(\alpha - 1)}\right) - \log 5 - 2\log(\sigma')}{\log \left(1 + \frac{1}{pq(\alpha - 1)}\right) + \log(pq\alpha) + \frac{1}{2\sigma'^2}}.
\]

**Proof:** Consider Eq. 2 and define \( f[t] \triangleq y[t] - n[t] - z[t] = \frac{1}{b_t} \sum_{i \in A_t} \sum_{j \in B_i} g_{i,j}(t) \). Remember that \( \Delta f = \max_{D,D'} \| f(D) - f(D') \|_2 \), where \( D \) and \( D' \) are two datasets with the same size, differing only in one data point. For the same realization of the random batching and device selection processes, \( B_{i,t}(D) \) and \( B_{i,t}(D') \) also differ at most in one element, and so do \( A_i(D) \) and \( A_i(D') \). Since \( \| g_{i,j}(t) \|_2 \leq L \), we find \( \Delta f = \frac{2L}{b_t} \). With GM, according to Lemma 1, our mechanism is \((\alpha, \alpha\Delta f/(2\sigma'^2))\)-RDP without sampling. Moreover, by Corollary Corollary 1 the sampling probability of each sample is \( p \cdot q \). The remaining of the lemma follows from the direct application of [22, Theorem 11].

If the conditions in Theorem 1 are not satisfied, the RDP of the sampled Gaussian mechanism can still be computed numerically and the improvement due to sampling is still valid. The numerical procedure for this is given in [22, Section 3.3] and it is demonstrated in Section V.

**Process to compute \((\epsilon, \delta)\)-DP after \( T \) iterations:** Given an \( \alpha \) value, in the composition of the RDP mechanism the corresponding \( \epsilon \) values add up as stated in Lemma 3. Thus, if the received message at each iteration is \((\alpha, \epsilon)\)-RDP, then after \( T \) iterations, the mechanism is \((\alpha, T\epsilon)\)-RDP. According to Lemma 2 it is \((T\epsilon + \log(1/\delta)/(\alpha - 1), \delta)\)-DP. We observe that although \( \epsilon \) depends on the parameters in our mechanism, \( \alpha \) does not. It is a parameter we choose to minimize \((T\epsilon + \log(1/\delta))/(\alpha - 1), \delta)\), which we compute for several \( \alpha \) values and take the best \( \alpha \) among them. Since both the analytical (if it exists) and the numerical computation of the composed \((\alpha, \epsilon)\)-RDP is computationally cheap, this is feasible.

V. Numerical Results

In this section, we numerically calculate the composite \((\epsilon, \delta)\)-DP of a learning task with \( \delta = 10^{-5} \) for different sampling rates, given by \( pq \), where \( pq = 1 \) means there is no sampling, i.e., all the devices participate and they use all their local datasets. We assume \( \Delta f = 1 \) and \( \sigma = 1 \). For \( pq = 1 \), we computed the total privacy spent, i.e., the composite \( \epsilon \) by using both advanced composition theorem [5, Theorem 3.20], which is denoted by ‘act’ in the figure, and by using the RDP approach given in Lemma 3. Since we numerically verify that RDP accounting does much better for the same sampling rate, for the other values of \( pq \), we use RDP accounting. For numerically computing the composition with RDP, we used...
Opacus library \(^1\), and as suggested in their tutorials, we used \(\alpha\) values from 1 to 64, and pick the one minimizing the composite \(\epsilon\) value.

We observe in Fig. 1 that for \(p \cdot q = 1\) and \(p \cdot q = 0.5\), the resultant \(\epsilon\) values are very high, namely more than 100. This is, in fact, very weak privacy. Roughly speaking, it means that the output distribution of the final learned model can change by a factor of 100 when only one element is changed in the dataset. However, we see that when we have smaller sampling rates, then the privacy guarantee comes into the acceptable range. In an edge network setting, we expect many devices to participate in learning, with quite large local datasets. Therefore, \(p \cdot q\) values on the order of 0.01 are not unreasonable.

**VI. Conclusion**

In this work, we have exploited the anonymity in wireless transmissions to enable private wireless FL across edge devices. This is achieved by exploiting OAC rather than orthogonal transmissions, which would reveal the identity of the devices. In particular, we exploit random subsampling of both the devices and the local data samples in achieving reasonable DP guarantees. As opposed to recent works on the topic, we do not depend on the channel gain or the noise at the PS for privacy guarantees as these values are prone to attacks; although our method is orthogonal to these techniques and can be combined with them. Our results demonstrate yet another favorable property of OAC in wireless edge learning. We will evaluate the benefits of the proposed privacy mechanism and its comparison with other alternatives on a practical learning problem in our future work.

\(^1\)https://github.com/pytorch/opacus
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