PP-wave/Yang-Mills Correspondence:
An Explicit Check

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Abstract

We present an explicit evidence that shows the correspondence between the type IIB supergravity in the pp-wave background and its dual supersymmetric Yang-Mills theory at the interaction level. We first construct the cubic term of the light-cone interaction Hamiltonian for the dilaton-axion sector of the supergravity. Our result agrees with the corresponding part of the light-cone string field theory (SFT) and furthermore fixes its previously undetermined $p^+$-dependent normalization. Adopting thus fixed light-cone SFT, we compute the matrix elements of light-cone Hamiltonian involving three chiral primary states and find an agreement with a prediction from the dual Yang-Mills theory.
1 Introduction

While the duality between type IIB string theory in a pp-wave background \cite{1, 2, 3, 4} and a sector of $N = 4$ supersymmetric Yang-Mills theory with large R charge \cite{5} is in a sense a part of the AdS\_5/CFT\_4 correspondence, its many novel features make one to approach it from a different angle. The authors of Ref. \cite{5} succeeded in reproducing the string spectrum \cite{6} from perturbative Yang-Mills theory, thereby putting the duality on a firm ground at the free theory level. Subsequent papers made some progress on the important question of string interactions both on the Yang-Mills theory side \cite{7, 8, 9, 10} and on the string theory side \cite{11}.

String spectrum in the pp-wave background can be obtained from that of the full AdS, at least in principle, by simply taking a limit. However, the dictionary between AdS and CFT at the interaction level is seemingly lost. In the AdS/CFT correspondence, the correlators of the CFT are recovered from string theory (supergravity in practice) by using boundary-to-bulk propagators and bulk supergravity interaction vertices. The pp-wave limit magnifies the neighborhood of a null geodesic and pushes away the original boundary, so that after taking the limit the notion of the bulk-to-boundary propagator becomes unclear.

Without clear understanding of how holography is realized for the pp-wave case (see however Refs. \cite{8, 26, 33, 35}), one should first specify what physical quantities can be computed and compared between the two theories. The string theory in the pp-wave background takes the simplest form in the light-cone gauge. As shown in Ref. \cite{5}, the light-cone Hamiltonian is related to the AdS variables as $H_{\text{lc}} = (J)$ where $J$ is the AdS energy and $J$ is a component of angular momentum along $S^5$. In the language of Yang-Mills defined on $S^3 \times \mathbb{R}$, $J$ is again energy and $J$ is a $U(1)$ R charge. Since $J$ does not change due to perturbation, the interaction Hamiltonians of the two theories are expected to be the same once appropriate observables are identified. That is, if matrix elements among well-defined states can be computed reliably in both theories, that will provide a non-trivial check of the duality at the interaction level. Note that scattering amplitudes are not well-defined in a pp-wave since the strings are confined by an effective gravitational potential; we are solving a quantum mechanics for ‘particles in a box.’ Note also that the quantum mechanical approach proved useful already in recovering the string spectrum from Yang-Mills theory \cite{5}.

In Ref. \cite{11} the cubic term of the interaction Hamiltonian of light-cone string field theory in a pp-wave background was determined up to an overall function of light-cone momenta\footnote{See Refs. \cite{12, 53} for related recent developments.}.
Once one fixes the normalization, one can in principle compute arbitrary matrix elements among three single particle states. On the other hand, in the course of computing the second order correction to the energy spectrum in the Yang-Mills theory, the authors of Ref. [10] made a concrete proposal for a set of matrix elements.

In this paper, we make an attempt to build a bridge between the two computations. Specifically, we fix the normalization of the cubic Hamiltonian of Ref. [11] by comparing it with a supergravity calculation. We then use it to compute the matrix elements of three chiral primary states and find that the result agrees with the prediction of Ref. [10]. Strictly speaking, the supergravity analysis in the present paper is valid for $p^+ 0 1$ while the Yang-Mills analysis of Ref. [10] is valid for $p^+ 0 1$. However, our experience in AdS/CFT [54] tells us that the cubic interactions involving chiral primaries are strongly protected by supersymmetry. It is plausible that the nearly chiral string states of Ref. [5] show only a small deviation from their chiral cousins. In any case, confirmation of the proposal of Ref. [10] for chiral primaries should be a first consistency check as a prelude to a full-fledged comparison involving the string states of Ref. [5]. Verification of the proposal for them will be reported elsewhere.

This paper is organized as follows. In section 2, we carefully perform the light-cone quantization of the dilaton-axion system of IIB supergravity in a pp-wave background. The fact that the cubic Hamiltonian is nearly the same as the one in the flat spacetime enables us to determine it uniquely using an argument partially based on Lorentz symmetry. In Section 3, we compare the cubic Hamiltonian $H_3$ of Ref. [11] with that of section 2 and fix the normalization of the latter. We then compute the matrix elements of $H_3$ for three chiral primary states, and compare them with the proposal of Ref. [10]. In an appendix, we present a computation of the same matrix elements in the full AdS space. We again find an agreement, but the derivation is not fully faithful.

2 Light-cone quantization of IIB supergravity in a pp-wave background

One of the most efficient ways of constructing the light-cone Hamiltonian of superstring theory is to solve the constraints from (super)symmetries, which uniquely fixes the Hamiltonian in the flat background [55, 56]. In pp-wave backgrounds whose symmetry generators are inherited from those of AdS space [18, 32], however, the counterparts of the flat space generators $J^+$
and $J^I$ are not present. This absence leaves the overall $p^+$-dependent factor of the interaction Hamiltonian undetermined, barring us from performing the comparison to the corresponding proposals based on Yang-Mills theory. It thus appears necessary to directly construct the light-cone Hamiltonian of the pp-wave supergravity from the known covariant action and to compare it with the zero mode part of the string interaction Hamiltonian.

We begin with the determination of the cubic interaction Hamiltonian of the IIB supergravity in a pp-wave background. The final answer that we get should be the same as the one obtained from the light-cone string field theory computations after fixing its overall $p^+$-dependent normalization. For this purpose, it is enough to work out the simplest non-trivial case, namely, the dilaton-axion system. In the Einstein frame, the bosonic action of type IIB supergravity with manifest $SL(2;\mathbb{R})$ invariance is given as follows:

$$S_{\text{IIB}} = \frac{1}{2} \int_{\mathbb{R}^{10}} dx^{10} \sqrt{g} \left( \frac{\theta}{2} \frac{\theta}{\bar{\eta}} - \frac{M}{2} f_{ij} \hat{f}^i \hat{f}^j \right)$$

(2.1)

where

$$e = + i e \quad ; \quad M_{ij} = \frac{1}{\text{Im}} \quad ; \quad j \hat{j} \quad ; \quad \text{Re} \quad \text{Re} \quad 1 \quad ; \quad F^i_3 = \frac{H_3}{F_3} ;$$

(2.2)

and

$$F^*_5 = F_5 \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 ;$$

(2.3)

and the coupling constant is given by $\mathcal{2}^2 = (\mathcal{2})^2 g_s^2 \alpha$. As is well-known, the action given in the above is incomplete, for the self-duality condition

$$F^*_5 = F_5$$

should be separately imposed. While this procedure is easy to perform on the classical solutions, it is difficult to implement at the quantum level. Concentrating on the dilaton-axion sector allows us to bypass this subtlety. Note that there is no coupling in (2.1) between the dilaton-axion and $F_5^*$.

Expanding (2.1) up to cubic orders for the dilaton-axion sector around $= 0$, $i = i( = 0$, $= 0$) and setting other perturbations to be zero, we have

$$S = \frac{1}{2} \int_{\mathbb{R}^{10}} dx^{10} \sqrt{g} \left( \frac{1}{2} r \quad r \quad \frac{i}{2} ( \quad ) r \quad r \quad ; \right)$$

(2.4)
where we suppressed the prime in $\tilde{\alpha}$. Rescaling $p_2$ and $p_2'$ shows that $p_2$ is the cubic coupling constant. The background metric $g$ is set to that of the pp-waves in IIB supergravity

$$ds^2 = g \, dx^i dx^i = 4 dx^i dx^j (dx^i)^2 + (dx^i)^2; \quad P \frac{\partial}{\partial g} = 2; \quad (2.5)$$

where $x^+$ is the light-cone ‘time’ and $I = 1, 2$. Written explicitly, the action is

$$S = dx^+ dx^i \frac{i}{2} \partial^+ \partial^i - \frac{1}{2} \partial^i \partial^i + \frac{1}{2} \partial^+ \partial^i \partial^i + \frac{1}{4} (dx^i)^2 \partial^i \partial^i \partial^i \partial^i + \frac{1}{2} \partial^i \partial^i; \quad (2.6)$$

One might try to construct the quantizable light-cone Hamiltonian from (2.6) treating $p_2$ as a perturbation expansion parameter. This approach, however, has a subtle problem. The light-cone canonical momenta are computed to be

$$p_2 = \partial^+ \partial^i + \frac{1}{2} \partial^i \partial^i; \quad (2.7)$$

$$p_{2'} = \partial^+ \partial^i + \frac{1}{2} \partial^i \partial^i; \quad (2.8)$$

The cubic interaction terms of supergravity action involve two derivatives. Due to them, the canonical momentum gets the second, interaction-dependent term. This behavior greatly complicates the imposition of the standard canonical commutation relations when we try to quantize (2.6). Specifically, the cubic part of the commutation relation yields nonlinear constraints between the operator and the operator. In the case of nonabelian gauge theories having single derivative cubic interactions, this difficulty can be overcome as follows; the interaction dependent part of the canonical momentum (proportional to $\partial^+ \partial^i \partial^i \partial^i \partial^i + \frac{1}{2} \partial^i \partial^i$) vanishes upon choosing the light-cone gauge $A = 0$. In our case, a simple cure is to introduce a nonlocal field redefinition (and its complex conjugated version)

$$! \frac{P - i \frac{1}{2} \partial^+}{2} \left( \partial^i \right) \partial^i + \frac{1}{2} \partial^i \partial^i; \quad (2.9)$$

$$! \frac{P - i \frac{1}{2} \partial^+}{2} \left( \partial^i \right) \partial^i; \quad (2.10)$$

that deletes the part of the cubic action that contributes to $\partial^i$. We will then try to quantize the resulting system. In (2.9), we understand

$$\frac{1}{\partial^+ f (x^0)} = \int_x^0 f (x^0) \quad (2.11)$$
or its momentum space version
\[
\frac{i}{\theta}! \frac{1}{2p^+} = \frac{1}{\theta};
\] (2.12)
where \( p^+ = 2p^+ \) throughout the rest of this paper. Strictly speaking, the field redefinition (2.9) should be applied to nonzero modes with \( p^+ \neq 0 \). One can show that a slightly modified version of (2.9) for the zero mode \( p^+ = 0 \) part may be used to delete the cubic terms involving \( \theta^+ \), -derivative and zero modes.

Upon performing the field redefinition, the action (2.6) changes to:
\[
S = \int \sum \left( \frac{1}{2} \theta \cdot \partial \cdot \theta + \frac{1}{4} \theta^i \theta^j \theta^k \theta^\ell \right) + \text{quartic and higher terms};
\] (2.13)
where we have formally used integration by parts. The cubic interaction terms in (2.13) do not involve any \( \theta^+ \) -derivatives. An important point is that they do not involve any \( \theta^+ \) -dependent parts either and, in fact, they are identical to the \textit{flat space} supergravity results. The field redefinition that removes interaction terms having \( \theta^+ \) -derivatives also removes the \( \theta^+ \) -dependent interaction terms (without using integration by parts). We propose to take (2.13) as the starting point for further analysis.

The main subtlety of light-cone Hamiltonian construction is the careful treatment of \( p^+ = 0 \) zero modes and the implementation of the extra global constraint from them. In the flat space-time case, Lorentz invariance emerges only after properly incorporating the zero mode effect, as reported for example in [57]. What makes our problem solvable is that the zero mode part of the action (2.13) is \( \theta^+ \) -independent. The only \( \theta^+ \) -dependent term in (2.13) involves \( \theta^+ \) -derivative and vanishes for the zero modes. Taken together, these mean that the extra \( \theta^+ \) -independent contribution to the light-cone Hamiltonian coming from the zero modes should make the total Hamiltonian compatible with Lorentz invariance when \( \theta^+ = 0 \).

Fortunately, for the theory at hand (2.13), the zero-mode contribution to the light-cone Hamiltonian which ensures Lorentz invariance can be inferred from an earlier work by Goroff and Schwarz [58]. The light-cone Hamiltonian thus obtained from (2.13) is given as follows:
\[
H = H_2 + H_3;
\] (2.14)
where
\[
H_2 = \int \sum \left( \frac{1}{4} x^i x^j \theta^i \theta^j + \theta \cdot \theta \right); \quad \] (2.15)
and
\[ H_3 = \frac{p}{2} \int \frac{Z}{i} \partial_i \partial^i \theta \partial^I \partial_I \left( \frac{1}{2} \theta_I \frac{1}{\bar{\theta}} (\bar{\theta}) \partial_I \right) + \text{c.c.:} + H_{GS}, \tag{2.16} \]

The extra term \( H_{GS} \) guarantees the Lorentz invariance up to cubic terms when \( = 0 \) and is given by
\[ H_{GS} = \frac{p}{4i} \theta \partial_i \theta \partial^i \partial_I \partial^I + \frac{\theta_I \partial_I}{\bar{\theta}^2} \theta \partial_i + \text{c.c.:} \] \( H_{GS} \). \tag{2.17} \]

In the above equations \( \text{c.c.:} \) represents the complex conjugate. We note that the relative normalization between \( H_{GS} \) and the other terms in \( H_3 \) is uniquely fixed by requiring the eventual Lorentz invariance when \( = 0 \).

It is worthwhile to give a brief review of the work by Goroff and Schwarz [58]. They considered the construction of light-cone Hamiltonian for a nonlinear sigma model with \( SL(2; \mathbb{R}) \) invariance to mimic the \( D \)-dimensional ‘gravity’ theory:
\[ S = \int d^p x K + \frac{1}{2} \partial_i K_{ij} + \frac{\theta_i}{\bar{\theta}} \partial_j K_{ij} \frac{1}{2} \frac{\theta_j \partial_j}{\bar{\theta}^2} K_{ij} \tag{2.18} \]

The field \( K_{ij} \) is a traceless symmetric ‘metric’ with \( i = 1; \quad ; D = 2 \), and
\[ K_{ij} \]

where \( = +; \quad ; 1; \quad ; D = 2 \). Since our dilaton-axion system is a nonlinear sigma model with \( SL(2; \mathbb{R}) \) invariance, the model (2.18) when \( D = 4 \) should be similar to our system. This becomes clear when we consider the following two points. First, we introduce two fields \( C \) and \( \bar{C} \) with opposite helicities for two physical modes of four-dimensional gravitons. The first term of (2.18) then produces \( \theta \partial_i \theta \partial^i \partial_I \partial^I \partial_I \theta \) and nothing else, up to cubic terms, analogous to our action (2.13). While this term alone is not \( J^{\mu} \) invariant, adding three extra terms in (2.18) makes it invariant. Secondly, the action of \( J^{\mu} \) on fields in flat space-time is given entirely by orbital parts with no extra spin contributions; the index structure of \( K_{ij} \) is irrelevant as far as the action of \( J^{\mu} \) is concerned, making it easier to apply to the dilaton-axion system. The extra term \( H_{GS} \) can be read off from (2.18) and it can be shown that \( H_{GS} \) makes the action (2.13) Lorentz invariant when \( = 0 \).

The cubic Hamiltonian \( H_3 \) is consistent with the light-cone superstring field theory construction in the pp-wave background of Ref. [11]; as a functional of classical fields, \( H_3 \) of
The supergravity light-cone Hamiltonian is identical to the flat space \((\ell = 0)\) result. Of course, this does not mean that the matrix elements of the quantum operator \(H_3\) are \(\ell\)-independent. As we have emphasized in the introduction, quantum mechanics of free particles \((\ell = 0)\) and that of bounded particles \((\ell \neq 0)\) are quite different. For nonzero \(\ell\), the Hilbert space consists of harmonic oscillator modes whereas the \(\ell = 0\) Hilbert space is a collection of free particles. As we will see in the next section, the matrix elements of \(H_3\) actually have some explicit dependence due to their dependence on the frequency of harmonic oscillators.

To further compare with the analysis of \([11]\), we write down the classical Hamiltonian in momentum space,

\[
H_3 = \sum_{r=1}^{Z} \frac{Y^3}{2} \frac{d}{dr} \left( r \right) h_3 (r) \left( 1_{2} 2_{3} \right) + \mathcal{C}: \quad (2.20)
\]

There are two contributions to \(h_3\):

\[
h_3^{(0)} = \frac{p_2}{2} \frac{1}{4} \left( \frac{1}{1} \frac{1}{2} \right) \left( 1_{2} 2_{3} \right) (1_{1} 2_{1})^2 \quad (2.21)
\]

from \(H_3\) other than \(H_{GS}\) and

\[
h_3^{(GS)} = \frac{p_2}{1} \frac{1}{8} \left( \frac{1}{2} \frac{2}{3} \right) \left( 1_{2} 2_{3} \right) (1_{1} 2_{1})^2 \quad (2.22)
\]

from \(H_{GS}\), which sum up to yield

\[
h_3 = \frac{p_2}{1} \frac{1}{8} \left( \frac{2}{1} \frac{3}{2} \right) \left( 1_{2} 2_{3} \right) (1_{1} 2_{1})^2 : \quad (2.23)
\]

Here, \(p_c\) are the transverse momenta of the \(r\)-th particle.

### 3 Supergravity sector of light-cone string field theory in a pp-wave background

Spradlin and Volovich \([11]\) adopted the light-cone string field theory of \([56]\) to study string interactions in the pp-wave background. They showed that the prefactor of the cubic Hamiltonian \(H_3\) is the same as the one in the flat space up to an overall function \(f \left( 1_{1} 2_{2} 3_{3} \right)\). We compute the dilaton-axion sector of \(H_3\) and compare it to our supergravity result in the previous section. It will be shown that the function \(f\) is indeed a constant, and the precise value of the normalization constant will also be determined. We then proceed to compute the matrix elements of \(H_3\) for three chiral primary fields.
3.1 Dilaton-axion system and normalization of $H_3$

We begin with a brief review of IIB supergravity in the light-cone gauge [55] following the notation of [56]. A single superfield contains the 256 physical degrees of freedom of IIB supergravity in the light-cone gauge:

\[
(\mathcal{I}p; ) = \frac{1}{4} A (\mathcal{I}p) + \frac{1}{8!} 2 A (\mathcal{I}p) \alpha^a h_a + \frac{1}{4!} A^{abcd} (\mathcal{I}p) a b c d \\
+ \frac{1}{4} A^{ab} (\mathcal{I}p) a b \frac{1}{6!} A^{ab} (\mathcal{I}p) h^c h + (\text{fermions}) ;
\]  

(3.1)

where $a$ is an $SO(8)$ spinor with positive chirality. Only the fields on the first line will concern us in this paper. The quadratic Hamiltonian in the pp-wave background is given by [6]

\[
H_2 = \frac{1}{2} \sum d^2 (2 \theta^2) d^6 (\mathcal{I}p; ) (4_b + 4_F) (\mathcal{I}p; ) ;
\]  

(3.2)

$$4_B = p^2 \frac{1}{4} 2^2 \theta^2 \theta p ;$$  

$$4_F = \frac{\theta}{\theta} ;$$  

(3.3)

where $= 1 2 3 4$ is the $SO(4)$ chirality operator. Expansion of $H_2$ in component fields is straightforward. The $SO(8)$ scalars $A;A$ give

\[
H_2 = \frac{1}{2} A (\mathcal{I}p) 4_B A ;
\]  

(3.4)

This is exactly the same as the quadratic Hamiltonian of field in (2.13) upon identifying $(\mathcal{I}p) = A (\mathcal{I}p)$. For the reasons explained in the previous section and in Ref. [11], we expect that ‘classically’ $H_3$ is essentially the same as in the flat spacetime. In terms of the light-cone superfields, $H_3$ is given by

\[
H_3 = N \sum p^2 \sum d^3 v^{1^J} (\mathcal{I}p) p^1 p^3 (1) (2) (3) ;
\]  

(3.5)

$$d_3 = \sum r^{-1} \frac{d^r \theta^2 \theta^r d^6 (\mathcal{I}p; ) (\mathcal{I}p) (\mathcal{I}p) ;}$$  

(3.6)

$$p^1 = \sum p^1 I^J 2 p^1 J ;$$  

(3.7)

$$a = \sum 1^a 2^a$$  

(3.8)

where we have introduced a normalization constant $N$. Furthermore, the prefactor $v^{1^J}$ is given by

\[
v^{1^J} = \sum I^J + \frac{1}{6} A_{ab} A^{cd} a b c d + \frac{16}{8!} 4 I^J a h_a h + \frac{4i}{6!} 3 v^{1^J} a b c h ;
\]  

(3.9)
where \( \alpha = 1 \ 2 \ 3 \). The terms on the second line are irrelevant for supergravity but become important for string states.

The dilaton-axion part of the interaction Hamiltonian is obtained from \( H_3 \) in (3.5) by performing the fermionic integrals and collecting terms involving \( A \) and \( \tilde{A} \). The \( {}^0 \) and the \( {}^8 \) terms of (3.9) give nonvanishing answer (the \( {}^8 \) term producing the result shown below and the \( {}^0 \) term producing its complex conjugate):

\[
H_3 = 12N \frac{p_1}{2} \ Z \ d_3 \ \frac{1}{2} \ \frac{1}{2} p^2_1 \ 2 \ 3 \ 2 + c \chi : ; 
\]

(3.10)

where \( d_3 \) is the bosonic part of \( d_3 \) defined in (3.6). We have \( H_3 \) from (2.23), which was obtained from the construction of light-cone Hamiltonian from the covariant action. After the 90-degree rotation of (2.23) on the complex plane, under which \( \! \! \! \! i \) and \( \! \! \! \! i \) \( H \) remains invariant, we have

\[
H_3 = \frac{p_2}{2} \ Z \ d_3 \ \frac{1}{2} \ \frac{1}{2} p^2_1 \ 2 \ 3 \ 2 + c \chi : ; 
\]

(3.11)

Comparing (3.10) and (3.11), we find that the normalization constant should be

\[
N = \frac{1}{3} \frac{1}{2} . 
\]

(3.12)

In conclusion, the possible function \( f ( 1 ; 2 ; 3 ) \) undetermined from the symmetry consideration should be set to a constant.

### 3.2 Matrix elements of \( H_3 \) for chiral primaries

#### 3.2.1 Interaction Hamiltonian

It is convenient to divide the \( SO(8) \) spinor \( ^a \) into two groups depending on their \( SO(4) \) chirality. For example, we may choose a basis of gamma matrices in which \( = 1 \ 2 \ 3 \ 4 \) is diagonal, so that \( ^a s ( a = 1; 2; 3; 4 ) \) have positive chirality and \( ^a s ( a = 5; 6; 7; 8 ) \) have negative chirality. In such a basis, the chiral primary field is related to the components of the superfield as

\[
s ( ) = \frac{1}{2} A^{5678} ( ); \quad s ( ) = [ s ( ) ] = \frac{1}{2} A^{1234} ( ) ( > 0) ; 
\]

(3.13)

The factor of \( \frac{p_2}{2} \) was introduced to normalize the quadratic Hamiltonian in the standard way:

\[
H_2 = \ Z \ d_3 \ s ( 1 4_5 \ 4_6 \ j \ s ( ) ; 
\]

(3.14)
In order to obtain the cubic Hamiltonian $H_3$ for $s(\cdot)$, one has to expand (3.3) and collect $s^3$ terms. It is clear that only the middle component

$$\frac{1}{6(1\ 2\ 3)^2} \delta_{IK} \delta_{JK} \delta_{ab} \delta_{cd} \quad (3.15)$$

do $s^3$ of $v_{IJ}$ will contribute to $H_3$ for chiral primaries since other terms simply do not have the right number of $s$ to saturate the integral in (3.5). Moreover, the relation (3.13) implies that it is sufficient to pick out the terms with all $a$'s having the same chirality, i.e., $1\ 2\ 3\ 4 \quad 1234$ and $5\ 6\ 7\ 8 \quad 5678$. When the spinor indices are restricted to the same $SO(4)$ chirality subspace, say $a = 1; 2; 3; 4$ space, it can be shown that

$$i_{IJK} = i_{abc} ; \quad i_{IJK}^0 = i_{abcd} ; \quad i_{IJK} = 0 ; \quad (3.16)$$

First, note that the $\delta_{IK}$ vanishes if $I$ and $J$ do not belong to the same $SO(4)$. This fact can be verified by using the aforementioned basis of gamma matrices where the matrix is diagonal. Second, since there are only four components of a spinor with the same chirality, the LHS of (3.16) must be proportional to $abcd$. With a suitable choice of the basis for matrices, one can use the self-dual property of

$$i_{ab} = \frac{1}{2} \delta_{abcd} ; \quad i_{ab}^0 = \frac{1}{2} \delta_{abcd} \quad (3.17)$$

to show that the RHS of (3.16) is correctly normalized and that there is a relative minus sign between the two $SO(4)$s.

Without loss of generality, one may assume that $(1) > (2) > 0$ and $(3) < 0$. In such a case, one encounters the following expression,

$$(1\ 2\ 3\ 4)_{1234} (1) + (2)_{1234} ; \quad (3.18)$$

where $= 1 (2) 2 (1)$. Only the terms proportional to $(1)_{1234} (2)_{1234}$ saturate the integral. Collecting all such terms, one finds

$$(1 + 2)^4 (1)_{1234} (2)_{1234} = \frac{1}{16} (j_{1} j_{2} j_{3} j_{4})^4 (1)_{1234} (2)_{1234} ; \quad (3.19)$$

All in all, we find that $H_3$ for chiral primaries is given by

$$H_3 = \frac{2^{3-2} \mathcal{P}_k}{3 \ 2^2} \mathcal{Z} \ \mathcal{C} \ 3 \ (j_{1} j_{2} j_{3} j_{4})^4 (1)_{1234} (2)_{1234} \mathcal{P}_k^2 \mathcal{P}_k^2 \mathcal{S}_1 \mathcal{S}_2 \mathcal{S}_3 ; \quad (3.20)$$

where $k$ and $\ ?$ denote the four transverse directions coming from $AdS_5$ and the other four from $S^5$, respectively.
3.2.2 Quantization

Consider a massless scalar in the pp-wave background in the light-cone gauge.\footnote{For simplicity, we consider a massless scalar in place of the chiral primary field $s$. The difference between the two fields will not be important in what follows.}

\[
S = \int d^{10}x \frac{\mathcal{P}}{g} \frac{1}{2} (\nabla \phi)^2
\]

\[
= \int dx^+ dx \, d^8x, \quad @+ @ = \frac{1}{4} x^2 (@^2) + (@^2) : \quad (3.21)
\]

The Dirac quantization procedure for a constrained system gives the quantum commutation relation

\[
(\phi; \phi); @ \quad (\psi; \psi) = \frac{i}{2} (\phi \quad \psi)(\psi \quad \phi) : \quad (3.22)
\]

The normalizable on-shell wavefunctions are labeled by \( \phi; m \)

\[
(\phi^+; \phi; \phi) = e^{iE \phi^-} \phi \phi \phi : \quad (3.23)
\]

Here, \( \phi_n \) is the \( n \)-excited state wave function of an eight dimensional harmonic oscillator with \( n = j \geq 2 \) and the energy is given by \( E = (j j + 4) \quad (3.28) \)

where \( \phi_n \) is the plane wave in the momentum space. We find it convenient to focus on the \( x \) (or \( p \) in the momentum space) dependence, suppressing the transverse directions as long as no confusion arises. Going to the momentum space only in the \( x \) direction via Fourier-transform

\[
\phi(x) = \frac{1}{2} (\phi) \phi \phi \phi : \quad (3.24)
\]

the commutation relation becomes

\[
[ (1); (2)] = \frac{1}{2} (1 + 2); \quad (3.25)
\]

which is solved by

\[
(\phi) = \frac{1}{2} \phi = \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi : \quad (3.26)
\]

where the oscillators satisfy

\[
\phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi : \quad (3.27)
\]

The corresponding single particle states are normalized accordingly,

\[
\phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi : \quad (3.28)
\]

In terms of the oscillators the quadratic Hamiltonian is expressed as

\[
H_2 = \frac{1}{2} \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi \phi : \quad (3.29)
\]
3.2.3 Matrix elements of $H_3$

Suppose we have a cubic Hamiltonian of the form

$$H_3 = \frac{\mathcal{O}}{2^3} \left( \begin{array}{ccc} d_1 & d_2 & d_3 \\ \end{array} \right) (r_1)h_3(r_1)s(1)s(2)s(3):$$

(3.30)

It is easy to show that the matrix element is given by

$$\langle H_3 \rangle_{\{1\}^3_2}^{2i} = \frac{2}{2^{3-2}} \text{Im} h_3(r) \langle \mathbf{H} \; \mathbf{E} \; \mathbf{F} \rangle :$$

(3.31)

Note that we have included the symmetry factor $3!$ and factors of $(2^3)s$ and $P\bar{s}$ that we introduced when writing fields in terms of creation/annihilation operators. The dimensionless factor $\langle \mathbf{H} \; \mathbf{E} \; \mathbf{F} \rangle$ arises from the overlap integral of the wave function in the transverse directions. In the language of quantum mechanics of a harmonic oscillator, we have the following expressions:

$$f_n(\lambda) = \langle n\lambda\rangle = \langle n\lambda\rangle \hat{\mathcal{O}}(\lambda)\rangle;$$

(3.32)

$$\hat{\mathcal{O}}(\lambda) = \frac{2}{2} \exp \frac{1}{2} \lambda^2 \eta_{\alpha \beta}^{\gamma} \eta_{\alpha \beta}^{\gamma} P \frac{1}{4} \gamma_0^{\alpha \beta} \frac{1}{4} \gamma_0^{\alpha \beta}$$

(3.33)

Integration over the eight transverse directions becomes a simple Gaussian integral which produces

$$\frac{2^4}{(\mathcal{O})^2} \left( j_1 j_2 j_3 \right)^2 \langle n\lambda\rangle \hat{\mathcal{O}}(\lambda)\rangle;$$

(3.34)

where the operator $E_{\{p\}}$ defined by

$$E = \exp \frac{1}{2} X^3 \alpha_{\{p\}}^{\gamma} \eta_{\alpha \beta}^{\gamma} ; \quad M = \frac{1}{\sqrt{r^2-1}} \frac{1}{b_2} \frac{1}{b_2} \frac{1}{b_3} \frac{1}{b_3}$$

(3.35)

and $P = \frac{2}{2} j_1 j_2 j_3 \frac{1}{4} \gamma_0^{\alpha \beta} \frac{1}{4} \gamma_0^{\alpha \beta} \frac{1}{4} \gamma_0^{\alpha \beta}$ is precisely the same as the operator $E_0^{\alpha}$ defined in Eqs. (4.10-11) of [11]. Before using $h_3$ for the chiral primaries in (3.20), it is useful to realize that within the momentum integral one can write

$$P_k^2 P_{\gamma}^2 = 1 2 3 (E_{123}^k E_{123}^\gamma);$$

(3.36)

where $E_{123}^k E_{123}^\gamma$ and $E_{123}^\gamma E_{123}^k$ are the contributions to the energy difference from the two $SO(4)$ directions. This identity [11] follows from the definition of the free part of the light-cone Hamiltonian for a chiral primary field,

$$H = \frac{1}{4} (p^2 + \frac{1}{4} (j^2 x^2 - 4);$$

(3.37)
and the fact that the sum of three $x$ terms vanishes due to momentum conservation ($1 + 2 + 3 = 0$). The constant term in (3.37) originates from the $4F$ term in (2.15) and cancels the zero point energy of the bosonic harmonic oscillator part. Since one has only the difference between the two $SO(4)$ directions, the zero point energy cancels out automatically and the term $4$ is irrelevant. Note also that in (3.31), $1$ and $2$ are positive while $3 = (1 + 2)$ is negative. Since the eigenvalue of Hamiltonian (3.37) has the same sign as $\omega$, if one defines $E_r$ to be the absolute value of the energy of the $r$-th state, one gets

$$P^2_{\omega} = 1 \cdot 2 \cdot 3 E_{123}^k$$

Taking everything into account, we insert the $\hbar$ of (3.20) into (3.31) to find

$$h_3 J^3 = \left( \begin{array}{c} \hbar \\ J^1 \\ J^2 \end{array} \right) = C_{123} \ S \left( \begin{array}{c} J^1 \\ J^2 \\ J^3 \end{array} \right) \hbar^2$$

The authors of Ref. [10] made a proposal for the matrix elements that is supposed to be valid when the energy difference between the in-state and the out-state is nearly zero. In the Yang-Mills variables, the proposal reads

$$h_3 J^3 = \left( \begin{array}{c} \hbar \\ J^1 \\ J^2 \end{array} \right) = C_{123} \ S \left( \begin{array}{c} J^1 \\ J^2 \\ J^3 \end{array} \right) \hbar^2$$

The value of $C_{123}$ depends on the states $f^i j^i p^i q^i$. To compare with the results of Ref. [10], we introduce the following operators in Yang-Mills and their counterparts in supergravity,

$$A^J = \frac{1}{JN^J} \text{Tr} Z^J \ S \left( \begin{array}{c} J^1 \\ J^2 \end{array} \right)$$

$$B^J = \frac{1}{N^J} \text{Tr} ( Z^J \ S \left( \begin{array}{c} a^J \end{array} \right) \ h)$$

$$C^J = \frac{1}{JN^J} \text{Tr} ( Z^J \ S \left( \begin{array}{c} a^J \end{array} \right) \ h)$$

There are four processes with $E_{123} = 0$. The value of $C_{123}$ for each case can be read off from section 3.2 of Ref. [10]:

$$A A ! A : C_{123} = \frac{p_{J_1 J_2 J_3}}{N} \ C_{123}^{(0)}$$

$$A B ! B : C_{123} = C_{123}^{(0)} \frac{r_{J_2}}{J_3}$$

$$A C ! C : C_{123} = C_{123}^{(0)} \frac{J_2}{J_3}$$

For the supergravity modes, the energy difference is always an integer (times $\hbar$), so one may think that we are comparing a zero with another zero in the rest of the section. A related subtlety that arises in the computation of extremal correlators in AdS [54] was circumvented by using analytic continuation. In the same spirit, we use analytic continuation to give a meaning to the coefficient multiplying the ‘zero.’
Note that since $J$ is proportional to $C$, the relative factor matches precisely with $(n_r E \phi_2) \bar{J}$.
Therefore, it suffices to check the correspondence for the case $AA \nrightarrow A$.

Following Ref. [10], we switch from the unit normalization $h_{ij} = \delta_{ij}$ to a normalization suitable for the continuum limit, $h_{ij} = J_{1ij} = \delta_{ij} (1 \ j)$, and use $J = \hat{R}^2 = 2$ and $R^4 = 4 \ g_{\alpha \beta}$ to find

$$h_{ij} \bar{J}_{3j} = 2i \ g_{\alpha \beta} \ e_{123} (1 \ j)$$

which indeed agrees with the supergravity result (3.38) with $n_r = 0$ (up to a numerical factor of two). In fact, the authors of Ref. [10] considered only the processes with strictly vanishing $E_{123}$ and slightly nonzero $E^2_{123}$. Our result (3.38) suggests that the Yang-Mills computation should also distinguish the two $SO(4)$ directions and $E_{123}$ in (3.47) be replaced by $E_{123}^2 \ E^k_{123}$. It would be interesting to verify this expectation explicitly on the Yang-Mills side.

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## A AdS computation

In this appendix, we compute the matrix elements for three chiral primary states ($= J$). The AdS supergravity action for the fields corresponding to the chiral primaries is known to be [54]

$$S = \int \d^5 x \sqrt{g} \ \gamma_{\alpha \beta} \ s^\alpha s^\beta \ m_{2}^2 B^{IJ} \ 1 \ G_{\alpha \beta} \ s^I s^J + \cdots$$

(A.1)

---

After submitting this paper, we were informed that the relative minus sign between the two $SO(4)$ directions and its implications have been noticed independently in Refs. [10, 59].
When all three $s$ fields have $J = 0$, the coupling constant is

$$G_{123} = \frac{p}{2} \frac{2^{1-2} \delta_{123}}{(J_1^2 + 1)(J_2^2 + 1)(J_3^2 + 1)(J_1 + 2)(J_2 + 2)(J_3 + 2)} F_{123};$$  \hspace{1cm} (A.2)

$$F_{123} = \frac{1}{2} \frac{p}{2} \frac{(J_1 + 1)(J_1 + 2)(J_2 + 1)(J_2 + 2)}{(J_3 + 1)(J_3 + 2)};$$  \hspace{1cm} (A.3)

The coefficient $F_{1JK}$ comes from the overlap integral of spherical harmonics on $S^5$. This factor gets modified when one considers fields with different values of $J$, but as we discussed in section 4, the change again matches $(\alpha E_0)$. It is straightforward to compute the matrix element of the cubic Hamiltonian from the action (A.1). Each on-shell wavefunction to the linearized equation of motion corresponds to a single particle state in the quantum theory. For simplicity, we restrict ourselves to the ground state ($E = $ ) for which the normalized wave-packet is given by

$$s = \frac{p}{(\cosh \cdot)};$$  \hspace{1cm} (A.4)

where is the radial variable of the AdS global coordinates. Following the standard recipe of quantum field theory, we find

$$h^3 \mathcal{H}_3 \mathcal{J}_2^1 = \frac{1}{2^{3-2}} \frac{p}{2} \frac{(1)(1)(2)(1)(3)(1)}{(3)(1)(3)(2)} G_{123};$$  \hspace{1cm} (A.5)

Using the $G_{123}$ written above, we find the following remarkably simple result:

$$h^3 \mathcal{H}_3 \mathcal{J}_2^1 = \frac{p}{2} \frac{1^2}{3^3};$$  \hspace{1cm} (A.6)

which is in agreement with the proposal of [10].

All of the above appear sensible, but we have to admit that there are a few reasons to doubt the validity of this computation. First, in the process of obtaining the cubic term in the action (A.1), one made a nonlinear field redefinition in which the on-shell condition $r^2 s = m^2 s$ was used. In particular, on-shell condition removed cubic couplings with time derivatives, which may cause trouble in quantization. Second, the method used in this appendix gives answers that are finite in the pp-wave limit only when $J_{123} = 0$. If the cubic Hamiltonian we used were the correct one, the matching between the AdS Hamiltonian and the pp-wave Hamiltonian would be valid for arbitrary values of $J_{123}$. 

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