An analysis of the *dee* voltage of DECY-13 cyclotron based on a simple model

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**Abstract.** The analytical solution of *dee* voltage of a cyclotron based on a simple model has been derived. This calculation is needed, since the measurement of *dee* voltage is difficult to do, because the introduction of measuring device will disturb the impedance of the system. Using such a model, the calculation of *dee* voltage can be done by solving two coupled and driven differential equations. The derivation is carried using Laplace transformation by neglecting the transient solution of the differential equations. The analytical solution is then compared with the solution obtained using the numerical method, specifically using the fourth order Runge-Kutta method to make sure that the solution is indeed correct. The result can therefore be used to estimate the *dee* voltage for a given set of parameters. In particular, it will be shown that it is possible to get a high value of *dee* voltage for a small driving voltage.

1. Introduction

The cyclotron is a type of cyclic accelerator which is widely used to produce radioisotopes [1]. A cyclotron works by accelerating charged particles using alternating and time-dependent electric fields. The accelerated particle is then guided by a strong magnetic field to make a circular trajectory, making it possible for the particle to be accelerated many times [2]. The accelerated particle is then bombarded to the target material to create the intended radioisotope.

The Center of Accelerator Science and Technology is currently developing a 13 MeV cyclotron based on the design that was first developed by the cyclotron team on Korea [3,4], called DECY-13, which is planned to produce fluorine-18 isotope for medical purposes [5,6]. Below are the schematics of DECY-13
The particles are accelerated in a small gap between *dee* and *liner*, which is shown below.

The form of the trajectory of the accelerated particle, which could affect the resulting current at the target, depends on the electric potential between *dee* and *liner* and the shape and strength of the magnet [7]. Thus, the connection between given parameter values (such as the capacitance, inductance, and resistance of the cyclotron) and the output voltage between *dee* and *liner* must be understood. Therefore, the purpose of this paper is to calculate the connection between *dee* voltage and all involved parameters for a simple model of the cyclotron.

2. **The analytical calculation of *dee* voltage**

A cyclotron with inductive coupling, can be roughly modeled as follows,
Figure 3. A simple electric circuit representing a cyclotron

With $V$ is the voltage of an AC source, $R_1$ represents the impedance of a transmission cable (assumed to be lossless), $L_1$ represents the inductance of the coupler, $L_2$ represents the inductance of the dee stem, $R_2$ is the resistance of the dee (which must be very small, since the dee is made of the conductor), while $C$ represents the capacitance of the dee (with liner). The emergence of the mutual inductance $M$ above is caused by the closeness of two inductors, the dee stem and the coupler, which will cause each of the induced magnetic field to be felt by the other inductor, generating a voltage with each other.

Using all of the parameters presented above, the electric potential of the capacitor (which represents the dee voltage) is then calculated. The calculation is carried by solving differential equations obtained by using Kirchoff’s Law on the circuit. The differential equations of the circuit by using Kirchoff’s Law on each loop (using the concept of mutual inductance) are

$$-V + i_1 R_1 + L_1 \frac{di_1}{dt} + M \frac{di_2}{dt} = 0$$  \hspace{1cm} (1)

$$L_2 \frac{di_2}{dt} + M \frac{di_1}{dt} + i_2 R_2 + \frac{q_2}{C} = 0$$  \hspace{1cm} (2)

with the electric potential of the source $V$ is equal to $V = V_0 \cos \omega_d t$, where $V_0$ is the maximum value of the voltage of the source and $\omega_d$ is the angular frequency of the source. The quantity that is needed to be calculated as a function of time is $q_2 = q_2(t)$, since $\frac{q_2}{C} = V_C$ represents the value of dee voltage. To determine $q_2(t)$, we will use the Laplace transform method. Assuming that $q_1(0) = q_2(0) = i_1(0) = i_2(0) = 0$, the Laplace transformed equations are (by multiplying every term with $e^{-pt}$ and integrating all of them with respect to time from $t = 0$ to $t = \infty$)

$$L_1 p^2 \tilde{q}_1 + M p^2 \tilde{q}_2 + R_1 p \tilde{q}_1 = V_0 \frac{p}{p^2 + \omega_d^2}$$  \hspace{1cm} (3)

$$L_2 p^2 \tilde{q}_2 + M p^2 \tilde{q}_1 + R_2 p \tilde{q}_2 = -\frac{q_2}{C}$$  \hspace{1cm} (4)

$\tilde{q}_1$ needs to be eliminated from equations above. To do so, equation (3) is multiplied with $\frac{M p^2}{L_1 p^2 + R_1 p}$, and the result is then subtracted with equation (4) which will give

$$\left(\frac{M^2 p^4}{L_1 p^2 + R_1 p} - L_2 p^2 - R_2 p - \frac{1}{C}\right) \tilde{q}_2 = -\frac{M p^2}{L_1 p^2 + R_1 p} V_0 \frac{p}{p^2 + \omega_d^2}$$  \hspace{1cm} (5)
The result can be simplified further by multiplying equation (5) by $L_1 p^2 + R_1 p$, to get

$$\left[(M^2 - L_1 L_2)p^3 - (L_1 R_2 + L_2 R_1)p^2 - \left(R_1 R_2 + \frac{L_1}{C}\right)p - \frac{R_1}{C}\right] q_2 = V_0 M p^4 \frac{p}{p^2 + \omega_d^2} \tag{6}$$

To make things easier, several new parameters are defined as follows

$$\alpha = M^2 - L_1 L_2, \quad \beta = L_1 R_2 + L_2 R_1, \quad \gamma = R_1 R_2 + \frac{L_1}{C}, \quad \sigma = \frac{R_1}{C} \tag{7}$$

Which means that

$$q_2 = \frac{V_0 M p}{\alpha p^3 - \beta p^2 - \gamma p - \sigma} \frac{p}{p^2 + \omega_d^2} \tag{8}$$

The result that is obtained here is then transformed back to “time-space”, so that the charge as a function of time is obtained, which is then used to calculate the voltage of the capacitor as a function of time. In general, transforming the back Laplace transformed equation is rather difficult to do. But it can be seen that equation (8) can be separated into two-part, the first one is the term that contains $\omega_d$ while the other one does not. Specifically, those two terms are

$$\frac{1}{\alpha p^3 - \beta p^2 - \gamma p - \sigma} \frac{1}{p^2 + \omega_d^2} \tag{9}$$

$$\frac{\alpha p^2 + \beta + C}{\alpha p^3 - \beta p^2 - \gamma p - \sigma} + \frac{D p + E}{p^2 + \omega_d^2} \tag{10}$$

Considering the possibility of polynomial terms in the numerator of both terms, those two terms can be written as

$$\frac{Ap^2 + Bp + C}{\alpha p^3 - \beta p^2 - \gamma p - \sigma} + \frac{Dp + E}{p^2 + \omega_d^2} \tag{11}$$

The polynomial terms in the numerator above are chosen so that we get $p^4$ as the largest term when they are being added. This way, five terms of the polynomial will be obtained, which is the same as the number of unknown parameters, $A, B, C, D$ and $E$. The addition of those two terms must give equation (8) again, so

$$\frac{Ap^2 + Bp + C}{\alpha p^3 - \beta p^2 - \gamma p - \sigma} + \frac{D p + E}{p^2 + \omega_d^2} = V_0 M p^4 \frac{p}{p^2 + \omega_d^2} \tag{12}$$

Which basically means that

$$(Ap^2 + Bp + C) \left(p^2 + \omega_d^2\right) + (Dp + E)(\alpha p^3 - \beta p^2 - \gamma p - \sigma) = V_0 M p^2 \tag{13}$$

Next, each one of the terms is grouped according to the power of $p$ as follows

$$A + \alpha D = 0 \tag{14}$$
$$B + \alpha E - \beta D = 0 \tag{15}$$
$$A \omega_d^2 + C - D \gamma - E \beta = V_0 M \tag{16}$$
$$B \omega_d^2 - D \sigma - E \gamma = 0 \tag{17}$$
$$C \omega_d^2 - E \sigma = 0 \tag{18}$$
Using all of those four equations, then $A, B, C, D,$ and $E$ can be stated in terms of known parameters. The next step is to analyze two parts of the left side of equation (11) according to the separation that we have done. The first part, the one that does not have $\omega_d^2$ in it, is rather difficult to be transformed back into time-space. But, since the denominator involves several terms of polynomials, then it can be seen that this term will always have a damping factor, which means that for large enough time $t$, this term can be neglected. The remaining term is the only second term of equation (11), which is way easier to be transformed back compared to the first term of equation (11). Thus, in this paper, the solution of coupled and forced differential equations given by Kirchoff’s law is assumed to be able to be solved analytically at large enough $t$.

To get the explicit form of time-dependent solution of the differential equations for large $t$, $D$ and $E$ need to be written in terms of known parameters. They are given by

\[ D = \frac{V_0 M \omega_d^2 (\alpha \omega_d^2 + \gamma)}{(\alpha \omega_d^2 + \gamma)^2 \omega_d^2 (\beta \omega_d^2 - \sigma)^2} \]  

\[ E = \frac{V_0 M \omega_d^2 (\beta \omega_d^2 - \sigma)}{(\alpha \omega_d^2 + \gamma)^2 \omega_d^2 (\beta \omega_d^2 - \sigma)^2} \]  

So, the transformed charge of capacitor for $t \to \infty$

\[ \tilde{q}_2 = -\frac{V_0 M \omega_d^2 (\alpha \omega_d^2 + \gamma)}{(\alpha \omega_d^2 + \gamma)^2 \omega_d^2 (\beta \omega_d^2 - \sigma)^2} p \frac{p}{p^2 + \omega_d^2} - \frac{V_0 M \omega_d^2 (\beta \omega_d^2 - \sigma)}{(\alpha \omega_d^2 + \gamma)^2 \omega_d^2 (\beta \omega_d^2 - \sigma)^2} \frac{1}{p^2 + \omega_d^2} \]  

Transforming back $\tilde{q}_2$ using the table of Laplace transformation [11], we’d have

\[ q_2(t) = -\frac{V_0 M \omega_d^2 (\alpha \omega_d^2 + \gamma)}{(\alpha \omega_d^2 + \gamma)^2 \omega_d^2 (\beta \omega_d^2 - \sigma)^2} \cos \omega_d t - \frac{V_0 M \omega_d^2 (\beta \omega_d^2 - \sigma)}{(\alpha \omega_d^2 + \gamma)^2 \omega_d^2 (\beta \omega_d^2 - \sigma)^2} \sin \omega_d t \]  

Now, the result looks somewhat complicated. We can simplify it further by defining

\[ W^2 = (\alpha \omega_d^2 + \gamma)^2 \omega_d^2 + (\beta \omega_d^2 - \sigma)^2 \]  

Recalling that since $\frac{\omega_d (\alpha \omega_d^2 + \gamma)}{W}$ and $\frac{\beta \omega_d^2 - \sigma}{W}$ are dimensionless, it is possible to define

\[ \cos \phi = \frac{\omega_d (\alpha \omega_d^2 + \gamma)}{W} \]  

\[ \sin \phi = \frac{\beta \omega_d^2 - \sigma}{W} \]  

because of both of $\sin \phi$ and $\cos \phi$ are dimensionless quantity. It can be seen that the definition really satisfy the trigonometric identity $\sin^2 \phi + \cos^2 \phi = 1$. The charge of the capacitor can thus be written as

\[ q_2(t) = -\frac{V_0 M \omega_d}{W} (\cos \phi \cos \omega_d t + \sin \phi \sin \omega_d t) \]  

\[ = -\frac{V_0 M \omega_d}{W} \cos(\omega_d t - \phi) \]
To determine the electric potential of the capacitor, all we need to do is simply divide $q_2(t)$ with $C$, giving us

$$ V_C = -\frac{V_0 M \omega_d}{CW} \cos(\omega_d t - \phi) $$

(27)

This is the result that is obtained by solving the differential equations analytically while ignoring the suppressed term. To check whether the result is correct or not, it would be best to try to compare this result with the result obtained by using the numerical method.

3. The Numerical Calculation of Dee Voltage

As stated previously, the numerical calculation is needed to make sure that the analytical result that was just obtained is correct. In this section, the differential equation will be solved numerically by using the Fourth Order Runge-Kutta method [14]. Now, the Runge-Kutta method needs some initial value of the differential equation (we need $q_1(0), q_2(0), i_1(0)$, and $i_2(0)$) to be able to get the quantity that we need ($q_2(t)$ or $V_C(t)$). Fortunately, since the problem at hand is basically dealing with a forced oscillation problem, all of the initial conditions can be set to be equal to zero, because different initial conditions do not really matter, since eventually all of the solutions with different initial conditions will converge to a solution when $t \to \infty$. This assumption also makes sense, since the cyclotron at $t < 0$ is not turned on, which means that all of the initial conditions will be equal to zero.

The numerical calculation will be implemented using python, specifically numpy and matplotlib packages are needed to get several mathematical functions and to do plotting, respectively. The result is as follows

![Figure 4](image)

**Figure 4.** Comparison of dee voltage (V) vs time (s) between numerical (blue) and analytical (orange) method.

The result above was calculated by setting the value of parameters as follows

\[
\begin{align*}
V_0 &= 300 \text{ V} \\
\omega_d &= 2\pi \times 77.78 \text{ MHz} \\
R_1 &= 50 \text{ } \Omega \\
L_1 &= 0.1 \text{ mH} \\
R_2 &= 5 \text{ m}\Omega \\
L_2 &= 60 \text{ pH} \\
C_2 &= 1 \mu\text{F} \\
M &= 50 \text{ pH}
\end{align*}
\]

The value of the parameters was arbitrarily chosen, although not completely random, since some of their value is actually known, such as the value of the impedance of transmission line used in DECY-13 (represented by $R_1 = 50 \Omega$). The point is to show that the result between analytical and numerical calculation confirms each other. As can be seen from Figure 4, the result somehow does not agree with each other. This is because the plot is not done for large enough $t$, which causes the transient part of the
solution to be not suppressed well enough. It will be shown below that when the plot is done for large enough \( t \), the transient part will be suppressed, thus the analytical and numerical result will exhibit an identical result. Figure 4 is plotted when maximum time is set at \( t = 5 \times 10^{-8} \) s, so by plotting for \( t = 10^{-7} \) s.

![Figure 4](image)

**Figure 5.** The comparison of *dee* voltage (V) vs time (s) between the numerical (blue) and analytical (orange) method for maximum time \( t = 10^{-7} \) s.

It can be seen here, the numerical and analytical result starts to agree with each other. To make sure, a comparison plot is presented again, now for \( t = 5 \times 10^{-7} \) s.

![Figure 5](image)

**Figure 6.** The comparison of *dee* voltage (V) vs time (s) between the numerical (blue) and analytical (orange) method for maximum time \( t = 5 \times 10^{-7} \) s.

From Figure 5, it made sense to say that the analytical and numerical result indeed agrees with each other. Thus, the result indicates that the analytical derivation that has been carried is indeed correct. Now, there are several issues that need to be addressed before we move on. The first one is the arbitrary parameters that were being used. The parameters need to be carefully chosen to see whether the result is correct or not. It can be understood that *mutual inductance* is similar to *self-inductance* (or just inductance), the difference lies in the fact that mutual inductance emerges since two inductors were kept close to each other, so that the magnetic field produced by each inductor will induce electromotive force (Faraday’s law) on the other inductor. Thus, it won’t make sense if the value of the mutual inductance is larger than the value of the inductance of each inductor. The second one is that the choice of parameters will affect how long for the transient part of the solution to be suppressed. Generally, a small value of \( R_2 \) will make transient solution last longer, since small “friction” will lessen the heat that is
produced initially. This will also depend on the choice of the other parameters, but the relation is not known, since the back-transformation of the transient term is not easy.

The result that was obtained before indicates that for an arbitrary choice of parameters, a rather big voltage value of the driving voltage will only give a tiny dee voltage. This is because generally, the cyclotron (the RLC circuit after the coupler) is not in a resonance condition with the driving voltage. There are several approaches that can be used to get an understanding of the resonance condition in this case. First, for one or two varied parameters (say, the mutual inductance or the capacitance of the dee), the resonance condition is achieved when the highest possible value of the electric charge of the dee is obtained (which correspond to the highest possible value of dee voltage). The second approach is related to the transfer of electric power from the source via the transmission line to the system. The maximum amount of power transferred when the value of the impedance of the load (the RLC circuit including the coupler) is exactly the same with the value of the impedance of the transmission line [15]. When the power is maximally transferred to the system, the value of dee voltage is at the maximum (but, there are only two parameters that can be adjusted at most). Here, the second approach will be used. Using Figure 3, the impedance of the load is the equivalent impedance of all circuit elements at the right side of the transmission line (\(R_1\)). Using Kirchoff’s Law the potentials on each loop are

\[
-V_t + L_1 \frac{dI_1}{dt} + M \frac{dI_2}{dt} = 0
\]  

(28)

\[
M \frac{dI_1}{dt} + L_2 \frac{dI_2}{dt} + i_2 R_2 + \frac{q_2}{c} = 0
\]  

(29)

where \(V_t\) is the voltage of the load. Now, since \(q_2 = \int i_2 dt\), transforming all of the terms above using Fourier transform (using \(f(t) = \int_{-\infty}^{\infty} e^{-j\omega t} \tilde{f}(\omega) d\omega\) to all of the terms, will give

\[
\tilde{V}_t - j\omega dL_1 \tilde{I}_1 - j\omega M \tilde{I}_2 = 0
\]  

(30)

\[
-j\omega M \tilde{I}_1 - j\omega L_2 \tilde{I}_2 + R_2 \tilde{I}_2 - \frac{i_2}{j\omega c} = 0
\]  

(31)

The impedance of the load is defined as \(Z_l = \frac{\tilde{V}_t}{\tilde{I}_1}\) since \(\tilde{I}_1\) is the current that went through the load. Using the second equation to eliminate \(\tilde{I}_2\),

\[
\tilde{V}_t = \left[ \frac{R_2\omega^2 M^2}{R_2^2 + (\omega L_2 - \frac{1}{\omega C})^2} + j \left( \frac{(\omega L_2 - \frac{1}{\omega C})\omega^2 M^2}{R_2^2 + (\omega L_2 - \frac{1}{\omega C})^2} - \omega L_1 \right) \right] \tilde{I}_1
\]  

(32)

Which means that

\[
Z_l = \left[ \frac{R_2\omega^2 M^2}{R_2^2 + (\omega L_2 - \frac{1}{\omega C})^2} + j \left( \frac{(\omega L_2 - \frac{1}{\omega C})\omega^2 M^2}{R_2^2 + (\omega L_2 - \frac{1}{\omega C})^2} - \omega L_1 \right) \right]
\]  

(31)

From the result obtained above, there are two parts of the equation (the real and imaginary part) that can be used to constraints two parameters, which in this case are the mutual inductance and the capacitance. As mentioned before, to get a maximum value of transferred energy, the impedance of the load is needed to be equal to the impedance of the transmission line. Which means that we need these \(\text{Re}(Z_l) = R_1\) and \(\text{Im}(Z_l) = 0\) to be satisfied. First for the imaginary part,
\[ M^2 = \frac{\omega L_1 R_2^2 + \left(\frac{\omega L_2 - 1}{\omega C}\right)^2}{\left(\frac{\omega L_2 - 1}{\omega L_2 + 1}\right) \omega^2} \]  

which can be used to determine \( C \). Substituting this into the real part of the impedance of the load,

\[ R_1 = \frac{R_2 L_1 \omega}{\omega L_2 - \frac{1}{\omega C}} \Rightarrow C = \frac{R_1}{(R_1 L_2 - R_2 L_1) \omega^2} \]  

Note that the result indicates that for the constraints to be satisfied, \( R_1 L_2 < R_2 L_1 \) cannot be true. For the arbitrary parameters that were used above, this requirement is not satisfied (which means that those parameters will not give a maximum power transfer, no matter how hard the value of the capacitance and the mutual inductance is modified). Thus, from now on, the impedance of the stem and the resistance of the de will be set as

\[ L_2 = 6 \times 10^{-8} \text{H} \text{ and } R_2 = 10^{-7} \Omega \]

Substituting \( C \) back to the previous equation will give

\[ M^2 = \frac{R_2 L_1 \omega^2 + \omega R_2^2 \left(\frac{R_1 (L_2 - L_1)}{R_1 L_2 - R_2 L_1}\right)^2}{\omega^2 R_2} \]  

Now, plugging in the parameters that we were using before,

\[ C = 6.97838 \times 10^{-11} \text{F} \text{ and } M = 4.47214 \times 10^{-9} \text{H} \]

The reason that so much precision is used in writing the value of \( C \) and \( M \) is to demonstrate how hard it is in practice to get one hundred percent of efficiency in power transfer. Using those parameters, equation (26) predicts that

\[ V_C = 1,966,997.766 \text{ V} \]

which is a huge number. If the value of mutual inductance is slightly altered, say \( M_a = 5 \times 10^{-9} \), which correspond to not so efficient power transfer, which is

\[ V_C = 17,999.262 \text{ V} \]

It can be seen that just a slight variation of mutual inductance could alter the value of dee voltage by that much. For the altered mutual inductance, the impedance of the load changes from \( Z_l = 50 + j0 \, \Omega \) to \( Z_{l_a} = 62.5 + 12217.6 \, j \) which is way off the mark. So for this simple model, it was shown that the system is very sensitive to a change of the value of mutual inductance, since the imaginary part of the impedance change so much when the value of mutual inductance is slightly changed. Although, it is possible that this is because the value of the capacitance of the dee has been optimized.

4. Conclusion

The derivation of the analytical solution of differential equations which represent a simple model of cyclotron has been shown. It was seen that the numerical calculation agreed very well with the analytical result. It has also been shown that when the power transferred to the system is maximum, it is possible to get a much higher value of dee voltage. Unfortunately, this condition is very hard to achieve in practice, since the system is predicted to be very sensitive to a small change in the value of parameters. Alas, this is only a calculation of the simplest model of the cyclotron. In order to get a more realistic
result, the model needs to be refined, such as adding two separate inductors in the system which represents two dee stem and rf cavity.

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