Dirac and Lagrangian reductions in the canonical approach to the first order form of the Einstein-Hilbert action

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Abstract

It is shown that the Lagrangian reduction, in which solutions of equations of motion that do not involve time derivatives are used to eliminate variables, leads to results quite different from the standard Dirac treatment of the first order form of the Einstein-Hilbert action when the equations of motion correspond to the first class constraints. A form of the first order formulation of the Einstein-Hilbert action which is more suitable for the Dirac approach to constrained systems is presented. The Dirac and reduced approaches are compared and contrasted. This general discussion is illustrated by a simple model in which all constraints and the gauge transformations which correspond to first class constraints are completely worked out using both methods in order to demonstrate explicitly their differences. These results show an inconsistency in the previous treatment of the first order Einstein-Hilbert action which is likely responsible for problems with its canonical quantization.

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I. INTRODUCTION

Canonical quantization is the oldest, most rigorous, non-perturbative approach to quantization. It demands no new hypotheses (which is especially important in quantum gravity because of lack of experimental guides) and rests completely on the classical general theory of relativity and conventional methods of quantum field theory. For a discussion of the problems that one faces in trying to establish a connection between classical gravity and models built on new hypotheses, see the review [1]. The first step in the canonical approach to any theory is to cast it into Hamiltonian form; analysis of this step is the main subject of this article.

The search for a canonical formulation of the Einstein-Hilbert (EH) action began after initial developments in analyzing the dynamics of singular (gauge invariant) systems where constraints arise [2, 3, 4].

Almost immediately after Dirac presented his work on constraint dynamics the first attempt to apply his algorithm to the gravitational field was made by Pirani, Schild and Skinner [5, 6], and by Dirac himself [7, 8].

In the above mentioned articles, the metric formulation of the EH action was used

\[ S_d(g^{\alpha\beta}) = \int d^d x \sqrt{-g} g^{\mu\nu} R_{\mu\nu} (\Gamma, \partial\Gamma) , \]  

where \( d \) is dimension of spacetime, \( g = \det (g_{\alpha\beta}) \), affine connections \( \Gamma^\lambda_{\mu\nu} \) are equal to Christoffel symbols \( \left\{ \Gamma^\lambda_{\mu\nu} \right\} = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \) and \( R_{\mu\nu} \) is the Ricci tensor expressed in terms of \( \Gamma^\lambda_{\mu\nu} \) (see (3)). This is a “second order” formalism, as second derivatives of \( g_{\mu\nu} \) appear in (1).

Unlike ‘ordinary’ gauge theories, the Dirac analysis [9] cannot be applied directly to (1) because it is not known how to deal with second order derivatives using the Dirac procedure. (Both velocities and accelerations are explicitly present in (1) ) To avoid this problem, the so-called gamma-gamma form \( L'_d \) [10] was used as a starting point in obtaining the Hamiltonian for pure gravity

\[ L'_d \left( g^{\alpha\beta} \right) = \sqrt{-g} g^{\alpha\beta} \left( \Gamma^\lambda_{\sigma\alpha} \Gamma^\sigma_{\alpha\lambda} - \Gamma^\lambda_{\sigma\beta} \Gamma^\sigma_{\alpha\lambda} \right) . \]  

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1 The course of lectures given at Canadian Mathematical Seminar, Vancouver, August-September 1949 and later published in [3, 4].
The Lagrangian of (2) differs from that in (1) by a total divergence \[10\]. The elimination of such a term does not affect the field equations but the reduced Lagrangian of (2) is not relativistically invariant. (This was clearly stated in \[5\] and reflected in its title: “On the Quantization of Einstein’s Field Equations”, not action.) The role of surface terms in the Hamiltonian formulation of General Relativity (GR) was discussed in \[11\]. Recently, the peculiar features of surface terms were reconsidered from quite different perspective in \[12\] where it was demonstrated that it is not possible to obtain the full EH action (only its gamma-gamma part) starting from the standard graviton action built from non-interacting, massless, spin-2 tensor field, and iterating in the coupling constant by having an interaction between the tensor field and its own energy-momentum tensor.

The canonical approach based on (2), instead of the EH action, is different from the canonical approach to ordinary gauge theories. If we use (2), the invariance of original action is lost completely and not just its manifest form as in ordinary gauge theories after the time coordinate has been singled out.

Dirac started his analysis using (2) and later added the particular divergence term. According to \[13\], this is a logically incomplete procedure. He also introduced space-like surfaces and fixation of coordinates in order to keep a space-like surface always space-like. This obviously destroys general covariance. In the conclusion to his paper \[7\] which is, probably, not well-known, Dirac clearly stated what one gives up in his formulation: “One starts with ten degrees of freedom for each point in space, corresponding to the ten $g_{\mu\nu}$, but one finds with the method here followed that some drop out, leaving only six, corresponding to six $g_{\tau\sigma}$. This is a substantial simplification, but it can be obtained only at the expense of giving up four-dimensional symmetry. I am inclined to believe from this that four-dimensional symmetry is not a fundamental property of the physical world.” In the next paragraph he continued: “The present paper shows that Hamiltonian methods, if expressed in their simplest form, force one to abandon the four-dimensional symmetry.” (Italic of Dirac) This conclusion gives only the relationship between this simplest form of his Hamiltonian methods and four-dimensional symmetry. Accepting Dirac’s conclusion means that GR has to be finally reformulated without four-dimensional symmetry. This is what is done in \[14\], where GR is reexpressed as a theory of evolving 3-dimensional conformal Riemannian geometries obtained by imposing two general principles: 1) time is derived from change; 2) motion and size are relative.
In contrast, if one believes that four-dimensional symmetry is a fundamental property of Nature and wants to keep this symmetry with the intention of eventually quantizing the EH action, one has to abandon the simplest Hamiltonian methods and try to find a Hamiltonian formulation that does not destroy four-dimensional symmetry right from outset. Two possible ways of doing this exist. The first one is to modify the Dirac procedure and work with the explicit dependence of the EH action on acceleration. The second is to find an \textit{equivalent} formulation of the EH action that permits use of the standard Dirac procedure. (In addition to these orthodox approaches, there are a few more which are less developed; see p.54 of [15] and references therein.)

In the first case, we can consider the EH Lagrangian as a Lagrangian with higher derivatives and try to apply the Ostrogradsky Hamiltonian formulation [16] with appropriate adjustments to accommodate singular systems. (It was clearly indicated by Ostrogradsky that he considered only non-singular cases.) The first systematic generalization to singular cases was given by Gitman and Tyutin [17] (see also [18]). A full analysis of the EH action or some models where higher order derivatives enter \textit{only} in such a way that they do not affect the equations of motion, to the best of our knowledge, does not exist. (The EH action in this respect is a kind of “one and a half” order system which probably creates problems in applying the Ostrogradsky method.) An attempt in this direction is due to Dutt and Dresden [19].

The second approach which does not involve reduction of the EH action by the elimination of a total divergence makes the action first order in derivatives by introducing auxiliary fields. If by elimination of these fields, we can return exactly to the original action (including terms with second order derivatives), we have an equivalent form. This form of the EH action is Einstein’s affine-metric formulation [20]; it is just linear in first order derivatives, so the standard Dirac procedure can be applied similarly to the way it is applied to a first order formulation of ordinary gauge theories. Moreover, all terms of the first order action contribute to the equations of motion, as opposed, to the second order formulation, and so the effect of all terms can be studied on the same footing.

Einstein considered $g^{\alpha\beta}$ and $\Gamma^\sigma_{\alpha\beta}$ as independent fields without assuming $\Gamma^\sigma_{\alpha\beta} = \left\{\sigma_{\alpha\beta}\right\}$, since they are varied independently\textsuperscript{2}. In [20], he also proved that for symmetric $g^{\alpha\beta}$ and $\Gamma^\sigma_{\alpha\beta}$

\textsuperscript{2} This formulation was inspired by his search for unification of gravity and electromagnetism (he originally
this formulation is equivalent to (1) and said that this was “the most simple and consistent way” of obtaining the field equations from the action principle. He also noted that with this formulation, no variation of fields on boundaries is needed (see also [23], Appendix E). In this approach the division of variables into being bulk or surface, as in [11], is avoided and all variables are treated on the same footing with field variations vanishing on the boundary.

The Lagrange density of [20] (eq.(3)) is given by

$$L_d(g^{\alpha\beta}, \Gamma_\sigma^{\alpha\beta}) = h^{\alpha\beta} R_{\alpha\beta} = h^{\alpha\beta} \left( \Gamma^\lambda_{\alpha\beta,\lambda} - \Gamma^\lambda_{\alpha\lambda,\beta} + \Gamma^\lambda_{\sigma\lambda} \Gamma_\sigma^{\alpha\beta} - \Gamma^\lambda_{\sigma\beta} \Gamma_\sigma^{\alpha\lambda} \right),$$

(3)

where $h^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$ is just a simplifying notation and is not treated as an independent variable. (If we consider $h^{\alpha\beta} = \sqrt{-g} g^{\alpha\beta}$ as a change of variables, the functional Jacobian $\frac{\delta h^{\alpha\beta}}{\delta g^{\mu\nu}}$ is field dependent in all dimensions $d > 2$ and for $d = 2$ is singular.) Moreover, if we consider $h^{\alpha\beta}$ as an independent field in (3) without taking into account the field dependence of the Jacobian, we cannot return to $g^{\alpha\beta}$.

The equivalence of (3) to the second order form (1) ($d > 2$) follows from the solution of the field equations for $\Gamma^\lambda_{\alpha\beta}$, which is just the Christoffel symbol [20]. We then obtain the standard Einstein field equations in terms of $g^{\alpha\beta}$. The first order Lagrangian reduces to the second order Lagrangian by substitution of the solution $\Gamma^\lambda_{\mu\nu} = \{\lambda^{\mu\nu}\}$ into the first order Lagrangian.

A canonical analysis of the first order form of the EH action was given for the first time by Arnowitt, Deser and Misner (ADM) [24, 25]. (They also refer to some preliminary unpublished steps based on the first order action made by Schwinger.) However, in [24, 25] the Dirac procedure was not used and preliminary Lagrangian reduction was performed to obtain a reduced Lagrangian with fewer fields than are used in a canonical formulation. To do this reduction, the time independent equations of motion are solved to eliminate certain fields.

Straightforward application of the Dirac procedure in the case of the first order formulation of gauge theories such as Maxwell theory [26] is well-known. In this approach, conjugate momenta to all independent variables are introduced and this immediately produces an equivalent number of primary constraints as all velocities enter the Lagrangian.

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tried to use the Eddington, pure affine, formulation [21], so, the symmetry of $g^{\alpha\beta}$ and $\Gamma^\sigma_{\alpha\beta}$ in $\alpha\beta$ was also lifted. (For further developments along this line see [22].) Einstein considered this formulation as the best starting point for possible generalizations of GR.
only linearly. From this point, we follow the standard path by considering the conservation of constraints in time which produce secondary and higher constraints, until it is possible to have all constraints conserved. The 4D Maxwell Lagrangian gives 14 constraints, two of which are first class and the twelve remaining ones are second class [26]. However, this is only a demonstration of the consistency of the Dirac procedure. The next step is the elimination of all second class constraints by passing from Poisson brackets (PB) to Dirac brackets. The Dirac reduction of Maxwell Lagrangian in first order form is performed in Appendix B providing a proof of the equivalence of using the second order and first order actions in a canonical analysis.

A brief discussion of applying the Dirac approach to (3) can be found in [26] where expressions for the primary constraints are explicitly given, emphasizing that the first order formulation of the EH action in 4D results in 50 primary constraints, serving as an illustration of the complexity of the Dirac procedure. The number of independent field components of \( g^{\alpha\beta}, \Gamma^\lambda_{\alpha\beta} \) in (3) is \( \frac{1}{2} d (d + 1)^2 \) in \( d \) dimensions, and introducing conjugate momenta doubles the number of phase-space variables. This large number is a way of showing the complexity of the EH action, but this is not a real problem, as in the Hamiltonian analysis we separate only spatial and temporal indices of fields so that, in this case, we have only nine distinct fields for all \( d \): \( g^{00}, g^{0k}, g^{km}, \Gamma^0_{00}, \Gamma^0_{0k}, \Gamma^0_{km}, \Gamma^k_{00}, \Gamma^k_{0m}, \Gamma^k_{mn} \). This is not greatly different from using four fields in the first order formulation of electrodynamics \( A_0, A_k, F_{0k}, F_{km} \).

In the 2D limit, the first order action is not equivalent to the second order action, which is a total divergence [10]. This was analyzed at the level of the Lagrangian in [27]. The first order Lagrangian in 2D is not a total divergence and its canonical form can be discussed just like any other model of 2D gravity [28]. Moreover, the first order formulation as a general field-theoretical construction should be valid in all dimensions, with possibly special behaviour in some particular dimensions, but also with some similarities in all dimensions.

These considerations have motivated us to perform a canonical analysis of the 2D EH action using the Dirac procedure without any \textit{a priori} assumptions or restrictions. In par-

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3 The possibility of similarities of the 2D limit of the first order form of the EH action with the higher dimensional form has to be stronger than is possible in the case of electrodynamics. In the 2D limit of the first order form of the EH action we have nine distinct types of fields just as in all higher dimensions while in electrodynamics there are only three fields in 2D as opposed to four in higher dimensions (as in 2D, \( F_{km} = 0 \)).
ticular, it is important to find the algebra of constraints. In Dirac’s analysis of GR the PB algebra of constraints is non-local with field dependent structure constants \[ \{H_a(x), H_b(x')\} = H_b(x) \delta_a(x, x') - (ax \leftrightarrow bx'), \] \[ \{H_a(x), H(x')\} = H(x) \delta_a(x, x'), \] \[ \{H(x), H(x')\} = h^{ab}(x) H_a(x) \delta_b(x, x') - (x \leftrightarrow x'). \] (4) (5) (6)

This type of algebra (sometimes called a hypersurface deformation algebra) is not encountered in ‘ordinary’ gauge theories. This is not a true Lie algebra, this being the main obstacle to canonically quantizing GR. The question that arises is whether this is an intrinsic property of GR or the result of assumptions made in the course of analyzing the reduced Lagrangian in the approach of Dirac and ADM. It turns out that in the 2D case, the Dirac procedure gives a local algebra of constraints with field independent structure constants. In order to preserve ‘ordinary’ properties beyond locality of the PB and to have also off-shell closure of the PB algebra of generators and off-shell invariance of the Lagrangian, we have made a simple linear transformation of the affine connections. In such transformations were expressed in component form but in fact they can be recast in the covariant form

\[ \xi^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \frac{1}{2} \left( \delta^\lambda_{\alpha} \Gamma^\sigma_{\beta\sigma} + \delta^\lambda_{\beta} \Gamma^\sigma_{\alpha\sigma} \right). \] (7)

This covariant change of variables is quite different from the usual non-covariant change and it provides an alternative covariant formulation of the first order form of the EH action which is more suitable for canonical analysis than the form of \[ \text{[3]} \]. We have not been able to find any particular geometrical significance of the variables \( \xi^\lambda_{\alpha\beta} \) but it appears that they reflect the dynamical properties of fields of the first order EH action which is richer than the geometrodynamics of space-like surfaces. According to Hawking, using a family of space-like surfaces is in contradiction to the whole spirit of General Relativity and restricts the topology of spacetime. (This echoes Dirac’s conclusion in \[ \text{[7]} \], partially cited above.) This restriction, imposed by the slicing of spacetime, must be lifted at the quantum level; avoiding it at the outset seems to be the most natural cure of this problem. The idea of slicing spacetime originated in the attempt “to recover the old comforts of a Hamiltonian-like scheme: a system of hypersurfaces stacked in a well defined way in spacetime, with the system of dynamical variables distributed over these hypersurfaces and developing uniquely.
from one hypersurface to another" [32]. This, although ‘reasonable’ from the point of view of classical Laplacian determinism, is hard to justify from the standpoint of General Relativity [33]. In GR, an entire spatial slice can only be seen by an observer in the infinite future [34] and an observer at any point of a space-like surface cannot have information about the rest of a surface. (This actually follows from just the basic principles of relativity, those of locality and the finite speed of signals (e.g. see p.7 of [10]).) It would be unphysical to build any formalism by basing it on the development in time of data that can be available only in the infinite future and to try to fit GR into a scheme of classical determinism and non-relativistic Quantum Mechanics with its notion of a wave-function defined on a space-like slice. This idea also contradicts the canonical treatment of local relativistic field theories which do not make any explicit references to the ambient space-time by making use of a particular coordinate system or class of coordinate systems 4.

The change of variables of (7) can be used in any dimension and it is quite natural to explore this change in higher dimensions with hope that, as in the 2D case, it leads to an algebra of constraints that has the form of a Lie algebra or to see how the non-locality associated with the “hypersurface deformation algebra” appears in higher dimensions without imposing it from outset by choosing a particular slicing of spacetime.

In the next section we consider the effect of using $\xi^{\lambda}_{\alpha\beta}$ in place of $\Gamma^{\lambda}_{\alpha\beta}$ in any dimension and demonstrate that straightforward application of the Dirac procedure is considerably simplified by this choice of variables. After a few simple steps using Dirac reduction to eliminate second class constraints, we face sharp discrepancies with previous results [25] obtained by using Lagrangian reduction in which time independent equations of motion are used to eliminate some variables. The source of this difference and the conditions under which the two approaches are equivalent are analyzed. The next two sections provide the full canonical analysis of a simple model both using the Dirac approach (Sec.3) and using the Lagrangian reduction (Sec.4) in a way similar to [23] in order to illustrate the general considerations of Sec.2. The results are summarized in a conclusion. In Appendix A an alternative first order formulation of the EH action in which the variables $\xi^{\lambda}_{\alpha\beta}$ are used is demonstrated to be equivalent to the second order form of the EH action. In Appendix B

4 The condition that a space-like surface remains space-like obviously imposes restriction on possible coordinate transformations, thereby destroying four-symmetry.
we perform the Hamiltonian (Dirac) reduction with the first order formulation of Maxwell electrodynamics by eliminating those secondary constraints that are of a special form (this is an illustration of what was done in the EH action in Sec.2 and in a simple model in Secs.3 and 4) and prove in this way that, starting from the first order form, one can obtain all of the standard results usually derived using the second order form of the action. In Appendix C, Lagrangian reduction of the first order form of the EH action in any dimension based on the variables $\xi^\lambda_{\alpha\beta}$ is performed in a way consistent with the Dirac analysis.

II. CANONICAL ANALYSIS OF FIRST ORDER FORM OF THE EH ACTION IN ANY DIMENSION

In this section we discuss the Hamiltonian formulation of the EH action using the generalization of the transformation of (7) that produces canonical results similar to those of ordinary gauge theories in the 2D limit of the first order form of the EH action. The inverse transformation of (7) in $d$ dimensions is given by

$$\Gamma^\lambda_{\alpha\beta} = \xi^\lambda_{\alpha\beta} - \frac{1}{d-1} \left( \delta^\lambda_{\alpha\beta} \xi^\sigma_{\sigma\sigma} + \delta^\lambda_{\beta\alpha} \xi^\sigma_{\sigma\sigma} \right)$$

which upon substitution into (3) gives

$$\tilde{L}_d(g, \xi) = \hbar^{\alpha\beta} \left( \xi^\lambda_{\alpha\beta,\lambda} - \frac{1}{d-1} \xi^\lambda_{\alpha\beta} \xi^\sigma_{\sigma\sigma} \right),$$

an alternative first order form of the EH action. This is because the linear transformation used for the field $\Gamma^\lambda_{\alpha\beta}$ appears in (3) at most bilinearly and only linearly in their derivatives. It is also possible to prove the equivalence of the first order form (9) with second order form (1) by solving the equation of motion for $\xi^\lambda_{\alpha\beta}$, and substituting the resulting expression for $\xi^\lambda_{\alpha\beta}$ into the equation of motion for $g^{\mu\nu}$. As a result, we obtain the Einstein field equations without any reference to the affine connection. Actually, solving the equation of motion for $\xi^\lambda_{\alpha\beta}$ is simpler than solving that of $\Gamma^\lambda_{\alpha\beta}$. (Details are given in Appendix A.)

However, the main advantage of (9) is that it is extremely well suited for applying the canonical procedure. There is now nice separation of components of $\xi^\lambda_{\alpha\beta}$ into those which are dynamical and non-dynamical, as the only term with derivatives is of the form

$$\xi^\lambda_{\alpha\beta,\lambda} = \xi^0_{\alpha\beta} + \xi^k_{\alpha\beta,k}.$$  

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5 This is similar to what was done to prove the equivalence of (3) to (1) in [20].
(Latin indices are spatial and a dot represents a time derivative.) Following Dirac, the first step is to introduce momenta conjugate to all fields

\[ \pi_{\alpha\beta} \approx 0, \Pi^\alpha_k \approx 0, \Pi_{\alpha\beta}^0 - \sqrt{-g} g^{\alpha\beta} \approx 0 \]

which equals the number of fields in the Lagrangian.

The total Hamiltonian is

\[ H_T = H_c + \lambda^\alpha_{\alpha\beta} \pi_{\alpha\beta} + \Lambda^0_{\alpha\beta} \left( \Pi^\alpha_0 - h^{\alpha\beta} \right) + \Lambda^k_{\alpha\beta} \Pi^\alpha_k, \]

\[ H_c = -h^{\alpha\beta} \left( \xi^k_{\alpha\beta,k} - \xi^\lambda_{\alpha\sigma} \xi^\sigma_{\beta\lambda} + \frac{1}{d-1} \xi^\lambda_{\alpha\lambda} \xi^\sigma_{\beta\sigma} \right), \]

where \( \lambda^\alpha_{\alpha\beta} \) and \( \Lambda^\gamma_{\alpha\beta} \) are Lagrange multipliers associated with the primary constraints.

If the \( d(d+1) \) by \( d(d+1) \) matrix

\[ \tilde{M}_d = \left\{ \phi, \tilde{\phi} \right\} \]

built from the non-zero PB among the primary constraints \( \left( \phi, \tilde{\phi} \in \left( \pi_{\alpha\beta}, \Pi^\sigma_0 - \sqrt{-g} g^{\gamma\sigma} \right) \right) \) is invertible, we have a subset of constraints which are second class. Moreover, all these constraints are of a special form involving the canonical pair \( (g^{\alpha\beta}, \pi_{\alpha\beta}) \) for which \( \pi_{\alpha\beta} \approx 0 \) (see Dirac [9], Appendix B, and for more detailed and general discussion [35]). For such constraints, if \( \det \tilde{M}_d \neq 0 \), we can set all momenta \( \pi_{\alpha\beta} \) to zero and then solve \( \Pi^\sigma_0 = \sqrt{-g} g^{\gamma\sigma} \) for \( g^{\alpha\beta} = g^{\alpha\beta} \left( \Pi^\sigma_0 \right) \) and use this equality to eliminate \( g^{\alpha\beta} \) in both the Hamiltonian and the remaining constraints. This is Hamiltonian (Dirac) reduction in its simplest form. Dirac brackets are equal to PB for all the remaining variables. The use of this reduction is shown in Appendix B to lead to equivalence of the first and second order formulations for electrodynamics at the level of the Hamiltonian.

Actually, for \( \tilde{L}_d \) it is not even necessary to solve the equations \( \Pi^\sigma_0 = \sqrt{-g} g^{\gamma\sigma} \) for \( g^{\alpha\beta} \) as they enter the Hamiltonian in the particular combinations which are present in the second class primary constraints and the solution for such combinations are, of course, obvious if the condition \( \det \tilde{M}_d \neq 0 \) is fulfilled. In this case, the canonical analysis of the reduced Hamiltonian leads to the form of the gauge transformation of \( \Pi^\alpha_0 \) and so using the strong equality, \( \Pi^\alpha_0 = \sqrt{-g} g^{\alpha\beta} \), we can immediately find the gauge transformation of \( g^{\alpha\beta} \).
In 2D (and only in 2D) the matrix (14) is singular. The rank of \( \tilde{M}_2 \) is four and its dimension is 6 by 6, so only two pairs of constraints constitute a subset of second class constraints which are of a special form meaning they can be eliminated. (For more details see [36].)

Let us denote the number of independent components of a field \( \Phi \) by \([\Phi]\). At this stage, a reduction of the \([g^{\alpha\beta}] = \frac{1}{2}d(d+1)\) fields has been performed by eliminating the canonical pairs \((g^{\alpha\beta}, \pi_{\alpha\beta})\) using the primary second class constraints. The remaining primary constraints \(\Pi^\alpha_{\beta} \) will produce secondary constraints \(\chi^\alpha_{\beta}\)

\[
\dot{\Pi}^\alpha_{\beta} = \{\Pi^\alpha_{\beta}, H_c\} = -\frac{\delta H_c}{\delta \xi^\alpha_{\beta}} \equiv \frac{\delta \tilde{L}_d}{\delta \xi^\alpha_{\beta}} = \chi^\alpha_{\beta}
\]

with \([\chi^\alpha_{\beta}] = \frac{1}{2}(d-1)d(d+1)\).

Explicitly separating time and space indices, we obtain three secondary constraints \((\chi^m_{\kappa}, \chi^0_{\kappa}, \chi^0_{0})\)

\[
\frac{\delta \tilde{L}_d}{\delta \xi^m_{\kappa}} = \chi^m_{\kappa} = -h^m_{\kappa} - h^{\mu m}\xi^{\mu}_{\kappa} - h^{m\nu}\xi^m_{\kappa} + \frac{1}{d-1}\left(h^{\mu\nu}\xi^\lambda_{\kappa}\delta^n_{\mu\kappa} + h^{\mu\nu}\xi^\lambda_{\mu\kappa}\delta^m_{\nu\kappa}\right), \tag{16}
\]

\[
\frac{\delta \tilde{L}_d}{\delta \xi^0_{\kappa}} = \chi^0_{\kappa} = -h^0_{\kappa} - h^{\mu 0}\xi^{\mu}_{\kappa} - h^{0\nu}\xi^0_{\kappa} + \frac{1}{d-1}h^{0\nu}\xi^\lambda_{\mu\kappa}\delta^m_{\nu\kappa}, \tag{17}
\]

\[
\frac{\delta \tilde{L}_d}{\delta \xi^0_{0}} = \chi^0_{0} = -h^0_{0} - 2h^{0\nu}\xi^0_{\kappa} - 2h^0_{\nu m}\xi^0_{\kappa}. \tag{18}
\]

From the point of view of the Dirac procedure, these three constraints are quite different. The distinction that arises between these constraints is not taken into account when they are treated as time independent Lagrangian equations of motion in the ADM approach to the first order Lagrangian of (3) \([25, 41]\). The matrix of PB of \(\chi^m_{\kappa} \) with the corresponding primary constraints \(\Pi^m_{\kappa} \) is non-singular. These constraints form a second class subset and this is a subset of the same special form as the part of second class primary constraints that have been already considered. Following Dirac reduction, we have \(\Pi^m_{\kappa} = 0 \) and \(\xi^k_{mn} = \xi^k_{mn}(\xi^0_{\alpha\beta}, \xi^q_{\alpha\beta}) \) - solutions of the second class constraints \(\chi^m_{\kappa} = 0\) that are now substituted into the Hamiltonian and remaining constraints. For the second equation (17), the matrix of PB of \(\chi^0_{\kappa} \) with the correspondent primary constraints (\(\Pi^0_{\kappa}\)) is singular and this subset is not purely second class. According to Dirac, we have to find the maximum possible number of first class combinations for this subset and only the remaining constraints which are second
class can be eliminated. Among the constraints $\chi_k^{0m}$ of (17), only one first class combination exists and it is $\chi_k^{0k}$:

$$\chi_k^{0k} = -h_{k}^{0k} - h^{mk} \xi_{0k} + h^{00} \xi_{00}.$$  \hspace{1cm} (19)

Only the fields $\xi_{0}^{a \beta}$ are present in (19) and they have a vanishing PB with the primary constraints $\Pi_{k}^{a \beta}$. The remaining constraints are again of the special form, so we can further reduce our system by eliminating $[\chi_k^{0m}] - 1 = (d - 1)^2 - 1$ fields. Elimination of these fields without destroying tensorial notation is performed in Appendix C.

The last constraint (18) is first class as there are no components of $\xi_k^{a \beta}$ appearing in (18), but only these components give a non-zero PB with the primary constraints $\Pi_{k}^{a \beta}$. Dirac reduction using (16-18) leads to the following number of fields in the reduced Hamiltonian

$$[g^{a \beta}] + [\xi_{00}^{a \beta}] - [\chi_k^{mn}] - [\chi_k^{0m}] + 1 = d(d + 2).$$ \hspace{1cm} (20)

Taking into account this reduction using the second class subset of primary constraints, we have only $d$ primary constraints ($[\Pi_{k}^{00}] + 1$) and $d$ secondary constraints left.

The secondary constraints are now

$$\chi_k^{00} = -\Pi_{0,k}^{00} - 2\Pi_{0}^{00} \xi_{0k} - 2\Pi_{0}^{0m} \xi_{0m},$$

$$\chi_k^{0k} = -\Pi_{0,k}^{0k} - \Pi_{0}^{nm} \xi_{nm} + \Pi_{0}^{00} \xi_{00}$$  \hspace{1cm} (21)

where we have used the strong equality $h^{a \beta} = \Pi_{0}^{a \beta}$. These are a $d$ dimensional generalization of two of the three constraints found in 2D \cite{30}, having a simple local PB algebra

$$\{\chi_k^{0k}(x), \chi_n^{00}(y)\} = \chi_n^{00}(x) \delta^{d-1}(x - y), \{\chi_k^{0k}, \chi_k^{0k}\} = 0, \{\chi_k^{00}, \chi_n^{00}\} = 0$$ \hspace{1cm} (22)

and zero PB with primary constraints. At this stage the primary and secondary constraints form a first class system. We now continue the Dirac procedure; it leads to the existence of, at least, tertiary constraints.

Already after the first steps of Dirac reduction using $\xi_{0}^{a \beta}$ and $g^{a \beta}$ as independent variables, we see that primary and secondary constraints having a local PB algebra with field independent structure constants arise and that tertiary constraints must be present in the Hamiltonian, which is no longer a linear combination of secondary constraints as in the 2D case \cite{30,36}. This result is quite unlike the previous treatment of the first order EH Lagrangian \cite{25} where after Lagrangian reduction the Hamiltonian is a linear combination
of secondary constraints with a non-local hypersurface deformation algebra of constraints with field dependent structure constants. This has been viewed as an inconsistency in the constraint algebra and the main obstacle to canonically quantizing GR [37, 38] or as an indication of the non-locality of Nature and an inspiration for new ideas, such as promoting this algebra to being a first principle, more fundamental than the action principle or the equations of motion [40].

Recently, Kummer and Schütz have reconsidered the first order formulation of 4D GR using Cartan variables [39]. Their analysis is based on avoiding the ADM decomposition that has been almost exclusively used when discussing tetrad gravity. Their approach also leads to tertiary constraints and a local algebra of constraints.

Before continuing with the Dirac procedure it is necessary to understand why the two approaches, Dirac and Lagrangian reduction, that are supposed to be equivalent lead to different results. We attempt to answer this question in the rest of this paper.

First of all, let us note that the presence of tertiary constraints is not in contradiction with the number of degrees of freedom. For example, if tertiary constraints are all first class and the Dirac procedure is closed at this stage, we have $3d$ first class constraints. The number of fields in the reduced Hamiltonian is $d(d + 2)$ (see (20)) minus $\frac{1}{2}d(d + 1)$ because of the first reduction using the second class primary constraints. The result is $\frac{1}{2}d(d - 3)$, the number of degrees of freedom associated with a symmetric tensor gauge field in $d$ dimensions.

This expression works only for $d > 2$; 2D is a special case which cannot be described by this relation because there are no second class constraints among the secondary constraints. (See [36] for a full discussion of the 2D EH action with $g^{a\beta}$ being treated as an independent field.) Of course, having $3d$ first class constraints is not the only possibility that results in the expected number of degrees of freedom, but this demonstrates that the presence of tertiary constraints is not inconsistent. Moreover, it is necessary to have tertiary constraints, because without solving the first class constraints we have additional independent variables and extra constraints are needed to reduce the number of degrees of freedom to the expected value.

Secondly, Lagrangian and not Dirac reduction was used in [25]. (For a very clear exposition of this reduction see Appendix A of the review article [41].)

In the Dirac approach, after identifying all second class constraints of the special form among the constraints [15, 18] and then eliminating the corresponding variables, we reduce
the number of fields (see (20)) to just \(d(d + 2)\) which is 24 if \(d = 4\). By way of contrast, in [25] the solution of the 30 Lagrangian equations of motion that do not involve time derivatives of any component of the affine connection leads to a reduced Lagrangian with only 16 independent variables (see eq.(4.1) of [25] and also eq.(A.27) of [41]), so that when using Lagrangian reduction 34 variables have disappeared after solving only 30 equations of motion. This is clear indication that secondary first class constraints have been solved in this approach. When using the variables \(\xi_{\alpha\beta}\), if we were to use the solutions of the first class constraints of (21) to eliminate fields, we eliminate more variables from the Lagrangian than equations of motion that have been solved. This contradicts the Dirac prescription for treating constrained dynamical systems and illustrates the importance of classifying constraints into first and second class, though the importance of this classification has been deemphasized in [42].

The Lagrangian reduction of the Maxwell action written in first order form is, by way of contrast, fully justified as all time independent equations of motion in this case correspond to second class constraints \(^6\). The equations of motion which do not have time derivatives (the Lagrangian constraints) for the first order formulation of electrodynamics are

\[
\frac{\delta L_M}{\delta F_{km}} = F_{km} - (\partial_k A_m - \partial_m A_k)
\]

giving \(F_{km}\) immediately in terms of \(A_m\). This is in agreement with Dirac reduction, as the primary constraints \(\Pi_{km} \approx 0\) (where \(\Pi_{km}\) are the momenta conjugate to \(F_{km}\)) give a subset of the second class constraints of the special form which allows for Dirac elimination or ensures that the reduced Lagrangian is equivalent to the initial one (see Appendix B).

Full correspondence between two reduction procedures exists only if we eliminate variables using Lagrangian equations similar to (23) where the field being eliminated is the same as the field being varied. (This also is the situation for auxiliary fields in supersymmetric models.) Only in this case is the reduced Lagrangian equivalent to the original. We see therefore that elimination of variables in the original Lagrangian by merely solving the Lagrange constraints may not always be correct. For example, suppose we have an action functional \(S(Q,q)\) and that the equation \(\delta S/\delta Q = 0\) can be solved for the \(Q\)’s so that \(Q = Q(q)\). This is then substituted back into \(S\) and the new action \(S'(q) = S(q,Q(q))\)

---

\(^6\) The usual references (e.g., [24]) concerning the similarity of this reduction to reduction of the first order form of the EH action are not entirely correct.
implies a dynamical equation \( \delta S'/\delta q = 0 \) which is the same dynamical equation for \( q \) that follows from \( \delta S/\delta q = 0 \). However, if \( \delta S/\delta Q = 0 \) is solved for \( q \) instead of \( Q \) one does not, in general, obtain the same dynamical equations from the action obtained by substitution of this solution into the original action. (We illustrate this in Section 4.)

When using the variables \( \Gamma \), it is difficult to compare results of these two approaches since the straightforward Dirac procedure is not easy to apply because of the way the Lagrangian (2) depends on derivatives of \( \Gamma \). The subset of second class primary constraints used in [30], where the variables \( \Gamma \) are employed, leads to elimination of some of the variables, which affect the rest of the primary constraints and some primary constraints become combinations of momenta and not just simply \( \Pi^\alpha_\beta \), as in the case when using the variables \( \xi \). Consequently not all secondary constraints have the same simple special form structure with primary constraints. In Lagrangian reduction, solving those equations of motion without time derivatives for auxiliary fields such as is done in (23), is not as easy to analyze because the last equality in (15) is due to the “diagonal” form of terms with derivatives, which is not the case when one uses \( \Gamma \) instead of \( \xi \).

To illustrate this general discussion we do not need to consider the EH action in \( d \) dimensions (using either \( \Gamma \) or \( \xi \)); we just need a simple model in which the first class constraints are present and can be algebraically solved in order to compare the results of the two approaches. We consider \( \tilde{L}_2 (h, \xi) \) which is a simple model with first class constraints that can be algebraically solved 7. It has stronger connection with \( \tilde{L}_d (g, \xi) \) for \( d > 2 \) than \( \tilde{L}_2 (g, \xi) \). (To see this, compare the analysis of \( \tilde{L}_2 (h, \xi) \) in [30] with that of \( \tilde{L}_2 (g, \xi) \) in [36]).

In the next two sections we present the complete canonical analysis of this simplest Lagrangian using both approaches. We examine the constraint structure and analyze the invariance of the action under the gauge transformation that is implied by the full set of first class constraints using the approach of Castellani [43].

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7 The non-equivalence of this model to the second order EH action in 2D, the difference between the \( h \) and \( g \) formulations in 2D, etc. are irrelevant since we are only comparing different methods of reduction.
III. CANONICAL ANALYSIS OF A SIMPLE MODEL USING DIRAC REDUCTION

In this section, we give the full Dirac analysis of a slightly modified form of (9) which has been advocated by Faddeev [41]. It allows us to demonstrate the effect of neglecting the contributions of surface terms. We consider

\[ \tilde{L}'_{d} = \tilde{L}_{d} - (h^{\alpha\beta} \xi^{\lambda}_{\alpha\beta}) \chi = -h^{\alpha\beta} \xi^{\lambda}_{\alpha\beta} - \frac{1}{d-1} h^{\alpha\beta} \xi^{\lambda}_{\alpha\beta} + \frac{1}{d-1} h^{\alpha\beta} \xi^{\lambda}_{\alpha\lambda} \xi^{\sigma}_{\beta\sigma} \]  

(24)

that in the 2D case results in

\[ \tilde{L}'_{2} = -\dot{h}^{11} \xi^{0}_{11} - 2\dot{h}^{01} \xi^{0}_{01} - \dot{h}^{00} \xi^{0}_{00} - H_{c} \]  

(25)

where

\[ H_{c} = \xi^{1}_{11} (h^{11}_{,1} - 2h^{11} \xi^{0}_{01} - 2h^{01} \xi^{0}_{00}) + 2\xi^{1}_{01} \left( h^{01}_{,1} + h^{11} \xi^{0}_{11} - h^{00} \xi^{0}_{00} \right) + \xi^{1}_{00} \left( h^{00}_{,1} + h^{01} \xi^{0}_{11} + h^{00} \xi^{0}_{01} \right). \]  

(26)

Introducing conjugate momenta to all variables, \( \pi^{\alpha\beta} \) and \( \Pi^{\gamma}_{\alpha\beta} \), we immediately obtain the primary constraints

\[ \Phi_{\alpha\beta} = \pi^{0}_{\alpha\beta} + \xi^{0}_{\alpha\beta} \approx 0, \Pi^{\alpha\beta} \approx 0 \]  

(27)

and the total Hamiltonian

\[ H_{T} = H_{c} + \lambda^{\alpha\beta} \Phi_{\alpha\beta} + \Lambda^{\gamma}_{\alpha\beta} \Pi^{\alpha\beta}_{\gamma}. \]  

(28)

Among the primary constraints, we have a subset which is second class as

\[ \{\Phi_{\alpha\beta}, \Pi^{\mu\nu}_{0}\} = \Delta^{\mu\nu}_{\alpha\beta}. \]  

(29)

We are using the standard fundamental PB for independent fields

\[ \{h^{\alpha\beta}, \pi_{\mu\nu}\} = \Delta^{\alpha\beta}_{\mu\nu}, \{\xi^{\lambda}_{\alpha\beta}, \Pi^{\mu\nu}_{\gamma}\} = \delta^{\lambda}_{\gamma} \Delta^{\mu\nu}_{\alpha\beta}, \]  

where \( \Delta^{\alpha\beta}_{\mu\nu} = \frac{1}{2} (\delta^{\alpha}_{\mu} \delta^{\beta}_{\nu} + \delta^{\beta}_{\mu} \delta^{\alpha}_{\nu}) \). The constraints (28) are of a special form and, according to Dirac reduction, such constraints can be eliminated without affecting the PB of all the remaining variables. We have now two strong equalities

\[ \Pi^{0}_{\alpha\beta} = 0, \xi^{0}_{\alpha\beta} = -\pi_{\alpha\beta} \]  

(30)
and as a result, we have the reduced total Hamiltonian

\[ H^{(1)}_{\text{r}} = H^{(1)} + \Lambda_k^{\alpha\beta} \Pi_k^{\alpha\beta}, \]

\[ H^{(1)} = -\xi^1_{\alpha1} x^1_{\alpha1} - 2\epsilon^1_{\alpha0} \tilde{x}^0_{\alpha1} - \xi^1_{\alpha0} \tilde{x}^0_{\alpha0} \]  

(31)

where using (30)

\[ \tilde{x}^1_{\alpha1} = -\left(h^1_{\alpha1} + 2h^1_{\alpha1} \pi_{01} + 2h^0_{\alpha0} \pi_{00}\right), \]

\[ \tilde{x}^0_{\alpha1} = -\left(h^0_{\alpha0} - h^1_{\alpha1} \pi_{11} + h^0_{\alpha0} \pi_{00}\right), \]

\[ \tilde{x}^0_{\alpha0} = -\left(h^0_{\alpha0} - 2h^0_{\alpha0} \pi_{11} - 2h^0_{\alpha0} \pi_{01}\right). \]  

(32)

Conservation of the primary constraints in time leads to the secondary constraints

\[ \dot{\Pi}^1_{\alpha1} = \{\Pi^1_{\alpha1}, H\} = \tilde{x}^1_{\alpha1}, \dot{\Pi}^0_{\alpha1} = \{\Pi^0_{\alpha1}, H\} = \tilde{x}^0_{\alpha1}, \dot{\Pi}^0_{\alpha0} = \{\Pi^0_{\alpha0}, H\} = \tilde{x}^0_{\alpha0}. \]  

(33)

All secondary constraints have zero PB with the primary constraints and among themselves have the following PB 8

\[ \{\tilde{x}^0_{\alpha1}, \tilde{x}^0_{\alpha0}\} = \tilde{x}^0_{\alpha0}, \{\tilde{x}^0_{\alpha1}, \tilde{x}^1_{\alpha1}\} = -\tilde{x}^1_{\alpha1}, \{\tilde{x}^1_{\alpha1}, \tilde{x}^0_{\alpha1}\} = 2\tilde{x}^0_{\alpha1}. \]  

(34)

The Hamiltonian (31) is a linear combination of secondary constraints and, because of the PB of (34), the Dirac canonical procedure is completed by the presence of six first class constraints for the six remaining canonical pairs (already three pairs \(\xi^0_{\alpha\beta}, \Pi^0_{\alpha\beta}\) have been eliminated) resulting in there being zero degrees of freedom. The secondary constraints (32) in Lagrangian language correspond to equations of motion obtained by varying such non-dynamical variables as those of (18). These equations cannot be solved for the fields being varied and so they are not auxiliary fields. These equations correspond to first class constraints in Dirac language and are different in this respect from the algebraic constraints arising in Maxwell electrodynamics (23). (The effect of performing a reduction by using the solution of first class constraints will be considered in the next section.)

In the Dirac procedure we have to first find all constraints, then eliminate the second class constraints. Only after these steps have been performed can we discuss gauge fixing, etc.

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8 Terms with derivatives must be carefully treated taking into account the distributional character of PB in the infinite dimensional case, that is, field theory [26, 44].
To find the full gauge invariance of the action using the Castellani procedure \cite{43}, it is important to determine the complete set of first class constraints. If some of the first class constraints are solved, they will not be present in the gauge generator and the some of gauge symmetries cannot be restored. We can obtain, at most, partial gauge symmetries (e.g., only the spatial diffeomorphism) or even possibly the wrong gauge symmetries.

The generator $G$ of gauge transformation, following Castellani \cite{43}, is found by first setting $G_a^{(1)} = C^a_p$ for the primary constraints ($C^a_p = (\Pi^{11}_1, \Pi^{01}_1, \Pi^{00}_1)$) and then determining $G_a^{(0)}(x) = - \{C^a_p, H_c\}(x) + \int dy \alpha_c^a(x, y) C^c_p(y)$ where the functions $\alpha_c^a(x, y)$ are found by requiring that $\{G_a^{(0)}, H_c\} = 0$. The full generator of gauge transformation is given by

$$G(\varepsilon^a, \dot{\varepsilon}^a) = \int dx \left( \varepsilon^a(x) G_a^{(0)}(x) + \dot{\varepsilon}^a(x) G_a^{(1)}(x) \right).$$

In our case this leads to the following expression, using the three primary and three secondary first class constraints,

$$G(\varepsilon) = \int dx \left[ \varepsilon \left( -\tilde{\chi}^{01}_1 - \xi^{00}_1 \Pi^{11}_1 + \xi^{11}_1 \Pi^{11}_1 \right) + \dot{\varepsilon} \Pi^{01}_1 
+ \varepsilon_1 \left( -\tilde{\chi}^{11}_1 - 2\xi^{01}_1 \Pi^{11}_1 - 2\xi^{10}_1 \Pi^{01}_1 \right) + \dot{\varepsilon}_1 \Pi^{11}_1 + \varepsilon \left( -\tilde{\chi}^{00}_1 + 2\xi^{11}_1 \Pi^{11}_1 + 2\xi^{11}_1 \Pi^{01}_1 \right) + \dot{\varepsilon} \Pi^{00}_1 \right]. \quad (35)$$

The PB of generators (35) have a closed off-shell algebra similar to that of ordinary gauge theories:

$$\{G(\varepsilon), G(\eta)\} = G \left( \tau^c = C^{ab} \varepsilon^a \eta^b \right) \quad (36)$$

where $\varepsilon^a = (\varepsilon^1(\varepsilon), \varepsilon^2(\varepsilon), \varepsilon^3(\varepsilon))$ and the only non-zero structure constants $C^{ab}$ are $C^{132} = 2 = -C^{123}, C^{212} = 1 = -C^{221}, C^{331} = 1 = -C^{313}$. These reflect the structure of the algebra of the PB among the first class constraints. More explicitly, these relations are

$$\tau = 2\varepsilon^1 \eta_1 - 2\varepsilon_1 \eta^1, \tau_1 = \varepsilon \eta_1 - \varepsilon_1 \eta, \tau^1 = \varepsilon^1 \eta - \varepsilon \eta^1.$$

Now we can find the transformations for all fields appearing in the initial Lagrangian that follow from $\delta (\text{field}) = \{\text{field}, G\}$:

$$\delta \xi^{11} = \dot{\varepsilon}_1 - 2\varepsilon_1 \xi^{11}_0 + \varepsilon \xi^{11}_1,$$
$$\delta \xi^{01} = \frac{1}{2} \dot{\varepsilon}_1 - \varepsilon_1 \xi^{11}_0 + \varepsilon \xi^{01}_1,$$
$$\delta \xi^{00} = \dot{\varepsilon}_1 - \varepsilon_1 \xi^{11}_0 + 2\varepsilon \xi^{01}_1,$$
$$\delta h^{11} = -\varepsilon \xi^{11}_1 - 2\varepsilon^1 h^{01}_1,$$

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\[ \delta h^{01} = \varepsilon_1 h^{11} - \varepsilon^1 h^{00}, \quad (38) \]
\[ \delta h^{00} = \varepsilon h^{00} + 2 \varepsilon_1 h^{01}, \]
\[ \delta \pi_{11} = \varepsilon_{1,1} - 2 \varepsilon_1 \pi_{01} + \varepsilon \pi_{11}, \]
\[ \delta \pi_{01} = \frac{1}{2} \varepsilon_{1} - \varepsilon_1 \pi_{00} + \varepsilon^1 \pi_{11}, \quad (39) \]
\[ \delta \pi_{00} = \varepsilon^1 - \varepsilon \pi_{00} + 2 \varepsilon^1 \pi_{01}. \]

From (39) and using the strong equalities (30), we obtain
\[ \delta \xi_{0}^{0} = -\varepsilon_{1,1} - 2 \varepsilon_1 \xi_{01}^{0} + \varepsilon \xi_{11}^{0}, \]
\[ \delta \xi_{0}^{0} = -\frac{1}{2} \varepsilon_{1} - \varepsilon_1 \xi_{00}^{0} + \varepsilon^1 \xi_{11}^{0}, \]
\[ \delta \xi_{00} = -\varepsilon^1 - \varepsilon \xi_{00}^{0} + 2 \varepsilon^1 \xi_{01}^{0}. \quad (40) \]

One can easily check the gauge invariance of the Lagrangian \( \tilde{L}'_2 \) of (25) using the transformations of (37, 38, 40). It is
\[ \delta \tilde{L}'_2 = \left( h^{11} \varepsilon_{1,1} + h^{01} \varepsilon_{1,1} + h^{00} \varepsilon_{1,1} \right)_0 - \left( h^{11} \varepsilon_1 + h^{01} \varepsilon + h^{00} \varepsilon^1 \right)_1, \quad (41) \]
and so \( \tilde{L}'_2 \) is invariant up to total derivatives. However, a variation of the total derivatives appearing in (24) results in a contribution that exactly compensates (41). Keeping the initial form \( \tilde{L}_2 \) of the Lagrangian, we have exact invariance under the transformations of (37, 38, 40). This illustrates the importance of surface terms in retaining invariance of the Lagrangian and shows that the elimination of surface terms can affect its gauge invariance. (For a discussion of a similar occurrence in SUSY models, see [45].)

The transformations of (37, 38, 40) can be written in a compact form which is similar to one appearing in [46].
\[ \delta h^{\alpha \beta} = \left( \varepsilon^{\alpha \lambda} h^{\sigma \beta} + \varepsilon^{\beta \lambda} h^{\sigma \alpha} \right) \zeta_{\lambda \sigma}, \quad (42) \]
\[ \delta \xi_{\alpha \beta} = -\varepsilon^{\alpha \lambda} \zeta_{\alpha \beta \rho} - \varepsilon^{\rho \sigma} \left( \xi_{\alpha \rho} \zeta_{\beta \sigma} + \xi_{\beta \rho} \zeta_{\alpha \sigma} \right), \quad (43) \]

\[ \delta \xi_{\alpha \beta} \]
\[ \]
where $\epsilon^{\alpha\beta}$ is the antisymmetric tensor ($\epsilon^{01} = 1$) and $\zeta_{\alpha\beta}$ is a symmetric tensor with components $\zeta_{00} = \epsilon^1$, $\zeta_{11} = \epsilon_1$, $\zeta_{01} = \frac{1}{2}\epsilon$. From (43) using (7, 8) we obtain the transformation of $\Gamma^\lambda_{\alpha\beta}$

$$
\delta \Gamma^\lambda_{\alpha\beta} = -\frac{1}{2}\epsilon^\lambda_{\rho\sigma} \zeta_{\alpha\beta,\rho} + \delta^\lambda_{\alpha} \epsilon^\nu_{\rho} \zeta_{\nu\beta,\rho}$$

(44)

$$
-\epsilon^{\rho\sigma} \left[ \Gamma^\lambda_{\alpha\rho} \zeta_{\beta\sigma} - \frac{1}{2} \delta^\lambda_{\alpha} \Gamma^\nu_{\rho\beta} \zeta_{\nu\sigma} + \frac{1}{2} \delta^\lambda_{\alpha} \Gamma^\nu_{\rho\nu} \zeta_{\beta\sigma} + \frac{1}{2} \delta^\lambda_{\alpha} \Gamma^\nu_{\nu\rho} \zeta_{\beta\sigma} \right] + (\alpha \leftrightarrow \beta).
$$

The Einstein form of the Lagrangian $L_2(h, \Gamma)$ is invariant under the transformations of (42) and (44).

All the usual canonical properties of non-Abelian gauge theories are present in the approach outlined here. We have local PB with field independent structure constants, a closed off-shell algebra of generators, and exact invariance of the Lagrangian under gauge transformations of the original fields. In [30], only the surface term $- (h^{\alpha\beta} \zeta_{\alpha\beta})_{,k}$ was added to $\tilde{L}_2$ and so $\zeta^0_{\alpha\beta}$ played role of a generalized coordinate $^\circ 10$.

IV. LAGRANGIAN REDUCTION BASED ON SOLUTIONS OF FIRST CLASS CONSTRAINTS

Again, starting with the same Lagrangian (25), we find the equations of motion associated with the non-dynamical fields to obtain the Lagrangian constraints. These are identical to the PB of the primary constraints with the Hamiltonian (33). We find that

$$
\frac{\delta \tilde{L}_2'}{\delta \xi_{11}^0} = h_{,1}^{11} - 2h_{,1}^{11} \xi_{01}^0 - 2h_{,1}^{01} \xi_{00}^0 = 0, \quad (45)
$$

$$
\frac{\delta \tilde{L}_2'}{\delta \xi_{01}^0} = h_{,1}^{01} + h_{,1}^{11} \xi_{11}^0 - h_{,00} \xi_{00}^0 = 0, \quad (46)
$$

$$
\frac{\delta \tilde{L}_2'}{\delta \xi_{00}^1} = h_{,1}^{00} + 2h_{,1}^{01} \xi_{11}^0 + 2h_{,00} \xi_{01}^0 = 0. \quad (47)
$$

These are the only three equations of motion out of nine in total that have no time derivatives.

$^\circ 10$ This, of course, cannot affect the result as the roles of coordinates and momenta are interchangable in the Hamiltonian formulation. The transformations in both cases are the same and only from a purely computational point of view might some preference exist.
In the Dirac approach, we cannot solve them to eliminate any variables as they are all first class constraints. From the Lagrangian point of view, they are not equations for the auxiliary fields, since the fields being varied do not appear on right hand side of (45-47). If one were to use them anyway, as is done in ADM approach [25], to solve for two variables algebraically, such as from (46) and (47)

$$\xi^0_{00} = \frac{1}{h_{00}} \left( h_{11}^{01} + h_{00}^{01} \xi^0_{11} \right),$$

$$\xi^0_{01} = -\frac{1}{h_{00}} \left( \frac{1}{2} h_{11}^{00} + h_{00}^{01} \xi^0_{11} \right),$$

and then substitute these equations back into the original Lagrangian (25) to eliminate $\xi^0_{00}$ and $\xi^0_{01}$, one finally obtains the reduced Lagrangian

$$\tilde{L}'_2^{(1)} = -\dot{h}_{11}^0 \xi^0_{11} + 2 \dot{h}_{01}^0 h_{00}^0 \xi^0_{11} - \dot{h}_{00}^0 h_{11}^0 \xi^0_{11} - h_{11}^0 h_{00}^0 \xi^0_{11} \left( h_{11}^1 - 2 \dot{h}_{01}^0 h_{00}^1 + \dot{h}_{11}^0 h_{00}^1 \right) + S$$

where $S$ consists of two total derivatives

$$S = \left( \dot{h}_{11}^0 \ln h_{00}^0 \right)_{,1} - \left( \dot{h}_{01}^0 \ln h_{00}^0 \right)_{,0}.$$  

(This is similar to eq.(A.26) of [41].) Note that only two out of the three equations (48,49) can be solved because after substitution of (48,49) into (45) all the variables $\xi^0_{\alpha\beta}$ disappear, converting equation (45) into a differential constraint that is retained in the reduced Lagrangian (50). After this reduction, we have five fields out of the original nine left in $\tilde{L}'_2^{(1)}$ in (50) so that two equations of motion have served to eliminate four dynamical fields. This is a familiar feature of the Lagrangian reduction in 4D [25]; solving one first class constraint leads to the disappearance of two variables in the reduced Lagrangian. In [25], after solving 30 equations, 34 variables disappear from the reduced Lagrangian (as four equations out of the 30 that have no time derivatives are first class constraints).

Moreover, if we perform variation of the reduced Lagrangian (50) with respect to $\xi^0_{11}$, we obtain

$$\frac{\delta \tilde{L}'_2^{(1)}}{\delta \xi^0_{11}} = -\dot{h}_{11} + 2 \dot{h}_{01}^0 h_{00}^1 - \dot{h}_{00}^1 h_{00}^1.$$  

The variation of the original Lagrangian (25) with respect to the same variable gives

$$\frac{\delta \tilde{L}_2}{\delta \xi^0_{11}} = -\dot{h}_{11} - 2 \dot{h}_{11}^0 \xi^0_{11} - h_{00}^0 \xi^0_{11}.$$  

The use of (48,49) or any of the other equations of motion cannot account for the difference between (52) and (53). Thus, the reduced Lagrangian $\tilde{L}'_2^{(1)}$ is not equivalent to the original
one $\tilde{L}'_2$. If one, despite of this inconsistency, wants to find the Hamiltonian associated with this new Lagrangian (50), the Dirac procedure must now be repeated.

Introducing momenta conjugate to the five fields

\[ \Pi_{11}^1 (\xi_{11}^0), \Pi_{00}^1 (\xi_{11}^0), \pi_{\alpha\beta} (h^{\alpha\beta}) \] (54)

leads to five primary constraints

\[ \Pi_{11}^1 \approx 0, \Pi_{00}^1 \approx 0, \pi_{01} + \xi_{11}^0 \approx 0, \pi_{00} - \frac{h_{01}}{h_{00}} \xi_{11}^0 \approx 0, \pi_{00} + \frac{h_{11}}{h_{00}} \xi_{11}^0 \approx 0. \] (55)

Among them we have a pair which are second class of a special form (the second and third constraints of (55)\(^{11}\)) that allows us to set $\Pi_{00}^1 = 0$ and $\xi_{11}^0 = -\pi_{11}$. Substitution of these equalities into the Hamiltonian and the remaining constraints gives the reduced total Hamiltonian

\[ H^{(1)}_T = H^{(1)}_c + \chi^{01} \left( \pi_{01} + \frac{h_{01}}{h_{00}} \xi_{11}^0 \right) + \chi^{00} \left( \pi_{00} - \frac{h_{11}}{h_{00}} \xi_{11}^0 \right) + \Lambda_{11}^1 \Pi_{11}^1 \] (56)

where

\[ H^{(1)}_c = -\xi_{11}^1 \chi_{11}^{11} \] (57)

with

\[ \chi_{11}^{11} = -h_{11}^{11} + 2 h_{01} \frac{h_{01}}{h_{00}} - h_{01} \frac{h_{11}}{h_{00}}. \] (58)

Continuing with the Dirac procedure, we find the secondary constraints. The PB among all primary constraints are zero, the only non-obvious one being $\{ \pi_{01} + \frac{h_{01}}{h_{00}} \xi_{11}^0, \pi_{00} - \frac{h_{11}}{h_{00}} \xi_{11}^0 \}$.

The only secondary constraint that arises is

\[ \dot{\Pi}_{11}^1 = \left\{ \Pi_{11}^1, H^{(1)}_c \right\} = \chi_{11}^{11}, \] (59)

showing that the Hamiltonian is a constraint. It is straightforward to show that the PB among all constraints, both primary and secondary, are zero so that we have four first class constraints for four pairs of canonical variables leaving us with no net degrees of freedom. At this point everything looks consistent as there are zero degrees of freedom, a local algebra of first class constraints and closure of the Dirac procedure. The Dirac constraint formalism applied to the reduced Lagrangian gives consistent results even though it is not equivalent

\[^{11}\text{We can equally take the second and fourth (or alternatively the second and fifth) constraints without affecting the final result.}\]
to the original theory. As in the previous section, we can find a gauge transformation corresponding to this system of constraints. The resulting gauge generator is much simpler than that of (35). It has though the same number of gauge parameters because the reduced total Hamiltonian (56) also has three primary first class constraints. We find that

$$G(\varepsilon) = \int dx \left[ \varepsilon^1 \left( \pi_{01} + \frac{h_{01}}{h_{00}} \pi_{11} \right) + \varepsilon \left( \pi_{00} - \frac{h_{11}}{h_{00}} \pi_{11} \right) - \varepsilon_1 \chi_{11} + \dot{\varepsilon}_1 \Pi_{11} \right] \right]. \quad (60)$$

The algebra of this generator is closed even off-shell. This generator leads to the gauge transformations of the fields

$$\delta \xi_{11} = \dot{\varepsilon}_1, \delta \Pi_{11} = 0, \quad (61)$$

$$\delta h_{11} = \varepsilon^1 \frac{h_{01}}{h_{00}} - \varepsilon \frac{h_{11}}{h_{00}}, \delta h_{01} = \frac{1}{2} \varepsilon^1, \delta h_{00} = \varepsilon, \quad (62)$$

$$\delta \pi_{11} = \varepsilon \frac{1}{h_{00}} \pi_{11} + \varepsilon_{1,1} - \varepsilon_1 \frac{1}{h_{00}} h_{11}. \quad (63)$$

To check the invariance of the reduced Lagrangian (50), we also need the transformation of $\xi_{11}^0$ which can be easily restored by using the strong equality $\xi_{11}^0 = -\pi_{11}$ so that

$$\delta \xi_{11}^0 = \varepsilon \frac{1}{h_{00}} \xi_{11}^0 + \varepsilon_{1,1} - \varepsilon_1 \frac{1}{h_{00}} h_{11}. \quad (64)$$

Using (61,62,64), the variation of (50) is

$$\delta \tilde{L}_2^{(1)} = \left( \varepsilon \frac{1}{h_{00}} \xi_{11}^0 - \varepsilon \frac{1}{h_{00}} h_{00} \right) \frac{\delta \tilde{L}_2^{(1)}}{\delta \xi_{11}^0} + S, \quad (65)$$

where $S$ is a term with total derivatives. Hence the Lagrangian is invariant only on-shell, which is a familiar feature of the ADM approach (the gauge generator in 4D has a closed algebra only on-shell [43]). In the previous section we were able to determine the transformations of all fields appearing in the original Lagrangian. It is not possible to do so now; by going back we can only restore the transformations of $\xi_{00}^0$ and $\xi_{01}^0$ using eqs. (48) and (49) but we cannot do this for $\xi_{01}^1$ and $\xi_{00}^1$. Thus using a Lagrangian reduction which is based on employing solutions of the first class constraints leads to the gauge transformations for only some of the fields in the original Lagrangian and the Lagrangian is invariant only on-shell. This is to be compared with having exact invariance of the Lagrangian and gauge transformations.

\[\text{Of course, after the Lagrangian reduction based on solution of first class constraints we cannot return to the variables } \Gamma \text{ because not all the transformations of the fields } \xi \text{ can be found. It is not difficult to repeat such a reduction directly in the Lagrangian when it is written in terms of } \Gamma \text{ and compare with the results of the previous section; they will be also different.}\]
transformations for all the fields when one uses the Dirac approach to the Lagrangian before making a reduction based on all time independent equations of motion.

Moreover, if we consider the formulation of the 2D EH action using $g^{\alpha\beta}$ instead of $h^{\alpha\beta}$ as independent variables and perform the Lagrangian reduction using (46) and (47), the Lagrangian vanishes identically. To demonstrate this, we present the reduced Lagrangian (50) in the following form

$$\tilde{L}_2' = - \left( h^{11}h^{00} - h^{01}h^{01} \right) \frac{1}{h^{00}g^{01}} - \left( h^{11}h^{00} - h^{01}h^{01} \right) \frac{1}{h^{00}g^{11}}. \quad \text{(66)}$$

If we consider $g^{\alpha\beta}$ to be the independent variable, then $h^{\alpha\beta}$ is just a short form for $\sqrt{-g}g^{\alpha\beta}$ and the particular combination that enters (66) under derivatives is

$$h^{11}h^{00} - h^{01}h^{01} = -1, \quad \text{(67)}$$

and so, the Lagrangian (66) vanishes identically. However, Dirac analysis in this case leads to seven independent first class constraints, five of which are primary, and consequently there are five parameters characterizing the group of gauge transformations [36].

V. CONCLUSION

The second order form of the EH action (1) in which the metric is the only independent dynamical field is invariant under a general coordinate transformations if all terms, including the terms with second order derivatives are present. It is possible to apply the standard Dirac canonical analysis and to keep simultaneously the effect of all terms when using an equivalent first order formulation. The oldest first order formulation which is the closest in form to the second order EH action is the affine-metric formulation of Einstein [20].

This formulation when treated using the standard methods of quantum field theory should automatically retain the classical limit. Demonstration that such a limit exists in models based on new ideas constitutes a considerable problem in itself [1].

The advantage of using this first order Einstein formulation of the action was recognized a long time ago by ADM and was used by them as a starting point in their canonical analysis of GR [24, 25]. However, they did not apply the straightforward Dirac analysis and they performed a preliminary Lagrangian reduction using solutions of first class constraints [25] (see also [41]). The reduced Lagrangian found in this way is not equivalent to the original
Lagrangian and, to quote [47], "does not represent the full statement of general relativity". In the concluding remarks to the last paper of the ADM series [47] (see also remarks in [48]), the authors suggested that in view of the many ambiguities that could arise in an attempt to quantize consistently at reduced level, it would seem more fruitful to return to the original Lagrangian [3] and try to repeat reduction to the canonical form within the framework of quantum theory. To keep the gauge invariance of the original Lagrangian when quantizing, all first class constraints must be preserved and second class constraints can only be eliminated when they are of a special form, or used to modify the PB by passing to Dirac brackets.

The importance of preserving all first class constraints in the course of the Dirac quantization was analysed by Ashtekar and Horowitz [49]. They concluded that the Lagrangian reduced-space method is likely to yield an incomplete description of quantum gravity. (See also the subsequent discussions in [50]).

During the two decades between the last paper of the ADM series and the Ashtekar-Horowitz analysis, another result of [47] got a lot of attention (e.g., see [32]). This involves a geometrical interpretation of what remains after a Lagrangian reduction of field variables. The Hamiltonian obtained from the reduced Lagrangian has been emphasized stressing the geometrical significance of the reduced set of variables, leading also to a shift back from treating four-dimensional Einstein spacetime to just treating space by itself and from constraint dynamics of the full Hamiltonian to geometrodynamics of a reduced Hamiltonian. The reduced Lagrangian is invariant only under spatial coordinate transformations [51] and the disappearance of some symmetries is a strong indication of the inequivalence of the two approaches. The possibility of a 3D geometrical interpretation of the variables appearing in the reduced Lagrangian (corresponding to a particular slicing of spacetime) is a demonstration of the inconsistency of Lagrangian reduction because it contradicts the spirit of GR and furthermore introduces a restriction on the topology of spacetime [31]. This restriction originates in solving a part of the first class constraints. This ‘freezing’ of symmetries is a sort of partial gauge fixing used in the ADM treatment [25] right from outset. However, the correct procedure is to fix the gauge only after the constraint analysis is performed [1, 52]. This approach also imposes the coordinate conditions that space-like surfaces remain space-like as is explicitly pointed out by Dirac [7]. This condition, which restricts the form of the general coordinate transformations, obviously means abandoning four-dimensional spacetime.
symmetry as is clearly indicated in the conclusion of [7].

Slicing and the imposition of coordinate conditions also contradict the canonical procedure, since for relativistic field theories a fundamental tenet of the canonical formulation is to not refer to the ambient spacetime [53]. Any reference to a surface (or a particular subset of surfaces) already contradicts this, and it also implies the implicit introduction of extended objects into a local theory. It quite likely leads to non-locality, as models explicitly built using extended objects (such as string models) are essentially non-local by construction [54].

The common reference to similarities of the hypersurface deformation algebra (4-6) to string models as a sign of consistency of the constraint algebra of the reduced Hamiltonian is really a warning sign that GR, which is a local field theory, has somehow been converted into a non-local one.

The Hamiltonian built from the reduced Lagrangian leads to the well-known non-local Dirac constraint algebra, which is difficult to quantize as it is not a true Lie algebra. There are also problems associated with defining time, finding physical observables, etc.; and as a result, numerous attempts to improve this approach by modifying the choice of variables [55], reshuffling of the constraints [37, 38], etc. have been made.

However, once reduction has been performed any change of the reduced variables or reshuffling of the reduced constraints cannot cure these inconsistencies, as they are inherent to the framework of the reduced Lagrangian. The only possibility of resolving these problems is to not abandon the spirit of GR and the standard canonical procedure. The first order formulation of the EH action treated by the standard methods of constraint dynamics preserves all symmetries, as it does in ordinary gauge theories, and all results should be reconsidered prior to reduction. This was actually suggested by ADM (see [47] Sec.7-8.1., “Discussion of quantization”).

In [30, 36] an analysis of the 2D limit of the affine-metric first order formulation was performed without any a priori assumptions or restrictions such as those used in Lagrangian reduction, and the Dirac procedure was applied to see how some properties of GR that make it distinct from ordinary gauge theories might appear. However, it turned out that all properties of ordinary gauge theories remain manifestly intact if an alternative first order formulation is used based on a change of variables involving linear combinations of affine connections. This change of variables is easily generalized to any dimension by (8) and has been employed in this article. The variables $\xi_{\alpha\beta}$ of (8) provide the alternative
first order formulation of (9) that considerably simplifies straightforward application of the Dirac procedure. (For details, see Appendix A.) The first steps of the Dirac algorithm (Sec.2) give results that are different from the ADM analysis which is based on the reduced Lagrangian. The origin of such difference lies in using solutions of equations of motion which correspond to first class constraints and the reduced Lagrangian obtained in this way is not equivalent to the original Lagrangian even at a classical level. (An example of where the Lagrangian reduction is equivalent to the Dirac approach is presented in Appendix C.) The first class primary constraints obtained in the Dirac analysis \((118)\) (Appendix C) cannot be eliminated since they constitute a first class subset with secondary constraints \((21)\) and any further constraint of higher order (e.g., tertiary constraints, either first or second class) cannot affect the first class character of this subset of constraints. Moreover, higher order constraints do not involve field variables conjugate to variables appearing in the primary constraints \((118)\), i.e. the first class nature of primary constraints cannot be changed by occurrence of higher order constraints. Consequently the Hamiltonian obtained by the Dirac approach cannot be reconciled with the Hamiltonian obtained from the reduced Lagrangian. The Dirac separation of constraints into first and second class is not merely a technical trick; these two classes of constraints are essentially different, as the first class constraints are an indication of the presence of gauge invariance. The knowledge of the gauge degrees of freedom is important when quantizing a model and must be kept in the formalism \([56]\).

In Secs.3 and 4, we demonstrated by considering a simple example that Dirac reduction and Lagrangian reduction produce different results if Lagrangian constraints are solved without appropriate care. At first glance, the simple model which is treated in two different ways produces in both approaches the expected canonical results such as a local algebra of constraints with field independent structure constants, a closed off-shell algebra of gauge generators and the possibility of finding the gauge transformation of all fields. However, whereas the Dirac constraint analysis allows us to determine the gauge transformation of all fields from the original Lagrangian and to demonstrate exact gauge invariance of the Lagrangian, the Lagrangian reduction (based on eliminating variables by use of solutions of first class constraints) does not lead to well defined transformations of all the original fields and the reduced Lagrangian is invariant only on-shell.

Moreover, if in more complicated cases the canonical analysis of the reduced Lagrangian leads to an algebra of constraints which is not a true Lie algebra, the problem of quantization
arises but this may be a problem of the reduced Lagrangian but it is not necessarily a problem of the original Lagrangian.

In our simple 2D example we obtained quite different results using the two approaches showing that in the general case the canonical analysis of what is obtained after Lagrangian reduction can lead not only to gauge invariance on-shell but also, for example, to non-locality of PB, a closed algebra of generators only on-shell \cite{33}, and consequently to a wrong or, at most, only partially right description of the initial Lagrangian. We thus feel that for the EH action in higher dimensions it is natural to expect there to be an even more drastic deviations between the two approaches. What are they? We are not going to speculate here about all possibilities, but we hope that we have been able to convince the reader that the existing canonical formulation of the first order EH action has been obtained in a non-canonical way and its reduced, geometrodynamical, formulation is not equivalent to the original EH Lagrangian. This gives rise to the very important question of what we are trying to quantize in canonical gravity. Is it the full Einstein GR theory or only the spatial, geometrodynamical, part of it?

The solution of the first class constraints in the first order formulation \cite{25} is somehow related to a partial neglect of surface terms and imposing coordinate conditions in the second order analysis \cite{7}. It is natural to ask about the connection between solving first class constraints in first order formalism and breaking relativistic invariance by integrating out second order derivatives in the second order formulation. The full answer to this question can be given only if a generalization of the Dirac procedure is possible that allows us to deal directly with accelerations present in the second order EH action. Qualitatively, we expect that the term linear in acceleration corresponds to a primary constraint (see \cite{17,18,19}) which initiates a chain of higher order constraints. The elimination of such a term corresponds to cutting off the first term of a chain of constraints and has the same effect as solving a first class constraint in the first order formulation of the action.

The numerous problems associated with canonical geometrodynamics are quite likely just problems of using the reduced Lagrangian (with its reduced symmetries), not an intrinsic characteristic of GR, and can actually be considered as an illustration of the fact that having only spatial symmetry is not enough for a consistent formulation of GR. The simplest and most natural possibility for resolving these problems has not been fully explored: instead of trying to improve the reduced formulation or attempting to find some new physics in the in-
consistencies of canonical geometrodynamics, one should try to find a canonical formulation of GR by applying the Dirac procedure to its first order formulation without any a priori assumptions or restrictions. The use of an alternative first order formulation given in \( (9) \) that is based on a generalization of the transformation found in the 2\( D \) limit of the action provides an example of how Dirac or Lagrangian reduction can be performed consistently. Possibly the use of these new variables is not sufficient to ensure a canonical form of GR that allows for quantization, and further modifications are needed in order to find a first order formulation that preserves all the properties of ordinary gauge theories in higher dimensions. In particular, we want to find a formulation that leads to a local algebra of constraints with field independent structure constants (as in \( [30, 36] \) and eqs.\( [22, 34] \)). The existence of such an algebra is needed to pass a crucial consistency test (for Hamiltonians) recently emphasized in \([1]\). We believe that all possibilities have to be explored in this direction before any new physical hypothesis is introduced and before the question: “spacetime or space?” (once answered by Einstein) can be posed again.\(^{13}\)

We would like to conclude our discussion by the epigraph to the second lecture, “Geometrodynamics”, in the course on Canonical Quantization of Gravity at Banff Summer School \([32]\):

*There is only the fight to recover
what has been lost
And found and lost again and again*

* T.S.Eliot: Four Quartets
* East Coker, 186-7.

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\(^{13}\) The answers to some more restricted questions such as: “Geometrodynamics: Spacetime or Space?” is known \([57]\).
Institute, October 2004) where many questions that we partially try to address in this article had arisen.

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VII. APPENDIX A

Here the proof of the equivalence of the first order formulation $\tilde{L}_d(g, \xi)$ defined in (9) with $L_d(g)$ of (1) is presented for the case $d \neq 2$.

The variation of $\tilde{L}_d(g, \xi)$ with respect to $g^{\mu\nu}$ gives the standard result

$$\frac{\delta \tilde{L}_d}{\delta g^{\mu\nu}} = \left(\sqrt{-g} \Delta_{\mu\nu} - \frac{1}{2} \sqrt{-g} g^{\alpha\beta} g_{\mu\nu} \right) R_{\alpha\beta}(\xi)$$

(68)

where

$$R_{\alpha\beta}(\xi) = \xi^{\lambda}_{\alpha\beta,\lambda} - \xi^{\lambda}_{\alpha\sigma} \xi^{\sigma}_{\beta\lambda} + \frac{1}{d-1} \xi^{\lambda}_{\alpha\lambda} \xi^{\sigma}_{\beta\sigma}.$$  

(69)

From (68) it immediately follows that

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

(70)

or alternatively, by using the inverse of the expression multiplying $R_{\alpha\beta}$ in (68)

$$\left(\sqrt{-g} \Delta_{\mu\nu} - \frac{1}{2} \sqrt{-g} g^{\gamma\sigma} g_{\mu\nu} \right)^{-1} = \frac{1}{\sqrt{-g}} \left(\Delta_{\mu\nu} - \frac{1}{d-2} g^{\mu\nu} g_{\gamma\sigma} \right)$$

(71)

(not defined in $2D$), we obtain

$$R_{\gamma\sigma}(\xi) = 0.$$  

(72)

$R_{\gamma\sigma}(\xi)$, as in the case when considering $L_d(g, \Gamma)$, now has to be expressed in terms of $g^{\alpha\beta}$. Varying $\tilde{L}_d$ with respect to $\xi^{\nu}_{\sigma\rho}$ we have

$$\frac{\delta \tilde{L}_d}{\delta \xi^{\nu}_{\sigma\rho}} = -h^{\sigma\rho}_{,\nu} - h^{\mu\sigma} \xi^{\rho}_{\mu\nu} - h^{\mu\rho} \xi^{\sigma}_{\mu\nu} - \frac{1}{d-1} \left(h^{\mu\sigma} \xi^{\lambda}_{\mu\lambda} \delta^{\rho}_{\nu} + h^{\mu\rho} \xi^{\lambda}_{\mu\lambda} \delta^{\sigma}_{\nu} \right).$$

(73)

This equation is easier to solve than the analogous equation for $\Gamma$. First, we can obtain the trace of $\xi^{\lambda}_{\lambda\nu}$. Multiplying (73) by $h_{\sigma\rho}$ (where $h_{\alpha\beta} h^{\beta\gamma} = \delta^{\gamma}_{\alpha}$) we obtain

$$\xi^{\lambda}_{\lambda\nu} = -\frac{1}{2} \frac{d-1}{d-2} h_{\gamma\tau} h^{\gamma\tau}_{,\nu}.$$  

(74)
We see that here, as when using the variables $\Gamma$, 2D case is special. Substitution of (74) into (73) gives
\[ h^{\mu \sigma} \xi_{\mu \nu} + h^{\mu \rho} \xi_{\mu \nu} = D^\sigma_{\nu} \] (75)
where
\[ D^\sigma_{\nu} = -h^\sigma_{\nu} - \frac{1}{2 (d - 2)} (h^{\mu \sigma} \delta^\rho_{\nu} + h^{\mu \rho} \delta^\sigma_{\nu}) h_{\gamma \tau} h^\gamma^\tau_{\mu}. \] (76)

Multiplying (75) by $h^{\alpha \rho} h^{\beta \sigma}$ and performing a permutation of the indices $\alpha, \beta, \nu$ we obtain three equations; we add two (with the permutations ($\beta, \alpha, \nu$) and ($\nu, \beta, \alpha$)) and subtract the third ($\alpha, \nu, \beta$), so after multiplication by $\frac{1}{2} h^{\omega \beta}$ we obtain the solution for $\xi_{\nu \alpha}^{\omega}$
\[ \xi_{\nu \alpha}^{\omega} = \frac{1}{2} [h_{\alpha \mu} D^\mu_{\nu} + h_{\nu \mu} D^\mu_{\alpha} - h^{\omega \sigma} h_{\nu \mu} h_{\alpha \gamma} D^\gamma_{\sigma}] \] (77)

Substitution of (77) into (70) or (72) gives the Einstein equations for free space. (No reference to $\Gamma$ has been made.) Similarly, if we substitute this solution into the Lagrangian $\tilde{L}_d (g, \xi)$ of (9), we obtain the reduced Lagrangian which is equivalent to the second order form of EH action $L_d (g)$ (including terms with second order derivatives).

The appearance of explicit dimensional dependence in (74, 76) seems to be inconsistent with using the Christoffel symbol\textsuperscript{14}. To resolve this, let us consider the trace of $\Gamma$ expressed in terms of $\xi$. Using (8) we find
\[ \Gamma^\lambda_{\nu \lambda} = -\frac{2}{d - 1} \xi^\lambda_{\nu \lambda}, \] (78)
so that upon substitution of (74) into (78) and remembering that $h^{\alpha \beta}$ is only short for $\sqrt{-g} g^{\alpha \beta}$, we obtain
\[ \Gamma^\lambda_{\nu \lambda} = \frac{1}{d - 2} h_{\alpha \beta} h^{\alpha \beta}_{\nu \lambda} = \frac{1}{d - 2} \frac{1}{\sqrt{-g}} g_{\alpha \beta} \left( \sqrt{-g} g^{\alpha \beta} \right)_{\nu \lambda} = -\frac{1}{2} g_{\alpha \beta} g^{\alpha \beta}_{\nu \lambda} \] (79)
which is a well-known expression. Similarly, the general case for arbitrary $\Gamma^\gamma_{\alpha \beta}$ can be demonstrated using (8) and (77).

\textbf{VIII. APPENDIX B}

As an illustration of Dirac reduction (used in Sections 2-4) that employs elimination of only second class constraints that are of a special form, we prove the equivalence of the first

\textsuperscript{14} The expression for it has the coefficient $\frac{1}{2}$ in any dimension.
and second order formulation of Maxwell electrodynamics at the level of the Hamiltonian.

The first order form of the Maxwell Lagrangian is

\[ L_M = -\frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) F^{\mu\nu} + \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]  

(80)

where \( A_\mu \) and \( F_{\mu\nu} = -F_{\nu\mu} \) are treated as independent fields. This formulation is equivalent to the standard second order form. This is obvious at the Lagrangian level, as the auxiliary field \( F_{\mu\nu} \) can be easily eliminated. It is also not difficult to prove this at the Hamiltonian level by reducing the Hamiltonian that corresponds to (80) to the standard one by using the Dirac procedure.

Introducing momenta conjugate to all fields \( \pi^\mu (A_\mu), \Pi_{\mu\nu} (F^{\mu\nu}) \) we obtain

\[ \pi^\mu = \frac{\delta L}{\delta (\partial_0 A_\mu)} = F^{\mu0} \]  

(81)

and

\[ \Pi_{\mu\nu} = \frac{\delta L}{\delta (\partial_0 F_{\mu\nu})} = 0. \]  

(82)

In 4D, equations (81, 82) give ten primary constraints

\[ \phi^\mu = \pi^\mu - F^{\mu0} \approx 0, \Phi_{\mu\nu} = \Pi_{\mu\nu} \approx 0 \]  

(83)

and the total Hamiltonian is

\[ H_p = H_c + \lambda_\mu \phi^\mu + \Lambda^{\mu\nu} \Phi_{\mu\nu} \]  

(84)

where \( \lambda_\mu, \Lambda^{\mu\nu} \) are Lagrange multipliers and

\[ H_c = \partial_k A_0 F^{k0} + \partial_k A_m F^{km} - \frac{1}{2} F_{k0} F^{k0} - \frac{1}{4} F_{km} F^{km}. \]  

(85)

The proof that the Dirac procedure closes can be found in [26]. The first order formulation produces 14 constraints (2 first class and 12 second class). However, we will proceed differently by eliminating step by step the second class constraints that are of a special form.

The non-zero fundamental PB are

\[ \{ A_\mu, \pi^\nu \} = \delta^\nu_\mu, \{ F^{\rho\sigma}, \Pi_{\mu\nu} \} = \frac{1}{2} \left( \delta^\rho_\mu \delta^\sigma_\nu - \delta^\rho_\nu \delta^\sigma_\mu \right). \]  

(86)

It is not difficult to calculate the PB among primary constraints. The only non-zero brackets are

\[ \{ \phi^\mu, \Phi_{\rho\sigma} \} = -\frac{1}{2} \left( \delta_\rho^\mu \delta^0_\sigma - \delta_\rho^0 \delta^\mu_\sigma \right), \]  

(87)
so that there is a second class subset of primary constraints because

\[ \{ \pi^k - F^{k0}, \Pi_{0m} \} = \frac{1}{2} \delta^k_m. \]  

(88)

This subset is of a special form and allows one to solve these constraints leading to the reduced Hamiltonian (with a reduced number of variables) after substitution of

\[ \Pi_{0k} = 0, F^{k0} = \pi^k \]  

(89)

into the original Hamiltonian as well as into the remaining constraints. After the first stage of reduction, we have

\[ H^{(1)}_p = H^{(1)}_c + \lambda_0 \phi^0 + \Lambda^{km} \Phi_{km} \]  

(90)

where

\[ H^{(1)}_c = \partial_k A_0 \pi^k + \partial_k A_m F^{km} - \frac{1}{2} \pi_k \pi^k - \frac{1}{4} F_{km} F^{km}. \]  

(91)

We have now only seven independent fields and four primary constraints

\[ \phi^0 = \pi^0 \approx 0, \Phi_{km} = \Pi_{km} \approx 0. \]  

(92)

They have zero PB among themselves and conservation of these constraints now has to be considered. The conservation of the primary constraints gives rise to the secondary constraints \( \chi^0 \) and \( \chi_{km} \)

\[ \dot{\pi}^0 = \{ \pi^0, H^{(1)}_c \} = \{ \pi^0, -A_0 \partial_k \pi^k \} = \partial_k \pi^k = \chi^0 \]  

(93)

and

\[ \dot{\Phi}_{km} = \{ \Phi_{km}, H^{(1)}_c \} = -\frac{1}{2} (\partial_k A_m - \partial_m A_k) + \frac{1}{2} F_{km} = \chi_{km}. \]  

(94)

It is obvious that the constraints \( \chi_{km} \) constitute a second class subset of a special form with \( \Phi_{km} \) and, as was done at the previous stage, can be eliminated by solving them for \( \Pi_{km} \) and \( F^{km} \)

\[ \Pi_{km} = 0, F_{km} = (\partial_k A_m - \partial_m A_k). \]  

(95)

Upon substitution of (95) into \( H^{(1)}_p \), we obtain

\[ H^{(2)}_p = H^{(2)}_c + \lambda_0 \phi^0 \]  

(96)

with

\[ H^{(2)}_c = \partial_k A_0 \pi^k - \frac{1}{2} \pi_k \pi^k + \frac{1}{4} (\partial_k A_m - \partial_m A_k) (\partial^k A^m - \partial^m A^k) \]  

(97)
which is exactly the standard Hamiltonian in the second order formulation with only four fields and two first class constraints. This completes the proof of the equivalence between the two types of reduction at the pure Hamiltonian level. It is important to note that here we never ‘solve’ first class constraints, unlike what occurs in the EH action when treated using the ADM formalism and so the problem associated with ADM reduction do not arise.

IX. APPENDIX C

We perform Lagrangian reduction of the action of (9) in a way that is consistent with the Dirac procedure by eliminating only non-dynamical fields by solving equations of motion with respect to fields used in the variation that leads to these equations. The Lagrangian (9), after a complete separation of spatial and temporal components, can be presented as a sum of terms

\[ L_d = h^{\alpha \beta} \xi_0^\alpha \xi_0^\beta + L_1 \left( \xi_0^0 \xi_0^\alpha \right) + L_2 \left( \xi_0^k \xi_0^0 \right) + L_3 \left( \xi_0^k \xi_0^\alpha \xi_0^\beta \right) + L_4 \left( \xi_0^k \xi_0^0 \xi_0^\alpha \xi_0^\beta \right) \]  

(98)

where the purely dynamical part is

\[ L_1 = -\frac{d-2}{d-1} \left( h^{km} \xi_0^0 \xi_0^k + 2h^{0k} \xi_0^0 \xi_0^k + h^{00} \xi_0^0 \right). \]  

(99)

The term with the non-dynamical field \( \xi_0^k \) is

\[ L_2 = h^{00} \xi_0^k \xi_0^k - 2\xi_0^k \left( h^{00} \xi_0^k + h^{0p} \xi_0^p \right), \]  

(100)

while the term with the non-dynamical field \( \xi_0^k \) is

\[ L_3 = h^{km} \xi_0^k \xi_0^m + \frac{1}{d-1} h^{km} \xi_0^k \xi_0^p \xi_0^q + \frac{2}{d-1} \left( h^{mk} \xi_0^k + h^{0m} \xi_0^0 \right) \xi_0^q, \]  

(101)

and finally the term that is at least linear in non-dynamical field \( \xi_0^k \) is

\[ L_4 = 2h^{0m} \xi_0^k \xi_0^m - h^{00} \xi_0^0 \xi_0^k + \frac{1}{d-1} h^{00} \xi_0^p \xi_0^k \xi_0^q - 2h^{0k} \xi_0^p \xi_0^0 \xi_0^q + \frac{2}{d-1} h^{0k} \xi_0^p \xi_0^q \]  

(102)

\[-2h^{0k} \xi_0^p \xi_0^0 + \frac{2}{d-1} h^{0k} \xi_0^p \xi_0^0 + 2h^{km} \xi_0^p \xi_0^0 \xi_0^k + \frac{2}{d-1} h^{00} \xi_0^p \xi_0^0 \xi_0^k. \]

The 2D limit of (98) is obtained by setting \( d = 2 \) and putting all spatial indices equal to one, giving (25).

We have three “non-dynamical” fields among the fields \( \xi \) (i.e., fields that enter the Lagrangian without any time derivatives): \( \xi_0^k \), \( \xi_0^k \) and \( \xi_0^k \). The first field enters only linearly
and cannot be eliminated. (This term \(100\) corresponds to a first class constraint in the Dirac approach.) Variation of \(102\) with respect to \(\xi_{0b}\) gives

\[
-2h^{00}\xi_{0a} + \frac{2}{d - 1} h^{00}\xi_{0c}\delta_{a} = D_{a}^{0b} \left(h^{\alpha\beta}, \xi_{00}, \xi_{mn}\right).
\]

(103)

The left side of this equation is not invertible and not all components can be eliminated because the trace of the left side is zero. To preserve the tensorial character of variables in the reduced Lagrangian it is better to introduce an extra (pure auxiliary) field \(\theta\) by performing a change of variables in the following term

\[
\frac{1}{d - 1} h^{00}\xi_{0p}\xi_{0q} = \frac{1}{d - 1} h^{00}\xi_{0p}\theta - \frac{1}{4 d - 1} h^{00}\theta\theta.
\]

(104)

Introducing of this extra field \(\theta\) allows us to solve \(103\) for all components of \(\xi_{0m}\). Variation of \(102\), taking into account \(104\), gives

\[
\xi_{0a} = \frac{1}{h_{00}} \left[-h_{0b} - h^{0k}\xi_{ka} - h^{0k}\xi_{0a} - h^{mk}\xi_{a} + \frac{1}{d - 1} \delta_{a} \left(\frac{1}{2} h^{00}\theta + h^{0k}\xi_{kq} + h^{0k}\xi_{0k} + h^{00}\xi_{00}\right)\right].
\]

(105)

Substitution of this leads to the reduced Lagrangian (with the \((d - 1)^2\) components of \(\xi_{0m}\) completely eliminated)

\[
L_{d}^{(1)} = h^{\alpha\beta}\xi_{00}^{\alpha\beta} + L_{1}^{(1)}(\xi_{00}^{\alpha\beta}) + L_{2}^{(1)}(\xi_{00}^{\alpha\beta}) + L_{3}^{(1)}(\theta; \xi_{00}^{\alpha\beta}) + L_{4}^{(1)}(\xi_{mn}^{k}; \xi_{00}^{\alpha\beta})
\]

(106)

where now

\[
L_{1}^{(1)} = L_{1} - 2h^{0m} \left[\frac{1}{h_{00}} \left(h^{0k}_{,m} + h^{0k}_{,0m} + h^{mk}_{,0m}\xi_{mn}\right)\right]_{,k} + \frac{2}{d - 1} h^{0m} \left[\frac{1}{h_{00}} \left(h^{0k}_{,0m} + h^{00}_{,0m}\xi_{00}\right)\right]_{,m}
\]

(107)

\[
L_{2}^{(1)} = L_{2};
\]

(108)

\[
L_{3}^{(1)} = \frac{1}{d - 1} \left(h^{0m}_{,m} - \theta h^{km}_{,0m} + \theta h^{0m}_{,00}\xi_{00}\right);
\]

(109)

\[
L_{4}^{(1)} = h^{mn}_{,mn,k} + e^{km}_{,smp} - e^{km}_{,smp} h^{0m}_{,00}\xi_{sm}, q + \frac{1}{d - 1} e^{km}_{,smp} - 2h^{0m} \left(h^{0p}_{,00}\delta_{mp}\right)_{,k} - \frac{2}{d - 1} h^{0m} \left(h^{0p}_{,00}\xi_{pq}\right)_{,m}
\]

(110)
Note, that the terms quadratic in $\theta$ have cancelled out and that the terms linear in $\theta$ lead to the additional first class constraint (109). The terms quadratic in $\xi^k_{mn}$ are multiplied by

$$e^{km} = h^{km} - \frac{h^{0k}h^{0m}}{h^{00}}$$  \hspace{2cm} (111)

where $e^{km}$ has the property: $e^{km}h_{mn} = \delta^k_n$. We can now reduce the Lagrangian further by eliminating $\xi^a_{km}$. The variation of (110) with respect to $\xi^a_{ka}$ gives

$$e^{kb}\xi^c_{ka} + e^{kc}\xi^b_{ka} = \frac{1}{d-1}\xi^p_{kp} \left( e^{kb}\delta^c_a + e^{kc}\delta^b_a \right) = D^b_a$$  \hspace{2cm} (112)

where

$$D^b_a \equiv -\frac{1}{2}h^b_{,a} + \frac{h^{0b}}{h^{00}}h^0_{,a} + \frac{h^{0b}h^{0c}}{h^{00}}s^0_{a} + \frac{h^{kb}h^{0c}}{h^{00}}\xi^0_{ka}$$  \hspace{2cm} (113)

$$+ \frac{1}{d-1}e^{kb}\xi^0_{0a} = \frac{1}{d-1}h^{0b} \left( h^{0m} + h^{km}\xi^0_{km} - h^{00}\xi^0_{00} \right) \delta^b_0 + (b \leftrightarrow c).$$

The solution of (112) is similar to solution of (73) appearing in Appendix A. Multiplying (112) by $h^{bc}_{rb}h^{sc}_{sa}$ and performing a permutation of the indices $r, s, a$ (as in (75)) and then multiplication by $\frac{1}{2}e^{ns}$, we obtain the solution for $\xi^a_{sa}$

$$\xi^a_{sa} = \frac{1}{2}h^{rb}\tilde{D}^bn_a + h_{ab}\tilde{D}^bn_r - e^{ns}h_{ab}h_{re}\tilde{D}^bc_s$$  \hspace{2cm} (116)

or in terms of (113)

$$\xi^a_{sa} = \frac{1}{2} \left[ h^{rb}D^bn_a + h_{ab}D^bn_r - e^{ns}h_{ab}h_{re}D^bc_s + \frac{1}{d-2}h_{pq}D^pq_{ka}e^{km} \right].$$  \hspace{2cm} (117)

Substitution of (117) back into the Lagrangian $L^{(1)}_d$ produces the reduced Lagrangian with the non-dynamical fields $\xi^k_{mn}$ all absent. We can use this reduced Lagrangian to pass to a Hamiltonian formulation which is different from the ADM-reduced formulation, as all components of $\xi^0_{a\beta}$ are still present (and not only its spatial components). This should lead, in principle, to a restoration of full gauge invariance that will involve all components of $g^{\alpha\beta}$. The solution of the equations of motion corresponding to the first class constraints in the
Dirac approach leads to a non-equivalent reduced Lagrangian with a loss of the possibility of restoring full gauge invariance of the initial Lagrangian. This is illustrated for \( d = 2 \) in Sec.4. We note that our elimination of the non-dynamical variables produces an alternative formulation of the Einstein-Hilbert action that is linear in time derivatives of the dynamical fields and well suited for application of the standard Dirac procedure.

The Dirac analysis applied to the reduced Lagrangian \( L_d^{(1)} \) gives (as in Sec.2) the second class primary constraints

\[
\pi_{\alpha\beta} \approx 0, \quad \Pi_{0}^{\alpha\beta} - \sqrt{-g} g^{\alpha\beta} \approx 0
\]

as well as two first class primary constraints

\[
\Pi_{k}^{00} \approx 0, \quad \pi \approx 0,
\]

where \( \Pi_{k}^{00}, \pi \) are momenta conjugate to fields \( \xi_{k0}, \theta \). Conservation of the constraints of (118) in time (using (100, 109)) leads to secondary constraints which are equivalent to (21).

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