Field theory formulation of the Quantum Hall Effect

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Abstract

We consider the quantum Hall effect in terms of an effective field theory formulation of the edge states, providing a natural common framework for the fractional and integral effects.

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There has been a lot of interest in the last years on the Quantum Hall Effect (QHE)\(^1\). The integral effect (IQHE) has been given simple explanation\(^2\), but the fractional effect (FQHE) is more involved and requires the concept of collective behavior of the electrons\(^3\). In particular, Jain\(^4\) used a unified scheme in order to explain both effects, since the explanation of the integral effect was believed to be described in terms of non-interacting electrons\(^1\), while the fractional effect includes the above mentioned collective phenomenon. The idea is to introduce copies of the electron, and consider condensations of such fields\(^4\).

Further understanding can be achieved once one verifies that chirality plays an essential role in the explanation of the Hall effect, as stressed recently\(^5\),\(^6\),\(^7\),\(^8\). Our purpose here is to provide a field theoretical approach to the quantum Hall effect, in such a way that the collective electron phenomenon is naturally accommodated, being just a consequence of the description of electrons squeezed in a small space region, and the chiral nature of the interaction\(^5\)\(^−\)\(^8\), leading to chiral anomaly, which together with the soliton behavior inherited from the collective effect, naturally accommodates the fractional effect, and gives, as a byproduct, the integral effect as a particular case. We refrain from reviewing the subject, referring to the vast literature\(^1\),\(^9\). However, we have to explicitly quote some results which will be used in the following, as well as refer to as a check of the correctedness of the procedure.

We start out of the Lagrangean density of four-dimensional QED, which reads

\[
\mathcal{L} = \bar{\Psi}_i \Gamma_\mu \partial_\mu \Psi_i + e \bar{\Psi}_i \Gamma^\mu A_\mu \Psi_i \quad i = 1, \ldots, N \quad ,
\]

where the gamma matrices \(\Gamma_\mu\) and the \(\Psi\) field are four-dimensional and read

\[
\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad \Gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{pmatrix} \quad , \quad \Psi = \begin{pmatrix} \chi \\ \eta \end{pmatrix} \quad ,
\]

\(\sigma^a\) are the usual Pauli matrices with \(a = 1, 2, 3\) and \(\chi\) and \(\eta\) are bispinors. In this notation the Lagrangian is

\[
\mathcal{L} = \chi^+ \partial_0 \chi + \eta^+ \partial_0 \eta - \eta^+ \sigma^a \partial_a \eta - \chi^+ \sigma^a \partial_a \chi + \chi^+ \chi A_0 + \eta^+ \eta A_0 - e(\chi^+ \sigma^a \chi + \eta^+ \sigma^a \eta)A_a \\
= \bar{\chi} \partial \chi + \bar{\eta} \partial \eta + (\bar{\chi} \gamma^\mu \chi + \bar{\eta} \gamma^\mu \eta)A_\mu + \chi^+ \sigma_M \partial_M \chi + \eta^+ \sigma_M \partial_M \eta \\
= \bar{\psi}_i \partial \psi_i + \bar{\psi}_i \gamma^\mu \psi_i A_\mu + M\text{-derivatives} \quad , \quad I = 1, 2 \text{ and } M = 2, 3 \quad .
\]

In the above we have used the following notation: The spinor field \(\psi_{I=1}\) stands for \(\chi\), the first bispinor, while \(\psi_{I=2}\) corresponds to \(\gamma_1 \eta(t, -\vec{x})\), and the two-dimensional gamma matrices are \(\gamma_0 = \sigma^3\), \(\gamma_1 = \sigma^2\), and \(\gamma_5 = \gamma_0 \gamma_1\); the slash derivative is \(\partial = \gamma^\mu \partial_\mu\). Therefore, for \(\Gamma_\mu\) the index \(\mu\) goes from 0 to 3, while for \(\gamma_\mu\) it takes the values 0,1. We neglect the derivatives in the 2 and 3 directions \((M)\), considering only the two-dimensional problem.
Integrating over the gauge field $A_\mu(\mu = 0, \ldots, 3)$ we get

$$\mathcal{L} = \bar{\psi}_j i \partial_\psi_j + \sum_{\mu, \nu = 0}^1 \bar{\psi}_j \gamma^\mu \psi_j D_\mu \bar{\psi}_k \gamma^\nu \psi_k + \bar{\psi}_j \gamma^5 \psi_j D_2 \bar{\psi}_k \gamma^5 \psi_k + \bar{\psi}_j \psi_j \psi_k \psi_k + \bar{\psi}_j \gamma^5 \psi_j D_2 \bar{\psi}_k \gamma^5 \psi_k ,$$

(4)

where $j, k = 1, \ldots, 2N$ and $D_\mu$ (for all the indices $\mu, \nu = 0, \ldots, 3$), is the gauge field propagator and may be approximated by an effective constant in the strong coupling regime. We are left with the effective Lagrangean

$$\mathcal{L} = \bar{\psi}_j i \partial_\psi_j + g_{\text{eff}} \bar{\psi}_j \gamma^\mu \psi_j \bar{\psi}_k \gamma_\mu \psi_k + g_{\text{eff}} \left[ (\bar{\psi}_j \psi_j)^2 - (\bar{\psi}_j \gamma_5 \psi_j)^2 \right] ,$$

(5)

where the effective current vertex corresponding to $\mu = 2$ in two dimensions is $\bar{\psi} \gamma_5 \psi$ and for $\mu = 3$, it is $\bar{\psi} \psi$. We find also a trivial Thirring interaction, which is neglected in the following. The electromagnetic field is responsible for the internal interactions in the crystal, and for the formation of the two-dimensional layer. Now the effective two-dimensional interaction is the chiral Gross-Neveu (CGN) theory, which has been much studied in the literature (see [10] and references therein). Since the electromagnetic field has been used in order to form the system, we shall for the time being disregard its interaction. Later we have to reconsider it, in order to know the interaction of the sample with the external fields responsible for the Hall effect. As far as the CGN interaction is concerned, we recall the following facts. The solution of such a theory is given in terms of solitons $\hat{\psi}_j$, which satisfy the non-linear relation

$$\hat{\psi}_j^+ \sim \epsilon_{jj_1 \cdots j_2 N - 1} \hat{\psi}_j \cdots \hat{\psi}_j \hat{\psi}_j \cdots ,$$

(6)

where on the right hand side a suitable redefinition of the Klein factor and the normal product prescription are required.

Such solitons are chargeless fields, since the $U(1)$-charge has been separated in order that the above relation be valid. In fact, the gauge interaction has been used up in the quantum gauge degrees of freedom, in order that the two-dimensional infra-red behavior does not cause problems. As shown by Wen, the excitations responsible for the QHE contain several branches. In fact, the excitations in each branch carry a fractional charge. Such edge excitations form the Fermi liquid. However, as stressed by Wen, the fermions in the Fermi liquid are not the electrons, but describe the edge excitation, a possibility opened by the boson/fermion duality in two dimensions. This means that the Fermi liquid is built of solitons, described by generalized statistics.

The next procedure to be followed concerns the interaction of the solitons with the external gauge fields. The gauge interaction of the degenerate solitons displaying an apparent $SU(2N - 1)$ symmetry, with the gauge field is fundamentally important in order to explain the effect. The discussion of gauge invariance in this context is given in [15]. From the fact that we have a bound state structure, or collective approach of the type pictured by eq. (6), we can describe the physical electron field by the l.h.s. of (6) times the $U(1)$ factor ($\psi_j = e^{i\chi} \hat{\psi}_j$), since it is such field which has physical interpretation. This means that we can distribute the charge equally among the soliton fields, and try to find an
effective Lagrangean which displays the desired features of the Hall effect. As a pure two-dimensional theory, the chiral Gross Neveu model presents fields with generalized statistics. As a four-dimensional theory we interpret its meaning as the fact that only bound states of the form (6) describe the physical electron. Therefore, we suppose that the physical electron is of the above form, multiplied by the convenient exponential representing the \( U(1) \) charge, that is

\[
\left( \hat{\psi} e^{-i\chi/n} \right)^+ = e^{i\chi} \epsilon_{jj_1\ldots j_n} \left( \hat{\psi}_{j_1} e^{-i\chi/n} \right) \cdots \left( \hat{\psi}_{j_n} e^{-i\chi/n} \right),
\]

and each of the solitons carries a charge \( \frac{e}{n} \). Such a charge can be traced back to the incompressibility, as discussed by Laughlin, due to the Schieffer counting argument \(^{16}\).

We have started with \( N \) fields. In the reduction scheme, due to the fact that four-dimensional electrons have four components, while two-dimensional electrons have two, there is a doubling in the number of components. Therefore the number of independent fields is \( n = 2N - 1 \). This mirrors the relativistic content of the model, that is, although the spin does not play a direct role in the QHE, it enters through the above doubling of components. However, the incompressible Fermi fluid picture of Laughlin, has a consequence that the excitations are anyons with fractional statistics and charge \(^{13,14}\).

We now come to the effective action describing the solitonic fields. It may be obtained by the minimal coupling of such fields with the external electromagnetic field as

\[
\mathcal{L}_{eff} = \bar{\psi}_j i \not{\partial} \psi_j + \bar{\psi}_j \gamma^\mu A_\mu \psi_j + \frac{\hat{a}e^2}{4\pi} A_\mu A^\mu + (\psi - \text{self interactions}),
\]

where we followed \(^{[5,6]}\) and introduced the interaction of the electromagnetic field with the current of definite chirality, that is \( \bar{\psi}_j \gamma^\mu P_- \psi_j \equiv \bar{\psi}_j \gamma^\mu \frac{1-\gamma_5}{2} \psi_j \). Above, the charge of the soliton is \( \hat{e} = \frac{e}{2N - 1} \), and \( \hat{a} = (2N - 1) a \) represents the regularization ambiguity \(^{17}\) (see below). In the QHE chirality plays a fundamental role. It is well known that chiral gauge interactions are anomalous \(^{10,17,18}\), and two-dimensional chiral QED with anomalous breakdown of gauge invariance \(^{17}\) admits exact solutions in a positive metric Hilbert space respecting unitarity, provided the parameter \( a \), defined above (see \(^{[17]}\)) is restricted to \( a \geq 1 \). In fact, the JR \(^{17}\) term \( ae^2A_\mu^2 \) takes account of the ambiguity in the regularization procedure resulting from the lack of gauge invariance. It may also be obtained as the massless limit of the Proca theory \(^{10}\).

This is our proposal for the effective field theory Lagrangean to describe the QHE. Therefore we arrive at the chiral \( QED_2 \) interaction, together with chiral Gross-Neveu type self-interaction for the Fermi fields, as well as the constraint (6). Since the essence of the chiral Gross-Neveu self-interaction is to imply the relation (6) characteristic of the two-dimensional theory, while the Hall effect itself depends on the relation between the current and the external gauge field, we simplify matters, considering chiral \( QED_2 \) as the model for the QHE, together with relation (6). (In fact, in the presence of impurities the free electron picture holds true\(^{13,14}\).)

An effective bosonic theory is obtained by means of the computation of the fermionic determinant in two dimensions, through the Polyakov-Wiegman formula \(^{19}\). In such a case,
integration over the fermions leads to an external gauge field dependent partition function of the form

\[ e^{iW[A]} = e^{i \int d^2 x \frac{\hat{e}^2}{4 \pi} A^2 \int Dh \, e^{i \gamma_1[A,h^{-1}]}} , \tag{9} \]

where \( h \) is a bosonic field, \( L \) means that we are considering the interaction of the left moving currents with the gauge field as in (8) and \( \gamma_1[A,h^{-1}] \) is the effective bosonic action satisfying a 1-cocycle condition. It is defined in terms of the WZW functional as

\[ \gamma_1[A,h] = \Gamma[hV^{-1}] - \Gamma[V^{-1}] , \tag{10a} \]

\[ = \Gamma[h] - \frac{i \hat{e}}{4 \pi} \int d^2 x \, A^\mu h^{-1} (\partial_\mu - \tilde{\partial}_\mu) h \quad , \tag{10b} \]

where the WZW functional is given by the expression

\[ \Gamma[h] = \frac{1}{8 \pi} \int d^2 x \, tr \partial_\mu h^{-1} \partial^\mu h - \frac{i \hat{e}}{4 \pi} \int dr \int d^2 x \, \tilde{h}^{-1} \partial_r \tilde{h}^{-1} \partial_\mu \tilde{h}^{-1} \partial_r \tilde{h} . \tag{10c} \]

In the above we chose \( A_- = \frac{i \hat{e}}{\hat{e}} V^{-1} \partial_- V \) and \( A_+ = 0 \). In the WZW functional \( \Gamma[h] \) the general \( \tilde{h}(r,x) \) field is an extension of the field \( h(x) = \tilde{h}(1,x) \) to a manifold containing the Minkowski space as a boundary with \( \tilde{h}(0,x) = 1 \). In the abelian case \( (h_j = e^{2i \phi_j}) \) the effective action boils down to†

\[ \gamma_1[A,\phi_j] = \int d^2 x \, \frac{1}{2} \sum_j (\partial_\mu \phi_j)^2 + \frac{\hat{e}}{2 \pi} \int d^2 x \, \sum_j A^\mu (\partial_\mu - \tilde{\partial}_\mu) \phi_j . \tag{11} \]

Thus the bosonized form of two-dimensional chiral QED with flavour reads

\[ \mathcal{L} = \frac{1}{2} \sum_j (\partial_\mu \phi_j)^2 - \frac{1}{4} F_{\mu \nu}^2 + \frac{\hat{e}}{2 \pi} A^\mu \sum_j (\partial_\mu - \tilde{\partial}_\mu) \phi_j + \frac{\hat{e} \hat{e}}{4 \pi} A_\mu A^\mu . \tag{12} \]

We could also obtain this action with the adiabatic principle of form invariance. The equations of motion obtained from (12) are

\[ \partial_\mu F^{\mu \nu} + \frac{1}{2 \pi} \sum_i (\partial^\nu - \tilde{\partial}^\nu) \phi_i + \frac{\hat{e}}{2 \pi} A^\nu = 0 , \tag{13} \]

\[ \Box \phi_i + \frac{\hat{e}}{2 \pi} (\partial_\mu - \tilde{\partial}_\mu) A^\mu = 0 . \]

Consider now the (classically conserved) currents

\[ J^{\mu}_L = \frac{\hat{e}}{2 \pi} \sum_j (\partial_\mu - \tilde{\partial}_\mu) \phi_j + \frac{\hat{e} \hat{e}}{2 \pi} A^\mu , \tag{14} \]

\[ J^{\mu}_{iR} = -(g^{\mu \nu} + \epsilon^{\mu \nu}) (\partial_\nu \phi_i - \hat{e} A_\nu) . \]

† As a matter of fact, we are describing a torus \( U(1)^{2N-1} \).
In the quantum theory we have conservation of the right-moving current $J_{iR}^\mu$, as expected,
\[
\partial_\mu J_{iR}^\mu = 0 ,
\]
but the left current is anomalous! Indeed, we have
\[
\partial_\mu J_L^\mu = (2N - 1) \frac{e^2}{2\pi} \left[ (a - 1)\partial_\mu + \tilde{\partial}_\mu \right] A^\mu .
\]

The fact that the “right” current is conserved at the quantum level, while the “left” current is anomalous is a very “efficient” description of the physics of the QHE, and may also be related to its description in terms of chiral bosons interacting with matter fields\(^5\,^6\). In such a case, we also have higher algebras describing the theory\(^2^1\).

The question whether the anomalous conservation is compatible or not with the equation of motion has been answered in a series of papers (see [10]) for an extensive list). In the case we are considering a functional integration over the gauge field $A_\mu$, i.e., if the electromagnetic field is fully quantized, the r.h.s. of (16) vanishes. Here we are considering an external gauge field - the one originating the Hall effect - therefore the r.h.s. of (16) does not vanish, but there is no contradiction with the equations of motion. For the applications on the QHE we have an ($2N - 1$)-plet of fields, therefore
\[
\partial_\mu J_L^\mu = (2N - 1) \frac{e^2}{2\pi(2N - 1)} \left[ (a - 1)\partial_\mu + \tilde{\partial}_\mu \right] A^\mu ,
\]
(17)

Since $\tilde{\partial}_\mu A_\mu = \epsilon_{\mu\nu} \partial^\nu A_\mu = -\frac{1}{2} \epsilon_{\mu\nu} F^{\mu\nu}$, we read the Hall conductivity from the anomaly coefficient. The first term, namely $(a - 1)\partial_\mu A_\mu$ is a pure two-dimensional effect, and does not appear if the external gauge field describes the $\vec{E}, \vec{B}$ usual system. Therefore, the (minimal) Hall conductivity is given by the coefficient of the anomaly\(^2^2\),
\[
\sigma_H = \frac{e^2}{2\pi(2N - 1)} .
\]
(18)

We conclude remarking once more that the current explanation of the quantum Hall conductance is naturally accomodated in terms of the two-dimensional field theoretic language. The fact that we have a two-dimensional relativistic field theory instead of a three-dimensional (non-relativistic) theory has been discussed elsewhere\(^2^3\). The description of this phenomenon in terms of a conformal field theory has also been developed in references [24, 25, 26].

We should also quote that copies of a $U(1)$ Kac-Moody algebra are obtained by the currents $J^R_\mu$. Indeed,
\[
\left[ J^{R\mu}_i(t, x), J^{R\nu}_j(t, y) \right] = \frac{1}{2\pi} \epsilon^2 \delta_{ij} \delta'(x - y) ,
\]
(19)
which should be compared to [12]. Moreover, the $W_\infty$ algebra obtained for the non-relativistic electron gas description\textsuperscript{23} can be understood as the algebra of higher spin chiral current studied is [27] for the abelian case, and in [28] in the non-abelian case. Such higher-dimensional algebras underline the models considered\textsuperscript{21}.

Although we did not discuss the issue of the hierarchy of the different filling factors, it should be clear at this point that once we have gotten (18), and consequently the simplest filling factor $\nu = 1/3$ (for $N = 2$), the arguments used in [29] now apply.

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