FAMILIES OF DIVISORS

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Abstract. We establish a new moduli theory for divisors, that interpolates between the Hilbert scheme and the Cayley-Chow variety. This completes the last step in the construction of a good moduli theory for stable pairs $(X, \Delta)$.

A persistent problem in the moduli theory of pairs $(X, \Delta)$ is that, while the underlying varieties form flat families, the divisorial parts $\Delta$ do not. Neither of the two main traditional methods of parametrizing varieties or schemes gives the right answer for the divisorial part.

- Hilbert schemes take into account embedded points, but we need to ignore them entirely.
- Cayley-Chow varieties work well only over seminormal schemes, but we wish to have a theory over arbitrary base schemes.

Our aim is to develop a theory that interpolates between these two, managing to keep from both of them the properties that we need. The definition of Mumford divisors codifies basic properties of the divisorial part of families of stable pairs, though with a new name introduced in [Kol18]. The main new result is Definition 2. The rest of the paper is then devoted to proving that it has all the hoped-for properties.

**Definition 1** (Mumford divisors). Let $f : X \to S$ be a flat morphism with $S_2$ fibers of pure dimension $n$. A subscheme $D \subset X$ is a relative Mumford divisor or a generically flat family of Mumford divisors over $S$ if there is an open subset $U \subset X$ such that

1. $\text{codim}_{X_s} (X_s \setminus U) \geq 2$ for every $s \in S$,
2. $D|_U$ is a relative Cartier divisor,
3. $D$ is the closure of $D|_U$ and
4. $X_s$ is smooth at generic points of $D_s$ for every $s \in S$.

Contents

1. Infinitesimal study of Mumford divisors 6
2. Divisorial support 13
3. Variants of K-flatness 18
4. Cayley-Chow flatness 21
5. Ideal of Chow equations 27
6. Representability Theorems 31
7. Hypersurface singularities 36
8. Seminormal curves 39
References 47
If $U \subset X$ denotes the largest open set with the properties (1–2), then $U$ is called the Cartier locus and $Z := X \setminus U$ the non-Cartier locus of $D$.

Let $q : W \to S$ be any morphism. We have a fiber product diagram

$$
\begin{array}{ccc}
X_W & \xrightarrow{q_X} & X \\
\downarrow f_W & & \downarrow f \\
W & \xrightarrow{q} & S
\end{array}
$$

(1.5)

Then $q_X^*(D|U)$ is a relative Cartier divisor on $U_W := q_X^{-1}(U)$ and its closure is a relative Mumford divisor, called the divisorial pull-back of $D$ by $q$. It is denoted by $q^*D$ or simply by $D_W$. If $q$ is flat then $q^*D = q_X^*D = W \times_S D$.

**Definition 2** (K-flatness). Let $f : X \to S$ be a flat, projective morphism with $S_2$ fibers of pure dimension $n$. A relative Mumford divisor $D \subset X$ is K-flat over $S$ iff one the following—increasingly more general—conditions hold.

1. (S local with infinite residue field) For every finite morphism $\pi : X \to \mathbb{P}^n_S$, $\pi_*D \subset \mathbb{P}^n_S$ is a relative Cartier divisor.
2. (S local) $q^*D$ is K-flat over $S'$ for some (equivalently every) flat, local morphism $q : S' \to S$, where $S'$ has infinite residue field.
3. (S arbitrary) $D$ is K-flat over every localization of $S$.

Let us start with some comments on the definition.

4. Here K stands for the first syllable of Cayley. We use C-flat for a closely related (possibly equivalent) notion; see (52).
5. The definition of $\pi_*D$ is not always obvious; in essence Section 2 is entirely devoted to establishing it. However, $\pi_*D$ equals the scheme-theoretic image of $D$ if $\text{red} D \to \text{red}(\pi(D))$ is birational and $\pi$ is étale at every generic point of every fiber $D_s$ (44.2). It is sufficient to check condition (1) for such morphisms $\pi : X \to \mathbb{P}^n_S$.
6. If $S$ is not local then there may not be any finite morphisms $\pi : X \to \mathbb{P}^n_S$ (18); this is one reason for the 3 step definition.
7. The infinite residue field extensions in (2) are necessary in some cases; see (116.9).
8. The definition of K-flatness is global in nature, but we show that it is in fact étale local on $X$ (66).

**Good properties of K-flatness.**

K-flat families have several good properties. Some of them are needed for the moduli theory of stable pairs, but others, for example (7–9), come as bonus.

**3 (Functoriality).** Being K-flat is preserved by arbitrary base changes and it descends from faithfully flat base changes (64). Thus we get the functor $K\text{Div}(X/S)$ of K-flat, relative Mumford divisors on $X/S$. If we have a fixed relatively ample divisor $H$ on $X$, then $K\text{Div}_d(X/S)$ denotes the functor of K-flat, relative Mumford divisors of degree $d$.

We have a disjoint union decomposition $K\text{Div}(X/S) = \cup_d K\text{Div}_d(X/S)$.

The main result is the following, to be proved in (98).

**Theorem 4.** Let $f : X \to S$ be a flat, projective morphism with $S_2$ fibers of pure dimension $n$. Then the functor $K\text{Div}_d(X/S)$ of K-flat, relative Mumford divisors of degree $d$ is representable by a separated $S$-scheme of finite type $K\text{Div}_d(X/S)$.
Complement 4.1. KDiv/d(X/S) is proper over S (and non-empty) iff the fibers of f are normal. This is, however, not a problem for the moduli of stable pairs.

5 (Comparison with flatness). K-flatness is a generalization of flatness and it is equivalent to it for smooth morphisms.

(1) If D is K-flat and f is smooth at x ∈ X, then f|D is flat at x. Equivalently, D is a relative Cartier divisor in a neighborhood of x; see (67).

(2) If f|D is flat at x then f is K-flat at x; see (68).

In particular, the notion of K-flatness gives something new only at the points where f is not smooth and f|D is not flat.

6 (Reduced base schemes). If S is reduced then every relative Mumford divisor is K-flat; see (45), which in turn follows directly from [Kol17, 4.36]. In retrospect, this is the reason why the moduli theory of pairs could be developed over reduced base schemes without this notion in [Kol17, Chap.4].

So in practice the main task is to understand K-flatness over Artin base schemes S. This takes care of the general case since D ⊂ X is K-flat over S iff DA ⊂ DA is K-flat over A for every Artin subscheme A ⊂ S; see (59).

Theorem 7 (Bertini theorems, up and down). Assume that n ≥ 2 and let |H| be a basepoint-free linear system on X. Then D is K-flat iff D|H is K-flat for general H ∈ |H|.

This is established by combining (71–72) with (55). As a consequence, K-flatness is really a question about families of surfaces and curves on them. There are similar theorems for families of stable pairs, see [Kol17, Chaps.2 and 5] or the original papers [Kol13a, BdJ14, Kol16a].

This reduction to surfaces is very helpful conceptually, but also computationally since we have rather complete lists of singularities of log canonical surface pairs (X, ∆), at least when the coefficients of ∆ are not too small.

Another variant of the phenomenon, that higher codimension points sometimes do not matter much, is the Hironaka-type flatness theorem [Kol17, 10.68], which is a generalization of [Hir58]; see also [Har77, III.9.11].

8 (K-flatness does not depend on X). It is well understood that in the theory of pairs (X, ∆) one can not separate the underlying variety X from the divisorial part ∆. For example, if X is a surface with quotient singularities only, then the pair (X, D) is plt for some smooth curves D ⊂ X but not even lc for some other cases. It really matters how exactly D sits inside X.

Thus it is unexpected that K-flatness depends only on the divisor D, not on the ambient variety X, though maybe this is less surprising if one thinks of K-flatness as a variant of flatness.

On the other hand, not all K-flat deformations of D are realized in a given X. For example, if we start with (A^2, (xy = 0)) then every K-flat deformation of the pair induces a flat deformation of D_1 = (xy = 0) ⊂ A^2. If we have ((xy = z^2), (z = 0)) then there are K-flat deformations of the pair that induce a non-flat deformation of D_2 = (xy − z^2 = z = 0) ≅ D_1.

9 (Push-forward). Let f : X → S and g : Y → S be flat, projective morphisms with S fibers of pure dimension n and τ : X → Y a finite morphism. Let D ⊂ X be a K-flat relative Mumford divisor such that τ∗D is also a relative Mumford divisor.
(That is, none of the irreducible components of $D_s$ is mapped to $\text{Sing}(Y_s)$.) Then $\tau_* D$ is also $K$-flat. (See (2.7) and Section 2 for the correct definition of $\tau_* D$.)

**Application to moduli spaces of pairs.**

The construction of the moduli space of stable pairs given in [Kol17, Sec.4.9] relies on a suitable moduli theory of divisors. In characteristic 0, all restrictions on bases schemes come from the moduli theory of divisors. In [Kol17, Sec.4.7] we used Cayley-Chow theory, which was at that time worked out for seminormal base schemes, though later it was extended to reduced base schemes.

Using (4) and the methods of [Kol17, Chap.4], we get a moduli theory of stable pairs over arbitrary base schemes in characteristic 0.

**Definition 10.** A family of stable pairs is a morphism $f : (X, cD) \rightarrow S$ where

1. $f : X \rightarrow S$ is flat and projective,
2. $D$ is a $K$-flat family of divisors on $X$,
3. $K_X + cD$ is $\mathbb{Q}$-Cartier, relatively ample and
4. the fibers $(X_s, cD_s)$ are semi-log-canonical.

The basic invariants of the fibers are the dimension $n = \dim X_s$ and the volume $(K_{X_s} + cD_s)^n$.

The role of the coefficient $c$ is murkier. It is well understood that, in order to get finite dimensional moduli spaces, one needs to control the coefficients of the divisorial part of stable pairs $(X, \Delta)$; see for example [Kol17, 4.68]. Fixing $c$ is one of the easiest way to ensure this control.

If we work with $\mathbb{Q}$-divisors $\sum a_i D_i$, then a convenient choice of $c$ is the reciprocal of the least common denominator of the $a_i$. Thus $D := c^{-1}(\sum a_i D_i)$ is a $\mathbb{Z}$-divisor.

Fixing $c$ leads to the largest moduli spaces. In practice one may want to impose additional restrictions, handle the different divisors $D_i$ differently and allow real coefficients as well. One way to achieve these is the notion of marked pairs [Kol17, Sec.4.7].

For now we focus on the most general form of the basic existence theorem.

**Theorem 11.** Fix constants $n, c, v$ and work with schemes over $\mathbb{Q}$. Then the functor of families of stable pairs $f : (X, cD) \rightarrow S$ of dimension $n$ and volume $v$ has a coarse moduli space $\text{SP}(n, c, v)$ that is projective over $\mathbb{Q}$.

This theorem represents the culmination of the work of several decades; some of the main contributions are [KSB88, Kol90, Ale94, Ale96, Kol13b, Kol16b, KP17, Fuj18, HMX18]. Many parts of the proof work in positive characteristic, and even over $\mathbb{Z}$, but there are several fundamental unsolved questions.

I hope—but do not claim—that $K$-flatness gives the ‘optimal’ moduli theory for divisors. By ‘optimal’ I mean that

- it is defined over arbitrary schemes,
- it agrees with the notion of Mumford divisors over reduced schemes,
- it leads to moduli spaces that include all families that one would wish to consider.

$K$-flatness satisfies the first 2 and it has surprisingly nice additional features. Once its basic properties are established, it is quite easy to work with, since we can mostly ignore singularities that occur in codimension $\geq 3$. On the other hand, it is possible that ignoring codimension $\geq 3$ means that we allow too many infinitesimal deformations and there is a better, more restrictive theory.
Problems and questions about K-flatness.

There are also some difficulties with K-flatness. I believe that they do not effect the general moduli theory of stable pairs, but they may make explicit computations lengthy.

12 (Hard to compute). The definition of K-flatness is quite hard to check, since for \( X \subset \mathbb{P}^N \) we need to check not just linear projections \( \mathbb{P}^N_S \to \mathbb{P}^n_S \) (51) but all morphisms \( X \to \mathbb{P}_S^n \) involving all linear systems on \( X \).

On the other hand, at least in the examples in Sections 7–8, the computation of the restrictions imposed by linear projections is the hard part, the general cases then follow easier. It would be good to work out more space curves \( C \subset \mathbb{A}^3 \).

13 (Tangent space and obstruction theory). I do not know how to write down the tangent space of \( \text{KDiv}(X/S) \). A handful of examples are computed in Sections 7–8, but they do not seem to suggest any general pattern. The obstruction theory of K-flatness is completely open.

14 (The definition is not formal-local). One expects K-flatness to be a formal-local property on \( X \), but there are some (hopefully only technical) problems with this. See (56) and (76) for partial results.

Over a DVR, every K-flat deformation of a variety \( X \) is a flat deformation of some scheme \( X' \) such that \( \text{red} X' = X \). By (5.1), the torsion subsheaf of \( \mathcal{O}_{X'} \) is supported on \( \text{Sing} X \). It would be good to get a good a priori bound on the size of \( \text{tors} \mathcal{O}_{X'} \).

Question 15 (Bounding the torsion). Let \((A, m, k)\) be an Artin scheme and \( C_A \to \text{Spec} A \) a K-flat deformation of a pointed curve \((c, C)\) that is flat on \( C \setminus \{c\}\). Let \( C_k \) be the central fiber and \( I = \ker[\mathcal{O}_{C_A} \to \mathcal{O}_C] \). Thus \( \text{tors}_c C_k = I/m\mathcal{O}_{C_A} \) (and \( C_A \to \text{Spec} A \) is flat iff \( \text{tors}_c C_k = 0 \) by (24)).

What is the best bound for \( \text{tors}_c C_k \), depending only on \( C \)?

We are always interested in divisors that lie on a particular family of varieties \( X \to S \), but, in view of (8), the following seems also natural.

Question 16 (Universal deformation spaces). Let \( D \) be a reduced, projective scheme over a field \( k \). Is there a universal deformation space for its K-flat deformations?

Examples.

The first example shows that the space of first order deformations of the smooth divisor \((x = 0) \subset \mathbb{A}^2\), that are Cartier away from the origin, is infinite dimensional. Thus working with generically flat divisors does not give a sensible moduli space.

Example 17. For \( g \in k[x, y] \) consider the ideal

\[ I_g = (x^n, xy + cg, cx) \subset k[x, y, \epsilon] \]

and set \( D_g = \text{Spec} k[x, y, \epsilon]/I_g \).

It is easy to check directly that

1. \( D_g \) is Cartier away from the origin,
2. \( (I_g, \epsilon)/(\epsilon) = (x) \cap (x^n, y) \),
3. \( D_g \) has no embedded points iff \( g \notin (x^n, xy) \) and
(4) $D_{g_1} = D_{g_2}$ if $g_1 - g_2 \in \langle x^n, xy \rangle$.

More general computations are done in (33).

**Example 18.** Let $C$ be a smooth projective curve and $E$ a stable vector bundle over $E$ of rank $n + 1 \geq 2$ and of degree 0. Then there is no finite morphism $\mathbb{P} C(E) \to \mathbb{P}^n \times C$.

19 (Description of the sections). We start by reviewing the divisor theory over Artin schemes in Section 1. The key notion of divisorial support is introduced and studied in Section 2.

Several versions of K-flatness are investigated in Section 3. For our treatment, technically the most important is C-flatness, which is treated in detail in Section 4. The ideal of Chow equations is introduced in Section 5 and the main results are proved in Section 6.

Sections 7–8 are devoted to examples; we describe K-flat deformations of plane curves and of seminormal curves over $k[\varepsilon]$. While the computations are somewhat lengthy, the answers are quite nice in both cases.

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1. **Infinitesimal study of Mumford divisors**

20. The infinitesimal method to study families of objects in algebraic geometry posits that we should proceed in 3 broad steps.

- Study families over Artin schemes.
- Inverse limits then give families over complete local schemes.
- For arbitrary local schemes, descend properties from the completion.

This approach has been very successful for proper varieties and coherent sheaves on them. One of the problems we have with general (possibly non-flat) families of divisors is that the global and the infinitesimal computations do not match up; in fact they say the opposite in some cases. We discuss 2 instances of this:

- Relative Cartier divisors on non-proper varieties.
- Generically flat families of divisors on surfaces.

The surprising feature is that the two behave quite differently. We state 2 special cases of the results where the contrasts between Artin and DVR bases are especially striking.

**Claim 20.1.** Let $\pi : X \to (s, S)$ be a smooth, affine morphism to a local scheme.

(a) If $S$ is Artin then the restriction map $\text{Pic}(X) \to \text{Pic}(X_s)$ is an isomorphism.
(b) If $S = \text{Spec } k[[t]]$ then $\text{Pic}(X)$ is frequently infinite dimensional.

Thus there can be many nontrivial line bundles on $X$ over $\text{Spec } k[[t]]$, but we do not see them when working over $\text{Spec } k[[t]]/(t^n)$; see (21.3) and (26) for details.

**Claim 20.2.** Let $\pi : X \to (s, S)$ be a smooth morphism of relative dimension 2 to a local scheme $S$.

(a) If $S$ is Artin and non-reduced, then relative class group $\text{Cl}(X/S)$ (27) is infinite dimensional.
(b) If $S = \text{Spec } k[[t]]$ then every relative Mumford divisor $D \subset X$ is Cartier.
As an example, one easily computes that
\[ \text{Cl}(\mathbb{P}^1_{k[[t]]}) \cong \mathbb{Z}, \quad \text{but} \quad \text{Cl}(\mathbb{P}^1_{k[[t]]}/(t^m)) \cong \mathbb{Z} + k^\infty \quad \text{for} \quad m \geq 2. \]
Note that if \( D \subset X \) is a relative Mumford divisor over \( S = \text{Spec} \ k[[t]]/(t^m) \) then it is Cartier on an open set \( X^o \subset X \) whose complement is finite. We see that the study of \( \text{Cl}(X/S) \) is pretty much equivalent to the study of \( CDiv(X^o/S) \) for every \( X^o \).
Here \( X^o \) is not affine, but it is the next simplest scheme, as far as cohomological dimension is concerned. Indeed,
- If \( X \) is affine then \( H^i(X, F) = 0 \), for every \( i > 0 \) and for every coherent sheaf \( F \) on \( X \).
- If \( X \) is an affine surface and \( X^o \subset X \) is open then \( H^i(X^o, F^o) = 0 \), for every \( i > 1 \) and for every coherent sheaf \( F^o \) on \( X^o \).

**Relative Picard group, examples.**

21 (Picard group over Artin schemes). Let \( (A, m, k) \) be a local Artin ring and \( X_A \rightarrow \text{Spec} A \) a flat morphism. Let \( (e) \subset A \) be an ideal such that \( I \cong k \) and set \( B = A/(e) \). We have an exact sequence
\[ 0 \rightarrow \mathcal{O}_{X_A} \xrightarrow{e} \mathcal{O}_{X_A}^* \rightarrow \mathcal{O}_{X_B}^* \rightarrow 1, \quad (21.1) \]
where \( e(h) = 1 + he \) is the exponential map. We use its long exact cohomology sequence and induction on length \( A \) to compute \( \text{Pic}(X_A) \). There are 3 cases that are especially interesting for us.

**Claim 21.2.** Let \( X_A \rightarrow \text{Spec} A \) be a flat, affine morphism. Then the restriction map \( \text{Pic}(X_A) \rightarrow \text{Pic}(X_k) \) is an isomorphism.

**Proof.** We use the exact sequence
\[ H^1(X_k, \mathcal{O}_{X_k}) \rightarrow \text{Pic}(X_A) \rightarrow \text{Pic}(X_B) \rightarrow H^2(X_k, \mathcal{O}_{X_k}). \quad (21.3) \]
Since \( X \) is affine, the two groups at the ends vanish, hence we get an isomorphism in the middle. Induction completes the proof.

**Claim 21.4.** Let \( X_A \rightarrow \text{Spec} A \) be a flat, proper morphism. Assume that \( H^0(X_k, \mathcal{O}_{X_k}) = k \). Then the kernel of the restriction map \( \text{Pic}(X_A) \rightarrow \text{Pic}(X_k) \) is a unipotent group scheme of dimension \( \leq h^1(X_k, \mathcal{O}_{X_k}) \cdot (\text{length} \ A - 1) \). (If \( \text{char} \ k = 0 \) then the kernel is \( k \)-vector space.)

**Proof.** By [Har77, III.12.11], \( H^0(X_A, \mathcal{O}_{X_A}) \rightarrow H^0(X_B, \mathcal{O}_{X_B}) \) is surjective, and so is \( H^0(X_A, \mathcal{O}_{X_A}) \rightarrow H^0(X_B, \mathcal{O}_{X_B}) \). Thus we get the exactness of
\[ 0 \rightarrow H^1(X_k, \mathcal{O}_{X_k}) \rightarrow \text{Pic}(X_A) \rightarrow \text{Pic}(X_B) \rightarrow H^2(X_k, \mathcal{O}_{X_k}). \quad (21.5) \]

**Claim 21.6.** Let \( X_A \rightarrow \text{Spec} A \) be a flat, affine morphism and \( Z \subset X_A \) a closed subset of codimension \( \geq 2 \). Set \( X_A^o := X_A \setminus Z \). Assume that \( X_k \) is \( S_2 \). Then the kernel of the restriction map \( \text{Pic}(X_A^o) \rightarrow \text{Pic}(X_k^o) \) is a unipotent group scheme of dimension \( \leq h^1(X_k^o, \mathcal{O}_{X_k^o}) \cdot (\text{length} \ A - 1) \).

**Proof.** Since \( X_A \) is \( S_2 \), \( H^0(X_k^o, \mathcal{O}_{X_k^o}) \cong H^0(X_k, \mathcal{O}_{X_k}) \) and similarly for \( X_A \). Thus \( H^0(X_A^o, \mathcal{O}_{X_A^o}) \rightarrow H^0(X_B^o, \mathcal{O}_{X_B^o}) \) is surjective, and the rest of the argument works as in (21.5).
**Remark 21.7.** Although (21.6) is very similar to (21.4), a key difference is that in (21.6) the group \(H^1(X_k, \mathcal{O}_{X_k})\) can be infinite dimensional. Indeed, \(H^1(X_k, \mathcal{O}_{X_k}) \cong H^2_\mathbb{Z}(X_k, \mathcal{O}_{X_k})\) and it is

(a) infinite dimensional if \(\dim X_k = 2\),
(b) finite dimensional if \(X_k = S_2\) and \(\text{codim} X_k, Z \geq 3\), and
(c) 0 if \(X_k = S_3\) and \(\text{codim} X_k, Z \geq 3\).

See, for example, [Kol17, Sec.10.2] for these claims.

**Remark 21.8.** If \(H^2(X_k, \mathcal{O}_{X_k}) = 0\) then the \(\leq\) in (21.4) and (21.6) are equalities. If the characteristic is 0 then equality holds even if \(H^2(X_k, \mathcal{O}_{X_k}) \neq 0\); see [BLR90].

The following immediate consequence of (21.7.c) is especially useful for us; see also [Kol17, 4.36].

**Corollary 22.** Let \(X \to S\) be a smooth morphism, \(D \subset X\) a closed subscheme and \(Z \subset X\) a closed subset. Assume that

1. \(D\) is a relative Cartier divisor on \(X \setminus Z\),
2. \(D\) has no embedded points in \(Z\) and
3. \(\text{codim} X_s, Z_s \geq 3\) for every \(s \in S\).

Then \(D\) is a relative Cartier divisor. \(\square\)

The following is essentially in [Gro68, XIII], see also [Kol17, 2.93].

**Theorem 23.** Let \(X \to S\) be a flat morphism with \(S_2\) fibers and \(D\) a divisorial subscheme. Let \(U \subset X\) be an open subscheme such that \(D|_U\) is relatively Cartier and \(\text{codim} X_s, (X_s \setminus U_s) \geq 2\) for every \(s \in S\).

Then \(D\) is relatively Cartier iff the divisorial pull-back \(\tau^*D\) (1.5) is relatively Cartier for every Artin subscheme \(\tau : A \hookrightarrow S\). \(\square\)

Over Artin rings, we have the following flatness criterion. For a coherent sheaf \(F\) let \(\text{emb} F\) denote the largest subsheaf whose support is the union of the (closures of the) embedded points of \(F\).

**Lemma 24.** Let \((A, m, k)\) be an local Artin ring, \(g : X \to \text{Spec} A\) a morphism and \(F\) a coherent sheaf on \(X\). Assume that \(F\) is generically flat over \(A\) and \(\text{emb} F = 0\). Then \(F\) is flat over \(A\).

Proof. Choose \(c \in m\) such that \(m c = 0\). If \(F\) is flat over \(A\) then \(F \cong F_k\), thus we get an injection \(\epsilon : \text{emb}(F_k) \hookrightarrow \text{emb} F\). Thus if \(\text{emb} F = 0\) then so is \(\text{emb}(F_k)\).

Conversely, assume that \(\text{emb}(F_k) = 0\). We may assume that \(X\) is affine. By induction on length \(A\) we may assume that \((F/\epsilon F)/\text{emb}(F/\epsilon F)\) is flat over \(A/(\epsilon)\). We claim that \(\text{emb}(F/\epsilon F) \subset \text{emb} F\).

By assumption \((F/\epsilon F)/\text{emb}(F/\epsilon F)\) is a free \(A/(\epsilon)\) module, choose basis elements \(f_\lambda\) and lift them back to \(\tilde{f}_\lambda \in H^0(X, F)\).

Let \((\epsilon F)^{(1)} \subset F\) be the preimage of \(\text{emb}(F/\epsilon F)\). Pick now \(h \in (\epsilon F)^{(1)}\). The image of \(h\) in \((F/mF)\) is 0, so \(h = \sum a_i g_i\) for some \(a_i \in m, g_i \in F\). Write each \(g_i\) in the \(\tilde{f}_\lambda\) basis. Thus we have

\[
g_i = \sum \alpha_i c_\lambda \tilde{f}_\lambda \mod (\epsilon F)^{(1)}.
\]

Since \(m(\epsilon F)^{(1)} = 0\), we get that

\[
h = \sum \lambda (\sum_i a_i c_\lambda) \tilde{f}_\lambda.
\]
This is zero modulo \((\varepsilon F)^{(1)}\), so \(\sum \alpha_i c_\lambda \in (\varepsilon)\) for every \(\lambda\). Thus \(h \in \varepsilon F\).

Thus \(\varepsilon F \cong F/mF\) and \(\text{emb}(F/\varepsilon F) = 0\), so \(F\) is flat over \(A\). \(\square\)

Relative Cartier divisors also have some unexpected properties over non-reduced base schemes. These do not cause theoretical problems, but it is good to keep them in mind.

**Example 25** (Cartier divisors over \(k[\varepsilon]\)). Let \(R\) be an integral domain over a field \(k\). Relative principal ideals in \(R[\varepsilon]\) over \(k[\varepsilon]\) are given as \((f + \varepsilon g)\) where \(f, g \in R\) and \(f \neq 0\). We list some properties of such principal ideals that hold for any integral domain \(R\).

1. \((f + \varepsilon g) = (f + \varepsilon g'\varepsilon)\) iff \(g_1 - g_2 \in (f)\),
2. If \(u \in R\) is a unit then so is \(u + \varepsilon g\) since \((u + \varepsilon g)(u^{-1} - u^{-2}g) = 1\),
3. If \(f\) is irreducible then so is \(f + \varepsilon g\) for every \(g\),
4. \((f + \varepsilon g)(f - \varepsilon g) = f^2\) shows that there is no unique factorization.
5. If the \(f_i\) are pairwise relatively prime then

\[
\prod (f_i + \varepsilon g) = \prod (f_i + g_i\varepsilon) \quad \text{iff} \quad (f_i + \varepsilon g_i) = (f_i + g_i\varepsilon) \quad \forall i.
\]

The following concrete example illustrates several of the above features.

**Example 26** (Picard group of a constant elliptic curve). Let \((0, E)\) be a smooth, projective elliptic curve. Over any base \(S\) we have the constant family \(\pi : E \times S \to S\) with the constant section \(s_0 : S \cong \{0\} \times S\). Let \(L\) be a line bundle on \(E \times S\). Then \(L \otimes \pi^* s_0^* L^{-1}\) has a canonical trivialization along \(\{0\} \times S\), hence it defines a morphism \(S \to \text{Pic}(E)\). So the relative Picard group is computed by the formula

\[
\text{Pic}(E \times S/S) \cong \text{Mor}(S, \text{Pic}(E)). \tag{26.1}
\]

Two consequences are worth mentioning.

**Claim 26.2.** Let \((R, m)\) be a complete local ring. Set \(S = \text{Spec} R\) and \(S_n = \text{Spec} R/m^n\). Then

\[
\text{Pic}(E \times S/S) = \varprojlim \text{Pic}(E \times S_n/S_n). \quad \square
\]

**Claim 26.3.** Let \(S = \text{Spec} k[\varepsilon][t]\) be the local ring of the affine line at the origin and \(\hat{S} = \text{Spec} k[[t]]\) its completion. Then

\[
\text{Pic}(E \times S/S) \cong \text{Pic}(E) \quad \text{but} \quad \text{Pic}(E \times \hat{S}/\hat{S}) \quad \text{is infinite dimensional}. \quad \square
\]

Next consider the affine elliptic curve \(E^\circ = E \setminus \{0\}\) and the constant affine family \(E^\circ \times S \to S\). Note that \(\text{Pic}(E^\circ) \cong \text{Pic}(E)\).

If \(S\) is smooth and \(D^\circ\) is a Cartier divisor on \(E^\circ \times S\) then its closure \(D \subset E \times S\) is also Cartier. More generally, this also holds if \(S\) is normal, using [Kol17, 4.21]. Thus (26.1) gives the following.

**Claim 26.3.** If \(S\) is normal then

\[
\text{Pic}(E^\circ \times S/S) \cong \text{Mor}(S, \text{Pic}(E)). \quad \square
\]

By contrast, (21.3) gives the following.

**Claim 26.4.** If \(S = \text{Spec} A\) is Artin then

\[
\text{Pic}(E^\circ \times S/S) \cong \text{Pic}(E). \quad \square
\]

So \(\text{Pic}(E^\circ \times S/S)\) has dimension 1 but \(\text{dim}_{\text{kr}} \text{Mor}(S, \text{Pic}(E)) = \text{length } A\).

The following is a good illustration.
Concrete Example 26.5. Start with the plane cubic with equation \( Y^2 Z = X^3 - 3 \). In the affine plane \( Z = 1 \) we get \( y^2 = x^3 - 1 \) (where \( x = X/Z, y = Y/Z \)) and in the \( Y = 1 \) plane we get \( v = u^3 - v^3 \) (where \( u = X/Y, v = Z/Y \)). The diagonal in \((y^2 = x^3 - 1) \times (v = u^3 - v^3)\) is a Cartier divisor which is defined by 2 equations \( yv = 1 \) and \( yu = x \).

At \((u = v = 0)\) the local coordinate is \( u \). Note that \( u \) also vanishes at the points where \( v^2 + 1 = 0 \). If we invert it, then we get that

\[
(u^{3r}) = (v^r) \subset k[u,v,(v^2 + 1)^{-1}]/(u^3-v^3-v).
\]

What is the ideal

\[
(yv - 1, yu - x, u^r) \subset k[x,y,u,v,(v^2 + 1)^{-1}]/(y^2 - x^3 + 1, u^3 - v^3 - v).
\]

Note that it contains

\[
(yv - 1)(y^{-1}v^{-1} + \cdots + yv + 1) = y^r v^r - 1 = y^r(v^2 + 1)^{-r}u^{3r} - 1.
\]

Thus \( 1 \in (yv - 1, yu - x, u^r) \) and the ideal is the whole ring.

Relative Mumford divisors.

Definition 27. Let \( f : X \to S \) be a flat morphism. Two relative Mumford divisors \( D_1, D_2 \subset X \) are linearly equivalent if \( \mathcal{O}_X(-D_1) \cong \mathcal{O}_X(-D_2) \), and linearly equivalent over \( S \) if \( \mathcal{O}_X(-D_1) \cong \mathcal{O}_X(-D_2) \otimes f^*L \) for some line bundle \( L \) on \( S \). The linear equivalence classes over \( S \) of relative Mumford divisors generate the relative Mumford class group \( \text{MCl}(X/S) \).

By definition, if \( D \) is a Mumford divisor then there is a closed subset \( Z \subset X \) such that \( \mathcal{O}_X(-D)|_{X \setminus Z} \) is locally free and \( \text{codim}_X Z_s \geq 2 \) for every \( s \in S \). This gives a natural identification

\[
\text{MCl}(X/S) = \varprojlim_S \text{Pic}(X \setminus Z/S),
\]

(27.1)

where the limit is over all closed subsets \( Z \subset X \) such that \( \text{codim}_X Z_s \geq 2 \) for every \( s \in S \).

On a normal variety, a Mumford divisor is the same as a Weil divisor and the Mumford class group is the same as the class group. If \( f \) has normal fibers, then we get the relative class group \( \text{Cl}(X/S) := \text{MCl}(X/S) \).

As with the Picard group, this may not be the optimal definition when \( S \) is projective, but we will use this notion mostly when \( S \) is local, and then this seems the right definition.

Proposition 28. Let \((A,k)\) be a local Artin ring, \( k \cong (\epsilon) \subset A \) an ideal and \( B = A/(\epsilon) \). Let \((R_A,m)\) be a flat, local, \( S_2 \), \( A \)-algebra of dimension 2 and set \( X_A := \text{Spec}_A R_A \). Let \( f_B \in R_B \) be a non-zero divisor and set \( C_B := (f_B = 0) \subset X_B \).

Then the set of all relative Mumford divisors \( D_A \subset X_A \) such that \( \text{pure}(D_A)|_B = C_B \) is a torsor under the infinite dimensional \( k \)-vector space \( H^0(B,\mathcal{O}_{C_B}) \).

Proof. We can lift \( f_B \) to \( f_A \in R_A \). Choose \( y \in m \) that is not a zerodivisor on \( C_B \) and such that \( D_A \) is a principal divisor on \( X_A \setminus \{y = 0\} \). After inverting \( y \), we can write the ideal of \( D \) as

\[
I_{(y)} = (f_A + cy^{-r}g_k) \quad \text{where} \quad g \in R_k, r \in \mathbb{N}.
\]

(28.1)

We can multiply \( f_A + cy^{-r}g_k \) by \( u + cy^{-s}v \) where \( u \) is a unit in \( R \). This changes \( g_k \) to \( ug_k + vy^{-s}f_A \). Thus the relevant information is carried by the residue class

\[
y^{-r}g_k \in H^0(C_B^0,\mathcal{O}_{C_B^0}),
\]

(28.2)
where \( C_k^c \subset C_k \) denotes the complement of the closed point.

If the residue class is in \( H^0(C_k, \mathcal{O}_{C_k}) \) then we get a Cartier divisor. Thus the non-Cartier divisors are parametrized by

\[
H^0(C_k, \mathcal{O}_{C_k})/H^0(C_k, \mathcal{O}_{C_k}) \cong H^1_m(C_k, \mathcal{O}_{C_k}).
\]  

(28.3)

We compute in (31.2) that different elements of \( H^1_m(C_k, \mathcal{O}_{C_k}) \) give non-isomorphic divisors.

\[ \square \]

**Corollary 30.** Let \((A, k) \) be a local Artin ring, \( k \cong (\epsilon) \subset A \) an ideal and \( B = A/(\epsilon) \). Let \((R_A, m) \) be a flat, local, \( S_2 \), \( A \)-algebra of dimension 2. Let \( f_A \in R_A \) and \( g_k \in R_k \) be a non-zeros-divisors, and \( y \) a non-zero divisor modulo both \( f_A \) and \( g_k \).

For the divisorial ideal \( I := R_A \cap (f_A + \epsilon y^{-r} g_k)R_A[y^{-1}] \) the following are equivalent.

1. \( I \) is a principal ideal.
2. The residue class \( y^{-r} g_k \) lies in \( R_k/(f_k) \).
3. \( g_k \in (f_k, y^r) \).

Proof. \( I \) is a principal ideal iff it has a generator of the form \( f_A + \epsilon h_k \) where \( h_k \in R_k \). This holds iff

\[
f_A + \epsilon y^{-r} g_k = (1 + \epsilon y^{-s} b_k)(f_A + \epsilon h_k) \quad \text{for some} \quad b_k \in R_A.
\]

Equivalently, iff \( y^{-r} g_k = h_k + \epsilon y^{-s} b_k f_k \). If \( r > s \) then \( g_k = y^r h_k + \epsilon y^{-s} b_k f_k \) which is impossible since \( y \) is not a zero divisor modulo \( g_k \). If \( r < s \) then \( y^s r = y^s h_k + \epsilon b_k f_k \) is impossible since \( y \) is not a zero divisor modulo \( f_k \). Thus \( r = s \) and then \( g_k = y^s h_k + \epsilon b_k f_k \) is equivalent to \( g_k \in (f_k, y^r) \).

\[ \square \]

**Corollary 30.** Using the notation of (29), assume that \( f_A - f_A^e' \in e m R \) and \( g_k - g_k' \) are \( m N \) such \( N \gg 1 \) (depending on \( f_k, g_k \) and \( r \)). Then \( (f_A + \epsilon y^{-r} g_k) \) defines a relative Cartier divisor iff \( (f_A + \epsilon y^{-r} g_k') \) does.

Proof. Choose \( N \) such that \( m N \subset (f_k, y^r) \). Then \( (f_A, y^r) = (f_k, y^r) \) and \( g_k - g_k' \in (f_k, y^r) \).

Remark 30.1. If \( R_k \) is regular then we can choose \( y \) to be a general element of \( m \setminus m^2 \). Then \( \dim R_k/(f_k, y^r) = r \cdot \text{mult } f_k \), so \( N = r \cdot \text{mult } f_k \) works.

The connection between (28) and (21) is given by the following.

31. Let \( X \) be an affine, \( S_2 \) scheme and \( D := (s = 0) \subset X \) a Cartier divisor. Let \( Z \subset D \) be a closed subset that has codimension \( \geq 2 \) in \( X \). Set \( X^0 := X \setminus Z \) and \( D^0 := D \setminus Z \). Restricting the exact sequence

\[ 0 \to \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_D \to 0 \]

to \( X^0 \) and taking cohomologies we get

\[ 0 \to H^0(X^0, \mathcal{O}_{X^0}) \to H^0(X, \mathcal{O}_X) \to H^0(D^0, \mathcal{O}_{D^0}) \to H^1(X^0, \mathcal{O}_{X^0}). \]

Note that \( H^0(X^0, \mathcal{O}_{X^0}) = H^0(X, \mathcal{O}_X) \) since \( X \) is \( S_2 \) and its image in \( H^0(D^0, \mathcal{O}_{D^0}) \) is \( H^0(D, \mathcal{O}_D) \). Thus \( \partial \) becomes the injection

\[ \partial: H^1_2(D, \mathcal{O}_D) \cong H^0(D^0, \mathcal{O}_{D^0})/H^0(D, \mathcal{O}_D) \to H^1_2(X, \mathcal{O}_X). \]  

(31.1)

We are especially interested in the case when \((x, X)\) is local, 2-dimensional and \( Z = \{x\} \). In this case (31.1) becomes

\[ \partial: H^1_2(D, \mathcal{O}_D) \to H^1_2(X, \mathcal{O}_X). \]  

(31.2)
We can be especially explicit in the smooth case. (Note that, by the Weierstrass preparation theorem, almost every curve in \( \mathbb{A}^2_{u,v} \) is defined by a monic polynomial in \( v \).

**Lemma 32.** Let \( f \in k[[u]][v] \) be a monic polynomial in \( v \) of degree \( n \) defining a curve \( C_k \subset \mathbb{A}^2_{u,v} \). Let \( D \subset \mathbb{A}^2_{k,v} \) be a relative Mumford divisor such that \( \text{pure}(D_k) = C_k \). Then the restriction of \( D \) to the complement of \( (u = 0) \) can be uniquely written as

\[
f + \epsilon \sum_{i=0}^{n-1} v^i \phi_i(u) = 0 \quad \text{where} \quad \phi_i(u) \in u^{-1}k[u^{-1}].
\]

Thus the set of all such \( D \) is naturally isomorphic to the infinite dimensional \( k \)-vector space \( H^1_m(C_k, \mathcal{O}_{C_k}) \cong \oplus_{i=0}^{n-1} u^{-1}k[[u^{-1}]]. \)

Proof. Note that \( k[[u,v]]/(f) \cong \oplus_{i=0}^{n-1} v^i k[[u]] \) as a \( k[[u]] \)-module, giving isomorphism

\[
H^0(C_k, \mathcal{O}_{C_k}) \cong \oplus_{i=0}^{n-1} v^i k[[u]] \quad \text{and} \quad H^0(C_k, \mathcal{O}_{C_k}) \cong \oplus_{i=0}^{n-1} v^i k((u)).
\]  

(32.1)

That is, if \( g \in k((u))[v] \) is a polynomial of degree \( < n \) in \( v \) then \( g|_{C_k} \) extends to a regular function on \( C \) iff \( g \in k[[u]][v] \).

We can also restate (32.1) as

\[
H^1_m(C_k, \mathcal{O}_{C_k}) \cong \oplus_{i=0}^{n-1} v^i k((u))/k[[u]] \cong \oplus_{i=0}^{n-1} v^i u^{-1}k[u^{-1}].
\]  

(32.2)

**Example 33.** Consider next the special case of (32) when \( f = v \). We can then write the restriction of \( D \) as \( (v + \phi(u)\epsilon = 0) \) where \( \phi \in u^{-1}k[u^{-1}] \). Let \( r \) denote the pole-order of \( \phi \) and set \( q(u) := u^r \phi(u) \).

**Claim 33.1.** The ideal of \( D \) is

\[
I_D = (v^2, vu^r + q(u)\epsilon, ve).
\]

Thus the fiber over the closed point is \( k[[u,v]]/(v^2, vu^r) \). Its torsion submodule is isomorphic to \( k[[u,v]]/(v, u^r) \).

Proof. To see this note first that \( v^2 = (v + \phi(u)\epsilon)(v - \phi(u)\epsilon), vu^r + q(u) = (v + \phi(u)\epsilon)u^r \) and \( ve = (v + \phi(u)\epsilon)\epsilon \) are elements of \( I_D \). Next note that \( q(u) \) is a polynomial with nonzero constant term, hence invertible in \( k[[u,v]] \).

Therefore

\[
k[[u,v]]/(v^2, vu^r + q(u)\epsilon, ve) \cong k[[u,v]]/(v^2, v^2 u^r q(u)^{-1}) = k[[u,v]]/(v^2)
\]

has no embedded points.

The ideals of relative Mumford divisors in \( k[[u,v]][\epsilon] \) are likely to be more complicated in general. At least the direct generalization of (33.1) does not always give the correct generators.

For example, let \( f = v^2 - u^3 \) and consider the ideal \( I \subset k[[u,v]][\epsilon] \) extended from \( (v^2 - u^3, u^{-3} ve) \). The above procedure gives the elements

\[
(v^2 - u^3)^2, \quad u^3(v^2 - u^3) + ve, \quad (v^2 - u^3)\epsilon \in I.
\]

However, \( u^3(v^2 - u^3) + ve = v^2(v^2 - u^3) + ve \) and we can cancel the \( v \) to get that

\[
I = (v^2 - u^3)^2, \quad v^2(v^2 - u^3) + ve, \quad (v^2 - u^3)\epsilon.
\]  

(33.2)

Using the isomorphism \( R[\epsilon]/(f^2, fg + \epsilon, f\epsilon) \cong R/(f^2, -f^2 g) \cong R/(f^2) \), the above examples can be generalized to the non-smooth case as follows.

**Claim 33.3.** Let \( (R, m) \) be a local, \( S_2 \), \( k \)-algebra of dimension 2 and \( f, g \in m \) a system of parameters. Then \( J_{f,g} = (f^2, fg + \epsilon, f\epsilon) \) is (the ideal of) a relative
Mumford divisor in $R[\epsilon]$ whose central fiber is $R/(f^2, fg)$, with embedded subsheaf isomorphic to $R/(f, g)$.

\[ \square \]

2. Divisorial support

There are at least 3 ways to associate a divisor to a sheaf (34) but only one of them—the divisorial support—behaves well in flat families. In this Section we develop this notion and a method to compute it. The latter is especially important for the applications.

**Definition 34 (Divisorial support of a sheaf).** Let $X$ be a scheme and $F$ a coherent sheaf on $X$. One usually defines its support $\text{Supp}\ F$ and its scheme-theoretic support $\text{SSupp}\ F := \text{Spec}_X(O_X/\text{Ann}\ F)$.

Assume next that $X$ is regular at every generic point $x_i \in \text{Supp}\ F$ that has codimension 1 in $X$. Then there is a unique divisorial sheaf $\text{Kol17, 3.50}$ associated to the Weil divisor $\sum \text{length}(F_{x_i}) \cdot [\bar{x}_i]$. We call it the divisorial support of $F$ and denote it by $\text{DSupp}\ F$.

If every associated point of $F$ has codimension 1 in $X$ then we have inclusions of subschemes

$$\text{Supp}\ F \subset \text{SSupp}\ F \subset \text{DSupp}\ F.$$

(34.1)

In general all 3 subschemes are different, though with the same support.

Our aim is to develop a relative version of this notion and some ways of computing it in families. Let $X \to S$ be a morphism and $F$ a coherent sheaf on $X$. Informally, we would like the relative divisorial support of $F$, denoted by $\text{DSupp}_S F$, to be a scheme over $S$ whose fibers are $\text{DSupp}(F_s)$ for all $s \in S$. If $S$ is reduced, this requirement uniquely determines $\text{DSupp}_S F$ but in general there are 2 problems.

- Even in nice situations, this requirement may be impossible to meet.
- For non-reduced base schemes $S$, the fibers alone do not determine $\text{DSupp}_S F$.

The right concept is developed through a series of Definition-Lemmas. Each one is a definition, where we need to check that it is independent of the choices involved, and that it coincides with our naive definition over reduced schemes.

We start with a very elementary case which, however, turns out to be crucial.

**Definition–Lemma 35 (Divisorial support I).** Let $C$ be a smooth curve and $M$ a torsion sheaf on $C$. Thus it can be written as $M \cong \oplus_j O_C/O_C(\mathbb{N})$ where $P_j \in C$ and $n_j \in \mathbb{N}$ (repetitions allowed). Then

$$\text{DSupp}(M) = \text{Spec}_C(O_C/O_C(-\sum n_j P_j)).$$

(35.1)

Let $\pi : C_1 \to C_2$ be an étale morphism of smooth curves and $M$ a torsion sheaf on $C_2$. Then we get that

$$\text{DSupp}(\pi^*M) = \pi^* \text{DSupp}(M).$$

(35.2)

Thus the computation of $\text{DSupp}(M)$ is an étale-local question. In order to develop another formula, we can work on $\mathbb{A}^1$. Let $g_j \in k[x]$ be monic polynomials and set $M = \oplus_j k[x]/(g_j)$. Multiplication by $x$ is an endomorphism $\mu_x$ of the $k$-vector space $M$. We claim that its characteristic polynomial is

$$\chi(\mu_x)(t) = (-1)^d \prod g_j(t) \quad \text{where} \quad d = \deg \prod g_j.$$

(35.3)

Thus

$$\text{DSupp} M = (\prod g_j = 0) = (\chi(\mu_x)(x) = 0).$$

(35.4)
In particular, we could use any étale coordinate instead of $x$ in (35.4).

It is enough to do check these for 1 polynomial. Thus let $g = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial. In the module $k[x]/(g)$ choose a $k$-basis $e_i = x^i$ for $i = 0, \ldots, n - 1$. Thus multiplication by $\mu_x$ is

$$\mu_x(e_i) = e_{i+1} \quad \text{for } i < n - 1 \quad \text{and} \quad \mu_x(e_{n-1}) = -\sum a_i e_i.$$  

We compute that its characteristic polynomial is $(-1)^n g(x)$. For example, if $n = 4$ then the matrix of $\mu_x$ is

$$\begin{pmatrix}
0 & 0 & 0 & -a_0 \\
1 & 0 & 0 & -a_1 \\
0 & 1 & 0 & -a_2 \\
0 & 0 & 1 & -a_3
\end{pmatrix}.$$  

Expanding by the last column gives the characteristic polynomial

$$\det \begin{pmatrix}
-x & 0 & 0 & -a_0 \\
1 & -x & 0 & -a_1 \\
0 & 1 & -x & -a_2 \\
0 & 0 & 1 & -x - a_3
\end{pmatrix} = (-1)^4(x^4 + a_3 x^2 + \cdots + a_0). \quad \square$$  

**Definition–Lemma 36** (Divisorial support II). Let $X \to \mathbb{A}^1_S$ be an étale morphism. Let $F$ be a coherent sheaf on $X$ that is finite and flat (hence locally free) over $S$. Let $t$ denote a coordinate on $\mathbb{A}^1_S$. Multiplication by $t$ is an endomorphism $\mu_t$ of the locally free $\mathcal{O}_S$-module $F$; let $\chi(\mu_t)(*)$ be its characteristic polynomial. Then the divisorial support of $F$ over $S$ is

$$\text{DSupp}_S(F) = (\chi(\mu_t)(t) = 0) \subset X.$$  

It is a relative Cartier divisor.

Proof of consistence. We need to show that this is independent of the choice of $t$. This is an étale-local question, We may thus assume that $S$ is the spectrum of a Henselian local ring $(R, m, k)$, $X$ is the spectrum of $R[t]$ (the Henselization of $R[t]$) and $F$ is the sheafification of the free $R$-module $M$.

Multiplication by $t$ is an endomorphism of $M$, its characteristic polynomial is the defining equation of $\text{DSupp}(M)$. However, the choice of $t$ is not unique; we could have used any other local parameter $t' = a_1 t + a_2 t^2 + \cdots$ where $a_1 \notin m$.

Assume first that $S$ is reduced. Then the independence of $t$ can be checked over the generic fibers, where we recover the computation of (35).

If $S$ is arbitrary, then we use that the pair $(M, \mu_t : M \to M)$ is induced from the universal endomorphism $\mu^a$ of $M_S := \oplus_i R^n e_i$ over the Henselian ring $R^a := k\langle t_{ij} : 1 \leq i, j \leq r \rangle$ given by

$$(e_1, \ldots, e_r)^{tr} \mapsto (t_{ij}) \cdot (e_1, \ldots, e_r)^{tr}.$$  

We already noted that independence of $t$ holds over $\text{Spec } R^n$, hence it also holds after pulling back to $S$.  

\square

**Definition–Lemma 37** (Divisorial support III). Let $X \to S$ be a smooth morphism of pure relative dimension $n$. Let $F$ be a coherent sheaf on $X$ that is flat over $S$ with CM fibers of pure dimension $n - 1$. Then its divisorial support $\text{DSupp}_S(F)$ is defined and it is relatively Cartier over $S$. 
Proof of consistence. Being relatively Cartier can be checked étale-locally. Thus we may assume that $S$ is the spectrum of a Henselian local ring $(R, m, k)$ with infinite residue field and $M$ is a finite $R(x_1, \ldots, x_n)$-module that is flat over $R$ and such that $M_k$ is CM and of dimension $n - 1$.

After a linear coordinate change, we may also assume that $M_k$ is finite over $k(x_1, \ldots, x_{n-1})$. Since $M_k$ is CM, it is also free, thus $M$ is also free over $R(x_1, \ldots, x_{n-1})$.

Multiplication by $x_n$ is an endomorphism of the free $R(x_1, \ldots, x_{n-1})$-module $M$, its characteristic polynomial is the defining equation of $\text{DSupp}(M)$.

It remains to show that this equation is independent of the choices that we made, up to a unit. On a scheme two locally principal divisors agree if they agree at their generic points.

Thus after further localization at a generic point of $\text{Supp} M$ and flat base change [Kol17, 10.47], we may assume that the relative dimension is 1. This was already treated in (36).

The following properties are especially important.

**Corollary 38.** Continuing with the notation and assumptions of (37), let $h : S' \to S$ be a morphism. By base change we get $g' : X' \to S'$ and $h_X : X' \to X$. Then

$$h_X^*(\text{DSupp } F) = \text{DSupp}(h_X^* F).$$

**Proof.** This is an étale-local question on $X$, thus, as in the proof of (37) we may assume that $M$ is free over $R(x_1, \ldots, x_{n-1})$. The base change $S' \to S$ corresponds to a ring extension $R \to R'$, and the characteristic polynomial commutes with ring extensions.

**Corollary 39.** Let $X \to S$ be a smooth morphism of pure relative dimension $n$. Let $F$ be a coherent sheaf on $X$ that is flat over $S$ fibers of pure dimension $n - 1$. Then $\text{DSupp}_S F$ is a relative Cartier divisor.

**Proof.** Unlike in (37), we do not assume that the fibers are CM. However, if $x \in X_s$ is a point of codimension $\leq 2$, then $F_s$ is CM at $x$, hence $\text{DSupp}_S F$ is a relative Cartier divisor at $x$ by (37). Since $X \to S$ is smooth, $\text{DSupp}_S F$ is a relative Cartier divisor everywhere by (22).

**Corollary 40.** Continuing with the notation and assumptions of (37), let $D \subset X$ be a relative Cartier divisor that is also smooth over $S$. Assume that $D$ does not contain any generic point of $\text{Supp} F_s$ for any $s \in S$. Then

$$\text{DSupp}(F|_D) = (\text{DSupp } F)|_D.$$

**Proof.** As in the proof of (37), we can choose local coordinates such that $D = (x_1 = 0)$. Then $\text{DSupp } F$ is computed from the characteristic polynomial of $M$ over $R(x_1, \ldots, x_{n-1})$ and $\text{DSupp}(F|_D)$ is computed from the characteristic polynomial of $M/x_1 M$ over $R(x_2, \ldots, x_{n-1})$.

Now we are ready to define the sheaves for which the relative divisorial support makes sense, but first we have to distinguish associated points that come from the base from the other ones.

**Definition 41.** Let $g : X \to S$ be a morphism and $F$ a coherent sheaf on $X$ such that $\text{Supp } F \to S$ has pure relative dimension $d$. An associated point $x \in \text{Ass}(F)$ is called vertical if $x$ is not a generic point of $\text{Supp}(F_{g(x)})$.

We say that $F$ is vertically pure if it has no vertical associated points.
If $F$ is generically flat over $S$ (42), then there is a unique largest subsheaf $v$-tors$_S(F) \subset F$—called the \textit{vertical torsion} of $F$—such that every fiber of the structure map $\text{Supp}(v$-tors$_S(F)) \to S$ has dimension $< d$.

Then $v$-pure($F$) := $F/v$-tors$_S(F)$ has no vertical associated primes.

All these notions make sense for subschemes of $X$ as well.

\textbf{Definition 42.} Let $X \to S$ be a morphism and $F$ a coherent sheaf on $X$. We say that $F$ is a \textit{generically flat family of pure sheaves} of dimension $d$ over $S$ if the following hold.

1. $F$ is flat at every generic point of $F_s$ for every $s \in S$ and
2. $\text{Supp} F \to S$ has pure relative dimension $d$.

We usually do not care about vertical associated points on $F$, thus we frequently replace $F$ by $v$-pure($F$) = $F/v$-tors$_S(F)$ and then the following condition is also satisfied.

3. $F$ is vertically pure.

We say that $Z \subset X$ is a \textit{generically flat family of pure subschemes} if its structure sheaf $\mathcal{O}_Z$ has this property.

The following properties are clear from the definition.

4. Conditions (1–2) are preserved by any base change $S' \to S$ and (3) is preserved by flat base change.

5. If (3) holds then the generic fibers $F_g$ are pure of dimension $d$, but special fibers may have embedded points outside $\text{FlatCM}(F)$ (43).

\textbf{Definition 43.} Let $X \to S$ be a morphism and $F$ a coherent sheaf on $X$. The \textit{flat locus} of $F$ is the largest open subset $U \subset \text{Supp} F$ such that $F|_U$ is flat over $S$. We denote it by $\text{Flat}(F)$.

It is sometimes more convenient to work with the \textit{flat-CM locus} of $F$. It is the largest open subset $U \subset \text{Supp} F$ such that $F|_U$ is flat with CM fibers over $S$. We denote it by $\text{FlatCM}(F)$.

These properties are unchanged if we replace $X$ by $\text{SSupp} F$. Thus we may assume that $\text{Supp} F = X$, or even that $\text{Ann}(F) = 0$, whenever it is convenient.

\textbf{Definition–Lemma 44 (Divisorial support IV).} Let $g : X \to S$ be a flat morphism of pure relative dimension $n$ and $g^\circ : X^\circ \to S$ the smooth locus of $g$.

Let $F$ be a coherent sheaf on $X$ that is generically flat and pure over $S$ of dimension $n - 1$. Assume that for every $s \in S$, every generic point of $F_s$ is contained in $X^\circ$.

Set $Z := \text{Supp} F \setminus (\text{FlatCM}(F) \cap X^\circ)$, $U := X \setminus Z$ and $j : U \hookrightarrow X$ the natural injection. We define the \textit{divisorial support} of $F$ over $S$ as

$$\text{DSupp}_S(F) := \text{DSupp}_S(F|_U),$$

the scheme-theoretic closure of $\text{DSupp}_S(F|_U)$. This makes sense since the latter is already defined by (37).

Note that $\text{DSupp}_S(F)$ is a generically flat family of pure subschemes of dimension $n - 1$ over $S$ and it is relatively Cartier on $U$.

It is enough to check the following equalities at codimension 1 points, which follow from (35).

\textbf{Claim 44.2.} Let $X_i \to S$ be flat morphisms of pure relative dimension $n$ and $\pi : X_1 \to X_2$ a finite morphism. Let $D \subset X_1$ be a relative Mumford divisor.
Assume that \( \text{red} D_s \to \text{red} (\pi(D_s)) \) is birational and \( \pi \) is étale at every generic point of \( D_s \). Then
\[
\text{DSupp}(\pi_* O_D) = \pi(D), \quad \text{the scheme-theoretic image of } D.
\]

**Claim 44.3.** Let \( X_i \to S \) be flat morphisms of pure relative dimension \( n \) and \( \pi : X_1 \to X_2 \) a finite morphism. Let \( F_1 \) be a coherent sheaf on \( X_1 \) that is generically flat and pure over \( S \) of dimension \( n - 1 \). Set \( F_2 := \pi_* F_1 \). Assume that \( g_i \) is smooth at every generic point of \( (F_i)_s \) for every \( s \in S \). Then
\[
\text{DSupp}_S(\pi_* F_1) = \text{DSupp}_S(\pi_* \text{DSupp}_S(F_1)). \quad \square
\]

The next claim directly follows from [Kol17, 4.36].

**Lemma 45.** Let \( S \) be a reduced scheme and \( g : X \to S \) a smooth morphism of pure relative dimension \( n \). Let \( F \) be a coherent sheaf on \( X \) that is generically flat and pure over \( S \) of dimension \( n - 1 \). Then \( \text{DSupp}_S(F) \) is a relative Cartier divisor. \( \square \)

Divisorial support commutes with restriction to a divisor, whenever everything makes sense. We just need to make enough assumptions that guarantee that (40) applies on a dense set of every fiber.

**Corollary 46.** Continuing with the notation and assumptions of (44), let \( D \subset X \) be a relative Cartier divisor. Assume that there is an open set \( D^0 \subset D \) such that
\begin{enumerate}
  \item \( g|_D \) is smooth on \( D^0 \),
  \item \( D^0_s \) is dense in \( D_s \) for every \( s \in S \),
  \item \( D \) does not contain any generic point of \( \text{Supp} F_s \) for any \( s \in S \), and
  \item \( D^0 \subset \text{FlatCM}(F) \).
\end{enumerate}
Then
\[
\text{DSupp}(F|_D) = \text{v-pure}((\text{DSupp} F)|_D). \quad \square
\]

Various Bertini-type theorems show that the above assumptions are quite easy to satisfy, at least locally.

**Corollary 47.** Continuing with the above notation, let \( |D| \) be a linear system on \( X \) that is base point free in characteristic 0 and very ample in general. Fix \( s \in S \) and let \( D \in |D| \) be a general member. Then there is an open neighborhood \( s \in S^0 \subset S \) such that
\[
\text{DSupp}(F|_D) = \text{DSupp}(F)|_D \quad \text{holds over } S^0. \quad (47.1)
\]

Proof. We apply the usual Bertini theorems to \( X_s \). We get that \( D_s \) satisfies conditions (46.1–4), and then they also hold over some open neighborhood \( s \in S^0 \subset S \).

This gives (47.1), modulo vertical torsion. Finally note that there is no such torsion for general \( D \) by [Kol17, 10.9]. \( \square \)

**Lemma 48.** Divisorial support commutes with base change. That is, let \( g : X \to S \) be a flat morphism of pure relative dimension \( n \) and \( F \) a generically flat family of pure sheaves of dimension \( n - 1 \) over \( S \). Assume that for every \( s \in S \), every generic point of \( \text{Supp} F_s \) is contained in the smooth locus of \( g \). Let \( h : S' \to S \) be a morphism. By base change we get \( g' : X' \to S' \) and \( h_X : X' \to X \). Then
\[
h_X^*(\text{DSupp} F) = \text{DSupp}(h_X^* F).
\]
Proof. Set \( U := \text{FlatCM}(F) \subset X \) with injection \( j : U \hookrightarrow X \). Set \( U' := h_X^{-1}(U) \) and \( h_U : U' \to U \) the restriction of \( h_X \). Then \( h_U^*(\text{DSupp} F|_U) = \text{DSupp}(h_U^*(F|_U)) \) by (38).

By (42.4) \( h_X^*[\text{DSupp} F] \) is a generically flat family of pure divisors and it agrees with \( \text{DSupp}(h_X^*F) \) over \( U' \). Thus the 2 are equal. \( \square \)

**Definition 49** (Divisorial support of cycles). Let \( S \) be a seminormal scheme and \( Z \) a well defined family of \( d \)-cycles on \( \mathbb{P}^d_S \) as in [Kol96, I.3.10].

Let \( \rho : \text{Supp} Z \to \mathbb{P}^d_{S} \) be a finite morphism. Then \( \rho_*Z \) is a well defined family of \( d \)-cycles on \( \mathbb{P}^d_{S} \).

If all the residue characteristics are 0, or if \( Z \) satisfies the field of definition condition [Kol96, I.4.7], then there is a unique relative Cartier divisor \( D \subset \mathbb{P}^d_{S} \) whose associated cycle is \( \rho_*Z \); see [Kol96, I.3.23.2]. We denote it by \( \text{DSupp}(\rho_*Z) \). As a practical matter, we usually think of \( \rho_*Z \) and \( \text{DSupp}(\rho_*Z) \) as the same object.

Let \( F \) be a coherent sheaf on \( \mathbb{P}^d_S \) that is generically flat and pure over \( S \) of dimension \( d \). One can associate to it a cycle \( Z(F) \) that is a well defined family of \( d \)-cycles over \( S \) (cf. [Kol96, I.3.15]). Let \( \rho : \text{Supp} F \to \mathbb{P}^d_{S} \) be a finite morphism. As in (44.3) we get that

\[
\text{DSupp}_S(\rho_*F) = \text{DSupp}_S(\rho_*Z(F)).
\] (49.1)

Thus \( \text{DSupp}_S(\rho_*F) \) can be defined entirely in terms of cycles. Note, however, that here the right hand side is defined only for seminormal schemes.

One of the main aims defining the divisorial support for sheaves is to be able to work over arbitrary schemes.

3. Variants of K-flatness

We introduce 5 versions of K-flatness, which may well be equivalent to each other. From the technical point of view Cayley-Chow-flatness (or C-flatness) is the easiest to use, but a priori it depends on the choice of a projective embedding. Then most of the work in the next 2 sections goes to proving that a modified version (stable C-flatness) is equivalent to K-flatness, hence independent of the projective embedding.

**50** (Projections of \( \mathbb{P}^n \)). Projecting \( \mathbb{P}^n \) from the point \( (a_0 : \cdots : a_n) \) to the \( (x_n = 0) \) hyperplane is given by

\[
\pi : (x_0 : \cdots : x_n) \to (a_nx_0 - a_0x_n : \cdots : a_nx_{n-1} - a_{n-1}x_n). \tag{50.1}
\]

It is convenient to normalize \( a_n = 1 \) and then we get

\[
\pi : (x_0 : \cdots : x_n) \to (x_0 - a_0x_n : \cdots : x_{n-1} - a_{n-1}x_n). \tag{50.2}
\]

Similarly, a Zariski open set of projections of \( \mathbb{P}^n \) to \( L^r = (x_n = \cdots = x_{r+1} = 0) \) is given by

\[
\pi : (x_0 : \cdots : x_n) \to (x_0 - \ell_0(x_{r+1}, \ldots, x_n) : \cdots : x_r - \ell_r(x_{r+1}, \ldots, x_n)), \tag{50.3}
\]

where the \( \ell_i \) are linear forms.

Note that in affine coordinates, when we set \( x_0 = 1 \), the projections become

\[
\pi : (x_1, \ldots, x_n) \to \left( \frac{x_1 - \ell_1}{1 - \ell_0}, \ldots, \frac{x_r - \ell_r}{1 - \ell_0} \right), \tag{50.4}
\]
where again the $\ell_i$ are (homogeneous) linear forms in the $x_{r+1}, \ldots, x_n$. The coordinate functions have a non-linear expansion

$$\frac{x_i - \ell_i}{1 - \ell_0} = (x_i - \ell_i)(1 + \ell_0 + \ell_0^2 + \cdots). \quad (50.5)$$

Finally, non-linear projections are given as

$$\pi : (x_1, \ldots, x_n) \to (x_1 - \phi_1(x_1, \ldots, x_n), \ldots, x_r - \phi_r(x_1, \ldots, x_n)),$$

where $\phi_i(x_1, \ldots, x_r, 0, \ldots, 0) \equiv 0$ for every $i$.

**Definition 51.** Let $E$ be a vector bundle over a scheme $S$ and $F \subset E$ a vector subbundle. This induces a natural linear projection map $\pi : \mathbb{P}_S(E) \dashrightarrow \mathbb{P}_S(F)$. If $S$ is local then $E, F$ are free. After choosing bases, $\pi$ is given by a matrix of constant rank with entries in $\mathcal{O}_S$. We call these $\mathcal{O}_S$-projections if we want to emphasize this. If $S$ is over a field $k$, we can also consider $k$-projections, given by a matrix with entries in $k$. These, however, only make good sense if we have a canonical trivialization of $E$; this rarely happens for us.

We can now formulate various versions of K-flatness and their basic relationships.

**Definition 52.** Let $(s, S)$ first be a local scheme with infinite residue field and $F$ a generically flat family of pure, coherent sheaves of relative dimension $d$ on $\mathbb{P}^n_S$ (42), with scheme-theoretic support $Y := \text{SSupp } F$.

1. $F$ is $C$-flat over $S$ iff $\text{DSupp}(\pi_* F)$ is Cartier over $S$ for every $\mathcal{O}_S$-projection $\pi : \mathbb{P}^n_S \dashrightarrow \mathbb{P}^{d+1}_S$ (51) that is finite on $Y$.
2. $F$ is stably $C$-flat iff $(v_m)_* F$ is $C$-flat for every Veronese embedding $v_m : \mathbb{P}^n_S \hookrightarrow \mathbb{P}^N_S$ (where $N = (n + m) - 1$).
3. $F$ is K-flat over $S$ iff $\text{DSupp}(\rho_* F)$ is Cartier over $S$ for every finite morphism $\rho : Y \to \mathbb{P}^{d+1}_S$.
4. $F$ is locally K-flat over $S$ at $y \in Y$ iff $\text{DSupp}(\rho_* F)$ is Cartier over $S$ at $\rho(y)$ for every finite morphism $\rho : Y \to \mathbb{P}^{d+1}_S$ for which $\{y\} = \rho^{-1}(\rho(y))$.
5. $F$ is formally K-flat over $S$ at a closed point $y \in Y$ iff $\text{DSupp}(\rho_* F)$ is Cartier over $\hat{S}$ for every finite morphism $\rho : \hat{Y} \to \hat{\mathbb{P}}^{d+1}_S$, where $\hat{S}$ (resp. $\hat{Y}$) denotes the completion of $S$ at $s$ (resp. $Y$ at $y$).

**Base change properties 52.6.** We see in (64) that being C-flat is preserved by arbitrary base changes and the property descends from faithfully flat base changes. This then implies the same for stable C-flatness. Once we prove that the latter is equivalent to K-flatness, the latter also has the same base change properties. Most likely the same holds for formal K-flatness.

**General base schemes 52.7.** We say that any of the above notions (1–3) holds for a local base scheme $(s, S)$ (with finite residue field) if it holds after some faithfully flat base change $(s', S') \to (s, S)$, where $k(s')$ is infinite. Property (6) assures that this is independent of the choice of $S'$.

Finally we say that any of the above notions (1–3) holds for an arbitrary base scheme $S$ if it holds for all its localizations.

**Comment on the notation 52.8.** Here C stands for the initial of either Cayley or Chow and, as before, K stands for the first syllable of Cayley.

**Variants 53.** These definitions each have other versions and relatives. I believe that each of the above 5 are natural and maybe even optimal, though they may
not be stated in the cleanest form. Here are some other possibilities and equivalent versions.

1. It could have been better to define C-flatness using the Cayley-Chow form; the equivalence is proved in (61). The Cayley-Chow form version matches better with the study of Chow varieties; the definition in (52.1) emphasizes the similarity with the other 4.

2. In (52.3) we get an equivalent notion if we allow all finite morphisms \( ρ : Y \to W \), where \( W \to S \) is any smooth, projective morphism of pure relative dimension \( d + 1 \) over \( S \). Indeed, let \( π : W \to \mathbb{P}^{d+1}_S \) be a finite morphism.

   If \( F \) is K-flat then \( \text{DSupp}(π \circ ρ)_*F \) is a relative Cartier divisor, hence \( \text{DSupp}(ρ, F) \) is K-flat by (44.3). Since \( W \to S \) is smooth, \( \text{DSupp}(ρ, F) \) is a relative Cartier divisor by (67).

3. It would be natural to consider an affine version of C-flatness: We start with a coherent sheaf \( F \) on \( \mathbb{A}^n_S \) and require that \( \text{DSupp}(π)_*F \) be Cartier over \( S \) for every projection \( π : \mathbb{A}^n_S \to \mathbb{A}^{d+1}_S \) that is finite on \( Y \).

   The problem is that the relative affine version of Noether’s normalization theorem does not hold, thus there may not be any such projections; see [Kol17, 10.73.7]. This is why (52.4) is stated for projective morphisms only.

Nonetheless, the notions (52.1–4) are étale local on \( X \) and most likely the following Henselian version of (52.5) does work.

4. Assume that \( f : (y, Y) \to (s, S) \) is a local morphism of pure relative dimension \( d \) of Henselian local schemes such that \( k(y)/k(s) \) is finite. Let \( F \) be a coherent sheaf on \( X \) that is pure of relative dimension \( d \) over \( S \). Then \( F \) is K-flat over \( S \) if \( \text{DSupp}(ρ, F) \) is Cartier over \( S \) for every finite morphism \( ρ : Y \to \text{Spec} \mathcal{O}_S⟨x_0, \ldots, x_d⟩ \) (where \( R⟨x⟩ \) denotes the Henselization of \( R[x] \)).

   It is possible that in fact all 5 versions (52.1–5) are equivalent to each other, but for now we can prove only 8 of the 10 possible implications. Four of them are easy to see.

**Proposition 54.** Let \( F \) be a generically flat family of pure, coherent sheaves of relative dimension \( d \) on \( \mathbb{P}^n_S \). Then

\[
\text{formally K-flat} \Rightarrow \text{K-flat} \Rightarrow \text{locally K-flat} \Rightarrow \text{stably C-flat} \Rightarrow \text{C-flat}.
\]

**Proof.** A divisor \( D \) on a scheme \( X \) is Cartier iff its completion \( \hat{D} \) is Cartier on \( \hat{X} \) for every \( x \in X \) by (23). Thus formally K-flat \( \Rightarrow \) K-flat.

K-flat \( \Rightarrow \) locally K-flat is clear and locally K-flat \( \Rightarrow \) stably C-flat follows from (66). Finally stably C-flat \( \Rightarrow \) C-flat is proved in (70). \( \square \)

A key technical result of the paper is the following, proved in (78).

**Theorem 55.** K-flatness is equivalent to stable C-flatness.

It is quite likely that our methods will prove the following.

**Conjecture 56.** Formal K-flatness is equivalent to K-flatness.

We prove the special case of relative dimension 1 in (76); this is also a key step in the proof of (55).

The remaining question is whether C-flat implies stably C-flat. This holds in the examples that I computed, but I have not been able to compute many and I do not
have any conceptual argument why these 2 notions should be equivalent. See also (79) for a related question about the ideal of Chow equations.

**Question 57.** Is C-flatness equivalent to stable C-flatness?

All of the above properties are automatic over reduced schemes and they can be checked on Artin subschemes.

**Proposition 58.** Let $S$ be a reduced scheme and $F$ a generically flat family of pure, coherent sheaves of relative dimension $d$ on $\mathbb{P}^n_S$. Then $F$ is K-flat over $S$.

Proof. This follows from (45).

**Proposition 59.** Let $S$ be a scheme and $F$ a generically flat family of pure, coherent sheaves of relative dimension $d$ on $\mathbb{P}^n_S$. Then $F$ satisfies one of the properties (52.1–5) iff $\tau^* F$ satisfies the same property for every Artin subscheme $\tau: A \hookrightarrow S$.

Proof. Let $\pi: X \rightarrow \mathbb{P}^{d+1}_A$ be a finite morphism. By (23) $\text{DSupp}_S(\pi_* F)$ is Cartier iff $\text{DSupp}_A((\pi_A)_* \tau^* F)$ is Cartier for every Artin subscheme $\tau: A \hookrightarrow S$. Thus the Artin versions imply the global ones in all cases.

To check the converse, we may localize at $\tau(A)$. The claim is clear if every finite morphism $\pi_A: X_A \rightarrow \mathbb{P}^{d+1}_A$ can be extended to $\pi: X \rightarrow \mathbb{P}^{d+1}_S$. This is obvious for C-flatness and stable C-flatness, but it need not hold for K-flatness.

However, we see in (69) that it is enough to extend it after composition with a high enough Veronese embedding.

**4. Cayley-Chow flatness**

Let $Z \subset \mathbb{P}^n$ be a subvariety of dimension $d$. Cayley [Cay1860, Cay1862] associates to it a hypersurface

$$\text{Ch}(Z) := \{ L \in \text{Gr}(n-d-1, \mathbb{P}^n) : Z \cap L \neq \emptyset \} \subset \text{Gr}(n-d-1, \mathbb{P}^n),$$

called the Cayley or Chow hypersurface; its equation is called the Cayley or Chow form.

We extend this definition to coherent sheaves on $\mathbb{P}^n_S$ over an arbitrary base scheme. We use 2 variants, but the proof of (61) needs 2 other versions as well. All of these are defined in the same way, but $\text{Gr}(n-d-1, \mathbb{P}^n)$ is replaced by other universal varieties.

**Definition 60** (Cayley-Chow hypersurfaces). Let $S$ be a scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ on $\mathbb{P}^n_S$ (42). We define 4 versions of the Cayley-Chow hypersurface associated to $F$ as follows.

In all 4 versions the left hand side map $\sigma$ is a smooth fiber bundle.

**Grassmannian version 60.1.** Consider the diagram

$$\begin{array}{ccc}
\text{Flag}_S((\text{point}), n-d-1, \mathbb{P}^n) & \xrightarrow{\sigma} & \text{Gr}_S(n-d-1, \mathbb{P}^n) \\
\mathbb{P}^n_S & \downarrow_{\pi_g} & \\
& & \end{array}$$

where the flag variety parametrizes pairs (point) $\in L^{n-d-1} \subset \mathbb{P}^n$. Set

$$\text{Ch}_g(F) := \text{DSupp}_S((\pi_g)_* \sigma^*_g F).$$

1The titles of these articles are identical.
Product version 60.2. Consider the diagram

\[
\begin{array}{ccc}
\Inc_S((\text{point}), (\mathbb{P}^n)^{d+1}) & & \\
\sigma_p & \xrightarrow{\pi_p} & \mathbb{P}^n_S \\
\end{array}
\]

where the incidence variety parametrizes \((d+2)\)-tuples \((\text{point}), H_0, \ldots, H_d\) satisfying \((\text{point}) \in H_i\) for every \(i\). Set

\[\Ch_p(F) := \text{DSupp}_S((\pi_p)_* \sigma_p^* F).\]

Flag version 60.3. Consider the diagram

\[
\begin{array}{ccc}
\PFlag_S(0, n-d-2, n-d-1, \mathbb{P}^n) & & \\
\sigma_f & \xrightarrow{\pi_f} & \Flag_S(n-d-2, n-d-1, \mathbb{P}^n) \\
\end{array}
\]

where the PFlag parametrizes triples \((\text{point}), L^{n-d-2}, L^{n-d-1}\) such that \((\text{point}) \in L^{n-d-1}\) and \(L^{n-d-2} \subset L^{n-d-1}\) (but the point need not lie on \(L^{n-d-2}\)). Set

\[\Ch_f(F) := \text{DSupp}_S((\pi_f)_* \sigma_f^* F).\]

Incidence version 60.4. Consider the diagram

\[
\begin{array}{ccc}
\Inc_S((\text{point}), L^{n-d-1}, (\mathbb{P}^n)^{d+1}) & & \\
\sigma_i & \xrightarrow{\pi_i} & \Inc_S(L^{n-d-1}, (\mathbb{P}^n)^{d+1}) \\
\end{array}
\]

where the incidence variety parametrizes \((d+3)\)-tuples \((\text{point}), L^{n-d-1}, H_0, \ldots, H_d\) satisfying \((\text{point}) \in L^{n-d-1} \subset H_i\) for every \(i\). Set

\[\Ch_i(F) := \text{DSupp}_S((\pi_i)_* \sigma_i^* F).\]

Theorem 61. Let \(S\) be a scheme and \(F\) a generically flat family of pure, coherent sheaves of dimension \(d\) on \(\mathbb{P}^n_S\). The following are equivalent.

1. \(\Ch_p(F) \subset (\mathbb{P}^n)^{d+1}_S\) is Cartier over \(S\).
2. \(\Ch_f(F) \subset \Gr_S(n-d-1, \mathbb{P}^n)\) is Cartier over \(S\).
3. \(\text{DSupp}(\pi_* F)\) is Cartier over \(S\) for every \(\mathcal{O}_S\)-projection \(\pi : \mathbb{P}^n_S \rightarrow \mathbb{P}^{d+1}_S\) (51) that is finite on \(\text{Supp} F\).

Proof. The extreme cases \(d = 0\) and \(d = n-1\) are somewhat exceptional, so we deal with them first.

If \(d = n-1\) then \(\Gr_S(n-d-1, \mathbb{P}^n_S) = \Gr_S(0, \mathbb{P}^n_S) \equiv \mathbb{P}^n_S\) and the only projection is the identity. Furthermore \(\Ch_g(F) = \text{DSupp}_S(F)\) by definition, so (2–4) are equivalent. If these hold then \(\Ch_p(F) = \Ch_p(\text{DSupp}_S(F))\) is also flat by (37). Conversely, for (1) \(\Rightarrow\) (2) the argument in (62) works.

If \(d = 0\) then \(F\) is flat over \(S\) and (1–3) hold by (39).

We may thus assume from now on that \(0 < d < n-1\). These cases are discussed in (62–63).
62 (Proof of (61.1) ⇔ (61.2)). To go between the product and the Grassmannian versions, the basic diagram is the following.

\[
\begin{array}{ccc}
\text{Inc}_S(L^{n-d-1}, \mathbb{P}^n_S) & \xleftarrow{\rho} & \mathbb{P}^d S \\
\uparrow & & \downarrow \sigma \quad \text{bundle} \\
S(n-d-1, \mathbb{P}^n_S) & \xrightarrow{\rho} & Gr_S(n-d-1, \mathbb{P}^n_S)
\end{array}
\]

The right hand side projection

\[\pi_2 : \text{Inc}_S(L^{n-d-1}, \mathbb{P}^n_S) \to Gr_S(n-d-1, \mathbb{P}^n_S)\]

is a \((\mathbb{P}^d)^{d+1}\)-bundle. Therefore \(Ch_i(F) = \pi_2^* Ch_i(F)\). Thus \(Ch_i(F)\) is Cartier over \(S\) iff \(Ch_i(F)\) is Cartier over \(S\). It remains to compare \(Ch_i(F)\) and \(Ch_i(F)\).

The left hand side projection

\[\pi_1 : \text{Inc}_S(L^{n-d-1}, \mathbb{P}^n_S) \to (\mathbb{P}^d)^{d+1}\]

is birational. It is an isomorphism over \((H_0, \ldots, H_d) \in (\mathbb{P}^n)^{d+1}\) iff \(\dim(H_0 \cap \cdots \cap H_d) = n-d-1\), the smallest possible. That is, when the rank of the matrix formed from the equations of the \(H_i\) is \(\leq d\). Thus \(pi_1^*\) is an isomorphism outside a subset of codimension \(n+1-d\) in each fiber of \(\pi_2\).

Therefore, if \(Ch_i(F)\) is Cartier over \(S\) then \(Ch_i(F)\) is Cartier over \(S\), outside a subset of codimension \(n+1-d\) \(\geq 3\) on each fiber of \(\pi_2\). Then \(Ch_i(F)\) is Cartier over \(S\) everywhere by (22).

Conversely, if \(Ch_i(F)\) is a relative Cartier divisor then so is \(\pi_1^* Ch_i(F)\), which is the union of \(Ch_i(F)\) and of the exceptional divisors. The latter are all relatively Cartier hence so is \(Ch_i(F)\).

63 (Proof of (61.2) ⇔ (61.3–5)). To go between the Grassmannian version and the projection versions, the basic diagram is the following.

\[
\begin{array}{ccc}
\text{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S) & \xleftarrow{\rho} & \mathbb{P}^{d+1} \\
\uparrow & & \downarrow \sigma \quad \text{bundle} \\
Gr_S(n-d-1, \mathbb{P}^n_S) & \xrightarrow{\rho} & Gr_S(n-d-2, \mathbb{P}^n_S)
\end{array}
\]

The left hand side projection

\[\rho_1 : \text{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S) \to Gr_S(n-d-1, \mathbb{P}^n_S)\]

is a \(\mathbb{P}^{n-d-1}\)-bundle and \(Ch_f(X) = \rho_1^* Ch_g(X)\). Thus \(Ch_g(F)\) is Cartier over \(S\) iff \(Ch_f(F)\) is Cartier over \(S\).

The right hand side projection

\[\rho_2 : \text{Flag}_S(n-d-2, n-d-1, \mathbb{P}^n_S) \to Gr_S(n-d-2, \mathbb{P}^n_S)\]

is a \(\mathbb{P}^{d+1}\)-bundle. Pick \(L \in \text{Gr}(n-d-2, \mathbb{P}^n_S)\). The fiber of \(\rho_2\) over \([L]\) is the set of all \(n-d-1\)-planes that contain \(L\); we can identify this with the target of the projection \(\pi_L : \mathbb{P}^n \to L^\perp\). So, if \(Ch_f(F)\) is Cartier over \(S\) then \(\text{DSupp}(\pi_L)_*(F) = Ch_f(F)|_{L^\perp}\) is also Cartier over \(S\).

Conversely, assume that \(\text{DSupp}(\pi_L)_*(F)\) is Cartier over \(S\) for general \(L\).

This gives an \(L^\perp\) in \(Gr(n-d-1, \mathbb{P}^n_S)\) where \(Ch_g(F)\) is Cartier. Since \(\dim_L L^\perp = d+1 \geq 2\), this implies that \(Ch_g(F)\) is Cartier over \(S\) outside a subset of codimension \(\geq 3\) on each fiber of \(\rho_2\). Then \(Ch_g(F)\) is Cartier over \(S\) everywhere by (22).

Corollary 64. Let \(S\) be a scheme and \(F\) a generically flat family of pure, coherent sheaves of dimension \(d\) on \(\mathbb{P}^n_S\). Let \(h : S' \to S\) be a morphism. By base change we get \(g' : X' \to S'\) and \(F' = v\text{-pure}(h_X^*F)\) (41).
(1) If $F$ is C-flat, then so is $F'$.

(2) If $F$ is C-flat and $h$ is faithfully flat then $F$ is C-flat.

Proof. We may assume that $S$ is local with infinite residue field. Being C-flat is exactly (61.3) which is equivalent to (61.1). $F \mapsto \text{Ch}_p(F)$ commutes with base change by (48) and, if $h$ is faithfully flat, then a divisorial sheaf is Cartier iff its pull-back is (cf. [Kol17, 4.22]). □

Definition 65. Let $S$ be a local scheme with infinite residue field and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ over $S$ (42). $F$ is locally C-flat over $S$ at $y \in Y := \text{SSupp } F$ iff $\text{DSupp}(\pi_*F)$ is Cartier over $S$ at $\pi(y)$ for every $\mathcal{O}_S$-projection $\pi : \mathbb{P}^n_S \to \mathbb{P}^{d+1}_S$ that is finite on $Y$ for which $\{y \} = \pi^{-1}(\pi(y)) \cap Y$.

Lemma 66. Let $S$ be a local scheme with infinite residue field and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ on $\mathbb{P}^n_S$. Then $F$ is C-flat iff it is locally C-flat at every point.

Proof. It is clear that C-flat implies locally C-flat.

Conversely, assume that $F$ is locally C-flat. Set $Z_s := \text{Supp}(F_s) \setminus \text{FlatCM}(F)$ and pick points $\{y_i : i \in I\}$, one in each irreducible component of $Z_s$. If $\pi : \mathbb{P}^{n}_S \to \mathbb{P}^{d+1}_S$ is a general $\mathcal{O}_S$-projection, then $\{y_i\} = \pi^{-1}(\pi(y_i)) \cap Y$ for all $i \in I$.

Note that $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor along $\mathbb{P}^{d+1}_s \setminus \pi(Z_s)$ by (37) and it is also relative Cartier at the points $\pi(y_i)$ for $i \in I$ since $F$ is locally C-flat. Thus $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor outside a codimension $\geq 3$ subset of $\mathbb{P}^{d+1}_s$, hence a relative Cartier divisor everywhere by (22). □

Corollary 67. Let $(s, S)$ be a local scheme and $X \subset \mathbb{P}^n_S$ a closed subscheme that is flat over $S$ of pure relative dimension $d + 1$. Let $D \subset X$ be a relative Mumford divisor. Let $x \in X_s$ be a smooth point. Then $\mathcal{O}_D$ is locally C-flat at $x$ iff $D$ is a relative Cartier divisor at $x$.

Proof. We may assume that $S$ has infinite residue field. A general linear projection $\pi : X \to \mathbb{P}^{d+1}_S$ is étale at $x$. Thus $D$ is a relative Cartier divisor at $x$ iff $\pi(D)$ is a relative Cartier divisor at $\pi(x)$. By (44.2) the latter holds iff $\mathcal{O}_D$ is locally C-flat at $x$. □

Corollary 68. Let $S$ be a scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ over $S$ (42). If $F$ is flat at $y \in Y := \text{SSupp } F$ then it is also locally C-flat at $y$.

Proof. By [Kol17, 10.8], $F_s$ is CM outside a subset $Z_s \subset Y_s$ of codimension $\geq 2$. Let $W_s \subset Y_s$ be the set of points where $F$ is not flat.

Let $\pi : Y \to \mathbb{P}^{d+1}_S$ be a general linear projection. By (37) $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor outside $\pi(Z_s \cup W_s)$. Moreover, we may assume that $\pi(y) \notin \pi(W_s)$. Thus, in a neighborhood of $\pi(y)$, $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor outside the codimension $\geq 3$ subset of $\pi(Z_s)$. Thus $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor at $y$ by (22). □

Lemma 69. Let $S$ be a scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ on $\mathbb{P}^n_S$. Set $Y := \text{SSupp } F$ and let $\pi : Y \to \mathbb{P}^{d+1}_S$ be a finite morphism. Let $g_m : Y \hookrightarrow \mathbb{P}^{n}_S$ be an embedding such that $g_m^*\mathcal{O}_{\mathbb{P}^n_S}(1) \cong \pi^*\mathcal{O}_{\mathbb{P}^{d+1}_S}(m)$ for some $m \gg 1$.

If $(g_m)_*F$ is C-flat then $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor.
Proof. We may assume that $S$ is local with infinite residue field. Choosing $d+2$ general sections of $\mathcal{O}_{P^n_S}(m)$ gives a morphism $w_m : \mathbb{P}^d_S \rightarrow \mathbb{P}^d_S$ and there is a linear projection $\rho : \mathbb{P}^N_S \rightarrow \mathbb{P}^d_S$ such that $w_m \circ \pi = \rho \circ g_m$. By assumption $\text{DSupp}((\rho \circ g_m)_* F)$ is a relative Cartier divisor, hence so is

$$\text{DSupp}((w_m \circ \pi)_* F) = \text{DSupp}((w_m)_* \text{DSupp}(\pi_* F)),$$

where the equality follows from (44.3).

Pick a point $x \in \text{DSupp}(\pi_* F)$. A general $w_m$ is étale at $x$ and $\{x\} = w_m^{-1}(w_m(x)) \cap \text{DSupp}(\pi_* F)$. Thus $w_m : \text{DSupp}(\pi_* F) \rightarrow \text{DSupp}((w_m \circ \pi)_* F)$ is étale at $x$. Thus $\text{DSupp}(\pi_* F)$ is Cartier at $x$.

Corollary 70. Let $S$ be a scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ on $\mathbb{P}^n_S$ (42). Let $v_m : \mathbb{P}^n_S \rightarrow \mathbb{P}^n_S$ be the $m$th Veronese embedding. If $(v_m)_* F$ is $C$-flat then so is $F$.

Bertini theorems for C-flatness.

Lemma 71. Let $(s, S)$ be a local scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d \geq 1$ on $\mathbb{P}^n_S$ (42). Set $Z_s := \text{Supp}(F_s) \setminus \text{FlatCM}(F)$. Let $H \subset \mathbb{P}^n_S$ be a hyperplane that does not contain any irreducible component of $Z_s$. If $F$ is $C$-flat then so is $F|_H$.

Proof. We may assume that the residue field is infinite. Every projection $H \rightarrow \mathbb{P}^d_S$ is obtained as the restriction of a projection $\mathbb{P}^n_S \rightarrow \mathbb{P}^d_S$. The rest follows from (46).

Lemma 72. Let $(s, S)$ be a local scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d \geq 2$ on $\mathbb{P}^n_S$. Then $F$ is $C$-flat iff $F|_H$ is $C$-flat for an open, dense set of hyperplanes $H$.

Proof. One direction follows from (71). Conversely, if $F|_H$ is $C$-flat for an open, dense set of hyperplanes $H$ then there is an open, dense set of projections $\pi : \mathbb{P}^n_S \rightarrow \mathbb{P}^d_S$ such that for an open, dense set of hyperplanes $L \subset \mathbb{P}^d_S$, the restriction of $F$ to $\pi^{-1}(L)$ is $C$-flat. Thus $\text{DSupp}(\pi_* F)$ is a relative Cartier divisor in an open neighborhood of such an $L$, by (47). Since $d \geq 2$, this implies that $\text{DSupp}(\pi_* F)$ is a relative Cartier divisor everywhere by (22). Thus $F$ is $C$-flat by (61).

Lemma 73. Let $(s, S)$ be a local scheme and $F$ a stably $C$-flat family of pure, coherent sheaves of dimension $d \geq 1$ over $S$. Set $Y := \text{SSupp} F$, $Z_s := Y \setminus \text{FlatCM}(F)$ and let $D \subset Y$ be a relative Cartier divisor that does not contain any irreducible component of $Z_s$. Then $F|_D$ is also stably $C$-flat.

Proof. We may assume that the residue field is infinite. By (66) it is sufficient to prove that $F|_D$ is locally $C$-flat. Pick a point $y \in D$ and let $H \supset D$ be a hypersurface section of $Y$ that does not contain any irreducible component of $Z_s$ and such that $H$ equals $D$ in a neighborhood of $y$. After a Veronese embedding $H$ becomes a hyperplane section, and then (71) implies that $F|_H$ is stably $C$-flat. Hence $F|_H$ is locally $C$-flat and so $F|_D$ also locally $C$-flat at $y$.

Definition 74. Let $S$ be a local scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d \geq 1$ over $S$. Set $Y := \text{SSupp} F$ and let $L$ be a relatively ample line bundle on $Y$. We say that $F$ is stably $C$-flat for $L$ over $S$ iff
τ_× F is C-flat for every embedding τ : X → ℙ^n_S such that τ^* O_{ℙ^n_S}(1) ∼= L^m for some m ≥ 1.

By (69) this notion is unchanged if we replace L by L^r for some r > 0.

**Lemma 75.** Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension d ≥ 1 over S. Let L, M be relatively ample line bundles on X. Then F is stably C-flat for L if and only if M is stably C-flat for M.

Proof. Assume that F is stably C-flat for M. We may assume that L is very ample. Repeatedly using (73) we get that, for general L_i ∈ |L|, the restriction of F to the complete intersection curve L_1 ∩ ... ∩ L_{d-1} ∩ Y is stably C-flat for M. Thus the restriction of F to L_1 ∩ ... ∩ L_{d-1} ∩ Y is formally K-flat by (76). Using (76) in the other direction for L, we get that the restriction of F to L_1 ∩ ... ∩ L_{d-1} ∩ Y is stably C-flat for L. Now we can use (72) to conclude that F is stably C-flat for L.

**Proposition 76.** Let (s, S) be a local scheme and F a generically flat family of pure, coherent sheaves of dimension 1 over S. Then F is stably C-flat ⇔ K-flat ⇔ formally K-flat.

Proof. We already proved in (54) that formally K-flat ⇒ K-flat ⇒ stably C-flat.

Thus assume that F is stably C-flat. Set Y := SSupp F and pick a closed point p ∈ Y. We need to show that F is formally K-flat at p. By (59) it is enough to prove this for Artin base schemes and after a faithfully flat extension. We may thus assume that S = Spec A for a local Artin ring (A, m, k) with k infinite and p ∈ Y_s(k).

Let π : Y → ℋ^2_S = Spec A[[u, v]] be a finite morphism. After a linear coordinate change we may assume that the composite

$$
\hat{\rho} : \hat{Y} \to \text{Spec } A[[u, v]] \to \text{Spec } A[[u]]
$$

is also finite.

Thus \hat{\rho} \hat{F} is a coherent sheaf on Spec A[[u]]; let \hat{G} denote its restriction to the generic point. Since F is generically flat over A, \hat{G} is flat over A, hence we can write it as the sheafification of a free A((u))-module \oplus_j A((u)) e_j of rank n with basis \{e_j\}. Then multiplication by v is given by a matrix M = (m_{ij}(u)) where the m_{ij}(u) ∈ A((u)) are Laurent series in u. By (37) DSupp(\hat{\pi}, \hat{F}) is given by

$$
(det(M − v1_n) = 0) \subset \hat{Y}^2_S.
$$

(76.1)

The finite map \hat{\pi} is given by 2 power series u, v. Fix some m_0 ∈ ℤ, to be determined later. By (77), for m ≥ m_0 we can choose homogeneous polynomials g_1, g_2 ∈ H^0(ℙ^n_A, O_{ℙ^n_A}(m)) such that

$$
\pi : Y \to ℙ^2_S \text{ given by } (x_0^m : g_1 : g_2)
$$

is a finite morphism,

$$
g_1/x_0^m \equiv u \mod O_Y(-p)^{m_0} \text{ and } g_2/x_0^m \equiv v \mod O_Y(-p)^{m_0},
$$

(76.3)

where O_Y(-p) is the ideal sheaf of p ∈ Y. Since the relative dimension is 1, O_Y(-p)^c ⊂ uO_Y for some c ∈ ℤ, thus

$$
g_1/x_0^m \equiv u \mod u^{m_0/c}O_Y \text{ and } g_2/x_0^m \equiv v \mod u^{m_0/c}O_Y
$$

(76.4)

holds, whenever c | m_0.
We now compute $\text{DSupp}(\pi_*F)$. This leads to a matrix $M'$ using $u' := g_1/x_0^n, v' := g_2/x_0^n$, and we see that

$$M(u) \equiv M'(u') \mod u^{m_{o/c}}A[[u]].$$

Let $r$ be the maximal pole order of the $m_{ij}(u)$. Expanding the determinant we get terms whose maximal pole order is $\leq nr$. Thus (76.5) implies that

$$\det(M - v1_n) \equiv \det(M' - v'1_n) \mod u^{-nr+m_{o/c}}A[[u,v]].$$

If $m_0 \gg 1$ (depending on $c,n,r$ and $M$) then, by (30), $\det(M - v1_n)$ defines a Cartier divisor iff $\det(M' - v'1_n)$ does. Since $F$ is stably C-flat, $\det(M' - v'1_n)$ defines a Cartier divisor, hence so does $\det(M - v1_n)$. Thus $F$ is formally K-flat at $p$. \hfill $\square$

77 (Approximation of formal projections). Let $(s,S)$ be a local scheme and $Y \subset \mathbb{P}^d_S$ a closed subset of pure relative dimension $d$. Let $p \in Y_s$ be a closed point with maximal ideal $m_p$ such that $x_0(p) \neq 0$. Let $(g_1: \cdots: g_e) : \hat{Y}_p \to \hat{k}^e_S$ be a finite morphism. Fix $m_0 \in \mathbb{N}$.

Then, for every $m \gg m_0$ there are $g_1, \ldots, g_e \in H^0(\mathbb{P}^d_S, \mathcal{O}_{\mathbb{P}^d_S}(m))$ such that

\begin{enumerate}
\item $\pi : (x_0^m.g_1: \cdots: g_e) : Y \to \mathbb{P}^e_S$ is a finite morphism,
\item $\pi^{-1}(\pi(p)) \cap Y = \{p\}$ and
\item $g_i \equiv g_i/x_0^m \mod m_p^{m_0}$ for every $i$.
\end{enumerate}

\hfill $\square$

78 (Proof of (55)). We already noted in (54) that $K$-flat $\Rightarrow$ stably C-flat.

To see the converse, assume that $F$ is stably C-flat and let $\pi : X \to \mathbb{P}^{d+1}_S$ be a finite projection. Set $L := \pi^*\mathcal{O}_{\mathbb{P}^{d+1}_S}(1)$. By (75) $F$ is stably C-flat for $L$, hence $\text{DSupp}(\pi_*F)$ is a relative Cartier divisor by (69). \hfill $\square$

5. Ideal of Chow equations

The ideal of Chow equations was introduced in various forms in [Cat92, DS95, Kol99].

**Definition 79** (Ideal of Chow equations). Let $S$ be a local scheme with infinite residue field and $Y \subset \mathbb{P}^d_S$ a generically flat family of pure subschemes of dimension $d$ (42). Let $\pi : \mathbb{P}^d_S \dashrightarrow \mathbb{P}^{d+1}_S$ be a projection that is finite on $Y$ and let $f(\pi,Y)$ be an equation of $\text{DSupp}_S(\pi_*\mathcal{O}_Y)$. The **ideal of Chow equations** of $Y$ is

$$I^{ch}(Y) := (\pi^*f(\pi,Y) : \pi \text{ finite on } Y) \subset \mathcal{O}_S[x_0, \ldots, x_n].$$

We use 2 other versions of this concept. If $I_Y$ is the ideal sheaf or the homogeneous ideal of $Y$, then sometimes we write $I^{ch}(I_Y)$ instead of $I^{ch}(Y)$. If $S$ is reduced and $Z$ is well defined family of $d$-cycles, then $\text{DSupp}_S(\pi_*D)$ is defined in (79), hence $I^{ch}(Z)$ can be defined as in (79.1). We see that if $Z(Y)$ denotes the well defined family of $d$-cycles associated to $Y$ then

$$I^{ch}(Y) = I^{ch}(Z(Y)).$$

One can also define the obvious formal-local version of this notion. In the examples that I know of, the global version is compatible with the formal-local one, but this is not known in general. This is closely related to (57).

One difficulty is that a typical space curve has non-linear projections that are not isomorphic to linear projections, not even locally analytically; see for example (118).
We would like to compare the ideal of Chow equations with the ideal of $Y$. A rather straightforward result is the following.

**Proposition 80.** [Kol99, Sec.8] Let $Y \subset \mathbb{P}^n_S \to S$ be $C$-flat. Then

$$ I^{ch}(\text{pure}(Y_s)) \subset I(Y_s). \quad \square $$

To get a more precise answer, we need some definitions.

81 (Element-wise powers of ideals). Let $R$ be a ring, $I \subset R$ an ideal and $m \in \mathbb{N}$. Set

$$ I^{[m]} := (r^m : r \in I). $$

These ideals have been studied mostly when char $k = p > 0$ and $m = q$ is a power of $p$; one of the early occurrences is in [Kun76]. In these cases $I^{[q]}$ is called a Frobenius power of $I$. Other values of the exponent are also interesting, the following properties follow from (81.6). We assume for simplicity that $R$ is a $k$-algebra.

1. If $I$ is principal then $I^{[m]} = I^m$.
2. If char $k = 0$ then $I^{[m]} = I^m$.
3. If $m < \text{char } k$ then $I^{[m]} = I^m$.
4. If $k$ is infinite then $(r_1, \ldots, r_n)^{[m]} = \left( (\sum c_i r_i)^m : c_i \in k \right)$.
5. If $k$ is infinite and $U \subset k^n$ is Zariski dense then

$$ (r_1, \ldots, r_n)^{[m]} = \left( (\sum c_i r_i)^m : (c_1, \ldots, c_n) \in U \right). $$

Note that (2) is close to being optimal. For example, if $I = (x, y) \subset k[x, y]$ and char $k = p \geq 3$ then

$$ (x, y)^{[p+1]} = (x^{p+1}, x^p y, xy^p, y^{p+1}) \subsetneq (x, y)^{p+1}. $$

**Claim 81.6.** Let $k$ be an infinite field. Then

$$ \langle (c_1 x_1 + \cdots + c_n x_n)^m : c_i \in k \rangle = \langle x_1^{i_1} \cdots x_n^{i_n} : (i_1, \ldots, i_n) \neq \emptyset \rangle. $$

Here $\binom{m}{i_1, \ldots, i_n}$ denotes the multinomial coefficient, that is, the coefficient of $x_1^{i_1} \cdots x_n^{i_n}$ in $(x_1 + \cdots + x_n)^m$.

**Proof.** The containment $\subset$ is clear. If the 2 sides are not equal then the left hand side is contained in some hyperplane of the form $\sum \lambda_i x_i = 0$, but this would give a nontrivial polynomial identity $\sum (\binom{m}{i_1, \ldots, i_n}) \lambda_i x_i = 0$ for the $c_i$. $\square$

**Proposition 82.** Let $k$ be an infinite field and $Z_i \subset \mathbb{P}^n$ be distinct, geometrically irreducible and reduced $k$-cycles of dimension $d$. Then

$$ \text{pure}(\mathcal{O}_{\mathbb{P}^n}/I^{ch}(\sum m_i Z_i)) = \text{pure}(\mathcal{O}_{\mathbb{P}^n}/\cap_i I(Z_i)^{[m_i]}). \quad (82.1) $$

**Proof.** Assume first that $d = 0$, thus $Z_i = \{p_i\}$. Let $\ell_1$ be a linear form on $\mathbb{P}^n$ that vanishes at $p_i$ but not at the other points. Choosing a general $\ell_0$ we get a projection $\pi := (\ell_0, \ell_1) : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$. At $p_i$, the resulting Chow equation can be written as $I^{[m]}_i(\text{unit})$. By (81.5) these generate $I(p_i)^{[m]}$. If $d \geq 0$ then it is clear that

$$ I^{ch}(\sum m_i Z_i) \subset \cap_i I(Z_i)^{[m_i]} , \quad (82.2) $$

In order to check that they agree generically, choose a projection $\rho : \mathbb{P}^n \dashrightarrow \mathbb{P}^d$ that is finite on $\cup_i Z_i$, and a point $p \in \mathbb{P}^d$ such that $\rho$ is étale over $p$.

Let $L \cong \mathbb{P}^{n-d}$ denote the closure of $\rho^{-1}(p)$ and set $W_i := L \cap Z_i$. It is clear that

$$ I(Z_i)^{[m_i]}|_L = I(W_i)^{[m_i]} \quad \text{and} \quad I^{ch}(m_i W_i) \subset I^{ch}(m_i Z_i)|_L. \quad (82.3) $$
(See (88) for the second of these.) Thus we get that
\[ \cap_i I(Z_i)^{[m_i]}|_L = \cap_i I(W_i)^{[m_i]} = I^{ch}(\sum_i m_i W_i) \subset I^{ch}(\sum_i m_i Z_i)|_L. \] (82.4)
Therefore
\[ \cap_i I(Z_i)^{[m_i]}|_L = I^{ch}(\sum_i m_i Z_i)|_L. \] (82.5)
Note that \( \cap_i I(Z_i)^{[m_i]} \) is flat over \( p \), thus (82.1) and (82.3) together imply that
\[ \cap_i I(Z_i)^{[m_i]} = I^{ch}(\sum_i m_i Z_i) \quad \text{near } L \cap \text{Supp } Z; \]
see for example [Kol96, I.7.4.1]. \( \square \)

**Definition 83.** We call the subscheme (82.1) the Chow hull of the cycle \( Z = \sum m_i Z_i \), and denote it by \( \text{CHull}(Z) \).

The following consequence is key to our study of Mumford divisors.

**Corollary 84.** Let \( k \) be an infinite field, \( X \subset P^N_k \) a reduced subscheme of pure dimension \( n + 1 \) and \( D \subset X \) a Mumford divisor of degree \( d \). Then
\[ \text{pure}(X \cap \text{CHull}(D)) = D. \] (84.1)

Proof. After a field extension we can write \( D = \sum m_i D_i \) where the \( D_i \) are geometrically irreducible and reduced. Then (82) says that
\[ \text{CHull}(D) = \text{pure}(\text{Spec } O_{\mathbb{P}^N_k}/\cap_i I(D_i)^{[m_i]}). \]

Let \( g_i \in D_i \) be the generic point and \( R_i \) its local ring in \( P^N_k \). Let \( J_i \subset R_i \) be the ideal defining \( X \) and \( (J_i, h_i) \) the ideal defining \( D_i \). The ideal defining the left hand side of (84.1) is then \( (J_i + (J_i, h_i)^{[m_i]})/J_i \). This is the same as \( (h_i)^{[m_i]} \), as an ideal in \( R_i/J_i \), which equals \( h_i^{m_i} \) by (81.1). \( \square \)

Combining (80) and (82), we get the following partial answer to Question 15.

**Corollary 85.** [Kol99, Sec.8] Let \( C \to S \) be a \( C \)-flat family whose pure fibers are geometrically reduced curves. Then
\[ \text{tors } C_s \subset I(\text{pure } C_s)/I^{ch}(\text{pure } C_s). \] \( \square \)

It is not clear that (85) is optimal, but the next example shows that is close to it in some directions.

**Example 86.** Consider the monomial curve \( s \mapsto (s^a, s^{a+1}) \) with equation \( x^{a+1} = y^a \). The surface \( \{s, t) \mapsto (s^a, s^{a+1}, st, t) \) defines a non-flat deformation of it in \( A^3_{xyz} \times A^1_t \).

If \( t \neq 0 \) then we get a complete intersection with equations \( x^{a+1} - y^a = xz - ty = 0 \). Near \( t = 0 \) we need more equations. These are obtained by multiplying \( x^{a+1} = y^a \) with \( (t/x)^m = (z/y)^m \) for \( 0 < m \leq a \) to get \( x^{a+1-m}y^m = y^{a-m}z^m \). This shows that the central fiber is
\[ \text{Spec } k[x, y, z]/(x + (y, z)^{a-1}). \]

Thus the torsion at the origin is isomorphic to \( k[y, z]/(y, z)^{a-1} \).

We compute in (87.7) that the ideal of Chow equations is
\[ (x^{a+1} - y^a, z^i x^{a+1-i} - y^{a-i} : i = 1, \ldots, a). \]
Thus (85) is close to being optimal, as far as the \( y, z \) variables are concerned.
Example 87. [Kol99, 4.8] Using (50.1) we see that the ideal of Chow equations of the codimension 2 subvariety \((x_{n+1} = f(x_0, \ldots, x_n) = 0) \subset \mathbb{P}^{n+1}\) is generated by the forms
\[
f(x_0 - a_0 x_{n+1} : \cdots : x_n - a_n x_{n+1}) \text{ for all } a_0, \ldots, a_n. \tag{87.1}
\]
If the characteristic is 0 then Taylor’s theorem gives that
\[
f(x_0 - a_0 x_{n+1} : \cdots : x_n - a_n x_{n+1}) = \sum_I \frac{(-1)^I}{I!} a^I \frac{\partial^I f}{\partial x^I x_{n+1}}
\]
where \(I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}\). The \(a^I\) are linearly independent, hence we get that the ideal of Chow equations is
\[
I^\text{ch}(f(x_0, \ldots, x_n), x_{n+1}) = (f, x_{n+1} D(f), \ldots, x_{n+1}^m D^m(f)), \tag{87.3}
\]
where we can stop at \(m = \deg f\). Here we use the usual notation for derivative ideals
\[
D(f) := (f, \frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}). \tag{87.4}
\]
(Note that if we work with the ideal \((f)\) and not just the polynomial \(f\), then we must include \(f\) itself in its derivative ideal.)

If we want to work locally at the point \((x_1 = \cdots = x_n = 0)\), the we can set \(x_0 = 1\) to get the local version
\[
I^\text{ch}(f(x_1, \ldots, x_n), x_{n+1}) = (f, x_{n+1} D(f), \ldots, x_{n+1}^m D^m(f)), \tag{87.7}
\]
where we can now stop at \(m = \text{mult} f\). This also holds if \(f\) is an analytic function, though this needs to be worked out using the more complicated formulas (50.6) that for us become
\[
\pi: (x_1, \ldots, x_n, x_{n+1}) \rightarrow (x_1 - x_{n+1} \psi_1, \ldots, x_n - x_{n+1} \psi_n), \tag{87.8}
\]
where \(\psi_i = \psi_i(x_0, \ldots, x_{n+1})\) are analytic functions. Expanding as in (87.2) we see that
\[
f(x_1 - x_{n+1} \psi_1, \ldots, x_n - x_{n+1} \psi_n) \in I^\text{ch}(f(x_1, \ldots, x_n), x_{n+1}). \tag{87.9}
\]
Thus we get the same ideal if we compute \(I^\text{ch}\) using analytic or formal projections.

The next example shows that taking the ideal of Chow equations does not commute with general hyperplane sections; see [Kol99, 5.1] for a positive result.

Example 88. Start with \(F = f(x, y) + zg(x, y)\) where \(\text{mult}_0 f = \text{mult}_0 g = m\). So family is equimultiple along the \(z\)-axis.

We compute the ideal of Chow equations for \((F(x, y, z), t) \subset k[x, y, z, t]\) and also for its restrictions \((F(x, y, c), t) \subset k[x, y, t]\). The first one is
\[
(F, tD(F), t^2 D^2(F), \ldots).
\]
Here \(D(F) = (F, F_x, F_y, F_z)\), thus setting \(z = c\) we get
\[
D(F(x, y, z))|_{z=c} = (f + cg, f_x + cg_x, f_y + cg_y, g).
\]
By contrast
\[
D(F(x, y, c)) = (f + cg, f_x + cg_x, f_y + cg_y). \tag{88.2}
\]
We have an extra term \(g\) in (88.1), which shows the following.

Claim 88.3. If \(g \notin (f + cg, f_x + cg_x, f_y + cg_y)\) then taking \(I^\text{ch}\) does not commute with restriction to \((z = c)\). \qed
To get a concrete example, take \( f(x, y) = x^4 + y^4, g(x, y) = x^2y^2 \). The ideal \((f_x + cg_x, f_y + cg_y, f + cg)\) contains 5 obvious degree 4 elements, but Euler’s equation
\[
x(f_x + cg_x) + y(f_y + cg_y) = 4(f + cg)
\]
tells us that they are dependent. An easy explicit computation shows that \( x^2y^2 \) is not in the ideal.

Note also that we get the exact same computation if we restrict to some other plane \( z = c + ax + by \).

**Remark 88.4.** It is possible that this can be used to improve on (85) and get a better answer to (15).

### 6. Representability Theorems

#### Definition 89.
Let \( S \) be a scheme and \( F \) a generically flat family of pure, coherent sheaves of dimension \( d \) on \( \mathbb{P}^n_S \). The functor of C-flat pull-backs is
\[
CFlat_F(q : T \to S) = \begin{cases}
1 & \text{if } q^*F \to T \text{ is C-flat, and} \\
0 & \text{otherwise},
\end{cases}
\]
where \( q^*F := v\text{-pure}(q^*F) \) is the divisorial pull-back as in (1.5) or (41). We say that a monomorphism \( j_F^{\text{c-flat}} : S_F^{\text{c-flat}} \to S \) represents \( CFlat_F \), provided \( CFlat_F(q) = 1 \) if \( q \) factors as \( q : T \to S_F^{\text{c-flat}} \to S \).

If \( Y \subset \mathbb{P}^n_S \) is a generically flat family of pure subschemes of dimension \( d \) then we write \( CFlat_Y \) instead of \( CFlat_{\mathbb{Q}_Y} \).

One defines analogously the functor of stably C-flat pull-backs \( SCFlat_F \), and the functor of K-flat pull-backs \( KFlat_F \). The monomorphisms representing them are denoted by \( j_F^{\text{sc-flat}} : S_F^{\text{sc-flat}} \to S \) and \( j_F^{\text{k-flat}} : S_F^{\text{k-flat}} \to S \).

Let \( f : X \to S \) be a flat morphism and \( D \) a relative Mumford divisor on \( X \). The functor of Cartier pull-backs is
\[
Cartier_D(q : T \to S) = \begin{cases}
1 & \text{if } q^*(D) \to T \text{ is Cartier, and} \\
0 & \text{otherwise.}
\end{cases}
\]
By [Kol17, 4.35], the functor of Cartier pull-backs is represented by a monomorphism \( j_D^{\text{c-car}} : S_D^{\text{c-car}} \to S \). (Note. Unfortunately [Kol17, 4.35] is about reduced schemes, but this assumption is not necessary in the proof. This will be fixed later.)

An immediate consequence of (61) is the following.

**Proposition 90.** Let \( S \) be a scheme and \( F \) a generically flat family of pure, coherent sheaves of dimension \( d \) on \( \mathbb{P}^n_S \). Then the functor of C-flat pull-backs of \( F \) is represented by a monomorphism \( j_F^{\text{c-flat}} : S_F^{\text{c-flat}} \to S \).

Proof. By (61), \( j_F^{\text{c-flat}} : S_F^{\text{c-flat}} \to S \) is the same as \( j_{\text{Ch}_p(F)}^{\text{c-car}} : S_{\text{Ch}_p(F)}^{\text{c-car}} \to S \), with the Chow hypersurface \( \text{Ch}_p(F) \) as defined in (60.2). \(\square\)

**Corollary 91.** Let \( S \) be a scheme and \( F \) a generically flat family of pure, coherent sheaves of dimension \( d \) on \( \mathbb{P}^n_S \). Then the functor of stably C-flat pull-backs of \( F \) is represented by a monomorphism \( j_F^{\text{sc-flat}} : S_F^{\text{sc-flat}} \to S \).

Proof. Let \( F_m \) denote the push-forward of \( F \) by the \( m \)th Veronese embedding \( v_m \). Then \( S_F^{\text{sc-flat}} \) should be the fiber product
\[
S_{F_1}^{\text{sc-flat}} \times S_{F_2}^{\text{sc-flat}} \times \cdots.
\]
(91.1)
However, this need not make sense if the product is truly infinite. To understand this, we make a slight twist and as the first step we replace $S$ by $S_F^{c\text{flat}}$ and $F$ by $F^* := (j_F^{c\text{flat}})^{[*]} F$. Then we consider

$$j_F^{c\text{flat}} : (S_F^{c\text{flat}})_{F_m}^{c\text{flat}} \to S_F^{c\text{flat}}. \tag{91.2}$$

Note that $j_F^{c\text{flat}}$ is an isomorphism on the underlying reduced subschemes by (58). Thus $j_F^{c\text{flat}}$ is a proper monomorphism, hence a closed immersion [Kol17, 3.47.1].

Now the infinite fiber product (91.1) is an infinite intersection of closed subschemes, which always exists. Thus we get that

$$S_F^{c\text{flat}} = \cap_m (S_F^{c\text{flat}})_{F_m}^{c\text{flat}} \subset S_F^{c\text{flat}}, \tag{91.3}$$

and $j_F^{c\text{flat}}$ is the restriction of $j_F^{c\text{flat}}$ to it. \hfill \Box

A combination of (55) and (91) gives the following.

**Corollary 92.** Let $S$ be a scheme and $F$ a generically flat family of pure, coherent sheaves of dimension $d$ on $\mathbb{P}^n_S$. Then the functor of $K$-flat pull-backs of $F$ is represented by a monomorphism $j_F^{c\text{flat}} : S_F^{c\text{flat}} \to S$. \hfill \Box

**93** (Construction of the Chow variety I). In order to construct $\text{Chow}_{n,d}(\mathbb{P}^n_S)$, the Chow variety of degree $d$ cycles of dimension $n$ in $\mathbb{P}^n_S$, we start with the diagram (60.2)

$$\mathbb{P}^n_S \leftarrow \text{Inc}_S \xrightarrow{\pi} (\mathbb{P}^N)^{n+1}_S, \tag{93.1}$$

where the incidence variety $\text{Inc}_S := \text{Inc}_S((\text{point}), (\mathbb{P}^N)^{n+1})$ parametrizes $(n+2)$-tuples $(p, H_0, \ldots, H_n)$ satisfying $p \in H_i$ for every $i$, where $p$ is a point in $\mathbb{P}^n_S$ and we view the $H_i$ either as hyperplanes in $\mathbb{P}^N_S$ or points in $\mathbb{P}^N_S$.

Let $\mathbf{P}_{N,n,d} = |\mathcal{O}_{\mathbb{P}^N}^{n+1}(d, \ldots, d)|$ be the linear system of hypersurfaces of multidegree $(d, \ldots, d)$ in $(\mathbb{P}^N)^{n+1}$ with universal hypersurface $\text{CH}_{N,n,d} \subset (\mathbb{P}^N)^{n+1} \times \mathbf{P}_{N,n,d}$.

Thus (93.1) extends to

$$\sigma_{N,n,d} \times_{\mathbb{P}^n_S \times S \mathbf{P}_{N,n,d}} \text{Inc}_S \times S \mathbf{P}_{N,n,d} \xrightarrow{\pi_{N,n,d}^{-1}} (\mathbb{P}^N)^{n+1}_S \times S \mathbf{P}_{N,n,d} \tag{93.2}$$

Consider now the restriction of the left hand projection

$$\bar{\sigma}_{N,n,d} : (\text{Inc}_S \times S \mathbf{P}_{N,n,d}) \cap \pi_{N,n,d}^{-1} \text{CH}_{N,n,d} \to \mathbb{P}^n_S \times S \mathbf{P}_{N,n,d}. \tag{93.3}$$

Note that the preimage of a pair

$$(p = (\text{point}), CH = (\text{Cayley-Chow-type hypersurface}))$$

consists of all $(d+1)$-tuples $(H_0, \ldots, H_n)$ such that $p \in H_i$ for every $i$ and $(H_0, \ldots, H_n) \in CH$.

In particular, if $Z$ is an $n$-cycle of degree $d$ on $\mathbb{P}^n_S$ and $\text{Ch}_p(Z)$ is its Cayley-Chow hypersurface then $\bar{\sigma}_{N,n,d}$ is a $(\mathbb{P}^N)^{n+1}$-bundle over $\text{Supp} Z$. The key observation is that this property alone is enough to construct the Chow variety.

By the Flattening Decomposition Theorem [Mum66, Lec.8], there is a unique, largest, locally closed subscheme

$$W_{N,n,d} \hookrightarrow \mathbb{P}^n_S \times S \mathbf{P}_{N,n,d} \tag{93.4}$$
over which \( \sigma_{N,n,d} \) is a \((\mathbb{P}^N)^{n+1}\)-bundle. Note that if \((p, CH) \in W_{N,n,d}\) then 
\(\sigma_{N,n,d}^{-1}(p, CH)\) is the product of \(n + 1\) copies of the dual hyperplane \(H(p) \subset \mathbb{P}^N\), 
that is, the set of all hyperplanes that contain \(p\).

The set-theoretic behavior of the projection

\[
\rho_{N,n,d} : W_{N,n,d} \to \mathbb{P}_{N,n,d}
\]

is rather clear; see [Kol96, I.3.24.4]. The fiber dimension of \(\rho_{N,n,d}\) is \(\leq n\), and if 
\(CH \in \mathbb{P}_{N,n,d}\) is irreducible then \(\dim \rho_{N,n,d}^{-1}(CH) = n\) iff \(CH\) is the Cayley-Chow hypersurface of \(Z := \text{red} \rho_{N,n,d}^{-1}(CH)\).

In the reducible case one has to be more careful with the multiplicities; this was 
completed in [CvdW37]. A scheme-theoretic version of this is done in (82). The 
end result is that there is a closed subset \(\text{Chow}_{n,d}(\mathbb{P}^N_S) \hookrightarrow \mathbb{P}_{N,n,d}\) that parametrizes 
Cayley-Chow hypersurfaces of \(n\)-cycles of degree \(d\).

One would like this closed subset to be \(\text{Chow}_{n,d}(\mathbb{P}^N_S)\). Unfortunately, its scheme 
structure may change if we apply a Veronese embedding of \(\mathbb{P}^N_S\); see [Nag55] or 
[Kol96, I.4.2]. For this reason [Kol96] defines the Chow variety \(\text{Chow}_{n,d}(\mathbb{P}^N_S)\) as the 
seminormalization of \(\text{Chow}_{n,d}(\mathbb{P}^N_S)\).

In order to get a scheme-theoretic version of \(\text{Chow}_{n,d}(\mathbb{P}^N_S)\), one needs to understand 
the scheme-theoretic fibers of \(\rho_{N,n,d}\). We consider this next.

94 (Construction of the Chow variety II). Let \(S\) be a local scheme with residue 
field \(k\). Let \(Y \subset \mathbb{P}^N_S\) be a generically flat family of subschemes of dimension \(n\), 
degree \(d\) and \(\text{Ch}_p(Y) \subset (\mathbb{P}^N_S)^{n+1}\) its Cayley-Chow hypersurface.

Choose coordinates \((x_0 : \ldots : x_N)\) on \(\mathbb{P}^N_S\) and dual coordinates \((x^\vee_{0j} : \ldots : x^\vee_{Nj})\) on 
the \(j\)th copy of \(\mathbb{P}^N_S\) for \(j = 0, \ldots, n\). Let \(F_Y(x^\vee_{ij}) = 0\) be an equation of \(\text{Ch}_p(Y)\).

For notational simplicity we compute in the affine cart \(A^N_S = \mathbb{P}^N_S \setminus \{x_0 = 0\}\). 
Assume that none of the irreducible components of \(Y_k\) is contained in the coordinate 
hyperplane \((x_0 = 0)\).

For \((p_1, \ldots, p_N) \in A^N_S\), the hyperplanes \(H\) in the \(j\)th copy of \(\mathbb{P}^N_S\) that pass 
through \((p_1, \ldots, p_N)\) are all written in the form

\[
(-\sum_{i=1}^N p_i x^\vee_{ij} : x^\vee_{1j} : \ldots : x^\vee_{Nj});
\]

Thus \((p_1, \ldots, p_N) \in Y \cap A^N_S\) iff \(F\) identically vanishes after the substitutions

\[
x_{0j} \mapsto -\sum_{i=1}^N p_i x^\vee_{ij} \quad \text{for} \quad j = 0, \ldots, n.
\]

If \(M(x^\vee_{ij})\) are all the monomials in the \(x^\vee_{ij}\) for \(1 \leq i \leq N, 0 \leq j \leq n\) then, after the 
substitutions (94.1), we can write \(F_Y\) as

\[
\sum_M f_{Y,M}(p_1, \ldots, p_N) M(x^\vee_{ij}).
\]

Since the monomials \(M(x^\vee_{ij})\) are linearly independent, this vanishes for all \(x^\vee_{ij}\) iff 
\(f_{Y,M}(p_1, \ldots, p_N) = 0\) for every \(M\). Equivalently:

Claim 94.3. The subscheme

\[
W_{N,n,d} \cap (A^N_S \times_S \mathbb{P}_{N,n,d}) \subset A^N_S \times_S \mathbb{P}_{N,n,d}
\]

defined in (93.4) is given by the equations \(f_{Y,M}(x_1, \ldots, x_N) = 0\) for all monomials 
\(M\). \(\Box\)
If we fix \( x_i^j = c_{ij} \) then these give a linear projection \( \pi : \mathbb{A}^n_S \to \mathbb{A}^{n+1}_S \), and the corresponding Chow equation is

\[
\sum_M f_{Y,M}(x_1, \ldots, x_N) M(c_{ij}) = 0. \tag{94.4}
\]

Thus we get the following.

**Claim 94.5.** If the residue field of \( S \) is infinite then \( I^{ch}(Y) |_{\mathbb{A}^n_S} \) is generated by the Chow equations of the linear projections \( \pi : \mathbb{A}^n_S \to \mathbb{A}^{n+1}_S \). \( \square \)

Note that a priori we would need to use the more general projections (50.4).

Using (94.3), we can reformulate (82) as follows.

**Corollary 95.** Let \( k \) be an infinite field and \( Z = \sum m_i Z_i \) a sum of distinct, geometrically irreducible and reduced \( n \)-cycles of dimension \( d \) in \( \mathbb{P}^N_S \). Then the purified fiber of \( \rho_{N,n,d} : W_{N,n,d} \to \mathbb{P}^N_{n,d} \) over \( \text{Ch}_p(Z) \)

\[
\text{pure}(\rho_{N,n,d}^{-1}[\text{Ch}_p Z]) = \text{CHull}(Z). \tag{95.1}
\]

Combining (95) and (84) gives the following.

**Corollary 96.** Let \( k \) be an infinite field, \( X \subset \mathbb{P}^N_k \) a reduced subscheme of pure dimension \( n+1 \) and \( D \subset X \) a Mumford divisor of degree \( d \). Then

\[
\text{pure}(X \cap \rho_{N,n,d}^{-1}[\text{Ch}_p D]) = D. \tag{96.1}
\]

97 (Construction of the Chow variety III). Let us now return to (93.4)

\[
\rho_{N,n,d} : W_{N,n,d} \to \mathbb{P}^N_{n,d}. \tag{97.1}
\]

By (95) the (purified) fiber of \( \rho_{N,n,d} \) over \( \text{Ch}_p Z \) is the Chow hull of \( Z \), which is usually much larger than \( Z \). Therefore \( \rho_{N,n,d} \) is not even generically flat over \( \text{Ch}_p Z \in \mathbb{P}^N_{n,d} \) if \( Z \) is not geometrically reduced.

However, if \( Z \) is geometrically reduced then its Chow hull equals \( \mathcal{O}_Z \) by (82). Thus there is a chance that \( \rho_{N,n,d} \) is generically flat over \( \text{Ch}_p Z \).

Using (99) we get a generically flattening decomposition

\[
\beta^{g-\text{flat}}_{W_{N,n,d}} : (\mathbb{P}^N_{n,d})^{g-\text{flat}}_{W_{N,n,d}} \to \mathbb{P}^N_{n,d}, \tag{97.2}
\]

and then (92) gives

\[
\beta^{kflat}_{W_{N,n,d}} : (\mathbb{P}^N_{n,d})^{kflat}_{W_{N,n,d}} \to \mathbb{P}^N_{n,d}, \tag{97.3}
\]

which represents K-flatness. Finally, being generically geometrically reduced is an open condition, hence we get

\[
\text{Chow}^{g-\text{red}}_{n,d}(\mathbb{P}^N_S) \subset (\mathbb{P}^N_{n,d})^{kflat}_{W_{N,n,d}}. \tag{97.4}
\]

which represents the functor of K-flat families of geometrically reduced \( n \)-cycles of degree \( d \) in \( \mathbb{P}^N_S \).

**Remark 97.5.** It would be more in the spirit of classical Chow theory to use C-flatness in (97.3) instead of K-flatness. However, when one defines \( \text{Chow}^{g-\text{red}}_{n,d}(X/S) \) for some projective scheme \( X \to S \), we would like the result to be independent of the embedding \( X \to \mathbb{P}^N_S \). Thus K-flatness is more natural.

I do not know whether it is possible to push through the above approach for the whole Chow variety. Fortunately, the method works with minor changes for C-flat families of Mumford divisors. This then completes the proof of (4).
Theorem 98. Let $X \subset \mathbb{P}_S^N$ be a closed subscheme that is flat over $S$ with $S$ fibers of pure dimension $n$. Then the functor $K\text{Div}_d(X/S)$ of $K$-flat, relative Mumford divisors of degree $d$ is representable by a separated $S$-scheme of finite type $K\text{Div}_d(X/S)$.

Proof. Let $D_s \subset X_s$ be a Mumford divisor of degree $d$. We can also view it as an $(n-1)$-cycle of degree $d$ in $\mathbb{P}_S^N$. We proceed as in (93.1–4) to get

$$\mathbf{W}_{N,n-1,d} \hookrightarrow \mathbb{P}_S^N \times_S \mathbb{P}_{N,n-1,d}. \quad (98.1)$$

We are only interested in cycles that lie on $X$, hence we focus on the restriction of the coordinate projection

$$\tilde{\rho}_{N,n-1,d} : \mathbf{W}_{N,n-1,d} \cap (X \times_S \mathbb{P}_{N,n-1,d}) \to \mathbb{P}_{N,n-1,d}. \quad (98.2)$$

Let $D_s \subset X_s$ be a Mumford divisor of degree $d$. By (96),

$$D_s = \text{pure}(X_s \cap \text{Chull}(Z(D_s))) = \text{pure}(\tilde{\rho}_{N,n-1,d}^{-1}(\text{Chull} D_s)). \quad (98.3)$$

Let $F$ be the structure sheaf of $\mathbf{W}_{N,n-1,d} \cap (X \times_S \mathbb{P}_{N,n-1,d})$. By (99) there is a locally closed partial decomposition

$$j_F^g_{- \text{flat}} : \mathbb{P}_{N,n-1,d,F}^{g_{- \text{flat}}} \to \mathbb{P}_{N,n-1,d}, \quad (98.4)$$

such that $F_W$ is generically flat in dimension $n-1$ over an $S$-scheme $W$ iff $W \to S$ factors through $j_F^{g_{- \text{flat}}}$.

Thus $\mathbb{P}_{N,n-1,d,F}^{g_{- \text{flat}}}$ paramerizes generically flat families of divisorial subschemes of $X$ of degree $d$. Applying (92) now gives

$$j_{\text{flat}}^k : (\mathbb{P}_{N,n-1,d,F}^{g_{- \text{flat}}})^{\text{flat}} \to \mathbb{P}_{N,n-1,d,F}, \quad (98.5)$$

which paramerizes $K$-flat Mumford divisors.

We have used the following is a variant of the Flattening Decomposition Theorem of [Mum66, Lec.8].

Theorem 99. Let $f : X \to S$ be a projective morphism and $F$ a coherent sheaf on $X$. Let $n$ be the maximal fiber dimension of $\text{Supp} F \to S$. There is a locally closed decomposition $j_F^{g_{- \text{flat}}} : S_F^{g_{- \text{flat}}} \to S$ such that $F_W$ is generically flat over $W$ in dimension $n$ iff $W \to S$ factors through $j_F^{g_{- \text{flat}}}$.

Proof. We may replace $X$ by $\text{Supp} F$. The question is local on $S$. By Noether normalization we may assume that there is a finite morphism $\pi : X \to \mathbb{P}_S^n$. Note that $F_W$ is generically flat over $W$ in dimension $n$ if the same holds for $\pi_* F_W$. We may thus assume that $X = \mathbb{P}_S^n$.

Applying [Mum66, Lec.8] to the identity $X \to X$ and $F$, we get a decomposition $\Pi_i X_i \to X = \mathbb{P}_S^n$ where every $F|_{X_i}$ is flat, hence locally free of rank $i$.

Let $Z \subset \mathbb{P}_S^n$ be a closed subscheme. Applying [Mum66, Lec.8] to the projection $\mathbb{P}_S^n \to S$ and $\mathcal{O}_Z$, we see that there is a unique largest subscheme $S(Z) \subset S$ such that $S(Z) \times_S \mathbb{P}_S^n \subset Z$. For a locally closed subscheme $\tilde{Z} \subset \mathbb{P}_S^n$ set $S(\tilde{Z}) = S(\tilde{Z}) \setminus S(\tilde{Z} \setminus Z)$, where $\tilde{Z}$ denotes the scheme-theoretic closure of $Z \subset \mathbb{P}_S^n$.

Note that $S(Z)$ is the largest subscheme $T \subset S$ with the following property:

1. There is an open subscheme $\mathbb{P}_T^0 \subset \mathbb{P}_S^n$ that contains the generic point of $\mathbb{P}_T^t$ for every $t \in T$ and such that $\mathbb{P}_T^0 \subset Z$. 

We claim that $S_F^{\text{flat}} = \Pi S(X_i)$.

First, $F|_{X_i}$ is locally free of rank $i$, so the restriction of $F$ to $S(X_i) \times_S \mathbb{P}^n_q$ is locally free, hence flat, at every generic point of every fiber.

Conversely, let $W$ be a connected scheme and $q : W \to S$ a morphism such that $F_W$ is generically flat over $W$ in dimension $n$. Since $F_w$ is generically free for every $w \in W$, this implies that $F_W$ is locally free at the generic point of every fiber. Let $\Phi_W^0 \subset \mathbb{P}^n_W$ be the open set where $F_W$ is locally free. By assumption the closure of $\Phi_W^0$ equals $\mathbb{P}^n_W$.

Since $\Phi_W^0$ contains the generic point of every fiber $\mathbb{P}^n_w$, it is connected. Thus $F$ has constant rank, say $i$, on $\Phi_W^0$. Therefore, the restriction of $q$ to $\Phi_W^0$ lifts to $\bar{q} : \Phi_W^0 \to X_i$, which in turn extends to the closures $\bar{q} : \mathbb{P}^n_W \to \bar{X}_i$. Thus $\bar{q}$ gives $q_W : W \to S(X_i)$ in view of (1).

7. HYPERSURFACE SINGULARITIES

In this section we give a detailed description of $K$-flat deformations of hypersurface singularities over $k[\epsilon]$.

100 (Non-flat deformations). Let $X \subset \mathbb{A}^n$ be a reduced subscheme of pure dimension $d$. We aim to describe non-flat deformations of $X$ that are flat outside a subset $W \subset X$.

Choose equations $g_1, \ldots, g_{n-d}$ such that

$$(g_1 = \cdots = g_{n-d} = 0) = X \cup X',$$

where $Z := X \cap X'$ has dimension $< d$. Let $h$ be an equation of $X' \cup W$ that does not vanish on any irreducible component of $X$. Thus $X$ is a complete intersection in $\mathbb{A}^n \setminus \{h = 0\}$ with equation $g_1 = \cdots = g_{n-d} = 0$. Its flat deformations over an Artin ring $(A, m, k)$ are then given by

$$g_i(x) = \Psi_i(x) \quad \text{where} \quad \Psi_i \in m[x_1, \ldots, x_n, h^{-1}], \quad (100.1)$$

Note that we can freely change the $\Psi_i$ by any element of the ideal $(g_1 - \Psi_1, \ldots, g_{n-d} - \Psi_{n-d})$. We get especially simple normal forms if $A = k[\epsilon]$, that is, we look at first order deformations. In this case the equations can be written as

$$g_i(x) = \Phi_i(x)\epsilon \quad \text{where} \quad \Phi_i \in k[x_1, \ldots, x_n, h^{-1}]. \quad (100.2)$$

Now we can freely change the $\Phi_i$ by any element of the ideal $(g_1, \ldots, g_{n-d})$. Thus first order generically flat deformations can be given in the form

$$g_i = \phi_i \epsilon \quad \text{where} \quad \phi_i \in H^0(X, \mathcal{O}_X)[h^{-1}]. \quad (100.3)$$

Set $X^\circ := X \setminus (Z \cup W)$. By varying $h$ we see that in fact

$$g_i = \phi_i \epsilon \quad \text{where} \quad \phi_i \in H^0(X^\circ, \mathcal{O}_{X^\circ}). \quad (100.4)$$

This shows that the choice of $h$ is largely irrelevant.

If the deformation is flat then the equations defining $X$ lift, that is, $\phi_i \in H^0(X, \mathcal{O}_X)$. In some simple cases, for example if $X$ is a complete intersection, this is equivalent to flatness. In the examples that we compute the most important information is carried by the polar parts

$$\bar{\phi}_i \in H^0(X^\circ, \mathcal{O}_{X^\circ})/H^0(X, \mathcal{O}_X). \quad (100.5)$$

We study first order non-flat deformations of hypersurface singularities. Plane curves turn out to be the most interesting ones.
101. Consider a hypersurface singularity \( X := (f = 0) \subset \mathbb{A}^n_x \) and a generically flat deformation of it
\[
X \subset \mathbb{A}^{n+1}_{x,z} \to \text{Spec } k[\epsilon].
\]
(Aiming to work inductively, we assume that the deformation is flat outside the origin. Choose coordinates such that the \( x_i \) do not divide \( f \).

As in (100.3) any such deformation can be given as
\[
f(x) = \psi(x)\epsilon \quad \text{and} \quad z = \phi(x)\epsilon,
\]
where \( \psi, \phi \in H^0(X, \mathcal{O}_X)[x_n^{-1}] \). Note that the choice of \( x_n \) is not intrinsic, so in fact
\[
\psi, \phi \in \cap \epsilon H^0(X, \mathcal{O}_X)[x_n^{-1}].
\]
If \( n \geq 3 \) then \( \cap \epsilon H^0(X, \mathcal{O}_X)[x_n^{-1}] = \mathcal{O}_X \) and we get the following special case of [Kol17, 10.68].

**Claim 101.4.** Let \( X \subset \mathbb{A}^{n+1} \) be a first order deformation over \( k[\epsilon] \) of a hypersurface singularity \( X := (f = 0) \subset \mathbb{A}^n \) that is flat outside the origin. If \( n \geq 3 \) then \( X \) is flat over \( k[\epsilon] \).

If \( n = 2 \) then we have a curve singularity \( C = \{(f(x, y) = 0) \subset \mathbb{A}^2 \). Set \( C^\circ := C \setminus \{(0, 0)\} \). Then the deformation is given as
\[
f(x, y) = \psi \epsilon \quad \text{and} \quad z = \phi \epsilon,
\]
where the relevant information about \( \phi, \psi \) is carried by the polar parts
\[
\tilde{\psi}, \tilde{\phi} \in H^0(C^\circ, \mathcal{O}_{C^\circ})/H^0(C, \mathcal{O}_C).
\]

**Definition 102.** We say that a (flat resp. generically flat) deformation over \( k[\epsilon] \) **globalizes** if it is induced from a (flat resp. generically flat) deformation over \( k[[\epsilon]] \) by base change.

**Theorem 103.** Consider a generically flat deformation \( C \) of the reduced plane curve singularity \( C := (f = 0) \subset \mathbb{A}^{2+y} \) given in (101.5–6).

1. If \( C \) is \( C \)-flat then \( \psi \in H^0(C, \mathcal{O}_C) \).
2. If \( \psi \in H^0(C, \mathcal{O}_C) \) then the deformation is
   (a) flat iff \( \phi \in H^0(C, \mathcal{O}_C) \),
   (b) globalizes iff \( \phi \in H^0(\bar{C}, \mathcal{O}_C) \) where \( \bar{C} \to C \) is the normalization, and
   (c) \( C \)-flat iff \( f_x\phi, f_y\phi \in H^0(C, \mathcal{O}_C) \).

Proof. If \( \psi, \phi \in H^0(C, \mathcal{O}_C) \) then we can rewrite (101.5) as
\[
(f(x, y) - \psi(x, y))\epsilon - (z - \tilde{\phi}(x, y))\epsilon = 0
\]
where \( \tilde{\psi}, \tilde{\phi} \) are regular; this is a flat deformation and the converse is clear.

If \( \phi \in H^0(\bar{C}, \mathcal{O}_C) \) then it is integral over \( k[x, y]|_{(x, y)} \), so it satisfies an equation
\[
\phi^m + \sum_{j=0}^{m-1} r_j(x, y)\phi^j = 0.
\]
Consider now the surface \( S \) given by the equations
\[
\begin{align*}
  f(x, y) - \psi(x, y)s &= 0 \\
  x^{a_1}z - \Phi_1(x, y)s &= 0 \\
  y^{c_1}z - \Phi_2(x, y)s &= 0 \quad \text{and} \\
  z^m + \sum_{j=0}^{m-1} r_j(x, y)z^j s^{m-j} &= 0,
\end{align*}
\]
where \( a_1, c_1 \) are chosen so that \( \Phi_1(x, y) = x^{a_1}\phi, \Phi_2(x, y) = y^{c_1}\phi \) are regular. The equations in line 2 of (103.3) guarantee that the projection \( S \to (f(x, y) = \psi(x, y)s) \subset \mathbb{A}^3_{xyz} \) is birational and the last equation shows that it is finite. We also
see that \( z - \phi s \in (s^2) \), thus \( S \) is a globalization of \( C \). The converse assertion in (b) follows from (106).

As for (c), we write down the image of the projection
\[
(x, y, z) \mapsto (\bar{x}, \bar{y}) = (x - \alpha(x, y, z)z, y - \gamma(x, y, z)z).
\]
where \( \alpha, \gamma \) are constants for linear projections and power series that are nonzero at the origin in general.

Since \( z^2 = 0 \) we get that
\[
f(x, y) = f(\bar{x}, \bar{y}) + \alpha(x, y, z)f_x(x, y)z + \gamma(x, y, z)f_y(x, y)z
\]
holds in \( \mathcal{O}_C \). Similarly, for any polynomial \( F(x, y) \) we get that \( F(\bar{x}, \bar{y}) \equiv F(x, y) \mod \mathcal{O}_C \), hence \( F(\bar{x}, \bar{y})z = F(x, y)z \) in \( \mathcal{O}_C \) since \( z^2 = 0 \). (Here we use that \( \epsilon^2 = 0 \), we get other terms in (105.3) otherwise.)

Thus the equation of the projection is
\[
f(\bar{x}, \bar{y}) + \psi(\bar{x}, \bar{y}) + \alpha(\bar{x}, \bar{y}, 0)f_x(\bar{x}, \bar{y})\phi + \gamma(\bar{x}, \bar{y}, 0)f_y(\bar{x}, \bar{y})\phi \cdot \epsilon = 0. \tag{103.4}
\]
By (29) this defines a relative Cartier divisor for every \( \alpha, \gamma \) iff \( f, f_x\phi, f_y\phi \in \mathcal{O}_C \). (In particular, linear projections and formal projections give the same restrictions.) \( \square \)

**Remark 104.** Note that \( \Omega^1_C \) is generated by \( dx|_C, dy|_C \), while \( \omega_C \) is generated by \( f_y^{-1}dx = -f_x^{-1}dy \).

Since \( C \) is reduced, \( \Omega^1_C \) and \( \omega_C \) are naturally isomorphic over the smooth locus \( C^0 \). This gives a natural inclusion \( \text{Hom}(\Omega^1_C, \omega_C) \rightarrow \mathcal{O}^{C^0} \). Then ((103.2.c) says that \( C \)-flat deformations as in (101.5) are parametrized by \( \text{Hom}(\Omega^1_C, \omega_C) \). We describe this space for monomial curves next.

**Example 105 (Monomial curves).** We can be more explicit if \( C \) is the irreducible monomial curve \( C := (x^a = y^c) \subset A^2 \) where \( (a, c) = 1 \). It can be parametrized as \( t \mapsto (t^a, t^c) \). Thus \( \mathcal{O}_C = k[t^a, t^c] \). Let \( E_C = Na + Nc \subset N \) denote the semigroup of exponents. Then the condition (103.2.c) becomes
\[
t^{|ac-c|\phi(t), t^{|ac-a|\phi(t)} \in k[t^a, t^c].
\]
This needs to be checked one monomial at a time. For \( \phi = t^{-m} \) we get \( ac - c - m \in E_C \) and \( ac - a - m \in E_C \). By (105.1) these are equivalent to \( ac - a - m \in E_C \).

The largest value of \( m \) satisfying this condition gives the deformation
\[
(x^a - y^c = z - t^{-ac+a+c}\epsilon = 0) \quad \text{over} \quad k[\epsilon].
\]
Note also that for \( 0 \leq m \leq ac - a - c \), we have that \( ac - a - c - m \in E_C \) iff \( m \notin E_C \). Thus we see that the space of \( C \)-flat deformations that do not generalize has dimension \( \frac{1}{2}(a - 1)(c - 1) \). (This is an integer since one of \( a, c \) must be odd.)

The following is left as an exercise.

**Lemma 105.1.** For \( (a, c) = 1 \) set \( E = Na + Nc \subset N \). Then
(a) If \( 0 \leq m \leq \min\{ac - a, ac - c\} \) then \( ac - a - m, ac - c - m \in E \) iff \( ac - a - c - m \in E \).
(b) If \( 0 \leq m \leq ac - a - c \) then \( ac - a - c - m \in E \) iff \( m \notin E \). \( \square \)

**106 (S_2 hull of a deformation).** Let \( T \) be the spectrum of a DVR with maximal ideal \( (t) \) and residue field \( k \). Let \( g : X \rightarrow T \) be a flat morphism of pure relative dimension \( d \) and \( Z := \text{Supp } \text{tors}(X_k) \). Let \( j : X \setminus Z \rightarrow X \) the natural injection and set \( \bar{X} := \text{Spec}_X j_*\mathcal{O}_X \setminus Z \). If \( X \) is excellent then \( \pi : \bar{X} \rightarrow X \) is finite and \( \bar{X} \) is \( S_2 \).
By composition we get \( \bar{g} : \bar{X} \to T \). Note that \( \pi_k : \bar{X}_k to X_k \) is an isomorphism over \( X_k \setminus Z \) and \( \bar{X}_k \) is \( S_1 \). Thus if \( \text{pure}(X_k) \) is reduced then \( \bar{X}_k \) is dominated by the normalization \( X_k^{\text{nor}} \to X_k \).

Note that \( t^nO_X \) usually has some embedded primes contained in \( Z \). The intersection of its height 1 primary ideals (also called the \( n \)-th symbolic power of \( tO_X \)) is

\[
(tO_X)^{(n)} = O_X \cap t^nO_X = \ker [O_X \to \text{pure}(O_X/t^nO_X)]
\]

(106.1)

Multiplication by \( t \) gives injections

\[
\text{pure}(O_{X_k}) = O_X/(tO_X)^{(1)} \hookrightarrow (tO_X)^{(1)}/(tO_X)^{(2)} \hookrightarrow \ldots \hookrightarrow (tO_X)^{(n)}
\]

(106.2)

Note that

\[
(tO_X)^{(n)}/(tO_X)^{(n+1)} \hookrightarrow t^nO_X/t^{n+1}O_X \cong O_{X_k},
\]

(106.3)

thus the sequence (106.2) eventually stabilizes. We can thus view the quotients

\[
(tO_X)^{(n+1)}/t(tO_X)^{(n)}
\]

as graded pieces of two filtrations, one of tors\((X_k)\) and one of \( O_{X_k}/O_{X_k} \).

To formalize this, let us write \( M \leq N \) to mean that there are filtrations \( 0 = M_0 \subset \cdots \subset M_n = M, 0 = N_0 \subset \cdots \subset N_n = N \) and an injection \( \sigma : [1, \ldots, m] \hookrightarrow [1, \ldots, n] \) such that, \( M_i/M_{i-1} \cong N_{\sigma(i)}/N_{\sigma(i)+1} \) for every \( i = 1, \ldots, m \). If \( M, N \) are artinian modules over a local ring then this holds iff length \( M \leq \text{length} \ N \).

We have thus proved the following.

**Corollary 107.** Using the notation of (106), assume that \( \text{pure}(X_k) \) is reduced with normalization \( X_k^{\text{nor}} \to X_k \). Then

\[
\text{tors} O_{X_k} \leq O_{X_k}^{\text{nor}}/O_{X_k}.
\]

In particular, if \( \dim X_k = 1 \) then

\[
\text{length}(\text{tors} O_{X_k}) \leq \text{length}(O_{X_k}^{\text{nor}}/O_{X_k}). \quad \square
\]

8. **SEMINORMAL CURVES**

Over an algebraically closed field \( k \), every seminormal curve singularity is formally isomorphic to \( C_n \subset \mathbb{A}^2_k \), formed by the union of the \( n \) coordinate axes. Equivalently,

\[
C_n = \text{Spec} \ k[x_1, \ldots, x_n]/(x_ix_j : i \neq j).
\]

In this section we study deformations of \( C_n \) over \( k[\epsilon] \) that are flat outside the origin.

A normal form is worked out in (108.4), which shows that the space of these deformations is infinite dimensional. Then we describe the flat deformations (109) and their relationship to smoothings (112).

We compute C-flat and K-flat deformations in (114); these turn out to be quite close to flat deformations.

The ideal of Chow equations of \( C_n \) is computed in (119). For \( n = 3 \) these are close to C-flat deformations, but the difference between the two classes increases rapidly with \( n \).

108 (Generically flat deformations of \( C_n \)). Let \( C_n \subset \mathbb{A}^m_k[\epsilon] \) be a generically flat deformation of \( C_n \subset \mathbb{A}^m_k \) over \( k[\epsilon] \).

If \( C_n \) is flat over \( k[\epsilon] \) then we can assume that \( n = m \), but a priori we only know that \( n \leq m \).

Following (100), we can describe \( C_n \) as follows.
Along the $x_j$-axis and away from the origin, the deformation is flat and the $x_j$-axis is a complete intersection. Thus, in the $(x_j \neq 0)$ open set, $C_n$ can be given as
\[ x_i = \Phi_{ij}(x_1, \ldots, x_n)\epsilon \quad \text{where} \quad i \neq j \quad \text{and} \quad \Phi_{ij} \in k[x_1, \ldots, x_n, x_j^{-1}]. \quad (108.1) \]
Note that $(x_1, \ldots, \hat{x}_j, \ldots, x_n, \epsilon)^2$ is identically zero on $C_n \cap (x_n \neq 0)$, so the terms in this ideal can be ignored. Thus along the $x_j$-axis we can change (108.1) to the simpler form
\[ x_i = \phi_{ij}(x_j)\epsilon \quad \text{where} \quad i \neq j \quad \text{and} \quad \phi_{ij} \in k[x_j, x_j^{-1}]. \quad (108.2) \]
There is one more simplification that we can make. Write
\[ \phi_{ij} = \phi_{ij}^p + \gamma_{ij} \quad \text{where} \quad \phi_{ij}^p \in k[x_j^{-1}], \gamma_{ij} \in (x_j) \subset k[x_j], \]
and set $x_i' = x_i - \sum_{j \neq i} \gamma_{ij}(x_j)$. Then we get the description
\[ x_i' = \phi_{ij}^p(x_j')\epsilon \quad \text{where} \quad i \neq j \quad \text{and} \quad \phi_{ij}^p \in k[x_j'^{-1}]. \quad (108.3) \]
For most of our computations the latter coordinate change is not very important. Thus we write our deformations as
\[ C_n : \{ x_i = \phi_{ij}(x_j)\epsilon \quad \text{along the} \ x_j\text{-axis}\}, \quad (108.4) \]
where $\phi_{ij}(x_j) \in k[x_j, x_j^{-1}]$, but we keep in mind that we can choose $\phi_{ij}(x_j) \in k[x_j^{-1}]$ if it is convenient. In order to deal with the cases when $m > n$, we make the following

**Convention 108.5.** We set $\phi_{ij} \equiv 0$ for $j > n$.

Writing $C_n$ as in (108.4) is almost unique; see (111) for one more coordinate change that leads to a unique normal form.

We get the same result (108.4) if we work with the analytic or formal local scheme of $C_n$: we still end up with $\phi_{ij}(x_j) \in k[x_j^{-1}]$.

**Proposition 109.** For $n \geq 3$ the generically flat deformation $C_n \subset A^n_k[\epsilon]$ as in (108.4) is flat iff the $\phi_{ij}(x_j)$ have no poles. (See (113.5) for the $n = 2$ case.)

**Proof.** If the $\phi_{ij}(x_j)$ are regular then
\[ x_i x_j + (x_j \phi_{ij}(x_j) + x_i \phi_{ji}(x_i))\epsilon = 0 \quad (109.1) \]
is an equation for $C_n$. Thus every equation of $C_n$ lifts to an equation of $C_n$, hence $C_n$ is flat over $k[\epsilon]$ by (24).

Conversely, if the deformation is flat then the equations defining $C_n$ lift, so we have a set of defining equations for $C_n$ of the form
\[ x_i x_j = \Phi_{ij}(x_1, \ldots, x_n)\epsilon. \quad (109.2) \]
As in (108.2), this simplifies to
\[ x_i x_j = \psi_{ij}(x_j)\epsilon \quad \text{along the} \ x_j\text{-axis}. \]
Note that $x_i x_j$ vanishes along the other $n - 2$ axes, so we must have $\psi_{ij}(0) = 0$. (Here we use that $n \geq 3$.) Thus $\phi_{ij} := x_j^{-1} \psi_{ij}$ is regular as needed. \(

**Remark 110.** Choosing $r \leq n$ of the coordinate axes we get an embedding $\tau_r : C_r \to C_n$ and any generically flat deformation $C_n$ of $C_n$ induces a generically flat deformation $C_r := \tau_r^* C_n$ of $C_r$.

From (109) we conclude that $C_n$ is flat iff $\tau_3^* C_n$ is flat for every $\tau_3 : C_3 \to C_n$. Neither direction of this claim seems to follow from general principles. For example,
if $\tau_1^* C_n$ is flat for every $\tau_2 : C_2 \rightarrow C_n$ then $C_n$ need not be flat; see (113.5) and (114.5).

**Remark 111.** Putting (108.3) and (109) together we get that flat deformations can be given as

$$C_n : \{ x_i = e_{ij} \epsilon \text{ along the } x_j\text{-axis, where } e_{ij} \in k \}. \quad (111.1)$$

The constants $e_{ij}$ are not yet unique, translations

$$x_i \mapsto x_i - a_i \epsilon \text{ change } e_{ij} \mapsto e_{ij} - a_j. \quad (111.2)$$

So we get a first order deformation space of dimension $n(n - 1) - n = n(n - 2)$.

We can also think of $\mathcal{O}_{C_n}$ as a subring of $\oplus_j k[X_j, e_j]$ given by

$$x_i \mapsto (e_{i1} \epsilon_1, \ldots, e_{i,i-1} \epsilon_{i-1}, X_i, e_{i,i+1} \epsilon_{i+1}, \ldots, e_{in} \epsilon_n).$$

Strangely, (111.1) says that every flat first order deformation of $C_n$ is obtained by translating the axes independently of each other. These deformations all globalize in the obvious way, but the globalization is not a flat deformation of $C_n$ unless the translated axes all pass through the same point. If this point is $(a_1 \epsilon, \ldots, a_n \epsilon)$ then $e_{ij} = a_j$ and applying (109.3) we get the trivial deformation.

If $n = 2$ then the universal deformation is $x_1 x_2 + \epsilon = 0$. One may ask why this deformation does not lift to a deformation of $C_3$: smooth 2 of the axes to a hyperbola and just move the 3rd axis along. If we use $x_1 x_2 + t = 0$, then the $x_3$-axis should move to the line $(x_1 - \sqrt{t} = x_2 - \sqrt{t} = 0)$. This gives the flat deformation given by equations

$$x_1 x_2 + t = x_3(x_1 - \sqrt{t}) = x_3(x_2 - \sqrt{t}) = 0.$$  

Of course this only makes sense if $t$ is a square. Thus setting $\epsilon = \sqrt{t} \mod t$ the $t = \epsilon^2 \mod t$ term becomes 0 and we get

$$x_1 x_2 = x_3 x_1 - x_3 \epsilon = x_3 x_2 - x_3 \epsilon = 0,$$

which is of the form given in (109.1).

**Example 112 (Smoothing $C_n$).** Rational normal curves $R_n \subset \mathbb{P}^n$ have a moduli space of dimension $(n + 1)(n + 1) - 1 - 3 = n^2 + 2n - 3$. The $C_n \subset \mathbb{P}^n$ have a moduli space of dimension $n + n(n - 1) = n^2$. Thus the smoothings of $C_n$ have a moduli space of dimension $n^2 + 2n - 3 - n^2 = 2n - 3$. We can construct these smoothings explicitly as follows.

Fix distinct $p_1, \ldots, p_n \in k$ and consider the map

$$(t, z) \mapsto (\frac{1}{z-p_1}, \ldots, \frac{1}{z-p_n}).$$

Eliminating $z$ gives the equations

$$(p_i - p_j) x_j + (x_i - x_j) t = 0: 1 \leq i \neq j \leq n \quad (112.1)$$

for the closure of the image, which is an affine cone over a degree $n$ rational normal curve $R_n \subset \mathbb{P}^n$. So far this is an $(n - 1)$-dimensional space of smoothings.

Applying the torus action $x_i \mapsto \lambda_i^{-1} x_i$, we get new smoothings given by the equations

$$(p_i - p_j) x_j + (\lambda_j x_i - \lambda_i x_j) t = 0: 1 \leq i \neq j \leq n. \quad (112.2)$$

Writing it in the form (108.4) we get

$$x_i = \frac{\lambda_i}{p_i - p_j} \epsilon \quad \text{along the } x_j\text{-axis.} \quad (112.3)$$
This looks like a $2n$-dimensional family, but $\text{Aut}(\mathbb{P}^1)$ acts on it, reducing the dimension to the expected $2n - 3$. The action is clear for $z \mapsto \alpha z + \beta$, but $z \mapsto z^{-1}$ also works out using (111.2) since

$$\frac{\lambda_i}{p_i - p_j} = \frac{-\lambda_j p_i^2}{p_i} + \lambda_i p_i.$$  

Claim 112.4. For distinct $p_i \in k$ and $\lambda_j \in k^*$, the vectors $(\frac{\lambda_i}{p_i - p_j}; i \neq j)$ span $\langle e_{ij} \rangle \cong k_{(2)}$. So the flat infinitesimal deformations determined in (111.1) form the Zariski tangent space of the smoothings.

Proof. Assume that there is a linear relation

$$\sum_{ij} m_{ij} \frac{\lambda_i}{p_i - p_j} = 0.$$  

If we let $p_i \to p_j$ but keep the others fixed, we get that $m_{ij} = 0$. \hfill \Box

Remark 112.5. If $n = 3$ then $2n - 3 = n(n - 2)$ and the Hilbert scheme of degree 3 reduced space curves with $p_a = 0$ is smooth, see [PS85].

Example 113 (Simple poles). Among non-flat deformations, the simplest ones are given by $\phi_{ij}(x_j) = c_{ij} x_j^{-1} + e_{ij}$. Then we have

$$q_{ij} := x_i x_j - (e_{ij} x_i + e_{ji} x_j) \epsilon = \begin{cases} c_{ij} \epsilon & \text{along the } x_j\text{-axis}, \\ c_{ji} \epsilon & \text{along the } x_i\text{-axis}, \\ 0 & \text{along the other axes.} \end{cases}$$  

(113.1)

Thus we see that $\sum_{ij} \gamma_{ij} q_{ij}$ vanishes on $C_n$ iff

$$\sum_i \gamma_{ij} c_{ij} \text{ is independent of } j.$$  

(113.2)

These impose $n - 1$ linear conditions on the $\gamma_{ij}$, which are in general independent. Thus we get the following.

Claim 113.3. For general $c_{ij}$, the torsion in the central fiber has length $n - 1$. \hfill \Box

In special cases the torsion can be smaller, but if the $c_{ij}$ are not identically 0, then we get at least 1 nontrivial condition. This is in accordance with (109).

The $n = 2$ case is exceptional and is worth discussing separately. We get that

$$q_{12} := x_1 x_2 - (e_{12} x_1 + e_{21} x_2) \epsilon = \begin{cases} c_{12} \epsilon & \text{along the } x_2\text{-axis}, \\ c_{21} \epsilon & \text{along the } x_1\text{-axis.} \end{cases}$$  

(113.4)

This gives the following.

Claim 113.5. For $n = 2$ the deformation as in (108.4) is flat iff $\phi_{12}, \phi_{21}$ have only simple poles and with the same residue. \hfill \Box

The main result is the following.

Theorem 114. For a first order deformation of $C_n \subset k^m$ specified as in (108.4) by

$$C_n : \{ x_i = \phi_{ij}(x_j) \epsilon \text{ along the } x_j\text{-axis} \}$$  

(114.1)

the following are equivalent.

(2) $C_n$ is C-flat.

(3) $C_n$ is K-flat.

(4) The $\phi_{ij}$ have only simple poles and $\phi_{ij}, \phi_{ji}$ have the same residue.
(5) $C_n$ induces a flat deformation on any pair of lines $C_2 \leftrightarrow C_n$.

Proof. The proof consist of 2 parts. First we show in (116) that (2) and (4) are equivalent by explicitly computing the equations of linear projections.

We see in (117) that if the $\phi_{ij}$ have only simple poles then there is only 1 term of the equation of a non-linear projection that could have a pole, and this term is the same for the linearization of the projection. Hence it vanishes iff it vanishes for linear projections. This shows that (4) $\Leftrightarrow$ (3).

Finally (4) $\Leftrightarrow$ (5) follows from (13.5). □

Remark 115. If $j > n$ then $\phi_{ij} \equiv 0$ by (108.5), so $\phi_{ji}$ is regular by (114.4). Thus

$$x_j - \sum_{\ell=1}^n \phi_{j\ell} x_\ell$$

is identically on $C_n$. We can thus eliminate the $x_j$ for $j > n$. Hence we see that allowing $m > n$ did not result in more deformations. This is in contrast with (103).

116 (Linear projections). Recall that by our convention (108.5), $\phi_{ij} \equiv 0$ for $j > n$. Extending this, in the following proof all sums/products involving $i$ go from 1 to $m$ and sums/products involving $j$ go from 1 to $n$.

With $C_n$ as in (114.1) consider the special projections

$$\pi_\alpha : \mathcal{A}_u^2[\epsilon] \to \mathcal{A}_{uv}[\epsilon]$$

given by $u = \sum x_i, v = \sum a_i x_i$, (116.1)

where $a_i \in k[\epsilon]$. Write $a_i = \tilde{a}_i + a'_i \epsilon$. (One should think that $a'_i = \partial a_i / \partial \epsilon$.)

In order to compute the projection, we follow the method of (36). Since we compute over $k[u, u^{-1}, \epsilon]$, we may as well work with the $k[u, \epsilon]$-module $M := \oplus_i k[x_j, \epsilon]$ and write $1_j \in k[x_j, \epsilon]$ for the $j$th unit. Then multiplication by $u$ and $v$ are given by

$$u \cdot 1_j = (\sum_i x_i) 1_j = x_j + \sum_i \phi_{ij} \epsilon \text{ and } v \cdot 1_j = (\sum_i a_i x_i) 1_j = a_j x_j + \sum_i a_i \phi_{ij} \epsilon.$$  

Thus

$$v \cdot 1_j = (a_j u + \sum_i (a_i - a_j) \phi_{ij}(u) \epsilon) \cdot 1_j.$$  

Thus the $v$-action on $M$ is given by the diagonal matrix

$$\text{diag}(a_j u + \sum_i (a_i - a_j) \phi_{ij}(u) \epsilon),$$

and by (36) the equation of the projection is its characteristic polynomial

$$\prod_j (v - a_j u - \sum_i (a_i - a_j) \phi_{ij}(u) \epsilon) = 0.$$  

(116.3)

Expanding it we get an equation of the form

$$\prod_j (v - a_j u) - B(u, v, a, \phi) \epsilon = 0,$$  

(116.4)

where

$$B(u, v, a, \phi) = \sum_j \left( \prod_{i \neq j} (v - a_j u) \right) \cdot \left( a'_j u + \sum_i (\tilde{a}_i - a_j) \phi_{ij}(u) \right).$$  

(116.5)

This is a polynomial of degree $\leq n - 1$ in $v$, hence by (32) its restriction to the curve $\left( \prod_j (v - a_j u) = 0 \right)$ is regular iff $B(u, v, a, \phi)$ is a polynomial in $u$ as well. Let now $r$ be the highest pole order of the $\phi_{ij}$ and write

$$\phi_{ij}(u) = c_{ij} u^{-r} + \text{(higher terms)}.$$  

Then the leading part of the coefficient of $v^{n-1}$ is

$$\sum_j \sum_i (\tilde{a}_i - a_j) c_{ij} u^{-r} = u^{-r} \sum a_i (\sum_j (c_{ij} - c_{ji})).$$  

(116.6)
Thus, as in \((116.1)\), the equations \(x_i = \phi_{ij}(x_j)\epsilon\) become
\[
y_i = \lambda_i \phi_{ij}(\lambda_j^{-1}y_j)\epsilon
\]
and \(c_{ij}\) changes to \(\lambda_i \lambda_j c_{ij}\) Thus the equations \((116.7)\) become
\[
\sum_j (\lambda_i \lambda_j c_{ij} - \lambda_j \lambda_i c_{ji}) = 0 \quad \forall i.
\]
(116.8)
If \(r \geq 2\) this implies that \(c_{ij} = 0\) and if \(r = 1\) then we get that \(c_{ij} = c_{ji}\). This completes the proof of \((114.2) \Leftrightarrow (114.4)\).

**Remark 116.9.** Note that if we work over \(\mathbb{F}_2\) then necessarily \(\lambda_i = 1\), hence \((116.8)\) does not exclude the \(r \geq 2\) cases.

**117 (Non-linear projections).** Consider a general non-linear projection
\[
(x_1, \ldots, x_n) \mapsto (\Phi_1(x_1, \ldots, x_n), \Phi_2(x_1, \ldots, x_n)).
\]
After a formal coordinate change we may assume that \(\Phi_1 = \sum x_i\). Note that the monomials of the form \(x_ix_jx_k, x_i^2x_j^2, x_ix_jx_k\epsilon\) vanish on \(\mathbb{C}_u\), so we can discard these terms from \(\Phi_2\). Thus, in suitable local coordinates a general non-linear projection can be written as
\[
u = \sum_i \phi_i(x_i) + \sum_{i \neq j} x_i \beta_{ij}(x_j),
\]
in the notation of \((116)\). Now we get that
\[
u \cdot 1_j = x_j + \sum_i \phi_{ij}(x_j)\epsilon \quad \text{and} \quad v \cdot 1_j = \alpha_j(x_j) + \sum_{i \neq j} \alpha_i(\phi_{ij}(x_j)\epsilon) + \sum_{i \neq j} \phi_{ij}(x_j)\beta_{ij}(x_j)\epsilon.
\]
Note further that \(\alpha_i(\phi_{ij}(x_j)\epsilon) = \alpha_i(0)\phi_{ij}(x_j)\epsilon\) and
\[
\begin{align*}
\alpha_j(x_j) &= \alpha_j(u - \sum_i \phi_{ij}(x_j)\epsilon) = \alpha_j(u) - \alpha_j'(u)\sum_i \phi_{ij}(x_j)\epsilon.
\end{align*}
\]
Thus, as in \((116.4)\), the projection is defined by the vanishing of
\[
\prod_j \left( v - \alpha_j(u) - \sum_i (\beta_{ij}(u) + \alpha_i'(0) - \alpha_i'(u))\phi_{ij}(u)\epsilon \right)
= \prod_j (v - \alpha_j(u)) - B(u, v, \alpha, \beta, \phi)\epsilon.
\]
(117.3)
Let \(\bar{\beta}_{ij}, \bar{\alpha}_j\) denote the residue of \(\beta_{ij}, \alpha_j\) modulo \(\epsilon\) and write \(\alpha_j(u) = \bar{\alpha}_j(u) + \bar{\beta}_j\alpha_j(u)\). As in \((116.5)\), expanding the product gives that \(B(u, v, \alpha, \beta, \phi)\) equals
\[
\sum_j (\prod_{i \neq j}(v - \bar{\alpha}_i(u))) \cdot (\partial_u \alpha_j(u) + \sum_i (\bar{\beta}_{ij}(u) + \bar{\alpha}_i'(0) - \bar{\alpha}_i'(u))\phi_{ij}).
\]
(117.4)
We already know that \(\phi_{ij}(u) = c_{ij}u^{-1} + \text{higher terms}\), hence \(B(u, v, \alpha, \beta, \phi)\) has at most simple pole along \((u = 0)\). Computing its residue along \(u = 0\) we get
\[
v^n \sum_j (\bar{\beta}_{ij}(0) + \bar{\alpha}_i'(0) - \bar{\alpha}_i'(0))c_{ij} = v^n \sum_j (\bar{a}_i - \bar{a}_j)c_{ij}.
\]
(117.5)
These are the same as in \((116.6)\). Thus \(B(u, v, \alpha, \beta, \phi)\) is regular iff it is regular for the linearization of the projection. This completes the proof of \((114.4) \Rightarrow (114.3)\).
Example 118. The image of a general linear projection of $C_n \subset A^n$ to $A^2$ is $n$ distinct lines through the origin. Their equation is $g_n(x, y) = 0$ where $g_n$ is homogeneous of degree $n$ with simple roots only. A typical example is $g_n = x^n + y^n$.

A general non-linear projection to $A^2$ gives $n$ smooth curve germs with distinct tangent lines through the origin. The equation of the image is $g_n(x, y) + (\text{higher terms}) = 0$ where $g_n$ is homogeneous of degree $n$ with simple roots only.

The miniversal deformation of $(x^n + y^n = 0)$ is

$$(x^n + y^n + \sum_{i,j \leq n-2} t_{ij} x^i y^j = 0) \subset A^2_{xy} \times A_{t}^{(n-1)^2}. \quad (118.1)$$

A general deformation is a smoothing, but deformations that have $n$ smooth branches with the same tangents at $(x^n + y^n = 0)$ form the subfamily

$$(x^n + y^n + \sum_{i,j \leq n} t_{ij} x^i y^j = 0) \subset A^2_{xy} \times A_t^{(n-2)}, \quad (118.2)$$

where summation is over those pairs $(i,j)$ that satisfy $i,j \leq n - 2$ and $n < i + j$. For $n \leq 4$ there is no such pair $(i,j)$, which gives the following.

Claim 118.3. For $n \leq 4$ every analytic projection $\hat{C}_n \to \hat{A}^2$ is obtained as the composite of an automorphism of $\hat{C}_n$, followed by a linear projection and then by an automorphism of $\hat{A}^2$.

For $n = 5$ we get the deformations

$$(x^5 + y^5 + t x^3 y^3 = 0) \subset A^2_{xy} \times A_t. \quad (118.4)$$

For $t \neq 0$ we get curves that are images of $\hat{C}_n$ by a nonlinear projection, but not as a linear projection pre-composed/composed with automorphisms.

The following strengthens [Kol99, 4.11].

Proposition 119. The ideal of Chow equations of $C_n$ is generated by

1. all degree $n$ monomials, save the $x_1^n$, if $n$ is even, and
2. all degree $n$ monomials, save the $x_1^n$ and $x_1 \cdots x_n$, if $n$ is odd.

These hold both for linear, polynomial and analytic projections.

Note that we can write the even case as $I_{ch}^{A_n} = I_{C_n} \cap (x_1, \ldots, x_n)^n$.

Proof. Every Chow equation has multiplicity $\geq n$, and we get the same equations modulo $(x_1, \ldots, x_n)^{n+1}$, whether we use linear, polynomial and analytic projections (50).

In both of our cases, $I_{C_n} \cap (x_1, \ldots, x_n)^{n+1} \subset I_{C_n}^{ch}$, so the ideal of Chow equations coming from linear projections already contains every possible higher order monomial. Thus it is sufficient to prove (1–2) for linear projections.

The linear projections of $C_n$ to $A_{uv}^2$ are given by $u = \sum_i a_i x_i, v = \sum b_i x_i$. The image of the $x_j$-axis is $b_j u - a_j v = 0$. So the pull-back of their product is

$$\prod_j \sum_i (a_i b_j - a_j b_i) x_i. \quad (119.3)$$

Since $C_n$ is toric, $I_{C_n}^{ch}$ is a monomial ideal. Thus we need to understand which degree $n$ monomials in the $x_i$ have a nonzero coefficient in (119.3).

First, the coefficient of $x_j$ in $\sum_i (a_i b_j - a_j b_i) x_i$ is 0, so we never get $x_j^n$. Next consider $x_1 \cdots x_n$. Its coefficient is

$$\sum_{\sigma \in S_n} \prod_i (a_i b_{\sigma(i)} - a_{\sigma(i)} b_i). \quad (119.4)$$
Note that the product is 0 if $\sigma(i) = i$ for some $i$ and changes by $(-1)^n$ when $\sigma$ is replaced by $\sigma^{-1}$. Thus if $n$ is odd then (119.4) is identically zero. (More generally, the permanent of a skew-symmetric matrix of odd size is 0.) We have thus proved the following.

**Claim 119.5.** If $n$ is odd then the coefficient of $x_1 \cdots x_n$ in (119.3) is 0. □

It remains to show that all other degree $n$ monomials appear in (119.3) with nonzero coefficient.

To show this we choose specific values of the $a_i, b_i$ and hope to get enough nonzero terms. Thus fix $1 \leq r \leq n$, choose $a_1 = \cdots = a_r = 1, a_{r+1} = \cdots = a_n = 0$ and $b_1 = \cdots = b_r = 0, b_{r+1} = \cdots = b_n = 1$. Then

$$a_i b_j - a_j b_i = \begin{cases} 1 & \text{if } i \leq r < j, \\ -1 & \text{if } j \leq r < i, \\ 0 & \text{otherwise.} \end{cases} \quad (119.6)$$

Thus (119.3) becomes

$$(-1)^r (x_1 + \cdots + x_r)^{n-r} (x_{r+1} + \cdots + x_n)^r \quad (119.7)$$

Applying this to various permutations of the $x_i$ and choices of $r$ we get the following.

**Claim 119.8.** Let $M = \prod x_i^{w_i}$ be a degree $n$ monomial. Then $M \in I_{C_n}^{eh}$ if the following holds.

(a) There is a subset $I \subset \{1, \ldots, n\}$ such that $\sum_{i \in I} w_i = n - |I|$. □

While this is only a sufficient condition, we check in (120) that it applies to every monomial other than $x_i^n$ and $x_1 \cdots x_n$ for odd $n$. This completes the proof of (119).

**Lemma 120.** Let $M = \prod x_i^{w_i}$ be a degree $n$ monomial other than $x_i^n$ or $x_1 \cdots x_n$ for odd $n$. Then there is a subset $I \subset \{1, \ldots, n\}$ such that $\sum_{i \in I} w_i = n - |I|$.

**Proof.** We use induction on $n$, the case $n = 1$ is empty and $n = 2$ is obvious.

Assume first that $w_{n-1} = w_n = 1$. If $M = x_1^{n-2} x_{n-1} x_n$ then $I = \{1, 2\}$ works. Otherwise, by induction, there is a subset $J \subset \{1, \ldots, n-2\}$ such that $\sum_{i \in J} w_i = n - 2 - |J|$. Set $I = J \cup \{n\}$. Then $\sum_{i \in I} w_i = n - 2 - |J| + 1 = n - |I|$ and we are done.

If the inductive step does not apply, then there is at most one $w_i = 1$, hence at least $n-1$ of the $w_i = 0$.

Reorder the $x_i$ such that $w_i$ is a decreasing function and take $r$ such that $w_1 + \cdots + w_{r-1} < n/2$ but $w_1 + \cdots + w_r \geq n/2$. If $w \leq n - r$ then we take $I = \{1, \ldots, r, n - s, \ldots, n\}$ where $s = n - r - w - 1$. Since $w_i = 0$ for $i \geq n-1$,

$$\sum_{i \in I} w_i = w_1 + \cdots + w_r = w \quad \text{and} \quad |I| = r + s + 1 = n - w.$$  

What happens if $w > n - r$? Note that then $r \geq 2$ and $w_1 \geq \cdots \geq w_r \geq 2$ so $(r-1)w_r < n/2$ and $2(r-1) < n/2$. On the other hand, $w_1 + \cdots + w_r < n/2 + w_r < n/2 + n/(2r-2)$. One checks that $n/2 + n/(2r-2) > n - r$ and $2(r-1) < n/2$ both hold only for $r = 2$. Furthermore, the only monomial for which the above choice of $I$ does not work is $x_1^{(n-1)/2} x_2^{(n-1)/2} x_3$ for $n$ odd. In this case we can take $I = \{1, 3, n - s, \ldots, n\}$ where $s = n-5/2$.

**Proposition 121.** Consider the deformation $C_n$ as in (114.1). Assume that $n \geq 3$. Then $I^{ch}(C_n)$ vanishes on the central fiber $(C_n)_k$ iff $\ord \phi_{ij} \leq n-2$ for every $i \neq j$. □
Proof. Note that \(x_i x_j x_k, x_i^2 x_j^2 \in I_{C_n}^{(2)}\) hence the only condition is the liftability of \(x_i x_j^{n-1}\).

If \(\text{ord} \phi_{ij} \leq n - 2\) then \(x_i x_j^{n-1} - x_j^{n-1} \phi_{ij}(x_j)\) vanishes along the \(x_j\)-axis and everywhere else.

Conversely, assume that we have equations

\[x_i x_j^{n-1} - \Psi_{ij}(x_1, \ldots, x_n) = 0.
\]

As in (108.2), they simplify to

\[x_i x_j^{n-1} - \psi_{ij}(x_j) = 0\]

along the \(x_j\)-axis.

Note that \(x_i x_j^{n-1}\) vanishes along the other \(n - 2\) axes, so we must have \(\psi_{ij}(0) = 0\).

Thus \(\phi_{ij} := x_j^{1-n} \psi_{ij}\) has a pole of order at most \(n - 2\).

\[\square\]

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