Buckling of chiral rods due to coupled axial and rotational growth

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Abstract

We present a growth model for special Cosserat rods that allows for induced rotation of cross-sections. The growth law considers two controls, one for lengthwise growth and other for rotations. This is explored in greater detail for straight rods with helical and hemitropic material symmetries by introduction of a symmetry preserving growth to account for the microstructure. The example of a guided-guided rod possessing a chiral microstructure is considered to study its deformation due to growth. We show the occurrence of growth induced out-of-plane buckling in such rods.

Keywords: Cosserat rod, Hemitropy, Helical symmetry, Growth, Bifurcation

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1 Introduction

Several theoretical models for elastic rods have been around for a while now. Starting from the Euler’s elastica to Kirchhoff rods, a very rich literature is available including the general model developed by Green, Naghdi and their collaborators. The general rod theory proposed by Green and Naghdi subsumes classical theories like the Cosserat rod theory as special cases under appropriate constraints. A comprehensive description of different rod theories is provided by \cite{Antman2005} and \cite{OREILLY2017}.

Rod theories have been employed in many interesting applications in the last few decades, for example in DNA biophysics \cite{Manning1996}, marine cables \cite{Goyal2005}, tendril perversion in plants \cite{Goriely1998, McMillen2002}, surgical filaments \cite{Nuti2014}, slender viscous jets \cite{Arne2010}, hair curls \cite{Miller2014} and carbon nanotubes \cite{Chandraseker2009, Kumar2011}.

Growing filamentary structures are ubiquitous in nature. Plant organs such as tendrils, roots and stem tend to twist while growing axially \cite{Wada2018}.

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There are studies with helical growth models where straight axial growth is accompanied by rotation of cross-sections (Wada, 2012; Goriely and Tabor, 2011). In this paper, we focus on this type of twisting growth, which can lead to non-planar configurations if the material of the rod exhibits some sort of twist-extension coupling.

The standard multiplicative decomposition (Rodriguez et al., 1994) used to model biological growth has been specialised for one-dimensional growth by Moulton et al. (2013). A recent study by Moulton et al. (2020) gives the reduction of three-dimensional energy for a tubular structure to a one-dimensional equivalent via minimization in cross-sections and subsequent averaging; it further demonstrates the generation of intrinsic twists and curvatures due to differential growth. A diverse account on biological growth is available in (Goriely, 2017), containing both mathematical and biomechanical aspects.

Euler buckling of filaments evolving their shape under time varying loads has been considered by Goldstein and Goriely (2006). Works like (McMillen et al., 2002) consider plant tendrils as Kirchhoff rods, straight in their initial states, which subsequently develop intrinsic curvatures in the grown equilibrium states. Another evolution law for intrinsic curvatures has been proposed by OReilly and Tresierras (2011) with a focus on tip growth. Guillon et al. (2012) modelled tree growth by considering the branch to be a special Cosserat rod growing in both length and diameter. They modelled the reference, relaxed and current configuration of the growing rod with separate base curves and director fields.

There have been several attempts to understand material symmetries in rods. Different types of chiral material symmetries applicable to initially straight rods with uniform circular cross-section have been discussed to great depth in an interesting work by Healey (2002). Other treatments of material symmetry in the context of rods include those by Luo and O’Reilly (2000); Lauderdale and OReilly (2006). In particular, Lauderdale and O’Reilly (2007) draw a few parallel comparisons with results by Healey (2002). In this manuscript, we follow the definitions and ideas of material symmetries introduced in (Healey, 2002, 2011).

Energy representations for helical symmetry and hemitropy have been derived in (Healey, 2002). Multi-fold helical symmetry is useful in modelling rods whose micro-structure mimics the symmetries of a rope made up of helices entwined together. Hemitropic rods possess the centre-line rotational symmetry of an isotropic rod but lack the reflection symmetries with respect to the longitudinal planes. Energy functions for rods with such chiral symmetries are typically characterized by coupled stretch, twist, shear and curvature terms. These couplings physically manifest as different types of non-traditional Poisson effects (Papadopoulos, 1999). Moreover, the conventional quadratic energy densities associated with linear elasticity is incapable of distinguishing between different orders helical symmetries and hemitropy.

Out-of-plane deformations are yet another feature of rods with such symmetries. Unshearable hemitropic rods can give rise to out-of-plane buckling when subjected to end displacements with fixed-fixed boundary condition, but on the other hand an axial load applied to a fixed-free rod always results in a planar solution (Healey and Papadopoulos, 2013). Similar bifurcation analysis has also been replicated for chiral rings with circular cross-sections under central loading (Hoang, 2019). Both in-plane and out-of-plane buckling of isotropic rods embedded in elastomeric matrix have been examined by Su et al. (2014), revealing that non-planar configurations are obtained whenever the matrix is stiff enough, compared to the bending stiffness of the rod. Primary root growth of certain plants has been investigated by Silverberg et al. (2012), drawing analogies from mechanical buckling of a metal filament embedded in a matrix comprising of two different
gels whose interface is transverse to the filament.

In this work, we study the growth induced deformation in rods possessing chiral material symmetries. It is natural to expect rods with helical symmetry to twist while growing length-wise, however the exact relationship between the growth law and rod’s microstructure is not well established. We assume growth and constitutive laws to be independent in general. Additionally, for rods with helical symmetry we postulate the growth law to be symmetry preserving, so that any imaginary helix associated with the microstructure remains unaltered as the rod grows. Such a growth problem depends only on the microstructural pitch and the constitutive laws, keeping aside the boundary conditions and other external factors.

A rod constrained to grow (or decay) in a guided-guided environment is considered, with a chiral constitutive law that is applicable to both helical symmetry and transverse hemitropy. Out-of-plane buckling is observed to occur at certain growth (or atrophy) stages, corresponding to the bifurcation modes. We demonstrate that an exact reversal in chirality of these non-planar solutions requires us to mirror the chiral parameters in both growth and constitutive laws simultaneously. Comparisons are made for the end-to-end distance in the buckled configuration with that in the virtual state to see if the ends have come closer or moved apart, than what they would have been in the absence of the guides. We also show that total growth induced extension in rod does not depend monotonically on the degree of chirality, that is, total extension in an isotropic rod need not lie between the total extension of rods with opposite material chirality.

This paper is organised as follows. We begin with a theoretical background of material symmetries in the context of special Cosserat rods in Section 2. A twisting growth law with two control parameters is systematically derived using certain kinematic assumptions such as homogeneity in length-wise growth and relative rotation of cross-sections in Section 3. In Section 4 we solve the problem of growth induced out-of-plane bifurcation in a chiral rod with guided-guided boundary conditions to study the interplay between chiralities in growth and material laws. We present our conclusions in Section 5.

1.1 Notation
Throughout this text, the indices $i, j, k \in \{1, 2, 3\}$ and $\alpha, \beta \in \{1, 2\}$, unless mentioned otherwise. We let $\{e_1, e_2, e_3\}$ to be a right-handed, fixed, orthonormal basis for the Euclidean space $E^3$. Boldface symbols are used to denote tensors, lowercase letters for first order e.g. $v$ and uppercase letters for second order tensors e.g. $T$. Underlined symbols such as $\underline{v}$ and $\underline{T}$ denote matrix representation of tensors with respect to a basis.

2 Special Cosserat rod formulation
Consider a straight rod of unit length in its stress-free reference configuration as shown in Figure 1. Assumption of special Cosserat rod behaviour requires the transverse cross-sections to stay rigid during the deformation. Let $s \in [-\frac{1}{2}, \frac{1}{2}]$ denote a signed arc-length parameter of the centre-line in the reference configuration. Let $r(s)$ define the centre-line of the deformed rod. Let $R(s) \in SO(3)$ be the rotation of transverse cross-sections in the reference configuration of the rod, mapping the fixed basis $\{e_1, e_2, e_3\}$ to a triad of orthonormal directors given by

$$d_i(s) = R(s)e_i.$$  \hspace{1cm} (1)
Figure 1: Kinematics of a special Cosserat rod—depicting the deformed centre-curve and the triad of orthonormal directors.

The vector fields

\[ \nu := r', \quad \kappa := \text{axial}(R'R^T), \]

define convected coordinates \( \nu = \nu_i d_i \) and \( \kappa = \kappa_i d_i \) with respect to the director frame field, along with the ordered triples \( \nu := (\nu_1, \nu_2, \nu_3) \) and \( \kappa := (\kappa_1, \kappa_2, \kappa_3) \). The strains \( \nu_\alpha \) correspond to shear, \( \nu_3 \) corresponds to stretch, \( \kappa_\alpha \) correspond to curvatures, and \( \kappa_3 \) corresponds to twist.

We further assume the rod to be hyperelastic with a differentiable energy density (per unit length) function \( \Phi(r', R, R', s) \). Material objectivity allows for a simpler version of energy function in terms of strains (Healey, 2002), given by

\[ \Phi = W(\nu, \kappa, s), \]

where \( W \) is another differentiable scalar valued function.

The internal force and moment on the transverse cross-section are denoted by \( n(s) = n_i d_i \) and \( m(s) = m_i d_i \), respectively, along with the corresponding triples \( \underline{n} := (n_1, n_2, n_3) \) and \( \underline{m} := (m_1, m_2, m_3) \). The components \( n_\alpha \) are essentially the shear forces, \( n_3 \) is axial force, \( m_\alpha \) are bending moments and \( m_3 \) is the torsional moment. These are related to the strain components as

\[ \underline{n} = \frac{\partial W}{\partial \nu}, \quad \underline{m} = \frac{\partial W}{\partial \kappa}. \]

To prevent self penetration, we require

\[ \nu_3 = r' \cdot d_3 > 0, \]

and the unshearability constraint is expressed as

\[ \nu_\alpha = r' \cdot d_\alpha = 0. \]
2.1 Material symmetry in Rods

In this section, we present a brief overview of certain classes of material symmetry for special Cosserat rods (Healey, 2002, 2011).

2.1.1 Helical Symmetry

Consider a straight rod possessing helical material symmetry with a signed pitch $\mathcal{M} \neq 0$ with $\mathcal{M} > 0$ for right-handed helices. Every transverse cross-section has a unique flip axis (or symmetry axis) which rotates as the section plane moves along the length of the rod. A 180-degree rotation (flip) about this renders the rod same as before. Unlike flips, reflections about a transverse plane do not result in a coincident helix, neither do the reflections through longitudinal planes. In fact, these reflections change the sign of $\mathcal{M}$, keeping its magnitude, the same.

We introduce a rotating basis $\{\mathbf{e}^*_1(\phi), \mathbf{e}^*_2(\phi), \mathbf{e}^*_3(\phi)\}$ and a corresponding triad of director fields given by

$$
\mathbf{e}^*_i(\phi) = Q_\phi \mathbf{e}_i, \quad 0 \leq \phi < 2\pi
$$

$$
\mathbf{d}^*_\alpha(s) = R(s) \mathbf{e}^*_\alpha \left( \frac{s}{\mathcal{M}} \right),
$$

where $Q_\phi$ is a proper orthogonal tensor with matrix representation

$$
Q_\phi = \begin{bmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{bmatrix},
$$
in the fixed basis.

Assuming $\mathbf{e}^*_1(\phi)$ to be the rotating flip axis, we denote by $H^*_\pi$ the flip about $\mathbf{e}^*_1(\phi)$. Material properties with respect to the symmetry axis $\mathbf{e}^*_1(\frac{s}{\mathcal{M}})$ are assumed not to change as the cross-section ‘$s$’ moves along the rod. This motivates the definition of a symmetry adapted energy function (Healey, 2002) independent of $s$, given by

$$
W^*(v^*, k^*) = \Phi = W^*(v^*, k^*),
$$

where $v^* = (\nu^*_1, \nu^*_2, \nu^*_3)$ and $k^* = (\kappa^*_1, \kappa^*_2, \kappa^*_3)$ emerge from the change of coordinates

$$
\kappa = \kappa^*_\alpha \mathbf{d}^*_\alpha + \kappa^*_3 \mathbf{d}_3, \quad \nu = \nu^*_\alpha \mathbf{d}^*_\alpha + \nu^*_3 \mathbf{d}_3.
$$

Helical symmetry is characterized by the following equation

$$
W^*(\nu^*_1, \nu^*_2, \nu^*_3, \kappa^*_1, \kappa^*_2, \kappa^*_3) = W^*(-\nu^*_1, \nu^*_2, \nu^*_3, -\kappa^*_1, \kappa^*_2, \kappa^*_3),
$$
in terms of the new energy function without ‘$s$’ as an argument.

2.1.2 n-fold Helical Symmetry

Consider a rod with a symmetry analogous to $n \geq 2$ helices entwined together, such that each cross-section at $s$ has $n$ equally spaced flip axes. A 180-degree rotation about each of these gives a symmetry. Such a rod is said to have a n-fold dihedral helical symmetry which is characterized by the condition

$$
W^* \left( -H^*_\frac{2\pi}{n} v^*, -H^*_\frac{2\pi}{n} k^* \right) = W^*(v^*, k^*),
$$
in addition to (12), where \( H^{\pi \ast}_{\pi n} \) is the matrix of \( H^{\pi \ast}_{\pi n} \) with respect to the rotating basis (7).

\[
H^{\pi \ast}_{\pi n} = \begin{bmatrix}
\cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\
\sin\left(\frac{2\pi}{n}\right) & -\cos\left(\frac{2\pi}{n}\right) & 0 \\
0 & 0 & -1
\end{bmatrix}.
\] (14)

### 2.1.3 Continuous Helical Symmetry

For \( n \gg 1 \), a straight rod with \( n \)-fold dihedral helical symmetry approaches to what is called continuous helical symmetry. In this type of symmetry all vectors of the cross section act as symmetry axis, or equivalently any fixed flip axis, say \( \mathbf{e}_1 \) acts as a symmetry axis for all cross sections. Continuous helical symmetry can be characterized by

\[
W(-H^{\pi \ast}_\phi \mathbf{v}, -H^{\pi \ast}_\phi \mathbf{k}) = W(\mathbf{v}, \mathbf{k}), \quad \forall \phi \in [0, \pi).
\] (15)

### 2.1.4 Transverse hemitropy and isotropy

Let \( \mathbf{E} \) denote a reflection with a matrix with respect to the fixed basis as

\[
\mathbf{E} = \begin{bmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\] (16)

A homogeneous hyperelastic straight rod with energy function \( W(\mathbf{v}, \mathbf{k}) \) is transversely hemitropic if

\[
W(\mathbf{Q}\mathbf{v}, \mathbf{Q}\mathbf{k}) = W(\mathbf{v}, \mathbf{k}), \quad \forall \phi \in [0, 2\pi),
\] (17)

and flip-symmetric if

\[
W(\mathbf{E}\mathbf{v}, \mathbf{E}\mathbf{k}) = W(\mathbf{v}, \mathbf{k}).
\] (18)

Note that flip-symmetry does not belong to the class of transverse symmetry, defined by Healey (2002). A straight rod is transversely isotropic if in addition to (17), it also satisfies

\[
W(\mathbf{v}, \mathbf{k}) = W(\mathbf{E}\mathbf{v}, -\mathbf{E}\mathbf{k}).
\] (19)

Flip-symmetric hemitropy is equivalent to continuous helical symmetry (Healey, 2002). Another way to obtain flip-symmetric hemitropy, is to consider a rod with helical symmetry and take the limit \( \mathcal{M} \to 0 \) (Healey, 2011).

### 2.2 Energy function

The energy density per unit length of unshearable hemitropic rods can be expressed as

\[
W = \Upsilon(\kappa_\alpha \kappa_\alpha, \nu_3, \kappa_3),
\] (20)
where $\Upsilon$ is a scalar valued function. This representation is also valid for flip-symmetry.

For calculations in this paper, we adopt a model considered by Papadopoulos (1999); Healey and Papadopoulos (2013) defined as

$$\Upsilon = \frac{1}{2} \left[ \Phi(\nu_3) + 2A(\nu_3 - 1)\kappa_3 + B\kappa_3^2 + C\kappa_3\kappa_\alpha \right], \quad (21)$$

where $\Phi : (0, \infty) \to \mathbb{R}$ is a function such that $g := \frac{1}{2} \Phi'$ obeys $g(\nu_3) \to -\infty$ as $\nu_3 \to 0$. The function $g(\cdot)$ allows us to modify the axial force response of the model, and it must satisfy $g(1) = 0$. The constant $C$ corresponds to bending stiffness, $B - \frac{A^2}{g'(1)}$ is equivalent to torsional rigidity and $g'(1) - \frac{A^2}{B}$ to axial stiffness, where $A$ is the degree of hemitropy. We assume $B > 0$, $C > 0$ and $Bg'(\nu_3) > A^2$ for all $\nu_3$ to ensure convexity. This in turn implies that $g(\cdot)$ should be monotonic and hence invertible. For example, a response function satisfying all our criteria can be chosen as (Papadopoulos, 1999)

$$g(\nu_3) = F \ln(\nu_3) + \frac{A^2}{B}(\nu_3 - 1), \quad (22)$$

where $F > 0$ is a constant. This energy allows for infinite compressive axial force $\nu_3 \to -\infty$ whenever an unrealistically extreme strain $\nu_3 \to 0$ is present.

As demonstrated in Healey (2002), quadratic energy functions are incapable of distinguishing between different types of $n$-fold helical symmetry ($n \geq 3$) and hemitropy. On similar lines, the energy function (21) can be shown to be applicable to $n$-fold helical symmetry.

### 3 Growth Formulation

Growth in elastic bodies is typically modelled by introduction of a multiplicative decomposition of the deformation gradient into pure growth and pure elastic deformation parts (Rodriguez et al., 1994; Ambrosi et al., 2011). This decomposition assumes a virtual stress-free incompatible configuration. For one-dimensional structures where growth manifests as increase in overall length, first the stress-free rod isolated from its environment and boundary conditions can be allowed to grow free into a virtual state, and then the boundary and environmental factors can be forcibly imposed (Goriely, 2017; O’Keefe et al., 2013).

#### 3.1 General Framework for Growing Rods

Let $\mathcal{R}_o$ denote the initial stress-free reference configuration of the rod, occupying $\{ \mathbf{Se}_3 : -\frac{1}{2} \leq S \leq \frac{1}{2} \}$ and denote by $S$ a signed arc length parameter of the centre-line in $\mathcal{R}_o$. Let $\tilde{\mathbf{r}}(S)$ be the curve taken by the centre-line in the virtual grown configuration $\tilde{\mathcal{R}}$, such that the point $\mathbf{Se}_3$ in $\mathcal{R}_o$ gets mapped to $\tilde{\mathbf{r}}(S)$ in $\tilde{\mathcal{R}}$ (Figure 2). The virtual configuration is assumed to be stress-free. We define a signed arc-length $s(S)$ in $\tilde{\mathcal{R}}$ by

$$s(S) := \int_0^S ||\tilde{\mathbf{r}}'(\tau)||d\tau. \quad (23)$$
Figure 2: Kinematics of an initially straight rod growing from origin $S_0$, depicting the configurations— reference $\mathcal{R}_o$, virtual $\tilde{\mathcal{R}}$ and current $\mathcal{R}$; along with the multiplicative decomposition $Q = RW$.

We denote the transverse cross-section at $S$ in $\mathcal{R}_o$ by $\Gamma_o(S)$ and let it get mapped to $\tilde{\Gamma}(S)$ in the virtual configuration $\tilde{\mathcal{R}}$. Define $W(S) \in SO(3)$ to be the rotation of $\tilde{\Gamma}(S)$ with respect to $\Gamma_o(S)$, and let it map the fixed basis $\{e_1, e_2, e_3\}$ to a virtual director field given by

$$\tilde{e}_i(S) = W(S)e_i.$$  \hspace{1cm} (24)

When the boundary conditions and environmental factors are imposed, let the centre-line take the curve $r(S)$ in the current configuration $\mathcal{R}$, and the cross-section $\Gamma(S)$ in $\tilde{\mathcal{R}}$ be mapped to $\Gamma(S)$ in $\mathcal{R}$. Define $R(S) \in SO(3)$ to be the rotation of $\Gamma(S)$ with respect to $\Gamma(S)$ and $Q(S) \in SO(3)$ to be the rotation of $\Gamma(S)$ with respect to $\Gamma_o(S)$, so that

$$Q(S) = R(S)W(S).$$  \hspace{1cm} (25)

The virtual director field is transformed into another director field in the current configuration given by

$$d_i(S) = R(S)\tilde{e}_i(S) = Q(S)e_i.$$  \hspace{1cm} (26)

All the maps we have introduced are assumed to be smooth for the sake of convenience. Analogous to $r : [-\frac{1}{2}, \frac{1}{2}] \rightarrow \mathbb{E}^3$, we define another map $\tilde{r} : [s(-\frac{1}{2}), s(\frac{1}{2})] \rightarrow \mathbb{E}^3$ to denote the same curve via another parametrization,

$$r(S) = (\tilde{r} \circ s)(S).$$  \hspace{1cm} (27)

This implies

$$r'(S) = ||\tilde{r}'(S)||\tilde{r}'(s(S)).$$  \hspace{1cm} (28)
Similarly, define \( \hat{R} : [s(-\frac{1}{2}), s(\frac{1}{2})] \rightarrow SO(3) \) by

\[
R(S) = (\hat{R} \circ s)(S).
\]

We assume the transverse cross-sections to remain orthogonal to centre-line in both virtual and current configurations, hence the conditions

\[
\tilde{r}' \cdot \tilde{e}_a = 0 \quad \text{and} \quad r' \cdot d_a = 0,
\]

where (31) is equivalent to the unshearability constraint (6). The symbols and notations introduced in this section are pictorially represented in Figure 2.

### 3.1.1 Homogeneous growth kinematics

We consider the growth to be homogeneous throughout the rod. This assumption leads to the following constraints:

- The length-wise growth parameter denoted by \( \gamma := ||\tilde{r}'(S)|| \) is a constant, that is, it is independent of \( S \).
- Let \( h \in \mathbb{R} \) be such that \( 0 < |h| < 1 \). Consider the relative rotation of cross-section \( \bar{\Gamma}(S + h) \) with respect to \( \tilde{\Gamma}(S) \).

\[
\tilde{e}_i(S + h) = W(S + h)W(S)^{-1}\tilde{e}_i(S).
\]

For all permissible \( h \), the relative rotation \( W(S + h)W(S)^{-1} \) is assumed to be independent of \( S \), and hence can be denoted as a function of \( h \) only.

\[
W(S + h)W(S)^{-1} =: \Pi(h).
\]

This gives us the decomposition \( W(S + h)W(S)^{-1} =: \Pi(h)W(S) \) which leads to

\[
\Pi(h)^T \frac{\partial \Pi(h)}{\partial h} = \frac{\partial W(S)}{\partial S}W(S)^T. \tag{34}
\]

Another way to interpret (33) is to set

\[
\frac{\partial}{\partial S}\left\{ W(S + h)W(S)^{-1} \right\} = O, \tag{35}
\]

which implies

\[
W(S + h)^T \frac{\partial}{\partial S}W(S + h) = W(S)^T \frac{\partial}{\partial S}W(S). \tag{36}
\]

We define the tensor fields for later use

\[
\Lambda(S) := W(S)^T \frac{\partial W(S)}{\partial S}, \quad \text{and} \quad \Omega(S) := \frac{\partial W(S)}{\partial S}W(S)^T. \tag{37}
\]

Equations (34) and (36) imply that \( \Lambda \) and \( \Omega \) are constant skew-symmetric tensors (The proofs are detailed in Appendix A).
We fix a point on the centre-line which gets mapped to itself under the growth transformation, along with its corresponding cross-section. Thus, we assume the existence of a point $S_o \in \left[ -\frac{1}{2}, \frac{1}{2} \right]$ satisfying
\[
\tilde{r}(S_o) = S_o e_3 \quad \text{and} \quad W(S_o) = I. \tag{38}
\]
This can also be interpreted as if the rod is allowed to grow while being held at $S_o$ (origin of growth). It is held in such a way that no incompatibility or stress is caused due to growth. Define vectors $a := \text{axial}(\Lambda)$ and $\omega := \text{axial}(\Omega)$, these are actually constant vectors and can be related by
\[
\omega = W(S)a. \tag{39}
\]
Since this is also satisfied for the specific point $S = S_o$, we imply $a = \omega$ and $\Lambda = \Omega$. This also means that $\text{axis}(W(S)) = \omega$ for all $S$. Thus one can solve (37) for $W(S)$ as a differential equation to obtain
\[
W(S) = e^{(S-S_o)\Omega}, \tag{40}
\]
where tensor exponential is defined by the usual series definition. The mathematical details for derivations in this section are provided in Appendix A.

### 3.1.2 Extension to a general growing curve

Consider a general scenario where the initial configuration $\mathcal{R}_o$ is a special Cosserat rod. Let $\tilde{r} : \left[ -\frac{1}{2}, \frac{1}{2} \right] \to \mathbb{E}^3$ be its centre-curve, where $\tilde{r}(S)$ is arc-length parametrized. Let $W(S) \in SO(3)$ denote the orientation of $\Gamma_o(S)$ with respect to the fixed basis, mapping those to an orthonormal director field $\tilde{e}_i(S) := W(S)e_i$ associated with initial configuration. Homogeneous growth law still requires $\gamma$ to be constant while equation (32) is modified as
\[
\tilde{e}_i(S + h) = W(S + h)\tilde{W}(S + h)\tilde{W}(S)^{-1}W(S)^{-1}\tilde{e}_i(S). \tag{41}
\]
The tensor $W(S + h)\tilde{W}(S)^{-1}$ is again independent of $S$. In addition, the rod is assumed to be held at $S_o \in \left[ -\frac{1}{2}, \frac{1}{2} \right]$ while growing, so that we have
\[
\tilde{r}(S_o) = \tilde{r}(S_o) \quad \text{and} \quad W(S_o) = I. \tag{42}
\]
This assumption along with the kind of homogeneity used in induced rotations gives such a $W(S)$ that makes all the cross-sections rotate about the particular axis $a$. Moreover, the solution is given by (40) which in turn implies
\[
\tilde{e}_i(S) = e^{(S-S_o)\Lambda}W(S)e_i, \tag{43}
\]
In fact, the constant vector $\omega = a$ can be treated as the growth parameter controlling relative rotation of cross-sections while $\gamma$ controls the length-wise growth as in the former case. Whenever the centre-curves are normal to the cross-sections throughout $\mathcal{R}_o$ and $\tilde{R}$, we deduce
\[
\tilde{r}(S) = \tilde{r}(S_o) + \gamma \int_{S_o}^{S} e^{(\tau-S_o)\Lambda}\tilde{r}'(\tau) d\tau. \tag{44}
\]
We emphasise that (43) and (40) do not assume the respective centre-curves to be normal to the cross-sections neither in $\mathcal{R}_o$ nor in $\tilde{R}$.
3.2 Growth in straight rods

Consider a straight rod with flip-symmetric hemitropy in its reference configuration. A straight virtual configuration condenses to

$$\tilde{r}(S) = \{S_o + \gamma(S - S_o)\}e_3,$$

(45)

which with the aid of (30) results in

$$W(S)e_3 = e_3 \quad \forall S \in \left[-\frac{1}{2}, \frac{1}{2}\right].$$

(46)

This indicates that $\omega$ is along $e_3$. We introduce another growth parameter $\omega$ defined by

$$\omega = \omega e_3,$$

(47)

so that its corresponding skew tensor is

$$\Omega = \omega A, \quad \text{with} \quad A = e_2 \otimes e_1 - e_1 \otimes e_2.$$

(48)

Since rotation tensor can also be expressed as

$$Q_\phi = e^{\phi A},$$

(49)

we get

$$W(S) = Q_{(S - S_o)\omega}.$$

(50)

The parameters $\gamma$ and $\omega$ capture all the necessary information regarding growth. It’s evident that $\gamma > 1$ reflects growth while $\gamma < 1$ denotes atrophy. Similarly, $\omega$ and $-\omega$ signify two opposite cross-sectional rotations caused by growth while $\omega = 0$ indicates no growth induced rotation.

3.2.1 Growth law

The growth law adopted here considers rotation of cross sections with respect to each other in the due course of growth. Consider a rotating basis field $\{e_1^*(S), e_2^*(S), e_3^*(S) = e_3\}$ given by

$$e_1^*(S) = Q_{\frac{\gamma}{3\pi}} e_1,$$

(51)

representing a helix embedded in the initial configuration of a rod. As the rod grows this transforms into $W(S)e_1^*(S)$ in the virtual configuration. Let us denote this by a basis field $\{f_1^*(s), f_2^*(s), f_3^*(s)\}$ defined on the virtual arc-length parameter by

$$W(S)e_1^*(S) =: (f_1^* \circ s)(S).$$

(52)

This is equivalent to

$$f_1^*(s) = Q_{\frac{\gamma}{3\pi} + (\frac{S}{S_o} - 1)\omega} e_1.$$

(53)

Let $h \neq 0$ be such that $e_1^*(S + h)$ and $f_1^*(s + h)$ are well defined, then we obtain

$$e_1^*(S + h) = Q_{\frac{\gamma}{3\pi}} e_1^*(S),$$

(54)

$$f_1^*(s + h) = Q_{\frac{\gamma}{3\pi} + \omega} f_1^*(s).$$

(55)
This shows that our chosen growth map transforms the initial helix with pitch $M$ into another helix with pitch, say $\mu$, which can be expressed as

$$\mu = \frac{\gamma M}{1 + \omega M}.$$  \hfill (56)

This motivates us to define a symmetry preserving growth law for rods possessing helical symmetry.

**Rods with helical symmetry** Consider a rod which due to its microstructure possesses simple helical symmetry or $n$-fold helical symmetry. Let $M$ be the pitch associated with its microstructure. Once growth parameters $\gamma$ and $\omega$ are known, (56) serves as an evolution law for the pitch of its microstructure.

We introduce the idea of symmetry preserving growth – wherein the growth map fixes all helices with pitch same as that of the microstructure ($\mu = M$). Thus, for rods with a pitch associated with their microstructure we have the following helical growth law

$$\gamma = 1 + \omega M,$$  \hfill (57)

where $\gamma$ is the only growth parameter and $M$ comes from the material symmetry. For rods having helical symmetry, this assumption of symmetry preserving growth provides a rationale for relative rotation of cross-sections during growth.

**Hemitropic rods** Although there are different versions (Healey, 2002, 2011) of how helical symmetry can be used to arrive at hemitropy, there is no pitch directly associated with transverse hemitropy (17). So, for hemitropic rods (and even isotropic) one may use the same helical growth law (57) without any notion of microstructural pitch, in which case both $\gamma$ and $M$ are independent growth parameters. For such a growth law all helices with pitch $M$ remain unaltered under the growth map, so we denote it as the characteristic pitch of growth.

### 3.2.2 Calculation of Strains

The grown configuration is obtained by imposing environmental and boundary effects on the virtual stress-free configuration. Hence the strain energy is a function of $\hat{r}'(s)$, $\hat{R}(s)$ and $\hat{R}'(s)$. We define the vector fields

$$\hat{\nu} = \hat{r}' \quad \text{and} \quad \hat{\kappa} = \text{axial}(\hat{R}' \hat{R}^T).$$  \hfill (58)

Let their components be $\hat{\nu} = \hat{\nu}_i d_i$ and $\hat{\kappa} = \hat{\kappa}_i d_i$ with respect to the director frame in current configuration. Consider the derivative

$$\frac{\partial d_i}{\partial S} = \frac{\partial Q^{-1}}{\partial S} Q^{-1} d_i = \left[ \frac{\partial R}{\partial S} W + R \frac{\partial W}{\partial S} \right] W^{-1} R^{-1} d_i$$  \hfill (59a)

$$= \left( \frac{\partial R}{\partial s} \right) \left( \frac{\partial s}{\partial S} \right) R^{-1} d_i + R \frac{\partial W}{\partial S} W^{-1} \bar{e}_i$$  \hfill (59b)

$$= \gamma \hat{\kappa} \times d_i + R (\omega \times \bar{e}_i)$$  \hfill (59c)

$$= (\gamma \hat{\kappa} + R \omega) \times d_i.$$  \hfill (59d)
Now define the axial vector $\beta := \text{axial}\left(\frac{\partial Q}{\partial S}Q^{-1}\right)$ which along with the straight growth assumption implies

$$\beta = \gamma \hat{\kappa} + \omega d_3. \tag{60}$$

Given the growth parameters, this relation will be used in retracting the actual strains from the apparent curvature $\beta$. Corresponding to $\hat{\nu}$ and $\hat{\kappa}$ we define

$$\nu = r' \quad \text{and} \quad \kappa = \text{axial}(R'R^T), \tag{61}$$

along with their convected components $\nu = \nu_id_i$ and $\kappa = \kappa_id_i$. These speeds and curvatures can be related to the actual strains by

$$\nu_i = \gamma \hat{\nu}_i \quad \text{and} \quad \kappa_i = \gamma \hat{\kappa}_i. \tag{62}$$

Upon use of the energy density function (21), the internal force $n(S) = n_i(S)d_i(S)$ and moment $m(S) = m_i(S)d_i(S)$ in the current configuration can be related to the strains as follows:

$$n_3 = g(\hat{\nu}_3) + A\hat{\kappa}_3, \tag{63}$$
$$m_3 = A(\hat{\nu}_3 - 1) + B\hat{\kappa}_3, \tag{64}$$
$$m_\alpha = C\hat{\kappa}_\alpha. \tag{65}$$

### 3.2.3 Equilibrium equations

The local linear and angular momentum balance equations for static equilibrium (O’Keeffe et al., 2013; Goriely, 2017) are as follows:

$$\frac{\partial n}{\partial s} + f = 0, \tag{66}$$
$$\frac{\partial m}{\partial s} + \frac{\partial r}{\partial s} \times n + l = 0. \tag{67}$$

where $f$ and $l$ respectively denote the body force and body moment per unit virtual arc-length. A change of variable to reference coordinates results in

$$n' + \gamma f = 0, \tag{68}$$
$$m' + r' \times n + \gamma l = 0. \tag{69}$$

### 4 Growing rod with guided-guided ends

In this section we consider the example of a growing rod with guided ends as shown in Figure 3. A guided boundary condition is equivalent to fixing the end of the rod to a block constrained by a slot to translate only along the rod’s axis. We use the energy function (21) and the growth law (57) to model the rod. Even though all the calculations would be similar, the results can be discussed separately for two different problems – first, a hemitropic rod and second, a rod with $n$-fold helical symmetry.
The linear and angular momentum balance equations are

\[
\frac{d}{ds}\left[n_\alpha Q_{e\alpha} + \{g(\hat{\nu}_3) + A\hat{\kappa}_3\} Q_{e3}\right] = 0, \tag{70}
\]

\[
\frac{d}{ds}\left[C\hat{\kappa}_\alpha Q_{e\alpha} + [A\{\hat{\nu}_3 - 1\} + B\hat{\kappa}_3] Q_{e3}\right] + \hat{r}' \times \left[n_\alpha Q_{e\alpha} + \{g(\hat{\nu}_3) + A\hat{\kappa}_3\} Q_{e3}\right] = 0, \tag{71}
\]

along with the boundary conditions

\[
n\left(\pm \frac{1}{2}\right) \cdot e_3 = 0, \tag{72}
\]

\[
r\left(\pm \frac{1}{2}\right) \cdot e_\alpha = 0 \tag{73}
\]

and

\[
Q\left(\pm \frac{1}{2}\right) = I. \tag{74}
\]

The unshearability constraint \[31\] results in

\[
r' \cdot Q_{e\alpha} = 0. \tag{75}
\]

Equations \[70\] - \[75\] comprise our boundary value problem to be solved for the fields \(r\), \(R\), and \(n_\alpha\). Since we have not imposed any sort of axial constraint, with these set of boundary conditions we will get a family of solutions differing by a scalar multiple of \(e_3\).

The rod is assumed to be of unit length; thus, all the kinematic quantities are dimensionless by default. The components of internal force, internal moment, material constants \(A\), \(B\) and the response function \(g(\cdot)\) can be all non-dimensionalized against \(C\) by either dividing the concerned quantities in \[63\] - \[65\] by \(C\), or equivalently setting \(C = 1\) in the boundary value problem \[85\] - \[92\]. We follow the bifurcation analysis methodology presented by \[Smith and Healey\] \[2008\]; \[Healey and Papadopoulos\] \[2013\] wherein first a primary solution is determined which is then perturbed and the boundary value problem is rederived in terms of the perturbations to get linearized equations.
4.1 The primary solution

Let us consider a simple solution where the rod always remains straight while growing. A straight solution is of the form

\[ r(S) = \lambda S e_3 , \quad Q(S) = I , \quad n_\alpha(S) = 0. \]  

where \( S \in \left[ -\frac{1}{2}, +\frac{1}{2} \right] \). This solution has its local force, moment and strain fields as follows:

\[ \tilde{\nu}(s) = \frac{\lambda}{\gamma} e_3 , \]  

\[ \tilde{\kappa}(s) = -\frac{\omega}{\gamma} e_3 , \]  

\[ n(S) = \left[ g\left( \frac{\lambda}{\gamma} \right) - A \frac{\omega}{\gamma} \right] e_3 , \]  

\[ m(S) = \left[ A \left\{ \frac{\lambda}{\gamma} - 1 \right\} - B \frac{\omega}{\gamma} \right] e_3 . \]

For such a solution to comply with the force boundary condition (72) we require \( \lambda \) to satisfy

\[ g\left( \frac{\lambda}{\gamma} \right) = A \frac{\omega}{\gamma} . \]  

4.2 Perturbed Solution

Consider a first order perturbation of the straight solution (with \( 0 < \epsilon \ll 1 \)) given by

\[ r(S) = \lambda S e_3 + \epsilon \rho(S) , \]  

\[ Q(S) = e^{\epsilon \Psi(S)} , \]  

\[ n_\alpha(S) = \epsilon \eta_\alpha(S) , \]

where \( \Psi(S) \) is skew symmetric with \( \text{axial}(\Psi) =: \psi \). We require these perturbed fields to satisfy our boundary value problem. Plugging in the perturbations (82)-(84) into our boundary value problem (70)-(75) results in the following linearized problem.

\[ \eta'_\alpha e_\alpha = 0 , \]  

\[ (\psi'' + \omega e_3 \times \psi') \cdot e_\alpha e_\alpha + \left[ A \{ \lambda - \gamma \} - B \omega \right] \psi' \times e_3 + \gamma \lambda e_3 \times \eta_\alpha e_\alpha = 0 , \]  

\[ \left\{ g'\left( \frac{\lambda}{\gamma} \right) \rho'' + A \psi'' \right\} \cdot e_3 = 0 , \]  

\[ (A \rho'' + B \psi'') \cdot e_3 = 0 , \]  

\[ (\rho' - \lambda \psi \times e_3) \cdot e_\alpha = 0 , \]  

\[ \psi\left( \pm \frac{1}{2} \right) = 0 , \]  

\[ \rho\left( \pm \frac{1}{2} \right) \cdot e_\alpha = 0 , \]  

\[ \left[ g'\left( \frac{\lambda}{\gamma} \right) \rho'\left( \pm \frac{1}{2} \right) + A \psi'\left( \pm \frac{1}{2} \right) \right] \cdot e_3 = 0 , \]
with details provided in Appendix B. Since \( Bg'(\frac{\lambda}{\gamma}) - A^2 \) is non-zero (assumed to be positive), equations (87) and (88) imply
\[
\rho'' \cdot e_3 = 0 \quad \text{and} \quad \psi'' \cdot e_3 = 0.
\] (93)
Boundary condition (90) forces us to have \( \psi(S) \in \text{span}\{e_1, e_2\} \), which motivates the introduction of the decomposition
\[
\rho(S) = \rho_t(S) + \rho_a(S),
\] (94)
where \( \rho_t(S) \in \text{span}\{e_1, e_2\} \) and \( \rho_a(S) \in \text{span}\{e_3\} \).
Equations (85)-(92) can now be reduced to the following (details in Appendix C):
\[
\psi'' + \zeta \psi' \times e_3 = \psi'(1 + \frac{1}{2}) - \psi'(-\frac{1}{2}),
\] (95)
\[
\rho_t' = \lambda \psi \times e_3,
\] (96)
\[
\rho_a'' = 0,
\] (97)
accompanied by the boundary conditions
\[
\rho_t\left( \pm \frac{1}{2} \right) = 0,
\] (98)
\[
\rho_a'\left( \pm \frac{1}{2} \right) = 0.
\] (99)
The new parameter \( \zeta \) appearing in (95) is defined as
\[
\zeta := A(\lambda - \gamma) - (B + 1)\omega.
\] (100)
It is clear that \( \rho_a(S) = C_6 e_3 \) for all \( S \), where \( C_6 \) is a constant that appears because we have put no physical constraint in axial direction. As the rod can slide in the axially without causing any strain, we can fix \( C_6 = 0 \).
For \( \zeta = 0 \), the problem admits only trivial solutions (Appendix C). Now assuming \( \zeta \neq 0 \), the differential equations (95) and (96) admit general solutions of the form
\[
\psi(S) = \frac{C_1}{\zeta} \begin{pmatrix} \sin(\zeta S) \\ -\cos(\zeta S) + 2S \sin\frac{\zeta}{2} \end{pmatrix} + \frac{C_2}{\zeta} \begin{pmatrix} -\cos(\zeta S) - 2S \sin\frac{\zeta}{2} \\ -\sin(\zeta S) \end{pmatrix} + \begin{pmatrix} C_3 \\ C_4 \end{pmatrix},
\] (101)
\[
\rho_t(S) = \frac{\lambda}{\zeta^2} \begin{pmatrix} -\sin(\zeta S) + \zeta S^2 \sin\frac{\zeta}{2} \\ \cos(\zeta S) \end{pmatrix} + \frac{\lambda}{\zeta^2} \begin{pmatrix} \cos(\zeta S) \\ \sin(\zeta S) + \zeta S^2 \sin\frac{\zeta}{2} \end{pmatrix}
+ \lambda \begin{pmatrix} C_5 + C_4 S \\ C_6 - C_3 S \end{pmatrix},
\] (102)
where \( C_1, C_2, \cdots, C_6 \) are generic integration constants in \( \mathbb{R} \). The representations \( \psi \) and \( \rho_t \) are with respect to the fixed basis. The boundary conditions (90) when invoked into the solution (101) leads to
\[
(C_1 - C_2) \sin \frac{\zeta}{2} = 0,
\] (103)
simultaneously giving
\[ C_3 = \frac{C_2}{\zeta} \cos \frac{\zeta}{2}, \quad C_4 = \frac{C_1}{\zeta} \cos \frac{\zeta}{2}. \] (104)

The values of \( \zeta \neq 0 \) for which \( \sin \frac{\zeta}{2} = 0 \) eventually lead to the trivial solution (Appendix C). Therefore, we assume \( C_1 = C_2 \), which when plugged into the general solution (102) and forced to satisfy (98), leads to the condition
\[ \frac{1}{\zeta} \sin \frac{\zeta}{2} - \frac{1}{2} \cos \frac{\zeta}{2} = 0. \] (105)

It simultaneously leads to the constants
\[ C_5 = -\frac{C_1}{\zeta^2} \left( \frac{\zeta}{4} \sin \frac{\zeta}{2} + \cos \frac{\zeta}{2} \right) = C_6. \] (106)

Hence we have an out-of-plane solution,
\[
\begin{align*}
\rho(S) &= C_1 \frac{\lambda}{\zeta^2} \left\{ \cos(\zeta S) + \left(S^2 - \frac{1}{4}\right) \zeta \sin \frac{\zeta}{2} - \cos \frac{\zeta}{2} \right\} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \\
&\quad + C_1 \frac{\lambda}{\zeta^2} \left\{ S \zeta \cos \frac{\zeta}{2} - \sin(\zeta S) \right\} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},
\end{align*}
\] (107)

whose existence is subject to the condition that parameters \( \gamma \) and \( \lambda \) admit sensible solutions (\( \gamma > 0 \) and \( \lambda > 0 \)). A positive increasing sequence \( (a_n)_{n=1}^\infty \) satisfying \( \tan(a_n) = a_n \) can be defined. The values taken by \( \zeta \in \{ \pm 2a_n : n \in \mathbb{N} \} \) correspond to the discrete bifurcation modes.

## 4.3 Results and Discussion

In view of the equivariance properties of our problem (Appendix D), any rotation of (107) about \( e_3 \) is an acceptable solution, hence the solution can be simplified to

\[
\begin{align*}
\rho(S) &= C_1 \frac{\lambda}{\zeta^2} \begin{pmatrix} \cos(\zeta S) + \left(S^2 - \frac{1}{4}\right) \zeta \sin \frac{\zeta}{2} - \cos \frac{\zeta}{2} \\ \sin(\zeta S) - S \zeta \cos \frac{\zeta}{2} \\ 0 \end{pmatrix}, \\
\psi(S) &= C_1 \frac{\lambda}{\zeta} \begin{pmatrix} \cos \frac{\zeta}{2} - \cos(\zeta S) \\ 2S \sin \frac{\zeta}{2} - \sin(\zeta S) \\ 0 \end{pmatrix}, \\
\eta_1(S) &= 0, \\
\eta_2(S) &= C_1 \frac{\zeta \cos \frac{\zeta}{2}}{\gamma}.
\end{align*}
\] (108-111)
where representations (108) and (109) are with respect to the fixed basis. This solution is clearly flip symmetric about \( e_1 \), thus making it clear that (107) was also flip symmetric, but about an axis different from \( e_1 \). The deformed centre-line \( r(S) \) for this solution is

\[
\mathbf{r}(S) = \frac{\lambda}{\zeta^2} \begin{pmatrix}
\cos(\zeta S) + \left( S^2 - \frac{1}{4} \right) \zeta \sin\frac{\zeta}{2} - \cos\frac{\zeta}{2} \\
\sin(\zeta S) - S \zeta \cos\frac{\zeta}{2} \\
\zeta^2 S
\end{pmatrix},
\]

represented with respect to the fixed basis, wherein \( \varepsilon C_1 = 1 \) is set for the sake of simplicity.

For a particular \( \zeta \in \{ \pm 2a_n : n \in \mathbb{N} \} \), the end-to-end distance \( \lambda \) and growth stage \( \gamma \) can be found by solving the system

\[
g(\lambda, \gamma) = A M \left( 1 - \frac{1}{\gamma} \right),
\]

\[
\zeta = A(\lambda - \gamma) - \frac{B + 1}{M} (\gamma - 1),
\]

simultaneously (Table 1, Appendix E). Equations (113)-(114) couple the axial force response of the rod with the bifurcation mode caused due to growth, via the kinematic constraint of symmetry preserving growth (57). Whenever this system does not admit a solution \( \gamma > 0 \) and \( \lambda > 0 \), the perturbation chosen gives only trivial solutions, indicating that out-of-plane buckling is not guaranteed.

An inspection of (112) reveals that the sign change \( \zeta \mapsto -\zeta \) reverses the chirality of solution curve, reflecting it about \( e_1 - e_3 \) plane (Figure 4). Moreover, since our solution is flip symmetric, this is equivalent to reflection in \( e_1 - e_2 \) plane. These centre-line solutions with handedness are similar to those obtained by Healey and Papadopoulos (2013) for a fixed-fixed rod under axial compression.

Internal chirality of the rod is taken care of by the constants \( M \) and \( A \). In case of hemitropic rods, \( A \) captures chirality in load response of the rod while \( M \) contains information regarding the chiral growth law. For rods with \( n \)-fold helical symmetry, \( A \) denotes the same thing, but with the assumption of symmetry preserving growth in place, \( M \) captures chirality in microstructure.

Consider two rods with opposite internal chirality with all other material properties as same. Let one of them with chiral constants \( A, M \) have a solution with bifurcation mode \( \zeta \), end-to-end distance \( \lambda \) and growth stage \( \gamma \). Naturally the second rod with opposite internal chirality is expected to give rise to a reflected solution with bifurcation mode \( -\zeta \) while end-to-end distance and growth stage are still the same. Thus equations (113)-(114) imply that the chiral constants associated with the second rod are \( -A \) and \( -M \). We infer that the complete reversal of internal chirality in rods requires the transformations \( M \leftrightarrow -M \) and \( A \leftrightarrow -A \) to be taken simultaneously. In addition, the \( \zeta \) solution of a rod with internal chirality \( M, A \) and the \( -\zeta \) solution of a rod with opposite internal chirality \( -M, -A \) are mirror images with respect to \( e_1 - e_2 \) and \( e_1 - e_3 \) planes.

Assuming that (113)-(114) admit an acceptable solution, the monotonicity of \( g(\cdot) \) and the condition \( g(1) = 0 \) reveal the following observations:

**Growth** \( \gamma > 1 \)

- \( A \) and \( M \) are of same sign if and only if \( \lambda > \gamma \), signifying that the ends in current configuration have moved away from each other, as compared to both initial and virtual configurations.
Figure 4: Out-of-plane bifurcated solution for the case $M = -0.1$, $A = -8$, $B = 1.2$ and $F = 10^5$. The two graphs correspond to projection of the rod centre-line on $X_1 - X_3$ and $X_2 - X_3$ planes.
Figure 5: Variation of $\lambda$ and $\gamma$ with $A$ for the first mode ($\zeta = 8.986$).

- $A$ and $M$ are of opposite if and only if $\lambda < \gamma$, signifying that the ends in current configuration have come closer as compared to virtual configuration, but no guaranteed comparison can be made with the initial configuration.

**Atrophy $\gamma < 1$**

- $A$ and $M$ are of opposite sign if and only if $\lambda > \gamma$, signifying that the ends in current configuration have moved apart as compared to virtual configuration, but no guaranteed comparison can be made with the initial configuration.

- $A$ and $M$ are of same if and only if $\lambda < \gamma$, signifying that the ends in current configuration have come closer as compared to both initial and virtual configurations.

For a rod with $n$-fold helical symmetry with growth law assumed to be symmetry preserving, these results reveal an interesting interplay between chiralities in microstructure and load response of the rod. But for a hemitropic rod, the growth law allowing cross-section to rotate makes the guided-guided problem similar to a non-growing rod subject to a axial twist at one end while the other end is free to move axially. And the results above directly reflect the *twist-extension type Poisson effect* expected in hemitropic rods.
Case of Isotropy $A = 0$

In this case, the solutions have $n_3 = 0$ with

$$\gamma = \lambda = 1 - \frac{\zeta M}{B + 1}. \quad \text{(115)}$$

A growing isotropic rod has an out-of-plane solution with sign of $\zeta$ opposite to that of $\mathcal{M}$. But for a decaying isotropic rod, (115) guarantees an out-of-plane solution only if $|\mathcal{M}| < \frac{B + 1}{2a_1}$, and hence such solutions exist only up to the first few modes (for the chosen perturbation), with sign of $\zeta$ same as that of $\mathcal{M}$.

For small $A \neq 0$, the solution is close (in terms of $\gamma$ and $\lambda$) to that of the isotropic case with $B$, $g(\cdot)$, $\mathcal{M}$ and $\zeta$ kept same. In addition, the chirality of these solutions are same as that of the corresponding isotropic case. With $A \neq 0$, growing rods admit $\zeta \mathcal{M} > 0$ and atrophying rods admit $\zeta \mathcal{M} < 0$ only if $A$ is taken to be very large, which in turn may be unrealistic.

Consider two rods with degrees of hemitropy $A^+ > 0$ and $A^- < 0$, such that $A^+ + A^- = 0$, everything else being kept same. Then one of these cases gives a solution where ends come closer, while the ends move apart in the other case (comparisons made here are with respect to the virtual configuration). Let $\lambda^+$ and $\lambda^-$ denote the respective solutions for $A^+$ and $A^-$, whereas $\lambda^o$ denotes the same for the isotropic case. While $\lambda^o$ may lie between $\lambda^+$ and $\lambda^-$, it is also a possibility that both $\lambda^+$ and $\lambda^-$ might lie on the same side of $\lambda^o$ (Figure 5), thus suggesting that no definitive comment can be made on this.

5 Conclusion

In this work we study the growth of slender elastic rods with chiral material symmetries—transverse hemitropy and multi-fold dihedral helical symmetry. Based on the intuitive notion that rods with helical symmetry should twist during growth, we propose a homogeneous growth law that allows for relative rotation of cross-sections. A guided-guided rod set-up is considered to illustrate the occurrence of out-of-plane buckling at certain stages of growth (or atrophy). These solutions obtained are flip symmetric and chiral in nature. A complete mirroring of the rod, including both growth and constitutive properties gives a solution with opposite chirality, under the same deformation. We show that the end-to-end distance at bifurcation modes for the isotropic case need not lie between those for rods of opposite material chiralities, with rest of the elastic and growth properties kept same. End-to-end distance for different combinations of growth (atrophy) and material chiralities have also been examined to understand the effect of twisting growth on the constitutive twist-extension coupling.

Embedding our biologically active (growth or atrophy) chiral rod set-up in an elastomeric matrix and introducing inhomogeneities similar to [Almet et al., 2019], can be an interesting direction to explore. One can also consider a ply of biologically active rods, like growing bi-rod in [Lessinnes et al., 2017], to study the effect of growth and material chiralities of individual rods on the total deformation.
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A The Growth Map

Λ and Ω are skew-symmetric

Since \( W(S) \in SO(3) \), we get

\[
WW^T = I = W^T W \quad \text{(116a)}
\]

\[
\Rightarrow \frac{\partial W}{\partial S} W^T + W \frac{\partial (W^T)}{\partial S} = 0 = \frac{\partial (W^T)}{\partial S} W + W^T \frac{\partial W}{\partial S} \quad \text{(116b)}
\]

\[
\Rightarrow \frac{\partial W}{\partial S} W^T + W \left( \frac{\partial W}{\partial S} \right)^T = 0 = \left( \frac{\partial W}{\partial S} \right)^T W + W^T \frac{\partial W}{\partial S} \quad \text{(116c)}
\]

Thus \( \Lambda^T = -\Lambda \) and \( \Omega^T = -\Omega \).

Relation between \( a \) and \( \omega \)

We observe that \( W \) being a proper rotation must satisfy \( W = W^* \), where \( W^* \) denotes the cofactor of \( W \). Now for any \( v \in \mathbb{R}^3 \),

\[
\omega \times v = \Omega v = W \Lambda W^T v = W(a \times W^T v) = (W^*a) \times (W^*W^T v) = Wa \times v. \quad \text{(117c)}
\]

This implies \( \omega = Wa \).

Λ and Ω are constant

We start with the decomposition

\[
W(S + h) = \Pi(h)W(S) \quad \text{(118)}
\]

Let \( \varepsilon \neq 0 \) be such that all the tensor fields appearing in the following calculation make sense.

\[
\left\{ \Pi(h + \varepsilon) - \Pi(h) \right\}W(S) = W(S + h + \varepsilon) - W(S + h) = \Pi(h) \left\{ W(S + \varepsilon) - W(S) \right\}. \quad \text{(119b)}
\]

Dividing by \( \varepsilon \) and taking the limit \( \varepsilon \to 0 \) yields

\[
\frac{\partial \Pi(h)}{\partial h} W(S) = \Pi(h) \frac{\partial W(S)}{\partial S}, \quad \text{(120)}
\]
which in turn implies
\[ \Pi(h)^T \frac{\partial \Pi(h)}{\partial h} = \Omega(S). \] (121)

Since \( h \) and \( S \) can be chosen arbitrarily, independent of each other, we conclude that \( \Omega(S) \) is constant. Now we expand (35) as
\[
O = \frac{\partial}{\partial S} \left\{ W(S + h)W(S)^{-1} \right\} \\
= \frac{\partial}{\partial S} \left\{ W(S + h)W(S)^T - W(S + h)W(S)^T \frac{\partial W(S)}{\partial S} W(S)^T \right\} \\
= W(S + h) \left\{ \Lambda(S + h) - \Lambda(S) \right\} W(S)^T. \] (122a)

This implies \( \Lambda(S + h) = \Lambda(S) \) for all choices of \( S \) and \( h \), chosen independent of each other, which means \( \Lambda(S) \) is constant.

**Solving for \( W(S) \)**

This boils down to solve (37) for \( W(S) \). Define orthogonal tensor fields \( \Phi := e^{S\Lambda} \) and \( U := \Phi W^{-1} \). Then we have the following:
\[
\frac{\partial \Phi}{\partial S} = \Lambda \Phi = \Phi \Lambda, \quad (123)
\]
\[
\Phi^T \frac{\partial \Phi}{\partial S} = W^T U^T \frac{\partial U}{\partial S} W + \Lambda. \quad (124)
\]

thus implying that \( U(S) \) is a constant equal to \( e^{S_o \Lambda} \), which results in
\[
W(S) = e^{(S - S_o) \Lambda}. \quad (125)
\]

**B Derivation of Perturbed Equations**

**Perturbations**

\[
\nu_3 = r' \cdot d_3 = (\lambda e_3 + \varepsilon \rho') \cdot (e^{\varepsilon \psi} e_3) \\
= (\lambda e_3 + \varepsilon \rho') \cdot (e_3 + \varepsilon \psi \times e_3 + \cdots) \\
= (\lambda + \varepsilon \rho' \cdot e_3 + \cdots). \quad (126a)
\]

\[
\frac{\partial Q}{\partial S} Q^T v = \frac{\partial Q}{\partial S} \left( v - \varepsilon \psi \times v + \cdots \right) \\
= \varepsilon \psi' \times \left( v - \varepsilon \psi \times v + \cdots \right) + \cdots \\
= \varepsilon \psi' \times v + \cdots. \quad (127b)
\]
\[
\frac{\partial R R^T}{\partial S} = \frac{\partial}{\partial S} (QW^T)(QW^T)^T
\]
\[
= \left( \frac{\partial Q}{\partial S} W^T + Q \frac{\partial W^T}{\partial S} \right) W^T \quad (128a)
\]
\[
= \frac{\partial Q}{\partial S} Q^T - QAQ^T. \quad (128c)
\]

\[
QAQ^T v = Q \left\{ a \times (v - \varepsilon \psi \times v + \cdots) \right\} \quad (129a)
\]
\[
= (I + \varepsilon \Psi + \cdots) \left\{ a \times v - \varepsilon a \times (\psi \times v) + v + \cdots \right\} \quad (129b)
\]
\[
= a \times v - \varepsilon a \times (\psi \times v) + \varepsilon \psi \times (a \times v) + \cdots \quad (129c)
\]
\[
= a \times v + \varepsilon (\psi \times a) \times v + \cdots. \quad (129d)
\]

\[
\kappa = \text{axial} \left( \frac{\partial R R^T}{\partial S} \right) = -a + \varepsilon (\psi' + a \times \psi) + \cdots. \quad (130)
\]

The perturbed strain fields are as follows:

\[
\hat{\nu}_3 = \frac{1}{\gamma} \left( \lambda + \varepsilon \rho' \cdot e_3 + \cdots \right), \quad (131)
\]
\[
\hat{\kappa}_\alpha = \varepsilon \frac{1}{\gamma} (\psi' + \omega e_3 \times \psi) \cdot e_\alpha + \cdots \quad (132)
\]
and

\[
\hat{\kappa}_3 = -\frac{\omega}{\gamma} + \varepsilon \frac{1}{\gamma} \psi' \cdot e_3 + \cdots. \quad (133)
\]

Thus

\[
g(\hat{\nu}_3) = g \left( \frac{\lambda}{\gamma} \right) + \frac{1}{\gamma} \left\{ \varepsilon \rho' \cdot e_3 + \cdots \right\} g' \left( \frac{\lambda}{\gamma} \right) + \cdots \quad (134)
\]
and

\[
n_3 = g(\hat{\nu}_3) + A\hat{\kappa}_3 = \varepsilon \frac{1}{\gamma} \left\{ g' \left( \frac{\lambda}{\gamma} \right) \rho'' + A\psi'' \right\} \cdot e_3 + \cdots. \quad (135)
\]

**Linearization**

**Linear momentum**

\[
\frac{dn}{dS} = \frac{d}{dS} \left[ n_\alpha Q e_\alpha + \left\{ g(\hat{\nu}_3) + A\hat{\kappa}_3 \right\} Q e_3 \right]
\]
\[
= \varepsilon \left[ \eta'' e_\alpha + \frac{1}{\gamma} \left\{ g' \left( \frac{\lambda}{\gamma} \right) \rho'' + A\psi'' \right\} \cdot e_3 + \cdots \right]. \quad (136)
\]

Equating the \( \varepsilon \) term to zero gives (85) and (87).
Angular momentum

\[ \mathbf{r}' \times \mathbf{n} = (\lambda \mathbf{e}_3 + \varepsilon \mathbf{e}') \times \varepsilon \left[ \eta_a \mathbf{e}_a + \frac{1}{\gamma} \{ \mathbf{g}' \left( \frac{\lambda}{\gamma} \right) \mathbf{e}' + A \mathbf{\psi}' \} \cdot \mathbf{e}_3 \mathbf{e}_3 \right] \]

(137a)

And

\[ \varepsilon \lambda \mathbf{e}_3 \times \eta_a \mathbf{e}_a. \]

(137b)

And

\[ \frac{d \mathbf{m}}{dS} = \frac{d}{dS} \left[ C \tilde{\kappa}_a \mathbf{Q}_e + \left[ A \{ \tilde{\nu}_3 - 1 \} + B \tilde{\kappa}_3 \right] \mathbf{Q}_3 \right] \]

\[ = \varepsilon \frac{1}{\gamma} \left[ C (\psi'' + \omega \mathbf{e}_3 \times \mathbf{\psi}') \cdot \mathbf{e}_a \mathbf{e}_a + \left[ A (\lambda - \gamma) - B \omega \right] \mathbf{\psi}' \times \mathbf{e}_3 + (A \rho'' + B \psi'') \cdot \mathbf{e}_3 \mathbf{e}_3 \right]. \]

(138)

Plugging these into (71) and equating the \( \varepsilon \) term to zero gives (86) and (88).

Unshearability

\[ \mathbf{r}'(S) \cdot \mathbf{Q}(S) \mathbf{e}_a = (\lambda \mathbf{e}_3 + \varepsilon \mathbf{e}') \cdot \left( \mathbf{e}_a + \varepsilon \mathbf{e} \times \mathbf{e}_a + \cdots \right) \]

(139a)

\[ = \varepsilon \left( \mathbf{e}' - \lambda \mathbf{e} \times \mathbf{e}_3 \right) \cdot \mathbf{e}_a + \cdots. \]

(139b)

C Solution for Perturbations

Proceeding on similar lines as that of Healey and Papadopoulos (2013), we eliminate \( \eta_a \) to obtain a differential equation in \( \mathbf{\psi} \) alone. Integrating (85) we get

\[ \eta_a \mathbf{e}_a = c, \]

(140)

for some constant \( c \in \text{span}\{\mathbf{e}_1, \mathbf{e}_2\} \). Having introduced the parameter \( \zeta \) in (100), equation (86) transforms into

\[ \mathbf{\psi}'' + \zeta \mathbf{\psi}' \times \mathbf{e}_3 = \gamma \lambda \mathbf{c} \times \mathbf{e}_3, \]

(141)

which upon integration and application of boundary condition (90) gives

\[ \gamma \lambda \mathbf{c} \times \mathbf{e}_3 = \mathbf{\psi}' \left( + \frac{1}{2} \right) - \mathbf{\psi}' \left( - \frac{1}{2} \right), \]

(142)

thus leading to (95).

Solution for \( \mathbf{\psi} \)

Denote by \( \mathbf{y} \) the two-component representation of \( \mathbf{\psi}' \) with respect to \( \{\mathbf{e}_1, \mathbf{e}_2\} \) and let \( \mathbf{b} \) denote a similar representation for \( \mathbf{\psi}' \left( + \frac{1}{2} \right) - \mathbf{\psi}' \left( - \frac{1}{2} \right) \). Define matrix \( \mathbf{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) so that (95) can be rewritten as

\[ \mathbf{y}' = \zeta \mathbf{M} \mathbf{y} + \mathbf{b}. \]

(143)
Assume $\zeta \neq 0$ for time being. Observe that solving (143) is equivalent to solving
\[ \dot{x} = \zeta M x, \]
so that the general solution of (143) would be given by
\[ y = x - \frac{1}{\zeta} M^{-1} b, \]
\[ b = y\left( + \frac{1}{2} \right) - y\left( - \frac{1}{2} \right) = x\left( + \frac{1}{2} \right) - x\left( - \frac{1}{2} \right). \]
Thus we have general solutions for $x$ and $y$ given by
\[ x(S) = C_1 \begin{pmatrix} \cos(\zeta S) \\ \sin(\zeta S) \end{pmatrix} + C_2 \begin{pmatrix} \sin(\zeta S) \\ -\cos(\zeta S) \end{pmatrix}, \]
\[ y(S) = C_1 \begin{pmatrix} \cos(\zeta S) \\ \sin(\zeta S) + \frac{2}{\zeta} \sin \frac{\zeta}{2} \end{pmatrix} + C_2 \begin{pmatrix} \sin(\zeta S) - \frac{2}{\zeta} \sin \frac{\zeta}{2} \\ -\cos(\zeta S) \end{pmatrix}, \]
where $C_1$ and $C_2$ are constants in $\mathbb{R}$.

This gives the solution for $\psi$ as (101). Finally all trivial and non-trivial solutions discussed in section 4.2 can be summarized as follows:

**Case-I** Assume $\zeta = 0$. Equation (95) with boundary condition (91) invoked gives
\[ \psi(S) = \frac{1}{2} \left( S^2 - \frac{1}{4} \right) \left\{ \psi\left( + \frac{1}{2} \right) - \psi\left( - \frac{1}{2} \right) \right\}, \]
substituting which into (96) gives the following relation between boundary values
\[ \rho_t\left( + \frac{1}{2} \right) - \rho_t\left( - \frac{1}{2} \right) = -\frac{\lambda}{12} \left\{ \psi\left( + \frac{1}{2} \right) - \psi\left( - \frac{1}{2} \right) \right\} \times e_3. \]
Invoking the boundary condition (98), we imply
\[ \psi\left( + \frac{1}{2} \right) = \psi\left( - \frac{1}{2} \right), \]
thus resulting in the trivial solution $\psi(S) = 0 = \rho(S)$.

**Case-II** Assume $\zeta \neq 0$. In this case, a general solution (102) is obtained, which subsequently gives rise to the following sub-cases based on (103).

- Let $\sin \frac{\zeta}{2} = 0$ with $\zeta \neq 0$. This implies $\zeta = 2n\pi$ where $n \in \mathbb{Z} \setminus \{0\}$. Each such value of $\zeta$ gives a solution
  \[ \rho(S) = \frac{C_1 \lambda}{\zeta^2} \begin{pmatrix} -\sin(\zeta S) + (-1)^n \zeta S \\ \cos(\zeta S) \\ 0 \end{pmatrix} + \frac{C_2 \lambda}{\zeta^2} \begin{pmatrix} \cos(\zeta S) \\ -\sin(\zeta S) - (-1)^n \zeta S \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} C_5 \\ C_6 \end{pmatrix}. \]
  But for this to agree with (98), we require $C_1 = 0 = C_2$ and $C_5 = 0 = C_6$, thus leading to a trivial solution.
- Let $C_1 = C_2$ with $\zeta \neq 0$. This leads to non-trivial out-of-plane solutions (107), which are discussed further in section 4.3.
D Equivariance Properties of Solutions

Let $F$ be the tensor defined by flip action – a 180-degree rotation– about $e_1$ axis and $Q_\theta$ denote the rotation tensor about $e_3$ axis as defined in (13).

For any solution $\left( r(S), R(S), n_\alpha(S) \right)$ of the boundary value problem (70)-(75), the tuple

$$\left( Q_\theta r(S), Q_\theta R(S)Q_\theta^T, (Q_\theta)_{\alpha\beta}n_\beta(S) \right)$$

also solves the system (70)-(75) for all $0 \leq \theta < 2\pi$ and so does

$$\left( Fr(-S), FR(-S)F, -E_{\alpha\beta}n_\beta(-S) \right)$$

Equivalently in terms of perturbations, any solution $\left( \rho(S), \psi(S), \eta_\alpha(S) \right)$ of the boundary value problem (85)-(92), generates an entire class of solutions comprising of

$$\left( Q_\theta \rho(S), Q_\theta \psi(S), (Q_\theta)_{\alpha\beta}\eta_\beta(S) \right)$$

for all $0 \leq \theta < 2\pi$ and

$$\left( F\rho(-S), F\psi(-S), -E_{\alpha\beta}\eta_\beta(-S) \right).$$

Our boundary value problem is equivariant with respect to the action of a group generated by rotations about $e_3$ axis and flip about $e_1$ axis.

A solution is said to be flip symmetric if

$$\left( Fr(-S), FR(-S)F, -E_{\alpha\beta}n_\beta(-S) \right) = \left( r(S), R(S), n_\alpha(S) \right),$$

or equivalently if the perturbations satisfy

$$\left( F\rho(-S), F\psi(-S), -E_{\alpha\beta}\eta_\beta(-S) \right) = \left( \rho(S), \psi(S), \eta_\alpha(S) \right)$$

for all $S \in \left[ -\frac{1}{4}, +\frac{1}{4} \right]$.

These equivariance properties of solutions are explained in much greater detail in [Papadopoulos, 1999].

E Calculation of $\lambda$ and $\gamma$

First of all, numerical values of $A, B, F$ and $M$ are fixed. Inspired by the calibration calculations present in [Papadopoulos, 1999], for a rod of length $L = 1$ with circular cross section and material constant $C = 1$, radius $r$ of the cross-section can be shown to be

$$r = \frac{2}{\sqrt{F}}$$

where both $r$ and $F$ are dimensionless. For instance, $F = 10^6$ is equivalent to consider a 1 metre rod with diameter 4 millimetres. In addition, we have the following values of $\zeta$ corresponding to different bifurcation modes

$$\zeta \in \left\{ \pm 8.986 , \pm 15.45 , \pm 21.808 , \pm 28.132 , \pm 34.442 , \cdots \right\}.$$
Introduce variables $x = \frac{\lambda}{\gamma}$ and $y = \frac{1}{\gamma}$. For a particular $\zeta$, equations (113)-(114) require us to solve
\[
F \ln(x) + \left( \frac{A^2}{B} + \frac{A^2 \zeta}{M \zeta - B - 1} \right) x = \left( \frac{A^2}{B} + \frac{A(A + \zeta)}{M \zeta - B - 1} \right)
\] (161)
for $x$. Define the following solution set.

\[
S(m,c) := \{ x : \ln(x) = mx + c, x \in (0, \infty) \}. \tag{162}
\]

We observe that,
\[
|S(m,c)| = \begin{cases} 
1 & \text{if } m \leq 0 \\
0 & \text{if } m > 0 \text{ and } \ln(m) + c + 1 > 0 \\
1 & \text{if } m > 0 \text{ and } \ln(m) + c + 1 = 0 \\
2 & \text{if } m > 0 \text{ and } \ln(m) + c + 1 < 0
\end{cases}
\] (163)

where $m, c \in \mathbb{R}$ and $| \cdot |$ denotes the cardinality of a set. We set
\[
m = -\frac{A^2}{F} \left( \frac{1}{B} + \frac{1}{M \zeta - B - 1} \right) \quad \text{and} \quad c = \frac{A}{F} \left( \frac{A}{B} + \frac{A + \zeta}{M \zeta - B - 1} \right). \tag{164}
\]

Clearly $m$ is positive only when $1 < \zeta M < 1 + B$. Thus if $\zeta$ and $M$ have opposite sign (161) has a guaranteed solution. Whenever they are of same sign, the choice $|M| < \frac{1}{2a_1}$ guarantees a solution to (161), although there may be several other scenarios leading to a solution.

Once we have a solution $x_o \in S(m,c)$, we have corresponding
\[
y_o = \frac{MA(x_o - 1) - B - 1}{M \zeta - B - 1}
\] (165)

and $\lambda_o = \frac{x_o}{y_o}$, $\gamma_o = \frac{1}{y_o}$ would give the complete solution (Table 1).

**Table 1:** Sample calculation for $F = 10^5$ and $\zeta = 8.986$

| $M$   | $A$ | $B$ | $\lambda - 1$ | $\gamma - 1$ |
|-------|-----|-----|----------------|--------------|
| Growth| -0.1| -8  | 1.2            | 0.4089       |
|       | -0.1| 8   | 1.2            | 0.4082       |
|       | -2 \times 10^{-4} | 24 | 0.2            | -2.5 \times 10^{-4} | 1.5 \times 10^{-3} |
| Atrophy| $10^{-4}$ | -16 | 0.4            | 3.8 \times 10^{-4} | -0.64 \times 10^{-3} |
|       | $10^{-2}$ | -16 | 0.32           | -0.0671      | -0.0682 $^\dagger$ |
|       | $10^{-2}$ | 16  | 0.32           | -0.0693      | -0.0682 $^\dagger$ |

$^\dagger,\ddagger$ Values are really close.

Note that we sometimes we may get an absurd solution $y_o < 0$.

For example, the case $M = 0.16$, $A = -8$, $B = 0.4$ and $F = 10^5$ when solved with $\zeta = 8.986$ gives $x_o = 0.9815$, $y_o = -36.4484$, an invalid solution. Moreover, in this case we have $m = -0.0185$, indicating that there is no other valid out-of-plane deformation arising from the chosen perturbation.
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