Quantum critical points in metallic systems have generated considerable theoretical and experimental interest in recent years. They are realized by tuning the finite temperature critical point of a phase transition in a metal to zero temperature. A variety of tuning parameters have been used including hydrostatic pressure, chemical composition and quantum criticality can occur serendipitously. In insulating quantum magnets, magnetic fields often provide a "handle" by which they may be made quantum critical and there has recently been much interest in doing the same in the metallic case. Notable examples include the metamagnetic quantum critical endpoint seen in Sr$_2$RuO$_4$ and also the antiferromagnetic quantum critical point in YbRh$_2$Si$_2$. The collapse of the characteristic energy scale for fluctuations can induce deviations from Landau Fermi liquid theory seen, for example, in transport quantities. These are usually computed by assuming linear response to driving fields, yet the vanishing energy scale could invalidate that assumption.

In this paper we explicitly demonstrate such a breakdown by considering magneto-transport in the simplest class of quantum critical metal tuned by magnetic field. The presence of the small but finite magnetic field at the transition point leads to modified transport response. This has assumed added significance recently because of the suggestion that the discrepancies between theory and experiment in quantum critical matter may originate in part from a transition at the critical point between localized and de-localized spins and with a concomitant change in the Fermi surface volume. It is suggested that the Hall coefficient would reveal this volume change and recently evidence for such a change has been presented at the field driven quantum critical point in YbRh$_2$Si$_2$. Here, we consider the $T \to 0$ limit of a spin or charge-density wave transition in finite magnetic field, so the quantum critical fluctuations play a small role in quasi-particle scattering when compared with elastic scattering from impurities. This limit has been considered by Bazaliy et al. recently for non-field driven quantum critical points under the assumption that a weak-field expansion in magnetic field can be made. They conclude that there are no anomalies in the Hall conductivity and that, in the absence of perfect nesting, changes in other transport quantities are generally linear in the energy gap so change continuously at the critical point.

The essential result of our work is that at a density wave (DW) quantum critical point the weak-field (Jones-Zener) expansion—upon which previous magnetotransport work was based—breaks down because of the vanishing DW gap $\Delta$. This breakdown leads, in the relaxation time approximation, to discontinuities in the magnetoconductivities: jumps in the resistivity and Hall coefficient along the line $\tau\Delta = 0$ due to the collapse of the weak-field regime. Disorder and magnetic breakdown will smooth this discontinuity into a crossover. A field-driven quantum critical point is characterized by a trajectory through this phase diagram: (b) a magnetic field suppressing the DW phase, or (b) stabilizing the DW phase.
sume particular significance at a field driven quantum critical point where $\Delta$ is tuned to zero at finite magnetic field. The interdependence of the gap and magnetic field define a trajectory through Fig. 1(a) [shown schematically in Figs. 1(b) or (c)] which inevitably explores the non-analytic region. We also go beyond the relaxation time approximation to consider the effects of disorder and magnetic breakdown very close to the critical point. While these smooth the discontinuity they do not alter the magnitude of the changes in conductivity across the transition at a finite magnetic field.

We consider the following highly simplified model of transport near a density-wave transition (DW). We take the Fermi surface in the paramagnetic state to be circular and (and assume two dimensionality for ease of computation). We then consider density wave formation to induce a periodic potential on the otherwise free electrons and assume this potential to be proportional to the mean-field order parameter: $V_{\pm \vec{Q}}(B) = \Delta \pi R \sqrt{\frac{(B_c-B)/B_c}{}}$. Here, the periodic potential (assumed real so both signs of $Q$ are present) has Fourier components at the order of wave-vector $Q$ of the DW and its perturbation on the free electrons describes Bragg scattering off the DW. Degenerate perturbation theory in the presence of this potential will gap the dispersion whenever a resonance condition is met: $\epsilon^2 = \epsilon^2_{k+\vec{Q}}$. Since transport in the $T \to 0$ limit that we will be considering is dominated by the Fermi surface, we need only consider the most important Bragg scattering matrix element. For example, near the point $\epsilon_k = \epsilon_{k-Q}$ the dispersion is modified to be

$$E_k = \frac{1}{2} (\epsilon_k + \epsilon_{k-Q}) \pm \sqrt{\frac{1}{4} (\epsilon_k - \epsilon_{k-Q}) + V^2_Q}.$$ (1)

When the chemical potential falls in one of the gaps, as it will if $2k_F > Q$, then the Fermi surface is modified by the density wave and it is the change in transport properties induced by this modification that we wish to study.

In Fig. 2 we illustrate the change in Fermi surface topology that we envisage. There are a number of possible scenarios

- $2k_F < Q$: no gaps appear at the Fermi surface and transport coefficients will change smoothly through the critical point. We therefore do not consider this case here.
- $2k_F > Q$, $Q$ incommensurate: gaps appear on the Fermi surface and lead to an open section of Fermi surface [Fig. 2(b)]. In the absence of twinning, transport properties would discriminate between currents parallel and perpendicular to $\vec{Q}$.
- $2k_F > Q$, $Q$ commensurate with the lattice (e.g. an antiferromagnet). Transport is isotropic through the transition and the Fermi surface remains closed. We treat this case by considering two real DWs

![FIG. 2: Fermi surface reconstructions at a density wave transition. The solid lines indicate the Fermi surface to be used for computing transport, while the dashed Fermi surfaces show the dispersion in an extended zone scheme. (a) An incommensurate DW transition leads to open Fermi surface sheets. (b) At a commensurate DW transition ($Q_2 = Q_1 + \vec{g}$) the Fermi surface can break into closed pockets. The small gap near the DW transition gives a local radius of curvature of the Fermi surface $\sim \Delta / v_F$ which leads to a breakdown of the weak-field expansion in the limit $\Delta \to 0^+$.]

with $\vec{Q} = (\pm Q_x, \pm Q_y)$ and $\vec{Q} = (\pm Q_x, \mp Q_y)$ (see Fig. 2(b)).

- $2k_F = Q$ (nested) [15]. Since the perfect nesting not be met very close to the critical point [14] this case will ultimately reduce to one of the cases above and is therefore not considered here.

For the initial analysis we use the classical Boltzmann equation in the relaxation time approximation $\tau$: we are envisaging the $T = 0$ limit of the conductivities and ignore inelastic processes. Rather than solve the Boltzmann equation order by order in the magnetic field [15] we solve to all orders in the field directly [16] using the Chamber’s formula [17]:

$$\sigma_{ij} = \frac{\epsilon^2}{4\pi^3} \int \frac{dS}{h|\vec{v}|} \int_0^\infty v_i(0)v_j(t) e^{-t/\tau} dt.$$ (2)

For each area element of the Fermi surface, $dS$, we integrate the velocity, $\vec{v}(t)$ measured along a semiclassical quasiparticle orbit. These orbits are defined by the Lorentz equation of motion

$$\hbar \frac{d\vec{k}}{dt} = -e\vec{v} \times \vec{B},$$ (3)

where $\vec{v} = \nabla_\vec{k} \epsilon(\vec{k})/\hbar$. In this paper we will always assume the magnetic field is perpendicular to the 2D electron fluid.

Our main numerical results are illustrated in Fig. 3. The circles illustrate the conductivities obtained by integrating the Chambers formula when an DW gap with
This deflects the current, leading to a reduction of the longitudinal conductivity $\Delta \sigma_{xx}/\sigma_{xx}(0) \sim \cos \theta_H - 1 \sim \theta_H^2$, giving the usual magnetoconductance quadratic in applied field. The Hall angle must be small for weak-field response. However at a DW transition a gap opens in the Fermi surface leading to a local radius of curvature of the Fermi surface of order $\Delta/eV_F$ which vanishes at the quantum critical point itself [see Fig. 2(c)]. The condition for being in the weak-field regime is therefore that $B_{\text{weak-field}} < \Delta/(e^2V_F^2)$ or equivalently $\omega_0 \tau < \tau \Delta/(k_F l)$ where $l$ is the mean free path. This upper bound vanishes at the critical point so magnetotransport there is never weak-field. Thus the results of Bazaliy et al. are valid only provided the limit $B/\Delta \rightarrow 0$ can be taken which is certainly not the case at a field-driven DW quantum critical point.

Instead magneto-transport is dominated by a fraction of the quasiparticles (proportional to $eV_F B \tau$) that are deflected around the cusp in the Fermi surface between scattering events. This leads to a magnetoconductance proportional to $|B|$—i.e. non-analytic in the field and thus beyond any weak-field expansion. This physics is beautifully illustrated by the magnetoresistance of a square Fermi surface calculated by Pippard.

We can use Pippard’s method to compute the size of the discontinuities in all three independent components of the conductivity tensor. In the limit that $\Delta \rightarrow 0^+$ we can solve the Boltzmann transport equation on each segment of the Fermi surface and then combine the solutions by ensuring that the magnitude of the distribution function is continuous. The result is that

$$\sigma_{xx} = \frac{ne^2 \tau}{m} \left[ \frac{1}{1 + (\omega_0 \tau)^2} - \frac{Q^2 \omega_0 \tau P(\alpha, \omega_0 \tau)}{k_F \left[ 1 + (\omega_0 \tau)^2 \right]^2} \right],$$

$$\sigma_{xy} = \frac{\omega_0 \tau \sigma_{xx}},$$

$$\sigma_{yy} = \frac{ne^2 \tau}{m} \left[ \frac{1}{1 + (\omega_0 \tau)^2} + \frac{Q^2 (\omega_0 \tau)^3 P(\alpha, \omega_0 \tau)}{k_F \left[ 1 + (\omega_0 \tau)^2 \right]^2} \right],$$

where $\alpha = \cos^{-1}(Q/2k_F)$ and

$$P(\alpha, \omega_0 \tau) = \frac{\cosh \frac{\pi}{\omega_0 \tau}}{\pi \cosh \frac{\alpha}{\omega_0 \tau} \sinh \frac{\tau/2-\alpha}{\omega_0 \tau}}.$$  

On the paramagnetic side ($\Delta \rightarrow 0^-$) the conductivities are as above but without the terms proportional to $P(\alpha, \omega_0 \tau)$. Thus the all three magnetoconductivities, $\sigma_{xx}$, $\sigma_{xy}$, $\sigma_{yy}$, show discontinuities with magnitude of order, $\omega_0 \tau$, $(\omega_0 \tau)^2$ and $(\omega_0 \tau)^3$ respectively so the most dramatic jump is the magnetoresistance footnote. These analytic solutions are shown as the dashed lines in Fig. 3.

We now extend our calculation beyond the relaxation time approximation. If $\tau \Delta < 1$ this calculation cannot be valid since the quasiparticles would be unable to notice the Bragg scattering from the DW above the scattering from impurities. One would expect this to washout the discontinuities in magnetoconductivities since quasi-particles will tend to remain on original Fermi surface.

**FIG. 3:** The components of the conductivity tensor (a) $\sigma_{xx}$, (b) $\sigma_{xy}$ and (c) $\sigma_{yy}$ (in units of $\sigma_0 = ne^2 \tau/m$) computed when a mean-field like DW gap is pushed to zero at a magnetic field equivalent to $\omega_0 \tau = 0.2$. The circles are the numerical solution of the Chamber’s formula within the relaxation time approximation by including the role of disorder in washing out the DW gap.
Magnetic (Zener) breakdown when \( B > 2\Delta^2/(e\tau v_F) \) as the same effect. (See Green and Sondhi for \( E \) field breakdown.) We include this phenomenologically in our calculation in the following fashion. Rather than hybridize the dispersion (in Eq. 1) we maintain a circular Fermi surface and treat the DW potential in the collision integral as a resonant scatterer between points of the Fermi surface that satisfy the resonance condition. This gives the following transport equation for the out of equilibrium distribution function \( g(\epsilon, \theta) \)

\[
-e\mathbf{v} \cdot \mathbf{E} \partial_\epsilon f_0 + \omega_0 \tau \frac{\partial g}{\partial \theta} = -g - (\tau \Delta)^2 [\delta(\cos \theta + \eta) + \delta(\cos \theta - \eta)] [g(\epsilon, \theta) - g(\epsilon, \pi - \theta)], \tag{9}
\]

where \( f_0 \) is the Fermi function and \( \eta = Q/2k_F \). Since the solution is periodic around the Fermi surface it may be solved by Fourier transform and, for a circular Fermi surface gives the following expression for the conductivities

\[
\sigma_{xx} = \frac{ne^2\tau}{m} \left[ \frac{1}{1 + (\omega_0\tau)^2} - \frac{K(\alpha, \omega_0\tau)}{(1 + (\omega_0\tau)^2)^2} \right], \tag{10}
\]

\[
\sigma_{xy} = \omega_0\tau \sigma_{xx}, \tag{11}
\]

\[
\sigma_{yy} = \frac{ne^2\tau}{m} \left[ \frac{1}{1 + (\omega_0\tau)^2} + \frac{(\omega_0\tau)^2 K(\alpha, \omega_0\tau)}{(1 + (\omega_0\tau)^2)^2} \right], \tag{12}
\]

where

\[
K(\alpha, \omega_0\tau) = \frac{2A\omega_0\tau P(\alpha, \omega_0\tau) \cos^2 \alpha}{\omega_0\tau P(\alpha, \omega_0\tau) + A}, \tag{13}
\]

and \( A = \frac{4(\pi\Delta)^2}{\pi\omega_0\tau}. \) These conductivities are shown as solid lines on Figs. 3. Note how these expressions interpolate between the paramagnetic solution when \( \tau\Delta = 0 \) (no DW scattering), and the Pippard result in the limit \( \tau\Delta \gg 1 \) where the angle dependent scattering effectively mimics the reconstructed Fermi surface. Thus we see that this disorder washes out the discontinuity over a region in field determined by \( \tau\Delta < 1 \).

We have also considered the case of closed Fermi surfaces. In that case \( \sigma_{xx} = \sigma_{yy} \) and both show a discontinuity of fractional order \( \omega_0\tau \). The other difference from the case of open Fermi surfaces is that, as for all open Fermi surfaces, \( \sigma_{yy} \) remains finite in the high field limit: \( \omega_0\tau \gg 1 \). This is the regime where Landau level quantization of the closed Fermi surfaces would also become important and is not considered here.

In summary, we have shown that at a simple density wave (DW) quantum critical point the weak-field regime of magnetotransport collapses to zero field with the size of the gap. At finite field in a clean metal one would expect to see discontinuities in the magnetoresistance of order of the magnitude of the Hall angle: \( \omega_0\tau \). This effect will be significant at a field driven quantum critical point where by definition the field is finite at the transition point. The case of YbRh$_2$Si$_2$ already shows features in the Hall effect which suggest that it falls outside the class of DW quantum critical points. A prediction from this work is that non-field driven DW quantum critical points should show a low field cross-over to a transverse magnetoresistance that is linear in the applied field. It would be interesting to look for such an effect in the \( Cr_1-V_x \) system under pressure where we estimate that \( \tau\Delta = 1 \) at \( (x-x_c)/x_c \sim 0.1 \) and \( \omega_0\tau \sim (k_F\delta)^{-1} \) at \( B \sim 1T \). Very recently we have learnt that Ca$_3$Ru$_2$O$_7$ shows exactly the linear magnetoresistance we predict and is argued to be a small gap density wave system.

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