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Key Points:
• The inferred $b$ value for acoustic emission apparent amplitude is equal to that at source irrespective of the form of attenuation law
• The finite size of the sample leads to a finite range of applicability of the Gutenberg-Richter law
• A new method is used to determine the finite range, and its application confirms that $b$ depends on material heterogeneity and stress

Supporting Information:
• Supporting Information S1
• Data Set S1
• Data Set S2
• Data Set S3

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A New $b$ Value Estimation Method in Rock Acoustic Emission Testing

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Abstract The Gutenberg-Richter (G-R) relationship has been regarded as the fundamental description of the size distribution from large-scale-earthquake to small-scale-laboratory rock ruptures. However, because deviation from the G-R relationship has often been reported, especially when the amplitude distribution is used for $b$ value estimation in rock acoustic emission testing, the effect of attenuation should be considered. Here, we perform a detailed analysis on the deviation of the size distribution from a G-R law and discuss modification of the attenuation on the doubly truncated size distribution from a more general perspective. We find that the attenuation will not modify the size distribution, and the $b$ value is theoretically verified to be unchanged within an interval specified by the minimum and maximum amount of attenuation. Based on these discussions, we propose a new $b$ value estimation procedure for rock acoustic emission testing and apply it to a dilation rupturing test; the results confirm that $b$ value depends on material heterogeneity and stress.

Plain Language Summary The Gutenberg-Richter relationship is an intrinsic characteristic of the frequency-size distribution of earthquake and laboratory rock ruptures, indicating that the number of small-scale ruptures is much greater than the number of large-scale ones. Nevertheless, due to many reasons such as the statistical methods, inadequate data acquisition, and data truncation in cumulative frequency distribution, deviation from a Gutenberg-Richter relationship of the frequency-size distribution is inevitable in practice. In order to understand the deviation of the frequency-size distribution in the laboratory rock fracture testing, we perform a detailed analysis of the statistical methods and the attenuation effect on the Gutenberg-Richter relationship of the frequency-size distribution and propose a new procedure for scaling parameter estimation of Gutenberg-Richter relationship. This newly proposed procedure shows its reliability in a dilation rupturing test and can well ensure the robustness of the scaling parameter estimation.

1. Introduction

In seismology, source energy $E$ and seismic moment $M_0$ are the fundamental physical measure of seismic event; $M_0$ can be transformed to moment magnitude $M_w$ by a logarithmic relation to get uniform and consistent magnitudes for earthquake catalog. As a widely used single parameter to characterize the earthquake, the local magnitude $M$ can also be scaled by $M_w$, and the proper scaling between $M$ and $M_w$ is an important prerequisite for any seismic hazard assessment (Deichmann, 2006, 2017). The source energy $E$ and seismic moment $M_0$ of earthquakes commonly follows a power law size distribution which is an intrinsic characteristic of the frequency-size distribution for earthquake, while the exponential Gutenberg-Richter (G-R) law in magnitude $M$ as a proxy source parameter is equivalent to the power law in a true source parameter. The G-R law is expressed on a logarithmic scale given by

$$\log_{10}(N) = a - b M,$$  

(1)

where $N$ is the number of earthquakes with a magnitude $\geq M$, $a$ and $b$ are constants (Gutenberg & Richter, 1944), and magnitude $M$ is a logarithmic measure of the radiated energy or the seismic moment of the source. Here, the parameter $b$ describes the size distribution scaling, which is often referred to as the $b$ value, and the spatial and temporal variation of the $b$ value is an important parameter for seismic hazard assessment.

As $b$ value is a statistical parameter, its inferred value is influenced by many factors, and the errors in estimation can result in potentially misleading $b$ value variations (Roberts et al., 2015). One of the most
Acoustic emission (AE) generated by fracturing of rock has been found to obey the same form of size distribution, and there have been numerous studies indicating that in many ways the mechanism of earthquake foreshock sequences could therefore be reproduced through AE from the statistical behavior of microfracture activity observed in laboratory fracture experiments (Goebel et al., 2014; Lei, 2003; Lockner, 1993; Mogi, 1962; Scholz, 1968, 2015; Vorobieva et al., 2016). Moreover, the variation of b value obtained in rock AE deformation tests and earthquakes has always been compared to establish analogies in the damage process and precursory analysis (Goebel et al., 2013; Lennartz-Sassinek et al., 2014; Lockner et al., 1991; Main et al., 1989; Scholz, 1968), and specifically the effects of pore fluid pressure on b value and precursors have also been focused through laboratory experiments (Sammonds et al., 1992).

Because of the acquisition threshold and the limitation of the maximum output voltage in some AE equipment, a similar deviation from the G-R relationship of the size distribution will also appear in laboratory rock AE tests, and the influence of data volume, cumulative or incremental data counting, bin width, and estimation algorithm on the b value result is obvious (Cosentino et al., 1977; Liakopoulou et al., 1994; Lockner, 1993). Furthermore, it is worth noting that unlike the b value in earthquakes which uses the frequency-magnitude distribution for estimation, the apparent frequency-amplitude distribution is mostly used for b value estimation in rock AE tests (Cox & Meredith, 1993; Liakopoulou et al., 1994; Weiss, 1997), and this apparent amplitude is directly derived from the sensor-collected signal that is attenuated from the source. Consequently, the distribution of these apparent amplitudes is not the physical size distribution of the sources, so attenuation could also modify the AE b value (Lavrov, 2005; Lockner et al., 1991; Unander, 1993; Weiss, 1997). Although some researchers have made provisions when using apparent amplitude to estimate b value by adjusting the AE amplitude for geometric spreading and averaging over transducers to give an equivalent event amplitude (Lockner et al., 1991) or by using the root-mean-square principle to obtain a relative AE magnitude (Kwiatek et al., 2014), the amplitude distribution actually cannot fully represent the crack size distribution. The attenuation effect was first proposed by Lockner et al. (1991), and then the attenuation effect on b value has been theoretically discussed by Unander (1993) and Weiss (1997). They assumed a constant attenuation coefficient and uniformly distributed sources in the medium, and they considered that the attenuation is composed of a geometric term and a “dissipative” exponential term. In fact, the attenuation coefficient is not constant in the rock material. Not only is it a frequency-dependent parameter but it also varies with the direction of elastic wave propagation because of the anisotropy of the rock, making it extremely difficult to theoretically describe the attenuation in the general case.

Since estimation result of b value can be influenced by many factors especially the deviation of frequency-magnitude distribution from G-R relationship. Therefore, a reliable b value estimation procedure is critical in rock AE testing which can well ensure the robustness of comparative precursory analysis between laboratory AE tests and earthquakes. Here we perform a detailed analysis of the deviation of the size distribution from G-R relationship and intend to examine the influence of statistical method on size distribution measurement. Meanwhile, we investigate the modification of attenuation on doubly truncated size distribution from a more general perspective, where we assume that sources in any distribution are attenuated to a certain extent and amplitudes are measured on the sample boundary by acoustic sensors, then
location of the sources and sensors and the amplitudes and locations of the sources are independent. Based on the deviation discussion, we propose a new $b$ value estimation procedure which specify the minimum data volume and statistical method and employ Fisher optimal split and Global Search algorithm (here we named this procedure as FGS) to determine the logarithmic linear segment in the frequency-amplitude distribution. In order to verify the reliability of FGS method, we design a static dilation rock rupturing AE test by injecting nonexplosive cracking agent into three predrilled boreholes to form a specific fracture surface in cubic rock specimen (as shown in Figure 1). This experimental design is to ensure that the sensor-collected AE signals are all generated by expansion rupturing in rock specimen and does not rely on source location to identify valid rupturing data. The results of this paper are of great help for deep understanding of attenuation effect on $b$ value which show that the inferred $b$ value for AE apparent amplitude is equal to that at source in a finite interval, and the data truncation actually is the fundamental cause of attenuation modification of the size distribution. Furthermore, the proposed FGS method in this paper also provides a new way for $b$ value estimation through apparent amplitude in rock AE tests.

2. Deviation From the G-R Law of Frequency-Size Distribution

The G-R relationship of the size distribution is an intrinsic characteristic of earthquakes and rock AE events that indicates that the number of small-scale ruptures is much greater than the number of large-scale ones. The G-R relationship can be interpreted as a manifestation of the self-organized critical behavior of earth dynamics (Bak & Tang, 1989; Christensen et al., 2002; Clauset, 2007; Newman, 2005). However, deviation from the G-R relationship has always been observed in earthquake and rock AE

Figure 1. Schematic of the expansion rupturing tests. A nonexplosive expansion agent is injected into three boreholes of each rock specimen and a fracture surface along the center of the three boreholes is formed. The specimen surfaces labeled by $a'$ and $b'$ are the surfaces parallel to $a$ and $b$, respectively.
expressed as follows: the density function of a doubly truncated distribution generated by an underlying exponential one can be estimated in a doubly truncated exponential distribution (Cosentino et al., 1977; Page, 1968). Generally, the modiﬁcation of attenuation on size distribution. Here we perform an analysis from another more general

\[ f(x) = \begin{cases} \alpha e^{-\beta x} & \text{if } a_0 \leq x \leq a_c \\ \text{other} & \text{otherwise} \end{cases} \]

where \( \alpha \) and \( \beta \) are constants, \( \beta = \ln(10) \), and “other” is any other distribution that is excluded from consideration. Then, the cumulative frequency distribution in the interval \([a_0, a_c]\) is

\[ N(x) = N_{\text{total}} \int_x^{a_c} \alpha e^{-\beta x} \, dx = N_{\text{total}} \frac{\alpha}{\beta} (e^{-\beta x} - e^{-\beta a_c}). \]

where \( N_{\text{total}} \) is the total number of events in the catalog. If the cumulative frequency distribution is expressed again on a logarithmic scale, the probability function becomes

\[ \log_{10} N(x) = \log_{10} N_{\text{total}} + \log_{10} \frac{\alpha}{\beta} - \beta x + \frac{\beta}{\ln(10)} (1 - e^{-\beta(a_c - x)}). \]

Accordingly, it is found from this equation that when \( x \) is closer to the upper limit of the magnitude \( a_c \), \( \log_{10}(1 - e^{-\beta(a_c - x)}) \rightarrow -\infty \), \( x \) is not linearly related to \( \log_{10} N(x) \), and the logarithmic cumulative frequency distribution behavior can be signiﬁcantly altered. However, if the incremental statistical method is used, the incremental frequency distribution in the interval is

\[ N(x_i - 1) = N_{\text{total}} \int_{x_i - 1}^{x_i} \alpha e^{-\beta x} \, dx = N_{\text{total}} \frac{\alpha}{\beta} (e^{-\beta x_i} - e^{-\beta(a_c - 1)}) = N_{\text{total}} \frac{\alpha}{\beta} e^{\beta x_i} (1 - e^{-\beta a_c}). \]

And again the probability function on a logarithmic scale will be

\[ \log_{10} N(x_i - 1) = \log_{10} N_{\text{total}} + \log_{10} \frac{\alpha}{\beta} + \log_{10}(1 - e^{-\beta a_c}) - \frac{\beta x_i}{\ln(10)}, \]

which shows that, for a certain bin width, \( x_i - 1 \) is linearly related to \( \log_{10} N(x_i - 1) \). The above analysis means that, theoretically, the cumulative frequency distribution used for \( b \) value estimation will inevitably result in deviation from an exponential distribution while the incremental frequency distribution will not.

In addition, owing to the natural smoothing effect of plotting cumulative frequency data, a regression analysis based on the cumulative frequency distribution will systematically increase the goodness of fit and affect the computation of the magnitude of completeness, \( Mc \) (Amorese et al., 2010; Main, 2000; Schorlemmer et al., 2004; Wiemer & Wyss, 2000). Moreover, the inaccurate magnitude determinations and breaks in a logarithmic frequency-magnitude distribution slope will possibly be smoothed out in cumulative frequency distribution, which cause the estimated \( b \) value to incorrectly describe the size distribution scaling in practice.

### 3. Effect of Attenuation on AE Frequency-Amplitude Distribution

The attenuation of elastic waves in rock material is hard to describe theoretically, so it is difﬁcult to analyze the modiﬁcation of attenuation on size distribution. Here we perform an analysis from another more general
3.1. Continuous Conditional Probability Density Function of the Attenuated Source Amplitude

Suppose the AE source obeys any distribution in a bounded region $\Omega$ ($\Omega$ maybe a 3-D solid, surface, or curve) and assume that the $b$ value and the region remain unchanged in some given time interval discussed. Let the source decibel amplitude ($Z$) probability density function in exponential distribution which truncated at an interval $[a_0, a_x]$ be as follows:

$$f(z) = \begin{cases} \alpha e^{-\beta z} & \text{other} \\ a_0 \leq z \leq a_x \\ \text{otherwise} \end{cases}$$

(7)

where $\alpha$ and $\beta$ are constants and “other” means any other distribution that is excluded from consideration.

We define $(x_1, y_1, z_1)$ in $\Omega$ and the origin $(0, 0, 0)$ as the coordinates of the source and sensor, respectively. In addition, the attenuation amplitude can be summarized by the relation $A = a_0 \log(x_1, y_1, z_1)$, where $0 < g(x_1, y_1, z_1) < 1$; $g(x_1, y_1, z_1)$ is the percentage attenuation in some given manner and $g$ can be any expression, $A$ and $A_0$ are the voltage amplitude of the AE source after and before attenuation respectively, and the attenuation relation on a logarithmic scale is

$$20 \log_{10}(A/10^{-6}) = 20 \log_{10}(A_0/10^{-6}) + 20 \log_{10}g(x_1, y_1, z_1).$$

(8)

Letting $X = 20 \log_{10}(A/10^{-6})$ and $Z = 20 \log_{10}(A_0/10^{-6})$ be the decibel amplitudes, we can simplify Equation 8 as $X = Z + 20 \log_{10}g(x_1, y_1, z_1)$.

The density function of the coordinate of the source is $f(x_1, y_1, z_1)$, which is in any distribution. Suppose that the source amplitude $Z$ and its location $(x_1, y_1, z_1)$ are independent of each other, so that $f_Z(z)f(x_1, y_1, z_1)$ is the joint density function. The probability of the attenuated decibel amplitude $X$ can be given by

$$F_X(x) = P(X \leq x) = P(Z + 20 \log_{10}g(x_1, y_1, z_1) \leq x, (x_1, y_1, z_1) \in \Omega).$$

(9)

We assume the region $\Omega$ be a subspace of three-dimension space and $\Omega$ could be any given 1 to 3 dimensional bounded subspace. In order to facilitate the discussion, the integral of $\Omega$ is denoted as $\iiint_{\Omega} f \, d\omega$. Then the probability of the attenuated decibel amplitude $X$ can be calculated as follows:

$$F_X(x) = \iiint_{x \leq \frac{20 \log_{10}(a_0/10^{-6})}{20 \log_{10}a_0} + \frac{20 \log_{10}g(x_1, y_1, z_1)}{20 \log_{10}a_0} f(Z) \, dZ \
\quad = \iiint_{x \geq \frac{20 \log_{10}(a_0/10^{-6})}{20 \log_{10}a_0} + \frac{20 \log_{10}g(x_1, y_1, z_1)}{20 \log_{10}a_0} + \frac{20 \log_{10}g(x_1, y_1, z_1)}{20 \log_{10}a_0} f(Z) \, dZ = \frac{\min(-20 \log_{10}g(x_1, y_1, z_1, a_0))}{\alpha x} \leq a_x$$

$$f(x_1, y_1, z_1) \leq a_x$$

$$f(x_1, y_1, z_1) \leq a_x$$

where $\gamma$ is a constant.

Denote $M = \max_{(x_1, y_1, z_1) \in \Omega} (-20 \log_{10}g(x_1, y_1, z_1))$ and $m = \min_{(x_1, y_1, z_1) \in \Omega} (-20 \log_{10}g(x_1, y_1, z_1))$ as the maximum and minimum attenuation amplitudes from the source to sensor, respectively, so $-20 \log_{10}g(x_1, y_1, z_1) \geq m \geq 0$ and $-20 \log_{10}g(x_1, y_1, z_1) \geq M \leq 0$. When $a_0 - m \leq x \leq a_x - M$, we have $0 \leq a_0 \leq a_x - m = -20 \log_{10}g(x_1, y_1, z_1) \leq x - 20 \log_{10}g(x_1, y_1, z_1) \leq a_x - M = -20 \log_{10}g(x_1, y_1, z_1) \leq a_x$, so $a_0 \leq x = -20 \log_{10}g(x_1, y_1, z_1) \leq a_x$. While $a_0 \leq x \leq -20 \log_{10}g(x_1, y_1, z_1)$, we obtain $a_0 \leq x \leq a_x$. 
Then we can continue to calculate the $F_X(x)$ as follows:

$$F_X(x) = \begin{cases} \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} e^{\alpha e^{-\beta x}} dz + \gamma & a_0 - m \leq x \leq a_x - M \\ \text{other} & \end{cases}$$

$$= \begin{cases} \frac{\alpha}{\beta} \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(e^{-\beta a_0} - e^{-\beta a_0 - x} \log_{10} \phi_{(x_1, y_1, z_1)} \right) dx + \gamma & a_0 - m \leq x \leq a_x - M \\ \text{other} & \end{cases}$$

$$= \begin{cases} \frac{\alpha}{\beta} e^{-\beta a_0} \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(e^{-\beta a_0} - e^{-\beta a_0 - x} \log_{10} \phi_{(x_1, y_1, z_1)} \right) dx + \gamma & a_0 - m \leq x \leq a_x - M \\ \text{other} & \end{cases}$$

The probability of the attenuated amplitude $X$ in interval $[a_0 - m, a_x - M]$ can be given by

$$P(a_0 - m \leq x \leq a_x - M) = F(a_x - M) - F(a_0 - m)$$

$$= \frac{\alpha}{\beta} e^{-\beta a_0} - \frac{\alpha}{\beta} e^{-\beta a_0 - m} \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(g(x_1, y_1, z_1) \right) dx + \gamma - \frac{\alpha}{\beta} e^{-\beta a_0} - \frac{\alpha}{\beta} e^{-\beta a_0 - m} \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(g(x_1, y_1, z_1) \right) dx + \gamma$$

$$= \frac{\alpha}{\beta} \left(e^{-\beta a_0} - e^{-\beta a_0 - m} \right) \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(g(x_1, y_1, z_1) \right) dx + \gamma$$

Now we only consider the interval $[a_0 - m, a_x - M]$ of $X$; it means that we will calculate the conditional probability distribution of $X$ known that $X$ is in the interval $[a_0 - m, a_x - M]$. In order to facilitate the discussion, $X$ in the interval $[a_0 - m, a_x - M]$ is replaced by $X_0$. Then, the conditional probability distribution function for the attenuated amplitude $X_0$ can be given by

$$F_{X_0}(x) = P(X \leq x | a_0 - m \leq X \leq a_x - M) = \frac{P(X \leq x, a_0 - m \leq X \leq a_x - M)}{P(a_0 - m \leq X \leq a_x - M)}$$

$$= \begin{cases} 1 & x > a_x - M \\ \frac{\alpha}{\beta} \left(e^{-\beta a_0} - e^{-\beta a_0 - m} \right) \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(g(x_1, y_1, z_1) \right) dx + \gamma & a_0 - m \leq x \leq a_x - M \\ 0 & \text{other} \end{cases}$$

$$= \begin{cases} 1 & x > a_x - M \\ \frac{\alpha}{\beta} \left(e^{-\beta a_0} - e^{-\beta a_0 - m} \right) \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(g(x_1, y_1, z_1) \right) dx + \gamma & a_0 - m \leq x \leq a_x - M \\ 0 & \text{other} \end{cases}$$

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$$= \begin{cases} 1 & x > a_x - M \\ \frac{\alpha}{\beta} \left(e^{-\beta a_0} - e^{-\beta a_0 - m} \right) \mathcal{J}_1^{\alpha} \phi_{(x_1, y_1, z_1)} \left(g(x_1, y_1, z_1) \right) dx + \gamma & a_0 - m \leq x \leq a_x - M \\ 0 & \text{other} \end{cases}$$
Then the conditional probability density function of $X_0$ is

$$f_{X_0}(x) = \begin{cases} \frac{\beta e^{-\beta x}}{\beta e^{\beta (m - x)} - e^{\beta (a_x - M)}} & a_0 - m \leq x \leq a_x - M \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

### 3.2. Discrete Probability Density Function of the Attenuated Source Amplitude

Similarly, suppose AE source $Z_2$ obeys any distribution in a region $\Omega$ with exponentially distributed amplitude in decibels. If $Y$ is the attenuation amplitude in decibels from source to sensor, then its probability can be given as

$$P(Y = a_i) = p_i, \quad i = 1, 2, \ldots, n, \quad \text{where} \quad \sum_{i=1}^{n} p_i = 1,$$ \quad (15)

where $a_i$ is a discrete attenuation value and $a_1 < a_2 < \cdots < a_n$. Then, the amplitude $X$ after attenuation will be

$$X = Z_2 - Y, \quad Z_2 \sim f_{Z_2}(x) = \begin{cases} \alpha e^{-\beta x} & a_0 \leq z \leq a_x \\ \text{other} & \text{otherwise} \end{cases} \quad (16)$$

Suppose that the source amplitude $Z_2$ and $Y$ are independent of each other, then the probability of the attenuated amplitude $X$ can be given by

$$F_X(x) = P(X \leq x) = P(Z_2 - Y \leq x) = \sum_{i=1}^{n} p_i F_{Z_2}(x + a_i). \quad (17)$$

The density function is

$$f_X(x) = \begin{cases} [p_1 f_{Z_2}(x + a_1) + p_2 f_{Z_2}(x + a_2) + \cdots + p_n f_{Z_2}(x + a_n)] \\ \text{other} \end{cases} \quad (18)$$

Similarly, let $X_1 = X \mid a_0 - \min_{1 \leq i \leq n} \{a_i\} \leq X \leq a_x - \max_{1 \leq i \leq n} \{a_i\}$. Then, the conditional probability distribution function for the attenuated amplitude $X_1$ can be given by

$$F_{X_1}(x) = P\left(X \leq a_0 - \min_{1 \leq i \leq n} \{a_i\} \leq X \leq a_x - \max_{1 \leq i \leq n} \{a_i\} \mid X \leq x\right)$$

$$= \begin{cases} 0 & x < a_0 - \min_{1 \leq i \leq n} \{a_i\} \\ \frac{P(X \leq x)}{P(a_0 - \min_{1 \leq i \leq n} \{a_i\} \leq X \leq a_x - \max_{1 \leq i \leq n} \{a_i\})} & a_0 - \min_{1 \leq i \leq n} \{a_i\} \leq x \leq a_x - \max_{1 \leq i \leq n} \{a_i\} \\ 1 & x > a_x - \max_{1 \leq i \leq n} \{a_i\} \end{cases} \quad (19)$$
And the conditional probability density function of \( X_i \) is

\[
 f_{X_i}(x) = \begin{cases} 
 0 & \text{otherwise} \\
 e^{-\beta(x - \min \{a_i\})} & \text{otherwise} \\
 e^{-\beta(a_i - \max \{a_i\})} & 1 \leq i \leq n 
\end{cases}
\]

As shown in Equations 7 and 14 or Equations 16 and 20, when the exponential source distribution is doubly truncated at a finite interval, the distribution function of the attenuated amplitudes obeys the same exponential distribution at a certain interval, and the \( \beta \) value is theoretically verified to be unchanged. This means that, if the amplitudes of AE sources obey an exponential distribution and the AE sources are in any distribution in a bounded region, attenuation will not affect the \( b \) value of the sources inferred from the AE amplitude data within a certain interval. That is, if the attenuated amplitude distribution measured on the sample boundary obeys the G-R relationship in a certain interval, its \( b \) value is the same as that of the AE source amplitude distribution.

4. A New \( b \) Value Estimation Procedure for Rock AE and Its Application

Based on the discussion above, we propose a new procedure when using apparent amplitude for rock AE \( b \) value estimation with the following steps:

Step 1: Minimum number of events for \( b \) value estimation.

From the statistical point of view, the minimum amount of data for estimation does not have a universal set value. It can be selected after the level of significance is specified in practical applications. A commonly recognized axiom is that the power of a statistical test increases with the increase in sample size (Siegel, 1956). For logarithmic linear fitting of a G-R law distribution, the average error of the estimated scaling parameter becomes <1% when the sample size exceeds 50 (Clauset, 2007), and 50 events were also adopted as the minimum number of events for stable \( b \) value estimation in various studies (Kurz et al., 2006; Murru et al., 1999; Schorlemmer et al., 2004), while the stability estimation of \( b \) value needs to include the space-time window of 100 earthquakes, as illustrated by Shi and Blot (1982). Amorese et al. (2010) and Robert et al. (2015) also stated that the computation of \( b \) value includes at least ~200 events. In comparison to the cumulative frequency distribution, larger data dispersion will occur if the incremental frequency distribution is used for \( b \) value estimation. Therefore, it is necessary to increase the data volume to ensure estimation robustness, so a minimum of 200 events should be utilized in the incremental frequency distribution.

Step 2: Perform incremental data counting and choose the bin width.

As discussed in section 2, the cumulative frequency distribution can inevitably deviate from a G-R relationship compared with the incremental frequency distribution, and the cumulative frequency distribution will also smooth the distribution breaks during the statistics process. Therefore, to display frequency distribution characteristics of AE data more realistically, the AE data should be counted incrementally.

Another parameter used for AE frequency-amplitude distribution statistics is bin width. In seismology, the selection of bin width is critical to grouping magnitudes. Improper choice of the bin width may introduce a significant bias in the magnitude of completeness and \( b \) value estimation (Marzocchi & Sandri, 2009; Schorlemmer et al., 2004; Wiemer & Wyss, 2000). Because earthquake magnitudes are given with one digit after the decimal point, many researches have demonstrated that it is reasonable to set the bin width as the magnitude round-off interval of 0.1 (Bender, 1983; Lasocki & Papadimitriou, 2006; Main, 2000). As AE amplitudes were measured in decibels, and the amplitudes are usually divided by 20 to produce a \( b \) value comparable to that reported in the seismic literature (Cox & Meredith, 1993; Liakopoulou et al., 1994; Sagar et al., 2012; Weiss, 1997), the divided value will have two digits after the decimal point with a 0.05 round-off interval. Therefore, the better bin width for amplitude grouping needs to be set to 0.05; this means that the number of amplitudes should be counted in steps of 1 dB. In fact, owing to the large amount of AE data collected in rock deformation tests, the coarser bin width will smooth the distribution breaks, while the finer bin width can demonstrate the frequency-amplitude distribution characteristics in more detail. Therefore, no matter what kind of equivalent amplitude or AE magnitude is used for \( b \) value estimation in the rock AE test, the bin width should be set to the size of the round-off interval at most.
Step 3: Determine the logarithmic linear segment in the frequency–amplitude distribution.

As discussed earlier, the deviation of the size distribution from the G–R law will appear at both lower and upper amplitude ends. However, the analysis in section 3 demonstrated that, if the attenuated amplitude distribution is exponential in a certain interval, its \( b \) value is the same as that of the AE source amplitude distribution. Therefore, a specific method is needed for determining the logarithmic linear segment in the incremental frequency-amplitude distribution. Here, we propose a method based on the Fisher optimal split and global search algorithm; the main idea of this method is as follows:

First, the slopes of any two successive points are calculated, and we want to find a benchmark slope \( (S(A_0)) \) in the linear segment and an interval width centered on \( S(A_0) \). By referencing the idea of the 3\( \sigma \) principle of mean value in statistics, the relation between that interval width and the standard deviation will be established.

Since the points at both ends of the logarithmic frequency-amplitude distribution deviate significantly from the linearity, the standard deviation \( (std0) \) of all slopes is large, so only the standard deviation \( (std1) \) with slopes \(<0\) is considered. We believe that as long as \( S(A_0) \) is found to be substantially accurate, the slopes in the linear segment should fall in the interval \([S(A_0)−std1, S(A_0)+std1]\). In order to make this interval more accurate, the scale parameter \( r_0 \) is introduced. Then the slopes in the linear segment will fall in the interval \([S(A_0)−r_0×std1, S(A_0)+r_0×std1]\), and \( S(A_0) \) is the slope corresponding to the maximum number of slopes that fall in the interval. In order to correct \( S(A_0) \) and the corresponding interval width, another scale parameter \( u \) is introduced, and a step size \( h \) is obtained from \( u \) and \( std0 \). The interval is adjusted to \([S(A_0)−kh, S(A_0)+kh]\), \( k = 1, 2, 3, 4, \ldots, m \). According to the change in the number of slopes falling in each interval at various \( k \), Fisher optimal split algorithm (Fisher, 1958) is used to divide the number into two categories to obtain the interval \([S(A_0)−kh, S(A_0)+kh]\). To further optimize \( S(A_0) \), a finer interval \([S(A_0)−std2, 0]\) is used to remove the abnormal slopes at both ends, where \( std2 \) is the standard deviation of slopes falling in the interval \([S(A_0)−kh, 0]\). Then, \( S(A_0) \) is replaced by the slope of the points within interval \([S(A_0)−std2, 0]\) obtained through the generalized linear regression with Poisson errors which was proved to be more appropriate in \( b \) value estimation (Greenhough & Main, 2008); Fisher optimal split algorithm is used again to correct the interval, and the corresponding left and right endpoints of the reasonable linear segment can be obtained.

According to the instructions above, scale parameters \( r_0 \) and \( u \) which used to adjust internal width are set between 0.1–1 and 1–100, respectively. The fitting error variance of the linear segment calculated under the given initial values of \( r_0 \) and \( u \) is taken as the target value. Because any values of \( r_0 \) and \( u \) can get a target value, a global search algorithm is used to run 1,000 times to determine the optimal value of \( r_0 \) and \( u \) by finding the minimum error variance of generalized linear regression. Finally, the linear segment of the logarithmic frequency-amplitude distribution is screened out and the corresponding \( b \) value can be obtained by generalized linear regression. The specific algorithm of this method is demonstrated in the following seven steps:

(1) Suppose that there are \( n \) points in the logarithmic frequency-amplitude distribution, \( A_{i-1} \) and \( A_i \) are the amplitudes of two successive points, and the slope for \( A = A_i \) is defined as

\[
S(A_i) = \frac{\log(N_i) - \log(N_{i-1})}{A_i - A_{i-1}}
\]

Then, the \( n-1 \) slopes are computed for each amplitude increment and the corresponding standard deviation is expressed by \( std0 \). Similarly, the standard deviation of all \( S(A_i) < 0 \) is expressed by \( std1 \).

(2) We define an interval \([S(A_i)−r_0std1, S(A_i)+r_0std1]\), where \( r_0 \) is a scaling parameter that is initially set to 0.1. For each slope where \( S(A_i) < 0 \), if the maximum number of slopes fall within that defined interval at a certain point \( i \), then \( S(A_i) \) is replaced by \( S(A_0) \), which is defined as the benchmark slope, and the number of slopes falling within that defined interval at point \( i \) is marked as \( s \). A step size \( h \) is defined as

\[
h = \frac{\max\{S(A_0)−\min(S(A_i))\}, \max(S(A_i))−S(A_0)\}}{u \times \frac{std0}{std1}}
\]

and \( u \) is another scaling parameter that is initially set to 10.
Then, the interval defined in (2) is adjusted to \([S(A_i^0) - k_ih, S(A_i^0) + k_ih]\), \(k = 1, 2, 3, 4, ..., m, m = \lceil u \times s / \text{std}0 \rceil\) is the ceil function value of \(u \times s / \text{std}0\). The number of slopes falling in each interval at various \(k\) can form an \(m\)-tuple sequence, marked as \(\{\text{temp1}_m\}\), and the first-order difference of \(\{\text{temp1}_m\}\) can form an \((m-1)\)-tuple sequence, marked as \(\{\text{temp}2_{m-1}\}\).

(4) The Fisher optimal split algorithm is used to divide \(\{\text{temp}2_{m-1}\}\) into two categories, each with a minimum sum of squared deviation. If \(k = k_i\) is the optimal split point, the corresponding interval \([S(A_i^0) - k_ih, S(A_i^0) + k_ih]\) will be obtained. Because slopes less than zero are the ones to be considered, the interval can be adjusted to \([S(A_i^0) - k_ih, 0)\).

(5) To further optimize the benchmark slope \(S(A_i^0)\), a finer interval \([S(A_i^0) - \text{std}2, 0)\), where \(\text{std}2\) is the standard deviation of slopes falling in the interval \([S(A_i^0) - k_ih, 0)\), is used to remove the abnormal slopes at both ends. Then, a new benchmark slope for points within interval \([S(A_i^0) - \text{std}2, 0)\) can be obtained using generalized linear regression.

(6) The new benchmark slope obtained in (5) is assigned to \(S(A_i^0)\), and the calculations in (3) and (4) are repeated to get the final optimal split point. Then the smallest amplitude and the largest amplitude in the interval \([S(A_i^0) - k_ih, 0)\) are the left and right endpoints, respectively, of the logarithmic linear segment.

(7) A global search algorithm is used to run 1,000 calculations to determine the optimal value of \(r_0\) and \(u\) by finding the minimum error variance of the generalized linear regression, and finally, the logarithmic linear segment of the amplitude distribution is screened out.

Here we call this newly proposed \(b\) value estimation procedure the FGS method, and the complete workflow of FGS is shown in Figure 2.

To investigate the performance of FGS, a rupturing scheme using a non-explosive expansion agent was designed (as shown in Figure 1) to conduct an AE test on granite, limestone, and red sandstone, which is intended to ensure that the sensor-collected AE signals are all generated by expansion rupturing in rock specimens. The AE activities were recorded by six
sensors with a resonant frequency of 140 kHz at a 5 MHz sampling frequency, and the signals collected by
the sensor with largest data volume were used for b value estimation. The estimated b value using FGS
method and corresponding standard error of each rock specimen are given in Table 1 and shown in
Figure 3a. The temporal variations of the b values of granite, limestone, and red sandstone are shown in
Figures 3b–3d. These were estimated using a running window of 4,916, 200, and 280 events and a running
step of 2,458, 100, and 100 events, respectively.

Figure 3a shows that the FGS method visually identifies the linear segment in the logarithmic incremental frequency-amplitude distribution of three rock specimens (a). Temporal variation of b value, amplitude, and energy of the AE signal with respect to time (b–d). The vertical bars (likely to be minimum estimates) are the standard error of the b value.

Figure 3. Estimated b value obtained by using the FGS method of each rock specimen. Linear segment in logarithmic incremental frequency-amplitude distribution of three rock specimens (a). Temporal variation of b value, amplitude, and energy of the AE signal with respect to time (b–d). The vertical bars (likely to be minimum estimates) are the standard error of the b value.

5. Conclusions

1. The G-R relationship is an intrinsic characteristic of the frequency-size distribution from earthquake to
laboratory rock ruptures, indicating that the number of small-scale ruptures is much greater than the
number of large-scale ones. Nevertheless, deviation from G-R relationship is inevitable in practice. In
fact, whether small-scale events cannot be fully recorded owing to limited equipment accuracy or
small-scale ruptures are masked by larger ones that mainly occur in the avalanche destruction stage,
the cause of the deviation of the size distribution from G-R law can be ultimately attributed to
inadequate data acquisition. Furthermore, from our discussion of attenuation effect on the amplitude distribution, if the source amplitude distribution interval in Equations 7 and 16 is set as infinite (−∞, +∞), a finite attenuation actually cannot cause deviation on both ends of the amplitude distribution; however, once the size distribution is truncated at a certain interval, deviation will occur at both ends. This indicates that data truncation actually is the fundamental cause of attenuation modification of the size distribution. Inadequate data acquisition and data truncation will both cause deviation at both ends of the size distribution; therefore, the data catalog is critical, and the b value should be estimated in a finite interval. The proposed b value estimation procedure (FGS) in this study is designed to search that finite interval, and the application also shows its reliability.

2. Because the b value is a statistical value, its estimation result is significantly influenced by the data volume. Generally, a sufficiently large number of events can improve the robustness of the b value. However, when spatial or temporal variations in b value are investigated, the pursuit of high resolution of the variations in b values will reduce the number of events, which correspondingly leads to a decrease in reliability and robustness of the estimation. In other words, the more we want to increase the resolution of the variations in b values, the less precisely we are able to simultaneously estimate the b value because fewer events are used. Nevertheless, a reliable estimation is crucial in b value analysis. Therefore, the robustness of b value estimation should be guaranteed in priority even at the sacrifice of resolution of the spatial or temporal variation.

Data Availability Statement

The original AE data of dilation rupturing tests used in this work will be uploaded to a public repository called Digital Rocks Portal. The data archiving is underway now; we temporarily provide a copy of these AE data as supporting information named Data Sets S1–S3 for review purposes.

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