Affine Λ-buildings, ultrapowers of Lie groups and Riemannian symmetric spaces: an algebraic proof of the Margulis conjecture

Linus Kramer and Katrin Tent *
Universität Würzburg, Am Hubland, D–97074 Würzburg, Germany
{kramer,tent}@mathematik.uni-wuerzburg.de
Dedicated to the memory of Reinhold Baer on his 100th birthday

Introduction

Let $X$ and $X'$ be Riemannian symmetric spaces of noncompact type, and let $G = I(X)$ and $G' = I(X')$ denote their respective isometry groups. Then $G$ and $G'$ are (finite extensions of) semisimple real Lie groups. Suppose furthermore that $\Gamma$ is a (torsion free) group which injects as a cocompact lattice into $G$ and into $G'$,

$$G \xrightarrow{\subseteq} \Gamma \xhookrightarrow{} G'.$$

Then $\Gamma$ acts cocompactly on $X$ and $X'$, and the orbit spaces $X/\Gamma$ and $X'/\Gamma$ are closed manifolds with contractible universal covers. The only nontrivial homotopy groups are thus the fundamental groups,

$$\pi_1(X/\Gamma) \cong \Gamma \cong \pi_1(X'/\Gamma).$$

It follows that there is a homotopy equivalence $X/\Gamma \xrightarrow{\simeq} X'/\Gamma$ which can be lifted to a map $f : X \longrightarrow X'$. This map $f$ can then be shown to be a quasi-isometry. Recall that a (not necessarily continuous) map $f : X \longrightarrow X'$ between metric spaces if called an $(L,C)$-quasi-isometry if there exist constants $L \geq 1$ and $C \geq 0$ such that

$$L^{-1} \cdot d(x,y) - C \leq d'(f(x),f(y)) \leq L \cdot d(x,y) + C.$$

holds for all $x, y \in X$, and such that every $x' \in X'$ is $C$-close to some image point $f(x)$. Special instances of quasi-isometries are

1. isometries (where $(L,C) = (1,0)$)
2. bi-Lipschitz maps (where $C = 0$)
3. coarse isometries (where $L = 1$).

*Both supported by Heisenberg fellowships of the DFG
Mostow’s Rigidity Theorem \[16\] If the Riemannian symmetric spaces \(X\) and \(X'\) have no de Rham factors of rank 1, then there exists a Lie group isomorphism \(G \cong G'\) such the diagram

\[
\begin{array}{ccc}
\Gamma & \cong & G \\
\downarrow & & \downarrow \cong \\
G' & & \\
\end{array}
\]

commutes. In particular, there is an isometry \(X \cong X'\) (after rescaling the metrics on the de Rham factors of \(X'\), if necessary.)

The key idea of Mostow’s proof was to show that \(f\) induces an isomorphism between the spherical buildings attached to \(X\) and \(X'\). This is not trivial, and the group \(\Gamma\) plays an important rôle in Mostow’s argument. The Margulis conjecture may be stated as follows:

**Margulis Conjecture** If \(f : X \to X'\) is a quasi-isometry, then (after rescaling the de Rham factors) \(X\) and \(X'\) are isometric, and there exists an isometry \(\bar{f} : X \to X'\) at bounded distance from \(f\).

Kleiner-Leeb \[12\] proved the Margulis conjecture in 1997; later, Eskin-Farb \[9\] gave another proof. Kleiner-Leeb used Gromov’s technique of asymptotic cones \[10\]. The asymptotic cone is a very powerful functor on the category of pointed metric spaces. Kleiner-Leeb showed that the asymptotic cone of a Riemannian symmetric space is an affine \(\mathbb{R}\)-building, and they used this building systematically in their work \[12\]. (Note that they work with a class of metric spaces they call euclidean buildings; their system of axioms is different from Tits’ systèmes d’appartements \[22\]. Parreau \[17\] proved that their spaces are a subcase of Tits’ systèmes d’appartements.) The point is that the quasi-isometry between the Riemannian symmetric spaces induces a homeomorphism of their respective affine \(\mathbb{R}\)-buildings, and that this homeomorphism can be shown to be an isomorphism.

However, it does not follow from their paper which building the asymptotic cone of a given Riemannian symmetric space really is. In this paper, we give a general group-theoretic construction of affine \(\mathbb{R}\)-buildings, and more generally, of affine \(\Lambda\)-buildings, associated to semisimple Lie groups over nonarchimedean real closed fields. This result is new. The construction of Kleiner-Leeb using the asymptotic cone appears only as a special case. Also, we give a new, sheaf-theoretic proof for the topological rigidity of affine \(\mathbb{R}\)-buildings. The explicit knowledge of the building arising here as the asymptotic cone simplifies the proof of the Margulis conjecture; it also sheds some light on the algebraic background of Mostow rigidity and the Margulis conjecture.

The present report is meant as a survey without detailed proofs. After completing it, we learned that Thornton, a student of Kleiner, independently proved Corollary \[21\]. He writes in his introduction: ‘In the case of [...] the group \(SL_n\mathbb{R}\), Leeb identified the asymptotic cone as a homogeneous space over an algebraic group. Parreau showed that the asymptotic cone for [...] \(SL_n\mathbb{R}\) fits a certain model for Euclidean buildings [...]’. The case \(G = SL_n\mathbb{R}\) is in fact rather elementary; Bennett \[3\] showed that one obtains from this group an affine \(\Lambda\)-building over any valued field.
1 Λ-metric spaces

Throughout this section, \((\Lambda, +, \leq)\) is an ordered abelian group; everything we need can be found in [18]. As usual, we define \(|\lambda| = \max\{\pm \lambda\} \).

1.1 We use the following slight generalization of a metric space. A \(\Lambda\)-pseudometric on a set \(X\) is a function \(d : X \times X \rightarrow \Lambda\) which satisfies the usual axioms of a pseudometric for all \(x, y, z \in X\):

\[
\begin{align*}
  d(x, x) &= 0 \quad d(x, y) = d(y, x) \geq 0 \quad d(x, y) + d(y, z) \geq d(x, z).
\end{align*}
\]

If \(d(x, y) = 0\) implies that \(x = y\), then the \(\Lambda\)-pseudometric is called a \(\Lambda\)-metric. A \(\Lambda\)-pseudometric defines a topology on \(X\) in the usual way, which is Hausdorff if and only if \(d\) is a \(\Lambda\)-metric. For \(\Lambda = (\mathbb{R}, +)\), we have of course the traditional version of a real valued (pseudo)metric.

The following easy lemma will be used later.

1.2 Lemma Let \(d : X \times X \rightarrow \Lambda\) be a function. Suppose that for any two elements \(x, y \in X\), there exists a subset \(A \subseteq X\), containing \(x\) and \(y\), such that the restriction \(d|_{A \times A}\) is a \(\Lambda\)-(pseudo)metric. If, for every such \(A\), there exists a retraction \(\rho_A : X \rightarrow A\) (i.e. \(\rho_A(X) = A\) and \(\rho_A(a) = a\) for all \(a \in A\)) which diminishes \(d\) (i.e. \(d(\rho_A(x), \rho_A(y)) \leq d(x, y)\) for all \(x, y \in X\)), then \(d\) is a \(\Lambda\)-(pseudo)metric.

Proof. The only point to check is the triangle inequality. So let \(x, y, z \in X\), with \(x, z \in A\) and \(\rho_A : X \rightarrow A\) as above. Then

\[
  d(x, z) = d(\rho_A(x), \rho_A(z)) \leq d(\rho_A(x), \rho_A(y)) + d(\rho_A(y), \rho_A(z)) \leq d(x, y) + d(y, z).
\]

Recall that \(\Lambda\) is archimedean if the following is true: for any two elements \(x, y > 0\), there exists an \(n \in \mathbb{N}\) such that \(nx \geq y\). An ordered abelian group is archimedean if and only if its only \(o\)-convex (order convex) subgroups are 0 and \(\Lambda\). Any archimedean \(\Lambda\) admits an \(o\)-embedding into \((\mathbb{R}, +)\), which is unique up to a scaling factor; thus, our concept of a \(\Lambda\)-metric is broader than the usual concept of a metric only in the nonarchimedean case.

1.3 In the nonarchimedean case, suppose that \(\Omega \subseteq \Lambda\) is an \(o\)-convex subgroup, and that \(d : X \times X \rightarrow \Lambda\) is a \(\Lambda\)-pseudometric. Put

\[
  x \approx_\Omega y \quad \text{if} \quad d(x, y) \in \Omega.
\]

This is an equivalence relation on \(X\), and we put \(X_\Omega^x = \{y \in X \mid x \approx_\Omega y\}\). The composite pseudometric \(X \times X \xrightarrow{d} \Lambda \rightarrow \Lambda/\Omega\) induces a \(\Lambda/\Omega\)-metric on the set

\[
\{X_\Omega^x \mid x \in X\} = X/\Omega
\]

of \(\approx_\Omega\)-equivalence classes in \(X\). In other words, we identify points whose distance is in \(\Omega\). This generalizes the well-known process of making a pseudometric into a metric (the case \(\Omega = 0\)).
1.4 Given two elements $\alpha, \beta \geq 0$ in $\Lambda$, we write $\alpha \gg \beta$ if $\alpha > n\beta$ holds for all $n \in \mathbb{N}$. Let $\alpha \in \Lambda$ be a positive element ($\alpha > 0$), and let

$$\Lambda^{(\alpha)} = \{ \lambda \in \Lambda | |\lambda| \leq n\alpha \text{ for some } n \in \mathbb{N} \} \quad \text{and} \quad \Lambda_{(\alpha)} = \{ \lambda \in \Lambda | |\lambda| \ll \alpha \}$$

Intuitively, we truncate the $\Lambda$ behind the multiples of $\alpha$ and before $\alpha$, respectively. Then $\Lambda_{(\alpha)}$ is a maximal $o$-convex subgroup of $\Lambda^{(\alpha)}$ and the quotient

$$\Lambda^{(\alpha)} = \Lambda^{(\alpha)}/\Lambda_{(\alpha)}$$

is archimedean; there exists a unique $o$-homomorphism $\phi : \Lambda^{(\alpha)} \to \mathbb{R}$ with $\phi(\alpha) = 1$, whose kernel is precisely $\Lambda_{(\alpha)}$,

$$0 \to \Lambda_{(\alpha)} \to \Lambda^{(\alpha)} \to \mathbb{R}.$$ 

1.5 We can make a very similar construction with any $\Lambda$-pseudometric space $X$. Choose a basepoint $o \in X$ and let

$$X^{(\alpha)}_o = \{ x \in X | x \approx \Lambda^{(\alpha)} o \} = \{ x \in X | d(x, o) \leq n\alpha \text{ for some } n \in \mathbb{N} \}.$$ 

The composite $\phi \circ d : X^{(\alpha)}_o \times X^{(\alpha)}_o \to \mathbb{R}$ is an $\mathbb{R}$-pseudometric, and induces an $\mathbb{R}$-metric on the quotient

$$X^{(\alpha)}_o = X^{(\alpha)}_o / \Lambda_{(\alpha)}.$$ 

While all these metric concepts are rather simple and basic, we will see that the construction of Gromov’s asymptotic cones [10] is just a special instance of this basic method, applied to ultrapowers of metric spaces.

2 Valuations on real closed fields

The basic reference for this section is again [18]; the ideas date back to Baer and Artin-Schreier.

2.1 Let $R$ be a real closed field, i.e. an ordered field where every positive element is a square, and where every odd polynomial has a zero. Let $O \subseteq R$ be an $o$-convex subring (containing 0 and 1). Then $O$ is a valuation ring: if $\alpha \in R \setminus O$, then $\alpha^{-1} \in O$. The ring $O$ thus has a unique maximal ideal $M$ consisting of all nonunits. The quotient $O/M$ is again a real closed field. If $K$ is any maximal subfield of $O$, then $K$ is real closed and $o$-projects onto $O/M$, so $O$ splits as a direct sum $R = K \oplus M$ (note however that there is in general no canonical choice for $K$),

$$0 \to M \to O \to O/M \to 0.$$ 

Of course, an archimedean real closed field $R$ (such as $\mathbb{R}$) contains no proper $o$-convex subring, but there is an abundance of nonarchimedean examples; in the next section, we will have a construction using ultraproducts.
2.2 Let \( R \) be a nonarchimedean real closed field, let \( \alpha \gg 1 \), and let \( O^{(\alpha)} = \{ r \in R \mid |r| \leq \alpha^n \text{ for some } n \in \mathbb{N} \} \) denote the \( o \)-convex subring generated by \( \alpha \). The maximal ideal is \( M^{(\alpha)} = \{ r \in R \mid |r| \leq \alpha^{-n} \text{ for all } n \in \mathbb{N} \} \), and we put \( R^{(\alpha)} = O^{(\alpha)}/M^{(\alpha)} \).

Intuitively, we have truncated the field \( R \) behind the powers of \( \alpha \). If \( R = R^{(\alpha)} \), we call \( R \) an \( \alpha \)-archimedean field (because \( R \) is archimedean over the subfield \( \mathbb{Q}(\alpha) \)). The real closed field \( R^{(\alpha)} \) is \( \bar{\alpha} \)-archimedean, where \( \bar{\alpha} \) is the image of \( \alpha \in O^{(\alpha)} \) in \( R^{(\alpha)} \).

2.3 In general, a real closed field need not have a logarithm or exponential function (i.e. an \( o \)-isomorphism \( (R,+) \cong (R_{>0},\cdot) \)). In fact, no \( \alpha \)-archimedean real closed field can have an exponential function. However, we may define a formal logarithm as follows: we take for \( \Lambda = (\mathbb{R}_{>0},\cdot) \) an \( o \)-isomorphic copy of the ordered abelian (multiplicative) group \( (\mathbb{R}_{>0},\cdot) \) of positive elements. For this, we rewrite this group additively, putting \( \lg : (\mathbb{R}_{>0},\cdot) \cong (\Lambda,+) \).

Note that then \( |\lg r| \) corresponds to \( \max\{r,r^{-1}\} \). If \( O \subseteq R \) is an \( o \)-convex subring, we may consider the \( o \)-convex subgroup

\[ \Omega = \{ \lg r \mid r > 0 \text{ is a unit of } O \} \subseteq \Lambda. \]

If we put \( \nu(0) = \infty \), then the map

\[ \nu : r \mapsto -\lg |r| + \Omega \in \Lambda/\Omega \]

is precisely the valuation determined by \( O \subseteq R \); the value group is \( \Gamma = \Lambda/\Omega \).

2.4 Suppose that \( R = R^{(\alpha)} \) is \( \alpha \)-archimedean, and consider the \( o \)-convex subring

\[ O^{(\alpha)} = \{ r \in R \mid |r|^n \leq \alpha \text{ for all } n \in \mathbb{N} \}. \]

Then the value group \( \Gamma = \Lambda/\Omega \) is archimedean, because \( \Lambda = \Lambda^{(\lg \alpha)} \) and \( \Omega = \Lambda^{(\lg \alpha)} \), so we have a (nondiscrete, rank 1) real-valued \( o \)-valuation \( \nu \) on the field \( R \). A special instance of this construction is Robinson’s asymptotic field \( \rho_R \) \([19][15]\), which we introduce in the next section.

3 Ultraproducts

A basic reference for this section is \([7]\).

3.1 Let \( I \) be an (infinite) set, and let \( \mu : 2^I \rightarrow \{0,1\} \) be a finitely additive probability measure. The collection of all sets with \( \mu \)-measure 1 is a nonprincipal ultrafilter. Using the axiom of choice, the existence of such a measure can be proved without difficulty. (Note that every subset of \( I \) is required to be \( \mu \)-measurable! Clearly, every finite set has measure 0, and every cofinite set has measure 1.)

Suppose that \( (Q_i)_{i \in I} \) is a family of first order structures: a family of groups, rings, fields, or metric spaces. The direct product \( \prod Q_i \) of these structures will in general be of a weaker type;
for example, the direct product of fields is a ring with zero-divisors. This deficiency can be corrected using the measure $\mu$: two elements $(q_i)_{i \in I}, (q'_i)_{i \in I}$ of the direct product are identified if their difference set $\{ j \in I \mid q_j \neq q'_j \}$ has $\mu$-measure 0. The resulting collection of equivalence classes is the ultraproduct $\prod_\mu Q_i$.

An ultraproduct has the same kind of first-order properties as its factors; for example, an ultraproduct of fields is again a field. Note that this construction is completely analogous to the construction of the Hilbert space $L^2$, where integrable functions are identified if they differ on a set of Lebesgue measure 0.

3.2 An important special case is when all structures $Q_i$ are equal to one fixed structure $Q$; in this case, one obtains an ultrapower $^*Q = \prod_\mu Q$ which is also called a nonstandard model of $Q$. If $Q = (\mathbb{R}, +, \cdot, \leq)$ is the field of real numbers, the resulting field $\prod_\mu (\mathbb{R}, +, \cdot, \leq) = (^*\mathbb{R}, +, \cdot, \leq)$ is the field of nonstandard real numbers. This is a nonarchimedean real closed field.

3.3 Let $\alpha \in ^*\mathbb{R}$ be an infinitely large nonstandard real, and let $\rho \alpha = \alpha^{-1}$. The fields $^\rho \mathbb{R} = (^*\mathbb{R})^{(\alpha)}$ were first studied by Robinson [19], and are sometimes called Robinson’s asymptotic fields.

3.4 Suppose now that each $Q_i = (d_i : X_i \times X_i \to \mathbb{R})$ is an $\mathbb{R}$-metric space. Then the ultraproduct $\prod_\mu (d_i : X_i \times X_i \to \mathbb{R}) = (d : X \times X \to ^*\mathbb{R})$ is a $\Lambda$-metric space, with $\Lambda = (^*\mathbb{R}, +)$. If we pick a point $o \in X$, a number $\alpha \gg 1$ in $^*\mathbb{R}$, then $X_{o}^{(\alpha)} = \{ x \in X \mid \alpha^{-1}d(x, o) \leq n \text{ for some } n \in \mathbb{N} \}$ and the $\mathbb{R}$-metric space $X_{o}^{(\alpha)} = X_{o}^{(\alpha)} / \Lambda_{(\alpha)}$ is precisely Gromov’s asymptotic cone of the family $(X_i)_{i \in I}$, where all points $x, y$ with infinitesimal distance $\alpha^{-1}d(x, y)$ are identified [10]. The general construction is due to Van den Dries and Wilkie [23].

4 Riemannian symmetric spaces over real closed fields

For the geometry of Riemannian symmetric spaces, see [8], [11], [2], and [3].
4.1 Let $P_n$ denote the set of all symmetric positive definite real $n \times n$-matrices with determinant 1. Every such matrix is of the form $X = gg^T$, for some $g \in \text{SL}_n \mathbb{R}$. The set $P_n$ can be identified with the Riemannian symmetric space $\text{SL}_n \mathbb{R}/\text{SO}(n)$ consisting of all elliptic polarities of real projective $n - 1$-space. The group $\text{SL}_n \mathbb{R}$ acts as 

$$(g, X) \mapsto gXg^T$$

(the geodesic reflection at $X \in P_n$ is $Y \mapsto XY^{-1}X$). Up to a real scaling factor, the metric $d_R$ induced by the (unique invariant) Riemannian metric of $P_n$ is given as follows. For $X \in P_n$, let $Y = \log(X)$ denote the unique traceless symmetric matrix with $\exp(Y) = X$. Then

$$d_R(1, X)^2 = \text{tr}(Y^2).$$

4.2 Now $P_n$ is an algebraic variety defined over $\mathbb{Q}$, and we may consider its set of $R$-points $P_n(R)$ over any real closed field $R$. The metric $d_R$, however, involves an analytic function, the logarithm, which need not be defined over an arbitrary real closed field $R$. We fix this problem, using the formal logarithm $\lg : R_{>0} \to \Lambda$ defined in 2.3. Given a diagonal matrix $X = \text{diag}(x_1, \ldots, x_n) \in P_n(R)$, we define the $\Lambda$-valued distance

$$d(1, X) = |\lg x_1| + \cdots + |\lg x_n| \in \Lambda.$$

This function is obviously invariant under coordinate permutations (the Weyl group action for $\text{SL}_n R$); it follows that we can use the $\text{SL}_n R$-action to extend $d$ to a well-defined distance function on $P_n(R)$, setting

$$d(gg^T, gXg^T) = d(1, X)$$

(where $X$ is a diagonal matrix). On the set $A$ of all diagonal matrices in $P_n(R)$, this is clearly a $\Lambda$-metric (essentially, it is the Manhattan Taxi Metric). By Lemma 1.2 and Kostant’s Convexity Theorem [13], $d$ is indeed an $\text{SL}_n R$-invariant metric on $P_n(R)$. Over the reals, we may take $\Lambda = \mathbb{R}$ and $\lg = \log$; then $d$ and $d_R$ are different, but (bi-Lipschitz) equivalent metrics.

4.3 We have successfully made $X = P_n(R)$ into a $\Lambda$-metric space. Suppose now that $O \not\subseteq R$ is an $o$-convex subring. Put as before $\Omega = \{\lg r| r > 0 \text{ and } r \text{ is a unit of } O\} \subseteq \Lambda$. Then $\Gamma = \Lambda/\Omega$ is the value group of the valuation determined by $O$. The group $\text{SL}_n R$ acts by isometries on the $\Lambda/\Omega$-metric space $P_n(R)/\Omega$, and it is not difficult to check the following result:

$$P_n(R)/\Omega = \text{SL}_n(R)/\text{SL}_n(O).$$

4.4 Suppose now that $R = R^{(\alpha)}$ is $\alpha$-archimedean, see 2.2, and that $O = O^{(\alpha)}$. Then the value group $\Gamma = \Lambda/\Omega = \Lambda^{(\alpha)}$ is archimedean (we could take for example $R = \mathbb{R}$, Robinson’s asymptotic field). Consequently,

$$P_n(R)/\Omega = \text{SL}_n(R)/\text{SL}_n(O)$$

is an $\mathbb{R}$-metric space. The main result of Section 3 is that this quotient is a (nondiscrete) affine $\mathbb{R}$-building (of type $\tilde{A}_{n-1}$, in fact). The group $\text{SL}_n(R)$ acts as a transitive automorphism group on its vertices. More generally, if $O \subseteq R$ is any $o$-convex valuation ring, then $\text{SL}_n(R)/\text{SL}_n(O)$ is an affine $\Lambda/\Omega$-building in the sense of Bennett [3].
4.5 In this section, we have concentrated on the Riemannian symmetric space \( SL_n\mathbb{R}/SO(n) \). Everything we have done works in the same generality for any Riemannian symmetric space of noncompact type. In fact, any irreducible Riemannian symmetric space can be equivariantly embedded in \( P_n \) as a totally geodesic submanifold; the embedding can be chosen to be algebraic over \( \mathbb{Q} \). Then one obtains without much difficulty the following result.

4.6 Theorem Let \( R \) be a (nonarchimedean) real closed field, let \( O \) be an \( o \)-convex subring, let \( X \) be a Riemannian symmetric space of noncompact type. Let \( G \) be the corresponding semisimple real Lie group, with maximal compact subgroup \( K \). Then we can view \( G \) and \( K \) as real algebraic groups defined over \( \mathbb{Q} \), and \( X = G/K \) as a real algebraic variety. Define \( \Lambda \) and \( \Omega \) as above. Then there exists a \( G(\mathbb{R}) \)-invariant \( \Lambda \)-metric on \( X(\mathbb{R}) = G(\mathbb{R})/K(\mathbb{R}) \) (semialgebraic over \( \mathbb{Q} \)), and there is a natural equivariant identification \( X(\mathbb{R})/\Omega = G(\mathbb{R})/G(O) \) of \( \Lambda/\Omega \)-metric spaces. If \( R \) is in addition \( \alpha \)-archimedean, and \( O = O_{(\alpha)} \), then \( X(\mathbb{R})/\Omega \) is an \( R \)-metric space. \( \square \)

Now we consider as a special case the ultrapower \( \prod\mu X \times X \xrightarrow{\ast} \mathbb{R} \). Pick a basepoint \( o \in X \) and an infinitely large nonstandard real \( \alpha \gg 1 \). Let \( \rho = 1/\alpha \).

4.7 Corollary For the asymptotic cone of the Riemannian symmetric space \( X \), we have \( X^{(\alpha)}_\rho = G(\mathbb{R})/G(O) \), where \( R = o\mathbb{R} = (\ast\mathbb{R})^{(\alpha)} \) is Robinson’s asymptotic field, and \( O = O_{(\bar{\alpha})} \). \( \square \)

5 Affine \( \Lambda \)-buildings

5.1 Suppose that \( W \) is a finite Coxeter group satisfying the crystallographic condition. Then \( W \) acts naturally on a \( \mathbb{Z} \)-lattice \( L \cong \mathbb{Z}^n \) in \( \mathbb{R}^n \). Let \( \Lambda \) be an ordered abelian group; then \( W \) acts naturally on \( A = L \otimes \Lambda \cong \Lambda^n \). Let \( \overline{W} = W \rtimes (A,+ \rangle \) denote the corresponding affine reflection group. Using the \( W \)-invariant inner product \( L \otimes L \to \mathbb{Z} \), it is possible to construct a \( \overline{W} \)-invariant \( \Lambda \)-metric on \( d : A \times A \to \Lambda \). (For the case of the symmetric group, the metric – the Manhattan Taxi Metric – was given in 4.2 above.) Using the action of \( W \) on \( A \), it makes sense to talk about affine reflection hyperplanes and closed halfspaces. A subset \( B \subseteq A \) is called \( W \)-convex if it is the intersection of finitely many halfspaces.

5.2 Suppose that \( X \) is a \( \Lambda \)-metric space. An atlas on \( X \) is a collection \( \mathcal{A} \) of \( \Lambda \)-isometric injections \( \phi : A \to X \), called coordinate charts, with the following properties.

(A1) If \( \phi \) is in \( \mathcal{A} \) and \( w \in \overline{W} \), then \( \phi \circ w : A \to X \) is in \( \mathcal{A} \).

(A2) Given two charts \( \phi_1, \phi_2 \), the set \( B = \phi_1^{-1}(\phi(A)) \) is \( W \)-convex, and there exists a \( w \in \overline{W} \) with \( \phi_1|_B = \phi_2 \circ w|_B \).

The sets \( F = \phi(A) \) are called apartments; the image \( S = \phi(S_0) \) of the basic Weyl cone \( S_0 \subseteq A \) is called a sector.

(A3) Given \( x, y \in X \), there exists an apartment \( F = \phi(A) \) containing \( x \) and \( y \).
Given two sectors \( S_1, S_2 \subseteq X \), there exist subsectors \( S'_1 \subseteq S_1 \) and \( S'_2 \subseteq S_2 \) and an apartment \( F \) containing \( S'_1 \cup S'_2 \).

(A5) If \( F_1, F_2, F_3 \) are apartments such that each of the three sets \( F_i \cap F_j, i \neq j \), is a halfapartment (i.e. the \( \phi \)-image of a halfspace), then \( F_1 \cap F_2 \cap F_3 \neq \emptyset \).

(A6) For any apartment \( F \) and any \( x \in F \), there exists a retraction \( \rho_{x,F} : X \rightarrow F \) which diminishes distances, with \( \rho_{x,F}^{-1}(x) = \{x\} \).

The pair \((X, A)\) is called an affine \( \Lambda \)-building \( [3] \). The atlas \( A \) is, in general, not unique, but it is always contained in a unique maximal atlas.

If we take for \( X \) a Riemannian symmetric space and the maximal flats as the apartments, then \( X \) satisfies axioms (A1) and (A3), and, rather trivially, axioms (A2) and (A5), while axiom (A4) is only approximately true.

5.3 For \( n = 1 \), an affine \( \Lambda \)-building is the same as a \( \Lambda \)-tree \([3, 1]\); affine \( \Lambda \)-buildings are in a sense higher-dimensional versions of (\( \Lambda \))-trees. The notion of affine \( \Lambda \)-buildings is due to Bennett \([3]\); affine \( \mathbb{R} \)-buildings are the same as a Tits’ systèmes d’appartements \([22]\). Kleiner-Leeb \([12]\) give a different set of axioms for spaces they call euclidean buildings; Parreau \([17]\) proved that their spaces are special cases of affine \( \mathbb{R} \)-buildings. Finally, there are the affine buildings in the proper sense of building theory \([1, 20]\). One has the following inclusions:

\[
\{\text{affine } \Lambda \text{-buildings}\} \supseteq \{\text{affine } \mathbb{R} \text{-buildings}\} \supseteq \{\text{Kleiner-Leeb euclidean buildings}\} \supseteq \{\text{affine buildings}\}
\]

5.4 Let \( X \) be an affine \( \Lambda \)-building, let \( o \in X \). One can construct two spherical buildings from \( X \). Consider the collection \( \text{Sec}_o \) of all sectors \( \phi(S_0) \) whose tip \( \phi(0) \) is \( o \) (recall that \( S_0 \subseteq L \otimes \Lambda \) is the basic Weyl cone). Then \( \text{Sec}_o \) generates in a rather natural way a poset, the spherical building at infinity. Using the axioms above, one can show that different basepoints lead to canonically isomorphic buildings at infinity, and we denote the resulting building by

\[
\Delta^A_{\infty} X.
\]

This building depends on the atlas \( A \); if \( A \) is maximal, we omit it and write \( \Delta_{\infty} X \).

5.5 One can construct another spherical building from \( \text{Sec}_o \); here, we identify two sectors \( S_1, S_2 \in \text{Sec}_o \) if they agree inside an open ball around \( o \) (with respect to the \( \Lambda \)-metric). The corresponding spherical building is denoted \( \Delta_o X \); it is independent of the apartment system (because it is defined by local data). There is a canonical building epimorphism

\[
\Delta^A_{\infty} X \rightarrow \Delta_o X.
\]

5.6 Let \( R \) be a real closed field and let \( O \subseteq R \) be an \( o \)-convex subring. Let \( X = G/K \) be a Riemannian symmetric space of noncompact type. Our first main result is as follows.

5.7 Theorem The quotient \( G(R)/G(O) \) is an affine \( \Gamma \)-building, where \( \Gamma = \Lambda/\Omega \) is the value group of the valuation determined by \( O \), see \([2, 3]\). In particular, if \( R = R^{(\alpha)} \) is \( \alpha \)-archimedean and \( O = O^{(\alpha)} \), then \( G(R)/G(O) \) is an affine \( \mathbb{R} \)-building. The automorphism group of the building at infinity (with respect to the apartment system we construct) is the group \( G(R) \), provided that no simple factor of \( G \) has rank 1.
The proof depends on various classical results about semisimple Lie groups. As the apartment system, we take the images of the maximal flats in the Riemannian symmetric space \( G(R)/K(R) \) under the canonical map

\[
G(R)/K(R) = G(R)/K(O) \rightarrow G(R)/G(O).
\]

Let \( G = KAU \) be an Iwasawa decomposition. For example, axiom \((A3)\) is a consequence of the \( KAK\)-decomposition, \( G(R) = K(R)A(R)K(R) \), while \((A2)\) follows from Kostant’s Convexity Theorem. The proof of \((A6)\) depends also on the Convexity Theorem, and the Iwasawa projection \( g = kau \rightarrow a \).

5.8 Corollary In the special case of Robinson’s asymptotic fields \( ρ\mathbb{R} \), we obtain in this way the full (maximal) apartment system. Thus, if \( G \) has no simple factor of rank 1, then \( G(ρ\mathbb{R}) \) is the full automorphism group of \( X \).

The corollary uses the model-theoretic fact that ultrapowers are saturated.

6 Topological rigidity of affine \( \mathbb{R} \)-buildings

Let \( Δ \) be the underlying simplicial complex of a (thick) rank \( k \) building \([6] [20]\), and let \(|Δ|\) denote its geometric realization (topologized in any sensible way). It is not difficult to recover the combinatorial structure of \( Δ \) from the topological space \(|Δ|\); the \( k-1 \)-skeleton of \(|Δ|\) is precisely the set of all points in \(|Δ|\) which do not have a locally euclidean neighborhood. It follows that buildings whose geometric realizations are homeomorphic are isomorphic. This is considerably more involved for nondiscrete affine \( \mathbb{R} \)-buildings, since here, it may happen that no point has a locally euclidean neighborhood. Topological rigidity of affine \( \mathbb{R} \)-buildings was proved first by Kleiner-Leeb \([12]\). The appearance of local homology groups in their argument suggests that there should be a simple sheaf-theoretic proof. This is indeed the case. The amount of sheaf theory needed here is very modest. The first introductory pages of \([4]\) completely suffice; in particular, we do not need such sophisticated tools as sheaf-theoretic (co)homology, although some ideas are inspired by \([14]\).

6.1 To any Hausdorff space \( X \), one can associate two basic presheaves: firstly, the graded \( \mathbb{Z}/2 \)-module valued presheaf \( U \rightarrow H_\bullet(X, X \setminus U; \mathbb{Z}/2) \) (ordinary singular homology with \( \mathbb{Z}/2 \)-coefficients) and secondly, the presheaf \( U \rightarrow 2^U \) which assigns to \( U \) the boolean algebra of all subsets of \( U \). Let

\[
\mathcal{H}_\bullet = \text{Sheaf}(U \rightarrow H_\bullet(X, X \setminus U; \mathbb{Z}/2)) \quad \text{and} \quad \mathcal{S} = \text{Sheaf}(U \rightarrow 2^U)
\]

denote the corresponding sheaves. The stalks of the first sheaf are the local homology groups,

\[
\mathcal{H}_\bullet_x = H_\bullet(X, X \setminus \{x\}; \mathbb{Z}/2)
\]

(this follows from the axiom of compact supports for singular homology), while the stalk \( \mathcal{S}_x \) consists of germs of subsets of \( X \) near \( x \). Note that the closure operation for subsets is compatible with restriction and thus induces a closure operation on germs of subsets near \( x \). Given a local section \( s : U \rightarrow \mathcal{H}_\bullet(U) \) of \( \mathcal{H}_\bullet \) over \( U \), there is the closed subset \( \text{supp}(s) = \{x \in U | s_x \neq 0\} \).
This yields a natural map \( ssg : \mathcal{H}_{\ast,x} \rightarrow S_x \) on the stalks: any element \( \xi \in H_{\ast}(X, X \setminus \{o\}; \mathbb{Z}/2) \) determines a subset germ \( ssg(\xi) \) at \( x \). For \( \xi, \eta \in \mathcal{H}_{\ast,x} \), we define

\[
\xi \ast \eta = ssg(\xi) \cap ssg(\eta) \setminus ssg(\xi + \eta) \in S_x.
\]

**6.2** Suppose now that \( X \) is an affine \( \mathbb{R} \)-building. Let \( o \in X \) and recall from \[5.4\] and \[5.5\] the definition of \( \text{Sec}_o \) and the building \( \Delta_o X \). Each sector \( S \in \text{Sec}_o \) determines a germ \( l(S) \in S_o \); the resulting sub-poset of \( S_o \) generated by the intersections of these germs is precisely \( \Delta_o X \),

\[
\Delta_o X \longrightarrow S_o.
\]

Now let \( F \) be an apartment of \( X \) containing \( o \). Using the retraction \( \rho_{o,F} : X \rightarrow F \), one sees that \( F \) determines an element \( \xi_F \in H_{\ast}(X, X \setminus \{o\}; \mathbb{Z}/2) \), given by the image of \( H_{\ast}(F, F \setminus \{o\}; \mathbb{Z}/2) \). To simplify the argument, we assume that all the buildings \( \Delta_o X \) are thick; in the setting of the proof of the Margulis conjecture, this is the case. (The general case, where the \( \Delta_o X \) may be weak buildings requires some minor refinements.)

**6.3 Proposition** There is an isomorphism

\[
H_{\ast+1}(X, X \setminus \{o\}; \mathbb{Z}/2) \cong H_{\ast}(X \setminus \{o\}; \mathbb{Z}/2) \cong H_{\ast}(\Delta_o X; \mathbb{Z}/2).
\]

The first isomorphism comes from the contractibility of \( X \). The main point of the proof of the second isomorphism is to show that every element \( \xi \in H_{\ast}(X \setminus \{o\}; \mathbb{Z}/2) \) arises as above from (finitely many) apartments. Here, the key idea is to use the fact that every chamber in \( \Delta_o X \) has a small neighborhood which lifts isometrically into the neighborhood of some chamber of \( \Delta_{\infty} X \). \( \square \)

Using this result, we can characterize the germs \( l(S) \in S_o \).

**6.4 Theorem** Every chamber \( l(S) \) of \( \Delta_o X \) is of the form \( \xi \ast \eta \), for suitably chosen \( \xi, \eta \). Every element \( \xi \ast \eta \) which is minimal (i.e. there exist no \( \xi', \eta' \) with \( \emptyset \neq \xi' \ast \eta' \subset \xi \ast \eta \)) represents a chamber \( l(S) \) of \( \Delta_o X \). \( \square \)

**6.5 Corollary** Let \( f : X \rightarrow X' \) be a homeomorphism of affine \( \mathbb{R} \)-buildings. Then \( f \) induces an isomorphism between \( \Delta_o X \) and \( \Delta_{f(o)} X' \), for every \( o \in X \).

From this local result, one can without difficulty derive the following.

**6.6 Theorem** Let \( s : X \rightarrow \mathcal{H}_{\ast} \) be a global section of the sheaf \( \mathcal{H}_{\ast} \rightarrow X \) over the affine \( \mathbb{R} \)-building \( X \). Then \( A = \text{supp}(s) \subset X \) is an apartment if and only if (1) \( A \) is homeomorphic to \( \mathbb{R}^n \), for some \( n \), and (2) \( ssg(s_a) \) is an apartment in \( \Delta_a \) for every \( a \in A \).

**6.7 Corollary** Let \( f : X \rightarrow X' \) be a homeomorphism of affine \( \mathbb{R} \)-buildings. Then \( f \) carries apartments to apartments.

**6.8 Corollary** A homeomorphism of affine \( \mathbb{R} \)-buildings \( X, X' \) induces an isomorphism (in a sense to be made precise); moreover, \( \Delta_{\infty} X \cong \Delta_{\infty} X' \).
7 Proof of the Margulis conjecture

Now we outline a proof of the Margulis conjecture, using the previous results.

7.1 Let $X, X'$ be Riemannian symmetric spaces of noncompact type, without de Rham factor of rank 1, and let

$$f : X \longrightarrow X$$

be an $(L, C)$-quasi-isometry,

$$L^{-1} \cdot d(x, y) - C \leq d'(f(x), f(y)) \leq L \cdot d(x, y) + C.$$ 

Taking ultrapowers of $X$ and $X'$, we obtain an $(L, C)$ quasi-isometry

$$*f : *X \longrightarrow *X'$$

between the $\Lambda$-metric spaces $*X, *X'$, where $(\Lambda, +) \cong (\mathbb{R}_0^+, \cdot)$ (we use the metric $d$ introduced in 4.2, which is bi-Lipschitz equivalent to the metric $d_R$ induced by the Riemannian metric).

7.2 Pick any point $o \in *X$, and an element $\alpha \in *\mathbb{R}$, with $\alpha \gg 1$. The properties of a quasi-isometry and the finiteness of the constants $L, C$ imply that we have an $(L, C)$-quasi-isometry

$$(*X)_{\alpha}^{(\alpha)} \longrightarrow (*X')_{f(o)}^{(\alpha)}$$

of $\Lambda^{(\alpha)}$-metric spaces, which descends to a (bi-Lipschitz) homeomorphism

$$(*X)_{\alpha} \longrightarrow (*X')_{f(o)}$$

on the asymptotic cones.

7.3 Let $R = \rho\mathbb{R} = (*\mathbb{R})^{(\alpha)}$ denote Robinson’s asymptotic field, and let $O = O_{(\alpha)}$. By the results in Section 5, $Z = (*X)_{\alpha}^{(\alpha)} = G(R)/G(O)$ and $Z' = G'(R)/G'(O) = (*X')_{f(o)}^{(\alpha)}$ are affine $\mathbb{R}$-buildings. By 6.8, we have an isomorphism

$$\Delta_{\infty}Z \cong \Delta_{\infty}Z',$$

whence a group isomorphism $G(\rho\mathbb{R}) \cong G'(\rho\mathbb{R})$. This implies that there is a Lie group isomorphism $G \cong G'$, and so, an equivariant diffeomorphism

$$X = G/K \cong G'/K' = X'.$$

It follows that the metrics on the de Rham factors of $X$ can be rescaled such that $X$ and $X'$ are isometric.

7.4 A careful analysis of the quasi-isometry shows that a stronger result holds, see [12]: there exists a (necessarily unique) isometry $\tilde{f} : X \longrightarrow X'$ such that $\tilde{f}$ has bounded distance from $f$. For the proof, one has to consider different choices for $\alpha$ and the base point $o$; the main step is to show that $f$ is (after rescaling the metrics on the de Rham factors) a coarse isometry. The original proof of this by Kleiner-Leeb can be simplified in a substantial way, using $\Lambda$-buildings and model theoretic methods.
References

[1] R. Alperin and H. Bass, Length functions of group actions on Λ-trees. *Combinatorial group theory and topology* (Alta, Utah, 1984), 265–378, Princeton Univ. Press, Princeton, NJ, 1987.

[2] W. Ballmann, M. Gromov and V. Schroeder, *Manifolds of nonpositive curvature*. Birkhäuser Boston, Inc., Boston, MA, 1985.

[3] C. Bennett, Affine A-buildings. I. Proc. London Math. Soc. 68 (1994) 541–576.

[4] G. Bredon, *Sheaf theory*, 2nd ed. Springer-Verlag, New York, 1997.

[5] M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*. Springer-Verlag, Berlin, 1999.

[6] K. Brown, *Buildings*. Springer-Verlag, New York (1989).

[7] C.C. Chang and H.J. Keisler, *Model theory*. 3rd ed. North-Holland Publishing Co., Amsterdam, 1990.

[8] P. Eberlein, *Geometry of nonpositively curved manifolds*. University of Chicago Press, Chicago, IL, 1996.

[9] A. Eskin and B. Farb, Quasi-flats and rigidity in higher rank symmetric spaces. J. Amer. Math. Soc. 10 (1997) 653–692.

[10] M. Gromov, Asymptotic invariants of infinite groups. *Geometric group theory, Vol. 2* (Sussex, 1991), 1–295, Cambridge Univ. Press, Cambridge, 1993.

[11] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*. Academic Press, Inc., New York-London, 1978.

[12] B. Kleiner and B. Leeb, *Rigidity of quasi-isometries for symmetric spaces and Euclidean buildings*. Inst. Hautes Études Sci. Publ. Math. No. 86 (1997) 115–197.

[13] B. Kostant, On convexity, the Weyl group and the Iwasawa decomposition. Ann. Sci. École Norm. Sup. (4) 6 (1973), 413–455.

[14] R. Löwen, Topology and dimension of stable planes: on a conjecture of H. Freudenthal. J. Reine Angew. Math. 343 (1983) 108–122.

[15] W. Luxemburg, On a class of valuation fields introduced by A. Robinson. Israel J. Math. 25 (1976) 189–201.

[16] G.D. Mostow, *Strong rigidity of locally symmetric spaces*. Princeton University Press, Princeton, N.J., 1973.

[17] A. Parreau, Immeubles affines: construction par les normes et étude des isométries. *Crystallographic groups and their generalizations* (Kortrijk, 1999), 263–302, Contemp. Math., 262, Amer. Math. Soc., Providence, RI, 2000.
[18] S. Prieß-Crampe, *Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen*. Springer-Verlag, Berlin, 1983.

[19] A.H. Lightstone and A. Robinson, *Nonarchimedean fields and asymptotic expansions*. North-Holland Publishing Co., New York, 1975.

[20] M. Ronan, *Lectures on buildings*. Academic Press, Inc., Boston, MA, 1989.

[21] B. Thornton, *Asymptotic cones of symmetric spaces*, PhD Thesis, Univ. Utah, 2002.

[22] J. Tits, Immeubles de type affine. *Buildings and the geometry of diagrams (Como, 1984)*, 159–190, Lecture Notes in Math. 1181, Springer, Berlin, 1986.

[23] L. van den Dries and A. Wilkie, Gromov’s theorem on groups of polynomial growth and elementary logic. *J. Algebra* 89 (1984) 349–374.