THE OPTIMAL UPPER BOUND FOR THE FIRST EIGENVALUE OF THE FOURTH ORDER EQUATION

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Abstract. In this paper we will give the optimal upper bound for the first eigenvalue of the fourth order equation with integrable potentials when the $L^1$ norm of potentials is known. Combining with the results for the corresponding optimal lower bound problem in [12], we have completely obtained the optimal estimation for the first eigenvalue of the fourth order equation.

1. Introduction. Given an integrable potential $q \in L^1 := L^1([0,1], \mathbb{R})$, we consider an eigenvalue problem of the fourth order beam equation

$$y^{''''}(t) + q(t)y(t) = \lambda y(t), \quad t \in [0,1],$$

with the Lidstone boundary condition

$$y(0) = y^{''}(0) = 0 = y(1) = y^{''}(1).$$

It is well-known that problem (1) has a sequence of (real) eigenvalues $-\infty < \lambda_1(q) < \lambda_2(q) < \cdots < \lambda_m(q) < \cdots$, satisfying $\lim_{m \to \infty} \lambda_m(q) = +\infty$. See [4].

In this paper we will give the optimal upper bound for the first eigenvalue $\lambda_1(q)$ of the fourth order equation (1) when the $L^1$ norm $\|q\|_1 = \|q\|_{L^1([0,1])}$ is known. To this end, we will solve the following maximization problem

$$M(r) := \sup \{ \lambda_1(q) : q \in B_1[r] \},$$

where $r \in (0, +\infty)$ and

$$B_1[r] := \{ q \in L^1 : \|q\|_1 \leq r \}.$$

In a recent paper [12], the present author and his collaborator have solved the minimization problem

$$L(r) := \inf \{ \lambda_1(q) : q \in B_1[r] \}.$$
Once maximization problem (3) is solved, one has the following the upper and lower bounds for $\lambda_1(q)$
\[ \mathbf{L}(\|q\|_1) \leq \lambda_1(q) \leq \mathbf{M}(\|q\|_1) \quad \forall q \in \mathcal{L}^1, \] (6)
which will be shown to be optimal in a certain sense.

Minimization and maximization problems for eigenvalues are important in applied sciences like quantum mechanics [2], population dynamics [5] and propagation speeds of traveling waves [8]. These are also interesting mathematical problems [1, 6, 7, 9, 18, 19, 20, 21], because the solutions are involved of many different branches of mathematics.

Since the $L^1$ ball $B_1[r]$ is not compact even in the weak topology of $\mathcal{L}^1$, we usually do not know if maximization problem (3) can be attained by some potentials from $B_1[r]$. To overcome this, different from the approach in [13, 17], here we will extend the problem to the measure case. More precisely, let $\mu : [0, 1] \to \mathbb{R}$ be a measure.

Firstly, we will find the explicit optimal upper bound for the first eigenvalue $\lambda_1(\mu)$ of the corresponding measure differential equation (MDE)
\[ dy^{(3)}(t) + y(t) \, d\mu(t) = \lambda y(t) \, dt, \quad t \in [0, 1], \] (7)
with the corresponding Lidstone boundary condition when the total variation of measure $\mu$ is known. Secondly, Based on the relationship between maximization problem of ODE and of MDE, we can obtain the main result of this paper as follows.

**Theorem 1.1.** Let $A_1 (\approx 3803.53)$ be the unique root of the equation
\[ \tanh \frac{x}{2} - \tan \frac{x}{2} = 0, \quad x \in ((2\pi)^4, (3\pi)^4), \] (8)
and $B_1 (\approx 500.56)$ be the unique root of the equation
\[ \tanh \frac{x}{2} + \tan \frac{x}{2} = 0, \quad x \in (\pi^4, (2\pi)^4). \] (9)

One has the following conclusions.
(i) When $0 < r \leq H(B_1) (\approx 215.42)$, it holds that
\[ \mathbf{M}(r) = H^{-1}(r). \]
Here, the invertible elementary function $H : (\pi^4, A_1) \to (0, +\infty)$ is defined as
\[ H(x) := \frac{4x^\frac{3}{2}}{\tanh \frac{x}{2} - \tan \frac{x}{2}}, \quad x \in (\pi^4, A_1), \] (10)
(ii) When $r > H(B_1)$, it holds that
\[ \mathbf{M}(r) = \frac{B_1}{16t_0^3}. \]
Here, $t_0 \in (0, 1/2)$ is the unique root of polynomial
\[ F_t(x) := rx^4 + \frac{B_1 \tanh \frac{B_1}{2}}{8 \tanh \frac{B_1}{2}} x - \frac{B_1}{16} = 0, \quad x \in (0, 1/2). \] (11)
2. Auxiliary results. Let $I = [0, 1]$. For a function $\mu : I \to \mathbb{R}$, the total variation of $\mu$ (over $I$) is defined as

$$V(\mu, I) := \sup \left\{ \sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 < t_1 < \cdots < t_n = 1, \ n \in \mathbb{N} \right\}.$$

Let

$$\mathcal{M}(I, \mathbb{R}) := \{ \mu : I \to \mathbb{R} : \mu(0+) \exists, \ \mu(t+) = \mu(t) \ \forall t \in (0, 1), \ V(\mu, I) < \infty \}$$

be the space of non-normalized $\mathbb{R}$-valued measures of $I$. Here, for any $t \in [0, 1)$, $\mu(t+) := \lim_{s \downarrow t} \mu(s)$ is the right-limit. The space of (normalized) $\mathbb{R}$-valued measures is

$$\mathcal{M}_0(I, \mathbb{R}) := \{ \mu \in \mathcal{M}(I, \mathbb{R}) : \mu(0+) = 0 \}.$$ 

For simplicity, we write $V(\mu, I)$ as $\|\mu\|_V$. For $t \in (0, 1]$, let

$$V(\mu, (0, t]) := \sup \left\{ \sum_{i=0}^{n-1} |\mu(t_{i+1}) - \mu(t_i)| : 0 < t_0 < \cdots < t_n < t, \ n \in \mathbb{N} \right\}.$$

It is known that for $\mu \in \mathcal{M}_0(I, \mathbb{R})$, $V(\mu, I) = |\mu(0)| + V(\mu, (0, 1])$. See, e.g., [15].

**Lemma 2.1.** ([16]) Let $\mu \in \mathcal{M}_0(I, \mathbb{R})$. Define

$$\tilde{\mu}(t) := \begin{cases} \frac{1}{2} |\mu(0)| & \text{for } t = 0, \\ V(\mu, (0, t]) & \text{for } t \in (0, 1]. \end{cases}$$

Then $\tilde{\mu} \in \mathcal{M}_0(I, \mathbb{R})$ satisfies $\|\mu\|_V = \tilde{\mu}(1) - \tilde{\mu}(0)$ and

$$\int_{[a,b]} f(s) \, d\mu(s) \leq \int_{[a,b]} |f(s)| \, d\tilde{\mu}(s) \quad \forall f \in C(I, \mathbb{R}), \ [a, b] \subset I,$$

which refers to the Riemann-Stieltjes integral, or the Lebesgue-Stieltjes integral [3].

Typical examples of measures are as follows.

- Let $\ell : I \to \mathbb{R}$ be $\ell(t) \equiv t$. Then $\ell$ yields the Lebesgue measure of $I$ and the Lebesgue integral. More generally, any $q \in L^1(I, \mathbb{R})$ induces an absolutely continuous measure defined by

$$\mu_q(t) := \int_{[0,t]} q(s) \, ds, \quad t \in I.$$ \hspace{1cm} (14)

In this case, one has

$$\|\mu_q\|_V = \|q\|_1 = \|q\|_{L^1(I, \mathbb{R})},$$

and

$$\int_{I_0} f(t) \, d\mu_q(t) = \int_{I_0} f(t) q(t) \, dt = \int_{I_0} f(t) \, d\mu_q(t)$$

for any $f \in C(I, \mathbb{R})$ and subinterval $I_0 \subset I$. 

• For \( a = 0 \), the unit Dirac measure at \( t = 0 \) is
\[
\delta_0(t) = \begin{cases} 
-1 & \text{for } t = 0, \\
0 & \text{for } t \in (0, 1]. 
\end{cases}
\]

• For \( a \in (0, 1] \), the unit Dirac measure at \( t = a \) is
\[
\delta_a(t) = \begin{cases} 
0 & \text{for } t \in [0, a), \\
1 & \text{for } t \in [a, 1]. 
\end{cases}
\]

By the Riesz representation theorem [10], \( (\mathcal{M}_0(I, \mathbb{R}), \| \cdot \|_\mathcal{V}) \) is the same as the dual space of the Banach space \( (C(I, \mathbb{R}), \| \cdot \|_\infty) \) of continuous \( \mathbb{R} \)-valued functions of \( I \) with the supremum norm \( \| \cdot \|_\infty \). In fact, \( \mu \in (\mathcal{M}_0(I, \mathbb{R}), \| \cdot \|_\mathcal{V}) \) defines \( \mu^* \in (C(I, \mathbb{R}), \| \cdot \|_\infty)^* \) by
\[
\mu^*(f) = \int_I f(t) \, d\mu(t), \quad \forall f \in C(I, \mathbb{R}).
\]
Moreover, one has
\[
\|\mu\|_\mathcal{V} = \mathcal{V}(\mu, I) = \sup \left\{ \int_I f \, d\mu : f \in C(I, \mathbb{R}), \quad \|f\|_\infty = 1 \right\}.
\]

In the space \( \mathcal{M}_0(I, \mathbb{R}) \) of measures, one has the usual topology induced by the norm \( \| \cdot \|_\mathcal{V} \). Due to duality relation (15), one has also the following weak* topology \( w^* \). Let \( \mu_0, \mu_n \in \mathcal{M}_0(I, \mathbb{R}) \), \( n \in \mathbb{N} \). We say that \( \mu_n \) is weakly* convergent to \( \mu_0 \) iff, for each \( f \in C(I, \mathbb{R}) \), one has
\[
\lim_{n \to \infty} \int_I f \, d\mu_n = \int_I f \, d\mu_0.
\]
We remark that in some literature, this topology is just called the weak topology for measures.

In general, a measure cannot be a limit of smooth measures in the norm \( \| \cdot \|_\mathcal{V} \). However, in the \( w^* \) topology, the following conclusion holds.

**Lemma 2.2.** [11] Given \( \mu_0 \in \mathcal{M}_0(I, \mathbb{R}) \), there exists a sequence of measures \( \{\mu_n\} \subset C^\infty(I, \mathbb{R}) \cap \mathcal{M}_0(I, \mathbb{R}) \) such that
\[
\mu_n \rightharpoonup \mu_0 \quad \text{in } (\mathcal{M}_0(I, \mathbb{R}), w^*).
\]
Moreover, if \( \mu_0 \) is increasing (decreasing) on \( I \), then the sequence \( \{\mu_n\} \) above can be chosen such that for each \( n \in \mathbb{N} \), \( \mu_n \) is increasing (decreasing) on \( I \) and \( \|\mu_n\|_\mathcal{V} = \|\mu_0\|_\mathcal{V} \).

Considering \( q \in L^1(I, \mathbb{R}) \) as a density, one has the measure or distribution given by (14). Since \( \|\mu_q\|_\mathcal{V} = \|q\|_1 \),
\[
(L^1(I, \mathbb{R}), \| \cdot \|_1) \hookrightarrow (\mathcal{M}_0(I, \mathbb{R}), \| \cdot \|_\mathcal{V}) \text{ is an isometric embedding.} \tag{17}
\]

For \( r \in [0, \infty) \), denote
\[
B_0[r] := \{ \mu \in \mathcal{M}_0(I, \mathbb{R}) : \|\mu\|_\mathcal{V} \leq r \}. \tag{18}
\]
Via (14), by the Hölder inequality and the isometrical embedding (17), the following inclusion is proper
\[
B_1[r] \subset B_0[r].
\]
As for the compactness of these balls in weak* topology, we have that \( B_0[r] \subset (\mathcal{M}_0(I, \mathbb{R}), w^*) \) is sequentially compact. See [10].

Given a measure \( \mu \in \mathcal{M}_0 := \mathcal{M}_0(I, \mathbb{R}) \), we will write the fourth order linear MDE with the measure \( \mu \) as
\[
dy^{(4)}(t) + y(t) \, d\mu(t) = 0, \quad t \in [0, 1]. \tag{19}
\]
Definition 2.3. A function \( y : [0, 1] \to \mathbb{R} \) is called a solution to the equation (19) on the interval \([0, 1]\) if

- \( y \in \mathcal{C}([0, 1], \mathbb{R}) \), and
- there exist \((y_0, y_1, y_2, y_3) \in \mathbb{R}^4\) and functions \( y^{(1)}, y^{(2)}, y^{(3)} : [0, 1] \to \mathbb{R} \) such that the following are satisfied

\[
\begin{align*}
y(t) &= y_0 + \int_{[0, t]} y^{(1)}(s) \, ds, \quad t \in [0, 1], \\
y^{(1)}(t) &= y_1 + \int_{[0, t]} y^{(2)}(s) \, ds, \quad t \in [0, 1], \\
y^{(2)}(t) &= y_2 + \int_{[0, t]} y^{(3)}(s) \, ds, \quad t \in [0, 1], \\
y^{(3)}(t) &= \begin{cases} y_3, & t = 0, \\ y_3 - \int_{[0, t]} y(s) \, d\mu(s), & t \in (0, 1]. \end{cases}
\end{align*}
\]

The initial condition of MDE (19) can be written as

\[
(y(0), y^{(1)}(0), y^{(2)}(0), y^{(3)}(0)) = (y_0, y_1, y_2, y_3).
\]

Since we have assumed that \( y \in \mathcal{C}([0, 1], \mathbb{R}) \), the right-hand sides of (20), (21), (22) are the Lebesgue integral and (23) Lebesgue-Stieltjes integral respectively. For each \((y_0, y_1, y_2, y_3) \in \mathbb{R}^4\), problem (19), (24) has the unique solution \( y(t) \) defined on \([0, 1]\). See [12].

For \( p \in [1, \infty] \), let \( \mathcal{L}^p := L^p([0, 1], \mathbb{R}) \) be the Lebesgue space of real-valued functions with the \( L^p \) norm \( \| \cdot \|_p \). For \( n \in \mathbb{N} \), let \( \mathcal{W}^{n,p} := W^{n,p}([0, 1], \mathbb{R}) \) and

\[
\mathcal{W}^{n,p}_0 := W^{n,p}_0([0, 1], \mathbb{R}) = \{ y \in \mathcal{W}^{n,p} : y(0) = y(1) = 0 \}
\]

be the Sobolev spaces with the norm \( \| \cdot \|_{\mathcal{W}^{n,p}} \). For \( p = 2 \), \( \mathcal{W}^{n,2} \) and \( \mathcal{W}^{n,2}_0 \) are denoted simply by \( \mathcal{H}^n \) and \( \mathcal{H}^n_0 \), respectively, with the norm \( \| \cdot \|_{\mathcal{H}^n} \).

By the properties of Lebesgue integral and Lebesgue-Stieltjes integral, some regularity results for solutions \( y(t) \) are as follows.

Lemma 2.4. Let \( y(t) \) be the solution of (19). Then \( y \in \mathcal{H}^3 \) and \( y^{(3)} \in \mathcal{M} := \mathcal{M}([0, 1], \mathbb{R}). \) Hence,

\[
\begin{align*}
y^{(1)}(t) &= y'(t) \in \mathcal{C}^1 := \mathcal{C}^1([0, 1], \mathbb{R}), \\
y^{(2)}(t) &= y''(t) \in \mathcal{AC} := \mathcal{AC}([0, 1], \mathbb{R}),
\end{align*}
\]

and \( y^{(3)}(t) = y'''(t) \) a.e. \( t \in [0, 1] \). Here ' denotes the derivative with respect to \( t \) and \( \mathcal{AC}([0, 1], \mathbb{R}) \) is the space of absolutely continuous functions.

We use \( y(t, y_0, y_1, y_2, y_3) \) to denote the unique solution of (19) and (24). Let

\[
\varphi_1(t) := y(t, 1, 0, 0, 0), \quad \varphi_2(t) := y(t, 0, 1, 0, 0), \\
\varphi_3(t) := y(t, 0, 0, 1, 0), \quad \varphi_4(t) := y(t, 0, 0, 0, 1),
\]
called the fundamental solutions of (19). By the linearity of (19) and the uniqueness of solution, one has that, for \( t \in [0, 1] \),
\[
\begin{pmatrix}
    y(t, y_0, y_1, y_2, y_3) \\
    y^{(1)}(t, y_0, y_1, y_2, y_3) \\
    y^{(2)}(t, y_0, y_1, y_2, y_3) \\
    y^{(3)}(t, y_0, y_1, y_2, y_3)
\end{pmatrix} = 
\begin{pmatrix}
    \varphi_1(t) & \varphi_2(t) & \varphi_3(t) & \varphi_4(t) \\
    \varphi_1^{(1)}(t) & \varphi_2^{(1)}(t) & \varphi_3^{(1)}(t) & \varphi_4^{(1)}(t) \\
    \varphi_1^{(2)}(t) & \varphi_2^{(2)}(t) & \varphi_3^{(2)}(t) & \varphi_4^{(2)}(t) \\
    \varphi_1^{(3)}(t) & \varphi_2^{(3)}(t) & \varphi_3^{(3)}(t) & \varphi_4^{(3)}(t)
\end{pmatrix} 
\begin{pmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    y_3
\end{pmatrix} =: N_\mu(t) \begin{pmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    y_3
\end{pmatrix}.
\]

We consider eigenvalue problem of the fourth order equation (7) with the Lidstone boundary condition (2). Given \( \mu \in M_0 \), we say that \( \lambda \in \mathbb{R} \) is an eigenvalue of the Lidstone problem (7) and (2), if MDE (7) with such a parameter \( \lambda \) has non-zero solutions \( y(t) \) satisfying (2). The corresponding solutions \( y(t) \) are called eigenfunctions associated with \( \lambda \).

Besides the Sobolev spaces \( \mathcal{H}^3_0 \) and \( \mathcal{H}^3_0 \), let us introduce
\[
\mathcal{H}^3_{00} := \{ y \in \mathcal{H}^3 : y \text{ satisfies (2)} \} = \{ y \in \mathcal{H}^3 : y(0) = y(1) = y''(0) = y''(1) = 0 \}.
\]

One has the proper inclusions \( \mathcal{H}^3_{00} \subset \mathcal{H}^3_0 \subset \mathcal{H}^3_0 \).

In [12], the authors have established the minimization characterization for the first eigenvalue of the measure differential equation, which plays an important role in the extremal problem of ordinary differential equation.

**Lemma 2.5.** ([12]) Given \( \mu \in M_0 \), problem (7) and (2) admits the first eigenvalue \( \lambda_1(\mu) \), which has the following minimization characterizations
\[
\lambda_1(\mu) = \min_{u \in \mathcal{H}^3_{00}\setminus\{0\}} R(u) = \min_{u \in \mathcal{H}^3_{00}\setminus\{0\}} R(u) = \min_{u \in \mathcal{H}^3_{00}\setminus\{0\}} R(u). \quad (25)
\]

Here,
\[
R(u) = R_\mu(u) := \frac{\int_{[0,1]} (u'')^2 \, dt + \int_{[0,1]} u^2 \, d\mu(t)}{\int_{[0,1]} u^2 \, dt} \quad \text{for } u \in \mathcal{H}^3_{00}\setminus\{0\}. \quad (26)
\]

Moreover, we have the following variational characterizations of the higher order eigenvalues, which is a limiting case of the minimax principle [4].

**Lemma 2.6.** For \( m \in \mathbb{N} \) with \( m \geq 2 \), \( \lambda_m(\mu) \) is determined by
\[
\lambda_m(\mu) = \min_{u \in S \setminus \{0\}} R(u), \quad (27)
\]

where \( S \) is defined to be the class of piecewise twice differentiable functions satisfying
\[
u(\eta_i) = u''(\eta_i) = 0, \quad 0 < \eta_i < 1, \quad i = 1, 2, ..., m - 1
\]

and
\[
\begin{align*}
u(0) &= u(1) = u''(0) = u''(1) = 0.
\end{align*}
\]

Let us introduce the following ordering for measures. We say that measures \( \mu_2 \geq \mu_1 \) if
\[
\int_{[0,1]} f(t) \, d\mu_2(t) \geq \int_{[0,1]} f(t) \, d\mu_1(t) \quad \forall f \in C_+ := \{ f \in C : f(t) \geq 0, \ t \in [0, 1] \}.
\]

As a consequence of (25) in Lemma 2.5, we can obtain the following result.
Lemma 2.7. Let $\mu_1, \mu_2 \in \mathcal{M}_0$. Then

$$\mu_2 \geq \mu_1 \Rightarrow \lambda_1(\mu_2) \geq \lambda_1(\mu_1).$$

From the Lemma above, if $\mu$ is increasing, then $\lambda_1(\mu) \geq \lambda_1(0) = \pi^4$.

The continuity of the first eigenvalue in measures with the weak$^*$ topology can be introduced as follows [12].

Lemma 2.8. As a nonlinear functional, $\lambda_1(\mu)$ is continuous in $\mu \in (\mathcal{M}_0, w^*)$.

To solve the maximization problem (3), we need the following properties about some elementary functions.

Lemma 2.9. (i) For each $n \in \mathbb{N}$, the equation

$$\tanh \frac{x^4}{2} - \tan \frac{x^4}{2} = 0, \quad x \in (0, \infty),$$

has the unique root $A_n$ on the interval $((2n\pi)^4,(2n+1\pi)^4)$.

(ii) For each $n \in \mathbb{N}$, the equation

$$\tanh \frac{x^4}{2} + \tan \frac{x^4}{2} = 0, \quad x \in (0, \infty),$$

has the unique root $B_n$ on the interval $((2n-1\pi)^4,(2n\pi)^4)$.

(iii) For each $n \in \mathbb{N}$, the function

$$(2n-1\pi)^4, A_n) \rightarrow (0, +\infty), \quad x \rightarrow \frac{4x^\frac{1}{4}}{\tanh \frac{x^\frac{1}{4}}{2} - \tan \frac{x^\frac{1}{4}}{2}}$$

is strictly increasing. Hence, for each $r \in (0, +\infty)$, the following equation

$$\frac{4x^\frac{1}{4}}{\tanh \frac{x^\frac{1}{4}}{2} - \tan \frac{x^\frac{1}{4}}{2}} = r,$$

has the unique solution $X_n(r)$ on the interval $((2n-1\pi)^4, A_n)$.

Lemma 2.10. For each $r > H(B_1)$, the following equation

$$rx^4 + \frac{B_1 \tanh \frac{B_1^4}{2} - 2B_1^3}{8 \tanh \frac{B_1^4}{2}} x - \frac{B_1}{16} = 0$$

has the unique root on the interval $(0, 1/2)$. Here, $B_1$ and $H$ are defined in (9) and (10), respectively.

For each $r > H(B_1)$, let

$$F(x) := rx^4 + \frac{B_1 \tanh \frac{B_1^4}{2} - 2B_1^3}{8 \tanh \frac{B_1^4}{2}} x - \frac{B_1}{16}.$$

We have that $F(0) = -\frac{B_1}{16} < 0$ and
\[ F(1/2) = \frac{r}{16} + \frac{B_1 \tanh \frac{B_1^4}{2} - 2B_1^2}{8 \tanh \frac{B_1^4}{2}} \\
= \frac{r}{16} - \frac{2B_1^2}{16 \tanh \frac{B_1^4}{2}} \]

which implies that there exists a root of \( F(x) \) on \((0, 1/2)\).

Moreover, we have that

\[ F'(x) = 4rx^3 + \frac{B_1 \tanh \frac{B_1^4}{2} - 2B_1^2}{8 \tanh \frac{B_1^4}{2}} \geq \frac{B_1^2}{8\tanh \frac{B_1^4}{2}} (B_1^4 \tanh \frac{B_1^4}{2} - 2) > 0. \]

So, \( F(x) \) has the unique root on the interval \((0, 1/2)\).\[\square\]

3. **Proof of main result.** Firstly, we study the following maximization problem

\[ \tilde{M}(r) := \sup \{ \lambda_1(\mu) : \mu \in B_0[r] \} = \max \{ \lambda_1(\mu) : \mu \in B_0[r] \}. \]

because \( B_0[r] \) is sequentially compact in \( (M_0, w^*) \) and Lemma 2.8.

We need the following result about the problem (7) and (2).

**Lemma 3.1.** Suppose that \( \hat{\mu} \in B_0[r] \) has a normalized eigenfunction \( \hat{y} \) corresponding to \( \lambda_1(\hat{\mu}) \) such that

\[ \int_{[0,1]} \hat{y}^2(t) \, d\mu(t) \leq \int_{[0,1]} \hat{y}^2(t) \, d\hat{\mu}(t) \forall \mu \in B_0[r]. \] (33)

Then one has

\[ \tilde{M}(r) = \lambda_1(\hat{\mu}). \] (34)

**Proof.** Since \( \hat{\mu} \in B_0[r] \), one has \( \lambda_1(\hat{\mu}) \leq \tilde{M}(r) \). On the other hand, for any \( \mu \in B_0[r] \), one has

\[ \lambda_1(\mu) = \min_{s \in H^2_0} \left( \int_{[0,1]} (u'')^2 \, dt + \int_{[0,1]} u^2 \, d\mu(t) \right) \] (by [25])

\[ \leq \int_{[0,1]} (\hat{y}'')^2 \, dt + \int_{[0,1]} \hat{y}^2 \, d\mu(t) \]

\[ \leq \int_{[0,1]} (\hat{y}'')^2 \, dt + \int_{[0,1]} \hat{y}^2 \, d\hat{\mu}(t) \] (by assumption [33])

\[ = \lambda_1(\hat{\mu}), \]

which implies \( \tilde{M}(r) \leq \lambda_1(\hat{\mu}) \). Hence we have (34).\[\square\]

We will find the explicit optimal upper bound of the first eigenvalue of MDE.
Lemma 3.2. When \( 0 < r \leq H(B_1) \), one has
\[
\tilde{M}(r) = \lambda_1(r\delta_{1/2}) = H^{-1}(r),
\]  
where \( B_1, H \) and \( \delta_{1/2} \) are as in [9], [10] and [13], respectively.

Proof. Step 1. We will prove \( \lambda_1(r\delta_{1/2}) = H^{-1}(r) \) and obtain the first eigenfunction \( y(t) \).

To this end, we need to solve the following equation
\[
dy^{(3)}(t) + y(t) \, d(r\delta_{1/2}(t)) = \lambda y(t) \, dt, \quad t \in [0, 1].
\]  
From the explanation to solutions of MDE, one knows that solutions \( y(t) \) of (30) satisfies the classical ODE
\[
y''''(t) = \lambda y(t)
\]
for \( t \) on the intervals \([0, 1/2]\) and \((1/2, 1]\). At \( t = 1/2 \), one has the following relations
\[
\begin{align*}
{y^{(1/2+)}(1/2)} &= y^{(1/2-)}(1/2), \\
{y^{''(1/2+)}(1/2)} &= y^{''(1/2-)}(1/2), \\
{y^{''''(1/2+)}(1/2)} &= y^{''''(1/2-)}(1/2) - ry^{(1/2-)}(1/2).
\end{align*}
\]
Denote
\[
\omega = \sqrt{\lambda_1(r\delta_{1/2})} \in [\pi, \infty).
\]
From ODE (37) and the first two conditions of (2), we obtain
\[
y(t) = c_1 \sin \omega t + c_2 \sinh \omega t, \quad t \in [0, 1/2),
\]
for some \((c_1, c_2) \neq 0\) and \( c_1 \geq 0 \).

By (38), we have
\[
\begin{align*}
y^{(1/2+)}(1/2) &= z_0 := c_1 \sin \frac{\omega}{2} + c_2 \sinh \frac{\omega}{2}, \\
y^{''(1/2+)}(1/2) &= z_1 := c_1 \omega \cos \frac{\omega}{2} + c_2 \omega \cosh \frac{\omega}{2}, \\
y^{''''(1/2+)}(1/2) &= z_3 := c_1 \left( -\omega^3 \cos \frac{\omega}{2} - r \sin \frac{\omega}{2} \right) + c_2 \left( \omega^3 \cosh \frac{\omega}{2} + r \sinh \frac{\omega}{2} \right).
\end{align*}
\]
By using this as the initial value at \( t = 1/2 \), we obtain from ODE (37) that
\[
y(t) = \frac{z_0 - \frac{z_1}{\omega} \cos \omega \left( t - 1/2 \right) + \frac{z_3}{\omega^2} - \frac{z_4}{2\omega^3} \sin \omega \left( t - 1/2 \right)}{+ z_0 + \frac{z_1}{\omega} \cosh \omega \left( t - 1/2 \right) + \frac{z_3}{\omega^2} + \frac{z_4}{2\omega^3} \sinh \omega \left( t - 1/2 \right)}
\]
\[
= c_1 \sin \omega t + c_2 \sinh \omega t
\]
\[
- r \frac{c_2 \sinh \frac{\omega}{2} + c_2 \sinh \frac{\omega}{2}}{2\omega^3} \left( \sinh \omega \left( t - 1/2 \right) - \sin \omega \left( t - 1/2 \right) \right)
\]
\[
\text{for } t \in (1/2, 1].
\]
Now the last two conditions \( y(1) = y''(1) = 0 \) of (2) are the following linear system for \((c_1, c_2)\)
\[
\begin{align*}
{c_1 \sin \omega + c_2 \sinh \omega - r \sinh \frac{\omega}{2} \sin \frac{\omega}{2}} &= 0, \\
{\omega^2 \left( -c_1 \sin \omega + c_2 \sinh \omega \right) - r \sin \frac{\omega}{2} \sinh \frac{\omega}{2}} &= 0.
\end{align*}
\]
This can yield the relation
\[
c_2 = \frac{2\omega^3 \sin \omega + r \left( \sin \frac{\omega}{2} - \sin \frac{\omega}{2} \right) \sin \frac{\omega}{2}}{2\omega^3 \sinh \omega + r \left( \sin \frac{\omega}{2} - \sin \frac{\omega}{2} \right) \sinh \frac{\omega}{2}} c_1 = -\frac{\cos \frac{\omega}{2}}{\cos \frac{\omega}{2}} c_1.
\]
In order that system (11) has non-zero solutions \((c_1, c_2)\), the corresponding determinant of (11) is necessarily zero. This yields the following equation

\[
4\omega^3 \tanh \frac{\omega}{2} - \tan \frac{\omega}{2} = r.
\]

Then, by the existence of the first eigenvalue, we conclude that

\[
\lambda_1(r\delta_1/2) = \min \left\{ \lambda \in (\pi^4, +\infty) : \frac{4\omega^3}{\tanh \frac{\omega}{2} - \tan \frac{\omega}{2}} = r \right\} = H^{-1}(r) \in (\pi^4, A_1),
\]

Here \(A_1\) and \(H\) are defined in (8) and (10), respectively.

**Step 2.** We assert that the eigenfunction \(y(t)\) is increasing in \([0, 1/2]\) and satisfies

\[
y(1 - t) = y(t) \text{ for all } t \in [0, 1].
\]

By Lemma 2.9 one has that \(\omega = \sqrt[3]{\lambda_1(r\delta_1/2)} \in (\pi, \sqrt{B_1}]\). Notice that for \(t \in [0, 1/2),\)

\[
y'(t) = c_1 \omega \cos \omega t + c_2 \omega \cosh \omega t = c_1 \omega \cos \omega t - \frac{\cos \frac{\omega}{2}}{\cosh \frac{\omega}{2}} c_1 \omega \cosh \omega t
\]

and

\[
y''(t) = -\frac{c_1 \omega^2}{\cosh \frac{\omega}{2}} \left( \cosh \frac{\omega}{2} \sin \omega t + \cos \frac{\omega}{2} \sinh \omega t \right).
\]

We claim that when \(\omega \in (\pi, \sqrt{B_1}] \subset (\pi, 2\pi)\) and \(t \in [0, 1/2)\), it holds that \(y''(t) \leq 0\).

In fact, when \(\omega t \geq \pi/2\), we have

\[
cosh \frac{\omega}{2} \sin \omega t \geq \cosh \frac{\omega}{2} \sin \frac{\omega}{2},
\]

and

\[
\cos \frac{\omega}{2} \sinh \omega t \geq \cos \frac{\omega}{2} \sinh \frac{\omega}{2}.
\]

So

\[
cosh \frac{\omega}{2} \sin \omega t + \cos \frac{\omega}{2} \sinh \omega t \geq \cosh \frac{\omega}{2} \sin \frac{\omega}{2} + \cos \frac{\omega}{2} \sinh \frac{\omega}{2}
\]

\[
= \cosh \frac{\omega}{2} \cos \frac{\omega}{2} \left( \tan \frac{\omega}{2} + \tanh \frac{\omega}{2} \right) \geq 0,
\]

Since \(\tan \frac{\omega}{2} + \tanh \frac{\omega}{2} < 0\) and \(\cos \frac{\omega}{2} < 0\). Then,

\[
y''(t) = -\frac{c_1 \omega^2}{\cosh \frac{\omega}{2}} \left( \cosh \frac{\omega}{2} \sin \omega t + \cos \frac{\omega}{2} \sinh \omega t \right) \leq 0.
\]

When \(\omega t < \pi/2\), we have that

\[
\frac{\sin \omega t}{\sinh \omega t} \geq \frac{\sin \pi/2}{\sinh \pi/2} = \frac{1}{\sinh \frac{\pi}{2}} \geq \frac{-\cos(\frac{\pi}{2})}{\cosh \frac{\pi}{2}},
\]

because

\[
\left( \frac{\sin x}{\sinh x} \right) = \frac{\cos x \sin x - \sin x \cosh x}{\sinh^2 x} = \frac{\cos x \cosh x (\tan x - \tan x)}{\sinh^2 x} \leq 0,
\]

for \(x \in [0, \pi/2]\). Then, \(y''(t) = -c_1 \omega^2 \sinh \omega t \left( \frac{\sin \omega t}{\sinh \omega t} + \frac{\cos \frac{\omega}{2}}{\cosh \frac{\omega}{2}} \right) \leq 0.\)
Since \( y'(1/2) = 0 \) and \( y''(t) \leq 0 \), we have that \( y'(t) \geq 0 \) for \( t \in [0, 1/2] \) and then \( y(x) \) is increasing in \( t \in [0, 1/2] \).

From (43), (44) and (47), we can obtain by computing directly that \( y(t) = y(1-t) \) for \( t \in [0, 1] \).

**Step 3.** We assert that

\[
\hat{M}(r) = \lambda_1(r \delta_{1/2}).
\]  

(43)

Let \( \hat{y} = \frac{y}{\|y\|_{L^2}} \) be a normalized eigenfunction corresponding to \( \lambda_1(r \delta_{1/2}) \). For each \( \mu \in B_0 \), \( r \), one has that

\[
\int_{[0,1]} \hat{y}^2 \, d\mu(t) \leq \|\mu\|_{L^1} \|\hat{y}\|_{L^2}^2 \leq r \hat{y}(1/2)^2 = \int_{[0,1]} \hat{y}^2 \, d(r \delta_{1/2}(t)).
\]

By Lemma 3.3, (43) holds. □

**Lemma 3.3.** When \( r > H(B_1) \), one has

\[
\hat{M}(r) = \lambda_1(\mu_0) = \frac{B_1}{16t_0}.
\]  

(44)

Here, \( t_0 \) is defined in (11) and

\[
\mu_0(t) := \begin{cases} 
0, & \text{for } t \in [0, t_0), \\
\frac{B_1}{16t_0}(t - \frac{1}{2}) + \frac{1}{2}, & \text{for } t \in [t_0, 1 - t_0), \\
r, & \text{for } t \in [1 - t_0, 1].
\end{cases}
\]

(45)

**Proof.** Let \( \Omega := \frac{B_1^2}{2t_0} \). From (10) and (11), we have by computing directly that

\[
\tanh \Omega t_0 + \tan \Omega t_0 = 0,
\]

and

\[
\frac{4\Omega^3}{(\tanh \Omega t_0 - \tan \Omega t_0)} + (1 - 2t_0)\Omega^4 = r.
\]

(46)

(47)

Let

\[
y(t) := \begin{cases} 
\sinh \Omega t_0 \sin \Omega t + \sin \Omega t_0 \sinh \Omega t, & t \in [0, t_0), \\
2 \sinh \Omega t_0 \sin \Omega t_0, & t \in [t_0, 1/2], \\
y(1-t), & t \in (1/2, 1],
\end{cases}
\]

(48)

\[
y^{(1)}(t) := \begin{cases} 
\Omega \sinh \Omega t_0 \cos \Omega t + \Omega \sin \Omega t_0 \cosh \Omega t, & t \in [0, t_0), \\
0, & t \in [t_0, 1/2], \\
y^{(1)}(1-t), & t \in (1/2, 1],
\end{cases}
\]

(49)

\[
y^{(2)}(t) := \begin{cases} 
-\Omega^2 \sinh \Omega t_0 \sin \Omega t + \Omega^2 \sin \Omega t_0 \sinh \Omega t, & t \in [0, t_0), \\
0, & t \in [t_0, 1/2], \\
y^{(2)}(1-t) & t \in (1/2, 1],
\end{cases}
\]

(50)

and

\[
y^{(3)}(t) := \begin{cases} 
-\Omega^3 \sinh \Omega t_0 \cos \Omega t + \Omega^3 \sin \Omega t_0 \cosh \Omega t, & t \in [0, t_0), \\
0, & t \in [t_0, 1/2], \\
-y^{(3)}(1-t) & t \in (1/2, 1].
\end{cases}
\]

(51)

Then, by (46) and (47), we can check that

\[
y \in H^3, \quad y^{(1)} = y' \in C^1, \quad y^{(2)} = y'' \in AC, \quad y^{(3)} \in M
\]

and \( y^{(3)}(t) = y'''(t) \) for \( t \in [0, 1] \setminus \{t_0, 1 - t_0\} \).
Notice that
\[ \frac{dy^{(3)}(t)}{dt} = \Omega^4 y(t), \quad t \in [0, 1) \backslash \{t_0, 1 - t_0\}. \]

and
\[
y^{(3)}(t_0) - y^{(3)}(t_0) = 0 - (-\Omega^3 \sinh \Omega t_0 \cos \Omega t_0 + \Omega^3 \sin \Omega t_0 \cosh \Omega t_0)
\]
\[
= \frac{4\Omega^3}{(\tanh \Omega t_0 - \tan \Omega t_0)} (\tanh \Omega t_0 - \tan \Omega t_0) (\sinh \Omega t_0 \cos \Omega t_0 - \sin \Omega t_0 \cosh \Omega t_0)
\]
\[
= (r - (1 - 2t_0)\Omega^4) \frac{(-2 \sin \Omega t_0 \cosh \Omega t_0)(2 \sin \Omega t_0 \cosh \Omega t_0)}{4 \cosh \Omega t_0 \cos \Omega t_0}
\]
\[
= -(r - (1 - 2t_0)\Omega^4) \sin \Omega t_0 \sinh \Omega t_0
\]
\[
= -(\sinh \Omega t_0 \sin \Omega t_0 + \sin \Omega t_0 \sinh \Omega t_0) r - (1 - 2t_0)\Omega^4
\]
\[
= -y(t_0)(\mu(t_0) - \mu(t_0 -))
\]
\[
= - \int_{[t_0, 1 - t_0]} y(s) d\mu_0(s).
\]

By Definition 2.3, we have that \( y(t) \) satisfies the equation
\[ dy^{(3)}(t) + y(t) d\mu_0(t) = \Omega^4 y(t) dt, \quad t \in [0, 1]. \]

It is obvious that \( y(0) = y''(0) = y(1) = y''(1) = 0 \). So, \( y(t) \) is an eigenfunction corresponding to the eigenvalue \( \Omega^4 \). Moreover, by (46), one has that \( y'(t) > 0 \) when \( t \in [0, t_0) \), and then the eigenfunction \( y(t) \geq 0 \) when \( t \in [0, 1] \), which implies by Lemma 2.6 that \( \Omega^4 \) is the first eigenvalue of \( \mu_0 \), i.e., \( \Omega^4 = \lambda_1(\mu_0) \).

Let \( \hat{y} = \frac{y}{\|y\|_2} \) be a normalized eigenfunction corresponding to \( \lambda_1(\mu_0) \). Notice that \( \hat{y}(t) \) is increasing when \( t \in [0, t_0] \) and constant when \( t \in (t_0, 1/2] \). For each \( \mu \in B_0[r] \), one has that
\[
\int_{[0,1]} \hat{y}^2(t) d\mu(t) \leq \|\mu\| \|\hat{y}^2\|_{\text{max}}
\]
\[
\leq r \hat{y}^2(t_0)
\]
\[
= 2 \left( \frac{r - (1 - 2t_0)\Omega^4}{2} \hat{y}^2(t_0) + \frac{(1 - 2t_0)\Omega^4}{2} \hat{y}^2(t_0) \right)
\]
\[
= 2 \left( \int_{[0,t_0]} \hat{y}^2(t) d\mu_0(t) + \int_{(t_0,1/2]} \hat{y}^2(t) d\mu_0(t) \right)
\]
\[
= \int_{[0,1]} \hat{y}^2(t) d\mu_0(t).
\]

By Lemma 3.1, (44) holds.

Secondly, we can obtain the relationship between maximization problem of ODE and of MDE as follows.
Lemma 3.4. Given $r > 0$, one has that
\[ M(r) = \tilde{M}(r). \]  

Proof. Given $q \in B_1[r]$, the measure $\mu_q \in \mathcal{M}_0$ is defined as (14). By (17), we have that $\mu_q \in B_0[r]$ is absolutely continuous with respect to the Lebesgue measure.

So for any $q \in B_1[r]$,
\[ \tilde{M}(r) \geq \lambda_1(\mu_q) = \lambda_1(q), \]
which implies that
\[ \tilde{M}(r) \geq M(r). \]  
\[ (53) \]

On the other hand, there exists $\bar{\mu} \in B_0[r]$ such that $\lambda_1(\bar{\mu}) = \tilde{M}(r)$. By the property of measures in Lemma 2.1 and the monotonicity of $\lambda_1(\mu)$ in Lemma 2.7, without loss of generality, we can assume that $\bar{\mu}$ is increasing. By Lemma 2.2, there exists a sequence of measures $\{\bar{\mu}_n\} \subset C_\infty \cap \mathcal{M}_0$ such that
\[ d\bar{\mu}_n(t) = \bar{q}_n(t), \]
\[ ||\bar{\mu}_n||_V = ||\bar{\mu}||_1 = ||\bar{\mu}||_V \leq r, \]
\[ \bar{\mu}_n \rightarrow \bar{\mu} \text{ in } (\mathcal{M}_0, \ast). \]

Therefore, by Lemma 2.8, we have
\[ \tilde{M}(r) = \lambda_1(\bar{\mu}) = \lim_{n \to \infty} \lambda_1(\bar{\mu}_n) = \lim_{n \to \infty} \lambda_1(\bar{q}_n) \leq \lim_{n \to \infty} M(r) = M(r). \]  
\[ (54) \]
Now (53) and (54) imply that $M(r) = \tilde{M}(r)$. \qed

Finally, we will prove the main conclusion of this paper.

The proof of Theorem 1.1. By Lemma 3.2, Lemma 3.3 and Lemma 3.4, the conclusion holds directly. \qed

Remark 1. In a recent paper [12], the present author and his collaborator have solved the minimization problem [5].

Here, $Y_r : (0, 1) \to (-\infty, \pi^4]$ is defined as $Y_r(a) := \lambda_a$, where $\lambda_a \in (-\infty, \pi^4]$ is the unique root of
\[ G(\lambda, a) - r = 0, \]
and $G : (-\infty, \pi^4] \times (0, 1) \to [0, +\infty)$ is defined as
\[ G(\lambda, a) := \left\{ \begin{array}{ll}
\frac{2a^3 \sinh \omega \sin \omega}{a^2 (1-a)^3} & \text{for } \lambda \neq 0, \\
\frac{\sin(\omega a) \sin(\omega(1-a)) \sinh \omega - \sinh(\omega a) \sin(\omega(1-a)) \sin \omega}{a^2 (1-a)^3} & \text{for } \lambda = 0,
\end{array} \right. \]
with
\[ \omega := \left\{ \begin{array}{ll}
\sqrt[4]{\lambda} \in \mathbb{R} & \text{for } \lambda \geq 0, \\
\sqrt[4]{|\lambda|} e^{\pi i} & \text{for } \lambda < 0.
\end{array} \right. \]

Hence, we have completely obtained the optimal estimation for the first eigenvalue $\lambda_1(q)$ of the fourth order equation (1)-(2) when the $L^1$ norm of potentials $q$ is known.

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