Threshold Effects in Heavy Quarkonium Spectroscopy and Decays

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The possible importance of threshold effects in heavy quarkonium spectroscopy is discussed. The starting point is the calculation of the spectrum of heavy quarkonium-like states with self-energy/threshold corrections. Two different approaches are compared: I) The Unquenched Quark Model (UQM); II) A novel coupled-channel model, based on the UQM formalism. The latter provides a possible solution to the long-standing problem of convergence in UQM calculations; it also makes it possible to distinguish between states which are almost pure quarkonia and exotic states, characterized by non-negligible threshold (or continuum) components in their wave functions. The UQM-based coupled-channel model is used to study the $\chi_c(2P)$ and $\chi_b(3P)$ multiplets: $\chi_c(2P)$’s are described as charmonium-like states with non-negligible molecular-type components in their wave functions, $\chi_b(3P)$’s as almost pure bottomonia. Other possible applications of the UQM and the UQM-based coupled-channel model formalisms to the calculation of other observables, like the strong decay amplitudes, are also discussed.

I. INTRODUCTION

In 2003 Belle discovered a meson, the $X(3872)$ [now $\chi_c(3872)$] [1, 2], whose properties are incompatible with a pure quark-antiquark interpretation. The $X(3872)$ was the first example of an exotic hadron, namely a baryon/meson whose description requires the introduction of multiquark (tetra- and pentaquark) or gluonic degrees of freedom [3–9]. If we restrict to tetraquark or quarkonium-like exotics, several theoretical interpretations can be mentioned, including: a) Compact tetraquark (or diquark-antidiquark) states [10–26]; b) Meson-meson molecular states [27–36]; c) Structures resulting from kinematic or threshold effects caused by virtual particles [37–47]; Hadro-quarkonium states [48–57]. Here, we focus on the Unquenched Quark Model (UQM) [37, 38, 58–60] description of heavy quarkonium-like states, which are interpreted as heavy quarkonium cores, $|A\rangle = |Q\bar{Q}\rangle$, plus meson-meson higher Fock (or continuum) components, $|BC\rangle = |Q\bar{q} - q\bar{Q}\rangle$, due to virtual particle/loop effects.

The UQM has a long story. Since its formulation in the early 80s [37], a wide variety of applications has been explored, both in baryon/meson spectroscopy and structure. The UQM formalism is useful and elegant but, during the years, has shown some weaknesses and ambiguities. They include: I) The lack of convergence of its results when one considers large towers of meson-meson/baryon intermediate states $|BC\rangle$; II) Ambiguities related to the nature of the $|BC\rangle$ continuum or higher Fock components, especially if the $|BC\rangle$ intermediate states are far from a meson-meson/baryon threshold; III) Difficulties in the interpretation of the UQM results.

Hadron observables in the UQM are proportional to the “propagator” $\frac{1}{E_{B,C} - E_{A}}$, with $E_{B,C} = \sqrt{k^2 + M_{B,C}^2}$. It can be easily shown that the form of the UQM pair-creation operator, which couples the $|A\rangle$ and $|BC\rangle$ states by creating a light quark-antiquark pair from the vacuum, cannot ensure a sharp distinction between intermediate states whose energies $E_B + E_C$ are close to $M_A$ or far from it. Even the introduction of form factors in the pair-creation operator, in order to suppress $A, B$ and $C$ hadron wave function overlaps, do not provide a solution to the previous issue. This strongly affects the convergence of the UQM.

To improve the reliability of the UQM, these difficulties should be urgently addressed. A possible solution to the previous inconsistencies is discussed in the present contribution. See also Refs. [47] and [61]. Other possible applications of the UQM and the UQM-based coupled-channel model formalisms to the calculation of other observables, like meson strong decay amplitudes, are discussed.

II. UNQUENCHING THE QUARK MODEL FOR HEAVY QUARKONIUM-LIKE MESONS

The procedure for “unquenching the quark model” relies on the introduction of higher Fock components in quark-antiquark bare meson wave functions. The first step is to introduce tetraquark or meson-meson molecular-type components, $|Q\bar{q} - q\bar{Q}\rangle$, where $Q = c$ or $b$ is a heavy quark and $q = u, d$ or $s$ a light one [37, 38, 41–43, 44, 47, 58–61]. The introduction of intermediate hybrid components, namely containing quark and gluonic degrees of freedom, will be discussed in the future. The “unquenched” meson wave function can be written as

$$|\psi_A\rangle = N \left[ |A\rangle + \sum_{BCLI} \int k^2 dk |BCk\ell J\rangle \frac{\langle BCI|T^\dagger|A\rangle}{M_A - E_B - E_C} \right].$$

(1)

Here, $N$ is a normalization factor, $|\psi_A\rangle$ the superposition of a quark-antiquark configuration, $|A\rangle$, plus a sum over
all the possible higher Fock components, \(|BC\rangle\), due to the creation of a \(3P_0\) light \(q\bar{q}\) pair. The sum is extended over a complete set of intermediate meson-meson states, \(|BC\rangle\); \(M_A\) is the physical mass of the meson \(A\); \(k\) and \(\ell\) are the relative radial momentum and orbital angular momentum of \(B\) and \(C\), and \(J\) is the total angular momentum, with \(J = J_B + J_C + \ell + \ell^*\) in Eq. \(|A\rangle\); is the pair-creation operator of Refs. \[47\]. \(T^\dagger\) is the pair-creation operator of Refs. \[42, 43, 47, 59, 61\]. See also Refs. \[52, 53\].

Below threshold, \(T^\dagger\) can couple a bare meson, \(|A\rangle\), to four-quark meson-meson continuum components, \(|BC\rangle\); above threshold, it can trigger \(A \rightarrow BC\) open-flavor strong decays; see \[63\] and references therein. In the former case, the expectation value of a meson observable, \(\langle \psi_A | \hat{O}_n | \psi_A \rangle\), Eq. \(|\psi_A\rangle\), is given by

\[
\langle \psi_A | \hat{O}_m | \psi_A \rangle = \langle \hat{O}_m \rangle_{\text{val}} + \langle \hat{O}_m \rangle_{\text{cont}},
\]

where \(\langle \hat{O}_m \rangle_{\text{val}}\) and \(\langle \hat{O}_m \rangle_{\text{cont}}\) are the expectation values on the valence, \(|A\rangle\), and continuum components, \(|BC\rangle\), respectively. Typical observables which can be calculated within the UQM formalism are the physical masses of quarkonium-like states, \(M_A\), with self-energy corrections. They are related to the bare and self-energies via

\[
M_A = E_A + \Sigma(M_A),
\]

where the bare energies of pure quark-antiquark states, \(E_A\), have to be evaluated in a specific quark model like, for example, the relativized QM of Ref. \[62\]. In the UQM, the self-energies, \(\Sigma(M_A)\), are computed according to

\[
\Sigma(M_A) = \sum_{BCJ} \int_0^\infty k^2 dk \frac{|(BCk\ell J)| T^\dagger |A\rangle|^2}{M_A - E_B - E_C}.
\]

The UQM calculation of the heavy quarkonium spectrum via Eq. \(|\psi_A\rangle\) has problems of convergence. Below, we discuss a simple procedure to “renormalize” the UQM results. More details can be found in Refs. \[47\] and \[61\].

III. A NOVEL COUPLED-DERIVATIVE APPROACH BASED ON THE UQM FORMALISM

Here, we briefly describe the UQM-based coupled-channel approach of Ref. \[47\]. In particular, we focus on the calculation of quarkonium-like meson masses with threshold corrections. This formalism can be easily used to compute other observables without major modifications.

The main difference with respect to the standard UQM approach of Refs. \[42, 43\], is that the coupled-channel model of Ref. \[47\] is not used to fit the whole heavy quarkonium spectrum, but it is applied to a single quarkonium multiplet at a time. Some examples include the \(\chi_c(2P)\) and \(\chi_b(3P)\) multiplets, which were studied in Ref. \[47\]. Moreover, a “subtraction” or “renormalization” prescription for the UQM results is introduced \[47\]. This consists in the substitution of Eq. \(|\psi_A\rangle\) with

\[
M_A = E_A + \Sigma(M_A) + \Delta,
\]

where \(\Delta\) is the only one free parameter for each multiplet. It is defined as the smallest self-energy correction (in terms of absolute value) of a multiplet member \[47\]. Its importance and impact on the UQM results are clarified by the examples provided below.

A. Threshold mass-shifts in \(\chi_c(2P)\) and \(\chi_b(3P)\) multiplets

As a possible application of the UQM-based coupled-channel model, we briefly discuss the results of Ref. \[47\] for the masses of \(\chi_c(2P)\) and \(\chi_b(3P)\) heavy quarkonia with threshold corrections.

The threshold mass shifts were computed by considering a complete set of \(1S1S\) meson-meson intermediate states, like \(DD, DD^* (BB, BB^*)\), and so on. If one is interested in the study of a different quarkonium multiplet, like the \(\chi_c(3P)\), one should substitute the previous \(1S1S\) loops with the set of meson-meson intermediate states which is closer to the multiplet of interest, like \(1S1P\) or \(1S2S\). The values of the bare masses, \(E_A\), were directly extracted from the relativized model of Refs. \[64, 66\], without retuning the potential model parameters to the spectrum via Eq. \(|\psi_A\rangle\); the values of the physical meson masses, \(M_A\), were taken from the PDG \[4\]. The UQM parameters, used in the calculation of the \(|BCk\ell J T^\dagger A\rangle\) vertices of Eq. \(|\psi_A\rangle\), were extracted from Refs. \[42, 43, 63\]. Finally, our UQM-based coupled-channel model results for the self-energy/threshold corrections and physical masses of \(\chi_c(2P)\) and \(\chi_b(3P)\) multiplets are reported in Table \[II\] and Figure \[I\].

It is worth to observe that: I) Our theoretical predictions agree with the data within the error of a QM calculation, of the order of 30 – 50 MeV; II) Among \(\chi_c(2P)\) multiplet members, the \(\chi_c(2P)\) is subject to the largest threshold correction, \(\sim 65 MeV\), which brings its bare mass value, 3953 MeV, towards the experimental one, 3871.69 MeV \[4\]; III) In the \(\chi_c(2P)\) case, threshold effects break the peculiar mass pattern of a \(\chi\)-type multiplet, i.e. \(M_{\chi_0} < M_{\chi_1} \approx M_{\chi_2} < M_{\chi_3}\). On the contrary, the previous mass pattern is respected in the \(\chi_b(3P)\) case; IV) Threshold effects are negligible in the \(\chi_b(3P)\) case. Because of this, we interpret \(\chi_b(3P)\) states as almost pure bottomonia. This purely bottomonium description of \(\chi_b(3P)\) states agrees with the experimental masses of the \(\chi_b(3P)\) and \(\chi_{b2}(3P)\); the latter was recently reported by the PDG \[4\], 10524.02 ± 0.57 ± 0.53 MeV.
TABLE I: Comparison between the experimental masses [4] of \( \chi_c(2P) \) and \( \chi_b(3P) \) states, \( M_{\text{exp}}^{\chi} \), and theoretical predictions from Ref. [47], \( M_A^{\chi} \). The bare masses, \( E_A \), are taken from Refs. [64–66]. The experimental results denoted by \( \dagger \) are from Ref. [66], where \( \chi_b(3P) \) predicted multiplet mass splittings were used in combination with the experimental value of the \( \chi_b(3P) \) mass. Due to the lack of experimental data, in the \( h_c(2P) \) case we use the same value for the physical mass as the bare one [64].

| State                  | \( E_A \) [MeV] | \( \Sigma(M_i) + \Delta \) [MeV] | \( M_{\text{th}}^{\chi} \) [MeV] | \( M_{\text{exp}}^{\chi} \) [MeV] |
|------------------------|----------------|----------------------------------|---------------------------------|---------------------------------|
| \( h_c(2P) \)         | 3956           | −16                               | 3940                            | 3940                            |
| \( \chi_c(3915) \) or \( \chi_c(2P) \) | 3916           | 0                                | 3916                            | 3916                            |
| \( \chi_c(3872) \) or \( \chi_c(2P) \) | 3953           | −65                              | 3888                            | 3872                            |
| \( \chi_c(3930) \) or \( \chi_c(2P) \) | 3979           | −30                              | 3949                            | 3927                            |
| \( h_b(3P) \)         | 10541          | −4                               | 10538                           | 10519\dagger                   |
| \( \chi_b(3P) \)      | 10522          | 0                                | 10522                           | 10500\dagger                   |
| \( \chi_b(3P) \)      | 10538          | −2                               | 10537                           | 10513                            |
| \( \chi_b(3P) \)      | 10550          | −7                               | 10543                           | 10524                            |

B. Continuum components

The norm of the continuum (or molecular-type) component of a quarkonium-like state is given by [47, 59, 60]

\[
P_A^{\text{cont}} = \sum_{BC} \int_0^\infty q^2 dq \left| \langle BC q \ell J | T^\dagger | A \rangle \right|^2 \frac{1}{(M_A - E_B - E_C)^2},
\]

(6)

where the probability to find the meson in its valence component, \( P_A^{\text{val}} \), is given by \( P_A^{\text{val}} = 1 - P_A^{\text{cont}} \). In the coupled-channel model of Ref. [47], the procedure to compute the norm of Eq. (6) is slightly different than in the UQM [52, 60], as discussed in the following.

For each multiplet member, we define the ratio

\[
R_i = \frac{\Sigma(M_i) - \Delta}{\Sigma(M_i)},
\]

(7)

where \( \Sigma(M_i) \) and \( \Delta \) are defined in Eqs. (1) and (4), and the index \( i = 1, ..., N_{\text{mult}} \) runs over the meson multiplet members. For each member of the multiplet, we also de-

TABLE II: Calculated probabilities to find the \( X(3872) \) in its valence, \( P_A^{\text{val}} \), or meson-meson continuum component, \( P_A^{\text{cont}} \); see Eq. (4) and Ref. [47]. The two probabilities are related via \( P_A^{\text{val}} = 1 - P_A^{\text{cont}} \).

| State                  | Component        | Probability |
|------------------------|------------------|-------------|
| \( X(3872) \) or \( \chi_c(1277) \) | Continuum        | 0.853       |
| \( \chi_c(3930) \)     | Valence          | 0.147       |

FIG. 1: \( \chi_c(2P) \) multiplet: masses with threshold corrections. Boxes, dashed and continuous lines correspond to the experimental [4], calculated bare and physical masses, respectively. See Table I and Ref. [47].

![Graph](image-url)

### IV. HEAVY QUARKONIUM STRONG DECAYS IN THE UQM-BASED COUPLED CHANNEL MODEL FORMALISM

The decay amplitudes of heavy quarkonia can be calculated in the UQM [12, 43] or the UQM-based coupled channel model formalism of Ref. [47] by making use of Eq. (2):

\[
\Gamma_A^{\text{UQM}} = P_A^{\text{val}} \Gamma_A^{\text{val}} + \sum_{BC} P_{BC}^{\text{cont}} \Gamma_{BC}^{\text{cont}}.
\]

(9)

In the particular case of open-flavor strong decays, one has

\[
\Gamma_A^{\text{UQM}} = P_A^{\text{val}} \Gamma_A^{\text{val}} + \sum_{BC} P_{BC}^{\text{cont}} \Gamma_{BC}^{\text{cont}} + \Gamma_{A \rightarrow BC}^{\text{cont}}.
\]

(10)
where $P_{\text{val}}^A$ and $P_{BC}^\text{cont}$ are the probability to find the meson $A$ in its valence and $BC$ continuum or molecular-type components, respectively. $\Gamma_{A\to BC}$ is the $A \to BC$ open-flavor strong decay width of Figure 2, left panel, which can be computed in the $^3P_0$ pair-creation model \cite{63, 65, 67-70}; $\Gamma_{BC\to BC}^\text{cont}$ is the $BC \to BC$ dissociation width of the molecular-type component $BC$, whose diagram is depicted in Figure 2, right panel. The calculation of the open-flavor strong decay amplitudes of heavy quarkonia in the UQM-based coupled channel model formalism of Ref. \cite{47} will be the subject of a subsequent paper.

Below, we briefly discuss a calculation of the $J/\psi\omega$ and $J/\psi\rho$ hidden-flavor strong decays of the $X(3872)$ in the UQM \cite{47}. In this particular case, the decay width has no contribution associated with the valence component.

A. Hidden-flavor strong decays of the $X(3872)$ in the UQM

In the UQM, the $J/\psi\omega$ and $J/\psi\rho$ hidden-flavor transitions of the $X(3872)$ can be seen as two-step processes \cite{47}. Firstly, the initially pure $c\bar{c}$ state, $\chi_{c1}(2^3P_1)$, is “dressed” with $1S1S$ open-charm molecular-type components, including $D\bar{D}$, $D\bar{D}^*$, ..., by means of the UQM formalism \cite{12, 13, 59, 60}. The wave function of the $X(3872)$ is thus made up of a $\chi_{c1}(2^3P_1)$ core, $|A\rangle$, plus open-charm $DD$, $DD^*$, ... higher Fock components, $|BC\rangle$. Secondly, the $D\bar{D}^{**)}$ molecular-type component of the $X(3872)$ dissociates into $J/\psi$ plus $\rho$ or $\omega$, which are indicated as $|D\rangle$ and $|E\rangle$ in Eq. (11). The $\sigma_{BC\to DE}$ dissociation cross-sections are calculated by means of the non-relativistic formalism of Refs. \cite{71}.

By analogy with the positronium case \cite{72}, the hidden-flavor decay width is given by

$$\Gamma = \left[ \int P_{B}^2 dP_B \left| \langle A | T | BC \rangle \right|^2 \frac{1}{(M_A - E_{BC})^2 + \Gamma_A^2} \right] \left| v_B - v_C \right| \left| \sigma_{BC\to DE} \right| \left| \Psi_{BC}(0) \right|^2 .$$

The term in square brackets in Eq. (11) is the product of the $BC \to DE$ dissociation cross-section, $\sigma_{BC\to DE}$, and $|v_B - v_C|$, which is the difference between the velocities of the mesons $B$ and $C$. The previous quantity is convoluted with a distribution function, which describes the probability to find the $|BC\rangle = |D\bar{D}^{**)}\rangle$ component in the wave function of the $X(3872)$. The term in square brackets is multiplied by $|\Psi_{BC}(0)|^2$, which is the squared wave function of the $BC$ molecular-type component, evaluated in the origin.

Finally, the results of the UQM calculation of Ref. \cite{47} are: $\Gamma_{X(3872)\to J/\psi\omega}^{\text{UQM}} = 10 \text{ keV}$, $\Gamma_{X(3872)\to J/\psi\rho}^{\text{UQM}} = 6 \text{ keV}$. It is worth noting that the ratio between the previous calculated amplitudes,

$$\frac{\Gamma_{X(3872)\to J/\psi\omega}^{\text{UQM}}}{\Gamma_{X(3872)\to J/\psi\rho}^{\text{UQM}}} = 0.6 ,$$

is compatible with the present experimental data \cite{4, 72},

$$\frac{\Gamma_{X(3872)\to J/\psi\omega}}{\Gamma_{X(3872)\to J/\psi\rho}} = 0.8 \pm 0.3 ,$$

within the experimental error.

V. CONCLUSION

In this contribution, we discussed the possible importance of threshold effects in heavy quarkonium spectroscopy. We compared two different approaches: I) The Unquenched Quark Model (UQM) \cite{12, 13}; II) A novel coupled-channel model, based on the UQM formalism \cite{47}. The latter suggests a possible solution to the problem of convergence in UQM calculations; it also allows to distinguish between states which are almost pure quarkonia and exotic states, whose wave functions contain non-negligible threshold (or continuum) components. The coupled channel approach was used to study the quark
structure of the $\chi_c(2P)$ and $\chi_b(3P)$ multiplets: the former was described as a multiplet of charmonium-like states with non-negligible molecular-type components in their wave functions, the latter as an almost pure bottomonium multiplet [17].

Other possible applications of the model, including the calculation of the strong decay amplitudes of quarkonium-like states, were also discussed. As an example, we summarized the UQM calculation of the $J/\psi\omega$ and $J/\psi\rho$ hidden-flavor strong decays of the $X(3872)$ of

Ref. [47].

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