Propagation of the Free Dirac Wavefunctions

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The free time evolution of a Dirac wavefunction is studied in detail. One finds that at any time the border of a bounded localized wavefunction moves at the speed of light. There is no zitterbewegung, but there is a period of trembling of the carrier, which is limited in time and space in the order of the diameter of the carrier at its minimal extension. Before the phase of trembling the carrier shrinks isotropically and similarly it expands afterwards. The various parts of the border reverse their move only once, but they do it abruptly and not simultaneously, thus creating the trembling. Asymptotically, regarding the past and the future as well, the probability of position concentrates up to 1 within any spherical shell whose outer radius increases at light speed. The mathematical tools employed for the proofs are the Fourier analysis of entire functions of exponential type and the non-stationary phase method.

Keywords: Dirac wavefunction, free time evolution, causality, zitterbewegung

1 Introduction

To start some common notations are fixed. For \( x, y \in \mathbb{R}^3 \) let \( xy := \sum_{k=1}^{3} x_k y_k \) and \( x^2 = xx, |x| = \sqrt{x^2} \). \( 1_\Delta \) is the indicator function of \( \Delta \subset \mathbb{R}^3 \). Let \( B_r := \{ x \in \mathbb{R}^3 : |x| \leq r \} \) be the ball with center 0 and radius \( r > 0 \). \( \mathcal{F} \) denotes the unitary Fourier transformation on \( L^2(\mathbb{R}^d, \mathbb{C}^n) \). For open \( U \subset \mathbb{R}^d, \mathcal{C}_\infty^c(U, \mathbb{C}^n) \) is the space of all infinitely differentiable functions on \( \mathbb{R}^d \) in \( \mathbb{C}^n \) with compact support in \( U \). Moreover, throughout the units are such that \( c = 1, \hbar = 1 \). \( m > 0 \) denotes a mass.

The free Dirac operator in position representation is

\[
H = \sum_{k=1}^{3} \alpha_k \frac{1}{i} \partial_k + \beta m
\]

acting in \( L^2(\mathbb{R}^3, \mathbb{C}^4) \). It determines the time evolution of the Dirac wavefunctions. If \( \psi \in L^2(\mathbb{R}^3, \mathbb{C}^4) \) is the wavefunction at time 0, then the wavefunction at time \( t \) is \( \psi_t = e^{-itH} \psi \). As known, the Dirac time evolution is causal. This means, if \( \psi \) is localized in the region (measurable subset) \( \Delta \subset \mathbb{R}^3 \), i.e., \( 1_\Delta \psi = \psi \) a.e., then \( \psi_t \) is localized in the region of influence

\[
\Delta_t := \{ y \in \mathbb{R}^3 : \exists x \in \Delta \text{ with } |y - x| \leq |t| \}
\]

which is the set of points reached from \( \Delta \) within time \( |t| \) with velocity of light. Usually, as in [1, Remark on Theorem 1.2], this is inferred from the fact that the propagator \( S \), satisfying

\[
\psi_t(x) = \int_{\mathbb{R}^3} S(t, x-y) \psi(y) \, d^3 y
\]

for Schwartz functions \( \psi \), is \( S(t, x) = (-\partial_t + H) \Delta(t, x) \), where the Pauli-Jordan commutator distribution \( \Delta(t, x) = \frac{\text{sgn}(t)}{2\pi} \delta(t^2 - x^2) - \frac{m \text{sgn}(t)}{4\pi} \Theta(t^2 - x^2) \frac{J_1 \left( m \sqrt{t^2 - x^2} \right)}{\sqrt{t^2 - x^2}} \) depends only on \( \text{sgn}(t) \) and \( t^2 - x^2 \) and vanishes...
if \( t^2 - x^2 < 0 \) \([1, \text{(1.86)}]\), \([2, \text{Appendix E}]\). Here, we like to cite \([3, \text{Theorem 10(b)}]\), which infers causal time evolution from the fundamental fact that the entire matrix-valued function \( z \mapsto e^{i \theta h(z)} \) on \( \mathbb{C}^3 \)

\[
h(z) := \sum_{k=1}^{3} \alpha_k z_k + \beta m
\]

is exponentially bounded.

Causality together with homogeneity of time gives a first idea of how Dirac wavefunctions propagate in space. The spreading to infinity all over the space is limited by the velocity of light. However causality implies also the non-superluminal shrinking of wavefunctions as the following simple consideration tells.

Let \( R > 0 \) and let \( \psi \) be a wavefunction localized in the ball \( B_R \). Let \( t_0 := R \). Then, due to causality, \( \psi_{t_0} \) is localized in \( B_{2R} \), but this does not exclude that actually \( \psi_{t_0} \) is localized in a smaller ball \( B_r \).

Indeed, here every \( r > 0 \) occurs: Choose \( \rho \in [0, R] \) and let the wavefunction \( \chi \neq 0 \) be localized in \( B_\rho \), then \( \psi := \chi_{\rho - R} \) is localized in \( B_{R} \), and \( \psi_{t_0} = \chi_{t_0 + \rho - R} = \chi_{\rho} \) is localized in \( B_{2\rho} \).

Exploiting further \((1.3)\) we study in sec. 2 in detail the free motion of the border of a bounded localized (i.e., localized in a bounded region) wavefunction. One finds that at any time the border moves at the speed of light. As expected, over the long term it moves isotropically to infinity. In the short term however the movement of the wave border is more complicated as parts of it may move in the opposite direction. The intriguing temporal behavior of the border is shown in \((4)\).

Another aspect of time evolution is the long-term behavior of the position probability density \( |\psi_t(x)|^2 \) across the carrier. In sec. 3 it is shown that in the past as in the future the probability of localization concentrates up to 1 in the spherical shell \( B_{|t|} \setminus B_r \) for every radius \( r > 0 \). In conclusion the so-called asymptotic causality is briefly discussed.

Obviously these findings about the propagation of a free Dirac wavefunction concern the basics of relativistic quantum theory, which we imagine could find their way into the textbooks.

In the sections 2 and 3 the results are presented. Their proofs are postponed to sec. 4.

## 2 Motion of the border of the wave function

**1) Definition.** Let \( e \in \mathbb{R}^3 \) be a unit vector and \( \alpha \in [-\infty, \infty] \). They determine the half-space \( \{x \in \mathbb{R}^3 : xe \leq \alpha\} \) (which equals \( \emptyset \) or \( \mathbb{R}^3 \) if \( \alpha = -\infty \) or \( \alpha = \infty \)). For every \( \psi \in L^2(\mathbb{R}^3, \mathbb{C}^4) \setminus \{0\} \) let \( e(\psi) \in [-\infty, \infty] \) denote the maximal \( \alpha \) satisfying \( 1_{\{x \in \mathbb{R}^3 : xe \leq \alpha\}} \psi = 0 \) a.e. Put \( \overline{e} := -e \).

The meaning of \( e(\psi) \) is best elucidated by

**2) Lemma.** \( \psi \neq 0 \) is localized in \( \{x \in \mathbb{R}^3 : e(\psi) \leq xe \leq -\overline{e}(\psi)\} \) with \( |e(\psi), -\overline{e}(\psi)| \) the smallest interval with this property. In particular, \( -\overline{e}(\psi) - e(\psi) > 0 \) is the width of the carrier of \( \psi \) in direction \( e \).

\[
x e \leq e(\psi) \\
\text{carrier of } \psi \\
e \\
\overline{e} \\
x e \geq -\overline{e}(\psi)
\]
In the following we are interested in the temporal behavior of \( e(\psi) \), i.e., in the functions \( \mathbb{R} \rightarrow \mathbb{R}, t \mapsto e(\psi_t) \).

(3) **Theorem.** Let \( \psi \neq 0 \) be a Dirac wavefunction localized in a bounded region. Then

\[
e(\psi_t) \leq -2 \varphi(\psi) - e(\psi) - |t|
\]

holds for all directions \( e \) and all times \( t \). If \( \psi \in \text{dom}(H) \) or if more generally \( hF\psi \) is bounded on \( \mathbb{R}^3 \) then the inequality holds even with \( < \) in place of \( \leq \).

For the proof of (4), Theorem (3) turns out to be decisive as it shows that \( t \mapsto e(\psi_t) \) is bounded above. Also, together with (4) it implies the important estimations in (5). Afterwards the bound \(-2 \varphi(\psi) - e(\psi)\) in (3) can be improved to \(-2 \varphi(\psi) - e(\psi) - |t|\) by (5)(a),(b) and (4).

(4) **Theorem.** Let \( \psi \neq 0 \) be a Dirac wave function localized in a bounded region. Then there exists a unique time \( t_e = t_e(\psi) \in \mathbb{R} \) such that

\[
e(\psi_t) = e(\psi) + |t_e| - |t - t_e|
\]

for all times \( t \in \mathbb{R} \) and directions \( e \). In particular \( e(\psi_{t_e}) = e(\psi) + |t_e| \) is the maximum of \( e(\psi_t) \).

As long as \( t < t_e \), one has \( e(\psi_t) = e(\psi_{t_e}) - t_e + t \) by (4), which means the retreat at the speed of light of the carrier of \( \psi_t \) in direction \( e \). Only after time \( t_e \) the carrier advances in direction \( -e \) at the speed of light as \( e(\psi_t) = e(\psi_{t_e}) + t_e - t \). Only then the wavefunction expands in the direction \( -e \) as expected. The abrupt change at the time \( t_e \) of the direction of the motion with light velocity to the opposite direction reminds of the phenomenon of the zitterbewegung. But this behavior is not paradoxical and is easy to understand. Let \( \psi' := \psi_{t_e} \). Then by (5)(d), which is due to homogeneity of time, i.e., the translational symmetry of time evolution, \( \psi' \) satisfies \( e(\psi'_t) = e(\psi') - |t| \) according to (4). So, as maximal permissible by causality, \( \psi' \) expands in the future as well as in the past in direction \( -e \) at the speed of light. In particular the result in (4) does not single out some direction of time. On the contrary, the reversal of motion is required by time reversal symmetry. Nevertheless in the short term the picture is complicated as the time of change \( t_e \) depends in general on the direction \( e \) (see (7)). Therefore the carrier of the wavefunction performs the changes from shrinking to expanding not isotropically. According to (5)(a), in every direction \( e \) the retreat equals at most the width of the carrier. But after, respectively before, the time corresponding to the diameter of the carrier a simultaneous isotropic expansion of the wavefunction with light velocity takes
place in the future respectively in the past (see (6)). So the trembling is limited in time and space in the order of the diameter of the carrier at its minimal extension.

(5) **Corollary.** Let \( \psi \neq 0 \) be a Dirac wavefunction localized in a bounded region. Then

(a) \( e(\psi_t) \leq -\tau(\psi) - |t - t_\sigma| \)
(b) \( |t_\sigma| + |\tau(\psi) - e(\psi)| \)
(c) \( 2|t_\sigma| < -\tau(\psi) - e(\psi) \) if \( t_\sigma = t_\tau \)
(d) \( t_\sigma(\psi_t) = t_\sigma(\psi) - t \) for \( t \in \mathbb{R} \)
(e) If \( h\mathcal{F}\psi \) is bounded on \( \mathbb{R}^3 \) then the inequalities in (a), (b) hold even with \( < \) in place of \( \leq \).

(6) **Corollary.** Let \( \psi \neq 0 \) be a Dirac wavefunction localized in \( B_R \) for some \( R > 0 \). Then

\[
e(\psi_t) = e(\psi_{2R}) + 2R - t \quad \forall t \geq 2R, \quad e(\psi_t) = e(\psi_{-2R}) + 2R + t \quad \forall t \leq -2R
\]

This section is concluded by some existence proofs for bounded localized wavefunctions \( \psi \neq 0 \) regarding the data \( e(\psi), t_\sigma, \tau(\psi), t_\tau \). Fix a direction \( e \).

(7) **Lemma.** Let \( t_1, t_2 \in \mathbb{R} \). Then there is a bounded localized wavefunction \( \psi \neq 0 \) with \( t_\sigma = t_1 \) and \( t_\tau = t_2 \).

This means that the shrinking-expanding point may take place in direction \( e \) and opposite direction \(-e\) at different times causing the trembling of the wavefunction described above.

(8) **Lemma.** For every \( a, b \in \mathbb{R} \) with \( a < b \) and \( |\tau| < \frac{1}{2}(b - a) \) there is a bounded localized wavefunction \( \psi \neq 0 \) such that \( a \leq e(\psi) < -\tau(\psi) \leq b \) and \( t_\sigma = t_\tau = \tau \).

So the estimation given in (5) cannot be improved.

(9) **Lemma.** Let \( a, b \in \mathbb{R} \) with \( a < b \). Let \( \varsigma \in \{+,-\} \). Then there is a bounded localized wavefunction \( \psi \neq 0 \) such that \( a \leq e(\psi) < -\tau(\psi) \leq b \) and \( \varsigma t_\tau \geq \frac{1}{2}( -\tau(\psi) - e(\psi)) \).

One verifies explicitly that it is not possible to obtain such a wavefunction \( \psi \) by the construction used for (7) (except in the trivial case that the initial state \( \psi^{(1)} \) already satisfies this condition). We do not know whether there exists a bounded localized wavefunction \( \psi \) such that \( |t_\tau| > \frac{1}{2}( -\tau(\psi) - e(\psi)) \).

Applying a spatial translation (see (4.7)) on \( \psi \) in (9) one achieves \( e(\psi) = 0, 2\varsigma t_\tau \geq -\tau(\psi) \). Wavefunctions satisfying \( e(\psi) \geq 0 \), \( 2\varsigma t_\tau \geq -\tau(\psi) \) are particularly interesting. We call them late-change states. They are thoroughly studied in [4]. Using properties of these wavefunctions, in a forthcoming contribution the phenomenon of Lorentz contraction is shown, i.e., for every \( \delta > 0 \)
\[
\|1_{\{x \in \mathbb{R}^3 : -\delta \leq x_e \leq \delta\}} \psi^{\rho e}\| \rightarrow 1, \quad |\rho| \rightarrow \infty
\]
holds, where \( \psi^{\rho e} \) denotes the Dirac wavefunction \( \psi \) boosted in direction \( e \) with rapidity \( \rho \in \mathbb{R} \).
3 Long-term behavior of the probability of localization

Let \( \psi \) be a normalized Dirac wave function. There are good reasons to interpret \( |\psi(x)|^2 \) as a position probability density of the system. (See [3] and the literature cited therein for entering into the details.) Then obviously the probability of localization within the carrier of the wave function evolving in time is constantly 1. Insomuch the foregoing results on the movement of the border of the wave function yield also an information about the time dependence of the probability of localization. However the probability stays not equally distributed across the carrier. It turns out that for every Dirac wavefunction (not necessarily bounded localized) in the long term the probability of localization concentrates up to 1 in the spherical shell \( B_{3r} \setminus B_r \) for every radius \( r > 0 \). More precisely one has the results in (10), (11).

(10) Theorem. Let \( \psi \) be a Dirac wavefunction. Let \( \varepsilon > 0 \). Then there are \( v \in [0, 1] \) and \( \tau > 0 \) such that \( \|1_{B_{3r}} \psi_t\| \leq \varepsilon \) for all \( |t| \geq \tau \). In particular

\[
1_{B_r} \psi_t \rightarrow 0, \quad |t| \rightarrow \infty
\]

holds for every radius \( r > 0 \).

(11) Theorem. Let \( \psi \) be a Dirac wavefunction. Then

\[
1_{\mathbb{R}^3 \setminus B_1} \psi_t \rightarrow 0, \quad |t| \rightarrow \infty
\]

If \( \mathcal{F}\psi \in C^\infty_c(\mathbb{R}^3, \mathbb{C}^4) \) holds, then for every \( N > 0 \) there is a finite constant \( C_N \) such that \( \|1_{\mathbb{R}^3 \setminus B_1} \psi_t\| \leq C_N(1 + |t|)^{-N} \) for all \( t \in \mathbb{R} \).

(12) Asymptotic causality. Actually both results (10), (11) are valid also for every massive system and antisystem \([m, j, \eta] (m > 0, \text{spin } j \in \mathbb{Z}/2, \eta = +, -)\) if endowed with the Newton-Wigner localization \( E_{\text{NW}} \) [7], although with respect to \( E_{\text{NW}} \) time evolution is not causal. (The latter fact is known for a long time [8] and studied in detail in [9].) So one has \( E_{\text{NW}}(B_r) \psi_t \rightarrow 0, \quad |t| \rightarrow \infty \) for every \( r > 0 \) and the asymptotic causality

\[
E_{\text{NW}}(\mathbb{R}^3 \setminus B_1) \psi_t \rightarrow 0, \quad |t| \rightarrow \infty
\]

(3.1)

Indeed, these results hold true since the evolution of a state \( \psi \) in Newton-Wigner position representation is \( \psi_t(x) = (2\pi)^{-3/2} \int e^{i(px + t\mathcal{F}\psi(p))} \mathcal{F}\psi(p) \, d^3 p \), which for every spinor component equals (4.10).

The asymptotic causality of Newton-Wigner localization is shown in [9, Proposition]. In [9] it is also pointed out that (3.1) is false for the massless system \([0, 0, \eta]\). But the failure of (3.1) must not mean at all an acausal behavior. Indeed, although radially symmetric Weyl wavefunctions satisfy \( \lim_{|t| \rightarrow \infty} \|1_{\mathbb{R}^3 \setminus B_1} \psi_t\| \geq 1/2 \), the Weyl systems are causal [4, (99)(b), (95)].

4 Proofs

Proof of (2) Lemma. Note that \( \{x \in \mathbb{R}^3 : x = \gamma\} \) is a Lebesgue null set. By definition \( 1_{\{x \in \mathbb{R}^3 : x \leq \alpha\}} \psi = 0, 1_{\{x \in \mathbb{R}^3 : x \leq \beta\}} \psi = 0 \) exactly for all \( \alpha \leq \varepsilon(\psi) \) and \( \beta \leq \varepsilon(\psi) \). From this it follows \( 1_{\{x \in \mathbb{R}^3 : x \leq -\varepsilon(\psi)\}} \psi = 0 \) and \( 1_{\{x \in \mathbb{R}^3 : x \leq \varepsilon(\psi)\}} \psi = 0 \).
ψ, whence the assertion.

The proof of (3) needs some preparation.

Referring to (1) define $c(\eta)$ for $\eta \in L^2(\mathbb{R}^d, \mathbb{C})$ quite analogously. Obviously $e(\psi) = \min_l e(\psi_l)$. — Recall the support function $H_C$ for a convex set $C \subset \mathbb{R}^d$ given by

$$H_C(\lambda) = \sup\{x\lambda : x \in C\}, \lambda \in \mathbb{R}^d$$

(4.1)

Let $C(\psi)$ denote the smallest convex set outside which $\psi$ vanishes almost everywhere. Clearly, $\{x \in \mathbb{R}^3 : x e \leq e(\psi)\} \cap C(\psi) = \emptyset$ and $\{x \in \mathbb{R}^3 : x e \leq \alpha\} \cap C(\psi)$ is not a null set if $\alpha > e(\psi)$. Hence $e(\psi) = \inf\{xe : x \in C(\psi)\}$. These considerations are applicable as well to every component $\psi_l$ of $\psi$. Therefore

$$e(\psi) = -H_{C(\psi)}(-e), \quad e(\psi_l) = -H_{C(\psi_l)}(-e)$$

(4.2)

The P-indicator (i.e., the Pólya-Plancherel indicator) $h_f$ of an entire function $f$ on $\mathbb{C}^d$ is

$$h_f(\lambda) = \sup\{h_f(\lambda, x) : x \in \mathbb{R}^d\}, \lambda \in \mathbb{R}^d \text{ with } h_f(\lambda, x) = \lim_{r \to \infty} \frac{1}{r} \ln |f(x + i\lambda r)|$$

(4.3)

An entire matrix-valued function $f$ on $\mathbb{C}^d$ is called exponentially bounded or of exponential type with exponent $\delta \geq 0$ if there is a finite constant $C_\delta$ such that $\|f(z)\| \leq C_\delta e^{\delta|z|}$ with $|z|^2 = \sum_{j=1}^d |z_j|^2$ for $z \in \mathbb{C}^d$. The type $\tau$ of $f$ is the infimum of all its exponents.

The main mathematical tool for the proof of (3) is (13) (see [5], [6]).

(13) Theorem of Plancherel and Pólya. A function $f : \mathbb{C}^d \to \mathbb{C}$ is entire and exponentially bounded with $f|_{\mathbb{R}^d} \in L^2$ if and only if there is $g \in L^2(\mathbb{R}^d)$ vanishing outside a bounded set with

$$f(z) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-iqz} g(q) \, dq$$

i.e., $f$ is the Fourier-Laplace transform of $g$. Then

$$h_f = H_C(g)$$

where $C(g)$ is the smallest convex set outside of which $g$ vanishes almost everywhere.

Moreover, $f|_{\mathbb{R}^d}$ is bounded by $(2\pi)^{-d/2} \int_{\mathbb{R}^d} |g(q)| \, dq$ and, by the Riemann-Lebesgue lemma, it vanishes at infinity. Also, for each $\lambda$ one has $h_f(\lambda, x) = h_f(\lambda)$ for almost all $x$, and $h_{k_f} = h_k + h_f$ holds for any exponentially bounded entire function $k$.

Let $\psi$ be a wavefunction localized in $B_R$. The Fourier-Laplace transform of $\psi_l$ localized in $B_{R+|t|}$ is an entire function $\varphi_l$, which is exponentially bounded with exponent $R + |t|$, i.e., $|\varphi_l(z)| \leq C e^{(R+|t|)|z|}$, $z \in \mathbb{C}^3$. Recall (1.3). Due to $h(z)^2 = (z_1^2 + z_2^2 + z_3^2 + m^2)I_4$, the time evolution yields

$$\varphi_t(z) = e^{iht(z)} \varphi(z) = \cos (t\varepsilon(z)) \varphi(z) + it \sin (t\varepsilon(z)) h(z) \varphi(z)$$

(4.4)
for every $z \in \mathbb{C}^3$. Here $\epsilon$ satisfies $\epsilon(z)^2 = z_1^2 + z_2^2 + z_3^2 + m^2$, and $\text{sinc}(w) = \sin(w)/w$ for $w \neq 0$, $\text{sinc}(0) = 1$. From (4.4) one obtains

$$\varphi_t + \varphi_{-t} = 2 \cos(te) \varphi, \quad \varphi_t - \varphi_{-t} = 2it \text{sinc}(te) h \varphi$$

and $(\varphi_t)k \varphi_t = \cos(te)\varphi_k \varphi_t + it \text{sinc}(te)(h \varphi)_k \varphi_t$ and hence $\phi_{kl} = it \text{sinc}(te)\chi_{kl}$ for $\phi_{kl} := (\varphi_t)_k \varphi_t - (\varphi_{-t})_k \varphi_t$ and $\chi_{kl} := (h \varphi)_k \varphi_t - (h \varphi)_k \varphi_t$, where $k, l = 1, \ldots, 4$ enumerate the spinor components.

There are $k, l$ such that $\chi_{kl} \neq 0$. (Indeed, assume $\chi_{kl} = 0$ for all $k, l$. Then $\varphi_k h \varphi = (h \varphi)_k \varphi$ and hence $\epsilon \varphi_k \varphi = (h \varphi)_k \varphi$. Then $(h \varphi)_k^2 - \epsilon \varphi_k^2 = 0$. Fix $k, z_2, z_3$ such that $f(z) := \varphi_k(z, z_2, z_3)$ is not the null function. Set $g(z) := (h \varphi)_k(z, z_2, z_3)$. Choose the square root $\mu(z, \phi)$ of $\epsilon \varphi_k^2 + z_2^2 + z_3^2$ such that $g = \mu f$. This, however, is impossible as $g/f$ is meromorphic whereas $\mu$ is not.) Thus $\chi := \chi_{kl}, \phi := \phi_{kl}$ are non-zero entire exponentially bounded functions with exponents $2R$ and $2R + |t|$, respectively, satisfying

$$\phi = it \text{sinc}(te) \chi$$

We are going to exploit the relations (4.5) and (4.6). The following elementary estimations are used to compute the P-indicator for $\cos(te)$ and $\text{sinc}(te)$ in (15).

(14) Lemma. Let $\mu, t, u, v$ be real, $\mu \geq 0$. Then there are finite constants $A_t > 0, B_t, C_t$ independent of $u, v$ such that

(a) $A_t e^{tv} \leq \left| \cos \left( t \sqrt{\mu^2 + (u + i v)^2} \right) \right| \leq B_t e^{tv}$

(b) $A_t |u + iv|^{-1} e^{tv} \leq \left| \text{sinc} \left( t \sqrt{\mu^2 + (u + i v)^2} \right) \right| \leq B_t |u + iv|^{-1} e^{tv}$

for all $u$ and $|v| > C_t$.

Proof. First we show

$$z \in \mathbb{C}, |z| \leq \frac{1}{2} \Rightarrow \sqrt{1 + z} = 1 + \zeta \text{ with } |\zeta| \leq \frac{3}{4} |z| \quad (\star)$$

Indeed, let $f : [0, 1] \to \mathbb{C}, f(r) := (1 + rz)^{1/2}$. As $f'(r) = \frac{1}{2}(1 + rz)^{-1/2} z$ one has $f(1) = 1 + \zeta$ with $\zeta := \int_0^1 f'(r) \, dr$ and $|\zeta| \leq \frac{1}{2} (1 - \frac{1}{2})^{-1/2} |z|$, whence $(\star)$.

Now assume at once $t \neq 0$ and let in the following $|v| > \sqrt{2}\mu$. Put $s(u, v) := \sqrt{\mu^2 + (u + iv)^2}$. More precisely, $s(u, v) := (u + iv) \sqrt{1 + z}$ for $z := \mu^2(u + iv)^2 - 1$ with $|z| = \mu^2 (u^2 + v^2)^{-1} \leq |z| \leq \frac{1}{2} \mu^2 v^{-2} \leq \frac{1}{2}$. By $(\star)$, $s(u, v) = (u + iv) (1 + \zeta)$ with $|u + iv| \zeta| \leq \frac{3}{4} |u + iv| \leq \frac{3}{4} |u + iv| \leq \mu v^{-1} \leq \mu$. This implies $|e^{\pm i ts(u, v)}| = |e^{\pm i t(u + iv) (1 + \zeta)}| = e^{tv} |e^{\pm i t(u + iv) \zeta}|$. Hence concludes

$$e^{-|t| \mu} e^{-tv} \leq |e^{i ts(u, v)}| \leq e^{|t| \mu} e^{-tv}, \quad e^{-|t| \mu} e^{tv} \leq |e^{-i ts(u, v)}| \leq e^{|t| \mu} e^{tv} \quad (\star\star)$$

for all $v \in \mathbb{R}$ with $|v| > \sqrt{2}\mu$.

(a) $|\cos(w)| = \frac{1}{2} |e^{i w} + e^{-i w}| \leq \frac{1}{2} (|e^{i w}| + |e^{-i w}|)$. For $w = ts(u, v)$ this yields by $(\star\star) |\cos(ts(u, v))| \leq \frac{1}{2} (e^{t|\mu|} e^{-tv} + e^{t|\mu|} e^{tv}) \leq e^{t|\mu|} e^{tv}$. Hence the right part of the inequality of (a) holds for $B_t := e^{t|\mu|}$ and $C_t := \sqrt{2} \mu$.

For the left part of the inequality use $|\cos(w)| = \frac{1}{2} |e^{i w} + e^{-i w}| \geq \frac{1}{2} |e^{i w} - |e^{-i w}||$. Then for $w = ts(u, v)$ one gets by $(\star\star) |\cos(ts(u, v))| \geq \frac{1}{2} (e^{-|t| \mu} e^{-tv} - e^{t|\mu|} e^{-tv}) = \text{sins} (|t|(|v| - \mu))$. Check $\sinh(x) \geq \frac{1}{2} e^x$ for $x \geq \frac{\ln(2)}{2}$. Thus we conclude that the left part of the inequality holds for $A_t := \frac{1}{4} e^{-|t| \mu}$.
and $C'' := \sqrt{2} \mu + \frac{\ln(2)}{2} \mu$.

(b) Check first $|t \sin(u, v)| \geq \frac{1}{\sqrt{2}} |u + i v|$, using $|\sqrt{1 + z}| \geq \frac{1}{\sqrt{2}}$ for $|z| \leq \frac{1}{2}$. Furthermore, $|\sin(w)| = \frac{1}{2} |e^{i w} - e^{-i w}| \leq \frac{1}{2} (|e^{i w}| + |e^{-i w}|)$. Hence, as for (a), the right part of the inequality holds for $B_t := \frac{1}{|t|} e^{\sqrt{2} \mu}$ and $C'_t := \sqrt{2} \mu$.

Regarding the left part of the inequality of (b), we estimate $|t \sin(u, v)|^{-1} \geq \left( \frac{2}{3} \right)^{1/2} |t| |u + i v|^{-1}$, as $|\sin(u, v)| = |u + i v| |\sqrt{1 + z}| \leq \left( \frac{2}{3} \right)^{1/2} |u + i v|$. Furthermore, one has $|\sin(w)| = \frac{1}{2} |e^{i w} - e^{-i w}| \geq \frac{1}{2} |e^{i w}| - |e^{-i w}|$. Hence, proceeding as in (a), it follows that the left part of the inequality holds for $A_t := \left( \frac{1}{|t|} \right)^{1/2} \frac{1}{|t|} e^{-\sqrt{2} \mu}$ and $C''_t := \sqrt{2} \mu + \frac{\ln(2)}{2} \mu$.

(15) Lemma. For $t \in \mathbb{R}$ the functions $z \mapsto \cos(te(z))$ and $z \mapsto \sin(te(z))$ are bounded on $\mathbb{R}^3$ and entire on $\mathbb{C}$ with exponent $|t|$, which is minimal. Moreover, $h_{\cos(te)}(\lambda) = h_{\sin(te)}(\lambda) = |t| |\lambda|$ holds for $\lambda \in \mathbb{R}^3$. More precisely one has $|t| |\lambda| = \lim_{r \to \infty} \frac{1}{|t|} \ln |f(p + i \lambda r)|, p \in \mathbb{R}^3$ for $f \in \{\cos(te), \sin(te)\}$.

Proof. We show the assertion for $\sin(te)$. Regarding $\cos(te)$ the proof is analogous. Assume at once $t \neq 0$.

Obviously, $\sin(te)$ is bounded on $\mathbb{R}^3$ and entire on $\mathbb{C}$. Also, there is an entire function $s$ satisfying $s'(z) = \sin(te(z))$ with $z^2 = z_1^2 + z_2^2 + z_3^2$ for all $z \in \mathbb{C}^3$. Now $|e^{\pm i z}| = |e^{\pm i z^2}| \leq |z^2| + m^2 = |z|^2 + m^2 \geq (|z|^2 + m^2)^{1/2} \geq m$. Hence, $|\sin(te(z))| \leq m \mu$. Therefore, $|\sin(e^{\pm i t})| = |e^{\pm i t}| |e^{\pm i t}| \leq |e^{\pm i t}| \leq e^{m \mu} |e^{\pm i t}|$.

If $|z|^2 \leq 2m^2 + 1$ then $|\sin(te(z))| = |s(z^2)| \leq C$ for some finite constant $C$. For $|z|^2 > 2m^2 + 1$ one has $|e^{\pm i z}| = |e^{\pm i z^2}| \geq |z|^2 - m^2 \geq m^2 + 1$, whence $|e^{\pm i t}| \leq 1$. Hence $|\sin(te(z))| \leq C' e^{m |\lambda|}$ for all $z$, where $C' := C + e^{m \mu}$. So $|t|$ is an exponent for $\sin(te)$.

In order to show that $|t|$ is minimal assume that $0 \leq \delta < |t|$ is an exponent for $\sin(te)$. Let $\delta < \delta' < |t|$. Then obviously $|\sin(te(z))| \leq C e^{\delta |\lambda|}, z \in \mathbb{C}^3$ for some finite constant $C$. Let $w \in \mathbb{C}$. Choose $e \in \mathbb{C}$ with $e^2 = w^2 - m^2$. Then $w \in \{ \pm e(0, 0, \zeta) \}$ and $|\zeta| \leq |w| + m$. Hence $|\sin(tw)| \leq C e^{\delta |\lambda|} \leq C' e^{\delta |w|}$ with $C' := C' e^{\delta |\lambda|}$. Therefore also $|\cos(tw)| = |\sin(tw + \frac{\pi}{2})| \leq C' e^{\frac{3}{2} \delta |\lambda|}$, whence finally $|e^{tw}| \leq C'' e^{\delta |w|}, w \in \mathbb{C}$ for some finite constant $C''$. This implies the contradiction $e^{(\delta' - \delta)r} \leq C''$ for all $r > 0$.

We turn to the P-indicator of $\sin(te)$. Assume at once $\lambda \neq 0$. Then $e^{(p + i \lambda)r} = (\mu^2 + (\frac{p^2}{\lambda^2} + (\frac{p^2}{\lambda^2})^2 \geq 0$ independent of $r$. Hence by (14) there are finite constants $A_t > 0, B_t$ independent of $r$ that such $A_t |\frac{p^2}{\lambda^2} + i |\lambda| r |^{-1} e^{(\delta' - \delta)|r|} \leq \sin(te(p + i \lambda r)) \leq B_t |\frac{p^2}{\lambda^2} + i |\lambda| r |^{-1} e^{(\delta' - \delta)|r|}$, whence the assertion.

Proof of (3) Theorem. Start from (4.6) $\phi = it \sin(te) \lambda$. Put here $\phi_{kl} := (\varphi_k)_k \varphi_l, \lambda_{kl} := (h\varphi)_k \varphi_l$, whence $\phi = \phi_{kl} - \phi_{lk}$ and $\lambda = \lambda_{kl} - \lambda_{lk}$.

As $\varphi_{kl} \in \mathbb{L}^2$ and $(\varphi_k)_k \in \mathbb{L}^2$ so that (13) applies to $\phi_{kl}$. Let $\theta := F^{-1}\phi_{kl}, \theta_{kl} := F^{-1}\phi_{kl}$. Obviously, $\mathcal{E}(\theta) = \min(\mathcal{E}(\theta_{kl}), \mathcal{E}(\theta_{lk}))$. Using (4.2) one gets $\mathcal{E}(\theta_{kl}) = -H C_{(\psi_k, \psi_l)}(-e) = -h_{\phi_{kl}}(-e) - h_{\phi_{kl}}(-e) - H C_{(\psi_k, \psi_l)}(-e) = -e(\psi_k) + e(\psi_l) \geq e(\psi_l) + e(\psi_l)$. It follows $\mathcal{E}(\theta) \geq e(\psi_l) + e(\psi_l)$.

We turn to the right hand side $it \sin(te) \lambda$ of (4.6). Recall $it \sin(te) \lambda_{kl} = \phi_{kl} - \cos(te) \varphi_k \varphi_l$ by (4.4). Note that $\cos(te) \varphi_k \varphi_l$ is bounded. Hence $\sin(te) \lambda_{kl} \in \mathbb{L}^2$. However, $\lambda_{kl} \in \mathbb{L}^2$ need not be square-integrable. Therefore we consider instead $\lambda_{kl} := s_{kl} \lambda_{kl}$ with $s_k := \sin(te)$ for $\delta > 0$. Then $\phi' = it \sin(te) \lambda'$ for $\phi' := s_{kl} \phi'$. As $h_{s_k}(e) = \delta$ by (15), the analogous computation for
\[ \theta' := \mathcal{F}^{-1} \phi' |_{R^3} \] in place of \( \theta \) yields \( e(\theta') \geq -\delta + e(\psi_t) + e(\psi) \). Moreover, \( (13) \) applies to \( \chi' \). Let \( \xi' := \mathcal{F}^{-1} \chi' |_{R^3} \). Then again, in the same way \( e(\theta') = -|t| + e(\xi') \) follows.

Next we examine \( -\nabla(\xi') \). Obviously \( -\nabla(\xi') \leq \max(-\nabla(\xi_t), -\nabla(\xi_t)) \). By \( (4.2) \) and \( (13) \) one has \( -\nabla(\xi_t) = H_{C(\xi_t)}(e) = h_{\xi_t}(e) = h_{s_{e}(\varphi)}(e) + h_{\varphi}(e) \), as \( s_{e}(\varphi) \) is exponentially bounded. Note \( |(h_{\varphi})_{k}(z)| \leq q(z) \max_{m} |\varphi_{m}(z)| \) with \( q(z)^2 := 4 \sum_{m=1} \|h(z)_{km}\|^2 \), where \( h(z)_{km} \) is linear. Therefore \( h_{s_{e}(\varphi)}(e, x) = \lim_{t_{k} \to \infty} \frac{1}{t_{k}} \{ - \ln |s_{e}(z_{k} + i e r)| + \ln |(h_{\varphi})_{k}(z_{k} + i e r)| \} = \delta + \lim_{t \to \infty} \frac{1}{t_{k}} \ln |(h_{\varphi})_{k}(z_{k} + i e r)| \)
\( (15) \leq \delta + \lim_{t \to \infty} \frac{1}{t_{k}} \{ - \ln |q(x + i e r)| + \ln(\max_{m} |\varphi_{m}(z_{k} + i e r)|) \} \) \( (\xi_t) = \delta + \max \max_{e} |\varphi_{m}(e, x)| \). Furthermore, \( \max_{e} h_{\varphi}(e) = \max_{e} H_{C(\psi_{m})}(e) = \max_{e} \{ -\nabla(\psi_{m}) \} = -\nabla(\varphi) \). Also \( h_{\varphi}(e) \leq -\nabla(\varphi) \). It follows \( -\nabla(\xi') \leq \delta - 2 \nabla(\varphi) \).

Now, using \( e(\xi') < -\nabla(\xi') \), one has the chain of inequalities \( -\delta + e(\psi_t) + e(\varphi) \leq e(\theta') = -|t| + e(\xi') < -|t| - \nabla(\xi') \leq -|t| + \delta - 2 \nabla(\varphi) \).

The next two lemmas serve for the proof of \( (4) \).

\[(16) \text{ Lemma. Let } \psi \neq 0 \text{ be a Dirac wave function localized in a bounded region. Then}
\]
\[ \min \{e(\psi_{t}), e(\psi_{t-})\} = e(\psi) - |t| \]
holds for every direction \( e \) and all times \( t \in \mathbb{R} \).

\[ \text{Proof. By causality } e(\psi_{t}) \geq e(\psi) - |t| \text{ for all } t, \text{ whence } \min \{e(\psi_{t}), e(\psi_{t-})\} \geq e(\psi) - |t|. \]

We prove now the reverse inequality. Recall \( \phi = 2 \cos(\theta) \varphi \) for \( \theta := \varphi_{t} + \varphi_{t} \) from \( (4.5) \). Let \( \theta := \mathcal{F}^{-1} \phi |_{R^3} \). Theorem \( (13) \) applies to the components of \( \varphi \) and, due to \( (15) \), also to those of \( \cos(\theta) \varphi \).

Hence, using \( (4.2) \) and \( (15) \), \( e(\theta) = -H_{C(\psi_{t})}(-e) = -H_{C(\psi_{t})}(e) = -H_{C(\psi_{t})}(e) = -|t| - \nabla(\psi_{t}) \) \( (\xi_t) = e(\psi_{t}) - |t| + e(\psi) \). Therefore \( e(\theta) = \min_{e} e(\theta_{t}) = -|t| + e(\psi) \).

It remains to show \( \alpha := \min \{e(\psi_{t}), e(\psi_{t-})\} \) \( \leq e(\psi + \psi_{t}) \). Put \( \chi_{0} := \min_{t \in \mathbb{R}: x \leq \alpha} \). Then \( \chi_{0} e(t) = 0 \)
and \( \chi_{0} e(t_{0}) = 0 \). Hence \( \chi_{0} e(t_{0}) = 0 \), whence the claim.

\[(17) \text{ Lemma. Let } \psi \text{ be a Dirac wavefunction. Then } \mathbb{R} \to \mathbb{R}, t \to e(\psi_{t}) \text{ is continuous.} \]

\[ \text{Proof. Let } t, t_{0} \in \mathbb{R} \text{. By causality (see \( (1.2) \)) \( e(\psi_{t}) \geq e(\psi_{t_{0}}) - |t - t_{0}| \). This implies } \lim_{t_{0} \to t} e(\psi_{t}) \geq e(\psi_{t_{0}}). \]

Furthermore, for \( \chi_t := 1_{t \leq t_{0}} \) one has \( 0 = \chi_{t} e(\psi_{t}) = \chi_{t} e(\psi_{t_{0} +} + \chi_{t_{0} +} - e(\psi_{t_{0}}), \text{ whence } \lim_{t_{0} \to t} \chi_{t} e(\psi_{t_{0} +}) = 0 \)
as \( \psi \to \psi_{t} \). This implies \( \lim_{t_{0} \to t} e(\psi_{t}) \leq e(\psi_{t_{0}}) \). Thus continuity of \( t \to e(\psi_{t}) \) at \( t_{0} \) holds.

\[ \text{Proof of } (4) \text{ Theorem. Since } t \to e(\psi_{t}) \text{ is continuous by } (17) \text{ and bounded above by } (3) \text{ there is } t_{e} \in \mathbb{R} \text{ with } e(\psi_{t_{e}}) = \sup_{t \in \mathbb{R}} e(\psi_{t}). \text{ Fix } t > 0. \]

Now we apply \( (16) \) to \( \psi' := \psi_{t_{e}} - t/2 \). Then \( \min \{e(\psi'), e(\psi_{t_{e}}')\} = e(\psi') - |t'| \) for all \( t' \in \mathbb{R} \). As \( e(\psi_{t_{e}}') = e(\psi_{t_{e}} - t/2) \) it follows \( e(\psi_{t_{e}} - t/2) = e(\psi_{t_{e}} - t/2) - t/2 \). For \( t/2 \) in place of \( t \) this reads \( e(\psi_{t_{e}} - t/2) = e(\psi_{t_{e}} - t/4) - t/4 \). Hence \( e(\psi_{t_{e}} - t) = e(\psi_{t_{e}} - t/4) - t/4 - t/4 \).

From this one obtains in the same way \( e(\psi_{t_{e}} - n) = e(\psi_{t_{e}} - t/4) - t/4 - t/4 \) and finally \( e(\psi_{t_{e}} - n) = e(\psi_{t_{e}} - t/4) - t/4 - t/4 \).

By continuity \( (17) \) the limit \( n \to \infty \) yields \( e(\psi_{t_{e}} - t) = e(\psi_{t_{e}} - t) \) \( - |t| \). — Analogously, applying \( (16) \) to \( \psi' := \psi_{t_{e}} + t/2 \) one obtains \( e(\psi_{t_{e}} + t/2) \).

Thus \( e(\psi_{e}) = e(\psi_{t_{e}}) - |t - t_{e}| \) holds for all \( t \in \mathbb{R} \). In particular \( e(\psi_{e}) = e(\psi_{t_{e}}) - |t_{e}| \), whence the
This implies (c).

Proof of (5) Corollary. (a) By (4) and (3) one has $e(\psi_t) = e(\psi) + t_{\tau_e} - |t - t_{\tau_e}| \leq -2\tau_e - e(\psi - |t|)$. For $t = t_{\tau_e}$ this yields $|t_{\tau_e}| \leq -\tau_e - e(\psi)$ and consequently $e(\psi_t) \leq -\tau_e - e(\psi - |t_{\tau_e}|)$.

(b) Let $s,t \in \mathbb{R}$ and consider $e(\psi_{t+s})$. One the one hand, by (4), $e(\psi_{t+s}) = e(\psi) + |t_{\tau_e} - |t_{\tau_e}| - |t - t_{\tau_e}| - |s|$. Hence $-2(\tau_e + e(\psi)) \geq 2|t_{\tau_e}| + 2|t_{\tau_e}| - |t - t_{\tau_e}| - |s| - |t - t_{\tau_e}| - |s| - |t + s - t_{\tau_e}|$. For $s = t_{\tau_e} - t$ this yields $-\tau_e - e(\psi) \geq |t_{\tau_e}| + |t_{\tau_e}| - |t - t_{\tau_e}|$. Then $|t_{\tau_e}| + |t_{\tau_e}| \leq -\tau_e - e(\psi)$ follows for $t = t_{\tau_e}$.

(c) By (2), $0 < -\tau_e - e(\psi_t)$ for all $t$. Hence (4) yields $|t_{\tau_e}| + |t_{\tau_e}| - |t - t_{\tau_e}| - |t - t_{\tau_e}| < -\tau_e - e(\psi)$. This implies (c).

(d) Let $\tau \in \mathbb{R}$, $\psi':= \psi + \tau$, and $t_{\tau_e} := t_{\tau_e}(\psi')$. Then $e(\psi_t) = e(\psi_t') + |t_{\tau_e}' - |t_{\tau_e}'| - |\tau - t_{\tau_e}|$ and $e(\psi_t') = e(\psi) + |t_{\tau_e} - |t_{\tau_e}| - |\tau - t_{\tau_e}|$. As $\psi_t = \psi_t(\tau)$ also $e(\psi_t) = e(\psi) + |t_{\tau_e} - |t_{\tau_e}| - |\tau - t_{\tau_e}|$ holds. Therefore $|t + \tau - t_{\tau_e} - |t_{\tau_e}'| = |t - t_{\tau_e} - |t_{\tau_e}'|\}$ for all $t$, whence $t_{\tau_e}' = t_{\tau_e} - \tau$.

(e) follows from the last part of (3).

Proof of (6) Corollary. By (4) one has $e(\psi_t) = e(\psi_{t_{\tau_e}}) - |t - t_{\tau_e}|$. For $t \geq 2R \geq |t_{\tau_e}|$ by (5)(b) it follows $e(\psi_t) = e(\psi_{t_{\tau_e}}) + t_{\tau_e} - t$ and in particular $e(\psi_{2R} = e(\psi_{t_{\tau_e}}) + 2R - t_{\tau_e}$. Hence $e(\psi_t) = e(\psi_{t_{\tau_e}}) - t_{\tau_e} + t$ and in particular $e(\psi_{2R}) = e(\psi_{t_{\tau_e}}) - 2R$, whence $e(\psi_t) = e(\psi_{2R}) + 2R + t$.

The space translations $b \in \mathbb{R}^3$ act on the Dirac wavefunctions $\psi$ by $(W(b)\psi)(x) := \psi(x - b)$. For the following construction we use the easily verifiable formulae

$$e(W(\lambda e)\psi) = e(\psi) + \lambda, \quad t_{\tau_e}(W(\lambda e)\psi) = t_{\tau_e}(\psi)$$

(4.7)

for all directions $e$ and $\lambda \in \mathbb{R}$.

Proof of (7) Lemma. Let $\tau \in \mathbb{R} \setminus \{0\}$ and $\delta > 0$. Let $\psi^{(1)} \neq 0$ be any bounded localized wavefunction. Set $\psi^{(2)} := W(\delta e)\psi^{(1)}$ and put

$$\psi := \psi^{(1)} + \psi^{(2)}$$

In the following we express the characteristic dates $e(\psi), t_{\tau_e}, \tau(\psi), \tau(\psi)$ referring to $\psi$ by the input dates

$e(\psi^{(1)}), t_{\tau_e}^{(1)}, \tau(\psi^{(1)}), \tau(\psi)$

and the parameters $\tau, \delta$.

By (4), (5) and Eq. (4.7) one has $t_{\tau_e}^{(1)} = t_{\tau_e} - \tau$ and $t_{\tau_e}^{(1)} = t_{\tau_e}^{(1)} - \tau$, and $e(\psi^{(1)}(1)) = e(\psi^{(1)}(1)) + |t_{\tau_e}^{(1)} - |t - t_{\tau_e}^{(1)}|, e(\psi^{(2)}(1)) = e(\psi^{(1)}(1)) + |t_{\tau_e}^{(1)} - |t + \tau - t_{\tau_e}^{(1)}| + \delta$ and similarly $\tau(\psi^{(1)}(1)) = \tau(\psi^{(1)}(1)) + |t_{\tau_e}^{(1)} - |t - t_{\tau_e}^{(1)}|, \tau(\psi^{(2)}(1)) = \tau(\psi^{(1)}(1)) + |t_{\tau_e}^{(1)} - |t + \tau - t_{\tau_e}^{(1)}| - \delta$.

Obviously $e(\psi^{(1)}) = \{\psi^{(1)}(1)\}$ and $\tau(\psi^{(1)}) = \{\tau(\psi^{(1)}(1))\}$. Hence $t_{\tau_e}$ and $t_{\tau_e}$ are determined by (4). Write $e(\psi^{(2)}(1)) = d(t - t_{\tau_e}^{(1)}$) with $d(x) := |x| - |x + \tau + \delta| + \delta$ and $\tau(\psi^{(2)}(1)) = \tau(\psi^{(2)}(1)) = d(t - t_{\tau_e}^{(1)}$) with $d(x) := |x| - |x + \tau - \delta| - \delta$. Note

$$|\tau| \leq \delta \Leftrightarrow d(t - t_{\tau_e}^{(1)}) \geq 0 \forall t \leftrightarrow d(t - t_{\tau_e}^{(1)}) \leq 0 \forall t$$

(4.7)

Indeed, $d(t - t_{\tau_e}^{(1)}) \geq 0$ is equivalent to $|\tau| \leq \delta$ as $d$ takes its minimum $-|\tau| + \delta$ at $x = 0$. Similarly, $d$ takes its maximum $|\tau| - \delta$ at $x = -\tau$. 

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Now consider the case $|\tau| \leq \delta$. By (\textstar) one has $e(\psi_t) = e(\psi_t^{(1)})$, $\tau(\psi_t) = \tau(\psi_t^{(2)})$, whence $t_\tau = t_\tau^{(1)}$ and $t_e = t_e^{(2)} = t_e^{(1)} - \tau$. So one obtains the given value of $t_\tau - t_e$ by choosing $\tau = (t_\tau^{(1)} - t_e^{(1)}) - (t_\tau - t_e)$. By a subsequent time translation, according to (5)(d) one gets the prescribed values of $t_\tau$ and $t_e$.

The construction in (7) for $\tau := t_\tau^{(1)} - t_e^{(1)}$, $\delta := |\tau|$ with subsequent time translation by $t_\tau = t_\tau^{(1)}$ yields a Dirac wavefunction $\psi$ satisfying

$$t_e = t_\tau = 0 \quad \text{and} \quad -\tau(\psi) - e(\psi) = -\tau(\psi^{(1)}) - e(\psi^{(1)}) + |t_\tau^{(1)} - t_e^{(1)}| - |t_\tau^{(1)}| - |t_e^{(1)}| \tag{4.8}$$

The width of the carrier in direction $e$ in not increased since $|t_\tau^{(1)} - t_e^{(1)}| - |t_\tau^{(1)}| - |t_e^{(1)}| \leq 0$.

**Proof of (8) Lemma.** Due to Eq. (4.7) it is no restriction to assume $a = -b$. Let $0 < \rho < b$. By Eq. (4.8) there is a Dirac wavefunction $\eta$ localized in a bounded region contained in $\{-\rho \leq xe \leq \rho\}$ with $t_\tau(\eta) = t_\tau^{(1)} = 0$. Let $\varsigma$ denote the sign of $\tau$. Then, by causality, $\psi := \eta(-b + \rho)$ is localized in $\{-b \leq xe \leq b\}$. Moreover, $t_\tau = t_\tau^{(1)} = \varsigma(b - \rho)$ holds by (5)(d). The assertion follows for $\rho := b - |\tau|$.

Time reversal is represented by the antunitary operator $T \psi = \omega \overline{\psi}$ with $\omega := \text{diag}(\sigma_2, \sigma_2)$ in the Weyl representation. It satisfies $(T\psi)_t = T\psi_{t-}$. One has for all directions $e$

$$e(\psi) = e(T\psi) \quad \text{and} \quad t_\tau(T\psi) = -t_e(\psi) \tag{4.9}$$

Indeed, the first formula is obvious. Hence on the one hand $e((T\psi)_t) = e(\psi) + |t_\tau(T\psi)| + |t - t_\tau(T\psi)|$ and on the other hand $e((T\psi)_t) = e(\psi - i) = e(\psi) + |t_e(\psi)| + |t + t_e(\psi)|$ for all $t$, whence the second formula.

**Proof of (9) Lemma.** Because of Eq. (4.9) it suffices to prove the case $\varsigma = +$. By (8) there is a bounded localized wavefunction $\eta$ with $a \leq e(\eta) < -\tau(\eta) \leq b$ such that $v := t_\tau(\eta) > 0$ and $v < -\tau(\eta) - e(\eta)$. Let $0 < \delta < v$. Set $\psi := 1(\tau(\eta) - \delta \leq xe \leq \tau(\eta))\eta$, $\tau := t_\tau$ and define $\eta' := \eta - \psi$. Obviously $\psi$ is bounded localized, $a \leq e(\psi) < -\tau(\psi) \leq b$, and

$$-\tau(\eta) = -\tau(\psi), \quad -\tau(\eta') \leq -\tau(\eta) - \delta \leq e(\psi) < -\tau(\psi) \tag{\textstar}$$

Therefore and by (4), $-\tau(\psi_t) = -\tau(\psi) - |\tau|$ and $-\tau(\psi_t') = -\tau(\psi') - |\tau| + |\tau - \psi'| \leq -\tau(\eta') + |\tau| \leq e(\psi) + |\tau|$. Assume $|\tau| < \frac{1}{2}( - \tau(\psi) - e(\psi))$, which means $e(\psi) + |\tau| < -\tau(\psi) - |\tau|$. Hence $-\tau(\eta'') < -\tau(\psi_t)$. Since $(\tau, \tau(\psi_t))$ is the apex of $t \mapsto (t, \tau(\psi_t))$ the former implies by (4) that $-\tau(\eta'') < -\tau(\psi_t')$ for all $t$. Now note $\eta_t = \eta'_t + \psi_t$. Hence $-\tau(\eta_t) = -\tau(\psi_t)$ for all $t$ implying $v = \tau$. Therefore, using (\textstar), $\tau > \delta \geq -\tau(\psi) - e(\psi)$. This contradicts $|\tau| = -\tau(\psi) - e(\psi)$ by (5)(b).

Thus $|\tau| \geq \frac{1}{2}( - \tau(\psi) - e(\psi))$ holds. It remains to show $\tau \geq 0$. Assume the contrary. We start from $-\tau(\eta') < -\tau(\psi)$ by (\textstar). Put $v' := t_\tau(\eta')$. Then by (4), for all $t \geq 0$ one has $-\tau(\eta_t') = -\tau(\eta') - |\tau'| + |t - \tau'| \leq -\tau(\eta') + \tau < -\tau(\psi) + \tau - \tau = -\tau(\psi_t)$. So $-\tau(\eta_t') < -\tau(\psi_t)$ for all $t \geq 0$. By $\eta_t = \eta'_t + \psi_t$ this implies $-\tau(\eta_t') = -\tau(\psi_t)$ for all $t \geq 0$. For $t = v' > 0$ this means $-\tau(\eta') = -\tau(\psi) + v$. Since $-\tau(\eta) = -\tau(\psi)$ it contradicts $v \neq 0$.

The main mathematical tool for the proofs of the claims in sec. 3 is an application of the non-stationary phase method as shown in [1, Theorem 1.8.] estimating (4.10) for large $|x| + |t|$. The result in (10), according to which the spatial probability in $B_t$ tends to zero, essentially is a corollary to [1, Corollary 1.9.]. Rather analogously we prove in (11) the fact that asymptotically the spatial probability vanishes outside $B_{|t|}$.
In the following the obvious reduction to scalar-valued wavefunctions is used. Let $\psi \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ be a Dirac wavefunction and let $\varphi = \mathcal{F}\psi$ be its momentum representation. Regarding the time evolution one has $\varphi_t = e^{it\varphi}$, i.e., $\varphi_t(p) = e^{it\varphi(p)} = \varphi^{\epsilon}(p) = \frac{1}{2}(I + \frac{\epsilon}{i\varphi(p)}h(p))$ with $\epsilon(\varphi) = \sqrt{|p|^2 + m^2}$ is the projection in $\mathbb{C}^4$ onto the 2-dimensional eigenspace of $h(p)$ with eigenvalue $\eta(p)$. Then $\varphi^n := \pi^n\varphi$ is the projection of $\varphi$ onto the positive, respectively negative, energy eigenspace. Analogously $(\varphi^n)^n := \pi^n\pi^m\varphi$, Note that $(\varphi^n)^n = (\varphi^n)^0$, as $e^{it\varphi}$ and $\pi^n\pi^m$ commute. One concludes $(\psi^n)_l = \sum_i (\psi^n_i)_l$ with $(\psi^n_i)_l := (\mathcal{F}^{-1}\varphi^n)_l = \mathcal{F}^{-1}(\pi^n\varphi)_l$ for the $l$-th component of $\psi^n$, $l = 1, \ldots, 4$.

If $\varphi$ is also integrable, then so is $\varphi^n$ and for each $l$ one has

$$
(\psi^n_l(x))(t) = (2\pi)^{-3/2} \int e^{ipx + it\varphi^n(p)}(\varphi^n(p))_l \, d^3p
$$

(4.10)

**Proof of (10) Theorem.** Recall $\varphi = \mathcal{F}\psi$ and choose $\varphi' \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}^4)$ with $\|\varphi - \varphi'\| \leq \varepsilon/2$. Hence $0 < \varepsilon < 1$. Choose $\varepsilon \in \inf\{\frac{\varepsilon}{2} : p \in \text{supp}(\varphi')\}$. Let $\chi_t := 1_{B_{2\varepsilon}}$. Now, according to [1, Corollary 1.9], there is a constant $C_1$ such that $\|\chi_t\psi^n_t\| \leq C_1(1 + |t|)^{-1}$ for all $t$. Let $\tau := 2C_1/\varepsilon$. Then $\|\chi_t\psi^n_t\| \leq C_1(1 + |t|)^{-1}$ for $|t| \geq \tau$. — Now fix $r > 0$. Then for $|t| \geq \tau$ one has $\|\chi_t\psi^n_t\| \leq C_1(1 + |t|)^{-1} \leq \varepsilon$. —

**Proof of (11) Theorem.** Suppose first $\varphi := \mathcal{F}\psi \in \mathcal{C}^\infty_c(\mathbb{R}^3, \mathbb{C}^4)$. Let $K := \text{supp}(\varphi)$. Set $\gamma := \max\{\frac{1}{2r} : p \in K\}$. Clearly $0 < \gamma < 1$. For the estimation of the integral in Eq. (4.10) consider $\phi^n(p) := \|x + t|t|^{-1}(px - t\varphi(p))\). Then $\nabla \phi^n(p) = (|x| + |t|)^{-1}(x - \frac{\epsilon}{r}\epsilon(p))p$ and $|\nabla \phi^n(p)| \geq (|x| + |t|)^{-1}(|x| - |t|) \frac{1}{|x| + |t|} \geq \frac{|x - \gamma|}{|x| + |t|}$. Hence $\|\chi_t\psi^n_t\| \leq C_N(1 + |t|)^{-N}$ for $|t| \geq \tau$.

Now consider a general Dirac wavefunction $\psi$. Let $\varepsilon > 0$. Set $\varphi := \mathcal{F}\psi$ and choose $\varphi' \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{C}^4)$ with $\|\varphi - \varphi'\| \leq \varepsilon/2$. Hence $|\psi - \psi'| \leq \varepsilon/2$ for $\psi := \mathcal{F}^{-1}\varphi'$. The foregoing result has a constant $C_1$ such that $\|\chi_t\psi^n_t\| \leq C_1(1 + |t|)^{-1}$ for all $t$. Let $\tau := 2C_1/\varepsilon$. Then $\|\chi_t\psi^n_t\| \leq \|\chi_t(\psi - \psi')_l\| + \|\chi_t\psi^n_t\| \leq \|\psi - \psi'\| + C_1(1 + |t|)^{-1} \leq \varepsilon$ for $|t| \geq \tau$.

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