Low and high energy constraints in AdS/QCD: not synergetic, but interchangeable?

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Abstract

The AdS/QCD models are believed to interpolate between low and high energy sectors of QCD. This belief is usually based on observations that many phenomenologically reasonable predictions follow from bounds imposed at high energies although the hypothetical range of applicability of semiclassical bottom-up holographic models is restricted by the gauge/gravity duality to low energies where QCD is strongly coupled. For testing the feasibility of high energy constraints it is interesting to calculate holographically some observable constants at low and high momenta independently and compare. We consider an AdS/QCD model describing the Regge like linear spectrum of spin-1 mesons in a general form and show that under definite physical assumptions, the low-energy constraints on 2-point correlation functions lead to nearly the same numerical values for the parameters of linear radial spectrum as the high energy ones. The found approximate coincidence looks surprising in view of the fact that such a property for observables is natural for conformal theories while real strong interactions are not conformal.

1 Introduction

The bottom-up holographic models for QCD (often referred to as AdS/QCD models) have proven to be an interesting approach to the phenomenology of strong interactions [1]. Being inspired by the gauge/gravity duality in string theory [2,3], they boldly apply the holographic methods developed for conformal theories to the case of real QCD which is not conformal. A theoretical justification for such an extension is still lacking (see, however, the pioneering work [4]), nevertheless the bottom-up holographic approach resulted in construction of some useful phenomenological models which turned out to be
unexpectedly successful [5–7] and triggered a large activity in the field (the Refs. [8–18] constitute only a tiny part).

According to the principles of gauge/gravity duality, the AdS/QCD models must be viewed as models for QCD constructed (i) in the large-$N_c$ limit [19, 20] and (ii) in the low energy domain where QCD is strongly coupled. The point (i) implies that only 2-point correlation functions can be described (the higher $n$-point functions vanish in the large-$N_c$ limit [20]) while the point (ii) ensures that a putative 5D dual gravitational theory is weakly coupled, hence, can be treated semiclassically. Unfortunately, the both requirements are often violated in phenomenological applications. For instance, the descriptions of spontaneous Chiral Symmetry Breaking (CSB) and hadron formfactors involve higher $n$-point correlators [1, 5, 6]. This is hardly compatible with (i). Also applications to high energy QCD become questionable in view of (ii). As an example one can mention holographic derivations of QCD sum rules [10–13]. Among the practitioners of AdS/QCD models there is, however, a widespread belief that these models efficiently interpolate between low energy and high energy QCD. In certain sense, they can be seen as a "meromorphization" of the perturbative QCD expression for 2-point correlation functions [1]. Since essentially the same is pursued in the large-$N_c$ QCD sum rules [21–24], the holographic analogues of these sum rules provide the same level of predictiveness.

The correct description of 2-point QCD correlators at large Euclidean momentum $Q$ (perturbative logarithm plus power corrections in $Q^2$) is provided by the Soft-Wall (SW) holographic model [7] (an earlier variant was suggested in Ref. [9]). This model describes the Regge like radial meson spectrum that in the case of spin-1 mesons reads $m_n^2 = \mu^2(n + 1)$, where $n = 0, 1, 2, \ldots$. The SW model can be generalized towards inclusion of arbitrary intercept parameter $b$, $m_n^2 = \mu^2(n + 1 + b)$ [13]. The holographic sum rules yield definite predictions for $b$ in the vector and axial case.

The aim of the present paper is to demonstrate how almost the same numerical predictions can be obtained in the opposite limit of low $Q^2$ expansion of correlators. This looks really surprising since the corresponding equations are very different from the case of high $Q^2$ expansion. In addition, we will reproduce an expression for the slope $\mu^2$ known from QCD sum rules.

As a byproduct of our analysis, we will show how the effects of CSB can be embedded into holographic models on the level of 2-point correlators, i.e. not violating (i). The usual bottom-up holographic description of CSB is based on $ad$ $hoc$ merging of chiral effective field theory with AdS/QCD models [5, 6]. This inevitably results in emergence of higher $n$-point functions. It is known that this procedure works much better in the case of Hard Wall (HW) holographic model [5, 6] which, however, does not reproduce the Regge like
spectrum and power like corrections in high $Q^2$ expansion of correlators. We will effectively take into account the CSB effects via different conditions on 2-point correlators at zero momentum using some physical motivations. Within this scheme, even the simplest SW holographic setup becomes a working model, at least after a certain reformulation.

The mentioned reformulation constitutes another one, albeit secondary, objective of our work. We will argue that the replacement of enigmatic "dilaton background" by 5D mass depending on the fifth coordinate looks more natural from a physical viewpoint and leads to a simpler model. The proposals to introduce similar infrared modifications of 5D mass in AdS/QCD models have appeared in the literature from time to time since Ref. [14], we will try to put these attempts on a firmer ground.

The paper is organized as follows. In order to make our analysis and arguments self-contained, in Section 2 we first briefly review the idea of AdS/QCD approach, reformulate the SW model and reproduce predictions for Regge spectral parameters of vector mesons from high $Q^2$ expansion of 2-point correlator. The main results are contained in Section 3, where near the same predictions are obtained from the low $Q^2$ expansion. Some relevant discussions are given in Section 4 and we conclude in Section 5.

2 Two-point vector correlators and OPE

We first recall briefly the formalism of effective action and its holographic realization. Within the functional approach to quantum field theory, the primary object is the partition function $Z[\phi]$ of Green functions which has the physical meaning of vacuum-to-vacuum transition amplitude in the presence of external source $\phi$, $Z[\phi] = \langle 0_{\text{out}} | 0_{\text{in}} \rangle_\phi$. The effective action $S_{\text{eff}} \{ \phi \}$ is defined by $Z[\phi] = \exp \left( i S_{\text{eff}} \{ \phi \} \right)$. The coefficients of expansion of $S_{\text{eff}} \{ \phi \}$ in the external field $\phi$ are the connected correlation functions of currents coupled to $\phi$. In the momentum space, the effective Lagrangian density of Lorentz invariant theory is

$$L_{\text{eff}} = \frac{1}{2} \hat{P} \text{Tr} \left[ \phi \Pi_{\phi}(q^2) \phi \right] + \mathcal{O} \left( \phi^3 \right), \quad (1)$$

where the term linear in $\phi$ disappears due to equation of motion, $\hat{P}$ denotes the corresponding polarization tensor if the field has Lorentz indices, and $\Pi_{\phi}$ represents the two-point correlator of currents coupled to $\phi$ (becoming the propagator in perturbation theory). In the large-$N_c$ limit of QCD, the correlator $\Pi_{\phi}$ is a meromorphic function and the higher n-point functions vanish [20]. It means in particular that in the limit $N_c \to \infty$, $\Pi_{\phi}$ represents
a sum of infinite number of pole terms corresponding to contributions of infinitely narrow hadrons with quantum numbers of field $\phi$,

$$\Pi_\phi(q^2) \sim \sum_{n=0}^{\infty} \frac{F_n^2}{q^2 - m_n^2},$$

where contact terms needed for regularization are omitted.

In the bottom-up holographic approach, following the ideas of gauge/gravity correspondence one assumes the existence of 5D dual theory (namely S-dual) for QCD in the large-$N_c$ limit and tries to build a phenomenologically useful gravitational 5D model. This putative 5D theory is usually constructed in 5D Anti-de Sitter (AdS$_5$) space or asymptotically AdS$_5$ space. A convenient parametrization of the AdS$_5$ metric is given by the Poincaré patch with the line element

$$ds^2 = \frac{R^2}{z^2} \left( \eta^{\mu\nu} dx_\mu dx_\nu - dz^2 \right),$$

where $R$ is the radius of AdS$_5$ space and $z \geq 0$ represents the fifth holographic coordinate that has the physical meaning of inverse energy scale. The 4D Minkowski space becomes the ultraviolet boundary of AdS$_5$ residing at $z = 0$. The relation between a 4D gauge theory and its dual 5D gravitational theory is given by a concise statement

$$S_{\text{eff}}\{\phi\} = S_{5D}\left(\phi(x_\mu, 0)\right).$$

If the 4D gauge theory is in the strong coupling regime, the 5D theory must be weakly coupled due to strong-weak duality. This general idea paved the way for building semiclassical 5D models which describe the low energy QCD and are often interpolated to higher energies.

We will consider the case of vector mesons, $V_\mu = \phi \epsilon_\mu$, where $\epsilon_\mu$ denotes polarization of $V_\mu$. The corresponding polarization tensor is

$$\hat{P} = P_{\mu\nu} = \sum_\lambda \epsilon^{(\lambda)}_\mu \epsilon^{(\lambda)}_\nu = \eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}.$$

The simplest action of 5D dual vector model can be written as

$$S_{5D} = \frac{1}{2g_5^2} \int d^4x \, dz \, \sqrt{g} \, e^z \left( -\partial^M V^N \partial_M \partial_N + m_5^2 V^N V_N \right),$$

with the usual conditions $\partial^\mu V_\mu = 0$ and $V_z = 0$ for physical 4D excitations. The 5D coupling $g_5$ plays the role of normalization constant for the field $V_M$. The quadratic in field structure of action ensures disappearance of three and higher point correlation functions as expected in the large-$N_c$ QCD.
The AdS/CFT prescription for 5D mass of $p$-form field reads \[ m_5^2 R^2 = (\Delta - p)(\Delta + p - 4), \] where $\Delta$ is the canonical dimension of operator dual to the field $V_M$ at the AdS$_5$ boundary. For vector fields ($p = 1$ as the action (6) can be written in terms of the field strength tensor $F = dV$) the prescription takes the form
\[ m_5^2 R^2 = (\Delta - 1)(\Delta - 3). \] (7)

The usual quark vector current has $\Delta = 3$, hence, $m_5^2 = 0$. Vector operators of higher dimensions correspond to massive 5D vector fields.

The dilaton like quantity $\varphi$ in the action (6) dictates a background. The standard SW holographic model is defined by \[ \varphi = c z^2, \] (8)
where the constant $c$ provides a mass scale and can be both positive and negative. The choice (8) yields the Regge like spectrum \[ m_n^2 = 4|c|(n + 1), \quad n = 0, 1, 2, \ldots, \] (9)
and what is very important it leads to a correct analytical structure of OPE.

The physical origin of (8) remains an open problem. As was first noticed in Ref. [11], the dilaton like background can be absorbed into a infrared modification of $m_5^2$. Indeed, using the metric (3) the term with $z$-derivative in action (6) can be written as
\[ \sqrt{g} e^{cz^2} \partial^2 V^N \partial_z V_N = \left( \frac{R}{z} \right)^5 e^{cz^2} g^{\mathcal{R}S} g^{R} \partial_R V_S \partial_z V_N = \frac{R}{z} e^{cz^2} (\partial_z V_N)^2, \] (10)
where the inverse to metric $g_{MN}$ in (3) tensor is $g^{MN} = \frac{z^2}{R^2} \eta^{MN}$ and we write only lower indices when contraction with flat metric is understood, e.g., $V_N^2 = \eta^{MN} V_M V_N$. Now we redefine the field $V_N = e^{-cz^2/2} v_N$ and arrive at
\[ \frac{R}{z} (\partial_z v_N - c z v_N)^2 = \sqrt{g} \left( \partial^2 v^N \partial_z v_N + \frac{c^2 z^4}{R^2} v^N v_N \right) - 2 R c v_N \partial_z v_N. \] (11)

We see that a $\mathcal{O}(z^4)$ mass term emerged. The last term will not contribute to the Equation Of Motion (e.o.m.). It is easy to show, however, that for scalar and tensor cases the analogous term contains a $z$-dependent factor and does contribute to the e.o.m. resulting in $\mathcal{O}(z^2)$ mass term.

The standard SW model can be thus reformulated without $z$-dependent dilaton background (8) if we instead use the following ansatz for $z$-dependent 5D mass,
\[ m_5^2(z) R^2 = a + b z^2 + c^2 z^4, \] (12)
which stays in the action

\[ S_{5D} = \frac{1}{2g_5^2} \int d^4x \, dz \sqrt{g} \left( -\partial^M v^N \partial_M v_N + m_5^2(z) v^N v_N \right), \quad (13) \]

Any infrared modifications of dilaton background or metric can be translated into corresponding modifications of (12). The constant \( a \) may be identified with (7), the further terms are infrared corrections. It should be emphasized that modifications of \( z \)-dependence in (12) will, generally speaking, spoil the structure of standard OPE for two-point correlators. This issue is closely related with the known fact that non-linear corrections to Regge-like spectrum (9) after summation over resonances in (2) generically lead to analytical structures incompatible with the OPE \[22\]. The compatibility can be achieved only if such corrections degrease with \( n \) exponentially or faster \[22, 23\]. A form of potential in the corresponding e.o.m. (written in a Schrödinger like form) that would generate these corrections is unknown.

In the present study, we will adhere to the ansatz (12) describing the linear spectrum in the most general form.

For phenomenological description of real meson spectra with different quantum numbers one should introduce different intercepts in the spectrum (9). Within the SW model, this can be achieved by different infrared modifications of dilaton background or AdS metric. This fine tuning looks as if we constructed different dual models for different quantum numbers. The reformulation above looks nicer in this respect: A dual holographic model is unique but infrared modifications of \( m_5^2 \) are different. Note that the parameter \( c \) in (12) is independent of quantum numbers since it dictates the slope of radial trajectories which phenomenologically is indeed approximately universal \[25\]. The given universality seems to be an important consequence of confinement and appears naturally in hadron string approaches and some related quark models \[26\]. The intercept will be determined by parameters \( a \) and \( b \). The first one can be fixed by (7). One of our goals will be determination of the second intercept parameter \( b \).

The e.o.m. ensuing from the action (13) with \( m_5^2(z) \) from (12) after 4D Fourier transform \( v_\mu(q, z) = \int d^4x \, e^{iqx} v_\mu(x, z) \) takes the form

\[
\left[ -q^2 - z \partial_z \left( \frac{1}{z} \partial_z \right) + \frac{m_5^2(z) R^2}{z^2} \right] v_\mu(q, z) = 0.
\]

(14)

In the present work, we will consider the case \( a = 0 \) in (12). This is an important case of twist 2 vector current according to relation (7) and is the most studied in the literature. In order to make comparisons with the OPE in QCD, we must calculate the two-point correlator in Euclidean space,
i.e. introducing Euclidean momentum $Q^2 = -q^2$. Following the standard holographic procedure, we should find the solution of Eq. (14) in the form $v_\mu(q, z) = v_\mu(q) v(q, z)$ with the boundary condition $v(q, 0) = 1$. Then $v_\mu(q)$ can be interpreted as the source, $v_\mu(q) = \epsilon_\mu \phi$. The corresponding solution for the scalar shape function $v(q, z)$, satisfying also $v(q, \infty) = 0$, is (we pass to Euclidean space in what follows)

$$
v(Q, \zeta) = \Gamma \left( 1 + Q^2 + \beta \right) e^{-\zeta^2/2} U \left( Q^2 + \beta, 0; \zeta^2 \right),
$$

(15)

where $U$ is the Tricomi confluent hypergeometric function and for simplicity of further relations we defined the dimensionless quantities

$$
Q^2 = \frac{Q^2}{4|c|}, \quad \beta = \frac{b}{4|c|}, \quad \zeta^2 = |c|z^2.
$$

(16)

Note that the Eq. (14) has the second solution which, however, diverges as $\zeta \to \infty$,

$$
v_2(Q, \zeta) \sim \zeta^2 e^{-\zeta^2/2} L_{1-(1+Q^2+\beta)}^1(\zeta^2) \sim e^{\zeta^2/2} \zeta^{2(\Delta+\beta)}.
$$

(17)

Here $L_\alpha^\beta(x) = (1+S)_1 F_1 (-S, 2, x)$ denotes the corresponding Laguerre function. We discard this solution since it does not satisfy the holographic “regularity in the bulk” condition. The solution (15) is regular in the bulk as $U \left( Q^2 + \beta, 0; \zeta^2 \right) \sim \zeta^{-2(\Delta+\beta)}$ at large $\zeta$.

Making use of e.o.m. (14) in the action (13) and taking the second functional derivative with respect to $v_\mu(q)$ we get the usual AdS/QCD expression for the vector two-point correlator in Euclidean space,

$$
\Pi_V(Q^2) = \lim_{\zeta \to 0} \left( -\frac{R}{g_5^2} \frac{\partial^2 v(Q, \zeta)}{\partial \zeta^2} \right).
$$

(18)

The expansion of solution (15) at $\zeta \to 0$ yields

$$
v(Q, \zeta) = 1 + \left\{ \left( Q^2 + \beta \right) \left[ \ln \zeta^2 + \psi \left( 1 + Q^2 + \beta \right) + 2\gamma - 1 \right] - \frac{1}{2} \right\} \zeta^2,
$$

(19)

where $\psi$ denotes the digamma function and $\gamma \approx 0.577$ is the Euler’s constant.

Following the standard procedure, we should substitute (19) into (18) and take the limit $\zeta \to 0$. The logarithmic term will give the divergent contribution proportional to $(Q^2 + \beta) \ln \zeta^2$ which is interpreted as a contact term and discarded. We will prefer, however, another regularization: The logarithm in (19) is discarded before insertion into (18). This way looks more compatible with the AdS/CFT duality according to which the correction to (19) must behave as $\zeta^{\Delta - J}$, that for dimension of vector current
Δ = 3 and spin \( J = 1 \) is ζ². The term \( ζ² \ln ζ² \) violates this AdS/CFT prescription. The given term appears because the Tricomi function \( U(S, 0; z) \) is not holomorphic in the point \( z = 0 \). In our regularization, we subtract this singularity to make the regularized solution holomorphic at \( z = 0 \). Note that in the ultraviolet limit \( z \to 0 \), the applicability of semiclassical holographic models to QCD looks questionable and the whole approach seems to need modifications [15].

Inserting the regularized expansion (19) into (18) we obtain the two-point correlator without diverging constants,

\[
Π(Q^2) = -\frac{2R}{g_s^5} |c| \left\{ (Q^2 + β) \left[ ψ \left( 1 + Q^2 + β \right) + 2γ - 1 \right] - \frac{1}{2} \right\}.
\] (20)

The expression (20) at \( β = 0 \) coincides with the result of ”No-wall” holographic model of Ref. [11]. In fact this expression can be obtained directly from the \( β = 0 \) case by observing that \( b \neq 0 \) corresponds just to the shift of momentum squared \( q^2 \to q^2 - b \) in the e.o.m. (14), i.e. \( Q^2 \to Q^2 + β \).

The digamma function in (20) has poles at

\[
-Q_n^2 = \frac{m_n^2}{4|c|} = n + 1 + β, \quad n = 0, 1, 2, \ldots,
\] (21)

which correspond to the mass spectrum of the model. The substitution of known pole representation for this function,

\[
ψ(1 + x) = \sum_{n=1}^{∞} \frac{1}{n} - \sum_{n=1}^{∞} \frac{1}{n + x} - γ,
\] (22)

into (20) leads to the expected analytical structure (2) written in the Euclidean space.

With the help of asymptotic representation of digamma function for large argument \( x \to ∞ \),

\[
ψ(1 + x) \simeq \ln x + \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-3}),
\] (23)

we can expand (20) at large \( Q^2 \),

\[
Π(Q^2) \simeq -\frac{2R}{g_s^5} |c| \left\{ (Q^2 + β) \ln Q^2 + \frac{β^2 - 1/6}{2Q^2} + \frac{β(1 - 2β^2)}{12Q^4} + \ldots \right\},
\] (24)

where the contact and \( O(Q^{-6}) \) terms are omitted. The expansion (24) coincides with the result of generalized SW model of Ref. [13], where the arbitrary
intercept $\beta$ in the linear spectrum \([21]\) was introduced via the modified dilaton background $e^{cz^2} \to U^2(\beta, 0; cz^2)e^{cz^2}$.

The relevant form of OPE in QCD in the chiral and large-$N_c$ limits reads \([27]\)

$$\Pi^{OPE}(Q^2) \simeq -\frac{N_c}{6\pi^2}c|Q^2\ln Q^2 + \frac{\langle \frac{\alpha_s}{\pi}G^2 \rangle}{96|c|Q^2} + \frac{\xi \pi \alpha_s \langle \bar{q}q \rangle^2}{98c^2Q^4} + \ldots,$$

\((25)\)

where $\langle \frac{\alpha_s}{\pi}G^2 \rangle$ and $\langle \bar{q}q \rangle$ are the gluon and quark condensates, respectively, and $\xi$ depends on the space and charge parities ($\xi = -7$ for the vector case and $\xi = 11$ for the axial-vector one). The omitted constant contribution in \((25)\) (part of contact terms) allows to change the renormalization scale $\mu$ in the perturbative logarithm, we used this freedom to set $\mu^2 = 4|c|$.

The expansions \((24)\) and \((25)\) can be now matched. The matching of coefficients in front of the leading logarithms gives the standard normalization factor for the 5D vector fields,

$$\frac{R}{g_5^2} = \frac{N_c}{12\pi^2}.$$ \((26)\)

The matching of $O(Q^{-4})$ terms may be questionable since in the large-$N_c$ limit, a contribution of dimension 6 gluon condensate $\varepsilon^{abc}\langle G_a G_b G_c \rangle$ can be sizeable or even dominating but it is not present in the phenomenological expansion \((25)\). The matching of $O(Q^{-2})$ terms relates the gluon condensate to the spectral parameters as

$$\langle \frac{\alpha_s}{\pi}G^2 \rangle = \frac{N_c}{2\pi^2} \left(1 - \beta^2\right)(4c)^2.$$ \((27)\)

As was shown in Ref. \([13]\), a good agreement for the phenomenological slope of radial trajectories \([25]\)

$$4|c| \approx 1.2\text{ GeV}^2$$ \((28)\)

is achieved for the intercept parameter $|\beta| \approx 0.3$, where $\beta = \mp 0.3$ refers to the vector and axial cases, correspondingly. The arguments were the following: (i) $|\beta| \approx 0.3$ leads to a correct value of the gluon condensate in relation \((27)\), $\langle \frac{\alpha_s}{\pi}G^2 \rangle \approx (360\text{ MeV})^4$; (ii) $\beta \approx -0.3$ gives a satisfactory description the spectrum of radially excited $\omega$-mesons within the accuracy of large-$N_c$ limit; (iii) $\beta \approx -0.3$ reproduces the observed electromagnetic decay width of $\omega$-meson; (iv) $\beta \approx 0.3$ describes reasonably the spectrum of radially excited axial $f_1$-mesons. The description of spectrum of excited isovector $\rho$ and $a_1$ mesons with $|\beta| \approx 0.3$ is even better. In addition, a remarkable qualitative agreement takes place: The first non-perturbative correction to the parton logarithm does not depend on parity while the next one is of opposite sign for different parities.
3 Predictions from correlators at zero momentum

Now we will demonstrate how the same prediction $|\beta| \approx 0.3$ can follow from a simple analysis of vector correlator at zero momentum. Our key proposal is to treat the linear spectrum of SW model (9) as dual (in the sense of quark-hadron duality) to perturbation theory corrected by gluonic power terms in the OPE (25). These terms are known to appear from breaking of conformal symmetry. In other words, the spectrum (9) does not yet correspond to real resonances but appears as a result of ”meromorphization” of perturbative background corrected at low energies by conformal symmetry breaking power terms. The hadron resonances are associated with deviations from a background. In our model, this means then that they are associated with non-zero intercept parameter $\beta$ in (21).

Simultaneously $\beta \neq 0$ signifies the breaking of chiral symmetry — the appearance of contributions from quark condensate in the expansion (24) and of mass splitting between parity partners.

The given philosophy can be converted into a predictive calculational scheme. Consider the vector correlator (20) at zero momentum,

$$\Pi(0) = -\frac{2R}{g_s^2}|c| \left\{ -\frac{1}{2} + \beta [\psi (1 + \beta) + 2\gamma - 1] \right\}. \quad (29)$$

It is important to emphasize the role of our regularization: $\Pi(0)$ is a finite quantity with all constants fixed. In the standard regularization, the subtraction of infinite constant makes $\Pi(0)$ ambiguous.

The successful old hypothesis of Partial Conservation of Axial Current predicts in the chiral limit that the value of axial-vector correlator is

$$\Pi_A(0) = f_\pi^2, \quad (30)$$

where $f_\pi$ is the weak pion decay constant emerging from the pion pole. In essence, one may regard (30) as an alternative definition of $f_\pi$ — if the spontaneous chiral symmetry breaking is assumed, the r.h.s. of (30) must give the residue of the corresponding Goldstone boson pole. The critical

\textsuperscript{1}In other models, the spectrum corresponding to perturbative background can be different. For instance, the spectrum of S-wave spin-1 mesons in light front holographic QCD behaves as $m_n^2 \sim n + 1/2$ [1], i.e. one should look for deviations from $\beta = -1/2$. It is curious to observe that this spectrum minimizes the power contributions to perturbative logarithm in the 2-point vector correlator within the class of linear spectra [23]. The value of $\beta = -1/2$, however, is incompatible with positivity of gluon condensate (27) in our approach.
observation is that the axial-vector resonances should not contribute to $\Pi_A(0)$ because of a large mass gap in the axial channel — $f_2$ absorbs effectively all contributions to $\Pi_A(0)$. According to our philosophy, this means that terms with $\beta$ do not contribute. We get thus from (29) the equation for the axial intercept parameter $\beta_a$,

$$\beta_a [\psi (1 + \beta_a) + 2\gamma - 1] = 0. \quad (31)$$

This equation has two numerical solutions: $\beta_a = 0$ (no axial mesons) and $\beta_a \approx 0.31$ which is almost exactly the value extracted above from the large $Q^2$ limit.

When the equation (31) holds, we can obtain from (29), (30) and (26) the following relation for the slope of linear spectrum (21),

$$4|c| = \frac{48\pi^2}{N_c} f_2^2. \quad (32)$$

This relation for linear spectrum was derived in QCD sum rules in the large-$N_c$ limit under various assumptions (see, e.g., Refs. [24]). The relation (32) meets well the phenomenology — in the real world with $N_c = 3$, the empirical value (28) is reproduced for $f_\pi = 87$ MeV, which is the value of $f_\pi$ in the chiral limit according to the chiral perturbation theory [30]. We get the relation (32) in the opposite to the OPE based sum rules limit $Q^2 \to 0$. In principle, we could act in the reverse direction — require (32) and obtain Eq. (31) as a consequence.

In the vector channel, the value of $\Pi_V(0)$ is also non-zero but the physical reason must be completely different — a conversion of $\omega$ and neutral $\rho$ mesons into massless photons and back is possible (the effect underlying the famous hypothesis of Vector Meson Dominance) leading to a kind of effective ”photon” contribution. We do not know this contribution apriori, however, we can use the relation (31) (a form of the ”Das-Mathur-Okubo sum rule” [32])

$$-4L_{10} = \frac{d}{dQ^2} (\Pi_V - \Pi_A)|_{Q^2=0}, \quad (33)$$

where $L_{10}$ is one of constants of $SU_f(3)$ chiral Lagrangian [30,33]. From (20) we get

$$\frac{d}{dQ^2} \Pi|_{Q^2=0} = -\frac{R}{2g_5^2} [\psi (1 + \beta) + 2\gamma - 1 + \beta \psi (1, 1 + \beta)]. \quad (34)$$

The relation (32) can be simply obtained [23] by combining the slope $2m_\rho^2$ of spectra of Veneziano like dual amplitudes with the relation $m_\rho^2 = (24\pi^2/N_c) f_\pi^2$ which often holds in models respecting the Vector Meson Dominance [25] including the classical QCD sum rules [27]. It also emerges in attempts to incorporate the spontaneous CSB into the hadron string framework [29].
Using Eq. (31) and normalization (26) we finally obtain an equation for the vector intercept parameter $\beta_v$,

$$\psi (1 + \beta_v) + 2 \gamma - 1 + \beta_v \psi (1, 1 + \beta_v) - \beta_a \psi (1, 1 + \beta_a) = \frac{96 \pi^2 L_{10}}{N_c}. \quad (35)$$

The typical values of $L_{10}$ extracted in the phenomenology lie near $L_{10} \approx -5.5 \cdot 10^{-3}$ [33]. One should keep in mind, however, that the chiral constants are scale dependent and the aforementioned values refer to the scale of $\rho$-meson mass, $L_{10}(m_{\rho})$. But we should substitute to Eq. (35) the value at zero momentum. Fortunately the value of $L_{10}(0)$ can be determined in a scale-independent manner from hadronic $\tau$-decays. The extracted value is $L_{10}(0) = (-6.36 \pm 0.09)_{\text{expt}} \pm (0.16)_{\text{theor}} \cdot 10^{-3}$ [34]. With this value$^3$ and $\beta_a = 0.31$ the numerical solution of Eq. (35) yields $\beta_v \approx -0.26$ which is close to $\beta_v \approx -0.3$ estimated above (the exact agreement $\beta_v = -\beta_a$ is achieved at $L_{10} = -7.5 \cdot 10^{-3}$).

An alternative strategy for making fits can consist in writing equation for $N_c = 3$ from (33) and (34),

$$\psi (1 + \beta_v) + \beta_v \psi (1, 1 + \beta_v) - \psi (1 + \beta_a) - \beta_a \psi (1, 1 + \beta_a) = 32 \pi^2 L_{10}, \quad (36)$$

and imposing $\beta_a = -\beta_v$ to have a universal gluon condensate in the OPE (24). This would give $\beta_a = -\beta_v \approx 0.27$. After that we could reproduce a reasonable value of gluon condensate and observe a very small contribution of terms with $\beta_a$ to the axial correlator at zero momentum,

$$\Pi_A(0) \sim -\frac{1}{2} + \beta_a [\psi (1 + \beta_a) + 2 \gamma - 1] \approx -\frac{1}{2} - 0.01, \quad (37)$$

while the corresponding contribution to the vector correlator would be relatively large,

$$\Pi_V(0) \sim -\frac{1}{2} + \beta_v [\psi (1 + \beta_v) + 2 \gamma - 1] \approx -\frac{1}{2} + 0.27. \quad (38)$$

The given simple calculations demonstrate phenomenologically how the constant contributions to correlators which are interpreted as part of "contact" terms and neglected in high $Q^2$ expansions play a decisive role at low $Q^2$.

$^3$An order by magnitude estimation of $SU_f(3)$ chiral constants is $L_i \sim \frac{f^2}{\Lambda_{\text{CSB}}^2}$ [33], where $\Lambda_{\text{CSB}}$ is the scale of spontaneous chiral symmetry breaking. This scale is usually estimated from variation of contributions of chiral loops, the result is $\Lambda_{\text{CSB}} \approx 4 \pi f_{\pi}$ [35] and leads to $L_i \sim 1/(4 \pi^2) \approx 6.3 \cdot 10^{-3}$. Note that the slope of radial trajectories $\beta_v$ at $N_c = 3$ is just $\Lambda_{\text{CSB}}^2$. The given coincidence certainly should have deep physical roots.
4 Discussions

There is a widespread opinion (see, e.g., discussions in Ref. [16]) that the phenomenology of bottom-up holographic models has much in common with QCD sum rules in the large-$N_c$ limit (sometimes called planar QCD sum rules) which were a fruitful phenomenological tool in the past [21–24]. Within this method, one takes the pole representation of two-point correlators \( \Pi(Q^2) \) in the Euclidean space,

\[
\Pi(Q^2) \sim Q^2 \sum_{n=0}^{\infty} \frac{F_n^2}{Q^2 + m_n^2},
\]

assumes some ansatz for the spectrum, sums over all states, matches the result with the corresponding OPE and low-energy constraints and finally derives phenomenological predictions. In the case of simple linear spectrum,

\[
m_n^2 = \mu^2 (n + b),
\]

with constant residues \( F_n^2 \) one obtains the usual representation via digamma function,

\[
\Pi(Q^2) \sim -Q^2 \psi(Q^2 + b) + \text{const},
\]

where \( Q^2 = Q^2/\mu^2 \).

We wish to emphasize an important technical distinction between this approach and our holographic one. In the vector case, we obtain a structure of the kind

\[
\Pi(Q^2) \sim -\left(Q^2 + \beta\right) \psi(Q^2 + 1 + \beta) + \text{const}.
\]

Now changing intercept we change also the coefficient in front of \( \psi \)-function. As a result, the expansion in large \( Q^2 \) of (42) becomes different from the expansion of (41) and consequently leads to a different set of sum rules when one matches to the OPE. The reason of arising distinction lies in the fact that the holographic models represent a dynamical approach where \( Q^2 = -q^2/\mu^2 \) appears in holographic e.o.m. like Eq. (14): Any shift of constant intercept \( b \rightarrow b + \Delta b \) can be interpreted as a shift \( Q^2 \rightarrow Q^2 + \mu^2 \Delta b \) in the corresponding e.o.m., i.e. as a redefinition of what we call \( Q^2 \). Physically this means, of course, a fine tuning of mass gap. It is interesting to note that this shift (i.e. \( \beta \neq 0 \) in (42)) leads to the appearance of contribution \( \beta \ln Q^2 \) in the large \( Q^2 \) expansion (as in expansion (24)). Such a contribution was often interpreted as a contribution from the effective non-local dimension-two gluon condensate [36].

\footnote{Actually this remark is valid for any extension of Klein–Gordon equation preserving the 4D relativistic invariance.}
A remark on definition of vector correlator is in order. This object is usually defined with a general factor $Q^2$ extracted, $\Pi_V(Q^2) \rightarrow Q^2\tilde{\Pi}_V(Q^2)$. For dimension 3 vector currents like $\bar{\psi}\gamma_\mu\psi$ the quantity $\tilde{\Pi}_V(Q^2)$ becomes dimensionless. The general proportionality to $Q^2$ appears in QED and perturbative QCD due to the gauge invariance. The use of this special (with respect to non-vector cases) definition is unfavorable in our approach because it is easy to lose the physics related with the shift of $Q^2$ discussed above. It is also inconvenient for low-$Q^2$ expansion which is in focus of the present work. In addition, this special definition caused much confusion in AdS/QCD models. The underlying reason is that after extraction of $Q^2$ the last term in the expression (20) looks as a massless pole. Accidentally this does not happen in the SW model when $c < 0$ (i.e. with the dilaton background $e^{-|c|z^2}$) because this last term $\frac{1}{2}$ in (20) is replaced by $\left(1 + \frac{c}{|c|}\right)$, all other terms are identical [12]. The existence of this "pole" was among the reasons to reject the SW model with positive dilaton background in the pioneering paper [7]. Meanwhile some other authors insisted that the background $e^{c|z^2}$ looks more physical. For relevant discussions see, e.g., Refs. [1, 12, 16–18]. In our approach, both signs of the slope parameter $c$ are equally acceptable since the effective mass $m_5^2$ depends on $c^2$.

In our point of view, the $z$-dependent effective mass [12] is primary while the dilaton background is secondary, i.e. the standard SW holographic model follows after appropriate redefinition of fields. If this is true, a question appears concerning the physical origin of $z$-dependence from a more fundamental theory. Its origin might be inevitable when passing to non-conformal theories in the gauge/gravity correspondence. We remind the reader that a 4D physical state in a dual 10D theory emerges in the form

$$\Phi = e^{iqx}\psi(z, \Omega),$$

(43)

where $\Omega$ are coordinates on the 5D transverse space which usually represents sphere $S_5$. In conformal case of pure $AdS_5$ space, a further factorization takes place,

$$\psi(z, \Omega) = C v(z) g(\Omega),$$

(44)

where $g(\Omega)$ is a normalized harmonic in the angular directions which can be safely integrated out. In non-conformal case, the factorization (44) is valid only at small $z$ because the infrared dynamics at large $z$ will in general induce mixing between different harmonics [4]. Nevertheless in AdS/QCD models, the factorization (44) is tacitly assumed for all $z$. If we effectively parametrize this mixing via the replacement

$$g(\Omega) \rightarrow g(\Omega) + f(z, \Omega),$$

(45)

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where \( f(z, \Omega) \) is a growing function of \( z \) with condition \( f(0, \Omega) = 0 \), then we naturally get a \( z \)-dependent contribution to five-dimensional mass,

\[
\Delta m_5^2(z) = \frac{\int f(z, \Omega) d\Omega}{\int g(\Omega) d\Omega}.
\]

(46)

A holographic derivation of the form of this contribution from first principles remains of course an open problem.

5 Conclusions

The main result of our work consists in an explicit demonstration of the fact that the AdS/QCD predictions for radial Regge spectrum of spin-1 mesons following from expansion of correlators at high momentum can be reproduced from expansion at low momentum. We provided a set of physical assumptions for which the whole scheme works successfully. Our analysis places on a new quantitative level the general idea that bottom-up holographic models interpolate between high and low energy sectors of QCD.

It is important to emphasize that according to the ideas of gauge/gravity duality, the semiclassical holographic dual models should describe the low energy domain where QCD is strongly coupled. This entails a much better conceptual justification for low energy holographic predictions in comparison with the usual high energy ones. The fact that such low energy predictions can describe the whole radial spectrum looks encouraging.

Our approach can be extended to vector mesons interpolated by higher twist operators. This is partly tantamount to considering non-zero constant \( a \) in the ansatz \([12]\). The resulting structure of 2-point correlators becomes different and deserves a separate study. An extension to the scalar and tensor cases is straightforward. The development of corresponding phenomenology is in progress.

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