INvariance principle in the singular perturbations limit

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The paper is dedicated to my friend Peter Kloeden
(Communicated by Tomas Caraballo)

Abstract. We examine the invariance principle in the stability theory of differential equations, within a general singularly perturbed system. The limit dynamics of such a system is depicted by the evolution of a Young measure whose values are invariant measures of the fast equation. We establish an invariance principle for the limit dynamics, and examine the relations, at times subtle, with the singularly perturbed system itself.

1. Introduction. The LaSalle invariance principle provides a refinement of the direct method, via Lyapunov functions, of analyzing asymptotic stability. Rather than requiring that the Lyapunov function be strictly decreasing along solutions, it is demanded that the function is non-increasing. The conclusion then is that the solution converges to an invariant part of a level set of the Lyapunov function. This by itself is of interest. If, moreover, such level sets do not exist, asymptotic stability follows. The principle generated a lot of applications. Indeed, non-increasing Lyapunov functions are, in many applications, natural and easy to come up with. For a general reference to the principle and its applications in ordinary differential equations and in difference equations see LaSalle [20]. The principle has been refined, extended to, and applied in a variety of other types of dynamics. A small sample of such extensions and applications of the principle is Afonso, Bonotto, Federson and Schwabik [1], Alvarez, Orlov and Acho [2], Bacciotti and Mazzi [8], Barkana [10], Bonoto [12], Byrnes and Martin [13], Chen, Zhou and Čelikovský [14], Hespanha [15], Iggidr, Kalitine and Outbib [17], Kloeden and Lorenz [18], Potzsche [24].

In the present paper we examine the invariance principle within a general singularly perturbed differential equation of the form

\[ \frac{dx}{dt} = \frac{1}{\varepsilon} F(x) + G(x), \]

with \( \varepsilon > 0 \) a small real parameter, and \( x \in \mathbb{R}^n \). Of interest is the case where \( \varepsilon \) is very small, and to that end we study the limit, as \( \varepsilon \to 0 \), of the dynamics generated by (1.1). Extending the invariance principle to the limit of (1.1) is not straightforward since there may be no ordinary differential equation that governs the
limit of trajectories of (1.1). In fact, the limit of trajectories may not be described as a trajectory in $\mathbb{R}^n$. Rather, it may be depicted as a trajectory in the space of probability measures on $\mathbb{R}^n$. We elaborate on this phenomenon in the next section.

There are two aspects to the examination of the invariance principle in the singular perturbations limit. The first is to find what form does the principle get when applied to the limit dynamics, which is probability measure-valued. The second is to unravel the information one can get from the invariance principle of the limit, on the behavior of solutions of the perturbed equation (1.1), when $\varepsilon$ is small. The latter behavior may be subtle. We display an example where the limit dynamics converges to an invariant level set, yet each solution of the perturbed dynamics penetrates through the level set.

In the next section we recall, telegraphically though, the structure of the limit dynamics of (1.1). The Invariance Principle of this limit is examined in Section 3. The singularly perturbed system (1.1) does not exhibit a split into fast and slow dynamics. Such a split is explicit in the Tikhonov model, and invariance in this model is examined in Section 4. The closing section analyzes the inference from the invariance principle of the limit dynamics, to the perturbations (1.1) for $\varepsilon$ small.

2. The limit dynamics. We first display the basic notions needed for the definition of the limit dynamics, as $\varepsilon \to 0$, of solutions of (1.1). Then we introduce the limit dynamics itself. The description here is terse. More details can be found in Artstein, Kevrekidis, Slemrod and Titi [5], in Artstein [4], and in the references that we mention below.

We employ probability measures on $\mathbb{R}^n$, and the weak convergence in the space of probability measures. The support of a probability measure $\mu$ on $\mathbb{R}^n$ is the smallest closed set $C$ such that $\mu(C) = 1$. The support of $\mu$ is denoted $\text{supp}(\mu)$. Recall that the family of measures supported on a compact set in $\mathbb{R}^n$ is compact with respect to the weak convergence. The support of the measure is not a continuous mapping of the measure itself, say with respect to the Hausdorff distance. It is, however, lower semi-continuous, namely, if $\mu_k$ converges to $\mu_0$, and if $x_0 \in \text{supp}(\mu_0)$, then there is a sequence $x_k \in \text{supp}(\mu_k)$ such that $x_k \to x_0$. A reference to probability measures and weak convergence is Billingsley [11].

We also need the notion of a Young measure. In our context a Young measure is a measurable mapping from a time interval into the space of probability measures on $\mathbb{R}^n$. A Young measure $\mu(\cdot)$, defined on an interval $[t_1, t_2]$, can be viewed as a measure on $[t_1, t_2] \times \mathbb{R}^n$ (this via integrating the measures over the time interval). The convergence in the space of Young measures is taken to be the weak convergence of these measures. For a later reference we spell out a useful criterion of this convergence. The sequence $\mu_k(\cdot)$ of Young measures on $[t_1, t_2]$ converges to $\mu_0(\cdot)$, if

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^n} b(t, x) \mu_k(t)(dx)dt \to \int_{t_1}^{t_2} \int_{\mathbb{R}^n} b(t, x) \mu_0(t)(dx)dt,$$

(2.1)
as $k \to \infty$, this for every bounded function $b(t, x) : [t_1, t_2] \times \mathbb{R}^n \to \mathbb{R}$, continuous in $x$ and measurable in $t$.

The convergence of Young measures on an unbounded interval is determined by the convergence in the Young measures sense on every compact sub-interval. The support of a Young measure is, naturally, the graph of the supports of the values of the measure-valued mapping (save sets of $t$ of Lebesgue measure zero, see Remark 2.5 below). As in the case of measures, the convergence of a sequence of Young measures does not imply the convergence of the supports.
A point-valued map can be viewed as a Young measure whose values are supported on a singleton. In particular, point-valued functions may converge, in the sense of Young measures, to a Young measure which is not point-valued. A useful property is that a family of uniformly bounded Young measures (or functions) is compact with respect to convergence in the Young measures sense.

References to general Young measures and the convergence in the sense of Young measures are Balder [9], Pedregal [23], Valadier [27].

Other notions that we need in order to define the limit dynamics are that of an invariant measure, and a limit occupational measure of a differential equation, say of \( \frac{dx}{ds} = h(x) \). We denote by \( x(s, x_0) \) the solution of the differential equation with initial condition \( x(0) = x_0 \). We assume that the solution is unique, and exists for all \( s \).

A probability measure \( \mu \) is an invariant measure of the differential equation if for every closed set \( C \) and every time \( s \) the value \( \mu(C) \) is equal to \( \mu(\{x(s, x_0) : x_0 \in C\}) \).

Of a special type of invariant measures are the limit occupational measures. For an initial condition \( x_0 \) at the initial time \( s_0 \), define \( \mu(x_0, s_0, s_1) \) to be the probability measure on \( \mathbb{R}^n \) which assigns to a set \( B \) the proportion of time in \([s_0, s_1]\) that the values of the solution that starts at the time \( s_0 \) and the state \( x_0 \), is in \( B \). If \( \mu(x_i, s_i, s_j) \) converge in the space of probability measures, and \( s_j - s_i \) tend to infinity, then the limit is an invariant measure of the equation. It is then referred to as a limit occupational measure. Note that not every invariant measure is a limit occupational measure. Also note that while a limit occupational measure may be generated by one trajectory on the half time line, there may be limit occupational measures that are not generated by a single trajectory. The support of an invariant measure of an equation is invariant with respect to the dynamics generated by the equation.

Classical references to the basic theory behind the notions of invariant measures and limit occupational measures are Kryloff and Bogoliuboff [19], Nemytskii and Stepanov [21].

We turn now to the definition of the limit dynamics of (1.1). A prime role in the analysis of the limit behavior of (1.1) is played by the fast component or the fast equation of (1.1). It is the differential equation

\[
\frac{dx}{ds} = F(x).
\]  

(2.2)

Notice that we use a different time frame when alluding to the fast equation, namely, the small parameter \( \varepsilon \) is not employed here. We think of the time frame \( s \) as magnifying the time variable \( t \), namely, \( t = \varepsilon s \).

We state now an assumption under which we work throughout the paper (we do not aim at the most general assumptions, in order not to blur the arguments).

**Assumption 2.1.** The functions \( F(\cdot) \) and \( G(\cdot) \) are continuous. Solutions of (1.1) are uniquely determined by their initial conditions, and exist on the entire time line. The solutions, say \( x(\cdot) \), of the fast equation (2.2) are also determined uniquely by the initial data, say \( x(s_0) = x_0 \), and stay bounded for \( s \geq s_0 \), uniformly for \( x_0 \) in a bounded set.

We state now the result underlying the structure of the limit dynamics of (1.1).

**Theorem 2.2.** For every sequence \( \varepsilon_i \to 0 \) and solutions \( x_{\varepsilon_i}(\cdot) \) of the perturbed equation (1.1), defined on \([0, \infty)\) with \( x_{\varepsilon_i}(0) \) in a bounded set, there exists a subsequence \( \varepsilon_j \) such that \( x_{\varepsilon_j}(\cdot) \) converges as \( j \to \infty \), in the sense of Young measures, to
A Young measure whose values are limit occupational measures of the fast equation (2.2).

A proof (for split dynamics) can be found in Artstein and Vigodner [7]. The general situation is addressed in Artstein, Kevrekidis, Slemrod and Titi [5], and in Artstein [4].

**Terminology 2.3.** A Young measure $\mu(\cdot)$ obtained as a limit as described in the previous theorem, is referred to as a *limit solution* of the singularly perturbed equation (1.1). The value $\mu(0)$ is its initial condition. The family of all limit solutions is referred to as the *limit dynamics* of (1.1). Notice that trajectories of the limit dynamics could be probability measure-valued but can also be point-valued or a combination of both. When the limit is point-valued, we view the values as Dirac probability measures.

**Example 2.4.** Although solutions of (1.1) and of (2.2) are uniquely determined by their initial conditions, this may not be the case with the limit dynamics. Two distinct limit solutions, generated of course by different converging sequences, may emerge from the same initial condition. Indeed, the small parameter $\varepsilon$ may cause an abrupt shift in the state, that in the limit may even show discontinuity. The phenomenon occurs even for point-valued limits. Here is an example in $\mathbb{R}^3$ (with scalar coordinates $(x,y,z)$). Consider the system

\begin{align*}
\frac{dx}{dt} &= 2 - x \\
\frac{dy}{dt} &= -\frac{1}{\varepsilon} y r(x,y,z) \\
\frac{dz}{dt} &= 0,
\end{align*}

(2.3)

where $r(x,y,z)$ is defined as follows. The value $r(x,y,z) = 0$ in two cases. One is when $y = 1$, and $z \geq 1$, and the second is when $y = 1, z \geq 0$, and $x \leq 1$. Otherwise $r(x,y,z) > 0$, and bounded. Consider the trajectories emanating from the initial conditions $(0,1,1 + \eta_j)$ with $\eta_j > 0$ and $\eta_j$ converging to 0. The limit, as $j \to \infty$ and $\varepsilon \to 0$, of these solutions is the trajectory given by $x(t) = 2 - 2e^{-t}$, $y(t) = 1$ and $z(t) = 1$. The limit as $t \to \infty$ of this trajectory is the vector $(2,1,1)$. When, however, we change the initial conditions and start with $(0,1,1 - \eta_j)$, the limit trajectory is $x(t) = 2 - 2e^{-t}$, $z(t) = 1$ as before, but with a discontinuity in the $y$-variable that occurs when $x(t)$ reaches the value 1, namely when $t = \log 2$.

At that point of time the $y$-coordinate of the limit solution jumps from the value 1 to the value 0, and stays there indefinitely. The limit vector is now $(2,0,1)$. Both trajectories share the same values on the initial time interval $[0, \log 2]$.

**Remark 2.5.** The discussion above ignores a subtle point. Being required only to be measurable, a Young measure is determined only up to a set of Lebesgue measure zero. Abusing rigor, we ignore the subtlety, and when in the sequel we refer to properties of a Young measure, we mean, but do not always state it explicitly, that the property holds almost everywhere. Likewise, since a Young measure is not affected by a change of its value at a single point, including the initial condition, we always assume that the initial value, say $\mu(t_0)$, is a limit of values $\mu(t_i)$ as $t_i \to t_0$, each of which is a continuity (Lebesgue) point of the Young measure. No confusion should arise.
3. Invariance principle for the limit dynamics. The LaSalle Invariance Principle establishes, in particular, that the limit of a trajectory is included in a level set of a Liapunov function. To this end consider a Liapunov function of the form

\[ V(x) : \mathbb{R}^n \to \mathbb{R}, \]

assuming it is continuously differentiable and, say, \( V(x) \geq 0 \) for all \( x \in \mathbb{R}^n \), we then call it positive definite. We work under the following condition.

**Assumption 3.1.** The function \( V(\cdot) \) is non-increasing along solutions of the fast equation (2.2), namely, for every solution \( x(\cdot) \) of (2.2) the real-valued function \( V(x(s)) \) is non-increasing in \( s \).

Notice that we demand that \( V(\cdot) \) is non-increasing only along solutions of the fast equation. It may not be non-increasing along solutions of (1.1).

The following is a direct consequence of the previous assumption. Recall that a continuity point, or a Lebesgue point, of the measurable map \( \mu(\cdot) \), is a point in time, say \( t_0 \), where, roughly, for small intervals around it, most of the values \( \mu(t) \) are close to \( \mu(t_0) \). The smaller the interval, the closer these values are to \( \mu(t_0) \) and the greater the proportion of these points is. A point at which \( \mu(\cdot) \) is continuous is in particular a Lebesgue point. Recall that the measure of points that are not continuity points is zero.

**Lemma 3.2.** Let \( \mu(\cdot) \) be a limit solution of (1.1), and let \( t_0 \) be such that there exists a sequence \( t_k \) of continuity points of \( \mu(\cdot) \) that converge to \( t_0 \) such that \( \mu_k(t) \to \mu(t_0) \). Under Assumption 3.1, the Liapunov function \( V(\cdot) \) is constant on \( \text{supp}(\mu(t_0)) \).

**Proof.** Suppose the conclusion is not correct and let \( x_0 \) and \( x_1 \) belong to \( \text{supp}(\mu(t_0)) \) and such that \( V(x_1) > V(x_0) \). Denote \( d = V(x_1) - V(x_0) \). Let \( x_{\varepsilon_j}(\cdot) \) be a sequence of solutions of (1.1) that converge in the Young measures sense to \( \mu(\cdot) \) on an interval, say on \([\tau_1, \tau_2]\), that contains the sequence \( t_k \) mentioned in the claim. Let \( B \) be a bounded set in \( \mathbb{R}^n \) that contains all the values \( x_{\varepsilon_j}(t) \) for \( t \in [\tau_1, \tau_2] \) and all \( j \). Such a bounded set exists due to the standing Assumption 2.1. Let \( \beta \) be a bound on \(|\nabla V(x)|\) for \( x \in B \). Let \( \theta \) be a bound on \(|\nabla V(x)|\) for \( x \in B \). Consider now one trajectory \( x_{\varepsilon_j}(\cdot) \) on a sub-interval \([s_0, s_1]\) of \([\tau_1, \tau_2]\). On this sub-interval we have the equality

\[
\frac{d}{dt} V(x_{\varepsilon_j}(t)) = (\nabla V(x_{\varepsilon_j}(t))) \cdot (G(x_{\varepsilon_j}(t)) + \frac{1}{\varepsilon_j} F(x_{\varepsilon_j}(t))). \tag{3.2}
\]

Since the Liapunov function is not increasing along solutions of (2.2), it follows that \( (\nabla V(x_{\varepsilon_j}(t))) \cdot \frac{1}{\varepsilon_j} F(x_{\varepsilon_j}(t)) \leq 0 \), hence, in view of the equality we get the estimate

\[
\frac{d}{dt} V(x_{\varepsilon_j}(t)) \leq \beta \theta. \tag{3.3}
\]

Therefore

\[
V(x_{\varepsilon_j}(s_1)) - V(x_{\varepsilon_j}(s_0)) \leq \beta \theta (s_1 - s_0). \tag{3.4}
\]

This holds for every solution \( x_{\varepsilon_j}(\cdot) \). In other words, over an interval of length \( s_2 - s_1 \) the value of the Liapunov function cannot increase by more than the right hand side term in (3.4). Let \( s_1 \) and \( s_0 \) be two points in the sequence \( t_k \) such that \( s_1 > s_0 \) and \((s_1 - s_0)\beta \theta < \frac{1}{2}d \). For \( \varepsilon_j \) small enough in the sequence, there are points \( \sigma_1 \) and \( \sigma_0 \) close enough to \( s_1 \) and \( s_0 \) respectively, such that \( x_{\varepsilon_j}(\sigma_1) \) and \( x_{\varepsilon_j}(\sigma_0) \) are close enough to \( x_1 \) and \( x_0 \) respectively, such that the values of the Liapunov function on these points approximate, say up to \( \frac{1}{2}d \), the values on \( x_1 \) and \( x_0 \). Such time points exist since the sequence of solutions of (1.1) converge to the Young measure. But
then the value of the Lyapunov function $V$ cannot increase on the short interval from $V(x_0) + \frac{1}{d}$, to the value $V(x_1) - \frac{1}{d}$. Since this holds for every solution in the sequence $x_{\varepsilon_j}(\cdot)$, it forms a contradiction to the convergence in the Young measures sense. This completes the proof. □

**Corollary 3.3.** Let $\mu(\cdot)$ be a limit solution (1.1). Under Assumption 3.1, for a set of full measure in the time line $t$, the Lyapunov function $V(\cdot)$ is constant on $\text{supp}(\mu(t))$.

**Proof.** The result follows from the previous lemma by noting that the continuity points of a measurable map have a full measure. □

**Remark 3.4.** As has been just proved, under Assumption 3.1, almost everywhere in time, the Lyapunov function is constant on the supports of the limit solution. Yet, it may be worth noting that the claim may not hold when $x(t_0)$ is not a continuity point of the limit solution, even if the value $\mu(t_0)$ can be defined in a natural way. As an example consider again Example 2.4. The mapping $V(x, y, z) = (2 - x)^2 + y^2 + z^2$ is a Lyapunov function of (2.3), and it satisfies Assumption 3.1. The limit solution determined by the initial conditions $(0, 1, 1 - \eta_j)$, is not continuous at the time when $x(t) = 1$. This limit solution has a natural measure-valued limit $\mu(1)$ at $t = 1$, namely, the measure that assigns equal probabilities to $y = 1$ and $y = 0$. The Lyapunov function is not constant on the support of this measure.

The following result is the main tool for establishing the invariance principle.

**Theorem 3.5.** Let $V(\cdot)$ be positive definite on $\mathbb{R}^n$. Suppose that $V(\cdot)$ is non-increasing along solutions of the fast equation (2.2), and in addition suppose that

$$\int_{\mathbb{R}^n} \nabla V(x) \cdot G(x) \nu(dx) \leq 0, \quad (3.5)$$

for every invariant measure $\nu$ of (2.2). Let $\mu(\cdot)$ be a limit solution of (1.1) and denote by $U(t)$ the common value of $V(x)$ for $x$ in the support of $\mu(t)$ (defined almost everywhere). Then $U(t)$ is non-increasing.

**Proof.** Although we know that $V(\cdot)$ is constant on the supports of the values $\mu$ of the limit dynamics, we denote $V(\mu) = \int_{\mathbb{R}^n} V(x) \mu(dx)$. Let the limit solution on an interval $[t_1, t_2]$ be the limit in the Young measures sense of a sequence $x_{\varepsilon_j}(\cdot)$. Consider the evaluation of the Lyapunov function $V(x_{\varepsilon_j}(t))$. Then

$$\frac{d}{dt} V(x_{\varepsilon_j}(t)) = (\nabla V)(x_{\varepsilon_j}(t)) \cdot \frac{d}{dt} x_{\varepsilon_j}(t)$$

$$= (\nabla V)(x_{\varepsilon_j}(t)) \cdot (G(x_{\varepsilon_j}(t)) + \frac{1}{\varepsilon_j} F(x_{\varepsilon_j}(t))) \quad (3.6)$$

$$\leq (\nabla V)(x_{\varepsilon_j}(t)) \cdot G(x_{\varepsilon_j}(t)).$$

The last inequality holds since $V(\cdot)$ is non-increasing over solutions of the fast equation. The inequality in (3.6) implies the following inequality.

$$V(x_{\varepsilon_j}(t_2)) \leq V(x_{\varepsilon_j}(t_1)) + \int_{t_1}^{t_2} (\nabla V)(x_{\varepsilon_j}(s)) \cdot G(x_{\varepsilon_j}(s))ds. \quad (3.7)$$

The continuity of $\nabla V(\cdot)$ and $G(\cdot)$ allows us to invoke the weak convergence in the sense of Young measures, see (2.1), and deduce that in the limit

$$V(\mu(t_2)) \leq V(\mu(t_1)) + \int_{t_1}^{t_2} \int_{\mathbb{R}^n} (\nabla V)(x) \cdot G(x) \mu(s)(dx)ds. \quad (3.8)$$
Together with (3.5) we get that
\[ V(\mu(t_2)) \leq V(\mu(t_1)), \quad (3.9) \]
and since \( V(\mu(t)) = U(t) \) the conclusion follows.

Following the standard terminology in dynamical systems we say that \( \mu_0 \) is an \( \omega \)-limit point of the limit solution \( \mu(\cdot) \) if a sequence \( t_k \to \infty \) exists, such that \( \mu(t_k) \) converge weakly to \( \mu_0 \).

**Theorem 3.6.** An Invariance Principle. Let \( V(\cdot) \) be positive definite on \( \mathbb{R}^n \). Suppose that \( V(\cdot) \) is non-increasing along solutions of the fast equation (2.2), and in addition suppose that (3.5) holds for every invariant measure \( \nu \) of (2.2). Let \( \mu(\cdot) \) be a limit solution of (1.1) and denote by \( U(t) \) the common value of \( V(x) \) for \( x \) in the support of \( \mu(t) \). Then all \( \omega \)-limit points of \( \mu(\cdot) \) are supported on the same level set of \( V(\cdot) \), specifically, the level set is determined by the limit of \( U(t) \) as \( t \to \infty \).

**Proof.** By Theorem 3.5 the function \( U(t) \) is non-increasing in \( t \), and since it is bounded from below, it has a limit, say \( v_0 \). Since \( U(t) = V(x) \) for every \( x \in \text{supp}(\mu(t)) \) and \( V(x) \) is continuous in \( x \), and since the support of a probability measure is lower semi-continuous with respect to the weak convergence, it follows that \( V(x) = v_0 \) whenever \( x \) is on the support of an \( \omega \)-limit point of \( \mu(\cdot) \). The family of invariant measures is closed under weak convergence, therefore any \( \omega \)-limit point \( \mu_0 \) of \( \mu(\cdot) \) is an invariant measure of (2.2), and in particular its support is invariant with respect to the fast equation. This completes the proof.

The invariance principle that we stated refers to \( \omega \)-limit points, but does not infer even the existence of such points, or convergence to such points. A standard condition will imply the existence and the convergence.

**Theorem 3.7.** In addition to the conditions in Theorem 3.6 assume that \( V(x) \to \infty \) as \( |x| \to \infty \). Then \( \mu(\cdot) \) converges, in the metric of weak convergence, as \( t \to \infty \) to the family of \( \omega \)-limit points of the limit solution.

**Proof.** The growth condition on the Liapunov function together with \( U(t) \) being non-increasing, imply that the supports of \( \mu(t) \) for \( t \geq 0 \) are all contained in a compact set of \( \mathbb{R}^n \). Hence the measures themselves are contained in a compact set in the space of measures. The claimed existence and convergence then is a simple consequence of the compactness.

The invariance alluded to in the previous results is the invariance of the limit level set of the Liapunov function, with respect to the fast equation. One may inquire about invariance with respect to the slow contribution to the equation, namely the element \( G(\cdot) \) in (1.1). To that end, notice that condition (3.5), that played a role in the former three results, is an assumption on the averaged derivative, in particular it allows that \( (\nabla V)(x) \cdot G(x) > 0 \) for some points in the support of the invariant measure, even on a set of positive measure. In that respect the condition is weaker than the classical condition in the Liapunov direct method. The following two items examine the extent to which (3.5) holds on the invariant level set.

**Remark 3.8.** A reasonable conjecture would be that (3.5) with equality, namely,
\[ \int_{\mathbb{R}^n} (\nabla V)(x) \cdot G(x) \, \mu_0(dx) = 0, \quad (3.10) \]
would hold for every \( \mu_0 \) in the \( \omega \)-limit set of \( \mu(\cdot) \). This, however, may not hold, as Example 5.2 below demonstrates. What can be established is that (3.10) holds for at least one measure \( \mu \) in the \( \omega \)-limit set of \( \mu(\cdot) \), as the next result assures. On the other hand, the level set may not be invariant with respect to the entire dynamics even if (3.10) holds for all the measures that are invariant with respect to the fast dynamics. This is demonstrated in Example 5.3 below.

**Theorem 3.9.** Under the conditions of the Invariance Principle, let \( \mu(\cdot) \) be a limit solution. Then (3.10) holds for at least one \( \omega \)-limit point \( \mu_0 \) in the \( \omega \)-limit set of \( \mu(\cdot) \). In particular, if (3.5) holds with strict inequality, namely,

\[
\int_{\mathbb{R}^n} (\nabla V)(x) \cdot G(x) \nu(dx) < 0,
\]

(3.11)

for all the measures \( \nu \) that are invariant with respect to the fast equation (2.2) and with support included in a level set \( C \) of the Liapunov function, then \( C \) is not a level set to which a limit solution converges.

**Proof.** The second claim has been, essentially, established in the proof of Theorem 5.1 in Artstein [4]. The latter assures that if (3.11) holds on a level set, and if \( \mu(\cdot) \) gets close to the level set, then it crosses the level set in a prescribed finite time. (The statement of the theorem in Artstein [4] is global, but the proof applies to a neighborhood of such a level set.) The idea behind the proof is that if inequality in (3.11) holds on the level set, it holds uniformly on a neighborhood of it. Then estimates similar to (3.8), and consequently (3.9), hold with uniformly strict inequality. The argument is valid also to (3.11) being valid on an \( \omega \)-limit set of a limit solution, hence proving the claimed result.

4. **Invariance principle for split dynamics.** A particular case of the singular perturbations model (1.1) is the classical Tikhonov model. It takes the form

\[
\frac{dz}{dt} = f(z, y) \\
\frac{dy}{dt} = \frac{1}{\varepsilon} g(z, y),
\]

(4.1)

with \( z \in \mathbb{R}^k \) and \( y \in \mathbb{R}^m \). In this model the state space is split into a slow variable, here \( z \), and a fast variable, here \( y \). See O’Malley [22], Tikhonov, Vasil’ev and Svishnikov [26], Verhulst [28]. In these references, as well as in the vast literature on the model (4.1), the prevailing assumption is that for a fixed slow variable \( z \), the dynamics of the \( y \)-equation tends to an equilibrium point of the fast equation of (4.1), namely, an equilibrium of

\[
\frac{dy}{ds} = g(z, y).
\]

(4.2)

This equilibrium point is a solution \( y(z) \) of the algebraic equation \( 0 = g(z, y) \). This approach has been applied in a large number of applications, as the convergence to equilibrium assumption holds in many real-life situations. Asymptotic stability of the limit trajectory of (4.1) under the convergence to equilibrium assumption, was also examined, see, e.g., Hoppensteadt [16]. The Liapunov function then takes the form \( V(z, y) \) and for the limit solutions it has the form \( V(z, y(z)) \). The convergence to equilibrium assumption is then satisfied when \( V(z, \cdot) \) is strictly decreasing, along solutions of (4.2), except at the equilibrium \( y(z) \). The invariance principle is then straightforward, along the lines of the classical formulation of LaSalle.
There are examples, however, where solutions of (4.2) do not converge to an equilibrium, but rather, keep rapidly oscillating, indefinitely. See, e.g., Tao, Owhadi and Marsden [25]. Within the model (4.1) this means that the solutions converge to Young measures. The corresponding theory has been developed, see Artstein and Vigodner [7], Artstein [3], and was applied in a variety of situations, see, e.g., Artstein and Slemrod [6].

It is clear that (4.1) is a particular case of (1.1) where \( x = (z, y), \ F(z, y) = (0, g(z, y)) \) and \( G(z, y) = (f(z, y), 0) \). Under our assumptions on the equation (4.1), the existence of a Young measure limit takes a form as follows.

**Theorem 4.1.** For every sequence \( \varepsilon_i \to 0 \) and solutions \((z_{\varepsilon_i}(\cdot), y_{\varepsilon_i}(\cdot))\) of the perturbed equation (4.1), defined on \([0, \infty), \) with \((z_{\varepsilon_i}(0), y_{\varepsilon_i}(0))\) in a bounded set, there exists a sub-sequence \( \varepsilon_j \) such that as \( j \to \infty \) the sequence \( z_{\varepsilon_j}(\cdot) \) converges, uniformly on bounded intervals, to a trajectory \( z_0(\cdot) \), and the sequence \( y_{\varepsilon_j}(\cdot) \) converges in the sense of Young measures, to a Young measure \( \nu_0(\cdot) \) with values being probability measures on \( \mathbb{R}^m \), such that for every \( t \) the value \( \nu_0(t) \) is an invariant measures of the fast equation (4.2) with \( z = z(t) \) fixed. Furthermore, the ordinary differential equation

\[
\frac{dz}{dt} = \int_{\mathbb{R}^m} g(z, y) \nu_0(t)(dy),
\]

(4.3)

governs the evolution of the slow trajectory \( z_0(\cdot) \).

Notice that the statement of the theorem refers to an invariant measure, rather than to a limit occupational measure as in the statement concerning the general case. Indeed, the value of the Young measure \( \nu_0(\cdot) \) are limit occupational measures of the system with \((z, y)\) being the state. Namely, limits of measures \( \mu_{(z_j, y_j, s_j)} \) (see Section 2) where \( x_j = (z_j, y_j) \) and where \( z_j \) is fixed when solving (4.2), yet \( z_j \to z_0 \). However, one cannot guarantee that the invariant measure in the statement is a limit occupational measure with \( z = z(t) \) fixed.

The additional components in the preceding theorem, relative to the existence result in Theorem 2.2, are the uniform convergence of the slow variable and the averaging equation (4.3). The result has been verified in Artstein and Vigodner [7], see also Artstein [3].

Since (4.1) is a particular case of (1.1), the invariance principle established for the latter applies also in the present case. We state now the form it takes. In what follows \( V(z) = V(z, y) \) is the Liapunov function. Note that the values of the limit solutions of the system are of the form \((z, \nu)\) with \( \nu \) being an invariant measure of (4.2) with \( z \) fixed. In what follows we adopt both Assumption 2.1 and Assumption 3.1, as applied to the system (4.1).

**Corollary 4.2.** Let \((z(\cdot), \nu(\cdot))\) be a limit solution of (4.1). For a set of full measure in the time line \( t \), the Liapunov function \( V(z(t), y) \), as a function of \( y \), is constant on \( \text{supp}(\nu(t)) \).

**Proof.** This is a particular case of Corollary 3.3. \( \square \)

In what follows we denote by \( \nabla_z V \) the gradient of \( V(z, y) \) with respect to the \( z \)-variable for \( y \) fixed, and by \( \nabla_y V \) the gradient of \( V(z, y) \) with respect to the \( y \)-variable for \( z \) fixed.

**Theorem 4.3.** Let \( V(\cdot, \cdot) \) be a positive definite on \( \mathbb{R}^k \times \mathbb{R}^m \). Suppose that \( V(z, \cdot) \) is non-increasing along solutions of the fast equation (4.2), and in addition suppose
that
\[
\int_{\mathbb{R}^m} \nabla_z V(z, y) \cdot f(z, y) \nu(dy) \leq 0
\] (4.4)
for every invariant measure \(\nu\) of (4.2) with \(z\) fixed. Let \((z(\cdot), \nu(\cdot))\) be a limit solution of (4.1). Denote by \(U(t)\) the common value of \(V(z(t), y)\) for \(y\) in the support of \(\nu(t)\). Then \(U(t)\) is non-increasing.

**Proof.** This is the appearance of Theorem 3.5 in the present framework. \(\square\)

Notice that the Liapunov function depends on both the slow and the fast variable. Thus, even if \(V\) is separable, namely \(V(z, y) = V_1(z) + V_2(y)\), the conclusion of the theorem does not imply that either \(V_1(z(t))\) or \(V_2(\nu(t))\) (the latter being the common value on the support of \(\nu(t)\)), do not increase.

As in the general case, we shall use \(\omega\)-limit sets of limit solutions of (4.1). Here an \(\omega\)-limit point has the form \((z_0, \nu_0)\) which is a limit in \(\mathbb{R}^k\) in the first coordinate, and a weak limit in the second, of a sequence \((z_{0j}, \nu_{0j})\) of values on the limit solution.

**Theorem 4.4.** Let \(V(\cdot, \cdot)\) be a positive definite on \(\mathbb{R}^k \times \mathbb{R}^m\). Suppose that \(V(z, \cdot)\) is non-increasing along solutions of the fast equation (4.2), and in addition suppose that (4.4) holds for every invariant measure \(\nu\) of (4.2) with \(z\) fixed. Let \((z(\cdot), \nu(\cdot))\) be a limit solution of (4.1). Denote by \(U(t)\) the common value of \(V(z(t), y)\) for \(y\) in the support of \(\nu(t)\). Then all \(\omega\)-limit points of the limit solution are supported on the level set of \(V\) determined by the limit of \(U(t)\) as \(t \to \infty\). The union of these supports is invariant with respect to the fast equation (4.2) with the respective slow coordinate \(z\). If, in addition, \(V(z, y) \to \infty\) as \(|z| + |y| \to \infty\), then the limit dynamics converges, as \(t \to \infty\), to its \(\omega\)-limit set.

**Proof.** This is the appearance of Theorems 3.6 and 3.7 in the present framework. \(\square\)

**Theorem 4.5.** Under the conditions of Theorem 4.4 suppose a strict inequality
\[
\int_{\mathbb{R}^m} \nabla_z V(z, y) \cdot f(z, y) \nu(dy) < 0
\] (4.5)
holds for every invariant measure \(\nu\) supported on a given level set of the Liapunov function. Then this level set is not one that supports the \(\omega\)-limit set of a limit solution of (4.1). In particular, for any limit solution, there is at least one \(\omega\)-limit point where (4.4) holds with equality.

**Proof.** This is the appearance of Theorem 3.9 in the present framework. \(\square\)

The next section includes examples demonstrating the limit behavior just established.

5. The limit versus the perturbed dynamics. Let \(\mu(\cdot)\) be a limit solution of (1.1), generated by a sequence \(x_{\epsilon_j}(\cdot)\) of solutions of (1.1), as \(j \to \infty\). Suppose the initial conditions \(x_{\epsilon_j}(0)\) are all contained in a compact set, say \(x_{\epsilon_j}(0) \to x_0\). What can we learn about the solutions of the perturbed equation from the structure of the limit?

Recall that in the definition of the limit we first fix the time \(t\), then define the Young measure \(\mu(\cdot)\) on the interval \([0, t]\) as the limit of \(x_{\epsilon_j}(\cdot)\). Then we may prolong the limit solution and consider the limit on a larger interval, even letting \(t \to \infty\). The essence of the construction is that we let \(\epsilon_j \to 0\) for \(t\) fixed, then let \(t\) grow. The implication to the perturbed equation is straightforward: For \(t\) fixed, when \(\epsilon_j\)
is small enough, the solution is close, in the Young measures sense, to the limit dynamics. Note, however, that being close in the Young measure sense does not imply that the graphs are near by, as the support of the Young measure may not depend continuously on the measure itself. What we know, however, is that for $\varepsilon_j$ small, most of the time the solution $x_{\varepsilon_j}(\cdot)$ spends near the graph of the limit solution. The limit solution itself tends to a level set of the Liapunov function, as guaranteed by the results in previous sections.

However, if $\varepsilon_j \to 0$ while $t_j \to \infty$, the behavior may exhibit different phenomena, at times counter-intuitive. Indeed, the limit behavior depends on the manner in which $\varepsilon_j$ tends to 0 while $t_j$ tends to infinity. In fact, it is possible that each individual solution goes through the level set to which the limit solution converges. We demonstrate such a behavior through examples, as follows.

We display four examples, demonstrating both the regular behavior as implied by the results above, and the peculiar behavior just mentioned. The four examples will be variations of the system (of the type (4.1)), written in polar coordinates, and given by

$$\begin{align*}
\frac{d\rho}{dt} &= -\rho(1 - \rho) - r(\rho, \theta) \\
\frac{d\theta}{dt} &= \frac{1}{\varepsilon} h(\rho, \theta).
\end{align*}$$

(5.1)

The term $h(\rho, \theta)$ will be the same for the four examples. We set $h(\rho, \theta) > 0$ except for a sequence of interval in the $$(\rho, \theta)$$ plane defined as follows. Consider the sequence of real intervals $I_j = [1 + 2^{-(j+1)}, 1 + 2^{-j}]$, for $j = 0, 1, 2, \ldots$. Then $h(\rho, \frac{1}{2}\pi) = 0$ for $\rho < c$ and $j$ even, and $h(\rho, \frac{3}{2}\pi) = 0$ for $\rho < c$ and $j$ odd. Also (by continuity) $h(1, \frac{1}{2}\pi) = 0$ and $h(1, \frac{3}{2}\pi) = 0$.

The term $r(\rho, \theta)$ will vary from one example to another, but it will always be non-negative. Thus, the function $V(\rho, \theta) = \rho^2$ is a Liapunov function for the system.

**Example 5.1.** Consider (5.1) with $r(\rho, \theta) = 0$.

If the initial condition $(\rho_0, \theta_0)$ satisfies $\rho_0 < 1$, then each solution of (5.1) emanating from this initial condition is spiraling toward the origin, with increasing $\theta$-velocity as $\varepsilon \to 0$. The limit solution consists then of measures $\nu(t)$ each distributed on a circle $\rho = c$ with $0 < c \leq \rho_0$ (the exact distribution depends on the positive function $h(\cdot, \cdot)$). These measures converge to the origin, thus the origin is its unique $\omega$-limit point.

If the initial condition $(\rho_0, \theta_0)$ satisfies $\rho_0 = 1$, then each solution of (5.1) emanating from the initial condition stays on the invariant circle $\rho = 1$, and converges to either $(1, \frac{1}{2}\pi)$ or $(1, \frac{3}{2}\pi)$, depending on the initial state $\theta_0$. Consequently the limit solution is the fixed Dirac measure supported on the corresponding limit point.

Suppose now that the initial condition satisfies $\rho_0 > 1$, say $\rho_0 < 2$. The solution will then spiral, fast as $\varepsilon$ is small, toward one of the intervals $I_j$, say with $\theta = \frac{1}{2}\pi$, then slide slowly close to the interval toward $\rho = 1$, until the lower end of the interval, then will circle fast toward the interval $I_{j+1}$, and so on, slowly approaching the circle $\rho = 1$. Consequently, the limit solution consists of a point-valued map, sliding along the $I_j$ intervals toward the unit circle, with discontinuities at the end of each interval. There are two $\omega$-limit points of the limit solution, namely, the points $(1, \frac{1}{2}\pi)$ and $(1, \frac{3}{2}\pi)$, each, of course, being a fixed point of the fast dynamics. In this case these limit points are also fixed points of the entire system. (Convex combinations in the space of measures, of the Dirac measures supported on these two
points, are also limit points, but of points of discontinuity of the limit dynamics, see Remark 2.5.) This is also an example where the graphs of the perturbed trajectories may not be close to the graph of the limit solution.

**Example 5.2.** Consider now (5.1) with \( r(\rho, \theta) > 0 \) inside a small circle, say of radius \( \frac{1}{10} \), around the point \((1, \frac{1}{2}\pi)\), and \( r(\rho, \theta) = 0 \) otherwise.

As in the last case of the previous example, suppose now that the initial condition satisfies \( \rho_0 > 1 \), again say \( \rho_0 < 2 \). The solution will then spiral, fast when \( \varepsilon \) is small, toward one of the intervals \( I_j \), say with \( \theta = \frac{1}{2}\pi \), then slide on the interval toward \( \rho = 1 \) until the lower end of the interval, then will circle with high speed toward the interval \( I_{j+1} \), and so on. When on the branch \( \theta = \frac{1}{2}\pi \), the sliding on the interval toward \( \rho = 1 \) stays away from zero when \( \rho < 1 + \frac{1}{10} \), but the smaller the \( \varepsilon \), the faster the solution will cross this region where sliding to \( \rho = 1 \) is not negligible, and will continue toward the interval on the other branch \( \theta = \frac{3}{2}\pi \). On the latter branch the convergence toward \( \rho = 1 \) will still becoming slow when close to \( \rho = 1 \). All in all, for \( \varepsilon \) small, on a long time interval the solution will slowly approach the circle \( \rho = 1 \).

Consequently, the limit solution is the same as in the former example, namely, it consists of a point-valued map, sliding along the \( I_j \) intervals toward the unit circle, with discontinuities at the end of each interval. The \( \omega \)-limit points are then, again, the points \((1, \frac{1}{2}\pi)\) and \((1, \frac{3}{2}\pi)\), each, of course, a fixed point of the fast dynamics. But note that this time the point \((1, \frac{3}{2}\pi)\), while still being a fixed point of the fast dynamics, is not a fixed point of the entire perturbed dynamics (see Remark 3.8).

In addition, however, each individual trajectory \((\rho_t(\cdot), \theta_t(\cdot))\), with \( \varepsilon \) fixed, will cross the circle \( \rho = 1 \) at a certain time. The crossing time is large for a smaller \( \varepsilon \). Thus, although the limit solution stays in the region \( \rho > 1 \) for all \( t \), every trajectory converges to the origin as \( t \to \infty \). This is the subtle behavior we alluded to. In fact, it is easy to determine sequences \((\varepsilon_k, t_k, T_k)\), with \( \varepsilon_k \to 0 \) and both \( t_k \) and \( T_k \) converging to \( \infty \), such that the limit occupational measure determined by the trajectories on the time intervals \([t_k, T_k]\), will be a measure supported on a circle \( \rho = c \), with an arbitrary \( c \in [0, 1] \).

It is easy to see that if the initial condition satisfies \( \rho_0 < 1 \), the limit dynamics is the same as in the previous example. If \( \rho_0 = 1 \) then the limit dynamics is also exactly as in the previous example. However, individual solutions with \( \theta_0 \) between \( \frac{1}{2}\pi \) and \( \frac{3}{2}\pi \) (but not within a distance \( \frac{1}{10} \) of \( \frac{1}{2}\pi \)), will converge to the fixed point of the fast dynamics, while those emanating from the complement of the unit circle will converge to the origin.

**Example 5.3.** In the previous example the inequality in (3.5) is strict on an \( \omega \)-limit point of the limit solution. One may ask whether this is the reason for the penetration of individual solutions through the limit level set. The answer is negative. Indeed, we can modify a bit the example by letting \( r(\rho, \theta) > 0 \) inside a small circle, again, say of radius \( \frac{1}{10} \), but this time around the point \((1, 0)\) (rather than around \((1, \frac{1}{2}\pi)\)), and let \( r(\rho, \theta) = 0 \) otherwise. The limit solution is not affected. Now the inequality (3.5) is equality on every \( \omega \)-limit point. Yet, each solution of (5.1) crosses the level set \( \rho = 1 \). (Of course, if we construct the example such that the limit level set is invariant for both the fast dynamics and the entire dynamics, then no solution will cross it.)

**Example 5.4.** Consider now (5.1) with \( r(\rho, \theta) > 0 \) inside two small circles, again say of radius \( \frac{1}{10} \), around, respectively, the point \((1, \frac{1}{2}\pi)\) and \((1, \frac{3}{2}\pi)\), and \( r(\rho, \theta) = 0 \) otherwise.
Suppose now that the initial condition satisfies $\rho_0 > 1$, again say $\rho_0 < 2$. The solution will then spiral, fast as $\varepsilon$ is small, toward one of the intervals $I_j$, say with $\theta = \frac{1}{2}\pi$, then slide on the interval toward $\rho = 1$ until the lower end of the interval, then will circle, with high speed, toward the interval $I_{j+1}$, and so on. Now, however, the velocity toward the unit circle is positive and bounded from below on each of the two branches determined by $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$. Hence, in a finite time shared by all trajectories, the trajectory will cross the unit circle, as guaranteed by Theorem 3.9 or Theorem 4.5. All in all, on a finite time interval, limit solution is the same as in the previous two examples, namely, consisting of alternating fixed points of the fast dynamics, sliding toward $\rho = 1$. Past this crossing time of the unit circle, the limit solution exhibits invariant measures with respect to the fast dynamics, each distributed on a circle with radius $c$ and $c \to 0$ as $t \to \infty$. The origin is then the unique $\omega$-limit point of both the limit solution and each of the trajectories of the perturbed system.

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