Eigen-functional bosonization and Eikonal-type equations in one-dimensional strongly correlated electron systems

Yu-Liang Liu

Center for Advanced Study, Tsinghua University, Beijing 100084, People’s Republic of China

Abstract

With the eigen-functional bosonization method, we study one-dimensional strongly correlated electron systems with large momentum ($2k_F$ and/or $4k_F$) transfer term(s), and demonstrate that this kind of problems ends in to solve the Eikonal-type equations, and these equations are universal, and independent of whether or not the system is integrable. In contrast to usual perturbation theory, this method is valid not only for weak electron interaction, but also for strong electron interaction. Comparing with exact solution of some integrable models, it can give correct results in one-loop approximation. This method can also be used to study electron-phonon interaction systems, and two coupled spin chain or quantum wire systems.

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I. INTRODUCTION

In contrast to two- and three-dimensional electron systems, one-dimensional electron system has a prominent feature that its Fermi “surface” is composed of two points $\pm k_F$, defined by the Fermi momentum $k_F$, and its Hilbert space is strongly suppressed. For a one-dimensional electron gas, there only exist two kinds of electron excitations: one is near these two Fermi levels $\pm k_F$, respectively, usually called as small momentum excitation, and another one is the excitation between these two Fermi levels, called as large (crossover) momentum excitation. If there is only the former excitation, the system generally shows the Luttinger liquid behavior. After turning on the latter excitation, the system becomes an insulator for repulsive electron interaction.

In low energy regime, the physics property of a one-dimensional electron system is determined by the electron states near its two Fermi levels $\pm k_F$. Therefore, it is reasonable to linearize electron energy spectrum near these two Fermi levels with a band-width parameter $D$, and the electrons are divided into two parts: one presents the electrons near the Fermi level $+k_F$, called right-moving electrons with the energy spectrum $\epsilon_R(k) = v_F k$, and another one presents the electrons near the Fermi level $-k_F$, called left-moving electrons with the energy spectrum $\epsilon_L(k) = -v_F k$, where $v_F$ is the electron Fermi velocity. It is shown that the high order terms like $k^n, n = 2,3,...$, only contribute high order irrelevant terms to the effective action of the system, and can be neglected. The crossover (large momentum) excitation corresponds to the excitation between the right- and left-moving electrons with $2k_F$ and/or $4k_F$ (or higher) momentum transfer. In some special cases, the crossover excitation is very important, and determines the low energy behavior of the system, because the crossover excitation term is relevant, and strongly affected by the electron interaction. Generally, usual perturbation methods fail to treating the crossover excitation term for the strong electron correlation.

The bosonization method is very powerful in studying of one-dimensional interacting electron systems, such as quantum wires, spin chains and impurity scattering in one-dimensional
fermion systems, where it can exactly treat electron’s density-density interactions \cite{1-5}, such as the interaction \(V(x-y)\rho_{R\sigma}(x)\rho_{L\sigma}(y)\). However, for example, if there meantime appear an interaction term \(V\rho_{R\sigma}(x)\rho_{L\sigma}(x)\) and a crossover term \(\lambda[\psi_{R\sigma}^\dagger(x)\psi_{L\sigma}(x)+\psi_{L\sigma}^\dagger(x)\psi_{R\sigma}(x)]\) in the Hamiltonian of the system, it is hard to use the usual bosonization and functional bosonization methods to effectively treat these two terms as a whole, because the \(\lambda\)-term mixes the right- and left-moving electrons which induces a highly relevant term, in the bosonization representation of the electron fields, it reduces to some cosine- and/or sine- terms. The usual bosonization methods can only tell us which ones are relevant or irrelevant, but cannot be used to effectively treat these terms, such as calculating spin and charge collective excitation spectrums, correlation functions, effective action of the system, and so on. Therefore, the usual bosonization methods do not provide more useful informations than usual perturbation methods. It is a big challenge to effectively treat these crossover excitation terms which often appear in one-dimensional strongly correlated electron systems, and desires to find a new method which can be used to exactly and effectively treat them.

In this paper, we introduce a new method \cite{6}, called the eigen-functional bosonization method, which is very simple and effective to treating these crossover terms, and show that the crossover terms can be exactly treated by solving the Eikonal-type equations. It is wonderful that the problems of one-dimensional strongly correlated electron systems end in to solve the Eikonal-type equations that can be exactly treated by a series expansion and/or computer calculations. This method is universal, and independent of whether or not the system is integrable. For example, it can also be used to treat electron-phonon interaction systems, and two- and three-dimensional electron systems \cite{7}. In Sect. II, for simplicity, we introduce the eigen-functional bosonization method by studying a one-dimensional spinless fermion system with a staggered chemical potential or a dimerized lattice sites, and show that this kind of problem ends in to solve simple Eikonal-type equations. At one-loop approximation, our calculation is consistent with the exact solution of the quantum sine-Gordon model. In Sect. III, with this method, we study the Hubbard model at half filling, and calculate charge collective excitation gap which is agreement with the exact solution of
the Hubbard model at half filling. In Sect. IV, we study the Hubbard model at half filling with a staggered chemical potential, and show that there is a phase transition from band insulator to Mott-type insulator at the electron interaction strength $V_T$ where the charge collective excitation gap is zero, and a spontaneous lattice dimerization takes place. This property of the system can be used to explain the origin of the ferroelectricity of transition metal oxides [8]. The conclusion and discussions are given in Sect. V.

II. EIGEN-FUNCTIONAL BOSONIZATION AND ITS APPLICATION

In this section, we introduce the eigen-functional bosonization method by studying a one-dimensional spinless fermion system with a staggered chemical potential or a dimerized lattice sites. It is well-known that in usual bosonization representation of fermion fields, the system reduces into the quantum sine-Gordon model (QSGM) which is integrable, and has been exactly solved [9]. The exact solution of the QSGM is a criterion for this new method. We show that in one-loop approximation this method can give correct results for any fermion interaction strength, whereas usual perturbation methods are invalid for strong fermion interaction.

Generally, we consider a strongly correlated spinless fermion system with the Hamiltonian

$$H = -i \hbar v_F \int dx [\psi_R^\dagger(x) \partial_x \psi_R(x) - \psi_L^\dagger(x) \partial_x \psi_L(x)]$$
$$- \int dx [\lambda \psi_R^\dagger(x) \psi_L(x) + \lambda^* \psi_L^\dagger(x) \psi_R(x)] + V \int dx \rho_R(x) \rho_L(x) \quad (1)$$

where $\psi_{R(L)}(x)$ is the right(left)-moving fermion field, and $\rho_{R(L)}(x) = \psi_{R(L)}^\dagger(x) \psi_{R(L)}(x)$ is the right(left)-moving fermion density. The $\lambda$-term represents a staggered chemical potential ($\lambda = \lambda^* = \Delta_0$), or a lattice dimerization parameter ($\lambda = -\lambda^* = iu$). Generally, we can write $\lambda$ as $\lambda = \Delta_0 + iu$. We can also study the influence of quantum fluctuation of electron-acoustic-phonon interaction on the lattice dimerization by adding the term, $u_a(x)[\rho_R(x) + \rho_L(x)]$. In bosonization representation, $\psi_{R(L)}(x) = (\frac{1}{2\pi\alpha})^{1/2} exp\{-i\Phi_{R(L)}(x)\}$, where $\partial_x \Phi_{R(L)}(x) = \pm 2\pi \rho_{R(L)}(x), \alpha \sim 1/D$, and $D$ is the band-width, we have the standard QSGM [11] ($u = 0$),
\[ H_{SC} = \frac{\hbar v_c}{2} \int dx \{ [\partial_x \Phi(x)]^2 + \Pi^2(x) + \frac{m^2}{\beta^2} \cos(\beta \Phi(x)) \} \]

where, \( m^2/\beta^2 = \lambda/(\pi \alpha \hbar v_c), \) \( \beta^2/(4\pi) = (1 - \gamma)^{1/2}/(1 + \gamma)^{1/2} = g, \) \( v_c = \sqrt{v_F(1 - \gamma^2)}, \)
\( \gamma = V/(2\pi \hbar v_F), \) \( \Phi(x) = (\Phi_R(x) - \Phi_L(x))/\beta, \) \( \Pi(x) = \beta [\partial_x \Phi_R(x) + \partial_x \Phi_L(x)]/(4\pi), \) and \( [\Pi(x), \Phi(y)] = i\delta(x - y). \) The parameter \( m \) is related to the mass of the solitons and anti-solitons or breathers \( [\Pi]. \) According to the relation \( \beta^2 = 4\pi g, \) at \( \beta^2 = 4\pi, \) the Hamiltonian \( [\Pi] \) describes a free massive fermion system \( (V = 0). \) As \( \beta^2 > 4\pi (g > 1), \) there is an attractive interaction between the fermions \( \psi_R(x) \) and \( \psi_L(x). \) However, due to the crossover term \( (\lambda \text{-term}) \) opens a gap in fermion excitation spectrum, the system is stable. As \( \beta^2 < 4\pi (g < 1), \) the fermion interaction is repulsive. In the fermionization representation \( [\Pi], [\Pi], \) it is very clear that in the regime \( 0 < \beta^2 < 4\pi \) the QSGM represents different physics from that in the regime \( 4\pi < \beta^2 < 8\pi. \) The critical point \( \beta^2 = 4\pi \) is a phase transition point which separates one phase from another one.

After introducing auxiliary boson fields \( \phi_R(x, t) \) and \( \phi_L(x, t) \) which decouple the four-fermion interaction by adding the constraints, \( \rho_{R(L)}(x, t) = \psi^\dagger_{R(L)}(x, t) \psi_{R(L)}(x, t), \) we only have the quadratic terms of the fermion fields \( \psi_{R(L)}(x, t). \) Integrating out them, we obtain an effective action,
\[
S[\rho, \phi] = W[\phi] - \int dt dx \{ \phi_R(x, t) \rho_R(x, t) + \phi_L(x, t) \rho_L(x, t) + V\rho_R(x, t) \rho_L(x, t) \}
\]
\[
W[\phi] = i \text{Tr} \ln(M_0) + i \int_0^1 d\xi \int dt dx \text{Tr} \left( \phi(x, t) \tilde{G}(x, t; x', t', [\xi \phi]) \right) |_{x' \to x} \]
\[
(2)
\]
where \( M = M_0 + \phi, \) \( \mathcal{D}_{R(L)} = i \partial_t \pm i \hbar v_F \partial_x, \) \( M \tilde{G}(x, t; x', t', [\phi]) = \delta(x - x') \delta(t - t'), \) and
\[
M_0 = \begin{pmatrix} \mathcal{D}_R, \lambda \\ \lambda^*, \mathcal{D}_L \end{pmatrix}, \quad (x, t) = \begin{pmatrix} \phi_R(x, t), 0 \\ 0, \phi_L(x, t) \end{pmatrix}.
\]

In order to calculate the Green’s functional \( \tilde{G}(x, t; x', t', [\phi]), \) we solve the eigen-functional equation \([3]\) of the operator \( M, \)
\[
M \Psi^{(i)}_{k, \omega}(x, t, [\phi]) = E^{(i)}_{k, \omega} \Psi^{(i)}_{k, \omega}(x, t, [\phi])
\]
(3)

With the orthogonality and normalization of the eigen-functionals,
\[
\sum_{k,\omega} \Psi_{k,\omega}^{(+)}(x, t, [\phi]) \Psi_{k,\omega}^{(+\dagger)}(x', t', [\phi]) = \delta_{ii'} \delta(x - x') \delta(t - t'),
\]

\[
\int dt dx \Psi_{k,\omega}^{(+\dagger)}(x, t, [\phi]) \Psi_{k',\omega'}^{(+)}(x, t, [\phi]) = \delta_{kk'} \delta_{\omega\omega'}
\]

the Green’s functional can be written as,

\[
\tilde{G}(x, t; x', t', [\phi]) = \sum_{i} \sum_{k,\omega} \frac{1}{E_{k,\omega}^{(i)}[\phi]} \Psi_{k,\omega}^{(i)}(x, t, [\phi]) \Psi_{k,\omega}^{(i\dagger)}(x', t', [\phi]) = \delta_{ii'} \delta_{kk'} \delta_{\omega\omega'}
\]

If we can exactly solve the eigen-functional equation (3), generally we can calculate the Green’s functional (4), and the effective action (2) which only includes the boson fields. This process is called the eigen-functional bosonization.

At \(\phi(x, t) = 0\) (interacting-free case), the system becomes a band insulator, and the eigen-equation (3) can be exactly solved. It has two sets of solutions corresponding to the eigen-values \(E_{k,\omega}^{(\pm)}[0] = \omega \mp E_k\), respectively, where \(E_k = \sqrt{k^2 + |\lambda|^2}\) (choosing \(\hbar = v_F = 1\)),

\[
\Psi_{k,\omega}^{(\pm)}(x, t, [0]) = \left(\frac{1}{TL}\right)^{1/2} \begin{pmatrix} u_k e^{i\theta} \\ -v_k e^{-i\theta} \end{pmatrix} e^{ikx - i\omega t}
\]

where \(u_k^2 = 1 - v_k^2 = (1 + k/E_k)/2\), \(\tan(2\theta) = u/\Delta_0\), and \(T\) and \(L\) are the time and space lengths of the system, respectively.

At \(\lambda = 0\), the Hamiltonian (1) represents the Luttinger liquid, and the eigen-functional equation (3) has the exact solutions with the eigen-values \(E_{k,\omega}^{(\pm)}[0] = \omega \mp k\), respectively,

\[
\Psi_{k,\omega}^{(\pm)}(x, t, [0]) = \left(\frac{1}{TL}\right)^{1/2} \begin{pmatrix} v_k e^{i\theta} \\ u_k e^{-i\theta} \end{pmatrix} e^{ikx - i\omega t}
\]

where \(D_{R(L)} Q_{R(L)}^0(x, t) = -\phi_{R(L)}(x, t)\), ant \(\theta(x)\) is a step function, \(\theta(x) = 1\) for \(x > 0\), and \(\theta(x) = 0\) for \(x < 0\). With the exact solution (3), we can easily obtain the effective action of the system, and the fermion Green’s function which are the same as that in Ref. [4,5].

In general, for \(\phi(x, t) \neq 0\), we can choose the following formal solutions with the eigenvalues \(E_{k,\omega}^{(\pm)}[\phi] = \omega \mp E_k + \Sigma_k^{(\pm)}[\phi]\), respectively,
\[
\Psi_{k,\omega}^{(+)}(x,t,[\phi]) = A_k \left( \frac{1}{T L} \right)^{1/2} \begin{pmatrix}
    u_k e^{i\theta} e^{Q_R(x,t)} \\
    -v_k e^{-i\theta} e^{Q_L(x,t)}
\end{pmatrix} e^{ikx-i(\omega+\Sigma^{(+)}_k[\phi])t},
\]

\[
\Psi_{k,\omega}^{(-)}(x,t,[\phi]) = A_k \left( \frac{1}{T L} \right)^{1/2} \begin{pmatrix}
    u_k e^{i\theta} e^{\tilde{Q}_R(x,t)} \\
    -v_k e^{-i\theta} e^{\tilde{Q}_L(x,t)}
\end{pmatrix} e^{ikx-i(\omega+\Sigma^{(-)}_k[\phi])t},
\]

where \(\Sigma^{(\pm)}_k[\phi] = \int_0^1 d\xi \int dt dx \Psi_{k,\omega}^{(\pm)}(x,t,[\xi \phi])\phi(x,t)\Psi_{k,\omega}^{(\pm)}(x,t,[\xi \phi])\) are independent of \(\omega\), and \(|A_k| \sim 1\) is a normalization constant. The boson fields \(Q_R(x,t)\) and \(\tilde{Q}_R(x,t)\) satisfy the differential equations,

\[
\begin{align*}
(D_R + \phi_R(x,t))e^{Q_R(x,t)} - \frac{|\lambda|u_k}{u_k}(e^{Q_L(x,t)} - e^{Q_R(x,t)}) &= 0, \\
(D_L + \phi_L(x,t))e^{Q_L(x,t)} - \frac{|\lambda|u_k}{u_k}(e^{Q_R(x,t)} - e^{Q_L(x,t)}) &= 0, \\
(D_R + \phi_R(x,t))e^{\tilde{Q}_R(x,t)} + \frac{|\lambda|u_k}{u_k}(e^{\tilde{Q}_L(x,t)} - e^{\tilde{Q}_R(x,t)}) &= 0, \\
(D_L + \phi_L(x,t))e^{\tilde{Q}_L(x,t)} + \frac{|\lambda|u_k}{u_k}(e^{\tilde{Q}_R(x,t)} - e^{\tilde{Q}_L(x,t)}) &= 0,
\end{align*}
\]

With the following definitions

\[
Q_R(x,t) = Q^0_R(x,t) + f_R(x,t), \quad \tilde{Q}_R(x,t) = Q^0_R(x,t) + \tilde{f}_R(x,t)
\]

the differential equation \((8)\) can be re-written as,

\[
\begin{align*}
(D_L D_R + iE_k \partial_t + ik \partial_x) f_R + D_R f_R D_L f_R &= - \left( \frac{|\lambda|u_k}{u_k} + D_R f_R \right) D_L Q^0, \\
(D_R D_L + iE_k \partial_t + ik \partial_x) f_L + D_R f_L D_L f_L &= \left( \frac{|\lambda|u_k}{u_k} + D_L f_L \right) D_R Q^0, \\
(D_R D_L - iE_k \partial_t + ik \partial_x) \tilde{f}_R + D_R \tilde{f}_R D_L \tilde{f}_R &= \left( \frac{|\lambda|u_k}{u_k} - D_R \tilde{f}_R \right) D_L Q^0, \\
(D_R D_L - iE_k \partial_t + ik \partial_x) \tilde{f}_L + D_R \tilde{f}_L D_L \tilde{f}_L &= \left( \frac{|\lambda|u_k}{u_k} - D_L \tilde{f}_L \right) D_R Q^0
\end{align*}
\]

where \(Q^0(x,t) = Q^0_R(x,t) - Q^0_L(x,t)\). This is usual Eikonal-type equations \([14,15]\), and can be solved by a series expansion of \(Q^0(x,t)\), or by computer calculations. These equations are universal, and independent of whether or not the system is integrable. For example, after including the electron-acoustic-phonon interaction, they keep invariance if one replaces \(\phi_{R(L)}(x,t)\) by \(\phi_{R(L)}(x,t) - u_d(x,t)\).
The non-linear terms will produce the high-order terms of \((|\lambda|/\hbar v_F)Q^0(x, t)\), and contribute cubic or higher order terms of the boson fields \(\phi_{R(L)}(x, t)\) to the effective action, as \(|\lambda|/\hbar v_F \ll 1\) they can be neglected. Only keeping the linear terms (one-loop approximation), we can obtain

\[
\begin{pmatrix}
    f_R(q, \Omega) \\
    f_L(q, \Omega)
\end{pmatrix} = \frac{1}{\Omega^2 - q^2 + 2(E_k\Omega - kq)} \begin{pmatrix}
    \frac{|\lambda|u_k}{v_k}(\Omega + q) \\
    \frac{|\lambda|u_k}{v_k}(\Omega - q)
\end{pmatrix} Q(q, \Omega)
\]

\[
\begin{pmatrix}
    \tilde{f}_R(q, \Omega) \\
    \tilde{f}_L(q, \Omega)
\end{pmatrix} = \frac{1}{\Omega^2 - q^2 - 2(E_k\Omega + kq)} \begin{pmatrix}
    \frac{|\lambda|u_k}{v_k}(\Omega + q) \\
    \frac{|\lambda|u_k}{v_k}(\Omega - q)
\end{pmatrix} Q(q, \Omega)
\]

where \(Q(q, \Omega) = \phi_R(q, \Omega)/(\Omega - q) - \phi_L(q, \Omega)/(\Omega + q)\). Generally, these boson fields depend on the fermion momentum \(k\). However, the low energy physical property is determined by the fermions near the Fermi levels \(\pm k_F\), we therefore can take the approximation in (10), \(k \sim 0\) and \(E_k \sim |\lambda|\). Under this approximation, these boson fields are independent of the fermion momentum \(k\).

According to equations (4) and (7), the Green’s functional can be written as,

\[
\tilde{G}(x, t; x', t, [\phi]) = -i \begin{pmatrix}
    G_R(x - x')e^{Q_R(x,t)-Q_R(x',t)} & Z(x - x')e^{Q_R(x,t)-Q_L(x',t)} \\
    Z(x - x')e^{Q_L(x,t)-Q_R(x',t)} & G_L(x - x')e^{Q_L(x,t)-Q_L(x',t)}
\end{pmatrix}
\]

where \(G_{R(L)}(x) = G^0_{R(L)}(x) \pm g(x)\), \(G^0_{R(L)}(x) = \pm 1/(2\pi i x)\), \(g(x) = \frac{1}{4\pi} \int_{-D}^0 dk (\frac{k}{|k|} - \frac{k}{E_k})e^{ikx}\), and \(Z(x) = \frac{1}{4\pi} \int_{-D}^0 dk \frac{|\lambda|}{E_k}e^{ikx}\), where \(D\) is the band-width. Here we have used \(Q^*_{R(L)}(x, t) = -Q_{R(L)}(x, t)\), at one-loop approximation this relation is exact. With the relation (4),

\[
\tilde{G}(x, t; x', t', [\phi])_{x'\rightarrow x} = \frac{1}{2} \lim_{\eta \rightarrow 0} [\tilde{G}(x, t; x + \eta, t, [\phi]) + \tilde{G}(x, t; x - \eta, t, [\phi])]
\]

we can obtain the following effective action (keeping the same zero energy level as fermions, we replace \(\Omega + |\lambda| \) by \(\Omega\)),

\[
S[\phi, \rho] = \frac{1}{TL} \sum_{q, \Omega} \left\{ -\frac{q}{4\pi} \frac{\Omega + q}{D} |\phi_R(q, \Omega)|^2 + \frac{q}{4\pi} \frac{\Omega - q}{D} |\phi_L(q, \Omega)|^2 \right.
\]

\[
- \phi_R(-q, -\Omega)\rho_R(q, \Omega) - \phi_L(-q, -\Omega)\rho_L(q, \Omega) - V\rho_R(-q, -\Omega)\rho_L(q, \Omega) \right\}
\]

(13)
where \( D = \Omega^2 - q^2 - |\lambda|^2 \). With the functional expressions (7) of the fermion fields and the effective action (13), we can calculate any correlation functions of the system by taking functional average, for example, the fermion Green’s function is,

\[
\tilde{G}(x, t; x', t') = \frac{1}{Z} \int \prod_{i=R,L} D\phi_i D\rho_i \tilde{G}(x, t; x', t', [\phi]) e^{-iS[\phi, \rho]} \tag{14}
\]

where \( Z = \int \prod_{i=R,L} D\phi_i D\rho_i e^{-iS[\phi, \rho]} \). As \( \lambda = 0 \), we can obtain the usual fermion Green’s function [1–5]. At \( V = 0 \), we can obtain the expectation,

\[
\langle e^{i\alpha(\Phi_R(x,t) - \Phi_L(x,t))} \rangle \sim (\frac{|\lambda|}{\alpha D + \sqrt{\alpha^2 D^2 + |\lambda|^2}})^{\alpha^2}, \quad \alpha^2 = 1 - (\frac{V}{2\pi h v_F})^2,
\]

which are consistent with the exact results [16] where \( \varphi = \sqrt{2/g(\Phi_R - \Phi_L)} \).

After integrating out the auxiliary boson fields \( \phi_{R(L)}(x, t) \), we can obtain the collective excitation spectrums of the fermion density \( (\lambda \neq 0) \) from the effective action,

\[
\Omega_1 = \sqrt{(1 - \frac{1}{2} \frac{V}{2\pi})^2 q^2 + |\lambda|^2 + F(q, \lambda)}
\]

\[
\Omega_2 = \sqrt{(1 - \frac{1}{2} \frac{V}{2\pi})^2 q^2 + |\lambda|^2 - F(q, \lambda)} \tag{15}
\]

where \( F(q, \lambda) = \frac{1}{2} |\lambda| \sqrt{(\frac{V}{2\pi})^2 q^2 - 4|\lambda|^2} \). As \( V = 0 \), the energy spectrums reduces to \( \Omega_1 = \Omega_2 = \sqrt{q^2 + |\lambda|^2} \); as \( \lambda = 0 \), we only have one-branch energy spectrum \( \Omega_2 = \sqrt{1 - (\frac{V}{2\pi})^2 |q|} \), which is well-known in the usual bosonization representation. Generally, for a finite \( \lambda \), the function \( F(q, \lambda) \) becomes imaginary for small \( q \) \( (|\frac{V}{2\pi}| < 1) \), therefore, the small \( q \) excitation modes are forbidden. The physics meaning of this phenomenon is unclear, it maybe derives from the finite-size and weak-localization of the (anti-) solitons and breathers.

As \( \lambda = iu \) and including the static tension potential energy of the lattices, the dimerization parameter \( u \) can be determined by taking the minimum of the ground state energy of the system [17]. When \( \lambda = \Delta_0 \) and taking \( \Delta_0 \) as a free parameter, we can obtain its renormalization group equation by using the effective action (13), and study its low energy flow behavior in strong repulsive and attractive interaction regions [1]. On the other hand, the renormalized parameter \( \Delta_{0R} \) is related to the mass of the (anti-)solitons and breathers in the exact solution of the QSGM. It is clearly shown [1] that the fermion interaction dependence of the renormalized parameter \( \Delta_{0R} \) is completely consistent with the exact solution.
of the QSGM. Therefore, we demonstrate that the eigen-functional bosonization method is powerful, and the low energy behavior of the strongly correlated systems is controlled by the Eikonal-type equations. Even if at one-loop approximation it can give the correct results in treating the one-dimensional strongly correlated systems, where in general usual perturbation theory is invalid.

III. CHARGE COLLECTIVE EXCITATION GAP OF THE HUBBARD MODEL AT HALF-FILLING

For the Hubbard model at half filling, it is well-known \[18\] that at half-filling, the spin collective excitation gap is zero, and the charge collective excitation gap is not zero, and increases with the on-site repulsive Coulomb interaction potential \(U\) of electrons. In this section, we use the eigen-functional bosonization method to calculate the effective action of the electron density fields, so that we can determine the electron interaction dependence of the charge collective excitation gap of the Hubbard model at half-filling. This system represented by the Hubbard model with electron half-filling has a metal-Mott insulator transition at \(U = 0\). As \(U > 0\) it is a Mott insulator, and at \(U = 0\) it is a metal.

In low energy region, and with the linearization of electron spectrum near the Fermi levels \(\pm k_F\), the Hamiltonian of the Hubbard model at half-filling can be written as in the continuum limit,

\[
H = -i\hbar v_F \sum_\sigma \int dx \left[ \psi_{R\sigma}^\dagger(x) \partial_x \psi_{R\sigma}(x) - \psi_{L\sigma}^\dagger(x) \partial_x \psi_{L\sigma}(x) \right] \\
+ V \int dx \left[ \rho_{R\uparrow}(x) + \rho_{L\uparrow}(x) \right] \left[ \rho_{R\downarrow}(x) + \rho_{L\downarrow}(x) \right] \\
+ \lambda \int dx \left[ \psi_{R\uparrow}^\dagger(x) \psi_{L\uparrow}(x) \psi_{R\downarrow}(x) \psi_{L\downarrow}(x) + \psi_{L\uparrow}^\dagger(x) \psi_{R\uparrow}(x) \psi_{L\downarrow}(x) \psi_{R\downarrow}(x) \right]
\]

where \(\psi_{R\sigma}(x)\) (\(\psi_{L\sigma}(x)\)) is the right (left) moving electron field with spin \(\sigma\), \(\rho_{R(L)\sigma}(x) = \psi_{R(L)\sigma}^\dagger(x) \psi_{R(L)\sigma}(x)\) is the electron density field, \(V = aU\), and \(\lambda = aU\), where \(a\) is the lattice constant. Here we have used \(\lambda\) to present the coefficient of the Umklapp scattering which is relevant for repulsive electron Coulomb interaction \((U > 0)\), therefore, in the low energy
region renormalized $\lambda$ is different from $aU$, and neglected the backward scattering term which is irrelevant for the repulsive interaction $V > 0$.

With auxiliary fields $\phi_{R\sigma}(x, t)$ and $\phi_{L\sigma}(x, t)$ which introduce the constraints $\rho_{R\sigma}(x, t) = \psi_{R\sigma}^\dagger(x, t)\psi_{R\sigma}(x, t)$ and $\rho_{L\sigma}(x, t) = \psi_{L\sigma}^\dagger(x, t)\psi_{L\sigma}(x, t)$, respectively, and the Hubbard-Stratonovich field $\Delta_\sigma(x, t) = \psi_{R\sigma}^\dagger(x, t)\psi_{L\sigma}(x, t)$, the action of the system can be written as,

$$S = \sum_\sigma \int dt dx \left\{ \Psi_\sigma^\dagger(x, t)M_\sigma(x, t)\Psi_\sigma(x, t) - \phi_{R\sigma}(x, t)\rho_{R\sigma}(x, t) - \phi_{L\sigma}(x, t)\rho_{L\sigma}(x, t) \right\}$$

$$+ \int dt dx \left\{ \frac{1}{\lambda} \left[ \Delta_+^\dagger(x, t)\Delta_-(x, t) + \Delta_-^\dagger(x, t)\Delta_+^\dagger(x, t) \right] - V [\rho_{R\sigma}^\dagger(x, t) + \rho_{L\sigma}^\dagger(x, t)] [\rho_{R\sigma}(x, t) + \rho_{L\sigma}(x, t)] \right\} \quad (17)$$

where $\tilde{M}_\sigma(x, t) = \tilde{M}_0 + \phi_\sigma(x, t)$, $\Psi_\sigma^\dagger(x, t) = \left( \psi_{R\sigma}^\dagger(x, t), \psi_{L\sigma}^\dagger(x, t) \right)$, and

$$\tilde{M}_0 = \begin{pmatrix} D_{R\sigma}, \Delta \\ \Delta^*, D_{L\sigma} \end{pmatrix}, \quad \phi_\sigma(x, t) = \begin{pmatrix} \phi_{R\sigma}(x, t), \Delta_\sigma(x, t) - \Delta \\ \Delta_\sigma^*(x, t) - \Delta^*, \phi_{L\sigma}(x, t) \end{pmatrix}$$

where $D_{R\sigma(L)} = i\partial_t \pm i\partial_x$ (choosing $\hbar = v_F = 1$), and $\Delta$ is a parameter (see below). Integrating out the electron field $\Psi_\sigma(x, t)$, we obtain an effective potential, $W[\phi] = i Tr \ln(\tilde{M}_\sigma)$ which can be represented by using eigen-functionals,

$$W[\phi] = i \sum_\sigma \int_0^1 d\xi \int dt dx \text{tr} \left[ \phi_\sigma G_\sigma(x, t; x', t', [\xi\phi]) \right] \bigg|_{x' \rightarrow x}$$

$$G_\sigma(x, t; x', t', [\phi]) = \sum_\nu \sum_{k, \omega} \frac{1}{E_{\sigma k\omega}^{(\nu)}[\phi]} \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) \Psi_{\sigma k\omega}^{(\nu)*}(x', t', [\phi]) \quad (18)$$

where the eigen-functional $\Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi])$ satisfies the eigen-equation,

$$\tilde{M}_\sigma(x, t)\Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) = E_{\sigma k\omega}^{(\nu)}[\phi]\Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) \quad (19)$$

Using Hellmann-Feynman theorem, the eigen-value $E_{\sigma k\omega}^{(\nu)}[\phi]$ reads $E_{\sigma k\omega}^{(\nu)}[\phi] = E_{k\omega}^{(\nu)}[0] + \Sigma_{\sigma k}^{(\nu)}[\phi]$, where $\Sigma_{\sigma k}^{(\nu)}[\phi] = \int_0^1 d\xi \int dt dx \Psi_{\sigma k\omega}^{(\nu)*}(x, t, [\xi\phi])\phi_\sigma(x, t)\Psi_{\sigma k\omega}^{(\nu)}(x, t, [\xi\phi])$, $E_{k\omega}^{(\nu)}[0] = \omega \mp E_k$, and $E_k = \sqrt{k^2 + |\Delta|^2}$. 

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At $\phi_{R(L)\sigma}(x, t) = 0$ (i.e., $V = 0$), we have the solutions corresponding to the eigen-values $E_{k\omega}^{(\pm)}[0]$, respectively,

$$\Psi_{\sigma k\omega}^{(\pm)}(x, t) = \left(\frac{1}{TL}\right)^{1/2} \begin{pmatrix} a_k^{(\pm)} e^{i\theta} \\ b_k^{(\pm)} e^{-i\theta} \end{pmatrix} e^{ikx - i\omega t},$$

where $a_k^{(\pm)} = b_k^{(-)} = u_k$, $-b_k^{(+)} = a_k^{(-)} = v_k$, $\Delta = \Delta^{(R)} + i\Delta^{(I)}$, and $\tan(2\theta) = \Delta^{(I)} / \Delta^{(R)}$. The Hubbard-Stratonovich field becomes a constant under this condition, $\Delta_{\sigma}(x, t) = \Delta$, and the parameter $\Delta$ is determined by the self-consistent equation,

$$\Delta = -\frac{\lambda}{2L} \sum_k \frac{\Delta^*}{\sqrt{k^2 + |\Delta|^2}}$$

which only has imaginary solution, $\Delta^{(R)} = 0$, and $\Delta^{(I)} \sim De^{-2\pi h\nu_F / \lambda}$, where $D$ is the bandwidth of the system. The parameter $\Delta^{(I)}$ is a charge gap parameter, and at $\lambda = 0$ (i.e., $U = 0$) it is zero. Therefore, it can be taken as an order parameter of the Mott insulator.

Generally, we can choose the ansatz,

$$\Psi_{\sigma k\omega}^{(\pm)}(x, t, [\phi]) = A_k \left(\frac{1}{TL}\right)^{1/2} \begin{pmatrix} a_k^{(\pm)} e^{i\theta} e^{Q_{R\sigma}^{(\pm)}(x, t)} \\ b_k^{(\pm)} e^{-i\theta} e^{Q_{L\sigma}^{(\pm)}(x, t)} \end{pmatrix} e^{ikx - i(\omega + \tilde{\tau}_{\sigma}^{(\pm)}(\phi))t}$$

where $A_k \sim 1$ is a normalization constant, and the Hubbard-Stratonovich field reads $\Delta_{\sigma}(x, t) = i\Delta^{(I)} e^{Q_{L\sigma}^{(-)}(x, t) - Q_{R\sigma}^{(-)}(x, t)}$. Substituting (21) into the eigen-equation (19), we can obtain,

$$D_R f_{R\sigma}^{(\pm)} + \frac{\Delta^{(I)} b_k^{(\pm)}}{a_k^{(\pm)}} \left(e^{Q_{L\sigma}^{(-)} - Q_{R\sigma}^{(-)}} e^{Q_{L\sigma}^{(\pm)} - Q_{R\sigma}^{(\pm)}} - 1\right) = 0$$

$$D_L f_{L\sigma}^{(\pm)} + \frac{\Delta^{(I)} a_k^{(\pm)}}{b_k^{(\pm)}} \left(e^{Q_{R\sigma}^{(-)} - Q_{L\sigma}^{(-)}} e^{Q_{R\sigma}^{(\pm)} - Q_{L\sigma}^{(\pm)}} - 1\right) = 0$$

where $Q_{R(L)\sigma}^{(\pm)}(x, t) = Q_{R(L)\sigma}^{0}(x, t) + f_{R(L)\sigma}^{(\pm)}(x, t)$. These differential equations can be simplified to the Eikonal-type equations,

$$\begin{cases}
(D_L D_R - \Delta^{(I)} b_k^{(\pm)}) D_L - \Delta^{(I)} a_k^{(\pm)} D_R) f_{R\sigma}^{(\pm)} = \left(\frac{\Delta^{(I)} b_k^{(\pm)}}{a_k^{(\pm)}} - D_R f_{R\sigma}^{(\pm)}\right) D_L Q_{\sigma}^{0} + Z_{\sigma}^{(\pm)} \\
(D_R D_L - \Delta^{(I)} b_k^{(\pm)}) a_k^{(\pm)} - \Delta^{(I)} a_k^{(\pm)} D_R) f_{L\sigma}^{(\pm)} = -\left(\frac{\Delta^{(I)} a_k^{(\pm)}}{b_k^{(\pm)}} - D_L f_{L\sigma}^{(\pm)}\right) D_R Q_{\sigma}^{0} + \tilde{Z}_{\sigma}^{(\pm)}
\end{cases}$$

(23)
\[
Z_\sigma^{(\pm)} = \frac{\Delta^{(I)} b_k^{(\pm)}}{\alpha_k^{(\pm)}} e^{-Q_{\sigma}^{(\pm)}} (D_L + D_L f_{L\sigma}^{(\pm)})(e^{-Q_{-\sigma}^{(-)}} - 1) + (\Delta^{(I)})^2 (e^{-Q_{-\sigma}^{(-)}} - 1)(e^{Q_{-\sigma}^{(-)}} - e^{-Q_{-\sigma}^{(\pm)}}) - D_R f_{R\sigma}^{(\pm)} D_L f_{L\sigma}^{(\pm)}
\]

\[
\tilde{Z}_\sigma^{(\pm)} = \frac{\Delta^{(I)} a_k^{(\pm)}}{b_k^{(\pm)}} e^{Q_{\sigma}^{(\pm)}} (D_R - D_R f_{R\sigma}^{(\pm)})(e^{Q_{-\sigma}^{(-)}} - 1) + (\Delta^{(I)})^2 (e^{Q_{-\sigma}^{(-)}} - 1)(e^{-Q_{-\sigma}^{(-)}} - e^{Q_{-\sigma}^{(\pm)}}) - D_R f_{R\sigma}^{(\pm)} D_L f_{L\sigma}^{(\pm)}
\]

where \(Q_0^\sigma = Q_{R\sigma}^0 - Q_{L\sigma}^0\), and \(Q_\sigma^{(\pm)} = Q_{R\sigma}^{(\pm)} - Q_{L\sigma}^{(\pm)}\).

In order to solve these equations, we only keep the linear terms of \(f_{R(L)\sigma}(x, t)\) and \(Q_R^{(\pm)}\).

Under this approximation, we obtain the effective actions \(S_{c,\sigma}^{(s)}[\phi, \rho]\) and \(S_{s,\sigma}^{(s)}[\phi, \rho]\) of the spin and charge parts, respectively,

\[
S_{c,\sigma}^{(s)}[\phi, \rho] = \frac{1}{TL} \sum_{q, \Omega} \left\{ -\frac{q}{8\pi} \frac{\Omega + q}{D_{c(s)}(q, \Omega)} |\phi_{c(s)}(q, \Omega)|^2 - \frac{q}{8\pi} \frac{\Omega - q}{D_{c(s)}(q, \Omega)} |\phi_{c(s)}(q, \Omega)|^2 \right. \\
- \frac{1}{2} \left[ \phi_{R(s)}(-q, -\Omega) \rho_{R(s)}(q, \Omega) + \phi_{L(s)}(-q, -\Omega) \rho_{L(s)}(q, \Omega) \right] \\
= \left. \frac{1}{4} \left[ \frac{V}{4A_{c(s)}} \frac{V}{4A_{c(s)}} - q(q + q) \right] \right\}
\]

where \(D_{c(s)} = \Omega^2 - q^2 - (\Delta^{(I)} \pm \Delta^{(I)})^2\), and \(\chi_{c(s)} = \chi_\uparrow \pm \chi_\downarrow\) where \(\chi = \phi_{R(L)}, \rho_{R(L)}\). Integrating out the auxiliary fields \(\phi_{R(s)}(q, \Omega)\) and \(\phi_{L(s)}(q, \Omega)\), we obtain the effective actions,

\[
\begin{align*}
S_{c,\sigma}^{(s)}[\rho] &= \frac{1}{TL} \sum_{q, \Omega} \rho_{c(s)}(q, \Omega) X_{c(s)}(q, \Omega) \rho_{c(s)}(q, \Omega) \\
X_{c(s)}(q, \Omega) &= A_{c(s)} \left( q(\Omega - q) \pm \frac{V}{4A_{c(s)}} \frac{V}{4A_{c(s)}} \right)
\end{align*}
\]

where \(\rho_{c(s)} = (\rho_{R(s)}, \rho_{L(s)})\), \(A_{c(s)} = -\frac{\pi D_{c(s)}(q^4 - q^2\Omega^2)}{2}\). With these effective action, we can obtain the gaps in the spin and charge collective excitation spectrum, respectively,

\[
\begin{align*}
\Delta_s &= 0 \\
\Delta_c &= 2\Delta^{(I)}
\end{align*}
\]

At zero-order approximation (i.e., \(V = 0\)), the parameter \(\Delta^{(I)}\) is determined by equation (20), and it increases with the electron interaction strength \(\lambda = aU\). However, the parameter \(\Delta^{(I)}\) can be more rigorously determined by taking the minimum of the ground state energy of the system, where the ground state energy reads,
\[ E_g = \frac{2(\Delta^{(I)})^2}{\lambda} - iT\ln(\hat{M}_0) + \frac{i}{2}T\ln\left(\frac{D_c(V)}{4VD_c}\right) \]  

(27)

where \( D_c(V) = \Omega^2 - (1 + V/\pi)q^2 - 4(\Delta^{(I)})^2 \). By simple calculation, we can obtain,

\[
\Delta^{(I)} = \frac{\lambda\Delta^{(I)}}{\pi} \ln \left(\frac{D + \sqrt{D^2 + (\Delta^{(I)})^2}}{\Delta^{(I)}}\right) 
- \frac{\lambda\Delta^{(I)}}{2\pi} \frac{1}{\sqrt{1 + V/\pi}} \ln \left(\frac{\sqrt{1 + V/\pi D + \sqrt{(1 + V/\pi)D^2 + (\Delta^{(I)})^2}}}{\Delta^{(I)}}\right) 
\]  

(28)

At \( V = 0 \), it reduces into the equation (21). We now can approximately take \( \lambda = V \) in the low energy region, and use equations (28) and (24) to determine the order parameter \( \Delta^{(I)} \) of the Mott insulator and the charge collective excitation gap, respectively, where the only parameter is the band-width \( D \) of electrons. This result is qualititively consistent with the exact solution of the Hubbard model at half-filling, where the charge gap increases with the on-site Coulomb interaction strength of electrons [18].

**IV. PHASE TRANSITION FROM BAND INSULATOR TO MOTT-TYPE INSULATOR**

In this section, we study the influence of strong electron correlation on the spin and charge collective excitation gaps with a prototype one-dimensional model [8]. This model can be used to qualititively explain the origin of ferroelectricity of some transition metal oxides, because of a spontaneous lattice dimerization in the strong electron correlation regime. By analytical calculations of the spin and charge gaps, we can clearly demonstrate that the system has a quantum critical point defined by \( \Delta_c = 0 \) at the interaction strength \( V = V_T(\Delta_0) \), where the order parameter \( \Delta^{(I)} \) of Mott insulator and dimerization parameter \( u \) turns on, and increase with \( V \) as \( V > V_T(\Delta_0) \), and the order parameter \( \Delta^{(R)} \) of band insulator goes to a constant as \( V > V_T(\Delta_0) \). This quantum critical point represents the phase transition from band insulator to Mott-type insulator, and as \( \Delta_0 = 0 \), it reduces into the usual metal-Mott insulator transition of the Hubbard model at half-filling. Another
quantum critical point defined by $\Delta_s = 0$ is reached as $\Delta^{(I)} \to \infty$ asymptotically for a finite $\Delta_0$. Qualititively, the phenomenal results of Ref. [19] are consistent with our calculations.

The Hamiltonian of this prototype one-dimensional model reads,

$$H = \sum_{i\sigma} \left\{ -t \left( c_{i\sigma}^\dagger c_{i+1\sigma} + c_{i+1\sigma}^\dagger c_{i\sigma} \right) + \Delta_0 (-1)^i n_{i\sigma} \right\} + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

where $c_{i\sigma}(c_{i\sigma}^\dagger)$ is the electron annihilation (creation) operator with spin $\sigma (=\uparrow, \downarrow$) at site $i$, and $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$ is the electron density operator. The odd and even sites represent oxygen atoms (O) and a generic cation (C), respectively, with the energy difference, $E_C - E_O = 2\Delta_0$.

It can be easily seen that the Hamiltonian (29) describes different low energy physical behavior for the on-site Coulomb interaction $U \to 0$ and $U \to \infty$, respectively. At $U = 0$, it represents a simple band insulator where there is an energy gap $2\Delta_0$ in the electron excitation spectrum. However, as $U \gg \Delta_0$, it represents a Mott insulator where charge density excitation spectrum has a gap and spin density excitation spectrum is gapless. Therefore, there is a transition (mixed-valence) region between the band insulator and Mott-type insulator as $U \sim 2\Delta_0$. The finite-size simulations [8,20,21] suggest that the mixed-valence region is accompanied by a spontaneous dimerized phase which turns on electron-phonon interaction,

$$H_{ep} = -u \sum_{i\sigma} (-1)^i \left( c_{i\sigma}^\dagger c_{i+1\sigma} + c_{i+1\sigma}^\dagger c_{i\sigma} \right)$$

where $u$ is the dimerization parameter.

In low energy region, and with the linearization of electron spectrum near the Fermi levels $\pm k_F$, the Hamiltonian (29) can be written as in the continuum limit,

$$H = -i\hbar v_F \sum_\sigma \int dx \left[ \psi_{R\sigma}^\dagger (x) \partial_x \psi_{R\sigma} (x) - \psi_{L\sigma}^\dagger (x) \partial_x \psi_{L\sigma} (x) \right]$$

$$+ V \int dx \left[ \rho_{R\uparrow} (x) + \rho_{L\uparrow} (x) \right] \left[ \rho_{R\downarrow} (x) + \rho_{L\downarrow} (x) \right]$$

$$- \Delta_0 \sum_\sigma \int dx \left[ \psi_{R\sigma}^\dagger (x) \psi_{L\sigma} (x) + \psi_{L\sigma}^\dagger (x) \psi_{R\sigma} (x) \right]$$

$$+ \lambda \int dx \left[ \psi_{R\uparrow}^\dagger (x) \psi_{L\uparrow} (x) \psi_{R\downarrow}^\dagger (x) \psi_{L\downarrow} (x) + \psi_{L\uparrow}^\dagger (x) \psi_{R\uparrow} (x) \psi_{L\downarrow}^\dagger (x) \psi_{R\downarrow} (x) \right]$$

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where $\psi_{R\sigma}(x)$ ($\psi_{L\sigma}(x)$) is the right (left) moving electron field with spin $\sigma$, $\rho_{R(L)\sigma}(x) = \psi_{R(L)\sigma}^\dagger(x)\psi_{R(L)\sigma}(x)$ is the electron density field, $V = aU$, and $\lambda = aU$, where $a$ is the lattice constant. Here we have used $\lambda$ to present the coefficient of the Umklapp scattering which is relevant for repulsive electron Coulomb interaction ($U > 0$), therefore, in the low energy region renormalized $\lambda$ is different from $aU$, and neglected the backward scattering term which is irrelevant for the repulsive interaction $V > 0$. The spontaneous electron-phonon coupling term reads in the continuum limit,

$$H_{ep} = iu \sum_\sigma \int dx \left[ \psi_{R\sigma}^\dagger(x)\psi_{L\sigma}(x) - \psi_{L\sigma}^\dagger(x)\psi_{R\sigma}(x) \right]$$

(32)

The dimerization parameter $u$ can be determined by taking the minimum of ground state energy after including static tension energy of the lattices.

With auxiliary fields $\phi_{R\sigma}(x, t)$ and $\phi_{L\sigma}(x, t)$ which introduce the constraints $\rho_{R\sigma}(x, t) = \psi_{R\sigma}^\dagger(x, t)\psi_{R\sigma}(x, t)$ and $\rho_{L\sigma}(x, t) = \psi_{L\sigma}^\dagger(x, t)\psi_{L\sigma}(x, t)$, respectively, and the Hubbard-Stratonovich field $\Delta_{-\sigma}(x, t) = \psi_{R\sigma}^\dagger(x, t)\psi_{L\sigma}(x, t)$, the action of the system without the lattice dimerization can be written as,

$$S = \sum_\sigma \int dt dx \left\{ \Psi^\dagger_\sigma(x, t)\hat{M}_\sigma(x, t)\Psi_\sigma(x, t) \\
- \phi_{R\sigma}(x, t)\rho_{R\sigma}(x, t) - \phi_{L\sigma}(x, t)\rho_{L\sigma}(x, t) \right\} \\
+ \int dt dx \left\{ \frac{1}{\lambda} \left[ \Delta^\dagger_\sigma(x, t)\Delta_\sigma(x, t) + \Delta^\dagger_\sigma(x, t)\Delta_\sigma(x, t) \right] \\
- V \left[ \rho_{R\uparrow}(x, t) + \rho_{L\uparrow}(x, t) \right] \left[ \rho_{R\downarrow}(x, t) + \rho_{L\downarrow}(x, t) \right] \right\}$$

(33)

where $\Delta_\sigma(x, t) = \hat{M}_0 + \phi_\sigma(x, t)$, $\Psi^\dagger_\sigma(x, t) = \left( \psi_{R\sigma}^\dagger(x, t), \psi_{L\sigma}^\dagger(x, t) \right)$, and

$$\hat{M}_0 = \begin{pmatrix} D_{R} & \Delta_0 + \Delta \\ \Delta_0^* + \Delta^* & D_{L} \end{pmatrix}, \quad \phi_\sigma(x, t) = \begin{pmatrix} \phi_{R\sigma}(x, t), \Delta_\sigma(x, t) - \Delta \\ \Delta_\sigma^*(x, t) - \Delta^*, \phi_{L\sigma}(x, t) \end{pmatrix}$$

where $D_{R(L)} = i\partial_t \pm i\partial_x$ (choosing $\hbar = v_F = 1$), and $\Delta$ is a parameter (see below). Integrating out the electron field $\Psi_\sigma(x, t)$, we obtain an effective potential, $W[\phi] = iTr \ln(\hat{M}_\sigma)$ which can be represented,
\[ W[\phi] = i \sum_{\sigma} \int_{0}^{1} d\xi \int dt dx \text{tr} (\phi_{\sigma} G_{\sigma}(x, t; x', t'\{\xi\phi\})) |_{\nu \to \nu_{1}} \]

\[ G_{\sigma}(x, t; x', t', [\phi]) = \sum_{\nu} \sum_{k, \omega} \frac{1}{E_{\sigma k\omega}^{(\nu)} [\phi]} \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) \Psi_{\sigma k\omega}^{(\nu)\dagger}(x', t', [\phi]) \] (34)

where the eigen-functional \( \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) \) satisfies the eigen-equation,

\[ \hat{M}_{\sigma}(x, t) \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) = E_{\sigma k\omega}^{(\nu)} [\phi] \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\phi]) \] (35)

Using Hellmann-Feynman theorem, the eigen-value \( E_{\sigma k\omega}^{(\nu)} [\phi] \) reads \( E_{\sigma k\omega}^{(\nu)} [\phi] = E_{k\omega}^{(\nu)}[0] + \Sigma_{\sigma k}^{(\nu)}[\phi] \), where \( \Sigma_{\sigma k}^{(\nu)}[\phi] = \int_{0}^{1} d\xi \int dt dx \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\xi\phi]) \phi_{\sigma}(x, t) \Psi_{\sigma k\omega}^{(\nu)}(x, t, [\xi\phi]) \), independent of \( \omega \).

In order to solve the eigen-equation (35), we can choose the ansatz,

\[ \Psi_{\sigma k\omega}^{(\pm)}(x, t, [\phi]) = A_{k} \left( \frac{1}{TL} \right)^{1/2} \begin{pmatrix} \alpha_{k}^{(\pm)} e^{i\theta} e^{Q_{Re}^{(\pm)}(x,t)} \\ \beta_{k}^{(\pm)} e^{-i\theta} e^{Q_{Le}^{(\pm)}(x,t)} \end{pmatrix} e^{ikx - i(\omega + \Sigma_{\sigma k}^{(\pm)}[\phi])t} \] (36)

where \( \tan(2\theta) = \Delta^{(R)}/(\Delta_{0} + \Delta^{(R)}) \), the eigen-value is \( E_{\sigma k\omega}^{(\pm)}[\phi] = \omega \mp E_{k} + \Sigma_{\sigma k}^{(\pm)}[\phi] \), \( E_{k} = \sqrt{k^{2} + |\Delta_{0} + \Delta|^{2}} \), \( \alpha_{k}^{(+)} = \beta_{k}^{(-)} = \sqrt{\frac{1}{2}(1 + k/E_{k})} \), \( -\beta_{k}^{(+)} = \alpha_{k}^{(-)} = \sqrt{\frac{1}{2}(1 - k/E_{k})} \), \( A_{k} \sim 1 \) is a normalization constant, and \( \Delta_{\sigma}(x, t) = \Delta e^{Q_{L-\sigma}^{(-)}(x,t) - Q_{R-\sigma}^{(-)}(x,t)} \), where \( \Delta = \Delta^{(R)} + i\Delta^{(I)} \) is determined by the self-consistent equation,

\[ \Delta^{(R)} = -\frac{\lambda}{2L} \sum_{k} \frac{\Delta_{0} + \Delta^{(R)}}{E_{k}} \] (37)

\[ \Delta^{(I)} = \frac{\lambda}{2L} \sum_{k} \frac{\Delta^{(I)}}{E_{k}} \]

For a finite \( \Delta_{0} \), the real part \( \Delta^{(R)} \) has a non-zero solution \( (\Delta^{(R)} < 0) \), and \( \Delta^{(I)} = 0 \) for \( V < V_{T}(\Delta_{0}) \), and at \( \Delta_{0} = 0 \) it only has a zero solution \( \Delta^{(R)} = 0 \), where the system reduces to the usual Hubbard model at half-filling. Therefore, the parameter \( \Delta^{(R)} \) characterizes the band insulator, and \( \Delta^{(I)} \) characterizes the Mott insulator. The parameters \( \Delta^{(R)} \) and \( \Delta^{(I)} \) can be taken as the order parameters of band insulator and Mott insulator, respectively.

Substituting (36) into the eigen-equation (35), we can obtain,

\[ D_{R} f_{R\sigma}^{(\pm)} + \frac{m^{2}}{\alpha_{k}^{(\pm)}} \left( e^{Q_{L_{\sigma}}^{(-)} - Q_{R_{\sigma}}^{(-)} - 1} \right) + \Delta^{(R)} \frac{e^{-2\theta}}{\alpha_{k}^{(\pm)}} \left( e^{Q_{L_{\sigma}}^{(-)} - Q_{R_{\sigma}}^{(-)} - 1} \right) e^{Q_{L_{\sigma}}^{(\pm)} - Q_{R_{\sigma}}^{(\pm)}} = 0 \] (38)

\[ D_{L} f_{L\sigma}^{(\pm)} + \frac{m^{2}}{\beta_{k}^{(\pm)}} \left( e^{Q_{R_{\sigma}}^{(-)} - Q_{L_{\sigma}}^{(-)} - 1} \right) + \Delta^{(R)} \frac{e^{2\theta}}{\beta_{k}^{(\pm)}} \left( e^{Q_{R_{\sigma}}^{(-)} - Q_{L_{\sigma}}^{(-)} - 1} \right) e^{Q_{R_{\sigma}}^{(\pm)} - Q_{L_{\sigma}}^{(\pm)}} = 0 \]
where \( m = [(\Delta_0 + \Delta^{(R)})^2 + (\Delta^{(I)})^2]^{1/2} \), and \( Q^{(\pm)}_{R(L)\sigma}(x, t) = Q^{0}_{R(L)\sigma}(x, t) + f^{(\pm)}_{R(L)\sigma}(x, t) \). These differential equations can be simplified to the Eikonal-type equations,

\[
\begin{align*}
\left\{ \begin{array}{l}
D_L D_R - \frac{m \beta_k^{(\pm)}}{\alpha_k^{(\pm)}} D_L - \frac{m \alpha_k^{(\pm)}}{\beta_k^{(\pm)}} D_R \\
D_R D_L - \frac{m \beta_k^{(\pm)}}{\alpha_k^{(\pm)}} D_L - \frac{m \alpha_k^{(\pm)}}{\beta_k^{(\pm)}} D_R 
\end{array} \right\} f^{(\pm)}_{R \sigma} = \left\{ \begin{array}{l}
\frac{m \beta_k^{(\pm)}}{\alpha_k^{(\pm)}} - D_R f^{(\pm)}_{R \sigma} \\
\frac{m \alpha_k^{(\pm)}}{\beta_k^{(\pm)}} - D_L f^{(\pm)}_{L \sigma} 
\end{array} \right\} D_L Q^{0}_\sigma + Z^{(\pm)}
\end{align*}
\]

(39)

where \( Q^{0}_\sigma = Q^{0}_{R \sigma} - Q^{0}_{L \sigma} \), \( F^{(\pm)}_\sigma = D^{(\pm)}_0 - \frac{m \alpha_k^{(\pm)}}{\beta_k^{(\pm)}} F^{(\pm)}_0 \), and \( \bar{F}^{(\pm)}_\sigma = D^{(\pm)}_0 - \frac{m \alpha_k^{(\pm)}}{\beta_k^{(\pm)}} \bar{F}^{(\pm)}_0 \). At \( \Delta = 0 \), it is reduced to the same Eikonal-type equations as that in the equation (I).

In order to solve these Eikonal-type equations, we only keep the linear terms of \( f^{(\pm)}_{R(L)\sigma}(x, t) \) and \( Q^{0}_{R(L)\sigma} \). Under this approximation, we obtain the effective actions \( S^{c}_{eff.}[\phi, \rho] \) and \( S^{s}_{eff.}[\phi, \rho] \) of the spin and charge parts, respectively,

\[
S^{c(s)}_{eff.}[\phi, \rho] = \frac{1}{TL} \sum_{q, \Omega} \left\{ -\frac{q}{8\pi D_{c(s)}} |\phi_{Rc(s)}(q, \Omega)|^2 + \frac{q}{8\pi D_{c(s)}} |\phi_{Lc(s)}(q, \Omega)|^2 \right\} - \frac{1}{2} \left[ \phi_{Rc(s)}(-q, -\Omega) \rho_{Rc(s)}(q, \Omega) + \phi_{Lc(s)}(-q, -\Omega) \rho_{Lc(s)}(q, \Omega) \right] + \frac{V}{4} |\rho_{Rc(s)}(q, \Omega) + \rho_{Lc(s)}(q, \Omega)|^2
\]

(40)

where \( D_{c(s)} = \Omega^2 - q^2 - (m + \Delta)^2, \Delta = \sqrt{(\Delta^{(R)})^2 + (\Delta^{(I)})^2 \cos (2(\theta - \theta))}, \tan(2\theta) = \Delta^{(I)}/\Delta^{(R)}, \) and \( \chi_{c(s)} = \chi_{c} \pm \chi_{s} \) where \( \chi = \phi_{R(L)}, \rho_{R(L)} \). Integrating out the auxiliary fields \( \phi_{Rc(s)}(q, \Omega) \) and \( \phi_{Lc(s)}(q, \Omega) \), we obtain the effective actions,

\[
S^{c(s)}_{eff.}[\rho] = \frac{1}{TL} \sum_{q, \Omega} \rho_{c(s)}^T(-q, -\Omega) X_{c(s)}(q, \Omega) \rho_{c(s)}(q, \Omega)
\]

(41)

where \( \rho_{c(s)}^T = (\rho_{Rc(s)}, \rho_{Lc(s)}) \), \( A_{c(s)} = -\frac{\pi D_{c(s)}}{2} (q^4 - q^2 \Omega^2)^{-1} \).
With the effective actions $S_{\text{eff.}}[\rho]$ and $S_{\text{eff.}}[\rho]$ (11), we can obtain the gaps in the spin and charge collective excitation spectrums, respectively,

\[
\begin{align*}
\Delta_s^2 &= (m - \bar{\Delta})^2 \\
\Delta_c^2 &= (m + \bar{\Delta})^2
\end{align*}
\] (42)

At $\Delta_0 = 0$, it reduces to the well-known result of the Hubbard model at half-filling that $\Delta_s = 0$ and $\Delta_c^2 = 4(\Delta^{(I)})^2$, where the charge gap is increased with the electron interaction strength $U$. This can be seen from the self-consistent equation of $\Delta^{(I)}$ (37) where $\Delta^{(I)}$ increases with $\lambda$. Generally, the parameter $\lambda$ can be determined by the minimum of the ground state energy, and it is a function of $V$, $\lambda(V)$. However, we can approximately take $\lambda = V$ which does not qualitatively change the behavior of the spin and charge gaps determined by equations (37) and (12), and further simplifies our calculation. Equation (12) clearly shows that the system has two quantum critical points defined by the charge and spin gaps, one is at $\Delta_c = 0$, and another is at $\Delta_s = 0$. The physical behavior around the quantum critical points can be understood by solving the self-consistent equation (37). By simple calculation, we can obtain the following solutions,

\[
\begin{align*}
\Delta^{(I)} &= \begin{cases} 
0, & V \leq V_T(\Delta_0) \\
\left[ 2De^{2\pi\hbar v_F/V} \left( e^{4\pi\hbar v_F/V - 1} \right)^2 - \frac{1}{4} \Delta_0^2 \right]^{1/2}, & V > V_T(\Delta_0)
\end{cases}
\end{align*}
\] (43)

\[
\begin{align*}
\Delta^{(R)} &= \begin{cases} 
\Delta^{(R)}(V), & V \leq V_T(\Delta_0) \\
-\frac{1}{2} \Delta_0, & V > V_T(\Delta_0)
\end{cases}
\end{align*}
\]

where $V_T(\Delta_0) = 2\pi\hbar v_F/\ln[2(D + \sqrt{D^2 + \Delta_0^2}/4)/\Delta_0]$, and $\Delta^{(R)}(V)$ is determined by equation (37) with $\Delta^{(I)} = 0$. The interaction $V = V_T(\Delta_0)$ corresponds to the quantum critical point $\Delta_c = 0$. It means that there exists a phase transition at $\Delta_c = 0$ where the parameter $\Delta_c$ shows different behavior on the two sides of the point $V = V_T(\Delta_0)$. As $V < V_T(\Delta_0)$, the parameter $\Delta^{(I)}$ is zero, and $\Delta^{(R)}$ decreases with increasing $V$. In this case, the parameter $\Delta$ is real. On the other hand, as $V > V_T(\Delta_0)$, the parameter $\Delta^{(R)}$ becomes a constant, and $\Delta^{(I)}$ is not zero and increases with $V$. The parameter $\Delta$ becomes a complex quantity. For the strong interaction $V > V_T(\Delta_0)$, the charge and spin gaps can be written as, respectively,
\[
\begin{align*}
\Delta_c &= \frac{4(\Delta^{(I)})^2}{\sqrt{\Delta_0^2 + 4(\Delta^{(I)})^2}} \\
\Delta_s &= \frac{\Delta_0^2}{\sqrt{\Delta_0^2 + 4(\Delta^{(I)})^2}}
\end{align*}
\] 

(44)

It is noted that for weak interaction \( V < V_T(\Delta_0) \) the spin gap retain invariant, and the quantum critical point \( \Delta_s = 0 \) corresponds to \( \Delta^{(I)} \to \infty \) (i.e., \( V, D \to \infty \)). In fact, this is due to our simple approximations that the self-consistent equation (37) is obtained at \( \phi_a(x, t) = 0 \) (i.e., \( V = 0 \)), it does not include the high-order correction of \( V \). However, our results are qualitatively correct after including this influence.

Including the electron-phonon interaction, the equation (42) keeps invariant, but the parameters \( m, \tilde{\theta} \) and the self-consistent equation of \( \Delta \) are changed as,

\[
m = [(\Delta_0 + \Delta^{(R)})^2 + (\Delta^{(I)} - u)^2]^{1/2}, \quad \tan(2\tilde{\theta}) = \frac{\Delta^{(I)} - u}{\Delta_0 + \Delta^{(R)}}
\]

\[
\Delta^{(R)} = -\frac{\lambda}{2L} \sum_k \frac{\Delta_0 + \Delta^{(R)}}{E_k}, \quad \Delta^{(I)} = \frac{\lambda}{2L} \sum_k \frac{\Delta^{(I)} - u}{E_k},
\]

where \( E_k = [k^2 + m^2]^{1/2}, \Delta = |\Delta| \cos \left(2(\theta - \tilde{\theta})\right) \), and \( \tan(2\tilde{\theta}) = \Delta^{(I)} / \Delta^{(R)} \). Here we only consider the static uniform dimerization of the lattices. Including the static tension energy of the lattices \( \frac{1}{2} \alpha u^2 \), the dimerization parameter \( u \) can be determined by the self-consistent equation,

\[
-\alpha u = \frac{2}{L} \sum_k \frac{(\Delta^{(I)} - u)}{E_k} - m + \tilde{\Delta} \left(\frac{\Delta^{(I)} - u}{m} - z\right) \sum_q \left(\frac{1}{E_q^c(V)} - \frac{1}{E_q^c}\right)
\]

\[
- \frac{m - \tilde{\Delta}}{2L} \left(\frac{(\Delta^{(I)} - u)}{m} + z\right) \sum_q \left(\frac{1}{E_q^c(V)} - \frac{1}{E_q^c}\right),
\]

where \( z = |\Delta| \sin \left(2(\tilde{\theta} - \theta)\right) \cos^2(2\tilde{\theta}) / (\Delta_0 + \Delta^{(R)}) \), \( E_q^c(V) = [(1 \pm V/\pi)q^2 + \Delta^{(c,s)}]^{1/2} \), and \( E_q^c(s) = [q^2 + \Delta^{(c,s)}]^{1/2} \). This equation is derived from the minimum of the ground state energy, \( \partial_u E_g[u] = 0 \). According to equations (45) and (46), we see that the dimerization parameter \( u \), similar to \( \Delta^{(I)} \), is zero for weak interaction \( V < V_T(\Delta_0) \), and turns on at \( V = V_T(\Delta_0) \). However, for strong interaction \( V > V_T(\Delta_0) \), the parameter \( \Delta^{(R)} \) basically keeps a constant, because the dimerization parameter \( u \) is a small quantity, and it goes to zero as \( V \gg V_T(\Delta_0) \).
In usual transition metal oxides, there is the strong hybridization between d-orbit electrons of cation and p-orbit electrons of oxygen atom. Therefore, at enough strong electron correlation, there spontaneously takes place the lattice dimerization and deformation-induced charge transfer, which is possibly the origin of ferroelectricity of the transition metal oxides. Our calculations demonstrate that in the strong interaction region $V > V_T(\Delta_0)$, the system has the spontaneous lattice dimerization, which is consistent with the previous numerical calculations [8,20(21).

V. DISCUSSION AND CONCLUSION

In general, for a one-dimensional strongly correlated electron system, if there is only low energy excitation modes with small momentum near the Fermi levels $\pm k_F$, it shows the Luttinger liquid behavior. If there are disorder impurities which induces the new low energy excitation modes with large momentum transfer between two Fermi levels $\pm k_F$, the electrons are localized, and the system becomes an insulator. At electron half-filling, there appears the Umklapp scattering which represents the low energy excitation modes with large momentum $4k_F$ transfer between the Fermi levels $\pm k_F$, the system becomes a Mott insulator. Therefore, the low energy excitation with large momentum transfer between the Fermi levels $\pm k_F$ (i.e., crossover excitation) makes the system become the insulator for repulsive electron interaction.

Usual perturbation methods fail to treat such the system, because the crossover excitation term(s) is relevant, and determines the low energy behavior of the system. It is very desirable to find a new method which can exactly and effectively treat this kind of systems. The eigen-functional bosonization is a good candidate, because it has more advantages than usual perturbation theory: a). Without the crossover excitation term(s), just as usual bosonization methods [1-3], it can exactly treat the system. b). For an interacting-free electron system, it is exact for dealing with the crossover excitation term(s). c). With this method, the problem of the strongly correlated electron systems ends in to solve the Eikonal-
type equations, which can be exactly solved by a series expansion of the boson fields $Q_\sigma^0(x,t)$, and/or by a computer calculations. d). For some integrable systems, such as the quantum sine-Gordon model and the Hubbard model at half-filling, at one-loop approximation (i.e., only keeping the linear terms of the Eikonal-type equations), it can give the correct results that are qualitatively consistent with the exact solution of the systems for both the weak and strong electron interactions.

It is the key point of the eigen-functional bosonization method to solve the Eikonal-type equations of the eigen-functional, and use these eigen-functionals to calculate the electron Green’s functional. With the electron Green’s functional, we can calculate the ground state energy and the effective action of the system as well as correlation functions. This method can also be used to treat one-dimensional electron-phonon interaction systems, two coupled spin-chain or quantum-wire systems, and one-dimensional disorder interacting electron systems. It can also be extended to two- and three-dimensional electron systems.
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