The weak equivalence principle with a quantum particle in a gravitational wave

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We show that the weak equivalence principle (WEP) is violated for a quantum particle in a gravitational wave (GW) background. In a freely-falling frame, the expected trajectory of a quantum particle is independent of the GW, but its probability distribution is not. By monitoring the position of the particle, extra mass information can be extracted due to the GW, in violation of the WEP.

We then ask, can the probability distribution of a quantum particle be used as a GW detector? In principle yes, but in practice unlikely, due to the unfeasibly high accuracy of particle detection required.

I. INTRODUCTION

The weak equivalence principle states that point particles in free-fall will follow trajectories that are independent of their mass. This principle underpins classical gravitational theory. In the context of classical theory the WEP is well defined; in quantum theory however, the WEP is ill-defined. This is because under the Heisenberg’s uncertainty principle, point particles and trajectories are ambiguous concepts. The problem is further highlighted when one compares the classical action of a particle with mass $m$ in a gravitational field with the quantum action of a wavefunction $\psi$ of a massive spin-$\frac{1}{2}$ particle. The classical action is

$$S_C = -mc \int ds ,$$  \hspace{1cm} (1)

where $ds^2 = g_{\mu \nu}dx^\mu dx^\nu$. As $m$ appears simply as a multiplicative factor, it does not feature in the equations of motion. This is consistent with the WEP. In comparison, the quantum action is

$$S_Q = \int \left( \frac{i\hbar c}{2} \bar{\psi} \gamma^\mu D_\mu \psi - \bar{\psi}mc^2\psi \right) dx ,$$  \hspace{1cm} (2)

where $D_\mu$ is the covariant derivative in curved space-time. In this case $m$ simply is not a multiplicative factor, and features in the Dirac equation [Eq. (2)].

Given the difficulties with interpreting the WEP in a quantum context, an alternative formulation has been offered by Seveso and Paris [1]. In their formulation, they encode an object’s trajectories in the Fisher information. The WEP’s notion that free-falling trajectories should be independent of mass, is reformulated as the statement that the Fisher information of a free-falling object is invariant with mass. In this information-theoretic framework, violation of the WEP means that one may extract information about an object’s mass in free-fall. This information-theoretic formulation of the WEP has the advantage that it is extendable to quantum objects in an unambiguous manner.

Under this formulation Seveso and Paris showed that quantum objects in a static uniform gravitation field does not violate the WEP, whereas in a non-uniform gravitational field they do. The classical analogue of this is that extended classical objects in non-uniform gravitational fields do not follow geodesics [2]. In this paper we show that quantum objects in a uniform but time-dependent gravitational field violates the WEP; in particular, we look at GWs. This does not have a classical analogue, as extended classical objects in the GW co-ordinates with uniform but time-dependent gravitational fields, do follow geodesics. As such, this work reinforces the idea that quantum objects violate the WEP in fundamental ways. We will then consider whether such violations can be used as a basis for GW detectors. In Sec. III we define the Fisher information. In Sec. III we look at quantum particles in static gravitational fields. In Sec. IV we look quantum particles in GWs.

II. FISHER INFORMATION

The Fisher information gives the amount of information that an observable random variable provides about an unknown parameter. In our case, the random variable is the position of the particle $x$, and the unknown parameter is its mass $m$. For a particle with wave function $\psi(x, t)$, the Fisher information is

$$F_x(m) = \int dx |\psi(x, t)|^2 |\partial_m \log |\psi(x, t)||^2|^2 .$$  \hspace{1cm} (3)

In the absence of gravity, observation of the position of the particle can betray information about its mass. For example, a free Gaussian wave packet spreads with variance $\sigma^2(t) = \sigma^2(0) + \hbar t/2m$; one may extract information about its mass by monitoring its position. Formulation of the WEP in terms of Fisher information states that the presence of a gravitational field should not produce more information about the mass of a particle, \textit{i.e.} $F_x(m) = F_x(m)^{\text{free}}$, where $F_x^{\text{free}}(m)$ is the Fisher information in the absence of a gravitational field.
III. SCHWARZSCHILD FIELD

Prior work took the non-relativistic limit of the Klein-Gordon equation in a curved space-time background, to derive the Hamiltonian in a weak gravitational field [1]. As an alternative derivation, we begin with the Dirac equation in curved space-time, which describes a spin-1/2 particle of rest mass $m$ in a gravitational field,

$$i\hbar\gamma_a\partial_\mu \psi = mc\psi \ . \quad (4)$$

The spacetime metric $g_{\mu\nu}$ can be related at every point to a tangent Minkowski space $\eta_{ab}$ via tetrads $e^a_\mu$, $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$. The tetrads obey the orthogonality conditions $e^a_\mu e^a_\nu = \delta_\mu^\nu$, $e^a_\mu e^b_\nu = \delta^\mu_b$. We use the convention that Latin indices represent components in the tetrad frame. The spinorial affine connection $\Gamma_\mu = \frac{1}{2}e^a_\mu (\partial_a e^b_\nu + \Gamma^\nu_{\mu a} e^b_\nu) \sigma_{ab}$, where $\Gamma^\nu_{\mu a}$ are the generators of the Lorentz group. $\gamma_a$ are gamma matrices defining the Clifford algebra $\{ \gamma_a, \gamma_b \} = -2\eta_{ab}$, with spacetime metric signature $(-, +, +, +)$. We use the Einstein summation convention where repeated indices $(\mu, \nu, \sigma, a, b = \{0, 1, 2, 3\})$ are summed.

We will consider the Schwarzschild metric in isotropic coordinates ($x^b = ct$),

$$ds^2 = V^2(dx^0)^2 - W^2(dx \cdot dx) \ , \quad (5)$$

where $(r \equiv \sqrt{x^2 - x^0})$

$$V = \left(1 - \frac{GM}{2c^2r}\right) \left(1 - \frac{GM}{2c^2r}\right)^{-1} \ , \quad (6)$$

$$W = \left(1 + \frac{GM}{2c^2r}\right) \ . \quad (7)$$

Under this metric Eq. (3) can be written in the familiar Schrödinger picture $i\hbar \partial_t \psi = H\psi$, where $(\mathbf{a} \equiv \gamma^0 \gamma^\mu \beta, \beta \equiv \gamma^0, \mathbf{p} \equiv -i\hbar \nabla, F \equiv V/W$, and indices $i, ..., n = \{1, 2, 3\}$

$$H = \beta mc^2 V + \frac{c}{2} [\mathbf{a} \cdot \mathbf{p}] F - F(\mathbf{a} \cdot \mathbf{p}) \ . \quad (8)$$

A means by which to write down the non-relativistic limit of the Dirac Hamiltonian with relativistic correction terms is provided by the Foldy-Wouthuysen (FW) transformation [4]. The FW transformation is a unitary transformation which separates the upper and lower spinor components. In the FW representation, the Hamiltonian and all operators are block-diagonal (diagonal in two spinors). There are two types of the FW transformation known as the standard FW (SFW) [4] and exact FW (EFW) [3] transformations. We will use here the EFW transformation.

Central to the EFW transformation is the property that when $H$ anti-commutes with $J = i\gamma^5 \beta$, $\{H, J\} = 0$, under the unitary transformation $U = U_2 U_1$, where $(\Lambda \equiv H/\sqrt{H^2})$

$$U_1 = \frac{1}{\sqrt{2}}(1 + JA), \quad U_2 = \frac{1}{\sqrt{2}}(1 + \beta J) \ , \quad (9)$$

the transformed Hamiltonian is even (even terms do not mix the upper and lower spinor components, odd terms do),

$$UHU^+ = \frac{1}{2}(\sqrt{H^2} + \beta \sqrt{H^2} \beta) + \frac{1}{2}(\sqrt{H^2} - \beta \sqrt{H^2} \beta)J = \{\sqrt{H^2}\}_{\text{even}} \beta + \{\sqrt{H^2}\}_{\text{odd}} J \ . \quad (10)$$

Note that as $\beta$ is an even operator and $J$ is an odd operator, Eq. (10) is an even expression which does not mix the positive and negative energy states.

Our Hamiltonian satisfies the EFW anti-commutation property. Using the identity $\alpha^\dagger \alpha^3 = ic^{ijk}\sigma_{i2} + \delta^3 I_4$, the perturbative expansion of $\sqrt{H^2}$ yields to first order,

$$H \approx mc^2V + \frac{1}{4m}(W^{-1}p^2F + Fp^2W^{-1}) \ . \quad (11)$$

Note that $\sqrt{H^2} = \{\sqrt{H^2}\}_{\text{even}} = H I_2$ contains only even terms, and therefore $\{\sqrt{H^2}\}_{\text{odd}} = 0$ in Eq. (10).

Taking the weak-limit gravitational field limit so that,

$$V \approx 1 - \frac{GM}{c^2r} \ , \quad W \approx 1 + \frac{GM}{c^2r} \ , \quad (12)$$

we get $(g \equiv -GMr/r^3)$

$$H = mc^2 + \frac{g^2}{2m} + mg \cdot x \ . \quad (13)$$

In Eq. (13) we arrive at a Hamiltonian of particle in a static gravitational well, which one may have simply written down by intuition. However, we have chosen to follow the formal derivation of taking the FW transformation of the Dirac equation in curved spacetime, as we will use this formalism to derive the Hamiltonian of a particle in GW background in Sec. IV, which cannot be simply written down by intuition.

The evolution of a quantum particle is governed by the time-evolution operator $U = e^{-iH t}$. Taking the Baker-Campbell-Hausdorff expansion of $U$ to second-order, the time-evolution operator in a Schwarzschild field is ($\hbar = 1$)

$$U \approx \exp\left(\frac{im^3}{3} g^2\right) \exp\left(\frac{i\gamma^3}{6m} \nabla \cdot \nabla - \frac{g^2}{2} \cdot \nabla\right) \left(\exp\left(-im^2 g \cdot x\right) U_{\text{free}}\right) \quad (14)$$

where $U_{\text{free}} = \exp(-im^2 t) \exp(-i\gamma^3 t)$ is the free time-evolution operator in the absence of any gravitational field. Note that the $\exp(-im^2 t)$ term only acts as a constant phase factor in the non-relativistic limit, and therefore can be ignored.

A. Uniform gravitational field

As our gravitational field is spherically symmetric, we can reduce our problem to one spatial dimension in the
radial direction. We consider a Gaussian wave packet,
\[
\psi(x, 0) = \left(\frac{2}{\pi}\right)^{1/4} e^{-r^2},
\]
as this is most amenable to comparison with a classical particle; our results however are generalisable to any wave function. For probe particles travelling over small distances, it is usual to take the terrestrial gravitational field as uniform. In the uniform gravitational case,
\[
U = \exp \left(-\frac{m g^2 t^2}{3}\right) \exp \left(-\frac{g^2 t^2}{2} \cdot \nabla\right) \psi(x, t) = U \psi(x, 0)
\]
(16)

where \(\psi(x, t)\) free = \(U\psi(x, 0)\) is the free wave function in the absence of a gravitational field.

Using the fact that the momentum operator is a translation operator in the conjugate position space, the time evolution of the wave function \(\psi(x, t)\) is
\[
\psi(x, t) = \psi(x - g t^2 / 2, t),
\]
(17)

where \(\psi(x, t)\) free = \(U\psi(x, 0)\) is the free wave function in the absence of a gravitational field.

Substituting Eq. (15) into Eq. (17), the expected position of the wave packet in a uniform gravitational field is
\[
\langle x \rangle = \int_{-\infty}^{\infty} \psi(x, t)^* x \psi(x, t) dx = \frac{gt^2}{2}.
\]
(18)

This is the geodesic of a freely-falling classical particle with no initial momentum in a uniform gravitational field \(g\). As with the classical case, the expected trajectory of the quantum particle is independent of its mass, in alignment with the WEP.

Along with translating the wave function by \(g t^2 / 2\), the uniform gravitational field induces a mass-dependent phase factor in Eq. (17). This mass-dependent phase factor however, is not present in the probability distribution, \(\psi(x, t)^2 = |\psi(x - g t^2 / 2, t)|^2\). Therefore, by a change of variable \((u = x - g t^2 / 2)\), we see that the uniform gravitational field does not produce any extra mass information, i.e.
\[
F_x(m) = \int d|\psi(x, t)|^2 |\psi(u, t)|^2 \partial_m \log |\psi(u, t)|^2
= F_x(m)\text{ free}.
\]
(19)

B. Non-uniform gravitational field

If we do not make the approximation that \(g\) is uniform, then we must use the time-evolution operator of Eq. (14). In this case the wave function is
\[
\psi(x, t) \approx \exp \left(-i m g^2 \frac{t^3}{3}\right) \exp \left(-i m g^2 \nabla\right) \psi(x, 0)
\]
(20)

\[
\exp \left(-i m g \cdot x\right) \psi_{\text{free}}(x - g t^2 / 2 + d, t),
\]
where
\[
d = \frac{t^2}{2}(x \cdot \nabla g - g) + \frac{t^3}{3m} p \cdot \nabla g + \frac{5t^4}{48} \nabla g^2.
\]
(21)

The wavefunction consists of simple phase factors in front of the \(\psi_{\text{free}}\). However, unlike the uniform gravitational case, the translation of \(\psi_{\text{free}}\) is mass dependent owing to the second term in \(d\). This is in violation of the WEP, which states that the trajectory of a test particle should be mass independent in a gravitational field. The mass Fisher information cannot be reduced to that of the free case. Interestingly, one notes that if operator \(p\) was approximated with its classical counter part \(mv\), the mass dependence once again disappears.

The violation of the WEP of a quantum particle in a non-uniform gravitational field can find analogy in the extended classical body. For an extended classical body in a non-uniform gravitational field, different parts of the body will experience different gravitational field strengths. This will cause it to deviate from the trajectory of a point mass located at its centre of mass.

IV. GRAVITATIONAL WAVE

The metric for a generally polarised linear plane GW is
\[
ds^2 = -c^2 dt^2 + dz^2 + (1 - 2v) dx^2 + (1 + 2v) dy^2 - 2udx dy,
\]
(22)

where \(u = u(t - z)\) and \(v = v(t - z)\) are functions which describe a wave propagating in the \(z\)-direction. We will consider the case of a circularly polarised GW travelling along the \(z\)-direction, i.e. \(v = f = f_0 \cos(kz - \omega t)\) and \(u = zf\). Under this metric Eq. (11) can be written in the familiar Schrödinger picture
\[
H = \beta mc^2 + \alpha(\hat{\delta}_i + T^i_j)p_i,
\]
(23)

with
\[
T = \begin{pmatrix} v & -u & 0 \\ -u & -v & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
(24)

Applying the EFW (or SFW) transformation and ignoring higher-order terms one arrives at
\[
H_{GW} = \frac{1}{2m}(\delta^{ij} + 2T^{ij})p_i p_j + mc^2.
\]
(25)

We would like to know how a Gaussian wave packet behaves in a GW background. We will consider the wave packet located at \(z = 0\), in one spatial dimension \(x\), without loss of generality in our conclusions.

\[
\psi(x, 0) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2/\sigma^2}.
\]
(26)

We apply the unitary transformation operator \(U = e^{-iH_{GW}t}\) to Eq. (26) to get the time-evolution of a wave
packet in a GW background (see Appendix A for derivation),

$$\psi(x,t) = \left(\frac{2}{\pi}\right)^{1/4} e^{-(x-x_0)^2/b} \sqrt{b}$$ (27)

where

$$b = 1 + \frac{2iht}{m} (1 + f_0 \cos \omega t)$$ (28)

The expected position of the wavepacket in a GW background is

$$\langle x \rangle = x_0$$ . (29)

In other words, the particle is expected to remain at rest in the co-ordinate system of Eq. (22). This actually is not surprising as this is also what happens in the classical case. In the classical case, the presence of a GW is measured in the change of the proper distance between two particles.

Using Eq. (28) we calculate the mass Fisher information of the particle in a GW, and compare it to the free case. For convenience we assume units where $\hbar, m, f_0, \omega$ are unity. Fig. 1 plots the difference in the mass Fisher information in a GW background from the free case, showing that in general it is different from zero. This means that one can extract mass information of the particle from the GW, in violation of the WEP.

Classical case, there is no such violation as the extended classical body experiences a uniform gravitational field.

As a GW can betray the mass of a quantum particle, it is tantalising to ask whether one could use this fact to detect the presence of a GW. The answer is: in principle yes, but in practice unlikely or at least with great difficulty. Let us consider a particle detector that is at rest in the local co-ordinate system, measuring the probability of a particle being at position $x_0$. At time $t_0$, we place a particle at $x_0$. In the absence of the GW, the probability of detecting the particle always decreases as the wave function spreads according to Schrödinger’s equation, as shown by the dotted line in Fig. 2. In the presence of the GW, the behaviour of the particle is distinctly different; in particular the probability of detecting the particle at $x_0$ can increase, as shown by the solid line in Fig. 2. Therefore, in principle one could simply look for these characteristic increases in the probability of particle detection as signatures of a GW. In practice however, these characteristic increases in the probability of detection is restrictively small for known GW sources. For example, for a rubidium atom ($m = 1.4 \times 10^{-25}$ kg) in GW with a generous amplitude of $f_0 = 10^{-14}$ (Hz) ($\hbar = 1.05 \times 10^{-34}$ Js) one would require a probability detection resolution of $10^{-32}$ to detect the effects of the GW. Given that this is only 2 orders of magnitude larger than the ultimate Planck scale of $10^{-34}$, it would seem that using the violation of the WEP in quantum particles to detected GWs in the manner described, is unlikely.

V. CONCLUSION

We have shown that quantum particles violate the WEP in GW background. We also argue, although in principle this violation could be used to detect the
presence of GWs, in practice it is unlikely to be feasible due to the high accuracy of particle detection required.

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Appendix A: Time-evolution of a wave packet in a gravitational wave

We Fourier transform the wave packet [Eq. (20)] into momentum space,

$$\psi(k,0) = \left(\frac{1}{2\pi}\right)^{1/4} e^{-k^2/4} e^{-ikx_0} . \quad (A1)$$

This allows us to easily write down the time-evolution of the wave pack in momentum space,

$$\psi(k,t) = e^{-iH_{GW}t/\hbar}\psi(k,0) = \left(\frac{1}{2\pi}\right)^{1/4} e^{-i(1+f_0 \cos \omega t)t/2m} e^{-k^2/4} e^{-ikx_0} . \quad (A2)$$

where we have used the fact that $H_{GW}$ in one dimension is

$$H_{GW} = \frac{1 + f_0 \cos \omega t}{2m} p^2 . \quad (A3)$$

We Fourier transform back into position space to arrive at Eq. (27).

[1] Luigi Seveso and Matteo G.A. Paris, “Can quantum probes satisfy the weak equivalence principle?” Annals of Physics 380, 213 – 223 (2017).
[2] Robert Geroch and Pong Soo Jang, “Motion of a body in general relativity,” Journal of Mathematical Physics 16, 65–67 (1975), https://doi.org/10.1063/1.522416.
[3] Yuri N. Obukhov, “Spin, gravity, and inertia,” Phys. Rev. Lett. 86, 192–195 (2001).
[4] Leslie L. Foldy and Siegfried A. Wouthuysen, “On the dirac theory of spin 1/2 particles and its non-relativistic limit,” Phys. Rev. 78, 29–36 (1950).
[5] Erik Eriksen, “Foldy-wouthuysen transformation. exact solution with generalization to the two-particle problem,” Phys. Rev. 111, 1011–1016 (1958).
[6] Anatoly G Nikitin, “On exact foldy-wouthuysen transformation,” Journal of Physics A: Mathematical, Nuclear and General 31, 3297–3300 (1998).
[7] UD Jentschura and JH Noble, “Foldy–wouthuysen transformation, scalar potentials and gravity,” Journal of Physics A: Mathematical and Theoretical 47, 045402 (2014).
[8] Bruno Gonçalves, Yuri N. Obukhov, and Ilya L. Shapiro, “Exact foldy-wouthuysen transformation for gravitational waves and magnetic field background,” Phys. Rev. D 75, 124023 (2007).
[9] James Q. Quach, “Spin gravitational resonance and graviton detection,” Phys. Rev. D 93, 104048 (2016).
[10] One notes that cigar shape potential traps could confine a particle to an effective one spatial dimension.