Proportional 2-Choosability with a Bounded Palette

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Abstract
Proportional choosability is a list coloring analogue of equitable coloring. Specifically, a k-assignment L for a graph G associates a list $L(v)$ of k available colors to each $v \in V(G)$. An L-coloring assigns a color to each vertex v from its list $L(v)$. A proportional L-coloring of G is a proper L-coloring in which each color $c \in \bigcup_{v \in V(G)} L(v)$ is used $\lceil \eta(c)/k \rceil \text{ or } \lfloor \eta(c)/k \rfloor$ times where $\eta(c) = \{|v \in V(G) : c \in L(v)\}|$. A graph G is proportionally k-choosable if a proportional L-coloring of G exists whenever $L$ is a k-assignment for G. Motivated by earlier work, we initiate the study of proportional choosability with a bounded palette by studying proportional 2-choosability with a bounded palette. In particular, when $\ell \geq 2$, a graph G is said to be proportionally $(2, \ell)$-choosable if a proportional L-coloring of G exists whenever $L$ is a 2-assignment for G satisfying $|\bigcup_{v \in V(G)} L(v)| \leq \ell$. We observe that a graph is proportionally $(2, 2)$-choosable if and only if it is equitably 2-colorable. As $\ell$ gets larger, the set of proportionally $(2, \ell)$-choosable graphs gets smaller. We show that whenever $\ell \geq 5$ a graph is proportionally $(2, \ell)$-choosable if and only if it is proportionally 2-choosable.

Keywords Graph coloring · Equitable coloring · List coloring

Mathematics Subject Classification 05C15

1 Introduction

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [18] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, \ldots, m\}$. For graph G we write $\Delta(G)$ for the maximum degree of a vertex in G. We write $K_{n,m}$ for the complete bipartite graph with partite sets of size n and m.

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When $C$ is a cycle on $n$ vertices ($n \geq 3$ since $C$ is simple), $V(C) = \{v_1, \ldots, v_n\}$, and $E(C) = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$, then we say the vertices are written in cyclic order when we write $v_1, \ldots, v_n$. When $G_1$ and $G_2$ are vertex disjoint graphs, we write $G_1 + G_2$ for the disjoint union of $G_1$ and $G_2$.

In 2019 a new notion combining the notions of list coloring and equitable coloring called proportional choosability was introduced [8]. In this paper, we study proportional choosability with a bounded palette. We begin by briefly reviewing some important notions.

### 1.1 List Coloring with a Bounded Palette

In the classical vertex coloring problem we wish to color the vertices of a graph $G$ with colors from the set $[k]$ so that adjacent vertices receive different colors, a so-called proper $k$-coloring. We say $G$ is $k$-colorable when a proper $k$-coloring of $G$ exists. The chromatic number of $G$, denoted $\chi(G)$, is the smallest $k$ such that $G$ is $k$-colorable.

List coloring is a variation on classical vertex coloring, and it was introduced independently by Vizing [17] and Erdős, Rubin, and Taylor [4] in the 1970’s. For list coloring, we associate with a graph $G$ a list assignment $L$ that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. We say $G$ is $L$-colorable if there exists a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a proper $L$-coloring of $G$). A list assignment $L$ is called a $k$-assignment for $G$ if $|L(v)| = k$ for each $v \in V(G)$. We say $G$ is $k$-choosable if $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$.

The study of list coloring with a bounded palette began in 2005 [11]. Suppose that $L$ is a list assignment for a graph $G$. The palette of colors associated with $L$ is $\bigcup_{v \in V(G)} L(v)$. From this point forward, we use $L$ to denote the palette of colors associated with $L$ whenever $L$ is a list assignment. Suppose $1 \leq k \leq \ell$. A list assignment $L$ for a graph $G$ is a $(k, \ell)$-assignment for $G$ if $L$ is a $k$-assignment for $G$ and $L \subseteq \ell$. Notice that if $L$ is a $(k, \ell)$-assignment for $G$, we can view $L$ as a function with domain $V(G)$ and codomain equal to the set of $k$-element subsets of $\ell$. We say $G$ is $(k, \ell)$-choosable if $G$ is $L$-colorable whenever $L$ is a $(k, \ell)$-assignment for $G$. Clearly, a graph is $(k, k)$-choosable if and only if it is $k$-colorable. In [2] the complexity of $(k, \ell)$-choosability is studied for grids (i.e., the Cartesian product of two paths), subgrids (i.e., induced subgraphs of grids), 3-colorable planar graphs, and triangle-free planar graphs.

In [11] it is shown that for any $k \geq 2$, there is a $C \in \mathbb{N}$ satisfying $C = O(k16^k \ln k)$ as $k \to \infty$ such that if $G$ is $(k, 2k - 1)$-choosable, then $G$ is $C$-choosable. In 2015, it was subsequently demonstrated that this constant $C$ must also satisfy $C = \Omega(4^k / \sqrt{k})$ as $k \to \infty$ (see [1]). Importantly, results like this show that understanding list coloring with a bounded palette can provide us with information about list coloring in general. On the other hand, graphs that fail to be $k$-choosable can be $(k, \ell)$-choosable. Indeed, for each $k$ and $\ell$ satisfying $3 \leq k \leq \ell$, there is a graph $G$ that is $(k, \ell)$-choosable but not $(k, \ell + 1)$-choosable (see [11]).
1.2 Equitable Coloring and Proportional Choosability

1.2.1 Equitable Coloring

Equitable coloring is another variation on the classical vertex coloring problem that began with a conjecture of Erdős in 1964 [3]. Equitable coloring was formally defined by Meyer in 1973 [12]. Specifically, an equitable \( k \)-coloring of a graph \( G \) is a proper \( k \)-coloring \( f \) of \( G \) such that the sizes of the color classes differ by at most one (where a proper \( k \)-coloring has exactly \( k \) color classes). In an equitable \( k \)-coloring, the color classes associated with the coloring are each of size \( \lfloor \frac{|V(G)|}{k} \rfloor \) or \( \lceil \frac{|V(G)|}{k} \rceil \). We say that a graph \( G \) is equitable \( k \)-colorable if there exists an equitable \( k \)-coloring of \( G \). Equitable coloring has been applied in various contexts (for example, see [6, 7, 15, 16]). Furthermore, in 1970 Hajnal and Szemerédi [5] proved the 1964 conjecture of Erdős: every graph \( G \) has an equitable \( k \)-coloring when \( \Delta(G) \leq \frac{1}{2}k \).

Unlike classical vertex coloring, increasing the number of colors can make equitable coloring more difficult. For example, for any \( m \in \mathbb{N} \), \( K_{2m+1,2m+1} \) is equitably \( 2m \)-colorable, but it is not equitably \( (2m + 1) \)-colorable. Moreover, unlike classical vertex coloring, the property of being equitably \( k \)-colorable is not monotone. For example, \( K_{3,3} \) is equitably 2-colorable, but \( K_{1,3} \) is not equitably 2-colorable.

1.2.2 Proportional Choosability

In 2003, Kostochka, Pelsmajer, and West [10] introduced a list version of equitable coloring called equitable choosability which has received quite a bit of attention in the literature. If \( L \) is a \( k \)-assignment for the graph \( G \), a proper \( L \)-coloring of \( G \) is an equitable \( L \)-coloring of \( G \) if each color in \( L \) appears on at most \( \frac{|V(G)|}{k} \) vertices. We say that \( G \) is equitably \( k \)-choosable if an equitable \( L \)-coloring of \( G \) exists whenever \( L \) is a \( k \)-assignment for \( G \). While equitable choosability is a useful notion in many contexts, it does not place a lower bound on how many times a color must be used, whereas in an equitable \( k \)-coloring of \( G \) each color must be used at least \( \lceil \frac{|V(G)|}{k} \rceil \) times.

Kaul, Pelsmajer, Reiniger, and the first author [8] introduced a new list analogue of equitable coloring called proportional choosability which places both an upper and lower bound on how many times a color must be used in a list coloring. Specifically, suppose that \( L \) is a \( k \)-assignment for a graph \( G \). For each color \( c \in L \), the multiplicity of \( c \) in \( L \) is the number of vertices \( v \) whose list \( L(v) \) contains \( c \). The multiplicity of \( c \) in \( L \) is denoted by \( \eta_L(c) \) (or simply \( \eta(c) \) when the list assignment is clear). So, \( \eta_L(c) = |\{ v \in V(G) : c \in L(v) \}| \). A proper \( L \)-coloring \( f \) for \( G \) is a proportional \( L \)-coloring of \( G \) if for each \( c \in L \), \( f^{-1}(c) \), the color class of \( c \), is of size \( \left\lfloor \frac{\eta(c)}{k} \right\rfloor \) or \( \left\lceil \frac{\eta(c)}{k} \right\rceil \).

We say that \( G \) is proportionally \( L \)-colorable if a proportional \( L \)-coloring of \( G \) exists,
and we say \( G \) is proportionally \( k \)-choosable if \( G \) is proportionally \( L \)-colorable whenever \( L \) is a \( k \)-assignment for \( G \). Proportional choosability has some beautiful properties, some of which, at first glance, may seem quite surprising.

**Proposition 1** [8] If \( G \) is proportionally \( k \)-choosable, then \( G \) is both equitably \( k \)-choosable and equitably \( k \)-colorable.

**Proposition 2** [8] If \( G \) is proportionally \( k \)-choosable, then \( G \) is proportionally \( (k+1) \)-choosable.

**Proposition 3** [8] Suppose \( H \) is a subgraph of \( G \). If \( G \) is proportionally \( k \)-choosable, then \( H \) is proportionally \( k \)-choosable.

Notice that Propositions 2 and 3 are particularly interesting since they do not hold in the contexts of equitable coloring and equitable choosability. Recently, a nice characterization of the proportionally 2-choosable graphs was discovered; this characterization inspired the questions that lead to this paper. Recall that a linear forest is a disjoint union of paths.

**Theorem 4** [9, 13] A graph \( G \) is proportionally 2-choosable if and only if \( G \) is a linear forest such that the largest component of \( G \) has at most five vertices and all the other components of \( G \) have two or fewer vertices.

### 1.3 Proportional Choosability with a Bounded Palette

Having defined proportional choosability, it is natural to consider proportional choosability with a bounded palette. Suppose \( 1 \leq k \leq \ell \). We say a graph \( G \) is proportionally \((k, \ell)\)-choosable if \( G \) is proportionally \( L \)-colorable whenever \( L \) is a \((k, \ell)\)-assignment for \( G \). Two properties of proportional \((k, \ell)\)-choosability are easy to immediately prove.

**Proposition 5** For each \( k \in \mathbb{N} \), \( G \) is proportionally \((k, k)\)-choosable if and only if \( G \) is equitably \( k \)-colorable.

**Proof** Suppose \( G \) is proportionally \((k, k)\)-choosable. Let \( L \) be a \( k \)-assignment for \( G \) such that \( L(v) = [k] \) for all \( v \in V(G) \). Note that \( \eta(1) = \cdots = \eta(k) = |V(G)| \). Since \( L \) is a \((k, k)\)-assignment for \( G \), we know there is a proportional \( L \)-coloring \( f \) of \( G \). Clearly, \( f \) is also an equitable \( k \)-coloring of \( G \).

Conversely, suppose \( G \) is equitably \( k \)-colorable and \( L \) is an arbitrary \((k, k)\)-assignment for \( G \). Notice that an equitable \( k \)-coloring of \( G \) exists, and \( L(v) = [k] \) for each \( v \in V(G) \). The result follows since an equitable \( k \)-coloring of \( G \) is also a proportional \( L \)-coloring of \( G \).

**Proposition 6** Suppose \( 1 \leq k \leq \ell \). If \( G \) is proportionally \((k, \ell + 1)\)-choosable, then \( G \) is proportionally \((k, \ell)\)-choosable.

**Proof** Suppose \( G \) is proportionally \((k, \ell + 1)\)-choosable, and suppose \( L \) is an arbitrary \((k, \ell)\)-assignment for \( G \). Clearly, \( L \) is also a \((k, \ell + 1)\)-assignment for \( G \).
Since $G$ is proportionally $(k, \ell + 1)$-choosable, we know that $G$ is proportionally $L$-colorable. 

The following question lead to the results in this paper.

**Question 7** For each $\ell \geq 2$, what graphs are proportionally $(2, \ell)$-choosable?

Suppose $G$ is the set of proportionally 2-choosable graphs. Notice that if $i \geq 2$ and $G_i$ is the set of graphs that are proportionally $(2, i)$-choosable, then by Proposition 6,

$$G_2 \supseteq G_3 \supseteq G_4 \supseteq \cdots.$$  

By Theorem 4, for every $\ell \in \mathbb{N}$, $G_\ell$ contains all linear forests such that the largest component has at most five vertices and all the other components have two or fewer vertices (i.e. $G$ is a subset of $G_\ell$ for each $\ell \in \mathbb{N}$). Furthermore, Proposition 5 tells us that $G_2$ is exactly the set of equitably 2-colorable graphs. Since an $n$-vertex graph is proportionally $k$-choosable if and only if it is proportionally $(k, kn)$-choosable the following question and its generalization are natural.

**Question 8** Is there a constant $\mu$ such that any graph $G$ is proportionally 2-choosable if and only if $G$ is proportionally $(2, \mu)$-choosable?

**Question 9** For each $k \geq 2$, is there a constant $\mu_k$ such that any graph $G$ is proportionally $k$-choosable if and only if $G$ is proportionally $(k, \mu_k)$-choosable?

Question 9 is open for each $k \geq 3$. The answer to Question 8 is yes, and interestingly, the smallest such $\mu$ for which the answer is yes is 5. Specifically, using the notation above, we will see below that

$$G_2 \supseteq G_3 \supseteq G_4 \supseteq G_5 \text{ and } G_\ell = G \text{ for each } \ell \geq 5.$$  

1.4 Outline of Results and an Open Question

We will answer Question 7 for $\ell = 2$ and each $\ell \geq 5$ which will give us an answer to Question 8. The proofs of many of our results rely on finding ways to extend Proposition 3 to the bounded palette context. This presents some difficulties as the proof of Proposition 3 relies on the construction of a list assignment that may have a large palette size (cf. the proof of Proposition 21 in [8]). The characterization of the proportionally $(2, 2)$-choosable graphs below follows immediately from Proposition 5 and a simple characterization of equitably 2-colorable graphs.

**Observation 10** A graph $G$ is proportionally $(2, 2)$-choosable if and only if $G$ is a bipartite graph with a bipartition $X, Y$ satisfying $|X| - |Y| \leq 1$.

Our next result answers Question 7 for each $\ell \geq 5$.

**Theorem 11** For each $\ell \geq 5$, a graph $G$ is proportionally $(2, \ell)$-choosable if and only if $G$ is a linear forest such that the largest component of $G$ has at most 5 vertices and all other components of $G$ have at most 2 vertices.

With Observation 10 and Theorem 11 in mind, the following question is natural.
**Question 12** For $\ell = 3, 4$ what graphs are proportionally $(2, \ell)$-choosable?

Question 12 is pursued in [14] which contains long arguments that prove the following two results: (1) A connected graph $G$ is proportionally $(2, 4)$-choosable if and only if $G = P_n$ where $n \leq 5$ or $n = 7$, and (2) A connected graph $G$ is proportionally $(2, 3)$-choosable if and only if $G = P_n$ for some $n \in \mathbb{N}$.

One might conjecture that a graph $G$ is proportionally $(2, 4)$-choosable (resp. $(2, 3)$-choosable) if and only if the components of $G$ are proportionally $(2, 4)$-choosable (resp. $(2, 3)$-choosable). It is shown in [14] that this conjecture however is not correct in both directions for proportional $(2, 4)$-choosability, and the “only if” direction of this conjecture is not correct for proportional $(2, 3)$-choosability. Thus, Question 12 is open in general.

### 2 Proving Theorem 11

We begin by proving three lemmas.

**Lemma 13** If $G$ contains a copy of $K_{1,3}$ as a subgraph, then $G$ is not proportionally $(2, 3)$-choosable. Consequently, if a graph $G$ is proportionally $(2, \ell)$-choosable for some $\ell \geq 3$, then $\Delta(G) \leq 2$.

**Proof** Suppose $H$ is a subgraph of $G$ such that $H = K_{1,3}$, and suppose $H$ has bipartition $\{a\}$ and $\{b_1, b_2, b_3\}$. To prove the desired, we will construct a $(2, 3)$-assignment, $L$, for $G$ such that there is no proportional $L$-coloring of $G$. Suppose $L$ is the $(2, 3)$-assignment for $G$ such that for each $v \in V(H), L(v) = \{1, 2\}$, and for each $v \in V(G) - V(H), L(v) = \{2, 3\}$. For the sake of contradiction, suppose that $f$ is a proportional $L$-coloring of $G$. Note that $\eta(1) = 4$, so $|f^{-1}(1)| = 2$. Clearly, $f(a) = 1$ or $f(a) = 2$. If $f(a) = 1$, then $f(b_i) = 2$ for each $i \in [3]$, and $|f^{-1}(1)| = 1$. If $f(a) = 2$, then $f(b_i) = 1$ for each $i \in [3]$, and $|f^{-1}(1)| = 3$. In either case we have a contradiction.

**Lemma 14** If a graph contains a cycle, then it is not proportionally $(2, \ell)$-choosable for each $\ell \geq 4$.

**Proof** Suppose $G$ is an arbitrary graph that contains a cycle $C$. By Proposition 6, it suffices to show that $G$ is not proportionally $(2, 4)$-choosable. If $C$ is an odd cycle, then $G$ is not 2-colorable; thus, $G$ is not proportionally $(2, 4)$-choosable. So, we may suppose that $C$ is an even cycle.

Suppose the vertices of $C$ written in cyclic order are: $v_1, \ldots, v_{2k+2}$ where $k \in \mathbb{N}$. We will now construct a $(2, 4)$-assignment, $L$, for $G$ such that there is no proportional $L$-coloring of $G$. Suppose $L$ is the $(2, 4)$-assignment for $G$ given by $L(v_{2i-1}) = \{1, 2\}$ and $L(v_{2i}) = \{1, 3\}$ for each $i \in [k + 1]$, and $L(v) = \{3, 4\}$ if $v \in V(G) - V(C)$. Notice that $\eta(1) = 2k + 2$ and $\eta(2) = k + 1$. For the sake of contradiction, suppose $f$ is a proportional $L$-coloring of $G$. This implies that $|f^{-1}(1)| = k + 1$ and
\[0 < \left\lfloor \frac{k + 1}{2} \right\rfloor \leq |f^{-1}(2)| \leq \left\lceil \frac{k + 1}{2} \right\rceil < k + 1.\]

Since \( C \) contains exactly two independent sets of size at least \( k + 1 \) and \( 1 \notin L(v) \) for each \( v \in V(G) - V(C) \), either \( f(v_{2i}) = 1 \) for each \( i \in [k + 1] \) or \( f(v_{2i-1}) = 1 \) for each \( i \in [k + 1] \). This implies that \( |f^{-1}(2)| = k + 1 \) or \( |f^{-1}(2)| = 0 \) which in either case is a contradiction. \( \square \)

**Lemma 15** If a graph contains a copy of \( K_{1,2} + K_{1,2} \), then it is not proportionally \((2, \ell)-choosable\) for each \( \ell \geq 5 \).

**Proof** Suppose \( G \) is a graph that contains two vertex disjoint graphs \( H_1 \) and \( H_2 \) that are copies of \( K_{1,2} \). By Proposition 6, it suffices to show that \( G \) is not proportionally \((2, 5)-choosable\). Suppose \( H_1 \) has bipartition \( A_1, B_1 \), where \( A_1 = \{a_1\} \) and \( B_1 = \{b_0, b_1\} \). Suppose \( H_2 \) has bipartition \( A_2, B_2 \), where \( A_2 = \{a_2\} \) and \( B_2 = \{b_2, b_3\} \). We will now construct a \((2, 5)-assignment\) \( L \) for \( G \) such that there is no proportional \( L\)-coloring of \( G \). Suppose \( L \) is the \((2, 5)-assignment\) for \( G \) given by \( L(a_1) = L(a_2) = \{1, 2\} \), \( L(b_0) = L(b_1) = \{1, 3\} \), \( L(b_2) = L(b_3) = \{1, 4\} \), and \( L(v) = \{1, 5\} \) if \( v \in V(G) - V(H_1 + H_2) \). Notice that \( \eta(i) = 2 \) for \( i = 2, 3, 4 \).

For the sake of contradiction, suppose \( f \) is a proportional \( L\)-coloring of \( G \). This means that \( |f^{-1}(i)| = 1 \) for \( i = 2, 3, 4 \). Thus, \( f(a_1) = 1 \) or \( f(a_2) = 1 \). This implies that \( |f^{-1}(3)| = 2 \) or \( |f^{-1}(4)| = 2 \) respectively which in either case is a contradiction. \( \square \)

We are now ready to prove Theorem 11 which we restate.

**Theorem 11** For each \( \ell \geq 5 \), a graph \( G \) is proportionally \((2, \ell)-choosable\) if and only if \( G \) is a linear forest such that the largest component of \( G \) has at most 5 vertices and all other components of \( G \) have at most 2 vertices.

**Proof** Throughout the proof, suppose \( \ell \) is a fixed natural number satisfying \( \ell \geq 5 \). Suppose that \( G \) is a linear forest such that the largest component of \( G \) has at most 5 vertices and all other components of \( G \) have at most 2 vertices. By Theorem 4, we know \( G \) is proportionally \((2, \ell)-choosable\).

Conversely, suppose that \( G \) is proportionally \((2, \ell)-choosable\). By Lemma 13 we know that \( \Delta(G) \leq 2 \), and by Lemma 14 we know that \( G \) can not contain a cycle. This means that \( G \) must be a linear forest. Finally, by Lemma 15 we know that \( G \) can not contain a copy of \( K_{1,2} + K_{1,2} \) (i.e., \( P_3 + P_3 \)). Thus, \( G \) must be a linear forest such that the longest path has at most 5 vertices and all other paths have at most 2 vertices. \( \square \)

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