Quantum counter erasure

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Interference comes from coherent mixing. It can be suppressed by entanglement, and the latter can be erased so as to revive interference. If the entanglement is a minimal-term one (with minimal-term mixing), as is the case in most thought and real experiments reported, there appears the possibility of counter erasure and counter interference. This peculiar phenomenon of minimal-term mixing and minimal-term entanglement is investigated in detail. In particular, all two-term mixings of an (arbitrary) given minimal-term mixed state are explicitly exhibited. And so are their possible laboratory realizations in terms of distant ensemble decomposition.

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I. INTRODUCTION

In order to gain concrete experimental notions, we start by discussing the well-known two-slit interference experiment [1], which is theoretically the simplest and best known example of interference.

Let the indices 1 and 2 refer to the two slits, and let \( |\psi_1\rangle \) and \( |\psi_2\rangle \) be the spatial state vectors of the photon having traversed only the first or only the second slit respectively. Then the superposition (also called coherent mixture) of these two state vectors, i.e.,

\[
|\psi\rangle \equiv (1/2)^{1/2} \left( |\psi_1\rangle + |\psi_2\rangle \right)
\]  

(1)

is the interference state vector corresponding to both slits being open. (The term "interference" actually refers to the interference pattern on the detection screen.)

To bring in entanglement, we assume that the photons pass a horizontal linear polarizer at slit 1 and a vertical one at slit 2 [2]. The entangled two-subsystem (but one-photon) state vector is then, in obvious notation:

\[
|\chi\rangle \equiv \left( |H\rangle |\psi_1\rangle + |V\rangle |\psi_2\rangle \right).
\]  

(2)

We are dealing with a minimal-term entanglement (two terms only). The state of the subsystem of spatial degrees of freedom is now an improper mixture [3]

\[
\rho_s \equiv Tr_p |\chi\rangle \langle \chi| = (1/2) \left( |\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2| \right)
\]  

(3)

as easily seen. The symbol \( \rho_s \) denotes the state operator (reduced statistical operator) of the spatial subsystem, and "\( Tr_p \)" denotes the partial trace over the (linear) polarization degree of freedom of the photon. Also the mixture in (3) is a minimal-term one.

The entanglement in (2) suppresses the interference replacing the interference state (1) by the noninterference one given by (3).

The entanglement (2) contains the so-called "which-path" memory, because, in principle, measuring only if the linear polarization is horizontal or vertical, one reestablishes \( |\psi_1\rangle \) or \( |\psi_2\rangle \) respectively. For example, if the polarization turns out to be horizontal, then, according to the so-called Lüders formula for ideal measurement [4], [5], one has the following disentanglement:

\[
|\chi\rangle \rightarrow c \left( |H\rangle \langle H| \otimes 1 \right) |\chi\rangle = |H\rangle |\psi_1\rangle,
\]  

(4)

where \( c \) is a normalization constant.

Before this "which-path" measurement is performed, there is a (potential) complementarity in \( |\chi\rangle \) because it provides also a complementary memory, on ground of which one can revive the suppressed interference in the distant subsystem [6], [7].
This revival is possible because, as easily checked, one can rewrite the same composite-system state vector given by (2) as follows:

\[ |\chi\rangle = (1/2)^{1/2} \left( |45^0\rangle |\psi\rangle + |−45^0\rangle |\psi^c\rangle \right), \]  

(5)

where, e.g., \(|45^0\rangle\) is the polarization state at \(45^0\) between horizontal and vertical, \(|\psi\rangle\) is given by (1), and, what we call the counter-interference state (for reasons seen below), \(|\psi^c\rangle\) is defined by

\[ |\psi^c\rangle \equiv (1/2)^{1/2} \left( |\psi_1\rangle - |\psi_2\rangle \right). \]  

(6)

Further, the corresponding linear polarization state vectors at the given angles, evidently, satisfy

\[ |45^0\rangle = (1/2)^{1/2} \left( |H\rangle + |V\rangle \right), \quad |−45^0\rangle = (1/2)^{1/2} \left( |H\rangle - |V\rangle \right). \]  

(7a, b)

If one measures the linear polarization at \(45^0\) or at \(−45^0\) (since \(⟨45^0|45^0⟩ = 0\), this is, essentially, an observable), and if the former result is obtained, then, on account of (5), the following disentanglement takes place (cf. [4] or [5]):

\[ |\chi\rangle \rightarrow |45^0\rangle |\psi\rangle. \]

Thus, the spatial interference state \(|\psi\rangle\) is revived. This phenomenon is called quantum erasure [8], because the “which-path memory” in the entanglement in \(|\chi\rangle\), which suppresses the interference, is erased.

If in the \(45^0\)-angle linear polarization measurement the result is \(45^0\), then the Lüders formula gives \(|\psi^c\rangle\), i.e., it is the counter-interference state that is revived.

As to what is actually observed in the laboratory, one cannot "see" the interference state \(|\psi\rangle\) itself (cf. (1)) in full. One usually observes an interference pattern implied by \(|\psi\rangle\) (on a detection screen). The pattern is actually the localization probability distribution:

\[ p_i(\mathbf{r}) ≡ |\psi(\mathbf{r})|^2 = (1/2) \left( |\psi_1(\mathbf{r})|^2 + |\psi_2(\mathbf{r})|^2 + ψ_1^*(\mathbf{r})ψ_2(\mathbf{r}) + ψ_1(\mathbf{r})ψ_2^*(\mathbf{r}) \right). \]  

(8a)

where "i" refers to interference, and \(ψ(\mathbf{r}) ≡ χ(\mathbf{r})\) is determined by (1), etc.

It follows from (5) that the state \(ρ_\text{s}\) of the spatial subsystem (cf (3)) can also be written as

\[ ρ_\text{s} = (1/2) \left( |\psi\rangle⟨\psi| + |\psi^c\rangle⟨\psi^c| \right). \]  

(9)

The probability distribution defined by the counter-interference state \(|\psi^c\rangle\), what we call counter interference, and the one defined by the incoherent mixture \(ρ_\text{s}\) are respectively:

\[ p^c_i(\mathbf{r}) ≡ |\psi^c(\mathbf{r})|^2 = \]

\[ (1/2) \left( |ψ_1(\mathbf{r})|^2 + |ψ_2(\mathbf{r})|^2 + ψ_1^*(\mathbf{r})ψ_2(\mathbf{r}) - ψ_1(\mathbf{r})ψ_2^*(\mathbf{r}) \right), \]  

(8b)

\[ p(\mathbf{r}) ≡ (\mathbf{r}|ρ_\text{s}|\mathbf{r}) = (1/2) \left( p_i(\mathbf{r}) + p^c_i(\mathbf{r}) \right) = (1/2) \left( |ψ_1(\mathbf{r})|^2 + |ψ_2(\mathbf{r})|^2 \right). \]  

(8c)

The empirical, i.e., ensemble view of the phenomena of interference and counter-interference consists in realizing that, on account of the measurement of \(⟨45^0|45^0⟩ \otimes 1\) on each individual photon of a laboratory ensemble that is described by \(|\chi\rangle\) given by (2), the (improper [3]) ensemble of spatial subsystems described by \(ρ_\text{s}\) breaks up into two subensembles (cf. the first sum in (8c)), each of which causes interference on the detection screen (cf. (8a) and (8b) respectively), but which are counter cases of each other in the sense that the two interferences cancel (cf (8c)).

A thought experiment in which the above mentioned linear polarizers at the slits are replaced by maser cavities was given by Scully et al. in [8]. The authors actually introduce quantum erasure in a pioneering way explaining the revival of the interference state \(|\psi\rangle\). With a slight modification of the experiment one can revive the counter-interference state \(|\psi^c\rangle\) instead of \(|\psi\rangle\). The first real experiment of quantum erasure was attempted in [9]. It turned out [10] that it was erasure in a somewhat broader sense.

Actually, the entangled composite-system state \(|\chi\rangle\) contains a nonenumerable infinity of spatial states that can, in principle, be revived. The revival takes place via the measurement of an opposite-subsystem observable. Since the entanglement is a minimal-term one, this observable is a yes-no measurement, and the revived states appear in pairs, the counter states of each other. The “which-way” states \(|\psi_1\rangle\) and \(|\psi_2\rangle\) on the one hand and the interference state \(|\psi\rangle\) and the counter-interference state \(|\psi^c\rangle\) on the other are examples of counter states of each other.

We explore this phenomenon in detail in this study.
II. COUNTER STATES IN MINIMAL-TERM MIXTURES

In this section the following question is given an answer:

*How to classify, i.e., enumerate (in a bijective way) explicitly the set of all mathematically possible decompositions of a given minimal-term mixture (like $\rho_s$ in (3)) into two pure states?*

This question is studied with a view to find out (in the next section) how one can revive any of the two pure states of any of the mentioned decompositions by a yes-no measurement on the opposite subsystem.

Let $\rho$ be a given minimal-term mixture state operator, i.e., one that can be written in the spectral form:

$$
\rho = r|1\rangle\langle1| + (1-r)|2\rangle\langle2|,
$$  
\[(10)\]

where $0 < r \leq (1/2)$. It is known [11] that each state vector from the range of $\rho$, and only such state vectors, can appear in a decomposition of $\rho$. We want to find out about the counter state vectors and the corresponding statistical weights.

Our answer to the above question goes as follows:

Let (10) be given. Let, further,

$$
|\phi\rangle \equiv p|1\rangle + (1-p^2)^{1/2}\text{e}^{i\vartheta}|2\rangle,
$$  
\[(11a)\]

with any values from the intervals

$$
0 \leq p \leq 1, \quad 0 \leq \vartheta < 2\pi,
$$  
\[(11b)\]

be an (up to a phase factor) arbitrary state vector from the range of $\rho$. Then there exists one and only one decomposition of $\rho$ into two pure states in which $|\phi\rangle\langle\phi|$ appears. It is

$$
\rho = w|\phi\rangle\langle\phi| + (1-w)|\phi^c\rangle\langle\phi^c|,
$$  
\[(12)\]

where

$$
w \equiv r(1-r)/\left(p^2(1-r)+(1-p^2)r\right),
$$  
\[(13)\]

and

$$
|\phi^c\rangle \equiv \left[(r-wp^2)/(1-w)\right]^{1/2}|1\rangle + \left[\left((1-r)-w(1-p^2)\right)/(1-w)\right]^{1/2}\text{e}^{i(\vartheta+\pi)}|2\rangle.
$$  
\[(14)\]

The claims made are shown to follow as an immediate consequence of a wider lemma stated and proved in Appendix 1.

To be practical, we shall call decomposition (12) of $\rho$ in the context of (10)-(14) ”the $p,\vartheta$-decomposition”. Thus, all decompositions of a given minimal-term mixture $\rho$ can be classified or enumerated by the two parameters $p$ and $\vartheta$.

The counter state $|\phi^c\rangle$ and the statistical weight $w$ are uniquely implied by the state operator $\rho$ and $|\phi\rangle$. The state vectors $|\phi\rangle$ and $|\phi^c\rangle$ are counter states of each other, i.e., if one is written as (11a), then the other takes the form (14).

Further, a more detailed examination of the answer given reveals the following peculiarities in the above relations:

(i) If the characteristic value $r$ of $\rho$ is nondegenerate, or, equivalently, if $r < (1/2)$, then relation (13) establishes a monotonously decreasing bijection of the interval $[0,1]$ of the values of $p$ onto the interval $[r, (1-r)]$ of the values of $w$. (Namely, $dw/dp^2 < 0$.)

(ii) If $r = (1-r) = 1/2$, then $p$ and $\vartheta$ can still take all values from their respective intervals (11b), but always $w = 1 - w = 1/2$. In this case the counter state takes the simple form

$$
|\phi^c\rangle = (1-p^2)^{1/2}|1\rangle - pe^{i\vartheta}|2\rangle
$$  
\[(15)\]

and $|\phi^c\rangle$ is orthogonal to $|\phi\rangle$. Decomposition (12) is now a spectral form of $\rho$ (just like (10)). In this case, every decomposition of $\rho$ into pure states is an orthogonal one (a spectral form), and there are no other decompositions into
pure states. Further, every orthogonal decomposition of the range $R(\rho)$ gives also a decomposition of $\rho$, and vice versa.

(iii) Always $r \leq w \leq (1 - r)$. The equality $r = w$ is observed if and only if $p = 1$, then $|\phi\rangle = |1\rangle$; whereas $w = (1 - r)$ if and only if $p = 0$, and then $|\phi\rangle = |2\rangle$. (These are consequences of (i) and (11a).)

In case of nondegenerate $r$, peculiarity (iii) implies that the spectral form (10) is the mixture in which the most dominant pure state (i.e., the one with the largest statistical weight) and the least dominant one are exhibited. All other mixture forms (i.e., $p, \vartheta$-decompositions) of the given state operator $\rho$ are less extreme.

In case of nondegenerate $r$, it, further, ensues from peculiarity (i) that for any $a$ priori given $w \in (r, 1 - r)$, there exists a family of $p, \vartheta$-decompositions that give this $w$ value. The (unique) value of $p$ is obtained by solving (13) for $p$, and $\vartheta$ is arbitrary. In particular, $w = 1/2$ is obtained with $p = r$.

Before we tackle (in the next section) the problem of how to perform empirically decomposition (12) of an empirically given (subsystem) state $\rho$, it should be noted that this decomposition may find application in various problems. For instance, $\rho$ may be the state operator of a composite system, and $|\phi\rangle$ (cf (11a)) an uncorrelated state vector. The evaluation of the counter state $|\phi^c\rangle$ (cf (14)) is then of interest because it decomposes $\rho$ into a separable and an inseparable state (cf [12] and [13]).

### III. WHICH YES-NO MEASUREMENT GIVES RISE TO A GIVEN DECOMPOSITION?

The state vectors $|\phi\rangle$ and $|\phi^c\rangle$ in decomposition (12) are in general not orthogonal. Hence, one cannot produce decomposition (12) by measurement in the laboratory because this always ends up in orthogonal states.

Nevertheless, these decompositions do have physical meaning in terms of so-called distant state decomposition (empirically distant ensemble decomposition):

One views the system on hand as a subsystem of a two-subsystem composite system, and one envisages the state vector $|\omega\rangle$ of the latter that implies the $a$ priori given state operator $\rho$ (cf (10)) as its subsystem state operator $\rho = Tr_\rho|\omega\rangle\langle\omega|$ (the letter “$o$” in the index of the partial trace applies to the “opposite” subsystem). Then, arguing along the lines presented in the Introduction for the Young two-slit interference, an opposite-subsystem yes-no measurement on the composite system in the state $|\omega\rangle$ may leave the subsystem state $\rho$ decomposed precisely as given in the $p, \vartheta$-decomposition (12). This is what we investigate in detail in this section.

We have the given minimal-term mixture $\rho$ in spectral form (10). We write $|\omega\rangle$ expanded in the characteristic basis \{1,2\} of $\rho$ with positive expansion coefficients:

$$|\omega\rangle = r^{1/2}|1_o\rangle + (1 - r)^{1/2}|2_o\rangle.$$

This is a so-called Schmidt biorthogonal expansion (cf section 4 in [14] or see [15]). The vectors \{1,2\}_o are orthogonal state vectors in the state space of the opposite subsystem. (One may define $|\omega\rangle$ via (16) by choosing any such subbasis.)

A suitable observable on the opposite subsystem that is a yes-no one on $|\omega\rangle$ has the following spectral form:

$$A_o = a_1|\mu_1\rangle_o\langle\mu_1|_o + a_2|\mu_2\rangle_o\langle\mu_2|_o, \quad a_1 \neq a_2,$$  \hspace{1cm} (17)

where \{(state vectors) $|\mu_1\rangle_o$ and $|\mu_2\rangle_o$ are required to be (mutually orthogonal) linear combinations of $|1\rangle_o$ and $|2\rangle_o$.

One should note that one of the exhibited characteristic values of $A_o$ can be zero if the opposite-subsystem state space is two dimensional. But if it is three- or more dimensional, then both $a_1$ and $a_2$ must be nonzero. Then, as evident from (17), $A_o$ necessarily has zero in its spectrum (though it is not exhibited in (17)).

We treat the characteristic values $\{a_1, a_2\}$ as irrelevant, i.e., we consider the whole class of observables having the same characteristic vectors $\{|\mu_1\rangle_o, |\mu_2\rangle_o\}$ as (essentially) one observable (as it is often done).

Further, the characteristic vectors can be written in the following suitable form:

$$|\mu_1\rangle_o = q|1\rangle_o + \left(1 - q^2\right)^{1/2} e^{i\lambda}|2\rangle_o,$$

$$0 \leq q \leq 1, \quad 0 \leq \lambda < 2\pi;$$  \hspace{1cm} (18a)

$$|\mu_2\rangle_o = \left(1 - q^2\right)^{1/2}|1\rangle_o - q e^{i\lambda}|2\rangle_o;$$  \hspace{1cm} (19)
where \{ |1⟩_o, |2⟩_o \} are determined by (or determine) the composite-system state vector \( |\omega⟩ \) (cf (16)).

We call the \( (A_o \otimes 1) \) measurement on the composite system in the state \( |\omega⟩ \) the \( q, \lambda \)-measurement.

Now, one can make the following **claim**, which answers the question from the title of the section:

If a \( p, \vartheta \)-decomposition (12) of a given minimal-term mixture state operator \( \rho \) (cf (10)) is given and a minimal-term entanglement composite system state vector \( |\omega⟩ \) (cf (16)) implying \( \rho \) as its subsystem state operator is also given, then the following \( q, \lambda \)-measurement performed on \( |\omega⟩ \), and no other one, gives rise to the mentioned \( p, \vartheta \)-decomposition:

\[
q = \left( \frac{w}{r} \right)^{1/2} p, \tag{20a}
\]

where \( w \) is the statistical weight of \( |\phi⟩⟨\phi| \) in decomposition (12) given by (13), and

\[
\lambda = 2\pi - \vartheta. \tag{20b}
\]

The claim is proved in Appendix 2.

Inverting the question from the title of the section, the (second) answer is as follows:

A given \( q, \lambda \)-measurement (cf (17)-(19)) on the composite-system state \( |\omega⟩ \) given by (16) gives rise to the following \( p, \vartheta \)-decomposition (12) of the subsystem state operator \( \rho \) implied by \( |\omega⟩ \):

\[
p = \left( \frac{r}{w} \right)^{1/2} q, \tag{21a}
\]

where \( w \) is the statistical weight given by (13), which can now more suitably be written as

\[
w = (2r - 1)q^2 + (1 - r); \tag{21b}
\]

and, finally,

\[
\vartheta = 2\pi - \lambda. \tag{21c}
\]

The validity of this claim is proved in Appendix 3.

The two claims made establish a correspondence between the set of all decompositions (12) and the set of all suitable yes-no measurements (cf (17)) on the opposite subsystem. "Suitability" here means that the two characteristic state vectors \( |\mu_1⟩_o \) and \( |\mu_2⟩_o \) exhibited in (17) span the range of the opposite-subsystem state operator of \( |\omega⟩ \) (cf (16)), and, as a consequence, one can expand \( |\omega⟩ \) in them (cf (22) below).

**IV. CONCLUDING REMARKS**

Let us return from detail to the global conceptual view.

If a composite-system state vector \( |\omega⟩ \) is given in a two-term Schmidt biorthogonal expansion (16), we can expand it in any orthonormal basis \( \{ |\mu_1⟩_o, |\mu_2⟩_o \} \) in the range of the opposite-subsystem state operator \( \rho_o \) \( \equiv \text{Tr}|\omega⟩⟨\omega| \) ("\text{Tr}" denotes here the partial trace over the subsystem at issue):

\[
|\omega⟩ = |\mu_1⟩_o|φ'_1⟩ + |\mu_2⟩_o|φ'_2⟩. \tag{22}
\]

(The vectors \( |φ'_i⟩ \), \( i=1,2 \), are, in general, not normalized, i.e., they are not state vectors). Then the nonselective (or all-results) version of ideal measurement of the observable \( (A_o \otimes 1) \), where \( A_o \) is given by (17) in terms of the basis considered, converts \( |\omega⟩ \) into the mixed state

\[
⟨φ'_1|φ'_1⟩|\mu_1⟩_o⟨\mu_1|_o + \langle φ'_2|φ'_2⟩|\mu_2⟩_o⟨\mu_2|_o \tag{23}
\]

\[
\left. + \langle φ'_1|φ'_2⟩|\mu_1⟩_o⟨\mu_2|_o + \langle φ'_2|φ'_1⟩|\mu_2⟩_o⟨\mu_1|_o \right.
\]
with $\langle \phi_i | \phi_j \rangle$, $i=1,2$, as the statistical weights (cf the Lüders formula (4) that applies to selective or particular-result measurement). This composite-system mixture implies the same subsystem state $\rho$ as $| \phi \rangle$ does (as one can see from (22) and (23)), and it also implies its decomposition into pure states:

$$\rho = \langle \phi'_1 | \phi'_1 \rangle \left( | \phi'_1 \rangle \langle \phi'_1 | / \langle \phi'_1 | \phi'_1 \rangle \right) + \langle \phi'_2 | \phi'_2 \rangle \left( | \phi'_2 \rangle \langle \phi'_2 | / \langle \phi'_2 | \phi'_2 \rangle \right).$$

(24)

In the two answers in the preceding section we have

$$\langle \phi'_1 | \phi'_1 \rangle = w, \quad \langle \phi'_2 | \phi'_2 \rangle = 1-w; \quad (25a, b)$$

$$| \phi'_{1/2} \rangle / \left( | \phi'_{1/2} \rangle^{1/2} \right) = | \phi \rangle, \quad | \phi'_{2/2} \rangle / \left( | \phi'_{2/2} \rangle^{1/2} \right) = | \phi' \rangle. \quad (25c, d)$$

The state decompositions (23) and (24) are actual (not just potential or mathematically possible like, e. g., the expansion (22)) because if one takes into account the (suppressed) states of the measuring instrument that has performed the $\left( A_{\mu} \otimes 1 \right)$-measurement, different "positions" of the "pointer" (symbolically stated) correspond to the two terms.

Finally, let us discuss the special case when (24) is an orthogonal decomposition of $\rho$, hence, in principle, a measurement. It is called distant measurement [15], [16], because the subsystem is not dynamically influenced by the opposite-subsystem measurement.

If the characteristic value $r$ of $\rho$ is not degenerate , (10) is the only orthogonal decomposition of $\rho$. In this case distant measurement takes place if and only if $| \mu_i \rangle_\alpha = | i \rangle_\alpha$, $i = 1, 2$ (cf (16)), and we are dealing with a common characteristic subbasis of $A_\alpha$ and $\rho_\alpha$.

Commutation of $A_\alpha$ with $\rho_\alpha$ is a necessary and sufficient condition for distant measurement for a general entangled two-subsystem state vector as proved in [15] and [16].

If $r$ is degenerate , every choice of $A_\alpha$ (as long as $| \mu_1 \rangle_\alpha$ given by (18a) and $| \mu_2 \rangle_\alpha$ given by (19) span the range of $\rho_\alpha$) leads to distant measurement because the state operator is a constant in $R(\rho_\alpha)$, and, hence, $A_\alpha$ always commutes with it.

A beautiful realization of essentially the entangled composite state vector $| \chi \rangle$ given by (2) in a real experiment has been reported [17]:

Instead of two slits, there are two processes of parametric down conversion. We’ll disregard, say, the so-called signal out of the pair of down-converted photons, and speak only about the so-called idler. The idler from the first process is reflected back so that it may spatially overlap with the idler created in the second process and thus approach a detector. Writing the state vector of the former as $| \psi_1 \rangle$, and that of the latter as $| \psi_2 \rangle$, the photon may stem either from the first or from the second process, and thus one obtains the above interference state $| \psi \rangle$ given by (1). The phenomenon of interference is observed by moving the mentioned reflecting mirror, and thus changing $| \psi_1 \rangle$ and changing the detection probability.

Both signal and idler are vertically polarized in the very processes of down-conversion. The role of the (mutually orthogonal) polarizers at the slits (see the Introduction) is here played by a quarter-wave plate that is put in the way of the idler from the first process (to be traversed to the mirror and back). It serves to rotate the polarization from vertical to horizontal. Thus, essentially the above entangled state $| \chi \rangle$ (cf. (2)) comes about.

Putting an analyzer at 45$^\circ$ in front of the detector, erasure is observed on the photons that pass the analyzer and reach the detector (cf. (5)). If the analyzer is at $-45^\circ$, then the counter-interference state $| \psi' \rangle$ is obtained out of $| \chi \rangle$. Other angles of the analyzer would, if the photon passes, give rise to, or distantly prepare, the spatial state in other linear combinations of $| \psi_1 \rangle$ and $| \psi_2 \rangle$.

And all this is only a small part of the mentioned experiment [17]. Incidentally, it may be compared, at least partially, with a previous experiment [18], because they both give realization to Franson’s idea [19] of superposing (coherently mixing), essentially, different instants of creation of the photon, which comes about due to some spatial detour that exceeds the coherence length. But in the recent experiment [17] polarization is included and manipulated in a practical way, and thus Ryff’s idea [20] of observing quantum erasure in Franson’s experiment can be considered realized.

As a matter of fact, the experiment [17] seems to be independent of these ideas, because the corresponding articles are not among the references of [17].
APPENDIX A

We rewrite the relations (10), (11a), (13) and (14) in a redundant, but more compact and for proof more suitable form:

\[ \rho = r|1\rangle\langle 1| + r'|2\rangle\langle 2|, \quad r' = 1 - r; \quad (A.1) \]

\[ |\phi\rangle \equiv p|1\rangle + p'e^{i\vartheta}|2\rangle, \quad p' \equiv \left(1 - p^2\right)^{1/2}; \quad (A.2) \]

\[ w \equiv \frac{rr'}{p^2 + p'^2}, \quad w' \equiv 1 - w; \quad (A.3) \]

\[ |\phi^c\rangle \equiv \left[\left(r - wp^2\right)/w'\right]^{1/2}|1\rangle + \left[\left(r' - wp^2\right)/w'\right]^{1/2}e^{i(\vartheta + \pi)}|2\rangle. \quad (A.4) \]

**Lemma A1.** Let a parameter \( s \) be given such that \( 0 < s \leq 1 \). Then for each value of \( s \) from the given interval, one can decompose \( \rho \) uniquely as follows:

\[ \rho = ws|\phi\rangle\langle \phi| + (1 - ws)\rho', \quad (A.5) \]

\[ \rho' \equiv \left((w - ws)/(1 - ws)\right)|\phi\rangle\langle \phi| + \left((1 - w)/(1 - ws)\right)|\phi^c\rangle\langle \phi^c|. \quad (A.6) \]

If \( s > 1 \), then there exists no statistical operator \( \rho' \) such that decomposition (A.5) is valid.

**Proof.** Replacing (A.6) in (A.5), the latter reduces to (12):

\[ \rho = w|\phi\rangle\langle \phi| + w'|\phi^c\rangle\langle \phi^c|. \quad (A.7) \]

Evidently, (A.5) is valid if and only if so is (A.7). Checking this relation, one easily obtains

\[ \langle 1|LHS|1 \rangle = \langle 1|RHS|1 \rangle \]

and

\[ \langle 2|LHS|2 \rangle = \langle 2|RHS|2 \rangle. \]

Further, \( \langle 1|LHS|2 \rangle = 0 \), and

\[ \langle 1|RHS|2 \rangle = wpp'e^{-i\vartheta} - \left(r - wp^2\right)^{1/2}\left(r' - wp^2\right)^{1/2}e^{-i\vartheta} = \]

\[ wpp'e^{-i\vartheta} - \left(rr' - rwp^2 - r'wp^2 + w^2p^2p'^2\right)^{1/2}e^{-i\vartheta}. \]

Substituting here \( rr' \) from (A.3), one obtains

\[ \langle 1|RHS|2 \rangle = \]

\[ wpp'e^{-i\vartheta} - \left(r'rwp^2 + rwp^2 - r'wp^2 - r'wp^2 + w^2p^2p'^2\right)^{1/2}e^{-i\vartheta} = 0. \]

The operator \( \rho' \) is unique because it is determined by (A.5) in terms of the rest of the entities in this relation. Assuming \( s' > 1 \) and the validity of (A.5) with \( s \equiv s' \) and \( \rho' \equiv \rho'' \), where \( \rho'' \) is some hypothetical statistical operator, we can write (A.5) as follows:

\[ \rho = w|\phi\rangle\langle \phi| + (ws' - w)|\phi\rangle\langle \phi| + (1 - ws')\rho''. \]
Subtracting (A.7) from this, one obtains
\[
\left(\frac{(ws' - w)}{w'}\right)|\phi\rangle\langle\phi| + \left(\frac{(1 - ws')}{w'}\right)\rho'' = |\phi^c\rangle\langle\phi^c|.
\]
This is not possible due to the homogeneity of the state on the RHS and the fact that \(|\phi^c\rangle \neq |\phi\rangle\) (or else \(\rho = |\phi\rangle\langle\phi|\), which is not true because \(\rho\) is assumed to be a mixture). This reductio ad absurdum argument proves that decomposition (A.5) with \(s > 1\) is not possible. \(\square\)

Corollary A1. Decomposition (A.7) is the only one that decomposes the mixture \(\rho\) into two pure states one of which is \(|\phi\rangle\langle\phi|\).

Proof. Let us assume ab contrario that there exists another decomposition
\[
\rho = w'|\phi\rangle\langle\phi| + (1 - w')|\phi''\rangle\langle\phi''|.
\]
If \(w' > w\), then we can rewrite this in the form of (A.5) with \(s > 1\), but, according to lemma A1, this is not possible. If \(w' < 1\), then we can, again, put this in the form of (A.5), but this time with \(s < 1\). Then, one, further, obtains
\[
|\phi''\rangle\langle\phi''| = \rho'.
\]
This is not possible because \(\rho'\) is a mixture (cf (A.6)). Finally, if \(w' = w\), then \(|\phi''\rangle\langle\phi''|\) is determined by the rest of the entities in the above decomposition. Thus, it cannot differ from \(|\phi'\rangle\langle\phi'|\) (cf (A7)). \(\square\)

**APPENDIX B**

Let a \(p, \vartheta\)-decomposition of \(\rho\) (cf (10)-(14)) be given together with a composite-system state vector \(|\omega\rangle\) that implies \(\rho\) as its subsystem state operator (cf (16)). To evaluate the corresponding yes-no measurement, we write (22) with (25a-d) substituted in it:
\[
|\omega\rangle = w^{1/2}|\mu_1\rangle_o|\phi\rangle + (1 - w)^{1/2}|\mu_2\rangle_o|\phi^c\rangle. \tag{A.8}
\]
Substituting here \(|\omega\rangle\) from (16), partial scalar product (see section 2 in [14] or see [15]) with \(|\mu_1\rangle_o\) from the left gives
\[
\langle\mu_1|_o\mid 1\rangle_1 + (1 - r)^{1/2}\langle\mu_1|_o\mid 2\rangle_2 = w^{1/2}\langle\phi\rangle
\]
on account of \(|\langle\mu_1|_o\mid \mu_2\rangle_o\rangle = 0\). Inserting the explicit forms of \(|\phi\rangle\) and \(|\mu_1\rangle_o\), i.e., (11a) and (18a) respectively, one further has
\[
r^{1/2}q|1\rangle + (1 - r)^{1/2}(1 - q^2)^{1/2}e^{-i\lambda}|2\rangle = w^{1/2}p|1\rangle + w^{1/2}(1 - p^2)^{1/2}e^{i\vartheta}|2\rangle
\]
or, putting the corresponding expansion coefficients on the two sides equal, one obtains
\[
r^{1/2}q = w^{1/2}p, \quad (1 - r)^{1/2}(1 - q^2)^{1/2}e^{-i\lambda} = w^{1/2}(1 - p^2)^{1/2}e^{i\vartheta}. \tag{A.9a, b}
\]
Relation (A.9a) can be rewritten as
\[
q = (w/r)^{1/2}p. \tag{A.10a}
\]
To evaluate \(w\), we utilize relation (A.9b), where equality of the norms, upon squaring, implies
\[
(1 - r)(1 - q^2) = w(1 - p^2). \tag{A.11}
\]
Replacing here \(q^2\) from (A.10a), one derives
\[
w = r(1 - r)/(p^2(1 - r) + (1 - p^2)r), \tag{A.10b}
\]
which is, actually, relation (13). In relations (A.10a) and (A10b) the dependence of \(q\) on \(p\) is expressed via \(w\).

The phase factors in (A.9b) give the second part of our unique solution:
\[
\lambda = 2\pi - \vartheta. \tag{A.10c}
\]
APPENDIX C

Let a \(q, \lambda\)-measurement (cf (17)-(19)) be given together with the composite-system state vector \(|\omega\rangle\) determined by (16) in which the measurement is to be performed. To evaluate the corresponding \(p, \vartheta\)-decomposition of \(\rho\), the state operator of the second subsystem, we return to the argument presented in Appendix 2 leading to (A.9a) and (A.9b). These relations connect \(p, \vartheta\) and \(q, \lambda\) independently of the fact which of them is given \textit{a priori}.

Solving (A.10a) for \(p\), we obtain

\[
p = \left( \frac{r}{w} \right)^{1/2} q, \tag{A.12a}
\]

and solving (A.11) with (A.12a) for \(w\), we end up with

\[
w = (2r - 1)q^2 + (1 - r). \tag{A.12b}
\]

The second part of our unique solution comes from inverting (A.10c):

\[
\vartheta = 2\pi - \lambda. \tag{A.12c}
\]