An Importance Sampling Scheme on Dual Factor Graphs. I. Models in a Strong External Field

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Abstract—We propose an importance sampling scheme to estimate the partition function of the two-dimensional ferromagnetic Ising model and the two-dimensional ferromagnetic $q$-state Potts model, both in the presence of an external magnetic field. The proposed scheme operates on the dual Forney factor graph and is capable of efficiently computing an estimate of the partition function under a wide range of model parameters. In particular, we consider models that are in a strong external magnetic field.

I. INTRODUCTION

In [1], the authors showed that for two-dimensional (2D) Ising models, at low temperature Monte Carlo methods mix much faster on the dual Forney factor graph than on the original factor graph. Monte Carlo methods on the dual factor graph were also proposed in [1] to estimate of the partition function of 2D Ising models (with constant or with spatially varying couplings) in the absence of an external magnetic field.

In the absence of an external field, the exact value of the partition function of 2D Ising models with constant coupling was first calculated by Onsager [2] [3, Chapter 7]. However, the 2D Ising model in an arbitrary non-zero external field and the three-dimensional (3D) Ising model have remained unsolved [4], [5].

In general, quantities of interest in statistical physics, e.g., the partition function and the mean magnetization of 2D models, can be estimated using Markov chain Monte Carlo methods [6]—[9]. At low temperatures, however, Monte Carlo methods usually suffer from critical slowing down. It is well known that at a certain critical temperature, the 2D ferromagnetic Ising model undergoes a phase transition; below this temperature, variables (spins) have long-range dependencies and Monte Carlo methods (based on single spin-flips) do not mix rapidly [8].

We propose an importance sampling algorithm [6], [7] which can be used to compute the partition function of models with pairwise interactions. In this paper, we are mainly concerned with computing the partition function of finite-size 2D ferromagnetic Ising models and $q$-state Potts models [10], when the models are under the influence of an external field. In our numerical experiments, we will also consider 3D ferromagnetic Ising models. The importance sampling scheme operates on the dual of the Forney factor graphs representing the models. Our numerical results show that the scheme performs well in a wide range of model parameters.

It must be emphasized that, unlike well-known algorithms, e.g., Gibbs sampling [11] and the Swendsen-Wang algorithm [12], the proposed scheme does not suggest a method to draw samples according to the Boltzmann distribution on factor graphs, as sampling is done in the dual domain.

The rest of the paper is organized as follows. In Section II we review the Ising model and graphical model representations in terms of Forney factor graphs. Dual Forney factor graphs and the normal factor graph duality theorem are discussed in Section III. The importance sampling algorithm on the dual Forney factor graph is described in Section IV. In Section V we briefly discuss generalizations to the $q$-state Potts model. Numerical experiments are reported in Section VI.

II. THE ISING MODEL IN AN EXTERNAL MAGNETIC FIELD

Let $X_1, X_2, \ldots, X_N$ be random variables arranged on the sites of a 2D lattice, as illustrated in Fig. [1] where interaction is restricted to adjacent (nearest-neighbor) variables. Suppose each random variable takes its value in a finite set $X$. Let $x_i$ represent a possible realization of $X_i$, let $x$ stand for a configuration $(x_1, x_2, \ldots, x_N)$, and let $X$ stand for $(X_1, X_2, \ldots, X_N)$.

We start with the 2D Ising model, generalizations to the $q$-state Potts model are deferred to Section V. In a 2D Ising model, $X = \{0, 1\}$ and the Hamiltonian is defined as [13]

$$
\mathcal{H}_{\text{Ising}}(x) \triangleq - \sum_{\langle k, \ell \rangle \in B} J_{k, \ell} \left( [x_k = x_\ell] - [x_k \neq x_\ell] \right) - \sum_{m=1}^{N} H_{m} \left( [x_m = 1] - [x_m = 0] \right)
$$

(1)

where the set $B$ contains all the pairs (bonds) $(k, \ell)$ with non-zero interactions and $[\cdot]$ denotes the Iverson bracket [14], which evaluates to 1 if the condition in the bracket is satisfied and to 0 otherwise.

The real coupling parameter $J_{k, \ell}$ controls the strength of the interaction between adjacent variables $(x_k, x_\ell)$. The real parameter $H_m$ corresponds to the presence of an external magnetic field and controls the strength of the interaction between $X_m$ and the field.

The model is called ferromagnetic if all coupling parameters are positive, i.e., $J_{k, \ell} > 0$, for each $(k, \ell) \in B$. In this paper, we concentrate on ferromagnetic models. If $H_m > 0$, variable $X_m$ tends to have value 1, while $X_k$ tends to have value 0 if $H_m < 0$.

In thermal equilibrium, the probability that the model is in configuration $x$, is given by the Boltzmann distribution

$$
p_B(x) \triangleq \frac{e^{-\beta \mathcal{H}(x)}}{Z}
$$

(2)
Here, $Z$ is the partition function (the normalization constant) and $\beta \triangleq \frac{1}{k_B T}$, where $T$ denotes the temperature and $k_B$ is the Boltzmann constant \cite{13, 15}.

The Helmholtz free energy is defined as

$$F_H \triangleq -\frac{1}{\beta} \ln Z$$

(3)

In the rest of this paper, we will assume $\beta = 1$. With this assumptions, e.g., large values of $J$ and $|H|$ correspond to models at low temperature and in a strong external field.

For each adjacent pair $(x_k, x_\ell)$, let

$$\kappa_{k,\ell}(x_k, x_\ell) = e^J_{k,\ell} \left(\delta_{x_k = x_\ell} - \delta_{x_k \neq x_\ell}\right)$$

(4)

and for each $x_m$

$$\tau_m(x_m) = e^{H_m}\left(\delta_{x_m = 1} - \delta_{x_m = 0}\right)$$

(5)

We can then define $f : \mathcal{X}^N \to \mathbb{R}$ as

$$f(x) \triangleq \prod_{(k, \ell) \in \mathcal{B}} \kappa_{k,\ell}(x_k, x_\ell) \prod_{m=1}^N \tau_m(x_m)$$

(6)

The corresponding Forney factor graph for the factorization in \cite{16} is shown in Fig. 1, where the boxes labeled “=” are equality constraints.

From \cite{16}, the partition function \cite{22} can be expressed as

$$Z = \sum_{x \in \mathcal{X}^N} f(x)$$

(7)

At high temperatures (small $J$) and in weak external fields (small $|H|$), the Boltzmann distribution \cite{22} tends to a uniform distribution and Monte Carlo methods generally perform well. To compute an estimate of $Z$ in more challenging situations, we propose an importance sampling scheme that operates on the dual of the Forney factor graph representing the factorization in \cite{22}. In particular, we apply the proposed method to models in a strong external field (large $|H|$).

### III. The Dual Model

Starting from a Forney factor graph, as in Fig. 1, we can obtain its dual by replacing each variable $x$ with its dual variable $\tilde{x}$, each factor $\kappa_{k,\ell}$ with its 2D Discrete Fourier transform (DFT)\cite{17} each factor $\tau_m$ with its one-dimensional (1D) DFT, and each equality constraint with an XOR factor, see \cite{17}–\cite{20}.

In the dual domain, random variables $\tilde{X}$ also take their values in $\mathcal{X}$. The partition function is denoted by $Z_d$ and the number of edges by $E$. For the models that we consider in this paper, the normal factor graph duality theorem states that\cite{22}

$$Z_d = |\mathcal{X}|^E Z$$

(8)

see \cite{19} Theorem 2 and \cite{18} especially for the normal factor graph duality theorem in the context of linear codes.

For variables $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k$, in the dual Forney factor graph of the Ising model, XOR factors are defined as

$$g(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_k) \triangleq [\tilde{x}_1 \oplus \tilde{x}_2 \oplus \ldots \oplus \tilde{x}_k = 0]$$

(9)

where $\oplus$ denotes addition in GF(2).

The 1D DFT of factors as in \cite{17} will have the following form

$$\lambda_m(\tilde{x}_m) = \begin{cases} 2 \cosh H_m, & \text{if } \tilde{x}_m = 0 \\ -2 \sinh H_m, & \text{if } \tilde{x}_m = 1 \end{cases}$$

(10)

\footnote{Here, $\gamma(\tilde{x}_1, \tilde{x}_2)$, the 2D DFT of $\kappa(x_1, x_2)$, is defined as

$$\gamma(\tilde{x}_1, \tilde{x}_2) \triangleq \sum_{x_1 \in \mathcal{X}} \sum_{x_2 \in \mathcal{X}} \kappa(x_1, x_2) e^{-i2\pi(x_1 \tilde{x}_1 + x_2 \tilde{x}_2)/|\mathcal{X}|}$$

where $i$ is the unit imaginary number \cite{22}.}

\footnote{To be more precise, in our models there are no variables involved with only one factor. In Forney factor graphs such variables are represented by half-edges \cite{17}. For the general form of the normal factor graph duality theorem see \cite{18}, \cite{19}.}
and each factor (14) is replaced by its 2D DFT, which has the following form

\[
\gamma_{k,\ell}(\tilde{x}_k, \tilde{x}_\ell) = \begin{cases} 
4 \cosh J_{k,\ell}, & \text{if } \tilde{x}_k = \tilde{x}_\ell = 0 \\
4 \sinh J_{k,\ell}, & \text{if } \tilde{x}_k = \tilde{x}_\ell = 1 \\
0, & \text{otherwise.}
\end{cases}
\]  

(11)

The corresponding dual Forney factor graph with factors as in (10) and (11) is shown in Fig. 2.

All the factors in (11) are diagonal, therefore it is possible to simplify the dual factor graph in Fig. 2 to construct the modified dual factor graph depicted in Fig. 3 with factors attached to each equality constraint as

\[
\gamma_k(\tilde{x}_k) = \begin{cases} 
4 \cosh J_k, & \text{if } \tilde{x}_k = 0 \\
4 \sinh J_k, & \text{if } \tilde{x}_k = 1 
\end{cases}
\]  

(12)

Here, \(J_k\) is the coupling parameter associated with each bond \(\Pi\).

The corresponding modified dual Forney factor graph with factors as in (10) and (12) is shown in Fig. 3.

In this paper, we concentrate on ferromagnetic models, therefore all the factors as in (12) are positive. Since in a 2D Ising model, the value of \(Z\) is invariant under the change of sign of the external magnetic field \(H\), without loss of generality, we assume \(H_m < 0\). With this assumption, all the factors as in (10) will also be positive.

As a side remark, we point out that by looking at the dual of the modified dual Forney factor graph in Fig. 3, we can obtain the modified Forney factor graph of a 2D Ising model in an external field, with factors attached to each equality constraint as

\[
\tau_m(x_m) = \begin{cases} 
e^{-H_m}, & \text{if } x_m = 0 \\
e^{H_m}, & \text{if } x_m = 1
\end{cases}
\]  

(13)

and with factors attached to each XOR factor as

\[
\kappa_k(x_k) = \begin{cases} 
e^{J_k}, & \text{if } x_k = 0 \\
e^{-J_k}, & \text{if } x_k = 1
\end{cases}
\]  

(14)

The corresponding modified Forney factor graph is shown in Fig. 4.

In Section IV, we use the dual Forney factor graph representation of the Ising model to propose an importance sampling scheme to compute an estimate of \(Z\), as in (7).

IV. AN IMPORTANCE SAMPLING SCHEME ON DUAL FACTOR GRAPHS

We describe our importance sampling scheme on the modified dual Forney factor graph of the 2D Ising model in an external field shown in Fig. 3. The scheme can be described analogously for the 2D q-state Potts model, see Section V.

Let us partition the set of random variables \(X\), into \(X_A\) and \(X_B\), with the restriction that the random variables in \(X_B\) are linear combinations (involving the XOR factors) of the random variables in \(X_A\).

In our framework, we let \(X_B\) be the set of all variables represented by the edges connected to the small unlabeled boxes in Figs. 3 and 5 which are the variables involved in factors as in (10). With this choice, \(X_A\) will contain all the variables involved in factors as in (12), which are associated with each bond on the modified dual Forney factor graph and are marked by thick edges in Fig. 5.

As will be discussed, this choice of partitioning is appropriate for models in a strong external field. It also simplifies the notations in the sequel.
Let us define
\[
\Gamma(\tilde{X}_A) \triangleq \prod_{X_k \in \tilde{X}_A} \gamma_k(\tilde{X}_k) \quad (15)
\]
\[
\Lambda(\tilde{X}_B) \triangleq \prod_{X_m \in \tilde{X}_B} \lambda_m(\tilde{X}_m) \quad (16)
\]

We use the following probability mass function as the auxiliary distribution in our importance sampling scheme
\[
q(\tilde{x}_A) \triangleq \frac{\Gamma(\tilde{x}_A)}{Z_q} \quad (17)
\]

The auxiliary distribution (17) has two key properties. First, its partition function \( Z_q \) is analytically available as
\[
Z_q = \sum_{\tilde{x}_A} \Gamma(\tilde{x}_A) = \prod_{k=1}^{\lvert B \rvert} 4(\cosh J_k + \sinh J_k) \quad (19)
\]
\[
= 4^{|B|} \exp \left( \sum_{k=1}^{|B|} J_k \right) \quad (20)
\]
where \( \lvert B \rvert \) is the cardinality of \( B \), which is equal to the number of bonds (the number of interacting pairs) in the lattice (cf. Section [1]). The value of \( Z_q \) is thus a function of the sum of all the coupling parameters.

Second, it is straightforward to draw independent samples \( \tilde{x}_A^{(1)}, \tilde{x}_A^{(2)}, \ldots, \tilde{x}_A^{(\ell)} \), according to \( q(\tilde{x}_A) \). To draw \( \tilde{x}_A^{(\ell)} \), we use the following algorithm
1. Draw \( u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_{\lvert B \rvert}^{(\ell)} \sim \mathcal{U}[0, 1] \)
2. For \( k = 1, 2, \ldots, |B| \), set
\[
\tilde{x}_{A,k}^{(\ell)} = \begin{cases} 0, & \text{if } u_k^{(\ell)} < \frac{1}{2}(1 + e^{-2J_k}) \\ 1, & \text{otherwise} \end{cases} \quad (21)
\]
Note that, the quantity \( \frac{1}{2}(1 + e^{-2J_k}) \) in (21) is equal to
\[
\gamma_k(0)/\gamma_k(0) + \gamma_k(1)).
\]

Random variables in \( \tilde{X}_B \) are linear combinations of those in \( \tilde{X}_A \), therefore after drawing \( \tilde{x}_A^{(\ell)} \), updating \( \tilde{x}_B^{(\ell)} \) can be done in a straightforward manner. To estimate the ratio \( Z_d/Z_q \), we can use the following importance sampling algorithm
1. Draw \( \tilde{x}_A^{(1)}, \tilde{x}_A^{(2)}, \ldots, \tilde{x}_A^{(L)} \) according to \( q(\tilde{x}_A) \).
2. Update \( \tilde{x}_B^{(1)}, \tilde{x}_B^{(2)}, \ldots, \tilde{x}_B^{(L)} \).
3. Compute
\[
\hat{r}_{\text{IS}} = \frac{1}{L} \sum_{\ell=1}^L \Lambda(\tilde{x}_B^{(\ell)}) \quad (22)
\]
It immediately follows that
\[
E[\hat{r}_{\text{IS}}] = \frac{Z_d}{Z_q} \quad (23)
\]
Since \( Z_q \) is analytically available (20), the proposed importance sampling scheme can yield an estimate of \( Z_d \), which can then be used to estimate the partition function (7), using the normal factor graph duality theorem (cf. Section [1]).

The accuracy of \( \hat{r} \) in (22) depends on the fluctuations of \( \Lambda(\tilde{x}_B) \). If \( \Lambda(\tilde{x}_B) \) varies smoothly, \( \hat{r} \) will have a small variance. With our choice of partitioning in (15) and (16), we expect to observe a small variance if the Ising model is in a strong (negative) external magnetic field, see (10).

The choice of partitioning on the dual graph is arbitrary, as long as \( \tilde{X}_B \) can be computed as linear combinations of \( \tilde{X}_A \). Our choice of partitioning is suitable for models in a strong external magnetic field. Depending on the value of the model parameters and their spatial distributions, different choices of partitioning will yield schemes with different dynamics.

If the model is not in a very strong external field, we can consider applying annealed importance sampling ([24], [9]). For model in a weak external field, the efficiency of the importance sampling algorithm on the dual factor graph should be compared to the efficiency of Monte Carlo methods applied directly to the original factor graph, as in Figs. 1 and 4.

We can design a uniform sampling scheme by drawing each \( x_{A,k}^{(\ell)} \) in (21) uniformly and independently from \( \mathcal{X} \) and using
\[
\hat{r}_{\text{Unif}} = \frac{1}{L} \sum_{\ell=1}^L \Lambda(\tilde{x}_B^{(\ell)}) \quad (24)
\]
It is easy to verify that, \( E[\hat{r}_{\text{Unif}}] = Z_d \).

The efficiency of the uniform sampling algorithm and the importance sampling scheme will be close if the model is at very low temperature, i.e., \( J_k \) is very large. However, for a wider range of model parameters, importance sampling outperforms uniform sampling, as will be illustrated in our numerical experiments in Section [VI]. Applying uniform sampling and Gibbs sampling in the dual domain to 2D Ising models in the absence of an external field are discussed in [1].
V. THE q-STATE POTTS MODEL IN AN EXTERNAL MAGNETIC FIELD

In a 2D $q$-state Potts model, $\mathcal{X} = \{0, 1, \ldots, q - 1\}$, where $q$ is an integer greater than or equal to 2. The energy of a configuration $\mathbf{x}$ is given by the Hamiltonian

$$
\mathcal{H}_{\text{Potts}}(\mathbf{x}) \doteq - \sum_{(k, \ell) \in \mathcal{B}} J_{k,\ell}[x_k = x_\ell] - \sum_{m=1}^N H_m[x_m = 0] \quad (25)
$$

Here, $J_{k,\ell}$ controls the strength of the interaction between adjacent variables $(x_k, x_\ell)$ and $H_m$ corresponds to the presence of an external magnetic field.

For $q = 2$, the Potts model is equivalent to the Ising model. Also note that, our definition of the Hamiltonian in (25) is based on the assumption that $H_m$ applies only if $x_m = 0$. The Hamiltonian of the Potts model can be defined in other ways, e.g., the external field can apply when $x_m = 1$ or when $x_m$ is in more than one state.

Similar to the 2D Ising model in Section II for each adjacent pair $(x_k, x_\ell)$, we let

$$
\kappa_{k,\ell}(x_k, x_\ell) = e^{J_{k,\ell}[x_k = x_\ell]} \quad (26)
$$

and for each $x_m$

$$
\tau_m(x_m) = e^{H_m[x_m = 0]} \quad (27)
$$

The Forney factor graph of a 2D $q$-state Potts model is similar to the factor graph in Fig. 1 where the unlabeled normal-size boxes represent factors as in (26), and the small boxes represent factors as in (27).

In the dual Forney graph, the XOR factors are as in (9), where $\oplus$ denotes addition in GF($q$). The 1D DFT of factors as in (27) are

$$
\lambda_m(\hat{x}_m) = \begin{cases} 
  e^{H_m + q - 1}, & \text{if } \hat{x}_m = 0 \\
  e^{H_m - 1}, & \text{otherwise}
\end{cases} \quad (28)
$$

and the 2D DFT of (28) will have the following form

$$
\gamma_k(\hat{x}_k, \hat{x}_\ell) = \begin{cases} 
  q(e^{J_{k,\ell} + q - 1}), & \text{if } \hat{x}_k = \hat{x}_\ell = 0 \\
  q(e^{J_{k,\ell} - 1}), & \text{if } \hat{x}_k \oplus \hat{x}_\ell = 0 \\
  0, & \text{otherwise}
\end{cases} \quad (29)
$$

where $\oplus$ denotes addition in GF($q$).

The corresponding dual Forney factor graph is shown in Fig. 2, where the small boxes represent factors as in (28) and the unlabeled normal-size boxes represent factors as in (29).

By adding extra XOR factors on each bond, we can obtain the modified dual Forney factor graph of the $q$-state Potts model with factors attached to each equality constraint as

$$
\gamma_k(\hat{x}_k) = \begin{cases} 
  q(e^{J_k + q - 1}), & \text{if } \hat{x}_k = 0 \\
  q(e^{J_k - 1}), & \text{otherwise}
\end{cases} \quad (30)
$$

Here, $J_k$ is the coupling parameter associated with each bond. Fig. 6 shows the corresponding modified dual Forney factor graph with factors as in (28) and (30).4

In this paper, we only consider ferromagnetic Potts models in a positive external field, characterized by $J_k > 0$ and $H_m > 0$, respectively. Therefore, all the factors in (28) and (30) will be positive.

The importance sampling scheme can be generalized to the $q$-state Potts model with little effort. We are not going to repeat the complete scheme here. We only point out that following the set-up of Section IV we have

$$
Z_q = q^{|B|} \exp \left( \sum_{k=1}^{|B|} J_k \right) \quad (31)
$$

In order to draw $\hat{x}_A^{(\ell)}$ according to $q(\hat{x}_A)$, we can use the following algorithm

1) Draw $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_8^{(\ell)} \overset{i.i.d}{\sim} \mathcal{U}[0, 1]$

2) For $k = 1, 2, \ldots, 8$

   a) If $u_k^{(\ell)} \leq \frac{1}{q} (1 + (q-1)e^{J_k})$, set $\hat{x}_A^{(\ell)} = 0$.

   b) Otherwise, draw $\hat{x}_A^{(\ell)}$ uniformly and independently from $\{1, 2, \ldots, q-1\}$.

The estimator in (22) is expected to have a small variance if the Potts model is in a strong (positive) external magnetic field, see (28).

If the Potts model is at very low temperature, one might instead want to use the following algorithm based on uniform sampling

1) Draw $u_1^{(\ell)}, u_2^{(\ell)}, \ldots, u_8^{(\ell)} \overset{i.i.d}{\sim} \mathcal{U}[0, 1]$

2) For $k = 1, 2, \ldots, 8$, set $\hat{x}_A^{(\ell)} = \lfloor q \cdot u_k^{(\ell)} \rfloor$

Here, $\lfloor \cdot \rfloor$ denotes the floor function. For known analytical results regarding the Potts model see [3] Chapter 12 and [23].
VI. NUMERICAL EXPERIMENTS

We apply the importance sampling and the uniform sampling schemes of Section V to estimate the free energy $\frac{1}{N} \ln Z$ of 2D and 3D ferromagnetic Ising models and 2D ferromagnetic Potts models.

In Section VI-A we consider 2D ferromagnetic Ising models in an external field with spatially varying model parameters. We recall that the value of $Z$ is invariant under the change of sign of the external field (cf. Section II), therefore we assume $H_m < 0$ to make all the factors as in (10) positive. In Section VI-B we consider 3D Ising models defined on a cubic lattice, 2D ferromagnetic Potts models with spatially varying couplings in a positive external magnetic field, $H_m > 0$, are considered in Section VI-C.

All simulation results show $\frac{1}{N} \ln Z$ vs. the number of samples for models with periodic boundary conditions, where to create periodic boundary conditions we need to add extra edges (with appropriate factors) to connect the sites on opposite sides of the boundary.

A. 2D Ising model

We consider 2D Ising models of size $N = 30 \times 30$ in a strong external field and set $H_m \overset{i.i.d.}\sim \mathcal{U}[-1.25, -1.0]$ in all the experiments. Note that, we have $|\mathcal{B}| = 2N$.

The coupling parameters are set to $J_{k,\ell} \overset{i.i.d.}\sim \mathcal{U}[1.3, 1.5]$ in the first experiment, and to $J_{k,\ell} \overset{i.i.d.}\sim \mathcal{U}[0.75, 1.5]$ in the second experiment. Simulation results for one instance of the Ising models obtained from importance sampling (solid lines) and uniform sampling (dashed lines) on the dual factor graph are shown in Figs. 7 and 8, the estimated free energy per site is about 3.926 and 3.381, respectively.
Note that, for very large coupling parameters (corresponding to models at very low temperature), convergence of uniform sampling is comparable to the convergence of the importance sampling algorithm, see Fig. 7. However, in Fig. 8 we observe that uniform sampling has issues with slow convergence for a wider range of coupling parameters, while the proposed importance sampling scheme performs well in all the ranges.

Fig. 9 shows simulation results obtained from importance sampling for one instance of the Ising model, where $J_{k,\ell} \sim U[0.25, 1.5]$. The estimated free energy per site is about 2.886.

**B. 3D Ising model**

The method can be applied to ferromagnetic 3D Ising models in an external field. In a model of size $N = 10 \times 10 \times 10$, we set $J_{k,\ell} \sim U[1.0, 2.0]$ and $H = -1.5$. For one instance of the Ising model, simulation results obtained from importance sampling (solid lines) and uniform sampling (dashed lines) on the dual factor graph are shown in Fig. 10. where the estimated free energy per site, i.e., $\frac{1}{N} \ln Z$, is about 5.451.

**C. 2D Potts model**

We consider a 2D 3-state Potts model of size $N = 30 \times 30$ in an external field, with $J_{k,\ell} \sim U[0.25, 2.5]$ and $H_m \sim U[2.25, 2.5]$. Fig. 11 shows simulation results obtained from importance sampling on the dual factor for one instance of the model. The estimated free energy per site is about 5.147.

**VII. CONCLUSION**

An importance sampling scheme on the dual Forney factor graph was proposed to estimate the partition function of 2D and 3D ferromagnetic Ising and 2D ferromagnetic $q$-state Potts models, when the models are in the presence of an external magnetic field. We described a method to partition the variables on the dual graph and introduced an auxiliary importance sampling distribution accordingly. The method can efficiently compute an estimate of the partition function under a wide range of model parameters, in particular (with our choice of partitioning), when the models are in a strong external field. Depending on the value of the model parameters and their spatial distributions, different choices of partitioning yield schemes with different convergence properties.

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