The solution of the $\infty$-Laplace equation in the square

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Abstract. By using the hodograph transform we are able to solve a Dirichlet problem for the $\infty$-Laplace equation $\Delta_{\infty} u := \nabla u^H \nabla u^T = 0$.

Figure 1. The graph of the solution over the square $[0,2]^2$.

1. Introduction

Let $\Omega$ be the square $\Omega := \{(x,y) \mid 0 < x < 2, 0 < y < 2\}$. We shall show that the viscosity solution of the Dirichlet problem

$$
\begin{aligned}
\Delta_{\infty} u &= 0 \quad \text{in } \Omega \setminus \{(1,1)\}, \\
u &= 0, \quad \text{on } \partial \Omega \\
u &= 1, \quad \text{at } (1,1),
\end{aligned}
$$

(1.1)

is

$$
u(x,y) = \min_{\theta \in [0,\pi/2]} \max_{r \in [0,1]} \left\{ r (x \cos \theta + y \sin \theta) - W(r, \theta) \right\}
$$

(1.2)

where

$$W(r, \theta) := \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{r^{m_n^2}}{[m_n^2 - 1] m_n} \sin (m_n \theta), \quad m_n := 4n - 2,$$

$$= \frac{8}{\pi} \left[ \frac{r^4}{6} \sin(2\theta) + \frac{r^{36}}{210} \sin(6\theta) + \frac{r^{100}}{990} \sin(10\theta) + \cdots \right].$$
The formula is valid for \((x, y) \in \overline{\Omega}_1\) where

\[
\Omega_1 := \{(x, y) \mid 0 < x < 1, \ 0 < y < 1\}.
\]

The solution in either of the other three quadrants of \(\Omega\) is the obvious translation and rotation of (1.2).

The problem (1.1) is investigated in [LL19] and [LL21] as a special case of the Dirichlet problem for the \(\infty\)-Laplace equation in convex rings.

Our claim is proved in the next Section. The argument is rigorous, except for Lemma 1.1 below, which we shall take for granted. Its validity is plausible from numerical investigations, but we shall not include a proof of the Lemma in this version of the paper.

In order to see how the solution is derived, the reader may want to skip directly to Section 4. The idea is to swap the roles of the independent variables \((x, y)\) with the dependent variables \((p, q) = \nabla u(x, y)\), and by this produce a linear equation for a function \(w = w(p, q)\). When transforming back, we get (1.2) where \(W\) is \(w\) in polar coordinates. The coefficients \(\frac{1}{m^2_n-1}m_n\) and the exponents \(m^2_n\) on \(r\) in \(W\) are chosen so that \(W(r, \theta)\) is the Fourier series for the odd and \(\pi\)-periodic extension of \(\cos \theta + \sin \theta - 1\), and so that \(rW_r + W_{\theta\theta} = 0\), respectively. Thus, \(W\) is the solution of the PDE

\[
\begin{cases}
    rW_r + W_{\theta\theta} = 0 & \text{in } D_1, \\
    W(r, 0) = W(r, \pi/2) = 0, & 0 \leq r \leq 1, \\
    W(0, \theta) = 0, \\
    W(1, \theta) = \cos \theta + \sin \theta - 1, & 0 \leq \theta \leq \pi/2,
\end{cases}
\]

(1.3)

where \(D_1 := \{(r, \theta) \mid 0 < r < 1, \ 0 < \theta < \pi/2\}\). 

**Lemma 1.1.** For every \((x, y) \in \Omega_1\) the minimax in (1.2) is obtained at a unique point \((r, \theta) \in D_1\). Moreover, the function \(W\) and its partial derivatives \(W_r\) and \(W_{rr}\) are positive in \(D_1\).

2. Proof of claim

We prove first that (1.2) is a viscosity solution to the problem

\[
\begin{cases}
    \Delta_\infty u = 0 & \text{in } \Omega_1, \\
    u(0, t) = u(t, 0) = 0, & \text{and} \\
    u(1, t) = u(t, 1) = t, & \text{for } 0 \leq t \leq 1.
\end{cases}
\]

(2.1)

It is then shown that the gluing (2.6) of the translations and rotations of \(u\) is the solution of (1.1).

For each \((x, y) \in \overline{\Omega}_1\) denote the objective function \(f_{(x,y)} : D_1 \to \mathbb{R}\) in (1.2) by

\[
f_{(x,y)}(r, \theta) := r(x \cos \theta + y \sin \theta) - W(r, \theta).
\]

Since \(\sin((4n - 2)(\pi/2 - \theta)) = \sin((4n - 2)\theta), \cos(\pi/2 - \theta) = \sin \theta,\) and \(\sin(\pi/2 - \theta) = \cos \theta,\) we have

\[
f_{(x,y)}(r, \pi/2 - \theta) = f_{(y,x)}(r, \theta).
\]

(2.2)
This implies the symmetry
\[ u(x, y) = u(y, x). \]

Note that we immediately have \( u(x, y) \geq \min_{\theta \in [0, \pi/2]} f(x, y)(0, \theta) = 0. \) The additional following bounds then proves that \( u \) attains the correct boundary values.

**Proposition 2.1.** For all \( (x, y) \in \Omega_1 \),
\[ 1 - \sqrt{(1-x)^2 + (1-y)^2} \leq u(x, y) \leq \text{dist}((x, y), \partial \Omega). \]

**Proof.**
\[
\begin{align*}
u(x, y) &= \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} f(x, y)(r, \theta) \\
&\geq \min_{\theta \in [0, \pi/2]} f(x, y)(1, \theta) \\
&= \min_{\theta \in [0, \pi/2]} \{x \cos \theta + y \sin \theta - (\cos \theta + \sin \theta - 1)\} \\
&= 1 - \max_{\theta \in [0, \pi/2]} \{(1-x) \cos \theta + (1-y) \sin \theta\} \\
&= 1 - \sqrt{(1-x)^2 + (1-y)^2} \max_{\theta \in [0, \pi/2]} \cos(\theta - \phi)
\end{align*}
\]
where the last line is due to a well known trigonometric identity. The lower bound is proved as the maximum is at \( \theta = \phi := \arctan \frac{1-y}{1-x} \in [0, \pi/2] \).

Since \( W \geq 0 \),
\[
\begin{align*}
u(x, y) &= \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} f(x, y)(r, \theta) \\
&\leq \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} \{r (x \cos \theta + y \sin \theta)\} \\
&= \min_{\theta \in [0, \pi/2]} \{x \cos \theta + y \sin \theta\} \\
&= \min \{x, y\} = \text{dist}((x, y), \partial \Omega).
\end{align*}
\]

\[ \square \]

**Proposition 2.2.** Let \( (x, y) \in \Omega_1 \). Then
\[ u(x, y) = f_{(x, y)}(r, \pi/4) \text{ for some } r \in (0, 1) \iff x = y. \]

**Proof.** Assume first that \( x = y \in (0, 1) \) and let \((r, \theta) \in D_1 \) be the unique minimax point. By the symmetry (2.2) we get
\[ u(x, x) = f(x, x)(r, \pi/4) = f_{(x, x)}(r, \pi/2 - \theta). \]
Thus \( \theta = \pi/2 - \theta \), i.e. \( \theta = \pi/4 \).

Assume now that
\[ u(x, y) = \min_{\theta \in [0, \pi/2]} f(x, y)(r_0, \theta) = f_{(x, y)}(r_0, \pi/4) \]
for some $r_0 \in (0,1)$. Then we must have

$$0 = \left|_{\theta = \pi/4} \frac{\partial}{\partial \theta} f(x,y)(r_0, \theta) \right|$$

$$= \left|_{\theta = \pi/4} r_0 (-x \sin \theta + y \cos \theta) - W_\theta(r_0, \theta) \right|$$

$$= \frac{r_0}{\sqrt{2}} (y - x) - 0.$$

□

Let $B_1 := \{(p,q) \mid 0 < p^2 + q^2 < 1, p > 0, q > 0\}$ be the first quadrant in the unit disk and define the function $w: \overline{B}_1 \to \mathbb{R}$ as $W$, but in Cartesian coordinates, i.e., $w(r \cos \theta, r \sin \theta) := W(r, \theta)$. We shift to vector notation

$$x := [x,y]^T \in \Omega_1, \quad p := [p,q]^T \in \overline{B}_1, \quad r := [r,\theta]^T \in \overline{D}_1,$$ and define the mapping $\Phi: \overline{D}_1 \to \overline{B}_1$,

$$\Phi(r) = \Phi(r, \theta) := [r \cos \theta, r \sin \theta]^T = [p,q]^T = p.$$

Now,

$$W(r) = w(\Phi(r))$$

and $\nabla W(r) = \nabla w(\Phi(r)) \nabla \Phi(r)$ where

$$\nabla \Phi(r) = \begin{bmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{bmatrix}$$

is the Jacobian matrix of $\Phi$.

Denote the diagonal in $\overline{\Omega}_1$ by $\delta := \{(x,x) \mid 0 \leq x \leq 1\}$.

**Proposition 2.3.** The function (1.2) has the alternative formula

$$u(x) = x^T g(x) - w(g(x)), \quad x \in \Omega_1,$$

where $g \in C(\Omega_1)$ is the inverse of $\nabla w: B_1 \to \mathbb{R}^2$. Moreover, $u$ is real-analytic in (each connected component of) $\Omega_1 \setminus \delta$.

**Proof.** As the objective function is smooth and the minimax is attained at an interior point of $D_1$, that point must be a saddle point of

$$f_\mathbf{x}(r) = x^T \Phi(r) - W(r).$$

That is,

$$0 = \nabla f_\mathbf{x}(r) = x^T \nabla \Phi(r) - \nabla W(r)$$

$$= \left[ x^T - \nabla w(\Phi(r)) \right] \nabla \Phi(r).$$

Since $\nabla w$ is analytic, there exists by the implicit function theorem, an analytic function $g$ in a neighbourhood of $x$ such that $\Phi(r) = g(x)$ provided $\nabla w$ is non-singular at $p = \Phi(r)$. Furthermore, $r$ is unique by Lemma [1.1] which is the same as saying that $\nabla w$ is one-to-one. The non-differentiable version
of the implicit function theorem then extends \( g \) continuously to the regions where \( \mathcal{H}w \) is singular. That is, \( \nabla w \) has a continuous inverse \( g : \Omega_1 \to \mathbb{R}^2 \) and

\[
u(x) = \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} f_r(r)
= \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} \{ x^T \Phi(r) - W(r) \}
= \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} \{ x^T \Phi(r) - w(\Phi(r)) \}
= x^T g(x) - w(g(x)).
\]

Since \( \nabla W^T(r) = \nabla \Phi^T(r) \nabla w^T(\Phi(r)) \), we have

\[
\mathcal{H}W = \nabla \Phi^T \mathcal{H}w \nabla \Phi + \nabla \nabla w \nabla \Phi^T
\]

and

\[
\mathcal{H}w = \nabla \Phi^{-T} [\mathcal{H}W - \nabla \nabla w \nabla \Phi^T] \nabla \Phi^{-1},
\]

which is non-singular if and only if \( \mathcal{H}W - \nabla \nabla w \nabla \Phi^T \) is non-singular. After some calculations one can check that

\[
\nabla \nabla w \nabla \Phi^T := \begin{bmatrix}
x = \nabla w = \nabla W \nabla \Phi^{-1}
\nabla [\nabla \Phi^T x]
\end{bmatrix}
= \begin{bmatrix}
0 & \frac{1}{r} W_\theta \\
\frac{1}{r} W_\theta & -r W_r
\end{bmatrix}.
\]

Thus

\[
\mathcal{H}W - \nabla \nabla w \nabla \Phi^T = \begin{bmatrix}
W_{rr} & W_{r\theta} - \frac{1}{r} W_\theta \\
W_{r\theta} - \frac{1}{r} W_\theta & W_{r\theta} + W_{\theta\theta}
\end{bmatrix} = \begin{bmatrix}
W_{rr} & W_{r\theta} - \frac{1}{r} W_\theta \\
W_{r\theta} & \frac{1}{r} W_\theta
\end{bmatrix}
\]

with determinant

\[
\det (\mathcal{H}W - \nabla \nabla w \nabla \Phi^T) = - \left( W_{r\theta} - \frac{1}{r} W_\theta \right)^2 \leq 0.
\]

We have

\[
W_{r\theta} - \frac{1}{r} W_\theta = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{m^2 r m^2 - 1}{m^2 - 1} \cos(m\theta) - \frac{r^m - 1}{m^2 - 1} \cos(m\theta)
= \frac{8}{\pi} \sum_{n=1}^{\infty} r^{m-1} \cos(m\theta), \quad m := 4n - 2,
= \frac{8}{\pi} r^3 \left( \cos(2\theta) + r^{32} \cos(6\theta) + r^{96} \cos(10\theta) + \cdots \right),
\]

which we claim is zero in \( D_1 \) if and only if \( \theta = \pi/4 \). Indeed, write \( V := rW_r - W \). Then \( V_r = rW_{rr} + W_r - W_r = rW_{rr} \) and since \( V \) is a solution to the PDE \([1.3]\),

\[
V_{\theta\theta} = -rV_r = -r^2 W_{rr} < 0
\]

by Lemma 1.1 Thus \( rW_{r\theta} - W_{\theta\theta} = V_\theta \) has at most one zero for each \( r \), which obviously is at \( \theta = \pi/4 \).

By Proposition 2.2 \( \theta = \pi/4 \) if and only if \( x \in \delta \). It follows that \( g \), as well as \( u(x) = x^T g(x) - w(g(x)) \), is analytic in \( \Omega \setminus \delta \). □
Proposition 2.4. The function (1.2) is infinity-harmonic in the smooth sense away from the diagonal, i.e.,
$$\nabla u(x)\mathcal{H}u(x)\nabla u^\top(x) = 0 \quad \forall x \in \Omega_1 \setminus \delta.
$$

Proof. By Proposition 2.3, $u(x) = x^\top g(x) - w(g(x))$ where $g^\top(x) := (\nabla w)^{-1}(x)$ is real-analytic in $\Omega_1 \setminus \delta$. Thus,
$$x = \nabla w^\top(g(x)) \quad \text{and} \quad I = \mathcal{H}w(g(x))\nabla g(x).$$

Moreover,
$$\nabla u(x) = g^\top(x) + x^\top \nabla g(x) - \nabla w(g(x))\nabla g(x) = g^\top(x),$$

which by the formulas (2.3) and (2.4) yield
$$\mathcal{H}u(x) = \nabla g(x)$$
$$= \mathcal{H}w^{-1}(g(x))$$
$$= [\nabla \Phi^{-T} \left[ \mathcal{H}W - \nabla \nabla w \nabla \Phi^\top \right] \nabla \Phi^{-1}]^{-1}$$
$$= \nabla \Phi \left[ \begin{array}{cc}
W_{rr} & W_{r\theta} - \frac{1}{r} W_{\theta} \\
W_{r\theta} - \frac{1}{r} W_{\theta} & 0
\end{array} \right]^{-1} \nabla \Phi^\top$$
$$= -\frac{1}{(W_{r\theta} - \frac{1}{r} W_{\theta})^2} \nabla \Phi \left[ \begin{array}{cc}
\frac{1}{r} W_{\theta} - W_{r\theta} & 0 \\
0 & W_{rr}
\end{array} \right] \nabla \Phi^\top$$
at $r = \Phi^{-1}(g(x))$. Since
$$g^\top(x) \nabla \Phi(r) = \Phi^\top(r) \nabla \Phi(r) = r[\cos \theta, \sin \theta] \left[ \begin{array}{cc}
\cos \theta & -r \sin \theta \\
\sin \theta & r \cos \theta
\end{array} \right] = r[1, 0],$$
it follows that
$$\nabla u(x)\mathcal{H}u(x)\nabla u^\top(x) = g^\top(x)\mathcal{H}w^{-1}(g(x))g(x)$$
$$= \frac{r^2}{(W_{r\theta} - \frac{1}{r} W_{\theta})^2} \left[ \begin{array}{cc}
0 & \frac{1}{r} W_{\theta} - W_{r\theta} \\
\frac{1}{r} W_{\theta} - W_{r\theta} & W_{rr}
\end{array} \right] \left[ \begin{array}{c}
1 \\
0
\end{array} \right]$$
$$= 0. \quad \square$$

Proposition 2.5. The function (1.2) is $C^1$ in $\Omega_1$ with gradient
$$\nabla u(x) = g^\top(x)$$

where, as always, $g$ is the inverse of $\nabla w: B_1 \to \mathbb{R}^2$.

By Proposition 2.3, the function $g$ is continuous in $\Omega_1$ and $u$ is given by $u(x) = x^\top g(x) - w(g(x))$. However, (as we shall see) $g$ is not differentiable over the diagonal and we cannot differentiate the formula for $u$ directly to obtain $\nabla u = g^\top$, as we did in Proposition 2.4.

Lemma 2.1. For every $x_0 \in \Omega_1$ we have
$$\lim_{y \to 0} \frac{(g(x_0 + y) - g(x_0))^\top \mathcal{H}w(g(x_0))(g(x_0 + y) - g(x_0))}{|y|} = 0.$$
Proof. Write $b_y := g(x_0 + y) - g(x_0)$. Evaluating $\nabla w$ at $g(x_0 + y) = g(x_0) + b_y$ and making a Taylor expansion yields

$$x_0 + y = \nabla w^\top(g(x_0) + b_y) = \nabla w^\top(g(x_0)) + \mathcal{H}w(g_y)b_y = x_0 + \mathcal{H}w(g_y)b_y$$

for some $g_y$ on the line segment between $g(x_0)$ and $g(x_0 + y)$. As $g$ is continuous and $w$ is smooth, $b_y \to 0$ and $\mathcal{H}w(g_y) \to \mathcal{H}w(g(x_0))$ when $y \to 0$. It follows that

$$\lim_{y \to 0} \frac{b_y^\top \mathcal{H}w(g(x_0))b_y}{|y|} = \lim_{y \to 0} \frac{b_y^\top \mathcal{H}w(g_y)b_y}{|y|} = \lim_{y \to 0} \frac{b_y^\top y}{|y|} = 0.$$

$\square$

Proof of Proposition. We need to show that $u(x_0 + y) = u(x_0) + g^\top(x)y + o(|y|)$ as $y \to 0$. We first note that

$$w(g(x_0 + y)) - w(g(x_0)) = w(g(x_0) + b_y) - w(g(x_0)) = \nabla w(g(x_0)) b_y + \frac{1}{2} b_y^\top \mathcal{H}w(g_y) b_y = x_0^\top b_y + o(|y|)$$

by the Lemma. Again, $b_y := g(x_0 + y) - g(x_0)$. Thus,

$$u(x_0 + y) - u(x_0) - g^\top(x_0)y = (x_0 + y)^\top g(x_0 + y) - w(g(x_0 + y)) - (x_0^\top g(x_0) - w(g(x_0))) - g^\top(x_0)y = x_0^\top b_y + y^\top g(x_0 + y) - x_0^\top b_y + o(|y|) - g^\top(x_0)y = y^\top b_y + o(|y|) = o(|y|).$$

$\square$

It is convenient to introduce the notation

$$\mathbb{1} := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbb{1}_\perp := \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then $C \mathbb{1} = \mathbb{1}$, $C \mathbb{1}_\perp = -\mathbb{1}_\perp$, and the symmetry $u(x, y) = u(y, x)$ can be written as $u(x) = u(Cx)$. Thus, $\nabla u(x) = \nabla u(Cx)C$ which will imply that $\nabla u$ is parallel to $\mathbb{1}$ on the diagonal. That is,

$$\nabla u(x_0) = \nabla u(x_0) \mathbb{1} \cdot \mathbb{1}, \quad x_0 \in \delta. \quad (2.5)$$

We show next that $u$ is not twice differentiable on the diagonal. In particular,

Proposition 2.6. Let $x_0 \in \delta$. Then

$$\lim_{t \to 0} \frac{[\nabla u(x_0 + t \mathbb{1}_\perp) - \nabla u(x_0)] \mathbb{1}_\perp}{t} = -\infty.$$

Note that if $u$ were $C^2$ at $x_0$, the limit would equal $\mathbb{1}_\perp^\top \mathcal{H}u(x_0) \mathbb{1}_\perp$. That is, the second order directional derivative $\frac{d^2}{dt^2}u(x_0 + t \mathbb{1}_\perp)$ at $t = 0$. 

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Proof. By (2.5), the second term in the nominator vanish and l'Hôpital yields

\[
\lim_{t \to 0} \frac{[\nabla u(x_0 + t1_\perp) - \nabla u(x_0)]1_\perp}{t} = \lim_{t \to 0} \frac{\nabla u(x_0 + t1_\perp)1_\perp}{t} = \lim_{t \to 0} \frac{d}{dt} \nabla u(x_0 + t1_\perp)1_\perp = \lim_{t \to 0} \nabla u(x_0 + t1_\perp)1_\perp.
\]

By the formula for \( \mathcal{H}u \) from Proposition 2.4 and the fact that \( \nabla \Phi1_\perp = \frac{1}{\sqrt{2}}[\sin \theta - \cos \theta r(\cos \theta + \sin \theta)] \), we get

\[
1_\perp \mathcal{H}u(x_0 + t1_\perp)1_\perp = -\left(\frac{\sin^2 \theta - \cos^2 \theta}{2} (W_{r\theta} - \frac{1}{r}W_\theta)^2 + r^2 (\cos \theta + \sin \theta)^2 W_{rr}\right)\nabla \Phi1_\perp
\]

at \( r = \Phi^{-1}(g(x_0 + t1_\perp)) \). This concludes the proof since \( \theta \to \pi/4 \) as \( t \to 0 \). The nominator then goes to \(- (0 + 2r^2 W_{rr}(r_0, \pi/4)) < 0 \) while the denominator goes to 0 from the positive side. \( \square \)

By this last result it is clear that no \( C^2 \) test function can touch \( u \) from below on the diagonal. In order to complete the proof of \( u \) being a viscosity solution to the Dirichlet problem (2.1), it only remains to consider test functions touching \( u \) on the diagonal from above.

**Proposition 2.7.** For \( x \in \delta \) we have

\[
\nabla u(x) = g(|x|)1_\perp := \frac{g(|x|)}{\sqrt{2}}[1, 1]
\]

where \( g: [0, \sqrt{2}] \to \mathbb{R} \) is the inverse of the function

\[
r \mapsto W_r(r, \pi/4) = \frac{8}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{m_n}{m_n^2 - 1} r^{m_n^2 - 1}, \quad m_n := 4n - 2
\]

\[
= \frac{8}{\pi} \left[ \frac{2}{3} r^3 - \frac{6}{35} r^{35} + \frac{10}{99} r^{99} - \ldots \right].
\]

Moreover, \( u(x) = |x|g(|x|) - W(g(|x|), \pi/4) \) and

\[
t \mapsto u(t1) = tg(t) - W(g(t), \pi/4)
\]

is smooth, strictly increasing, and convex.

**Proof.** Let \( x \in \delta \), i.e., \( x = |x|1 \). We know that \( g(x) \) is parallel to \( 1 \), so it must be on the form \( \nabla u1(x) = g(x) = g(|x|)1 \) for some continuous scalar function \( g \). We also know that \( \theta = \pi/4 \) at the minimax point \( r = (r, \theta) \) of the
objective function $f_x$. Since $g$ is the inverse of $\nabla w$, it follows that $g(|x|)1^\top$ is the inverse of
\[
\nabla w(\Phi(r)) = \nabla W(r)\nabla \Phi^{-1}(r),
\]
which can be computed to $W_r(r, \pi/4)1^\top$ when $\theta = \pi/4$. The formula for $u$ on
the diagonal then follow from Proposition 2.3 and the fact that $\sin(m_n\pi/4) = (-1)^{n-1}$.

The function $h(t) := u(t11)$ is strictly increasing since $h'(t) = g(t)$ and since the inverse
of a positive increasing function with $W_r(0, \pi/4) = 0$ is positive. Recall that $W_r, W_{rr} > 0$ from Lemma 1.1. Finally, $t \mapsto h(t) = u(t11)$ is smooth and convex since $W_{rr} > 0$ and $h''(t) = g'(t) = 1/W_{rr}(g(t), \pi/4) \geq 0$. □

If a $C^2$ test function $\phi$ touches $u$ from above at $x_0 = t_011 \in \delta$, then
$\nabla \phi(x_0) = \nabla u(x_0) = g(t_0)1^\top$ since $u$ is $C^1$ and
\[
\Delta_\infty \phi(x_0) = g^2(t_0) \lim_{t \to 0} \frac{\phi(x_0 - t11) - 2\phi(x_0) + \phi(x_0 + t11)}{t^2}
\geq g^2(t_0) \lim_{t \to 0} \frac{u(x_0 - t11) - 2u(x_0) + u(x_0 + t11)}{t^2}
= g^2(t_0)h''(t_0) \geq 0.
\]
It is proved that $u$ is a viscosity solution of the Dirichlet problem (2.1).

We now glue the translations and reflections of $u$ together to make a
solution of the problem (1.1). More specifically, temporarily rename $u$ as
\[
u_1(x, y) := \min_{\theta \in [0, \pi/2]} \max_{r \in [0, 1]} \left\{ r(x \cos \theta + y \sin \theta) - \frac{8}{\pi} \sum_{n=1}^\infty \frac{r m_n^2}{[m_n^2 - 1] m_n} \sin(m_n\theta) \right\},
\]
m_n := 4n - 2, for $0 \leq x, y \leq 1$ and define $u : \overline{\Omega} \to \mathbb{R}$ as
\[
u(x, y) := \begin{cases}
u_1(x, y), & \text{for } 0 \leq x \leq 1, 0 \leq y \leq 1, \\
u_1(2 - x, y), & \text{for } 1 \leq x \leq 2, 0 \leq y \leq 1, \\
u_1(2 - x, 2 - y), & \text{for } 1 \leq x \leq 2, 1 \leq y \leq 2, \\
u_1(x, 2 - y), & \text{for } 0 \leq x \leq 1, 1 \leq y \leq 2.
\end{cases}
\]

The bounds
\[1 - \sqrt{(1 - x)^2 + (1 - y)^2} \leq u(x, y) \leq \text{dist}((x, y), \partial\Omega)\]
then holds for all $(x, y) \in \overline{\Omega}$. In particular, $u \in C^1(\Omega \setminus \{(1, 1)\})$ since it is
squeezed between smooth functions along the gluing edges, i.e, the medians
in the square $\Omega$. Furthermore, either of the bounds are $\infty$-harmonic and any
test function $\phi$ touching $u$ from above or below at the medians will have the
correct sign of $\Delta_\infty \phi$. 

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3. A first approximation

If we truncate the series $W$ at some finite partial sum, we should expect to get an approximation of the solution $u$. Since $W$ converge very fast for small $r \geq 0$, it is dominated by its first term

$$W(r, \theta) \approx \frac{4}{3\pi} r^4 \sin(2\theta).$$

In Cartesian coordinates $(p, q) = r(\cos \theta, \sin \theta)$ this is

$$w(p, q) \approx \frac{4}{3\pi} r^4 \sin(2\theta) = \frac{8}{3\pi} r^2 r^2 \sin \theta \cos \theta = \frac{8}{3\pi} (p^2 + q^2)pq$$

with gradient

$$\nabla w(p, q) \approx \frac{8}{3\pi} [3p^2q + q^3, p^3 + 3pq^2].$$

We want to find the inverse of this function. That is, to solve the system $\nabla w(p, q) = (x, y)$ for $p$ and $q$. Adding and subtracting yields

$$\frac{3\pi}{8} (y + x) = p^3 + 3p^2q + 3pq^2 + q^3 = (p + q)^3,$$

$$\frac{3\pi}{8} (y - x) = p^3 - 3p^2q + 3pq^2 - q^3 = (p - q)^3,$$

so

$$p = \frac{1}{2} \left( c(x + y)^{1/3} + c(y - x)^{1/3} \right),$$

$$q = \frac{1}{2} \left( c(x + y)^{1/3} - c(y - x)^{1/3} \right),$$

where $c := (3\pi)^{1/3}/2$. This defines the function $g(x, y)$, and

$$\nabla u(x, y) = g^\top (x, y) \approx \frac{c}{2} \left[ (x + y)^{1/3} + (y - x)^{1/3}, (x + y)^{1/3} - (y - x)^{1/3} \right],$$

which we recognise as the gradient of

$$u(x, y) \approx \frac{3c}{8} \left( (x + y)^{4/3} - (y - x)^{4/3} \right), \quad 0 \leq x \leq 1, 0 \leq y \leq 1. \quad (3.1)$$

Namely, a rotation of Aronsson’s function. It will be a very good approximation to the solution $u$ when $r$ – that is, the length of the gradient of $u$ – is small. Near the boundary point $(1, 1)$ the length of $\nabla u$ tends to its maximal value $1$. Even so, the formula $(3.1)$ yields the acceptable estimate

$$1 = u(1, 1) \approx \frac{3}{8} (6\pi)^{1/3} = 0.99800...$$

4. Deriving the solution

In this Section we shall formally derive the solution (1.2) by considering the Dirichlet problem

$$\begin{cases}
\Delta_\infty u = 0 & \text{in } S, \\
u(t, 1) = u(1, t) = t, \\
u(t, -1) = u(-1, t) = -t, & \text{for } -1 \leq t \leq 1,
\end{cases} \quad (4.1)$$
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where

$$S := \{(x, y) \mid -1 < x < 1, -1 < y < 1\}.$$

Figure 2. The graph of the solution of (4.1) over the square $S = [-1, 1]^2$.

The symmetry of the boundary conditions implies $u = 0$ on the coordinate axis, and $u|_{\partial S}$ will therefore be the solution of (2.1).

We make the following ansatz: The gradient $\nabla u : S \to B$ to the solution of (4.1) is one-to-one and onto $B := \{(p, q) \in \mathbb{R}^2 \mid p^2 + q^2 < 1\}$ – the unit disk. Denote by $f : B \to S$ the inverse of $\nabla u$ and define $w : B \to \mathbb{R}$ as

$$w(p) := p^\top f(p) - u(f(p)).$$

If $f$ is differentiable, then

$$p = \nabla u^\top(f(p)) \quad \text{implies} \quad I = \mathcal{H}u(f(p))\nabla f(p)$$

and

$$\nabla w(p) = f^\top(p) + p^\top \nabla f(p) - \nabla u(f(p))\nabla f(p) = f^\top(p),$$

$$\mathcal{H}w(p) = \nabla f(p) = \mathcal{H}u^{-1}(f(p)).$$

Thus,

$$0 = \nabla u(f(p))\mathcal{H}u(f(p))\nabla u^\top(f(p))$$

$$= p^\top \mathcal{H}w^{-1}(p)p$$

$$= [p, q] \begin{bmatrix} w_{pp} & w_{pq} \\ w_{pq} & w_{qq} \end{bmatrix}^{-1} \begin{bmatrix} p \\ q \end{bmatrix}$$

$$= \frac{1}{w_{pp}w_{qq} - w_{pq}^2} [p, q] \begin{bmatrix} w_{qq} & -w_{pq} \\ -w_{pq} & w_{pp} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix},$$
which, assuming that $Hw$ is non-singular, leads to the linear homogeneous equation

$$0 = [p, q] \begin{bmatrix} w_{qq} & -w_{pq} \\ -w_{pq} & w_{pp} \end{bmatrix} [p, q] = p^2 w_{qq} - 2pqw_{pq} + q^2 w_{pp}.$$ 

The equation is degenerate elliptic, as can be seen from writing the above as

$$0 = [-q, p] \begin{bmatrix} w_{pp} & w_{pq} \\ w_{pq} & w_{qq} \end{bmatrix} [-q, p] = \text{tr}(A(p)Hw)$$

where $A: B \to S^2_+$ is

$$A(p) := QpQ^\top, \quad Q := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

Since the domain of $w$ is the unit disk, it is natural to introduce polar coordinates. Define

$$W(r, \theta) := w(r \cos \theta, r \sin \theta).$$

Then $W_\theta = -w_p r \sin \theta + w_q r \cos \theta$ and

$$W_{\theta\theta} = -[-w_{pp} r \sin \theta + w_{pq} r \cos \theta] r \sin \theta - w_p r \cos \theta + [w_{pq} r \sin \theta + w_{qq} r \cos \theta] r \cos \theta - w_q r \sin \theta = q^2 w_{pp} - 2pqw_{pq} + p^2 w_{qq} - pw_p - qw_q.$$

Since $rW_r = r(w_p \cos \theta + w_q \sin \theta) = pw_p + qw_q$ it follows that

$$rW_r + W_{\theta\theta} = 0.$$ 

A separation of variables $W(r, \theta) = F(r)G(\theta)$ yields

$$\frac{rF'(r)}{F(r)} = n^2 = -\frac{G''(\theta)}{G(\theta)}, \quad n = 0, 1, 2, \ldots$$

and we will look for solutions on the form

$$W(r, \theta) = \sum_{n=0}^{\infty} r^{n^2} (A_n \cos(n\theta) + B_n \sin(n\theta)). \quad (4.2)$$

We need to establish the boundary values of $w$ at $\partial B$. That is, the values $W(1, \theta)$ for $\theta \in [0, 2\pi]$. From geometric considerations we expect that $\nabla u(x) \to e_1^\top := [1, 0]$ as $x \in S$ approaches the upper boundary $\ell_u := \{(t, 1) \mid -1 < t < 1\} \subseteq \partial S$. Thus, $\nabla u$ is certainly not invertible in the closure of $S$. Nevertheless, allowing $f$ to be multivalued at $e_1$ as $f(e_1) := \ell_u$, 

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still produces the single value

$$w(e_1) = e_1^\top f(e_1) - u(f(e_1)) = e_1^\top \begin{bmatrix} t \\ 1 \end{bmatrix} - u(t, 1) = t - t = 0$$

of $w$ at $e_1$. Likewise, $\nabla u(x) \to e_1^\top := [0, 1]$ as $x \in S$ approaches the right boundary $\ell_r := \{(1, t) \mid -1 < t < 1\}$ and setting $f(e_2) := \ell_r$ again yields $w(e_2) = e_2^\top f(e_2) - u(f(e_2)) = e_2^\top \begin{bmatrix} 1 \\ t \end{bmatrix} - u(1, t) = t - t = 0$.

The same calculations apply for the remaining two edges and we should therefore require that $w(\pm e_i) = 0$, $i = 1, 2$. Equivalently, $W(1, k\pi/2) = 0$ for $k = 0, 1, 2, 3$.

On the other hand, it makes sense to consider $\nabla u$ as multivalued at the corners of $S$: When $x \in S$ approaches the corner point $(1, 1)$ we know from Corollary 10 in [LL19] that $|\nabla u(x)| \to 1$. Also, the gradient is continuous and since $\nabla u(\ell_u) = e_1^\top$ and $\nabla u(\ell_r) = e_2^\top$, it must take on all the values $\nabla u(x) = (\cos \theta, \sin \theta)$, $0 \leq \theta \leq \pi/2$, in the limit for some $x$ close to $(1, 1)$. We therefore define $f(\cos \theta, \sin \theta) := \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for all $0 \leq \theta \leq \pi/2$. This produces the boundary values

$$W(1, \theta) = w(\cos \theta, \sin \theta) = [\cos \theta, \sin \theta] f(\cos \theta, \sin \theta) - u(f(\cos \theta, \sin \theta)) = [\cos \theta, \sin \theta] \begin{bmatrix} 1 \\ 1 \end{bmatrix} - u(1, 1) = \cos \theta + \sin \theta - 1, \quad \theta \in [0, \pi/2].$$

We now continue around the square $S$ in a clockwise manner in order to derive the boundary values also for $\theta \in [\pi/2, 2\pi]$. At the lower edge $\ell_l := \{(t, -1) \mid -1 < t < 1\}$ we have $\nabla u = -e_1$, so near the corner $(1, -1)$ the gradient takes on the values $(\cos \theta, \sin \theta)$, $\pi/2 \leq \theta \leq \pi$. We define $f(\cos \theta, \sin \theta) := \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for those values and get

$$W(1, \theta) = w(\cos \theta, \sin \theta) = [\cos \theta, \sin \theta] f(\cos \theta, \sin \theta) - u(f(\cos \theta, \sin \theta)) = [\cos \theta, \sin \theta] \begin{bmatrix} 1 \\ -1 \end{bmatrix} - u(1, -1) = \cos \theta - \sin \theta + 1, \quad \theta \in [\pi/2, \pi].$$

At the left edge $\ell_f := \{(-1, t) \mid -1 < t < 1\}$ we have $\nabla u = -e_2$, so near the corner $(-1, -1)$ the gradient sweeps $(\cos \theta, \sin \theta)$, $\pi \leq \theta \leq 3\pi/2$. We

...
define \( f(\cos \theta, \sin \theta) := \begin{bmatrix} -1 \\ -1 \end{bmatrix} \) and get

\[
W(1, \theta) = w(\cos \theta, \sin \theta) = \begin{bmatrix} \cos \theta, \sin \theta \end{bmatrix} f(\cos \theta, \sin \theta) - u(f(\cos \theta, \sin \theta))
\]
\[
= \begin{bmatrix} \cos \theta, \sin \theta \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} - u(-1, -1)
\]
\[
= -\cos \theta - \sin \theta - 1, \quad \theta \in [\pi, 3\pi/2).
\]

Finally, near the corner \((-1, 1)\) the gradient is \((\cos \theta, \sin \theta)\) for some \(3\pi/2 \leq \theta \leq 2\pi\). We define \( f(\cos \theta, \sin \theta) := \begin{bmatrix} -1 \\ 1 \end{bmatrix} \) and get

\[
W(1, \theta) = w(\cos \theta, \sin \theta) = \begin{bmatrix} \cos \theta, \sin \theta \end{bmatrix} f(\cos \theta, \sin \theta) - u(f(\cos \theta, \sin \theta))
\]
\[
= \begin{bmatrix} \cos \theta, \sin \theta \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} - u(-1, 1)
\]
\[
= -\cos \theta + \sin \theta + 1, \quad \theta \in [\pi, 3\pi/2).
\]

To summarize, we must have

\[
W(1, \theta) = \begin{cases} 
\cos \theta + \sin \theta - 1, & \text{for } 0 \leq \theta \leq \pi/2, \\
\cos \theta - \sin \theta + 1, & \text{for } \pi/2 \leq \theta \leq \pi, \\
-\cos \theta - \sin \theta - 1, & \text{for } \pi \leq \theta \leq 3\pi/2, \\
-\cos \theta + \sin \theta + 1, & \text{for } 3\pi/2 \leq \theta \leq 2\pi.
\end{cases}
\]

Note that \( W(1, k\pi/2) = 0 \) for integers \( k \), as it should. The function is well defined and continuous. In fact, its derivative is

\[
W_\theta(1, \theta) = \begin{cases} 
-\sin \theta + \cos \theta, & \text{for } 0 \leq \theta \leq \pi/2, \\
-\sin \theta - \cos \theta, & \text{for } \pi/2 \leq \theta \leq \pi, \\
\sin \theta - \cos \theta, & \text{for } \pi \leq \theta \leq 3\pi/2, \\
\sin \theta + \cos \theta, & \text{for } 3\pi/2 \leq \theta \leq 2\pi.
\end{cases}
\]

for all \( \theta \). As this function is continuous, even, and \( \pi \)-periodic, it follows that \( W(1, \theta) \) is \( C^1 \), odd, and \( \pi \)-periodic.

The Fourier series of \(|\cos \theta|\) and \(|\sin \theta|\) can be calculated to

\[
|\cos \theta| = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2 - 1} \cos(2n\theta), \quad |\sin \theta| = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cos(2n\theta),
\]
\[ W_\theta(1, \theta) = |\cos \theta| - |\sin \theta| \]
\[ = 4 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} + 1}{4n^2 - 1} \cos(2n\theta) \]
\[ = 8 \sum_{n=1}^{\infty} \frac{\cos(2(2n-1)\theta)}{4(2n-1)^2 - 1} \]
\[ = 8 \sum_{n=1}^{\infty} \frac{\cos(m_n \theta)}{m_n^2 - 1} \]

where \( m_n := 4n-2 \). From (4.2) it follows that \( W_\theta(r, \theta) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{r^{m_n^2}}{m_n^2 - 1} \cos(m_n \theta) \) and
\[ w(r \cos \theta, r \sin \theta) = W(r, \theta) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{r^{m_n^2}}{(m_n^2 - 1)m_n} \sin(m_n \theta). \]

There is no integration constant because the average of \( W(1, \theta) \) is zero.

By transforming back and letting \( g: S \to B \) be the inverse of \( \nabla w^\top = f \) – that is, \( f(p) = x \) if and only if \( p = g(x) \) – the identity \( w(p) := p^\top f(p) - u(f(p)) \) yields
\[ u(x) = x^\top g(x) - w(g(x)), \]
which is the formula also derived from (1.2) in Proposition 2.3.

References

[LL19] Erik Lindgren and Peter Lindqvist. Infinity-harmonic potentials and their streamlines. *Discrete and Continuous Dynamical Systems*, 39(8):4731–4746, 2019.

[LL21] Erik Lindgren and Peter Lindqvist. The gradient flow of infinity-harmonic potentials. *Advances in Mathematics*, 378:107526, 2021.

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