Nearly integrable nonlinear equations on the half line

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Abstract

We employ the Riemann-Hilbert problem for solution of the initial-boundary value problems for nearly integrable equations on the half line which have important applications in physics. The detailed derivation of the integrable and perturbation-induced evolutions of the spectral data are given for the two most important models, the self-induced transparency and the stimulated Raman scattering. In particular, we prove that dephasing stabilizes the spike phenomenon in the stimulated Raman scattering.

1 Introduction

Nonlinear integrable models of resonant interaction of radiation with matter possess a number of features which distinguish them from a variety of integrable systems. First of all, these equations determine initial-boundary-value problems, because, along with an initial condition, some boundary conditions are necessary to properly formulate the problem. Secondly, they have singular dispersion relations [1, 2]. As examples of such integrable equations we mention the Maxwell-Bloch equations of self-induced transparency [3], equations of stimulated Raman scattering [4], the Karpman-Kaup equations [5, 6], the equations of the interaction of the electrostatic high-frequency wave with the ion-acoustic wave [7], and the equations of the resonant interaction between optical solitons and impurity atoms in fibres [8].

There exists a well-developed formalism for solution of the integrable equations with the singular dispersion relations when the boundary conditions are asymptotic, i. e. they are posed at the plus or/and minus infinity [1, 9, 10]. Along with the fact that the semi-infinite interval, when one of the boundaries is placed at a finite point, e. g. zero, is physically more appropriate, the formalism of the inverse scattering has to be substantially modified as compared with the whole line case [11, 12].

Recently significant progress has been achieved in solving the nonlinear evolution equations with singular dispersion relations posed on the semi-infinite [13, 14] and finite [15, 16] intervals. Also it was realised that the Riemann-Hilbert (RH) problem is a natural setting for analysis of the above nonlinear models. The data uniquely characterising the solution to the RH problem serve as parameters of the solution to these nonlinear equations.

All the above-mentioned papers deal with the integrable evolutions. On the other hand, in the experimental observations one has various additional (frequently small enough) effects which destroy the integrability property but must be accounted for complete explanation of the experimental results. For instance, the phenomenon of the spike formation in the pump depletion zone in the stimulated Raman scattering [17] is directly related with the dephasing effect in the medium [18]. Hence, a problem arises to describe evolution of the RH (or spectral) data in the presence of small perturbations, i. e., to develop a perturbation theory on the half line.
In the present paper we extend the perturbation theory developed for the whole line \([20, 21]\), to the half line. We take the equations of the self-induced transparency (SIT) and stimulated Raman scattering (SRS) posed on the half line as the examples. In both cases the dephasing effect, which destroys the integrability of the models, is considered as the perturbation. We show that for the half line, as for the whole line, construction of the perturbation theory amounts to introduction of an additional matrix functional, which we call the evolution functional. Moreover, it turns out that the evolution functional has the explicit expression very much similar to that in the case of the whole line (see, for instance \([21]\)).

The plan of the paper is the following. In section 2 we briefly remind the formulation of the initial-boundary value problems for the SIT and SRS equations. In sections 3 and 4 we present necessary results on the direct and, respectively, inverse spectral problems for the half line. In particular, we demonstrate the advantage of dealing with the analytic functions with constant determinant for derivation of the evolution equations for the spectral data in the case of the half line. Our main section is section 5, where the perturbation-induced evolution equations for the RH data are derived.

This paper is dedicated to Professor Pierre Sabatier whose versatile activities have determined in many respects the contemporary status of inverse methods in science and technology, in honor of his 65th birthday.

2 Nonlinear models with singular dispersion relations

The SIT phenomenon consists in a soliton-like \(2\pi\)-pulse propagation in a medium without energy loss and is described by the Maxwell-Bloch equations \([3]\)

\[
\mathcal{E}_x = \langle \rho \rangle, \quad \rho_t + 2i\lambda \rho + \Gamma \rho = N \mathcal{E}, \quad N_t = -\frac{1}{2} (\mathcal{E} \bar{\rho} + \bar{\mathcal{E}} \rho). \tag{1}
\]

Here \(\mathcal{E}(x,t)\) is the envelope of the light-pulse electric field with the carrier frequency close to the particular transition frequency of an atom,

\[
M = \begin{pmatrix} -N & \rho \\ \bar{\rho} & N \end{pmatrix}
\]

is the density matrix of the atomic ensemble (medium) with \(\rho(x,t,\lambda)\) being induced polarisability and \(N(x,t,\lambda)\) being normalised population difference between the chosen atomic levels. We have \(\det M = -1\) and for atoms initially on the lower level \(N(t = 0) = -1\) and \(M(t = 0) = \sigma_3\). \(\lambda\) is the normalised frequency mismatch between the carrier frequency and atomic transition frequency, \(\Gamma\) is the dephasing factor. We think of the value of \(\Gamma\) as being small, \(\Gamma \ll 1\), in order to justify the applicability of the perturbation theory. The coordinate \(x\) is the resonant medium extension and \(t\) is time. The symbol \(\langle \ldots \rangle\) means the averaging over the frequency mismatch with the normalised distribution function \(g(\lambda)\):

\[
\langle \rho \rangle = \int_{-\infty}^{\infty} d\lambda \rho(x,t,\lambda)g(\lambda), \quad \int_{-\infty}^{\infty} d\lambda g(\lambda) = 1.
\]

As regards the SRS equations, we follow the approach by Leon and Mikhailov \([14]\): \(u_t + \Gamma u = -g \int_{-\infty}^{\infty} dka_1 \bar{a}_2 e^{-2ikx}, \quad a_{1x} = ua_2 e^{2ikx}, \quad a_{2x} = -\bar{u}a_1 e^{-2ikx}\). \(a_1(x,t,k)\) and \(a_2(x,t,k)\) are the electric field envelopes of the pump and Stokes pulses, respectively, the parameter \(k\) describes a spectral extension of the input pulses, \(u(x,t)\) is related with the polarisability of a
Raman-active medium, $g$ is a coupling constant. It should be noted that the parameter $k$ can be connected with the group-velocity dispersion of the pulses in the medium \[14\].

Both nonlinear systems are considered in the quadrant $0 \leq x < \infty$, $0 \leq t < \infty$ with the initial and boundary values:

**a)** SIT: $E(0,t) = E_0(t)$, $M(x,0,\lambda) \equiv M_0 = \left( \begin{array}{c} -N_0 \\ \rho_0 \\ \rho_0 \\ N_0 \end{array} \right)$, $E \to 0$ as $t \to \infty$; \[ 3 \]

**b)** SRS: $u(x,0) = 0$, $a_1(0,t,k) = I_1(t,k)$, $a_2(0,t,k) = I_2(t,k)$, \[ 4 \]

Note that for the SIT equations we have an initial-boundary-value problem, while for the SRS equations we consider a pure boundary-value problem.

For $\Gamma = 0$ both systems (1) and (2) are integrable and admit the Lax representation \[3, 22\]. In both cases the dispersion relation is non-analytic (singular) function on the complex plane of the spectral parameter.

### 3 Direct spectral problem

Because both the SIT and SRS equations are solved by means of one and the same Zakharov-Shabat spectral problem \[23\], we will analyze below the direct spectral problem for both systems together (omitting below the explicit dependence on the other coordinate). The spectral problem is written as

$$\chi_\xi = -ik[\sigma_3, \chi] + Q \chi. \quad (5)$$

Here $\chi$ is the $2 \times 2$ matrix function,

$$Q = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$$

is a potential with $E = 2q$ for the SIT and $u = q$ for the SRS. The coordinate $\xi$ stands for $t$ in the case of the SIT and for $x$ in the SRS case, $k$ is a spectral parameter coinciding with the mismatch parameter for SRS. It should be stressed that the physical status of the potential $q$ is different for the SIT and SRS. Whereas for the SIT the potential $q$ (or $E$) is the main experimentally measured quantity, for the SRS the potential $u$ could be reconstructed only indirectly from the measured intensities $|a_1|^2$ and $|a_2|^2$.

Let us define the matrix Jost solutions $J_{\pm}(\xi,k)$ of the spectral equation (5) by means of the following conditions:

$$J_-(0,k) = I, \quad J_+(\xi,k) \to I, \quad \xi \to \infty, \quad (6)$$

where $I$ is the identity matrix. The scattering matrix is defined in the usual way:

$$S = E^{-1}J_+^{-1}J_-E, \quad E(\xi,k) = e^{-ik\xi\sigma_3}, \quad \text{Im}k = 0 \quad (7)$$

and has the structure

$$S = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1. \quad (8)$$

The scattering matrix is connected with the boundary value of $J_+$, $S(k) = J_+^{-1}(0,k)$. Note that the potential is anti-Hermitian: $Q^\dagger = -Q$. Hence, we have the involutions (Im$k = 0$):

$$J_{\mp}^\dagger(\xi,k) = J_{\pm}^{-1}(\xi,k), \quad S^\dagger(k) = S^{-1}(k). \quad (9)$$
The spectral equation \( E(\xi) \) can be represented in the integral form:

\[
J(\xi, k) = E(\xi) \left[ J_0 + \int_{\xi_0}^{\xi} \text{d}\xi' E^{-1}(\xi') Q(\xi') J(\xi', k) E(\xi') \right] E^{-1}(\xi),
\]

where \( J_0 = J(\xi = \xi_0) \). The Jost solution \( J_- (J_+) \) corresponds to the choice \( \xi_0 = 0 \) (\( \xi_0 = \infty \)). It follows from \((14)\) that the matrix functions \((J_-(1), J_+(2))\) and \((J_+(1), J_-(2))\) are solutions to equation \((\tilde{H})\) and moreover holomorphic in \( k \) in the upper and lower half-planes, \( \pm \text{Im} k \geq 0 \), respectively. Here \( J^{(\alpha)} \) stands for the corresponding column vector. By noticing that a solution to \((\tilde{H})\) transforms to another solution if multiplied on the right by a diagonal \( \xi \)-independent matrix and that \( \det(J_-, J_+^\dagger) = a \) and \( \det(J_+, J_-^\dagger) = \bar{a} \) we introduce the solutions to \((\tilde{H})\) having unit determinant:

\[
\Psi_+ = \left( J_+(1), a^{-1} J_+(2)^\dagger \right), \quad \Psi_- = \left( a^{-1} J_-(1), J_-(2)^\dagger \right).
\]

By definition the functions \( \Psi_+(\xi, k) \) and \( \Psi_-(\xi, k) \) are analytic in the upper and lower half-planes of the complex \( k \)-plane, respectively. On the real line \( \text{Im} k = 0 \) they can be given in terms of the matrix Jost solutions and entries of the scattering matrix:

\[
\Psi_+ = J_+ E G_+ E^{-1} = J_- E G_- E^{-1}, \quad \Psi_- = J_+ E H_+^{-1} E^{-1} = J_- E H_-^{-1} E^{-1}.
\]

Here

\[
G_+ = \begin{pmatrix} a & 0 \\ b & 1/a \end{pmatrix}, \quad G_- = \begin{pmatrix} 1 & \bar{b}/a \\ \bar{b}/a & 1 \end{pmatrix},
\]

\[
H_+ = \begin{pmatrix} \bar{a} & \bar{b} \\ 0 & 1/\bar{a} \end{pmatrix}, \quad H_- = \begin{pmatrix} 1 & 0 \\ b/\bar{a} & 1 \end{pmatrix}.
\]

As follows from \((\tilde{H})\) and \((12)\), the matrix functions \( \Psi_{\pm}(\xi, k) \) satisfy the involution:

\[
\Psi_+^\dagger(k) = \Psi_-^{-1}(k).
\]

The representation \((12)\) involves the factorizations of the scattering matrix with division: \( S = G_+ G_-^{-1} \) and \( S = H_+^{-1} H_- \). The advantage of using such factorizations for the half line stems from the fact that both \( G_- \) and \( H_- \) contain a single quantity: \( \bar{b}/a \) and \( b/\bar{a} \), respectively. On the other hand they provide the values of \( \Psi_{\pm} \) at the boundary: \( \Psi_+(0, k) = G_- \) and \( \Psi_-(0, k) = H_-^{-1} \). In view of this, it is convenient to introduce the following notations:

\[
\beta(k) = \frac{\bar{b}(k)}{a(k)}, \quad \alpha(k) = \frac{1}{a(k)}.
\]

The functions \( \beta(k) \) and \( \alpha(k) \) are usually referred to as the reflection and transmission coefficients, respectively. Then the boundary values of the solutions \( \Psi_{\pm}(\xi, k) \) read

\[
\Psi_+(\xi = 0) = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}, \quad \Psi_-(\xi = 0) = \begin{pmatrix} 1 & 0 \\ -\bar{\beta} & 1 \end{pmatrix}.
\]

For completeness, let us present also the asymptotic values of \( \Psi_{\pm}(\xi, k) \) as \( \xi \to \infty \):

\[
\Psi_+(\xi) \to \begin{pmatrix} 1/\alpha & 0 \\ \bar{\beta}/\alpha e^{2ik\xi} & \alpha \end{pmatrix}, \quad \Psi_-(\xi) \to \begin{pmatrix} \bar{\alpha} & -\beta/\alpha e^{-2ik\xi} \\ 0 & 1/\bar{\alpha} \end{pmatrix}.
\]
Now, let us consider in more detail the analyticity properties and the \(k\)-asymptotics of the matrix functions \(\Psi_\pm\). These properties are easy to deduce from the integral equations for these functions:

\[
\begin{pmatrix}
\Psi_{+11} \\
\Psi_{+21}
\end{pmatrix} = 
\begin{pmatrix}
1 \\
0
\end{pmatrix} + \left(1 + \int_0^\infty d\xi' \frac{q(\xi')}{\bar{\Psi}(\xi')} \right) \bar{\Psi}_{11}.
\]

\[
\begin{pmatrix}
\Psi_{-11} \\
\Psi_{-21}
\end{pmatrix} = 
\begin{pmatrix}
1 \\
0
\end{pmatrix} + \left(1 + \int_0^\infty d\xi' \frac{q(\xi')}{\bar{\Psi}(\xi')} \right) \bar{\Psi}_{11}.
\]

Evidently the first column \(\Psi_{-11}^{(1)}\) and the second column \(\Psi_{-21}^{(2)}\) are entire functions of \(k\), while \(\Psi_{+11}^{(2)}\) and \(\Psi_{+21}^{(1)}\) are meromorphic functions in the upper and lower half-planes of the complex \(k\)-plane, respectively (see also Ref. [14]). By application of (15) we conclude that the reflection coefficient \(r(k)\) is meromorphic in the upper half of the complex \(k\)-plane. Moreover, as \(\beta/\alpha = \bar{b}\), by using the identity \(b_{+12}(0,k) = \bar{b}(k)\) (see (9) and (10)) we conclude that the quotient \(\beta(k)/\alpha(k)\) is holomorphic in the upper half plane. Equivalently, \(a(k)\) and \(\bar{b}(k)\) are holomorphic in the upper half of the complex \(k\)-plane. Note that the poles of \(\beta(k)\) coincide with zeros of \(a(k)\).

From the above integral equations it is seen that \(\Psi_+ \to I\) as \(k \to \infty\). Integration by parts in the integrals containing \(\exp(\pm 2ik\xi')\) gives the generalized asymptotic expansions:

\[
\begin{pmatrix}
\Psi_{+11} \\
\Psi_{+21}
\end{pmatrix} = 
\begin{pmatrix}
1 \\
0
\end{pmatrix} + \frac{1}{2ik} \left(\int_0^\infty d\xi' \frac{|q(\xi')|^2}{\bar{q}(\xi')} - \bar{q}(0) \int_0^\infty d\xi' e^{2ik\xi'} q(\xi')\right) + O\left(\frac{1}{k^2}\right),
\]

\[
\begin{pmatrix}
\Psi_{+12} \\
\Psi_{+22}
\end{pmatrix} = 
\begin{pmatrix}
0 \\
1
\end{pmatrix} + \frac{1}{2ik} \left(- \int_0^\infty d\xi' |q(\xi')|^2\right) + O\left(\frac{1}{k^2}\right),
\]

\[
\begin{pmatrix}
\Psi_{-11} \\
\Psi_{-21}
\end{pmatrix} = 
\begin{pmatrix}
1 \\
0
\end{pmatrix} + \frac{1}{2ik} \left(\int_0^\infty d\xi' |q(\xi')|^2\right) + O\left(\frac{1}{k^2}\right),
\]

\[
\begin{pmatrix}
\Psi_{-12} \\
\Psi_{-22}
\end{pmatrix} = 
\begin{pmatrix}
0 \\
1
\end{pmatrix} + \frac{1}{2ik} \left(- \int_0^\infty d\xi' |q(\xi')|^2 + q(0) \int_0^\infty d\xi' e^{-2ik\xi'} \bar{q}(\xi')\right) + O\left(\frac{1}{k^2}\right).
\]

These formulae give triangular matrices for \(\Psi_\pm(0,k)\) in agreement with the the boundary values (13). For \(\text{Im}k = 0\) and \(\xi \neq 0\) \(\Psi_\pm(\xi,k)\) do not have asymptotic expansions in the usual sense due to the presence of the terms \(\exp(2ik\xi)\bar{q}(0)\) and \(\exp(-2ik\xi)q(0)\).

Now we are in a position to express the potential \(Q\) through the matrix elements of \(\Psi_\pm\):

\[
q(\xi) = 2i \lim_{k \to \infty} k \Psi_{+12}(\xi,k), \quad \bar{q}(\xi) = 2i \lim_{k \to \infty} k \Psi_{-21}(\xi,k).
\]

Alternatively, one can use the following formulae:

\[
q(\xi) = 2i \lim_{k \to \infty} k \left(\Psi_{-12}(\xi,k) + e^{-2ik\xi} \beta(k)\right),
\]

\[
\bar{q}(\xi) = 2i \lim_{k \to \infty} k \left(\Psi_{+21}(\xi,k) + e^{-2ik\xi} \beta(k)\right).
\]
\[ \dot{q}(\xi) = 2i \lim_{k \to \infty} k \left( \Psi_{+21}(\xi, k) - e^{2ik\xi} \tilde{\beta}(k) \right). \]

Here we have taken into account (17) for \( \xi = 0 \) with \( \Psi_{+21}(0, k) = \beta(k) \) and \( \Psi_{-21}(0, k) = -\tilde{\beta}(k) \) from (13).

The reconstruction of the potential requires knowledge of the functions \( \Psi_{\pm} \). The latter can be obtained via solution of the RH problem, which we consider in the next section.

### 4 The Riemann-Hilbert problem

We have from (12)
\[ \Psi_{+} = \Psi_{-} E G E^{-1}, \quad \text{Im} k = 0, \]
where
\[ G(k) = H_{-} G_{-} = H_{+} G_{+} = \begin{pmatrix} \frac{1}{\beta} & \beta \\ \frac{1}{1 + |\beta|^2} & 1 \end{pmatrix}. \]

This is the matrix RH problem, i.e., the problem of analytic factorization a nondegenerate matrix \( G(k) \), given on the real line \( \text{Im} k = 0 \), into a product of two matrices analytic in the upper and lower half-planes, respectively. The above RH problem has the canonical normalization condition:
\[ \Psi_{\pm} \to I \quad \text{as} \quad k \to \infty. \]

To completely characterize the solution to the RH problem (18)-(20) one needs to specify the spectral data. First, it is the value of \( \beta(k) \) on the real line \( \text{Im} k = 0 \), which enters \( G(k) \) (19) and represents the continuous part of the spectral data. Second, the discrete spectral data should be given. The latter include zeros, \( k_j, \text{Im} k_j > 0 \), of the function \( \beta^{-1}(k) [\beta^{-1}(k_j) = 0] \) and the transition coefficients \( c_j \). The transition coefficients are given by the residues of \( \Psi_{+} \) or \( \Psi_{-} \) at the points \( k_j \) and \( \bar{k}_j \), respectively. Using the identity \( \det \Psi_{\pm}(\xi, k) = 1 \), the involution (13) and formula (11) it is not difficult to obtain (see also [4])
\[ \text{Res} \left[ \Psi_{+}^{(2)}(k), k_j \right] = c_j \Psi_{+}^{(1)}(k_j), \quad \text{Res} \left[ \Psi_{-}^{(1)}(k), \bar{k}_j \right] = -\bar{c}_j \Psi_{-}^{(2)}(\bar{k}_j). \]

The \( \xi \)-dependence of \( c_j \) follows from the equation
\[ \partial_{\xi} \left[ \text{Res} \Psi_{+}^{(2)}(k), k_j \right] = -ik \left( \sigma_3 + I \right) \left[ \text{Res} \Psi_{+}^{(2)}(k), k_j \right] + Q \left[ \text{Res} \Psi_{+}^{(2)}(k), k_j \right]. \]

Substituting here (21), we get \( c_j(\xi) = c_j^{(0)} e^{-2ik\xi} \).

The last step towards the complete description of the RH data involves the analysis of their temporal behavior. Here we analyse in detail the two models of section 2, the SIT and SRS equations. These models are integrable and hence represent the compatibility condition of a linear system, i.e., the Lax pair. The first Lax equation is the spectral equation (3). The time evolution is determined by the second Lax equation:
\[ \partial_{\tau} \Psi_{\pm} = \Psi_{\pm} E \Omega_{\pm} E^{-1} + V_{\pm} \Psi_{\pm}. \]

Here \( \tau = x \) for the SIT and \( \tau = t \) for the SRS models. The matrices \( V_{\pm}(\xi, \tau, k) \) are the limiting values, as \( k \to \text{Re} k \pm i0 \), of a piece-wise holomorphic function \( V(\xi, \tau, k) \). The choice of \( V(\xi, \tau, k) \) depends on the model.

In the same way, the dispersion relation \( \Omega(k, \tau) \) is also a piece-wise holomorphic function with the limiting values
Ω±(k, τ) defined for real k. (Hence the name “singular”, i. e., non-analytic dispersion relation). The matrices $V$ and $Ω$ satisfy the involutions

$$V_+^T(k) = -V_-(k), \quad Ω_+^T(k) = -Ω_-(k),$$

which follow from those of $Ψ(k)$ (13). Note that the compatibility of the Lax pair (3) and (22) is guaranteed if $V$ is chosen in such a way that the jump matrix $ΔV(ξ, τ, k) = V_+(ξ, τ, k) - V_-(ξ, τ, k)$ satisfies

$$∂_ξΔV(ξ, τ, k) = [-ikσ_3 + Q, ΔV(ξ, τ, k)], \quad \text{(23)}$$

while $Ω(k, τ)$ is still arbitrary. It means that the dispersion relation is not defined by the model. In fact, as it was realized in Ref. [11], it is specified by the imposed boundary conditions (see also [14]). Let us first obtain $V$ for the models we consider. It is convenient to represent a $V$ satisfying (23) in the form

$$V(ξ, τ, k) = \frac{1}{2πi} \int_{−∞}^{∞} \frac{dλ}{λ - k} J_-(ξ, τ, λ)ΔV(0, τ, λ)E(ξ, λ)ΔV(0, τ, λ)^{-1}E(ξ, λ), \quad \text{(24)}$$

Formula (24) trivially transforms in the SIT case by letting $ΔV(ξ = 0) = -\frac{πg(k)}{2}M_0$, \quad \text{(25)}

while for the SRS equations we have

$$\begin{pmatrix} a_1 \\ a_2 e^{2ikξ} \end{pmatrix} = I_1J_1(1) + I_2J_2(2)e^{2ikξ} \equiv I_1Ψ_1(1) + I_2Ψ_2(2)e^{2ikξ}, \quad \text{(26)}$$

thereby

$$ΔV(ξ = 0) = \frac{πg}{2} \left( \frac{|I_1|^2 - |I_2|^2}{2I_1I_2}, \quad \frac{2I_1I_2}{|I_2|^2 - |I_1|^2} \right). \quad \text{(27)}$$

Note that in both cases $\text{tr}V = \text{tr}ΔV = 0$.

To derive evolution equations for the spectral data and the dispersion relation $Ω$ we recall that the analyticity properties of $Ψ_{±}(k)$ depend on the conditions (13) defining the Jost solutions to the spectral equation (3). Hence, to have analyticity of $Ψ_{±}(k)$ for arbitrary $τ$ one must demand that those conditions hold for all $τ$. Equivalently, we can deal with the boundary and asymptotic values of $Ψ_{±}(ξ, τ, k)$, which must satisfy (13) and (16) for arbitrary $τ$. Substitution of formulae (13) and (14) into the second Lax equation (22) supplies one with the evolution of the spectral data $β$ and $α$ together with the dispersion relation $Ω$. After somewhat lengthy but straightforward calculations (in computing the limits of $V_{±}$ as $ξ → ∞$ one uses the identity: $V.p.\{\exp(±2ikξ)/k\} → ±iπδ(k)$) we obtain (see also Ref. [14])

$$β_τ = -V_{+21}β^2 + 2V_{+11}β + V_{+12}, \quad \text{(28)}$$

$$α_τ = \left((SΔV^{(0)}S^{-1})_{22} - βV_{+21} - V_{+22}\right)α. \quad \text{(29)}$$

Here $V_{+ij}(ξ = 0)$. For completeness we give also the dispersion relation:

$$Ω_+ = \begin{pmatrix} βV_{+21} - V_{+11} & 0 \\ -V_{+21} & −βV_{+21} - V_{+22} \end{pmatrix}, \quad \text{(20)}$$
Here we note that the corresponding evolution equations for the entries (a and b) of the scattering matrix are highly nonlinear coupled equations; while for the quotient $\beta = \tilde{b}/a$ one gets a simple Riccati equation. This is the advantage of using the matrices $\Psi_\pm$ defined by (11). The relevance of the Riccati equation for equations with the singular dispersion relations was noted previously (see, for example, [25] for SIT and [14, 26] for the SRS).

The time evolution of the discrete RH data is found as follows. First, let us find the time dependence of $c_j$. It follows from (21) taken at $\xi = 0$ that

$$
c_j^{(0)}(\tau) = \text{Res}\left[\beta(k, \tau), k_j\right],
$$

duly

$$
c_j(\xi, \tau) = e^{-2i k_j \xi} \text{Res}\left[\beta(k, \tau), k_j\right].
$$

Evolution of the poles of $\beta$ is derived in the following way. Inasmuch as $\beta^{-1}(k_j, \tau) = 0$ for all $\tau$, we have

$$
\frac{dk_j}{d\tau} = \left.\frac{(\partial/\partial\tau)\beta^{-1}(k)}{(\partial/\partial k)\beta^{-1}(k)}\right|_{k=k_j}
$$

which, in view of (28) and the identity $(\partial/\partial k)\beta^{-1}(k_j) = 1/c_j^{(0)}$, gives

$$
\frac{dk_j}{d\tau} = -\frac{V_{+21}^{(0)}(k_j)}{\partial k \beta^{-1}(k_j, \tau)} = -V_{+21}^{(0)}(k_j)c_j^{(0)}.
$$

We remind that

a) SIT : $V_{+21}^{(0)}(x, k_j) = \frac{i}{4} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - k_j} g(\lambda) \tilde{\rho}(x, \lambda)$

b) SRS : $V_{+21}^{(0)}(t, k_j) = \frac{g}{2\tau} \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda - k_j} I_1(t, \lambda)I_2(t, \lambda)$.

Hence, $k_j = \text{const}$ for the “causal” solutions (in Zakharov’s terminology [27] for the SIT or in the absence of the Stokes pulse for the SRS.

In this paper we are interested primarily in description of the effect of a perturbation on the evolution of the spectral data rather than characterization of specific solutions to the integrable nonlinear equations. Therefore, we proceed to the derivation of the perturbation-induced evolution of the spectral data. This is done in the next section.

5 Perturbation-induced evolution of RH data

Let us attribute the “variational” derivatives to the perturbation-induced evolution. For instance, we introduce the perturbation matrix $R$ as the variation of the potential,

$$
R = \frac{\delta Q}{\delta \tau} = \begin{pmatrix} 0 & \delta q / \delta \tau \\ -\delta \bar{q} / \delta \tau & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & r \\ -\bar{r} & 0 \end{pmatrix}.
$$
A variation of $Q$ leads to the corresponding variations $\delta J_\pm$ of the Jost solutions. We have

$$(J_\pm^{-1}\delta J_\pm)_\xi = -ik \left[ \sigma_3, J_\pm^{-1}\delta J_\pm \right] + J_\pm^{-1}\delta QQJ_\pm.$$  

Solving this equation we get

$$\delta J_\pm = J_\pm E(\xi) \left( \int_{\xi_0}^\xi d\xi' E^{-1}(\xi') J_\pm^{-1}\delta QQJ_\pm E(\xi') \right) E^{-1}(\xi)$$

with $\xi_0 = 0$ for $\delta J_-$ and $\xi_0 = \infty$ for $\delta J_+$. Recalling the definition of the scattering matrix (7) and using the relations (12), we obtain its variation:

$$\frac{\delta S}{\delta \tau} = G_+ Y_+(k) G_-^{-1} = H_-^{-1} Y_-(k) H_-, \quad (34)$$

where $Y_\pm(k) \equiv Y_\pm(0, \infty; k)$ and

$$Y_\pm(q, p; k) = \int_q^p d\xi E^{-1} \Psi_\pm^{-1} R \Psi_\pm E.$$  

By variation of the relations (12) and using the above formulae one easily derives the perturbation-induced evolution of the matrices $\Psi_\pm$:

$$\frac{\delta \Psi_\pm}{\delta \tau} = \Psi_\pm E \Pi_\pm E^{-1}, \quad (35)$$

where the functional $\Pi$ is defined as follows (omitting the explicit $\tau$-dependence)

$$\Pi_+(\xi, k) = \begin{pmatrix} Y_{+11}(0, \xi; k) & -Y_{+12}(\xi, \infty; k) \\ Y_{+21}(0, \xi; k) & -Y_{+11}(0, \xi; k) \end{pmatrix}, \quad \Pi_-(\xi, k) = \begin{pmatrix} Y_{-11}(0, \xi; k) & Y_{-12}(0, \xi; k) \\ -Y_{-21}(\xi, \infty; k) & -Y_{-11}(0, \xi; k) \end{pmatrix}. \quad (36)$$

The r.h.s. of (35) should be added to the second (evolutional) Lax equation (22) to account for the perturbation-induced evolution of $\Psi_\pm$. Note that $\text{tr}\Pi_\pm = 0$ in agreement with the identity $\det \Psi_\pm = 1$ and $\Pi_+(\bar{k}) = -\Pi_-(k)$ due to the involution (13).

Formula (35) accounts for the perturbation exactly. However direct application of (35) is not possible due to the fact that $\Psi_\pm$ should be obtained from the RH problem, solution of which requires knowledge of the spectral data, while the perturbation-induced evolution of the latter explicitly depends on $\Psi_\pm$ (see below). On the other hand, if the perturbation is small, one can expand the exact evolution equations for the spectral data into series of approximate equations which are solvable.

5.1 Evolution of the continuous datum

In view of the boundary values of $\Psi$ (13), the perturbation-induced evolution (variation) of $\beta(k)$ obtains from (36) taken at the boundary $\xi = 0$. We have

$$\begin{pmatrix} 0 & \delta \beta / \delta \tau \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -Y_{+12}(k) \\ 0 & 0 \end{pmatrix}.$$  

Hence the variation of $\beta(k)$ satisfies

$$\frac{\delta \beta}{\delta \tau} = -Y_{+12}(k) = -\int_0^\infty d\xi e^{2ik\xi} (\Psi_+^{-1} R \Psi_+^{-1})_{12}. \quad (37)$$
Substitution of the expression for $R$ from (33) gives
$$
\frac{\delta \beta}{\delta \tau} = \int_0^\infty d\xi e^{2ik\xi} \left[ \bar{r} (\Psi_{+12})^2 + r (\Psi_{+22})^2 \right].
$$

(38)

The integrand in (38) is meromorphic for $\text{Im} k \geq 0$ by the definition of $\Psi_{\pm}(k)$ (11) and this is in agreement of the analytic properties of $\beta(k)$.

Equation (38) allows us to prove the applicability of the linear limit in accounting for the effect of non-zero dephasing on the spike formation in the SRS equations (24). Moreover, the argument we present below is valid for non-small dephasing parameter $\Gamma$. Following the arguments of Ref. [24] (see also Ref. [14]), we note that the spike of pump radiation, first observed in Ref. [18] (see also more recent Refs. [28] and [29]), occurs at a time $\tau_0 \equiv t_0$ when $\beta(k_0, \tau_0) = 0$. Hence, the limit $\beta \ll 1$ serves as an approximation when dealing with the spike formation. For small $\beta$, solving the RH problem (18) in the linear limit and using (17) we get (in the SRS notations (2))
$$
u = -\frac{1}{\pi} \int dke^{-2ikx} \beta(k) + O(\beta^2).
$$

(39)

Here the integration is in the complex plane above all poles of $\beta(k)$. On the other hand, substitution of $\delta u/\delta \tau = -\Gamma u$ in equation (38) gives
$$
\frac{\delta \beta}{\delta t} = \int_0^\infty dx e^{2ikx} \frac{\delta u}{\delta t} + O(\beta^3) = -\Gamma \int_0^\infty dx e^{2ikx} u + O(\beta^3) = -\Gamma \beta + O(\beta^3).
$$

(40)

Here we have used formula (39).

Equation (40) is a quite remarkable result. Indeed, the integrable part of the evolution of $\beta(k, \tau)$ is given by the Riccati equation (28), which is quadratic in $\beta$. When $\beta$ is small (spike formation), addition of the perturbation-induced evolution in the same order in $\beta$ to the integrable limit requires keeping the terms linear and quadratic in $\beta$. However, in equation (40) the quadratic term is equal to zero. Thereby, the only effect of the dephasing term is stabilization of the spike of pump radiation in the SRS as it was realized, though without a rigorous argument, in Refs. [24].

5.2 Evolution of the discrete data

Here we describe the action of a perturbation on the evolution of the discrete RH data. A perturbation-induced evolution of $c_j^{(0)}$ is calculated from (33) and (38). Namely,
$$
\frac{\delta c_j^{(0)}}{\delta \tau} = \text{Res} \left\{ \int_0^\infty d\xi e^{2ik\xi} \left[ \bar{r} (\Psi_{+12})^2 + r (\Psi_{+22})^2 \right], k_j \right\}.
$$

(41)

Separating out the singular parts of the functions inside the integrand by using formula (21),
$$
\Psi_{+12}(k) = \frac{c_j^{(0)} e^{-2ik_j \xi}}{k - k_j} \Psi_{+11}(k_j) + \Psi_{+12}^{\text{reg}}(k),
$$

(42)
$$
\Psi_{+22}(k) = \frac{c_j^{(0)} e^{-2ik_j \xi}}{k - k_j} \Psi_{+21}(k_j) + \Psi_{+22}^{\text{reg}}(k),
$$
we can write equation (41) as follows
\[ \frac{\delta c_j^{(0)}}{\delta \tau} = \text{Res} \left[ \frac{c_j^{(0)} e^{2ik_1 \xi - 4ik_1 \xi}}{(k - k_1^2)} \int_0^\infty \text{d} \xi e^{2ik_1 \xi - 4ik_1 \xi} F_1(\xi) \right] + 2 \frac{c_j^{(0)}}{k - k_1} \int_0^\infty \text{d} \xi e^{2ik_1 \xi - 2ik_1 \xi} F_2(k, \xi, k_j). \]

Here we have introduced a $k$-independent, $F_1(\xi)$, and a regular function of $k$, $F_2(k, \xi, k)$, by (omitting the explicit $\tau$-dependence for simplicity)

\[ F_1(\xi) = \bar{r}(\xi) (\Psi_{+11}(\xi, k_j))^2 + r(\xi) (\Psi_{+21}(\xi, k_j))^2, \]

\[ F_2(k, \xi, k_j) = \bar{r}(\xi) \Psi_{+11}(\xi, k_j) \Psi_{+12}^{reg}(\xi, k) + r(\xi) \Psi_{+21}(\xi, k_j) \Psi_{+22}^{reg}(\xi, k). \]

Calculating the residues in (42), we obtain the perturbation-induced evolution of the coefficient $c_j^{(0)}$:

\[ \frac{\delta c_j^{(0)}}{\delta \tau} = 2c_j^{(0)} \left( \int_0^\infty \text{d} \xi F_2(k_j, \xi) + i \int_0^\infty \text{d} \xi e^{-2ik_1 \xi} F_1(\xi) \right). \]

Finally, the perturbation-induced evolution of $k_j$ is found as follows. We start with the formula similar to (31) but with the variational derivatives, i. e.,

\[ \frac{\delta k_j}{\delta \tau} = - \left( \frac{\delta}{\delta \tau} \beta^{-1}(k) \right) \bigg|_{k = k_j}. \]

Using the relations

\[ \beta^{-1}(k) = \frac{k - k_j}{c_j^{(0)}} + \mathcal{O}((k - k_j)^2), \quad \frac{\partial \beta^{-1}}{\partial k}(k) = \frac{1}{c_j^{(0)}} + \mathcal{O}(k - k_j), \quad k \to k_j, \]

which follow from (34), and taking the perturbation-induced evolution of $\beta$ in the form of (37), we rewrite (43) as

\[ \frac{\delta k_j}{\delta \tau} = \frac{(k - k_j)^2 \Upsilon_{+12}(k)(1 + \mathcal{O}(k - k_j))}{c_j^{(0)} + \mathcal{O}(k - k_j)} \bigg|_{k = k_j}. \]

Note that the numerator is non-zero because $\Upsilon_{+12}(k)$ is meromorphic function having poles of the second order, as it is seen from (24). Using (38) and (44), we obtain

\[ \Upsilon_{+12} = - \frac{c_j^{(0)} e^{2ik_1 \xi} \left[ \bar{r}(\Psi_{+11}(k_j))^2 + r(\Psi_{+21}(k_j))^2 \right] + \mathcal{O}((k - k_j)^{-1})}{(k - k_j)^2} \bigg|_{k = k_j}. \]

Substituting this result into (46), we arrive at the perturbation-induced evolution of $k_j$:

\[ \frac{\delta k_j}{\delta \tau} = c_j^{(0)} \int_0^\infty \text{d} \xi e^{-2ik_1 \xi} \left[ \bar{r}(\Psi_{+11}(k_j))^2 + r(\Psi_{+21}(k_j))^2 \right]. \]

We conclude this section with the note that the general (i. e., with account for the perturbation) evolution equations for the spectral data $\beta(\tau, k)$, $k_j(\tau)$, and $c_j^{(0)}(\tau)$ are given by addition of their perturbation-induced evolution (represented by the variational derivatives (38), (44), and (47) to the evolution equations of these quantities in the integrable limit.
6 Conclusion

We have shown that the Riemann-Hilbert problem is a natural setting for dealing with both integrable and nearly integrable nonlinear equations with the singular dispersion relations. In particular, for small perturbations of the integrable models, we have explicit perturbation-induced evolution equations for the RH data, which are necessary for analysis and approximate solution for the physically interesting quantities.

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