Learning Financial Network with Focally Sparse Structure

[Preliminary and Incomplete Version]

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Abstract

This paper studies the estimation of network connectedness with focally sparse structure. We try to uncover the network effect with a flexible sparse deviation from a predetermined adjacency matrix. To be more specific, the sparse deviation structure can be regarded as latent or misspecified linkages. To obtain high-quality estimator for parameters of interest, we propose to use a double regularized high-dimensional generalized method of moments (GMM) framework. Moreover, this framework also facilitates us to conduct the inference. Theoretical results on consistency and asymptotic normality are provided with accounting for general spatial and temporal dependency of the underlying data generating processes. Simulations demonstrate good performance of our proposed procedure. Finally, we apply the methodology to study the spatial network effect of stock returns.

Keywords: Machine learning, GMM, financial network analysis, misspecification, time series.

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1 Introduction

Network analysis has gained significant interest recently. In particular, measuring the connectedness within a network structure has become a central task for econometricians. In this paper we consider a double regularized GMM estimator for a system of regression equations to flexibly learn a focally sparse network structure. The moment equations cover the linear models for dependent variables that may reply on the outcomes and characteristics of others in the network. For any pre-specified network structure, we allow a sparse deviation and learn the network structure incorporating some given structural restrictions. We provide high-quality inference procedure for the parameters of interest in the debiasing step.

There have been extensive studies on understanding the network connectedness within a complex financial system. Diebold and Yılmaz (2014) develop a network topology analysis using forecast variance decomposition in a dynamic vector autoregressive setup. To extend the estimation framework, Hautsch et al. (2015) estimate the financial network using quantile LASSO proposed in Belloni and Chernozhukov (2011). Härdle et al. (2016) study the calibration for nonlinear network effects in high-dimensional single index regression. Chen et al. (2019) estimate the system connectedness of a financial network driven by the tail events following a correlation methodology proposed by Ng (2006). Chernozhukov et al. (2018b) study the network estimation in a high-dimensional time series model without incorporating the contemporaneous relationships of the firms. Financial networks shall be regarded as a particular kind of complex dynamic systems. The unique feature of financial networks is the nonexistence of a natural connectedness structure. That explains the unstructured nature of the above mentioned papers. Nevertheless, one can still infer the connectedness using the information from data (such as credit chain, common ownership etc.). However, the estimation is subject to misspecification error.

In the meanwhile, the celebrated literature on social networks favors a predetermined structure. Dating back to Lee (1983), it is very common to use a pre-defined network structure to study the peer effects in social networks, and as a particular application, to analyze the financial network connectedness. Zhu et al. (2019) look at a nonlinear quantile model for tail event driven network analysis with a pre-specified share ownership structure. Xu et al. (2020) recently extend it to a dynamic spatial quantile framework with utilizing the instrumental variable (IV) quantile regressive method as
in Chernozhukov and Hansen (2006). They document the importance of incorporating contemporaneous lags in understanding the connectedness of a financial network. Paralleling to the financial network framework, there is a big literature studying the social interactions with pre-specified network structure, see for example De Paula (2017) for a comprehensive review. Kuersteiner and Prucha (2020) consider a class of GMM estimators for general dynamic panel models, allowing for cross-sectional dependence due to spatial lags and unspecified common shocks. Our model can be viewed as a high-dimensional generalization to Kuersteiner and Prucha (2020), where we discuss in a different perspective on how to learn the high-dimensional misspecification effect of the network structure.

It is worth noting that structure imposed on the model might induce the risk of misidentification of the network. When we try to impose less structure on the network, we run into the challenge of estimating many parameters. The machine learning literature using for example LASSO (Least Absolute Shrinkage and Selection Operator) often requires the parameters to be sparse, which is not necessarily the case for most of the application scenarios. To compromise the two approaches between estimating a fully unspecified and fixing a completely given network structure, we propose an estimation method with a pre-specified network structure and some sparse unknown misspecification errors. For example, in the financial system, it is not hard to obtain pre-specified network information, such as, credit chain, common ownership etc. We allow the links that connect firms to be estimated from a procedure. We focus on the estimation of the misspecification error matrix (assumed to be sparse) in the context of a complex financial network. We propose to use a Danzig type of estimation in a high-dimensional GMM estimation framework. The method has been studied in Belloni et al. (2018) in the i.i.d. equation system. Caner and Kock (2018) estimate the high dimension linear GMM with inference using LASSO. Manresa (2013); Lam and Souza (2014); De Paula et al. (2018) estimate networks using LASSO for panel data. We target on the specific financial network estimation and formulate the problem into a system of dynamic regression equations. The parameters are reparameterized so that we can shrink towards a pre-specified network structure. We show in the simplest case the requirement of the design matrix as well as the link to the causal graph inference in the dynamic setting. Our theoretical framework fits into more general dynamic panel models, with heterogeneity reflected in the individual based parameters.

Furthermore, the issue of high quality estimation and inference for parame-
ters in a system of regression equations with endogeneity is involving. There are many works in the simple setup. For the case of i.i.d. data, Belloni et al. (2014); Zhang and Zhang (2014); van de Geer et al. (2014); Chernozhukov et al. (2018a); Zhu and Bradic (2018), among others, develop the confidence intervals of low-dimensional parameters in high-dimensional models with various forms of de-biased/orthogonalization methods. Still in the case of i.i.d. data, Belloni et al. (2015) establish a uniform post-selection inference for the target parameters defined via de-biased Huber’s Z-estimators, where they utilize the multiplier bootstrap to the estimated residuals. And more recently, Belloni et al. (2018) give detailed ways of high quality estimation and inference in high-dimensional GMM framework.

We formulate our theory in a general system of regression equations. To appreciate the intuition of our motivating example, we show a simple setup as follows. Let $t = 1, \ldots, n$, $j = 1, \ldots, p$, both $p, n$ tend to infinity. $y_t = (y_{j,t})_{j=1}^p$ is a $p$-dimensional time series expressed by $y_t = \rho Wy_t + \Delta y_t + \epsilon_t$, where $W$ is a pre-specified network structure, which does not need to be sparse. $\Delta$ is the specification error matrix with diagonal entries to be 0, and $|\rho| < 1$ is the network effect. Obviously we need restrictions on $\Delta$ to ensure the regression does not suffer from multicollinearity. What is known from the literature is the classical spatial estimator for $\rho$ e.g. the IV estimator would not be consistent if the misspecification error $\Delta$ is too large, see a recent work by Lewbel et al. (2019). De Paula et al. (2018) study the network estimation of adjacency matrix $W$ using the GMM LASSO technique with important contribution to the identification of the structural parameters. Ata et al. (2018) consider a reduced form estimation with the innovative discovery of the algebraic results, i.e., a sparsity assumption on $W$ would lead to approximate sparsity for the matrix $(I - \rho W)^{-1}$. These two studies somehow all need to assume the sparsity of $W$ for the estimation to work. Thus it is critical to account for the misspecification. However, it is not preferable to estimate totally an unstructured network effect as most of applications would give us a predetermined $W$. Different from the usual proposal, we shall try to uncover the misspecification error $\Delta$ and thus recover the true network structure. As a first step we would like to treat $\rho$ as a nuisance parameter and try to estimate $\Delta$ using a regularized GMM method.

We contribute to the literature in three aspects. First, we develop a method to estimate parameters in high-dimensional endogenous equation system with incorporating dynamics. Secondly, we suggest a way to adapt to pre-information of a model structure and provide theoretical results on how the restricted
eigenvalue conditions changes. Thirdly, we propose a debiasing and inference procedure for the estimated high-dimensional misspecification parameters. Lastly, we illustrate the usefulness of our method in a financial network context empirically.

The following notations are adopted throughout the paper. For a vector \( v = (v_1, \ldots, v_p)^\top \), let \( |v|_k = (\sum_{i=1}^p |v_i|^k)^{1/k} \) with \( k \geq 1 \), and \( |v|_\infty = \max_{1 \leq i \leq p} |v_i| \).

For a random variable \( X \), let \( \|X\|_r \overset{\text{def}}{=} \left( E|X|^r \right)^{1/r}, r > 0 \). For a matrix \( A = (a_{ij}) \in \mathbb{R}^{p \times q} \), we define \( |A|_1 = \max_{1 \leq j \leq q} \sum_{i=1}^p |a_{ij}|, |A|_\infty = \max_{1 \leq i \leq p} \sum_{j=1}^q |a_{ij}| \), and the spectral norm \( |A|_2 = \sup_{||v||_2 \leq 1} |Av|_2 \). Moreover, let \( \lambda_i(A) \) and \( \sigma_i(A) \) be the \( i \)-th largest eigenvalues and singular values of \( A \), respectively. Let \( I_p \): \( p \times p \) denote the identity matrix. For any function on a measurable space \( g : \mathcal{W} \to \mathbb{R} \), \( E_n(g) \overset{\text{def}}{=} n^{-1} \sum_{t=1}^n g(\omega_t) \). Given two sequences of positive numbers \( a_n \) and \( b_n \), write \( a_n \lesssim b_n \) (resp. \( a_n \asymp b_n \)) if there exists constant \( C > 0 \) (does not depend on \( n \)) such that \( a_n/b_n \leq C \) (resp. \( 1/C \leq a_n/b_n \leq C \)) for all large \( n \). For a sequence of random variables \( x_n \), we use the notation \( x_n \lesssim_P b_n \) to denote \( x_n = O_P(b_n) \).

The rest of the article is organized as follows. Section 2 shows the model specification with a simple example as well as the general system model and the estimation steps. Section 3 presents the main theoretical results in the linear moment case. In Section 4 and 5 we deliver the simulation studies and an empirical application on financial network analysis with possible misspecification. The technical proofs and other details are given in the Appendix.

## 2 Model and Estimation

### 2.1 Simple example

Consider a model with high-dimensional covariates

\[
y_t = \rho x_t^\top h + \varepsilon_t, \quad x_t \in \mathbb{R}^p, \, t = 1, \ldots, n, \, p \gg n, \tag{1}
\]

where \( h = (h_k)_{k=1}^p \) is a \( p \times 1 \) column vector and \( \rho \) is a scalar effect. Our goal is to estimate \( \rho \) and recover the unobserved \( h \). In practice, \( h \) can be misspecified as \( w = (w_k)_{k=1}^p \). The model can be re-written as

\[
y_t = x_t^\top (\rho w + \delta) + \varepsilon_t, \tag{2}
\]

where \( \delta = (\delta_k)_{k=1}^p = \rho(h - w) \) reflects the misspecification errors, which is more likely to be sparse. Without loss of generality, we assume there exists
\(k \in \{k : w_k \neq 0\}\) such that \(\delta_k = 0\) (at least one of such \(k\)'s is known and to avoid multicollinearity the corresponding element will be eliminated from the covariates of \(x_i^\top \delta\) in the regression \((2)\)). Furthermore, by setting \(B_p \times \beta^0_p = \rho w + \delta\), we get the linear model

\[
y_t = x_i^\top B \beta^0 + \varepsilon_t,
\]

and in the matrix form we have

\[
Y = XB \beta + \varepsilon,
\]

where \(Y = (y_1, \ldots, y_n)^\top\), \(X = (x_1, \ldots, x_n)^\top\), and \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)^\top\). Note that the first column of the matrix \(B\) is given as \(w\) and the first element in the vector \(\beta^0\) is \(\rho\). In particular, \(\beta^0\) measures the extend of deviation. It is more natural to assume that \(\beta^0\) is sparse but not necessarily for \(B \beta^0\) given \(h\) (and \(w\)) are not sparse.

We shall allow endogeneity for the model with \(E x_t \varepsilon_t \neq 0\). This falls into the IV estimation framework. There are many application scenarios in econometrics with endogenous \(x_t\), see e.g., Arellano and Bond (1991); Blundell and Bond (2000) in the dynamic panel data case. We assume that \(\varepsilon_t\) is martingale difference sequence and let \(x_t\) be a process with spatial and temporal dependency.

We shall estimate \(\beta^0\) by regularization. Now the question is to which condition we need to impose on \(X\) to ensure a restricted isometry property (RIP) or restricted eigenvalue (RE) condition on the design matrix \(XB\). Also it may be helpful to understand the format of \(B\) as well. For example, when \(p = 4\), \(B^\top\) can take the form as

\[
\begin{pmatrix}
1/4 & 1/4 & 1/4 & 1/4 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

**REMARK 2.1.** We notice that for the full rank matrix \(X\), if there exists a full rank matrix \(A_{p \times (p-n)}\), such that \(XA = 0\) (i.e., the columns of \(A\) form the null space of \(X\)), then for each \(\xi \neq \xi^0\) (\(\xi, \xi^0 \in \mathbb{R}^p\)), we can find a nonzero vector \(\eta \in \mathbb{R}^{p-n}\) such that \(\xi = A\eta + \xi^0\), if we have \(X\xi = X\xi^0\). Thus we shall restrict the columns of \(B\) do not belong to the space spanned by the columns of \(A\), namely there does not exist a column of \(B\), \(B_i\), such that \(B_i = A\eta\).

The RIP for \(XB\) in the case that \(X\) is deterministic is discussed in the following lemma.
LEMMA 2.1. Let $B_I$ denote the sub-matrix of $B$ with columns indexed by the set $I$ and the cardinality $|I|$ is given by $s$ ($s \leq n$). Let $\mathcal{V}_{B_I} \overset{\text{def}}{=} \{\delta \in \mathbb{R}^p : \xi = B_I \xi_I, \xi_I \in S^{s-1}\}$, where $S^{s-1}$ denotes the unit Euclidean sphere, i.e., $\xi_I$ is a unit vector with $|\xi_I|_2 = 1$. If $B_I$ is of rank $s$ for any $I$, and $c \leq \lambda_{s,B} \leq \lambda_1(X^\top X) \leq C$, $\tilde{\lambda}_{s,B} \overset{\text{def}}{=} \min_{I, |I|=s} \min_{\xi \in \mathcal{V}_{B_I}} \frac{\xi^\top X^\top X \xi}{\xi^\top \xi}$, then we have the RIP for $XB$.

Proof. Note that to prove the RIP of $XB$ is equivalent to show that $c' \leq \sigma_s(XB_I) \leq \sigma_1(XB_I) \leq C'$.

Let $\mathcal{V} : \dim(\mathcal{V}) = i$ be a subspace of $\mathbb{R}^s$ of dimension $i$, $i = 1, \ldots, s$. Due to the Min-max theorem for singular values, we have

$$\lambda_s(B_I^\top X^\top XB_I) = \sigma_s^2(XB_I) = \max_{\mathcal{V} : \dim(\mathcal{V}) = s} \min_{\xi_I \in \mathcal{V}, \xi_I \xi_I = 1} \xi_I^\top B_I^\top X^\top XB_I \xi_I$$

$$= \min_{\mathcal{V} : \dim(\mathcal{V}) = 1} \max_{\xi_I \in \mathcal{V}, \xi_I \xi_I = 1} \xi_I^\top B_I^\top X^\top XB_I \xi_I.$$

For any fixed $\mathcal{V} : \dim(\mathcal{V}) = s$, we have

$$\min_{\xi_I \in \mathcal{V}, \xi_I \xi_I = 1} \frac{\xi_I^\top B_I^\top X^\top XB_I \xi_I}{\xi_I \xi_I}$$

$$= \min_{\xi_I \in B_I} \xi_I^\top B_I^\top X^\top XB_I \xi_I \xi_I^\top B_I \xi_I$$

$$\geq \min_{\xi \in \mathcal{V}_{B_I}} \frac{\xi^\top X^\top X \xi}{\xi^\top \xi} \lambda_s(B_I^\top B_I),$$

$$\geq \tilde{\lambda}_{s,B} \lambda_s(B_I^\top B_I),$$

where the last inequality is due to the definition of $\tilde{\lambda}_{s,B}$ and the full rank property of $B_I^\top B_I$, which implies $\lambda_{\min}(B_I^\top B_I) = \lambda_s(B_I^\top B_I)$ is positive. As the above inequality holds for any subspace $\mathcal{V}$ of dimension $s$, thus we have $\lambda_s(B_I^\top X^\top XB_I) \geq \tilde{\lambda}_{s,B} \lambda_s(B_I^\top B_I)$. Similarly, we have $\lambda_1(B_I^\top X^\top XB_I) \leq \lambda_1(X^\top X) \lambda_1(B_I^\top B_I)$.

Given $\lambda_s(B_I^\top X^\top XB_I) = \sigma_s^2(XB_I)$, we have proved that if $B_I$ is of rank $s$ for any $I$, and $c \leq \tilde{\lambda}_{s,B} \leq \lambda_1(X^\top X) \leq C$, then the RIP for $XB$ follows. \qed

Now we provide another Lemma for the random design $X$ with i.i.d. sub-Gaussian entries.
We define \( \|Z\|_{\psi_1} = \inf\{t < 0 : E\exp(|Z|/t) \leq 2\} \) and \( \|Z\|_{\psi_2} = \inf\{t < 0 : E\exp(|Z|^2/t^2) \leq 2\} \) as the sub-exponential norm and sub-Gaussian norm of the random variable \( Z \).

**Lemma 2.2.** Let \( X \) be an \( n \times p \) matrix whose rows \( X_i \) are independent mean-zero sub-Gaussian isotropic random vectors in \( \mathbb{R}^p \). Suppose \( c \leq \sigma_s(B_I) \leq \sigma_1(B_I) \leq C \) and \( n \gg s \log(pe/s) \). Then we have

\[
c_K \sqrt{n} \leq \sigma_s(X^\top B_I) \leq \sigma_1(X^\top B_I) \leq C K \sqrt{n},
\]

with probability approaching one, where \( c_K, C_K \) are positive constants related to \( K = \max_i \|X_i\|_{\psi_2} \).

**Proof.** **Step 1:**

For \( \xi \in S^{s-1} \), where \( S^{s-1} \) denotes the unit Euclidean sphere, i.e. \( |\xi|_2 = 1 \), we first show that \( \xi^\top B_I^\top X_i X_i^\top B_I \xi \) is concentrated around its mean \( \xi^\top B_I^\top B_I \xi \).

Let \( U_i \) defined as \( \xi^\top B_I^\top X_i X_i^\top B_I \xi - \xi^\top B_I^\top B_I \xi \). By Bernstein inequality, we have

\[
P\left( \left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \varepsilon / 2 \right) \leq 2 \exp \left( -c \min \left( \frac{\varepsilon^2 n^2}{\sum_{i=1}^n \|U_i\|_{\psi_1}^2}, \frac{\varepsilon n}{\max_{1 \leq i \leq n} \|U_i\|_{\psi_1}} \right) \right).
\]

By applying the properties of sub-Gaussian and sub-exponential random variables, we have

\[
\|U_i\|_{\psi_1} \leq C_1 \|\xi^\top (B_I^\top X_i)\|_{\psi_1}^2 = C_1 \|\xi^\top B_I^\top X_i\|_{\psi_2}^2 \leq C_1 \sum_{j=1}^s \|\xi_j B_I^\top X_i\|_{\psi_2}^2 \leq C_1 \sum_{j=1}^s \|\xi_j B_{I,j}^\top X_i\|_{\psi_2}^2 \leq C_1 \max_{1 \leq j \leq s} \|B_{I,j}^\top X_i\|_{\psi_2}^2 \leq C_2 \lambda_{\max}(B_I^\top B_I) =: K,
\]

where \( B_{I,j}, j = 1, \ldots, s \) is the \( j \)-th column vector of \( B_I \) and the last inequality follows given \( \max_{1 \leq j \leq s} \|B_{I,j}^\top X_i\|_{\psi_2}^2 \leq \max_{1 \leq j \leq s} |B_{I,j}|_2^2 \max_{1 \leq k \leq p} \|X_{i,k}\|_{\psi_2}^2 \leq C \lambda_{\max}(B_I^\top B_I) \).

It follows that

\[
P\left( \left| \frac{1}{n} \sum_{i=1}^n U_i \right| \geq \varepsilon / 2 \right) \leq 2 \exp \left( -c \min \left( \frac{\varepsilon^2 n^2}{K^2}, \frac{\varepsilon n}{K} \right) \right),
\]

8
Step 2:
Let \( \bar{\sigma} \overset{\text{def}}{=} \sigma_1(B_I) \), and \( \underline{\sigma} = \sigma_s(B_I) \), which are bounded positive constants.

Note that
\[
|n^{-1} B_I^T X^T X B_I - B_I^T B_I|_2 = \sup_{\xi \in S^{s-1}} |n^{-1} \xi^T B_I^T X^T X B_I \xi - \xi^T B_I^T B_I \xi|.
\]

Moreover, for any \( \xi \in S^{s-1} \), we have
\[
|n^{-1}|\xi^T B_I^T X^T|_2^2 - |\xi^T B_I^T|_2^2| \geq |\xi^T B_I^T|_2 \left( \frac{1}{\sqrt{n}}|\xi^T B_I^T X^T|_2 + |\xi^T B_I^T|_2 \right) \geq 2 \sup_{\xi \in S^{s-1}} |n^{-1} \xi^T B_I^T X^T \xi - \xi^T B_I^T \xi|.
\]

Therefore, we have shown that \( |n^{-1} B_I^T X^T X B_I - B_I^T B_I|_2 \leq \varepsilon \) holds with high probability implies \( \varepsilon \sqrt{n}(\bar{\sigma} - \varepsilon/\underline{\sigma}) \leq \sigma_s(XB_I) \leq \sigma_1(XB_I) \leq \varepsilon \sqrt{n}(\bar{\sigma} + \varepsilon/\underline{\sigma}) \) holds with the same probability.

Step 3:
By applying the Corollary 4.2.13 of \cite{Vershynin2019}, we can find a \( 1/4 \)-net \( \mathcal{N} \) of the unit sphere \( S^{s-1} \) with cardinality \( |\mathcal{N}| \leq 9^s \). By the discretized property of the net, we have
\[
|n^{-1} B_I^T X^T X B_I - B_I^T B_I|_2 = \sup_{\xi \in S^{s-1}} |n^{-1} \xi^T B_I^T X^T X B_I \xi - \xi^T B_I^T B_I \xi| 
\leq 2 \sup_{\xi \in \mathcal{N}} |n^{-1} \xi^T B_I^T X^T X B_I \xi - \xi^T B_I^T B_I \xi|.
\]

Using the union bounds, we obtain
\[
P\left( \sup_{\xi \in \mathcal{N}} |n^{-1} \xi^T B_I^T X^T X B_I \xi - \xi^T B_I^T B_I \xi| \geq \varepsilon/2 \right) \leq 2 \cdot 9^s \exp\left(-c \min(\varepsilon^2/K^2, \varepsilon/K)n\right).
\]

We have proved that the pointwise concentration in Step 1 implies that \( |n^{-1} B_I^T X^T X B_I - B_I^T B_I|_2 \leq \varepsilon \) holds with high probability.

Step 4:
By Step 2 and 3 we know that provided \( n \min(\varepsilon^2/K^2, \varepsilon/K) \gg s \log 9 \) we can get
\[
\varepsilon \sqrt{n}(\bar{\sigma} - \varepsilon/\underline{\sigma}) \leq \sigma_s(XB_I) \leq \sigma_1(XB_I) \leq \varepsilon \sqrt{n}(\bar{\sigma} + \varepsilon/\underline{\sigma})
\]
holds with probability \( 2 \exp\left(-c' \min(\varepsilon^2/K^2, \varepsilon/K)n\right) \). In addition, we know that there are \( \binom{p}{s} \lesssim (pe/s)^s \) set of \( I \) among the \( p \) dimensional covariates. Thus, by the union bounds, we can bound the probability by
\[
1 - 2 \exp\left(-c' \min(\varepsilon^2/K^2, \varepsilon/K)n + s \log(pe/s)\right).
\]
We have shown in a simple high dimensional linear regression case, our framework goes through with modified design matrix. In the following we will set the general framework and discuss the property of the estimated network. In a second step we do high quality inference of our procedure covering spatial and temporal dependence.

2.2 General model

We consider the following model, with $t = 1, \ldots, n, j = 1, \ldots, p$.

$$ y_{j,t} = x_{j,t}^\top b_j + \epsilon_{j,t}, \quad (6) $$

where $b_j = (b_{j,k})_{k=1}^{K'_j}$ is a $K'_j$-dimensional vector. We assume that $\epsilon_{j,t}$ are martingale difference sequences and allow for dependency in $x_{j,t}$. Moreover, we allow the model to be dynamic such that the lagged values of $y_{j,t}$ can be included in the covariates.

We can also rewrite the above model as

$$ y_{j,t} = x_{j,t}^\top B_j \beta^0_j + \epsilon_{j,i}, \quad (7) $$

where $B_j$ is a $K'_j \times K_j$ matrix and $\beta^0_j$ is a $K_j \times 1$ vector ($K_j \leq K'_j$). We shall discuss how $b_j$ is expressed by $B_j \beta_j$ under several examples below. We allow for endogeneity in the covariates $x_{j,t}$, i.e. $\mathbb{E} x_{j,t} \epsilon_{j,t} \neq 0$.

**Example 1** (Network formation). The simple case in (1) can be extended to the model with multiple equations:

$$ y_{j,t} = \rho x_{j,t}^\top h_j + \epsilon_{j,t}, \quad j = 1, \ldots, p. $$

In particular, under a network framework with $x_{j,t} = x_t \in \mathbb{R}^p$ and $h_j = (h_{j,k})_{k=1}^p \in \mathbb{R}^p$, for $j = 1, \ldots, p$,

$$ y_{j,t} = \rho x_t^\top h_j + \epsilon_{j,t}, \quad (8) $$

where $|\rho| < 1$ measures the network effect and $h_{j,k}$ ($k \neq j$) is referred to as the actual, unobserved spillover effect from individual $k$ to $j$.

Suppose one observe $w_j = (w_{j,k})_{k=1}^p \in \mathbb{R}^p$ instead of $h_j$ and let $\delta_j = (\delta_{j,k})_{k=1}^p = \rho (h_j - w_j)$ measures the misspecification errors. The model can be re-written by

$$ y_{j,t} = x_t^\top (\rho w_j + \delta_j) + \epsilon_{j,t}. \quad (9) $$
Without loss of generality, we assume there exists \( k \in \{ k : w_{j,k} \neq 0 \} \) such that \( \delta_{j,k} = 0 \) (one of such \( k \)'s is known and to avoid multicollinearity the corresponding element is eliminated from the covariates of \( x_t^T \delta_j \) in the regression \((9)\)). Similarly, by letting \( B_j \beta_j^0 = \rho w_j + \delta_{j} \), we have a linear model in the form of \((7)\) given by

\[
y_{j,t} = x_t^T B_j \beta_j^0 + \varepsilon_{j,t}, \quad j = 1, \ldots, p.
\]

In this case, \( K'_j = \dim(x_{j,t}) = p \), \( B_j \) is a \( p \times p \) matrix and \( \beta_j^0 \) is a \( p \times 1 \) vector, for \( j = 1, \ldots, p \).

**Example 2** (Spatial network). Consider \( x_{j,t} = y_{-j,t} = (y_{k,t})_{k \neq j} \in \mathbb{R}^{p-1} \) and suppose we have a prefixed network structure \( w_j^T y_t \) with \( w_j = (w_{j,k})_{k=1}^p \), for all \( j = 1, \ldots, p \). The spatial network model is given by

\[
y_{j,t} = \rho w_j^T y_t + \delta_j^T y_t + \varepsilon_{j,t}, \quad j = 1, \ldots, p,
\]

where \( |\rho| < 1 \) and \( \delta_j^T = (\delta_{j,k})_{k=1}^p \) measures the misspecification error. By convention we let \( w_{j,j} = 0 \) and assume \( \delta_{j,j} = 0 \), for all \( j \). Let \( y_t = (y_{1,t}, \ldots, y_{p,t})^T \), \( \varepsilon_t = (\varepsilon_{1,t}, \ldots, \varepsilon_{p,t})^T \), we can re-write the model as

\[
y_t = \rho W y_t + \Delta y_t + \varepsilon_t,
\]

where \( W \) and \( \Delta \) are \( p \times p \) matrices with the \( j \)-th row given by \( w_j^T \) and \( \delta_j^T \), respectively. Without loss of generality, suppose there exist \( j, j' \) (\( j \neq j' \)) such that \( W_{j,j'} \neq 0 \) and \( \Delta_{j,j'} = 0 \).

Denote by \( \varepsilon_j \) the \( p \times 1 \) unit vector with the \( j \)-th element equals 1. Define \( X_t = [e_j^T \otimes y_t]^p_{j=1} (p \times p^2) \), \( \tilde{B}_p^x \times (p^2 + 1) = ([e_j^T \otimes 1_p]_{j=1}^p, [w_j^T]_{j=1}^p)^T \), \( \beta = (\delta_1^T, \ldots, \delta_p^T, \rho)^T \), where the notation \( [A_{j}]^p_{j=1} \) indicates we stack \( A_j \) by rows over \( j = 1, \ldots, p \). The model can be expressed by

\[
y_t = X_t B \beta^0 + \varepsilon_t,
\]

where \( B \) is \( \tilde{B} \) with the \( (jp+j') \)-th column eliminated and \( \beta^0 \) is \( \tilde{\beta} \) with the \( (jp+j') \)-th element removed. In this example, \( X_t \) are the original covariates and \( X_t B \) are the transformed covariates.

**Example 3** (Multiple regression with autoregressive lags). Consider \( x_{j,t} = y_{t-1} = (y_{j,t-1})_{j=1}^p \in \mathbb{R}^p \) and suppose we have a prefixed lagged network structure \( w_j^T y_{t-1} \) with \( w_j = (w_{j,k})_{k=1}^p \), for all \( j = 1, \ldots, p \). The regression model is given by

\[
y_{j,t} = \rho_j y_{t-1}^T w_j + y_{t-1}^T \delta_j + \varepsilon_{j,t}, \quad j = 1, \ldots, p,
\]
where $\delta_j = (\delta_{j,k})_{k=1}^p$ reflects the misspecification error. Suppose it is known that $\delta_{j,j} = 0$ while $w_{j,j} \neq 0$, for all $j$. Then we have a linear model in the form of (7) given by

$$y_{jt} = y_{t-1}B_j\beta_0^j + \varepsilon_{jt}, \quad j = 1, \ldots, p.$$  

In this case, $B_j = (w_j, I_{p,j})$ is a $p \times p$ matrix where $I_{p,j}$ is $I_p$ with the $j$-th column eliminated, and $\beta_0^j = (\rho_j, \delta_{j,j}^\top)$ is a $p \times 1$ vector with $\delta_{j,j} = (\delta_{j,k})_{k \neq j} \in \mathbb{R}^{p-1}$.

Let $\varepsilon_t = (\varepsilon_{jt})_{j=1}^p \in \mathbb{R}^p$, $X_t = [e_j^\top \otimes y_{t-1}]_{j=1}^p$, where $e_j$ is the $p \times 1$ unit vector with the $j$-th element equals 1. The model can be expressed as

$$y_t = X_tB\beta_0 + \varepsilon_t,$$

where $B$ is a block diagonal matrix whose $j$-th block is given by the $p \times p$ matrix $B_j$ for $j = 1, \ldots, p$, and $\beta_0 = [\beta_{j1}]_{j=1}^p \in \mathbb{R}^{p^2}$. Again, in this multiple regression model $X_t$ and $X_tB$ are the original and transformed covariates respectively.

REMARK 2.2 (Unobserved heterogeneity in the error terms). Suppose the error term $\varepsilon_{jt}$ contains an unobserved component $\alpha_j$, $\varepsilon_{jt} = \alpha_j + u_{jt}$, where the idiosyncratic error $u_{jt}$ is assumed to be uncorrelated with $\alpha_j$ for all $j$ and $t$. It is known that any attempt to estimate $\beta_0^j$ directly will render inconsistent estimators if $E(\alpha_j|x_{jt}) \neq 0$. Standard panel data transformation techniques can be used to remove the heterogeneity term from the statistical model of interest. We note that the assumptions (A5)-(A6) will remain to be held under such transformation. In particular, if the dependence adjusted norm (defined in Definition 3.1) of $x_{jt}$ has a decay rate, we can preserve this property after taking difference.

Moreover, in some cases, estimating $\alpha_j$ in terms of $x_{jt}$ is of special interests, e.g. in the correlated random effects models. One can follow the method of Chamberlain (1982) by considering the specification:

$$E(\alpha_j|x_{j1}, \ldots, x_{jn}, \pi_{j0}, \ldots, \pi_{jL}, \nu_j) = \sum_{\ell=0}^L \pi_{j,\ell}x_{j,\ell-t} + \nu_j, \quad E(\nu_j|x_{jt}) = 0, \quad t = L + 1, \ldots, n.$$

REMARK 2.3 (Connection to Directed Acyclic Graph (DAG)). Our model is connected to the causal graph inference. In a system of equations, a structural causal model (SCM) can be linked to our setting when we express
in the form of an additive noise model (ANM), $y_{j,t} = \sum_{k=1}^{K_j} x_{jk,t} \beta_{j,k} + \epsilon_{j,t} =: G_j(x_{j,t}; \epsilon_{j,t})$ where $x_{j,t} = (x_{jk,t})_{k=1}^{K_j}$, with the identification assumption $E(x_{j,t} \epsilon_{j,t}) = 0$. In this way we specify a linear SCM. And $x_{jk,t}$ can also denote the basis functions in series expansion. The model class is restricted to linear and nonlinear ANMs. The graphs are composed by nodal responses $y_{j,t}$ and $x_{jk,t}$. For example, suppose $x_{j,t}$ is only a vector of the lags $y_{j,t-1}$. The distribution induced by the model is Markov to a DAG. Suppose that there is no instantaneous effect, namely $x_{j,t}$ only consists of lagged information compared to $y_{j,t}$. In the Gaussian case, $b_{j,k}$’s directly link to a causal relationship with conditions $E(x_{j,t} \epsilon_{j,t}) = 0$ for both linear and nonlinear models. In the non-Gaussian case, since the causal relationship is not reversed in time, we can directly learn a unique DAG utilizing conditional independence (CI) test. There are many ways to conduct CI test, and one way is based on the residuals obtained from Danzig regression. As we have an additive noise model, we can test the independence of $\hat{\epsilon}_{j,t}$ with respect to $x_{j,t}$, the details can be found in Heinze-Deml et al. (2018).

When there are instantaneous effects, for example $x_{j,t}$ includes the set of variables $(y_{j',t})_{j' \neq j}$. In terms of models with spatial lags included, the distribution induced by the structural model is the same as the distribution induced from the reduced form model. See De Paula et al. (2018) for the detailed conditions of identification of the parameters. Thus the reduced form model induce a distribution which can be Markov and faithful to a DAG. Thus, the causal relationships are better drawn from the reduced form model. Models with distributed lags are also tricky since the CI test will not be able to pin down the directions of the instantaneous effects. One can restrict to nonlinear ANM models or non-Gaussian ANM models to obtain the DAG identifiability, see Chapter 10 of Peters et al. (2017).

2.3 Estimation

Recall the system of linear regression equations given in (7). Denote by $\tilde{x}_{j,t}^\top \in \mathbb{R}^{K_j}$ the transformed covariates $x_{j,t}^\top B_j$ in the $j$-th equation. We shall estimate the unknown parameters $\theta^0 = [\beta_j^0]_{j=1}^{p_j} \in \mathbb{R}^K$ ($K = \sum_{j=1}^{p_j} K_j$), where $\theta^0$ is assumed to be sparse.

As we allow for endogeneity in $x_{j,t}$, we need to introduce the instrument variables (IVs) $z_t = [z_{j,t}]_{j=1}^{p} \in \mathbb{R}^q$ with $q = \sum_{j=1}^{p_j} q_j \geq K$. In particular, $z_{j,t} \in \mathbb{R}^{q_j}$ contains the IVs for the $j$-th equation such that $E(\epsilon_{j,t}^\top z_{j,t}) = 0$.

For each $j = 1, \ldots, p$, we consider a vector valued score function $g_j(D_{j,t}, \theta)$
mapping $\mathbb{R}^{K_j+q_j+1} \times \mathbb{R}^{q_j}$ into $\mathbb{R}^{q_j}$, where $D_{j,t} \defeq (y_{j,t}, \tilde{x}_{j,t}, z_{j,t})^\top$. Thus, the moment functions are given by

$$g_j(\theta) = E[g_j(D_{j,t}, \theta)],$$

and $g_j(\theta^0) = 0$. In particular, for the case with linear moments, we have $g_j(D_{j,t}, \theta) = z_{j,t} \varepsilon_j(D_{j,t}, \theta) = y_{j,t} - \tilde{x}_{j,t} \beta_j$. By stacking the moment functions over equations by rows, we let $g(\theta) = [g_j(D_{j,t}, \theta)]_{j=1}^p$.

Suppose there are two parts in $\theta^0$: the parameters of interests $\theta_0^1 \in \mathbb{R}^{K(1)}$ and the nuisance parameters $\theta_0^2 \in \mathbb{R}^{K(2)}$. Let $G_1 = \partial_{\theta_1^1} g(\theta_1^1, \theta_2^0)_{|\theta_2^0=\theta_0^2}$ and $G_2 = \partial_{\theta_2^0} g(\theta_1^1, \theta_2^0)_{|\theta_2^0=\theta_0^2}$. Denote the covariance matrix of the scores by $\Omega = E[g(D_t, \theta_1^1, \theta_2^0)g(D_t, \theta_1^0, \theta_2^0)]^\top$, where $D_t = [D_{j,t}]_{j=1}^p \in \mathbb{R}^{K}$ and $g(D_t, \theta_1, \theta_2) = [g_j(D_{j,t}, \theta_1, \theta_2)]_{j=1}^p \in \mathbb{R}^{q}$. The estimation will be carried out by two steps:

1. Define $\Theta$ as a $s$-sparse parameter space with $\theta \neq 0$. We consider a Dantzig type of regularization to estimate $\theta_0^1$ by following Belloni et al. (2018). Let $\lambda_n > 0$, the Regularized Minimum Distance (RMD) estimator $\hat{\theta} = (\hat{\theta}_1^1, \hat{\theta}_2^0)^\top$ is given by

$$\hat{\theta} = \arg \min_{\theta \in \Theta} \|\theta\|_1 \quad \text{such that} \quad |\hat{g}(\theta)|_\infty \leq \lambda_n, \tag{10}$$

where $\hat{g}(\theta) = \hat{g}(\theta_1, \theta_2) = [n^{-1} \sum_{t=1}^n g_j(D_{j,t}, \theta_1, \theta_2)]_{j=1}^p$.

2. For partialling out the effect of the nuisance parameters, we first consider the moment functions given by

$$\tilde{g}(\theta_1, \theta_2) = \{I_q - G_2 P(\Omega, G_2)\} g(\theta_1, \theta_2),$$

where $P(\Omega, G_2) = (G_2^\top \Omega^{-1} G_2)^{-1} G_2^\top \Omega^{-1}$. It follows that $\tilde{g}(\theta_1^0, \theta_0^2) = 0$ and the Neyman orthogonality property $\partial_{\theta_2^0} \tilde{g}(\theta_1, \theta_2)_{|\theta_2^0=\theta_0^2} = 0$ is satisfied. Moreover, to construct the approximate mean estimators, we further consider the moment functions given by

$$M(\theta_1, \theta_2; \gamma) = G_1^\top \Omega^{-1} \{I_q - G_2 P(\Omega, G_2)\} G_1 (\theta_1 - \gamma) + G_1^\top \Omega^{-1} \tilde{g}(\gamma, \theta_2)$$

$$= G_1^\top \Omega^{-1} \{I_q - G_2 P(\Omega, G_2)\} \{G_1 (\theta_1 - \gamma) + g(\gamma, \theta_2)\},$$

which satisfy $M(\theta_1^0, \theta_0^1; \theta_0^1) = 0$ and $\partial_{\gamma} M(\theta_1^0, \theta_0^1; \gamma)|_{\gamma=\theta_0^0} = 0$. This motivates us to update the estimator on the target parameters in the form of

$$\hat{\theta}_1 - [\hat{G}_1^\top \hat{\Omega}^{-1} \{I_q - \hat{G}_2 P(\hat{\Omega}, \hat{G}_2)\} \hat{G}_1]^{-1} \hat{G}_1^\top \hat{\Omega}^{-1} \{I_q - \hat{G}_2 P(\hat{\Omega}, \hat{G}_2)\} \hat{g}(\hat{\theta}_1, \hat{\theta}_2),$$

14
where $\hat{\Omega} = n^{-1} \sum_{t=1}^{n} \{g(D_t, \hat{\theta}_1, \hat{\theta}_2)g(D_t, \hat{\theta}_1, \hat{\theta}_2)^\top\}$, $\hat{G}_1$ and $\hat{G}_2$ are thresholding estimators for $G_1$ and $G_2$, respectively. In particular, let $\hat{G}_{1,ij} = \hat{G}_{1,ij}^1 I\{|\hat{G}_{1,ij}^1| > T_1\}$ with $\hat{G}_{1,ij}^1 = \partial_{\theta_{ij}} \hat{g}(\hat{\theta}_1, \hat{\theta}_2)|_{\theta_1=\hat{\theta}_1}$ (the selection of the threshold will be discussed in the proof of Lemma A.10), and similarly for $\hat{G}_2$ with $\hat{G}_{2,ij}^1 = \partial_{\theta_{ij}} \hat{g}(\hat{\theta}_1, \hat{\theta}_2)|_{\theta_2=\hat{\theta}_2}$.

It is worth noting that in the high-dimensional setting $\hat{\Omega}$ is singular due to the rank deficiency. A regularized estimator should be used. In particular, we shall consider the constrained $\ell_1$-minimization for inverse matrix estimation (CLIME, see Cai et al., 2011). Define $\Upsilon_0 \overset{\text{def}}{=} \Omega^{-1}$ and let $\hat{\Upsilon}_1 = (\hat{\upsilon}_{ij}^1)$ be the solution of

$$
\min_{\Upsilon \in \mathbb{R}^{q \times q}} \sum_{i=1}^{q} \sum_{j=1}^{q} |\Upsilon_{ij}| : \quad |\hat{\Upsilon} \Upsilon - I_q|_{\max} \leq \ell_n^\Upsilon,
$$

(11)

where $|\cdot|_{\max}$ is the element-wise max norm of a matrix, and $\ell_n^\Upsilon > 0$ is a tuning parameter. A further symmetrization step is taken by

$$
\hat{\Upsilon} = (\hat{\upsilon}_{ij}), \quad \hat{\upsilon}_{ij} = \hat{\upsilon}_{ji} = \hat{\upsilon}_{ij}^1 I\{\hat{\upsilon}_{ij}^1 \leq |\hat{\upsilon}_{ji}^1|\} + \hat{\upsilon}_{ji}^1 I\{|\hat{\upsilon}_{ij}^1| > |\hat{\upsilon}_{ji}^1|\}.
$$

(12)

Likewise, define $\Pi_0 \overset{\text{def}}{=} (G_1^\top \Upsilon_0 G_1)^{-1}$, $\Xi_0 \overset{\text{def}}{=} (G_2^\top \Upsilon_0 G_2)^{-1}$. We shall use the same approach to approximate the inverse of $\hat{G}_1^\top \hat{\Upsilon} \hat{G}_1$ and $\hat{G}_2^\top \hat{\Upsilon} \hat{G}_2$ by $\hat{\Pi}$ and $\hat{\Xi}$, respectively.

Finally, we let $G_1^\top \Upsilon_0 (I_q - G_2 \Xi_0 G_2^\top \Upsilon_0) G_1 =: D + F$, where $D \overset{\text{def}}{=} G_1^\top \Upsilon_0 G_1 = (\Pi_0)^{-1}$ and $F \overset{\text{def}}{=} G_1^\top \Upsilon_0 G_2 \Xi_0 G_2^\top \Upsilon_0 G_1$. By using the formula $(D + F)^{-1} = D^{-1} - D^{-1}(I + FD^{-1})^{-1}FD^{-1}$, the debiased estimator $\hat{\theta}_1$ is obtained by

$$
\hat{\theta}_1 = \hat{\theta}_1 - \{\hat{\Pi} - \hat{\Pi}(I_q + \hat{F} \hat{\Pi})^{-1} \hat{F} \hat{\Pi}\} \hat{G}_1^\top \hat{\Upsilon} (I_q - G_2 \Xi_2 \hat{\Upsilon} \hat{G}_2 \hat{\Upsilon}) \hat{g}(\hat{\theta}_1, \hat{\theta}_2),
$$

(13)

where $\hat{F} = \hat{G}_1^\top \hat{\Upsilon} \hat{G}_2 \Xi_2 \hat{G}_2^\top \hat{\Upsilon} \hat{G}_1$. We shall analyze the convergence rates of the estimators involved in handling the rank deficiency issues in Appendix A.2.

In this step, we will also conduct simultaneous inference on the parameters of interests $\theta_1^0$.

REMARK 2.4 (Common parameters across equations). In some cases, the parameters of interests are shared across equations. We therefore propose to add a third step to achieve a $\sqrt{np}$ rate. In this case, the parameters of interests are the common shared parameters, and we can leave the other
parameters to the nuisance parameters. The endogeneity assumption can be specified as \( E(\varepsilon_{j_1,t} | z_{j_2,t}) = 0 \) for all \( j_1, j_2 = 1, \ldots, p \).

**Example 4** (Spatial network (Example 2 continued)). We extend the spatial network model in Example 2 by including a set of equation specific exogenous variables \( X_{j,t} \), which is of a fixed dimension \( L \), for \( j = 1, \ldots, p \). The model then becomes

\[
y_{j,t} = \rho w_{j}^{\top} y_t + \delta_j^T y_t + \gamma^T X_{j,t} + \varepsilon_{j,t}, \quad j = 1, \ldots, p.
\]

In the first step, the target moment equations in the linear case are given by \( g_j(\theta_0^1, \theta_0^2) = E\{ (y_{j,t} - \rho w_{j}^{\top} y_t - \delta_j^T y_t - \gamma^T X_{j,t}) z_{j,t} \} = 0 \). As in Example 2 we assume there exist \( j, j' \) (\( j \neq j' \)) such that \( w_{j,j'} \neq 0 \) and \( \delta_{j,j'} \neq 0 \). Here \( \theta_1^1 = [\delta_j]_{j=1}^p \) with \((jp + j')\)-th element eliminated and \( \theta_2^1 = (\rho, \gamma^\top)^\top \). Let \( \tilde{G}_1 \) be a block diagonal matrix whose \( j \)-th block is given by \(-E\{z_{j,t} y_{j,t}\}\) for \( j = 1, \ldots, p \). In this model the gradients \( G_1 \) is just \( \tilde{G}_1 \) with the \((jp + j')\)-th column eliminated, and \( G_2 = -E\{z_{j,t}(X_{j,t}^\top, w_{j}^{\top} y_t)\}_{j=1}^p \).

Once we finish the two-step estimation and obtain the debiased estimator \( \tilde{\theta}_1 \), we need to re-estimate the common parameters with incorporating the misspecification error back. We shall introduce another set of IVs \( \tilde{z}_{j,t} \), which is of dimension no less than \( K^{(2)} = L + 1 \), for \( j = 1, \ldots, p \). And then implement the third step as follows.

3. Plug in the debiased estimator \( \tilde{\theta}_1 \) and re-estimate the common parameters by

\[
\tilde{\theta}_2 = \left\{ \left( \sum_{i,j} \tilde{X}_{j,t} \tilde{z}_{j,t}^\top \right)^{-1} \left( \sum_{i,j} \tilde{z}_{j,t} \tilde{X}_{j,t}^\top \right) \left( \sum_{i,j} \tilde{z}_{j,t} \tilde{y}_{j,t}^\top \right) \right\}^{-1}
\]

where \( \tilde{X}_{j,t} = (X_{j,t}^\top, w_{j}^{\top} y_t)^\top \) and \( \tilde{y}_{j,t} = y_{j,t} - \delta_j y_t \), with \( \delta_j \) achieved as part of \( \tilde{\theta}_1 \) from step 2.

**REMARK 2.5.** (Endogenous networks) There is a literature in spatial econometrics addressing endogenous networks. In particular, it is assume that \( W \) is endogenously generated, see e.g. Qu and Lee (2015). If \( W \) is endogenously generated. As a result, it may rend the standard IV in the spatial literature invalid. Using the control function approach will effectively address the endogeneity of the adjacency matrices. There is also a significant literature modeling the endogenous network formation, see e.g. Auerbach (2016).
Our framework can use these pre-estimated network structures and shrink sparsely to the pre-estimated structure.

3 Main results

3.1 Consistency of the RMD estimator \( \hat{\theta} \)

To establish the consistency of the RMD estimator \( \hat{\theta} \), we will use the following assumptions, which follow directly from Belloni et al. (2018).

We first denote by \( R(\theta_0) \) \( \overset{\text{def}}{=} \{ \theta \in \Theta : |\theta|_1 \leq |\theta_0|_1 \} \) the restricted set. Let \( \epsilon_n \downarrow 0, \delta_n \downarrow 0 \) be sequences of positive constants.

(A1) (Concentration)

\[
\sup_{\theta \in R(\theta_0)} |\hat{g}(\theta) - g(\theta)|_\infty \leq \epsilon_n
\]

holds with probability at least \( 1 - \delta_n \).

(A2) (Identification) The target moment function \( g \) satisfies the identification condition:

\[
\{ \theta \in R(\theta_0), |g(\theta) - g(\theta_0)|_\infty \leq \epsilon \} \text{ implies } |\theta_0 - \theta|_a \leq \rho(\epsilon; \theta_0, a),
\]

for all \( \epsilon > 0 \), where \( \epsilon \mapsto \rho(\epsilon; \theta_0, a) \) is a weakly increasing function mapping from \( [0, \infty) \) to \( [0, \infty) \) such that \( \rho(\epsilon; \theta_0, a) \to 0 \) as \( \epsilon \to 0 \).

(A3) The regularized parameter \( \lambda_n \) is selected so that

\[
|\hat{g}(\theta_0)|_\infty \leq \lambda_n
\]

holds with probability at least \( 1 - \alpha \).

We note that the assumption [A3] implies that \( \theta_0 \) is feasible for the problem in (10) with probability at least \( 1 - \alpha \), and thus, \( \hat{\theta} \in R(\theta_0) \), if a solution \( \hat{\theta} \) to the problem exists.

Consider the event \( \{ |\hat{g}(\theta_0)|_\infty \leq \lambda_n, \hat{\theta} \in R(\theta_0), |\hat{g}(\hat{\theta}) - g(\hat{\theta})|_\infty \leq \epsilon_n \} \). By (A1) [A3] and the union bound, we have this event holds with probability at least \( 1 - \alpha - \delta_n \). Moreover, on this event, by the definition of the RMD estimator in (10), it follows that

\[
|g(\hat{\theta}) - g(\theta_0)|_\infty = |g(\hat{\theta})|_\infty \\
\leq |g(\hat{\theta}) - \hat{g}(\hat{\theta})|_\infty + |\hat{g}(\hat{\theta})|_\infty \\
\leq \epsilon_n + \lambda_n,
\]
where the first equality is due to \( g(\theta^0) = 0 \). Provided (A2) is satisfied for some \( a \), we obtain the bounds on the estimation error \( |\hat{\theta} - \theta^0|_a \leq \rho(\epsilon_n + \lambda_n; \theta^0, a) \) with probability \( 1 - \alpha - \delta_n \).

To further analyze the convergence rate of the RMD estimator \( \hat{\theta} \), we shall consider two different assumptions on the sparsity of the true parameter \( \theta^0 \).

\((A4.i)\) (Exactly Sparse) There exists \( T \subset \{1, \ldots, K\} \) with cardinality \( |T| = s = o(n) \) such that \( \theta^0_j \neq 0 \) only for \( j \in T \).

\((A4.ii)\) (Approximately Sparse) For some \( A > 0 \) and \( \bar{a} > 1/2 \), the absolute values of the parameters \( (|\theta^0_j|)_{j=1}^K \) can be rearranged in a non-increasing order \( (|\theta^0_j^*|)_{j=1}^K \) such that \( |\theta^0_j^*| \leq Aj^{-\bar{a}}, j = 1, \ldots, K \).

**Remark 3.1.** We note that the case \((A4.ii)\) can be reformulated to \((A4.i)\). Suppose \( \theta^0 \) is approximately sparse and denote by \( \theta^0[j] \) the value of the true parameter that corresponds to \( |\theta^0_j^*| \) which is defined in \((A4.ii)\). We shall sparsify \( \theta^0 \) to \( \theta^0(\tau) \). In particular, for each \( j = 1, \ldots, K, \theta^0_j(\tau) = \text{sign}(\theta^0[j])\hat{\theta}_j(\tau), \) where \( \tau \) makes \( s = \lfloor (A/\tau)^{1/\bar{a}} \rfloor = o(n) \) and \( s > 1, \delta = \sum_{j=1}^K |\theta^0[j]| \mathbb{1}(Aj^{-\bar{a}} \leq \tau) \). Then, we have

\[
|\theta^0(\tau)|_1 = \sum_{j=1}^s |\theta^0[j]^* + \frac{\delta s}{s-1} = \sum_{j=1}^s |\theta^0[j]^* + \frac{s}{s-1} \sum_{j=s+1}^K |\theta^0[j]^* \geq |\theta^0|_1.
\]

It follows that \( \mathcal{R}(\theta^0) \subseteq \mathcal{R}(\theta^0(\tau)) \).

Suppose we focus on the case of linear moment with \( g(\theta) = G\theta + g(0) \) and \( \hat{g}(\theta) = \hat{G}\theta + \hat{g}(0) \), where \( G = \partial_{\theta^\top} g(\theta)|_{\theta = \theta^0} \) and \( \hat{G} = \partial_{\theta^\top} \hat{g}(\theta)|_{\theta = \theta^0} \). We shall verify the conditions on concentration and identification in the following two subsections.

### 3.1.1 Concentration

In this subsection, we discuss the condition needed to ensure the concentration condition in the previous subsection for the linear case. We first observe
\[
\sup_{\theta \in \mathcal{R}(\theta^0)} |\hat{g}(\theta) - g(\theta)|_{\infty},
\]
\[
= \sup_{\theta \in \mathcal{R}(\theta^0)} |(\hat{G} - G)\theta|_{\infty} + |\hat{g}(0) - g(0)|_{\infty},
\]
\[
\leq \sup_{\theta \in \mathcal{R}(\theta^0)} |\theta|_1 |\hat{G} - G|_{\text{max}} + |\hat{g}(0) - g(0)|_{\infty},
\]
\[
\leq |\theta^0|_1 |\hat{G} - G|_{\text{max}} + |\hat{g}(0) - g(0)|_{\infty},
\]
where $|\cdot|_{\text{max}}$ is the element-wise max norm of a matrix.

To analyze the rate of $|\hat{G} - G|_{\text{max}}$ and $|\hat{g}(0) - g(0)|_{\infty}$, a few assumptions and definitions are required to characterize the dependency observed in the data processes. We shall impose a few conditions on the aggregated dependence adjusted norm as follows.

\section*{Remark 3.2.} The above condition restricts the dependency structure of the error term. For simplicity we assume that the error term behaves like a martingale difference with respect to the filtration $\mathcal{F}_{t-1}$. Moreover, we pose some structure on the conditional variance-covariance matrix to simplify the derivation. It would be possible to extend the setting to a more complicated structure, e.g. factor structure. If the known factor is not correlated with the instrumental variables, the steps remains to be the same as in Section 2.3.

Alternatively, we can partial out the known factors as follows.

As an example, we extend the spatial network model in Example 3 by including some common factor $f_t$, which is of dimension $L$. Denote $\mathbb{Y}_{p \times n} \overset{\text{def}}{=} (y_1, \cdots, y_n)$, $\boldsymbol{\varepsilon}_{p \times n} \overset{\text{def}}{=} (\varepsilon_1, \cdots, \varepsilon_n)$, $\mathbb{F}_{L \times n} \overset{\text{def}}{=} (f_1, \cdots, f_n)$. The model then becomes
\[
\mathbb{Y} = \rho \mathbb{W} \mathbb{Y} + \Delta \mathbb{Y} + \Gamma \mathbb{F} + \boldsymbol{\varepsilon},
\]
where $\Gamma_{p \times L}$ contains the factor loadings. Denote the projection matrix

$$P_F = I_n - F^T (FF^T)^{-1} F.$$  

Then, to partial out $F$, we transform the model by

$$YP_F = \rho W Y P_F + \Delta Y P_F + F F P_F + \epsilon P_F,$$

due to $FP_F = 0$. Another alternative approach one can use is the generalized Helmert transformation as considered in Kuersteiner and Prucha (2020).

The proof shall be extended to conditional on the filtration corresponding to the factors.

**Definition 3.1.** Let $\xi_0, \eta_0$ be replaced by their i.i.d. copies $\xi^*_0, \eta^*_0$, and $
\tilde{x}_{j,k,t} = f_{jk}(\ldots, \xi_{t-1}^*, \ldots, \eta_{t-1}, \xi_t, \xi_t)$. For $r \geq 1$, define the functional dependence measure $\delta_{r,j,k,t} = \|\tilde{x}_{j,k,t} - \tilde{x}_{j,k,t}^*\|_r$, which measures the dependency of $\xi_0$ and $\eta_0$ on $\tilde{x}_{j,k,t}$. Also define $\Delta_{d,r,j,k} = \sum_{t=0}^{\infty} \delta_{r,j,k,t}$, which accumulates the effects of $\xi_0$ and $\eta_0$ on $\tilde{x}_{j,k,t \leq d}$. Moreover, the dependence adjusted norm of $\tilde{x}_{j,k,t}$ is denoted by $\|\tilde{x}_{j,k,t}\|_{r,\xi} = \sup_{d \geq 0} (s+1)^{\xi} \Delta_{d,r,j,k}$ where $\xi > 0$. Similarly, we can define $\|z_{jm}\|_{r,\xi}$ and $\|\tilde{x}_{j,k,m}\|_{r,\xi}$ in the same fashion.

(A6) $\|\tilde{x}_{j,k,t}\|_{r,\xi} < \infty$ and $\|z_{jm}\|_{r,\xi} < \infty$ ($r \geq 8$) for all $j = 1, \ldots, p$, and $k = 1, \ldots, K_j, m = 1, \ldots, q_j$.

The above condition is essentially assuming sufficiency decay rate of dependence.

**Remark 3.3.** We note that there is a more general way to define the dependence adjusted norm. Let $
\tilde{x}_{j,k,t}^*(\ell) = f_{jk}(\ldots, \xi_{t-\ell}^*, \eta_{t-\ell}, \ldots, \eta_t, \xi_t)$ where $\xi_{t-\ell}, \eta_{t-\ell}$ are replaced by their i.i.d. copies $\xi^*_{t-\ell}, \eta^*_{t-\ell}$. The functional dependence measure is denoted by $\delta_{r,j,k,t}(\ell) = \|\tilde{x}_{j,k,t} - \tilde{x}_{j,k,t}^*(\ell)\|_r$ and define $\Delta_{d,r,j,k} = \max_t \sum_{\ell=0}^{\infty} \delta_{r,j,k,t}(\ell)$ which measure the cumulative effects. Some non-stationary time series cases can also be covered under the assumption that $\|\tilde{x}_{j,k,t}\|_{r,\xi} = \sup_{d \geq 0} (d+1)^{\xi} \Delta_{d,r,j,k} < \infty$.

For each equation $j$, we aggregate the dependence adjusted norm of the vector of processes $\tilde{x}_{j,t}$ by $\|\tilde{x}_{j,\cdot}\|_{r,\xi} = \sup_{d \geq 0} (d+1)^{\xi} \Delta_{d,r,j}$ where $\Delta_{d,r,j} = \sum_{t=0}^{\infty} \|\tilde{x}_{j,t} - \tilde{x}_{j,t}^*\|_{r,\xi}$. Likewise, we can define $\|\tilde{x}_{j,\cdot}, z_{jm}\|_{r,\xi}$. Moreover, we aggregate over $j = 1, \ldots, p$ by $\max_j \|\tilde{x}_{j,\cdot}\|_{r,\xi} = \sup_{d \geq 0} (d+1)^{\xi} \Delta_{d,r}$ where $\Delta_{d,r} = \sum_{t=0}^{\infty} \max_j \|\tilde{x}_{j,t} - \tilde{x}_{j,t}^*\|_{r,\xi}$. The definition for $\max_j \|\tilde{x}_{j,\cdot}, z_{jm}\|_{r,\xi}$ follows similarly.
To apply the concentration inequality in Lemma A.4, we define the following quantities: $\Phi^x_{r,\varsigma} = \max_{j,k} \|\tilde{x}_{jk,\cdot}\|_{r,\varsigma}$, $\Phi^{xz}_{r,\varsigma} = \max_{j,m} \|\tilde{x}_{jm,\cdot}\|_{r,\varsigma}$, and $\Phi^{yz}_{r,\varsigma} = \max_{j,k,m} \|\tilde{x}_{jm,\cdot}\|_{r,\varsigma}$, which are all assumed to be bounded by constants. Let $\Phi^{xz}_{r,\varsigma} = \max_{j,m} \|y_j,z_{jm,\cdot}\|_{r,\varsigma}$. Recall the system of regression equations given by $y_{j,t} = \tilde{x}_{j,t} \beta_j(0) + \epsilon_{j,t}$. It is not hard to see that $\|y_j,z_{jm,\cdot}\|_{r,\varsigma} \leq \|\tilde{x}_{j,\cdot}\|_{r,\varsigma} |\beta_j(0)|_1 + \|\epsilon_{j,\cdot}\|_{r,\varsigma}$, which implies

$$\Phi^{yz}_{r,\varsigma} \leq \max_{j,m} \|y_j,z_{jm,\cdot}\|_{r,\varsigma} |\beta_j(0)|_1 + \Phi^{xz}_{r,\varsigma},$$

$$\|\max_{j,m} |y_j,z_{jm,\cdot}\|_{r,\varsigma} \leq \|\max_{j,m} |\tilde{x}_{j,\cdot}\|_{r,\varsigma} |\beta_j(0)|_1 + \|\max_{j} \epsilon_{j,\cdot}\|_{r,\varsigma},$$

where $|\beta_j(0)|_1 \leq s$ given the sparsity assumption.

We define $b_n = cn^{-1/2}(\log P_n)^{1/2} \Phi^{xz}_{r,\varsigma} + cn^{-1} c_{n,\varsigma}(\log P_n)^{3/2} \max_{j,m} |\tilde{x}_{j,\cdot}|_{r,\varsigma}$, where $P_n = (K \cup q \cup n)$, $c_{n,\varsigma} = n^{1/r}$ for $\varsigma > 1/2 - 1/r$ and $c_{n,\varsigma} = n^{1/2-\varsigma}$ for $0 < \varsigma < 1/2 - 1/r$. By applying Lemma A.4 we obtain that $|G^1 - G|_{\max} \lesssim b_n$ holds with probability $1 - o(1)$ with sufficiently large $c$, where $G^1$ is the sample estimator of $G$ without thresholding. It can be easily seen that the same conclusion for $|\hat{G} - G|_{\max}$ follows given $|G|_{\max}$ is a constant. Similarly, we define $b_n' = cn^{-1/2}(\log P_n)^{1/2} \Phi^{xz}_{r,\varsigma} + cn^{-1} c_{n,\varsigma}(\log P_n)^{3/2} \max_{j,m} |y_j,z_{jm,\cdot}|_{r,\varsigma}$. It follows that $|\hat{g}(0) - g(0)|_{\infty} \lesssim b_n'$ holds with probability $1 - o(1)$ with sufficiently large $c$. The follow Lemma provides the desired concentration inequality in the linear case.

**Lemma 3.1** (Concentration for the linear moments model). Assume (A5) (A6) and (A4.i) (or (A4.ii)), then we have

$$\sup_{\theta \in \mathcal{R}(\theta^0)} |\hat{g}(\theta) - g(\theta)|_{\infty} \lesssim b_n s + b_n'|. \quad (14)$$

**Remark 3.4** (Discussion on the concentration rate). Suppose the dependence adjust norms $\Phi^{xz}_{r,\varsigma}$, $\max_{j,m} |\tilde{x}_{j,\cdot}|_{r,\varsigma}$, $\max_{j,m} \|y_j,z_{jm,\cdot}\|_{r,\varsigma}$ are all bounded by constants. For $\varsigma > 1/2 - 1/r$ (weak dependence case), if $n^{-1/2+1/r}(\log P_n) = O(1)$ for sufficiently large $r$, we have the concentration rate $b_n s + b_n' \lesssim (s+1)n^{-1/2}(\log P_n)^{1/2}$, which is of the same order as the rate shown in Lemma 3.3 of Belloni et al. (2018) for the i.i.d. data.

### 3.1.2 Identification

In this subsection, we show the necessary conditions of our estimation framework to ensure the identification condition. Denote $G_{H,I}$ as the sub-matrix of $G$ with rows and columns indexed respectively by the sets $H \subseteq \{1, \ldots, q\}$ and
$I \subseteq \{1, \ldots, K\}$, where $|I| \leq |H|$. Let $\sigma_{\min}(m, G) = \min_{|I| \leq m} \max_{|H| \leq m} \sigma_{\min}(G_{H,I})$ and $\sigma_{\max}(m, G) = \max_{|I| \leq m} \max_{|H| \leq m} \sigma_{\max}(G_{H,I})$ be the $m$-sparse smallest and largest singular values of $G$ ($m \geq s$), where $\sigma_{\min}(G_{H,I})$ and $\sigma_{\max}(G_{H,I})$ are the smallest and largest singular values of $G_{H,I}$ respectively. Recall that $G$ is a block diagonal matrix whose $j$-th block is given by the $q_j \times K_j$ matrix $G_{[j]} = -\mathbf{E}(z_{j,t} \tilde{x}_{j,t}^\top)$. In the following lemma, we show the singular values of sub-matrices of the transformed matrix $G$ are bounded under some conditions.

**Lemma 3.2.** Suppose we can express $G$ by $G = \Sigma^{xz} B$, where $\Sigma^{xz}$ is $q \times K$ and $B$ is $K \times K$. Let $V_B = \{ \xi : \xi = B I \xi_t, \xi_t^\top \xi = 1 \}$. Assume that there exist $c_1, c_2 > 0$ such that $\min_{|I| \leq m} \lambda_{\min}(B_I^\top B_I) > c_1$ and $\sigma_{\min}(m, \Sigma_{H,H}^{xz}) \defeq \min_{|I| \leq m} \max_{|H| \leq m} \xi_{\top} \Sigma_{H,H}^{xz} \xi$. Moreover, assume that there exist constants $C_1, C_2 > 0$ such that $\max_{|I| \leq m} \lambda_{\max}(\Sigma_{H,H}^{xz} \Sigma_{H,H}^{xz}) < C_1$ and $\max_{|I| \leq m} \lambda_{\max}(B_I^\top B_I) < C_2$.

Then, we have $\sigma_{\min}(m, G) > c'$ and $\sigma_{\max}(m, G) \leq C'$ for some constants $c', C' > 0$.

**Proof.** Similar to the proof of Lemma 2.1, we observe that

$$
\sigma_{\min}^2(\Sigma_{H}^{xz} B_I) = \lambda_{\min}(B_I^\top \Sigma_{H}^{xz} \Sigma_{H}^{xz} B_I) \geq \min_{\xi \in V_B} \frac{\xi_{\top} \Sigma_{H}^{xz} \Sigma_{H}^{xz} \xi}{\xi_{\top} \xi} \lambda_{\min}(B_I^\top B_I)
$$

$$
\sigma_{\max}^2(\Sigma_{H}^{xz} B_I) = \lambda_{\max}(B_I^\top \Sigma_{H}^{xz} \Sigma_{H}^{xz} B_I) \leq \lambda_{\max}(B_I^\top B_I) \lambda_{\max}(\Sigma_{H}^{xz} \Sigma_{H}^{xz}).
$$

Consequently, we have

$$
\min_{|I| \leq m} \max_{|H| \leq m} \sigma_{\min}^2(\Sigma_{H}^{xz} B_I) \geq \sigma_{\min}(m, \Sigma_{H}^{xz}) \min_{|I| \leq m} \lambda_{\min}(B_I^\top B_I)
$$

$$
\max_{|I| \leq m} \sigma_{\max}^2(\Sigma_{H}^{xz} B_I) \leq \max_{|I| \leq m} \lambda_{\max}(B_I^\top B_I) \max_{|H| \leq m} \lambda_{\max}(\Sigma_{H}^{xz} \Sigma_{H}^{xz}).
$$

It follows that for some constants $c', C' > 0$, we have $\sigma_{\min}(m, G) > c'$ and $\sigma_{\max}(m, G) < C'$.

For $a \geq 1$, we define $\kappa_A^G(s, u) \defeq \min_{|I| \leq s} \min_{\theta \in C_I(u), |\theta|_1 = 1} |G \theta|_{\infty}$, where $C_I(u) = \{ \theta \in \mathbb{R}^K : |\theta_{IC}|_1 \leq u |\theta_I|_1 \}$ with $u > 0$ and $I^C = \{1, \ldots, K\} \setminus I$. Given the boundedness of the singular values of sub-matrices of the transformed matrix $G$, we can show the identification condition, which is crucial for guaranteeing the rate of consistency.

22
Lemma 3.3 (Identification). Assume i) there exist constants $c', C' > 0$ such that $\sigma_{\min}(m, G) > c'$ and $\sigma_{\max}(m, G) < C'$, for $m \leq s(1 + u)^2 \log n$ with $u > 0$; ii) $b_n(1 + u)s \lesssim C(u)$ holds for large enough $n$, where $C(u) = \tilde{c}/(1 + u)^2$ and $\tilde{c}$ only depends on $c'$ and $C'$. Then, under $[A5]$, with probability approaching 1, we have

$$\kappa^G_a(s, u) \geq s^{-1/4}C(u), \ a \in \{1, 2\}.$$  

Proof. The proof follows that of Corollary 2 of Belloni et al. (2017) with the concentration inequality therein replaced by applying Lemma A.4 on the matrix $G$.

According to the triangle inequality, we have

$$|\hat{G}\theta|_{\infty}/|\theta|_a \geq -|\hat{G} - G|_{\infty}/||\theta||_a + |G\theta|_{\infty}/||\theta||_a =: -T_{n,1} + T_{n,2}.$$  

$T_{n,1}$ is handled by applying the concentration inequality in Lemma A.4. In particular, $|(\hat{G} - G)\theta|_{\infty}/|\theta|_a \leq |\hat{G} - G|_{\max}|\theta|_1/|\theta|_a$. Note that if $a = 1$, $|\theta|_1/|\theta|_a = 1$, and if $a = 2$, $|\theta|_1/|\theta|_a \leq (1 + u)s^{1/2}$. As we have shown in Section 3.1.1, $|\hat{G} - G|_{\max} \lesssim_P b_n$, thus we have $-T_{n,1} \lesssim_P b_n(1 + u)s^{-1/4}$.

The rest of the proof follows from Theorem 1 and Corollary 2 of Belloni et al. (2017). Provided $\sigma_{\min}(m, G) > c'$ and $\sigma_{\max}(m, G) < C'$, we can obtain the conclusion that $\kappa^G_a(s, u) \geq s^{-1/4}C(u)$ with $a \in \{1, 2\}$, given $b_n(1 + u)s \lesssim C(u)$ holds for sufficiently large $n$ and $C(u) = \tilde{c}/(1 + u)^2$. □

Lemma 3.3 implies that (A2) is satisfied immediately if $\rho(\epsilon_n + \lambda_n, \theta^0, a) \leq (\epsilon_n + \lambda_n)s^{1/4}C(u)^{-1}$ holds.

Theorem 3.1. Under the conditions of Lemma 3.3 and 3.1, we have the consistency of the estimator defined in (10) in the linear moments case

$$||\hat{\theta} - \theta^0||_a \lesssim (b_n s + b_n^r + \lambda_n)s^{1/4}C(u)^{-1} =: d_{n,a}, \ a \in \{1, 2\}$$  

holds with probability $1 - o(1)$, where $b_n(1 + u)s \lesssim C(u)$ (recall the assumption ii) in Lemma 3.3) for sufficiently large $n$ and $u > 0$. According to Corollary 5.1 of Chernozhukov et al. (2018b), the order of $\lambda_n$ is given by

$$n^{-1}\max_{j,m} \left( \|z_{jm, \varepsilon_j, c} \|_{2, c} (n \log q)^{1/2} \lor \|z_{jm, \varepsilon_j, c} \|_{r, c} (n \varpi_{n, c} q)^{1/r} \right),$$

where for $\varsigma > 1/2 - 1/r$, $\varpi_{n, c} = 1$; for $\varsigma < 1/2 - 1/r$, $\varpi_{n, c} = n^{r/2 - 1 - \varsigma r}$.

Proof. The conclusions is implied by combining the results in Lemma 3.1 and 3.3. □
**Remark 3.5** (Discussion on the consistency rate). As a continuation of Remark 3.4, additionally assume $\max_{j,m} \|z_{jm, \varepsilon_j} \|_{r, \varsigma}$ is bounded by constant. Again, for $\varsigma > 1/2 - 1/r$, we have $\lambda_n \lesssim n^{-1/2} (\log q)^{1/2}$, given that $(nq)^{1/r} \lesssim (n \log q)^{1/2}$, which implies if $r$ is large enough then $q$ can diverge as a polynomial rate of $n$ (there is a better dimension allowance of $q$ under stronger exponential moment conditions, see e.g. Comment 5.5 in Chernozhukov et al. (2018)). It follows that $d_{n,a} \lesssim (s + 2)s^{1/2} n^{-1/2} (\log P_n)^{1/2}$, which is of the same order as the rate for i.i.d. case studied in Theorem 3.1 of Belloni et al. (2018).

### 3.2 Inference on the debiased estimator $\hat{\theta}_1$

In this subsection we show the asymptotic properties of the debiased estimator $\hat{\theta}_1$ obtained in the second step as in (13). In particular, a key representation which linearizes the estimator for a proper application of the central limit theorem for inference is provided.

#### 3.2.1 Linearization

Define $A \overset{\text{def}}{=} G_1^\top \Omega^{-1} (I_q - G_2 P(\Omega, G_2))$ and $B \overset{\text{def}}{=} (A G_1)^{-1}$, where $P(\Omega, G_2) = (G_2^\top \Omega^{-1} G_2)^{-1} G_2^\top \Omega^{-1}$. As discussed in Section 2.3, we consider an estimator of $A$ given by $\hat{A} = \hat{G}_1^\top \hat{Y} (I_q - \hat{G}_2 \hat{\Xi} \hat{G}_2 \hat{Y})$ and an approximate of $B$ by $\hat{B} = \hat{\Pi} - \hat{\Pi} (I_q + \hat{F} \hat{\Pi})^{-1} \hat{F} \hat{\Pi}$.

We denote by $\hat{G}_1 \overset{\text{def}}{=} \partial_{\theta_1} \hat{g}(\theta_1, \theta_2)|_{\theta_1 = \hat{\theta}_1}$ the partial derivative of $\hat{g}(\theta_1, \theta_2)$ with respect to $\theta_1$ valued at $\hat{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$, which is the corresponding point lying in the line segment between $\bar{\theta} = (\hat{\theta}_1, \hat{\theta}_2)$ and $\theta^0 = (\theta^0_1, \theta^0_2)$. In the case of linear moment models $\hat{G}_1 = \hat{G}_1$.

We shall analyze the accuracy of estimator $\hat{\theta}_1$ in (13). Observe that

$$\hat{\theta}_1 - \theta^0_1 = \hat{\theta}_1 - \theta^0_1 - \hat{B} \hat{A} \hat{g}(\hat{\theta}) = -BA \hat{g}(\theta^0) + r_n,$$

where $r_n = r_{n,1} + r_{n,2} + r_{n,3}$, and

$$r_{n,1} = (I - \hat{B} \hat{A} \hat{G}_1) (\hat{\theta}_1 - \theta^0_1), \quad r_{n,2} = \hat{B} \hat{A} (\hat{G}_1 - \hat{G}_1) (\hat{\theta}_1 - \theta^0_1), \quad r_{n,3} = (BA - \hat{B} \hat{A}) \hat{g}(\theta^0).$$

By applying the triangle inequality and the Hölder’s inequality, we have the
following bounds for the three terms respectively,

\[
|r_{n,1}|_{\infty} \leq |I - \hat{B} \hat{A} \hat{G}_1|_{\max} |\hat{\theta}_1 - \theta_1^0|_1 \\
\leq |B|_{\infty} |AG_1 - \hat{A} \hat{G}_1|_{\max} |\hat{\theta}_1 - \theta_1^0|_1 + |\hat{B} - B|_{\max} |\hat{A} \hat{G}_1|_1 |\hat{\theta}_1 - \theta_1^0|_1, \\
|r_{n,2}|_{\infty} \leq |\hat{B}|_{\infty} |\hat{A}(\hat{G}_1 - \hat{G}_1)|_{\max} |\hat{\theta}_1 - \theta_1^0|_1, \\
|r_{n,3}|_{\infty} \leq |\hat{B} - B|_{\infty} |A|_{\infty} |\hat{g}(\theta^0)|_{\infty} + |B|_{\infty} |\hat{A} - A|_{\infty} |\hat{g}(\theta^0)|_{\infty}.
\]

A (high dimensional) Gaussian approximation on the leading term $BA \hat{g}(\theta^0) = (AG_1)^{-1} \hat{A} \hat{g}(\theta^0)$ follows as we shall show in Section 3.2.2, given that $|r_n|_{\infty}$ is of small order. We now provide a theorem for the debiased estimator under the linear case.

**THEOREM 3.2 (Linearization of debiased estimator).** Under the conditions in Lemma 3.3.3.1, A.8, A.11, and given the Gaussian approximation assumptions (as in (A7)) for $g(D_t, \theta^0)$, suppose that there exists constant $C$ such that $|A|_{\max} \leq C$ and $|A|_{\infty} \leq \iota$. Moreover, assume that $|AG_1|_{\infty} \leq \omega_1^{1/2} |AG_1|_2 \asymp \omega_1^{2/3}$, $|(AG_1)^{-1}|_{\infty} \leq \vartheta \asymp \omega_1^{-1}$ if $K^{(1)}$ is fixed, $|(AG_1)^{-1}|_{\infty} \leq \omega_1^{1/2} |(AG_1)^{-1}|_2 \leq \vartheta \asymp \omega_1^{2/3}$ while $K^{(1)}$ is diverging, where $\omega_1 = o(n)$. We have

\[
\hat{\theta}_1 - \theta_1^0 = -(AG_1)^{-1} \hat{A} \hat{g}(\theta^0) + r_n,
\]

with $|r_n|_{\infty} \lesssim \varrho_{n,1} + \varrho_{n,3}$, where $\varrho_{n,1}$ and $\varrho_{n,3}$ are defined in (16).

**Proof.** We know from Lemma A.8 and A.11 that $|\hat{A} \hat{G}_1 - AG_1|_{\max} \lesssim \varrho_{n,2}^\ell / M + \rho_{n,2}^F$ and $|\hat{B} - B|_{\max} \lesssim \varrho_{n,2}^\theta + \rho_{n,2}^B \vee \rho_{n,2}^{\ell, G}_n$. According to the Gaussian approximation results as discussed in Section 3.2.2, we have $|\hat{g}(\theta^0)|_{\infty} \lesssim \varrho_{n,2}^F \lesssim n^{-1/2}(\log n)^{1/2}$. On the event $\{|\hat{A} \hat{G}_1|_1 \lesssim \omega_1^{3/2}\}$, which holds with probability approaching 1, applying the results in (15) as well as Remarks A.1 and A.7 shows that

\[
|r_{n,1}|_{\infty} \lesssim \varrho_{n,1} (\ell^\ell / M + \rho_{n,2}^F) d_{n,1} + \rho_{n,2}^B \omega_{1} d_{n,1} =: \varrho_{n,1}, \\
|r_{n,3}|_{\infty} \lesssim \varrho_{n,3} \{\rho_{n,2}^B + (\varrho_{n,2}^\theta + \rho_{n,2}^B) \rho_{n,2}^A\} n^{-1/2}(\log n)^{1/2} =: \varrho_{n,3}.
\]

We note that $r_{n,2} = 0$ in linear moment models. \hfill \Box

We shall discuss the detailed rates of $\ell_n^\ell, \ell_n^\Pi, \rho_{n,2}^F$ which are involved in the rate of $|r_n|_{\infty}$ in Appendix A.3.
REMARK 3.6 (Discussion on the rate of $|r_{n}|_{\infty}$). Continued to Remarks 3.3 and 3.5, we set up a special case with all the dependence adjusted norms involved bounded by constants and specifically discuss the relevant rates of $\ell_{n}^{\prime}$, $\ell_{n}^{\Pi}$, $\rho_{n,2}^{G}$, $\rho_{n,2}^{U}$, $\rho_{n,2}^{F}$ in Remarks A.5 - A.6. By summarizing all the results together, we have $\rho_{n,2}^{B} \leq s^{5}n\log P_{n}^{-1/2} + s^{6}b(n^{-1}\log P_{n})^{1/2}$, which implies that $\rho_{n,2} \leq (s^{5}n^{1/2}(\log P_{n})^{1/2} + s^{6}b(1-n\log P_{n})^{1/2})s + 2sn^{-1/2}(\log P_{n})^{1/2}$, for some $0 \leq b < 1$.

Additionally, suppose that $(\nu \vee \iota) \leq s$. It follows that $\rho_{n,2}^{B} = \nu^{3}(\rho_{n,2}^{E} \vee \rho_{n,2}^{II}) \leq s^{5}n\log P_{n}^{-1/2}$ and $\rho_{n,2}^{V} \leq s^{6}b(1-n\log P_{n})^{1/2} \to 0$ as $n \to \infty$. Moreover, according to Remark A.1, given $\rho_{n,2}^{II}, \rho_{n,2}^{V} \to 0$ as $n \to \infty$. Finally, we get $\varrho_{n,3} \leq s^{10}b(n^{-1}\log P_{n})^{1/2}$, given $\rho_{n,2}^{B} \to 0$ as $n \to \infty$.

3.2.2 Simultaneous inference

In this subsection, we cite a high dimensional central limit theorem to facilitate the simultaneous inference of the parameters. The theorem is adapted from [Zhang and Wu (2017)]. Consider the inference on $H_{0}: \theta_{1,j}^{0} = 0, \forall j \in S$, with $S \subseteq \{1, \ldots, \Lambda^{(n)}\}$. Define the vector $G_{i} = (G_{j,t})_{j \in S}, G_{j,t} = -\zeta_{j}g(D_{t}, \theta_{1}^{0}, \theta_{2}^{0})$ where $\zeta_{j}$ is the $j$-th row of the matrix $(AG_{1})^{-1}A$. Define the aggregated dependence adjusted norm as

$$\|G_{i}\|_{r,\varsigma} := \sup_{s \geq 0} (s + 1)^{\varsigma} \sum_{t=s}^{\infty} ||G_{i} - G_{i}^{*}||_{r},$$

where $r \geq 1, \varsigma > 0$. Moreover, define the following quantities

$$\Phi_{r,\varsigma}^{G} := \max_{j \in S} \|G_{j,}\|_{r,\varsigma}, \Gamma_{r,\varsigma}^{G} := \left( \sum_{j \in S} \|G_{j,}\|_{r,\varsigma}^{r} \right)^{1/r},$$

$$\Theta_{r,\varsigma}^{G} := \Gamma_{r,\varsigma}^{G} \Lambda \{ \|G_{i,}\|_{r,\varsigma}(\log |S|)^{3/2} \}.$$

Let $L_{1}^{G} = (\Phi_{2,\varsigma}^{G})^{1/\varsigma}, W_{1}^{G} = \{(\Phi_{3,0}^{G})^{2} + \Phi_{4,0}^{G}\}^{1/2}, W_{2}^{G} = (\Phi_{2,\varsigma}^{G})^{1/\varsigma}, W_{3}^{G} = \{n^{-1}(\log |S|)^{1/2}\}^{1/2}, N_{1}^{G} = n(\log |S|)^{1/2}(\Theta_{r,\varsigma}^{G})^{r}, N_{2}^{G} = n(\log |S|)^{-2}(\Phi_{2,\varsigma}^{G})^{-2}, N_{3}^{G} = \{n^{1/2}(\log |S|)^{-1/2}(\Theta_{r,\varsigma}^{G})^{1/2}\}^{1/2}.$

(A7) i) (weak dependency case) Given $\Theta_{r,\varsigma}^{G} < \infty$ with $r \geq 2$ and $\varsigma > 1/2 - 1/r$, then
Denote by $c_{\alpha}$ the $(1 - \alpha)$ quantile of the $\max_{j \in S} |Z_j|$, where $Z_j$ are the standard normal random variables. Let $\sigma_j$ be the $j$-th diagonal element of the covariance matrix $(AG_1)^{-1}A\Omega A^T((AG_1)^{-1})^T$. Under $[A7]$ and the same conditions as in Theorem 3.2, for each $j \in S$ assume that there exists a constant $c > 0$ such that $\min \{n^{-1/2} \sum_{t=1}^{n} \theta_{j,t} \} \geq c$, with probability $1 - o(1)$, we have
\[
\lim_{n \to \infty} \left| \mathbb{P}(\sqrt{n}D_1 - \theta_1) \leq c_{\alpha}\sigma_j, \forall j \in S) - (1 - \alpha) \right| = 0.
\] (17)

The results also hold when $\sigma_j$ is replaced by the consistent estimator $\hat{\sigma}_j$.

Define the vector $\hat{T}$ as
\[
\hat{T}_j = \frac{1}{\sqrt{n}} \sum_{i}^{l_n} e_i \sum_{t=1}^{i b_n} \hat{\sigma}_j g(D_t, \hat{\theta}_1, \hat{\theta}_1), \quad j \in S,
\]
where $\hat{\sigma}_j$ is the $j$-th row of the matrix $(\hat{A}G_1)^{-1} \hat{A}$ and $e_i$ are independently drawn from $\mathcal{N}(0, 1)$; $l_n$ and $b_n$ are the numbers of blocks and block size, respectively.

**THEOREM 3.3.** Denote by $c_{\alpha,S}$ the $(1 - \alpha)$ conditional quantile of $\max_{j \in S} |\hat{T}_j|$. Under $[A7]$ and the same conditions as in Theorem 3.2, assume $\Phi_{r,c}^{\theta} < \infty$ with $r > 4$, $b_n = \mathcal{O}(n^\eta)$ for some $0 < \eta < 1$, we have,
\[
\lim_{n \to \infty} \left| \mathbb{P}(\hat{\theta}_{1,j} \leq \theta_{1,j} \leq \hat{\theta}_{1,j} + n^{-1/2}c_{\alpha,S}\hat{\sigma}_j, \forall j \in S) - (1 - \alpha) \right| = 0.
\] (18)

In particular, the following conditions on $b_n$ are required:
\[
b_n = \sigma\{n\log |S|\}^{-4}(\Phi_{r,c}^{\theta})^{-4} \wedge n\log |S|^{-5}(\Phi_{4,\alpha}^{\theta})^{-4}, \quad F_\varsigma = \sigma\{n^{r/2} \log |S|\}^{-r} |S|^{-1}(\Gamma_{4,\alpha}^{\theta})^{-r},
\]
\[
\Phi_{2,0}^{\theta} \Phi_{2,\varsigma}^{\theta} \{b_n^{\varsigma-1} + \log(n/b_n) / n + (n-b_n) \log b_n/(nb_n)/(\log |S|)^2 = o(1), \text{ if } \varsigma = 1;
\]
\[
\Phi_{2,0}^{\theta} \Phi_{2,\varsigma}^{\theta} \{b_n^{\varsigma-1} + n^{-\varsigma} + (n-b_n)b_n^{-\varsigma+1}/(nb_n)/(\log |S|)^2 = o(1), \text{ if } \varsigma < 1;
\]
\[
\Phi_{2,0}^{\theta} \Phi_{2,\varsigma}^{\theta} \{b_n^{\varsigma-1} + n^{-1}b_n^{\varsigma+1} + (n-b_n)/(nb_n)/(\log |S|)^2 = o(1), \text{ if } \varsigma > 1.
\]
(19)

where $F_\varsigma = n$, for $\varsigma > 1 - 2/r$; $F_\varsigma = l_n b_n^{\varsigma/2 - cr/2}$, for $1/2 - 2/r < \varsigma < 1 - 2/r$; $F_\varsigma = l_n^{4/\varsigma - cr/2} b_n^{\varsigma/2 - cr/2}$, for $\varsigma < 1/2 - 2/r$.

The above results are similar to Theorem 5.8 of Chernozhukov et al. (2018b), which is proved by applying Theorem 5.1 of Zhang and Wu (2017).
4 Simulation study

In this section, we illustrate the finite sample properties of our proposed methodology under different simulation scenarios.

4.1 Single equation model

Consider a model given by

\[ Y_t = \rho h^\top X_t + \varepsilon_t, \quad X_t \in \mathbb{R}^p, \quad t = 1, \ldots, n, \quad p \gg n, \]

where \(|\rho| < 1\) and \(h\) is referred to as the actual, unobserved effect of \(X\) on \(Y\). In particular, \(h_i = 1\) \((i = 1, \ldots, p)\) if there exists a causal effect from \(X_i\) to \(Y\) and \(h_i = 0\) otherwise.

Our goal is to estimate \(\rho\) and recover the unobserved \(h\). In practice, \(h\) can be misspecified as \(w\), with \(w_i \in \{0, 1\}\) for \(i = 1, \ldots, p\). The model can be re-written as

\[ Y_t = \rho w^\top X_t + \rho(h^\top - w^\top)X_t + \varepsilon_t. \]

We assume the error \(\delta \overset{\text{def}}{=} (h - w)\) is a sparse vector to be estimated via regularization, while \(h\) and \(w\) might not be sparse. In particular, we can generate \(h\) by independent Bernoulli random variables with probability 0.8 of equaling one. And we let the misspecification occurs randomly with probability \(P\). The multicollinearity can be ruled out if \(P\) is relatively small.

In our setting, we allow \(X_t\) to be endogenous and is generated by

\[ X_t = \pi^\top Z_t + v_t, \]

where the instruments \(Z_t \sim_{i.i.d.} \mathcal{N}_q(0, \Sigma), \) with \(\Sigma_{i,j} = \rho_z |i-j|, \rho_z = 0.5\). We choose the \(q \times p\) matrix \(\pi = [2 + 2\rho_z^2]^{-1/2}(\iota_2 \otimes I_{q/2})\) (in this case \(q = q/2\)), where \(\iota_2\) is a \(2 \times 1\) vector of ones. The errors \(\varepsilon_t\) and \(v_t\) are generated as follows:

\[ \varepsilon_t = \sqrt{\kappa} u_{1t} + \sqrt{1 - \kappa} u_{2t}, \quad v_t = \sqrt{\kappa} u_{1t} u_{1p} + \sqrt{1 - \kappa} u_{3t}, \]

where \(u_t = (u_{1t}, u_{2t}, u_{3t}) \sim_{i.i.d.} \mathcal{N}_{p+2}(0, I_{p+2})(u_{1t}\) and \(u_{2t}\) are scalars and \(u_{3t}\) is a \(p \times 1\) vector), \(\kappa = 0.25\).

We take \(n = 100\) and repeat the designs for 100 times. We consider the cases of \(p = 120, 150\) (accordingly \(q = 240, 300\), \(P = 0.2\) and \(\rho = 0.5, 0.9\). The estimation performance is evaluated by calculating the mean square error of
\( \hat{\rho} \) and the average (mean and median) of \( |\hat{\delta} - \delta|_2 \) over replications. As a comparison, we also implement a one-dimension regression of \( Y_t \) on \( w^\top X_t \). The MSE of the OLS estimate of \( \rho \) is reported in parentheses.

\[
\begin{array}{cccc}
\text{MSE of } \hat{\rho} & p = 120, q = 240 & p = 150, q = 300 \\
\rho = 0.5 & 1.7e-04 & 2.2e-04 & 3.8e-03 \\
\rho = 0.9 & 8.5e-04 & (2.6e-03) & (4.8e-02) \\
\text{Mean of } |\hat{\delta} - \delta|_2 & 1.2026 & 1.3369 & 1.3203 \\
\text{Median of } |\hat{\delta} - \delta|_2 & 1.1431 & 1.1855 & 1.3353 \\
\end{array}
\]

Table 1. Estimation performance.

More importantly, we also examine the inference performance by computing the empirical power and size. In particular, denote \( \beta \overset{\text{def}}{=} (\rho, \delta^\top) \). For \( j \in \{ j : \beta_j = 0 \} \), the averaged rejection rate on the null hypothesis \( H_{j0} : \beta_j = 0 \) reflects the size performance, while for \( j \in \{ j : \beta_j \neq 0 \} \) the power results would be illustrated. We make an comparison with the benchmark where Dantzig without debiasing is used (namely RMD). The false positive rate under the RMD selection is also reported.

\[
\begin{array}{cccc}
\text{Size (false positive rate)} & p = 120, q = 240 & p = 150, q = 300 \\
\rho = 0.5 & 0.0293 & 0.0280 & 0.0269 \\
\rho = 0.9 & 0.0170 & 1.3369 & 1.3203 \\
\text{Power} & 0.9575 & 0.9050 & 0.9038 \\
\end{array}
\]

Table 2. Power and size.

To incorporate the data dependency, we shall make \( Z_t \) follows a linear process such that \( Z_t = \sum_{\ell=0}^\infty A_\ell \xi_{t-\ell} \), with \( A_\ell = (\ell + 1)^{-\tau-1} M_\ell \), where \( M_\ell \) are independently drawn from Ginibre matrices, i.e. all the entries of \( M_\ell \) are i.i.d. \( N(0,1) \), and in practice the sum is truncated to \( \sum_{\ell=0}^{1000} \). We set \( \tau \) to be 1.0 for the weaker dependence and 0.1 for the stronger dependence cases respectively. Let \( \xi_{k,t} = e_{k,t}(0.8e_{k,t-1}^2 + 0.2)^{1/2} \) where \( e_{k,t} \) are i.i.d. distributed
as $t(d)/\sqrt{d/(d-2)}$ and $t(d)$ is the Student’s $t$ with degree of freedom $d$ (take $d = 8$ for example). The residuals are still generated by samples over $t$ independently.

\begin{table}[h]
\centering
\begin{tabular}{lcccc}
\hline
 & $p = 120, q = 240$ & & $p = 150, q = 300$ & \\
 & $\rho = 0.5$ & $\rho = 0.9$ & $\rho = 0.5$ & $\rho = 0.9$ \\
\hline
MSE of $\hat{\rho}$ & 1.9e-06 & 7.2e-06 & 5.1e-06 & 6.3e-06 \\
& (1.1e-03) & (2.1e-03) & (9.8e-04) & (3.1e-03) \\
Mean of $|\hat{\delta} - \delta|_2$ & 0.1223 & 0.0878 & 0.1697 & 0.1153 \\
Median of $|\hat{\delta} - \delta|_2$ & 0.1178 & 0.0777 & 0.1648 & 0.1141 \\
Size of DRGMM & 0.0458 & 0.0317 & 0.0807 & 0.0620 \\
False positive rate of RMD & 0.3463 & 0.2924 & 0.2954 & 0.2548 \\
Power of DRGMM & 1.0000 & 1.0000 & 1.0000 & 1.0000 \\
\hline
\end{tabular}
\caption{Results with time dependency}
\end{table}

4.2 Multiple equations model

Consider a linear network model

\[ Y_{j,t} = \rho h_j^T D_t + \gamma^T X_{j,t} + \varepsilon_{j,t}, \quad j = 1, \ldots, p, t = 1, \ldots, n, \ D_t \in \mathbb{R}^p, X_{j,t} \in \mathbb{R}^m, \]

where $|\rho| < 1$ and $h_{j,i}$ ($i \neq j$) is referred to as the actual, unobserved spillover effect of $i$ on $j$ ($h_{jj} = 0$ by convention). In particular, $h_{j,i} = 1$ if there is a link from $D_i$ to $Y_j$ and $h_{j,i} = 0$ otherwise.

Our goal is to estimate the network effect $\rho$ under the fact that the spillover effects $h_j$ can be misspecified as $w_j$ practically, where $w_{j,i} \in \{0, 1\}$, for $i = 1, \ldots, p$. We randomly generate the actual links by independent Bernoulli random variables with probability 0.5 of equaling one. And assume the
misspecification occurs randomly with probability 0.9 if the actual link is nonzero.

Here we allow $X_{j,t}$ to be endogenous. $X_{j,t}$ and $\varepsilon_{j,t}$ are generated by following the same way as above.

In addition, we also consider the spatial model given by

$$Y_{j,t} = \rho h_j^\top Y_t + \gamma^\top X_{j,t} + \varepsilon_{j,t},$$

(21)

with $\gamma = (1, 1, 1, 1, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, \ldots, 0)$.

In this case, $h_{ij}$ ($i \neq j$) reflects the peer effect of $i$ on $j$ ($h_{jj} = 0$ by convention) and normalization on $h_j$ for each $j$ is required.

Similar to the single equation model, (20) and (21) can be re-written as

$$Y_{j,t} = \rho w_j^\top D_t + \rho (h_j^\top - w_j^\top) D_t + \gamma^\top X_t + \varepsilon_{j,t},$$

$$Y_{j,t} = \rho w_j^\top Y_t + \rho (h_j^\top - w_j^\top) Y_t + \gamma^\top X_t + \varepsilon_{j,t}.$$

And we shall estimate $\delta \overset{\text{def}}{=} [\delta_{ij}]_{i=1}^p$, $\delta_j \overset{\text{def}}{=} (h_j - w_j)$, and $\gamma$ by regularization.

Let $n = 100$, $p = 10$, $m = 100$, $q_j = 200$, and $\rho = 0.7, 0.9$. We focus on comparing the estimation accuracy of $\rho$, $\gamma$, and $\delta$ with/without debiasing.

| $\rho$ | DGP (20) | DGP (21) |
|-------|----------|----------|
| MSE of $\hat{\rho}$ | 0.0690 | 0.0077 |
| Mean of $|\hat{\gamma} - \gamma|_2$ | 0.7283 | 0.9763 |
| Median of $|\hat{\gamma} - \gamma|_2$ | 0.7120 | 0.9653 |
| Mean of $|\hat{\delta} - \delta|_2$ | 2.2361 | 0.4044 |
| Median of $|\hat{\delta} - \delta|_2$ | 2.2361 | 0.4002 |

| $\rho$ | RMD | DRGMM |
|-------|-----|-----|
| MSE of $\hat{\rho}$ | 0.1640 | 0.0069 |
| Mean of $|\hat{\gamma} - \gamma|_2$ | 0.7487 | 0.7385 |
| Median of $|\hat{\gamma} - \gamma|_2$ | 0.7212 | 0.7212 |
| Mean of $|\hat{\delta} - \delta|_2$ | 1.7321 | 0.4044 |
| Median of $|\hat{\delta} - \delta|_2$ | 1.7321 | 0.4002 |

| $\rho$ | RMD | DRGMM |
|-------|-----|-----|
| MSE of $\hat{\rho}$ | 0.0001 | 0.0001 |
| Mean of $|\hat{\gamma} - \gamma|_2$ | 0.3552 | 0.1674 |
| Median of $|\hat{\gamma} - \gamma|_2$ | 0.3557 | 0.1674 |
| Mean of $|\hat{\delta} - \delta|_2$ | 0.3770 | 0.3786 |
| Median of $|\hat{\delta} - \delta|_2$ | 0.1921 | 0.1934 |

Table 4. Estimation performance.
5 Empirical Analysis: Spatial Network of Stock Returns

In this section our proposed methodology is employed to study the spatial network effect of stock returns. We use the public cross ownership information as the pre-specified social network structure, however, there might be misspecification in the network given some of the cross shareholding information is not published. Our purpose is to analyze the network effect and recover the unobserved linkages simultaneously.

5.1 Data and model setting

Our empirical illustration is carried out on a dataset consists of 100 individual stocks traded on the Chinese A share market (Shanghai Stock Exchange and Shenzhen Stock Exchange) from 14 sectors (according to the guidelines for the Industry Classification by the China Securities Regulatory Commission). The time span we consider is from January 2, 2019 to December 31, 2019 (244 trading days). The daily stock returns and the annual cross ownership data were obtained from Wind Data Service.

The spatial network model is construed by

\[ r_{j,t} = \rho h_{j,t}^\top r_t + \gamma^\top z_t + \varepsilon_{j,t} \]
\[ = \rho w_{j,t}^\top r_t + \rho (h_{j,t}^\top - w_{j,t}^\top) r_t + \gamma^\top X_t + \varepsilon_{j,t} \]  

(22)

where \( j = 1, \ldots, J \) indicate the stock individuals, \( r_t = (r_{1,t}, \ldots, r_{J,t})^\top \) are the daily log returns, and \( X_t \) contains the control variables, including the log returns of Wind entire-A index and Chicago Board Options Exchange China ETF volatility index. \( w_{j,i} \) is referred to as the public cross ownership between stock \( i \) and \( j \), i.e., \( w_{j,i} = 1 \) if company \( j \) holds share of company \( i \) according to the accessible information and \( w_{j,i} = 0 \) otherwise. The network structure given by \( w_{j,i} \) is depicted in Figure 1. We note that the cross ownerships are observed cross sectors.
It might be possible that $w_{j,i} = 0$ while $h_{j,i} = 1$, if some shareholders of company $j$ are not revealed publicly. By convention, we set $w_{j,j} = h_{j,j} = 0$. We aim at estimating on the network effect $\rho$ and the misspecification errors $h_{j,i} - w_{j,i}$ by regularization using our proposed approach, where the lags $r_{t-1}, r_{t-2}$ are chosen as the instruments variables. In the end, we would like to recover the linkage $h_{j,i}$ based on the inference results on the errors $h_{j,i} - w_{j,i}$.

The results are shown in Figure 2. In addition, we get $\hat{\rho} = 0.2434$ and $\hat{\gamma} = (0.7276, 0.0003)^\top$. We discover that the network accounting for latent link structure is sufficiently different from the pre-specified one. The results show the necessity of accounting for misidentification.
Figure 2. The recovered network structure by $\hat{h}_{j,i}$.

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Appendix

A.1 Some Useful Lemmas

**LEMMA A.1** (Weyls’ inequality for Hermitian Matrix). We let $H$ be the exact matrix and $P$ be a perturbation matrix that represents the uncertainty. Consider the matrix $M = H + P$. If any two of $M$, $H$ and $P$ are $n$ by $n$ Hermitian matrices, where $M$ has eigenvalues $\mu_1 \geq \cdots \geq \mu_n$, and $H$ has eigenvalues $\nu_1 \geq \cdots \geq \nu_n$, and $P$ has eigenvalues $\rho_1 \geq \cdots \geq \rho_n$. Then the following inequalities hold for $i = 1, \ldots, n$,

$$\nu_i + \rho_n \leq \mu_i \leq \nu_i + \rho_1.$$  

More generally, if $j + k - n \geq i \geq r + s - 1$, we have

$$\nu_j + \rho_k \leq \mu_i \leq \nu_r + \rho_s.$$  

**LEMMA A.2** (Weyls’ inequality for SVD). Let $H$ ($m \times n$) be the exact matrix and $P$ ($m \times n$) be a perturbation matrix that represents the uncertainty. Consider the matrix $M = H + P$. If any two of $M$, $H$ and $P$ are $m$ by $n$ real matrices, where $M$ has singular values $\mu_1 \geq \cdots \geq \mu_{\min(m,n)}$, $H$ has singular values $\nu_1 \geq \cdots \geq \nu_{\min(m,n)}$, and $P$ has singular values $\rho_1 \geq \cdots \geq \rho_{\min(m,n)}$. Then the following inequalities hold for $i = 1, \ldots, \min(m,n)$, $1 \leq k \leq \min(m,n)$,

$$\max_{0 \leq i \leq \min(m,n) - k} \{\nu_{k+i} - \rho_{i+1} - \nu_{i+1} + \rho_{i+k}, 0\} \leq \mu_k \leq \min_{1 \leq i \leq k} (\nu_i + \rho_{k-i+1}).$$
Proof. The results is a direct consequence of Theorem 2 of [Queiró and Sá (1995)] with completion of the $m \times n$ matrix to square matrix by letting the zero entries and the nonzero singular values stay the same.

**Lemma A.3** (Corollary 3.3 of [Lu and Pearce (2000)]). Suppose that $B$ and $A$ are $m \times l$ and $l \times n$ matrices respectively, and let $p = \max\{m, n, l\}$ and $q = \min\{m, n, l\}$. Then for each $k = 1, \ldots, q$,

$$
\sigma_k(BA) \leq \min_{1 \leq i \leq k} \sigma_i(B)\sigma_{k+1-i}(A).
$$

If $p < 2q$, then for each $k = 1, \ldots, 2q - p$,

$$
\max_{k+p-q \leq i \leq q} \sigma_i(B)\sigma_{p+k-i}(A) \leq \sigma_k(BA).
$$

**Lemma A.4** (Theorem 6.2 of [Zhang and Wu (2017)] Tail probabilities for high dimensional partial sums). For a mean zero $p$-dimensional random variable $X_t \in \mathbb{R}^p$ ($p > 1$), let $S_n = \sum_{t=1}^n X_t$ and assume that $\|X_n\|_{q, \lambda} < \infty$, where $q > 2$ and $\lambda \geq 0$, and $\Phi_{2, \lambda} = \max_{1 \leq j \leq p} \|X_j\|_{2, \lambda} < \infty$. i) If $1/2 - 1/\lambda > 0$, then for $x \geq \sqrt{n\log p}\Phi_{2, \lambda} + n^{1/\lambda}(\log p)^{3/2}||X||_{q, \lambda}$,

$$
P(|S_n| \geq x) \leq \frac{C_{q, \lambda}n^{q/2}||X||_{q, \lambda}}{x^q} + C_{q, \lambda} \exp\left(-\frac{C_{q, \lambda}x^2}{n\Phi_{2, \lambda}^2}\right).
$$

ii) If $0 < \lambda < 1/2 - 1/\lambda$, then for $x \geq \sqrt{n\log p}\Phi_{2, \lambda} + n^{1/2-\lambda}(\log p)^{3/2}||X||_{q, \lambda}$,

$$
P(|S_n| \geq x) \leq \frac{C_{q, \lambda}n^{q/2-\lambda}||X||_{q, \lambda}}{x^q} + C_{q, \lambda} \exp\left(-\frac{C_{q, \lambda}x^2}{n\Phi_{2, \lambda}^2}\right).
$$

**Lemma A.5.** Consider a $p \times p$ positive semi-definite random matrix $H_1$ and a $p \times p$ deterministic positive definite matrix $H_2$. Assume that $|H_1 - H_2|_2 = \mathcal{O}_p(c_n)$, $c_n \to 0$. Then, we have

$$
\lambda_{\min}(H_1) = \lambda_{\min}(H_2) - \mathcal{O}_p(c_n).
$$

Proof. The results are implied by

$$
\lambda_{\min}(H_1) = \min_{v \in \mathbb{R}^p, ||v||_2 = 1} v^\top H_1 v \geq \min_{v \in \mathbb{R}^p, ||v||_2 = 1} v^\top H_2 v - \max_{v \in \mathbb{R}^p, ||v||_2 = 1} v^\top (H_1 - H_2)v
$$

$$
= \min_{v \in \mathbb{R}^p, ||v||_2 = 1} v^\top H_2 v - |H_1 - H_2|_2
$$

$$
\geq \lambda_{\min}(H_2) - \mathcal{O}_p(c_n).
$$
A.2 Convergence Rates of the Approximate Inverse Matrices

Define the class of matrices

\[ \mathcal{U} \overset{\text{def}}{=} \mathcal{U}(b, s_0(q)) = \{ \Upsilon : \Upsilon \succ 0, |\Upsilon|_1 \leq M, \max_{1 \leq i \leq q} \sum_{j=1}^{q} |\Upsilon_{ij}|^b \leq s_0(q) \} \]

for \(0 \leq b < 1\), where \(\Upsilon = (\Upsilon_{ij})\) and the notation \(\Upsilon \succ 0\) indicates that \(\Upsilon\) is positive definite.

**Lemma A.6.** Assume that \(\Upsilon_0 = \Omega^{-1} \in \mathcal{U}(b, s_0(q))\). Select \(\ell_\Upsilon^n\) such that

\[ |\hat{\Upsilon} - \Upsilon_0|_{\max} \leq 4M\ell_\Upsilon^n =: \rho_\Upsilon^n \]

holds with probability approaching 1. Moreover, with probability approaching 1, we have

\[ |\hat{\Upsilon} - \Upsilon_0|_2 \leq C_b(4M\ell_\Upsilon^n)^{1-b} s_0(q) =: \rho_\Upsilon^{n,2}, \]

where \(C_b\) is a positive constant only depends on \(b\).

**Proof.** Recall that \(\hat{\Upsilon}^1\) is the solution of (11). We first observe that

\[ |\hat{\Upsilon}^1 - \Upsilon_0|_{\max} \leq |\Upsilon_0\Omega(\hat{\Upsilon}^1 - \Upsilon_0)|_{\max} \leq |\Omega(\hat{\Upsilon}^1 - \Upsilon_0)|_{\max} |\Upsilon_0|_1 \]

\[ |\Omega(\hat{\Upsilon}^1 - \Upsilon_0)|_{\max} \leq |(\Omega - \hat{\Omega})(\hat{\Upsilon}^1 - \Upsilon_0)|_{\max} + |\hat{\Omega}(\Omega^{-1} - \hat{\Omega}^{-1})|_{\max} =: R_{n,1} + R_{n,2}. \]

In particular, we have \(R_{n,1} \leq |\Omega - \hat{\Omega}|_{\max} 2M \lesssim_p 2\ell_\Upsilon^n\) and \(R_{n,2} \leq |\hat{\Omega}\Upsilon^0 - \I_q|_{\max} + |\hat{\Omega}\hat{\Upsilon}^1 - \I_q|_{\max} \lesssim_p 2\ell_\Upsilon^n\). According to the definition given by (12), it follows that \(|\hat{\Upsilon} - \Upsilon_0|_{\max} \leq 4M\ell_\Upsilon^n\) with probability approaching 1. The rate of \(\ell_\Upsilon^n\) will depend on the concentration inequalities we use.

Moreover, with probability approaching 1, we have

\[ |\hat{\Upsilon} - \Upsilon_0|_2 \leq \sqrt{|\hat{\Upsilon} - \Upsilon_0|_1 |\hat{\Upsilon} - \Upsilon_0|_\infty} = |\hat{\Upsilon} - \Upsilon_0|_1 \leq C_b(4M\ell_\Upsilon^n)^{1-b} s_0(q), \]

where \(C_b\) is a positive constant only depends on \(b\). The rate of \(|\hat{\Upsilon} - \Upsilon_0|_1\) follows from the proof of Theorem 6 in [Cai et al. (2011)].
Similarly, we define the class of matrices
\[
\tilde{U} \overset{\text{def}}{=} \tilde{U}(b, s_0(K^{(1)})) = \left\{ \Pi : \Pi \succ 0, |\Pi|_1 \leq M, \max_{1 \leq i \leq K^{(1)}} \sum_{j=1}^{K^{(1)}} |\Pi_{ij}|^b \leq s_0(K^{(1)}) \right\}
\]
for \(0 \leq b < 1\), where \(\Pi = (\Pi_{ij})\). The lemma below follows.

**Lemma A.7.** Assume that \(\Pi^0 = (G_1^\top \Upsilon^0 G_1)^{-1} \in \tilde{U}(b, s_0(K^{(1)}))\). Select \(\ell_n^\Pi\) such that \(|G_1^\top \Upsilon^0 G_1 - \hat{G}_1^\top \hat{\Upsilon}_1|_{\max} M \leq \ell_n^\Pi\) with probability approaching 1 (see Lemma A.11 for the specific rate of \(\ell_n^\Pi\)). Then, we have
\[
|\hat{\Pi} - \Pi^0|_{\max} \leq 4M\ell_n^\Pi =: \rho_n^\Pi
\]
and
\[
|\hat{\Pi} - \Pi^0|_2 \leq C_6(4M\ell_n^\Pi)^{1-b}s_0(K^{(1)}) =: \rho_{n,2}^F
\]
hold with probability approaching 1, respectively.

**Proof.** The proof is similar to that of Lemma A.6 and thus is omitted. \(\square\)

Recall that \(D \overset{\text{def}}{=} G_1^\top \Upsilon^0 G_1 = (\Pi^0)^{-1}\) and \(F \overset{\text{def}}{=} G_1^\top \Upsilon^0 G_2 \Xi^0 G_2^\top \Upsilon^0 G_1\). Next, we show the rate of the estimator of \(B = (D + F)^{-1} = ((\Pi^0)^{-1} + F)^{-1}\) given by \(\hat{B} = \hat{\Pi} - \hat{\Pi}(I_q + \hat{\Pi})^{-1}\hat{\Pi}\). Denote by \(\rho_{n,2}^F\) the rate such that \(|\hat{F} - F|_2 \lesssim_P \rho_{n,2}^F\). We shall discuss the conditions on this rate in Lemma A.11.

**Lemma A.8.** Under the conditions of Lemma A.7, suppose that there exist constants \(c_1, c_2, c_3\) such that \(c_1 \leq \sigma_{\min}(F\Pi^0) \leq \sigma_{\max}(F\Pi^0) \leq c_2\) and \(\sigma_{\max}(F) \vee \sigma_{\max}(\Pi^0) \leq c_3\). Assume that there exists a constant \(C > 0\) such that \(|(I - F\Pi^0)^{-1}|_2 \leq C\) and \(|(I - \hat{F}\Pi)^{-1}|_2 \leq C\). Then, we have
\[
|\hat{\Pi} - I|_{\max} \lesssim_P (\rho_n^\Pi \lor \rho_{n,2}^F) =: \rho_n^B.
\]

**Proof.** We first observe that
\[
|\hat{B} - B|_{\max} \leq |\hat{\Pi} - I|_{\max} + |(\hat{\Pi} - \Pi^0)(I - F\Pi^0)^{-1}F\Pi^0|_{\max} + |\hat{\Pi}(I - \hat{F}\Pi)^{-1}(\hat{F}\Pi - F\Pi^0)|_{\max}
\]
\[
\lesssim_P \rho_n^\Pi + |\hat{\Pi} - I|_2|F\Pi^0|_2 + |\hat{\Pi}(I - \hat{F}\Pi)^{-1} - (I - F\Pi^0)^{-1}|_2|F\Pi^0|_2 + |\hat{\Pi}|_2|(I - \hat{F}\Pi)^{-1} - (I - F\Pi^0)^{-1}|_2|F\Pi^0|_2.
\]
By applying Lemma A.7 we obtain that
\[
|\hat{\Pi}|_2 \leq |\hat{\Pi} - I|_2 + |I - \Pi^0|_2 \lesssim_P \rho_{n,2}^F + c_3
\]
Besides, we have
\[ |\hat{F}\tilde{\Pi} - F\Pi^0|_2 \leq |\hat{F} - F|_2|\tilde{\Pi}|_2 + |\tilde{\Pi} - \Pi^0|_2|F|_2 \leq_p \rho_{n,2}^F c_3 + \rho_{n,2}^{\Pi} c_3 \leq \rho_{n,2}^F \lor \rho_{n,2}^{\Pi}, \]
and
\[ |(I - \hat{F}\tilde{\Pi})^{-1} - (I - F\Pi^0)^{-1}|_2 \leq |(I - F\Pi^0)^{-1}|_2 |(I - \hat{F}\tilde{\Pi})^{-1} - |F\tilde{\Pi} - F\Pi^0|_2 \leq_p \rho_{n,2}^F \lor \rho_{n,2}^{\Pi}. \]
Finally, the desired conclusion follows by collecting all the results above.

\[ \square \]

In this Lemma we assume that \(|(I - F\Pi^0)^{-1}|_2 \leq C\), which can be implied by the condition \(\sigma_{\min}(F\Pi^0) > 1\) or \(\sigma_{\max}(F\Pi^0) < 1\). For example, given \(1 < c_1 \leq \sigma_{\min}(F\Pi^0)\), we have
\[ |(I - F\Pi^0)^{-1}|_2 \leq (\sigma_{\min}(I - F\Pi^0))^{-1} \leq (\sigma_{\min}(F\Pi^0) - 1)^{-1} \leq (c_1 - 1)^{-1}, \]
where the first inequality is implied by Lemma \(\text{A.2}\) and the second one is due to Lemma \(\text{A.3}\). Additionally, based on Lemma \(\text{A.5}\) on the event \(\{\sigma_{\min}(\hat{F}\tilde{\Pi}) > 1\}\), which holds with probability approaching 1, it follows that
\[ |(I - \hat{F}\tilde{\Pi})^{-1}|_2 \leq (\sigma_{\min}(I - \hat{F}\tilde{\Pi}))^{-1} \leq (\lambda_{\min}(\hat{F})\lambda_{\min}(\tilde{\Pi}) - 1)^{-1} \leq_p (\lambda_{\min}(F) - \rho_{n,2}^F)(\lambda_{\min}(\Pi^0) - \rho_{n,2}^{\Pi}) - 1)^{-1} \leq C. \]

**Remark A.1.** The rate of \(|\hat{B} - B|_\infty\) shall follow similarly once we have dealt with the rate of \(|\hat{V} - V|_\infty\), where \(V \overset{\text{def}}{=} (I - F\Pi^0)^{-1}\) and \(\hat{V} \overset{\text{def}}{=} (I - \hat{F}\tilde{\Pi})^{-1}\). In particular, provided \(|\hat{V} - V|_{\max} \leq_p \rho_n^V = o(1)\), analogue to Lemma \(\text{A.10}\), we have \(|\hat{V} - V|_{\infty} \leq_p \rho_n^V\), with \(\rho_n^V = s(V)\rho_n^V\) if we assume \(|V|_0 = s(V)\) while \(\rho_n^V = \nu(\rho_n^V)^{-1}\) in the case of \(|V|_{\infty} \lor |\hat{V}|_{\infty} \leq \nu\) for some \(0 \leq l < 1\). Finally, given \(\max\{|\Pi|_\infty, |F|_\infty, |\hat{V}|_\infty\} \leq \nu\), applying the results in Lemma \(\text{A.7}\) shows that \(|\hat{B} - B|_\infty \leq_p \nu^3(\rho_n^F \lor \rho_n^{\Pi}) =: \rho_n^B\), provided that \(\rho_{n,2}^F, \rho_{n,2}^{\Pi} \to 0\) as \(n \to \infty\).

**A.3 Detailed Rate of \(|r_n|_\infty\) for Linear Case**

Recall that in the case of linear moment models, the score functions are given by \(g_j(D_{j,t}, \theta) = z_{j,t}\varepsilon_j(D_{j,t}, \theta)\), where \(\varepsilon_j(D_{j,t}, \theta) = y_{j,t} - \hat{x}_{j,t}^T \beta_j\). To
Proof. We first observe that (15) defined in simplify the notations, we shall denote \( \hat{\gamma}_{jm,t} \equiv z_{jm,t}\varepsilon_j(D_{jt}, 0) \) and \( \tilde{\gamma}_{jm,t} \equiv z_{jm,t}\varepsilon_j(D_{jt}, \tilde{\theta}) \), for all \( j = 1, \ldots, p \), and \( m = 1, \ldots, q_j \). We note that when the time series is non-stationary and the mean varies with respect to \( t \), we can replace \( \mathbb{E} g_{it,t}\hat{\gamma}_{jm,t} \) by \( \mathbb{E}_n \mathbb{E} g_{it,t}\tilde{\gamma}_{jm,t} \).

Let \( C_{xx} \) and \( C_{xee} \) be constants such that \( \max_{i,j,l,m} |\mathbb{E}(\hat{x}_{it}z_{jm,t}^Tz_{it,t}\varepsilon_{i,t})|_{\max} \leq C_{xx} \) and \( \max_{i,j,l,m} |\mathbb{E}(\tilde{x}_{it}z_{jm,t}z_{it,t}\varepsilon_{i,t})|_{\max} \leq C_{xee} \), respectively.

**Lemma A.9** (Rate of \( \ell_n^T \)). Under conditions in Lemma 3.1 and 3.3, we have

\[
|\tilde{\Omega} - \Omega|_{\max} \lesssim \ell_n^T / M,
\]

given \( d_{n,1}^2(C_{xx} + \gamma_n) + d_{n,1}(C_{xx} + \gamma'_n) + \gamma_{n,1} + \gamma_{n,2} \lesssim \ell_n^T / M \), where \( d_{n,1} \) is defined in (15), \( \gamma_n, \gamma'_n, \gamma_{n,1}, \gamma_{n,2} \) are specified in (A.1), (A.2) and (A.3).

Proof. We first observe that

\[
|\tilde{\Omega} - \Omega|_{\max} = \max_{i,j,l,m} |\mathbb{E}_n(\hat{\gamma}_{it,t}\hat{\gamma}_{jm,t}) - \mathbb{E}(\hat{\gamma}_{it,t}\hat{\gamma}_{jm,t})| \\
\leq \max_{i,j,l,m} |\mathbb{E}_n(\hat{\gamma}_{it,t}\hat{\gamma}_{jm,t})| + 2 \max_{i,j,l,m} |\mathbb{E}_n(\hat{\gamma}_{it,t}\hat{\gamma}_{jm,t})| \\
=: I_{n,1} + I_{n,2} + I_{n,3}
\]

For \( I_{n,1} \), it can be seen that

\[
I_{n,1} \leq \max_{i,j,l,m} |\hat{\beta}_i - \beta'_i|_1 |\hat{\beta}_j - \beta'_j|_1 |\mathbb{E}_n(\hat{x}_{it}z_{jm,t}^Tz_{it,t}\varepsilon_{i,t})|_{\max} \\
\leq |\theta - \theta'|_2^2 \{C_{xx} + |\mathbb{E}_n(\hat{x}_{it}z_{jm,t}^Tz_{it,t}\varepsilon_{i,t})| - |\mathbb{E}_n(\hat{x}_{it}z_{jm,t}^Tz_{it,t}\varepsilon_{i,t})|_{\max}\}
\]

Let \( \chi_{ijlm} \equiv \text{vec}(\hat{x}_{it}z_{jm,t}^Tz_{it,t}\varepsilon_{i,t}) = (\chi_{ijlm}, K_{i} K_{j}) \) and define

\[
\gamma_n \equiv cn^{-1/2}(\log P_n)^{1/2} \max_{i,j,l,m,k} \|\chi_{ijlm} \|_{2,\infty} + cn^{-1/2} c_n(\log P_n)^{1/2} \max_{i,j,l,m} |\chi|_{r,\infty},
\]

(A.1)

with \( P_n = (K \lor q \lor n) \), \( c_n \equiv n^{1/r} \) for \( \zeta > 1/2 - 1/r \) and \( c_n \equiv n^{1/2 - \zeta} \) for \( 0 < \zeta < 1/2 - 1/r \). By applying Lemma A.4 and the results in [15], we have \( I_{n,1} \lesssim \mathbb{P} d_{n,1}^2(C_{xx} + \gamma_n) \), for sufficiently large \( c \).

Similarly,

\[
I_{n,2} \leq 2 \max_{i,j,l,m} |\hat{\beta}_j - \beta'_j|_1 |\mathbb{E}_n(\hat{x}_{it}z_{jm,t}z_{it,t}\varepsilon_{i,t})|_{\infty} \\
\leq 2 |\theta - \theta'|_1 \{C_{xx} + |\mathbb{E}_n(\hat{x}_{it}z_{jm,t}z_{it,t}\varepsilon_{i,t}) - |\mathbb{E}_n(\hat{x}_{it}z_{jm,t}z_{it,t}\varepsilon_{i,t})|_{\infty}\}
\]

43
Let $c_{ijlm} \overset{\text{def}}{=} x_{i,t}z_{jm,t}z_{il,t}^2 = (c_{ijlm})_k^{K_j}$ and define

$$
\gamma_n \overset{\text{def}}{=} cn^{-1/2}(\log P_n)^{1/2} \max_{i,j,l,m,k} \|c_{ijlm}^k\|_{\ell_2} + cn^{-1}c_{n,\gamma}(\log P_n)^{3/2} \max_{i,j,l,m,k} \|\gamma_i\|_{\ell_\gamma}.
$$

(A.2)

It follows that $I_{n,2} \lesssim d_{n,1}(C_{x\varepsilon} + \gamma_n')$, for sufficiently large $c$.

Lastly, $I_{n,3}$ is handled by pointwise concentration for two parts as

$$
I_{n,3} \leq \max_{i \neq j} \left| E_n(g_{il,t}g_{jm,t}) - E(g_{il,t}g_{jm,t}) \right| + \max_{j,m} \left| E_n\gamma_j^2 - E\gamma_j^2 \right|
$$

where Hölder’s inequality is applied when dealing with the first part.

Let

$$
\gamma_{n,1} \overset{\text{def}}{=} cn^{-1/2}(\log P_n)^{1/2}(\Phi^{\varepsilon}_{4,\gamma})^2 + cn^{-1}c_{n,\gamma}(\log P_n)^{3/2} \max_j \|\varepsilon_j, z_j, \|_{\ell_\infty}^2;
$$

$$
\gamma_{n,2} \overset{\text{def}}{=} cn^{-1/2}(\log P_n)^{1/2} \max_{j,m} \|\varepsilon_j^2 z_j^2, \|_{\ell_2} + cn^{-1}c_{n,\gamma}(\log P_n)^{3/2} \max_j \|\varepsilon_j^2 z_j^2, \|_{\ell_\infty}.
$$

(A.3)

Then, we have $I_{n,3} \lesssim d_{n,1} + \gamma_{n,2}$ for sufficiently large $c$.

By collecting all the results above, we can claim that $|\Omega - \Omega|_{\max} \lesssim \log P_n$ by selecting $\ell_n$ such that $d_{n,1} + \gamma_{n,2} \lesssim \ell_n / M$.

**REMARK A.2** (Admissible rate of $\ell_n$). Suppose that $M \lesssim s$ and assume all the dependence adjusted norms involved in $\gamma_n$, $\gamma_n'$, $d_{n,1}$ are bounded by constants. For the weak dependence case where $\varepsilon > 1/2 - 1/r$, if $n^{-1/2+1/r}(\log P_n) = O(1)$ for sufficiently large $r$, we have $\gamma_n, \gamma_n', \gamma_{n,1}, \gamma_{n,2} \lesssim n^{-1/2}(\log P_n)^{1/2}$. Moreover, according to Remark 7.5, we know that $d_{n,1} \lesssim (s + 2)sn^{-1/2}(\log P_n)^{1/2}$. Therefore, an admissible rate of $\ell_n$ is given by $sn^{-1/2}(\log P_n)^{1/2}$, provided that $d_{n,1} \to 0$ as $n \to \infty$.

By applying Lemma A.6 under this rate we have $\rho_n \lesssim s^2n^{-1/2}(\log P_n)^{1/2}$ and $\rho_n^2 \lesssim s^{3-2b}(n^{-1/2}(\log P_n)^{1/2})$ for some $0 \leq b < 1$ and $s_0(q) \lesssim s$ such that $\mathcal{Y}^0 \in \mathcal{U}(b, s_0(q))$.

Next we analyze the rate $\ell_n^P$. For this purpose, we introduce the following definitions.

Let the subset $\mathcal{P}^{(1)} \subseteq \{1, \ldots, p\}$ be the equation index space related to $\theta_2^0$. And for each $j \in \mathcal{P}^{(1)}$, the subset $\mathcal{K}_j^{(1)} \subseteq \{1, \ldots, K_j\}$ is the parameter index space related to $\theta_2^0$ in the $j$-th equation. Let

$$
\rho_n^{G_j} \overset{\text{def}}{=} cn^{-1/2}(\log P_n)^{1/2}(\Phi_{2,\gamma}^\varepsilon + cn^{-1}c_{n,\gamma}(\log P_n)^{3/2} \max_{j \in \mathcal{P}^{(1)}, k \in \mathcal{K}_j^{(1)}} |\bar{x}_{jk}, z_j, \|_{\ell_\gamma}.
$$

44
Define the matrix norms $|G_1|_{l,1} = \max_j \sum_i |G_{1,ij}|^l$, $|G_1|_{\infty,1} = \max_i \sum_j |G_{1,ij}|^l$, and $|G_1|_0$ is the number of nonzero components in $G_1$.

**Lemma A.10.** Assume that $|\hat{G}_1 - G_1|_{\max} \preceq_P \rho_n^{G_1}$. Then, we have

$$|\hat{G}_1 - G_1|_1 \preceq_P \rho_n^{G_1}, \quad |\hat{G}_1 - G_1|_2 \preceq_P \rho_n^{G_1},$$

where $\rho_n^{G_1} = s(G_1) \rho_n^{G_1}$ in the sparse case with $|G_1|_0 = s(G_1)$ and $\rho_n^{G_1} = L(\rho_n^{G_1})^{1-l}$ in the dense case with max\{|$G_1$|_{1,l}, |$G_1$|_{\infty,l}, |$G_1$|_{1}, |$G_1$|_{\infty} \} \leq L$ for some $0 \leq l < 1$.

**Proof.** Recall that $\hat{G}_1 = (\hat{G}_{1,ij})$ is a thresholding estimator with $\hat{G}_{1,ij} = \hat{G}_{1,ij} 1\{|\hat{G}_{1,ij}| > T\}$, $\hat{G}_1 = (\hat{G}_{1,ij}) = \partial_{\theta_{ij}} \hat{g}(\theta_1, \theta_2)|_{\theta_{ij} = \hat{\theta}_{ij}}$. Consider the event $\mathcal{A}$ defined by

$$\mathcal{A} \overset{\text{def}}{=} \{G_{1,ij} - \rho_n^{G_1} \leq \hat{G}_{1,ij} \leq G_{1,ij} + \rho_n^{G_1}, \text{ for all } i, j\}.$$

Let $T \geq \rho_n^{G_1}$. On the event $\mathcal{A}$, which holds with probability approaching one, we have

$$\max_j \sum_i |\hat{G}_{1,ij} - G_{1,ij}|$$

$$\leq \max_j \sum_i |\hat{G}_{1,ij} - G_{1,ij}| 1\{|\hat{G}_{1,ij}| > T\} + \max_j \sum_i |G_{1,ij}| 1\{|\hat{G}_{1,ij}| \leq T\}$$

$$\leq \max_j \sum_i |\hat{G}_{1,ij} - G_{1,ij}| 1\{|G_{1,ij}| > T + \rho_n^{G_1}\} + \max_j \sum_i |G_{1,ij}| 1\{|G_{1,ij}| \leq T - \rho_n^{G_1}\}$$

$$\preceq_P s(G_1) \rho_n^{G_1} + (T - \rho_n^{G_1}) s(G_1),$$

in the sparse case. By picking $T = 2\rho_n^{G_1}$, we obtain that $|\hat{G}_1 - G_1|_1 \preceq_P \rho_n^{G_1} = s(G_1) \rho_n^{G_1}$. Similarly, we can prove that $|\hat{G}_1 - G_1|_{\infty} \preceq_P \rho_n^{G_1}$ and it follows that $|\hat{G}_1 - G_1|_2 \preceq_P \rho_n^{G_1}$ by Hölder’s inequality.

Likewise, for the dense case, on the event $\mathcal{A}$, we have

$$\max_j \sum_i |\hat{G}_{1,ij} - G_{1,ij}|$$

$$\leq \max_j \sum_i |\hat{G}_{1,ij} - G_{1,ij}| / |G_{1,ij}| 1\{|G_{1,ij}| > T + \rho_n^{G_1}\} + L(T - \rho_n^{G_1})$$

$$\preceq_P L \rho_n^{G_1} / (T + \rho_n^{G_1}) + L(T - \rho_n^{G_1}).$$

It follows that $\rho_n^{G_1} = L(\rho_n^{G_1})^{1-l}$ in this case, if we select $T = 2\rho_n^{G_1}$. \qed
REMARK A.3 (Discussion on the rates of \( \rho_{n}^{G_1} \) and \( \rho_{n,2}^{G_2} \)). Consider again the special case discussed in Remark A.2, here we have \( \rho_{n}^{G_1} \lesssim n^{-1/2}(\log P_n)^{1/2} \). Assume that \( L \lesssim s \) and \( s(G_1) \lesssim s \). It follows that \( \rho_{n,2}^{G_2} \lesssim s(n^{-1} \log P_n)^{(1-l)/2} \) (\( l = 0 \) for the sparse case).

We denote \( U \overset{\text{def}}{=} G_2 P(\Omega, G_2) \). Note that \( |U|_2 = 1 \) as it is an idempotent matrix. When \( K^{(1)} \) is of high dimension potentially larger than \( n \), we need to consider a regularized estimator given by \( \hat{U} = G_2 \hat{\varepsilon} G_2^\top \hat{\Theta} \). Denote by \( \rho_{n,2}^{U} \) the rate such that \( |\hat{U} - U|_2 \lesssim_P \rho_{n,2}^{U} \). To further discuss the conditions on this rate, we assume that \( |G_2|^2 \lesssim \omega_2, \sigma_{\text{min}}(G_2) \geq \omega_2^{-1/2}, \) and there exists constants \( c \) and \( C \) such that \( 0 < c \leq \sigma_{\text{min}}(\Theta^0) \) and \( |\Theta^0|_2 \leq C \). It is not hard to see that

\[
|\hat{U} - U|_2 \\
\leq |\hat{G}_2 - G_2|^2 (|\hat{\varepsilon} G_2^\top \hat{\Theta} - \Theta^0 G_2^\top \Theta^0|_2 + \omega_2^{3/2}) + \omega_2^{1/2} |\hat{\varepsilon} G_2^\top \hat{\Theta} - \Theta^0 G_2^\top \Theta^0|_2 \\
\leq |\hat{\varepsilon} - \Theta^0|_2 (|G_2^\top \hat{\Theta} - G_2^\top \Theta^0|_2 + \omega_2^{1/2}) + \omega_2 |\hat{G}_2^\top \hat{\Theta} - G_2^\top \Theta^0|_2 \\
\leq |\hat{G}_2 - G_2|^2 + \rho_{n,2}^{Y} (|\hat{G}_2 - G_2|^2 + \omega_2^{1/2}),
\]

where we have applied the results in Lemma A.6 (where the rate of \( \rho_{n,2}^{Y} \) is defined) in the last inequality. In particular, the rates of \( |\hat{G}_2 - G_2|_2 \lesssim_P \rho_{n,2}^{G_2} \) and \( |(\hat{\varepsilon} - \Theta^0)|_2 \lesssim_P \rho_{n,2}^{\bar{\varepsilon}} \) can be derived similarly as in Lemma A.10 and A.7 with the same assumptions with respect to \( G_2 \) instead of \( G_1 \).

REMARK A.4. [Discussion on the rate of \( \rho_{n,2}^{U} \)] Consider a similar special case as specified in Remark A.3 and A.5, we have \( \rho_{n,2}^{G_2} \lesssim s(n^{-1} \log P_n)^{(1-l)/2} \) (\( l = 0 \) for the sparse case), and \( \rho_{n,2}^{\bar{\varepsilon}} \lesssim s^{6-5b}(n^{-1} \log P_n)^{(1-b)/2} \). Suppose \( \omega_2 \) is given by a constant, it follows that \( \rho_{n,2}^{U} \lesssim s^{6-5b}(n^{-1} \log P_n)^{(1-b)/2} \), given \( \rho_{n,2}^{G_2}, \rho_{n,2}^{Y}, \rho_{n,2}^{\bar{\varepsilon}} \to 0 \) as \( n \to \infty \).

LEMMA A.11 (Rates of \( \ell_n^{II} \) and \( \rho_{n}^{F} \)). Under the conditions of Lemma A.6 and A.10, assume that there exists a constant \( C > 0 \) such that \( |\Theta^0|_2 \leq C \). In addition, suppose that \( |G_1|_1 \vee |G_1|_\infty \leq \mu, \ |G_1|_{\text{max}} \leq \bar{\mu}, \) and \( |G_1|_2 \leq \omega_1 \). Then, we have

\[
|\hat{G}_1 Y\hat{G}_1 - G_1^\top \Theta^0 G_1|_{\text{max}} \lesssim_P \ell_n^{II} \mu / M,
\]
given \( \rho_{n}^{G_1}(\rho_{n}^{Y} + M)(\rho_{n,2}^{G_2} + \mu) + \mu \rho_{n}^{Y}(\bar{\mu} + \rho_{n}^{G_1}) + \mu M \rho_{n}^{G_1} \leq \ell_n^{II} / M \). Moreover, we have

\[
|\hat{F} - F|_2 \lesssim_P \rho_{n,2}^{F},
\]

46
provided \((\rho_{n,1}^G + \omega_1^{1/2})^2 \rho_{n,2}^U + (\rho_{n,2}^G + \omega_1^{1/2})^2 \rho_{n,2}^\Upsilon + \omega_1^{1/2} \rho_{n,2}^G \leq \rho_{n,2}^F\).

**Proof.** By applying the results in Lemma \(A.6\) and \(A.10\) we have

\[
|\hat{G}_1 \hat{G}_1 - G_1^T \Upsilon G_1|_{\max,}
\leq |\hat{G}_1 - G_1|_{\max} (|\hat{T} - \Upsilon|_1 + |\Upsilon|_1) (\hat{G}_1 - G_1 + |G_1|_1)
\]

\[
+ |G_1|_\infty (|\hat{T} - \Upsilon|_\infty (|G_1|_{\max} + |\hat{G}_1 - G_1|_{\max}) + |G_1|_\infty |\Upsilon|_\infty |\hat{G}_1 - G_1|_{\max},
\]

\[
\lesssim \rho (\rho^\Upsilon + M)(\rho^G + \mu) + \mu \rho^\Upsilon (\hat{\rho} + \rho^G) + \mu M \rho^G.
\]

Finally, recall that \(F = G_1^T \Upsilon G_2 \Xi G_2^T \Upsilon G_1 = G_1^T \Upsilon U G_1\) and a regularized estimator is given by \(\hat{F} = \hat{G}_1^T \hat{\Upsilon} \hat{U} \hat{G}_1\). Again, applying the results in Lemma \(A.6\) and \(A.10\) yields that

\[
|\hat{F} - F|_2
= |\hat{G}_1^T \hat{\Upsilon} \hat{U} \hat{G}_1 - G_1^T \Upsilon U G_1|_2,
\]

\[
\leq |\hat{G}_1^T \hat{T} - G_1^T \Upsilon U|_2 (|\hat{G}_1 - G_1|_2 + |G_1|_2)
\]

\[
+ |G_1^T \Upsilon U|_2 (|\hat{G}_1 - G_1|_2 + |G_1|_2) + |G_1^T \Upsilon^0 U|_2 |\hat{G}_1 - G_1|_2
\]

\[
\lesssim \rho (\rho^\Upsilon + \omega_1^{1/2})^2 \rho_{n,2} + (\rho_{n,2}^G + \omega_1^{1/2})^2 \rho_{n,2}^\Upsilon + \omega_1^{1/2} \rho_{n,2}^G.
\]

\(\square\)

**Remark A.5 (Admissible rate of \(\ell_n^\Pi\)).** Assume that \(\mu \lesssim s\) and \(\hat{\mu}\) is bounded by a constant. As a continuation of Remark \(A.2\), an admissible rate \(\ell_n^\Pi\) is provided by \(s^4 n^{-1/2} (\log P_n)^{1/2}\), given \(\rho_{n,2}^G, \rho_{n,2}^\Upsilon \to 0\) as \(n \to \infty\).

Consequently, applying Lemma \(A.7\) yields that \(\rho_{n,2}^G \lesssim s^5 n^{-1/2} (\log P_n)^{1/2}\) and \(\rho_{n,2}^\Upsilon \lesssim s^6 - 5b (n^{-1} \log P_n)^{(1-b)/2}\) for some \(0 \leq b < 1\) and \(s_0(K^{(1)}) \lesssim s\) such that \(\Pi \in \mathcal{U}(b, s_0(K^{(1)}))\).

**Remark A.6 (Discussion on the rate of \(\rho_{n,2}^U\)).** Suppose \(\omega_1\) is given by a constant. As a continuation of Remark \(A.3\) and \(A.4\) here we have \(\rho_{n,2}^U \lesssim s (n^{-1} \log P_n)^{(1-\ell)/2}\) (\(\ell = 0\) for the sparse case), given \(\rho_{n,2}^G, \rho_{n,2}^\Upsilon \to 0\) as \(n \to \infty\).

**Remark A.7.** Recall that \(A = G_1^T \Omega^{-1} (I - G_2 P(\Omega, G_2)) = G_1^T \Upsilon (I - U)\) and we consider the regularized estimator \(\hat{A} = \hat{G}_1^T \hat{\Upsilon} (I - \hat{U})\). Given \(|\Upsilon|_2 \leq C\)
and $|G_1|^2 \leq \omega_1$, by applying the results in Lemma A.6, we obtain

$$|\hat{A} - A|_{\text{max}} \leq |\hat{G}_1^\top \hat{\Upsilon}(I - \hat{U}) - G_1^\top \Upsilon^0(I - U)|_2$$

$$\leq |\hat{G}_1^\top \hat{\Upsilon} - G_1^\top \Upsilon^0|_2 + |\hat{U} - U|_2|G_1^\top \Upsilon^0|_2 + |\hat{G}_1^\top \hat{\Upsilon} - G_1^\top \Upsilon^0|_2(|\hat{U} - U|_2 + 1)$$

$$\leq |\hat{G}_1 - G_1|_2\Upsilon^0|_2 + (|\hat{G}_1 - G_1|_2 + |G_1|_2)|\hat{\Upsilon} - \Upsilon^0|_2 + |\hat{U} - U|_2|G_1^\top \Upsilon^0|_2$$

$$+ |\hat{G}_1 - G_1|_2|\Upsilon^0|_2 + (|\hat{G}_1 - G_1|_2 + |G_1|_2)|\hat{\Upsilon} - \Upsilon^0|_2(|\hat{U} - U|_2 + 1)$$

$$\lesssim_p \{\rho_{n,2}^{G_1} + (\rho_{n,2}^{G_1} + \omega_1^{1/2})\rho_{n,2}^\Upsilon\}(\rho_{n,2}^{U'} + 2) + \omega_1^{1/2}\rho_{n,2}^{U'} =: \rho_n^A.$$

Analogue to Lemma A.10, we have $|\hat{A} - A|_{\infty} \lesssim_p \rho_n^{A,1}$, with $\rho_n^{A,1} = s(A)\rho_n^A$ if we assume $|A|_0 = s(A)$ while $\rho_n^{A,2} = \iota(\rho_n^A)^{1-l}$ in the case of $(|A|_{\infty,\iota} \vee |A|_{\infty}) \leq \iota$ for some $0 \leq l < 1.$