Khovanov homology: torsion and thickness

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Abstract. We partially solve the conjecture by A. Shumakovitch that the Khovanov homology of a prime, non-split link in $S^3$ has a non-trivial torsion part. We give a size restriction on the Khovanov homology of almost alternating links. We relate the Khovanov homology of the connected sum of a link diagram with the Hopf link to the Khovanov homology of the diagram via a short exact sequence of homology and prove that this sequence splits. Finally, we show that our results can be adapted to reduced Khovanov homology and that there is a long exact sequence connecting reduced Khovanov homology with unreduced homology.

Introduction

Khovanov homology offers a nontrivial generalization of the Jones polynomials of links in $S^3$ (and of the Kauffman bracket skein modules of some 3-manifolds). In this paper we use Viro’s approach to construction of Khovanov homology, and utilize the fact that one works with unoriented diagrams (unoriented framed links) in which case there is a long exact sequence of Khovanov homology. Khovanov homology, over the field $\mathbb{Q}$, is a categorification of the Jones polynomial (i.e. we represent the Jones polynomial as the generating function of Euler characteristics). However, for integral coefficients Khovanov homology almost always has torsion. The first part of the paper is devoted to the construction of torsion in Khovanov homology. In the second part of the paper we analyze the thickness of Khovanov homology and reduced Khovanov homology.

The paper is organized as follows. In the first section we recall the definition of Khovanov homology and its basic properties.

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In the second section we prove that adequate link diagrams with an odd cycle property have \( \mathbb{Z}_2 \)-torsion in Khovanov homology.

In the third section we discuss torsion in the Khovanov homology of an adequate link diagram with an even cycle property.

In the fourth section we prove Shumakovitch’s theorem that prime, non-split alternating links different from the trivial knot and the Hopf link have \( \mathbb{Z}_2 \)-torsion in Khovanov homology. We generalize this result to a class of adequate links.

In the fifth section we generalize result of E.S.Lee about the Khovanov homology of alternating links (they are \( H \)-thin\(^2 \)). We do not assume rational coefficients in this generalization and we allow alternating adequate links on a surface. We use Viro’s exact sequence of Khovanov homology to extend Lee’s results to almost alternating diagrams and \( H\)-\( k \)-thick links.

In the sixth section we compute the Khovanov homology for a connected sum of \( n \) copies of Hopf links and construct a short exact sequence of Khovanov homology involving a link and its connected sum with the Hopf link. By showing that this sequence splits, we answer the question asked by Shumakovitch.

In the seventh section we notice that the results of sections 5 and 6 can be adapted to reduced Khovanov homology. Finally, we show that there is a long exact sequence connecting reduced Khovanov homology with unreduced homology.

1 Basic properties of Khovanov homology

The first spectacular application of the Jones polynomial (via Kauffman bracket relation) was the solution of Tait conjectures on alternating diagrams and their generalizations to adequate diagrams. Our method of analysing torsion in Khovanov homology has its root in work related to solutions of Tait conjectures \([\text{Ka}, \text{Mu}, \text{Th}])\.

Recall that the Kauffman bracket polynomial \(< D >\) of a link diagram \( D \) is defined by the skein relations \(< \begin{xy} <(1) > = A < (1) > + A^{-1} < (1) > \) and \(< D \cup \bigcirc > = (-A^2 - A^{-2}) < D > \) and the normalization \(< \bigcirc > = 1 \). The categorification of this invariant (named by Khovanov reduced homology) is discussed in Section 7. For the (unreduced) Khovanov homology we use the

\(^2\)We also found a simple proof of Lee’s result \([\text{Lee-1}, \text{Lee-2}]\) that for alternating links Khovanov homology yields the classical signature, see Remark 1.6.
version of the Kauffman bracket polynomial normalized to be 1 for the empty link (we use the notation $[D]$ in this case).

**Definition 1.1 (Kauffman States)**

Let $D$ be a diagram\(^3\) of an unoriented, framed link in a 3-ball $B^3$. A Kauffman state $s$ of $D$ is a function from the set of crossings of $D$ to the set $\{+1, -1\}$. Equivalently, we assign to each crossing of $D$ a marker according to the following convention:

\[
\begin{align*}
\begin{array}{c}
+1 \text{ marker} \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
-1 \text{ marker} \\
\end{array}
\end{align*}
\]

*Fig. 1.1; markers and associated smoothings*

By $D_s$ we denote the system of circles in the diagram obtained by smoothing all crossings of $D$ according to the markers of the state $s$, Fig. 1.1. By $|s|$ we denote the number of components of $D_s$.

Using this notation we have the Kauffman bracket polynomial given by the state sum formula: $[D] = (-A^2 - A^{-2}) < D > = \sum_s A^{\sigma(s)}(-A^2 - A^{-2})^{|s|}$, where $\sigma(s)$ is the number of positive markers minus the number of negative markers in the state $s$.

To define Khovanov homology it is convenient (as noticed by Viro) to consider enhanced Kauffman states.

\(^3\)We think of the 3-ball $B^3$ as $D^2 \times I$ and the diagram is drawn on the disc $D^2$. In [APS], we have proved that the theory of Khovanov homology can be extended to links in an oriented 3-manifold $M$ that is the bundle over a surface $F$ ($M = F \tilde{\times} I$). If $F$ is orientable then $M = F \times I$. If $F$ is unorientable then $M$ is a twisted $I$ bundle over $F$ (denoted by $F \tilde{\times} I$). Several results of the paper are valid for the Khovanov homology of links in $M = F \tilde{\times} I$. 
Definition 1.2 An enhanced Kauffman state $S$ of an unoriented framed link diagram $D$ is a Kauffman state $s$ with an additional assignment of $+$ or $-$ sign to each circle of $D_s$.

Using enhanced states we express the Kauffman bracket polynomial as a (state) sum of monomials which is important in the definition of Khovanov homology we use. We have $[D] = (-A^2-A^{-2}) < D > = \sum_S (-1)^{\tau(S)} A^{\sigma(s)+2\tau(S)}$, where $\tau(S)$ is the number of positive circles minus the number of negative circles in the enhanced state $S$.

Definition 1.3 (Khovanov chain complex)

(i) Let $S(D)$ denote the set of enhanced Kauffman states of a diagram $D$, and let $S_{i,j}(D)$ denote the set of enhanced Kauffman states $S$ such that $\sigma(S) = i$ and $\sigma(S) + 2\tau(S) = j$. The group $C(D)$ (resp. $C_{i,j}(D)$) is defined to be the free abelian group spanned by $S(D)$ (resp. $S_{i,j}(D)$).

(ii) For a link diagram $D$ with ordered crossings, we define the chain complex $(C(D), d)$ where $d = \{d_{i,j}\}$ and the differential $d_{i,j} : C_{i,j}(D) \to C_{i-2,j}(D)$ satisfies $d(S) = \sum S'(-1)^{t(S:S')} [S : S'] S'$ with $S \in S_{i,j}(D)$, $S' \in S_{i-2,j}(D)$, and $[S : S']$ equal to 0 or 1. $[S : S'] = 1$ if and only if markers of $S$ and $S'$ differ exactly at one crossing, call it $c$, and all the circles of $D_S$ and $D_{S'}$ not touching $c$ have the same sign\(^4\). Furthermore, $t(S : S')$ is the number of negative markers assigned to crossings in $S$ bigger than $c$ in the chosen ordering.

(iii) The Khovanov homology of the diagram $D$ is defined to be the homology of the chain complex $(C(D), d)$: $H_{i,j}(D) = \ker(d_{i,j})/d_{i+2,j}(C_{i+2,j}(D))$. The Khovanov cohomology of the diagram $D$ are defined to be the cohomology of the chain complex $(C(D), d)$.

Below we list a few elementary properties of Khovanov homology following from properties of Kauffman states used in the proof of Tait conjectures [Ka, Mu, Thi].

The positive state $s_+ = s_+(D)$ (respectively the negative state $s_- = s_-(D)$) is the state with all positive markers (resp. negative markers). The alternating diagrams without nugatory crossings (i.e. crossings in a diagram

\(^4\)From our conditions it follows that at the crossing $c$ the marker of $S$ is positive, the marker of $S'$ is negative, and that $\tau(S') = \tau(S) + 1$. 

of the form \( \square \times \square \) are generalized to adequate diagrams using properties of states \( s_+ \) and \( s_- \). Namely, the diagram \( D \) is \( + \)-adequate (resp. \( - \)-adequate) if the state of positive (resp. negative) markers, \( s_+ \) (resp. \( s_- \)), cuts the diagram to the collection of circles, so that every crossing is connecting different circles. \( D \) is an adequate diagram if it is \( + \)- and \( - \)-adequate \( [L-T] \).

**Property 1.4**

If \( D \) is a diagram of \( n \) crossings and its positive state \( s_+ \) has \( |s_+| \) circles then the highest term (in both grading indexes) of Khovanov chain complex is \( C_{n,n+2|s_+|}(D) \); we have \( C_{n,n+2|s_+|}(D) = \mathbb{Z} \). Furthermore, if \( D \) is a \( + \)-adequate diagram, then the whole group \( C_{n,n+2|s_+|}(D) = \mathbb{Z} \) and \( H_{n,n+2|s_+|}(D) = \mathbb{Z} \). Similarly the lowest term in the Khovanov chain complex is \( C_{n,n+2|s_-|}(D) \). Furthermore, if \( D \) is a \( - \)-adequate diagram, then the whole group \( C_{n,n+2|s_-|}(D) = \mathbb{Z} \) and \( H_{n,n+2|s_-|}(D) = \mathbb{Z} \). Assume that \( D \) is a non-split diagram then \( |s_+| + |s_-| \leq n + 2 \) and the equality holds if and only if \( D \) is an alternating diagram or a connected sum of such diagrams (Wu’s dual state lemma \( [Wu] \)).

**Property 1.5** Let \( \sigma(L) \) be the classical (Trotter-Murasugi) signature\(^5\) of an oriented link \( L \) and \( \hat{\sigma}(L) = \sigma(L) + \text{lk}(L) \), where \( \text{lk}(L) \) is the global linking number of \( L \), its Murasugi’s version which does not depend on an orientation of \( L \). Then

(i) \( [\text{Traczyk’s local property}] \) If \( D_0^v \) is a link diagram obtained from an oriented alternating link diagram \( D \) by smoothing its crossing \( v \) and \( D_0^v \) has the same number of (graph) components as \( D \), then \( \sigma(D) = \sigma(D_0^v) - \text{sgn}(v) \). One defines the sign of a crossing \( v \) as \( \text{sgn}(v) = \pm 1 \) according to the convention \( \text{sgn}(\sesideration) = 1 \) and \( \text{sgn}(\sesideration) = -1 \).

(ii) \( [\text{Traczyk Theorem} \ [Tr, Pr]] \)

The signature, \( \sigma(D) \), of the non-split alternating oriented link diagram \( D \) is equal to \( n_+ - |s_-| + 1 = -n_+ + |s_+| - 1 = -\frac{1}{2}(n_+ - n_- - (|s_+| - |s_-|)) = n_- - n_+ + d^+ - d^- \), where \( n_+(D) \) (resp. \( n_-(D) \)) is the number of positive (resp. negative) crossings of \( D \) and \( d^+ \) (resp. \( d^- \)) is the number of positive (resp. negative) edges in a spanning forest of the Seifert graph\(^6\) of \( D \).

\(^5\)One should not mix the signature \( \sigma(L) \) with \( \sigma(s) \) which is the signed sum of markers of the state \( s \) of a link diagram.

\(^6\)The Seifert graph, \( GS(D) \), of an oriented link diagram \( D \) is a signed graph whose
(iii) [Murasugi’s Theorem \cite{Mu1, Mu2}]
Let $D$ be a non-split alternating oriented diagram without nugatory crossings or a connected sum of such diagrams. Let $V_D(t)$ be its Jones polynomial, then the maximal degree $\max V_L(t) = n^+(L) - \frac{\sigma(L)}{2}$ and the minimal degree $\min V_L(t) = -n^-(L) - \frac{\sigma(L)}{2}$.

(iv) [Murasugi’s Theorem for unoriented link diagrams]. Let $D$ be a non-split alternating unoriented diagram without nugatory crossings or a connected sum of such diagrams. Then the maximal degree $\max <D> = \max [D] - 2 = n + 2|s_+| - 2 = 2n + sw(D) + 2\delta(D)$ and the minimal degree $\min <D> = \min [D] + 2 = -n - 2|s_-| + 2 = -2n + sw(D) + 2\delta(D)$. The self-twist number of a diagram $sw(D) = \sum v \text{sgn}(v)$, where the sum is taken over all self-crossings of $D$. A self-crossing involves arcs from the same component of a link. $sw(D)$ does not depend on orientation of $D$.

Remark 1.6
In Section 5 we reprove the result of Lee \cite{Lee1} that the Khovanov homology of non-split alternating links is supported by two adjacent diagonals of slope 2, that is $H_{i,j}(D)$ can be nontrivial only for two values of $j - 2i$ which differ by 4 (Corollary 5.5). One can combine Murasugi-Traczyk result with Viro’s long exact sequence of Khovanov homology and Theorem 7.3 to recover Lee’s result \cite{Lee2} that for alternating links Khovanov homology has the same information as the Jones polynomial and the classical signature (see Chapter 10 of \cite{Pr}). From properties 1.5 and 1.6 it follows that for non-split alternating diagram without nugatory crossings $H_{n,2n+sw+2\delta+2}(D) = H_{-n,-2n+sw+2\delta-2}(D) = \mathbb{Z}$. Thus diagonals which support nontrivial $H_{i,j}(D)$ satisfy $j - 2i = sw(D) + 2\delta(D) \pm 2$. If we consider Khovanov cohomology $H^{i,j'}(D)$, as considered in \cite{Kh, BN}, then $H^{i,j'}(D) = H_{i,j}(D)$ for vertices are in bijection with Seifert circles of $D$ and edges are in a natural bijection with crossings of $D$. For an alternating diagram the 2-connected components (blocks) of $GS(D)$ have edges of the same sign which makes $d^+$ and $d^-$ well defined.

7Recall that if $\bar{D}$ is an oriented diagram (any orientation put on the unoriented diagram $D$), and $w(\bar{D})$ is its writhe or Tait number, $w(\bar{D}) = n^+ - n^-$, then $V_{\bar{B}}(t) = A^{-3w(\bar{D})} < D >$ for $t = A^-$. P.G.Tait (1831-1901) was the first to consider the number $w(\bar{D})$ and is often called the Tait number of the diagram $\bar{D}$ and denoted by $Tait(\bar{D})$.

8The beautiful paper by Jacob Rasmussen \cite{Ras} generalizes Lee’s results and fulfil our dream (with Paweł Traczyk) of constructing a “supersignature” from Jones type construction \cite{Pr}.
\[ i' = \frac{w(D) - i}{2}, \quad j' = \frac{3w(D) - j}{2} \]
and thus
\[ j' - 2i' = \frac{1}{2}(j - 2i - w(D)) = \sigma(D) \mp 1 \]
as in Lee’s Theorem.

**Remark 1.7**
The definition of Khovanov homology extends to links in \( I \)-bundles over surfaces \( F \) \((F \neq \mathbb{RP}^2)\) [APS]. In the definition we must differentiate between trivial curves, curves bounding a Möbius band, and other non-trivial curves. Namely, we define \( \tau(S) \) as the sum of signs of circles of \( D_S \) taken over all trivial circles of \( D_S \). Furthermore, to have \([S : S'] = 1\), we assume additionally that the sum of signs of circles of \( D_S \) taken over all nontrivial circles which do not bound a Möbius band is the same for \( S \) and \( S' \).

### 2 Diagrams with odd cycle property

In the next few sections we use the concept of a graph, \( G_s(D) \), associated to a link diagram \( D \) and its state \( s \). The graphs corresponding to states \( s_+ \) and \( s_- \) are of particular interest. If \( D \) is an alternating diagram then \( G_{s_+}(D) \) and \( G_{s_-}(D) \) are the plane graphs first constructed by Tait.

**Definition 2.1**

(i) Let \( D \) be a diagram of a link and \( s \) its Kauffman state. We form a graph, \( G_s(D) \), associated to \( D \) and \( s \) as follows. Vertices of \( G_s(D) \) correspond to circles of \( D_s \). Edges of \( G_s(D) \) are in bijection with crossings of \( D \) and an edge connects given vertices if the corresponding crossing connects circles of \( D_s \) corresponding to the vertices.\(^9\)

(ii) In the language of associated graphs we can state the definition of adequate diagrams as follows: the diagram \( D \) is \(+\)-adequate (resp. \(-\)-adequate) if the graph \( G_{s_+}(D) \) (resp. \( G_{s_-}(D) \)) has no loops.

In the language of associated graphs we can state the definition of adequate diagrams as follows:

\(^9\)If \( S \) is an enhanced Kauffman state of \( D \) then, in a similar manner, we associate to \( D \) and \( S \) the graph \( G_S(D) \) with signed vertices. Furthermore, we can additionally equip \( G_S(D) \) with a cyclic ordering of edges at every vertex following the ordering of crossings at any circle of \( D_s \). The sign of each edge is the label of the corresponding crossing. In short, we can assume that \( G_S(D) \) is a ribbon (or framed) graph. We do not use this additional data in this paper but we plan to utilize this in a sequel paper.
Theorem 2.2
Consider a link diagram $D$ of $N$ crossings. Then

(+) If $D$ is $+$-adequate and $G_{s_+}(D)$ has a cycle of odd length, then the Khovanov homology has $\mathbb{Z}_2$ torsion. More precisely we show that $H_{N-2, N+2|s_+|-4}(D)$ has $\mathbb{Z}_2$ torsion.

(-) If $D$ is $-$-adequate and $G_{s_-}(D)$ has a cycle of odd length, then $H_{N-2, N-2|s_-|+4}(D)$ has $\mathbb{Z}_2$ torsion.

Proof: (+) It suffices to show that the group $C_{N-2, N+2|s_+|-4}(D)/d(C_{N, N+2|s_+|-4}(D))$ has 2-torsion.

Consider first the diagram $D$ of the left handed torus knot $T_{2,n}$ (Fig. 2.1 illustrates the case of $n = 5$). The associated graph $G_n = G_{s_+}(T_{2,n})$ is an $n$-gon.

For this diagram we have $C_{n,n+2|s_+|-4}(D) = \mathbb{Z}$, $C_{n,n+2|s_+|-4}(D) = \mathbb{Z}^n$ and $C_{n-2,n+2|s_+|-4}(D) = \mathbb{Z}^n$, where enhanced states generating $C_{n,n+2|s_+|-4}(D)$ have all markers positive and exactly one circle (of $D_S$) negative\textsuperscript{10}. Enhanced states generating $C_{n-2,n+2|s_+|-4}(D)$ have exactly one negative marker and all positive circles of $D_S$. The differential $d : C_{n,n+2|s_+|-4}(D) \rightarrow C_{n-2,n+2|s_+|-4}(D)$ can be described by an $n \times n$ circulant matrix (for the ordering of states corresponding to the ordering of crossings and regions as in Fig. 2.1)).

\textsuperscript{10}In this case $s_+ = n$ but we keep the general notation so the generalization which follows is natural.
Clearly the determinant of the matrix is equal to 2 (because \( n \) is odd; for \( n \) even the determinant is equal to 0 because the alternating sum of columns gives the zero column). To see this one can consider for example the first row expansion\(^{11}\). Therefore the group described by the matrix is equal to \( \mathbb{Z}_2 \) (for an even \( n \) one would get \( \mathbb{Z} \)). One more observation (which will be used later).

To see this one can consider for example the first row expansion

\[
\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & 1 & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}
\]

The sum of rows of the matrix is equal to the row vector \((2, 2, 2, \ldots, 2, 2)\) but the row vector \((1, 1, 1, \ldots, 1, 1)\) is not an integral linear combination of rows of the matrix. In fact the element \((1, 1, 1, \ldots, 1, 1)\) is the generator of \( \mathbb{Z}_2 \) group represented by the matrix. This can be easily checked because any sum of the odd number of states \( S_i \) represents the generator of \( \mathbb{Z}_2 \).

Now consider the general case in which \( G_{s+1}(D) \) is a graph without a loop and with an odd polygon. Again, we build a matrix presenting the group \( C_{N-2, N+2|s+1| - 4}(D) / d(C_{N,N+2|s+1| - 4}(D)) \) with the north-west block corresponding to the odd \( n \)-gon. This block is exactly the matrix described previously. Furthermore, the submatrix of the full matrix below this block is the zero matrix, as every column has exactly two nonzero entries (both equal to 1). This is the case because each edge of the graph (generator) has two endpoints (belongs to exactly two relations). If we add all rows of the matrix we get the row of all two’s. On the other hand the row of one’s cannot be created, even in the first block. Thus the row of all one’s representing the sum of all enhanced states in \( C_{N-2, N+2|s+1| - 4}(D) \) is \( \mathbb{Z}_2 \)-torsion element in the quotient group (presented by the matrix) so also in \( H_{N-2, N+2|s+1| - 4}(D) \).

\(-\) This part follows from the fact that the mirror image of \( D \), the diagram \( \bar{D} \), satisfies the assumptions of the part \((+)\) of the theorem. Therefore the quotient \( \mathcal{C}_{N-2, N+2|s+1| - 4}(D) / d(\mathcal{C}_{N,N+2|s+1| - 4}(D)) \) has \( \mathbb{Z}_2 \) torsion. Furthermore,

\[\[\begin{pmatrix}
1 & 1 & 0 & \ldots & 0 & 0 \\
0 & 1 & 1 & \ldots & 0 & 0 \\
\ldots & \ldots & 1 & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}\]

\(^{11}\)Because the matrix is a circulant one we know furthermore that its eigenvalues are equal to \( 1 + \omega \), where \( \omega \) is any \( n \)'th root of unity \( (\omega^n = 1) \), and that \( \prod_{\omega^n=1}(1 + \omega) = 0 \) for \( n \) even and 2 for \( n \) odd.
the matrix describing the map \( d : \mathcal{C}_{n-2,N-2|s_-|+4}(D) \to \mathcal{C}_{n-2,N-2|s_-|+4}(D) \) is (up to sign of every row) equal to the transpose of the matrix describing the map \( d : \mathcal{C}_{N,N+2|s_+|-4}(\bar{D}) \to \mathcal{C}_{N-2,N+2|s_+|-4}(\bar{D}) \). Therefore the torsion of the group \( \mathcal{C}_{n-2,N-2|s_-|+4}(D)/d(\mathcal{C}_{n-2,N-2|s_-|+4}(D)) \) is the same as the torsion of the group \( \mathcal{C}_{N-2,N+2|s_+|-4}(\bar{D})/d(\mathcal{C}_{N,N+2|s_+|-4}(\bar{D})) \) and, in conclusion, \( H_{n-2,N-2|s_-|+4}(D) \) has \( \mathbb{Z}_2 \) torsion\(^{12}\).

**Remark 2.3** Notice that the torsion part of the homology, \( T_{n-2,N+2|s_+|-4}(D) \), depends only on the graph \( G_{s_+}(D) \). Furthermore if \( G_{s_+}(D) \) has no 2-gons then \( H_{n-2,N+2|s_+|-4}(D) = \mathcal{C}_{n-2,N+2|s_+|-4}(D)/d(\mathcal{C}_{n,N+2|s_+|-4}(D)) \) and depends only on the graph \( G_{s_+}(D) \). See a generalization in Remark 3.6.

### 3 Diagrams with an even cycle property

If every cycle of the graph \( G_{s_+}(D) \) is even (i.e. the graph is a bipartite graph) we cannot expect that \( H_{n-2,N+2|s_+|-4}(D) \) always has nontrivial torsion. The simplest link diagram without an odd cycle in \( G_{s_+}(D) \) is the left handed torus link diagram \( T_{2,n} \) for \( n \) even. As mentioned before, in this case \( \mathcal{C}_{n-2,n+2|s_+|-4}(D)/d(\mathcal{C}_{n,n+2|s_+|-4}(D)) = \mathbb{Z} \), and, in fact \( H_{n-2,n+2|s_+|-4}(D) = \mathbb{Z} \) except \( n = 2 \), i.e. the Hopf link, in which case \( H_0(D) = 0 \).

To find torsion we have to look “deeper” into the homology. We will find a condition for which \( H_{n-4,N+2|s_+|-8}(D) \) has \( \mathbb{Z}_2 \) torsion, where \( N \) is the number of crossings of \( D \).

Analogously to the odd case, we will start from the left handed torus link \( T_{2,n} \) and associated graph \( G_{s_+}(D) \) being an \( n \)-gon with even \( n \geq 4 \); Fig. 3.1.

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\(^{12}\)Our reasoning reflects a more general fact observed by Khovanov \[Kh-1\] (see \[APS\] for the case of \( F \times I \)) that Khovanov homology satisfies “duality theorem”, namely \( H^{ij}(D) = H_{-i,-j}(\bar{D}) \). This combined with the Universal Coefficients Theorem saying that \( H^{ij}(D) = H_{ij}(D)/T_{ij}(D) \oplus T_{i-2,j}(D) \), where \( T_{ij}(D) \) denote the torsion part of \( H_{ij}(D) \) gives: \( T_{n-2,N-2|s_-|+4}(D) = T_{n-2,N+2|s_+|-4}(\bar{D}) \) (notice that \( |s_-| \) for \( D \) equals to \( |s_+| \) for \( \bar{D} \)).
Lemma 3.1
Let $D$ be the diagram of the left-handed torus link of type $(2,n)$ with $n$ even, $n \geq 4$.
Then $H_{n-4,n+2|s_+| - 8}(D) = H_{n-4,3n-8}(D) = C_{n-4,3n-8}(D)/d(C_{n-2,3n-8}(D)) = \mathbb{Z}_2$.
Furthermore, every enhanced state from the basis of $C_{n-4,3n-8}(D)$ (or an odd sum of such states) is the generator of $\mathbb{Z}_2$.

Proof: We have $n = |s_+|$. The chain group $C_{n-4,3n-8}(D) = \mathbb{Z}^{n(n-1)/2}$ is freely generated by enhanced states $S_{i,j}$, where exactly $i$th and $j$th crossings have negative markers, and all the circles of $D_{S_{i,j}}$ are positive (crossings of $D$ and circles of $D_{s_+}$ are ordered in Fig. 3.1). We have to understand the differential $d : C_{n-2,3n-8}(D) \to C_{n-4,3n-8}(D)$. The chain group $C_{n-2,3n-8}(D) = \mathbb{Z}^{n(n-1)}$ is freely generated by enhanced states with $i$th negative marker and one negative circle of $D_S$. In our notation we will write $S_{i,(i-1,i)}$ if the negative circle is obtained by connecting circles $i-1$ and $i$ in $D_{s_+}$ by a negative marker. Notation $S_{i,(j)}$ is used if we have $j$th negative circle, $j \neq i-1, j \neq i$. The states $S_{1,(2)}$, $S_{1,(3)}$ and $S_{1,(4,1)}$ are shown in Fig. 3.1 ($n = 4$ in the figure). The quotient group $C_{n-4,3n-8}(D)/d(C_{n-2,3n-8}(D))$ can be presented by a $n(n-1) \times n(n-1)/2$ matrix, $E_n$. One should just understand the images of enhanced states of $C_{n-2,3n-8}(D)$. In fact, for a fixed crossing $i$ the corresponding $n-1 \times n-1$ block is (up to sign of columns\textsuperscript{13}) the circulant

\textsuperscript{13}In the $(n-1) \times (n-1)$ block corresponding to the $i$th crossing (i.e. we consider only states in which $i$th crossing has a negative marker), the column under the generator $S_{i,j}$ of $C_{n-4,3n-8}$ has $+1$ entries if $i < j$ and $-1$ entries if $i > j$. 

11
matrix discussed in Section 2. Our goal is to understand the matrix $E_n$, to show that it represents the group $\mathbb{Z}_2$ and to find natural representatives of the generator of the group. For $n = 4$, $d : \mathbb{Z}^6 \to \mathbb{Z}^6$ and it is given by: $d(S_{1(2)}) = S_{12} + S_{13}$, $d(S_{1(3)}) = S_{13} + S_{14}$, $d(S_{1,1(4)}) = S_{12} + S_{14}$, $d(S_{2,1(2)}) = -S_{21} + S_{23}$, $d(S_{2,3}) = S_{23} + S_{24}$, $d(S_{2,4}) = -S_{21} + S_{24}$, $d(S_{3,1}) = -S_{31} - S_{32}$, $d(S_{3,2}) = -S_{32} + S_{34}$, $d(S_{3,4}) = -S_{31} + S_{34}$, $d(S_{4,1}) = -S_{41} - S_{42}$, $d(S_{4,2}) = -S_{42} - S_{43}$, $d(S_{4,3}) = -S_{41} - S_{43}$.

Therefore $d$ can be described by the $12 \times 6$ matrix. States are ordered lexicographically, e.g. $S_{i,j}$ ($i < j$) is before $S_{i',j'}$ ($i' < j'$) if $i < i'$ or $i = i'$ and $j < j'$.

$$\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 \\
0 & 0 & -1 & 0 & 0 & -1
\end{pmatrix}$$

In our example the rows correspond to $S_{1,2}, S_{1,3}, S_{1,1(4)}, S_{2,1(2)}, S_{2,3}, S_{2,4}, S_{3,1}, S_{3,2}, S_{3,4}, S_{4,1}, S_{4,2}, S_{4,3}$, and $S_{4,4(3)}$, the columns correspond to $S_{1,2}, S_{1,3}, S_{1,4}, S_{2,3}, S_{2,4}, S_{3,4}$ in this order. Notice that the sum of columns of the matrix gives the non-zero column of all $\pm 2$ or $0$. Therefore over $\mathbb{Z}_2$ our matrix represents a nontrivial group. On the other hand, over $\mathbb{Q}$, the matrix represent the trivial group. Thus over $\mathbb{Z}$ the group represented by the matrix has $\mathbb{Z}_2$ torsion. More precisely, we can see that the group is $\mathbb{Z}_2$ as follows: The row relations can be expressed as: $S_{1,2} = -S_{1,3} = S_{1,4} = -S_{1,2}$, $S_{2,1} = S_{2,3} = -S_{2,4} = -S_{2,1}$, $S_{3,1} = -S_{3,2} = -S_{3,4} = -S_{3,1}$ and $S_{4,1} = -S_{4,2} = S_{4,3} = -S_{4,1}$. $S_{i,j} = S_{j,i}$ in our notation. In particular, it follows from these equalities that the group given by the matrix is equal to $\mathbb{Z}_2$ and is generated by any basic enhanced state $S_{i,j}$ or the sum of odd number of $S_{i,j}$’s.

Similar reasoning works for any even $n \geq 4$ (not only $n = 4$).

Furthermore, $C_{n-6,3n-8} = 0$, therefore $H_{n-4,3n-8} = C_{n-4,3n-8}/d(C_{n-2,3n-8}) = \mathbb{Z}_2$. □
We are ready now to use Lemma 3.1 in the general case of an even cycle.

**Theorem 3.2**

Let $D$ be a connected diagram of a link of $N$ crossings such that the associated graph $G_{s_+}(D)$ has no loops (i.e. $D$ is $+$-adequate) and the graph has an even $n$-cycle with a singular edge (i.e. not a part of a 2-gon). Then $H_{N-4,N+2|s_+|-8}(D)$ has $\mathbb{Z}_2$ torsion.

**Proof:** Consider an ordering of crossings of $D$ such that $e_1, e_2, ..., e_n$ are crossings (edges) of the $n$-cycle. The chain group $C_{N-2,N+2|s_+|-8}(D)$ is freely generated by $N(V-1)$ enhanced states, $S_{i,(c)}$, where $N$ is the number of crossings of $D$ (edges of $G_{s_+}(D)$) and $V = |s_+|$ is the number of circles of $D_{s_+}$ (vertices of $G_{s_+}(D)$). $S_{i,(c)}$ is the enhanced state in which the crossing $e_i$ has the negative marker and the circle $c$ of $D_{s_i}$ is negative, where $s_i$ is the state which has all positive markers except at $e_i$. The chain group $C_{N-4,N+2|s_+|-8}(D)$ is freely generated by enhanced states which we can partition into two groups.

(i) States $S_{i,j}$, where crossings $e_i$, $e_j$ have negative markers and corresponding edges of $G_{s_+}(D)$ do not form part of a multi-edge (i.e. $e_i$ and $e_j$ do not have the same endpoints). All circles of the state $S_{i,j}$ are positive.

(ii) States $S'_{i,j}$ and $S''_{i,j}$, where crossings $e_i$, $e_j$ have negative markers and corresponding edges of $G_{s_+}(D)$ are parts of a multi-edge (i.e. $e_i$, $e_j$ have the same endpoints). All but one circle of $S'_{i,j}$ and $S''_{i,j}$ are positive and we have two choices for a negative circle leading to $S'_{i,j}$ and $S''_{i,j}$, i.e. the crossings $e_i$, $e_j$ touch two circles, and we give negative sign to one of them.

In our proof we will make the essential use of the assumption that the edge (crossing) $e_1$ is a singular edge.

We analyze the matrix presenting the group $C_{N-4,N+2|s_+|-8}(D)/d(C_{N-2,N+2|s_+|-8}(D))$.

By Lemma 3.1, we understand already the $n(n-1) \times \frac{1}{2}n(n-1)$ block corresponding to the even $n$-cycle. In this block every column has 4 non-zero entries (two +1 and two −1), therefore columns of the full matrix corresponding to states $S_{i,j}$, where $e_i$ and $e_j$ are in the $n$-gon, have zeros outside our block. We use this property later.

We now analyze another block represented by rows and columns associated to states having the first crossing $e_1$ with the negative marker. This $(V-1) \times (N-1)$ block has entries equal to 0 or 1. If we add rows in this block we obtain the vector row of two’s (2, 2, ..., 2), following from the fact that every edge of $G_{s_+}(D)$ and of $G_{s_1}(D)$ has 2 endpoints (we use the fact that $D$ is $+$-adequate and $e_1$ is a singular edge). Consider now the bigger submatrix of the full matrix composed of the same rows as our block.
but without restriction on columns. All additional columns are 0 columns as our row relations involve only states with negative marker at $e_1$. Thus the sum of these rows is equal to the row vector $(2, 2, ..., 2, 0, ..., 0)$. We will argue now that the half of this vector, $(1, 1, ..., 1, 0, ..., 0)$, is not an integral linear combination of rows of the full matrix and so represents $\mathbb{Z}_2$-torsion element of the group $C_{N-4,N+2|s_+|-8}(D)/d(C_{N-2,N+2|s_+|-8}(D))$. For simplicity assume that $n = 4$ (but the argument holds for any even $n \geq 4$). Consider the columns indexed by $S_{1,2}, S_{1,3}, S_{1,4}, S_{2,3}, S_{2,4}$ and $S_{3,4}$. The integral linear combination of rows restricted to this columns cannot give a row with odd number of one’s, as proven in Lemma 3.1. In particular we cannot get the row vector $(1, 1, 1, 0, 0, 0)$. This excludes the row $(1, 1, ..., 1, 0, ..., 0)$, as an integral linear combination of rows of the full matrix. Therefore the sum of enhanced states with the marker of $e_1$ negative is 2-torsion element in $C_{N-4,N+2|s_+|-4}(D)$ and therefore in $H_{N-4,N+2|s_+|-8}(D)$. □

Similarly, using duality, we can deal with $-+$-adequate diagrams.

**Corollary 3.3**

Let $D$ be a connected, $-+$-adequate diagram of a link and the graph $G_{s_+}(D)$ has an even $n$-cycle, $n \geq 4$, with a singular edge. Then $H_{-N+2,-N-2|s_-|+8}(D)$ has $\mathbb{Z}_2$ torsion.

**Remark 3.4**

The restriction on $D$ to be a connected diagram is not essential (it just simplifies the proof) as for a non-connected diagram, $D = D_1 \sqcup D_2$ we have “Künneth formula” $H_*(D) = H_*(D_1) \otimes H_*(D_2)$ so if any of $H_*(D_i)$ has torsion then $H_*(D)$ has torsion as well.

We say that a link diagram is doubly $+-$adequate if its graph $G_{s_+}(D)$ has no loops and 2-gons. In other words, if a state $s$ differs from the state $s_+$ by two markers then $|s| = |s_+| - 2$. We say that a link diagram is doubly $-+$-adequate if its mirror image is doubly $+-$adequate.

**Corollary 3.5**

Let $D$ be a connected doubly $+-$adequate diagram of a link of $N$ crossings, then either $D$ represents the trivial knot or one of the groups $H_{N-2,N+2|s_+|-4}(D)$ and $H_{N-4,N+2|s_+|-8}(D)$ has $\mathbb{Z}_2$ torsion.

**Proof:** The associated graph $G_{s_+}(D)$ has no loops and 2-gons. If $G_{s_+}(D)$ has an odd cycle then by Theorem 2.2 $H_{N-2,N+2|s_+|-4}(D)$ has $\mathbb{Z}_2$ torsion. If
\(G_{s+}(D)\) has an even \(n\)-cycle, \(n \geq 4\) then \(H_{N-4,N+2|s+|4m}(D)\) has \(\mathbb{Z}_2\) torsion by Theorem 3.2 (every edge of \(G_{s+}(D)\) is a singular edge as \(G_{s+}(D)\) has no \(2\)-gons). Otherwise \(G_{s+}(D)\) is a tree, each crossing of \(D\) is a nugatory crossing and \(D\) represents the trivial knot. □

We can generalize and interpret Remark 2.3 as follows.

**Remark 3.6**
Assume that the associated graph \(G_{s+}(D)\) has no \(k\)-gons, for every \(k \leq m\). Then the torsion part of Khovanov homology, \(T_{N-2m,N+2|s+|4m}(D)\) depends only on the graph \(G_{s+}(D)\). Furthermore, \(H_{N-2m+2,N+2|s+|4m+4}(D) = C_{N-2m+2,N+2|s+|4m+4}(D)/d(C_{N-2m+2,N+2|s+|4m+4}(D))\) and it depends only on the graph \(G_{s+}(D)\). On a more philosophical level\(^{14}\) our observation is related to the fact that if the edge \(e_c\) in \(G_{s+}(D)\) corresponding to a crossing \(c\) in \(D\) is not a loop then for the crossing \(c\) the graphs \(G_{s+}(D_0)\) and \(G_{s+}(D_\infty)\) are the graphs obtained from \(G_{s+}(D)\) by deleting \((G_{s+}(D) - e_c)\) and contracting \((G_{s+}(D)/e_c)\), respectively, the edge \(e_c\) (compare Fig. 3.2).

\[\text{Fig. 3.2}\]

\(^{14}\)In order to be able to recover the full Khovanov homology from the graph \(G_{s+}\) we would have to equip the graph with additional data: ordering of signed edges adjacent to every vertex. This allows us to construct a closed surface and the link diagram \(D\) on it so that \(G_{s+} = G_{s+}(D)\). The construction imitates the 2-cell embedding of Edmonds (but every vertex corresponds to a circle and signs of edges regulate whether an edge is added inside or outside of the circle). If the surface we obtain is equal to \(S^2\) we get the classical Khovanov homology. If we get a higher genus surface we have to use APS theory. This can be utilised also to construct Khovanov homology of virtual links (via Kuperberg minimal genus embedding theory [Ku]). For example, if the graph \(G_{s+}\) is a loop with adjacent edge(s) ordered \(e,-e\) then the diagram is composed of a meridian and a longitude on the torus.
Example 3.7
Consider the 2-component alternating link $6_2^2$ (10/3 rational link), with $G_{s_1}(D) = G_{s_2}(D)$ being a square with one edge tripled (this is a self-dual graph); see Fig. 3.3. Corollary 3.5 does not apply to this case but Theorem 3.2 guarantees $\mathbb{Z}_2$ torsion at $H_2,6(D)$ and $H_4,−6(D)$. In fact, the KhoHo \cite{Sh-2} computation gives the following Khovanov homology$^{15}$: $H_{6,14} = H_{6,10} = H_{4,10} = \mathbb{Z}$, $H_{2,6} = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_{2,2} = \mathbb{Z}$, $H_{0,2} = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_{0,−2} = \mathbb{Z}$, $H_{−2,−2} = \mathbb{Z} \oplus \mathbb{Z}_2$, $H_{−2,−6} = \mathbb{Z}$, $H_{−4,−6} = \mathbb{Z}_2$, $H_{−4,−10} = \mathbb{Z}$, $H_{−6,−10} = H_{−6,−14} = \mathbb{Z}$.

Fig. 3.3

4 Torsion in the Khovanov homology of alternating and adequate links

We show in this section how to use technical results from the previous sections to prove Shumakovitch’s result on torsion in the Khovanov homology of alternating links and the analogous result for a class of adequate diagrams.

Theorem 4.1 (Shumakovitch) The alternating link has torsion free Khovanov homology if and only if it is the trivial knot, the Hopf link or the connected or split sum of copies of them. The nontrivial torsion always contains the $\mathbb{Z}_2$ subgroup.

$^{15}$Tables and programs by Bar-Natan and Shumakovitch \cite{BN-3, Sh-2} use the version of Khovanov homology for oriented diagrams, and the variable $q = A^{−2}$, therefore their monomial $q^{i,b}$ corresponds to the free part of the group $H_{i,j}(D;\mathbb{Z})$ for $j = −2b + 3w(D)$, $i = −2a + w(D)$ and the monomial $Q^{i,b}$ corresponds to the $\mathbb{Z}_2$ part of the group again with $j = −2b + 3w(D)$, $i = −2a + w(D)$. KhoHo gives the torsion part of the polynomial for the oriented link $6_2^2$, with $w(D) = −6$, as $Q^{−6t^{−1}} + Q^{−8t^{−2}} + Q^{−10t^{−3}} + Q^{−12t^{−4}}$. 

16
The fact that the Khovanov homology of the connected sum of Hopf links is a free group, is discussed in Section 6 (Corollary 6.6).

We start with the “only if” part of the proof by showing the following geometric fact.

**Lemma 4.2** Assume that $D$ is a link diagram which contains a clasp: either $T_{[-2]} = \raisebox{-5pt}{\includegraphics[height=10pt]{clasp.png}}$ or $T_{[2]} = \raisebox{-5pt}{\includegraphics[height=10pt]{clasp.png}}$. Assume additionally that the clasp is not a part of the Hopf link summand of $D$. Then if the clasp is of $T_{[-2]}$ type then the associated graph $G_{s_+}(D)$ has a singular edge. If the clasp is of $T_{[2]}$ type then the associated graph $G_{s_-}(D)$ has a singular edge. Furthermore the singular edge is not a loop.

**Proof:** Consider the case of the clasp $T_{[-2]}$, the case of $T_{[2]}$ being similar. The region bounded by the clasp corresponds to the vertex of degree 2 in $G_{s_+}(D)$. The two edges adjacent to this vertex are not loops and they are not singular edges only if they share the second endpoint as well. In that case our diagram looks like on the Fig. 4.1 so it clearly has a Hopf link summand (possibly it is just a Hopf link) as the north part is separated by a clasp from the south part of the diagram. □

![Fig. 4.1](image.png)

**Corollary 4.3** If $D$ is a $+\text{-adequate diagram}$ (resp. $-\text{-adequate diagram}$) with a clasp of type $T_{[-2]}$ (resp. $T_{[2]}$), then Khovanov homology contains $\mathbb{Z}_2$-torsion or $T_{[-2]}$ (resp. $T_{[2]}$) is a part of a Hopf link summand of $D$.

**Proof:** Assume that $T_{[-2]}$ is not a part of Hopf link summand of $D$. By Lemma 4.2 the graph $G_{s_+}(D)$ has a singular edge. Furthermore, the graph $G_{s_+}(D)$ has no loops as $D$ is $+\text{-adequate}$. If the graph has an odd cycle then $H_{N-2,N+2|s_+|−4}(D)$ has $\mathbb{Z}_2$ torsion by Theorem 2.2. If $G_{s_+}(D)$ is bipartite (i.e. it has only even cycles), then consider the cycle containing the singular edge. It is an even cycle of length at least 4, so by Theorem 3.2 $H_{N-4,N+2|s_+|−8}(D)$ has $\mathbb{Z}_2$ torsion. A similar proof works in $-\text{-adequate case}$. □
With this preliminary result we can complete our proof of Theorem 4.1.

Proof: First we prove the theorem for non-split, prime alternating links. Let $D$ be a diagram of such a link without a nugatory crossing. $D$ is an adequate diagram (i.e. it is $+$ and $-$ adequate diagram), so it is enough to show that if $G_{s+}(D)$ (or $G_{s-}(D)$) has a double edge then $D$ can be modified by Tait flypes into a diagram with $T_{[-2]}$ (resp. $T_{[2]}$) clasp. This is a standard fact, justification of which is illustrated in Fig. 4.2.

If we do not assume that $D$ is a prime link then we use the theorem by Menasco [Me] that prime decomposition is visible on the level of a diagram. In particular the Tait graphs $G_{s+}(D)$ and $G_{s-}(D)$ have block structure, where each block (2-connected component) corresponds to prime factor of a link. Using the previous results we see that the only situation when we didn’t find torsion is if every block represents a Hopf link so $D$ represents the sum of Hopf links (including the possibility that the graph is just one vertex representing the trivial knot).

If we relax condition that $D$ is a non-split link then we use the fact, mentioned before, that for $D = D_1 \sqcup D_2$, Khovanov homology satisfies Künneth’s formula, $H(D) = H(D_1) \otimes H(D_2)$. □

Example 4.4 (The 8_19 knot)
The first entry in the knot tables which is not alternating is the (3, 4) torus knot, $8_{19}$. It is $+$-adequate as it is a positive 3-braid, the closure of $(\sigma_1 \sigma_2)^4$.

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16For alternating diagrams, $G_{s+}(D)$ and $G_{s-}(D)$ are Tait graphs of $D$. These graphs are plane graphs and the only possibilities when multiple edges are not “parallel” is if our graphs are not 3-connected (as $D$ is not a split link, graphs are connected, and because $D$ is a prime link, the graphs are 2-connected). Tait flype corresponds to the special case of change of the graph in its 2-isomorphic class as illustrated in Fig. 4.2.
Every positive braid is +–-adequate but its associated graph $G_{s_+}(D)$ is composed of 2-gons. Furthermore the diagram $D$ of $8_{19}$ is not +–-adequate, Fig. 4.3. KhoHo shows that the Khovanov homology of $8_{19}$ has torsion, namely $H_{2,2} = \mathbb{Z}_2$. This torsion is hidden deeply inside the homology spectrum\(^{17}\), which starts from maximum $H_{8,14}(D) = \mathbb{Z}$ and ends on the minimum $H_{-2,-10}(D) = \mathbb{Z}$.

$$D = 8_{19}$$

$$G_{s_+}(D)$$

$$G_{s_-}(D)$$

Fig. 4.3

The simplest alternating link which satisfies all conditions of Theorem 3.2 except for the existence of a singular edge, is the four component alternating link of 8 crossings $8_3^4$ [Rol]; Fig. 4.4. We know that $H_{s_+}(8_3^4)$ has torsion (by using duality) but Theorem 3.2 does not guarantee torsion in $H_{N-4,N+2|s_+|-8}(D) = H_{4,8}(D)$, the graph $G_{s_+}(D)$ is a square with every edge doubled; Fig. 4.3. We checked, however using KhoHo the torsion part and in fact $T_{4,8}(D) = \mathbb{Z}_2$. This suggests that Theorem 3.2 can be improved\(^{18}\).

\(^{17}\)The full graded homology group is: $H_{8,14}(D) = H_{8,10}(D) = H_{4,6}(D) = \mathbb{Z}$, $H_{2,2} = \mathbb{Z}_2$, $H_{0,2}(D) = H_{2,-2}(D) = H_{0,-2}(D) = H_{-2,-4}(D) = H_{2,-10}(D) = \mathbb{Z}$.

\(^{18}\)In [BN2] the figure describes, by mistake, the mirror image of $8_3^4$. The full homology is as follows: $H_{8,16} = \mathbb{Z}$, $H_{8,12} = \mathbb{Z}$, $H_{6,12} = \mathbb{Z}_2 \oplus \mathbb{Z}^4$, $H_{4,8} = \mathbb{Z}_2 \oplus \mathbb{Z}^4$, $H_{2,4} = \mathbb{Z}_2$, $H_{2,0} = \mathbb{Z}_4$, $H_{0,0} = \mathbb{Z}_7$, $H_{0,-4} = \mathbb{Z}^6$, $H_{-2,-4} = \mathbb{Z}_2^4 \oplus \mathbb{Z}^3$, $H_{-2,-8} = \mathbb{Z}$, $H_{-4,-8} = \mathbb{Z}^2$, $H_{-4,-12} = \mathbb{Z}^3$, $H_{-6,-12} = \mathbb{Z}_2^3$, $H_{-6,-16} = \mathbb{Z}^3$, $H_{-8,-16} = \mathbb{Z}$, $H_{-8,-20} = \mathbb{Z}$. In KhoHo the generating polynomials, assuming $w(8_3^4) = -8$, are: $\text{KhPol}(8_3^4,21) = \{(q^{18} + q^{16}) * t^8 + 3 * q^{16} * t^7 + (3 * q^{14} + 3 * q^{12}) * t^6 + (q^{12} + 3 * q^{10}) * t^5 + (6 * q^{10} + 7 * q^8) * t^4 + 4 * q^8 * t^3 + (q^8 + 4 * q^6) * t^2 + q^2 * t + (q^2 + 1)/(q^{20} * t^8),$

$(3 * Q^{10} * t^5 + 3 * Q^8 * t^4 + Q^6 * t^3 + 4 * Q^2 * t + 1)/(Q^{16} * t^6))$. 

19
5 Thickness of Khovanov homology and almost alternating links

We define, in this section, the notion of an $H$-$k$-thick link diagram and relate it to $(k - 1)$-almost alternating diagrams. In particular we give a short proof of Lee’s theorem [Lee-1] (conjectured by Khovanov, Bar-Natan, and Garoufalidis) that alternating non-split links are $H$-$1$-thick ($H$-thin in Khovanov terminology).

Definition 5.1 We say that a link is $k$-almost alternating if it has a diagram which becomes alternating after changing $k$ of its crossings.

As noted in Property 1.4 the “extreme” terms of Khovanov chain complexes are $C_{N,N+2|s_+|}(D) = C_{-N,-N-2|s_-|}(D) = \mathbb{Z}$. In the following definition of a $H$-$(k_1, k_2)$-thick diagram we compare indices of actual Khovanov homology of $D$ with lines of slope 2 going through the points $(N, N + 2|s_+|)$ and $(-N, -N - 2|s_-|)$.

Definition 5.2 (i) We say that a link diagram, $D$ of $N$ crossings is $H$-$(k_1, k_2)$-thick if $H_{i,j}(D) = 0$ with a possible exception of $i$ and $j$ satisfying:

$$N - 2|s_-| - 4k_2 \leq j - 2i \leq 2|s_+| - N + 4k_1.$$
(ii) We say that a link diagram of $N$ crossings is $H$-$k$-thick\footnote{Possibly, the more appropriate name would be $H$-$k$-thin diagram, as the width of Khovanov homology is bounded from above by $k$. Khovanov \cite{Kh-2}, page 7 suggests the term homological width; $\text{hw}(D) = k$ if homology of $D$ lies on $k$ adjacent diagonals (in our terminology, $D$ is $k - 1$ thick).} if, it is $H$-$(k_1, k_2)$-thick where $k_1$ and $k_2$ satisfy:

$$k \geq k_1 + k_2 + \frac{1}{2}(|s_+| + |s_-| - N).$$

(iii) We define also $(k_1, k_2)$-thickness (resp. $k$-thickness) of Khovanov homology separately for the torsion part (we use the notation $\text{TH}-(k_1, k_2)$-thick diagram), and for the free part (we use the notation $\text{FH}-(k_1, k_2)$-thick diagram).

Our $\text{FH}-1$-thick diagram is a $H$-thin diagram in \cite{Kh-2, Lee-1, BN-1, Sh-1}.

With the above notation we are able to formulate our main result of this section.

\textbf{Theorem 5.3}

If the diagram $D_\infty = D(\bigotimes)$ is $H$-$(k_1(D_\infty), k_2(D_\infty))$-thick and the diagram $D_0 = D(\bigotimes)$ is $H$-$(k_1(D_0), k_2(D_0))$-thick, then the diagram $D_+ = D(\bigotimes)$ is $H$-$(k_1(D_+), k_2(D_+))$-thick where

$$k_1(D_+) = \max(k_1(D_\infty) + \frac{1}{2}(|s_+(D_\infty)| - |s_+(D_\infty)| + 1), k_1(D_0))$$

and

$$k_2(D_+) = \max(k_2(D_\infty), k_2(D_0)) + \frac{1}{2}(|s_-(D_0)| - |s_-(D_\infty)| + 1)).$$

In particular

(i) if $|s_+(D_+)| - |s_+(D_\infty)| = 1$, as is always the case for a $+$-adequate diagram, then $k_1(D_+) = \max(k_1(D_\infty), k_1(D_0))$,

(ii) if $|s_-(D_+)| - |s_-(D_\infty)| = 1$, as is always the case for a $-$-adequate diagram, then $k_2(D_+) = \max(k_2(D_\infty), k_2(D_0))$.

\textbf{Proof:} We formulated our definitions so that our proof follows almost immediately via the Viro’s long exact sequence of Khovanov homology:

$$\cdots \to H_{i+1,j-1}(D_0) \xrightarrow{\partial} H_{i+1,j+1}(D_\infty) \xrightarrow{\alpha} H_{i,j}(D_+) \xrightarrow{\beta} H_{i-1,j+1}(D_\infty) \to \cdots$$
If $0 \neq h \in H_{i,j}(D_\pm)$ then either $h = \alpha(h')$ for $0 \neq h' \in H_{i+1,j+1}(D_\infty)$ or $0 \neq \beta(h) \in H_{i-1,j-1}(D_0)$. Thus if $H_{i,j}(D_\pm) \neq 0$ then either $H_{i+1,j+1}(D_\infty) \neq 0$ or $H_{i-1,j-1}(D_0) \neq 0$. The first possibility gives the inequalities involving $(j + 1) - 2(i + 1)$:

$$N(D_\infty) - 2|s_-(D_\infty)| - 4k_2(D_\infty) \leq j - 2i - 1 \leq 2|s_+(D_\infty)| - N(D_\infty) + 4k_1(D_\infty)$$

which, after observing that $|s_-(D_\pm)| = |s_-(D_\infty)|$, leads to:

$$N(D_\pm) - 2|s_-(D_\pm)| - 4k_2(D_\infty) \leq j - 2i \leq 2|s_+(D_\infty)| - N(D_\infty) + 4k_1(D_\infty) + 1).$$

The second possibility gives the inequalities involving $(j - 1) - 2(i - 1)$:

$$N(D_0) - 2|s_-(D_0)| - 4k_2(D_0) \leq j - 2i + 1 \leq 2|s_+(D_0)| - N(D_0) + 4k_1(D_0)$$

which, after observing that $|s_+(D_\pm)| = |s_+(D_0)|$, leads to:

$$N(D_\pm) - 2|s_-(D_\pm)| - 4k_2(D_0) - 2(|s_-(D_0)| - |s_-(D_\pm)| + 1) \leq j - 2i \leq 2|s_+(D_\pm)| - N(D_\pm) + 4k_1(D_0).$$

Combining these two cases we obtain the conclusion of Theorem 5.3. □

**Corollary 5.4** If $D$ is an adequate diagram such that, for some crossing of $D$, the diagrams $D_0$ and $D_\infty$ are $H$-$(k_1,k_2)$-thick (resp. $H$-$k$-thick) then $D$ is $H$-$(k_1,k_2)$-thick (resp. $H$-$k$-thick).

**Corollary 5.5** Every alternating non-split diagram without a nugatory crossing is $H$-$(0,0)$-thick and $H$-1-thick.

**Proof:** The $H$-$(k_1,k_2)$-thickness in Corollary 5.4 follows immediately from Theorem 5.3. To show $H$-$k$-thickness we observe additionally that for an adequate diagram $D_\pm$ one has $|s_+(D_0)| + |s_-(D_0)| - N(D_0) = |s_+(D_\pm)| + |s_-(D_\pm)| - N(D_\pm) = |s_+(D_\infty)| + |s_-(D_\infty)| - N(D_\infty).

We prove Corollary 5.5 using induction on the number of crossings a slightly more general statement allowing nugatory crossings.

+ If $D$ is an alternating non-split $+$-adequate diagram then $H_{i,j}(D) \neq 0$ can happen only for $j - 2i \leq 2|s_+(D)| - N(D)$.

- If $D$ is an alternating non-split $-$-adequate diagram then $H_{i,j}(D) \neq 0$ can happen only for $N(D) - 2|s_-(D)| \leq j - 2i$.  

22
If the diagram $D$ from $(+)$ has only nugatory crossings then it represents the trivial knot and its nontrivial Khovanov homology are $H_{N,3N-2}(D) = H_{N,3N+2}(D) = \mathbb{Z}$. Because $|s_+(D)| = N(D) + 1$ in this case, the inequality $ (+)$ holds. In the inductive step we use the property of a non-nugatory crossing of a non-split $+$-adequate diagram, namely $D_0$ is also an alternating non-split $+$-adequate diagram and inductive step follows from Theorem 5.3. Similarly one proves the condition $(-)$. Because the non-split alternating diagram without nugatory crossings is an adequate diagram, therefore Corollary 5.5 follows from Conditions $(+)$ and $(-)$. □

The conclusion of the theorem is the same if we are interested only in the free part of Khovanov homology (or work over a field). In the case of the torsion part of the homology we should take into account the possibility that torsion “comes” from the free part of the homology, that is $H_{i+1,j+1}(D_\infty)$ may be torsion free but its image under $\alpha$ may have torsion element.

**Theorem 5.6** If $T_{i,j}(D_+) \neq 0$ then either

1. $T_{i+1,j+1}(D_\infty) \neq 0$ or $T_{i-1,j-1}(D_0) \neq 0$,

or

2. $FH_{i+1,j+1}(D_\infty) \neq 0$ and $FH_{i+1,j-1}(D_0) \neq 0$.

**Proof:** From the long exact sequence of Khovanov homology it follows that the only way the torsion is not related to the torsion of $H_{i+1,j+1}(D_\infty)$ or $H_{i-1,j-1}(D_0)$ is the possibility of torsion created by taking the quotient $FH_{i+1,j+1}(D_\infty)/\partial(FH_{i+1,j-1}(D_0))$ and in this case both groups $FH_{i+1,j+1}(D_\infty)$ and $FH_{i+1,j-1}(D_0)$ have to be nontrivial. □

**Corollary 5.7** If $D$ is an alternating non-split diagram without a nugatory crossing then $D$ is $TH-(0,-1)$-thick and $TH$-$0$-thick. In other words if $T_{i,j}(D) \neq 0$ then $j - 2i = 2|s_+(D)| - N(D) = N(D) - 2|s_-(D)| + 4$.

**Proof:** We proceed in the same (inductive) manner as in the proof of Corollary 5.5, using Theorem 5.7 and Corollary 5.5. In the first step of the induction we use the fact that the trivial knot has no torsion in Khovanov homology. □

The interest in $H$-thin diagrams was motivated by the observation (proved by Lee) that diagrams of non-split alternating links are $H$-thin (see Corollary 5.5). Our approach allows the straightforward generalization to $k$-almost alternating diagrams.
Corollary 5.8 Let $D$ be a non-split $k$-almost alternating diagram without a nugatory crossing. Then $D$ is $H-(k, k)$-thick and $TH-(k, k-1)$-thick.

Proof: The corollary holds for $k = 0$ (alternating diagrams) and we use an induction on the number of crossings needed to change the diagram $D$ to an alternating diagram, using Theorem 5.3 in each step. □

We were assuming throughout the section that our diagrams are non-split. This assumption was not always necessary. In particular even the split alternating diagram without nugatory crossings is $H-(0, 0)$-thick as follows from the following observation.

Lemma 5.9 If the diagrams $D'$ and $D''$ are $H-(k', k''_1)$-thick and $H-(k''_2, k''_2)$-thick, respectively, then the diagram $D = D' \sqcup D''$ is $H-(k'+k'_1, k''_2+k''_2)$-thick.

Proof: Lemma 5.9 follows from the obvious fact that in the split sum $D = D' \sqcup D''$ we have $N(D) = N(D') + N(D'')$, $|s_+(D)| = |s_+(D')| + |s_+(D'')|$ and $|s_-(D)| = |s_-(D')| + |s_-(D'')|$. □

Khovanov observed ([Kh-2, Proposition 7]) that adequate non-alternating knots are not $H-1$-thick. We are able to proof the similar result about torsion of adequate non-alternating links.

Theorem 5.10 Let $D$ be a connected adequate diagram which does not represent an alternating link and such that $G_{s_+}(D)$ and $G_{s_-}(D)$ have either an odd cycle or an even cycle with a singular edge, then $D$ is not $TH-0$-thick diagram. More generally, $D$ is at best $TH-\frac{1}{2}(N + 2 - (|s_+(D)| + |s_-(D)|))-thick$.

Proof: The first part of Theorem 5.10 follows from the second part because by Proposition 1.4 (Wu’s Lemma), $\frac{1}{2}(N + 2 - (|s_+(D)| + |s_-(D)|)) > 0$ for a diagram which is not a connected sum of alternating diagrams. By Theorems 2.2, 3.2 and Corollary 3.3, $TH_{i,j}(D)$ is nontrivial on slope 2 diagonals $j-2i = 2|s_+| - N$ and $N - 2|s_-| + 4$. The $j$ distance between these diagonals is $N - 2|s_-| + 4 - (2|s_+| - N) = 2(N + 2 - (|s_+(D)| + |s_+(D)|))$, so the theorem follows. □

Example 5.11 Consider the knot $10_{153}$ (in the notation of [Rol]). It is an adequate non-alternating knot. Its associated graphs $G_{s_+}(10_{153})$ and $G_{s_-}(10_{153})$ have triangles (Fig. 5.1) so Theorem 5.10 applies. Here $|s_+| = 6, |s_-| = 4$ and by
**Theorem 2.2**, \( H_{8,18}(10_{153}) \) and \( H_{-10,-14}(10_{153}) \) have \( \mathbb{Z}_2 \) torsion. Thus support of torsion requires at least 2 adjacent diagonals.

![Fig. 5.1](image)

**Corollary 5.12** Any doubly adequate link which is not an alternating link is not \( TH-0 \)-thick.

### 6 Hopf link addition

We find, in this section, the structure of the Khovanov homology of connected sum of \( n \) copies of the Hopf link, as promised in Section 5. As a byproduct of our method, we are able to compute Khovanov homology of a connected sum of a diagram \( D \) and the Hopf link \( D_h \), Fig 6.1, confirming a conjecture by A. Shumakovitch that the Khovanov homology of the connected sum of \( D \) with the Hopf link, is the double of the Khovanov homology of \( D \).

![Fig. 6.1](image)

\(^{20}\)Checking [Sh2], gives the full torsion of the Khovanov homology of \( 10_{153} \) as: \( T_{8,18} = T_{4,10} = T_{2,6} = T_{0,6} = T_{-2,-2} = T_{-4,-2} = T_{-6,-6} = T_{-10,-14} = \mathbb{Z}_2. \)
**Theorem 6.1** For every diagram $D$ we have the short exact sequence of Khovanov homology

$$0 \rightarrow H_{i+2,j+4}(D) \xrightarrow{\alpha_h} H_{i,j}(D \# D_h) \xrightarrow{\beta_h} H_{i-2,j-4}(D) \rightarrow 0$$

where $\alpha_h$ is given on a state $S$ by Fig.6.2(a) and $\beta_h$ is a projection given by Fig.6.2(b) (and 0 on other states). The theorem holds for any ring of coefficients, $R$, not just $R = \mathbb{Z}$.

![Fig. 6.2](image)

**Theorem 6.2**

The short exact sequence of homology from Theorem 6.1 splits, so we have

$$H_{i,j}(D \# D_h) = H_{i+2,j+4}(D) \oplus H_{i-2,j-4}(D).$$

**Proof:** To prove Theorem 6.1 we consider the long exact sequence of the Khovanov homology of the diagram $D \# D_h$ with respect to the first crossing of the diagram, $e_1$ (Fig.6.1). To simplify the notation we assume that $R = \mathbb{Z}$ but our proof works for any ring of coefficients.

$$\cdots \rightarrow H_{i+1,j-1}((D \# D_h)_0) \xrightarrow{\partial} H_{i+1,j+1}((D \# D_h)_\infty) \xrightarrow{\alpha} H_{i,j}(D \# D_h) \xrightarrow{\beta}$$

$$H_{i-1,j-1}((D \# D_h)_0) \xrightarrow{\partial} H_{i-1,j+1}((D \# D_h)_\infty) \rightarrow \cdots$$

We show that the homomorphism $\partial$ is the zero map. We use the fact that $(D \# D_h)_0$ differs from $D$ by a positive first Reidemeister move $R_{+1}$ and that $(D \# D_h)_\infty$ differs from $D$ by a negative first Reidemeister move $R_{-1}$; Fig.6.1. We know, see [APS] for example, that the chain map

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21 Theorems 6.1 and 6.2 hold for a diagram $D$ on any surface $F$ and for any ring of coefficients $R$ with the restriction that for $F = RP^2$ we need $2R = 0$. In this more general case of a manifold being $I$-bundle over a surface, we use definitions and setting of [APS].
\( r_{\cdot 1} : \mathcal{C}(D) \to \mathcal{C}(R_{\cdot 1}(D)) \) given by \( r_{\cdot 1}(\varepsilon) = \varepsilon \) yields the isomorphism of homology:

\[
\begin{aligned}
\quad r_{\cdot 1} : H_{i,j}(D) & \to H_{i-1,j-3}(R_{\cdot 1}(D)) \\
\end{aligned}
\]

and the chain map \( \bar{r}_{\cdot 1}(\mathcal{C}(R_{\cdot 1}(D)) = \mathcal{C}((D) \) given by the projection with \( \bar{r}_{\cdot 1}(\varepsilon) = (\varepsilon) \) and 0 otherwise, induces the isomorphism of homology:

\[
\begin{aligned}
\quad \bar{r}_{\cdot 1} : H_{i+1,j+3}(R_{\cdot 1}(D)) & \to H_{i,j}(D). \\
\end{aligned}
\]

From these we get immediately that the composition homomorphism:

\[
\begin{aligned}
\quad r_{\cdot 1}^{-1} \partial \bar{r}_{\cdot 1}^{-1} : H_{i,j-4}(D) & \to H_{i+2,j+4}(D) \\
\end{aligned}
\]

is the zero map by considering the composition of homomorphisms

\[
\begin{aligned}
H_{i,j-4}(D) \xrightarrow{r_{\cdot 1}^{-1}} H_{i+1,j-1}((D\#D_h)_0) \xrightarrow{\partial} H_{i+1,j+1}((D\#D_h)_\infty) \xrightarrow{\bar{r}_{\cdot 1}^{-1}} H_{i+2,j+4}(D). \\
\end{aligned}
\]

\( \square \)

Let \( h(a,b)(D) \) (resp. \( h_F(a,b)(D) \) for a field \( F \)) be the generating polynomial of the free part of \( H_{\ast\ast}(D) \) (resp. \( H_{\ast\ast}(D;F) \)), where \( kb^j a^j \) (resp. \( k_F b^j a^j \)) represents the fact that the free part of \( H_{i,j}(D), F H_{i,j}(D) = \mathbb{Z}^k \) (resp. \( H_{i,j}(D;F) = F^k \)).

Theorem 6.2 will be proved in several steps.

**Lemma 6.3**

If the module \( H_{i-\cdot,j-\cdot}(D;\mathcal{R}) \) is free (e.g. \( \mathcal{R} \) is a field) then the sequence from Theorem 6.1 splits and \( H_{i,j}((D\#D_h);\mathcal{R}) = H_{i-\cdot,j-\cdot}(D;\mathcal{R}) \oplus H_{i+2,j+4}(D;\mathcal{R}) \) or shortly \( H_{\ast\ast}(D\#D_h;\mathcal{R}) = H_{\ast\ast}(D;\mathcal{R})(b^2 a^4 + b^{-2} a^{-4}). \)

For the free part we have always \( F H_{i,j}((D\#D_h) = F H_{i+2,j+4}(D) \oplus F H_{i-\cdot,j-\cdot}(D)) \) or in the language of generating functions: \( h(a,b)(D\#D_h) = (b^2 a^4 + b^{-2} a^{-4}) h(a,b)(D). \)

**Proof:** The first part of the lemma follows immediately from Theorem 6.1 which holds for any ring of coefficients, in particular \( \text{rank}(F H_{i,j}(D\#D_h) = \text{rank}(F H_{i+2,j+4}(D)) + \text{rank}(F H_{i-\cdot,j-\cdot}(D)). \) \( \square \)

**Lemma 6.4**

There is the exact sequence of \( \mathbb{Z}_p \) linear spaces:

\[
0 \to H_{i+2,j+4}(D) \otimes \mathbb{Z}_p \to H_{i,j}(D\#D_h) \otimes \mathbb{Z}_p \to H_{i-\cdot,j-\cdot}(D) \otimes \mathbb{Z}_p \to 0.
\]

27
Proof: Our main tool is the universal coefficients theorem (see, for example, [Ha]; Theorem 3A.3) combined with Lemma 6.3. By the second part of Lemma 6.3 it suffices to prove that:

\[ T_{i,j}(D \# D_h) \otimes \mathbb{Z}_p = T_{i+2,j+4}(D) \otimes \mathbb{Z}_p \oplus T_{i-2,j-4}(D) \otimes \mathbb{Z}_p. \]

From the universal coefficients theorem we have:

\[ H_{i,j}(D \# D_h); \mathbb{Z}_p) = H_{i,j}(D \# D_h) \otimes \mathbb{Z}_p \oplus \text{Tor}(H_{i-2,j}(D \# D_h), \mathbb{Z}_p) \]

and \( \text{Tor}(H_{i-2,j}(D \# D_h), \mathbb{Z}_p) = T_{i-2,j}(D \# D_h) \otimes \mathbb{Z}_p \) and the analogous formulas for the Khovanov homology of \( D \). Combining this with both parts of Lemma 6.3, we obtain:

\[ T_{i,j}(D \# D_h) \otimes \mathbb{Z}_p \oplus T_{i-2,j}(D \# D_h) \otimes \mathbb{Z}_p = (T_{i+2,j+4}(D) \otimes \mathbb{Z}_p \oplus T_{i+4}(D) \otimes \mathbb{Z}_p) \oplus (T_{i-2,j-4}(D) \otimes \mathbb{Z}_p \oplus T_{i-4,j-4}(D) \otimes \mathbb{Z}_p). \]

We can express this in the language of generating functions assuming that \( t(b, a)(D) \) is the generating function of dimensions of \( T_{i,j}(D) \otimes \mathbb{Z}_p \):

\[(1 + b^{-2})t(b, a)(D \# D_h) = (1 + b^{-2})(b^2a^4 + b^{-2}a^{-4}t(b, a)(D)).\]

Therefore \( t(b, a)(D \# D_h) = (b^2a^4 + b^{-2}a^{-4})t(b, a)(D) \) and \( \text{dim}(T_{i,j}(D \# D_h) \otimes \mathbb{Z}_p) = \text{dim}(T_{i+2,j+4}(D) \otimes \mathbb{Z}_p + \text{dim}(T_{i,j+4}(D) \otimes \mathbb{Z}_p). \)

The lemma follows by observing that the short exact sequence with \( \mathbb{Z} \) coefficients leads to the sequence

\[ 0 \to \ker(\alpha_p) \to H_{i+2,j+4}(D) \otimes \mathbb{Z}_p \xrightarrow{\alpha_p} H_{i,j}(D \# D_h) \otimes \mathbb{Z}_p \to H_{i-2,j-4}(D) \otimes \mathbb{Z}_p \to 0. \]

By the previous computation \( \text{dim}(\ker(\alpha_p)) = 0 \) and the proof is completed.

To finish our proof of Theorem 6.2 we only need the following lemma.

Lemma 6.5 Consider a short exact sequence of finitely generated abelian groups:

\[ 0 \to A \to B \to C \to 0. \]

If for every prime number \( p \) we have also the exact sequence:

\[ 0 \to A \otimes \mathbb{Z}_p \to B \otimes \mathbb{Z}_p \to C \otimes \mathbb{Z}_p \to 0 \]

then the exact sequence \( 0 \to A \to B \to C \to 0 \) splits and \( B = A \oplus C. \)
Proof: Assume, for contradiction, that the sequence $0 \to A \xrightarrow{\alpha} B \to C \to 0$ does not split. Then there is an element $a \in A$ such that $\alpha(a)$ is not $p$-primitive in $B$, that is $\alpha(a) = pb$ for $b \in B$ and $p$ a prime number and $b$ does not lies in the subgroup of $B$ span by $\alpha(a)$ (to see that such an $a$ exists one can use the maximal decomposition of $A$ and $B$ into cyclic subgroups (e.g. $A = \mathbb{Z}^k \oplus Z_{p,i}^{k_p,i}$)). Now comparing dimensions of linear spaces $A \otimes \mathbb{Z}_p, B \otimes \mathbb{Z}_p, C \otimes \mathbb{Z}_p$ (e.g. $\dim(A \otimes \mathbb{Z}_p) = k + \sum_{i=1}^{p} k_i + \sum_{i=2}^{p} k_i^2 + \ldots$ we see that the sequence $0 \to A \otimes \mathbb{Z}_p \to B \otimes \mathbb{Z}_p \to C \otimes \mathbb{Z}_p \to 0$ is not exact, a contradiction. □

Corollary 6.6 For the connected sum of $n$ copies of the Hopf link we get

$$H_{*,*}(D_h \# \ldots \# D_h) = h(a,b)(D) = (a^2 + a^{-2})(a^4b^2 + a^{-4}b^{-2})^n$$

Remark 6.7 Notice that $h(a,b)(D_h) - h(a,b)(OO) = (a^2 + a^{-2})(a^4b^2 + a^{-4}b^{-2}) - (a^2 + a^{-2})^2 = b^{-2}a^{-4}(a^2 + a^{-2})^2(1 + ba)(1 - ba)(1 - ba^3)(1 - ba^3)$. This equality may serve as a starting point to formulate a conjecture for links, analogous to Bar-Natan-Garoufalidis-Khovanov conjecture [Kh-2, Ga, BN-1] (Conjecture 1), formulated for knots and proved for alternating knots by Lee [Lee-1].

7 Reduced Khovanov homology

Most of the results of Sections 5 and 6 can be adjusted to the case of reduced Khovanov homology\(^{23}\). We introduce the concept of $H^r-(k_1, k_2)$-thick diagram and formulate the result analogous to Theorem 5.3. The highlight of this section is the exact sequence connecting reduced and unreduced Khovanov homology.

Choose a base point, $b$, on a link diagram $D$. Enhanced states, $S(D)$ can be decomposed into disjoint union of enhanced states $S_+(D)$ and $S_-(D)$, where the circle containing the base point is positive, respectively negative. The Khovanov abelian group $\mathcal{C}(D) = \mathcal{C}_+(D) \oplus \mathcal{C}_-(D)$ where $\mathcal{C}_+(D)$ is spanned by $S_+(D)$ and $\mathcal{C}_-(D)$ is spanned by $S_-(D)$. $\mathcal{C}_+(D)$ is a chain subcomplex of $\mathcal{C}(D)$. Its homology, $H^r(D)$, is called the reduced Khovanov homology of

\(^{22}\)In the oriented version (with the linking number equal to $n$, so the writhe number $w = 2n$) and with Bar Natan notation one gets: $q^{3n}t^n(q+q^{-1})(q^2t+q^{-2}t^{-1})^n$, as computed first by Shumakovitch.

\(^{23}\)Introduced by Khovanov; we follow here Shumakovitch’s approach adjusted to the framed link version.
$D$, or more precisely of $(D, b)$ (it may depend on the component on which the base point lies). Using the long exact sequence of reduced Khovanov homology we can reformulate most of the results of Sections 5 and 6.

**Definition 7.1**

We say that a link diagram, $D$ of $N$ crossings is $H_r$-$(k_1, k_2)$-thick if $H_{i,j}^r(D) = 0$ with a possible exception of $i$ and $j$ satisfying:

$$N - 2|s_-| - 4k_2 + 4 \leq j - 2i \leq 2|s_+| - N + 4k_1.$$

With this definition we have

**Theorem 7.2**

If the diagram $D_{\infty}$ is $H_r$-$(k_1(D_{\infty}), k_2(D_{\infty}))$-thick and the diagram $D_0$ is $H_r$-$(k_1(D_0), k_2(D_0))$-thick, then the diagram $D_+$ is $H_r$-$(k_1(D_+), k_2(D_+))$-thick where

$$k_1(D_+) = \max(k_1(D_{\infty}) + \frac{1}{2}(|s_+(D_{\infty})| - |s_+(D_+)| + 1), k_1(D_0))$$

and

$$k_2(D_+) = \max(k_2(D_{\infty}), k_2(D_0)) + \frac{1}{2}(|s_-(D_0)| - |s_-(D_+) + 1|).$$

Every alternating non-split diagram $D$ without a nugatory crossing is $H_r$-$(0,0)$-thick, and $H_{*}^*(D)$ is torsion free [Lee-1, Sh-1].

The graded abelian group $C_-(D) = \bigoplus_{i,j} C_{i,j; -}(D)$ is not a sub-chain complex of $C(D)$, as $d(S)$ is not necessary in $C_-(D)$, for $S \in S_-(D)$. However the quotient $C^-(D) = C(D)/C_+(D)$ is a graded chain complex and as a graded abelian group it can be identified with $C_-(D)$.

**Theorem 7.3**

(i) We have the following short exact sequence of chain complexes:

$$0 \to C_+(D) \xrightarrow{\phi} C(D) \xrightarrow{\psi} C^-(D) \to 0.$$

(ii) We have the following long exact sequence of homology:

$$\ldots \to H^r_{i,j}(D) \xrightarrow{\phi} H_{i,j}(D) \xrightarrow{\psi} H^r_{i,j}(D) \xrightarrow{\partial} H^r_{i-2,j}(D) \to \ldots$$

where $H^r_{i,j}(D)$ is the homology of $C^-(D)$. The boundary map can be roughly interpreted for a state $S \in S_-(D)$ as $d(S)$ restricted to $C_+(D)$.

Applications of Theorem 7.3 will be the topic of a sequel paper, here we only mention that the group $H_{*}^*(D)$ is related to the group of the mirror image of $D$, $H_{*}^*(\bar{D})$.
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