Comparison of methods for estimating the reliability of arriving for the Burr distribution

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Abstract: The paper deals with the comparison between the methods of estimating the reliability of arriving at the time of the trip for the Burr distribution, namely (Maximum likelihood method, the Bayes method, the non-linear least squares method). We are comparing among the three estimation methods using the simulation method. To make an understanding of the best method for estimating the reliability of arriving at the Burr distribution through a comparison between the three methods, an integrative mean square error standard (IMSE) was used, as the simulation results showed the preference of the Bayes method over the other methods.

1-Introduction:
Many people expect that their trip times do not exceed the scheduled time, or that there will be a slight delay in the trip time, but what breaks this expectation is the generation of many factors that arise suddenly and at any time and without preempting my perception of their occurrence, to affect negatively flowing Traffic, which leads to traffic congestion and delay in arriving at the specified time, among these factors the occurrence of a breakdown of the car, road accidents, initiation of road maintenance work, the closure of the street for security purposes, and the movement in which the holy streets of Karbala are crowded during holidays, religious and national events and a holiday weekend. In this research, a sample was taken for Al-Iskan Street in the holy city of Karbala, and after data collection, and with the help of statistical programs to find out the distribution that the real data follow, as it was observed from reading the sample data of the study, the Burr distribution was taken and methods were used to estimate the reliability (reliability) of arriving on time using estimation methods. (The greatest possibility, BES method, nonlinear least squares method) to distribute Burr and then compare between estimation methods using simulation and choose the best method using the average integral error squares (IMSE). As for the application side, the real data, which represents the flight time data for Al Iskan Street, was applied.
2- Research problem:
Study the phenomenon of traffic congestion in the main streets of the holy city of Kerbala, which causes delays in the time of arrival time, so we find it necessary to study trip times to estimate the reliability of arriving on time.

3- The objective of the research
Comparison of methods of estimating reliability of flight time by using the Maximum likelihood method, the Bayes method, the non-linear least squares method for the Burr distribution and determining the best methods for estimating reliability of access using the statistical standard (IMSE)

4- The theoretical side

4-1 Burr Distribution
Scientist Burr in 1942 proposed a distribution family that was named after him for use in data modeling. The scientist suggested twelve models for the Burr distribution.

The Burr distribution is a right skew distribution. This distribution is flexible, and has a flexible range and location to control which makes it data fitting. It has been used extensively in recent years because of its great importance in reliability studies and failure time analyzes.

The probability density function of a Burr distribution is:

\[ f(x, \alpha, k, \beta) = \frac{a k}{\beta} \left( \frac{x}{\beta} \right)^{a-1} \left[ 1 + \left( \frac{x}{\beta} \right)^a \right]^{-k-1}; \beta > 0, k > 0, \alpha > 0, x > 0 \] \hspace{0.5cm} \ldots (1)

Where:

k > 0 and \( \alpha > 0 \) are shape parameters
\( \beta > 0 \) is the scaling parameter for the distribution

And that the cumulative distribution function cdf is given as following:

\[ F(x, \alpha, k, \beta) = 1 - \left[ 1 + \left( \frac{x}{\beta} \right)^a \right]^{-k}; \beta > 0, k > 0, \alpha > 0, x > 0 \] \hspace{0.5cm} \ldots (2)

The reliability function of the distribution is:

\[ R(x) = \int_{x}^{\infty} f(x, \alpha, k, \beta) dx = \left[ 1 + \left( \frac{x}{\beta} \right)^a \right]^{-k}; \beta > 0, k > 0, \alpha > 0, x > 0 \] \hspace{0.5cm} \ldots (3)

4-2 Methods for estimating the reliability of arrival to a Burr distribution:

4-2-1 The Maximum likelihood method:
Assume that \( x_1, x_2, x_3, \ldots, x_n \) are independent random variables subject to the Burr distribution, so the likelihood function of the distribution parameters is expressed in the following equation:

\[ L = \prod_{i=1}^{n} f(x, \alpha, k) \] \hspace{0.5cm} \ldots (4)

The function of Maximum likelihood will be as follows:

\[ L(x_1, x_2, \ldots, x_n; \alpha, k) = (\alpha, k)^n \prod_{i=1}^{n} \left[ \frac{x_i^{a-1}}{(1 + x_i)^{k+1}} \right] \]
And by taking the natural logarithm to convert the function into linear form:

$$\log L(x_1, x_2, \ldots, x_n; \alpha, k) = L^*$$

$$= n\alpha + nk - (\alpha - 1)\log x_i - (k + 1)\log (1 + x_i^\alpha)$$

We find the partial derivatives of the above equation to find the estimated values of the two parameters ($k, \alpha$)

$$\frac{\partial L^*}{\partial k} = n - \sum_{i=1}^{n} \log(1 + x_i^\alpha)$$

We find the value of $k$ after the derivative is equal to zero:

$$\hat{k}_{mle} = \frac{n}{\sum_{i=1}^{n} \log x_i} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\frac{\partial L^*}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log x_i - (k + 1) \sum_{i=1}^{n} \frac{x_i^\alpha \log x_i}{1 + x_i^\alpha}$$

To find the value of $\alpha$, equate the derivative to zero:

$$\frac{n}{\hat{\alpha}} + \sum_{i=1}^{n} \log x_i - (k + 1) \sum_{i=1}^{n} \frac{x_i^\alpha \log x_i}{1 + x_i^\alpha} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

From the above two equations, we note that it cannot be solved by the linear method, so one of the numerical methods is used to solve nonlinear equations, such as the (Newton-Raphson) method, where the value of $\alpha$ is found when iterating $J$.

To obtain an estimate of the reliability function of the Burr distribution, we substitute the estimates obtained into the reliability function as follows:

$$R_{MLE} = [1 + (x)^\alpha]^{-k} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

**4.2-2 Bayesian Estimation Method:**

We assume that the prior information as following:

$$g_1(\alpha) = \alpha^a e^{-ba} \quad \ldots \quad \ldots \quad (8)$$

$$b, a > 0 \quad , \quad \alpha > 0$$

$$g_2(k) = \alpha^c e^{-dk} \quad \ldots \quad \ldots \quad (9)$$

$$c, d > 0 \quad , \quad k > 0$$

Where ($\alpha$) and ($k$) are independent.
\[ f(t \ldots \ldots t_n) = \int_0^\tau \int_0^\tau \alpha L(\alpha,k) \ g(\alpha,k) \ d\alpha \ dk \]

\[ \hat{g} = g(\hat{\alpha}, \hat{k}) + 0.5[A + L_{30}B_{12} + L_{03}B_{21} + L_{21}C_{12} + L_{12}C_{21} + P_1A_{12} + P_2A_{21}] \]

Where:

\[ A = \sum_{i=1}^{2} \sum_{j=1}^{2} W_{ij} T_{ij} \]

\[ L_{ij} = \frac{\partial^2 L(\lambda_1, \lambda_2)}{\partial \lambda_1^i \partial \lambda_2^j}; \ i + j = 3 , \ \alpha = \lambda_1 , \ \ k = \lambda_2 \]

\[ P = \frac{\partial P}{\partial \lambda_i}; \ \ W_{i} = \frac{\partial g}{\partial \lambda_i}, \ \ P = Log \pi(\lambda_1, \lambda_2) \]

\[ B_{ij} = (W_{i} T_{ij} + W_{j} T_{ij}) \ T_{ii} \]

\[ C_{ij} = (W_{i} T_{ii} + W_{j} T_{ji}) \]

\[ W_{ij} = \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j}; \ L = \]

Tij: it is the Cij component in the inverse Fisher information matrix

\[ I = -E\left[ \frac{\partial L}{\partial \alpha \partial k} \right] ; \]

And by maximizing the function for CDF:

\[ Log \ L = L' = n \log \log \alpha + n \log k + (\alpha - 1) \sum \log \log x - (k + 1) \sum \log \log (1 + x^\alpha) \]

\[ \frac{\partial L'}{\partial \alpha} = \frac{n}{\alpha} + \sum \log \log x - (k + 1) \sum x_i^\alpha \log x_i \left( \frac{1}{1 + x_i^\alpha} \right) \]

\[ \frac{\partial L'}{\partial k} = \frac{n}{k} - \sum \log \log (1 + x_i^\alpha) \]
\[
\frac{\partial^2 L^*}{\partial \alpha^2} = -\frac{n}{\alpha^2} - (k + 1) \sum \frac{(1 + x_i^a) \{x_i^a (\log x_i)^2\} - (x_i^a \log x_i)^2}{(1 + x_i^a)}
\]

\[
\frac{\partial^2 L^*}{\partial \alpha^2} = -\frac{n}{\alpha^2} - (k + 1) \sum \frac{x_i^a (\log x_i)^2}{(1 + x_i^a)^2}
\]

\[
\frac{\partial^2 \log L(\alpha, k)}{\partial k \partial \alpha} = -\sum \frac{x_i^a \log x_i}{(1 + x_i^a)^2}
\]

The Fisher information matrix is as follows:

\[
I = -E \left[ \frac{\partial^2 \log L(\alpha, k)}{\partial \alpha \partial k} \right]
\]

\[
= - \begin{bmatrix}
\frac{\partial \log L(\alpha, k)}{\partial \alpha^2} & \frac{\partial \log L(\alpha, k)}{\partial \alpha \partial k} & \frac{\partial \log L(\alpha, k)}{\partial k \partial \alpha} & \frac{\partial \log L(\alpha, k)}{\partial k^2} \\
\end{bmatrix}
\]

\[
I = [I_{11} \ I_{12} \ I_{21} \ I_{22}]
\]

\[
I_{11} = \frac{n}{\alpha^2} + (k + 1) \sum \frac{(1 + x_i^a) \{x_i^a (\log x_i)^2\} - (x_i^a \log x_i)^2}{(1 + x_i^a)}
\]

\[
I_{22} = \frac{n}{k^2}
\]

\[
T = I^{-1} = \frac{1}{[I_{11}I_{22} - I_{12}^2]} \begin{bmatrix}
I_{22} & -I_{12} & I_{21} & I_{11} \\
\end{bmatrix}
\]

\[
T_{11} = \frac{I_{22}}{[I_{11}I_{22} - I_{12}^2]}
\]

\[
T_{12} = T_{21} = \frac{-I_{12}}{[I_{11}I_{22} - I_{12}^2]}
\]

\[
T_{22} = \frac{I_{11}}{[I_{11}I_{22} - I_{12}^2]}
\]

\[
L_{30} = \frac{\partial^3 \log L(\alpha, k)}{\partial \alpha^3}
\]

\[
= \frac{2^n}{\alpha^3} - (k + 1) \sum \frac{(1 + x_i^a)^2 \{x_i^a (\log x_i)^3\} - x_i^{2a} (\log x_i)^3}{(1 + x_i^a)^2}
\]

\[
= \frac{2^n}{\alpha^3} - (k + 1) \sum \frac{(1 + x_i^a) \{x_i^a (\log x_i)^3\} - 2x_i^{2a} (\log x_i)^3}{(1 + x_i^a)^3}
\]
\[
\frac{2^n}{\alpha^3} - (k + 1) \sum \frac{x_i^\alpha (\log x_i)^3 [(1 + x_i^\alpha) - 2x_i^\alpha]}{(1 + x_i^\alpha)^3}
\]

\[L_{30} = \frac{2^n}{\alpha^3} - (k + 1) \sum \frac{x_i^\alpha (\log x_i)^3 (1 - x_i^\alpha)}{(1 + x_i^\alpha)^3}\]

\[L_{03} = \frac{\partial^3 \log L(\alpha, k)}{\partial k^3}\]

\[L_{03} = \frac{2^n}{k^3}\]

\[L_{21} = \frac{\partial^3 \log L(\alpha, k)}{\partial \alpha^2 \partial k}\]

\[L_{21} = -\sum \frac{x_i^\alpha (\log x_i)^2}{(1 + x_i^\alpha)^2}\]

\[L_{12} = \frac{\partial \log L(\alpha, k)}{\partial \alpha \partial k^2} = 0\]

\[P = \log \left[ g_1(\alpha) \ g_2(k) \right]\]

\[p = \log \left[ g(\alpha, k) \right]\]

\[P = a \log \alpha + c \log k - (ba - dk)\]

\[P_1 = \frac{\partial P}{\partial \alpha}\]

\[= \frac{a}{\alpha} - b\]

\[P_2 = \frac{\partial P}{\partial k}\]

\[= \frac{c}{k} - d\]

An estimate of \(\alpha\) is:

\[f(t \ldots \ldots t_n) = \frac{\int_0^r \int_0^r \alpha L(\alpha, k) \ g(\alpha, k) \ da \ dk}{\int_0^r \int_0^r L(\alpha, k) \ g(\alpha, k) \ da \ dk}\]

\[\hat{\alpha}_{Lindley} = \hat{\alpha}_{MLE} + 0.5 \left[ L_{30}B_{12} + L_{03}B_{21} + L_{21}C_{12} + L_{21}C_{21} \right] + p_1A_{12} + P_2A_{21}\]

Where:

\[B_{12} = (W_1T_{12} + W_2T_{12}) \ T_{11} = T_{12}T_{11}\]

\[B_{21} = (W_2T_{22} + W_1T_{21}) \ T_{22} = T_{21}T_{22}\]

\[C_{12} = 3W_1T_{11}T_{22} = 3T_{11}T_{22}\]
\[ C_{21} = 3W_2T_{22} + W_1(T_{22}T_{11} + 2T_{21}) \]
\[ A_{12} = W_1T_{11} = T_{11} \]
\[ A_{21} = W_2T_{22} + W_1T_{12} = T_{12} \]
\[ W_i = \frac{\partial g(\lambda_i)}{\partial \lambda_i} \quad i = 1, 2 \]
\[ W_{ij} = 0 \quad \forall \ i, j \]
\[ A = \sum_{i=1}^{2} \sum_{j=1}^{2} W_{ij}T_{ij} \]
\[ = W_{11}T_{11} + W_{12}T_{12} + W_{21}T_{21} + W_{22}T_{22} \]
\[ W_{ij} = \frac{\partial^2 g}{\partial \alpha \partial k} \]
\[ g = \alpha \]
\[ A = 0 \]
\[ \hat{k} = \hat{k}_{MLE} + 0.5[L_{30}B_{12} + L_{03}B_{21} + L_{21}C_{12} + L_{12}C_{21}] + p_1A_{12} + p_2A_{21} \]
\[ g(\alpha, k) = k \]
\[ W_1 = 0 \quad ; \quad W_2 = 1 \]
\[ B_{12} = T_{12}T_{11} \]
\[ B_{21} = T_{21}T_{22} \]
\[ C_{12} = T_{11}T_{22} + 2T_{12}^2 \]
\[ C_{21} = 3T_{22}T_{21} \]
\[ A_{12} = T_{21} \quad ; \quad A_{21} = T_{22} \]

Find an estimate of the reliability function of the Burr distribution:
\[ \hat{g}_{\text{linlay}}(\alpha, k) = g(\hat{\alpha}_{\text{MLE}}, \hat{k}_{\text{MLE}}) + 0.5[A + L_{30}B_{12} + L_{03}B_{21} + L_{21}C_{12} + L_{12}C_{21}] + p_1A_{12} + p_2A_{21} \]

Where:
\[ W_{ij} = \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \quad ; \quad \lambda_i = \alpha , \lambda_j = k \]
\[ W_i = \frac{\partial g}{\partial \lambda_i} \quad i = 1, 2 \]
\[ W_1 = \frac{\partial g}{\partial \lambda_1} = \frac{\partial g}{\partial \alpha} = (-k)(1 + t^\alpha)^{-k-1}t^\alpha \log(t) \]
\[ W_2 = \frac{\partial g}{\partial \lambda_2} = \frac{\partial g}{\partial k} = (1 + t^\alpha)^{-k} \log (1 + t^\alpha)(-1) \]

Then:

\[ B_{12} = (W_1 T_{12} + W_2 T_{12}) T_{11} \]
\[ B_{21} = (W_2 T_{22} + W_1 T_{21}) T_{22} \]
\[ C_{12} = 3W_1 T_{11} + W_2 (T_{11} T_{22} + 2T_{21}^2) \]
\[ C_{21} = 3W_2 T_{22} T_{21} + W_1 (T_{22} T_{11} + 2T_{21}^2) \]
\[ A_{12} = W_1 T_{11} + W_2 T_{21} \]
\[ A_{21} = W_2 T_{22} + W_1 T_{12} \]
\[ W_{ij} = \frac{\partial^2 g}{\partial \lambda_i \partial \lambda_j} \quad i = 1, 2 \]
\[ A = \sum_{i=1}^{2} \sum_{j=1}^{2} W_{ij} T_{ij} \]
\[ = W_{11} T_{11} + W_{12} T_{12} + W_{21} T_{21} + W_{22} T_{22} \]
\[ W_{11} = \frac{\partial^2 g}{\partial \alpha^2} \]
\[ = [-k \log(t)][1 + t^\alpha]^{-k-1} t \log(t) + t^\alpha (-k - 1)(-k \log(t))[1 + t^\alpha]^{-k-2} t^\alpha \log(t) \]
\[ W_{21} = \frac{\partial^2 g}{\partial k \partial \alpha} \]
\[ = [1 + t^\alpha]^{-k} \frac{-t^\alpha \log(t)}{1 + t^\alpha} - \log(1 + t^\alpha)(-k)(1 + t^\alpha)^{-k-1} t^\alpha \log(t) \]
\[ W_{22} = \frac{\partial^2 g}{\partial k^2} \]
\[ = [1 + t^\alpha]^{-k} \left[ \log(1 + t^\alpha) \right]^2 \]

Where:

\[ t = 1, 2, 3, \ldots, \ldots, 10 \]

Then \( \hat{\alpha}_{MLE} \) and \( \hat{k}_{MLE} \) are maximum likelihoods estimates for the \( \alpha \) and \( k \) respectively.

4-2-3 Nonlinear least squares method: \(^{[437]}\)

The cumulative distribution function is given as follows:

\[ F(x, \alpha, k, \beta) = 1 - \left[ 1 + (x^\alpha) \right]^{-k}; \quad k > 0, \alpha > 0, x > 0 \]

Then the cumulative distribution function will be:

\[ (1 - F_i) = (1 + x^\alpha)^{-k} \]
\( Y_i = (1 + x^α)^{-k} \)

When adding \( \varepsilon_i \) to the above equation:
\[
Y_i = (1 + x^α)^{-k} + \varepsilon_i \quad \ldots (11)
\]

When \( Y_i = (1 - \hat{F}_i) \), the probability density function is as follows:
\[
f(x; \alpha, k) = (1 + x^α)^{-k} \varepsilon_i \sim N(0,1)
\]

Equation (2-27) is a nonlinear equation. To obtain an estimate of the two parameters \((x; \alpha, k)\) we use the nonlinear regression method.

Then the sum of squares of errors:
\[
Q = \sum [Y_i - (1 + x^α)^{-k}]^2 \quad \ldots \ldots (12)
\]

By differentiating the above equation with respect to \( \alpha \) times and with respect to \( k \) times we get:
\[
\frac{\partial Q}{\partial \alpha} = 2 \sum [Y_i - (1 + x^α)^{-k}] k(1 + x^α)^{-k-1} x_i^α a^{-1}
\]
\[
\frac{\partial Q}{\partial k} = 2 \sum [Y_i - (1 + x^α)^{-k}] [-(1 + x_i^α)^{-k} \log (1 + x_i^α) - 1]
\]

To find the value of \( \alpha \) and \( k \), we set the above two derivatives down to zero:
\[
\sum \left( \hat{a} \hat{k} \right) [Y_i - (1 + x^α)^{-k}] \left( 1 + x^α \right)^{-k-1} x_i^α a^{-1} = 0 \quad \ldots (2 - 29)
\]
\[
\sum \left[ Y_i - (1 + x^α)^{-k} \right] \left[-(1 + x_i^α)^{-k} \log (1 + x_i^α) - 1 \right] = 0 \quad \ldots (13)
\]

Equations (12) and (13) are solved using the Newton - Raphson iterative method as follows:

\[
\left( \hat{a}_{(k)} \hat{k}_{(k)} \right) = \left( \hat{a}_{(k+1)} \hat{k}_{(k+1)} \right) \left( J \right)^{-1} \left( \hat{a}_{(k-1)} \hat{k}_{(k-1)} \right) \ldots \ldots (2 - 31)
\]

\[
J = \begin{pmatrix} \frac{\partial Q_1}{\partial \alpha} & \frac{\partial Q_1}{\partial k} & \frac{\partial Q_2}{\partial \alpha} & \frac{\partial Q_2}{\partial k} \\ \frac{\partial Q_1}{\partial k} & \frac{\partial Q_1}{\partial k} & \frac{\partial Q_2}{\partial k} & \frac{\partial Q_2}{\partial k} \end{pmatrix}
\]

Where:
\[
k = 1, 2, 3, \ldots
\]

\[
\left( \hat{a}_{(k)} \hat{k}_{(k)} \right) = \left( \hat{a}_{(k-1)} \hat{k}_{(k-1)} \right) = (0.001 \ 0.001) \quad \ldots \ldots (14)
\]

To obtain an estimate of the reliability function of the Burr distribution, we substitute the estimates obtained into the reliability function as follows:

5- Experimental side & the applied ide:

5-1 Simulation: [1][2]
In this paper, the Simulation-Monte Carlo method was used for the generating data of different sizes used in estimating the reliability function of the Burr distribution as well as to explain the effect of the methods of estimating the reliability function of the Burr distribution towards the change in the sample size and the change in the relationship between the two shape parameters. (Shape Parameter) α and k to obtain estimates of reliability, the Burr distribution parameters (α) and (k) will first be estimated by choosing Sample Sizes (n = 10, 30, 50, 100) and default values for the Burr distribution parameters (α = 2,1,0.9; k = 2,2,0.8) as a variable that follows a uniform distribution u ~ U (0,1) with the help of the Rand instruction and generating data that follows the distribution of Burr by applying the inverted transformation method and using the following formula:

\[ t_i = \beta \left[ (1 - u)^{-\frac{1}{\alpha}} - 1 \right]^\frac{1}{k} \]  

; \( i = 1,2, \ldots, n \)  

When \( \beta = 1 \), the formula becomes:

\[ t_i = \left[ (1 - u)^{-\frac{1}{\alpha}} - 1 \right]^\frac{1}{k} \]  

; \( i = 1,2, \ldots, n \)  

Estimates of \( \alpha \) and were obtained for the sample by using Newton Raphson's iterative method in the Maximum Likelihood method and Non-liner least square method and Lindely approximates in Bayes method under quadratic loss function by using iterated (1000) times for each simulation experiment for the purpose of obtaining high homogeneity in estimating the reliability function of the Burr distribution. The simulation experiments were carried out using the Mat lab 2015 programming language, and by using the statistical criterion (IMSE), which is calculated for each \( (t_i) \) representing integral the total area and reducing it to a single value is general for time or expressing the total time for the purpose of comparison between the methods of estimating the reliability function of the Burr distribution for the purpose of reaching the best estimate through a comparison between the studied estimation methods and the formula for this scale as follows:

\[ IMSE[\hat{R}(t)] = \frac{1}{n} \sum_{i=1}^{r} \left\{ \frac{1}{n_t} \sum_{j=1}^{n_t} \left[ \hat{R}_i(t_j) - R(t_j) \right] \right\} \]  

\[ = \frac{1}{n_t} \sum_{j=1}^{n_t} MSE[\hat{R}(t)] , \quad i = 1, \ldots, r \]  

Where:

r: represents the times of replantation of the experiment, and the simulation results were obtained using (Matlab 2015) program. All results were presented in the simulation experiments tables.
Table (1) the default (initial) values for the Burr distribution parameters

| Experiment | 1    | 2    | 3    |
|------------|------|------|------|
| \(\alpha\) | 2    | 1    | 0.9  |
| \(k\)      | 2    | 2    | 0.8  |

Table (2) shows the values of the reliability function and the mean integral error squares (IMSE) for the different estimation methods when \((\alpha = 2, k = 2)\) at the assumed sample sizes

| n  | \(R(t_i)\) | \(\hat{R}_{Mle}\) | \(\hat{R}_{NOLS}\) | \(\hat{R}_{Bayes}\) |
|----|-----------------|-----------------|-----------------|-----------------|
|    | \(\hat{\alpha} = 1.96\) | \(\hat{\alpha} = 2.48\) | \(\hat{\alpha} = 2.28\) |
|    | \(\hat{k} = 2.11\) | \(\hat{k} = 2.58\) | \(\hat{k} = 2.01\) |
| 10 | 0.8858          | 0.9260          | 0.8816          | 0.8885          |
|    | 0.6400          | 0.6728          | 0.6439          | 0.6489          |
|    | 0.4096          | 0.3900          | 0.4248          | 0.4103          |
|    | 0.2500          | 0.2118          | 0.2759          | 0.2440          |
|    | 0.1523          | 0.1202          | 0.1846          | 0.1413          |
|    | 0.0947          | 0.0732          | 0.1289          | 0.0867          |
|    | 0.0606          | 0.0476          | 0.0938          | 0.07001         |
|    | 0.0400          | 0.0328          | 0.0707          | 0.04901         |
|    | 0.0272          | 0.0237          | 0.0549          | 0.03372         |
| 30  | 0.0190          | 0.0177          | 0.0438          | 0.0134          |
| IMSE |                |                |                |                  |
|   | 0.0003          | 0.0004          | 0.00022         |
| Best | Bayes          |                |                |                  |
|      | \(\hat{\alpha} = 2.05\) | \(\hat{\alpha} = 2.36\) | \(\hat{\alpha} = 2.02\) |
|      | \(\hat{k} = 1.85\) | \(\hat{k} = 2.23\) | \(\hat{k} = 2.19\) |
| 30 | 0.8858          | 0.9243          | 0.9166          | 0.8898          |
|    | 0.6400          | 0.6988          | 0.7007          | 0.6513          |
|    | 0.4096          | 0.4395          | 0.4625          | 0.4752          |
|    | 0.2500          | 0.2532          | 0.2853          | 0.2644          |
|    | 0.1523          | 0.1458          | 0.1756          | 0.1411          |
|    | 0.0947          | 0.0871          | 0.1115          | 0.0864          |
|    | 0.0606          | 0.0545          | 0.0737          | 0.0743          |
|    | 0.0400          | 0.0356          | 0.0506          | 0.0481          |
|    | 0.0272 | 0.0241 | 0.0360 | 0.0381 |
|----|--------|--------|--------|--------|
| IMSE | 0.0003 | 0.0005 |        | 0.00023 |
| Best | Bayes  |        |        |        |
| 50  |        | $\hat{\alpha} = 2.14$ | $\hat{\alpha} = 2.06$ | $\hat{\alpha} = 2.41$ |
|     |        | $\hat{k} = 2.13$    | $\hat{k} = 2.10$    | $\hat{k} = 2.31$    |
|     | 0.8858 | 0.8990 | 0.8967 | 0.9803 |
|     | 0.6400 | 0.6546 | 0.6506 | 0.9704 |
|     | 0.4096 | 0.4106 | 0.4060 | 0.8665 |
|     | 0.2500 | 0.2405 | 0.2372 | 0.77628 |
|     | 0.1523 | 0.1395 | 0.1381 | 0.6547 |
|     | 0.0947 | 0.0827 | 0.0829 | 0.5776 |
|     | 0.0606 | 0.0508 | 0.0518 | 0.5065 |
|     | 0.0400 | 0.0323 | 0.0337 | 0.4436 |
|     | 0.0272 | 0.0213 | 0.0228 | 0.3893 |
|     | 0.0190 | 0.0146 | 0.0160 | 0.3652 |
| IMSE | 0.0001 | 0.0000 |        | 0.0011 |
| Best | NOLS   |        |        |        |
| 100 |        | $\hat{\alpha} = 1.98$ | $\hat{\alpha} = 2.16$ | $\hat{\alpha} = 2.44$ |
|     |        | $\hat{k} = 1.92$    | $\hat{k} = 2.05$    | $\hat{k} = 2.38$    |
|     | 0.8858 | 0.9043 | 0.8946 | 0.9913 |
|     | 0.6400 | 0.6697 | 0.6602 | 0.9511 |
|     | 0.4096 | 0.4285 | 0.4310 | 0.8781 |
|     | 0.2500 | 0.2559 | 0.2678 | 0.7777 |
|     | 0.1523 | 0.1515 | 0.1664 | 0.6619 |
|     | 0.0947 | 0.0918 | 0.1059 | 0.5833 |
|     | 0.0606 | 0.0576 | 0.0696 | 0.5082 |
|     | 0.0400 | 0.0376 | 0.0473 | 0.5082 |
|     | 0.0272 | 0.0254 | 0.0332 | 0.4392 |
|     | 0.0190 | 0.0177 | 0.0240 | 0.3778 |
| IMSE | 0.0001 | 0.0002 |        | 0.1199 |
| Best | MLE    |        |        |        |
Table (3) shows the values of the reliability function and the mean integral error squares (IMSE) for the different estimation methods when ($\alpha = 1, k = 2$) at the assumed sample sizes.

| N   | $R(t_i)$ | $\hat{R}_{MLE}$ | $\hat{R}_{NOLS}$ | $\hat{R}_{Bayes}$ |
|-----|----------|------------------|------------------|--------------------|
|     | $\hat{\alpha} = 1.21$ | $\hat{\alpha} = 1.13$ | $\hat{\alpha} = 1.13$ | $\hat{\alpha} = 1.13$ |
|     | $\hat{k} = 2.32$    | $\hat{k} = 2.20$    | $\hat{k} = 2.20$    | $\hat{k} = 2.20$    |
| 10  | 0.6400    | 0.6198           | 0.6061           | 0.6661             |
|     | 0.4444    | 0.4152           | 0.4279           | 0.4046             |
|     | 0.3265    | 0.2908           | 0.3210           | 0.3274             |
|     | 0.2500    | 0.2121           | 0.2512           | 0.2402             |
|     | 0.1975    | 0.1603           | 0.2030           | 0.1791             |
|     | 0.1600    | 0.1249           | 0.1681           | 0.1677             |
|     | 0.1322    | 0.0998           | 0.1420           | 0.1289             |
|     | 0.1111    | 0.0815           | 0.1219           | 0.1110             |
|     | 0.0947    | 0.0677           | 0.1061           | 0.0871             |
|     | 0.0816    | 0.0572           | 0.0934           | 0.0797             |
| IMSE|          | 0.0006           | 0.0002           | 0.0001             |
| Best| Bayes    | $\hat{\alpha} = 1.11$ | $\hat{\alpha} = 1.26$ | $\hat{\alpha} = 1.10$ |
|     |          | $\hat{k} = 2.12$ | $\hat{k} = 2.17$ | $\hat{k} = 2.16$ |
| 30  | 0.6400    | 0.6605           | 0.6594           | 0.6804             |
|     | 0.4444    | 0.4686           | 0.4715           | 0.4164             |
|     | 0.3265    | 0.3490           | 0.3540           | 0.3345             |
|     | 0.2500    | 0.2702           | 0.2763           | 0.2437             |
|     | 0.1975    | 0.2158           | 0.2227           | 0.1816             |
|     | 0.1600    | 0.1767           | 0.1841           | 0.1601             |
|     | 0.1322    | 0.1476           | 0.1553           | 0.1357             |
|     | 0.1111    | 0.1255           | 0.1333           | 0.1109             |
|     | 0.0947    | 0.1081           | 0.1160           | 0.0834             |
|     | 0.0816    | 0.0943           | 0.1021           | 0.0765             |
| IMSE|          | 0.0002           | 0.0004           | 0.0001             |
| Best| Bayes    | $\hat{\alpha} = 1.30$ | $\hat{\alpha} = 1.14$ | $\hat{\alpha} = 1.24$ |
|     |          | $\hat{k} = 2.19$ | $\hat{k} = 2.09$ | $\hat{k} = 2.43$ |
|     | 0.6400    | 0.6112           | 0.6097           | 0.9820             |
| N  | R(t_i) | $\hat{\alpha}_{\text{MLE}}$ | $\hat{\alpha}_{\text{NOLS}}$ | $\hat{\alpha}_{\text{Bayes}}$ |
|----|--------|-------------------------|-------------------------|-------------------------|
| 50 | 0.8171 | 0.8315                  | 0.7865                  | 0.8255                  |
|    | 0.7094 | 0.7200                  | 0.6824                  | 0.7141                  |
|    | 0.6328 | 0.6317                  | 0.6064                  | 0.6308                  |
| 100| 0.6400 | 0.6225                  | 0.6164                  | 0.9831                  |
|    | 0.4444 | 0.4298                  | 0.4230                  | 0.9195                  |
|    | 0.3265 | 0.3155                  | 0.3089                  | 0.8368                  |
|    | 0.2500 | 0.2418                  | 0.2358                  | 0.7454                  |

Table (4) shows the values of the reliability function and mean integral error squares (IMSE) for the different estimation methods when ($\alpha = 0.9$ $k = 0.8$) at the assumed sample sizes.
| \( \alpha \) | \( \hat{\alpha} \) | \( \hat{k} \) | \( \alpha \) | \( \hat{\alpha} \) | \( \hat{k} \) | \( \alpha \) | \( \hat{\alpha} \) | \( \hat{k} \) |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 10      | 0.5743  | 0.5616  | 0.5481  | 0.5430  |
|         | 0.5279  | 0.5054  | 0.5019  | 0.52160 |
|         | 0.4898  | 0.4597  | 0.4644  | 0.4708  |
|         | 0.4579  | 0.4222  | 0.4333  | 0.4352  |
|         | 0.4307  | 0.3908  | 0.4071  | 0.4017  |
|         | 0.4071  | 0.3642  | 0.3847  | 0.3659  |
|         | 0.3865  | 0.3415  | 0.3653  | 0.3378  |
|         | 0.0016  | 0.0004  | 0.00001 |         |
| 30      | \( \hat{\alpha} = 0.93 \) | \( \hat{k} = 0.83 \) | \( \hat{\alpha} = 0.78 \) | \( \hat{k} = 0.76 \) | \( \hat{\alpha} = 1.31 \) | \( \hat{k} = 1.44 \) |
|         | 0.8171  | 0.8081  | 0.7732  | 0.9392  |
|         | 0.7094  | 0.6951  | 0.6624  | 0.8369  |
|         | 0.6328  | 0.6147  | 0.5866  | 0.7405  |
|         | 0.5743  | 0.5536  | 0.5300  | 0.6494  |
|         | 0.5279  | 0.5053  | 0.4856  | 0.5668  |
|         | 0.4898  | 0.4659  | 0.4496  | 0.4949  |
|         | 0.4579  | 0.4331  | 0.4197  | 0.4340  |
|         | 0.4307  | 0.4053  | 0.3943  | 0.3953  |
|         | 0.4071  | 0.3814  | 0.3724  | 0.3516  |
|         | 0.3865  | 0.3605  | 0.3532  | 0.3156  |
|         | 0.0006  | 0.0012  | 0.1991  |         |
| 50      | \( \hat{\alpha} = 0.91 \) | \( \hat{k} = 0.84 \) | \( \hat{\alpha} = 0.99 \) | \( \hat{k} = 0.88 \) | \( \hat{\alpha} = 1.34 \) | \( \hat{k} = 1.47 \) |
|         | 0.8171  | 0.8284  | 0.8251  | 0.9521  |
|         | 0.7094  | 0.7233  | 0.7223  | 0.8398  |
|         | 0.6328  | 0.6475  | 0.6486  | 0.7384  |
|         | 0.5743  | 0.5893  | 0.5920  | 0.6458  |
|         | 0.5279  | 0.5428  | 0.5468  | 0.5631  |
|         | 0.4898  | 0.5046  | 0.5096  | 0.4910  |
|         | 0.4579  | 0.4725  | 0.4783  | 0.4291  |
|         | 0.4307  | 0.4451  | 0.4515  | 0.3889  |
|         | 0.4071  | 0.4214  | 0.4283  | 0.3428  |
Table (5) shows the number of times the preference of each of the estimation methods and the preference percentage. As is evident, the Bayes method is superior to the rest of the estimation methods with a preference percentage (42%), followed by the Maximum likelihood method with a preference percentage (25%) and finally the non-linear least squares method with a preference ratio (33 %).

Table (5) the preference for estimation methods and the preference for reliability ratio

| Method  | Preference percentage No. | Ratio |
|---------|---------------------------|-------|
| \( \hat{R}_{MLE} \)   | 4                          | 33    |
| \( \hat{R}_{NOLS} \)  | 3                          | 25    |
| \( \hat{R}_{Bayes} \) | 5                          | 42    |

It is evident from Table (1) and (2) and at the default values of the parameters of the Burr distribution, that at a sample size of (n = 10.30), the Bayes method was the best of the rest in estimating the reliability function of the Burr distribution \( \hat{R}_{Bayes} \) with the lowest mean integral error squares. At a sample size (50), the nonlinear least squares method was the best of the rest in estimating the reliability function of the Burr \( \hat{R}_{Nlos} \) distribution with the lowest mean integral error squares and at a sample size
(n = 100) the maximum likelihood method was the best of the rest of the methods for estimating the reliability function Burr distribution with the lowest mean integral error squares. In Table (3), and at the default values for the parameters of the Burr distribution, at a sample size of (n = 10), the Bayes method was the best of the rest of the methods in estimating the reliability function of the Burr distribution $\hat{R}_{Bayes}$ with the lowest mean integral error squares. At a sample size (n = 30,50,100), the maximum likelihood method was preferable to the rest of the methods for estimating the reliability function of the Burr distribution with the lowest mean squares of integral error.

5-2 Data Description

The data for the field study was obtained with the help of a staff previously trained in the method of data collection as a main and important street in the holy Karbala governorate was chosen in addition to determining the starting and ending points of the trip for this street as the journey starts from the intersection point of the ship to the intersection point of Karbala stadium, and to impose collection The data The test vehicle was provided with a GPS device and conducted a number of field trips on these roads for a period of one week, starting from Wednesday until Sunday and at times (3:00 - 5:00) in the evening (8:00 - 10:00) in the evening.

5-3 Estimate of Parameters

As same below shows the values of the estimated parameters of the Burr distribution using the maximum likelihood method, the nonlinear least squares method, and the Burr method as follows:

| Method          | $\hat{\alpha}$ | $\hat{k}$  |
|-----------------|-----------------|------------|
| Maximum likelihood | 2.42422         | 2.33287    |
| Non-Linear Least square | 2.47631         | 2.33675    |
| Bayes           | 2.21371         | 2.13543    |

We note from Table (5) that when comparing the values of the estimated parameters by the methods (the greatest possibility, nonlinear least squares - Bayes) with the values of the default parameters on the experimental side (simulation), when assuming ($\alpha = 2, k = 2$) and at the sample size (10) The values of the estimated parameters were ($\hat{\alpha} = 2.28, \hat{k} = 2.01$), when the sample size (30) the values of the estimated parameters were ($\hat{\alpha} = 2.02, \hat{k} = 2.19$). The parameters estimated for the real data are closest to the parameters estimated by the Bayes method.

5-4 Criteria for selecting the best reliability of distribution: \[58]\]

As same below Aakei criteria (AIC criteria, BIC criteria, akaiki consistent criteria) (CAIC criteria) to choose the best distribution reliability.
The results were included in a table representing the results of using the criteria, and as shown below:

Table No. (6) Criteria for choosing the best reliability of the distribution for Al-Iskan Street

| Method                  | AIC         | BIC         | CAIC        |
|-------------------------|-------------|-------------|-------------|
| Maximum likelihood      | 25649.766   | 23231.239   | 10.08976    |
| Non-Linear Least square | 28323.545   | 24544.765   | 15.45333    |
| Bayes                   | 24567.321   | 2034.762    | 8.67347     |

Table (6) shows that by using the comparison criteria (AIC), Bayes akayki (BIC), and akayki's consistent criteria ((CAIC) to choose the best method for estimating the reliability of accessibility for the Burr distribution, it was found that the Bayes method had the lowest value for the three criteria.

5.5 Data analysis:

As same above the results of the simulation experiments showed that the best way to estimate the reliability of the Burr distribution function is the Bayes method at small sample sizes n = 10.30, so this method will be applied to the real data to measure the reliability of arrival to Al-Iskan Street, and by using the Mat lab program a special program was created to implement method and the results were as shown in Table (7):

Table (7) shows the estimated reliability function for the real data using the Maximum Likelihood method for Al-Iskan Street, which has a Burr distribution in the period (3-5) in the evening, which is an off-peak period.

| Day       | Trip Time | R(t)  |
|-----------|-----------|-------|
| Wednesday | 1.39      | 0.9202|
|           | 1.4       | 0.9300|
| Thursday  | 2.52      | 0.5178|
|           | 3.11      | 0.3457|
| Friday    | 2.38      | 0.5615|
|           | 4.39      | 0.1926|
| Saturday  | 3.38      | 0.3103|
|           | 1.57      | 0.9120|
| Sunday    | 2.2       | 0.6521|
|           | 1.55      | 0.9201|
Table (8) the estimated reliability function for the real data using the Bayes method for Al-Iskan Street, which has a land distribution in the period (8-10) in the evening, which is the peak period.

| Day    | Trip Time | R(t)  |
|--------|-----------|-------|
| Monday | 1.15      | 0.9546|
|        | 1.05      | 0.9678|

Table (8) the estimated reliability function for the real data for Al-Iskan Street.

From the tables (7) (8), we note that the estimated reliability of arrival by Bayes method for the real data of Al-Iskan Street for the off-peak period (3-5) and the peak period (8-10), which had a Burr distribution, decreases with increasing flight time. For example, in the off-peak period, the probability of arriving in time (1.05), which represents the shortest flight time is (0.9678), and the probability of arrival in time (4.39), which represents the longest flight time within the period (0.1926), while in the peak period, the probability of arrival in time (1.19) is (0.9501) and the probability of arrival in time is (3.53), which is the longest flight time within the period (0.2811).

6- Conclusions

As same above The estimated parameters converge with the default values, in addition to that reliability of access converges with the default reliability of access as the sample size increases, and this is consistent with the statistical theory which states that the sample size increases, the values of the estimates approach the real values and behave according to the normal distribution.
1. The Bayes method is superior to the three methods in estimating the reliability of the flight time, followed by the greatest possibility method and finally the nonlinear least squares method.

2. The appropriateness of all three methods in estimating the distribution parameters and in estimating the reliability of access where the results were close.

3. In Al-Iskan Street, we notice that the more time the trip increases, the more reliable it will arrive on time.

4. The Bayes method is superior to the remaining methods in estimating the reliability function at small sample sizes, and the greatest possibility method surpasses the rest of the methods in estimating the reliability function at sample size (n = 100) and for all simulation experiments.

7. Recommendations:
   As same above we using the Bayes method in estimating the reliability of arrival at the specified time for distributions and for all sample sizes.
   Using the Maximum Likelihood method in estimating the reliability of flight time data.
   The researcher recommends applying the Burr distribution to estimate the reliability of the arrival time.
   Expanding the use of other distributions in estimating the reliability of arrival time.
   Conducting future studies and research in estimating the reliability of the flight time, and measuring the departure time and arrival time of the vehicle

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