Ratcheting and tumbling motion of Vibrots

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Abstract
Systems of granular rotors (Vibrots), that is, small devices that convert linear vibrational motion into rotation by frictional impact, are of scientific interest since they could reveal various types of collective behavior. Looking at an isolated Vibrot, we note at least two different dynamical modes, depending on the parameters of the vibrational driving. By means of finite element simulations, we reveal the driving mechanism for both cases which may be correspondingly identified as ratcheting and tumbling. The transition between both modes resembles period doubling in certain bouncing ball problems leading eventually to chaotic motion in such systems.

1. Introduction
Systems of self-propelled particles are of large interest because of emergent collective behavior evidenced by many examples from physics, biology, chemistry [1] and even robotics [2]. Macroscopic amounts of such particles constitute a material, sometimes called active matter [3] with characteristic macroscopic properties.

In the context of granular matter, energy is provided by vibrating boundaries and transferred into motion of the particle that is directed in the reference frame of the particle, e.g. along its axis of symmetry. Directed motion requires anisotropy of the particle, e.g. shape, mass distribution or elasticity. Most self-propelled granular particles reported in the literature are dumbbells [4–7] or rods [8–10] or other objects with linear asymmetry [11]. Their mechanism of energy transfer relies on an interplay between friction, inertial forces, and inelastic interaction between the particle and the vibrating wall [4–7]. Systems of self-propelled dumbbells or rod-shaped particles show interesting collective behavior, e.g. [8, 9, 12–16], frequently reminiscent to swarm-like motion in biological systems [1].

The particles considered in this paper do also transfer the energy of the vibrating floor into directed motion, however, here we consider rotation rather than translation. Numerical simulations of self-spinning particles predict interesting statistical [17] and collective behavior and pattern formation, e.g. [15, 16, 18]. Experiments using self-spinning particles had been reported before, [7, 19] however, in these cases the result of the energy transfer was not pure rotation but the particles perform more complex motion patterns.

The type of self-spinning particles we consider here, called Vibrot, was proposed by Altshuler et al [20]. Vibrots consist of a disk resting on circularly aligned tilted elastic legs (figure 1(a)). The advantage of these particles is that they transfer the vibrational motion of the floor almost completely into rotational motion without any other linear or more complex component [7, 19]. As such, they are experimental implementations corresponding to many simulations, e.g. [15–18]. Vibrots are simple enough to be produced in large quantities by rapid prototyping [21] to study collective behavior and exhibit a simple motion pattern which allows for analytic treatment based on statistical physics and kinetic theory [17].

While the directed motion of Vibrots can be understood already from the asymmetry of the particle, the microscopic details of the propulsion mechanism and their relation to the parameters of the vibrational excitation and material properties are not so obvious. However, in order to derive a kinetic theory of self-spinning granular particles [22–24], an exact knowledge of the particle motion is required [25]. A first attempt to explain the motion of Vibrots based on elastic deformation of the legs and viscous friction between particle and
vibrating floor [26] yielded already quantitative agreement with the experiment [20]. This theory relies on some tunable parameters and is, moreover, unable to explain the transition between different modes of motion of the particles, depending on the parameters of driving.

In the present paper, we investigate the self-propelled spinning of particles of Vibrot type, produced by rapid prototyping [21]. By means of numerical simulations using the finite element method (FEM) we study the microscopic mechanics of the particles which is not easily accessible in experiments. The simulation results agree very well with the experiment and deliver detailed insight into the mechanics of self-spinning Vibrots, including the transition between the dynamical modes which can be identified as a ratcheting and a tumbling mode due to our findings, where the transition resembles the phase doubling scenario known from bouncing ball experiments [27–30].

2. Materials and methods

2.1. Experimental setup and results

Figure 1(a) shows the Vibrot particle used in the experiments. It consists of a central cylindrical cap to which seven legs are attached circularly, tilted tangentially to the cylinder by $\alpha = 18^\circ$ (the number of legs was chosen to be odd, to prevent waggling of the particles which would be observed for four or six legs due to mirror symmetry). A second cylinder of smaller diameter is attached to lower the center of mass of the particle and, thus, stabilize the motion. The particles was manufactured from acrylonitrile butadiene styrene (ABS) using a 3D printer. The particle is placed on a polymethyl methacrylate (PMMA) plate which performs vertical vibrations at $f_D = 50 \text{ s}^{-1}$ excited by an electromagnetic shaker. The amplitude of the vibrations is measured using a magnetic stripe sensor with an accuracy of $3 \mu\text{m}$. The setup was previously used in [21], from where experimental results are reproduced in figure 2. The figure shows the mean angular velocity $\omega_{\text{rot}}$ as a function of the amplitude of excitation. Rotation starts at $A_D \approx 0.1 \text{ mm}$ and the angular velocity increases with $A_D$. At $A_D \approx 0.16 \text{ mm}$ a transition to a different mode is observed.

2.2. Details of the FEM simulation

The setup of the FEM model is sketched in figure 1(b). The geometrical parameters are given in the caption of figure 1 and the material parameters used in the simulation are provided in table 1. The data for ABS are used for the particle material and PMMA data are relevant for the vibrating table. The material characteristics are taken from [31] except for the density $\rho$ of the particle material which was deduced from the weight and dimensions of the particle to account for cavities due to the manufacturing process (see figure 3) and the Coulomb friction coefficient for which we used $\mu = 0.25$ which is a typical value for smooth plastic–plastic contact [32].

For the evaluation of the average angular velocity for a certain amplitude, we simulated the Vibrot’s motion for up to $1 \text{ s}$ real time using Ansys Mechanical [33]. In order to accelerate the FEM simulation performed, we applied two simplifications: (a) the lower cylinder is modeled as a rigid body since its deformation is assumed unimportant to the Vibrot’s dynamics, (b) The legs are modeled as bars with hexagonal cross-section. The fine mesh used for the FEM simulations is shown in figure 1(b). Inhomogeneous meshing allows for finer detail in

![Figure 1](image-url)

**Figure 1.** Vibrot shape in experiment and simulation (a) self-spinning particle of Vibrot type manufactured by rapid prototyping (mass $= 0.91$ g). The cap (diameter 15 mm, height 2 mm) rests on seven inclined legs (angle 18°, length 8.5 mm, diameter 1 mm, center distance 6.25 mm). The lower cylinder (diameter 11 mm, height 6 mm) stabilizes the Vibrot by lowering the center of mass. (b) Meshed model used for FEM simulations. The mesh size is inhomogeneous allowing for finer detail in sensitive regions. The central cylinder is modeled as a rigid body.
sensitive regions, in particular close to the contact points between particle and vibrating table. The mesh shown in figure 1(b) comprises 8851 elements. By means of stabilizing the number of elements could be reduced to 284 without noticeable changes in the results. In particular, we applied augmented Lagrange contact treatment, which prevents large penetration [34, 35] and a viscous contact stabilization [36]. The integration time step is chosen variable between $10^{-5}$ to $10^{-4}$ s and is updated iteratively. Each time step is integrated using the Newton–Raphson method [37, 38].

Figure 2. Mean angular velocity $\omega_{\text{rot}}$ of the Vibrot as a function of the amplitude of excitation at frequency $f_0 = 50$ s$^{-1}$. The error bars show the standard deviation of $\omega_{\text{rot}}$ computed from several measurement intervals of one second duration. Rotation starts at $A_D \approx 0.1$ mm and the angular velocity increases with $A_D$. At $A_D \approx 0.16$ mm a transition to a different mode is observed (‘ratcheting’ $\rightarrow$ ‘tumbling’). * experiment [21], o FEM simulation. The width of the error bars does hardly change when $A_D$ approaches the critical value indicating that there are no long-lasting transients and, thus, the duration of measurement is sufficient. The inset shows a magnification of the ratcheting regime. See also supplementary information S1 and S2.

Table 1. Material parameters for the FEM simulations.

| Symbol | ABS      | PMMA     | Description  |
|--------|----------|----------|--------------|
| $\rho$ | 1040 kg m$^{-3}$ | 1170 kg m$^{-3}$ | Density     |
| $E$    | 2.3 GPa  | 2.5 GPa  | Young’s modulus |
| $\nu$  | 0.35     | 0.37     | Poisson’s ratio |
| $K$    | 2.56 GPa | 3.21 GPa | Bulk modulus |
| $G$    | 0.85 GPa | 0.91 GPa | Shear modulus |
| $\sigma$ | 43.0 MPa | 51.7 MPa | Yield strength |

Figure 3. X-ray tomography cross-section of a Vibrot particle manufactured my means of rapid prototyping. The object reveals cavities and other imperfections which do not affect the functionality, however, the value of material density is different from the bulk density of the material and was, therefore, computed from the mass of the particle.
Figure 2 shows the average angular velocity determined from the particle trajectory (excluding the initial part, where the particle is accelerated from rest) obtained from the simulations in comparison to the experiments. For amplitude $A_D < 0.16$ mm and $A_D > 0.16$ mm, we find quantitative agreement with the experiment, including onset of motion at $A_D \approx 0.1$ mm. Close to the onset the rotational velocity is small but negative, which was previously observed in experiments [21, 39]. Since the value is very small this was blamed on inaccurate manufacturing and not explicitly mentioned. However, this behavior has been predicted from analytical models [40]. Our simulations reproduce this behavior and provide an explanation for the change in the direction of rotation, as explained in the following section. The transition between the modes of motion is slightly shifted from $A_D \approx 0.15$ mm in simulations to $A_D \approx 0.16$ mm in experiments, which is not too surprising due to our simplified model of the Vibrot shape.

3. Microscopic mechanism of Vibrot dynamics

3.1. Modes of dynamics

From the average angular velocity as a function of the amplitude of vibration shown in figure 2 and the corresponding supplementary movies S1 and S2, we can distinguish two different modes of dynamical behavior. For $A_D \leq 0.16$ mm we observe slow and very regular rotation while for $A_D \geq 0.16$ mm we see fast but somewhat irregular rotation. In order to understand these modes and the transition between them, let us first consider the motion of a perfectly inelastic particle on a table vibrating with $z(t) = A_D \sin(\omega t)$, where $\omega = 2\pi f$, $f$ is the frequency of vibration. If the amplitude of the acceleration does not exceed gravity, $A_D \omega^2 < -g$ (with $g = -9.81 \text{ m s}^{-2}$), the particle would rest on the table.

For more intense driving, at some time, $t_d$, when $z = g$ the particle would detach from the table, that is

$$t_d = \frac{1}{\omega} \arcsin\left(\frac{-g}{A_D \omega^2}\right).$$

At this instant, the particle’s velocity is

$$v_d = A_D \omega \cos(\omega t_d) = A_D \omega \sqrt{1 - \frac{g^2}{A_D^2 \omega^4}}.$$  

If the particle would re-touch the table before $t'_d \equiv t_d + \frac{2\pi}{\omega}$, it would stay in contact with the table till $t'_d$ and then detach again (perfectly inelastic contact is assumed here). This condition is fulfilled if $\pi/\omega \leq -v_d/g$, thus

$$-\frac{A_D \omega^2}{g} < \sqrt{\pi^2 + 1}.$$  

If this condition holds true, the motion of the particle is regular, that is, the particle detaches each period at $t = t_d + \frac{2\pi}{\omega}$, $i = 0, 1, 2, \ldots$. This motion would be stable against small perturbations in $v_d$ and, thus, the flight time would not change, provided, the particle re-touches the table before $t'_d$.

If condition (3) does not hold, the particle would re-touch at an instant, when $z(t) < g$ such that the particle would jump off immediately. It was shown that this motion is unstable against small perturbations and leads eventually to chaotic motion via a period doubling scenario [27–30].

In a very simple approach, we consider the legs of a Vibrot as inelastic particles moving on a vibrated table. From the discussion above, we conclude that for

$$1 < -\frac{A_D \omega^2}{g} < \sqrt{\pi^2 + 1}$$

all particle would move in perfect synchrony with the vibrating table and, thus, with one another, irrespective of small perturbations which are always present due to imperfections of the particles, see figure 3. For larger amplitude, that is, more intense forcing, the motion of the balls would be irregular, thus, an ensemble of balls would rapidly de-synchronize. Since the legs of a Vibrot are not independent, de-synchronized motion of the legs corresponds to irregular (wagglng) motion of the entire Vibrot.

The situation resembles the motion of granular particles in a vibrated container in the absence of gravity [41, 42] or in a horizontally vibrated container [43] where a collective synchronous motion of the particles (termed collect and collide mode) is observed for parameters of driving in a certain interval. In this mode, the particles are collectively ejected at the inward stroke of the container wall and then ‘collected’ at the opposite wall. For other parameters, the particles are scattered individually and a disordered, gas-like state is observed.

Note that a subharmonic motion when $t_d + \frac{3\pi}{\omega} < t'_d < t_d + \frac{4\pi}{\omega}$ would also allow for stable motion, however, it was shown in [44] that subharmonic motion requires special initial conditions and unrealistic material properties.
of restitution applies only to spheres: the typical amplitude and velocity in our experiment are the viscous relaxation time for typical plastics materials, and\textit{f}. In the limit of perfectly inelastic contact, restitution made for spheres, is appropriate here. Consider a sphere of the size of a Vibrot dissipative properties and, thus, the approximative characterization of the collision by means of a coefficient of restitution, \(\varepsilon = 0\). For the contact of plastic material with a hard wall, a value of \(\varepsilon \in (0.4, 0.7)\) may be more appropriate. When we assume a finite coefficient of restitution, the inequality condition for the interval of stable periodic motion, equation (4), turns into [30, 45]

\[-\frac{A_D \omega^2}{g} < \pi \left[\frac{1 - \varepsilon}{1 + \varepsilon}\right]^2 + \frac{2(1 + \varepsilon^2)}{\pi(1 + \varepsilon)^2}.\]

In the limit of perfectly inelastic contact, \(\varepsilon \to 0\), equation (3) is recovered. For the plausible value, \(\varepsilon = 0.47\), this condition would predict \(A_D^{(1)} \approx 0.16\) mm, in perfect agreement with the experiment.

The introduction of the coefficient of restitution, \(\varepsilon\), needs a justification since in strict sense, the coefficient of restitution applies only to spheres: the typical amplitude and velocity in our experiment are \(A \approx 0.15\) mm and \(f = 50\) s\(^{-1}\), resulting in the velocity amplitude \(v_{\text{max}} \approx 0.47\) m s\(^{-1}\), corresponding to a free fall from height 11 mm. Using the Hertz contact law, it can be checked that the maximal contact circle of a sphere made of the same material as a Vibrot (cf table 1) colliding at this velocity with a hard plane would not exceed the cross section of a leg. Therefore, we believe that for the given value of \(v_{\text{max}}\) the shape of the particle is irrelevant for the dissipative properties and, thus, the approximative characterization of the collision by means of a coefficient of restitution made for spheres, is appropriate here. Consider a sphere of the size of a Vibrot (\(R = 7.5\) mm) colliding at rate \(v_{\text{max}}\) with a solid plane. For this system, and the material parameters summarized in table 1, and the viscous relaxation time for typical plastics materials, \(\tau_A = 0.075\) s, we can compute the coefficient of restitution by means of equations (42) from [46], with the series truncated after the second term. We obtain \(\varepsilon \approx 0.53\), in agreement with the estimate, \(\varepsilon \in (0.4, 0.7)\), given above.

The discussion in this section concerns only the vertical motion of the Vibrot, the mechanism of propulsion is discussed in the following sections, separately for both dynamical modes.

### 3.2. Ratcheting mode

Let us first look at the motion of the Vibrot at low amplitude. In this regime, the legs move in synchrony, such that we can restrict the discussion to the motion of a single leg. Figure 4 shows snapshots of the tip of a leg (\(f = 50\) s\(^{-1}\), \(A_D = 0.135\) mm), at representative times before and after impact at the table: after detachment (a) while in the free falling regime (b) almost no rotational motion is observed. When the leg contacts the plate (c), it is bent due to inertia of the Vibrot. In a short interval after impact, when the contact force is small, the leg slides slightly backwards (d). Later, when the contact force is larger due to increased bent of the leg (d), Coulomb

**Figure 4.** Magnification of the leg’s motion. (a)–(f): sequence of snapshots of the leg’s motion during one period of oscillation. A small backward stroke in the interval (c)–(d) is followed by a larger forward stroke in (d)–(e). The vertical dashed line in each sub-figure (a)–(f) indicates a fixed position at the plate to emphasize the leg’s progress in horizontal direction. The color codes for the strain, \(\varepsilon\). Figure (g) shows the summarizes the trace of the leg’s tip during one cycle. See also supplementary information S3 and S4.
friction law prohibits sliding of the legs for forces of revolution of the Vibrot’s body. Thus, the component of the elastic force parallel to the surface $F_{\parallel} \propto \sin(\theta)$ drags the particle forward. When the normal force decreases, the leg relaxes its stored energy due to bending and moves with respect to the table. This observation obtained from the FEM calculation agrees nicely with the assumptions made in [26]. Afterwards the particle follows the motion of the table until the point of detachment, repeating this cycle (f). Accelerated motion parallel to the plate, i.e., rotation, occurs only during the intervals (c)–(e) and forward motion only during (d)–(e), see figure 4(g), where the trajectory of the tip is drawn. Clearly this mechanism is due to the friction at the surface and the elasticity of the legs, in agreement with experimental observations [20] and analytical modeling [26, 40]. A rigid particle would perform only vertical motion. Due to the characteristic, ratchet like motion of the Vibrot, we term this dynamical mode ratcheting mode.

To explain the occurrence of global forward and backward rotation depending on $A_D$, a closer look at the dynamics is necessary. Figure 5 illustrates the time resolved motion of the Vibrot including the rotational velocity of its cap, $\omega$, the position of the tip of a leg (angular, $\phi$, and vertical, $z$) and vertical position of the vibrating table for $A_D = 0.11$ mm and $A_D = 0.135$ mm. Careful analysis of $\omega$ and $\Phi$ shows a small linear backwards motion during the free fall regime (a)–(c) after the leg detaches from the plate. This apparent negative $\omega_{tot}$ is caused by the force balance between friction force and the elastic deformation of the leg, which relax after the particle jumps off the surface. For weak excitation close to the onset of motion (figure 5(b)) the forward rotation does not compensate the backward rotation, and in total the particle rotates backwards, i.e., $\omega_{tot} < 0$. For higher amplitudes (figure 5(c)) the forward motion surpasses the backward motion and $\omega_{tot} > 0$ is observed. The motion mechanism persists for all amplitudes in the ratcheting regime up to $A_D = 0.14$ mm.

3.3. Tumbling mode

According to the arguments given in the previous section, for larger amplitude, according to equation (5) the ratcheting mode is unstable, that is, small fluctuations amplify. Such fluctuations are always present due to material inhomogeneities (see figure 3) or imperfections of the geometrical shape. It was shown that even for the case of apparently smooth steel spheres microscopic impurities can significantly determine the collisional behavior [47–49].

As a result of this instability, the dynamic turns into the less regular tumbling mode which is characterized by the time between consecutive impacts being larger than the period of excitation. The tumbling regime is fundamentally different from the ratcheting regime. In this regime, the legs do not any longer move synchronously leading to less regular motion. The enhanced flight time implies larger relative velocity of the leg impacting the table which renders the previously assumed assumption of perfectly inelastic impact invalid and leads, in turn, to yet larger flight times such that impacts may take place at any phase of the periodic motion of the table, $z(t)$, leading to enhanced or reduced flight time of the subsequent trajectory. The described dynamics resembles the dynamics of a bouncing ball on a vibrating table beyond the regime of periodic stability. This system reveals a rich dynamics which eventually leads to chaotic behavior. There is an extended literature on the problem of a partly inelastic bouncing ball on a periodically driven platform which is an interesting problem by its own. For a detailed discussion of the bifurcation diagram, the transition to chaos via a period doubling scenario, the strange attractor, etc we refer to, e.g., [27–30, 50, 51].
Figure 6 shows the angular velocity and the vertical position of all Vibrot’s tips of the legs when moving in the tumbling regime, $A_0 = 0.18 \text{ mm}$. Obviously, the motion is rather irregular, indicated by large fluctuations of the angular velocity. The vertical positions, $z(t)$ (dashed lines shown in the bottom panel) starting synchronously at $t = 0$ desynchronize already after the first jump. This result of the FEM simulation agrees with experimental results [21] where large fluctuations of $\omega$ have been observed in the tumbling mode, suggesting chaotic motion.

The different lines correspond to the vertical trajectories of all seven legs and are drawn simultaneously into one panel to illustrate the dispersion.

We wish to note that the arguments presented here rely on the assumption that the legs of the Vibrot move independently. Although this strong simplification led to quantitative agreement between theory, experiment and FEM simulation, the motion is certainly more complex since rotational and translational motion of the Vibrot is coupled, such that energy is converted back and forth between the two degrees of freedom, similar to a double pendulum [52–54].

4. Summary

By means of FEM simulations, we investigate the microscopic mechanism underlying the motion of Vibrots, that is simple devices that are able to transform linear motion in rotational motion when placed on a vibrating table. Such particles allow experimental verification of many existing numerical studies. We identify an interplay between elastic deformations of the Vibrot’s legs and dynamic/static friction between the vibrating table and the Vibrot as origin of its dynamics.

The rotational motion of the Vibrot changes its characteristics with the parameters of driving. From a simple model assuming independent motion of the Vibrot’s legs, we obtain estimates for the onset of rotational motion (ratcheting regime) $\frac{A_0 \omega^2}{\xi} \gtrsim 1$ and an upper limit for the transition from the smooth motion due to ratcheting into the tumbling regime at $\frac{A_0 \omega^2}{\xi} \gtrsim \sqrt{\pi^2 + 1}$, characterized by an incoherent leg motion and chaotic fluctuations. The first estimate agrees well with experiments and with FEM simulations, the latter overestimates the transition due to the assumption of a perfectly inelastic contact. This deviation can be explained quantitatively by assuming a realistic value for the coefficient of restitution.

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