Hyperplane conjecture for quotient spaces of $L_p$

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Abstract

We give a positive solution for the hyperplane conjecture of quotient spaces $F$ of $L_p$, where $1 < p \leq \infty$.

$$\text{vol}(B_F)^{\frac{n-1}{n}} \leq c_0 \, p' \, \sup_{H \text{ hyperplane}} \text{vol}(B_F \cap H).$$

This result is extended to Banach lattices which does not contain $\ell_1^n$'s uniformly. Our main tools are tensor products and minimal volume ratio with respect to $L_p$-sections.

Introduction:

An open problem in the theory of convex sets is the so called

Hyperplane problem: Does there exist a universal constant $c > 0$ such that for all $n \in \mathbb{N}$ and all convex, symmetric bodies $K \subset \mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leq c \sup_{H \text{ hyperplane}} |K \cap H|?$$

For some classes of convex bodies this problem has a positive solution. For example, for convex bodies with unconditional basis a positive solution was first given by Bourgain. He also proved

$$|K|^{\frac{n-1}{n}} \leq c_0 \, n^{\frac{1}{4}} \, (1 + \ln n) \, \sup_{H \text{ hyperplane}} |K \cap H|,$$

which is still the best known estimate for arbitrary convex bodies. Another class consists of convex bodies with small volume ratio with respect to the ellipsoid of minimal volume. This includes the class of zonoids. K. Ball [BA] solved the problem for the duals of zonoids, i.e. unit balls of subspaces of an $L_1$-space, briefly $L_1$-sections.

Theorem 1 (K. Ball) For a convex, symmetric body $K \subset \mathbb{R}^n$ one has

$$|K|^{\frac{n-1}{n}} \leq 2 \inf_{H \text{ hyperplane}} \left\{ \frac{|S|}{|K|} \right\}^\frac{1}{2} \sup \left\{ \frac{|K \cap S|}{|S|} \mid K \subset S, \ S L_1\text{-section} \right\}$$

This theorem implies all positive solutions listed before (except Bourgain’s $n^{\frac{1}{4}}$ estimate). A further application of Ball’s theorem for the hyperplane problem can be deduced from the following
Theorem 2 Let $Y$ be a Banach space with the Gordon-Lewis property. Then the following conditions are equivalent.

i) $Y$ does not contain $\ell_\infty^n$’s uniformly.

ii) There exists a constant $c > 0$ such that for all $n \in \mathbb{N}$ and all $n$-dimensional subspace $F \subset X$ there is a $L_1$-section $S \subset F$ with

$$\left(\frac{|S|}{|B_F|}\right)^{\frac{1}{n}} \leq c.$$  

The Gordon-Lewis property was introduced in connection with a problem of Grothendieck. A Banach space $X$ has the Gordon-Lewis property if every absolutely 1-summing operator acting on $X$ factors through $L_1$. This is an operator ideal property which typically holds in Banach lattices. Unfortunately, Gordon and Lewis [GL] discovered spaces without this property. A combination of theorem 1 and theorem 2 gives a positive solution of the hyperplane conjecture for subspaces of a Banach lattice with finite cotype. This was proved independently by J. Zinn (still unpublished). Since we are using the Gordon-Lewis property it is not surprising that the proof of theorem 2 is based on the theory of absolutely summing operators. But in this framework one can also replace 1-summing operators by $p$-summing operators. Thereby one obtains the theory of minimal $L_p$-sections, i.e. affine images of finite dimensional sections of the unit ball of $L_p$, containing a certain convex body. In this way a connection to the case $p = 2$ which is about minimal ellipsoids containing a convex body is established. This was discussed intensively in the literature under the name of weak type 2 spaces. We will prove

Theorem 3 Let $Y$ be a Banach space with the Gordon-Lewis property and not containing $\ell_\infty^n$’s uniformly. For a subspace $X \subset Y$ the following conditions are equivalent.

i) $X$ does not contain $\ell_1^n$’s uniformly.

ii) There exists $1 < p \leq s < \infty$ and a constant $c > 0$ such that for all $T : L_p \to L_s$ one has

$$\|T \otimes Id_X : L_p(X) \to L_s(X)\| \leq c \|T\|.$$  

iii) There exists a $1 < p \leq 2$ and a constant $c_p > 0$ such that for all $n \in \mathbb{N}$ and all $n$-dimensional subspace $F \subset X$ there is a $L_p$-section $S_p \subset F$ with

$$\left(\frac{|S_p|}{|B_F|}\right)^{\frac{1}{p}} \leq c.$$  

Analyzing the proof of theorem 3 it turns out that the type index of $X$ coincides with the supremum over all $p$ such that $X$ has the $L_p$-section property. Examples show that a type $p$ condition in Banach lattices does not imply the $L_p$-section property. On the other hand $p$-convex Banach lattices with finite cotype have the $L_p$-section property. Condition ii) is related to Bourgain’s Hausdorff-Young inequalities for spaces which does not contain $\ell_\infty^n$’s uniformly. In the presence of the Gordon-Lewis property the proof is considerably easier and extends to arbitrary operators. But this phenomena also indicates a limitation of the method which is motivated by results of Pisier [PS3, PS4]. Namely, the Schatten classes $S_p$, $p \neq 2$, can not be embedded in a quotient of a subspace $X \subset Y$ which satisfies one of the above conditions. But on the other hand condition ii) is extremely useful for the solution of the hyperplane conjecture. The main idea consists in comparing gaussian variables and coordinate functionals on convex bodies.
Theorem 4 Let $Y$ be a Banach space with the Gordon-Lewis property and not containing $\ell_1^n$’s uniformly. Then there exists a constant $c_Y > 0$ such that for all $n$-dimensional quotients of subspaces $F$ of $Y$ one has
\[
\left| B_F \right|^{\frac{1}{n}} \leq c_Y \sup_{H \text{ hyperplane}} \left| B_F \cap H \right| .
\]
In particular, the hyperplane conjecture is uniformly satisfied for quotients of $L_p$ for $1 < p \leq \infty$.

In section 1 we develop the tensor product techniques used for the geometric applications in section 2. Parts of the results are contained in the author’s PHD-Thesis. An investigation of $L_q$-sections contained in convex bodies is planned in a further publication.

Preliminaries

We are only dealing with Banach spaces over the scalar field $\mathbb{R}$ of real numbers. In the text standard Banach space notation will be used. In particular, $c_0, c_1, \ldots$ always denote universal constants. Banach spaces will be denoted by $E, F, Y, X, X_0, X_1, \ldots$ The symbols $E, F$ are reserved for finite dimensional spaces. Given a closed subspace $X$ of $Y$ there is a natural injection $\iota_X : X \to Y, x \mapsto x$. We use the same notation $\iota_X$ for the isometric embedding of $X$ in it’s bidual. The unit ball of a Banach space $X$ is denoted by $B_X$. In contrast to this we denote by $B_p^n (1 \leq p \leq \infty, n \in \mathbb{N})$ the unit ball of the space $\ell_p^n$. This space as well as $\ell_p, L_p, c_0$ and the vector valued spaces $L_p(\Omega, \mu; X), \text{ or briefly } L_p(X)$, are defined in the usual way, where $(\Omega, \mu)$ is a measure space. In the following $p'$ denotes the conjugate index $p$, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

A standard reference on operator ideals is the monograph of Pietsch [PIE]. The ideal of all linear bounded, finite rank operators is denoted by $\mathcal{L}, \mathcal{F}$, respectively. For a Banach ideal $(\mathcal{A}, \alpha)$ the component $\mathcal{A}^d(X, Y)$ of the conjugate ideal $(\mathcal{A}^*, \alpha^*)$ is the class of all operators $T \in \mathcal{L}(X, Y)$ such that
\[
\alpha^*(T) := \sup \left\{ \left| \text{tr} TS \right| \bigm| S \in \mathcal{F}(Y, X), \alpha(S) \leq 1 \right\} < \infty .
\]

The component $\mathcal{A}^d(X, Y)$ consists of all operators $T \in \mathcal{L}(X, Y)$ such that $T^* \in \mathcal{A}(Y^*, X^*)$. Equipped with the norm $\alpha^d(T) := \alpha(T^*)$ the pair $(\mathcal{A}^d, \alpha^d)$ is again a Banach ideal. For $1 \leq p \leq \infty$ an operator $T \in \mathcal{L}(X, Y)$ is said to be (absolutely) $p$-summing ($T \in \Pi_p(X, Y)$) if there is a constant $c > 0$ such that for all $n \in \mathbb{N}, (x_k)^n \subset X$
\[
\left( \sum_{1}^{n} \left| Tx_k \right|^p \right)^{\frac{1}{p}} \leq c \sup_{x \in B_{X^*}^p} \left( \sum_{1}^{n} |(x_k, x^*)|^p \right)^{\frac{1}{p}} .
\]

We denote by $\pi_p(T) := \inf \{ c \}$, where the infimum is taken over all $c$ satisfying the above inequality. An operator $T \in \mathcal{L}(X, Y)$ is $p$-integral ($T \in \mathcal{I}_p(X, Y)$), if there is a factorization $\iota_X T = SIR$, where $R \in \mathcal{L}(X, L_\infty(\Omega, \mu)), \ (\Omega, \mu)$ a probability space, $I \in \mathcal{L}(L_\infty(\Omega, \mu), L_p(\Omega, \mu))$ the formal identity and $S \in \mathcal{L}(L_p(\Omega, \mu), Y^{**})$. The $p$-integral norm $\iota_p(T)$ is defined as $\inf \{ \| S \| \| R \| \}$, where the infimum is taken over all such factorizations. Let us note that if one of the spaces $X, Y$ is finite dimensional one has, see [PIE]
\[
\Pi_p(X, Y) = \mathcal{I}_p(X, Y) \quad \text{and} \quad \mathcal{I}_p(X, Y) = \Pi_p(X, Y) .
\]

An operator is called $p$-factorable ($T \in \Gamma_p(X, Y)$) if there are a measure space $(\Omega, \mu)$ and operators $R \in \mathcal{L}(X, L_p(\Omega, \mu)), S \in \mathcal{L}(L_p(\Omega, \mu), Y^{**})$ such that $\iota_Y T = SR$. Here $\iota_Y : Y \to Y^{**}$ denotes the canonical embedding from $Y$ in it’s bidual. It is well known that $\Gamma_p^d = \Gamma_p^d$. 

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In the following \((g_k)_{k \in \mathbb{N}}\) denotes a sequence of independent, normalized gaussian variables on a probability space \((\Omega, \mathbb{P})\). With this notion we define for \(u \in \mathcal{L}(\ell_2, X)\) the ideal norm

\[
\ell(u) := \sup_{n \in \mathbb{N}} \left\| \sum_{k=1}^{n} g_k u(e_k) \right\|_{L_2(X)},
\]

where \((e_k)_{k \in \mathbb{N}}\) denotes the unit vectors in \(\ell_2\) (or \(\mathbb{R}^n\)). In this context Kahane’s inequality is of particular interest. There exists an absolute constant such that for all \(1 \leq p < \infty\) one has

\[
\left( \int_{\Omega} \left\| \sum_{k=1}^{n} g_k x_k \right\|^p d\mathbb{P} \right)^{\frac{1}{p}} \leq c_0 \sqrt{p} \int_{\Omega} \left\| \sum_{k=1}^{n} g_k x_k \right\| d\mathbb{P}.
\]

Finally, we define a volume number \(v_n(T)\) of an operator \(T \in \mathcal{L}(X, Y)\). This notion is helpful to connect volume properties of Banach spaces with the theory of operator ideals. A closely related notion was introduced in Mascioni [MAS].

\[
v_n(T) := \sup \left\{ \left( \frac{|T(B_E)|}{|B_F|} \right)^{\frac{1}{n}} \mid E \subset X, \ T(E) \subset F \subset Y, \ dimE = dimF = n \right\}.
\]

Here and in the following \(| |\) denotes the translation invariant Lebesgue measure. If we consider the Lebesgue measure of \(k\)-dimensional sections of a convex body this will be denoted by \(||k\)\.

The following multiplication formula for \(S \in \mathcal{L}(X_1, X)\) is completely elementary

\[
v_n(TS) \leq v_n(T) v_n(S).
\]

Certainly, equality holds if \(X\) is \(n\) dimensional.

A Banach lattice is a Banach space with an order satisfying the same properties as a function spaces. For a precise definition of this and for further notations see [LTII]. For \(1 \leq p < \infty\) a Banach lattice \(Y\) is said to be \(p\)-convex, resp. \(p\)-concave if there exists a constant \(c > 0\) such that for all \(n \in \mathbb{N}, \ (x_k)_1^n \subset Y\)

\[
\left\| \sum_{k=1}^{n} |x_k|^p \right\|^\frac{1}{p} \leq c \left( \sum_{k=1}^{n} \|x_k\|^p \right)^\frac{1}{p}, \quad \left( \sum_{k=1}^{n} \|x_k\|^p \right)^\frac{1}{p} \leq c \left( \sum_{k=1}^{n} |x_k|^p \right)^\frac{1}{p}, \quad \text{resp.}
\]

The best possible constant will be denoted by \(K^p(Y), K_p(Y)\) respectively. Let us note that \(Y\) is \(p\)-convex if and only if \(Y^*\) is \(p'\)-concave. By a characterization of Maurey a Banach lattice is \(q\)-concave if and only if every positive operator \(T \in \mathcal{L}(\ell_\infty, Y)\) is \(q\)-summing, see again [LTII].

Closely connected with the notion of concavity is the notion of cotype in arbitrary Banach spaces. A Banach space \(Y\) has cotype \(q\) \((2 \leq q < \infty)\) if there is a constant \(c > 0\), such that for all \(n \in \mathbb{N}, \ (x_k)_1^n \subset Y\)

\[
\left( \sum_{k=1}^{n} \|x_k\|^q \right)^\frac{1}{q} \leq c \int_{\Omega} \left\| \sum_{k=1}^{n} g_k x_k \right\| d\mathbb{P}.
\]

The best constant is denoted by \(C_q(Y)\). \(Y\) has finite cotype if it has cotype \(q\) for some \(q < \infty\). We will frequently use the following fact, which is a combination of Maurey’s theorem and Pietsch-Grothendieck’s factorization theorem, see [PIE]. Let \(Y\) be a Banach space with cotype \(q\), then for all \(q < s < \infty\) and \(T \in \mathcal{L}(\ell_\infty, Y)\) one has
Proposition 1.1

Let $Y$ be a Banach space.

conditions can be used to improve the Gordon-Lewis-propert y.

that for all $1 \leq p < q'$ and $v \in \mathcal{L}(X,Y)$ one has

$$\pi_1(v) \leq c(p', X) \pi_p(v).$$

1 Gordon-Lewis property and vector-valued extensions

In this chapter we establish the connection between the Gordon-Lewis property with additional cotype conditions and vector valued extension of operators between $L_p$-spaces. In a way we continue the ideas developed in the work of Pisier [PSE]. It will be distinguish between the Gordon-Lewis property and a restricted version $gl_2$. A Banach space $Y$ is said to have the Gordon-Lewis property ( GLP), if for every absolutely 1-summing operator $v$ the operator $\iota_Y v$ admits a factorization through some $L_1$ space. More precisely, there exists a constant $c > 0$ such that for all $v \in \Pi_1(Y, Z)$

$$\gamma_1(v) \leq c \pi_1(v).$$

The best possible constant is denoted by $gl(Y)$. If this inequality only holds for operators $v \in \Pi_1(Y, \ell_2)$ the Banach space is said to have $gl_2$ with constant $gl_2(X)$. Let us note that the GLP and $gl_2$ are self dual properties. In the next proposition we indicate how cotype and type conditions can be used to improve the Gordon-Lewis-property.

Proposition 1.1 Let $Y$ be a Banach space.

1. The following conditions are equivalent

1i) $Y$ has the Gordon-Lewis property ($gl_2$) and $Y^*$ is of finite cotype.

1ii) There exists a $1 < p < \infty$ and a constant $c_p$ such that for every absolutely 1-summing operator $v \in \Pi_1(Y, Z)$ ($v \in \Pi_1(Y, \ell_2)$) the operator $\iota_Y v$ admits a factorization through an identity $I_p : L_p(\Omega, \mu) \to L_1(\Omega, \mu)$, $(\Omega, \mu)$ a probability space. In other terms

$$\iota_p(v^*) \leq c_p \pi_1(v).$$

2. The following conditions are equivalent

2i) $Y$ has the Gordon-Lewis property ($gl_2$) and $Y$ as well as $Y^*$ is of finite cotype.

2ii) There exists $1 < p, r < \infty$ and a constant $c_{pr}$ such that for every absolutely $r$-summing operator $v \in \Pi_r(Y, Z)$ ($v \in \Pi_r(Y, \ell_2)$) the operator $\iota_Y v$ admits a factorization through an identity $I_p : L_p(\Omega, \mu) \to L_1(\Omega, \mu)$, $(\Omega, \mu)$ a probability space. In other terms

$$\iota_p(v^*) \leq c_p \pi_r(v).$$

Proof: i) $\Rightarrow$ ii) By the definition of $p'$-integral operators it is sufficient to prove the corresponding norm inequalities. We assume $Y^*$ of cotype $q < \infty$ and choose $p = s'$ for some $q < s < \infty$. Let $v \in \mathcal{L}(Y, Z)$, $(v \in \mathcal{L}(Y, \ell_2))$ be an absolutely 1-summing operator. By definition there exists a factorization $\iota_Y v = SR$, $R \in \mathcal{L}(Y, L_1)$, $S \in \mathcal{L}(L_1, Z^{**})$. Using $(\ast)$ in the preliminaries we see that $R^*$ is s integral and get

$$\iota_s(v^*) \leq \iota_s(R^*) \|S\| \leq c(s, Y^*) \|R\| \|S\|.$$
Taking the infimum over all factorization we obtain $c_p \leq gl(Y) C_s(Y^*)$. If $Y$ has in addition some cotype $\bar{q}$ we can apply ($**$) in the preliminaries for all $1 < r < \bar{q}'$.

(ii) $\Rightarrow$ (i) Since the identity $I_p : L_p(\Omega, \mu) \to L_1(\Omega, \mu)$ trivially factors through some $L_1$ space we only have to check the corresponding cotype conditions. Now $1 < p \leq 2$ be given by condition (ii) and $(y_i^*)_1 \subset Y^*$. As a consequence of Kintchine’s inequality one can easily see that for $v := \sum_1^n y_i^* \otimes e_i \in \mathcal{L}(Y, \ell_2^n)$ one has

$$\pi_1(v) \leq \sqrt{\frac{\pi}{2}} \left( \sum_1^n \|y_i^*\|^{p'} \right)^{\frac{1}{p'}} \leq \pi_{p'}(v^*) \leq c_p \pi_1(v) \leq c_p \sqrt{\frac{\pi}{2}} \left( \sum_1^n \|y_i^*\| \right) d\mathbb{P}.$$

Hence we deduce from the injectivity of the $p'$-summing norm

$$\left( \sum_1^n \|y_i^*\|^{p'} \right)^{\frac{1}{p'}} \leq \pi_{p'}(v^*) \leq \iota_{p'}(v^*) \leq c_p \pi_1(v) \leq c_p \sqrt{\frac{\pi}{2}} \left( \sum_1^n \|y_i^*\| \right) d\mathbb{P}.$$

This means that $Y^*$ has cotype $p'$. If condition 2(ii) is satisfied we use trace duality to get for all $u \in \mathcal{L}(\ell_2^n, Y)$

$$\iota_{p'}(u) \leq c_p \pi_p(u^*) .$$

The same argument above yields that $Y$ is of cotype $r'$.

\[\square\]

**Remark 1.2** Since the GLP and $gl_2$ are self dual proposition \([\square]\) implies that a Banach space has GLP ($gl_2$) and finite cotype if and only if there exists an $2 \leq s < \infty$ and a constant $c_s$ such that every operator $u \in \mathcal{L}(X, Y)$ ($u \in \mathcal{L}(\ell_2, Y)$) whose dual is absolutely 1-summing is even $s$-integral with

$$\iota_s(u) \leq c_s \pi_1(u^*) .$$

If $Y^*$ has in addition some finite cotype this improves to

$$\iota_s(u) \leq c_s \pi_p(u^*)$$

for some $p > 1$.

The results in this chapter are motivated by the phenomena in Banach lattices. In this case we can prove a sharp formula.

**Proposition 1.3** Let $1 \leq p, q \leq \infty$ and $Y$ a Banach lattice which is $p$-convex and $q$-concave. Then we have for all Banach space $X$ and $u \in \mathcal{L}(X, Y)$

$$\iota_q(u) \leq K^p(Y) \, K_q(Y) \, \pi_p(u^*) .$$

\[\text{Proof:}\] Having Maurey’s proof of the local unconditional structure of Banach lattices in mind, \([\text{MAU}]\), there is no restriction to assume that $Y$ has an unconditional, normalized Basis $(x_i)_{i \in \mathbb{N}}$ with coordinate functionals $(x_i^*)_1 \subset Y^*$. For $u \in \mathcal{L}(X, Y)$ we define

$$S := \sum_{i \in \mathbb{N}} \frac{u^*(x_i^*)}{\|u^*(x_i^*)\|} \otimes e_i \in \mathcal{L}(X, \ell_\infty) ,$$
where \((e_i)_{i \in \mathbb{N}}\) denotes the usual unit vector basis in \(\ell_p\). We want to show that the operator 
\[ R := \sum_{i \in \mathbb{N}} e_i \otimes \Vert u^*(x_i^*) \Vert \] 
\(x_i\) is a positive continuous operator from \(\ell_\infty\) to \(Y^{**}\). Clearly, \(R\) is positive.

For the continuity let \(\alpha = (\alpha_i)_{i \in \mathbb{N}} \in \ell_\infty\) and \(x^* \in Y^*\) with coefficients \(\beta_i := \langle x, x_i \rangle\). In this situation the diagonal operator \(D_\beta \in L(c_0, Y^*)\) defines a positive lattice homomorphism of norm at most 1. By assumption \(Y\) is \(p\)-convex, hence \(Y^*\) is \(p'\)-concave. Therefore \(D_\beta\) is \(p'\)-summing and by Pietsch’s factorization theorem there are positive diagonal operators \(D_\sigma \in L(\ell_{p'}, Y^*)\) with \(D_\beta = D_\sigma D_\tau\), \(\|D_\sigma\| \leq 1\) and
\
\[ \|R\|_{p'} \leq \pi_{p'}(D_\beta) \leq K_{p'}(Y^*) \|\beta\|_{Y^*} \leq K_p(Y) \|x^*\| . \]
\
From this we obtain with Hölder’s inequality
\
\[ |(R(\alpha), x^*)| \leq \sum_{i \in \mathbb{N}} \|u^*(x_i^*)\| |\beta_i| \]
\[ = \sum_{i \in \mathbb{N}} \|u^*(x_i^*)\| \, \sigma_i \, \tau_i \]
\[ \leq \left( \sum_{i \in \mathbb{N}} \|u^*(x_i^*)\| \sigma_i \right)^{\frac{1}{p'}} \|\tau\|_{p'} \]
\[ \leq \pi_p(u^*) \|D_\sigma\| \|\tau\|_{p'} \]
\[ \leq \pi_p(u^*) K_p(Y) \|x^*\| . \]
\
Let us note that \(Y^{**}\) is also a \(q\)-concave Banach lattice. Therefore Maurey’s characterization implies together with Pietsch’s factorization theorem applied the operator \(R\) together with \(\iota_Y u = RS\)
\
\[ \iota_q(u) \leq \iota_q(R) \|S\| \leq K_q(Y) \|R\| \|S\| \leq K_q(Y) K_p(Y) \pi_p(u^*) . \]

At the end of this chapter we show how inequalities between summing operators and integral operators can be characterized in terms of vector-valued extensions of operators between \(L_p\)-spaces. This is connected to Kwapien’s characterization of quotients of subspaces of \(L_p\)-spaces.

**Proposition 1.4** Let \(1 \leq p < \infty\), \(1 \leq s \leq \infty\), \((A, \alpha)\) an operator ideal and \(T \in L(X, Y)\). Then the following assertions are equivalent.

i) There exists a constant \(c_1 > 0\) such that for all \(R \in A(X_0, X_1)\), \(u \in \mathcal{T}_p^{\ell}(X_1, X)\)
\
\[ \pi_s(T u R) \leq c_1 \iota_p(u^*) \alpha(R) . \]

ii) There exists a constant \(c_2 > 0\) such that for all \(R \in A^{\ell}(\ell_p, \ell_s)\)
\
\[ \|R \otimes T : \ell_p(X) \to \ell_s(Y)\| \leq c_2 \alpha(R^*) . \]

Moreover, the best constants in i) and ii) coincide.
Proof: \(i \Rightarrow ii\) We consider an element \(u = \sum_{k} e_k \otimes x_k \in \ell_p^n(X)\) as an operator in \(\mathcal{L}(\ell_p^n, X)\) which satisfies \(\iota_p(u^*) \leq \|u\|_{\ell_p^n(X)}\). In the same way an element \(w \in \ell_p^n(Y^*)\) defines a \(\delta\)-integral operator \(w \in \mathcal{I}_s(Y, \ell_p^n)\) with \(\iota_s(w) \leq \|w\|_{\ell_p^n(Y^*)}\). With an elementary computation for traces we immediately get

\[|\langle R \otimes T(u), w \rangle| = |\text{tr}(w TuR^*)| \leq \iota_s(w) \pi_s(TuR^*) \leq c_1 \|w\| \|u\| \alpha(R^*).\]

\(ii \Rightarrow i\) By maximality we can assume that \(ii\) also holds for arbitrary \(L_p, L_s\)-spaces and for \(T^{**}\) instead of \(T\). Let \((x_i)^n_i \subset E\) with \(\sup_{e^* \in B_{E^*}} \sum_1^n |\langle x_i, e^* \rangle|^s \leq 1\). Then the operator

\[O := \sum_1^n e_i \otimes x_i \in \mathcal{L}(\ell_p^n, E)\]

has norm less than one. We choose an element \(w = (y^*_i)^n_i \in B_{\ell_p^n(Y^*)}\), which corresponds to an operator from \(Y\) to \(\ell_p^n\), such that

\[\left( \sum_1^n \|TuR(x_i)\|^s \right)^{\frac{1}{s}} = \sum_1^n \langle TuR(x_i), y^*_i \rangle = \text{tr}(w TuRO).\]

By the definition of \(p\)-integral operators there is a factorization \(u^* = SIQ\) where \(Q \in \mathcal{L}(X^*, L_{\infty})\), \(L_{\infty}\) is defined on a probability space, \(I \in \mathcal{L}(L_{\infty}, L_p)\) the formal identity and \(S \in \mathcal{L}(L_p, F^*)\) such that \(\|Q\| \leq 1, \|S\| \leq (1 + \varepsilon) \iota_p(u^*)\). By approximation we can even assume that the image of \(IQT^{**}w^*\) is contained in the span of a finite sequence \((\chi_{A_j})_1^m\) of mutually disjoint characteristic functions. Then

\[u := \sum_1^m \frac{Q^*(\chi_{A_j})}{\mu(A_j)} \otimes \chi_{A_j}\]

is an element of norm at most 1 in \(L_p(X^{**})\). We apply \(ii\) for the operator \(\bar{R} := O^*R^*\) and deduce

\[\text{tr}(w TuRO) = \langle \bar{R} \otimes T^{**}(u), w \rangle \leq \|\bar{R} \otimes T\| \|u\|_{L_p(X^{**})} \leq c_2 \alpha(\bar{R}^*) \leq c_2 \alpha(R) \|O\| \|S\| \leq c_2 \alpha(R) (1 + \varepsilon) \iota_p(u^*).\]

Letting \(\varepsilon\) to zero yields the assertion. \(\square\)

Remark 1.5 For a subspace \(S_p \subset L_p\) and a Banach space \(X\) we denote by \(S_p(X)\) the closure of \(\{\sum_1^m f_i \otimes x_i | f_i \in S_p, x_i \in X\}\) in \(L_p(X)\). This space consists of typical \(p\)-summing operators from \(X^*\) to \(S_p\). In a similar way the space \(Q_s(X)\) is defined for a quotient space \(Q_{s}\) of \(L_s\). Following the same pattern as the proof of proposition 4.4 it can be proved that the following assertions \(iii\) and \(iv\) as well as \(v\) and \(vi\) are equivalent for an operator \(T \in \mathcal{L}(X, Y)\).

\[\Rightarrow\]

\(iii\) There exists a constant \(c_3 > 0\) such that for all \(R \in \mathcal{A}(X_0, X_1), u \in \Pi_p^d(X_1, X)\)

\[\pi_s(TuR) \leq c_3 \pi_p(u^*) \alpha(R).\]

\(iv\) There exists a constant \(c_4 > 0\) such that for all subspaces \(S_p \subset \ell_p, R \in \mathcal{A}^d(S_p, \ell_s)\)

\[\|R \otimes T : S_p(X) \rightarrow \ell_s(Y)\| \leq c_4 \alpha(R^*).\]
v) There exists a constant $c_5 > 0$ such that for all $R \in \mathcal{A}(X_0, X_1)$, $u \in \Pi^d_p(X_1, X)$
\[ \iota_s(TuR) \leq c_5 \pi_p(u^*) \alpha(R) . \]

vi) There exists a constant $c_6 > 0$ such that for all subspaces $S_p \subset \ell_p$, all quotients $Q_s$ of $\ell_s$ and $R \in \mathcal{A}^d(S_p, Q_s)$
\[ \| R \otimes T : S_p(X) \to Q_s(Y) \| \leq c_6 \alpha(R^*). \]

Remark 1.6 As a consequence of the preceding remark 1.5 and remark 1.2 we deduce the following characterization for Banach spaces with GLP $(gl_2)$ and non-trivial type, namely for a Banach space $Y$ the following are equivalent.

i) $Y$ has the GLP $(gl_2)$ and $Y$ as well as $Y^*$ are of finite cotype.

ii) There exists $1 < p \leq s < \infty$ and a constant $c > 0$ such that for all subspaces $S_p \subset L_p$, quotients $Q_s$ of $L_s$ and all $T \in \mathcal{L}(S_p, Q_s)$ ($T \in \Gamma_2(S_p, Q_s)$)
\[ \| T \otimes Id_Y : S_p(Y) \to Q_s(Y) \| \leq c \| T \| (\gamma_2(T)) . \]

In particular, in this situation $Y$ is K-convex and does not contain $\ell_1^n$'s uniformly, see [PS2].

Proof: We will shortly indicate why ii) implies the K-convexity of $Y$. Indeed we denote by
\[ P = \sum_{n \in \mathbb{N}} g_n \otimes g_n, \]
the orthogonal projection onto the span of a sequence of independent normalized gaussian variables. By Kintchine’s or Kahane’s inequality $P : L_p \to L_s$ can be factorized in the form $P = u_s(u'p)^*$ where
\[ u_s := \sum_k e_k \otimes g_k \in \mathcal{L}(\ell_2, L_s). \]
Therefore, we get
\[ \gamma_2(P : L_p \to L_s) \leq c_0^2 \sqrt{sp}. \]
Condition ii) implies together with Kahane’s inequality
\[ \| P \otimes Id_Y : L_2(Y) \to L_2(Y) \| \leq c_0^2 c \gamma_2(P) \leq c c_0^4 \sqrt{sp}. \]
Therefore, $Y$ is K-convex and does not contain $\ell_1^n$’s uniformly, see [PS2].

2 Geometric applications

In the following a convex body will be a convex, symmetric, compact set $K \subset \mathbb{R}^n$ with non-empty interior. By $X_K := (\mathbb{R}^n, \| \cdot \|_K)$ we denote the $n$-dimensional Banach space whose unit ball is $K$. The following lemma is well-known and can be deduced from Pajor-Tomczak’s inequality and Kahane’s inequality, for more information and constants see [PS2] and [SCH]. We want to formulate this lemma because of the frequent use.
Lemma 2.1 Let \( 1 \leq s < \infty \) then for all Banach space \( X \) and \( u \in \Pi_s(\ell_2, X) \) one has
\[
\sup_{k \in \mathbb{N}} \sqrt{k} v_k(u) \leq \ell(u) \leq \sqrt{s} \pi_s(u). 
\]
Now we are able to prove the connection between minimal \( L_p \)-sections and volume estimates for \( p \)-summing operators. This is a generalization of Ball’s characterization of the weak-right-hand Gordon-Lewis property.

Proposition 2.2 There is a constant \( c_0 > 0 \) such that for \( 1 \leq p < \infty \) and for all convex bodies \( K \subset \mathbb{R}^n \) one has
\[
\sqrt{n} \pi_p(u) \leq \inf \left\{ \left( \frac{|S_p|}{|K|} \right)^{\frac{1}{p}} \mid K \subset S_p, S_p L_p \text{-section} \right\} \leq c_0 \sqrt{n} \pi_{p(u^*)} \sup \pi_p(u^*),
\]
where the supremum is taken over all operators \( u \in \mathcal{L}(\ell_2^n, X_K) \).

Proof: Let \( K \subset S \) where \( S \) is an \( L_p \)-section, i.e. \( S = T^{-1}(B_{L_p}) \) where \( T \in \mathcal{L}(X_K, L_p) \) is a rank \( n \) operator of norm at most 1. For an operator \( u \in \mathcal{L}(\ell_2^n, X_K) \) we consider the composition \( U := Tu \) which satisfies \( \pi_p(U^*) \leq \pi_p(u^*) \). By lemma 2.4 and proposition 1.3
\[
\sqrt{n} v_n(u) = \sqrt{n} \left( \frac{|U(B^n_2)|}{|T(\mathbb{R}^n) \cap B_{L_p}|} \right)^{\frac{1}{p}} \left( \frac{|S|}{|K|} \right)^{\frac{1}{p}} \leq \sqrt{p} \pi_p(U) \left( \frac{|S|}{|K|} \right)^{\frac{1}{p}} \leq \sqrt{p} \pi_p(u^*) \left( \frac{|S|}{|K|} \right)^{\frac{1}{p}} \]
Taking the infimum over all \( L_p \)-sections yields the first estimate. For the second one we apply Lewis lemma [LEW] to find an isomorphism \( u^{-1} = VIR \) where \( R \in \mathcal{L}(X_K, L_p) \), \( I \in \mathcal{L}(L_p, L_1) \) the formal identity and \( V \in \mathcal{L}(L_1, \ell_2^n) \) with \( ||R|| \leq 1 \) and \( ||V|| \leq \sqrt{n} \). Clearly, \( S : = R^{-1}(B_{L_p}) \) is an \( L_p \) section which contains \( K \). As a consequence of Grothendieck’s inequality and the fact that \( \ell_2 \) is of (weak) cotype 2 we deduce
\[
\sup_{k \in \mathbb{N}} \sqrt{k} v_k(U) \leq c_0 \pi_2(v) \leq c_0 K_G ||V|| \leq c_0 \sqrt{n}. 
\]
If we denote the supremum on the right hand side of our assertion by \( Sup \) we obtain
\[
1 = v_n(id_{\ell_2^n}) = v_n(u) v_n(u^{-1}) \leq v_n(u) v_n(VI) v_n(R) \leq \sqrt{n} Sup c_0 \left( \frac{|K|}{|S|} \right)^{\frac{1}{p}} \]

Remark 2.3 From Kwapien’s inequality between \( p \)-summing operators it is evident that the supremum on the right hand side of the proceeding proposition is minimal for \( p = 1 \). Nevertheless, random quotients of \( \ell_q^n \) \( (1 \leq q \leq 2) \) with proportional dimension \( k = \delta n \) yield examples of spaces where the minimal volume ratio with respect to \( L_p \)-sections is worst possible. This was discovered by K. Ball in the case \( p = 1 \). More precisely, for such a random quotient \( Q \) one has
\[
\inf \left\{ \left( \frac{|S|}{|B_Q|} \right)^{\frac{1}{p}} \mid B_Q \subset S, S L_1 \text{-section} \right\} \sim_{c_p} k^{\frac{p-1}{2}}. 
\]
Proof: We will show that for a random subspace $E \subset \ell^m_p$ of dimension $k = \delta n$, $1 \leq p < \infty$

$$k^{\frac{1}{p} - \frac{1}{q}} \leq c_0 \delta - \frac{1}{p} \sup \{ v_k(w) | \pi_p(w) \leq \sqrt{k} \},$$

where the supremum is taken over all operators $w \in \mathcal{L}(E, \ell^m_2)$. Then the assertion follows from proposition 2.2 and the inverse of Santaó’s inequality [PS2]. By [FIJ] a random subspace of $\ell^m_q$ satisfies

$$\pi_1(\ell^m_q \mathcal{L}E) \leq c_0 \pi_2(\ell^m_q) \leq c_0 \sqrt{n}.$$  

Here $\ell^m_q$ denotes the formal identity from $\ell^m_q$ to $\ell^m_2$. On the other hand a result of Meyer and Pajor, see [MEP], implies

$$1 = \left( \frac{|E \cap B^0_n|}{|E \cap B^0_q|} \frac{|E \cap B^0_n|}{|E \cap B^0_2|} \right)^{\frac{1}{2}} \leq c k^{\frac{1}{p} - \frac{1}{q}} v_k(\ell^m_q \mathcal{L}E).$$

Since $\ell_q$ is a (weak) type $q$ space and ellipsoids are $L_p$-sections this estimate is best possible. □

**Corollary 2.4** Let $1 \leq p < \infty$ and $Y$ a $p$-convex Banach lattice. $Y$ is of finite cotype if and only if there is a constant $c > 0$ such that for all $n \in \mathbb{N}$, and $n$-dimensional subspace $F$ there is a $L_p$-section $S_p \subset F$ containing $B_F$ with

$$\left( \frac{|S_p|}{|B_F|} \right)^{\frac{1}{p}} \leq c.$$  

**Proof:** If $Y$ has finite cotype it is $s$-concave for some $s < \infty$, see [LIT]. Using lemma 2.1 and proposition 1.3 we deduce for all $u \in \mathcal{L}(\ell^m_2, F)$

$$\sqrt{n} v_n(u) \leq \ell(t_Fu) \leq \sqrt{s} \pi_s(u) \leq \sqrt{s} K^p(Y) K_s(Y) \pi_p((t_Fu)^*) \leq \sqrt{s} K^p(Y) K_s(Y) \pi_p(u^*).$$

By proposition 2.2 $B_F$ is contained in a $L_p$ section with small volume. On the other hand a Banach lattice which is not of finite cotype contains $\ell_p^\infty$’s uniformly by Maurey/Pisier’s theorem [MP]. Following the proceeding remark there are $n$-dimensional Banach spaces such that the volume ratio with respect to minimal $L_p$-sections is of order $n^{\frac{1}{p}}$ (as far as $p < \infty$). Since every $n$-dimensional Banach space can be embedded 2-isomorphic into some $\ell_p^n$ we deduce that $Y$ does not have the $L_p$-section property. □

The proof of corollary 2.4 contains the main idea of this paper. First we establish an inequality like

$$\pi_s(u) \leq c_{ps} \pi_p(u^*) \leq c_{ps} t_p(u^*)$$

for a Banach space $X$ with the help of the Gordon-Lewis property. By the first chapter this corresponds to a Fubini type inequality. (That’s how it is proved for $L_p$.) Using Lemma 2.1 we use this inequality for abstract volume estimates to deduce the $L_p$-section property and estimates for the hyperplane problem.

**Remark 2.5** In the case $p = 1$ we can apply exactly the same proof as in corollary 2.4 for a Banach space with $gl_2$, simply by replacing proposition 1.3 by remark 1.2. In particular, this yields as a proof of theorem 2. Moreover, if $Y$ is a Banach space with $gl_2$ and $Y^*$ has cotype $q$ then $Y$ itself has finite cotype if and only if $Y$ has the $L_p$-section property for some (for all) $1 < p < q'$.  

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Proof of theorem 3: Let $Y$ be a Banach space which does not contain $\ell_\infty$’s uniformly and has GLP. By Maurey/Pisier’s theorem $[MP]$ $Y$ is of finite cotype. Following proposition $[1]$ and remark $[2]$ there exists an $2 < s < \infty$ such that for every Banach space $X_1$ and every operator $u \in \mathcal{L}(X_1,Y)$ whose dual is absolutely $1$-summing the operator $u$ is already absolutely $s$-summing with

$$
\pi_s(u) \leq \iota_s(u) \leq c_s \pi_1(u^*) .
$$

\(i) \Rightarrow ii), iii)\) Let $X \subset Y$ a space not containing $\ell_1^n$’s uniformly. From Maurey/Pisier’s characterization of non trivial type $[MP]$ the dual space $X^*$ has some finite cotype $q$, say. By the injectivity of the absolutely $s$-summing operators and $(\ast \ast)$ in the preliminaries we deduce for all Banach spaces $X_1$ and all $u \in \mathcal{L}(X_1,X)$

$$
\pi_s(u) \leq c_s \pi_1((1_Y u)^*) \leq c_s \pi_1(u^*) \leq c_s c(p,X) \pi_p(u^*) \leq c_s c(p,X) \iota_p(u^*) ,
$$

for all $1 \leq p < q'$. \(ii)\) follows from proposition $[1]$ and \(iii)\) from lemma $[2]$ and proposition $[2]$ as in the proof of corollary $[2]$. \(ii) \Rightarrow i)\) has already been noticed in remark $[1]$. \(iii) \Rightarrow i)\) We only have to show that for $1 \leq p \leq 2$

$$
\inf \left\{ \left( \frac{|S|}{|B^n_1|} \right)^{\frac{1}{p}} \mid B^n_1 \subset S, S \text{ L}_p\text{-section} \right\} \geq c_p n^{\frac{1}{p'}} ,
$$

which is a well-known fact due to Maurey/Carl $[CA]$ in the theory of (weak) type $p$ spaces.

Remark 2.6 For a fixed subspace $X \subset Y$, where $Y$ has $gl_2$ and finite cotype, the proof of theorem 3 yields the following implications given $1 < r < p < q \leq 2$

$$
X \text{ type } q \Rightarrow X \text{ L}_p\text{-section property} \Rightarrow X \text{ weak type } p \Rightarrow X \text{ type } r .
$$

This means that the type index coincides with the supremum over all $p$ satisfying the $L_p$-section property. For Banach lattices $X$ with finite cotype the situation is slightly better ($1 < p \leq 2$).

$$
X \text{ p-convex} \Rightarrow X \text{ L}_p\text{-section property} \Rightarrow X \text{ weak type } p \Rightarrow X \text{ r-convex} .
$$

Nevertheless, the Lorentz spaces $\ell_{pq}$ with $p < q < \infty$ yield examples of Banach lattices with type $p$, not having the $L_p$-section property. This can be proved using proposition $[2]$ and \(+)\)

$$
\pi_p^d(\ell^0_2 \to \ell^0_{pq}) \sim c_0 n^{\frac{1}{p}} (1 + \ln n)^{\frac{1}{p} - \frac{1}{q}} .
$$

Proof of \(+): It is clearly enough to prove

$$
\pi_p(\ell^0_{p'} \to \ell^0_2) \sim c_0 \left( \frac{n}{1 + \ln n} \right)^{\frac{1}{p'}} .
$$

For this we note that

$$
\|\alpha\|_{p'} = \sup \sum_{k=1}^n \alpha_k \varepsilon_k \pi(k)^{-\frac{1}{p'}} ,
$$

where the supremum is taken over all signs $\varepsilon_k = \pm 1$ and all permutations $\pi$ of the number $\{1, \ldots, n\}$. We consider the group of signs $\mathbb{D}_n$, the group of permutations $\mathbb{P}_n$ with Haar measure
\(\mu, \nu,\) respectively. From the triangle inequality in \(\ell_2^p\) and Kintchine’s inequality we deduce

\[
\left(\frac{1}{n} \sum_{1}^{n} \frac{1}{k}\right)^{\frac{1}{p}} \|\alpha\|_2 = \left(\sum_{1}^{n} \left(\int_{\mathbb{F}_n} \pi(k)^{-1} |\alpha_k|^p \, d\nu(\pi)\right)^{\frac{2}{p}}\right)^{\frac{1}{2}} \\
\leq \left(\int_{\mathbb{F}_n} \left(\sum_{1}^{n} \pi(k)^{-\frac{1}{p}} |\alpha_k|^2\right)^{\frac{4}{p}} \, d\nu(\pi)\right)^{\frac{1}{2}} \\
\leq c_0 \left(\int_\pi \left(\sum_{1}^{n} \pi(k)^{-\frac{1}{p}} \varepsilon_k |\alpha_k|^p \, d\mu(\varepsilon)\, d\nu(\pi)\right)^{\frac{1}{p}}\right)
\]

By the trivial part of Pietsch factorization theorem this implies

\[
\pi_p(\ell_{\rho}^n \to \ell_2^n) \leq c_0^2 \left(\frac{n}{1 + \ln n}\right)^{\frac{1}{p}}.
\]

This estimate is optimal since \(\pi_p(\ell_{\rho}^n \to \ell_2^n) \sim n^{\frac{1}{p}},\) see [PIE]. □

For the hyperplane problem the isotropic position of a convex body is of particular interest. \(K \subset \mathbb{R}^n\) is said to be in isotropic position if there exists a constant \(L\) such that

\[
\int_{K} |\langle x, \theta \rangle|^2 \frac{dx}{|K|} = L^2 \|\theta\|_2^2
\]

holds for all vectors \(\theta \in \mathbb{R}^n\). If in addition \(|K| = 1\) then \(L_K := L\) is called the constant of isotropy, a detailed discussion is contained in [MIPA]. For further applications we will give a slight generalization of Hensley’s result [HEN].

**Proposition 2.7** Let \(K \subset \mathbb{R}^n\) be a convex body.

i) If \(K\) is in isotropic position and \(H\) is a \(n - k\) dimensional subspace of \(\mathbb{R}^n\) one has

\[
|K|^{\frac{n-k}{nk}} \left(\frac{k + 2}{k}\right)^{\frac{1}{2}} \leq \sqrt{2\pi} e L_K |K \cap H|^{\frac{1}{n-k}}.
\]

ii) There exists an absolute constant \(c > 0\) and an orthogonal matrix \(O\) such that

\[
\frac{1}{c} \left(\prod_{\text{card}A=k} |O(K) \cap H_A|_{n-k}\right)^{\frac{1}{(k)^n}} \leq |K|^{\frac{n-k}{nk}} \leq c L_K \left(\prod_{\text{card}A=k} |O(K) \cap H_A|_{n-k}\right)^{\frac{1}{(k)^n}},
\]

where \(H_A\) is the span of \(\{e_i \mid i \notin A\}\).

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Proof: i) W.l.o.g. we can assume that $K$ has volume 1. By [J2] we get
\[
\left( \frac{k}{k + 2} \right)^\frac{s}{2} \leq \left( \int_k \| P_{H^*} (x) \|^2_2 \, dx \right)^\frac{s}{2} \frac{\|B_2^n \cap H^*\|^2}{|H \cap K|^{\frac{s}{2}}} = L_K \sqrt{k} \frac{|B_2^n \cap H^*|^{\frac{s}{2}}}{|H \cap K|^{\frac{s}{2}}} \leq \sqrt{2\pi e} L_K \frac{|H \cap K|^{\frac{s}{2}}}{|H^*|^{\frac{s}{2}}},
\]
where $H^*$ denotes the orthogonal complement of $H$.

ii) The inequality on the left side follows from a result of Meyer [MEY]. Now let us assume that $|K| = 1$. Then there exists a selfadjoint transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ with $det(T) = 1$ such that $T(K)$ is in isotropic position. By spectral decomposition there are orthogonal matrices $O, P$ such that $T = PD_T O$ where $D_T$ is a positive diagonal operator. Since the isotropic position is invariant under orthogonal transformations we can assume $T = D_T O$. By the transformation formula we deduce from i) $(c = \sqrt{6\pi e})$
\[
(cL_k)^{-k} \leq |D_T O(K) \cap H_A| = \left( \prod_{i \notin A} \tau_i \right) |O(K) \cap H_A|.
\]
If we take the product over all $A$ with cardinality $k$ we obtain
\[
(cL_k)^{-1} \leq \left( \prod_{\text{card} A = k} \prod_{i \notin A} \right) \left( \prod_{\text{card} A = k} \left| \frac{1}{\prod_{i \notin A} \tau_i} \right| \right) \left( \prod_{\text{card} A = k} \left| \frac{1}{\prod_{i \notin A} \tau_i} \right| \right) \leq (det(T))^{\frac{n-k}{nk}} \left( \prod_{\text{card} A = k} \left| \frac{1}{\prod_{i \notin A} \tau_i} \right| \right) \leq 1.
\]
But $det(T) = 1$ and we have proved the assertion.

Theorem 2.8 Let $Y$ be a Banach space with $gl_2$ and such that $Y$ and $Y^*$ have finite cotype. Then there exists a constant $c_Y$ such that for all quotients of a subspace $X$ and all convex bodies $K \subset \mathbb{R}^n$ one has
\[
|K|^{\frac{n-1}{n}} \leq c_Y \inf \left\{ \left( \frac{\sup_{H \text{hyperplane}} \left| K \cap H \right|}{|K|} \right)^{\frac{1}{n}} \right\} T : \mathbb{R}^n \to X, T(K) \subset B_X \right\} \sup_{H \text{hyperplane}} |K \cap H|.
\]
In particular, the hyperplane conjecture is uniformly satisfied for quotients of subspaces of $Y$.

Proof: Since $Y$ and $Y^*$ have finite cotype we can apply remark 1.2 to deduce the existence of $1 < p \leq s < \infty$ such that for all $u \in \mathcal{L}(\ell_2, Y)$
\[
i_s(u) \leq c_Y \pi_p(u^*).
\]
In particular, we obtain for all Banach spaces $X_0$, all Hilbert space factorizing operator $R \in \Gamma_2(X_0, X_1)$ and all $u \in \mathcal{L}(X_1, Y)$
\[
\pi_s(uR) \leq c_Y \iota_p(u^*) \gamma_2(R).
\]
Functionals of isotropy of a convex body $K$ constant of isotropy of a convex body $g$ a sequence of independent, normalized gaussian variables $(g_n)_{n=1}^\infty$ for all $P$. From proposition 1.4 we deduce

$$\|P \otimes Id_X : L_p(Y) \to L_s(Y)\| \leq cy \gamma_2(P)$$

for all $P \in \Gamma_2(L_p, L_s)$. By proposition 2.7 the assertion is proved if we give an estimate for the constant of isotropy of a convex body $K \subset \mathbb{R}^n$ with $|K| = 1$. For this we want to compare a sequence of independent, normalized gaussian variables $(g_k)_{k=1}^n$ on $(\Omega, \mathcal{P})$ with the coordinate functionals $(x_k)_{k=1}^n$ on $K$. Therefore let us consider

$$P := \sum_{k=1}^n \frac{x_k}{L_K} \otimes g_k \in \mathcal{L}(L_p(K), L_s(\Omega)).$$

An appropriate factorization of $P$ is given by $SR$, where

$$R := \sum_{k=1}^n \frac{x_k}{L_K} \otimes e_k \in \mathcal{L}(L_p(K), \ell_2^n)$$

and the isotropic position of $K$ yields

$$\|R\| = \|R^*\| = \sup_{\|\beta\|_2 \leq 1} \left( \int \langle \beta \frac{L_K}{\|L_K\|}, x \rangle^p \frac{dx}{\|L_K\|} \right)^{\frac{1}{p}} \leq c_0 p' \sup_{\|\beta\|_2 \leq 1} \left( \int \langle \beta \frac{L_K}{\|L_K\|}, x \rangle^2 \frac{dx}{\|L_K\|} \right)^{\frac{1}{2}} \leq c_0 p'.$$

Hence we get $\gamma_2(P) \leq c_0 \sqrt{s} p'$. By $(++)$ we have $\|P \otimes Id_Y\| \leq c_X c_0 \sqrt{s} p'$. Clearly, such an estimate is also valid for subspaces of $Y$ and, by duality, for quotients of subspaces. For more precise information on the injective and surjective ideal of operators tensoring with $P$ see [DEF].

Now let $X$ be a subspace of a quotient of $Y$ and $T : \mathbb{R}^n \to X$ with $T(K) \subset B_X$. We define $f := \sum_{k=1}^n T(e_k) \otimes x_k \in L_p(K; X)$ and $S := T^{-1}(B_X)$. With lemma 2.1 we conclude

$$2 L_K \leq \left( \frac{|S|}{|K|} \right) \frac{1}{n} \frac{L_K \sqrt{n}}{\|T(\mathbb{R}^n) \cap B_Y\|} \leq \left( \frac{|S|}{|K|} \right) \frac{1}{n} \left\| \sum_{k=1}^n T(e_k) \otimes L_K g_k \right\|_{L_2(\Omega; X)} \leq c_0 c_X \sqrt{s} p' \left( \frac{|S|}{|K|} \right) \frac{1}{n} \|f\|_{L_p(K; X)} \leq c_0 c_X \sqrt{s} p' \left( \frac{|S|}{|K|} \right) \frac{1}{n} \left( \int \|T(x)\|^p \frac{dx}{\|L_K\|} \right)^{\frac{1}{p}} \leq c_0 c_X \sqrt{s} p' \left( \frac{|S|}{|K|} \right) \frac{1}{n}.$$

With the second part of proposition 2.7 we immediately get the following

**Corollary 2.9** Let $Y$ be a Banach space with $\ell_2^n$ and such that $Y$ and $Y^*$ have finite cotype. Then there exists a constant $c_Y$ such that for all $n$ dimensional quotients of subspaces $F$ one has

$$|B_F|^{\frac{n-k}{n}} \leq c_Y \sup_{H \subset F, \text{codim} H = k} |B_F \cap H|^{\frac{1}{n-k}}.$$
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