The marginalization paradox does not imply inconsistency for improper priors

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Summary.
The marginalization paradox involves a disagreement between two Bayesians who use two different procedures for calculating a posterior in the presence of an improper prior. We show that the argument used to justify the procedure of one of the Bayesians is inapplicable. There is therefore no reason to expect agreement, no paradox, and no evidence that improper priors are inherently inconsistent. We show further that the procedure in question can be interpreted as the cancellation of infinities in the formal posterior. We suggest that the implicit use of this formal procedure is the source of the observed disagreement.

Keywords: Marginalization paradox; improper prior; reduction principle; noninformative prior

1. Introduction.
An important question in statistics is whether Bayesian inference can be extended to the setting of improper priors in a consistent and intuitively viable manner. The use of improper priors was common throughout much of the twentieth century, and appears to be a useful idealization for many applications. In the 1970s, however, two influential arguments appeared against the use of improper priors: the “marginalization paradox,” and “strong inconsistency.” These arguments appear to have convinced most statisticians that improper priors must be abandoned.

In this paper we discuss the marginalization paradox, due to Dawid, Stone, and Zidek [1973] (DSZ73). Let \( p(x|\theta) \) be a normalized sampling distribution with parameter \( \theta = (\eta, \zeta) \) and data \( x = (y, z) \), and let \( p(\theta) \) be a prior, which may be improper, i.e., of infinite total probability. The marginalization paradox concerns the problem of calculating \( p(\zeta|z) \), under a certain set of assumptions. A first Bayesian, \( B_1 \), eliminates \( \eta \) and then \( y \); a second Bayesian, \( B_2 \), eliminates \( y \) and then \( \eta \). The details of the procedures are given in DSZ73. It is claimed that these procedures rely only on principles that would have to hold in any intuitively viable theory of inference. If \( p(\theta) \) is improper, however, \( B_1 \) and \( B_2 \) generally get incompatible answers. It has been widely inferred that any extension of Bayesian inference to the context of improper priors will be inconsistent.

The purpose of this paper is to show that the marginalization paradox does not imply that the use of improper priors will lead to inconsistency. First, we show that the argument used to justify \( B_1 \)'s elimination of \( y \) is invalid, because it is based on the application of probabilistic intuitions to a formal quantity whose probabilistic meaning has not been justified. The “paradox” is thereby resolved, since we now have no reason to believe that \( B_1 \)'s answer is correct, and no reason to insist that the answers of \( B_1 \) and \( B_2 \) be compatible.

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Second, we analyze $B_1$’s procedure on its own terms, to get a better sense for what is being assumed. The posterior $p(\zeta|z)$ is defined as a ratio, which is only formal when the prior is improper because there are infinities in the numerator and denominator. $B_1$’s procedure is equivalent to the assumption that these infinities will cancel. What DSZ73 have shown, therefore, is that there is no consistent extension of Bayesian inference in which the cancellation law, assumed implicitly by $B_1$, holds when the prior is improper. But this is only to be expected: it is analogous to the well-known fact that there is no consistent extension of arithmetic to the extended real numbers in which the cancellation law holds for infinity. The proposal that we abandon improper priors because of the marginalization paradox is analogous to the proposal that we abandon the use of infinity because it does not obey the laws of arithmetic.

In brief, the inconsistency of the marginalization paradox is based on an assumption that has not been justified intuitively and that is unreasonable mathematically. There is nothing in the marginalization paradox to preclude the existence of a formalism that justifies the careful use of improper priors.

2. The intuitive argument.

In this section we show that the validity of $B_1$’s argument has not been established, because it is based on an intuitive probabilistic argument, and the distribution to which it is applied has not been shown to have a probabilistic meaning. In other words, we show that DSZ73 have not made their case, because their argument contains a gap.

In addition to the assumptions described in Section 1, we assume the following:

(1) The formal posterior, defined as

$$p(\zeta|y, z) = \frac{\int p(y, z|\eta, \zeta) p(\eta, \zeta) d\eta}{\int p(y, z|\eta, \zeta) p(\eta, \zeta) d\eta d\zeta},$$

is independent of $y$. We denote the common value by $p_1(\zeta|z)$. Note that the value of $p(\zeta|y, z)$ and the validity of the assumption itself depend on the prior.

(2) The marginalized sampling distribution,

$$p(z|\eta, \zeta) = \int p(y, z|\eta, \zeta) dy$$

is independent of $\eta$. We denote the common value by $p_2(z|\zeta)$.

(3) For each value of $\zeta$, the prior is improper in $\eta$: $\int p(\eta, \zeta) d\eta = \infty$.

Assumptions 1 and 2 enable $B_1$ and $B_2$, respectively, to invoke intuitive arguments to determine $p(\zeta|z)$, even though the formal calculations would lead to infinities. Assumption 3 is satisfied by all of the examples in DSZ73, and reflects the fact that we are really interested in impropriety in $\eta$.

We focus on only one aspect of the analysis in DSZ73, because we believe that aspect to be the source of all of the difficulties. The aspect in question is $B_1$’s elimination of $y$, which occurs after he has already marginalized over $\eta$. $B_1$ assumes that since $p(\zeta|y, z)$ is independent of $y$, then $p(\zeta|z)$ must be equal to the $y$-independent value of this function.
The justification that DSZ73 give for this assumption is intuitive, and has been formalized as the “reduction principle,” which is stated as follows in Dawid, Stone, and Zidek (1996): “Suppose that a general method of inference, applied to data \((y, z)\), leads to an answer that in fact depends on \(z\) alone. Then we should obtain the same answer if we apply the method to \(z\) alone.” The principle enables one to determine the answer to the problem with data \(z\) from the answer to the problem with data \((y, z)\), provided that the latter answer depends only on \(z\). We have no objection to this principle as stated. We wish to emphasize, however, that in order to apply the principle (or invoke the intuition behind the principle), we must first have the “answer” to a problem of inference, given data \((y, z)\).

The problem with \(B_1\)’s argument is that \(p(\zeta|y, z)\) has not been shown to be the “answer” to a problem of inference, so the reduction principle is inapplicable. We show below that in the context of the marginalization paradox, any sampling distribution \(p(y, z|\zeta)\) associated with \(p(\zeta|y, z)\) is necessarily improper, so that it has no inherent probabilistic meaning. There is no reason to assume that the associated formal posterior will have any probabilistic meaning, even if that posterior is proper. In the absence of such a meaning, \(p(\zeta|y, z)\) is not the answer to a problem of inference, \(B_1\) is unable to use the reduction principle to complete his argument, and the inconsistency vanishes.

We are not claiming that it is impossible to provide a meaning for an improper distribution. Indeed, such an assumption would preclude the use of improper priors and prejudge the whole issue. We are merely observing that in order to use the reduction principle, a probabilistic meaning must be provided for \(p(y, z|\zeta)\), and this has not been done. Even if a meaning is provided, any manipulations of the distribution must be justified in terms of that meaning, and there is no guarantee that the resulting procedures will be the formal analogs of valid procedures for proper distributions.

We now establish the impropriety of the sampling distribution.

**Proposition:** Let \(p(\eta, \zeta)\) be given, and let \(p(\eta, \zeta) = p(\eta|\zeta) p(\zeta)\) be any factorization of \(p(\eta, \zeta)\) such that \(0 < p(\zeta) < \infty\). Under the above assumptions we have, for each \(\zeta\),

\[
\int p(y, z|\zeta) \, dy = \infty. \tag{2}
\]

**Proof:**

\[
\int p(y, z|\zeta) \, dy = \frac{1}{p(\zeta)} \int p(y, z|\eta, \zeta) p(\eta, \zeta) \, d\eta \, dy = \frac{p(z|\zeta)}{p(\zeta)} \int p(\eta, \zeta) \, d\eta = \infty.
\]

The interchange in the order of integration is justified by Tonelli’s theorem.

An immediate corollary is that \(\int p(y, z|\zeta) \, dy \, dz = \infty\). The factorization of \(p(\eta, \zeta)\) is nonunique, and this implies a nonuniqueness in the definition of \(p(y, z|\zeta)\). The proposition shows, however, that impropriety of the conditional distribution is independent of the choice of factorization. Note also that although we are evaluating \(B_1\)’s argument, the proof depends on assumption (2), which was made for \(B_2\)’s benefit.

3. **The formal argument.**

We now consider \(B_1\)’s procedure on its own terms, as a formal procedure. We find that in the case of a proper prior, \(B_1\)’s use of the reduction principle is equivalent to the cancellation of a finite factor in a ratio defining \(p(\zeta|z)\), and in the case of an improper prior, to the cancellation of an infinite factor. It is well-known that the formal cancellation of infinities will generally
lead to inconsistencies. We conclude that when viewed formally, \( B_1 \)'s procedure is highly suspect.

In general, the posteriors of \( \zeta \) given \( (y, z) \) and given \( z \) are given formally by the following expressions:

\[
p(\zeta | y, z) = \frac{p(y, z, \zeta)}{\int p(y, z, \zeta) d\zeta}, \quad \text{and} \quad (3)
\]

\[
p(\zeta | z) = \frac{\int p(y, z, \zeta) dy}{\int p(y, z, \zeta) dy d\zeta}.
\]

(4)

Under Assumption 1, \( p(\zeta | y, z) \) is independent of \( y \). Then

\[
p(y, z, \zeta) = p(y, z) p_1(\zeta | z),
\]

(5)

where \( p(y, z) = \int p(y, z, \zeta) d\zeta \). Substituting Eq. (5) into Eq. (4), we obtain

\[
p(\zeta | z) = \frac{\int p(y, z) p_1(\zeta | z) dy}{\int p(y, z) p_1(\zeta | z) dy d\zeta}.
\]

(6)

When \( \int p(y, z) dy \) is finite, then \( p(\zeta | z) = p_1(\zeta | z) \).

If we also make Assumptions 2 and 3, the proposition implies that \( \int p(y, z) dy = \infty \). The assumption that \( p(\zeta | z) = p_1(\zeta | z) \) is now equivalent, as claimed, to the assumption that it is permissible to cancel infinite factors of \( \int p(y, z) dy \) from the ratio defining \( p(\zeta | z) \).

4. Discussion.

We have observed that the inconsistencies uncovered in DSZ73 depend on formal manipulation on the part of \( B_1 \). We have shown, in Sections 2 and 3, respectively, that \( B_1 \)'s procedure has not been justified intuitively, and is suspect mathematically. We therefore see no reason to accept \( B_1 \)'s reasoning, or to regard the validity of this reasoning as necessary or desirable in any extension of Bayesian inference to improper priors. Once \( B_1 \)'s reasoning is rejected, the marginalization paradox disappears.

The core of our argument is the observation that \( B_1 \)'s argument is formal because the sampling distribution \( p(y, z | \zeta) \) is improper. To the best of our knowledge, this observation has not been made previously. The impropriety of the sampling distribution has perhaps been obscured by its nonuniqueness and by the fact that the formal posterior can be calculated from Eq. (1) without ever computing the sampling distribution explicitly.

Previous analyses of the marginalization paradox generally accepted the validity of both Bayesians’ arguments. The problem then becomes one of understanding when and why the two Bayesians will agree. This analysis was initiated in DSZ73, which is mostly dedicated to this question. It turns out that for problems amenable to group analysis, consistency may be achieved by a uniquely determined prior. The priors determined by this constraint, however, are unsatisfactory for a variety of reasons, which DSZ73 explore in detail. They conclude that an acceptable theory is elusive or unachievable.

The most persistent and insightful critic of the marginalization paradox has been the late E. T. Jaynes. Cf. Jaynes [1980a]; Dawid et al. [1980]; Jaynes [1980b]; Dawid et al. [1990]; Jaynes [2003], for his extended debate with the authors of DSZ73. We believe that at the conceptual level, Jaynes’ critique was fundamentally correct, in that he identified
the source of the inconsistencies as the formal manipulation of completed infinities. A particularly elegant statement of this view can be found in Jaynes (2003). At the technical level, Jaynes did not recognize that \( B_1 \)'s argument was invalid, so he was forced to try to determine how the two Bayesians could be reconciled. His thesis was that the disagreement between the Bayesians reflected differences in their prior information. In our opinion, this analysis was not entirely successful, and the correct approach is to reject \( B_1 \)'s reasoning.

For general background on the marginalization paradox and related issues, we refer the reader to the excellent review article of Kass and Wasserman (1996).

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