RETHINKING POLYHEDRALITY FOR LINDENSTRAUSS SPACES

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Abstract. A recent example by the authors (see [3]) shows that an old result of Zippin about the existence of an isometric copy of $c$ in a separable Lindenstrauss space is incorrect. The same example proves that some characterizations of polyhedral Lindenstrauss spaces, based on the result of Zippin, are false. The main result of the present paper provides a new characterization of polyhedrality for the preduals of $\ell_1$ and gives a correct proof for one of the older. Indeed, we prove that for a space $X$ such that $X^* = \ell_1$ the following properties are equivalent:

(1) $X$ is a polyhedral space;
(2) $X$ does not contain an isometric copy of $c$;
(3) $\sup \{x^*(x) : x^* \in \text{ext}(B_{X^*}) \setminus D(x) \} < 1$ for each $x \in S_X$, where $D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}$.

By known theory, from our result follows that a generic Lindenstrauss space is polyhedral if and only if it does not contain an isometric copy of $c$. Moreover, a correct version of the result of Zippin is derived as a corollary of the main result.

1. Introduction

Let $B_X$ ($S_X$) denote the closed unit ball (sphere) in a real Banach space $X$ and $X^*$ denote the dual of $X$. If $K$ is a closed convex subset of $X$, then by $\text{ext}(K)$ we denote the set of all extreme points of $K$. A closed convex subset $F$ of $B_X$ is called a face of $B_X$ if for every $x, y \in B_X$ and $\lambda \in (0, 1)$ such that $(1 - \lambda)x + \lambda y \in F$ we have $x, y \in F$. A face $F$ of $B_X$ is named a proper face if $F \neq B_X$. Here $c$ denotes the Banach space of all real convergent sequences. It is well known that $c^* = \ell_1$ and the duality inducing the standard $w^*$-topology in $c^*$ is given by:

$$f(x) = f(1) \lim_{i \to \infty} x(i) + \sum_{i=1}^{+\infty} f(i+1)x(i)$$

where $f = (f(1), f(2), \ldots) \in \ell_1$ and $x = (x(1), x(2), \ldots) \in c$. In the sequel by $\{e^*_n\}_{n=1}^{+\infty}$ we denote the standard basis of $\ell_1$. Let $H \subseteq X$, then we denote by $\overline{H}$ the norm closure of $H$. If $A$ is a set in $X^*$, then $A'$ denotes the set of all $w^*$-limit points of $A$:

$$A' = \{x^* \in X^* : x^* \in \overline{w^*(A \setminus \{x^*\})}\}.$$ 

For $x \in S_X$, $D(x)$ is defined as

$$D(x) = \{x^* \in S_{X^*} : x^*(x) = 1\}.$$ 

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We recall that $D(x)$ is a $w^*$-compact face for every $x \in S_X$ and consequently $\operatorname{ext}(D(x)) = D(x) \cap \operatorname{ext}(B_{X^*}) \neq \emptyset$ by the Krein-Milman theorem.

A normed space is polyhedral if the unit balls of all its finite-dimensional subspaces are polytopes. This definition was originally introduced by Klee in [11] and now it can be seen as the classical definition of polyhedrality. Nevertheless, a lot of different notions of polyhedrality are appeared in the literature (see [5] and [7]). The relationships between each of them are studied in [5, 7] from the isometric point of view whereas their isomorphic classification is established in [7]. To our aims it is important to recall two alternative notions of polyhedrality. By following the notations used in [7] we list the following properties:

(GM) $x^*(x) < 1$ whenever $x \in S_X$ and $x^* \in (\operatorname{ext}(B_{X^*}))'$ (originally introduced in [9]);

(BD) $\sup \{x^*(x) : x^* \in \operatorname{ext}(B_{X^*}) \setminus D(x)\} < 1$ for each $x \in S_X$ (originally introduced in [2]).

In [5], Theorem 1 is shown (see also Theorem 1.2 in [7]) that for a general Banach space $X$ we have

$$\text{property (GM)} \Rightarrow \text{property (BD)} \Rightarrow X \text{ is a polyhedral space.}$$

Without additional assumptions none of the implications above can be reversed. On the other hand both the papers [5] and [7] recall that, if $X$ is a Lindenstrauss space, the three properties are all equivalent (see also Proposition 6.22 in [6]). This result essentially amount to the paper [9] by Gleit and McGuigan where some structural properties of polyhedral Lindenstrauss space are studied. For the convenience of the reader we explicitly recall the theorem by Gleit and McGuigan.

**Theorem 1.1.** (Theorem 1.2 in [9]) Let $X$ be a Lindenstrauss space. Then the following properties are equivalent:

1. $X$ enjoys property (GM);
2. $X$ is a polyhedral space;
3. $X$ does not contain an isometric copy of $c$.

As an immediate consequence of the previous result and of the implications holding for a generic Banach space, we obtain that properties (1), (2) and (3) in Theorem 1.1 and property (BD) should be all equivalent whenever we consider a Lindenstrauss space. The scheme of the proof of Theorem 1.1 given in [9] is the following: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). It is important to remark that the proof of Theorem 1.1 is based on the following result of Lazar in turn.

**Theorem 1.2.** (Theorem 3 in [13]) Let $X$ be a Lindenstrauss space. Then the following properties are equivalent:

1. $X$ is a polyhedral space;
2. $X$ does not contain an isometric copy of $c$;
3. there is not infinite dimensional $w^*$-closed proper faces of $B_{X^*}$;
4. for every Banach spaces $Y \subset Z$ and every compact operator $T : Y \to X$ there exists a compact extension $\tilde{T} : Z \to X$ with $\|\tilde{T}\| = \|T\|$.

The proof of Theorem 1.2 given in [13], proceeds as follows: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1).

Recently some of the implications of the two quoted above theorems have been disproved in [3] by an example of a suitable space. Indeed, we found an hyperplane
W of c such that W is a polyhedral predual of $\ell_1$ with an infinite dimensional $w^*$-closed proper face of $B_{W^*}$ and not enjoying property (GM). Thus, implications (2) $\Rightarrow$ (3) in Theorem 1.2 and (3) $\Rightarrow$ (1) in Theorem 1.1 are false. The failure of these results are essentially due to an incorrect result about the presence of an isometric copy of c in a Lindestrauss space stated in [16] (see below the comments after the proof of Theorem 2.1) and subsequently used by Lazar to prove the implications (2) $\Rightarrow$ (3) in Theorem 1.2. As a consequence of these faults, the equivalence between the polyhedrality of $X$ and the lack of an isometric copy of c in $X$ remains unproved.

Let us recall that, in view of the known results, it is enough to solve this problem in the case of $\ell_1$-preduals. The main aim of the present paper is to provide a correct proof of this equivalence by finding a counterpart of property (GM) in Theorem 1.1 that works as intermediate step in the proof that the lack of an isometric copy of c in a Lindenstrauss space implies the polyhedrality of $X$. Indeed, under the assumption that $X$ is a predual of $\ell_1$, the main result of the paper shows that the correct property to substitute property (GM) is property (BD) mentioned above. This result completely characterizes polyhedrality of $\ell_1$-preduals by means of the structural and geometrical properties. It also provides a correct reformulation (see Corollary 2.3) of the result, stated in [16], about the existence of an isometric copy of c in a separable Lindenstrauss space. Finally, it is worth to mention that polyhedrality plays an important role in the theory of Lindestrauss spaces. Indeed, it is deeply linked with the norm preserving extension property for compact operators.

It is well known that a Lindenstrauss space is polyhedral if and only if for every Banach spaces $Y \subset Z$ and every operator $T : Y \to X$ with $\dim T(Y) \leq 2$ there exists a compact extension $\tilde{T} : Z \to X$ with $\|\tilde{T}\| = \|T\|$ (combine Theorem 7.9 in [15] and Theorem 4.7 in [10]). However, the more general extension property stated in Theorem 1.2 (see also Proposition 6.23 in [6]) now reveals to be unproven since the chain of implications between (1) and (4) has an interruption as remarked above.

2. Characterization of polyhedral $\ell_1$-preduals

This section is devoted to the main result of the paper. In order to deal with a characterization of polyhedral Lindenstrauss spaces, by known results (see, e.g. §22 in [12]), the only interesting situation to deal with is that of $\ell_1$-preduals.

**Theorem 2.1.** Let $X$ be a predual of $\ell_1$. The following properties are equivalent:

1. $X$ is a polyhedral space;
2. $X$ does not contain an isometric copy of c;
3. $\sup \{x^*(x) : x^* \in \text{ext}(B_{X^*}) \setminus D(x)\} < 1$ for each $x \in S_X$ (property (BD)).

**Proof.** The implication (1) $\Rightarrow$ (2) is straightforward to prove since it is well known that c is not a polyhedral space. The implication (3) $\Rightarrow$ (1) is proved in Theorem 1 in [15] (see also Theorem 1.2 in [12]). Therefore it remains to prove only the implication (2) $\Rightarrow$ (3).

By contradiction let us suppose that property (3) does not hold. Therefore there exist a sequence $\{x_n^*\}_{n=1}^{+\infty} \subset \text{ext}(B_{X^*}) \setminus D(x)$ and a point $x \in S_X$ such that

$$x_n^*(x) \xrightarrow{n \to +\infty} 1.$$
Without loss of generality we can consider a subsequence \( \{x_{n_k}^*\} \) of \( \{x_n^*\} \) such that \( x_{n_k}^* = e_{n_k}^* \) for every \( k \in \mathbb{N} \) and an element \( e^* \in X^* \) such that

\[
e_{n_k}^* \xrightarrow{w^*} e^*.
\]

It is easily seen that \( e^*(x) = 1 \) and \( \|e^*\| = 1 \) and hence \( e^* \in D(x) \).

Let us recall that the set \( D(x) \) is a weak*-compact face. By the Krein-Milman Theorem we have

\[
D(x) = \left\{ \sum_{i \in \Delta} \delta_i d_i^* : \sum_{i \in \Delta} \delta_i = 1; \delta_i \geq 0 \text{ for all } i \in \Delta \right\},
\]

where \( \{d_i^*\}_{i \in \Delta} = \text{ext } (D(x)) \) and \( \Delta \) is a finite or countable set of indexes. Let us introduce the set

\[
K = \{e_{n_k}^*\}_{k=1}^{+\infty} \cup \text{ext } (D(x)).
\]

We see at once that:

(i) \( K \) is a countable subset of \( \text{ext } (B_{X^*}) \);
(ii) \( K \cap (-K) = \emptyset \);
(iii) the set \( Y = \text{span}(K) \) is a weak*-closed set in \( X^* \) (see Lemma 1 in [1]).

We are now in position to apply Theorem 1.1 in [8]. By using this result we obtain that there exists a \( w^* \)-continuous contractive projection \( P_1 \) from \( X^* \) onto \( Y \).

The task is now to find a \( w^* \)-continuous contractive projection \( P_2 \) from \( Y \) onto a subspace \( V \) of \( Y \) itself, such that \( V \) is isometric to \( e^* \) (endowed with its natural \( w^* \)-topology). In order to define such a projection, it is convenient to describe the subspace \( Y \) as:

\[
Y = \left\{ y^* \in X^*: y^* = \sum_{k=1}^{+\infty} \alpha_k e_{n_k}^* + \sum_{i \in \Delta} \beta_i d_i^*; \sum_{k=1}^{+\infty} |\alpha_k| < +\infty, \sum_{i \in \Delta} |\beta_i| < +\infty \right\}.
\]

Let us define the linear map \( P_2: Y \to X^* \) by

\[
P_2(y^*) = \sum_{k=1}^{+\infty} \alpha_k e_{n_k}^* + \left( \sum_{i \in \Delta} \beta_i \right) e^*.
\]

We remark that

(iv) \( P_2 \) is a projection from \( Y \) onto its closed subspace \( V \) defined by

\[
V = \left\{ v \in X^*: v = \sum_{k=1}^{+\infty} \alpha_k e_{n_k}^* + \theta e^*, \text{where } \sum_{k=1}^{+\infty} |\alpha_k| < +\infty, \theta \in \mathbb{R} \right\}.
\]

Indeed, \( e^* \in D(x) \). Hence there exists \( \{\varepsilon_i\}_{i \in \Delta} \subset [0,1] \) such that \( \sum_{i \in \Delta} \varepsilon_i = 1 \) and \( \sum_{i \in \Delta} \varepsilon_i d_i^* = e^* \). Let us consider \( v = \sum_{k=1}^{+\infty} \alpha_k e_{n_k}^* + \theta e^* \in V \), then

\[
P_2(v) = P_2 \left( \sum_{k=1}^{+\infty} \alpha_k e_{n_k}^* + \theta \left( \sum_{i \in \Delta} \varepsilon_i d_i^* \right) \right) = \sum_{k=1}^{+\infty} \alpha_k e_{n_k}^* + \theta \left( \sum_{i \in \Delta} \varepsilon_i \right) e^* = v.
\]

(v) \( V \) is a \( w^* \)-closed subspace of \( X^* \) (to prove this fact it is sufficient to remember that \( w^* = \lim_{k \to +\infty} e_{n_k}^* = e^* \));
(vi) \( P_2 \) is a \( w^* \)-continuous map;
(vii) \( \|P_2\| = 1 \) because \( \Delta \cap \{n_k\}_{k=1}^{+\infty} = \emptyset \).
By combining the two projections $P_1$ and $P_2$ introduced above we now define the linear map $P : X^* \rightarrow V$ such that $P(x^*) = P_2(P_1(x^*))$ for every $x^* \in X^*$. The map $P$ enjoys the following properties:

(viii) $P$ is a $w^*$-continuous and contractive projection from $X^*$ onto $V$; the subspace $V$ is isometric to $c^*$ (endowed with its standard $w^*$-topology) by means of the $w^*$-continuous isometry $T : V \rightarrow c^*$ defined by

$$T(e^*_n) = f^*_1$$

and

$$T(e^*_{nk}) = f^*_k + 1$$

for all $k \geq 1$, where $\{f^*_n\}_{n=1}^{+\infty}$ is the standard basis of $c^* = \ell_1$.

Let us denote by $\perp V$ the annihilator of $V \subseteq X^*$ in $X$; since $P$ and $T$ are $w^*$-continuous, there exist two linear bounded operators

$$S : X/\perp V \rightarrow X$$

and

$$R : c \rightarrow X/\perp V$$

such that $S^* = P$, $R^* = T$ and $R$ is an isometry. The linear map $S \circ R : c \rightarrow X$ is an isometric injection. This fact concludes the proof.

As a consequence of our results, Theorem 2.3 in [14], Theorem 5 in [12], p.226, and Lemma 1 in [12], p.232, we obtain a complete characterization of Lindenstrauss spaces that are polyhedral.

**Corollary 2.2.** Let $X$ be a Lindenstrauss space. The following properties are equivalent:

1. $X$ does not contain an isometric copy of $c$;
2. $X$ is a polyhedral space.

In *Concluding remarks* of [16], Zippin stated that a separable Lindenstrauss space $X$ contains a contractively complemented isometric copy of $c$ whenever the unit ball of $X$ has at least one extreme point. In [3] an example, built upon a careful study of the hyperplanes of $c$, disproves this result. By recalling Proposition 3.1 in [4] and considering the previous results, we obtain something more about the presence of an isometric copy of $c$ in a separable Lindenstrauss space. In particular, we show the existence of a contractively complemented copy of $c$. The next result can be seen as a correct version of the Zippin’s result.

**Corollary 2.3.** Let $X$ be a separable Lindenstrauss space. The following properties are equivalent:

1. $X$ contains a contractively complemented isometric copy of $c$;
2. there exists $x \in S_X$ such that $\sup \{x^*(x) : x^* \in \text{ext} (B_X^*) \setminus D(x)\} = 1$ .

It is worth to underline that the assumption, assumed in [16], of the mere existence of an extreme point of $B_X$ is far from to be sufficient in order to ensure the existence of an isometric copy of $c$ in $X$, as shown by the example considered in [3]. Indeed, condition (2) in Corollary 2.3 gives a much more detailed description of the geometry of $B_X$, and the point $x \in S_X$ is not necessarily an extreme point (for instance, let us consider the space $X$ given by $c$ itself).

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