An Optimality Proof for the PairDiff operator for Representing Relations between Words

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Abstract
Representing the semantic relations that exist between two given words (or entities) is an important first step in a wide-range of NLP applications such as analogical reasoning, knowledge base completion and relational information retrieval. A simple, yet surprisingly accurate method for representing a relation between two words is to compute the vector offset (PairDiff) between the corresponding word embeddings. Despite its empirical success, it remains unclear whether PairDiff is the best operator for obtaining a relational representation from word embeddings. In this paper, we conduct a theoretical analysis of the PairDiff operator. In particular, we show that for word embeddings where cross-dimensional correlations are zero, PairDiff is the only bilinear operator that can minimise the \( \ell_2 \) loss between analogous word-pairs. We experimentally show that for word embedding created using a broad range of methods, the cross-dimensional correlations in word embeddings are approximately zero, demonstrating the general applicability of our theoretical result. Moreover, we empirically verify the implications of the proven theoretical result in a series of experiments where we repeatedly discover PairDiff as the best bilinear operator for representing semantic relations between words in several benchmark datasets.

1 Introduction
Different types of semantic relations exist between words such as HYPERNYMY between ostrich and bird, or ANTONYMY between hot and cold. If we consider entities\(^1\), we can observe even a richer diversity of relations such as FOUNDER-OF between Bill Gates and Microsoft, or CAPITAL-OF between Tokyo and Japan. Identifying the relations between words and entities is important for various Natural Language Processing (NLP) tasks such as automatic knowledge base completion [Socher et al., 2013], analogical reasoning [Turney and Littman, 2005; Bollegala et al., 2009] and relational information retrieval [Duc et al., 2010]. For example, to solve a word analogy problem of the form “a is to b as c is to ?”, the relationship between the two words in the pair \((a, b)\) must be correctly identified in order to find candidates \(d\) that have similar relations with \(c\). For example, given the query “Bill Gates is to Microsoft as Steve Jobs is to ?”, a relational search engine must retrieve Apple Inc. because the FOUNDER-OF relation exists between the first and the second entity pairs.

Two main approaches for creating relation embeddings can be identified in the literature. In the first approach, from given corpora or knowledge bases, word and relation embeddings are jointly learnt such that some objective is optimised [Guo et al., 2016; Yang et al., 2015; Nickel et al., 2016; Bordes et al., 2013; Rocktäschel et al., 2016; Miervini et al., 2017; Trouillon et al., 2016]. In this approach, word and relation embeddings are considered to be independent parameters that must be learnt by the embedding method. For example, TransE [Bordes et al., 2013] learns the word and relation embeddings such that we can accurately predict relations (links) in a given knowledge base using the learnt word and relation embeddings. Because relations are learnt independently from the words, we refer to methods that are based on this approach as independent relational embedding methods.

A second approach for creating relation embeddings is to apply some operator on two word embeddings to compose the embedding for the relation that exists between those two words, if any. In contrast to the first approach, we do not have to learn relation embeddings and hence this can be considered as an unsupervised setting when the compositional operator is predefined. A popular operator for composing a relation embedding from two word embeddings is \textit{PairDiff}, which is the vector difference (offset) of the word embeddings. Specifically, given two words \(a\) and \(b\) represented by their word embeddings respectively \(a\) and \(b\), the relation between \(a\) and \(b\) is given by \(a - b\) under the PairDiff operator. Mikolov et al. [2013b] showed that PairDiff can accurately solve analogy equations such as \textit{king} − \textit{man} + \textit{woman} = \textit{queen}, where we have used the top arrows to denote the embeddings of the corresponding words. Bollegala et al. [2015a] showed that PairDiff can be used as a proxy for learning better word embeddings and Vylomova et al. [2016] conducted an extensive empirical comparison of PairDiff using a dataset containing 16 different relation types. Besides PairDiff, vector addition, elementwise multiplication [Hakami and Bollegala, 2017], concatenation [Yin and Schütze, 2016], circular correlation and con-

\(^1\)We interchangeably use the terms word and entity to represent both unigrams as well as a multi-word expressions including named entities.
volution [Nickel et al., 2016] have been used in prior work for representing the relations between words. Because the relation embedding is composed using word embeddings instead of learning as a separate parameter, we refer to methods that are based on this approach as compositional relational embedding methods. Note that in this approach it is implicitly assumed that there exist only a single relation between two words.

In this paper, we focus on the operators that are used in compositional relation embedding methods. In particular, we consider the question:

“What is the best operator for obtaining an accurate representation for the relation between two given words using only the embeddings of the two words?”

If we assume that the words and relations are represented by vectors embedded in some common space, then the operator we are seeking must be able to produce a vector representing the relation between two words given their word embeddings as the input. Although there has been different proposals for computing relational embeddings from word embeddings, it remains unclear what is the best operator for this task. The space of operators that can be used to compose relational embeddings is open and vast. A space of particular interest from a computational point-of-view is the second-order linear operators that can be parametrised using matrices. Specifically, we consider operators that consider pairwise interactions between two word embeddings (second-order terms) and contributions from individual word embeddings towards their relational embedding (first-order terms). The appropriateness of an operator under these settings can be evaluated using some loss function over word-pairs for which we know a particular relation type to exist. As a concrete example of such a loss function, we use $\ell_2$ loss defined over a set of positive word-pairs (e.g. word-pairs that represent a particular semantic relation).

If we assume that the cross-dimensional correlations in word embeddings to be negligible, we prove in §3 that the optimal bilinear operator for minimising an upper bound of the $\ell_2$ relation prediction loss is PairDiff. In §4.1, we empirically show that the cross-dimensional correlations are small for a broad range of word embeddings such as the continuous bag-of-words model (CBOW) [Mikolov et al., 2013a], skip-gram with negative sampling (SG) [Mikolov et al., 2013a], Global vectors (GloVe) [Pennington et al., 2014], word embeddings created using Latent Semantic Analysis (LSA) [Deerwester et al., 1990], Sparse Coding (HSC) [Faruqui et al., 2015; Yogatama et al., 2015], and Latent Dirichlet Allocation (LDA) [Blei et al., 2003a]. This empirical evidence implies that our theoretical analysis is applicable to a wide range of word embeddings. Moreover, our experimental results show that a bilinear operator reaches its optimal performance in two different word-analogy benchmark datasets, when it satisfies the requirements of the PairDiff operator. We hope that our theoretical analysis will expand the understanding of relation embedding methods, and inspire future research on accurate relation embedding methods using word embeddings as the input.

## 2 Related Work

As already mentioned in §1, methods for representing a relation between two words can be broadly categorised into two groups depending on whether the relation embeddings are learnt independently of the word embeddings, or they are composed from the word embeddings, in which case the relation embeddings fully depend on the input word embeddings. Next, we briefly overview the different methods that fall under each category. For a detailed survey of relation embedding methods see [Nickel et al., 2015].

Given a knowledge base where an entity $h$ is linked to an entity $t$ by a relation $r$, the TransE model [Bordes et al., 2013] scores the tuple $(h, t, r)$ by the $\ell_1$ or $\ell_2$ norm of the vector $(h + r - t)$. Nickel et al. [2011] proposed RESCAL, which uses $h^\top M_r t$ as the scoring function, where $M_r$ is a matrix embedding of the relation $r$. Similar to RESCAL, Neural Tensor Network [Socher et al., 2013] also models a relation by a matrix. However, compared to vector embeddings of relations, matrix embeddings increase the number of parameters to be estimated, resulting in an increase in computational time and space. To overcome these limitations, DistMult [Yang et al., 2015] models relations by vectors and use elementwise multilinear dot product $r \odot h \odot t$. Unfortunately, DistMult cannot capture directionality of a relation. Complex Embeddings [Trouillon et al., 2016] overcome this limitation of DistMult by using complex embeddings and defining the score to be the real part of $r \odot h \odot t$, where $t$ denotes the complex conjugate of $t$.

The observation made by Mikolov et al. [2013b] that the relation between two words can be represented by the difference between their word embeddings sparked a renewed interest in methods that compose relation embeddings using word embeddings. Word analogy datasets such as Google dataset [Mikolov et al., 2013b], SemEval 2012 Task2 dataset [Jurgens et al., 2012], BATS [Drozd et al., 2016] etc. have established as benchmarks for evaluating word embedding learning methods. Different methods have been proposed to measure the similarity between the relations that exist between two given word pairs such as CosMult, CosAdd and PairDiff [Levy and Goldberg, 2014; Bollegala et al., 2015a]. Vylomova et al. [2016] studied as to what extent the vectors generated using simple PairDiff encode different relation types. Under supervised classification settings, they conclude that PairDiff can cover a wide range of semantic relation types. Holographic embeddings proposed by Nickel et al. [2016] use circular convolution to mix the embeddings of two words to create an embedding for the relation that exist between those words. It can be showed that circular correlation is indeed an elementwise product in the Fourier space and is mathematically equivalent to complex embeddings [Hayashi and Shinbo, 2017].

Although PairDiff operator has been widely used in prior work for computing relation embeddings from word embeddings, to the best of our knowledge, no theoretical analysis has been conducted so far explaining why and under what conditions PairDiff is optimal, which is the focus of this paper.
3 Bilinear Relation Representations

Let us consider the problem of representing the semantic relation $r(h, t)$ between two given words $h$ and $t$. We assume that $h$ and $t$ are already represented in some $d$-dimensional space respectively by their word embeddings $h, t \in \mathbb{R}^d$. The relation between two words can be represented using different linear algebraic structures. Two popular alternatives are vectors [Nickel et al., 2016; Bordes et al., 2013; Minervini et al., 2017; Trouillon et al., 2016] and matrices [Socher et al., 2013; Bollegala et al., 2015b]. Vector representations are preferred over matrix representations because of the smaller number of parameters to be learnt [Nickel et al., 2015]. Let us assume that the relation $r$ is represented by a vector $r \in \mathbb{R}^n$ in some $n$-dimensional space. Therefore, we can write $r(h, t)$ as a function that takes two vectors (corresponding to the embeddings of the two words) as the input and returns a single vector (representing the relation between the two words) as given in (1).

$$r: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^n$$ (1)

Having both words and relations represented in the same $n = d$ dimensional space is useful for performing linear algebraic operations using those representations in that space. For example, in TransE [Bordes et al., 2013], the strength of a relation $r$ that exists between two words $h$ and $t$ is computed as the $\ell_2$ norm of the vector $(h + r - t)$ using the word and relation embeddings. Such direct comparisons between word and relation embeddings would not be possible if words and relations were not embedded in the same vector space. Nevertheless, the theoretical result we derive in this paper is still valid in the $n \neq d$ case.

Different functions can be used as $r(h, t)$ that satisfy the domain and range requirements specified by (1). If we limit ourselves to bilinear functions, the most general functional form is given by (2).

$$r(h, t) = h^T A t + Ph + Qt + b$$ (2)

Here, $A \in \mathbb{R}^{d \times d \times n}$ is a 3-way tensor in which each slice is a $d \times d$ real matrix. Let us denote the $k$-th slice of $A$ by $A^{(k)}$ and its $(i, j)$ element by $A^{(k)}_{ij}$. The first term in (2) corresponds to the pairwise interactions between $h$ and $t$. $P, Q \in \mathbb{R}^{d \times n}$ are the projection matrices involving first-order contributions respectively of $h$ and $t$ towards $r$. $b \in \mathbb{R}^n$ is a bias vector representing any contributions to $r$ not involving $h$ or $t$.

Let us consider the problem of learning the best bilinear functional form according to (2) from a given dataset of analogous word-pairs $D_+ = \{(h, t), (h', t')\}$. Specifically, we would like to learn the parameters $A, P, Q$ and $b$ such that some distance (loss) between analogous word-pairs is minimised. As a concrete example of a loss function, let us consider the squared $\ell_2$ loss given by (3).

$$J((h, t), (h', t'): A, P, Q, b) = \|r(h, t) - r(h', t')\|^2_2$$ (3)

If we were provided only analogous word-pairs (i.e. positive examples), then this task could be trivially achieved by setting all parameters to zero. However, such a trivial solution would not generalise to unseen test data. Therefore, in addition to $D_+$ we would require a set of non-analogous word-pairs $D_-$ as negative examples. Such negative examples are often generated in prior work by randomly corrupting positive relational tuples [Nickel et al., 2016; Bordes et al., 2013; Trouillon et al., 2016] or by training an adversarial generator [Minervini et al., 2017].

The total loss over both positive and negative training data can be compactly written as follows:

$$J = \sum_{((h, t), (h', t'): D_+ \cup D_-)} \theta_i \|r(h, t) - r(h', t')\|^2_2$$ (4)

Here, $\theta_i = 1$ if $((h, t), (h', t')) \in D_+$ and $-1$ otherwise.

Assuming that training word-pairs are randomly sampled from a distribution $\phi$, we can compute the expected loss, $\mathbb{E}_\phi[J]$, as follows:

$$\mathbb{E}_\phi[J] = \mathbb{E}_\phi \left[ \theta_i \|r(h, t) - r(h', t')\|^2_2 \right]$$ (5)

For relation representations given by (2), the following theorem states that PairDiff is the only possible operator that would minimise an upper bound of the expected squared $\ell_2$ loss over a set of word-pairs.

**Theorem 1.** Consider the bilinear relational embeddings defined by (2) computed using uncorrelated word embeddings. Then the upper bound of the the expected loss given by (5) is minimised when $A = 0, P = p, Q = q$ and $b = 0$, for some $p, q \in \mathbb{R}$.

Here, $0$ is a tensor with all elements set to zero. One might think that the uncorrelation of word embedding dimensions to be a strong assumption, but we later show its validity empirically in §4.1.

**Proof.** We will first note that for zero word embeddings we would not expect a relation between them. This would require that $r(h, t) = 0$ when $h = t = 0$. For (2) to satisfy this requirement we must have $b = 0$.

Next, let us consider the term $Ph$ in (2). This term produces linear combinations of the dimensions of $h$. Recall that word embeddings are typically randomly initialised and there is no prior ordering among the different dimensions of a word embedding. For example, a particular dimension $i$ might be closely related to a relation $r$, whereas a different dimension $j(\neq i)$ might be closely related to a different relation $r'(\neq r)$. Therefore, there is no justification to consider one dimension of a word embedding to be superior to another for the purpose of relation representation. This permutative symmetry in the dimensions of the word embeddings require that we treat all dimensions of $h$ in the same manner when considering their contributions towards a relation embedding. To achieve this $P$ must be a diagonal matrix with all diagonal elements set to the same value $p \in \mathbb{R}$. This multiplication is equivalent to multiplying $h$ by a scalar $p$. A similar argument could be used to show that $Q$ degenerates to a scalar $q \in \mathbb{R}$.

Before we further analyse the relationship between $p$ and $q$, let us consider the pairwise term in (2). Because $i$ and $j(\neq i)$ dimensions of word embeddings are uncorrelated by the
assumption (i.e. \( \text{corr}(u_i, u_j) = 0 \)), then from the definition of correlation we have,
\[
\text{corr}(u_i, u_j) = \mathbb{E}[u_i u_j] - \mathbb{E}[u_i] \mathbb{E}[u_j] = 0 \tag{6}
\]
\[
\mathbb{E}[u_i u_j] = \mathbb{E}[u_i] \mathbb{E}[u_j]. \tag{7}
\]
Without loss of generality, we can assume that all dimensions are scaled such that they have zero means. In other words, \( \mathbb{E}[u_i] = 0 \), \( \forall i \). From (7) it follows that
\[
\mathbb{E}[u_i u_j] = 0 \tag{8}
\]
for \( i \neq j \) dimensions.

Let us consider the implications of this uncorrelation on the expected loss given by (5). Because \( \theta_i \in \{-1, 1\} \) and the squared loss terms for both positive and negative examples are nonnegative, an upper bound on the expected loss can be obtained as follows:
\[
\mathbb{E}_\phi[J] \leq \mathbb{E}_\phi \left[ ||r(h, t) - r(h', t')||_2^2 \right] \tag{9}
\]

We will next analyse the contribution of \( A \) towards this upper bound. For this purpose, let us write the \( k \)-th dimension of \( r(h, t) \) using \( A^{(k)} \), \( p \) and \( q \) as follows:
\[
\sum_{i,j} \left( A^{(k)}_{ij} h_i t_j + ph_k + qt_k \right) \tag{10}
\]
Plugging (10) in (3) and computing the loss over all training instances we get,
\[
J = \sum_k \left( \sum_{i,j} \left( A^{(k)}_{ij} (h_i t_j - h'_i t'_j) \right) + ph_k + qt_k \right)^2 \tag{11}
\]
Terms that involve only elements in \( A^{(k)} \) take the form:
\[
\mathbb{E}_\phi \left[ A^{(k)}_{ij} A^{(k)}_{lm} (h_i t_j - h'_i t'_j) (h_l t_m - h'_l t'_m) \right] = A^{(k)}_{ij} A^{(k)}_{lm} \left( \mathbb{E}_\phi[h_i t_j h_l t_m] - \mathbb{E}_\phi[h_i t_j h'_l t'_m] - \mathbb{E}[h'_i t'_j h_l t_m] + \mathbb{E}[h'_i t'_j h'_l t'_m] \right) \tag{12}
\]
In cases where \( i \neq j \) and \( l \neq m \), each of the four expectations in (12) contains the product of different dimensionalities, which is zero from (8). For \( i = j = l = m = \) case we have,
\[
A^{(k)}_{ij} \left( \mathbb{E}_\phi[h_i^2 t^2_l] - \mathbb{E}_\phi[h_i t_i h'_l t'_i] - \mathbb{E}_\phi[h'_i t'_i h_l t_m] + \mathbb{E}_\phi[h'_i t'_i h'_l t'_m] \right) \tag{13}
\]
Note that we are taking the expectation over all word-pairs. Therefore, \( h, h' \) and \( t, t' \) are interchangeable for the purpose of computing expectations as long as we are considering the same dimensions. From this fact it follows that (13) is also zero. Therefore, none of the terms arising purely from \( A \) will remain in the upper bound of the expected loss.

Next, lets consider the \( A^{(k)}_{ij} p \) terms in the expansion of (11) given by,
\[
A^{(k)}_{ij} p(h_i t_j - h'_i t'_j) (h_k - h'_k). \tag{14}
\]
Taking the expectation of (14) w.r.t. \( \phi \) we get,
\[
A^{(k)}_{ij} p \left( \mathbb{E}_\phi[h_i t_j h_k] - \mathbb{E}_\phi[h_i t_j h'_k] - \mathbb{E}_\phi[h'_i t'_j h_k] + \mathbb{E}_\phi[h'_i t'_j h'_k] \right). \tag{15}
\]
From (8) it follows that all the expectations in (15) are zero. A similar argument can be used to show that terms that involve \( A^{(k)}_{ij} q \) disappear from (11). Therefore, \( A \) does not play any part in the upper bound of the expected loss, and can be set to an arbitrary value. If we attempt to minimise the upper bound of the expected loss under some regularisation on \( A \) such as Frobenius norm regularisation, then this can be achieved by sending \( A \) to zero tensor.

With \( A = 0 \) and \( b = 0 \), (2) simplifies to:
\[
r(h, t) = ph + qt \tag{16}
\]

Two important cases of relations can be considered: symmetric and anti-symmetric. For symmetric relations we have,
\[
r(h, t) = r(t, h)
\]
\[
ph + qt = pt + qh
\]
\[
(p - q)(h - t) = 0. \tag{17}
\]
For this requirement to be satisfied for \( h \neq t \) general case, we must have \( p = q \).

For anti-symmetric relations we have,
\[
r(h, t) = -r(t, h)
\]
\[
ph + qt = -pt - qh
\]
\[
(p + q)(h + t) = 0. \tag{18}
\]
In the general case where \( h \neq -t \), for this to be satisfied we must have \( p = -q \). Most of the relations we encounter in practice can be seen as an intermediate case between these two extremes of symmetry and anti-symmetry.

The relation representation given in (16) can be seen as the weighted average of \( h \) and \( t \), weighted by respectively \( p \) and \( q \). If we further assume that \( p \) and \( q \) are not independent variables and \( p + q = 1 \), then we have
\[
r(h, t) = ph + (1 - p)t
\]
\[
= p(h - t) + t. \tag{19}
\]
(19) shows that the relations between \( h \) and \( t \) can be represented by the difference between the two corresponding word embeddings, scaled by a factor \( p \) and shifting the origin to \( t \). Therefore, the popular PairDiff operator for representing relations can be discovered as a special case of Theorem 1.

4 Experimental Results

4.1 Cross-dimensional Correlations

A key assumption in our theoretical analysis is the uncorrelations between different dimensions in word embeddings. To empirically validate this assumption we compute cross-correlations for word embeddings created from several different word embedding learning methods.

We create SG, CBOW and GloVe embeddings from the ukWaC corpus.\(^2\) We use a context window of 5 tokens and

\(^2\)http://wacky.sslmit.unibo.it/doku.php?id=corpora
select words that occur at least 6 times in the corpus. We use the publicly available implementations for those methods by the original authors and set the parameters to the recommended values in [Levy et al., 2015]. As a representative of counting-based word embeddings, we create a word co-occurrence matrix weighted by the positive pointwise mutual information and apply singular value decomposition to obtain low-dimensional embeddings, which we refer to as the Latent Semantic Analysis (LSA) embeddings.

We use Latent Dirichlet Allocation (LDA) [Blei et al., 2003b] to create a topic model, and represent each word by its distribution over the set of topics. Ideally, each topic will capture some semantic category and the topic distribution provides a semantic representation for a word. We use gensim to extract topics from a 2017 January dump of English Wikipedia. In contrast to above-mentioned word embeddings, which are dense and flat structured, we used Hierarchical Sparse Codind\textsuperscript{4} (HSC) [Yogatama et al., 2015] to produce sparse and hierarchical word embeddings.

Given a word embedding matrix $\mathbf{W} \in \mathbb{R}^{m \times d}$, where each row correspond to the $d$-dimensional embedding of a word in a vocabulary containing $m$ words, we compute a correlation matrix $\mathbf{C} \in \mathbb{R}^{d \times d}$, where the $(i,j)$ element, $C_{ij}$, denotes the Pearson correlation coefficient between the $i$-th and $j$-th dimensions in the word embeddings for the $m$ words. By construction $C_{ii} = 1$ and the histograms of the cross-dimensional correlations ($i \neq j$) are shown in Figure 1 for 50 dimensional word embeddings obtained from the six methods described above. The mean of the absolute pairwise correlations for each embedding type and the standard deviation (sd) are indicated in the figure.

From Figure 1, irrespective of the word embedding learning method used, we see that cross-dimensional correlations are distributed in a narrow range with an almost zero mean. This result empirically validates the uncorrelation assumption we used in our theoretical analysis. Moreover, this result indicates that Theorem 1 can be applied to a wide-range of existing word embeddings.

### 4.2 Learning Relation Representations

Our theoretical analysis in §3 implies that the performance of the bilinear relational embedding given by (20) is upper bounded by PairDiff and it reaches this bound when $\mathbf{A} = \mathbf{0}$ and $p = -q = c$ for some real-valued constant $c$.

$$r(h, t) = \mathbf{h}^\top \mathbf{A} \mathbf{t} + ph + qt \tag{20}$$

To verify this claim empirically we conduct the following experiment.

We use the BATS dataset [Gladkova et al., 2016] that contains 40 semantic and syntactic relation types\textsuperscript{5}, and generate positive examples by pairing word-pairs that have the same relation types. Approximately each relation type has 1,225 word-pairs, which enables us to generate a total of 48k positive training instances (analogous word-pairs) of the form $((h, t), (h', t'))$. Because word-pairs representing different relation types can be considered to be non-analogous, using word-pairs representing multiple relation types provides us a natural method for obtaining negative training instances. Let us refer to this collection of word-pairs as the training dataset.

Next, we train $d = 50$ dimensional word embeddings using CBOW, SG, GloVe, LSA, LDA, and HSC methods. We then compute relational embeddings for the word pairs in our training dataset and minimise the $\ell_2$ loss given by (4). We add an $\ell_2$ regulariser on $\mathbf{A}$ to avoid overfitting due to the large number of parameters ($50^3$) in the tensor, whereas no regularisation is imposed on $p$ and $q$.\textsuperscript{6} We use AdaGrad [Duchi et al., 2011] to find $\mathbf{A}, p$ and $q$ that minimise the overall objective function. All parameters are initialised uniformly at random to the range $[-1, +1]$ and the initial learning rate is set to 0.01.

Figure 2 shows the Frobenius norm of the tensor $\mathbf{A}$ (on the left vertical axis) and the values of $p$ and $q$ (on the right vertical axis) for the six word embeddings. In all cases, we see that as the training progresses, $\mathbf{A}$ goes to zero, whereas $p = -q = c$ is reached, as predicted by our theoretical analysis.

So far we have seen that the bilinear relational representation given by (20) does indeed converge to the form predicted by our theoretical analysis for different types of word embeddings. However, it remains unclear whether the learnt relational embeddings from the training instances generated from the BATS dataset accurately generalises to other benchmark datasets for analogy detection. To measure the

\textsuperscript{3}http://radimrehurek.com/gensim/wiki.html
\textsuperscript{4}http://www.cs.cmu.edu/~ark/dyogatam/wordvecs/
\textsuperscript{5}http://vsm.blackbird.pw/bats
\textsuperscript{6}If $p + q$ converges to zero as implied by the theory, then regularising these two terms would result in each term separately converging to zero, which is a trivial solution.
generalisation capability of the learnt relational embeddings, we measure their performance on two other benchmark datasets: SAT and SemEval 2012-Task\textsuperscript{7}. In SAT analogical questions, given a stem word-pair \((a, b)\) with five candidate word-pairs \((c, d)\), the task is to select the word-pair that is relationally similar to the stem word-pair. The relational similarity between two word-pairs \((a, b)\) and \((c, d)\) is computed by the cosine similarity between the corresponding relational embeddings \(r(a, b)\) and \(r(c, d)\). The candidate word-pair that has the highest relational similarity with the stem word-pair is selected as the correct answer to a word analogy question. The reported accuracy is the ratio of the correctly answered questions to the total number of questions. On the other hand, SemEval dataset has 79 semantic relations, with each relation having ca. 41 word-pairs and four prototypical examples. The task is to assign a score for each word pair which is the average of the relational similarities between the given word-pair and prototypical word-pairs in a relation. Maximum difference scaling (MaxDiff) is used as the evaluation measure in this task.

Figure 3 shows the performance of the relational embeddings composed from 50-dimensional CBOW embeddings.\textsuperscript{8} The level of performance reported by PairDiff on SAT and SemEval datasets are respective 35.16\% and 41.98\%, and are shown by horizontal dashed lines. From Figure 3, we see that the training loss gradually decreases with the number of training epochs and the performance of the relational embeddings on SAT and SemEval datasets reach that of the PairDiff operator. This result indicates that the relational embeddings learnt not only converge to PairDiff operator on training data but also generalise to unseen relation types in SAT and SemEval test datasets.

\section{Conclusion}

We conducted a theoretical analysis on bilinear relational embeddings that are composed from pre-trained word embeddings. In particular, our analysis showed that the upper bound of the expected \(\ell_2\) loss over a set of analogous and non-analogous word-pairs is minimised by the PairDiff operator when the cross-dimensional correlations in the input word embeddings can be ignored. We then provided empirical evidence to the effect that indeed for a wide-range of word embeddings, their cross-dimensional correlations are narrowly distributed around a zero mean. Moreover, we empirically showed that we can discover the PairDiff operator from a training dataset containing a diverse set of semantic relations as predicted by our theoretical analysis.

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\textsuperscript{7}https://sites.google.com/site/semeval2012task2/

\textsuperscript{8}Similar trends were observed for all six word embedding types but not shown here due to space limitations.
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