TWO PROBLEMS IN THE THEORY OF DIFFERENTIAL EQUATIONS

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ABSTRACT. 1) The differential equation considered in terms of exterior differential forms, as É. Cartan did, singles out a differential ideal in the supercommutative superalgebra of differential forms, hence an affine supervariety. In view of this observation, it is evident that every differential equation has a supersymmetry (perhaps trivial). Superymmetries of which (systems of) classical differential equations are missed yet?

2) Why criteria of formal integrability of differential equations are never used in practice?

To the memory of Ludvig Dmitrievich Faddeev

1. Introduction

While the Large Hadron Collider is busy testing (among other things) the existence of supersymmetry in the high energy physics (more correctly, supersymmetry’s applicability to the models), the existence of supersymmetry in the solid body theory (more correctly, its usefulness for describing the models) is proved beyond any doubt, see the excitatory book [Ef], and later works by K. Efetov.

Nonholonomic nature of our space-time, suggested in models due to Kaluza–Klein (and seldom remembered Mandel) was usually considered separately from its super nature. In this note I give examples where the supersymmetry and non-holonomicity manifest themselves in various, sometimes unexpected, instances. In the models of super space-time, and super strings, as well as in the problems spoken about in this note, these notions appear together and interact.

L.D. Faddeev would have liked these topics, each of them being related to what used to be of interest to him. Many examples will be only mentioned in passing.

1.1. History: Faddeev–Popov ghosts as analogs of momenta in “odd” mechanics. Being elected Member of the USSR Academy of Science, L.D. Faddeev got the right to submit papers to Soviet Mathematics Doklady, a rather prestigious journal at that time. The first paper he presented having become Academician was my note [L2] in which I gave not only an interpretation of the Schouten bracket (lately often called antibracket, and sometimes Buttin bracket, in honor of the first person who proved that the antibracket satisfies the super Jacobian identity) as an analog of the Poisson bracket (this analogy was manifest from the moment the Schouten bracket was discovered, but without important examples of application, like the equation $\dot{f} = \{f, h\}_{a.b.}$ and its interpretation). I also interpreted the quotient of the Buttin superalgebra $b(n)$, on which the Schouten bracket is the multiplication, by the center as preserving an odd (in both senses) analog of the...
symplectic form. This quotient, \( \mathfrak{le}(n) \), preserves an odd analog of the symplectic form, and is an “odd” superization of the Lie algebra \( \mathfrak{h}(2n) \) of Hamiltonian vector fields, whereas the Lie superalgebra \( \mathfrak{h}(2n|m) \) — the quotient by the center of the Poisson Lie superalgebra \( \mathfrak{po}(2n|m) \), —preserving an even (ortho)symplectic (nondegenerate and closed) even 2-form is a “direct” superization.

This latter interpretation was new.

The Lie superalgebras \( \mathfrak{le}(n) \), the corresponding “odd” analog \( \mathfrak{m}(n) \) of the Lie algebra of contact vector fields, preserving a distribution with an “odd time” \( \tau \) singled out by the Pfaff equation with the form

\[
d\tau - \sum \pi_i dq_i,
\]

together with the divergence-free analogs of these superalgebras were the first of several series of counterexamples to a “Theorem” and a Conjecture in [K] Th.10, p.92; Conj. 1, p.93, claiming the classification of simple (and even primitive) Lie superalgebras of vector fields. Grozman found a deformation of the Schouten bracket and more counterexamples, see [GrI], but his result was appreciated only much later, see [LSh5].

It then took 2 decades to formulate the classification of simple vectorial Lie superalgebras without mistakes, and without claiming having solved a wild problem of classification of primitive Lie superalgebras as in [K] [K7], see [Sh14]; it took 10 more years to completely prove it, see [LSh5], and [K7] with numerous rectifications, see, e.g., [CaKa2, CaKa4, CK1a, CCK]. Observe that classification of deformations (the results of deformations) of simple vectorial Lie superalgebras listed in [LSI5], especially with odd parameters, was never published to this day, bar occasional examples, see [LSh3], whereas the ones rediscovered in [CK1], were discarded by the authors. People (even mathematicians) still look askance at the odd parameters of deformations, although these parameters constitute the very times of supersymmetry, whereas the odd central extensions are welcome, despite the likeness of cohomology that describes each of them.

The note [L2] also corrected the formulation of the super version of Darboux’s theorem (on the normal shape of the non-degenerate closed differential 2-form) in [Kos], where odd forms were not even considered.

Most importantly, [L2] proclaimed the existence of an “odd” analog of mechanics. Speaking at that time about the two analogs of symplectic forms — and ensuing mechanics — I used to joke that I could not believe that God blessed only the Poisson one and not both of them.

In 1976–77, M. Marinov told me, leaving theology for future (which did not hesitate to come), that unless I demonstrate how to quantize this “odd” mechanics, no physicist will listen to me, especially since the “Planck’s constant” for the “odd” mechanics, if exists, should be odd, which is so odd he did not believe it could be so.

Marinov was too pessimistic. This “odd” mechanics, soon rediscovered by I.Batalin and G.Vilkovisky, drew a huge interest of physicists, thanks to applications demonstrated already in their first paper [BV]. This happened despite the fact that the “odd” mechanics were quantized much later (and to an extent; I think this is still an open problem since it is unclear how to interpret multiple analogs of Fock spaces and work with them, see [LSh3]).
In any case, “Faddeev–Popov’s ghosts” got an interpretation (e.g., as “coordinates” \( \pi_i \) in eq. (1)). A bit later, mathematicians got interested in derived of the Schouten construction, called BV-algebras; among numerous works, let me point at a recent one: \[ DSV \].

1.2. **Towards future.** In this note I want to point at two open problems, or rather topics for research, in the area of differential equations. More or less implicitly, these problems were being observed for decades, but their formulations was published only sketchily and only in Russian. Clearly, there are more than two problems of importance in the theory of differential equations; my choice of the two problems discussed below reflects only my taste and understanding, not claiming for more.

The problems to be discussed here, although important, were beyond reach until recently for the lack of adequate tools and even terms, so these problems did not draw any attention, to say nothing about these attention they deserve in my opinion. Now the time is more propitious for solving these problems: we have at our disposal not only the vocabulary but also a technique which, speaking of the second topic, is implemented, to an extent, in a Mathematica-based package \[ Gr \].

To make presentation reasonably short (as a speech at the wake), all the excruciating details are replaced by references to illuminating text-books and monographs \[ BCG \, DcL \, LSoS \, Ber \], and papers \[ L3 \, Shch \, Sha \].

For a criticism of epygony and wrong texts that should not have been published, see one of Appendices to the paper \[ Mo \], devoted to infinite-dimensional supermanifolds, cf. Wheeler’s conjecture in epigraph to the next section.

2. **Supersymmetries of differential equations**

“The stage on which the space of the Universe moves is certainly not space itself. Nobody can be a stage for himself; he has to have a larger arena in which to move. The arena in which space does its changing is not even the space-time of Einstein, for space-time is the history of space changing with time. The arena must be a larger object: Superspace ... It is not endowed with three or four dimensions — it is endowed with an infinite number of dimensions.” (J.A. Wheeler: *Superspace*, Harper’s Magazine, July 1974, p. 9, see \[ Giu \].)

2.1. **Sophus Lie, Élie Cartan and supersymmetry.** As is well-known, Sophus Lie introduced the groups, now bearing his name, in order to carry out for differential equations the analog of what Galois did for algebraic equations. Lie’s work was the beginning of the systematic study of symmetries of differential equations and their solutions.

Several monographs and text-books, and countless research papers are devoted to the study symmetries of differential equations.

Nobody, however, observed (at least, this observation was never made public, except briefly in \[ LSoS \]) that every differential equation possesses a supersymmetry (SUSY for short, as physicists say), and hence nobody tried to describe this SUSY. (I am speaking about DE on any manifold, not on a supermanifold where a SUSY, but not the one I am now writing about, is present by definition.)

Since ca 1974, i.e., shortly after I’ve understood how to define what in modern terms is called affine superscheme or supervariety, see \[ L1 \], I was planning to rectify this oversight.
The discovery of supersymmetries by J. Wess and B. Zumino explained that the Maxwell equation and the Dirac equation should not be considered as separate equations, the Lie algebra of symmetries of each of them being the Poincaré algebra (see, e.g., [FN]). They should be considered as a system of equations tied by a supersymmetry whose Lie superalgebra is currently called $N = 1$ Poincaré superalgebra, see [DBS] [BLS] where the extensions for $N > 1$ are described. This discovery made my observation, described a bit further in this section, obvious, I thought. Obviously, it was not as obvious as I thought it was; for almost 40 years it was never rediscovered.

Later, I intended to delegate the problem of finding the supersymmetry of classical equations of mathematical physics to somebody among my students, but failed to find a sufficiently interested one. Perhaps, my own enthusiasm was insufficiently contagious, or the problem looked too vague, or time was not ripe yet.

The current occasion seems to me appropriate to make researchers interested in this problem, and somewhat related problems I will describe in §3.

É. Cartan performed a step most important from super point of view: he reformulated the notion “differential equation” in terms of exterior differential forms. For an exposition of this approach, see the books [BCG] and [C].

For simplicity, consider an ODE: the passage to PDE, and to systems of equations is evident. Recall that to a given differential equation of order $k$ for an unknown function $u$ depending on $x$, i.e., to the expression of the form

$$F(x, u, \frac{du}{dx}, \ldots, \frac{d^k u}{dx^k}) = 0,$$

one can assign a differential ideal (i.e., an ideal closed with respect to the exterior differential $d$) in the superalgebra of functions in even (commuting) indeterminates $x, u, p$, and odd (anti-commuting) indeterminates $dx, du, dp$, where $p = (p_1, \ldots, p_k)$ and $dp = (dp_1, \ldots, dp_k)$, generated by the function $F(x, u, p)$ and exterior (odd, anti-commuting) forms $\omega_0 = du - p_1 dx$, $\omega_1 = dp_1 - p_2 dx$, $\ldots$, $\omega_{k-1} = dp_{k-1} - p_k dx$.

The only thing that É. Cartan did not do was to say what does this differential ideal single out and where. To be able to say this, he needed the definition of supervarieties (given in 1972, published in [L1]), and basics of supersymmetries, see lectures by J. Bernstein (notes taken by P. Deligne and J. Morgan) in [Del, pp.41–96] and a more detailed text-book [LSoS]. Equipped with this knowledge we can say: the ideal in question singles out a certain subsuperscheme or subsupervariety in the affine superspace whose algebra of functions is the supercommutative superalgebra with even (topological) generators (“coordinates”) $x, u, p$ and odd ones $dx, du, dp$.

There are several text-books teaching us the art of seeking symmetries of differential equations, e.g., see the book [FN] and references therein. Superizations of certain facts and notions of algebraic and differential geometry are rather intricate, see [LSoS] and [Sha]; in this particular case, however, superization of the technique describing symmetries of differential
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equations seems to be straightforward. Therefore, we are fully equipped for solving the following problem.

2.1.1. Problem. Describe supersymmetry groups (infinitesimally: Lie superalgebras) of differential equations of mathematical physics. Which of them do not reduce to groups (Lie algebras)?

Remarks. 1) Physicists (pioneers of supersymmetry) discovered supersymmetries that in-termix vector and spinor fields together with space-time coordinates. In the light of the above interpretation of any differential equation as something that singles out a subsuper-variety, this discovery does not look as astounding as it is still represented in the literature. However, these physicists were the first to observe SUSY and this is what they are being praised for. Remarkably, some of these pioneers do not understand the importance of their own discoveries even now, see their prefaces in [DBS].

2) It was Wess and Zumino who used the catchy term supersymmetry. Soon after, Salam and Strathdee termed the object, on which supergroup of supersymmetry acts — superspace — as is now customary to call the pair \( M = (M, \mathcal{O}_M) \) consisting of the space (manifold) \( M \) ringed by the sheaf of supercommutative superalgebras \( \mathcal{O}_M \). Two years earlier, in the purely algebraic setting, where the only functions considered are polynomial and rational, I used for what is now called superscheme or algebraic supervariety the dull and boring term spectrum of graded-commutative ring, see [L1].

2.1.2. Gauge fields from super point of view. Let \( \mathcal{F} \) be the sheaf (this is not yet popular term among the physicists; locally, it is the algebra) of functions on \( M \), and \( V = \Gamma(M, E) \) is the \( \mathcal{F} \)-module of sections of the bundle \( E \). On the level of points, a connection on a vector bundle \( E \) over a (super)manifold \( M \) is an odd map \( \nabla: V \to \Omega^1(M) \otimes \mathcal{F} V \), by the formula

\[
\nabla(fv) = df \otimes v + (-1)^{p(f)} f \nabla(v) \text{ for any } f \in \mathcal{F} \text{ and } v \in V.
\]

Having fixed a flat connection \( \nabla_0 := d \), meaning here, by the usual abuse of notation, \( d \otimes 1_{\dim V} \) rather than \( d \), we can represent any connection in the shape

\[
\nabla = d + \alpha, \text{ where } \alpha \in (\mathfrak{gl}(rk V) \otimes \Omega^1(M))_1.
\]

The form \( \alpha \) is called the form of \( \nabla \) or a gauge field with gauge group \( GL(rk V) \) or any smaller sub(super)group \( G \subset GL(rk V) \), if \( \alpha \in (\mathfrak{g} \otimes \Omega^1(M))_1 \) for the Lie (super)algebra \( \mathfrak{g} \) of \( G \).

The form \( F_\nabla = \nabla^2 \in (\mathfrak{g} \otimes \Omega^2(M))_0 \) is called the curvature form of the connection \( \nabla \) or the stress tensor of the field \( \alpha \).

\footnote{From somewhere in the 1950s on, John Wheeler repeatedly urged people who were interested in the quantum-gravity program to understand the structure of a mathematical object that he called Superspace [W]. The intended meaning of Superspace was that of a set, denoted by \( S(\Sigma) \), whose points faithfully correspond to all possible Riemannian geometries on a given three-manifold \( \Sigma \), see [Gin]. Thus, Weeler (known not only as Feynman’s teacher, but also for having coined the terms “black hole” and “wormhole”) used the term “superspace” in a sense absolutely different from what is now customary to use by everybody, bar geometro-dynamists.

It is a unfortunate that the same term has two completely different meanings because both meanings of the term can meet in one sentence, e.g., when the points of Wheeler’s “superspace” are generalized to consist of super-Riemann manifolds.}
2.1.2a. **Super specifics, unexplored to this day, cf. [MaG, Del]**. 1) Observe that the definitions in the above paragraph describe only **points** of the linear supermanifold of connections with the gauge superalgebra \( \mathfrak{g} \) on \( \mathcal{M} \) and the **points** of the linear supermanifold of curvature forms of these connections. To describe the odd parameters of these supermanifolds we use the functors of points (“Grassmann envelopes” in Berezin’s words, see [Ber]) that to every supercommutative superalgebra \( C \) assign the sets of \( C \)-points of connections \( \mathfrak{g} \otimes \Omega^1(\mathcal{M}) \otimes C \) and curvature forms \( \mathfrak{g} \otimes \Omega^2(\mathcal{M}) \otimes C \).

2) The map \( \nabla \) can be extended to the higher differential forms (sections of higher exterior powers of the tangent bundle), where the \( \mathcal{F} \)-module of pseudoforms is defined to be \( \Sigma_{-i}(\mathcal{M}) := (\Omega^i(\mathcal{M}))^* \otimes \mathcal{F} \operatorname{Vol}(\mathcal{M}) \), where \( \operatorname{Vol}(\mathcal{M}) \) is the \( \mathcal{F} \)-module of volume forms (aka Berezinian):

\[
\nabla : \Omega^i(\mathcal{M}) \otimes \mathcal{F} V \rightarrow \Omega^{i+1}(\mathcal{M}) \otimes \mathcal{F} V; \quad \nabla : V \otimes \mathcal{F} \Sigma_j(\mathcal{M}) \rightarrow V \otimes \mathcal{F} \Sigma_{j+1}(\mathcal{M}).
\]

Let \( T : \Omega^i(\mathcal{M}) \otimes \mathcal{F} V \simeq V \otimes \mathcal{F} \Omega^i(\mathcal{M}) \) be a twisting isomorphism

\[
T(\omega \otimes v) = (-1)^p(\omega)p(v) \otimes \omega \text{ for any } \omega \in \Omega^i(\mathcal{M}) \text{ and } v \in V
\]

and \( \alpha \) be defined in [3]. We define \( \nabla \) in eq. (4) by means of the following formulas

\[
\nabla(\omega \otimes v) = d\omega \otimes v + (-1)^p(\omega)\alpha(v) \text{ for any } \omega \in \Omega^i(\mathcal{M}) \text{ and } v \in V;
\]

\[
\nabla(\alpha \otimes \sigma) = T(\nabla(\alpha))(\sigma) + (-1)^p(\alpha)v \otimes d\sigma \text{ for any } \sigma \in \Sigma_j(\mathcal{M}) \text{ and } v \in V.
\]

Observe that

The de Rham cohomology of any supermanifold is isomorphic to the de Rham cohomology of the underlying manifold, see [MaG].

Let pseudoforms be a common name for various types of smooth functions in coordinates \( x_i \) and their differentials \( \hat{x}_i := dx_i \). There are several types of pseudoforms: of rapid descent at infinity, homogeneous (such that \( f(x, t\hat{x}) = t^a f(x, \hat{x}) \) for any \( t \in \mathbb{R} \) and some \( a \in \mathbb{R} \)), etc.

The pseudo(co)homology on the space of pseudoforms is NOT isomorphic, generally, to the de Rham cohomology of the underlying manifold, see [Za].

The fact (8) is most interesting since \( \nabla \) can be extended by the operators given by eqs. (1) — (6) from the space of forms, which are polynomials in the (even) differentials of the odd coordinates, to the space of pseudoforms.

Given two \( \mathcal{F} \)-modules with connections \( (V_i, \nabla_i) \), where \( i = 1, 2 \), define their tensor product \( (V_1 \otimes \mathcal{F} V_2, \nabla_{1 \otimes 2}) \) and the module of homomorphisms \( \operatorname{Hom}(V_1, V_2, \nabla_{1 \otimes 2}) \) by setting for any \( v_1 \in V_1 \) and \( F \in \operatorname{Hom}_\mathcal{F}(V_1, V_2) \):

\[
\nabla_{1 \otimes 2}(v_1 \otimes v_2) = \nabla_1(v_1) \otimes v_2 + (-1)^{p(v_1)}(T \otimes 1)v_1 \otimes \nabla_2(v_2) \text{ for any } \omega \in \Omega^i(\mathcal{M}) \text{ and } v \in \mathcal{V};
\]

\[
\nabla_{1 \otimes 2}(F)(v_1) = \nabla_2(F(v_1)) - (-1)^{p(F)}(1 \otimes F)(\nabla_1(v_1)).
\]

If \( V_1 = V_2 = V \), we set

\[
(\operatorname{End}(V), \nabla^{\operatorname{End}}) := (\operatorname{Hom}(V_1, V_2, \nabla_{1 \otimes 2})).
\]

3) One more aspect of super specifics is that the physically meaningful supermanifolds, e.g., \( N \)-extended Minkowski superspace for any \( N \), and each of the superstrings considered
usually, is endowed with a non-integrable distribution and is neither real, nor complex supermanifold but a real supermanifold with a complex structure on subspaces that define the distribution, see [BGLS].

The connection on such real-complex supermanifold is what is called circumcised (or reduced) connection; the form of such a connection is defined by its values on the subspaces that define the distribution.

4) Quillen was the first to apply “superconnections” to prove the index theorem, see [Q1] and a later exposition in the book [BGV]. Quillen’s proof was based on the isomorphism spoken about in (7). For details of an open problem “how to superize the index theorem”, see [Li].

2.1.3. Well-known example: SUSY of Maxwell and Dirac equations. Let $M$ be a manifold locally diffeomorphic to the Minkowski space with coordinates we arrange in a $2 \times 2$ matrix $(x_{ab}) = \begin{pmatrix} x_0 - x_3 & x_2 - i x_1 \\ x_2 + i x_1 & x_0 + x_3 \end{pmatrix}$, where $x_0, \ldots, x_3$ are the usual local coordinates in the Minkowski space with the metric $\det(dx_{ab})$, where $a, b = 0, 1$ and the differentials $dx_{ab}$ are considered even (commuting).

Let $A = \{A_\mu(x) \mid \mu = 0, \ldots, 3\}$ be the vector-potential of an electro-magnetic field. For mathematicians, the coordinates of this $A$ are components of the form of the connection (a.k.a. gauge field) $\nabla := d + \sum A_\mu dx_\mu$, where the differentials are considered odd (anti-commuting). Let $F_\nabla = \sum F^{\mu\nu} dx_\mu \wedge dx_\nu$, and $*F_\nabla$ its Hodge dual. The Maxwell equation is the simplest Yang-Mills equation, the one with the smallest (1-dimensional) gauge group for the case where $rk V = 1$.

Namely, let $\psi := \{\psi^a \mid a = 0, 1\}$ and $\bar{\psi} := \{\bar{\psi}^b \mid b = 0, 1\}$ be the spinor wave-function and its Dirac-adjoint, i.e., $\psi \in \mathbb{C}^2$ on which $\mathfrak{o}(3, 1) \simeq \mathfrak{sl}(2)$ acts via the spinor representation, and $\bar{\psi} \in (\mathbb{C}^2)^* \simeq \mathbb{C}^2$ is an element of the hermitian dual (dual and complex conjugate) spinor representation. The Maxwell equation with the source given by an electron-positron field is

$$\nabla_{\text{End}}(*F_\nabla) = \bar{\psi} \otimes \psi$$

(10) (for the definition of $\nabla_{\text{End}}$, see eq. (9)).

Let $\hbar = 1$ and $c = 1$ whereas $e$ and $m$ are the charge and mass of the electron, respectively. The Dirac equation for the electron in the electro-magnetic field can be written as

$$\sum_\mu \gamma^\mu (i \partial_\mu + eA_\mu) \psi = m\psi, \quad \text{where} \quad \gamma^\mu$$

are the Dirac matrices,
and similarly for the Dirac-adjoint spinor $\bar{\psi}$. The symmetry group of each of the equations (10) and (11) is the same — the Poincaré group, i.e. $O(3,1) \ltimes \mathbb{R}^4$, see, e.g., [FN].

Now, consider the tangent space to the Minkowski superspace $\mathcal{M}_1 = (x_{ab}, \psi^a, \bar{\psi}^b), a, b = 0, 1$, realized as a Lie subsuperalgebra of one of the real forms of the complex Lie superalgebra $\mathfrak{sl}(4|1)$ by supermatrices in the nonstandard format $\mathfrak{sl}(2|1|2)$, and vector fields, as follows, see [WB, eq. (I), p.3, and p.231]:

$$\begin{pmatrix} 0 & 0 & 0 \\ Q & 0 & 0 \\ T & \bar{Q} & 0 \end{pmatrix},$$

where $Q_a = \partial_{\psi^a} + \psi^b \partial_{x_{ab}}$ and $\bar{Q}_b = \partial_{\bar{\psi}^b} + \bar{\psi}^a \partial_{x_{ab}}, T_{ab} = \partial_{x_{ab}}$.

Let $\sigma_{\mu\nu} := \frac{i}{2} [\gamma^\mu, \gamma^\nu]$ and let $\varepsilon^a$ and $\bar{\varepsilon}^b$ are the odd parameters forming 2-component spinors, like $\psi$ and $\bar{\psi}$, respectively. The direct verification shows that the transformations generated by odd generators $Q_a$ and $\bar{Q}_b$ act as follows (only the linear part of the increment is shown):

$$(\psi, \bar{\psi}, A) \mapsto (\sigma_{\mu\nu} F^\mu_{\nu\varepsilon}, -\varepsilon \sigma_{\mu\nu} F^\mu_{\nu}, i(\bar{\varepsilon} \gamma^\mu \psi - \psi \gamma^\mu \varepsilon)).$$

Hence,

the symmetry of the system, consisting of (10), and (11) with its Dirac-adjoint, is the Lie subsuperalgebra of $\mathfrak{sl}(2|1|2)$ whose elements are supermatrices of the form

$$\begin{pmatrix} A & 0 & 0 \\ Q & 0 & 0 \\ T & \bar{Q} & -A^T \end{pmatrix},$$

where $A \in \mathfrak{sl}_C(2)$.

For a description of $N$-extended Minkowski supermanifolds $\mathcal{M}_N$, and related structures, like the curvature tensor taking into account the non-integrable distribution on $\mathcal{M}_N$, see [BGLS].

2.1.4. Witten’s discoveries: two more types of SUSY related to differential equations. These SUSYs are close to those considered above but differ from them.

1) Witten’s interpretation of integrability criteria of Yang–Mills equations, see [W1] and an extensive comment in the book [MaG].

2) SUSY of the “conventional” Schrödinger equation with matrix potential, see [W2], see a lucid review [GK] which inspired numerous works with keyword “shape invariance”. This SUSY explains degeneration of the non-zero spectrum of the equation and is a realization (representaion) of $\mathfrak{q}(1)$, a queer analog of $\mathfrak{gl}(1)$. For an explanation why this Lie superalgebra is an analog of $\mathfrak{gl}(n)$, see [LSoS]; for a description of irreducible representations of $\mathfrak{q}(n)$, see [Br, ChK].

2.1.5. Kirillov’s observation: a SUSY-related interpretation of a known property of solutions of the Sturm–Liouville equation. A.A.Kirillov noticed, see [Kir], that the known fact the product of any two solutions of the Hill (Sturm–Liouville) equation

$$2y'' + p(x)y = 0$$

is a solution of the equation

$$y''' + 2p(x)y' + p'(x)y = 0$$
follows from the description of the stationary Lie superalgebra of the point in the co-adjoint representation of the Neveu–Schwarz Lie superalgebra, one of the 12 non-trivial central extensions of simple stringy (aka superconformal) Lie superalgebras, see [CLS]. A similar investigation of the stationary Lie superalgebras of the points in the co-adjoint representations in 9 of the remaining 11 cases, see [OOCh Lkur].

3. Formal integrability of differential systems and non-holonomic structures

In [BCG] one can find, among other things, criteria of formal integrability of differential equations. There are several known ways to solve a given DE or systems of DE; the method offered in [BCG] differs from the other methods I heard about in that it is NEVER used in practice (by engineers, physicists, chemists, biologists, geologists, etc.). It is still used sometimes in grant applications by mathematicians, but this is hardly what taxpayers usually have in mind speaking about “applications” of the method.

It seems to me that the trouble with the problems discussed in the book [BCG] was the same as with the problem described in §2: time was not ripe yet, several vital notions were lacking.

3.1. Criteria for formal integrability of differential equations. These criteria are expressed in [BCG] in terms of the Lie algebra $\mathfrak{g}$ of local symmetries of the DE and the certain Spencer cohomology of $\mathfrak{g}$, where $\mathfrak{g}$ is supposed to be a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{i \geq -d} \mathfrak{g}_i$ of depth $d = 1$. Among the manifolds with a $G$-structure, the simplest ones are flat, see [BGLS]. The cocycles representing nontrivial classes of the same Spencer cohomology are the obstructions to flatness of the $G$-structure, where $G$ is the Lie group whose Lie algebra is the 0th component $\mathfrak{g}_0$ of $\mathfrak{g}$.

Examples of such obstructions well-known before they were expressed in terms of Spencer cohomology (note that examples (a) and (b) are meaningful over any field of characteristic $\neq 2$):

(a) the Riemann tensors for metrics, here $\mathfrak{g} = \mathfrak{o}(n)$;
(b) $d\omega$ for the nondegenerate exterior 2-form $\omega$, here $\mathfrak{g} = \mathfrak{sp}(2m)$;
(c) the Nijenhuis tensor for a given almost complex structure, here $\mathfrak{g} = \mathfrak{gl}_C(m) \subset \mathfrak{gl}_R(2m)$.

Question No.1: why nobody among all those who solve DEs every day — ever use this approach and use instead numerical approximations and scores of other methods?

Various methods work under different conditions; e.g., it is more reasonable to approximate $\sin x$ by the Fourier series, not Taylor one, but even the latter works in a neighborhood. It is very difficult to imagine a method that NEVER works. So, when and for which DE should one use Spencer cohomology as means for getting a solution?

Although essentially all method for solving a given (system of) DE (should) lead to the same answer, computing (co)homology is at the moment not efficiently organized. Even the most efficient code known to me (SuperLie, see [Gr]) is applicable to a small portion of the Lie algebras whose (co)homology we’d like to compute, cf. [MF]; for a comparison of SuperLie efficiency with an ad hoc code written by a professional, see [Kor].

It seems strange to me that although to define Lie algebra cohomology $H^i(\mathfrak{g}; M)$ is possible very succinctly, by means of structure constants of the actions of $\mathfrak{g}$ in the adjoint
module and the module of coefficients $M$, computing cohomology is — at the moment — a task formulated everywhere so clumsily (from the point of view of the programmer and computer) that the time needed grows exponentially with \( \dim \mathfrak{g}, \dim M \) and \( i \). One has to use the same data again and again. It is hopeless to directly compute \( H^i(\mathfrak{g}; M) \) starting with \( \dim \mathfrak{g} > 15 \) and \( i > 4 \) using any of the widely known codes/platforms.

*Is the complexity of computing (co)homology of the Lie algebra \( \mathfrak{g} \) in-built indeed, and the volume of computations grows exponentially with \( \dim \mathfrak{g} \) and \( i \)? (We’d like to compute, e.g., \( H^i(\mathfrak{g}) \) for \( \dim \mathfrak{g} < 300 \) and \( i < 4 \).)

**Question No.2:** According to a well-known theorem, the symmetries of a given DE are induced by either point or contact transformations. Why the integrability criteria of DE given in [BCG] and later works, only deal with DEs of the first type.

The answer to this question is clear to me: there are several snags.

First, the symmetry algebras, are, strictly speaking, filtered, not graded.

Second, the integrability criteria given in [BCG] are expressed in terms of Spencer cohomology, see [Po].

Until recently, this cohomology was only defined for the \( \mathbb{Z} \)-graded Lie algebras \( \mathfrak{g} = \oplus_{i \geq -d} \mathfrak{g}_i \) of depth \( d = 1 \), whereas the analogs of Spencer cohomology for the DE whose symmetries are induced from contact transformations should be computed for the graded Lie algebra of depth \( > 1 \) associated with filtered Lie algebra of symmetries of the given DE.

The definition of analogs of Spencer cohomology for the graded Lie (super)algebras of any depth associated with a filtered one, see in [GrL], where there are considered various versions of supergravity equations (SUGRA) for any \( N \leq 8 \), cf. with the best for today book about SUGRA, [GIOS], where SUGRA are deduced for \( N \leq 3 \). Regrettably, [GrL] is written in the jargon alien to physicists and differential geometers: without Christoffel symbols but in terms of Lie superalgebra cohomology instead; that is why nobody, it seems, read the paper.

The difference between cohomology of the filtered and associated with it graded Lie superalgebra is controlled by what, in the particular case of \( N = 1 \) SUGRA equations, physicists call Wess-Zumino constraints.

To generalize the criteria for formal integrability to embrace the DEs whose symmetries are induced by contact transformations, new notions and results are needed. Now we have them.

First: we need the generalization of Cartan prolongs for Lie (super)algebras of depth \( > 1 \), due to Shchepochkina [Shch]. Instead, people often refer to N.Tanaka. It is instructive to compare [Shch] with the earlier, but less useful, works by other authors, e.g., by N. Tanaka (not with the first one, where the square of the differential, by means of which one was supposed to compute cohomology, did not vanish, but with reasonable later ones), and other meaningful ones, e.g., [Ze, AD], in which [Shch] is not even mentioned. The paper [Shch] embraces the super case, and characteristic \( p > 0 \) case, and *partial prolongs* — all these for the first time, and gives the most convenient algorithm for realization of Lie (super)algebras by creation and annihilation operators (i.e., by vector fields, speaking prose).

Second: we need the definition of an analog of Spencer cohomology for Lie (super)algebras of depth \( > 1 \), see [L3].

3.1.1. **Hertz, É.Cartan, Carathéodori, V.Sergeev, and non-holonomic structures.** A nonintegrable distribution (meaning a subbundle of the tangent bundle, not a generalized
The term non-holonomic was coined by H. Hertz who considered mechanical systems with linear constraints on velocities, such as a ball on a rough plane, or any vehicle (at the point of tangency with asphalt the wheel’s velocity vanishes).

Examples of manifestation of non-holonomic nature of certain mechanical systems are ubiquitous: from the cat’s ability to land on its feet having fallen out of the window (“Cat’s problem”), to guided missiles, to skates, to basket balls refusing to fall into the basket; Internet returns tens of thousands of examples.

More generally, a non-holonomic manifold (no dynamics) is the one endowed with a non-integrable distribution. A natural way to define a distribution is to single it out as a set of zeros of a system of Pfaff equations whose left hand sides are differential 1-forms. The simplest and most known non-holonomic distribution is singled out by the Pfaff equation for vector fields \( X \), given by the contact form \( \alpha \):

\[
\alpha(X) = 0, \quad \text{where} \quad \alpha = dt - \sum p_idq_i.
\]

Interesting examples of non-holonomic structures are provided by modern high energy physics, e.g., Minkowski superspaces and superstrings with a contact structure, see (L3) and [BGLS] (whereas arbitrary superstrings or supermanifolds are not necessarily endowed with a non-holonomic distribution).

Note that the first definitions of the exceptional simple Lie algebras É.Cartan and Killing gave in terms of non-holonomic distributions these algebras preserve, see [Shch], not in terms of Cartan matrices and diagrams bearing the name of Dynkin (who was not yet born then). Lately people started to return to the Cartan-Killing way of looking at the Lie algebras due to its applications in the study of PDEs, see [The].

In 1920s, C. Carathéodory reformulated thermodynamics as a non-holonomic mechanical model provided we do not require that the temperature or energy of the gas is a constant.

In 1980s, V. Sergeev rewrote several first pages of the text-book by Landau and Lifshitz on statistical physics in terms of market economy, see [S], with words “market agent” (seller/buyer) instead of “particle”. This is one of the first, if not the first, monographs on what nowadays is called “econophysics”, more interesting, in my opinion, than other books on the topic I read so far. The main mathematical observation of Sergeev’s book (for me; the book contains other results, e.g., deducing Chatelier’s principle) is the same as Carathéodory’s: the heart of the matter lies in non-holonomic nature of the object of the study (see also [Pal [PS]]); nobody yet studied markets in this way.

3.2. Nonholonomic structures and the corresponding analogs of the curvature tensor. For a long time nobody could define an analog of the curvature tensor (field, more correctly say) that would take into account presence of a non-holonomic distribution. In his review of mathematics of non-holonomic dynamics studied together with L. Faddeev (with possible ramifications to other types of nonintegrable constraints, such as in linear programming), Vershik even conjectured that, despite several examples where such tensor...
fields were defined and computed, there is probably no general definition, see Vershik’s Appendix in [Seng] — the translation of [S] with appendices.

In [L3], the definition of the non-holonomic curvature tensor was, however, given together with a reformulation of “Spencer cohomology” in equivalent, but much more convenient terms of Lie algebra cohomology.

As a result, we get a generalization of criteria for formal integrability of DEs, given in [BCG] for the DE whose symmetries are induced by point transformations, to all DEs. Regrettably, it is unclear how to use these criteria. Observe that the nonlinear constraints are also quite real.

Nonlinear constraints on velocities are represented, e.g., by the car with cruise control switched on (so the velocity vector runs over a sphere of radius, say, 55 mph). To define integrability of such distributions, we have to consider infinite dimensional manifolds of “curved Grassmannians” whose points are submanifolds of given dimension passing through a given point (see [MaG], where finite-dimensional super models of curved Grassmannians of \( 0|k \)-dimensional subsupervariety in the \( 0|n \)-dimensional linear supervariety \( (k \leq n) \) are briefly discussed).

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4To compute Spencer cohomology is a grievous job to perform: there are no general theorems on such cohomology, whereas for Lie algebra cohomology there are several general theorems (well, at least one), and a powerful trick called “spectral sequence”, see [FT]. The theorem I have in mind states that the toral (i.e., maximal diagonalizing) subalgebra of the Lie (super)algebra \( g \) acts by zero on any (co)homology of \( g \) with any coefficients (at least, in finite-dimensional cases). This theorem greatly simplifies computations, but with an infinite number even of finite-dimensional cases the theorem fails, cf. [MF].

In my opinion, the notion of Spencer cohomology is “the opiate of the masses” (das Opium des Volkes), as K. Marx would say quoting (without due reference) Marquis de Sade. (Having learned this by chance from Wiki, I now better understand the roots of Marx’s teaching than during the compulsory courses of history of the communist party of the Soviet Union.)
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