VIRTUAL LOCALIZATION IN EQUIVARIANT WITT COHOMOLOGY

MARC LEVINE

Abstract. We prove an analog of the virtual localization theorem of Graber-Pandharipande, in the setting of an action by the normalizer of the torus in \( SL_2 \), and with the Chow groups replaced by the cohomology of a suitably twisted sheaf of Witt groups.

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Introduction

The Bott residue theorem for a torus action on a variety \( X \) over \( \mathbb{C} \), as formulated by Edidin-Graham [9], gives an explicit formula for a class in the equivariant Chow groups of \( X \) in terms of the restriction of the class to the fixed point locus, after inverting a suitable product of equivariant Euler classes of line bundles in the equivariant Chow ring of a point. Graber and Pandharipande [10] have extended this to the case of a similar expression for the equivariant virtual fundamental class of \( X \) associated to an equivariant perfect obstruction theory on \( X \), in terms of the virtual fundamental class of the fixed point locus with respect to a certain perfect obstruction theory induced by the one on \( X \).

For applications to “quadratically enhanced” theories, such as the Chow-Witt groups or cohomology of the sheaf of Witt groups, torus localization is not useful,
as inverting the Euler class of line bundles for such theories kills all the quadratic information. Instead, we take $G = \text{SL}_2$ or the normalizer $N$ of the diagonal torus $T_1$ in $\text{SL}_2$. We developed in [15] versions of Atiyah-Bott localization and a Bott residue formula for schemes with an action by these groups. Here the Euler class of equivariant line bundles gets replaced by the Euler class of the tautological rank two bundles induced by natural representations $G \to \text{GL}_2$, which preserve at least some of the quadratic information after localization.

The main goal of this paper is to extend the virtual localization theorem of Graber-Pandharipande to this setting. We consider a $G$-linearized perfect obstruction theory $\phi \colon E_\bullet \to L_{X/B}$ on some $B$-scheme $X$ with $G$-action (with some additional assumptions which are omitted here), and to this associate a virtual fundamental class $[X, \phi_j]_{vir}^{N}$ in the $G$-equivariant Borel-Moore homology $\mathbb{A}^{BM}_G(X/B, E_\bullet)$. The equivariant Borel-Moore homology is defined as a limit, using the algebraic Borel construction of Totaro, Edidin-Graham and Morel-Voevodsky, and the virtual fundamental class is defined at each finite stage of the limit by a modification, following Graber-Pandharipande, of the method we used in [16] in the non-equivariant case.

For the virtual localization theorem, we restrict to the case of $G = N$ acting on a $k$-scheme $X$. To explain the main players in the statement below, $\text{Sch}^N/k$ is the category of quasi-projective $k$-schemes with an $N$-action, and $\text{Sm}^N/k$ is the full subcategory of smooth $k$-schemes. $|X|^N \subset X^{T_1}$ is the union of the $N$-stable irreducible components of $X^{T_1}$, with $i_j : |X|^N \to X$ a certain union of connected components of $|X|^N$ determined by an equivariant embedding of $X$ in a smooth $Y$. The given $N$-linearized perfect obstruction theory $\phi \colon E_\bullet \to L_{X/k}$ on $X$ induces a perfect obstruction theory $\phi_j : i_j^*E_\bullet \to L_{|X|^N/j/k}$ on each $|X|^N_j$, and we have the Euler class of the virtual normal bundle, $e_N(N^\nu_i) := e_N(\mathbb{A}(i_j^*E^m_0)) \cdot e_N(\mathbb{A}(i_j^*E^m_1))^{-1}$. Here $\nu$ and $m$ refer to the “fixed” and “moving” parts of the corresponding sheaves with respect to the $T_1$-action. Also, the integer $M_0 > 0$ below is chosen so that the localization theorem [15] Introduction, Theorem 5] for the $N$-action on $X$ holds after inverting $M_0 e \in H^*(BN, W)$, where $e$ is the Euler class of the tautological rank two bundle on $BN$ corresponding to the inclusion $N \subset \text{SL}_2 \subset \text{GL}_2$. See Definition 6.3 for the definition of a strict $N$-action; for the other notations used in the statement, please see [16]. In the statement of our main theorem, we use the motivic spectrum $EM(W_\bullet) \in \text{SH}(k)$ representing cohomology of the sheaf of Witt groups, via

$$EM(W_\bullet)^{a,b}(X) = H^{a-b}(X, W).$$

**Theorem 1** (Virtual localization, see Theorem [6.7]. Let $k$ be a field and let $E \in \text{SH}(k)$ be the Eilenberg-MacLane spectrum $E := EM(W_\bullet)$. Let $i : X \to Y$ be a closed immersion in $\text{Sch}^N/k$, with $Y \in \text{Sm}^N/k$, and let $\phi : E_\bullet \to L_{X/k}$ be an $N$-linearized perfect obstruction theory on $X$. Suppose the $N$-action on $X$ is strict. We have the positive integer $M = M_0 \prod_i M_i^N \cdot M_i^Y$, where the $M_i^N$, $M_i^Y$ are as in Definition 6.6 and $M_0$ is described above. Let $[X]^N_j, i_j^*E_\bullet^N$ be the $E$-equivariant virtual fundamental class for the $N$-linearized perfect obstruction theory $\phi_j : i_j^*E_\bullet \to L_{|X|^N/j/k}$ on $|X|^N_j$. Then

$$[X, \phi_j]^N = \sum_{j=1}^s i_{j*}([X]^N_j, \phi_j)^N \cap e_N(N^\nu_i)^{-1} \in E^{BM}_N(X/BN, E_\bullet)[1/Me].$$
We recall the basic constructions and operations for cohomology and Borel-Moore homology in the setting of the theories represented by a motivic ring spectrum \( \mathcal{E} \in \text{SH}(k) \) in \[\text{1}\] following \[\text{6}\] and \[\text{15}\]. This background is used in the construction of the specialization map, fundamental classes, and the construction of Gysin maps and refined Gysin maps; we also recall some facts about cup and cap products. The main technical heart of this paper is \[\text{2}\] where we prove a version (Proposition 2.1) of a result of Vistoli \[\text{23, Lemma 3.16}\]. Vistoli’s lemma is about the Chow groups of DM stacks, whereas Proposition 2.1 is suitable for application to the \( G \)-equivariant motivic theory. We imagine that, with the help of the recent technology in the motivic stable homotopy theory of stacks provided by Chowdhury \[\text{4}\] and Khan-Ravi \[\text{13}\], Proposition 2.1 could be extended to motivic theories on a suitable type of stack. Kresch \[\text{14, Proposition 4}\] has given a proof of Vistoli’s lemma in the setting of the Chow groups of Artin stacks locally of finite type over a field, and perhaps his line of argument would be helpful in extending our result to the motivic setting for the type of Artin stacks considered by Chowdhury and Khan-Ravi.

One consequence of our motivic version of Vistoli’s lemma is the commutativity of refined motivic Gysin pull-back, Corollary \[\text{2.3}\] which as far as we are aware was not available in the literature up to now.

In \[\text{3}\] we recall the construction of equivariant cohomology and Borel-Moore homology in the style of Totaro \[\text{21}\], Edidin-Graham \[\text{8}\] and Morel-Voevodsky \[\text{20}\], following the work of Di Lorenzo and Mantovani \[\text{7, §1.2}\]. In \[\text{4}\] we show how the construction of these classes as presented by Graber-Pandharipande also works in the motivic setting, and we apply this in \[\text{5}\] to the construction of virtual fundamental classes in equivariant Borel-Moore homology. We assemble all the ingredients in \[\text{6}\] with the statement and proof of our main result.

Theorem 1 has been used by Anneloes Viergever \[\text{22}\] to compute some Witt ring-valued Donaldson-Thomas invariants for zero-dimensional subschemes of \( \mathbb{P}^3 \). Alessandro D’Angelo shows in his Ph.D. thesis \[\text{5, Theorem 4.3.10}\] how to extend Theorem 1 from the case \( \mathcal{E} = \text{EM}(\mathcal{W}_n) \) to an arbitrary SL-oriented theory \( \mathcal{E} \) on which the algebraic Hopf map \( \eta \) acts invertibly.

I wish to thank Fangzhou Jin for his very useful explanations of how to apply a number of results from \[\text{6}\].

1. Preliminaries and background

1.1. Cohomology, Borel-Moore homology and the specialization map. Fix a noetherian separated base-scheme \( B \) of finite Krull dimension, and an affine group-scheme \( G \) over \( B \). We let \( \text{Sch}^G/B \) denote the category of quasi-projective \( B \)-schemes with a \( G \)-action and let \( \text{Sm}^G/B \) be the full subcategory of smooth \( B \)-schemes. In this section, we assume that \( G \) is tame \[\text{11, Definition 2.26}\]; for our ultimate purpose, the case of the trivial \( G \) will suffice, so this is not an essential limitation.

Following Hoyois \[\text{11}\], we have the \( G \)-equivariant motivic stable homotopy category \( X \mapsto \text{SH}^G(X) \), \( X \in \text{Sch}^G/B \), together with its Grothendieck six-functor formalism. We refer the reader to \[\text{10}\] for our notation and basic facts about \( \text{SH}^G(-) \), extracted from Hoyois. For example, we have the \( K \)-theory space of \( G \)-linearized perfect complexes on \( X \), \( \mathcal{K}^G(X) \), and for \( v \in \mathcal{K}^G(X) \), the associated suspension functor \( \Sigma^v : \text{SH}^G(X) \rightarrow \text{SH}^G(X) \). For \( \mathcal{V} \) a locally free sheaf on \( X \), we
have the vector bundle $\mathcal{V}(\mathcal{V}) := \text{Spec} \mathcal{O}_X(\text{Sym}^* \mathcal{V})$ with projection $p: \mathcal{V}(\mathcal{V}) \to X$ and zero-section $s: X \to \mathcal{V}(\mathcal{V})$. The suspension functor $\Sigma^\mathcal{V}$ is defined as $p_# \circ s_*$ and its quasi-inverse $\Sigma^{-\mathcal{V}}$ is given by the adjoint $s^! \circ p^*$.

For each morphism $f: Y \to X$ in $\text{Sch}^G/B$ we have the natural transformation $\alpha_f: f_! \to f_*$, which is a natural isomorphism if $f$ is proper.

We also have the twisted Borel-Moore homology

$$H_{a,b}^{BM}(X/B, v) := \text{Hom}_{\mathcal{SH}^G(B)}(\Sigma_{a,b}^* p_{X/\Sigma^1 X, 1_B});$$

if we need to include mention of $G$, we will write this as $H_{a,b}^{BM}(X/B, G, v)$. Replacing $1_B$ with an arbitrary object $\mathcal{E} \in \mathcal{SH}^G(B)$ gives us $E_{a,b}^{BM}(X/B, v)$. We will write $\mathcal{E}^{BM}(X/B, v)$ for $E_{a,b}^{BM}(X/B, v)$. For a morphism $f: Y \to X$ and a $v \in K^G(X)$, we often write $\mathcal{E}^{BM}(Y/B, v)$ for $E_{a,b}^{BM}(Y/B, f^*v)$ and often write $E_{a,b}^{BM}(X, v)$ for $E_{a,b}^{BM}(X/B, v)$ if the base-scheme $B$ is clear from the context.

A proper map $f: Y \to X$ in $\text{Sch}^G/B$ gives rise to the natural transformation $f_{BM}: pX! \to pYf^*$, defined as the composition

$$pX! \xrightarrow{\alpha^*} pX!/f_* f^* \xrightarrow{\alpha_f^{-1}} pX! f_* f^* = pYf^*.$$

This induces the proper push-forward map

$$f_*: \mathcal{E}^{BM}_{a,b}(Y, f^*v) \to \mathcal{E}^{BM}_{a,b}(X, v)$$

by pre-composition with $f^*_{BM}$, applied to $\Sigma_{a,b}^n 1_X$.

We recall the construction of the specialization map. Let $\iota: X \hookrightarrow Y$ be a closed immersion of $B$-schemes with associated ideal sheaf $I_X \subset \mathcal{O}_Y$. We have the blow-up $\text{Bl}_{X \times_0 Y} A^1 \to X \times A^1$ and the deformation space $\text{Def}_X Y := \text{Bl}_{X \times_0 Y} X \times A^1 \setminus \text{Bl}_{X \times_0 Y} X \times 0$, with open immersion $j: Y \times A^1 \setminus \{0\} \to \text{Def}_X Y$ and closed complement $i: C_{X/Y} \to \text{Def}_X Y$; here $C_{X/Y}$ is the normal cone of the immersion $\iota$,

$$C_{X/Y} := \text{Spec} \mathcal{O}_X \bigoplus_{n \geq 0} T^a_X / T^{a+1}_X.$$

Take $v$ in $K^G(Y)$. Let

$$\mathcal{S}_{\iota/B, v}: pC_{X/Y} \circ \Sigma^v \circ pC_{X/Y} \to pY^! \circ \Sigma^v \circ pY^!$$

be the natural transformation defined as follows. Start with the localization distinguished triangle

$$j!j^* \to \text{Id}_{\mathcal{SH}^G(\text{Def}_X Y)} \to i_* i^* \xrightarrow{\theta} j!j^*[1].$$

Composing $\partial$ with $p_{\text{Def}_X Y}$ on the left and $\Sigma^v \circ p_{\text{Def}_X Y}$ on the right gives

$$\partial: pC_{X/Y} \circ \Sigma^v \circ pC_{X/Y} \to pY \times \Omega^v \circ \Sigma S^1 \circ \Sigma^v \circ pY^! \times (A^1 \setminus \{0\}).$$

We have $p_{Y \times (A^1 \setminus \{0\})^n} = p_{Y^!} \circ p_{1^n} = p_{Y^!} \circ p_{1^n} \circ \Omega_{\Pi_{1^n}}$. Writing $Y \times (A^1 \setminus \{0\}) = \text{Spec} \mathcal{O}_Y \{x \pm 1\}$ gives us the generator $dx$ for $\Omega_{p_{1^n}}$, which gives us the canonical isomorphism $\Sigma^{-\Omega_{p_{1^n}}} \cong \Sigma_{p_{1^n}}^{-1}$, and yields the identity

$$p_Y \times (A^1 \setminus \{0\})^n \circ \Sigma S^1 \circ \Sigma^v \circ p_Y \times (A^1 \setminus \{0\}) = p_{Y^!} \circ (p_{1^n} \circ p_{1^n}^{-1}) \circ \Sigma S^1 \circ \Sigma^v \circ p_{Y^!}.$$ 

The quotient map $(A^1 \setminus \{0\})_+ \to \mathcal{G}_m := (A^1 \setminus \{0\})/\{1\}$ induces the map $p_{1^n} \circ p_{1^n}^{-1} \cong \Sigma S^1 \circ \Sigma^v \circ p_{Y^!} \circ \Sigma S^1 \circ \Sigma^v$.

Since

$$\Sigma S^1 = \Sigma S_{\mathcal{G}_m} = \Sigma S_{\mathcal{G}_m} \circ \Sigma S^1,$$
this gives the map (1.3)
\[ \partial_{Y,v} : p_{Y \times \mathbb{A}^1 \setminus \{0\}} \circ \Sigma_{S^1} \circ \Sigma^{0} \circ p_{Y \times \mathbb{A}^1 \setminus \{0\}} \rightarrow p_{Y!} \circ \Sigma_{S^1}^{-1} \circ \Sigma_{S^1}^{0} \circ p_Y = p_{Y!} \circ \Sigma^{0} \circ p_Y, \]
and we set
\[ sp_{i/B,v} := \partial_{Y,v} \circ \partial : p_{C_{X/Y!}} \circ \Sigma^{0} \circ p_{C_{X/Y}} \rightarrow p_{Y!} \circ \Sigma^{0} \circ p_Y. \]

Applying this to \( 1_Y \) and taking \( \text{Hom}_{SH^G(B)}(-, \mathcal{E}) \) gives the specialization map in \( \mathcal{E} \)-Borel-Moore homology,
\[ sp_{i/B,v} : \mathcal{E}^{B, M}_{a,b}(Y/B, v) \rightarrow \mathcal{E}^{B, M}_{a,b}(C_{X/Y}/B, v), \]
and the commutative diagram
\[ \begin{array}{ccc}
\mathcal{E}^{B, M}_{a+1,b}(Y \times \mathbb{A}^1 \setminus \{0\}/B, v) & \xrightarrow{\partial_{Y,v}} & \mathcal{E}^{B, M}_{a,b}(C_{X/Y}/B, v) \\
\downarrow \partial_{Y,v} & & \downarrow \partial_{Y,v} \\
\mathcal{E}^{B, M}_{a,b}(Y/B, v) & & \mathcal{E}^{B, M}_{a,b}(Y/B, v).
\end{array} \]

We will often drop the \(-/B\) in the notation if the base-scheme is clear, writing \( sp_{i,v} \), \( sp_{i,v} \) for \( sp_{i/B,v} \), \( sp_{i/B,v} \).

The specialization map commutes with smooth pull-back in cartesian squares. Let \( f : Y' \rightarrow Y \) be a smooth morphism, let \( \Omega_f \) denote the sheaf of relative Kähler differentials, and take \( v \in K^G(Y) \). We have the natural transformation
\[ f_!: p_{Y!} \circ \Sigma^{0} \circ p_Y \rightarrow p_{Y!} \circ \Sigma^{0} \circ p_Y, \]
defined by the composition
\[ p_{Y!} \circ \Sigma^{0} \circ p_Y = p_{Y!} \circ f_! \circ \Sigma^{0} \circ p_Y, \]
\[ = p_{Y!} \circ f_! \circ \Sigma^{0} \circ p_Y \rightarrow p_{Y!} \circ \Sigma^{0} \circ p_Y. \]

This gives the smooth pull-back map
\[ f_! : \mathcal{E}^{B, M}_{a,b}(Y, v) \rightarrow \mathcal{E}^{B, M}_{a,b}(Y', v + \Omega_f). \]

**Proposition 1.1.** Let \( i : X \rightarrow Y \) be a closed immersion and let \( f : Y' \rightarrow Y \) be a smooth morphism, both in \( \text{Sch}^G/B \). Form the cartesian diagram
\[ \begin{array}{ccc}
X' & \xrightarrow{f'} & X \\
\downarrow i' & & \downarrow i \\
Y' & \xrightarrow{f} & Y.
\end{array} \]

Then for \( v \in K^G(Y) \) and \( \mathcal{E} \in \text{SH}^G(B) \), the diagram
\[ \begin{array}{ccc}
\mathcal{E}^{B, M}_{a,b}(Y, v) & \xrightarrow{sp_{i,v}} & \mathcal{E}^{B, M}_{a,b}(C_{X/Y}, v) \\
\downarrow f_! & & \downarrow f_! \\
\mathcal{E}^{B, M}_{a,b}(Y', v + \Omega_f) & \xrightarrow{sp_{i,v} + \Omega_f} & \mathcal{E}^{B, M}_{a,b}(C_{X'/Y'}, v + \Omega_f)
\end{array} \]
commutes.
Proof: This follows from the fact that the localization sequence is compatible with smooth pull-back. 

Remark 1.2. For $\mathcal{N}$ a locally free sheaf on some $Y$ with associated vector bundle $p: N \to Y$, $N := \mathbb{V}(\mathcal{N})$, the smooth pull-back

$$p^! : \mathcal{E}^{B,M}_{a,b}(Y, v) \to \mathcal{E}^{B,M}_{a,b}(N, v + \mathcal{N})$$

is an isomorphism and one defines the pull-back by the zero section

$$0^!_{N} : \mathcal{E}^{B,M}_{a,b}(N, v + \mathcal{N}) \to \mathcal{E}^{B,M}_{a,b}(Y, v)$$

to be the inverse of $p^!$. Thus, specialization commutes with pull-back by zero-sections in cartesian squares.

1.2. Gysin morphisms. Let $D_G(X)$ denote the derived category of $G$-linearized complexes of quasi-coherent sheaves on $X$. For a morphism $f : X \to Y$ in $\text{Sch}^G/B$, we have the cotangent complex $L_f \in D_G(X)$. $L_f$ agrees with the class of the relative differentials $\Omega_f$ in case $f$ is smooth, and with $N_f[1]$ if $f$ is a regular embedding with conormal sheaf $N_f := \mathcal{I}_X/\mathcal{I}_X^2$; in this case, the normal bundle $N_f$ is $\mathbb{V}(N_f)$.

In what follows, we fix $\mathcal{E} \in SH^G(B)$. We simply the notation a bit by writing everything for $\mathcal{E}^{B,M}_{a,b}(X, v) := \mathcal{E}^{B,M}_{a,b}(X, v)$, the general case $\mathcal{E}^{B,M}_{a,b}(X, v)$ follows from this by changing $v$ suitably. Unless mentioned to the contrary, we work over the fixed base-scheme $B$.

Let $\iota : Y_0 \to Y$ be a regular embedding in $\text{Sch}^G/B$. Then the normal cone is the normal bundle $C_{Y_0/Y} = N_\iota$ and one defines the Gysin map

$$\iota! : \mathcal{E}^{B,M}_{a,b}(Y, v) \to \mathcal{E}^{B,M}_{a,b}(Y_0, v + L_\iota)$$

as the composition

$$\mathcal{E}^{B,M}_{a,b}(Y, v) \xrightarrow{sp^*} \mathcal{E}^{B,M}_{a,b}(N_\iota, v) \xrightarrow{\partial^!} \mathcal{E}^{B,M}_{a,b}(Y_0, v - N_\iota = \mathcal{E}^{B,M}_{a,b}(Y_0, v + L_\iota).$$

More generally, suppose we are given a cartesian square

$$\begin{array}{ccc}
X_0 & \xrightarrow{\iota'} & X \\
\downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\
Y_0 & \xrightarrow{\iota} & Y \\
\end{array}$$

in $\text{Sch}^G/B$, with $\iota$ a regular immersion. Letting $N_\iota \to Y_0$ be the normal bundle, this diagram defines a closed immersion $\alpha : C_{X_0/X} \to q^*N_\iota$. For $v \in K^G(X)$, we thus have the proper push-forward map

$$\alpha_* : \mathcal{E}^{B,M}_{a,b}(C_{X_0/X}, v) \to \mathcal{E}^{B,M}_{a,b}(q^*N_\iota, v)$$

and the Gysin pull-back by the zero-section

$$0^!_{q^*N_\iota} : \mathcal{E}^{B,M}_{a,b}(q^*N_\iota, v) \to \mathcal{E}^{B,M}_{a,b}(X_0, v - N_\iota = \mathcal{E}^{B,M}_{a,b}(X_0, v + L_\iota).$$

Define

$$\iota_p^! : \mathcal{E}^{B,M}_{a,b}(X, v) \to \mathcal{E}^{B,M}_{a,b}(X_0, v + L_\iota)$$
to be the composition
\[ \mathcal{E}^{B.M.}(X, v) \xrightarrow{\alpha} \mathcal{E}^{B.M.}(C_{X_0/X}, v) \]
\[ \xrightarrow{\alpha} \mathcal{E}^{B.M.}(q^* N_0, v) \xrightarrow{q_0^* N_0} \mathcal{E}^{B.M.}(X_0, v + L_\iota). \]

Note that \( \iota_{id_Y}^! \) is the Gysin map \( \iota^! \).

**Proposition 1.3.** Take \( Y \in \text{Sch}^G/B \) with \( v \in K^G(Y) \).

1 (compatibility with proper push-forward and smooth pull-back). Let

\[
\begin{array}{ccc}
W_0 & \xrightarrow{\iota''} & W \\
\downarrow q_2 & & \downarrow p_2 \\
X_0 & \xrightarrow{\iota'} & X \\
\downarrow q_1 & & \downarrow p_1 \\
Y_0 & \xrightarrow{\iota} & Y
\end{array}
\]

be a commutative diagram in \( \text{Sch}^G/B \), with both squares cartesian, with \( \iota \) a regular immersion.

1a. Suppose \( p_2 \) is proper. Then the diagram

\[
\begin{array}{ccc}
\mathcal{E}^{B.M.}(W, v) & \xrightarrow{\iota''_p} & \mathcal{E}^{B.M.}(W_0, v + L_\iota) \\
\downarrow p_2^* & & \downarrow q_2^* \\
\mathcal{E}^{B.M.}(X, v) & \xrightarrow{\iota'_p} & \mathcal{E}^{B.M.}(X_0, v + L_\iota)
\end{array}
\]

commutes.

1b. Suppose \( p_2 \) is smooth. Then the diagram

\[
\begin{array}{ccc}
\mathcal{E}^{B.M.}(W, v + L_{p_2}) & \xrightarrow{\iota''_p} & \mathcal{E}^{B.M.}(W_0, v + L_\iota + L_{q_2}) \\
\downarrow p_2^* & & \downarrow q_2^* \\
\mathcal{E}^{B.M.}(X, v) & \xrightarrow{\iota'_p} & \mathcal{E}^{B.M.}(X_0, v + L_\iota)
\end{array}
\]

commutes.

2 (functoriality). Let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{\iota'_1} & X_0 & \xrightarrow{\iota'_0} & X \\
\downarrow p_1 & & \downarrow p_0 & & \downarrow p \\
Y_1 & \xrightarrow{\iota_1} & Y_0 & \xrightarrow{\iota_0} & Y
\end{array}
\]

be a commutative diagram in \( \text{Sch}^G/B \), with both squares cartesian, and with \( \iota_0 \) and \( \iota_1 \) regular immersions. Then

\[ \iota'_1 \circ \iota'_0 = (\iota_0 \circ \iota_1)' : \mathcal{E}^{B.M.}(X, v) \to \mathcal{E}^{B.M.}(X_1, v + L_{\iota_0 \circ \iota_1}). \]

Here we use the exact sequence

\[ 0 \to \iota^*_1 N_{\iota_0} \to N_{\iota_0 \circ \iota_1} \to N_{\iota_1} \to 0 \]
to identify $\mathcal{E}^{B,M}(X_1, v + L_{i_1} + L_{i_2})$ with $\mathcal{E}^{B,M}(X_1, v + L_{i_0 i_1})$.

Let $f$ be a commutative diagram in $\text{Sch}^G/B$, with both squares cartesian, and with $i$ and $i'$ both regular immersions. We have a natural surjection of locally free sheaves $q^*_1 \mathcal{N}_i \to \mathcal{N}_v$ with kernel $\mathcal{E}$. Letting $E := \mathcal{V}(\mathcal{E})$, this gives the Euler class $e(q^* E) \in H(W_0, q^* E)$ (§3.4) and the cap product (see Remark 1.7 below)

$\mathcal{E}^{B,M}(W_0, v + L_{i'}) \times H(W_0, q^* E) \to \mathcal{E}^{B,M}(W_0, v + L_{i'} - q^* E)$.

Then for $\alpha \in \mathcal{E}^{B,M}(W, v)$, we have

$i'^!(\alpha) = i^!(\alpha) \cap e(q^* E) \in \mathcal{E}^{B,M}(W_0, v + L_{i'}) = \mathcal{E}^{B,M}(W_0, v + L_{i'} - q^* E),$

where we use the exact sequence

$0 \to \mathcal{N}_{i'} \to q^*_1 \mathcal{N}_i \to \mathcal{E} \to 0$

to give the canonical isomorphism

$\mathcal{E}^{B,M}(W_0, v + L_{i}) : = \mathcal{E}^{B,M}(W_0, v - \mathcal{N}_i')$

$\cong \mathcal{E}^{B,M}(W_0, v - (\mathcal{N}_{i'} + q^* E)) =: \mathcal{E}^{B,M}(W_0, v + L_{i'} - q^* E)$.

Proof. See [6, Proposition 2.4.2, Theorem 4.2.1, Proposition 4.2.2].

Remark 1.4. The paper [6] is set in the framework of the motivic stable category $\text{SH}(-)$, not the $G$-equivariant version of Hoyois. However, all the constructions and results of [6] carry over to the equivariant case (still for $G$ tame), and we will use this extension throughout this section. As we have mentioned already, for our ultimate purpose, we only use the theory for $G$ the trivial group, so the cautious reader may safely restrict to that case.

Let $f : X \to Y$ be an lci morphism in $\text{Sch}^G/B$, that is, $f$ admits a factorization as $p \circ i$, with $i : X \to P$ a regular closed immersion and $p : P \to Y$ a smooth morphism (both in $\text{Sch}^G/B$). We have the composition

$\mathcal{E}^{B,M}(Y, v) \xrightarrow{p^*} \mathcal{E}^{B,M}(P, v - \Omega_p^1) \xrightarrow{i_1^*} \mathcal{E}^{B,M}(X, v + \Omega_p - \mathcal{N}_i) = \mathcal{E}^{B,M}(X, v + L_f).

It is shown in [6, Theorem 3.3.2] that this composition is independent of the choice of factorization of $f$, giving the map

$f^!: \mathcal{E}^{B,M}(Y, v) \to \mathcal{E}^{B,M}(X, v + L_f).

In addition, for composable lci morphisms $f : X \to Y$, $g : Y \to Z$, one has

$(gf)^! = f^! g^!$
after the identification $E^{B,M}(X, v + L_{gf}) \cong E^{B,M}(X, v + L_f + L_g)$ given by the distinguished triangle

$$f^*L_g \to L_{gf} \to L_f \to f^*L_g[1].$$

As proper push-forward is compatible with smooth pull-back in cartesian squares, proper push-forward is also compatible with lci pull-back in cartesian squares.

**Remark 1.5 (Relative pull-back).** 1. Let

$$\begin{array}{ccc}
X_0 & \xrightarrow{q} & X_1 \\
\downarrow f_0 & & \downarrow f_1 \\
B_0 & \xrightarrow{p} & B_1
\end{array}$$

be a Tor-independent cartesian square in $\text{Sch}/B$, with $B_0$ and $B_1$ smooth over $B$. Take $v \in K(X)$ and $E \in SH^G(B)$. We define the relative pull-back

$$p^!: E^{B,M}(X_1/B_1, v) \to E^{B,M}(X_0/B_0, v)$$

as follows. Since $B_0$ and $B_1$ are smooth over $B$, the map $p$ is lci. Since the square is Tor-independent and cartesian, $q$ is an lci morphism and $L_q = f^*L_p$, giving the lci pull-back $q^!: E^{B,M}(X_1/B_1, v) \to E^{B,M}(X_0/B_1, v + L_p)$.

The purity isomorphisms

$$p_{X_0} = p_{B_1} \circ (pf_0)! \cong p_{B_1} \circ \Sigma^{-L_{B_1}/B} \circ (pf_0)!$$

and

$$p_{X_0} = p_{B_0} \circ f_0! \cong p_{B_0} \circ \Sigma^{-L_{B_0}/B} \circ f_0!$$

give us the isomorphisms

$$E^{B,M}(X_0/B_1, v + L_p) \cong E^{B,M}(X_0/B_1, v + L_p + L_{B_1}/B)$$

and

$$E^{B,M}(X_0/B_0, v) \cong E^{B,M}(X_0/B_0, v + L_{B_0}/B).$$

The distinguished triangle

$$p^*L_{B_1}/B \to L_{B_0}/B \to L_p \to p^*L_{B_1}/B$$

gives the isomorphism

$$E^{B,M}(X_0/B_0, v + L_p + L_{B_1}/B) \cong E^{B,M}(X_0/B_0, v + L_{B_0}/B).$$

Putting these together gives the isomorphism

$$\vartheta_p: E^{B,M}(X_0/B_1, v + L_p) \xrightarrow{\sim} E^{B,M}(X_0/B_0, v)$$

and we define $p^! := \vartheta_p \circ q^!$.

2. The relative pull-back maps are functorial with respect to adjacent cartesian squares

$$\begin{array}{ccc}
\bullet & \xrightarrow{q_2} & \bullet \\
\downarrow p_2 & & \downarrow p_1 \\
\bullet & \xrightarrow{q_1} & \bullet
\end{array}$$

when defined. This follows from the functoriality of pull-back for lci morphisms and the identities

$$q_2^! \circ \vartheta_{p_1} = \vartheta_{p_1} \circ q_1^!$$
and

\[ \partial_{p_1p_2} = \partial_{p_2} \circ \partial_{p_1}, \]

this latter after using the canonical isomorphism \( E^B(M)(-/B, v + L_{p_1} + L_{p_2}) \cong E^B(M)(-/B, v + L_{p_1p_2}) \). The easy proofs of these identities are left to the reader.

**1.3. External products and refined intersection product.** Take \( Z, X \in \text{Sch}^G/B \) and let \( E \) be a commutative monoid in \( \text{SH}^G(B) \) (i.e., \( E \) is a “motivic commutative ring spectrum”). We have the external product

\[ \boxtimes_{Z,X} : E^B_{a,b}(Z/B, v) \times E^B_{c,d}(X/B, w) \to E^B_{a+c,b+d}(Z \times_B X/B, v + w) \]

defined as follows. We have the cartesian diagram

\[
\begin{array}{ccc}
Z \times_B X & \longrightarrow & Z \\
\downarrow & \Delta & \downarrow \\
X & \longrightarrow & B.
\end{array}
\]

The base-change construction of \( \boxtimes_{Z,X} \) gives us the map

\[ \Delta^* : E^B_{a,b}(Z/B, v) \to (f^*E^B_{a,b})(Z \times_B X/X, v) \]

We also have the composition product \( \boxtimes_{Z,X} \)

\[ (f^*E^B_{a,b})(Z \times_B X/X, v) \times E^B_{c,d}(X/B, w) \to (f^*E^B_{a,b}) \circ (f^*E^B_{c,d})(Z \times_B X/B, v + w) \]

and we define \( \boxtimes_{Z,X} := (- \bullet -) \circ (\Delta^* \times \text{Id}) \).

Now suppose we have a \( Y \in \text{Sm}^G/B \) and morphisms \( f : Z \to Y, g : X \to Y \). Let \( \delta_Y : Y \to Y \times_B Y \) be the diagonal. This gives us the cartesian diagram

\[
\begin{array}{ccc}
Z \times_Y X & \longrightarrow & Z \times_B X \\
\downarrow \delta_Y & \downarrow f \times g & \downarrow \\
Y & \longrightarrow & Y \times_B Y.
\end{array}
\]

The refined intersection product

\[ \cdot_{f,g} : E^B_{a,b}(Z/B, v) \times E^B_{c,d}(X/B, w) \to E^B_{a+c,b+d}(Z \times_Y X/B, v + w - \Omega_{Y/B}) \]

is defined as the composition

\[ E^B_{a,b}(Z/B, v) \times E^B_{c,d}(X/B, w) \xrightarrow{\boxtimes_{Z,X}} E^B_{a+c,b+d}(Z \times_B X/B, v + w) \xrightarrow{\delta_Y} E^B_{a+c,b+d}(Z \times_Y X/B, v + w - \Omega_{Y/B}). \]

**Lemma 1.6.** Let \( i : Z' \to Z, j : X' \to X \) be a closed immersions in \( \text{Sch}^G/B \).

1. For \( \alpha \in E^B_{a,b}(Z', v), \beta \in E^B_{c,d}(X', w) \), we have

\[ i_*(\alpha) \boxtimes_{Z',W} j_*(\beta) = \text{is} (i \times j)_*(\alpha \boxtimes_{Z',W} \beta). \]

and

\[ i_*(\alpha) \cdot_{f,g} j_*(\beta) = (i \times_Y j)_*(\alpha \cdot_{f',g'} \beta). \]
2. Suppose that the cartesian squares
\[
\begin{array}{ccc}
Z \times_B W' & \longrightarrow & W' \\
\Delta_1 & \quad & \\
Z \times_B W & \longrightarrow & W
\end{array}
\]
and
\[
\begin{array}{ccc}
Z' \times_B W' & \longrightarrow & Z' \\
\Delta_2 & \quad & \\
Z \times_B W' & \longrightarrow & Z
\end{array}
\]
are both Tor-independent (for example if $Z \to B$ and $W' \to B$ are both flat). Then for $\alpha \in \mathcal{E}^a_{a,b}(Z, v)$, $\beta \in \mathcal{E}^b_{c,d}(X, w)$, we have
\[
i^j_!(\alpha \boxtimes_{Z', W', j}_! \beta) = (i \times j)_!(\alpha \boxtimes_{Z, W} \beta).
\]

Proof. (1) By [6, Proposition 2.2.12], we have
\[
i^j_!(\alpha \boxtimes_{Z', W', j}_! \beta) = (i \times j)_!(\alpha \boxtimes_{Z, W} \beta).
\]

We then apply Proposition 1.3(1a) to the diagram
\[
\begin{array}{ccc}
Z' \times_Y X' & \longrightarrow & Z' \times_B X' \\
i \times j & \quad & \\
Z \times_Y X & \longrightarrow & Z \times_B X \\
\Delta & \quad & \\
Y & \longrightarrow & Y \times_B Y
\end{array}
\]

(2) We have the cartesian diagrams
\[
\begin{array}{ccc}
Z \times_B W & \longrightarrow & W \\
\Delta & \quad & \\
Z & \longrightarrow & B
\end{array}
\]
and
\[
\begin{array}{ccc}
Z \times_B W' & \longrightarrow & W' \\
\Delta' & \quad & \\
Z & \longrightarrow & B.
\end{array}
\]

The Tor-independence condition implies that $i \times \text{Id}_W$ and $\text{Id}_Z \times j$ are both lci morphisms, and that $\Delta_1^*(\eta_j) = \eta_{d_2 \times j}$. Thus
\[
(\text{Id}_Z \times j)_!(\alpha \boxtimes_{Z, W} \beta) = \Delta_1^*(\eta_j) \bullet \Delta^*(\alpha) \bullet \beta
\]
\[
\Delta_1^*(\Delta^*(\alpha)) \bullet \eta_j \bullet \beta
\]
\[
\Delta^*(\alpha) \bullet \eta_j \bullet \beta
\]
\[
\alpha \boxtimes_{Z, W, j} \beta.
\]
Similarly, we have
\[(i \times \text{Id}_{W'})^! (\alpha \boxtimes_{Z, W'} \beta') = i'(\alpha) \boxtimes_{Z', W'} \beta',\]
so
\[(i \times j)^! (\alpha \boxtimes_{Z, W} \beta) = (i \times \text{Id}_{W'})^! \circ (\text{Id}_{Z} \times j)^! (\alpha \boxtimes_{Z, W} \beta)\]
\[= (i \times \text{Id}_{W'})^! (\alpha \boxtimes_{Z, W} \beta') = i'(\alpha) \boxtimes_{Z', W'} \beta'.\]

Remark 1.7. For \(f : Z \to B\) in \(\text{Sch}^G/B\) and \(E \in \text{SH}^G(B)\) a commutative monoid, the cap product action of \(E^*\cdot(Z, v)\) on \(E_{a, b}^*(Z/B, w)\) is given by the identity
\[E^*\cdot(Z, v) = (f^*E)^{-a, b}_{a, b}(Z/B, w)\]
and the composition map
\[(f^*E)^{-a, b}_{a, b}(Z/Z, -v) \times E_{c, d}^*(Z/B, w) \to E_{c-a, b-d}^*(Z/B, w-v).\]

1.4. **Fundamental classes of lci schemes and of cones.** The construction of fundamental classes plays a central role in [6]. Here we take \(\text{SH}^G(B)\) as a commutative monoid with unit map \(1_B \to \mathcal{E}\). We have the identity \(H^{0, 0}_B(B) = \text{End}_{\text{SH}^G(B)}(1_B)\); set set \([B] = \text{Id}_1 \in H^{0, 0}_B(B)\) and let \([B]_\mathcal{E} := u([B]) \in \mathcal{E}_{B, b}^*(B)\).

Let \(p_X : X \to B\) be in \(\text{Sch}^G/B\) and suppose that \(p_X\) is an lci morphism (since we are considering only quasi-projective \(B\)-schemes, \(p_X\) is a smoothable lci morphism in the sense of [6]). The fundamental class \([X]_\mathcal{E}\) is defined as
\[[X]_\mathcal{E} := p_X^!([B]_\mathcal{E}) \in \mathcal{E}^{B, b}_X(X, L_{X/B}).\]

Since \(\mathcal{E}\) is fixed in this section, we will drop the subscript \(\mathcal{E}\) and write \([X]\) for \([X]_\mathcal{E}\).

Let \(f : Y \to X\) be an lci morphism in \(\text{Sch}^G/B\). By the functoriality of \((-)_!\), we have
\[f^!([X]) = f^!p_X^!([B]) = p_Y^!([B]) = [Y]\]
in \(\mathcal{E}^{B, b}_Y(Y, L_{Y/B}).\)

For \(p_X : X \to B\) lci, and \(v \in K^G(X)\), cap product with \([X]\) gives the map
\[[X] \cap - : E^{a, b}_X (X, v) \to E^{B, b}_{-a, -b}(X, L_{X/B} - v);\]
in case \(X\) is smooth over \(B\), this is the Poincaré duality isomorphism.

**Lemma 1.8.** Let \(f : Y \to X\) be a morphism in \(\text{Sm}/B\) (\(f\) is automatically an lci morphism). Then the diagram
\[\begin{array}{ccc}
E^{a, b}_X (X, v) & \xrightarrow{[X] \cap -} & E^{B, b}_{-a, -b}(X, L_{X/B} - v) \\
\downarrow f^* & & \downarrow f^* \\
E^{a, b}_Y (Y, f^*v) & \xrightarrow{[Y] \cap -} & E^{B, b}_{-a, -b}(Y, L_{Y/B} - v)
\end{array}\]
commutes.

Let \(\iota : X \to Y\) be a closed immersion in \(\text{Sch}^G/B\), with \(Y\) smooth over \(B\). We have the cone \(C_{X/Y} := \text{Spec} \oplus_{n \geq 0} \mathcal{I}_X^n / \mathcal{I}_X^{n+1}\). Define the fundamental class \([C_{X/Y}]\) by
\[[C_{X/Y}] = sp_{\iota_! \Omega_{Y/B}}([Y]) \in \mathcal{E}^{B, b}_{C_{X/Y}}(C_{X/Y}, \Omega_{Y/B}).\]
Remark 1.9. Let \( \iota: X \to Y \) be a regular closed immersion in \( \text{Sch}/B \) with \( Y \) smooth over \( B \). We have the conormal sheaf \( N_i := \mathcal{I}_X/\mathcal{I}_X^2 \) and corresponding normal bundle \( N_i := \mathcal{V}(N_i) \). Then \( C_{X/Y} = N_i \). Since \( X \) is lci over \( B \), so is \( N_i \) and \( [C_{X/Y}] = [N_i] \). Indeed,

\[
[X] = \iota^!(Y) = 0^!(\text{sp}_i([Y])) = 0^!(\mathcal{C}_{X/Y}),
\]

\[
0^!(N_i) = 0^!(p_{N_i}^!([B])) = p_X^!(B) = [X] = 0^!(\mathcal{C}_{X/Y})
\]

and \( 0^! : \mathcal{E}^{B,M}(N_i, \Omega_{N_i/B}) \to \mathcal{E}^{B,M}(X, \Omega_{X/B}) \) is an isomorphism.

Remark 1.10. It was shown in [16, Theorem 3.2] that, given two \( G \)-equivariant closed immersions \( \iota_j: X \to Y_j, j = 1, 2, Y_j \in \text{Sm}^G/B \), there is a canonical isomorphism

\[
\psi_{i_2, i_1} : \mathcal{E}^{B,M}_{\ast\ast}(C_{X/Y_1}, \Omega_{Y_1/B}) \xrightarrow{\sim} \mathcal{E}^{B,M}_{\ast\ast}(C_{X/Y_2}, \Omega_{Y_2/B})
\]

satisfying \( \psi_{i_2, i_1}[C_{X/Y_1}] = [C_{X/Y_2}] \) and \( \psi_{i_3, i_1} \circ \psi_{i_2, i_1} = \psi_{i_3, i_1} \). This says that \( [C_{X/Y}] \in \mathcal{E}^{B,M}(C_{X/Y}, \Omega_{Y/B}) \) is canonically defined, independent of the choice of immersion \( \iota \).

2. VISTOLI'S LEMMA

In this section we state and prove our version of Vistoli’s lemma in the \( G \)-equivariant motivic setting, for tame \( G \) over a base-scheme \( B \). As in the previous section, we assume that \( B \) is a noetherian separated scheme of finite Krull dimension. We fix a commutative monoid \( E \in \text{SH}^G(B) \) and use the \( E \)-valued twisted Borel-Moore homology \( \mathcal{E}^{B,M}_{\ast\ast}(-, -) \). To simplify the notation, we sometimes state results for \( \mathcal{E}^{B,M}_{\ast\ast}(-, -) := \mathcal{E}^{B,M}_{0,0}(-, -) \), but the results are valid for \( \mathcal{E}^{B,M}_{\ast\ast}(-, -) \) by applying a suitable twist. As we fix the base-scheme \( B \) for the entire section, we will drop the \(-/B\) from the notation for Borel-Moore homology, unless we are using additional base-schemes.

Consider a cartesian square in \( \text{Sch}^G/B \)

\[
\begin{array}{ccc}
Z & \xrightarrow{i_2} & X_2 \\
\downarrow & & \downarrow i_1 \\
X_1 & \xrightarrow{i_1} & Y
\end{array}
\]

with \( i_1, i_2 \) closed immersions. This gives rise to the commutative diagram with all squares cartesian

\[
\begin{array}{ccc}
C_{\alpha_2} & \xrightarrow{\beta_2} & i_2^*C_{i_1} \\
\downarrow & & \downarrow \\
\beta_1 & & \downarrow \\
C_{\alpha_1} & \xrightarrow{\beta_1} & i_1^*C_{i_2} \\
\downarrow & & \downarrow \\
C_{\alpha} & \xrightarrow{\beta} & i_1^*C_{i_2} \times_Y i_2^*C_{i_1} \\
\downarrow & & \downarrow \\
\alpha_2 & \xrightarrow{\gamma} & i_1^*C_{i_2} \\
\downarrow & & \downarrow \\
C_{\alpha} & \xrightarrow{\gamma} & i_1^*C_{i_2} \times_Y i_2^*C_{i_1} \\
\downarrow & & \downarrow \\
Z & \xrightarrow{i_2} & X_2 \\
\downarrow & & \downarrow i_2 \\
X_1 & \xrightarrow{i_1} & Y
\end{array}
\]
To explain the notation: given a pair of maps $f: S \to T$, $U \to T$ in $\text{Sch}^G/B$ we write $f^*U$ for the fiber product $S \times_T U$. If $f$ is a closed immersion, we write $C_f \to T$ for the cone of $f: S \to T$. We identify a closed immersion $f: S \to T$ with the corresponding closed subscheme of $T$, which we write as $S \subset T$, and often write the cone $C_f$ as $C_{S/T}$.

The maps $\beta_1, \beta_2$ are closed immersions defined as follows. If $X_j \subset Y$ is defined by an ideal sheaf $I_j$, then $C_{ij} = \text{Spec} \mathcal{O}_Y \oplus_n T^i_j/T^i_j$. For $j \neq j'$, $i^*_{ij}C_{ij}$ is $\text{Spec} \mathcal{O}_Y \oplus_n (T^i_j/T^i_j)^{-1} \otimes \mathcal{O}_Y$ and

$$C_{ij} = \text{Spec} \mathcal{O}_Y \oplus_m T^m_j \cdot (\oplus_n T^m_j/I_j^{n+1})/T^m_j \cdot (\oplus_n T^m_j/I_j^{n+1}),$$

while

$$i^*_1C_{12} \times Y i^*_2C_{12} = \text{Spec} \mathcal{O}_Y \oplus_{n,m} T^m_j/T^m_j \otimes \mathcal{O}_Y T^n_j/T^n_j.$$
be a commutative diagram in \( \text{Sch}^G/B \) with all squares cartesian. Suppose that \( i \) and \( j \) are regular immersions. Then the diagram

\[
\begin{align*}
\mathcal{E}^{B,M}(Y, v) & \xrightarrow{i^!} \mathcal{E}^{B,M}(X_1, v + L_i) \\
\downarrow j^! & \quad \downarrow j^!
\end{align*}
\]

\[
\mathcal{E}^{B,M}(X_2, v + L_j) \xrightarrow{i^!} \mathcal{E}^{B,M}(Z, v + L_i + L_j)
\]

commutes.

Proof. We refer to the diagram (2.1), and let \( t_2 : Z \to S, t_1 : Z \to T \) be the respective compositions, \( t_2 := q' \circ i_2', t_1 := p' \circ i_1' \), giving the commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{i_2} & X_1 & \xrightarrow{q} & S \\
\downarrow t_1 & & \downarrow i_1 & & \downarrow i \\
X_2 & \xrightarrow{i_2'} & Y & \xrightarrow{q'} & T \\
\downarrow p' & & \downarrow p & & \downarrow T' \\
S' & \xrightarrow{j} & T'
\end{array}
\]

Let \( \mathcal{I}_i \subset \mathcal{O}_T \) be the ideal sheaf of the closed immersion \( i \), and let \( N_i \to S \) be the normal bundle, \( N_i = \text{Spec} \mathcal{O}_S \text{Sym}^* \mathcal{I}_i/\mathcal{I}_i^2 \). Letting \( \mathcal{I}_{i_1} \subset \mathcal{O}_S \) be the ideal sheaf of the closed immersion \( i_1 \), we have the surjection \( \mathcal{O}_S \otimes_{\mathcal{O}_T} \mathcal{I}_i \to \mathcal{I}_{i_1} \), which induces the closed immersion

\[
\gamma_{i_1} : C_{i_1} = \text{Spec} \mathcal{O}_X \oplus \mathcal{T}_i^n/\mathcal{T}_i^{n+1} \to \text{Spec} \mathcal{O}_X \text{Sym}^* \mathcal{I}_i/\mathcal{I}_i^2 \otimes_{\mathcal{O}_S} \mathcal{O}_{X_1} = q'^* N_i
\]

over \( X_1 \). The closed immersion \( i_2' : Z \to X_1 \) induces the closed immersions \( i_2'^* N_i \to q'^* N_i \) and \( \alpha_1 : i_2'^* C_{i_1} \to C_{i_1} \); let \( C_{\alpha_1} \) be the cone of \( \alpha_1 \). This gives us the commutative diagram

(2.2)
with all squares cartesian; here the closed immersion \(\gamma_{i_1}\) is defined similarly to \(\gamma_{i_1}\), using the cartesian diagram

\[
\begin{array}{ccc}
i_2^*C_{i_1} & \longrightarrow & C_{i_1} \\
\downarrow_{h_1} & & \downarrow_{h} \\
S' & \longrightarrow & T
\end{array}
\]

instead of the cartesian diagram

\[
\begin{array}{ccc}
X_1 & \longrightarrow & S \\
\downarrow & & \downarrow j \\
Y & \longrightarrow & T
\end{array}
\]

used to construct \(\gamma_{i_1}\).

Let \(C_{i_2'} \to t^*_2N_i\) be the corresponding cone

Let \(0_{q',i}: X_1 \to q'^*N_i\), \(0_{i_2,i}: Z \to t^*_2N_i\), and \(0_{h_1,j}: i_2^*C_{i_1} \to h^*_1N_j\) be the respective 0-sections. For \(y \in c_{n,b}(Y, v)\) we have

\[
j'^i(y) = j'[0_{q',i}(\gamma_{i_1}, sp^*_i(y))].
\]

Since the refined Gysin map commutes with proper push-forward, smooth pull-back, and pull-back with respect to 0-sections (Proposition 1.3) we have

\[
j'^i(y) = j'[0_{q',i}(\gamma_{i_1}, sp^*_i(y))] \\
= 0_{t_2,i}'j^i(\gamma_{i_1}, sp^*_i(y)) \\
= 0_{t_2,i}'\gamma_{i_1}' j^i sp^*_i(y) \\
= 0_{t_2,i}'\gamma_{i_1}' 0_{h_1,j} \alpha_{i_1} sp^*_i(y).
\]

Exchanging the role of \(i\) and \(j\), \(p\) and \(q\), etc., in (2.2) gives us the corresponding diagram

and we have the respective 0-sections \(0_{p',j}: X_2 \to p'^*N_j\), \(0_{i_1,j}: Z \to t^*_1N_j\), and \(0_{h_2,i}: i_1^*C_{i_2} \to h^*_2N_i\).
We have the commutative diagram
\[
\begin{array}{ccc}
C_{\alpha_1} & \xrightarrow{\gamma_{\alpha_1}} & h^*_1 N_j \\
\downarrow{\gamma_1} & & \downarrow{\gamma''_1} \\
t^*_2 N_i \times_Z t^*_1 N_j & \xrightarrow{p_1} & t^*_2 N_i,
\end{array}
\]
with the square cartesian, \(\gamma''_1\) the pull-back of the closed immersion \(\gamma'_{\alpha_1}\), and \(\gamma_1 := \gamma''_1 \circ \gamma_{\alpha_1}\). Let \(0_1 : t^*_2 N_i \to t^*_2 N_i \times_Z t^*_1 N_j\) be the 0-section to \(p_1\). Since the pull-back by a 0-section commutes with proper push-forward in a cartesian square, we have
\[
j''_1^*(y) = 0''_{t_2,i} \gamma'_{\alpha_1} \circ h_{t_1,j} \gamma_{\alpha_1} \circ sp_{\alpha_1} sp_{t_1}^*(y)
= 0'_t h_{t_1,j} \gamma_{\alpha_1} \circ sp_{\alpha_1} sp_{t_1}^*(y)
= 0'_{t_2,i} \gamma_{\alpha_1} \circ sp_{\alpha_1} sp_{t_1}^*(y),
\]
where \(0'_{t_2,i} = 0_1 \circ 0_{t_2,i} : Z \to t^*_2 N_i \times_Z t^*_1 N_j\) is the zero-section.

By symmetry, we have the commutative diagram, with the square being cartesian,
\[
\begin{array}{ccc}
C_{\alpha_2} & \xrightarrow{\gamma_{\alpha_2}} & h^*_2 N_i \\
\downarrow{\gamma_2} & & \downarrow{\gamma''_2} \\
t^*_2 N_i \times_Z t^*_1 N_j & \xrightarrow{p_1} & t^*_2 N_i,
\end{array}
\]
and with \(\gamma_2 := \gamma''_2 \circ \gamma_{\alpha_2}\). Letting \(0_2 : t^*_1 N_j \to t^*_2 N_i \times_Z t^*_1 N_j\) be the 0-section to \(p_2\), we have the commutative diagram
\[
\begin{array}{ccc}
Z & \xrightarrow{0_{1,j}} & t^*_1 N_j \\
\downarrow{0_{t_2,i}} & & \downarrow{0_2} \\
t^*_2 N_i & \xrightarrow{0_1} & t^*_2 N_i \times_Z t^*_1 N_j,
\end{array}
\]
that is,
\[
0_{1,2} = 0_1 \circ 0_{t_2,i} = 0_2 \circ 0_{t_1,j}.
\]
Arguing as above gives us the identity
\[
i''_1 j''(y) = 0''_{t_2,i} \gamma_{\alpha_1} \circ sp_{\alpha_2} sp_{t_2}^*(y).
\]
We have the closed immersion
\[
\gamma_{1,2} := \gamma'_{t_1} \times \gamma'_{t_2} : i''_1 C_{t_1} \times_Z i''_2 C_{t_2} \to t^*_2 N_i \times_Z t^*_1 N_j.
\]
Recalling the closed immersions
\[
\beta_1 : C_{\alpha_1} \to i''_1 C_{t_1} \times_Z i''_2 C_{t_2}, \quad \beta_2 : C_{\alpha_2} \to i''_2 C_{t_1} \times_Z i''_1 C_{t_2}
\]
one checks that \(\gamma_i = \gamma_{1,2} \circ \beta_i\), \(i = 1, 2\), so we have
\[
j''_1 j''(y) = 0''_{t_2,i} \gamma_{\alpha_1} \circ \beta_1 \circ sp_{\alpha_1} sp_{t_1}^*(y)
\]
and
\[
i''_1 j''(y) = 0''_{t_2,i} \gamma_{\alpha_1} \circ \beta_2 \circ sp_{\alpha_2} sp_{t_2}^*(y),
\]
so the identity \(i''_1 j'' = j''_1 i''\) follows from Proposition 2.1 \(\square\)
For the proof of Proposition 2.1, we use a “double deformation diagram”,

\[ \pi_{12} : \text{Def}_{12} \to Y \times \mathbb{A}^1 \times \mathbb{A}^1, \]

which we now proceed to construct.

Let \( \text{Def}_1 = (\text{Def}_{X_1} Y) \times \mathbb{A}^1 \) and let \( \text{Def}_2 = \text{Def}_{X_2 \times \mathbb{A}^1} (Y \times \mathbb{A}^1) \). \( \text{Def}_1 \) is an open subscheme of \( \text{Bl}_{X_1 \times 0} Y \times \mathbb{A}^1 \times \mathbb{A}^1 \) and \( \text{Def}_2 \) is an open subscheme of \( \text{Bl}_{X_2 \times \mathbb{A}^1} Y \times \mathbb{A}^1 \times \mathbb{A}^1 \), giving structure maps \( \pi_i : \text{Def}_i \to Y \times \mathbb{A}^1 \times \mathbb{A}^1, i = 1, 2 \).

Let \( \text{Def}_{12} \) be the fiber product

\[ \text{Def}_{12} = \text{Def}_1 \times Y \times \mathbb{A}^1 \times \mathbb{A}^1 \]

with structure morphism

\[ \pi_{12} : \text{Def}_{12} \to Y \times \mathbb{A}^1 \times \mathbb{A}^1, \]

and projections

\[ p_j : \text{Def}_{12} \to \text{Def}_j. \]

We have \( \pi_1^{-1}(Y \times 0 \times \mathbb{A}^1) = C_{i_1} \times \mathbb{A}^1 \) and \( \pi_2^{-1}(Y \times \mathbb{A}^1 \times 0) = C_{i_2} \times \mathbb{A}^1 \), giving closed immersions

\[ \sigma_j : C_{i_j} \times \mathbb{A}^1 \to \text{Def}_j \]

with respective open complements

\[ \eta_1 : Y \times (\mathbb{A}^1 \setminus \{0\}) \times \mathbb{A}^1 \to \text{Def}_1, \quad \eta_2 : Y \times \mathbb{A}^1 \times (\mathbb{A}^1 \setminus \{0\}) \to \text{Def}_2. \]

The proper transform of \( \sigma_1(C_{i_1} \times \mathbb{A}^1) \) to \( \text{Def}_{12} \) is an open subscheme of \( \text{Bl}_{i_2} C_{i_1} \times 0 C_{i_1} \times \mathbb{A}^1 \), namely \( \text{Def}_{i_2} C_{i_1} C_{i_1} \), giving the closed immersion

\[ \hat{\sigma}_1 : \text{Def}_{i_2} C_{i_1} C_{i_1} \to \text{Def}_{12} \]

over the closed immersion \( Y \times 0 \times \mathbb{A}^1 \to Y \times \mathbb{A}^1 \times \mathbb{A}^1 \). This gives us the commutative diagram

\[
\begin{array}{c}
\text{Def}_{i_2} C_{i_1} C_{i_1} \setminus C_{\alpha_1} \downarrow \rho_i^1 \\
\uparrow \pi_i^1 \quad \text{Def}_{i_2} C_{i_1} C_{i_1} \quad \hat{\sigma}_1 \downarrow \rho_1 \\
C_{i_1} \times (\mathbb{A}^1 \setminus \{0\}) \downarrow \pi_{i_1} \quad C_{i_1} \times \mathbb{A}^1 \quad \sigma_1 \downarrow \pi_1 \quad \text{Def}_1 \quad \pi_{12} \\
\uparrow \pi_i \quad \uparrow \pi_1 \quad \uparrow \pi_1 \\
Y \times 0 \times (\mathbb{A}^1 \setminus \{0\}) \quad Y \times 0 \times \mathbb{A}^1 \quad Y \times \mathbb{A}^1 \times \mathbb{A}^1,
\end{array}
\]

with \( \hat{\sigma}_1 \) the canonical morphism.

Similarly, we have the closed immersion

\[ \hat{\sigma}_2 : \text{Def}_{i_1} C_{i_2} C_{i_2} \to \text{Def}_{12} \]
over the closed immersion $Y \times \mathbb{A}^1 \times 0 \to Y \times \mathbb{A}^1 \times \mathbb{A}^1$, which gives us the commutative diagram

$$
\begin{array}{ccc}
\text{Def}_{i_1^* C_{i_2}} C_{i_2} \setminus C_{\alpha_2} & \xrightarrow{\sigma_2} & \text{Def}_{i_2^* C_{i_2}} C_{i_2} \\
\downarrow \rho_2 & & \downarrow \rho_2 \\
C_{i_2} \times (\mathbb{A}^1 \setminus \{0\})^2 & \xrightarrow{\pi_2} & C_{i_2} \times \mathbb{A}^1 \\
\downarrow \pi_2 & & \downarrow \pi_2 \\
Y \times (\mathbb{A}^1 \setminus \{0\}) \times 0 \times 0 & \xrightarrow{\pi_{12}} & Y \times \mathbb{A}^1 \times 0 \times 0 \\
\end{array}
$$

with $\sigma_2$ the canonical morphism.

Thus, the closed immersions

$$
\text{Def}_{i_2^* C_{i_1}} C_{i_1} \hookrightarrow \pi_{12}^{-1}(Y \times 0 \times \mathbb{A}^1), \quad \text{Def}_{i_2^* C_{i_2}} C_{i_2} \hookrightarrow \pi_{12}^{-1}(Y \times \mathbb{A}^1 \times 0)
$$

are isomorphisms over $Y \times \mathbb{A}^1 \times \mathbb{A}^1 \setminus Y \times 0 \times 0$.

We have

$$
\pi_{12}^{-1}(Y \times 0 \times 0) = \pi_1^{-1}(Y \times 0 \times 0) \times_Y \pi_2^{-1}(Y \times 0 \times 0) = i_2^* C_{i_1} \times_Y i_1^* C_{i_2}
$$

so

$$
\pi_{12}^{-1}(Y \times 0 \times \mathbb{A}^1) = \text{Def}_{i_2^* C_{i_1}} C_{i_1} \cup i_2^* C_{i_1} \times_Y i_1^* C_{i_2}
$$

and

$$
\pi_{12}^{-1}(Y \times \mathbb{A}^1 \times 0) = \text{Def}_{i_2^* C_{i_2}} C_{i_2} \cup i_2^* C_{i_2} \times_Y i_1^* C_{i_2}.
$$

The fiber of $\text{Def}_{i_2^* C_{i_1}} C_{i_1} \to Y \times \mathbb{A}^1 \times \mathbb{A}^1$ over $Y \times 0 \times 0$ is $C_{\alpha_1} \subset \text{Def}_{i_2^* C_{i_1}} C_{i_1}$, and the inclusion of this fiber in $\pi_{12}^{-1}(Y \times 0 \times 0)$ is the closed immersion $\beta_1$. Similarly, the inclusion of $C_{\alpha_2} \subset \text{Def}_{i_2^* C_{i_2}} C_{i_2}$ in $\pi_{12}^{-1}(Y \times 0 \times 0)$ is the closed immersion $\beta_2$.

We fit this all together in the following commutative diagram

(2.3)
Moreover, $\pi_{12}$ restricts to an isomorphism
\[
\text{Def}_{12}\setminus\pi_{12}^{-1}(Y\times\mathbb{A}^1\cup Y\times\mathbb{A}^1\times 0) \xrightarrow{\pi_{12}} Y\times\mathbb{A}^1\setminus(Y\times 0\times\mathbb{A}^1\cup Y\times\mathbb{A}^1\times 0),
\]
we have
\[
\pi_{12}^{-1}(Y\times 0\times\mathbb{A}^1\cup Y\times\mathbb{A}^1\times 0) = \hat{\delta}_1(\text{Def}_{i_2}C_{i_1}) \cup i_2^*C_{i_1} \times Y\ i_2^*C_{i_2} \cup \hat{\delta}_2(\text{Def}_{i_2}C_{i_2}),
\]
and
\[
i_2^*C_{i_1} \times Y\ i_2^*C_{i_2} \cap \hat{\delta}_1(\text{Def}_{i_2}C_{i_1}) = \beta_1(C_{i_1}),
\]
\[
i_2^*C_{i_1} \times Y\ i_2^*C_{i_2} \cap \hat{\delta}_2(\text{Def}_{i_2}C_{i_2}) = \beta_2(C_{i_2}).
\]

With these preparations, we now proceed to the proof of Proposition 2.1. As above, for $(W \to B) \in \mathbf{Sch}^G/B$, we let $p_W: W \to B$ denote the structure morphism.

**Proof of Proposition 2.1** We first define a 2-variable version of the map $\vartheta_{Y,v}(1.3)$:
\[
\vartheta_{Y,v}: p_Y\times(A^1\setminus\{0\})^2 \circ \sum_i^2 \circ \Sigma^v \circ p_Y^v \circ (p_Y\times(A^1\setminus\{0\})^2) \to p_Y \circ \Sigma^v \circ p_Y^v.
\]

Here it is important to keep track of the order of the two factors in the $Y$-scheme $Y \times (A^1 \setminus \{0\})^2$, so we will write this as
\[
Y \times (A^1 \setminus \{0\})^2 = \text{Spec}_{\mathcal{O}_Y} \mathcal{O}_Y[x_1^{\pm 1}, x_2^{\pm 1}],
\]
and we fix an isomorphism $\Omega_{(A^1\setminus\{0\})^2/k} \cong \mathcal{O}_{(A^1\setminus\{0\})^2}$ by using the (ordered) basis $dx_1, dx_2$. We will distinguish the two factors by writing
\[
A_i^1 = \text{Spec}_{\mathcal{O}_Y} \mathcal{O}_Y[x_i], \quad i = 1, 2.
\]
and define the pointed $Y$-scheme $\mathbb{G}_{m,i}$ by
\[
\mathbb{G}_{m,i} := (A^1_i \setminus \{0\}, \{1\})
\]
We have the commutative diagram of projections
\[
\begin{array}{ccc}
Y \times (A^1_1 \setminus \{0\}) \times (A^1_2 \setminus \{0\}) & \xrightarrow{p^1_1} & Y \times (A^1_1 \setminus \{0\})\\
Y \times (A^1_1 \setminus \{0\}) & \xrightarrow{p^2_1} & Y \times (A^1_2 \setminus \{0\})\\
Y & \xrightarrow{p_Y} & Y
\end{array}
\]
We let
\[
\vartheta_{p^1_2 , n[1]} : p_Y \times (A^1 \setminus \{0\})^2 \circ \sum_i^2 \circ \sum^v \circ p_Y^v \circ (p_Y \times (A^1 \setminus \{0\})^2) \to p_Y \times (A^1_2 \setminus \{0\}) \circ \sum_i^2 \circ \sum^v \circ p_Y^v \circ (p_Y \times (A^1_2 \setminus \{0\}))
\]
be the composition
\[
\begin{align*}
    p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 & \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \times \left(p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right) \\
    & = \left(p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \right)
\end{align*}
\]

We define
\[
\vartheta_{Y,v} : p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \rightarrow p_{Y^1} \Sigma^v p^*_Y
\]
as \vartheta_{Y,v} where we use our coordinate $x_2$ for $\mathbb{A}^1 \setminus \{0\}$. We let
\[
\vartheta_{Y,v,2} : p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \rightarrow p_{Y^1} \Sigma^v p^*_Y
\]
be the composition
\[
\vartheta_{Y,v,2} := \vartheta_{Y,v} \circ \vartheta_{p^1_{Y,v},v[1]}.
\]
Writing $p^1_{Y^1} = p^1_{Y} \circ p^1_{v}$ and proceeding as above gives us the maps
\[
\vartheta_{p^1_{Y,v},v[1]} : p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \rightarrow p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2
\]
and
\[
\vartheta_{p^1_{Y,v},v} : p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \rightarrow p_{Y^1} \Sigma^v p^*_Y,
\]
and we define
\[
\vartheta_{Y,v,1} := \vartheta_{Y,v} \circ \vartheta_{p^1_{Y,v},v[1]}.
\]

**Lemma 2.4.**
\[
\vartheta_{Y,v,1} = -\vartheta_{Y,v,2}
\]

**Proof.** We can write $\vartheta_{Y,v,2}$ as a composition
\[
p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \rightarrow p_{Y^1} \left[\Sigma_{G_m,1} \circ (\Sigma^{O_Y dx_1})^{-1} \right] \circ \left[\Sigma_{G_m,2} \circ (\Sigma^{O_Y dx_2})^{-1} \right] \Sigma^2 S^1 \Sigma^v p^*_Y
\]
\[
\cong p_{Y^1} \Sigma^{-1}_{S^1} \circ \Sigma_{S^1} \Sigma^v p^*_Y \cong p_{Y^1} \Sigma^v p^*_Y
\]
and $\vartheta_{Y,v,1}$ similarly as
\[
p_{Y \times (\mathbb{A}^1 \setminus \{0\})^2}^2 \Sigma^2 S^1 \Sigma^v p^*_{Y \times (\mathbb{A}^1 \setminus \{0\})}^2 \rightarrow p_{Y^1} \left[\Sigma_{G_m,1} (\Sigma^{O_Y dx_2})^{-1} \right] \circ \left[\Sigma_{G_m,2} (\Sigma^{O_Y dx_2})^{-1} \right] \Sigma^2 S^1 \Sigma^v p^*_Y
\]
\[
\cong p_{Y^1} \Sigma^{-1}_{S^1} \circ \Sigma_{S^1} \Sigma^v p^*_Y \cong p_{Y^1} \Sigma^v p^*_Y.
\]

Acting by the respective exchange of factors maps gives a commutative diagram
\[
\begin{array}{ccc}
[\Sigma_{G_m,1} (\Sigma^{O_Y dx_1})^{-1}] \circ [\Sigma_{G_m,2} (\Sigma^{O_Y dx_2})^{-1}] & \rightarrow & \Sigma^{-1}_{S^1} \circ \Sigma_{S^1}^{-1} \\
\tau & \rightarrow & \tau \\
[\Sigma_{G_m,1} (\Sigma^{O_Y dx_2})^{-1}] \circ [\Sigma_{G_m,2} (\Sigma^{O_Y dx_1})^{-1}] & \rightarrow & \Sigma^{-1}_{S^1} \circ \Sigma_{S^1}^{-1}
\end{array}
\]
and transforms \( \partial_{Y,v,2} \) into \( \partial_{Y,v,1} \). As \( \tau : S^1 \land S^1 \to S^1 \land S^1 \) is multiplication by \(-1\), this proves the lemma. \(\square\)

**Remark 2.5.** As a reality check, we take \( Y = \text{Spec} \, k \) and \( v = 0 \), and look at the action of \( \partial_{Y,v,1}^* \) and \( \partial_{Y,v,2}^* \) on \( \text{EM}(K_{\ast}^{MW})_{B,M} \). Letting \( X = \text{Spec} \, k[x_1, x_2] \), we have

\[
\text{EM}(K_{\ast}^{MW})_{B,M}^+(X) = H^0(X, K_{2}^{MW}((\omega_{X/k})))
\]

and we have the following identity in \( H^0(X, K_2^{MW}((\omega_{X/k}))) \),

\[
\partial_{Y,v,2}^*(1) = [x_2][x_1] \otimes_{\mathbb{Z}[G_m]} dx_1 \land dx_2 = (-1)[x_2][x_1] \otimes_{\mathbb{Z}[G_m]} dx_2 \land dx_1 = (-1)[x_1][x_2] \otimes_{\mathbb{Z}[G_m]} dx_2 \land dx_1 = (-1)[x_1][x_2] \otimes_{\mathbb{Z}[G_m]} dx_1
\]

The second equality follows from the Hopkins-Morel presentation of \( K_2^{MW} \), see [18, Lemma 3.7], or, if one prefers, from the fact that the exchange of factors

\[
\mathbb{G}_{m,1} \land \mathbb{G}_{m,2} \to \mathbb{G}_{m,2} \land \mathbb{G}_{m,1}
\]

induces multiplication by \(-1\) in \( GW(k) = \text{End}_{SH(k)}(1_k) \) [19, Remark 6.3.5].

Let \( \text{Def}^0_{i_2} = \text{Def}_{12} \setminus \pi_{12}^{-1}(Y \times 0 \times 0) \), let \( \text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_i \) and let \( \text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_2 \). This gives us the closed immersion

\[
i^0 : \text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_i \to \text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_2
\]

with open complement

\[
\eta^0 : Y \times (\mathbb{A}^1 \setminus \{0\})^2 \to \text{Def}^0_{i_2}
\]

The localization sequence for

\[
\text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_i \to \text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_2 \xrightarrow{i^0} Y \times (\mathbb{A}^1 \setminus \{0\})^2
\]

gives the boundary map

\[
(\partial_1^0, \partial_2^0) : \epsilon_{a+2,b}^{B,M}(Y \times (\mathbb{A}^1 \setminus \{0\})^2, v) \to \epsilon_{a+1,b}^{B,M}(\text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_i, v) \oplus \epsilon_{a+1,b}^{B,M}(\text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_2, v)
\]

We claim that diagram (2.4)

\[
\begin{array}{ccc}
\epsilon_{a,b}^{B,M}(Y, v) & \xrightarrow{(\text{sp}_{i_1}^*, \text{sp}_{i_2}^*)} & \epsilon_{a,b}^{B,M}(Y \times (\mathbb{A}^1 \setminus \{0\})^2, v) \\
\partial_{Y,v,2}^* & & \epsilon_{a,b}^{B,M}(C_i, v) \oplus \epsilon_{a,b}^{B,M}(C_2, v) \\
\epsilon_{a+2,b}^{B,M}(Y \times (\mathbb{A}^1 \setminus \{0\})^2, v) & \xrightarrow{(\partial_1^*, \partial_2^*)} & \epsilon_{a+1,b}^{B,M}(\text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_i, v) \oplus \epsilon_{a+1,b}^{B,M}(\text{Def}^0_{i_2} \circ Y, v, 1 \setminus C_2, v)
\end{array}
\]

commutes.
To see this, let
\[
\partial_{i_1} : p_{C_{i_1}}^\Sigma Y_{C_{i_1}}^\ast \to p_{Y \times (A^1 \setminus \{0\})!} \circ \Sigma S^! Y_{C_{i_1}}^\ast
\]
be the map induced from the boundary map in the localization triangle for
\[
C_{i_1} \hookrightarrow \text{Def}_{i_1} Y \hookleftarrow Y \times (A^1 \setminus \{0\}).
\]
We recall that \(\pi_{12}^{-1}(Y \times A^1 \times (A^1 - \{0\})) \cong \text{Def}_{i_1} Y \times (A^1 \setminus \{0\})\) and this isomorphism restricts to the isomorphism \(\bar{p}_1 : \text{Def}_{i_1}^\Sigma C_{i_1} \to C_{i_1} \times (A^1 \setminus \{0\})\), lying over \(Y \times 0 \times (A^1 \setminus \{0\})\). It then follows directly from the definitions that the diagram
\[
\begin{array}{ccc}
p_{\text{Def}_{i_1}^\Sigma C_{i_1}} : \Sigma S^! Y_{C_{i_1}}^\ast & \xrightarrow{\partial_{C_{i_1}} \circ \bar{p}_1^\ast} & p_{C_{i_1}}^\Sigma Y_{C_{i_1}}^\ast \\
\downarrow \partial_1 & & \downarrow \partial_1 \\
p_{Y \times (A^1 \setminus \{0\})^2} \circ \Sigma S^! Y_{C_{i_1} \times (A^1 \setminus \{0\})^2} & \xrightarrow{\partial_{\text{Def}_{i_1}^\Sigma} \circ [1]} & p_{Y \times (A^1 \setminus \{0\})!} \circ \Sigma S^! Y_{C_{i_1} \times (A^1 \setminus \{0\})^2}
\end{array}
\]
commutes.

Recall that the specialization map \(sp_{i_1,v} : p_{C_{i_1}}^\Sigma Y_{C_{i_1}}^\ast \to p_{Y^!} Y_{C_{i_1}}^\ast\) is defined as
\[
sp_{i_1,v} := \partial_{Y,v} \circ \partial_{i_1}.
\]

Using Lemma 2.4 and the commutativity of (2.5), we have
\[
\partial_{Y,v,2} \circ \partial_1 = -\partial_{Y,v,1} \circ \partial_1
\]
\[
= -\partial_{Y,v} \circ \partial_{p_1^\ast,v[1]} \circ \partial_1
\]
\[
= -\partial_{Y,v} \circ \partial_{C_{i_1}} \circ \partial_{i_1} \circ \bar{p}_1^\ast
\]
\[
= -sp_{i_1,v} \circ (\partial_{C_{i_1}} \circ \bar{p}_1^\ast).
\]

The argument for \(C_{i_2}\) is similar, where just use the definition of \(\partial_{Y,v,2} \circ \partial_{p_2^\ast,v[1]}\), and the identity
\[
\partial_{p_2^\ast,v[1]} \circ \partial_2 = \partial_{i_2} \circ \partial_{C_{i_2}} \circ \bar{p}_2^\ast,
\]
alogous to the commutativity of (2.5), where \(\partial_{i_2}\) is induced by the boundary map in the localization triangle for
\[
C_{i_2} \hookrightarrow \text{Def}_{i_2} Y \hookleftarrow Y \times (A^1 \setminus \{0\}).
\]

This gives
\[
\partial_{Y,v,2} \circ \partial_2 = \partial_{Y,v} \circ \partial_{p_2^\ast,v[1]} \circ \partial_2
\]
\[
= \partial_{Y,v} \circ \partial_{i_2} \circ \partial_{C_{i_2}} \circ \partial_{i_1} \circ \bar{p}_2^\ast
\]
\[
= sp_{i_2,v} \circ (\partial_{C_{i_2}} \circ \bar{p}_2^\ast),
\]
which completes the proof of the commutativity of (2.4).

Now let \(\text{Def}_{i_2}^\Sigma = \text{Def}_{i_2} \setminus (\beta_1(C_{i_1}) \cup \beta_2(C_{i_2}))\) and let \(i_2^* C_{i_1} \times_Y i_2^* C_{i_2}' = i_2^* C_{i_1} \times_Y i_2^* C_{i_2} \setminus (\beta_1(C_{i_1}) \cup \beta_2(C_{i_2}))\). The closed immersion \(i^\circ\) extends to a closed immersion
\[
i^\circ : \text{Def}_{i_2}^\Sigma C_{i_1} \amalg \text{Def}_{i_2}^\Sigma C_{i_2} \amalg i_2^* C_{i_1} \times_Y i_2^* C_{i_2}' \to \text{Def}_{i_2}'
\]
with open complement
\[
\eta^\circ : Y \times (A^1 \setminus \{0\})^2 \to \text{Def}_{i_2}'.
\]
The corresponding localization sequence gives us the boundary map

\[ \mathcal{E}_{a+2,b}^{\text{B.M.}}(Y \times (\mathbb{A}^1 \setminus \{0\})^2, v) \xrightarrow{(\partial_1, \partial_2, \partial_3')} \left[ \begin{array}{c} \mathcal{E}_{a+1,b}^{\text{B.M.}}(\text{Def}_{i_2^{\ast}}C_{i_1} C_{i_2}, v) \\ \oplus \\ \mathcal{E}_{a+1,b}^{\text{B.M.}}(\text{Def}_{i_1^{\ast}}C_{i_1} C_{i_2}, v) \\ \oplus \\ \mathcal{E}_{a+1,b}^{\text{B.M.}}(i_2^{\ast}C_{i_1} \times_Y i_1^{\ast}C_{i_2}', v) \end{array} \right]. \] 

We have the closed immersion

\[ i: \text{Def}_{i_2^{\ast}}C_{i_1} C_{i_1} \cup \text{Def}_{i_1^{\ast}}C_{i_1} C_{i_2} \cup i_2^{\ast}C_{i_1} \times_Y i_1^{\ast}C_{i_2} \to \text{Def}_{12}, \]

with open complement

\[ \eta: Y \times (\mathbb{A}^1 \setminus \{0\})^2 \to \text{Def}_{12} \]

and the closed immersion

\[ \bar{i}: \beta_1(C_{\alpha_1}) \cup \beta_2(C_{\alpha_2}) \to \text{Def}_{i_2^{\ast}}C_{i_1} C_{i_1} \cup \text{Def}_{i_1^{\ast}}C_{i_1} C_{i_2} \cup i_2^{\ast}C_{i_1} \times_Y i_1^{\ast}C_{i_2} \]

with open complement

\[ \text{Def}_{i_2^{\ast}}C_{i_1} C_{i_1} \cup \text{Def}_{i_1^{\ast}}C_{i_1} C_{i_2} \cup i_2^{\ast}C_{i_1} \times_Y i_1^{\ast}C_{i_2} \]

This gives us the boundary maps

\[ \mathcal{E}_{a+1,b}^{\text{B.M.}}(\text{Def}_{i_2^{\ast}}C_{i_1} C_{i_2}, v) \to \mathcal{E}_{a+1,b}^{\text{B.M.}}(\text{Def}_{i_2^{\ast}}C_{i_1} C_{i_2}, v) \]

and

\[ \left[ \begin{array}{c} \mathcal{E}_{a+1,b}^{\text{B.M.}}(\text{Def}_{i_2^{\ast}}C_{i_1} C_{i_2}, v) \\ \oplus \\ \mathcal{E}_{a+1,b}^{\text{B.M.}}(\text{Def}_{i_2^{\ast}}C_{i_2}, v) \\ \oplus \\ \mathcal{E}_{a+1,b}^{\text{B.M.}}(i_2^{\ast}C_{i_1} \times_Y i_1^{\ast}C_{i_2}', v) \end{array} \right] \xrightarrow{\partial} \mathcal{E}_{a,b}^{\text{B.M.}}(\beta_1(C_{\alpha_1}) \cup \beta_2(C_{\alpha_2}), v), \]

fitting into their respective long exact localization sequences.

Now take \( y \in \mathcal{E}_{a+2,b}^{\text{B.M.}}(Y \times (\mathbb{A}^1 \setminus \{0\})^2, v) \). We claim that

\[ (2.6) \quad \tilde{\partial}((\partial_1, \partial_2, \partial_3')(y)) = 0 \in \mathcal{E}_{a,b}^{\text{B.M.}}(\beta_1(C_{\alpha_1}) \cup \beta_2(C_{\alpha_2}), v). \]
Indeed, we have the commutative diagram

\[ \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_1} \cup \text{Def}_{i_1}^\beta C_{i_2} \cup i_{12}^* C_{i_2} \times Y i_{12}^* C_{i_2}, v) \]

\[ \xrightarrow{\partial} \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_1}, v) \]

\[ \xrightarrow{\partial'} \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_1}, v) \]

\[ \mathcal{E}_{a+1,b}(i_{12}^* C_{i_2} \times Y i_{12}^* C_{i_2}', v) \]

with the right-hand column a part of the long exact localization sequence for \( \bar{\eta}, \bar{\eta} \).

Let

\[ \bar{\partial}_1: \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_1} C_{i_2}, v) \rightarrow \mathcal{E}_{a,b}(C_{i_1}, v) \]

be the boundary map in the localization sequence for

\[ C_{i_1} \mapsto \text{Def}_{i_2}^\beta C_{i_1} C_{i_2} \leftarrow \text{Def}_{i_2}^\beta C_{i_1} C_{i_2} \]

and define

\[ \bar{\partial}_2: \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_1}, v) \rightarrow \mathcal{E}_{a,b}(C_{i_2}, v) \]

similarly. Let

\[ \bar{\partial}_3: \mathcal{E}_{a+1,b}(i_{12}^* C_{i_2} \times Y i_{12}^* C_{i_2}', v) \rightarrow \mathcal{E}_{a,b}(\beta_1(C_{i_1}) \cup \beta_2(C_{i_2}), v) \]

be the boundary map for the localization sequence for

\[ \beta_1(C_{i_1}) \cup \beta_2(C_{i_2}) \mapsto i_{12}^* C_{i_1} \times Y i_{12}^* C_{i_2} \leftarrow i_{12}^* C_{i_1} \times Y i_{12}^* C_{i_2}' \]

Let \( \bar{\partial}_3: C_{i_2} \rightarrow \beta_1(C_{i_1}) \cup \beta_2(C_{i_2}) \) be the closed immersion induced by \( \beta_j \). Then

\[ \bar{\partial}_3((\bar{\partial}_1(x_1))) = \bar{\partial}(x_1, 0, 0), \quad \bar{\partial}_3((\bar{\partial}_2(x_2))) = \bar{\partial}(0, x_2, 0), \quad \bar{\partial}_3((x_3)) = \bar{\partial}(0, 0, x_3) \]

for

\[ x_1 \in \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_2} C_{i_1}, v), \quad x_2 \in \mathcal{E}_{a+1,b}(\text{Def}_{i_2}^\beta C_{i_2} C_{i_1}, v), \]

\[ x_3 \in \mathcal{E}_{a+1,b}(i_{12}^* C_{i_1} \times Y i_{12}^* C_{i_2}', v). \]

Take \( y \in \mathcal{E}_{a+1,b}(Y, v) \). Putting this all together with (2.6) and the commutativity of (2.4), and inserting the definition of the various specialization maps, we arrive at the identity

\[ \bar{\beta}_{2*}(\text{sp}_{\alpha_2}^*(\text{sp}_{i_2}^*(y))) = \bar{\beta}_{1*}(\text{sp}_{\alpha_1}^*(\text{sp}_{i_1}^*(y))) = \bar{\partial}_3((y)) \]

in \( \mathcal{E}_{a,b}(\beta_1(C_{i_1}) \cup \beta_2(C_{i_2}), v) \). Pushing forward to \( \mathcal{E}_{a,b}(i_{12}^* C_{i_2} \times Y i_{12}^* C_{i_2}, v) \), and applying the localization sequence for \( \beta_1(C_{i_1}) \cup \beta_2(C_{i_2}) \mapsto i_{12}^* C_{i_1} \times Y i_{12}^* C_{i_2} \) yields the identity

\[ \beta_{1*}(\text{sp}_{\alpha_1}^*(\text{sp}_{i_1}^*(y))) = \beta_{2*}(\text{sp}_{\alpha_2}^*(\text{sp}_{i_2}^*(y))) \]
in $\mathcal{E}_{a,b}^{B,M}(i_2^*C_{i_2} \times_Y i_1^*C_{i_1}, v)$, which completes the proof.

We conclude this section with a computation of the refined Gysin pull-back of a normal cone. Since this involves a change of base-scheme, we reintroduce the $-/B$ to the notation.

Consider a cartesian square

\begin{equation}
\begin{array}{c}
X_0 \rightarrow X \\
i_0 \downarrow \downarrow \\
Y_0 \rightarrow Y \\
\pi_0 \downarrow \downarrow \\
B_0 \rightarrow B
\end{array}
\end{equation}

as above, with $i$ a closed immersion and with $Y \rightarrow B$ in $\text{Sm}/B$. We assume that the bottom square is Tor-independent and that $i_0$ is a regular embedding.

This gives the diagram with all squares cartesian

:\begin{equation}
\begin{array}{c}
t^*C_{X/Y} \rightarrow C_{X/Y} \\
\downarrow \downarrow \\
q \\
\downarrow \\
\phi
\end{array}
\end{equation}:

\begin{equation}
\begin{array}{c}
X_0 \rightarrow X \\
i_0 \downarrow \downarrow \\
Y_0 \rightarrow Y \\
\pi_0 \downarrow \downarrow \\
B_0 \rightarrow B
\end{array}
\end{equation}

\begin{equation}
\begin{array}{c}
t^*C_{X/Y} \rightarrow C_{X/Y} \\
\downarrow \downarrow \\
q \\
\downarrow \\
\phi
\end{array}
\end{equation}

We also have the closed immersion $\beta: C_{X_0/Y_0} \rightarrow t^*C_{X/Y}$; using the outer rectangle and the distinguished triangle

\begin{equation}
\begin{array}{c}
N_i \rightarrow t^*\Omega_{Y/B} \rightarrow L_{Y_0/B} \rightarrow N_i[1]
\end{array}
\end{equation}

gives the refined Gysin pull-back

\begin{equation}
\begin{array}{c}
t^! : \mathcal{E}_{a,b}^{B,M}(C_{X/Y}/B, \Omega_{Y/B}) \rightarrow \mathcal{E}_{a,b}^{B,M}(t^*C_{X/Y}/B, \Omega_{Y/B} - N_i) \cong \mathcal{E}_{a,b}^{B,M}(t^*C_{X/Y}/B, L_{Y_0/B}).
\end{array}
\end{equation}

Similarly, the distinguished triangle

\begin{equation}
\begin{array}{c}
N_i \rightarrow \Omega_{Y_0/B_0} \rightarrow L_{Y_0/B} \rightarrow N_i[1]
\end{array}
\end{equation}

and isomorphism $N_i \cong \pi_0^*N_i$ gives the isomorphism

\begin{equation}
\begin{array}{c}
\mathcal{E}_{a,b}^{B,M}(t^*C_{X/Y}/B, L_{Y_0/B}) \cong \mathcal{E}_{a,b}^{B,M}(t^*C_{X/Y}/B_0, \Omega_{Y_0/B_0}).
\end{array}
\end{equation}

We have the fundamental classes

\begin{equation}
\begin{array}{c}
[C_{X/Y}] \in \mathcal{E}_{a,b}^{B,M}(C_{X/Y}/B, \Omega_{Y/B}), \ [C_{X_0/Y_0}] \in \mathcal{E}_{a,b}^{B,M}(C_{X_0/Y_0}/B_0, \Omega_{Y_0/B_0})
\end{array}
\end{equation}

and the closed immersion $\beta: C_{X_0/Y_0} \rightarrow t^*C_{X/Y} := C_{X/Y} \times_B B_0$, inducing the map

\begin{equation}
\beta_* : \mathcal{E}_{a,b}^{B,M}(C_{X_0/Y_0}/B_0, \Omega_{Y_0/B_0}) \rightarrow \mathcal{E}_{a,b}^{B,M}(t^*C_{X/Y}/B_0, \Omega_{Y_0/B_0}).
\end{equation}
Proposition 2.6. We have the identity
\[ \beta_\ast [C_{X_0/Y_0}] = \iota^! [C_{X/Y}] \]
in \( \mathcal{E}^{B,M}(\iota^* C_{X/Y}/B_0, \Omega_{Y_0/B_0}) \).

Proof. After making the identifications of the various Borel-Moore homology groups, this comes down to a special case of Proposition 2.1. We take \( i_1 := i, i_2 := i_0, \) so \( C_{i_1} = C_{X/Y}, C_{i_2} = C_{Y_0/Y} = N_i \) and \( \alpha_1 = \bar{\gamma} \).

The map \( \alpha_2 \) is the projection \( i_0^* N_i \to N_i \), so letting \( \gamma : C_{X_0/Y_0} \to X_0 \) be the structure morphism, \( C_{i_2} = (i_0 \circ \gamma)^* N_i \) with structure map to \( C_{X_0/Y_0} \). The fiber product \( i_2^2 C_{i_1} \times_Y i_1^* C_{i_2} \) is \( q^* N_i \) and \( \beta_2 : C_{i_2} \to i_2^2 C_{i_1} \times_Y i_1^* C_{i_2} \) is the map \( (i_0 \circ \gamma)^* N_i \to q^* N_i \) induced by \( \beta : C_{X_0/Y_0} \to \iota^* C_{X/Y} \).

Finally, \( C_{i_1} = C_{i^* C_{X/Y}}(C_{X/Y}) \) and the map \( \alpha : C_{i^* C_{X/Y}}(C_{X/Y}) \to q^* N_i \) used to define \( \iota \) is the map \( \beta_1 : C_{i_1} \to i_2^2 C_{i_1} \times_Y i_1^* C_{i_2} \).

Via the Poincaré duality isomorphism
\[ \mathcal{E}^{0,0}(Y) \cong \mathcal{E}^{B,M}(Y/B, \Omega_{Y/B}) \]
the unit \( 1_Y \) gives the fundamental class \([Y] \in \mathcal{E}^{B,M}(Y/B, \Omega_{Y/B}) \). Applying Proposition 2.1 to \([Y]\), we have
\[ \alpha_* (\text{sp}^*_\gamma([C_{X/Y}])) = \beta_2_* (\text{sp}^*_\gamma([Y])) \]
in \( \mathcal{E}^{B,M}(q^* N_i/B, \Omega_{Y/B}) \).

Since \( i \) is a regular embedding, we have \( \text{sp}^*_\gamma([Y]) = [N_i] \) and with \( \bar{\pi} : q^* N_i \to C_{X_0/Y_0}, \pi : N_i \to Y_0 \) the structure morphisms, we have
\[ \text{sp}^*_\gamma([N_i]) = \text{sp}^*_\gamma(\pi^![Y_0]) = \bar{\pi}^! (\text{sp}^*_\gamma([Y_0])) = \bar{\pi}^! ([C_{X_0/Y_0}]). \]

Applying \( 0^0_{q^* N_i} \) to (2.8), we find
\[ \iota^! [C_{X/Y}] = 0^0_{q^* N_i} \alpha_* (\text{sp}^*_\gamma([C_{X/Y}])) \]
\[ = 0^0_{q^* N_i} \beta_2_* (\text{sp}^*_\gamma([C_{X_0/Y_0}])) \]
\[ = \beta_* 0^0_{q^* N_i} \bar{\pi}^! ([C_{X_0/Y_0}]) \]
\[ = \beta_* [C_{X_0/Y_0}]. \]

The first identity is the definition of \( \iota^! \), and the third identity is the compatibility of \( \iota^! \) and \( \ast \) in Tor-independent cartesian squares (Proposition 1.3(1, 3)). \( \square \)

Remark 2.7. Suppose that the top square in (2.7) is also Tor-independent. Then
\[ [C_{X_0/Y_0}] = \iota^! [C_{X/Y}] \]
in \( \mathcal{E}^{B,M}(C_{X_0/Y_0}/B_0, \Omega_{Y_0/B_0}) \). Indeed, this additional Tor-independence implies that the map \( \beta : C_{X_0/Y_0} \to \iota^* C_{X/Y} := C_{X/Y} \times_X X_0 \) is an isomorphism.

Remark 2.8. We may apply Proposition 2.6 in the case \( Y \to B \) is the identity map on \( Y \), giving the identity
\[ \beta_\ast [C_{X_0/Y_0}] = \iota^! [C_{X/Y}] \]
in \( \mathcal{E}^{B,M}(\iota^* C_{X/Y}/Y_0) \). If now we have a base-scheme \( A \), with \( Y_0 \to Y \) a morphism in \( \text{Sm}^G/A \), and take \( \mathcal{E} \in \text{SH}^G(A) \), we then have the identity
\[ \beta_\ast [C_{X_0/Y_0}] = \iota^! [C_{X/Y}] \]
in $E_B^{*}(X/Y/A, \Omega_{Y_0/A})$. Indeed, letting $\pi_{*}c_{X/Y} : \pi_{*}C_{X/Y} \to Y_0$ and $\pi_{Y_0} : Y_0 \to A$ be the structure morphisms, we have

$$E_B^{*}(X/Y/A, \Omega_{Y_0/A}) = [\pi_{Y_0} ! \pi_{*}c_{X/Y} ! (\Sigma \Omega_{Y_0/A} 1 \cdot c_{X/Y}), \mathcal{E}]_{(A)}$$

$$= [\pi_{Y_0} ! \Sigma \Omega_{Y_0/A} \pi_{*}c_{X/Y} ! (\Sigma \Omega_{Y_0/A} 1 \cdot c_{X/Y}), \mathcal{E}]_{(A)}$$

$$= [\pi_{*}c_{X/Y} ! (1 \cdot c_{X/Y}), \pi_{Y_0} ! \mathcal{E}]_{\mathcal{E}[Y_0]}$$

$$= E_B^{*}(X*Y/ Y_0).$$

3. EQUIVARIANT (CO)HOMOLOGY VIA THE ALGEBRAIC BOREL CONSTRUCTION

For $G \to B$ tame, $\mathcal{E} \in \mathcal{SH}^{G}(B), X \in \text{Sch}^{G}/B$, we have been using the $\mathcal{E}$-valued Borel-Moore homology defined by

$$E_B^{*}(X/BG, v) := \text{Hom}_{\mathcal{SH}^{G}(B)}(\Sigma_{a,b} p_{X!}(\Sigma^a \mathcal{E}), \mathcal{E}).$$

For $\mathcal{E} \in \mathcal{SH}(B)$, we also have the equivariant Borel-Moore homology defined by the algebraic version of the Borel construction. We have discussed this construction in [15] §4, following the construction in [17] §1, yielding a theory $E_B^{*}(X/BG, v)$; we use the same construction here, which we briefly recall.

We assume that $G$ is a closed subgroup scheme of $\text{GL}_n / B$ for some $n$ and that $G$ is smooth over $B$; we no longer need to assume that $G$ is tame. Let $E_j \text{GL}_n$ be the $B$-scheme of $n \times n + j$ matrices of rank $n$, with the action of $\text{GL}_n$ by left multiplication.

We let $E_j G$ denote the scheme $E_j \text{GL}_n$ with the $G$-action by restriction, and let $B_j G = G \backslash E_j G$; note that this quotient exists as a quasi-projective $B$-scheme and the quotient map $E_j G \to B_j G$ is a $G$-torsor for the étale topology. Let $F = \mathbb{A}_B^1$ with the left action of $\text{GL}_n$ via the usual matrix multiplication; we consider $F$ as a $G$-representation via the inclusion $G \hookrightarrow \text{GL}_n$.

We have the $G$-equivariant closed immersion $i_j : E_j G \to E_{j+1} G$ defined by adding a $j + 1$ column of zeroes, and the $G$-equivariant open immersion $\eta_j : E_j G \times F \to E_{j+1} G$ by considering $F$ as the $j + 1$ column. Let $p_j : E_j G \times F \to E_j G$ be the projection with 0-section $i_0^j$.

Let $B_j G = G \backslash E_j G, N_j G = G \backslash (E_j G \times F)$. We have the corresponding regular embeddings $i_{G,j} : B_j G \to B_{j+1} G, i_0^0 G,j : B_j G \to N_j G$, open immersion $\eta_{G,j} : N_j G \to B_{j+1} G \times F$ and projection $p_{G,j} : N_j G \to B_j G$, making $N_j G$ a vector bundle over $B_j G$ with zero-section $i_0^0 G,j$; isomorphic to the normal bundle of $i_{G,j}$. This gives us the Ind-objects $E_G := \{E_G\}_j, B_G := \{B_G\}_j$ in $\text{Sm}^{G}/B$, with transition maps $i_j, i_{G,j}$ all regular immersions, and the morphism of ind-objects $p_G : E_G \to B_G$, making $E_G$ an ind-$G$-torsor over $B_G$.

We will work in the full subcategory $\text{Sch}^{G}_B$ with objects the $X \in \text{Sch}^{G}/B$ for which the fppf quotients $X \times_G E_j G := G \backslash X \times B E_j G$ exist in $\text{Sch}/B$; this the whole of $\text{Sch}^{G}/B$ if $B = \text{Spec} k, k$ a field, (see [3] Lemma 9, Proposition 23).

Take $X \in \text{Sch}^{G}_B$. We have the maps $i_{X,j} : X \times_B E_j G \to X \times_B E_{j+1} G, i_0^j : X \times_B E_j G \to X \times_B E_j G \times F, \eta_{X,j} : X \times_B E_j G \times F \to X \times_B E_{j+1} G, p_{X,j} : X \times_B E_j G \times F \to X \times_B E_j G$, with $i_{X,j} = 1d \times_i i_{j}$, etc. These yield the corresponding maps on the quotients, with $G$ acting diagonally, $i_{X,G,j} : X \times G E_j G \to X \times G E_{j+1} G, i_0^0 X,G,j, \eta_{X,G,j} \text{ and } p_{X,G,j}$. The projections $\pi_{X,j} : X \times_B E_j G \to E_j G$ induce the projections $\pi_{X,G,j} : X \times G E_j G \to B_j G$. We denote the ind-object $\{X \times G E_j G, i_{X,G,j}\}_j$ by $X \times G E_G$. The projection $\pi_X : X \to B$ induces maps $\pi_{X,G,j} : X \times G E_j G \to B_j G$, and the corresponding map of ind-objects $\pi_{X,G} : X \times G E_G \to B_G$. 
We have the cartesian, Tor-independent diagram

\[
\begin{array}{c}
X \times G E_j G \\ \downarrow \pi_{X,G,j} \\
B_j G \\
\end{array} \quad \begin{array}{c}
\downarrow \pi_{X,G,j+1} \\
B_{j+1} G \\
\end{array}
\]

Given \( w \in \mathcal{K}^G(X) \), we have the corresponding perfect complex \( w_j \) on \( X \times G E_j \) defined by applying \( G \)-descent to the \( G \)-linearized perfect complex \( p_1^*(w) \) on \( X \times_B E_j G \). This gives us the map of groupoids

\[
(-)_j : \mathcal{K}^G(X) \to \mathcal{K}(X \times G E_j G)
\]

with canonical natural isomorphisms \( w_j \cong i_{X,G,j}^* \). Given an \( \mathcal{E} \in \text{SH}(B) \), the regular immersion \( i_{X,G,j} \) thus gives us the natural map

\[
i_{X,G,j}^! : \mathcal{E}_{**}^{BM}(X \times G E_{j+1} G / B, w_{j+1}) \to \mathcal{E}_{**}^{BM}(X \times G E_j G / B, w_j - N_{i_{G,j}}).
\]

We have the exact sequence of sheaves on \( B_j G \)

\[
0 \to N_{i_{G,j}} \to i_{G,j}^* \Omega_{B_{j+1} G / B} \to \Omega_{B_j G / B} \to 0.
\]

Using this, we may twist (3.2) to give

\[
\mathcal{E}_{**}^{BM}(X \times G E_{j+1} G / B, w_{j+1} + \Omega_{B_{j+1} G}) \xrightarrow{i_{X,G,j}^!} \mathcal{E}_{**}^{BM}(X \times G E_j G / B, w_j + \Omega_{B_j G}).
\]

Lemma 3.1. 1. The projection \( p_{B_j G} : B_j G \to B \) gives rise to a canonical natural isomorphism

\[
p_{B_j G}^! : \mathcal{E}_{**}^{BM}(X \times G E_j G / B, w_j + \Omega_{B_j G}) \to \mathcal{E}_{**}^{BM}(X \times G E_j G / B, w_j).
\]

2. We have the relative pull-back maps of Remark 1.3 with respect to the diagram (3.1),

\[
i_{G,j}^! : \mathcal{E}_{**}^{BM}(X \times G E_{j+1} G / B_{j+1} G, w_{j+1}) \to \mathcal{E}_{**}^{BM}(X \times G E_j G / B_j G, w_j),
\]

giving the commutative diagram

\[
\begin{array}{c}
\mathcal{E}_{**}^{BM}(X \times G E_{j+1} G / B, w_{j+1} + \Omega_{B_{j+1} G}) \\
\downarrow \phi_{B_{j+1} G} \\
\mathcal{E}_{**}^{BM}(X \times G E_j G / B_{j+1} G, w_{j+1}) \\
\downarrow \phi_{B_j G} \\
\end{array} \quad \begin{array}{c}
\xrightarrow{i_{X,G,j}^!} \\
\xrightarrow{i_{G,j}^!} \\
\end{array}
\]

\[
\mathcal{E}_{**}^{BM}(X \times G E_j G / B, w_j + \Omega_{B_j G}) \xrightarrow{i_{X,G,j}^!} \mathcal{E}_{**}^{BM}(X \times G E_j G / B, w_j).
\]

Proof. For \( Y \in \text{Sch}/B \), we let \( p_Y : Y \to B \) denote the structure morphism.

(1) It suffices to consider the case \( \ast, \ast = 0, 0 \). We recall [15, Lemma 4.1] that \( B_j G \) is smooth over \( B \). We have isomorphisms (the third is the purity isomorphism)

\[
p_{X \times G E_j G}(\Sigma^{w_j, \Omega_{B_j G}} 1_{X \times G E_j G}) \cong p_{B_j G} \circ \pi_{X,G,j}^!(\Sigma^{w_j, \Omega_{B_j G}} 1_{X \times G E_j G})
\]

\[
\cong p_{B_j G} \Sigma^{\Omega_{B_j G}} \circ \pi_{X,G,j}^!(\Sigma^{w_j} 1_{X \times G E_j G})
\]

\[
\cong p_{B_j G} \# \circ \pi_{X,G,j}^!(\Sigma^{w_j} 1_{X \times G E_j G})
\]

\[
p_{X \times G E_j G}(\Sigma^{w_j, \Omega_{B_j G}} 1_{X \times G E_j G}) \cong p_{B_j G} \circ \pi_{X,G,j}^!(\Sigma^{w_j, \Omega_{B_j G}} 1_{X \times G E_j G})
\]

\[
\cong p_{B_j G} \circ \pi_{X,G,j}^!(\Sigma^{w_j} 1_{X \times G E_j G})
\]

\[
\cong p_{B_j G} \# \circ \pi_{X,G,j}^!(\Sigma^{w_j} 1_{X \times G E_j G})
\]
inducing the isomorphisms
\[ E^{B,M}(X \times^G E_j G/B, w_j + \Omega B_j G) = \text{Hom}_{SH(B)}(p_{X \times^G E_j G}(\Sigma^{w_j + \Omega B_j G}1_{X \times^G E_j G}), E) \]
\[ \cong \text{Hom}_{SH(B)}(p_{B_j G} \circ \pi_{X,G,j}(\Sigma^{w_j}1_{X \times^G E_j G}), E) \]
\[ \cong \text{Hom}_{SH(B,G)}(\pi_{X,G,j}(\Sigma^{w_j}1_{X \times^G E_j G}), \pi^*_{B_j G}E) \]
\[ = E^{B,M}(X \times^G E_j G/B_j G, w_j). \]

We define \( p_{B_j G} \) to be the composition of these isomorphisms.

(2) follows from the definition of the relative pull-back in Remark 1.5 \( \square \)

This gives us the pro-system \( \{ E^{B,M}_{a,b}(X \times^G E_j G/B_j G, w_j), i_{X,G,j}^! \} \). Similarly, using the maps \( i_{X,G,j}^*: E^{a,b}(X \times^G E_{j+1} G, w_{j+1}) \to E^{a,b}(X \times^G E_j G, w_j) \) defines the pro-system \( \{ E^{a,b}(X \times^G E_j G, w_j) \} \). This justifies the following definition.

**Definition 3.2.** 1. For \( X \in \text{Sch}^G/B \) and \( w \in K^G(X) \), define the \( w \)-twisted \( G \)-equivariant Borel-Moore homology by
\[ E_{a,b}^{B,M}(X/BG, w) := \lim_{j} E_{a,b}^{B,M}(X \times^G E_j G/B_j G, w_j), \]
and define the \( w \)-twisted \( G \)-equivariant cohomology as
\[ E^{a,b}_G(X, w) := \lim_{j} E^{a,b}(X \times^G E_j G, w_j). \]

2. Given a motivic ring spectrum \( E \in \text{SH}(B) \), we call \( E \) bounded if, for each \( w \in K^G(X) \) and each \( (a,b) \), the pro-system \( \{ E_{a,b}^{B,M}(X \times^G E_j G/B_j G, w_j) \} \) is eventually constant.

**Remark 3.3.** Taking \( B = \text{Spec } k \), if \( E = \text{EM}(M_\ast) \) for \( M_\ast \) a homotopy module \([19, \text{Definition 5.2.4}] \), the description of \( \text{EM}(M_\ast)^{B,M}(-,-) \) using the Rost-Schmid complex implies that \( \text{EM}(M_\ast) \) is bounded. See \([7, \text{or } 15, \text{§4}] \) for details of the proof.

**Remark 3.4.** Morel-Voevodsky \([20, \text{§4.3}] \) give a general version of the construction outlined above. Their construction involves some choices, one of which is the one we use above, and it is shown in loc. cit. that the resulting ind-object \( BG \), considered as a presheaf on \( \text{Sm}/B \), gives an object in the unstable motivic homotopy category \( \mathcal{H}(B) \) that is independent of choices. The same arguments, as used in \([8, \text{show that the ind-object } X \times^G EG \text{ yields a well-defined object of } \mathcal{H}(B), \text{independent of choices.} \]

**Remark 3.5.** As we have discussed in \([15, \text{Remark 4.11}] \), the operations of pull-back for lci morphisms and proper push-forward for Borel-Moore homology, and the usual pull-back for cohomology, as well as the corresponding identities relating these operations for twisted \( E \)-Borel-Moore homology and twisted \( E \)-cohomology described in the previous sections, extend to the equivariant setting defined using the algebraic Borel construction described above; essentially, one just applies the functor \( X \mapsto X \times^G E_j G \) for \( j \gg 0 \). For the extension to the equivariant setting of the operations \( f^! \), \( f_* \), products in cohomology and cap product action of cohomology on Borel-Moore homology, one needs to show that these are compatible with respect to change in \( j \); this is \([15, \text{Proposition 4.10}] \). The case of the external products in Borel-Moore homology follows from Lemma \([1.6] \).
Similarly, we have available Vistoli’s lemma Proposition 2.1 and all of its consequences as detailed in §2 in the setting of equivariant theories defined using the algebraic Borel construction.

The exact localization sequence for a closed immersion \( i : Z \to X \) with open complement \( X \setminus U \to X \) gives a corresponding exact localization sequence for \( Z \times^G E_j G \to X \times^G E_j G \) and its open complement \( (X \setminus Z) \times^G E_j G \to X \times^G E_j G \). In case \( E \) is bounded, these yield an exact localization sequence on the limit.

However, if the given theory is not bounded, the possible lack of an exact localization sequence is a serious drawback. This has been rectified in the work of D’Angelo [5], which replaces the limit of \( E \)-valued Borel-Moore homology with the \( E \)-valued cohomology of a “Borel Borel-Moore motive” formed by taking an appropriate colimit of the objects \( \pi_{X,G,j}(\Sigma^w 1_{X \times^G E_j G}) \). This preserves all the exactness properties one would like to have, and agrees with the naive theory presented here in the case of a bounded theory.

Remark 3.6. The sheaf of relative differentials \( \Omega_{E_j G/B} \) has a canonical \( G \)-linearization, defining the locally free sheaf \( \Omega^G_{E_j G} \) on \( B_j G \) by descent. We have the exact sequence on \( B_j G \)

\[
0 \to N_{i_{G,j}} \to i_{G,j}^* \Omega^G_{E_{j+1} G} \to \Omega^G_{E_j G} \to 0,
\]

giving us the canonical isomorphism

\[
\mathcal{E}^{B,M}_{\ast}(X \times^G E_j G/B, w_j + \Omega^G_{E_{j+1} G} - N_{i_{G,j}}) \cong \mathcal{E}^{B,M}_{\ast}(X \times^G E_j G/B, w_j + \Omega^G_{E_j G}),
\]

and thus giving the map

\[
\mathcal{E}^{B,M}_{\ast}(X \times^G E_{j+1} G/B, w_{j+1} + \Omega^G_{E_{j+1} G}) \xrightarrow{i_{X,G,j}} \mathcal{E}^{B,M}_{\ast}(X \times^G E_j G/B, w_j + \Omega^G_{E_j G})
\]

as above. The \( G \)-equivariant Borel-Moore homology \( \mathcal{E}^{B,M}_{G,\ast}(X/B, w) \) is defined in [7 §1] as

\[
\mathcal{E}^{B,M}_{G,a,b}(X/B, w) = \lim_j \mathcal{E}^{B,M}_{a,b}(X \times^G E_j G/B, w_j + \Omega^G_{E_j G/B}).
\]

Also, we have the exact sequence of \( G \)-linearized sheaves on \( E_j G \)

\[
0 \to \pi_*^e \Omega_{B_j G/B} \to \Omega^G_{E_j G/B} \to g^\vee \otimes \mathcal{O}_B \mathcal{O}_{E_j G} \to 0,
\]

where \( g^\vee \) is the co-Lie algebra of \( G \) over \( B \), with the co-adjoint action. Letting \( g^\vee \) be the sheaf on \( B_j G \) defined by \( g^\vee \otimes_B \mathcal{O}_{E_j G} \) via descent, this gives us the exact sequence on \( B_j G \)

\[
0 \to \Omega_{B_j G/B} \to \Omega^G_{E_j G/B} \to g^\vee \to 0,
\]

from which we have the natural isomorphism

\[
\mathcal{E}^{B,M}_{a,b}(X \times^G E_j G/B, w_j + \Omega^G_{E_j G/B}) \cong \mathcal{E}^{B,M}_{a,b}(X \times^G E_j G/B, w_j + \Omega_{B_j G/B} + g^\vee),
\]

compatible with the Gysin maps \( i_{X,G,j}^* \). This defines the natural isomorphism

\[
\mathcal{E}^{B,M}_{G,a,b}(X/B, w) \cong \mathcal{E}^{B,M}_{G,a,b}(X/BG, w + g^\vee \otimes \mathcal{O}_B \mathcal{O}_X).
\]

In other words, the two theories \( \mathcal{E}^{B,M}_{G,-}(B/-, -) \) and \( \mathcal{E}^{B,M}_{G,-}(B/-BG, -) \) differ only by the twist by \( g^\vee \).

Finally, if \( G \) is a subgroup of \( \text{SL}_n \subset \text{GL}_n \), then the \( G \)-representation \( \text{det} g^\vee \) is trivial. Thus, if \( E \) is \( \text{SL} \)-oriented, we have

\[
\mathcal{E}^{B,M}_{G,a,b}(X/B, w) \cong \mathcal{E}^{B,M}_{G,a,b}(X/BG, w + g^\vee \otimes \mathcal{O}_B \mathcal{O}_X) \cong \mathcal{E}^{B,M}_{G,a+2d,b+d}(X/BG, w),
\]
where \( d = \dim_B G \).

**Proposition 3.7** (Equivariant refined Gysin pull-back). Let

\[
\begin{array}{ccc}
X_0 & \xrightarrow{i'} & X \\
q & & p \\
Y_0 & \xrightarrow{i} & Y
\end{array}
\]

be a cartesian square in \( \mathbf{Sch}^G / B \), with \( i \) a regular immersion, and take \( v \in K^G(Y) \).

Then for each \( j \), the map

\[
\iota \times^G \text{Id}_{E^G_j} : Y_0 \times^G E^G_j \rightarrow Y \times^G E^G_j
\]

is a regular immersion with conormal sheaf \( N_{i \times \text{Id}_{E^G_j}} \cong (N_i)_j \), the diagram

\[
\begin{array}{ccc}
X_0 \times^G E^G_j \times_{\text{Id}_{E^G_j}} p & \xrightarrow{q \times \text{Id}} & X \times^G E^G_j \\
q \times \text{Id} & & p \times \text{Id} \\
Y_0 \times^G E^G_j \times_{\text{Id}_{E^G_j}} p & \xrightarrow{q \times \text{Id}} & Y \times^G E^G_j
\end{array}
\]

is cartesian and the sequence of refined Gysin maps

\[
\varepsilon^{B,M}_{\ast \ast}(X \times^G E^G_j / B^j, v_j) \xrightarrow{(t \times^G \text{Id}_{E^G_j})} \varepsilon^{B,M}_{\ast \ast}(X_0 \times^G E^G_j / B^j, v_j - (N_i)_j)
\]

gives rise to a well-defined map of pro-systems

\[
\{\varepsilon^{B,M}_{\ast \ast}(X \times^G E^G_j / B^j, v_j)\}_j \rightarrow \{\varepsilon^{B,M}_{\ast \ast}(X_0 \times^G E^G_j / B^j, v_j - (N_i)_j)\}_j.
\]

This in turn gives a well-defined map

\[
\iota^! : \varepsilon^{B,M}_{G, \ast \ast}(X / BG, v) \rightarrow \varepsilon^{B,M}_{G, \ast \ast}(X_0 / BG, v - N_i).
\]

**Proof.** The first two assertions are easily verified and are left to the reader. To show that the maps \((t \times^G \text{Id}_{E^G_j})\)\(^{!}\) give rise to a well-defined map of pro-systems, we need only that that

\[
(t \times^G \text{Id}_{E^G_j})^{!} \circ i_{X,G,j}^{!} = i_{X,G,j}^{!} \circ (t \times^G \text{Id}_{E^G_{j+1}})^{!}
\]

as maps

\[
\varepsilon^{B,M}_{\ast \ast}(X \times^G E^G_{j+1} / B, v_{j+1} + \Omega_{B^j+1}) \rightarrow \varepsilon^{B,M}_{\ast \ast}(X_0 \times^G E^G_j / B, v_j + \Omega_{B^j} - (N_i)_j).
\]

For this, consider the commutative diagram

\[
\begin{array}{ccc}
X_0 \times^G E^G_j & \xrightarrow{i' \times \text{Id}} & X \times^G E^G_j & \xrightarrow{\pi_{X,G,j}} & B^j \\
\downarrow{i_{X,G,j}} & & \downarrow{i_{X,G,j}} & & \downarrow{i_{G,j}} \\
X_0 \times^G E^G_{j+1} & \xrightarrow{i' \times \text{Id}} & X \times^G E^G_{j+1} & \xrightarrow{\pi_{X,G,j+1}} & B_{j+1} \\
\downarrow{q \times \text{Id}} & & \downarrow{p \times \text{Id}} & & \\
Y_0 \times^G E^G_{j+1} & \xrightarrow{i \times \text{Id}} & Y \times^G E^G_{j+1}.
\end{array}
\]

By Corollary 2.3 we have the identity of refined Gysin maps

\[
(t \times^G \text{Id})^{!} \circ i_{G,j}^{!} = i_{G,j}^{!} \circ (t \times^G \text{Id})^{!}.
\]
But the top right-hand square and top rectangle are cartesian and Tor-independent, so the excess intersection formula of Proposition 1.3 rewrites this as the desired identity
\[(i \times G \text{Id})^! \circ i_{X,G,j}^! = i_{X_0,G,j}^! \circ (i \times G \text{Id})^! \].
\[\Box\]

Remark 3.8. As above, once we have the well-defined refined Gysin map on equivariant Borel-Moore homology, all the relations satisfied by the refined Gysin maps for Borel-Moore homology, as in Proposition 1.3 and Corollary 2.3, are also satisfied in the equivariant setting.

4. Virtual fundamental classes

In [16] we showed how, given a motivic commutative ring spectrum $E \in \text{SH}^G(B)$, a $G$-linearized perfect obstruction theory $\phi_\bullet : E_\bullet \to L_{Z/B}$ on some $Z \in \text{Sch}^G/B$ gives rise to a virtual fundamental class in twisted Borel-Moore homology, $\left[ Z, \phi_\bullet \right]_{\text{vir}} \in E_\text{B}^G(Z/B, G, E_\bullet)$.

In the next section, we define a virtual class $\left[ Z, \phi_\bullet \right]_{\text{vir},G}$ in the $G$-equivariant (Tate, Edidin-Graham) Borel-Moore homology $E_\text{B}^G(Z/BG, E_\bullet)$ for a motivic commutative ring spectrum $E \in \text{SH}(B)$. In this section, we use the description of the Behrend-Fantechi virtual fundamental class given by Graber-Pandharipande in [10] to simplify our earlier construction of $\left[ Z, \phi_\bullet \right]_{\text{vir}}$, as an aid to defining $\left[ Z, \phi_\bullet \right]_{\text{vir},G}$. We fix a tame group-scheme $G$ over $B$, and we take $B$ to be affine.

Remark 4.1. In the next section, we will use the Graber-Pandharipande construction to define an equivariant virtual fundamental class using the algebraic Borel construction of §3. For this, we only need the results of this section for the trivial group, but as the added generality of a general tame $G$ does not require any essential difference in the construction or arguments, we will work in this more general setting, with an eye to other possible future applications.

As above, we let $D^\text{perf}_G(X)$ denote the derived category of $G$-linearized perfect complexes on $X$, and $D_G(X)$ the derived category of complexes of $G$-linearized quasi-coherent sheaves on $X$. For $X \in \text{Sch}^G/B$, the cotangent complex $L_{X/B}$ has a canonical $G$-linearization, so defines an object $L_{X/B}$ in $D_G(X)$.

Definition 4.2. A $G$-linearized perfect obstruction theory on $X$ relative to $B$ is a $G$-linearized perfect complex $E_\bullet$ and a map $\phi : E_\bullet \to L_{X/B}$ in $D_G(X)$ that defines a perfect obstruction theory after forgetting the $G$-linearizations. In other words, $E_\bullet$ is supported in (homological) degrees $[0,1]$, $h_0(\phi)$ is an isomorphism and $h_1(\phi)$ is a surjection.

As defined in [16], the virtual fundamental class associated to a perfect obstruction theory only depends on the truncation $\phi : E_\bullet \to \tau_{\leq 1} L_{X/B}$ (we use the homological convention for the truncation); in what follows, we will abuse notation and refer to a map $\phi : E_\bullet \to \tau_{\leq 1} L_{X/B}$ in $D_G(X)$, with $E_\bullet \in D^\text{perf}_G(X)$ supported in $[0,1]$, and with $h_0(\phi)$ an isomorphism and $h_1(\phi)$ a surjection, as a $G$-linearized perfect obstruction theory relative to $B$. Throughout this section, all the obstruction theories will be relative to the fixed base-scheme $B$, so we will drop the phrase “relative to $B$".
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Given a $G$-linearized perfect obstruction theory $\phi: E_\bullet \to (\tau_{\leq 1}L_{X/B})$ on some $X \in \text{Sch}^G/B$, as $X$ is quasi-projective over $B$, $G$ is tame and $B$ is affine, $\phi$ has a representatives $(E_1 \to E_0) \xrightarrow{(\phi_1, \phi_0)} (\tau_{\leq 1}L_{X/B})$ with $(E_1 \to E_0)$ a $G$-linearized two-term complex of locally free coherent sheaves on $X$.

We first recall the construction of Graber-Pandharipande. For a $G$-linearized coherent sheaf $F$ on a scheme $X$, we let $C(F) \to X$ denote the associated abelian cone $\text{Spec}_X \text{Sym}^*F$; in case $F$ is locally free, this is just the $G$-linearized vector bundle $\text{Sym}^*F \to X$.

Let $\phi_\bullet$ be a $G$-linearized perfect obstruction theory on $Z$. We assume that $Z$ admits a closed immersion $i: Z \to M$ in $\text{Sch}^G/B$, with $M$ smooth over $B$. Thus $\phi_\bullet$ admits a representative as a map of $G$-linearized complexes $\phi_\bullet: (F_1 \xrightarrow{\partial} F_0) \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} \Omega_{M/B})$ with $F_1$ $G$-linearized locally free coherent sheaves on $Z$. The assumption that $\phi_\bullet$ defines a perfect obstruction theory is equivalent to the exactness of the sequence $F_1 \to F_0 \oplus \mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{\gamma} i^*\Omega_{M/B} \to 0$.

Let $Q = \ker(\gamma)$ with surjection $\pi: F_1 \to Q$. Then we have the exact sequence of abelian cones $0 \to i^*T_{M/B} \to C(\mathcal{I}_Z/\mathcal{I}_Z^2) \times_Z \mathbb{V}(F_0) \to C(Q) \to 0$, and the surjection $\pi$ gives rise to a closed immersion $i_\pi: C(Q) \hookrightarrow \mathbb{V}(F_1)$.

The surjection $\text{Sym}^*\mathcal{I}_Z/\mathcal{I}_Z^2 \to \oplus_{n \geq 0} \mathcal{I}_Z^n/\mathcal{I}_Z^{n+1}$ gives the closed immersion $C_{Z/M} \hookrightarrow C(\mathcal{I}_Z/\mathcal{I}_Z^2)$, inducing the closed immersion $C_{Z/M} \times_Z \mathbb{V}(F_0) \to C(\mathcal{I}_Z/\mathcal{I}_Z^2) \times_Z \mathbb{V}(F_0)$ with image containing the sub-vector bundle $i^*T_{M/B}$. Let $D := C_{Z/M} \times_Z \mathbb{V}(F_0)$, giving the quotient cone $D^{\text{vir}} := i^*T_{M/B}\backslash D$, which is naturally a closed sub-cone of $C(Q)$. The closed immersion $i_\pi$ thus induces a closed immersion $i_{F_\bullet}: D^{\text{vir}} \hookrightarrow \mathbb{V}(F_1)$.

We have the projections $\pi_1: D \to C_{Z/M}, \quad \pi_2: D \to D^{\text{vir}}$, respectively, and are both maps over $Z$.

The smooth pull-back maps $\pi_1^*: \mathcal{E}_{a,b}^{B,M}(C_{Z/M}/B, v) \to \mathcal{E}_{a,b}^{B,M}(D/B, v + F_0)$ and $\pi_2^*: \mathcal{E}_{a,b}^{B,M}(D^{\text{vir}}/B, v) \to \mathcal{E}_{a,b}^{B,M}(D/B, v + i^*\Omega_{M/B})$ are both isomorphisms, by $A^1$-homotopy invariance.

We have defined in [16] Definition 4.1] the fundamental class $[C_{Z/M}] \in \mathcal{E}_{a,b}^{B,M}(C_{Z/M}/B, i^*\Omega_{M/B})$. Using the isomorphisms $\pi_1^*, \pi_2^*$, the fundamental class $[C_{Z/M}] \in \mathcal{E}_{a,b}^{B,M}(C_{Z/M}/B, i^*\Omega_{M/B})$ gives rise to the class $[D^{\text{vir}}] := (\pi_2^*)^{-1}(\pi_1^*[C_{Z/M}]) \in \mathcal{E}_{a,b}^{B,M}(D^{\text{vir}}/B, F_0)$. 


Definition 4.3. Let $E$ be in $\text{SH}^G(B)$ and let $\phi_* : F_* \to (\mathcal{I}_Z/T_Z^2 \xrightarrow{d} i^*\Omega_{M/B})$ be a representative of a $G$-linearized perfect obstruction theory on a $(G$-equivariant) embedded $B$-scheme $i : Z \to M$ with $M$ smooth over $B$. Let $0_{\nu(F_1)} : Z \to \nu(F_1)$ be the zero-section. The $E$-valued Graber-Pandharipande virtual fundamental class is defined as

$$ [Z, \phi, i]_{E, GP} := 0_{\nu(F_1)}(i_*[D^{\text{vir}}]) \in E^B(M)(Z/B, F_*) $$

Remark 4.4. Taking $E = H\mathbb{Z}$, the motivic spectrum representing motivic cohomology, $B = \text{Spec} k$ for some perfect field $k$, and $G$ the trivial group, we have the natural isomorphism $H\mathbb{Z}^B(M)(Z/B, F_*) \cong \text{CH}_r(Z)$, where $r$ is the virtual rank of $F_*$, and the Graber-Pandharipande class $[Z, \phi, i]_{H\mathbb{Z}, GP} \in \text{CH}_r(Z)$ is equal to the virtual fundamental class as defined by Behrend-Fantechi [2].

We have the virtual fundamental class $[Z, [\phi]]^{\text{vir}}_E \in E^B(M)(Z/B, F_*)$, as defined in [10]. Note that $[Z, [\phi]]^{\text{vir}}_E$ depends only on the choice of perfect obstruction theory $[\phi] : E_* \to \tau_{\leq 1}L_{Z/B}$ in $D_G(Z)$.

Proposition 4.5. Let $E$, $i : Z \to M$ and $\phi$ be as in Definition 4.3. Then

$$ [Z, [\phi]]^{\text{vir}}_E = [Z, \phi, i]^{\text{vir}}_{E, GP} \in E^B(M)(Z/B, F_*) $$

Proof. We prove the result by tracing through the step-by-step construction of $[Z, [\phi]]^{\text{vir}}_E$. This consists of starting with a representative $\phi_* : F_* \to (\mathcal{I}_Z/T_Z^2 \to i^*\Omega_{M/B})$ of $[\phi]$, and modifying $\phi_*$ and $Z$ step-by-step, until we arrive at a particularly simple case. At each stage, there is a canonical isomorphism of the corresponding Borel-Moore homology groups and at the final stage, we have an explicit construction of a class. The class $[Z, [\phi]]^{\text{vir}}_E$ is then defined in this explicit class through the sequence of canonical isomorphisms of Borel-Moore homology groups. The Graber-Pandharipande class, however, is directly defined at each stage of the process.

Here we will go through this procedure in reverse order, starting with the special situation needed for the explicit construction of $[Z, [\phi]]^{\text{vir}}_E$, and going back step-by-step until we reach the general situation. We will identify the explicit construction of $[Z, [\phi]]^{\text{vir}}_E$ with the Graber-Pandharipande construction at the initial stage of our reverse-order process, and show that the classes one constructs by the Graber-Pandharipande method are compatible with the canonical isomorphisms at each stage in the reverse process, which will prove the result. Now for the details.

First suppose that the obstruction theory $\phi_* : F_* \to (\mathcal{I}_Z/T_Z^2 \to i^*\Omega_{M/B})$ is normalized, meaning that $\phi_* : F_1 \to \mathcal{I}_Z/T_Z^2$ is surjective, and is reduced, meaning $F_0 = i^*\Omega_{M/B}$ and $\phi_0 = \text{Id}$. In this case we have the canonical isomorphism

$$ D^{\text{vir}} = C_{Z/M} \times Z V(F_0)/i^*T_{M/B} = C_{Z/M} \times Z i^*T_{M/B}/i^*T_{M/B} = C_{Z/M}, $$

and via this isomorphism we have

$$ [D^{\text{vir}}] = [C_{Z/M}] \in E^B(M)(D^{\text{vir}}/B, F_0) = E^B(M)(C_{Z/M}/B, i^*\Omega_{M/B}). $$

Also in this case, the surjection $\phi_1$ gives the closed immersion $i : C_{Z/M} \to V(F_1)$, which gets identified with the closed immersion $i_{F_*} : D^{\text{vir}} \to V(F_1)$. If in addition we assume that $Z$ and $M$ are affine, then $[Z, [\phi]]^{\text{vir}}_E$ is defined by

$$ [Z, [\phi]]^{\text{vir}}_E := 0_{\nu(F_1)}(i_*[C_{Z/M}]) \in E^B(M)(Z/B, F_*), $$

where $0_{\nu(F_1)} : Z \to \nu(F_1)$ is the zero-section.
which thus agrees with \([Z, \phi, \tilde{\iota}]_{\mathrm{vir}} := 0_{\psi(F_1)}(\iota_{F_*}[D^{\mathrm{vir}}]).\)

Now assume that \(Z\) and \(M\) are affine and \(\phi_1\) is surjective. In this case \(\phi_0\) is also surjective; let \(K_i\) be the kernel of \(\phi_i\). Then \(K_0\) is locally free and lifts isomorphically via the differential \(\partial: F_1 \to F_0\) to a locally free summand of \(K_1\) (see [15] §6 for proofs of these assertions). Taking the quotient of \(F_*\) by the subcomplex \((K_0 \xrightarrow{\mathbf{1d}} K_0)\) gives us the normalized perfect obstruction theory \(\phi'_*\colon F'_* \to (\mathcal{I}_{Z}/\mathcal{I}_{Z}^2 \to i^*\Omega_{M/B})\), and our original obstruction theory is \(F'_* \oplus (K_0 \xrightarrow{\mathbf{1d}} K_0)\) with \(\phi_0\) and \(\phi_1\) the zero map on \(K_0 \subset F_0\) and on \(K_0 \subset F_1\). Also, this says that \(\phi'_0\colon F'_0 \to i^*\Omega_{M/B}\) is an isomorphism, so by a "change of coordinates", we may assume that \(\phi'_0\colon F'_0 \to (\mathcal{I}_{Z}/\mathcal{I}_{Z}^2 \to i^*\Omega_{M/B})\) is reduced and normalized.

The quotient map \(\pi\colon F_* \to F'_*\) is a quasi-isomorphism of perfect complexes, inducing a canonical isomorphism \(\mathcal{E}^{B.M.}(Z/B, F_*) \cong \mathcal{E}^{B.M.}(Z/B, F'_*)\). We have our cone \(D\) and closed immersion \(\iota_{F_*}: D^{\mathrm{vir}} \to \mathcal{V}(F_1)\) and the corresponding cone \(D'\) and closed immersion \(\iota_{F'_*}: D'^{\mathrm{vir}} \to \mathcal{V}(F'_1)\). The surjection \(\pi\) induces closed immersions \(C(\pi): D'^{\mathrm{vir}} \to D^{\mathrm{vir}}\) and \(\mathcal{V}(\pi): \mathcal{V}(F'_1) \to \mathcal{V}(F_1)\). We claim that

i. \(C(\pi)^!\left[D^{\mathrm{vir}}\right] = \left[D^{\mathrm{vir}}\right]\),
ii. \(\mathcal{V}(\pi)^!\left(\iota_{F_*}[D^{\mathrm{vir}}]\right) = \iota_{F'_*}[D'^{\mathrm{vir}}]\),
iii. \(0_{\psi(F_1)}(\iota_{F_*}[D^{\mathrm{vir}}]) = 0_{\psi(F'_1)}(\iota_{F'_*}[D'^{\mathrm{vir}}])\).

To prove (i) and (ii), let \(p_{D^{\mathrm{vir}}} : D^{\mathrm{vir}} \to Z\), \(p_{\psi(F'_1)} : \mathcal{V}(F'_1) \to Z\) be the structure morphisms. Then we have

\[
\begin{align*}
\mathcal{V}(F_1) &= \mathcal{V}(F'_1) \oplus \mathcal{V}(K_0), \\
D^{\mathrm{vir}} &= D'^{\mathrm{vir}} \oplus \mathcal{V}(K_0),
\end{align*}
\]

via which \(C(\pi)\) and \(\mathcal{V}(\pi)\) become the inclusion as the first summand. The respective projections onto the first summands,

\[
\pi_{\psi(F'_1)} : \mathcal{V}(F'_1) \to \mathcal{V}(F'_1), \quad \pi_{D^{\mathrm{vir}}} : D^{\mathrm{vir}} \to D'^{\mathrm{vir}},
\]

thus exhibit \(D^{\mathrm{vir}}\) as the vector bundle \(p_{D^{\mathrm{vir}}}^!(\mathcal{V}(K_0))\) over \(D'^{\mathrm{vir}}\), and \(\mathcal{V}(F_1)\) as the vector bundle \(p_{\psi(F'_1)}^!(\mathcal{V}(K_0))\) over \(\mathcal{V}(F'_1)\); the inclusions \(C(\pi)\) and \(\mathcal{V}(\pi)\) are the respective zero-sections. Thus \(C(\pi)^!\) is the inverse to the smooth pull-back map \(\pi_{D'}^!\) and \(\mathcal{V}(\pi)^!\) is inverse to the smooth pull-back map \(\mathcal{V}(\pi)^!\).

The identity

\[
[D^{\mathrm{vir}}] = \pi_{D^{\mathrm{vir}}}^!(\left[D'^{\mathrm{vir}}\right])
\]

follows immediately from the definition of \([D^{\mathrm{vir}}]\) and \([D'^{\mathrm{vir}}]\) and the functoriality of smooth pull-back, proving (i). The identity in (ii) is equivalent to the identity

\[
\iota_{F'_*}(\pi_{D^{\mathrm{vir}}}^!(\left[D'^{\mathrm{vir}}\right])) = \pi_{\psi(F'_1)}^!(\iota_{F'_*}[D'^{\mathrm{vir}}]).
\]

Since the square

\[
\begin{array}{ccc}
D^{\mathrm{vir}} & \xrightarrow{\iota_{F_*}} & \mathcal{V}(F_1) \\
\pi_{D'} & \downarrow & \mathcal{V}(F'_1) \\
D'^{\mathrm{vir}} & \xrightarrow{\iota_{F'_*}} & \mathcal{V}(F'_1)
\end{array}
\]

is cartesian, this latter identity follows from the compatibility of proper push-forward and smooth pull-back in cartesian squares: \(\iota_{F_*} \circ \pi_{D^{\mathrm{vir}}}^! = \pi_{\psi(F'_1)}^! \circ \iota_{F'_*}\). Proposition 1.3 This proves (ii).
The proof of (iii) is similar: Let $p_{\Psi(F_1)}: \Psi(F_1) \to Z$ be the structure map. The identity is equivalent to

$$p_{\Psi(F_1)}^! (p_{\Psi(F_1)}^! (\{D^{vir}\})) = \iota_{F_1^*} [D^{vir}],$$

and we have

$$\iota_{F_1^*} (\{D^{vir}\}) = \pi_{\Psi(F_1)}^! (\iota_{F_1^*} [D^{vir}]).$$

By functoriality of smooth push-forward, we have

$$p_{\Psi(F_1)}^! = p_{\Psi(F_1)}^! \circ \pi_{\Psi(F_1)}^!,$$

and thus

$$p_{\Psi(F_1)}^! (p_{\Psi(F_1)}^! (\{D^{vir}\})) = p_{\Psi(F_1)}^! \circ (p_{\Psi(F_1)}^! \circ \iota_{F_1^*} [D^{vir}]) = \iota_{F_1^*} [D^{vir}],$$

proving (iii). In other words, $[Z, \phi, i]_{vir} = [Z', \phi', i']_{vir}$ in $\mathcal{E}^{B,M}(Z/B, F_*) = \mathcal{E}^{B,M}(Z/B, F_*)$.

Now suppose that $\phi_\bullet$ is normalized, but $M$ is only quasi-projective over the affine base-scheme $B$. Let $q: \tilde{M} \to M$ be a Jouanolou cover of $M$, that is, $q$ is a (Zariski) torsor for a $G$-linearized vector bundle on $M$ and $\tilde{M}$ is affine. Letting $\tilde{Z} = Z \times_M \tilde{M}$, we have the Jouanolou cover $q_Z: \tilde{Z} \to Z$ and the closed immersion $i := p_2: \tilde{Z} \to \tilde{M}$. Since $\tilde{M}$ is affine, we can split the canonical surjection $\Omega_{\tilde{M}/B} \to \Omega_{\tilde{M}/M}$, giving the isomorphism $\Omega_{\tilde{M}/B} = q^* \Omega_{\tilde{M}/B} \oplus \Omega_{\tilde{M}/M}$. We define $\tilde{F}_1 = q_Z^* F_1$, $\tilde{F}_0 := q_Z^* F_0 \oplus \Omega_{\tilde{M}/M}$, with differential $\partial_{\tilde{F}}: \tilde{F}_1 \to \tilde{F}_0$ to be $q_Z^* (\partial_F)$ followed by the inclusion $q_Z^* F_0 \to \tilde{F}_0$.

Similarly, define $\tilde{\phi}_1 := q_Z^* \phi_1$ and $\tilde{\phi}_0 = q_Z^* \phi_0 \oplus \text{Id}_{\Omega_{\tilde{M}/M}}$.

This gives us the normalized perfect obstruction theory

$$\tilde{\phi}_\bullet: \tilde{F}_\bullet \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \to i^* \Omega_{\tilde{M}/B})$$

on $\tilde{Z}$. Since $q_Z: \tilde{Z} \to Z$ is the pull-back to $Z$ of the vector bundle torsor $\tilde{M} \to M$, we see that the relative tangent bundle $T_{q_Z}$ is canonically isomorphic to $\Psi(i^* \Omega_{\tilde{M}/M})$. By homotopy invariance, the smooth pull-back map

$$q_Z: \mathcal{E}^{B,M}(Z/B, F_*) \to \mathcal{E}^{B,M}(\tilde{Z}/B, q_Z^* F_*)$$

is an isomorphism and the isomorphism $L_{q_Z} \cong i^* \Omega_{\tilde{M}/M}$ gives us the canonical isomorphism

$$\mathcal{E}^{B,M}(\tilde{Z}/B, q_Z^* F_*) \cong \mathcal{E}^{B,M}(\tilde{Z}/B, \tilde{F}_*)^\vee.$$

Let $\iota_{\tilde{F}_1}: \tilde{D}^{vir} \to \Psi(\tilde{F}_1)$ be the closed immersion associated to the perfect obstruction theory $\tilde{\phi}_\bullet$. We have the commutative diagram

$$\begin{array}{ccc}
\Psi(\tilde{F}_1) & \xrightarrow{\iota_{\tilde{F}_1}} & \tilde{Z} \\
q_{\Psi(\tilde{F}_1)} \downarrow & & \downarrow \pi_{\Psi(\tilde{F}_1)} \\
\tilde{D}^{vir} & \xrightarrow{PD} & \tilde{Z} \\
q_{\tilde{D}^{vir}} \downarrow & & \downarrow \pi_{\tilde{D}^{vir}} \\
D^{vir} & \xrightarrow{PD} & Z
\end{array}$$
with all squares cartesian. Arguing as in the previous case, we see that
\[ q'_2(0_{\nu(F_\bullet)}(\iota_{F_\bullet,*}[D_{vir}])) = 0_{\nu(F_\bullet)}(\iota_{F_\bullet,*}[\tilde{D}_{vir}]), \]
in other words \( q'_2([Z, \phi, i]_{E,GP}^{vir} = [\tilde{Z}, \tilde{\phi}, \tilde{i}]_{E,GP}^{vir} \) in \( E^{B,M}(\tilde{Z}/B, \tilde{F}_\bullet) \).

We conclude the argument with the general case. If
\[ \phi_\bullet : F_\bullet \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{M/B}) \]
is an arbitrary \( G \)-linearized perfect obstruction theory on our quasi-projective \( Z \), with \( i : Z \to M \) a closed immersion in \( \text{Sch}^G/B, M \in \text{Sm}^G/B \), there is a \( G \)-linearized locally free sheaf \( F \) and a \( G \)-equivariant surjection \( a : F \to \mathcal{I}_Z/\mathcal{I}_Z^2 \). Replace \( \phi_\bullet \) with \( \phi'_\bullet : F'_\bullet \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \xrightarrow{d} i^*\Omega_{M/B}) \) with \( F'_i := F_i \oplus F, \partial F'_i = \partial F_i \oplus \text{Id}_F \), \( \phi'_i = \phi_i + a \) and \( \phi'_0 = \phi_0 + d \circ a \). Then \( \phi'_\bullet \) is normalized. As above, we have a canonical isomorphism \( E^{B,M}(Z, F_\bullet) = E^{B,M}(Z, F'_\bullet) \) and via this isomorphism, we have
\[ 0_{\nu(F_\bullet)}(\iota_{F_\bullet,*}[D_{vir}]) = 0_{\nu(F'_\bullet)}(\iota_{F'_\bullet,*}[D_{vir}]), \]
that is, \([Z, \phi, i]_{E,GP}^{vir} = [Z', \phi', i]_{E,GP}^{vir} \) in \( E^{B,M}(Z/B, F'_\bullet) = E^{B,M}(Z/B, F_\bullet) \).

Thus, we have verified the identity of the two classes at the initial stage of our reverse process, and have shown that the Graber-Pandharipande classes are compatible with the canonical isomorphisms of Borel-Moore homology at each stage of the process. This completes the proof.

\[ \square \]

Remark 4.6. The main result of [16] is that the virtual fundamental class \([Z, [\phi]]_{E,GP}^{vir} \in E^{B,M}(Z/B, F_\bullet) \) depends only on the perfect obstruction theory \([\phi] \) on \( Z \) and is independent of the various choices made in its construction. Thus, the Graber-Pandharipande class \([Z, \phi, i]_{E,GP}^{vir} \) is also independent of the choice of embedding \( i \) and the choice of representative \( \phi_\bullet \) for \([\phi] \). One can also show this directly by using a small modification of the proof of this independence for the Behrend-Fantechi classes [2 Proposition 5.3].

5. Equivariant virtual fundamental classes via the algebraic Borel-construction

In this section, we discuss how a \( G \)-linearized perfect obstruction theory gives rise to a virtual fundamental class in the equivariant Borel-Moore homology defined by the algebraic Borel-construction of [3]. We fix a motivic commutative ring spectrum \( E \in \text{SH}(B) \); note that we do not use the equivariant category \( \text{SH}^G(B) \).

We retain the assumptions on \( B \) and \( G \) from the previous section. In particular, a \( G \)-linearized perfect obstruction \( \phi \) on some \( X \in \text{Sch}^G/B \) admits a representative \( (E_1 \to E_0) \xrightarrow{\phi \circ \alpha_0} \tau_{\leq 1} L_{X/B} \) with \( (E_1 \to E_0) \) a \( G \)-linearized two-term complex of locally free coherent sheaves on \( X \). We will assume that \( X \) is in the subcategory \( \text{Sch}^G_q/B \) and admits a \( G \)-equivariant closed immersion \( \iota : X \to Y \) for some \( Y \in \text{Sm}^G_q/B \). This gives us for each \( j \) the \( G \)-equivariant closed immersion \( \iota_j := \iota \times \text{Id}_{E_j G} : X \times E_j G \to Y \times E_j G \) and induced closed immersion \( \alpha_j : X \times^G E_j G \to Y \times^G E_j G \).

We note that for \( p_1 : X \times_B E_j G \to X \) the projection, \( p_1^* L_{X/B} = L_{X \times^B E_j G} \) and that the complex \( L_{X/B,j} \) on \( X \times^G E_j G \) induced by \( p_1^* L_{X/B} \) via descent is
canonically isomorphic to \( L_{X \times^G E_j G/B_j G} \). Letting

\[
\phi_j : E_{\bullet,j} \to \tau_{\leq 1} L_{X \times^G E_j G/B_j G}
\]

be the map induced by \( p^*_i \phi \) by descent, \( \phi_j \) defines a perfect obstruction theory on \( X \times^G E_j G \) relative to \( B_j G \), giving us the virtual fundamental class

\[
[X \times^G E_j G, \phi_j]_{\text{vir}}^* \in \mathcal{E}^B_{\text{vir}}(X \times^G E_j G/B_j G, E_{\bullet,j}).
\]

**(Lemma 5.1)** \( i : B_0 \to B_1 \) be a regular closed immersion in \( \text{Sm}_B \). Let \( \psi : (F_1 \to F_0) \to \tau_{\leq 1} L_{Z_1/B_1} \) be a perfect obstruction theory on some \( Z_1 \in \text{Sch}_B/B_1 \), relative to \( B_1 \). Let \( Z_0 = Z_1 \times_{B_1} B_0 \) and assume that the cartesian square

\[
\begin{array}{ccc}
Z_0 & \to & Z_1 \\
\downarrow & & \downarrow \\
B_0 & \to & B_1
\end{array}
\]

is Tor-independent, giving the canonical isomorphism

\[
i^* L_{Z_1/B_1} \cong L_{Z_0/B_0}
\]

and the induced perfect obstruction theory \( i^* \psi : (i^* F_1 \to i^* F_0) \to \tau_{\leq 1} L_{Z_0/B_0} \).

Choose a closed immersion \( i : Z_1 \to Y_1 \), with \( Y_1 \in \text{Sm}_B/B_1 \), giving the closed immersion \( i_0 : Z_0 \to Y_0 := Y_1 \times_{B_1} B_0 \). We assume that the canonical map \( \beta : C_{Z_0/Y_0} \to i^* C_{Z_1/Y_1} \) is an isomorphism. Then the refined Gysin morphism

\[
i^i : \mathcal{E}^B_{\text{vir}}(Z_1/B_1, F_*) \to \mathcal{E}^B_{\text{vir}}(Z_0/B_0, i^* F_*) \cong \mathcal{E}^B_{\text{vir}}(Z_0/B_0, i^* F_*)
\]

satisfies

\[
i^i([Z, \psi]_{\text{vir}}^*) = [Z_0, \psi_0]_{\text{vir}}^*.
\]

**Proof.** As in Remark 1.5, the purity isomorphisms for \( B_0 \to B \), \( B_1 \to B \) give us the canonical isomorphism

\[
\partial_i : \mathcal{E}^B_{\text{vir}}(Z_0/B_1, v - N_i) \xrightarrow{\sim} \mathcal{E}^B_{\text{vir}}(Z_0/B_0, v)
\]

for \( v \in \mathcal{K}(Z_0) \).

Letting \( \mathcal{I}_Z \) be the ideal sheaf of \( Z \) in \( Y \), we have the map \( \tilde{\psi} : (F_1 \to F_0) \to (\mathcal{I}_Z/\mathcal{I}_Z^2 \to i^* \Omega_{Y/B}) \) induced by \( \psi \), the corresponding closed immersion \( i_0 : Z_0 \to Y_0 := Y \times_B B_0 \) and the corresponding map \( \tilde{\psi} : (i^* F_1 \to i^* F_0) \to (\mathcal{I}_{Z_0}/\mathcal{I}_{Z_0}^2 \to i^* \Omega_{Y_0/B_0}) \) induced by \( \psi \).

Following the Graber-Pandharipande construction given in \( \text{[4]} \) we have the cone \( D_{\text{vir}} \) with fundamental class \( [D_{\text{vir}}] \), and closed immersion \( i_{F_*} : D_{\text{vir}} \to \mathcal{V}(F_1) \), giving the virtual fundamental class \( [Z, \psi]_{\text{vir}}^* \) as

\[
[Z, \psi]_{\text{vir}}^* = 0_{\mathcal{V}(F_1)}(i_{F_*}([D_{\text{vir}}])).
\]

We also have the corresponding cone \( D^0_{\text{vir}} \) with fundamental class \( [D^0_{\text{vir}}] \), and closed immersion \( i_{F_*} : D^0_{\text{vir}} \to \mathcal{V}(i^* F_1) \), giving the virtual fundamental class \( [Z_0, i^* \psi]_{\text{vir}}^* \) as

\[
[Z_0, i^* \psi]_{\text{vir}}^* = 0_{\mathcal{V}(i^* F_1)}(i_{F_*}([D^0_{\text{vir}}])).
\]

By assumption, the canonical map \( \beta : C_{Z_0/Y_0} \to i^* C_{Z/Y} \) is an isomorphism. By Proposition 2.6 this gives the identity

\[
[C_{Z_0/Y_0}] = i^i[C_{Z/Y}] \in \mathcal{E}^B_{\text{vir}}(C_{Z_0/Y_0}/B_0, \Omega_{Y_0/B_0}).
\]
Using the zig-zag of isomorphisms of Borel-Moore homology used to define $[D]$, $[D_{\text{vir}}]$, and $[D_0]$, $[D_{\text{vir}}^0]$, and using the compatibility of the refined Gysin map with smooth pull-back (Proposition 1.3) this implies

$$[D_{\text{vir}}^0] = i^![D_{\text{vir}}] \in E^{B,M}(D_{\text{vir}}^0/B_0, i^*F_0).$$

Thus

$$[Z_0, i^*\psi]_{E_G} = 0^j_{V(i^*F)}(i^*F_{\bullet*}([D_{\text{vir}}^0]))$$

$$= 0^j_{V(i^*F)}(i^*F_{\bullet*}([D_{\text{vir}}]))$$

$$= 0^j_{V(i^*F)}(i^*F_{\bullet*}([D_{\text{vir}}^0]))$$

$$= i^j(0^j_{V(F)}(i^*F_{\bullet*}([D_{\text{vir}}])))$$

$$= i^j(Z, \psi)_{E_G}.$$

The third identity is Proposition 1.3 and the fourth is Corollary 2.3.

**Definition/Proposition 5.2.** Let $\phi_{\bullet}: E_{\bullet} \to \tau_{\leq 1}L_{X/B}$ be a $G$-linearized perfect obstruction theory on $X \in \text{Sch}_q^G/B$. Then the family of virtual fundamental classes of (5.1),

$$[X \times^G E_jG, \phi_j]_{E_G} \in E^{B,M}(X \times^G E_jG/B_jG, E_{\bullet,j})$$

gives a well-defined element

$$[X, \phi]_{E_G} \in E^{B,M}_G(X/B, E_{\bullet}).$$

**Proof.** We have the cartesian, Tor-independent diagram

$$\begin{array}{ccc}
X \times^G E_jG & \longrightarrow & X \times^G E_{j+1}G \\
\downarrow & & \downarrow \\
B_jG & \longrightarrow & B_{j+1}G,
\end{array}$$

and the perfect obstruction theory $\phi_{j+1}$ on $X \times^G E_{j+1}G$, relative to $B_{j+1}G$. Choose a $G$-equivariant closed immersion $\iota: X \to Y$ with $Y \in \text{Sm}_q^G/B$, giving the closed immersions

$$\iota_j: X \times^G E_jG \to Y \times^G E_jG$$

with $Y \times^G E_jG$ smooth over $B_jG$ and $\iota_j \cong \iota_{j+1} \times_{B_{j+1}G} B_jG$. This gives the isomorphism

$$C_{\iota_j} \cong C_{X/Y \times^G E_jG}$$

and thus the canonical map $\beta: C_{\iota_j} \to C_{\iota_{j+1} \times_{B_{j+1}G} B_jG}$ is an isomorphism. We also have the canonical isomorphism of perfect obstruction theories $i^*\phi_{j+1} \cong \phi_j$. Thus, we may apply Lemma 5.1 to yield the result.

We will sometimes drop the subscripts $E, G$ from $[X, \phi]_{E_G}$ if there is no cause for confusion.
6. Virtual localization

With the machinery of \cite{35} and \cite{36} together with our version of Vistoli’s lemma, we can essentially copy the constructions and arguments of Graber-Pandharipande to prove our virtual localization theorem. We concentrate on the case $G = N$, with torus $T_1 \subset N$, $T_1 = \mathbb{G}_m$, and work over a perfect field $k$.

Fix an $X \in \text{Sch}^N/k$, a $Y \in \text{Sm}^N/k$ and a closed immersion $\iota: X \hookrightarrow Y$ in $\text{Sch}^N/k$. We also fix a $N$-equivariant perfect obstruction theory on $X$, represented by a two-term complex $E_* := (E_1 \to E_0)$ of $N$-linearized locally free sheaves on $X$, together with a $N$-equivariant map $\phi_*: E_* \to \tau_{\leq 1}L_{X/k} = (I_X/T_X^2 \to i^*\Omega_Y/k)$.

We recall from \cite{10} §1 some facts about the $T_1$-fixed loci of $X$ and $Y$. We have the $T_1$-fixed subschemes $X^{T_1} \subset Y^{T_1}$, with $X^{T_1} = X \cap Y^{T_1}$, $Y^{T_1}$ is smooth; let $Y_1, \ldots, Y_s$ be the irreducible components of $Y^{T_1}$ with inclusion maps $i^T_j : Y_j \to Y$, and let $X_j = X \cap Y_j$, so $X^{T_1} = \coprod_j X_j$. Let $\tilde{i}: X^{T_1} \to X$, $i_j : X_j \to X$ be the inclusions.

Let $\mathcal{F}$ be a $T_1$-linearized coherent sheaf on $X$. Since the $T_1$-action on $X$ is trivial, we can decompose $\mathcal{F}$ into weight spaces for the $T_1 = \mathbb{G}_m$-action

$$\mathcal{F} = \oplus_r \mathcal{F}_r$$

where $t \in \mathbb{G}_m$ acts on $\mathcal{F}_r$ by the character $\chi_r(t) = t^r$. if $\mathcal{F}$ is locally free, so is each $\mathcal{F}_r$. The subsheaf $\mathcal{F}^m := \oplus_{r \neq 0} \mathcal{F}_r$ is the moving part of $\mathcal{F}$ and $\mathcal{F}_1^* := \mathcal{F}_0$ is the fixed part of $\mathcal{F}$.

The map $\phi_*$ induces the map

$$\tilde{\phi}: i^*E_* \to \tau_{\leq 1}L_{X^{T_1}/k}$$

By \cite{10} Proposition 1 $\tilde{\phi}$ defines a perfect obstruction theory on $X^{T_1}$.

**Definition 6.1.** The virtual conormal sheaf of $X^{T_1}$ in $X$, $N^{vir}$, is defined to be the perfect complex $i^*E_*^m$.

We have the order-four element $\sigma = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right) \in N$, which together with $T_1$ generates $N$. Letting $\bar{\sigma}$ denote the image of $\sigma$ in $N/T_1$, $N/T_1$ is the cyclic group of order two generated by $\bar{\sigma}$.

**Lemma 6.2.** $\tilde{\phi}: i^*E_* \to \tau_{\leq 1}L_{X^{T_1}/k}$ and $N^{vir}$ inherit from $\phi$ and $E_*$ natural $N$-linearizations.

**Proof.** For $t \in T_1$, we have $t \cdot \sigma = \sigma \cdot t^{-1}$. If $\mathcal{F}$ is an $N$-linearized coherent sheaf on $X$, $i^*\mathcal{F}$ inherits an $N$-linearization on $X^{T_1}$, and this relation implies that the canonical isomorphism $\sigma \cdot \sigma^*(\mathcal{F}) \to \mathcal{F}$ given by the $N$-linearization restricts to $\sigma \cdot \sigma^*(i^*\mathcal{F}_r) \to i^*\mathcal{F}_{r^{-1}}$. Thus $i^*\mathcal{F}_r \oplus i^*\mathcal{F}_{r^{-1}}$ is an $N$-linearized subsheaf of $i^*\mathcal{F}$ for each $r \neq 0$, and $i^*\mathcal{F}_0$ is also an $N$-linearized subsheaf of $i^*\mathcal{F}$.

Thus $\tilde{\phi} : i^*E_* \to \tau_{\leq 1}L_{X^{T_1}/k}$ and $N^{vir} = i^*E_*^m$ have canonical $N$-linearizations. $\square$

We recall that Witt sheaf cohomology is represented in $\text{SH}(k)$ by the Eilenberg-MacLane spectrum $\text{EM}(\mathbb{W}_*)$, via

$$\text{EM}(\mathbb{W}_*)^a(X) = H^{a-b}(X, W)$$

for $X \in \text{Sm}_k$. Here $\mathbb{W}_*$ is the homotopy module which is $W$ in each degree; formally $\mathbb{W}_* := \mathbb{K}^{MW}_*[\eta^{-1}]$, see \cite{19} §5, §6 for details. We use this formalism to
be able to apply the machinery in $\text{SH}(k)$ to Witt sheaf cohomology and its related constructions, such as the Borel-Moore homology and the equivariant theory.

Given $X \in \text{Sch}^N/k$, we write $H^*_{N,a}(X, \mathcal{W}(\mathcal{L}))$ for the corresponding $N$-equivariant Borel-Moore homology, as defined in [3] via

$$H^*_{N,a}(X, \mathcal{W}(\mathcal{L})) := EM(\mathcal{W}_a)_{a,0}(X/k, N, \mathcal{L} - \mathcal{O}_X).$$

We similarly have the equivariant Witt sheaf cohomology $H^*_N(X, \mathcal{W}(\mathcal{L}))$, defined by

$$H_N^*(X, \mathcal{W}(\mathcal{L})) := EM(\mathcal{W}_a)_{a,0}(X/k, N, \mathcal{L} - \mathcal{O}_X).$$

We have shown in [15, Theorem 8.6] that there is a positive integer $M_0$ such that, for each $N$-linearized invertible sheaf $\mathcal{L}$ on $X$, the push-forward by the inclusion $i : X^{T_1} \to X$ induces an isomorphism

$$i_* : H^*_{N,a}(X^{T_1}, \mathcal{W}(i^*\mathcal{L}))[1/M_0] \xrightarrow{\sim} H^*_{N,a}(X, \mathcal{W}(\mathcal{L}))[1/M_0].$$

**Definition 6.3.** Let $|X|^N \subset X^{T_1}$ be the union of the $N$-stable irreducible components of $X^{T_1}$, and let $X^{T_1}_{\text{ind}}$ be the union of the irreducible components $C$ of $X^{T_1}$ such that $N \cdot C = C \cup C'$ with $C' \neq C$ and $C \cap C' = \emptyset$. We call the action semi-strict if $X^{T_1} = |X|^N \cup X^{T_1}_{\text{ind}}$ and strict if the action is semi-strict, $|X|^N \cap X^{T_1}_{\text{ind}} = \emptyset$, and $|X|^N$ decomposes as a disjoint union of two $N$-stable closed subschemes

$$|X|^N = X^N \amalg X^{T_1}_{\text{fr}},$$

where the $N/T_1$-action on $X^{T_1}_{\text{fr}}$ is free.

If the $N$-action is semi-strict, then the inclusion $|X|^N \to X$ induces an isomorphism

$$H^*_{N,a}(|X|^N, \mathcal{W}(i^*\mathcal{L}))[1/M_0] \to H^*_{N,a}(X, \mathcal{W}(\mathcal{L}))[1/M_0].$$

and

$$H^*_{N,a}(X^{T_1} \setminus |X|^N, \mathcal{W}(\mathcal{L}))[1/M_0] = 0.$$  

This is [15, Theorem 8.6].

**Remark 6.4.** Suppose that the $N$-action on $Y$ is strict. Then the $N$-action on $X$ is also strict, with $X^N = Y^N \cap X$, with $Y^{T_1}_{\text{ind}} \cap X \subset X^{T_1}_{\text{ind}}$, and with $Y^{T_1}_{\text{fr}} \cap X = X^{T_1}_{\text{fr}} \amalg X'$, with $X^{T_1}_{\text{ind}} = X' \amalg Y^{T_1}_{\text{ind}} \cap X$.

Assuming the $N$-action on $X$ is strict, we set $|X|^N_{j} := |X|^N \cap X_j$. This writes $X_j$ as a disjoint union

$$X_j = |X|^N_{j} \amalg (X_j \cap X^{T_1}_{\text{ind}})$$

with $|X|^N_{j}$ and $N$-stable closed subscheme of $X^{T_1}$. This decomposes $X^{T_1}$ as a disjoint union

$$X^{T_1} = (\amalg_j |X|^N_{j}) \amalg X^{T_1}_{\text{ind}},$$

and similarly decomposes $|X|^N$ as a disjoint union

$$|X|^N = \amalg_j |X|^N_{j}.$$  

Decomposing each $|X|^N_{j}$ further, we have the connected components $i_{j,\ell} : |X|^N_{j,\ell} \to X$ of $|X|^N_{j}$, which are all $N$-stable.

Let $i_j : |X|^N_{j} \to X$ be the inclusion. The $N$-linearized perfect obstruction theory $\phi$ on $X^{T_1}$ restricts to an $N$-linearized perfect obstruction theory $\phi_j : (i^*_j E_\nu)^l \to \tau_{\leq 1} L_{|X|^N_{j}/k}$ on $|X|^N_{j}$, and $N$-linearized the virtual conormal sheaf $N^{\text{vir}}$ on $X^{T_1}$
Lemma 6.5. (see [15, Construction 2.7] for details). 

ϕ may further restrict to |1. There are positive integers N restricts to the X virtual vector bundles. 

Virtual localization

Theorem 6.7 Q X

Let Y \subseteq X be a closed immersion in Sch^N/k, with Y \subseteq Sm^N/k, and let \phi: E_\bullet \rightarrow \tau_{\leq 1}L_{X/k} be an N-linearized perfect obstruction theory on X. Suppose that the action on X is strict. For each \iota: |X|_N \rightarrow X of |X|^N, write e_N((i^*_\iota E_0)gen) \cdot e_N((i^*_\iota E_1)gen) as M^{N}e^{\kappa \iota}, following Lemma 6.5. We define 

\[ e_N(N^{vir}_{j,\iota}) \in H^*_N(|X|_N, W(d\det^{-1} V))[1/M^X_{\iota}] \]

by 

\[ e_N(N^{vir}_{j,\iota}) = e_N(V((i^*_\iota E_0)^m)) \cdot e_N(V((i^*_\iota E_1)^m))^{-1}. \]

Let M^X = \prod_\iota M^X_\iota and let 

\[ e_N(N^{vir}_{j,\iota}) = \{e_N(N^{vir}_{j,\iota})\}_\iota \in H^*_N(|X|^N, W(d\det^{-1} N_0))[1/M^X] \]

\[ := \prod_\iota H^*_N(|X|_N, W(d\det^{-1} N_\iota))[1/M^X_{\iota}]. \]

Similarly, noting that each irreducible component Y_j of Y is smooth, we can write e_N((i^*_Y T^Y_\iota)^gen) as M^Y_{\iota} e^{\kappa \iota}. 

It follows from Lemma 6.5 that e_N(N^{vir}_{j,\iota}) is well-defined and is invertible in H^*_N(|X|^N, W(d\det^{-1} N^{vir}_{j,\iota}))[1/M^X_{\iota}]. Similarly, the Euler class e_N(i^*_Y T^Y_\iota) is invertible in H^*_N(Y_j, W(d\det^{-1} i^*_Y T^Y_\iota))[1/M^Y_{\iota}].

Recall the integer M_0 > 0 defined above, just before Definition 6.3.

Theorem 6.7 (Virtual localization). Let i: X \rightarrow Y be a closed immersion in Sch^N/k, with Y \subseteq Sm^N/k, and let \phi: E_\bullet \rightarrow L_{X/k} be an N-linearized perfect obstruction theory on X. Suppose that the N-action on X is strict. Let M = M_0 \cdot \prod_{\iota} M^{X}_{\iota} \cdot M^{Y}_{\iota}, where the M^{X}_{\iota}, M^{Y}_{\iota} are as in Definition 6.6. Let \|[X|_{i,\iota}, \phi_\iota]^{vir}_{N} \in
EM(\(W_\bullet\))_{N_{i_j}}^{B, M}(\{X\}^N_j, i_j^* E_i) be the equivariant virtual fundamental class for the N-linearized perfect obstruction theory \(\phi_j : i_j^* E_i \to \tau_{\leq 1} L_{\{X\}^N_j} \) on \(\{X\}^N_j\). Then

\[
[X, \phi]|_{vir}^{vir} = \sum_{j=1}^r i_{j*}(\{X\}^N_j, \phi_j)|_{vir}^{vir} \cap e_N(N_{i_j}|_{vir}^{-1}) \in EM(\mathcal{W}_\bullet)^{B, M}(X, E_\bullet)[1/M \epsilon]
\]

Remarks 6.8. 1. In our notation \(H_{N, a}^{B, M}(-, -)\) for the Witt-sheaf equivariant Borel-Moore homology, we have

\[
EM(\mathcal{W}_\bullet)^{B, M}(X, E_\bullet) = H_{N, r}^{B, M}(X, \mathcal{W}(\mathcal{L})),
\]

with \(r\) the virtual rank of \(E_\bullet\) and \(\mathcal{L}\) the virtual determinant, and similarly for \(EM(\mathcal{W}_\bullet)^{B, M}(\{X\}^N_j, i_j^* E_i)\).

2. One is ultimately not just interested in the virtual fundamental class \([X, \phi]|_{vir}^{vir}\), but rather its “quadratic degree”. In the classical case, with \([X, \phi]|_{vir}^{vir} \in \text{CH}_0(Spec k) = Z\), \(r\) is the virtual rank of the perfect obstruction theory, and if \(X\) is proper over \(k\) and \(r = 0\), one can get the numerical invariant \(\deg_k[X, \phi]|_{vir}^{vir} \in \text{CH}_0(Spec k) = Z\). If one computes this by torus localization (and \(r = 0\)), one will arrive at a torus equivariant class, whose degree will land in \(\text{CH}_0(BT)[1/P] = Z[x_1, \ldots, x_n][1/P]\), where \(n\) is the rank of the torus, and \(P\) is some non-zero homogeneous element of positive degree. However, one knows that this is the image of the non-equivariant degree in \(Z\) that one wishes to compute, so the rational function \(\deg_{BT}[X, \phi]|_{vir}^{vir}\) will actually be the integer \(\deg_k[X, \phi]|_{vir}^{vir}\).

In the case of Witt sheaf Borel-Moore homology, and \(N\)-localization, this situation is similar, except that the virtual class \([X, \phi]|_{vir}^{vir}\) will land in \(H_{0, r}^{B, M}(X, \mathcal{W}(\mathcal{L}))\), where as before \(r\) is the virtual rank of the perfect obstruction theory and now \(\mathcal{L}\) is the virtual determinant. In order to have a well-defined push-forward to the point, landing in \(H_0^{B, M}(Spec k, \mathcal{W}) = W(k)\), one needs \(r = 0\) and an isomorphism \(\mathcal{L} \cong O_X\); this second condition is usual referred to as an orientation. If one wants to compute \(\deg_k[X, \phi]|_{vir}^{vir} \in W(k)\) by localization, one needs to have \(N\) acting, and then one will arrive at \(\deg_{BN}[X, \phi]|_{vir}^{vir} \in W(BN)[1/P]\). Here \(W(BN)[1/P] = W(k)[x][1/P(x)]\) for some non-zero homogeneous polynomial \(P(t) \in Z[t]\) of positive degree, and as before \(\deg_{BN}[X, \phi]|_{vir}^{vir} \in W(k)[x][1/P(x)]\). What is different is that \(W(k)\) will in general have a lot of 2-torsion, which may get killed by inverting the polynomial \(P\), so one will lose more information that one does in the classical case. What will in any case remain are the \(Z\)-valued invariants given by the signature map \(W(\mathbb{R}) \to Z\) for each real embedding \(k \hookrightarrow \mathbb{R}\), or more generally, for each embedding of \(k\) into a real closed field.

There are some tricks one can use to improve the situation, for example, if everything is defined over \(Z[1/m]\) for some \(m > 0\), then one gets a lifting of \(\deg_k[X, \phi]|_{vir}^{vir}\) to \(W(Z[1/2m])\). As the kernel of \(W(Z[1/2m]) \to W(\mathbb{R})\) is generated by the quadratic forms \(x \to \pm ax^2\) with \(a\) a product of primes dividing \(2m\), knowing the signature recovers \(\deg_k[X, \phi]|_{vir}^{vir}\) modulo this kernel. See [12, 17] for examples of this method.

The proof of Theorem 6.7 is accomplished by following the argument used in [10] §3 for the proof of their main result. The argument there is essentially formal, relying on the various operations for the equivariant Chow groups, which as we have explained in §3 extend to the setting of equivariant Borel-Moore homology.
and equivariant cohomology. The main non-formal ingredient in their proof is their use of of Vistoli’s lemma, which we have extended here to the setting of Borel-Moore homology, and which thus extends to equivariant Borel-Moore homology as defined in \(\mathbb{E}\) via the algebraic Borel construction. We give here a step-by-step sketch of our extension of their proof. We set \(\mathcal{E} = \text{EM}(\mathcal{W}_*)\).

**Step 1** We note that the localized equivariant Borel-Moore homology of \(X_{\text{ind}}^{T_1}\) is zero \([15]\) Lemma 8.5(2)]. Using the assumption that the \(N\)-action is strict, we have \(X = |X|^N \cup (X_j \cap X_{\text{ind}}^N)\), so using the localization sequence in equivariant Borel-Moore homology, we may replace \(X\) with \(X \setminus X_{\text{ind}}^N\), \(Y\) with \(Y \setminus X_{\text{ind}}^N\), and thus we may assume that \(X_{\text{ind}}^N = \emptyset\) and \(|X|^N = X_j\). We have the inclusions \(\iota: X \to Y\), \(\iota_j: X_j \to Y_j\), \(i^Y_j: Y_j \to Y\) and \(\iota_j: X_j \to X\).

We will use all the results of §1.1 and Step 2 promoted to the equivariant setting, following Remark 3.5. We write \(T_Y\) for \(T_{Y/B}\), etc.

We have the equivariant fundamental class \([Y]_N \in \mathcal{E}_{\mathcal{B}^M}(Y/BN, \Omega_{Y/k})\), with \([Y]_{\mathcal{N}}\) the image of \(1 \in \mathcal{E}_{\mathcal{N},0,0}(Y)\) under the Poincaré duality isomorphism

\[
\mathcal{E}_{\mathcal{N},0,0}(Y) \cong \mathcal{E}_{\mathcal{B}^M}(Y/BN, \Omega_{Y/k}).
\]

We recall that \(Y^{T_1}\) is smooth over \(k\) with irreducible components \(Y_1, \ldots, Y_s\). We have fundamental classes \([Y_j]_N \in \mathcal{E}_{\mathcal{B}^M}(Y_j/BN, \Omega_{Y_j/B})\) defined as for \([Y]_N\).

For each irreducible component \(i^Y_j: Y_j \to Y\) of \(Y^{T_1}\), the normal bundle \(N_{Y_j}Y\) of \(i^Y_j\) is exactly \(i^Y_j* T^\text{m}_{Y_j}\). Since \(i^Y_j[Y]_N = [Y_j]_N\), applying the Bott residue theorem \([15]\) Theorem 10.5, Remark 10.6 for the \(N\)-action on \(Y\) gives

\[
[Y]_N = \sum_{j=1}^s i^Y_{j*} ([Y_j]_N \cap e_N(i^Y_j* T^\text{m}_{Y_j})^{-1}).
\]

\(\mathcal{E}_{\mathcal{B}^M}(Y/BN, \Omega_{Y/k})\) [1/\(M_{Y/k}\)].

Note that, although the Bott residue theorem of \([15]\) is written for the Borel-Moore homology \(\mathcal{E}_{\mathcal{B}^M}(-/B, -)\), the fact that

\[
\mathcal{E}_{\mathcal{B}^M}(-/B, -) = \mathcal{E}_{\mathcal{B}^M}(-/BN, - + g^\vee)
\]

(see Remark 3.6) allows us to apply the results of \([15]\) to \(\mathcal{E}_{\mathcal{B}^M}(-/BN, -)\) by simply substituting \(\mathcal{E}_{\mathcal{B}^M}(-/BN, -)\) for \(\mathcal{E}_{\mathcal{B}^M}(-/B, -)\) in all statements to be found in \([15]\).

**Step 2.** We apply the refined intersection product (§1.3) on \(Y\) with respect to \(\iota: X \to Y\), \(\text{Id}: Y \to Y\), and the \(i^Y_j: Y_j \to Y\), and use the compatibility with proper push-forward (Lemma 1.6(1)) to give the identity

\[
[X, \phi]_{N, \iota*} = [X, \phi]_{N, \iota^Y_j* \text{Id}} [Y]_N = \sum_{j=1}^s i^Y_{j*} \left( [X, \phi]_{N, \iota^Y_j* \text{Id}} ([Y_j]_N \cap e_N(i^Y_j* T^\text{m}_{Y_j})^{-1}) \right).
\]

in \(\mathcal{E}_{\mathcal{B}^M}(X/BN, E_*)[1/M_{Y/k}]\), We will show that

\[
[X, \phi]_{N, \iota*} = [X, \phi]_{N, \iota^Y_j* \text{Id}} ([Y_j]_N \cap e_N(i^Y_j* T^\text{m}_{Y_j})^{-1}) = [X, \phi]_{N, \iota^Y_j* \text{Id}} ([Y_j]_N \cap e_N(N_{i^Y_j}))^{-1}
\]

in \(\mathcal{E}_{\mathcal{B}^M}(X_j/BN, E_*)[1/M_e]\) for suitable \(M\), which will yield the formula of Theorem 6.7.

**Step 3.** We have the cone \(D = C_{X/Y} \times \mathbb{V}(E_0)\) over \(X\) and its quotient cone.
$D^{vir} : = 0^* T_Y \backslash D$, with closed immersion $\iota_{E^*} : D^{vir} \hookrightarrow \mathcal{V}(E_1)$, all of these being maps in $\text{Sch}^N/k$.

Referring to Definition/Proposition 5.2, we have

$$[X, \phi]_N^{vir} = 0^!_{\mathcal{V}(E_1)}(\iota_{E^*} [D^{vir}]_N) \in \mathcal{E}^{BM}_N(X/BN, E_*)$$

For each $j$, we have a similar description of the virtual class $[X_j, \phi_j]^{vir}$ for the $N$-linearized perfect obstruction theory $\phi_j : i_j^* E^!_j \rightarrow \tau_{L X/j/k}$ on $X_j$ with corresponding cones $D_j$ and $D_j^{vir}$, namely,

$$[X_j, \phi_j]_N^{vir} = 0^!_{\mathcal{V}(i_j^* E^!_j)}(i_j^* E^!_j [D_j^{vir}]_N) \in \mathcal{E}^{BM}_N(X/BN, i_j^* E^!_j).$$

Here $D_j = C_{X_j/Y_j} \times \mathcal{V}(i_j^* E^!_j)$ and $D_j^{vir} = i_j^* T_{Y_j} \backslash D_j$, with closed equivariant immersion $i_j^* E^!_j : D_j^{vir} \hookrightarrow \mathcal{V}(i_j^* E^!_j)$.

**Step 4.** We consider the closed immersions $\iota : X \rightarrow Y$ and $\iota_j : X_j \rightarrow Y_j$, with their corresponding cones $C_i := C_{X/Y}$ and $C_{i_j} := C_{X_j/Y_j}$. We have the closed immersion $\beta_j : C_{i_j} \hookrightarrow i_j^* C_i$. We now apply Vistoli’s lemma, using Remark 2.8 to apply this in the context of equivariant Borel-Moore homology defined using the algebraic Borel construction. Since $i_j^* \beta_j$ is a regular embedding, Proposition 2.6 and Remark 2.8 give us the identity

$$\beta_{i_j}([C_{i_j}]_N) = i_j^* Y_C|_N$$

in $\mathcal{E}^{BM}_N(i_j^* C_i/BN)$; we suppress the twist here and in the remainder of the proof to simplify the notation.

The closed immersion $\beta_j$ and the inclusion $i_j^* E^!_0 \subseteq i_j^* E^!_0 \oplus i_j^* E^m_0 = i_j^* E_0$ gives the closed immersion $\beta_j^D : D_j \times \mathcal{V}(i_j^* E^m_0) \rightarrow D$, and (6.3) gives us the relation

$$(6.4) \quad \beta_{i_j}^D([D_j \times \mathcal{V}(i_j^* E^m_0)]_N) = i_j^* Y_C[D]_N$$

in $\mathcal{E}^{BM}_N(i_j^* D/BN)$. We also have the fundamental class $[D_j]$ induced from $[C_{i_j}]$ via the projection $D_j \rightarrow C_{i_j}$.

We have the quotient cones

$$D^{vir} := 0^* T_Y \backslash D, \quad D_j^{vir} := i_j^* T_{Y_j} \backslash D_j,$$

with their fundamental classes $[D^{vir}]$ and $[D_j^{vir}]$ induced from $[D]$ and $[D_j]$. The closed immersion $\iota_{E^*} : D^{vir} \hookrightarrow \mathcal{V}(E_1)$ induces the cartesian diagram

$$\begin{array}{ccc}
\iota^* T_Y & \rightarrow & D \\
\downarrow & & \downarrow \\
X & \rightarrow_{0^!_{\mathcal{V}(E_1)}} & \mathcal{V}(E_1),
\end{array}$$

which gives the identity

$$[X, \phi]_N^{vir} := 0^!_{\mathcal{V}(E_1)}(\iota_{E^*} [D^{vir}]_N) = 0^!_{T_Y} 0^!_{\mathcal{V}(E_1)}([D]_N).$$
Combined with (6.4) and using the commutativity of refined Gysin pull-back, this gives

\[ [X, \phi]^\text{vir}_{i, \gamma_j} [Y_j]_N = i_j^* [X, \phi]^\text{vir}_N \]

\[ = i_j^* Y_j^! 0_{V(E_1)}([D]_N) = 0_{i_j^* T_Y}^! 0_{V(i_j^* E_1)}(i_j^*[D]_N) \]

\[ = 0_{i_j^* T_Y}^! 0_{V(i_j^* E_1)}([D_j \times \mathbb{V}(i_j^* E_0^m)]_N). \]

Similarly, the closed immersion \( \iota_{E_{*, j}} : D_j^{vir} \hookrightarrow \mathbb{V}(i_j^* E_1^f) \) gives the cartesian diagram

\[
\begin{array}{ccc}
i_j^* T_Y & \longrightarrow & D_j \\
\downarrow & & \downarrow \\
X_j & \longrightarrow & \mathbb{V}(i_j^* E_1^f)
\end{array}
\]

and the identity

\[ [X_j, \phi_j]_{N}^{\text{vir}} := 0_{V(E_{1,j})}(i_{E_{*, j}}^* [D_j^{vir}]_N) = 0_{i_j^* T_Y}^! 0_{V(i_j^* E_1)}([D]_N). \]

We also note that

\[ T_Y = i_j^* T_Y \]

giving

\[ i_j^* T_Y = T_Y \oplus i_j^* T_Y^m, \]

and in turn

\[ 0_{i_j^* T_Y}^! 0_{V(i_j^* E_1)}([D_j \times \mathbb{V}(i_j^* E_0^m)]_N) \]

\[ = 0_{i_j^* T_Y}^! 0_{V(i_j^* E_1)}([i_j^* T_Y \downarrow D_j \times \mathbb{V}(i_j^* E_0^m)]_N) \]

\[ = 0_{i_j^* T_Y}^! 0_{V(i_j^* E_1)}([D_j^{vir} \times \mathbb{V}(i_j^* E_0^m)]_N). \]

Following the discussion on [10] pg. 498, we have the commutative diagram

\[
\begin{array}{ccc}
i_j^* t^* T_Y & \longrightarrow & i_j^* D \\
\downarrow & & \downarrow \\
i_j^* t^* T_Y^m & \longrightarrow & i_j^* D / i_j^* T_Y \\
\downarrow & & \downarrow \\
X_j & \longrightarrow & \mathbb{V}(i_j^* E_1) \times_{\mathbb{V}(i_j^* E_1^1)} \mathbb{V}(i_j^* E_1^f) \times_{\mathbb{V}(i_j^* E_1^f)} \mathbb{V}(i_j^* E_1^m),
\end{array}
\]

with the “hook” arrows all closed immersions and all squares cartesian, except for the one marked “●”. The map \( \mathbb{V}(d^m) : \mathbb{V}(i_j^* E_1^m) \to \mathbb{V}(i_j^* E_1^f) \) is induced from the “moving part” of the differential \( d_j : i_j^* E_1 \to i_j^* E_0 \).

Let \( 0_{V(i_j^* E_1)}(\_\_\_\_\_\_\_\_) \) denote the scheme-theoretic pull-back with respect to the zero-section \( 0_{V(i_j^* E_1)} \). From the above diagram, we have the closed immersion

\[ b : 0^{-1}_{V(i_j^* E_1)}([D_j^{vir} \times X_j \mathbb{V}(i_j^* E_0^m)]) \hookrightarrow i_j^* t^* T_Y / i_j^* T_Y = i_j^* t^* T_Y^m \]
Since $0 \nu^{-1}(D_j \nu^!) \subset X_j$, we also have the closed immersion

$$a: 0 \nu^{-1}(D_j \nu^!) \to \nu(i_j^* E_\nu^m).$$

This gives us the commutative diagram in $\text{Sch}^N/k$

$$\begin{array}{c}
0 \nu^{-1}(D_j \nu^!) \to \nu(i_j^* E_\nu^m) \\
\downarrow a \\
i_j^* T_Y^m \to X_j.
\end{array}$$

Let $\gamma = 0 \nu^{-1}(D_j \nu^!) \nu(i_j^* E_\nu^m)],$ this class living in the equivariant Borel-Moore homology $E_N^{BM}(0 \nu^{-1}(D_j \nu^!) \nu(i_j^* E_\nu^m))/BN$. Using the excess intersection formula as in [10], §3, Lemma 1, we have

$$0 \nu^{-1}(\nu(i_j^* E_\nu^m)] \cap e_N(\nu(i_j^* E_\nu^m)) = 0 \nu^{-1}(\nu(i_j^* E_\nu^m)] \cap e_N(i_j^* T_Y^m).$$

This gives

$$[X, \phi_N^\nu \nu^!, i_j^* Y_j] \cap e(\nu(i_j^* E_\nu^m)) = 0 \nu^{-1}(\nu(i_j^* E_\nu^m)] \cap e_N(i_j^* T_Y^m).$$

By using $A^1$-homotopy invariance, it follows that the right-hand side of this identity is not changed if we replace the differential $d_j: i_j^* E_\nu \to i_j^* E_0$ with the 0-map, so that the map

$$(i_j^* E_\nu, \nu(d_0^m)): D_j \nu^! \nu(i_j^* E_\nu^m) \to \nu(i_j^* E_\nu)$$

factors as the projection $D_j \nu^! \nu(i_j^* E_\nu^m) \to D_j \nu^!$ followed by the canonical closed immersion $D_j \nu^! \to \nu(i_j^* E_\nu^m) \subset \nu(i_j^* E_\nu).$ Using the equivariant excess intersection formula, this yields

$$0 \nu^{-1}(\nu(i_j^* E_\nu^m)] \cap e_N(\nu(i_j^* E_\nu^m)) = [X, \phi_N^\nu \nu^!, i_j^* Y_j] \cap e_N(i_j^* T_Y^m).$$

or

$$[X, \phi_N^\nu \nu^!, i_j^* Y_j] \cap e_N(i_j^* T_Y^m)^{-1} = [X, \phi_N^\nu \nu^!, i_j^* Y_j] \cap e_N(i_j^* T_Y^m)^{-1}.$$

This is exactly the desired formula (6.2).

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Universität Duisburg-Essen, Fakultät Mathematik, Campus Essen, 45117 Essen, Germany

Email address: marc.levine@uni-due.de