GÖDEL’S THEOREM IS INVALID

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Abstract

Gödel’s results have had a great impact in diverse fields such as philosophy, computer sciences and fundamentals of mathematics.

The fact that the rule of mathematical induction is contradictory with the rest of clauses used by Gödel to prove his undecidability and incompleteness theorems is proved in this paper. This means that those theorems are invalid.

In section 1, a study is carried out on the mathematical induction principle, even though it is not directly relevant to the problem, just to familiarize the reader with the operations that are used later; in section 2 the rule of mathematical induction is introduced, this rule has a metamathematical character; in section 3 the original proof of Gödel’s undecidability theorem is reproduced, and finally in section 4 the same proof is given, but now with the explicit and formal use of all the axioms; this is needed to be able to use logical resolution. It is shown that the inclusion of the mathematical induction rule causes a contradiction.

Keywords: Gödel, Godel, Goedel, incompleteness, undecidability, theorem, logic, mathematical induction, resolution.

1 INTRODUCTION

Even though mathematics itself has never lost its power, the limitative theorems need the conceptual admission that there are unanswerable questions. As Howard Delong [3] relates, “previously it was thought that if a question was well-defined, that question had an answer”. An illustration of this attitude may be found in Hilbert’s address “on the infinite”, delivered in 1925:

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"As an example of the way in which fundamental questions can be treated I would like to choose the thesis that every mathematical problem can be solved. We are all convinced of that. After all, one of the things that attract us most when we apply ourselves to a mathematical problem is precisely that within us we always hear the call: here is the problem, search for the solution; you can find it by pure thought, for in mathematics there is no ignorabimus."

In the same way, Nagel and Newman [4] in his book published in 1958 confirm that “until recently it was taken as a matter of course that a complete set of axioms for any branch of mathematics can be assembled. In particular, mathematicians believed that the set proposed for arithmetic in the past was in fact complete, or, at worst, could be made complete simply by adding a finite number of axioms to the original list. The discovery that this will not work is one of Gödel’s major achievements”.

In another place of their book, Nagel and Newman [4] evaluate the effects of those theorems declaring that: “the import of Gödel’s conclusions is far reaching, though it has not been fully fathomed. These conclusions show that the prospect of finding for every deductive system (and in particular, for a system in which the whole arithmetic can be expressed) an absolute proof of consistency that satisfies the finitistic requirements of Hilbert’s proposal, though not logically impossible, is most unlikely”.

There are consequences in several fields, from philosophy to computer sciences, but the above mentioned are enough to justify the need to make sure that Gödel’s theorems are free of any kind of faults.

In the present paper a contradiction is exposed, between those theorems and an accepted and well known rule of mathematics, which allows us to conclude that such incompleteness and undecidability theorems are invalid.

In section 1, a revision is made of the mathematical induction principle, even though it is not directly relevant to the problem, just to get the reader acquainted with the operations used later; the rule of mathematical induction, which has a metamathematical character, is introduced in section 2; in section 3 the proof of Gödel’s undecidability theorem is exhibited again but now with the explicit use of all the axioms, which are used with the objective of employing logical resolution; the further inclusion of the mathematical
induction rule causes the cited contradiction.

2 THE MATHEMATICAL INDUCTION PRINCIPLE

The mathematical induction principle can be expressed in the following way:

Let $P$ be any property of natural numbers.

Suppose that:

A) $0$ (zero) has the property $P$.

B) If any natural number $x$ has property $P$, then its successor $s(x)$ also has property $P$.

Then:

C) All natural numbers $x$ have property $P$.

This “mathematical induction principle” can be expressed more concisely by using the logical connectives for conjunction ($\land$) and implication ($\rightarrow$) and the universal quantifier ($\forall$), as follows:

$$P(0) \land (\forall x)(P(x) \rightarrow P(s(x))) \rightarrow (\forall x)P(x)$$  \hspace{1cm} (1)

This formula is referred to by some authors as an *axiom*, because it corresponds to the fifth Peano’s axiom for the numbers (See Gödel [1], p.61). In the APPENDIX of this paper the author presents a prove that it is a *theorem*.

Let us work for a moment with premises (A) and (B) only:

$$P(0)$$  \hspace{1cm} (2)

$$(\forall x)(P(x) \rightarrow P(s(x)))$$  \hspace{1cm} (3)
If we assign to \( x \) the value 0 in (3), it can be proved, by means of a truth table, that (2) and (3) are logically equivalent to:

\[
P(0) \land P(s(0))
\]  

(4)

With a new value for \( x \) in (3), now of \( s(0) \), clauses (2) and (3) are equivalent to:

\[
P(0) \land P(s(0)) \land P(s(s(0)))
\]  

(5)

Expression (5) is an expansion of:

\[
(\forall x)P(x)
\]  

(6)

where the universal quantifier “for all \( x \)” refers to the elements of the set:

\[
\{ 0, s(0), s(s(0)) \}
\]  

(7)

The same operations could be successively applied to all the integers, with the result that the premises (A) and (B) are equivalent to \((\forall x)P(x)\), this is identical to the conclusion (C). So the mathematical induction principle is logically equivalent to the clause:

\[
(\forall x)P(x) \rightarrow (\forall x)P(x)
\]  

(8)

which is a tautology, and as such it does not provide any new useful information to number theory, so it is superfluous and can be removed.

3 **THE MATHEMATICAL INDUCTION RULE**

The previous conclusion does not mean, however, that Peano didn’t have something to say. But, in order to express that “something”, it is not enough a mathematical principle but a metamathematical rule. In fact, Kleene [2] has the “induction rule” expressed as:

\[
\Gamma \vdash P(0), \text{ and } \Gamma, P(x) \vdash P(s(x)), \text{ then } \Gamma \vdash P(x)
\]  

(9)
Where the turnstile symbol “$\vdash$” represents the metamathematical concept “then it is deducible”.

Let us use Gödel’s predicate “Bew”, which means “deducible formula”, instead of $\vdash$. Kleene’s induction rule can then be reexpressed as:

$$Bew(P(0)) \land (\forall x)(Bew(P(x)) \rightarrow Bew(P(s(x))))$$

$$\rightarrow Bew((\forall x)P(s(x)))$$

(10)

If the value 0 is assigned to x, as we did in the case of the induction axiom, it can be found that the antecedent of the implication (10):

$$Bew(P(0)) \land (Bew(P(0)) \rightarrow Bew(P(s(0))))$$

is equivalent to $Bew(P(0)) \land Bew(P(s(0)))$, and in this fashion, continuing successively with these operations, it is found that the antecedent rule (10) is logically equivalent to:

$$(\forall x)Bew(P(x))$$

(11)

After each operation, the universal quantifier has a range in a set that contains one additional successor than the previous set, and eventually could be applied to a set that has whatever number of elements.

Therefore, Kleene’s induction rule, (10), is logically equivalent to:

$$(\forall x)Bew(P(x)) \rightarrow Bew((\forall x)P(x))$$

(12)

Where the quantifiers on each side of the implication should have their ranges within the same set of elements.

The implication in the opposite sense:

$$Bew((\forall x)P(x)) \rightarrow (\forall x)Bew(P(x))$$

(13)

was used by Gödel, implicitly, in the first part of his proof of the undecidability theorem ([1], p.76). It also has its non-metamathematical counterpart among the universally valid rules of formal systems with the name of “specialization rule”:

$$(\forall x)P(x) \rightarrow P(c)$$

(14)
where “c” is an arbitrary constant of the domain ([1], axiom III.1, p. 62).

This “specialization rule”, as remarked before for the mathematical induction principle, has the logical deficiency of trying to express a metalogical concept within the object language. In fact, if we expand the antecedent of the sentence (14), we find, among other conjunctive expressions, the expression to the right of the implication; in other words, we attain a trivial tautology. But the real purpose can only be described by a metalogical sentence which asserts that, if the formula \((\forall x)P(x)\) can be deduced then the formula \(P(x)\) can be deduced for any particular value of \(x\) (see formula (13)).

Combining the two sentences (12) and (13) we finally find:

\[(\forall x)\text{Bew}(P(x)) \leftrightarrow \text{Bew}((\forall x)P(x))\]  

(15)

Later it will be explained how a couple of sentences, postulated by Gödel in the course of the proof of the undecidability theorem, and which predicate its own indeducibility, are in contradiction with the set of axioms used in that proof, when putting them together with the current theorem, that is, sentences (12) or (13).

The author would have preferred to propose rule (15) as a metamathematical axiom, because it seems to have a more fundamental nature than the mathematical induction rule. However, in this paper, it is better to consider it as a theorem within the normal or current mathematics. It is also possible to propose another metamathematical axiom, similar to rule (15), but using the existential quantifier instead of the universal quantifier.

4 GÖDEL’S UNDECIDABILITY THEOREM

Gödel’s undecidability and incompleteness theorems impose the mathematicians the conclusion that the axiomatic methods have some intrinsic limitations that state, for example, that even the ordinary arithmetic cannot be fully axiomatized, or that most of the more significant fields of mathematics cannot be free of internal contradiction. If we can refute the limitative theorems, we could restore the bright alternatives proposed by Leibniz and
Hilbert.

The first of Gödel’s limitative theorem, or undecidability theorem, has the number VI in the referenced author’s original paper [1], since, to arrive to it, he shows a long development within the theory of “primitive recursive functions”.

This theorem claims that, in system P (from Principia Mathematica augmented with Peano’s axioms), there is always some sentence such that neither it nor its negation is deducible in the system.

The “primitive recursive functions” play a fundamental role in mathematics, since it is generally acknowledged that its use constitutes the formal equivalent of a “finite effective method” for computing or proving something; in other words, it means the same as what we are used to calling “algorithm”. The notion of mathematical truth has this character, so any human being is capable of reproducing a (mathematical) result.

If some theorem has been proved by using the tools of the “primitive recursive functions” theory, it is totally acceptable in principle; it cannot be said the same about “non finitistic” proofs which require the use of the “infinite” in order to prove something.

Preceding the proof of his theorem, Gödel develops 46 relations and primitive recursive functions. For our current purposes we require only the last three definitions which are re-expressed in a little different notation; $\wedge$ between term represents logical conjunction, $\lor$ represents disjunction, $(\exists x)$ and $(\forall x)$ represent the existential and universal quantifiers respectively.

Gödel’s relation 46 expounds that a formula “$x$” is deducible if there exist a sequence “$y$” of formulas which represents the deduction of $x$:

$$
\text{deducible}$\text{-}$\text{formula}(x) \leftrightarrow (\exists y)(y \text{ is a deduction of formula } x)
$$

Gödel’s relation 45 is a definition of “is a deduction of formula” in terms of “deduction”:
His relation 44 is a definition of “deduction”:

\[
deduction(y) \iff (\forall x)((f \text{ is first term of } y \wedge l \text{ is last term of } y \wedge f \leq x \leq l) \rightarrow is\_axiom(x) \vee (\exists p)(\exists q)(f \leq p, q < x \wedge x \text{ is an immediate inference of } p, q))
\]  

(18)

We will not present here the definitions of “is first term of”, “is last term of”, “is axiom” and “is an immediate inference of”, appealing to its intuitive interpretation or to the original Gödel’s paper [1]. The three definitions mentioned explicitly above express that a formula is deducible if that formula is the last term of a deduction, where a deduction is a series of formulas in which each one of them is either an axiom or is obtained by an immediate inference of two preceding formulas.

Next, Gödel sketches the proof of a theorem (theorem V in the original paper [1]) which expresses that, for any n-ary relation R, it can be found a relational symbol “r” which can be deduced:

\[
R(x_1, ..., x_n) \rightarrow deducible\_formula(r(x_1, ..., x_n))
\]

(19)

and also:

\[
\sim R(x_1, ..., x_n) \rightarrow deducible\_formula(\sim r(x_1, ..., x_n))
\]

(20)

Later, Gödel defines the relation:

\[
Q(x, y) \iff x \text{ is deduction of formula } y
\]

(21)

which is constructed by using only primitive recursive relations, so Gödel concludes that Q(x, y) is also primitive recursive. This relation says that, for all “x” and for all “y”, x is not a deduction of formula y.
Consequently, by relation (21), and theorem V, there must be a relational symbol “q” such that

\[ \sim x \text{ is a deduction of formula } y \rightarrow \text{deducible formula } (q(x, y)) \quad (22) \]

\[ x \text{ is a deduction of formula } y \rightarrow \text{deducible formula}(\sim q(x, y)) \quad (23) \]

Following this, Gödel proposes the formula:

\[ p = (\forall x)q(x, p) \quad (24) \]

and calls:

\[ q(x, p) = r(x) \quad (25) \]

which, after replacing in (24), gives:

\[ p = (\forall x)q(x, p) = (\forall x)r(x) \quad (26) \]

This formula “p” declares about itself that it is not deducible. Take notice that the relation is not demonstrated and it is very strange to mathematics. It is in some sense similar to the equality relation \( z = z \) where we postulate that \( z \) in the right hand side is an arbitrary function of itself, such as \( f(x) = z + 1 \); after replacing in the equality, we get: \( z = f(z) = z + 1 \). With this kind of operations we could prove anything.

Next, Gödel substitutes “y” for “p” in (22) and (23), and applies the equivalences (25) and (26), obtaining:

\[ \sim x \text{ is a deduction of formula } (\forall x)r(x) \rightarrow \text{deducible formula } r(x) \quad (27) \]

\[ x \text{ is a deduction of formula } (\forall x)r(x) \rightarrow \text{deducible formula } (\sim r(x)) \quad (28) \]
Now Gödel is prepared to prove the undecidability theorem (theorem VI in the original paper), which states that neither formula $(\forall x)r(x)$, nor formula $\sim (\forall x)r(x)$ are deducible in the system.

Thus, in the first part he proves in the following way that:

1. $\sim$ deducible_formula $(\forall x)r(x)$

   For, if it were deducible, in other words if:
   deducible_formula $(\forall x)r(x)$,
   then, by (16), there would be an “n” such that
   n is a deduction of formula $(\forall x)r(x)$.
   But then, by (28),
   deducible_formula $(\sim r(n))$,
   Whereas, by the same:
   deducible_formula $(\forall x)r(x)$
   it is also concluded that:
   deducible_formula $(r(n))$.
   And the system would be inconsistent.

Next, in the second part, Gödel proves that:

2. $\sim$ deducible_formula $(\sim (\forall x)r(x))$

   It was just proved that:
   $\sim$ deducible_formula $(\forall x)r(x)$
   then, by (16),
   $(\forall n) \sim n$ is a deduction of formula $(\forall x)r(x)$.
   From here it follows, by (27), that
   $(\forall n)$ deducible_formula $(r(n))$,
   which, together with
   deducible_formula $(\sim (\forall x)r(x))$
   would be incompatible with the consistency of the system.
   So, he concludes, $(\forall x)r(x)$ is undecidable, finishing the proof of theorem VI.

In Gödel’s proofs in general, and in the present paper in particular, we should be aware that the arguments within the predicates are always Gödel’s numbers, which we have represented, for the sake of clarity, with the character series of the formulas they represent. The only operation allowed upon
those arguments is the substitution of “free variables” when the unification
is done in order to apply resolution.

5 DISCOVERING THE CONTRADICTION

Let us put together the set of rules, sentences or postulates used by Gödel in
the proof of theorem VI, not including yet the mathematical induction rule
\[.\] The names of the predicates will be abridged by employing the original
Gödel's terminology, where “deducible_formula” is represented by “Bew” and
“is_a_deduction_of_formula” is represented by “B”.

From (16):
\[\text{Bew}(x) \leftrightarrow (\exists y)(yBx)\]  \hspace{1cm} (29)

From (27):
\[\sim xB(\forall x)r(x) \rightarrow \text{Bew}(r(x))\]  \hspace{1cm} (30)

From (28):
\[xB(\forall x)r(x) \rightarrow \text{Bew}(\sim r(x))\]  \hspace{1cm} (31)

To this group of three clauses we have to add three sentences that Gödel
uses implicitly, without having proved or written previously. In part 1 of the
proof of his theorem VI, he uses the two rules:

\[\text{Bew}(\forall x)r(x) \rightarrow \text{Bew}(r(y))\]  \hspace{1cm} (32)

\[\sim (\text{Bew}(x) \land \text{Bew}(\sim x))\]  \hspace{1cm} (33)

Rule (32) expresses that if a relation or property “r(x)” can be deduced
for all values of variable x, then that relation can also be deduced for a
particular value of “y” (Gödel uses “n” as the value of “y”).
Rule (33) expresses the consistency of the system, in the sense that it cannot
simultaneously be deduced a formula and its negation if the system is to be
consistent.

In part 2 of his proof, Gödel uses the following rule, also without having
deduced or proved previously:

\[\sim ((\forall y)\text{Bew}(r(y)) \land \text{Bew}(\sim (\forall x)\text{Bew}(r(x))))\]  \hspace{1cm} (34)
which expresses that, if the system is going to preserve consistency, it is not possible to deduce “r(y)”, for all values of “y”, and also the negation of (\(\forall x)\text{Bew}(r(x))\).

Now we re-express the premises in clausal form:

From (2916) are obtained

\[ \sim \text{Bew}(x) \lor nBx \]  
(35)

\[ \text{Bew}(x) \lor \sim \text{yBx} \]  
(36)

From (3026):

\[ xB(\forall x)r(x) \lor \text{Bew}(r(x)) \]  
(37)

From (3127):

\[ \sim xB(\forall x)r(x) \lor \text{Bew}(\sim r(y)) \]  
(38)

From (3228):

\[ \sim \text{Bew}(\forall x)r(x) \lor \text{Bew}(r(y)) \]  
(39)

From (3329):

\[ \sim \text{Bew}(x) \lor \sim \text{Bew}(\sim x) \]  
(40)

and, from (3430):

\[ \sim \text{Bew}(r(n1)) \lor \sim \text{Bew}(\sim (\forall x)r(x)) \]  
(41)

Skolem constants n and n1 have been used to eliminate the existential quantifiers.

Let us call “S” to this set of clauses. By using the set S of clauses it is easy to reproduce Gödel’s proof by logical resolution:

To prove that (\(\forall x)r(x)\) is not deducible, we assume that it is deducible:

\[ \text{Bew}(\forall x)r(x) \]  
(42)
resolving:

\[ n \in B (\forall x) r(x) \]  \hspace{1cm} (43)

\[ \text{Bew}(\sim r(y)) \]  \hspace{1cm} (44)

\[ \sim \text{Bew}(r(y)) \]  \hspace{1cm} (45)

\[ \text{Bew}(r(y)) \]  \hspace{1cm} (46)

\[ \text{EMPTY CLAUSE} \]

As there is contradiction, it has been proved that:

\[ \sim \text{Bew}(\forall x) r(x) \]  \hspace{1cm} (47)

From here, the proof can be followed with:

\[ y \in B (\forall x) r(x) \]  \hspace{1cm} (48)

\[ \text{Bew}(r(x)) \]  \hspace{1cm} (49)

\[ \sim \text{Bew}(\sim (\forall x) r(x)) \]  \hspace{1cm} (50)

With this, Gödel finishes the proof of theorem VI, for he wanted to come to sentences (47) and (50).

Now, sentence (49) is interpreted by Gödel as:

\[ (\forall x) \text{Bew}(r(x)) \]  \hspace{1cm} (51)

which declares that, for all x, the relation r(x) is deducible.

In the following, let us consider the theorem we had obtained from the rule of induction (sentence (12)):

\[ (\forall x) \text{Bew}(r(x)) \rightarrow \text{Bew}(\forall x) r(x) \]  \hspace{1cm} (52)
Resolving between (51) and (52) we deduce:

\[ Bew(\forall x)r(x) \]  

(53)

and the inconsistency is revealed, precisely between sentences (47) and (53).

It would be a meager favor to our formal systems if similar contradictions were to appear when we introduce the mathematical induction rule (52). But we have to remark that the set S includes two special Gödel’s clauses, besides the rest which are perfectly acceptable and valid for any formal system. Those special clauses are (37) and (38), or similarly (27) and (28), which are arbitrarily postulated by Gödel and which do not need to be present in our formal systems. If clauses (37) and (38) are suppressed then it is not anymore possible to prove the contradiction among the rest of clauses belonging to the set S, which now includes the rule of induction.

6 CONCLUSION

One of the corollaries of the undecidability theorem, the incompleteness theorem, establishes that the axiomatization, of any formal system which contains, at least, the elementary arithmetic, cannot be completed, unless it becomes inconsistent. The influence of this theorem in computer science lies in the fact that a computer program is directly expressible in, or translatable to, logic (for example PROLOG), which is a formal system. The answer, to the question of what would happen if, by accident, we had completed the axiomatization of arithmetic within a program, is that we could get absolutely whatever response, since that is what happens when logical contradiction is present in a formal system. So we would always have the doubt about the reliability of computation.

The dilemma of either stop using the mathematical induction rule or stop accepting Gödel’s theorem is established. It is suggested to use a very natural rule within the formal systems, which is the rule of induction, to avoid the effect of Gödel’s theorems.
7 APPENDIX

The following short note has the purpose of revealing the logical origin of the mathematical induction principle. This provides some useful insight and generalization for its use.

8 Definitions

The mathematical induction principle is usually expressed in the following way:

Let P be any property of natural numbers.
Suppose that:
Q) 0 (zero) has the property P.
R) If any natural number x has property P, then its successor s(x) also has property P.
Then:
S) All natural numbers x have property P.

This “mathematical induction principle” can be expressed more concisely by using the logical connectives for conjunction ( ∧ ) and implication ( → ) and the universal quantifier ( ∀ ), as follows:

\[ P(0) \land (\forall x)(P(x) \rightarrow P(s(x))) \rightarrow (\forall x)P(x) \]  (54)

This formula is referred to by some authors as an “axiom”, because it corresponds to the fifth Peano’s axiom for the numbers (See Gödel [1], p.61).

9 The normal use of the mathematical induction principle

With the previous definitions, the form of the induction schema is:  \( Q \land R \rightarrow S \)

If the induction principle is to be applied to some problem, you have to ascertain that it is true Q (the base case) and, independently, it must be true R (the induction step). This means that you have the following three clauses:
1. \( Q \land R \rightarrow S \) (induction principle)
2. \( Q \) (base case)
3. \( R \) (induction step)

Then, by using resolution between 1, 2 and 3, it is concluded “S”. This is the classic or normal use of induction.

10 The logical origin of the mathematical induction principle

If the mathematical induction principle were in fact an “axiom”, it could not be demonstrated. But, in the following, it will be suggested an argument which seems to be a proof of it.

What I am going to do is to take only clauses 2 and 3, namely Q and R, or the base case and the induction step, and apply resolution between them. Obviously, we cannot use only the propositional letters but the full predicate notation. Let us assume that we apply the mathematical induction principle to the set of non-negative integers:

\[
P(0) \tag{55}
\]

\[
(\forall x)(P(x) \rightarrow P(s(x))) \tag{56}
\]

Next, assign to \( x \) the value 0 in (56). It can be proved, by means of a truth table that the conjunction of clause (55) and the cited instantiation of (56) is logically equivalent to:

\[
P(0) \land P(s(0)) \tag{57}
\]

With a new value for \( x \) in (56), now of \( s(0) \), clauses (56) and (57) are logically equivalent to:

\[
P(0) \land P(s(0)) \land P(s(s(0))) \tag{58}
\]

Expression (58) can be recognized that is an expansion of:

\[
(\forall x)P(x) \tag{59}
\]
where the universal quantifier “for all x“ refers to the elements of the set:

\{0, \ s(0), \ s(s(0))\} \quad (60)

This is a growing set of elements. For, in each resolution step, a new element appears, and the cardinality of the set is increased by one. So, it is concluded that the left part of the mathematical induction principle, expressions (55) and (56) is logically equivalent to \((\forall x)P(x)\), expression (59), where the universal quantifier applies to a potentially infinite set of numbers.

This is the origin of the mathematical induction principle.

11 What to think about this

A “potential infinite” is, by definition, a “growing but never completed set of elements”; and, according to Gauss, this is the only allowable infinite, because he didn’t thought that the infinite could be closed.

In the previous section it has been shown that the left hand side of the mathematical induction principle implies the same result in the right hand side: \((\forall x)P(x)\).

Consequently, we have found a tautology of the form: \((\forall x)P(x) \rightarrow (\forall x)P(x)\)

But many current mathematicians do not accept this.

Because, when they conclude the right hand side, \((\forall x)P(x)\), they are probably thinking in a closed infinite, as they are cantorian.

Moreover, the mathematicians feel “entitled to stipulate what the domain is”, in the right hand side.

I see no reason to imply that the conclusion applies to some other set of elements, such as a set of characters, a set of lists, etc., different from the set of numbers with which I was working in the left hand side.

However, in some interesting mails received by me, this is precisely the objection given by a mathematician from the University of Illinois, to whom
I will refer in the following by his initials “HD”. He concludes that the mathematical induction principle cannot be a theorem (or a tautology).

Let us analyze his example: Assume that the predicate “P” means “even” and that the function “successor” is defined as s(s(x)) (or “x+2”); then, if we apply the induction principle, we obtain that the left hand side of the induction principle (namely Q ∧ R) is true; but, according to HD, the conclusion isn’t true because not every number is even; or, in other words, the clause (∀x)P(x) isn’t true.

Let us analyze it in more detail. The individual resolutions between the base case and the induction step have shown us what the set of elements we are dealing with is. In fact, repeating the above argument, each instantiation of x, in each induction step (resolution), is increasing by one element the set of elements to which induction applies. Consequently, the interpretation of “(∀x)” should be: “all the elements for which either the base case applies (zero in the example) or the numbers obtained in each induction step”. We obtain the set: {0, s(s(0)), s(s(s(s(0)))),...}, which is the set of even numbers, to which the conclusion does apply. For this set it is true that (∀x)P(x).

But, according to HD:

“You are right that the set {0, s(s(0)), etc.} is the set of even numbers. But the domain of my model is the set of all natural numbers. I am entitled to stipulate what the domain is. In that event, the universal quantifier says “all natural numbers” and the formula (∀x)P(x) says “all natural numbers are even.” I would add that it is well-known among mathematical logicians that the induction schema is falsifiable. HD”

What are the implications of this kind of reasoning?

First, it seems that the mathematicians jump to one conclusion that is not logically given by the premises. For, although in the previous example it is possible to continue indefinitely producing resolutions until the property for any given number is proved, the mathematicians conclude the same thing, and maybe faster, but for a different set of elements (a closed infinite) and, probably, because the god given law called mathematical induction principle says so.
Moreover, they are going a further step by supposing that, in the right hand side, they are entitled to stipulate what the domain is, with which the logical conclusion is not only extrapolated but lost.

With the use of the mathematical induction rule, which is a little different to what has already been said, and which has a metamathematical character, the author has been able to develop the refutation to the Gödel incompleteness theorems, contained in the main body of this paper.

Such refutation consists in reexpressing, the left hand side of the mathematical induction rule (see Kleene [2]), by using the classical mathematical induction principle that has been reviewed in this appendix. It is obtained the following formula, which is logically equivalent to the induction rule:

\[(∀x)\text{Bew}(P(x)) \rightarrow \text{Bew}((∀x)P(x))\]

Where \“(∀x)\text{Bew}(P(x))” means that there are proofs for all: \(\vdash P(0),\ \vdash P(1),\ \vdash P(2),\) etc.

“\text{Bew}” is a predicate used by Gödel with the meaning of “is deducible”.

This formula enters in contradiction with the premises of Gödel incompleteness and undecidability theorems, with the consequence that such theorems are invalid.

References

[1] Gödel, Kurt, Obras completas. Alianza Editorial. 1981.

[2] Kleene, Stephen. Introducción a la metamatemática. Editorial TEC-NOS. 1974.

[3] Delong, Howard. A profile of mathematical logic. Addison Wesley. 1970.
[4] Nagel & Newman. Gödel’s Proof. New York University Press. 1958.