On uniformly $S$-coherent rings

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Abstract

In this paper, we introduce and study the notions of uniformly $S$-finitely presented modules and uniformly $S$-coherent rings (modules) which are “uniform” versions of ($c$-) $S$-finitely presented modules and ($c$-) $S$-coherent rings (modules) introduced by Bennis and Hajoui [3]. Among the results, uniformly $S$-versions of Chase’s result, Chase Theorem and Matlis Theorem are obtained.

Key Words: uniformly $S$-coherent ring; uniformly $S$-finitely presented module; uniformly $S$-coherent modules; uniformly $S$-flat module; uniformly $S$-injective module.

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1. Introduction

Throughout this paper, all rings are commutative with identity. Let $R$ be a ring. For a subset $U$ of an $R$-module $M$, we denote by $\langle U \rangle$ the submodule of $M$ generated by $U$. A subset $S$ of $R$ is called a multiplicative subset of $R$ if $1 \in S$ and $s_1s_2 \in S$ for any $s_1 \in S$, $s_2 \in S$.

The study of commutative rings in terms of multiplicative sets began with Anderson and Dumitrescu [1], who introduced the notion of $S$-Noetherian rings. Recall that a ring $R$ is called an $S$-Noetherian ring if for any ideal $I$ of $R$, there is a finitely generated sub-ideal $K$ of $I$ such that $sI \subseteq K$ for some $s \in S$. Cohen’s Theorem, Eakin-Nagata Theorem and Hilbert Basis Theorem for $S$-Noetherian rings are also given in [1]. However, the element $s \in S$ in the definition of $S$-Noetherian rings is not “uniform” in general. This situation make it difficult to study $S$-Noetherian rings via module-theoretic methods. To overcome this difficulty, Qi et al. [16] defined uniformly $S$-Noetherian rings as $S$-Noetherian rings in which definition the choice of $s$ is fixed. Then they characterized uniformly $S$-Noetherian rings using $u$-$S$-injective modules.

Recall from [7] that a ring $R$ is said to be a coherent ring provided that any finitely generated ideal is finitely presented. The notion of coherent rings, which is a
generalization of Noetherian rings, is another important rings defined by finiteness condition. Many algebraists studied coherent rings in terms of various of modules. Early in 1960, Chase [5, Theorem 2.1] showed that a ring is coherent exactly when the class of flat modules is closed under direct product. In 1970 Stenström [19, Theorem 3.2] obtained coherent rings are rings when any direct limits of absolutely pure modules is absolutely pure. In 1982, Matlis [14, Theorem 1] proved that a ring $R$ is coherent if and only if $\text{Hom}_R(M, E)$ is flat for any injective modules $M$ and $E$.

To extend coherent rings by multiplicative sets, Bennis et al. [3] introduced the notions of $S$-coherent rings and $c\cdot S$-coherent rings. They also gave an $S$-version of Chase’s result to characterize $S$-coherent rings using ideals. Recently, the authors in this paper et al. [17] characterized $S$-coherent rings in terms of $S$-Mittag-Leffler modules and $S$-flat modules (which can be seen as flat modules by localizing at $S$). The main motivation of this paper is to introduce and study the “uniform” version of $S$-coherent rings for extending uniformly $S$-Noetherian rings. The organization of the paper is as follows: In Section 2, we introduce and study uniformly $S$-finitely presented modules and their connections with $u\cdot S$-flat modules and $u\cdot S$-projective modules (see Proposition 2.8). In Section 3, we introduce uniformly $S$-coherent modules and uniformly $S$-coherent rings. In particular, we study the ideal-theoretic characterizations of uniformly $S$-coherent rings (see Proposition 3.11). Moreover examples of $S$-coherent rings and $c\cdot S$-coherent rings which are not uniformly $S$-coherent of are provided (see Example 3.15). In Section 4, Chase Theorem and Matlis Theorem for uniformly $S$-coherent rings are obtained (see Theorem 4.4 and Theorem 4.7).

Since the paper involves uniformly torsion theory, we give a quick review (see [24, Lemma 2.1] for more details). An $R$-module $T$ is called $u\cdot S$-torsion (with respect to $s$) provided that there exists $s \in S$ such that $sT = 0$. An $R$-sequence $\cdots \rightarrow A_{n-1} \xrightarrow{f_n} A_n \xrightarrow{f_{n+1}} A_{n+1} \rightarrow \cdots$ is $u\cdot S$-exact, if for any $n$ there is an element $s \in S$ such that $s\text{Ker}(f_{n+1}) \subseteq \text{Im}(f_n)$ and $s\text{Im}(f_n) \subseteq \text{Ker}(f_{n+1})$. An $R$-sequence $0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$ is called a short $u\cdot S$-exact sequence (with respect to $s$), if $s\text{Ker}(g) \subseteq \text{Im}(f)$ and $s\text{Im}(f) \subseteq \text{Ker}(g)$ for some $s \in S$. An $R$-homomorphism $f : M \rightarrow N$ is an $u\cdot S$-monomorphism (resp., $u\cdot S$-epimorphism, $u\cdot S$-isomorphism) (with respect to $s$) provided $0 \rightarrow M \xrightarrow{f} N$ (resp., $M \xrightarrow{f} N \rightarrow 0$, $0 \rightarrow M \xrightarrow{f} N \rightarrow 0$) is $u\cdot S$-exact (with respect to $s$). Suppose $M$ and $N$ are $R$-modules. We say $M$ is $u\cdot S$-isomorphic to $N$ if there exists a $u\cdot S$-isomorphism $f : M \rightarrow N$. A family $\mathcal{C}$ of $R$-modules is said to be closed under $u\cdot S$-isomorphisms if $M$ is $u\cdot S$-isomorphic to $N$ and $M$ is in $\mathcal{C}$, then $N$ is also in $\mathcal{C}$. One can deduce from the following [24, Lemma 2.1] that the existence of $u\cdot S$-isomorphisms of two $R$-modules is actually an equivalence relationship.
2. Uniformly S-finitely presented modules

Recall from [1] that an R-module M is called S-finite (with respect to s) provided that there is an element s ∈ S and a finitely generated R-module F such that sM ⊆ F ⊆ M. Trivially, S-finite modules are generalizations of finitely generated modules. For generalizing finitely presented R-modules, Bennis et al. [3] introduced the notions of S-finitely presented modules and c-S-finitely presented modules. Following [3, Definition 2.1] that an R-module M is called S-finitely presented provided that there exists an exact sequence of R-modules 0 → K → F → M → 0 with K S-finite and F finitely generated free. Certainly, an R-module M is S-finitely presented if and only if there exists an exact sequence of R-modules

0 → T_1 → N → M → 0

with N finitely presented and sT_1 = 0 for some s ∈ S. Following [3, Definition 4.1] that an R-module M is called c-S-finitely presented provided that there exists a finitely presented submodule N of M such that sM ⊆ N ⊆ M for some s ∈ S. Trivially, an R-module M is called c-S-finitely presented if and only if there exists an exact sequence of R-modules 0 → N → M → T_2 → 0 with N finitely presented and sT_2 = 0 for some s ∈ S. Next we will give the notion of uniformly S-finitely presented modules which generalize both S-finitely presented modules and c-S-finitely presented modules.

Definition 2.1. Let R be a ring, S a multiplicative subset of R and s ∈ S. An R-module M is called u-S-finitely presented (abbreviates uniformly S-finitely presented) (with respect to s) provided that there is an exact sequence

0 → T_1 → F → M → 0

with F finitely presented and sT_1 = sT_2 = 0.

Trivially, S-finitely presented modules and c-S-finitely presented modules are all u-S-finitely presented. Certainly, every u-S-finitely presented R-module is S-finite. Indeed, since in definition [2.1] we have sT_2 = 0, so sM ⊆ Im(f). Note that the fact that Im(f) is finitely generated implies M is S-finite.

By [24, Lemma 2.1], an R-module M is u-S-finitely presented if and only if there is an exact sequence 0 → T_1 → M  f → F → T_2 → 0 with F finitely presented and s'T_1 = s'T_2 = 0 for some s' ∈ S. So an R-module M is u-S-finitely presented if and only if it is u-S-isomorphic to a finitely presented R-module.

Theorem 2.2. Let Φ : 0 → M  f → N  q → L → 0 be a u-S-exact sequence of R-modules. The following statements hold.

1. The class of u-S-finitely presented modules is closed under u-S-isomorphisms.
2. If M and L are u-S-finitely presented, so is N.
(3) Any finite direct sum of $u$-$S$-finitely presented modules is $u$-$S$-finitely presented.

(4) If $N$ is $u$-$S$-finitely presented, then $L$ is $u$-$S$-finitely presented if and only if $M$ is $S$-finite. Moreover, if $\Phi$ is an exact sequence, the both side of conditions in (2) and (4) can be taken to be “uniform” with respective to a same $s \in S$.

Proof. (1) It follows from the fact that an $R$-module $M$ is $u$-$S$-finitely presented if and only it is $u$-$S$-isomorphic to a finitely presented $R$-module.

(2) Since $u$-$S$-finitely presented modules are closed under $u$-$S$-isomorphisms, we may assume $\Phi$ is an exact sequence by (1). Consider the following push-out:

$$
\begin{array}{cc}
0 & M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{h} 0 \\
0 & F_1 \xrightarrow{m} X \xrightarrow{n} L \xrightarrow{l} 0.
\end{array}
$$

with $F_2$ finitely presented, Ker$(h)$ and Coker$(h)$ $u$-$S$-torsion. So $l$ is also a $u$-$S$-isomorphism. Consider the following pull-back:

$$
\begin{array}{cc}
0 & F_1 \xrightarrow{m} X \xrightarrow{n} L \xrightarrow{k} Z \xrightarrow{j} 0 \\
0 & F_1 \xrightarrow{j} Y \xrightarrow{k} F_2 \xrightarrow{l} 0.
\end{array}
$$

with $F_2$ finitely presented, Ker$(j)$ and Coker$(j)$ $u$-$S$-torsion. So $k$ is also a $u$-$S$-isomorphism. Since $F_1$ and $F_2$ are finitely presented, $Y$ is also finitely presented. Hence $N$ is $u$-$S$-isomorphic to a finitely presented $R$-module, and thus is $u$-$S$-finitely presented.

(3) Follows from (2).

(4) Since $u$-$S$-finitely presented modules and $S$-finite modules are closed under $u$-$S$-isomorphisms respectively, we may assume $\Phi$ is an exact sequence by (1). Suppose $M$ is $S$-finite. Since $N$ is $u$-$S$-finitely presented, there is an exact sequence $0 \rightarrow T_1 \rightarrow F \xrightarrow{l} N \rightarrow T_2 \rightarrow 0$ with $F$ finitely presented and $sT_1 = sT_2 = 0$ for some $s \in S$. Consider the following pull-back of $f$ an $l$:

$$
\begin{array}{cc}
0 & M \xrightarrow{f} N \xrightarrow{g} L \xrightarrow{s} Z \xrightarrow{t} F \xrightarrow{l} K \xrightarrow{t} 0 \\
0 & Z \xrightarrow{t} F \xrightarrow{l} K \xrightarrow{t} 0.
\end{array}
$$

Since $l$ is a $u$-$S$-isomorphism, $s$ and $t$ are both $u$-$S$-isomorphisms. So $Z$ is also $S$-finite. Note that $L$ is $u$-$S$-isomorphic to $K$ which is $u$-$S$-finitely presented (see
So $L$ is $u$-$S$-finitely presented. Suppose $L$ is $u$-$S$-finitely presented. Considering the above pull-back, we have $K$ is also $S$-finitely presented. Hence $Z$ is $S$-finite by [3, Theorem 2.4(5)] which implies that $M$ is also $S$-finite.

The “Moreover” part can be checked by the proof of (2) and (4). □

Recall from [4] that an $R$-module $M$ is said to be $S$-Noetherian provided that any submodule of $M$ is $S$-finite. A ring $R$ is called $S$-Noetherian if $R$ itself is $S$-Noetherian.

**Proposition 2.3.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then a ring $R$ is $S$-Noetherian if and only if any $S$-finite module is $u$-$S$-finitely presented.

**Proof.** For necessity, let $M$ be an $S$-finite module. Then there is a $u$-$S$-epimorphism $f : F \to M$ with $F$ finitely generated free. Since $R$ is an $S$-Noetherian ring, we have $F$ is also $S$-Noetherian (see [4]). Hence $M$ is $u$-$S$-finitely presented by Theorem 2.2(4). For sufficiency, let $I$ be an ideal of $R$. Then $R/I$ is $S$-finite, and thus $u$-$S$-finitely presented. By Theorem 2.2(4) again, $I$ is $S$-finite. □

**Proposition 2.4.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ consisting of finite elements. Then an $R$-module $M$ is a $u$-$S$-finitely presented $R$-module if and only if $M_S$ is a finitely presented $R_S$-module.

**Proof.** Suppose $M$ is a $u$-$S$-finitely presented $R$-module. there is an exact sequence $0 \to T_1 \to N \xrightarrow{f} M \to T_2 \to 0$ with $N$ finitely presented and $sT_1 = sT_2 = 0$. Localizing at $S$, we have $0 \to (T_1)_S \to N_S \xrightarrow{f_S} M_S \to (T_2)_S \to 0$. Since $sT_1 = sT_2 = 0$, $(T_1)_S = (T_2)_S = 0$. So $M_S \cong N_S$ is a finitely generated $R_S$-module. On the other hand, suppose $M_S$ is a finitely generated $R_S$-module. Suppose $S = \{s_1, \ldots, s_n\}$ and set $s = s_1 \cdots s_n$. We may assume that $M_S$ is generated by $\{\frac{m_1}{s}, \ldots, \frac{m_n}{s}\}$. Consider the $R$-homomorphism $f : R^n \to M$ satisfying $f(e_i) = m_i$ for each $i = 1, \ldots, n$. It is easy to verify $f$ is a $u$-$S$-epimorphism. Consider the exact sequence $0 \to \text{Ker}(f_S) \to R^n_S \xrightarrow{f_S} M_S \to 0$. Then $\text{Ker}(f_S)$ is a finitely generated $R_S$-module, and thus $\text{Ker}(f)$ is $S$-finite. By Theorem 2.2(2), $M$ is $u$-$S$-finitely presented. □

Let $p$ be a prime ideal of $R$. We say an $R$-module $M$ is (simply) $p$-finite provided $R$ is $(R \setminus p)$-finite. We always denote by Spec($R$) the spectrum of all prime ideals of $R$, and Max($R$) the set of all maximal ideals of $R$, respectively.

**Lemma 2.5.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. The following statements are equivalent:

1. $M$ is finitely generated $R$-module;
Proposition 2.6. The following statements are equivalent:

(2) $M$ is $p$-finite for any $p \in \text{Spec}(R)$;

(3) $M$ is $m$-finite for any $m \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) For each $m \in \text{Max}(R)$, there exists an element $s^m \in R \setminus m$ and a finitely generated submodule $F^m$ of $M$ such that $s^m M \subseteq F^m$. Since $\{s^m | m \in \text{Max}(R)\}$ generated $R$, there exist finite elements $\{s^{m_1}, ..., s^{m_n}\}$ such that $\langle s^{m_1}, ..., s^{m_n} \rangle = R$. So $M = \langle s^{m_1}, ..., s^{m_n} \rangle M \subseteq F^{m_1} + ... + F^{m_n} \subseteq M$. Hence $M = F^{m_1} + ... + F^{m_n}$. It follows that $M$ is finitely generated.

Let $p$ be a prime ideal of $R$. We say an $R$-module $M$ is (simply) $u$-$p$-finitely presented provided $R$ is $u-(R \setminus p)$-finitely presented.

Proposition 2.6. Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. The following statements are equivalent:

(1) $M$ is a finitely presented $R$-module;

(2) $M$ is $u$-$p$-finitely presented for any $p \in \text{Spec}(R)$;

(3) $M$ is $u$-$m$-finitely presented for any $m \in \text{Max}(R)$.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) By Lemma 2.5, $M$ is finitely generated. Consider the exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with $F$ finitely generated free. By Theorem 2.2, $K$ is $m$-finite for any $m \in \text{Max}(R)$. So $K$ is also finitely generated, and thus $M$ is finitely presented.

Let $\{M_j\}_{j \in \Gamma}$ be a family of $R$-modules and $N_j$ a submodule of $M_j$ generated by $\{m_{i,j} \}_{i \in \Lambda_j} \subseteq M_j$ for each $j \in \Gamma$. Recall from [21] that a family of $R$-modules $\{M_j\}_{j \in \Gamma}$ is $u$-$S$-generated (with respective to $s$) by $\{\langle m_{i,j} \rangle_{i \in \Lambda_j} \}_{j \in \Gamma}$ provided that there exists an element $s \in S$ such that $sM_j \subseteq N_j$ for each $j \in \Gamma$, where $N_j = \langle \{m_{i,j} \}_{i \in \Lambda_j} \rangle$. We say a family of $R$-modules $\{M_j\}_{j \in \Gamma}$ is $u$-$S$-finite (with respective to $s$) if the set $\{m_{i,j} \}_{i \in \Lambda_j}$ can be chosen as a finite set for each $j \in \Gamma$, that is, there is $s \in S$ such that $\{M_j\}_{j \in \Gamma}$ are all $S$-finite with respect to $s$. Recall from [16] that an $R$-module $M$ is called a $u$-$S$-Noetherian module provided the set of all submodules of $M$ is $u$-$S$-finite. A ring $R$ is called to be a $u$-$S$-Noetherian ring provided that $R$ itself is a $u$-$S$-Noetherian $R$-module.

Theorem 2.7. Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then the following statements are equivalent:

(1) A ring $R$ is $u$-$S$-Noetherian;

(2) Any $S$-finite module is $u$-$S$-Noetherian;
(3) Any finitely generated module is $u$-$S$-Noetherian;
(4) There is $s \in S$ such that any finitely generated module is $u$-$S$-finitely presented with respective to $s$.

**Proof.** (1) $\Rightarrow$ (2) Let $M$ be an $S$-finite module. Then there is a $u$-$S$-epimorphism $f : F \to M$ with $F$ finitely generated free. Since $R$ is $u$-$S$-Noetherian, we have $F$ is also $u$-$S$-Noetherian, and so is $M$ (see [16, Proposition 2.13]).

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) Trivial.

(4) $\Rightarrow$ (1) Let $I$ be an ideal of $R$. Then $R/I$ is $u$-$S$-finitely presented with respective to $s$. So $I$ is $S$-finite with respective to $s$ by Theorem 2.2, implying $R$ is $u$-$S$-Noetherian. □

Recall from [21, 24] that an $R$-module $P$ is called $u$-$S$-projective (resp., $u$-$S$-flat) provided that the induced sequence $0 \to \text{Hom}_R(P, A) \to \text{Hom}_R(P, B) \to \text{Hom}_R(P, C) \to 0$ (resp., $0 \to P \otimes_R A \to P \otimes_R B \to P \otimes_R C \to 0$) is $u$-$S$-exact for any $u$-$S$-exact sequence $0 \to A \to B \to C \to 0$. It was proved in [24, Proposition 2.9] that any $u$-$S$-projective module is $u$-$S$-flat.

**Proposition 2.8.** Let $R$ be a ring and $S$ a multiplicative subset of $R$. Then the following statements hold.

(1) Every $S$-finite $u$-$S$-projective module is $u$-$S$-finitely presented.
(2) Every $u$-$S$-finitely presented $u$-$S$-flat module is $u$-$S$-projective.

**Proof.** (1) Let $P$ be an $S$-finite $u$-$S$-projective module, then there is a $u$-$S$-exact sequence $\Psi : 0 \to \ker(f) \xrightarrow{i} F \xrightarrow{j} P \to 0$ with $F$ finitely generated free. Since $P$ is $u$-$S$-projective, the sequence $\Psi$ is $u$-$S$-split by [24, Theorem 2.7]. So there is a $u$-$S$-epimorphism $i' : F \to \ker(f)$ such that $i' \circ i = s\text{Id}_{\ker(f)}$ for some $s \in S$. Hence $\ker(f)$ is $S$-finite, and so $P$ is $u$-$S$-finitely presented by Theorem 2.2.

(2) Let $M$ be a $u$-$S$-finitely presented $u$-$S$-flat module. Then there is a $u$-$S$-exact sequence $\Upsilon : 0 \to \ker(f) \xrightarrow{i} F \xrightarrow{j} M \to 0$ with $F$ finitely generated free and $\ker(f)$ $S$-finite. Since $M$ is $u$-$S$-flat, $\Upsilon$ is $u$-$S$-pure by [22, Proposition 2.4]. It follows from [22, Theorem 2.2] that $\Upsilon$ is $u$-$S$-split. Thus $M$ is $u$-$S$-projective. □

**3. Uniformly $S$-Coherent Modules and Uniformly $S$-Coherent Rings**

Recall that an $R$-module is said to be a coherent module if it is finitely generated and any finitely generated submodule is finitely presented. A ring $R$ is said to be a coherent ring if $R$ is a coherent $R$-module. In this section, we will introduce a “uniform” version of coherent rings and coherent modules.
Definition 3.1. Let $R$ be a ring and $S$ a multiplicative subset of $R$. An $R$-module $M$ is called a $u$-$S$-coherent module (abbreviates uniformly $S$-coherent) (with respective to $s$) provided that there is $s \in S$ such that it is $S$-finite with respect to $s$ and any finitely generated submodule of $M$ is $u$-$S$-finitely presented with respective to $s$.

Theorem 3.2. Let $\Phi : 0 \to M \xrightarrow{f} N \xrightarrow{g} L \to 0$ be a $u$-$S$-exact sequence of $R$-modules. The following statements hold.

1. The class of $u$-$S$-coherent modules is closed under $u$-$S$-isomorphisms.
2. If $L$ is $u$-$S$-coherent, then $M$ is $u$-$S$-coherent if and only if $N$ is $u$-$S$-coherent.
3. Any finite direct sum of $u$-$S$-coherent modules is $u$-$S$-coherent.
4. If $N$ is $u$-$S$-coherent and $M$ is $S$-finite, then $L$ is $u$-$S$-coherent.

Proof. (1) Let $h : A \to B$ be a $u$-$S$-isomorphism with $s_1\ker(h) = s_1\coker(h) = 0$. Suppose $B$ is $u$-$S$-coherent with respective to $s_2$, then one can check $A$ is $u$-$S$-coherent with respective to $s_1s_2$. Similarly, if $A$ is $u$-$S$-coherent, then $B$ is also $u$-$S$-coherent (see [24, Lemma 2.1]).

(2) By (1), we can assume that $\Phi$ is an exact sequence. Suppose $M$ and $L$ are $u$-$S$-coherent with respective to $s$. Then one can check $N$ is $u$-$S$-coherent with respective to $s$ from the proof of Theorem [22(2)]. Suppose $N$ and $L$ are $u$-$S$-coherent with respective to $s$. Then $M$ is $S$-finite with respective to some $s \in S$ by Theorem [22(4)]. Since $N$ is $u$-$S$-coherent with respective to $s$, $M$ is $u$-$S$-coherent with respective to $s$.

(3) Follows by (2).

(4) Assume that $\Phi$ is an exact sequence. Suppose $N$ is $u$-$S$-coherent with respective to $s$ and $M$ is $S$-finite with respective to $s$ for some $s \in S$. Then $L$ is also $S$-finite with respective to $s$. Let $K$ be a finitely generated submodule of $L$. Then the sequence $0 \to M \to g^{-1}(K) \to K \to 0$ is exact. So $g^{-1}(K)$ is $S$-finite. Consider the following commutative diagram with rows and columns exact:

```
0 \to \ker(m) \to \ker(n) \to K_1 \to 0
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow
0 \to R^n \to R^{n+s} \to R^s \to 0
\downarrow m \quad \downarrow n \quad \downarrow
0 \to M \to g^{-1}(K) \to K \to 0
\downarrow \quad \downarrow \quad \downarrow
0 \quad 0
```
where $m$ and $n$ are $u$-$S$-epimorphisms. Since $N$ is $u$-$S$-coherent, Ker($n$) is $S$-finite, and so is $K_1$. Thus $L$ is $u$-$S$-coherent (with respective to $s$).

**Corollary 3.3.** Let $f : M \to N$ be an $R$-homomorphism of $u$-$S$-coherent modules $M$ and $N$. Then Ker($f$), Im($f$) and Coker($f$) are also $u$-$S$-coherent.

*Proof.* Using Theorem 3.2 and the exact sequences $0 \to$ Ker($f$) $\to$ $M$ $\to$ Im($f$) $\to$ 0 and $0 \to$ Im($f$) $\to$ $N$ $\to$ Coker($f$) $\to$ 0. □

**Corollary 3.4.** Let $M$ and $N$ be $u$-$S$-coherent sub-modules of a $u$-$S$-coherent module. Then $M + N$ is $u$-$S$-coherent if and only if so is $M \cap N$.

*Proof.* Following by Theorem 3.2 and the exact sequence $0 \to$ $M \cap N$ $\to$ $M \oplus N$ $\to$ $M + N$ $\to$ 0. □

Let $p$ be a prime ideal of $R$. We say an $R$-module $M$ is (simply) $u$-$p$-coherent provided $R$ is $u$-($R \setminus p$)-coherent.

**Proposition 3.5.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $M$ an $R$-module. The following statements are equivalent.

1. $M$ is a coherent $R$-module.
2. $M$ is $u$-$p$-coherent for any $p \in \text{Spec}(R)$.
3. $M$ is $u$-$m$-coherent for any $m \in \text{Max}(R)$.

*Proof.* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Trivial. 

(3) $\Rightarrow$ (1) By Lemma 2.5, $M$ is finitely generated. Let $N$ be a finitely generated of $M$, then $M$ is $u$-$m$-finitely presented for any $m \in \text{Max}(R)$. So $M$ is finitely presented by Proposition 2.6. □

**Definition 3.6.** Let $R$ be a ring, $S$ a multiplicative subset of $R$ and $s \in S$. Then $R$ is called a $u$-$S$-coherent ring (abbreviates uniformly $S$-coherent) ring (with respective to $s$) provided that $R$ itself is a uniformly $S$-coherent $R$-module with respective to $s$.

Trivially, every coherent ring is $u$-$S$-coherent for any multiplicative set $S$. And if $S$ is composed of units, then $u$-$S$-coherent rings are exactly coherent rings.

The proof of the following result is easy and direct, so we omit it.

**Lemma 3.7.** Let $R = R_1 \times R_2$ be direct product of rings $R_1$ and $R_2$, $S = S_1 \times S_2$ a multiplicative subset of $R$. Then $R$ is $u$-$S$-coherent if and only if $R_i$ is $u$-$S_i$-coherent for any $i = 1, 2$.

The following example shows that not every $u$-$S$-coherent rings is coherent.
Example 3.8. Let \( R_1 \) be a coherent ring and \( R_2 \) a non-coherent ring, \( S_1 = \{1\} \) and \( S_2 = \{0\} \). Set \( R = R_1 \times R_2 \) and \( S = S_1 \times S_2 \). Then \( R \) is \( u\)-\( S \)-coherent non-coherent ring.

Let \( \mathfrak{p} \) be a prime ideal of \( R \). We say a ring \( R \) is (simply) \( u\)-\( \mathfrak{p} \)-coherent provided \( R \) is \( u\)-(\( R \setminus \mathfrak{p} \))-coherent.

Proposition 3.9. Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). The following statements are equivalent.

(1) \( R \) is a coherent ring.
(2) \( R \) is a \( u\)-\( \mathfrak{p} \)-coherent ring for any \( \mathfrak{p} \in \text{Spec}(R) \).
(3) \( R \) is a \( u\)-\( \mathfrak{m} \)-coherent ring for any \( \mathfrak{m} \in \text{Max}(R) \).

Proof. Follows by Proposition 3.5.

Proposition 3.10. Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). If \( R \) is a \( u\)-\( S \)-Noetherian ring, then \( R \) is \( u\)-\( S \)-coherent.

Proof. Follows from Theorem 2.7.

Trivially, \( u\)-\( S \)-coherent rings are not \( u\)-\( S \)-Noetherian in general. Indeed, we can find a non-Noetherian coherent ring in the case that \( S = \{1\} \).

In 1960, Chase characterized coherent rings by considering annihilator of elements and intersection of finitely generated ideals in [5, Theorem 2.2]. Now, we give a “uniform” version of Chase’s result.

Proposition 3.11. (Chase’s result for \( u\)-\( S \)-coherent rings) Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \). Then the following assertions are equivalent:

(1) \( R \) is a \( u\)-\( S \)-coherent ring;
(2) there is \( s \in S \) such that \((0 :_R r)\) is \( S \)-finite with respective to \( s \) for any \( r \in R \), and the intersection of two finitely generated ideals of \( R \) is \( S \)-finite with respective to \( s \);
(3) there is \( s \in S \) such that \((I :_R b)\) is \( S \)-finite with respective to \( s \) for any element \( b \in R \) and any finitely generated ideal \( I \) of \( R \).

Proof. (1) \( \Rightarrow \) (2): Suppose \( R \) is \( u\)-\( S \)-coherent with respective to \( s \). Considering the exact sequence \( 0 \rightarrow (0 :_R r) \rightarrow R \rightarrow Rr \rightarrow 0 \), we have \((0 :_R r)\) is \( S \)-finite with respective to \( s \) by Theorem 2.2. For any two finitely generated ideals \( I, J \) of \( R \), we have \( I \cap J \) is \( S \)-finite with respective to \( s \) by Corollary 3.4 and Theorem 2.2.

(2) \( \Rightarrow \) (1): Let \( I = (a_1, \ldots, a_n) \) be a finitely generated ideal of \( R \). We claim that \( I \) is \( u\)-\( S \)-finitely presented with respective to \( s \) by induction on \( n \). Suppose \( n = 1 \).
the claim follows by the exact sequence \( 0 \to (0 :_R r) \to R \to Rr \to 0 \). Suppose \( n = k \), the claim holds. Suppose \( n = k + 1 \). the claim holds by the exact sequence \( 0 \to \langle a_1, \cdots, a_k \rangle \cap \langle a_{k+1} \rangle \to \langle a_1, \cdots, a_k \rangle \oplus \langle a_{k+1} \rangle \to \langle a_1, \cdots, a_{k+1} \rangle \to 0 \). So the claim holds for all \( n \).

(1) \( \Rightarrow \) (3): Suppose \( R \) is \( u \)-\( S \)\(-\)coherent with respective to \( s \). Let \( I \) be a finitely generated ideal of \( R \) and \( b \) an element in \( R \). Consider the following commutative diagram with exact rows:

\[
\begin{array}{c}
0 & \to & I & \to & Rb + I & \to & (Rb + I)/I & \to & 0 \\
0 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & (I :_R b) & \to & R & \to & R/(I :_R b) & \to & 0.
\end{array}
\]

Since \( R \) is \( u \)-\( S \)\(-\)coherent with respective to \( s \), we have \( Rb + I \) is \( u \)-\( S \)\(-\)finitely presented with respective to \( s \). Since \( I \) is finitely generated, \( (Rb + I)/I \) is \( u \)-\( S \)\(-\)finitely presented with respective to \( s \) by Theorem 2.2. Thus \( (I :_R b) \) is \( S \)\(-\)finite is with respective to \( s \) by Theorem 2.2 again.

(3) \( \Rightarrow \) (1): Let \( I \) be a finitely generated ideal of \( R \) generated by \( \{a_1, ..., a_n\} \). We will show \( I \) is \( u \)-\( S \)\(-\)finitely presented by induction on \( n \). The case \( n = 1 \) follows from the exact sequence \( 0 \to (0 :_R a_1) \to R \to Ra_1 \to 0 \). For \( n \geq 2 \), let \( L = \langle a_1, ..., a_{n-1} \rangle \). Consider the exact sequence \( 0 \to (L :_R a_n) \to R \to (Ra_n + L)/L \to 0 \). Then \( (Ra_n + L)/L = I/L \) is \( u \)-\( S \)\(-\)finitely presented with respective to \( s \) by (3) and Theorem 2.2. Consider the exact sequence \( 0 \to L \to I \to I/L \to 0 \). Since \( L \) is finitely presented by induction and \( I/L \) is \( u \)-\( S \)\(-\)finitely presented with respective to \( s \), \( I \) is also \( u \)-\( S \)\(-\)finitely presented with respective to \( s \) by Theorem 2.2.

Recall from [3] that a ring \( R \) is \( S \)\(-\)coherent (resp., \( c \)-\( S \)\(-\)coherent) provided that any finitely generated ideal is \( S \)\(-\)finitely presented (resp., \( c \)-\( S \)\(-\)finitely presented).

**Proposition 3.12.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). If \( R \) is a \( u \)-\( S \)\(-\)coherent ring, then \( R \) is both \( S \)-coherent and \( c \)-\( S \)\(-\)coherent.

**Proof.** Let \( I \) be a finitely generated ideal and \( 0 \to K \to F \to I \to 0 \) an exact sequence with \( F \) finitely generated free. Then \( K \) is \( S \)-finite by Theorem 2.2(4). Thus \( I \) is \( S \)\(-\)finitely presented, and so \( R \) is \( S \)\(-\)coherent. Consider the exact sequence \( 0 \to T_1 \to N \xrightarrow{f} I \to T_2 \to 0 \) with \( N \) finitely presented and \( sT_1 = sT_2 = 0 \). Note that \( \text{sin } sT_2 = 0 \), we have \( sI \subseteq \text{Im}(f) \cong N/T_1 \). Since \( sT_1 = 0 \), \( s^2I \) can be seen as a submodule of \( N \). Hence \( I \) is \( c \)-\( S \)\(-\)finitely presented. Consequently, \( R \) is \( c \)-\( S \)\(-\)coherent.

**Proposition 3.13.** Let \( R \) be a ring and \( S \) a multiplicative subset of \( R \) consisting of finite elements. Then the following statements are equivalent.
(1) $R$ is a $u$-$S$-coherent ring.

(2) $R$ is an $S$-coherent ring.

(3) $R$ is a $c$-$S$-coherent ring.

Proof. Suppose $S = \{s_1, \ldots, s_n\}$ and set $s = s_1 \cdots s_n$.

(1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) Follows by Proposition 3.12.

(2) $\Rightarrow$ (1) Let $I$ be a finitely generated ideal of $R$. Then there is an exact sequence $0 \to K \to F \to I \to 0$ with $F$ finitely generated free and $K$ $S$-finite. Let $X$ be a submodule of $K$ such that $s_iK \subseteq X$ for some $s_i \in S$. So $sK/X = 0$ Then the exact sequence $0 \to K/X \to F/X \to I \to 0$ makes $I$ $u$-$S$-finitely presented with respective to $s$. So $R$ is a $u$-$S$-coherent ring.

(3) $\Rightarrow$ (1) Let $I$ be a finitely generated ideal of $R$. Then there is a finitely presented sub-ideal $J$ of $R$ such that $s_iI \subseteq J = 0$. So $s(I/J) = 0$. Then the exact sequence $0 \to I \to J \to I/J \to 0$ makes $I$ $u$-$S$-finitely presented with respective to $s$. So $R$ is a $u$-$S$-coherent ring.

Let $R$ be a ring, $M$ an $R$-module and $S$ a multiplicative subset of $R$. For any $s \in S$, there is a multiplicative subset $S_s = \{1, s, s^2, \ldots\}$ of $S$. We denote by $M_s$ the localization of $M$ at $S_s$. Certainly, $M_s \cong M \otimes_R R_s$

Proposition 3.14. Let $R$ be a ring and $S$ a multiplicative subset of $R$. If $R$ is a $u$-$S$-coherent ring with respective to some $s \in S$, then $R_s$ is a coherent ring.

Proof. Suppose $R$ is $u$-$S$-coherent ring with respective to $s \in S$. Let $J$ be a finitely generated ideal of $R_s$. Then $J \cong I_s$ for some finitely generated ideal $I$ of $R$. So there is an exact sequence $0 \to T_1 \to K \to I \to T_2 \to 0$ with $K$ finitely presented and $sT_1 = sT_2 = 0$. Localizing at $S_s$, we have $(T_1)_s = (T_2)_s = 0$. So $J \cong I_s \cong K_s$ which is finitely presented over $R_s$. So $R_s$ is a coherent ring.

Next, we will give an example of a ring which is both $S$-coherent and $c$-$S$-coherent, but not $u$-$S$-coherent.

Example 3.15. Let $R$ be a domain. Set $S = R - \{0\}$. First, we will show $R$ is $c$-$S$-coherent. Let $I$ be a nonzero finitely generated ideal of $R$. Suppose $0 \neq r \in I$. Then we have $rI \subseteq Rr \subseteq I$. Since $Rr \cong R$ is finitely presented, $R$ is a $c$-$S$-coherent ring.

Next we will show $R$ is $S$-coherent. Let $I$ be a nonzero finitely generated ideal of $R$ generated by nonzero elements $\{a_1, \ldots, a_n\}$. Set $a = a_1 \cdots a_n$. Consider the natural exact sequence $0 \to K \to R^n \xrightarrow{f} I \to 0$ satisfying $f(e_i) = a_i$ for each $i$. Claim $K$ is $S$-finite with respective to $a$ by induction on $n$. Set $I_k = \langle a_1, \ldots, a_k \rangle$.

Suppose $n = 1$. Then $K = 0$ as $a_1$ is a non-zero-divisor. So the claim trivially holds.
Suppose the claim holds for \( n = k \). Now let \( n = k + 1 \). Consider the following commutative diagram with exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & K_k & \rightarrow & R^k & \rightarrow & I_k & \rightarrow & 0 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & K_{k+1} & \rightarrow & R^{k+1} & \rightarrow & I_k + R_{k+1} & \rightarrow & 0 \\
\uparrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & (I_k :_R R_{k+1}) & \rightarrow & R & \rightarrow & (I_k + R_{k+1})/I_k & \rightarrow & 0,
\end{array}
\]

Since \( a(I_k :_R R_{k+1}) \subseteq aR \subseteq (I_k :_R R_{k+1}) \), So \((I_k :_R R_{k+1})\) is \( S \)-finite with respective to \( a \). By induction, \( K_k \) is \( S \)-finite with respective to \( a \). It is easy to check \( K_{k+1} \) is also \( S \)-finite with respective to \( a \). So the claim holds. Consequently, \( R \) is \( S \)-coherent.

Now, let \( R \) is a domain such that \( R_s \) is not coherent for any \( s \neq 0 \). For example, \( R = \mathbb{Q} + x\mathbb{R}[[x]] \) be the subring of formal power series ring \( T = \mathbb{R}[[x]] \) with constants in real numbers \( \mathbb{R} \), where \( \mathbb{Q} \) is the set of all rational numbers. Indeed, let \( 0 \neq s = a + xf(x) \in R \). We divide it into two cases. Case I: \( a \neq 0 \). In this case, \( s \) is a unit in \( R \), and so \( R_s \cong R \) which is not coherent by [7, Theorem 5.2.3]. Case II: \( a = 0 \). In this case, \( R_s \cong \mathbb{Q} + (x\mathbb{R}[[x]])_x f(x) \cong \mathbb{Q} + (x\mathbb{R}[[x]])_x \). So \( R_s \) can fit into a Milnor square of type II:

\[
\begin{array}{ccc}
R_s & \rightarrow & \mathbb{R}[[x]][x^{-1}] \\
\uparrow & & \downarrow \\
\mathbb{Q} & \rightarrow & \mathbb{R}.
\end{array}
\]

Hence \( R_s \) is not a coherent domain by [20, Theorem 8.5.17]. We will show \( R \) is not a \( u \)-\( S \)-coherent ring. On contrary, suppose \( R \) is \( u \)-\( S \)-coherent. Then there is a \( s \neq 0 \) such that \( R_s \) is a coherent ring by Proposition 3.14 which is a contradiction.

4. Module-theoretic characterizations of uniformly \( S \)-coherent rings

In this section, we will characterize uniformly \( S \)-coherent rings in terms of \( u \)-\( S \)-flat modules and \( u \)-\( S \)-injective modules. The following lemma is basic and of independent interest.

**Lemma 4.1.** Let \( R \) be a ring, \( r \in R \) and \( M \) an \( R \)-module. Suppose \( N \) is a pure submodule of \( M \). Then we have the following natural isomorphism

\[
\frac{rM}{rN} \cong r\left(\frac{M}{N}\right).
\]
Consequently, suppose \( \{M_i \mid i \in \Lambda\} \) is a direct system of \( R \)-modules. Then
\[
\varprojlim M_i \cong \varprojlim (rM_i).
\]

Proof. Consider the surjective map \( f : \frac{rM}{rN} \to \frac{M}{N} \) defined by \( f(rm + rN) = r(m + N) \). It is certainly \( R \)-linear. We will check it is also well defined. Indeed, \( f(rn + rN) = r(n + N) = r(0 + N) = 0 \). So \( f \) is an \( R \)-epimorphism. Let \( rm + rN \in \text{Ker}(f) \). Then \( rm \in N \). Since \( N \) is a pure submodule of \( M \), there is \( n \in N \) such that \( rm = rn \). So \( rm + rN = rn + rN = 0 \). Hence \( f \) is an isomorphism. Suppose \( \{ (M_i, f_{ij}) \mid i, j \in \Lambda \} \) is a direct system of \( R \)-modules. Then there is a pure exact sequence
\[ 0 \to K \to \bigoplus M_i \to \varprojlim M_i \to 0 \]
where \( K = \langle x - f_{ij}(x) \mid x \in M_i, i \leq j \in I \rangle \) (see [8, (2.1.1)]). Note that \( \{ (rM_i, f_{ij}) \mid i, j \in \Lambda \} \) is also a direct system of \( R \)-modules. We have the following equivalence
\[
\varprojlim (rM_i) \cong \frac{\bigoplus M_i}{K'} = \frac{r \bigoplus M_i}{rK} \cong r \frac{\bigoplus M_i}{K} \cong \varprojlim rM_i
\]
where \( K' = \langle rx - f_{ij}(rx) \mid rx \in rM_i, i \leq j \in I \rangle \). \( \square \)

**Lemma 4.2.** Let \( E \) be an injective cogenerator. Then the following assertions are equivalent.

1. \( T \) is uniformly \( S \)-torsion with respect to \( s \).
2. \( \text{Hom}_R(T, E) \) is uniformly \( S \)-torsion with respect to \( s \).

Proof. (1) \( \Rightarrow \) (2): Follows from [16, Lemma 4.2].

(2) \( \Rightarrow \) (1): Let \( f : sT \to E \) be an \( R \)-homomorphism and \( i : sT \to T \) the embedding map. Since \( E \) is injective, there exists an an \( R \)-homomorphism \( g : T \to E \) such that \( f = gi \). Let \( st \in sT \), we have \( f(st) = sg(t) = 0 \) as \( s\text{Hom}_R(T, E) = 0 \). So \( \text{Hom}_R(sT, E) = 0 \). Hence \( sT = 0 \) as \( E \) is an injective cogenerator. \( \square \)

A multiplicative subset \( S \) of \( R \) is said to be regular if it is composed of non-zero-divisors. Next, we give some new characterizations of \( u \)-\( S \)-flat modules

**Proposition 4.3.** Let \( R \) be a ring, \( S \) a multiplicative subset of \( R \). Then the following assertions are equivalent.

1. \( F \) is \( u \)-\( S \)-flat.
2. there exists an element \( s \in S \) satisfying that \( \text{Tor}_1^R(N, F) \) is uniformly \( S \)-torsion with respect to \( s \) for any finitely presented \( R \)-module \( N \).
3. \( \text{Hom}_R(F, E) \) is \( u \)-\( S \)-injective for any injective module \( E \).
4. \( \text{Hom}_R(F, E) \) is \( u \)-\( S \)-absolutely pure for any injective module \( E \).
5. if \( E \) is an injective cogenerator, then \( \text{Hom}_R(F, E) \) is \( u \)-\( S \)-injective.
6. if \( E \) is an injective cogenerator, then \( \text{Hom}_R(F, E) \) is \( u \)-\( S \)-absolutely pure.
Moreover, if $S$ is regular, then all above are equivalent to the following assertions.

(7) there exists $s \in S$ satisfying that $\text{Tor}_1^R(R/I, F)$ is uniformly $S$-torsion with respect to $s$ for any ideal $I$ of $R$.

(8) there exists $s \in S$ satisfying that, for any ideal $I$ of $R$, the natural homomorphism $\sigma_I : I \otimes_R F \to IF$ is a $u$-$S$-isomorphism with respect to $s$.

(9) there exists $s \in S$ satisfying that $\text{Tor}_1^R(R/K, F)$ is uniformly $S$-torsion with respect to $s$ for any finitely presented ideal $K$ of $R$.

(10) there exists $s \in S$ satisfying that, for any finite generated ideal $K$ of $R$, the natural homomorphism $\sigma_K : K \otimes_R F \toKF$ is a $u$-$S$-isomorphism with respect to $s$.

Proof. (1) $\Rightarrow$ (2): Set the set $\Gamma = \{(K, R^n) \mid K$ is a finitely generated submodule of $R^n$ and $n < \infty\}$. Define $M = \bigoplus_{(K, R^n) \in \Gamma} R^n/K$. Then $\text{Tor}_1^R(M, F)$ is a finitely presented $R$-module, then $N \cong R^n/K$ for some $(K, R^n) \in \Gamma$. Hence $\text{Tor}_1^R(N, F) = 0$ is uniformly $S$-torsion with respect to $s$.

(2) $\Rightarrow$ (1): Let $M$ be an $R$-module. Then $M = \lim \longrightarrow N_i$ for some direct system of finitely presented $R$-modules $\{N_i\}$. So $\text{Tor}_1^R(M, F) = \text{Tor}_1^R(\lim \longrightarrow N_i, F) \cong \lim \longrightarrow (\text{Tor}_1^R(N_i, F)) \cong \lim \longrightarrow (\text{Tor}_1^R(N_i, F)) = 0$ by Lemma 4.1. Hence $F$ is $u$-$S$-flat by [21 Theorem 3.2]

(1) $\Rightarrow$ (3): Let $M$ be an $R$-module and $E$ an injective $R$-module. Since $M$ is $u$-$S$-flat, then $\text{Tor}_1^R(M, F)$ is uniformly $S$-torsion. Thus $\text{Ext}_R^1(M, \text{Hom}_R(F, E)) \cong \text{Hom}_R(\text{Tor}_1^R(M, F), E)$ is also uniformly $S$-torsion by [16 Lemma 4.2]. Thus $\text{Hom}_R(F, E)$ is $u$-$S$-injective by [16 Theorem 4.3].

(3) $\Rightarrow$ (4) and (3) $\Rightarrow$ (5) $\Rightarrow$ (6): Trivial.

(6) $\Rightarrow$ (2): Let $E$ be an injective cogenerator. Since $\text{Hom}_R(F, E)$ is $u$-$S$-absolutely pure, there exists $s \in S$ such that $\text{Hom}_R(\text{Tor}_1^R(N, F), E) \cong \text{Ext}_R^1(N, \text{Hom}_R(F, E))$ is uniformly $S$-torsion with respect to $s$ for any finitely presented $R$-module $N$. Since $E$ is an injective cogenerator, $\text{Tor}_1^R(N, F)$ is uniformly $S$-torsion with respect to $s$ for any finitely presented $R$-module $N$ by Lemma 4.2.

(2) $\Rightarrow$ (9), (7) $\Rightarrow$ (9), (7) $\iff$ (8) and (9) $\iff$ (10): Obvious.

(10) $\Rightarrow$ (8): Let $\sum_{i=1}^n a_i \otimes x_i \in \text{Ker}(\sigma_I)$. Let $K$ be the finitely generated ideal generated by $\{a_i \mid i = 1, \ldots, n\}$. Consider the following commutative diagram:
\[
\begin{array}{c}
\begin{array}{c}
K \otimes_R F \xrightarrow{i \otimes 1} I \otimes_R F \\
K F \xrightarrow{i'} I F
\end{array}
\end{array}
\]

Let \( \sum_{i=1}^{n} a_i \otimes x_i \) be the element in \( K \otimes_R F \) such that \( i \otimes 1(\sum_{i=1}^{n} a_i \otimes x_i) = \sum_{i=1}^{n} a_i \otimes x_i \in I \otimes_R F \). Since \( i' \sigma_K(\sum_{i=1}^{n} a_i \otimes x_i) = \sigma_I(\sum_{i=1}^{n} a_i \otimes x_i) = 0 \), we have \( \sum_{i=1}^{n} a_i \otimes x_i \in \ker(\sigma_K) \) since \( i' \) is a monomorphism. Then \( s \sum_{i=1}^{n} a_i \otimes x_i = 0 \in K \otimes_R F \).

So \( \sum_{i=1}^{n} a_i \otimes x_i \in \ker(\sigma_I) \).

Now assume the multiplicative subset \( S \) is regular.

(7) \( \Rightarrow \) (5) Let \( E \) be an injective cogenerator. Since \( \text{Tor}^R_1(R/I, F) \) is uniformly \( S \)-torsion with respect to \( s \), we have \( \text{Hom}_R(\text{Tor}^R_1(R/I, F), E) \cong \text{Ext}^1_R(R/I, \text{Hom}_R(F, E)) \) is uniformly \( S \)-torsion with respect to \( s \) by Lemma 4.2. Since \( s \) is regular and \( E \) is injective, we have \( E \) is \( s \)-divisible. Hence \( \text{Hom}_R(F, E) \) is also \( s \)-divisible. Hence \( \text{Hom}_R(F, E) \) is \( u \)-\( S \)-injective by [16, Proposition 4.9].

In 1960, Chase also characterized coherent rings in terms of flat modules (see [3, Theorem 2.1]). Now, we are ready to give a “uniform” \( S \)-version of Chase Theorem.

**Theorem 4.4. (Chase Theorem for \( u \)-\( S \)-coherent rings)** Let \( R \) be a ring and \( S \) is a regular multiplicative subset of \( R \). Then the following assertions are equivalent:

1. \( R \) is a \( u \)-\( S \)-coherent ring.
2. there is \( s \in S \) such that any direct product of flat modules is \( u \)-\( S \)-flat with respective to \( s \).
3. there is \( s \in S \) such that any direct product of projective modules is \( u \)-\( S \)-flat with respective to \( s \).
4. there is \( s \in S \) such that any direct product of \( R \) is \( u \)-\( S \)-flat with respective to \( s \).

**Proof.** (2) \( \Rightarrow \) (3) \( \Rightarrow \) (4) Trivial.

(1) \( \Rightarrow \) (2) Suppose \( R \) is \( u \)-\( S \)-coherent with respective to some \( s \in S \). Let \( \{F_i | i \in \Lambda\} \) be a family of flat \( R \)-modules and \( I \) a finitely generated ideal of \( R \). Then \( I \) is \( u \)-\( S \)-finitely presented with respective to \( s \). So we have an exact sequence \( 0 \to T' \to K \xrightarrow{f} I \to T \to 0 \) with \( K \) finitely presented and \( sT = sT' = 0 \). Set
\( \text{Im}(f) = K' \). Consider the following commutative diagrams with rows exact:

\[
\begin{array}{c}
\text{Im}(f) = K'. \\
\text{Consider the following commutative diagrams with rows exact:} \\
T' \otimes_R \prod_{i \in I} F_i \longrightarrow K \otimes_R \prod_{i \in I} F_i \longrightarrow K' \otimes_R \prod_{i \in I} F_i \longrightarrow 0 \\
\alpha \downarrow \quad \gamma \downarrow \cong \quad \beta \\
0 \longrightarrow \prod_{i \in I}(T' \otimes_R F_i) \longrightarrow \prod_{i \in I}(K \otimes_R F_i) \longrightarrow \prod_{i \in I}(K' \otimes_R F_i) \longrightarrow 0,
\end{array}
\]

and

\[
\begin{array}{c}
K' \otimes_R \prod_{i \in I} F_i \longrightarrow I \otimes_R \prod_{i \in I} F_i \longrightarrow T \otimes_R \prod_{i \in I} F_i \longrightarrow 0 \\
\beta \downarrow \theta \\
0 \longrightarrow \prod_{i \in I}(K' \otimes_R F_i) \longrightarrow \prod_{i \in I}(I \otimes_R F_i) \longrightarrow \prod_{i \in I}(T \otimes_R F_i) \longrightarrow 0,
\end{array}
\]

By [8, Lemma 3.8(2)], \( \gamma \) is an isomorphism. Then \( \text{Ker}(\beta) \cong \text{Coker}(\alpha) \) which is \( u\)-\( S \)-torsion with respective to \( s \). Since \( K' \) is finitely generated, we have \( \beta \) is an epimorphism by [8, Lemma 3.8(1)]. Since \( T \otimes_R \prod_{i \in I} F_i \) and \( \text{Ker}(\beta) \) are all \( u\)-\( S \)-torsion with respective to \( s \), so \( \text{Ker}(\theta) \) is also \( u\)-\( S \)-torsion with respective to \( s \).

Now we consider the following commutative diagram with rows exact:

\[
\begin{array}{c}
0 \longrightarrow \text{Tor}_1^R(R/I, \prod_{i \in I} F_i) \longrightarrow I \otimes_R \prod_{i \in I} F_i \longrightarrow R \otimes_R \prod_{i \in I} F_i \\
\text{Tor}_1^R(R/I, F_i) \downarrow \theta \downarrow \\
0 \longrightarrow \prod_{i \in I}(\text{Tor}_1^R(R/I, F_i)) \longrightarrow \prod_{i \in I}(I \otimes_R F_i) \longrightarrow \prod_{i \in I}(R \otimes_R F_i),
\end{array}
\]

Note \( \text{Tor}_1^R(R/I, \prod_{i \in I} F_i) \subseteq \text{Ker}(\theta) \). So \( \text{Tor}_1^R(R/I, \prod_{i \in I} F_i) \) is \( u\)-\( S \)-torsion with respective to \( s \), Hence \( \prod_{i \in I} F_i \) is \( u\)-\( S \)-flat (with respective to \( s \)) by Proposition 4.3.

(4) \( \Rightarrow \) (1) Let \( I \) be a finitely generated ideal of \( R \). Consider the following commutative diagram with rows exact:

\[
\begin{array}{c}
I \otimes_R \prod_{i \in I} R \longrightarrow R \otimes_R \prod_{i \in I} R \longrightarrow R/I \otimes_R \prod_{i \in I} R \longrightarrow 0 \\
g \downarrow \quad \cong \quad \cong \\
0 \longrightarrow \prod_{i \in I}(I \otimes_R R) \longrightarrow \prod_{i \in I}(R \otimes_R R) \longrightarrow \prod_{i \in I}(R/I \otimes_R R) \longrightarrow 0.
\end{array}
\]

Since \( \prod_{i \in I} R \) is a \( u\)-\( S \)-flat module with respective to \( s \), then \( f \) is a \( u\)-\( S \)-monomorphism. So \( g \) is also a \( u\)-\( S \)-monomorphism with respective to \( s \).
Let $0 \rightarrow L \rightarrow F \rightarrow I \rightarrow 0$ be an exact sequence with $F$ finitely generated free. Consider the following commutative diagram with rows exact:

\[
\begin{array}{ccccccccc}
L \otimes_R \prod_{i \in I} R & \longrightarrow & F \otimes_R \prod_{i \in I} R & \longrightarrow & I \otimes_R \prod_{i \in I} R & \longrightarrow & 0 \\
0 & \longrightarrow & \prod_{i \in I} (L \otimes_R R) & \longrightarrow & \prod_{i \in I} (F \otimes_R R) & \longrightarrow & \prod_{i \in I} (I \otimes_R R) & \longrightarrow & 0.
\end{array}
\]

Since $g$ is a $u$-$S$-monomorphism with respective to $s$, $h$ is a $u$-$S$-epimorphism with respective to $s$. Set $\Lambda$ equal to the cardinal of $L$. We will show $L$ is $S$-finite with respective to $s$. Indeed, consider the following exact sequence

\[
L \otimes_R R^L \longrightarrow L^L \longrightarrow T \longrightarrow 0
\]

with $T$ a $u$-$S$-torsion module with respective to $s$. Let $x = (m)_{m \in L} \in L^L$. Then $sx \subseteq \text{Im} h$. Subsequently, there exist $m_j \in L, r_{j,i} \in R, i \in L, j = 1, \ldots, n$ such that for each $t = 1, \ldots, k$, we have

\[
sx = h(\sum_{j=1}^{n} m_j \otimes (r_{j,i})_{i \in L}) = (\sum_{j=1}^{n} m_j r_{j,i})_{i \in L}.
\]

Set $U = \langle m_j \mid j = 1, \ldots, n \rangle$ be the finitely generated submodule of $L$. Now, for any $m \in L$, $sm \in \langle \sum_{j=1}^{n} m_j r_{j,m} \rangle \subseteq U$, thus the embedding map $U \hookrightarrow L$ is a $u$-$S$-isomorphism with respective to $s$ and so $L$ is $S$-finite with respective to $s$. Consequently, $I$ is $u$-$S$-finitely presented with respective to $s$. Hence, $R$ is $u$-$S$-coherent with respective to $s$. □

In 1982, Matlis [14, Theorem 1] showed that a ring $R$ is coherent if and only if $\text{Hom}_R(M, E)$ is flat for any injective modules $M$ and $E$. The rest of this paper is devoted to obtain a “uniform” $S$-version of this result.

**Lemma 4.5.** Let $R$ be a ring, $S$ is a regular multiplicative subset of $R$ and $E$ an injective cogenerator over $R$. Suppose $\text{Hom}_R(E, E)$ is $u$-$S$-flat with respective to $s \in S$, then $\text{Hom}_R(E, E)/R$ is also $u$-$S$-flat with respective to $s$.

**Proof.** Let $I$ be an ideal of $R$. Set $H = \text{Hom}_R(E, E)$. Let $i : R \rightarrow H$ be the multiplicative map. Suppose $H$ is $u$-$S$-flat with respective to $s \in S$. Then there is a long exact sequence

\[
\text{Tor}^R_1(R/I, H) \rightarrow \text{Tor}^R_1(R/I, H/R) \rightarrow R/I \otimes_R R^{R/I \otimes i} \rightarrow R/I \otimes H.
\]

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Note that $\text{Ker}(R/I \otimes i) \cong (HI \cap R)/I = 0$ by [14, Proposition 1(2)]. Since $\text{Tor}^1_0(R/I, H)$ is $u$-S-torsion with respective to $s \in S$, $\text{Tor}^1_0(R/I, H/R)$ is $u$-S-torsion with respective to $s \in S$, which implies that $H/R$ is also $u$-S-flat with respective to $s$. \hfill \square

**Lemma 4.6.** Let $R$ be a ring, $S$ is a regular multiplicative subset of $R$. Suppose that $\{A_\lambda \mid \lambda \in \Lambda\}$ is a family of $u$-S-flat modules with respective to $s \in S$, and that $B_\lambda$ is a submodule of $A_\lambda$ such that $A_\lambda/B_\lambda$ is $u$-S-flat with respective to $s$ for each $\lambda \in \Lambda$. Then $\prod_{\lambda \in \Lambda} A_\lambda$ is $u$-S-flat with respective to $s$ if and only if so is $\prod_{\lambda \in \Lambda} B_\lambda$ and $\prod_{\lambda \in \Lambda} A_\lambda/B_\lambda$.

**Proof.** Let $I$ be a finitely generated ideal of $R$. Then there is an exact sequence

$$\text{Tor}^2_0(R/I, \prod_{\lambda \in \Lambda} A_\lambda/B_\lambda) \to \text{Tor}^1_0(R/I, \prod_{\lambda \in \Lambda} B_\lambda) \to \text{Tor}^1_0(R/I, \prod_{\lambda \in \Lambda} A_\lambda).$$

By [21 Theorem 3.2], we just need to show $\prod_{\lambda \in \Lambda} A_\lambda/B_\lambda$ is $u$-S-flat with respective to $s$. Consider the following exact sequence

$$\text{Tor}^1_0(R/I, \prod_{\lambda \in \Lambda} A_\lambda) \to \text{Tor}^1_0(R/I, \prod_{\lambda \in \Lambda} A_\lambda/B_\lambda) \to R/I \otimes_R \prod_{\lambda \in \Lambda} B_\lambda \xrightarrow{f} R/I \otimes_R \prod_{\lambda \in \Lambda} A_\lambda.$$

Since $\text{Tor}^1_0(R/I, \prod_{\lambda \in \Lambda} A_\lambda)$ is $u$-S-torsion with respective to $s$, to show $\prod_{\lambda \in \Lambda} B_\lambda$ is $u$-S-flat with respective to $s$, we just need to show $\text{Ker}(f)$ is $u$-S-torsion with respective to $s$. Note that $\text{Ker}(f) \cong (\prod_{\lambda \in \Lambda} B_\lambda \cap I(\prod_{\lambda \in \Lambda} A_\lambda))/I \prod_{\lambda \in \Lambda} B_\lambda \cong \prod_{\lambda \in \Lambda} (B_\lambda \cap IA_\lambda)/IB_\lambda$ as $I$ is finitely generated. Consider the following exact sequence $\text{Tor}^0_0(R/I, A_\lambda) \to \text{Tor}^1_0(R/I, A_\lambda/B_\lambda) \to R/I \otimes_R B_\lambda \xrightarrow{f_\lambda} R/I \otimes_R \prod_{\lambda \in \Lambda} A_\lambda$. We have $\text{Ker}(f_\lambda) \cong (B_\lambda \cap IA_\lambda)/IB_\lambda$ is $u$-S-torsion with respective to $s$ since $A_\lambda/B_\lambda$ is $u$-S-flat with respective to $s$. So $\text{Ker}(f) \cong \prod_{\lambda \in \Lambda} \text{Ker}(f_\lambda)$ is $u$-S-torsion with respective to $s$. \hfill \square

**Theorem 4.7.** (*Matlis Theorem for u-S-coherent rings*) Let $R$ be a ring and $S$ is a regular multiplicative subset of $R$. Then the following statements are equivalent.

1. $R$ is a $u$-S-coherent ring.
2. there are $s_1, s_2 \in S$ such that $\text{Hom}_R(M, E)$ is $u$-S-flat with respective to $s_1$ for any $M$ a $u$-S-absolutely pure module with respective to $s_2$ and any injective module $E$.
3. there are $s_1, s_2 \in S$ such that $\text{Hom}_R(M, E)$ is $u$-S-flat with respective to $s_1$ for any $M$ $u$-S-injective module with respective to $s_2$ and any injective module $E$. 

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(4) there is $s_1, s_2 \in S$ such that if $E$ is injective cogenerators, then $\text{Hom}_R(M, E)$ is $u$-$S$-flat with respective to $s_1$ for any $M$ $u$-$S$-injective module with respective to $s_2$.

(5) there are $s_1, s_2 \in S$ such that $\text{Hom}_R(\text{Hom}_R(M, E_1), E_2)$ is $u$-$S$-flat with respective to $s_1$ for any $M$ a $u$-$S$-flat module with respective to $s_2$ and any injective modules $E_1, E_2$.

(6) there are $s_1, s_2 \in S$ such that if $E_1$ and $E_2$ are injective cogenerators, then $\text{Hom}_R(\text{Hom}_R(M, E_1), E_2)$ is $u$-$S$-flat with respective to $s_1$ for any $M$ a $u$-$S$-flat module with respective to $s_2$.

(7) there is $s \in S$ such that if $E_1$ is an injective cogenerator, then $\text{Hom}_R(E_1, E_2)$ is $u$-$S$-flat with respective to $s$ for any injective cogenerator $E_2$.

Proof. (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (7) and (5) \Rightarrow (6): Trivial.

(3) \Leftrightarrow (5) and (4) \Leftrightarrow (6): Follows from Proposition 1.3

(1) \Rightarrow (2): Suppose $R$ is a uniformly $S$-coherent ring with respective to some element $s \in S$. Let $I$ be a finitely generated ideal of $R$. Then we have an exact sequence $0 \rightarrow T' \rightarrow K \xrightarrow{f} I \rightarrow T \rightarrow 0$ with $K$ finitely presented and $sT = sT' = 0$. Set $\text{Im}(f) = K'$. Consider the following commutative diagrams with exact rows $((- -) = \text{instead of } \text{Hom}_R(-, -))$:

\[
\begin{array}{c}
\begin{array}{c}
(M, E) \otimes_R T' \rightarrow \quad (M, E) \otimes_R K \rightarrow \quad (M, E) \otimes_R K' \rightarrow 0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
(\quad (T', M), E) \quad \rightarrow \quad ((K, M), E) \quad \rightarrow \quad ((K', M), E) \rightarrow 0,
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow \text{Tor}_1^R((M, E), R/K') \rightarrow \quad (M, E) \otimes_R K' \rightarrow \quad (M, E) \otimes_R R \rightarrow \quad (M, E) \otimes_R R/K' \rightarrow 0
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\psi_{K'} \quad \psi_K \quad \psi_{K'}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow \text{Tor}_1^R((M/K', M), E) \rightarrow \quad ((K', M), E) \rightarrow \quad ((R, M), E) \rightarrow \quad ((R/K', M), E) \rightarrow 0
\end{array}
\]

and

\[
\begin{array}{c}
\begin{array}{c}
\text{Tor}_1^R((M, E), T) \rightarrow \quad \text{Tor}_1^R((M, E), R/K') \rightarrow \quad \text{Tor}_1^R((M, E), R/I) \rightarrow \quad (M, E) \otimes_R T
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\psi_{K'} \quad \psi_K \quad \psi_{R/I}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
(\text{Ext}_1^R(T, M), E) \rightarrow \quad (\text{Ext}_1^R(R/K', M), E) \rightarrow \quad (\text{Ext}_1^R(R/I, M), E) \rightarrow \quad ((T, M), E)
\end{array}
\]

Since $\psi_K$ is an isomorphism by [2 Proposition 8.14(1)] and [10 Theorem 2], $\psi_{K'}$ is a $u$-$S$-isomorphism with respective to $s$, and so is $\psi_{R/I}$. Then $\psi_{R/I}$ is a $u$-$S$-isomorphism with respective to $s^3$ (see the proof of [23 Theorem 1.2]). Since $M$ is $u$-$S$-absolutely pure, $\text{Ext}_1^R(R/I, M)$ is $u$-$S$-torsion with respective to $s_2$ ($s_2$ is independent of $I$). Then $\text{Tor}_1^R(\text{Hom}_R(M, E), R/I)$ is $u$-$S$-torsion with respective to
$s_1 := s^3 s'$, and thus $\Hom_R(M, E)$ is $u$-$S$-flat with respective to $s_1$ by Proposition 4.3.

(7) $\Rightarrow$ (1): Let $E$ be an injective cogenerator and set $H = \Hom_R(E, E)$. Then $H$ is $u$-$S$-flat with respective to $s$ by assumption. Since $R \subseteq H$, we have that $H/R$ is $u$-$S$-flat with respective to $s$ by Lemma 4.5. Let $\Lambda$ be an index set. Set $H_\lambda = H$, $R_\lambda = R$ and $E_\lambda = E$ for any $\lambda \in \Lambda$. Since $\prod_{\lambda \in \Lambda} E_\lambda$ is also a injective cogenerator, $\prod_{\lambda \in \Lambda} H_\lambda \cong \Hom_R(E_\lambda, \prod_{\lambda \in \Lambda} E_\lambda)$ is $u$-$S$-flat with respective to $s$ by assumption. Hence $\prod_{\lambda \in \Lambda} R_\lambda$ is $u$-$S$-flat with respective to $s$ by Lemma 4.6. So $R$ is a $u$-$S$-coherent ring by Theorem 4.4.

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