Abstract. We show that the two complementary parts of the dynamics associated to the Feigenbaum attractor, inside and towards the attractor, form together a \(q\)-deformed statistical-mechanical structure. A time-dependent partition function produced by summing distances between neighboring positions of the attractor leads to a \(q\)-entropy that measures the ratio of ensemble trajectories still away at a given time from the attractor (and the repellor). The values of the \(q\)-indexes are given by the attractor’s universal constants, while the thermodynamic framework is closely related to that first developed for multifractals.

The Feigenbaum attractor, an icon of the historical developments in the theory of nonlinear dynamics [1], is getting renewed attention [2]. This is because it offers a convenient model system to explore features that might reflect those of statistical mechanical systems under conditions of phase space mixing and ergodicity breakdown. It therefore offers insights on the limits of validity of ordinary statistical mechanics. Recently a thorough description with newly revealed features has been given [3] [4] of the intricate dynamics that takes place both inside and towards this famous multifractal attractor. Here we show that these two types of dynamics are related to each other in a statistical-mechanical fashion, i.e. the dynamics at the attractor provides the ‘microscopic configurations’ in a partition function while the approach to the attractor is economically described by an entropy obtained from it. As we show below, this property conforms to a \(q\)-deformation [5] [6] of the ordinary exponential (Boltzmann) weight statistics.

Furthermore, this novel statistical-mechanical property arises in many other cases, some very familiar, that involve multifractal attractors with vanishing Lyapunov exponents. We have already uncovered [7] the prerequisites for the quasiperiodic transition to chaos (specifically along the golden mean) to be another example of the same kind of \(q\)-statistics [8]. Features in problems that relate to the period-doubling or the quasiperiodic routes to chaos (with general nonlinearities differing from the usual quadratic or cubic) acquire thermodynamic \(q\)-deformed structures. These include, for instance, phase transitions in spin or plaquette models of systems with many degrees of freedom, when built-in scaling properties lead to iteration techniques, and their singular behavior appears linked to Mandelbrot-like sets [9]. The broader issue of incidence of \(q\)-statistical properties in the complex dynamics of Julia sets associated to neutral or indeferent fixed points is an interesting question to go in for.

Trajectories within the Feigenbaum attractor show self-similar temporal structures, they preserve memory of their previous locations and do not have the mixing property of chaotic...
trajectories [10]. The fluctuating sensitivity to initial conditions has the form of infinitely many interlaced \( q \)-exponential functions that fold into a single one with use of a two-time scaling property [3] [6] [7]. More precisely, there is a hierarchy of such families of interlaced \( q \)-exponentials; an intricate (and previously unknown) state of affairs that befits the rich scaling features of a multifractal attractor. Furthermore, the entire dynamics is made of a family of pairs of Mori’s dynamical \( q \)-phase transitions [10] [3] [6] [7].

On the other hand, the process of convergence of trajectories into the Feigenbaum attractor is governed by another unlimited hierarchy feature built into the preimage structure of the attractor and its counterpart repellor [4]. The overall rate of approach of trajectories towards the attractor (and repellor) is conveniently measured by the fraction of (fine partition) bins \( W_{t_1} \) still occupied at time \( t_1 \) by an ensemble of trajectories with initial positions uniformly distributed over phase space [11] [4]. For the first few time steps the rate \( W_{t_1} \) remains constant, \( W_{t_1} \approx \Delta \), \( 1 \leq t_1 \leq t_0 \), \( t_0 = O(1) \) [12], after which a power-law decay with log-periodic modulation sets in, a signature of discrete-scale invariance [13]. This property of \( W_{t_1} \) is explained in terms of a sequential formation of gaps in phase space, and its self-similar features are seen to originate in the ladder feature of the preimage structure [4]. The rate \( W_{t_1} \) was originally presented in Ref. [11] where the power law exponent \( \varphi \) in

\[
W_{t_1} \approx \Delta \ h \left( \frac{\ln t}{\ln \Lambda} \right) t^{-\varphi}, \ t = t_1 - t_0, \tag{1}
\]

was obtained numerically. Above, \( h(x) \) is a periodic function with \( h(1) = 1 \), and \( \Lambda \) is the scaling factor between the periods of two consecutive oscillations [14].

We proceed now to demonstrate the connection between the aforementioned dynamical properties. We recall [1] the definition of the interval lengths or diameters \( d_{n,m} \) that measure the bifurcation forks that form the period-doubling cascade sequence in unimodal maps, here represented by the logistic map \( f_\mu(x) = 1 - \mu x^2 \), \(-1 \leq x \leq 1, 0 \leq \mu \leq 2\). These quantities are measured when considering the superstable periodic orbits of lengths \( 2^n \), \( n = 1, 2, \ldots \); i.e. the \( 2^n \)-cycles that contain the point \( x = 0 \) at \( \mu_n < \mu_\infty \), where \( \mu_\infty = 1.401155189 \ldots \) is the value of the control parameter \( \mu \) at the period-doubling accumulation point [15]. The positions of the limit \( 2^\infty \)-cycle constitute the Feigenbaum attractor. The \( d_{n,m} \) in these orbits are defined (here) as the (positive) distances of the elements \( x_{n,m} \), \( m = 0, 1, \ldots, 2^n - 1 \), to their nearest neighbors.

\[
f_{\mu_n}^{(2^n-1)}(x_{n,m}), \quad d_{n,m} = \left| f_{\mu_n}^{(2^n-1)}(0) - f_{\mu_n}^{(m)}(0) \right|.
\]

For large \( n \), \( d_{n,0}/d_{n+1,0} \approx \alpha \), where \( \alpha \) is Feigenbaum’s universal constant \( \alpha \approx 2.5091 \).

I nnermost to our arguments is the following comprehensive property: Time evolution at \( \mu_\infty \) from \( t = 0 \) up to \( t \rightarrow \infty \) traces the period-doubling cascade progression from \( \mu = 0 \) up to \( \mu_\infty \). Not only is there a close resemblance between the two developments but also asymptotic quantitative agreement. For example, the trajectory inside the Feigenbaum attractor with initial condition \( x_0 = 0 \), takes positions \( x_t \) such that the distances between nearest neighbor pairs of them reproduce the diameters \( d_{n,m} \) defined from the supercycle orbits with \( \mu_n < \mu_\infty \). See Fig. 1. This property has been central to obtain rigorous results for the fluctuating sensitivity to initial conditions \( x(t) \) within the Feigenbaum attractor, as separations at chosen times \( t \) of pairs of trajectories originating close to \( x_0 \) can be obtained as diameters \( d_{n,m} \) [3] [6].

Further, the complex dynamical events that fix the decay rate \( W_{t_1} \) can be understood in terms of the correlation between time evolution at \( \mu_\infty \) from \( t = 0 \) up to \( t \rightarrow \infty \) and the ‘static’ period-doubling cascade progression from \( \mu = 0 \) up to \( \mu_\infty \). As shown recently [16] each doubling of the period (obtained through the shift \( \mu_n \rightarrow \mu_{n+1} \)) introduces additional elements in the hierarchy of the preimage structure and in the family of sequentially-formed phase space gaps in the finite period cycles. The complexity of these added elements is similar to that of the total period \( 2^n \) system. Also, the shift \( \mu_n \rightarrow \mu_{n+1} \) increases in one unit the number of undulations in the
transitory log-periodic power law decay found for the corresponding rate $W_{n,t_1}$ of approach to the $2^n$-supercycle attractor [16]. As a consequence of this we have obtained detailed understanding of the mechanism by means of which the discrete scale invariance implied by the log-periodic property in $W_{t_1} \equiv \lim_{n \to \infty} W_{n,t_1}$ arises. What is more, the rate $W_{t_1}$, at the values of time for period doubling, can be obtained quantitatively from the supercycle diameters $d_{n,m}$, that is [4],

$$W_{t_1} = \Delta Z_t, \, t = t_1 - t_0,$$

and for these we have, respectively, $d_{n,0} \simeq \alpha_{y}^{-n+1}$ and $d_{n,1} \simeq \alpha_{y}^{-2(n-1)}$. (The 1st diameter $d_{0,0} = 1$ and the $d_{n,m}$ converge rapidly to $\alpha_{y}^{-n+1}$ as $n$ increases). With use of the identity $A^{-n+1} \equiv (1 + \beta)^{-\ln A/\ln 2}$, $\beta = 2^{n-1} - 1$, the power law $d_{n,m} \simeq \alpha_{y}^{-n+1}$ can be rewritten as a $q$-exponential ($\exp_q(x) \equiv \left[1 - (q - 1)x\right]^{-1/(q-1)}$, i.e., $d_{n,m} \simeq \exp_q(-\beta \nu_y)$, where $q_y = 1 + \nu_y^{-1}$, $\nu_y = \ln \alpha_y/\ln 2$, and $\beta = t - 1 = 2^{n-1} - 1$. Likewise, the partition function $Z_t \simeq t^{-\varphi}$ (or $Z_t \simeq e^{-n+1}$), with $\varphi = \ln \epsilon/\ln 2$ and $t = 2^{n-1}$, can be written as $Z_t \simeq \exp_q(-\beta \varphi)$, $Q = 1 + \varphi^{-1}$ and again $\beta = t - 1 = 2^{n-1} - 1$.

Figure 1. Left panel: Absolute value of attractor positions for the logistic map $f_\mu(x)$ in logarithmic scale as a function of $-\ln (\mu_\infty - \mu)$. Right panel: Absolute value of trajectory positions for $f_\mu(x)$ at $\mu_\infty$ with initial condition $x_0 = 0$ in logarithmic scale as a function of the logarithm of time $t$, also shown by the numbers close to the circles. The arrows indicate the equivalence between the diameters $d_{n,0}$ in the left panel, and position differences $D_n$ with respect to $x_0 = 0$ in the right panel.
Our main point becomes apparent when the above $q$-exponential forms for $d_{n,m}$ and $Z_t$ are used in Eq. (2), to yield
\[
\exp_Q(-\beta \varphi) \simeq \sum_y \exp_{q_y}(-\beta \nu_y). \tag{3}
\]

Eq. (3) resembles a basic statistical-mechanical expression where the quantities in it play the following roles: $\beta$ an inverse temperature, $\varphi$ a free energy (or the product $s = -\beta \varphi$ a Massieu thermodynamic potential, or entropy), and the $\nu_y$ configurational energies. However, the equality involves $q$-deformed exponentials in place of ordinary exponentials that would be recovered when $Q = q_y = 1$. It is worth noticing that there is a multiplicity of $q$-indexes associated to the configurational weights in Eq. (3), however their values form a well-defined family [6] determined by the discontinuities of Feigenbaum’s function $\sigma$. To substantiate the usefulness and appropriateness of this identification we present in the remaining part of this paper: 1) A ‘mean field’ evaluation of $Z_t$ and a thermodynamic interpretation of the time evolution process. 2) The relationship of $Z_t$ with the familiar partition function developed for the description of multifractal geometry. 3) A crossover to $q = 1$ ordinary statistics.

Figure 2. Fig. 2. Sector of the bifurcation tree for the logistic map $f_\mu(x)$ that shows the formation of a Pascal triangle of diameter lengths according to the scaling approximation explained in the text, $\alpha \simeq 2.5091$ is the pertinent universal constant.

Akin to a mean field approximation we assume that, for a given value of $n$, e.g., $n = 3$, the diameters $d_{n,m}$ that are of comparable lengths have equal length and this is obtained from those of the shortest or longest diameters via a simple scale factor; e.g., $d_{3,3} = d_{3,2} = \alpha^{-1}d_{3,0} = \alpha d_{3,1}$. This introduces some degeneracy in the lengths that propagates across the bifurcation tree. See Fig. 2 [18]. Specifically, the $d_{n,m}$ scale now with increasing $n$ according to a binomial combination of the scaling of those diameters that converge to the most crowded and most sparse regions of the multifractal attractor. This is to consider that the $2^{n-1}$ diameters at the $n$-th supercycle have lengths equal to $\alpha^{-(n-1-l)}\alpha^{-2l}$ and occur with multiplicities $\binom{n-1}{l}$, where $l = 0, 1, \ldots, n-1$. As seen in Fig. 2 the diameters form a Pascal triangle across the bifurcation cascade. The partition function can be immediately evaluated to yield
\[
Z_t = \sum_{l=0}^{n-1} \binom{n-1}{l} \alpha^{-(n-1-l)}\alpha^{-2l} = (\alpha^{-1} + \alpha^{-2})^{n-1}, \tag{4}
\]
The 'static' partition function \( Z(\tau, q) \) length contracts under backward iteration as cover the multifractal set and the \( \sum \) mechanical framework the so-called thermodynamic formalism \([b]\). The partition function \( Z(\tau, q) \) formed under backward iteration so there are \( d \) all diameters \( \neq \) procedure consists of requiring that

\[
\exp_Q(-\beta \varphi) = \sum_{l=0}^{n-1} \Omega(n-1,l) \exp_q(-\beta \nu),
\]

where \( \Omega(n-1,l) = \alpha^{-l} \binom{n-1}{l}. \) Thermodynamically, the approach to the attractor described by Eqs. \((3)\) or \((5)\) is a cooling process \( \beta \to \infty \) in which the free energy (or energy) \( \varphi \) is fixed and therefore the entropy \( s = -\beta \varphi \) is linear in \( \beta \). It is illustrative to define an ‘energy landscape’ for the Feigenbaum attractor as being composed by an infinite number of ‘wells’ whose equal-valued minima at \( \beta \to \infty \) coincide with the points of the attractor on the interval \([\alpha^{-1}, 1]\) \([14]\). When \( \beta = 2^{n-1} - 1 \), \( n \) finite, the wells merge into \( 2^{n-1} \) intervals of widths equal to the diameters \( d_{n,m}. \)

As it is well known the geometric properties of multifractals conform to a statistical mechanical framework, the so-called thermodynamic formalism \([15]\). The partition function devised to study their properties, such as the spectrum of singularities \( f(\alpha) \) \([15]\), is 

\[
Z(\tau, q) = \sum_{m} \sum_{l} p_{m}^{l} m_{q}^{l},
\]

where the \( l_{m} \) (in one-dimensional systems) are \( M \) disjoint interval lengths that cover the multifractal set and the \( p_{m} \) are probabilities assigned to these intervals. The usual procedure consists of requiring that \( Z(\tau, q) \) neither vanishes nor diverges in the limit \( l_{m} \to 0 \) for all \( m \) (and \( M \to \infty \)). In this case the exponents \( \tau \) and \( q \) define a function \( \tau(q) \) from which \( f(\alpha) \) is obtained via Legendre transformation \([15]\). When the multifractal is an attractor its elements become ordered dynamically, and for the Feigenbaum attractor the trajectory with initial condition \( x_{0} = 0 \) generates sequentially the positions that form the diameters, producing all diameters \( d_{n,m} \) for \( n \) fixed between times \( t = 2^{-n} \) and \( t = 3 \cdot 2^{-n} \). Since the diameters cover the attractor it is natural to choose the covering lengths at stage \( n \) to be \( l_{m}(n) = d_{n,m} \) and to assign to each of them the same probability \( p_{m}^{(n)} = 1/2 \). Within the two-scale approximation to the Feigenbaum multifractal \([15]\), \( l_{k}^{(n)} = \alpha^{-(n-1-k)} \alpha^{-2k} \), the condition \( Z(\tau, q) = 1 \) reproduces Eq. \((4)\) when \( p_{m}^{(n)} = t^{-1} = 2^{-n+1} \), with \( \tau = 1 \) and \( q = -\varphi \). It should be kept in mind that the ‘static’ partition function \( Z(\tau, q) \) is not meant to distinguish between chaotic and critical (vanishing Lyapunov exponent \( \lambda \)) multifractal attractors as we do here. It is the functional form of the link between the probabilities \( p_{m}^{(n)} \) and actual time \( t \) that determines the nature of the statistical mechanical structure of the dynamical system.

The recursive method of backward iteration of chaotic maps provides a convenient way for reconstructing multifractal sets and obtaining their underlying statistical mechanics \([19]\). A chaotic unimodal map has a two-valued inverse and given a position \( x = x_{n} \) a binary tree is formed under backward iteration, so there are \( 2^{n} \) initial conditions \( x_{0} \) for trajectories that lead to \( x_{n} \). Since now \( \lambda > 0 \), lengths expand under forward iteration according to \( l \sim \exp(\lambda n) \) and contract under backward iteration as \( t \sim \exp(-\lambda n) \). We can define, as above, a set of covering lengths \( D_{n,m} = \delta_{m} \exp(-\lambda n) \), where \( m \) relates to the initial condition \( x_{0} \) and use of them in a partition function like that in Eq. \((2)\) gives \( \exp(-\beta \varphi) = \sum_{m} \delta_{m} \exp(-\beta \lambda) \), where now \( \beta = n \). Recalling Pesin’s theorem, \( \varphi \) is clearly identified as the Kolmogorov-Sinai entropy. The crossover from \( q \)-deformed statistics to ordinary \( q = 1 \) statistics can be observed for control parameter values in the vicinity of the Feigenbaum attractor, \( \mu \gtrsim \mu_{\infty} \), when the attractor consists of \( 2^{\pi} \) bands, \( \pi \) large. The Lyapunov coefficient \( \lambda \) of the chaotic attractor decreases with \( \Delta \mu = \mu_{\infty} - \mu_{\infty} \) as \( \lambda \propto 2^{-\pi} \sim \Delta \mu^{\kappa} \), \( \kappa = \ln 2/\ln \delta_{F}(\kappa) \), where \( \delta_{F} \) is the Feigenbaum constant that measures the rate of development of the bifurcation tree in control parameter space \([1]\). The chaotic orbit consists of an interband periodic motion of period \( 2^{\pi} \) and an intraband chaotic motion. The expansion rate \( \sum_{i=0}^{t-1} \ln |df_{\mu}(x_{t})/dx_{t}| \) fluctuates with increasing amplitude as \( \ln t \) for \( t < 2^{\pi} \).
but converges to a fixed number that grows linearly with $t$ for $t \gg n$ [10]. This translates as dynamics with $q \neq 1$ for $t < 2\pi$ but ordinary dynamics with $q = 1$ for $t \gg 2\pi$.

We have shown that there is a statistical-mechanical property lying beneath the dynamics of an ensemble of trajectories en route to the Feigenbaum attractor. The fraction of phase space still occupied at time $t$ is a partition function $Z_t$ made up of exponential weighted configurations, while $Z_t$ itself is the $q$-exponential of a thermodynamic potential function. This is a clear signature of $q$-deformation of ordinary statistical mechanics, and, to our knowledge, it is the first \textit{bona fide} concrete instance (anticipated or not in the form presented here) where arguments can be made explicit and rigorous. There is a close parallel with the thermodynamic formalism for multifractal sets, but it should be emphasized that the departure from the usual exponential statistics is dynamical in origin, and, due to the vanishing of the (only) Lyapunov exponent. Our results translate to other multifractal critical attractors.

Although there should not be any confusion that our discussion presented here refers to the dynamical properties of low-dimensional multifractal critical attractors, and explicitly only to the Feigenbaum attractor, we would like to remind the casual reader that our development is not intended to apply to thermal systems, Hamiltonian systems composed of many degrees of freedom, the customary subject of ordinary statistical mechanics. Rather, our analysis relates to the statistical-mechanical structure exhibited by static or dynamical properties of multifractal sets.

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References

[1] See, for example, H.G. Schuster 1988 \textit{Deterministic Chaos. An Introduction} 2nd Revised Edition (Weinheim: VCH Publishers)

[2] A. Robledo 2005 \textit{Europhys. News} 35 214

[3] E. Mayoral and A. Robledo 2005 \textit{Phys. Rev. E} 72 026209

[4] A. Robledo and L.G. Moyano 2008 \textit{Phys. Rev. E} 77 036213 ; 2009 Braz. J. Phys. 39 364

[5] C. Tsallis 1988 \textit{J. Stat. Phys.} 52 479

[6] A. Robledo 2006 \textit{Physica A} 370 449

[7] H. Hernández-Saldana and A. Robledo 2006 \textit{Physica A} 370 286

[8] Technically, translating properties from one route to chaos to the other entails little more than replacement of universal constants and period-doublings by Fibonacci number sequences.

[9] See, for example, N.S. Ananikian \textit{et al} 1998 \textit{Phys. Lett. A} 248 381 ; N.S. Ananikian and R.G. Ghulghazaryan 2000 \textit{Phys. Lett. A} 277 249 and references therein.

[10] H. Mori, H. Hata, T. Horita and T. Kobayashi 1989 \textit{Prog. Theor. Phys. Suppl.} 99 1

[11] F.A.B.F. de Moura, U. Tirnakli and M.L. Lyra 2000 \textit{Phys. Rev. E} 62 6361

[12] The constant $\Delta$ is given by $\Delta = (1 + \alpha^{-1})/2$, where $\alpha$ is Feigenbaum’s universal number $\alpha \approx 2.5091$. This follows from the fact that all initial conditions out of the interval $[-\alpha^{-1}, 1]$ take a value inside it in the 1st iteration. See Fig. (21) in Ref. [4].

[13] D. Sornette 1998 \textit{Phys. Rep.} 297 239

[14] Discrete scale invariance due to period doubling implies $\Lambda = 2$ but use of a finite number of trajectories $N_t$ in the numerical evaluation of $W_t$ yields $\Lambda < 2$. See Ref. [4] for the convergence $\Lambda \to 2$ as $N_t \to \infty$.

[15] See, for example, C. Beck and F. Schlogl 1993 \textit{Thermodynamics of Chaotic Systems} (UK: Cambridge University Press)

[16] L.G. Moyano, D. Silva and A. Robledo 2009 \textit{Cent. Eur. J. Phys.} 7 591

[17] P. Grassberger 2005 \textit{Phys. Rev. Lett.} 95 140601

[18] G. Linage, F. Montoya, A. Sarmiento, K. Showalter and P. Parmananda 2006 \textit{Phys. Lett. A} 359 638

[19] J.L. McCauley 1989 \textit{Int. J. Mod. Phys.} B3 821