Dispersion estimates for one-dimensional Schrödinger and Klein–Gordon equations revisited

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Abstract. It is shown that for a one-dimensional Schrödinger operator with a potential whose first moment is integrable the elements of the scattering matrix are in the unital Wiener algebra of functions with integrable Fourier transforms. This is then used to derive dispersion estimates for solutions of the associated Schrödinger and Klein–Gordon equations. In particular, the additional decay conditions are removed in the case where a resonance is present at the edge of the continuous spectrum.

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Contents

1. Introduction 391
2. Continuity properties of the scattering matrix 394
3. The Schrödinger equation 399
4. The Schrödinger equation (non-resonant case) 401
5. The Klein–Gordon equation 405
   5.1. Low-energy decay 405
   5.2. High-energy decay 407
6. The Klein–Gordon equation (non-resonant case) 409
7. A decay estimate 411
Bibliography 413

1. Introduction

We are concerned with the one-dimensional Schrödinger equation

\[ i \dot{\psi}(x, t) = H \psi(x, t), \quad H := -\frac{d^2}{dx^2} + V(x), \quad (x, t) \in \mathbb{R}^2, \quad (1.1) \]
with a real integrable potential $V$, and the Klein–Gordon equation
\[ \ddot{\psi}(x,t) = -(H + m^2)\psi(x,t), \quad (x,t) \in \mathbb{R}^2, \quad m > 0. \] (1.2)
In vector form equation (1.2) is
\[ i\dot{\Psi}(t) = H\Psi(t), \] (1.3)
where
\[ \Psi(t) = \begin{pmatrix} \psi(t) \\ \dot{\psi}(t) \end{pmatrix}, \quad H = i \begin{pmatrix} 0 & 1 \\ -H - m^2 & 0 \end{pmatrix}. \] (1.4)

More specifically, our goal is to provide dispersion decay estimates for these equations. This is a well-studied area, and one of our main contributions is a strikingly simple proof which at the same time improves the previous results. This approach depends on the fact that the scattering matrix minus the identity matrix is in the Wiener algebra (that is, its Fourier transform is integrable). Since this result is of independent interest we prove it first in the separate §2. Based on this we then establish our main results. To formulate them we introduce the weighted spaces $L^p_\sigma = L^p_\sigma(\mathbb{R})$, $\sigma \in \mathbb{R}$, associated with the norm
\[ \|\psi\|_{L^p_\sigma} = \begin{cases} \left( \int_{\mathbb{R}} (1 + |x|)^p |\psi(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ \sup_{x \in \mathbb{R}} (1 + |x|)^\sigma |\psi(x)|, & p = \infty. \end{cases} \]

Of course, the case $\sigma = 0$ corresponds to the usual $L^p$ spaces without a weight. We recall (see [14] or [23], §9.7) that for $V \in L^1_1$ the operator $H$ has a purely absolutely continuous spectrum on $[0, \infty)$ plus a finite number of eigenvalues in $(-\infty, 0)$. At the edge of the continuous spectrum there can be a resonance if there is a corresponding bounded solution of $-\psi'' + V\psi = 0$ (equivalently, if the Wronskian of the two Jost solutions vanishes at this point).

The following two results hold for the Schrödinger equation.

**Theorem 1.1.** Let $V \in L^1_1(\mathbb{R})$. Then the following decay holds:
\[ \|e^{-itH}P_c\|_{L^1 \to L^\infty} = O(t^{-1/2}), \quad t \to \infty, \] (1.5)
where $P_c = P_c(H)$ is the orthogonal projection in $L^2(\mathbb{R})$ onto the continuous spectrum of $H$.

**Theorem 1.2.** Let $V \in L^2_2(\mathbb{R})$. Then in the non-resonant case the following decay holds:
\[ \|e^{-itH}P_c\|_{L^1_1 \to L^\infty_1} = O(t^{-3/2}), \quad t \to \infty. \] (1.6)

Note that for the free Schrödinger equation (1.1) with $V = 0$ the estimate (1.5) is immediate from the explicit formula for the time evolution (see [23], §7.3, for example). The dispersion decay (1.5) for the perturbed Schrödinger equation was established by Goldberg and Schlag [8], improving earlier results of Weder [25] in the non-resonant case for $V \in L^1_1$ and in the resonant case under the more restrictive condition $V \in L^3_2$ (see also [4]). We emphasize that our approach does not require
this additional decay in the resonant case. Moreover, our proof for Theorem 1.1 is a simple application of Fubini’s theorem. To show that the extra decay in the resonant case is not needed, we generalize an old (but obviously not so well known) result due to Guseinov [9]. We also remark that in the half-line case the analogous result for the scattering data is well known (see Problem 3.2.1 in [14]) and was used by Weder [27] to prove a corresponding result in the half-line case.

Interpolating between the unitarity of \( \exp\{-itH\} : L^2 \to L^2 \) and (1.5), we get from the Riesz–Thorin theorem that

\[
\|e^{-itH}P_c\|_{L^{p'} \to L^p} = \mathcal{O}(t^{-1/2+1/p}) \tag{1.7}
\]

for any \( p \in [2, \infty) \) with \( 1/p + 1/p' = 1 \). Using (1.7), we can also deduce the corresponding Strichartz estimates (see [10], Theorem 1.2).

The dispersion decay (1.6) was established by Schlag [21] in the case \( V \in L^1_4 \) and later refined by Goldberg [7] to the case \( V \in L^1_3 \). For \( V \in L^1_2 \) the estimate (1.6) was first obtained by Mizutani in [17]. Here we propose a proof of (1.6) based on a somewhat different approach.

The decay (1.6) immediately implies the following long-time asymptotics in weighted norms:

\[
\|e^{-itH}P_c\|_{L^2_{\sigma} \to L^2_{-\sigma}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty, \tag{1.8}
\]

for any \( \sigma > 3/2 \). Asymptotics of the type (1.8) in the non-resonant case were obtained by Murata [18] for more general (for instance, multidimensional) Schrödinger-type operators. In particular, in the one-dimensional case these asymptotics were established for \( \sigma > 5/2 \) and \( |V(x)| \leq C(1+|x|)^{-\rho} \) with some \( \rho > 4 \) (see also [21] for an up-to-date survey in this direction).

In the second part of our paper we obtain some dispersion estimates for the Klein–Gordon equation. To this end we introduce the Bessel potential

\[
J_\alpha = \mathcal{F}^{-1}(1 + |\cdot|^2)^{\alpha/2},
\]

where \( \mathcal{F} \) is the Fourier transform. Then the generalized Sobolev space \( H^\alpha_{\sigma,1}(\mathbb{R}) \) (cf. [1], Definition 6.2.2) is the space of tempered distributions \( f \in \mathcal{S}'(\mathbb{R}) \) for which the norm

\[
\|f\|_{H^\alpha_{\sigma,1}} = \|J_\alpha f\|_{L^p_{\sigma}}, \quad \alpha, \sigma \in \mathbb{R},
\]

is finite. As before \( H^\alpha_{\sigma,1} = H^\alpha_{0,1} \).

**Theorem 1.3.** (i) Let \( V \in L^1_1(\mathbb{R}) \). Then the following decay holds:

\[
\| [e^{-itH}P_c]^{12} \|_{H^{1/2,1}_1 \to L^{\infty}} = \mathcal{O}(t^{-1/2}), \quad t \to \infty.
\]

(ii) Let \( V \in L^1_2(\mathbb{R}) \). Assume that there is no resonance at the edge of the spectrum of \( H \). Then the following decay holds:

\[
\| [e^{-itH}P_c]^{12} \|_{H^{1/2,1}_{1} \to L^{\infty}_{-2}} = \mathcal{O}(t^{-3/2}), \quad t \to \infty.
\]

Here \( P_c \) is the orthogonal projection in \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) onto the continuous spectrum of \( H^2 \), and \([\cdot]^{ij}\) denotes the \((i,j)\)-entry of the corresponding matrix operator.
We remark that for the other entries of the matrix operator $e^{-itH}P_c$ the corresponding decay can be obtained similarly.

We note that Theorem 1.3, (i) is frequently stated in terms of the Besov space $B^{1/2}_{1,1}(\mathbb{R})$ defined in [1], Definition 6.2.2. Namely, recalling that $B^{1/2}_{1,1} \subset H^{1/2,1}$ (see [1], Theorem 6.2.4) shows that (1.10) holds with $B^{1/2}_{1,1}$ in place of $H^{1/2,1}$. Similarly, (1.11) holds with $B^{1/2}_{1,1,1}$ in place of $H^{1/2,1}$, where $B^{1/2}_{1,1,1}$ is the corresponding weighted Besov space (to define $B^{1/2}_{1,1,1}$ one needs to replace $L^1$ by $L^1$ in the definition of $B^{1/2}_{1,1}$). As before, this follows from $B^{1/2}_{1,1,1} \subset H^{1/2,1}$ (see, for instance, [16], Proposition 3.12).

Moreover, as a consequence of (1.10) we obtain

$$
|||e^{-itH}P_c|||^{12}_{B^{1/2-3/p}_{p',p'}} = O(t^{-1/2+1/p}), \quad t \to \infty, \quad \frac{1}{p'} + \frac{1}{p} = 1, \tag{1.12}
$$

for any $p \in [2, \infty]$ under the same assumption $V \in L^1_2(\mathbb{R})$ (see Corollary 5.3).

In three space dimensions $W^{k,p} \to L^q$ estimates for the Klein–Gordon equation were established by Soffer and Weinstein [22] (see also [29] for general space dimensions $n \geq 3$). In the one-dimensional case $W^{k,p} \to W^{k,q}$ estimates were obtained by Weder [26] for $V \in L^1_\gamma$, where $\gamma > 3/2$ in the non-resonant case and $\gamma > 5/2$ in the resonant case. The dispersion estimate of type (1.12) (with $B^{1/2-3/p}_{p',p'}(V)$ instead of $B^{1/2-3/p}_{p',p'}$) were shown in [4], but again requiring that $V \in L^1_2$ in the resonant case.

For the one-dimensional Klein–Gordon equation the decay $t^{-3/2}$ in the weighted energy norms $H^{1,2}_\sigma \otimes L^2_\sigma \to H^{1,2}_\sigma \otimes L^2_\sigma$ with $\sigma > 5/2$ was obtained by Komech and Kopylova [11] (see also the survey [12]).

We note that dispersion estimates of type (1.5)–(1.8) and (1.10)–(1.12) play an important role in proving the asymptotic stability of solitons in the associated one-dimensional non-linear equations [2], [13]. For the discrete Schrödinger and wave equations we refer to [6].

2. Continuity properties of the scattering matrix

We first introduce the Banach algebra $\mathcal{A}$ of Fourier transforms of integrable functions

$$
\mathcal{A} = \left\{ f : f(k) = \int_\mathbb{R} e^{ikp} \hat{f}(p) \, dp, \, \hat{f} \in L^1(\mathbb{R}) \right\}
$$

with the norm $\|f\|_{\mathcal{A}} = \|\hat{f}\|_{L^1}$, and also the corresponding unital Banach algebra

$$
\mathcal{A}_1 = \left\{ f : f(k) = c + \int_\mathbb{R} e^{ikp} \hat{g}(p) \, dp, \, \hat{g} \in L^1(\mathbb{R}), \, c \in \mathbb{C} \right\}
$$

with the norm $\|f\|_{\mathcal{A}_1} = |c| + \|\hat{g}\|_{L^1}$. Obviously, $\mathcal{A}$ is a subalgebra of $\mathcal{A}_1$. The algebra $\mathcal{A}_1$ can be treated as an algebra of Fourier transforms of functions $c\delta(\cdot) + \hat{g}(\cdot)$, where $\delta$ is the Dirac delta distribution and $\hat{g} \in L^1(\mathbb{R})$. Note that if $f \in \mathcal{A}_1 \setminus \mathcal{A}$ and $f(k) \neq 0$ for all $k \in \mathbb{R}$ then $f^{-1}(k) \in \mathcal{A}_1$ by the Wiener theorem [28].
Next we recall a few facts from scattering theory [5], [14] of the Schrödinger operator $H$ defined by (1.1). Under the assumption $V \in L^1$, there exist Jost solutions $f_\pm(x,k)$ of the equation

$$H \psi = k^2 \psi, \quad k \in \overline{\mathbb{C}}_+,$$

where $\mathbb{C}_+ = \{k: \text{Im} \ k > 0\}$, and they are normalized so that

$$f_\pm(x,k) \sim e^{\pm ikx}, \quad x \to \pm \infty.$$

These solutions can be written in the form

$$f_\pm(x,k) = e^{\pm ikx}h_\pm(x,k), \quad h_\pm(x,k) = 1 \pm \int_0^{\pm \infty} B_\pm(x,y)e^{\pm 2iky} dy, \quad (2.1)$$

where the $B_\pm(x,y)$ are real-valued and satisfy the conditions

$$|B_\pm(x,y)| \leq e^{\gamma_\pm(x)}\eta_\pm(x + y), \quad (2.2)$$

$$\left| \frac{\partial}{\partial x} B_\pm(x,y) + V(x+y) \right| \leq 2e^{\gamma_\pm(x)}\eta_\pm(x+y)\eta_\pm(x), \quad (2.3)$$

with

$$\gamma_\pm(x) = \int_x^{\pm \infty} (y-x)|V(y)| dy \quad \text{and} \quad \eta_\pm(x) = \pm \int_x^{\pm \infty} |V(y)| dy \quad (2.4)$$

(see [5], §2, or [14], §3.1). Since $\eta_\pm(x+\cdot) \in L^1(\mathbb{R})$, we clearly have

$$h_\pm(x, \cdot) - 1, h_\pm'(x, \cdot) \in \mathcal{A} \quad \forall x \in \mathbb{R}. \quad (2.5)$$

Let

$$W(\varphi(x,k), \psi(x,k)) = \varphi(x,k)\psi'(x,k) - \varphi'(x,k)\psi(x,k)$$

be the usual Wronskian, and let

$$W(k) = W(f_-(x,k), f_+(x,k)) \quad \text{and} \quad W_\pm(k) = W(f_+(x,k), f_\pm(x,-k)).$$

The Jost solutions $f_\pm(x,k)$ and their derivatives do not belong to $\mathcal{A}_1$, nor does their Wronskian $W(k)$. However, the entries of the scattering matrix, that is, the transmission and reflection coefficients

$$T(k) = \frac{2ik}{W(k)} \quad \text{and} \quad R_\pm(k) = \mp \frac{W_\pm(k)}{W(k)},$$

turn out to be elements of this algebra.

**Theorem 2.1.** If $V \in L^1$, then $T(k) - 1 \in \mathcal{A}$ and $R_\pm(k) \in \mathcal{A}$.

**Proof.** Since $|T(k)| \leq 1$ for $k \in \mathbb{R}$, the Wronskian $W(k)$ can vanish only at the edge $k = 0$ of the continuous spectrum, which is known as the resonant case. Moreover, the zero has at most first order.
Step (i). We first consider the non-resonant case $W(0) \neq 0$. We use the abbreviated notation $h_{\pm}(k) := h_{\pm}(0, k)$ and $h'_{\pm}(k) := h'_{\pm}(0, k)$. Then (2.1) implies that

$$W(k) = 2ikh_+(k)h_-(k) + \tilde{W}(k), \quad \tilde{W}(k) := h_-(k)h'_+(k) - h'_-(k)h_+(k),$$

(2.6)

$$W_\pm(k) = h_\mp(k)h'_\pm(-k) - h_\pm(-k)h'_\mp(k).$$

(2.7)

Moreover, $\tilde{W}(k), W_\pm(k) \in \mathcal{A}$. Let

$$\nu(k) := \frac{1}{ik - 1} = -\int_0^\infty e^{iky}e^{-y} dy$$

(2.8)

and observe that $\nu(k) \in \mathcal{A}$, $k\nu(k) \in \mathcal{A}_1$, and therefore $\nu(k)W(k) \in \mathcal{A}_1$. Since $\nu(k)W(k) \to 2$ as $k \to \infty$, it follows that $\nu(k)W(k) \in \mathcal{A}_1 \setminus \mathcal{A}$. Moreover, $\nu(k)W(k) \neq 0$ for all $k \in \mathbb{R}$, whence $(\nu(k)W(k))^{-1} \in \mathcal{A}_1$. Furthermore, $\nu(k)W_\pm(k) \in \mathcal{A}$, and we get that

$$R_\pm(k) = 1 + \frac{\nu(k)W_\pm(k)}{\nu(k)W(k)} \in \mathcal{A} \quad \text{and} \quad T(k) = \frac{2ik\nu(k)}{\nu(k)W(k)} \in \mathcal{A}_1.$$

Moreover, since $T(k) \to 1$ as $k \to \infty$, we have $T(k) - 1 \in \mathcal{A}$.

Step (ii). In the resonant case we need to work a bit harder. We introduce the functions

$$\Phi_\pm(k) := h_\pm(k)h'_\pm(0) - h'_\pm(k)h_\pm(0),$$

(2.9)

$$K_\pm(x) := \pm \int_{-\infty}^{x} B_\pm(0, y) dy \quad \text{and} \quad D_\pm(x) := \pm \int_{x}^{\infty} \frac{\partial}{\partial x} B_\pm(0, y) dy,$$

(2.10)

where the $B_\pm(x, y)$ are the transformation operators in (2.1). Integrating in (2.1) formally by parts, we obtain

$$h'_\pm(k) = \pm \int_0^{\pm\infty} \frac{\partial}{\partial x} B_\pm(0, y)e^{2iky} dy = D_\pm(0) + 2ik \int_0^{\pm\infty} D_\pm(y)e^{2iky} dy$$

$$= h'_\pm(0) + 2ik \int_0^{\pm\infty} D_\pm(y)e^{2iky} dy,$$

$$h_\pm(k) = h_\pm(0) + 2ik \int_0^{\pm\infty} K_\pm(y)e^{2iky} dy.$$

We emphasize that the above integrals have to be understood as improper integrals. Inserting them into (2.9) gives

$$\Phi_\pm(k) = 2ik\Psi_\pm(k), \quad \Psi_\pm(k) := \int_0^{\pm\infty} (D_\pm(y)h_\pm(0) - K_\pm(y)h'_\pm(0))e^{2iky} dy.$$

Lemma 2.2. If $V \in L^1$ then $\Psi_\pm(k) \in \mathcal{A}$.

Proof. Following [9], we will prove that the functions

$$H_\pm(y) := D_\pm(y)h_\pm(0) - K_\pm(y)h'_\pm(0)$$

...
are in the space $L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$. We simplify the original proof in [9] using the Gelfand–Levitan–Marchenko equation in the form proposed in [5]. Namely, as is known (see §3.5 in [14]), the kernels $B_\pm(x,y)$ solve the equations

$$F_\pm(x+y) + B_\pm(x,y) \pm \int_0^{\pm\infty} B_\pm(x,t)F_\pm(x+y+z)\,dz = 0,$$  

(2.11)

where the functions $F_\pm(x)$ are absolutely continuous with $F'_\pm \in L^1(\mathbb{R}_+)$ and they satisfy

$$|F_\pm(x)| \leq C\eta_\pm(x), \quad \pm x > 0,$$  

(2.12)

with $\eta_\pm$ from (2.4). Now we differentiate (2.11) with respect to $x$ and set $x = 0$. Furthermore, we set $x = 0$ in equation (2.11) itself and then integrate both equations with respect to $y$ from $x$ to $\pm\infty$. Then (2.10) implies that

$$\pm \int_x^{\pm\infty} F_\pm(y)\,dy + K_\pm(x) + \int_0^{\pm\infty} B_\pm(0,z) \int_x^{\pm\infty} F_\pm(y+z)\,dy\,dz = 0$$

and

$$\mp F_\pm(x) + D_\pm(x) + \int_0^{\pm\infty} \frac{\partial}{\partial x} B_\pm(0,z) \int_x^{\pm\infty} F_\pm(y+z)\,dy\,dz$$

$$- \int_0^{\pm\infty} B_\pm(0,z)F_\pm(x+z)\,dz = 0.$$  

To get rid of the double integration here, we use (2.10) and the equalities

$$\frac{\partial}{\partial z} \int_x^{\pm\infty} F_\pm(y+z)\,dy = -F_\pm(x+z).$$

Integration by parts yields

$$\pm (1 + K_\pm(0)) \int_x^{\pm\infty} F_\pm(y)\,dy + K_\pm(x) \mp \int_0^{\pm\infty} K_\pm(z)F_\pm(x+z)\,dz$$

$$= K_\pm(x) \pm h_\pm(0) \int_x^{\pm\infty} F_\pm(y)\,dy \mp \int_0^{\pm\infty} K_\pm(z)F_\pm(x+z)\,dz = 0$$  

(2.13)

and

$$\mp F_\pm(x) + D_\pm(x) \pm h'_\pm(0) \int_x^{\pm\infty} F_\pm(y)\,dy \mp \int_0^{\pm\infty} D_\pm(z)F_\pm(x+z)\,dz$$

$$- \int_0^{\pm\infty} B_\pm(0,z)F_\pm(x+z)\,dz = 0.$$  

(2.14)

Multiplying (2.13) by $h'_\pm(0)$ and (2.14) by $h_\pm(0)$ and subtracting the first equation from the second, we get the integral equations

$$H_\pm(x) \mp \int_0^{\pm\infty} H_\pm(y)F_\pm(x+y)\,dy = G_\pm(x),$$  

(2.15)
where
\[ G_\pm(x) = h_\pm(0) \left( \int_0^{\pm \infty} B_\pm(0, y) F_\pm(x + y) \, dy \pm F_\pm(x) \right). \]

The estimates (2.2) and (2.12) imply that
\[ |G_\pm(x)| \leq C \eta_\pm(x), \quad \pm x \geq 0. \tag{2.16} \]

Furthermore, for sufficiently large \( N > 0 \) we represent (2.15) in the form
\[ H_\pm(x) \mp \int_{\pm N}^{\pm \infty} H_\pm(y) F_\pm(x + y) \, dy = G_\pm(x, N), \tag{2.17} \]
where
\[ G_\pm(x, N) = G_\pm(x) \pm \int_0^{\pm N} H_\pm(y) F_\pm(x + y) \, dy. \]

The formulae (2.10) and the estimates (2.2)–(2.4) give us that \( H_\pm \in L^\infty(\mathbb{R}_\pm) \cap C(\mathbb{R}_\pm) \). Then \( |G_\pm(x, N)| \leq C(N) \eta_\pm(x) \) by (2.16) and the monotonicity of \( \eta_\pm(x) \).
Applying the method of successive approximations (see [14], Chap. 3, §2) to (2.17), we find that \( H_\pm \in L^1(\mathbb{R}_\pm) \). Lemma 2.2 is proved. \( \square \)

Now we can continue the proof of Theorem 2.1 in the resonant case. Since the Jost solutions are linearly dependent for \( k = 0 \), that is, \( h_+(x, 0) = ch_-(x, 0) \), we distinguish two cases: \( h_+(0)h_-(-0) \neq 0 \) and \( h_+(0) = h_-(-0) = 0 \). In the first case we have
\[ \widetilde{W}(k) = \widetilde{W}(k) - \widetilde{W}(0) = \frac{h_+(k)}{h_-(0)} \Phi_-(k) - \frac{h_-(-k)}{h_+(0)} \Phi_+(k) \]
\[ = 2ik \left( \frac{h_+(k)}{h_-(0)} \Psi_- (k) - \frac{h_-(-k)}{h_+(0)} \Psi_+ (k) \right), \]
and similarly in the second case when \( h_+(0) = h_-(-0) = 0 \) (so \( h'_+(0)h'_-(-0) \neq 0 \)) we have \( \Phi_\pm(k) = h_\pm(k)h'_\pm(0) = 2ik \Psi_\pm(k) \), so that
\[ \widetilde{W}(k) = 2ik \left( \frac{h'_+(k)}{h'_-(0)} \Psi_- (k) - \frac{h'_-(k)}{h'_+(0)} \Psi_+ (k) \right). \]

In summary,
\[ \frac{W(k)}{2ik} = h_-(k)h_+(k) + \begin{cases} h_+(k) \Psi_-(k) - \frac{h_-(-k)}{h_+(0)} \Psi_+(k), & h_+(0)h_-(-0) \neq 0, \\ h'_+(k) \Psi_- (k) - \frac{h'_-(k)}{h'_+(0)} \Psi_+ (k), & h_+(0)h_-(-0) = 0, \end{cases} \]
where the right-hand side is in \( \mathcal{A}_1 \) by (2.5) and Lemma 2.2. Since \( W(k)/(2ik) = T(k)^{-1} \neq 0 \), we conclude that \( T(k) - 1 \in \mathcal{A} \). Similarly,
\[ \frac{W_\pm(k)}{2ik} = \begin{cases} \frac{h_\pm(-k)}{h_\mp(0)} \Psi_\mp (k) - \frac{h_\pm(k)}{h_\mp(0)} \Psi_\pm (-k), & h_+(0)h_-(-0) \neq 0, \\ \frac{h'_\pm(-k)}{h'_\mp(0)} \Psi_\mp (k) - \frac{h'_\pm(k)}{h'_\mp(0)} \Psi_\pm (-k), & h_+(0)h_-(-0) = 0, \end{cases} \]
where the right-hand side is again in \( \mathcal{A} \), and hence

\[
R_\pm(k) = \pm \frac{W_\pm(k)}{2ik} T(k) \in \mathcal{A}.
\]

Theorem 2.1 is proved. □

Finally, we investigate the function defined as follows:

\[
\psi(x, y, k) = h_+(y, k)h_-(x, k)T(k) - 1 \quad \text{for } y \geq x
\]

and \( \psi(x, y, k) = \psi(y, x, k) \) for \( y < x \). From Theorem 2.1 and (2.5) it follows that \( \psi(x, y, \cdot) \in \mathcal{A} \).

**Lemma 2.3.** The estimate

\[
\| \psi(x, y, \cdot) \|_{\mathcal{A}} \leq C
\]

is valid with some constant \( C \) which does not depend on \( x \) and \( y \).

**Proof.** We introduce the quantities

\[
\sup_{\pm x \geq 0} \left( \pm \int_0^{\pm \infty} |B_\pm(x, y)| \, dy \right) = C_\pm,
\]

which are finite by (2.2). Then

\[
\| h_\pm(x, \cdot) \|_{\mathcal{A}^1} \leq 1 + C_\pm, \quad \| h_\pm(x, \cdot) - 1 \|_{\mathcal{A}} \leq C_\pm \quad \text{for } \pm x \geq 0.
\]

Now consider the three possibilities: (a) \( x \leq y \leq 0 \), (b) \( 0 \leq x \leq y \), and (c) \( x \leq 0 \leq y \). In the case (c) the estimate \( \| \psi(x, y, \cdot) \|_{\mathcal{A}} \leq C \) follows immediately from (2.20) and Theorem 2.1. In the other two cases we use the scattering relations

\[
T(k)f_\pm(x, k) = R_\mp(k)f_\mp(x, k) + f_\mp(x, -k)
\]

to get the representation

\[
\psi(k, x, y) = \begin{cases} 
    h_-(x, k)(R_-(k)h_-(y, k)e^{-2i\nu k} + h_-(y, -k)) - 1, & x \leq y \leq 0, \\
    h_+(y, k)(R_+(k)h_+(x, k)e^{2i\nu k} + h_+(x, -k)) - 1, & 0 \leq x \leq y.
\end{cases}
\]

Observing that \( g(k)e^{iks} \in \mathcal{A} \) for any function \( g(k) \in \mathcal{A} \) and any real \( s \), with norm independent of \( s \), we have (2.19). □

### 3. The Schrödinger equation

Now we are ready to prove the dispersion decay estimate (1.5) for the Schrödinger equation (1.1). The spectral theorem implies that

\[
e^{-itH}P_c = \frac{1}{2\pi i} \int_0^\infty e^{-it\omega} (\Re(\omega + i0) - \Re(\omega - i0)) \, d\omega,
\]

where the right-hand side is again in \( \mathcal{A} \), and hence

\[
R_\pm(k) = \pm \frac{W_\pm(k)}{2ik} T(k) \in \mathcal{A}.
\]
where \( \mathcal{R}(\omega) = (H - \omega)^{-1} \) is the resolvent of the Schrödinger operator \( H \) and the limit is understood in the strong sense \([23]\). Given the Jost solutions, we can express the kernel of the resolvent \( R(\omega) \) for \( \omega = k^2 \pm i0 \) with \( k > 0 \) as (see \([5], [23]\))

\[
[\mathcal{R}(k^2 \pm i0)](x, y) = -f_+(y, \pm k)f_-(x, \pm k) \frac{W(\pm k)}{2ik} = \mp f_+(y, \pm k)f_-(x, \pm k)T(\pm k)
\]

for all \( x \leq y \) (and with the positions of \( x \) and \( y \) reversed if \( x > y \)). Therefore, in the case \( x \leq y \) the integral kernel of the operator \( e^{-itH}P_{k_0}(H) \) is given by

\[
[e^{-itH}P_{k_0}](x, y) = \frac{1}{\pi} \int_{-k_0}^{k_0} e^{-ikt^2} \frac{f_+(y, k)f_-(x, k)T(k)}{2ik} k \, dk
\]

where \( P_{k_0} = P_H([0, k_0^2]) \) is the projection onto the energies in the interval \([0, k_0^2]\). Taking the limit as \( k_0 \to \infty \), we get that

\[
[e^{-itH}P_c](x, y) = \lim_{k_0 \to \infty} [e^{-itH}P_{k_0}](x, y)
= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(tk^2 - |y-x|k)} h_+(y, k)h_-(x, k)T(k) \, dk,
\]

where the integral is understood as an improper integral for \( t \in \mathbb{R} \) (if \( \text{Im}(t) < 0 \), then the integral converges absolutely and we do not need to take the limit, of course). In fact, the convergence of the integral for \( t \in \mathbb{R} \) will follow from Lemma 3.1 below, and Lemma 2.3 will imply that \( |[e^{-itH}P_{k_0}](x, y)| \leq C|t|^{-1/2} \). Thus, we can use the dominated convergence theorem to conclude that the right-hand side of (3.2) is indeed the kernel of \( e^{-itH}P_c \) on \( L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \).

**Lemma 3.1.** Let \( \psi(x, y, k) \) be defined by (2.18) and let \( \hat{\psi}(x, y, p) \) be its Fourier transform with respect to \( k \). Then the following representation is valid for \( \text{Im}(t) \leq 0 \):

\[
[e^{-itH}P_c](x, y) = \frac{1}{\sqrt{4\pi i t}} \left( \exp\left\{ -\frac{|x-y|^2}{4it} \right\} \right.
+ \int_{\mathbb{R}} \exp\left\{ -\frac{(p+|x-y|)^2}{4it} \right\} \hat{\psi}(x, y, p) \, dp \bigg). \tag{3.3}
\]

**Proof.** By (3.2) we have

\[
[e^{-itH}P_c](x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(tk^2 - |y-x|k)(1 + \psi(x, y, k))} dk.
\]
Since the first part of the integral is easy to compute, we focus only on the second part containing $\psi$. By Fubini’s theorem this integral is given by

$$
\frac{1}{2\pi} \lim_{k_0 \to \infty} \int_{-k_0}^{k_0} \exp \left\{ -i(tk^2 - |y - x|k - kp) \right\} \hat{\psi}(x, y, p) \, dp \, dk
$$

$$
= \frac{1}{2\pi} \lim_{k_0 \to \infty} \int_{-k_0}^{k_0} \exp \left\{ -i \frac{(p + |y - x|)^2}{4t} \right\} \, dp \, dk \hat{\psi}(x, y, p)
$$

$$
= \frac{1}{2\sqrt{4\pi t}} \lim_{k_0 \to \infty} \int_{-k_0}^{k_0} \exp \left\{ i \frac{(p + |y - x|)^2}{4t} \right\} \left( \text{erf}(q_+) + \text{erf}(q_-) \right) \hat{\psi}(x, y, p) \, dp,
$$

where

$$q_\pm = \frac{k_0}{2} \sqrt{\frac{4it}{4p + |y - x|}},$$

and $\text{erf}(z)$ is the error function ([19], §7.2). Using the fact that $\text{erf}(z) = 1 + \mathcal{O}(e^{-z^2})$ as $z \to \infty$ with $|\arg(z)| < 3\pi/4$ (see [19], formula (7.12.1)), we get the theorem from dominated convergence.

**Proof of Theorem 1.1.** Since

$$\|e^{-itH}P_c\|_{L^1 \to L^\infty} = \sup_{\|f\|_{L^1} = 1, \|g\|_{L^1} = 1} \langle f, e^{-itH}P_c g \rangle = \sup_{x,y} \langle e^{-itH}P_c \rangle(x, y),$$

the theorem follows from Lemmas 3.1 and 2.3. □

In fact, we have established a slightly stronger result.

**Corollary 3.2.** Let $V \in L^1_+(\mathbb{R})$. Then

$$\|e^{-itH}P_{k_0}\|_{L^1 \to L^\infty} \leq C|t|^{-1/2}, \quad \text{Im } t \leq 0,$$

for every $0 \leq k_0 \leq \infty$.

**Proof.** Using the representation for $[e^{-itH}P_{k_0}](x, y)$ in the proof of Lemma 3.1, together with the boundedness of $\text{erf}(q_\pm)$ in the region under consideration and the estimate (2.19), we get the required estimate. □

### 4. The Schrödinger equation (non-resonant case)

In this section we consider the non-resonant case and prove the dispersion decay estimate (1.6). We begin by representing the jump of the resolvent across the spectrum as

$$[\mathcal{R}(k^2 + i0) - \mathcal{R}(k^2 - i0)](x, y) = \frac{T(k) f_+(y, k) f_-(x, k) + T(k) f_+(y, k) f_-(x, k)}{-2ik}$$

for $x \leq y$ and $k > 0$. The scattering relations (2.21) imply that

$$f_-(x, k) = T(-k) f_+(x, -k) - R_-(-k) f_-(x, -k),$$

$$f_+(y, k) = T(k) f_-(y, k) - R_+(k) f_+(y, k),$$
and using the consistency relation $T\overline{R}_- + \overline{T}R_+ = 0$, we arrive at the formula

$$
[\mathcal{R}(k^2 + i0) - \mathcal{R}(k^2 - i0)](x, y) = \frac{|T(k)|^2}{2ik} [f_+(y, k)f_+(x, -k) + f_-(y, k)f_-(x, -k)]
$$

(4.1)

(see [21], p. 13). Inserting this into (3.1) gives us that

$$
[e^{-itH_0}]_n(x, y) = [\mathcal{X}_+(t)](x, y) + [\mathcal{X}_-(t)](x, y),
$$

$$
[\mathcal{X}_\pm(t)](x, y) = \frac{1}{4\pi} \int_{-\infty}^\infty e^{-i(tk^2 + |y - x|k)} |T(k)|^2 h_\pm(y, k)h_\pm(x, -k) dk.
$$

Integrating by parts, we get that

$$
[\mathcal{X}_\pm(t)](x, y) = \pm \frac{|y - x|}{8\pi t} \int_{-\infty}^\infty e^{-i(tk^2 + |y - x|k)} \frac{|T(k)|^2}{k} h_\pm(y, k)h_\pm(x, -k) dk
$$

$$
- \frac{1}{8\pi it} \int_{-\infty}^\infty e^{-i(tk^2 + |y - x|k)} \frac{|T(k)|^2}{k^2} h_\pm(y, k)h_\pm(x, -k) dk
$$

$$
+ \frac{1}{8\pi it} \int_{-\infty}^\infty e^{-i(tk^2 + |y - x|k)} \frac{1}{k} \frac{\partial}{\partial k} [ |T(k)|^2 h_\pm(y, k)h_\pm(x, -k) ] dk.
$$

Applying the arguments from the proof of Lemma 3.1, we obtain

$$
[\mathcal{X}_\pm(t)](x, y) = \frac{t^{-3/2}}{8\sqrt{\pi} i} \int_{\mathbb{R}} \exp \left\{ \frac{i(p + |x - y|)^2}{4t} \right\} \sum_{j=1}^3 \hat{\psi}_j^\pm(x, y, p) dp,
$$

(4.2)

where $\hat{\psi}_j^\pm(x, y, p)$, $j = 1, 2, 3$, are the Fourier transforms of the respective functions

$$
\psi_1^\pm(x, y, k) = \pm |y - x| \frac{|T(k)|^2}{k} h_\pm(y, k)h_\pm(x, -k),
$$

$$
\psi_2^\pm(x, y, k) = i \frac{|T(k)|^2}{k^2} h_\pm(y, k)h_\pm(x, -k),
$$

$$
\psi_3^\pm(x, y, k) = -i \frac{1}{k} \frac{\partial}{\partial k} [ |T(k)|^2 h_\pm(y, k)h_\pm(x, -k) ].
$$

To estimate their $\mathcal{A}$-norms we first prove the following lemma.

**Lemma 4.1.** Let $V \in L^1_2$ and $W(0) \neq 0$. Then $T(k)h_\pm(x, k)/k \in \mathcal{A}$, and

$$
\left\| \frac{T(k)h_\pm(x, k)}{k} \right\|_\mathcal{A} \leq C(1 + |x|), \quad x \in \mathbb{R}.
$$

(4.3)

**Proof.** Since

$$
\frac{T(k)}{k} = \frac{2\sqrt{\nu(k)}}{\nu(k)W(k)} \in \mathcal{A}
$$

(see (2.8)), for $x \in \mathbb{R}_+$ the bound (4.3) follows from (2.20). Consider the case $x \in \mathbb{R}_-$. The scattering relations (2.21) imply that

$$
T(k)h_\pm(x, k) = (R_+(k) + 1)h_\pm(x, k)e^{2ikx} - (h_\pm(x, k) - h_\mp(x, -k))e^{2ikx}
$$

$$
+ h_\mp(x, -k)(1 - e^{2ikx}).
$$

(4.4)
By (2.1),
\[
\frac{h_\mp(x, k) - h_\mp(x, -k)}{k} = \mp \int_{0}^{\mp \infty} B_\mp(x, r) \frac{e^{\mp ikr} - e^{\pm ikr}}{k} \, dr \\
= i \int_{0}^{\mp \infty} B_\mp(x, r) \int_{-r}^{r} e^{iky} \, dy \, dr \\
= i \int_{-\infty}^{\infty} \left( \int_{\mp |y|}^{\mp \infty} B_\mp(x, r) \, dr \right) e^{iky} \, dy.
\]

Next, observe that the formula (2.2) implies that if \( V \in L^1 \), then \( B_\mp(x, \cdot) \in L^1(\mathbb{R}_\mp) \) for any fixed \( x \), and consequently
\[
S_\mp(x, y) = \int_{y}^{\mp \infty} |B_\mp(x, r)| \, dr \in L^1(\mathbb{R}_\mp).
\]

Based on this observation, we get that
\[
\frac{h_\mp(x, k) - h_\mp(x, -k)}{k} \leq C, \quad x \in \mathbb{R}_\mp. \tag{4.5}
\]

For the same reasons (2.3) implies that
\[
\frac{h'_\mp(0, k) - h'_\mp(0, -k)}{k} \in \mathcal{A}. \tag{4.6}
\]

Next, it follows from (2.6) and (2.7) that
\[
\frac{W(k) \mp W_\pm(k)}{k} = 2i h_\mp(k) h_\pm(k) \\
+ \frac{h_\mp(k) h'_\pm(k) - h'_\mp(k) h_\pm(k) \mp h_\pm(k) h'_\mp(-k) \mp h_\pm(-k) h'_\mp(k)}{k} \\
= 2i h_\pm(k) h_\mp(k) \pm \frac{h'_\pm(k) - h'_\mp(-k)}{k} \mp \frac{h'_\mp(k) h_\pm(k) - h_\pm(-k)}{k}.
\]

Using (2.5), (4.5), and (4.6), we find that
\[
\frac{W(k) \mp W_{\pm}(k)}{k} - 2i \in \mathcal{A}.
\]

As shown in Theorem 2.1, \( W^{-1}(k) \in \mathcal{A} \) in the non-resonant case. Thus,
\[
\frac{R_{\pm}(k) + 1}{k} = \frac{1}{W(k)} \left( \frac{W(k) \mp W_\mp(k)}{k} \right) \in \mathcal{A}. \tag{4.7}
\]

Next, the function \( (1 - e^{\mp 2ikx})/(ik) \) is the Fourier transform of the indicator function of the interval \([0, 2x]\), therefore
\[
\left\| \frac{1 - e^{\pm 2ikx}}{k} \right\| \leq 2|x|. \tag{4.8}
\]

Finally, substituting (4.5), (4.7), and (4.8) into (4.4), we obtain (4.3). \( \square \)
Since we have already obtained the estimate \( \|T(k)h_{\pm}(x, k)\|_{A} \leq C \) in the proof of Theorem 1.1, Lemma 4.1 immediately implies that
\[
\|\hat{\psi}_{j}^{\pm}(x, y, \cdot)\|_{L^{1}} \leq C(1 + |x|)(1 + |y|), \quad j = 1, 2. \tag{4.9}
\]
To estimate \( \|\hat{\psi}_{3}^{\pm}(x, y, \cdot)\|_{L^{1}} \) we need one more lemma.

**Lemma 4.2.** Let \( V \in L^1_2 \) and \( W(0) \neq 0 \). Then \( \frac{\partial}{\partial k}(T(k)h_{\pm}(x, k)) \in \mathcal{A} \) and
\[
\left\| \frac{\partial}{\partial k}(T(k)h_{\pm}(x, k)) \right\|_{\mathcal{A}} \leq C(1 + |x|), \quad x \in \mathbb{R}.
\]

**Proof.** The representation (2.1) and the estimates (2.2) and (2.3) imply that
\[
\frac{\partial}{\partial k}h_{\pm}(x, k) = \dot{h}_{\pm}(x, k) \in \mathcal{A} \quad \text{and} \quad \frac{\partial}{\partial k}h'_{\pm}(x, k) \in \mathcal{A} \quad \text{if} \quad V \in L^1_2, \tag{4.10}
\]
with
\[
\left\| \frac{\partial}{\partial k}h'_{\pm}(x, \cdot) \right\|_{\mathcal{A}} + \|\dot{h}_{\pm}(x, \cdot)\|_{\mathcal{A}} \leq C, \quad x \in \mathbb{R}_{\pm}. \tag{4.11}
\]
Therefore,
\[
d\frac{d}{dk} W_{\pm}(k) := \dot{W}_{\pm}(k) \in \mathcal{A}. \quad \text{Further, from (2.6) and (4.10) it follows that}
\]
\[
\nu(k)\dot{W}(k) \in \mathcal{A}, \quad \text{where } \nu(k) \text{ is defined by (2.8).}
\]
Since in the non-resonant case \((\nu(k)W(k))^{-1} \in \mathcal{A}_1 \) and \(W^{-1}(k) \in \mathcal{A} \), it follows that
\[
\dot{T}(k) = \frac{1}{W(k)}(2i - \dot{W}(k)T(k)) \in \mathcal{A}, \quad \dot{R}_{\pm}(k) \in \mathcal{A}. \tag{4.12}
\]
Thus, for \( x \in \mathbb{R}_{\pm} \) the statement of the lemma is evident in view of (2.20), (4.12), and (4.11). To get it for \( x \in \mathbb{R}_{\mp} \) we use (2.21), (4.11), and (4.12), which gives
\[
\frac{\partial}{\partial k}(T(k)h_{\pm}(x, k)) = e^{\mp 2ikx} \left( \frac{\partial}{\partial k}(R_{\mp}(k)h_{\mp}(x, k)) \mp 2ixR_{\mp}(k)h_{\mp}(x, k) \right) + \dot{h}_{\mp}(x, -k).
\]
The lemma is proved. \( \square \)

As pointed out in the proof of Theorem 1.1, the estimate \( \|T(k)h_{\pm}(x, k)\|_{A} \leq C \) is valid for \( x \in \mathbb{R} \). This and Lemma 4.2 imply that
\[
\|\hat{\psi}_{3}^{\pm}(x, y, \cdot)\|_{L^{1}} \leq C(1 + |x|)(1 + |y|). \tag{4.13}
\]
Finally, combining (4.2), (4.9), (4.13), and Lemma 3.1, we obtain
\[
|K_{\pm}(t)(x, y)| \leq Ct^{-3/2}(1 + |x|)(1 + |y|), \quad t \geq 1,
\]
which proves (1.6) and finishes the proof of Theorem 1.2.
5. The Klein–Gordon equation

In this section we prove Theorem 1.3, (i), that is, the estimate (1.10), for the Klein–Gordon equation (1.3). We estimate the low-energy and high-energy components of the solution separately. The relation (1.10) will immediately follow from the two theorems below.

Theorem 5.1. Assume that \( V \in L_1^1(\mathbb{R}) \). Then for any smooth function \( \zeta \) with compact support the following decay holds:

\[
\left\| e^{-itH}P_c \zeta(H^2) \right\|_{L^1 \to L^\infty} = O(t^{-1/2}), \quad t \to \infty.
\]

Theorem 5.2. Assume that \( V \in L_1^1(\mathbb{R}) \) and let \( \xi(x) \) be a smooth function such that \( \xi(x) = 0 \) for \( x \leq m^2 + 1 \) and \( \xi(x) = 1 \) for \( x \geq m^2 + 2 \). Then

\[
\left\| e^{-itH} \right\|^{12}_{H^{1/2,1} \to L^\infty} = O(t^{-1/2}), \quad t \to \infty.
\]

The next result is a consequence of Theorem 1.3, (i).

Corollary 5.3. Assume that \( V \in L_1^1(\mathbb{R}) \). Then (1.12) holds for any \( p \in [2, \infty] \). Namely,

\[
\left\| e^{-itH} [c]^{12} \right\|_{B^{1/2-3/p,1}_p \to L^p} = O(t^{-1/2+1/p}), \quad t \to \infty, \quad \frac{1}{p'} + \frac{1}{p} = 1. \tag{5.1}
\]

Proof. Recall that the Klein–Gordon equation preserves energy:

\[
\| \dot{\psi} \|^2_{L^2} + \langle \psi, H\psi \rangle_{L^2} + m^2 \| \psi \|^2_{L^2} = \text{const}.
\]

Since \( e^{-itH}P_c \) \( \xi_0 \) corresponds to the initial condition \( \langle \psi(0), \dot{\psi}(0) \rangle = (0, \pi_0) \) with \( \pi_0 = P_c(H)\pi_0 \), we obtain the estimate \( \langle \psi, H\psi \rangle_{L^2} + m^2 \| \psi \|^2_{L^2} \leq \| \pi_0 \|^2_{L^2} \) in this case. Moreover, since for \( V \in L^1 \) the operator of multiplication by \( V \) is relatively form-bounded with bound 0 with respect to \( H_0 = -\frac{d^2}{dx^2} \) (see [23], Lemma 9.33), the graph norms of \( H \) and \( H_0 \) are equivalent, and we obtain \( \| \psi \|_{H^1} \leq C \| \pi_0 \|_{L^2} \). Hence by duality we also obtain

\[
\left\| e^{-itH} [c]^{12} \right\|_{H^{-1} \to L^2} = O(1), \quad t \to \infty, \quad H^{-1} = H^{-1,2}. \tag{5.2}
\]

Since \( H^{-1} = B^{1/2}_2 \) due to Theorem 2.3.2, (d) in [24], real interpolation between (1.10) and (5.2) gives (5.1). \( \square \)

5.1. Low-energy decay. Here we prove Theorem 5.1. We will need a variant of the van der Corput lemma of independent interest.

Lemma 5.4. Consider the oscillatory integral

\[
I(t) = \int_a^b e^{i\phi(k)} f(k) \, dk,
\]

where \( \phi(k) \) is real-valued function. If \( \phi''(k) \neq 0 \) on \( [a, b] \) and \( f \in A_1 \), then

\[
|I(t)| \leq C_2 \left[ t \min_{a \leq k \leq b} |\phi''(k)| \right]^{-1/2} \| f \|_{A_1}, \quad t \geq 1,
\]

where \( C_2 \leq 2^{8/3} \) is the optimal constant from the van der Corput lemma.
Proof. Writing $f(k) = c + \int_{\mathbb{R}} e^{iky} \widehat{g}(y) \, dy$, we have

$$I(t) = \int_{\mathbb{R}} \widehat{g}(y) I_{y/t}(t) \, dy + cI_0(t) \quad \text{and} \quad I_v(t) = \int_{\mathbb{R}} e^{it(\phi(k) + vk)} \, dk.$$ 

By the van der Corput lemma,

$$|I_v(t)| \leq C_2 \left( t \min_{a \leq k \leq b} |\phi''(k)| \right)^{-1/2}, \quad t \geq 1,$$

where $C_2 \leq 2^{8/3}$ (see [20]), and the lemma now follows from the definition of the norm in $\mathcal{A}_1$. \hfill \Box

We note that this lemma extends to higher derivatives and to unbounded intervals (with the integral understood as an improper Riemann integral).

The resolvent $R(\omega)$ of the operator $H$ in (1.4) can be expressed in terms of the resolvent $\mathcal{R}(\omega) = (H - \omega)^{-1}$ of the Schrödinger operator as

$$R(\omega) = \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix} + \begin{pmatrix} \omega & i \\ -i\omega & \omega \end{pmatrix} \mathcal{R}(\omega^2 - m^2).$$

For $e^{-itH}P_c \zeta(H^2)$ the spectral representation of type (3.1) holds:

$$e^{-itH}P_c \zeta(H^2) = \frac{1}{2\pi i} \int_{\Gamma} e^{-it\omega} \zeta(\omega^2)(R(\omega + i0) - R(\omega - i0)) \, d\omega$$

$$= \frac{1}{2\pi i} \int_{\Gamma} e^{-i\omega t} \zeta(\omega^2) \begin{pmatrix} \omega & i \\ -i\omega & \omega \end{pmatrix} \mathcal{R}((\omega + i0)^2 - m^2)$$

$$- \mathcal{R}((\omega - i0)^2 - m^2) \, d\omega, \quad (5.3)$$

where $\Gamma = (-\infty, -m) \cup (m, \infty)$. Let

$$\mathcal{M}_t(k) = \begin{pmatrix} \cos(t\sqrt{k^2 + m^2}) & \sin(t\sqrt{k^2 + m^2}) \\ -\sqrt{k^2 + m^2} \sin(t\sqrt{k^2 + m^2}) & \cos(t\sqrt{k^2 + m^2}) \end{pmatrix}.$$ 

Then (5.3) can be rewritten as

$$[e^{-itH}P_c \zeta(H^2)](x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{M}_t(k)e^{i|y-x|k} \zeta(k^2 + m^2)(\psi(x, y, k) + 1) \, dk, \quad (5.4)$$

where the function $\psi(x, y, k)$ is defined by (2.18). We obtain oscillatory integrals with the phase functions $\phi_{\pm}(k) = \pm \sqrt{k^2 + m^2} - vk$, where $v = |y-x|/t$. The second derivative of $\phi_{\pm}(k)$ satisfies

$$|\phi''_{\pm}(k)| = \frac{m^2}{\sqrt{(k^2 + m^2)^3}} \geq C(m, \zeta), \quad k^2 + m^2 \in \text{supp} \zeta.$$ 

Since $(k^2 + m^2)^{j/2} \zeta(k^2 + m^2) \in \mathcal{A}$ for $j = -1, 0, 1$ and $\|\psi(x, y, k)\|_{\mathcal{A}} \leq C$ due to (2.19), Lemma 5.4 implies that

$$\max_{x, y \in \mathbb{R}} \left| [e^{-itH}P_c \zeta(H^2)](x, y) \right| \leq Ct^{-1/2}, \quad t \geq 1.$$
5.2. High-energy decay. Here we prove Theorem 5.2. Our proof is based on the following version of Lemma 2 from [15].

**Lemma 5.5.** Let \( \eta(k), k \geq 1 \), be a smooth function such that \( |\eta^{(j)}(k)| \leq k^{-j} \) for \( j = 0, 1 \). Then for any \( g(k) \in \mathcal{A}_1 \), \( \alpha > 3/2 \), and \( t \geq 1 \),

\[
\sup_{p \in \mathbb{R}} \left| \int_1^\infty \eta(k) e^{\pm i \sqrt{k^2 + m^2 + ikp}/k^\alpha} g(k) \, dk \right| \leq C \| g \|_{\mathcal{A}_1} t^{-1/2}. \tag{5.5}
\]

Moreover,

\[
\sup_{p \in \mathbb{R}} \left| \int_1^\infty \eta(k) e^{\pm i \sqrt{k^2 + m^2 + ikp}/k^{3/2}} \, dk \right| \leq Ct^{-1/2}. \tag{5.6}
\]

Here the constants \( C \) depend only on the parameters \( m \) and \( \alpha \).

**Proof.** Consider the \(+\)-case and let \( v = -p/t \). To prove (5.5) we have to estimate the oscillatory integral

\[
I_\alpha(t) = \int_1^\infty k^{-\alpha} \eta(k) e^{i \phi(k)} g(k) \, dk
\]

with the phase function \( \phi(k) = \sqrt{k^2 + m^2} - vk \). We split the integral into two parts:

\[
I_\alpha(t) = I^1_\alpha(t) + I^2_\alpha(t) = \int_1^t + \int_t^\infty.
\]

Since \( \| g \|_{\infty} \leq \| g \|_{\mathcal{A}_1} \),

\[
|I^2_\alpha(t)| \leq \| g \|_{\mathcal{A}_1} \int_t^\infty k^{-\alpha} \, dk \leq C \| g \|_{\mathcal{A}_1} t^{1-\alpha}. \tag{5.7}
\]

We estimate \( I^1_\alpha(t) \). Let

\[
\Psi(k, t) = \int_1^k e^{i \phi(\tau)} g(\tau) \, d\tau.
\]

Since

\[
\min_{1 \leq \tau \leq k} \phi''(\tau) = \phi''(k) = \frac{m^2}{(\sqrt{k^2 + m^2})^3} \geq C k^3,
\]

Lemma 5.4 implies that

\[
|\Psi(k, t)| \leq C \| g \|_{\mathcal{A}_1} t^{-1/2} k^{3/2}. \tag{5.8}
\]

Integrating \( I^1_\alpha(t) \) by parts, we get that

\[
|I^1_\alpha(t)| \leq |\Psi(t, t)| t^{-\alpha} + \int_1^t |\Psi(k, t)| |\Lambda(k)| \, dk,
\]

where \( \Lambda(k) = (k \eta'(k) - \alpha \eta(k))/k^{\alpha+1} \) is a smooth bounded function, and \( \Lambda(k) = \mathcal{O}(k^{-\alpha-1}) \) as \( k \to \infty \). By (5.8),

\[
|I^1_\alpha(t)| \leq C \| g \|_{\mathcal{A}_1} \left( t^{1-\alpha} + (1 + \alpha) t^{-1/2} \int_1^t k^{1/2-\alpha} \, dk \right) \leq C \| g \|_{\mathcal{A}_1} t^{-1/2}.
\]

Together with (5.7) this proves (5.5).
Next we turn to (5.6). Since (5.7) is valid for $\alpha = 3/2$ and $g(k) = 1$, this estimate follows from Lemma 7.1. □

To prove Theorem 5.2 we have to show that for any smooth function $f$ with compact support

$$\left\| [e^{-itH}]^{12} \xi(H^2)f \right\|_{L^\infty} \leq C t^{-1/2} \|f\|_{H^{1/2,1}}, \quad t \geq 1. \tag{5.9}$$

The kernel of the resolvent of the free Schrödinger operator is (see [23], §7.4)

$$[\mathcal{R}_0(k^2 \pm i0)](x, y) = \frac{\pm ie^{\pm ik|x-y|}}{2k}, \quad k > 0.$$ 

Substituting the second resolvent identity $\mathcal{R}(\lambda) = \mathcal{R}_0(\lambda) - \mathcal{R}_0(\lambda)V\mathcal{R}(\lambda)$ into the 1,2-entry of (5.4) and taking into account the equality $\xi(x) = 0$ for $x \leq m^2 + 1$, we get that

$$[e^{-itH}]^{12} \xi(H^2) = K_0(t) + K_1(t),$$

where the kernels of the operators $K_0(t)$ and $K_1(t)$ are

$$[K_0(t)](x, y) = \frac{1}{2\pi} \int_{|k| \geq 1} \xi(k^2 + m^2) \frac{\sin(t\sqrt{k^2 + m^2})}{\sqrt{k^2 + m^2}} e^{ik(x-y)} \, dk, \tag{5.10}$$

$$[K_1(t)](x, y) = \frac{i}{4\pi} \int_{\mathbb{R}} V(z) \left( \int_{|k| \geq 1} \xi(k^2 + m^2) \right. \times \frac{\sin(t\sqrt{k^2 + m^2})}{\sqrt{k^2 + m^2}} e^{ik(|x-z|+|z-y|)} k (\psi(y,z,k) + 1) \, dk \bigg) \, dz. \tag{5.11}$$

We note that the derivative $\xi'(x)$ has support in the interval $[m^2 + 1, m^2 + 2]$. Therefore, the function

$$\eta(k) := \frac{i}{4\pi} \xi(k^2 + m^2) \frac{k}{\sqrt{k^2 + m^2}} \tag{5.12}$$

satisfies the assumptions of Lemma 5.5. Applying this lemma with $\alpha = 2$, $g(k) = \psi(y,z,k) + 1$, and $p = |x-z| + |z-y|$ and taking (2.19) into account, we obtain

$$\|K_1(t)f\|_{L^\infty} \leq C t^{-1/2} \|f\|_{L^1} \leq C |t|^{-1/2} \|f\|_{H^{1/2,1}}, \quad t \geq 1, \tag{5.13}$$

since $H^{1/2,1} \subset L^1$ due to Theorem 6.2.3 in [3]. It remains to get an estimate of the type (5.9) for $K_0(t)$.

**Lemma 5.6.** Assume that $V \in L^1_1$. Then

$$\|K_0(t)f\|_{L^\infty} \leq C |t|^{-1/2} \|f\|_{H^{1/2,1}}, \quad |t| \geq 1.$$

**Proof.** For any $f \in C_0^\infty$, (5.10) implies that

$$\int_{\mathbb{R}} [K_0(t)](x, y) f(y) \, dy = \sum_{k \geq 1} \int_{|k| \geq 1} \frac{\eta(k)}{k(1+k^2)^{1/4}} e^{\pm t\sqrt{k^2 + m^2} + ikx} (1 + k^2)^{1/4} f(k) \, dk,$$
where $\eta(k)$ is defined by (5.12). Let $g = \mathcal{J}_{1/2}f$. By the definition (1.9),
\[ \|g\|_{L^1} = \|f\|_{H^{1/2,1}}. \] (5.14)
Thus,
\[ \|K_0(t)f\|_{L^\infty} \leq C \int_{\mathbb{R}} |g(y)| \sup_{x,y \in \mathbb{R}} \left| \int_1^\infty \eta(k) e^{\pm it\sqrt{k^2+m^2+ik(x-y)}} \frac{k^{3/2}}{k^3/2} \right| dk \ dy, \]
which, together with (5.14) and (5.6), implies the required estimate for $K_0$. □

Together with (5.13), this establishes Theorem 5.2, and completes the proof of Theorem 1.3, (i).

6. The Klein–Gordon equation (non-resonant case)

Suppose that the operator $H$ in (1.4) does not have a resonance at 0. To prove Theorem 1.3, (ii), consider first the low-energy part of the solution. Using the representation (4.1), we rewrite (5.4) as
\[ [e^{-itH}P_c\zeta(H^2)](x, y) = \sum_{\sigma_1, \sigma_2 \in \{\pm\}} \frac{1}{2\pi} \int_{-\infty}^\infty A_{\sigma_1}(k)e^{it\sqrt{k^2+m^2}} \]
\[ \times e^{i|y-x|k} \zeta(k^2+m^2) T_{\sigma_2}(x, y, k) \ dk, \]
where
\[ A_{\pm}(k) = \begin{pmatrix} 1 & \mp \frac{i}{\sqrt{k^2+m^2}} \\ \pm i\sqrt{k^2+m^2} & 1 \end{pmatrix} \]
and $T_{\pm}(x, y, k) = |T(k)|^2 f_{\pm}(k) f_{\pm}(-k)$. Using integration by parts, we get for the summand $[e^{-itH}P_c\zeta(H^2)]_{++}(x, y)$ with $A_+$ and $T_+$ that
\[ [e^{-itH}P_c\zeta(H^2)]_{++}(x, y) = -\frac{1}{2\pi it} \int_{-\infty}^\infty e^{it\sqrt{k^2+m^2}} \]
\[ \times \frac{\partial}{\partial k} \left[ e^{i|y-x|k} \zeta(k^2+m^2) \frac{\sqrt{k^2+m^2}}{k} A_+(k) T_+(x, y, k) \right] dk. \]
By the same arguments as in the proof of Theorem 1.2 (see §4), we find that
\[ \left| [e^{-itH}P_c\zeta(H^2)]_{++}(x, y) \right| \leq Ct^{-3/2}(1+|x|)(1+|y|), \quad t \geq 1, \]
and hence
\[ \left\| [e^{-itH}P_c\zeta(H^2)]_{++} \right\|_{L^1 \to L^\infty} = \mathcal{O}(|t|^{-3/2}), \quad t \to \infty. \]
The other summands can be estimated similarly, and we get that
\[ \left\| [e^{-itH}P_c\zeta(H^2)]_{++} \right\|_{L^1 \to L^\infty} = \mathcal{O}(t^{-3/2}), \quad t \to \infty. \] (6.1)
It remains to consider the high-energy part. To simplify the notation let
\[ c(k, t) := \cos(t \sqrt{k^2 + m^2}) \quad \text{and} \quad \chi(k) := \xi(k^2 + m^2). \]

Integrating by parts in (5.10) and (5.11), we find that
\[
[K_0(t)](x, y) = \frac{1}{2\pi t} \int_{|k| \geq 1} c(k, t) \frac{\partial}{\partial k} \frac{\chi(k)e^{ik(x-y)}}{k} \, dk,
\]
\[
[K_1(t)](x, y) = \frac{i}{4\pi t} \int_{\mathbb{R}} V(z) \int_{|k| \geq 1} c(k, t) \times \frac{\partial}{\partial k} \frac{\chi(k)e^{ik(|x-z|+|z-y|)+\psi(y,z,k)}}{k^2} \, dk \, dz. \quad (6.2)
\]

To estimate (6.2) recall that
\[
\left\| \frac{\partial}{\partial k} \psi(z, y, k) \right\|_{\mathcal{A}} \leq C(1 + |z|)(1 + |y|).
\]
by (4.9) and (4.13). Moreover, \(|x - z| + |z - y| \leq (1 + |x|)(1 + |y|)(1 + 2|z|).\) Hence, the \(\mathcal{A}\)-norm of the derivative with respect to \(k\) in the integrand in (6.2) does not exceed \(C(1 + |x|)(1 + |y|)(1 + |z|).\) Furthermore, \(\chi'(k)\) is a smooth function with finite support. Applying Lemma 5.5 to the integral with respect to \(k\) in (6.2) and taking into account that \(|V(z)| \in L^1_1(\mathbb{R})\), we arrive at the estimate
\[
[K_1(t)](x, y) \leq Ct^{-3/2}(1 + |x|)(1 + |y|), \quad t \geq 1. \quad (6.3)
\]

Further,
\[
[K_0(t)](x, y) = \frac{1}{2\pi t} \int_{|k| \geq 1} c(k, t)\chi'(k)k^{-1}e^{ik(x-y)} \, dk - \frac{1}{2\pi t} \int_{|k| \geq 1} c(k, t)\chi(k)e^{ik(x-y)}k^{-2} \, dk + \frac{i}{2\pi t} \int_{|k| \geq 1} (x-y)c(k, t)\chi(k)e^{ik(x-y)}k^{-1} \, dk
\]
\[= [K_{01}(t)](x, y) + [K_{02}(t)](x, y) + [K_{03}(t)](x, y). \]

Lemma 5.5 applied to \(K_{01}\) and \(K_{02}\) implies that
\[
\|K_{0j}(t)f\|_{L^\infty} \leq Ct^{-3/2}\|f\|_{L^1}, \quad j = 1, 2, \quad t \geq 1. \quad (6.4)
\]

It remains to estimate \(K_{03}\). Letting \(g = \mathcal{J}_{1/2}f\), we have
\[
\hat{f}(k) = (1 + k^2)^{-1/4}\tilde{g}(k).
\]

Since
\[
\mathcal{F}[f(\cdot)] = \hat{f}' \quad \text{and} \quad \hat{f}'(k) = (1 + k^2)^{-1/4}\tilde{g}'(k) - (k/2)(1 + k^2)^{-5/4}\tilde{g}(k),
\]
we get that
\[
\int_{\mathbb{R}} [K_{03}(t)](x, y) f(y)\, dy
= \frac{ix}{4\pi t} \sum_{|k| \geq 1} \int \frac{\chi(k)}{k(1 + k^2)^{1/4}} e^{\pm t \sqrt{k^2 + m^2 + ikx}} (1 + k^2)^{1/4} \hat{f}(k)\, dk
+ \frac{i}{4\pi t} \sum_{|k| \geq 1} \int \frac{\chi(k)}{k(1 + k^2)^{1/4}} e^{\pm t \sqrt{k^2 + m^2 + ikx}} \hat{g}(k)\, dk
- \frac{i}{8\pi t} \sum_{|k| \geq 1} \int \frac{\chi(k)}{(1 + k^2)^{5/4}} e^{\pm t \sqrt{k^2 + m^2 + ikx}} \hat{g}(k)\, dk.
\]

By the same arguments as in the proof of Lemma 5.6,
\[
\|(1 + |\cdot|)^{-1} K_{03}(t) f\|_{L^\infty} \leq C t^{-3/2} (\|g\|_{L^1} + \|\cdot\|_{L^1})
\leq C t^{-3/2} (\|f\|_{H^{1/2,1}} + \|f\|_{H^{1/2,1}}).
\]

The last estimate and the estimate (6.4) then imply that
\[
\|K_0(t)\|_{H^{1/2,1} \rightarrow L^\infty} \leq C t^{-3/2}.
\]
Together with (6.3) this gives us that
\[
\|[e^{-it\mathcal{H}}]^2 \xi(\mathcal{H}^2) f\|_{L^\infty_{-1}} \leq C |t|^{-3/2} \|f\|_{H^{1/2,1}}.
\]
Combination of this with (6.1) completes the proof of Theorem 1.3, (ii), and thus Theorem 1.3 is proved.

7. A decay estimate

The following is Lemma 6.7 from [3], which is an adapted version of Lemma 2 in [15]. We include a proof here for the sake of completeness.

**Lemma 7.1** (see [3] and [15]). Let $\Lambda(k), k \geq 0$, be a smooth function such that $\Lambda(k) = O(k^{-5/2})$ as $k \to \infty$, and let
\[
\Psi(k, t) := \int_{0}^{k} e^{it\phi(\tau)}\, d\tau, \quad t \geq 1, \quad k \geq 0,
\]
where $\phi(\tau) = \sqrt{\tau^2 + 1} + v\tau$ with some $v \in \mathbb{R}$. Then the following estimate is valid uniformly with respect to $v$:
\[
J(t) := \int_{1}^{t} |\Psi(k, t)\Lambda(k)|\, dk \leq C t^{-1/2}.
\]

**Proof.** For brevity we refer to the van der Corput lemmas for the first and second derivative as vdC-1 and vdC-2, respectively (see [20], Corollary 5 and Lemma 7).
First of all, we observe that the second derivative of the phase function \( \phi(\tau) \) admits the estimate
\[
\min_{0 \leq \tau \leq k} \phi''(\tau) = \min_{0 \leq \tau \leq k} (1 + \tau^2)^{-3/2} = (1 + k^2)^{-3/2}.
\] (7.2)
Hence, vdC-2 implies that
\[
|\Psi(k, t)| \leq Ct^{-1/2}(k + 1)^{3/2}, \quad k \geq 0, \quad t \geq 1.
\] (7.3)

The first derivative of the phase function \( \phi(\tau) \) is a monotonically increasing function satisfying the lower estimates
\[
|\phi'(\tau)| \geq \begin{cases} 
2^{-1/2}, & v \geq 0, \ \tau \geq 1, \\
\frac{1}{2(\tau^2 + 1)}, & v \leq -1, \ \tau \geq 0.
\end{cases}
\] (7.4)
For \( v \in (-1, 0) \) the function \( \phi' \) has a zero at \( \tau_0 = -v(1 - v^2)^{-1/2} \). We study the three regions \( v \geq 0, v \leq -1, \) and \( v \in (-1, 0) \) separately.

For \( v \geq 0 \) and \( k \geq 1 \) we use vdC-1 and (7.4) to get the estimate
\[
|\Psi(k, t) - \Psi(1, t)| \leq Ct^{-1}.
\]
Since \( |\Psi(1, t)| \leq Ct^{-1/2} \) by (7.3), (7.1) follows immediately for \( v \geq 0 \). Similarly, for \( v \leq -1 \) it follows from (7.4) and (7.3) that
\[
|\Psi(k, t)| \leq Ck^2t^{-1} + |\Psi(1, t)| \leq C(k^2t^{-1} + t^{-1/2}), \quad k \geq 1.
\] (7.5)
Thus, (7.1) holds in this case also.

It remains to consider the case \( v \in (-1, 0) \), or equivalently, \( \tau_0 \in (0, \infty) \). In particular, we will estimate \( J(t) \) in terms of \( \tau_0 \) rather than \( v \). From the monotonicity of \( \phi' \) and (7.2) it follows that for all \( \tau \in (0, \tau_0/2] \)
\[
|\phi'(\tau)| = \frac{\tau_0}{\sqrt{\tau_0^2 + 1}} - \frac{\tau}{\sqrt{\tau^2 + 1}} \geq \phi'(2\tau) - \phi'(\tau) \geq \phi''(2\tau)\tau \geq \frac{C}{\tau^2}.
\]
Thus, in the same way as for (7.5) we get that for \( \tau_0/2 \geq 1 \)
\[
|\Psi(k, t)| \leq C(k^2t^{-1} + t^{-1/2}), \quad 1 \leq k \leq \frac{\tau_0}{2}.
\] (7.6)
Furthermore, for all \( \tau \geq 2\tau_0 \)
\[
\phi'(\tau) = \frac{\tau}{\sqrt{\tau^2 + 1}} - \frac{\tau_0}{\sqrt{\tau_0^2 + 1}} \geq \phi'(\tau) - \phi'(\frac{\tau}{2}) \geq \frac{1}{2}\phi''(\tau)\tau \geq \frac{C}{\tau^2}.
\]
Therefore,
\[
|\Psi(k, t)| \leq C(k^2t^{-1} + t^{-1/2}), \quad \max\{1, 2\tau_0\} \leq k.
\] (7.7)
Moreover, (7.3) implies that
\[
\int_{\tau_0/2}^{2\tau_0} |\Psi(k)\Lambda(k)|\,dk \leq Ct^{-1/2},
\] (7.8)
since $\int_{y/2}^{2y} k^{-1} \, dk$ does not depend on $y > 0$. It is proved similarly that $J(t) - J(t/4) \leq C t^{-1/2}$. Moreover, in the case $4 \leq t \leq 2\tau_0$ it follows from (7.6) that $J(t/4) \leq C t^{-1/2}$, and we obtain (7.1) for $4 \leq t \leq 2\tau_0$. We note that (7.1) holds for all $\tau_0 \in (0, \infty)$ in the case $1 \leq t \leq 4$.

Next, consider the case $1 \leq 2\tau_0 \leq t$. If, in addition, $\tau_0/2 \leq 2\tau_0 \leq t$ we find that

$$J(t) \leq \int_{\tau_0/2}^{2\tau_0} |\Psi(k)\Lambda(k)| \, dk + \int_{2\tau_0}^{t} |\Psi(k)\Lambda(k)| \, dk \leq C t^{-1/2}$$

by (7.8) and (7.7). In the case $1 \leq \tau_0/2 \leq 2\tau_0 \leq t$ we find that

$$J(t) \leq \int_{1}^{\tau_0/2} |\Psi(k)\Lambda(k)| \, dk + \int_{\tau_0/2}^{2\tau_0} |\Psi(k)\Lambda(k)| \, dk + \int_{2\tau_0}^{t} |\Psi(k)\Lambda(k)| \, dk \leq C t^{-1/2}$$

by (7.6), (7.8), and (7.7). Finally, in the case $2\tau_0 \leq 1$ the estimate (7.1) follows from (7.7).

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Bibliography

[1] J. Bergh and J. Lofstrom, *Interpolation spaces. An introduction*, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, Berlin–New York 1976, x+207 pp.

[2] V.S. Buslaev and C. Sulem, “On asymptotic stability of solitary waves for nonlinear Schrödinger equations”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 20:3 (2003), 419–475.

[3] S. Cuccagna, “On dispersion for Klein Gordon equation with periodic potential in 1D”, *Hokkaido Math. J.* 37:4 (2008), 627–645.

[4] P. D’Ancona and L. Fanelli, “$L^p$-boundedness of the wave operator for the one dimensional Schrödinger operator”, *Comm. Math. Phys.* 268:2 (2006), 415–438.

[5] P. Deift and E. Trubowitz, “Inverse scattering on the line”, *Comm. Pure Appl. Math.* 32:2 (1979), 121–251.

[6] I. Egorova, E.A. Kopylova, and G. Teschl, “Dispersion estimates for one-dimensional discrete Schrödinger and wave equations”, *J. Spectr. Theory* 5:4 (2015), 663–696.

[7] M. Goldberg, “Transport in the one-dimensional Schrödinger equation”, *Proc. Amer. Math. Soc.* 135:10 (2007), 3171–3179.

[8] M. Goldberg and W. Schlag, “Dispersive estimates for Schrödinger operators in dimensions one and three”, *Comm. Math. Phys.* 251:1 (2004), 157–178.

[9] И. М. Гусейнов, “О непрерывности коэффициента отражения одномерного уравнения Шредингера”, *Дифференц. уравнения* 21:11 (1985), 1993–1995. [H. M. Huseinov, “Continuity of the coefficient of reflection of a one-dimensional Schrodinger equation”, *Differentsial’nye Uravneniya* 21:11 (1985), 1993–1995.]

[10] M. Keel and T. Tao, “Endpoint Strichartz estimates”, *Amer. J. Math.* 120:5 (1998), 955–980.

[11] A.I. Komech and E.A. Kopylova, “Weighted energy decay for 1D Klein–Gordon equation”, *Comm. Partial Differential Equations* 35:2 (2010), 353–374.
I. E. Egorova, E. A. Kopylova, V. A. Marchenko, and G. Teschl

[12] Е. А. Копылова, “Дисперсионные оценки для уравнений Шредингера и Клейна–Гордона”, УМН 65:1(391) (2010), 97–144; English transl., E. A. Kopylova, “Dispersive estimates for the Schrödinger and Klein–Gordon equations”, Russian Math. Surveys 65:1 (2010), 95–142.

[13] E. Kopylova and A. I. Komech, “On asymptotic stability of kink for relativistic Ginzburg–Landau equation”, Arch. Ration. Mech. Anal. 202:1 (2011), 213–245.

[14] В. А. Марченко, Операторы Штурма–Лиувилля и их приложения, Наукова думка, Киев 1977, 331 с.; English transl., V. A. Marchenko, Sturm–Liouville operators and applications, rev. ed., AMS Chelsea Publishing, Providence, RI 2011, xiv+396 pp.

[15] B. Marshall, W. Strauss, and S. Wainger, “$L^p$–$L^q$ estimates for the Klein–Gordon equation”, J. Math. Pures Appl. (9) 59:4 (1980), 417–440.

[16] M. Meyries and M. Veraar, “Sharp embedding results for spaces of smooth functions with power weights”, Studia Math. 208:3 (2012), 257–293.

[17] H. Mizutani, “Dispersive estimates and asymptotic expansions for Schrödinger equations in dimension one”, J. Math. Soc. Japan 63:1 (2011), 239–261.

[18] M. Murata, “Asymptotic expansions in time for solutions of Schrödinger-type equations”, J. Funct. Anal. 49:1 (1982), 10–56.

[19] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and Ch. W. Clark (eds.), NIST handbook of mathematical functions, Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge Univ. Press, Cambridge 2010, xvi+951 pp.

[20] K. M. Rogers, “Sharp van der Corput estimates and minimal divided differences”, Proc. Amer. Math. Soc. 133:12 (2005), 3543–3550.

[21] W. Schlag, “Dispersive estimates for Schrödinger operators: a survey”, Mathematical aspects of nonlinear dispersive equations, Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ 2007, pp. 255–285.

[22] A. Soffer and M. I. Weinstein, “Resonances, radiation damping and instability in Hamiltonian nonlinear wave equations”, Invent. Math. 136:1 (1999), 9–74.

[23] G. Teschl, Mathematical methods in quantum mechanics. With applications to Schrödinger operators, 2nd ed., Grad. Stud. Math., vol. 157, Amer. Math. Soc., Providence, RI 2014, xiv+358 pp.

[24] H. Triebel, Interpolation theory, function spaces, differential operators, North-Holland Math. Library, vol. 18, North-Holland Publishing Co., Amsterdam–New York 1978, 528 pp.; VEB Deutscher Verlag der Wissenschaften, Berlin 1978, 528 pp.

[25] R. Weder, “$L^p$–$L^p$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential”, J. Funct. Anal. 170:1 (2000), 37–68.

[26] R. Weder, “Inverse scattering on the line for the nonlinear Klein–Gordon equation with a potential”, J. Math. Anal. Appl. 252:1 (2000), 102–123.

[27] R. Weder, “The $L^p$–$L^p$ estimate for the Schrödinger equation on the half-line”, J. Math. Anal. Appl. 281:1 (2003), 233–243.

[28] N. Wiener, “Tauberian theorems”, Ann. of Math. (2) 33:1 (1932), 1–100.
[29] K. Yajima, “The $W^{k,p}$-continuity of wave operators for Schrödinger operators”, *J. Math. Soc. Japan* **47**:3 (1995), 551–581.

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