An algebraic study of the first order version of some implicational fragments of the three-valued Lukasiewicz logic

Aldo Figallo-Orellano and Juan Sebastián Slagter

Abstract

MV-algebras are an algebraic semantics for Lukasiewicz logic and MV-algebras generated by a finite chain are Heyting algebras where the Gödel implication can be written in terms of De Morgan and Moisil’s modal operators. In our work, a fragment of trivalent Lukasiewicz logic is studied. The propositional and first-order logic is presented. The maximal consistent theories are studied as Monteiro’s maximal deductive systems of the Lindenbaum-Tarski algebra, in both cases. Consequently, the adequacy theorem with respect to the suitable algebraic structures is proven.

1 Introduction and Preliminaries

In 1923, David Hilbert proposed studying the implicative fragment of classical propositional calculus. This fragment is well-known as positive propositional calculus and its study started in 1934 by D. Hilbert and P. Bernays. The following axiom schemas define this calculus

\[(E1) \alpha \rightarrow (\beta \rightarrow \alpha), \]
\[(E2) (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)). \]

and the inference rule modus ponens is

\[(MP) \quad \frac{\alpha, \alpha \rightarrow \beta}{\beta}. \]

In 1950, L. Henkin introduced the implicative models as algebraic models of the positive implicative calculus. Later, A. Monteiro renamed them as Hilbert algebras and his Ph. D. student A. Diego ([9]) made one of the most important contributions to these algebraic structures.

On the other hand, I. Thomas in [21] considered the \(n\)-valued positive implicative calculus, with signature \(\{\rightarrow, 1\}\), as a calculus that has a characteristic matrix \(\langle A, \{1\} \rangle\) where \(\{1\}\) is the set of designated elements and the algebra \(A = (C_n, \rightarrow, 1)\) is defined as follows

\[C_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\} \]

and
\[ x \to y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } y < x \end{cases}, \]

This author proved that for this calculus, we have to add the following axiom to the positive implicative calculus:

\[(E3) \ T_n(\alpha_0, \ldots, \alpha_{n-1}) = \beta_{n-2} \to (\beta_{n-3} \to (\cdots \to (\beta_0 \to \alpha_0) \cdots)),\]

where

\[ \beta_i = (\alpha_i \to \alpha_{i+1}) \to \alpha_0 \quad \text{for all } i, 0 \leq i \leq n - 2. \]

The algebraic counterpart of \( n \)-valued positive implicative calculus was studied by Luiz Monteiro in [18] where the axiom \((E3)\) is translated by the equation \( T_n = 1 \) to Hilbert algebras. In particular, in the \( n = 3 \) case, the variety is generated by an algebra that has this set \( C_3 = \{0, \frac{1}{2}, 1\} \) as support and an implication \( \to \) defined by the following table:

| \( \to \) | 0 | 1 | 1 |
|---|---|---|---|
| 0 | 1 | 1 | 1 |
| \( \frac{1}{2} \) | 0 | 1 | 1 |
| 1 | 0 | \( \frac{1}{2} \) | 1 |

Table 1

It is clear that 3-valued Hilbert algebras are Hilbert algebras that verify the following identity:

\[(IT3) \ (\Box x \to y) \to ((\Box y \to z) \to \Box ((\Box z \to x) \to z)) = 1.\]

It is important to note that the implication defined in table 1 characterizes the implicative fragment of 3-valued Gödel logic.

On the other hand, infinite-valued Lukasiewicz logic \( L \), introduced for philosophical reasons by Jan Lukasiewicz, is among the most important and widely studied of all non-classical logics. Later, MV-algebras were introduced by C. Chang in order to prove completeness with respect to the calculus \( L \). These algebras are term equivalent to Wajsberg algebras. Besides, Komori introduced \( CN \)-algebra as algebraic models to \( L \) in terms of implication and negation.

Recall that an algebra \( A = (A, \to, \neg, 1) \) is said to be a Wajsberg algebra if it satisfies the following identities (see [13, 6]):

\[(w1) \ 1 \to x = x,\]

\[(w2) \ (x \to y) \to ((y \to z) \to (x \to z)) = 1,\]

\[(w3) \ (x \to y) \to y = (y \to x) \to x,\]

\[(w4) \ (\neg y \to \neg x) \to (x \to y) = 1.\]
We can define other operations in a Wajsberg algebra. Indeed, \( 1 = \sim 0, x \oplus y = \sim x \Rightarrow y, \)
\( x \odot y = \sim (x \oplus \sim y), x \vee y = \sim (x \oplus y) \odot y = (x \Rightarrow y) \Rightarrow y, x \wedge y = \sim (x \vee \sim y), \) where \( \wedge \) and \( \vee \) are lattice-operations. If we consider the operations \( \oplus \) and \( \odot \) as primitive operations, then we have \((A, \oplus, \odot, \sim, 0)\) is an MV-algebra in Chang’s formulation. Conversely, any MV-algebra in Chang’s formulation produces one Wajsberg algebra by the appropriate definitions of \( \sim \) and \( \Rightarrow \).

\( (I3) \), where \( x \Rightarrow y = \sim x \oplus y \). Besides, it is well-known that the category of MV-algebra is equivalent to the category of \( l \)-groups with strong unit.

Let us remark that the variety of MV-algebras generated by an MV-chain of length \( n < \omega \) is often denoted by \( \text{MV}_n \)-algebras. This notion was introduced by Grigolia and can be axiomatized by adding two new axioms \((w5)\) and \((w6)\) to the axioms of MV-algebras:

\[(w5)\] \(x^{n-1} = x^n,\)

\[(w6)\] \(n(x^j \odot (\sim x \circ \sim x^{j-1})) = 1\) for \(1 < j < n\) and \(j\) does not divide \(n\).

Besides, we denote by \( C_n \) the \( \text{MV}_n \)-algebra whose universe is \( \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\} \) endowed with the operations \( x \Rightarrow y := \min\{1,1-x+y\}, \sim x := 1-x \). Also, it is well known that if \((A, \oplus, \sim, 1)\) is an MV\(_n\)-algebra then \( L_n(A) = (A, \wedge, \vee, \sim, \sigma_0, \ldots, \sigma_{n-1}, 0, 1)\) is an \( n \)-valued Lukasiewicz-Moisil algebra (see [2]), where the operators \( \sigma_i : A \rightarrow A \) are lattice-homomorphisms, for \( 1 \leq i \leq n \), are defined in terms of the MV-operations and are called Moisil operators. On the other hand, it is well-known that the Gödel implication can be written in terms of the Moisil and De Morgan’s operations as follows

\[x \Rightarrow y = x \vee \sim \sigma_{n-1}y \vee (\sigma_{n-1}y \wedge \sigma_{n-1}x \wedge \sim \sigma_n y) \vee \cdots \vee (\sigma_1 y \wedge \sigma_1 x \wedge \sim \sigma_1 y) \vee (\sigma_0 x \wedge \sigma_0 y)(\ast).\]

This implication was discovered by Cignoli in his Ph. D. thesis, as we can see in the following papers [3] [4] [5]. It is worth mentioning that Cignoli in [5] studied the first-order \( n \)-valued Lukasiewicz logic. In that work, it was presented the \( n \)-valued Lukasiewicz logic as extension of intuitionistic calculus, this fact was commented in the abstract of [4]. Indeed, Cignoli was based on the fact that \( \text{MV}_n \)-algebras can be defined in terms of symmetric Heyting algebras with Moisil operations adding a special set of operations (see [5] Definition 2.1); that he called them \( n \)-valued proper Lukasiewicz algebras; that is to say, obviously, these algebras are term equivalent to \( \text{MV}_n \)-algebras. The latter facts allowed him to present soundness and completeness Theorems for first-order \( n \)-valued Lukasiewicz logic ([5]) by means of the Rasiowa’s technique for the standard models. Much more recently, Iorgulescu studied the connection between \( \text{MV}_n \)-algebras and \( n \)-valued Lukasiewicz-Moisil algebra in [14].

On the other hand, in 1941, G. Moisil introduced 3-valued Lukasiewicz algebras (or 3-valued Lukasiewicz-Moisil algebras) as algebraic models of 3-valued logic proposed by Łukasiewicz ([20]). It is well-known, and part of folklore, that the class of 3-valued Lukasiewicz algebras is term equivalent to the one of 3-valued MV-algebras (see, for instance, [2]). Recall that an algebra \((A, \wedge, \vee, \sim, \nabla, 0, 1)\) is a 3-valued Lukasiewicz algebras if the following conditions hold: \((L0)\) \( x \vee 1 = 1, (L1) x \wedge (x \vee y) = x, (L2) x \wedge (y \vee z) = (z \wedge x) \vee (y \wedge x), (L3) \sim \sim x = x, (L4) \sim (x \wedge y) = \sim x \vee \sim y, (L5) x \vee \nabla x = 1, (L6) \sim x \wedge x = \sim x \vee \nabla x, \) and \((L7) \nabla (x \wedge y) = \nabla x \wedge \nabla y.\)
Besides, it is well-known that the algebra \((A, \wedge, \vee, \sim, 0, 1)\) is a De Morgan algebra if \((L0)\) to \((L4)\) hold \([2, \text{Definition 2.6}]\). On the other hand, the characteristic matrix of logic from trivalent Lukasiewicz algebras has the operator \(\land, \lor, \sim, \nabla\) (possibility operator) and \(\Delta\) (necessity operator) over the chain \(C_3 = \{0, \frac{1}{2}, 1\}\), and they are defined by the next table:

| \(x\) | \(\sim x\) | \(\nabla x\) | \(\Delta x\) |
|-------|------------|------------|------------|
| 0     | 1          | 0          | 0          |
| \(\frac{1}{2}\) | \(\frac{1}{2}\) | 1          | 0          |
| 1     | 0          | 1          | 1          |

Table 2

In addition, the implication \(\Rightarrow\) defined above can be obtained from the operator \(\land, \lor, \sim, \nabla\) and \(\Delta\) by the following formula:

\[
x \Rightarrow y = \Delta \sim x \lor y \lor (\nabla \sim x \land \nabla y).
\]

Besides, it is easy to check that \(\nabla x = (x \rightarrow \Delta x) \rightarrow \Delta x\). From latter and the fact that the implication can be written in terms of operations from 3-valued Lukasiewicz algebras, the authors of \([7, 8]\) were motivated to study the interesting implicational fragments of the 3-valued Lukasiewicz logic. In general, for some technical aspects of Lukasiewicz-Moisil algebras, the reader can consult \([2]\).

The rest of the paper is organized as follows. In the next section, we introduce the class of modal 3-valued Hilbert algebras with infimum, where the modal operator is the same considered by Moisil. Besides, we prove that variety of these algebras is semisimple and we determine the generating algebras. Later on, using our algebraic results, we present Hilbert Calculus has algebraic counterparts to these algebras introduced in this section. In section 3, we introduce and study the class of modal 3-valued Hilbert algebra with supremum and also, as application of our algebraic work, we present Hilbert calculus for the fragment with disjunction soundness and completeness, in a strong version, with respect to this class of algebras. Finally, in Section 4, we study the first order logic for the fragment with disjunction by means of an adaptation of the Rasiowa’s technique \([23]\) using our algebraic work for the propositional case.

For the sake of motivating these notes and roughly speaking, this work is developed using Henkin’s notion of maximal consistent theory as Monteiro’s maximal deductive system of Lindenbaum-Tarski algebra. Monteiro named it \(Systèmes deductifs liés à "a"\), where \(a\) is an element of some given algebra such that the congruences are determined by deductive systems \([16, \text{pag. 19}]\). We use this notion, applying Monteiro’s technique, in the Section \([23]\) in order to prove this variety is semisimple and in the proof of the completeness. It is important to note that the relation of Henkin’s maximal consistent theories and Monteiro’s maximal deductive systems is only verified in some semisimple varieties of algebras studied in Monteiro’s school. In addition, for instance, Nelson algebras, Heyting algebras, Hilbert algebras, residuated lattices, the implicational algebraic systems so-called standard models \([23]\) and others classes of algebras from non-semisimple varieties, this relation is not verified. This fact was one...
of our reasons for studying the algebraic systems introduced in this note. In the Rasiowa’s book, one can see the algebraic study of first-order of the logics of standard models and in order to present the algebraic models as models for these first-order logics. This work needs to prove the existence of the complete structures such as the Dedekind-Macneille completion for Boolean algebras or Heyting algebras in order to interpret the quantified formulas. Using this method Cignoli needed to find the completion for $n$-valued Lukasiewic algebras. In contrast, our technique simplifies the proof of the completeness theorem using the fact that the simple algebras are complete lattice. Moreover, we can apply the technique to the Cignoli’s works, what is more, it is possible to apply to several semisimple varieties of algebras studied in Monteiro’s school. By the way, these observations will be part of the future works.

2 Trivalent modal Hilbert algebras with infimum

In this section, we introduce trivalent modal Hilbert algebras with infimum, for short $iH_3^\triangle$-algebra. Using Monteiro’s characterization of maximal congruences (see Definition 10), we prove that the variety of $iH_3^\triangle$-algebra is semisimple. Then, it will be presented a propositional calculi that has the class of $iH_3^\triangle$-algebra as algebraic counterparts.

For the sake of brevity, in what follows, we only introduce those essential notions of Hilbert algebras that we need, thought not in full detail. Anyway, for more information about these algebras the reader can consult the bibliography.

Now, recall that a Hilbert algebra is an algebra $(A, \to, 1)$ such that for all $x, y, z \in A$ verifies:

(H1) $x \to (y \to x) = 1$,
(H2) $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$,
(H3) if $x \to y = 1$, $y \to x = 1$, then $x = y$.

The following lemma is well-known

**Lemma 1** Let $A$ be a Hilbert algebra. The following properties are satisfied for every $x, y, z \in A$:

(H4) If $x = 1$ and $x \to y = 1$, then $y = 1$,
(H5) the relation $\leq$ defined by $x \leq y$ iff $x \to y = 1$, it is an order on $A$ and $1$ is the last element,
(H6) $x \to x = 1$, (H7) $x \leq y \to x$, (H8) $x \to (y \to z) \leq (x \to y) \to (x \to z)$, (H9) $x \to 1 = 1$,
(H10) $x \leq y$ implies $z \to x \leq z \to y$, (H11) $x \leq y \to z$ implies $y \leq x \to z$,
(H12) $x \to ((x \to y) \to y) = 1$, (H13) $1 \to x = x$,
(H14) $x \leq y$ implies $y \to z \leq x \to z$, (H15) $x \to (y \to z) = y \to (x \to z)$,
(H16) $x \to (x \to y) = x \to y$, (H17) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$,
(H18) $x \to (y \to z) = (x \to y) \to (x \to z)$, (H19) $(x \to y) \to y \to x \to y$. 

5
The proof of last lemma can be found in [13]. Now, recall that ([7])

**Definition 2** An algebra \((A,\rightarrow,\triangle,1)\) is said to be a 3-valued modal Hilbert algebra if its reduct \((A,\rightarrow,1)\) is a 3-valued Hilbert algebra and \(\triangle\) verifies the following identities:

\[(M1) \triangle x \rightarrow x = 1,\]
\[(M2) ((y \rightarrow \triangle y) \rightarrow (x \rightarrow \triangle \triangle x)) \rightarrow \triangle(x \rightarrow y) = \triangle x \rightarrow \triangle \triangle y, \text{ and}\]
\[(M3) (\triangle x \rightarrow \triangle y) \rightarrow \triangle x = \triangle x.\]

Besides, we define a new connective by \(\nabla x = (x \rightarrow \triangle x) \rightarrow \triangle x.\)

On the other hand, in [10], the authors introduced and studied the class of Hilbert algebras such that each pair of elements has infimum. Then,

**Definition 3** An algebra \((A,\rightarrow,\land,1)\) is said to be an \(iH_3\)-algebra if the following conditions hold:

1. the reduct \((A,\rightarrow,1)\) is a Hilbert algebra such that the axiom (IT3) is satisfied.
2. the following identities hold: (\(iH_1\)) \(x \land (y \land z) = (x \land y) \land z\), (\(iH_2\)) \(x \land x = x\), (\(iH_3\)) \(x \land (x \rightarrow y) = x \land y\), and (\(iH_4\)) \((x \rightarrow (y \land z)) \rightarrow ((x \rightarrow z) \land (x \rightarrow y)) = 1.\)

Let us observe that all \(iH_3\)-algebra \(A\) and every \(x, y \in A\), we can define the supremum of \(\{x, y\}\) in the following way:

\[x \lor y \overset{\text{def}}{=} ((x \rightarrow y) \rightarrow y) \land ((y \rightarrow x) \rightarrow x).\]

Indeed, let \(a, b \in A\) and put \(c = ((a \rightarrow b) \rightarrow b) \land ((b \rightarrow a) \rightarrow a).\) Since \(x \leq (x \rightarrow y) \rightarrow y\) and \(x \leq (y \rightarrow x) \rightarrow x\) hold and there exists the infimum \(((x \rightarrow y) \rightarrow y) \land ((y \rightarrow x) \rightarrow x)\), then \(c\) is upper bound of the set \(\{a, b\}\). Now, let us suppose that \(d\) is another upper bound of \(\{a, b\}\) such that \(c \not\leq d.\) Thus, there exists an irreducible deductive system \(P\) such that \(c \in P\) and \(d \not\in P\) [?, Corolario 1]. Besides, since \(a, b \leq d\) then \(a, b \not\in P\). On the other hand, as \(A\) is a trivalent Hilbert algebra and according to [15, Théorème 4.1], we have \(a \rightarrow b \in P\) or \(b \rightarrow a \in P\). Now, if we suppose that \(a \rightarrow b \in P\) and since \(c \leq (b \rightarrow a) \rightarrow a\), then we can infer that \(a \in P\), which is a contradiction. If we consider the case \(b \rightarrow a \in P\), we obtain again a contradiction. Thus, \(c\) is the supremum of \(\{a, b\}\). Therefore, all \(iH_3\)-algebra is a relatively pseudocomplemented lattice (see [23]), being as \(x \land z \leq y\) iff \(x \leq z \rightarrow y\). From the latter, we have that each \(iH_3\)-algebra is a distributive lattice.

**Definition 4** An algebra \((A,\rightarrow,\land,\triangle,1)\) is called a trivalent modal Hilbert algebra with infimum (for short, \(iH_3^\triangle\)-algebra) if the reduct \((A,\rightarrow,\land,1)\) is an \(iH_3\)-algebra and the reduct \((A,\rightarrow,\triangle,1)\) is a \(\triangle H_3\)-algebra.

We note with \(iH_3^\triangle\) the variety of \(iH_3^\triangle\)-algebras.

**Lemma 5** Let \(A\) be a \(iH_3^\triangle\)-algebra. The following properties are satisfied for every \(x, y, z \in A:\)
1. \( x \leq y \iff x \to y = 1 \iff x \land y = x \), 2. \( x \to (y \to z) = (x \land y) \to z \), 3. \( x \to (x \land y) = x \to y \),
4. \( (x \land y) \to (x \to y) = 1 \), 5. \( (x \to y) \to ((z \land x) \to (z \land y)) = 1 \), 6. \( (x \land y) \to x = 1 \),
7. \( (x \land y) \to y = 1 \), 8. \( 1 \land x = x \), 9. \( x \to (y \to (x \land y)) = 1 \), 10. \( \triangle 1 = 1 \),
11. \( \triangle (x \to y) \to (\triangle x \to \triangle y) = 1 \), 12. \( \nabla (x \land y) = \nabla x \land \nabla y \),
13. \( \triangle (x \land y) = \triangle x \land \triangle y \), 14. \( (\nabla x \to x) \land \nabla x = x \), 15. \( x \to (x \land y) = x \to y \).

**Proof.** Taking into account the fact all trivalent Hilbert algebras are distributive lattices and the Representation Theorem 2.6 from [18], we can prove the condition (1.) to (9.). The rest of the proof follows the very definitions. \(\square\)

**Definition 6** For a given \(iH_3^\Delta\)-algebra \(A\) and \(D \subseteq A\). Then, \(D\) is said to be a deductive system if (D1) \(1 \in D\), and (D2) if \(x, x \to y \in D\) imply \(y \in D\). Besides, we say that \(D\) is a modal if:

(D3) \(x \in D\) implies \(\triangle x \in D\).

Given a \(iH_3^\Delta\)-algebra \(A\) and \(\{H_i\}_{i \in I}\) a family of modal deductive systems of \(A\), then it is easy to see that \(\bigcap_{i \in I} H_i\) is a modal deductive system. Thus, we can consider the notion of modal deductive system generated by \(H\), and we denote \([H]_m\), as an intersection of all modal deductive system \(D\) such that \(D \subseteq H\). It is well-known that \([H]_m = \{x \in A : \text{exist } h_1, \ldots, h_k \in H : h_1 \to (h_2 \to \cdots \to (h_k \to x) \cdots) = 1\}\) where \(k\) is a finite integer. Now, we will introduce the following notation:

\[
(x_1, \ldots, x_{n-1}; x_n) = \begin{cases} 
  x_n & \text{if } n = 1 \\
  x_1 \to (x_2, \ldots, x_{n-1}; x_n) & \text{if } n > 1
\end{cases}
\]

Hence, we can write:

\([H] = \{x \in A : \text{there exist } h_1, \ldots, h_k \in D_1 : (h_1, \ldots, h_k; x) = 1\}\).

Then, we have the following result

**Proposition 7** Let \(A\) be a \(iH_3^\Delta\)-algebra-algebra, suppose that \(H \subseteq A\) and \(a \in A\). Then the following properties hold:

(i) \([H]_m = \{x \in A : \text{there exist } h_1, \ldots, h_k \in H : (\triangle h_1, \ldots, \triangle h_k; x) = 1\}\),
(ii) \([a]_m = [\triangle a], \text{ where } [b] \text{ is the set } \{|b|\}\),
(iii) \([H \cup \{a\}]_m = \{x \in A : \triangle a \to x \in [H]_m\}\).
Proof. It is a routine. □

Besides, we denote by \( D_m(\mathcal{A}) \) the set of modal deductive systems of \( iH^3_\triangle \)-algebra \( \mathcal{A} \), and by \( \text{Con}_{iH^3_\triangle}(\mathcal{A}) \) the set of congruence relations of a given \( iH^3_\triangle \)-algebra \( \mathcal{A} \).

Lemma 8 For all \( \mathcal{A} \in iH^\triangle \), we have that the poset \( D_m(\mathcal{A}) \) is lattice-isomorphic to \( \text{Con}_{iH^3_\triangle}(\mathcal{A}) \).

Proof. It is well-known that the set of congruences of Hilbert algebra \( \mathcal{A} \) is lattice-isomorphic to the set of all deductive systems. This bijection is given for each deductive system \( D \), we have the relation \( R(D) = \{(x, y) : x \rightarrow y, y \rightarrow x \in D\} \) is a congruence of \( \mathcal{A} \) such that the class of 1 verifies \( |1| R(D) = D \). Besides, for each congruence \( \theta \) of \( \mathcal{A} \) the class of \( |1| \theta \) is a deductive system and \( R(|1| \theta) = \theta \). From the latter and Lemma 5 (10.) and (11.), we can infer that every congruence \( \theta \) for a given \( \mathcal{A} \) respect \( \triangle \) and \( |1| \theta \) is a modal deductive system. □

2.1 Weak deductive systems

For each \( iH^\triangle \)-algebra \( \mathcal{A} \), we can define a new binary operation \( \rightarrow \) named weak implication such that:

\[
(x \rightarrow y) = \triangle x \rightarrow y.
\]

Lemma 9 Let \( \mathcal{A} \in iH^\triangle \), for any \( x, y, z \in \mathcal{A} \) the following properties hold:

\begin{enumerate}
\item[(wi1)] \( 1 \rightarrow x = x \),
\item[(wi2)] \( x \rightarrow x = 1 \),
\item[(wi3)] \( x \rightarrow \triangle x = 1 \),
\item[(wi4)] \( x \rightarrow (y \rightarrow z) = (x \rightarrow y) \rightarrow (x \rightarrow z) \),
\item[(wi5)] \( x \rightarrow (y \rightarrow x) = 1 \),
\item[(wi6)] \( (x \rightarrow y) \rightarrow x \rightarrow 1 \).
\end{enumerate}

Let \( \mathcal{A} \in iH^\triangle \) and suppose a subset \( D \subseteq \mathcal{A} \), we say that \( D \) is a weak deductive system (w.d.s.) if \( 1 \in D \), and \( x, x \rightarrow y \in D \) imply \( y \in D \). It is not hard to see that the set of modal deductive systems is equal to the set of weak deductive systems. We denote by \( D_w(\mathcal{A}) \) the set of weak deductive systems of a Hilbert algebra.

Now, for a given \( iH^\triangle \)-algebra \( \mathcal{A} \) and a (weak) deductive system \( D \) of \( \mathcal{A} \) is said to be a maximal if for every (weak) deductive system \( M \) such that \( D \subseteq M \) implies \( M = \mathcal{A} \) or \( M = D \). Besides, let us consider the set of all maximal w.d.s. \( E_w(\mathcal{A}) \). A. Monteiro gave the following definition in order to characterize maximal deductive systems:

Definition 10 (A. Monteiro) Let \( \mathcal{A} \) be a \( iH^\triangle \)-algebra, \( D \in D_w(\mathcal{A}) \) and \( p \in \mathcal{A} \). We say that \( D \) is a weak deductive system tied to \( p \) if \( p \notin D \) and for any \( D' \in D(\mathcal{A}) \) such that \( D \subseteq D' \), then \( p \in D' \).
The importance for introducing the notion of weak deductive systems is to prove that every maximal weak deductive system is a weak deductive system tied to some element of a given \( iH^\triangle_3 \)-algebra, \( A \). Conversely, and using (wi6), we can prove every w.d.s is a maximal weak deductive systems. Moreover, from (wi4), (wi5) and (wi1) and using A. Monteiro’s techniques, we also can prove that \( \{1\} = \bigcap_{M \in E_w(A)} M \) and so, we have that the following lemma holds.

First, in what follows, it will be considered the quotient algebra \( A/M \) defined by \( a \equiv_M b \) iff \( a \rightarrow b, b \rightarrow a \in M \), see Lemma 8 and the canonical projection \( q_M : A \rightarrow A/M \) defined by 

\[
q_M(x) = |x|_M
\]

where \( |x|_M \) denotes the equivalence class of \( x \) generated by \( M \).

**Lemma 11** Let \( A \) be a \( iH^\triangle_3 \)-algebra then map \( \Phi : A \rightarrow \prod_{M \in E_w(A)} A/M \) such that \( \Phi(x)(M) = q_M(x) \) is a one-to-one homomorphism; that is to say, the variety of \( iH^\triangle_3 \)-algebras is semisimple.

**Proof.** It is routine. \( \square \)

The construction of the following homomorphism is fundamental to obtaining the generating algebras of the variety of \( iH^\triangle_3 \)-algebra. First, we denote by \( C_{\rightarrow, \wedge, \triangle} \) the \( iH^\triangle_3 \)-algebra with support is a chain \( C_3 \) \( (0 < \frac{1}{2} < 1) \) and \( \wedge \) is a lattice operation, and \( \rightarrow, \triangle \) are defined in the Table 1 and 2 form Section 1. In addition, it is easy to see that the algebra \( C_{\rightarrow, \wedge, \triangle} \) has a unique subalgebra \( C_{\rightarrow, \wedge} = \langle \{0, 1\}, \rightarrow, \wedge, \triangle \rangle \).

**Theorem 12** Let \( M \) be a non-trivial maximal modal deductive system of \( iH^\triangle_3 \)-algebra \( A \). Let us consider the sets \( M_0 = \{ x \in A : \neg x \notin M \} \) and \( M_{1/2} = \{ x \in A : x \notin M, \neg x \in M \} \), and the map \( h : A \rightarrow C_3 \) defined by 

\[
h(x) = \begin{cases} 
0 & \text{if } x \in M_0 \\
1/2 & \text{if } x \in M_{1/2} \\
1 & \text{if } x \in M.
\end{cases}
\]

Then, \( h \) is a \( iH^\triangle_3 \)-homomorphism such that \( h^{-1} \{1\} = M \).

**Proof.** We shall prove only that \( h(x \wedge y) = h(x) \wedge h(y) \), for the rest of the proof can be done in a similar manner. We will show:

1) if \( x \in M_0 \) and \( y \in A \), then \( x \wedge y \in M_0 \),
2) if \( x \in M_{1/2} \) and \( y \in A \), then \( x \wedge y \in M_{1/2} \),
3) if \( x, y \in M \), then \( x \wedge y \in M \),

Indeed,

1) Let \( x \in M_0, y \in A \), then \( x \notin M, \neg x \notin M \). From Lemma 5 (12.), and the fact \( x \notin M \), we have that \( x \wedge y \notin M \). Since \( \neg (\neg x \wedge \neg y) \rightarrow \neg x \in M \) but \( \neg x \notin M, \neg x \wedge \neg y \notin M, \) we infer that, by Lemma 5 (12.), \( \neg (x \wedge y) \notin M \) and therefore, \( x \wedge y \in M_0 \).
2) Assume that $x \in M_{1/2}$ and $y \notin M_0$. Thus, $x \notin M$, and $\nabla x, \nabla y \in M$ and then, $x \land y \notin M$. From the latter and Lemma 5 (9.), we can write $\nabla x \to (\nabla y \to (\nabla x \land \nabla y)) \in M$. Since $\nabla x, \nabla y \in M$ we have that $\nabla (x \land y) \in M$. So, $x \land y \in M_{1/2}$.

3) It follows immediately from Lemma 5 (9.), which completes the proof.

□

It is worth mentioning that the last theorem is an important tool for the algebraic study of the class of these algebras. Moreover, this also is an important tool to prove the completeness theorem for the associated logic. It is not possible to have this homomorphism a general context such as Universal Algebra, we have to find it in order to show the generating algebras and to give the canonical model. The definition of this homomorphism is not the same for 3-valued Lukasiewicz algebras or for MV$_3$-algebras. By the way, according to Lemma 11 and Theorem 23 and an adaptation of the first isomorphism theorem of Universal Algebra, we have proved the following theorem and corollary.

**Theorem 13** The variety $i_{H^3_3}$ is semisimple. Besides, the algebras $C_{3}^\to,^\land,^\Delta,^1 = \langle\{0, \frac{1}{2}, 1\}, \to, \land, \Delta, 1 \rangle$ and $C_{2}^\to,^\land,^\Delta,^1 = \langle\{0, 1\}, \to, \land, \Delta, 1 \rangle$ are the unique simple algebras.

### 2.2 Hilbert calculus for $iH^3_3$-algebras

In the sequel, we are going to exhibit a calculus $iH^3_3$. Now, consider the signature $\Sigma = \{\to, \land, \Delta\}$, and let $Var = \{p_1, p_2, \ldots\}$ a numerable set of propositional variables. The propositional language generated by $\Sigma$ and $Var$ will be denoted by $\mathfrak{Fm}_i$. It is clear that $\mathfrak{Fm}_i$ is the absolutely free algebra of formulas generated by $Var$.

**Definition 14** The calculus $iH^3_3$ defined over the language $\mathfrak{Fm}_i$ is the Hilbert calculus obtained from the following axiom schemas and inference rules:

**Axioms**

(A1) $\alpha \to (\beta \to \alpha)$,

(A2) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$,

(A3) $((\alpha \to \beta) \to \gamma) \to (((\gamma \to \alpha) \to \gamma) \to \gamma)$,

(A4) $(\alpha \land \beta) \to \beta$,

(A5) $(\alpha \land \beta) \to \alpha$,

(A6) $\alpha \to (\beta \to (\alpha \land \beta))$,

(A7) $\Delta \alpha \to \alpha$,

(A8) $\Delta (\Delta \alpha \to \beta) \to (\Delta \alpha \to \Delta \beta)$,
(A.9) \(((\beta \to \Delta \beta) \to (\alpha \to \Delta(\alpha \to \beta))) \to \Delta(\alpha \to \beta),\)

(A.10) \(((\Delta \alpha \to \beta) \to \gamma) \to ((\Delta \alpha \to \gamma) \to \gamma)\).

Assume that \(\nabla \alpha := (\alpha \to \Delta \alpha) \to \Delta \alpha\).

Inference rules

\[
\frac{\alpha, \alpha \to \beta}{\beta} \quad \text{(MP)} \quad \frac{\alpha}{\Delta \alpha} \quad \text{(NEC)} \quad \frac{\alpha}{\alpha \to (\alpha \land \beta)} \quad \text{(R\alpha)}
\]

We are going to consider the usual notion of derivation of a formula \(\alpha\) of \(i\mathcal{H}_\Delta^3\), and we shall denote by \(\vdash_i \alpha\). Now, let us consider the relation \(\equiv_i\) defined by \(\alpha \equiv_i \beta\) iff \(\vdash_i \alpha \to \beta\) and \(\vdash_i \beta \to \alpha\). Then, we have the following technical result

Lemma 15 The following properties and rules are verified in \(i\mathcal{H}_\Delta^3\).

\[
(M1) \vdash_i \alpha \to \alpha, \quad (M2) \{\gamma\} \vdash_i \alpha \to \gamma, \quad (M3) \{\alpha \to (\beta \to \gamma)\} \vdash_i (\alpha \to \beta) \to (\alpha \to \gamma),
\]

\[
(M4) \vdash_i (\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma)), \quad (M5) \{\alpha \to (\beta \to \gamma)\} \vdash_i \beta \to (\alpha \to \gamma),
\]

\[
(M6) \{\alpha \to \beta\} \vdash_i (\beta \to \gamma) \to (\alpha \to \gamma), \quad (M7) \{\alpha \to \beta\} \vdash_i (\gamma \to \alpha) \to (\gamma \to \beta),
\]

\[
(M8) \{\alpha \to \beta, \beta \to \gamma\} \vdash_i \alpha \to \gamma, \quad (M9) \{\alpha \equiv_i \beta, \theta \equiv_i \eta\} \vdash_i (\alpha \to \theta) \equiv_i (\beta \to \eta),
\]

\[
(R6) \{\alpha \to \beta\} \vdash_i \{(\gamma \land \alpha) \to (\gamma \land \beta)\}, \quad (R10) \{\alpha \to \beta, \theta \to \eta\} \vdash_i \{(\alpha \land \theta) \to (\beta \land \eta)\},
\]

\[
(M10) \{\alpha \equiv_i \beta, \theta \equiv_i \eta\} \vdash_i (\alpha \land \theta) \equiv_i (\beta \land \eta), \quad (M11) \vdash_i (\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta),
\]

\[
(M12) \vdash_i ((\alpha \to \beta) \to (\beta \to \gamma)) \to (\alpha \to \gamma),
\]

\[
(M13) \{((\alpha \to \beta) \to \beta) \to \beta \equiv_i \alpha \to \beta, \quad (M14) \vdash_i ((\alpha \to (\beta \land \gamma)) \equiv_i \beta,
\]

\[
(M15) \vdash_i (\alpha \land (\alpha \to \beta)) \equiv_i \alpha \land \beta, \quad (R11) \{\alpha \to \beta, \alpha \to \gamma\} \vdash_i \{\alpha \to (\beta \land \gamma)\},
\]

\[
(M16) \vdash_i (\alpha \to (\beta \land \gamma)) \to (\alpha \to \beta) \land (\alpha \to \gamma), \quad (M17) \vdash_i \Delta(\alpha \to \beta) \to (\Delta \alpha \to \Delta \beta),
\]

\[
(M18) \vdash_i \Delta \alpha \equiv_i \Delta \alpha \equiv_i \Delta \alpha, \quad (M19) \vdash_i (((\beta \to \Delta \beta) \to (\alpha \to \Delta \alpha)) \to \Delta(\alpha \to \beta)) \to (\Delta \alpha \to \Delta \beta),
\]

\[
(M20) \vdash_i (((\beta \to \Delta \beta) \to (\alpha \to \Delta \alpha)) \to \Delta(\alpha \to \beta)) \equiv_i \Delta \alpha \equiv_i \Delta \beta,
\]

\[
(M21) \{\alpha \to \beta\} \vdash_i \Delta \alpha \to \Delta \beta, \quad (M22) \{\alpha \equiv_i \beta\} \vdash_i \Delta \alpha \equiv_i \Delta \beta, \quad (M23) \Delta(\alpha \to \Delta \alpha) \equiv_i \alpha \to \Delta \alpha,
\]

\[
(M24) \vdash_i (\Delta \alpha \to \Delta \beta) \to (((\beta \to \Delta \beta) \to (\alpha \to \Delta \alpha)) \to \Delta(\alpha \to \beta)),
\]

\[
(M25) \vdash_i (((\beta \to \Delta \beta) \to (\alpha \to \Delta \alpha)) \to \Delta(\alpha \to \beta) \equiv_i \Delta \alpha \to \Delta \beta,
\]

\[
(M26) \vdash_i ((\beta \to \Delta \beta) \to \Delta \alpha) \to (((\alpha \to \Delta \alpha) \to \Delta \beta) \to \Delta(\alpha \to \beta)), \quad (M27) \vdash_i \alpha \to \nabla \alpha,
\]

\[
(M28) \vdash_i \nabla \alpha \equiv_i (\alpha \to \Delta \alpha) \to \Delta \alpha, \quad (M29) \vdash_i \nabla(\alpha \to \beta) \equiv_i \nabla \alpha \to \nabla \beta, \quad (M30) \vdash_i \nabla \alpha \equiv_i \Delta \alpha,
\]

11
\[(M_{31}) \nabla (\alpha \land \beta) \equiv_i \nabla \alpha \land \nabla \beta.\]

**Proof.** It is routine. \[\square\]

The last result gives the final ingredients to obtain the soundness and completeness Theorem. Is is important to note we only need a few properties of this lemma but in order to prove these properties we need the rest of properties.

**Lemma 16** The relation \(\equiv_i\) is a congruence on \(\mathfrak{Fm}\).

**Proof.** The relation \(\equiv_i\) is reflexive, symmetric and transitive follows immediately from \((M_{1})\) and \((M_{7})\). Let us suppose that \(\alpha \equiv_i \beta\) and \(\gamma \equiv_i \delta\) and taking into account \((M_{6})\) and \((M_{7})\), we can prove that \(\alpha \rightarrow \gamma \equiv_i \beta \rightarrow \delta\). Besides, suppose that \(\alpha \equiv_i \beta\), the from \((M_{17})\), \((MP)\) and \((NEC)\) we have that \(\triangle \alpha \equiv_i \triangle \beta\).

Since the \(\equiv_i\) is a congruence, it allows to define the quotient algebra \(\mathfrak{Fm}/\equiv_i\) that is so-called as Lindenbaum-Tarski algebra.

**Theorem 17** The algebra \(\mathfrak{Fm}/\equiv_i\) is a \(iH^3_\triangle\)-algebra in which the operators are generated as follows: \(|\alpha| \rightarrow |\beta| = |\alpha \rightarrow \beta|\), \(|\alpha| \land |\beta| = |\alpha \land \beta|\) and \(|\beta \rightarrow \beta| = \{\alpha \in \mathfrak{Fm} \vdash_i \alpha\}\) and where \(|\delta|\) denotes the equivalence class of the formula \(\delta\). Besides, \(|\alpha| \leq |\beta|\) if \(\vdash_i \alpha \rightarrow \beta\).

**Proof.** It is easy to see that the relation "\(\leq\)" is an order relation on \(\mathfrak{Fm}/\equiv_i\). Thus, it is clear that \(|\alpha| \leq |\beta \rightarrow \beta|\) for every \(\alpha, \beta\) and so, \(|\beta \rightarrow \beta|\) is the last element of \(\mathfrak{Fm}/\equiv_i\) that we denote with 1 and also, it is not hard to see that \(1 = \{\alpha \in \mathfrak{Fm} \vdash_i \alpha\}\).

On the other hand, taking into account above remarks of \(\leq\), the axioms \((A1)\) and \((A2)\), and \((M_{1})\), \((M_{5})\), \((M_{6})\) and \((M_{7})\), we verify that conditions \((H1)\), \((H2)\) and \((H3)\) are valid on \(\mathfrak{Fm}/\equiv_i\), see definition of Hilbert algebras of section 1. Also, form \((A3)\) we can prove the condition \((IT3)\) holds, see Definition 3. Besides, taking \((A_{4})\) \((A_{5})\) \((A_{7})\), we have the condition 2. of Definition 4 is verified. Now, from \((A_{7})\), \((A_{17})\), \((A_{20})\), \((A_{24})\) we can prove the axiom \((M1)\) \((M2)\) \((M3)\) of Definition 2 are verified for \(\mathfrak{Fm}/\equiv_i\), which completes the proof. \[\square\]

Remember that \(C^\rightarrow_3^\land\) is a \(iH^3_\triangle\)-algebra where the support is a chain \(C_3\) and \(\land\) is a lattice operation, and \(\rightarrow, \triangle\) are defined in the Table 1 and 2 from Section 1. In addition, for us a logical matrix for \(iH^3_\triangle\) is a pair \(\langle C^\rightarrow_3^\land, \{1\}\rangle\) where \(\{1\}\) is the set of designated elements.

A function \(v : \mathfrak{Fm} \rightarrow C^\rightarrow_3^\land\) is a valuation for \(iH^3_\triangle\) if it satisfies \(v(\alpha \# \beta) = v(\alpha) \# v(\beta)\) with \(# \in \{\rightarrow, \triangle\}\), \(v(\triangle \alpha) = \triangle v(\alpha)\). Besides, we say that \(\alpha\) is valid semantically if \(v(\alpha) = 1\) for all valuation \(v\) and, in this case, we denote \(\vdash \alpha\).

**Theorem 18** (Soundness and Completeness Theorem) Let \(\alpha\) be a formula in \(\mathfrak{Fm}\).

Then, \(\vdash \alpha\) if and only if \(\vdash_i \alpha\).

**Proof.** It is easy to see that every axiom in \(iH^3_\triangle\) is valid semantically and the satisfaction is preserved by the inference rules.

Conversely, let us suppose that \(\alpha\) is valid semantically; that is to say, for every valuation \(v : \mathfrak{Fm} \rightarrow C^\rightarrow_3^\land\) we have \(v(\alpha) = 1\). On the other hand, according to Theorem 17, we have \(\mathfrak{Fm}/\equiv_i\) is a \(iH^3_\triangle\)-algebra and now, consider the canonical projection \(q : \mathfrak{Fm} \rightarrow \mathfrak{Fm}/\equiv_i\) defined by \(q(\alpha) = |\alpha|\). From the latter and the hypothesis, \(q(\alpha) = 1\) which imply \(\vdash_i \alpha\). \[\square\]
3 Trivalent modal Hilbert algebras with supremum

In this section, we introduce and study trivalent modal Hilbert algebra with supremum, for short $H_3^{∨,△}$-algebras. We are going to study the class of $H_3^{∨,△}$-algebras in order to present a calculi sound and complete w.r.t. class of $H_3^{∨,△}$-algebras in propositional and first-order version. Our adequacy theorems are based on algebraic previous results and well-known results of Universal Algebra as the first isomorphism Theorem. Now, consider the following

**Definition 19** An algebra $⟨A,→,∨,△,1⟩$ is said to be a trivalent modal Hilbert algebra with supremum (for short, $H_3^{∨,△}$-algebra) if the following properties hold:

1. The reduct $⟨A,∨,1⟩$ is a join-semilattice with greatest element 1, and the conditions (a) $x→(x∨y)=1$ and (b) $(x→y)→((x∨y)→y)=1$ hold. Besides, given $x,y∈A$ such that there exists the infimum of $\{x,y\}$, denote by $x∧y$, then $△(x∧y)=△x∧△y$.

2. The reduct $⟨A,→,△,1⟩$ is a $△H_3$-algebra.

Next we are going to show some properties that will be very useful for the rest of this section.

**Lemma 20** For a given $H_3^{∨,△}$-algebra $A$ and $a,b,c∈A$, then the following holds:

(H$_3^{∨,△}$1) If $a→b=1$, then $a∨b=b$, (H$_3^{∨,△}$2) If $a→c=1$ and $b→c=1$, then $(a∨b)→c=1$,

(H$_3^{∨,△}$3) $a→(a∨b)=1$, (H$_3^{∨,△}$4) $(a→c)→((b→c)→((a∨b)→c))=1$,

(H$_3^{∨,△}$5) $△(a∨b)=△a∨△b$, (H$_3^{∨,△}$6) $∇(a∨b)=∇a∨∇b$.

**Proof.** It is routine. □

For a given $H_3^{∨,△}$-algebra $A$, we are going to consider the notion of modal deductive system, see Definition 6.

**Lemma 21** Given a $H_3^{∨,△}$-algebra $A$, there exists a lattice-isomorphism between the poset of congruences of $A$ and the poset of the modal deductive systems of $A$.

**Proof.** The proof is similar to the Lemma □

The following lemma can be proved in a similar way that was made in the Section 2.1 using the notion of weak deductive system and the notion of deductive system tied to some element.

**Lemma 22** Let $A$ be a $H_3^{∨,△}$-algebra then map $Φ : A → \prod_{M∈E_w(A)} A/M$ such that $Φ(x)(M) = q_M(x)$ is a homomorphism; that is to say, the variety of $H_3^{∨,△}$-algebras is semisimple.
The construction of the following homomorphism is fundamental to obtaining the generating algebras of the variety of $iH_3^{\sqcap,\wedge}$-algebra. Moreover, this homomorphism will play a central role in the adequacy theorems in a propositional and first-order version of logic. First, we denote by $\mathbb{C}_3^{\to, \sqcap, \wedge}$ the $iH_3^{\sqcap,\wedge}$-algebra where the support is a chain $\mathbb{C}_3$ ($0 < \frac{1}{2} < 1$) and $\sqcup$ is a lattice operation, and $\to, \Delta$ are defined in the Table 1 and 2 form Section 1. In addition, it is easy to see that the algebra $\mathbb{C}_3^{\to, \sqcap, \wedge}$ has a unique subalgebra $\mathbb{C}_2^{\to, \sqcap, \wedge} = \{\{0,1\}, \to, \sqcup, \Delta, 1\}$.

**Theorem 23** Let $M$ be a non-trivial maximal modal deductive system of $H_3^{\vee,\sqcap,\wedge}$-algebra $A$. Let us consider the sets $M_0 = \{x \in A : \nabla x \notin M\}$ and $M_{1/2} = \{x \in A : x \notin M, \nabla x \in M\}$, and the map $h : A \rightarrow \mathbb{C}_3$ defined by

$$h(x) = \begin{cases} 
0 & \text{if } x \in M_0 \\
1/2 & \text{if } x \in M_{1/2} \\
1 & \text{if } x \in M.
\end{cases}$$

Then, $h$ is a homomorphism such that $h^{-1}(\{1\}) = M$.

**Proof.** We shall prove only that $h(x \lor y) = h(x) \lor h(y)$, for the rest of the proof can be done in a similar manner.

1. Let $x \in M$ and $y \in A$. Taking into account $(H_3^{\vee,\sqcap,\wedge}3)$, we have that $x \rightarrow (x \lor y) = 1$. Thus, from $D_1$ and $D_2$ then $x \lor y \in M$.

2. Let us consider $x, y \in M_0$ and suppose that $\nabla (x \lor y) \in M$. Then, by $(H_3^{\vee,\sqcap,\wedge}6)$ we have that $\nabla x \lor \nabla y \in M$. Thus, according to $(H_3^{\vee,\sqcap,\wedge}4)$ we infer that $((\nabla y \rightarrow \nabla x) \rightarrow ((\nabla y \rightarrow \nabla x) \rightarrow ((\nabla x \lor \nabla y) \rightarrow \nabla x)) = 1$. So, from $D_1$, $D_2$ and (H6) we can obtain that $((\nabla y \rightarrow \nabla x) \rightarrow ((\nabla x \lor \nabla y) \rightarrow \nabla x)) \in M$. Since $\nabla x \notin M$, we can infer that $\Delta \nabla y \rightarrow \nabla x \in M$ and so, we have $\nabla y \rightarrow \nabla x \in M$. Form the latter and $D_2$, we can write $((\nabla x \lor \nabla y) \rightarrow \nabla x) \in M$. Therefore, $\nabla x \in M$ which is impossible, then $\nabla (x \lor y) \notin M$.

3. If $x \in M_0$ and $y \in M_{1/2}$, since $\nabla y \rightarrow (\nabla x \lor \nabla y) = 1$ and $\nabla y \in M$ we can infer that $\nabla (x \lor y) \in M$. Now, let us suppose that $x \lor y \in M$. From $(H_3^{\vee,\sqcap,\wedge}4)$ we can write $(x \rightarrow y) \rightarrow ((y \rightarrow y) \rightarrow ((x \lor y) \rightarrow y)) = 1$. Thus, $x \rightarrow y \in M$ and then, $y \in M$ which is a contradiction. Therefore, $x \lor y \in M_{1/2}$.

4. If $x \in M_{1/2}$ and $y \in M_0$ we can prove that $x \lor y \in M_{1/2}$ in a similar way to (4).

5. Suppose that $x \in M_{1/2}$ and $y \in M_{1/2}$, then from $(H_3^{\vee,\sqcap,\wedge}6)$ we have that $\nabla (x \lor y) \in M$. On the other hand, let us suppose $x \lor y \in M$, thus by $(H_3^{\vee,\sqcap,\wedge}4)$ we infer that $(x \rightarrow x) \rightarrow ((x \rightarrow y) \rightarrow ((x \lor y) \rightarrow x)) = 1$. Hence, since $x \rightarrow y \in M$ we can write $x \in M$ which is a contradiction. Therefore, $x \lor y \in M_{1/2}$.

According to Lemma 22 and Theorem 23, and well-known facts about universal algebra, we have proved the following theorem and corollary.
Corollary 24 The variety of $H_3^\lor,\land$-algebras is semisimple. Besides, the algebras $C_3^\lor,\land = \langle\{0, \frac{1}{2}, 1\}, \lor, \land, \lor, \land, \lor, \land, \lor, \land\rangle$ and $C_2^\lor,\land = \langle\{0, 1\}, \lor, \land, \lor, \land\rangle$ are the unique simple algebras.

Let us notice that not every $H_3^\lor,\land$-algebra has infimum. To seeing that, it is enough to see some subalgebras of $C_3^\lor,\land \times C_3^\lor,\land$ where $\times$ is the direct product.

3.1 Propositional calculus for $H_3^\lor,\land$-algebras

Let $\mathfrak{F}_{m,s} = \langle F_m, \lor, \land, \lor, \land \rangle$ be the absolutely free algebra over $\Sigma = \{\lor, \land, \lor, \land\}$ generated by a set $\text{Var} = \{p_1, p_2, \ldots\}$ of numerable variables. Also, sometimes we say that $\mathfrak{F}_{m,s}$ is a language over $\text{Var}$ and $\Sigma$. Consider now the following logic:

Definition 25 We denote by $H_3^\lor,\land$ the Hilbert calculus determined by the following axioms and inference rules, where $\alpha, \beta, \gamma, \ldots \in F_m$:

Axiom schemas

$(Ax1)$ $\alpha \to (\beta \to \alpha)$,
$(Ax2)$ $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)),$
$(Ax3)$ $((\alpha \to (\beta \to \gamma)) \to ((\gamma \to \alpha) \to \gamma)),$
$(Ax4)$ $\alpha \to (\alpha \lor \beta),$
$(Ax5)$ $\beta \to (\alpha \lor \beta),$
$(Ax6)$ $(\alpha \to \gamma) \to ((\beta \to \gamma) \to ((\alpha \lor \beta) \to \gamma)),$
$(Ax7)$ $\Delta \alpha \to \alpha,$
$(Ax8)$ $\Delta(\Delta \alpha \to \beta) \to (\Delta \alpha \to \Delta \beta),$
$(Ax9)$ $((\beta \to \Delta \beta) \to (\alpha \to \Delta(\alpha \to \beta))) \to \Delta(\alpha \to \beta),$
$(Ax10)$ $((\Delta \alpha \to \beta) \to \gamma) \to ((\Delta \alpha \to \gamma) \to \gamma).$

Inference rules

$(MP) \frac{\alpha, \alpha \to \beta}{\beta},$ $(NEC) \frac{\alpha}{\Delta \alpha \to \Delta \alpha}.$

Assume that $\nabla \alpha := (\alpha \to \Delta \alpha) \to \Delta \alpha.$

Let $\Gamma \cup \{\alpha\}$ be a set formulas of $H_3^\lor,\land$, we define the derivation of $\alpha$ from $\Gamma$ in usual way and denote by $\Gamma \vdash \lor \alpha$.

Lemma 26 The following rules are derivable in $H_3^\lor,\land$:

$(P_s I) \vdash \lor \{(x \lor y) \to (y \lor x)\}$
\((P_s2)\) \(\{x \to y\} \vdash \{(x \lor z) \to (y \lor z)\}\)

\((P_s3)\) \(\{x \to y, u \to v\} \vdash \{(x \lor u) \to (y \lor v)\}\)

\((R_\lor 3)\)

\[
\frac{\alpha \to \beta}{(\alpha \lor \beta) \to \beta}
\]

**Proof.** It is routine. \(\square\)

Now, we denote by \(\alpha \equiv \lor \beta\) if conditions \(\vdash \lor \alpha \to \beta\) and \(\vdash \lor \beta \to \alpha\) hold. Then,

**Lemma 27** \(\equiv \lor\) is a congruence on \(\mathfrak{Fm}_s\).

**Proof.** We only have to prove that if \(\alpha \equiv \lor \beta\) and \(\gamma \equiv \lor \delta\), then \(\alpha \lor \gamma \equiv \lor \beta \lor \delta\), which follows immediately from \((P_s3)\). \(\square\)

Since the \(\equiv \lor\) is a congruence, it allows to define the quotient algebra \(\mathfrak{Fm}_s/\equiv \lor\) that is so-called the Lindenbaum-Tarski algebra.

**Theorem 28** The Lindenbaum algebra \(\mathfrak{Fm}_s/\equiv \lor\) of \(H_3^\lor,\Delta\) is a \(H_3^\lor,\Delta\)-algebra by defining: \(|\alpha| \to |\beta| = |\alpha \to \beta|, |\alpha| \lor |\beta| = |\alpha \lor \beta|\) and \(1 = |\beta \to \beta| = \{\alpha \in \mathfrak{Fm}_s : \vdash \lor \alpha\}\), where \(|\delta|\) denotes the equivalence class of the formula \(\delta\).

**Proof.** We only have to prove \(\mathfrak{Fm}_s/\equiv \lor\) is a join-semilattice and the axioms (a) and (b) from Definition [19](2). So, the first part follows from (Ax4), (Ax5) and (Ax6), and the second one follows from axioms (Ax4) and \((R_\lor 3)\). \(\square\)

On the other hand, let us remark that for the propositional calculus \(iH_3^\lor\) it is possible to define the conjunction connective by \(\alpha \lor \beta := ((\alpha \to \beta) \to \beta) \land ((\beta \to \alpha) \to \alpha)\) (see [21], pag. 170). Thus, taking into account the Theorem 17, it is easy to see that \(\mathfrak{Fm}_i/\equiv_i\) is a Hilbert algebra with supremum where \(|\alpha| \lor |\beta| = |\alpha \lor \beta|\) for every formula \(\alpha\) and \(\beta\). Therefore, the calculus \(iH_3^\lor\) is a \(\{\to, \land, \lor, \Delta\}\)-fragment of a 3-valued Lukasiewicz logic where \(\to\) is the 3-valued Gödel implication.

Now, we are going to introduce some useful notions in order to prove a strong Completeness Theorem for \(H_3^\lor,\Delta\) w.r.t. the class of \(H_3^\lor,\Delta\)-algebras.

Recall that a logic defined over a language \(S\) is a system \(L = \langle For, \vdash \rangle\) where For is the set of formulas over \(S\) and the relation \(\vdash \subseteq P(For) \times For, P(A)\) is the set of all subsets of \(A\). The logic \(L\) is said to be tarskian if it satisfies the following properties, for every set \(\Gamma \cup \Omega \cup \{\varphi, \beta\}\) of formulas:

1. if \(\alpha \in \Gamma\), then \(\Gamma \vdash \alpha\),
2. if \(\Gamma \vdash \alpha\) and \(\Gamma \subseteq \Omega\), then \(\Omega \vdash \alpha\),
3. if \(\Omega \vdash \alpha\) and \(\Gamma \vdash \beta\) for every \(\beta \in \Omega\), then \(\Gamma \vdash \alpha\).

A logic \(L\) is said to be finitary if it satisfies the following:
(4) if $\Gamma \vdash \alpha$, then there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\Gamma_0 \vdash \alpha$.

**Definition 29** Let $\mathcal{L}$ be a tarskian logic and let $\Gamma \cup \{\varphi\}$ be a set of formulas, we say that $\Gamma$ is a theory. Besides, $\Gamma$ is said to be a consistent theory if there is $\varphi$ such that $\Gamma \not\vdash \varphi$. Besides, we say that $\Gamma$ is a maximal consistent theory if $\Gamma, \psi \vdash \varphi$ for any $\psi \notin \Gamma$ and in this case, we say $\Gamma$ non-trivial maximal respect to $\varphi$.

A set of formulas $\Gamma$ is closed in $\mathcal{L}$ if the following property holds for every formula $\varphi$: $\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$. It is easy to see that any maximal consistent theory is a closed one.

**Lemma 30 (Lindenbaum-Los)** Let $\mathcal{L}$ be a tarskian and finitary logic. Let $\Gamma \cup \{\varphi\}$ be a set of formulas such that $\Gamma \not\vdash \varphi$. Then, there exists a set of formulas $\Omega$ such that $\Gamma \subseteq \Omega$ with $\Omega$ maximal non-trivial with respect to $\varphi$ in $\mathcal{L}$.

**Proof.** It can be found [24 Theorem 2.22].

It is worth mentioning that, by the very definitions, $\mathcal{H}^3_{\varphi, \Delta}$ is a tarskian and finitary logic and then, we have the following

**Theorem 31** Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{m}_s$, with $\Gamma$ non-trivial maximal respect to $\varphi$ in $\mathcal{H}^3_{\varphi, \Delta}$. Let $\Gamma / \equiv_{\varphi} = \{\overline{\alpha} : \alpha \in \Gamma\}$ be a subset of the trivalent modal Hilbert algebra with supremum $\mathfrak{m}_s / \equiv_{\varphi}$, then:

1. If $\alpha \in \Gamma$ and $\overline{\alpha} = \overline{\beta}$ then $\beta \in \Gamma$,

2. $\Gamma / \equiv_{\varphi}$ is a modal deductive system of $\mathfrak{m}_s / \equiv_{\varphi}$. Also, if $\overline{\alpha} \notin \Gamma / \equiv_{\varphi}$ and for any modal deductive system $\overline{\mathcal{D}}$ which contains properly to $\Gamma / \equiv_{\varphi}$, then $\varphi \in \overline{\mathcal{D}}$.

**Proof.** Taking into account $\alpha \in \Gamma$ and $\alpha \equiv_{\varphi} \beta$, we have that $\Gamma \vdash \alpha \rightarrow \beta$ and $\Gamma \vdash \beta \rightarrow \alpha$. Therefore, $\beta \in \Gamma$. Besides, it is not hard to see that $D_1$, $D_2$ and $D_3$ are valid.

On the other hand, let $\overline{\mathcal{D}}$ be mds that contains $\Gamma / \equiv_{\varphi}$ and so, there is $\overline{\alpha} \in \overline{\mathcal{D}}$ such that $\overline{\alpha} \notin \Gamma / \equiv_{\varphi}$. Now, we have that $\gamma \notin \Gamma$ and therefore, $\Gamma \cup \{\gamma\} \vdash \varphi$. From the latter and taking into account $D = \{\alpha : \overline{\alpha} \in \overline{\mathcal{D}}\}$ we can infer that $D \vdash \varphi$. Now, let us suppose that $\alpha_1, ..., \alpha_n$ is a derivation from $D$. We shall prove by induction over the length of the derivation that $\overline{\alpha_n} \in \overline{\mathcal{D}}$.

If $n = 1$ then $\alpha_1$ is an instance of an axiom or otherwise $\alpha_1 \in D$. If $\Gamma \vdash \alpha_1$ is the case, then $\Gamma \vdash \alpha_1$ which is a contradiction. Then, we only have $\alpha_1 \in D$ which implies $\overline{\alpha_1} \in \overline{\mathcal{D}}$.

Suppose that $\overline{\alpha_k} \in \overline{\mathcal{D}}$ if $k$ is less than $n$. Then, we have the following cases:

1. If $\varphi$ be the instance of an axiom, then $\Gamma \vdash \varphi$ which is a contradiction.

2. If $\varphi \in D$, then $\overline{\varphi} \in \overline{\mathcal{D}}$.

3. If there exists $\{j, t_1, ..., t_m\} \subseteq \{1, ..., k-1\}$ such that $\alpha_{t_1}, ..., \alpha_{t_m}$ is a derivation of $\alpha_j \rightarrow \varphi$, then we have $\overline{\alpha_j} \rightarrow \overline{\varphi} \in \overline{\mathcal{D}}$ by induction hypothesis. So, $\overline{\alpha_j} \rightarrow \overline{\varphi} \in \overline{\mathcal{D}}$. From the latter and since $j < k$, we have $\overline{\alpha_j} \rightarrow \overline{\varphi} \in \overline{\mathcal{D}}$ and therefore, $\overline{\varphi} \in \overline{\mathcal{D}}$.

4. If there exists $\{j, t_1, ..., t_m\} \subseteq \{1, ..., k-1\}$ such that $\alpha_{t_1}, ..., \alpha_{t_m}$ is a derivation of $\alpha_j$ and suppose that $\alpha_n$ is $\Delta \alpha_j$, then $\overline{\alpha_n} \in \overline{\mathcal{D}}$. Now, since $\overline{\mathcal{D}}$ is a mds, we have that $\overline{\Delta \alpha_j} \in \overline{\mathcal{D}}$. Thus, $\overline{\varphi} \in \overline{\mathcal{D}}$, which completes the proof. □
The notion of deductive systems considered in the last Theorem, part 2, was named Systèmes deductifs liés à “a” by A. Monteiro, where a is an element of some given algebra such that the congruences are determined by deductive systems [16, pag. 19].

Recall that for a given iH$_3$-algebra A, a logical matrix for H$_3$ is a pair ⟨A, {1}⟩ where {1} is the set of designated elements. In addition, For a given H$_3$-algebra A, we say that an homomorphism v : 3m$_a$ → A is a valuation. Then, we say that φ is a semantical consequence of Γ, and we denote by Γ ⊨H$_3$ ϕ, if for every H$_3$-algebra A and every valuation v, v(Γ) = {1} then v(φ) = 1. Beside, we say that α is valid in A if v(α) = 1 for every valuation.

**Corollary 32** Let Γ ∪ {φ} ⊆ 3m$_s$, with Γ non-trivial maximal respect to φ in H$_{\vee, \wedge}^3$. Then, there exists a valuation v : 3m$_s$ → C$_{\vee, \wedge}^3$ such that v(φ) = 1 if and only if α ∈ Γ.

**Proof.** Taking into account Theorem 31, we known that Γ/ ≡ is a maximal modal deductive system of 3m$_s$/ ≡. Then, by Theorem 23, we have there is an homomorphism h : 3m$_s$/ ≡ → C$_{\vee, \wedge}^3$ (see Corollary 24) such that h$^{-1}$(1) = Γ/ ≡. Now, consider the canonical projection π : 3m$_s$ → 3m$_s$/ ≡ defined by π(α) = |α|, see Theorem 28. Now, it is enough to take v = h o π. □

**Theorem 33** (Soundness and completeness of H$_{\vee, \wedge}^3$ w.r.t. H$_{\vee, \wedge}^3$-algebras) Let Γ ∪ {φ} ⊆ 3m$_s$, Γ ⊢H$_{\vee, \wedge}^3$ φ if and only if Γ ⊨H$_{\vee, \wedge}^3$ φ.

**Proof.** Only if part (Soundness): It is not hard to see that every axiom is valid for every H$_{\vee, \wedge}^3$-algebra A. In addition, satisfaction is preserved by the inference rules

If part (Completeness): Suppose Γ ⊨H$_{\vee, \wedge}^3$ φ and Γ ⊬H$_{\vee, \wedge}^3$ φ. Then, according to Lemma 30, there is maximal consistent theory M such that Γ ⊆ M and M ⊬H$_{\vee, \wedge}^3$ φ. From the latter and Corollary 32 there is a valuation µ : 3m$_s$ → C$_{\vee, \wedge}^3$ such that µ(∆) = {1} but µ(φ) ≠ 1. □

4 Model Theory and first order logics of H$_{\vee, \wedge}^3$ without identities

In this section, we define the first order logic of H$_{\vee, \wedge}^3$. Let Σ = {¬, ∨, ∆} be the propositional signature of H$_{\vee, \wedge}^3$, the symbols ∀ (universal quantifier) and ∃ (existential quantifier), with the punctuation marks (commas and parentheses). Let Var = {v$_1$, v$_2$, ...} a numerable set of individual variables. A first order signature Θ is composed by the following elements:

- a set C of individual constants,
- for each n ≥ 1, F a set of functions of arity n,
- for each n ≥ 1, P a set of predicates of arity n.
The notions of bound and free variables inside a formula, closed terms, closed formulas (or sentences), and of term free for a variable in a formula are defined as usual. It will be denoted by $T_\Theta$ and $Fm_\Theta$ the sets of all terms and formulas, respectively. Given a formula $\varphi$, the formula obtained from $\varphi$ by substituting every free occurrence of a variable $x$ by a term $t$ will be denoted by $\varphi(x/t)$.

**Definition 34** Let $\Theta$ be a first order signature. The logic $QH^\land_3$ over $\Theta$ is defined by Hilbert calculus obtained by extending $H^\land_3$ expressed in the language $Fm_\Theta$ by adding the following:

**Axioms Schemas**

(Ax11) $\varphi(x/t) \rightarrow \exists x \varphi$, if $t$ is a term free for $x$ in $\varphi$,

(Ax12) $\forall x \varphi \rightarrow \varphi(x/t)$, if $t$ is a term free for $x$ in $\varphi$,

(Ax13) $\Delta \exists x \varphi \leftrightarrow \exists x \Delta \varphi$,

(Ax14) $\Delta \forall x \varphi \leftrightarrow \forall x \Delta \varphi$,

**Inferences Rules**

(R3) $\varphi \rightarrow \psi \quad \exists x \varphi \rightarrow \psi$ where $x$ does not occur free in $\psi$,

(R4) $\varphi \rightarrow \psi \quad \varphi \rightarrow \forall x \psi$ where $x$ does not occur free in $\varphi$.

We denote by $\vdash \alpha$ to a derivation of a formula $\alpha$ in $QH^\land_3$ and with $\Gamma \vdash \alpha$ to the derivation of $\alpha$ from a set of premises $\Gamma$. These notions are defined as the usual way. Besides, we denote $\vdash \varphi \leftrightarrow \psi$ as an abbreviation of $\vdash \varphi \rightarrow \psi$ and $\vdash \varphi \rightarrow \psi$.

**Definition 35** Let $\Theta$ be a first-order signature. A $\Theta$-structure is a triple $S = \langle A, S, \cdot^S \rangle$ such that $A$ is a complete $H^\land_3$-algebra, and $S$ is a non-empty set and $\cdot^S$ is an interpretation mapping defined on $\Theta$ as follows:

1. for each individual constant symbol $c$ of $\Theta$, $c^S$ of $S$,

2. for each function symbol $f$ $n$-ary of $\Theta$, $f^S : S^n \rightarrow S$,

3. for each predicate symbol $P$ $n$-ary of $\Theta$, $P^S : S^n \rightarrow A$.

Given $\Theta$-structure $S = \langle A, S, \cdot^S \rangle$, a $S$-valuation is a function $v : Var \rightarrow S$. Given $a \in S$ and $S$-valuation $v$, by $v[x \rightarrow a]$ we denote the following $S$-valuation, $v[x \rightarrow a](x) = a$ and $v[x \rightarrow a](y) = v(y)$ for any $y \in V$ such that $y \neq x$.

Let $S = \langle A, S, \cdot^S \rangle$ be a $\Theta$-structure and $v$ a $S$-valuation. A $\Theta$-structure $S = \langle A, S, \cdot^S \rangle$ and a $S$-valuation $v$ induce an interpretation map $\| \cdot \|_S^S$ for terms and formulas defined as follows.
We say that $\mathcal{G}$ and $v$ satisfy a formula $\varphi$, denoted by $\mathcal{G} \models \varphi[v]$, if $||\varphi||^v = 1$. Besides, we say that $\varphi$ is true $\mathcal{G}$ if $||\varphi||^\mathcal{G} = 1$ for each a $\mathcal{G}$-valuation $v$ and we denote by $\mathcal{G} \models \varphi$. We say that $\varphi$ is a semantical consequence of $\Gamma$ in $QH^\mathcal{G}$, if, for any structure $\mathcal{G}$: if $\mathcal{G} \models \gamma$ for each $\gamma \in \Gamma$, then $\mathcal{G} \models \varphi$. For a given set of formulas $\Gamma$, we say that the structure $\mathcal{G}$ is a model of $\Gamma$ if $\mathcal{G} \models \gamma$ for each $\gamma \in \Gamma$.

Now, it is worth mentioning the following property that is different to the propositional case because if one uses the definition of propositional case we are unable to prove an important rule as $\alpha(x) \models \forall \alpha(x)$.

In addition, we need to exhibit some important property of complete $H^\mathcal{G}$-algebra.

**Lemma 36** [19] Lemma 0.1.21, see also [22] Let $A$ be a complete $H^\mathcal{G}$ and the set $\{a_i\}_{i \in I}$ of element of $A$ for any non-empty set $I$. Then if there exist $\bigvee_{i \in I} a_i$ (or $\bigwedge_{i \in I} a_i$) then there exist $\bigvee_{i \in I} \Delta a_i$ (or $\bigwedge_{i \in I} \Delta a_i$) and also, $\bigvee_{i \in I} \Delta a_i = \Delta \bigvee_{i \in I} a_i$ and $\bigwedge_{i \in I} \Delta a_i = \Delta \bigwedge_{i \in I} a_i$.

This property is useful to prove the following theorem.

**Theorem 37** Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{M}_\Theta$, if $\Gamma \models \varphi$ then $\Gamma \models \varphi$.

**Proof.** In what follows we will consider a fixed structure $\mathcal{G} = \langle A, S, \cdot^\mathcal{G} \rangle$. It is clear that the propositional axioms are true in $\mathcal{G}$. Now, we have to prove the new axioms (Ax11) and (Ax12) are true in $\mathcal{G}$, and the new inference rules (R3) and (R4) preserve trueness in $\mathcal{G}$.

(Ax11) Suppose that $\varphi$ is $\alpha(x/t) \rightarrow \exists \alpha$. Then, $||\varphi||^\mathcal{G} = ||\alpha||^\mathcal{G}_v \rightarrow ||\exists \alpha||^\mathcal{G}_v$. It is clear that $||\alpha||^\mathcal{G}_v \rightarrow ||\exists \alpha||^\mathcal{G}_v$ and then, $||\alpha||^\mathcal{G}_v \rightarrow ||\exists \alpha||^\mathcal{G}_v$. Therefore $||\alpha(x/t) \rightarrow \exists \alpha||^\mathcal{G}_v = 1$ for every $\mathcal{G}$-valuation $v$. (Ax12) is analogous to (Ax11). Now, according to Lemma $\mathcal{G}$ the axioms (Ax13) (Ax14) are true in $\mathcal{G}$.
(R4) Let $\alpha \to \beta$ such that $x$ is not free in $\alpha$, and let $\alpha \to \forall x \beta$. Let us suppose that $\|\alpha \to \beta\|^S_v = 1$ for every $\mathcal{S}$-valuation $v$. Now, consider a fix valuation $v$ then $\|\alpha \to \forall x \beta\|^S_v = \|\alpha\|^S_v \to \|\forall x \beta\|^S_v = \|\alpha\|^S_v \to \bigwedge_{a \in S} \|\beta\|^S_{v[x \to a]}$. On the other hand, By hypothesis we know that $\|\alpha\|^S_v \leq \|\beta\|^S_v$ for every $\mathcal{S}$-valuation $u$. In particular, $\|\alpha\|^S_v = \|\alpha\|^S_{v[x \to a]} \leq \|\beta\|^S_{v[x \to a]}$ for every $\mathcal{S}$-valuation $v$. Then, $\|\alpha\|^S_v \leq \bigwedge_{a \in S} \|\beta\|^S_{v[x \to a]}$ and so, $\|\alpha\|^S_v \to \bigwedge_{a \in S} \|\beta\|^S_{v[x \to a]} = 1$ for every $\mathcal{S}$-valuation $v$. The proof of preservation of trueness for (R3) is analogous to (R4). \hfill $\Box$

In what follows, we will prove a strong version of completeness Theorem for $QH^3_{\vee, \Delta}$ using the Lindenbaum-Tarski algebra in a similar way the propositional case. Let us observe the algebra of formulas is an absolutely free algebra generated by the atomic formulas and its quantified formulas.

Now, let us consider the relation $\equiv$ defined by $\alpha \equiv \beta$ iff $\vdash \alpha \to \beta$ and $\vdash \alpha \to \beta$, then we have the algebra $\mathfrak{Fm}_\Theta / \equiv$ is a $H^3_{\vee, \Delta}$-algebra and the proof is exactly the same as in the propositional case (see, for instance, [1]). On the other hand, it is clear that $QH^3_{\vee, \Delta}$ is a tarskian and finitary logic. So, we can consider the notion of (maximal) consistent and closed theories with respect to some formula in the same way as the propositional case. Therefore, we have that Lindenbaum-Los’ Theorem holds for $QH^3_{\vee, \Delta}$. Then, we have the following

**Theorem 38** Let $\Gamma \cup \{\varphi\} \subseteq \mathfrak{Fm}_\Theta$, with $\Gamma$ non-trivial maximal respect to $\varphi$ in $QH^3_{\vee, \Delta}$. Let $\Gamma / \equiv = \{[\alpha] : \alpha \in \Gamma\}$ be a subset of $\mathfrak{Fm}_\Theta / \equiv$, then:

1. If $\alpha \in \Gamma$ and $[\alpha] = [\beta]$, then $\beta \in \Gamma$. Besides, it is verified that $\Gamma / \equiv = \{[\alpha] : \alpha \in \Gamma\}$ in this case we say that it is closed.

2. $\Gamma / \equiv$ is a modal deductive system of $\mathfrak{Fm}_\Theta / \equiv$. Also, if $\varphi \notin \Gamma / \equiv$ and for any modal deductive system $D$ being closed in the sense of 1 and containing properly to $\Gamma / \equiv$, then $\varphi \in D$.

**Proof.** According to the proof of Theorem 31, we only have to consider the rules (R3) and (R4). The fact that $\Gamma / \equiv$ is closed follows immediately.

In order to complete the proof we have to consider two new cases. It is clear that $\Gamma / \equiv$ is a subset of $D$. Now, let us consider $\phi \notin D$ then $\phi \notin \Gamma / \equiv$ and remember $D = \{\alpha : [\alpha] \in D\}$. 5. There exists $\{j, t_1, \ldots, t_m\} \subseteq \{1, \ldots, k - 1\}$ such that $\alpha_{t_1}, \ldots, \alpha_{tm}$ is a derivation of $\alpha_j = \theta \to \beta$. Let us suppose that $\alpha_n = \exists x \theta \to \beta$ is obtained by $\alpha_j$ applying (R3). From induction hypothesis, we have that $\theta \to \beta \in D$. From the latter, we obtain $\exists x \theta \to \beta \in D$. 6. There exists $\{j, t_1, \ldots, t_m\} \subseteq \{1, \ldots, k - 1\}$ such that $\alpha_{t_1}, \ldots, \alpha_{tm}$ is a derivation of $\alpha_j = \theta \to \beta$. Let us suppose that $\alpha_n = \theta \to \forall x \beta$ is obtained by $\alpha_j$ applying (R4). From induction hypothesis, we have $\theta \to \beta \in D$ and then, $\theta \to \forall x \beta \in D$. \hfill $\Box$

We note that for a given maximal consistent theory $\Gamma$ of $\mathfrak{Fm}_\Theta$ we have $\Gamma / \equiv$ is a maximal modal deductive system of $\mathfrak{Fm}_\Theta / \equiv$. If we denote $A := \mathfrak{Fm}_\Theta / \equiv$ and $\theta := \Gamma / \equiv$ by well-known
results of Universal algebras, we have the quotient algebra \( A/\theta \) is a simple algebra, see Corollary 24. From the latter and by adapting the first isomorphism theorem for Universal Algebras, we have that \( A/\theta \) is an isomorphic to \( \mathfrak{m}_\Theta/\Gamma \) where it is defined by the congruence \( \alpha \equiv_\Gamma \beta \) iff \( \alpha \to \beta, \beta \to \alpha \in \Gamma \).

**Theorem 39** Let \( \Gamma \cup \{ \varphi \} \) be a set of sentences, then \( \Gamma \models \varphi \) if \( \Gamma \not\models \varphi \).

**Proof.** Let us suppose \( \Gamma \models \varphi \) and \( \Gamma \not\models \gamma \). Then, by Lindenbaum- Los’ Lemma, there exists \( \Delta \) maximal consistent theory such that \( \Gamma \subseteq \Delta \). Now, consider \( \mathfrak{m} \) the algebra of closed formulas and the algebra \( \mathfrak{m}_\Theta/\Delta \) defined by the congruence \( \alpha \equiv_\Delta \beta \) iff \( \alpha \to \beta, \beta \to \alpha \in \Delta \). We know that \( \mathfrak{m}_\Theta/\Delta \) is isomorphic to a subalgebra of \( C_{3\times \cdots \times 3} \) and so, complete as lattice, in view of the above observations. Thus, taking the canonical projection \( \pi_\Delta : \mathfrak{m} \to \mathfrak{m}_\Theta/\Delta \).

On the other hand, consider the structure \( \mathcal{M} = (\mathfrak{m}_\Theta/\Delta, T_\Theta, T_\Theta) \) where \( T_\Theta \) is a set of terms. Then, it is clear that for every \( t \in T_\Theta \) we have a constant \( t \) of \( \Theta \). Now, we can consider a function \( \mu : \text{Var} \to T_\Theta \) defined by \( v(x) = x \). Besides, we have the interpretation \( ||| \cdot ||| : \mathfrak{m} \to \mathfrak{m}_\Theta/\Delta \) defined by if \( \bar{t} \) is a constant then \( |||\bar{t}|||_\mu := t \), if \( f \in F \) then \( |||f(t_1, \ldots, t_n)|||_\mu = f(t_1, \ldots, t_n) \); if \( P \in \mathcal{P} \) then \( |||P(t_1, \ldots, t_n)|||_\mu = \pi_\Delta(P(t_1, \ldots, t_n)) \). Our interpretation is defined for atomic formulas; but it is easy to see that \( |||\alpha|||_\mu = \pi_\Delta(\alpha) \) for every quantifier-free formula \( \alpha \). Moreover, it is easy to see that for every formula \( \phi(x) \) and every term \( t \) we have \( |||\phi(x/t)|||_\mu = |||\phi(x'|| t)|||_\mu \). Therefore, from the latter property and by (Ax12) and (R4), we have \( |||\forall x \alpha|||_\mu = \bigwedge_{\alpha \in T_\Theta} |||\alpha|||_\mu[x \to a] \) and now using (Ax11) and (R3), we obtain \( |||\exists x \alpha|||_\mu = \bigvee_{a \in T_\Theta} |||\alpha|||_\mu[x \to a] \). So, \( |||\cdot|||_\mu \) is an interpretation map such that \( |||\alpha|||_\mu = 1 \) iff \( \alpha \in \Delta \). On the other hand, it is not hard to see for every formula \( \beta \), we have \( |||\beta|||_\mu = |||\beta|||_\mu \) for every \( \mathcal{M} \)-valuation \( v \). Therefore, \( \mathcal{M} \models \gamma \) for every \( \gamma \in \Gamma \) but \( \mathcal{M} \not\models \varphi \). \( \square \)

Given a formula \( \varphi \) and suppose \( \{ x_1, \ldots, x_n \} \) is the set of variable of \( \varphi \), the **universal closure** of \( \varphi \) is defined by \( \forall x_1 \cdots \forall x_n \varphi \). Thus, it is clear that if \( \varphi \) is a sentence then the universal closure of \( \varphi \) is itself. Now, we are in condition to proving the following completeness theorem for formulas:

**Theorem 40** Let \( \Gamma \cup \{ \varphi \} \) be a set formulas, then \( \Gamma \models \varphi \) then \( \Gamma \mid \varphi \).

**Proof.** Let us suppose \( \Gamma \models \varphi \) and consider the set \( \forall \Gamma \) all universal closure of \( \Gamma \). From the latter and definition of \( \models \), we have \( \forall \Gamma \models \forall x_1 \cdots \forall x_n \varphi \). Then, according to Theorem 39 \( \forall \Gamma \mid \forall x_1 \cdots \forall x_n \varphi \). Now, from latter and (Ax12) and (R4), we have \( \Gamma \mid \varphi \) as desired. \( \square \)

**Theorem 41** (Compactness Theorem) Let \( \Omega \) be a subset of \( \mathfrak{m}_\Theta \). \( \Omega \) has a model if and only if any finite subtheory of \( \Omega \) has a model.

**References**

[1] J. Bell and A. Slomson, *Models and Ultraproducts: An Introduction*, North Holland, Amsterdam, 1971
[2] V. Boicescu and A. Filipoiu and G. Georgescu and S. Rudeanu, *Lukasiewicz - Moisil Algebras*, Annals of Discrete Mathematics 49, North - Holland, 1991.

[3] R. Cignoli, *Estudio algebraico de lógicas polivalentes. Algebras de Moisil de orden n*, Ph. D. thesis, Universidad Nacional del Sur, Bahía Blanca, 1969.

[4] R. Cignoli, *An algebraic approach to elementary theories based on n-valued Lukasiewicz logics*. Z. Math. Logik Grundlag. Math. 30 (1984), no. 1, 87–96.

[5] R. Cignoli, *Proper n-valued Lukasiewicz algebras as S-algebras of Lukasiewicz n-valued propositional calculi*. Studia Logica 41 (1982), no. 1, 3–16.

[6] R. Cignoli, I. D’Ottaviano and D. Mundici, *Algebraic foundations of many-valued reasoning*, Trends in Logic Studia Logica Library, 7. Kluwer Academic Publishers, Dordrecht, 2000. x+231 pp.

[7] M. Canals Frau, A. V. Figallo and S. Saad, *Modal three valued Hilbert algebras*, Preprints Del ICB. Universidad Nacional de San Juan, Argentina, p.1 - 21, 1990.

[8] M. Canals Frau and A. V. Figallo, *Modal 3-valued implicative semilattices*, Preprints Del Instituto de Ciencias Básicas. U. N. de San Juan, Argentina, p.1 - 24, 1992.

[9] A. Diego, *Sur les algèbres de Hilbert*, Colléction de Logique Mathématique, ser. A, fasc. 21. Gouthier-Villars, Paris (1966)

[10] A. V. Figallo, G. Ramón and S. Saad, *A note on the distributive Hilbert algebras*, Proceedings of the Fifth “Dr. Antonio A. R. Monteiro” Congress on Mathematics, Bahía Blanca, (1999), 139–152.

[11] A. V. Figallo, G. Ramón and S. Saad, *iH-Propositional calculus*, Bull. Sect. Logic Univ. Lódz 35 (2006), no. 4, 157–162.

[12] A. Figallo Jr. and A. Zilian, *Remarks on Hertz algebras and implicative semilattices*, Bull. Sect. Logic Univ. Lódz, 34, 1 (2005), 37–42.

[13] J. M. Font, A.J. Rodriguez and A. Torrens, *Wajsberg algebras*, Stochastica 8 (1984), Nro. 1, 5–31.

[14] A. Iorgulescu, *Connections between MVn-algebras and n-valued Lukasiewicz–Moisil algebras Part I*, Discrete Math. 181, 155–177 (1998)

[15] A. Monteiro, *Les algèbres de Hilbert linéaires, Unpublished papers I*, Notas de Lógica Matemática, Univ. Nac. del Sur, Bahía Blanca, Vol. 40, (1996), 114–127.

[16] A. Monteiro, *Sur les algèbres de Heyting simetriques*, Portugaliae Math., 39, 1-4 (1980), 1–237.
[17] A. Monteiro, *Cálculo proposicional implicativo*, Informe técnico N° 90. INMABB-Conicet, Universidad Nacional del Sur, 2005.

[18] L. Monteiro, *Algèbres de Hilbert n−valentes*, Portugaliae Math. 36(1977), 159–174.

[19] L. Monteiro, *Algebras de Lukasiewicz trivalentes monádicas*. Ph. D. thesis, Universidad Nacional del Sur, 1973.

[20] Gr. C. Moisil, *Recherches sur logiques non-chrysippiennes*, Ann. Sc. de l’Université de Yassy, 27 (1941), 60-90.

[21] I. Thomas, *Finite limitations on Dummett’s LC*, Notre Dame Journal of Formal Logic, 3 (1962), 170 – 174.

[22] A. Petrovich and M. Lattanzi, *An alternative notion of quantifiers on three-valued Lukasiewicz algebras*, Mult.-Val. Logic Soft Comput. 28(4–5), 335–360 (2017)

[23] H. Rasiowa, *An algebraic approach to non-clasical logics*, Studies in logic and the foundations of mathematics, vol. 78. North-Holland Publishing Company, Amsterdam and London, and American Elsevier Publishing Company, Inc., New York, 1974.

[24] R. Wójcicki, *Lectures on propositional calculi*, Ossolineum, Warsaw, 1984.