Landau Quantization for An electric quadrupole moment of Position-Dependent Mass Quantum Particles interacting with Electromagnetic fields

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Abstract: Analogous to Landau quantization related to a neutral particle possessing an electric quadrupole moment, we generalize such a Landau quantization to include position-dependent mass (PDM) neutral particles. Using cylindrical coordinates, the exact solvability of PDM neutral particles with an electric quadrupole moment moving in electromagnetic fields is reported. The interaction between the electric quadrupole moment of a PDM neutral particle and a magnetic field in the absence of an electric field is analyzed for two different radial cylindrical PDM settings. Next, two particular cases of radial electric fields \( \vec{E} = \frac{\lambda}{\rho} \hat{\rho} \) and \( \vec{E} = \frac{\lambda \rho}{r^2} \hat{\rho} \) are considered to investigate their influence on the Landau quantization (of this system using the same models of PDM settings). The exact eigenvalues and eigenfunctions for each case are analytically obtained.

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I. INTRODUCTION

The interaction of multipole moments with the electromagnetic fields has attracted a lot of attention and produced fundamental quantum effects. For example, the Aharonov-Bohm effect [1–4] for a charged particle, the scalar Aharonov-Bohm and He-Mckellar-Willens effects [5–11], for bound states [12], and Landau quantization [13–16] for an electric dipole moment of a neutral particle. Furthermore, recent studies have investigated the interaction between the quadrupole moments of neutral particle and external fields in several quantum systems such as geometric quantum phases [17], noncommutative quantum mechanics [19], nuclear structure [20, 21], atomic systems [22–27], molecular systems [28–30], and Landau quantization [18, 31–33].

In particular, the study of the Landau quantization has recently been applied to the quantum dynamics of an electric quadrupole moment [18, 31]. It investigated the possibility of achieving the Landau quantization for neutral particles, resulting from the coupling of the electric quadrupole moment with a magnetic field, making a similar minimal coupling with a constant magnetic field [1, 2]. Moreover, they have discussed the conditions necessary for the field configuration in order to achieve the Landau quantization for neutral particles possessing an electric quadrupole moment [14, 18]. It is shown [18] that the field configuration in the quadrupole system is dependent on the structure of the quadrupole tensor (i.e., diagonal or non-diagonal), and has to be different in each case. However, all the previous studies have used different methods for quantum systems of multipole moments with constant mass. Such methods need to be modified to include the spatial dependence of the mass.

On the other hand, quantum mechanical systems with position-dependent mass (PDM) have attracted attention over the years. Namely, the von Roos Hamiltonian [34] has been extensively investigated in the literature [35–55]. Not only because of its ordering ambiguity associated with the non-unique representation of the kinetic energy operator, but also because of its feasible applicability in many fields of physics. Recent studies on such PDM charged particles in constant magnetic fields [54, 60], and position-dependent magnetic fields [61] are carried out (using different interaction potentials). To the best of our knowledge, however, no studies have ever been considered to discuss the quantum mechanical effects on PDM neutral particles possessing an electric quadrupole moment. To fill this gap, we discuss in this work a quantum system that consists of a PDM neutral particle with an electric quadrupole moment interacting with external fields. We follow the discussion in [18] and extend their idea for PDM systems.

This paper is organized as follows. In section II, we start giving a brief description of the quantum dynamics for a moving electric quadrupole moment interacting with external fields with a constant mass as done in [18], and extend it to include the PDM case. In so doing, we use the very recent result suggested by [58, 59] for the
PDM-minimal-coupling and the PDM-momentum operator. Furthermore, we discuss the possibility of achieving the Landau quantization for such a system, and the separability of the problem in the cylindrical coordinates \((\rho, \phi, z)\), under azimuthal symmetrization, by considering that the field configurations and the PDM settings are purely radial dependent as in [54, 55, 59–61]. In section III, we discuss a Landau levels analog for an electric quadrupole moment interacting with an external magnetic field in the absence of electric field. In the same section we obtain exact eigenfunctions and eigenvalues for different PDM settings. We take into account, in section IV, the effect of an electric field on the problem at hand, by choosing two models for the radial electric field, a Coulomb-type electric field \(\vec{E} = \frac{\lambda}{\rho} \hat{\rho}\) and a linear-type electric field \(\vec{E} = \frac{2\lambda}{\rho^2} \hat{\rho}\). Finally, we report exact solutions of the radial Schrödinger equation for both case of an electric field and for the same PDM settings presented in previous section. Our conclusion is given in section V.

II. ANALOGOUS TO THE LANDAU-TYPE QUANTIZATION:

In this section, we start our discussion by describing the quantum dynamics of a moving electric quadrupole moment interacting with electromagnetic fields as suggested in [17]. By considering an electric quadrupole moment as a scalar particle, the potential energy of a multipole expansion in the rest frame of a particle can be written as

\[
U = q\Phi - \vec{d} \cdot \nabla \Phi + \sum_{i,j} Q_{ij} \partial_i \partial_j \Phi
\]  

(1)

where \(q\) is the electric charge, \(\vec{d}\) is the electric dipole moment, \(Q_{ij}\) is the electric quadrupole moment tensor, and \(\Phi\) is the electric potential.

In order to study the dynamics of an electric quadrupole moment, we consider \(q = 0\), \(\vec{d} = 0\) and \(\vec{E} = -\nabla \Phi\), where \(\vec{E}\) is the electric field. Therefore, the equation (1) reads

\[
U = -\sum_{i,j} Q_{ij} \partial_i E_j
\]  

(2)

For a moving quadrupole, Lagrangian of this system (a constant mass system) becomes

\[
L = \frac{1}{2} mv^2 + \sum_{i,j} Q_{ij} \partial_i E_j
\]  

(3)

Now we must apply the Lorentz transformation of the electromagnetic fields. Therefore, we replace the electric field in (3) by

\[
\vec{E} \rightarrow \frac{\vec{E}}{c} + \frac{1}{c} \vec{v} \times \vec{B}
\]  

(4)

where \(\vec{E}\) and \(\vec{B}\) are the electric and magnetic fields, respectively. Thus, Lagrangian (3) becomes

\[
L = \frac{1}{2} mv^2 + \vec{Q} \cdot \vec{E} - \frac{1}{c} \vec{v} \cdot (\vec{Q} \times \vec{B})
\]  

(5)

where we used

\[
Q_i = \sum_{i,j} Q_{ij} \partial_j, \quad \vec{Q} = \sum_i Q_i \hat{e}_i
\]  

(6)

as in [18, 31]. Using the canonical momentum

\[
\vec{P} = m \vec{v} - \frac{1}{c} (\vec{Q} \times \vec{B})
\]  

(7)
the classical Hamiltonian of a constant mass reads

\[
H = \frac{1}{2m} \left[ \vec{P} + \frac{1}{c} (\vec{Q} \times \vec{B}) \right]^2 - \vec{Q} \cdot \vec{E} \tag{8}
\]

To write the quantum Hamiltonian operator, we replace the canonical momentum \(\vec{P}\) by the operator \(\hat{P} = -i \vec{\nabla}\) for constant mass settings. However, in this work we are interested to study the PDM system. Thus, we rewrite the PDM-non relativistic Hamiltonian (in \(\hbar = 2m_c = c = 1\) units) as

\[
\hat{H} = \left( \frac{\hat{P}(\vec{r}) + \hat{A}_{\text{eff}}(\vec{r})}{\sqrt{m(\vec{r})}} \right)^2 - \vec{Q} \cdot \vec{E} \tag{9}
\]

where the kinetic energy term was proposed by Mustafa and Al gadhi \[59\] along with the definition of PDM-momentum operator (which resulted from a factorization recipe of Mustafa and Mazharimousavi in \[46\]):

\[
\hat{P}(\vec{r}) = -i \left[ \vec{\nabla} - \frac{1}{4} \left( \frac{\vec{\nabla} m(\vec{r})}{m(\vec{r})} \right) \right] \tag{10}
\]

where

\[
\hat{A}_{\text{eff}}(\vec{r}) = \vec{Q} \times \vec{B}, \quad V_{\text{eff}}(\vec{r}) = -\vec{Q} \cdot \vec{E} \tag{11}
\]

In this way, the corresponding time-independent Schrödinger equation is written in the form

\[
\left[ \left( \frac{\hat{P}(\vec{r}) + \hat{A}_{\text{eff}}(\vec{r})}{\sqrt{m(\vec{r})}} \right)^2 - \vec{Q} \cdot \vec{E} \right] \psi(\vec{r}) = \varepsilon \psi(\vec{r}). \tag{12}
\]

Hence,

\[
\left[ \left( \frac{\hat{P}(\vec{r})}{\sqrt{m(\vec{r})}} \right)^2 - \frac{2i}{m(\vec{r})} \hat{A}_{\text{eff}}(\vec{r}) \cdot \vec{\nabla} - \frac{i}{m(\vec{r})} \left( \vec{\nabla} \cdot \hat{A}_{\text{eff}}(\vec{r}) \right) + i \hat{A}_{\text{eff}}(\vec{r}) \cdot \left( \frac{\vec{\nabla} m(\vec{r})}{m(\vec{r})^2} \right) + \frac{\hat{A}_{\text{eff}}(\vec{r})^2}{m(\vec{r})} - \vec{Q} \cdot \vec{E} \right] \psi(\vec{r}) = \varepsilon \psi(\vec{r}). \tag{13}
\]

in which the vector potential satisfies the Coulomb gauge \(\vec{\nabla} \cdot \hat{A}_{\text{eff}} = 0\). Moreover, using the momentum operator in equation\(10\) would imply:

\[
\left[ \frac{1}{m(\vec{r})} \vec{\nabla}^2 + \left( \frac{\vec{\nabla} m(\vec{r})}{m(\vec{r})^2} \right) \cdot \vec{\nabla} + \frac{1}{4} \left( \frac{\vec{\nabla}^2 m(\vec{r})}{m(\vec{r})^2} \right) - \frac{7}{16} \left( \frac{\vec{\nabla} m(\vec{r})}{m(\vec{r})^3} \right)^2 - \frac{2i}{m(\vec{r})} \hat{A}_{\text{eff}}(\vec{r}) \cdot \vec{\nabla} + i \hat{A}_{\text{eff}}(\vec{r}) \cdot \left( \frac{\vec{\nabla} m(\vec{r})}{m(\vec{r})^2} \right) + \frac{\hat{A}_{\text{eff}}(\vec{r})^2}{m(\vec{r})} - \vec{Q} \cdot \vec{E} \right] \psi(\vec{r}) = \varepsilon \psi(\vec{r}). \tag{14}
\]

The discussion of the possibility of achieving the Landau quantization for an electric quadrupole moment was done in \[18\], where they found that the Landau quantization can be achieved by imposing these two conditions: the first one is that the tensor \(Q_{ij}\) must be symmetric and tracless. And the second one is that the field configuration must be chosen in such a way that there exists a uniform effective magnetic field given by
\[ \overrightarrow{B}_{\text{eff}} = \nabla \times \overrightarrow{A}_{\text{eff}} = \nabla \times (\overrightarrow{Q} \times \overrightarrow{B}) = \text{constant vector} \]  \hspace{1cm} (15)

perpendicular to the plane of motion of the electric quadrupole moment. Thus, it is clear that the field configuration depends on the choice of the components of the tensor \( Q_{ij} \) that describes the electric quadrupole moment. Moreover, \( \overrightarrow{E} \) must satisfy the electrostatic conditions \( (\nabla \times \overrightarrow{E} = 0, \partial_t \overrightarrow{E} = 0) \).

In the following, we present the field configurations and structures of the tensor \( Q_{ij} \). Thus, we choose the case when the tensor \( Q_{ij} \) has the non-null components:

\[ Q_{\rho\rho} = Q_{\phi\phi} = Q, \quad Q_{zz} = -2Q \]  \hspace{1cm} (diagonal form)  \hspace{1cm} (16)

which was studied by [17, 18], where \( Q \) is a constant. It is notable that this choice satisfies the properties of the tensor \( Q_{ij} \).

Moreover, we consider a magnetic field given by [18, 31]

\[ \overrightarrow{B} = \frac{1}{2} B_0 \rho^2 \hat{z} \]  \hspace{1cm} (17)

where \( B_0 \) is a constant. By using the the definitions of (6) and the assumption in (16) we obtain the electric quadrupole moment as

\[ \overrightarrow{Q} = (Q \partial_\rho) \hat{\rho} + (Q \partial_\phi) \hat{\phi} - (2Q \partial_z) \hat{z} \]  \hspace{1cm} (18)

At this point, we can find the effective vector potential \( \overrightarrow{A}_{\text{eff}} \) as

\[ \overrightarrow{A}_{\text{eff}} (\overrightarrow{r}) = \overrightarrow{Q} \times \overrightarrow{B} = -QB_0 \rho \hat{\phi} \]  \hspace{1cm} (19)

Consequently, the effective magnetic field reads

\[ \overrightarrow{B}_{\text{eff}} (\overrightarrow{r}) = \nabla \times \overrightarrow{A}_{\text{eff}} (\overrightarrow{r}) = -2QB_0 \hat{z} \]  \hspace{1cm} (20)

which satisfies the second condition (15), where \( \overrightarrow{B}_{\text{eff}} \) is a uniform effective magnetic field.

We may now discuss the separability of the PDM Schrödinger equation (14) in the cylindrical coordinates \((\rho, \phi, z)\) and under azimuthal symmetrization. By assuming that the field configurations and that the PDM functions are only radially dependent [54, 55, 59–61] (i.e., \( m (\overrightarrow{r}) = M (\rho, \phi, z) = g (\rho) \)), the wavefunction can be written as

\[ \psi (\rho, \phi, z) = e^{im\phi} e^{ikz} R (\rho) \]  \hspace{1cm} (21)

where \( m = 0, \pm 1, \pm 2, \ldots, \pm \ell \) is the magnetic quantum number. Thereby, and with the substitution of (19), (20) and (21) into (14), we obtain the radial equation:

\[ \frac{R'' (\rho)}{R (\rho)} - \left( \frac{g' (\rho)}{g (\rho)} - \frac{1}{\rho} \right) \frac{R' (\rho)}{R (\rho)} - \frac{1}{4} \left( \frac{g'' (\rho)}{g (\rho)} - \frac{g' (\rho)}{pg (\rho)} \right) + \frac{7}{16} \left( \frac{g' (\rho)}{g (\rho)} \right)^2 + g (\rho) (\varepsilon + \overrightarrow{Q} \cdot \overrightarrow{E}) - \frac{m^2}{\rho^2} - Q^2 B_0^2 \rho^2 + 2QB_0m - k_z^2 = 0. \]  \hspace{1cm} (22)

Which is to be solved for no electric field \( \overrightarrow{E} = 0 \) and for a different choice of radial electric fields \( \overrightarrow{E} \neq 0 \), with suitable PDM settings, to find the exact eigenvalues and eigenfunctions of the system.
III. PDM PARTICLES WITH AN ELECTRIC QUADRUPOLE MOMENT IN A MAGNETIC FIELD:

In this section, we focus on the discussion of Landau quantization for an electric quadrupole moment interacting with an external magnetic field, and a vanishing electric field $\vec{E} = 0$ (i.e. $V_{eff} = 0$). At this point, equation (22) would read

$$R''(\rho) + \left[ -\frac{g''(\rho) - \frac{1}{\rho} g'(\rho)}{g(\rho)} R'(\rho) - \frac{1}{4} \left( \frac{g''(\rho)}{g(\rho)} - \frac{g'(\rho)}{\rho g(\rho)} \right) + \frac{7}{16} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 + g(\rho) \varepsilon - \frac{m^2}{\rho^2} - Q^2 B_o^2 \rho^2 + 2Q B_o m - k_z^2 \right] R(\rho) = 0. \quad (23)$$

In the following examples, we use some power-law PDM type in the radial Schrödinger equation (23) and report their exact-solutions.

A. Model-I: A linear-type PDM $g(\rho) = \eta \rho$:

Consider a neutral particle with the radial cylindrical PDM setting, $g(\rho) = \eta \rho$, and the electric quadrupole moment of (18) in presence of the magnetic field in (17). Then, the Schrödinger equation (23) would read

$$R''(\rho) + \left[ -\frac{(m^2 - 3/16)}{\rho^2} - Q^2 B_o^2 \rho^2 + \eta \rho \varepsilon + 2Q B_o m - k_z^2 \right] R(\rho) = 0 \quad (24)$$

Now, let us make a simple change of variables in equation (24) and use $r = \sqrt{Q B_o} \rho$. Then equation (24) becomes

$$R''(r) + \left[ -\frac{(m^2 - 3/16)}{r^2} - r^2 + \frac{\eta \varepsilon}{(Q B_o)^{3/2}} r + \frac{2Q B_o m - k_z^2}{Q B_o} \right] R(r) = 0, \quad (25)$$

which implies the one-dimensional Schrödinger form of the Biconfluent Heun equation (see, [63, 64]) reads

$$R''(r) + \left[ \frac{(1 - \alpha^2)}{4r^2} - \frac{1}{2} \frac{\delta - \beta r - r^2 + \gamma - \frac{\beta^2}{4}}{} \right] R(r) = 0 \quad (26)$$

where

$$\frac{1}{4} (1 - \alpha^2) = 3/16 - m^2, \quad \beta = \frac{-\eta \varepsilon}{(Q B_o)^{3/2}}, \quad \frac{\delta}{2} = 0, \quad \gamma - \frac{\beta^2}{4} = \frac{2Q B_o m - k_z^2}{Q B_o} \quad (27)$$

To solve the above equation (26), we consider the asymptotic behavior for $r \to 0$ and $r \to \infty$, the function $R(r)$ can be written in terms of an unknown function $u(r)$ as follows

$$R(r) = r^{(1+\alpha)/2} e^{-(\beta r + r^2)/2} u(r) \quad (28)$$

that transforms equation (26) into a simpler form

$$ru''(r) + \left[ 1 + \alpha - \beta r - 2r^2 \right] u'(r) + \left[ (\gamma - 2 - \alpha) r - \frac{1}{2} (\delta + (1 + \alpha) \beta) \right] u(r) = 0 \quad (29)$$

which is the Biconfluent Heun-type equation (BHE) [63], where $\alpha, \beta, \gamma$ and $\delta$ are arbitrary parameters. The polynomial solutions of this equation (c.f, e.g., [62, 66]) is
\[ u(r) = H_B(\alpha, \beta, \gamma, \delta; r) \quad (30) \]

where \( H_B(\alpha, \beta, \gamma, \delta; r) \) are the Heun polynomials of degree \( n_\rho \) such that

\[ \gamma - 2 - \alpha = 2n_\rho, \quad \text{where} \quad n_\rho = 0, 1, 2, \ldots, \text{ and } a_{n_\rho+1} = 0. \quad (31) \]

Here, \( n_\rho \) is the radial quantum number and \( a_{n_\rho+1} \) is a polynomial of degree \( n_\rho + 1 \) defined by the recurrent relation (see [64–66] for more details). By substituting (27) into (31), we get the exact eigenvalues

\[ \epsilon_{n_\rho,m} = \frac{(2QB_c)^{3/2}}{\eta} \left[ 1 + n_\rho - m + \sqrt{m^2 + 1/16} + \frac{k_z^2}{2QB_c} \right]^{1/2} \quad (32) \]

where the cyclotron frequency is \( \omega = \frac{(2QB_c)^{3/2}}{\eta} \), and the exact normalized eigenfunctions is

\[ R(\rho) = N_{n_\rho}|\tilde{\ell}|^{1/2}e^{-|\tilde{\ell}|} H_B(\alpha, \beta, \gamma, 0; \sqrt{QB_c \rho}) \quad (33) \]

where \( N \) is the normalization constant, \( |\tilde{\ell}| = \sqrt{m^2 + 1/16} \), and \( \alpha, \beta \) and \( \gamma \) are defined respectively in (27).

Comparing with [31], the eigenvalues are changed due to the effect of the PDM of the system, where the spectrum of energy (32) is proportional to \( n^{1/2} \) and removes degeneracies (associated with the magnetic quantum number) similar to the energy levels reported in [31], where they are proportional to \( n \). Furthermore, the frequency is also modified.

**B. Model-II: A harmonic-type PDM** \( g(\rho) = \eta \rho^2 \):

A PDM neutral particle with \( g(\rho) = \eta \rho^2 \), and an electric quadrupole moment interacting with the external magnetic field (17) would imply that equation (23) be rewritten as

\[ R''(\rho) - \frac{1}{\rho} R'(\rho) + \left[ -\frac{(m^2 - 3/4)}{\rho^2} - (Q^2B_c^2 - \eta \varepsilon) \rho^2 + 2QB_c m - k_z^2 \right] R(\rho) = 0 \quad (34) \]

To determine the radial part \( R(\rho) \) of the wave function and the energy spectrum, we follow the same analysis of Gasiorowicz [68] and start by using the change of variable \( x = (Q^2B_c^2 - \eta \varepsilon)^{1/4} \rho \) in (34) to obtain

\[ R''(x) - \frac{1}{x} R'(x) - \frac{L^2}{x^2} R(x) - x^2 R(x) + \mu R(x) = 0 \quad (35) \]

where

\[ L^2 = m^2 - 3/4 \quad \text{and} \quad \mu = \frac{2QB_c m - k_z^2}{(Q^2B_c^2 - \eta \varepsilon)^{1/2}} \quad (36) \]

Next, we consider the asymptotic solutions \((x \to 0 \text{ and } x \to \infty)\) of the radial wavefunction \( R(x) \) to come out with

\[ R(x) = x^{1+|\tilde{\ell}|}e^{-x/2}G(x) \quad (37) \]

with
\[ \bar{\ell}^2 = L^2 + 1 \implies |\bar{\ell}| = \sqrt{m^2 + 1/4}, \text{ with } \bar{\ell} > 0. \] (38)

Substituting (37) in (35) would imply

\[ G''(x) + \left( \frac{1 + 2|\bar{\ell}|}{x} - 2x \right) G'(x) + \left( \mu - 2 - 2|\bar{\ell}| \right) G(x) = 0 \] (39)

Which, in turn, with \( y = x^2 \) yields

\[ yG''(y) + \left( 1 + |\bar{\ell}| - y \right) G'(y) + \left( \frac{\mu}{4} - \frac{|\bar{\ell}|}{2} - \frac{1}{2} \right) G(y) = 0 \] (40)

This equation is the confluent hypergeometric equation, the series of which is a polynomial of degree \( n_\rho \) (finite everywhere) when

\[ n_\rho = \frac{\mu}{4} - \frac{|\bar{\ell}|}{2} - \frac{1}{2}. \] (41)

Consequently, (36) and (38) would give the eigenvalues as

\[ \varepsilon_{n_\rho,m} = \frac{Q^2B_0^2}{\eta} \left[ 1 - \left( \frac{k^2 - m}{1 + 2n_\rho + \sqrt{m^2 + 1/4}} \right)^2 \right] \] (42)

and the eigenfunctions as

\[ R(\rho) = N_\rho |\bar{\ell}|^{1+\mu} e^{-\sqrt{Q^2B_0^2-\eta\rho^2}/2} \binom{-n_\rho; 1; \sqrt{Q^2B_0^2 - \eta\rho^2}}{1}. \] (43)

In this case, the effect of PDM setting produces a new contribution to the non-degenerate energy levels (42), where they are proportional to \( n^{-2} \) and the frequency is modified as \( \omega = \frac{Q^2B_0^2}{\eta} \).

IV. PDM-PARTICLES WITH AN ELECTRIC QUADRUPOLE MOMENT AND ELECTROMAGNETIC FIELDS:

In this section, we study PDM-particles with an electric quadrupole moment and electromagnetic fields. However, we focus on analysis of the effect of the electric field on a PDM particle with an electric quadrupole moment in the presence of a magnetic field (on the Landau-type system reported in the previous section). For this purpose, we choose a sample of radial electric fields (c.f., e.g., [14–16, 24, 31]), in the following illustrative examples.

A. The influence of a Coulomb-type electric field on the Landau-type system:

Consider a radial electric field in the form of

\[ \vec{E} = \frac{\lambda}{\rho^2} \] (44)
where \( \lambda \) is a constant \[31\].

Thus, we can see that the interaction between the electric quadrupole moment (18) and the electric field (44) leads to an effective scalar potential

\[
V_{\text{eff}} (\rho) = -Q \cdot E = \frac{Q \lambda}{\rho^2}
\]

which plays the role of a scalar potential in the PDM-Schrödinger equation (22) to imply

\[
R''(\rho) - \left( \frac{g''(\rho)}{g(\rho)} - \frac{1}{\rho} \right) R'(\rho) + \left[ -\frac{1}{4} \left( \frac{g''(\rho)}{g(\rho)} - \frac{g'(\rho)}{\rho g(\rho)} \right) + \frac{7}{16} \left( \frac{g'(\rho)}{g(\rho)} \right)^2 \right. \\
+ g(\rho) \varepsilon - g(\rho) \frac{Q \lambda}{\rho^2} - \frac{m^2}{\rho^2} - \frac{Q^2 B_2^2 \rho^2 + 2 Q B_2 m - k_z^2}{Q B_0} \right] R(\rho) = 0.
\]

Hereby, we again consider the same examples used in the previous section to find exact solutions of equation (46):

1. **Model-I :** \( g(\rho) = \eta \rho \):

The PDM radial Schrödinger equation (46) with, \( g(\rho) = \eta \rho \), reads

\[
R''(\rho) + \left[ -\left( \frac{m^2 - 3/16}{\rho^2} - Q^2 B_2^2 \rho^2 + \eta \rho \varepsilon - \frac{\eta \lambda Q}{\rho} + 2 Q B_2 m - k_z^2 \right) \right] R(\rho) = 0
\]

with the change of variable \( r = \sqrt{Q B_0} \rho \), equation (47) becomes

\[
R''(r) + \left[ -\frac{m^2 - 3/16}{r^2} - r^2 + \frac{\eta \varepsilon}{(Q B_0)^{3/2} r} - \frac{\eta \lambda Q}{(Q B_0)^{1/2} r} + \frac{2 Q B_2 m - k_z^2}{QB_0} \right] R(r) = 0
\]

To find its solutions, we define these parameters

\[
\frac{1}{4} (1 - \alpha^2) = 3/16 - m^2, \quad \beta = \frac{-\eta \varepsilon}{(Q B_0)^{3/2}}, \quad \delta = \frac{\eta \lambda Q}{(Q B_0)^{1/2}}, \quad \gamma - \frac{\beta^2}{4} = \frac{2 Q B_2 m - k_z^2}{Q B_0},
\]

and follow the same steps as in (28) to (31). Thus the exact eigenvalues are

\[
\varepsilon_{n_\rho,m} = \frac{(2QB_0)^{3/2}}{\eta} \left[ \left( 1 + n_\rho + m + \sqrt{m^2 + 1/16} \right) + \frac{1}{(2QB_o)^{1/2}} \right]^{1/2}
\]

and the exact eigenfunctions are

\[
R(\rho) = N\rho^{1/2} e^{-\left( \frac{Q^2 B_2^2 \rho^2 - m \varepsilon}{2QB_0 m} \right)} H_B \left( \alpha, \beta, \gamma; \sqrt{QB_0} \rho \right)
\]

It is obvious that these eigenvalues (50) are the same as the eigenvalues in the absence of an electric field for the same PDM setting given in (32) but with different eigenfunctions. Thus, the effective potential with the PDM setting does not effect the eigenvalues of the system.
2. Model-II: \( g(\rho) = \eta \rho^2 \)

The substitution of \( g(\rho) = \eta \rho^2 \), in the PDM radial Schrödinger equation (46) would yield

\[
R''(\rho) - \frac{1}{\rho} R'(\rho) + \left[ -\frac{(m^2 - 3/4)}{\rho^2} - \left(Q^2 B_0^2 - \eta \varepsilon\right) \rho^2 - \eta \lambda Q + 2QB_0 m - k_z^2 \right] R(\rho) = 0
\]

(52)

We repeat the same procedure as in the previous section and immediately write the corresponding eigenvalues and radial wave functions, respectively, as

\[
\varepsilon_{n,\rho,m} = \frac{Q^2 B_0^2}{\eta} \left[ 1 - \frac{\sqrt{\lambda Q + k_z^2 - m}}{2QB_0} \right]^{2n} (53)
\]

and

\[
R(\rho) = N\rho^{|\tilde{\ell}|+1} e^{-\sqrt{Q^2 B_0^2 - \eta \varepsilon \rho^2}} F_1\left(-n_{\rho}; |\tilde{\ell}|+1; \sqrt{Q^2 B_0^2 - \eta \varepsilon \rho^2}\right) (54)
\]

where \( |\tilde{\ell}| = \sqrt{m^2 + 1/4} \), with \( \tilde{\ell} > 0 \). In this case, the influence of the PD-effective potential is appeared by making a shift in the energy levels of (42) and producing new eigenvalues (53), therefore.

B. The influence of A linear-type electric field on the Landau-type system:

Now, let us consider another radial electric field (e.g., \([14, 16, 24]\)) as

\[
\vec{E} = \frac{\lambda \rho}{2} \hat{\rho}
\]

(55)

Using the same components of the the electric quadrupole moment tensor defined in (16), the effective scalar potential given in the PDM-Schrödinger equation(22) becomes

\[
V_{\text{eff}}(\rho) = -\vec{Q} \cdot \vec{E} = -\frac{Q\lambda}{2}
\]

(56)

Hence, equation(22) reads

\[
R''(\rho) - \left(\frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho}\right) R'(\rho) + \left[ -\frac{1}{4} \left(\frac{g''(\rho)}{g(\rho)} - \frac{g'(\rho)}{\rho g(\rho)}\right) + \frac{7}{16} \left(\frac{g'(\rho)}{g(\rho)}\right)^2 + g(\rho) \varepsilon + g(\rho) \frac{Q\lambda}{2} - \frac{m^2}{\rho^2} - Q^2 B_0^2 \rho^2 + 2QB_0 m - k_z^2 \right] R(\rho) = 0.
\]

(57)

In the two examples below, we investigate the influence of this effective potential using the same PDM setting that used in the previous sections:

1. Model-I: \( g(\rho) = \eta \rho \)

With \( g(\rho) = \eta \rho \) in (57), we obtain
\[ R''(\rho) + \left[ -\frac{(m^2 - 3/16)}{\rho^2} - Q^2 B_0^2 \rho^2 + \left( \eta \varepsilon + \frac{\eta \lambda Q}{2} \right) \rho + 2 Q B_0 m - k_z^2 \right] R(\rho) = 0 \quad (58) \]

Using the previous technique for the linear-type PDM to get the exact solutions for this case. Hence, this would correspond to the exact eigenvalues and eigenfunctions given, respectively,

\[ \varepsilon_{n,\rho,m} = \frac{(2QB_0)^{3/2}}{\eta} \left[ 1 + n_\rho - m + \sqrt{m^2 + 1/16 + \frac{k_z^2}{2QB_0}} \right]^{1/2} - \frac{\lambda Q}{2} \quad (59) \]

\[ R(\rho) = N \rho^{|\ell| + 1/2} e^{-\frac{QB_0^2}{2QB_0}} H_B(\alpha, \beta, \gamma, 0; \sqrt{QB_0} \rho) \quad (60) \]

where, \( |\ell| = \sqrt{m^2 + 1/16} \) and the parameters \( \alpha, \beta, \gamma \) and \( \delta \) are defined as

\[ \frac{1}{4} \left( 1 - \alpha^2 \right) = 3/16 - m^2, \quad \beta = -\frac{\left( \eta \varepsilon + \frac{\eta \lambda Q}{2} \right)}{(QB_0)^{3/2}}, \quad \gamma = -\frac{\beta^2}{4} = \frac{2QB_0 m - k_z^2}{QB_0}, \quad \delta = 0. \quad (61) \]

2. Model-II: \( g(\rho) = \eta \rho^2 \):

Considering this radial cylindrical PDM would imply that equation (57) be rewritten as

\[ R''(\rho) - \frac{1}{\rho} R'(\rho) + \left[ -\frac{(m^2 - 3/4)}{\rho^2} - \left( Q^2 B_0^2 - \frac{\eta \lambda Q}{2} - \eta \varepsilon \right) \rho^2 + 2Q B_0 m - k_z^2 \right] R(\rho) = 0 \quad (62) \]

Equation (62) is again in the same form of equation (34) and admits the exact solution of the eigenvalues and the corresponding radial eigenfunctions as

\[ \varepsilon_{n,\rho,m} = \frac{Q^2 B_0^2}{\eta} \left[ 1 - \left( \frac{k_z^2}{2QB_0} - \frac{m}{1 + 2n_\rho + \sqrt{m^2 + 1/4}} \right)^2 \right] - \frac{\lambda Q}{2} \quad (63) \]

and

\[ R(\rho) = N \rho^{|\ell| + 1} e^{-\sqrt{QB_0^2 - \frac{\eta \lambda Q}{2} - \eta \varepsilon \rho^2}} F_1\left( -n_\rho, |\ell| + 1; \sqrt{QB_0^2 - \frac{\eta \lambda Q}{2} - \eta \varepsilon \rho^2} \right) \quad (64) \]

where \( |\ell| = \sqrt{m^2 + 1/4} \), with \( |\ell| > 0 \).

It is shown that the effective potential generated by the interaction between the electric quadrupole moment and the radial electric field given by (55) produces constant potential \( -\frac{2\lambda}{2} \). Thus, the effect of mass settings on the effective potential in the term \( (g(\rho) Q \cdot \vec{E}) \) yields only a constant shift in the energy levels given in (32) and (42) creating a new set of energies given in (59) and (63).

V. CONCLUDING REMARKS

In this paper, we have started with a quantum system of an electric quadrupole moment interacting with magnetic and electric fields of a constant mass, as has been previously reported in the literature \[18, 31\]. Next, we have extended
this procedure to study PDM systems by using recent results of [58, 59] for the PDM- minimal-coupling and the PDM-momentum operator given by (9) and (10), respectively. We have discussed the possibility of achieving the Landau quantization following [18] and modified this discussion to include a PDM case. Thus, we have recollected the most important and essential relations (equations (15)-(20) above), that have been reported in [18]. Moreover, we have studied this problem within the context of cylindrical coordinates and investigated the exact solvability of the PDM radial Schrödinger equation of a neutral particle possessing an electric quadrupole moment interacting with external fields, where we have considered PDM settings \( m(\vec{r}) = g(\rho) \), along with the field configurations (documented in (17) for the magnetic field \( \vec{B}(\rho) \) and (44),(55) for electric fields \( \vec{E}(\rho) \)), which exhibits a pure radial cylindrical dependence. We have shown that the Landau quantization is produced from the interaction between the magnetic field and the electric quadrupole moment given in (17) and (18) respectively, where this has yielded the PDM radial Schrödinger equation of this system in the absence of an electric field (documented in (23) above). However, comparing with [18], the energy levels of the Landau quantization are modified because of the influence of the spatial dependence of the mass, where we have obtained different eigenvalues for the two examples of PDM settings (i.e. \( g(\rho) = \eta \rho \) and \( g(\rho) = \eta \rho^2 \)) given in (32) and (42), respectively.

Furthermore, we have analyzed the effect of the interaction of radial electric fields with the electric quadrupole moment of a PDM neutral particle by choosing two particular cases of the electric field \( \vec{E} = \frac{\lambda}{2} \hat{\rho} \) and \( \vec{E} = \frac{\lambda}{2} \hat{\rho} \), which have produced the effective potentials. It is shown that the structure of the PD effective potential term, plays the role of scalar potentials in the PDM radial Schrödinger equation (22), producing same or an energy shift in energy levels of the systems with the same mass settings and in the absence of the effective potential. We have observed that the difference in energy levels depends on the mass structure. The more complex the chosen mass, the more this difference will be over Landau levels. However, the exact eigenvalues and eigenfunctions for all these cases are obtained.

Finally, although the presence of an effective uniform magnetic field produces the Landau quantization, the influence of the spatial dependence of the mass of the system yields a new contribution to energy levels creating a set of new eigenvalues.

This study has investigated, for the first time, that the PDM quantum particle that possesses multipole moments under the influence of external fields. Thus, this work opens new discussions regarding the position-dependent concept and provides a good starting point for future research.
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