A kind of Upwind Finite Element Approximations for Compressible Flow of Contamination from Nuclear Waste in Porous Media

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ABSTRACT

A non-linear parabolic system is derived to describe compressible nuclear waste disposal contamination in porous media. Galerkin method is applied for the pressure equation. For the concentration of the brine of the fluid, a kind of partial upwind finite element scheme is constructed. A numerical application is included to demonstrate certain aspects of the theory and illustrate the capabilities of the kind of partial upwind finite element approach.

Keywords: Galerkin method; A kind of partial upwind finite element scheme; Non-linear parabolic system

1. Introduction

The proposed disposal of high-level nuclear waste in underground repositories is an important environmental topic for many countries. Decisions on the feasibility and safety of the various sites and disposal methods will be based, in part, on numerical models for describing the flow of contaminated brines and groundwater through porous or fractured media under severe thermal regimes caused by the radioactive contaminants.
A fully discrete formulation is given in some detail to present key ideas that are essential in code development. The non-linear couplings between the unknowns are important in modeling the correct physics of flow.

In this model one obtain a convection-diffusion equations which represent a mathematical model for a case of diffusion phenomena in which underlying flow is present; \( \Delta w \) and \( b \nu w \) correspond to the transport of \( w \) through the diffusion process and the convection effects, respectively, where \( \nabla \) and \( \Delta \) denoted respectively the gradient operator and the Laplacian operator in the spatial coordinates.

In this paper we will consider the fluid flow in porous media using a Galerkin method for the pressure equation and a kind of partial upwind finite element scheme is constructed for the convection dominated saturation (or concentration) equation. For more details of this subject see \([7, 6, \text{and } 4]\).

2. Model Equations

The model for compressible flow and transport of contaminated brine in porous media can be described by a differential system that can be put into the following form \([2]\).

Fluid:
\[
\phi_1 \frac{\partial p}{\partial t} + \nabla \cdot u = -q + R_s
\]
\[
u = -\frac{k(x)}{\mu(c)} \nabla p = -a(c) \nabla p
\]  

... (1)

Brine:
\[
\phi \frac{\partial c}{\partial t} + u \nabla c - \nabla (E_c \nabla c) = g(c)
\]  

... (2)

Radionuclide:
\[
\phi K_i \frac{\partial c_i}{\partial t} + u \nabla c_i - \nabla (E_c \nabla c_i) + d_3 (c_i) \frac{\partial p}{\partial t} = f_i (c, c_1, \ldots, c_N)
\]  

... (3)

Heat:
\[
d_1 (p) \frac{\partial p}{\partial t} + d_2 \frac{\partial T}{\partial t} + e \rho u \nabla T - \nabla (E_H \nabla T) = Q(u, T, c, p)
\]  

... (4)

Where \( x \in \Omega \subset \mathbb{R}^2, t \in (0, T] \), \( p = p(x, t) \) is the pressure, \( T = T(x, t) \) is the temperature of the fluid, \( c = c(x, t) \) is the concentration of the brine in the
fluid, \( c_i = c_i(x,t) \) is the trace concentration of the \( i \)-th radionuclide, \( i=1,2,...,N \), \( u \) is the Darcy velocity, \( \phi_i = \phi\phi_i \), \( \phi = \phi(x) \), \( c_w \) is the concentration of water, \( k = k(x) \) is the permeability of the rock, \( \mu(c) \) is the viscosity of the fluid, \( q = q(x,t) \) is a production term, \( R'_i = R'_i(c) = [c_i \phi K c_i f_i / (1 + c_i)](1-c) \) is a salt dissolution term, see [3].

\[
d_2 = \phi c_p + (1-\phi) \rho_p c_p R_t, \quad d_1(p) = -\phi c_w [U_0 + (p / \rho)], \quad d_3(c_i) = \phi c_i (K_i - 1)
\]

\( E_H = D c_m + K_m I \), \( K_m = k_m / \rho_0 \), \( D = (D_y) = (\alpha_T u) \delta_y + (\alpha_L - \alpha_T) u, j / |u| \).

and \( Q(u,T,c,p) = -[(\nabla U_0 - c_p \nabla T_0) u + [U_0 + c_p(T - T_0)] + (p / \rho)[-q + R'_i]] \)

\( -q_L - q_H - q_H \).

\( E_c = D + D_m I \), and \( g(c) = c_i ([c_i \phi K c_i f_i / (1 + c_i)](1-c)) - q_{c_i} + R'_i \),

and \( f_i(c,c_1,c_2,...,c_N) = c_i [q - [c_i \phi K c_i f_i / (1 + c_i)](1-c)] - q_{c_i} - q_{c_i} + q_{c_i} \)

\[ \sum_{j=1}^N k_{ij} c_j (\phi K c_j - \phi K c_i) \]

Shifting with the boundary conditions

(a) \( u.n = 0 \) on \( \Gamma \)
(b) \( (E \nabla c - c u).n = 0 \) on \( \Gamma \)
(c) \( (E \nabla c_i - c_i u).n = 0 \) on \( \Gamma \)
(d) \( (E \nabla T - c_p T u).n = 0 \) on \( \Gamma \)
(e) \( \partial \rho / \partial n = \alpha \); \( (x,t) \in \Gamma \times (0, T) \)

\[ \text{(5)} \]

Shifting and the initial conditions

(a) \( p(x,0) = p_0(x) \); \( x \in \Omega \)
(b) \( c(x,0) = c_0(x) \); \( x \in \Omega \)
(c) \( c_i(x,0) = c_{0i}(x) \); \( x \in \Omega \)
(d) \( T(x,0) = T_0 \); \( x \in \Omega \)

\[ \text{(6)} \]

The reservoir \( \Omega \) will be taken to be of unit thickness and will be identified with a bounded domain in \( \mathbb{R}^2 \). We shall omit gravitational terms for simplicity of exposition, no significant mathematical questions arise the lower order terms are included.

We assume that

(A1) \( \phi(c), R'_i(c), g(c), E_c \in C^1(\mathbb{R}) \).

\( \phi(x), \phi_i(x), E_c \in H^1(\Omega), q \in L^\infty(0,T;H^1(\Omega)) \)

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0 < c_0 \leq a(c), R(c), g(c), \phi_1, E, c \leq c_1, \forall c_0, c_1 \in R, x \in \Omega
\quad D_m > 0 \ and \ \alpha_i \geq \alpha, > 0

(A2) The solution of the problem (from eq.(1) to eq.(6)) is regular:
\quad c(x,t) \in L^2(0,T;H^2(\Omega)) \cap L^\infty(0,T;\omega^1_\infty(\Omega))
\quad p(x,t) \in L^p(0,T;H^{r+1}(\Omega)), (r \geq 2)
\quad c_i, c_m, c_{m,} \in L^\infty(0,T;H^1(\Omega)) \ : p_i, p_m \in L^\infty(0,T;L^\infty(\Omega))

(A3) For any \( \phi \in L^2(\Omega) \), the boundary value problem:
\quad -\Delta \phi + \phi = \phi, \ x \in \Omega
\quad \frac{\partial \phi}{\partial n} = 0, \ x \in \Gamma

there exists a unique solution \( \phi \in H^2(\Omega) \) and a positive constant M
such that \( \|\phi\|_2 \leq M\|\phi\| \).see [8]

3. Finite Element Spaces

Consider a regular family \( \{T_h\} \) of triangulation defined over \( \Omega \), where
h is the longest diameter of a triangular element with the triangular \( T_h \), we
have a set of close triangles \( \{e_i\} (1 \leq i \leq N_e) \) and a set of nodes
\( \{p_i\} (1 \leq i \leq N_p + M_p) \) where \( p_i \) (1 \leq i \leq N_p) are interior nodes in \( \Omega \) and
\( p_j \) (1 \leq j \leq N_p + M_p) are boundary nodes on \( \Gamma \). We put \( h_s \) to be the
maximum side length of triangles and k to be minimum perpendicular
length of triangles for all \( e \in T_h \).

Definition(3.1): A family \( T_h \) of triangulation is of weakly acute type, if
there exists a constant \( \theta_0 > 0 \) independent of \( h \) such that, the internal angle
\( \theta \) of any triangle \( e_i \in T_h \) satisfies \( \theta_0 \leq \theta \leq \pi/2 \).

Let \( \phi_i(p_i) (1 \leq i \leq M) \), be the continuous function in \( \Omega \) s.t. \( \phi_i(p_i) \) is linear
on each \( e \in T_h \) and \( \phi_i(p_j) = \delta_{ij} \) for any nodal point \( p_i \). We denote \( M_h \) the linear
span of \( \phi_i (1 \leq i \leq M) \), i.e., a finite dimensional subspace of \( H^1(\Omega) \)
\( M_h = \{z_h | z_h \in C(\Omega); z_h \ is \ a \ linear \ functionone, \forall e \in T_h \} \).

And a subspace of \( H^1_0(\Omega) \)
\( M_{0h} = \{z_h | z_h \in M_h; z_h(p_j) = 0, k = M + 1, \ldots, K \} \).
We associate the index set \( \Lambda = \{ j \neq i: P_j \ is \ adjacent \ to \ P_i \} \) Let \( P, P_j, P_k \), be three
vertices of triangular element \( e \) and \( \lambda_i, \lambda_j, \lambda_k \) be barycentric coordinates. We
have the following definitions see [5].
Definition (3.2): with each vertex $P_i$ belonging to triangle $e$, the barycentric subdivision $\Omega_i$ is given by:

$$\Omega_i^e = \{ P \mid P \in e : \lambda_i(P) \geq \lambda_j(P), \lambda_i(P) \geq \lambda_k(P), \forall P \in e \},$$

and the barycentric domain $\Omega_i$ associated with vertex $P_i$ is given by $\Omega_i = \cup \Omega_i^e, e \in T_h$.

Definition (3.3): with the characteristic function $\mu_i(x)$ of barycentric domain $i$,

$$\text{mass lumping operator } \wedge : w \in C(\Omega) \to \hat{w} \in L_\infty(\Omega) \text{ is defined by } \hat{w}(p) = \sum_i w(p_i)\mu_i(p).$$

Definition (3.4): Let $\{M_h\}$ be a family of finite dimensional subspaces of $C(\Omega)$, which is piecewise polynomial space of degree less or equal to $r$ with step length $h_P$ and the following property: for $P \in [1, \infty)$, $r \geq 2$, there exists a constant $M$ such that for $0 \leq q \leq 2$ and $\phi \in H^{r+1}(\Omega)$:

$$\inf_{x \in M_h} \| \phi - x \|_{q,p} \leq M h^{r+1-q} \| \phi \|_{r+1,p}.$$

Similarly, we define $\{N_h\}$ be a family of finite-dimensional subspace of $C(\Omega) \times C(\Omega)$, which is piecewise polynomial space of degree less or equal to $r-1$ with the similar property as $M_h$ and $0 \leq q \leq r-1$. We also assume the families $\{M_h\}$ and $\{N_h\}$ satisfy inverse inequalities:

$$\| \phi \|_{L^2} \leq M h^{-1} \| \phi \| \quad \| \nabla \phi \|_{L^2} \leq M h^{-1} \| \nabla \phi \| \quad \forall \phi \in M_h$$

see [8].

Lemma (3.1): [5]

There exists a constant $C$ such that:

$$\| w - \hat{w} \|_{0,p} \leq Ch_p \| w \|_{1,p}, \quad \forall w \in M_h, p \geq 1 \quad \ldots(7)$$

$$\| w_h \|_{0} \leq M h^{-1} \| w_h \|, \quad \forall w_h \in M_{0h} \quad \ldots(8)$$

Lemma (3.2): [8]

There exists constants $C_1, C_2 > 0$ such that:

$$C_1 \| w \|_{0,p} \leq \| \hat{w} \|_{0,p} \leq C_2 \| w \|_{0,p}, \quad \forall w \in M_h \quad \ldots(9)$$

Lemma (3.3): [10]

Let $\overline{p} \in V_h$ be the elliptic projection of $p \in H^1(\Omega)$ into $V_h$ defined by $$(a(c)\nabla \overline{p}, \nabla v) = (a(c)\nabla p, \nabla v), \quad \forall v \in V_h \quad \text{then there exists a constant } k_1 \text{ such that}$$

$$\| p - \overline{p} \| + h_p \| \nabla p - \nabla \overline{p} \| \leq k_1 \| p \|_{r+1,h_p}.$$
4. Error Estimates

Let $\tau > 0$ be time step and $N_\tau = \frac{T}{\tau}$. We use a Galerkin finite element method for the pressure and velocity and partial upwind finite element scheme for brine.

Let $c^0 \in M_h$ be a $L^2(\Omega)$ - projection of $c^0$ in $M_h$:

$$(c^0 - C^0, z_h) = 0 \quad \forall z_h \in M_h.$$

We can get $P^0 \in V_h$ such that

$$\int_\Omega P^0 dx = 0 \quad (\frac{k(x)}{\mu(c^0)}) \nabla P^0 \cdot \nabla v = (-q^0, v) + (R^0, v) \quad \forall v \in V_h \text{ and } U^0 \in W_h$$

from

$$U^0 = -\frac{k(x)}{\mu(c^0)} \nabla P^0 = -a(C^0) \nabla P^0.$$

If the approximate solution $\{P^m, U^m, C^m\} \in V_h \times W_h \times M_h$ is known, we want to find $\{P^{m+1}, U^{m+1}, C^{m+1}\} \in V_h \times W_h \times M_h$ at $t = t^{m+1}$, with three steps. Let $(...)_\alpha$ denote the inner product in $L^2(\Omega)$.

Step 1. Find $C^{m+1}$ for $m = 0, 1, \ldots, N_\tau - 1$, such that

$$(\Delta c^m, \hat{z}_h) + (E_c, \nabla C^{m+1/2}) + R(U^m, C^{m+1/2}, \hat{z}_h) = (\Delta(c^m), \hat{z}_h) \quad \forall \hat{z}_h \in M_h \quad \text{...(10)}$$

where $D_c C^m = (C^{m+1} - C^m)/\tau$, $C^{m+1/2} = (C^{m+1} + C^m)/2$ and

$$R(U^m, C^{m+1/2}, \hat{z}_h) = \sum_{i=1}^{N_\tau} \sum_{j \in \Lambda_i} \beta_{ij}^m (\alpha_{ij}^m C_i^{m+1/2} + \alpha_{ij}^m C_j^{m+1/2})$$

with $z_i = \beta_i^m(P_i)$, $C_i^{m+1/2} = C^{m+1/2}(P_i)$, and $\beta_{ij}^m = \int_{\Gamma_{ij}} U^m_n d\Gamma$, here $n_{ij}$ is the unit outer normal to $\Gamma_{ij}$. The partial upwind coefficients should be required that [5].

(a) $\alpha_{ij}^m + \alpha_{ij}^m = 1$

(b) $\max\{1/2, 1 - \rho_{ij}^{-1}\} \leq \alpha_{ij} \leq 1, \text{iff } \beta_{ij} \geq 0,$

$$\max\{1/2, 1 - \rho_{ij}^{-1}\} \leq \alpha_{ij} \leq 1, \text{iff } \beta_{ij} < 0,$$

...(11)

Step 2. Find $P^{m+1}$ such that:

$$\int_\Omega P^{m+1} dx = 0 \quad (\frac{k(x)}{\mu(C^{m+1})}) \nabla P^{m+1} \cdot \nabla v = (-q^{m+1}, v) + (R^m, v) \quad \forall v \in V_h \quad \text{...(12)}$$

Step 3. Find $U^{m+1}$ as:

$$U^{m+1} = -a(C^{m+1}) \nabla P^{m+1} \quad \text{...(13)}$$

Let $\bar{c}: J \rightarrow M_h$ be determined by the relations

$$(E_c, \nabla (c - \bar{c}), \nabla c) + \lambda(c - \bar{c}, z) = 0 \quad \forall z \in M_h \quad \text{...(14)}$$

$$...$$
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for \( t \in J \), where the constant \( \lambda \) is chosen to be large enough to insure the coercivity of the bilinear form over \( H^1(\Omega) \). Similarly, let \( \tilde{p}: J \rightarrow V_h \) satisfy
\[
a(\cdot, \nabla (p - \tilde{p}), \nabla v) + \mu (p - \tilde{p}, v) = 0, \quad \forall v \in V_h
\]
where \( \mu \) assures coercivity over \( H^1(\Omega), \text{and} \ (\tilde{p}, 1) = (p, 1) \). Let \( \zeta = c - \vec{c}, \ \xi = \vec{c} - C, \ \eta = p - \tilde{p}, \ \pi = \tilde{p} - P \)
\[
(15)
\]
A standard result in the theory of Galerkin methods give [2].

\[
\| \xi \|_{0} + h_{\mu} \| \xi \|_{0} \leq M \| \xi \|_{2} h_{\xi}^{2}
\]
\[
\| \eta \|_{0} + h_{\mu} \| \eta \|_{0} \leq M \| \eta \|_{r+1} h_{\eta}^{r+1}
\]
\[
\| \pi \|_{\infty} \leq M
\]

for \( t \in J \), where the constant \( M \) depends on bounds for lower order derivatives of \( p, c \). And also
\[
\| \frac{\partial \xi}{\partial t} \|_{0} + h_{\mu} \| \frac{\partial \xi}{\partial t} \|_{0} \leq M (\| \xi \|_{2} + \| \frac{\partial \xi}{\partial t} \|_{2}) h_{\xi}^{2}
\]
\[
(18a)
\]
\[
\| \frac{\partial \eta}{\partial t} \|_{0} + h_{\mu} \| \frac{\partial \eta}{\partial t} \|_{0} \leq M (\| \eta \|_{r+1} + \| \frac{\partial \eta}{\partial t} \|_{r+1}) h_{\eta}^{r+1}
\]
\[
(18b)
\]
where \( M \) now depends on bounds for lower order derivatives of \( p, c \) and their first derivatives with respect to time.

**Lemma (4.1):** There exists a positive constant \( k_2 \) such that:
\[
\| \nabla p_{m+1} - \nabla \tilde{p}_{m+1} \| \leq k_2 \| c_{m+1} - c_{m+1} \|
\]
**Proof:** see [1].

**Lemma (4.2):** For all \( z_h \in M_h \) and \( \zeta = c - \vec{c}, \ \xi = \vec{c} - C, \)
\[
(\mu \nabla c_{m+1/2}, z_h) - R(U^m, C_{m+1/2}, z_h) \leq M (h_{\xi}^{2} + \| \frac{\partial \xi}{\partial t} \|_{2}^{2} + \| \frac{\partial \xi}{\partial t} \|_{2}^{2}) + \| \nabla \xi_{m+1/2} \|_{2}^{2} + \| u^m - U^m \|_{2}^{2} + \| z_h \|_{2}^{2} + \varepsilon \| \nabla z_h \|_{2}^{2}
\]
where \( \varepsilon > 0 \) is arbitrary small constant
**Proof:** [1].

**Notes:**
1. The inductive assumption [10],\( \| U^{l} \|_{L_{\infty}} \leq k^* \ (0 \leq l \leq m) \), then \( \| U^{m+1} \|_{L_{\infty}} \leq k^* \)
2. If \( T_h \) is regular triangulation of weakly acute type we have
\[ |w_h| \leq \sqrt{\frac{\delta}{\kappa}} \|w_h\|, \quad \forall w_h \in M_h, \quad \text{see [8].} \]

**Lemma (4.3):** There exists a positive constant \( k_3 \) such that
\[
\| u^{m+1} - u^{m+1} \| \leq k_3 \| c^{m+1} - c^{m+1} \| + h_p^r .
\]

**Proof:**
\[
\| u^{m+1} - u^{m+1} \| = \| a(c^{m+1})V c^{m+1} - a(c^{m+1})V c^{m+1} \| \\
\leq \| a(c^{m+1})V (p^{m+1} - p^{m+1}) \| + \| a(c^{m+1}) - a(c^{m+1}) \| \| V p^{m+1} \| 
\]
From (A1) and (A2) we have
\[
\leq \text{const} \| V p^{m+1} - V p^{m+1} \| + \| c^{m+1} - c^{m+1} \| \| V p^{m+1} \|_{L_0}
\]
we have \( \| V p^{m+1} - V p^{m+1} \| \leq \| V p^{m+1} - V p^{m+1} \| + \| V p^{m+1} - V p^{m+1} \| \)
using lemma (3.3) and lemma (4.1), we get
\[
\| V p^{m+1} - V p^{m+1} \| \leq k_2 \| c^{m+1} - c^{m+1} \| + k_1 \| p^{m+1} \|_{L_1} h_p^r , \quad \text{from (A2)},
\]
\[
\| u^{m+1} - u^{m+1} \| \leq k_3 \| c^{m+1} - c^{m+1} \| + h_p^r .
\]

**Theorem (4.1):** For all \( m \leq l \leq N_\tau \) if \( \tau \leq \tau_0 \), then
\[
\| e^{l+1} - C^{l+1} \| \leq M (\tau + h_c + h_p^r) , \quad \text{where M is independent of } \tau \text{ and } h_c .
\]

**Proof:**
Multiply eq. (2) by \( z_h \) and integrating by parts we obtain for \( t = (m + 1/2) \tau \). Let \( w_{m+1/2} = w_{t, (m + 1/2) \tau} \) and \( w_{m+1/2} = (w_{m+1} + w_m)/2 \), then
\[
(\phi D^m c, z_h) + (Ec V c^{m+1/2}, z_h) + (u^m V c^{m+1/2}, z_h) = \\
(\phi (D^m c, z_h) + (Ec V c^{m+1/2}, z_h) + \\
(2/1 + (Ec V c^{m+1/2} - Ec V c^{m+1/2}, z_h) + ((u^m V c^{m+1/2} - u^m V c^{m+1/2}, z_h)
\]
Let \( e = c - C = c - C = c + \xi \), and subtract (10) from (19), we obtain:
\[
(\phi D^m c, z_h) + (Ec V c^{m+1/2}, z_h) = (R(U^m, c^{m+1/2}, z_h) \\
- (u^m V c^{m+1/2}, z_h)) + ((\phi D^m c, z_h) - (\phi D^m c, z_h)) \\
+ ((\phi D^m c, z_h) - (\phi D^m c, z_h)) + (\phi D^m c, z_h) + (\phi D^m c, z_h)
\]
Hence
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\[
(\phi D \tilde{z}_h) + (Ec \nabla z_m, \nabla z_h) = -(\phi D \tilde{z}_h) - (Ec \nabla z_m, \nabla z_h) - 
\]

\[
((u^m \nabla c_m, z_h) - R(U^m, C^{m+1/2}, z_h)) - ((\tilde{g}(c_{m+1/2}), z_h) - (g(c_{m+1/2}), z_h)) - 
\]

\[
(\phi D \epsilon_m, z_h) - (\phi D \epsilon_h, z_h) + (\phi D \epsilon_m - \epsilon_h, z_h) + 
\]

\[
((Ec \nabla c_{m+1/2} - Ec \nabla c_{m+1}), \nabla z_h) + ((u^m \nabla c_{m+1/2} - u^{m+1} \nabla c_{m+1/2}), z_h) = I1 + I2 + I3 + I4 + I5 + I6
\]

In (20), let \(z_h = \tilde{z}_{m+1/2} \in M_h\), and using (A1) the left – hand side is

\[
\geq \frac{1}{2} \phi (\xi_{m+1/2}^2 - \xi_m^2) + c_0 \|\nabla z_{m+1/2}\|^2
\]

from (A1), we have:

\[
I1 = (\phi D \tilde{z}_h) \leq M \left( \|D \tilde{z}_h\|^2 + \|\tilde{z}_{m+1/2}\|^2 \right)
\]

using (7) and (8), we have

\[
\|D \tilde{z}_h\|^2 \leq \|D \tilde{z}_h\|^2 + \|D \tilde{z}_h - D z_h\| \leq \|D \tilde{z}_h\|^2 + M h \|D \tilde{z}_h\|
\]

From (9) we have

\[
\|\tilde{z}_{m+1/2}\| \leq M \|z_{m+1/2}\| \leq M \left( \|z_{m+1/2}\| + \|z_m\| \right)
\]

\[
I1 \leq M \left( \|z_{m+1}\|^2 + \|z_m\|^2 + \|D \tilde{z}_h\|^2 + h^2 \|D \tilde{z}_h\|^2 \right)
\]

Using (A1), we have:

\[
I2 \leq M \|\nabla z_{m+1/2}\|^2 + \epsilon \|z_{m+1/2}\|^2
\]

From lemma (4.2)

\[
I3 \leq M (h^2 + \|z_{m+1/2}\|^2 + \|z_{m+1/2}\|^2 +
\]

\[
\|\nabla z_{m+1/2}\|^2 + \|z_{m+1}\|^2 + \epsilon \|\nabla z_{m+1/2}\|^2
\]

let \(\theta = \phi \frac{c_{m+1} - c_m}{c} = \phi \left( \frac{c}{c} \right) + 1/24 \frac{\beta^3}{c} \|z_{m+1}\|^2\)

using (A1), (A2) and [11], we get

\[
I4 \leq M h \|z_h\| + M h \|z_{m+1}\| + M h \|z_{m+1}\| + M h \|z_{m+1}\| + M h \|z_{m+1}\|
\]

\[
\leq M \left( h^2 + h^4 \right) + \epsilon \|\nabla z_{m+1/2}\|^2
\]

\[
I5 \leq M \left( h^2 + \|z_{m+1/2}\|^2 + \|z_{m+1/2}\|^2 \right) + \epsilon \|\nabla z_{m+1/2}\|^2
\]

let I6 = K1 + K2 + K3, so

\[
K1 \leq M (\tau^4 + \|z_{m+1/2}\|^2 + \|z_{m+1/2}\|^2), K2 \leq M \tau^2 + \epsilon \|z_{m+1/2}\|^2, K3 \leq M (\tau^2 + \|z_{m+1/2}\|^2 + \|z_{m+1/2}\|^2)
\]

then we get

...
Equation (20) can be written now

$$\phi\left(\frac{\|\xi^{m+1}\|}{2} - \frac{\|\xi^m\|^2}{2}\right) + c_0\|\nabla \xi^{m+1/2}\|^2 \leq M \left(\|\xi^{m+1}\|^2 + \|\xi^m\|^2 + \|D_x \xi^{m+1}\|^2 + \|D_x \xi^m\|^2 + \|\nabla \xi^{m+1}\|^2 + \|\nabla \xi^m\|^2 + \|\nabla^2 \xi^{m+1}\|^2 + \|\nabla^2 \xi^m\|^2\right)$$

Take the summation from 0 to 1, where $m \leq l \leq N_{x}$, and $C_0 = \tilde{c}^0$ so $\xi^0 = 0$

$$\|D_x \xi^{l+1}\|^2 \leq M_1 \sum_{m=0}^{l+1} \|\xi^m\|^2 \tau + M_2 \|D_x \xi^{l+1}\|^2 + M_3 h_2^2 \|D_x \xi^{l+1}\|^2$$

$$+ M_2 \sum_{m=0}^{l} \|\xi^m\|^2 \tau + M_3 (h_2^2 + \tau^2) + M_3 \sum_{m=0}^{l} \|\mu^m - U^m\|^2 \tau$$

where $\|\xi^m\|^2_{\ell^2} = \sum_{m=0}^{N_x} \|\xi^m\|^2 \tau$, using (7) and (A1), we get

$$\|D_x \xi^{l+1}\|^2 \leq \tilde{c}^{l+1} + M_4 \|\xi^m\|^2 \tau + \tilde{c}^{l+1} + M_4 \|\xi^m\|^2 \tau + \|\mu^m - U^m\|^2 \tau$$

$$\|D_x \xi^{l+1}\|^2 \geq M_0 \|\xi^{l+1}\|^2$$

From lemma (4.3) we have

$$\sum_{m=0}^{l} \|\mu^m - U^m\|^2 \tau \leq M \sum_{m=0}^{l} \|\xi^m\|^2 \tau + \|\xi^m\|^2 \tau + \|\mu^m - U^m\|^2 \tau$$

From (21) and (22), we have

$$\|\xi^{l+1}\|^2 \leq M_1 \sum_{m=0}^{l+1} \|\xi^m\|^2 \tau + M_2 (h_2^2 + \tau^2 + h_2^2 \tau)$$

from Gronwall inequality, we get:

$$\|\xi^{l+1}\|^2 \leq M (h_2 + \tau + h_2 \tau)$$

it is $\|\xi^{l+1} - C^{l+1}\|^2 \leq M (h_2 + \tau + h_2 \tau)$

**Theorem (4.2):** For all $m \leq l \leq N_{x}$ if $\tau \leq \tau_0$, then

$$\|\xi^{l+1}\| + \|\mu^{l+1}\| + \sum_{m=0}^{l+1} \|\nabla \xi^m\|^2 + \|\nabla \mu^m\|^2 \tau \leq M (\tau + h_2 + h_2 \tau)$$

where $M$ independent of $\tau$, $h_2$, and $h_2$.

**Proof:**

We shall begin by deriving an evolution inequality for the difference $\mu$ between the projection $\tilde{p}$ and the approximation solution $P$. The weak form of eq.(1) is
As the test function in (23), we have from (17)

\[(\phi_1 \frac{\partial p}{\partial t}, v) + (a(c)\nabla p, \nabla v) = (q, v) + (R_i, v) ; \forall v \in H^1(\Omega)\]

so there can be difference between equations (12) and (15) to show that.

\[(\phi_1 D_T \pi^m, v) + (a(C^m+1)\nabla \pi^{m+1}, \nabla v) = -(a(c^m)) \nabla p^{m+1}, \nabla v) -
\]

\[(\phi_1 D_T \eta^{m+1}, v) + \mu (\eta^{m+1}, v) + (\phi_1 (D_T p^m - \frac{\partial p}{\partial t}, \nabla v) \quad \cdots \cdots \cdots \cdots \cdots \cdots (23)\]

\[= I1 + I2 + I3 + I4\]

Select $\pi^{m+1}$ as the test function in (23), we have from (17-c) and (A1)

\[I1 \leq M \| \pi^{m+1} - C_{m+1} \nabla \pi^{m+1} \| \]

\[\leq M \| \pi^{m+1} - C_{m+1} \nabla \pi^{m+1} \| \]

\[\leq M \| \pi^{m+1} \| + \mu \| \nabla \pi^{m+1} \|^2 \]

From (A1), we have

\[I2 \leq M \| D_T \pi^m - \frac{\partial p}{\partial t} \| \]

\[\leq M \| D_T \pi^m - \frac{\partial p}{\partial t} \| \]

and

\[I3 \leq M \| \eta^{m+1} \| \pi^{m+1} \| \]

\[\leq M \| \eta^{m+1} \| + \| \pi^{m+1} \|^2 \]

From (A1),(A2) and using [9], we get

\[I4 \leq M \| D_T p^m - \frac{\partial p}{\partial t} (t_{m+1}) \| \]

\[\leq M \| \pi^{m+1} \|^2 \leq M (\tau^2 + \| \pi^{m+1} \|^2) \]

Now, (23) can be written as

\[\frac{c_0}{\tau} \| \pi^{m+1} \|^2 - \| \pi^m \|^2 + c_0 \| \nabla \pi^{m+1} \|^2 \leq c_0 \| D_T \pi^{m+1} \|^2 + \| a(C^{m+1}) \nabla \pi^{m+1} \|^2 \]

\[= M \| \pi^{m+1} \|^2 + \| D_T \pi^{m+1} \|^2 + \| a(C^{m+1}) \nabla \pi^{m+1} \|^2 \]

Now by using prove of theorem (4.1) we get

\[\frac{c_0}{2\tau} \| \pi^{m+1} \|^2 - \| \pi^m \|^2 + c_0 \| \nabla \pi^{m+1} \|^2 \leq M \| \pi^{m+1} \|^2 + \| D_T \pi^{m+1} \|^2 + \| a(C^{m+1}) \nabla \pi^{m+1} \|^2 \]

\[+ \| D_T \pi^{m+1} \|^2 + \| a(C^{m+1}) \nabla \pi^{m+1} \|^2 + \| \pi^{m+1} \|^2 \]

\[+ M \| \pi^{m+1} \|^2 + \| D_T \pi^{m+1} \|^2 + \| a(C^{m+1}) \nabla \pi^{m+1} \|^2 \]

\[\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdOTS
Combine equation (24) and (25) and where \( \varepsilon' > 0 \) is arbitrary small constant, move \( \frac{c_0}{2} \| \nabla \varepsilon_{m+1} \|^2 \) into left side and take \( \varepsilon' > 0 \) small enough such that \( c_0 - \varepsilon' > 0 \):

\[
\frac{\varepsilon_0}{\tau} \left[ \| \mu^{m+1} \|^2 - \| \pi^m \|^2 \right] + \frac{\varepsilon_0}{2\tau} \left[ \| \varepsilon_{m+1} \|^2 - \| \varepsilon_m \|^2 \right] + c_0 \| \nabla \pi^{m+1} \|^2 + \frac{\varepsilon_0}{2} \| \nabla \varepsilon_{m+1/2} \|^2 \leq M \left( \| \varepsilon_{m+1} \|^2 + \| \pi^m \|^2 + \| \pi^{m+1} \|^2 \right) + D_\tau \varepsilon^m \|
\]

\[
+ h_2^2 D_\tau \varepsilon_{m+1/2} \|^2 + \| D_\tau \eta^m \|^2 + \| \eta^{m+1} \|^2 + \| \mu^m - U^m \|^2 + h_2^2 + \tau^2 \)
\]

\[
+ \varepsilon' \left( \| \nabla \pi^{m+1} \|^2 + \| \nabla \varepsilon_{m+1} \|^2 \right)
\]

Then

\[
\| \varepsilon_{m+1} \|^2 + \| \pi^{m+1} \|^2 + \sum_{m=0}^{l+1} \| \nabla \varepsilon^m \|^2 + \| \nabla \pi^m \|^2 \leq M_i \left( \| \varepsilon^m \|^2 + \| \pi^m \|^2 \right) + M_2 \left( \| \varepsilon_{m+1} \|^2 + \| \pi^{m+1} \|^2 \right) + D_\tau \varepsilon \|
\]

\[
+ h_2^2 D_\tau \varepsilon_{m+1/2} \|^2 + \| D_\tau \eta^m \|^2 + \| \eta^{m+1} \|^2 + \| \mu^m - U^m \|^2 + h_2^2 + \tau^2 \tau
\]

Take the summation from 0 to 1, where \( m \leq l \leq N_z \), and \( C^0 = \dot{c} \) so \( \dot{c} = 0 \), \( \pi^0 = \pi^0 \) so \( \pi^0 = 0 \)

\[
\| \varepsilon^{l+1} \|^2 + \| \pi^{l+1} \|^2 + \sum_{m=0}^{l+1} \| \nabla \varepsilon^m \|^2 + \| \nabla \pi^m \|^2 \leq M_i \left( \sum_{m=0}^{l+1} \| \varepsilon^m \|^2 + \| \pi^m \|^2 \right) + M_2 \left( \sum_{m=0}^{l+1} \| \mu^m - U^m \|^2 \right) + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)} + \| \varepsilon \|_{L^2(H^1)}
\]

where

\[
\| \varepsilon \|_{L^2(H^1)} = \sum_{m=0}^{N_z} \| \varepsilon^m \|^2 \leq k_3 \left( \sum_{m=0}^{l+1} \| \varepsilon^m \|^2 + \| \varepsilon \|^2 \right) \| \eta \|^2 \|
\]

from lemma (4.3), we have

\[
\sum_{m=0}^{l+1} \| \varepsilon^m \|^2 \| \pi^m \|^2 \leq k_3 \left( \sum_{m=0}^{l+1} \| \varepsilon^m \|^2 + \| \varepsilon \|^2 \right) \| \eta \|^2 \|
\]

using (17)(18) we get

\[30\]
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\[ \| \xi^{t+1} \|^2 + \| \pi^{t+1} \|^2 + \sum_{m=0}^{l+1} \| \nabla \xi^m \|^2 + \sum_{m=0}^{l+1} \| \nabla \pi^m \|^2 \leq \]
\[ M_1 \sum_{m=0}^{l+1} (\| \xi^m \|^2 + \| \pi^m \|^2) + M_2(h^2 + \eta^2) + \tau^2 + \epsilon \sum_{m=0}^{l+1} (\| \nabla \xi^m \|^2 + \| \nabla \pi^m \|^2) \tau \]

From Gronwall inequality, for \( \tau \) and \( \epsilon \) small enough, we get

\[ \| \xi^{t+1} \| + \| \pi^{t+1} \| + \sum_{m=0}^{l+1} \| \nabla \xi^m \| + \sum_{m=0}^{l+1} \| \nabla \pi^m \| \tau \leq M(h_c + h_p + \tau) \]

where \( M \) independent of \( \tau, h_c \) and \( h_p \).

**Theorem (4.5):** with the assumption (A1)~(A3), if \( h_c = O(h_p), \tau = O(h_p) \), then

\[ \| c - C \|_{L^2(i)} + \| p - P \|_{L^2(i)} + \| u - U \|_{L^2(i)} + \| c - C \|_{L^2(H)} + \| p - P \|_{L^2(H)} \leq M(\tau + h_c + h_p) \]

**Proof:**

With \( c - C = \zeta + \eta \) and \( p - P = \eta + \pi \), we have

\[ \| c - C \|_{L^2(i)} \leq \| \zeta \|_{L^2} + \| \eta \|_{L^2} \]
\[ \| p - P \|_{L^2(i)} \leq \| \eta \|_{L^2} + \| \pi \|_{L^2} \]

From lemma (4.3) and theorem (4.1)

\[ \| u - U \|_{L^2(i)} \leq M(\tau + h_c + h_p) \]

By using (Raviart, 1979), we get

\[ \| c - C \|_{L^2(H)} \leq \| \zeta \|_{L^2} + \| \eta \|_{L^2} \leq M(\| \zeta \|_{L^2} + \| \nabla \zeta \|) \] and

\[ \| p - P \|_{L^2(H)} \leq M(\| \eta \|_{L^2} + \| \nabla \pi \|) \]

then by using theorem (4.2) and eq. (17) we get

\[ \| c - C \|_{L^2(H)} + \| p - P \|_{L^2(H)} + \| u - U \|_{L^2(H)} \leq M(\tau + h_c + h_p) \]

which complete the proof.

5. **Numerical Application**

In this example, we solve a purely convective problem in one dimension [12]

\[ -u \frac{\partial c}{\partial y} = \frac{\partial c}{\partial t} \]

where \( c \) is the concentration in the region \( -2 \leq y \leq 0 \), subject to the boundary conditions
we apply two methods: Galerkin and a kind of partial upwind finite element for this example. We discrete the region \(-2 \leq y \leq 0\) into at first 100 quadrilateral elements and second 200 triangular elements with 202 nodes, also the distance between any two nodes is 0.02 and take \(u_y = 1.0\), \(\theta = 0.5\), \(\tau = 0.04\), and the number of steps (N) = 25 on a 1*100 mesh for quadrilateral element and 1*200 mesh for triangular element.

The correct solution to the problem is described by a rectangular pulse moving with unit velocity in the y direction. Table (1) contains numerical results where \(\theta = 1/2\) at all nodes lie on the right-hand side of the finite element mesh in this example. From the boundary conditions we can see that the solution at nodes 1 and 2 is held at the value 1.0 for the first 0.2 seconds of convection. Figure (1) shows the computed solution after one second and it draws between the concentration and coordinate y, we can see that while y convergence to zero the value of concentration convergence oscillation to solution.
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Table (1): The numerical solutions at all nodes of the right - hand side of mesh, where \( \theta = 0.5, \ \tau = 0.04, \ \text{N} = 25 \)

| Coordinate \( Y \) | Galerkin method (Quadrilateral Element) | Galerkin method (Triangular Element) | Kind of Partial Upwind F.E.M. |
|-------------------|----------------------------------------|-------------------------------------|-----------------------------|
| 0.0000E+00       | 0.3289E-24                             | -0.3987E-24                        | -0.3987E-24                 |
| -0.2000E-01      | -0.4934E-02                            | 0.4238E-02                         | -0.4238E-02                 |
| -0.4000E-01      | 0.1548E-01                             | -0.1405E-01                        | -0.1405E-01                 |
| -0.6000E-01      | -0.4225E-02                            | 0.2256E-01                         | 0.2256E-01                  |
| -0.8000E-01      | -0.2975E-01                            | -0.7050E-02                        | -0.7051E-02                 |
| -0.1000E+00      | 0.3345E-01                             | -0.2730E-01                        | -0.2730E-01                 |
| -0.1200E+00      | 0.2570E-01                             | 0.5020E-01                         | 0.5020E-01                  |
| -0.1400E+00      | -0.6509E-01                            | -0.2636E-01                        | -0.2636E-01                 |
| -0.1600E+00      | -0.1126E-01                            | -0.5949E-01                        | -0.5949E-01                 |
| -0.1800E+00      | 0.9229E-01                             | 0.8407E-01                         | 0.8406E-01                  |
| -0.2000E+00      | 0.4207E-02                             | 0.5155E-01                         | 0.5155E-01                  |
| -0.2200E+00      | -0.1170E+00                            | -0.1160E+00                        | -0.1160E+00                 |
| -0.2400E+00      | -0.2221E-01                            | -0.5481E-01                        | -0.5481E-01                 |
| -0.2600E+00      | 0.1329E+00                             | 0.1267E+00                         | 0.1267E+00                  |
| -0.2800E+00      | 0.7632E-01                             | 0.8523E-01                         | 0.8523E-01                  |
| -0.3000E+00      | -0.1111E+00                            | -0.1125E+00                        | 0.1125E+00                  |
| -0.3200E+00      | -0.1485E+00                            | -0.1465E+00                        | 0.1465E+00                  |
| -0.3400E+00      | 0.1559E-01                             | 0.3309E-01                         | 0.3309E-01                  |
| -0.3600E+00      | 0.1643E+00                             | 0.1784E+00                         | 0.1784E+00                  |
| -0.3800E+00      | 0.1244E+00                             | 0.1156E+00                         | 0.1156E+00                  |
| -0.4000E+00      | -0.4353E-01                            | -0.6673E-01                        | -0.6673E-01                 |
| -0.4200E+00      | -0.1594E+00                            | -0.1742E+00                        | -0.1742E+00                 |
| \( \cdot \)       | \( \cdot \)                             | \( \cdot \)                         | \( \cdot \)                  |
| \( \cdot \)       | \( \cdot \)                             | \( \cdot \)                         | \( \cdot \)                  |
| \( \cdot \)       | \( \cdot \)                             | \( \cdot \)                         | \( \cdot \)                  |
| \( \cdot \)       | \( \cdot \)                             | \( \cdot \)                         | \( \cdot \)                  |
| -0.1860E+01      | 0.6962E-11                             | 0.4492E-10                         | 0.4492E-10                  |
| -0.1880E+01      | 0.3435E-11                             | 0.2433E-10                         | 0.2427E-10                  |
| -0.1900E+01      | 0.1688E-11                             | 0.1313E-10                         | 0.1320E-10                  |
| -0.1920E+01      | 0.8274E-12                             | 0.7031E-11                         | 0.7033E-11                  |
| -0.1940E+01      | 0.4025E-12                             | 0.3846E-11                         | 0.3703E-11                  |
| -0.1960E+01      | 0.1974E-12                             | 0.2032E-11                         | 0.2236E-11                  |
| -0.1980E+01      | 0.9360E-13                             | 0.1000E-11                         | 0.9707E-12                  |
| -0.2000E+01      | 0.4779E-13                             | 0.7114E-12                         | 0.3765E-12                  |
6. Conclusions

We used the system with large coupled of strongly non-linear partial differential equations which arise from the contamination of nuclear waste in porous media. We used a Galerkin method for the pressure equation and a kind of partial upwind finite element method for the concentration. For the compressible case, we obtained the error estimates for approximate Darcy velocity $U$, concentrations $C$ in $L^r(0,T,L^2(\Omega))$. From the numerical results presented in this application, we have got a kind of partial upwind finite element method for triangular element convergent to the exact solution and in comparison with Galerkin method, we found that a kind of

Figure (1): The solutions in example after one second for $\theta = 0.5$, $\tau = 0.04$, $N = 25$. 

Exact solution at $t=1$ second
partial upwind finite element method much more accurate than Galerkin
method.

**NOTATIONS**

- $c$: Concentration of the brine in fluid
- $c_p$: Specific heat
- $c_R$: Compressibility of rock formation
- $c_T$: Coefficient of thermal expansion
- $c_w$: Compressibility of the fluid
- $D$: Dispersion tensor
- $D_m$: Molecular diffusion
- $E$: Dispersivity tensor (hydrodynamic + molecular)
- $k$: Permeability tensor
- $N$: Number of nuclei
- $p$: Pressure
- $q$: Rate of fluid withdrawal
- $R_f$: Brine source rate due to salt dissolution
- $R_s$: Fluid source rate due to salt dissolution
- $u$: Darcy velocity vector
- $\phi$: Porosity
- $\phi_0$: Porosity at the reference pressure
- $\mu$: Viscosity
- $R$: Rock (formation)
- $w$: Water (fluid)
- $s$: Salt (brine)
- $\tau$: Time step
- $W_s^k$: Sobolev space
- $e$: closed triangle element in $T_h$
- $h_c$: Step length for the concentration
- $h_p$: Step length for the pressure
- $U$: Approximate velocity
- $P$: Approximate pressure
- $C$: Approximate concentration
- $\bar{u}$: Projection of velocity
- $\bar{p}$: Projection of pressure
- $\bar{c}$: Projection of concentration
REFERENCES

[1] Al-Bayati, A.Y., Manaa, S.A. and Saadalah, E.A. (2003) “On Upwind Galerkin Finite Element Method for Incompressible Flow of Contamination, from Nuclear Waste, in Porous Media”, Al-Rafidain J. of Computer Sci. and Mathematics, Vol.1, No.2, pp.107-126.

[2] Ewing, R.E., Yuan, Y., and Li, G. (1989) “Timestepping Along Characteristics for a Mixed Finite – Element approximation for Compressible Flow”. SIAM J. NUMER. ANAL, Vol. 26, No.6, pp. 1513-1524.

[3] Gaohong, W. and Cheng Aijie (1999) “Convergence of finite Element Method for Compressible flow of contamination from nuclear waste”, (Chinese Series), 3, pp. 21-31.

[4] Huang, Chieh-Sen (2000), “Implementation of a locally conservative Eulerian-Lagrangian method applied to nuclear contaminant transport”, "Numerical Treatment of Multiphase Flow in Porous Media, State of Art", Lecture Notes in Physics, vol. 552, 179-189, Springer-Verlag, Heidelberg.

[5] Hu, J. W. and Tian, C. S. (1992), “A Galerkin Partial Upwind Finite Element Method for the Convection Diffusion Equations” (Chinese Series), pp. 446-459.

[6] Jim Douglas, Jr. (2001) “The transport of nuclear contamination in fractured porous media”, Journal of the Korean Mathematical Society, 38, pp. 723-761.

[7] Jim Douglas, Jr. (2002), "Parameter Estimates for a Limit Model for High-Level Nuclear Waste Transport in Porous Media”, Mathematical and Numerical Treatment, Contemporary Mathematics, vol. 295.

[8] Manaa, S. A. (2000), “A kind of Upwind F.E.M. for Numerical Reservoir Simulation”, Ph.D Thesis, Nankai University, China.

[9] Peaceman, D.W. (1977), "Fundamentals of numerical reservoir simulation", Elsevier Scientific Publishing Company, Amsterdam, New York.

[10] Quarteroni, A. and A. Valli (1997), “Numerical Approximation of partial differential Equations”, Springer-Verlag.
[11] Raviart, P. A. and V. Girault (1979), “Finite Element Approximation of the Navier-Stokes Equations”, Lecture notes in mathematics, Springer-Verlag.

[12] Smith, I. M., and Griffiths, D. V. (1998), “Programming The Finite Element Method”, Third Edition, John Wiley and Sons.