Binary Tree Arithmetic with Generalized Constructors

Abstract

We describe arithmetic computations in terms of operations on some well known free algebras (S1S, S2S and ordered rooted binary trees) while emphasizing the common structure present in all them when seen as isomorphic with the set of natural numbers.

Constructors and deconstructors seen through an initial algebra semantics are generalized to recursively defined functions obeying similar laws. Implementations using Scala’s apply and unapply are discussed together with an application to a realistic arbitrary size arithmetic package written in Scala, based on the free algebra of rooted ordered binary trees, which also supports rational number operations through an extension to signed rationals of the Calkin-Wilf bijection.

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1. Introduction

Classical mathematics frequently uses functions defined on equivalence classes (e.g. modular arithmetic, factor objects in algebraic structures) provided that it can prove that the choice of a representative in the class is irrelevant.

On the other hand, when working with proof assistants like Coq [9], based on type theory and its computationally refined extensions like the Calculus of Construction [4], one cannot avoid noticing the prevalence of data types corresponding to free objects, on top of which everything else is built in the form of canonical representations.

Category-theory based descriptions of Peano arithmetic fit naturally in the general view that data types are initial algebras - in this case the initial algebra generated by the successor function, as a provider of the canonical representation of natural numbers. Of course, a critical element in choosing such free algebras is computational efficiency of the operations one wants to perform on them, in terms of low time and space complexity. For instance Coq formalizations of natural numbers typically use binary representations while keeping the Peano arithmetic view when more convenient in proofs [9]. Note also that free algebras corresponding to one and two successor arithmetic (S1S and S2S) have been used as a basis for decidable weak arithmetic systems like [2] and [11]. It has been shown recently in [16, 17] that the initial algebra of ordered rooted binary trees corresponding to the language of Gödel’s System T types [7] can be used as the language of arithmetic representations, with hyper-exponential gains when handling numbers built from “towers of exponents” like $2^{2^2}$. Independently, this view is confirmed by the suggestion to use $\lambda$-terms as a form of universal data compression tool [8] and by deriving bijective encodings of data types using a game-based mechanism [18].

These results suggest a free algebra based reconstruction of fundamental data types that are relevant as building blocks for finite mathematics and computer science. We will sketch in this paper an (elementary, not involving category theory) foundation for arithmetic computations with free algebras, in which construction of sets, sequences, graphs, etc. can be further carried out along the lines of [14, 15, 17].

The paper is organized as follows. We define in section 2 isomorphisms between the free algebras of signatures consisting of one constant and respectively, of one successor (S), two successors (0 and 1) and a free magma constructor (C).

To enable computations with the objects of the free algebras, we discuss the use of generalized constructors / destructors derived from these free algebras using the apply / unapply constructs available in Scala (section 3). As an application, a complete arbitrary size rational arithmetic package using the Calkin-Wilf bijection between positive rationals and natural numbers is described in section 4. Sections 5 and 6 discuss related work and our conclusions.

2. Free Algebras and Data Types

Definition 1. Let $\sigma$ be a signature consisting of an alphabet of constants (called generators) and an alphabet of function symbols (called constructors) with various arities. We define the free algebra $A_\sigma$ of signature $\sigma$ inductively as the smallest set such that:

1. if $c$ is a constant of $\sigma$ then $c \in A_\sigma$
2. if $f$ is an n-argument function symbol of $\sigma$, then $\forall i, 0 \leq i < n, t_i \in A_\sigma \Rightarrow f(t_0, \ldots, t_i, \ldots, t_{n-1}) \in A_\sigma$.

We will write c/0 for constants and f/n for function symbols with n arguments belonging to a given a signature.

More general definitions, e.g. as initial objects in the category of algebraic structures, are also used in the literature and a close relation exists with term algebras distinguishing between function constructors (generating the Herbrand Universe) and predicate constructors (generating the Herbrand Base).

Recursive data types in programming languages like Haskell, ML, Scala can be seen as a notation for free algebras. We refer to [19] for a clear and convincing description of this connection.

For instance, the Haskell declarations

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correspond, respectively to:

- the free algebra \( \mathsf{Alg} \) with a single generator \( T \) and unary constructor \( S \) (that can be seen as part of the language of Peano or Robinson arithmetic, or the decidable (\(\mathsf{WS1S} \) system, \([2]\))

- the free algebra \( \mathsf{Alg} \) with single generator \( B \) and two unary constructors \( 0 \) and \( I \) (corresponding to the language of the decidable system (\(\mathsf{WS2S} \) \([11]\)), as well as “bijective base-2” number notation \([20]\)

- the free algebra \( \mathsf{Alg} \) with single generator \( T \) and one binary constructor \( C \) (essentially the same thing as the free magma generated by \( T \)).

The set-theoretical construction corresponding to the “1” operation is disjoint union and the data types correspond to infinite sets generated by applying the respective constructors repeatedly. The set-theoretical interpretation of “self-reference” in such data type defined by applying the respective constructors repeatedly. The set-theoretical definitions such as fixpoint operation on sets of natural numbers and so as in the \(\mathsf{P} \) construction used by Dana Scott in defining the denotational semantics for various \(\mathcal{L} \) constructs \([13]\).

We will next “instantiate” some general results to make the underlying mathematic as elementary and self-contained as possible. While category theory is frequently used as the mathematical backbone for data-types, we will provide here a simple set-theory-based formalism, along the lines of \([1]\).

We will start with the elementary mathematics behind the \( \mathsf{Alg} \) data type and follow with an outline for a similar treatment of \( \mathsf{Alg} \) and \( \mathsf{Alg} \).

### 2.1 The free magma of ordered rooted binary trees with empty leaves

**Definition 2.** A set \( M \) with a (total) binary operation \( * \) is called a magma.

**Definition 3.** A morphism between two magmas \( M \) and \( M' \) is a function \( f : M \to M' \) such that \( f(x * y) = f(x) * f(y) \).

Let \( X \) be a set. We define the sets \( M_n(X) \) inductively as follows: \( M_1(X) = X \) and for \( n > 1 \), \( M_n(X) \) is the disjoint union of the sets \( M_{n-1}(X) \times M_{n-k}(X) \) for \( 0 < k < n \). Let \( M(X) \) be the disjoint union of the family of sets \( M_n(X) \) for \( n > 0 \). We identify each set \( M_n \) with its canonical image in \( M(X) \). Then for \( w \in M_n(X) \), we call \( n \) the length of \( w \) and denote it \( |w| \). Let \( w, w' \in M(X) \) and let \( p = |w| \) and \( q = |w'| \). The image of \( (w, w') \in M_p \times M_q \) under the canonical injection in \( M(X) \) is called the composition of \( w \) and \( w' \) and is denoted \( w * w' \).

When \( X = \{ T \} \) where \( T \) is interpreted as the “empty” leaf of ordered rooted binary trees, the elements of \( M_n \) can be seen as ordered rooted binary trees with \( n \) leaves while the composition operation “\( * \)” represents joining two trees at their roots to form a new tree.

**Definition 4.** The set \( M(X) \) with the composition operation \( (w, w') \to w * w' \) is called the free magma generated by \( X \).

**Proposition 1.** Let \( M \) be a magma. Then every mapping \( u : X \to M \) can be extended in a unique way to a morphism of \( M(X) \) into \( M \).

**Proof.** We define inductively the mappings \( f_n : M_n(X) \to M \) as follows: For \( n = 1 \), \( f_1 = f \). For \( n > 1 \), \( \forall \rho \in \{ 1, \ldots, n - 1 \} \), \( f_n(u * w') = f_\rho(w) * f_{n-\rho}(w') \). Let \( g : M(X) \to M \) such that \( \forall n \geq 0, \forall x \in M_n(X), g(x) = f_n(x) \). Then \( g \) is the unique morphism of \( M(X) \) into \( M \) which extends \( f \).

Note that this property corresponds of the initial algebra \([19]\) view of the corresponding (ordered, rooted) binary tree data type.

**Definition 5.** If \( u : X \to Y \), we denote \( M(u) : M(X) \to M(Y) \) the unique morphism of magmas defined by the construction in **Proposition 1**.

If \( v : Y \to Z \) then the morphism \( M(v) \circ M(u) \) extends \( v \circ u : X \to Z \) and therefore \( M(v) \circ M(u) = M(v \circ u) \).

**Proposition 2.** If \( u : X \to Y \) is respectively injective, surjective, then so is \( M(u) \).

It follows that

**Proposition 3.** If \( X = \{ x \} \) and \( Y = \{ y \} \) and \( u : X \to Y \) is the bijection such that \( f(x) = y \), then \( M(u) : M(X) \to M(Y) \) is a bijection morphism (i.e. an isomorphism) of free magmas.

**Proof.** If \( X \) is empty so is \( M(X) \), hence \( u \) is injective. If \( u \) is injective, then \( \exists u' : Y \to X, u' \circ u = id_{M(X)} \) where \( id_{M(X)} \) denotes the identity mapping of \( M(X) \). Then \( M(u') \circ M(u) = M(u' \circ u) = id_{M(Y)} \) and hence \( M(u) \) is injective. If \( u \) is surjective, then \( \exists u' : Y \to X, u \circ u' = id_{M(Y)} \). Then \( M(u) \circ M(u') = M(u \circ u') = id_{M(Y)} \) and hence \( M(u) \) is surjective. If \( u \) is bijective, then it is injective and surjective and so is \( M(u) \).

We will identify the data type \( \mathsf{Alg} \) with the free magma generated by the set \( \{ T \} \) and denote its binary operation \( * \) as \( C \times C \). It corresponds to the free algebra (that we will also denote \( \mathsf{Alg} \)) defined by the signature \( \{ T, C, \} \).

We can now instantiate the results described by the previous propositions to \( \mathsf{Alg} \):

**Proposition 4.** Let \( X \) be an algebra defined by a constant \( c \) and a binary operation \( \circ \). Then there’s a unique morphism \( f : \mathsf{Alg} \to X \) that verifies

\[
\begin{align*}
  f(T) & = \circ \\
  f(C(x, y)) & = c(f(x), f(y))
\end{align*}
\]

Moreover, if \( X \) is a free algebra then \( f \) is an isomorphism.

**Proof.** It follows from **Proposition 2** and equation (1) that \( f(T) = \circ \), that is \( f \) is a bijection between the singleton sets \( \{ T \} \) and \( \{ \circ \} \).

### 2.2 The One Successor and Two Successors Free Algebras

The one successor free algebra (also known as unary natural numbers or Peano algebra, as well as the language of the monoid \( \{ 0 \}^* \) and the decidable systems \(\mathsf{WS1S} \) and \(\mathsf{S1S} \)) is defined by the signature \( \{ U, 0, S \} \), where \( U \) is a constant (seen as zero) and \( S \) is the unary successor function. We will denote \( \mathsf{Alg} \) this algebra and identify it with its corresponding data type.

We state an analogue of **Proposition 4** for the free algebra \( \mathsf{Alg} \):

**Proposition 5.** Let \( X \) be an algebra defined by a constant \( u \) and a unary operation \( s \). Then there’s a unique morphism \( f : \mathsf{Alg} \to X \) that verifies

\[
\begin{align*}
  f(U) & = u \\
  f(S(x)) & = s(f(x))
\end{align*}
\]

Moreover, if \( X \) is a free algebra then \( f \) is an isomorphism.

Note that following the usual identification of data types and initial algebras, \( \mathsf{Alg} \) corresponds to the initial algebra \( \{ 1, +, \} \) through the operation \( g = \text{<U,S>} \) seen as a bijection \( g : 1 + N \to N \).

The two successor free algebra (also known as bijective base-2 natural numbers or Peano algebra, as well as the language of the monoid \( \{ 0, 1 \}^* \) and the decidable systems \(\mathsf{WS2S} \) and \(\mathsf{S2S} \)) is
defined by the signature \{ B/0, O/1, I/1 \} where B is a constant (seen as the empty sequence) and O, I are two unary successor functions. We will denote AlgB this algebra and identify it with its corresponding data type.

We can state an analogue of Proposition 4 for the free algebra AlgB.

**Proposition 6.** Let X be an algebra defined by a constant b and a two unary operations o, i. Then there’s a unique morphism \( f : \text{AlgB} \to X \) that verifies

\[
\begin{align*}
  f(B) & = b \\
  f(O(x)) & = o(f(x)) \\
  f(I(x)) & = i(f(x))
\end{align*}
\]

Moreover, if X is a free algebra then \( f \) is an isomorphism.

These observations suggest that for defining isomorphisms between AlgU, AlgB and AlgT that enable a complete set of equivalent arithmetic (and later set-theoretic) operations on each of them, we will need a mechanism to prove such equivalences. To this end, it will be enough to prove that such non-constructor operations also form free algebras of matching signatures.

We will call terms the elements of our initial algebras.

### 3. Generalized Constructors

The iso-functors supporting the equivalence between actual constructors and their recursively defined function counterparts suggest exploring programming language constructs that treat them in a similar way. For instance it makes sense to extend “constructor-only benefits” like pattern matching to their function counterparts.

Fortunately, constructors/deconstructors generalized to arbitrary functions are available in Scala through apply/unapply methods and in Haskell through a special notation implementing views, under the implicit assumption that they define inverse operations.

One can immediately notice that our free algebras provide sufficient conditions under which this assumption is enforced. This suggests the possibility that such generalized constructor/deconstructor pairs could provide the combined benefits of pattern matching and data abstraction, with the implication that direct syntactic support for such constructs can bring significant expressiveness to functional programming languages.

#### 3.1 Generalized Constructors with apply/unapply in Scala

Besides supporting case classes and case objects that are used (among other things) to implement pattern matching, Scala’s apply and unapply methods [5, 10] allow definition of customized constructors and destructors (called extractors in Scala).

We will next describe how arithmetic operations with our AlgT terms, represented as ordered rooted binary trees, can benefit from the use such “generalized constructors”.

Our AlgT free algebra will correspond in Scala to a case object / case class definition, combined with a mechanism to share actual code, encapsulated in the AlgT trait.

```
case object T extends AlgT

trait AlgT {
  def s(z: AlgT): AlgT = z match {
    case T => C(T, T)
    case C(T, y) => d(s(y))
    case z => C(T, h(z))
  }

  def p(z: AlgT): AlgT = z match {
    case C(T, T) => T
    case C(T, y) => d(y)
    case z => C(T, p(h(z)))
  }
}
```

Note the predecessor function called p and our auxiliary functions named d (which “doubles” its input, assumed different from T) and h (which “halves” its input, assumed “even” and different from T).

```
object S extends AlgT {
  def apply(x: AlgT) = s(x)
  def unapply(x: AlgT) = x match {
    case C(_, _) => Some(h(x))
    case _ => None
  }
}
```

The definition of the generalized constructor/destructor D representing double / half is similar. Note the use of the method d defined in the trait AlgT.

```
object D extends AlgT {
  def apply(x: AlgT) = d(x)
  def unapply(x: AlgT) = x match {
    case C(_, _, _) => Some(h(x))
    case _ => None
  }
}
```

The definition of the generalized constructor/destructor 0 can be seen as corresponding to \( \lambda x.2x+1 \) and its inverse.

```
object O extends AlgT {
  def apply(x: AlgT) = C(T, x)
  def unapply(x: AlgT) = x match {
    case C(T, b) => Some(b)
    case _ => None
  }
}
```

The definition of the generalized constructor/destructor 0 can be seen as corresponding to \( \lambda x.2x+2 \) and its inverse. Note the use of the generalized constructors S, D and O, both on the left and right side of match statements, illustrating their usefulness both as constructors and as extractors.

```
object I extends AlgT {
  def apply(x: AlgT) = S(0(x))
  def unapply(x: AlgT) = x match {
    case D(a) => Some(p(a))
    case _ => None
  }
}
```
3.2 A Scala-based Natural Number Arithmetic Package using AlgT Terms

We will now illustrate how the use of generalized constructors helps writing a fairly complete set of arithmetic algorithms on terms of AlgT seen as natural numbers. For comparison purposes, the reader might want to look at the Haskell code in [17] where similar algorithms are expressed using a type class-based mechanism. However, while the use of type classes comes with the benefits of data abstraction it needs separate functions for constructing, deconstructing and recognizing terms to express the equivalent of the generalized constructors used here.

We start with a comparison function returning LT, EQ, and GT and supporting a total order relation on AlgT, isomorphic to the one on \( \mathbb{N} \). Note here the use of the generalized constructors \( 0 \) and \( I \) providing a view of the terms of AlgT as terms of the free algebra \( \text{BinT} \).

```
trait Tcompute extends AlgT {
  def cmp(u: AlgT, v: AlgT): Int = (u, v) match {
    case (T, T) => EQ
    case (T, _) => LT
    case (_, T) => GT
    case (0(x), 0(y)) => cmp(x, y)
    case (I(x), I(y)) => cmp(x, y)
    case (O(x), I(y)) => strengthen(cmp(x, y), LT)
    case (I(x), 0(y)) => strengthen(cmp(x, y), GT)
  }

  val LT = -1
  val EQ = 0
  val GT = 1

  private def strengthen(rel: Int, from: Int) = rel match {
    case EQ => from
    case _ => rel
  }
}
```

Addition is expressed compactly in terms of the generalized constructors \( 0, I \) and \( S \).

```
def add(u: AlgT, v: AlgT): AlgT = (u, v) match {
  case (T, y) => y
  case (x, T) => x
  case (0(x), 0(y)) => I(add(x, y))
  case (0(x), I(y)) => O(S(add(x, y)))
  case (I(x), 0(y)) => 0(S(add(x, y)))
  case (I(x), I(y)) => I(add(x, y))
}
```

The definition of subtraction is similar, except that the code of the predecessor function \( p \) is conveniently inherited directly from the trait AlgT, given that the trait Tcompute extends it.

```
def sub(u: AlgT, v: AlgT): AlgT = (u, v) match {
  case (x, T) => x
  case (0(x), 0(y)) => p(0(sub(x, y)))
  case (0(x), I(y)) => p(0(sub(x, y))))
  case (I(x), 0(y)) => p(0(sub(x, y)))
  case (I(x), I(y)) => p(0(sub(x, y)))
}
```

The multiplication operation is similar to the Haskell code in section ??, except for the use of the generalized constructor \( 0 \).

```
def multiply(u: AlgT, v: AlgT): AlgT = (u, v) match {
  case (T, _) => T
  case (_, T) => T
  case (0(x), tx, C(hx, ty)) => {
    val z = add(tx, ty)
    case (C(hx, hy), (C(hx, ty), C(hy, ty))) => {
      val v = add(tx, ty)
      C(add(hx, hy), add(v, z))
    }
  }
}
```

Similarly, a constant time complexity definition is given here for the exponent of 2 operation, by using the “real” constructor \( C \).

```
def exp2(x: AlgT) = C(x, T)
```

The power operation pow takes advantage of the generalized constructors \( 0 \) and \( I \) on the left side of a case statement through the AlgB view of AlgT.

```
def pow(u: AlgT, v: AlgT): AlgT = (u, v) match {
  case (_, T) => C(T, T)
  case (x, 0(y)) => multiply(x, pow(multiply(x, x), y))
  case (x, I(y)) => {
    val xx = multiply(x, x)
    multiply(xx, pow(xx, y))
  }
}
```

Efficient division with remainder is a slightly more complex algorithm, where we take advantage of generalized constructors, direct inheritance from trait AlgT as well as number of previously defined functions:

```
def div_and_rem(x: AlgT, y: AlgT): (AlgT, AlgT) = if (cmp(x, y) == LT) (T, x)
else if (T = y) null // division by zero
else {
  def try_to_double(x:AlgT, y:AlgT, k:AlgT) = if (cmp(x, y) == LT) p(k)
else try_to_double(x, D(y), S(k))

  def divstep(n: AlgT, m: AlgT): (AlgT, AlgT) = {
    val q = try_to_double(n, m, T)
    val p = multiply(exp2(q), m)
    (q, sub(n, p))
  }

  val (qt, rm) = divstep(x, y)
  val (z, r) = div_and_rem(rm, y)
  val dv = add(exp2(qt), z)
  (dv, r)
}
```

Division and reminder can be separated using Scala’s projection functions:

```
def divide(x: AlgT, y: AlgT) = div_and_rem(x, y)_1
```

```
def reminder(x: AlgT, y: AlgT) = div_and_rem(x, y)_2
```

Finally, the greatest common divisor \( \text{gcd} \) and the least common multiplier \( \text{lcm} \) are defined as follows:

```
def gcd(x: AlgT, y: AlgT) = if (y == T) x else gcd(y, reminder(x, y))
```

```
def lcm(x: AlgT, y: AlgT) = multiply(divide(x, gcd(x, y)), y)
```

The trait Tconvert implements efficiently conversion to/from Scala’s BigInt arbitrary size integers using bit-level operations corresponding to power of 2 and recognition of odd and even natural numbers. The function fromN builds an AlgT tree representation equivalent to a BigInt.

```
trait Tconvert {
  def fromN(i: BigInt): AlgT = {
    def oddN(i: BigInt) = if (i.testBit(0)) i else oddN(i) & i & !i.testBit(0)
    def evenN(i: BigInt) = i
  }
  def lcm(x: AlgT, y: AlgT): AlgT = multiply(divide(x, gcd(x, y)), y)
  def exp2(x: AlgT) = C(x, T)
  def pow(u: AlgT, v: AlgT): AlgT = (u, v) match {
    case (_, T) => C(T, T)
    case (x, 0(y)) => multiply(x, pow(multiply(x, x), y))
    case (x, I(y)) => {
      val xx = multiply(x, x)
      multiply(xx, pow(xx, y))
    }
  }
  def cmp(u: AlgT, v: AlgT): Int = (u, v) match {
    case (T, T) => EQ
    case (T, _) => LT
    case (_, T) => GT
    case (0(x), 0(y)) => cmp(x, y)
    case (I(x), I(y)) => cmp(x, y)
    case (O(x), I(y)) => strengthen(cmp(x, y), LT)
    case (I(x), 0(y)) => strengthen(cmp(x, y), GT)
  }
  def subtract(u: AlgT, v: AlgT): AlgT = (u, v) match {
    case (T, T) => T
    case (T, _) => T
    case (_, T) => T
    case (x, y, C(hx, ty)) => {
      val v = add(tx, ty)
      val z = p(0(multiply(tx, ty)))
      C(add(hx, hy), add(v, z))
    }
  }
  def divide(x: AlgT, y: AlgT) = div_and_rem(x, y)_1
  def reminder(x: AlgT, y: AlgT) = div_and_rem(x, y)_2
  def fromN(i: BigInt): AlgT = {
    def oddN(i: BigInt) = if (i.testBit(0)) i else oddN(i) & i & !i.testBit(0)
    def evenN(i: BigInt) = i
  }
  def lcm(x: AlgT, y: AlgT): AlgT = multiply(divide(x, gcd(x, y)), y)
  def exp2(x: AlgT) = C(x, T)
}
```
The function `toN` converts an `AlgT` tree representation to a `BigInt`.

```scala
def toN(z: AlgT): BigInt =
  z match {
    case O(T) ⇒ 0
    case C(x, y) ⇒
      (BigInt(1) ⋆ toN(x).intValue()) *
      (BigInt(2) ⋆ toN(y) + 1)
  }
```

Note that for both these conversions we have used, for efficiency reasons, the “real constructors” `T` and `C`, although much simpler (and slower) converters can be built using either the `AlgB` or `AlgU` view of `AlgT` terms.

The use of Scala’s generalized constructors inspired by our free algebra isomorphisms has shown the combined flexibility of inheritance as a mechanism for data abstraction and convenient pattern matching allowing the design of our algorithms in a functional style. The implicit use of `apply` and `unapply` methods in combination with our simple free algebra semantics has facilitated the safe use of fairly complex (mutually) recursive functions in the definition of the generalized constructors. The use of Scala’s traits has facilitated flexible inheritance mechanisms supporting shared definitions without any additional syntactic clutter.

### 4. An Application: Rational Arithmetic in Scala with Calkin-Wilf Trees

We will extend our Scala code snippet described in subsection 3.1 to a realistic arbitrary size arithmetic package. It is somewhat unconventional, as it is based on the Calkin-Wilf bijection [3, 6] between `N` and the set of positive rational numbers `Q+`, rather than more typical representations like the arrays of long words used in Java’s `BigDecimal` package, also adopted through a wrapper class with the same name by Scala (which runs on top of the Java Virtual Machine).

Among its advantages, division (with non-zero) always returns a finitely represented rational and “no bit is lost” in the representation as canonical rational numbers with co-prime numerator/denominator pairs are bijectively mapped to natural numbers. Our approach emphasizes the fact that a mathematical concept defined traditionally through equivalence classes and quotients, can be expressed entirely in terms of a free algebra-based mechanism.

The trait `Q` representing our rational number data type contains distinct constructors for positive (P), negative numbers (M) and zero (Z).

```scala
trait Q extends Qcode
    case object Z extends Q
    case class P(x: (AlgT, AlgT)) extends Q
    case class M(x: (AlgT, AlgT)) extends Q
```

The actual code will be shared through the trait `Qcode` that also mixes-in functionality from the natural number operations defined in the traits `Tcompute` and `Tconvert`.

We start with a type definition for ordered pairs of natural numbers `PQ` represented as terms of `AlgT` and the conversion function to a conventional fraction represented as an ordered pair of `BigInt` objects. The conversion function `toFraq` uses the `AlgT` to `BigInt` converter `toN`.

```scala
trait Qcode extends Tcompute with Tconvert {
  type PQ = (AlgT, AlgT)
}
```

```scala
def toFraq(): (BigInt, BigInt) =
  case Z ⇒ (0, 1)
  case M((a, b)) ⇒ (-toN(a), toN(b))
  case P((a, b)) ⇒ (toN(a), toN(b))
```

The function `t2pq` splits its argument `u` seen as a natural number into its corresponding Calkin-Wilf rational, represented as a pair of positive natural numbers of type `PQ`. Note the use of our generalized constructors `O` and `I` distinguishing between odd and even numbers.

The algorithm uses an encoding of the path in the Calkin-Wilf tree as a member of `AlgB`, where `O` is interpreted as a command to take the left branch and `I` is interpreted as a command to take the right branch at a node of the Calkin-Wilf tree (shown in Fig. 1, for a few small positive rationals, represented as conventional fractions).

![Figure 1. The Calkin-Wilf Tree](image)

```scala
def t2pq(u: AlgT): PQ =
  case O(T) ⇒ (S(T), S(T))
  case 0(n) ⇒ {
    val (x, y) = t2pq(n)
    (x, add(x, y))
  }
  case I(n) ⇒ {
    val (x, y) = t2pq(n)
    (add(x, y), y)
  }
```

The function `pq2t` fuses back into a “natural number” represented as a term of `AlgT` corresponding to the path in the Calkin-Wilf tree, a pair of co-prime natural numbers representing the `(numerator, denominator)` pair defining a positive rational number.

```scala
def pq2t(uv: PQ): AlgT =
  uv match {
    case (0(T), 0(T)) ⇒ T
    case (a, b) ⇒
      cmp(a, b) match {
        case GT ⇒ I(pq2t(sub(a, b), b))
        case LT ⇒ O(pq2t(a, sub(b, a)))
      }
  }
```

This brings us to the definition of the bijection between `signed` rationals and terms seen through the use of our generalized constructors `O` and `I` as terms of `AlgB` representing natural numbers.
The bijection between Scala's BigInt, seen as a natural number type and signed rationals, is defined as the pair of functions \( \text{rat2nat} \) and \( \text{nat2rat} \).

\[
\text{rat2nat}(q) = (\text{toN}(\text{toT}(q)), \text{fromN}(\text{fromT}(q)))
\]

Next we define a simplifier of positive fractions represented as a pair, to facilitate arithmetic operations on our rationals.

\[
\text{pqsimpl}(xy: \text{PQ}, uv: \text{PQ}) = (\text{divide}(xy._1, uv._1), \text{divide}(xy._2, uv._2))
\]

We are now ready for our arithmetic operations. The template function \( \text{pqop} \) parameterized by a function \( f \), will be shared between addition and subtraction. Note that it also involves simplification, to ensure that the results are in a canonical co-prime numerator/denominator form.

\[
\text{pqop}(f: \text{AlgT, AlgT}) = \text{AlgT, xy: PQ, uv:PQ): PQ} = \{ \\
\text{val} (x, y) = xy \\
\text{val} (u, v) = uv \\
\text{val} z = \text{gcd}(y, v) \\
\text{val} v1 = \text{divide}(y, z) \\
\text{val} num = f(multiply(x, v1), multiply(u, v1)) \\
\text{val} den = multiply(z, multiply(y1, v1)) \\
\text{pqimpl}((\text{num}, \text{den})) \\
\}
\]

We can use it to define addition and subtraction of positive rationals by simply instantiating our function parameter \( f \) to add and sub operating on terms of \( \text{AlgT} \).

\[
\text{def pqadd(a: PQ, b: PQ) = pqop(add, a, b)}
\]

\[
\text{def pqsub(a: PQ, b: PQ) = pqop(sub, a, b)}
\]

The comparison operation providing a total ordering of \( \mathbb{Q}^+ \) relies on the function \( \text{cmp} \) comparing terms of \( \text{AlgT} \) seen as natural numbers.

\[
\text{def pqcmp(xy: PQ, uv: PQ) = \{} \\
\text{val} (x, y) = xy \\
\text{val} (u, v) = uv \\
\text{cmp(multiply(x, v), multiply(y, u)) \\
\}
\]

Multiplication, inverse and division on \( \mathbb{Q}^+ \) are defined as usual.

\[
\text{def pqmultiply(a: PQ, b: PQ) = pqimpl(multiply(a._1, b._1), multiply(a._2, b._2))}
\]

\[
\text{def pqinverse(a: PQ) = (a._1, a._2)}
\]

\[
\text{def pqdivide(a: PQ, b: PQ) = pqmultiply(a, pqinverse(b))}
\]

We are ready to define arithmetic operations on the set of signed rationals \( \mathbb{Q} \), by case analysis on their sign. We start with the opposite of a rational.

\[
\text{def ropposite(x: Q) = x match \{} \\
\text{case Z} \Rightarrow Z \\
\text{case M(a) ⇒ P(a)} \\
\text{case P(a) ⇒ M(a)} \\
\text{\}}
\]

Addition is defined by case analysis on the sign and calls to the appropriate operations on positive rationals.

\[
\text{def radd(a: Q, b: Q) = (a, b) match \{} \\
\text{case Z, y) ⇒ y} \\
\text{case (M(x), M(y)) ⇒ M(pqadd(x, y))} \\
\text{case (P(x), P(y)) ⇒ P(pqadd(x, y))} \\
\text{case (P(x), M(y)) ⇒ pqcmp(x, y) match \{} \\
\text{case LT ⇒ M(pqsub(y, x))} \\
\text{case EQ ⇒ Z} \\
\text{case GT ⇒ P(pqsub(y, x))} \\
\text{\}} \\
\text{case (M(x), P(y)) ⇒ ropposite(radd(P(x), M(y))))} \\
\text{\}}
\]

Subtraction is defined similarly.

\[
\text{def rsub(a: Q, b: Q) = radd(a, ropposite(b))}
\]

\[
\text{def rmultiply(a: Q, b: Q) = (a, b) match \{} \\
\text{case Z, _) ⇒ Z} \\
\text{case (Z, Z) ⇒ Z} \\
\text{case (M(x), M(y)) ⇒ M(pqmultiply(x, y))} \\
\text{case (M(x), P(y)) ⇒ M(pqmultiply(x, y))} \\
\text{case (P(x), M(y)) ⇒ M(pqmultiply(x, y))} \\
\text{case (P(x), P(y)) ⇒ P(pqmultiply(x, y))} \\
\text{\}}
\]

Finally we define the inverse on non-zero rationals.

\[
\text{def rinverse(a: Q) = a match \{} \\
\text{case M(z) ⇒ M(pqinverse(z))} \\
\text{case P(z) ⇒ P(pqinverse(z))} \\
\text{\}}
\]

and use it to derive from it a division operation on \( \mathbb{Q} \).

\[
\text{def rdivide(a: Q, b: Q) = rmultiply(a, rinverse(b))}
\]

These operations conclude the trait \( \text{Qcode} \). While this complete arithmetic package was built mostly as a proof of concept for the expressiveness of our free algebra based approach on progressively more interesting mathematical objects, future work is planned for turning this package into a practical tool. A first observation toward this end is that, like in the case of Java’s BigIntegers or the C-based GMP package, one needs to use a hybrid approach, taking advantage of actual machine words (64 bits at this point), to store and operate on numbers that fit in a machine word.
5. Related Work

Numeration systems on regular languages have been studied recently, e.g. in [12] and specific instances of them are also known as bijective base-k numbers [20]. Arithmetic packages similar to $\text{AlgU}$ and $\text{AlgB}$ are part of libraries of proof assistants like Coq [9] and the corresponding regular languages have been used as a basis of decidable arithmetic systems like $(W)S1S$ [2] and $(W)S2S$ [11].

Arithmetic computations based on more complex recursive data types like the free magma of binary trees (essentially isomorphic to the context-free language of balanced parentheses) are described in [17] and [16], where they are seen as Gödel’s System T types, as well as combinator application trees. In [15] a type class mechanism is used to express computations on hereditarily finite sets and hereditarily finite functions. However, none of these papers provides proofs of the properties of the underlying free algebras or uses mechanisms similar to the generalized constructors described in this paper.

A very nice functional pearl [6] has explored in the past (using Haskell code) algorithms related to the Calkin-Wilf bijection [3]. While using the same underlying mathematics, our Scala-based package works on terms of the $\text{AlgT}$ free algebra rather than conventional numbers, and provides a complete package of arbitrary size rational arithmetic operations taking advantage of our generalized constructors.

6. Conclusion

We have shown that free algebras corresponding to some basic data types in programming languages can be used for arithmetic computations isomorphic to the usual operations on $\mathbb{N}$.

As a new theoretical contribution, we have worked-out details of proofs, based only on elementary mathematics, of essential properties of the mutually recursive successor and predecessor functions, on the free algebra $\text{AlgT}$ of ordered rooted binary trees.

A concept of generalized constructor, for which we have found simple implementations in Scala, has been introduced. By working in synergy with our free algebra isomorphisms we have described, using language constructs like Scala’s $\text{apply} / \text{unapply}$, simple and safe means to combine data abstraction and pattern matching in modern-day functional and object oriented languages.

As a new practical contribution, a complete arbitrary size signed rational number package written in Scala has been derived working with terms of the $\text{AlgT}$ free algebra of rooted ordered binary trees with empty leaves.

Future work is planned to investigate possible practical applications of our algorithms to symbolic and/or arbitrary length integer arithmetic packages and to parallel execution of arithmetic computations on $\text{AlgT}$.

The code snippet showing the use of Scala’s $\text{apply}$ and $\text{unapply}$ methods to support generalized constructors as well as the arithmetic on rationals is available as a separate file at http://jeremy.gibbons/publications/rationals.pdf.

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