On the generation of three-dimensional disturbances from two-dimensional nonlinear instabilities in shear flows

J T C Liu
School of Engineering, Brown University, 184 Hope Street, Providence, Rhode Island 02912-9104, USA
joseph_liu@brown.edu

Abstract. The generation of three-dimensional disturbances from nonlinear two-dimensional ones is central to the transition of an otherwise laminar flow to turbulence as well as being a fundamental mechanism in turbulence flows. A generic energy balance between three- and two-dimensional disturbances is discussed, followed by specific application to the generation of three-dimensional disturbances and an explanation of the origins of such disturbances in the far wake behind a flat plate. The similarities with bluff body far wakes are discussed.

1. Introduction
The interaction between two- and three-dimensional motions is one of central problems in turbulent flows. The situation is similar in unstable laminar flows and the transition to turbulent flow. But these transitional events are more tractable due to the presence of coherent motions, Liu [1]. In this paper we first discuss in general terms the interaction of three-dimensional spanwise-periodic motions and two-dimensional motions that are modulated in time and streamwise-development direction as in spatially developing shear flows. These discussions are not confined to linear situations. In the next section, the characteristics obtained from a linear instability study of a strongly nonlinear base flow, developing out of the laminar wake behind a flat plate and the accompanying fundamental instability, are used to interpret experiments that involve cross-hatch patterns in the far wake. These are used to obtain the far downstream oblique wave angles and speculation is made that the quasi-equilibrium far wake oblique-wave angles (Cimbala, et al. [2]) may well be more universal than its initiation upstream, thus bringing in aspects of bluff body far wake behavior discussed by Williamson & Prasad [3],[4].

2. Physical mechanism of three-dimensional disturbances developing from two-dimensional nonlinear unstable flow
We discuss first the generalities of the interaction between two-dimensional and three-dimensional disturbances for spatially developing shear flows and begin with the dimensionless continuity and Navier-Stokes equations for an incompressible fluid,

\[ \nabla \cdot \vec{u} = 0 \]  
\[ \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \frac{1}{R} \nabla^2 \vec{u} \]

where the velocities \( \vec{u} \), coordinates and time \( t \) are made dimensionless by the free stream velocity \( U_0 \), and a physical length scale (e.g., the wake half width) \( h_0 \) at the streamwise location \( x_0 \); \( R = U_0 h_0 / \nu \)
is the Reynolds number and $\nu$ is the kinematic viscosity. The pressure is made dimensionless by the free stream dynamical pressure. We then split any total flow quantity $q(x,y,z,t)$ into

$$q(x,y,z,t) = q_2(x,y,t) + q_3(x,y,z,t),$$

where $x$ is the developing streamwise coordinate, $y$ is the normal coordinate, $z$ is the spanwise variable and $t$ is the time. The subscript notation follows that of Herbert [5] in his depiction of a two-dimensional flow consisting of the sum $q_2 = q_0 + q_1$, where $q_0$ is the steady, time-mean flow and $q_1$ is the two-dimensional disturbance; $q_3$ is the three-dimensional, spanwise periodic secondary disturbance. In the following consideration, no presumptions about linearities (Herbert) or weak nonlinearities are made, but only that the three-dimensional disturbances are spanwise periodic.

Upon substituting (2.3) into (2.1) and (2.2) and spanwise $z$-averaging, denoted by an over bar, the two-dimensional primary flow equations are obtained

$$\nabla \cdot \bar{u}_2 = 0$$

$$\frac{\partial \bar{u}_2}{\partial t} + (\bar{u}_2 \cdot \nabla)\bar{u}_2 = -\nabla p_2 + \frac{1}{R} \nabla^2 \bar{u}_2 - \nabla \cdot \bar{u}_3 \bar{u}_3$$

In the short hand notation, the second vector velocity in the Reynolds stresses is indicative of the vector representation of the two-dimensional flow and thus becomes a scalar in the component form of (2.5); while the divergence operates on the vector quantity formed by the first of the velocities. The two-dimensional flow, including the spanwise-averaged Reynolds stresses, is modulated in both $(x,t)$ because of the nonlinear or linear wavy disturbance $\bar{u}_1(x,y,t)$ upon the mean shear flow $\bar{u}_0(x,y)$. Upon subtracting (2.4) and (2.5) from the respective full equations (2.1) and (2.2), the nonlinear form of the three-dimensional disturbance equations are obtained. As in hydrodynamic stability theory, it is preferred to write the advective effect in terms of the spanwise-averaged flow $\bar{u}_2$,

$$\frac{\partial \bar{u}_3}{\partial t} + (\bar{u}_2 \cdot \nabla)\bar{u}_3 = -(\bar{u}_3 \cdot \nabla)\bar{u}_2 - \nabla p_3 + \frac{1}{R} \nabla^2 \bar{u}_3 - \nabla \cdot (\bar{u}_3 \bar{u}_3 - \bar{u}_3 \bar{u}_3)$$

The short hand notation used in (2.5) is again applied to the excess Reynolds stresses in (2.8). In general, the two-dimensional nonlinear system (2.4), (2.5) and the three-dimensional nonlinear system (2.7), (2.8) are interacting and are coupled through the Reynolds stresses and the advective effects $(\bar{u}_2 \cdot \nabla)\bar{u}_1$, $-(\bar{u}_3 \cdot \nabla)\bar{u}_2$ and the pressure. They describe the nonlinear interaction between the spanwise-mean two-dimensional field $\bar{u}_2$, $p_2$ and the three-dimensional field $\bar{u}_3$, $p_3$. The source of hydrodynamic instability of $\bar{u}_3$, i.e., momentum (and energy) exchanges with $\bar{u}_2$, lies in its advection of momentum sources coming from $\bar{u}_2$. The “internal interactions” within the two-dimensional system between $q_0$ and $q_1$ is the linear or nonlinear two-dimensional instability problem.

The physical mechanisms are best understood by considering the energy exchanges between the primary and secondary instability flows. The kinetic energy of the two-dimensional primary flow is obtained from (2.5), using (2.4),

$$\left( \frac{\partial}{\partial t} + \bar{u}_2 \cdot \nabla \right) k_{e_2} = \left[ - \bar{u}_3 \frac{\partial \bar{u}_3}{\partial x} - \bar{u}_3 v_3 \left( \frac{\partial \bar{u}_2}{\partial x} + \frac{\partial v_3}{\partial x} \right) - \bar{v}_3 \frac{\partial v_3}{\partial y} \right] - \nabla \left( p_2 \bar{u}_2 \right) + \frac{1}{R} \nabla^2 k_{e_2} + \Phi_2$$

The kinetic energy per unit mass (in dimensionless form) is written as $k_{e_2} = \bar{u}_2 \cdot \bar{u}_2 / 2$. The energy conversion mechanisms involve the Reynolds stresses of the three-dimensional disturbances doing work against the rates of strain of the two-dimensional disturbances; the first two terms in the square bracket represent exchange between $u_2^2 / 2$ and $u_3^2 / 2$, whereas the last two represent exchanges
between $v_2^2/2$ and $v_2^2/2$. The pressure work against the normal rates of strain through $p, \partial u_2/\partial x$ and $p, \partial v_2/\partial y$ sums to zero for an incompressible fluid in the overall energy balance, but in the internal balance it has the tendency to isotropize $u_2^2/2$ and $v_2^2/2$. The rate of viscous dissipation of is symbolically represented by $\Phi_2$.

The kinetic energy equation for the three-dimensional disturbance, $ke_3 = \bar{u}_3 \cdot \bar{u}_3/2$ after spanwise averaging is

$$
\left( \frac{\partial}{\partial t} + \bar{u}_3 \cdot \nabla \right) ke_3 = \left[ -u_3 \frac{\partial u_2}{\partial x} - u_3 v_3 \left( \frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) - v_3 \frac{\partial v_2}{\partial y} \right] - \nabla \cdot \left( p, \bar{u}_3 \right) 
$$

(2.10)

The energy exchange between the two-dimensional modulated flow and the three-dimensional disturbance, which is the group of bracketed terms on the right hand side of (2.10), appears as equal but opposite in sign as that appearing in (2.9). The first and third terms in the exchange mechanism are the work done by the normal Reynolds stresses of the three-dimensional disturbance against the normal rates of strain of the two-dimensional disturbance; the second group is the work done by the Reynolds shear stresses against the shear rates of strain.

In an unmodulated two-dimensional shear flow, the only energy conversion mechanism comes from the Reynolds shear stress of the three-dimensional disturbance doing work against the rate of strain of the two-dimensional flow, $-u_3 v_3 \partial u_2/\partial y$, which plays the only role towards instability development (Lin 1955). Since expressing the advection of $ke_3$ by $\bar{u}_3$ (“the mean flow”) is the preferred on the LHS of (3.2), the nonlinear self advection effect would then appear as the second term on the RHS of (3.2); together with $-\nabla \cdot \left( p, \bar{u}_3 \right)$, they are referred broadly as “self transport” of energy by the disturbances and do not contribute to net energy transfer; similarly with the viscous diffusion effect. The rate of viscous dissipation of the secondary instability energy is written in short as $\Phi_3$.

It is clear from (3.2) that the common reference to resonance generation of three-dimensional disturbance is meant that the net (volume integrated) energy exchange mechanism is positive and can overwhelm the effects of viscous dissipation. In order for this to happen, the net effect of energy exchange has to favour the instability to begin with and thus depends on the relative phase between the Reynolds stress and the appropriate rate of strain which it is working against.

In the discussion of (3.1) and (3.2), it is noticed that only $ke_2$ exchanges energy with $\bar{u}_3^2 + v_3^2/2$, whereas $w_3^2/2$ receives its sustainable development from the isotropizing mechanism of pressure velocity strain: $p, \partial w_3/\partial z = -p, (\partial u_3/\partial x + \partial v_3/\partial y)$.

The origin of the three-dimensional disturbance $\bar{u}_3$ is attributable to a stability analysis of the two-dimensional modulated nonlinear flow. This provides the most amplified modes within a band of spanwise wavenumbers $\beta$ and therefore the selectivity of naturally or externally excited disturbances within that band. The analysis is usually performed at the same frequency and streamwise wavenumber of the modulated two-dimensional flow $\bar{u}_3$. The spirit is similar to the receptivity of the two-dimensional shear flows where the eigenvalues are given by the Orr-Sommerfeld equation where the advecting velocity is simply the steady mean shear flow (Lin [6]).

3. Some comments on oblique wave disturbances in the far wake of two-dimensional bluff bodies: similarities with the far wake behind a flat plate
The work on a spatially developing two-dimensional unstable laminar wake behind a flat plate indicate that the basic $\vec{u}_2$ flow (Liu & Yu [7]; see also Maekawa, et al. [8], Ko, et al. [9], Sato & Kuriki [10]) strongly resembles that of bluff body wakes (Williamson & Prasad, Cimbala) in the two-dimensional shedding region. The nonlinear development of the two-dimensional wake ($\vec{u}_2$) behind a flat plate is illustrated in Figure 1 in terms of the spanwise vorticity contours. It has been termed a “spatially developing von Kármán vortex street” by Liu & Yu in their context. Bluff body wakes (Cimbala, et al., Williamson & Prasad) involve the shedding frequency, whereas in the incipient nonlinear region of the flat plate wake (Liu & Yu) the frequency is that of the fundamental disturbance. Bluff body wakes also may involve oblique shedding (Williamson & Prasad (1993a)). In incipient instability region of flat plate wakes, oblique-fundamental instabilities are much weaker than two-dimensional ones according to Squire’s theorem and thus were not observed. Despite of the differences of the initial development region, similarities do exist in the far wake regions of bluff body and flat plate wakes. The downstream region would be connected with the initial region as spatially developing flows do dictate the upstream initial value dependence, but different initial situation may well develop into similar quasi-equilibrium similarities as we shall see in the following in terms of cross-hatching oblique wave angles.

![Figure 1. The spanwise vorticity contours from computed two-dimensional nonlinear $\vec{u}_2$, initiated by the laminar wake behind a flat plate with an initial small amplitude fundamental mode for $Re=818$, wake centerline velocity defect of 0.55, frequency $\Omega=0.65$, fundamental streamwise wavenumber $\alpha=0.857$. The bottom figure is the streamwise continuation of the top figure. From Liu & Yu.](image)

Liu & Yu (2011) studied three-dimensional instability of the above strongly nonlinear base flow ($\vec{u}_2$). That is, it is used as the advecting velocity, after local parallel flow assumptions, in the three-dimensional disturbance stability analysis. This is in contrast to the simple Floquet analyses where the advecting velocity comes from the superposition of a laminar flow and its linear fundamental instability mode with an assigned amplitude as a parameter (Fleming [11]; see also Noaek & Eckelmann [12], Barkley, et al. [13]). The “selectivity” obtained from studies of the nonlinear base flow is illustrated in Figure 2, where naturally occurring or externally excited disturbances within a band of spanwise wavenumbers surrounding $\beta \approx 0.80$ would likely develop downstream. Previous work confirms that wavy characteristics obtained from upstream linear theory persist downstream,
thus it is possible to obtain oblique angles even for nonlinear three-dimensional flows downstream. The oblique angle to the two-dimensional “line” in a plan view is \( \tan \theta_{\text{sub}} = \beta / \alpha_{\text{sub}} \), where the subscript “sub” denotes the subharmonic streamwise wavenumber \( \alpha_{\text{sub}} = \alpha / 2 \). Its amplification rates are much stronger than that of the weaker fundamental at \( \alpha \). This does not imply that there is visually a wavelength doubling, it simply means that the three-dimensional subharmonic is the stronger disturbance than the three-dimensional fundamental. For the cases shown in Figures 1 and 2, the oblique angle is about \( \theta_{\text{sub}} \approx 62^0 \), so that the wave front propagation angle is about \( 28^0 \). Although the conditions in terms of parameter ranges, geometry and intrinsic nature are very much different between bluff body wakes and the present flat plate far wake, relatively large wave angles have been observed by Williamson & Prasad (Figure 7 of [4] at \( 64^0 \)) in the range \( 43^0 - 74^0 \) for different ratios of two-dimensional to oblique-wave frequencies.

![Figure 2. The three-dimensional secondary instability (\( \tilde{u}_1 \)) of the basic nonlinear two-dimensional wake flow (\( \tilde{u}_2 \)).](image)

From the general discussions of Section 2, the basic equations for the three-dimensional disturbances, (2.7), (2.8) do reflect the dependence of initial starting conditions, while their development must necessarily be “sparked” by natural or external circumstances and selected by the characteristics in Figure 2 in the flat plate case, the far downstream situation might well have a “universal” behavior in terms of oblique wave angles, though without rigorous proof. We emphasize that the observations drawn here are differential equation(s) based and do bring the aforementioned different situation of bluff body wakes, as described by Cimbala, et al., and Williamson & Prasad and the present flat plate wake closer together.

**References**

[1] Liu, J.T.C.: Contributions to the understanding of large-scale coherent structures in developing free turbulent shear flows. *Adv. Appl. Mech.* **26**: 183–309, 1988.

[2] Cimbala, J.M., Nagib, H.M. & Roshko, A.: Large structure in the far wakes of two-dimensional bluff bodies. *J. Fluid Mech.* **190**: 265–298, 1988.

[3] Williamson, C.H.K. & Prasad, A.: A new mechanism for oblique wave resonance in the ‘natural’ far wake. *J. Fluid Mech.* **256**: 269–313, 1993.

[4] Williamson, C.H.K. & Prasad, A.: Acoustic forcing of oblique wave resonance in the far wake. *J. Fluid Mech.* **256**: 315–341, 1993.

[5] Herbert, T.: Secondary instability of boundary layers. *Ann. Rev. Fluid Mech.* **20**: 487–526, 1988.
[6] Lin, C.C.: *The theory of hydrodynamic stability*. Cambridge, UK: Cambridge University Press, 1955.
[7] Liu, J.T.C. & Yu, X.: The three-dimensional secondary instability of a spatially developing von Kármán vortex street in a far wake. *Proc. R. Soc. A* 467: 675-694, 2011.
[8] Maekawa, H., Mansour, N. & Buell, J. C.: Instability mode interactions in a spatially developing plane wake. *J. Fluid Mech.* 235: 223–254, 1992.
[9] Ko, D.R., Kubota, T. & Lees, L.: Finite disturbance effect on the stability of a laminar incompressible wake behind a flat plate. *J. Fluid Mech.* 40: 315–341, 1970.
[10] Sato, H. & Kuriki, K.: The mechanism of transition in the wake of a thin flat plate placed parallel to uniform flow. *J. Fluid Mech.* 11: 321–352, 1961.
[11] Fleming, M.F.: Secondary instability on the far wake. MS thesis, Illinois Institute of Technology, USA, 1987.
[12] Noack, B.N. & Eckelmann, H.: A global analysis of the steady and periodic cylinder wake. *J. Fluid Mech.* 270: 297–330, 1994.
[13] Barkley, D. & Henderson, R.: Three-dimensional Floquet stability analysis of the wake of a circular cylinder. *J. Fluid Mech.* 322: 215–241, 1996.