Reflection Groups and Polytopes over Finite Fields, II

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Abstract
When the standard representation of a crystallographic Coxeter group $\Gamma$ is reduced modulo an odd prime $p$, a finite representation in some orthogonal space over $\mathbb{Z}_p$ is obtained. If $\Gamma$ has a string diagram, the latter group will often be the automorphism group of a finite regular polytope. In Part I we described the basics of this construction and enumerated the polytopes associated with the groups of rank 3 and the groups of spherical or Euclidean type. In this paper, we investigate such families of polytopes for more general choices of $\Gamma$, including all groups of rank 4. In particular, we study in depth the interplay between their geometric properties and the algebraic structure of the corresponding finite orthogonal group.

Key Words: reflection groups, abstract regular polytopes

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1 Introduction

The regular polytopes are a rich and ongoing source of mathematical ideas. Their combinatorial features, for instance, have been beautifully generalized in the theory of abstract regular polytopes.

In [17], the precursor to this paper, we surveyed some of the essential properties of an abstract regular polytope $\mathcal{P}$, referring to [13] for details. Then, reframing the key results

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in [25], we outlined an abbreviated classification of finite, irreducible groups generated by reflections in \( n \)-space \( V \), over a field of odd characteristic \( p \) (see [17] Thm. 3.1).

When \( G \) is a (possibly infinite) crystallographic Coxeter group with string diagram, reduction modulo an odd prime \( p \) of the standard real representation yields a finite reflection group \( G^p \), which we could then classify and which is often the automorphism group of a finite, abstract regular \( n \)-polytope \( P \). (If this is so, we say that \( G^p \) is a string C-group.)

Next we established two useful criteria for \( G^p \) to be a string C-group: Theorems 4.1 and 4.2 of [17] concern the features of \( V \) as an orthogonal space, as well as the action of standard subgroups of \( G^p \) on \( V \). With this, we were able to classify all groups \( G^p \), and their polytopes, whenever \( n \leq 3 \), as well as when \( G \) is of spherical or Euclidean type, for all ranks \( n \).

Here, we begin by summarizing in Section 2 some key notation. Next, in Section 3, we extend our criteria for \( G^p \) to be a string C-group. Finally, in Sections 4 and 5, we discuss and completely classify all 4-polytopes which arise from our construction.

## 2 Notation

We refer the reader to the notation and basic set up in [17]. Throughout, \( G = \langle r_0, \ldots, r_{n-1} \rangle \) will be a crystallographic Coxeter group \([p_1, p_2, \ldots, p_{n-1}]\) with a string Coxeter diagram \( \Delta_s(G) \) (with branches labeled \( p_1, p_2, \ldots, p_{n-1} \), respectively), obtained from the corresponding abstract Coxeter group \( \Gamma = \langle \rho_0, \ldots, \rho_{n-1} \rangle \) via the standard representation on real \( n \)-space \( V \).

For any odd prime \( p \), we may reduce \( G \) modulo \( p \) to obtain a subgroup \( G^p \) of \( GL_n(\mathbb{Z}_p) \) generated by the modular images of the \( r_i \)'s. We shall abuse notation by referring to the modular images of objects by the same name (such as \( r_i, b_i, B = [b_{ij}], V, \) etc.). In particular, \( \{b_i\} \) will denote the standard basis for \( V = \mathbb{Z}_p^n \). In any event, \( G^p \) is a subgroup of the orthogonal group \( O(\mathbb{Z}_p^n) \) of isometries for the (possibly singular) symmetric bilinear form \( x \cdot y \), the latter being defined on \( \mathbb{Z}_p^n \) by means of the Gram matrix \( B \); in particular, \( r_i \) is the orthogonal reflection with root \( b_i \) if \( b_i^2 \neq 0 \).

Next we make a convenient definition: if \( p \geq 5 \), or \( p = 3 \) but no branch of \( \Delta_s(G) \) is marked 6, then we say that \( p \) is generic for \( G \). In such cases, no node label of the diagram \( \Delta(G) \) (for a basic system) is zero mod \( p \) and a change in the underlying basic system for \( G \) has the effect of merely conjugating \( G^p \) in \( GL_n(\mathbb{Z}_p) \). On the other hand, in the non-generic case, in which \( p = 3 \) and \( \Delta_s(G) \) has branches marked 6, the group \( G^p \) may depend essentially on the actual diagram \( \Delta(G) \) taken for the reduction mod \( p \). (Note that \( p \) generic does not necessarily mean that \( p \nmid |G| \), or that certain subspaces of \( V \) are non-singular, etc.)

Recall from [17] Thm. 3.1] that an irreducible group \( G^p \) of the above sort, generated by \( n \geq 3 \) reflections, must necessarily be one of the following:

- an orthogonal group \( O(n, p, \epsilon) = O(V) \) or \( O_2(n, p, \epsilon) = O_2(V) \), excluding the cases \( O_1(3, 3, 0), O_2(3, 5, 0), O_2(5, 3, 0) \) (supposing for these three that \( \text{disc}(V) \sim 1 \)), and also excluding the case \( O_2(4, 3, -1) \); or
- the reduction mod \( p \) of one of the finite linear Coxeter groups of type \( A_n \) \((p \nmid n + 1), B_n, D_n, E_6 \) \((p \neq 3) \), \( E_7, E_8, F_4, H_3 \) or \( H_4 \).
We shall say in these two cases that $G^p$ is of orthogonal or spherical type, respectively, although there is some overlap for small primes. Our description rests on the classification of the finite irreducible reflection groups over any field, obtained in Zalesskiĭ & Sereţkin [25] (see also [9, 20, 21, 24]). It is only a slight abuse of notation to let $[p_1, \ldots, p_{n-1}]^p$ denote the modular representation of a group $[p_1, \ldots, p_{n-1}]$, so long as $p$ is generic for the group.

The generators $r_j$ of $G^p$ satisfy the Coxeter-type relations inherited from $G$. Our main problem is to determine when $G^p$ has the intersection property (1) for its standard subgroups. For any $J \subseteq \{0, \ldots, n-1\}$, we let $G^p_J := \langle r_j \mid j \not\in J \rangle$; in particular, for $k, l \in \{0, \ldots, n-1\}$ we let $G^p_k := \langle r_j \mid j \neq k \rangle$ and $G^p_{k,l} := \langle r_j \mid j \neq k, l \rangle$. Then $G^p$ is a string C-group if and only if $G^p$ satisfies the intersection property

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle;$$

and in this case $G^p$ is the automorphism group of a finite regular polytope denoted by $\mathcal{P}(G^p)$ (see [13, §2E]). Note as well that $G^p$ is a string C-group if and only if $G^p_0$ and $G^p_{n-1}$ are string C-groups and $G^p_0 \cap G^p_{n-1} = G^p_{0,n-1}$. We also let $V_J$ be the subspace of $V = \mathbb{Z}_p^n$ spanned by $\{b_j \mid j \notin J\}$, and similarly for $V_k, V_{k,l}$. Note that $V_J$ is $G^p_J$-invariant.

### 3 The Intersection Property

The goal of this section is to assess when the modular reduction $G^p$ of a crystallographic string Coxeter group $G$ satisfies the intersection property (1). In [17] we established a number of sufficient conditions and verified that $G^p$, with $p \geq 3$, has the intersection property whenever $G$ has rank at most 3, or whenever $G$ is of spherical or Euclidean type. However, the situation changes drastically for more general groups of higher ranks, with obstructions already occurring for rank 4. We already know that $G^p$ is a string C-group if one of the subgroups $G^p_0$ or $G^p_{n-1}$ is spherical and the other is a string C-group (see [17, Thm. 4.2]). Moreover, $G^p$ also is a string C-group if both $G^p_0$ and $G^p_{n-1}$ are C-groups, $V_{0,n-1}$ is a non-singular subspace of $V$, and $G^p_{0,n-1}$ is the full orthogonal group $O(n - 2, p, \epsilon)$ on $V_{0,n-1}$ (see [17, Thm. 4.1]).

The criteria established here will settle the Coxeter groups $[k, l, m]$ of rank 4 completely. Before we move on, note two simple cases. If $l = 2$ and $k, m < \infty$, then

$$G \cong [k] \times [m] \cong G^p,$$

so $G^p$ certainly is a C-group. Similarly, if say $m = 2$ (but $k$ or $l = \infty$ is allowed), then $G \cong [k, l] \times C_2$, and

$$G^p \cong [k, l]^p \times C_2.$$

Thus the intersection property of $G^p$ follows directly from that of its subgroup $[k, l]^p$ (see [17, Thm. 5.1]). More generally, if $[p_1, \ldots, p_{n-1}]^p$ is a string C-group, then so is

$$[p_1, \ldots, p_{n-1}, 2]^p \cong [p_1, \ldots, p_{n-1}]^p \times C_2.$$

After a preliminary lemma, we shall continue to build upon the known results concerning C-groups mentioned above.
Lemma 3.1 Suppose that $G$ has rank $n$ and that $(\text{rad} V) \cap V_j = \{o\}$. Then the subgroup $G_j^p$ is, by restriction to the invariant subspace $V_j$, isomorphic to $H^p$, the reduction modulo $p$ of the group of rank $n-1$ defined from the subdiagram of $\Delta(G)$ which results from the deletion of node $j$. In particular, when $j = 0$ or $n-1$ this holds if $p$ is generic for $G$.

Proof. This result is well known in characteristic 0 [8, §5.5]. Here we restrict $g \in G_j^p$ to the invariant subspace $V_j$, and so obtain a homomorphism

$$\varphi : G_j^p \rightarrow O(V_j),$$

$$g \mapsto g|_{V_j}.$$

Of course, as a subspace of $V$, $V_j$ is isometric to $\mathbb{Z}^{n-1}_p$, with the metric structure obtained from the subdiagram of $\Delta(G)$ obtained by deleting node $j$. Clearly the image group $\varphi(G_j^p)$ is isomorphic to the reflection group $H^p$ of rank $n-1$ defined directly from the subdiagram.

Suppose $g \in \ker \varphi$. Then $g(b_k) = b_k$ for all $k \neq j$, whereas $g(b_j) = b_j + x$ for some $x \in V_j$. Thus for any $k \neq j$

$$b_j \cdot b_k = g(b_j) \cdot g(b_k) = (b_j + x) \cdot b_k = b_j \cdot b_k + x \cdot b_k,$$

so that $x \cdot b_k = 0$, and $x \in \text{rad} V_j$. But then $x \cdot x = 0$ and so

$$b_j \cdot b_j = g(b_j) \cdot g(b_j) = b_j \cdot b_j + 2x \cdot b_j + x \cdot x,$$

whence $x \cdot b_j = 0$. Thus $x \in (\text{rad} V) \cap V_j$, so that $x = o$ when this subspace is trivial. Hence $\varphi$ is injective. When $p$ is generic for $G$, a direct calculation in coordinates along the string diagram shows that $(\text{rad} V) \cap V_j = \{o\}$ for $j = 0, n-1$. □

Remark. Informally, the Lemma asserts that reduction by a generic prime commutes with the deletion of a node from $\Delta(G)$. Note that

$$G_j^p \simeq [p_1, \ldots, p_{j-1}]^p \times [p_{j+2}, \ldots, p_{n-1}]^p \simeq [p_1, \ldots, p_{j-1}, 2, p_{j+2}, \ldots, p_{n-1}]^p.$$

Concerning the non-generic cases, there are examples showing the necessity of the hypotheses. For example, the group $G \simeq [4, 3, 6]$ with diagram

$$2 \hspace{1cm} 1 \hspace{1cm} 1 \hspace{1cm} 3$$

yields, as we observe below, a $C$-group $G^3$. Here the subgroup $G_0^3$ is the automorphism group of order 108 for the toroidal polyhedron $\{3, 6\}_{(3,0)}$. However, the subdiagram

$$1 \hspace{1cm} 1 \hspace{1cm} 3$$

yields the smaller group of order 36 for $\{3, 6\}_{(1,1)}$. Thus the map $\varphi$ of the Lemma is here not injective.

Theorem 3.1 Suppose that $G \simeq [k, l, m]$ is crystallographic and that the subgroup $[k, l]$ or $[l, m]$ is spherical. Then $G^p$ is a $C$-group for any prime $p \geq 3$. 

4
Proof. Let \([k, l]\) (say) be spherical, so that \(G^p_3 \simeq G_3 = [k, l]\). First suppose that \(p\) is generic for \(G\). Since \(G^p_0\) is a \(C\)-group by Lemma 3.1, the proof follows directly from [17] Thm. 4.2. Moreover, even in non-generic cases of rank 4 (so that \(p = 3\)), \(G^3\) turns out to be a \(C\)-group when \([k, l]\) is spherical. This is routinely verified using the computer algebra system GAP [4]. The pertinent examples are \(G \simeq [3, 3, 6], [3, 4, 6] \) or \([4, 3, 6]\), each with two essentially distinct diagrams \(\Delta(G)\) (for the basic systems). \(\square\)

We now establish two general results, of which the first allows us to reject large classes of groups \(G^p\) as \(C\)-groups because of the size of their subgroups \(G^p_0 \cap G^p_{n-1}\). First we deal with the fully non-singular case.

**Theorem 3.2** Let \(G = \langle r_0, \ldots, r_{n-1} \rangle\) be a crystallographic linear Coxeter group with string diagram. Suppose that \(n \geq 3\) and that the prime \(p\) is generic for \(G\). Let the subspaces \(V, V_0, V_{n-1}\) and \(V_{0,n-1}\) be non-singular, and let \(G^p_0, G^p_{n-1}\) be of orthogonal type. Suppose as well that there is a square among the labels of the nodes \(1, \ldots, n-2\) of the diagram \(\Delta(G)\) (this can be achieved by readjusting the node labels).

(a) Then \(G^p_0 \cap G^p_{n-1}\) acts trivially on \(V^\perp_{0,n-1}\), and

\[
O_1(V^\perp_{0,n-1}) \leq G^p_0 \cap G^p_{n-1} \leq O(V_{0,n-1}),
\]

where we have identified the \((n-2)\)-dimensional groups \(O(V_{0,n-1})\) and \(O_1(V_{0,n-1})\) with the pointwise stabilizers of \(V^\perp_{0,n-1}\) in the \(n\)-dimensional groups \(O(V)\) and \(O_1(V)\), respectively.

(b) If \(G^p_0 = O(V_0)\) and \(G^p_{n-1} = O(V_{n-1})\), with a similar interpretation as stabilizers, then

\[
G^p_0 \cap G^p_{n-1} = O(V_{0,n-1}).
\]

Proof. Since all four subspaces of \(V = \langle b_0, \ldots, b_{n-1} \rangle\) are non-singular, we have the orthogonal sums

\[
V = V_{n-1} \oplus \langle v \rangle = V_0 \oplus \langle v' \rangle, \quad V_{n-1} = V_{0,n-1} \oplus \langle w \rangle, \quad V_0 = V_{0,n-1} \oplus \langle w' \rangle,
\]

for non-isotropic vectors \(v, v', w, w'\). Then,

\[
\langle v, v' \rangle = V^\perp_{0,n-1} = \langle w, w' \rangle.
\]

Since \(p\) is generic for \(G\), each reflection \(r_j\) actually has \(b_j\) as a root, and \(v \perp b_j\) for \(j \leq n-2\), while \(v' \perp b_j\) for \(j \geq 1\). Hence the subgroups \(G^p_{n-1}, G^p_0\) and \(G^p_0 \cap G^p_{n-1}\) stabilize the vectors \(v, v'\) or \(v, v', v', v'\), respectively. In particular,

\[
G^p_0 \cap G^p_{n-1} \leq O(V_{0,n-1}),
\]

with \(O(V_{0,n-1})\) identified with the pointwise stabilizer of \(V^\perp_{0,n-1}\) in \(O(V)\). Note that the restrictions of \(G^p_{n-1}, G^p_0\) and \(G^p_0 \cap G^p_{n-1}\) to the subspaces \(V_{n-1}, V_0\) or \(V_{0,n-1}\), respectively, are faithful, by Lemma 3.1.

Since \(G^p_0, G^p_{n-1}\) are of orthogonal type and there is a square among the labels of the nodes \(1, \ldots, n-2\), we must have \(O_1(V_{n-1}) \leq G^p_{n-1}\) and \(O_1(V_0) \leq G^p_0\) (that is, a group merely of type \(O_2\) cannot occur). Now, if \(g \in O_1(V_{0,n-1})\), then \(g(v) = v\), so that \(g \in O(V_{n-1})\); but the spinor norm is invariant under orthogonal embedding ([17] Thm. 5.13]), so actually
Let $g \in O_1(V_{n-1})$. Similarly, $g \in O_1(V_0)$, and hence $g \in G^p_0 \cap G^p_{n-1}$. This completes the proof of part (a).

Now let $G^p_0 = O(V_0)$ and $G^p_{n-1} = O(V_{n-1})$. Once again, if $g \in O(V_{0,n-1})$, then $g(v) = v$, so now $g \in O(V_{n-1}) = G^p_{n-1}$. Similarly, $g \in O(V_0) = G^p_0$, and hence $g \in G^p_0 \cap G^p_{n-1}$, as required.

We note an immediate corollary to Theorem 3.2. It shows that many groups $G^p$ of rank 4 fail to satisfy the intersection property for large primes $p$. However, those primes for which $G^p$ actually is a C-group lead to interesting polytopes, which we investigate in later sections.

**Corollary 3.1** Suppose the prime $p$ is generic for the crystallographic group $G = [k, l, m]$. Let $V$, $V_0$, $V_3$, $V_{0,3}$ be non-singular, and let $G^p_0$, $G^p_3$ be of orthogonal type.
(a) Then $G^p$ is not a C-group if $p > 2l + \epsilon(V_{0,3})$, where $\epsilon(V_{0,3}) = \pm 1$ is the parameter associated with the plane $V_{0,3}$.
(b) If $G^p_0 = O(V_0)$ and $G^p_3 = O(V_3)$, then $G^p$ is a C-group if and only if $p = l + \epsilon(V_{0,3})$.

**Proof.** We apply Theorem 3.2 with $n = 4$. The subgroups $G^p_0$ and $G^p_3$ are known to be C-groups ([17] Thm. 5.1), so it suffices to determine when

$$G^p_0 \cap G^p_3 = G^p_{0,3}.$$ 

Now $G^p_{0,3} = \langle r_1, r_2 \rangle$ is a dihedral group of order $2l = 6, 8$ or $12$; the case $l = \infty$ is excluded, as $V_{0,3}$ is then a non-singular plane. Note that we may assume that there is a square (in fact, a 1) among the labels of the nodes 1 or 2 of the diagram; this can be achieved by readjusting the node labels as described earlier. Then, by Theorem 3.2 we have

$$O_1(V_{0,3}) \leq G^p_0 \cap G^p_3,$$

so $G^p_0 \cap G^p_3$ is larger than $G^p_{0,3}$ if the order of $O_1(V_{0,3})$, which is $p - \epsilon(V_{0,3})$, exceeds $2l$. Hence the intersection property certainly fails if $p > 2l + \epsilon(V_{0,3})$. Moreover, by Theorem 3.2 if $G^p_0 = O(V_0)$ and $G^p_3 = O(V_3)$, then

$$G^p_0 \cap G^p_3 = O(V_{0,3}),$$

so $G^p_0 \cap G^p_3 = G^p_{0,3}$ if and only if $2(p - \epsilon(V_{0,3})) = 2l$, or equivalently, $p = l + \epsilon(V_{0,3})$. □

Corollary 3.1 immediately implies (in fully non-singular cases) that $G^p$ is not a C-group if $p > 13$. However, for the primes $p = 5, 7, 11, 13$ (and 3), the outcome is less predictable and actually depends on the group $G = [k, l, m]$ as well as the diagram $\Delta(G)$ chosen for the reduction modulo $p$. For example, $G^{13}$ can only be a C-group if $l = 6$ and $G^{13}_0 = O_1(V_0)$ or $G^{13}_3 = O_1(V_3)$. Similarly, if $G^p_0 = O(V_0)$ and $G^p_3 = O(V_3)$, then $G^p$ is not a C-group if $p > 7$; moreover, $G^7$ can then be a C-group only if $l = 6$.

Next we study the case when the middle section of the diagram for $G$ determines a singular space $V_{0,n-1}$, while $V$, $V_0$ and $V_{n-1}$ still are non-singular, again with $G^p_0, G^p_{n-1}$ of orthogonal type. In a singular space $W$ over a field $\mathbb{K}$, the isometry group $O(W)$ leaves invariant the radical subspace $\text{rad} W$, thereby providing a natural epimorphism $\eta: O(W) \to O(W/\text{rad} W)$. Since $W/\text{rad} W$ is non-singular, we may define a ‘spinor norm’ $\theta$ on $W$, sufficient for our needs, by

$$\theta(g) := \theta_{W/\text{rad} W}(\eta(g)), \; g \in O(W).$$
Now let $\hat{O}(W)$ denote the subgroup of $O(W)$ consisting of those isometries $g$ which act trivially on $\text{rad } W$. It is not hard to show that $\hat{O}(W)$ contains all reflections (with non-isotropic roots in $W$) and is even generated by them. The key observation here is that any transvection in the kernel of the action of $O(W)$ on $\text{rad } W$ can be factored as a product of reflections (cf. [1, Thm. 3.20 and p. 133]).

It is easy to see that if $g$ is the product of reflections with non-isotropic roots $a_1, \ldots, a_k$, then $\theta(g) = a_1^2 \cdots a_k^2 \mathbb{K}^2$. Naturally, by $\hat{O}_1(W)$ (or $\hat{O}_2(W)$) we mean the subgroup of $\hat{O}(W)$ generated by the reflections in $O(W)$ whose spinor norm is a square (or non-square, respectively).

**Theorem 3.3.** Let $G = \langle r_0, \ldots, r_{n-1} \rangle$ be a crystallographic linear Coxeter group with string diagram, and suppose the prime $p$ is generic for $G$. Let $V$, $V_0$, $V_{n-1}$ be non-singular, let $V_{0,n-1}$ be singular, and let $G^p_0, G^p_{n-1}$ be of orthogonal type. Suppose there is a square among the labels of the nodes $1, \ldots, n-2$ of the diagram $\Delta(G)$ (this can be achieved by readjusting the node labels).

(a) Then $G^p_0 \cap G^p_{n-1}$ acts trivially on $V^\perp_{0,n-1}$, and

$$\hat{O}_1(V_{0,n-1}) \leq G^p_0 \cap G^p_{n-1} \leq \hat{O}(V_{0,n-1}),$$

where $\hat{O}(V_{0,n-1})$ has been identified with the pointwise stabilizer of $V^\perp_{0,n-1}$ in $O(V)$, and $\hat{O}_1(V_{0,n-1})$ with the subgroup of $O_1(V)$ generated by the reflections with roots in $V_{0,n-1}$ and square spinor norm.

(b) If $G^p_0 = O(V_0)$ and $G^p_{n-1} = O(V_{n-1})$, then also

$$\hat{O}(V_{0,n-1}) = G^p_0 \cap G^p_{n-1}.$$ 

**Proof.** As before we have

$$V = V_{n-1} \oplus \langle v \rangle = V_0 \oplus \langle v' \rangle$$

with non-isotropic vectors $v, v'$. The subspace $V^\perp_{0,n-2}$ still is 2-dimensional, so necessarily $V^\perp_{0,n-1} = \langle v, v' \rangle$. Moreover,

$$V_{0,n-1} \cap V^\perp_{0,n-1} = \text{rad } V_{0,n-1} \neq \{0\},$$

so the vectors in $V_{0,n-1} \cup V^\perp_{0,n-1}$ span a singular hyperplane $U$ in $V$ with 1-dimensional radical $\text{rad } U = \text{rad } V_{0,n-1}$. For the same reason as before, $G^p_{n-1}$, $G^p_0$ and $G^p_0 \cap G^p_{n-1}$ stabilize the vectors $v, v'$ or $v, v'$, respectively, and yield faithful restrictions to the subspaces $V_{n-1}$, $V_0$ and $V_{0,n-1}$. In particular,

$$G^p_0 \cap G^p_{n-1} \leq H := \{g \in O(V) \mid g(x) = x, \forall x \in V^\perp_{0,n-1}\}.$$

Note also that $G^p_0 \cap G^p_{n-1}$ leaves $U$ invariant because it leaves $V_{0,n-1}$ invariant.

We claim that we may identify $H$ with $\hat{O}(V_{0,n-1})$; this would settle the inclusion on the right in part (a) of the theorem. Now, since each element of $H$ leaves $V_{0,n-1}$ invariant, while fixing $\text{rad } V_{0,n-1}$, we can consider the restriction mapping to $V_{0,n-1}$,

$$\kappa : H \rightarrow \hat{O}(V_{0,n-1})$$

$$g \mapsto g|_{V_{0,n-1}}.$$  

(2)
We prove that $\kappa$ is an isomorphism. If $g \in \ker(\kappa)$, then $g$ acts trivially on $V_{0,n-1}$, and hence also on $U$ because $g \in H$. It follows that $g = e$, the identity mapping on $V$. Here we have used the fact that an isometry of a non-singular space is uniquely determined by its effect on a hyperplane, $H$ in this case, if this hyperplane is singular ([1, Thm. 3.17]). This shows that $\kappa$ is injective. Now let $h \in \hat{O}(V_{0,n-1})$. Then $h$ acts trivially on rad $V_{0,n-1}$, so we can extend $h$ to an isometry $h'$ (say) of $U = V_{0,n-1} \oplus \langle v \rangle$ by setting $h'(v) := v$. We now apply Witt’s extension theorem for isometries between subspaces of a non-singular space ([1, Thm. 3.9]) and conclude that $h'$ extends further to an isometry $g$ of the entire space $V$. Then $g$ must be in $H$ and so $\kappa$ is also surjective. By our earlier remarks, $\hat{O}(V_{0,n-1})$ is generated by all reflections $r_a$, for non-isotropic roots $a \in V_{0,n-1}$; pulling back, a similar claim is true for $H$.

Continuing along these lines, we now prove the inclusion on the left in part (a) of the theorem. For a non-isotropic vector $a \in V_{0,n-1}$, let $r_{a,V}$, $r_{a,V_0}$, $r_{a,V_0^\perp}$ and $r_{a,V_0,n-1}$ denote the reflections with root $a$ in $V$, $V_{n-1}$, $V_0$ or $V_{0,n-1}$, respectively. Then $r_{a,V} \in H$ because $a \perp V_{0,n-1}$, and

$$r_{a,V_0,n-1} = \kappa(r_{a,V}).$$

It follows that the subgroup $H_1$ of $H$ generated by the reflections $r_{a,V}$, with $a \in V_{0,n-1}$ and $a^2$ a square, is isomorphic, under $\kappa$, to $\hat{O}_1(V_{0,n-1})$. We have to show that $H_1 \leq G^p_0 \cap G^p_{n-1}$.

Now, by assumption, the subgroups $G^p_0$, $G^p_{n-1}$ of $G$ are of orthogonal type, and there is a square among the labels of the nodes 1, . . . , $n - 2$ of the diagram. It follows that $G^p_0$ and $G^p_{n-1}$, when restricted to the subspaces $V_0$ or $V_{n-1}$, respectively, must contain the groups $O_1(V_0)$ or $O_1(V_{n-1})$. Hence, if we identify the restricted groups with the stabilizers of $v'$ or $v$, respectively, in $O_1(V)$, then we see that $O_1(V_0)$ and $O_1(V_{n-1})$ are actually subgroups of $G^p_0$ and $G^p_{n-1}$. In particular, if $a \in V_0$ and $a^2$ is a square, then $r_{a,V}$ belongs to $O_1(V_j)$, for $j = 0, n - 1$, so $r_{a,V} \in G^p_0 \cap G^p_{n-1}$. Now it follows that $H_1$ is a subgroup of $G^p_0 \cap G^p_{n-1}$. This settles part (a).

Finally, suppose that $G^p_0 = O(V_0)$ and $G^p_{n-1} = O(V_{n-1})$. Much of the same analysis carries over, but now it is applied to all reflections, including those whose spinor norm is a non-square. In particular, if $a$ is any non-isotropic vector in $V_{0,n-1}$, then $r_{a,V} \in G^p_0 \cap G^p_{n-1}$. Hence we also have

$$\hat{O}(V_{0,n-1}) \leq G^p_0 \cap G^p_{n-1},$$

since $\hat{O}(V_{0,n-1})$ is generated by its reflections. Now part (a) gives the equality of these groups.

□

Remark. It is not difficult to show that $\hat{O}_1(V_{0,n-1})$ can also be identified with the pointwise stabilizer of $V_{0,n-1}^\perp$ in $O_1(V)$.

For a crystallographic Coxeter group $[k, l, m]$, the middle section of the diagram determines a singular subspace if and only if $l = \infty$. In this case, the reduced group $G^p$ is always a C-group:

Corollary 3.2 Let $G \simeq [k, \infty, m]$ be crystallographic. Then $G^p$ is a C-group for any prime $p \geq 3$.

Proof. We know that $G^p_0$ and $G^p_3$ are C-groups ([17 Thm. 5.1]), so again it suffices to check
that
\[ G^p_0 \cap G^p_3 = G^p_{0,3}. \]
Suppose for the moment that \( p \) is generic for \( G \), so that we may apply Theorem 3.3 with \( n = 4 \). Then, with few exceptions, \( V \) is still non-singular. Moreover, \( V_0 \) and \( V_3 \) correspond to the subgroups \([\infty, m]\) and \([k, \infty]\) of \( G \), and hence are known to be non-singular as well (except in the non-generic case with \( p = 3, k, m = 6; \) see [17, Sect. 5]). However, \( V_{0,3} \) is a singular plane, so Theorem 3.3 implies that
\[ \hat{O}(V_{0,3}) \leq G^p_0 \cap G^p_3 \leq \hat{O}(V_{0,3}), \]
provided \( V \) is non-singular. If the labels of the nodes 1 and 2 of the diagram are 1 and 4, respectively, as we may assume, then \( V_{0,3} \) is a singular plane in which the squared norm of each non-isotropic vector is a square. In particular,
\[ \hat{O}(V_{0,3}) = \hat{O}(V_{0,3}) \cong [p] \cong G^p_{0,3}, \]
and hence \( G^p_0 \cap G^p_3 = G^p_{0,3} \). In fact, since \( 2b_1 + b_2 \) spans \( \text{rad} \, V_{0,3} \), a matrix representing an element of \( \hat{O}(V_{0,3}) \) in the basis \( b_1, 2b_1 + b_2 \) must necessarily have the form
\[ \begin{pmatrix} \pm 1 & 0 \\ \mu & 1 \end{pmatrix} \quad (\mu \in \mathbb{Z}_p), \]
so there are at most \( 2p \) of them. On the other hand, the restrictions of \( r_1 \) and \( r_2 \) to \( V_{0,3} \) already generate a dihedral group \([p]\) contained in \( \hat{O}(V_{0,3}) \), so the three groups must coincide.

The space \( V \) is singular for the following groups \( G \) (up to duality) and primes \( p (> 3): [4, \infty, 3] \) and \([6, \infty, 6]\), with \( p = 5; [3, \infty, 3] \) and \([4, \infty, 6]\), with \( p = 7; \) and \([3, \infty, 6]\), with \( p = 13 \). In each of these cases, as well as for \( p = 3 \) for any of the crystallographic groups \([k, \infty, m]\), computations in GAP confirm that \( G^p \) also is a C-group.

We now concentrate entirely on the groups \( G = [k, l, m] \) which are not yet covered by the previous results. These groups have a Euclidean subgroup \([k, l]\) or \([l, m]\). The following theorem settles the case when \( l = 4 \) or 6.

**Theorem 3.4** Let \( G \cong [k, l, m] \) be crystallographic. Suppose the subgroup \([k, l]\) or \([l, m]\) is Euclidean, and that \( l = 4 \) or 6. Then \( G^p \) is a C-group for any prime \( p \geq 3 \).

**Proof.** Suppose \( G_3 = [k, l] \) (say) is Euclidean. The case \( m = 2 \) was already settled, so let \( m \neq 2 \). For the moment, let \( p \) be generic for \( G \). Then \( V = \langle b_0, \ldots, b_3 \rangle \) is non-singular and \( V_3 \) is singular, so that every isometry of \( V \) is uniquely determined by its effect on \( V_3 \). Let \( \text{rad} \, V_3 = \langle c \rangle \) (say).

Let \( g \in G^p_0 \cap G^p_3 \). Then \( g \) leaves \( V_3 \) and \( V_{0,3} \) invariant, and fixes \( c \). Since \( l = 4 \) or 6, we necessarily have \( G_3 = [3, 6] \) or \([4, 4]\), so \( G^p_3 \) is a semi-direct product of its “translation subgroup” \( T^p \) (of order \( p^2 \)) by \( G^p_{0,3} \). In particular, since we are allowed to multiply \( g \) by an element in \( G^p_{0,3} \), we may assume that \( g \in T^p \). We prove that this forces \( g = e \), hence the desired conclusion.

When \( G_3 = [4, 4] \), we may assume that the labels of the nodes 0, 1, 2 of the diagram of \( G \) are 1, 2, 1, respectively. Then \( c = b_0 + b_1 + b_2 \) and
\[ T^p = \langle r_0 r_1 r_2, r_1 r_0 r_1 r_2 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p. \]
Let \( M^p \) denote the group of \( p^2 \) matrices of the form

\[
M(\lambda, \mu) := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lambda & \mu & 1 \end{bmatrix} \quad (\lambda, \mu \in \mathbb{Z}_p). \tag{3}
\]

In the basis \( b_0, b_1, c \) of \( V_3 \), each element of \( T^p \), when restricted to \( V_3 \), is represented by a matrix in \( M^p \). \( T^p \) acts faithfully on \( V_3 \) by Lemma 3.1. For example, we have

\[
(r_0 r_1 r_2 r_1)^3 (r_1 r_0 r_1 r_2)^k \mapsto M(2j, 2(k - j))
\]

so \( T^p \) and \( M^p \) are clearly isomorphic. Now suppose that \( M(\lambda, \mu) \) is the matrix for \( g \). Then

\[
g(b_1) = b_1 + \mu c = \mu b_0 + (1 + \mu)b_1 + \mu b_2,
\]

so we must have \( \mu = 0 \) because \( V_{0,3} \) is invariant under \( g \). Similarly,

\[
g(b_2) = g(c - b_0 - b_1) = g(c) - g(b_0) - g(b_1) = c - (b_0 + \lambda c) - b_1 = -\lambda b_0 - \lambda b_1 + (1 - \lambda)b_2,
\]

so also \( \lambda = 0 \), for the same reason. It follows that \( g \) acts trivially on \( V_3 \) and hence also on \( V \), that is, \( g = e \).

When \( G_3 = [3, 6] \), we may take the labels of nodes 0, 1, 2 of the diagram to be 1, 1, 3, respectively. Then \( c = b_0 + 2b_1 + b_2 \) and

\[
T^p = \langle r_0 r_1 (r_2 r_1)^2, r_1 r_0 (r_1 r_2)^3 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.
\]

The elements of \( T^p \) still are represented by the matrices \( M(\lambda, \mu) \) in \( M^p \) (in the basis \( b_0, b_1, c \) of \( V_3 \)), so we can proceed in a similar fashion as above. In particular, if \( M(\lambda, \mu) \) is the matrix for \( g \), then we obtain \( \mu = 0 \) from

\[
g(b_1) = b_1 + \mu c = \mu b_0 + (1 + 2\mu)b_1 + \mu b_2,
\]

and \( \lambda = 0 \) from

\[
g(b_2) = g(c - b_0 - 2b_1) = g(c) - g(b_0) - 2g(b_1) = c - (b_0 + \lambda c) - 2b_1 = -\lambda b_0 - 2\lambda b_1 + (1 - \lambda)b_2,
\]

in each case using the invariance of \( V_{0,3} \) under \( g \). Hence \( g = e \), as required.

Only a few non-generic cases remain, which are not covered by previous theorems: \( p = 3 \) for \([4, 4, 6], \ [3, 6, 3], \ [3, 6, 4], \ [3, 6, 6] \) or \([3, 6, \infty] \). But for each of the eleven possibly distinct basic systems here, we easily verify the intersection condition with the help of \textit{GAP}.

This, then, leaves us with the groups \( G = [6, 3, m] \). If \( m = 3 \) or 4, then \( G_0 \) is spherical, so Theorem 3.1 applies and proves that \( G^p \) is a C-group. It remains to investigate the cases \( m = 6 \) or \( \infty \).

\textbf{Theorem 3.5} Let \( G = [6, 3, m] \) with \( m = 6 \) or \( \infty \). Then for \( m = 6 \), \( G^p \) is a C-group only for \( p = 3 \); and for \( m = \infty \), \( G^p \) is a C-group if and only if \( p = 3 \) or \( p \equiv \pm 5 \pmod{12} \). 

10
Proof. As in the previous theorem, we use GAP to verify the intersection condition in all cases with $p = 3$. So suppose $p > 3$. We may assume that the nodes 0, 1, 2, 3 have the set of labels 3, 1, 1, 3, or 3, 1, 1, 4 according as $m = 6$ or $\infty$. Since $V$ is non-singular, each isometry of $V$ is uniquely determined by its effect on the singular subspace $V_3$. In particular,

$$G_3^p \cong [6,3]^p \cong T^p \times \langle r_0, r_1 \rangle,$$

with

$$T^p = \langle r_1 r_2 (r_1 r_0)^2, r_2 r_1 (r_0 r_1)^2 \rangle \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$ 

In the basis $b_1, b_2, c$ of $V_3$, with $c := b_0 + 2b_1 + b_2$ (generating rad $V_3$), each element of $T^p$, when restricted to $V_3$, is represented by a matrix $M(\lambda, \mu)$ as in (3). In particular,

$$r_2 r_1 (r_0 r_1)^2 \mapsto M(-1, 2), \quad r_1 r_2 (r_1 r_0)^2 \mapsto M(1, 1).$$

Each element in $G_3^p$ is of the form $r_i^j (r_0 r_1)^t$ with $i = 0, 1, j = 0, \ldots, 5$ and $t \in T^p$. Inspection shows that the elements with $i = 0$ have the following matrix representation in the basis $b_1, b_2, c$: if a matrix $M(a, b)$ represents an element $t = t(a, b)$ of $T^p$, then the matrix of $(r_0 r_1)^j t(a, b)$ is obtained from the matrix of $(r_0 r_1)^j$ by simply adding $a$ or $b$, respectively, to the first or second entry in its last row.

Now suppose an element $g \in G_3^p$ leaves $V_{0,3}$ invariant. Multiplying by $r_1$ if need be, we may assume that $i = 0$, that is, $g = (r_0 r_1)^j t$ with $j = 0, \ldots, 5$ and $t = t(a, b) \in T^p$. The invariance of $V_{0,3}$ considerably restricts the possibilities for $j$ and $t$. In fact, inspection of the matrices for $(r_0 r_1)^j t(a, b)$ shows that we must have $(j, a, b) = (0, 0, 0), (1, 1, -1), (2, 1, -2), (3, 0, -2), (4, -1, -1)$ or $(5, -1, 0)$. Bearing in mind an initial multiplication by $r_1$, this leaves 12 choices for $g$.

In particular, since $G_0^p \cap G_3^p$ leaves $V_{0,3}$ invariant, it has order at most 12 and contains $G_{0,3}^p$ as a subgroup of order 6. Comparison of the matrices shows that $(r_0 r_1)^j t(a, b)$ with $(j, a, b) = (1, 1, -1), (3, 0, -2)$ or $(5, -1, 0)$ is not contained in $G_{0,3}^p$. It follows that $G^p$ fails to be a $C$-group if and only if one of these (and then all) three elements also lies in $G_0^p$. Now consider

$$g := (r_0 r_1)^3 t(0, -2)$$

obtained for $(j, a, b) = (3, 0, -2)$; its matrix in the basis $b_1, b_2, c$ is

$$M = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (4)$$

Note that

$$g = (r_0 r_1)^3 (r_1 r_2 (r_1 r_0)^2)^\ell (r_2 r_1 (r_0 r_1)^2)^\ell, \quad 3\ell \equiv -2 \mod p. \quad (5)$$

First, if $G = [6,3,6]$, then $V_0$ is also singular, $d := b_1 + 2b_2 + b_3$ generates $\text{rad} V_0$, and $b_1, b_2, d$ is a basis of $V_0$. Now it is straightforward to check that $g(d) = d$; in fact, the equations

$$g(b_3) \cdot g(b_j) = b_3 \cdot b_j \quad (j = 0, \ldots, 3)$$

have the unique solution $g(b_3) = 2b_1 + 4b_2 + b_3$. We now exploit duality. In fact, by the symmetry of the diagram of $G$, we also have an element

$$g' = (r_3 r_2)^3 t'(0, -2)$$
(say) in $G_0^p$, which is obtained from $G_0^p$ in the same way as $g$ from $G_3^p$. Since $g$ and $g'$ have the same $(4 \times 4)$-matrices in the full basis $b_1, b_2, c, d$ of $V$, we must have $g = g'$. Hence $g \in G_0^p \cap G_3^p$ but $g \not\in G_{0,3}^p$, so $G^p$ is not a C-group. In particular, from Lemma 3.1 and a similar expression for $g'$ we obtain the relation

$$(r_0 r_1)^3 (r_1 r_2 (r_1 r_0)^2)^l (r_2 r_1 (r_0 r_1)^2)^l = (r_3 r_2)^3 (r_2 r_1 (r_2 r_3)^2)^l (r_1 r_2 (r_3 r_2)^2)^l,$$

with $3l \equiv -2 \pmod{p}$.

Finally, we consider $G = [6, 3, \infty]$. Here it is a little easier to work with $r := r_1 g$, still with $g$ defined in (3). $G^p$ will be a C-group exactly when $r \not\in G_0^p$. But $r$ acts on $V_3$ as a reflection with root $a := b_1 + 2b_2$; thus, since $V_3$ is a singular subspace of $V$, $r$ actually is a reflection in $G^p$. On the other hand, $G_0^p = O_1(V_0) \simeq O_1(3, p, 0)$, for $p > 3$ (see [17, 5.7]); and $a^2 = 3$. Hence, for $p > 3$, $G^p$ is a C-group if and only if 3 is a non-square modulo $p$, i.e. if and only if $p \equiv \pm 5 \pmod{12}$. \qed

We conclude this section with a look back at the peculiar role of the non-generic prime $p = 3$, which is a frequent irritant in proofs but never an obstruction to polytopality. We have seen for $p = 3$ that when one of $k, l, m$ is 6, different basic systems for the crystallographic group $G = [k, l, m]$ can result in non-isomorphic reduced groups $G^3$. In all cases, however, $G^3$ happens to be a C-group. Although this fact can be checked by hand, we have often resorted to computer verification.

In fact, we can refine our description of the group $G^p$ in singular cases. Indeed, if $n = \dim(V)$ and $r = \dim(\operatorname{rad}(V))$, then it is easy to see that

$$\hat{O}(V) \simeq \mathbb{Z}_p^{r(n-r)} \rtimes O(V/\operatorname{rad} V).$$

In particular, if $n = 4$ and $r = 1, 2$ or 3, then it follows from the above isomorphism that (in non-generic cases), $G^3$ must have order of the form $2^m 3^n$ (see [17, p. 301] for the orders of the groups of orthogonal type). Several such groups appear in the following sections.

### 4 Groups $[k, l, m]$ with a spherical or Euclidean subgroup $[l, m]$

In the previous section we determined the crystallographic groups $G = [k, l, m]$ and primes $p \geq 3$ for which the modular reduction $G^p$ is C-group. We now investigate the corresponding regular 4-polytope $P = \mathcal{P}(G^p)$, whose automorphism group $\Gamma(P)$ is $G^p$. When $p$ is generic for $G$, Lemma 3.1 implies that $G_3^p \simeq [k, l]^p$ and $G_0^p \simeq [l, m]^p$. (In fact, this often holds in non-generic situations, too.) Thus the facets and vertex-figures of $\mathcal{P}(G^p)$ are usually isomorphic to the regular maps associated with the reduced groups $[k, l]^p$ and $[l, m]^p$, respectively, as described in [17, §5]. These maps are orientable, because the even subgroups of $G_0^p$ and $G_3^p$ have index 2.

The groups $G$ with disconnected diagrams $\Delta(G)$ were described early in Section 3 and the corresponding polytopes are easily classified. Thus we shall assume from here on that $\Delta(G)$ is connected. It then follows from Lemma 3.2 of [17] that $G^p$ acts irreducibly on $V$,
so long as det \(m_{ij} \equiv 0 \pmod{p} \). (Recall that the \(m_{ij}\) are the Cartan integers appearing in [17 §4]. We take this opportunity to correct an oversight in part (a) of [17 Lemma 3.2], where we should have stated that the roots \(a_j\) form a basis for \(V\). This actually is the case in all applications here and in [17].)

Before proceeding to specifics, we indicate how to decide whether \(G^p\) is of orthogonal or spherical type (see Section 2). If, for example, \([k, l]\) is Euclidean, it is very easy to check that a basic translation is a product of two reflections in \([k, l]^p\) and has period \(p\). By scanning the parameter \(d(G)\) in Table 1 of [17], we see that the product of two reflections has period at most 5 in spherical cases. Thus \(G^p\) is of orthogonal type for \(p > 5\), and perhaps also for \(p = 3\) or 5, cases which can be directly checked in GAP. We will employ this sort of analysis without much comment in what follows.

We break the discussion down into three cases according as the (vertex-figure) subgroup \([l, m]\) of \(G\) is spherical, Euclidean or hyperbolic, respectively. In this section we treat the groups \(G\) with a spherical or Euclidean subgroup \([l, m]\); groups with hyperbolic subgroups \([l, m]\) are studied in the next section. We begin with the spherical case.

4.1. Groups with a spherical subgroup \([l, m]\)

Let \(G = [k, l, m]\) be crystallographic, let \([l, m]\) be spherical, and let \(p \geq 3\). Then \(G^p\) is a C-group by Theorem 3.1. Moreover, \([l, m]^p \cong [l, m]\), so the vertex-figures of the corresponding polytope \(P\) are isomorphic to Platonic solids \([l, m]\). The finite or Euclidean groups \(G\) were already discussed in [17, §5-6], so we may assume here that \([k, l]\) is not spherical.

If \([k, l]\) is Euclidean, then \(G = [4, 4, 3], [6, 3, 3]\) or \([6, 3, 4]\), and \(P\) is locally toroidal. Recall that a regular 4-polytope is locally toroidal if its facets and vertex-figures are toroidal or spherical, but not all spherical.

For \(G = [4, 4, 3]\) with diagram

\[\cdot -- \cdot -- \cdot -- \cdot,\]

the facets of \(P\) are toroidal maps \(\{4, 4\}_{(p, 0)}\) and the vertex-figures are 3-cubes \(\{4, 3\}\). Now \(\text{disc}(V) \sim -1\), independent of \(p\), so \(e(V) = 1\) if and only if \(p \equiv 1 \pmod{4}\). Moreover, \(G^p = O_1(V)\) or \(O(V)\) according as 2 is a square or non-square, that is, \(p \equiv \pm 1 \pmod{8}\) or \(p \equiv \pm 3 \pmod{8}\). Hence \(P\) has automorphism group

\[\Gamma(P) = G^p = \begin{cases} O_1(4, p, 1), & \text{if } p \equiv 1 \pmod{8} \\ O_1(4, p, -1), & \text{if } p \equiv 7 \pmod{8} \\ O(4, p, 1), & \text{if } p \equiv 5 \pmod{8} \\ O(4, p, -1), & \text{if } p \equiv 3 \pmod{8}. \end{cases}\] (6)

In particular, for \(p = 3\) we obtain the group

\[\Gamma(P) = O(4, 3, -1) \cong O_1(4, 3, -1) \times C_2 \cong S_6 \times C_2\] (7)

([17 §3]). In this case,

\[P = \{\{4, 4\}_{(3, 0)}, \{4, 3\}\},\]

(8) the universal regular 4-polytope with facets \(\{4, 4\}_{(3, 0)}\) and vertex-figures \(\{4, 3\}\) (see [13 Thm. 10B3], which also implies that the product in (7) indeed is direct). Recall that \(\{P_1, P_2\}\)
denotes the universal regular \((n+1)\)-polytope (if it exists) with facets isomorphic to \(P_1\) and vertex-figures isomorphic to \(P_2\) (see [13 Ch. 4]).

For \(G = [6, 3, m]\), with \(m = 3\) or 4, we take the diagram
\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]
\text{or}
\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]
respectively. Now the vertex-figures of \(P\) are tetrahedra \(\{3, 3\}\) or octahedra \(\{3, 4\}\), respectively, and the facets are toroidal maps \(\{6, 3\}_{(p,0)}\) for all \(p \geq 3\) (see [17, §5.6]). Note that \(\text{disc}(V) \sim -3\), so \(V\) is non-singular if \(p > 3\), and \(\epsilon(V) = 1\) if and only if \(-3\) is a square; furthermore, \(G^p = O_1(V)\) or \(O(V)\) depending on the quadratic character of 3 (or 2 and 3).

In particular, if \(G = [6, 3, 3]\) and \(p > 3\), we have
\[
\Gamma(P) = \begin{cases} 
  O_1(4, p, 1), & \text{if } p \equiv 1 \pmod{12} \\
  O_1(4, p, -1), & \text{if } p \equiv 11 \pmod{12} \\
  O(4, p, 1), & \text{if } p \equiv 7 \pmod{12} \\
  O(4, p, -1), & \text{if } p \equiv 5 \pmod{12}.
\end{cases}
\]
When \(p = 3\) we also obtain a C-group \(G^3\), which acts on a singular space \(V\) and has order 1296. The corresponding subgroup \(G^3_3\) has order 108, and the facets are toroidal maps \(\{6, 3\}_{(3,0)}\) (see [17 §5.6]). Note that Lemma 5.1 does not apply; indeed the subdiagram on nodes 0, 1, 2 of \(\Delta(G)\) defines the rank 3 group \(H^3\), of order just 36, for the map \(\{6, 3\}_{(1,1)}\).

In a sense, the subspace \(V_3\) of \(V\) cannot fully represent the structure of the facet. The alternative diagram
\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]
represents the same group, but without this deficiency. By comparing group orders, we find that we have the universal polytope
\[
P = \{\{6, 3\}_{(3,0)}, \{3, 3\}\}
\]
with
\[
G^3 = \Gamma(P) \cong [1 1 2]^3 \rtimes C_2,
\]
where \([1 1 2]^3\) is a certain unitary reflection group in \(C^4\) (see [13 Thm. 11B5]). (The superscript on \([1 1 2]^3\) signifies a group relation, not reduction modulo 3).

When \(G = [6, 3, 4]\) and \(p > 3\), we similarly find that
\[
\Gamma(P) = \begin{cases} 
  O_1(4, p, 1), & \text{if } p \equiv 1 \pmod{24} \\
  O_1(4, p, -1), & \text{if } p \equiv 23 \pmod{24} \\
  O(4, p, 1), & \text{if } p \equiv 7, 13, 19 \pmod{24} \\
  O(4, p, -1), & \text{if } p \equiv 5, 11, 17 \pmod{24}.
\end{cases}
\]
For \(p = 3\) and for either of the diagrams
\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]
\text{or}
\[
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\]
\(G^3\) has order 2592; we obtain the same polytope \(P\) of type \(\{6, 3\}_{(3,0)}, \{3, 4\}\). However, \(P\) is not the (infinite!) universal polytope of this type [13 Thm. 11B5].
We now consider the case that the subgroup \([k, l]\) is hyperbolic (and \([l, m]\) is still spherical). The corresponding groups are \(G = [\infty, 3, 3], [\infty, 3, 4], [6, 4, 3]\) or \([\infty, 4, 3]\).

For \(G = [\infty, 3, m]\), with \(m = 3\) or \(4\), we employ the diagram

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
4 & 1 & 1 & 1 & 1
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
4 & 1 & 1 & 1 & 1 & 2
\end{array}
\]

respectively. In either case, \(G^p_3 = [\infty, 3]^p\), so that the facets of \(\mathcal{P}\) are the regular maps \(\mathcal{M}_{p,3}\) of type \(\{p, 3\}\) described in \([17, \S 5.7]\) (see also \([12]\)); the vertex-figures are tetrahedra \(\{3, 3\}\) or octahedra \(\{3, 4\}\), respectively. In particular, if \(p = 3\), we obtain the 4-simplex \(\{3, 3, 3\}\) or 4-cross-polytope \(\{3, 3, 4\}\), respectively. Now \(\text{disc}(V) \sim -1\) or \(-2\), respectively, independent of \(p\); and \(G^p = O_1(V)\) unless \(m = p = 3\), or \(m = 4\) and \(p = 2\) is a non-square. Thus, when \(m = 3\) we have

\[
\Gamma(\mathcal{P}) = \begin{cases} 
S_5, & \text{if } p = 3 \\
O_1(4, p, 1), & \text{if } p \equiv 1 \pmod{4} \\
O_1(4, p, -1), & \text{if } p \equiv 3 \pmod{4}, \quad (p \neq 3),
\end{cases}
\]

and when \(m = 4\) and \(p > 3\), we have

\[
\Gamma(\mathcal{P}) = \begin{cases} 
O_1(4, p, 1), & \text{if } p \equiv 1 \pmod{8} \\
O_1(4, p, -1), & \text{if } p \equiv 7 \pmod{8} \\
O(4, p, 1), & \text{if } p \equiv 3 \pmod{8} \\
O(4, p, -1), & \text{if } p \equiv 5 \pmod{8}.
\end{cases}
\]

(When \(m = 4\) and \(p = 3\), the group is \(B_3\), which is of index 3 in \(O(4, 3, 1) \cong F_4\).) For both \(m = 3\) and \(m = 4\), and all \(p \geq 3\), we have \(G_3^p = O_1(3, p, 0)\), of order \(p(p^2 - 1)\). For \(p = 5\) the facets are isomorphic to dodecahedra \(\{5, 3\}\) and for \(p = 7\) they are isomorphic to Klein’s map \(\{7, 3\}_8 = \mathcal{M}_{7,3}\), of genus 3 (see \([6, \S 8.6]\)).

With \(m = 3\), we here encounter, for the first time, the classical regular 4-polytopes of ‘pentagonal type’. For future reference, let us denote \(\mathcal{P}\) by \(C_{p,3,3}\). Likewise, as suggested above, it suits us to denote the regular maps arising from

\[
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet \\
4 & 1 & 1 & 1 & 1
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
4 & 1 & 1 & 1 & 4
\end{array}
\]

by \(\mathcal{M}_{p,3}\) and \(\mathcal{M}_{p,p}\), respectively (see \([17, \S 5.7]\)).

Thus, for \(p = 5\), \(C_{5,3,3}\) is the 120-cell \(\{5, 3, 3\}\) (isomorphic to both the convex regular polytope of this type and to the star-polytope \(\{\frac{5}{2}, 3, 3\}\); see \([13, 7D]\)). For \(p = 7\), we find that \(C_{7,3,3}\) is the universal regular polytope

\[
\{\{7, 3\}_8, \{3, 3\}\},
\]

first described in \([16]\). Rephrasing the conjectured presentation in \([16, 3.1]\), we now offer

**Conjecture 1:** For primes \(p \geq 3\), \(C_{p,3,3}\) is the universal polytope

\[
\{ \mathcal{M}_{p,3}, \{3, 3\}\}.
\]

(This has been verified by coset enumeration for \(p \leq 31\). We cannot, in general, expect the facets to be determined in some elegant way, say by their Petrie polygons or holes.)
Moving on to the next class of groups, let \( G = [k, 4, 3] \), with \( k = 6 \) or \( \infty \), and let the diagram be

\[
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\text{or} & \text{or} & \text{or} & \text{or} & \text{or} & \text{or} & \text{or}
\end{array}
\]

respectively. Then \( G_p^p = [k, 4]^p \), so the facets are maps of type \( \{6, 4\} \) or \( \{p, 4\} \), respectively; see [17] §5.8-5.9] for details. Of course, the vertex-figures are cubes \( \{4, 3\} \). Since \( \text{disc}(V) \sim -15 \) or \(-2\), respectively, \( V \) is singular if \( k = 6 \) and \( p = 3 \) or \( 5 \).

Suppose \( k = 6 \). For \( p = 3 \), the group \( G^3 \) has order 2592 and appeared earlier in [13] with different generators. In this new guise, \( G^3 \) is the group of the universal polytope of type

\[
\{\{6, 4\}_4, \{4, 3\}\}.
\]

Indeed, for \( p = 3 \), each basic system gives just this polytope, whose facets are isomorphic to the map \( \{6, 4\}_4 \) (the Petrial of the toroidal map \( \{4, 4\}_{(3, 3)} \)). (See [13] 7B2] for a description of \( Q^5 \), the Petrial or Petrie dual of a map \( Q \).)

For \( p = 5 \), the facets are Coxeter-Petrie polyhedra \( \{6, 4\}_{3 6} \) (see [3]). Since \( G^5 \) has order 30000, the corresponding polytope has 125 of these facets and 625 vertices.

Otherwise, if \( G = [6, 4, 3] \) and \( p > 5 \), then we have

\[
\Gamma(\mathcal{P}) = \begin{cases} 
O_1(4, p, 1), & \text{if } p \equiv 1, 23, 47, 49 \pmod{120} \\
O_1(4, p, -1), & \text{if } p \equiv 71, 73, 97, 119 \pmod{120} \\
O(4, p, 1), & \text{if } p \equiv 17, 19, 31, 53, 61, 77, 79, 83, 91, 107, 109, 113 \pmod{120} \\
O(4, p, -1), & \text{if } p \equiv 7, 11, 13, 29, 37, 41, 43, 59, 67, 69, 89, 101, 103 \pmod{120}.
\end{cases}
\]

Finally, when \( G = [\infty, 4, 3] \), the polytopes are of type \( \{p, 4, 3\} \) and their groups are those described in [15] above (now allowing \( p = 3 \)), with new generators of course. For \( p = 3 \) we obtain the 24-cell \( \{3, 4, 3\} \). When \( p = 5 \), the polytope has facets isomorphic to Gordan’s map \( \{5, 4\}_{6} \) of genus 4 (see [3]).

4.2. Groups with a Euclidean subgroup \([l, m]\)

Next we consider the crystallographic groups \( G = [k, l, m] \) with a Euclidean subgroup \([l, m]\). The groups with a spherical subgroup \([k, l]\) have already been discussed in the dual setting, so we may further assume that \([k, l]\) is Euclidean or hyperbolic. By Theorems [3.3] and [3.3] \( G^p \) is a C-group for all \( p \geq 3 \), with these exceptions: for \( G = [6, 3, 6] \) only \( p = 3 \) is acceptable, and for \( G = [\infty, 3, 6] \), only \( p = 3 \) and \( p \equiv \pm 5 \pmod{12} \). The corresponding regular 4-polytopes \( \mathcal{P} \) all have toroidal vertex-figures.

For the group \( G = [k, 4, 4] \), with \( k = 4, 6 \) or \( \infty \), we take the diagram

\[
\begin{array}{ccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\text{or} & \text{or} & \text{or} & \text{or} & \text{or} & \text{or} & \text{or}
\end{array}
\]

respectively. Their polytopes \( \mathcal{P} \) have toroidal vertex-figures \( \{4, 4\}_{(p, 0)} \) and facets isomorphic to the maps of type \( \{k, 4\} \) described in [17] §5.5,5.8,5.9. Now \( \text{disc}(V) \sim -1 \) in each case, except when \( k = 6 \) and \( p = 3 \); in this latter case, \( V \) is singular (for each admissible diagram).

When \( G = [4, 4, 4] \), the polytope is self-dual and its facets are also maps \( \{4, 4\}_{(p, 0)} \). The group \( \Gamma(\mathcal{P}) \) is the same as in [3]; that is, we have

\[
[4, 4, 4]^p \cong [4, 4, 3]^p
\]

16
for each \( p \geq 3 \). In fact, the Coxeter group \([4,4,4]\) is known to be a subgroup of index 3 in \([4,4,3]\), and under the modular reduction this index collapses to 1; see, for example, [13] §10E, which also explains the corresponding relationship between the polytopes. In particular, if \([4,4,3] = \langle r_0, \ldots, r_3 \rangle \) (say), then we can identify \([4,4,4]\) with the subgroup \( \langle r_1, r_0, r_2r_1r_2, r_3 \rangle \), with the generators taken in this order; modulo \( p \), this subgroup is the group itself.

For \( p = 3 \) we obtain the universal regular polytope

\[
\{\{4,4\}_{(3,0)}, \{4,4\}_{(3,0)}\}
\]

with group \( S_6 \times C_2 \), which is related to the polytope in [8] (see [13] 10E6 and Thm. 10C12).

For \( G = [6,4,4] \) and \( p > 3 \), we have

\[
\Gamma(\mathcal{P}) = \begin{cases} 
O_1(4,p,1), & \text{if } p \equiv 1 \pmod{24} \\
O_1(4,p,-1), & \text{if } p \equiv 23 \pmod{24} \\
O(4,p,1), & \text{if } p \equiv 5,13,17 \pmod{24} \\
O(4,p,-1), & \text{if } p \equiv 7,11,19 \pmod{24}.
\end{cases}
\]

When \( p = 5 \) the facets of \( \mathcal{P} \) are Coxeter-Petrie polyhedra \( \{6,4|3\} \) (see [6]). For \( p = 3 \), the facets are maps \( \{6,4\}_4 \), the vertex-figures are maps of type \( \{4,4\}_{(3,0)} \), and the diagram given above yields a group \( G^3 \) with order 1296. However, the alternate diagram

\[
\bullet - \bullet - \bullet - \bullet - \bullet
\]

results instead in a group \( G^3 \) of order 3888. We thus obtain different polytopes with the same local structure.

When \( G = [\infty,4,4] \), the polytopes are of type \( \{p,4,4\} \), and although differently generated, the group \( \Gamma(\mathcal{P}) \) is again described by [6] when \( p \geq 3 \). For \( p = 3 \) we obtain the universal regular polytope

\[
\{\{3,4\}, \{4,4\}_{(3,0)}\},
\]

the dual of [8], with group \( S_6 \times C_2 \). For \( p = 5 \) the polytope \( \mathcal{P} \) has facets isomorphic to Gordan’s map \( \{5,4\}_6 \) of genus 4.

Next we investigate the groups \([k,3,6]\). By Theorem 3.22 when \( k = 6 \) we need only consider \( p = 3 \). The diagrams

\[
\bullet - \bullet - \bullet - \bullet - \bullet \quad \text{and} \quad \bullet - \bullet - \bullet - \bullet - \bullet
\]

(17)

give isomorphic groups \( G^3 \) of order 1944, yet distinct polytopes of type \( \{\{6,3\}_{(3,0)}, \{3,6\}_{(3,0)}\} \) (with the same type of facets and vertex-figures, the latter type being the dual of the former). However, only the first of these two 4-polytopes is in itself self-dual. The other essentially distinct diagram

\[
\bullet - \bullet - \bullet - \bullet - \bullet
\]

(18)

yields a group \( G^3 \) of order 216 and the universal self-dual polytope of type
\[
\{\{6,3\}_{(1,1)}, \{3,6\}_{(1,1)}\}
\]

(see [13 11C8]).

Let us now turn to the case \(k = \infty\), with diagram
\[
\begin{array}{c}
\bullet & - & 1 & - & 1 & - & 3 \\
\end{array}
\]

(20)

Now \(\text{disc}(V) \sim -3\), so \(V\) is singular if \(p = 3\). When \(p > 3\), the vertex-figures of \(\mathcal{P}\) are toroidal maps \(\{3,6\}_{(p,0)}\) (see [17 §5.6]), and the facets are the maps of type \(\{p,3\}\) described in [17 §5.7]. Keeping in mind Theorem 3.5 we have just two possibilities:

\[
\Gamma(\mathcal{P}) = \begin{cases} 
O(4, p, 1), & \text{if } p \equiv 7 \pmod{12} \\
O(4, p, -1), & \text{if } p \equiv 5 \pmod{12}.
\end{cases}
\]

(21)

For \(p = 5\) the facets of \(\mathcal{P}\) are dodecahedra \(\{5,3\}\), and for \(p = 7\) they are isomorphic to Klein’s map \(\{7,3\}_8\) of genus 3. When \(p = 3\), both the diagram in (20) and its alternative yield the universal regular polytope

\[
\mathcal{P} = \{\{3,3\}, \{3,6\}_{(3,0)}\}
\]

namely the dual of (11). Again the group is \([1 1 2]^3 \rtimes C_2\) of order 1296.

It remains to study the groups \(G = [k, 6, 3]\) with \(k = 3, 4, 6 \text{ or } \infty\), where we take the diagrams

\[
\begin{array}{c}
1 & - & 1 & - & 3 & - & 3 \\
2 & - & 1 & - & 3 & - & 3 \\
3 & - & 1 & - & 3 & - & 3 \\
\text{or} & 4 & - & 1 & - & 3 & - & 3,
\end{array}
\]

respectively. When \(p > 3\) the vertex-figures of \(\mathcal{P}\) are toroidal maps \(\{6,3\}_{(p,0)}\), and the facets are the maps of type \(\{k,6\}\) or \(\{p,6\}\), for \(k = 3,4,6\) or \(k = \infty\), as described in [17 §5.6,5.8,5.10,5.11]. Since \(\text{disc}(V) \sim -3\) in each case, \(V\) is singular if \(p = 3\).

When \(G = [3,6,3]\) and \(p > 3\), the polytope \(\mathcal{P}\) is self-dual and its facets are toroidal maps \(\{3,6\}_{(p,0)}\). The group \(\Gamma(\mathcal{P})\) is the same as in (10); that is, we have

\[
[3,6,3]^p \cong [6,3,3]^p
\]

(22)

for \(p > 3\). In fact, the Coxeter group \([3,6,3]\) is a subgroup of index 4 in \([6,3,3]\), and under reduction modulo \(p\) this index becomes 1, again so long as \(p > 3\); see, for example, [13 Sect. 11G,11H], which also describes the relationship between the polytopes. In particular, if \([6,3,3] = \langle r_0, \ldots, r_3 \rangle\) (say), then \([3,6,3]\) can be identified with the subgroup \(\langle r_0, r_1 r_0 r_1, r_2, r_3 \rangle\), with the generators taken in this order; modulo \(p\), this is the whole group.

When \(p = 3\), we obtain the (combinatorially flat) universal regular polytope

\[
\mathcal{P} = \{\{3,6\}_{(1,1)}, \{6,3\}_{(3,0)}\}
\]

(23)

with group

\[
\Gamma(\mathcal{P}) \cong [1 1 1]^3 \rtimes S_3,
\]
of order 324, where \([1 1 1]^3\) denotes a certain unitary reflection group in \(C^3\) (see [13, Thm. 11E7]). Of course, \(\mathcal{P}\) is not self-dual, although the alternative diagram

\[
\begin{array}{cccc}
3 & 3 & 1 & 1 \\
\end{array}
\]

does yield the dual. Note also that the isomorphism in [22] must fail for \(p = 3\), regardless of choice of diagrams. Indeed, \([3, 6, 3]^3\) does have index 4 in \([6, 3, 3]^3\).

Incidentally, we also have

\[ [6, 3, 6]^p \cong [6, 3, 3]^p \]

for \(p > 3\); but then, as we have seen, \([6, 3, 6]^p\) is not a \(C\)-group with its natural generators. (In this context, note that [13, Thm. 11H10] is incorrect for the parameter vectors \((s, 0)\) with \(s\) not divisible by 3, so does not yield any polytopes of type \([6, 3, 6]\)).

If \(G = [4, 6, 3]\) and \(p > 3\), then \(\Gamma(\mathcal{P})\) is the group described earlier in (12), though again with new generators. For \(p = 5\) the facets of \(\mathcal{P}\) are Coxeter-Petrie polyhedra \([4, 6 \mid 3]\) (see [6]). For \(p = 3\), the facets are maps \([4, 6]_4\); but the diagrams

\[
\begin{array}{cccc}
2 & 1 & 3 & 3 \\
\end{array},
\begin{array}{cccc}
6 & 3 & 1 & 1 \\
\end{array}
\]

lead respectively to vertex-figures \([6, 3]_{(3,0)}\) and group order 3888, or to vertex-figures \([6, 3]_{(1,1)}\) and group order 432.

When \(G = [6, 6, 3]\) and \(p > 3\), the facets of \(\mathcal{P}\) are self-dual maps of type \([6, 6]\), with group \(O_1(3, p, 0)\) or \(O(3, p, 0)\) according as \(p \equiv \pm 1 \mod 12\) or \(p \not\equiv \pm 1 \mod 12\). In fact, \(\Gamma(\mathcal{P})\) is a newly generated version of the group described in (10). When \(p = 3\), we find that the type of facets depends on the diagram chosen; they are isomorphic to the Petrials of two maps first described by Sherk in [18]: \([6, 6]_{(1,1)}\), with group of order 72 and genus 4, and \([6, 6]_{(3,0)}\), with group 216 and genus 10. Note that \([6, 6]_{(1,1)} \cong [6, 6 \mid 2]\) (see [6, § 8.5]). The various diagrams yield four distinct universal polytopes, summarized in the chart below:

| \(\Delta(G)\) | \(|G^3|\) | The Universal Polytope |
|-----------------|---------|-------------------------|
| \(\begin{array}{cccc}
1 & 3 & 1 & 1 \\
\end{array}\) | 216 | \([6, 6]_{(1,1)}^\pi, [6, 3]_{(1,1)}\) |
| \(\begin{array}{cccc}
9 & 3 & 1 & 1 \\
\end{array}\) | 648 | \([6, 6]_{(3,0)}^\pi, [6, 3]_{(1,1)}\) |
| \(\begin{array}{cccc}
3 & 1 & 3 & 3 \\
\end{array}\) | 648 | \([6, 6]_{(1,1)}^\pi, [6, 3]_{(3,0)}\) |
| \(\begin{array}{cccc}
1 & 3 & 9 & 9 \\
\end{array}\) | 5832 | \([6, 6]_{(3,0)}^\pi, [6, 3]_{(3,0)}\) |

(The two groups of order 648 are isomorphic, though differently generated.)
Finally, if \( G = [\infty, 6, 3] \) and \( p > 3 \), we have the same group as in \([10]\), now yielding a polytope \( P \) of type \( \{p, 6, 3\} \). In particular, if \( p = 5 \), the facets are maps \( \{5, 6\}_4 \), with group \( S_5 \times C_2 \). When \( p = 3 \), we again obtain the universal polytope \([23]\), or its dual, depending on choice of diagram.

5 Groups \([k, l, m]\) with a hyperbolic subgroup \([l, m]\)

We now consider the crystallographic groups \( G = [k, l, m] \), for which \([l, m]\) is of hyperbolic type. The groups with a spherical or Euclidean subgroup \([k, l]\) have already occurred in the previous section in the dual setting, so we may assume that \([k, l]\) is also of hyperbolic type. Then, except possibly for small primes \( p \), all three subspaces \( V \), \( V_0 \) and \( V_3 \) are non-singular, but \( V_{03} \) is non-singular only if \( l \neq \infty \). By Corollary 3.2, if \( l = \infty \), then \( G^p \) is a C-group for \( p \geq 3 \). On the other hand, by Corollary 3.1 if \( l \neq \infty \), then \( G^p \) can only be a C-group if \( p \) is small.

5.1. The groups \([k, \infty, m]\)

We begin by discussing the groups \( G = [k, l, m] \) with \( l = \infty \) and with diagrams specified below. Inspection of the discriminants shows that \( V \) is non-singular, except when

\[
(k, m, p) \text{ or } (m, k, p) = \begin{cases} 
(3, 3, 7), (4, 3, 5), (6, 3, 3), (6, 3, 13), (4, 4, 3), (6, 4, 3), (6, 4, 7), \\
(6, 6, 3), (6, 6, 5), (\infty, 6, 3).
\end{cases}
\]

Moreover, \( V_0 \) and \( V_3 \) are non-singular except occasionally when \( p = 3 \). If all three spaces \( V \), \( V_0 \), \( V_3 \) are non-singular, then the vertex-figures of the polytope \( P \) with group \( G^p \) are the maps of type \( \{p, m\} \) or \( \{p, p\} \) associated with \([\infty, m]^p\) (see \([17]\) 5.7,5.9,5.11,5.12), and the facets are duals of such maps (with \( m \) replaced by \( k \)).

When \( G = [k, \infty, 3] \), with \( k = 3, 4, 6 \) or \( \infty \), we take the diagram

\[
\begin{align*}
1 & \bullet \bullet \bullet \bullet, \\
2 & \bullet \bullet \bullet \bullet, \\
3 & \bullet \bullet \bullet \bullet \bullet \bullet, \\
4 & \bullet \bullet \bullet \bullet, \\
\end{align*}
\]

respectively. Then \( \text{disc}(V) \sim -7, -5, -39 \) or \(-1 \), respectively, so \( V \) may be singular for \( p = 3, 5, 7 \) or \( 13 \).

From \([3, \infty, 3]\) we obtain a self-dual regular 4-polytope \( P \) of type \( \{3, p, 3\} \) for all \( p \geq 3 \). In the non-singular cases with \( p \neq 3, 7 \) we have

\[
\Gamma(P) = \begin{cases} 
O_1(4, p, 1), & \text{if } p \equiv 1, 9, 11, 15, 23, 25 \pmod{28} \\
O_1(4, p, -1), & \text{if } p \equiv 3, 5, 13, 17, 19, 27 \pmod{28}.
\end{cases} \tag{25}
\]

In particular, if \( p = 5 \), then \( \Gamma(P) = O_1(4, 5, -1) \), and \( P \) has 130 icosahedral facets and 130 dodecahedral vertex-figures. For \( p = 7 \) the \( 7^3 \) facets are maps \( \{7, 3\}_8 \), and so dually the vertex-figures are maps \( \{7, 3\}_8 \). Of course, for \( p = 3 \) we get the 4-simplex with group \( S_5 \).

The group \( G = [4, \infty, 3] \), with \( p > 5 \), yields regular polytopes \( P \) of type \( \{4, p, 3\} \) with group

\[
\Gamma(P) = \begin{cases} 
O_1(4, p, 1), & \text{if } p \equiv 1, 7, 9, 23 \pmod{40} \\
O_1(4, p, -1), & \text{if } p \equiv 17, 31, 33, 39 \pmod{40} \\
O(4, p, 1), & \text{if } p \equiv 3, 21, 27, 29 \pmod{40} \\
O(4, p, -1), & \text{if } p \equiv 11, 13, 19, 37 \pmod{40}.
\end{cases} \tag{26}
\]
For \( p = 3 \) we obtain the 4-cube \( \{4,3,3\} \). When \( p = 7 \), the vertex-figures of \( P \) are isomorphic to Klein’s map \( \{7,3\}_8 \) of genus 3. Lastly, when \( p = 5 \) we obtain a polytope with 125 facets of type \( \{4,5\}_6 \) and with 250 dodecahedral vertex-figures.

When \( G = [6, \infty, 3] \), with \( p \neq 3, 13 \), we obtain regular polytopes \( P \) of type \( \{6, p, 3\} \) with group

\[
\Gamma(P) = \begin{cases}
O_1(4, p, 1), & \text{if } p \equiv 1, 11, 25, 47, 49, 59, 61, 71, 83, 119, 121, 133 \pmod{156} \\
O_1(4, p, -1), & \text{if } p \equiv 23, 35, 73, 85, 95, 97, 107, 109, 131, 145, 155 \pmod{156} \\
O(4, p, 1), & \text{if } p \equiv 5, 41, 43, 55, 79, 89, 103, 125, 127, 137, 139, 149 \pmod{156} \\
O(4, p, -1), & \text{otherwise.}
\end{cases}
\]

(27)

For \( p = 5 \) we have a polytope with facets isomorphic to \( \{6, 5\}_4 \) and with dodecahedral vertex-figures. When \( p = 3 \), both the original diagram and the alternative yield the universal polytope described earlier in [11]. Finally, even though \( V \) is singular for \( p = 13 \), we still obtain a C-group \( G^{13} \) of order \( 13^4 \cdot 7 \cdot 4! \).

Suppose now that \( G = [\infty, \infty, 3] \). For \( p \geq 3 \), we obtain a regular polytope of type \( \{p, p, 3\} \), which we shall denote \( C_{p,p,3} \), again to suggest connections with the classical star-polytopes. To explain this, we resurrect the earlier group \( [\infty, 3, 3] \), now with generators \( \langle s_0, s_1, s_2, s_3 \rangle \), to which we apply the mixing operation

\[
(s_0, s_1, s_2, s_3) \rightarrow (s_0, s_1 s_0 s_1, s_2, s_3) =: (r_0, r_1, r_2, r_3).
\]

From this construction we find that the reflection group \( G = [\infty, \infty, 3] := \langle r_0, r_1, r_2, r_3 \rangle \) is a subgroup of \( [\infty, 3, 3] \), with index 4 in characteristic 0. (See [14] Lemma 2.1 or [15] for an equivalent geometric dissection.) Now, as explained in the proof of [13] Thm. 7D16(a), the index collapses to 1 in characteristic \( p \geq 3 \). Thus \( G^p \) is the group described in [14], though with the new generators.

Again for \( p = 3 \), \( C_{3,3,3} \) is the 4-simplex with group \( S_5 \). If \( p = 5 \), then

\[
\Gamma(C_{5,5,3}) = O_1(4, 5, 1) \cong H_4,
\]

which is isomorphic to the symmetry group of the 120-cell \( \{5,3,3\} \) (or any regular star-polytope in Euclidean 4-space associated with it). The facets of \( C_{5,5,3} \) are maps \( \{5,5|3\} ( = \mathcal{M}_{5,5}) \) of genus 4, which can be metrically realized in Euclidean 3-space by the small stellated dodecahedron \( \{\frac{5}{2}, 5\} \); the vertex-figures of \( C_{5,5,3} \) are dodecahedra \( \{5,3\} ( = \mathcal{M}_{5,3}) \). We know from [13] Thm. 7D16 (or [11]) that the universal abstract polytope \( \{\{5,5|3\}, \{5,3\}\} \) is isomorphic to the regular star-polytope \( \{\frac{5}{2}, 5,3\} \) in Euclidean 4-space. Therefore, since they have the same group, we must have

\[
C_{5,5,3} \cong \{\{5,5|3\}, \{5,3\}\} \cong \{\frac{5}{2}, 5,3\}.
\]

More generally, our earlier conjecture can be restated as:

**Conjecture 2:** For primes \( p \geq 3 \), \( C_{p,p,3} \) is the universal polytope of type

\[
\{\mathcal{M}_{p,p}, \mathcal{M}_{p,3}\}.
\]
Geometrically, it is useful to view this mixing of new generators for the same group as a stellation of the polygonal faces of $C_{p,3,3}$, thereby yielding $C_{p,p,3}$. In a sense, the two polytopes share the same edges, which are, however, allocated differently to form the $p$-gons in $C_{p,p,3}$. For $p > 3$, the two polytopes have equal numbers of facets, polygons and edges, though rather different numbers of vertices.

For the groups $G = [k, \infty, 4]$, with $k = 3, 4, 6$ or $\infty$, our diagrams are

1–1–4–2, 2–1–4–2, 3–1–4–2 or 4–1–4–2,

respectively. Now $\text{disc}(V) = -5, -3, -21$ or $-2$, respectively, so $V$ may be singular for the primes $p = 3, 5$ or $7$. We have already discussed the case $k = 3$ in the dual setting.

From $[4, \infty, 4]$, with $p > 3$, we obtain a self-dual regular 4-polytope $\mathcal{P}$ of type $\{4, p, 4\}$ with group

$$\Gamma(\mathcal{P}) = \begin{cases} O(4, p, 1), & \text{if } p \equiv 1, 7 \pmod{24} \\ O(4, p, -1), & \text{if } p \equiv 17, 23 \pmod{24} \\ O(4, p, 1), & \text{if } p \equiv 13, 19 \pmod{24} \\ O(4, p, -1), & \text{if } p \equiv 5, 11 \pmod{24} \end{cases}$$

(28)

For $p = 5$, the polytope $\mathcal{P}$ has facets isomorphic to $\{4, 5\}_6$ and vertex-figures isomorphic to $\{5, 4\}_6$. When $p = 3$, the space $V$ is singular, and we obtain a regular toroid $\{3, 3, 4\}_{(3,0,0)}$ of rank 4 with automorphism group $G^3 \cong C_3^3 \rtimes [3, 4]$. The group $G = [6, \infty, 4]$, with $p \neq 3, 7$, yields regular polytopes $\mathcal{P}$ of type $\{6, p, 4\}$ with group

$$\Gamma(\mathcal{P}) = \begin{cases} O(4, p, 1), & \text{if } p \equiv 1, 23, 25, 71, 95, 121 \pmod{168} \\ O(4, p, -1), & \text{if } p \equiv 47, 73, 97, 143, 145, 167 \pmod{168} \\ O(4, p, 1), & \text{if } p \equiv \{5, 11, 17, 31, 37, 41, 55, 85, 89, 101, 103, 107, 109, 115, 125, 139, 155\} \pmod{168} \\ O(4, p, -1), & \text{other wise} \end{cases}$$

(29)

For $p = 5$ we have a polytope with facets isomorphic to $\{6, 5\}_4$ and vertex-figures isomorphic to $\{5, 4\}_6$. When $p = 3$ both the given diagram and its alternate yield groups $G^3$ which are similar in $GL_4(\mathbb{Z}_3)$ to the group of order 2592 defined by the diagrams in [13]. We get the same non-universal polytope of type $\{(6,3)_{(3,0)},\{3,4\}\}$. Lastly, in the singular case with $p = 7$, we still obtain a $C$-group of order $2^5 \cdot 3 \cdot 7^4$.

When $G = [\infty, \infty, 4]$, with $p > 3$, the corresponding polytopes $\mathcal{P}$ are of type $\{p, p, 4\}$ and their groups are once more newly generated versions of those described in [15]. For $p = 3$ we obtain the cross-polytope $\{3, 3, 4\}$, whose group has index 3 in $O(4,3,1) \cong F_4$. For $p = 5$, the polytope $\mathcal{P}$ has facets isomorphic to $\{5, 5\}_3 \cong \{\frac{\Phi}{2}, 5\}$ and vertex-figures isomorphic to $\{5, 4\}_6$.

We now turn to the groups $G = [k, \infty, 6]$, with $k = 3, 4, 6$ or $\infty$. We have already investigated the cases $k = 3$ or 4 in the dual setting, so we need only consider the diagrams

3–1–4–12 or 4–1–4–12,

for which $\text{disc}(V) \sim -15$ or $-3$, respectively.
From $G = [6, \infty, 6]$, with $p \neq 3, 5$, we obtain self-dual regular polytopes of type $\{6, p, 6\}$ with group

$$\Gamma(P) = \begin{cases} O_1(4, p, 1), & \text{if } p \equiv 1, 23, 47, 49 \pmod{60} \\ O_1(4, p, -1), & \text{if } p \equiv 11, 13, 37, 59 \pmod{60} \\ O(4, p, 1), & \text{if } p \equiv 17, 19, 31, 53 \pmod{60} \\ O(4, p, -1), & \text{otherwise.} \end{cases}$$

(30)

For $p = 3$ or 5, each possible diagram $\Delta(G)$ gives a singular space $V$, though we still obtain polytopes. When $p = 5$, $P$ is self-dual with facets $\{6, 5\}_4$ and vertex-figures $\{5, 6\}_4$ and a group of order $2^4 \cdot 3 \cdot 5^4$. For $p = 3$, the essentially distinct diagrams

$$\begin{align*}
4 & \longrightarrow 12 & 3 & \longrightarrow 1 \\
& \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{align*}$$

describe exactly the same groups $G^3$ as the diagrams displayed in (17) and (18). Thus only the first and last diagrams yield self-dual polytopes.

When $G = [\infty, \infty, 6]$ and $p > 3$, the polytope $P$ is of type $\{p, p, 6\}$ and its group is given by (11). For $p = 5$, $P$ has facets $\{5, 5 \mid 3\} \cong \{\frac{5}{2}, 5\}$ and vertex-figures $\{5, 6\}_4$, and $\Gamma(P) = O(4, 5, -1)$. For $p = 3$ each admissible diagram yields the universal polytope $\{\{3, 3\}, \{3, 6\}_{(3,0)}\}$ (dual to the polytope described in (11)).

It remains to investigate the group $G = [\infty, \infty, \infty]$, with diagram

$$\begin{align*}
4 & \longrightarrow 1 & 4 & \longrightarrow 1 \\
& \bullet & \bullet & \bullet & \bullet
\end{align*}$$

Then $\text{disc}(V) \sim -1$, independent of $p$. In each case we obtain a regular polytope $C_{p,p,p}$ of type $\{p, p, p\}$, again related to a classical star-polytope. Note that $C_{p,p,p}$, along with its facets and vertex-figures, is self-dual. Taking an approach similar to that for $C_{p,p,3}$, we begin with the group $[\infty, 3, 3] = \langle s_0, s_1, s_2, s_3 \rangle$, to which we apply the mixing operation

$$\begin{align*}(s_0, s_1, s_2, s_3) & \rightarrow (s_1, s_0, s_2s_1s_0s_1s_2, s_3) =: (r_0, r_1, r_2, r_3).
\end{align*}$$

Then in characteristic 0, $G = [\infty, \infty, \infty] := \langle r_0, r_1, r_2, r_3 \rangle$ is a subgroup of index 6 in $[\infty, 3, 3]$ (see [12] part (6)). Again following the proof of [13] Thm. 7D16(c)], we find that the index collapses to 1 in characteristic $p \geq 3$. Once more $G^p$ is the group described in (14), though with the new generators.

For $p = 3$ we again get the regular simplex $\{3, 3, 3\}$. If $p = 5$, the facets and vertex-figures are maps $\{5, 5 \mid 3\} (= M_{p,p})$ of genus 4, and

$$\Gamma(C_{5,5,5}) = O_1(4, 5, 1) \cong H_4.$$ 

(17)

We know from [13] Thm. 7D16 (or 10]) that the universal abstract regular 4-polytope $\{\{5, 5 \mid 3\}, \{5, 5 \mid 3\}\}$ is isomorphic to the regular star-polytope $\{\frac{5}{2}, 5, \frac{5}{2}\}$ in Euclidean 4-space, which also has group $H_4$. Then it follows that

$$C_{5,5,5} \cong \{\{5, 5 \mid 3\}, \{5, 5 \mid 3\}\} \cong \{\frac{5}{2}, 5, \frac{5}{2}\}.$$ 

This suggests a final variant of our earlier conjecture:
**Conjecture 3:** For primes $p \geq 3$, $C_{p,p,p}$ is the universal polytope of type

$$\{ M_{p,p}, M_{p,p} \} .$$

5.2. The groups $[k, l, m]$ with $l = 3, 4$ or 6

By Corollary 3.1, the remaining groups $G = [k, l, m]$ with $l = 3, 4$ or 6 do not generally yield a C-group $G^p$. In particular, if $V$, $V_0$, $V_3$, $V_0,3$ are non-singular and $G_0^p$, $G_3^p$ are of orthogonal type, then $G^p$ fails to be a C-group if $p > 13$. Hence only small primes need to be considered.

Every group $G = [k, 3, m]$, except $[\infty, 3, \infty]$, has a spherical or Euclidean subgroup $[k, 3]$ or $[3, m]$, so $G$ has already been investigated in the previous section.

When $G = [\infty, 3, \infty]$, with diagram

```
    4
   / \   \
  1   1  1
```

the three subspaces $V$, $V_0$, $V_4$ have discriminant $-1$, and $\text{disc}(V_0,3) \sim 3$, so all four subspaces are certainly non-singular for $p > 3$. Moreover, $G_0^p$, $G_3^p$ are of orthogonal type for $p > 3$, and $\epsilon(V_0,3) = 1$ if and only if $p \equiv 1, 7 \text{ mod } 12$. (Recall that $\epsilon(V_0,3) = 1$ or $-1$ according as $\text{disc}(V_0,3) \sim -1$ or $-\gamma$, with $\gamma$ a non-square.) Now Corollary 3.1(a) shows that $G^p$ can only be a C-group if $p \leq 7$. Computations with GAP confirm that $G^p$ is indeed a C-group for $p = 3, 5$ or 7. The corresponding polytope has type $\{ p, 3, p \}$, and we shall denote it by $C_{p,3,p}$, again to suggest a connection with the classical cases. Even though there are just three polytopes in this family, the group $G^p$ is still described by $[14]$ for any prime $p \geq 3$.

Clearly, when $p = 3$ we reacquire the simplex $\{ 3, 3, 3 \}$. For $p = 5$ we obtain a self-dual polytope $C_{5,3,5}$ with 120 dodecahedral facets $\{ 5, 3 \}$ and 120 icosahedral vertex-figures $\{ 3, 5 \}$, and with

$$\Gamma(C_{5,3,5}) = O_1(4, 5, 1) \cong H_4.$$  

We know from [13] Thm. 7D16 that the regular star-polytope $\{ 5, 3, \frac{5}{2} \}$ in $E^4$ is isomorphic to the abstract regular polytope $\{ 5, 3, 5 | 3 \}$, which is the quotient of the hyperbolic tessellation $\{ 5, 3, 5 \}$ obtained by imposing the extra relation

$$\langle \rho_0 \rho_1 \rho_2 \rho_3 \rho_2 \rho_1 \rangle^3 = 1$$

on its group $\langle \rho_0, \ldots, \rho_3 \rangle$ (the “3” in the symbol for the quotient signifies this relation). It is straightforward to check that the corresponding element $r_0 r_1 r_2 r_3 r_2 r_1$ of $G^5$ indeed has order 3, so we also have

$$\text{C}_{5,3,5} \cong \{ 5, 3, \frac{5}{2} \} \cong \{ 5, 3, 5 | 3 \}.$$  

(In fact, this isomorphism proves directly that $G^5$ is a C-group.)

When $p = 7$, the self-dual polytope $C_{7,3,7}$ has 350 facets isomorphic to $\{ 7, 3 \}_8$ and 350 vertex-figures isomorphic to $\{ 3, 7 \}_8$, and $\Gamma(C_{7,3,7}) = O_1(4, 7, -1)$, of order $7^2(7^4 - 1)$. Thus Klein’s dual pair of maps $\{ 7, 3 \}_8$ and $\{ 3, 7 \}_8$ of genus 3 occur as facets and vertex-figures of a self-dual regular 4-polytope. (In contrast to the case $p = 5$, we have no interesting presentation for $G^7$.)
All but three groups \([k, 4, m]\) have a spherical or Euclidean subgroup \([k, 4]\) or \([4, m]\), so their polytopes have already been discussed in the previous section. The three exceptions are \(G = [6, 4, 6], [\infty, 4, 6]\) (or \([6, 4, \infty]\)) and \([\infty, 4, \infty]\), with respective diagrams

\[
\begin{align*}
3 & \quad 1 \quad 2 \quad 6 , \\
4 & \quad 1 \quad 2 \quad 6 \quad 3 , \\
4 & \quad 1 \quad 2 \quad 8 .
\end{align*}
\]

When \(G = [6, 4, 6]\), the four subspaces \(V, V_0, V_3, V_{0,3}\) have discriminants \(\sim -7, -6, -6 \text{ and } 1\), respectively, and hence are non-singular for \(p \neq 3, 7\). Moreover, \(G_{5}^{p}\), \(G_{3}^{p}\) are of orthogonal type for \(p > 3\), and \(\epsilon(V_{0,3}) = 1\) if and only if \(p \equiv 1 \mod 4\). Then Corollary 3.1(b) implies that \(G^{p}\) can only be a C-group if \(p \leq 7\). In fact, \(G^{7}\) also fails to be a C-group.

However, \(G^{5}\) is a C-group by Corollary 3.1(b), because \(G_{6}^{0}\) and \(G_{3}^{0}\) are full orthogonal groups. The corresponding regular polytope \(\mathcal{P}\) is self-dual of type \([6,4,6]\). A dual pair of Petrie-Coxeter polyhedra \([6,4\mid 3]\) and \([4,6\mid 3]\) occur as the types of the 130 facets and 130 vertex-figures of \(\mathcal{P}\), respectively; and \(\Gamma(\mathcal{P}) = O(4,5, -1)\).

When \(p = 3\), each possible diagram yields a polytope with facets isomorphic to \([6, 4]\) and vertex-figures isomorphic to \([4, 6]\). Both the diagram given above for this case and the alternate

\[
\begin{align*}
1 & \quad 3 \quad 6 \quad 2 ,
\end{align*}
\]

yield isomorphic, self-dual polytopes with group order 2592. (This is a new group of that order, not isomorphic for example to the group defined for \(p = 3\) by the diagrams in (13).) For the remaining diagram

\[
\begin{align*}
1 & \quad 3 \quad 6 \quad 18 ,
\end{align*}
\]

we find that \(G^{3}\) has order 7776 and is the automorphism group of the universal self-dual polytope

\[
\{[6, 4]_4, [4, 6]_4\} .
\]

For \(G = [\infty, 4, 6]\), the discriminants of \(V, V_0, V_3, V_{0,3}\) are \(\sim -6, -3, -1 \text{ and } 1\), respectively. Hence, if \(p > 3\), then all four subspaces are non-singular, \(G_{5}^{p}\), \(G_{3}^{p}\) are of orthogonal type, and \(\epsilon(V_{0,3}) = 1\) if and only if \(p \equiv 1 \mod 4\). Then Corollary 3.1(b) shows again that \(G^{p}\) can only be a C-group if \(p \leq 7\). Indeed, for \(p = 7\) we do obtain a C-group of order \(2^9 \cdot 3^2 \cdot 7^2\).

We see also that \(G^{5}\) is a C-group by Corollary 3.1(b). The corresponding polytope \(\mathcal{P}\) has 120 facets isomorphic to \([5, 4]_6\) and 120 vertex-figures isomorphic to \([4, 6\mid 3]\), and

\[
\Gamma(\mathcal{P}) = O(4,5, 1) \cong H_4 \rtimes C_2.
\]

When \(p = 3\), both the given diagram and its alternate produce the dual of the universal polytope \((16)\), with group order 2592.

For \(G = [\infty, 4, \infty]\), the discriminants of \(V, V_0, V_3, V_{0,3}\) are \(\sim -2, -2, -1 \text{ and } 1\), respectively, so these spaces are non-singular for each \(p\). Corollary 3.1(b) once again implies that \(G^{p}\) can only be a C-group if \(p \leq 7\), and Corollary 3.1(b) confirms that \(G^{5}\) actually is a C-group. Moreover, computation with GAP shows that \(G^{3}\) and \(G^{7}\) also are C-groups. The
corresponding regular polytopes $\mathcal{P}$ are self-dual and are of type $\{p, 4, p\}$ in each case. In particular, for $p = 3$ we have

$$\Gamma(\mathcal{P}) \cong O(4, 3, 1) \cong F_4,$$

so $\mathcal{P}$ is the 24-cell $\{3, 4, 3\}$. When $p = 5$, the polytope $\mathcal{P}$ has 130 facets $\{5, 4\}_6$ and 130 vertex-figures $\{4, 5\}_6$, and $\Gamma(\mathcal{P}) \cong O(4, 5, -1)$. Finally, if $p = 7$, then $\mathcal{P}$ has 350 vertices and 350 facets, and $\Gamma(\mathcal{P}) = O_1(4, 7, -1)$; the vertex-figures are isomorphic to the map R10.9 of $\mathcal{B}$.

For the groups $[k, 6, m]$ we may assume that $k, m \neq 3$, as the other groups have already been studied previously. Now disc($V_{0,3}$) $\sim 3$ in each case, so $\epsilon(V_{0,3}) = 1$ if and only if $p \equiv 1, 7 \mod 12$.

When $G = [k, 6, 4]$, with $k = 4, 6$ or $\infty$, we take the diagram

$$
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
\quad \text{or} \quad
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
$$

respectively. Then disc($V$) $\sim -2, -15$ or $-6$, and disc($V_3$) $\sim -6, -2$ or $-1$, respectively. Moreover, disc($V_0$) $\sim -2$ in each case. It follows that the subspaces $V, V_0, V_3$ and $V_{0,3}$ certainly are non-singular for $p > 3$, except when $k = 6$ and $p = 5$; in particular, $G_0^p$ and $G_3^p$ are then of orthogonal type. Now Corollary 3.1(a) eliminates all primes $p > 13$.

For $G = [4, 6, 4]$, the subgroups $G_0^p$ and $G_3^p$ are full orthogonal for all primes $p \leq 13$, so Corollary 3.1(b) shows (for $p > 3$) that $G^p$ is a C-group only when $p = 5, 7$. For $p = 5$, we obtain a self-dual regular polytope $\mathcal{P}$ with 130 facets isomorphic to $\{4, 6\}_3$ and 130 vertex-figures isomorphic to $\{6, 4\}_3$, and with group $\Gamma(\mathcal{P}) = O(4, 5, -1)$. Thus $\mathcal{P}$ has a dual pair of Petrie-Coxeter polyhedra as its facets and vertex-figures. For $p = 7$, we have another self-dual polytope $\mathcal{P}$ of type $\{4, 6, 4\}$, now with 350 vertices, 350 facets, and automorphism group $O(4, 7, -1)$. Finally, when $p = 3$, $G^3$ has order 5184 and is the automorphism group of the universal, self-dual polytope

$$\{\{4, 6\}_4, \{6, 4\}_4\},$$

with 36 facets (and vertices).

Let $G = [6, 6, 4]$. Now Corollary 3.1(b) applies only to $p = 7$ among primes $p \leq 13$, so in particular $G^7$ is a C-group. The corresponding regular polytope $\mathcal{P}$ is of type $\{6, 6, 4\}$, has 350 vertices and 350 facets, and has $O(4, 7, -1)$ as its group. Using GAP we find that $G^p$ is also a C-group for the remaining primes $p \leq 13$, thereby giving a few regular polytopes $\mathcal{P}$ of type $\{6, 6, 4\}$.

For $p = 3$, the group $G^3$ depends on the diagram used for reduction. Both the middle diagram in 32 and

$$
\begin{array}{c}
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{array}
$$

give the group of order 864 for the universal polytope

$$\{\{6, 6\}^7_{(1,1)}, \{6, 4\}_4\}$$

whose 12 facets are isomorphic to the Petrial of Sherk’s map $\{6, 6\}^7_{(1,1)}$, and with 6 vertex-figures isomorphic to $\{6, 4\}_4$. On the other hand, both
yield the group of order 7776 for the universal polytope
\[ \{\{6,6\}^\pi_{(3,0)} , \{6,4\}_4\}, \]
whose 36 facets isomorphic to the Petrial of \{6,6\}_{(3,0)}. (This is not the group for the polytope in (31).)

When \( p = 5 \), the underlying space \( V \) is singular and \( G^5 \) is of order 30000; the corresponding polytope \( P \) has 125 facets isomorphic to the map R11.5 of \( [3] \), and 125 vertex-figures isomorphic to \{6,4|3\}. Finally, we have the groups \( G^{11} = O(4,11,-1) \) and \( G^{13} = O(4,13,-1) \), each contributing a regular polytope of type \{6,6,4\}, with 2684 or 4420 facets, respectively, and with half as many vertices.

Next let \( G = [\infty,6,4] \). The primes 5 and 7 are easily seen to yield C-groups by Corollary 3.11(b); in fact, \( G^p \) is a C-group for the remaining primes \( p \leq 13 \). In particular, when \( p = 5 \), we have a polytope \( P \) with 120 facets \{5,6\}_4 and 120 vertex-figures \{6,4|3\}, and with
\[ \Gamma(P) = O(4,5,1) \cong H_4 \rtimes C_2. \]

When \( p = 7 \), \( P \) is of type \{7,6,4\} and has 336 facets and 336 vertices, and \( \Gamma(P) = O(4,7,1) \). Similarly, we find that \( G^{11} = O(4,11,+1) \) and \( G^{13} = O(4,13,-1) \). These groups yield regular polytopes of type \{11,6,4\} or \{13,6,4\}, respectively, with 2640 or 4420 facets and half as many vertices.

When \( p = 3 \), the right-most diagram in (32) yields the group \( G^3 \) of order 432 for the universal regular polytope
\[ \{\{3,6\}_{(1,1)} , \{6,4\}_4\}. \]
This polytope has with 12 toroidal facets \{3,6\}_{(1,1)} and 3 vertex-figures \{6,4\}_4. However, the alternate diagram
\[ \begin{array}{c}
\bullet & 3 & 1 & 2
\end{array} \]
gives the group of order 3888 for the universal regular polytope
\[ \{\{3,6\}_{(3,0)} , \{6,4\}_4\}, \]
now with 36 toroidal facets \{3,6\}_{(3,0)} and 27 vertex-figures.

At last then, we are left to consider only the groups \( G = [6,6,6] , [\infty,6,6] \) and \([\infty,6,\infty]\), with diagrams
\[ \begin{array}{c}
3 & 1 & 3 & 1 & 4 & 1 & 3 & 1 & 4 & 1 & 3 & 1 & 12
\end{array} \]
and disc\( (V) \sim -11, -1 \) and \(-3\), respectively. Again the subspaces \( V, V_0, V_3 \) and \( V_{0,3} \) are non-singular for \( p > 3 \) (excluding \( p = 11 \) for the left diagram); and then \( G^p_0 \) and \( G^p_3 \) are of orthogonal type. By Corollary 3.11(a) we exclude all \( p > 13 \). For each diagram, \( \epsilon(V_{0,3}) = 1 \) if and only if \( p \equiv 1, 7 \mod 12 \). Thus, by Corollary 3.11(b), we do get a C-group for \( p = 5, 7 \).
In fact, from GAP we find for each diagram that $G^p$ is a C-group for $p = 3, 11, 13$, too. We describe some features of the corresponding polytopes in the more notable cases only.

Let us first consider $G = [6, 6, 6]$. The resulting polytope $P$ is self-dual for $p = 5, 7, 11, 13$. When $p = 5$, the facets (and vertex-figures) are copies of the map R11.5 listed in [3]. For $p = 11$, $V$ is singular and $G^{11}$ has order $11^4 \cdot 5!$. For $p = 3$, the facets and vertex-figures are the Petrials of Sherk’s maps, as described earlier. The diagrams

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
27
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
9
\end{array}
\begin{array}{c}
\bullet \\
27
\end{array}
\end{array}$$

yield, respectively, the groups of orders 432 and 11664 for the self-dual, universal regular polytopes

$$\{\{6, 6\}^{\pi}_{(1,1)}, \{6, 6\}^{\pi}_{(1,1)}\} \quad \text{and} \quad \{\{6, 6\}^{\pi}_{(3,0)}, \{6, 6\}^{\pi}_{(3,0)}\} .$$

The remaining pertinent diagrams

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
9
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
9
\end{array}
\end{array}$$

yield the group of order 1296 for the universal regular polytope

$$\{\{6, 6\}^{\pi}_{(3,0)}, \{6, 6\}^{\pi}_{(1,1)}\} ,$$

and its dual, respectively.

Next suppose $G = [\infty, 6, 6]$. For $p = 5$ we reobtain $\Gamma(P) = O(4, 5, 1) \cong H_4 \rtimes C_2$ as the group of order 28800 for a polytope $P$ whose 120 facets are isomorphic to $\{5, 6\}_4$ and whose 120 vertex-figures are copies of map R11.5 in [3].

For $p = 3$, the situation is quite analogous to that in the previous case. The diagrams

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
4
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
36
\end{array}
\begin{array}{c}
\bullet \\
9
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\end{array}$$

yield, respectively, the groups of orders 216 and 5832 for the duals of the first and last of the universal regular polytopes displayed in [24]. The two other pertinent diagrams

$$\begin{array}{c}
\begin{array}{c}
\bullet \\
12
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\bullet \\
4
\end{array}
\begin{array}{c}
\bullet \\
1
\end{array}
\begin{array}{c}
\bullet \\
3
\end{array}
\begin{array}{c}
\bullet \\
9
\end{array}
\end{array}$$

yield isomorphic groups of order 648. The given generators then provide the (non-isomorphic) duals of the second and third of the universal regular polytopes in [24].

Finally, consider $G = [\infty, 6, \infty]$. When $p = 3$ we again get the universal polytope described in [24]. For $p = 5, 7, 11, 13$, we obtain self-dual polytopes of type $\{p, 6, p\}$, with groups $O(4, 5, -1)$, $O(4, 7, +1)$, $O_4(4, 11, -1)$ and $O_4(4, 13, +1)$, respectively. In particular, for $p = 5$, we get the universal self-dual regular polytope

$$\{\{5, 6\}_4, \{6, 5\}_4\} .$$

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