Employing feedback in adiabatic quantum dynamics

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Abstract – We study quantum adiabatic dynamics, where the slowly moving field is influenced by the system’s state (feedback). The feedback is achieved either via mean-field quantum-classical interaction, or, alternatively, via non-disturbing measurements done on an ensemble of identical non-interacting systems. The situation without feedback is governed by the adiabatic theorem: adiabatic energy level populations stay constant, while the adiabatic eigenvectors get a specific phase contribution (Berry phase). However, under feedback the adiabatic theorem does not hold: the adiabatic populations satisfy a closed equation of motion that coincides with the replicator dynamics known by its numerous applications in evolutionary game theory. The feedback generates a new gauge-invariant adiabatic phase, which is free of the constraints on the Berry phase (e.g., the new phase is non-zero even for real adiabatic eigenfunctions). In a particular case the adiabatic theorem can still hold, but the new phases are non-trivial.

The adiabatic theorem of quantum mechanics governs the evolution of a quantum system subject to slowly varying external fields. Its applications span a vast array of fields, such as two-level systems (nuclei undergoing magnetic resonance or atoms interacting with a laser field), quantum phase transitions, quantum field theory (where a low-energy theory is derived by tracing out fast, high-energy degrees of freedom), and Berry’s phase [1–3]. This phase and the adiabatic theorem also find applications in quantum information processing [4,5]. For a recent discussion on the validity of the adiabatic approach see [6].

A general perspective of the quantum adiabatic physics is that it studies a system subject to a slow, open-loop (i.e., no feedback) control, where the evolution of the external fields is given a priori via time-dependent parameters of the system Hamiltonian. In view of numerous application of this setup, it is natural to wonder about the quantum adiabatic closed-loop control, where the external controlling fields evolve under feedback from the controlled quantum system. We expect to find in this case a much richer dynamic behaviour, keeping simultaneously the system-independent features of the adiabatic approach. Now any feedback needs information on the evolution of the system. This information is to be gained via measurements, which in the quantum situation are normally related with unpredictable disturbances and irreversibility. Thus quantum control has so far been restricted to open-system dynamics [7].

However, also quantum measurements can be non-disturbing, if N non-interacting quantum particles (spins, etc.) couple to the proper measuring apparatus. For \( N \gg 1 \) (analog of the classical limit) one can measure single-particle observables (almost) without disturbing the single-particle density matrix, since the disturbance caused by such measurements scales as \( 1/N^2 \), where \( \sigma \) is the measurement precision [8]. The knowledge of these observables allows to implement feedback [9].

Non-disturbing measurements on ensembles of few-level systems are routinely employed in NMR physics (e.g., in ensemble computation) and quantum optics [10–12]. An experimental example of realizing feedback in NMR physics is presented in [12].

Here we develop an adiabatic theory under feedback obtained via such non-disturbing measurements. We intend to show that the inclusion of the feedback in quantum adiabatic dynamics leads to a much richer dynamic behavior (non-linear replicator equation for the adiabatic probabilities, new gauge-invariant adiabatic phase), while still keeping the main attractive point of the adiabatic approach that is the ability to gain important information without actually solving the equations of motion.

Basic equations. – Consider a \( d \)-level quantum system described by a pure state \( |\psi\rangle \) (the generalization
to mixed states is indicated below). The system evolves according to the Schrödinger equation with the Hamiltonian $H[R(t)]$, where $R(t)$ is a classical controlling parameter ($\hbar = 1$):

$$i\dot{\psi}(t) = H[R(t)]\psi(t).$$

By means of a continuous non-disturbing measurement performed on an ensemble of identical, non-interacting systems (each one described by $|\psi(t)\rangle$) one finds the average $\langle \psi(t)|A|\psi(t)\rangle$ of a monitoring observable $A$ (in NMR physics $A$ typically corresponds to the magnetization). This average enters the feedback dynamics of $R$

$$\dot{R} = \varepsilon F(R, \langle \psi(t)|A|\psi(t)\rangle),$$

where $\varepsilon \ll 1$ is a small dimensionless parameter. We assume that $|F(\ldots)|$ is bounded from above, which means that $R$ is a slow variable: its derivative is bounded by a small number. For $F = F(R)$ (no feedback) we recover the standard adiabatic setup. The dynamics (1) conserves the purity of $|\psi(t)\rangle$, but the overlap $\langle \phi(t)|\psi(t)\rangle$ between two different wave functions is not conserved in time, since $H$ depends on $|\psi(t)\rangle/\psi(t)$ via (2).

We assume that $R$ is a scalar variable, though most of our results can be generalized directly to the case of several variables (vector-valued $R$).

The way we introduced (2) referred to the situation, where the feedback is engineered artificially. However, there is an important case, where the above scheme occurs naturally. For a particular case

$$F = -\langle \psi |\partial RH|\psi \rangle,$$

Equations (1), (2) can be viewed as a hybrid dynamics, where a classical particle with coordinate $R$ performs an overdamped motion and couples to the quantum system. Then $\varepsilon$ in (2) corresponds to an inverse damping constant, while $\langle \psi |\partial RH|\psi \rangle$ is the average (mean-field) force acting on $(\psi)$. (Overdamping means that the acceleration is irrelevant and the friction force is directly equated to the average potential force.) Hybrid theories are frequently employed in optics, atomic physics and chemical physics; see [13] for the current state of art. In this context it is useful to recall that also the non-feedback (adiabatic) situation, where a quantum system interacts with a given external classical parameter, refers to an interaction of the quantum system with a classical one (the parameter is then the coordinate of this classical system). However, in this standard case the feedback effect is neglected.

Let us now introduce the adiabatic eigensolution of the Hamiltonian for a fixed value of $R$ ($n = 1, \ldots, d$):

$$H[R] |n[R]\rangle = E_n[R] |n[R]\rangle, \quad \langle n[R]|m[R]\rangle = \delta_{mn}. \quad (4)$$

For simplicity we assume that the adiabatic energy levels are non-degenerate. The representation (4) has a gauge freedom:

$$|n[R]\rangle \rightarrow e^{i\alpha_n[R]} |n[R]\rangle,$$

where $\alpha_n[R]$ is an arbitrary single-valued function depending on $n$ and $R$. All observables should be gauge invariant. Now let us expand $|\psi(t)\rangle$ as

$$\psi(t) = \sum_n c_n(t) e^{i\gamma_n(t)} |n[R(t)]\rangle,$$

$$\gamma_n(t) \equiv -\int_{t_0}^t dt' E_n[R(t')], c_n(t) \equiv \langle n[R(t)] |\psi\rangle e^{-i\gamma_n(t)}, \quad (6)$$

where $\gamma_n(t)$ are the dynamical phases, $c_n$ are the adiabatic amplitudes, and where all summation indices run from 1 to $d$ (the total number of energy levels), if not stated otherwise.

One gets from (1), (2), (6):

$$\dot{c}_n = -\varepsilon \sum_k c_k \langle n|k]\rangle F(R, \{c\}, \{e^{i\Delta\gamma(t)}\}) e^{i(\gamma_k(t)-\gamma_n(t))}, \quad (7)$$

and where $F(R, c, e^{i\Delta\gamma(t)})$ is already reparametrized in terms of all $c_i$ and all phase differences $e^{i\gamma_i(t)-\gamma_j(t)}$. To make this point clear we indicated this dependence via curly brackets in $F(R, \{c\}, \{e^{i\Delta\gamma(t)}\})$. The amplitudes $c_n$ and $R$ are slow variables, since, e.g., $|c_n|$ is bounded from above by $O(\varepsilon)$ provided that the feedback $F(R)$ and the Hamiltonian $H[R]$ are bounded functions of $R$. However, the contribution from the dynamical phases $\gamma_n$ changes fast, since on the slow time $\tau = \varepsilon t$ it behaves as $\sim e^{it/\varepsilon}$; see (6). If the spacings between the adiabatic energy levels $E_n[R]$ remain large enough, the existence of some intermediate time $\tau_1$ is guaranteed, over which the dynamical phase contribution performs many oscillations, but $c_n$ and $R$ do not change appreciably. Note that in particular we assume that the adiabatic energy levels do not cross as functions of $R$ (in addition to these levels being non-degenerate).

The adiabatic approximation divides $c_n$ into the time-averaged (over $\tau_1$) part $\bar{c}_n$ and the small (at least as $O(\varepsilon)$) oscillating part: $c_n = \bar{c}_n + \delta c_n$ [16]. Leading order we neglect in the RHS of (7) all the oscillating factors and substitute $c \rightarrow \bar{c}$ and $R \rightarrow R$:

$$\bar{c}_n^* = -\sum_k c_k \langle n|k]\rangle F(R, \{\bar{c}\}, \{e^{i\Delta\gamma}\}) e^{i(\gamma_k-\gamma_n)}, \quad (9)$$

where $\tau = \varepsilon t, \bar{X} \equiv \int_0^{\tau} \frac{ds}{\tau} X(s)$, and where $X^* \equiv dX/d\tau$. Likewise, eq. (2) produces for the averaged motion of $R$

$$R^* = \bar{F}(R, \{\bar{c}\}, \{e^{i\Delta\gamma}\}), \quad \footnotemark[1]$$

\footnotetext[1]{The ordinary (non-feedback) adiabatic approach can be generalized to the case, where some level crossings are allowed; see [14] for detailed references. However, for the present feedback-driven adiabatic dynamics allowing level crossings requires more involved analysis [15].}
Linear feedback. – The simplest example of feedback is

\[ F = \langle \psi | A | \psi \rangle = \sum_{nm} c_n^* c_m A_{nm} e^{i(\gamma_m - \gamma_n)}, \]

(10)

where \( \langle n | A | m \rangle \equiv A_{nm} \). Equation (10) can be regarded as the first term of the Taylor expansion assuming that \( F(x) \) depends weakly on its argument. Equation (9) leads to

\[ \bar{c}_l^* = \sum_{k,m} c_k \langle l | k \rangle^* c_m A_{nm} e^{i(\gamma_m - \gamma_n + \delta_l)}, \]

(11)

In working out (11) we shall assume that the time-integrated energy level differences are distinct:

\[ \gamma_n(t) - \gamma_m(t) \neq \gamma(t) - \gamma_k(t), \quad \text{if} \ m \neq n \quad \text{and} \ m \neq l. \]

(12)

This condition is generic for few-level systems. It does not hold for cases like harmonic oscillator, which should be separately worked out from (11). Now in the RHS of (11) the non-zero terms are those with \( m = n \) and \( l = k \), and those with \( m = l \) and \( k = n \) (but \( n \neq l \), not to count twice the term \( m = n = k = l \):

\[ \bar{c}_l^* = -c_l \sum_{n(\neq l)} |c_n|^2 \langle l | n \rangle^* A_{nl}, \]

(13)

where \( \langle l | n \rangle^* \) is imaginary, since \( \partial_R \langle l | n \rangle = 0 \). The non-trivial (second) term in the RHS of (13) is due to non-diagonal elements of \( A \). Defining the phase and modulus of \( \bar{c}_n \),

\[ \bar{c}_n = \sqrt{p_n} e^{i\phi_n}, \quad \sum_n p_n = 1, \]

(14)

we get from (13) (and likewise from (2), (10))

\[ p_l^* = -2p_l \sum_{n(\neq l)} p_n \Re \langle l | n \rangle^* A_{nl}, \]

(15)

\[ \phi_l^* = i\langle l | l \rangle^* R^* - \sum_{n(\neq l)} p_n \Im \langle l | n \rangle^* A_{nl}, \]

(16)

\[ R^* = \sum_n p_n A_{nn}. \]

(17)

Equations (15)–(17) are our central results. Before exploring them in more detail let us discuss the standard (openloop, i.e., no feedback) adiabatics, where \( A = A(R) \) is a c-number. Now \( R \) moves in a prescribed way according to \( R^* = A(R) \). Equation (15) leads to the conservation of the probabilities \( p_l^* = 0 \) (adiabatic theorem): the system does not get enough energy to move out of the given energy level [1]. The RHS of (16) reduces to Berry’s factor \( \phi_{B,l}^* = i\langle l | l \rangle^* R^* \). As seen from (6), though \( \phi_{B,l} \) is by itself not gauge invariant, it does bring an observable (Berry phase) contribution to the average of an observable (Hermitean operator) over the state \( | \psi(t) \rangle = \sum_n c_n(0) e^{i\phi_n A(R) + i\gamma_n(t)} \), provided that this operator is not diagonal over the adiabatic basis (4).

The Berry phase was observed in numerous experiments; see [2,3] for review. It is constrained by the following conditions.

1) The Berry phase nullifies, \( \langle l | l \rangle = 0 \), if the adiabatic eigenvectors \( | l \rangle \) can be made real via a gauge transformation, e.g., a spinless particle without magnetic field. (This statement does not hold if there are level crossings.)

2) \( \phi_{B,l} = 0 \) for a cyclic motion of a single slow parameter \( R \), where \( R \) is switched on at the initial time and then switched off at the final time. The Berry phase may be different from zero if there is more than one slow parameter \( R = (R_1, R_2, \ldots) \) on a closed curve \( \mathcal{C} : R(0) = R(t) \) [2]. Then one gets a gauge-invariant expression \( \phi_{B,l} = i \oint_C dR \langle l | \partial_R l \rangle \) [2,3].

To our opinion these constraints seriously limit applications of the Berry phase. We shall see below that the adiabatic phases generated by the feedback-driven adiabatic dynamics are free of these limitations.

Closed-loop adiabatics. – Equation (15) for \( p_l \) arises out of the averaging over the fast dynamic phases under condition (12). Equation (15) is non-linear over \( p_n \) due to the feedback. The probabilities \( p_n \) are no longer conserved (due to the resonance between the oscillations of \( c_n \) and those of \( R \), see (11)), and if \( p_n \)'s are known, the phases \( \phi_l \) are obtained directly from (16). The matrix

\[ a_{ln} = -2R\langle l | \partial_R n \rangle \langle n | A | l \rangle, \quad a_{ln} = -a_{nl}, \]

(18)

in (15) is antisymmetric; in particular, \( a_{ll} = 0 \), which means \( \sum_l p_l \phi_l = 1 \). The edges of the probability simplex, e.g. \( p_l = \delta_{l1} \), are (possibly unstable) stationary solutions of (15), and \( p_l \phi_l \) is always non-negative.

It is noteworthy that (15) coincides with the replicator equation for a zero-sum population game [17,18]. Consider a population of agents that consists of groups \( l = 1, \ldots, d \). The fraction \( p_l \) of each group in the total population changes due to interaction between the groups, so that \( p_l^* \) is proportional to \( p_l \) itself, while the proportionality coefficient is the average payoff of the group \( l \): \( p_l^* = p_l \sum_{n} a_{ln}p_n \) [17,18]. Here the payoff matrix \( a_{ln} \) determines the fitness increase of the group \( l \) in its interaction with the group \( n \). The actual mechanism of this interaction depends on the concrete implementation of the model (inheritance, learning, imitation, infection, etc.) [18]. The condition \( a_{ll} = a_{ln} \) means a zero-sum game (e.g., poker); the gain of one group equals to the loss of the other. Thus in (15) the population game, with (in general) \( \tau \)-dependent payoffs \( a_{ln} \), is now played by the energy levels. Interesting features of the replicator equation can be found without solving it; see (20)–(23).

We note that this seems to be the first physical realization of the replicator dynamics, which so far was limited to the phenomenology of population dynamics. This, in particular, opens up a way for its precise experimental investigation (something that arguably is hardly possible in a field-like population dynamics).

For the open-loop control changing of \( R \) on the slow time scale is mandatory, otherwise no adiabatic motion occurs at all. The closed-loop situation is different, since now for \( \langle n | A | n \rangle = A_{nn} \neq 0 \) the slow motion of \( R \) is absent,

\[ R^* = 0, \]

(19)
(see (17)), with still non-trivial adiabatic dynamics. Equation (19) implies that the fast motion of $R$ averages out on the slow time. Let us focus on this situation, since we cannot study (15)–(17) in full generality.

Equations (15), (18), now with $\tau$-independent $a_{l,n}$, is conveniently solved via the time averages [18]:

$$
\frac{1}{T} \ln \frac{p_l(T)}{p_l(0)} = \sum_n a_{l,n} \tilde{p}_n(T),
\tilde{p}_n(T) = \int_0^T d\tau p_n(\tau), \quad (20)
$$

where $T$ defines the time window of the averaging. We shall normally take $T \to \infty$; see below.

There are now two different dynamic scenarios depending on the concrete form of $\tau$-independent $a_{l,n}$ in (15), (18).

1) If all $p_l(t)$ (which were non-zero at the initial time $\tau = 0$) stay non-zero for all times, $\tilde{p}(T)$ in the LHS of (20) is limited, which means that this LHS can be neglected for $T \to \infty$. We then get from (20) [17,18]

$$
\sum_n a_{l,n} \tilde{p}_n(\infty) = 0. \quad (21)
$$

Thus all $p_l(t)$ may remain non-zero for all times provided that there is a probability vector $\tilde{p}(\infty)$ that satisfies (21). Clearly, $\tilde{p}(\infty)$ is a stationary state of (15), (18). Recall that the (non-negative) relative entropy is defined as

$$
S[\tilde{p}(\infty)|p(\tau)] = \sum_l \tilde{p}_l(\infty) \ln \frac{\tilde{p}_l(\infty)}{p_l(\tau)}, \quad (22)
$$

where $p(t)$ is a time-dependent solution of (15). $S[\tilde{p}(\infty)|p(\tau)]$ is equal to zero if and only if $\tilde{p}(\infty) = p(\tau)$. Due to (21), $S[\tilde{p}(\infty)|p(t)]$ is a constant of motion (thus an adiabatic invariant), since

$$
S^*[\tilde{p}(\infty)|p(t)] = \sum_n p_l(\tau) a_{l,n} \tilde{p}_n(\infty). \quad (23)
$$

Equation (15) can be recast into a Hamiltonian form [17], where the constant of motion $S[\tilde{p}(\infty)|p(\tau)]$ becomes the Hamiltonian. The non-linearity of this dynamics is essential, since it can demonstrate chaos for $d \geq 5$ [15]. In some related systems the chaotic behavior was seen in [19].

2) If the matrix $a_{l,n}$ is such that (21) does not have any probability vector solution, $\frac{1}{T} \ln \frac{p_l(T)}{p_l(0)}$ in (20) is necessarily finite for at least one $l$. The corresponding probability $p_l(T)$ goes to zero (for a large $T$); $p_l(T) \to p_l(\infty) = 0$, so that for all $k$ one has $\sum_n a_{k,n} \tilde{p}_n(\infty) < 0$. This inequality is strict at least for $k = l$. Equation (23) shows that $S[\tilde{p}(\infty)|p(\tau)]$ now decays to zero meaning that $p(\tau)$ relaxes to $\tilde{p}(\infty)$. This relaxation is due to the non-linearity of (15); it is impossible without feedback.

Equation (16) for the phases integrates as

$$
\phi_l(\tau) = -\tau \sum_{n \neq l} \tilde{p}_n(\tau) b_{ln}, \quad b_{ln} \equiv \Im \{ |l| n' \langle n|A|l \rangle \}, \quad (24)
$$

where $\tilde{p}_n(\tau)$ satisfies the algebraic equation (20), and $b_{ln}$ is symmetric: $b_{ln} = b_{nl}$. Equation (24) gives the phases of the adiabatic feedback control. Clearly, $\phi_l(\tau)$ is free of the constraints for the open-loop (Berry) phase $\phi_{B,l}$: i) it is gauge invariant together with $n_{ln}$, see (5), (24); ii) its existence does not require complex adiabatic eigenvectors $|l\rangle$, provided that the monitoring observable $A$ has at least some complex elements $\langle n|A|l \rangle$; iii) it does not require several control parameters for cyclic processes; iv) even if $a_{ln}$, defined via (18), is zero, i.e., if the probabilities $p_n$ are conserved, the feedback-driven phases $\phi_l$ in (24) can be non-zero. Here we have an important situation, where the adiabatic theorem holds, but the new adiabatic phases are non-trivial.

Note that $\phi = 0$ if the evolution starts from one adiabatic eigenvector $p_n(0) = \delta_{nk}$, i.e., in the expansion (6) only one adiabatic eigenfunction with index $k$ is present: $\bar{c}_n(0) = \delta_{nk}$ (however, this stationary state of (15) need not be stable, as we saw above). In contrast, the Berry phase may be non-zero even for this case, although its observation does require interference with another eigenstate.

Examples. – We now apply our findings to two simple examples. For a two-level system eqs. (15), (18) reduce to (recall that $R^* = 0$, and thus $a_{12}$ does not depend on time $\tau$)

$$
p_1(\tau) = \frac{p_1(0)e^{a_{12}\tau}}{1 + p_1(0)e^{a_{12}\tau} - 1}, \quad (25)
$$

which means that independent of the initial value $p_1(0)$, $p_1 \to 1$ ($p_1 \to 0$, if $a_{12} > 0$ ($a_{12} < 0$), Properly choosing the time $\tau$ and $a_{12}$, and knowing $p_1(0)$, we can reach any value $0 \leq p_1(\tau) \leq 1$. For this simplest two-level example the oscillatory behavior (described after (23)) is absent.

Equation (24) produces for the phases

$$
\phi_{1,2}(\tau) = \frac{a_{12}}{a_{12}} \ln [p_1(0)(e^{a_{12}\tau} - 1) + 1], \quad (26)
$$

Two basic examples of two-level systems are the spin-$\frac{1}{2}$ and the polarization states of light. The standard Berry phase was observed in both these cases [2].

For the three-level situation the internal stationary vector is obtained from (21) (up to normalization)

$$
\tilde{p}_1(\infty) = \frac{a_{23}}{a_{12} + a_{23} - a_{13}}, \quad \tilde{p}_2(\infty) = -\frac{a_{13}}{a_{12} + a_{23} - a_{13}}, \quad (27)
$$

provided all these probabilities are positive as, e.g., for $a_{12} > 0$, $a_{23} > 0$, $a_{13} < 0$. In game-theoretic terms this means that 1 beats 2, 2 beats 3, but 3 beats 1 (cyclic dominance or rock-scissor-paper game) [18]. Now the $\tau$-dependent solution $p(\tau)$ of (15) oscillates around the stable point given by (27), (28) above [18].

The relaxation regime is realized if one of the probabilities in (27), (28) is not positive. For example, if 1 beats both others ($a_{12} > 0$, $a_{13} > 0$), the only attractor of (15) is $\tilde{p}(\infty) = (1,0,0)$. The latter conclusion holds also for a $\tau$-dependent $R$, if the conditions $a_{12}(\tau) > 0$ and $a_{13}(\tau) > 0$ are satisfied for all $\tau$’s. However, the general arguments (20)–(23) do not hold for $\tau$-dependent $a_{ln}$. 40007-p4
In the last example (as well as in the two-level example given by (25)) we meet a situation, where for an arbitrary initial pure state, the dynamics leads to relaxation to a definite state, i.e., to the state \( p_1 = 1 \) with the example given by (25). The reason for this relaxation behavior, which is clearly absent for the usual linear Schrödinger dynamics, is the underlying non-linearity of the feedback-driven adiabatic dynamics, as expressed by the dependence of \( F \) on \(|\psi\rangle\langle\psi|\) in (2). To some extent the above relaxation effect resembles the collapse of the wave function known from the phenomenology of quantum measurement. Recall from our discussion around (3) that this non-linearity may be caused by a mean-field interaction with a classical (i.e., macroscopic) system. However, the studied situation does not correspond to any measurement, since the “collapse” is not probabilistic and does not correspond to a macroscopic change in any measuring apparatus.

The slow motion of the controlling parameter \( R \) is allowed for \(|n⟩A|m⟩ ≠ 0\). Now (15) becomes a driven replicator equation, since \( a_{nm} \) are \( \tau \)-dependent. The general theory for a time-dependent replicator is (to our knowledge) lacking. There are two cases, where the above results suffice for analyzing the driven situation. In the above three-level example assume that conditions \( \frac{a_{21}(\tau)}{a_{12}(\tau)} < 0 \) or \( \frac{a_{21}(\tau)}{a_{12}(\tau)} > 0 \), are satisfied for all \( \tau \). Then the same argument on the relaxation to a single state applies. If the opposite conditions \( \frac{a_{21}(\tau)}{a_{12}(\tau)} > 0 \) and \( \frac{a_{21}(\tau)}{a_{12}(\tau)} < 0 \) hold for all \( \tau \), all \( p_1(\tau) \) are non-zero for all times \([15]\).

Mixed states. – So far we focused on pure states of the quantum system. Now we assume that the quantum state \( \rho \) is mixed and the feedback goes via the average \( \text{tr}(A\rho) \); compare with the closed-loop equation (2). Since the closed-loop equation (1) is not linear, the mixed-state dynamics (in general) does not reduce to the pure case. Starting from the feedback-driven von Neumann equation (compare with (1))

\[
i \dot{\rho}(t) = [H[R(t)], \rho], \quad \dot{R} = \varepsilon F \left( R, \text{tr}\{A \rho\} \right),
\]

and defining for the adiabatic amplitude

\[
c_{nm} \equiv \langle n|\rho|m⟩e^{i\gamma_{nm}},
\]

(compare with (6)), and proceeding along the lines of (4)–(12), we obtain

\[\dot{c}_{nm}^\ast + R^\ast c_{nm}(⟨n|n’⟩ + ⟨m|n⟩) = - \sum_{l(l≠n)} (⟨n|l’⟩A_{ln}c_{nl}c_{lm} - \sum_{l(l≠m)} (⟨l|m⟩c_{nl}c_{lm}A_{ml}),\]

(31)

\[
\dot{R}^\ast = \sum_{l} c_{ll}A_{ll}.
\]

(32)

There is a case where the pure-state analysis applies directly: pseudo-pure states in NMR are important for ensemble computation and are given as

\[
\rho = (1 - \eta) \frac{1}{d} + \eta |\psi⟩⟨\psi|,
\]

where \( \hat{1} \) is the unit matrix, and where \( 0 < \eta < 1 \) is a parameter \([11]\). Since \( \hat{1} \) is an invariant of (31), eq. (31) reduces to (13), but with \( A_{nl} \rightarrow \eta^2 A_{nl} \). Thus for the pseudo-pure case we get the same (though rescaled in time with the factor \( \eta^2 \)) adiabatic dynamics.

In general the phases of \( c_{nm} \) do not decouple from \( |c_{nm}| \), and we do not have a general theory for mixed states.

Hybrid dynamics. – Let us study in more detail the hybrid (quantum-classical) dynamics \( A = -i\partial_t H \); see our discussion after (3). Let us first of all address the pure-state dynamics. Equation (4) implies for \( n ≠ l \)

\[A_{nl} \equiv ⟨n|A|m⟩ = (E_n - E_l)⟨n|l’⟩,\]

(34)

\[⟨l|n’⟩A_{nl} = (E_l - E_n)|l⟩⟨n’|l’⟩.\]

(35)

Substituting (35) into (24) we see that the new phases \( \phi_l \) nullify. Equations (35), (15) predict for the probabilities

\[p_l^\ast = 2p_l \sum_{n(≠l)} p_n(E_n - E_l)|l⟩⟨n’|l’⟩.\]

(36)

We shall order the energies as

\[E_1 < ... < E_d.\]

(37)

Since we assumed that there are no level crossings (see our discussion after (8)), this ordering is conserved at all times, once it was imposed initially.

Considering (36) for \( l = d \) we see that \( p_d \) goes to zero for large times \( \tau → \infty \). Continuing this logic for \( l = d - 1, ... , 1 \) we see that all \( p_l \) with \( l > 1 \) go to zero for large times, while the probability \( p_1 \) for the lowest level goes to one. The origin of this phenomenon of relaxation to the ground state (or cooling) is related to (3), which implies that the external parameter \( R \) interacts with a zero-temperature thermal bath.

Let us now turn to the mixed-state dynamics, which will be seen to be more interesting. Equation (31) implies

\[c_{nm} = 2 ∑_{l(≠n)} (E_l - E_n)|l⟩⟨n’|l’⟩.\]

(38)

Let \( n = 1 \) be the lowest energy level. If all \( |⟨l’|1⟩| \) differ from zero, the non-diagonal elements \( c_{l≠1} \) have to nullify for large \( \tau \), since \( c_{11} \) should be limited. Continuing this reasoning for \( n > 1 \), we get that all non-diagonal elements \( c_{n≠l} \) nullify for large times, if all \( |⟨l’|n⟩| \) are positive.

Once the non-diagonal elements \( c_{n≠l} \) nullify, the large-time behavior of the diagonal elements \( c_{n} \) is determined by the fact that the dynamics (29) conserves the eigenvalues of the density matrix \( \rho \). Inspecting (31) and using (35), (37) one can see that the long-time diagonal values of \( c_{n}(∞) \) are equal to the eigenvalues \( λ_n(0) \) of the initial matrix \( {c_{nm}(0)}_{n,m=1}^{N} \) : the lowest energy gets the largest eigenvalue, \( c_{11}(∞) = max(λ_1(0), ... , λ_d(0)) \), the second energy level gets the second largest eigenvalues among \( \{λ_n(0)\} \), and so on till finally the highest energy level gets the smallest eigenvalue:

\[c_{dd}(∞) = min(λ_1(0), ... , λ_d(0)).\]

The resulting long-time state with the density matrix \( δ_{nm}(E_n(∞)) \) has the lowest energy among all density matrices with energies (37) and eigenvalues \( λ_n(0) \). Thus what we have here is an effect of controlled decoherence.
The above analysis required that all \(|\langle l'|n\rangle|\) are non-zero. If, however, \(|\langle l'|n\rangle| = 0\) for some \(n \neq l\), the element \(c_{nl}\) survives at long times and undergoes a non-trivial evolution. An example of such evolution is presented in fig. 1 under the following simplifying conditions (which are in a sense minimal for the existence of this effect): 1) The number of energy levels is equal to three: \(d = 3\). 2) The field \(R\) acts only on energy levels 2 and 3; the level 1 does not feel \(R\). Thus \(|\langle 2'|1\rangle| = |\langle 3'|1\rangle| = 0\) and \(|\langle 2'|3\rangle| > 0\). 3) The adiabatic eigenvectors do not depend on \(R\). Thus \(\dot{A}_{ll} = -\partial_R E_l = 0\) and \(R\) is also a constant of the averaged dynamics, (32). The \(R\)-dependence is restricted to the eigenvectors of the Hamiltonian \(\hat{H}[R]\) in (4). 4) These adiabatic eigenvectors \(|\phi\rangle\) are real (no Berry phases).

Equations (31), (35) now read

\[
\begin{align*}
\dot{c}_{22} &= 2\beta \tilde{c}_{23}^2, \\
\dot{c}_{23}^* &= \beta(1 - \tilde{c}_{11} - 2\tilde{c}_{22}), \\
\dot{c}_{11}^* &= \beta \tilde{c}_{13} \tilde{e}_{23} e^{-i\varphi_{23}}, \\
\dot{c}_{13}^* &= -\beta \tilde{c}_{12} \tilde{e}_{23} e^{i\varphi_{23}},
\end{align*}
\]

where \(\varphi_{23} = 0\), and where \(\beta \equiv (E_3 - E_2)/|\langle 2'|3\rangle|^2\) and \(\tilde{e}_{23} \equiv |\tilde{e}_{23} e^{i\varphi_{23}}|\). The numerical solution of (31), (35) under the above conditions 1)–4) is presented in fig. 1. The fact that \(c_{13}\) is constant is natural, since the external field \(R\) does not act on the first energy level due to condition 2). However, we see that \(c_{12}\) and \(c_{13}\) do change in time, an effect that is impossible for pure states; see (15), (16), which show that under \(|\langle 2'|1\rangle| = |\langle 3'|1\rangle| = 0\) both \(p_1\) and \(\phi_1\) are constant (in the slow time). Another interesting aspect seen in fig. 1, where all \(c_{|\phi|}\) are real, is that \(c_{13}\) changes its sign. This is an example of the adiabatic phase for the considered case. Note that though \(c_{23}\) decays to zero (according to the argument presented after (38)), it increases in the intermediate times.

In summary, we studied how the feedback generated by non-disturbing (ensemble) measurements affects the adiabatic (i.e., slowly driven) quantum dynamics. For the simplest linear feedback we have found that i) the populations are no longer constant. Instead, they satisfy the canonical (replicator) equation of the population game theory, allowing us to visualize the corresponding dynamics as a zero-sum game played by the adiabatic energy levels. The (non-linear) replicator equation generates a non-trivial (possibly chaotic) Hamiltonian motion, or alternatively, relaxation toward a certain state. ii) In addition to the Berry phase, the feedback generates a new, explicitly gauge-invariant phase, which exists under a wider range of conditions. There are scenarios of feedback, where the probabilities are constant (resembling the ordinary situation), but the new phases are still non-trivial. All these results extend directly to pseudo-pure quantum states. For the properly mixed-state situation we analyzed in detail the hybrid (quantum-classical) case.

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