Criteria for a certain class of the Carathéodory functions and their applications

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Abstract
In this paper, we obtain some potentially useful conditions (or criteria) for the Carathéodory functions as a certain class of analytic functions by applying Nunokawa’s lemma. We also obtain several conditions for strong starlikeness and close-to-convexity as special cases of the main results presented here.

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1 Introduction and preliminaries
Let \( A \) be a class of functions \( f \) of the following normalized form:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]

which are analytic in the open unit disk \( U \) given by

\[
U := \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}.
\]

Let \( \tilde{P}(\alpha) \) be a class of functions \( p \) of the form

\[
p(z) = \sum_{n=0}^{\infty} p_n z^n,
\]

which are analytic in \( U \) with \( p(0) = 1 \) and

\[
|\arg(p(z))| < \frac{\alpha \pi}{2} \quad (z \in U; 0 < \alpha \leq 1).
\]

Then, in the special case when \( \alpha = 1 \), \( \tilde{P}(1) \) is the well-known class of Carathéodory functions in \( U \) (see [8] and [9]; see also the recent developments on this subject in, for example, [19, 20, 23], and [28]).
For two functions \( f \) and \( F \), which are analytic in \( U \), we say that the function \( f \) is subordinated to the function \( F \) in \( U \) and we write \( f(z) \prec F(z) \) if there exists a Schwarz function \( \omega \), which is analytic in \( U \) with

\[
\omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \quad (z \in U),
\]

such that \( f(z) = F(\omega(z)) \) for all \( z \in U \). In particular, if the function \( F \) is univalent in \( U \), then we have the following equivalence:

\[
f(z) \prec F(z) \iff f(0) = F(0) \quad \text{and} \quad f(U) \subset F(U).
\]

Several recent investigations on various applications of differential subordination and differential superordination were reported in, for example, \([21, 22, 26, 27]\) (see also \([4, 5]\), and \([6]\)).

We denote by \( \tilde{S}^*(\alpha) \) the subclass of \( A \) consisting of functions which are strongly starlike of order \( \alpha \) in \( U \), that is,

\[
\tilde{S}^*(\alpha) := \left\{ f : f \in A \text{ and } \arg\left(\frac{zf'(z)}{f(z)}\right) < \frac{\alpha \pi}{2} \quad (z \in U; 0 < \alpha \leq 1) \right\}.
\]

Thus, in particular, \( S^* := \tilde{S}^*(1) \) is the class of starlike functions in the open unit disk \( U \).

By means of the principle of subordination between analytic functions, the above definition is equivalent to

\[
\tilde{S}^*(\alpha) := \left\{ f : f \in A \text{ and } \frac{zf'(z)}{f(z)} < \left(\frac{1 + z}{1 - z}\right)^\alpha \quad (z \in U; 0 < \alpha \leq 1) \right\}.
\]

We also denote by \( \tilde{CC}(\alpha) \) the subclass of \( A \) consisting of functions that are strongly close-to-convex of order \( \alpha \) in \( U \) if there exists a function \( g \in S^* \) such that

\[
\left| \arg\left(\frac{zf'(z)}{g(z)}\right) \right| < \frac{\alpha \pi}{2} \quad (z \in U; 0 \leq \alpha < 1).
\]

In particular, \( CC := \tilde{CC}(1) \) is the class of close-to-convex functions in the open unit disk \( U \).

Furthermore, we denote by \( \tilde{C}(\alpha) \) the subclass of \( A \) consisting of functions satisfying the following condition:

\[
\left| \arg(f'(z)) \right| < \frac{\alpha \pi}{2} \quad (z \in U; 0 \leq \alpha < 1).
\]

In particular, \( C := \tilde{C}(1) \) is a subclass of close-to-convex functions in the open unit disk \( U \).

In the year 1978, Miller and Mocanu \([14]\) introduced the method of differential subordinations. Then, in recent years, several authors have obtained several applications of the method of differential subordinations in geometric function theory by using differential subordination associated with starlikeness, convexity, close-to-convexity, and so on (see, for example, \([1–3, 7, 10–13, 17, 18, 24, 25]\)). The object of the present paper is to derive various potentially useful conditions (or criteria) for the Carathéodory functions as a certain class of analytic functions in the open unit disk \( U \) by using a lemma given by
Nunokawa (see [15] and [16]). Further, we give some applications to strong starlikeness and close-to-convexity.

The following lemma will be used in proving our main result.

**Lemma 1.1** (see [15] and [16]) Let the function $p(z)$ given by

$$p(z) = 1 + \sum_{n=m}^{\infty} c_n z^n \quad (c_n \neq 0)$$

be analytic in $U$ with $p(0) = 1$ and $p(z) \neq 0 \ (z \in U)$.

If there exists a point $z_0$ (with $|z_0| < 1$) such that

$$|\arg(p(z))| < \frac{\beta \pi}{2} \quad (|z| < |z_0|)$$

and

$$|\arg(p(z_0))| = \frac{\beta \pi}{2}$$

for some $\beta > 0$, then

$$\frac{zp'(z_0)}{p(z_0)} = ik\beta \quad (i = \sqrt{-1}),$$

where

$$k \geq m(a + a^{-1})^2 \geq m \quad \text{when } \arg(p(z_0)) = \frac{\beta \pi}{2} \quad (1.2)$$

and

$$k \leq -m(a + a^{-1})^2 \leq -m \quad \text{when } \arg(p(z_0)) = -\frac{\beta \pi}{2}, \quad (1.3)$$

where

$$[p(z_0)]^{1/\beta} = \pm ia \quad \text{and} \quad a > 0.$$

2 **Sufficient conditions for strong starlikeness and close-to-convexity**

**Theorem 2.1** Let $p$ be an analytic function in $U$, with $p(0) = 1$, $p'(0) \neq 0$, and $p(z) \neq 0$ for $z \in U$, that satisfies the following inequality:

$$\left| \left[ p(z) \right]^2 + \frac{zp'(z)}{p(z)} \right| < A(\alpha) |p(z)|, \quad (2.1)$$

where

$$A(\alpha) = \left\{ \begin{array}{ll}
\frac{\sqrt{2}}{4} \left( \frac{1 + \alpha}{1 - \alpha} \right)^{1-\alpha/2} + \left( \frac{1 + \alpha}{1 - \alpha} \right)^{-1-\alpha/2} + 2\alpha \sin(\pi \alpha) & \text{if } 0 < \alpha < 1, \\
\frac{\sqrt{2}}{4} & \text{if } \alpha = 1.
\end{array} \right. \quad (2.2)$$

Then $p \in \hat{P}(\alpha)$. 
Proof  To prove the result asserted by Theorem 2.1, we suppose that there exists a point \( z_0 \in \mathbb{U} \) such that

\[
|\arg(p(z))| < \frac{\alpha \pi}{2} \quad \text{for } |z| < |z_0|
\]

and

\[
|\arg(p(z_0))| = \frac{\alpha \pi}{2}.
\]

Then, from Lemma 1.1, it follows that

\[
z p'(z_0) \quad p(z_0) = i k \alpha,
\]

where \([p(z_0)]^{\frac{1}{2}} = \pm ia \quad (a > 0)\) and \(k\) is given by (1.2) or (1.3) for \( m = 1\).

For the case when

\[
[p(z_0)]^{\frac{1}{2}} = ia \quad (a > 0),
\]

we have

\[
|p(z_0)|^2 + \left| \frac{zp'(z_0)}{p(z_0)} \right| = \left| p(z_0) \right| \left| p(z_0) \right| + \left| \frac{zp'(z_0)}{p(z_0)} \right| \frac{1}{|p(z_0)|} = \left| p(z_0) \right| \left| (ia)^{\alpha} + ik \alpha \frac{1}{(ia)^{\alpha}} \right|
\]

\[
= \left| p(z_0) \right| \left| a^\alpha e^{i\pi \alpha/2} + \frac{k \alpha}{a^\alpha} e^{i\pi (1-\alpha)/2} \right|. \tag{2.3}
\]

From (2.3) for \( \alpha = 1 \), we find that

\[
\left| p(z_0) \right|^2 + \left| \frac{zp'(z_0)}{p(z_0)} \right| = \left| p(z_0) \right| \left| a^\alpha e^{i\pi \alpha/2} + \frac{k \alpha}{a^\alpha} e^{i\pi (1-\alpha)/2} \right| = \left| p(z_0) \right| \left| a^\alpha e^{i\pi/2} + \frac{k}{a} \right|
\]

\[
= \left| p(z_0) \right| \left| ia + \frac{k}{a} \right| = \left| p(z_0) \right| \sqrt{a^2 + \left( \frac{k}{a} \right)^2} \left( \frac{k}{a} \leq \frac{a^2 + 1}{2a^2} \geq \frac{1}{2} \right)
\]

\[
\geq \left| p(z_0) \right| \sqrt{\frac{1}{4}} = \left| p(z_0) \right| \frac{1}{2}.
\]

Also, since

\[
\frac{k \alpha}{a^\alpha} \geq \frac{a}{2} \left(a^{1-\alpha} + a^{-1-\alpha}\right),
\]

by applying (2.3) for \( 0 < \alpha < 1 \) with

\[
k \geq \frac{(a + a^{-1})}{2} \geq 1,
\]

we deduce that

\[
\left| p(z_0) \right|^2 + \left| \frac{zp'(z_0)}{p(z_0)} \right| = \left| p(z_0) \right| \left| a^\alpha e^{i\pi \alpha/2} + \frac{k \alpha}{a^\alpha} e^{i\pi (1-\alpha)/2} \right|
\]
\[= |p(z_0)| \sqrt{\left(\frac{k\alpha}{a^{\alpha}}\right)^2 + (\alpha^a)^2 + 2k\alpha \cos \left(\frac{\pi}{2}(1 - 2\alpha)\right)}\]

\[\geq |p(z_0)| \sqrt{\left(\frac{\alpha}{2} \left(a^{1-\alpha} + a^{\alpha-1}\right)\right)^2 + 0 + 2\alpha \cos \left(\frac{\pi}{2}(1 - 2\alpha)\right)}.
\]

We now define a real function \(g\) by

\[g(a) = a^{1-\alpha} + a^{\alpha-1} \quad (a > 0).
\]

Then this function \(g\) takes the minimum value for \(a\) given by

\[a = \sqrt{\frac{1 + \alpha}{1 - \alpha}}.
\]

Therefore, from the above equality in the case when \(0 < \alpha < 1\), we obtain

\[\left|p(z_0)^2 + \frac{zp'(z_0)}{p(z_0)}\right| \geq |p(z_0)| \sqrt{\frac{\alpha^2}{4} \left[\left(\frac{1 + \alpha}{1 - \alpha}\right)^{(1-\alpha)/2} + \left(\frac{1 + \alpha}{1 - \alpha}\right)^{(-1-\alpha)/2}\right]^2 + 2\alpha \sin(\pi \alpha)},\]

which contradicts our hypothesis of Theorem 2.1.

For the case when \(\left|p(z_0)\right|^2 = -ia\) \((a > 0)\),

by utilizing the same method as above, Lemma 1.1 for \(\alpha = 1\) yields

\[\left|p'(z_0)\right|^2 + \frac{zp'(z_0)}{p(z_0)} = |p(z_0)| \left|\alpha^a e^{-i\pi a/2} + \frac{k\alpha}{a^a} e^{i\pi(1+\alpha)/2}\right|
\]

\[= |p(z_0)| \left|\alpha e^{-i\pi a/2} - \frac{k}{a}\right|
\]

\[= |p(z_0)| \left|-ia - \frac{k}{a}\right| = |p(z_0)| \sqrt{a^2 + \left(\frac{k}{a}\right)^2}
\]

\[\geq |p(z_0)| \sqrt{\frac{1}{4}} = \frac{1}{2} |p(z_0)|.
\]

Also, for \(0 < \alpha < 1\), it follows for \(k \leq -1\) that

\[\left|p(z_0)^2 + \frac{zp'(z_0)}{p(z_0)}\right| = |p(z_0)| \left|\alpha^a e^{-i\pi a/2} + \frac{k\alpha}{a^a} e^{i\pi(1+\alpha)/2}\right|
\]

\[= |p(z_0)| \sqrt{\frac{k^2}{a^2} + (\alpha^a)^2 + 2k\alpha \cos \left(\frac{\pi}{2}(1 + 2\alpha)\right)}
\]

\[\geq |p(z_0)| \sqrt{\frac{\alpha^2}{4} \left[\left(\frac{1 + \alpha}{1 - \alpha}\right)^{(1-\alpha)/2} + \left(\frac{1 + \alpha}{1 - \alpha}\right)^{(-1-\alpha)/2}\right]^2 + 2k\alpha \cos \left(\frac{\pi}{2}(1 + 2\alpha)\right)}.
\]
\[ \geq |p(z_0)| \sqrt{\frac{\alpha^2}{4} \left[ \left( \frac{1 + \alpha}{1 - \alpha} \right)^{(1-\alpha)/2} + \left( \frac{1 + \alpha}{1 - \alpha} \right)^{(-1-\alpha)/2} \right]^2 + 2\alpha \sin(\pi \alpha)}, \]

which also contradicts our hypothesis of Theorem 2.1. From the two above-discussed contradictions, it follows that

\[ |\text{arg}(p(z))| < \frac{\alpha \pi}{2} \quad (\forall z \in \mathbb{U}). \]

This completes the proof of Theorem 2.1. □

**Remark 2.1** If

\[ f \in \mathcal{A} \quad \text{and} \quad p(z) := \frac{zf'(z)}{f(z)} \neq 0, \]

then \( p'(0) \neq 0 \) is equivalent to \( f''(0) \neq 0 \) and Theorem 2.1 leads to the following result, which gives a sufficient condition for strong starlikeness of order \( \alpha \).

**Corollary 2.1** Let the function \( f \in \mathcal{A} \), with \( f''(0) \neq 0 \), satisfy the following inequality:

\[ \left| 1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right| < A(\alpha) \left| \frac{zf'(z)}{f(z)} \right|, \]

where \( A(\alpha) \) is given by (2.2). Then \( f \in \tilde{S}^*(\alpha) \).

**Remark 2.2** For \( f \in \mathcal{A}, \alpha = 1 \) and \( p(z) := f'(z) \neq 0 \), Theorem 2.1 leads to the following result which gives a sufficient condition for the close-to-convexity (univalence) of the function \( f \).

**Corollary 2.2** If the function \( f \in \mathcal{A} \), with \( f''(0) \neq 0 \), satisfies the following inequality:

\[ \left| \left[ f'(z) \right]^2 + \frac{zf'(z)}{f(z)} \right| < \frac{1}{2} |f'(z)|, \]

then \( f \in \mathcal{C} \).

We now state and prove the following result.

**Theorem 2.2** Let \( p \) be an analytic function in \( \mathbb{U} \), with \( p(0) = 1, p'(0) \neq 0, \) and \( p(z) \neq 0 \) for \( z \in \mathbb{U} \), that satisfies the following inequality:

\[ \left| p(z) + \frac{zp'(z)}{|p(z)|^2} \right| < B(\alpha)|p(z)|, \]

where

\[ B(\alpha) = \begin{cases} \sqrt{1 + \frac{\alpha^2}{4} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2})^2 + \alpha (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \sin(\pi \alpha)} & \text{if } 0 < \alpha < \frac{1}{2}, \\ \sqrt{\frac{5}{4}} & \text{if } \alpha = \frac{1}{2} \end{cases}, \quad (2.4) \]
and
\[ \tilde{\alpha} = \frac{1 + 2\alpha}{1 - 2\alpha}. \]

Then \( p \in \tilde{\mathcal{P}}(\alpha) \).

**Proof** If we suppose that there exists a point \( z_0 \in \Omega \) such that
\[ |\arg(p(z))| < \frac{\alpha \pi}{2} \quad \text{for } |z| < |z_0| \]
and
\[ |\arg(p(z_0))| = \frac{\alpha \pi}{2}, \]
we find from Lemma 1.1 that
\[ \frac{zp'(z_0)}{p(z_0)} = i\alpha, \]
where
\[ \left[ p(z_0) \right]^{\frac{1}{\alpha}} = \pm ia \quad (a > 0) \]
and \( k \) is given by (1.2) or (1.3) for \( m = 1 \).

For the case when
\[ p(z_0)^{\frac{1}{2}} = ia \quad (a > 0), \]
we have
\[ \left| p(z_0) + \frac{zp'(z_0)}{[p(z_0)]^2} \right| = \left| p(z_0) \right| \cdot \left| 1 + \frac{zp'(z_0)}{p(z_0)} \cdot \frac{1}{[p(z_0)]^2} \right| = \left| p(z_0) \right| \cdot \left| 1 + i\alpha \cdot \frac{1}{(ia)^{2\alpha}} \right| \]
\[ = \left| p(z_0) \right| \cdot \left| 1 + \frac{k\alpha}{a^{2\alpha}} e^{i\pi(1-2\alpha)/2} \right|. \quad (2.5) \]

Now, from (2.5) for \( \alpha = \frac{1}{2} \), we get
\[ \left| p(z_0) + \frac{zp'(z_0)}{[p(z_0)]^2} \right| = \left| p(z_0) \right| \cdot \left| 1 - i\frac{k}{a} \right| = \left| p(z_0) \right| \cdot \sqrt{1 + \left( \frac{k}{a} \right)^2} \]
\[ \geq \sqrt{\frac{5}{4}} \left| p(z_0) \right|. \]

Also, from (2.5) for \( 0 < \alpha < \frac{1}{2} \), we deduce that
\[ \left| p(z_0) + \frac{zp'(z_0)}{[p(z_0)]^2} \right| \]
\[ = \left| p(z_0) \right| \cdot \left| 1 + \frac{k\alpha}{a^{2\alpha}} e^{i\pi(1-2\alpha)/2} \right| \]
\[ = \left| p(z_0) \right| \sqrt{1 + \left( \frac{k\alpha}{a^{2\alpha}} \right)^2 + 2 \frac{k\alpha}{a^{2\alpha}} \cos \left( \frac{\pi}{2} (1 - 2\alpha) \right)} \]
\[ \geq \left| p(z_0) \right| \sqrt{1 + \left( \frac{\alpha}{2} \left( a^{1-2\alpha} + a^{-1-2\alpha} \right)^2 + \alpha \left( a^{1-2\alpha} + a^{-1-2\alpha} \right) \cos \left( \frac{\pi}{2} (1 - 2\alpha) \right) \right).} \]

We now define a real function \( h \) by
\[ h(a) = a^{1-2\alpha} + a^{-1-2\alpha} \quad (a > 0). \]

Then this function takes the minimum value for \( a \) given by
\[ a = \sqrt{\frac{1 + 2\alpha}{1 - 2\alpha}}. \]

Therefore, from the above equality, when
\[ 0 < \alpha < \frac{1}{2} \quad \text{for} \quad \tilde{\alpha} = \frac{1 + 2\alpha}{1 - 2\alpha}, \]
we obtain
\[ \left| p(z_0) + \frac{zp'(z_0)}{\left| p(z_0) \right|^2} \right| \]
\[ \geq \left| p(z_0) \right| \]
\[ \times \sqrt{1 + \frac{\alpha^2}{4} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2})^2 + \alpha (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \cos \left( \frac{\pi}{2} (1 - 2\alpha) \right)}, \]

which contradicts our hypothesis in Theorem 2.2.

Next, for the case when
\[ \left[ p(z_0) \right]^{\frac{1}{2}} = -ia \quad (a > 0), \]
using the same method as before, we can obtain a contradiction to the assumption in Theorem 2.2.

From the two above-discussed contradictions, it follows that
\[ \left| \arg\left( p(z) \right) \right| < \frac{\alpha \pi}{2} \quad (\forall z \in \mathbb{U}). \]

This completes the proof of Theorem 2.2. \( \square \)

**Corollary 2.3** Let the function \( f \in \mathcal{A} \), with \( f''(0) \neq 0 \), satisfy the following inequality:
\[ \left| \frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) + \frac{zf'(z)}{f(z)} - 1 \right| < B(\alpha) \left| \frac{zf'(z)}{f(z)} \right|, \]
where \( B(\alpha) \) is given by (2.4). Then \( f \in \tilde{S}^*(\alpha) \).
Theorem 2.3  Let $p$ be an analytic function in $U$, with $p(0) = 1$, $p'(0) \neq 0$, and $p(z) \neq 0$ for $z \in U$, that satisfies the following inequality:

$$\left| \arg\left( p(z) + \frac{zp'(z)}{|p(z)|^2} \right) \right| < \frac{\delta\pi}{2},$$

where

$$\delta = \alpha + \frac{2}{\pi} \arctan\left( \frac{\frac{\alpha^2(1-2\alpha)^2}{2} + \alpha^2(-1-2\alpha)^2 \cos(\pi\alpha)}{1 + \frac{\alpha^2(1-2\alpha)^2}{2} + \alpha^2(-1-2\alpha)^2 \sin(\pi\alpha)} \right)$$

(2.6)

and

$$\tilde{\alpha} = \frac{1 + 2\alpha}{1 - 2\alpha} \left( 0 < \alpha < \frac{1}{2} \right).$$

Then $p \in \tilde{P}(\alpha)$.

Proof  Using similar arguments as in the proof of Theorem 2.1, for the case when

$$\left[ p(z_0) \right]^{\frac{1}{2}} = ia \quad (a > 0),$$

we have

$$\arg\left( p(z_0) + \frac{zp'(z_0)}{|p(z_0)|^2} \right) = \arg\left( p(z_0) \left( 1 + \frac{zp'(z_0)}{p(z_0)} \frac{1}{|p(z_0)|^2} \right) \right)$$

$$= \arg(p(z_0)) + \arg\left( 1 + ik\alpha \frac{1}{(ia)^{2\alpha}} \right)$$

$$= \arg(p(z_0)) + \arg\left( 1 + \frac{k\alpha}{a^{2\alpha}} e^{i\pi(1-2\alpha)/2} \right).$$

Since

$$\frac{k\alpha}{a^{2\alpha}} \geq \frac{\alpha}{2} \left( a^{1-2\alpha} + a^{-1-2\alpha} \right),$$

we now define a real function $h$ by

$$h(a) = a^{1-2\alpha} + a^{-1-2\alpha} \quad (a > 0).$$

Then this function takes on the minimum value for $a$ given by

$$a = \sqrt{\frac{1 + 2\alpha}{1 - 2\alpha}}.$$  

Therefore, from the above inequality, when

$$0 < \alpha < \frac{1}{2} \quad \text{for } \tilde{\alpha} = \frac{1 + 2\alpha}{1 - 2\alpha},$$
we obtain
\[
\frac{k\alpha}{a^{2\alpha}} \geq \frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}).
\]

Therefore
\[
\arg\left(\frac{p(z_0) + zp'(z_0)}{|p(z_0)|^2}\right) \geq \frac{\alpha \pi}{2} + \arctan\left(\frac{\frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \cos(\pi\alpha)}{1 + \frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \sin(\pi\alpha)}\right)
\]
\[
= \frac{\delta\pi}{2},
\]
which contradicts our hypothesis in Theorem 2.3.

Next, for the case when
\[
\left[\frac{p(z_0)}{z_0}\right]^{\frac{1}{2}} = -ia \quad (a > 0),
\]
with
\[
\frac{k\alpha}{a^{2\alpha}} \leq -\frac{\alpha}{2} (a^{1-2\alpha} + a^{-1-2\alpha}),
\]
using the same method as before, we can obtain
\[
\arg\left(\frac{p(z_0) + zp'(z_0)}{|p(z_0)|^2}\right) = \arg(p(z_0)) + \arg\left(1 + ik\alpha \frac{1}{(-ia)^{2\alpha}}\right)
\]
\[
= \arg(p(z_0)) + \arg\left(1 + \frac{k\alpha}{a^{2\alpha}} e^{\pi(1+2\alpha)/2}\right)
\]
\[
= -\frac{\alpha \pi}{2} + \arctan\left(\frac{\frac{k\alpha}{a^{2\alpha}} \sin\left(\frac{\pi}{2} (1 + 2\alpha)\right)}{1 + \frac{k\alpha}{a^{2\alpha}} \cos\left(\frac{\pi}{2} (1 + 2\alpha)\right)}\right)
\]
\[
= -\frac{\alpha \pi}{2} + \arctan\left(\frac{k\alpha}{a^{2\alpha}} \cos(\pi\alpha)\right)
\]
\[
\leq -\frac{\alpha \pi}{2} - \arctan\left(\frac{\frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \cos(\pi\alpha)}{1 + \frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \sin(\pi\alpha)}\right)
\]
\[
= -\frac{\delta\pi}{2},
\]
which is a contradiction to the assumption of Theorem 2.3.

From the two above-discussed contradictions, it follows that
\[
|\arg(p(z))| < \frac{\alpha \pi}{2} \quad (\forall z \in \mathbb{U}).
\]

This completes the proof of Theorem 2.3.

\[\square\]

**Corollary 2.4** Let the function \(f \in \mathcal{A}\), with \(f'''(0) \neq 0\), satisfy the following inequality:
\[
\left|\arg\left(\frac{f(z)}{zf'(z)} \left(1 + \frac{zf''(z)}{f'(z)}\right) + \frac{zf'(z)}{f(z)} - 1\right)\right| < \frac{\delta\pi}{2},
\]
where \(\delta\) is given by (2.6). Then \(f \in \tilde{S}_+^\alpha\).
Theorem 2.4 Let \( p \) be an analytic function in \( U \), with \( p(0) = 1, \ p'(0) \neq 0 \), and \( p(z) \neq 0 \) for \( z \in \mathbb{U} \), that satisfies the following inequality:

\[
\left| \arg\left( \left[ p(z) \right]^2 + \frac{zp'(z)}{p(z)} \right) \right| < \frac{\gamma \pi}{2},
\]

where

\[
\gamma = 2\alpha + \frac{2}{\pi} \arctan\left( \frac{\frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \cos(\pi \alpha)}{1 + \frac{\alpha}{2} (\tilde{\alpha}^{(1-2\alpha)/2} + \tilde{\alpha}^{(-1-2\alpha)/2}) \sin(\pi \alpha)} \right)
\]

and

\[
\tilde{\alpha} = \frac{1 + 2\alpha}{1 - 2\alpha} \quad (0 < \alpha < \frac{1}{2})
\]

Then \( p \in \tilde{P} (\alpha) \).

Proof By using a similar method as in the proof of Theorem 2.1, for the case when

\[
\left[ p(z_0) \right]^{\frac{1}{2}} = ia \quad (a > 0),
\]

with

\[
\frac{k \alpha}{a^{2\alpha}} \geq \frac{\alpha}{2} (a^{1-2\alpha} + a^{-1-2\alpha}),
\]

we have

\[
\arg\left( \left[ p(z_0) \right]^2 + \frac{z_0 p'(z_0)}{p(z_0)} \right) = \arg\left( \left[ p(z_0) \right]^2 \left( 1 + \frac{zp'(z_0)}{p(z_0)} \frac{1}{[p(z_0)]^2} \right) \right)
\]

\[
= \arg\left( \left[ p(z_0) \right]^2 \right) + \arg\left( 1 + ik \alpha \frac{1}{(ia)^{2\alpha}} \right)
\]

\[
= 2 \arg(p(z_0)) + \arg\left( 1 + \frac{k \alpha}{a^{2\alpha}} e^{i(1-2\alpha)/2} \right)
\]

\[
= \alpha \pi + \arctan\left( \frac{k \alpha}{a^{2\alpha}} \sin\left( \frac{\pi}{2} (1 - 2\alpha) \right) \right) \cos\left( \frac{\pi}{2} (1 - 2\alpha) \right).
\]

We now define a real function \( h \) by

\[
h(a) = a^{1-2\alpha} + a^{-1-2\alpha} \quad (a > 0).
\]

Then this function takes on the minimum value for \( a \) given by

\[
a = \sqrt{\frac{1 + 2\alpha}{1 - 2\alpha}}.
\]

Therefore, from the above equality, when

\[
0 < \alpha < \frac{1}{2} \quad \text{for} \quad \tilde{\alpha} = \frac{1 + 2\alpha}{1 - 2\alpha},
\]
we obtain
\[
\text{arg}\left(\left[p(z_0)\right]^2 + \frac{zp'(z_0)}{p(z_0)}\right) \geq \alpha \pi + \arctan\left(\frac{\frac{q}{2}(\tilde{\alpha}^{1-2\alpha})/2 + \tilde{\alpha}^{1-2\alpha}}{1 + \frac{q}{2}(\tilde{\alpha}^{1-2\alpha})/2 + \tilde{\alpha}^{1-2\alpha}} \cos(\pi \alpha)\right)
\]
\[
= \frac{\gamma \pi}{2},
\]
which contradicts our hypothesis in Theorem 2.4.

For the case when
\[
\left[p(z_0)\right]^2 = -ia \quad (a > 0),
\]
by using the same method as before, we can obtain
\[
\text{arg}\left(\left[p(z_0)\right]^2 + \frac{zp'(z_0)}{p(z_0)}\right) \leq -\frac{\gamma \pi}{2},
\]
which is a contradiction to the assumption in Theorem 2.4.

From the two above-discussed contradictions, it follows that
\[
\left|\text{arg}(p(z))\right| < \frac{\alpha \pi}{2} \quad (\forall z \in \mathbb{U}).
\]
This completes the proof of Theorem 2.4. \(\square\)

**Corollary 2.5** Suppose that the function \(f \in \mathcal{A}\), with \(f''(0) \neq 0\), satisfies the following inequality:
\[
\left|\text{arg}\left(1 + \frac{zf''(z)}{f'(z)} + \frac{zf'(z)}{f(z)}\left(\frac{zf''(z)}{f'(z)} - 1\right)\right)\right| < \frac{\gamma \pi}{2},
\]
where \(\gamma\) is given by (2.7). Then \(f \in \tilde{S}^*(\alpha)\).

**Remark 2.3** For \(g \in S^*\) and \(f \in \mathcal{A}\) such that \(2f''(0) \neq g''(0)\), by setting
\[
p(z) := \frac{zf'(z)}{g(z)} \neq 0
\]
in the above theorems, we will obtain a sufficient condition for strong close-to-convexity.

### 3 Conclusion
In the present paper, we have derived some sufficient conditions (or criteria) for the Carathéodory functions as a certain class of analytic functions in the open unit disk \(\mathbb{U}\). We have also deduced various sufficient conditions for the univalence, strong starlikeness, and strong close-to-convexity of functions in the normalized analytic function class \(\mathcal{A}\). We have considered several other related results as well. Also, with a view to motivating further research on the subject-matter of this investigation, we have included the citations of other closely-related recent developments as well.
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