Abstract

We consider the distributed version of the Multiple Knapsack Problem (MKP), where \( m \) items are to be distributed amongst \( n \) processors, each with a knapsack. We propose different distributed approximation algorithms with a tradeoff between time and message complexities. The algorithms are based on the greedy approach of assigning the best item to the knapsack with the largest capacity. These algorithms obtain a solution with a bound of \( \frac{1}{n+1} \) times the optimum solution, with either \( O(m \log n) \) time and \( O(mn) \) messages, or \( O(m) \) time and \( O(mn^2) \) messages.

1 Introduction

The Multiple Knapsack Problem (MKP) is a well known optimization problem which has been studied extensively \[12, 5\]. This problem is NP hard, and can be solved exactly with dynamic programming. The standard MKP is, however, only studied in the centralized settings, and its analogue for distributed algorithms \[2, 11\] is heretofore unknown.

In the distributed setting where knapsacks are dispersed, computation is divided among different processors that can only communicated by message-passing, this problem can be a useful model for certain problems arising in distributed systems, e.g., a data center where various jobs of different lengths and priority are delegated to machines with limited resources \[10, 9, 1\]. Then, the process of selecting the optimum set of jobs to complete with these limited resources is equivalent to the MKP. Here, jobs are equivalent to items, where the processing time of a job is equivalent to the weight of the item and the priority of the job is equivalent to the price of the item. The limited machine resources (like processing time or processing power) is analogous to the fixed capacity of knapsacks. The question of distributing load across multiple servers is a matter of practical interest \[21\].
While there is little pertinent work on distributed algorithms for the MKP, there is pertinent literature on the centralized problem as arising in various application domains. Nogueira et al. [14] attempt to schedule real time parallel jobs which are not all known beforehand. This work attempts to schedule all jobs arriving real time, in the most efficient way possible. We however propose a model for a different scenario, where the server resources (i.e., time) are fixed and the objective is to complete as many jobs as possible (with allowance for weights for each job). Most scheduling algorithms assume that balancing load across a servers in a distributed network is a good approach to obtain an optimal or close to optimal schedule. This however is not always the case (especially in the cases where job distributions are heavy tailed) [7]. This finding supports the choice of MKP as a model for job scheduling, as solving the MKP is inherently different from balancing the load across all servers.

Islam et al. [8] look at scheduling jobs as a multidimensional knapsack problem, where each dimension is associated with a resource and where each job with some revenue. This work follows a divide and conquer approach and tries to combine individual solutions obtained. However, this model is suited for a single processor with multiple resources rather than a model with multiple processors, which is the problem we attempt to solve. Another application of the MKP is the Multiple Subtopic Knapsack problem, to achieve search result diversification [22]. A part of the knapsack is allocated for and filled with relevant results while the remaining capacity is used to show diverse results. Each subtopic is treated as one of the multiple knapsacks in the standard MKP.

There exist several approximation algorithms based on dynamic programming after rounding, integer linear programming (ILP) or various greedy approaches to solve this problem in polynomial time which obtain solutions within a certain bound of the optimum solution [4]. However, LP/ILP approaches to solve the problem are not apt in a distributed system, as they lead to non-polynomial message complexity. Similarly, Bersekas [3] has proposed an algorithm for dynamic programming on a distributed system, but this method also has exponential time/message complexity in the worst case. These methods are not particularly suited for the distributed setting. This conclusion is echoed again by Paschalidis et al. [15] who formulate job scheduling as a Maximum Weighted Independent Set problem. They use a relaxed linear programming approach to solve it. The solution obtained is close to optimal; however this method requires a non-trivial number of iterations to converge.

We hence attempt to develop a distributed approximation algorithm
which achieves a trade-off between optimal performance and time and message complexity. We aim for an algorithm that has a message complexity close to $O(mn)$, as anything higher is unacceptable in large networks (large $n$) or with a large number of items (large $m$). Thus, we attempt to solve this problem in a distributed system with a focus on low message complexity.

We consider a generalized MKP in a distributed setting, where $n$ processors $p_j$ each own a single knapsack $k_j$ and have access to a common pool of items. Each knapsack has a capacity $W_j$. $K_j$ denotes the set of items assigned to the knapsack $k_j$. There are $m$ items indexed by $i$, each having a fixed weight $w_i$ and profit or cost $c_i$ associated with it. The objective here is then to assign each item uniquely to at most one knapsack in such a way that the sum of prices of all the items across all knapsacks is maximized, and the sum of weights of the items assigned to every knapsack is less than the capacity of that knapsack. Mathematically, the objective function

$$C = \sum_j \sum_i c_i x_{ij}$$

has to be maximized, under the constraints

$$\forall j, \sum_i w_i x_{ij} < W_j$$
$$x_{ij} \leq 1$$

where

$$x_{ij} = \begin{cases} 1 & \text{if item } i \text{ is assigned to } k_j \\ 0 & \text{otherwise} \end{cases}$$

It will also be useful to define the profit of a knapsack as the sum of the profits of all the items assigned to that knapsack, i.e., $c(K_j) = \sum_i c_i x_{ij}$. Further, we define the notion of the remaining capacity of a knapsack, $r_j$, as the difference of the capacity of the knapsack and the sum of the weights of the items assigned to this knapsack, i.e., $r_j = W_j - \sum_i w_i x_{ij}$.

### 1.1 The Model

There are $n$ processors $p_j$, $j = 1$ to $n$, fully connected to each other. We assume there is a distinguished node $S$ (which can be thought of as the source of these items or a dispatcher for jobs within a distributed system), with $(c_i, w_i)$ for each item. This node stores for the $i^{th}$ item, the index $j$ of the processor $p_j$ to which the item is assigned or $\perp$ if it is unassigned. We also assume that the node $S$ has the items sorted in the ratio of $\frac{c_i}{w_i}$. This node is also connected to all other processors $p_j$. We further assume that this model is failure free and synchronous.
1.2 Organization

Section 2 describes a greedy approach to solving the MKP, along with the analysis of its performance. In Section 3 we present and analyze two distributed algorithms which obtain the same solution to MKP, with differing time and message complexities. In Section 4 we discuss the downsides of adapting optimization and dynamic programming to this setting. In Section 5 we finally present our conclusion and the scope for future work.

2 Greedy Approach

The obvious greedy approach to solve the centralised single knapsack problem is to assign the “best” item (the item with highest $\frac{c_i}{w_i}$ ratio) to the knapsack and repeat till no more items fit into the knapsack. In the MKP, this approach would imply assigning the “best” item to any knapsack it fits in. Martello and Toth [12] show that the choice of the knapsack in this case is irrelevant and leads to the same approximation factor for the worst case. However, for simplicity, in case the “best” item can be assigned to multiple different knapsacks, we choose the convention to assign it to the knapsack with the largest remaining capacity, $r_j$. This approach is described in Algorithm 1. In each round, each processor $p_j$ sends its remaining capacity $r_j$ to $S$. $S$ assigns one item to each $p_j$ from the sorted list in order of decreasing capacity. Each processor $p_j$ updates its capacity after receiving an item. This is repeated till no items can be fit into any knapsack.

Lines 4–6 show the procedure followed by each processor $p_j$. Each processor sends its remaining capacity to $S$, receives a new item and updates its capacity accordingly. The source $S$ (lines 7–18) repeats its procedure as long as there are items left unassigned (line 7). It receives from each processor $p_j$ its remaining capacity $r_j$ (line 8) and then sorts them by decreasing order of $r_j$ (line 9). For each processor in this sorted list, $S$ sends the next item if it fits (lines 11–15).

Theorem 1. At each step, Algorithm 1 assigns the best available item $i$, to the knapsack $p_j$, currently having the largest capacity.

Proof. The proof is by contradiction. Suppose that this was not the case and the algorithm assigns an item $i$ to a knapsack $p_j$ where either the knapsack or the item is not the optimal choice (i.e., the item with the highest ratio of cost to weight and the knapsack with the largest capacity). This would
Algorithm 1 Simple Greedy Approach

1: ItemList ← ItemList.SortDecreasingBy($\frac{c_i}{w_i}$)  
2: $i = 0$  \stepcounter{algorithm} \hfill \text{\textit{S has all items sorted by}} \frac{c_i}{w_i} 
3: $\forall j, r_j = W_j$

For the processor $p_j$ in each round:
4: Send $\langle r_j \rangle$ to $S$
5: Receive $\langle c_i, w_i \rangle$ from $S$
6: $r_j \leftarrow r_j - w_i$

For the source $S$:
7: while $i \leq \text{length(ItemList)}$ do
8: Receive $\langle r_j \rangle$ from all $p_j$
9: $l = (p_j, r_j).\text{SortDecreasingBy}(r_j)$ \hfill \text{Sort by remaining capacity}
10: for $p_j$ in $l$ do
11: if $w_i \leq r_j$ then
12: Send item $\text{ItemList}[i]$ to $p_j$ \hfill \text{Send next item}
13: else
14: Send $\langle \bot \rangle$ to $p_j$
15: end if
16: $i \leftarrow i + 1$
17: end for
18: end while
imply that either

$$\exists i_2 : \frac{c_{i_2}}{w_{i_2}} \geq \frac{c_i}{w_i} \text{ OR } \exists j_2 : r_{j_2} \geq r_j$$

But this is not possible as both the items and the knapsacks are considered in the decreasing order of the ratio of cost to weight or remaining capacity. Thus, both the item $i$ and the knapsack $p_j$ are optimal, at every step of the algorithm.

2.1 Analysis

Algorithm 1 takes $\lceil \frac{m}{n} \rceil$ rounds where $m$ is the number of items and $n$ is the number of processors. This comes from the fact that there are $m$ items in all and 1 item is dispatched to each of the $n$ processors in each round. The number of messages is exactly 2 for every item assigned to some knapsack, which means we have at most $2m$ messages. However, this the fatal flaw of this algorithm is that it performs arbitrarily bad in the worst case [12], as shown below.

Consider $n$ knapsacks all of capacity $W$. Consider $2n$ of items, the first $n$ of which have cost 2 and weight 1 and the remaining $n$ items having cost and weight both equal to $W$. Using the previous algorithm, the first set of $n$ items are chosen, whereas the optimum solution is to pick the second set of $n$ items. The ratio of the solution to the optimum is $\frac{2n}{2n} = \frac{2}{W}$, which can be arbitrarily bad depending on the value of $W$.

There is a simple remedy to this problem [12]. At the end of the previous algorithm for each knapsack, we pick the best of the following two options:

- the solution obtained by the previous algorithm
- the most profitable unassigned item $i$, with maximum $c_i$ and with $w_i \leq W_j$

This can be represented as

$$\text{arg max} \left( c(K_j), \max_{i: w_i \leq W_j \text{ and } \sum_j x_{ij} = 0} (c_i) \right)$$

Martello and Toth [12] prove that the centralised version of Algorithm 2 gives a factor $\frac{1}{2}$ approximation scheme in the case of a single knapsack problem, and a factor $\frac{1}{n+1}$ approximation scheme in the case of a multiple knapsack problem.
Algorithm 2 Modified Greedy Approach

1: \( \text{ItemList} \leftarrow \text{ItemList}.\text{SortDecreasingBy}(\frac{c_i}{w_i}) \)
2: \( i = 0 \) \( \triangleright \) \( S \) has all items sorted by \( \frac{c_i}{w_i} \)
3: \( \forall j, r_j = W_j \)

For the processor \( p_j \):
4: Send \( \langle r_j \rangle \) to \( S \)
5: Receive \( \langle c_i, w_i \rangle \) from \( S \)
6: \( r_j \leftarrow r_j - w_i \)

For the source \( S \):
7: while \( i \leq \text{length(ItemList)} \) do
8: Receive \( \langle r_j \rangle \) from all \( p_j \)
9: \( l = (p_j, r_j).\text{SortDecreasingBy}(r_j) \) \( \triangleright \) Sort by remaining capacity
10: for \( p_j \) in \( l \) do
11: if \( w_i \leq r_j \) then
12: Send item \( \text{ItemList}[i] \) to \( p_j \) \( \triangleright \) Send next item
13: else
14: Send \( \langle \bot \rangle \) to \( p_j \)
15: end if
16: \( i \leftarrow i + 1 \)
17: end for
18: end while

procedure Final() : \( \triangleright \) Executed after initial assignment of items
19: for \( j = 1 \) to \( n \) do
20: Pick max \( \left( K_j, \max_{i:w_i \leq W_j} (c_i) \right) \)
21: end for
Lines 1–18 of Algorithm 2 remain the same as in Algorithm 1. The new refinement is implemented as the procedure labelled Final (shown in lines 19–21). At the end of the initial assignment of all items, the maximum of the current contents of the knapsack $K_j$ and the single costliest item which has not been assigned is picked as the final content of that knapsack. The correctness properties for Algorithm 2 are the similar to those of Algorithm 1. Explicitly, we can say that

**Theorem 2.** Algorithm 2 assigns the best of either:

1. the single largest unassigned item; or
2. the set of items obtained by the greedy approach of assigning the best item to the largest knapsack.

**Proof.** (1) follows trivially from line 19. (2) is equivalent to the correctness property of Algorithm 1 and has been proven as Theorem 1. The same proof holds here.

This performance bound of $\frac{1}{n+1}$ for Algorithm 2 is for the centralized version of the Algorithm 2. Algorithm 2 has the same correctness properties as the centralized algorithm [12]. Thus, the same proof for the performance bound holds here as well. Both the algorithms presented do not inherently exploit the distributed nature of the system; they are very similar to the centralized greedy approach. In the next section, we modify the simple greedy algorithm to exploit the distributed setting.

### 3 Distributed Greedy Algorithm

The previous algorithms presented did not exploit the distributed nature of the setting. The source assigned all items to the knapsacks in which case the method proposed by Chekuri [4] can be used to obtain better performance factor of $\frac{1}{e-1}$. However, these algorithms require the single node $S$ to carry out all the computation.

In the following algorithms, we present a method in which the nodes in the network themselves decide the assignment of items to knapsacks. The source $S$ does not have to perform any major computation during the algorithm; it only broadcasts details of items and receives the ID of the knapsack to which that item is assigned. This assignment is decided by the processors achieving consensus on which processor has the largest capacity left. This algorithm still follows the greedy approach for assigning items, which was outlined in the previous sections.
Each round is split into two phases, the first in which $S$ broadcasts the details for an item, and the second in which the nodes choose which knapsack the item is assigned to. This is repeated for each item. The knapsack is, as before, chosen to be the one with the largest remaining capacity. Algorithms 3 and 4 differ only in the way in which the nodes identify this knapsack.

The source simply broadcasts each item and receives the ID of the processor to which the item is assigned (lines 4–8). Each processor $p_j$ broadcasts $\langle j, r_j \rangle$ or $\langle j, \bot \rangle$ depending on whether $r_j \geq w_i$ (lines 10–14). Each processor then picks the maximum capacity of all the knapsack capacities received (lines 15–16). It then checks if its capacity is the maximum and if so, notifies the source $S$ (line 18) and updates its capacity (line 19). This process is repeated for each item. Finally, the procedure Final (lines 21–23) is called for each processor. This is exactly the same as in the previous algorithm.

**Theorem 3.** Algorithm 5 assigns each item $i$ to the largest knapsack $p_m$ in each round.

**Proof.** The proof is by contradiction. Assume that the “best” item $i$ in each round is assigned to a non-optimal knapsack $p_k$, where $p_k \neq p_m$ and $p_m$ is the knapsack with the largest remaining capacity. However, all processors broadcast their capacities and each processor picks the maximum from this set. Since this is a failure-free model, all processors pick the maximum from the same set. Thus, the item $i$ cannot be assigned to anything but $p_m$. 

This algorithm is runs in exactly $m$ rounds, one for each item. The number of messages is $n^2$ is each round for consensus and $n$ for the initial broadcast. Thus, the algorithm requires $m(n + n^2) = O(mn^2)$ messages. As before, this algorithm obtains a solution at least as good as $\frac{1}{n+1}$ times the optimum.

Algorithm 3 takes $O(m)$ time instead of $O(\frac{m}{n})$, and has a high message complexity of $O(mn^2)$. This can be improved at a further cost to time, using consensus. Currently consensus is $O(1)$ in time in each round 2.

In the next algorithm we present a slightly different approach to identify the knapsack with the largest remaining capacity. This is done with $n$ messages in each round. This will however require $O(\log n)$ time in each round. We try to exploit the synchronous properties of this setting. To do this:

- First create a rooted binary tree by identifying some edges in the network as tree edges, either to a parent or a child
Algorithm 3 Distributed Greedy Approach

1: ItemList ← ItemList.SortDecreasingBy($\frac{c_i}{w_i}$)
2: $i = 0$ ▷ $S$ has all items sorted by $\frac{c_i}{w_i}$
3: $\forall j, r_j = W_j$

For the source $S$:
4: for item $i = 1$ to $m$ do
5: Broadcast $\langle w_i \rangle$ to all $p_j$ ▷ Details of the next item
6: Receive $\langle j \rangle$
7: Assign $i$ to $j$
8: end for

For the processor $p_j$:
9: Receive $\langle w_i \rangle$ from $S$
10: if $r_j \geq w_i$ then
11: Broadcast $\langle j, r_j \rangle$ to all $p'_j$
12: else
13: Broadcast $\langle j, \bot \rangle$ to all $p'_j$
14: end if
15: Receive $\langle j, r_j \rangle$ from all $p'_j$
16: $m = \arg \max_{j'} (r'_j)$ from $S$ ▷ Reach consensus
17: if $m = j$ then
18: Send $\langle j \rangle$ to all $S$
19: $r_j \leftarrow r_j - w_i$
20: end if

procedure Final() : ▷ Executed after initial assignment of items
21: for $j = 1$ to $n$ do
22: Pick $\max \left( K_j, \max_{i:w_i \leq W_j} (c_i) \right)$
23: end for
• Use only the tree edges to achieve consensus. Only the root needs to know which processor picks the next item.

The main algorithm remains the same as before. A tree is constructed at the start, and only the consensus part changes. To construct the tree:

• $p_1$ is chosen as root
• The children of $p_j$ are taken to be $p_{2j}$ and $p_{2j+1}$.
• The parent of $p_j$ is $p_{\lfloor j/2 \rfloor}$.

To achieve consensus, after receiving item details from $S$, each node $p_j$ will pick the maximum capacity from all the capacities in the nodes of the subtree rooted at $p_j$ itself and send this to its parent. Finally $p_1$ sends the ID of the processor with the largest remaining capacity to $S$. This is described in Algorithm 4.

11
Algorithm 4 GREEDY APPROACH WITH MODIFIED CONSENSUS

1: ItemList ← ItemList.SortDecreasingBy(\( \frac{c_i}{w_i} \))
2: \( i = 0 \) \( \triangleright S \) has all items sorted by \( \frac{c_i}{w_i} \)
3: \( \forall j, r_j = W_j \)
4: \( \forall j, parent = left = right = \bot \)

Tree construction for \( p_j \):
5: \( parent = p_{\lfloor j/2 \rfloor} \)
6: \( left = p_{2j} \)
7: \( right = p_{2j+1} \)

For the source \( S \):
8: for item \( i = 1 \) to \( m \) do
9: Broadcast \( \langle w_i \rangle \) to all \( p_j \) \( \triangleright \) Details of the next item
10: Receive \( \langle j \rangle \)
11: Assign \( i \) to \( j \)
12: end for

For the processor \( p_j \):

upon receiving \( w_i \) from \( S \):
13: execute Consensus()

upon receiving \( i \) from \( S \) \( \triangleright \) Item received
14: \( r_j \leftarrow r_j - w_i \)

procedure Consensus() \( \triangleright \) To reach consensus on for a particular item
15: for \( k = \log n \) to 1 do
16: if \( k = \log j \) then
17: Receive \( \langle id_1, cap_1 \rangle \) from \( left \)
18: Receive \( \langle id_2, cap_2 \rangle \) from \( right \)
19: \( \langle id, cap \rangle = \langle \arg \max(r_j, cap_1, cap_2), \max(r_j, cap_1, cap_2) \rangle \)
20: Send \( \langle id, cap \rangle \) to \( parent \)
21: end if
22: end for
23: if \( j = 1 \) then
24: Send \( \langle id \rangle \) to \( S \)
25: end if
Algorithm 4 Greedy Approach with Modified Consensus-Contd.

procedure Final() :
\[ \triangleright \text{Executed after initial assignment of items} \]
26: \textbf{for} \ $j = 1$ to $n$ \ \textbf{do}
27: \quad \text{Pick } \max \left( K_j, \max_{i:w_i \leq W_j} (c_i) \right)
28: \textbf{end for}

Each processor identifies tree edges at the start of the procedure (lines 5–7). As before the source simply broadcasts each item and receives the ID of the processor to which the item is assigned (lines 8–12). Each processor $p_j$, upon receiving $w_i$ from $S$, starts the consensus subroutine (line 13). To achieve consensus (lines 15–25) each processor receives the maximum capacity from its left and right sub trees (lines 17–18). It then picks the maximum of these two capacities and its own capacity and sends this to its parent node (lines 19–20). This repeats for all processors at each of the log $n$ levels of the binary tree, starting bottom up (line 15–22). Finally, the root, $p_1$ sends to $S$ the ID of the processor with the largest capacity (lines 23–25). The source $S$ then sends the item $i$ to this processor, say $p_j$. This processor then updates its capacity accordingly (line 14). This process is repeated for each of the $m$ items. Finally, the procedure Final (lines 21–23) is called for each processor. This is exactly the same as in the previous algorithm.

**Theorem 4.** Algorithm 4 assigns each item $i$ to the largest knapsack $p_m$ in each round.

*Proof.* The proof is by induction. We will prove that in each round of the consensus subroutine, each node $p_j$, sends the maximum capacity of all the nodes present in the sub tree rooted at $p_j$ to its parent. The base case is for the nodes at the lowest level which simply transmit their capacities to their parent nodes. For the induction step, assume that this property is satisfied at level $k$ of the binary tree. Then, each node at level $k$ receives the maximum from its left and right children (for the left and right sub trees). It then picks the maximum capacity from amongst these and its own capacity and transmits it to its parent. Thus, the maximum capacity of the all the nodes in the sub tree rooted at this node is sent to its parent at the next level. Thus, this property now holds for the next level as well. Hence, it also holds for the root node $p_1$, which transmits the maximum capacity of all the nodes to the source (as all the nodes are children of $p_1$). This completes the proof for this theorem. \[ \square \]
3.1 Analysis

Algorithm 4 is \( O(m \log n) \) in time. There are \( m \) rounds, one for each item, and consensus takes \( \log n \) phases in each round. The number of messages is now \( O(n) \) in each round for consensus (as each node transmits only a single message to its parent) and the initial broadcast. Thus, the algorithm requires \( O(mn) \) messages. This solution obtained is at least as good as \( \frac{1}{n+1} \) times the optimum, as before.

It should be noticed here that each node sending its remaining capacity directly to the root node \( p_1 \) is not as efficient as the method described in Algorithm 4. If one node has to pick the maximum value of remaining capacity from a list of \( n \) elements, then it would require \( O(n) \) comparisons and \( O(n) \) time. Our method requires \( O(n) \) comparisons but \( O(\log n) \) time since these comparisons happen in parallel.

4 Further Improvements

In this section we look at other methods to improve the performance with reference to the optimum.

4.1 Heuristics

Heuristics involve switching items between knapsacks to fit more items in. Items can be switched one for one, one for two or two for one. We can even consider more cases of switching—three for one, and so on. If we do this for all possible combinations of items, we will eventually achieve the optimum. This will however take exponential time. Thus we have to restrict ourselves to some limit. However no performance guarantee can be achieved unless all possible switches are considered.

Other centralized heuristics for MKP are similar, one of which involves setting up D-sets (Dominating sets) for every element. A D-set for an item is the set of all items that are dominated by it, i.e., the set of items which cannot be included in the solution if the first item is not included in the solution. This is otherwise the set of items which have a higher weight and lesser cost than this item. Once, the D-set is found for each item, optimised selection is used, where an item and its D-set can be eliminated from consideration, which takes \( O(m) \) time. However, computing these D-sets is still expensive and the overall time complexity remains exponential.
4.2 Distributed LP Approaches

A LP problem can be solved on a distributed system in the following way—the variables whose values are to be found are split across all nodes [20]. In each iteration, only one variable is updated on one node and all the other variables are kept fixed. At the end of the iteration, this value is updated in all nodes. This means that we have $O(n)$ messages for each iteration. Further, we also have $mn$ variables for the LP. We will therefore have $O(mn^2)$ message complexity at the very least assuming one iteration for every variable. This method also assumes the diagonal dominance condition for the constraints (which we have not verified for the MKP). We also do not know of a good rounding scheme from a LP solution obtained to an ILP solution required for the MKP, making this approach infeasible.

The MKP can also be posed as a convex optimization problem with linear constraints can be solved to obtain close to optimum values [13]. This assumes that the constraints are positive and the objective function is separable (which is true for the MKP). This algorithm uses gradient descent, which may not be easily calculable for the MKP. This algorithm has inner and outer iterations: the inner iterations apply gradient descent on a given set of parameters, and these parameters are chosen by binary search by the outer iterations. The algorithm also calls as a subroutine, the “gossip” algorithm to communicate across the network at the end of each inner iteration. Like before, the gossip subroutine will lead to a high message complexity, making this approach infeasible.

4.3 Distributed Dynamic Programming

Chekuri [4] suggests that MKP can be solved within an approximation factor of $1 - 1/e \approx 0.63$ for uniform knapsack capacities and $1/2$ for non-uniform knapsack capacities. This is a far better bound than what we have obtained. This scheme uses a PTAS (Polynomial Time Approximation Scheme) for solving single knapsack problems with an approximation factor of $1 - \epsilon$ for each knapsack. The bound of $1/2$ remains irrespective of the order that knapsacks are considered.

This scheme implies a DP problem for each knapsack, but solving a DP problem in a distributed setting is not known to be efficient. Bertsekas [3] proposes an algorithm that has exponential-time convergence in bad cases. Even with constant message passing per round, this would still have exponential message complexity in the worst case.
5 Conclusion

We have presented distributed approximation algorithms for the MKP, the best of which has a message complexity of $O(mn)$, time complexity of $O(m \log n)$, and a performance bound of $\frac{1}{n+1}$. The currently existing methods to obtain better performance cannot be feasibly implemented on a distributed system with low message/time complexity (in $O(n)$ or $O(n \log n)$).

We believe that the MKP can be used as an alternative approach to scheduling and allocation in distributed systems such as data centers used in cloud computing. Our focus on a low message complexity is of particular importance when the number of items or jobs to be assigned is very high, as in the case of modern web servers. A low message complexity is also necessary when the number of processors is high, as in a large data center.

The MKP also has applications in other systems such as allocation of spectra in radio networks [19], so it stands to reason that distributed versions of the same would also be of much interest for similar reasons.

References

[1] Antonios Antoniadis, Chien-Chung Huang, Sebastian Ott, and José Verschae. How to pack your items when you have to buy your knapsack. In 38th International Symposium on Mathematical Foundations of Computer Science (MFSC 2013), August 2013. doi:10.1007/978-3-642-40313-2_8.

[2] Hagit Attiya and Jennifer Welch. Distributed Computing: Fundamentals, Simulations, and Advanced Topics. Wiley-Interscience, second edition, 2004.

[3] Dimitri P. Bertsekas. Distributed dynamic programming. AC-27:610–616, June 1982.

[4] Chandra Chekuri and Sanjeev Khanna. A PTAS for the Multiple Knapsack Problem. In Proceedings of the Eleventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA ’00), pages 213–222, 2000.

[5] M. Dawande, J. Kalagnanam, P. Keskinocak, F.S. Salman, and R. Ravi. Approximation algorithms for the multiple knapsack problem with assignment restrictions. Journal of Combinatorial Optimization, 4(2):171–186, June 2000.
[6] Stefka Fidanova. Heuristics for multiple knapsack problem. In *Proceedings of the IADIS International Conference on Applied Computing*, pages 255–260, February 2005.

[7] M Harchol-Balter, M Crovella, and C Murta. Task assignment in a distributed system: Improving performance by load unbalancing. Technical Report TR-97-018, Department of Computer Science, Boston University, October 1997.

[8] Md Imdadul Islam and Mostofa Akbar. Heuristic algorithm of the multiple-choice multidimensional knapsack problem (mmkp) for cluster computing. In *Computers and Information Technology, 2009. IC-CIT’09. 12th International Conference on*, pages 157–161. IEEE, 2009.

[9] Madhukar Korupolu, Aameek Singh, and Bhuvan Bamba. Coupled placement in modern data centers. In *IEEE International Symposium on Parallel & Distributed Processing (IPDPS 2009)*, May 2009. doi:10.1109/IPDPS.2009.5161067.

[10] Jiaxin Li, Dongsheng Li, Yuming Ye, and Xicheng Lu. Efficient multi-tenant virtual machine allocation in cloud data centers. *Tsinghua Science and Technology*, 20(1):81–89.

[11] Nancy A. Lynch. *Distributed Algorithms*. Morgan Kaufmann, 1996.

[12] Silvano Martello and Paolo Toth. *Knapsack Problems*. Wiley, 1990.

[13] Damon Mosk-Aoyama, Tim Roughgarden, and Devavrat Shah. Fully distributed algorithms for convex optimization problems. *SIAM Journal on Optimization*, 20(6):3260–3279, 2010. doi:10.1137/080743706.

[14] Luís Nogueira and Luís Miguel Pinho. Server-based scheduling of parallel real-time tasks. In *Proceedings of the Tenth ACM International Conference on Embedded Software (EMSOFT ’12)*, pages 73–82, October 2012. doi:10.1145/2380356.2380374.

[15] Ioannis Ch. Paschalidis, Fuzhuo Huang, and Wei Lai. A message-passing algorithm for wireless network scheduling. *IEEE/ACM Trans. Netw.*, 23(5):1528–1541, October 2015. doi:10.1109/TNET.2014.2338277.

[16] David Pisinger. *Algorithms for knapsack problems*. PhD thesis, University of Copenhagen, February 1995.
[17] David Pisinger. An exact algorithm for large multiple knapsack problems. *European Journal of Operational Research*, 114(3):528–541, 1999.

[18] David Pisinger. Where are the hard knapsack problems? *Computers & Operations Research*, 32(9):2271–2284, 2005.

[19] Yang Song, Chi Zhang, and Yuguang Fang. Multiple multidimensional knapsack problem and its applications in cognitive radio networks. In *IEEE Military Communications Conference (IEEE MILCOM 2008)*, November 2008. doi:10.1109/MILCOM.2008.4753629.

[20] Paul Tseng. Distributed computation for linear programming problems satisfying a certain diagonal dominance condition. *Mathematics of Operations Research*, 15(1):33–48, 1990. doi:10.1287/moor.15.1.33.

[21] S. Willehadson, A. Danne, and M. Blomme. Method and apparatus for load sharing and data distribution in servers, November 30 2010. US Patent 7,844,708.

[22] Hai-Tao Yu and Fuji Ren. Search result diversification via filling up multiple knapsacks. In *Proceedings of the 23rd ACM International Conference on Conference on Information and Knowledge Management (CIKM ’14)*, pages 609–618, November 2014. doi:10.1145/2661829.2661933.