Tutte polynomial of a small-world Farey graph

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Abstract – In this paper, we find recursive formulas for the Tutte polynomials of a family of small-world Farey graphs, which are modular, and has an exponential degree hierarchy. As applications of the recursive formula, the exact expressions for the chromatic polynomial and the reliability polynomial of Farey graphs are derived and the number of connected spanning subgraphs is also obtained.

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Introduction. – The Tutte polynomial of a graph, also known as the partition function of the Potts model [1,2], is a renown tool for analyzing the properties of graphs and networks. The two-variable polynomial, due to Tutte [3,4], plays an important role in several areas of sciences, for instance, combinatorics, statistical mechanics and biology. In a strong sense, it contains every graphical invariant that can be computed by deleting and contraction operations which are natural reductions for many networks model. The Tutte polynomial $T(G; x, y)$ can be evaluated at particular points $(x, y)$ to give numerical graph invariants, including the number of spanning trees, the number of connected spanning subgraphs, the dimension of the bicycle space. The Tutte polynomial also specializes to a variety of single-variable graphical polynomials, including the chromatic polynomial, the reliability polynomial and the flow polynomial. Furthermore, the Tutte polynomial has been widely studied in the field of statistical physics where it appears as the Potts model partition function $Z_G(q, v)$. In fact, let $G$ be a graph with $n$ vertices and $k$ connected components, then

$$T(G; x, y) = (x - 1)^{-k} (y - 1)^{-n} Z_G((x - 1)(y - 1), (y - 1))$$

and so the partition function of the Potts model is simply the Tutte polynomial expressed in different variables. For overviews, see refs. [5–7]. However, there are no widely available effective computational tools to compute the Tutte polynomial of a general graph of reasonable size.

In this paper, we follow a combinatorial approach and use the self-similar structure to investigate the Tutte polynomials of a family of recursive graphs which are called Farey graphs. The Farey graph was first introduced by Matula and Kornerup in 1979 and further studied by Colbourn in 1982. Recently, this graph was used as a deterministic network model by Zhang and Comellas [8,9]. This network model exhibits some remarkable properties of real networks. It is small-world with its average distance increasing logarithmically with its vertex number, and its clustering coefficient converges to a large constant ln 2 [8]. The Farey graph also has many interesting graphical properties, e.g., it is minimally 3-colorable, uniquely Hamiltonian, maximally outerplanar and perfect, see [10–12]. It is worth mentioning that the Potts model on recursively defined graphs was studied first by Dhar [13] and some related work can be found in refs. [14,15].

By analyzing all the spanning subgraphs of the Farey graph, we give recursive formulas for the Tutte polynomial (see Theorem 2 and Corollary 3). In particular, as special cases of the general Tutte polynomials, we get

- the number of connected spanning subgraphs (see eq. (14));
- the number of spanning trees (see eq. (17));
- the reliability polynomial $R(G; p)$ (see eq. (18));
- the chromatic polynomial $P(G; \lambda)$ (see eq. (19)).

Preliminaries. – In this section, we briefly discuss some necessary background that will be used for our calculations. We use standard graph terminology and the words “network” and “graph” indistinctly.
Tutte polynomial. For a graph $G$, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. Let $k(G)$ be the number of connected components of the graph $G$. A spanning subgraph $H = (V(H), E(H))$ is a subgraph of $G$ such that $H$ has the same vertex set as $G$ and a subset $E(H)$ of the edge set $E(G)$. There are several definitions for the Tutte polynomial of a graph. We introduce the definition in terms of spanning subgraphs:

$$T(G; x, y) = \sum_{H \subseteq G} (x - 1)^{r(G) - r(H)}(y - 1)^{n(H)},$$

where $r(H) = |V(H)| - k(H)$ is the rank of $H$ and $n(H) = |E(H)| - r(H)$ is the nullity of $H$. Recall that a one-point join $G \ast H$ of two graphs $G$ and $H$ is formed by identifying a vertex $u$ of $G$ and a vertex $v$ of $H$ into a single vertex $w$ of $G \ast H$. It is well known that the Tutte polynomial fulfills the following property:

$$T(G \ast H; x, y) = T(G; x, y)T(H; x, y).$$

In this paper, we are also interested in special evaluations of the Tutte polynomial at some particular points $(x, y)$, which allow us to deduce many combinatorial and algebraic properties of the graphs considered. We recall some graphic terminology: A spanning forest of a graph $G$ is a spanning subgraph which is a forest. A connected spanning subgraph is a spanning subgraph that is connected. The special evaluations of interest are i) $T(G; 1, 1) = N_{ST}(G)$, the number of spanning trees of $G$; ii) $T(G; 2, 1) = N_{SF}(G)$, the number of spanning forests of $G$; iii) $T(G; 1, 2) = N_{CSSG}(G)$, the number of connected spanning subgraphs of $G$ [6].

The Tutte polynomial also contains several other graphical polynomials as partial evaluations, such as the reliability polynomial, the flow polynomial and the chromatic polynomial. Considering a related graph $H$ obtained from $G$ by going through the full edge set of $G$ and, for each edge, randomly retaining it with probability $p$ (thus deleting it with probability $1 - p$), where $0 \leq p \leq 1$. The reliability polynomial $R(G; p)$ gives the resultant probability that any two vertices in $H$ are connected, i.e., that for any two such vertices, there is a path between them consisting of a sequence of connected edges of $G$ [6]. The chromatic polynomial $P(G; \lambda)$ of a graph $G$ is an important specialization of the Tutte polynomial, which counts the number of ways of coloring the vertices of $G$ subject to the constraint that no adjacent pairs of vertices have the same color [16,17]. In a different form, the chromatic polynomial arise in statistical physics as the zero-temperature limit of the anti-ferromagnetic Potts model, and in that context its complex roots are of particular interest [18,19]. The connection with the Tutte polynomial is given by the following equations:

$$R(G; p) = q^{n(G)}p^{r(G)}T(G; 1, q^{-1}),$$

$$P(G; \lambda) = (-1)^{r(G)}\lambda^{n(G)}T(G; 1 - \lambda, 0),$$

where $q = 1 - p$, $n(G)$ and $r(G)$ are the nullity and the rank of $G$, respectively.

**Farey graphs.** The Farey graph [8–11] is derived from the famous Farey sequence [20] and this graph can be created in the following recursive way, see fig. 1. Let $G_0$ denote the Farey graph after $n$ ($n \geq 0$) generations. $G_n$ has two special vertices which are represented by $X_n$ and $Y_n$, respectively. For $n = 0$, $G_0$ is an edge connecting two special vertices. For $n \geq 1$, $G_n$ can be obtained by joining two copies of $G_{n-1}$ which are labeled by $G_{n-1}^i$ ($i = 1, 2$). The two special vertices of $G_{n-1}^i$ are represented by $X_{n-1}^i$ and $Y_{n-1}^i$, respectively. In the constructing process, $X_{n-1}^1$ and $Y_{n-1}^1$ are identified as a new vertex $Z_n$ of $G_n$, $X_{n-1}^2$ and $Y_{n-1}^2$ are connected by a new edge $e_n$. Moreover, $X_{n-1}^1$ and $Y_{n-1}^2$ become $X_n$ and $Y_n$, respectively.

It is clear that the order and size of the Farey graph $G_n = (V_n, E_n)$ are, respectively,

$$|V_n| = 2^n + 1, \quad |E_n| = 2^{n+1} - 1.$$  

**Remark 1.** There is another definition of the Farey graph given in [8] as Definition 2.1, which seems simpler and easier to imagine. In fact, the Farey graph $G_n = (V_n, E_n)$ can be constructed in the following iterative way: For $n = 0$, $G_0$ has two vertices and an edge joining them. For $n \geq 1$, $G_n$ is obtained from $G_{n-1}$ by adding to every edge introduced at step $n-1$ a new vertex adjacent to the endvertices of this edge, see fig. 2.

![Fig. 1: (Color online) Recursive construction of Farey graphs that highlights their self-similarity. In this figure, red vertices represent the special vertices, the new edge is denoted by a green line.](image1)

![Fig. 2: Farey graphs $G_0$ to $G_3$.](image2)

**The Tutte polynomial of the Farey graph $G_n$.** We divide the set of spanning subgraphs of $G_n$ into two disjoint subsets $G_{1,n}$ and $G_{2,n}$, see fig. 3.

- $G_{1,n}$ denotes the set of spanning subgraphs of $G_n$, where the special vertices of $G_n$ belong to the same connected component;
red vertices represent the special vertices of $G_n$ and two special vertices joined by two blue lines belong to the same connected component. Dotted lines indicate that the two special vertices belong to different connected components.

$G_{2,n}$ denotes the set of spanning subgraphs of $G_n$, where the special vertices of $G_n$ belong to two different connected components.

Observe that, for each $n \geq 0$, we have the partition $G_1,n \cup G_2,n$ of the set of spanning subgraphs of $G_n$. Next let us simply denote by $T_n(x,y)$ the Tutte polynomial $T(G_n; x,y)$ and define, for every $n \geq 1$, the following polynomials:

$T_{1,n}(x,y) = \sum_{H \in G_{1,n}} (x-1)^{r(G_n)-r(H)}(y-1)^{n(H)}$;

$T_{2,n}(x,y) = \sum_{H \in G_{2,n}} (x-1)^{r(G_n)-r(H)}(y-1)^{n(H)}$.

We have

$$T_n(x,y) = T_{1,n}(x,y) + T_{2,n}(x,y). \tag{6}$$

In order to give a recursive formula for $T_n(x,y)$, we provide recursive formulas for $T_{1,n}$ and $T_{2,n}$, respectively.

From the construction of $G_n$, it is easy to see that any spanning subgraph of $G_n$ can be obtained from two spanning subgraphs of $G_{n-1}$ and a new edge. Let $H_{a,n-1}^a$, $H_{b,n-1}^b$ be two spanning subgraphs of $G_{n-1}$, we denote $H_{a,n-1}^a \ast H_{b,n-1}^b$ as the graph obtained from $H_{a,n-1}^a$ and $H_{b,n-1}^b$ by identifying the vertex $Y_{n-1}$ of $H_{b,n-1}^b$ and the vertex $X_{n-1}$ of $H_{a,n-1}^a$. Let $H_n = H_{a,n-1}^a \ast H_{b,n-1}^b \cup C$, where $C$ may be $\{e_n\}$ or an empty set $\emptyset$. Thus, $H_n$ is a spanning subgraph of $G_n$. Let $(G_{i,n-1},G_{i,n-1})$ be the set of $H_{a,n-1}^a \ast H_{b,n-1}^b$ with $H_{a,n-1}^a \in G_{i,n-1}$ and $H_{b,n-1}^b \in G_{j,n-1}$, for $i,j = 1,2,3$.

Table 1: (Color online) All combinations and corresponding contributions.

| $H_{a,n-1}^a \ast H_{b,n-1}^b$ | $C$ | $H_n$ | Contribution | Graphic |
|-----------------------------|-----|-------|--------------|---------|
| $(G_{1,n-1},G_{1,n-1})$    | $\{e_n\}$ | $G_{1,n}$ | $(y-1)T_{1,n-1}^2$ | $G_{1,n-1},G_{1,n-1}$ |
| $(G_{1,n-1},G_{2,n-1})$    | $\{e_n\}$ | $G_{1,n}$ | $\frac{1}{x-1}T_{1,n-1}T_{2,n}$ | $G_{1,n-1},G_{2,n-1}$ |
| $(G_{2,n-1},G_{1,n-1})$    | $\{e_n\}$ | $G_{1,n}$ | $\frac{1}{x-1}T_{1,n-1}T_{2,n}$ | $G_{2,n-1},G_{1,n-1}$ |
| $(G_{2,n-1},G_{2,n-1})$    | $\{e_n\}$ | $G_{1,n}$ | $\frac{1}{x-1}T_{2,n-1}^2$ | $G_{2,n-1},G_{2,n-1}$ |

Theorem 2. For each $n \geq 1$, the Tutte polynomial $T_n(x,y)$ of $G_n$ is given by

$$T_n(x,y) = T_{1,n}(x,y) + T_{2,n}(x,y),$$

where the polynomials $T_{1,n}(x,y)$, $T_{2,n}(x,y)$ satisfy the following recursive relations:

$$T_{1,n}(x,y) = yT_{1,n-1}^2 + \frac{2}{x-1}T_{1,n-1}T_{2,n-1} + \frac{1}{x-1}T_{2,n-1}^2,$$

$$T_{2,n}(x,y) = 2T_{1,n-1}T_{2,n-1} + 2T_{2,n-1}.$$

Proof. The initial conditions are easily verified. The strategy of the proof is to study all the possible combinations of $H_{a,n-1}^a \ast H_{b,n-1}^b$ and $C$, and analyze their contributions to $T_{1,n}(x,y)$ and $T_{2,n}(x,y)$, respectively. See table 1 for details.

As shown in table 1, we have,

$$T_{1,n}(x,y) = \sum_{H_n \in G_{1,n}} (x-1)^{r(G_n)-r(H_n)}(y-1)^{n(H_n)},$$

$$T_{2,n}(x,y) = \sum_{H_n \in G_{2,n}} (x-1)^{r(G_n)-r(H_n)}(y-1)^{n(H_n)}.$$

It can be proven by induction that $x-1$ divides $T_{2,n}$ for each $n \geq 0$. As a consequence, we can write $T_{2,n}(x,y) = (x-1)N_n(x,y)$, with $N_n(x,y) \in \mathbb{Z}[x,y]$. Next corollary is useful in the derivation of special evaluations of the Tutte polynomial $T_n(x,y)$.

Corollary 3. For each $n \geq 1$, the Tutte polynomial $T_n(x,y)$ of $G_n$ is given by

$$T_n(x,y) = T_{1,n}(x,y) + (x-1)N_n(x,y),$$

where the polynomials $T_{1,n}(x,y)$, $N_n(x,y)$ satisfy the following recursive relations:

$$T_{1,n}(x,y) = yT_{1,n-1}^2 + 2T_{1,n-1}N_{n-1} + (x-1)N_{n-1}^2,$$

$$N_n(x,y) = 2T_{1,n-1}N_{n-1} + (x-1)N_{n-1}^2,$$

with initial conditions $T_{1,0}(x,y) = 1$, $N_0(x,y) = 1$. 38001-p3
Simple applications. — The problem of spanning trees is closely related to various aspects of networks, such as dimer coverings [21] and random walks [22,23]. Thus it is of great interest to determine the exact number of spanning trees \( N_{ST}(G) \) [9,24,25]. Two well-known methods for computing \( N_{ST}(G) \) are as follows: i) via the Laplacian matrix [26] and ii) as a special evaluation of the Tutte polynomial. With the above-obtained results, we go on to compute \( N_{ST}(G_n) \) by the second method. First, according to Corollary 3, let \( x = 1 \), we have \( T_n(1,y) = T_{1,n}(1,y) \), and
\[
T_{1,n}(1,y) = yT_{1,n-1}^2 + 2T_{1,n-1}N_{n-1},
\]
(7)
\[
N_{n}(1,y) = 2T_{1,n-1}N_{n-1}.
\]
Equations (7) and (8) together yield a useful relation given by
\[
\frac{T_{n+1}(1,y)}{T_n(1,y)} = (2 + y) - \frac{2yT_{n-1}^2}{T_n}.
\]
(9)
For \( t \geq 2 \), we introduce
\[
a_t(1,y) = \frac{T_t(1,y)}{T_{t-1}(1,y)^2}, \quad \text{with } a_1(1,y) = 2 + y.
\]
(10)
Thus, eq. (9) can be further written as
\[
a_{n+1}(1,y) = (2 + y) - \frac{2y}{a_1(1,y)}.
\]
(11)
If \( y = 2 \), eq. (11) leads to
\[
\frac{1}{a_{n+1} - 2} = \frac{1}{a_n - 2} + \frac{1}{2}.
\]
(12)
Since \( a_1(1,2) = 4 \), we have
\[
a_n(1,2) = \frac{2(n + 1)}{n}.
\]
(13)
Substituting the above-obtained expression of \( a_n(1,2) \) into eq. (10) and using the initial condition \( T_0(1,2) = 1 \) yields
\[
T_n(1,2) = \frac{2(n + 1)}{n}T_{n-1}^2 = 2^{n-1}(n + 1)\prod_{i=2}^{n}2^{a_i}.
\]
(14)
It is well known that \( N_{CSSG}(G_n) = T_n(1,2) \). Therefore, we obtain explicit expression of the number of connected spanning subgraphs of the Farey graph \( G_n \).

If \( y \neq 2 \), eq. (11) can be solved to obtain
\[
a_n(1,y) = \frac{2^{n+1} - y^{n+1}}{2^n - y^n}.
\]
(15)
Using the initial condition \( T_0(1,y) = 1 \) and the expression for \( a_n(1,y) \) provided by eq. (15), eq. (10) can be solved inductively to obtain
\[
T_n(1,y) = \frac{2^{n+1} - y^{n+1}}{2^n - y^n}T_{n-1}^2 = \frac{2^{n+1} - y^{n+1}}{(2 - y)^{2^n} - (2 - y)^{2^n}}\prod_{i=2}^{n}2^{a_i}.
\]
(16)
Notice that, the eq. (16) is also right when \( y = 2 \), since
\[
2^i - y^i = (2 - y)(2^{i-1} + 2^{i-2}y + \cdots + 2^{i-k-1}y^{k} + \cdots + y^{i-1})
\]
and
\[
(2 - y)\prod_{i=2}^{n}(2 - y)^{2^n-i} = (2 - y)^{2^n-1}.
\]
Let \( y = 1 \), eq. (16) leads to
\[
N_{ST}(G_n) = T_n(1,1) = (2^{n+1} - 1)\prod_{i=2}^{n}(2^{i} - 1)^{2^n-1}.
\]
(17)
Thus far, we have derived the exact number of spanning trees of the Farey graph \( G_n \). Note that this number was first obtained by Zhang et al. in ref. [9], where the authors used the Laplacian matrix.

In the general case, the calculation of the reliability polynomial is NP-hard [27]. But for the Farey graph, inserting eqs. (16) and (5) into eq. (3), we obtain the explicit expression for the reliability polynomial \( R(G_n;p) \) as
\[
R(G_n;p) = q^{2^n-1}p^2q^{n+1} - q^{n-1} \sum_{i=2}^{n}(2^i - q^i)^{2^n-1},
\]
where \( q = 1 - p \).

Finally, we will determine the chromatic polynomial \( P(G_n;\lambda) \). Since \( P(G_n;\lambda) \) satisfies the following relation:
\[
P(G_n;\lambda) = (-1)^{|V_n| - k(G_n)}\lambda^{k(G_n)}T_n(1 - \lambda, 0)
\]
\[
= \lambda T_n(1 - \lambda, 0),
\]
we turn our aim to finding \( T_n(x, 0) \). From Corollary 3, let \( y = 0 \), we can easily get the following recursive equations:
\[
T_n(x, 0) = T_{n+1}(x, 0) + (x - 1)N_n(x, 0),
\]
(20)
\[
T_1(n, 0) = 2T_{1,n-1}N_{n-1} + (x - 1)N_{n-1}^2,
\]
(21)
\[
N_n(x, 0) = 2T_{1,n-1}N_{n-1} + (x - 1)N_{n-1}^2.
\]
(22)
It follows from eqs. (21) and (22) that
\[
T_1(n, x) = N_n(x, 0).
\]
(23)
Substituting eq. (23) into eq. (22), we obtain
\[
N_n(x, 0) = (x + 1)N_{n-1}^2.
\]
(24)
Considering the initial condition \( N_0(x, 0) = 1, \) eq. (20) is solved to yield
\[
T_n(x, 0) = xN_n(x, 0) = x(x + 1)^{2^n-1}.
\]
(25)
Plugging the last expression into eq. (19), we arrive at the explicit formula for the chromatic polynomial,
\[
P(G_n;\lambda) = \lambda(1 - \lambda)(2 - \lambda)^{2^n-1}.
\]
(26)
Then, it is clear that \( G_n \) is minimally 3-colorable.
Conclusions. – In this paper, we find recursive formulas for the Tutte polynomial of the Farey graph family by using a method, based on their self-similar structure. We also evaluate special cases of these results to compute the corresponding reliability polynomial, chromatic polynomial, the number of spanning trees and the number of connected spanning subgraphs.

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Appendix: Tutte polynomials of other self-similar networks. – The computation of the Tutte polynomial of a graph is NP-hard in general. Indeed, even evaluating $T(G; x, y)$ for some specific values $(x, y)$ is also $\#P$-complete, except for a few special points and a special hyperbola [28]. In this appendix we gain explicit expressions of the Tutte polynomials of some other self-similar networks with different degree distributions, exponential or scale-free.

Koch network. We first study the self-similar Koch network family [29,30]. The Koch networks are derived from the Koch fractals [31] and can be constructed recursively. Let $K_{m,n}$ ($m$ is a natural number) be the Koch network in generation $n$. Then, the family of Koch networks can be generated following a recursive-modular method. For $n = 0$, $K_{m,0}$ consists of a triangle with three vertices labeled by $X$, $Y$ and $Z$, respectively, and called them hub vertices, which have the highest degree among all vertices in the networks. For $n \geq 1$, $K_{m,n}$ is obtained from $3m + 1$ copies of $K_{m,n-1}$ by joining them at the hub vertices. As shown in Fig. 4, the network $K_{m,n+1}$ may be obtained by the juxtaposition of $3m + 1$ copies of $K_{m,n}$ which are labeled as $K_{1,n}^1, K_{2,n}^1, \ldots, K_{3,n}^1$ and $K_{m,n}^{3m+1}$, respectively.

The number of vertices and edges of $K_{m,n} = (V_n, E_n)$ are $|V_n| = 2(3m + 1)^n + 1$ and $|E_n| = 3(3m + 1)^n$, respectively. The Koch networks present some typical properties of real-world networks. They are scale-free with vertex degree distribution $P(k)$ in a power-law form, $P(k) \sim k^{-\gamma}$, where $\gamma = 1 + \frac{\ln(3m+1)}{\ln(3m+1)}$. They also have an obvious small-world characteristic. Their average path length increases logarithmically with the network size and their clustering coefficient is very high.

We now determine the Tutte polynomial $T(K_{m,n}; x, y)$. According to the construction of the Koch network, $K_{m,n}$ can be obtained from $3m + 1$ copies of $K_{m,n-1}$ through one-point join. From eq. (2), we have

$$T(K_{m,n}; x, y) = T(K_{m,n-1}; x, y)^{3m+1}.$$  \hfill (A.1)

Considering the initial condition $T(K_{m,0}; x, y) = x^2 + x + y$, we obtain explicit expression of the Tutte polynomial of the Koch network $K_{m,n}$ as

$$T(K_{m,n}; x, y) = (x^2 + x + y)^{(3m+1)^n}. \hfill (A.2)$$

Hence some special valuations of $T(K_{m,n}; x, y)$ can easily be computed:

$$N_{ST}(K_{m,n}) = T(K_{m,n}; 1, 1) = 3^{(3m+1)^n}; \hfill (A.3)$$

$$N_{CSSC}(K_{m,n}) = T(K_{m,n}; 1, 2) = 4^{(3m+1)^n}; \hfill (A.4)$$

$$N_{SF}(K_{m,n}) = T(K_{m,n}; 2, 1) = 7^{(3m+1)^n}. \hfill (A.5)$$

Note that these quantities coincide with the result obtained in refs. [29,30] without using the Tutte polynomial.

Small-world exponential network. In this subsection we discuss a self-similar network which is observed from some real-life systems. This network is constructed iteratively [32], denoted by $S_n$, after $n$ ($n \geq 0$) iterations. The network starts from a triangle $S_0$, with three vertices called initial vertices. For $n \geq 1$, $S_n$ is obtained from $S_{n-1}$: for each existing vertex in $S_{n-1}$, two new vertices are generated, which and their mother vertex together form a new triangle, see fig. 5. It is clear that the number of vertices and edges of $S_n$ are $|V_n| = 3^{n+1}$ and $|E_n| = 3(3^{n+1} - 1)/2$, respectively. The resultant network has a degree distribution decaying exponentially with the degree. In addition, it is a small-world network with the average distance growing logarithmically with its size.

As shown in fig. 5, $S_n$ can be obtained from three copies of $S_{n-1}$ and a triangle through one-point join. Since the Tutte polynomial of a triangle is equal to $x^2 + x + y$, we have

$$T(S_n; x, y) = (x^2 + x + y)T(S_{n-1}; x, y). \hfill (A.6)$$

Fig. 4: (Color online) Recursive construction of Koch networks that highlights the self-similarity of its members. The right one is a particular Koch network $K_{2,2}$.

Fig. 5: (Color online) Recursive construction of the small-world network family that highlights the self-similarity of its members. The right one is a particular network $S_2$. 

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Additionally, it is clearly that 
\[ T(S_n;x,y) = x^2 + x + y. \]
We can therefore obtain the exact expression of the Tutte polynomial of \( S_n \),
\[ T(S_n;x,y) = (x^2 + x + y)^{(3^n+1)/2}. \] (A.7)

Based on the above equation, we can give some interesting quantities for this network, that is
\[ N_{ST}(S_n) = 3^{(3^n+1)/2}, \] \[ N_{CSSG}(S_n) = 4^{(3^n+1)/2}, \] \[ N_{SF}(S_n) = 7^{(3^n+1)/2}. \] (A.8) (A.9) (A.10)

The same result about the number of spanning trees was obtained in ref. [24], where the authors used the Laplacian matrix.

REFERENCES

[1] Potts, R. B., Proc. Cambridge Philos. Soc., 48 (1952) 106.
[2] Wu F. Y., Rev. Mod. Phys., 54 (1982) 235.
[3] Tutte W. T., Can. J. Math., 6 (1954) 80.
[4] Tutte W. T., J. Comb. Theory, 2 (1967) 301.
[5] Welsh D. J. A. and Merino C., J. Math. Phys., 41 (2000) 1127.
[6] Ellis-Monaghan J. A. and Merino C., in Structural Analysis of Complex Networks, edited by Dehmer Matthias (Birkhäuser, Boston) 2011, pp. 219–255.
[7] Ellis-Monaghan J. A. and Merino C., in Structural Analysis of Complex Networks, edited by Dehmer Matthias (Birkhäuser, Boston) 2011, pp. 257–292.
[8] Zhang Z. Z. and Comellas F., Theor. Comput. Sci., 412 (2011) 865.
[9] Zhang Z. Z., Bin W. and Yuan L., Physica A, 391 (2012) 3342.
[10] Matula D. W. and Kornerup P., in A Graph Theoretic Interpretation of Fractions and the GCD Algorithm, edited by Mullin R. C. and Stanlom R. G. (Utilitas Mathematica Publishing Inc., Winnipeg) 1979, p. 932.
[11] Colbourn C. J., Siam J. Algebr. Discrete Methods, 3 (1982) 187.
[12] Biggs N. L., Ars Comb., 25C (1988) 73.
[13] Dhar D., J. Math. Phys., 18 (1977) 577.
[14] Alvarez P. D., Canfora F., Reyes S. A. and Riquelme S., Eur. Phys. J. B, 85 (2012) 89.
[15] Simoi J. De., J. Phys. A: Math. Theor., 42 (2009) 095002.
[16] Read R. C., J. Comb. Theory, 4 (1968) 52.
[17] Whitney H., Ann. Math., 33 (1932) 688.
[18] Chang S. C. and Shrock R., J. Stat. Phys., 112 (2003) 815.
[19] Salas J. and Sokal A. D., J. Stat. Phys., 104 (2001) 609.
[20] Hardy G. H. and Wright E. M. (Editors), An Introduction to the Theory of Numbers (Oxford University Press, London) 1979.
[21] Tseng W. J. and Wu F. Y., J. Stat. Phys., 110 (2003) 671.
[22] Dhar D. and Dhar A., Phys. Rev. E, 55 (1997) 2093(R).
[23] Noh J. D. and Rieger H., Phys. Rev. Lett., 92 (2004) 118701.
[24] Lin Y., Zhang Z. Z. and Chen G. R., J. Math. Phys., 52 (2011) 113303.
[25] Zhang Z. Z., Liu H. X., Wu B. and Zhou S. G., EPL, 90 (2010) 68002.
[26] Biggs N. L. (Editor), Algebraic Graph Theory (Cambridge University Press, Cambridge) 1993.
[27] Ball M. O., Networks, 10 (1980) 153.
[28] Jaeger F., Vertigan D. L. and Welsh D. J. A., Math. Proc. Cambridge Philos. Soc., 108 (1990) 35.
[29] Wu B., Zhang Z. Z. and Chen G. R., J. Phys. A: Math. Theor., 45 (2012) 025102.
[30] Zhang Z. Z., Gao S. Y., Chen L. C., Zhou S. G., Zhang H. J. and Guan J. H., J. Phys. A: Math. Theor., 43 (2010) 395101.
[31] Schneider J. E., Math. Mag., 38 (1965) 144.
[32] Barrière L., Comellas F., Dalfó C. and Fiol M. A., Discrete Appl. Math., 157 (2009) 36.