A Liouville Theorem for the Higher Order Fractional Laplacian

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Abstract

We deal with the higher-order fractional Laplacians by two methods: the integral method and the system method. The former depends on the integral equation equivalent to the differential equation. The latter works directly on the differential equations. We first derive nonexistence of positive solutions, often known as the Liouville type theorem, for the integral and differential equations. Then through an delicate iteration, we show symmetry for positive solutions.

Key words: Higher-order Fractional Laplacian, Green’s function, the method of moving planes, nonexistence, symmetry

1 Introduction

Let $t = m + \frac{\alpha}{2}$ and $0 < \alpha < 2$. The higher-order fractional Laplacian $(-\Delta)^t$ with $m \in \mathbb{N}$, and the usual fractional Laplacian $(-\Delta)^{\alpha/2}$, usually take the form of

$$(-\Delta)^t u(x) = C_{n,t} P.V. \int_{\mathbb{R}^n} \frac{\sum_{k=0}^{m} H_k \Delta^k u(x) - u(y)}{|x - y|^{n+2t}} dy$$

$$= C_{n,t} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} \frac{\sum_{k=0}^{m} H_k \Delta^k u(x) - u(y)}{|x - y|^{n+2t}} dy,$$

where

$$H_k = \frac{1}{2^k k! n(n+2) \cdots (n+2k-2)}.$$
An equivalent approach to define it is via the difference quotient:

\[ (-\Delta)^t u(x) = C_{n,t} \int_{\mathbb{R}^n} \frac{\sum_{k=0}^{2m+2} (-1)^{k-m-1} C^k_{2m+2} u(x + (k - m - 1)y)}{|y|^{n+2t}} dy, \]

with

\[ C^k_{2m+2} = \frac{(2m + 2)!}{k!(2m + 2 - k)!}. \]

From the defining integral in (1), we can see that both \((-\Delta)^t\) and \((-\Delta)^{\alpha/2}\) are nonlocal pseudo-differential operators. Such non-locality brings in the real difference between the usual Laplacians and the fractional Laplacians, and poses a strong barrier in the generalization of many useful results from the Laplacians to the fractional ones. For example, when considering a Dirichlet problem involving \(-\Delta\) in a bounded domain \(\Omega\), it is sufficient to know the behavior of solution on \(\partial \Omega\). But when it comes to \((-\Delta)^{\alpha/2}\), we need to gather information on both \(\partial \Omega\) and \(\mathbb{R}^n \setminus \Omega\).

Due to its non-local nature, the fractional Laplacian have been receiving intense attention from researchers in physics, astrophysics, mechanics, mathematics and economics. Scientists have been using integrations and differentiation of fractional orders to describe the behavior of objects and systems. In physics, it is used to derive heat kernel estimates for a large class of symmetric jump-type processes (see [7], [2]). Its application is also seen in the study of the acoustic wave equation, which was an important point of reference in the development of the electromagnetic wave equation. In astrophysics, researchers use it to model the dynamics in the Hamiltonian chaos (see [20]). In probability, it is defined as the generator of \(\alpha\)-stable Lévy processes that represents random motions, such as the Brownian motion and the Poisson process (see [12] and [14]). In finance, it models jump processes (see [19]). For more details, please see [1], [3], [4] and the references there in.

In this paper, we consider the Navier boundary value problem:

\[
\begin{cases}
(-\Delta)^t u(x) = u^p(x), & x \in \mathbb{R}^n_+ \\
(-\Delta)^i u(x) = 0, & x \notin \mathbb{R}^n_+,
\end{cases}
\tag{1}
\]

with \(i = 0, 1, 2, \ldots, m\). We want to derive the nonexistence of positive solutions.

The nonexistence result is often famously known as the Liouville-type theorem. Interestingly enough, in PDE analysis, by applying the nonexistence to blowed-up-and-rescaled equations, one is able to derive the a priori
estimates for the solutions of the original equations. Then, combining the estimates with methods like the topological degree and continuation method, it forms a powerful tool to derive the existence of solutions.

Considerable work have been devoted to the study of the Liouville theorems associated with the usual fractional Laplacian \((-\Delta)^{\alpha/2}\) and interesting results have been obtained.

In [22], the authors proved that the only solution for
\[
\begin{cases}
(-\Delta)^{\alpha/2}u(x) = 0, & x \in \mathbb{R}^n, \\
u(x) \geq 0, & x \in \mathbb{R}^n,
\end{cases}
\]
is constant.

In [6], Chen, Fang and Yang studied the following Dirichlet problem involving the fractional Laplacian:
\[
\begin{cases}
(-\Delta)^{\alpha/2}u = u^p, & x \in \mathbb{R}^n_+ \\
u \equiv 0, & x \notin \mathbb{R}^n_+.
\end{cases}
\]

By studying its equivalent integral equation
\[
u(x) = \int_{\mathbb{R}^n_+} G(x, y)u^p(y)dy,
\]
they obtained the non-existence of positive solutions in the critical and sub-critical cases \(1 < p \leq \frac{n+\alpha}{n-\alpha}\) under no restrictions on the growth of the solutions.

In [5], the authors considered an \(\alpha\)-harmonic problem:

\[
\begin{cases}
(-\Delta)^{\alpha/2}u(x) = 0, & x \in \mathbb{R}^n \\
\lim_{|x| \to \infty} \frac{u(x)}{|x|^{\gamma}} \geq 0,
\end{cases}
\]
for some \(0 \leq \gamma \leq 1\) and \(\gamma < \alpha\). There they proved that \(u\) must be constant throughout \(\mathbb{R}^n\).

For more results on the Liouville theorems, please see [13], [15], [21] and the references therein). To our best knowledge, the Liouville theorems concerning the higher-order fractional Laplacian are few and new. In [16], among which, we derived the equivalent integral form of the above differential equation:
Lemma 1 Suppose $u$ is a positive solution of (1). If $(-\Delta)^m u \geq 0$ and $2m < n$, then $u$ must satisfy the integral equation

$$u(x) = \int_{R^n_+} G^+_t(x, y) u^p(y) dy.$$  

(2)

And vice versa.

Based on the equivalence result, at the end of the paper we conjectured that

Assume that $u \in L^{n(p-1)\frac{n-2t}{n}}_{loc}(R^n_+)$ for $p > 1$ is a nonnegative solution to (1). Then $u$ must be trivial.

As a continuation of [16], here we partially prove the conjecture under a stronger assumption that $u \in L^{n(p-1)\frac{n-2t}{n}}_{loc}(R^n_+)$. Let $R^+ = \{x = (x_1, \ldots, x_n) \in R^n \mid x_n > 0\}$.

Theorem 1 Assume that $(-\Delta)^m u \geq 0$ and $2m < n$. If $u \in C^{2m, 1}_{loc} \cap L^{2t}$ is a nonnegative solution for

$$u(x) = \int_{R^n_+} G^+_t(x, y) u^p(y) dy,$$  

(3)

then $u$ must be trivial.

As an immediate consequence of Lemma 1 and Theorem 1 we have

Theorem 2 For $p > 1$ and $n > 2t$, suppose $u \in L^{n(p-1)\frac{n-2t}{n}}_{loc}(R^n_+)$ is a nonnegative solution for (1). If $(-\Delta)^m u \geq 0$ in $R^n_+$, $u$ must be trivial.

The pseudo-differential operator $(-\Delta)^t$ can also be defined inductively (see [18]) as:

$$(-\Delta)^t = (-\Delta)^{\alpha/2} \circ (-\Delta)^m,$$  

(4)

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. This allows us to split a single higher-order fractional equation into a system. We will develop an iteration method based on the following narrow region principle to deal with the differential problem directly.
Theorem 3 Assume that $\Omega$ is a bounded narrow region in $\Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \}$. Without loss of generality, we may assume that $\Omega$ is contained in the slab $\{ x \in \mathbb{R}^n \mid \lambda - l < x_1 < \lambda \}$ with $\varepsilon > 0$ small. Consider

$$\begin{cases} -\Delta U(x) + c_1(x)V(x) \geq 0, & x \in \Omega, \\
(\Delta)^\alpha_2 V(x) + c_2(x)U(x) \geq 0, & x \in \Omega, \\
V(x^\lambda) = -V(x), & x \in \Sigma_\lambda, \\
U(x^\lambda a) = -U(x), & x \in \Sigma_\lambda, \\
U(x), V(x) \geq 0, & x \in \Sigma_\lambda \setminus \Omega, \end{cases}$$

where $c_i(x) \leq 0$ in $\Omega$ and are bounded for $i = 1, 2$, $U \in C^2$ and $V \in C^{1,1}_{\text{loc}}(\Omega \cap L_\alpha)$ are lower semi-continuous in $\Omega$.

Then for $l$ sufficiently small, we have

$$U(x), V(x) \geq 0, x \in \Omega.$$

For unbounded $\Omega$, the above result still holds on condition that

$$U(x), V(x) \rightarrow 0, \ |x| \rightarrow \infty.$$

Further, if either $U(x)$ or $V(x)$ equals $0$ at some point in $\Omega$, then

$$U(x), V(x) \equiv 0, x \in \mathbb{R}^n.$$

Then we are able to prove that

Theorem 4 Let $t = 1 + \alpha/2$ for $0 < \alpha < 2$. Assume that $u \in C^{3,1}_{\text{loc}}(B_1(0))$ is a positive solution of

$$\begin{cases} (-\Delta)^t u = f(u), & x \in B_1, \\
u = \Delta u = 0, & x \in \mathbb{R}^n \setminus B_1, \end{cases}$$

where $f(t)$ is Lipschitz continuous and increasing in $t$. Then $u$ must be symmetric about the origin.

With the symmetry result, one can continue to carry out estimates on the regularity results and derive useful Sobolev inequalities. For readers who are interested please see [10], [11] and the references therein.

This paper is organized as follows: In Section 2, we obtain some essential inequalities for the Green’s function related to $(-\Delta)^t$; In Section 3, we prove Theorem $\Pi$ through the method of moving planes in integral forms. We close the paper with a proof of the narrow region principle and Theorem $\Pi$. 5
2 Properties of the Green’s Function

First we derive some properties of the Green’s function $G^{+}_{2t}(x, y)$, which is essential in the process of moving the planes.

For any real number $\lambda$, let

$$\Sigma_{\lambda} = \{x = (x', x_n) \in R_{+}^{n} \mid 0 < x_n < \lambda\},$$

$$T_{\lambda} = \{x \in R_{+}^{n} \mid x_n = \lambda\},$$

and let

$$x^{\lambda} = (x_1, x_2, \cdots, 2\lambda - x_n)$$

be the reflection of the point $x = (x', x_n)$ about the plane $T_{\lambda}$.

Lemma 2.1 The Green’s function $G^{+}_{2t}(x, y)$ satisfies the following properties:

1. For any $x, y \in \Sigma_{\lambda}$ and $x \neq y$, we have

$$G^{+}_{2t}(x^{\lambda}, y^{\lambda}) > \max\{G^{+}_{2t}(x^{\lambda}, y), G^{+}_{2t}(x, y^{\lambda})\}$$

and

$$G^{+}_{2t}(x^{\lambda}, y^{\lambda}) - G^{+}_{2t}(x, y) > |G^{+}_{2t}(x^{\lambda}, y) - G^{+}_{2t}(x, y^{\lambda})|.$$  \hspace{1cm} (5)

2. For any $x \in \Sigma_{\lambda}$, $y \in \Sigma^{C}_{\lambda}$, it holds

$$G^{+}_{2t}(x^{\lambda}, y) > G^{+}_{2t}(x, y) \quad \text{and} \quad G^{+}_{2t}(y, x^{\lambda}) > G^{+}_{2t}(y, x).$$  \hspace{1cm} (6)

3. For $x, y \in R_{+}^{n}$,

$$G^{+}_{2t}(x, y) \leq \frac{C}{|x - y|^{n-2t}}.$$  \hspace{1cm} (7)

Proof. It’s well known that the above inequalities are true for the Green’s functions associated with $-\Delta$ and $(-\Delta)^{m}$. In the proof below, we will make use of this fact.

Let $\tilde{\Sigma}_{\lambda}$ be the reflection of $\Sigma_{\lambda}$ about $T_{\lambda}$, and $R_{+}^{n} = \Sigma_{\lambda} \cup \tilde{\Sigma}_{\lambda} \cup D_{\lambda}$. To prove the lemma, we recall that $G^{+}_{2m}(x, y)$ and $G^{+}_{\alpha}(x, y)$ satisfy inequality (6) as well. Therefore,
1.

\[ G_{2i}^+(x^\lambda, y^\lambda) - G_{2t}^+(x^\lambda, y) = \int_{R_i^t} G_{2m}^+(x^\lambda, z) \left( G_\alpha^+(z, y^\lambda) - G_\alpha^+(z, y) \right) dz \]

\[ = \int_{\Sigma\lambda \cup \tilde{\Sigma}\lambda \cup D_\lambda} G_{2m}^+(x^\lambda, z) \left( G_\alpha^+(z, y^\lambda) - G_\alpha^+(z, y) \right) dz \]

\[ = \int_{\Sigma\lambda} \left( G_{2m}^+(x^\lambda, z) \left( G_\alpha^+(z, y^\lambda) - G_\alpha^+(z, y) \right) \right) dz \]

\[ + \int_{D_\lambda} G_{2m}^+(x^\lambda, z) \left( G_\alpha^+(z, y^\lambda) - G_\alpha^+(z, y) \right) dz \]

\[ = I_1 + I_2. \] (9)

Since \( G_\alpha^+ \) satisfies (6), we have

\[ G_\alpha^+(z^\lambda, y) - G_\alpha^+(z, y^\lambda) \geq G_\alpha^+(z, y^\lambda) - G_\alpha^+(z^\lambda, y^\lambda). \]

Thus

\[ I_1 = \int_{\Sigma\lambda} \left( G_{2m}^+(x^\lambda, z) \left( G_\alpha^+(z, y^\lambda) - G_\alpha^+(z, y) \right) \right) dz \]

\[ \geq \int_{\Sigma\lambda} \left( - G_{2m}^+(x^\lambda, z) \left[ G_\alpha^+(z^\lambda, y^\lambda) - G_\alpha^+(z^\lambda, y) \right] \right) dz \]

\[ + \int_{D_\lambda} G_{2m}^+(x^\lambda, z) \left[ G_\alpha^+(z^\lambda, y^\lambda) - G_\alpha^+(z^\lambda, y) \right] dz \]

\[ = \int_{\Sigma\lambda} \left[ G_{2m}^+(x^\lambda, z^\lambda) - G_{2m}^+(x^\lambda, z) \right] \left[ G_\alpha^+(z^\lambda, y^\lambda) - G_\alpha^+(z^\lambda, y) \right] dz \]

\[ > 0. \]

Because (6) is true for \( G_\alpha^+ \), it implies that

\[ I_2 = \int_{D_\lambda} G_{2m}^+(x^\lambda, z) \left( G_\alpha^+(z, y^\lambda) - G_\alpha^+(z, y) \right) dz > 0. \]
Together with (10), it yields

$$G_{2t}^+(x^\lambda, y^\lambda) \geq G_{2t}^+(x^\lambda, y).$$

Similarly, one can show that

$$G_{2t}^+(x^\lambda, y^\lambda) \geq G_{2t}^+(x, y^\lambda).$$

This proves (5). To prove (6), let

$$[G_{2t}^+(x^\lambda, y^\lambda) - G_{2t}^+(x, y)] - [G_{2t}^+(x^\lambda, y) - G_{2t}^+(x, y^\lambda)]$$

$$= \int_{\Sigma^\lambda} Kdz + \int_{D^\lambda} Qdz,$$  \hspace{1cm} (10)

where

$$K = [G_{2m}^+(x^\lambda, z) + G_{2m}^+(x, z)][G_{\alpha}^+(z, y^\lambda) - G_{\alpha}^+(z, y)]$$

$$+ [G_{2m}^+(x^\lambda, z^\lambda) + G_{2m}^+(x, z^\lambda)][G_{\alpha}^+(z^\lambda, y^\lambda) - G_{\alpha}^+(z^\lambda, y)]$$

$$> -[G_{2m}^+(x^\lambda, z) + G_{2m}^+(x, z)][G_{\alpha}^+(z^\lambda, y^\lambda) - G_{\alpha}^+(z^\lambda, y)]$$

$$+ [G_{2m}^+(x^\lambda, z^\lambda) + G_{2m}^+(x, z^\lambda)][G_{\alpha}^+(z^\lambda, y^\lambda) - G_{\alpha}^+(z^\lambda, y)]$$

$$= [G_{2m}^+(x^\lambda, z^\lambda) + G_{2m}^+(x, z^\lambda) - G_{2m}^+(x^\lambda, z) - G_{2m}^+(x, z)]$$

$$\cdot [G_{\alpha}^+(z^\lambda, y^\lambda) - G_{\alpha}^+(z^\lambda, y)]$$

$$> 0.$$  \hspace{1cm} (11)

By (7), we also have

$$Q = [G_{2m}^+(x^\lambda, z) + G_{2m}^+(x, z)][G_{\alpha}^+(z, y^\lambda) - G_{\alpha}^+(z, y)] > 0.$$  \hspace{1cm} (12)

Combining (10), (11), and (12), we arrive at

$$G_{2t}^+(x^\lambda, y^\lambda) - G_{2t}^+(x, y) > G_{2t}^+(x^\lambda, y) - G_{2t}^+(x, y^\lambda).$$

Similarly, one can show that

$$G_{2t}^+(x^\lambda, y^\lambda) - G_{2t}^+(x, y) > G_{2t}^+(x, y^\lambda) - G_{2t}^+(x^\lambda, y).$$

This verifies (6).
2. For \( x \in \Sigma \) and \( y \in \Sigma^C \), let

\[
G_{2m}^+(x^\lambda, y) - G_{2m}^+(x, y) = \int_{\Sigma} Mdz + \int_{D^\lambda} Nz,
\]

where

\[
M = \left[ G_{2m}^+(x^\lambda, z) - G_{2m}^+(x, z) \right] G_{\alpha}^+(z, y) + \left[ G_{2m}^+(x^\lambda, z^\lambda) - G_{2m}^+(x, z^\lambda) \right] [G_{\alpha}^+(z^\lambda, y) - G_{\alpha}^+(z, y)] > 0.
\]

Meanwhile, we have

\[
N = \left[ G_{2m}^+(x^\lambda, z) - G_{2m}^+(x, z) \right] G_{\alpha}^+(z, y) > 0.
\]

Combining (13), (14) and (15), we arrive at (7).

3. It’s well known that for \( 0 < \alpha \leq 2, \ x, y \in \mathbb{R}^n \),

\[
G_{\alpha}^+(x, y) = \frac{A_{n,\alpha}}{s(n-\alpha)/2} \left[ 1 - \frac{B_{n,\alpha}}{(s+t)(n-2)/2} \int_0^\frac{s}{t} \frac{(s-tb)^{(n-2)/2}}{b^{\alpha/2}(1+b)} db \right],
\]

is the Green’s function for \((-\Delta)^{\alpha/2}\) in \(\mathbb{R}^n_+\). Here

\[
s = |x - y|^2 \quad \text{while} \quad t = 4x_ny_n.
\]

We know that

\[
G_{2m}^+(x, y) = \frac{1}{|x - y|^{n-2m}} - \frac{1}{|x^* - y|^{n-2m}},
\]

thus

\[
G_{2m}^+(x, y) \leq \frac{C}{|x - y|^{n-2m}}, \quad x, y \in \mathbb{R}^n_+.
\]

and

\[
G_{\alpha}^+(x, y) \leq \frac{C}{|x - y|^{n-\alpha}}, \quad x, y \in \mathbb{R}^n_+.
\]
Hence
\[ G^+_t(x, y) = \int_{R^n_+} G^+_2(x, z)G^+_\alpha(z, y)dz \]
\[ \leq \int_{R^n_+} \frac{C}{|x - z|^{n-2m}|z - y|^{n-\alpha}}dz \]
\[ = \int_{\tilde{z} > -y} \frac{C}{|x - y - \tilde{z}|^{n-2m}|\tilde{z}|^{n-\alpha}}, \quad \tilde{z} + y = z \]
\[ = \frac{C}{|x - y|^{n-2t}} \int_{\tilde{z} > -y} \frac{d\tilde{z}}{|\tilde{z}|^{n-\alpha}|x - y| - \tilde{z}|^{n-2m}}d\tilde{z}, \quad \tilde{z} = |x - y|\tilde{z} \]
\[ = \frac{C}{|x - y|^{n-2t}}. \]

3 Nonexistence of Positive Solutions for the Integral Equation

Proof of Theorem 1

Let
\[ u_\lambda(x) = u(x^\lambda), \quad w_\lambda(x) = u_\lambda(x) - u(x). \]

Then

Lemma 3.1

\[ w_\lambda(x) \geq \int_{\Sigma_\lambda} [G_{2t}(x^\lambda, y^\lambda) - G_{2t}(x, y^\lambda)](u^p_\lambda(y) - u^p(y))dy. \quad (16) \]
Proof By (2) and Property 2 in Lemma 2.1, we have
\[ w_{\lambda}(x) = u_{\lambda}(x) - u(x) \]
\[ = \int_{R^+_t} G^+_t(x, y)u^p(y)dy - \int_{R^+_t} G^+_t(x, y)u^p(y)dy \]
\[ = \int_{\Sigma_\lambda} G^+_t(x, y)u^p(y)dy - \int_{\Sigma_\lambda} G^+_t(x, y)u^p(y)dy \]
\[ = \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy \]
\[ + \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy \]
\[ + \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy \]
\[ \geq \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy \]
\[ + \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy. \]

By Property 1 in Lemma 2.1, one can see that for \( x, y \in \Sigma_\lambda \),
\[ G^+_t(x, y) - G^+_t(x, y) \geq G^+_t(x, y) - G^+_t(x, y). \]

Therefore,
\[ w_{\lambda}(x) \geq \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy \]
\[ + \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy \]
\[ \geq \int_{\Sigma_\lambda} [G^+_t(x, y) - G^+_t(x, y)]u^p(y)dy. \]

This proves the lemma.

Next we use the method of moving planes to derive a contradiction assuming that (1) has positive solutions.

Step 1. Start moving the plane \( T_\lambda \) from near \( x_n = 0 \) to the right along the \( x_n \) axis.

Let \( \Sigma_{\lambda^-} = \{ x \in \Sigma_\lambda \mid w_{\lambda}(x) < 0 \} \). We show that
\[ w_{\lambda}(x) \geq 0, \quad a.e. \Sigma_\lambda. \]
If not, then for any $x^o \in \Sigma^-\lambda$, by (10) and the mean value theorem, one have

$$0 < -w_\lambda(x^o) \leq \int_{\Sigma^-\lambda} [G_{2t}(x^\lambda, y^\lambda) - G_{2t}(x, y^\lambda)](u^\lambda(y) - u^p(y)) \, dy$$

$$\leq C \int_{\Sigma^-\lambda} \frac{1}{|x^\lambda - y^\lambda|^{n-2t}} (u^\lambda(y) - u^p(y)) \, dy$$

$$\leq C \int_{\Sigma^-\lambda} \frac{1}{|x - y|^{n-2t}}pu^{p-1}(y)(-w_\lambda(y)) \, dy.$$

Next we need an equivalent form of the Hardy-Little-Sobolev inequality.

**Lemma 3.2** Assume $0 < \alpha < n$ and $\Omega \subset \mathbb{R}^n$. Let $g \in L^{np_{n+\alpha p}}(\Omega)$ for $\frac{n}{n-\alpha} < p < \infty$. Define

$$Tg(x) := \int_{\Omega} \frac{1}{|x - y|^{n-\alpha}} g(y) \, dy.$$

Then

$$\|Tg\|_{L^p(\Omega)} \leq C(n, p, \alpha) \|g\|_{L^{np_{n+\alpha p}}(\Omega)}, \quad (18)$$

The proof of this lemma is standard and can be found in [8] or [9]. By Lemma 3.2 and the Hölder inequality, it follows that

$$\|w_\lambda\|_{L^q(\Sigma^-\lambda)} \leq C\|u^{p-1}w_\lambda\|_{L^{np_{n+\alpha p}}(\Sigma^-\lambda)}$$

$$\leq C\|u^{p-1}\|_{L^{\frac{n}{n-2t}}(\Sigma^-\lambda)} \|w_\lambda\|_{L^q(\Sigma^-\lambda)},$$

for any $\frac{n}{n-2t} < q < \infty$. Since $u \in L^{\frac{n(p-1)}{2t}}(R^n_+)$, for $\lambda$ sufficiently negative, we have

$$C\|u^{p-1}\|_{L^{\frac{n}{n-2t}}(\Sigma^-\lambda)} < \frac{1}{2}. \quad (19)$$

Thus

$$\|w_\lambda\|_{L^q(\Sigma^-\lambda)} < \frac{1}{2} \|w_\lambda\|_{L^q(\Sigma^-\lambda)},$$

i.e. $\Sigma^-\lambda$ is measure 0.

This proves (17) and completes step 1.
Step 2. Keep moving the plane until the limiting position

\[ \lambda_0 = \sup \{ \lambda \leq \infty \mid w_\mu(x) \geq 0, \quad \forall x \in \Sigma_\mu, \quad \mu \leq \lambda \}.\]

We claim that

\[ \lambda_0 = \infty. \tag{20} \]

If not, then for \( \lambda_0 < \infty \), we can show that

\[ w_{\lambda_0}(x) \equiv 0. \tag{21} \]

Thus for any \( x^0 \in \partial R^n_+ \), we have

\[ 0 < u_{\lambda_0}(x^0) = u(x^0) = 0. \]

The contradiction establishes (20).

To prove (21), we suppose its contrary is true, i.e. there exists a non-measure-zero set \( D \subseteq \Sigma_{\lambda_0} \) such that

\[ w_{\lambda_0}(x) > 0, \quad x \in D. \tag{22} \]

Thus, for any \( x \in \Sigma_{\lambda_0} \), by (16),

\[
w_{\lambda_0}(x) \geq \int_{\Sigma_{\lambda_0}} \left[ G_{2t}(x^{\lambda_0}, y^{\lambda_0}) - G_{2t}(x, y^{\lambda_0}) \right] (u_{\lambda_0}^p(y) - u^p(y)) dy \\
\geq \int_D \left[ G_{2t}(x^{\lambda_0}, y^{\lambda_0}) - G_{2t}(x, y^{\lambda_0}) \right] (u_{\lambda_0}^p(y) - u^p(y)) dy \\
> 0.
\]

Such strict positivity allows us to keep moving \( T_{\lambda_0} \) to the right while preserving (17). In other words, there exists some \( \varepsilon > 0 \) small such that for any \( \lambda \in (\lambda_0, \lambda_0 + \varepsilon) \), it holds that

\[ w_\lambda(x) \geq 0, \quad a.e. \Sigma_\lambda. \tag{23} \]

This is a contradiction with the definition of \( \lambda_0 \). It thus verifies (21).

To prove (23), for any small \( \eta > 0 \), we can choose \( R \) sufficiently large, so that

\[ \left( \int_{R^n \setminus B_R(0)} u^{r+1}(y) dy \right)^\frac{1}{r} < \eta. \tag{24} \]
Fix this $R$, we will show that the measure of $\Sigma^-_\lambda \cap B_R(0)$ is sufficiently small as $\lambda$ close to $\lambda_o$.

By the definition of $\lambda_o$ and Lemma [16] it is trivial to deduce that $w_{\lambda_o} (x) > 0$ in the interior of $\Sigma_{\lambda_o}$. For any $\delta > 0$, let

$$E_\delta = \{ x \in \Sigma_{\lambda_o} \cap B_R(0) | w_{\lambda_o} (x) > \delta \} \quad \text{and} \quad F_\delta = (\Sigma_{\lambda_o} \cap B_R(0)) \setminus E_\delta.$$  

Then obviously

$$\lim_{\delta \to 0^+} \mu(F_\delta) = 0.$$  

For $\lambda > \lambda_o$, let

$$D_\lambda = (\Sigma_\lambda \setminus \Sigma_{\lambda_o}) \cap B_R(0).$$  

Then it is easy to see that

$$(\Sigma^-_\lambda \cap B_R(0)) \subset (\Sigma^-_\lambda \cap E_\delta) \cup F_\delta \cup D_\lambda.$$  

(25) 

Apparently, the measure of $D_\lambda$ gets small as $\lambda$ approaches $\lambda_o$. We show that the measure of $\Sigma^-_\lambda \cap E_\delta$ can also be sufficiently small as $\lambda$ close to $\lambda_o$. In fact, for any $x \in \Sigma^-_\lambda \cap E_\delta$, we have

$$w_\lambda (x) = u(x^\lambda) - u(x_{\lambda_o}^\lambda) + u(x_{\lambda_o}^\lambda) - u(x) < 0.$$  

Hence

$$u(x_{\lambda_o}^\lambda) - u(x^\lambda) > w_{\lambda_o} (x) > \delta.$$  

It follows that

$$(\Sigma^-_\lambda \cap E_\delta) \subset G_\delta \equiv \{ x \in B_R(0) | u(x_{\lambda_o}^\lambda) - u(x^\lambda) > \delta \}.$$  

(26) 

While by the well-known Chebyshev inequality, we have

$$\mu(G_\delta) \leq \frac{1}{\delta^{\tau+1}} \int_{G_\delta} |u(x_{\lambda_o}^\lambda) - u(x^\lambda)|^{\tau+1} dx$$

$$\leq \frac{1}{\delta^{\tau+1}} \int_{B_R(0)} |u(x_{\lambda_o}^\lambda) - u(x^\lambda)|^{\tau+1} dx.$$  

For each fixed $\delta$, as $\lambda$ close to $\lambda_o$, the right hand side of the above inequality can be made as small as we wish. Therefore by (26) and (25), the measure of $\Sigma^-_\lambda \cap B_R(0)$ can also be sufficiently small. Combining this with (24), we arrive at (23).

It follows from $\lambda_0 = \infty$ that $u(x)$ is monotone increasing in the $x_n$ direction. This violates our assumption that $u \in L^{\frac{n(p-1)}{2}} (R_+^n)$. Therefore, (1) has no positive solution.

This completes the proof of the theorem.
4  Symmetry of Positive Solutions

To derive the symmetry through the iteration method, we first prove a key ingredient – the narrow region principle.

4.1 Narrow Region Principle

Theorem 4.1 Assume that \( \Omega \) is a bounded narrow region in \( \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \} \). Without loss of generality, we may assume that \( \Omega \) is contained in the slab \( \{ x \in \mathbb{R}^n \mid \lambda - l < x_1 < \lambda \} \) with \( \varepsilon > 0 \) small.

Consider

\[
\begin{align*}
-\Delta U(x) + c_1(x)V(x) & \geq 0, \quad x \in \Omega, \\
(-\Delta)^{\alpha/2}V(x) + c_2(x)U(x) & \geq 0, \quad x \in \Omega, \\
V(x^\lambda) & = -V(x), \quad x \in \Sigma_\lambda, \\
U(x^\lambda) & = -U(x), \quad x \in \Sigma_\lambda, \\
U(x), V(x) & \geq 0, \quad x \in \Sigma_\lambda \setminus \Omega, \\
\end{align*}
\]

(27)

where \( c_i(x) \leq 0 \) in \( \Omega \) and are bounded for \( i = 1, 2 \), \( U \in C^2 \) and \( V \in C^{1,1}_{loc}(\Omega \cap L_\alpha) \) are lower semi-continuous in \( \bar{\Omega} \).

Then for \( l \) sufficiently small, we have

\[
U(x), V(x) \geq 0, \quad x \in \Omega. \quad (28)
\]

For unbounded \( \Omega \), (28) still holds on condition that

\[
U(x), V(x) \rightarrow 0, \quad |x| \rightarrow \infty.
\]

Further, if either \( U(x) \) or \( V(x) \) equals 0 at some point in \( \Omega \), then

\[
U(x), V(x) \equiv 0, \quad x \in \mathbb{R}^n. \quad (29)
\]

Proof. If \( V(x) \geq 0 \) is not true, then there must exist some \( x^0 \in \Omega \) such that

\[
V(x^0) = \min_{\Sigma_\lambda} V(x) < 0.
\]
Then

\[
(-\Delta)^{\alpha/2} V(x^0) = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{V(x^0) - V(y)}{|x^0 - y|^{n+\alpha}} dy
\]

\[
= C_{n,\alpha} PV \left\{ \int_{\Sigma_\lambda} \frac{V(x^0) - V(y)}{|x^0 - y|^{n+\alpha}} dy + \int_{\mathbb{R}^n \setminus \Sigma_\lambda} \frac{V(x^0) - V(y)}{|x^0 - y|^{n+\alpha}} dy \right\}
\]

\[
= C_{n,\alpha} PV \left\{ \int_{\Sigma_\lambda} \frac{V(x^0) - V(y)}{|x^0 - y|^{n+\alpha}} dy + \int_{\Sigma_\lambda} \frac{V(x^0) + V(y)}{|x^0 - y|^{n+\alpha}} dy \right\}
\]

\[
\leq C_{n,\alpha} \int_{\Sigma_\lambda} \left\{ \frac{V(x^0) - V(y)}{|x^0 - y|^{n+\alpha}} + \frac{V(x^0) + V(y)}{|x^0 - y|^{n+\alpha}} \right\} dy
\]

\[
= C_{n,\alpha} \int_{\Sigma_\lambda} \frac{2V(x^0)}{|x^0 - y|^{n+\alpha}} dy
\]

\[
\leq CV(x^0) \int_{B_1(x^0) \setminus B_1(x^0)} \frac{1}{|x^0 - y|^{n+\alpha}} dy
\]

\[
\leq \frac{CV(x^0)}{l^\alpha}.
\]

Together with

\[
(-\Delta)^{\alpha/2} V(x) + c_2(x) U(x) \geq 0,
\]

we deduce that

\[
U(x^0) < 0.
\]

It thus implies that there exists some \( \bar{x} \in \Sigma_\lambda \cap B_1(0) \) such that

\[
U(\bar{x}) = \min_{\Sigma_\lambda} U(x) < 0.
\]

For \( \delta > 0 \) small, let \( \phi(x) = \sin \left( \frac{2\lambda + \delta}{l} \right) \). Then \( \phi(x) \) has positive bounds and satisfies \( \Delta \phi(x) = -\frac{\phi(x)}{l} \). Let \( w(x) = \frac{U(x)}{\phi(x)} \). It follows from (31) that there exists some \( \xi \) such that

\[
w(\xi) = \min_{\Sigma_\lambda} w < 0.
\]

At the negative minimum point of \( w \), we have

\[
\Delta U(\xi) = \Delta w(\xi) \phi(\xi) + 2\nabla w(\xi) \cdot \nabla \phi(\xi) + w(\xi) \Delta \phi(\xi)
\]

\[
\geq -w(\xi) \frac{\phi(\xi)}{l^2}
\]

\[
\geq -U(\bar{x}) \frac{\phi(\xi)}{\phi(\bar{x})} \frac{1}{l^2}.
\]
On the other hand, by (27) and (30), we know
\[ \Delta U(\xi) \leq c_1(\xi) V(\xi) \leq -c_1(\xi) c_2(x^0) U(x^0) l^\alpha. \] (32)

Combining (??) and (32), it yields
\[ l^{2+\alpha} \frac{\phi(x) c_1(\xi) c_2(x^0)}{\phi(\xi)} \geq 1. \]

The inequality above will certainly fail to hold for \( l \) sufficiently small. This shows that \( V(x) \) must be non-negative. Applying the maximum principle to (27), it yields
\[ U(x) \geq 0, \quad x \in \Omega. \]

Next we argue (29) by a contradiction.
Suppose for some \( \eta \in \Omega \), \( U(\eta) = 0 \). Then \( \eta \) is the minimum point of \( U \).
Thus
\[ 0 \geq -\Delta U(\eta) \geq c_1(\eta) V(\eta). \]

Meanwhile,
\[ c_1(\eta) V(\eta) \leq 0. \]

Hence
\[ V(\eta) = 0 = \min_{\Sigma_\lambda} V, \]
and
\[ (-\Delta)^{\alpha/2} V(\eta) \]
\[ = C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-V(y)}{|\eta - y|^{n+\alpha}} dy \]
\[ = C_{n,\alpha} PV \int_{\Sigma_\lambda} \frac{-V(y)}{|\eta - y|^{n+\alpha}} dy + \int_{\Sigma_\lambda} \frac{-V(y)}{|\eta - y^\lambda|^{n+\alpha}} dy \]
\[ = C_{n,\alpha} PV \int_{\Sigma_\lambda} \left( \frac{1}{|x^0 - y^{\lambda}|^{n+\alpha}} - \frac{1}{|x^0 - y|^n} \right) V(y) dy. \] (33)

If \( V(x) \neq 0 \), then (33) implies that
\[ (-\Delta)^{\alpha/2} V(\eta) < 0. \]
Together with (27), it shows that
\[ U(\eta) < 0. \]
This is a contradiction with (28). Hence \( V(x) \) must be identically 0 in \( \Sigma_\lambda \).
Since
\[ V(x^\lambda) = -V(x), x \in \Sigma_\lambda, \]
it shows that
\[ V(x) \equiv 0, \quad x \in \mathbb{R}^n. \]
Again with (27), one can easily deduce that \( U(x) \leq 0, x \in \Sigma_\lambda \). Since we already know that \( U(x) \geq 0, x \in \Sigma_\lambda \), it must hold that \( U(x) = 0, x \in \Sigma_\lambda \). Together with \( U(x^\lambda) = U(x) \), we arrive at
\[ U(x) \equiv 0, x \in \mathbb{R}^n. \]
If there exists a \( \xi \in \Omega \) such that \( V(\xi) = 0 \). Then from (30) and (27) it follows
\[ -c_1(\xi)V(\xi) \leq (-\triangle)^{\alpha/2}V(\xi) \leq \frac{V(\xi)}{l^\alpha} < 0. \]
It thus implies that some \( \eta \in \Omega, U(\eta) = 0 \). The rest is the same as the previous argument. It thus proves (29).

4.2 The Iteration Method

**Theorem 4.2** Let \( s = 1 + \alpha/2 \) for \( 0 < \alpha < 2 \). Assume that \( u \in C^{3,1}_{loc}(B_1(0)) \) is a positive solution of
\[
\begin{cases}
(-\triangle)^s u = f(u), & x \in B_1, \\
u = \Delta u = 0, & x \in \mathbb{R}^n \setminus B_1,
\end{cases}
\]
where \( f(t) \) is Lipschitz continuous and increasing in \( t \). Then \( u \) must be symmetric about the origin.

**Proof.** Let \(-\triangle u = v\). Then (34) can be split into two equations:
\[
\begin{cases}
(-\triangle)^{\alpha/2} v = f(u), & x \in B_1, \\
v = 0, & x \in \mathbb{R}^n \setminus B_1,
\end{cases}
\]
and
\[
\begin{cases}
-\triangle u = v, & x \in B_1, \\
u = 0, & x \in \mathbb{R}^n \setminus B_1.
\end{cases}
\]
We carry out the proof via the method of moving planes.
We first choose any direction to be the \(x_1\)-direction and let

\[ T_\lambda = \{ x \in \mathbb{R}^n \mid x_1 = \lambda \}, \quad \Sigma_\lambda = \{ x \in \mathbb{R}^n \mid x_1 < \lambda \}, \]

and

\[ x^\lambda = \{ (2\lambda - x_1, x') \mid x = (x_1, x') \in \mathbb{R}^n \} \]

be the reflection of \(x\) about the plane \(T_\lambda\).

Let

\[ u_\lambda(x) = u(x^\lambda) \]

and

\[ U_\lambda(x) = u_\lambda(x) - u(x), \quad V_\lambda(x) = v_\lambda(x) - v(x). \]

Then

\[ \begin{cases} 
-\Delta U_\lambda(x) = V_\lambda(x), & x \in \Sigma_\lambda, \\
(\Delta)^{\alpha/2} V_\lambda(x) = f(u_\lambda) - f(u), & x \in \Sigma_\lambda \cap B_1(0), \\
U_\lambda(x) \geq 0, & x \in \partial(\Sigma_\lambda \cap B_1(0)), \\
V_\lambda(x) \geq 0, & x \in \Sigma_\lambda \setminus (\Sigma_\lambda \cap B_1(0)).
\end{cases} \tag{37} \]

**Step 1. Moving the plane \(T_\lambda\) from \(-1\) to the right.**

For \(\lambda\) near \(-1\), we claim that

\[ U_\lambda(x) \geq 0, \quad x \in \Sigma_\lambda. \tag{38} \]

Notice that

\[ f(u_\lambda) - f(u) = \frac{f(u_\lambda) - f(u)}{u_\lambda(x) - u(x)} U_\lambda(x), \]

this allows us to obtain desired result by applying Lemma \(27\) (narrow region principle) to (37) with \(c_1(x) = -1\) and

\[ c_2(x) = -\frac{f(u_\lambda) - f(u)}{u_\lambda(x) - u(x)}. \]

**Step 2. Keep moving the plane \(T_\lambda\) until the limiting position**

\[ \lambda_0 = \sup\{ \lambda \leq 0 \mid U_\mu(x), V_\mu(x) \geq 0, x \in \Sigma_\mu, \forall \mu < \lambda \}. \]

We claim that

\[ \lambda_0 = 0. \tag{39} \]
If not, then for \( \lambda_0 < 0 \) we can show that

\[
U_{\lambda_0}(x) \equiv 0, \quad x \in \Sigma_{\lambda_0}.
\] (40)

We postpone the proof of (40) for the moment. From (39), we know that

\[
0 = u_{\lambda_0}(x) = u(x) > 0, \quad x = 1 + \lambda_0.
\]

A contradiction. This proves (39).

To prove (40), we argue by contradiction. If (40) does not hold, then by (29) (the strong maximum principle), we have

\[
U_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0} \cap B_1(0).
\]

This enables us to keep moving the plane \( T_{\lambda_0} \) to the right. Precisely speaking, for \( \varepsilon > 0 \) small such that \( \lambda_0 + \varepsilon < 0 \), it holds

\[
U_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda} \cap B_1(0), \quad \lambda \in [\lambda_0, \lambda_0 + \varepsilon).
\] (41)

The inequality above contradicts the definition of \( \lambda_0 \). Therefore (40) must be true.

Now we show (41) by constructing a narrow region. For \( \delta > 0 \) small, there exists a constant \( C \) such that

\[
U_{\lambda_0}(x) \geq C > 0, \quad x \in \Sigma_{\lambda_0} \cap B_1(0).
\]

By the continuity of \( U \) in \( \lambda \), we have

\[
U_{\lambda}(x) \geq 0, \quad x \in \Sigma_{\lambda_{-\delta}} \cap B_1(0).
\] (42)

It’s easy to see that

\[
U_{\lambda}(x) \geq 0, \quad x \in \partial((\Sigma_{\lambda} \setminus \Sigma_{\lambda_{-\delta}}) \cap B_1).
\]

Since \( \Sigma_{\lambda} \setminus \Sigma_{\lambda_{-\delta}} \) is a narrow region, with Lemma 27 we derive that

\[
U_{\lambda}(x) \geq 0, \quad x \in (\Sigma_{\lambda} \setminus \Sigma_{\lambda_{-\delta}}) \cap B_1.
\]

Together with (42), we arrive at (41). This completes with proof of (39).

Now we have

\[
U_0(x) \geq 0, \quad x \in \Sigma_0.
\]

Similarly, one can move \( T_\lambda \) from near 1 to the left and show that

\[
U_0(x) \leq 0, \quad x \in \Sigma_0.
\]

Thus

\[
U_0(x) \equiv 0, \quad x \in \Sigma_0.
\]

Since the direction of the \( x_1 \)-axis is arbitrary, we have actually proved that \( u \) is symmetric about the origin.
References

[1] J. Bertoin, Lévy Processes, Cambridge Tracts in Mathematics, 121 Cambridge University Press, Cambridge, 1996.

[2] M. T. Barlow, R. F. Bass, Z.-Q. Chen and M. Kassmann, Trans. Amer. Math. Soc. 361 (2009), 1963-1999.

[3] K. Bogdan, T. Grzywny and M. Ryznar, Heat kernel estimates for the fractional Laplacian with Dirichlet conditions, Ann. of Prob. 38(2010) 1901-1923.

[4] J. P. Bouchard and A. Georges, Anomalous diffusion in disordered media, Statistical mechanics, models and physical applications, Physics reports, 195(1990) 127-293.

[5] W. Chen, L. D’Ambrosio and Y. Li, Some Liouville theorems for the fractional Laplacian, Nonlinear Analysis: Theory, Methods & Applications, 121, 7 (2015) 370-381.

[6] W. Chen, Y. Fang and Y. Yang, Nonlinear equations involving fractional Laplacians in domains, to appear, Adv. Math.

[7] Z.-Q. Chen, P. Kim and T. Kumagai, Global heat kernel estimates for symmetric jump processes, Trans. Amer. Math. Soc. 363 (2011) 5021-5055.

[8] W. Chen and C. Li, Methods on Nonlinear Elliptic Equations, AIMS. Ser. Differ. Equ. Dyn. Syst, vol.4 2010.

[9] W. Chen and C. Li, Regularity of solutions for a system of integral equation, Comm. Pure Appl. Anal. 4(2005) 1-8.

[10] W. Chen, C. Li and B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59(2006) 330-343.

[11] W. Chen, C. Li and B. Ou, Classification of solutions for a system of integral equations, Comm. Partial Differential Equations, 30(2005) 59-65.
[12] A. David, Lévy Processes – From Probability to Finance and Quantum Groups, Notices of the American Mathematical Society. Providence, RI: American Mathematical Society. 51 (11): 13361347.

[13] M. Fall, Entire s-harmonic functions are affine, Proc. Amer. Math. Soc. 144 (2016) 2587-2592.

[14] S. Ken-Iti, Lévy Processes and Infinitely Divisible Distributions, 2011, Cambridge University Press. ISBN 978-0521553025.

[15] Y. Li, Nonexistence of positive solutions for a semilinear equation involving the fractional Laplacian in $R^n$, Acta Mathematica Scientia, 36(3), 5(2016) 666-682.

[16] Y. Li and R. Zhuo, Symmetry of Positive Solutions for Equations Involving Higher Order Fractional Laplacian, Proc. Amer. Math. Soc. Proc. Amer. Math. Soc. 144 (2016) 4303-4318.

[17] L. Ma and D. Chen, A Liouville type theorem for an integral system, Comm. Pure Appl. Anal. 5(2006) 855-859.

[18] X. Ros-Oton and J. Serra, The Pohozaev Identity for the Fractional Laplacian, Archive for Rational Mechanics and Analysis, 213(2014) 587-628.

[19] C. Rama, T. Peter, Financial Modeling with Jump Processes, 2003. CRC Press. ISBN 978-1584884132..

[20] G.M. Zaslavsky. Hamiltonian Chaos and Fractional Dynamics [2] Oxford University Press, 2008.

[21] R. Zhuo R, F. Li and B. Lv, Liouville type theorems for Schrödinger system with Navier boundary conditions in a half space, Comm. Pure Appl. Anal. 13(2014) 977-990.

[22] R. Zhuo, W. Chen, X. Cui and Z. Yuan, A Liouville theorem for the fractional Laplacian, accepted by Disc. Cont. Dyn. Sys..

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