FINITE GENERATION
OF
CANONICAL RINGS

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Abstract
We prove the finite generation of canonical rings of smooth projective varieties of general type defined over complex numbers. MSC 32J25

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1 Introduction

Let $X$ be a smooth projective variety and let $K_X$ be the canonical bundle of $X$. The graded ring:

$$R(X, K_X) := \bigoplus_{m=0}^{\infty} H^0(X, \mathcal{O}_X(mK_X))$$

is called the canonical ring of $X$. $R(X, K_X)$ is a birational invariant of $X$. It is conjectured that for every smooth projective variety $X$, $R(X, K_X)$ finitely generated. The purpose of this article is to give the following partial answer to the conjecture.

**Theorem 1.1** Let $X$ be a smooth projective variety of general type defined over complex numbers.

Then the canonical ring $R(X, K_X)$ is finitely generated.

This theorem has already been known in the case of dim $X \leq 3$ ([13], [12]).

Y. Kawamata pointed that the existence of a Zariski decomposition of $K_X$ implies the finite generation of $R(X, K_X)$ ([9]). This is our starting point of the proof of Theorem 1.1. In this case the finite generation is equivalent to the stable base point freeness of the nef part of the Zariski decomposition. In this case the nef part is not ample in general. Hence to prove the stable base point freeness we need to use some additional positivity. One of the important observation in [9] is the fact that on the stable fixed component of $K_X$, we may use the positivity coming from the conormal bundle of it. In [1], using a variant of Shokurov’s nonvanishing theorem ([16], see Theorem
4.5 below), he proved the finite generation of canonical rings of smooth projective varieties of general type by Noetherian induction under the assumption that there exists a Zariski decomposition of $K_X$. The main difficulty to prove the finite generation lies in such delicate semipositivity.

The proof of Theorem 1.1 consists of the following five steps.

1. construct an AZD $h$ of $K_X$ to distinguish the positive part of $K_X$,
2. construct the nontrivial numerically trivial fibration associated with $(K_X, h)$ by the first nonvanishing theorem (Theorem 4.1) on every stable fixed components,
3. find an effective $\mathcal{R}$-divisor on a very general fiber of the numerically trivial fibration by using the structure theorem for numerically trivial singular hermitian line bundles (cf. Theorem 4.3).
4. construct the formal canonical model and prove the virtual base point freeness (cf. Definition 6.1) of $R(X, K_X)$, using the second nonvanishing theorem (Theorem 4.4),
5. prove finite generation of $R(X, K_X)$ by showing that the formal canonical model is a projective variety and is the canonical model of $X$.

Let us briefly explain each steps.

**Step 1.** An **AZD** $h$ of $K_X$ is a singular hermitian metric such that the curvature $\Theta_h$ is semipositive and

$$H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m)) \simeq H^0(X, \mathcal{O}_X(mK_X))$$

holds for every $m \geq 0$. The AZD $h$ exists (more generally for any pseudoeffective line bundle) by [18, 19, 6]. The singular hermitian line bundle $(K_X, h)$ is considered as an analogue of the nef part of a Zariski decomposition.

**Step 2.** In general $\Theta_h$ is not strictly positive. $(K_X, h)$ has only weak positivity on every stable fixed components. This is one of the main difficulty of the proof. To distinguish the null direction of $(K_X, h)$ on every stable fixed component, we construct the nontrivial **numerically trivial fibrations** (cf. Theorem 4.2) associated with $(K_X, h)$. Here the essential ingredient is the first nonvanishing theorem (Theorem 4.1) which is a generalization of Shokurov’s nonvanishing theorem ([16]).

**Step 3.** On a very general fiber of the above numerically trivial fibration, by the structure theorem for numerically trivial singular hermitian line bundles
(cf. Theorem 4.3), we may distinguish a sum of at most countably many prime divisors with nonnegative coefficients. In the next step, the number of the positive coefficients turns out to be finite.

**Step 4.** Taking a successive resolution of $B_s \mid m!K_X \mid (m \geq 1)$, and identifying the fiber of the numerically trivial fibrations to a point, we construct a formal canonical model $\hat{X}_{can}$. At this stage $\hat{X}_{can}$ is only a set. But we may consider $(K_X, h)$ as a numerically positive “$\mathbb{R}$-line bundle” on $\hat{X}_{can}$. Using effective base point freeness argument as in [1] and the second non-vanishing theorem (Theorem 4.4), we may prove the virtual base point freeness of $R(X, K_X)$. The advantage of this argument is that we can specify a point where we want to prove the stably base point freeness of $(K_X, h)$ on $\hat{X}_{can}$. Hence we do not need to use the Noetherian induction argument as in [8]. This is the effect of the fact that we may consider $(K_X, h)$ to be numerically positive on $\hat{X}_{can}$.

**Step 5.** The last step is to prove that $\hat{X}_{can}$ is actually the canonical model of $X$. To prove this we use a topological argument using the virtual base point freeness of $R(X, K_X)$.

We may prove Theorem 1.1 without using singular hermitian metrics. In this sense our proof is essentially algebraic. Although for the better presentation I decided to present the proof in the complex analytic language, one may easily transcript the proof in algebro-geometric language. In fact the transcription follows from the fact that one can approximate any plurisubharmonic functions by plurisubharmonic functions with algebraic singularities ([6, Section 3]).

In this paper “very general” means outside of at most countably many union of proper Zariski closed subsets and “general” means in the sense of usual Zariski topology.

## 2 Multiplier ideal sheaves

In this section, we shall review the basic definitions and properties of multiplier ideal sheaves.

### 2.1 Multiplier ideal sheaves

**Definition 2.1** Let $L$ be a line bundle on a complex manifold $M$. A singular hermitian metric $h$ on $L$ is given by

\[ h = e^{-\varphi} \cdot h_0, \]

where $\varphi$ is a plurisubharmonic function on $M$. The multiplier ideal sheaf $\mathcal{J}(L)$ is defined as the sheaf of ideals of the ring $\mathcal{O}(M)$ such that

\[ \mathcal{J}(L) = \{ f \in \mathcal{O}(M) \mid f \cdot e^{-\varphi} \in \mathcal{O}_X \}. \]
where \( h_0 \) is a \( C^\infty \)-hermitian metric on \( L \) and \( \varphi \in L^1_{\text{loc}}(M) \) is an arbitrary function on \( M \). We call \( \varphi \) a weight function of \( h \).

The curvature current \( \Theta_h \) of the singular hermitian line bundle \((L, h)\) is defined by

\[
\Theta_h := \Theta_{h_0} + \sqrt{-1} \partial \bar{\partial} \varphi,
\]

where \( \partial \bar{\partial} \) is taken in the sense of a current. The \( L^2 \)-sheaf \( \mathcal{L}^2(L, h) \) of the singular hermitian line bundle \((L, h)\) is defined by

\[
\mathcal{L}^2(L, h) := \{ \sigma \in \Gamma(U, \mathcal{O}_M(L)) \mid h(\sigma, \sigma) \in L^1_{\text{loc}}(U) \},
\]

where \( U \) runs open subsets of \( M \). In this case there exists an ideal sheaf \( \mathcal{I}(h) \) such that

\[
\mathcal{L}^2(L, h) = \mathcal{O}_M(L) \otimes \mathcal{I}(h)
\]

holds. We call \( \mathcal{I}(h) \) the multiplier ideal sheaf of \((L, h)\). If we write \( h \) as

\[
h = e^{-\varphi} \cdot h_0,
\]

where \( h_0 \) is a \( C^\infty \) hermitian metric on \( L \) and \( \varphi \in L^1_{\text{loc}}(M) \) is the weight function, we see that

\[
\mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_M, e^{-\varphi})
\]

holds.

If \( \{\sigma_i\} \) are finite number of global holomorphic sections of a line bundle \( L \), for every positive rational number \( \alpha \) and a \( C^\infty \)-function \( \phi \)

\[
h := e^{-\phi} \cdot \frac{1}{(\sum |\sigma_i|^2)^\alpha}
\]

defines a singular hermitian metric on the \( \mathbb{Q} \)-line bundle \( \alpha L \). Here \( |\sigma_i|^2 \) is defined by

\[
|\sigma_i|^2 = \frac{h_0(\sigma_i, \sigma_i)}{h_0},
\]

where \( h_0 \) is an arbitrary \( C^\infty \)-hermitian metric on \( L \) (the righthandside is independent of the choice of \( h_0 \)). We call such a metric \( h \) a singular hermitian metric on \( \alpha L \) with algebraic singularities. Singular hermitian metrics with algebraic singularities is particulary easy to handle, because its multiplier ideal sheaf or that of the multiple of the metric can be controlled by taking suitable successive blowing ups such that the total transform of the divisor \( \sum_i(\sigma_i) \) is a divisor with normal crossings.

Let \( D \) be an effective \( \mathbb{R} \)-divisor on \( M \) and let

\[
\sum_i a_i D_i
\]
be the irreducible decomposition of $D$. Let $\sigma_i$ be a global section of $\mathcal{O}_M(D_i)$ with divisor $D_i$. Then

$$h = \frac{1}{\prod_i |\sigma_i|^{2a_i}}$$

is a singular hermitian metric on the $\mathbb{R}$-line bundle $\mathcal{O}_M(D)$. We define the multiplier sheaf $\mathcal{I}(D)$ associated with $D$ by

$$\mathcal{I}(D) = \mathcal{I}(h) = \mathcal{L}^2(\mathcal{O}_X, \frac{1}{\prod_i ||\sigma_i||^{2a_i}}),$$

where $||\sigma_i||$ is the hermitian norm of $\sigma_i$ with respect to a $C^\infty$-hermitian metric on $\mathcal{O}_M(D_i)$.

If $\text{Supp} \, D$ is a divisor with normal crossings,

$$\mathcal{I}(D) = \mathcal{O}_M(-[D])$$

holds, where $[D] := \sum_i [a_i]D_i$ (for a real number $a$, $[a]$ denotes the largest integer smaller than or equal to $a$).

### 2.2 Nadel’s vanishing theorem

The following vanishing theorem plays a central role in this paper.

**Theorem 2.1** (Nadel’s vanishing theorem [14, p.561]) Let $(L, h)$ be a singular hermitian line bundle on a compact Kähler manifold $M$ and let $\omega$ be a Kähler form on $M$. Suppose that $\Theta_h$ is strictly positive, i.e. there exists a positive constant $\varepsilon$ such that

$$\Theta_h \geq \varepsilon \omega$$

holds. Then $\mathcal{I}(h)$ is a coherent sheaf of $\mathcal{O}_M$ ideal and for every $q \geq 1$

$$H^q(M, \mathcal{O}_M(K_M + L) \otimes \mathcal{I}(h)) = 0$$

holds.

**Remark 2.1** The word “a closed positive $(1,1)$ current” does not mean a closed strictly positive current. For example the $0$-current is closed positive. This terminology might be misleading for algebraic geometers.

By the definition of a multiplier ideal sheaf we have the following lemma which will be used later.

**Lemma 2.1** Let $(L, h)$ be a singular hermitian line bundle on a complex manifold $M$. Let $f : N \rightarrow M$ be a modification. Then $(f^*L, f^*h)$ is a singular hermitian line bundle on $N$ and

$$f_*\mathcal{I}(f^*h) \subseteq \mathcal{I}(h)$$

holds.
2.3 Lelong numbers and structure of closed positive (1,1)-currents

A closed positive (1,1)-current is considered as a (1,1)-form whose coefficients are distributions. Hence by the Lebesgue decomposition of the coefficients, every closed positive (1,1)-current $T$ on a complex manifold $M$ is uniquely decomposed as:

$$T = T_{abc} + T_{sing},$$

where $T_{abc}$ denotes the absolutely continuous part and $T_{sing}$ denotes the singular part. We call this decomposition the **Lebesgue decomposition** of $T$. It is important to note that $T_{abc}$ and $T_{sing}$ are not closed in general. To measure the magnitude of the singular part, the following definition is fundamental.

**Definition 2.2** Let $T$ be a closed positive (1,1)-current on a unit open polydisk $\Delta^n$ with center $O$. Then by $\partial\bar{\partial}$-Poincaré lemma there exists a plurisubharmonic function $\varphi$ on $\Delta^n$ such that

$$T = \frac{\sqrt{-1}}{\pi} \partial\bar{\partial}\varphi.$$

We define the Lelong number $\nu(T, O)$ at $O$ by

$$\nu(T, O) = \liminf_{x \to O} \frac{\varphi(x)}{\log |x|},$$

where $|x| = (\sum |x_i|^2)^{1/2}$. It is easy to see that $\nu(T, O)$ is independent of the choice of $\varphi$ and local coordinates around $O$. For an analytic subset $V$ of a complex manifold $X$, we set

$$\nu(T, V) = \inf_{x \in V} \nu(T, x).$$

**Remark 2.2** More generally the Lelong number is defined for a closed positive $(k, k)$-current on a complex manifold.

Let us consider a singular hermitian metric on the trivial bundle with algebraic singularities

$$h = \left(\sum_{i=1}^k |f_i|^2\right)^{-1} (f_i \in \mathcal{O}(\Delta^n))$$

on $\Delta^n$. Then we see that $\Theta_h$ is positive and for every $x \in \Delta^n$.

$$\nu(\Theta_h, x) = 2 \min_{i} \text{mult}_x(f_i)$$
holds, where \((f_i)\) denotes the divisor of \(f_i\) for every \(i\). This implies that for a singular hermitian metric with algebraic singularities, the Lelong number of the curvature is essentially the infimum of the vanishing order of the defining (multi)sections.

In this paper we only deal with singular hermitian metrics which is a limit of singular hermitian metrics with algebraic singularities. Hence in this paper we may consider that the **Lelong number is nothing but the limit vanishing order of (multi)sections** in an obvious manner.

The following theorem is fundamental.

**Theorem 2.2** ([17, p.53, Main Theorem]) Let \(T\) be a closed positive \((k,k)\)-current on a complex manifold \(M\). Then for every \(c > 0\)

\[
\{ x \in M \mid \nu(T, x) \geq c \}
\]

is a subvariety of codimension \(\geq k\) in \(M\).

Let \((L, h)\) be a singular hermitian line bundle on a smooth projective variety \(X\) such that \(\Theta_h\) is a positive current. The following lemma shows a rough relationship between the Lelong number of \(\nu(\Theta_h, x)\) at \(x \in X\) and the stalk of the multiplier ideal sheaf \(I(h)_x\) at \(x\).

**Lemma 2.2** ([2, p.284, Lemma 7], [17, p.85, Lemma 5.3]) Let \(\varphi\) be a plurisubharmonic function on the open unit polydisk \(\Delta^n\) in \(C^n\) with center \(O\). Suppose that \(e^{-\varphi}\) is not locally integrable around \(O\). Then we have that

\[
\nu(\sqrt{-1} \partial \bar{\partial} \varphi, O) \geq 2
\]

holds. And if

\[
\nu(\sqrt{-1} \partial \bar{\partial} \varphi, O) > 2n
\]

holds, then \(e^{-\varphi}\) is not locally integrable around \(O\).

Let \(T\) be a closed positive \((1,1)\)-current on a complex manifold \(X\). Let \(\mathcal{U} = \{U_\alpha\}\) be an open covering of \(X\) such that for every \(\alpha\) there exists a plurisubharmonic function \(\varphi_\alpha\) such that

\[
T \mid U_\alpha = \sqrt{-1} \partial \bar{\partial} \varphi_\alpha
\]

holds. We define the singular set \(\text{Sing} T\) by

\[
\text{Sing} T \cap U_\alpha = \{ x \in U_\alpha \mid \varphi_\alpha(x) = -\infty \}.
\]

\(\text{Sing} T\) is well defined and independent of the choice of \(\{U_\alpha\}\) and \(\{\varphi_\alpha\}\). Let \(Y\) be a complex manifold and let \(f : Y \to X\) be a holomorphic map such that

\[
f(Y) \not\subseteq \text{Sing} T.
\]
Then the pullback $f^*T$ is defined by

$$f^* | f^{-1}(U_\alpha) = \sqrt{-1} \partial \bar{\partial} (f^* \varphi_\alpha).$$

If $Y$ is a submanifold of $X$ and $f$ is the canonical immersion, then we denote $f^*T$ by $T \mid_Y$ and call it the restriction of $T$ to $Y$. To compute the Lelong number the following lemma is useful.

**Lemma 2.3** \cite{[17]} Let $T$ be a closed positive $(1,1)$-current on the open unit polydisk $\Delta^n$ with center $O$. Let us parametrize the lines passing through $O$ by $\mathbb{P}^{n-1}$ in the standard way. Then there exists a set $E$ of measure 0 in $\mathbb{P}^{n-1}$ such that for every $[L] \in \mathbb{P}^{n-1} - E$, $T \mid_{L \cap \Delta^n}$ is well defined and

$$\nu(T,O) = \nu(T \mid_{L \cap \Delta^n}, O)$$

holds.

The next corollary is analogous to the corresponding fact about multiplicities of divisors.

**Corollary 2.1** Let $M$ be a complex manifold and let $T$ be a closed positive $(1,1)$-current on $M$. Let $f : N \to M$ be a composition of successive blowing ups with smooth centers. Then for every $x \in M$ and $y \in f^{-1}(x)$,

$$\nu(f^*T, y) \geq \nu(T, x)$$

holds.

## 3 Analytic Zariski decomposition

In this section we shall introduce the notion of analytic Zariski decompositions which play essential roles in this paper. By using analytic Zariski decompositions, we can handle a big line bundles as if it were a nef and big line bundles.

### 3.1 Definition of AZD

**Definition 3.1** Let $M$ be a compact complex manifold and let $L$ be a line bundle on $M$. A singular hermitian metric $h$ on $L$ is said to be an analytic Zariski decomposition (AZD), if the followings hold.

1. $\Theta_h$ is a closed positive current,
2. for every $m \geq 0$, the natural inclusion

$$H^0(M, \mathcal{O}_M(mL) \otimes \mathcal{I}(h^m)) \to H^0(M, \mathcal{O}_M(mL))$$

is an isomorphism.
Remark 3.1 If an AZD exists on a line bundle $L$ on a smooth projective variety $M$, $L$ is pseudoeffective by the condition 1 above.

As for the existence the following theorems are known.

Theorem 3.1 ([18, 19]) Let $L$ be a big line bundle on a smooth projective variety $M$. Then $L$ has an AZD.

More generally the existence for general pseudoeffective line bundles, now we have the following theorem.

Theorem 3.2 ([7], cf. [22, Theorem 2.4]) Let $X$ be a smooth projective variety and let $L$ be a pseudoeffective line bundle on $X$. Then $L$ has an AZD.

3.2 An explicit construction of AZD

In the case of big line bundles, we have an explicit construction of an AZD ([18]). Here we shall review the construction.

Definition 3.2 Let $(L, h)$ be a line bundle on a compact Kähler manifold $(X, \omega)$. Let $\{\phi_0, \ldots, \phi_{N(m)}\}$ be an orthonormal basis of $\Gamma(X, \mathcal{O}_X(mL))$ with respect to the $L^2$-inner product

$$(\phi, \phi') := \int_X h^m \phi \cdot \bar{\phi}' \frac{\omega^n}{n!} (\phi, \phi' \in \Gamma(X, \mathcal{O}_X(mL))).$$

We define the $m$-th Bergman kernel $K_m(z, w)$ of $(L, h)$ by

$$K_m(z, w) := \sum_{i=0}^{N(m)} \phi_i(z) \bar{\phi}_i(w).$$

Then it is trivial to see that $K_m(z, w)$ is independent of the choice of the orthonormal basis. For simplicity we denote the restriction of $K_m(z, w)$ to the diagonal of $X \times X$ by $K_m$.

Theorem 3.3 ([18]) Let $L$ be a big line bundle on a compact Kähler manifold $(X, \omega)$. Let $h_0$ be a $C^\infty$-hermitian metric on $L$. Let $K_m(z, w)$ be the $m$-the Bergman kernel of $(L, h_0)$. Then

$$h := (\lim_{m \to \infty} \sqrt[2]{K_m})^{-1}$$

is an AZD of $L$. 

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The reason why we presented this explicit construction is to show that **the AZD is a limit of singular hermitian metrics** \( \{h_m\} \) **with algebraic singularities** on \( L \) defined by

\[
h_m := 1/ \sqrt[4]{K_m}.
\]

The Lelong number of \( \Theta_h \) is considered as a limit of the Lelong number of \( \Theta_{h_m} \) which is nothing but

\[
m^{-1} \text{mult}_{B_s} |mL|.
\]

Hence for every \( x \in X \)

\[
\nu(\Theta_h, x) = \lim_{m \to \infty} m^{-1} \text{mult}_{B_s} |mL|
\]

holds, where \( \text{mult}_{B_s} |mL| \) denotes the multiplicity of the general member of \( |mL| \) at \( x \). Hence \( \nu(\Theta_h, x)(x \in X) \) is essentially an algebro-geometric number.

### 3.3 Some properties of AZD

Let \((L, h)\) be a singular hermitian line bundle on a smooth projective variety \(X\). We denote the linear system \(|H^0(X, \mathcal{O}_X(mL) \otimes I(h^m))| \) by \(|m(L, h)|\).

**Theorem 3.4** Let \(L\) be a big line bundle on a smooth projective variety \(X\) and let \(h\) be an AZD of \(L\). Then there exists a positive constant \(C\) such that

\[
0 \leq \text{mult}_{B_s} |m(L, h)| - m \cdot \nu(\Theta_h, x) \leq C
\]

holds for every \(m\) and \(x \in X\).

**Proof.** The first inequality is trivial by the definition of an AZD and the fact that \(R(X, L)\) is a ring. In fact by Lemma 2.2 we see that

\[
\text{mult}_{B_s} |m(L, h)| \geq m \cdot \nu(\Theta_h, x) - n
\]

holds. Since \(h\) is an AZD of \(L\), \(|m(L, h)| = |mL|\) holds for every \(m \geq 0\). Hence for every \(\sigma \in \Gamma(X, \mathcal{O}_X(mL)) \setminus \{0\}\) and a positive integer \(\ell\),

\[
\ell \cdot \text{mult}_{B_s}(\sigma) = \text{mult}_{B_s}(\sigma^\ell) \geq \ell m \cdot \nu(\Theta_h, x)
\]

holds. Dividing both sides by \(\ell\) and letting \(\ell\) tend to infinity, we see that

\[
\text{mult}_{B_s}(\sigma) \geq m \cdot \nu(\Theta_h, x)
\]

holds.
Next we shall verify the second inequality. Let \( x \) be a point on \( X \). Let \( \pi : \tilde{X} \to X \) be the blowing up of \( X \) at \( x \). Since \( L \) is big, by Kodaira’s lemma (cf. [11, Appendix]) there exists an effective \( \mathbb{Q} \)-divisor \( E \) such that \( \pi^*L - E \) is ample. Let \( r \) be a sufficiently large positive integer such that

\[
H := r(\pi^*L - E)
\]

is Cartier and \( H - K_{\tilde{X}} \) is ample. Let \( \tilde{x} \) be a very general point on the exceptional divisor \( \pi^{-1}(x) \). Then by Theorem 2.2 and Lemma 2.2 we may assume that for every \( m \geq 0 \) the multiplier ideal sheaf \( \mathcal{I}(\pi^*h^m) \) is locally free on a neighbourhood of \( \tilde{x} \) (the neighbourhood may depend on \( m \)). Let \( U \) be a small neighbourhood of \( \tilde{x} \) and let \( \rho \) be a \( C^\infty \)-function on \( \tilde{X} \) such that

1. \( \text{Supp} \rho \subset \subset U \),
2. \( \rho \equiv 1 \) on a neighbourhood of \( \tilde{x} \),
3. \( 0 \leq \rho \leq 1 \)

hold. Let \( d_{\tilde{x}} \) denote the distance function from \( \tilde{x} \) with respect to a fixed Kähler metric on \( \tilde{X} \). If we take \( r \) sufficiently large we may assume that there exists a \( C^\infty \) hermitian metric \( \tilde{h} \) on \( H - K_{\tilde{X}} \) such that

\[
\Theta_{\tilde{h}} + 2n\sqrt{-1}\partial\bar{\partial} \log(\rho \cdot d_{\tilde{x}})
\]

is strictly positive on \( \tilde{X} \). Then by Nadel’s vanishing theorem we see that

\[
H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(H + \pi^*(mL)) \otimes \mathcal{I}(\pi^*h^m)) \to \mathcal{O}_{\tilde{X}}(H + \pi^*(mL)) \otimes \mathcal{I}(\pi^*h^m) \otimes \mathcal{O}_{\tilde{X}} / \mathcal{M}_{\tilde{x}}
\]

is surjective for every \( m \geq 0 \), where \( \mathcal{M}_{\tilde{x}} \) denotes the maximal ideal sheaf at \( \tilde{x} \). Since \( h \) is an AZD of \( L \), we see that there exists a canonical injection

\[
H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(H + \pi^*(mL)) \otimes \mathcal{I}(\pi^*h^m)) \hookrightarrow \pi^*H^0(X, \mathcal{O}_X(mL)).
\]

Since

\[
\pi_*(\mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + mL) \otimes \mathcal{I}(\pi^*h^m)) = \mathcal{O}_X(K_X + mL) \otimes \mathcal{I}(h^m)
\]

holds by the definition of multiplier ideal sheaves, we see that

\[
\mathcal{I}(h^m) \otimes \mathcal{M}_{\tilde{x}}^n \subset \pi_*\mathcal{I}(\pi^*h^m)
\]

holds for every \( m \). Hence by the above argument, there exists a positive constant \( C \) such that

\[
\text{mult}_xBs \mid m(L, h) \mid -m \cdot \nu(\Theta_{\tilde{h}}, x) \leq C
\]

holds for every \( m \). It is easy to see that \( C \) can be taken independent of \( x \in X \). This completes the proof of Theorem 3.4. Q.E.D

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Corollary 3.1 Let $P$ be a nef and big line bundle on a smooth projective variety $X$, then there exists a singular hermitian metric $h_P$ on $P$ such that $\Theta_{h_P}$ is a closed positive current on $X$ and $\nu(\Theta_{h_P})$ is identically 0 on $X$.

We shall discuss about the uniqueness of the multiplier ideal sheaves associated with an AZD. First we introduce the following terminology.

Definition 3.3 Let $h_L$ be a singular hermitian metric on a line bundle $L$ on a complex manifold $X$. Suppose that the curvature of $h_L$ is a positive current on $X$. We set

$$\bar{\mathcal{I}}(h_L) := \lim_{\varepsilon \downarrow 0} \mathcal{I}(h^{1+\varepsilon}_L)$$

and call it the closure of $\mathcal{I}(h_L)$.

Let us explain the reason why we take the closure. Let $h_L$ be a singular hermitian metric on a line bundle $L$ on a complex manifold $X$ with positive curvature current. Then $\bar{\mathcal{I}}(h_L)$ is coherent ideal sheaf on $X$ by Theorem 2.1. Let $f : Y \to X$ be a modification such that $f^*\bar{\mathcal{I}}(h_L)$ is locally free. If we take $f$ properly, we may assume that there exists a divisor $F = \sum F_i$ with normal crossings on $Y$ such that

$$K_Y = f^*K_X + \sum a_i F_i$$

and

$$f^*\bar{\mathcal{I}}(h_L) = \mathcal{O}_Y(-\sum b_i F_i)$$

hold on $Y$ for some nonnegative integers $\{a_i\}$ and $\{b_i\}$. Then by Lemma 2.2,

$$b_i = [\nu(f^*\Theta_{h_L}, F_i) - a_i]$$

holds for every $i$. In this way $\bar{\mathcal{I}}(h_L)$ is determined by the Lelong numbers of the curvature current on some modification. This is not the case, unless we take the closure as in the following example.

Example 1 Let $h_P$ be a singular hermitian metric on the trivial line bundle on $\mathbb{C}$ defined by

$$h_P = \frac{1}{|z|^2 (\log |z|)^2}.$$ 

Then $\nu(\Theta_{h_P}, 0) = 1$ holds. But $\mathcal{I}(h_P) = \mathcal{O}_\mathbb{C}$ holds. On the other hand $\bar{\mathcal{I}}(h_P) = \mathcal{M}_0$ holds.

Remark 3.2 Theorem 2.1 still holds, even if we replace the multiplier ideal by its closure. This can be verified as follows. Let $X$ be a compact Kähler manifold and let $(L, h_L)$ be a singular hermitian line bundle on $X$ with strictly positive curvature. Let $h_\infty$ be a $C^\infty$-hermitian metric on $L$. Then
$h^{1+\varepsilon}_L \cdot h^{-\varepsilon}_\infty$ has strictly positive curvature for every sufficiently small positive number $\varepsilon$. By Theorem 2.1, we see that

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h^{1+\varepsilon}_L)) = 0$$

holds for every $q \geq 1$ and every sufficiently small positive number $\varepsilon$. Letting $\varepsilon \downarrow 0$ we obtain that

$$H^q(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}(h_L)) = 0$$

holds for every $q \geq 1$.

Now we shall prove the following uniqueness theorem for the multiplier ideal sheaves associated with an AZD.

**Proposition 3.1** Let $L$ be a big line bundle on a smooth projective variety $X$. Let $h$ be an AZD of $L$. For any positive integer $m$, $\mathcal{I}(h^m)$ is independent of the choice of the AZD $h$.

**Proof of Proposition 3.1** Let $L, h$ be as above. By Theorem 3.4, for any modification $f : Y \to X$ and $y \in Y$, we see that

$$\nu(f^*\Theta_h, y) = \lim_{m \to \infty} m^{-1} \text{mult}_y f^*Bs | mL |$$

holds. This implies that for any positive integer $m \mathcal{I}(h^m)$ is independent of the choice of $h$. Q.E.D.

**Definition 3.4** Let $L$ be a pseudoeffective line bundle on a smooth projective variety $X$ and let $h$ be a singular hermitian metric on $L$ with positive curvature current. Let $x$ be a point on $X$. $| m(L, h) |$ is said to be base point free at $x$, if

$$\text{mult}_x Bs | m(L, h) | = m \cdot \nu(\Theta_h, x)$$

holds.

Let $h$ be an AZD on a big line bundle $L$ on a smooth projective variety $X$. Then by Theorem 3.4, we see that $(L, h)$ is asymptotically base point free in the context of Definition 3.4.
3.4 Volume of subvarieties

Let $L$ be a big line bundle on a smooth projective variety $X$. To measure the total positivity of $L$ on a subvariety of $X$, we define the following notion.

Definition 3.5 Let $L$ be a big line bundle on a smooth projective variety $X$ and let $h$ be an AZD of $L$. Let $Y$ be a subvariety of $X$ of dimension $r$. We define the volume $\mu(Y, L)$ of $Y$ with respect to $L$ by

$$\mu(Y, L) := r! \cdot \lim_{m \to \infty} m^{-r} \dim H^0(Y, \mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)/\text{tor}),$$

where $\text{tor}$ denotes the torsion part of $\mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)$.

Remark 3.3 If we define $\mu(Y, L)$ by

$$\mu(Y, L) := r! \cdot \lim_{m \to \infty} m^{-r} \dim H^0(Y, \mathcal{O}_Y(mL))$$

then it is totally different unless $Y = X$. The above definition is meaningful when $\mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)$ is generically rank one for $m >> 1$. Otherwise $\mu(Y, L)$ may be infinity. If $\mu(Y, L)$ is finite, by Proposition 3.1, it is easy to see that $\mu(Y, L)$ is independent of the choice of the AZD $h$. In fact if $\mu(Y, L) > 0$, then there exists a positive integer $m_0$ and an effective divisor $E$ on $Y$ so that for every nonnegative integer $m$, the natural injection

$$\mathcal{O}_Y((m + m_0)L - E) \otimes \mathcal{I}(h^{m+m_0}) \to \mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)$$

exists.

If $L$ is a nef and big line bundle on a smooth projective variety, then by Corollary 3.1 and Lemma 2.2, for every subvariety $Y$ in $X$,

$$\mu(Y, L) = L^{\dim Y} \cdot Y$$

holds. For a general singular hermitian line bundle with positive curvature, we define the volume as follows.

Definition 3.6 Let $L$ be a pseudoeffective line bundle on a smooth projective variety $X$ and let $h$ be a singular hermitian metric on $X$ such that $\Theta_h$ is a closed positive current. Let $Y$ be a subvariety of $X$. We define the volume $\mu(Y, (L, h))$ of $Y$ with respect to $(L, h)$ by

$$\mu(Y, (L, h)) := (\dim Y)! \cdot \lim_{m \to \infty} m^{-\dim Y} \dim H^0(Y, \mathcal{O}_Y(mL) \otimes \mathcal{I}(h^m)/\text{tor}).$$
3.5 Intersection theory for singular hermitian line bundles

In this subsection we review the definition an intersection number for a singular hermitian line bundle with positive curvature current on a smooth projective variety and an irreducible curve on it (cf. [22]). This intersection number is different from the usual intersection number of the underlying line bundle and the curve. The new intersection number measures the intersection of the positive part of the singular hermitian line bundle and the curve. Next we shall consider the restriction of singular hermitian line bundles to subvarieties.

**Definition 3.7** Let $L$ be a line bundle on a complex manifold $M$. Let $h$ be a singular hermitian metric on $L$ given by

$$h = e^{-\varphi} \cdot h_0,$$

where $h_0$ is a $C^\infty$-hermitian metric on $L$ and $\varphi \in L^1_{\text{loc}}(M)$. Suppose that the curvature current $\Theta_h$ is bounded from below by some $C^\infty$-(1,1)-form. For a subvariety $V$ of $M$, we say that the restriction $h|_V$ is well defined, if $\varphi$ is not identically $-\infty$ on $V$.

Let $(L, h), h_0, V, \varphi$ be as in Definition 3.7. Then $\varphi$ is an almost plurisubharmonic function i.e. locally a sum of a plurisubharmonic function and $C^\infty$-function. Let $\pi : \tilde{V} \rightarrow V$ be an arbitrary resolution of $V$. Then $\pi^*(\varphi|_V)$ is locally integrable on $\tilde{V}$, since $\varphi$ is almost plurisubharmonic. Hence

$$\pi^*(\Theta_h|_V) := \Theta_{\pi^*h_0|_\tilde{V}} + \sqrt{-1} \partial \bar{\partial} \pi^*(\varphi|_V)$$

is well defined. We shall define the intersection number for a singular hermitian metric with positive curvature current and an irreducible curve such that the restriction of the singular hermitian metric is well defined.

**Definition 3.8** Let $(L, h)$ be a singular hermitian line bundle on a smooth projective variety $X$ such that the curvature current $\Theta_h$ is closed positive. Let $C$ be an irreducible curve on $X$ such that $h|_C$ is well defined. The intersection number $(L, h) \cdot C$ is defined by

$$(L, h) \cdot C := \lim_{m \to \infty} m^{-1} \dim H^0(C, \mathcal{O}_C(mL) \otimes \mathcal{I}(h^m)\text{tor}),$$

where tor denotes the torsion part of $\mathcal{O}_C(mL) \otimes \mathcal{I}(h^m)$.

Let $(L, h), C$ be as above. Let

$$\pi : \tilde{C} \rightarrow C$$
be the normalization of $C$. We define the multiplier ideal sheaf $\mathcal{I}(h^m |_C)(m \geq 0)$ on $C$ by

$$\mathcal{I}(h^m |_C) := \pi_* \mathcal{I}(\pi^* h^m |_C).$$

And the Lelong number $\nu(\Theta h |_C, x)(x \in C)$ by

$$\nu(\Theta h, x) = \sum_{\tilde{x} \in \pi^{-1}(x)} \nu(\pi^* \Theta h |_C, \tilde{x}).$$

**Lemma 3.1** (\cite[Lemma 2.4]{22}) Let $(L, h)$ be a singular hermitian line bundle on a smooth projective variety $X$ such that $\Theta h$ is closed positive. Let $C$ be an irreducible curve on $X$ such that $h |_C$ is well defined. Suppose that $(L, h) \cdot C = 0$ holds. Then

$$\Theta h |_C = \sum_{x \in C} \nu(\Theta h |_C, x)x$$

holds in the sense that

$$\pi^* (\Theta h |_C) = \sum_{\tilde{x} \in C} \nu(\pi^* \Theta h |_C, \tilde{x})\tilde{x}$$

holds.

**Definition 3.9** Let $(L, h)$ be a singular hermitian line bundle on a smooth projective variety $X$ such that $\Theta h$ is positive. $(L, h)$ is said to be numerically trivial, if for every irreducible curve $C$ on $X$ such that $h |_C$ is well defined,

$$(L, h) \cdot C = 0$$

holds.

**3.6 Restriction of the intersection theory to divisors**

In the previous subsection we define an intersection number of a singular hermitian line bundle with positive curvature and an irreducible curve on which the restriction of the singular hermitian metric is well defined. In this subsection, we shall extend the definition of the intersection number.

Let $(L, h)$ be a singular hermitian line bundle on a smooth projective variety $X$ such that $\Theta h$ is positive.

Let $D$ be a smooth divisor on $X$. We set

$$v_m(D) := \text{mult}_D \text{Spec}(\mathcal{O}_X/\mathcal{I}(h^m))$$
\[ \mathcal{I}_D(h^m) := \mathcal{O}_D(v_m(D)D) \otimes \mathcal{I}(h^m). \]

Then \( \mathcal{I}_D(h^m) \) is an ideal sheaf on \( D \), since \( \mathcal{O}_D(mL) \otimes \mathcal{I}(h^m) \) is a subsheaf of the locally free sheaf \( \mathcal{O}_D(mL - v_m(D)D) \) on the smooth variety \( D \). We define the ideal sheaf \( \sqrt[m]{\mathcal{I}_D(h^m)} \) on \( D \) by

\[ \sqrt[m]{\mathcal{I}_D(h^m)}_x := \mathcal{I}(\frac{1}{m}(\sigma))_x(x \in D), \]

where \( \sigma \) runs all the germs of \( \mathcal{I}_D(h^m)_x \). And we set

\[ \mathcal{I}_D(h) := \bigcap_{m \geq 1} \sqrt[m]{\mathcal{I}_D(h^m)} \]

and call it the **multiplier ideal sheaf of** \( h \) **on** \( D \). Also we set

\[ \bar{\mathcal{I}}_D(h) := \lim_{\epsilon \downarrow 0} \mathcal{I}_D(h^{1+\epsilon}). \]

If \( h |_D \) is well defined, then

\[ \bar{\mathcal{I}}_D(h) = \bar{\mathcal{I}}(h |_D) \]

holds (\([22, \text{Theorem 2.8}]\)). For every irreducible curve \( C \) on \( D \), we say that the intersection number \( (L, h) \cdot C \) is well defined, if \( \nu(\Theta_h, x) = \nu(\Theta_h, D) \) holds for a very general point \( x \) on \( C \). In this case \( \mathcal{I}_D(h^m) |_C \) is an ideal sheaf on \( C \).

We define the **intersection number** \( (L, h) \cdot C \) by

\[ (L, h) \cdot C := \lim_{m \to \infty} m^{-1} \dim H^0(C, \mathcal{O}_C(mL - v_m(D)D) \otimes \mathcal{I}_D(h^m)/\text{tor}). \]

Then we see that

\[ (L, h) \cdot C = (L - \nu(\Theta_h, D)D) \cdot C + \lim_{m \to \infty} m^{-1} \deg \mathcal{I}_D(h^m) \]

holds.

We can also define the **volume** of \( r \)-dimensional subvariety \( Y \) of \( D \) with respect to \( (L, h) \) by using \( \mathcal{I}_D(h^m) \) as

\[ \mu(Y, (L, h)) := r! \lim_{m \to \infty} m^{-r} \dim H^0(Y, \mathcal{O}_Y(mL - v_m(D)D) \otimes \mathcal{I}_D(h^m)/\text{tor}). \]

But this coincides the definition before as is easily be seen.

We may define the **Lelong number** \( \nu_D(\Theta_h, x)(x \in D) \) by

\[ \nu_D(\Theta_h, x) := \lim_{m \to \infty} m^{-1} \text{mult}_x \text{Spec}(\mathcal{O}_D/\mathcal{I}_D(h^m)), \]

where \( \text{mult}_x \) denotes the multiplicity on \( D \). Then we see that the set

\[ S_D := \{ x \in D \mid \nu_D(\Theta_h, x) > 0 \} \]

is at most countable union of subvarieties on \( D \). This follows from the approximation theorem ([3], p.380,Proposition 3.7). Also this is obvious, if \( h \) is an AZD constructed as in Section 3.2.
3.7 Another definition of the intersection number

Let \((L, h)\) be a singular hermitian line bundle on a smooth projective variety \(X\) such that \(\Theta_h\) is positive. And let \(C\) be an irreducible curve on \(X\) such that the restriction \(h|_C\) is well defined. The another candidate of the intersection number of \((L, h)\) and \(C\) is:

\[
(L, h) \ast C := L \cdot C - \sum_{x \in C} \nu(\Theta_h|_C, x).
\]

But we have the following theorem.

**Theorem 3.5** ([22, Theorem 2.7])

\[
(L, h) \cdot C = (L, h) \ast C
\]

holds.

3.8 Limit multiplicities

Let \((L, h)\) be a singular hermitian line bundle on a smooth projective variety \(X\). Suppose that \(\Theta_h\) is strictly positive. In this subsection, we shall consider the behavior of

\[
\text{mult}_x Bs | m(L, h) |
\]

as \(m\) goes to infinity. We shall prove the following theorem.

**Theorem 3.6** Let \((L, h)\) be a singular hermitian line bundle on a smooth projective variety \(X\). Suppose that \(\Theta_h\) is strictly positive. Let \(x_0 \in X\) be a point such that \(\nu(\Theta_h, x_0) = 0\). Let \(c\) be a positive number such that

\[c < \mu(X, (L, h)).\]

Then for every \(x \in X\)

\[
\nu(x) := \lim_{m \to \infty} m^{-1} \text{mult}_x Bs | \ m(L, h) \otimes \mathcal{M}_{x_0}^{[cm]} |
\]

exists, where \(| m(L, h) \otimes \mathcal{M}_{x_0}^{[cm]} |\) denotes

\[| H^0(X, \mathcal{O}_X(mL) \otimes \mathcal{I}(h^m) \otimes \mathcal{M}_{x_0}^{[cm]} | .\]

Moreover for any modification

\[f : Y \to X\]

and \(y \in Y\),

\[
\nu(y) := \lim_{m \to \infty} m^{-1} \text{mult}_y f^* Bs | \ m(L, h) \otimes \mathcal{M}_{x_0}^{[cm]} |
\]

exists.
Proof of Theorem 3.6. For $x \in X$ we set

$$\tilde{\nu}(x) := \lim_{m \to \infty} m^{-1} \text{mult}_x Bs | m(L, h) \otimes M_{x_0}^{[cm]} | .$$

We claim that for any $\epsilon > 0$ and $x \in X$, there exists a positive integer $m(\epsilon)$ such that for every $m \geq m(\epsilon)$

$$\text{mult}_x Bs | m(L, h) \otimes M_{x_0}^{[cm]} | \geq (1 - \epsilon) \tilde{\nu}(x)m$$

holds.

Let $\delta$ be a small positive number such that

$$\mu(X, (L, h)) > c + \delta$$

and $c + \delta$ is a rational number. Let us fix $x \in X$. Let

$$\pi : \tilde{X} \to X$$

be the blowing up at $\{x, x_0\}$. We set

$$E := \pi^{-1}(x)$$

and

$$E_0 := \pi^{-1}(x_0).$$

We shall prove the following lemma.

**Lemma 3.2** There exists a singular hermitian metric $\tilde{h}_\delta$ on $\pi^*L$ such that

1. $\pi_*\mathcal{I}(h_0^m) \subseteq \mathcal{I}(h^m) \otimes M_{x_0}^{[(c+\frac{1}{2}\delta)m]}$ holds for every sufficiently large $m$,

2. $\Theta_{h_\delta}$ is strictly positive on $\tilde{X}$.

**Proof.** Let $H$ be a very ample divisor on $\tilde{X}$ and let $h_H$ be a $C^\infty$-hermitian metric on $H$ with strictly positive curvature. Let $\varepsilon_H$ be a sufficiently small positive rational number such that

$$\mu(\tilde{X}, (L - \varepsilon_H H, h \cdot h_H^{-\varepsilon_H}))) > 0$$

holds. For every sufficiently large $\ell$, let

$$\tilde{\sigma}_\ell \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\ell!(\pi^*L - \varepsilon_H H - (c + \delta)E_0) \otimes \pi^*\mathcal{I}(h^\ell)))$$

be a nontrivial section and set

$$\tilde{h}_\ell = \frac{1}{|\tilde{\sigma}_\ell|^{2/\ell!}}.$$
Since for every sufficiently large \( \ell \), \( \Theta_{h_{\ell}} \) is a closed positive current which represents \( 2\pi c_{1}(\pi^{*}L - \varepsilon_{H}H) \), we may assume that there exists a subsequence \( \{\Theta_{h_{\ell_{j}}}\} \) of \( \{\Theta_{h_{\ell}}\} \) such that

\[
\Theta_{\infty} := \lim_{j \to \infty} \Theta_{h_{\ell_{j}}}
\]

exists as a closed positive current. Let \( \tilde{h} \) be a singular hermitian metric on \( \pi^{*}L - \varepsilon_{H}H \) such that

\[
\Theta_{\tilde{h}} = \Theta_{\infty}
\]

holds. Then

\[
\tilde{h}_{\delta} := \tilde{h} \cdot h_{H}^{\varepsilon_{H}}
\]

is a singular hermitian metric on \( \pi^{*}L \) with strictly positive curvature current. By the construction we see that for every modification

\[
f : Y \longrightarrow X
\]

and \( y \in Y \),

\[
\nu(f^{*}\Theta_{h_{\delta}}, y) \geq \nu(f^{*}\Theta_{h}, y) + (c + \delta) \cdot \text{mult}_{y} f^{*}E_{0}
\]

holds. Hence by Lemma 2.1 and Lemma 2.2 we see that

\[
\pi_{*}\mathcal{I}(h_{\delta}^{m}) \subseteq \mathcal{I}(h_{m}^{m}) \otimes \mathcal{M}_{c_{0}}^{[\lfloor c + \frac{1}{2}\delta \rfloor m]}
\]

holds for every sufficiently large \( m \). This completes the proof of Lemma 3.2.

Q.E.D.

Suppose there exists a point \( x \in X \) such that for some \( \epsilon > 0 \), there exists an increasing sequence of positive integers \( \{m_{j}\} \) such that

\[
\text{mult}_{x} \text{Bs} \mid m_{j}(L, h) \otimes \mathcal{M}_{c_{0}}^{[c_{m_{j}}]} \mid < (1 - \epsilon) \tilde{\nu}(x)m_{j}
\]

holds. Let

\[
\sigma_{j} \in H^{0}(X, O_{X}(m_{j}L) \otimes \mathcal{I}(h_{m}^{m}) \otimes \mathcal{M}^{[c_{m_{j}}]})
\]

(here for a real number \( a \), \( \lfloor a \rfloor \) denotes the smallest integer larger or equal to \( a \)) be a nonzero element such that

\[
\text{mult}_{x}(\sigma_{j}) \leq (1 - \epsilon) \tilde{\nu}(x)m_{j}
\]

holds. We define the singular hermitian metric \( h_{j} \) of \( L \) by

\[
h_{j} := \frac{1}{|\sigma_{j}|^{2/m_{j}}}
\]
Let \( \tilde{x} \in \tilde{X} \) be a point on \( E \) such that for every \( m \), \( \mathcal{I}(\pi^*h^m) \) is locally free on a neighbourhood of \( \tilde{x} \) (the neighbourhood may depend on \( m \)).

Let \( U \) be a small neighbourhood of \( \tilde{x} \) and let \( \rho \) be a \( C^\infty \)-function on \( \tilde{X} \) such that

1. \( \text{Supp} \rho \subset \subset U \),
2. \( \rho \equiv 1 \) on a neighbourhood of \( \tilde{x} \),
3. \( 0 \leq \rho \leq 1 \)

hold. Let \( d_{\tilde{x}} \) denote the distance function from \( \tilde{x} \) with respect to a fixed Kähler form \( \omega \) on \( \tilde{X} \).

Let \( \nu_0 \) be a sufficiently large positive integer such that

\[
\nu_0 \Theta_{\tilde{h}} + \text{Ric}_\omega + 2n\sqrt{-1}\partial\bar{\partial}(\rho \log d_{\tilde{x}})
\]

is strictly positive and

\[
\pi_* \mathcal{I}(h_{\delta}^{\nu_0}) \subseteq \mathcal{I}(h^{\nu_0}) \otimes \mathcal{M}_{x_0}^{[c + \frac{1}{2}(d)\nu_0]}
\]

holds. Let \( h_0 \) be a \( C^\infty \)-hermitian metric on \( L \). Let \( \epsilon_0 \) be a sufficiently small positive number such that

\[
\nu_0 \Theta_{\tilde{h}} + \text{Ric}_\omega + 2n\sqrt{-1}\partial\bar{\partial}(\rho \log d_{\tilde{x}}) - \epsilon_0 \pi^* \Theta_{h_0}
\]

is strictly positive. Then by Nadel’s vanishing theorem (Theorem 2.1),

\[
H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*L) \otimes \mathcal{I}(\pi^*h_j^{m+\epsilon_0} \cdot \tilde{h}_x^{\nu_0} e^{-2\rho \log d_{\tilde{x}}})) = 0
\]

holds for every \( j \) and \( m \geq 0 \). This implies that there exists

\[
\tilde{\sigma} \in H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\pi^*(m + \nu_0)L) \otimes \mathcal{I}(\pi^*h_j^{m+\epsilon_0} \cdot \tilde{h}_x^{\nu_0}))
\]

which generates

\[
\mathcal{O}_{\tilde{X}}(\pi^*(m + \nu_0)L) \otimes \mathcal{I}(\pi^*h_j^{m+\epsilon_0} \cdot \tilde{h}_x^{\nu_0})
\]

at \( \tilde{x} \). Let us fix an arbitrary positive integer \( m \). We note that for every sufficiently large \( j \),

\[
\pi_*(\mathcal{I}(\pi^*h_j^{m+\epsilon_0} \cdot \tilde{h}_x^{\nu_0})) \subseteq \mathcal{I}(h^{m+\nu_0}) \otimes \mathcal{M}_{x_0}^{[c + \nu_0]}
\]

holds by the construction of \( \{h_j\} \). Hence we see that

\[
\text{mult}_x B_s | (m + \nu_0)(L, h) \otimes \mathcal{M}_{x_0}^{[cm]} | < (1 - \epsilon)(m + \nu_0)\tilde{\nu}(x)
\]
holds. Since $m$ is an arbitrary positive integer, this is the contradiction. Hence we conclude that for every $\epsilon > 0$, there exists a positive integer $\nu_0$ such that for every $m \geq \nu_0$

$$\text{mult}_x B_s |_m (L, h) \otimes \mathcal{M}_{x_0}^{[cm]} | \geq (1 - \epsilon)m \tilde{\nu}(x)$$

holds. This implies that

$$\lim_{m \to \infty} m^{-1} \text{mult}_x B_s |_m (L, h) \otimes \mathcal{M}_{x_0}^{[cm]} | = \tilde{\nu}(x)$$

holds. By the definition of $\tilde{\nu}(x)$ we see that

$$\nu(x) := \lim_{m \to \infty} m^{-1} \text{mult}_x B_s |_m (L, h) \otimes \mathcal{M}_{x_0}^{[cm]} |$$

exists. Since $x$ is arbitrary, this completes the proof of Theorem 3.6 except the last statement. The proof of the last statement is similar. Q.E.D.

4 Fibration theorem

4.1 The first nonvanishing theorem

Let $X$ be a smooth projective variety. Let $D$ be a divisor on $X$ and let $A = \sum a_i A_i$ be a $\mathbb{Q}$-divisor on $X$. Assume the following conditions:

1. $D$ is nef,
2. $[A] := \sum_i [a_i] A_i$ is effective,
3. $\text{Supp}\{A\}$ is a divisor with normal crossings, where $\{A\}$ denotes the fractional part of $A$, i.e., $\{A\} := A - [A]$.
4. there exists a positive integer $a$ such that $aD + A - K_X$ is nef and big.

In [16], Shokurov proved that under these conditions, for every sufficiently large positive integer $b$,

$$H^0(X, \mathcal{O}_X(bD + [A])) \neq 0$$

holds. In this section, we shall prove a similar nonvanishing theorem which plays essential roles in this paper.

Theorem 4.1 (The first nonvanishing theorem) Let $X$ be a smooth projective variety and let $(L, h_L)$ be a singular hermitian line bundle on $X$ such that the curvature current $\Theta_L$ is positive. Let $(A, h_A)$ be a singular hermitian line bundle on $X$ with strictly positive curvature current $\Theta_A$. Then one of the followings holds.
1. $H^0(X, \mathcal{O}_X(K_X + A + mL) \otimes \mathcal{I}(h_A h_L^m)) \neq 0$ holds for every sufficiently large $m$,

2. there exists a nontrivial numerically trivial fiber space structure

$$f : X \to Y,$$

i.e.,

(a) $f$ is regular over the generic point of $Y$,

(b) for a very general fiber $F$ the restriction $(L, h_L)|_F$ is numerically trivial,

(c) for a very general point $x \in X$ and every irreducible horizontal curve $C$ containing $x$, $(L, h) \cdot C > 0$ holds,

(d) $\dim Y < \dim X$ is minimal among such fibrations.

Let us compare Theorem 4.1 with Shokurov’s nonvanishing theorem above. The positivity of $\Theta_L$ in Theorem 4.1 corresponds to the nefness of $D$ in Shokurov’s theorem. The strict positivity of $(A, h_A)$ corresponds to the third condition in Shokurov’s theorem. But the second condition in Shokurov’s theorem does not have a counterpart in Theorem 4.1. That is why we have two cases. Roughly speaking Theorem 4.1 tells us what happens, if we drop the second condition in Shokurov’s theorem. To construct a nontrivial holomorphic section of $\mathcal{O}_X(K_X + A + mL)$ on $X$ in the second case, we need to construct a section on a general fiber $F$ of the numerically trivial fibration $f : X \to Y$. This problem will be treated in the second nonvanishing theorem (Theorem 4.4) later.

The following (more algebraic) corollary follows from the proof of Theorem 4.1 (Corollary 4.1 is a corollary of the proof of Theorem 4.1. See Remark 4.1 below.).

**Corollary 4.1 ([20, Corollary 8.1])** Let $X$ be a smooth projective variety and let $L$ be a nef line bundle on $X$. Let $A$ be a big line bundle on $X$. Then one of the followings holds.

1. 

$$H^0(X, \mathcal{O}_X(K_X + A + mL)) \neq 0$$

holds for every sufficiently large $m$,

2. there exists a rational fibration

$$f : X \to Y$$

such that
(a) \( f \) is regular over the generic point of \( Y \),
(b) for a very general fiber \( F \) the restriction \( L|_F \) is numerically trivial,
(c) for every general point \( x \) on \( X \) and every irreducible horizontal
(with respect to \( f \)) curve \( C \) containing \( x \), \( L \cdot C > 0 \) holds,
(d) \( \dim Y < \dim X \) is minimal among such fibrations.

**Remark 4.1** In Corollary 4.1 \( L \) may not admit a singular hermitian metric \( h \) such that \( \Theta_h \) is positive and \( \mathcal{I}(h^m) = \mathcal{O}_X \) for every \( m \geq 0 \) on \( X \). But the proof is parallel to that of Theorem 4.1, if we change the volume \( \mu(X, (A + mL,h_A h_L^m)) \) of a subvariety \( V \) in \( X \) with respect to \( A + mL \) (see Lemma 4.1 below) by the intersection number \( (A + mL)^{\dim V} \cdot V \).

**Example 2** To illustrate our method let us consider the following example. Let \( X \) be an irreducible quotient of the open unit bidisk \( \Delta^2 \) in \( \mathbb{C}^2 \), i.e.
\[
X = \Delta^2 / \Gamma,
\]
where \( \Gamma \) is an irreducible cocompact torsion free lattice. Let \( L \) denotes the line bundle such that whose curvature form comes from the Poincaré metric on the first factor. Then one see that \( L \) is nef and \( L^2 = 0 \) holds. In particular \( L \) is not big. Let \( c_1(L) \) be the first Chern form of \( L \) induced by the Poincaré metric on the first factor. On the other hand for every ample line bundle \( A \), \( K_X + mL + A \) is very ample for \( m >> 1 \). Moreover for every singular hermitian line bundle \( (A,h_A) \) with strictly positive curvature (in the sense of current),
\[
H^0(X, \mathcal{O}_X(K_X + mL + A) \otimes \mathcal{I}(h_A))
\]
gives a birational rational map from \( X \) into a projective space and even it separates jets of any fixed order \( k \) at very general points on \( X \) for every sufficiently large \( m \) (of course such \( m \) depends on \( k \)). In a sense \( L \) behaves more or less like an ample line bundle.

### 4.2 Numerically trivial fibrations

The following theorems are key ingredients for our proof of Theorem 4.1 and Theorem 1.1.

**Theorem 4.2** ([23, Theorem 1.1]) Let \( (L, h) \) be a singular hermitian line bundle on a smooth projective variety \( X \). Suppose that the curvature current \( \Theta_h \) is positive. Then there exists a unique (up to birational equivalence) rational fibration
\[
f : X \to Y
\]
such that
1. $f$ is regular over the generic point of $Y$,

2. for every very general fiber $F$, $(L, h) |_F$ is well defined and is numerically trivial (cf. Definition 3.9),

3. $\dim Y$ is minimal among such fibrations,

4. for a very general point $x \in X$ and any irreducible horizontal curve (with respect to $f$) $C$ containing $x$, $(L, h) \cdot C > 0$ holds.

We call the above fibration the **numerically trivial fibration** associated with $(L, h)$.

**Remark 4.2** Let $X, (L, h)$ be as above. Then for any smooth divisor $D$ on $X$, there exists a numerically trivial fibration

$$f_D : D - \cdots \to W.$$

This is simply because the restriction of the intersection theory on $D$ exists and the proof of the above theorem essentially does not require the existence of the restriction of $\Theta_h$ on $D$.

The structure of numerically trivial singular hermitian line bundles with positive curvature current is given as follows.

**Theorem 4.3** ([22, Theorem 1.2]) Let $(L, h)$ be a singular hermitian line bundle on a smooth projective variety $X$. Suppose that $(L, h)$ is numerically trivial on $X$. Then there exist at most countably many prime divisors $\{D_i\}$ and nonnegative numbers $\{a_i\}$ such that

$$\Theta_h = 2\pi \sum_i a_i D_i$$

holds. More generally let $Y$ be a subvariety of $X$ such that the restriction $h |_Y$ is well defined. Suppose that $(L, h)$ is numerically trivial on $Y$. Then the restriction $\Theta_h |_Y$ is a sum of at most countably many prime divisors with nonnegative coefficients on $Y$.

Theorem 4.3 gives an information on the restriction of the singular hermitian metric on a very general fiber of a numerically trivial fibration.

**Corollary 4.2** ([22, Corollary 3.2]) Let $X$ be a smooth projective variety and let $(L, h)$ be a singular hermitian line bundle on $X$ such that $\Theta_h$ is positive. Let $D$ be a smooth divisor on $X$. Suppose that $(L, h)$ is numerically trivial on $D$. Then

$$S_D := \{x \in D \mid \nu_D(\Theta_h, x) > 0\}$$
is a sum of at most countably many prime divisors on \( D \), where \( \nu_D(\Theta_h, x) \) is the Lelong number defined as in Section 3.6. Also

\[
(L - \nu(\Theta_h, D) \cdot D)_D - \sum_E \nu_D(\Theta_h, E) \cdot E
\]

is numerically trivial on \( D \), where \( E \) runs all the prime divisors on \( D \).

**Remark 4.3** Corollary 4.2 still holds for a subvariety \( V \) on \( D \), if there exists a curve on \( V \) such that \((L, h) \cdot C \) is well defined (cf. [22, Remark 3.1]).

### 4.3 Proof of Theorem 4.1

Let \( X, (L, h_L), (A, h_A) \) be as in Theorem 4.1. Let

\[ f : X \to Y \]

be the numerically fibration associated with \((L, h_L)\). If \( \dim Y < \dim X \) holds, then this is the desired fibration. Hence we shall assume that \( f \) is the identity morphism. In other words, for a very general point \( x \) on \( X \) and any irreducible curve \( C \) containing \( x \), \( h_L |_C \) is well defined and

\[(L, h_L) \cdot C > 0\]

holds. We say that \((L, h_L)\) is **very generically numerically positive**.

**Lemma 4.1** ([23, Lemma 4.1]) Suppose that \((L, h_L)\) is not numerically trivial. Then for every ample line bundle \( H \) on \( X \)

\[ \lim_{m \to \infty} m^{-1} \mu(X, (H + mL, h_H h_L^m)) > 0 \]

holds, where \( h_H \) is any \( C^\infty \) hermitian metric with strictly positive curvature on \( H \).

**Proof.** Let \( n \) be the dimension of \( X \). We prove this lemma by induction on \( n \). If \( n = 1 \), Lemma 4.1 is trivial. Let \( \pi : \tilde{X} \to \mathbb{P}^1 \) be a Lefschetz pencil associated with a very ample linear system say \( H \) on \( X \). If we take the pencil very general, we may assume that \( \mathcal{I}(h_L^\ell) \) is an ideal sheaf on all fibers of \( \pi \) for every \( \ell \geq 1 \). And let

\[ b : \tilde{X} \to X \]

be the modification associated with the pencil and let \( E \) be the exceptional divisor of \( b \). Then by the inductive assumption for a very general fiber \( F \) of \( \pi \), we see that

\[ \lim_{m \to \infty} m^{-1} \mu(F, b^*(H + mL, h_H h_L^m)) > 0 \]
holds. Let us consider the direct image
\[ E_{m, \ell} := \pi_* \mathcal{O}_{\tilde{X}}(\ell b^*(H + mL)) \otimes \mathcal{I}(b^*h_{L}^{m\ell}) \].

By Grothendieck’s theorem, we see that
\[ E_{m, \ell} \simeq \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i) \]
for some \( a_i = a_i(m, \ell) \) and \( r = r(m, \ell) \). By the inductive assumption, we see that
\[ \lim_{m \to \infty} m^{-1} (\lim_{\ell \to \infty} \ell^{-1} r(m, \ell)) > 0 \]
holds. We note that \( \ell_0 b^*H - E \) is ample for some positive integer \( \ell_0 \). Hence we see that
\[ \mathcal{O}_{\tilde{X}}(\ell_0 b^*H - E) \]
admits a \( C^\infty \)-hemitian metric with strictly positive curvature. Hence by Nadel’s vanishing theorem [14, p.561] there is a positive constant \( c \) such that
\[ H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}(\ell b^*(H + mL - \frac{1}{\ell_0}E)) \otimes \mathcal{I}(b^*h_{L}^{m\ell}) \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(-\lceil c \ell \rceil)) = 0 \]
holds for every sufficiently large \( \ell \) divisible by \( \ell_0 \). This implies that
\[ \lim_{\ell \to \infty} \ell^{-1} (\min_i a_i) \geq c \]
holds and
\[ \lim_{\ell \to \infty} \ell^{-n} \dim H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\ell p^*(H + mL)) \otimes \mathcal{I}(p^*(h_{L}^{m\ell}))) \geq c \cdot \lim_{\ell \to \infty} \ell^{-(n-1)} r(m, \ell) \]
holds. Hence we see that
\[ \lim_{m \to \infty} m^{-1} (\lim_{\ell \to \infty} \ell^{-n} \dim H^0(\tilde{X}, \mathcal{O}_{\tilde{X}}(\ell b^*(H + mL)) \otimes \mathcal{I}(b^*h_{L}^{m\ell}))) > 0 \]
holds. Since
\[ b_* \mathcal{I}(b^*h_{L}^{m\ell}) \subseteq \mathcal{I}(h_{L}^{m\ell}) \]
holds by Lemma 2.1, we see that
\[ \lim_{m \to \infty} m^{-1} \mu(X, (H + mL, h_{H}^{m\ell})) > 0 \]
holds. Here we have assumed that \( H \) to be sufficiently very ample. To prove the general case of Lemma 4.1, we argue as follows. Let \( H \) be any ample line bundle on \( X \). Then
\[ \mu(X, (a(H + mL), h_{H}^{m\ell})) = a^n \cdot \mu(X, (H + mL, h_{H}^{m\ell})) \]
holds for every positive integer $a$. Now it is clear that Lemma 4.1 holds for any ample line bundle $H$. This completes the proof of Lemma 4.1. Q.E.D.

Let $H$ be a very ample line bundle on $X$ and let $h_H$ be a $C^\infty$-hermitian metric on $H$. Since the curvature $\Theta_A$ of $h_A$ is strictly positive, we see that for every sufficiently small positive number $\epsilon$.

$$\mu(X, (A - \epsilon H, h_A h_H^{-\epsilon})) > 0$$

holds by Theorem 2.1.

By the assumption $(L, h_L)$ is very generically numerically positive on $X$. Let $\nu_0$ be a positive integer and let us consider the singular hermitian line bundle $(\nu_0 L + A, h_L^{\nu_0} h_A)$ on $X$. By Lemma 4.1 we can take a sufficiently large $\nu_0$ so that

$$\mu(X, (\nu_0 L + A, h_L^{\nu_0} h_A)) > \epsilon^n \mu(X, (\nu_0 L + \epsilon H, h_L^{\nu_0} h_H^{-\epsilon})) > 2^n (n + 1)^{2n}$$

hold.

**Lemma 4.2** Let $x \in X$ be a very general point such that $\nu(\Theta_L, x) = \nu(\Theta_A, x) = 0$ hold.

Then for every sufficiently large positive integer $m$,

$$H^0(X, \mathcal{O}_X (m(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m) \otimes \mathcal{M}^{\otimes 2(n+1)^2m}) = 0$$

holds, where $\mathcal{M}_x$ denotes the maximal ideal sheaf at $x$.

**Proof.** Let us consider the following morphism

$$H^0(X, \mathcal{O}_X (m(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m) \otimes \mathcal{M}^{\otimes 2(n+1)^2m}))$$

The kernel of this morphism is exactly

$$H^0(X, \mathcal{O}_X (m(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m) \otimes \mathcal{M}^{\otimes 2(n+1)^2m}).$$

If we take $x$ very general we may assume that $\nu(\Theta_L, x) = 0$ holds. Hence $\mathcal{I}(h_L^m)_x = \mathcal{O}_{X,x}$ holds for every $m \geq 0$ by Lemma 2.2. Since

$$\dim H^0(X, \mathcal{O}_X (m(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m) \otimes \mathcal{M}^{\otimes 2(n+1)^2m}) = \frac{2^n (n + 1)^{2n}}{n!} m^n + O(m^{n-1})$$

and

$$\mu(X, (\nu_0 L + A, h_L^{\nu_0} h_A)) > 2^n (n + 1)^{2n}$$

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hold, for every sufficiently large $m$,
\[
\dim H^0(X, \mathcal{O}_X(m\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m)) > \\
\dim H^0(X, \mathcal{O}_X(m\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m)/\mathcal{M}_x^{\otimes 2(n+1)^2 m})
\]
holds. This completes the proof of Lemma 4.2. Q.E.D.

Let $x$ be a very general point of $X$ such that $\nu(\Theta_L, x) = \nu(\Theta_A, x) = 0$ hold and for every irreducible curve $C$ containing $x$ $h_L|_C$ is well defined and satisfies
\[
(L, h_L) \cdot C > 0.
\]
Let $\varepsilon$ be a sufficiently small positive number. Let $m_0$ be a sufficiently large positive number and let
\[
\sigma_0 \in H^0(X, \mathcal{O}_X(m_0(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^{m_0}) \otimes \mathcal{M}_x^{\otimes 2(n+1)^2 m_0})
\]
be a general nonzero section.
Let $h_0$ be the singular hermitian metric on $\nu_0 L + A$ defined by
\[
h_0 = \frac{1}{|\sigma_0|^{2/m_0}}.
\]
Let $\alpha_0$ be the positive number defined by
\[
\alpha_0 := \inf\{\alpha \mid (\mathcal{O}_X/\mathcal{I}(h_0^\alpha))_x \neq 0\}.
\]
We set
\[
V_1 := \lim_{\delta \downarrow 0} \text{Spec}(\mathcal{O}_X/\mathcal{I}(\alpha_0 + \delta)).
\]
And let $X_1$ be a branch of $V_1$ containing $x$. Then since $\sigma_0$ is an element of
\[
H^0(X, \mathcal{O}_X(m_0(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^{m_0}) \otimes \mathcal{M}_x^{\otimes 2(n+1)^2 m_0}),
\]
we see that
\[
\alpha_0 < \frac{1}{2n}
\]
holds. Let us take $m_0$ sufficiently large and $\sigma_0$ very general.

**Remark 4.4** By Theorem 3.6 for every $m'_0 \geq m_0$ and very general
\[
\sigma'_0 \in H^0(X, \mathcal{O}_X(m'_0(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^{m'_0}) \otimes \mathcal{M}_x^{\otimes 2(n+1)^2 m'_0})
\]
we see that
\[
X_1 \subseteq \text{Spec}(\mathcal{O}_X/\mathcal{I}((h_0')^{\alpha_0 + \delta_0}))
\]
holds, where $\delta_0$ is a positive number which tends to 0 as $m_0$ tends to infinity. Hence even if we move $m_0$, only finitely many subvarieties appears as $X_1$ as far as we take $\sigma_0$ very general.
We set 
\[ n_1 := \dim X_1. \]

We note that \( h_L |_{X_1} \) is well defined and \((L, h_L)\) is not numerically trivial on \( X_1 \) by the choice of \( x \). In this case by Lemma 4.1 we take a sufficiently large positive integer \( \nu_1 \) so that
\[ \mu(X_1, (\nu_1 L + A, h_L^{\nu_1} h_A)) > 2^{n_1} (n_1 + 1)^n n_1. \]

Then we have the following lemma.

**Lemma 4.3** Let \( x_1 \) be a very general point on \( X_{1, \mathrm{reg}} \). Then for every sufficiently large positive integer \( m \),
\[ H^0(X_1, (\mathcal{O}_{X_1}(m(\nu_0 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_1})^m)))/\text{tor} \otimes \mathcal{M}_{x_1}^{\otimes 2(n+1)^2 m} \neq 0 \]
holds.

**Proof.** Let \( x_1 \) be a very general point on \( X_{1, \mathrm{reg}} \) such that \( \nu(\Theta_{h_A h_L^{\nu_1}}, x_1) = 0 \) holds. Then for every \( m \), \( \mathcal{I}((h_A h_L^{\nu_1})^m)_{x_1} = \mathcal{O}_{X,x_1} \) holds. Then the proof of Lemma 4.3 is parallel to that of Lemma 4.2. Q.E.D.

Let \( E \) be an effective \( \mathbb{Q} \)-divisor such that \( A + \nu_1 L - E \) is ample (such a divisor exists by Kodaira’s lemma [11, Appendix]). We set
\[ H_1 = r(A + \nu_1 L - E), \]
where \( r \) is a positive integer such that \( H \) is an integral divisor on \( X \). Then by Nadel’s vanishing theorem, we have the following lemma.

**Lemma 4.4** If we take \( r \) sufficiently large, then
\[ \phi_m : H^0(X, \mathcal{O}_X(m(\nu_1 L + A) + H) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m)) \rightarrow H^0(X_1, \mathcal{O}_{X_1}(m(\nu_1 L + A) + H_1) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m)) \]
is surjective for every \( m \geq 0 \).

**Proof.** Let us take a locally free resolution of the ideal sheaf \( \mathcal{I}_{X_1} \) of \( X_1 \).
\[ 0 \leftarrow \mathcal{I}_{X_1} \leftarrow \mathcal{E}_1 \leftarrow \mathcal{E}_2 \leftarrow \cdots \leftarrow \mathcal{E}_\ell \leftarrow 0. \]

Then by the trivial extension of Nadel’s vanishing theorem to the case of vector bundles, if \( r \) is sufficiently large, we have:
Sublemma 4.1

\[ H^q(X, \mathcal{O}_X (m(\nu L + A) + H_1) \otimes \mathcal{I}((h_A h_L^{m(\nu)})^m) \otimes \mathcal{E}_j) = 0 \]

holds for every \( m \geq 1, q \geq 1 \) and \( 1 \leq j \leq k \).

In fact if we take \( r \) sufficiently large, we see that for every \( j \), \( \mathcal{O}_X (H - K_X) \otimes \mathcal{E}_j \) admits a \( C^\infty \)-hermitian metric \( g_j \) such that

\[ \Theta_{g_j} \geq \text{Id}_{\mathcal{E}_j} \otimes \omega \]

holds, where \( \omega \) is a Kähler form on \( X \). By [3, Theorem 4.1.2 and Lemma 4.2.2], we completes the proof of Sublemma 4.1. Q.E.D.

Let

\[ p_m : Y_m \rightarrow X \]

be a composition of successive blowing ups with smooth centers such that \( p_m^* \mathcal{I}((h_A h_L^{m(\nu)})^m) \) is locally free on \( Y_m \).

Sublemma 4.2

\[ R^p p_m^* p_m^* (\mathcal{O}_{Y_m} (K_{Y_m}) \otimes \mathcal{I}(p_m^* (h_A h_L^{m(\nu)}))) = 0 \]

holds for every \( p \geq 1 \) and \( m \geq 1 \).

Proof. This sublemma follows from Theorem 2.1. Q.E.D.

We note that by the definition of the multiplier ideal sheaves

\[ p_m^* (\mathcal{O}_{Y_m} (K_{Y_m}) \otimes \mathcal{I}(p_m^* (h_A h_L^{m(\nu)}))) = \mathcal{O}_X (K_X) \otimes \mathcal{I}((h_A h_L^{m(\nu)})^m) \]

holds. Hence by Sublemma 4.1, Sublemma 4.2 and the Leray spectral sequence, we see that

\[ H^q(Y_m, \mathcal{O}_{Y_m} (K_{Y_m} + p_m^* (m(\nu L + A) + H_1 - K_X)) \otimes \mathcal{I}(p_m^* (h_A h_L^{m(\nu)}))^m) \otimes p_m^* \mathcal{E}_j) = 0 \]

holds for every \( q \geq 1 \) and \( m \geq 1 \). Hence

\[ H^1(Y_m, \mathcal{O}_{Y_m} (K_{Y_m} + p_m^* (m(\nu L + A) + H_1 - K_X)) \otimes p_m^* \mathcal{I}((h_A h_L^{m(\nu)})^m)) \otimes p_m^* \mathcal{I}_X(1) = 0 \]

holds. Hence every element of

\[ H^0(Y_m, \mathcal{O}_{Y_m} (K_{Y_m} + p_m^* (m(\nu L + A) + H_1 - K_X)) \otimes \mathcal{I}(p_m^* (h_A h_L^{m(\nu)}))^m)) \otimes \mathcal{O}_{Y_m}/p_m^* \mathcal{I}_X(1) \]

extends to an element of

\[ H^0(Y_m, \mathcal{O}_{Y_m} (K_{Y_m} + p_m^* (m(\nu L + A) + H_1 - K_X)) \otimes \mathcal{I}(p_m^* (h_A h_L^{m(\nu)}))^m)) \]
Also there exists a natural map
\[ H^0(X_1, \mathcal{O}_{X_1}(m(\nu_1 L + A) + H_1)) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m) \rightarrow \]
\[ H^0(Y_m, \mathcal{O}_{Y_m}(K_{Y_m} + p_m^*(m(\nu_1 L + A) + H_1 - K_X)) \otimes \mathcal{I}(p_m^*(h_A h_L^{\nu_0})^m)) \otimes \mathcal{O}_{Y_m}/p_m^* \mathcal{I}_{X_1}). \]
Hence we can extend every element of
\[ p_m^* H^0(X_1, \mathcal{O}_{X_1}(m(\nu_1 L + A) + H_1)) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m)) \]
to an element of
\[ H^0(Y_m, \mathcal{O}_{Y_m}(K_{Y_m} + p_m^*(m(\nu_1 L + A) + H_1 - K_X)) \otimes \mathcal{I}(p_m^*(h_A h_L^{\nu_0})^m)) \]
Since
\[ H^0(Y_m, \mathcal{O}_{Y_m}(K_{Y_m} + p_m^*(m(\nu_1 L + A) + H_1 - K_X)) \otimes \mathcal{I}(p_m^*(h_A h_L^{\nu_0})^m))) \simeq \]
\[ H^0(X, \mathcal{O}_X(m(\nu_1 L + A) + H_1) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m)) \]
holds by the isomorphism
\[ p_m^*(\mathcal{O}_{Y_m}(K_{Y_m}) \otimes \mathcal{I}(p_m^*(h_A h_L^{\nu_0})^m))) = \mathcal{O}_X(K_X) \otimes \mathcal{I}((h_A h_L^{\nu_0})^m), \]
this completes the proof of Lemma 4.4. Q.E.D.

Let \( \tau_1 \) be a nonzero element of \( H^0(X, \mathcal{O}_X(H_1)) \). Let \( x_1 \) be a very general point on \( X_{1,reg} \) such that \( \nu(\Theta_{h_A h_L^{\nu_1}}, x_1) = 0 \) holds. Let \( m_1 \) be a sufficiently large positive integer and let
\[ \sigma'_1 \in H^0(X_1, \mathcal{O}_{X_1}(m_1(\nu_1 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^{m_1})/tor \otimes M_{x_1}^{\otimes 2(n+1)^2 m_1}) \]
be a nonzero element. We note that if \( X_1 \) is smooth (and if we take \( x \) very general),
\[ \mathcal{O}_{X_1}(m_1(\nu_1 L + A) \otimes \mathcal{I}((h_A h_L^{\nu_0})^{m_1}) \]
is torsion free, since it is a subsheaf of a locally free sheaf on a smooth variety.
Let
\[ p : \tilde{X} \rightarrow X \]
be an embedded resolution and let \( X'_1 \) be the strict transform of \( X_1 \). We may consider \( \sigma'_1 \) as an element of
\[ H^0(X'_1, \mathcal{O}_{X'_1}(p^*(m_1(\nu_1 L + A)) \otimes p^*(\mathcal{I}((h_A h_L^{\nu_0})^{m_1}) \otimes M_{x_1}^{\otimes 2(n+1)^2 m_1})). \]
Hence \( \sigma'_1 \) can be lifted to an element of

\[
H^0(X_1, \mathcal{O}_{X_1}(m_1(\nu_1 L + A) \otimes \mathcal{I}((h_A h^m_L)^m_1) \otimes \mathcal{M}^{\otimes 2(n+1)^2}_x),
\]

if it vanishes on \((p^* X_1 - X'_1) \cap X'_1\). Such a nonzero element \( \sigma'_1 \) certainly exists, if \( m_1 \) is sufficiently large. Hence we may assume that \( \sigma'_1 \) is an element of

\[
H^0(X'_1, \mathcal{O}_{X'_1}(p^*(m_1(\nu_1 L + A)) \otimes p^*(\mathcal{I}((h_A h^m_L)^m_1) \otimes \mathcal{M}^{\otimes 2(n+1)^2}_x)).
\]

Let \( \sigma_1 \) be an extension of

\[
\sigma'_1 \otimes \tau'_1 \in H^0(X_1, \mathcal{O}_{X_1}(m_1(\nu_1 L + A) + H_1) \otimes \mathcal{I}((h_A h^m_L)^m_1))
\]
to \( X \). This extension is possible by Lemma 4.4. Then we set

\[
h_1 := \frac{1}{| \sigma_1 |^{2n+1}}.
\]

Then \( h_1 \) is a singular hermitian metric of \( A + \nu_1 L \) with positive curvature current.

Suppose that \( x \) is a regular point of \( X_1 \). In this case we shall take \( x_1 = x \). Let \( \varepsilon_0 \) be a sufficiently small positive number. We define a positive number \( \alpha_1 \) by

\[
\alpha_1 = \inf \{ \alpha > 0 \mid (\mathcal{O}_X/\mathcal{I}(h^0_0 - \varepsilon_0 \cdot h^0_1))_x \neq 0 \}.
\]

Let us recall the following lemma.

**Lemma 4.5** (\[21, p.12, Lemma 6\]) Let \( a, b \) be positive numbers. Then

\[
\int_0^1 \frac{r_2^{2n-1}}{(r_1^2 + r_2^2)^b} dr_2 = \int_0^{r_1^a - 2b} \frac{r_3^{2n-1}}{(1 + r_3^2)^b} dr_3
\]

holds, where

\[
r_3 = r_2/r_1^{1/a}.
\]

By Lemma 4.5 (if we take \( \varepsilon_0 \) sufficiently small), we see that

\[
\alpha_1 \leq \frac{1}{2n}
\]

holds.

Suppose that \( x \) is a singular point of \( X_1 \). In this case letting \( x_1 \) tend to \( x \), we define the singular hermitian metric \( h_1 \). To estimate \( \alpha_1 \), we use the following lemma (\[9\]).

**Lemma 4.6** Let \( \varphi \) be a plurisubharmonic function on \( \Delta^n \times \Delta \). Let \( \varphi_t (t \in \Delta) \) be the restriction of \( \varphi \) on \( \Delta^n \times \{ t \} \). Assume that \( e^{-\varphi_t} \) does not belong to \( L^1_{\text{loc}}(\Delta^n, \mathcal{O}) \) for every \( t \in \Delta^* \).

Then \( e^{-\varphi_0} \) is not locally integrable at \( O \in \Delta^n \).
Lemma 4.6 is an immediate consequence of the $L^2$-extension theorem ([13, p.200, Theorem]). By Lemma 4.6 we have the same estimate

$$\alpha_1 \leq \frac{1}{2n}$$

also in the case that $x$ is a singular point of $X_1$. We define

$$V_2 = \lim_{\delta \downarrow 0} \text{Spec}(\mathcal{O}_X/\mathcal{I}(h_0^{-\varepsilon_0} \cdot h_1^{\frac{\alpha_1}{1} + \delta}))$$

and let $X_2$ be a branch of $V_2$ containing $x$. By the choice of $x$, $h_L|_{X_2}$ is well defined and $(L, h_L)$ is not numerically trivial on $X_2$.

By the above argument, inductively we obtain the strictly decreasing sequence of subvarieties:

$$X = X_0 \supset X_1 \supset \cdots X_r \supset X_{r+1} = \{x\}$$

(the last subvariety $X_{r+1}$ is a point by the choice of $x$, i.e. by the numerical positivity of $(L, h_L)$ at $x$) and the positive numbers $\{\alpha_i\}_{i=0}^{r-1}$ depending on small positive numbers $\{\varepsilon_i\}_{i=0}^{r-1}$. Since

$$\sum_{i=0}^{r} \alpha_i \leq \frac{1}{2}$$

holds by the construction, we can define a singular hermitian metric $\tilde{h}_x$ on $mL + A$ for every $m > \sum_{i=0}^{r} \alpha_i \nu_i$ by

$$\tilde{h}_x = (\prod_{i=0}^{r-1} h_i^{\alpha_i - \varepsilon_i} \cdot h_r^{\alpha_r + \varepsilon_r} \cdot h_A^{1 - (\sum_{i=0}^{r-1} (\alpha_i - \varepsilon_i)) - (\alpha_r + \varepsilon_r)}) \cdot h_L^{m - \sum_{i=0}^{r} \alpha_i \nu_i},$$

where $\varepsilon_0, \ldots, \varepsilon_r$ are sufficiently small positive numbers. Then the curvature $\Theta_{\tilde{h}_x}$ is a closed strictly positive $(1,1)$-current on $X$ since

$$\Theta_{\tilde{h}_x} = \sum_{i=0}^{r-1} (\alpha_i - \varepsilon_i) \Theta_{h_i} + (\alpha_r + \varepsilon_r) \Theta_{h_r} + (1 - (\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i)) - (\alpha_r + \varepsilon_r)) \Theta_{h_A} + (m - \sum_{i=0}^{r} \alpha_i \nu_i) \Theta_{h_L},$$

with

$$\Theta_{h_i}(0 \leq i \leq r), \Theta_L(= \Theta_{h_L}) \geq 0, \Theta_A(= \Theta_{h_A}) > 0,$$

$$1 - (\sum_{i=0}^{r-1}(\alpha_i - \varepsilon_i)) - (\alpha_r + \varepsilon_r) > 0, m - \sum_{i=0}^{r} \alpha_i \nu_i > 0$$

hold.

And moreover $\mathcal{I}(\tilde{h}_x)$ defines a subscheme of isolated support at $x$, if we have taken $x$ to be a very general point on $X$. 

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If 
\[ \mathcal{I}(\tilde{h}_x) \subseteq \mathcal{I}(h_A h_{L}^{m_i}) \]
holds, applying Nadel’s vanishing theorem (Theorem 2.1), we see that for every \( m > \sum \alpha_i \nu_i \) there exists a section 
\[ \sigma \in H^0(X, \mathcal{O}_X(K_X + A + mL) \otimes \mathcal{I}(h_A h_{L}^{m_i})) \]
such that \( \sigma(x) \neq 0 \) holds. But at this stage it is not clear that the above inclusion holds. The reason is that \( h_0, \ldots, h_r \) may have weaker singularities than \( h_A h_{L}^{\nu_0}, \ldots, h_A h_{L}^{\nu_r} \) at some points on \( X \) respectively, i.e. \( h_A h_{L}^{\nu_i}/h_i(i = 0, \ldots, r) \) may not be bounded on \( X \).

Suppose that 
\[ \mathcal{I}(\tilde{h}_x) \not\subseteq \mathcal{I}(h_A h_{L}^{m_i}) \]
holds. Let us fix \( m \) such that
\[ m > \sum_{i=0}^{r} \alpha_i \nu_i. \]
and 
\[ \rho_m : X^{(m)} \rightarrow X \]
be a modification such that \( \rho_m^* \mathcal{I}(h_{L}^{m_i} h_A) \) is locally free on \( X^{(m)} \). Let \( D_m = \sum_k a_{m,k} D_{m,k} \) be the integral divisor such that
\[ \mathcal{O}_{X^{(m)}}(-D_m) = \rho_m^* \mathcal{I}(h_{L}^{m_i} h_A). \]
As is seen in Remark 4.4, we have only finitely many choices of \( X_1, \ldots X_r \).
Hence we may assume that \( X_1, \ldots, X_r \) are independent of \( m_0, \ldots, m_{r-1} \).

**Sublemma 4.3** There exists a positive constant \( C \) such that
\[ \nu(\rho_m^* \Theta_{h_A h_{L}^{\nu_i}}, D_{m,k}) - \nu(\rho_m^* \Theta_{h_i}, D_{m,k}) \leq \frac{C}{m_i} \]
holds for every \( y \in X^{(m)}, 0 \leq i \leq r \) and \( k \).

**Proof.** Since 
\[ \sigma_i \in H^0(X, \mathcal{O}_X((m_i + r_i)(\nu_i L + A)) \otimes \mathcal{I}((h_A h_{L}^{\nu_i})^{m_i})) \]
holds, by Lemma 2.2, by the definition of \( h_i \) we see that
\[ \nu(\rho_m^* \Theta_{h_i}, D_{m,k}) \geq \frac{m_i}{m_i + r_i} \nu(\rho_m^* (\Theta_A + \nu_i \Theta_{L}), D_{m,k}) - \frac{1}{m_i + r_i} (a_k + 1) \]
holds, where \( a_k \geq 0 \) is the coefficient of \( D_{m,k} \) in the discrepancy \( K_{X^{(m)}} - \rho_{m}^* K_X \), i.e.

\[
K_{X^{(m)}} - \rho_{m}^* K_X = \sum_k a_k D_{m,k}
\]

holds. Hence there exists a positive constant \( C \) such that

\[
\nu(\rho_{m}^* \Theta_{h_A h_{L_i}^{m}}, D_{m,k}) - \nu(\rho_{m}^* \Theta_{h_i}, D_{m,k}) \leq \frac{C}{m_i}
\]

holds for every \( y \in X^{(m)} \), \( 0 \leq i \leq r \) and \( k \). Q.E.D.

Sublemma 4.3 means that if we take \( m_i \) very large, then \( h_A h_{L_i}^{m_i} / h_i \) has very small singularities on \( X \), even if it is not bounded on \( X \).

To assure the inclusion

\[
I(h_x) \subseteq I(h_A h_{L_i}^{m_i})
\]

we modify the argument as follows. Let us fix \( m > \sum_{i=0}^r \alpha_i \nu_i \). We set

\[
S(= S_m) := \text{Spec}(\mathcal{O}_X/I(h_{L_i}^{m_i} h_A))_{\text{red}}.
\]

Let \( \varphi \) be an almost plurisubharmonic function which is expressed locally:

\[
\varphi = \log \sum_j |f_j|^2 + C^\infty\text{-function},
\]

where \( \{f_j\} \) is a finite set of local generators of the ideal of \( S \). We set

\[
h_x := \tilde{h}_x \cdot e^{-\delta \varphi},
\]

where \( \delta \) is a small positive number so that \( \Theta_{h_x} \) is strictly positive.

By this modification and Sublemma 4.3, we see that

\[
I(h_x) \subseteq I(h_{L_i}^{m_i} h_A)
\]

holds, if \( m_0, \ldots, m_r \) are sufficiently large.

In fact this can be verified as follows. If we take \( \delta \) so that

\[
\delta > > \sum_{i=0}^r \frac{C}{m_i}
\]

holds (this does not violate the fact that we need to take \( \delta \) sufficiently small, because \( \{m_i\}'s \) can be arbitrary large),

\[
\nu(\rho_{m}^* \Theta_{h_x}, D_{m,i}) > \nu(\rho_{m}^* (\Theta_A + m\Theta_L), D_{m,i})
\]

holds for every \( i \), if we take \( m_0, \ldots, m_r \) sufficiently large.
Then by Lemma 2.2, we have the inclusion:
\[ \mathcal{I}(h_x) \hookrightarrow \mathcal{I}(h^m_L h_A). \]

Then by Nadel’s vanishing theorem (Theorem 2.1) we see that
\[ H^0(X, \mathcal{O}_X(K_X + mL + A) \otimes \mathcal{I}(h^m_L h_A)) \neq 0 \]
for every \( m > \sum_{i=0}^r \alpha_i \nu_i \). This completes the proof of Theorem 4.1. Q.E.D.

**Remark 4.5** By modifying the above proof in the first case of Theorem 4.1, it is not hard to show that
\[ H^0(X, \mathcal{O}_X(K_X + A + mL) \otimes \mathcal{I}(h_A h^m_L)) \]
gives a birational rational map from \( X \) into a projective space for every sufficiently large \( m \).

### 4.4 The second nonvanishing theorem

In this subsection we shall consider the existence of sections of a numerically trivial singular hermitian line bundles twisted by some line bundle.

Let \((L, h)\) be a singular hermitian line bundle on a smooth projective variety \( Y \) such that \( \Theta_h \) is closed positive on \( Y \). Let \( \sum_{i=1}^r Z_i \) be a divisor with normal crossings on \( Y \). Let \( X \) be a smooth subvariety defined by
\[ X = Z_1 \cap \cdots \cap Z_r. \]

We say such a subvariety \( X \) a **transverse complete intersection** in \( Y \).

Suppose that \((L, h)\) is **numerically trivial** on \( Y \). Then by Theorem 4.3, we see that there exists at most countably many prime divisors \( \{F_k\} \) and nonnegative real numbers \( \{a_k\} \) such that
\[ \Theta_h = 2\pi \sum_k a_k F_k \]
holds. Let \( \xi_k \) be a nonzero global section of \( \mathcal{O}_Y(F_k) \) with divisor \( F_k \). Then we see that there exists a positive constant \( C \) such that
\[ h = C \cdot \prod_k \frac{1}{|\xi_k|^{2a_k}} \]
holds.

We shall assume that \( \{Z_1, \ldots Z_r\} \) contains all the divisorial components of \( \{y \in Y \mid \nu(\Theta_h, y) > 0\} \) containing \( X \). In this case we say that \( X \) is...
a transverse complete intersection with respect to \((L, h)\). Then for every \(m \geq 1\), we see that

\[
\mathcal{O}_X(mL) \otimes \mathcal{I}(h^m) = \mathcal{O}_X(mL - \sum_{i=1}^{r}[m \cdot \nu(\Theta_h, Z_i)]Z_i) \otimes \tilde{\mathcal{I}}_X(h^m),
\]

holds for some ideal sheaf \(\tilde{\mathcal{I}}_X(h^m)\) on \(X\), since the left-hand side is a subsheaf of the locally free sheaf \(\mathcal{O}_X(mL - \sum_{i=1}^{r}[m \cdot \nu(\Theta_h, Z_i)]Z_i)\) on the smooth variety \(X\). We define the Lelong number \(\nu_X(\Theta_h, x)(x \in X)\) by

\[
\nu_X(\Theta_h, x) := \lim_{m \to \infty} m^{-1} \text{mult}_x \text{Spec}(\mathcal{O}_X/\tilde{\mathcal{I}}_X(h^m)).
\]

And define

\[
\mathcal{I}_X(h^m) := \bigcap_{m \geq 1} m^{1/2} \tilde{\mathcal{I}}_X(h^m),
\]

where \(m^{1/2} \tilde{\mathcal{I}}_X(h^m)\) is defined by

\[
\sqrt{m} \tilde{\mathcal{I}}_X(h^m)_x := \cup \frac{I}{m}(\sigma)_x \quad (x \in X),
\]

where \(\sigma\) runs all the germs of \(\tilde{\mathcal{I}}_X(h^m)_x\).

Then by successive use of Corollary 4.2 we see that

\[
S := \{x \in X \mid \nu_X(\Theta_h, x) > 0\}
\]

consists of countably many prime divisors on \(X\). We set

\[
S = \sum_j D_j,
\]

\[
d_j := \nu_X(\Theta_h, D_j)
\]

and

\[
D := \sum_j d_j D_j.
\]

Then since \(\Theta_h\) is a sum of at most countably prime divisors with nonnegative real coefficients by Theorem 4.3, we see that

\[
\mathcal{I}_X(h^m) = \mathcal{I}(mD)
\]

holds for every \(m \geq 0\), since \(X\) is a transverse complete intersection with respect to \((L, h)\).

The following theorem is as important as Theorem 4.1.
Theorem 4.4 (The second nonvanishing theorem) Let $Y$ be a smooth projective variety and let $(L, h)$ be a numerically trivial singular hermitian line bundle on $Y$. Let $X = Z_1 \cap \cdots \cap Z_r$ be a transverse complete intersection subvariety with respect to $(L, h)$. We set

$$\nu_i := \nu(\Theta_h, Z_i)$$

and let $\zeta_i$ be nonzero global section in $\Gamma(Y, \mathcal{O}_Y(Z_i))$ with divisor $Z_i$ for $1 \leq i \leq r$. For $1 \leq i \leq r$ let $h_{Z_i}$ be a $C^\infty$-hermitian metric on $\mathcal{O}_Y(Z_i)$ respectively. Let $\omega$ be a $C^\infty$-Kähler form on $X$. Let $A = \sum_k a_k A_k$ be a $\mathbb{R}$-divisor on $X$ and let $\tau_k$ denotes a nonzero global section of $\mathcal{O}_X(A_k)$ with divisor $A_k$ for every $k$. Suppose that the following conditions are satisfied.

1. $\operatorname{Supp} A = \sum_k A_k$ is a divisor with normal crossings,
2. $\lceil A \rceil$ is effective,
3. there exists a $C^\infty$-hermitian metric $h_{A-K_X}$ with strictly positive curvature on the $\mathbb{R}$-line bundle $A-K_X$ such that there exists a positive number $\delta$ such that for every $m \geq 1$ satisfying

$$\\{m\nu_i\} \leq \delta$$

the curvature current of the singular hermitian metric

$$h_m := h^m \cdot h_{A-K_X} \cdot \left(\prod_{i=1}^r h_{Z_i}^{\{m\nu_i\}} \cdot |\zeta_i|^{2m\nu_i} \cdot \prod_k |\tau_k|^{-2([a_i]-a_i)}\right)$$

on $\mathcal{O}_X([A] - K_X + mL - \sum_i [m\nu_i] Z_i)$, satisfies the inequality

$$\Theta_{h_m} > c \cdot \omega,$$

where $c$ is a positive number independent of such $m$.

Then there are finitely many positive numbers $t_0$, $\alpha_1, \ldots, \alpha_\ell$, $\beta_1, \ldots, \beta_p$ and a small positive number $\varepsilon$ such that for every $m \geq t_0$ satisfying the inequalities

$$|\langle m\alpha_s \rangle - m\alpha_s | < \varepsilon \quad (1 \leq s \leq \ell),$$

$$0 < |m\beta_q - m\beta_q | < \varepsilon \quad (1 \leq q \leq p)$$

and $\{m\nu_i\} \leq \delta$ (1 $\leq i \leq r$),

$$H^0(X, \mathcal{O}_X([A] + mL) \otimes \mathcal{I}_X(h_m)) \neq 0$$

holds. In other words for such $m \geq t_0$

$$H^0(X, \mathcal{O}_X([A] + mL) \otimes \mathcal{I}_X((\prod_{i=1}^r h_{Z_i}^{\{m\nu_i\}} \cdot |\zeta_i|^{2m\nu_i} \cdot h_{[A]} \cdot h^m)) \neq 0$$
holds, where $h_{[A]}$ denotes the singular hermitian metric on $\mathcal{O}_X([A])$ defined by
\[
h_{[A]} = \prod_k h_k^{a_i} | \tau_k |^{-2([a_i]-\alpha_i)}.
\]
Moreover the set of such $m$ is nonempty and infinite.

**Proof of Theorem 4.4.** Let $n$ denote $\dim X$. We prove this theorem by induction on $n$. If $n = 1$, by the assumption for every $m \geq 1$ such that
\[
\{m \nu_i\} < \delta \quad (1 \leq i \leq r)
\]
hold,
\[
de\dim X \mathcal{O}_X([A] + mL) \otimes \mathcal{I}(h_m) \geq \dim X \mathcal{K}_X
\]
holds (such a positive integer $m$ certainly exists by Lemma 4.7 below). Since $\dim X = 1$, $\mathcal{O}_X([A] + mL) \otimes \mathcal{I}(h_m)$ is an invertible sheaf on $X$. Hence for such $m$
\[
H^0(X, \mathcal{O}_X([A] + mL) \otimes \mathcal{I}(h_m)) \neq 0
\]
holds by the Kodaira vanishing theorem and the Riemann-Roch theorem.

Suppose that the theorem holds for every $X$ with $\dim X < n$. Let us consider the case that $\dim X = n$. We set
\[
S := \{x \in X \mid \nu_X(\Theta_h, x) > 0\}.
\]
Then $S$ consists of at most countably many prime divisors on $X$. Let
\[
S := \sum_{j \in J} D_j
\]
be the irreducible decomposition. We set
\[
d_j := \nu_X(\Theta_h, D_j).
\]
We shall consider the following two cases.

**Case 1:** $|J| = \infty$,
**Case 2:** $|J| < \infty$.

First let us consider Case 1. Let $H$ be a very ample smooth divisor on $X$. We may assume that $2\pi \omega$ is a 1-st Chern form of $\mathcal{O}_X(H)$. Now we have the following sublemma.

**Sublemma 4.4** There exists a positive number $\delta_H$ such that for every effective $\mathbb{R}$-divisor $E$ on $X$ such that
\[
H^{n-1} \cdot E < \delta_H
\]
$H - E$ is ample.
Proof of Sublemma 4.4. Since for every positive number $a$

$$\{c_1(E) \in H^2(X, \mathbb{R}) \mid E: \text{effective } \mathbb{R}\text{-divisor}, H^{n-1} \cdot E < a \}$$

is relatively compact in $H^2(X, \mathbb{R})$, This follows from Kleinman’s criterion for ampleness. Q.E.D.

Then there exists some $D_j$, say $D_0$ such that

1. $D_0 \not\subset \text{Supp } A$,
2. $d_0 \cdot H^{n-1} \cdot D_0 << \frac{1}{c} \cdot \delta_H$

hold. Since $H^{n-1} \cdot (\sum_{j \in J} d_j D_j)$ is finite and $\# J$ is infinite, there exists such $D_0$. We may assume that $D_0$ is a smooth divisor. In fact let

$$\pi_Y: \tilde{Y} \rightarrow Y$$

be an embedded resolution of $D_0$ obtained by successive blow ups with smooth centers. Let $\tilde{D}_0$ denote the strict transform of $D_0$ in $\tilde{X}$. Let $\tilde{X}$ be the strict transform of $X$ in $\tilde{Y}$ and let

$$\pi: \tilde{X} \rightarrow X$$

be the restriction of $\pi_Y$ to $\tilde{X}$. Let $E$ be the effective divisor defined by

$$E := K_{\tilde{X}} - \pi^* K_X.$$}

We define the divisor $\tilde{A}$ on $\tilde{X}$ by

$$\tilde{A} := \pi^* A + E.$$}

If we take $\pi_Y$ properly we may assume that $\text{Supp } \tilde{A}$ is a divisor with normal crossings. We note that $[\tilde{A}]$ is effective, if and only if $\mathcal{I}(-\tilde{A}) \simeq \mathcal{O}_{\tilde{X}}$ holds. We note that that

$$K_{\tilde{X}} - \tilde{A} = \pi^* (K_X - A)$$

holds by the definition of $\tilde{A}$. Hence $[\tilde{A}]$ is also effective. Also the above formula implies that for every $m \geq 1$ such that

$$\{m\nu_i\} < \delta \quad (1 \leq i \leq r),$$

there exists a $C^{\infty}$-hermitian metric $h_{\tilde{A}-K_{\tilde{X}}}$ on the $\mathbb{R}$-line bundle $A - K_{\tilde{X}}$ such that

$$\tilde{h}_m := \pi^* h^m \cdot h_{\tilde{A}-K_{\tilde{X}}} \cdot \pi^* (\prod_{i=1}^r h_{E_i}^{(m\nu_i)} \cdot |\zeta_i|^{2m\nu_i}) \cdot \pi^* (\prod_k |\tau_k|^{-2(\lfloor a_i \rfloor - a_i)})$$
is a singular hermitian metric on $O_X(\tilde{A} - K_X + m\pi^*L - \sum_i [m\nu_i]Z_i)$ such that the curvature current $\Theta_{\tilde{h}_m}$ satisfies the inequality

$$\Theta_{\tilde{h}_m} \geq c \cdot \pi^*\omega.$$

We note that there exists an effective divisor $E'$ supported on $E_{red}$ and a $C^\infty$-hermitian metric $h_{E'}$ on $O_{\tilde{X}}(E')$ such that $\tilde{\omega}(= \tilde{\omega}(\lambda)) = c \cdot \pi^*\omega + \lambda \cdot \Theta_{h_{E'}}$ is a $C^\infty$-Kähler form on $\tilde{X}$ for every sufficiently small positive number $\lambda$. Let $\lambda_0$ be a sufficiently small positive number such that $[\tilde{A} - \lambda_0 E']$ is effective and $\tilde{\omega}(\lambda_0)$ is a Kähler form on $\tilde{X}$. Then if we replace $(Y, X, (L, h), D_0, A)$ by $(\hat{Y}, \hat{X}, \pi_Y^*(L, h), \hat{D}_0, \hat{A} - \lambda_0 E')$, we may assume that $D_0$ is smooth, since by the definition of $\tilde{A}$

$$\pi^*H^0(\tilde{X}, O_{\tilde{X}}([\tilde{A} - \lambda_0 E'] + mL) \otimes I_{\tilde{X}}(\pi^*(\prod_{i=1}^r h_{Z_i}^{[m\nu_i]} \cdot |\zeta_i|^{2m\nu_i} \cdot h_{[A]} \cdot h^m)))$$

is contained in

$$H^0(X, O_X([A] + m\pi^*L) \otimes I_X((\prod_{i=1}^r h_{Z_i}^{[m\nu_i]} \cdot |\zeta_i|^{2m\nu_i} \cdot h_{[A]} \cdot h^m)))$$

holds. Now we assume that $D_0$ is smooth.

We need the following lemma.

**Lemma 4.7** Let $a_1, \ldots, a_\ell$ are positive numbers. Let us consider the sequence

$$\{(m[a_1], \ldots, m[a_\ell]) \mid m \in \mathbb{N}\}$$

in $(\mathbb{R}/\mathbb{Z})^\ell$. Then there exist a connected subgroup $T$ of $(\mathbb{R}/\mathbb{Z})^\ell$ such that

$$T \cap \{(m[a_1], \ldots, m[a_\ell]) \mid m \in \mathbb{N}\}$$

is dense in $T$.

**Proof of Lemma 4.7.** Let $T'$ denote the closure of $\{(m[a_1], \ldots, m[a_\ell]) \mid m \in \mathbb{N}\}$. Then $T'$ contains the origin. Let $T$ be the connected component containing the identity. Then by the definition of $T$, for every positive integer $m$, $(m[a_1], \ldots, m[a_\ell])$ has an inverse in $T$. This means that $T$ is a connected subgroup of $(\mathbb{R}/\mathbb{Z})^\ell$. It is clear that $T$ satisfies the desired property. Q.E.D.

Suppose that $m$ is a positive integer such that $\{-md_0\}$ is a nonzero number satisfying

$$\{-md_0\} \cdot (H^{n-1} \cdot D_0) < \frac{1}{2} c \cdot \delta_H$$

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and
\[\{m_{\nu_i}\} \leq \delta \quad (1 \leq i \leq r)\]
hold. By Lemma 4.7 such a positive integer \(m\) exists, if we take \(d_0\) sufficiently small. For such \(m\), by the assumption there exists a \(C^\infty\) hermitian metric \(h_{A-K_X}\) on the \(\mathbb{R}\)-line bundle \(A-K_X\) such that
\[
h_m := h^m \cdot h_{A-K_X} \cdot (\prod_{i=1}^r h_{Z_i}^{\{mv_i\}} \cdot \zeta_i \cdot |2mv_i|) \cdot (\prod_k |\tau_k|^{-2(|a_i|-a_i)})
\]
is a singular hermitian metric on \(\mathcal{O}_X([-A] - K_X + mL - \sum_{i=1}^r [mv_i]Z_i)\) such that \(\Theta_{h_m}\) satisfies the inequality
\[\Theta_{h_m} \geq c \cdot \omega.\]
Let \(h_0\) be a \(C^\infty\)-hermitian metric on \(\mathcal{O}_X(D_0)\) and let \(\sigma_0 \in \Gamma(X, \mathcal{O}_X(D_0))\) be a nonzero global section with divisor \(D_0\). Then if we take \(h_0\) properly, by the choice of \(D_0\) and Sublemma 4.4
\[
\hat{h}_m := h_m \cdot \frac{1}{h_0(\sigma_0, \sigma_0)^{-m_d_0}}
\]
is a singular hermitian metric on \(\mathcal{O}_X([-A] - K_X + mL - \sum_{i=1}^r [mv_i]Z_i)\) such that the curvature current \(\Theta_{\hat{h}_m}\) satisfies the inequality
\[\Theta_{\hat{h}_m} \geq \frac{1}{2} c \cdot \omega\]
on \(X\). This implies that
\[H^1(X, \mathcal{O}_X([-A] + mL) \otimes \mathcal{I}_X(\hat{h}_m)) = 0\]
holds. Hence
\[H^0(X, \mathcal{O}_X([-A] + D_0 + mL) \otimes \mathcal{I}_X(\hat{h}_m)) \rightarrow H^0(D_0, \mathcal{O}_{D_0}([-A] + D_0 + mL) \otimes \mathcal{I}_X(\hat{h}_m))\]
is surjective for such \(m\). By the construction of \(\hat{h}_m\) we see that
\[\mathcal{I}_X(\hat{h}_m) \subseteq \mathcal{I}_X(h_{[-A]} \cdot h^m) \otimes \mathcal{O}_X(-D_0)\]
holds. In fact this can be verified as follows. First by the definition of \(\hat{h}_m\), it is clear that
\[\mathcal{I}_X(\hat{h}_m) \subseteq \mathcal{I}_X(h_{[-A]} \cdot h^m)\]
holds. Let \(\varpi : X' \rightarrow X\) be a modification such that \(\varpi^* \mathcal{I}_X(\hat{h}_m)\) and \(\varpi^* \mathcal{I}_X(h_{[-A]} \cdot h^m)\) are locally free. Take a sufficiently small open set \(U\) in \(X\) and any local section \(\sigma\) of \(\mathcal{I}_X(\hat{h}_m)(U)\). Then \(\varpi^* \sigma\) is an element of
\(\varpi^*\mathcal{I}_X(h_{[A]} \cdot h^m)(\varpi^{-1}(U))\) which is identically 0 on the strict transform of \(D_0\) in \(\varpi^{-1}(U)\). Hence we have that \(\sigma \in (\mathcal{I}_X(h_{[A]} \cdot h^m) \otimes \mathcal{O}_X(-D_0))(U)\) holds.

Since \(D_0\) is not contained in the support of \(A\), we see that \([A \mid D_0]\) is effective. Since \(K_{D_0} = (K_X + D_0) \mid D_0\) holds by the adjunction formula, we see that for such \(m\), there exists a \(C^\infty\)-hermitian metric \(h_{[A] \mid D_0 - K_{D_0}}\) on the \(\mathbb{R}\)-line bundle \(A \mid D_0 - K_{D_0}\) such that

\[h_{m,D_0} := \hat{h}_m \mid D_0 = h^m \cdot h_{A \mid D_0 - K_{D_0}} \cdot \left(\prod_i h_{Z_i}^{m\nu_i} \cdot \left(\prod_k \left| \zeta_i \right|^{2m\nu_i} \cdot \left(\prod_k \left| \tau_k \right|^{-2(\nu_i - a_i)} \cdot \left(1 \cdot \sigma_0, \sigma_0\right)(\nu_i)\right)^{-1}\right)\]

is a singular hermitian metric on \(\mathcal{O}_{D_0}([A] - K_X + mL - \sum_{i=1}^r [m\nu_i]Z_i - [md_0] \cdot D_0)\) such that the curvature current \(\Theta_{h_{m,D_0}}\) satisfies the inequality

\[\Theta_{h_{m,D_0}} \geq \frac{1}{2} c\cdot \omega_{D_0}.\]

Since

\[\mathcal{I}(h_{m,D_0}) \subseteq \mathcal{I}(\hat{h}_m) \otimes \mathcal{O}_{D_0}\]

holds by the \(L^2\)-extension theorem ([13], cf. Lemma 4.6 above),

\[H^0(D_0, \mathcal{O}_{D_0}([A \mid D_0] + mL) \otimes \mathcal{I}(h_{m,D_0})) \subseteq H^0(D_0, \mathcal{O}_{D_0}([A \mid D_0] + mL) \otimes \mathcal{I}(\hat{h}_m))\]

holds. Hence if

\[H^0(D_0, \mathcal{O}_{D_0}([A \mid D_0] + mL) \otimes \mathcal{I}(h_{m,D_0})) \neq 0\]

holds, then by the above argument, we see that

\[H^0(X, \mathcal{O}_X([A] + mL) \otimes \mathcal{I}_X(h_m)) \neq 0\]

holds.

By the definition of \(D_0\), \([A \mid D_0]\) is effective. Hence repeating the same procedure, we may continue the argument and reduce the problem to the (1-dimension) lower dimensional case. We note that \(D_0\) may not be a transverse complete intersection with respect to \((L, h)\) in \(Y\), but it is a smooth divisor on \(X\). Hence essentially we may apply the induction. In fact only difference is that we should consider the \(\mathbb{R}\)-line bundle \(m(L - \sum_{i=1}^r \nu_iD_i)\) on \(X\) instead of \(mL\). But as above this does not matter, if the residual divisor \(\sum_{i=1}^r \{m\nu_i\}D_i\) is sufficiently small. By the inductive argument (see also the second case below), setting \(d_0\) to be one of \(\{\beta_1, \ldots, \beta_p\}\) in the statement of
Theorem 4.4, we completes the proof of Theorem 4.4 in this case.

Next let us consider the second case, i.e. the case that $\# J < \infty$. By taking a suitable modification of $X$ in $Y$, we may assume that

$$A + \sum_{j \in J} d_j D_j$$

is a divisor with normal crossings on $X$.

In this case we quote the following theorem.

**Theorem 4.5** ([9, p. 427, Theorem 3]) Let $M$ be a smooth projective variety and let $A$ be an divisor on $X$ with real coefficients such that

1. $\text{Supp} \{A\}$ is a divisor with normal crossings,
2. $\lceil A \rceil$ is effective,
3. $A - K_M$ is ample.

Let $L$ be a line bundle and let $D = \sum d_j D_j$ be an effective divisor with real coefficients on $M$ such that $L - D$ is nef and $\text{Supp} D$ is a divisor with normal crossings. Then there exist positive numbers $t_0$ and $\varepsilon$ such that for every integer $m$ satisfying

$$m \geq t_0 \text{ and } |\langle md_j \rangle - md_j| < \varepsilon,$$

$$H^0(M, \mathcal{O}_M(\lceil A \rceil - \langle mD \rangle + mL)) \neq 0$$

holds, where for a real number $d$, $\langle d \rangle$ denotes the integer such that

$$d - \frac{1}{2} \leq \langle d \rangle < d + \frac{1}{2}$$

and

$$\langle mD \rangle := \sum_j \langle md_j \rangle D_j.$$

**Remark 4.6** By Lemma 4.7, the set of $m$ satisfying the inequalities in Theorem 4.5 is nonempty and infinite.

Let us continue the proof of Theorem 4.4. We note that

$$L - \sum_{i=1}^r \nu_i Z_i - \sum_{j \in J} d_j D_j$$
is numerically trivial on $X$. Also by the assumption for every positive number $\varepsilon$ for every $1 \leq i \leq r$ and $j \in J$ there exists a positive integer $m$ such that
\[ |\langle md_j \rangle - md_j| < \varepsilon \quad (j \in J) \]
and
\[ |m\nu_i| < \delta \quad (1 \leq i \leq r) \]
hold. The existence of such $m$ follows from Lemma 4.7.

We cannot apply Theorem 4.5 directly in our situation, since $O_X(Z_i)|_X$ is not effective. To use the Cartier divisor $mL - \sum_{i=1}^{r} [m\nu_i]Z_i$ instead of the $\mathbb{R}$-divisor $mL - \sum_{i=1}^{r} m\nu_i Z_i$, we need to dispose of the residual divisor $\sum_{i=1}^{r} \{m\nu_i\}Z_i$. This residual divisor can be absorbed in $A$ in the following manner, if $\{m\nu_i\}, 1 \leq i \leq r$ are sufficiently small. Let us take a very ample divisor $H$ as above. We may assume that $A + H$ is a divisor with normal crossings. Let us take a positive rational number $\varepsilon_0$ so that $A - \varepsilon_0 H - K_X$ is ample. Then there exists a positive rational number $\delta_0$ such that if
\[ \{m\nu_i\} < \delta_0 (1 \leq i \leq r), \]
then
\[ \varepsilon_0 H - \sum_{i=1}^{r} \{m\nu_i\}Z_i \]
is $\mathbb{Q}$-linearly equivalent to an ample effective $\mathbb{Q}$-divisor $B$. We may assume that $A + H + B$ is a divisor with normal crossings. Then we see that
\[ \sum_{i=1}^{r} \{m\nu_i\}Z_i \sim_{\mathbb{Q}} \varepsilon_0 H - B \]
and
\[ A - \varepsilon_0 H + \sum_{i=1}^{r} \{m\nu_i\}Z_i \sim_{\mathbb{Q}} A - B \]
hold, where $\sim_{\mathbb{Q}}$ denotes the $\mathbb{Q}$-linear equivalence relation. Also we note that we may assume that $[A - B] = [A]$ holds. Thus if $\{m\nu_i\} < \delta_0$ holds for every $1 \leq i \leq r$, we may neglect the residual divisor $\sum_{i=1}^{r} \{m\nu_i\}Z_i$ by the perturbation of the divisor $A$. This argument has already been used in [3] to prove Theorem 4.5. The essential part of the proof of Theorem 4.5 is this argument and the rest of the proof is parallel to the proof of the Shokurov’s nonvanishing theorem ([10]). In this way we can dispose of $\mathbb{R}$-divisors, if the residual part is sufficiently small.

Then by Theorem 4.5 if we replace $\delta$ by $\min(\delta, \delta_0)$, we see that there exists a positive numbers $t_0$ and $\varepsilon$ such that
\[ H^0(X, O_X([A] + mL - [m \cdot \sum \nu_iZ_i] - [m \sum_{j} d_jD_j])) \neq 0 \]
holds, if
\[ |\langle md_j \rangle - md_j| < \varepsilon \quad (j \in J) \]
and
\[ \{m\nu_i\} < \delta \quad (1 \leq i \leq r) \]
and \( m \geq t_0 \). This completes the proof of Theorem 4.4.

Q.E.D.

Remark 4.7 We note that in Theorem 4.5 for every flat line bundle \( F \) on \( M \) and a positive integer \( m \) such that
\[ m \geq t_0 \quad \text{and} \quad |\langle md_j \rangle - md_j| < \varepsilon, \]
holds for every \( q \geq 1 \) by the assumption. And we note that the curvature of the singular hermitian line bundle is stable under tensoring a flat line bundle. Hence we see that Theorem 4.4 holds also for the nonvanishing of
\[ H^q(M, \mathcal{O}_M([A] - \langle mD \rangle + mL + F)) = 0 \]
holds for every \( q \geq 1 \) by the assumption. And we note that the curvature of the singular hermitian line bundle is stable under tensoring a flat line bundle. Hence we see that Theorem 4.4 holds also for the nonvanishing of
\[ H^0(X, \mathcal{O}_X([A] + mL + F_X) \otimes \mathcal{L}_X(h_m)), \]
where \( F_X \) is an arbitrary flat line bundle on \( X \).

4.5 Volume of the stable fixed component

We call the set
\[ \text{SBs}(K_X) := \cap \{ \text{Supp} \text{ Bs} | mK_X | \} \]
the stable base locus of \( K_X \).

Theorem 4.6 Let \( V \) be a divisorial component of \( \text{SBs}(K_X) \). Then
\[ \mu(V, K_X)(= \mu(V, (K_X, h))) = 0 \]
holds (for the definition of \( \mu(V, K_X) \) see Definition 3.5).

Proof. Let \( V \) be a divisorial component of \( \text{SBs}(K_X) \). Taking an embedded resolution of \( V \), we may assume that \( V \) is smooth. Suppose that \( \mu(V, K_X) > 0 \) holds.

Let \( m_0 \) be a positive integer such that \( \Phi_{|m_0K_X|} \) gives a birational rational map onto its image. By taking a suitable modification, we may assume that \( \text{Bs} | m_0K_X | \) is a divisor. Let
\[ |m_0K_X| = |P| + F \]
be the decomposition into the free part \( |P| \) and the fixed component \( F \).
Let \( h_P \) be a \( C^∞ \)-hermitian metric on \( \mathcal{O}_X(P) \) with semipositive curvature defined by a pull back of the Fubini-Study metric on \( \mathcal{O}(1) \) by \( \Phi_{|P|} \). We set
\[
r'_V = \text{mult}_V \text{Bs} \ | m_0 K_X \ | - m_0 \nu(\Theta_h, V)
\]
and
\[
r_V = \begin{cases} r'_V & \text{if } r'_V > 0 \\ 1 & \text{if } r'_V = 0 \end{cases}
\]
We note that if \( \nu(\Theta_h, V) \) is 0, \( r'_V \geq 1 \) holds by the assumption.

By Kodaira’s lemma ([11, Appendix]) there exists an effective \( \mathbb{Q} \)-divisor \( E \) such that \( P - E \) is positive. Let \( h_{P,E} \) be a \( C^∞ \)-hermitian metric on the \( \mathbb{Q} \)-line bundle \( \mathcal{O}_X(P - E) \) with strictly positive curvature. Then \( h_{P,E} \) is considered as a singular hermitian metric on \( m_0 K_X \) as follows. Let \( a \) be a positive integer such that \( aE \) is an integral divisor. Let \( \sigma \) be a global section of \( \mathcal{O}_X(aE) \) with divisor \( aE \). Then
\[
\hat{h}_{P,E} := \frac{h_{P,E}}{|\sigma|^{\frac{1}{a}}}
\]
is a singular hermitian metric on \( m_0 K_X \) with strictly positive curvature. Let \( \varepsilon \) be a sufficiently small positive number. For \( m > m_0/r_V \), we set
\[
h_{m,\varepsilon} = \frac{1}{r_V} \cdot h^{(m - \frac{m}{r_V} \cdot m_0\varepsilon)} \cdot \hat{h}_{P,E}.
\]

Then \( h_{m,\varepsilon} \) is a singular hermitian metric on \( mK_X \) with strictly positive curvature. By Nadel’s vanishing theorem (Theorem 2.1) we have that
\[
H^1(X, \mathcal{O}_X((m + 1)K_X) \otimes \mathcal{I}(h_{m,\varepsilon})) = 0
\]
holds. Hence
\[
H^0(X, \mathcal{O}_X((m+1)K_X + V) \otimes \mathcal{I}(h_{m,\varepsilon})) \to H^0(V, \mathcal{O}_V((m+1)K_X + V) \otimes \mathcal{I}(h_{m,\varepsilon}))
\]
is surjective. Let \( G \) be any effective divisor on \( V \). Since \( \mu(V, K_X) \) is positive,
\[
H^0(V, \mathcal{O}_V(mK_X - G) \otimes \mathcal{I}(h^m)) \neq 0
\]
holds for every sufficiently large \( m \). Since \( G \) is an arbitrary effective divisor on \( V \), this implies that for any fixed effective divisor \( G' \) on \( V \),
\[
H^0(V, \mathcal{O}_V(mK_X - G') \otimes \mathcal{I}(h_{m,\varepsilon})) \neq 0
\]
holds for every sufficiently large \( m \). Hence we see that
\[
H^0(V, \mathcal{O}_V((m + 1)K_X + V) \otimes \mathcal{I}(h_{m,\varepsilon})) \neq 0
\]
holds for every sufficiently large $m$. We note that by the definition of $h_{m,\varepsilon}$
on the generic point of $V$

$$\mathcal{O}_V((m+1)K_X + V) \otimes I_{(m,\varepsilon)}$$

is a subsheaf of

$$\mathcal{O}_V((m+1)K_X - [(m-1) \cdot \nu(\Theta_h, V)V]),$$

if $r'_V > 0$ and is a subsheaf of

$$\mathcal{O}_V((m+1)K_X - [(m-1) \cdot \nu(\Theta_h, V) - 1)V]),$$

if $r'_V = 0$.

If $\nu(\Theta_h, V)$ is positive, taking $\varepsilon$ sufficiently small, we see that

$$\text{mult}_V \text{Bs} | (m+1)K_X | < m \cdot \nu(\Theta_h, V)$$

holds. This is the contradiction. If $\nu(\Theta_h, V)$ is 0, then we see that

$$\text{mult}_V \text{Bs} | (m+1)K_X | = 0$$

holds. This also contradicts the assumption that $V$ is in $\text{SBs}(K_X)$. \textbf{Q.E.D.}

### 4.6 Fibration theorem

Using Theorem 4.1, we have the following theorem.

**Theorem 4.7** Let $X$ be a smooth projective variety of general type and let $h$
be an AZD of $K_X$. Let $F$ be a divisorial irreducible component of the stable
base locus of $K_X$. Then $(K_X, h)$ defines a nontrivial numerically trivial fiber
space structure on $F$, i.e. there exists a unique (up to birational equivalence)
rational fibration

$$f : F \rightarrow W$$

such that

1. for a very general fiber $V$, $(K_X, h)$ is numerically trivial on $V$, where
   $h$ is an AZD of $K_X$,  
2. $\dim W$ is minimal among such fibrations and is less than $\dim F$,  
3. for a very general point $x$ on $F$ and any irreducible horizontal curve
   $C \subset F$ containing $x$, $(K_X, h) \cdot C > 0$ holds.

Moreover if $F$ is smooth, then $f$ is regular over the generic point of $W$. 

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**Proof.** By taking an embedded resolution, we may assume that \( F \) is smooth. Let \( m_0 \) be a positive integer such that \( \Phi_{|m_0K_X|} \) is a birational rational map onto its image. Taking a suitable modification, if necessary, we may assume that \( Bs \mid m_0K_X \mid \) is a divisor with normal crossings. Let

\[
| m_0K_X | = P + \sum_i a_iD_i
\]

be the decomposition of \( | m_0K_X | \) into the free part \( | P | \) and the fixed component \( \sum_i a_iD_i \).

Taking a suitable modification, if necessary, there exists a divisor \( E = \sum_j E_j \) and positive numbers \( \{ \delta_i \}, \{ \delta_j \} \) such that

\[
P^* = P - \sum_i \delta_iD_i - \sum_j \delta_jE_j
\]

is ample and \( \sum_i \delta_iD_i + \sum_j \delta_jE_j \) is a divisor with normal crossings. By Kleiman’s criterion for ampleness we may assume that \( \{ \delta_i \}, \{ \delta_j \} \) are sufficiently small positive numbers. Let \( h_{P^*} \) be a \( C^\infty \) hermitian metric on the \( \mathbb{R} \)-line bundle \( O(P^*) \) with strictly positive curvature.

Let \( h \) be an AZD of \( K_X \). If \( F \) is not contained in \( Bs \mid m_0(K_X, h) \mid \) (cf. Definition 3.4), by Theorem 4.6, \( F \) is blown down by \( \Phi_{|m_0K_X|} \) and this defines a numerically trivial fiber space structure of \( F \). Hence \( (K_X, h) \) defines a nontrivial numerically trivial fiber space structure on \( F \).

Next suppose that \( F \) is contained in \( Bs \mid m_0(K_X, h) \mid \). By changing the indices we may assume that \( F = D_0 \) holds. We set

\[
r = \text{mult}_F Bs \mid m_0K_X \mid - m_0 \cdot \nu(\Theta_h, F) > 0,
\]
and

\[
c_F = \frac{1 + \nu(\Theta_h, F)}{r + \delta_0} + \delta,
\]

where \( \delta \) is a sufficiently small positive number. Then for \( b > c_Fm_0 + 1 \)

\[
c_F(P - \sum_i \delta_iD_i - \sum_j \delta_jE_j) + (b - c_Fm_0 - 1)(K_X - \sum_i \nu(\Theta_h, D_i)D_i)
\]

has a singular hermitian metric

\[
h_{P^*}^{c_F} \cdot h^{b-c_Fm_0-1}
\]

with strictly positive curvature. By Nadel’s vanishing theorem (Theorem 2.1), we have see that the homomorphism

\[
H^0(X, \mathcal{O}_X(K_X + F + (b - 1)K_X) \otimes \mathcal{I}(h_{P^*}^{c_F} \cdot h^{b-c_Fm_0-1})) \to \n
H^0(F, \mathcal{O}_F(K_F + (b - 1)K_X) \otimes \mathcal{I}(h_{P^*}^{c_F} \cdot h^{b-c_Fm_0-1}))
\]

is surjective. By Theorem 4.1 and its proof, we see that one of the followings holds.
1. \( H^0(F, \mathcal{O}_F(K_F + (b-1)K_X)) \otimes \mathcal{I}(h^{c_F}_{\mathcal{D}}, h^{b-cFM_0-1}) \neq 0 \) holds for infinitely many positive integers \( b \).

2. there exists a nontrivial rational fiber space structure

\[
f : F \to W
\]

such that for a very general fiber \( V \), \((K_X, h)\) is numerically trivial.

We note that we may not apply Theorem 4.1 directly, since \( h \mid_F \) is not well defined in general. But we shall modify the proof of Theorem 4.1 as follows.

Let us consider the case that \( F \) does not admit a nontrivial numerically trivial fibration associated with \((K_X, h)\). Let \( x \) be a very general point on \( F \). Then adding one more strata, i.e. constructing the stratification

\[
X \supset F \supset F_1 \supset \cdots \supset F_{r+1} = \{x\},
\]

starting from \( X \) (where the strata \( F \) is associated with \( h^{c_F}_{\mathcal{D}} \)) as in the proof of Theorem 4.1, we directly prove the nonvanishing

\[
H^0(X, \mathcal{O}_X(bK_X + F) \otimes \mathcal{I}(h^{c_F}_{\mathcal{D}}, h^{b-cFM_0-1}) \neq 0
\]

for every sufficiently large \( b \). In this case we construct a singular hermitian metric \( h_x \) with strictly positive curvature on \((b-1)K_X\) such that

1. \( \text{Spec}(\mathcal{O}_X(-[h\nu(\Theta_h, F)]F)/\mathcal{I}(h_x)) \) has isolated support at \( x \),

2. \( h_x \) is of the form :

\[
h_x = h^{(1-\varepsilon_0)c_F}_{\mathcal{D}} \prod_{i=0}^{r-1} h_i^{\alpha_i-\varepsilon_i} \cdot h^{\varepsilon_0-(\sum_{i=0}^{r-1}(\alpha_i-\varepsilon_i))-(\epsilon_0-\epsilon_i)}_i \cdot h^{b-(cFM_0+1)-\sum_{i=0}^{r} \alpha_i \nu_i}_P,
\]

where \( \{\nu_i\} \) are sufficiently large positive integers, \( \{h_i\} \) are singular hermitian metrics on \( \{\nu_iK_X + P^r\} \) constructed as in the proof of Theorem 4.1, \( \{\alpha_i\}, \alpha_i > 0 \) are invariants defined as in the proof of Theorem 4.1 such that

\[
\delta << \sum_{i=0}^{r} \alpha_i < \varepsilon_0
\]

holds. and \( \varepsilon_i(0 \leq i \leq r) \) are sufficiently small positive numbers.

As in the proof of Theorem 4.1, we may also need to consider the correction term \( e^{-\delta} \) in the proof of Theorem 4.1, but since we are constructing a section of \( bK_X \) with the desired properties, this is not essential. We note that by the definition of \( c_F \)

\[
\mathcal{O}_X(bK_X + F) \otimes \mathcal{I}(h^{c_F}_{\mathcal{D}}, h^{b-cFM_0-1})
\]
is a subsheaf of
\[ \mathcal{O}_X(bK_X + [A]) \otimes \mathcal{I}(h^b), \]
where
\[ A := \sum_{i \neq 0} (-c_F r_i + \nu(\Theta_h, D_i) - \delta_i) D_i. \]

Hence by Lemma 2.2, we see that
\[ \mathcal{O}_X(bK_X + F) \otimes \mathcal{I}(h^{c_F P} \cdot h^{b-c_{F \mathfrak{m}}a-1}) \]
is a subsheaf of \( \mathcal{O}_X(bK_X) \) for every sufficiently large \( b \) and is isomorphic to\( \mathcal{O}_F(bK_X) \otimes \mathcal{I}(h^b) \) or \( \mathcal{O}_F(bK_X + F) \otimes \mathcal{I}(h^b) \)
(in the latter case \( \nu(\Theta_h, F) > 0 \) holds) on the generic point of \( F \). By Nadel’s vanishing theorem
\[ H^1(X, \mathcal{O}_X(K_X + F + (b-1)K_X) \otimes \mathcal{I}(h_x)) = 0 \]
holds, we see that for every sufficiently large \( b \) there exists a section
\[ \sigma \in H^0(X, \mathcal{O}_X(K_X + F + (b-1)K_X) \otimes \mathcal{I}(h^{c_F P} \cdot h^{b-c_{F \mathfrak{m}}a-1})) \]
such that
\[ \text{mult}_F(\sigma) = [b \cdot \nu(\Theta_h, F)] \]
holds. By Theorem 3.4, we see that \( b \cdot \nu(\Theta_h, F) \) is an integer and by Theorem 4.6, \( F \) is blown down by \( \Phi_{|bK_X|} \) for some \( b \). This contradicts he very generic numerical positivity of \((K_X, h)\) on \( F \). Hence this case cannot occur and we see that \((K_X, h)\) defines a nontrivial numerically trivial fiber space structure on \( F \). The last statement follows from Theorem 4.2 and Remark 4.2. \textbf{Q.E.D.}

The following theorem follows from the proof of Theorem 4.6.

\textbf{Theorem 4.8} Let \( X, F \) be as in Theorem 4.6. If for a positive integer \( b \)
\[ \text{mult}_F Bs | bK_X | = b \cdot \nu(\Theta_h, F) \]
holds. Then \( \Phi_{|bK_X|} \) blows down \( F \).
5 Local base point freeness

The goal of this section is to prove the following proposition.

**Proposition 5.1** Let $X$ be a smooth projective variety of general type. Let $u : \tilde{X} \rightarrow X$ be an arbitrary composition of successive blowing ups with smooth centers. Then for every prime divisor $D$ on $\tilde{X}$, there exists a positive integer $m(D)$ depending on $D$ such that

$$\text{mult}_D BS | m(D)u^* K_X | = m(D) \cdot \nu(u^*\Theta_h, D)$$

holds.

Proposition 5.1 and Theorem 4.6 imply that every irreducible stable fixed component of $K_X$ is contracted by $| mK_X |$ for some $m > 0$.

5.1 Projective limit of projective varieties

In this section we deal with a projective limit of projective varieties. Usually it is not easy to handle such spaces because the usual algebro-geometric tools break down on such spaces. But in this paper, we only need such spaces to state the results. The actual proofs are carried out on usual algebraic varieties.

First we shall define useful objects. Let

$$\cdots \rightarrow M_m \xrightarrow{f_m} M_{m-1} \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} M_1 \xrightarrow{f_1} M_0 := M$$

be successive modifications of a projective variety $M$ such that every $M_m (m \geq 0)$ is smooth. Then we consider the projective limit

$$\hat{M} := \lim_{\leftarrow} M_m.$$

In general $\hat{M}$ is not a projective variety.

We define the topology on $\hat{M}$ as a minimal topology such that the natural map $\hat{M} \rightarrow M_m$ is continuous for every $m$.

Let $m_0$ be a nonnegative integer and let $\{D_m\}_{m=m_0}^\infty$ be a system of divisors such that

1. $D_m$ is a divisor on $X_m$,
2. $D_m = (f_{m+1})_* D_{m+1}$ holds for every $m \geq m_0$.

We note that $m_0$ can be taken to be 0 by setting

$$D_m := (f_{m_0,m})_* D_{m_0}.$$
for every $m \leq m_0$, where

$$f_{m_0, m} : M_{m_0} \longrightarrow M_m$$

be the composition of $f_{m+1} \circ \cdots \circ f_{m_0}$. In this case we may consider the projective limit

$$\hat{D} := \lim_{\leftarrow} D_m$$

and call it a divisor on $\hat{M}$. A prime divisor on $\hat{M}$ is a projective system of divisor $\{D_m\}$ such that every $D_m$ is a prime divisor (we consider 0 is a prime divisor).

Let $D$ be a divisor of some $M_m$. For $\ell \geq m$, let $D_\ell$ be the strict transform of $D$ in $M_\ell$. Then we see that

$$g_\ell, * D_\ell = D_{\ell-1}$$

holds for every $\ell > m$. Hence $\{D_\ell\}_{\ell \geq m}$ defines a divisor $\hat{D}$ in $\hat{M}$. We call $\hat{D}$ the strict transform of $D$ in $\hat{M}$. We note that every strict transform of a prime divisor on some $M_m$ is always a prime divisor on $\hat{M}$.

A sheaf $\hat{F}$ on $\hat{M}$ is a system of sheaves $\{F_m\}_{m \geq 0}$ such that

$$(f_{m+1})_* F_{m+1} = F_m$$

holds for every $m \geq 0$. In particular we can define the structure sheaf $\mathcal{O}_{\hat{M}}$ is defined as

$$\mathcal{O}_{\hat{M}} := \lim_{\leftarrow} \mathcal{O}_{M_m}.$$ 

Let $h$ be a singular hermitian metric on a line bundle $L$ on $M$ such that $\Theta_h$ is bounded from below by a $\mathcal{C}^\infty$-form on $M$.

Let

$$g_m : M_m \longrightarrow M$$

be the natural morphism. Then we see that

$$\mathcal{O}_{M_m}(K_{M_m}) \otimes \mathcal{I}(g_m^* h) = (f_{m+1})_*(\mathcal{O}_{M_{m+1}}(K_{M_{m+1}}) \otimes \mathcal{I}(g_{m+1}^* h))$$

holds for every $m \geq 0$ by the definition of multiplier ideal sheaves. Hence we may define

$$\mathcal{O}_{\hat{M}}(K_{\hat{M}}) \otimes \mathcal{I}(\hat{g}^* h)$$

as the projective limit

$$\lim_{\leftarrow} \mathcal{O}_{M_m}(K_{M_m}) \otimes \mathcal{I}(g_m^* h),$$

where

$$\hat{g} : \hat{M} \longrightarrow M$$

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is the natural morphism, i.e., the projective limit
\[ \hat{g} := \lim_{\leftarrow} g_m. \]
We note that \( \mathcal{I}(\hat{g}^* h) \) is not a well defined sheaf on \( \hat{M} \). In this paper we always consider \( \mathcal{O}_{\hat{M}}(K_{\hat{M}}) \otimes \mathcal{I}(\hat{g}^* h) \) instead of \( \mathcal{I}(\hat{g}^* h) \). We call \( \mathcal{O}_{\hat{M}}(K_{\hat{M}}) \otimes \mathcal{I}(\hat{g}^* h) \) the **multiplier canonical sheaf** of \( \hat{\pi}^*(L, h) \) on \( \hat{M} \). Also for a prime divisor \( \hat{D} = \lim_{\leftarrow} D_m \) on \( \hat{M} \), we define the **Lelong number** \( \nu(\hat{\pi}^* \Theta_h, \hat{D}) \) by
\[ \nu(\hat{g}^* \Theta_h, \hat{D}) = \nu(g_{\ell_0}^* \Theta_h, D_{\ell_0}), \]
where \( \ell_0 \) is a sufficiently large positive integer.

### 5.2 Formal canonical model
For \( m \geq 1 \) let
\[ \pi_m : X_m \rightarrow X \]
be a resolution of \( \text{Bs} | m!K_X | \) (we set \( X_0 = X \)). We may assume
1. for \( m \geq 2 \) there exists a morphism
   \[ \phi_m : X_m \rightarrow X_{m-1} \]
such that
   \[ \pi_m = \phi_m \circ \pi_{m-1} \]
holds,
2. \( \phi_m \) is a sequence of blowing ups with smooth centers contained in the indeterminacy locus of the rational map \( \Phi_{\pi_{m-1}^* | m!K_X |} \).
3. the exceptional divisor of \( \pi_m \) is a divisor with normal crossings,
4. \( \pi_m^*(\mathcal{O}_X(m!K_X) \otimes \mathcal{I}(h^m)) \) is locally free on \( X_m \)

Let \( F^{(m)} \) denote the exceptional divisor of \( \pi_m \). Let us consider the projective limit:
\[ \hat{X} = \lim_{\leftarrow} X_m. \]
\( \hat{X} \) is not a projective variety and depends on the choice of \( \{\pi_m\} \) (but we note that we are not considering all such choices at the same time). Let
\[ \hat{\pi} : \hat{X} \rightarrow X \]
be the natural morphism. We decompose \( \phi_{m+1} \) as a sequence of blowing ups:
\[
X_{m+1} \xrightarrow{p_{m}(\ell_m)} \cdots \xrightarrow{p_{m}^{\ell+2}} X_{m} \xrightarrow{p_{m}^{\ell+1}} X_{m}^{\ell+1} \xrightarrow{p_{m}^{\ell}} X_{m}^{\ell} \xrightarrow{p_{m}^{\ell-1}} X_{m}^{\ell-1} \xrightarrow{p_{m}^{\ell-2}} \cdots \xrightarrow{p_{m}^{1}} X_{m}
\]
with smooth centers. Let \( SE(K_X) \) be the subset of \( X - SBs(K_X) \) defined by

\[
SE(K_X) := \{ x \in X - SBs(K_X) \mid \Phi_{|m|K_X} \text{ is not local isomorphism onto its image on a neighbourhood of } x \text{ for every } m \geq 1 \}.
\]

We call \( SE(K_X) \) the **stable exceptional locus** of \( K_X \). \( SE(K_X) \) is a divisor in \( X - SBs(K_X) \). Let \( \hat{F} \) be the inverse image of \( SBs(K_X) \cup SE(K_X) \). Then we may and do assume that \( \hat{F} \) is a divisor on \( \hat{X} \). Let

\[
\hat{F} = \sum \hat{F}_\alpha
\]

be the irreducible decomposition of \( \hat{F} \). For each \( \hat{F}_\alpha \), there exists a rational fibration

\[
f_\alpha : \hat{F}_\alpha - \cdots \to \hat{W}_\alpha
\]

constructed as in Theorem 4.7 or the contraction morphism on \( SE(K_X) \) induced by \( \Phi_{|m|K_X} \) for every sufficiently large \( m \). We may assume that every \( f_\alpha \) is a morphism. In fact we construct \( \{ \phi_{m+1} \mid \phi_{m+1} : X_{m+1} \to X_m, m = 0, 1, 2, \ldots \} \) as follows. Let

\[
F^{(m)} = \sum_{\alpha \in I_m} F^{(m)}_\alpha
\]

be the irreducible decomposition of \( F^{(m)} \). Let

\[
f^{(m)}_\alpha : F^{(m)}_\alpha \to W^{(m)}_\alpha
\]

be the fibration constructed as in Theorem 4.7. By taking a composition of successive blowing ups

\[
w_m : \tilde{X}_m \to X_m
\]

with smooth centers, we may assume that for the strict transform \( \tilde{F}_\alpha^{(m)} \) of \( F^{(m)}_\alpha \) in \( X_m \), the induced rational map

\[
\tilde{f}_\alpha^{(m)} : \tilde{F}_\alpha^{(m)} - \cdots \to \tilde{W}_\alpha^{(m)}
\]

is actually a morphism for every \( \alpha \in I_m \). We shall take

\[
\phi_{m+1} : X_{m+1} \to X_m
\]

so that it factors through \( w_m \). Inductively we repeat the above procedure for all \( m \geq 1 \). Then \( \hat{\pi} : \hat{X} \to X \) has the desired property.

We consider the equivalence relation \( \sim \) generated by \( \{ f_\alpha \} \), i.e. we identify all the points on a fiber of every \( f_\alpha \). We set the quotient space

\[
\hat{X}_{can} = \hat{X} / \sim
\]

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and call it the formal canonical model of $X$. It is easy to see that $\hat{X}_{\text{can}}$ does not depend on the choice of $\{\pi_m\}$. Let
\[ \varpi : \hat{X} \rightarrow \hat{X}_{\text{can}} \]
be the natural map. The reason why we introduce $\hat{X}_{\text{can}}$ is that one may consider $(K_X, h)$ is numerically positive on $\hat{X}_{\text{can}}$ as we will see in the next subsection.

5.3 Concentration method on the formal canonical model

We shall prove Proposition 5.1 in this subsection. The proof is similar to that of Theorem 4.1. The only difference is that we construct the stratification as in Section 5 on the formal canonical model $\hat{X}_{\text{can}}$. But since we have not proved $\hat{X}_{\text{can}}$ is a (projective) variety, we cannot construct the stratification directly on $\hat{X}_{\text{can}}$. Hence we use the fiber space structure on the stable fixed components. Also we use $(K_X, h)$ ($h$ is the AZD of $K_X$ as before) as canonical divisor of $\hat{X}_{\text{can}}$.

Let $X$ and $\hat{X}$ be as in Proposition 5.1. It is sufficient to prove the case that $\hat{X} = X$ holds. Let $n$ be the dimension of $X$. Let $D$ be a prime divisor on $X$. Let $h$ be the analytic Zariski decomposition of $K_X$ as before. Let
\[ f_D : D \rightarrow \cdots \rightarrow W \]
be the rational fibration constructed as in Theorem 4.7. By successive blowing ups with smooth centers, we may assume that $f_D$ is a morphism. Let $x$ be a very general point on $D$, i.e. $x$ is outside of a union of at most countably many proper subvarieties of $D$. If we take $x$ very general we may assume that $\hat{\pi}^{-1}(x) \in \hat{X}$ is a point. We set
\[ x_{\text{can}} = \varpi(\hat{\pi}^{-1}(x)) \]
and
\[ \hat{x} = \varpi^{-1}(x_{\text{can}}). \]
Then $\hat{x}$ is a union of at most countably many of subvarieties in $\hat{X}$. We set
\[ \mu_0 = \mu(X, K_X). \]
We note that $\mathcal{I}(h^m)$ is locally free at $x$ for every $m$, if we take $x \in D$ very general. Then since
\[ \dim H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m)) = \frac{\mu_0}{n!} m^n + o(m^n) \]
holds, we see that for every $\varepsilon > 0$,
\[ H^0(X, \mathcal{O}_X(mK_X) \otimes \mathcal{I}(h^m) \otimes M_x^{[1-\varepsilon, \sqrt[\mu_0 m}]}) \neq 0 \]
holds for every sufficiently large $m$. Let $m_0$ be a sufficiently large positive integer and let
\[ \sigma_0 \in H^0(X, \mathcal{O}_X(m_0K_X) \otimes \mathcal{I}(h^{m_0}) \otimes \mathcal{M}_x^{[(1-\epsilon) \psi_{m_0}]} ) \]
be a general nonzero element. We define the singular hermitian metric $h_0$ on $K_X$ by
\[ h_0 := \frac{1}{| \sigma_0 |^{m_0}}. \]
We set
\[ \alpha_0 = \inf \{ \alpha > 0 \mid \hat{x} \cap \{ y \in \hat{X} \mid (\mathcal{O}_{\hat{X}}(K_{\hat{X}}) \otimes \mathcal{I}(\hat{\pi}^*(h_0^\alpha h^\beta)))_y \subseteq \mathcal{O}_{\hat{X}}(K_{\hat{X}}) \otimes \mathcal{I}(\hat{\pi}^*h^\alpha + h^\beta + 1)_y \otimes \mathcal{M}_y \} \neq \emptyset \quad \text{holds for every } \beta > 0 \} . \]
$\alpha_0$ is clearly finite. To consider $\mathcal{O}_{\hat{X}}(K_{\hat{X}}) \otimes \mathcal{I}(\hat{\pi}^*(h_0^\alpha h^\beta))$ instead of $\mathcal{O}_{\hat{X}}(K_{\hat{X}}) \otimes \mathcal{I}(\hat{\pi}^*K_X)$ reflects that we are using $\hat{\pi}^*(K_X, h)$ instead of $\hat{\pi}^*K_X$. Since $h_0$ has algebraic singularities as a singular hermitian metric of $K_X$, there exists a modification
\[ p_0 : Y_0 \longrightarrow X \]
such that the current $(\alpha_0 p_0^*\Theta_{h_0})_{\text{sing}} (= \alpha_0 p_0^*\Theta_{h_0})$ is a divisor with normal crossings $B = \sum b_iB_i$. Then if we define the numbers $\{ c_i \}$ by
\[ K_{Y_0} - p_0^*K_X - B = \sum c_iB_i, \]
\[ \min\{ \nu(p_0^*\Theta_h, B_i) + c_i \mid \hat{x} \cap \hat{B}_i \neq \emptyset \} = -1 \]
holds, where we have assumed that $\hat{\pi}$ factors through $p_0$ (this is clearly possible) and $\hat{B}_i$ denotes the strict transform of $B_i$ in $\hat{X}$. We note that if we replace $p_0$ by another $p_0'$ which factors through $p_0$, then by Corollary 2.1 the prime divisors which attain the above minimum are exactly the strict transforms of the ones associated with $p_0$.

By the above assumption there exists a morphism
\[ q_0 : \hat{X} \longrightarrow Y_0. \]
Since $p_0^*K_X$ is big, by Kodaira’s lemma, there exists an effective $\mathbf{Q}$-divisor $E_0$ on $Y_0$ such that $p_0^*K_X - E_0$ is ample. Let $h_{E_0}$ be a $C^\infty$-hermitian metric on $p_0^*K_X - E_0$ (this is a $\mathbf{Q}$-divisor on $Y_0$, but the hermitian metric is well defined) with strictly positive curvature. We may and do consider $h_{E_0}$ a singular hermitian metric on $p_0^*K_X$. If we perturb $h_0$ as
\[ h_0 := (\frac{1}{| \sigma_0 |^{m_0}})^{1-\delta_0} \cdot h_{E_0}^{\delta_0}, \]
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where $\delta_0$ is a sufficiently small positive number, perturbing also $E_0$, if necessary, we may assume that there exists a unique irreducible divisor $D_1 = B_{i_0}$ which belongs to $\{B_i \mid \hat{x} \cap \hat{B}_i \neq \emptyset\}$ such that
\[
\nu(p_0^*\mathcal{O}_h, D_1) + c_{i_0} = -1
\]
holds. We set
\[
Z_1 := \varpi(\hat{D}_1),
\]
where $\hat{D}_1$ is the strict transform of $D_1$ in $\hat{X}$. We define the nonnegative integer $n_1$ by
\[
n_1 := \dim \varpi(\hat{D}_1).
\]
We note that $n_1$ is nothing but the dimension of the base space of the numerically trivial fiber space structure on $D_1$ associated with $p_0^*(K_X, h)$. If $n_1$ is 0, then $p_0^*(K_X, h)$ is numerically trivial on $D_1$ (cf. Section 3.6 and Lemma 4.1, also [22, Theorem 4.1]). In this case we stop this process.

Suppose that $n_1 > 0$ holds. We set
\[
A_1 := r_1(p_0^*K_X - E_0) \mid_{D_1},
\]
where $r_1$ is a sufficiently large positive integer such that $r_1(p_0^*K_X - E_0)$ is Cartier. We set
\[
\mu_1 := (n-1)!\lim_{m \to \infty}m^{-(n-1)} \dim H^0(D_1, \mathcal{O}_{D_1}(m(A_1 + p_0^*(\ell_1 K_X)) \otimes \mathcal{I}(p_0^*h^{\ell_1 m})),
\]
where $\ell_1$ is a sufficiently large positive integer which will be specified later.

Let $y_1 \in D_1 \cap q_0(\varpi^{-1}(x_{can}))$ be a point. And we set $x_1 = p_0(y_1) \in X_1$. Then as before
\[
H^0(D_1, \mathcal{O}_{D_1}(m(A_1 + p_0^*(\ell_1 K_X))) \otimes \mathcal{I}(p_0^*h^m) \otimes \mathcal{M}_{y_1}^{(1-\varepsilon, n-\sqrt{m}m)} \neq 0
\]
holds for every sufficiently large $m$. Let $m_1 >>> r_1$ be a sufficiently large positive integer and let
\[
\sigma'_1 \in H^0(D_1, \mathcal{O}_{D_1}(m_1(A_1 + p_0^*(\ell_1 K_X))) \otimes \mathcal{I}(p_0^*h^{\ell_1 m_1}) \otimes \mathcal{M}_{y_1}^{(1-\varepsilon, n-\sqrt{m_1}m_1)}
\]
be a general nonzero element. We note that since $D_1$ is smooth,
\[
\mathcal{O}_{D_1}(p_0^*(m_1 \ell_1 K_X)) \otimes \mathcal{I}(p_0^*h^{m_1 \ell_1})
\]
is torsion free, since it is a subsheaf of a locally free sheaf on a smooth variety. Then as in Lemma 4.4, we see that the restriction map
\[
H^0(Y_0, \mathcal{O}_{Y_0}(m(A_1 + p_0^*(\ell_1 K_X))) \otimes \mathcal{I}(p_0^*h^{m \ell_1})) \to H^0(D_1, \mathcal{O}_{D_1}(m(A_1 + p_0^*(\ell_1 K_X))) \otimes \mathcal{I}(p_0^*h^{m \ell_1}))
\]

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is surjective for every $m > 0$, if we take $r_1$ sufficiently large. Then $\sigma'_1$ extends to an element $\sigma_1$ of

$$H^0(Y_0, \mathcal{O}_{Y_0}(m_1(A_1 + p_0^* (\ell_1 K_X)))) \otimes \mathcal{I}(p_0^* (\ell_1 m_1)).$$

We define the singular hermitian metric $h_1$ of $(r_1 + \ell_1)K_X$ by

$$h_1 = \frac{1}{|\sigma_1|}.$$  

(Originally $h_1$ is considered to be a singular hermitian metric on $p_0^* (r_1 + \ell_1)K_X$, but one may consider $h_1$ as a singular hermitian metric on $(r_1 + \ell_1)K_X$.) Let $\varepsilon_0$ be a sufficiently small positive number. We define the positive number $\alpha_1$ by

$$\alpha_1 := \inf \{ \alpha | \hat{x} \cap \{ y \in \hat{X} | (\mathcal{O}_{\hat{X}}(K_{\hat{X}}) \otimes \mathcal{I}(\hat{\pi}^*(h_0^{\alpha - \varepsilon_0} \cdot h_1^{\alpha} \cdot h_0^{\beta})))y \subseteq O_X(K_X) \otimes \mathcal{I}(\hat{\pi}^* h_0^{a_0 - \varepsilon_0 + a + \beta + 1}y \otimes M_y) \neq \emptyset \text{ holds for every } \beta > 0 \} \}.$$

Then as in Section 5 we have the estimate:

$$\alpha_1 \leq \frac{n-1}{\sqrt[n]{\mu_1}} + O(\varepsilon_0).$$

Taking $\ell_1$ to be sufficiently large, we may assume that

$$\mu_1 >> \left( \frac{(n-1)r_1 \cdot \text{mult}_{D_1} E_0}{\varepsilon_0} \right)^{n-1}$$

holds.

We take a modification

$$f_1 : Y_1 \to Y_0$$

such that the singular part of $f_1^* p_0^* ((a_0 - \varepsilon_0) \Theta h_0 + \alpha_1 \Theta h_1)$ is a divisor with normal crossings in $Y_1$. We may assume that $\hat{\pi} : \hat{X} \to X$ factors through $p_1 := f_1 \circ p_0$. Then by the procedure as before, we define a divisor $D_2$ as before in $Y_1$ and the subset $Z_2$ in $\hat{X}_{can}$ by

$$Z_2 = \varpi(\hat{D}_2),$$

where $\hat{D}_2$ is the strict transform of $D_2$ in $\hat{X}$. We note that since we have taken $\ell_1$ so that

$$\mu_1 >> \left( \frac{(n-1)r_1 \cdot \text{mult}_{D_1} E_0}{\varepsilon_0} \right)^{n-1}$$

holds, by the estimate of $\alpha_1$, we have that

$$\alpha_1 r_1 \cdot \text{mult}_{D_1} E_0 <<< \varepsilon_0.$$
holds. Hence the singularity of $h^{\alpha_1}_1$ coming from \( A_1 = r_1(p_0^*K_X - E_0) \) (roughly speaking the singularity is equal to $\alpha_1 r_1 E_0$) is enough small so that $Z_2$ is a proper subset of $Z_1$. Inductively we define a sequence of modifications

\[ X \leftarrow Y_0 \leftarrow Y_1 \leftarrow \cdots \leftarrow Y_r, \]

irreducible smooth divisors

\[ D_i \subset Y_{i-1} (i = 1, \ldots, r + 1), \]

points

\[ y_i \in D_i \cap q_{i-1}(\mathcal{w}^{-1}(x_{can})) \ (i = 1, \ldots, r), \]

where

\[ q_i : \hat{X} \rightarrow Y_i \]

are the natural morphisms, singular hermitian metrics

\[ h_0, \ldots, h_r, \]

small positive numbers

\[ \varepsilon_0, \ldots, \varepsilon_{r-1}, \]

positive integers

\[ \ell_1, \ldots, \ell_r, \]

positive numbers

\[ \alpha_0, \ldots, \alpha_r, \]

and nonnegative integers

\[ n_1, \ldots, n_{r+1}, \]

positive numbers

\[ \mu_0, \ldots, \mu_{r+1} \]

and strictly decreasing sequence of irreducible subsets

\[ Z_1 \supset Z_2 \supset \cdots \supset Z_{r+1} \]

in $\hat{X}_{can}$. By the construction of this process we see that $n_{r+1} = 0$ holds. This means that

\[ Z_{r+1} = x_{can} \]

holds. We define the singular hermitian metric $h_x$ on

\[(\sum_{i=1}^{r-1}(\alpha_i - \varepsilon_i) + (\alpha_r + \varepsilon_r))K_X\]

by

\[ h_x := h_0^{\alpha_0-\varepsilon_0} \cdot h_1^{\alpha_1-\varepsilon_1} \cdots h_r^{\alpha_r+\varepsilon_r}, \]
where $\varepsilon_r$ is a sufficiently small positive number. We set for every positive number $\beta > \sum_{i=0}^{r} \alpha_i (r_i + \ell_i)$ (where we have set $\ell_0 := 1$ and $r_0 := 0$),

$$h_x(\beta) = h^{\beta - \sum_{i=1}^{r-1} (\alpha_i - \varepsilon_i) - (\alpha_r + \varepsilon_r)} h_x.$$

We see that $p_\ast^r(K_X, h)$ is numerically trivial on $D_{r+1}$. Hence $\hat{D}_{r+1}$ is contained in $\hat{x}$, where $\hat{D}_{r+1}$ is the strict transform of $D_{r+1}$ in $\hat{X}$.

Let $p_i : Y_i \rightarrow X(i = 0, \ldots, r)$ be the natural morphisms.

**Lemma 5.1**

$$H^0(Y_r, \mathcal{O}_{Y_r}(K_{Y_r} + mp_\ast^r K_X + D_{r+1}) \otimes \mathcal{I}(p_\ast^r h_x(m))) \rightarrow H^0(D_{r+1}, \mathcal{O}_{D_{r+1}}(K_{D_{r+1}} + mp_\ast^r K_X) \otimes \mathcal{I}(p_\ast^r h_x(m)))$$ is surjective for every positive integer $m \geq \sum_{i=1}^{r-1} (\alpha_i - \varepsilon_i) - (\alpha_r + \varepsilon_r)$.

**Proof.** The assertion follows from Theorem 2.1, since

$$0 \rightarrow \mathcal{O}_{Y_r}(K_{Y_r} + mp_\ast^r K_X) \otimes \mathcal{I}(p_\ast^r h_x(m)) \rightarrow \mathcal{O}_{Y_r}(K_{Y_r} + mp_\ast^r K_X + D_{r+1}) \otimes \mathcal{I}(p_\ast^r h_x(m)) \rightarrow$$

$$\mathcal{O}_{D_{r+1}}(K_{D_{r+1}} + mp_\ast^r K_X) \otimes \mathcal{I}(p_\ast^r h_x(m)) \rightarrow 0$$

is exact. Q.E.D.

**5.4 Finding divisors on $D_{r+1}$**

On $D_{r+1}$, $p_\ast^r(K_X, h)$ is numerically trivial. By Theorem 4.3 and Corollary 4.2 we see that

$$S := \{x \in D_{r+1} \mid \nu_{D_{r+1}}(p_\ast^r \Theta_h, x) > 0\}$$

consists of at most countably many prime divisors on $D_{r+1}$. Let

$$S = \sum_{j \in J} E_j$$

be the irreducible decomposition of $S$. We set

$$e_j := \nu_{D_{r+1}}(p_\ast^r \Theta_h, E_j) \quad (j \in J).$$

By Theorem 4.3 and Corollary 4.2 we have the following lemma.

**Lemma 5.2**

$$(p_\ast^r K_X - \nu(p_\ast^r \Theta_h, D_{r+1}) D_{r+1}) \mid_{D_{r+1}} - \sum_{j \in J} e_j E_j$$

is numerically trivial on $D_{r+1}$. 63
5.5 Completion of the proof of Proposition 5.1.

Let

$$2\pi(p^*_r\Theta_{h_x})_{sing} = \sum_{i \in I} r_i F_i$$

be the decomposition into irreducible component of the singular part of the current $2\pi p^*_r\Theta_{h_x}$. We may assume that $\sum_i F_i$ is a divisor with normal crossings. Changing the indices if necessary, we may assume that $F_0 = D_{r+1}$.

We define the $\mathbf{R}$-divisor $A'_r$ on $Y_r$ by

$$A'_r := K_{Y_r} + \beta \cdot p^*_r K_X - \sum_{i \in I} r_i F_i + F_0,$$

where $\beta$ is the positive number defined by

$$\beta := \sum_{k=0}^{r-1} (\alpha_k - \epsilon_k) + (\alpha_r + \epsilon_r).$$

We set

$$\nu_i := \nu(p^*_r\Theta_{h_x}, F_i) \quad (i \in I)$$

and

$$K_{Y_r} = p^*_r K_X + \sum_{i \in I} a_i F_i.$$

Then by the definition of $h_x$ we see that

$$\min_{i \in I} (-r_i + a_i + \nu_i) = -r_0 + a_0 + \nu_0 = -1$$

and

$$\min_{i \neq 0} (-r_i + a_i + \nu_i) > -1$$

hold. If we set

$$\theta_i := -r_i + a_i + \nu_i$$

for every $i \in I$, it is easy to verify that $A'_r$ is numerically equivalent to the $\mathbf{R}$-divisor $A_r$ defined by

$$A_r = (\beta + 1)(p^*_r K_X - \sum_{i} \nu_i F_i) + \sum_{i \neq 0} \theta_i F_i.$$
is ample. We note that by Lemma 5.2

$$[(p^*_rK_X - \nu_0F_0)|_{F_0} \sim \sum_{j \in J} e_jE_j]$$

is numerically trivial on $F_0 (= D_{r+1})$. Let $J_0$ be the subset of the indices $J$ such that

$$\sum_{i \neq 0} F_i |_{F_0} = \sum_{j \in J_0} E_j$$

We set

$$E^* := \sum_{j \in J - J_0} e_jE_j.$$  

Then $E^*$ defines a point on the closure of the cone of the effective $\mathbb{R}$-divisors on $F_0$ in $H^2(F_0, \mathbb{R})$. Let $H_0$ be a smooth very ample divisor on $F_0$ such that

$$\sum_{i \neq 0} F_i |_{F_0} + H_0$$

is a divisor with normal crossings. Let $\epsilon$ be a sufficiently small positive number such that

$$A_r |_{F_0} - K_{F_0} - \epsilon H_0$$

is ample. Since

$$\epsilon H + E^*$$

is numerically equivalent to an effective $\mathbb{R}$-divisor in $H^2(F_0, \mathbb{R})$, taking a suitable modification, if necessary, we may assume that $A_r |_{F_0}$ is numerically equivalent to an $\mathbb{R}$-divisor $B_r$ with normal crossings such that $[B_r]$ is effective.

We note that

$$\mathcal{O}_{Y_r}(K_{Y_r} + m \cdot p^*_rK_X) \otimes \mathcal{I}(p^*_r h_x(m)) \simeq \mathcal{O}_{Y_r}([A_r] + (m - \beta)p^*_rK_X) \otimes \mathcal{I}(h_{[A_r]} \cdot p^*_r h^{m-\beta})$$

holds for every $m \geq \beta$, where the singular hermitian metric $h_{[A_r]}$ on $\mathcal{O}_{Y_r}([A_r])$ is defined as the singular hermitian metric $h_{[A]}$ in Theorem 4.4. We note that there exists a positive number $\epsilon_0$ such that if $\{m \cdot \nu(p^*_r \Theta_h, D_{r+1})\} \leq \epsilon_0$ holds,

$$\mathcal{O}_{D_{r+1}}([A_r] + (m - \beta)p^*_rK_X) \otimes \mathcal{I}(\tau_{r+1}^{(m \cdot \nu(p^*_r \Theta_h, D_{r+1}))} h_{[B_r]} \cdot h^{m-\beta}) \subseteq \mathcal{O}_{D_{r+1}}([A_r] + (m - \beta)p^*_rK_X) \otimes \mathcal{I}(h_{[A_r]} \cdot p^*_r h^{m-\beta})$$

holds by the perturbation of $A_r$ as in the proof of Theorem 4.4, where $\tau_{r+1}$ is a global section of $\mathcal{O}_{Y_r}(D_{r+1})$ with divisor $D_{r+1}$ and $h_{[B_r]}$ is the singular
hermitian metric on $O_{D_{r+1}}([B_r])$ defined as the singular hermitian metric $h_{[A]}$ in Theorem 4.4. We also note that

$$O_{D_{r+1}}([A_r] + (m-\beta)p^*_rK_X) \otimes \mathcal{I}(h_{[A_r]}^{-1}p^*_rh^{m-\beta}) \simeq O_{D_{r+1}}(K_{D_{r+1}} + mp^*_rK_X) \otimes \mathcal{I}(p^*_rh_x(m))$$

holds by the definition of $A_r$. Now we apply Theorem 4.4 and Remark 4.7 to our situation by setting $X = D_{r+1}$, $A = B_r$ and $L = p^*_r(K_X,h)$. Then we have the following lemma.

**Lemma 5.3** For some $m > \sum_{i=0}^r \alpha_i(r_i + \ell_i)$,

$$H^0(D_{r+1}, O_{D_{r+1}}(K_{D_{r+1}} + mp^*_rK_X) \otimes \mathcal{I}(p^*_rh_x(m))) \neq 0$$

holds.

By Lemma 5.1 and Lemma 5.3, there exists a positive integer $m(D_{r+1})$ such that

$$\text{mult}_{D_{r+1}} Bs | p^*_r(m(D_{r+1})K_X) | = m(D_{r+1}) \cdot \nu(p^*_r\Theta h, D_{r+1})$$

holds. Let

$$\hat{F} = \sum_{\alpha} \hat{F}_\alpha$$

be the inverse image of $\hat{\pi}^{-1}(SBs(K_X))$ as in Section 5.2. And let

$$\varphi_m : \hat{X} \rightarrow X_m$$

be the natural map. We note that since $\hat{\pi}^*(K_X,h)$ is numerically trivial on $\hat{x}$, the divisor

$$\hat{\pi}^*K_X - \sum_{\alpha} \nu(\hat{\pi}^*\Theta h, \hat{F}_\alpha)\hat{F}_\alpha$$

is numerically trivial on $\hat{x}$, i.e., for every $m \geq 0$, $\pi^*_m(K_X,h)$ is numerically trivial on every irreducible component of $\varphi_m(\hat{x})$. By Theorem 3.4, we see that for any positive integer $\ell$ and any $\sigma \in \Gamma(X,O_X(\ell K_X))$,

$$\text{mult}_{\hat{F}_\alpha} \hat{\pi}^*(\sigma) = \text{mult}_{F_{\alpha,m}} \pi^*_m(\sigma) \geq [\ell \cdot \nu(\pi^*_m(\Theta h), F_{\alpha,m})] = [\ell \cdot \nu(\hat{\pi}^*\Theta h, \hat{F}_\alpha)]$$

hold, where $m$ is a sufficiently large positive integer depending on $\alpha$ and $F_{\alpha,m}$ is the prime divisor whose strict transform in $\hat{X}$ is $\hat{F}_\alpha$.

Let $\sum_{j \in J} e_j E_j$ be the divisor on $D_{r+1}$ as in Lemma 5.2. Then the above formula implies that for every $j \in J$, $m(D_{r+1})e_j$ is an integer. Hence $J$ is a finite set and

$$m(D_{r+1})(p^*_rK_X |_{D_{r+1}} - \sum_{j \in J} e_j E_j)$$
is a Cartier divisor on \( D_{r+1} \) and is linearly equivalent to 0. Let us consider \( X_{m(D_{r+1})} \) and let \( D^*_{r+1} \) be the divisor on \( X_{m(D_{r+1})} \) defined by

\[
D^*_{r+1} := (\varphi_{m(D_{r+1})})_* \hat{D}_{r+1},
\]

where \( \hat{D}_{r+1} \) is the strict transform of \( D_{r+1} \) in \( \hat{X} \). Since \( m(D_{r+1})/(p^*_r K_X |_{D_{r+1}} - \sum_{j \in J} e_j E_j) \) is linearly equivalent to 0, we see that

\[
\text{Bs} \ | \ \pi^*_m(D^*_{r+1}) (K_X, h) | \cap D^*_{r+1} = \emptyset
\]

holds. Hence for any subvariety \( V \) on \( X_{m(D_{r+1})} \) such that \( D^*_{r+1} \cap V \neq \emptyset \),

\[
\text{mult}_V | \pi^*_m(D^*_{r+1})(m(D_{r+1}K_X) = m(D_{r+1}) \cdot \nu(\pi^*_m(D^*_{r+1})\Theta_h, V)
\]

holds. Let us define the analytic subset \( V_{m(D_{r+1})} \) in \( X_{m(D_{r+1})} \) by

\[
V_{m(D_{r+1})} := \varphi_{m(D_{r+1})}(\hat{x}).
\]

Then \( V_{m(D_{r+1})} \) is connected and contains \( D^*_{r+1} \).

**Lemma 5.4** \( \pi^*_m(D^*_{r+1})(K_X, h) \) is numerically trivial on \( V_{m(D_{r+1})} \).

**Proof of Lemma 5.4.** Suppose the contrary. Then \( \varphi^*_{m(D_{r+1})}((\pi^*_m(D^*_{r+1})(K_X, h)) |_{\hat{x}} \)

is not numerically trivial.

On the other hand by the definition \( \tilde{\pi}^*(K_X, h) \) is numerically trivial on \( \hat{x} \), i.e., for every \( m \geq 0 \), \( \pi^*_m(K_X, h) \) is numerically trivial on every irreducible component of \( \varphi_m(\hat{x}) \).

This is the contradiction. **Q.E.D.**

By Lemma 5.4, we see that

\[
\text{Bs} | \pi^*_m(D^*_{r+1})(K_X, h) | \cap V_{m(D_{r+1})} = \emptyset
\]

holds. Thus we see that

\[
[m(D_{r+1})(\tilde{\pi}^* K_X \sum_{\alpha} \nu(\tilde{\pi}^* \Theta_h, \hat{F}_\alpha) \hat{F}_\alpha) |_{\hat{x}} = [m(D_{r+1})(\tilde{\pi}^* K_X \sum_{\alpha} \nu(\tilde{\pi}^* \Theta_h, \hat{F}_\alpha) \hat{F}_\alpha) |_{\hat{x}}
\]

and

\[
\text{Bs} | \tilde{\pi}^*(m(D^*_{r+1})(K_X, h)) | \cap \hat{x} = \emptyset
\]

hold. Hence the base point freeness propagates through \( \hat{x} \) (in particular we may take \( \hat{X} \) so that \( \hat{x} \) consists of finitely many irreducible components). Since \( x \) is a very general point on \( D \), we see that there exists a positive integer \( m(D) \) such that

\[
\text{mult}_D \text{Bs} | m(D) K_X | = m(D) \cdot \nu(\Theta_h, D)
\]

holds. This completes the proof of Proposition 5.1. **Q.E.D.**
6 Completion of the proof of Theorem 1.1

We complete the proof of Theorem 1.1 by using a topological consideration. We use the same notations and conventions as in Section 5.

**Definition 6.1** Let $X$ be a smooth projective variety of dimension $n$ and let $L$ be a big line bundle on $X$. Let $R = \bigoplus_{m \geq 0} R_m$ be a subring of $R(X, L)$ such that

$$\lim_{m \to \infty} m^{-n} \dim R_m > 0.$$ 

For every subvariety $V$ in $X$, we set

$$\nu(R, V) := \lim_{m \to \infty} \frac{1}{m} \operatorname{mult}_V Bs | R_m |,$$

where $Bs | R_m |$ is the base scheme as a linear subsystem of $| mL |$. Suppose that for every modification $f : Y \to X$

and every prime divisor $D$ on $Y$, there exists a positive integer $m_D$ depending on $D$ such that

$$\nu(f^* R, D) = \frac{1}{m_D} \operatorname{mult}_D Bs | m_D f^* R_{m_D} |$$

holds.

In this case we call that $R$ is virtually base point free on $X$.

Proposition 5.1 implies that the canonical ring $R(X, K_X)$ of smooth projective variety of general type $X$ is virtually base point free.

Let $X$ be a smooth projective variety of general type and let $n$ denote the dimension of $X$. By the virtual base point freeness of $R(X, K_X)$, we see that $\hat{X}_{\text{can}}$ is a complex space (possibly noncompact). In fact by the construction and Proposition 5.1, for every compact subset $G$ of $\hat{X}_{\text{can}}$, there exists a positive integer $m(G)$ depending on $G$ such that $m(G)K_{\hat{X}_{\text{can}}}$ is Cartier on $G$ and $| m(G)K_{\hat{X}_{\text{can}}}$ is base point free on every compact subset of $\hat{X}_{\text{can}}$ and is numerically positive on $\hat{W \cap G}$ in the obvious sense. This implies that $\hat{X}_{\text{can}}$ is a complex space. Also it is easy to see that $\hat{X}_{\text{can}}$ is normal by showing that $\hat{X}_{\text{can}}$ is isomorphic to the normalization. Moreover since by the construction of $\hat{X}_{\text{can}}$ and the virtual base point freeness, $\bigoplus_{m \geq 0} O_{\hat{X}_{\text{can}}} (mK_{\hat{X}_{\text{can}}})$ is a finitely generated ring over $O_{\hat{X}_{\text{can}}}$ on $G$, $\hat{X}_{\text{can}}$ has only canonical singularities.

Let $\hat{W}$ be the subspace of $\hat{X}_{\text{can}}$ defined by

$$\hat{W} := \hat{\pi}(\hat{F}).$$
Then by Theorem 4.6 codim $\hat{W} \geq 2$ holds. We only need to consider the case: $\dim X \geq 3$. Now we consider the exact sequence:

$$H^2(\hat{X}_{\text{can}}, \mathbb{Z}) \to H^2(\hat{W}, \mathbb{Z}) \to H^3(\hat{X}_{\text{can}}, \hat{W}, \mathbb{Z}).$$

We note that since codim $\hat{W} \geq 2$, i.e., $\hat{\nu}$ contracts all the irreducible components of $\hat{F}$ in $\hat{X}$, $\dim H^2(\hat{X}_{\text{can}}, \mathbb{C})$ is finite. Hence we see that

$$\text{rank } H^3(\hat{X}_{\text{can}}, \hat{W}, \mathbb{Z}) = \infty$$

holds, if

$$\text{rank } \dim H^2(\hat{W}, \mathbb{Z}) = \infty$$

holds. We note that

$$H^3(\hat{X}_{\text{can}}, \hat{W}, \mathbb{Z}) \simeq H^3(X, S, \mathbb{Z})$$

holds, where $S$ denotes the union of the stable base locus $\text{SBs}(K_X)$ and the stable exceptional locus $\text{SE}(K_X)$. This means that $H^3(\hat{X}_{\text{can}}, \hat{W}, \mathbb{C})$ is an finitely generated abelian group. This implies that

$$\text{rank } H^2(\hat{W}, \mathbb{Z}) < \infty$$

holds (in the case of $\dim X = 3$, this immediately implies that $\hat{W}$ consists of finitely many irreducible components). By the universal coefficients theorem we see that

$$0 \to \text{Ext}(H_1(\hat{X}_{\text{can}}, \mathbb{Z}), \mathbb{Z}) \to H^2(\hat{X}_{\text{can}}, \mathbb{Z}) \to \text{Hom}(H_2(\hat{X}_{\text{can}}, \mathbb{Z}), \mathbb{Z}) \to 0$$

is exact. Since $H_1(\hat{X}_{\text{can}}, \mathbb{Z})$ is finitely generated (because $\text{codim } \hat{W} \geq 2$ holds), we see that the torsion part of $H^2(\hat{X}_{\text{can}}, \mathbb{Z})$ is finite.

Since $H^2(\hat{X}_{\text{can}}, \mathbb{Z})$ is finitely generated and rank $H^2(\hat{W}, \mathbb{Z})$ is finite, considering the maps:

$$H^2(\hat{X}_{\text{can}}, \mathbb{Z}) \to \text{Hom}(H_2(\hat{X}_{\text{can}}, \mathbb{Z}), \mathbb{Z}) \to \text{Hom}(H_2(\hat{W}, \mathbb{Z}), \mathbb{Z}),$$

we see that the images of some positive multiple of $c_1(K_{\hat{X}_{\text{can}}}) \in H^2(\hat{X}_{\text{can}}, \mathbb{R})$ under the maps:

$$H^2(\hat{X}_{\text{can}}, \mathbb{R}) \to \text{Hom}(H_2(\hat{W}, \mathbb{R}), \mathbb{R}),$$

and

$$H^2(\hat{X}_{\text{can}}, \mathbb{R}) \to \text{Hom}(H_2(\hat{X}_{\text{can}}, \mathbb{R}), \mathbb{R}),$$

are the images of elements of $\text{Hom}(H_2(\hat{W}, \mathbb{Z}), \mathbb{Z})$ and $\text{Hom}(H_2(\hat{X}_{\text{can}}, \mathbb{Z}), \mathbb{Z})$ respectively. This implies that some positive multiple of $c_1(K_{\hat{X}_{\text{can}}}) \in H^2(\hat{X}_{\text{can}}, \mathbb{R})$ is integral (i.e. it is in the image of the natural morphism $H^2(\hat{X}_{\text{can}}, \mathbb{Z}) \to H^2(\hat{X}_{\text{can}}, \mathbb{R})$) in $H^2(\hat{X}_{\text{can}}, \mathbb{R})$. Hence some positive multiple of $K_{\hat{X}_{\text{can}}}$ is a line bundle on $\hat{X}_{\text{can}}$.

Let $r$ be a positive integer such that $rK_{\hat{X}_{\text{can}}}$ is a line bundle.
Definition 6.2 Let $X$ be a normal complex space. We define the $L^2$-dualizing sheaf $K_X^{(2)}$ by

$$K_X^{(2)}(U) = \{ \eta \in \Gamma(U, \mathcal{O}_X(K_X)) \mid \eta \wedge \bar{\eta} \in L^1_{\text{loc}}(U) \}. $$

The following lemma is clear by the definition of canonical singularities.

Lemma 6.1 Let $X$ be a normal complex space with only canonical singularities. Then the canonical sheaf $K_X := i^*K_{X_{\text{reg}}}$ of $X$ is isomorphic to $K_X^{(2)}$, where $i : X_{\text{reg}} \to X$ is the canonical injection.

Lemma 6.2 Let $Z$ be a closed $n$-dimensional subvariety of the unit open polydisk $\Delta^N$ with only canonical singularities and let $\varphi$ be a pliusisubharmonic function on $Z \times \Delta$, where $\Delta$ is an open unit disk in $\mathbb{C}$. Let $D$ be a $\mathbb{Q}$-Cartier divisor on $Z$ such that $K_Z + D$ is Cartier. Let $h_D$ be a $C^\infty$-hermitian metric on the $\mathbb{Q}$-line bundle $\mathcal{O}_Z(D)$. Let $t$ be the standard coordinate on $\Delta$. Let $p_1 : Z \times \Delta \to Z$ be the first projection. Then there exists a positive constant $C_Z$ depending only on $Z$ such that for every $f \in \Gamma(Z, \mathcal{O}_Z(K_Z + D))$ such that

$$(\sqrt{-1})^{n(n-1)/2} \int_Z e^{-\varphi} \cdot h_D \cdot f \wedge \bar{f} < \infty$$

there exists a holomorphic section $F \in \Gamma(Z \times \Delta, \mathcal{O}_{Z \times \Delta}(K_{Z \times \Delta} + p_1^*D))$ such that

1. $F \mid_Z = dt \wedge f$,
2. $((\sqrt{-1})^{n(n+1)/2} \int_{Z \times \Delta} e^{-\varphi} \cdot h_D \cdot F \wedge \bar{F} \leq C_Z(\sqrt{-1})^{n(n-1)/2} \int_Z e^{-\varphi} \cdot h_D \cdot f \wedge \bar{f}$

This lemma is an immediate consequence of the $L^2$-extension theorem ([75, p. 200, Theorem]). Since $\hat{X}_{\text{can}} - \hat{W}$ is biholomorphic to $X - S$, it admits a complete Kähler metric. Hence we can apply $L^2$-estimates for $\bar{\partial}$-operator on $\hat{X}_{\text{can}} - \hat{W}$.

We note that for every compact positive dimensional subvariety $V$ in $\hat{X}_{\text{can}}$, $(rK_{\hat{X}_{\text{can}}})^{\dim V} : V := \mu(V, rK_{\hat{X}_{\text{can}}}) \geq 1$. Let $m_0$ be a positive integer such that $\text{Supp} B_s \mid m_0rK_{\hat{X}_{\text{can}}} \mid \subseteq \hat{W}$ holds. Such $m_0$ exists by Proposition 5.1. Let $\tau_0, \ldots, \tau_N$ be a basis of $\Gamma(\hat{X}_{\text{can}}, \mathcal{O}_{\hat{X}_{\text{can}}}(m_0rK_{\hat{X}_{\text{can}}}))$. We define the singular hermitian metric on $rK_{\hat{X}_{\text{can}}}$ by

$$h_0 := \frac{1}{(\sum_{i=0}^N |\tau_i|^2/m_0)^{1/m_0}}.$$
Let $x_0$ be an arbitrary point in $\text{Supp} \, B_s | m_0rK_{\hat{X}_{\text{can}}}$|. Then by Lemma 6.2, we see that for a local generator $\sigma$ of $rK_{\hat{X}_{\text{can}}}$ on a neighbourhood $U$ of $x_0$, for every $\alpha \geq m_0n$ the singular volume form

$$h_0^\alpha | \sigma |^{2\alpha + \frac{\hat{\phi}}{r}}$$

is not locally integrable on $U \cap (\hat{X}_{\text{can}} - \hat{W})$. In fact let $x(t)(t \in \Delta)$ be a local holomorphic curve on $\hat{X}_{\text{can}}$ such that $x(0) = x_0$ and $x(t) \in \hat{X}_{\text{can}} - \hat{W}(t \in \Delta^*)$. Then the limit $\lim_{t \to 0} n \cdot x(t)$ in the Douady space of $\hat{X}_{\text{can}}$ is contained in $n \cdot x(0)$. Hence by Lemma 6.2, the assertion follows.

By Lemma 6.1 and Lemma 6.2 instead of Lemma 4.6, using the parallel argument as in [1] or Section 4 we conclude that $| m(rK_{\hat{X}_{\text{can}}}) |$ is free at $x_0$ for every $m \geq m_0n + n(n - 1)/2 + 2$ (we note that every strata constructed as in Section 4 except $\hat{X}_{\text{can}}$ is contained in $\hat{W}$, hence it is compact). Since $m_0$ is independent of the choice of $x_0$, we see that $| m(rK_{\hat{X}_{\text{can}}}) |$ is free on $\hat{X}_{\text{can}}$ for every $m \geq m_0(n(n+1)/2+2)$. Since $K_{\hat{X}_{\text{can}}}$ is numerically positive, we see that $\hat{W}$ consists of finitely many irreducible components and $\hat{X}_{\text{can}}$ is a projective variety (with only canonical singularities). This implies that $\hat{X}_{\text{can}}$ is the canonical model of $X$. Hence $R(X, K_X)$ is finitely generated. This completes the proof of Theorem 1.1.

References

[1] U. Anghern-Y.-T. Siu, Effective freeness and point separation for adjoint bundles, Invent. Math. 122 (1995), 291-308.

[2] E. Bombieri, Algebraic values of meromorphic maps, Invent. Math. 10 (1970), 267-287.

[3] E. Bombieri, Addendum to my paper: Algebraic values of meromorphic maps, Invent. Math. 11, 163-166.

[4] E. Bombieri, Canonical models of surfaces of general type, Publ. I.H.E.S. 42 (1972), 171-219.

[5] M.A.A. de Cataldo, Singular hermitian metrics on vector bundles, J. f"ur Reine Angewande Math. 502 (1998), 93-122.

[6] J.P. Demailly, Regularization of closed positive currents and intersection theory, J. of Alg. Geom. 1 (1992) 361-409.

[7] J.P. Demailly : oral communucation, to appear in J.P. Demailly-T. Peternell-M. Schneider.
[8] L. Hörmander, An Introduction to Complex Analysis in Several Variables 3-rd ed., North-Holland (1990).

[9] Y. Kawamata, The Zariski decomposition of logcanonical divisors, Collection of Alg. geom. Bowdowin 1985, Proc. of Sym. Pure Math. 46 Part 1, (1987), 425-433.

[10] Y. Kawamata, Pluricanonical systems on minimal algebraic varieties, Invent. Math. 79 (1985), 567-588.

[11] S. Kobayashi-T. Ochiai, Mappings into compact complex manifolds with negative first Chern class, Jour. Math. Soc. Japan 23 (1971), 137-148.

[12] S. Mori, Flip conjecture and the existence of minimal model for 3-folds, J. of A.M.S. 1 (1988), 117-253.

[13] D. Mumford, Appendix to the paper of O. Zariski, The theorem of Riemann-Roch for high multiplicities of an effective divisor on an algebraic surface, Ann. of Math. 76 (1962), 560-615.

[14] A.M. Nadel, Multiplier ideal sheaves and existence of Kähler-Einstein metrics of positive scalar curvature, Ann. of Math. 132 (1990), 549-596.

[15] T. Ohsawa and K. Takegoshi, $L^2$-extension of holomorphic functions, Math. Z. 195 (1987), 197-204.

[16] V.V. Shokurov, The nonvanishing theorem, Izv. Nauk USSR 26 (1986), 510-519.

[17] Y.-T. Siu, Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, Invent. Math. 27 (1974), 53-156.

[18] H. Tsuji, Analytic Zariski decomposition, Proc. of Japan Acad. 61(1992) 161-163.

[19] H. Tsuji, Existence and Applications of Analytic Zariski Decompositions, Analysis and Geometry in Several Complex Variables (Komatsu and Kuranishi ed.), Trends in Math. 253-271, Birkhäuser (1999).

[20] H. Tsuji, On the structure of pluricanonical systems of projective varieties of general type, preprint (1997).

[21] H. Tsuji, Global generation of adjoint bundles, Nagoya Math. J. 142 (1996), 5-16.

[22] H. Tsuji, Numerically trivial fibrations, math.AG/0001023 (2000).
[23] O. Zariski, The theorem of Riemann-Roch for high multiplicities of an effective divisor on an algebraic surface, Ann. of Math. 76 (1962), 560-615.

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