Light bending in Schwarzschild–de Sitter: projective geometry of the optical metric

G W Gibbons¹, C M Warnick¹ and M C Werner¹,²

¹ DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK
² IoA, University of Cambridge, Madingley Road, Cambridge CB3 0HA, UK

E-mail: g.w.gibbons@damtp.cam.ac.uk, c.m.warnick@damtp.cam.ac.uk and mcw36@cam.ac.uk

Received 8 September 2008
Published 27 November 2008
Online at stacks.iop.org/CQG/25/245009

Abstract
We interpret the well-known fact that the equations for light rays in the Kottler or Schwarzschild–de Sitter metric are independent of the cosmological constant in terms of the projective equivalence of the optical metric for any value of $\Lambda$. We explain why this does not imply that lensing phenomena are independent of $\Lambda$. Motivated by this example, we find a large collection of one-parameter families of projectively equivalent metrics including both the Kottler optical geometry and the constant curvature metrics as special cases. Using standard constructions for geodesically equivalent metrics we find classical and quantum conserved quantities and relate these to known quantities.

PACS numbers: 04.70.−s, 98.62.SB

1. Introduction

Some time ago, Islam [1] observed that the differential equation

$$\frac{d^2u}{d\phi^2} + u = 3mu^2,$$

(1.1)

where $u = 1/r$, and $r$ is the usual Schwarzschild radial coordinate, which governs light rays moving in a Schwarzschild–de Sitter, or Kottler [2], metric, depends only on the mass $m$ and not the cosmological constant $\Lambda$.³

One might therefore be led into supposing that the light deflection and time-delay formulae, which relate observable quantities, would be the same as in the Schwarzschild metric, or because our present universe appears to be dominated by a cosmological term, that the usual formulae for gravitational lensing in Friedmann–Lemaître universes are not applicable if the cosmological constant is non-zero. More excitingly, one might suppose that

³ Throughout we take units such that $c = G = 1$. 
gravitational lensing observations could allow a new and independent measurement of the cosmological constant.

There has been a considerable amount of discussion of these possibilities recently and the consensus is emerging that as far as the observational situation is concerned the existing theory of gravitational lensing is perfectly adequate (see [3, 4] and references therein). There remain however questions about precisely what equation (1.1) is telling one and why it is misleading.

Roughly speaking, the origins of the confusion are two fold:

- Equation (1.1) merely governs the unparametrized projection of null rays onto the spatial sections of the Kottler metric and not more detailed geometrical features such as deflection angles and differences in duration of different paths joining a source to an observer.
- In the cosmological setting, the source and the observer both participate in the Hubble flow and hence are not at rest with respect to the static Kottler coordinates, but are in relative motion, and this must also be taken into account.

It is well known that according to Fermat’s principle (e.g., [5]), the projections of null geodesics onto a surface of constant time are geodesics of a conformally rescaled spatial metric called the optical metric. One of the principal purposes of this paper is to explain that Islam’s observation is essentially that the projective properties of these geodesics are independent of the cosmological constant, but not their lengths nor their conformal properties such as angles. In fact the optical metrics of the Kottler spacetimes provide a hitherto unrecognized example of a one-parameter family of projectively equivalent metrics, which may be of interest to geometers in its own right.

The second aspect of the problem relates to the full four-dimensional spacetime geometry. As was first realized by McVittie [6], the Kottler metric may be cast in a form which clearly exhibits it as the gravitational field of a point mass with respect to spatially flat Friedman–Lemaître expanding universe with de Sitter type exponentially expanding scale factor

$$a(t) = \exp(\mathcal{H}t), \quad \mathcal{H} = \sqrt{\frac{3}{\Lambda}}.$$  \hspace{1cm} (1.2)

Since, to lowest order in the mass $M$, this coincides precisely with the standard form of the approximate metric used in all studies of gravitational lensing, it is clear that no new or unaccounted effects can be obtained working to linear order in $M$ using the Kottler form which can also not be obtained using the standard linearized McVittie form. This point has been made forcibly recently by Park [4] who explicitly shows that the light deflection formulae, if derived correctly, are identical in both formalisms.

In this paper, we shall first establish the lensing problem in terms of sources and observers fixed in coordinates comoving with the Hubble flow and demonstrate that one may instead consider a moving source and an observer in the Kottler metric. By passing to the optical metric we will derive the standard equations governing light rays. We will then discuss the geometrical notion of projective equivalence and show that this explains the peculiar property of equation (1.1). Led by the Kottler metrics, we find a large class of families of geodesically equivalent metrics and, using results from the literature, exhibit conserved quantities of their geodesic flow.

2. McVittie’s formalism and the Kottler metric

In order to calculate the effects of gravitational lensing we need the metric due to a system of comoving mass points in a background Friedman–Lemaître universe. This was given to
lowest order by McVittie in 1964 [6]. McVittie’s approximate metric is
\[
d s^2 = -\left(1 + \frac{2U}{a^2}\right)dt^2 + a^2\left(1 - \frac{2U}{a^2}\right)\left\{\frac{dx^2}{(1 + k \frac{x^2}{l^2})}\right\},
\]
(2.1)
where \(k = -1, 0, 1\), depending on whether the metric inside the brace is hyperbolic space, Euclidean space or spherical space and \(l\) is the radius of curvature of these spaces. The Newtonian potential \(U(x)\) satisfies
\[
\nabla^2_k U = 0,
\]
(2.2)
where \(\nabla^2_k\) is the Laplacian with respect to the metric in the brace.

A typical physical question concerns the light deflection caused by a lensing galaxy in a space with a positive cosmological constant. In order to study this, one may make use of McVittie’s version of the Kottler or Schwarzschild–de Sitter metric. In McVittie’s coordinates the metric takes the form
\[
d s^2 = -\left(1 - \frac{mr}{a(t)x}\right)dt^2 + a(t)^2\left(1 + \frac{mr}{a(t)x}\right)^4\left\{dx^2 + x^2(d\theta^2 + \sin^2\theta d\phi^2)\right\},
\]
(2.3)
where \(a(t) = \exp(t/l)\). This represents an exact solution of Einstein’s equations with a cosmological constant \(\Lambda_1 = 3/l^2\). This form is of interest because for large \(x\) it approaches the standard FRLW form of the de Sitter metric familiar to cosmologists. Sources and observers are considered to be fixed in the comoving \(x, \theta, \phi\) coordinates. We may make a coordinate transformation to the static Kottler form of the metric as follows:
\[
T = (t + \tau(r)), \quad r = e^{\frac{t}{l}}x + m + \frac{m^2}{4e^{\frac{t}{l}}x},
\]
(2.4)
where the function \(\tau\) is defined by
\[
\frac{dr}{dr} = -\frac{r^2}{l(r - m)(1 - \frac{m}{r} - \frac{r^2}{l^2})}.
\]
(2.5)
This gives the metric form
\[
d s^2 = -\left(1 - \frac{2mr}{r^2 - l^2}\right)dt^2 + \frac{dr^2}{(1 - \frac{2mr}{r^2 - l^2})} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\]
(2.6)
The advantage of these coordinates is that they are static, making the problem of finding light rays simpler. The disadvantage is that observers ‘at rest’ in the comoving coordinates are in motion with respect to the Kottler coordinates. One may readily translate between the two pictures using equations (2.4) and (2.5).

It is convenient to pass to the optical metric in order to find the light rays. This is given by
\[
d s^2_{\text{opt}} = \frac{dr^2}{(1 - \frac{2mr}{r^2 - l^2})} + \frac{r^2}{(1 - \frac{2mr}{r^2 - l^2})}(d\theta^2 + \sin^2\theta d\phi^2).
\]
(2.7)
The geodesics of the optical metric are the projections of the light rays of the full spacetime onto a constant \(T\) hypersurface, and length in the optical metric corresponds to the coordinate time \(T\) taken by a light ray to traverse the line segment.
2.1. Light-ray equations

From the Kottler optical metric we deduce the equations governing the light rays. Without loss of generality we may take these to lie in the equatorial plane, giving

\[
\frac{d\phi}{dT} = \frac{\Delta_r h}{r^4}, \quad \frac{r^4}{\Delta_r^2} \left( \frac{dr}{dT} \right)^2 + \frac{\Delta_r}{r^4} h^2 = 1,
\]

where

\[
\Delta_r = r^2 - 2mr - \frac{r^4}{l^2}.
\]

Eliminating the time coordinate \(T\) we arrive at an equation for the projection of the light rays into the \((r, \phi)\) plane

\[
\left( \frac{dr}{d\phi} \right)^2 + \Delta_r = \frac{r^4}{h^2}.
\]

We note that by defining a modified angular momentum \(k = \hbar l / \sqrt{h^2 + l^2}\) we may write this as

\[
\left( \frac{dr}{d\phi} \right)^2 + r^2 - 2mr = \frac{r^4}{k^2},
\]

so that there is an equivalence between the light rays for different values of the cosmological constant, provided they give rise to the same value of \(k\). We make the standard substitution \(u = 1/r\) to obtain the equations we will require

\[
\left( \frac{du}{d\phi} \right)^2 = 2mu^3 - u^2 + \frac{1}{k^2}, \quad \frac{d\phi}{dT} = \left( u^2 - 2mu^3 - \frac{1}{l^2} \right) h
\]

so that although the rays are the same the length in the optical metric (i.e. the time to traverse the ray) does depend upon the value of the cosmological constant. There are other effects due to the fact that the source and observer may not be taken to be at rest in the static Kottler coordinates. We may in fact find solutions to equation (2.10) using elliptic functions

\[
m \frac{2r(\phi)}{m} - \frac{1}{12} = \wp(\phi + \text{constant}),
\]

where Weirstrass’s function \(\wp\) satisfies

\[
\wp'(\phi)^2 = 4\wp^3 - g_2 \wp - g_3,
\]

with

\[
g_2 = \frac{1}{12}, \quad g_3 = \frac{1}{216} - \frac{m^2}{4k^2}.
\]

One may check that

\[
u = \frac{1}{3m} - \frac{1}{m \cosh \phi + 1}
\]

and

\[
u = \frac{1}{m \cos \phi + 1}
\]

are exact solutions. The former (2.16) starts at infinity at \(\phi = \cosh^{-1}(2)\) and moves inwards, spirally around the circular orbit at \(r = 3m\).
In fact (2.16) is not a Weierstrass function but if one adds $\frac{i\pi}{2}$ to the argument $\phi$ one obtains

$$u = \frac{1}{3m} + \frac{1}{m \cosh \phi - 1}$$

which is a Weierstrass function. This orbit starts from $r = 0$ at $\phi = 0$ and moves outwards ultimately endlessly approaching the circle at $r = 3m$. In practice, it is not convenient to work with these exact solutions and an approximation is usually made.

3. Projective equivalence

As Islam observed, the problem of finding the paths of null geodesics in the Kottler spacetime is equivalent to the same problem in the Schwarzschild spacetime. The reason for this is the projective equivalence of the optical metrics, which we define as follows.

Given manifolds $\mathcal{M}^n, \mathcal{\bar{M}}^n$, two metrics $g, \bar{g}$ are said to be projectively equivalent in neighbourhoods $U, \bar{U}$ if there exist charts $\phi : U \rightarrow V, \bar{\phi} : \bar{U} \rightarrow V$ with $V \subset \mathbb{R}^n$ such that the geodesics of both metrics coincide as unparametrized curves on $V$. We will refer loosely to geodesics as curves without any prejudice as to their parametrization. For example, we may take $\mathcal{M} = \mathbb{R}^3, g = \delta$ the Euclidean metric and $\bar{g}$ to be the Beltrami metric for one-half of the sphere:

$$\bar{g} = \frac{dr^2}{(1 + r^2 \pi^2)^2} + \frac{r^2}{1 + r^2 \pi^2} (d\theta^2 + \sin^2 \theta d\phi^2).$$

One may verify that the geodesics of this metric are in fact straight lines. Of course if we parametrize these curves by arc length then they no longer agree with the Euclidean geodesics parametrized by arc length since, for example, the length of a geodesic with respect to $\bar{g}$ may not exceed $\pi R$. Thus $g$ and $\bar{g}$ are projectively equivalent.

We may also consider $R \rightarrow iR$ which takes the metric $\bar{g}$ to the Beltrami metric for hyperbolic 3-space, $\mathbb{H}^3$. In this case we should be careful as these metrics are only defined inside the ball $r < R$, however the geodesics are still straight lines, so with some care regarding the domain of definition, these metrics are projectively equivalent to one another and to the flat metric.

In general, finding the coordinate charts $\phi, \bar{\phi}$ is fundamental to constructing a projective equivalence, since two different coordinate representations of the same metric may look quite different. We will for the moment take the question of the existence of charts as solved and restrict ourselves to the simpler question of whether two metrics defined on some subset of $\mathbb{R}^n$ have the same geodesics. In this case, an answer has been given by Eisenhart [7, p 131]. He shows by considering the geodesic equations for both metrics that a necessary condition for the geodesics to coincide is that

$$W^l_{ijk}(g) = \bar{W}^l_{ijk}(\bar{g}),$$

where the projective curvature tensor $W$ of $g$ is defined by

$$W^l_{ijk} = R^l_{ijk} + \frac{1}{n-1} (\delta^l_k R_{ij} - \delta^l_j R_{ik}).$$

The index structure is important here, as we have two means of raising and lowering indices, with $g$ and $\bar{g}$, so should be careful. If $W$ vanishes for a spacetime, then by contracting indices one discovers that the space has constant curvature, so the only projectively flat spaces are those with constant curvature. In fact, as we saw above the spaces of constant curvature are indeed projectively equivalent.

4 We take the curvature conventions $[\nabla_i, \nabla_j]X^k = R^k_{i[j}X^{i]}$ and $R_{ij} = R^k_{ikj}$, which differ from that of Eisenhart, who accordingly has a formula with $R_{ij} \rightarrow -R_{ij}$. 
4. Projective tensor for a warped product metric

In addition to the projectively equivalent constant curvature metrics, we know that the Kottler metrics are a family of projectively equivalent metrics for all values of the cosmological constant. Motivated by this observation, we will consider a warped metric on the \((n + 1)\)-dimensional manifold \(\mathbb{R} \times M^n\) which may be cast locally into the form

\[
g = \frac{dr^2}{r^2 f(r)^2} + \frac{1}{f(r)} h_{\alpha \beta}(x^\gamma),
\]

where \(h\) is an \(n\)-dimensional metric on \(M\). We will take local coordinates \(x^\mu = (r, x^i)\), with Greek indices ranging over \(0, \ldots, n\) and Roman indices over \(1, \ldots, n\) and we will use the notation \((\alpha)^i \Gamma^j_k, (\alpha) R^i_{jkl}, (\alpha) R_{ij}\) to refer to objects intrinsic to \(M\). We will at this stage avoid the assumption that \(f\) is positive or that \(h\) is Riemannian. We wish to calculate the projective tensor:

\[
W^\mu_{\nu \sigma \rho} = R^\mu_{\nu \sigma \rho} + \frac{1}{n} \left( \delta^\mu_\rho R_{\nu \sigma} - \delta^\mu_\sigma R_{\nu \rho} \right).
\]

The non-zero components of the Riemann tensor are

\[
R^i_{jkl} = (\alpha) R^i_{jkl} + \frac{r^4 f'^2}{4f} \left( \delta^i_j h_{kl} - \delta^i_k h_{jl} \right),
\]

\[
R^0_{0ij} = \left( r^3 f' + \frac{1}{2} r^4 f'' - \frac{n r^4 f'^2}{4f} \right) h_{ij}.
\]

Contracting \(R_{\mu \nu} = R^\tau_{\mu \tau \nu}\) to get the Ricci tensor, we find that the non-zero components are

\[
R_{00} = n \left( \frac{f''}{2f} + \frac{f'}{fr} - \frac{f'^2}{4f^2} \right),
\]

\[
R_{ij} = (\alpha) R_{ij} + \left( r^3 f' + \frac{1}{2} r^4 f'' - \frac{n r^4 f'^2}{4f} \right). \tag{4.6}
\]

We have finally the necessary information to calculate the projective curvature (4.2) and we find that the non-zero components are

\[
W^i_{jkl} = (\alpha) R^i_{jkl} + \frac{1}{n} \left( \delta^i_j (\alpha) R_{jk} - \delta^i_k (\alpha) R_{jl} \right) + \frac{1}{n} \left( r^3 f' + \frac{1}{2} r^4 f'' \right) \left( \delta^i_j h_{kl} - \delta^i_k h_{jl} \right), \tag{4.7}
\]

\[
W^0_{0ij} = \left( 1 - \frac{1}{n} \right) \left( r^3 f' + \frac{1}{2} r^4 f'' \right) h_{ij} - \frac{1}{n} (\alpha) R_{ij}. \tag{4.8}
\]

The dependence on \(f\) is solely through \(f'\) and \(f''\), so that adding a constant to \(f\) does not change the projective tensor.

4.1. An application—metrics of constant curvature

Let us suppose that \(h\) is a metric of constant curvature, so that we may write

\[
(\alpha) R^i_{jkl} = K (\delta^i_j h_{kl} - \delta^i_k h_{jl}), \quad (\alpha) R_{ij} = (n - 1) K h_{ij}, \tag{4.9}
\]

then the projective tensor simplifies to give

\[
W^i_{jkl} = \frac{1}{n} \left( r^3 f' + \frac{1}{2} r^4 f'' - K \right) \left( \delta^i_j h_{kl} - \delta^i_k h_{jl} \right). \tag{4.10}
\]
$W_{i0j} = \left(1 - \frac{1}{n}\right) \left(r^3 f'' + \frac{1}{2} r^4 f'' - K\right) h_{ij}.$ \hspace{1cm} (4.11)

We already know that the projective tensor vanishes if and only if the metric $g$ is constant curvature. In our case, this happens when $f$ satisfies the differential equation

$$r^3 f' + \frac{1}{2} r^4 f'' = K.$$ \hspace{1cm} (4.12)

This linear equation is readily solved to give

$$f(r) = \frac{K}{r^2} - \frac{2a}{r} + b$$ \hspace{1cm} (4.13)

with $a$ and $b$ being arbitrary constants. Using the components of the Riemann tensor derived above (4.4) we check and find that $g$ is indeed constant curvature, with

$$R_{\mu\nu\sigma\tau} = (bK - a^2) \left(\delta_{\mu\nu} g_{\sigma\tau} - \delta_{\mu\tau} g_{\sigma\nu}\right)$$ \hspace{1cm} (4.14)

so we can construct constant curvature metrics sliced by other constant curvature metrics, and can rule out the possibility of such constructions. Some caution must be exercised here, since changing the signature changes the sign of the right-hand side of (4.14) but not the left. We will take the view that the sign of the curvature is determined by (4.14) when $g$ has mostly plus signature, since this means $A dS$ has negative constant curvature and $dS$ has positive. For example, if we take $K = 1, b = -1, a = 0$ then $f$ is negative. In three dimensions, after a signature change, we have the metric

$$ds^2 = -dr^2 + \left(r^2 - 1\right)^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right)$$ \hspace{1cm} (4.15)

which for $r > 1$ is Lorentzian, with mostly plus signature. Since we changed signature, this is a constant positive curvature ($-bK = 1$). Setting $r = \coth t$ we recover the standard FRLW $k = 1$ metric for de Sitter:

$$ds^2 = -dt^2 + \cosh^2 t \left(d\theta^2 + \sin^2 \theta d\phi^2\right).$$ \hspace{1cm} (4.16)

It is straightforward using the results of this section to find all of the standard metrics for constant curvature spaces.

5. Constants of the motion

In [8] Topalov and Matveev showed using results of Painlevé and Levi-Civita that if two metrics, $g$ and $\overline{g}$, on an $n$-dimensional manifold are projectively equivalent, then it is possible to construct $n$ integrals of the geodesic flow of each metric. In the case where the metrics are strictly non-proportional at some point $x$, i.e., $\det (g - \lambda \overline{g})$ has only simple roots at $x$, then they showed, in addition, that this implied that the geodesic flow is integrable for both metrics. We can explicitly construct these integrals in the case when the metrics may be cast into the form (4.1), with corresponding functions $f, \overline{f}$. We first briefly review the construction of [8] before stating the result.

Given $g, \overline{g}$ metrics on $M^{n+1}$, we form an endomorphism of $TM, G = g^{-1}\overline{g}$. The characteristic polynomial $\det(G - \mu I) = c_n \mu^{n+1} + c_1 \mu^n + \ldots + c_{n+1}$ has coefficients $c_i$ which are smooth functions on the manifold. From them we may construct $S_k = \sum_{i=0}^k c_i G^{k-i}$, for $k = 0, \ldots, n$, which are endomorphisms of $TM$. Finally, the functions

$$I_k = (\det G)^{-\frac{k+1}{n+1}} S_k(\dot{x}, \dot{x})$$ \hspace{1cm} (5.1)

are conserved along a geodesic $x(t)$ of the metric $g$ and mutually commute.
If we now specialize to the case of metrics of the form \((4.1)\), one may show after some calculation that the integrals \(I_k\) may be reduced to the following form:

\[
I_k = (-1)^{n+1} \left[ 2(-1)^k \binom{n}{k} E_g + (\bar{f} - f) J^2 \sum_{i=0}^{k} (-1)^i \binom{n}{i} \right]. \tag{5.2}
\]

where

\[
E_g = \frac{1}{2} \left( \frac{\dot{r}^2}{r^2 f(r)} + \frac{h_{ij} \dot{x}^i \dot{x}^j}{f} \right) \quad \text{and} \quad J^2 = \frac{h_{ij} \dot{x}^i \dot{x}^j}{f^2} \tag{5.3}
\]

are conserved quantities under the geodesic flow of \(g\) and \(\bar{f} - f\) must be a constant by the geodesic equivalence of the metrics \(g, \bar{g}\). Thus we see that by this construction, we recover the conserved quantities of the geodesic flow, but do not gain any extra.

In [9] the authors of [8] showed that these classical constants of the motion have quantum-mechanical analogues. This means that for each \(I_k\) which may be written in the form

\[
I_k = I_k^{\mu
u} p_\mu p_\nu, \tag{5.4}
\]

where \(p_\mu\) are the canonical momenta of the geodesic flow, there exists an operator

\[
\mathcal{I}_k = \nabla_\mu I_k^{\mu
u} \nabla_\nu \tag{5.5}
\]

which commutes with the scalar Laplacian. It is not true in general that a classical constant of the motion of the form (5.4) arising from a Stäckel–Killing tensor gives rise to such an operator. In general there is an anomaly, as found by Carter [10]. We may calculate the form of these operators for our family of metrics and we find that they take precisely the form (5.2) with \(E_g\) and \(J^2\) replaced by the operators

\[
\mathcal{E}_g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} \sqrt{g} g^{\mu
u} \frac{\partial}{\partial x^\nu} \quad \text{and} \quad \mathcal{J}^2 = \frac{1}{\sqrt{h}} \frac{\partial}{\partial x^i} \sqrt{h} h^{ij} \frac{\partial}{\partial x^j}, \tag{5.6}
\]

respectively. Here \(\mu, \nu\) range over \(0, \ldots, n\), where 0 refers to the (not necessarily timelike) \(r\) direction and \(i, j\) range over \(1, \ldots, n\). These clearly commute with the scalar Laplacian, which is in fact given by \(\mathcal{E}_g\) and hence \(\mathcal{I}_k\) give conserved quantum numbers.

6. Conclusion

We have shown that the fact that the light rays of the Kottler metric are independent of the cosmological constants follows from the projective equivalence of the optical metrics for all values of \(\Lambda\). Motivated by this we have found a large collection of one-parameter families of metrics which are projectively equivalent for any value of the parameter. As a sub-case we find the Kottler optical metrics together with the constant curvature metrics of any signature. Using results of Matveev and Topalov we showed that this implies the existence of classical and quantum conserved quantities and we related these to known quantities.

References

[1] Islam J N 1983 The cosmological constant and classical tests of general relativity Phys. Lett. 97A 239
[2] Kottler F 1918 Über die physikalischen Grundlagen der Einsteinschen Gravitationstheorie Ann. Phys. (Berlin) 56 401–61
[3] Sereno M 2008 Influence of the cosmological constant on gravitational lensing in small systems Phys. Rev. D 77 043004 (arXiv:0711.1802)
[4] Park M 2008 Rigorous approach to gravitational lensing Phys. Rev. D 78 023014 (arXiv:0804.4331)
[5] Synge J L 1960 Relativity: The General Theory (Amsterdam: North-Holland)
[6] McVittie G C 1933 Mon. Not. R. Astron. Soc. 93 325
See also McVittie G C 1965 General Relativity and Cosmology (London: Chapman and Hall)
[7] Eisenhart L P 1926 Riemannian Geometry (Princeton, NJ: Princeton University Press)
[8] Topalov P J and Matveev V S 1999 Geodesic equivalence and integrability arXiv:math.DG/9911062
[9] Matveev V S and Topalov P J 2001 Quantum integrability of Beltrami–Laplace operator as geodesic equivalence
Math. Z. no 4 238 833–66
[10] Carter B 1977 Killing tensor quantum numbers and conserved currents in curved space Phys. Rev. D 16 3395