Abstract: A few elementary estimates of a basic character sum over the prime numbers are derived here. These estimates are nontrivial for character sums modulo large $q$. In addition, an omega result for character sums over the primes is also included.

Mathematics Subject Classifications: Primary 11L20, 11L40; Secondary 11L03, 11L07.
Keyword: Character sums, Exponential sums, Gauss sums, Polya-Vinogradov inequality, Prime numbers.

1 Introduction
Let $q \geq 1$ be an integer, and let $\chi \neq 1$ be a nonprincipal character modulo $q$. The tasks of determining nontrivial estimates and explicit estimates of the basic character sums

$$\sum_{p \leq x} \chi(p), \quad \text{and} \quad \sum_{p \leq x} \chi(p)$$

over the primes, arithmetic progressions or the integers up to $x \geq 1$ are extensively studied problems in the analytic number theory literature, [KA], [GS], [GK], [GR], [PM], [TR], et alii.

The earliest nontrivial estimate of the basic character sum over the prime numbers, in the literature, seems to be the Vinogradov estimate

$$\sum_{p \leq x} \chi(p) = O(\pi(x)q^{-\delta})$$

for arbitrarily small real number $\varepsilon > 0$, all sufficiently large number $x > q^{1+\varepsilon}$, and $\delta = \delta(\varepsilon) > 0$, see [KA, p. 156]. This is slightly better than the trivial estimate $\sum_{p \leq x} \chi(p) \leq \pi(x)$, where $\pi(x) = \# \{ p \leq x : p \text{ prime} \}$.

Many other estimates of various forms such as $\sum_{p \leq x} \chi(p) = O(x / \log^B x)$, $B > 1$ constant, are available in the
literature, see [IK, p. 348], [FR], and similar references. A survey of open problems in exponential and character sums appears in [SK].

The elementary estimates and evaluation in Theorems 1, 2, 3, and 4 for character sums over the prime numbers are derived by elementary methods, and appear to be nontrivial for character sums over the prime numbers modulo large $q$.

2 Character Sums Over The Prime Numbers

There are various other related, and more sophisticated results available in the literature, confer the more recent works given in [KA], [FR], et alii. The estimates of character sums over the primes derived here, by elementary methods, are simpler, and similar to the Polya-Vinogradov inequality, and the Paley inequality in forms and the theoretical frameworks.

The proof uses a windowing technique to spread the calculations into two parts: the main component, and an error component. This spreading is intrinsic in the Fourier series of a window function. It closely follows the pattern of the proof of the Polya-Vinogradov inequality using a window function constructed in [PM], and attributed to Landau.

**Theorem 1.** Let $q \geq 3$ be a large integer, and let $\chi \neq 1$ be a nonprincipal character modulo $q$. Then

\[
\left| \sum_{p \leq x} \chi(p) \right| = O(q^{1/2+\varepsilon})
\]

for any large number $x \geq 1$ such that $x \leq q^{1-\varepsilon}$, and arbitrarily small $\varepsilon > 0$. In particular,

\[
\max_{1 \leq x \leq q^{1-\varepsilon}} \left| \sum_{p \leq x} \chi(p) \right| = O(q^{1/2+\varepsilon}).
\]

**Proof:** Utilize the window function $W(t)$, see Lemma 7, to rewrite the character sum over the primes as

\[
\sum_{p \leq x} \chi(p) = \frac{\chi(x)}{2} + \sum_{l \leq p < q} \chi(p) W \left( \frac{2\pi p}{q} \right)
\]

\[
= \frac{\chi(x)}{2} + \sum_{l \leq p < q} \chi(p) \left( a_0 + \sum_{m \geq 1} \left( a_m \cos \frac{2\pi mp}{q} + b_m \sin \frac{2\pi mp}{q} \right) \right)
\]

\[
= \frac{\chi(x)}{2} + \sum_{l \leq p < q} \chi(p) \left( a_0 + \sum_{k \geq 1} \left( a_m \cos \frac{2\pi mp}{q} + b_m \sin \frac{2\pi mp}{q} \right) \right)
\]

\[
+ \sum_{l \leq p < q} \chi(p) \left( \sum_{K \geq 1} \left( a_m \cos \frac{2\pi mp}{q} + b_m \sin \frac{2\pi mp}{q} \right) \right)
\]

\[
= P_K + Q_K + \frac{\chi(x)}{2} + \frac{x}{q} \sum_{l \leq p < q} \chi(p),
\]

where $P_K$ and $Q_K$ are the main and error components, respectively.
where the initial parameters are set to $M = 0$, $N = x$, and $a_0 = x/q$, see Lemma 7. Further, the parity of the character $\chi \neq 1$ classifies the character sum as one of the two types:

$$
\rho(\chi) \sum_{p \in \mathbb{P}} \chi(p) = \begin{cases} 
\sum_{p \in \mathbb{P}} \sum_{t \equiv a \mod q} \overline{\chi}(t) \cos \frac{2\pi pt}{q}, & \text{if } \chi(-1) = 1 \text{ is even}, \\
\sum_{p \in \mathbb{P}} \sum_{t \equiv a \mod q} \overline{\chi}(t) \sin \frac{2\pi pt}{q}, & \text{if } \chi(-1) = -1 \text{ is odd},
\end{cases}
$$

(6)

where $\rho(\chi) = \sum_{p \in \mathbb{P}} \chi(p) e^{2\pi p/q}$ is the complete prime exponential sum, see Theorem 2, and Lemmas 5 and 6. In synopsis, this simplifies the analysis since the sine terms vanish if the character is even; likewise, the cosine terms vanish if the character is odd.

**Case of Even Character $\chi(-1) = 1$.** In this case the expressions $P_K$ and $Q_K$ have the following upper bounds:

$$
P_K = \sum_{1 \leq p < q} \chi(p) \sum_{m \leq K} \left( a_m \cos \frac{2\pi mp}{q} + b_m \sin \frac{2\pi mp}{q} \right)
$$

$$
= \sum_{1 \leq p < q} \sum_{m \leq K} a_m \chi(p) \cos \frac{2\pi mp}{q}
$$

$$
= \rho(\chi) \sum_{m \leq K} a_m \overline{\chi}(m),
$$

(7)

refer to equations (5) and (6). Now, replacing the coefficients $a_m$, see Lemma 7, taking absolute value, and simplifying return

$$
|P_K| = \left| \rho(\chi) \sum_{m \leq K} a_m \overline{\chi}(m) \right|
$$

$$
= \left| \rho(\chi) \sum_{m \leq K} \left( \frac{1}{\pi m} \sin \frac{2\pi mx}{q} \overline{\chi}(m) \right) \right|
$$

$$
\leq q^{1/2} \sum_{m \leq K} \frac{1}{\pi m}
$$

$$
\leq c_0 q^{1/2} \log K,
$$

(8)

where $|\rho(\chi)| \leq \pi(q)^{1/2} \leq q^{1/2}$, see Theorem 2, and $c_0 > 0$ is a constant. A slightly different procedure is applied to the expression $Q_K$: replacing the coefficients $a_m$ yields

$$
Q_K = \sum_{1 \leq p < q} \chi(p) \sum_{K < m} \left( a_m \cos \frac{2\pi mp}{q} + b_m \sin \frac{2\pi mp}{q} \right)
$$

$$
= \sum_{1 \leq p < q} \chi(p) \sum_{K < m} \left( \frac{-1}{\pi m} \cos \frac{2\pi mx}{q} \right) \cos \frac{2\pi mp}{q}
$$

$$
= \frac{-1}{\pi} \sum_{1 \leq p < q} \chi(p) \sum_{K < m} \frac{1}{m} \left( \sin \frac{2\pi (x - p)m}{q} + \sin \frac{2\pi mx}{q} \right),
$$

(9)
see equation (5), and Lemma 7. Now, taking absolute value, and simplifying return

\[
\left| Q_K \right| = \left| \frac{1}{\pi} \sum_{1 \leq p \leq q} \chi(p) \sum_{k \leq m} \frac{1}{m} \left( \sin \frac{2\pi(x-p)m}{q} + \sin \frac{2\pi mp}{q} \right) \right|
\]

\[
\leq 2 \left| \sum_{1 \leq p \leq q} \chi(p) \sum_{k \leq m} \frac{1}{\pi m} \left( \sin \frac{2\pi mp}{q} \right) \right|
\]

\[
\leq \frac{4}{\pi} \sum_{1 \leq p \leq q} \sum_{k \leq m} \left| \frac{1}{m} \sin \frac{2\pi mp}{q} \right|
\]

\[
\leq c_1 \frac{q}{K+1} \log K,
\]

where \( c_1 > 0 \) is a constant, this follows from the estimate of Lemma 8. Rearrange (5) as

\[
\mathcal{S}(x) = \frac{q}{q-x} \left( P_k + Q_k + \frac{\chi(x)}{2} \right),
\]

and consider the list of all the estimates:

\[
\left| P_k \right| \leq c_0 q^{1/2} \log K, \quad \left| Q_K \right| \leq c_1 \frac{q}{K+1} \log K, \quad \left| \frac{\chi(x)}{2} \right| \leq 1.
\]

Put \( K = K(x) \), for example, \( K = x^2 \) if \( x \geq q^{1/2} \), or \( K = x^4 \) if \( x \geq q^{1/4} \), etc. Then, it quickly follows that \( q/(q-x) \leq c_2 \) for any large number \( x \geq 1 \) such that \( x \leq q^{1-\varepsilon} \), \( c_2 > 0 \) constant. Lastly, applying the triangle inequality yields

\[
\left| \sum_{p \leq x} \chi(p) \right| \leq \frac{q}{q-x} \left( P_k + Q_k + \frac{\chi(x)}{2} \right) \leq c_3 q^{1/2+\varepsilon},
\]

where \( c_3 > 0 \) constant.

**Case of Odd Character \( \chi(-1) = -1 \).** The proof is similar.

For a fixed \( q \geq 3 \), and nonprincipal character \( \chi \neq 1 \) modulo \( q \), the prime character sum \( \sum_{p \leq x} \chi(p) \) is unbounded as \( x \to \infty \), see Theorem 4. But it has an upper bound on the interval \([1, x], x \leq q\). The norm of the complete prime exponential sum \( \rho(\chi) = \sum_{p \leq q} \chi(p)e^{2\pi p/q} \) is calculated below.
**Theorem 2.** Let \( q \geq 3 \) be a large integer, and let \( \chi \neq 1 \) be a character modulo \( q \). Then \( |\rho(\chi)| \leq \sqrt{q} \).

**Proof:** Let \( t \not= 0 \) be a parameter. By definition, the norm is given by

\[
|\rho(t, \chi)|^2 = \left( \sum_{p \leq q} \chi(p) e^{i2\pi pt/q} \right) \left( \sum_{r \leq q} \overline{\chi}(r) e^{-i2\pi rt/q} \right)
= \pi(q) + a + \sum_{p, r \leq q, \ p \neq r} \chi(pr^{-1}) e^{i2\pi (p-r)t/q},
\]  

(14)

where \( p \) and \( r \) run over the prime numbers up to \( q \geq 3 \), \( \pi(q) \) denotes the number of primes counting function, \( \omega(n) = \# \{ p \mid n \} \), denotes the number of prime divisors counting function, and

\[
a = \begin{cases} 
\omega(q) - 1 & \text{if } q \text{ is prime}, \\
\omega(q) & \text{if } q \text{ is composite}.
\end{cases}
\]  

(15)

The averages of the norm \( |\rho(t, \chi)|^2 \) with respect to the variables \( t \) and \( \chi \) are:

\[
E_x = \frac{1}{q} \sum_{0 < \chi < q} |\rho(t, \chi)|^2 = \pi(q) + a, \quad \text{and} \quad E_y = \frac{1}{q(q-1)} \sum_{0 < \chi < q} |\rho(y, \chi)|^2 = \pi(q) + a
\]

(16)

respectively. These information shows that the norm \( |\rho(t, \chi)|^2 \) of any prime exponential sum \( \rho(t, \chi) \) is independent of the variables \( t \) and \( \chi \). Moreover, the oscillating error term in (14) is a complex number

\[
R(t, \chi) = \sum_{p, r \leq q, \ p \neq r} \chi(pr^{-1}) e^{i2\pi (p-r)t/q} = |R(t, \chi)| e^{i\theta}
\]

(17)

where the magnitude \( |R(t, \chi)| \geq 0 \), and the angle \( \theta = \theta(t, \chi) \) are functions of the variables \( t \) and \( \chi \). Since

\[
-|R(t, \chi)| \leq |R(t, \chi)| e^{i\theta} \leq |R(t, \chi)|,
\]

(18)

And the norm is nonnegative

\[
|\rho(t, \chi)|^2 = \pi(q) + a + |R(t, \chi)| e^{i\theta} \geq 0.
\]

(19)

It readily follows that the error term is bounded as \( |R(t, \chi)| \leq \pi(q) + a \). Therefore, the norm has the form

\[
|\rho(t, \chi)|^2 = \pi(q) + a + R(t, \chi) \leq q,
\]

(20)

with the error term \( |R(t, \chi)| = O(q / \log q) \).
3 Comparisons Of Character Sums Over The Primes And Integers

The two basic character sums (1) are linked via the formula

$$\sum_{p \leq x} \chi(p) = \sum_{n \leq x} \chi(n) \left( - \sum_{d|n} \mu(d) \frac{\log d}{\log n} \right) + O(x^{1/2}),$$

(21)

so it should not be surprising that these character sums have comparable upper bounds on the interval $[1, x]$. This relationship arises after an application of the Mobius inversion pair

$$\sum_{d|n} \Lambda(d) = \log n \quad \text{and} \quad \Lambda(n) = \sum_{d|n} \mu(d) \log n / d,$$

(22)

and the identity

$$\sum_{p \leq x} \chi(p) = \sum_{n \leq x} \chi(n) \frac{\Lambda(n)}{\log n} + O(x^{1/2}),$$

(23)

which decouples the complicated character sum over the primes into a product of two simpler character sums.

Furthermore, the mean square value basic character sum over the primes is given by

$$\sum_{\chi \in S(Q)} \left| \sum_{p \leq x} \chi(p) \right|^2 \leq c Q^{\beta+\epsilon},$$

(24)

where $S(Q)$ is the set of primitive characters of conductor up to $Q$, $\beta = \max \{ 1 + \alpha / 2, 1 / 2 + \alpha \}$, $0 < \alpha \leq 1$, and $c > 0$ constant, see [HB] for more general results. Compare this to (15).

The estimate of the basic character sum $\sum_{n \leq x} \chi(n)$ over the integers is given by the Polya-Vinogradov inequality.

**Theorem 3.** Let $q \in \mathbb{N}$ be a large integer, and let $\chi \neq 1$ be a character modulo $q$. Then

$$\left| \sum_{n \leq x} \chi(n) \right| \leq 2q^{1/2} \log q$$

(25)

for any real number such that $x > q^{1-\epsilon}$, $\epsilon > 0$.

**Proof:** Let $q \equiv 2 \mod 4$, and let $\chi \neq 1$ be a character modulo $q$. The basic character sum over the integers can be written as

$$\sum_{n \leq x} \chi(n) = \sum_{n \leq x} \frac{1}{\tau(\chi)} \sum_{t \leq q} \bar{\chi}(t) e^{2\pi i nt/q},$$

(26)

see Lemma 6. Rearrange and evaluate this representation into the form
\[ \sum_{n \leq x} \chi(n) = \frac{1}{\tau(\chi)} \sum_{t \equiv q} \mathcal{X}(t) \left( e^{i2\pi (x+1)/t} - 1 \right), \]

(27)

where \(|\tau(\chi)| = \sqrt{q}\), or \(|\tau(\chi)| = \sqrt{2q}\). An upper estimate is

\[ \left| \sum_{n \leq x} \chi(n) \right| \leq \frac{1}{\sqrt{q}} \sum_{t \equiv q} \frac{1}{|\sin \pi t / q|} \leq cq^{1/2} \log q, \]

(28)

where \(|\sin \pi t / q| \approx \pi t / q\) for \(1 \leq t < q\) was utilized, and \(c > 0\) is a constant.

The exceptional value \(q = 2 \mod 4\) stems from the vanishing of the Gaussian sum \(\sum_{0 \leq t < p} \chi(t) e^{i2\pi t / q} = 0\) whenever \(\chi \neq 1\) is a quadratic character modulo \(q = 2 \mod 4\), see Lemma 6. The numbers \(q = 2 \mod 4\) seem to be exceptional values of the Polya-Vinogradov inequality \(\sum_{n \leq x} \chi(n) \leq c q^{1/2} \log q, c > 0\) constant. But this seems to be irrelevant since there is different proof of the Polya-Vinogradov inequality, refer to [DW].

The Polya-Vinogradov estimate is nontrivial for any real number such that \(x > q^{1-x}, \varepsilon > 0\). The estimates of character sums over short intervals of length \(x < q^{1-x}\) are more delicate and complex. The analysis for the short range of values \(x < q^{1-x}\) are given in [GS], [GK], [GR], [BR], LZ and similar literature. The nontrivial estimates of \(\sum_{n \leq x, n \equiv a \mod q} \chi(n)\) over arithmetic progressions \(\{ p = qn + a : \gcd(a, q) = 1, n \geq 1 \}\) are given in [FR].

4 Omega Result For Character Sums Over The Prime Numbers

The Polya-Vinogradov inequality is about the best possible since the Paley inequality

\[ \sum_{n \leq x} \chi(n) \geq cq^{1/2} \log \log q \]

(29)

holds for some characters \(\chi \neq 1\), and infinitely many primes \(q\), and \(c > 0\) constant, confer [MV, p. 312] for a proof. A similar omega result can be obtained for the basic character sums over the prime numbers.

Let \(q\) be a large integer, and let \(g\) be a primitive root modulo \(q\). For \(q \neq 2^m\), a multiplicative character has the form \(\chi_k(n) = e^{i2\pi k \text{Ind}_n / q(\phi(q))}\), where \(0 \leq k < q\), and \(\text{Ind}_n\) is the discrete logarithm modulo \(q\), see Section 6. The character \(\chi\) is called even if \(\chi(-1) = 1\), otherwise, it is odd and \(\chi(-1) = -1\). For example, every character such that \(\gcd(2, k) = 1\) is odd.

The cancelation mechanism of some Dirichlet characters springs from some unique pairwise partitions of the multiplicative group of the integers \(\mathbb{Z}_q\) modulo \(q\). Let the congruence \(p \equiv g^v \mod q, 0 \leq v < \varphi(q) - 1\), specifies the \(\varphi(q)\) equivalent classes of primes modulo \(q\). Some unique pairwise partitions of the multiplicative group modulo \(q\) are as follows:
(i) For odd character $\chi(-1) = -1$ modulo $q = 2m + 1$, the pairwise partition is
\[
\left(g^v, g^{v+\phi(q)/2}\right), \quad \text{for } 0 \leq v < \varphi(q)/2,
\]
(30)
(ii) For quartic character $\chi(\pm i) = -1$ modulo $q = 4m + 1$, the pairwise partition is
\[
\left(g^v, g^{v+\phi(q)/4}\right), \quad \left(g^{v+1+\phi(q)/2}, g^{v+1+3\phi(q)/4}\right), \quad \text{for } 0 \leq v < \varphi(q)/4,
\]
and similar pairwise partitions for some other $q = 2^n m + 1$, and $a > 2$, mutatis mutandis. In addition, the prime counting function on the arithmetic progression $\{ p = qn + a : \gcd(a, q) = 1, \text{ and } n \geq 1 \}$ is defined by
\[
\pi(x, a, q) = \# \{ p \leq x : p \equiv a \mod q \} = \varphi(q)^{-1}li(x) + E(x, a, q),
\]
where $li(x) = \int_1^x (\log t)^{-1} \, dt$ is the logarithmic integral, and $E(x, a, q)$ is the prime number theorem error term.

**Theorem 4.** Let $q \in \mathbb{N}$ be a fixed integer, and let $\chi = 1$ be an odd character modulo $q$. Then
\[
\sum_{p \leq x} \chi(p) = \Omega_z \left( \frac{x^{1/2} \log \log \log x}{\log x} \right)
\]
for some sufficiently large real number $x \geq q^{1-\varepsilon}$, and $\varepsilon > 0$ arbitrarily small. In particular,
\[
\sum_{p \leq x} \chi(p) = \Omega_z \left( q^{1/2} \log \log q / \log q \right)
\]
for infinitely many large $q$ and odd character.

Proof: Let $g$ be a primitive root modulo $q$, and choose an odd character $\chi(g^{\varphi(q)/2}) = \chi(-1) = -1$. Since $\chi(g^{i\varphi(q)/2}) = \chi(g^v)\chi(g^{\varphi(q)/2}) = -\chi(g^v)$, the partition pairing $g^v$, $g^{v+\varphi(q)/2}$ of residue classes, see (10?), yields
\[
\pi(x, g^v, q)\chi(g^v) + \pi(x, g^{v+\varphi(q)/2}, q)\chi(g^{v+\varphi(q)/2}) - \pi(x, g^v, q)\chi(g^v) = \left( E(x, g^v, q) - E(x, g^{v+\varphi(q)/2}, q) \right) \chi(g^v).
\]
(34)
Thus, it follows that the main terms $\pi(x, g^v, q) \sim \varphi(q)^{-1}li(x)$ and $\pi(x, g^{v+\varphi(q)/2}, q) \sim \varphi(q)^{-1}li(x)$ cancel pairwise, $0 \leq v < \varphi(q)/2$. As a consequence, the basic character sum over the primes collapses to
\[
\sum_{p \leq x} \chi(p) = \sum_{0 \leq v < \varphi(q)/2} \sum_{p \equiv g^v \mod q} \chi(p)
\]
\[
= \sum_{0 \leq v < \varphi(q)/2} \chi(g^v)\pi(x, g^v, q)
\]
\[
= \sum_{0 \leq v < \varphi(q)/2} \chi(g^v)E(x, g^v, q),
\]
(35)
which is a complex linear combination of the error terms \( E(x, g^v, q) \) of the \( q(q) \) equivalent classes of primes modulo \( q \). Now, the result follows from the Littlewood form of the prime number theorem

\[
\pi(x, a, q) = \frac{li(x)}{q(q)} + \Omega_x \left( \frac{x^{1/2} \log\log\log x}{q(q) \log x} \right),
\]

on the arithmetic progression \( \{ p = qn + a : \gcd(a, q) = 1, \text{ and } n \geq 1 \} \). Refer to [MV, p. 479], [IV, p. 51], and similar literature.

A few examples were computed to demonstrate the concept, and the challenges faced in estimating small character sums over the primes. Here, the congruence \( p \equiv g^v \mod q, 0 \leq v < q(q) - 1 \), specifies the \( q(q) \) equivalent classes of primes modulo \( q \), and the corresponding odd character \( \chi(g^{q(q)/2}) = \chi(-1) = -1 \).

1. \[ \sum_{p \leq x} \chi(p) = \pi(x, 1, 3) - \pi(x, 2, 3) = \Omega_x \left( x^{1/2} \log\log\log x / \log x \right), \quad \text{for } q = 3, \ \chi_3(2^v) = (-1)^v, \ g^v \equiv 2^v \mod 3. \]

2. \[ \sum_{p \leq x} \chi(p) = \pi(x, 1, 4) - \pi(x, 3, 4) = \Omega_x \left( x^{1/2} \log\log x / \log x \right), \quad \text{for } q = 4, \ \chi_4(3^v) = (-1)^v, \ g^v \equiv 3^v \mod 4. \]

3. \[ \sum_{p \leq x} \chi(p) = \pi(x, 1, 5) + i\pi(x, 2, 5) - i\pi(x, 3, 5) - \pi(x, 4, 5) = \Omega_x \left( x^{1/2} \log\log\log x / \log x \right), \]

for \( q = 5, \ \chi_5(2^v) = i^v, \ g^v \equiv 2^v \mod 5 \).

Similar, and other related advanced topics in comparative number theory, are discussed in [RN, p. 275], [GM], [MJ], [RS], [SC], et alii.

**Theorem 5.** Let \( q \in \mathbb{N} \) be an integer, and let \( \chi \equiv 1 \) be a character modulo \( q \). Then

\[
\sum_{p \leq x} \chi(p) = \mathcal{O}\left( x e^{-c(\log x)^{3/2}} \right)
\]

for some sufficiently large real number \( x \geq x_0 \), and \( c > 0 \) constant.

Proof: Since \( \sum_{l \in \mathbb{Z} \cap p(q)} \chi(a) = 0 \), the basic character sum over the primes collapses to

\[
\sum_{p \leq x} \chi(p) = \sum_{l \in \mathbb{Z} \cap p(q)} \sum_{p \leq x, p \equiv a \mod q} \chi(p)
\]

\[= \sum_{l \in \mathbb{Z} \cap p(q)} \chi(a) \pi(x, a, q) \]

\[= \sum_{l \in \mathbb{Z} \cap p(q)} \chi(a) E(x, a, q), \]

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which is a complex linear combination of the error terms \( E(x, a, q) \) of the \( \phi(q) \) equivalent classes of primes modulo \( q \). Now, the result follows from the delaVallee Poussin form of the prime number theorem

\[
\pi(x, a, q) = \frac{li(x)}{q(q)} + O\left( \frac{x e^{-\text{erf}(\frac{1}{2})}}{q(q)} \right),
\]

(40)
on the arithmetic progression \( \{ p = qn + a : \gcd(a, q) = 1, \text{ and } n \geq 1 \} \). Refer to [MV, p. 479], [IV, p. 51], and similar literature.

5. Elementary Foundation

This Section serves as a reference of some of the concepts used to complete the proof of the main results in previous Section.

5.1 A Few Arithmetic Functions

Let \( n = p_1^{v_1} \cdot p_2^{v_2} \cdots p_r^{v_r} \), let \( \omega(n) = \#\{ p \mid n \} \) be the number of prime divisors counting function, and let \( \Omega(n) = v_1 + v_2 + \cdots + v_r \). The Mobius function defined by

\[
\mu(n) = \begin{cases} 
(-1)^{\omega(n)} & \text{if } \omega(n) = \Omega(n), \\
0 & \text{if } \omega(n) \neq \Omega(n),
\end{cases}
\]

(41)

the vonMangold function defined by

\[
\Lambda(n) = \begin{cases} 
\log p & \text{if } n = p^k, \\
0 & \text{if } n \neq p^k,
\end{cases}
\]

(42)

where \( p^k \) is a prime power. And let the Euler function be defined by

\[
\varphi(q) = \prod_{p \mid q} (1 - 1/p),
\]

(43)

where \( p \) ranges over the prime divisors of \( q \). A few other number theoretical functions are also used throughout the paper.

5.2 Multiplicative Characters

A multiplicative character \( \chi \) is a periodic, complex valued and completely multiplicative function \( \chi : \mathbb{Z} \rightarrow \mathbb{C} \) on the integers. For each \( q \in \mathbb{N} \), the set of characters \( \hat{G} = \{ 1, \chi_1, \ldots, \chi_{\varphi(q)-1} \} \) is a group of order \( \varphi(q) \).

**Lemma 6.** If a function \( f : \mathbb{Z} \rightarrow \mathbb{C} \) satisfies \( f(n) \equiv 0 \mod q \) for \( \gcd(n, q) \neq 1 \), is periodic, and completely multiplicative, then \( f(n) = \chi(n) \) is a character modulo \( q \).
Properties of Nontrivial Characters

(i) $\chi(1) = 1$ and $\chi(s) \neq 1$, for $\gcd(s, q) > 1$,
(ii) $\chi(st) = \chi(s)\chi(t)$, multiplicative,
(iii) $\chi(s + k) = \chi(s)$, periodic of period $k \geq 1$,
(iv) $|\chi(s)| = 1$, a point in unit circle.

There are several forms of the multiplicative characters depending on the decomposition of the integer $q$.

For $q \neq 2^m$, $m \geq 1$. There are $\varphi(q)$ characters in $\hat{G} = \{1 = \chi_0, \chi_1, \ldots, \chi_{\varphi(q)-1}\}$, the principal character $\chi = 1$ and nonprincipal characters $\chi \neq 1$ are defined by

$$\chi_0(n) = \begin{cases} 1 & \text{if } \gcd(n, q) = 1, \\ 0 & \text{if } \gcd(n, q) \neq 1, \end{cases} \quad (44)$$

and

$$\chi_k(n) = \begin{cases} e^{2\pi i k \log(n)/\varphi(q)} & \text{if } \gcd(n, q) = 1, \\ 0 & \text{if } \gcd(n, q) \neq 1, \end{cases} \quad (45)$$

respectively. The notation $\log_q(n) = \text{Ind}_g(n)$ denotes the discrete logarithm with respect to some primitive root $g \mod q$.

For $q = 2^v$, $v \geq 1$. A character is realized by one of the three forms described below.

Case $v = 1$, there is a single character in $\hat{G} = \{1 = \chi_0\}$, and it is defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 2, \\ 0 & \text{otherwise.} \end{cases} \quad (46)$$

Case $v = 2$. There are two characters in $\hat{G} = \{1 = \chi_0, \chi\}$, the nontrivial character is defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod 4, \\ -1 & \text{if } n \equiv 3 \mod 4, \\ 0 & \text{if } n \equiv 0, 2 \mod 4. \end{cases} \quad (47)$$

Case $v > 2$. There are $\varphi(q)$ characters in $\hat{G} = \{1 = \chi_0, \chi_1, \ldots, \chi_{\varphi(q)-1}\}$, and a nontrivial character is defined by

$$\chi_s(n) = \begin{cases} (-1)^s e^{2\pi i t/\varphi(q)} & \text{if } n \equiv 1 \mod 2, \\ 0 & \text{if } n \equiv 0 \mod 2, \end{cases} \quad (48)$$

for some $0 \leq s, t < \varphi(q)$. The integer $n$ is represented as $n \equiv (-1)^s 5^t \mod 2^v$ in the multiplicative group of units $\{-1, 1\} \times \{5^t : 0 \leq t < \varphi(2^v)\}$ of $\mathbb{Z}_q$, where $\delta = 0$ if $n \equiv 1 \mod 4$ or $\delta = 1$ if $n \equiv 3 \mod 4$. This is due to the fact that this multiplicative group is not cyclic.
A character $\chi$ is even if $\chi(t) = \chi(-t)$, otherwise $\chi(t) = -\chi(-t)$, and the character is odd. The binary variable $\delta_t = 0$, 1 tracks the even odd condition, specifically, $\chi(-n) = (-1)^{\delta_t} \chi(n)$. A character $\chi$ is primitive if no proper subgroup of the group $\hat{G} = \{1 = \chi_0, \chi_1, \ldots, \chi_{\varphi(q)-1}\}$ contains it. Under this condition the conductor of a character is the integer $f = q$.

**Lemma 7.** (Orthogonal relations) For $a \geq 1$, and the set of characters modulo $q$, the followings hold.

\begin{align*}
(i) & \sum_{1 \leq n < q} \chi(n) = \begin{cases} 
q & \text{if } \chi = \chi_0, \\
0 & \text{if } \chi \neq \chi_0.
\end{cases} \\
(ii) & \sum_{\chi \mod q} \chi(n) = \begin{cases} 
q & \text{if } n = 0 \mod q, \\
0 & \text{otherwise}.
\end{cases}
\end{align*}

A Gauss sum is defined by the exponential sum

$$
\tau_a(\chi) = \sum_{n=1}^{q} \chi(n)e^{\frac{2\pi i an}{q}}, 0 \leq a < q.
$$

**Lemma 8.** Let $\chi \neq \chi_0$ be a nontrivial character modulo $q$, and let $\tau(\chi) = \tau_1(\chi)$. Then

\begin{align*}
(i) & \quad \tau_a(\chi) = \chi(a)\tau(\chi), \\
(ii) & \quad \tau_a(\chi)\tau_a(\chi) = q, \\
(iii) & \quad \tau_a(\chi) = \begin{cases} 
(1+i)q^{1/2} & \text{if } q \equiv 0 \mod 4, \\
q^{1/2} & \text{if } q \equiv 1 \mod 4, \\
0 & \text{if } q \equiv 2 \mod 4, \\
iq^{1/2} & \text{if } q \equiv 3 \mod 4.
\end{cases}
\end{align*}

Let $q \equiv 2 \mod 4$, and let $\chi \neq 1$ be a nonprincipal character modulo $q$. The Fourier transform

$$
\chi(n) = \frac{1}{\tau(\chi)} \sum_{t=1}^{q} \overline{\chi}(t)e^{-\frac{2\pi i nt}{q}}
$$

is a complex-valued function that interpolates the Dirichlet character $\chi$ form $\mathbb{N}$ to $\mathbb{C}$. For $q \equiv 2 \mod 4$, it is undefined. The interpolation formula has a simpler form identified by the parity of the character:

$$
\tau(\chi)\chi(n) = \begin{cases} 
\sum_{t \equiv 0 \mod q} \overline{\chi}(t)\cos \frac{2\pi mt}{q} & \text{if } \chi(-1) = 1 \text{ is even}, \\
\sum_{t \equiv 0 \mod q} \overline{\chi}(t)\sin \frac{2\pi mt}{q} & \text{if } \chi(-1) = -1 \text{ is odd}.
\end{cases}
$$

### 5.3 Window Function, and Estimate of Trigonometric Series

Windowing schemes are widely used in signal analysis and number theory. For a pair of integers $M < N$, defines the window function
on the interval $[0, 2\pi)$.

**Lemma 9.** The Fourier series of the window function $W(x)$ is given by

$$W(x) = a_0 + \sum_{m=1} \left( a_m \cos mx + b_m \sin mx \right),$$

where the first coefficient is $a_0 = (N - M)/q$, and for $m \geq 1$, the coefficients are given by

$$a_m = \frac{1}{\pi m} \left( \sin \frac{2\pi Nm}{q} - \sin \frac{2\pi Mm}{q} \right), \quad \text{and} \quad b_m = -\frac{1}{\pi m} \left( \cos \frac{2\pi Nm}{q} - \cos \frac{2\pi Mm}{q} \right).$$

The pair of functions (48) and (49) is a Fourier pairs. Other well known Fourier pairs are the followings:

$$f(t) = \begin{cases} 0, & \text{if } |t| > 1, \\ 1, & \text{if } |t| < 1, \\ 1/2, & \text{if } |t| = 1, \end{cases} \quad \text{and} \quad \hat{f}(s) = \frac{\sin 2\pi s}{\pi s},$$

and

$$g(t) = \max \{ 1 - |t|, 0 \} \quad \text{and} \quad \hat{g}(s) = \left( \frac{\sin \pi s}{\pi s} \right)^2,$$

for $t \in \mathbb{R}$, and $s \in \mathbb{C}$. Other versions and proofs of Lemmas 7 and 8 are discussed in [PM, p. 4].

**Lemma 10.** Let $q \geq 3$ be an integer, and let $K > 1$ be a large number. Then

$$\sum_{1 \leq n \leq q} \left| \sum_{k \leq m/q} \frac{1}{m} \sin(mn/q) \right| \leq c \frac{q}{K+1} \log K,$$

where $c > 0$ is a constant.
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