Unified description of pairing and quarteting correlations within the particle-hole-boson approach

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We present a new bosonic approximation for the projected-BCS (PBCS) and Quartet Condensation Model (QCM). In each case, the starting point is the reformulation of the pair/quartet condensate state in terms of particle-hole excitations with respect to the completely occupied Fermi sea. The main simplification of our approach is the assumption that the pair operators corresponding to both particle and hole states obey bosonic commutation relations. This simplifies tremendously the computations and allows for an analytic derivation of the averaged Hamiltonian on the condensate state as a function of the mixing amplitudes. We study both the pure bosonic approach and the renormalized version. In the case of a picket fence model of doubly degenerate states, we find a good agreement between the fermionic and the renormalized bosonic case.

PACS numbers: 21.60.n, 21.60.Gx
Keywords: Boson approximation, Quartet Condensation, Isovector Pairing

I. INTRODUCTION

The $\alpha$-cluster model of the nucleus was proposed in order to explain the relative stability of $4\alpha$ light nuclei [1, 2]. The main theoretical difficulty is connected to strong antisymmetrisation effects between nucleons entering $\alpha$-like structures. Various microscopic $\alpha$-clustering models were proposed to account for it [3, 11]. At the same time, a simplified version considering proton and neutron pairs within the boson approximation was successful in explaining the even-odd pair staggering of binding energies [12]. In medium and heavy nuclei the $\alpha$-clustering can experimentally be related to the $\alpha$-decay phenomenon [13]. It became clear that an $\alpha$-clustering component was necessary in addition to the standard single-particle basis in order to describe the absolute value of the $\alpha$-decay width [14, 13]. This can be explained by the fact that $\alpha$-particles can appear only at relative low nuclear densities [16], a situation which may be realised on the nuclear surface of $\alpha$-decaying nuclei [17].

This may be also the case for special configurations like the Hoyle state in $^{12}$C, which may be seen as a loosely bound agglomerate of three $\alpha$ particles condensed, as bosons, in the 0S orbit of their own cluster mean field. The understanding of the dynamics of $\alpha$ clusters in such situations was greatly improved by the recent THSR approach [18], which also triggered a significant amount of new interest in the field; for a recent review, see e.g. [19]. A bosonic type condensation is however not guaranteed in a fixed number of particles approach, and the asessment of the way in which this phenomenon is realized is a subtle point. Nevertheless, an attractive feature of the THSR approach is that it exhibits the two opposite limits, namely that of a pure Slater determinant and the other in which "the $\alpha$ particles are so far apart from one another that the Pauli principle can be neglected leading to a pure product state of $\alpha$ particles, i.e., a condensate" [19].

Also recently, the Quartet Condensation Model (QCM) was proposed for the study of isovector pairing and quarteting correlations in $N=Z$ nuclei [20, 21] and was further developed in Refs. [22, 23] to the case of isoscalar pairing and $N>Z$ nuclei. The basic building blocks are not the Cooper pairs anymore, but four-body structures composed of two neutrons and two protons coupled to the isospin $T=0$ and to the angular momentum $J=0$, denoted "$\alpha$-like quartets". The QCM approach was proven to be a very precise tool for the description of the amount of correlations present the ground state of $N=Z$ nuclei. The antisymmetrization effects are significant in these configurations and thus an $\alpha$ condensation picture, in the sense mentioned above, is not appropriate. However, studies yet to be published [29] interestingly indicate the presence of "long-range correlations of condensate type" deduced from the behavior of the eigenvalues of the 4-body density matrix.

Having said this, let us specify that in this paper we will use the terms "pair condensate" and "quartet condensate" to denote the specific projected-BCS (PBCS) and QCM trial states of Eqs. (2) and (10), also due to their structural similarity. Anyway, we should keep in mind that an actual $\alpha$ condensate appears only at low densities, as opposed to the usual pair condensate.

It is noteworthy that there are inherent difficulties in describing even the simpler pairing correlations, which have led to significant efforts dedicated to formulate approximate descriptions, including RPA [30] and coupled clusters methods [31, 32]. More recently, an improved approximate treatment of pairing corre-
lations has been developed, in which the starting point is the reformulation of the PBCS condensate in the particle-hole basis. Particle-hole treatments have also been recently analyzed in the case of arbitrary generalized seniority cases.

In the present work we take the opportunity of generalizing these ideas to the more complicated quartet correlations. We will argue that the particle-hole description is natural for both PBCS and QCM models, as can be seen from the behavior of the mixing amplitudes solutions (see Fig. below). We are thus motivated to find the representation of the QCM quartet condensate state in terms of particle-hole excitations with respect to the completely occupied Fermi sea (see Eqs. below).

We also introduce a new approximate hybrid fermionic-bosonic approach, which we will refer to as the particle-hole bosons approach, applicable to both pairing and quarteting cases for the study of ground state correlations. In a first step, this approach requires the reformulation of the condensate state with respect to the correlated Fermi vacuum, as opposed to the empty vacuum state |0⟩. This ensures that a significant amount of fermionic correlations are already accounted for if we pass to bosonic degrees of freedom, but keep the same structure of the trial state. As it turns out, if we consider as a second step the simplest mapping of the individual pair operators to bosons, the ground state correlations in both pairing and quarteting cases are reproduced rather well as compared to the fully fermionic setting.

Although the basic ideas of treating quartet correlations in a boson formalism (see e.g. 12, 33, 37) and also considering the particle-hole excitations as bosons are certainly not new (see e.g. 33 for a thorough review on boson mappings), we are unaware of the two-step approach having been implemented in the specific way mentioned above.

Let us finally note that in both pairing and quarteting cases our formalism is structurally very similar, leading to the same functional form of the energy of the bosonic condensate, up to form factors (see Eq. below). It is rather pleasing that in this sense a unified description of the pairing and the (significantly more complicated) quarteting correlations has been possible.

Our work is structured as follows: in the following section we present the details regarding the reformulation of the pair and quartet condensates as particle-hole expansions. In section II B we develop the bosonic formalism, which is compared to the fully fermionic results in Section III, for the case of a picket fence model. Finally, in Section IV we draw Conclusions.

II. THEORETICAL BACKGROUND

A. Particle-hole representation of the pair and quartet condensates

Let us consider first a model of a number \( N_{\text{ev}} \) of doubly degenerate levels \( i, i' \) (the so-called picket fence model), with single particle energies \( \epsilon_i \), where the ground state of the standard pairing Hamiltonian

\[
H = \sum_{i=1}^{N_{\text{ev}}} \epsilon_i (c_i^\dagger c_i + c_i c_i^\dagger) + \sum_{i,j=1}^{N_{\text{ev}}} V_{ij} P_i^1 P_j^1 ,
\]

(1)

is taken to be the PBCS pair condensate of \( n_p \) pairs,

\[
| \text{PBCS} \rangle = (\Gamma^\dagger (x))^{n_p} |0\rangle .
\]

(2)

Here, the coherent pair is a superposition of single particle pairs \( P_i^1 = c_i^\dagger c_i' \),

\[
\Gamma^\dagger (x) = \sum_{i=1}^{N_{\text{ev}}} x_i P_i^1 ,
\]

(3)

and \( |0\rangle \) is the vacuum state with no particles. We assume \( N_{\text{ev}} > n_p \). All other notations are standard.

Following 33, 38, instead of expressing the \(| \text{PBCS} \rangle\) state with respect to the \(|0\rangle\) vacuum, we may find an equivalent form involving the completely occupied Fermi sea

\[
| \mathcal{F} \rangle = \left( \prod_{i=1}^{n_p} P_i^1 \right) |0\rangle .
\]

(4)

To this end, we first decompose the coherent pair on components below and above the Fermi level as follows

\[
\Gamma^\dagger (x) = \sum_{i=1}^{n_p} x_i P_i^1 + \sum_{i=n_p+1}^{N_{\text{ev}}} x_i P_i^1 \equiv \Gamma^\dagger_h (x) + \Gamma^\dagger_p (x) \quad (5)
\]

It is not difficult to show that the action of the hole component of the coherent pair of arguments \( x \) on the \(|0\rangle\) vacuum may be related to the action of the coherent pair of inverse arguments \( 1/x \) on the Fermi vacuum (see Appendix A for computational details). In this way, one may prove that the reformulation of the pair condensate reads

\[
| \text{PBCS} \rangle = n_p! \cdot \Pi_{1}^{n_p} \cdot \sum_{j=0}^{n_p} \frac{1}{(j!)^2} \left( \Gamma^\dagger_p (x) \Gamma^\dagger_h \left( \frac{1}{x} \right) \right)^j | \mathcal{F} \rangle ,
\]

(6)

where \( \Pi_1 = x_1 x_2 \cdots x_{n_p} \).

This approach can be generalised from pair to quartet correlations. To this purpose we consider the isovector pairing Hamiltonian applicable to both spherical and deformed nuclei

\[
H = \sum_{i=1}^{N_{\text{ev}}} \epsilon_i (N_i^\pi + N_i^\nu) + \sum_{\tau=0,\pm 1} \sum_{i,j=1}^{N_{\text{ev}}} V_{ij} P_i^{1\tau} P_j^{1\tau} ,
\]

(7)
where \( \tau = 0, \pm 1 \) is the isospin projection. All other notations are identical to the pairing case. Within the QCM, one first defines a set of collective \( \pi \pi, \nu \nu \) and \( \pi \nu \) Cooper pairs

\[
\Gamma^\dagger_\tau (x) \equiv \sum_{i=1}^{N_{\text{lev}}} x_i P^\dagger_{\tau,i},
\]

where the mixing amplitudes \( x_i \) are the same in all cases due to isospin invariance. A collective quartet operator is then build by coupling two collective pairs to the total isospin \( T = 0 \)

\[
Q^\dagger \equiv [\Gamma^\dagger \Gamma^\dagger]^T \mid_{S=0} = 2\Gamma^\dagger_1 \Gamma^\dagger_{-1} - (\Gamma^\dagger_0)^2.
\]

Finally, the ground state of the Hamiltonian (1) is described as a condensate of such \( \alpha \)-like quartets

\[
|\Psi_q(x)\rangle = (Q^\dagger)^q |0\rangle,
\]

where \( q \) is the number of quartets. By construction, this state has a well defined particle number and isospin. Its structure is defined by the mixing amplitudes \( x_i \), which are determined numerically by the minimization of the Hamiltonian expectation value, subject to the unit norm constraint.

In analogy with the standard pairing case described above, instead of expressing the quartet condensate state with respect to the \( |0\rangle \) vacuum, we may find an equivalent form involving the completely occupied Fermi sea, in this case given by

\[
|\mathcal{F}\rangle = \left( \prod_{i=1}^{q} P^\dagger_{1,i} P^\dagger_{-1,i} \right) |0\rangle.
\]

The coherent pairs may be decomposed on components below and above the Fermi level

\[
\Gamma^\dagger_\tau (x) = \sum_{i=1}^{q} x_i P^\dagger_{\tau,i} + \sum_{i=n_p+1}^{N_{\text{lev}}} x_i P^\dagger_{\tau,i} \equiv \Gamma^\dagger_{\tau,h}(x) + \Gamma^\dagger_{\tau,p}(x).
\]

As a consequence, the collective quartet decomposes as follows

\[
Q^\dagger(x) = 2\Gamma^\dagger_1 \Gamma^\dagger_{-1} - (\Gamma^\dagger_0)^2
= 2\Gamma^\dagger_{1,h} \Gamma^\dagger_{-1,h} - (\Gamma^\dagger_{0,h})^2 + 2\Gamma^\dagger_{1,p} \Gamma^\dagger_{-1,p} - (\Gamma^\dagger_{0,p})^2
+ 2 \left( \Gamma^\dagger_{1,p} \Gamma^\dagger_{-1,h} + \Gamma^\dagger_{1,h} \Gamma^\dagger_{-1,p} - \Gamma^\dagger_{0,p} \Gamma^\dagger_{0,h} \right)
\equiv Q^\dagger_h(x) + Q^\dagger_p(x) + 2 \left[ \Gamma^\dagger_h(x) \Gamma^\dagger_p(x) \right].
\]

Given the more complicated decomposition of the quartet operator with respect to the simple pairing case, it is remarkable that the quartet condensate state may also be expressed as a particle-hole expansion. The computational strategy is similar to the PBCS case, involving the introduction, for the hole subspace, of collective pair annihilation operators having as arguments the inverse amplitudes. As the derivation is rather long and tedious, only its main points being presented in Appendix A. The exact, fully fermionic, analytical expression for the quartet condensate of Eq. (10) as a particle-hole expansion reads

\[
|\Psi_q\rangle = 2^q q! \Pi_2 \sum_{a=0}^{q} \sum_{b=0}^{q} \lambda_{ab} \left( Q^\dagger_p(x) Q_h \left( \frac{1}{x} \right) \right)^a \times \left[ \Gamma^\dagger_p(x) \Gamma_h \left( \frac{1}{x} \right) \right]^b |\mathcal{F}\rangle,
\]

where

\[
\lambda_{ab} = \frac{1}{2^a b!} \sum_{r=\text{Max}(0,N_{ab}-q)}^{a} \frac{(q-N_{ab})_{a-r}}{2^r (a-r)! (r)!^2} \times \frac{\Gamma \left( \frac{3}{2} + q - r \right)}{\Gamma \left( \frac{3}{2} + N_{ab} - r \right)},
\]

and \( \Pi_2 = x_1^2 x_2^2 \cdots x_q^2 \). The above formula is expressed using the total number of pair excitations in a given term \( N_{ab} = 2a + b \), the Gamma function \( \Gamma(z) \) (not to be confused with the collective pair operator) and the Pochammer symbol \( (z)_k = z(z-1)...(z-k) \). We also used a similar notation to that in Eq. (13) for the coupling of two pairs to \( T = 0 \):

\[
\left[ \Gamma^\dagger_p(x) \Gamma_h \left( \frac{1}{x} \right) \right] = \sum_{\tau=\pm 1,0} \Gamma^\dagger_{\tau,p}(x) \Gamma_{\tau,h} \left( \frac{1}{x} \right).
\]

In the notation \( |\Psi_q\rangle = \Pi_2 \mathcal{O}_q |\mathcal{F}\rangle \), some particular expressions for the operators \( \mathcal{O}_q \) are
\[ \mathcal{O}_1 = 2 \left[ \Gamma_p^\dagger \Gamma_h \right] + \frac{1}{3} (Q_p^\dagger Q_h) + 3 \]
\[ \mathcal{O}_2 = 4 \left[ \Gamma_p^\dagger \Gamma_h \right]^2 + 20 \left[ \Gamma_p^\dagger \Gamma_h \right] + \frac{1}{30} (Q_p^\dagger Q_h)^2 + \frac{4}{5} \left[ \Gamma_p^\dagger \Gamma_h \right] (Q_p^\dagger Q_h) + 2 (Q_p^\dagger Q_h) + 30 \]
\[ \mathcal{O}_3 = 8 \left[ \Gamma_p^\dagger \Gamma_h \right]^3 + 84 \left[ \Gamma_p^\dagger \Gamma_h \right]^2 + 420 \left[ \Gamma_p^\dagger \Gamma_h \right] + \frac{1}{630} (Q_p^\dagger Q_h)^3 + \frac{3}{35} \left[ \Gamma_p^\dagger \Gamma_h \right] (Q_p^\dagger Q_h)^2 \]
\[ - \frac{99}{70} (Q_p^\dagger Q_h)^2 + \frac{12}{7} \left[ \Gamma_p^\dagger \Gamma_h \right]^2 (Q_p^\dagger Q_h) + 12 \left[ \Gamma_p^\dagger \Gamma_h \right] (Q_p^\dagger Q_h) + 114 (Q_p^\dagger Q_h) + 630. \]

The expressions for \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) have been checked by evaluating each individual term with the help of the symbolic computer algebra system Cadabra 2 [39–41]. It is very interesting to note that the quartet condensate actually arises as an interplay of isoscalar particle-hole excitations made out of coupled-pair type excitations and excitations of quartet-quartet type.

Notice that both Eqs. (6) and (14) suggest that the pair mixing amplitudes corresponding to hole states behave inversely to those corresponding to particle states. Indeed, it is interesting to notice that the mixing amplitudes corresponding to hole states and the inverse amplitudes corresponding to particle states both present an almost perfect linear behavior, as can be seen in Fig. 1. This was an early indication of the adequacy of a particle-hole description for both type of correlations.

![FIG. 1: The linear behavior of the normalized PBCS (a) and QCM (b) pair mixing amplitudes for hole states \( x_a \) (circles) and inverses of pair mixing amplitudes for particle states \( 1/x_i \) (squares) for a picket fence model of doubly degenerate levels \( \epsilon_k = k \text{ MeV}, k = 0, 1, \ldots, 19 \), for a coupling constant \( G = 0.5 \text{MeV} \). The linear fit for each set of amplitudes is shown with dashed lines.](image)

The expressions of Eq. (6) and those of Eqs. (14)- (15) are the starting point for the particle-hole-boson treatment of pairing and quartet correlation in the next section.

Before presenting the boson approximation, we need to complete the particle-hole description by also expressing the pairing Hamiltonian in terms of particle and hole degrees of freedom. We introduce particle \((i, j, k, \ldots)\) and hole indices \((a, b, c, \ldots)\). For hole the subspace, we introduce the pair creation operators as \( P_a^\dagger \equiv P_a^\dagger \). After a decomposition of the pairing Hamiltonian of Eq. (11) into particle and hole components we obtain

\[ H = \sum_{a=1}^{n_p} (2 \epsilon_a + V_{aa}) \]
\[ + \sum_{a=1}^{n_p} (-\epsilon_a - V_{aa}) \tilde{N}_a + \sum_{i=n_p+1}^{N_{lev}} \epsilon_i N_i \]
\[ + \sum_{a,b=1}^{n_p} V_{ab} P_a^\dagger P_b + \sum_{i,j=n_p+1}^{N_{lev}} V_{ij} P_i^\dagger P_j \]
\[ + \sum_{a=1}^{n_p} \sum_{j=n_p+1}^{N_{lev}} V_{ai} \left( \tilde{P}_a P_j + P_a^\dagger P_j \right). \]
expressed also in terms of the number of holes operator $\hat{N}_a = 2 - N_a$ which satisfies $[\hat{N}_a, \hat{P}_b^\dagger] = 2\delta_{ab}\hat{P}_b^\dagger$.

For the case of isovector pairing, we analogously introduce the pair creation operators for holes for a given isospin projection as $\hat{P}_{\tau,a}^\dagger \equiv P_{\tau,a}$ and the corresponding hole number operator $\hat{N}_{\tau,a} = 4 - N_{\tau,a}$. Thus, the Hamiltonian \[ H \] decomposes as follows

\[
H = \sum_{a=1}^q (4\epsilon_a + 3V_{aa}) + \sum_{a=1}^q (-\epsilon_a - \frac{3}{2}V_{aa})\hat{N}_{0,a} + \sum_{i=q+1}^{N_{lev}} \epsilon_i\hat{N}_{0,i} \\
+ \sum_{a,b=1}^q V_{ab} \sum_{\tau=\pm 1,0} \hat{P}_{\tau,a}^\dagger \hat{P}_{\tau,b} + N_{lev} \sum_{i,j=n_p+1} V_{ij} \sum_{\tau=\pm 1,0} P_{\tau,i}^\dagger P_{\tau,j} \\
+ \sum_{a=1}^q \sum_{j=n_p+1} V_{ai} \sum_{\tau=\pm 1,0} (\hat{P}_{\tau,a}^\dagger P_{\tau,i} + P_{\tau,i}^\dagger \hat{P}_{\tau,a}) (20)
\]

B. Particle-hole boson approximation

![Graph](image)

FIG. 2: The effect of the particle and hole degrees of freedom on the average level occupation fraction $\langle n \rangle$, shown for the QCM in the case of the picket fence model, for $q = 4$ quartets.

The basic idea of our bosonic approximation is to take as reference state the completely occupied Fermi sea and to describe the particle and hole degrees of freedom with a simplified bosonic treatment, the particle-hole bosons thus accounting for the deviation from the Fermi distribution (see Fig. (2)).

In the following we will first define the particle-hole-boson approximation for the standard pairing case, the generalization to the quarteting case being trivial. Our approximation is defined by the replacements for the pair operators and for the vacuum state

\[
P_i^\dagger \rightarrow p_i^\dagger , \quad \hat{P}_a^\dagger \rightarrow h_b^\dagger , \quad |F\rangle \rightarrow |0\rangle .
\]

where the particle and hole bosons annihilate the boson vacuum: $p_i|0\rangle = 0$ and $h_a|0\rangle = 0$, together with the mapping of the particle and hole number operators $\hat{N}_a \rightarrow \hat{N}_a , \quad N_i \rightarrow N_i$ . They obey a standard bosonic algebra

\[
\begin{align*}
[p_i, p_j^\dagger] &= \delta_{ij}\pi_j , \quad [h_a, h_b^\dagger] = \delta_{ab}\eta_a , \quad [p_i, h_j] = 0 , \\
[N_i, p_j^\dagger] &= 2\delta_{ij}p_j^\dagger , \quad [N_a, h_b^\dagger] = 2\delta_{ab}h_b^\dagger
\end{align*}
\]

where the coefficients $\pi_i$ and $\eta_i$ are c-numbers and all other commutators vanish. We also define the corresponding collective bosons

\[
\mathcal{H}^\dagger(y) \equiv \sum_{a=1}^{n_p} y_a h_i^\dagger , \quad \mathcal{P}^\dagger(x) \equiv \sum_{i=n_p+1}^{N_{lev}} x_i p_i^\dagger .
\]

The main point is that we consider the bosonic ground state to be of the same form as the fermionic PBCS condensate

\[
|\psi(x,y)\rangle \equiv \sqrt{\chi} \sum_{\eta} \frac{1}{(n!)^2} (\mathcal{P}^\dagger(x) \mathcal{H}^\dagger(y))^n |0\rangle ,
\]

where $\chi$ is a normalization constant. The hole amplitudes $y$ will be compared in the end to the inverse of the fermionic amplitudes corresponding to levels below the Fermi level. Let us first notice that the commutation relations involving the number of bosons in Eq.

\[ (22) \]

can also be realized by using the following replacements

\[
N_i \rightarrow \frac{2}{\pi_i} p_i^\dagger p_i , \quad N_a \rightarrow \frac{2}{\eta_a} h_a^\dagger h_a.
\]

The boson Hamiltonian can be obtained from Eq. \[ (18) \] as follows

\[
H_b = \sum_{a,b=1}^{n_p} \left( \frac{2\epsilon_a \delta_{ab} + V_{ab}}{\eta_a} \right) h_b^\dagger h_a \\
+ \sum_{i,j=n_p+1} N_{lev} \left( \frac{2\epsilon_i \delta_{ij} + V_{ij}}{\pi_i} \right) p_i^\dagger p_j \\
+ \sum_{a=1}^{n_p} \sum_{j=n_p+1} N_{lev} V_{ai} \left( h_a p_i + p_i^\dagger h_a^\dagger \right) + \sum_{a=1}^{n_p} (2\epsilon_a + V_{aa}) .
\]

Throughout this paper, we consistently define the single particle energy corresponding to holes degrees of freedom to be simply $\tilde{\epsilon}_a = -\epsilon_a$, as we neglect the respective interaction contribution appearing in the fully fermionic approach. In order to compute the averages of the boson operators on the state \[ (24) \] it is
very convenient to define first the norms of the collective boson pairs
\[ [\mathcal{P}(x), \mathcal{P}^\dagger(x)] = \sum_{i=n_p+1}^{N_{lev}} x_i^2 \pi_i \equiv S_p , \]
\[ [\mathcal{H}(y), \mathcal{H}^\dagger(y)] = \sum_{a=1}^{n_p} y_a^2 \eta_a \equiv S_h . \]

The product $S_p S_h$ will appear frequently in the following and we choose to denote it by $S_{ph} = S_p \cdot S_h$. The bosonic approximation is simple enough to allow for an analytical derivation for the norm of the bosonic pair condensate
\[ \langle \psi(x,y) | \psi(x,y) \rangle = \chi \sum_{n=0}^{N_{lev}} \frac{(S_{ph})^n}{(n!)^2} \equiv \nu(S_{ph}) . \]

The averages of bosonic pair bilinears are easily found to be
\[ \langle h_i^a h_i^b \rangle = \chi y_a \eta_a y_b \eta_b S_p \sum_{n=1}^{n_p} \frac{n}{(n!)^2} S_{ph}^{n-1} , \]
\[ \langle p_i^a p_j^b \rangle = \chi x_i \pi_i x_j \pi_j S_h \sum_{n=1}^{n_p} \frac{n}{(n!)^2} S_{ph}^{n-1} , \]
\[ \langle p_i^a h_i^b \rangle = \langle p_i^a p_i^b \rangle = \chi x_i \pi_i y_a \eta_a \sum_{n=0}^{n_p-1} \frac{1}{(n!)^2} (S_{ph})^n . \]

Finally, the average of the Hamiltonian over the bosonic pair condensate may thus be written as follows
\[ \langle H_b \rangle = (\mathcal{H}_{hh} S_p + \mathcal{H}_{pp} S_h) \cdot f_1(S_{ph}) + \mathcal{H}_{ph} \cdot f_2(S_{ph}) + E_0 \cdot \nu(S_{ph}) , \]
\[ \mathcal{H}_{hh} = \sum_{a=1}^{n_p} 2 \epsilon_a \eta_a y_a^2 + \sum_{a,b=1}^{n_p} V_{ab} y_a \eta_a y_b \eta_b , \]
\[ \mathcal{H}_{pp} = \sum_{i=1}^{N_{lev}} 2 \epsilon_i \pi_i x_i^2 + \sum_{i,j=n_p+1}^{N_{lev}} V_{ij} x_i \pi_i x_j \pi_j , \]
\[ \mathcal{H}_{ph} = \sum_{a=1}^{n_p} \sum_{j=n_p+1}^{N_{lev}} V_{a} x_i \pi_i y_a \eta_a , \]

in terms of the form factors
\[ f_1(z) = \sum_{n=1}^{n_p} \frac{y_a^2}{(n!)^2} z^{n-1} , \quad f_2(z) = \sum_{n=0}^{n_p-1} \frac{z^n}{(n!)^2} , \]
and the zero point energy $E_0 = \sum_{a=1}^{n_p} (2 \epsilon_a + V_{aa})$. It is important to remark that the bosonic approximation remains a highly nonlinear problem, the particle and hole bosons being coupled not only through the interaction terms $V_{aa}$, but also through the form factors. We may thus speak of dressed particle and holes degrees of freedom.

The ground state energy corresponding to the minimum of the energy function
\[ E(x,y) = \frac{\langle \psi(x,y) | H_b | \psi(x,y) \rangle}{\langle \psi(x,y) | \psi(x,y) \rangle} , \]

may be computed upon a minimization procedure with respect to the particle and hole amplitudes $x_i$ and $y_a$. We note that the energy function has a scaling symmetry $E(x,y) = E \left( \lambda x, \frac{y}{\lambda} \right)$, such that the number of independent parameters are actually $N_{lev} - 1$. We will analyze two choices for the commutator coefficients in Eqs. \[(30)\]

1. pure bosonic case: $\eta_a = 1, \pi_i = 1$.

2. renormalized bosonic case:
\[ \eta_a = 1 - \frac{1}{2} \langle N_a \rangle = 1 - y_a^2 \eta_a S_p f_1(S_{ph}) / \nu(S_{ph}) \]
\[ \pi_i = 1 - \frac{1}{2} \langle N_i \rangle = 1 - x_i^2 \pi_i S_h f_1(S_{ph}) / \nu(S_{ph}) \]

It follows that in this latter case the commutator coefficients satisfy the following self-consistency condition (as the $S_p$ and $S_h$ terms depend implicitly on them)
\[ \eta_a = (1 + y_a^2 S_p f_1(S_{ph}) / \nu(S_{ph}))^{-1} \]
\[ \pi_i = (1 + x_i^2 S_h f_1(S_{ph}) / \nu(S_{ph}))^{-1} . \]

Their precise values may be found, given a set of mixing amplitudes, by a straightforward and rapidly converging iterative procedure.

Let us mention that a renormalized procedure is preferred as to effectively take into account the finite maximum occupation of a given level as dictated by the Pauli exclusion principle. Indeed, it can be seen by combining Eqs. \[(33)\] and \[(34)\] that the average level occupation fraction satisfies, e.g. for hole states
\[ \langle n_a \rangle = \frac{1}{2} \langle N_a \rangle = 1 - \eta_a < 1 . \]

The same basic idea of the bosonic approximation for the standard pairing case is easily applicable to the isovector pairing situation. Each projection of the triplet of pair operators translates into a corresponding boson
\[ p_{\tau,i} \rightarrow p_{\tau,i}^\dagger , \quad \tilde{p}_{\tau,a} \rightarrow h_{\tau,a}^\dagger , \]

where we consider bosonic pairs of different isospin projection to commute:
\[ [p_{\tau,i} p_{\tau,j}^\dagger] = \delta_{\tau \sigma} \delta_{ij} \pi_j , \quad [h_{\tau,a} h_{\tau,b}^\dagger] = \delta_{\tau \sigma} \delta_{ab} \eta_b . \]

The expressions of Eqs. \[(23)\] and \[(27)\] are generalized accordingly for each member of the collective pair triplet. The bosonic isovector pairing Hamiltonian is basically identical with that of Eq. \[(20)\]
upon the replacement of the bosonic pair bilinears with the sum over the three isospin projections, and the redefinition of the constant energy term to $E_0 = \sum_{q=1}^3 (4e_q + 3V_{qq})$. The bosonic isovector pairing Hamiltonian may thus be cast into the same form as in Eq. (30). We present in Appendix A the details regarding the form factors in the isovector pairing case.

As in the PBCS case, we analyse both choices of pure bosonic commutation relations and their renormalized version. In the latter case, the coefficients are computed for the isovector pairing as

$$\eta_a = 1 - \frac{1}{4} \langle N_{0,a} \rangle = 1 - \frac{1}{2} q_a^2 \eta_a S_p f_1(S_{ph}) / \nu(S_{ph})$$

$$\pi_i = 1 - \frac{1}{4} \langle N_{0,i} \rangle = 1 - \frac{1}{2} q_i^2 \eta_i S_h f_1(S_{ph}) / \nu(S_{ph})$$

(37)

We have thus succeeded in applying the same basic idea of approximating as bosons the pairs in the particle-hole expansion of the pair and quartet condensates. We have obtained in both cases the same form of the average of the Hamiltonian on the bosonic version of the condensate, the only differences appearing in the so-called form factors.

### III. NUMERICAL APPLICATION

We have analyzed the projected-BCS cases of $n_p = 6$ and $n_p = 10$ pairs and the QCM cases of $q = 1, 2, 3, 4$ quartets distributed over 20 equally spaced, $\epsilon_k = (k - 1) \text{ MeV}$, doubly-degenerate single particle levels, interacting via a constant pairing force, $V_{ij} = -G$. The solutions for the QCM fermionic approach were obtained by using the analytical expression method described in [42]. For the PBCS case, we implemented the recurrence relations presented in [43]. In all cases the minimization of the energy function with respect to the mixing amplitudes was carried out by using the e04ucf routine of the NAG library.

The pairing strength $G$, for both standard pairing and isovector pairing cases, is given in units of the critical strength $G_{cr}$ for which the pairing gap vanishes in the standard BCS and, respectively, proton-neutron BCS [44, 46]. This allows to distinguish between the weak, medium, and strong pairing regimes which roughly correspond to $G < G_{cr}$, $G \sim G_{cr}$, and $G > G_{cr}$. For our particular picket-fence model, the specific values are $G^{(n_p=6)}_{cr} = 0.238 \text{ MeV}$, $G^{(n_p=10)}_{cr} = 0.234 \text{ MeV}$, and respectively $G^{(q=1)}_{cr} = 0.144 \text{ MeV}$, $G^{(q=2)}_{cr} = 0.132 \text{ MeV}$, $G^{(q=3)}_{cr} = 0.127 \text{ MeV}$, $G^{(q=4)}_{cr} = 0.123 \text{ MeV}$.

In all cases, we find excellent agreement between the fermionic and both bosonic approximations in the weak pairing regime. However, as the strength of the correlations increases the inadequacy of the pure bosonic approach is quickly revealed, the main reason being its inability to reproduce the finite level occupancy as dictated by the Pauli principle. Indeed, we observe from Fig. 3 (b) and (c) that the average level occupation fractions for the pure bosonic approach in the PBCS case, for a strong pairing scenario, exceed unity for the states close to the Fermi surface. As a consequence, the ground state energy is not correctly reproduced. These problems are however completely solved by the renormalization procedure described in the previous section. Indeed, the rbPBCS approach is in almost perfect agreement with the fully fermionic results regarding average level occupation fractions, as shown in the same figure. The renormalization restrictions also have the effect of bringing the ground state energy much closer to the fermionic value, as displayed in Fig. 3(a) and (b).

In order to emphasize the essential role of considering the bosonic ground state as having the same structure as the particle-hole version of the fermionic condensate (see Eqs. (6) and (24)), in Fig. 3 (a) and (b) we also plotted the ground state energy in the so-called "naive" renormalized bosonic approach. In this case, we applied the bosonization procedure (in the renormalized version) directly to the original PBCS condensate of Eq. (2) and to the original pairing Hamiltonian of Eq. (1). We notice strong discrepancies between the naive bosonic approach and the fermionic case, for all interaction strengths (except the $G = 0$ case which is reproduced due to the renormalization procedure). In this way we support the idea that the particle-hole expansion of the (pair) condensate contains a significant amount of information about the pure fermionic correlations in the ground state.

The situation is qualitatively similar in the case of quarteting correlations, as seen in Fig. 4. There is a very good agreement between the fermionic and bosonic approaches in the weak and medium pairing regimes. In the strong pairing case however, the correlation energy is strongly overestimated in the pure bosonic approach, and slightly underestimated within the renormalized bosonic rbQCM approximation.

We show in Fig. 5 the errors in the correlation energy of the renormalized bosonic approximation relative to the fermionic approaches, i.e. for PBCS in the left panel (a) and respectively for QCM in the right panel (b). We note the perfect agreement in the weak pairing regime in all cases. Slight discrepancies start to emerge in the medium pairing regime, but all errors are at most of the order of 10% even in the strong pairing scenario, up to $G/G_{cr} = 4$. 

Regarding the average level occupation fraction in the quarteting case, we notice in Fig. 6, as expected, unphysical values that exceed unity for the pure bosonic approach. On the other hand, the rbQCM approximation results slightly underestimate the exact values; this is to be traced back to the fact that in the bosonic QCM approaches we made the additional assumption of commuting pairs of different isospin projection (see Eq. (36)). While it is not difficult to conceive further improvements that take into account the pair mixing effects within a bosonic treatment, we limit ourselves in this work to the assessment of the consequences of the simplest kind of approximation.

We finally compare the results regarding the mixing amplitudes for the quarteting case in the strong pairing regime. One should recall that in the bosonic case they are defined up to an overall factor, due to the scaling symmetry of the bosonic ground state energy mentioned in the previous section, and to the lack of the unit norm constraint in the bosonic approach (the normalization constant is this case being a free parameter). We thus limit ourselves to a comparison of the relative behavior of the fermionic and bosonic amplitudes, by choosing the overall factor in the bosonic case as to give the best fit with respect to the fermionic case. In the following, for the purpose of this comparison, we need to work with the inverse amplitudes for the hole states (see also the discussion following Eq. (24)), thus we perform the replacement $x_a^{(b)} \rightarrow 1/x_a^{(b)}$ for the bosonic hole amplitudes. Explicitly, a least squares fit for $x(f) = \alpha x^{(b)}$ leads to the the expression

$$\alpha = \frac{\sum_i x_i^{(f)} x_i^{(b)}}{\left( \sum_i x_i^{(b)} \right)^2},$$

where $f$ stands for the fermionic QCM case and $b$ refers to each of the two bosonic approximations.

As seen from Fig. 7 a very good agreement regarding the behavior of the mixing amplitudes is found between the exact QCM result and the renormalized bosonic rbQCM approximation, even in the strong pairing regime. The pure bosonic theory, however, shows discrepancies, especially for the states around the Fermi level.

It is noteworthy that in the isovector pairing case, the QCM offers an almost perfect description of the ground state (the correlation energies are generally within a 1% error with respect to the exact shell model diagonalization [22]), even for the weak pairing regime (as opposed to the PBCS case (see e.g. [33]). As such, the renormalized bosonic treatment of the quartet condensates promises to be a simple but also quite a precise approach.

IV. CONCLUSIONS

We developed a rather accurate bosonic approximation for pair and quartet condensates, corresponding to the standard pairing and isovector pairing scenarios. The starting point was the reformulation of the condensates as a particle hole expansion. In particular, we derived the expression of the quartet condensate state as a particle-hole expansion, and found both quartet-quartet excitations and coupled pair excitations.

The bosonic formalism is straightforward and very similar in both pairing and quarteting cases. The average of the Hamiltonian on the condensate states has the same form in both cases, the only differences being in the expressions of the above defined form-factors.

In conclusion, we have found that the particle-hole expansion of the pair and quartet condensates contains a lot of information about the fermionic correlations in the ground state, which allows for a good description in terms of bosonic degrees of freedom (provided one effectively takes into account the exclusion principle via the renormalization procedure).

Appendix A: Particle-hole expansion of the condensate states

In order to derive the particle hole formulation of the pair condensate, we first express the completely occupied Fermi sea of Eq. (4) in terms of the hole component of the coherent pair as

$$|\mathcal{F}\rangle = \frac{1}{n_p!} \frac{1}{\Pi_1} \left( \Gamma_p(x) \right)^{n_p} |0\rangle,$$

(A1)
where $\Pi_1 = x_1 x_2 \cdots x_{n_p}$. The main trick is then to use a coherent pair of inverse arguments $\Gamma_h \left( \frac{1}{x} \right)$. Starting from the commutator $[P_i, P_j^\dagger] = \delta_{ij} (1 - \hat{N}_i)$, it is easy to compute

$$\left[ \Gamma_h \left( \frac{1}{x} \right), \Gamma_h^\dagger(x) \right] = n_p - \sum_{i=1}^{n_p} \hat{N}_i = n_p - \hat{N}_h .$$

From this it may be shown that

$$\left( \Gamma_h \left( \frac{1}{x} \right) \right)^j \left( \Gamma_h^\dagger(x) \right)^k |0\rangle = \frac{j! k!}{(k-j)!} \left( \Gamma_h^\dagger(x) \right)^{k-j} |0\rangle .$$

For the particular case of $k = n_p$, we may relate the action of the coherent pair of inverse arguments on the Fermi vacuum to the action of the original coherent pair on the $|0\rangle$ vacuum as

$$\left( \Gamma_h \left( \frac{1}{x} \right) \right)^j |\mathcal{F}\rangle = \frac{1}{\Pi (n_p - j)!} \left( \Gamma_h^\dagger(x) \right)^{n_p-j} |0\rangle .$$

By employing this expression in the expansion of the PBCS condensate, we arrive at the form mentioned in Eq. (6)

$$|PBCS\rangle = \left( \Gamma_h^\dagger(x) + \Gamma_p^\dagger(x) \right)^{n_p} |0\rangle = n_p! \cdot \Pi_1 \cdot \sum_{j=0}^{n_p} \frac{1}{(j!)^2} \left( \Gamma_p^\dagger(x) \right)^j \Gamma_h \left( \frac{1}{x} \right)^j |\mathcal{F}\rangle ,$$

For the quarteting case, we perform a similar maneuver. We start from the $q$-quartet condensate of Eq. (13) state which may be expanded as follows
FIG. 4: The ground state energies (in MeV) vs the ratio \( G/G_{cr} \) for \( q = 1 \) (a), \( q = 2 \) (b), \( q = 3 \) (c), \( q = 4 \) (d) the QCM, bQCM and rbQCM approaches of the picket fence model.

FIG. 5: The error in the correlation energy of the renormalized boson approximation rbPBCS (a) and, respectively, rbQCM (b), relative to the fermionic cases of PBCS (a) and QCM (b).

\[ |\Psi_q\rangle = \sum_{n=0}^{q} \sum_{j=0}^{n} \frac{q!}{(n-j)!j!(q-n)!} 2^j (Q_p^\dagger)^{n-j} \left[ \Gamma_p \Gamma_h^\dagger \right]^j (Q_h^\dagger)^{q-n} |0\rangle \]  

(A6)

We will perform the transition to the particle hole representation in two steps:

1. we express the \( n \)-hole-quartet state \( (Q_h^\dagger)^{q-n} |0\rangle \) as the annihilation of \( n \) quartets from the Fermi vacuum
of Eq. (11) and

2. perform a similar computation for the term involving the coupled pairs. Note that, as in the PBCS case, the collective annihilation operators will be dependent on the inverse amplitudes.

1. Relate \((Q_h^q(x))^{q-n} |0\rangle \sim \left( Q_h \left( \frac{1}{x} \right) \right)^n |\mathcal{F}\rangle\)

Using the hole quartet \(Q_h^q\) we may also express the occupied Fermi sea as

\[
|\mathcal{F}\rangle = \frac{1}{\Pi_2} \frac{2^q}{(2q+1)!} \left( Q_h^q(x) \right)^q |0\rangle , \tag{A7}
\]

where \(\Pi_2 = x_1^2 \cdot x_2^2 \cdots x_q^2\). Consider the action of the collective pair annihilation operators of inverse amplitudes on a \(n\)-quartet state on the hole subspace. From the SO(5) algebra \[21\] of the hole operators, it is not difficult to show that

\[
\Gamma_{\tau,h} \left( \frac{1}{x} \right) Q_h^q(x)^n |0\rangle = (-1)^{n-\tau} \cdot n \cdot (2q - 2n + 3) \cdot \Gamma_{-\tau,h}^\dagger (x) Q_h^q(x)^{n-1} |0\rangle . \tag{A8}
\]

From this relation it follows that the action of a collective quartet annihilation operator on an \(n\)-hole-quartet state is

\[
Q_h \left( \frac{1}{x} \right) Q_h^q(x)^n |0\rangle = n \cdot (2q - 2n + 3) \cdot [(n-1)(2q - 2n + 5) - 3(q - 2n + 2)] \cdot Q_h^q(x)^{n-1} |0\rangle . \tag{A9}
\]

By iterating this relation, we obtain

\[
\left( Q_h \left( \frac{1}{x} \right) \right)^a Q_h^q(x)^n |0\rangle = \frac{(2n+1)! \cdot (2a+1)!}{2^{2a} (2n-2a+1)!} Q_h^q(x)^{n-a} |0\rangle . \tag{A10}
\]
It is now possible to compute the action of hole-quartet annihilation operators on the Fermi vacuum

\[ \Pi_2 \frac{2^a}{(2a+1)!} (Q_h \left( \frac{1}{x} \right)^a |\mathcal{F}\rangle = \frac{2^k}{(2k+1)!} (Q_h^\dagger(x))^k |0\rangle, \quad a + k = q, \]

which allows for the first rewriting of the \(q\)-quartet condensate state as

\[ |\Psi_q\rangle = \Pi_2 \sum_{n=0}^{q} \sum_{j=0}^{n} \frac{q!}{(n-j)!j!(q-n)!} 2^{2n-q} (2q-2n+1)! (2n+1)! -2j (Q_p^\dagger(x))^{n-j} \left[ \Gamma_p(x) \Gamma_h^\dagger(x) \right]^j \left( Q_h \left( \frac{1}{x} \right) \right)^n |\mathcal{F}\rangle. \]

(A12)

2. Relate \[ \left[ \Gamma_p^\dagger \Gamma_h^\dagger \right] \left( Q_h \right)^n |\mathcal{F}\rangle \sim \left[ \Gamma_p \Gamma_h \right]^n \left( Q_h \right)^n |\mathcal{F}\rangle \]

In the following we denote the collective annihilation operators on the hole subspace simply by \(A_h \equiv A_h \left( \frac{1}{x} \right)\).

We also use the notation

\[ \left[ \Gamma_p^\dagger \Gamma_p \Gamma_h^\dagger \Gamma_h \right] \equiv \Gamma_1^\dagger \Gamma_1 \Gamma_{1,h} + \Gamma_{-1}^\dagger \Gamma_{-1} \Gamma_{-1,h} + \Gamma_0^\dagger \Gamma_0. \]

(A13)

Let us compute the commutator

\[ \left[ \left[ \Gamma_p^\dagger \Gamma_h, \Gamma_p^\dagger \Gamma_h^\dagger \right] \right] = Q_p^\dagger \left( q - \frac{1}{2} \hat{N}_{0,h} \right), \]

(A14)
where $\hat{N}_{0,h} = \sum_{i=1}^{p} \hat{N}_{0,i}$ is the total number of particles on the hole subspace. From the previous relation it follows that
\[
\left[ \Gamma_p^\dagger \Gamma_h \right] = Q_p^\dagger \left[ \Gamma_p^\dagger \Gamma_h \right]^{n-1} \cdot \frac{n}{2} (2q - n + 1 - \hat{N}_{0,h}).
\] (A15)

Combining Eqs. (A8) and (A15) we obtain the recurrence relation
\[
|j n\rangle = \left[ \Gamma_p^\dagger \Gamma_h \right]^{j} (Q_h)^n |F\rangle = n(2q - 2n + 3) \left( \left[ \Gamma_p^\dagger \Gamma_h \right] |j - 1, n - 1\rangle + \frac{j - 1}{2} (2q + j - 4n + 2)Q_p^\dagger |j - 2, n - 1\rangle \right).
\] (A16)

A careful analysis of this recurrence relation leads to the expression
\[
\left[ \Gamma_p^\dagger \Gamma_h \right]^{j} (Q_h)^n |F\rangle = \frac{n!}{(n - j)!} \frac{(2q - 2n + 2j + 1)!}{(2q - 2n + 1)!} \frac{(q - n)!}{2^j(n - j)!} \sum_{k=0}^{[j/2]} \frac{1}{2^k k!} \frac{j!}{(j - 2k)!} (q - 2n + j)_k \left[ \Gamma_p^\dagger \Gamma_h \right]^{j-2k} (Q_p^\dagger Q_h)^k (Q_h)^{n-j} |F\rangle.
\] (A17)

where the notation $(z)_n$ is the Pochhammer symbol $(z)_n = z(z - 1) \cdots (z - n + 1)$ and the $\lfloor n \rfloor$ is floor function.

By using the expressions (A12) and (A17) we obtain
\[
|\Psi_s\rangle = \prod_{i=1}^{p} \sum_{n=0}^{\hat{N}_{0,i}} \sum_{j=0}^{\hat{N}_{0,i} - n} \frac{q!}{(n - j)!} 2^{2n-a} \frac{n!(2q - 2n + 1)!}{(2q + 1)!} \frac{[j/2]}{2^k k!} \frac{(q - 2n + j)_k}{(q - n + j)!} \sum_{k=0}^{[j/2]} \frac{1}{2^k k!} \frac{j!}{(j - 2k)!} (Q_p^\dagger Q_h)^{n-j+k} \left[ \Gamma_p^\dagger \Gamma_h \right]^{j-2k} |F\rangle.
\] (A18)

Let us mention that Eqs. (A14-A15) follow after regrouping the terms of the quartet-quartet and coupled pairs excitations with the same powers.

**Appendix B: Pairing and quarteting bosonic form factors**

In the standard pairing case, the form factors and norm function entering the expression of the Hamiltonian expectation value on the bosonic version of the condensate (see Eq. (29)) may be computed analytically for any number of pairs. They can be read off the averages of bosonic pair bilinears of Eqs. (29),
\[
f_1(z) = \sum_{n=1}^{\hat{N}_{0,i}} \frac{n!}{(n!)}^2 z^{n-1}, \quad f_2(z) = \sum_{n=0}^{\hat{N}_{0,i} - a} \frac{z^n}{(n!)}^2, \quad \nu(z) = \sum_{n=0}^{\hat{N}_{0,i} - a} \frac{z^n}{(n!)}^2.
\] (B1)

However, due to the more complicated form of the quartet condensate particle-hole expansion of Eqs. (44)-(45) we are unable to provide general analytical expressions of the form factors for the isovector pairing case. For each particular number of quartets, they may be computed in a straightforward fashion by expanding the collective quartets and coupled bosonic pairs into individual pairs and then evaluating each expression, taking advantage of the bosonic character of the pairs.

We present below the formulas for the form factors and norm function in the isovector pairing case for a number of quartets ranging from one to four.
\[
f_1^{(q=1)}(z) = 12 + 8z, \quad f_2^{(q=1)}(z) = 18 + 8z, \quad \nu^{(q=1)}(z) = 9 + 12z + 4z^2.
\] (B2)
\[
f_1^{(q=2)}(z) = 1200 + 1440z + 576z^2 + 64z^3, \quad f_2^{(q=2)}(z) = 1800 + 2400z + 1056z^2 + 128z^3, \quad \nu^{(q=2)}(z) = 900 + 1200z + 720z^2 + 192z^3 + 16z^4.
\] (B3)
\( f_1^{(q=3)}(z) = 529200 + 1734048z + 302400z^2 + \frac{4262976}{49}z^3 + 8640z^4 + 384z^5, \)

\( f_2^{(q=3)}(z) = 793800 + 1421280z + 997920z^2 + 58752z^3 - 7776z^4 + 1152z^5, \)

\( \nu^{(q=3)}(z) = 396900 + 529200z + 867024z^2 + 100800z^3 + \frac{1065744}{49}z^4 + 1728z^5 + 64z^6. \)

\( f_1^{(q=4)}(z) = 2048z^7 + 86016z^6 + 14678016z^5 + \frac{179532800}{3}z^4 + 101606400z^3 + 1758827520z^2 + 4854753792z + 685843200, \)

\( f_2^{(q=4)}(z) = 8192z^7 - 718848z^6 + 11515904z^5 - 47715840z^4 + 539965440z^3 + 380346240z^2 + 2347107840z + 1028764800, \)

\( \nu^{(q=4)}(z) = 256z^8 + 12288z^7 + 2446336z^6 + \frac{35906560z^5}{3} + 25401600z^4 + 586275840z^3 + 2427376896z^2 + 685843200z + 514382400. \)

**Acknowledgments**

This work was supported by the grants of the Romanian Ministry of Research and Innovation, CNCS - UEFISCDI, PN-III-P4-ID-PCE-2016-0092, PN-III-P4-ID-PCE-2016-0792, within PNCDI III, and PN-19060101/2019.

[1] L.R. Hafstad and E. Teller, Phys. Rev. 54, 681 (1938).
[2] K. Ikeda, N. Tagakawa, and H. Horiuchi, Prog. Theor. Phys. Suppl. 464 (1968).
[3] B.H. Flowers and M. Vujčić, Nucl. Phys. 49, 586 (1963).
[4] D.M. Brink, Proceedings of the International School of Physics Enrico Fermi, Varenna Course 36, 247 (1966).
[5] A. Arima and V. Gillet, Ann. Phys. 66, 117 (1971).
[6] K. Wildermuth and Y.C. Tang, A Unified Theory of the Nucleus (Academic, New York, 1977).
[7] M. Freer, and A.C. Merchant, J. Phys. G 23, 261 (1997).
[8] D.S. Delion, G.G. Dussel, and R.J. Liotta, Rom. J. Phys. 47, 97 (2002).
[9] H. Horiuchi, Nucl. Phys. A 731, 329 (2004).
[10] Y. Funaki, H. Horiuchi, W. von Oertzen, G. Röpke, P. Schuck, A. Tohsaki, and T. Yamada, Phys. Rev. C 80, 064326 (2009).
[11] A. Tohsaki, H. Horiuchi, P. Schuck, and G. Röpke, Rev. Mod. Phys. 89, 011002 (2017).
[12] Y.K. Gambhir, P. Ring, and P. Schuck, Phys. Rev. Lett. 51, 1235 (1983).
[13] D.S. Delion, Theory of particle and cluster emission (Springer-Verlag, Berlin, 2010).
[14] K. Varga, R.G. Lovas, and R.J. Liotta, Phys. Rev. Lett. 69, 37 (1992); Nucl. Phys. 550, 421 (1992).
[15] D.S. Delion, A. Sandulescu, and W. Greiner, Phys. Rev. C 69, 044318 (2004).
[16] G. Röpke, A. Schnell, P. Schuck, and P. Nozieres, Phys. Rev. Lett. 80, 3177 (1998).
[17] D.S. Delion and R.J. Liotta, Phys. Rev. C 87, 041302(R) (2013).
[18] A. Tohsaki, H. Horiuchi, P. Schuck, G. Röpke, Phys. Rev. Lett. 87 192501 (2001).
[19] P. Schuck, Y. Funaki, H. Horiuchi, G. Röpke, A. Tohsaki, T. Yamada, Phys. Scr. 91 123001 (2016).
[20] N. Sandulescu, D. Negrea, J. Dukelsky, C. W. Johnson, Phys. Rev. C 85, 061303(R) (2012).
[21] D. Negrea, Proton-neutron correlations in atomic nuclei, Ph.D. thesis, University of Bucharest and University Paris-Sud, 2013, https://tel.archives-ouvertes.fr/tel-00870588/document.
[22] N. Sandulescu, D. Negrea, C. W. Johnson, Phys. Rev. C 86, 041302(R) (2012).
[23] D. Negrea, N. Sandulescu, Phys. Rev. C 90, 024322 (2014).
[24] N Sandulescu et al, J. Phys.: Conf. Ser. 533, 012018 (2014).
[25] N. Sandulescu, D. Negrea, D. Gambacurta, Phys. Lett. B. 751, 348 (2015).
[26] D. Negrea, N. Sandulescu, D. Gambacurta, Prog. Theor. Exp. Phys. 073D05 (2017).
[27] M. Sambataro, N. Sandulescu, Eur. Phys. J. A 53 47 (2017).
[28] D. Negrea, P. Buganu, D. Gambacurta, N. Sandulescu, Phys. Rev. C 98, 064319 (2018).
[29] M. Sambataro, N. Sandulescu, in preparation; N. Sandulescu, Other topics in nuclear physics, Conference Proceeding, 2018.
[30] J. Dukelsky, G. G. Dussel, J. C. Hirsch, P. Schuck, Nucl. Phys. A 714, 63 (2003).
[31] P. A. Johnson, P. W. Ayers, P. A. Limacher, S. De Baerdemacker, D. Van Neck, and P. Bultinck, Comp. Theor. Chem. 1003, 101 (2013)
[32] T. M. Henderson, G. E. Scuseria, J. Dukelsky, A. Signoracci, and T. Duguet, Phys. Rev. C 89, 054305 (2014).
[33] J. Dukelsky, S. Pittel, and C. Esebbag, Phys. Rev. C 93, 034313 (2016).
[34] L.Y. Jia, Phys. Rev. C 93, 064307 (2016).
[35] A. Klein, E. R. Marshalek, Rev. Mod. Phys. 63 2 (1991)
[36] J. Dobeš, S. Pittel, Phys. Rev. C 57 2 (1998)
[37] M. Sambataro, N. Sandulescu, Phys. Lett. B 786 1115 (2018)
[38] P. Ring and P. Schuck, The Many Body Nuclear Problem, Springer-Verlag, Berlin 1980.
[39] K. Peeters, hep-th/0701238
[40] K. Peeters, Journal of Open Source Software, 3(32), 1118 (2018)
[41] https://cadabra.science
[42] https://cadabra.science
[43] V.V. Baran, D.S. Delion, Analytical approach for the Quartet Condensation Model, arXiv:1901.04759
[44] J. Dukelsky, G. Sierra, Phys. Rev. B 61 18 (2000)
[45] P. Camiz, A. Covello, M. Jean, Nuovo Cimento 36, 663 (1965)
[46] H. T. Chen, A. Goswami, Phys. Lett. B 24, 257 (1967).
[47] D. S. Delion et al., Phys. Rev. C 82, 024307 (2010)