ON THE LOCALLY SELF-SIMILAR SINGULAR SOLUTIONS FOR THE INCOMPRESSIBLE EULER EQUATIONS

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Abstract. In this paper we focus on the backward self-similar solutions for the Euler system. The self-similar singular region is a local ball of \( \mathbb{R}^N \), and may shrink to a point as the time converges to the singular time. Under the assumptions that the velocity profile belongs to \( L^p(\mathbb{R}^N) \) or has the non-decaying asymptotics, we prove various nonexistence results and show some asymptotic property concerning the velocity profiles. The proof is mainly relying on the local energy inequality of velocity profile and the bootstrapping method.

1. Introduction

Perfect incompressible fluids are governed by the well-known Euler system

\[
\begin{aligned}
\frac{\partial v}{\partial t} + v \cdot \nabla v + \nabla p &= 0, \\
\nabla \cdot v &= 0, \\
v|_{t=0} &= v_0(x),
\end{aligned}
\]  

where \((x,t) \in \mathbb{R}^N \times \mathbb{R}^+\), \(N = 2, 3, \cdots\) is the spatial dimension, \(v = (v_1, v_2, \cdots, v_N)\) is the velocity vector field of \( \mathbb{R}^N \) and \(p\) is the scalar-valued pressure field. Assume \(v_0 \in H^s(\mathbb{R}^N), s > \frac{N}{2} + 1\), it has been known for decades (e.g. [14]) that there is a unique local-in-time smooth solution \(v \in C([0,T]; H^s)\) and the pressure can be expressed up to a constant by \(p = -\text{div} \text{div} \Delta^{-1}(v \otimes v)\), that is,

\[
p(x,t) = -\frac{1}{N} |v(x,t)|^2 p.v. \int_{\mathbb{R}^N} K_{ij}(x-y)v_i(y,t)v_j(y,t) \, dy,
\]  

where \(K_{ij}(y) = \frac{1}{N|S^{N-1}|} \frac{N\delta_{ij} - y_i y_j}{|y|^{N+2}}\) \((i,j=1,2,\cdots,N)\) is the Calderón-Zygmund kernel and the Einstein convention on repeated indices is used. However, for \(N \geq 3\), whether such smooth solutions have global regularity or they have finite-time blowup remains an outstanding open problem.

In this paper we address the problem of the existence or not of backward locally self-similar solutions for the Euler system. More precisely, we consider solutions that develop a finite-time self-similar singularity on a spacetime domain \([0,T] \times D(t)\) of the form

\[
v(x,t) = \frac{1}{(T-t)^{\alpha+1}} u\left(\frac{x-x_0}{(T-t)^{\frac{1}{\alpha+1}}}\right),
\]  

and

\[
p(x,t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} q\left(\frac{x-x_0}{(T-t)^{\frac{1}{\alpha+1}}}\right) + d(t),
\]  

where \(T > 0, \alpha > -1, x_0 \in \mathbb{R}^N, D(t) := \{x : |x-x_0| \leq \rho(T-t)^{\sigma}\}\) with \(\sigma \geq 0\) small number, \(\rho > 0\), and the solutions \(v, p\) remain regular outside \(D(t)\). If \(\sigma = 0\) and \(\rho = \infty\), this corresponds to the “globally” self-similar solutions and \(D(t) = \mathbb{R}^N\), \(d(t) \equiv 0\); while if \(\rho < \infty\) and \(\sigma \geq 0\), this is the

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“locally” self-similar solutions, and for the case $\sigma > 0$ the domain $D(t)$ dynamically shrinks to the point $x_0$ as $t$ converges to $T$, and $d(t)$ is a function depending only on $t$. For the locally self-similar solutions, from (1.2) and (1.3) it seems not obvious to get the expression (1.4), but which can indeed be justified by Lemma 2.2 (at least for the cases considered below). In terms of $(u, q)$, we formally have

$$
\begin{aligned}
\alpha &\frac{u}{\alpha + 1} + \frac{1}{\alpha + 2} y \cdot \nabla u + u \cdot \nabla u + \nabla q = 0, \\
\text{div } u &= 0,
\end{aligned}
$$

where $y \in \mathbb{R}^N$ and $q$ up to a constant is given by

$$
q(y) = - \frac{|u(y)|^2}{N} + p.v. \int_{\mathbb{R}^N} K_{ij}(y-z) u_i(z) u_j(z) dz,
$$

with $K_{ij} = N|N-1| y^{\alpha - j} |y|^{N+2}$ $(i, j = 1, 2, \ldots, N)$.

The globally self-similar solutions with $\alpha = 1$ were firstly proposed by J. Leray [17] as possible finite-time blowup phenomena for the 3D Navier-Stokes equations, and this scenario was excluded by J. Nečas et al. [18] for the self-similar velocities $u$ belonging to $L^3(\mathbb{R}^3)$, which further extended by T. Tsai [24] for velocities in $L^p(\mathbb{R}^3)$ $(p \in [3, \infty])$. For the locally self-similar solutions about the 3D Navier-Stokes system with $\sigma \geq 0$ small and $\alpha = 1$; T. Hou and R. Li in [12] proved some nonexistence result under suitable assumptions.

For the Euler system (1.1), the ansatz (1.3)-(1.4) is widely used in the numerical simulations, and several results by studying the vortex filament models or high-symmetric flows (see e.g. [1, 13, 15, 16, 19]), suggest that such backward self-similar solutions will exist at a finite time.

From the analytical viewpoint, X. He in [11] constructed nontrivial solutions to the 3D Euler equations (1.5) with $\alpha = 1$ on the exterior domain $\mathbb{R}^3 \setminus B_1(0)$, and the asymptotic decay of such solutions are $|u(y)| \lesssim \frac{1}{|y|}$ and $|\nabla u(y)| \lesssim \frac{1}{|y|^2}$. There are also some noticeable exclusion results on such self-similar solutions in the literature. In [3], D. Chae considered the globally self-similar solutions to the 3D Euler system and proved that if $u \in C^1(\mathbb{R}^3)$ and $\omega = \nabla \times u$ belongs to $\cap_{0 < r < r_0} L^r(\mathbb{R}^3)$ with some $r_0 > 0$, then $\omega \equiv 0$ for all $\alpha > -1$. This result was generalized in [4] for the Euler system in a bounded domain of $\mathbb{R}^3$. R. Takada [23] considered the strong solution to the self-similar Euler system (1.5) and proved $u \equiv 0$ for all $\alpha \in \mathbb{R} \setminus \{-1, N/2\}$ under the condition $u \in C^1_{loc} \cap X^{2, \infty} \cap L^p$ with $p \in [\frac{3N}{N-1}, \frac{4N}{N-2}]$ and $X^{2, \infty} = \{ f \in L^2_{loc} : sup_R \int_{\mathbb{R}^N \setminus |y| \leq 2R} |f(y)|^2 dy < \infty \}$. See also [10, 20] for similar but slightly weaker nonexistence results. For the locally self-similar solutions (1.3)-(1.4) with $\sigma = 0$ and $\rho > 0$, D. Chae and R. Shvydkoy [5] proved that if $u \in C^1_{loc} \cap L^p$ with $r \in [3, \infty)$, then $u \equiv 0$ for all $-1 < \alpha < \frac{N}{r}$ and $\alpha > \frac{N}{r}$. They also improved the result of [3] to get $u \equiv const$ for all $\alpha > -1$ under some suitable assumptions on $u$ and $\omega = \nabla \times u$. Very recently, A. Bronzi and R. Shvydkoy in [2] justified the expression formula of pressure (1.4) for the locally self-similar solutions at the case $\sigma = 0$ and $\rho > 0$, and under the mild assumptions they also proved that the possible nontrivial velocity profile behaves like (1.10) for $0 < \alpha < N/2$.

In this article we deal with the backward locally self-similar solutions (1.3)-(1.4) with $\sigma \geq 0$ small number and $\rho > 0$, to show some exclusion results and some properties of such solutions. More precisely, the first result reads as follows.

**Theorem 1.1.** Suppose that $u \in C^3_{loc}(\mathbb{R}^N)$, and $q$ is defined from $u$ by (1.6). Let $0 \leq \sigma \leq \sigma_\alpha$ with $0 < \sigma_\alpha < \min\{\frac{1}{1+\alpha}, 1\}$ a small number depending only on the coefficients $\alpha, N, \rho, p, \delta$. We have the following statements.
(1) If additionally $u \in L^p(\mathbb{R}^N)$ with some $p \in [3, \infty[$, then for $\alpha > \frac{N}{2}$ and $-1 < \alpha \leq \Lambda_\alpha$ with 
\[ \Lambda_\alpha := \frac{\alpha/2 + N(1-\alpha)}{N\alpha + p(1-\alpha)}, \]
we have $u \equiv 0$, while for $\Lambda_\alpha < \alpha \leq \frac{N}{2}$, we have
\[ \int_{|y| \leq L} |u(y)|^2dy \lesssim L^{1-\frac{2\alpha}{\alpha(1+\alpha)}+\epsilon}, \quad \forall \epsilon > 0, L \gg 1. \tag{1.7} \]

Besides, if $\sigma = 0$ and for $\frac{N}{p} = \Lambda_0 < \alpha \leq \frac{N}{2}$, we have
\[ \int_{|y| \leq L} |u(y)|^2dy \lesssim L^{N-2\alpha}, \quad \forall L \gg 1. \tag{1.8} \]

(2) Under the $L^p$-assumption of $u$ for some $p \in [3, \infty[$, and for every $0 \leq \sigma \leq \tilde{\sigma}_0$ with $\tilde{\sigma}_0$ a small number depending only on $p, N, \rho$, then for $\Lambda_\sigma < \alpha \leq \frac{N}{2}$, we have either $u \equiv 0$, or
\[ L^{\frac{N-2\alpha}{\alpha(1+\alpha)}} \lesssim \int_{|y| \leq L} |u(y)|^2dy \lesssim L^{1-\frac{2\alpha}{\alpha(1+\alpha)}+\epsilon}, \quad \forall \epsilon > 0, L \gg 1. \tag{1.9} \]

Moreover, if $\sigma = 0$ and for $\frac{N}{p} = \Lambda_0 < \alpha < \frac{N}{2}$, we have either $u \equiv 0$ or
\[ \int_{|y| \leq L} |u(y)|^2dy \sim L^{N-2\alpha}, \quad \forall L \gg 1. \tag{1.10} \]

(3) For $\alpha = \frac{N}{2}$, if $u \in L^2(\mathbb{R}^N)$ (which is consistent with (1.7) and (1.8)) and there exists some constant $\delta > 0$ such that
\[ |u(y)| \lesssim |y|^{-\delta}, \quad \forall |y| \gg 1, \tag{1.11} \]

then we have
\[ \int_{L \leq |y| \leq 2L} |u(y)|^2dy \lesssim \frac{1}{L^{N+2-\epsilon}}, \quad \forall L \gg 1, 0 < \epsilon \ll 1. \tag{1.12} \]

**Remark 1.2.** Note that the second part of Theorem 1.1 implies $u \equiv 0$ if $u$ satisfies
\[ \int_{|y| \leq L} |u(y)|^2dy \lesssim L^{1-\frac{2\alpha}{\alpha(1+\alpha)}} o_L(1), \quad \text{for } \Lambda_\sigma < \alpha < \frac{\alpha}{2}, \tag{1.13} \]

with $o_L(1) \to 0$ as $L \to \infty$; while for $-1 < \alpha \leq \Lambda_\sigma$, (1.12) automatically holds by the $L^p$-assumption of $u$. Thus at the case of $\sigma = 0$, the first and second parts of Theorem 1.1 improve the results of [10, 20, 23] dramatically, and they mostly reduce to the corresponding results of [2, 5]. For the third part of Theorem 1.1, although for $\sigma = 0$ it has appeared in [21] based on Theorem 3.1 of [5] (which is about the drain property of original velocity and needs additionally $v \in L^r([0,T]; L^\infty)$, $r > 1$), we here provide a different and more direct proof which only relies on the basic local energy inequality of the velocity profile.

**Remark 1.3.** So far it is not clear to derive the more natural bounds than (1.7) and (1.9) that
\[ \int_{|y| \leq L} |u(y)|^2dy \lesssim L^{\frac{N-2\alpha}{\alpha(1+\alpha)}}, \quad \forall L \gg 1, \sigma > 0 \text{ small enough}, \tag{1.13} \]

by only relying on the local energy inequality of the velocity profile (see (3.3) below) and the general $L^p$-assumption.

The next result is concerned with the situation that the velocity profile $u$ has non-decaying asymptotics. This is motivated by some numerical simulations and especially by several works on the 1D models of Euler equations: the 1D Burgers equation $\partial_t u + u \partial_x u = 0$ and the 1D CCF model $\partial_t v + (Hv) \partial_x v = 0$, with $v$ a 1D scalar function and $H$ the usual Hilbert transform. The Burgers equation develops shock singularity at finite time, while it is proved in [6] that the CCF equation forms finite-time cusp singularity for some smooth data. The further study [7, 9] shows that the finite-time
singularities of both equations are of (locally) self-similar type with some index \( \alpha \in [ -1, 0 ] \) and the corresponding profiles have growing asymptotics. Hence it deserves to consider such a scenario for Euler equations.

For this purpose, we need a more refined (and equivalent) version of the expression formula of the pressure profile. Since the pressure profile is defined by (1.6) up to a constant, we assume it is given by that for all \( y \in B_L(0) \),

\[
q(y) = q(y, L) = -\frac{1}{N} |u(y)|^2 + p.v. \int_{\mathbb{R}^N} K_{ij}(y - z)u_i(z)u_j(z) \, dz + \bar{q}(L)
\]

with

\[
\bar{q}(L) = - \int_{|z| \geq 2L} K_{ij}(z)u_i(z)u_j(z) \, dz;
\]

that is, for all \( y \in B_L(0) \),

\[
q(y) = q(y, L) = -\frac{1}{N} |u(y)|^2 + p.v. \int_{|y| \leq 2L} K_{ij}(y - z)u_i(z)u_j(z) \, dz + \\
+ \int_{|z| \geq 2L} (K_{ij}(y - z) - K_{ij}(z))u_i(z)u_j(z) \, dz.
\]

This is analogous with the introducing of a new expression formula (cf. [3 Eq. (6.4)]) of the singular integral operator when proving the property that it continuously maps \( L^\infty(\mathbb{R}^N) \) to \( BMO(\mathbb{R}^N) \). Note that for every \( 0 \leq L_1 \leq L_2 < \infty \) and \( y \in B_{L_2}(0) \), the difference between \( q(y, L_1) \) and \( q(y, L_2) \) is \( \int_{2L_1 \leq |x| \leq 2L_2} K_{ij}(z)u_i(z)u_j(z) \, dz \), which is a finite constant under the general assumption like (1.17), and thus on their common domain \( q(y, L_1) \) and \( q(t, L_2) \) can be seen as equal.

Our second main result reads as follows.

**Theorem 1.4.** Suppose that \( u \in C^3_{\text{loc}}(\mathbb{R}^N) \) satisfies that for some \( \delta \in [0, 1/2] \),

\[
1 \lesssim |u(y)| \lesssim |y|^\delta, \quad \forall |y| \gg 1.
\]

and up to a constant \( q \) is defined from \( u \) by (1.14). Then for \( 0 \leq \sigma \leq \sigma_0 \) with \( 0 < \sigma_0 < \min\{\frac{1}{1+\alpha}, 1\} \) a small number depending only on the coefficients \( \alpha, N, \rho, \delta \), the only possible scope of \( \alpha \) is \( \tilde{\Lambda}_\sigma \leq \alpha \leq \frac{N\sigma}{2-N\sigma} \), and \( \tilde{\Lambda}_\sigma := -\frac{\delta(1-\sigma)}{1-(\sigma + N/2)\sigma} \), and we have

\[
L^{\frac{N-2\alpha}{1-\sigma(1+\alpha)}} \lesssim \int_{|y| \leq L} |u(y)|^2 \, dy \lesssim L^{\frac{N-2\alpha}{1-\sigma(1+\alpha)+\epsilon}}, \quad \forall \epsilon > 0, L \gg 1.
\]

Besides, if \( \sigma = 0 \) and for \( -\delta = \tilde{\Lambda}_0 \leq \alpha \leq 0 \), we have

\[
\int_{|y| \leq L} |u(y)|^2 \, dy \sim L^{N-2\alpha}, \quad \forall L \gg 1.
\]

**Remark 1.5.** From (1.9)-(1.10) and (1.18)-(1.19), we can expect the typical possible velocity profile is that

\[
u(y) \sim |y|^{\frac{\sigma(1+\alpha)/2-\alpha}{1-\sigma(1+\alpha)}+l.o.t.}, \quad \forall |y| \gg 1, \sigma \geq 0 \text{ small enough},
\]

and by scaling, we can also expect the typical vorticity profile is

\[
\nabla \times u(y) \sim |y|^{\frac{\sigma(1+\alpha)/2-\alpha}{1-\sigma(1+\alpha)}-1+l.o.t.}, \quad \forall |y| \gg 1, \sigma \geq 0 \text{ small enough}.
\]
where l.o.t. is the abbreviation of the lower order terms. Thus in the considered blowup scenario, we have that for all \( (t, x) \in [0, T] \times (D(t) \setminus \{ x_0 \}) \),

\[
\nabla \times v(x, t) = \frac{1}{T-t} \nabla \times u \left( \frac{x-x_0}{(T-t)^{1+\alpha}} \right) \sim \frac{1}{(T-t)^{\frac{\sigma(1/2-\alpha)}{1-\sigma(1+\alpha)}}} \frac{1}{x-x_0^{\frac{(1+\alpha)(1-3\sigma/2)}{1-\sigma(1+\alpha)}}},
\]

(1.22)

Such typical finite-time blowup case is compatible with the Beale-Kato-Majda criterion since for any \( \alpha > -1 \) and some \( \sigma \in [0, \min\left\{ \frac{1}{\alpha+1}, \frac{2}{3} \right\}] \),

\[
\int_0^T \| \nabla \times v \|_{L^\infty} \ dt \sim T^{\frac{1-3\sigma/2}{1-\sigma(1+\alpha)}} \sup_{0 < |x-x_0| \leq \rho} |x-x_0| \frac{(1+\alpha)(1-3\sigma/2)}{1-\sigma(1+\alpha)} = \infty.
\]

But if one instead can manage to prove the a priori bound that \( \int_0^T \| \nabla \times v \|_{L^r} \ dt < \infty \) with some \( 1 \leq r < \infty \), there is still some possibility that such typical blowup scenario may happen at the range \(-1 < \alpha < -1 + \frac{N\sigma}{r(1-3\sigma/2+N\sigma/r)}\).

The starting point of the proofs for both Theorem 1.1 and 1.4 is the local energy inequality

\[
\left| \int_{l_2}^{l_1} \int_{|y| \leq l_2} |u(y)|^2 \phi(y/l_2) \, dy - l_1 \int_{|y| \leq l_1} |u(y)|^2 \phi(y/l_1) \, dy \right| \\
\leq C_0 \int_{l_1/2 \leq |y| \leq l_2} \frac{|u(y)|^2}{|y|^{N-2\alpha+\alpha+1}} \, dy + C \int_{l_1/2 \leq |y| \leq l_2} \frac{|u^3 + |q||u|}{|y|^{N-2\alpha+\alpha+1}} \, dy,
\]

(1.23)

where \( 0 < l_1 < l_2 \) and \( \phi(x) = \phi(|x|) \) is a suitable test function. The inequality (1.23) in turn is deduced from the local energy equality of the original velocity \( v \). If \( \sigma > 0 \), there is an additional integral term on the righthand side of (1.23) comparing with the \( \sigma = 0 \) case, and this will lead to much technical difficulty in the further deduction. Here for \( \sigma > 0 \) small enough, we can overcome it by applying the bootstrapping method.

The outline of this paper is as follows. In Section 2 we display two useful auxiliary lemmas, and the one justifying the expression of pressure is shown in the appendix section. By relying on the local energy inequality of \( u \) presented in Subsection 3.1 we give the detailed proofs of Theorem 1.1 and Theorem 1.4 in Section 3 and 4 respectively.

Throughout this paper, \( C \) stands for a constant which may be different from line to line, and \( X \leq Y \) means that there is a harmless constant \( C \) such that \( X \leq CY \). Denote \( B_r(x) := \{ y \in \mathbb{R}^N : |y-x| \leq r \} \) the ball of \( \mathbb{R}^N \) and \( B_r(x)^c := \mathbb{R}^N \setminus B_r(x) \) its complement set.

2. Auxiliary lemmas of the pressure profile

We collect two auxiliary lemmas in this section: one is useful in the main proof, and the other is about the justification of the expression formula of pressure.

Lemma 2.1. Assume that \( u \in C^1_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N) \) is a locally regular vector field.

(1) Suppose \( u \) additionally satisfies that for every \( 2 < p < \infty \) and \( L \gg 1 \),

\[
\int_{|y| \leq L} |u(y)|^p \, dy \lesssim L^a, \quad \text{with} \quad 0 \leq a < N.
\]

Let \( q \) be a scalar field defined from \( u \) by

\[
q(y) = c_0 |u(y)|^2 + p.v. \int_{\mathbb{R}^N} K_{ij}(y-z)u_i(z)u_j(z) \, dz,
\]
with \( c_0 \in \mathbb{R} \) and \( K_{ij}(z) \) \((i, j = 1, \ldots, N)\) some Calderón-Zygmund kernel, then we have

\[
\int_{|y| \leq L} |q(y)|^\frac{p}{2} \, dy \lesssim L^a. \tag{2.1}
\]

(2) Suppose \( u \) additionally satisfies that for every \( 0 \leq \delta < 1 \) and \( L \gg 1 \),

\[
|u(y)| \lesssim |y|^\delta, \quad \forall |y| \gg 1, \quad \text{and}
\]

\[
\int_{|y| \leq L} |u(y)|^2 \, dy \lesssim L^b, \quad \text{with} \quad 0 \leq b < N + 1.
\]

Let \( q \) be a scalar field defined from \( u \) by that for every \( |y| \leq L \),

\[
q(y) = c_0 |u(y)|^2 + p.v. \int_{|z| \leq 2L} K_{ij}(y - z)u_i(z)u_j(z) \, dz
\]

\[
+ \int_{|z| \geq 2L} (K_{ij}(y - z) - K_{ij}(z))u_i(z)u_j(z) \, dz
\]

\[
:= c_0 |u(y)|^2 + q_1(y, L) + q_2(y, L),
\]

with \( c_0 \in \mathbb{R} \) and \( K_{ij}(z) \) \((i, j = 1, \ldots, N)\) some Calderón-Zygmund kernel, then we have

\[
\int_{|y| \leq L} |q(y)||u(y)| \, dy \lesssim L^{b+\delta}. \tag{2.3}
\]

**Proof of Lemma 2.1.** (1) Denote by \( \tilde{q}(y) = p.v. \int_{\mathbb{R}^N} K_{ij}(y - z)u_i(z)u_j(z) \, dz \). We only need to treat the integral involving \( \tilde{q} \), and we have the following decomposition for \( \tilde{q} \)

\[
\tilde{q}(y) = p.v. \int_{|z| \leq 2L} K_{ij}(y - z)u_i(z)u_j(z) \, dz + \int_{|z| \geq 2L} K_{ij}(y - z)u_i(z)u_j(z) \, dz
\]

\[
:= \hat{q}_1(y, L) + \hat{q}_2(y, L). \tag{2.4}
\]

For the integral of \( q_1(y, L) \), by the Calderón-Zygmund inequality, we find

\[
\int_{|y| \leq L} |\hat{q}_1(y, L)|^\frac{p}{2} \, dy \leq \int_{|z| \leq 2L} |u(z)|^p \, dz \lesssim L^a.
\]

For the integral of \( \hat{q}_2(y, L) \), by the dyadic decomposition, we have

\[
\int_{|y| \leq L} |\hat{q}_2(y, L)|^\frac{p}{2} \, dy \leq C \int_{|y| \leq L} \left( \sum_{k=1}^{\infty} \int_{2^k L \leq |z| \leq 2^{k+1} L} \frac{1}{|y - z|^N} |u(z)|^2 \, dz \right)^{p/2} \, dy
\]

\[
\leq CL^N \left( \sum_{k=1}^{\infty} \frac{1}{(2^k N)^{2/p}} \left( \int_{|z| \leq 2^{k+1} L} |u(z)|^p \, dz \right)^{\frac{p}{2}} \right)
\]

\[
\leq CL^a \left( \sum_{k=1}^{\infty} \frac{1}{2^{(N-a)/p}} \right)^{p/2} \lesssim CL^a.
\]

Gathering the above estimates yields the desired result.

(2) It suffices to consider the terms containing \( q_1(y, L) \) and \( q_2(y, L) \). For the term involving \( q_1(y, L) \), by the Hölder inequality and Calderón-Zygmund theorem, we get

\[
\int_{|y| \leq L} |q_1(y, L)||u(y)| \, dy \leq \left( \int_{|y| \leq L} |q_1(y, L)|^\frac{p}{2} \, dy \right)^{\frac{2}{p}} \left( \int_{|y| \leq L} |u(y)|^3 \, dy \right)^{\frac{1}{3}}
\]

\[
\lesssim \int_{|y| \leq 2L} |u(y)|^3 \, dy \lesssim L^{b+\delta}.
\]
For the term containing \( q_2(y, L) \), using the support property and the dyadic decomposition again, we infer that
\[
\int_{|y| \leq L} |q_2(y, L)||u(y)| \, dy \lesssim L^{N+\delta} \sup_{|y| \leq L} |q_2(y, L)|
\]
\[
\lesssim L^{N+\delta} \sup_{|y| \leq L} \left( \sum_{k=1}^{\infty} \int_{|z| \leq 2^{k+1}L} \frac{|y|}{|z|^{N+1}} |u(z)|^2 \, dz \right)
\]
\[
\lesssim L^{N+\delta+1} \sum_{k=1}^{\infty} \frac{1}{(2^k L)^{N+1}} \int_{|z| \sim 2^k L} |u(z)|^2 \, dz
\]
\[
\lesssim L^{N+\delta+1} \sum_{k=1}^{\infty} (2^k L)^{b-N-1} \lesssim L^{b+\delta}.
\]
Collecting the above estimates leads to (2.3). □

Next we state the lemma that justifies the expression formula of pressure (1.4).

**Lemma 2.2.** Suppose \( v \) is a locally self-similar solution (1.3) to the Euler equations and \( \alpha > -1/2 \). Assume that \( u \in C^{3}_{\text{loc}}(\mathbb{R}^N) \) satisfies that for some \( \delta \in [0, 1/2] \),
\[
|u(y, t)| \lesssim |y|^\delta, \quad \forall|y| \gg 1,
\]
then the scalar function \( q(y) \) defined by
\[
q(y) = -\frac{1}{N} |u(y)|^2 + p \cdot v, \quad K_{ij}(y-z)u_i(z)u_j(z) \, dz + \bar{q}(L)
\]
with
\[
\bar{q}(L) = \begin{cases} -\int_{|z| \geq 2L} K_{ij}(z)u_i(z)u_j(z) \, dz, & \text{if } 1 \lesssim |u(y)| \lesssim |y|^\delta, \quad \forall|y| \gg 1, \\
0, & \text{if } u \in L^p(\mathbb{R}^N), \quad p \in ]2, \infty[, 
\end{cases}
\]
is \( C^2 \)-smooth on the domain \( B_L(0) \) and solves that
\[
\frac{\alpha}{\alpha+1} u + \frac{1}{\alpha+1} y \cdot \nabla u + u \cdot \nabla u + \nabla q = 0.
\]
Moreover, there is a function \( d(t) \) depending only on \( t \) and satisfying \( |d(t)| \lesssim (T-t)^{\frac{2\alpha}{\alpha+1} - N\sigma} + (T-t)^{-\frac{\sigma}{1+\alpha}} \), such that the pressure is expressed as
\[
p(x, t) = \frac{1}{(T-t)^{\frac{2\alpha}{\alpha+1}}} q \left( \frac{x-x_0}{(T-t)^{\frac{1}{\alpha+1}}} \right) + d(t),
\]
which holds in the ball \( |x-x_0| \leq \rho_0|T-t|^{\sigma} \) for all \( t \) near \( T \) and \( \sigma \geq 0 \) small.

We here mainly adapt the strategy in the proof of [2] Lemma 2.1 with suitable modification, and the detailed proof is given in the appendix section.

3. Proof of Theorem 1.1

3.1. Local energy inequality. We start with the following energy equality for the original velocity, at least on the region of self-similarity,
\[
\int_{\mathbb{R}^N} |v(t_2, x)|^2 \chi(t_2, x) \, dx - \int_{\mathbb{R}^N} |v(t_1, x)|^2 \chi(t_1, x) \, dx
\]
\[
= \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |v(t, x)|^2 \partial_t \chi(t, x) \, dx \, dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (|v|^2 v + 2(p-d(t))v) \cdot \nabla \chi(t, x) \, dx \, dt,
\]
where \( \chi \in \mathcal{D}(0,T \times \mathbb{R}^N) \) and \( 0 < t_1 < t_2 < T \). The equality can be satisfied if the velocity is regular enough, for instance, \( v \in C^1_{loc}(0,T \times \mathbb{R}^N) \cap L^\infty(0,T; L^2(\mathbb{R}^N)) \).

We only consider (3.1) on the region of self-similarity, and we assume that \( x_0 = 0, \rho = 1 \) without loss of generality. Let \( \phi \in \mathcal{D}(\mathbb{R}^N) \) be a radial smooth cutoff function such that \( 0 \leq \phi \leq 1, \phi \equiv 1 \) on \( B_{1/2}(0) \) and \( \phi \equiv 0 \) on \( B_1(0)^c \). Then set \( \chi(t,x) = \phi(t/T, x) \), we have

\[
\int_{\mathbb{R}^N} |v(t_2,x)|^2 \phi\left(\frac{x}{T-t_2}\right) dx - \int_{\mathbb{R}^N} |v(t_1,x)|^2 \phi\left(\frac{x}{T-t_1}\right) dx
= \sigma \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \frac{|v(t,x)|^2}{(T-t)^{\sigma+1}} x \cdot \nabla \phi\left(\frac{x}{T-t}\right) dx dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \frac{|v|^2 v + (p-d(t))v}{(T-t)^{\sigma}} \cdot \nabla \phi\left(\frac{x}{T-t}\right) dx dt,
\]

(3.2)

Thanks to (1.3), and by changing the variable \( \chi = (T-t)^{1+\sigma} y \), we see that

\[
(T-t_2)^{\frac{N-2\alpha}{1+\sigma}} \int_{\mathbb{R}^N} |u(y)|^2 \phi(y) (T-t_2)^{1+\sigma} dy - (T-t_1)^{\frac{N-2\alpha}{1+\sigma}} \int_{\mathbb{R}^N} |u(y)|^2 \phi(y) (T-t_1)^{1+\sigma} dy
= \sigma \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u(y)|^2 (T-t)^{-\frac{N-2\alpha}{1+\sigma} - 1 - \sigma} \cdot \nabla \phi(y) (T-t)^{1+\sigma} dy dt + \int_{t_1}^{t_2} \int_{\mathbb{R}^N} (T-t)^{-\frac{N-2\alpha}{1+\sigma} - \sigma} |u|^2 u(y) + 2q(y) u(y)) \cdot \nabla \phi(y) (T-t)^{1+\sigma} dy dt
\]

By denoting \( l_i = (T-t_i)^{-\frac{1}{1+\sigma}} \), \( i = 1, 2 \), and integrating on the \( t \)-variable in the last two integrals, we obtain that

\[
\frac{1}{l_2^{\gamma_\alpha}} \int_{|y| \leq l_2} |u(y)|^2 \phi(y) l_2^\gamma dy - \frac{1}{l_1^{\gamma_\alpha}} \int_{|y| \leq l_1} |u(y)|^2 \phi(y) l_1^\gamma dy
\leq C\sigma \int_{l_1 \leq |y| \leq l_2} \frac{|u(y)|^2}{|y|^{\gamma_\alpha}} dy + C \int_{l_1 \leq |y| \leq l_2} \frac{|u|^3 + |q||u|}{|y|^{\gamma_\alpha+1}} dy
\]

(3.3)

with

\[
\gamma_\alpha := \frac{N - 2\alpha}{1 - \sigma(1 + \alpha)}.
\]

3.2. **Proof of Theorem 1.1(1).** First we consider the case \( \alpha > N/2 \). From (3.3), and by setting \( l_1 = 2 \) and \( l_2 = 2L \gg 1 \), we have

\[
\frac{1}{L^{\gamma_\alpha}} \int_{|y| \leq L} |u(y)|^2 dy \leq C \int_{|y| \leq 1} |u(y)|^2 dy + C\sigma \int_{1 \leq |y| \leq 2L} \frac{|u(y)|^2}{|y|^{\gamma_\alpha}} dy + C \int_{1 \leq |y| \leq 2L} \frac{|u|^3 + |q||u|}{|y|^{\gamma_\alpha+1}} dy,
\]

that is,

\[
\int_{|y| \leq L} |u(y)|^2 dy \leq CL^{\gamma_\alpha} + E_1(L) + E_2(L),
\]

(3.5)

with

\[
E_1(L) := C\sigma L^{\gamma_\alpha} \int_{1 \leq |y| \leq 2L} \frac{|u(y)|^2}{|y|^{\gamma_\alpha}} dy, \quad \text{and} \quad E_2(L) := CL^{\gamma_\alpha} \int_{1 \leq |y| \leq 2L} \frac{|u|^3 + |q||u|}{|y|^{\gamma_\alpha+1}} dy.
\]
By the dyadic decomposition, Hölder’s inequality and Lemma 2.1, we find

\[
E_2(L) \leq CL^{\gamma_\alpha} \sum_{k=-1}^{[\log_2 L]} \int_{\frac{L}{2^{k+1}} \leq |y| \leq \frac{L}{2^k}} \frac{|u|^3 + |q||u|}{|y|^{\gamma_\alpha + 1}} \, dy \\
\leq \frac{C}{L} \sum_{k=-1}^{[\log_2 L]} 2^{k(\gamma_\alpha + 1)} \int_{\frac{L}{2^{k+1}} \leq |y| \leq \frac{L}{2^k}} (|u|^3 + |q||u|) \, dy \\
\leq CL^{N-3N/p-1} \sum_{k=-1}^{[\log_2 L]} 2^{k(\gamma_\alpha + 1 - N+3N/p)} \left( \int_{|y| \leq \frac{L}{2^k}} (|u|^3 + |q||u|)^{\frac{2}{p}} \, dy \right)^{\frac{3}{p}} \\
\leq CL^{N-1-3N/p} \sum_{k=-1}^{[\log_2 L]} 2^{-k(N-1-3N/p-\gamma_\alpha)},
\]

where \([\log_2 L]\) denotes the integer part of \(\log_2 L\). If \(N - 1 - 3N/p - \gamma_\alpha \leq 0\), we get

\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq CL^{\gamma_\alpha} [\log_2 L] + E_1(L) \\
\leq C_\epsilon L^{\gamma_\alpha + \epsilon} + E_1(L), \quad \forall \epsilon > 0.
\] (3.7)

Otherwise, we get

\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq CL^{\beta_p} + E(L), \quad \text{with} \quad \beta_p = N - 1 - 3N/p.
\] (3.8)

From (3.7), we intend to show that

\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} C_\epsilon L^{\gamma_\alpha + \epsilon}, \quad \forall \epsilon > 0, L \geq 1,
\] (3.9)

by letting \(\sigma\) small enough; which ensures that \(u \equiv 0\) for \(\epsilon\) small enough (e.g. \(\epsilon < \alpha - N/2\)). Indeed, we may assume a larger bound with replacing \(3/2\) by \(2\) in (3.9) is satisfied, then by the dyadic decomposition we have

\[
E_1(L) \leq C\sigma L^{\gamma_\alpha} \sum_{k=-1}^{[\log_2 L]} \int_{\frac{L}{2^{k+1}} \leq |y| \leq \frac{L}{2^k}} \frac{|u(y)|^2}{|y|^{\gamma_\alpha}} \, dy \\
\leq C\sigma \sum_{k=-1}^{[\log_2 L]} 2^{k\gamma_\alpha} \int_{|y| \leq \frac{L}{2^k}} |u(y)|^2 \, dy \leq C\sigma \sum_{k=-1}^{[\log_2 L]} 2CL^{\gamma_\alpha} (L/2^k)\epsilon \leq C''_\sigma L^{\gamma_\alpha + \epsilon}.
\]

Thus by choosing \(\sigma\) sufficiently small, and from (3.7), we get the improved bound (3.9). The bootstrapping method implies that (3.9) holds. Similarly, from (3.8), by letting \(\sigma\) small enough, we can obtain that

\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{\beta_p}.
\] (3.10)
Indeed, we only observe that

$$E_1(L) \leq C \sigma \sum_{k=-1}^{[\log_2 L]} 2^{k\gamma_\alpha} \int_{|y| \leq \frac{L}{2^k}} |u(y)|^2 \, dy$$

$$\leq C \sigma \sum_{k=-1}^{[\log_2 L]} 2^{k(\gamma_\alpha - \beta_p)} 2CL^{\beta_p} \leq C \frac{\sigma}{\beta_p - \gamma_\alpha} L^{\beta_p},$$

where in the last line we have used the facts that $\gamma_\alpha - \beta_p < 0$ and $\sum_{k=-1}^{\infty} 2^{-k\beta} = O\left(\frac{1}{\epsilon}\right)$. If $\beta_p < 0$, the proof is finished. Next we consider the case $\beta_p \geq 0$. By interpolation, we see that

$$\int_{|y| \leq L} |u(y)|^3 \, dy \leq \left( \int_{|y| \leq L} |u(y)|^2 \, dy \right)^{\mu_p} \left( \int_{|y| \leq L} |u(y)|^p \, dy \right)^{1-\mu_p} \leq CL^{\beta_p \mu_p}. \quad (3.11)$$

with $\mu_p = \frac{p-3}{p-2}$. Inserting the refined estimate (3.11) into (3.6) and using Lemma 2.1 again, we infer that

$$E_2(L) \leq C \sum_{k=-1}^{[\log_2 L]} 2^{k(\gamma_\alpha + 1)} \int_{2^{k+1} \leq |y| \leq \frac{L}{2^k}} (|u|^3 + |q||u|) \, dy$$

$$\leq CL^{\beta_p \mu_p - 1} \sum_{k=-1}^{[\log_2 L]} 2^{-k(\beta_p \mu_p - 1 - \gamma_\alpha)}.$$

If $\beta_p \mu_p - 1 - \gamma_\alpha \leq 0$, then we get

$$\int_{|y| \leq L} |u(y)|^2 \, dy \leq CL^{\alpha} [\log_2 L] + E_1(L) \leq C_\epsilon L^{\gamma_\alpha + \epsilon} + E_1(L).$$

Otherwise, we get

$$\int_{|y| \leq L} |u(y)|^2 \, dy \leq CL^{\beta_p \mu_p - 1} + E_1(L).$$

In a similar way as the treating of (3.7)-(3.8), if $\sigma$ is small enough, we can show $u \equiv 0$ except the case $\beta_p \mu_p - 1 \geq 0$. Then for such a case, we have

$$\int_{|y| \leq L} |u(y)|^2 \, dy \leq C L^{\beta_p \mu_p - 1}, \quad \text{and} \quad \int_{|y| \leq L} |u(y)|^3 \, dy \leq C L^{\beta_p \mu_p - \mu_p}.$$

We can repeat the above process until the power $\beta_p \mu_p - \mu_p^{n-1} - \cdots - 1 < 0$, or we can terminate before that, which ends the proof.

Now we tackle the case $-1 < \alpha < \Lambda_\sigma$. First note that we can eliminate the $l_2$-integral in (3.3) by sending $l_2$ to $\infty$. Indeed, by Hölder’s inequality, we deduce that for $0 < M < l_2$,

$$\frac{1}{l_2} \int_{|y| \leq l_2} |u(y)|^2 \phi(y l_2^{-1}) \, dy = \frac{1}{l_2} \int_{|y| \leq M} |u(y)|^2 \, dy + \frac{1}{l_2} \int_{M \leq |y| \leq l_2} |u(y)|^2 \, dy$$

$$\leq \frac{1}{l_2} \left( M^{N(1-2/p)} \|u\|_{L^p}^2 + C \frac{1}{l_2^2} l_2^{N(1-2/p)} \left( \int_{|y| \geq M} |u(y)|^p \, dy \right)^{2/p} \right) \rightarrow 0, \quad \text{by} \ l_2 \rightarrow \infty \quad \text{and then} \ M \rightarrow \infty,$$
where we have used the fact that the power $N - 2N/p - \gamma_\alpha \leq 0$ for all $\alpha \leq \Lambda_\sigma$. Thus in (3.3) by letting $l_2 \to \infty$ and $l_1 = 2L \gg 1$, we get
\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq C \sigma L^{\gamma_\alpha} \int_{|y| \leq L} \frac{|u(y)|^2}{|y|^\gamma_\alpha} \, dy + C L^{\gamma_\alpha} \int_{|y| \geq L} \frac{|u|^3 + |q||u|}{|y|^\gamma_\alpha+1} \, dy
\]
\[:= F_1(L) + F_2(L). \tag{3.12} \]

Thanks to the dyadic decomposition and Lemma 2.1, we find
\[
F_2(L) = CL^{\gamma_\alpha} \sum_{k=0}^\infty \int_{2^k L \leq |y| \leq 2^{k+1} L} \frac{|u|^3 + |q||u|}{|y|^\gamma_\alpha+1} \, dy
\]
\[\leq C L \sum_{k=0}^\infty \frac{1}{2^{k(\gamma_\alpha+1)}} \int_{2^k L \leq |y| \leq 2^{k+1} L} (|u|^3 + |q||u|) \, dy
\]
\[\leq C L \sum_{k=0}^\infty \frac{1}{2^{k(\gamma_\alpha+1)}} (2^k L)^N(1-3/p) \left( \int_{|y| \leq 2^{k+1} L} (|u|^3 + |q||u|)^{p/3} \, dy \right)^{3/p}
\]
\[\leq CL^{N-1-3N/p} \sum_{k=0}^\infty 2^{-k(\gamma_\alpha+1-N+3N/p)}
\]
\[\leq CL^{\beta_p}, \quad \text{with} \quad \beta_p = N - 1 - 3N/p,
\]

where we have used the fact that $\gamma_\alpha > \beta_p$ which is deduced from $\gamma_\alpha + 2N/p - N > 0$. Hence we obtain
\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq CL^{\beta_p} + F_1(L). \tag{3.14}
\]

If $\sigma$ is small enough, we claim that
\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{\beta_p}, \quad \forall L \gg 1. \tag{3.15}
\]

Indeed, we first assume that a larger bound with replacing $3/2$ by $2$ in (3.15) is satisfied, then using the dyadic decomposition yields
\[
F_1(L) = C \sigma L^{\gamma_\alpha} \sum_{k=0}^\infty \int_{2^k L \leq |y| \leq 2^{k+1} L} \frac{|u(y)|^2}{|y|^\gamma_\alpha} \, dy
\]
\[\leq C \sigma \sum_{k=0}^\infty 2^{-k\gamma_\alpha} \int_{|y| \leq 2^{k+1} L} |u(y)|^2 \, dy
\]
\[\leq C \sigma \sum_{k=0}^\infty 2^{-k(\gamma_\alpha - \beta_p)} 2CL^{\beta_p} \leq \frac{C \sigma}{\gamma_\alpha - \beta_p} L^{\beta_p}.
\]

Thus from (3.14), by choosing $\sigma$ small enough, we find an improved bound (3.15) holds, which concludes the claim by the bootstrapping method. Now from (3.15), if $\beta_p < 0$, the proof is over. Otherwise, by interpolation,
\[
\int_{|y| \leq L} |u(y)|^3 \, dy \leq CL^{\beta_p \mu_p}, \quad \text{with} \quad \mu_p = \frac{p - 3}{p - 2}. \tag{3.16}
\]
Plugging (3.16) into (3.13) and using Lemma 2.1 again, we have
\[
\int_{|y| \leq L} |u(y)|^2 dy \leq \frac{C}{L} \sum_{k=0}^{\infty} \frac{1}{2^{k(\gamma_{\alpha}+1)}} \int_{|y| \leq 2^{k+1}L} (|u|^3 + |q||u|) dy
\leq CL^{\beta_p \mu_p - 1} \sum_{k=0}^{\infty} \frac{1}{2^{k(\gamma_{\alpha} - \beta_p \mu_p + 1)}} \leq CL^{\beta_p \mu_p - 1}.
\]
If \( \beta_p \mu_p - 1 < 0 \), the proof is finished. Otherwise, we can repeat the process as above to show that
\[
\int_{|y| \leq L} |u(y)|^2 dy \leq CL^{\beta_p \mu_p^n - \mu_p^{-1} - \ldots - 1}, \quad n \in \mathbb{N}.
\]
For \( n \) large enough, the power becomes negative, and then concludes the proof.

Now we consider the case \( \Lambda_{\sigma} < \alpha < N/2 \). The procedure is quite similar to the case \( \alpha > N/2 \), and for every \( \epsilon > 0 \), as long as the power of \( L \) is larger than \( \gamma_{\alpha} + \epsilon \), we can go into the iterative process to reduce its value. If \( \sigma = \bar{\sigma} \), we can prove the improved estimate (1.8) in a similar way as the treating in [5] (or referring to the proof of (1.19) below).

3.3. Proof of Theorem 1.1-(2): the range \( \Lambda_{\sigma} \leq \alpha < N/2 \). To show (1.9), we only need to prove the the inequality on the left-hand-side
\[
\frac{1}{L^{\gamma_{\alpha}}} \int_{|y| \leq L} |u(y)|^2 dy \gtrsim 1, \quad \forall L \gg 1.
\]
(3.17)
Suppose (3.17) is not true, then there exists a sequence of numbers \( L_n \gg 1 \) such that
\[
\frac{1}{L_n^{\gamma_{\alpha}}} \int_{|y| \leq L_n} |u(y)|^2 dy \to 0, \quad \text{as } L_n \to \infty.
\]
(3.18)
Thus by letting \( l_2 = L_n \to \infty \) and \( l_1 = 2L > 0 \), we can eliminate the \( l_2 \)-integral in (3.3) to get
\[
\int_{|y| \leq L} |u(y)|^2 dy \leq F_1(L) + F_2(L),
\]
with \( F_1(L), F_2(L) \) introduced in (3.12).

Dyadic decomposition leads to
\[
F_2(L) \leq \frac{C}{L} \sum_{k=0}^{\infty} \frac{1}{2^{k(\gamma_{\alpha}+1)}} \int_{2^kL \leq |y| \leq 2^{k+1}L} (|u|^3 + |q||u|) dy,
\]
(3.19)
Since we do not have (3.17), that is, we instead have
\[
\int_{|y| \leq L} |u(y)|^2 dy \lesssim L^{\gamma_{\alpha}},
\]
then from this estimate and interpolation, we see that
\[
\int_{|y| \leq L} |u(y)|^3 dy \leq CL^{\gamma_{\alpha} \mu_p}, \quad \text{with } \mu_p = \frac{p-3}{p-2}.
\]
(3.20)
Inserting (3.20) into (3.19) and using Lemma 2.1 (needing that \( \gamma_{\alpha} < N \), i.e. \( \sigma < \frac{2\Lambda_{\sigma}}{N(1 + \Lambda_{\sigma})} \), equivalently, \( \sigma < \bar{\sigma}_0 \) with some small absolute constant \( \bar{\sigma}_0 > 0 \)), we get
\[
F_2(L) \leq CL^{\gamma_{\alpha}} \sum_{k=0}^{\infty} \frac{1}{(2^kL)^{\gamma_{\alpha}(1-\mu_p)+1}} \leq CL^{\gamma_{\alpha} \mu_p - 1}.
\]
From (3.14), and in a similar way as deriving (3.15) from (3.14), we get that for \( \sigma \) small enough,
\[
\int_{|y| \leq L} |u(y)|^2 dy \leq \frac{3}{2} C L^{\gamma_0 \mu_0 - 1}.
\]
If \( \gamma_0 \mu_0 - 1 < 0 \), the proof is finished. Otherwise,
\[
\int_{|y| \leq L} |u(y)|^3 dy \leq CL^{\gamma_0 \mu_0^2 - \mu_0 - 1}.
\]
By iteration, after a finite steps we have
\[
\int_{|y| \leq L} |u(y)|^2 dy \leq CL^{\gamma_0 \mu_0^{n+1} - \mu_0^{n+1} - 1}, \quad n \in \mathbb{N},
\]
and for \( n \) large enough, the power becomes negative, which ends the proof.

3.4. **Proof of Theorem 1.1 (3):** \( \alpha = \frac{N}{2} \). We begin with the local energy inequality (3.3) for \( \alpha = N/2 \):
\[
\int_{|y| \leq L} |u(y)|^2 \phi(l_2^{-1}y) dy \int_{|y| \leq L} |u(y)|^2 \phi(l_1^{-1}y) dy \leq C \sigma \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy + C \int_{L/2 \leq |y| \leq 4L} \frac{|u|^3 + |q||u|}{|y|} dy,
\]
which implies that
\[
\int_{|y| \leq L} |u(y)|^2 dy \leq \int_{|y| \leq L} |u(y)|^2 dy + C \sigma \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy + C \int_{L/2 \leq |y| \leq 4L} \frac{|u|^3 + |q||u|}{|y|} dy.
\]
By setting \( l_2 = 4l_1 = 4L > 0 \), we get
\[
\int_{L \leq |y| \leq 2L} |u(y)|^2 dy \leq C \sigma \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy + C \int_{L/2 \leq |y| \leq 4L} \frac{|u|^3 + |q||u|}{|y|} dy.
\]
From (1.11) we deduce that
\[
\int_{L \leq |y| \leq 2L} |u(y)|^2 dy \leq C \sigma \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy + \frac{C}{L^8} \int_{L/2 \leq |y| \leq 4L} |u|^2 dy + \frac{C}{L} \int_{L/2 \leq |y| \leq 4L} |q||u| dy.
\]
Next we treat the integral involving the pressure. We decompose the pressure as follows
\[
q(y) = \int_{|z| \leq L/4} K_{ij}(y - z)u_i(z)u_j(z)dz + \int_{L/4 \leq |z| \leq 8L} K_{ij}(y - z)u_i(z)u_j(z)dz + \int_{|z| \geq 8L} K_{ij}(y - z)u_i(z)u_j(z)dz - \frac{1}{N} |u(y)|^2
\]
\[
= q_1(y) + q_2(y) + q_3(y) + q_4(y)
\]
We only need to consider the first three terms. Due to that \( |y - z| \sim L \) for \( L/2 \leq |y| \leq 4L \) and \( |z| \leq L/4 \), we obtain
\[
|q_1(y)| \leq \frac{C}{L^N} \int_{|z| \leq L/4} |u(y)|^2 dy \leq \frac{C}{L^N} \left( \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy \right)^{1/2},
\]
and thus
\[
\frac{1}{L} \int_{L/2 \leq |y| \leq 4L} |q_1(y)||u(y)| dy \leq \frac{C}{L^{N+1}} \int_{L/2 \leq |y| \leq 4L} |u(y)| dy \leq \frac{C}{L^{N/2+1}} \left( \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy \right)^{1/2}.
\]
For $q_2$, by virtue of the Calderón-Zygmund inequality and (1.11), we get

$$
\frac{1}{L} \int_{L/2 \leq |y| \leq 4L} |q_2(y)||u(y)|dy \leq \frac{1}{L} \left( \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy \right)^{1/2} \left( \int_{L/4 \leq |z| \leq 8L} |u(z)|^4 dz \right)^{1/2} \\
\leq \frac{C}{L} \left( \int_{L/2 \leq |y| \leq 4L} |u(y)|^2 dy \right)^{1/2} \left( \int_{L/4 \leq |z| \leq 8L} |u(z)|^4 dz \right)^{1/2} \\
\leq \frac{C}{L^\delta} \int_{L/4 \leq |y| \leq 8L} |u(y)|^2 dy.
$$

Using the dyadic decomposition, we treat $q_3$ as follows

$$
|q_3(y)| \leq C \sum_{k=3}^{\infty} \int_{2^k L \leq |z| \leq 2^{k+1} L} \frac{1}{|y-z|^N} |u(z)|^2 dz \\
\leq C \sum_{k=3}^{\infty} \int_{2^k L \leq |z| \leq 2^{k+1} L} \frac{1}{|z|^{2N}} |u(z)|^2 dz \leq C \frac{1}{L^N} \sum_{k=3}^{\infty} \frac{1}{2^{kN}} \int_{2^k L \leq |z| \leq 2^{k+1} L} |u(z)|^2 dz,
$$

and thus

$$
\frac{1}{L} \int_{L/2 \leq |y| \leq 4L} |q_3(y)||u(y)|dy \leq \frac{C}{L^{\delta}} \sum_{k=3}^{\infty} \frac{1}{2^{kN}} \int_{2^k L \leq |y| \leq 2^{k+1} L} |u(y)|^2 dy.
$$

Gathering the above estimates yields

$$
\int_{L \leq |y| \leq 2L} |u(y)|^2 dy \leq \frac{C}{L^{N/2+1}} \sum_{j=-1}^{2} \left( \int_{2^j L \leq |y| \leq 2^{j+1} L} |u(y)|^2 dy \right)^{1/2} \\
+ \left( \frac{C}{L^\delta} + C\sigma \right) \sum_{k=-2}^{\infty} \frac{1}{2^{kN}} \int_{2^k L \leq |y| \leq 2^{k+1} L} |u(y)|^2 dy.
$$

(3.21)

Denote $A_k = A_k(L) = \int_{2^k L \leq |y| \leq 2^{k+1} L} |u(y)|^2 dy$, $k \in \mathbb{Z}$, then (3.21) can be expressed as

$$
A_1 \leq \frac{C}{L^{N/2+1}} \sum_{j=-1}^{2} A_j^{1/2} + \left( \frac{C}{L^\delta} + C\sigma \right) \sum_{k=-2}^{\infty} \frac{1}{2^{kN}} A_k,
$$

(3.22)

which also ensures that for every $i \in \mathbb{Z}$,

$$
A_i \leq \frac{C}{(2^i L)^{N/2+1}} \sum_{j=-1}^{2} A_j^{1/2} + \left( \frac{C}{(2^i L)^\delta} + C\sigma \right) \sum_{k=-2}^{\infty} \frac{1}{2^{kN}} A_{k+i}.
$$

(3.23)
Inserting (3.28) into (3.22), we get

\[
A_1 \leq \frac{C}{L^{N/2+1}} \sum_{j_1=-1}^{2} \left( \frac{C}{(2^j L)^N} \right)^{2/N} \sum_{j_2=-1}^{2} A_{j_1+j_2}^{1/2} + \left( \frac{C}{(2^j L)^N} \right)^{\infty} \sum_{j_2=-1}^{2} \left( \frac{C}{(2^j L)^N} \right)^{\infty} \sum_{k_2=-2}^{1} \frac{1}{2^{k_2 N}} A_{j_1+k_2}^{1/2} \right) \right)^{1/2} 
+ \left( \frac{C}{L^\delta} + C \sigma \right) \sum_{k_2=-2}^{1} \frac{1}{2^{k_2 N}} A_{j_1+k_2}^{1/2} 
+ \left( \frac{C}{L^\delta} + C \sigma \right) \sum_{k_2=-2}^{1} \frac{1}{2^{k_2 N}} A_{j_1+k_2}^{1/2} 
\leq \frac{C}{L^{N/2+1}(1+1/2^0)} \sum_{j_1,j_2=-1}^{2} \frac{C}{L^{N/2+1/2}} \sum_{j_1=-1}^{2} \frac{C}{L^{N/2+1/2}} \sum_{j_2=-1}^{2} \frac{C}{L^{N/2+1/2}} \sum_{k_2=-2}^{1} \frac{1}{2^{k_2 N}} A_{j_1+k_2}^{1/2} 
+ \left( \frac{C}{L^{N/2+1/2}} + \frac{C \sigma}{L^{N/2+1/2}} \right) \sum_{k_1=-2}^{2} \sum_{j_2=-1}^{2} \frac{1}{2^{k_1 N}} A_{j_1+k_2}^{1/2} + \left( \frac{C}{L^\delta} + C \sigma \right) \sum_{k_2=-2}^{1} \frac{1}{2^{k_2 N}} A_{j_1+k_2}^{1/2} \right) 
+ \left( \frac{C}{L^N} + C \sigma \right) \sum_{k_1=-2}^{2} \sum_{j_2=-1}^{2} \frac{1}{2^{k_1 N}} A_{j_1+k_2}^{1/2} + \left( \frac{C}{L^\delta} + C \sigma^2 \right) \sum_{k_2=-2}^{1} \frac{1}{2^{k_2 N}} A_{j_1+k_2}^{1/2} 
\right) 
\]

By repeating the above process, we have

\[
A_1 \leq I(\sigma, L) + J(L), 
\]

where \(I(\sigma, L)\) contains any term involving \(\sigma\) which is given by

\[
I(\sigma, L) \leq \frac{C \sigma^{1/2+\cdots+1/2^{n-1}}}{L^{N/2+1}(1+1/2^{n-2})} \sum_{j_1=-1}^{2} \sum_{j_2=-2}^{2} \sum_{j_3,\ldots,j_n=-1}^{2} \frac{1}{2^{k_2 N/2}} A_{j_1+j_2+j_3+\cdots+j_n}^{1/2} 
+ \cdots + \frac{C \sigma^{n-1/2}}{L^{N/2+1}} \sum_{j_1=-1}^{2} \sum_{j_2=-1}^{2} \sum_{j_3,\ldots,j_n=-1}^{2} \frac{1}{2^{k_2 N/2}} A_{j_1+j_2+j_3+\cdots+j_n}^{1/2} 
\]

and \(J(L)\) only contains the terms of \(L\) which is given by

\[
J(L) \leq \frac{C}{L^{N/2+1}(1+1/2^{n-2})} \sum_{j_1,\ldots,j_n=-1}^{2} A_{j_1+\cdots+j_n}^{1/2} 
+ \cdots + \frac{C}{L^{N/2+1}(1+1/2^{n-2})} \sum_{j_1,\ldots,j_n=-1}^{2} \sum_{j_2,\ldots,j_{n-1}=-1}^{2} \sum_{k_{n-1}=-2}^{1} \frac{1}{2^{k_{n-1} N/2}} A_{j_1+\cdots+j_{n-1}+j_n}^{1/2} 
\]

For every small number \(\epsilon > 0\), from \(A_k < \infty\) \((\forall k \in \mathbb{Z})\), we can choose \(n\) large enough so that

\[
J(L) \leq \frac{C}{L^{N/2+1}(1+1/2^{n-2})} + \cdots + \frac{C}{L^{n \delta}} \leq \frac{C}{L^{N+2-\epsilon}}. 
\]

Now for \(\sigma\) small enough, we can use the bootstrapping method to show that

\[
A_1(L) \leq \frac{3}{2} \frac{C}{L^{N+2-\epsilon}}, 
\]

that is, for every \(i \in \mathbb{Z}\),

\[
A_i(L) \leq \frac{3}{2} \frac{C}{(2^i L)^{N+2-\epsilon}}. 
\]
Indeed, we assume (3.26) holds with a larger bound, then for \( \sigma \) small enough, we infer that

\[
I(\sigma, L) \leq \frac{C \sigma^{1/2 + \cdots + 1/2^{n-1}}}{L(N/2+1)(1+\cdots+1/2^{n-1})} \sum_{j_1=-1}^{2} \sum_{j_2=-2}^{\infty} \sum_{j_3, \ldots, j_n=-1}^{2} \frac{1}{2^{k_2N/2}} \left( \frac{2C}{(2j_1+k_2+j_3+\cdots+j_n L)^{N+2-\epsilon}} \right)^{1/2^{n-1}} \\
+ \cdots + \frac{C \sigma^{(n-1)/2}}{L(N/2+1)} \sum_{j_1=-1}^{2} \sum_{j_2, \ldots, j_n=-1}^{\infty} \frac{1}{2^{k_2N/2}} \left( \frac{2C}{(2j_1+k_2+\cdots+j_n L)^{N+2-\epsilon}} \right)^{1/2} \\
+ \cdots + C \sigma^n \sum_{k_1, \ldots, k_n=-2}^{\infty} \frac{1}{2^{(k_1+\cdots+k_n)N/2}} \left( \frac{2C}{(2k_1+\cdots+k_n L)^{N+2-\epsilon}} \right)^{1/2} \\
\leq \frac{C}{L^{N+1}} \left( C \sigma^{1/2 + \cdots + 2^{n-1}} L^{1/2^{n-1}} + \cdots + C \sigma^{(n-1)/2} L^{1/2} + \cdots + \sigma^n L^{\epsilon} \right) \\
\leq \frac{1}{2} \frac{C}{L^{N+2-\epsilon}},
\]

which leads to (3.25). Then by the bootstrapping method, we proves (3.25).

4. Proof of Theorem 1.4

We first consider the case \(-1 < \alpha < \tilde{\Lambda}_\sigma = \frac{\delta(1-\sigma) - N\sigma/2}{1-(d+N/2)\sigma}\). Note that from (1.17), we have as \( l_2 \to \infty, \)

\[
\frac{1}{l_2^{\gamma_0}} \int_{|y| \leq l_2} |u(y)|^2 dy \lesssim l_2^{N+2\delta-\gamma_0} \to 0, \quad \forall \alpha \in [-1, \tilde{\Lambda}_\sigma[.
\]

Thus by letting \( l_1 = 2L \gg 1 \) and \( l_2 \to \infty \), similarly as obtaining (3.12), we get

\[
\int_{|y| \leq L} |u(y)|^2 dy \leq C \sigma |L\gamma_0 \int_{|y| \geq L} |u(y)|^2 dy + CL^{\gamma_0} \int_{|y| \geq L} \frac{|u|^3 + |q||u|}{|y|^{\gamma_0+1}} dy \\
:= G_1(L) + G_2(L).
\]

We will use the formula of \( q \) as (2.6) that for every \( y \in B_L(0), \)

\[
q(y) = -\frac{1}{N} |u(y)|^2 + p.v. \int_{|z| \leq 2L} K_{ij}(y-z)u_i(z)u_j(z) dz + \\
+ \int_{|z| \geq 2L} (K_{ij}(y-z) - K_{ij}(z))u_i(z)u_j(z) dz \\
:= -\frac{1}{N} |u(y)|^2 + q_1(y, L) + q_2(y, L).
\]

For \( G_2(L) \), by the dyadic decomposition and using the notation of \( q \) as \( q(y) = -\frac{1}{N} |u(y)|^2 + q_1(y, 2^{k+1}L) + q_2(y, 2^{k+1}L) \) for every \( y \in B_{2^{k+1}L}(0) \) and \( k \in \mathbb{N}, \) we infer that

\[
G_2(L) = CL^{\gamma_0} \sum_{k=0}^{\infty} \int_{2^k L \leq |y| \leq 2^{k+1}L} \frac{|u|^3 + |q||u|}{|y|^{\gamma_0+1}} dy \\
\leq C \sum_{k=0}^{\infty} 2^{-k(\gamma_0+1)} \int_{2^k L \leq |y| \leq 2^{k+1}L} (|u(y)|^3 + |q(y)||u(y)|) dy \\
\leq \frac{C}{L} \sum_{k=0}^{\infty} 2^{-k(\gamma_0+1)} \int_{2^k L \leq |y| \leq 2^{k+1}L} (|u(y)|^3 + |q(y)||u(y)|) dy
\]

By using the following estimate

\[
\int_{|y| \leq L} |u(y)|^2 dy \lesssim L^{N+2\delta}, \quad \forall L \gg 1,
\]

(4.3)
Lemma 2.1 ensures that
\[ \int_{|y| \leq 2^{k+1}L} |u(y)||q(y)| \, dy \lesssim (2^k L)^{N+3\delta}. \]
Thus for \( G_2(L) \), from the fact that \( \gamma_\alpha > N + 2\delta \) for all \( \alpha < \tilde{\Lambda}_\alpha \), we first obtain a rough bound
\[ G_2(L) \leq \frac{C}{L} \sum_{k=0}^{\infty} 2^{-k(\gamma_\alpha+1)} (2^k L)^{N+3\delta} \leq CL^{N+3\delta-1}. \] (4.4)
We claim that for sufficiently small \( \sigma \),
\[ \int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{N+3\delta-1}, \quad \forall L \gg 1. \] (4.5)
Indeed, if the bound on the right-hand side of (4.5) is replaced by a larger bound \( 2CL^{N+3\delta-1} \), we deduce that
\[ G_1(L) = C\sigma L^{\gamma_\alpha} \sum_{k=0}^{\infty} \int_{2^k L \leq |y| \leq 2^{k+1}L} \frac{|u(y)|^2}{|y|^\gamma_\alpha} \, dy \]
\[ \leq C\sigma \sum_{k=0}^{\infty} \frac{1}{2^{k\gamma_\alpha}} \int_{|y| \sim 2^k L} |u(y)|^2 \, dy \]
\[ \leq C\sigma \sum_{k=0}^{\infty} 2^{-k\gamma_\alpha} 2C(2^k L)^{N+3\delta-1} \leq C\sigma L^{N+3\delta-1}, \]
then for \( \sigma \) small enough, we get the improved estimate (4.5). The bootstrapping method implies that (4.5) in fact holds. Next we will use (4.5) to show an more refined bound. By applying Lemma 2.1 again, we get
\[ \int_{|y| \leq 2^{k+1}L} |u(y)||q(y)| \, dy \lesssim (2^k L)^{N+4\delta-1}. \] (4.6)
Plugging it into (4.2), we have
\[ G_2(L) \leq \frac{C}{L} \sum_{k=0}^{\infty} 2^{-k(\gamma_\alpha+1)} (2^k L)^{N+4\delta-1} \leq CL^{N+4\delta-2}. \] (4.7)
Similarly as obtaining (4.5), and noting that
\[ G_1(L) \leq C\sigma \sum_{k=0}^{\infty} \frac{1}{2^{k\gamma_\alpha}} \int_{|y| \sim 2^k L} |u(y)|^2 \, dy \]
\[ \leq C\sigma \sum_{k=0}^{\infty} 2^{-k\gamma_\alpha} 2C(2^k L)^{N+4\delta-2} \leq C\sigma L^{N+4\delta-2}, \]
we get that for sufficiently small \( \sigma \),
\[ \int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{N+4\delta-2}. \] (4.8)
We can repeat the above process for \( n \) times to show the improved estimate
\[ \int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{N+3\delta-1+(\delta-1)n}, \quad \forall L \gg 1, \] (4.9)
with \( C \) independent of \( L \). For \( n \) sufficiently large we conclude that \( u \equiv 0 \) and this clearly contradicts with the condition \( |u(y)| \gtrsim 1 \) for \( |y| \gg 1 \).
Next we consider the case \( \alpha \geq \hat{\Lambda}_\sigma \). By letting \( l_1 = 2 \) and \( l_2 = 2L \gg 1 \), we begin with (3.3) to get

\[
\int_{|y| \leq L} |u(y)|^2 dy \leq CL^{\alpha} + H_1(L) + H_2(L),
\]

with

\[
H_1(L) := C\sigma L^{\gamma_0} \int_{1 \leq |y| \leq 2L} \frac{|u(y)|^2}{|y|^{\gamma_0}} dy,
\]

and

\[
H_2(L) := CL^{\gamma_0} \int_{1 \leq |y| \leq 2L} \frac{|u| + |q||u|}{|y|^{\gamma_0 + 1}} dy.
\]

For \( H_2(L) \), by the dyadic decomposition, we infer that

\[
H_2(L) = CL^{\gamma_0} \sum_{k=1}^{[\log_2 L]} \int_{\frac{3k}{2k+1} \leq |y| \leq \frac{L}{2k+1}} \frac{|u(y)|^3 + |q(y)||u(y)|}{|y|^{\gamma_0 + 1}} dy.
\]

We use the expression of \( q \) as

\[
q(y) = -\frac{1}{N} |u(y)|^2 + q_1(y, L/2^k) + q_2(y, L/2^k) \text{ for } y \in B_{L/2^k}(0), \land (4.3) \text{ and Lemma 2.1 we have a rough bound for } H_2(L):
\]

\[
H_2(L) \leq C \sum_{k=1}^{[\log_2 L]} 2^{k(\gamma_0 + 1)} \left( \frac{L}{2^k} \right)^{N+3\delta}
\]

Similarly using the way of obtaining (3.9) and (3.10), we deduce that for sufficiently small \( \sigma \),

\[
\int_{|y| \leq L} |u(y)|^2 dy \leq \begin{cases} 
\frac{3}{2} C L^{\gamma_0 + \epsilon}, & \text{if } \gamma_0 + 1 - N - 3\delta \geq 0, \\
\frac{3}{2} C L^{N + 3\delta - 1}, & \text{if } \gamma_0 + 1 - N - 3\delta < 0,
\end{cases}
\]

and we only need to notice that

\[
H_1(L) \leq \begin{cases} 
C \sigma \sum_{k=1}^{[\log_2 L]} 2^{k\gamma_0} 2C(L/2^k)^{\gamma_0 + \epsilon} \leq C' \sigma L^{\gamma_0 + \epsilon}, & \text{if } \gamma_0 + 1 - N - 3\delta \geq 0 \\
C \sigma \sum_{k=1}^{[\log_2 L]} 2^{k\gamma_0} 2C(L/2^k)^{N + 3\delta - 1} \leq C' \sigma L^{N + 3\delta - 1}, & \text{if } \gamma_0 + 1 - N - 3\delta < 0.
\end{cases}
\]

If \( \gamma_0 + 1 - N - 3\delta < 0 \), we will use the iterative method to reduce the power of \( L \). By virtue of Lemma 2.1 again, we get

\[
\int_{|y| \leq \frac{L}{2^k}} |u(y)||q(y)| dy \leq L^{N + 4\delta - 1}.
\]

Using this refined estimate in (4.11), we see that

\[
H_2(L) \leq C \sum_{k=1}^{[\log_2 L]} 2^{-k(\gamma_0 + 1)} \left( \frac{L}{2^k} \right)^{N+4\delta-1}
\]

\[
\leq \begin{cases} 
CL^{\gamma_0} (\log_2 L), & \text{if } \gamma_0 + 2 - N - 4\delta \geq 0, \\
CL^{N+4\delta-2}, & \text{if } \gamma_0 + 2 - N - 4\delta < 0,
\end{cases}
\]

In a similar way as obtaining (4.12) we get that for \( \sigma \) small enough,

\[
\int_{|y| \leq L} |u(y)|^2 dy \leq \begin{cases} 
CL^{\gamma_0 + \epsilon}, & \text{if } \gamma_0 + 2 - N - 4\delta \geq 0, \\
CL^{N+4\delta-2}, & \text{if } \gamma_0 + 2 - N - 4\delta < 0.
\end{cases}
\]
We can repeat the above process, and for $n \in \mathbb{N}$, if $\gamma_\alpha + 1 - 3\delta - N + n(1 - \delta) < 0$, we can show that for $\sigma$ small enough,

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \leq \begin{cases} 
  C_L \sigma^{\gamma_\alpha + \epsilon}, & \text{if } \gamma_\alpha + 2 - 4\delta - N + n(1 - \delta) \geq 0, \\
  CL^{N+4\delta-2+n(\delta-1)}, & \text{if } \gamma_\alpha + 2 - 4\delta - N + n(1 - \delta) < 0.
\end{cases}
$$

(4.14)

Since $\delta \in [0, 1/2]$, for $n$ large enough, we will get $\gamma_\alpha + 1 - 3\delta - N + n(1 - \delta) \geq 0$, and thus for $\alpha \geq \tilde{\Lambda}_\sigma$,

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \leq C_L \sigma^{\gamma_\alpha + \epsilon}, \quad \forall \epsilon > 0.
$$

(4.15)

Since from (1.17), we have

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \geq cL^n,
$$

thus if $\gamma_\alpha < N$, equivalently, $\alpha > \frac{N_\sigma}{2 - N_\sigma}$, and for $\epsilon$ small enough so that $\gamma_\alpha + \epsilon < N$, it yields a contradiction with (4.15) as $L \to \infty$.

If $\sigma = 0$, there is no $H_1(L)$ in (4.10) and we can show the following improved estimate, i.e. a part of (1.19):

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \lesssim L^{\gamma_\alpha}.
$$

(4.16)

Indeed for any $\epsilon \in ]0, 1 - \delta[$, using the estimate (4.15) and Lemma 2.1, we have

$$
H_2(L) \leq \frac{C}{L} \sum_{k=-1}^{[\log_2 L]} 2^{k(\gamma_\alpha + 1)} \left( \frac{L}{2^k} \right)^{\gamma_\alpha + \epsilon + \delta} \lesssim L^{\gamma_\alpha}.
$$

Hence for $\sigma$ small enough, the desired estimate (4.16) is guaranteed from the bootstrapping method.

At last it suffices to prove the inequality on the left-hand-side of (1.18). Suppose it is not true, then there exists $L_n \gg 1$ such that

$$
\frac{1}{L_n} \int_{|y| \leq L_n} |u(y)|^2 \, dy \to 0, \quad \text{as } n \to \infty.
$$

Thus by setting $l_2 = L_n \to \infty$ and $l_1 = 2L > 0$, we get

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \leq G_1(L) + G_2(L),
$$

(4.17)

with $G_1(L), G_2(L)$ introduced in (4.11). For $G_2(L)$, from (4.15) with $\epsilon \in ]0, 1 - \delta[$, similarly as obtaining (4.4) and by using the decomposition (4.2), we have

$$
G_2(L) \leq \frac{C}{L} \sum_{k=0}^{\infty} \frac{1}{2^{k(\gamma_\alpha + 1)}} \left( \frac{L}{2^k} \right)^{\gamma_\alpha + \epsilon + \delta} \leq CL^{\gamma_\alpha + \epsilon + \delta - 1}.
$$

In a similar way as getting (4.5), we infer that for $\sigma$ small enough,

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{\gamma_\alpha + \epsilon + 2\delta - 1}.
$$

Using this rough estimate, similarly as deducing (4.7) and (4.8), we get

$$
G_2(L) \leq CL^{\gamma_\alpha + \epsilon + 2\delta - 2},
$$

and

$$
\int_{|y| \leq L} |u(y)|^2 \, dy \leq \frac{3}{2} CL^{\gamma_\alpha + \epsilon + 2\delta - 2}.
$$
By repeating the above process, we see that for sufficiently small $\sigma$ and any $n \in \mathbb{N}$, 
\[
\int_{|y| \leq L} |u(y)|^2 \, dy \leq CL^{\gamma_n + \epsilon + (\delta-1)n}.
\]
For $n$ large enough and as $L \to \infty$, we have $u \equiv 0$ for all $y \in \mathbb{R}^N$, which clearly contradicts with assumption (1.17).

5. Appendix: proof of Lemma 2.2

In this section we prove Lemma 2.2.

First we show that the integral in (2.6) is meaningful and is a tempered distribution. Let $\phi_0 \in \mathcal{D}(\mathbb{R}^N)$ be a cutoff function supported on $B_1(0)$ such that $\phi_0 \equiv 1$ on $B_{1/2}(0)$ and $0 \leq \phi_0 \leq 1$. For $L \geq 1$, set $\phi_L(z) = \phi_0(z/L)$, then we have
\[
\text{p.v.} \int_{\mathbb{R}^N} K_{ij}(y-z)u_i(z)u_j(z) \, dz = \text{p.v.} \int_{\mathbb{R}^N} K_{ij}(y-z)\phi_{4L}(z)u_i(z)u_j(z) \, dz \\
+ \int_{\mathbb{R}^N} K_{ij}(y-z)(1 - \phi_{4L}(z))u_i(z)u_j(z) \, dz \\
:= I_1(y, L) + I_2(y, L).
\]

Since $u \in C^2_{\text{loc}}(\mathbb{R}^3)$, from the Besov embedding, we infer that $I_1(y, L) \in C^\beta$ for all $\beta < 3$. For $I_2(y, L)$, we consider $I_2(y, L) + \bar{q}(L)$ act on the ball $B_L(0)$, and if $u \in L^p(\mathbb{R}^N)$ for some $p \in ]2, \infty[$, then
\[
I_2(y, L) \lesssim \sum_{k=0}^{\infty} \int_{2^k L \leq |z| \leq 2^{k+1} L} \frac{1}{|z|^N} |u(z)|^2 \, dz \\
\lesssim \sum_{k=0}^{\infty} (2^k L)^{-N+N(1-2/p)} \|u\|^{2/p}_{L^p} \lesssim L^{-2/p},
\]
and if $1 \lesssim |u(y)| \lesssim |y|^\delta$ for all $|y| \gg 1$, then
\[
I_2(y, L) + \bar{q}(L) \lesssim \int_{|z| \geq 2L} (K_{ij}(y-z) - K_{ij}(z))u_i(z)u_j(z) \, dz + C \int_{|z| \sim L} \frac{1}{|z|^N} |u(z)|^2 \, dz \\
\lesssim \int_{|z| \geq 2L} \frac{|y|}{|z|^{N+1}} |u(z)|^2 \, dz + \int_{|z| \sim L} \frac{1}{|z|^N} |u(z)|^2 \, dz \lesssim L^{2\delta}.
\]

For all $s > 0$, we also get that for all $y \in B_L(0)$,
\[
\partial_s^\alpha (I_2(y, L) + \bar{q}(L)) \lesssim \partial_s^\alpha \left( \int_{|z| \geq 4L} \int_{0}^{1} y \cdot \nabla K_{ij}(sy-z)u_i(z)u_j(z) \, ds \, dz \right) + C \int_{|z| \sim L} \frac{1}{|z|^{N+s}} |u(z)|^2 \, dz \\
\lesssim \int_{|z| \geq 4L} \frac{|y|}{|z|^{N+1+s}} |u(z)|^2 \, dz + \int_{|z| \sim L} \frac{1}{|z|^{N+s}} |u(z)|^2 \, dz \lesssim L^{-s+2\delta}.
\]
Hence the scalar function given by
\[
I(y) = -\frac{1}{N}|u(y)|^2 + \text{p.v.} \int_{\mathbb{R}^N} K_{ij}(y-z)u_i(z)u_j(z) \, dz + \bar{q}(L)
\]
is $C^2$-smooth on $B_L(0)$. Moreover, for all $y \in B_L(0)$, we have
\[
\Delta I = \Delta \left( -\frac{1}{N}|u|^2 \phi_{4L} + I_1 \right) + \Delta I_2 \\
= -\text{div} \text{div} (u \sqrt{\phi_{4L}} \otimes u \sqrt{\phi_{4L}}) = -\text{div} \text{div} (u \otimes u),
\]
where in the second line $\Delta I_2 = 0$ due to that $K_{ij}$ is harmonic away from the origin. Besides, it is not hard to show that $I$ is a tempered distribution, which can be seen from the following computation that for $L \gg 1$ and some $p > 2$,

$$\int_{|y|\leq L} |I_1(y, L)|^2 dy \lesssim \int_{|z|\leq 4L} |u(z)|^p dz \lesssim L^{N+p\delta},$$

and by (5.1)-(5.2),

$$\int_{|y|\leq L} |I_2(y, L) + \tilde{q}(L)|^2 dy \lesssim L^{N+p\delta}.$$

Next we intend to find a distributional pressure profile solving (2.7). Without loss of generality, we let $x_0 = 0$. Applying the ansatz (1.3) to (1.1), and by setting

$$p(x, t) := \bar{p}(y, t),$$

we obtain that for all $|y| \leq \rho(T-t)^{-(\frac{1}{1+\alpha}-\sigma)},$

$$\frac{\alpha}{1+\alpha} u(y) + \frac{1}{1+\alpha} y \cdot \nabla y u(y) + u \cdot \nabla_y u(y) + (T-t)^{\frac{2\alpha}{1+\alpha}} \nabla_y \bar{p}(y, t) = 0. \quad (5.5)$$

For some $t$ fixed, denoting $f(y, t) = (T-t)^{\frac{2\alpha}{1+\alpha}} \bar{p}(y, t)$, then the vector-valued function $\nabla_y f(y, t) =: g(y)$ depends only on $y$ on the domain $\tilde{D}(t) := \{|y| \leq \rho(T-t)^{-(\frac{1}{1+\alpha}-\sigma)}\}$. Thus from the fundamental theorem of calculus, we deduce that for all $y \in \tilde{D}(t)$,

$$f(y, t) - f(0, t) = \int_0^1 \frac{d}{ds} f(sy, t) ds = \int_0^1 y \cdot \nabla f(sy, t) ds = \int_0^1 y \cdot g(sy) ds =: q(y),$$

that is,

$$p(x, t) = \frac{1}{(T-t)^{\frac{2\alpha}{1+\alpha}}} q \left( \frac{x}{(T-t)^{\frac{1}{1+\alpha}}} \right) + c(t), \quad \forall x \in D(t). \quad (5.6)$$

Inserting (5.6) into (5.5) yields the equation (2.7) on $\mathbb{R}^N$. In a similar deduction as in [2, Lemma 2.1] (or simply from the expression of $q(y)$), we can see that $q(y)$ is a tempered distribution.

Now we show that $\bar{q}$ and $I$ are equal up to a constant. Since they both satisfy the Laplace equation $\Delta I = -\text{div}(u \otimes u) = \Delta q$, and are both distributions on $\mathbb{R}^N$, the difference $q - I =: h$ is a harmonic polynomial. In the following we prove $h$ is a constant. For all $|y| \leq \frac{\rho}{2(T-t)^{\frac{1}{1+\alpha}-\sigma}}$, we have

$$(T-t)^{\frac{2\alpha}{1+\alpha}} p(y(T-t)^{\frac{1}{1+\alpha}}, t) =$$

$$= -\frac{1}{N} |u(y)|^2 + p.v. \int_{|z|\leq \rho(T-t)^{1+\alpha}} K_{ij}(y(T-t)^{\frac{1}{1+\alpha}} - z) (u_i u_j) \left( \frac{z}{(T-t)^{\frac{1}{1+\alpha}}} \right) dz$$

$$+ (T-t)^{\frac{2\alpha}{1+\alpha}} \int_{|z|\geq \rho(T-t)^{1+\alpha}} K_{ij}(y(T-t)^{\frac{1}{1+\alpha}} - z) (v_i v_j)(z, t) dz$$

$$= -\frac{1}{N} |u(y)|^2 + p.v. \int_{|z|\leq \rho(T-t)^{1+\alpha}} K_{ij}(y - z) (u_i u_j)(z) dz + \bar{p}(y, t).$$

Thanks to (5.6) and (5.3), we moreover see that

$$(T-t)^{\frac{2\alpha}{1+\alpha}} p(y(T-t)^{\frac{1}{1+\alpha}}, t) = q(y) + d(t) = I(y) - \hat{I}(y, t) + \bar{p}(y, t),$$
with \( d(t) := (T - t)^{\frac{2\alpha}{1+\alpha}} c(t) \) and
\[
\tilde{I}(y, t) := \int_{|z| \geq \rho(T-t)^{\sigma}} K_{ij}(y - z) u_i(z) u_j(z) \, dz + \tilde{q}\left(\rho/2(T - t)^{\frac{1}{1+\alpha} - \sigma}\right).
\]
Hence, we have
\[
|h(y) - d(t)| \leq |\tilde{p}(y, t)| + |\tilde{I}(y, t)|, \quad \forall \|y\| \leq \frac{\rho}{2(T - t)^{\frac{1}{1+\alpha} - \sigma}}.
\]
For \( \tilde{p} \), from the separation of \( y(T - t)^{\frac{1}{1+\alpha}} \) and \( z \), we directly obtain
\[
|\tilde{p}(y, t)| \lesssim (T - t)^{\frac{2\alpha}{1+\alpha} - N\sigma} \|v\|_{L^2}^2.
\]
For \( \tilde{I} \), thanks to \([5,2]\), we get
\[
|\tilde{I}(y, t)| \lesssim (T - t)^{-(\frac{1}{1+\alpha} - \sigma)2\delta}.
\]
Since \( \frac{2\alpha}{1+\alpha} - N\sigma > -(\frac{1}{1+\alpha} - \sigma) \) for \( \sigma \) small enough and \( \delta \in [0, 1/2] \), we infer that \( h(y) \equiv \text{const} \) and
\[
|d(t)| \lesssim (T - t)^{\frac{2\alpha}{1+\alpha} - N\sigma} + (T - t)^{-(\frac{1}{1+\alpha} - \sigma)2\delta}, \text{ which proves } (2.8).
\]

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