Order-disorder phase transition in black-hole star clusters – III. A mono-energetic cluster

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ABSTRACT
Supermassive black holes at the centres of galaxies are often surrounded by dense star clusters. For a wide range of cluster properties and orbital radii the resonant relaxation times in these clusters are much shorter than the Hubble time. Since resonant relaxation conserves semimajor axes, these clusters should be in the maximum-entropy state consistent with the given semimajor axis distribution. We determine these maximum-entropy equilibria in a simplified model in which all of the stars have the same semimajor axes. We find that the cluster exhibits a phase transition from a disordered, spherical, high-temperature equilibrium to an ordered low-temperature equilibrium in which the stellar orbits have a preferred orientation or line of apsides. Here ‘temperature’ is a measure of the non-Keplerian or self-gravitational energy of the cluster; in the spherical state, temperature is a function of the rms eccentricity of the stars. We explore a simple two-parameter model of black-hole star clusters – the two parameters are semimajor axis and black-hole mass — and find that clusters are susceptible to the lopsided phase transition over a range of \( \sim 10^2 \) in semimajor axis, mostly for black-hole masses \( \lesssim 10^7 \) M\(_\odot\).

Key words: galaxies: kinematics and dynamics – galaxies: nuclei.

1 INTRODUCTION

This is one of a series of papers investigating the thermodynamic equilibria of a black-hole star cluster, by which we mean a stellar system of mass \( M_* \) orbiting a central black hole of mass \( M_\bullet \gg M_* \) (Touma, Tremaine & Kazandjian 2019; Tremaine 2019). We investigate the equilibria of these systems on time-scales that are much longer than the resonant-relaxation time but shorter than the two-body relaxation time. On these time-scales the semimajor axes of the stars are frozen, but the eccentricities and orbit orientations are distributed in a maximum-entropy state in the phase space at a given semimajor axis. For simplicity we shall make a number of assumptions and simplifications, of which the most important is that we focus on a cluster composed of stars at a single semimajor axis. We call this a ‘mono-energetic’ cluster since the Keplerian energy \( 1 \) is the same for all stars. We also assume that all stars have the same mass and ignore the destruction of stars by the black hole.

The assumption of a mono-energetic cluster is unrealistic, but such clusters provide a fairly simple limiting case that illuminates the fairly complex dynamics we shall encounter. Moreover, the mono-energetic cluster has a radial distribution of stars that is very different from the scale-free cluster investigated by Tremaine (2019), so we may expect that the behaviour common to these two over-simplified model systems is also found in clusters with more realistic radial profiles. Our focus on mono-energetic clusters was stimulated and informed by numerical simulations with mono-energetic clusters of wires, which were reported briefly in Touma, Tremaine & Kazandjian (2019) and which will be the subject of a forthcoming paper (Touma & Kazandjian, in preparation).

Sections 2 and 3 describe the analytic and numerical machinery we use to construct and describe maximum-entropy stellar systems. The properties of the equilibria, with and without corrections for relativistic precession, are derived in §4. Section

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1 We do not distinguish ‘energy’ and ‘energy per unit mass’ in this paper; in other words we often assume that the stellar mass is unity. The meaning should be clear from the context or dimensional analysis.
2 EQUILIBRIA OF MAXIMUM-ENTROPY SYSTEMS

2.1 Phase-space variables

Let the mass of the central object be $M_\star$. The usual Keplerian orbital elements include semimajor axis $a$, eccentricity $e$, inclination $I$, argument of periapsis $\omega$, and angle of the ascending node $\Omega$. The angular momentum per unit mass $L \equiv (G M_\star a)^{3/2}(1-e^2)^{1/2}$ and the $z$-component of the angular momentum $L_z \equiv L \cos I$. The position of a particle in its orbit can be specified by the mean anomaly $\ell$, the eccentric anomaly $u$, or the true anomaly $f$.

We define a set of action-angle variables by the actions $\Lambda \equiv (G M_\star a)^{3/2}$, $L$, and $L_z$ and the conjugate angles $\ell$, $\omega$, and $\Omega$. Let $(L, \theta) = (L, L_z, \omega, \Omega)$; these variables vary during resonant relaxation while $\Lambda$ is constant.

The canonical volume element in phase space is

$$d\mu = d\Lambda dL d\omega d\Omega df \equiv d\Lambda d\theta df = \frac{1}{2} (G M_\star)^{3/2} a^{1/2} d\alpha d\ell df.$$

The equilibrium mass distribution function (hereafter DF) in a Kepler potential may be written $F(\Lambda, L, \theta)$, since Jeans’s theorem ensures that it is independent of the mean anomaly $\ell$. In this paper, the DF is normalized such that $F(\Lambda, L, \theta)d\mu$ is the mass in the phase-space volume element $d\mu$.

2.2 General relativity

The most important effect of general relativity on stellar orbits is apsidal precession, which takes place at an orbit-averaged rate

$$\dot{\omega}_{GR} = \frac{3 (G M_\star)^{3/2}}{c^2 a^{5/2}(1-e^2)} = \frac{3 (G M_\star)^{5/2}}{c^2 a^{1/2} L^2}.$$

The corresponding Hamiltonian is found by integrating $\dot{\omega}_{GR} = \partial H_{GR}/\partial L$, which yields

$$H_{GR}(\Lambda, L) = \frac{3 (G M_\star)^{5/2}}{c^2 a^{1/2} L} = \frac{3 (G M_\star)^{4}}{c^2 a^3 L}$$

plus a term that is unimportant (for our purposes) depending on $a$ but not $L$.

Equation (2) shows that the apsidal precession due to general relativity $\dot{\omega} \propto L^{-2}$, which diverges for radial orbits. This divergence has two important and related consequences:

(i) The rapid precession suppresses resonant relaxation, which therefore becomes less effective than two-body relaxation for nearly radial orbits (the ‘Schwarzschild barrier’; see Merritt et al. 2011; Bar-Or & Alexander 2016; Alexander 2017). Thus the fundamental approximation on which our analysis is based – the existence of a metastable equilibrium on time-scales long compared to the resonant relaxation time but short compared to the two-body relaxation time – fails for high-eccentricity orbits.

(ii) In a canonical ensemble with inverse temperature $\beta$, the DF is proportional to $\exp(-\beta H)$ where $H$ is the Hamiltonian. Since $H_{GR}$ is negative and divergent as $L \to 0$, the DF also diverges if the temperature is positive.

The nature of the relaxation process near the Schwarzschild barrier has been described in detail for spherical systems in the references above. We shall not attempt a similar treatment for non-spherical systems. Instead, when relativistic effects are included we simply truncate the phase space at some maximum eccentricity $e_{\text{max}}$ that is intended to represent the upper limit to the phase-space region in which resonant relaxation is more effective than two-body relaxation.

2.3 Entropy and energy

The entropy is

$$S = -2\pi \int d\Lambda dL d\theta F(\Lambda, L, \theta) \log F(\Lambda, L, \theta).$$

Since semimajor axes are conserved in resonant relaxation, the Keplerian energy $E = -\frac{1}{2} G M_\star m/a$ of each star is conserved.
Therefore for brevity we use the term ‘energy’ to denote the non-Keplerian component of the total energy, which arises from the relativistic Hamiltonian (3) and from the orbit-averaged gravitational interactions between the stars:

\[ E \equiv 2\pi \int d\Lambda dLd\theta \, F(\Lambda, L, \theta) H_{\text{GR}}(\Lambda, L) + 2\pi^2 \int d\Lambda dLd\lambda dL'd\theta' \, F(\Lambda, L, \theta) K(\Lambda, \Lambda', L, L', \theta, \theta') F(\Lambda', L', \theta'). \]  

(5)

where \( K \) is the time-averaged gravitational potential energy between unit masses on distinct Keplerian orbits,

\[ K(\Lambda, \Lambda', L, L', \theta, \theta') = -G \int \frac{d\ell \, d\ell'}{2\pi} \frac{1}{|r - r'|}. \]  

(6)

The Hamiltonian of the system is\(^2\)

\[ H(\Lambda, L, \theta) = H_{\text{GR}}(\Lambda, L) + 2\pi \int d\Lambda' dL' d\theta' K(\Lambda, \Lambda', L, L', \theta, \theta') F(\Lambda', L', \theta'). \]  

(7)

The entropy at fixed energy and mass is extremized when the DF has the form

\[ F(\Lambda, L, \theta) = A(\Lambda) \exp[-\beta H(\Lambda, L, \theta)] \]  

(8)

where \( \beta \) is an inverse temperature. Since resonant relaxation does not affect semimajor axes, the mass per unit semimajor axis

\[ \rho(\Lambda) \equiv 2\pi \int d\Lambda d\theta \, F(\Lambda, L, \theta) \]  

(9)

is conserved. Thus the function \( A(\Lambda) \) in a maximum-entropy state is determined by the non-linear equation

\[ A(\Lambda) = \frac{\rho(\Lambda)}{2\pi \int d\Lambda d\theta \, \exp[-\beta H(\Lambda, L, \theta)]}, \]  

(10)

in which \( H(\Lambda, L, \theta) \) depends on \( A(\Lambda) \) through equation (7).

\[ \text{2.4 The averaged gravitational potential} \]

To evaluate the kernel \( K \) we use the standard expansion in spherical coordinates \( r = (r, \theta, \phi) \)

\[ \frac{1}{|r - r'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l + 1} \frac{\delta_{l}^{j}}{r_<^l} Y_{lm}^{1}(\theta', \phi') Y_{lm}(\theta, \phi), \]  

(11)

where \( r_< \) and \( r_> \) are the smaller and larger of \( r \) and \( r' \) and \( Y_{lm}(\theta, \phi) \) is a spherical harmonic. We also use the representation of a spherical harmonic in orbital elements

\[ Y_{lm}(\theta, \phi) = \sum_{n=-l}^{l} i^{n-l} d_{lm}^{n}(I) Y_{in}(1, \theta, \phi) \]  

(12)

Here \( f \) is the true anomaly and the Wigner d-matrix is

\[ d_{lm}^{n}(I) = \sum_{s} \left( \frac{(-1)^{s} \sqrt{(l + n)!(l - n)!(l + m)!(l - m)!}}{((l + m - s)!(l - m - s)!)} \right) \right) \frac{1}{2l} \left( \sin \theta \right)^{2l + n - 2} \frac{1}{2} \frac{r_{<}^{2l + n - 2}}{r_{>}^{2l + n - 2}}, \]  

(13)

where the sum is over all integer values of \( s \) for which the arguments of the factorials are non-negative. Later we shall use the orthogonality relation

\[ \int_{0}^{\pi} \sin \theta \, d\theta \, d_{lm}^{n}(I) d_{m'n'}^{l'}(I) = \frac{2}{2l_{1} + 1} \delta_{m_{1}m_{2}} \delta_{n_{1}n_{2}} \delta_{l_{1}l_{2}}, \]  

(14)

and the symmetry relation

\[ d_{l-m}^{n}(I) = (-1)^{m-n} d_{lm}^{n}(I). \]  

(15)

With these results,

\[ K(\Lambda, \Lambda', L, L', \theta, \theta') = \sum_{l=0}^{\infty} \frac{4\pi G}{2l + 1} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{\rho_{l}^{n-l} Y_{lm}(\theta, \phi) Y_{lm}(\theta', \phi')}{r_{<}^{l}} Q_{nm}^{l}(\Lambda, \Lambda', L, L'). \]  

(16)

where

\[ Q_{nm}^{l}(\Lambda, L, \Lambda', L') = -\int \frac{d\ell \, d\ell'}{2\pi} \frac{1}{2\pi} \left( \cos(n\ell) \cos(n'\ell') \right) \frac{r_{<}^{l}}{r_{>}^{l+1}} \]  

(17)

and

\[ Y_{in} = Y_{in}(1, \theta, \phi). \]  

(18)

\(^{2}\) Note that in our convention the Hamiltonian has units (velocity\(^2\)).
Note that $y_m = 0$ unless $l - n$ is even, so the only non-zero terms are those with $n$ and $n'$ both even if $l$ is even, or odd if $l$ is odd.

### 2.5 Mono-energetic, axisymmetric systems

We now take two steps to simplify the analysis: (i) We restrict ourselves to mono-energetic stellar systems, in which all the stars have the same semimajor axis $a_0$. Thus we assume that the DF has the form

$$F(A, L, \theta) = \delta(A - A_0) f(L, \theta), \quad \text{with} \quad A_0 = (GM_*/a_0)^{1/2}. \quad (19)$$

To make the notation more concise, we henceforth drop $A$ as an explicit argument of functions such as $K$ and $Q$. (ii) We assume that the DF is axisymmetric. We have conducted unsuccessful experiments to look for non-axisymmetric equilibria and the simplification to axisymmetry allows a more careful numerical exploration. Mathematically, this assumption means that only terms with $m = 0$ survive in equation (16) and $f(L, \theta)$ is independent of the angle variable $\Omega$.

After eliminating $L_z = L \cos \iota$ and integrating over $\Omega$, the total mass of the stars is

$$M_* = (2\pi)^2 \int L dL \sin L d\omega f(L, I, \iota, \omega). \quad (20)$$

The total energy may be written

$$E = (2\pi)^2 \int L dL \sin L d\omega f(L, I, \iota, \omega) H_{GR}(A, L) + (2\pi)^2 \int L dL' \sin L d\omega f(L, I, \iota, \omega) f(L', I', \omega') \times \sum_{l=0}^{\infty} \frac{\sin l\theta}{2l+1} d_l^e(l) d_{l'}^e(l') c_{inn} \Omega_{ln}(L, L'). \quad (21)$$

Using the relation

$$Y_{ln}(I, \omega) = \left(\frac{2l+1}{4\pi}\right)^{1/2} d_l^e(I) c_{inn} \Omega_{ln}(L, L'). \quad (22)$$

the expression for the energy simplifies to

$$E = (2\pi)^2 \int L dL \sin L d\omega f(L, I, \iota, \omega) H_{GR}(A, L) + 2\pi^2 G \int L dL' \sin L d\omega f(L, I, \iota, \omega) f(L', I', \omega') \times \sum_{l=0}^{\infty} \frac{\sin l\theta}{2l+1} y_{ln}(I, \omega) y_{ln}^e(I', \omega') \Omega_{ln}(L, L'). \quad (23)$$

The relative strength of relativistic precession and precession due to self-gravity can be parametrized by

$$\epsilon_{GR} = \frac{GM_*/c^2 a_0}{M_*} = \frac{r_{Sch}}{2a_0} M_*; \quad (24)$$

here $M_* = (2\pi)^2 \int L dL \sin L d\omega f(L, I, \iota, \omega)$ is the total mass of the stars and $r_{Sch}$ is the Schwarzschild radius of the black hole (eq. (25)).

We shall also use the mean eccentricity vector $\overline{e}$ of the system. Since the system is axisymmetric, we may assume that $\overline{e}$ points along the positive $z$-axis, and its value is

$$|\overline{e}| \equiv \frac{\overline{z}}{\int L dL \sin L d\omega f(L, I, \iota, \omega) \sin I \sin \omega/ \int L dL \sin L d\omega f(L, I, \iota, \omega)}. \quad (25)$$

### 2.6 Spherical systems

Spherically symmetric systems provide an important benchmark. In this case the DF $f(L, I, \omega)$ depends only on $L$ and hence only terms with $l = n = n' = 0$ contribute to the energy integral. Thus

$$E = 2\pi^2 \int L dL f(L) H_{GR}(A, L) + 2\pi^2 G \int L dL' f(L) f(L') Q_{00}(L, L'). \quad (26)$$

It is straightforward to show that

$$Q_{00}(L, L') = - \int \frac{dl}{2\pi^2} \frac{dl'}{2\pi^2} = \frac{1}{2\pi a_0} \left[ 4e_\perp E(e_\perp e_\perp) - 2e_\perp (1 - e_\perp^2) K(e_\perp e_\perp) - \pi^2 \right] \quad (27)$$

where $e_\perp$ and $e_\parallel$ are the smaller and larger of the eccentricities corresponding to $L = (GM_*/a_0)^{1/2}(1 - e^2)^{1/2}$ and $L'$, and $E(k), K(k) = \int_0^{\pi/2} \sin^2 \delta d\delta (1 - k^2 \sin^2 \theta)^{1/2}$ are complete elliptic integrals.

If all the stars are on circular orbits, $f(L) \propto (\delta(L - A_0))$ and

$$E = E_c = - \frac{GM_*^2}{a_0} \left( \frac{1}{2} + 3 \epsilon_{GR} \right). \quad (28)$$
If all the stars are on radial orbits with $\epsilon_{GR} = 0$ then
\[ E \approx E_r = -0.29736 \frac{GM_*^2}{a_0}; \] (29)
if $\epsilon_{GR} > 0$ then the energy diverges for radial orbits.

If the DF is ergodic $f(L)$ is constant for $e < \epsilon_{max}$ and
\[ E = \frac{GM_*^2}{a_0} \times \left\{ \begin{array}{ll} 0.3559 + 6 \epsilon_{GR} & \epsilon_{max} = 1, \\ 0.3703 + 4.1786 \epsilon_{GR} & \epsilon_{max} = 0.9. \end{array} \right. \] (30)
This energy is a useful reference point because some formation scenarios suggest, and most N-body simulations assume, that the initial state of star clusters is close to ergodic. In the ergodic state the DF is independent of the orbital elements other than semimajor axis, the inverse temperature (8) is zero, and the mean-square eccentricity is $\frac{1}{2}\epsilon^2_{max}$.

We have searched numerically for spherical maximum-entropy equilibria outside the energy bounds set by (28) and (29). When $\epsilon_{GR} = 0$ we have not found any, suggesting that systems with $e = 0$ and $e = \epsilon_{max}$ have the smallest and largest energies of any spherical systems. On the other hand, for $\epsilon_{GR} > 0$ the systems with extreme energies may have DFs that peak at intermediate eccentricities.

The linear stability of spherical systems is related to the existence of lopsided maximum-entropy equilibria with the same non-Keplerian energy. We distinguish two kinds of stability. A system is thermodynamically stable or metastable if its entropy is a local maximum relative to all nearby systems, spherical or non-spherical, having the same mass and non-Keplerian energy. We denote the non-Keplerian energy by $\epsilon_{GR}$, and we define the same non-Keplerian energy $E$ for that energy and the same distribution of semimajor axes. A system is dynamically stable if there are no growing modes of the collisionless Boltzmann equation. If such modes exist, the growth time will be of order $(a^2/GM_*)^{1/2}M_*/M_*$. Since the collisionless Boltzmann equation conserves entropy, thermodynamic stability implies dynamical stability. If the maximum-entropy state at a given non-Keplerian energy is lopsided, then the spherical equilibrium at that energy must either be unstable — a saddle point or a minimum of the entropy — or metastable — a local but not global maximum of the entropy at fixed energy.

The determination of the thermodynamic and dynamical stability of spherical equilibria is described in Appendix A.

3 NUMERICAL METHODS
To evaluate the entropy and energy integrals (4) and (23) we assume that the DF $f(L, I, \omega)$ is localized at a finite set of nodes, with eccentricities $\{\epsilon_j\}$, $j = 1, \ldots, J$, and inclinations and periapsis arguments $\{f_k, \omega_k\}$, $k = 1, \ldots, K$. Thus
\[ f(L, I, \omega) = \sum_{j=1}^{J} \sum_{k=1}^{K} f_{jk} \delta(e^2 - \epsilon_j^2)\delta(\cos I - \cos f_k)\delta(\omega - \omega_k) \] (31)
where the angular momentum $L$ is related to eccentricity $e$ through $L^2 = \Lambda^2 (1 - e^2)$. The mass associated with node $(j, k)$ is given by (cf. eq. 20)
\[ M_{jk} = 2\pi^2\Lambda^2 f_{jk}. \] (32)
The entropy (4) is approximated as
\[ S = -\sum_{j=1}^{J} \sum_{k=1}^{K} M_{jk} \log(M_{jk}/V_{jk}) \] (33)
where $V_{jk}$ is the phase-space volume associated with node $(j, k)$, which we determine below.

Equation (23) for the energy becomes
\[
E = -6\pi^2(GM_*)^3 \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{f_{jk}}{(1 - \epsilon_j^2)^{1/2}} + 3\pi^5 G^3 M_*^2 a_0^2
\times \sum_{j,j'=1}^{J} \sum_{k,k'=1}^{K} f_{jk} f_{j'k'} \sum_{l=0}^{\infty} \sum_{n,n'=\pm l}^{\infty} \frac{\sin^2 \theta_{jn} \sin^2 \theta_{j'n'}}{(2l+1)^2} \mathcal{Q}_{nn'}^{l}\mathcal{Q}_{j'n'}^{l}(\epsilon_j, \epsilon_{j'}); \] (34)
we have changed the arguments of $\mathcal{Q}_{nn'}$ from $L = \Lambda_0 (1 - e^2)^{1/2}$ to eccentricity $e$, with a similar change from $L'$ to $e'$.

We truncate the sum over $l$ at some maximum value $l_{max}$. The sums can be shortened to sums over non-negative $n$ by observing that $\mathcal{Q}_{nn'}^{l}$ is even in both $n$ and $n'$, that $Y_{l-n} = (-1)^n Y_{ln}$, and that $Y_{l+n}(l, \omega) = (-1)^n Y_{ln}^{*}(l, \omega)$. Thus
\[
E = -6\pi^2(GM_*)^3 \sum_{j=1}^{J} \sum_{k=1}^{K} \frac{f_{jk}}{(1 - \epsilon_j^2)^{1/2}} + 3\pi^5 G^3 M_*^2 a_0^2 \sum_{j,j'=1}^{J} \sum_{k,k'=1}^{K} f_{jk} f_{j'k'}
\times \sum_{l=0}^{l_{max}} \sum_{n,n'=\pm l}^{\infty} c_{ln} c_{l'n'} Y_{ln}(I_k, 0) Y_{l'n'}^{*}(I_{k'}, 0) \cos n(\omega_k + \frac{1}{2}\pi) \cos n'(\omega_{k'} + \frac{1}{2}\pi) \mathcal{Q}_{nn'}^{l}(\epsilon_j, \epsilon_{j'}). \] (35)
where
\[ \epsilon_{ln} = \frac{\mathcal{Y}_n}{(1 + \delta_{n0})(2l + 1)}. \]

Since the second term in (35) is symmetric in the primed and unprimed variables the summation over \((j, k)\) and \((j', k')\) can be shortened by roughly a factor of two, yielding further savings in the computation time.

We use a set of nodes \(\{e_j\}\) that are uniformly distributed in \(e^2\),
\[ e_j^2 = \left(\frac{j - \frac{1}{2}}{J}\right)^2 \epsilon_{\text{max}}, \quad j = 1, \ldots, J, \]
where \(\epsilon_{\text{max}}\) is defined in \(\S\,2.2\). The nodes in inclination and periapsis argument are assigned to a Lebedev quadrature grid. A Lebedev grid \(\{\theta_k, \phi_k, w_k\}\) of order \(P\) is a Gaussian quadrature algorithm in the sense that
\[ \int_0^\pi \int_0^{2\pi} h(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \sin \theta \cos \phi \sin \phi \cos \phi \cos \theta \, d\theta d\phi = 4\pi \sum_{k=1}^K w_k h(\sin \theta_k \cos \phi_k, \sin \theta_k \sin \phi_k, \cos \theta_k) \]
is exact when the function \(h(x, y, z)\) is any polynomial of order \(\leq P\). The weights \(w_k\) sum to unity and the number of points \(K\) is related to the order \(P\); for example, for \(P = 5, 7, 9, 11, 13, 15\) we have \(K = 14, 26, 38, 50, 74, 86\). We have checked that assigning the angular nodes to a Cartesian grid in \(\cos I \in [0, \pi]\) and \(\omega \in [0, 2\pi]\) yields the same results, although with lower accuracy for the same number of grid points.

The phase-space volume enclosed by a surface \(S\) is
\[ V_S = \int_S d\Lambda dI dL \cdot d\omega d\Omega d\ell \]
\[ = \frac{1}{2} \int_0^\pi \Lambda^2 d\lambda e^2 \sin I dI d\omega d\Omega d\ell. \]
The integrals over \(\Lambda, \ell,\) and \(\Omega\) are the same for all stars (because we consider mono-energetic, orbit-averaged, axisymmetric systems), so if we evaluate the remaining three integrals using the method we have described above we find
\[ V_S = \text{constant} \times \sum_{j=1}^J \sum_{k=1}^K W(j, k, S) w_k \]
where \(W(j, k, S)\) is 1 if \((e_j, I_k, \omega_k)\) is inside \(S\) and zero otherwise. Thus the volume associated with node \((j, k)\) is
\[ V_{jk} = \text{constant} \times w_k. \]
Since \(V_{jk}\) is only used in the expression for the entropy (33) and here it appears only in the argument of a logarithm, we can set the constant to unity if we assume that the entropy is only defined to within a constant.

The matrix \(Q_{nn}^j(e_j, e_j')\) (eq. 17) is computed once and for all at the start. The most convenient integration variable for this task is the eccentric anomaly \(u\), which is related to the variables in equation (17) by \(\ell = u - e \sin u, \ r = a(1 - e \cos u), \) and \(\cos f = (\cos u - e)/(1 - e \cos u)\).

We then maximize the entropy (33) subject to the non-linear constraint that the energy (35) is fixed and the linear constraints that the total mass \(\sum_{j} M_{jk} = 1\) and \(M_{jk} \geq 0\). The initial conditions for the optimization algorithm are chosen in one of two ways: (i) the periapsis directions are restricted to lie within \(45^\circ\) of the positive z-axis (\(\sin I_j \sin \omega_j \geq 2^{-1/2}\)); this encourages the routine to find lopsided states if they exist; (ii) if we are finding a sequence of equilibria, say for a set of energies \(E_n\), the equilibrium for energy \(E_n\) is used as the initial state when seeking the equilibrium for energy \(E_{n+1}\).

We use the optimization routine E04UCF from the NAG (Numerical Algorithms Group) library. Most of the computation time is spent on evaluating the sum (35) but this task is easy to parallelize.

The accuracy of the calculations depends on the number of eccentricity grid points \((J)\), the number of angular grid points \((K)\), and the maximum multipole \(l_{\text{max}}\). Increasing \(J\) and \(K\) improves the accuracy of the calculations but the required computing time grows rapidly with \(J\) and \(K\); moreover as the number of variables \(JK\) grows it becomes more difficult for the optimization routine to converge. The values we use are a compromise between these conflicting demands: typically \(l_{\text{max}} = 8\), \(J = 16\), and \(K = 50\) (corresponding to order \(P = 11\)). In this case we have an optimization problem with 800 variables.
Figure 1. Magnitude of the mean eccentricity vector in a maximum-entropy mono-energetic stellar system, as a function of the self-gravitational energy measured in units of $GM_\star^2/a_0$ (eq. 35). Relativistic precession is assumed to be negligible and the maximum allowed eccentricity is unity ($\epsilon_{GR} = 0$, $e_{max} = 1$). The ergodic DF (inverse temperature $\beta = 0$) is marked by a black circle at $E = -0.3559$. Note that there is a small interval ($-0.42 \lesssim E \lesssim -0.41$) in which both spherical and lopsided systems are local entropy maxima. Each set of colored symbols represents a specific combination of the number of eccentricity grid points $(J)$, the number of points in the Lebedev quadrature $(K)$, and the maximum multipole $l_{max}$, shown in the legend as $(J, K, l_{max})$. The dashed vertical lines denote the minimum and maximum energy of spherical equilibria, $E_c = -0.5$ and $E_r = -0.2974$ (cf. eqs. 28–30).

4 RESULTS

4.1 No relativistic precession

We first examine the properties of maximum-entropy states in the absence of relativistic precession ($\epsilon_{GR} = 0$). Figure 1 shows the mean eccentricity vector (eq. 25) as a function of the self-gravitational energy of the stars measured in units of $GM_\star^2/a_0$. In these units spherically symmetric systems have energies between the vertical dashed lines at $E = -0.5$ (circular orbits, eq. 28) and $-0.2974$ (radial orbits, eq. 29). Ergodic systems have $E = -0.3559$ (eq. 30) and are marked by a filled black circle. The colored symbols represent different choices for the integration parameters $J$, $K$, and $l_{max}$. All of the choices shown yield similar results even though the number of eccentricity grid points $J$ ranges from 16 to 32, the number of angular grid points $K$ ranges from 38 to 50 ($P$ between 9 and 11), and the maximum order of the spherical-harmonic expansion $l_{max}$ ranges from 5 to 8. Thus we are confident that the numerical methods have converged reasonably well.

The total angular momentum, which we assume to be zero, is also conserved. This constraint is automatically satisfied by our solutions, because the energy (35) is invariant when orbit directions are reversed. Thus maximum-entropy solutions have equal numbers of orbits going in opposite directions.
Equilibrium systems with energy $E > -0.41$ have zero mean eccentricity vector and are spherically symmetric\(^4\). For $E < -0.41$ the maximum-entropy systems acquire a non-zero mean eccentricity vector, indicating that they are lopsided. In the language of condensed-matter physics, there is an order-disorder phase transition as the stellar system is cooled.

Figures 2 and 3 show the rms eccentricity and entropy of these models as a function of energy. The solid black curve in each figure is the rms eccentricity or entropy of the maximum-entropy spherical model. In Figure 2, the rms eccentricity of the spherical model grows smoothly from 0 to 1 as the energy grows from its minimum value for spherical systems, $-0.5GM^2/\alpha_0$, to its maximum of $-0.2974 GM^2/\alpha_0$.

In Figure 3, the slope $dS/dE = \beta$ is negative for energies larger than that of the ergodic state (dotted line at $E = -0.3559$), which means that the equilibrium temperature $1/\beta$ is negative.

In both Figure 2 and 3, the black curves coincide with the colored markers for $E > -0.41$, confirming that in this region the maximum-entropy equilibrium is spherical. For $E < -0.41$ the maximum-entropy spherical states have smaller entropy than the lopsided states, which are global entropy maxima.

Some of this behaviour can be illuminated by examining the linear stability of spherical maximum-entropy systems using the methods of Appendix A. When $l = 1$ and relativistic precession is absent, it can be shown analytically that spherical equilibria are always dynamically stable or at least neutrally stable (Tremaine 2005; Polyachenko et al. 2007). Numerical solutions of the eigenvalue equation for the matrix $\mathbf{V}^l$ (eq. A24) over the range of energies $-0.310$ to $-0.489$ show that the

\(^4\) For spherical symmetry it is necessary but not sufficient that the eccentricity vector vanishes. We have checked explicitly that for $E > -0.41$ the DF $f_{jk}$ is independent of the angular node $k$ at given eccentricity $e_j$. 

**Figure 2.** Root mean square (rms) eccentricity in a maximum-entropy mono-energetic stellar system, as a function of the self-gravitational energy $E$ measured in units of $GM^2/\alpha_0$. The symbols have the same meaning as in Figure 1. The solid line shows the rms eccentricity for spherical models. The ergodic system (black circle) has rms eccentricity $2^{-1/2} = 0.7071$. The orange line with star symbols shows maximum-entropy models in which the mean eccentricity vector is constrained to be zero.
spherical equilibria are also dynamically stable for \( l = 2, 3, \ldots, 8 \). Numerical solutions of the eigenvalue equation for \( R^l \) (eq. A13) show that they are also thermodynamically stable for odd values of \( l = 1, 3, 5, 7 \). However, they can be thermodynamically unstable for even values of \( l \): for \( l = 2 \) the spherical system is unstable when \( E < -0.439 \), for \( l = 4 \) when \( E < -0.478 \), for \( l = 6 \) when \( E < -0.488 \), and for \( l = 8 \) when \( E < -0.491 \).

Since spherical systems are dynamically and thermodynamically stable (at least up to \( l = 8 \)) for energy \( E > -0.439 \), the phase transition at larger energies must arise because the spherical equilibrium is metastable, i.e., it is a local entropy maximum but not a global one. This conclusion is consistent with the observation that our numerical optimization algorithm finds both spherical and lopsided maximum-entropy states for energies in the range \( E = -0.405 \) to \( E = -0.418 \). In principle there are spherical maximum-entropy states up to the onset of the \( l = 2 \) instability at \( E = -0.439 \) but the maxima are very shallow, and difficult to detect without higher resolution simulations.

The orange curve in Figures 2 and 3 shows the maximum-entropy state when the mean eccentricity vector is constrained to be zero. For \( E > -0.439 \) the maximum-entropy state with \( \overline{e} = 0 \) is spherical, so the orange and black lines coincide. For \( E < -0.439 \) the spherical state is thermodynamically unstable to \( l = 2 \) perturbations, which leave the eccentricity vector unchanged, so the maximum-entropy state with \( \overline{e} = 0 \) is non-spherical (zero dipole but non-zero quadrupole moment), and the orange line has higher entropy than the black line.

4.2 Effects of relativistic precession

We parametrize the importance of relativistic precession compared to self-gravity by \( e_{\text{GR}} \), defined in equation (24). To avoid divergences in the Hamiltonian when \( e_{\text{GR}} \) is non-zero, we cut off the DF above some maximum eccentricity \( e_{\text{max}} \), which we arbitrarily set to be \( e_{\text{max}} = 0.9 \). In our plots we use \( E + 6e_{\text{GR}} \) as the independent variable. Here \( E \) is the non-Keplerian energy.
Figure 4. Inverse temperature in a maximum-entropy mono-energetic stellar system, as a function of the non-Keplerian energy $E$ measured in units of $GM^2/a_0$. Each curve represents a different value of the parameter $\epsilon_{\text{GR}}$, which measures the strength of relativistic precession relative to precession from self-gravity (eq. 24). On the horizontal axis, the energy $E$ is offset by $6\epsilon_{\text{GR}}$ to help all of the curves to fit on a single figure. The solid curves denote stable equilibria while the dashed curves denote unstable equilibria; the boundary between stability and instability is marked on each curve by a plus sign. The most unstable mode is an $l=2$ thermodynamic instability for $\epsilon_{\text{GR}} = 0$ and an $l=1$ dynamical instability for $\epsilon_{\text{GR}} > 0$.

Figure 4 shows the inverse temperature of spherical systems as a function of energy, for several values of the relativistic parameter $\epsilon_{\text{GR}}$. The inverse temperature declines with increasing energy, so all of these systems have positive heat capacity. The plus signs separate stable systems (solid lines) from unstable ones (dashed lines). For $\epsilon_{\text{GR}} = 0$ the transition occurs at $E = -0.454$ through an $l = 2$ thermodynamic instability (to be compared to $E = -0.439$ for the system examined in §4.1, which had $e_{\text{max}} = 1$ compared to $e_{\text{max}} = 0.9$). For $\epsilon_{\text{GR}} > 0$ the transition occurs through an $l = 1$ dynamical instability.

Figure 5, the analog to Figure 1, plots the mean eccentricity vector as a function of energy. The order-disorder phase transition is present for all five values of the relativistic parameter $\epsilon_{\text{GR}}$, although the transition is off the figure (at $E = -7.071$) for $\epsilon_{\text{GR}} = 1$. For energies below the phase transition the maximum-entropy equilibria are lopsided, while above the phase transition they are spherical.

The nature of the phase transition depends on the strength of the relativistic effects. As in Figure 1, when $\epsilon_{\text{GR}} = 0$ (i) spherically symmetric states are metastable entropy maxima for energies just below (or inverse temperatures above) the phase transition at $E \approx -0.41$; (ii) eventually, as the energy declines, an $l = 2$ thermodynamic instability sets in; (iii) spherical systems are always dynamically stable and always stable to $l = 1$ disturbances. In contrast, when $\epsilon_{\text{GR}} > 0$ spherical systems with sufficiently low energy are thermodynamically and dynamically unstable to $l = 1$ disturbances. As $\epsilon_{\text{GR}}$ grows the onset
Phase transition in black-hole star clusters

Figure 5. Magnitude of the mean eccentricity vector in a maximum-entropy mono-energetic stellar system, as a function of the non-Keplerian energy $E$ measured in units of $GM_s^2 / a_0$. Each set of colored symbols represents a different value of the parameter $\epsilon_{GR}$, which measures the strength of relativistic precession relative to precession from self-gravity (eq. 24). On the horizontal axis, the energy $E$ is offset by $6\epsilon_{GR}$ to help all of the curves to fit on a single figure. All spherical systems to the left of the plus signs for $\epsilon_{GR} = 0.03, 0.1,$ and $0.3$ are subject to $l = 1$ thermodynamic and dynamical instabilities. The models shown by symbols have numerical grid $(J, L, l_{\text{max}}) = (16, 11, 8)$.

of this instability, marked by colored plus signs in Figure 5, shifts closer and closer to the phase transition. Eventually the onset of the $l = 1$ instability coincides with the phase transition so there is no metastable spherical state at energies below the transition energy.

Figure 6 and 7 show the rms eccentricity and entropy for the same systems. When $\epsilon_{GR}$ is small, the rms eccentricity increases as the energy grows (or inverse temperature declines). However, between $\epsilon_{GR} = 0.03$ and $\epsilon_{GR} = 0.1$ this behaviour reverses, and the rms eccentricity declines as the energy grows for spherical systems. Below this transition the rms eccentricity of the lopsided equilibrium is larger than that of the spherical equilibrium with the same energy; above the transition this ordering is reversed.

5 DISCUSSION

We have shown that a phase transition from spherical to lopsided equilibria occurs in an idealized model of a black-hole star cluster dominated by a central black hole. We now ask what conditions are needed for this transition to be present in realistic clusters.
5.1 A simplified dynamical model of a black-hole star cluster

We consider a cluster containing a central black hole of mass $M_\bullet$. The Schwarzschild radius of the black hole is

$$r_{\text{Sch}} = \frac{2GM_\bullet}{c^2} = 9.57 \times 10^{-6} \text{ pc} \left( \frac{M_\bullet}{10^8 M_\odot} \right).$$  \hfill (42)

The black hole is surrounded by a spherical star cluster. The mass of stars interior to radius $r$ is $M_\star(r)$ and we define the dynamical radius of the black hole, $r_{\text{dyn}}$, by $M_\star(r_{\text{dyn}}) = M_\bullet$ (Binney & Tremaine 2008). For $r \lesssim r_{\text{dyn}}$ the gravitational force is dominated by the central black hole, and the orbital angular frequency of a star with semimajor axis $a$ is

$$\Omega^{-1}(a) = \left( \frac{a^3}{GM_\bullet} \right)^{1/2} = 1.49 \times 10^3 \, \text{yr} \left( \frac{a}{\text{pc}} \right)^{3/2} \left( \frac{10^8 M_\odot}{M_\bullet} \right)^{1/2}. \hfill (43)$$

The mass of stars inside radius $r$ is assumed to be a power law,

$$M_\star(r) = M_\bullet \left( \frac{r}{r_{\text{dyn}}} \right)^{3-\gamma}.$$ \hfill (44)

For numerical calculations we set $\gamma = 1.5$, a typical value for black-hole star clusters. Assuming that the velocity-dispersion tensor of the cluster is isotropic and solving the Jeans equation for the one-dimensional velocity dispersion $\sigma(r)$ (eq. 4.216 in...
Figure 7. Maximum entropy of a mono-energetic stellar system, as a function of the non-Keplerian energy $E$, measured in units of $GM_\bullet^2/a_0$. The symbols have the same meaning as in Figures 5 and 6. The small offset between the symbols and the solid curve for $\epsilon_{\text{GR}} = 0.3$ and 1 probably reflect the limited eccentricity resolution of the models ($J = 16$), which means that they are inaccurate when the rms eccentricity is close to unity.

We assume that the dispersion at the dynamical radius, $\sigma(r_{\text{dyn}})$, is equal to the dispersion of the central region of the galaxy outside the dynamical radius, which we denote $\sigma_g$. In early-type galaxies $\sigma_g$ is strongly correlated with the central black-hole mass and this correlation can be approximated as (e.g., Kormendy & Ho 2013)

$$M_\bullet \simeq 3.1 \times 10^8 M_\odot \left( \sigma_g / 200 \text{ km s}^{-1} \right)^{4.4}.$$  

(46)

With these assumptions the local properties of the cluster are fully described by two parameters, the black-hole mass and the radius or semimajor axis. The dynamical radius can be written

$$r_{\text{dyn}} = \frac{GM_\bullet}{2.5 \sigma_g^2} = 7.2 \text{ pc} \left( \frac{M_\bullet}{10^8 M_\odot} \right)^{0.545}.$$  

(47)

The mass density of stars at radii $r \ll r_{\text{dyn}}$ is

$$\rho(r) = \frac{1}{4\pi r^2} \frac{dM_\bullet}{dr} = \frac{3M_\bullet}{8\pi (r_{\text{dyn}}r)^{3/2}} = 6.19 \times 10^5 M_\odot \text{pc}^{-3} \left( \frac{M_\bullet}{10^8 M_\odot} \right)^{0.182} \left( \frac{1\text{ pc}}{r} \right)^{1.5}.$$  

(48)
Figure 8. Properties of the model of a black-hole star cluster described in §5.1. The horizontal axis is radius or semimajor axis and the vertical axis is the mass of the central black hole. The Schwarzschild radius of the black hole \( r_{\text{Sch}} \) (eq. 42) and the dynamical radius \( r_{\text{dyn}} \) at which the enclosed stellar mass equals the black-hole mass (eq. 47) are shown as dashed magenta lines. Above the horizontal dashed green line the two-body relaxation time \( t_{2b} \) (eq. 53) exceeds the typical galaxy age of \( 10^{10} \) yr. Within the shaded region the resonant relaxation time \( t_{\text{rr}} \) (eq. 53) is less than \( 10^{10} \) yr (at a typical eccentricity \( e = 0.5 \)). Apsidal precession is dominated by relativity to the left of this line and by the stellar self-gravity to the right, and on this line the relativistic parameter is \( \epsilon_{\text{GR}} = 0.1666 \). Finally, the dotted blue lines mark loci of constant \( f_{\text{lc}} \), the fraction of phase space occupied by the loss cone (eq. 56). All curves assume an effective stellar mass \( m_{\text{eff}} = \langle m^2 \rangle / \langle m \rangle = 1 M_\odot \).

We generalize the relativistic parameter \( \epsilon_{\text{GR}} \) (eq. 24) to a function of radius,

\[
\epsilon_{\text{GR}}(r) = \frac{GM^2_{\star}}{c^2 r M_\odot(r)} = \frac{r_{\text{Sch}}}{2 r_{\text{dyn}}^{1.5}} = 9.23 \times 10^{-5} \left( \frac{M_\star}{10^8 M_\odot} \right)^{1.818} \left( \frac{1 \text{ pc}}{r} \right)^{2.5}.
\]  

(49)

The apsidal precession rate for an orbit of semimajor axis \( a \) and eccentricity \( e \) is

\[
\dot{\omega}(a, e) = \frac{a}{r_{\text{dyn}}} \left( \frac{a}{r_{\text{dyn}}} \right)^{1.5} \frac{3 \Omega(a)GM_\star}{c^2 a(1 - e^2)} w(e) + \frac{3 \epsilon_{\text{GR}}(a)}{1 - e^2}, \quad r \ll r_{\text{dyn}}.
\]  

(50)

Here

\[
w(e) = \frac{2(1 + e)(1 - e)^{1/2}}{\pi e^2} \left[ (1 - e)K(k) - E(k) \right], \quad k^2 = \frac{2e}{1 + e},
\]  

(51)

where \( K \) and \( E \) are complete elliptic integrals. The function \( w(e) \) is negative (i.e., the precession due to self-gravity is retrograde) for all eccentricities \( e \) between 0 and 1; \( w(0) = -\frac{3}{4} \), \( w(0.5) = -0.66634 \), and \( w(e) \to -4(1 - e)^{1/2} \pi \) as \( e \to 1 \).
The two-body relaxation time is given by equation (7.106) of Binney & Tremaine (2008),

$$t_{2b} = \frac{0.34 \sigma^3}{G^2 m_{\text{eff}}^2 \log \Lambda}$$

$$= 1.38 \times 10^{11} \text{yr} \left( \frac{M_\star}{10^8 M_\odot} \right)^{1.318} \frac{15}{m_{\text{eff}}} \frac{1}{\log \Lambda} \quad r \ll r_{\text{dyn}}.$$  \hfill (52)

Note that for the value of \( \gamma \) we have chosen the relaxation time is independent of radius. The Coulomb logarithm is \( \log \Lambda \approx \log(M_\star/m_{\text{eff}}) \) and the effective stellar mass is \( m_{\text{eff}} = (m^2)/(m) \) where \( \langle \cdot \rangle \) represents a number-weighted average over the local stellar population. Unfortunately the appropriate value for \( m_{\text{eff}} \) is quite uncertain. For the solar neighborhood \( m_{\text{eff}} = 0.66 M_\odot \), while for a Salpeter mass function \( m_{\text{eff}} = 0.54 n_{\text{max}}^6 m_{\text{min}}^{0.35} \) where \( m_{\text{max}} \) and \( m_{\text{min}} \) are the upper and lower cutoffs to the distribution. For example, when \( m_{\text{max}} = 100 M_\odot \) and \( m_{\text{min}} = 0.1 M_\odot \), \( m_{\text{eff}} = 4.8 M_\odot \) (see Kocsis & Tremaine 2011 for a fuller discussion). For simplicity, in the estimates below we use \( m_{\text{eff}} = 1 M_\odot \), but the actual value of the effective mass is probably the largest single uncertainty in the estimates of this subsection.

The resonant relaxation time-scale at semimajor axis \( a \) may be written (Hopman & Alexander 2006; Kocsis & Tremaine 2011; Bar-Or & Fouvry 2018)

$$t_{\text{rr}}(a, e) = \frac{10^{8} M_\star \omega(a, e) a^3}{GM_\star(a) m_{\text{eff}}} \quad r \ll r_{\text{dyn}}$$

$$= 1.49 \times 10^{12} \text{yr} \left( \frac{M_\star}{10^8 M_\odot} \right)^{1/2} \left( \frac{a}{1 \text{pc}} \right)^{3/2} \frac{1}{m_{\text{eff}}} \left| w(e) + \frac{3 \epsilon_{\text{GR}}(a)}{1 - e^2} \right|.$$  \hfill (54)

The total precession rate can have either sign (retrograde if self-gravity dominates, prograde if relativistic effects dominate) and vanishes on the locus where \( w(e) < 0 \) and

$$\eta_0(M_\star, e) = \langle \frac{0.0378 \text{pc}}{[1 - e^2]|w(e)|^{0.4}} \left( \frac{M_\star}{10^8 M_\odot} \right)^{0.727} \rangle$$

(55)

On this locus the relativistic parameter is \( \epsilon_{\text{GR}} = \frac{1}{2}(1 - e^2)w(e) \).

Stars are lost from the cluster if they pass too close to the central black hole (Binney & Tremaine 2008, §7.5.9). A star crosses the event horizon of a non-rotating black hole if its pericentre distance \( q = a(1 - e) < 4 R_{\text{Sch}} \), where the Schwarzschild radius \( R_{\text{Sch}} \) is defined in equation (42)\(^5\). The star is tidally disrupted if \( q < g R_\star(M_\star/m)^{1/3} \) where \( R_\star \) is the stellar radius and \( g \) is a factor of order unity. The fraction of phase space occupied by orbits with pericentre distance less than \( q = 2q/a \) if \( q \ll a \). Thus the fraction of phase space on which stars are lost at their next pericentre passage (the ‘loss cone’) is

$$f_{\text{lc}}(a) = \max \left[ \frac{16 G M_\star}{c^2 a}, \frac{2g R_\star}{a} \left( \frac{M_\star}{m_{\text{eff}}} \right)^{1/3} \right].$$

(56)

When evaluating this formula, we shall assume \( g = 1.5, R_\star = 1 R_\odot \) and \( m = 1 M_\odot \).

The results in this subsection are illustrated in Figure 8, as described in the caption and the next subsection.

### 5.2 Conditions for a phase transition

We have shown that mono-energetic stellar systems can exhibit lopsided thermal equilibrium states. Here we use the simplified model of the preceding subsection to explore whether these equilibria are likely to be present in black-hole star clusters (and in simulations of them).

To establish a maximum-entropy equilibrium, the resonant relaxation time \( t_{\text{rr}} \) must be less than the age of the cluster, typically \( 10^{10} \text{yr} \) in an old galaxy (lopsided equilibria are possible even if this condition is not satisfied, but they reflect the initial conditions rather than the relaxation process). This region is shaded in light and dark gray in Figure 8. The derivations in this paper also assume that the resonant relaxation time is less than the two-body relaxation time \( t_{2b} \), a condition satisfied in the dark gray region; it is likely that our analysis remains approximately valid even if this condition is violated, since non-resonant relaxation leads to a steady state that can persist for many two-body relaxation times. Finally, when the relativistic parameter \( \epsilon_{\text{GR}} > 1 \) to the left of the white dashed line, which marks the locus \( \epsilon_{\text{GR}} = 0.1666 \) – most of the stars in the maximum-entropy state have eccentricities near unity (see Fig. 6). In this case our analysis is of limited value since (i) thermal equilibrium may not be achieved, since resonant relaxation is suppressed by rapid precession (the ‘Schwarzschild barrier’); (ii) the equilibria may be short-lived because stars on high-eccentricity orbits are likely to be consumed by the black hole.

These approximate arguments suggest that the maximum-entropy state is most likely to be established for black-hole masses \( M_\star \leq 10^{7.5} M_\odot \) at radii \( \sim 0.001 \text{–} 0.1 \text{pc} \). The maximum-entropy state will be lopsided if the system is sufficiently ‘cold’, i.e., if the non-Keplerian energy is small.

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5 Here the semimajor axis \( a \) and eccentricity \( e \) are determined from the position and velocity at radii much larger than \( R_{\text{Sch}} \).

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enough (see Figure 5). In other words, a cluster with an initial spherically symmetric DF $f(L)$ (eq. 19) may or may not suffer a transition to a lopsided state, depending on its energy $E$ as defined by equation (5). For brevity, focus on the case where relativistic precession is negligible, $T_{GR} = 0$. Then the maximum-entropy state is lopsided for $E < -0.41 \frac{G M^2}{a_0}$ (Fig. 1), corresponding to spherically symmetric maximum-entropy systems with rms eccentricity $(\epsilon^2)^{1/2} < 0.46$ (Fig. 2). We do not understand how black-hole star clusters form, so we cannot predict whether energies or rms eccentricities in this range are common. However, the following considerations are relevant:

(i) The usual default assumption is that the DF in black-hole star clusters is ergodic ($\beta = 0$), which corresponds to $E = -0.3559 \frac{G M^2}{a_0}$ or $(\epsilon^2)^{1/2} = 2^{-1/2} = 0.7071$ if $e_{\text{max}} = 1$ and the system is spherical. For this energy there is no phase transition. However, there is no compelling theoretical reason why the initial DF of a cluster should be ergodic.

(ii) The velocity distribution of the old stars in the black-hole star cluster of the Milky Way is close to isotropic (Schödel et al. 2009); however, the data are reliable only outside a few arcsec (1 arcsec=0.04 pc), well outside the region in which any lopsided transition is likely to occur. We are not aware of any direct evidence that the old stars near the centre of the Galaxy have a lopsided distribution.

(iii) Destruction of stars that pass too close to the black hole tends to reduce the cluster energy, since stars on high-eccentricity orbits have larger energies in the gravitational potential of the cluster than those on low-eccentricity orbits with the same semi-major axis, but in the shaded region of Figure 8 this effect is relatively small\(^6\).

(iv) If the stars in the cluster form in situ then they probably form in a disc, although the orientation of the disc and the star-formation rate may vary strongly with time. In this case the orientations of the stars relax through resonant relaxation much faster than the eccentricities (these separate processes are sometimes called vector and scalar resonant relaxation, respectively; see for example Fig. 1 of Kocsis & Tremaine 2011 or Fig. 4 of Bar-Or & Fouvry 2018). In this case the initial state for the cluster – on time-scales longer than the vector resonant relaxation time-scale but shorter than the scalar time-scale – would contain randomly oriented stellar orbits with low eccentricities, and thus would be susceptible to the phase transition.

(v) An alternative possibility is that black-hole star clusters form from the inspiral of globular clusters through dynamical friction and their subsequent tidal disruption (Tremaine et al. 1975; Antonini et al. 2012; Gnedin et al. 2014). Since dynamical friction tends to circularize the globular-cluster orbits, this mechanism would add stars to the black-hole star cluster on low-eccentricity orbits. Numerical simulations of this process (Antonini et al. 2012) show that the resulting cluster can have an anisotropy parameter $\beta_v = 1 - \frac{\sigma^2_{\text{r}}}{\sigma^2_{\text{t}}}$ (here $\sigma_{\text{r}}$ and $\sigma_{\text{t}}$ are the velocity dispersions in the radial and tangential directions; $\beta_v = 0$ for an ergodic system) as small as $\sim -0.4$, indicating a significant bias towards low-eccentricity orbits which would make the lopsided transition more likely.

Most of our theoretical understanding of the dynamics of black-hole star clusters is based on analytic arguments or numerical solutions of the Fokker–Planck equation. These assume spherical symmetry and therefore do not address the question of whether a lopsided transition occurs. Direct N-body simulations are much more challenging: only a few have been carried out, and these remain oversimplified in several respects. In particular they generally contain too few stars and do not span the full dynamical range of $\sim 10^6$ between the event horizon $r_{\text{Sch}}$ and the dynamical radius $r_{\text{dyn}}$ (Fig. 8). As an example we describe the recent state-of-the-art simulation by Baumgardt et al. (2018), which is scaled to the Milky Way’s black-hole star cluster with $M_\bullet = 4 \times 10^6 M_{\odot}$. The simulation contains $0.95 \times 10^5$ stars with a total mass of $4 \times 10^7 M_{\odot}$, so the mean stellar mass is $\langle m \rangle = 42 M_{\odot}$, which is unrealistically high. The dynamical radius, where the enclosed stellar mass equals the black-hole mass, is $r_{\text{dyn}} \approx 1.2$ pc at the start of the simulation, growing slowly to 1.5 pc over 5.5 Gyr. From Figure 8, we expect the resonant relaxation time-scale to be shorter than the two-body time-scale (dark gray band) at radii less than about 0.04 pc, and this is where any lopsided transition is expected to occur\(^7\). However, within this radius the Baumgardt et al. (2018) simulation has only $\sim 10 \sim 20$ stars. This unrealistically small number arises for two reasons: (i) a large assumed mean stellar mass, which reduces the number of stars at each radius and enhances the relaxation rate; (ii) an artificially large Schwarzschild radius for the black hole, $10^3$ times its actual value or 0.0004 pc, which enhances the consumption rate of the black hole and leads to a shallower central density cusp than would otherwise be present. Thus it is unlikely that a detectable lopsided region would arise in this simulation, even if one were to be expected in a fully realistic N-body simulation of the same cluster.

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\(^6\) The fractional area of the loss cone in phase space is $f_L(a) = 1 - e_{\text{max}}^2$, where $e_{\text{max}}$ is the maximum eccentricity if the loss cone is empty (eq. 56). Then the dependence of the steady-state DF on eccentricity is approximately given by $f(e^2) = \log[(1 - e^2)/(1 - e_{\text{max}}^2)]$ for $e < e_{\text{max}}$ and zero otherwise (e.g., Cohn & Kulsrud 1978). For this DF the rms eccentricity is 0.685 for $f_L = 10^{-4}$, 0.677 for $f_L = 10^{-3}$, and 0.657 for $f_L = 10^{-2}$.

\(^7\) This simulation does not include relativistic effects and so lopsided equilibria can persist at smaller radii than indicated in Figure 8.
6 SUMMARY

We have explored the equilibria of stellar systems orbiting in the gravitational field of a central massive object, typically a black hole. In particular we have focused on the thermal equilibria over time-scales long compared to the resonant relaxation time and short compared to the two-body relaxation time. These equilibria maximize the entropy subject to the usual constraints that the mass, energy, and angular momentum are conserved, and the additional constraint that the semimajor axes of the stars are conserved.

We have shown that this system exhibits a phase transition from a disordered high-temperature equilibrium state to an ordered low-temperature state. The disordered state is spherically symmetric, while in the ordered state the stellar orbits have higher eccentricities and nearly aligned apsides. The ‘temperature’ of the system is a measure of the non-Keplerian component of its total energy, which arises from the self-gravity of the system and any relativistic corrections to the Keplerian Hamiltonian. In the absence of relativistic precession, the lopsided states correspond to systems with small self-gravitational energy, $E < -0.41 \frac{GM_\odot^2}{a_0}$, which arise from initially spherical states with relatively small rms eccentricity, $(\epsilon)^{1/2} < 0.46$.

The existence of lopsided equilibria in stellar systems dominated by a central black hole is not too surprising: the black-hole star cluster in M31 is lopsided (Tremaine 1995; Peiris & Tremaine 2003; Brown & Magorrian 2013), both analytic and N-body models of nearly Keplerian discs exhibit lopsided secular instabilities (Jacobs & Sellwood 2001; Touma 2002; Sridhar & Saini 2010; Touma & Sridhar 2012), and it is straightforward to construct equilibrium models of collisionless fluid eccentric discs (Statler 2001; Ogilvie & Barker 2014; Davydenkova & Rafikov 2018; Lee et al. 2018). In contrast, the lopsided equilibria described here bifurcate from spherical systems rather than axisymmetric discs, and have no precession or rotation.

For simplicity we have specialized to the case of a single stellar mass, and to a mono-energetic system – by which we mean that all stars have the same semimajor axis – but many of our conclusions also hold for systems with a more realistic distribution of masses and semimajor axes (Touma, Tremaine & Kazandjian 2019, Tremaine 2019).

We have only looked carefully at axisymmetric lopsided systems, that is, systems that are axisymmetric around the $z$-axis but asymmetric with respect to the $z = 0$ plane. Non-axisymmetric maximum-entropy systems may also be present but we have not found any.

The lopsided equilibria persist when relativistic precession is present, but when the relativistic parameter $a_0^{\text{GR}} \gtrsim 1$ (eq. 24) these have rms eccentricity near unity and thus our models are unrealistic, since resonant relaxation is suppressed by the Schwarzschild barrier and we have not accounted for the loss of stars on high-eccentricity orbits that pass close to the black hole. The high rms eccentricities found in lopsided states should enhance the rate of tidal disruption events and extreme mass-ratio inspirals, which may be detected through optical, ultraviolet, or X-ray transient searches or gravitational-wave observatories.

The phase transition described here is driven by scalar resonant relaxation, in which the eccentricities and orientations of the orbits relax while their semimajor axes remain fixed. In vector resonant relaxation, the eccentricities and semimajor axes remain fixed and only the orientations of the orbital planes relax. Vector resonant relaxation can also drive phase transitions (Roupas et al. 2017).

The lopsided equilibria we have found are both dynamical (solutions of the collisionless Boltzmann equation) and thermal (global maxima of the entropy, subject to the constraint that the stellar semimajor axes are fixed in resonant relaxation). Thus the lopsided equilibria are possible even if resonant relaxation is not complete, but required if it is.

Important next steps are to establish that lopsided equilibria are found in simulations of secular dynamics (Touma & Kazandjian, in preparation) and in direct N-body simulations of star clusters containing central black holes. Realistic simulations of black-hole star clusters are challenging but the results of this paper can be used to guide the design of simpler N-body simulations that should still exhibit the relevant behaviour.

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8 Strictly, we have only established that equilibrium exists in the orbit-averaged sense, i.e., that the system satisfies the collisionless Boltzmann equation after averaging over the characteristic dynamical time $\Omega^{-1}$ (eq. 43), but this approximation should be benign so long as $M_\star \ll M_\bullet$.

9 Thus they are thermal equilibria on time-scales long compared to the resonant relaxation time-scale but short compared to the two-body relaxation time-scale.
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APPENDIX A: LINEAR STABILITY OF SPHERICAL MAXIMUM-ENTROPY SYSTEMS

To keep the derivations in this section as general as possible, we do not use the assumptions of a mono-energetic system and of axisymmetry that we introduced at the start of §2.5.

A1 Thermodynamic stability

In a spherically symmetric system the DF can depend only on the integrals of motion $\Lambda$ and $L$. Thus the DF of a perturbed spherical system can be written

$$F(\Lambda, L, \theta) = F_0(\Lambda, L) + \Delta F(\Lambda, L, \theta)$$  \hspace{1cm} (A1)

where $|\Delta F|/F_0 \ll 1$. Expanding equation (4) to $O(\Delta F)^2$ we find that the perturbation to the entropy is

$$\Delta S = -2\pi \int d\Delta L d\theta \left[ F(\Lambda, L, \theta) \log F(\Lambda, L, \theta) - F_0(\Lambda, L) \log F_0(\Lambda, L) \right]$$

$$= -2\pi \int d\Delta L d\theta \left[ \Delta F + \Delta F \log F_0 + \frac{\Delta F^2}{2F_0} \right] + O(\Delta F)^3.$$ \hspace{1cm} (A2)

Similarly, from equation (5) the perturbed energy is

$$\Delta E = 2\pi \int d\Delta L d\theta \Delta F(\Lambda, L, \theta) H_0(\Lambda, L) + 2\pi^2 \int d\Delta L d\theta d\Delta N' d\Delta L' d\theta' \Delta F(\Lambda, L, \theta) K(\Lambda, N', L, L', \theta', \theta') \Delta F(N', L', \theta') + O(\Delta F)^3.$$ \hspace{1cm} (A3)

in which we have defined the unperturbed Hamiltonian (cf. eq. 7)

$$H_0(\Lambda, L) = H_{GR}(\Lambda, L) + 2\pi \int d\Delta N' d\Delta L' d\theta' K(\Lambda, N', L, L', \theta', \theta') F_0(\Lambda', L').$$ \hspace{1cm} (A4)

The perturbation to the mass at a given semimajor axis (eq. 9) is

$$\Delta \rho(\Lambda) = 2\pi \int d\Delta \theta \Delta F(\Lambda, L, \theta).$$ \hspace{1cm} (A5)

Since the unperturbed DF is an equilibrium, it must be an extremum of the entropy at fixed energy ($\Delta E = 0$), and since resonant relaxation conserves semimajor axis we also require $\Delta \rho(\Lambda) = 0$. These requirements are satisfied if $\Delta S - \beta \Delta E =$
\[ \gamma(\Lambda) \Delta \rho(\Lambda) = O(\Delta F)^2 \] for all variations \( \Delta F(\Lambda, L, \theta) \), where \( \beta \) and \( \gamma(\Lambda) \) are Lagrange multipliers. This condition implies that
\[ F_0(\Lambda, L) = \exp[-\beta H_0(\Lambda, L) - \gamma(\Lambda) - 1] \] (cf. eq. 8). Substituting this result into equation (A2) yields
\[ \Delta S = 2\pi \int d\Lambda dL d\theta \left[ \gamma(\Lambda) \Delta F + \beta H_0 \Delta F - \frac{\Delta F^2}{2F_0} \right] + O(\Delta F)^3. \] (A6)
Using the conditions \( \Delta \rho(\Lambda) = 0 \) and \( \Delta E = 0 \) to eliminate the terms linear in \( \Delta F \), we find
\[ \Delta S = -2\pi^2 \beta \int d\Lambda dL d\theta d\lambda' d\theta' \Delta F(\Lambda, L, \theta) \mathcal{K}(\Lambda, \Lambda', L, L', \theta', \theta') \Delta F(\Lambda', L', \theta') - \pi \int d\Lambda dL d\theta \frac{\Delta F^2}{F_0} + O(\Delta F)^3. \] (A7)
The system is thermodynamically stable if the entropy is a local maximum, which requires that \( \Delta S \leq 0 \) for all variations \( \Delta F \) that conserve \( E \) and \( \rho(\Lambda) \).

We can write the dependence of \( \Delta F \) on the orientation angles \( \theta = (\omega, \Omega) \) and \( I = \cos^{-1} L_z / L \) as an expansion in Wigner d-matrices (13), since these provide a complete representation of the rotation group SO(3):
\[ \Delta F(\Lambda, L, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} B_{l m}^I(\Lambda, L) d_{lm}^I(\theta) e^{in\omega + im\Omega}. \] (A8)
Here \( \{B_{l m}^I(\Lambda, L)\} \) are the undetermined functions that specify \( \Delta F \). Since \( \Delta F \) is real, the relation (15) implies that
\[ B_{l-n-m}^I(\Lambda, L) = (-1)^{m-n} B_{l m}^I(\Lambda, L). \] (A9)
Using equation (16) and the orthogonality relation (14) we find
\[ \Delta S = -4\pi^3 \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \int d\Lambda dL^2 \left[ \frac{|B_{l m}^I(\Lambda, L)|^2}{F_0(\Lambda, L)} - (2\pi)^2 G \beta \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{\mu^{m-n}}{L} |\mathcal{Y}_m| |\mathcal{X}_n| \right] \]
\[ \times \int d\Lambda dL^2 d\lambda' d\lambda'^2 B_{l m}^{I*}(\Lambda, L) B_{l m}^I(\Lambda', L') \mathcal{Q}_{m n}(\Lambda, L, \Lambda', L'). \] (A10)
Thus the system is thermodynamically stable if
\[ 2^{5/2} \pi^4 G \beta \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} (2l + 1)^{1/2} \int d\Lambda dL^2 \left[ \frac{|B_{l m}^I(\Lambda, L)|^2}{F_0(\Lambda, L)} \right] \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{\mu^{m-n}}{L} |\mathcal{Y}_m| |\mathcal{X}_n| \]
\[ \times \int d\Lambda dL^2 d\lambda' d\lambda'^2 B_{l m}^{I*}(\Lambda, L) B_{l m}^I(\Lambda', L') \mathcal{Q}_{m n}(\Lambda, L, \Lambda', L') > -1 \] (A11)
for all trial functions \( B_{l m}^I(\Lambda, L) \). Note that (i) this is a sufficient condition for stability; a necessary condition is that the inequality is satisfied for all trial functions for which \( \Delta \rho(\Lambda) = 0 \), which in turn requires
\[ \int d\Lambda dL^2 \left( |\mathcal{Y}_m(\Lambda, L)|^2 + |\mathcal{X}_n(\Lambda, L)|^2 \right) / F_0(\Lambda, L) \]
\[ \times \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} \frac{\mu^{m-n}}{L} |\mathcal{Y}_m| |\mathcal{X}_n| \] \( \Delta \mathcal{Q}_{m n}(\Lambda, L, \Lambda', L') ) > -1, \] \( \forall \ell \),
(A12)
where \( \mathcal{Q}_{m n} \) is defined in equation (36). This formula shows that the most unstable perturbations are those with \( \mathcal{A}_n(\Lambda, L) = 0 \) so we assume this from now on.

For numerical work we discretize these integrals by covering \( (\Lambda, L^2) \) space with bins centred on \( (\Lambda_o, L_o^2) \) having area \( \Delta \Lambda_o \Delta L_o^2, \alpha = 1, 2, \ldots \). We define
\[ s_{an} \equiv \sum_{\Lambda_o} \mathcal{S}(\Lambda_o, L_o) \left[ \frac{\Delta \Lambda_o \Delta L_o^2}{(1 + \delta_{0})} \right]^{1/2} \]
\[ R_{a,n,m,n'}^I = 2^{5/2} \pi^4 G \beta \sum_{\Lambda_o} \mathcal{S}(\Lambda_o, L_o) \Delta \mathcal{A}_m \Delta L_m^2 F_0(\Lambda_o, L_o) \Delta \mathcal{A}_{n'} \Delta L_{n'}^2 F_0(\Lambda_{n'}, L_{n'}) \left[ \frac{1}{(1 + \delta_{0})(1 + \delta_{n'})} \right]^{1/2} \mathcal{Q}_{m n}(\Lambda_o, L_o, \Lambda_{n'}, L_{n'}). \] (A13)
The stability criterion (A12) becomes
\[
\frac{\sum_{m' \geq 0} \sum_{\alpha' \sigma' \gamma'} s_{m'}^\alpha R^I_{\alpha', \gamma'} s_{m'}^\sigma} {\sum_{m' \geq 0} \sum_{\alpha} |s_{m'}|^2} > -1. \tag{A14}
\]

Now \( R^I_{\alpha', \gamma'} \) is a real symmetric matrix with multi-index \((\alpha, n)\) so its eigenvalues are real. Since \( A^I_{\min} > -1 \), this sufficient condition is also necessary if the perturbation conserves mass, \( \int \partial_d L d\theta \Delta F = 0 \). This requirement is automatically satisfied if \( l > 0 \).

For the mono-energetic systems explored in this paper, the spherically symmetric equilibrium DF has the form \( F_0(\Lambda, L) = \delta(\Lambda - \Lambda_0)f_0(L) \) (cf. eq. 19). To evaluate the matrix \( R^I \) we need the equilibrium spherical DF \( f_0(L) \). We find this by evaluating \( f \) using equation \( (8) \) for an assumed Hamiltonian \( H \), then evaluating \( H \) using equation \( (7) \), and iterating to convergence.

### A2 Dynamical stability

The linearized collisionless Boltzmann equation that describes the evolution of small perturbations to an equilibrium DF is
\[
\frac{\partial \Delta F}{\partial t} + \frac{\partial H_0}{\partial J} \frac{\partial \Delta F}{\partial \omega} - \frac{\partial H}{\partial \omega} \frac{\partial \Delta F}{\partial J} = 0. \tag{A15}
\]

Here \( J = (\Lambda, L, \omega) \) is a vector of the three actions, \( \omega = (\ell, \omega, \Omega) \) is a vector of the angles, and \( \Delta H \) is the perturbed Hamiltonian. Since we are working in the secular approximation, \( \Delta F \) and \( \Delta H \) are independent of the mean longitude \( \ell \). Since the equilibrium DF is spherically symmetric, \( F_0 \) and \( H_0 \) are independent of \( L \). Moreover \( \Delta F \) can be represented as a linear combination of functions with time dependence \( \exp(-i\omega t) \). Thus equation \( (A15) \) simplifies to
\[
-i\omega \Delta F + \frac{\partial H_0}{\partial \omega} \frac{\partial \Delta F}{\partial J} - \frac{\partial H}{\partial \omega} \frac{\partial \Delta F}{\partial J} = 0. \tag{A16}
\]

The perturbed Hamiltonian is (cf. eq. 7)
\[
\Delta H(\Lambda, L, \theta) = 2\pi \int_{\Lambda'} d\Lambda' d\theta' K(\Lambda, \Lambda', L, L', \theta', \theta') \Delta F(\Lambda', L', \theta'). \tag{A17}
\]

Using equations \((A16)\) and \((A16)\) as well as the expansion of the perturbed DF \((A8)\), we find
\[
\Delta H(\Lambda, L, \theta) = 2\pi i^k G \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{n=-l}^{l} i^{l-n} y_{ln} y_{ln}' d_{ln}(l) e^{i\omega_G \delta_{ln}} \int d\Lambda' dL' Q^I_{ln}(\Lambda, L, \Lambda', L', \theta', \theta') B^I_{ln}(\Lambda', L', \theta') B^I_{n}(\Lambda, L, \theta). \tag{A18}
\]

Because of the orthogonality relation \((A16)\) we can examine terms of a single \( l \) and \( m \) in equation \((A16)\). Thus we can drop the indices \( l \) and \( m \) on \( B^I_{ln} \) and the linearized collisionless Boltzmann equation now reads
\[
-i\omega B_n + n \frac{\partial H_0}{\partial L} B_n - 2\pi i^k G \frac{\partial H_0}{\partial L} \frac{\partial F_0}{\partial L} \sum_{n' \geq 0} i^{l-n'} y_{ln} y_{ln}' d_{ln}(l) e^{i\omega_G \delta_{ln}} \int d\Lambda' dL' Q^I_{ln}(\Lambda, L, \Lambda', L', \theta', \theta') B^I_{ln}(\Lambda', L', \theta') = 0. \tag{A19}
\]

As in the preceding subsection, we can split this equation into components that are even and odd in \( n \) by writing \( B_{2n} = \frac{1}{2} [B_n + A_n] \) with \( n \geq 0 \). Since \( Q^I_{ln}, y_{ln}, y_{ln}' \) are all even in \( n \), for \( n \geq 0 \) we have
\[
\omega A_n = n \frac{\partial H_0}{\partial L} A_n,
\]
\[
\omega A_n = n \frac{\partial H_0}{\partial L} S_n - 2\pi i^k G \frac{\partial H_0}{\partial L} \frac{\partial F_0}{\partial L} \sum_{n' \geq 0} i^{l-n'} y_{ln} y_{ln}' \int d\Lambda' dL' Q^I_{ln}(\Lambda, L, \Lambda', L', \theta', \theta') S_{ln}(\Lambda', L'). \tag{A20}
\]

Eliminating \( A_n \),
\[
n^2 \left( \frac{\partial H_0}{\partial L} \right)^2 S_n - 2\pi i^k G \frac{\partial H_0}{\partial L} \frac{\partial F_0}{\partial L} \sum_{n' \geq 0} i^{l-n'} y_{ln} y_{ln}' \int d\Lambda' dL' Q^I_{ln}(\Lambda, L, \Lambda', L', \theta', \theta') S_{ln}(\Lambda', L') = \omega^2 S_n. \tag{A21}
\]

If the unperturbed DF is in thermal equilibrium then at a given semimajor axis \( F_0 = \exp(-\beta H_0) \), so \( \partial F_0/\partial L = -\beta F_0 \partial H_0/\partial L \) and
\[
n^2 \left( \frac{\partial H_0}{\partial L} \right)^2 S_n + 2\pi i^k G \beta \frac{\partial H_0}{\partial L} \frac{\partial F_0}{\partial L} \sum_{n' \geq 0} i^{l-n'} y_{ln} y_{ln}' \int d\Lambda' dL' Q^I_{ln}(\Lambda, L, \Lambda', L', \theta', \theta') S_{ln}(\Lambda', L', \theta') = \omega^2 S_n. \tag{A22}
\]

We now discretize this integral equation as in equations \((A13)\):
\[
\sum_{\alpha' \sigma'} V_{\alpha', \sigma'}^{I_{an}} s_{\alpha' \sigma'} = \omega^2 s_{an}, \tag{A23}
\]

where
\[
V_{\alpha', \sigma'}^{I_{an}} = n^2 \left( \frac{\partial H_0}{\partial L} \right)^2 \delta_{\alpha' \sigma'} + R^I_{\alpha', \sigma'} \tag{A24}
\]
Thus $\omega^2$ is an eigenvalue of the matrix $V^l$, which is closely related to the matrix $R^l$ whose eigenvalues determine thermodynamic stability. The matrix $V^l$ is real and although it is non-symmetric its eigenvalues are all real (see below for proof). Thus the system is dynamically unstable if and only if the minimum eigenvalue is negative, $\omega^2 < 0$.

### A3 Proof that eigenvalues of $V^l$ are real

Rewrite (A23) and (A24) as $Vs = \omega^2 s$ where $V = D + DR$, with $D$ a diagonal matrix having entries $n^2(\partial H_0/\partial L)^2_\alpha$. Rearrange the labeling of the rows and columns in $D$ and $R$ so the first $N$ diagonal elements of $D$ are non-zero and the last $M$ elements are zero. Then we can write $D$ and $R$ in block form as

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} R_a & R_b \\ R'_b & R_c \end{bmatrix}; \quad (A25)$$

Here $D_1$ is a positive-definite $N \times N$ diagonal matrix, $R_a$ is a symmetric real $N \times N$ matrix, $R_c$ is a symmetric real $M \times M$ matrix, $R_b$ is a real $M \times N$ matrix, and $R'_b$ is its transpose. Writing $s^t = [x^t, y^t]$ where $x$ and $y$ are $1 \times N$ and $1 \times M$ column matrices, the eigenvalue equation becomes

$$D_1 R_a x + D_1 x + D_1 R_b y = \omega^2 x, \quad 0 = \omega^2 y. \quad (A26)$$

Therefore either $\omega^2 = 0$ or $y = 0$. In the latter case, the eigenvalue equation becomes

$$D_1 R_a x + D_1 x = \omega^2 x. \quad (A27)$$

Since $D_1$ is diagonal with positive-definite diagonal elements, we can define a $1 \times N$ column matrix $z$ whose elements are $z_n = D^{-1/2}_{1nn} x_n$ and a symmetric, real $N \times N$ matrix $W$ by $W_{jk} = D^{1/2}_{1jj} R_{x,jk} D^{1/2}_{1kk}$. Then

$$(W + D_1)z = \omega^2 z. \quad (A28)$$

Since the matrix on the left side is symmetric, its eigenvalues $\omega^2$ must be real.