Recursive equations for arbitrary scattering processes

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The usefulness of recursive equations to compute scattering matrix elements for arbitrary processes is discussed. Explicit results at tree and one-loop order, obtained by the \textit{HELAC/PHEGAS} package that is based on the Dyson-Schwinger recursive equations approach, are briefly presented.

1. Introduction

Recursive equations to compute scattering matrix elements have been used extensively over the last years in order to obtain results for multi-leg amplitudes. Their history started essentially with the work of Berends and Giele \cite{1}, who were able to prove the conjectured simple all-$n$ form of Parke and Taylor \cite{2} for the MHV amplitudes in QCD. The recognition of their usefulness has been expanded recently by the discovery of a new class of recursive equations, by Britto, Cachazo and Feng \cite{3} and Witten \cite{4}.

In this paper we are considering the Dyson-Schwinger (DS) recursive approach \cite{5,6,7,8}, and show how this can be used as a general framework for scattering elements computation. We also present selected results for processes at tree order and at the one-loop level, obtained with the \textit{HELAC/PHEGAS} \cite{9} package, which is an implementation of the DS method.

2. The Dyson-Schwinger approach

The traditional representation of the scattering amplitude in terms of Feynman graphs results to a computational cost that grows like the number of those graphs, therefore as $n!$ (at tree order), where $n$ is the number of particles involved in the scattering process.

An alternative\footnote{1For other alternatives see \cite{10,11}.} to the Feynman graph representation is provided by the Dyson-Schwinger approach \cite{7}. Dyson-Schwinger equations express recursively the $n$-point Green's functions in terms of the $1-$, $2-$, $\ldots$, $(n-1)$-point functions. In the framework of a theory with three- and four-point vertices the DS equations are rather simple and
their diagrammatic representation is given below, for $1 \to n$ amplitude:

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram.png}
\end{array}
\]

Omitting the contribution of the second line in the above formula is equivalent to restrict ourselves at tree order. In order to get an idea of the actual mathematical form of these equations, let us consider the simplest case where we are interested to “count graphs”, so by dropping all propagators, couplings, wave-functions, etc, we end up with the following equation:

\[
a(n) = \delta_{n,1} + \frac{1}{2!} \sum_{n_1,n_2} \frac{n!}{n_1!n_2!} a(n_1)a(n_2)\delta_{n_1+n_2,n} + \frac{1}{3!} \sum_{n_1,n_2,n_3} \frac{n!}{n_1!n_2!n_3!} a(n_1)a(n_2)a(n_3)\delta_{n_1+n_2+n_3,n}
\]

with the initial condition $a(0) = 0$; $a(n)$ is nothing more than the number of Feynman graphs, contributing to the $1 \to n$ matrix element.

The computational cost of DS equations grows like $\sim 3^n$, which essentially counts the steps used to solve the recursive equations. Obviously for large $n$ there is a tremendous saving of computational time, compared to the $n!$ growth of the Feynman graph approach.

2.1. Color representation

Color representation or color decomposition of the amplitude is a major issue when dealing with multi-parton processes. Let us consider $n$-gluon scattering with external momenta $\{p_i\}_1^n$, helicities $\{\varepsilon_i\}_1^n$ and colors $\{a_i\}_1^n$ of gluons $i = 1, \ldots, n$. As is well known the total amplitude can be expressed as a sum of single trace terms [18]:

\[
\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{a_i\}_1^n) = 2ig^{n-2} \sum_{I \in \mathcal{P}(2, \ldots, n)} Tr(t^{a_1}t^{a_{\sigma_1}(2)} \ldots t^{a_{\sigma_1(n)}}) A_I(\{p_i\}_1^n, \{\varepsilon_i\}_1^n)
\]

where $\sigma_I(2 : n)$ represent the $I$-th permutation of the set $\{2, \ldots, n\}$ and $Tr(t^{a_1}t^{a_{\sigma_1}(2)} \ldots t^{a_{\sigma_1(n)}})$ represents a trace of generators of the $SU(N_c)$ gauge group in the fundamental representation. For processes involving quarks a similar but much more cumbersome expression can be derived [18].

One of the most interesting aspects of this decomposition is the fact that the $A_I(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{a_i\}_1^n)$ functions (called dual, partial or color-ordered amplitudes), which contain all the kinematic information, depend on the permutation and are gauge invariant and cyclically symmetric in the momenta and helicities of gluons. The color-ordered amplitudes are simpler than the full amplitude because they only receive contributions from diagrams with a particular cyclic ordering of the external gluons (planar graphs).

Of course to get the full amplitude one has to square the matrix element,

\[
\sum_{\{a_i\}_1^n, \{\varepsilon_i\}_1^n} |\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{a_i\}_1^n)|^2 = g^{2n-4} \sum_{I \neq J} C_{I,J} A_I A_J^*
\]

where the $(n - 1)! \times (n - 1)!$ dimensional color matrix can be written in the most general form as follows:

\[
C_{I,J} = \sum_{1 \ldots N_c} Tr(t^{a_{\sigma_1}(2)} \ldots t^{a_{\sigma_1(n)}}) Tr(I \leftrightarrow J)^*
\]

(1)

There exists a much simpler approach, in fact far superior from the point of view of an automatized numerical calculation, where the matrix element is represented as follows [19, 20, 8]:

\[
\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{a_i\}_1^n) = 2ig^{n-2} \sum_{I \in \mathcal{P}(2, \ldots, n)} D_I A_I(\{p_i\}_1^n, \{\varepsilon_i\}_1^n)
\]

with $c_i, a_i$ the color and anticolor indices for each external particle, i.e. $(c, 0)$ for quarks, $(0, a)$ for antiquarks, $(c, a)$ for gluons and $(0, 0)$ for non-colored particles, and

\[
D_I = \delta_{c_1,a_{\sigma_1(1)}} \delta_{c_2,a_{\sigma_1(2)}} \ldots \delta_{c_n,a_{\sigma_1(n)}}
\]
or in a more abstract notation

\[ D_I = \delta_{1,\sigma_I(1)}\delta_{2,\sigma_I(2)}\ldots\delta_{n,\sigma_I(n)} \]

where \( \sigma_I(1 : n) \) represent the \( I \)-th permutation of the set \( \{1, 2, \ldots, n\} \). The sequence of numbers \( i, \sigma_I(i) \), \( i = 1 \ldots n \), is identified as a color-connection configuration, describing the way the color connection is structured. In that sense, no explicit reference to ‘real’ color indices is made. Finally the color matrix takes a very simple form,

\[ \mathcal{C}_{IJ} = N_c^m(\sigma_I, \sigma_J) \]  

(2)

where \( 1 \leq m(\sigma_I, \sigma_J) \leq n \) counts how many common cycles the permutations \( \sigma_I \) and \( \sigma_J \) have. For a detailed description, see [8].

Recursive equations can be written both for the full amplitude, \( \mathcal{M} \), and for the color ordered, \( \mathcal{A} \). In the latter case the DS equations are identical to the Berends-Giele ones.

For numerical applications the computation of the color ordered amplitudes suffers from the growth related to the number of color-flow or color-connection configurations. In such cases it is preferable to write down DS equations for the full amplitude \( \mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{c_i, a_i\}_1^n) \) and then perform the incoherent sum

\[ \sum_{c_i, a_i = 1 \ldots 3} |\mathcal{M}(\{p_i\}_1^n, \{\varepsilon_i\}_1^n, \{c_i, a_i\}_1^n)|^2 \]

by Monte-Carlo methods. We have recently extended HELAC so that a Monte-Carlo over ‘real’ colors, or color-configurations can be performed [8]. A color configuration is identified by the sequence of numbers \( \{c_i, a_i\}_1^n, c_i, a_i = 1 \ldots 3 \). The details are given in [8].

Besides the problem related to the color treatment, the summation over different flavors is also a very important problem when the flavor of partons at the final state is unidentified, as usually. In that case a Monte Carlo treatment over flavor degrees of freedom has been proposed some time ago[21], showing that the purely gluonic contribution falls from 45.7% for 3-jet, to 26.6% for 8-jet production [21].

### 2.2. On-shell recursive equations

During the last year much progress has been made in the understanding of analytical calculations of color amplitudes in perturbative Yang-Mills theories. Led by an observation of Witten [22], Britto, Cachazo and Feng (BCF) have proposed a new recursion relation for tree amplitudes of gluons [3] that naturally arrives at the simplest known expressions for those amplitudes in terms of Weyl - Van der Waerden spinors, with Maximal Helicity Violating vertices as building blocks. Explicit calculations have been performed using this technique [23][24], extensions to amplitudes involving particles from the electroweak sector [25] have been pursued and a new approach to one loop amplitudes has been proposed [26] employing MHV vertices and unitarity arguments as well as the use of recursive equations [27][28].

The BCF recursion relation features some remarkable characteristics, among which the on-shell analytic continuation of selected off-shell propagators, the analytic continuation of two selected external momenta in the complex plane and a decomposition of a color helicity amplitude into smaller helicity amplitudes with complex external momenta that doesn’t appear to be in direct connection with the decomposition in Feynman diagrams.

For \( n \)-gluon amplitude the BCF equation, in a diagrammatic representation reads as,

\[ \mathcal{A}(1 \ldots n) = \sum_{j=2}^{n-2} \sum_{P_{1 \ldots j}}^{n-3} A_n(1, 2, \ldots, (n - 1)^-, n^+) = \]

\[ \sum_{i=1}^{n-3} \sum_{h=+, -} \left( A_{i+2}(\hat{n}, 1, 2, \ldots i, -\hat{P}_{n,i}) \frac{1}{P_{n,i}^2} \right) \]

or in mathematical terms,

\[ A_n(1, 2, \ldots, (n - 1)^-, n^+) = \]

\[ \sum_{i=1}^{n-3} \sum_{h=+, -} \left( A_{i+2}(\hat{n}, 1, 2, \ldots i, -\hat{P}_{n,i}) \frac{1}{P_{n,i}^2} \right) \]
where
\[ P_{n,i} = p_n + p_1 + \ldots + p_i, \]
\[ \hat{P}_{n,i} = P_{n,i} + \frac{p_i^2}{(n-1)P_{n,i}[n]} \lambda_{n-1} \hat{\lambda}_n, \]
\[ \hat{p}_{n-1} = p_{n-1} - \frac{p_{n-1}^2}{(n-1)P_{n-1,i}[n]} \lambda_{n-1} \hat{\lambda}_n, \]
\[ \hat{p}_n = p_n + \frac{p_n^2}{(n-1)P_{n,i}[n]} \lambda_{n-1} \hat{\lambda}_n. \]

where \( \lambda_i \) and \( \hat{\lambda}_i \) are spinors and anti-spinors corresponding to the momentum \( p_i \) and \( \langle i | P | j \rangle \equiv \delta_{ij} \lambda^a_i P_{ab} \tilde{\lambda}^b_j \).

The kinematical operation on the momenta of the 1st and \( n \)th particles, is called the ‘hat’ operation.

It can be proven that BCF equation can be obtained from the Berends-Giele (or DS) recursive equations, by making use of the following points \[29\]:

• a special gauge choice, that allows the cancellation of all contributions where diagrams with the first and the last leg meeting in a three-vertex, as well as with the first and the last leg meeting in a four-vertex with another external leg. These diagrams are obviously not-present in the BCF decomposition of the amplitude,

• a set of relations guaranteed by the kinematical transformation (the hat ‘operation’) applied to the chosen momenta, that exactly takes care of the apparent over-counting of certain Feynman diagrams within the BCF decomposition,

• and finally a gauge identity, that relates a ‘hatted’ contribution arising from a three vertex, with the un-hatted three- and four-vertex contributions.

\[ \begin{array}{c}
\begin{xy}
0.5*
"\fig{2.5}{hat}\fig{2.5}{hat}\fig{2.5}{hat}"
\end{xy}
\end{array} = \begin{array}{c}
\begin{xy}
0.5*
"\fig{2.5}{hat}\fig{2.5}{hat}"
\end{xy}
\end{array} + \begin{array}{c}
\begin{xy}
0.5*
"\fig{2.5}{hat}\fig{2.5}{hat}\fig{2.5}{hat}"
\end{xy}
\end{array} \end{array} \]

where

\[ \begin{array}{c}
\begin{xy}
0.5*
"\fig{2.5}{hat}\fig{2.5}{hat}"
\end{xy}
\end{array} = -\varepsilon \nu \rho \sigma V_{\nu \rho \sigma} \]

with \( V_{\nu \rho \sigma} \) the QCD four-vertex.

Although BCF equations are very powerful in order to obtain ‘analytical’ results, their numerical implementation does not show up any real gain as compared to the Berends-Giele (or DS) ones. In fact for a moderate number of external particles \( 8 < n < 12 \) their complexity and therefore their CPU-time consumption is substantially larger than that of the Berends-Giele equations\[2\]. Nevertheless their usefulness in computing tree- as well as one-loop amplitudes is still an unexplored territory.

3. HELAC/PHEGAS: results

HELAC/PHEGAS is a computer package that incorporates the DS approach to compute scattering cross section for arbitrary process. In the computation of the matrix elements, it includes all Standard Model particles and interactions, both in Feynman and Unitary gauges. There are two options to deal with the colored particles, namely the color-connection approach, in which all color ordered amplitudes are computed, and the color-configuration approach in which the full amplitude is given, followed by a Monte-Carlo treatment of the color summation. In the phase space generation and integration, sector, PHEGAS is using a multi-channel approach, each Feynman graph, identified with a potential generation channel, followed by an optimization of the a priori weights, entering the calculation of the global phase-space density. Moreover for multi-particle processes where the number of Feynman graphs makes the use of a multi-channel approach impossible, other phase-space generation methods and packages, like HAAG \[31\] and DURHAM \[8\] are used.

HELAC/PHEGAS has been used extensively to produce physically relevant results\[3\]. In the sequel we are going to restrict ourselves to two specific examples, in order to reveal its potential for physics studies.

The first example refers to the process \( p p \rightarrow t \bar{t} b \bar{b} b \bar{b} \). From the physics point of view it consists the irreducible background of \( t\bar{t}HH \) production, which seems interesting in a high-luminosity LHC (SLHC) version, for studying HHH cou-
pling [34]. From the computational point of view, it is a challenging process, and a nice example to demonstrate the ability of PHEGAS/HELAC to deal with QCD processes in a realistic setup.

The number of Feynman graphs contributing to this process is **1454** (for a \(gg\) initial state), with 5! color-connection configurations. We have used the structure functions and \(\alpha_s\) from [PDFLIB, CTEQ-4L (LO)]. Kinematical decays of \(t \to bW^+\) have been implemented and the following set of cuts has been used: \(p_T^b > 20\text{GeV}, |\eta_b| < 2.5, \Delta R > 0.5\). The result for the total cross section is \(1.053 \pm 0.073\) (fb) @ LHC energy.

The second example is the computation of Fermion-Loop (FL) contributions in six-fermion production processes in \(e^+e^-\) collisions [35,36,37]. It is an explicit example of the use of DS equations to compute one-loop amplitudes. This is achieved in a rather straightforward way, by adding to the tree-order SM vertices, contributions arising from 1PI graphs at one loop [38,39]. In such a way using FORM we were able to calculate all \(V_1V_2V_3\) and \(V_1V_2V_3V_4\) vertices at one loop, for arbitrary kinematical configuration (all particles off-shell) and for all gauge bosons \(V = \gamma, Z, W^\pm\). Then, as dictated by the well known quantum-field theoretic argument, the amplitude is just the tree-order one with the tree-order vertices replaced by the generalized one-loop vertices.

We take as an example the process \(e^-e^+ \to \mu^-\bar{\nu}_\mu ud\tau^-\tau^+\). The number of Feynman Graphs contributing to the process is 208, whereas the number of DS vertices (the steps needed to compute the amplitude) is 140. We have used the following set of kinematical cuts: \(E_{l}, E_q > 5\text{GeV}\) and \(m_{tt}, m_{qq} > 10\text{GeV}\), and the result for \(E = 500\text{GeV}\) is given by \(\sigma_0/ab = 54.96(26)\) at tree order \(\sigma_1/ab = 57.31(28)\) at the one loop, with a \(K\)-factor given by \(K/100 = 4.28(2)\). The MC data for this particular run are as follows: MC points generated: 1 Million (961792), MC points used after cuts: 404842, real time of running: 6 1/2 hours on a very basic PC.

**4. Outlook**

Recursive equations have been proven to be the framework for an efficient matrix element computation for arbitrary scattering processes. They are the basic ingredients towards the construction of an automatized generator including NLO corrections. The fusion with parton-shower generators and the understanding of the working of this fusion, will be one of the main tasks in the near future. Precision calculations will offer the solid basis needed for discoveries in future high-energy colliders.

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