INDEX OF EQUIVARIANT CALLIAS-TYPE OPERATORS AND INVARIANT METRICS OF POSITIVE SCALAR CURVATURE

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ABSTRACT. We formulate, for any Lie group $G$ acting isometrically on a manifold $M$, the general notion of a $G$-equivariant elliptic operator that is invertible outside of a $G$-cocompact subset of $M$. We prove a version of the Rellich lemma for this setting and use this to define the equivariant index of such operators. We show that $G$-equivariant Callias-type operators are self-adjoint, regular, and hence equivariantly invertible at infinity. Such operators explicitly arise from a pairing of the Dirac operator with the equivariant Higson corona. We apply the theory developed herein to obtain an obstruction to positive scalar curvature metrics on non-cocompact manifolds.

1. Introduction

It is well-known that a Dirac operator $D$ on a non-compact manifold $M$ is not in general Fredholm, since the usual version of the Rellich lemma fails in this setting. Nevertheless, it is possible to modify $D$ so as to make it Fredholm but still remain within the class of Dirac-type operators. One such modification is a Callias-type operator, which was initially studied by Callias [11] on $M$ a Euclidean space, before being generalised to the setting of Riemannian manifolds by others [7], [9], [10]. A Callias-type operator may be written as $B = D + \Phi$, where $\Phi$ is an endomorphism making $B$ invertible at infinity. As in [10], one may form the order-0 bounded transform $F$ of $B$, defined formally by

$$F := B(B^2 + f)^{-1/2},$$

where $f$ is a compactly supported function. The formal computation $F^2 - 1 = -f(B^2 + f)^{-1}$ shows that $F$ is a Fredholm operator, since multiplication by the compactly supported function $f$ defines a compact operator between Sobolev spaces $H^i \to H^j$ for all $i > j$.

More generally, if $M$ is a manifold with a proper, non-cocompact action of a Lie group $G$, a $G$-invariant Dirac operator on $M$ need not be Fredholm in the sense of $C^*$-algebras. From the point of view of the equivariant index map in $KK$-theory, the lack of a general index for elements of the equivariant analytic $K$-homology $K^G_*(M)$ can be traced back to the lack of a counterpart to the canonical projection in $KK(C, C_0(M) \rtimes G)$ defined by a compactly supported cut-off function when $M/G$ is compact.

In this paper we introduce $G$-equivariant analogues of the Sobolev spaces $H^i$ and the Rellich lemma. With these tools, the formal computation above can be made to work in the non-cocompact setting for any operator whose square is positive outside of a cocompact set. We establish that such operators have a $G$-equivariant index.

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This provides an equivariant generalization of the situation considered by Roe in [36]. It is possible to formulate the index theory of equivariant Callias-type operators in terms of the Roe algebra. We will do this, as well as consider applications, in forthcoming papers.

In section 3, we define $G$-Sobolev modules $E^i$ over the (maximal or reduced) group $C^*$-algebra $C^*(G)$. The main purpose of this construction is to establish the analogue of the Rellich lemma alluded to above.

In section 4 we define the notion of $G$-invertibility at infinity - an equivariant generalisation of the notion of invertibility at infinity (see for example [10] section 1).

**Theorem 4.19.** Suppose $B$ is $G$-invertible at infinity. Let $F$ be its bounded transform. Then $(E^0, F)$ is a Kasparov module over the pair of $C^*$-algebras $(\mathbb{C}, C^*(G))$. The class $[E^0, F] \in KK(\mathbb{C}, C^*(G))$ is independent of the choice of the cocompactly supported function $f$ used to define $F$.

This implies that $G$-invertible-at-infinity operators are equivariantly Fredholm. An example of such an operator is a $G$-Callias-type operator, defined in section 5.

We show that these operators are essentially self-adjoint and regular in the sense of Hilbert modules. We give an explicit construction of $G$-Callias-type operators in section 6 using the $K$-theory of an equivariant Higson corona of $M$, whose $K$-theory turns out to be highly non-trivial. In particular, we prove:

**Theorem 6.19.** Let $M$ be a complete $G$-Riemannian manifold with $M/G$ non-compact. Then the $K$-theory of the Higson $G$-corona of $M$ is uncountable.

In section 7 we give an application of the theory to $G$-invariant metrics of positive scalar curvature on non-cocompact $M$. We prove:

**Theorem 7.3.** Let $M$ be a $G$-equivariantly spin Riemannian manifold with $G$-spin-Dirac operator $\partial_0$. If $M$ admits a $G$-invariant metric with pointwise positive scalar curvature, then the $G$-index of every $G$-Callias-type operator on $M$ vanishes. Let $F$ be the bounded transform of the $G$-Callias-type operator defined by $\partial_0$ and any $G$-admissible endomorphism $\Phi$. Then

$$\text{index}_G(F) = 0 \in K_0(C^*(G)).$$

This result vastly generalises two existing results on obstructions to $G$-invariant positive scalar curvature on proper $G$-spin manifolds, where $G$ is a non-compact Lie group. The first is a recent result of Zhang (Theorem 2.2 of [38]), which was the first generalisation of Lichnerowicz' vanishing theorem to the cocompact-action case; the notion of index used there was the Mathai-Zhang index (see [31]). The second is Theorem 54 of [14], which states that the equivariant index of a $G$-invariant Dirac operator vanishes in the presence of $G$-invariant positive scalar curvature.

The methods and results of this paper can be contrasted with the equivariant index theory and applications studied in [14]. First, [14] deals exclusively with index theory in the cocompact case, where the $C^*(G)$-Fredholmness of the Dirac operator was known. In addition, [14] focused almost entirely on the case of almost-connected $G$, where a global slice of the manifold exists.

The paper [12] also deals with Callias-type operators in the setting of Hilbert modules on non-compact manifolds. The indices studied there and in the present paper
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are in general different generalisations of the classical Callias-type index, although the case when \( G \) is the fundamental group of a compact manifold can be approached from both directions. The technical difference between the two approaches is that whereas in [12], the Hilbert module structure arises by twisting the space of \( L^2 \)-sections of a vector bundle by a Hilbert module bundle, the \( G \)-Sobolev modules we define here arise from the \( G \)-action on the space of compactly supported smooth sections, and then completing with respect to a particular inner product with values in the group \( C^\ast \)-algebra. Notably, [12] establishes a twisted Callias-type index theorem that allows one to reduce the computation of a Callias-type index to an index on a compact hypersurface, using the machinery of \( KK \)-theory. The analogous theorem in the \( G \)-equivariant case, although expected to be true, does not follow directly from the results of [12]. We hope to address this in future work.

Finally, other versions of \( G \)-index theory, for non-compact \( M/G \) and \( G \), have been developed elsewhere. This includes the work of Hochs-Mathai [17],[18], Braverman [8], and Hochs-Song [21],[19],[20].

The results contained in this paper set the stage for a number of further questions. The first concerns the Baum-Connes conjecture. When \( M/G \) is compact, the equivariant index map for \( G \)-Callias-type operators reduces to the Baum-Connes assembly map. However, when \( M/G \) is non-compact, the class of operators encapsulated by Callias index theory is strictly larger, and an interesting open question arises:

**Open question:** Does the index of a \( G \)-Callias-type operator always lie in the image of the Baum-Connes assembly map?

Another direction in which we will look to apply the theory developed herein is the problem of *quantisation commutes with reduction* (or \([Q, R] = 0\)) for non-cocompact manifolds. In [29], Landsman used the language of noncommutative geometry to formulate a version of geometric quantization for \( G \)-cocompact \( \text{Spin}^c \)-manifolds. He defined quantization as taking equivariant index of a \( \text{Spin}^c \)-Dirac operator, via the equivariant index map \( \text{index}_G : K^G_0(M) \to K_0(C^\ast(G)) \), while reduction of a quantized system is defined to be the image of the index under a certain tracial map

\[
R_G : K_0(C^\ast(G)) \to K_0(\mathbb{C}) \cong \mathbb{Z}.
\]

Landsman conjectured that, under certain assumptions on the symmetry group \( G \) (such as semi-simplicity) \([Q, R] = 0\) holds:

\[
R_G(\text{index}_G(D_M)) = \text{index}(D_{M_0}).
\]

Here \( M_0 \) is a certain (compact) reduced space of \( M \) defined as the inverse image of the regular value 0 under the symplectic moment map \( \mu \), and the index on the right-hand side of the equation is the classical Fredholm index.

This conjecture was solved by Mathai and Zhang [31] in 2008, following work done in special cases by Hochs [15] and Hochs-Landsman [16].

The theory we formulate in this paper provides a natural class of operators with which to investigate the \([Q, R] = 0\) problem in the non-cocompact setting, namely \( G \)-equivariant Callias-type operators that arise from an equivariant \( \text{Spin}^c \) structure.
on $M$. In future work, we aim to address the following specific question.

**Open question:** Does $[Q, R] = 0$ hold for $G$-Spin$^c$ Callias-type operators?

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2. Notation and Terminology

Throughout this paper, $M$ will be a Riemannian manifold on which a Lie group $G$ acts smoothly, properly and isometrically. We shall call this set-up $G$-Riemannian manifold. If the orbit space $M/G$ is compact, we say that the $G$-action is cocompact, or that $M$ is a cocompact $G$-manifold. Except in section 7 we will suppress the Riemannian metric when mentioning $M$, to avoid unnecessary confusion with elements of $G$. 
Fix a left Haar measure \( dg \) on \( G \), with modular function \( \mu : G \to \mathbb{R}^+ \) given by \( d(gs) = \mu(s)dg \). We use \( d\mu \) to denote the smooth \( G \)-invariant measure on \( M \) induced by the Riemannian metric.

Let \( \pi : M \to M/G \) be the natural projection. A subset \( K \subseteq M \) is called cocompact if \( \pi(K) \) has compact closure in \( M/G \). We say that \( S \subseteq M \) is cocompactly compact if for any cocompact \( K \subseteq M \), \( S \cap K \) has compact closure in \( M \). A proper \( G \)-manifold \( M \) admits a smooth cut-off function \( c : M \to [0,1] \), with the property that for all \( x \in M \),

\[
\int_G c(g^{-1}x) \, dg = 1.
\]

Clearly, \( \text{supp}(c) \) is cocompactly compact.

Let \( \pi_1, \pi_2 : M \times M \to M \) be the projection maps onto the first and second factors. Given an operator \( A \) on \( M \) with Schwartz kernel \( k_A \), we say that \( k_A \) has compact (resp. cocompactly compact) support if there exists a compact (resp. cocompactly compact) subset \( K \subseteq M \) such that \( \text{supp}(k_A) \subseteq K \times K \).

For \( A \) a \( C^* \)-algebra, denote its positive (which we use to mean positive semi-definite) elements by \( A_+ \). For Hilbert \( A \)-modules \( \mathcal{M} \) and \( \mathcal{N} \), denote the bounded adjointable operators and compact operators \( \mathcal{M} \to \mathcal{N} \) by \( \mathcal{L}(\mathcal{M}, \mathcal{N}) \) and \( \mathcal{K}(\mathcal{M}, \mathcal{N}) \) respectively. If \( \mathcal{M} = \mathcal{N} \), we use \( \mathcal{L}(\mathcal{M}) \) and \( \mathcal{K}(\mathcal{M}) \) respectively.

### 3. Sobolev Modules

We first set up certain generalisations of Sobolev spaces that take into account the \( G \)-action. The definition is based on Kasparov’s definition of the module \( \mathcal{E} \) (see for instance [24] section 5), which is a \( C^*(G) \)-module analogue of \( L^2(E) \).

First let us recall the definition of \( \mathcal{E} \). Let \( E \) be a \( G \)-equivariant Hermitian vector bundle over a non-cocompact \( G \)-Riemannian manifold \( M \). The space of compactly supported smooth sections \( C^\infty_c(E) \) can be given a pre-Hilbert \( C^\infty_c(G) \)-module structure, with right \( C^\infty_c(G) \)-action and \( C^\infty_c(G) \)-valued inner product given by

\[
(e \cdot b)(x) = \int_G g(e)(x) \cdot b(g^{-1}) \cdot \mu(g)^{-1/2} \, dg \in C^\infty_c(E),
\]

\[
\langle e_1, e_2 \rangle(g) = \mu(g)^{-1/2} \int_M \langle e_1(x), g(e_2)(x) \rangle_E \, d\mu \in C^\infty_c(G),
\]

for \( e, e_1, e_2 \in C^\infty_c(E) \) and \( b \in C^\infty_c(G) \). Here \( G \) acts on \( C^\infty_c(E) \) by \( g(e)(x) := g(e(g^{-1}x)) \). Then \( \mathcal{E} \) is defined to be the completion of \( C^\infty_c(E) \) under the norm induced by the above inner product. The \( C^\infty_c(G) \)-action extends to a \( C^*(G) \)-action on \( \mathcal{E} \) and the inner product to a \( C^*(G) \)-valued inner product. Thus \( \mathcal{E} \) is a Hilbert \( C^*(G) \)-module.

We generalise this definition using Sobolev-type inner products defined using a Dirac-type operator, by which we mean an operator whose principal symbol is that of a Dirac operator. In particular, this includes the Dirac-plus-potential-type operators of section 5.

**Definition 3.1.** Let \( M \) be a (not necessarily cocompact) \( G \)-Riemannian manifold and \( E \) a Hermitian \( G \)-vector bundle over \( M \). Let \( B \) be a \( G \)-invariant, formally self-adjoint Dirac-type operator on \( E \) with initial domain \( C^\infty_c(E) \). For each integer \( i \geq 0 \),
let $C_{c}^{\infty}(E)$ be the pre-Hilbert $C_{c}^{\infty}(G)$-module with right $C_{c}^{\infty}(G)$-action and $C_{c}^{\infty}(G)$-valued inner product given by

$$(e \cdot b)(x) = \int_{G} g(e)(x) \cdot b(g^{-1}) \mu(g)^{-1/2} \, dg \in C_{c}^{\infty}(E),$$

for $e, e_{1}, e_{2} \in C_{c}^{\infty}(E)$ and $b \in C_{c}^{\infty}(G)$, and where we set $B^{0}$ equal to the identity operator. Positivity of these inner products is proved in Lemma 3.2. Denote by $\mathcal{E}^{i}(E)$, or simply $\mathcal{E}^{i}$, the vector space completion of $C_{c}^{\infty}(E)$ with respect to the norm induced by $\langle \, , \, \rangle_i$, and extend naturally the $C_{c}^{\infty}(G)$-action to a $C^{*}(G)$-action and $\langle \, , \, \rangle_i$ to a $C^{*}(G)$-valued inner product, to give $\mathcal{E}^{i}$ the structure of a Hilbert $C^{*}(G)$-module. Let $\| \cdot \|_i$ denote the associated norm. We call $\mathcal{E}^{i}$ the $i$-th $G$-Sobolev module with respect to $B$.

**Lemma 3.2.** For each $i \geq 0$, the $C_{c}^{\infty}(G)$-valued inner product $\langle \, , \, \rangle_i$ on the pre-Hilbert $C_{c}^{\infty}(G)$-module $C_{c}^{\infty}(E)$ is positive in $C^{*}(G)$.

**Proof.** For any $u \in C_{c}^{\infty}(E)$ we can write $\langle u, u \rangle_i = \sum_{k=0}^{i} \langle B^k u, B^k u \rangle_0 \in C_{c}^{\infty}(G) \subseteq C^{*}(G)$. Each summand is in $C^{*}(G)$, as shown in [24] section 5, and a sum of finitely many positive elements in a $C^{*}$-algebra is positive. $\square$

**Definition 3.3.** A $G$-triple $(G, M, E)$ consists of a Lie group $G$, a proper $G$-Riemannian manifold $M$ and a Hermitian $G$-vector bundle $E \rightarrow M$, together with a $G$-invariant Dirac-type operator $B$ on $E$ and the collection $\{ \mathcal{E}^{i} \}$ of $G$-Sobolev modules formed using $B$. If $E = E^{+} \oplus E^{-}$ is $\mathbb{Z}_2$-graded and $B = B^{-} \oplus B^{+}$ is an odd operator, we shall call $(G, M, E)$ a $\mathbb{Z}_2$-graded $G$-triple. If $M$ is cocompact, we shall say the triple $(G, M, E)$ is cocompact.

**Remark 3.4.** In general, the inner products and modules $\mathcal{E}^{i}$ in Definitions 3.1 and 3.3 depend on the choice of the operator $B$.

Given a $G$-triple with operator $B$, $\text{dom}(\overline{B^i})$ equipped with the graph norm is isomorphic to $\mathcal{E}^{i}$. Thus $\overline{B^i}$ is a bounded operator $\mathcal{E}^{i} \rightarrow \mathcal{E}^{0}$ for all $i \geq 0$; Proposition 3.8 implies that $\overline{B^i}$ is adjointable. Further, the results of subsection 5.2 imply that $B$ is regular and essentially self-adjoint. Therefore, except in those subsections, we will not make the distinction between $B$ and its closure $\overline{B}$. By $B^i$ we will mean the bounded adjointable operator $\overline{B^i} : \mathcal{E}^{i} \rightarrow \mathcal{E}^{0}$.

### 3.1. Boundedness and Adjointability

We now establish basic boundedness and adjointability results for $G$-Sobolev modules. When $M/G$ is compact, Kasparov ([24] Theorem 5.4) proved that an $L^{2}(E)$-bounded, $G$-invariant operator on $C_{c}(E)$ with properly supported Schwartz kernel defines a bounded adjointable operator on $\mathcal{E} = \mathcal{E}^{0}$. We now use the same method of proof to establish the following result.

**Proposition 3.5.** Let $(G, M, E)$ be a cocompact $G$-triple with operator $B$. Let $A$ be an operator on $C_{c}^{\infty}(E)$ that is $G$-invariant and bounded $H^{1}(E) \rightarrow H^{1}(E)$, where $H^{k}(E)$ denotes the completion of $C_{c}^{\infty}(E)$ with respect to the inner product $\langle B^k e_1, B^k e_2 \rangle_{L^2(E)}$, for $e_1, e_2 \in C_{c}^{\infty}(E)$. If $A$ has properly supported Schwartz kernel, then $A$ defines an
element of $\mathcal{L}(E^i, E^j)$ with norm $\leq C \cdot \|A\|_{B(H^i, H^j)}$, where $C$ is a constant that depends only on the supports of $cA^*A + A^*Ac$ and $cAA^* + AA^*c$.

**Remark 3.6.** The action of $G$ on $C_c^\infty(E)$ extends to a unitary action on each $H^i(E)$. For an operator $T$ and $g \in G$, we define

$$g(T) := gTg^{-1}.$$ 

In particular, we may speak of $G$-invariant operators $H^i(E) \to H^j(E)$.

The proof of Proposition 3.5 uses the following lemma.

**Lemma 3.7.** Let $(G, M, E)$ be a cocompact $G$-triple. Let $T$ be a bounded positive operator on $H^i(E)$ with compactly supported Schwartz kernel. Then for $e \in C_c^\infty(E)$,

$$\langle e, \left( \int_G s(T) \, ds \right)(e) \rangle_{E^i} \in C^*(G)_+.$$ 

**Proof.** The proof we give is a sketch based on [24] Lemma 5.3. Note that $T$ has a unique positive square root $T^{1/2} : H^i(E) \to H^i(E)$, so that

$$\langle s(e), T(s(e)) \rangle_{H^i(E)} = \langle T^{1/2}(s(e)), T^{1/2}(s(e)) \rangle_{H^i(E)}$$

for $e \in C_c(E)$ and $s \in G$. The function $G \to H^i(E)$ given by $s \mapsto T^{1/2}(s(e))$ has compact support in $G$. It can be shown as in [24] Lemma 5.3 that, for any unitary representation of $G$ on a Hilbert space $H$ and any $h \in H$,

$$v := \int_G \mu^{-1/2}(s) T^{1/2}(s(e)) \otimes s(h) \, ds$$

is a well-defined vector in $H^i(E) \otimes H$, and that $v$ has norm equal to

$$\int_G \int_G \mu^{-1/2}(t) \mu^{-1/2}(s) \langle T^{1/2}(s(e)), T^{1/2}(t(e)) \rangle_{H^i(E)} \cdot \langle s(h), t(h) \rangle_{H} \, ds \, dt$$

$$= \int_G \langle e, \left( \int_G s(T) \, ds \right)(e) \rangle_{E^i} (t) \cdot \langle h, t(h) \rangle_{H} \, dt.$$ 

Thus $\langle e, (\int_G s(T) \, ds)(e) \rangle_{E^i}$ is a positive operator on $H$ for all unitary representations of $G$, where we let $f \in C_c^\infty(G)$ act on $H$ by $f(h) := \int_G f(g)g(h) \, dg$. It follows that $\langle e, (\int_G s(T) \, ds)(e) \rangle_{E^i}$ is a positive element of $C^*(G)$.

**Proof of Proposition 3.5.** Let $c$ be a cut-off function on $M$. The operator $A_1 := (cA^*A + A^*Ac)/2$ is bounded on $H^i(E)$ by $\|A\|^2 \|c\|$, where $\|A\|$ is the norm of $A : H^i(E) \to H^i(E)$. Note that $A_1$ is self-adjoint with compactly supported Schwartz kernel. Let $c_1$ be a non-negative, compactly supported function on $M$, identically 1 on the support of the kernel of $A_1$. Then the operator $A_2 := c_1^2 \|A\|^2 \|c\| - A_1$ is positive, bounded and has Schwartz kernel with compact support. Now let $M_0$ be the least upper bound over all functions $c_1$ satisfying the above conditions of the function $x \mapsto \int_M g(c_1(x)) \, dg$ on $M$. Note that $M_0$ depends only on the support of $cA^*A + A^*Ac$. Since $\int_M g(A_1) \, dg = A^*A$, the previous lemma applied to $A_2$ shows that for any $e \in C_c^\infty(E)$,

$$\langle e, \left( \int_G g(A_2) \, dg \right)(e) \rangle_{E^i} = \left( \int_G g(c_1(x)) \|A\|^2 \|c\| \, dg \right) \langle e, e \rangle_{E^i} - \langle e, A^*A(e) \rangle_{E^i}.$$
is positive in $C^*(G)$. Hence $\langle A(c), A(e) \rangle_{\mathcal{E}^i} = \langle e, A^* A(e) \rangle_{\mathcal{E}^i} \leq M_0 \| A \|^2 \| e \| \langle e, e \rangle_{\mathcal{E}^i} \in C^*(G)$, and $A$ extends to an operator on all of $\mathcal{E}^i$. Similarly, $A^*: H^i(E) \to H^i(E)$ defines a bounded operator $\mathcal{E}^i(E) \to \mathcal{E}^i(E)$ that one checks is the adjoint of $A$. □

**Proposition 3.8.** Let $(G, M, E)$ be a (not necessarily cocompact) $G$-triple with $B$ as above. For $j \geq 0$, $B$ defines an element of $\mathcal{L}(\mathcal{E}^{j+1}, \mathcal{E}^j)$.

**Proof.** Boundedness is clear. Now since $G$ acts on $M$ properly, there exists a countable, locally finite open covering $\mathcal{U}$ of $M$ by $G$-stable open subsets $U_k$, $k \in \mathbb{N}$, such that for each $k$, $U_k$ is cocompact and $\overline{U_k}$ is a manifold with boundary (see [32] Lemma IV.4. for such an exhaustive sequence in the non-equivariant case). By [33] (see also [34] Theorem 5.2.5) one can find a $G$-invariant partition of unity $\{\rho_k\}$ subordinate to $\mathcal{U}$. Now one can form the modules $\mathcal{E}^i$ by first forming analogous local modules $\mathcal{E}_U^i$ on $U_k$, where $U_k$ is considered as an open $G$-submanifold of a cocompact $G$-manifold without boundary, namely the double $\overline{U_k}$ of the cocompact $G$-manifold with boundary $\overline{U_k}$. (This can be done since there exists a $G$-equivariant collar neighbourhood of $\partial \overline{U_k}$ inside $\overline{U_k}$, by Theorem 3.5 of [23].) One can then use $\{\rho_k\}$ to form the inner product on $\mathcal{E}^i$. For example, in the case of $\mathcal{E}^0$, we have

\[
\langle s, t \rangle_{\mathcal{E}^0}(g) = \mu(g)^{-1/2} \sum_k \langle \sqrt{\rho_k s}, \sqrt{\rho_k t} \rangle_{L^2(\mathcal{E}_{U_k})} \geq \sum_k \langle \sqrt{\rho_k s}, \sqrt{\rho_k t} \rangle_{\mathcal{E}^0_{U_k}}(g),
\]

for $s, t \in C_c^\infty(E)$ and $g \in G$, where we have used $G$-invariance of $\rho_k$. By Proposition 3.3 the operator $B$ restricted to sections supported on each neighbourhood $U_k$ is in $\mathcal{L}(\mathcal{E}_U^{j+1}, \mathcal{E}_U^j)$. By [24] Theorem 5.8, the local inverse $((B^2 + 1)U_k)^{-1}: \mathcal{E}^l_{U_k} \to \mathcal{E}^{l+2}_{U_k}$ exists for all $l \geq 0$. One can verify that $B_{U_k}((B^2 + 1)U_k)^{-1}$ is the adjoint of $B_{U_k}$, and that $\sum_k B_{U_k}((B^2 + 1)U_k)^{-1} \rho_k$ is the adjoint of $B$. □

**Corollary 3.9.** Let $(G, M, E)$ be a $G$-triple with $B$ as above. For $j \geq 0$, $\mathcal{E}^i$ defines an element of $\mathcal{L}(\mathcal{E}^{j+i}, \mathcal{E}^j)$.

**Proposition 3.10.** Let $(G, M, E)$ be a $G$-triple. Then multiplication by a $G$-invariant function $f: M \to \mathbb{C}$ for which $\| f \|_\infty < \infty$ is an element of $\mathcal{L}(\mathcal{E}^i, \mathcal{E}^0)$ for all $i \geq 0$.

**Proof.** Boundedness follows from

\[
\| \langle fe, fe \rangle_{\mathcal{E}^0} \|_{C^*(G)} \leq C^2 \| \langle e, e \rangle_{\mathcal{E}^0} \|_{C^*(G)} \leq C^2 \| \langle e, e \rangle_{\mathcal{E}^i} \|_{C^*(G)}.
\]

Now let $f^*: H^0(E) \to H^i(E)$ be the adjoint of $f: H^i(E) \to H^0(E)$. Since $f: H^i(E) \to H^0(E)$ is bounded and $G$-invariant, one sees that $f^*$ is $G$-invariant. For $e_1, e_2 \in C_c^\infty(E)$ and $g \in G$, we have

\[
\langle fe_1, fe_2 \rangle_{\mathcal{E}^0}(g) = \mu(g)^{-1/2} \langle fe_1, g(e_2) \rangle_{H^0(E)} = \langle e_1, f^* e_2 \rangle_{\mathcal{E}^i}(g).
\]

Hence $f^*$ is the adjoint for $f: \mathcal{E}^i \to \mathcal{E}^0$ and therefore bounded [28]. □
3.2. An Equivariant Rellich Lemma. Recall the following non-compact analogue of the Rellich lemma:

**Lemma 3.11.** Let $M$ be a non-compact manifold and $f : M \to \mathbb{C}$ a compactly supported function. Then multiplication by $f$ is a compact operator $H^s(M) \to H^t(M)$ if $s > t$.

Now suppose $(G, M, E)$ is a non-cocompact $G$-triple with operator $B$, and let $H^s(E)$ denote the completion of $C_c^\infty(E)$ with respect to the inner product $\langle B^i e_1, B^j e_2 \rangle_{L^2(E)}$, for $e_1, e_2 \in C_c^\infty(E)$.

One can verify, from the definition of rank-one operators between Hilbert modules (see [30]) that a rank-one element of $K(E^s, E^t)$ can be constructed by taking the $G$-average of a rank-one operator $H^s(E) \to H^t(E)$ in the sense of Hilbert spaces. More precisely, if $e_1$ and $e_2$ are compactly supported smooth sections of $E$, the $G$-average of the rank-one operator $\theta_{e_1 e_2} : H^s(E) \to H^t(E)$ is defined to be the operator

$$
\int_G g(\theta_{e_1 e_2}) \, dg : E^s \to E^t
$$

that takes an element $e \in C_c^\infty(E)$ to the element of $C_c^\infty(E)$ given by

$$
\left( \int_G g(\theta_{e_1 e_2}) \, dg \right) (e)(x) := \int_M \left( \int_G \theta_{g(e_1)(x), g(e_2)(y)} \, dg \right) e(y) \, d\mu
$$

$$
= \int_M \int_G g(e_1)(x) \langle gB^s e_2(y), B^t e(y) \rangle_{E_y} \, dg \, d\mu.
$$

More generally, if $\theta : H^s(E) \to H^t(E)$ is the sum of finitely many such rank-one operators, we can make sense of the $G$-average of $\theta$, denoted by

$$
\int_G g(\theta) \, dg.
$$

We now prove the following equivariant version of the Rellich lemma, which will be important for our subsequent analysis.

**Theorem 3.12.** Let $f$ be a cocompactly supported $G$-invariant function. Then multiplication by $f$ is an element of $\mathcal{K}(E^s, E^t)$ for $s > t$.

**Proof.** First observe that, by Proposition 3.10, multiplication by $f$ is an element of $\mathcal{L}(E^t, E^0)$. To see that it is compact, let $c$ be a cut-off function. By Lemma 3.11, multiplication by $cf$ is a compact operator $H^s(E) \to H^t(E)$ and hence the operator-norm limit of a sequence of finite-rank operators $(\theta_i)_{i \in \mathbb{N}}$. We may assume that the $\theta_i$ have continuous Schwartz kernels supported within a fixed compact subset $L \subseteq M \times M$ of diameter $r$. The averaged operator $\int_G g(\theta_i) \, dg$ is still bounded $H^s(E) \to H^t(E)$, has Schwartz kernel supported within an $r$-ball of the diagonal, and we have

$$
\int_G g(\theta_i) \, dg \to \int_G g(cf) \, dg = f
$$

in the operator norm on $B(H^s, H^t)$. Since both $f$ and $\int_G g(\theta_i) \, dg$ have properly and cocompactly supported Schwartz kernels, Proposition 3.5 applies and shows that the convergence also holds in the norm of $\mathcal{L}(E^t, E^t)$. \qed
4. G-invertibility at Infinity

In order to formulate index theory, we introduce a notion of invertibility at infinity [10] for the non-cocompact $G$-setting. Let $(G, M, E)$ be a $\mathbb{Z}_2$-graded non-cocompact $G$-triple with operator $B$. It follows from our previous results that $B^2 \in \mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$. Moreover, Proposition 3.10 gives:

**Lemma 4.1.** Let $f : M \to \mathbb{C}$ be a continuous $G$-invariant function for which $\|f\|_\infty < \infty$. Then $B^2 + f \in \mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$.

Next recall that an unbounded operator $A$ on a Hilbert module $\mathcal{M}$ is said to be regular if its graph is orthogonally complementable in $\mathcal{M} \oplus \mathcal{M}$. It can be shown that $A$ is both regular and self-adjoint if and only if there exists $\mu \in i\mathbb{R}$ such that both $A \pm \mu : \mathcal{M} \to \mathcal{M}$ have dense range [13].

**Definition 4.2.** Let $(G, M, E)$ be a $\mathbb{Z}_2$-graded $G$-triple equipped with a regular, self-adjoint operator $B$. Then $B : \mathcal{E}^0 \to \mathcal{E}^0$ is said to be $G$-invertible at infinity if there exists a non-negative, $G$-invariant, cocompactly supported smooth function $f$ on $M$ such that $B^2 + f \in \mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$ has an inverse $(B^2 + f)^{-1}$ in $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$.

**Remark 4.3.** When the acting group $G$ is trivial, we will use the term invertible at infinity; this is consistent with the usage in [10].

4.1. Equivariant Fredholmness. To prove that $G$-invertible-at-infinity operators have an equivariant index, we adapt Bunke’s estimates from [10]. The difference in our approach is that the Hilbert $C^*$-module structure of $\mathcal{E}$ arises from the $G$-action. Still, we find that most of the estimates in [10] carry over to our setting.

The next lemma is a $G$-equivariant analogue of Lemma 1.4 in [10].

**Lemma 4.4.**

$$d := \inf_{\psi \in \mathcal{E}^2, \|\psi\|_{\mathcal{E}^0} = 1} \left( \|B\psi\|_{\mathcal{E}^0}^2 + \|\sqrt{f}\psi\|_{\mathcal{E}^0}^2 \right) > 0.$$

**Corollary 4.5.** Let $d$ be as above and $\lambda \in \mathbb{R}$. Then

$$\|(B^2 + f + \lambda^2)\psi\|_{\mathcal{E}^0} \geq (d + \lambda^2) \|\psi\|_{\mathcal{E}^0} \quad \forall \psi \in \mathcal{E}^2.$$

Let us denote the resolvent by $R(\lambda) := (B^2 + f + \lambda^2)^{-1} : \mathcal{E}^0 \to \mathcal{E}^2$ whenever it exists. The next lemma is an analogue of Lemma 1.5 in [10]. We give a detailed proof for later reference.

**Lemma 4.6.** Suppose $B : \mathcal{E}^1 \to \mathcal{E}^0$ is $G$-invertible at infinity. Then

(a) for all $\lambda \geq 0$, $R(\lambda) \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$ exists, and

$$\|R(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq (d + \lambda^2)^{-1};$$

(b) there exists $C$ such that for all $\lambda \geq 0$,

$$\|B^2 R(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq C.$$

**Proof.** Assume that (a) is true for all $0 \leq \lambda \leq \Lambda$. That this is true for $\Lambda = 0$ follows from $G$-invertible at infinity and the corollary above. Indeed, since the inclusion $\mathcal{E}^2 \hookrightarrow \mathcal{E}^0$ is bounded adjointable, $R(\lambda) \in \mathcal{L}(\mathcal{E}^0)$ for such $\lambda$. To get the estimate,
notice that for any \( \phi \in \mathcal{E}^0 \), the above corollary with \( \lambda = 0 \) applied to the element \( R(0)\phi \in \mathcal{E}^2 \) gives
\[
\|R(0)\phi\|_{\mathcal{E}^0} \leq \frac{1}{d} \| (B^2 + f) R(0) \phi \|_{\mathcal{E}^0} = \frac{1}{d} \| \phi \|_{\mathcal{E}^0},
\]
which proves \( \|R(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq (d + \lambda^2)^{-1} \) for all \( \lambda \) in this range. With this in hand, we can show existence of \( R(\lambda) \) for \( \lambda \) in the range \( |\lambda^2 - \Lambda^2| < d + \Lambda^2 \). For such \( \lambda \) it is true that
\[
\| (\Lambda^2 - \lambda^2) R(\lambda) \|_{\mathcal{L}(\mathcal{E}^0)} \leq |\lambda^2 - \Lambda^2| \| R(\lambda) \|_{\mathcal{L}(\mathcal{E}^0)} < (d + \Lambda^2) \| R(\lambda) \|_{\mathcal{L}(\mathcal{E}^0)} \leq 1,
\]
where the final inequality follows from \( \| R(\lambda) \|_{\mathcal{L}(\mathcal{E}^0)} \leq \frac{1}{d + \Lambda^2} \). Thus the series
\[
\sum_{i=0}^{\infty} (\Lambda^2 - \lambda^2)^i R(\lambda)^i
\]
converges and defines an element of \( \mathcal{L}(\mathcal{E}^0) \) with adjoint \( \sum_{i=0}^{\infty} (\Lambda^2 - \lambda^2)^i (R(\lambda)^*)^i \). Note that we have
\[
R(\lambda) = R(\Lambda) \left( \sum_{i=0}^{\infty} (\Lambda^2 - \lambda^2)^i R(\lambda)^i \right).
\]
Thus for all \( \lambda \) such that \( |\lambda^2 - \Lambda^2| < d + \Lambda^2 \), \( R(\lambda) \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2) \) exists. We can now apply the above corollary to \( R(\lambda)\phi \in \mathcal{E}^2 \) for any \( \phi \in \mathcal{E}^0 \), which yields the desired estimate for \( \lambda \) in this interval:
\[
\|R(\lambda)\phi\|_{\mathcal{E}^0} \leq \frac{1}{d + \lambda^2} \| (B^2 + f + \lambda^2) R(\lambda) \phi \|_{\mathcal{E}^0} = \frac{1}{d + \lambda^2} \| \phi \|_{\mathcal{E}^0}.
\]
Iterating this argument countably many times, we exhaust the positive part of \( \mathbb{R} \) and get (a). (b) follows from (a) by the triangle inequality applied to
\[
B^2 (B^2 + f + \lambda^2)^{-1} = (B^2 + f + \lambda^2)(B^2 + f + \lambda^2)^{-1} - (f + \lambda^2)(B^2 + f + \lambda^2)^{-1},
\]
which shows that, for all \( \phi \in \mathcal{E}^0 \),
\[
\|B^2 R(\lambda)\phi\|_{\mathcal{E}^0} \leq \|\phi\|_{\mathcal{E}^0} + \|(f + \lambda^2) R(\lambda)\phi\|_{\mathcal{E}^0} \leq C \|\phi\|_{\mathcal{E}^0}.
\]
\( \square \)

**Remark 4.7.** The above proof also shows that \( R(\lambda) \) exists for all \( \lambda \in \mathbb{C} \) with \( \lambda^2 > -d \).

We would like to form the operator \( R(0)^{1/2} \) via functional calculus on \( R(\lambda) \in \mathcal{L}(\mathcal{E}^0) \), in order to define a bounded version of \( B \). Note that \( R(\lambda) \) is a self-adjoint element of the \( C^* \)-algebra \( \mathcal{L}(\mathcal{E}^0) \).

We have the following two estimates relating to \( B \) and \( R(\lambda) \), as equivariant analogues of Lemmas 1.6 and 1.7 of [10].

**Lemma 4.8.** We have \( BR(\lambda) \in \mathcal{L}(\mathcal{E}^0) \) and
\[
\|BR(\lambda)\|_{\mathcal{L}(\mathcal{E}^0)} \leq C (d + \lambda^2)^{-1/2}
\]
for some \( C < \infty \) independent of \( \lambda \geq 0 \), with adjoint
\[
(BR(\lambda))^* = BR(\lambda) + R(\lambda)c(df)R(\lambda),
\]
where \( c \) denotes Clifford multiplication.

**Lemma 4.9.** The commutator of \( B \) and \( R(\lambda) \) acts on \( \phi \in \mathcal{E}^1 \) by
\[
[B, R(\lambda)] \phi = -R(\lambda)c(df)R(\lambda)\phi.
\]
Lemma 4.10. The operator

\[(R(0) + \kappa)^{-1} \in \mathcal{L}(\mathcal{E}^0)\]

exists for all \(\kappa \in (-\infty, -\frac{1}{d}) \cup (0, \infty)\).

Proof. By Remark 4.7 that there exists a sufficiently small real number \(\mu\) such that \(B^2 + f + \mu i\) and \(B^2 + f - \mu i\) are both invertible. By Proposition 4.1 of [22], this means that \(B^2 + f\) is a regular self-adjoint operator with spectrum contained in \((-\infty, d]\). Upon taking the inverse, the continuous functional calculus for regular self-adjoint operators (Theorem 1.19 in [13]) implies that spectrum of \(R(0)\) is contained in \([0, 1/d]\). It follows that \((R(0) + \kappa)^{-1} \in \mathcal{L}(\mathcal{E}^0)\) exists for all \(\kappa \in (-\infty, -\frac{1}{d}) \cup (0, \infty)\). □

Definition 4.11. Suppose \((G, M, E)\) is a \(G\)-triple equipped with an operator \(B\) that is \(G\)-invertible at infinity. Then for any \(\psi \in \mathcal{E}^1\), the integral

\[\frac{2}{\pi} \int_0^\infty BR(\lambda) \psi d\lambda\]

converges in \(\mathcal{E}^0\) and defines a bounded operator \(\mathcal{E}^1 \to \mathcal{E}^0\), where elements of \(\mathcal{E}^1\) are given the \(\mathcal{E}^0\)-norm. This operator extends to an odd operator \(F \in \mathcal{L}(\mathcal{E}^0)\).

We now proceed as in [10] Lemma 1.8, with \(D\) replaced by \(B\), \(H^i\) replaced by \(\mathcal{E}^i\) and \(B(H^0)\) replaced by \(\mathcal{L}(\mathcal{E}^0)\). One verifies that

\[R(\lambda) = \frac{1}{\lambda^2} R(0) \left( R(0) + \frac{1}{\lambda^2} \right)^{-1},\]

where the inverse on the right-hand side exists by Lemma 4.10. A manipulation given in [10] Lemma 1.8 then shows that

\[\frac{2}{\pi} \int_0^\infty R(\lambda) d\lambda = R(0)^{1/2},\]

the right-hand side being defined by functional calculus in \(\mathcal{L}(\mathcal{E}^0)\). The operator \(R(0)^{1/2}B\) extends by continuity to an element \(L \in \mathcal{L}(\mathcal{E}^0)\) such that for \(\psi \in \mathcal{E}^1\),

\[F\psi = L\psi - \frac{2}{\pi} \int_0^\infty R(\lambda) c(df) R(\lambda) \psi d\lambda.\]

The continuous extension of this operator defines \(F \in \mathcal{L}(\mathcal{E}^0)\).

Proposition 4.12. The above definition of \(F\) is equivalent to

\[F\phi := \frac{2}{\pi} \int_0^\infty BR(\lambda) \phi d\lambda \quad \forall \phi \in \mathcal{E}^0.\]

Proof. Let \(\phi = \lim_{n \to \infty} \phi_n\), where \(\phi_n \in \mathcal{E}^1\). Then for all \(\psi \in \mathcal{E}^1\), we have

\[\langle \psi, F\phi \rangle_{\mathcal{E}^0} = \lim_{\psi \to \infty} \langle \psi, \frac{2}{\pi} \int_0^\infty BR(\lambda) \phi_n d\lambda \rangle_{\mathcal{E}^0} = \frac{2}{\pi} \int_0^\infty \langle R(\lambda)B\psi, \lim_{\phi_n} \rangle_{\mathcal{E}^0} d\lambda = \langle \psi, \frac{2}{\pi} \int_0^\infty BR(\lambda) \phi d\lambda \rangle_{\mathcal{E}^0}. \] □
The next result follows from the proof of Lemma 1.11 in [10] and Theorem 3.12 of the present paper.

**Proposition 4.13.** Let $B$ be $G$-invertible at infinity. Then $F^2 \sim 1$ modulo $K(E^0)$.

**Proof.** Note that by [10] Lemmas 1.8 and 1.9, $R(0)^{1/2}B$ extends by continuity to an operator $L \in \mathcal{L}(E^0)$ and that $F$ differs from $L$ by a compact operator. Furthermore, $F - F^* \in K(E^0)$. Thus it is sufficient to show that $LL^* - 1 \in K(E^0)$. For $\psi \in E^2$, we have

$$(LL^* - 1)\psi = R(0)^{1/2}B^2RR(0)^{1/2}(B^2 + f)\psi - \psi = -R(0)^{1/2}fR(0)^{1/2}\psi.$$

One calculates that $[f, R(\lambda)] = R(\lambda)(Bc(df) + c(df)B)R(\lambda) \in \mathcal{L}(E^0)$. Observing Theorem 3.12 one sees that $fR(0)^{1/2}$ differs from $R(0)^{1/2}f$ by a compact operator. □

This allows us to state our first main result:

**Theorem 4.14.** Let $(G, M, E)$ be a $\mathbb{Z}_2$-graded $G$-triple equipped with an odd operator $B$ that is $G$-invertible at infinity. Then the bounded transform of $B$, $F \in \mathcal{L}(E^0)$, is $C^*(G)$-Fredholm with an index in $K_0(C^*(G))$.

We shall write $\text{index}_G(F)$ for the $C^*(G)$-index of $F$.

4.2. **A Simplified Definition of $F$.** We now show that in fact

$$F = \frac{2}{\pi} \int_0^\infty BR(\lambda) d\lambda = BR(0)^{1/2}.$$

This will simplify certain calculations, for instance in section 7. The argument uses facts about regular operators and Bochner integration.

It follows from the proofs of Lemmas 9.1 and 9.2 in [28] that the operator $R(0)^{1/2}$ has range equal to $E^1$. Moreover, using the functional calculus for regular operators (see [27] section 7, [13] Theorem 1.19 and [28] Chapter 10), we deduce that $R(0)^{1/2} \in \mathcal{L}(E^0, E^1)$. Using these facts, we have that:

**Proposition 4.15.** Let $F \in \mathcal{L}(E^0)$ be as in the previous subsection. Then $F = BR(0)^{1/2}$.

**Proof.** Recall that bounded linear maps commute with Bochner integration ([2] Lemma 11.45). Since $F$ is the continuous extension of

$$E^1 \ni \psi \mapsto \frac{2}{\pi} \int_0^\infty BR(\lambda)\psi d\lambda \in E^0,$$

$F$ acts on a general element $\phi = \lim_{n \to \infty} \psi_n \in E^0$, where $\psi_n \in E^1$, by

$$F \phi = \lim_{n \to \infty} \frac{2}{\pi} \int_0^\infty BR(\lambda)\psi_n d\lambda.$$

Since $R(0)^{1/2}$ is bounded $E^0 \to E^1$,

$$BR(0)^{1/2} \phi = BR(0)^{1/2} \lim_{n \to \infty} \psi_n = B \lim_{n \to \infty} R(0)^{1/2} \psi_n,$$
where the second limit is taken in $\mathcal{E}^1$. This is equal to

$$\lim_{n \to \infty} BR(0)^{1/2} \psi_n = \lim_{n \to \infty} \frac{2B}{\pi} \left( \int_0^\infty R(\lambda) d\lambda \right)_0 \psi_n,$$

where the integration is performed in either $\mathcal{L}(\mathcal{E}^0)$ or $\mathcal{L}(\mathcal{E}^1)$. This equals

$$\lim_{n \to \infty} \frac{2B}{\pi} \int_0^\infty R(\lambda) \psi_n d\lambda,$$

since pairing with $\psi_n$ is a bounded linear map $\mathcal{L}(\mathcal{E}^1) \to \mathcal{E}^1$. Since the bounded operator $B : \mathcal{E}^1 \to \mathcal{E}^0$ commutes with integration in $\mathcal{E}^1$, this equals

$$BR(0)^{1/2} \phi = \frac{2}{\pi} \lim_{n \to \infty} \int_0^\infty BR(\lambda) \psi_n d\lambda. \quad \square$$

This gives another proof of:

**Corollary 4.16.** $F = BR(0)^{1/2} : \mathcal{E}^0 \to \mathcal{E}^0$ is an odd operator.

**Proof.** The functional calculus of an even operator is even, hence $R(0)^{1/2}$ is even. $F$ is the composition of the odd operator $B$ with $R(0)^{1/2}$. \qed

**Remark 4.17.** To make sense of the Bochner integrals of $R(\lambda)$ used above in the context of the non-separable Banach spaces $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$ and $\mathcal{L}(\mathcal{E}^1)$, it is necessary for the integrand to be a strongly measurable function of $\lambda \in [0, \infty)$. By Pettis’ measurability theorem (\cite{33} Theorem 1.1) and the fact that $\lambda \mapsto R(\lambda)$ is continuous, it suffices to show that the image $R(\lambda)$ for all $\lambda$ is contained in a closed separable subspace of the codomain. Indeed this follows by expanding $R(\lambda)$ as a Neumann series over countably many intervals, as shown in the following lemma, which we state for $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$, but also holds for $\mathcal{L}(\mathcal{E}^1)$.

**Lemma 4.18.** The image of the map $[0, \infty) \to \mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$, $\lambda \mapsto R(\lambda)$ lies in a separable subspace of $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^1)$.

**Proof.** Define a sequence $(\Lambda_k)_{k \in \mathbb{N}}$ by $\Lambda_k := \sqrt{(k-1)d \over 2}$, with $d$ as in Lemma 4.14. From the proof of Lemma 4.16 one sees that, for $k \geq 1$ and $\lambda \in [a_k, a_{k+1}] =: J_k$,

$$R(\lambda) = R(\Lambda_k) \left( \sum_{i=0}^{\infty} (\Lambda_k^2 - \lambda^2)^i R(\Lambda_k)^i \right).$$

Thus for $\lambda$ belonging to each of the countably many intervals $J_k$, $k \in \mathbb{N}$, the resolvent $R(\lambda)$ belongs to the closure of the span of $\mathcal{A}_k := \{R(\Lambda_k)^i : i \geq 0\}$. It follows that the closure of the span of $\bigcup_{k \in \mathbb{N}} \mathcal{A}_k$, contains $R(\lambda)$ for all $\lambda \in [0, \infty)$. \qed

4.3. $G$-invertible Operators as $KK$-elements. Given the results of the previous section, we can now complete the proof of:

**Theorem 4.19.** Let $(G, M, E)$ be a $G$-triple equipped with an operator $B$ that is $G$-invertible at infinity. Let $F$ be the bounded transform of $B$ defined using a cocompactly supported function $f$, as in Definition 4.14. Then $(\mathcal{E}^0, F)$ is a Kasparov module over the pair of $C^*$-algebras $(\mathbb{C}, C^*(G))$. The class $[\mathcal{E}^0, F] \in KK(\mathbb{C}, C^*(G))$ is independent of the choice of the function $f$. 
Proof. In view of the previous results, it remains to prove the last assertion. This follows from Theorem 3.12 and the computation in the proof of [10] Lemma 1.10. □

The image of $[E^0, F]$ under the isomorphism (see [5] 17.5.5)

$$KK(C, C^*(G)) \cong K_0(C^*(G))$$

coincides with $\text{index}_G(F)$ defined after Theorem 4.14, hence we will denote this map also by $\text{index}_G$.  

Remark 4.20. With $F$ as above, one can in fact show that $[F, h] \in K(E^0)$ for all $h \in C^*_g(M)$, where $C^*_g(M)$ is the space of continuous functions on the Higson $G$-compactification of $M$ (see section 5). Indeed, it suffices to establish this for all $h \in C^\infty(M)$. One can, for example, proceed as in [10] Lemma 1.12, replacing $C^\infty(M)$ with $C^\infty_g(M)$. However, we give a simpler proof as follows. First note that $hR(0)^{1/2}$ and $R(0)^{1/2}h$ differ by the operator

$$\frac{2}{\pi} \int_0^\infty R(\lambda)(Bc(dh) + c(dh)B)R(\lambda) d\lambda.$$

Since $h \in C^\infty_g(M)$, $\|dh\|_{S^*_M} \in C^\infty_g(M)$. Thus $dh$ can be approximated by cocompactly supported endomorphisms, for which the integral converges absolutely. On the other hand, $hB$ and $Bh$ differ by $c(dh)$, which is again a limit of cocompactly supported endomorphisms. Thus $hBR(0)^{1/2} - BR(0)^{1/2}h$ is compact.

5. $G$-Callias-type Operators

We now define, for $G$ a general Lie group, $G$-invariant Callias-type operators and prove that they are $G$-invertible at infinity. This notion generalises the Callias-type operators studied in [10] section 2 to the equivariant, non-cocompact setting.

Definition 5.1. Let $E \to M$ be a $\mathbb{Z}_2$-graded $G$-Clifford bundle with Dirac operator $D$. An odd-graded, $G$-invariant endomorphism $\Phi \in C^1(M, \text{End} E)$ is called $G$-admissible for $D$ (or simply $G$-admissible) if

(a) $\Phi D + D\Phi$ is a bounded, order-0 bundle endomorphism;  
(b) $\Phi$ is self-adjoint with respect to the inner product on $E$;  
(c) there exists a cocompact subset $K \subseteq M$ and $C > 0$ such that

$$\Phi D + D\Phi + \Phi^2 \geq C \text{ on } M \setminus K.$$

For example, let $N$ be a cocompact $G$-equivariantly spin manifold and $D$ the spin-Dirac operator. Let $\chi : \mathbb{R} \to \mathbb{R}$ be the identity function, and extend it naturally to a function $\tilde{\chi}$ on $N \times \mathbb{R}$. Then multiplication by $\tilde{\chi}$ is an endomorphism on the spinor bundle that is $G$-admissible for $D$.

In subsection 6.1 we will use an equivariant version of the Higson corona of $M$ to construct more examples of $G$-admissible endomorphisms.

Definition 5.2. Let $E$ and $D$ be as in Definition 5.1. A $G$-invariant Callias-type operator (or simply a $G$-Callias-type operator) is an operator of the form

$$B := D + \Phi,$$
where the endomorphism $\Phi$ is $G$-admissible for $D$.

We first show that such an operator is $G$-invertible at infinity and hence $C^*(G)$-Fredholm.

5.1. **Positivity of** $B^2 + f$. Let $B = D + \Phi$ be a $G$-Callias-type operator. We first show that there exists a cocompactly supported, $G$-invariant function $f$ such that $B^2 + f$ is a positive unbounded operator with respect to the $C^*(G)$-valued inner product on $\mathcal{E}^0$.

**Lemma 5.3.** Let $B = D + \Phi$ be a $G$-Callias-type operator on $E \rightarrow M$. Then there exist a $G$-invariant, cocompactly supported function $f$ and a constant $C > 0$ such that for all $s \in H^2(E)$,

$$\langle (B^2 + f)s, s \rangle_{H^0} \geq C \langle s, s \rangle_{H^0}.$$

**Proof.** Let $\pi: M \rightarrow M/G$ be the projection. The $G$-bundle $E$ descends to a topological vector bundle $\tilde{E}$ over $M/G$, while the $G$-invariant bundle map $\Phi D + D\Phi + \Phi^2$ descends to a continuous bundle map $\chi$ on $\tilde{E}$. Let $K$ be the cocompact subset in Definition 5.1 (c). Then $\Phi D + D\Phi + \Phi^2$ is bounded below by the same constant as for $\chi$ over $\pi(K)$, namely

$$\inf_{x \in \pi^{-1}(K)} \left( \inf_{v \in E_x} \left( \frac{\langle \chi v, v \rangle}{\|v\|^2} \right) \right) \geq \inf_{x \in \pi^{-1}(K)} (-\|\chi\|).$$

Adding a sufficiently large, compactly supported function $\tilde{f}: M/G \rightarrow [0, \infty)$ to $\Phi D + D\Phi + \Phi^2$ makes it positive on $K$. The result now follows by taking $f$ to be the pullback of $\tilde{f}$ to $M$, observing that on $M \setminus K$ we have $\Phi D + D\Phi + \Phi^2 \geq C$, where $C$ is the constant in Definition 5.1 (c). \qed

The above lemma and Corollary 3.9 imply that, given a $G$-Callias-type operator $B$, $B^2 + f$ defines an element of $\mathcal{L}(\mathcal{E}^2, \mathcal{E}^0)$. We now show that this operator is positive in the sense of $C^*(G)$. The proof is inspired by Kasparov’s proof of [24] Lemma 5.3.

**Proposition 5.4.** Let $(G, M, E)$ be a $\mathbb{Z}_2$-graded $G$-triple and $B$ a $G$-Callias-type operator on $E$. Then there exists a $G$-invariant cocompactly supported function $f$ and a constant $C > 0$ such that for all $s \in \mathcal{E}^2$,

$$\langle (B^2 + f)s, s \rangle_{\mathcal{E}^0} \geq C \langle s, s \rangle_{\mathcal{E}^0}.$$

**Proof.** Let $B = D + \Phi$ and $\mathfrak{c}$ be a cut-off function on $M$. Since $\langle D^2 e, e \rangle_{\mathcal{E}^0} \geq 0$ for all $e \in C_c^\infty(E)$, it suffices to show that there exist $f$ and $C > 0$ such that

$$\langle (B^2 + f - D^2)e, e \rangle_{\mathcal{E}^0} = \langle (D\Phi + \Phi D + \Phi^2 + f)e, e \rangle_{\mathcal{E}^0} \geq C \langle e, e \rangle_{\mathcal{E}^0}.$$

In other words, we want to show

$$\langle (D\Phi + \Phi D + \Phi^2 + f - C)e, e \rangle_{\mathcal{E}^0} \in C^*(G)_+. $$

Choose $f$ as in the previous lemma, so that $D\Phi + \Phi D + \Phi^2 + f - C$ is a bounded positive operator on $L^2(E)$. Since $G$ acts by unitaries on $L^2(E)$, and conjugating by unitaries preserves the functional calculus, $D\Phi + \Phi D + \Phi^2 + f - C$ has a bounded, $G$-invariant positive square root $Q$. Now the operator $QcQ$ has cocompactly compactly supported Schwartz kernel

$$k_{QcQ}(x, y) = \int_M k_Q(x, z)c(z)k_Q(z, y) \, d\mu(z),$$
since \( \iota \) has cocompactly compact support. Let \( \tilde{K} \) be a cocompactly compact subset of \( M \) such that \( \text{supp}(k_{Q\iota Q}) \subseteq \tilde{K} \times \tilde{K} \). Define \( a: G \to \mathbb{R} \) to be the function taking \( g \in G \) to:

\[
\langle Q\iota Q g(e), g(e) \rangle_{H^0(E)} = \int_M \left\langle \int_M k_{Q\iota Q}(x, y) g(e)(y) \, d\mu(y), g(e)(x) \right\rangle \, d\mu(x).
\]

Then \( \text{supp}(a) \) is contained in

\[
\{ g \in G \mid \text{supp}(g(e)) \cap \tilde{K} \neq \emptyset \} \subseteq \{ g \in G \mid \text{supp}(g(e)) \cap G \cdot \text{supp}(e) \cap \tilde{K} \neq \emptyset \}.
\]

Since the set \( G \cdot \text{supp}(e) \cap \tilde{K} \) is compact, \( \text{supp}(a) \) is a compact subset of \( G \), by properness of the \( G \)-action. Thus the map \( G \mapsto H^0(E), g \mapsto \sqrt{Q}(g(e)) \) has compact support in \( G \). It follows that for any unitary representation of \( G \) on a Hilbert space \((H, (\,, )_H)\) and \( h \in H \),

\[
v := \int_G \mu^{-1/2}(g) \sqrt{Q}(g(e)) \otimes g(h) \, dg
\]

is a well-defined vector in \( H^0(E) \otimes H \). Its norm \( \|v\|_{H^0(E) \otimes H} \) is equal to

\[
\int_G \int_G \mu^{-1/2}(g) \mu^{-1/2}(g') \langle \sqrt{Q}(g(e)), \sqrt{Q}(g'(e)) \rangle_{H^0} \cdot (g(h), g'(h))_H \, dg \, dg'.
\]

Since \( \langle \,, \rangle_{H^0(E)} \) and \( (\,, )_H \) are \( G \)-invariant (see Remark 3.6), this equals

\[
\int_G \int_G \mu^{-1/2}(g) \mu^{-1/2}(g') \langle g^{g^{-1}}(Q\iota Q)g(e), e \rangle_{H^0} \cdot (g^{g^{-1}}g(h), h)_H \, dg \, dg'.
\]

Substituting \( g' \mapsto gg' \) and following similar computations to the proof of [24] Lemma 5.3, one sees that this is equal to

\[
\int_G \left\langle \left\langle \int_G g(Q\iota Q) \, dg \right\rangle(e), e \right\rangle_{E^0} (g', (g'(h), h)_H) \, dg'.
\]

Thus \( \langle \int_G g(Q\iota Q) \, dg \rangle(e), e \rangle_{E^0} \) is a positive operator on \( H \) for all unitary representations of \( \tilde{G} \), where we let \( f \in C^\infty_c(G) \) act on \( H \) by

\[
f \cdot h := \int_G f(g) g(h) \, dg.
\]

It follows that the element

\[
\langle (D\Phi + \Phi D + \Phi^2 + f - C)e, e \rangle_{E^0} = \langle Q^2 e, e \rangle_{E^0} = \left\langle \left\langle \int_G g(Q\iota Q) \, dg \right\rangle(e), e \right\rangle_{E^0}
\]

is in \( C^*(G)_+ \), where we have used the fact that, since \( Q \) is \( G \)-invariant,

\[
\int_G g(Q\iota Q) \, dg = Q^2.
\]

Next, we show that \( B \) has a regular self-adjoint extension. In fact, we show that \( B \) is essentially self-adjoint with regular closure.
5.2. Essential Self-adjointness and Regularity of $B$. The notation in this subsection and the next will distinguish between the formally self-adjoint operator $B$ and its closure $\overline{B}$.

**Proposition 5.5.** Let $(G, M, E)$ be a $\mathbb{Z}_2$-graded $G$-triple with $M$ complete. Let $B$ be a $G$-Callias-type operator on $E$. Then $B$ is essentially self-adjoint in $\langle \cdot, \cdot \rangle_{\mathcal{E}^0}$, and $\overline{B}$ is regular.

**Proof.** When $(G, M, E)$ is cocompact with $M$ complete, the regularity and self-adjointness of $\overline{B}$ was established in [24] Theorem 5.8. Now suppose $M$ is complete and non-cocompact. There exists a family $\{a_\epsilon : \epsilon > 0\}$ of compactly supported smooth functions taking values in $[0, 1]$ satisfying:

1. $\bigcup_{\epsilon > 0} \{a_\epsilon^{-1}(1)\} = M$;
2. $\sup_{x \in M} \|da_\epsilon(x)\| \leq \epsilon$;
3. $a_\epsilon^{-1}(1) \subseteq a_\epsilon^{-1}(1)$ if $\epsilon_2 \leq \epsilon_1$.

Let $s \in C_0^\infty(E)$. Choose $\epsilon$ so that $a_\epsilon \equiv 1$ on supp$(s)$. Take an exhaustion of $M$ by cocompact, $G$-stable open subsets $\{U_k : k \in \mathbb{N}\}$ as in the proof of Proposition 3.8 and pick $i$ large enough so that $U_i$ contains supp$(a_\epsilon)$. Now form the double $M'$ of the closure $\overline{U_i}$, using a $G$-equivariant collar neighbourhood (which exists by [23] Theorem 3.5), and extend the action of $G$ to $M'$ naturally. Then $M'$ is a cocompact $G$-Riemannian manifold without boundary, and all $G$-structures on $U_i$ naturally extend to $M'$ also.

In particular, this gives a $G$-Callias-type operator $B'$, acting on $E' \rightarrow M'$. Since $M'$ is cocompact, the closure $\overline{B'}$ is regular and self-adjoint. Now use $\overline{B'}$ to form $G$-Sobolev modules $\{\mathcal{E}^{\epsilon, i}\}$ on $M'$, following section 3. Because $\overline{B'} + i : \mathcal{E}^{\epsilon, 1} \rightarrow \mathcal{E}^{\epsilon, 0}$ is onto, we can find $e \in \mathcal{E}^{\epsilon, 1}$ for which $(\overline{B'} + i)e = s$. We have $\langle s, s \rangle_{\mathcal{E}^{\epsilon, 0}} \geq \langle e, e \rangle_{\mathcal{E}^{\epsilon, 0}} \in C^*(G)_+$ and

\[
(\overline{B} + i)(a_\epsilon e) = (\overline{B'} + i)(a_\epsilon x) = [\overline{B'}, a_\epsilon]e + a_\epsilon(\overline{B'} + i)e = c(\nabla a_\epsilon)e + a_\epsilon(\overline{B'} + i)e.
\]

Note that $c(\nabla a_\epsilon)$ is a bounded operator on $\mathcal{E}^{\epsilon, 0}$, but since $a_\epsilon$ was not assumed to be $G$-invariant, it could fail to be adjointable. Nevertheless, boundedness is enough, since it enables us to conclude

\[
\| (\overline{B} + i)(a_\epsilon e) \|_{\mathcal{E}^{\epsilon, 0}} = \| c(\nabla a_\epsilon)e + a_\epsilon(\overline{B'} + i)e \|_{\mathcal{E}^{\epsilon, 0}} \leq (1 + \epsilon) \| s \|_{\mathcal{E}^{\epsilon, 0}}.
\]

Taking a sequence $\epsilon_i \rightarrow 0$, this implies that

\[
(\overline{B} + i)(a_\epsilon e) \rightarrow a_\epsilon(\overline{B'} + i)e = a_\epsilon s = s,
\]

thus $\overline{B} + i$ has dense range. \hfill $\square$

5.3. $G$-invertibility at Infinity of $B$. In this subsection we establish the key result:

**Theorem 5.6.** Let $(G, M, E)$ be a $\mathbb{Z}_2$-graded $G$-triple with $B = D + \Phi$ a $G$-Callias-type operator. Then $B$ is $G$-invertible at infinity.
**Proposition 5.7.** For $0 \neq \mu \in \mathbb{R}$, $B^2 + \mu^2 : \mathcal{E}^2 \to \mathcal{E}^0$ is bijective, with an inverse in $\mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$.

**Proof.** By part (e) of [12] 3.3, surjectivity follows from the fact that $B$ has regular closure, which was shown in the previous subsection. For all $v \in \mathcal{E}^2$, we have $\|(B^2 + \mu^2)v\|_{\mathcal{E}^0} \geq \mu^2 \|v\|_{\mathcal{E}^0}$. This implies injectivity. By the open mapping theorem, the inverse $(B^2 + \mu^2)^{-1}$ is bounded. Adjointability is guaranteed, since from general theory one knows that a bounded inverse of an invertible bounded adjointable operator $T$ between Hilbert $A$-modules $\mathcal{M}$, $\mathcal{N}$ is must be adjointable. Indeed, it follows from [6] Theorem II.7.2.9 that $\text{ran}(T^*) = \ker(T)^\perp = \{0\} = \mathcal{M}$ and $\ker(T^*) = \text{ran}(T)^\perp = F^\perp = \{0\}$. Hence $T^*$ has an inverse satisfying

$$\langle (T^{-1}y, x) \rangle_{\mathcal{M}} = \langle (T^{-1}y, T^*(T^*)^{-1}x) \rangle_{\mathcal{M}} = \langle y, (T^*)^{-1}x \rangle_{\mathcal{N}}.$$ 

Thus the inverse of $T^{-1}$ is $(T^*)^{-1}$. □

**Remark 5.8.** The formula $((B^2 + \mu^2)^{-1})^* = B^2 + (1 - \mu^2)(B^2 + \mu)^{-1}B^2 + (B^2 + \mu)^{-1}$ can be proved as in [12] Proposition 4.9.

**Proof of Theorem 5.6.** The analogue of the argument in [12] subsection 4.10 also works in our situation, thus we only sketch the argument. First one shows that for $\mathbb{R} \ni \mu \neq 0$ and $f : M \to \mathbb{R}$ a $G$-invariant uniformly bounded smooth function with $\mu^2 > \|f\|_\infty$, the operator $B^2 + \mu^2 + f$ has an inverse in $\mathcal{L}(\mathcal{E}^0) \cap \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$. This inverse is given by the Neumann series

$$(B^2 + \mu^2 + f)^{-1} = (B^2 + \mu^2)^{-1} \sum_{k=0}^{\infty} (-1)^k \{f(B^2 + \mu^2)^{-1}\}^k.$$

By Proposition 5.4, we can pick a $G$-invariant cocompactly supported function $f$ and a constant $C > 0$ such that for all $s \in \mathcal{E}^2$, $\langle (B^2 + f)s, s \rangle_{\mathcal{E}^0} \geq C \langle s, s \rangle_{\mathcal{E}^0}$. Pick $\mu \neq 0$ such that $\mu^2 > \|f\|_\infty$. We have

$$B^2 + f = (1 - \mu^2(B^2 + \mu^2 + f)^{-1})(B^2 + \mu^2 + f).$$

The Cauchy-Schwartz inequality then yields

$$\|\mu^2(B^2 + \mu^2 + f)^{-1}\|_{\mathcal{L}(\mathcal{E}^0)} \leq \frac{\mu^2}{\mu^2 + C} < 1,$$

so that the Neumann series

$$(B^2 + f)^{-1} = (B^2 + \mu^2 + f)^{-1} \sum_{k=0}^{\infty} \{\mu^2(B^2 + \mu^2 + f)^{-1}\}^k$$

converges in norm to an element of $\mathcal{L}(\mathcal{E}^0) \cap \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2)$. □
6. The Endomorphism $\Phi$ and the Higson Corona

We now construct $G$-admissible endomorphisms $\Phi$ using the $K$-theory of an equivariant version of the Higson corona used in [10].

Let $C^G_b(M)$ denote the algebra of bounded, continuous $G$-invariant functions on $M$. Let $C^\infty_g(M) \subseteq C^G_b(M)$ denote the subalgebra of smooth functions $f$ such that for all $\epsilon > 0$, there exists a cocompact subset $M_0 \subseteq M$ such that

$$\|df\|_{T^*M} < \epsilon$$

on $M \setminus M_0$. Let $C^G_0(M)$ be the algebra of $G$-invariant functions $f$ on $M$ such that for all $\epsilon > 0$ there exists a cocompact subset $M_1 \subseteq M$ such that $|f| < \epsilon$ on $M \setminus M_1$. Thus $C^\infty_g(M)$ consists of those bounded smooth $G$-invariant functions $f$ for which $\|df\|_{T^*M} \in C^G_0(M)$.

Remark 6.1. In [10], the notation $C_g(M)$ is used for the closure in $\| \cdot \|_\infty$ of the smooth functions $f$ on $M$ for which $\|df\|_{T^*M} \in C_0(M)$.

Definition 6.2. Let the algebra $C^G_g(M)$ be the completion of $C^\infty_g(M)$ in $C^G_b(M)$ with respect to $\| \cdot \|_\infty$. Define

$$C(\partial^G_h M) = C^G_g(M)/C^G_0(M).$$

The maximal ideal space $\overline{M}^G$ of $C^G_g(M)$ is called the Higson $G$-compactification of $M$. The maximal ideal space $\partial^G_h M$ of $C(\partial^G_h M)$ is called the Higson $G$-corona of $M$.

The Higson $G$-compactification for $G = \{e\}$ was studied in [10]. Note that there, the notation $\partial_h M$ was used where we have used $\partial^G_h M$.

Lemma 6.3. $\overline{M}^G$ is a compactification of $M/G$.

Proof. $C^G_g(M)$ contains $C^G_0(M) \cong C_0(M/G)$, which is the closure of the space of $G$-invariant cocompactly supported functions on $M$ under $\| \cdot \|_\infty$. Thus $C^G_g(M)$ separates points of $M/G$ from closed subsets and contains the constant functions. Thus $\overline{M}^G$ is a compactification of $M/G$.

We will show that the $K$-theory of the Higson $G$-corona is highly non-trivial, which provides motivation for the study of $G$-Callias-type index theory. First, we give a construction of $G$-admissible endomorphisms using the $K$-theory of $\overline{M}^G$.

6.1. Constructing $\Phi$. Suppose first that $M$ is an odd-dimensional complete non-cocompact $G$-Riemannian manifold. Let $E_0 \to M$ be a complex Clifford bundle with an ungraded Dirac operator $D_{E_0}$. Let $P_0$ be a projection in $\text{Mat}_i(C(\partial^G_h M))$ for some $i \geq 0$. Then $P_0$ lifts to an element $P \in \text{Mat}_i(C^\infty_g(M))$ such that $P = P^*$ and $P^2 - P \in \text{Mat}_i(C^0_g(M))$. Form the $\mathbb{Z}_2$-graded Clifford bundle $E := E_0 \otimes (\mathbb{C}^i \oplus (\mathbb{C}^i)^{op})$, with the associated Dirac operator

$$D := \begin{bmatrix} 0 & D_{E_0} \otimes 1 \\ D_{E_0} \otimes 1 & 0 \end{bmatrix},$$
where $D_{E_0}$ denotes $D_{E_0}$ twisted by the trivial connection $d^0$ on $\mathbb{C}^l \to M$. Define the endomorphism

$$\Phi := i \otimes \begin{bmatrix} 0 & 1 - 2P \\ 2P - 1 & 0 \end{bmatrix} \in C^1(M, \text{End}(E)).$$

**Proposition 6.4.** $\Phi$ is a $G$-admissible endomorphism.

**Proof.** Clearly $\Phi$ is odd-graded and self-adjoint. Let $\nabla^{E_0}$ denote the Clifford connection on $E_0$. Then the twisted connection on $E_0 \otimes \mathbb{C}^l$ is $\nabla^{E_0} \otimes 1 + 1 \otimes d^0$. Since $1 - 2P$ acts only on the factor $\mathbb{C}^l$, it commutes with $\nabla^{E_0} \otimes 1$. Also, $[1 \otimes d^0, 1 \otimes (1 - 2P)] = d^2 P$, where $d^2$ is the trivial connection on $\text{End}(\mathbb{C}^l)$. Thus we have

$$D\Phi + \Phi D = 2i \otimes \begin{bmatrix} -c(d^2 P) & 0 \\ 0 & c(d^2 P) \end{bmatrix}.$$

By definition of $C^G_{g,\infty}(M)$, $c(d^2 P) \to 0$ and $\Phi^2 \to 1$ at infinity in the direction transverse to the $G$-orbits. Thus there exists a cocompact subset $K \subseteq M$ such that

$$\text{on } M \setminus K, \quad D\Phi + \Phi D + \Phi^2 \geq c \text{ for some constant } c > 0. \quad \Box$$

By Theorem 5.6, the operator $B := D + \Phi$ is $G$-invertible at infinity. Thus $B$ has an index in $K_0(C^*(G))$ by Theorem 4.14. Let $F$ be the bounded transform of $B$. By the same reasoning as in [10] section 2, $\text{index}_G(F)$ depends only on the class of projection $P_0$ in $K_0(C(\partial^G_h(M)))$.

Note that the endomorphism $\Phi$ arising from the zero element of $K_0(C(\partial^G_h(M)))$ gives rise to the invertible operator

$$B = \begin{bmatrix} 0 & D + i \\ D - i & 0 \end{bmatrix},$$

for which $\text{index}_G(BR(0)^{1/2}) = 0 \in K_0(C^*(G))$.

Suppose instead that $M$ is an even-dimensional, complete non-cocompact $G$-Riemannian manifold. Let $E_0 \to M$ be a $\mathbb{Z}_2$-graded Clifford bundle with grading $z$. An element $[U_0] \in K_1(C(\partial^G_h(M)))$ is represented by a unitary $U_0 \in \text{Mat}_l(C(\partial^G_h(M)))$ for some $l \geq 0$. Let $U$ be a lift of $U_0$ to $\text{Mat}_l(C^\infty G(M))$. Form the $\mathbb{Z}_2$-graded bundle $E := E_0 \otimes (\mathbb{C}^l \oplus (\mathbb{C}^l)_{op})$ with Dirac operator $D$ formed from the connection on $E_0$ twisted by $d^0$. Let

$$\Phi := z \otimes \begin{bmatrix} 0 & U^* \\ U & 0 \end{bmatrix} \in C^1(M, \text{End}(E)),$$

and define $B = D + \Phi$.

**Proposition 6.5.** $\Phi$ is a $G$-admissible endomorphism.

**Proof.** Clearly $\Phi$ is odd-graded and self-adjoint. Similarly to the calculation in the previous proposition, one has

$$D\Phi + \Phi D = z \begin{bmatrix} 0 & -c(d^2 U^*) \\ -c(d^2 U) & 0 \end{bmatrix}.$$

By definition of $C^G_{g,\infty}(M)$, $c(d^2(U)) \to 0$ and $\Phi^2 \to 1$ at infinity in the direction transverse to the $G$-orbits. It follows that there is some cocompact subset $K \subseteq M$ such that on $M \setminus K$, $D\Phi + \Phi D + \Phi^2 \geq c > 0$ for some $c$. \quad \Box
As in the odd-dimensional case, one can verify that the index\(_G(F)\), where \(F\) is the bounded transform of \(B\), depends only on \([U_0] \in K_i(C(\partial^2 U_0))\).

For \(\dim M \equiv i \pmod{2} , i = 0, 1\), the map
\[
K_i(C(\partial^2 U_0)) \to K_0(C^*(G)),
\]

\([R] \mapsto \text{index}_G(F_R)\),

is a homomorphism of abelian groups. Here \(R\) denotes a projection or unitary matrix representative of \(K_0\) or \(K_1\), depending on \(\dim M\), and \(F_R\) denotes the bounded transform of \(B = D + \Phi R\), where \(\Phi R\) is formed using \(R\).

6.2. **Interpretation as a \(KK\)-product.** Let \(F\) be the bounded transform of a \(G\)-Callias-type operator \(B = D + \Phi\). In this subsection we interpret the cycle \([\mathcal{E}^0, F]\) in terms of a \(KK\)-pairing, similar to those constructed in [10] and [26].

Suppose first that \(M\) is even-dimensional, with a Dirac operator \(D_0\) acting on a \(\mathbb{Z}_2\)-graded \(G\)-Clifford bundle \(E_0 \to M\). Given a \(G\)-admissible \(\Phi\) of the kind constructed in the previous subsection, one can form the bundle \(E\) and the operator \(D\). Let \(B = D + \Phi\). Using the procedure in section 3, form the \(G\)-Sobolev modules \(\mathcal{E}_{D_0}^0\) and \(\mathcal{E}'\) associated to \(D_0\) and \(B\) respectively. Note that \(\mathcal{E}^0 = \mathcal{E}_{D_0}^0 \otimes \mathbb{C}^2\).

Let \(R_{D_0}(0) := (D_0^2 + 1)^{-1}\). Then by the same kind of analysis as in section 4, one knows that \(R_{D_0}(0)^{1/2}\) is a bounded adjointable operator \(\mathcal{E}_{D_0}^0 \to \mathcal{E}_{D_0}^0\). Define its bounded transform \(F' := D_0 R_{D_0}(0)^{1/2}\).

**Proposition 6.6.** The pair \((F', \mathcal{E}_{D_0}^0)\) defines a cycle
\[
[D_0] := [F', \mathcal{E}_{D_0}^0] \in KK(C_G^0(M), C^*(G)).
\]

**Proof.** Each \(a \in C_G^0(M)\) defines a bounded adjointable operator \(\mathcal{E}_{D_0}^0 \to \mathcal{E}_{D_0}^0\). Let \(C_G^0(M) \subseteq C_0(M)\) be the subring of \(G\)-invariant, cocompactly supported functions on \(M\), with closure \(C_G^0(M)\) (see subsection 4.3). It suffices to show that for all \(a \in C_G^0(M)\), we have
\[
a((F')^2 - 1) \in \mathcal{K}(\mathcal{E}_{D_0}^0), \quad F'a - aF' \in \mathcal{K}(\mathcal{E}_{D_0}^0).
\]

The second relation follows from the same sort of argument used to prove Theorem 4.19 applied to \(F'\) instead of \(F\). By functional calculus of the regular operator \(D_0\) one has
\[
(F')^2 - 1 = -R_{D_0}(0) \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2).
\]

Hence by Theorem 3.12 \(a((F')^2 - 1)\) is a compact operator \(\mathcal{E}_{D_0}^0 \to \mathcal{E}_{D_0}^0\). \(\square\)

From here, our construction is similar to [10] 2.3.3 and 2.4.3 for the non-equivariant case, so we will be rather brief. The differences are: instead of the algebra \(C_2(M)\) used in [10], we use \(\mathbb{C}\); instead of \(C_0(M)\) we use \(C_G^0(M)\); and the \(\mathbb{C}\)-Fredholm cycle \([h, F_2]\) used there is replaced by the \(C^*(G)\)-Fredholm cycle \([D] = [\mathcal{E}_{D_0}^0, F']\).

Define the \(\mathbb{Z}_2\)-graded Hilbert \(C_G^0(M)\)-module
\[
L := \mathcal{E}_{D_0}^0 \otimes \mathbb{C}^l \otimes \mathbb{C}^2,
\]
and consider the cycle \([\Phi] := [L, \Phi] \in KK(\mathbb{C}, C_G^0(M))\). We have:

**Proposition 6.7.** Let \(E, E_0, D, D_0, B\) be as before, with \(\dim M\) even. Then
\[
[\mathcal{E}^0, F] = [\Phi] \otimes_{C_G^0(M)} [D_0] \in KK(\mathbb{C}, C^*(G)).
\]
Next, we consider odd-dimensional $M$, with the initial Dirac operator $D_0$ being ungraded. Let $C^{1,0}$ be the Clifford algebra generated by a single element $X$ satisfying the relation $X^2 = -1$. Let $z$ be the grading of $C^{1,0}$, and form the operator

$$D = D_0 \otimes zX,$$

which defines a class $[D] \in KK(C^G_0(M) \otimes C^{1,0}, C^*(G))$. Consider

$$L := C^G_0(M) \otimes \mathbb{C}^l \otimes C^{1,0}$$

as a $\mathbb{Z}_2$-graded Hilbert $C^G_0(M) \otimes C^{1,0}$-module. Write

$$\Phi = i(1 - 2P) \otimes X,$$

which defines a bounded adjointable operator on $L$. One verifies as in [10] 2.3.3 that the pair $(L, \Phi)$ defines a Kasparov module

$$[\Phi] := [L, \Phi] \in KK(\mathbb{C}, C^G_0(M) \otimes C^{1,0}).$$

Let $\tau_{C^{1,0}}$ be the isomorphism

$$KK(\mathbb{C}, C^*(G)) \cong KK(\mathbb{C} \otimes C^{1,0}, C^*(G) \otimes C^{1,0})$$

defined in [5] 17.8.5. Then we have:

**Proposition 6.8.** Let $E, D, B$ be as before, with $\dim M$ odd. Then

$$[\mathcal{E}^0, F] = \tau_{C^{1,0}}^{-1}([\Phi] \otimes_{C_0(M)} [D]) \in KK(\mathbb{C}, C^*(G)).$$

### 6.3. The Higson $G$-corona

We now turn to the $K$-theory of the Higson $G$-corona $\overline{M}^G$. We are motivated by the work of Keesling [25] on $K$-theory of the Higson compactification of metric spaces. Recall that for $X$ a metric space and $\phi$ a function on $X$, the map $V_r(\phi) : M \to \mathbb{R}^+$ defined by

$$V_r(\phi)(x) = \sup\{|\phi(y) - \phi(x)| : y \in B_r(x)\}$$

is called the variation of $\phi$ at scale $r$. Define $C_h(X)$ to be the $C^*$-algebra of all bounded continuous functions $\phi$ on $X$ such that, for every $r$, $V_r(\phi) \to 0$ at infinity. The maximal ideal space $\overline{X}^d$ of $C_h(X)$ is the Higson compactification of $X$. In [25], it is proved that:

**Theorem 6.9.** Let $(X, d)$ be a non-compact connected metric space with proper metric $d$. Then $\check{H}^1(\overline{X}^d)$ contains a subgroup isomorphic to $(\mathbb{R}, +)$, where $\check{H}^1$ denotes the first Čech cohomology group.

**Corollary 6.10.** Let $(X, d)$ be as in the above proposition. Then $K^1(\overline{X}^d)$ is uncountable.

**Proof.** The Chern character gives an isomorphism

$$ch : K^1(\overline{X}^d) \otimes \mathbb{Q} \cong \check{H}^{odd}(\overline{X}^d, \mathbb{Q}) \cong \check{H}^{odd}(\overline{X}^d) \otimes \mathbb{Q}.$$ 

By the previous theorem, $\check{H}^1(X)^d \otimes \mathbb{Q}$, and hence $K^1(\overline{X}^d) \otimes \mathbb{Q}$, is uncountable. $\square$

Let us return to the setting of the smooth manifold $M$ and the Higson $G$-compactification $\overline{M}^G$. We can characterise $\overline{M}^G$ using functions with values in a compact submanifold $Y \subseteq \mathbb{R}^N$ for some $N$. Recall that $f \in C_g^{G, \infty}(M)$ if $f$ is bounded, smooth and $G$-invariant, and $\|df\|_{T^*M} \in C_0^G(M)$.
Definition 6.11. Let $M$ be a non-cocompact $G$-Riemannian manifold and $Y$ a compact submanifold of $\mathbb{R}^N$ for some $N \geq 0$. Let $\pi_i: \mathbb{R}^N \to \mathbb{R}$ be the projection map onto the $i$-th coordinate. Suppose $f: M \to Y$ is a $G$-invariant function. Then we say that $f \in C^G_g(M,Y)$ if $\pi_i \circ f \in C^G_g(M)$ for each $1 \leq i \leq N$.

The proofs of the next two propositions are adapted from the work of Keesling [25].

Proposition 6.12. Let $M$ be a complete non-cocompact $G$-Riemannian manifold. Let $Y \subseteq \mathbb{R}^N$ be a compact submanifold for some $N \geq 0$ and $f: M/G \to Y$ continuous. Then $f$ has a continuous extension to the Higson $G$-compactification $\overline{M}^G$ if and only if $f \in C^G_g(M,Y)$. Further, $\overline{M}^G$ is the unique such compactification of $M/G$.

Proof. Let $f \in C^G_g(M,Y)$. Without loss of generality, since $Y$ is compact, we may take $Y$ to be a submanifold of $[0,1]^N$. For each $j$, let $\pi_j: [0,1]^N \to [0,1]$ be the projection onto the $j$-th coordinate. Then $\pi_j \circ f \in C^G_g(M)$ and hence can be extended to a continuous function $\overline{\pi_j} \circ f$ on $\overline{M}^G$. The function $\overline{f}: \overline{M}^G \to [0,1]^N$, $x \mapsto (\overline{\pi_1}(x),\ldots,\overline{\pi_N}(x))$ is a continuous extension of $f$ to $\overline{M}^G$, taking values in $Y \subseteq [0,1]^N$. On the other hand, suppose $f: M \to Y$ is a continuous map not in $C^G_g(M,Y)$. Then there exists some $j$ for which $\pi_j \circ f \notin C^G_g(M)$. Thus $\pi_j \circ f$ does not extend continuously to $\overline{M}^G$. It follows that $f$ does not extend continuously to $\overline{M}^G$, for otherwise $\pi_j \circ f$ would also.

For uniqueness, suppose $\mathcal{C}$ is a compactification of $M/G$ with the above properties. Then the set of continuous functions $M/G \to \mathcal{C}$ that extend to $\mathcal{C}$ is the closed subring $C^G_g(M) \subseteq C^G_g(M)$. Taking maximal ideals of $C^G_g(M)$ with the convex-hull topology recovers $\overline{M}^G$. \qed

We apply this proposition to $Y = S^1 \subseteq \mathbb{R}^2$. The first Čech cohomology $\check{H}^1(\overline{M}^G)$ can be identified with the group $[\overline{M}^G,S^1]$ of homotopy classes of maps $\overline{M}^G \to S^1$, with a certain group operation derived from pointwise multiplication of functions. Let $e: \mathbb{R} \to S^1$ be the covering map $x \mapsto e^{2\pi i x} \in S^1 \subseteq \mathbb{C}$. Let the algebra $C^G_{g,u}(M)$ be defined by the same conditions as $C^G_g(M)$ except that the functions need not be bounded.

For clarity, let us denote $C^G_{g,b}(M) := C^G_g(M)$, and let $C^G_{g,b}(M,\mathbb{R})$ and $C^G_{g,u}(M,\mathbb{R})$ be the real-valued functions in $C^G_{g,b}(M)$ and $C^G_{g,u}(M)$.

If $f \in C^G_{g,u}(M,\mathbb{R})$, it is easy to see that $e \circ f \in C^G_g(M,S^1)$, and so by the above proposition, $f$ has an extension $e \circ f: \overline{M}^G \to S^1$. Define the map $a: C^G_{g,u}(M) \to \check{H}^1(\overline{M}^G)$ by

$$f \mapsto [e \circ \overline{f}] \in [\overline{M}^G,S^1] \equiv \check{H}^1(\overline{M}^G).$$

Proposition 6.13. There is an exact sequence of abelian groups

$$0 \to C^G_{g,b}(M,\mathbb{R}) \hookrightarrow C^G_{g,u}(M,\mathbb{R}) \overset{a}{\to} \check{H}^1(\overline{M}^G).$$

Proof. Clearly the sequence is exact at $C^G_{g,b}(M,\mathbb{R})$. To show that $C^G_{g,b}(M,\mathbb{R}) \subseteq \ker(a)$, let $f \in C^G_{g,b}(M,\mathbb{R})$. Then $f$ has an extension $\overline{f}: \overline{M}^G \to \mathbb{R}$. Since $\overline{f}$ is a lift for $e \circ \overline{f}$, the latter map is null-homotopic. Next, to show that $\ker(a) \subseteq C^G_{g,b}(M,\mathbb{R})$,
suppose \( f \in C^a_g(M, \mathbb{R}) \) and \( a(f) = 0 \in \overline{M}^G \). This implies that \( a(f) = [\delta \circ g] \) is null-homotopic and hence \( e \circ f \) has a lift \( q: \overline{M}^G \to \mathbb{R} \). Since both \( q \mid_{M/G} \) and \( f \) are lifts for \( e \circ f \mid_{M/G} \), they must be related by a deck transformation. Since \( q \) is bounded, \( f \) must be bounded, and \( f \in C^G_b(M, \mathbb{R}) \). Hence \( \ker(a) = C^G_b(M, \mathbb{R}) \). 

Corollary 6.14. \( \check{H}^1(M^G) \) contains a subgroup isomorphic to \( C^G_a(M, \mathbb{R})/C^G_b(M, \mathbb{R}) \).

We now show that \( C^G_a(M, \mathbb{R})/C^G_b(M, \mathbb{R}) \) is uncountable. Since the action of \( G \) on \( M \) is proper and isometric, \( M/G \) has a natural distance function \( d_{M/G} \) given by the minimal geodesic distance between orbits (see [1] p. 74). \( d_{M/G} \) lifts to a function \( d_M \) on \( M \) that is well-defined on pairs of orbits. For a fixed point \( x_0 \in M \), the function

\[
d_{x_0}: M \to \mathbb{R}, \quad x \mapsto d_M(x_0, x)
\]

is \( G \)-invariant and 1-Lipschitz. We now show that \( d_{x_0} \) can be approximated by a smooth \( G \)-invariant Lipschitz function.

Proposition 6.15. For any \( \epsilon > 0 \), there is a smooth \( G \)-invariant function \( d_{x_0, \epsilon} \) on \( M \) such that for any \( x \in M \),

\[
|d_{x_0, \epsilon}(x) - d_{x_0}(x)| < \epsilon, \quad \|d(d_{x_0, \epsilon})\|_{T^*M} \leq 1 + \epsilon.
\]

Proof. Since \( M \) is a complete Riemannian manifold, it is a proper metric space and hence separable. By [1], for any function \( \delta : M \to (0, \infty) \) and \( r > 0 \), one can construct a smooth approximation \( f_{x_0} \) such that for all \( x \in M \),

\[
|f_{x_0}(x) - d_{x_0}(x)| < \delta(x), \quad \|df_{x_0}(x)\|_{T^*M} < 1 + r.
\]

Since the \( G \)-action is proper, we can find a cut-off function \( \epsilon \) on \( M \). Let \( \tilde{f}_{x_0} \) be the \( G \)-average of \( f_{x_0} \), defined by \( \tilde{f}_{x_0}(x) := \int_G \epsilon(g^{-1}x) f_{x_0}(g^{-1}x) \, dg \). Clearly \( \tilde{f}_{x_0} \) is \( G \)-invariant.

We now argue that \( \tilde{f}_{x_0} \) is Lipschitz. Observe that \( \|df_{x_0}\|_{T^*M} \) is equal to

\[
\left\| \int_G (d\epsilon(g^{-1}x) f_{x_0}(g^{-1}x) + \epsilon(g^{-1}x)df_{x_0}(g^{-1}x) \, dg \right\|_{T^*M}
\]

\[
\leq \left\| \int_G d\epsilon(g^{-1}x) (d(g^{-1}x) + \delta_x) \, dg \right\|_{T^*M} + \int_G \epsilon(g^{-1}x) \|df_{x_0}(g^{-1}x)\|_{T^*M} \, dg,
\]

where \( \delta_x := f_{x_0}(g^{-1}x) - d(g^{-1}x) \). Suppose we choose \( \delta_x \leq \delta \) uniformly for some constant \( \delta > 0 \). Then the above expression is bounded by

\[
\left\| \int_G (d\epsilon(g^{-1}x)) d(g^{-1}x) \, dg \right\|_{T^*M} + \delta \int_G \|d\epsilon(g^{-1}x)\|_{T^*M} \, dg
\]

\[+ \int_G \epsilon(g^{-1}x) \|df_{x_0}(g^{-1}x)\|_{T^*M} \, dg.\]

The first summand vanishes because \( d \) is \( G \)-invariant, since for any \( G \)-invariant function \( l: M \to \mathbb{R} \), we have

\[
\int_G d\epsilon(g^{-1}x) l(g^{-1}x) \, dg = l(x) \int_G d\epsilon(g^{-1}x) \, dg = l(x) d(1) = 0.
\]
Proof. Fix \( r \) for each \( r \). Let \( \delta \) be an upper-bound for \( \delta \) on \( U \). For \( x \in U \),
\[
\delta_U \int_G \| dc(g^{-1}x) \|_{T^* M} \, dg \leq \delta_U \left( \sup_{x \in U} \| dc(x) \| \right) \left( \sup_{x \in U} \{ \text{vol} ( \text{supp} \, c(g^{-1}x) ) \} \right). 
\]
The function \( x \mapsto dc(x) \) has cocompactly compact support and hence is bounded above on \( U \). The function \( x \mapsto \text{vol} ( \text{supp} \, c(g^{-1}x) ) \) is \( G \)-invariant and so descends to a compact set in \( M/G \), whence it is also bounded above on \( U \). The above product is bounded by \( \delta_U C_U \), where the constant \( C_U > 0 \) depends only on the cocompact set \( U \). The third summand above is bounded by \( 1 + r \), given our choice of \( f_{x_0} \). Thus the whole expression is strictly less than \( 1 + \delta_U C_U + r \). By picking \( r < \frac{\epsilon}{2} \) and \( \delta_U \) such that \( \delta_U C_U < \frac{\epsilon}{2} \), this expression can be made to be less than \( 1 + \epsilon \). By further choosing \( \delta_U \leq \epsilon \), we have that for \( x \in U \),
\[
\left| \tilde{f}_{x_0}(x) - d_{x_0}(x) \right| = \int_G c(g^{-1}x) \left| f_{x_0}(g^{-1}x) - d_{x_0}(g^{-1}x) \right| \, dg \leq \delta_U \leq \epsilon.
\]
To obtain the estimate on all of \( M \), let \( U = \{ U_i : i \in \mathbb{N} \} \) be a locally finite, countable open cover of \( M \) by \( G \)-stable cocompact sets. There exists a smooth partition of unity subordinate to \( U \) consisting of \( G \)-invariant functions \( \psi_U \). Then
\[
\delta(x) := \sum_{i=1}^{\infty} \psi_i(x) \delta_{U_i}(x)
\]
is a well-defined smooth function \( M \to \mathbb{R} \), so we may choose the approximation \( f_{x_0} \) so that for all \( x \in M \), \( |f_{x_0} - d_{x_0}| < \delta(x) \). For each \( x \in M \), let
\[
C_x := \max \{ C_i : x \in U_i, \, i \in \mathbb{N} \}.
\]
Then it holds that for all \( x \in M \), \( \delta(x) C_x < \frac{\epsilon}{2} \). By our previous calculations, for all \( x \in M \) we have
\[
\left| \tilde{f}_{x_0}(x) - d_{x_0}(x) \right| \leq \epsilon,
\]
and \( \left\| d \tilde{f}_{x_0} \right\|_{T^* M} \leq 1 + \epsilon \). Finally, set \( d_{x_0, \epsilon} := \tilde{f}_{x_0} \), and we conclude. \( \square \)

Remark 6.16. This argument applies more generally to produce a \( G \)-invariant smooth approximation \( \tilde{f} \) starting from a \( G \)-invariant Lipschitz function \( f \).

Proposition 6.17. Let \( M \) be a non-cocompact \( G \)-Riemannian manifold. Then \( \hat{H}^1(M^G) \) contains a subgroup isomorphic to \( (\mathbb{R}, +) \).

Proof. Fix \( x_0 \in M \). Define \( d_M, d_{M/G}, d_{x_0} \) and \( d_{x_0, \epsilon} \) as in the proof of the previous proposition, so that for \( y \in M \),
\[
|d_{x_0, \epsilon}(y) - d_{x_0}(y)| < \epsilon, \quad \| d(d_{x_0, \epsilon})(y) \|_{T^* M} < 2.
\]
For each \( r \in \mathbb{R} \), consider the function \( \rho_r \in C^G_c(M) \) defined by
\[
\rho_r : M \to \mathbb{R}, \quad y \mapsto r \ln d_{x_0, \epsilon}(y).
\]
If \( r \neq s \), \( \rho_r - \rho_s \) is unbounded. Hence the subgroup
\[
\{ [\rho_r] : r \in \mathbb{R} \} \subseteq C^G_c(M, \mathbb{R})/C^G_c(M, \mathbb{R})
\]
is uncountable; thus $C^G_a(M, \mathbb{R})/C^G_b(M, \mathbb{R})$ has rank at least $2^{\aleph_0}$. Now $(M, d_M)$ is a proper non-compact metric space, hence separable. Thus $(M/G, d_{M/G})$ is separable. Thus the rank of $C^G_a(M, \mathbb{R})$, and hence that of

$$C^G_a(M, \mathbb{R})/C^G_b(M, \mathbb{R}),$$

is at most $2^{\aleph_0}$. Since $C^G_a(M, \mathbb{R})/C^G_b(M, \mathbb{R})$ is an abelian, divisible and torsion-free group, it is isomorphic to $(\mathbb{R}, +)$.

**Corollary 6.18.** Let $M$ be a complete non-cocompact $G$-Riemannian manifold. Then both $K_1(C^G_a(M))$ and $K_1(C(\partial^G_h(M)))$ are uncountable.

**Proof.** The first statement follows from the Chern character isomorphism

$$ch: K^1(M^G) \otimes \mathbb{Q} \cong H^1(M^G, \mathbb{Q}) \cong H^1(M^G) \otimes \mathbb{Q}$$

together with the above proposition, noting that $K_1(C^G_a(M)) \cong K^1(M^G)$. The second statement follows from the first by the Five Lemma. □

This shows that when $M$ is a complete non-cocompact $G$-Riemannian manifold, the group $K_1(C(\partial^G_h(M)))$ is uncountable. On the other hand $K_0(C^G_a(M))) \cong K^0(M^G)$ contains a copy of $\mathbb{Z}$, since $M^G$ is a compact space, so that $K^0(\partial^G_h(M))$ is also infinite.

In summary:

**Theorem 6.19.** Let $M$ be a complete non-cocompact $G$-Riemannian manifold. Then the $K$-theory of the Higson $G$-corona of $M$ is uncountable.

7. **Invariant Metrics of Positive Scalar Curvature**

Let $M$ be a $G$-spin manifold with spinor bundle $S$ and Dirac operator $\partial_0$. Let $\Phi$ be a $G$-admissible endomorphism, and form the $G$-Callias-type operator $B = \partial + \Phi$, where $\partial$ is a $\mathbb{Z}_2$-graded version of $\partial_0$ acting on the $\mathbb{Z}_2$-graded bundle $E$ constructed from $S = E_0$, in the notation of section 5. Form the $\mathbb{Z}_2$-graded $G$-Sobolev modules $\mathcal{E}^i = \mathcal{E}^i(E)$ as in section 3. Then by Proposition 3.8, $B$ extends to a bounded adjointable operator $\mathcal{E}^1 \to \mathcal{E}^0$. For $\lambda \in \mathbb{R}$, let

$$R(\lambda) = (B^2 + f + \lambda^2)^{-1},$$

which exists by Lemma 4.6. Normalising $B$ gives rise to the operator $F := BR(0)^{1/2} \in \mathcal{L}(\mathcal{E}^0)$, by subsection 4.2.

Denote the $G$-invariant Riemannian metric on $M$ by $g$, not to be confused with elements of the group $G$. Suppose the scalar curvature $\kappa^g$ associated to $g$ is everywhere positive. By Lichnerowicz’s formula,

$$B^2 = \partial^2 + \partial \Phi + \Phi \partial + \Phi^2 = \nabla^* \nabla + \frac{\kappa^g}{4} + \partial \Phi + \Phi \partial + \Phi^2,$$

where $\nabla$ is the connection on $E$. We now show that $g$ can be scaled by an appropriate constant to make $B$ invertible. For $r > 0$, define the metric $rg$ on $M$ by $rg(v, w) := r \cdot g(v, w)$, and let $D^{rg}$ be the associated Dirac operator. The $G$-admissible endomorphism $\Phi$ is still $G$-admissible for $D^{rg}$, so the operator $B^{rg} := D^{rg} + \Phi$ is again of $G$-Callias-type.
Proposition 7.1. There exists \( r > 0 \) for which \( B^rg \) is strictly positive.

Proof. \( \Phi \Phi + \Phi \Phi \) is bounded away from 0 outside a cocompact subset \( K \subseteq M \). It is equal to Clifford multiplication by a one-form \( dR \), where \( R \) is a projection or unitary depending on \( K \) and is bounded below by a constant \( C_B \) depending on \( \Phi \). Now if \( \frac{\kappa}{4} > -C_B + \epsilon \) everywhere on \( K \) for some \( \epsilon > 0 \), then \( B^2 \) is strictly positive. Otherwise, note that \( \kappa \) is \( \frac{\kappa^2}{2} \), and \( \epsilon \) is \( \frac{c}{r} \), where \( c \) means Clifford multiplication by a one-form.

The latter implies that \( C_B^2 = \frac{C_B}{\kappa^2} \), since, for a fixed \( \Phi \), the constant \( C_B \) scales with the metric in the same way as \( c \). As \( K \) is cocompact, \( \kappa \) is bounded below by some \( \kappa_0 > 0 \) on \( K \). Thus we can find \( r > 0 \) such that \( \frac{\kappa^2}{4} > -C_B^2 + \epsilon \). It follows that \( (B^rg)^2 \) is a strictly positive operator.

\( \square \)

Proposition 7.2. Let \( B \) be a \( G \)-Callias-type operator with \( B^2 \) strictly positive. Then \( B \) has a bounded adjointable inverse.

Proof. It suffices to show that \( B^2 : \mathcal{E}^2 \to \mathcal{E}^0 \) is invertible. By regularity of \( B \) and hence \( B^2, B^2 + \mu^2 \) is surjective for every positive number \( \mu^2 \). Further, since \( B^2 \) is strictly positive, \( B^2 + \mu^2 \) is injective with bounded inverse. To see that \( (B^2 + \mu^2)^{-1} \) is adjointable, let \( T := (B^2 + \mu^2)^{-1} \). Then \( T \) is self-adjoint as a bounded operator \( \mathcal{E}^0 \to \mathcal{E}^0 \), which follows from Lemma 4.1 in [28] and \( \langle u, Tu \rangle_{\mathcal{E}^0} = \langle (B + \mu^2)Tu, Tu \rangle_{\mathcal{E}^0} \geq \mu^2 \langle Tu, Tu \rangle_{\mathcal{E}^0} \geq 0 \). Next, for any \( w \in \mathcal{E}^0 \) and \( u \in \mathcal{E}^2 \), we have

\[
\langle Tu, w \rangle_{\mathcal{E}^2} = \langle B^2Tu, B^2w \rangle_{\mathcal{E}^0} + \langle BTu, Bw \rangle_{\mathcal{E}^0} + \langle Tu, w \rangle_{\mathcal{E}^0} = \langle (B^2 + \mu^2)Tu, B^2w \rangle_{\mathcal{E}^0} + (1 - \mu^2) \langle Tu, B^2w \rangle_{\mathcal{E}^0} + \langle Tu, w \rangle_{\mathcal{E}^0} = \langle u, B^2w \rangle_{\mathcal{E}^0} + (1 - \mu^2) \langle u, TB^2w \rangle_{\mathcal{E}^0} + \langle Tu, w \rangle_{\mathcal{E}^0} = \langle u, (B^2 + (1 - \mu^2)TB^2 + B)w \rangle_{\mathcal{E}^0},
\]

Thus \( (B^2 + \mu^2)^{-1} \in \mathcal{L}(\mathcal{E}^0, \mathcal{E}^2) \). Now note that \( B^2 = (1 - \mu^2 (B^2 + \mu^2)^{-1})(B^2 + \mu^2) \). Since \( B^2 \) is a strictly positive operator, there exists \( C > 0 \) such that for all \( s \in \mathcal{E}^2 \), we have \( \langle B^2s, s \rangle_{\mathcal{E}^0} \geq C \langle s, s \rangle_{\mathcal{E}^0} \). It follows from the Cauchy-Schwarz inequality for Hilbert modules that for any \( \psi \in \mathcal{E}^0 \),

\[
\frac{\mu^2 (B^2 + \mu^2)^{-1}}{\mu^2 + C} \frac{\mu^2 + \mu^2 T \psi}{\mu^2 + C} \Rightarrow \mathcal{E}^0.
\]

Hence \( (1 - \mu^2 (B^2 + \mu^2)^{-1}) \) has an adjointable inverse, and

\[
(B^2)^{-1} = T(1 - \mu^2 (B + \mu^2)^{-1})^{-1}.
\]

This yields the following application to \( G \)-invariant metrics of positive scalar curvature:

Theorem 7.3. Let \( M \) be a \( G \)-spin manifold with Dirac operator \( \phi_0 \). Suppose \( M \) admits a \( G \)-invariant positive scalar curvature metric. Let \( D \) be the \( \mathbb{Z}_2 \)-graded Dirac operator formed from \( D_0 = \phi_0 \) as in section 3. Then \( \text{index}_G F = 0 \in K_0(C^*(G)) \), where \( F \) is the bounded transform of any \( G \)-Callias-type operator defined by a \( G \)-admissible \( \Phi \).

Proof. Since \( B \) is \( G \)-invertible at infinity, it follows from the results in section 4 that \( (\mathcal{E}^0, F) \) defines a class

\[
[B] := [\mathcal{E}^0, F] \in KK(\mathcal{C}, C^*(G)).
\]
This class is independent of the choice of metric on $M$ (see [37]). In particular, let $F$ and $F^{rg}$ be the normalised Callias-type operators associated to the metrics $g$ and $rg$ respectively, for some $r > 0$. Since $[\mathcal{E}^0, F]$ and $[\mathcal{E}^0, F^{rg}]$ are related by an element of $\mathcal{K}(\mathcal{E}^0)$, $\text{index}_G F = \text{index}_G F^{rg}$. By Propositions 7.1 and 7.2, we can find an $r$ such that $F^{rg} = B^{rg} R^{rg}(0)^{1/2}$ is invertible. 

As immediate corollaries, we obtain:

**Corollary 7.4** ([14] Theorem 54). Suppose $M/G$ is compact, with $\varphi$ being the $G$-Spin-Dirac operator on $M$. Suppose that $M$ admits a Riemannian metric of positive scalar curvature. Then $\text{index}_G(\varphi) = 0 \in K_*(C^*(G))$.

**Corollary 7.5** ([38] Theorem 1.2). Suppose $M/G$ is compact, with $\varphi$ being the $G$-Spin-Dirac operator on $M$. Then $\text{ind}_G(\varphi) = 0$, where $\text{ind}_G$ denotes the Mathai-Zhang index.

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