POLYNOMIAL MAPS WITH HIDDEN COMPLEX DYNAMICS

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Abstract. The dynamics of a class of one-dimensional polynomial maps are studied, and interesting dynamics are observed under certain conditions: the existence of periodic points with even periods except for one fixed point; the coexistence of two attractors, an attracting fixed point and a hidden attractor; the existence of a double period-doubling bifurcation, which is different from the classical period-doubling bifurcation of the Logistic map; the existence of Li-Yorke chaos. Furthermore, based on this one-dimensional map, the corresponding generalized Hénon map is investigated, and some interesting dynamics are found for certain parameter values: the coexistence of an attracting fixed point and a hidden attractor; the existence of Smale horseshoe for a subshift of finite type and also Li-Yorke chaos.

1. Introduction. There are many interesting dynamics on polynomial maps. The Logistic map, discovered by the mathematical biologist May [20], is a polynomial map and a classical model for the dynamics of a population, which is well known because of its simple expression with complicated dynamical behavior. Another famous polynomial map is the two-dimensional Hénon map [10]. The elegant theory of Smale horseshoe is often applied to the study of the Hénon or generalized Hénon maps [5]. There exist many results on the polynomial maps with horseshoes, for instance, the investigation of Devaney and Nitecki on the two-dimensional Hénon map [3]; Friedland and Milnor’s study on the n-fold horseshoes of two-dimensional polynomial diffeomorphisms [5]; the work of Dullin and Meiss on two-dimensional cubic Hénon maps [4]; the investigations of Gonchenko et al. on several types of three-dimensional polynomial maps [6, 7, 8, 9]; some discussions on polynomial maps of any dimensions [25, 26].

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An equilibrium is often closely related to the dynamics of many interesting autonomous ordinary differential equations, where the differential equation is defined by \( \dot{x} = \Phi(x) \) with \( x \in \mathbb{R}^n \) and \( \Phi : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n \), and a real solution to \( \Phi(x) = 0 \) is an equilibrium. If the eigenvalues of the Jacobian matrix of the differential equation at the equilibrium have nonzero real parts, then the equilibrium is called hyperbolic, which is classified as a node, saddle, node-focus, or saddle-focus. For instance, the chaotic Lorenz system [19] and the chaotic Chen system [1] with the classical parameters have two unstable saddle-foci and one unstable node, and the multistability (the coexistence of several attractors) in the general Lorenz system with two stable equilibria could be observed [16].

There is a common characteristic of the Lorenz system, Chen system, and many other interesting polynomial systems, that is, the basin of attraction intersects with arbitrarily small neighborhoods of an equilibrium. This type of attractors is called self-excited, and the corresponding system is said to be a self-excited system. On the other hand, if the intersection between the basin of attraction and a small neighborhood of an equilibrium is empty, then this class of attractors is called hidden [17]. The hidden attractors have been found for the Chua circuit, and many interesting systems [15]. Several classes of chaotic autonomous systems with hidden attractors were reviewed in [12], showing that there exist several main groups of hidden attractors: (i) rare flows with no equilibrium [23], (ii) rare flows with a line of equilibrium points [21], and (iii) rare flows with one and only one equilibrium, which is stable [24], and so on. While hidden attractors have been found from various Lorenz-like systems [2], the existence of hidden attractors in the Lorenz and Chen systems is still an open problem.

The maps with hidden dynamics could be similarly defined. Some hidden attractors in one-dimensional maps were obtained by extending the Logistic map [11], and a class of two-dimensional quadratic maps with hidden dynamics was studied in [13].

In this paper, a type of one-dimensional polynomial maps and their corresponding two-dimensional generalized Hénon maps with hidden attractors are introduced and studied.

There are some interesting dynamics found for one-dimensional maps. First, the maps have only periodic points with even periods except for one attracting fixed point under certain conditions. Second, the coexistence of an attracting fixed point and a chaotic hidden attractor is observed under certain conditions. Third, a bifurcation diagram, referred to as a double period-doubling bifurcation, is found by numerical experiments, which is different from the classical period-doubling bifurcation of the Logistic map. Finally, the maps are proved to be chaotic in the sense of Li-Yorke under certain conditions.

For the two-dimensional maps, some strange dynamics are revealed. First, there exist a hidden attractor and an attracting fixed point, where the shape of the attractor is different from that of the Hénon map. Second, the maps have horseshoes for a subshift of finite type for certain parameter values, and are shown to be chaotic in the sense of Li-Yorke.

The rest of the paper is organized as follows. In Section 2, some basic concepts and useful results are introduced. In Section 3, some one-dimensional polynomial maps with hidden dynamics are studied. In Section 4, the dynamics of the two-dimensional generalized Hénon maps are investigated.
2. Preliminary. In this section, some basic concepts and useful results are presented.

Definition 2.1. [18] Let \((X, d)\) be a metric space, \(F : X \to X\) a map, and \(S\) a subset of \(X\) with at least two points. Then, \(S\) is called a scrambled set of \(F\), if for any two distinct points \(x, y \in S\),

\[
\liminf_{n \to \infty} d(F^n(x), F^n(y)) = 0, \quad \limsup_{n \to \infty} d(F^n(x), F^n(y)) > 0.
\]

The map \(F\) is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set \(S\) of \(F\).

Next, the symbolic dynamics are reviewed [22]. Set \(S_0 := \{1, 2, ..., m\}, m \geq 2\). Let

\[
\sum_m := \{\alpha = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) : a_i \in S_0, \ i \in \mathbb{Z}\}
\]

be the two-sided sequence space, where the metric on \(\sum_m\) is defined by \(d(\alpha, \beta) = \sum_{i=-\infty}^{\infty} d(a_i, b_i)/2^i\), with \(d(a_i, b_i) = 1\) if \(a_i \neq b_i\), and \(d(a_i, b_i) = 0\) if \(a_i = b_i, i \in \mathbb{Z}\), for \(\alpha = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots)\) and \(\beta = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)\) \(\in \sum_m\). The shift map \(\sigma : \sum_m \to \sum_m\) is defined by \(\sigma(\alpha) = (\ldots, b_{-2}, b_{-1}, b_0, b_1, b_2, \ldots)\), where \(b_i = a_{i+1}\) for any \(i \in \mathbb{Z}\), and \(\alpha = (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots) \in \sum_m\). Call \((\sum_m, \sigma)\) the two-sided symbolic dynamical system on \(m\) symbols, or simply two-sided full shift on \(m\) symbols. Similarly, define the one-sided fullshift on \(m\) symbols in the following space:

\[
\sum_m^+ := \{\alpha = (a_0, a_1, a_2, \ldots) : a_i \in S_0, \ i \in \mathbb{Z}^+ \cup \{0\}\}.
\]

A matrix \(A = (a_{ij})_{m \times m}\) (\(m \geq 2\)) is called a transition matrix if \(a_{ij} = 0\) or 1 for all \(i, j\). For a transition matrix \(A = (a_{ij})_{m \times m}\),

\[
\sum_m^+(A) := \{\beta = (b_0, b_1, \ldots) \in \sum_m^+ : a_{b_i b_{i+1}} = 1, \ i \geq 0\}.
\]

The map \(\sigma_A := \sigma|_{\sum_m^+(A)} : \sum_m^+(A) \to \sum_m^+(A)\) is said to be the one-sided subshift of finite type for \(A\). Similarly, define \(\sum_m(A)\) and the two-sided subshift of finite type \(\sigma_A\) on \(\sum_m(A)\).

Definition 2.2. [28, Definition 2.3] Let \((X, d)\) be a metric space, \(F : D \subset X \to X\) be a map, and \(A = (a_{ij})_{m \times m}\) be a transition matrix (\(m \geq 2\)). If there exist \(m\) subsets \(V_i\) (\(1 \leq i \leq m\)) of \(D\) with \(V_i \cap V_j = \partial_D V_i \cap \partial_D V_j\) for every pair \((i, j)\), \(1 \leq i \neq j \leq m\), where \(\partial_D V_i\) is the relative boundary of \(V_i\) with respect to \(D\), such that

\[
F(V_i) \supset \bigcup_{a_{ij}=1} V_j, \ 1 \leq i \leq m,
\]

then \(F\) is said to be coupled-expanding for matrix \(A\) (or simply, \(A\)-coupled-expanding) in \(V_i\), \(1 \leq i \leq m\). Further, the map \(F\) is said to be strictly coupled-expanding for matrix \(A\) (or strictly \(A\)-coupled-expanding) in \(V_i\), \(1 \leq i \leq m\), if \(d(V_i, V_j) > 0\) for all \(1 \leq i \neq j \leq m\). Moreover, if each entry of matrix \(A\) is equal to 1, then \(F\) is briefly called coupled-expanding in \(V_i\), \(1 \leq i \leq m\).

Lemma 2.3. [28, Theorem 3.1] Let \((X, d)\) be a metric space, \(V_1, \ldots, V_m\) (\(m \geq 2\)) be pairwise-disjoint compact subsets of \(X\), and \(A = (a_{ij})_{m \times m}\) be a transition matrix. If a continuous map \(F : D = \bigcup_{i=1}^m V_i \to X\) is strictly \(A\)-coupled-expanding in \(V_1, \ldots, V_m\), then there exists a compact subset \(\Lambda \subset D\) such that \(F(\Lambda) = \Lambda\) and \(F : \Lambda \to \Lambda\) is topologically semi-conjugate to \(\sigma_A : \sum_m^+(A) \to \sum_m^+(A)\). Furthermore, if the topological entropy of \(\sigma_A\) is positive, then \(F\) is chaotic in the sense of Li-Yorke.
3. One-dimensional maps. In this section, consider the following type of polynomial maps:

\[ P(x) = ax^m(x + b)(x - b), \quad x \in [-b, b], \]  

where \( a \) and \( b \) are real parameters, and \( m \) is a positive integer. The coexistence of two attractors is observed for certain parameter values; a strange bifurcation diagram is obtained; the maps are verified to be chaotic in the sense of Li-Yorke for certain parameter values.

**Proposition 1.** Assume that \( a > 0 \), \( b > 0 \), and \( m \) is a positive integer. There are three critical points for \( P(x) \):

\[-b\sqrt{\frac{m}{m+2}}, 0, \text{ and } b\sqrt{\frac{m}{m+2}}.\]

If \( m \) is odd, then the function \( P(x) \) is monotone increasing for \( x \in \left(-\infty, -b\sqrt{\frac{m}{m+2}}\right) \) or \( x \in \left[b\sqrt{\frac{m}{m+2}}, \infty\right) \), and monotone decreasing for \( x \in \left[-b\sqrt{\frac{m}{m+2}}, -\infty\right) \) or \( x \in \left[0, b\sqrt{\frac{m}{m+2}}\right] \).

If \( m \) is even, then the function \( P(x) \) is monotone increasing for \( x \in \left[-b\sqrt{\frac{m}{m+2}}, 0\right] \) or \( x \in \left[b\sqrt{\frac{m}{m+2}}, \infty\right) \), and monotone decreasing for \( x \in \left(-\infty, -b\sqrt{\frac{m}{m+2}}\right) \) or \( x \in \left[0, b\sqrt{\frac{m}{m+2}}\right] \).

**Proposition 2.** Assume that \( a > 0 \), \( b > 0 \), and \( m \) is a positive integer. If \( m \) is odd, then the function \( P(x) \) is odd; there is a local maximum at the point \( -b\sqrt{\frac{m}{m+2}} \) and a local minimum at the point \( b\sqrt{\frac{m}{m+2}} \) for \( P(x) \).

If \( m \) is even, then the function \( P(x) \) is even; there are two global minimum at the points \( \pm b\sqrt{\frac{m}{m+2}} \) and a local maximum at the origin for \( P(x) \).

By Propositions 1 and 2, illustration diagrams can be obtained for the polynomial function with \( a > 0 \). Figures 1 and 2 show the function \( P(x) \) with odd and even \( m \), respectively.

**Proposition 3.** If \( a > 0 \), \( b > 0 \), and

\[ 2ab^{m+1}\left(\frac{m}{m+2}\right)^m \leq m + 2, \]

then \( P([-b, b]) \subset [-b, b] \) for any odd number \( m \); \( P([-b, b]) \subset [-b, 0] \) for any even number \( m \).

**Proof.** This is an easy consequence of Proposition 2. The proof is omitted.

**Proposition 4.** Assume that \( a > 0 \), \( b > 0 \), and \( m \geq 2 \). Then, there are at most three fixed points for the map (1) with sufficiently large \( a \) and \( x \in (-\infty, \infty) \). The origin is an attracting fixed point for any \( m \geq 2 \). For odd \( m \), one fixed point is contained in the interval \((-\infty, -b)\), and another one is contained in the interval \((b, \infty)\). For even \( m \), other two fixed points are contained in the interval \((-b, 0)\).

**Proof.** The existence of the fixed points is equivalent to the solutions to the polynomial equation \( ax^m(x + b)(x - b) = x \). It is evident that \( x = 0 \) is a solution. This, together with the fact that \( P'(0) = 0 \) for \( m \geq 2 \), implies that the origin is an
attracting fixed point. If $x \neq 0$, the equation $ax^m(x + b)(x - b) = x$ can be reduced to the equation $ax^{m-1}(x + b)(x - b) = 1$. With this equation, by Propositions 1 and 2, the proposition is proved.

Next, numerical experiments are performed, which show that for the parameters satisfying the assumptions of Proposition 3, in particular, $a > 0$, $b > 0$, and $m \geq 2$ is an odd number, the map $P : [-b, b] \to [-b, b]$ has periodic points with even periods, hidden dynamics, and a new bifurcation diagram, double period-doubling
bifurcation. On the other hand, for \( a > 0, b > 0, m \geq 2 \) is an even number, and the hypothesis of Proposition 3 are satisfied, the map \( P : [-b, b] \rightarrow [-b, b] \) has the coexistence of an attracting fixed point and a self-excited attractor, and also the classical period-doubling bifurcation.

Figure 3 is the bifurcation diagram for map (1) with \( m = 2 \). This bifurcation diagram is similar with the period-doubling bifurcation of the Logistic map.

\[
\text{Figure 3. Bifurcation diagram of } P(x) = ax^2(x + 1)(x - 1) \text{ for } 3 \leq a \leq 4, \text{ where the initial value is 0.6.}
\]

Figure 4 is the bifurcation diagram for map (1) with \( m = 3 \). This diagram is different from the classical period-doubling bifurcation, which is referred to as a double period-doubling bifurcation.

For large enough \( a \), there exist chaotic dynamics according to the coupled-expansion theory described by Definition 2.2 above, which is corresponding to the dynamics of the Logistic map \( f(x) = ax(1 - x) \) for \( a > 4 \). The results are summarized as follows.

**Lemma 3.1.** The topological entropy for the subshift map \( \sigma_A : \sum_4^+ (A) \rightarrow \sum_4^+ (A) \) or \( \sigma_A : \sum_4 (A) \rightarrow \sum_4 (A) \) is \( \log 2 > 0 \), where

\[
A = \begin{pmatrix}
0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0
\end{pmatrix}.
\]

**Proof.** This is an easy consequence following some discussions in [22, 27]. The proof is omitted.

**Theorem 3.2.** Assume that \( m \geq 2 \) and

\[
2ab^{m+1}\left(\sqrt{\frac{m}{m+2}}\right)^m > m + 2.
\]

If \( m \) is odd, then there exist disjoint intervals \( V_1, V_2, V_3, \) and \( V_4 \), such that \( P : D = \bigcup_{i=1}^4 V_i \rightarrow \mathbb{R} \) is strictly \( A \)-coupled-expanding in \( V_1, ..., V_4 \), where \( A \) is specified in (2).
And, there exists a compact subset \( \Lambda \subset D \) such that \( P(\Lambda) = \Lambda \) and \( P : \Lambda \to \Lambda \) is topologically semi-conjugate to \( \sigma_A : \sum^+_{i=1}(A) \to \sum^+_{i=1}(A) \).

If \( m \) is even, then there exist two disjoint intervals \( U_1 \) and \( U_2 \) such that \( P : D = \bigcup_{i=1}^2 U_i \to \mathbb{R} \) is coupled-expanding in \( U_1 \) and \( U_2 \). And, there exists a compact subset \( \Lambda \subset D \) such that \( P(\Lambda) = \Lambda \) and \( P : \Lambda \to \Lambda \) is topologically semi-conjugate to \( \sigma : \sum^+_{2} \to \sum^+_{2} \).

Furthermore, \( P \) is chaotic in the sense of Li-Yorke.

**Proof.** Consider the case that \( m \) is odd.

By Proposition 4, there are two positive constants, \( 0 < \lambda < 1 \) and \( 0 < \delta < 1 \) (depending on \( a \) and \( m \)), such that \( |P(x)| \leq \lambda|x| \) for all \( x \in (-\delta, \delta) \). It follows from (3) and Proposition 2 that \( P\left(-b\sqrt{\frac{m+2}{m}}\right) > b \) and \( P\left(b\sqrt{\frac{m+2}{m}}\right) < -b \). This, together with the fact of \( P(b) = P(-b) = 0 \) and the Mean Value Theorem, shows that there are two numbers, \( 0 < x_1 < b\sqrt{\frac{m+2}{m}} < x_2 < b \), such that \( P(x_1) = P(x_2) = -b \) and \( P(-x_1) = P(-x_2) = b \). Set

\[
V_1 := [-b, -x_2], \quad V_2 := [-x_1, -\delta/2], \quad V_3 := [\delta/2, x_1], \quad V_4 := [x_2, b].
\]

Then, by the Mean Value Theorem, one has

\[
V_3 \cup V_4 \subset P(V_1), \quad V_3 \cup V_4 \subset P(V_2), \quad V_1 \cup V_2 \subset P(V_3), \quad V_1 \cup V_2 \subset P(V_4).
\]

So, \( P \) is strictly \( A \)-coupled-expanding in \( V_1, \ldots, V_4 \).

Now, consider the case that \( m \) is even.

By (3) and Proposition 2, one has that \( P\left(-b\sqrt{\frac{m+2}{m}}\right) < -b \). This, together with the fact of \( P(-b) = P(0) = 0 \) and the Mean Value Theorem, implies that there are two numbers, \( -b < y_1 < -b\sqrt{\frac{m+2}{m}} < y_2 < 0 \), such that \( P(y_1) = P(y_2) = -b \). Set

\[
U_1 := [-b, y_1] \quad \text{and} \quad U_2 := [y_2, 0].
\]
Then, by the Mean Value Theorem, one has
\[ U_1 \cup U_2 \subset P(U_1) = P(U_2) = [-b, 0]. \]
So, \( P \) is coupled-expanding in \( U_1 \) and \( U_2 \).

By Lemmas 2.3 and 3.1, \( P \) is chaotic in the sense of Li-Yorke.

This completes the proof. \( \square \)

4. Two-dimensional maps. In this section, consider the following type of generalized Hénon maps \( F = (f_1, f_2) \) on the plane:
\[
\begin{align*}
  f_1(x, y) &= dy \\
  f_2(x, y) &= P(y) + cx,
\end{align*}
\]
where \( P(x) \) is specified in (1), and \( c \) and \( d \) are real parameters with \( cd \neq 0 \). The coexistence of a hidden attractor and an attracting fixed point is observable under certain conditions. The maps have horseshoes for a subshift of finite type and are chaotic in the sense of Li-Yorke under certain conditions.

Next, recall the classical Smale horseshoe map. Consider a square, denoted by \( U \), a compact subset on a two-dimensional manifold. A horseshoe map \( F \) is defined as follows. The action of the map is defined geometrically by squishing the square along one direction, then stretching the result into a long strip along the perpendicular direction, and finally folding the strip into the shape of a horseshoe, where \( F(U) \cap U \neq \emptyset \). This operation is repeated for infinitely many times. An invariant set is naturally defined as \( \Lambda = \cap_{i \in \mathbb{Z}} F^i(U) \), and the dynamics on this invariant set are described by the two-sided fullshift on two symbols [22].

Now, introduce the Smale horseshoe to a subshift of finite type [22]. For brevity, only a special case is discussed, which will be used in the sequel. Consider two squares, denoted by \( U_1 \) and \( U_2 \), on a two-dimensional manifold, where the intersection of \( U_1 \) and \( U_2 \) is empty. The horseshoe map \( F \) defined on \( U_1 \cup U_2 \) is given as follows. The action of the map is defined geometrically by squishing the two squares along the same direction, then stretching the results into two long strips along the perpendicular direction, and finally folding the two strips into the shape of two horseshoes. \( F(U_1) \) and \( U_2 \) contribute to a horseshoe, and \( F(U_2) \) and \( U_1 \) form another horseshoe. Figure 5 is the illustration diagram for the Smale horseshoe for a subshift of finite type for the matrix \( A \) introduced in (2), \( U_1 \) and \( F(U_1) \) are represented by the green color, \( U_2 \) and \( F(U_2) \) are associated with the yellow color. Note that \( F \) is contracting along the horizontal direction and expanding along the vertical direction, \( F(U_1) \) and \( U_2 \) form a horseshoe, and \( F(U_2) \) and \( U_1 \) contribute to another horseshoe. An invariant set is naturally defined as \( \Lambda = \cap_{i \in \mathbb{Z}} F^i(U_1 \cup U_2) \), and the dynamics on this invariant set are described by the two-sided subshift of finite type for the matrix \( A \).

**Proposition 5.** For \( 0 < |cd| < 1 \) and \( m \geq 2 \), the origin \((0, 0)\) is an attracting fixed point of the map (4).

**Proof.** It is evident that \((0, 0)\) is a fixed point of the map (4).

The Jacobian of the map is
\[
\begin{pmatrix}
  0 & d \\
  c & P'(y)
\end{pmatrix}.
\]
By \( m \geq 2 \), one has
\[
\begin{pmatrix}
  0 & d \\
  c & 0
\end{pmatrix}.
\]
This, together with the assumption $|cd| < 1$, implies that $(0, 0)$ is an attracting fixed point.

Here, only the situation that $a > 0$ is discussed. There are four different cases: (i) $c > 0$ and $d > 0$; (ii) $c < 0$ and $d > 0$; (iii) $c < 0$ and $d > 0$; (iv) $c < 0$ and $d < 0$. For the case that $a < 0$, some discussions are contained in [25]. So, it only needs to study cases (i) and (ii), whereas for cases (iii) and (iv), it only needs to take two squares $S_1 = [0, b] \times [0, b]$ and $S_2 := [-b, 0] \times [-b, 0]$, and then apply similar arguments.

**Theorem 4.1.** Assume that $0 < d \leq 1$, $c \neq 0$, $b > 0$, and $a > 0$, and $m$ is an odd number. For fixed $d, c, b$, there is a sufficiently large $a_*$ such that, for any $a > a_*$, there is a Smale horseshoe for a subshift of finite type for the matrix $A$, and a compact invariant set $\Lambda$ on which $F$ is semi-conjugate with the subshift of finite type for the matrix $A$, where $A$ is specified in (2). Consequently, $F$ is chaotic in the sense of Li-Yorke.

In the following, let $S_1 := [0, b] \times [-b, 0]$ and $S_2 := [-b, 0] \times [0, b]$. 

**Remark 1.** Note that for these choices of $S_1$ and $S_2$, one has $S_1 \cap S_2 = \{(0, 0)\}$. This is different from the discussions above, where $S_1 \cap S_2 = \emptyset$. This does not bring too much trouble, since $(0, 0)$ is a fixed point of $F$ and $F$ is a diffeomorphism. The only difference is that the map on the invariant set $(\bigcap_{i \in \mathbb{Z}} F^i(S_1 \cup S_2)) \setminus \{(0, 0)\}$ is topologically semi-conjugate to $\sigma_A : \sum_4(A) \to \sum_4(A)$.

Set

$$\Lambda := \left(\bigcap_{i \in \mathbb{Z}} F^i(S_1 \cup S_2)\right) \setminus \{(0, 0)\}.$$

**Lemma 4.2.** Under the assumptions of Theorem 4.1 and for $a > a_*$, there are two connected components in $F(S_i) \cap S_j$, $1 \leq i \neq j \leq 2$. 

**Figure 5.** Illustration diagram of the horseshoe for a subshift of finite type for the matrix $A$. 


The vertices of the square $S_1$ are

$V_1 = (0, 0), V_2 = (0, -b), V_3 = (b, -b), V_4 = (b, 0)$;

the vertices of the square $S_2$ are

$V_5 = (-b, b), V_6 = (-b, 0), V_7 = (0, 0), V_8 = (0, b)$.

For notational simplicity, the notation $V_iV_j$ ($i < j$) is used to represent the line segment joining $V_i$ and $V_j$, which is a part of the boundary of one square, $S_1$ or $S_2$.

By direct calculation, one obtains

$F(V_1) = (0, 0), F(V_2) = (-db, 0), F(V_3) = (-db, cb), F(V_4) = (0, cb), F(V_5) = (db, -cb), F(V_6) = (0, -cb), F(V_7) = (0, 0), F(V_8) = (db, 0)$;

and

$F(V_1V_2) = \{(dy, P(y)) : y \in [-b, 0]\}, F(V_2V_3) = \{(-db, cx) : x \in [0, b]\}, F(V_3V_4) = \{(dy, P(y) + cb) : y \in [-b, 0]\}, F(V_4V_5) = \{(0, cx) : x \in [0, b]\}, F(V_5V_6) = \{(dy, P(y) - cb) : y \in [0, b]\}, F(V_6V_7) = \{(0, cx) : x \in [-b, 0]\}, F(V_7V_8) = \{(dy, P(y)) : y \in [0, b]\}, F(V_8V_1) = \{(db, cx) : x \in [-b, 0]\}$.

So, the images of these curves are classified into two categories: $F(V_1V_2), F(V_3V_4)$, $F(V_5V_6)$, and $F(V_7V_8)$ are parabolas; $F(V_2V_3), F(V_4V_5), F(V_6V_7)$, and $F(V_8V_1)$ are line segments.

Figure 6 is the illustration diagram for $a = 6, b = c = d = 1, m = 3$, where the green color is used for $S_1$ and $F(S_1)$, the cyan color is for $S_2$ and $F(S_2)$. A horseshoe of a subshift of finite type for the matrix $A$ is observable in this graph.

By the above arguments, for sufficiently large $a$, $F(S_1) \cap S_2$ and $F(S_2) \cap S_1$ have two connected components, respectively. This, together with the fact that $F$ is a diffeomorphism, implies that $S_1 \cap F^{-1}(S_2)$ and $S_2 \cap F^{-1}(S_1)$ have two connected components, respectively. By induction, $\cap_{n=0}^{\infty} F^{n}(S_1 \cup S_2)$ can be easily defined. With this information, one can define a semi-conjugacy between $F$ on $\Lambda$ and $\sigma_A$ on $\Sigma_4(A)$, which is onto, because $(0, 0)$ is an attracting fixed point and $F$ is a diffeomorphism.

Therefore, for sufficiently large $a$, there exists an invariant set on which $F$ is semi-conjugate with the two-sided subshift of finite type for the matrix $A$ in the space $\Sigma_4(A)$.

**Lemma 4.3.** Consider the generalized Hénon map (4), and suppose that

$$0 < d \leq 1, \quad 2ab^{m+1}\left(\sqrt{\frac{m}{m+2}}\right)^m < m + 2,$$

and $0 < c < 1 - ab^{m+1}\left(\sqrt{\frac{m}{m+2}}\right)^m \frac{2}{m + 2}$.

Then, $F$ maps $[-b, b]^2$ into $[-b, b]^2$, and is injective, that is, $F([-b, b]^2) \subset [-b, b]^2$.

**Proof.** This is an easy consequence of Proposition 3. \hfill $\Box$

The numerical experiments indicate the possible existence of strange attractors for the map $F : [-b, b]^2 \rightarrow [-b, b]^2$, satisfying the assumptions of Lemma 4.3, where $F$ is introduced in (4), and the shape of the attractors is different from that of the Hénon map. This, together with Proposition 5, proves the existence of hidden attractors for the two-dimensional maps under the assumed conditions. Figure 7
Figure 6. Illustration diagram for the map (4) with $a = 6$, $b = c = d = 1$, and $m = 3$.

is the simulation of the hidden attractor, with $a = 5$, $b = 1$, and $m = 3$, $d = 1$, $c = 0.005$, where the initial value is $(0.8, 0.8)$.

Finally, the case that $m$ is an even number is discussed.

**Theorem 4.4.** Assume that $0 < d \leq 1$, $c \neq 0$, $b > 0$, and $a > 0$, and $m$ is an even number. For fixed $d$, $c$, and $b$, there is a sufficiently large $a_*$ such that, for any $a > a_*$, there exist a classical Smale horseshoe and a compact invariant set $\Lambda$ on which $F$ is semi-conjugate with the fullshift on $\sum_2$. Consequently, $F$ is chaotic in the sense of Li-Yorke.

**Proof.** Consider the square

$$S = [-b, 0] \times [-b, 0].$$

The vertices of $S$ are

$$V_1 = (-b, 0), \ V_2 = (-b, -b), \ V_3 = (0, -b), \ V_4 = (0, 0).$$
The images of the vertices under the map $F$ are

\[ F(V_1) = (0, -cb), \quad F(V_2) = (-db, -cb), \quad F(V_3) = (-db, 0), \quad F(V_4) = (0, 0). \]

Let $V_iV_j$ be the line segment joining $V_i$ and $V_j$, which is a part of the boundary of the square $S$. Direct calculation yields

\[ F(V_1V_2) = \{(dy, P(y) - cb) : y \in [-b, 0]\}, \quad F(V_2V_3) = \{(-db, cx) : x \in [-b, 0]\}, \]
\[ F(V_3V_4) = \{(dy, P(y)) : y \in [-b, 0]\}, \quad F(V_1V_4) = \{(0, cx), x \in [-b, 0]\}. \]

Figure 8 is the illustration diagram for $a = 5$, $b = 1$, $c = 1$, $d = 1$, and $m = 2$, where the color of $S$ and $F(S)$ is cyan. A horseshoe is observable in this graph.

These conclusions could be drawn by applying similar arguments in [22, 25, 26]. So, the proof is omitted here.

**Remark 2.** For the generalized Hénon maps, the existence of the positive Lyapunov exponents is an interesting problem; another problem is the existence of non-symmetric chaotic attractors with different Lyapunov exponents and Lyapunov dimension. For more details, please see footnote 4 in [14].

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Figure 8. Illustration diagram for the map (4) with $a = 5$, $b = c = d = 1$, and $m = 2$.

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