Abstract We consider quantum analogs of the relativistic Toda lattices and give new $2 \times 2$ $L$-operators for these models. Making use of the variable separation the spectral problem for the quantum integrals of motion is reduced to solving one-dimensional separation equations.

1. Introduction

The relativistic Toda lattices (RTL’s) were originally introduced in [9] where the classical nonperiodic case has been solved by means of an explicit action-angle transformation with the help of the $N \times N$ matrix Lax representation for the model. For this RTL, which is naturally associated to the root system $A_{N-1}$, the Lax representation in terms of $N \times N$ matrices appeared to have more complicated structure in comparison with the Lax pair for corresponding non-relativistic Toda lattice [8]. In [1,2] there was introduced a Lax triad for the model, three matrices of which have simpler three-diagonal form. With the help of this representation the direct and inverse problems for the system have been solved and the separation of variables has been carried out in full analogy to the classical work [4].

In recent works [12-14] there was proved an equivalence, due to canonical transformation, of the RTL’s for general classical root systems and discrete time Toda lattices. There was given also an orbit interpretation of the Lax representations for all the systems. Following [3,7,11] in work [13] for all the RTL’s there were
introduced $2 \times 2$ $L$-operators which satisfy the Sklyanin quadratic algebra with non-standard trigonometric $r$-matrix.

In the present paper we construct, for all the RTL’s, new $2 \times 2$ quantum $L$-operators obeying the quadratic $R$-matrix relations with the standard trigonometric $R$-matrix. We give also the separation of variables for the $A_{N-1}$ RTL which is a generalization to relativistic case of the variable separation procedure given in [10] for non-relativistic Toda lattice.

The structure of the paper is as follows: in the Section 2 we recall the known facts about the classical RTL, which will be needed in the sequel. In the Section 3 there is introduced new $2 \times 2$ $L$-operator for the $A_{N-1}$ RTL, which is a nonsymmetric $L$-operator of the lattice sh-Gordon model, and there is given also the variable separation procedure indicating that our separation variables are equivalent to the ones introduced before [1]. In the Section 4 we consider the quantum $A_{N-1}$ RTL and construct the separation variables and corresponding separation equations for it. The final Section 5 describes the construction of the monodromy matrix for the quantum RTL’s for general classical root systems (including affine cases).

2. Description of the model

The relativistic generalizations of the classical Galilei-invariant Calogero-Moser systems and Toda lattices are characterized by the time and space translation generators

\[ T \equiv \frac{1}{2} (S_+ + S_-), \quad P \equiv \frac{1}{2} (S_+ - S_-), \]

(1)

where \( S_\pm \equiv \sum_{j=1}^{N} e^{\pm \theta_j} V_j (\tilde{q}_1, \ldots, \tilde{q}_N) \),

and the boost generator

\[ B \equiv - \sum_{j=1}^{N} \tilde{q}_j. \]

(2)

In this equations \( \theta_j \) and \( \tilde{q}_j \) are canonically conjugated momentum and coordinate of the \( j \)-th particle. One obtains a representation of the Lie algebra of the Poincare group if and only if \( V_j \) satisfy the functional equations \[9]

\[ V_i \partial_i V_j + V_j \partial_j V_i = 0, \quad i \neq j, \]

(3)

\[ \sum_{j=1}^{N} \partial_j V_j^2 = 0, \quad \partial_j = \frac{\partial}{\partial \tilde{q}_j}. \]

There are known two solutions to this equation \[9\]:

1. \( V_j (\tilde{q}_1, \ldots, \tilde{q}_N) = \prod_{i \neq j} f (\tilde{q}_j - \tilde{q}_i), \quad f^2 \equiv a + b \wp (\tilde{q}) \)

where \( \wp \) is the Weierstrass function.

2. \( V_j (\tilde{q}_1, \ldots, \tilde{q}_N) = f (\tilde{q}_{j-1} - \tilde{q}_j) f (\tilde{q}_j - \tilde{q}_{j+1}), \quad f = (1 + e^{2 \wp (\tilde{q})})^{1/2}, \quad a \in R. \)
The first case is called relativistic Calogero-Moser system and the second case is referred to as RTL.

As phase space we have

\[ \Omega \equiv \{ (\theta, \tilde{q}) \in \mathbb{R}^{2N} \} , \quad \omega \equiv \sum_{j=1}^{N} d\theta_j \wedge d\tilde{q}_j . \]

We will consider RTL as the integrable system of \( N \) one-dimensional particles with the Hamiltonian

\[ H = \sum_{j=1}^{M} \exp(\theta_j) \left[ \frac{1}{2} \ln \left( \frac{1 + \exp(\tilde{q}_j - \tilde{q}_{j-1})}{1 + \exp(\tilde{q}_{j+1} - \tilde{q}_j)} \right) \right]^{1/2} . \]

The RTL is open (non-periodic) when \( M = N - 1 \) and periodic in the case of

\[ M = N , \quad \tilde{q}_{j+kN} = \tilde{q}_j , \quad \theta_{j+kN} = \theta_j , \quad \text{where} \quad j = 1, \ldots , N \quad \text{and} \quad k \in \mathbb{Z} . \]

In this section we consider only periodic RTL.

According to [12,13] there is an equivalence between the RTL and the discrete time Toda lattice. It appeared to be more natural to introduce new canonical coordinates \((p_j, \tilde{q}_j)\) connected to the old coordinates \((\theta_j, \tilde{q}_j)\) as follows:

\[ p_j = \theta_j + \frac{1}{2} \ln \left( \frac{1 + \exp(\tilde{q}_j - \tilde{q}_{j-1})}{1 + \exp(\tilde{q}_{j+1} - \tilde{q}_j)} \right) . \]

Transformation \((\theta_j, \tilde{q}_j) \rightarrow (p_j, \tilde{q}_j)\) is a canonical transformation.

In new variables the Hamiltonian (7) has the form

\[ H = \sum_{j=1}^{N} \exp(p_j) \left[ 1 + \exp(\tilde{q}_{j+1} - \tilde{q}_j) \right] . \]

One can introduce a set of new variables \(c_j, d_j\) and \(f_j\) [1,2]

\[ c_j = \exp(p_j - \tilde{q}_j + \tilde{q}_{j+1}) , \quad d_j = \exp p_j , \quad f_j^2 = c_j . \]

Then the equations of motion given by the Hamiltonian (10) have the following form

\[ \dot{d}_j = -d_j (c_{j-1} - c_j) , \]
\[ \dot{c}_j = c_j (d_{j+1} - d_j + c_{j+1} - c_{j-1}) , \]

and can be rewritten as a compatibility condition for the two linear problems:

\[ (k^2 - d_j) \phi_j + k (f_j \phi_{j+1} + f_{j-1} \phi_{j-1}) = 0 , \]
\[ \dot{\phi}_j = -\frac{1}{2} (c_j - c_{j-1} + k^2) \phi_j - \frac{1}{2} k (f_j \phi_{j+1} - f_{j-1} \phi_{j-1}) , \]

where \( \phi_j = (\phi_1, \ldots , \phi_N)^t \) is so-called Baker-Akhiezer function.
Let us give the equations (13) in the matrix form introducing two $N \times N$ matrices $\tilde{L}(k)$ and $\tilde{M}(k)$

\begin{equation}
\tilde{L}(k) \phi = 0, \quad \dot{\phi} = -\tilde{M}(k) \phi.
\end{equation}

The equations of motion are thereby written in terms of the Lax triad $(\tilde{L}, \tilde{M}, \tilde{F})$ [1,2]:

\begin{equation}
\frac{d}{dt} \tilde{L} = [\tilde{L}, \tilde{M}] + \tilde{F} \tilde{L},
\end{equation}

where the symmetric matrix $\tilde{L}$ has the form

\begin{equation}
\tilde{L} = \begin{pmatrix}
k^2 - d_1 & k f_1 & 0 & \cdots & k f_N \\
k f_1 & k^2 - d_2 & k f_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
k f_N & 0 & \cdots & k f_{N-1} & k^2 - d_N
\end{pmatrix},
\end{equation}

and the matrix $\tilde{M}$ looks like

\begin{equation}
\tilde{M} = \frac{1}{2} \begin{pmatrix}
c_1 - c_N + k^2 & k f_1 & 0 & -k f_N \\
-k f_1 & c_2 - c_1 + k^2 & k f_2 & 0 \\
\vdots & \vdots & \ddots & \ddots \\
k f_N & 0 & -k f_{N-1} & c_N - c_{N-1} + k^2
\end{pmatrix}.
\end{equation}

The matrix $\tilde{F}$ is diagonal and has zero trace:

\begin{equation}
\tilde{F} = \begin{pmatrix}
f_1^2 - f_N^2 \\
f_2^2 - f_1^2 \\
\vdots \\
(f_{N-1}^2 - f_N^2)
\end{pmatrix}.
\end{equation}

From the Lax representation (15) there follows that

\begin{equation}
\text{tr} \ \tilde{L}^{-1} \dot{\tilde{L}} = \text{tr} \ \tilde{F} = 0,
\end{equation}

and, therefore, using the Abel identity

\begin{equation}
\frac{\partial \ln \det X}{\partial t} = \text{tr} \ X^{-1} \frac{\partial X}{\partial t},
\end{equation}

one concludes that the $\det(\tilde{L})$ is the generating function of the integrals of motion for the RTL. The Lax triad (15) and the related questions were studied in [1,2]. We are giving below one Proposition from there.

**Proposition 1.**

a). The determinant and all principal minors of the matrix $\tilde{L}(k)$ are polynomials on $k^2$. 

**Proof.**

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b). The zeros of determinant and of all principal minors of the matrix \( \tilde{L}(k) \) are simple and positive real.

According to [3,7] it is possible to rewrite the equations (13) in the matrix form with the help of the \( 2 \times 2 \) matrices \( \tilde{L}_j(k) \) [12]

\[
\tilde{L}_j(k) = \begin{pmatrix} k^2 - \exp(p_j), & -k \exp(q_j) \\ k \exp(p_j - q_j), & 0 \end{pmatrix}.
\]

The matrix \( \tilde{L}_N(k) \) is related to the \( 2 \times 2 \) monodromy matrix \( \tilde{T}_N(k) \) such that

\[
\text{det} \tilde{L}_N(k) - 2 = \text{tr} \tilde{T}_N(k) \equiv \text{tr} \prod_{j=1}^N \tilde{L}_j(k) = \text{tr} \begin{pmatrix} \tilde{A}_N & \tilde{B}_N \\ \tilde{C}_N & \tilde{D}_N \end{pmatrix}.
\]

Matrices \( \tilde{L}_j(\lambda) \) and monodromy matrix \( \tilde{T}_N(\lambda) \) satisfy the Sklyanin quadratic algebra [3,10]

\[
\{ \tilde{T}^{(1)}(\lambda), \tilde{T}^{(2)}(\mu) \} = [\tilde{r}(\lambda/\mu), \tilde{T}^{(1)}(\lambda) \tilde{T}^{(2)}(\mu)]
\]

where \( \tilde{T}^{(1)}(\lambda) = \tilde{T}(\lambda) \otimes I \) and \( \tilde{T}^{(2)}(\mu) = I \otimes \tilde{T}(\mu) \), with the following non-standard \( r \)-matrix

\[
\tilde{r}(\lambda/\mu) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -1/2 & c & 0 \\ 0 & c & 1/2 & 0 \\ 0 & 0 & 0 & a \end{pmatrix},
\]

where \( a(\lambda/\mu) = \frac{1}{2} \frac{\lambda^2 + \mu^2}{\lambda - \mu}, \ c(\lambda/\mu) = \frac{\lambda \mu}{\lambda^2 - \mu^2} \). Note that the algebra (21) can be rewritten in the form

\[
\{ \tilde{T}(\lambda) \otimes \tilde{T}(\mu) \} = [r(\lambda/\mu), \tilde{T}(\lambda) \otimes \tilde{T}(\mu)] + [s, \tilde{T}(\lambda) \otimes \tilde{T}(\mu)].
\]

In equation (23) \( r(\lambda/\mu) \) is the standard \( r \)-matrix of the \( XXZ \) type and \( s \) is the matrix of scalars:

\[
r(\lambda/\mu) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

In the quantum case the equation (23) goes to the nonstandard quadratic relation

\[
R_1(\lambda/\mu) T^{(1)}(\lambda) T^{(2)}(\mu) = T^{(2)}(\mu) T^{(1)}(\lambda) R_2(\lambda/\mu).
\]

We can express the entries of the monodromy matrix \( \tilde{T}(\lambda) \) in terms of the determinant and principal minors of the matrix \( \tilde{L}(k) \) in the same way as it was done in [7] for the non-relativistic Toda lattice. Then using the Proposition 1 we have the following properties for the entries of the \( \tilde{T}(\lambda) \).
Proposition 2.
a) $\tilde{A}_N(\lambda) = \lambda^N + \ldots$ is a polynomial of degree $N$ on the $\lambda = k^2$ where:

$$\tilde{A}_N(0) = (-1)^N \exp(P), \quad P = \sum_{j=1}^{N} p_j.$$ 

b) $\tilde{B}_N(\lambda) = k (\beta \lambda^{N-1} + \ldots), \tilde{C}_N(\lambda) = k (\gamma \lambda^{N-1} + \ldots), \tilde{D}_N(\lambda) = \delta \lambda^{N-1} + \ldots$.

c) Zeros of the polynomials $\tilde{A}(\lambda), \tilde{B}(\lambda)/k, \tilde{C}(\lambda)/k, \text{and } \tilde{D}(\lambda)$ are simple and positive real.

These facts will be used in the next section for doing the separation of variables. We will give also a new $2 \times 2$ $L$-operator and a new monodromy matrix for the RTL which are closely connected to the previous ones, but satisfy the standard $XXZ$-type quadratic $r$-matrix algebra.

3. Classical mechanics

It is well-known that the $2 \times 2$ $L$-operator for the non-relativistic Toda lattice can be obtained by contraction from the $L$-operator of the $XXX$-type Heisenberg magnet. For the $XXZ$-type Heisenberg magnet one can use the analogous contraction and construct the $L$-operator of the form:

$$(26) \quad L_j(u) = \begin{pmatrix} 2 \sinh(u - p_j/2) & -\exp(q_j) \\ \exp(-q_j) & 0 \end{pmatrix},$$

where the $p_j$ and $q_j$ are the canonically conjugated momentum and coordinate of $j$-th particle. This $L$-operator obeys the fundamental Poisson brackets (21) with the standard $r$-matrix $r(\lambda/\mu)$ of the $XXZ$-type (24) ($\lambda = e^u$) and it can be thought of as a nonsymmetric analog of the $L$-operator for the lattice sh-Gordon model [6].

The $L$-operator (26) and the monodromy matrix $T_N(u)$,

$$(27) \quad T_N(u) = \prod_{j=1}^{N} L_j(u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}(u),$$

have a simple connection to the $\tilde{L}_j(k)$-operator (19) and the monodromy matrix $\tilde{T}(k)$ (20) through the following change of variables:

$$q_j = \tilde{q}_j - \frac{p_j}{2}, \quad k = \exp(u);$$

$$(28) \quad L_j(u) = k^{-1} \exp(-\frac{P_j}{2}) \tilde{L}(u),$$

$$T_N(u) = k^{-N} \exp(-\frac{P}{2}) \tilde{T}_N(u),$$

where $P = \sum_{j=1}^{N} p_j$ is the total momentum of the system which is an integral of motion.

We can associate the following $N \times N$ matrix $L(u)$ to the monodromy matrix $T_N(u)$ (cf. with [7]):

$$L = \begin{pmatrix} 2 \sinh(u - p_1/2) & e_{21} & 0 & e_{1N} \\ e_{21} & 2 \sinh(u - p_2/2) & e_{32} & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 2 \sinh(u - p_N/2) \end{pmatrix}.$$
where \( e_{jn} = \exp\left(\frac{q_j - q_n}{2}\right) \). The matrices \( L(u) \) and \( \tilde{\mathcal{L}}(k) \) are connected by the formula

\[
L(u) = U \tilde{\mathcal{L}}(k) U
\]

where the matrix \( U \) has the form

\[
U = k^{-1/2} \text{diag}(\exp(-\frac{p_1}{4}), \exp(-\frac{p_2}{4}), \ldots, \exp(-\frac{p_N}{4})).
\]

Due to the relation (29) one can construct the Lax triad for the matrix \( L(u) \):

\[
\frac{dL(u)}{dt} = [L, M] + FL,
\]

where the matrices \( M \) and \( F \) are expressed in terms of the known matrices \( \tilde{\mathcal{M}} \) and \( \tilde{\mathcal{F}} \) (see (15)–(16))

\[
M = U^{-1} \tilde{\mathcal{M}} U + U^{-1} \dot{U},
\]
\[
F = U^{-1} \tilde{\mathcal{M}} U - U \tilde{\mathcal{M}} U^{-1} + \tilde{\mathcal{F}} + 2U^{-1} \dot{U}.
\]

Notice that in the limit to non-relativistic case the Lax triad \( L, M, F \) turns into well-known Lax pair for the usual Toda lattice [4,8].

The separation of variables for the periodic RTL was originally given in [1]. Let us carry out this procedure within the \( r \)-matrix method with an intention to generalise all the results afterwards to the quantum case.

We begin with the monodromy matrix \( \tilde{T}(k) \) (20)–(21), since due to the Proposition 2 the entries of it have simple dependence upon the spectral parameter \( k \). Let us take as new variables \( \mu_j \) the zeros of the off-diagonal entry \( \tilde{\mathcal{C}}(\lambda) \) (for such chosen, the variables \( \mu_j \) coincide up to a factor with the ones introduced earlier [2] and are zeros of the Baker-Akhiezer function):

\[
\tilde{\mathcal{C}}(\mu_j) = 0, \quad j = 1, \ldots, N - 1.
\]

By the Proposition 2 these variables are positive real and independent. Further, define a set of the variables \( \nu^\pm_j \) as follows:

\[
\nu^-_j = \tilde{A}_N(\mu_j), \quad \nu^+_j = \tilde{D}_N(\mu_j), \quad j = 1, \ldots, N - 1.
\]

The variables \( \mu_j, \nu^\pm_j, j = 1, \ldots, N - 1 \) obey the Poisson brackets:

\[
\{\mu_j, \nu^-_n\} = \pm \delta_{jn} \nu^-_n, \quad \{\nu^+_j, \nu^\pm_n\} = \{\mu_j, \mu_n\} = 0,
\]
\[
\{\nu^-_j, \nu^+_n\} = -\delta_{jn} \frac{d \det \tilde{T}(\lambda)}{d\lambda} \bigg|_{\lambda=\mu_j},
\]
\[
\nu^+_j \nu^-_j = \det \tilde{T}(\mu_j).
\]

These brackets follow from the fundamental Poisson brackets (21) and can be calculated along the lines of the work [10].
Introduce two more variables

$$\mu_N = \exp(p_N - \tilde{q}_N), \quad \nu_N^\pm = \exp(\pm P).$$

The set of the $\mu_j, \nu_j^\pm$, $j = 1, \ldots, N$ is complete, i.e. any dynamical variable can be expressed through them. To prove this let us restore the monodromy matrix $\tilde{T}(k)$ in terms of the new variables:

$$\tilde{A}_N(\lambda) = \lambda \nu_N^\pm \prod_{j=0}^{N-1} (\lambda - \mu_j) + \sum_{j=0}^{N-1} \prod_{i=0}^{N-1} \frac{\lambda - \mu_j}{\mu_i - \mu_j} \nu_j^-,$$

where $\mu_0 \equiv 0$ and $\nu_0^- \equiv 1$,

\begin{equation}
(34)
\end{equation}

$$\tilde{D}_N(\lambda) = \sum_{j=1}^{N-1} \prod_{i=1}^{N-1} \frac{\lambda - \mu_j}{\mu_i - \mu_j} \nu_j^+, \quad \tilde{C}_N(\lambda) = \mu_N k \prod_{j=0}^{N-1} (\lambda - \mu_j)$$

(recall that $\lambda = k^2$). The entry $\tilde{B}(k)$ of the matrix $\tilde{T}(k)$ can be obtained using the identity \(\det \tilde{T}(k) = k^N \exp(-P)\).

Now we introduce another set of variables $y_j, Y_j^\pm$ using the monodromy matrix $T(u)$ (26)–(27):

\begin{equation}
(35)
Y^- = A(y_j), \quad Y^+ = D(y_j), \quad j = 1, \ldots, N - 1.
\end{equation}

The matrix entry $C(u)$, considered as a function on the spectral parameter $u$, has infinite number of zeros, but using the connection between the matrices $T(u)$ and $\tilde{T}(k)$ (28) we can prove that these zeros have the following structure

\begin{equation}
(36)
y_{jn} = y_j + 2\pi i n \equiv \ln \mu_j + 2\pi i n, \quad n = 0, 1, 2, \ldots,
\end{equation}

hence the variables $y_j, \ j = 1, \ldots, N - 1$ can be chosen to be simple and real. Variables $y_j, Y_j^\pm$ obey the Poisson brackets:

$$\{y_j, y_n\} = \{Y_j^\pm, Y_n^\pm\} = 0, \quad Y_j^\pm Y_j^\mp = 1, \quad \{y_j, Y_n^\pm\} = \pm \delta_{jn} Y_n^\pm.$$

One can restore the monodromy matrix $T(u)$ in terms of these variables.

The set $y_j, Y_j^\pm$ gives the separation variables, since by their definition (35), they obey the equations

\begin{equation}
(37)
\text{tr} T_N(y_j) = Y_j^+ + Y_j^-, \quad j = 1, \ldots, N - 1.
\end{equation}
4. Separation of variables in the quantum case

In this section we apply the general scheme [10] of the quantum separation of variables to the quantum $A_{N-1}$ RTL model.

Quantum $L$-operator for the non-relativistic Toda lattice [3,10] has the form

$$L_j (u) = \begin{pmatrix} u - \frac{p_j}{2} & -\exp(q_j) \\ \exp(-q_j) & 0 \end{pmatrix}.$$ 

For the RTL the quantum $L$-operator and the monodromy matrix $T_N (u)$ are

$$L_j (u) = \begin{pmatrix} 2 \sinh(u - \frac{p_j}{2}) & -\exp(q_j) \\ \exp(-q_j) & 0 \end{pmatrix},$$

$$T_N (u) = \prod_{j=1}^{N} L_j (u) = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $[p_j, q_n] = i \eta \delta_{jn}$

and obey the quadratic $R$-matrix algebra

$$R (u - v) T^{(1)} (u) T^{(2)} (v) = T^{(2)} (v) T^{(1)} (u) R (u - v),$$

where $T^{(1)} (u) = T(u) \otimes I$ and $T^{(2)} (v) = I \otimes T(v)$ and $R(u)$ is the following trigonometric solution of the quantum Yang-Baxter equation [3,6,10]:

$$R (u) = \begin{pmatrix} a(u) & 0 & 0 & 0 \\ 0 & b(u) & c(u) & 0 \\ 0 & c(u) & b(u) & 0 \\ 0 & 0 & 0 & a(u) \end{pmatrix}$$

where

$$a(u) = \sinh(u + \eta), \quad b(u) = \sinh u, \quad c(u) = \sinh \eta.$$ 

It is natural now from the above formulation of the problem to call the RTL a $q$-deformation ($q = e^\eta$) of the ordinary Toda lattice.

Separation of variables for the quantum RTL is taken in a complete analogy to the separation for the ordinary Toda lattice [10]. One considers the operator roots of the equation $C_N (u) = 0$. They are $N - 1$ Hermitian operators $y_j$ which give quantum analogs of the corresponding real simple zeros in the classical case (36).

Let us introduce the operators $Y_j^\pm$, $j = 1, \ldots, N - 1$ by the rule:

$$Y_j^- = A_N (u \rightarrow y_j), \quad Y_j^+ = D_N (u \rightarrow y_j).$$

These operators are defined by the values of the functions $A(u)$ and $D(u)$ in $u = y_j$.

The substitution of operator values for $u$ will be defined correctly if one prescribes some rule for operator ordering. Here the ordering of $y$'s to the left chosen. We will call it substitution from the left and denote $u \rightarrow y_j$. The operators $y_j$, $Y_j^\pm$, $j = 1, \ldots, N - 1$ have nice commutation relations

$$[y_j, y_n] = [Y_j^\pm, Y_n^\pm] = 0, \quad Y_j^\pm Y_n^\mp - 1, \quad [y_j, Y_{\pm}] = \mp i \eta \delta_{jn}, \quad Y_j^\mp.$$
which follow from the basic commutation relation (39) [10].

Let we consider the spectral problem

\[ \text{tr} T(u) \psi = \tau(u) \psi \]

then substitute \( u = y_j \) (from the left) and use the definition of \( Y_j^\pm \). The resulting set of equations has the form

\[ \tau(y_j) \psi(y_j) = (\Delta_- Y_j^- + \Delta_+ Y_j^+) \psi(y_j) \]

where \( \Delta_\pm(u) \) give a factorization of the quantum determinant of the monodromy matrix [10] \( \Delta(u) = \text{det}_q T(u) = \Delta_+ \Delta_- = 1 \). Let \( \Delta_- = i^{-N}, \Delta_+ = i^N \). In \( y \)-representation the operators \( Y_j^\pm \) are acting as shift operators and the separation equations take the form

\[ \tau(y_j) \psi(y_j) = i^N \psi(y_j + i\eta) + i^{-N} \psi(y_j - i\eta) \]  

while the full eigenfunction can be represented in the factorised form (quantum separation of variables)

\[ \psi(y_1, \ldots, y_{N-1}) = \prod_{k=1}^{N-1} \psi(y_k). \]

The separation equations above are \( N-1 \) one-dimensional finite-difference multi-parameter spectral problems.

We can rewrite equations (43) as

\[ \sum_{k=0}^{N} \left( I_{N-2k} \exp \left( (N - 2k) y_j \right) \right) \psi(y_j) = \]

\[ = i^N \psi(y_j + i\eta) + i^{-N} \psi(y_j - i\eta) \]

where \( I_{\pm k} \) are the eigenvalues of the quantum integrals of motion. For ordinary Toda lattice these equations have the form

\[ \sum_{k=1}^{N} \left( I_k y_j^k \right) \psi(y_j) = i^N \psi(y_j + i\eta) + i^{-N} \psi(y_j - i\eta). \]

As for a solution of the above separation equations, there are more questions till now than answers, although in the non-relativistic case one can use the Gutzwiller algorithm. The RTL case provides even more difficult problem, for example, for \( N = 2 \) the separation equation

\[ (\cosh p + E) \psi = \sinh x \psi \]

becomes the famous Harper equation [5].
5. Monodromy matrices of the quantum RTL for general classical root systems

The monodromy matrices for such systems obey, as well as in the non-relativistic case [7,11], the reflection equations

\[ R (u-v) \hat{U}_{-}(u) R (u+v-\eta) \hat{U}_{-}(v) = \]
\[ 2 \hat{U}_{-}(v) R (u+v-\eta) \hat{U}_{-}(u) R (u-v), \]
\[ R (-u+v) \hat{U}_{+}^{t_{1}}(u) R (-u-v-\eta) \hat{U}_{+}^{t_{2}}(v) = \]
\[ 2 \hat{U}_{+}^{t_{2}}(v) R (-u-v-\eta) \hat{U}_{+}^{t_{1}}(v) R (-u+v), \]

where \( \hat{U}_{\pm} = U_{\pm} \otimes I, 2 U_{\pm} = I \otimes U_{\pm} \) and \( t_{1,2} \) are matrix transpositions in the first and second spaces, respectively.

We have the following rules for constructing the monodromy matrices \( \hat{T}(u) \) here [11]:

\[ U_{-}(u) = T_{-}(u) K_{-}(u - \frac{i\eta}{2}) T_{-}^{-1}(-u), \]
\[ U_{+}^{t}(u) = T_{+}^{t}(u) K_{+}(u + \frac{i\eta}{2}) (T_{+}^{-1}(-u))^{t}, \]
\[ \hat{T}(u) = U_{+}(u) U_{-}(u), \quad \text{for the closed lattices (interval)} \]
\[ \hat{T}(u) = U_{-}(u), \quad \text{for the open lattices (semi-axis)} \]

where the \( T_{\pm}(u) \) are matrices obeying the equation (39) and the \( K_{\pm} \) are simple solutions of the reflection equations (45). The generating function of the integrals of motion is [7,11]

\[ t(u) = \text{tr} \hat{T}(u) = \text{tr} U_{+} U_{-} \]
\[ = \text{tr} K_{+}(u + \frac{i\eta}{2}) T(u) K_{-}(u - \frac{i\eta}{2}) T^{-1}(-u), \]
\[ T(u) = \prod_{j=1}^{M} L_{j}(u). \]

For the root systems \( B_{N} \) and \( C_{N} \), \( M = N \) and matrices \( K_{\pm} \) are scalar solutions of the reflection equations (45); for the root system \( D_{N} \), \( M = N - 2 \) and matrices \( K_{\pm} \) depend on the dynamical variables.

We list below the matrices \( K_{\pm} \) and corresponding Hamiltonians for the quantum generalized RTL’s (we give also the corresponding \( K \)-matrices for the non-relativistic Toda lattices):

\( B_{N} \) and \( C_{N} \) series:

for RTL’s

\[ K_{-} = \left( \begin{array}{ccc}
\frac{\alpha_{1}}{\sinh u} + \frac{\beta_{1}}{\cosh u} & -\gamma_{1} \\
\frac{\alpha_{N}}{\sinh u} - \frac{\beta_{N}}{\cosh u} & \frac{\gamma_{N}}{\sinh u} + \frac{\beta_{N}}{\cosh u}
\end{array} \right), \]

\[ K_{+} = \left( \begin{array}{ccc}
\frac{\alpha_{1}}{\sinh u} + \frac{\beta_{1}}{\cosh u} & \frac{\gamma_{1}}{\sinh u} + \frac{\beta_{1}}{\cosh u} \\
\frac{\alpha_{N}}{\sinh u} - \frac{\beta_{N}}{\cosh u} & \frac{\gamma_{N}}{\sinh u} + \frac{\beta_{N}}{\cosh u}
\end{array} \right). \]
for non-relativistic Toda lattices
\[ K_- = \begin{pmatrix} \alpha_1 & -u \\ \beta_1 u & \alpha_1 \end{pmatrix}, \quad K_+ = \begin{pmatrix} \alpha_N & -\beta_N u \\ -u & \alpha_N \end{pmatrix}. \]

The Hamiltonian of the corresponding RTL's is
\[ H = H_0 + J_1 + J_N, \]
where we use the notations
\[ H_0 = \sum_{j=1}^{N} 2 \cosh(p_j) + 2 \sum_{j=1}^{N-1} \left[ \exp(q_{j+1} - q_j) \cdot \cosh\left( \frac{p_{j+1} + p_j}{2} \right) \right], \]
\[ J_1 = 2 \exp(q_1) \left[ \alpha_1 \cosh\left( \frac{p_1}{2} \right) + \beta_1 \sinh\left( \frac{p_1}{2} \right) \right] + \gamma_1 \exp(2q_1), \]
\[ J_N = 2 \exp(q_N) \left[ \alpha_N \cosh\left( \frac{p_N}{2} \right) + \beta_N \sinh\left( \frac{p_N}{2} \right) \right] + \gamma_N \exp(-2q_N). \]

We have here RTL's associated to the following root systems: \( C_N \) \((\alpha_1 = \alpha_N = \beta_1 = 0), B_N \) \((\alpha_1 = \beta_N = \beta_1 = 0), \) for the open lattices and \( C_N^{(1)} \) \((\alpha_1 = \alpha_N = 0), \) \( A_{2N}^{(2)} \) \((\beta_1 = \alpha_N = 0), \) \( D_{N+1}^{(2)} \) \((\beta_1 = \beta_N = 0)\) for the closed lattices.

\( D_N \):

Denote \( \hat{F}_j = \cosh(p_j + q_j) \) and \( \hat{G}_j = \cosh(p_j - q_j) \) then
\[ \hat{K}_- = \begin{pmatrix} \exp(u) \hat{F}_1 + \exp(-u) \hat{G}_1 & \sinh^2 q_1 - \sinh^2 (u) \\ \sinh^2 (u) - \sinh^2 p_1 & \exp(u) \hat{F}_1 - \exp(-u) \hat{G}_1 \end{pmatrix}, \]
\[ \hat{K}_+ = \begin{pmatrix} \exp(u) \hat{F}_N - \exp(-u) \hat{G}_N & \sinh^2 (u) - \sinh^2 p_1 \\ \sinh^2 q_1 - \sinh^2 (u) & \exp(u) \hat{F}_N + \exp(-u) \hat{G}_N \end{pmatrix}. \]

For ordinary Toda lattices
\[ \hat{K}_- = \begin{pmatrix} u \cosh q_1 - p_1 \sinh q_1 & \sinh^2 q_1 \\ u^2 - p_1^2 & u \cosh q_1 + p_1 \sinh q_1 \end{pmatrix}, \]
\[ \hat{K}_+ = \begin{pmatrix} u \cosh q_N + p_N \sinh q_N & u^2 - p_N^2 \\ \sinh^2 q_N & u \cosh q_N - p_N \sinh q_N \end{pmatrix}. \]

The Hamiltonian now is \( H = H_0 + \hat{J}_1 + \hat{J}_N \) where \( H_0 \) is given by (49) and
\[ \hat{J}_1 = 2 \exp(q_1 + q_2) \cosh\left( \frac{p_1 + p_2}{2} \right) + \exp(2q_2), \]
\[ \hat{J}_N = 2 \exp(-q_N - q_{N-1}) \cosh\left( \frac{p_N + p_{N-1}}{2} \right) + \exp(-2q_{N-1}). \]

We have here RTL's associated to the following root systems: \( D_N^{(1)}, B_N^{(1)} \) \((\beta_N = 0)\) and \( A_{2N}^{(2)} \) \((\alpha_1 = 0)\).
6. Acknowledgments

The authors are grateful to I.V. Komarov and E.K. Sklyanin for valuable discussions. AVT acknowledges the support from the Netherlands Organisation for Scientific Research (NWO) during one-week stay at the Universiteit van Amsterdam. VBK was supported by NWO under the Project # 611–306–540.

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