Existence of solutions to non-homogeneous higher order differential equation in the Schwartz space

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Abstract. There is studied problem on existence of solutions to non-homogeneous differential equation of higher even order. Similar problem arises while studying soliton and soliton-like solutions to partial differential equations of integrable type.
By means of Fourier transform and theory of pseudodifferential operators there is proved the theorem on necessary and sufficient conditions on existence of solutions to linear non-homogeneous differential equation of higher even order in the Schwartz space.

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1 Introduction and statement of problem

One of the most important problems of the qualitative theory of differential equations is the problem of existence and uniqueness of solutions to the
equations. Similar problems need to be studied frequently when considering the qualitative properties of differential equations, as well as of mathematical models of natural sciences. In this connection it should be noted the problem on existence of particular solutions with special properties that leads to searching the ones in certain functional spaces [1]. For example, while considering soliton solutions in hydrodynamical models there is appeared a problem on determining solutions in space of quickly decreasing functions. The same problems are also arisen when constructing asymptotic soliton like solutions for different partial differential equations of integrable type with singular perturbations [2].

While studying asymptotic solutions of soliton type to singular perturbed Korteweg-de Vries equation with variable coefficients [3] the problem on existence of a solution to the following equation

\[ \frac{d^2 v}{dx^2} + qv = f, \quad x \in \mathbb{R}, \]  

in Schwartz space has been generated.

On the other hand, when searching soliton solutions to the higher Korteweg-de Vries equations and KdV-like equations there is appeared the same problem for similar equations with differential operator of higher order. So, there is come up the problem on finding necessary and sufficient conditions on existence of a solution to the following equation

\[ Lv = f, \quad x \in \mathbb{R}, \]  

in the Schwartz space, where the differential operator \( L \) is written as

\[ L = - \sum_{m=1}^{n} a_{2m} \frac{d^{2m}}{dx^{2m}} + q \]  

with constant coefficients \( a_{2m}, m = 1, \ldots, n \), and \( q \) being a function of \( x \).

## 2 Main result

Let \( S(\mathbb{R}) \) be the Schwartz space.
The main result of the paper is the statement.

**Theorem 1.** Let the following conditions be fulfilled:

1. the coefficients \(a_{2m}, m = 1, n\), are nonnegative constants and \(a_{2n} > 0\);
2. \(q(x) = q_0 + q_1(x)\), where constant \(q_0 < 0\) and the function \(q_1(x) \in S(R)\);
3. the function \(f \in S(R)\).

If the kernel of the operator \(L : S(R) \to S(R)\) is trivial, then equation (2) has a solution in the space \(S(R)\) for any function \(f\).

Otherwise, if the kernel of the operator \(L : S(R) \to S(R)\) is not trivial, then equation (2) has a solution in the space \(S(R)\) if and only if the function \(f\) satisfies the condition of orthogonality in the form

\[
\int_{-\infty}^{+\infty} f(x)v_0(x)dx = 0 \tag{4}
\]

for any \(v_0 \in \text{ker } L\).

### 3 Necessary definitions and statements

To prove the theorem we need to remind some notations, definitions and results. For any function \(h \in S(R)\) there is denoted the Fourier transform as

\[
F[h](\xi) = \int_{-\infty}^{+\infty} e^{-ix\xi} h(x)dx.
\]

Due to properties of the Fourier transform for any differential operator

\[
p\left(x, \frac{d}{dx}\right) = \sum_{k=0}^{n} a_k(x) \frac{d^k}{dx^k}, \quad x \in R,
\]

it’s possible to define its action on function \(h \in S(R)\) as

\[
p\left(x, \frac{d}{dx}\right) h(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ix\xi} p(x, \xi) F[h](\xi) \, d\xi. \tag{5}
\]

Here

\[
p(x, \xi) = \sum_{k=0}^{n} a_k(x)(-i\xi)^k, \quad x, \xi \in R,
\]

is called a symbol of the differential operator \(p\left(x, \frac{d}{dx}\right)\).
Let $S^m$ be a set of symbols $p(x, \xi) \in C^\infty(\mathbb{R}^2)$ such that for any $k, l \in \mathbb{N} \cup \{0\}$ the inequality
\[
|p^{(k)}_{(l)}(x, \xi)| \leq C_{kl} (1 + |\xi|)^{m-k}, \quad (x, \xi) \in \mathbb{R}^2,
\]
is true, where
\[
p^{(k)}_{(l)}(x, \xi) = \frac{\partial^{k+l}}{\partial \xi^k \partial x^l} p(x, \xi), \quad (x, \xi) \in \mathbb{R}^2,
\]
and values $C_{kl}$, $k, l \in \mathbb{N} \cup \{0\}$, are some constants \[4\].

By $S^m_0$ denote a set of symbols $p(x, \xi) \in S^m$ such that
\[
|p(x, \xi)| \leq M(x) (1 + |\xi|)^m,
\]
where value $M(x) \to 0$ as $|x| \to +\infty$.

Let $H^s(\mathbb{R})$, $s \in \mathbb{R}$, be a Sobolev space \[5\], i.e. a space of generalized functions $g \in S^*(\mathbb{R})$ such that their Fourier transform $F[g](\xi)$ satisfies condition
\[
\|g\|^2_s = \int_{-\infty}^{+\infty} (1 + |\xi|^2)^s |F[g](\xi)|^2 d\xi < \infty. \quad (6)
\]

It is worthy to recall the following theorem.

**Theorem 2** (Grushin, \[3\]). Let $p(x, \xi) \in S^m$ be a symbol such that \[
\partial^l p(x, \xi)/\partial x^l \in S^m_0, \quad l \in \mathbb{N}, \quad \text{and inequality}
\]
\[
\lim_{(x, \xi) \to \infty} \frac{|p(x, \xi)|}{(1 + |\xi|)^m} > 0
\]
is true.

Then $p \left( x, \frac{d}{dx} \right) : H^{s+m}(\mathbb{R}) \to H^s(\mathbb{R})$, defined through formula \[5\], is the Noether operator for any $s \in \mathbb{R}$.

### 4 Proof of the main result

Proving the theorem 1 contains two steps. Firstly, we show that the operator $L : H^{s+2n}(\mathbb{R}) \to H^s(\mathbb{R})$ of form \[3\] is the Noether operator for any $s \in \mathbb{R}$. Later we prove that the solution to equation \[2\] belongs to the Schwartz space $S(\mathbb{R})$. 
Let us consider symbol of the differential operator $L$ having a form

$$p(x, \xi) = -\sum_{m=1}^{n} a_{2m} \xi^{2m} + q(x). \quad (7)$$

It’s obviously that $p(x, \xi)$ belongs to the set $S^{2n}$ due to inequality

$$\left| \frac{\partial^{k+l}}{\partial \xi^k \partial x^l} p(x, \xi) \right| \leq C_{kl} (1 + |\xi|)^{2n-k}, \quad k, l \in \mathbb{N} \cup \{0\}.$$ 

Moreover,

$$\frac{\partial^l}{\partial x^l} p(x, \xi) \in S^{2n}_0, \quad l \in \mathbb{N}.$$ 

According to assumptions of theorem 1 the operator $L : H^{s+2n}(\mathbb{R}) \to H^s(\mathbb{R})$ satisfies all conditions of theorem 2 for any $s \in \mathbb{R}$. So, it is the Noether operator. As consequence the operator $L : H^{s+2n}(\mathbb{R}) \to H^s(\mathbb{R})$ is normally solvable.

By $L^*$ denote an operator being adjoint to the operator $L$.

Let us assume kernel of the operator $L^*$ be nontrivial. Then differential equation $[2]$ has a solution in the space $H^s(\mathbb{R})$ if and only if the following condition of orthogonality

$$< f, \ker L^* > = 0 \quad (8)$$

holds.

Since

$$L^* = -\sum_{m=1}^{n} a_{2m} \frac{d^{2m}}{dx^{2m}} + q(x),$$

then $\ker (L^*) \subset \bigcap_{s \in \mathbb{R}} H^s(\mathbb{R}) [5]$. 

Using Sobolev embedding theorems for the spaces $H^s(\mathbb{R})$, $s \in \mathbb{R}$, we find $v_0^* \in \tilde{C}^\infty_0(\mathbb{R})$ for any element $v_0^* \in \ker L^*$.

As a consequence of the orthogonality condition [5] and theorem 2 one obtains that the solution $v(x)$ of equation [2] belongs to space $\bigcap_{s \in \mathbb{R}} H^s(\mathbb{R})$.

Arguing as above we get $v \in \tilde{C}^\infty_0(\mathbb{R})$.

Now let us show that moreover $v \in S(\mathbb{R})$. Indeed, since the function $v \in \tilde{C}^\infty_0(\mathbb{R})$ and it satisfies equation

$$-\sum_{m=1}^{n} a_{2m} \frac{d^{2m} v}{dx^{2m}} = -qv + f, \quad (9)$$
where the function \(-qv + f \in S(\mathbb{R})\), then due to properties of elliptic pseudodifferential operators with polynomial coefficients [6] we deduce that any solution to equation (9) from the space \(S^*(\mathbb{R})\) belongs to the space \(S(\mathbb{R})\). Thus, \(v \in S(\mathbb{R})\).

Continuing this line of reasoning we see \(v_0^* \in S(\mathbb{R})\). Kind of the operator \(L^*\) implies equality \(v_0^* = v_0\). It means that orthogonality condition (8) is equivalent to the one (4).

From the above consideration it also follows that if the kernel of the operator \(L : S(\mathbb{R}) \to S(\mathbb{R})\) is trivial, i.e. the homogeneous equation \(Lv = 0\) has the only trivial solution in the space \(S(\mathbb{R})\), then equation (2) has a solution in the space \(S(\mathbb{R})\) for any \(f \in S(\mathbb{R})\).

The theorem 1 is proved.

5 Conclusions

Theorem on existence of a solution to the linear non-homogeneous differential equation in the Schwartz space is proved. The theorem can be used while studying soliton solutions to integrable systems of modern mathematical physics [7] and asymptotic soliton like solutions to singular perturbed higher order partial differential equations of integrable type with variable coefficients.

The theorem 1 generalizes the statement on existence of a solution to the non-homogeneous equation with the one-dimensional Schrodinger operator in the space of quickly decreasing functions [8].

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