The Stable Random Matrix ensembles

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Abstract

We address the construction of stable random matrix ensembles as the generalization of the stable random variables (Levy distributions). With a simple method we derive the Cauchy case, which is known to have remarkable properties. These properties allow for such an intuitive method -that relies on taking traces- to hold. Approximate but general results regarding the other distributions are derived as well. Some of the special properties of these ensembles are evidenced by showing partial failure of mean-field approaches. To conclude, we compute the confining potential that gives a Gaussian density of states in the limit of large matrices. The result is an hypergeometric function, in contrast with the simplicity of the Cauchy case.

1 Introduction

The ensembles of random matrices have shown beyond doubt its usefulness in many physical applications [13]. The definition of the joint probability distribution $P(M)$ of the matrix elements of a $N$ by $N$ matrix $M$ is [13]:

$$P(M) = C_N \exp[-Tr(V(M))] ,$$

with an arbitrary $V(M)$, provided existence of the partition function $C_N^{-1}$. Integrating (1) over the parameters related to the eigenvectors, one obtains
the well-known joint probability distribution of the eigenvalues:

\[ P(x_1, \ldots, x_N) = C_N \exp\left[-\sum_{i=1}^{N} V(x_i)\right] \prod_{i<j} |x_i - x_j|^\beta. \] (2)

Level repulsion described by the Vandermonde determinant is originated from the Jacobian, that appears when passing from the integration over independent elements of the Hamiltonian to the integration over the smaller space of its \( N \) eigenvalues. The parameter \( \beta \), with values 1, 2 or 4 describe the symmetry of the ensemble (named orthogonal, unitary and symplectic, respectively).

If the elements of the random matrix are believed to be statistically independent from each other, one obtains the quadratic confinement potential, leading to the Gaussian ensembles of random matrices [13]. The requirement of statistical independence is not motivated from first principles. Thus, the influence of a particular form for the confinement potential in relationship with the Gaussian predictions is an interesting problem [11].

As it is well-known, the joint probability distribution can be mapped onto the Gibbs distribution of a classical one-dimensional gas of fictitious particles with a pair-wise logarithmic repulsion and a one-particle potential.

\[ P(\{x_i\}) = Z^{-1} \exp[-\beta H(\{x_i\})], \] (3)

\[ H(\{x_i\}) = -\sum_{i<j} \ln |x_i - x_j| + \sum_i V(x_i). \] (4)

This is usually called the Coulomb gas picture. Let us now present the mean-field approximation that was introduced by Dyson [8]. By substituting the density of states in the previous Hamiltonian, we obtain a continuous description of the energy functional \( H[\rho] \) in terms of \( \rho(x) \). The extremum of this functional corresponds to an equilibrium of the effective plasma expressed by the equation:

\[ \int dx' \rho(x') \ln |x - x'| = V(x) + c, \] (5)

where \( \rho \) is the mean density. This expression is usually known as Wigner equation. In [8], the next correction term was found using a hydrodynamical approximation:

\[ V(x') = \int_{-\infty}^{\infty} \rho(x) \ln |x - x'| dx + \frac{1}{2} (1 - \frac{\beta}{2}) \ln \rho(x'). \] (6)
Higher order terms involve a complicated combination of the density of levels with correlation functions [2]. In addition, in [8] it was shown that in this approximation the first correction term—the one that appears in (6)—is of order $\ln \frac{N}{N}$, while the next term is of order $N^{-2}$. This applies to systems with strong confining potential.

In this paper, we study features of stable random matrix ensembles, in analogy with the stable random variables. We provide here a derivation, based on simple arguments, of the Cauchy case for the three ensembles ($\beta = 1, 2$ and $4$). In this way, we obtain the so-called Lorentzian ensemble, that has been already studied with various points of view [2, 4, 10, 18]. Further work related with stable distributions can be found in [17], and from the point of view of free probability theory [16].

In section 3, this ensemble allows to provide explicit examples to the failure of Dyson equation, in the case of non strongly confining potentials. Additionally, we solve several inverse problems using Dyson equation and present a systematic comparison between all cases. In particular, we find the confining potential that gives a Gaussian density of states in the limit $N \rightarrow \infty$.

2 The stable random matrix ensembles. Derivation

We begin by showing a simple way to obtain the expression for the stable ensembles. We will focus on the Cauchy distribution, but all the stable distributions can be studied, in principle, with the same method. We look for ensembles that satisfy the following:

$$M = \sum_{i=1}^{N} M_i . \tag{7}$$

$M$ and $M_i$ denote the random matrices on the same ensemble. Put it simply, the matrix generalization of the stable random variables (see Appendix A). Notice the following:

$$\text{Tr}M = \text{Tr} \sum_{i=1}^{N} M_i = \sum_{i=1}^{N} \text{Tr}M_i . \tag{8}$$

By taking the trace of the matrix, we reduce the problem to a problem of random variables. In order that this last equality holds, it is evident that
the trace itself should be a stable random variable. More clearly, we make a mapping between the stable random variables and the stable matrices (the matrix counterpart). It is manifest now in which sense we say that we search for the matrix generalization of the stable random variables. Note that the expression for the Gaussian ensemble satisfy the relation (8), after taking the trace on its \( P(x_1, ..., x_N) \). Let us go now into the Cauchy case. We begin by writing again the general expression in terms of the eigenvalues:

\[
P(x_1, ..., x_N) = C_N \prod_{i<j} |x_i - x_j|^\beta \exp\left[-\sum_{i=1}^{N} V(x_i)\right].
\]  

(9)

For consistency with the case \( N = 1 \), we clearly need the following weight function (or confining potential):

\[
\omega(x) = \exp(-V(x)) = \frac{1}{\lambda^2 + x^2}.
\]  

(10)

It is also clear that we need some function of \( N \) in our weight function. Otherwise, the joint probability distribution would not even be normalizable. What we need to impose is:

\[
\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dx_1 \cdots dx_N P(x_1, ..., x_N) \delta(y-x_1-x_2-...-x_N) = \frac{\lambda N}{\pi} \frac{1}{(\lambda N)^2 + y^2},
\]  

(11)

with:

\[
P(x_1, ..., x_N) = C_N \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^{N} \left(\frac{1}{\lambda^2 + x^2}\right)^{\alpha(N)}.
\]  

(12)

From the case \( N = 1 \), we already know that \( \alpha(1) = 1 \). We can see that the function \( \alpha(N) \) should be a linear polynomial in \( N \). Notice that we are studying the modification we have to impose to the Cauchy distribution in order that the effect introduced by the correlation term does not modify the stability property. The Vandermonde determinant introduces the product of \( N(N-1)/2 \) polynomial factors, while the weight function term introduces \( N \) products of polynomial factors. Thus, we need each one of these factors to contain a term of order \( N \). So, we have to consider the following function:

\[
\alpha(N) = a_0 N + a_1.
\]  

(13)

Then, it is enough to look at the \( N = 2 \) case. Needless to say, it can be checked that, for example, the case \( N = 3 \) implies a zero coefficient for an
hypothetical quadratic term, and so on. Actually, in the next section we will present an alternative argument that also implies that the function should be of the form indicated above.

Doing the integral for the three ensembles, we finally arrive at:

\[ P(x_1, \ldots, x_N) = \frac{2^{\frac{3}{2}(N^2-N)}}{N!\pi^N} \prod_{i<j} |x_i - x_j|^\beta \prod_{k=1}^{N} \left( \frac{1}{\lambda^2 + x^2} \right)^{(1+\frac{\beta}{2}(N-1))}. \]  

We have arrived at a known and important random matrix ensemble \[4, 10\]. In the physics literature it is named Lorentzian ensemble \[4\]. The normalization constant can be computed with the Selberg integral \[13\].

2.1. The relationship with the circular ensemble and universality.

It has been shown that the Lorentzian ensemble \[14\] has a density of states with same form for any dimension of the matrix and for any \(\beta\) \[4\]. Recall that the same result holds for the circular ensembles: they have constant density on the circle for all \(N\) \[13\]. We know that there are several different parametrizations for a unitary matrix. A rather convenient specific parametrization is \(S = \frac{1-iA}{1+iA}\) (Cayley transform), since it is one on one. Notice the effect of any of this transformation on the Cauchy probability distribution:

\[ f(x) = \frac{1}{\pi} \frac{1}{1 + x^2} \text{ and } x = \cotg(\theta/2), \]  

then:

\[ f(\theta) = \left| \frac{dx}{d\theta} \right| \frac{1}{\pi(1 + \cotg(\theta/2))^2} = \left| \frac{-1}{2\cos^2 \theta} \right| \frac{\cos^2 \theta}{\pi} = \frac{1}{2\pi}. \]  

That is, the Cauchy distribution transforms into a uniform probability in the circle. In \[4, 10\], it is shown that under the Cayley Transform, the Circular ensemble maps into the Lorentzian ensemble. Consequently, taking into account the universality property for the correlation functions of the circular ensembles in the limit \(N \to \infty\) \[13\], the Lorentzian ensemble also exhibits universal behavior in this limit. That is to say, all the correlation functions of the Lorentzian ensemble behave as the ones from the Gaussian ensemble in the limit of large matrices \[4\].
Additionally, notice that if $X$ is a random variable with probability distribution function $f(x) = \frac{\lambda}{\pi(\lambda^2 + x^2)}$, then the random variable $Y = X^{-1}$ has the probability distribution $g(y) = \frac{\lambda}{\pi(\lambda^2 + y^2)}$. Namely, the inverse of a Cauchy distribution is a Cauchy distribution. Again, we see that the same property holds for the matrix generalization, and in [10] it is shown that if $M$ is a matrix in the Lorentzian ensemble, then $M^{-1}$ is also a matrix in the Lorentzian ensemble. This result readily follows by the corresponding change of variables in the multidimensional probability distribution function for the eigenvalues.

Therefore, a matrix in the Lorentzian ensemble shares many properties with a diagonal matrix, whose elements are identically distributed Cauchy distributions. First of all, the diagonal matrix, after trivial integration over $N - 1$ eigenvalues, also possesses a Cauchy density of states (independent of $N$). Secondly, at the level of the density of states it also has the correspondence with a uniform density. Furthermore, the diagonal matrix is equal to its inverse as well, as it readily follows from the property of the Cauchy distribution. Needless to say, the correlation properties of the eigenvalues are very different. These properties are not shared by a random matrix in general and make the Lorentzian ensemble very useful for physical applications [4].

It is also interesting to discuss a little bit more about universality. As it is well-known, any strongly confining potential (that is, growing at least as $V(x) \sim |x|$ for $x \to \infty$), gives the same Gaussian universality class [14] (see also [1]). Here, we deal with a special confining potential, that contains the dimension of the matrix $N$ in an explicit way:

$$V_\beta(x) = (\beta N + 2 - \beta) \ln(\lambda^2 + x^2) .$$

Note that the dependence with the variable $x$ is given by a very soft confining potential. Nevertheless, it is multiplied by a function that is linear in $N$, the dimension of the matrix. This is important, since any random matrix universality regime is understood in the limit $N \to \infty$. Thus, at the end, it is the term in $N$ the one that achieves the universal behavior. This clarifies the mechanism that makes this ensemble to belong to the Gaussian universality class. The consequence is that the ensemble exhibits very different global properties (the density of states), while it has the same behavior at the local scale (all the correlation functions).

It is patent that all the arguments above hold for any stable distribution (see Appendix A). As an extension, note that this matrix generalization of
the Levy distributions can be expressed, in general, as:

\[ P(x_1, \ldots, x_N) = C_N \prod_{i<j} |x_i - x_j|^{\beta} \prod_{k=1}^{N} \left( f_{\alpha, \beta}(x) \right)^{a_0 N + a_1} , \quad a_0 + a_1 = 1 . \tag{18} \]

 Needless to say, it is consistent that the only stable distribution with finite variance, the Gaussian distribution, is the only one that has different form for \( \alpha(N) \) (more precisely \( a_0 = 0 \) and \( a_1 = \beta \)) in the matrix generalization.

We see that the coefficient \( a_0 \) is actually equal to \( a_0 = \beta \mu = \frac{\beta}{\alpha + 1} \). Needless to say, as explained above, the value of the coefficient depends both on the power of the polynomial in the numerator (the Vandermonde determinant) and the one in the denominator (weight function). Thus, this particular value seems to be expected for all the Levy distributions. Since we know the asymptotic behavior of the distributions for \( x \) large (see Appendix A):

\[ f_{\alpha, \beta}(x) \sim \frac{A_{\alpha, \beta}}{|x|^{1+\alpha}} \quad \alpha < 2 , \tag{19} \]

we have the following confining potential for \( x \) large:

\[ V(x) = \left( \frac{\beta}{\alpha + 1} N + a_0 \right) \ln \left( 1 + x^{1+\alpha} \right) \approx \left( \beta N + \frac{a_0}{1+\alpha} \right) \ln x , \tag{20} \]

and, for \( N \) large and \( x \) large:

\[ V(x) = \beta N \ln x . \tag{21} \]

This coincides with the result found in [7] for the Hermitian case (\( \beta = 2 \)), the one considered there. Notice that, at this level of approximation, the confining potential is the same for any Levy distribution. Since this is precisely the universality limit, it is expected that they all share the same universal properties in the sense explained above.

### 3 The Dyson equation

We use now the previous results to illustrate the failure of Dyson’s mean-field approximation in certain cases. This has been recently discussed in the literature, and has proved to be an increasingly interesting topic, as soon as
new type of random matrix ensembles, different from the Gaussian ones, are studied. As mentioned, Dyson equation is (see also [2]):

\[ V(x') = \int_{-\infty}^{\infty} \rho(x) \ln |x - x'| dx + \frac{1}{2}(1 - \frac{\beta}{2}) \ln \rho(x') . \]  

(22)

We would like to study several nontrivial examples that allow to illustrate the situation. In the progress of doing this, we will also solve some interesting inverse problem in random matrix theory.

The Cauchy case allow us for a simple and enlightening example. First, we begin by asking which is the confining potential that give us a Cauchy density of states in the limit \( N \) very large. This reduces to compute:

\[ V(x) = \frac{\lambda}{\pi} \int_{-\infty}^{\infty} \frac{\ln |x - x'|}{\lambda^2 + x'^2} dx = \frac{\lambda}{2\pi} \ln(\lambda^2 + x^2) = \frac{N}{2} \ln(\lambda^2 + x^2) . \]  

(23)

To begin with, we find that for the Hermitian case (\( \beta = 2 \)), this result gives just half of the right value, since the first correction term is identically zero for \( \beta = 2 \). Thus, we already see that half of the value would be contained in the rest of the following correction terms and we should remember that the first one of these corrections is of order \( N^{-2} \) in the strongly confining paradigm. In fact, with the confining potential obtained in (23), we do not even have a normalizable probability distribution function, so it can not even define an ensemble. We see, in this way, a clear example for the failure of the mean-field approximation. It is interesting to summarize what we obtain for the three symmetries:

| \( \beta \) | Exact | Mean-field |
|-----|-------|-----------|
| 1   | \( (\frac{N+1}{2}) \ln(1 + x^2) \) | \( (\frac{N-1}{2}) \ln(1 + x^2) \) |
| 2   | \( N \ln(1 + x^2) \) | \( \frac{\lambda}{\sqrt{2}} \ln(1 + x^2) \) |
| 4   | \( (2N - 1) \ln(1 + x^2) \) | \( (\frac{N}{\sqrt{2}} + \frac{1}{4}) \ln(1 + x^2) \) |

(24)

Notice that the mean-field analysis is still able to give the right analytical expression for the part in \( x \) of the potential, and again hints that we should expect a linear polynomial for the part in \( N \). Nevertheless, it completely fails to give the right values for the coefficients of the polynomial.

The typical use of the Wigner integral is to show, for example, that a parabolic confining potential give rise to the Wigner semi-circle law [13].

\(^1\)Note that we already know the answer, is the confining potential in the previous section.
is standard material, but not completely trivial. Now, the reader might wonder what happens with some intermediate cases, like for example a density of eigenvalues without compact support but coming from a strongly confining potential. A natural choice is a Gaussian distribution for the density of states. This is also an interesting problem for other reasons: on one hand, since we know in detail the Cauchy case -that turns out to be rather simple- then, the Gaussian case naturally comes to mind. On the other hand, a Gaussian density of states has a long story in the theory of random matrices and it is indeed physical. We have to compute:

\[ V(x') = \int_{-\infty}^{\infty} e^{-x'^2} \ln |x - x'| \, dx . \]  

This is a non trivial computation, relevant in other fields as well (including disordered systems [5] and determinants of random Schrödinger operators [12]). For example, it appears in [12], where it is left unsolved. We carry out the computation in Appendix B. The result is:

\[ V(x) = x^2 F_2(1, 1; \frac{3}{2}, 2; -x^2) . \]  

Note that we need a rather complex expression for the confining potential in order to have an ensemble with Gaussian density of states in the limit \( N \to \infty \).

In spite of this complex solution -in great contrast to the simplicity of the Cauchy case-, it would be of interest to study it in connection with the two body random matrix ensembles. These are old ensembles, that have been the subject of considerable revision and renewed interest (see [6] for a recent review). They are designed to include correlations among the different matrix elements and one of their main features is that they show, in a certain regime, a Gaussian density of states. In contrast, we have obtained the confining potential that give us this density of states for ensembles with the typical symmetries (\( \beta = 1, 2, 4 \)) of random matrix theory.

To conclude this section, notice the different results given by the first corrective term in Dyson equation depending on the nature of the density of states:

\[ \rho(x) = C \exp(-\sigma x^\mu) \to \ln \rho(x) = \ln C - \sigma x^\mu , \]  
\[ \rho(x) = \frac{\lambda}{\pi \lambda^2 + x^2} \to \ln \rho(x) = \ln \frac{\lambda}{\pi} - \ln(\lambda^2 + x^2) . \]
The constant terms are dropped from the potential since they are reabsorbed in the normalization constant. Observe that the parameter appears explicitly multiplying the potential in the stretched exponential case (in this case, the parameter is essentially the variance and it is directly related to $N$). In contrast, in the Cauchy case, the parameter $^2$ drops out, and does not enter into the expression of the potential. We see then a big difference between the two cases and a complete and systematic study of all the possible cases seems worth studying.

4 Conclusions and Outlook

We have shown with simple arguments how to arrive to random matrix ensembles that are the matrix generalization, in the sense of random matrix theory, of the scalar Levy distributions. In the particular case where an exact solution is known (the Cauchy case), an explicit form for the ensemble can be obtained. We have arrived to known and well-established results in the mathematics and the physics literature $^4$. More important, our intuitive method of derivation clearly shows some of the particular features of these type of ensembles. It reveals that we need an explicit and, at most, linear dependence of the confining potential with the dimension of the matrix. Namely, the confining potential is so weak that we are forced to this dependence.

Thus, a purely logarithmic term (as exactly appears in the Cauchy case, and asymptotically in the others), yields an algebraic expression for the weight function, that has to battle in equal conditions with the Vandermonde determinant. It is precisely this point what makes this ensemble so special. It can also be seen within the Coulomb gas picture, where the Vandermonde determinant is a logarithmic two-body repulsive term and the weight function is essentially a logarithmic (one-body) confining potential, that needs the explicit dependence with $N$ to be able to compensate the repulsive term. Then, it is rather natural that we get the same expression for the density of eigenvalues for all $N$, since we have a modified confining potential for each $N$. This is exact for the Lorentzian ensemble and also for the tails of the distribution of the eigenvalues for the other Levy cases.

$^2$That in the Cauchy case is not the variance, since it is infinite. This is usually referred as lack of characteristic scale.
The known results on the correlation functions of the Lorentzian ensemble are also noteworthy. In addition with the previous arguments, they lead to the expectation that all the Levy ensembles possess the universal behavior.

We have also seen, in the discussion on mean-field approaches, the differences with the Gaussian ensembles and with the strongly confining paradigm in general. In particular, the computation of the confining potential that gives a Gaussian shape for the density of states in the limit $N \to \infty$ is also useful to illustrate this. In the Levy cases, due to its power-like behavior we need an explicit modification of the potential for each dimension of the matrix. This leads to a relative simple solution for the density of states that is essentially insensitive to the level repulsion (by construction). This is exact in the Lorentzian case, and in great contrast with the Gaussian ensembles, where its shape goes from a Gaussian for $N = 1$ to a semicircle when increasing $N$. We have also studied the inverse case: after all the influence of level repulsion among all the eigenvalues (in the limit $N$ very large), what do we need to end up with a Gaussian shape for the density of states? The solution is a very complex expression for the confining potential. This complexity is in contrast with the simplicity of the Cauchy case. One interesting remark is that a power-like weight function is a natural antagonist for a power-like repulsion term.

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A. The stable random variables

The fundamental importance of the normal distribution is due to the Central Limit Theorem which is a consequence of the Bernoulli and de Moivre-Laplace theorems, and the law of large numbers.

For our discussion it is important to have clear the following basic results:

According to Levy, a distribution $F$ is stable if and only if, for the two positive constants $c_1$ and $c_2$, there exists a positive constant $c$ such that $X$ given by:

$$c_1X_1 + c_2X_2 = cX,$$

is a random variable following the same distribution $F$, as the independent, identically distributed random variables $X_1$ and $X_2$. Alternatively, if:

$$\varphi(z) \equiv \langle e^{izX} \rangle = \int_{-\infty}^{\infty} e^{izX}dF(X),$$

denotes the characteristic function of the distribution $F$, then $F$ is stable if and only if:

$$\varphi(c_1z)\varphi(c_2z) = \varphi(cz).$$

The most general definition can be found in \[9]. Let $X, X_1, X_2, \ldots, X_n$ be iid random variables with a common distribution $F$. Then $F$ is called stable iff and only iff there exist constants $c_n > 0$ and $\gamma_n$ such that:

$$Y_n \equiv \sum_i X_i = c_nX + \gamma_n.$$

Then, the characteristic function according to the definition above satisfy the functional relation:

$$\varphi^n(z) = \varphi(c_nz)e^{i\gamma_nz},$$

which can be solved exactly, and the result is:

**Proposition 1**

$$\psi(z) = \log \varphi(z) = i\gamma z - c|z|^\alpha \left\{ 1 + i\beta' \frac{z}{|z|} w(z, \alpha) \right\},$$

where $\alpha, \beta', \gamma, c$ are constants ($\gamma$ is any real number, $0 < \alpha \leq 2, -1 < \beta' < 1$, and $c > 0$, and,
\[ \omega(z, \alpha) = \begin{cases} \tan \frac{\pi \alpha}{2} & \text{if } \alpha \neq 1 \\ \frac{2}{\pi} \log |z| & \text{if } \alpha = 1 \end{cases} \] (35)

\( \alpha \) is called the Levy index or characteristic exponent. The limiting case \( \alpha = 2 \) corresponds to the Gaussian. For \( \beta' = 0 \) the distribution is symmetric. \( \gamma \) translates the distribution, and \( c \) is a scaling factor for \( X \). So, these last two parameters are not essential and one can disregard them.

**Proposition 2.** The asymptotic behavior of a Levy stable distribution follows the inverse power-law:

\[ f_{\alpha, \beta'}(x) \sim \frac{A_{\alpha, \beta'}}{|x|^{1+\alpha}}, \quad \alpha < 2. \] (36)

**Proposition 3.** The analytic form of a stable law is given through the Fox function:

\[ f_{\alpha, \beta'}(x) = \epsilon H_{2,1}^{1,1} \begin{bmatrix} x & (1 - \epsilon, \epsilon) \begin{pmatrix} 1 - \gamma, \gamma \\ 0, 1 \end{pmatrix} \\ (0, 1) \end{bmatrix} \] (37)

for \( \alpha < 1 \).

\[ f_{\alpha, \beta'}(x^{-1}) = \epsilon x^2 H_{2,1}^{1,1} \begin{bmatrix} x & (-1, 1) \begin{pmatrix} -\gamma, \gamma \\ -\epsilon, \epsilon \end{pmatrix} \\ (-\epsilon, \epsilon) \end{bmatrix} \] (38)

for \( \alpha > 1 \). With the abbreviations \( \epsilon = 1/\alpha \) and \( \gamma = (\alpha - \beta')/2\alpha \).

**Examples:** For \( \alpha = 2 \), then \( \beta' \equiv 0 \), and the stable density is identical to the Gaussian distribution.

For \( \alpha = 1 \) and \( \beta' = 0 \), the stable density is identical to the Cauchy or Lorentz distribution:

\[ f_{1,0}(x) = \frac{\lambda}{\pi(\lambda^2 + x^2)}. \] (39)

Further examples can be found in [9, 15].
B. Integral computation.

We need to compute the following integral:

\[ V(t) = \int_{-\infty}^{\infty} \ln |x - t| e^{-x^2} dx. \] (40)

These type of integrals represent a characteristic polynomial computation (the characteristic polynomial associated to a certain continuous density of states). Its relevance in different fields of mathematics and physics can be appreciated by consulting [5, 12].

First, we consider the Taylor series expansion:

\[ V(t) = \sum_{k=0}^{\infty} \frac{V^{(k)}(0)}{k!} t^k. \] (41)

and we make the following change of variables \( x = y + t \), then we have:

\[ V(t) = \int_{-\infty}^{\infty} \ln |y| e^{-(y+t)^2} dy. \] (42)

and we consider:

\[ \frac{d^k}{dz^k} e^{-z^2} = (-1)^k e^{-z^2} H_k(z). \] (43)

Then the Taylor expansion looks:

\[ V(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \hat{V}_k(0)}{k!} t^k, \] (44)

where \( \hat{V}_k(0) = \int_{-\infty}^{\infty} \ln |x| e^{-x^2} H_k(x) dx \). Because of the symmetry of the Hermite polynomials, we arrive at:

\[ V(t) = \sum_{k=0}^{\infty} \frac{(-1)^k \hat{V}_{2k}(0)}{k!} t^k \quad \hat{V}_{2k}(0) = 2 \int_{0}^{\infty} H_{2k}(x)e^{-x^2} \ln x \, dx. \] (45)

The point is that last integral can be solved and gives:

\[ \hat{V}_{2k}(0) = -\frac{\sqrt{\pi}}{2} (-1)^k 2^{2k} \Gamma(k). \] (46)

We obtain then:

\[ V(t) = \frac{-\sqrt{\pi}}{2} \left( \gamma + 2 \ln 2 + \sum_{r=1}^{\infty} \frac{(-1)^k 2^{2k} \Gamma(k)}{\Gamma(2k+1)} r^{2k} \right). \] (47)
Using the recurrence and duplication formulas for the gamma function we can arrive at our final result:

\[ V(t) = -\frac{\sqrt{\pi}}{2} \left( \gamma + 2\ln 2 - 2t^2 \right) _2F_2(1, 1; \frac{3}{2}, 2; -t^2) \] . (48)