Grünwald Implicit Solution for Solving One-Dimensional Time-Fractional Parabolic Equations Using SOR Iteration

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Abstract. The aim of this paper is to examine the effectiveness of Successive Over-Relaxation (SOR) iterative method for solving one-dimensional time-fractional parabolic equations. The Grünwald fractional derivative operator and implicit finite difference scheme have been used to discretize the proposed linear time-fractional equations to construct system of Grünwald implicit approximation equation. The basic formulation and application of the SOR iterative method are also presented. To investigate the effectiveness of the proposed iterative method, numerical experiments and comparison are made based on the iteration numbers, time execution, and maximum absolute error. Based on numerical results, the accuracy of Grünwald implicit solution obtained by proposed iterative method is in excellent agreement, and it can be concluded that the proposed iterative method requires less number of iterations and execution time as compared to the Gauss-Seidel (GS) iterative method.

1. Introduction

For the past few decades, fractional differential equations (FDE) have caught the attention of researchers in diverse fields such as sciences, engineering, economics and finance. Examples of the applications have been mentioned in [1,2]. The ability of FDE to describe the non-Markovian random walks problems [3] may produce a more realistic model of real complex physical problems. Sparked by this matter, several discussions on solving time fractional parabolic equations (TFPE) using both analytical and numerical approaches could be found in the literature. However, while dealing with fractional order equations, solving it analytically has become almost impossible and not so cost-effective. For this reason, many researchers have devoted their time and effort in the development of numerical techniques for solving TFPE problems. Recent researches on finite difference methods for solving TFPE that have been initiated are, implicit scheme [4-12], Crank-Nicholson scheme [13], the compact difference scheme [14] and the alternating segment explicit-implicit/ implicit-explicit parallel difference method [15]. Their discussions are mostly described based on Caputo sense.

In this paper, we study one-dimensional time-fractional parabolic equations (TFPE) of order $0 < \alpha < 1$. The TFPE we consider in this paper being given as
\[
\frac{\partial^\alpha U(x,t)}{\partial x^\alpha} + p(x) \frac{\partial U(x,t)}{\partial x} + q(x) \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \quad t > 0, \quad x \in \Omega = [a,b], \quad 0 < \alpha \leq 1, \tag{1}
\]

subject to the initial condition

\[U(x,0) = f(x), \quad a \leq x \leq b,\]

and the boundary conditions

\[U(a,t) = g_1(t); \quad U(b,t) = g_2(t) \quad 0 \leq t \leq T,\]

where \(p(x)\) and \(q(x)\) are real parameters and \(f(x,t)\) is the source term. Note that for \(\alpha = 1\), equation (1) is the classical parabolic equation.

Solving TFPE numerically will form large sparse linear systems. In this case, the iterative method is usually the common approach adopted by researchers because of its reputation in providing some speed improvement compared to the direct method. However, the classical iterative method such as Gauss-Seidel (GS) iterative method still has a relatively slow convergence rate. Therefore, to enhance the classical iterative method performances, researchers such as [16-17] have devoted their studies to improve the convergence rate of large sparse linear systems. Prior to that, [18-19] has introduced and discussed another approach called as the Successive Over-Relaxation (SOR) iterative method, which can be applied to any existed slow converging iterative methods.

There are several definitions of a fractional derivative of order \(\alpha > 0\). The two most frequently used definitions are Riemann-Liouville and Caputo fractional derivatives, which can be defined as [20]

**Definition 1:** Riemann-Liouville fractional derivative

\[
D_{RL}^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \left( \frac{d}{dt} \right)^n \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{\alpha+n}}, \quad n-1 < \alpha < n \tag{2}
\]

**Definition 2:** Caputo fractional derivative

\[
D_C^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+n-1}}, \quad n-1 < \alpha < n \tag{3}
\]

Whereas, this paper focused on the Grünwald fractional derivative instead. The definition of the Grünwald fractional derivative of order \(\alpha\) can be defined as [21-22]

**Definition 3:** Grünwald fractional derivative

\[
D_G^\alpha f(t) = \frac{1}{(\Delta t)^\alpha} \lim_{N \to \infty} \sum_{k=0}^{N} g_{\alpha,k} f(t-k\Delta t), \quad 0 < \alpha < 1 \tag{4}
\]

where the Grünwald weights are

\[
g_{\alpha,k} = \frac{\Gamma(k-\alpha)}{\Gamma(-\alpha)\Gamma(k+1)}.\]

The formulation of the approximate equation is based on the finite grid network as depicted in Figure 1. The GS and SOR iterative methods are applied onto each interior node point until it converged.

![Figure 1](image_url)

**Figure 1.** The distribution of uniform node points for the solution domain at m=8.
2. Grünwald finite difference approximation

In this section, problem (1) is discretized using the Grünwald fractional derivative as stated in equation (4) and implicit difference scheme. To derive numerical approximations based on Grünwald derivative, first let the mesh point

\[ x_i = \alpha + i h \]

where \( i = 0,1,...,N \) and \( j = 0,1,...,M \) and where \( h = \Delta x = (b-a)/N \) and \( \Delta t = T/M \) denotes the uniform step-size at \( x \) and \( t \) directions respectively.

Hence, the discrete equation of problem (1) is given by

\[
\frac{1}{(\Delta t)^\alpha} \sum_{k=0}^{\alpha} g_{\alpha,k} U_{i,j-k} + \frac{p(x)}{2\Delta x} (U_{i+1,j} - U_{i-1,j}) + \frac{q(x)}{(\Delta x)^2} (U_{i+1,j} - 2U_{i,j} + U_{i-1,j}) = f_{i,j}
\]

Then, by letting

\[ G_k = \frac{g_{\alpha,k}}{(\Delta t)^\alpha}, \quad a_i = \frac{p(x)}{2\Delta x}, \quad b_j = \frac{q(x)}{(\Delta x)^2}, \]

the simplified approximation equation can be generally written as

\[ \alpha_i U_{i-1,j} + \beta_i U_{i,j} + \gamma_i U_{i+1,j} = F_{i,j} \]

where

\[ \alpha_i = b_i - a_i, \quad \beta_i = G_0 - 2b_i, \quad \gamma_i = a_i + b_i, \]

and

\[ F_{i,j} = \begin{cases} f_{i,j} - G_0, & j = 1 \\ f_{i,j} - G_k \sum_{k=0}^{\alpha} g_{\alpha,k} U_{i,j-k}, & j = 2,3,\ldots \end{cases} \]

Meanwhile, the illustration of their respective computational molecules when time level \( j=1 \) and \( j=2 \) as shown in Figures 2 and 3 respectively.

Based on Figures 2 and 3, it shows that the computational molecules for Grünwald time-fractional parabolic approximation equation are similar to the regular computational molecules of implicit finite difference approximation only at time level \( j=1 \). Whereas, as the time level increases, a series of points will appear as a tail-like at time level \( j=2,3,\ldots \). Hence, equation (6) will generate large sparse linear systems, which could be written in a matrix form as
where
\[ A \mathbf{U} = \mathbf{F} \]  
(7)

and
\[ A = D + V + L \] 
(8)

where \( D, V \) and \( L \) are diagonal, lower triangular and upper triangular matrices respectively. Hence, the general form of SOR iterative method can be defined as [19]-[20],[23]

\[ U_j^{(k+1)} = (1 - \omega)U_j^{(k)} + \omega(D + L)^{-1}(F - VL_j^{(k)}) \] 
(9)

where, \( \omega \) represent the relaxation factor and \( U_j^{(k+1)} \) as the unknown vector at \( k^{th} \) iteration. From equation (9), it is noted that at \( \omega = 1 \) the SOR iterative method is equivalent to the GS iterative method. Hence, to ensure SOR iterative method is converging faster, the usual range of the optimal value is set within \( 1 \leq \omega < 2 \).

Thus, the SOR algorithm to solve problem (1) is as summarized in Algorithm 1.

Algorithm 1: SOR scheme

1. Initialize \( U_0 \) and \( \varepsilon \leftarrow 10^{-10} \)
2. Assign the optimal value of \( \omega \),
3. For \( j=1,2,3,...,M \),
   a. Assign \( U_j^{(0)} \leftarrow 0 \)
   b. Solve equation (7) iteratively using equation (9)
   c. Perform the convergence test, \( |U_j^{(k+1)} - U_j^{(k)}| \leq \varepsilon = 10^{-10} \). If yes, go to step (d).
   Otherwise, repeat step (b).
   d. Check time level, \( j=M \). If yes, go to step (iv). Otherwise, repeat step (a).
4. Display approximate solutions.

4. Numerical experiments
In order to verify the effectiveness of the SOR iterative method over the GS iterative method, we tested three examples of one-dimensional inhomogeneous time-fractional parabolic equations for \( \alpha = 0.333 \), \( \alpha = 0.666 \) and \( \alpha = 0.999 \). For comparison, three criteria were considered which are the number of iterations,
execution time and maximum absolute error. In the implementation of the iterative methods, the convergence test considered the tolerance error \( \varepsilon = 10^{-10} \).

**Example 1:** Consider the following one-dimensional linear inhomogeneous fractional Burger’s equation [24]

\[
\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} + \frac{\partial U(x,t)}{\partial x} - \frac{\partial^2 U(x,t)}{\partial x^2} = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2,
\]

with \( p(x) = 1 \) and \( q(x) = -1 \), and subject to the initial condition \( U(x,0) = x^2 \). The exact solution is \( U(x,t) = x^2 + t^2 \).

**Example 2:** Consider the following one-dimensional inhomogeneous time-fractional parabolic equation [25]

\[
\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t),
\]

with \( p(x) = 0 \) and \( q(x) = -1 \), while \( f(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - t^2 \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x) \), and subject to the initial condition \( U(x,0) = 0 \). The exact solution is given by \( U(x,t) = t^2 \sin(2\pi x) \).

**Example 3:** Consider the following one-dimensional linear inhomogeneous time-fractional parabolic equation [3]

\[
\frac{\partial^\alpha U(x,t)}{\partial t^\alpha} - \frac{\partial^2 U(x,t)}{\partial x^2} = f(x,t), \quad t > 0, x \in \mathbb{R}, 0 < \alpha \leq 1,
\]

with \( p(x) = 0 \) and \( q(x) = -1 \), while \( f(x,t) = \frac{2e^t t^{2-\alpha}}{\Gamma(3-\alpha)} - t^2 e^x \), subject to the initial condition \( U(x,0) = 0 \). The exact solution is given by \( U(x,t) = t^2 e^x \).

Then, the numerical results obtained were recorded in Tables 1 to 3, which correspond to examples 1 to 3 respectively.

**Table 1.** Comparison of Number Iterations (k), execution time (in seconds) and maximum absolute error for the iterative methods using example 1 at \( \alpha = 0.333, 0.666, 0.999 \).

| M  | Method | \( \alpha = 0.333 \) | \( \alpha = 0.666 \) | \( \alpha = 0.999 \) |
|----|--------|----------------------|----------------------|----------------------|
|    | k      | Time                | Max Error            | k                    | Time                | Max Error            | k                    | Time                | Max Error            |
| GS | 18325  | 7.89                | 2.5972e-02           | 8957                 | 41.90               | 2.5973e-02           | 32944                | 11.42               | 2.5967e-02           |
| SOR| 411    | 3.02                | 2.5972e-02           | 286                  | 2.99                | 2.5972e-02           | 10244                | 11.42               | 2.5972e-02           |
| GS | 67139  | 41.90               | 2.5972e-02           | 32944                | 23.62               | 2.5972e-02           | 10244                | 11.42               | 2.5972e-02           |
| SOR| 826    | 6.40                | 2.5972e-02           | 583                  | 6.23                | 2.5972e-02           | 333                  | 6.07                | 2.5972e-02           |
| GS | 243922 | 276.40              | 2.5972e-02           | 120271               | 142.56              | 2.5972e-02           | 767                  | 12.67               | 2.5972e-02           |
| SOR| 1662   | 13.84               | 2.5972e-02           | 1194                 | 13.28               | 2.5972e-02           | 673                  | 12.67               | 2.5972e-02           |
| GS | 877165 | 1914.74             | 2.5981e-02           | 435083               | 965.24              | 2.5974e-02           | 137338               | 53.24               | 2.5974e-02           |
| SOR| 3603   | 32.32               | 2.5972e-02           | 2311                 | 29.30               | 2.5972e-02           | 1311                 | 26.75               | 2.5972e-02           |
| GS | 3114564| 13356.94            | 2.6008e-02           | 1556326              | 6730.67             | 2.6008e-02           | 496352               | 2216.78             | 2.6008e-02           |
| SOR| 7356   | 83.47               | 2.5972e-02           | 5253                 | 72.95               | 2.5972e-02           | 2643                 | 60.06               | 2.5972e-02           |
Table 2. Comparison of Number Iterations (k), execution time (in seconds) and maximum absolute error for the iterative methods using example 2 at $\alpha=0.333$, 0.666, 0.999.

| M  | Method | $\alpha=0.333$ | $\alpha=0.666$ | $\alpha=0.999$ |
|----|--------|----------------|----------------|----------------|
|    |        | k | Time | Max Error | k | Time | Max Error | k | Time | Max Error |
|    | GS     | 14148 | 6.40 | 3.4745e-04 | 7120 | 4.72 | 5.2947e-04 | 2358 | 3.57 | 4.5604e-04 |
| 128| GS     | 385   | 0.08 | 3.4734e-04 | 271 | 3.04 | 5.2935e-04 | 159 | 3.01 | 4.5694e-04 |
| 256| GS     | 47194 | 28.00 | 6.0203e-04 | 2425 | 17.50 | 3.8529e-04 | 8332 | 10.12 | 3.1294e-04 |
| 512| GS     | 151187 | 149.13 | 1.6691e-04 | 7910 | 86.99 | 3.5049e-04 | 29799 | 42.17 | 2.7708e-04 |
| 1024| GS    | 454367 | 812.72 | 1.6516e-04 | 251077 | 509.06 | 3.4871e-04 | 107228 | 240.64 | 2.6674e-04 |
| 2048| GS    | 6265 | 76.77 | 1.5379e-04 | 4097 | 66.99 | 3.3745e-04 | 2387 | 58.54 | 2.6568e-04 |

Table 3. Comparison of Number Iterations (k), execution time (in seconds) and maximum absolute error for the iterative methods using example 3 at $\alpha=0.333$, 0.666, 0.999.

| M  | Method | $\alpha=0.333$ | $\alpha=0.666$ | $\alpha=0.999$ |
|----|--------|----------------|----------------|----------------|
|    |        | k | Time | Max Error | k | Time | Max Error | k | Time | Max Error |
|    | GS     | 18947 | 7.94 | 1.1755e-03 | 9174 | 5.54 | 2.5778e-03 | 2824 | 3.78 | 2.2069e-03 |
| 128| GS     | 423   | 3.11 | 1.1757e-03 | 295 | 3.07 | 2.5780e-03 | 178 | 3.02 | 2.2070e-03 |
| 256| GS     | 69499 | 40.98 | 1.1744e-03 | 33775 | 23.09 | 2.5767e-03 | 10436 | 11.31 | 2.2058e-03 |
| 512| GS     | 855   | 6.43 | 1.1750e-03 | 602 | 6.26 | 2.5773e-03 | 344 | 6.15 | 2.2064e-03 |
| 1024| GS   | 252889 | 267.84 | 1.1724e-03 | 123467 | 138.85 | 2.5746e-03 | 38383 | 51.68 | 2.2036e-03 |
| 2048| GS   | 1241856 | 4591.56 | 1.1766e-03 | 839911 | 3310.98 | 3.3991e-04 | 384794 | 1578.31 | 2.5732e-04 |
|    | SOR    | 3035 | 30.43 | 1.5605e-04 | 2049 | 28.73 | 3.3966e-04 | 1264 | 26.52 | 2.6792e-04 |
|    | GS     | 454367 | 812.72 | 1.1744e-03 | 251077 | 509.06 | 3.4511e-04 | 107228 | 240.64 | 2.6674e-04 |
|    | GS     | 6265 | 76.77 | 1.5379e-04 | 4097 | 66.99 | 3.3745e-04 | 2387 | 58.54 | 2.6568e-04 |

Meanwhile, Table 4 briefly shows the percentage of improvement in the iteration numbers and execution time when the SOR iterative method is applied as compared to GS iterative Method. Based on Table 4, the approximate percentage reduction of the iteration numbers and execution time of the GS iterative method are from 93.26–99.76% and 15.69–99.38% respectively for the three different values of $\alpha=0.333$, 0.666, 0.999.
5. Conclusion

In this paper, we proposed the Grünwald implicit finite difference scheme to solve inhomogeneous time-fractional parabolic equations using the SOR iterative method. Three examples were presented for three different values of $\alpha = 0.333, 0.666, 0.999$. The numerical results recorded indicate that the accuracy of the Grünwald implicit solution obtained by the SOR iterative method is in excellent agreement with GS iterative method. Meanwhile, the introduction of the weighted parameter in the SOR iterative method has reduced the iteration numbers and the computation time of the GS iterative method. Hence, it can be concluded that the SOR iterative method is able to substantially improve the iteration numbers and execution time of the GS iterative method. For future work, further investigation on the other iterative method such as the Kaudd SOR (KSOR) iterative method, which discussed could be found in [26-28] will be applied to the proposed approximation equation in order to increase its smoothness.

6. References

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