Wilson lines and UV sensitivity in magnetic compactifications

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Abstract

We investigate the ultraviolet (UV) behaviour of 6D N=1 supersymmetric effective (Abelian) gauge theories compactified on a two-torus ($T^2$) with magnetic flux. To this purpose we compute offshell the one-loop correction to the Wilson line state self-energy. The offshell calculation is actually necessary to capture the usual effective field theory expansion in powers of ($\partial/\Lambda$). Particular care is paid to the regularization of the (divergent) momentum integrals, which is relevant for identifying the corresponding counterterm(s). We find a counterterm which is a new higher dimensional effective operator of dimension d=6, that is enhanced for a larger compactification area (where the effective theory applies) and is consistent with the symmetries of the theory. Its consequences are briefly discussed and comparison is made with orbifold compactifications without flux.

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1 Introduction

In this letter we explore the ultraviolet behaviour of supersymmetric models compactified to four dimensions on a two-torus $T_2$ in the presence of magnetic flux. Compactifications on tori with magnetic fluxes were investigated in string theory e.g. [1]-[6] (for a review [7, 8]) and are interesting because they can break supersymmetry and lead to chiral fermions [9]. This motivated the interest in effective theory approach to model building e.g. [10]-[24].

In this work we compute (offshell) the one-loop correction to a two-point Green function of the self-energy of a complex scalar field $\varphi$ in a compactification to four dimensions of a 6D $\mathcal{N}=1$ supersymmetric Abelian gauge theory on $T_2$ with magnetic flux. The scalar field $\varphi$ is actually a Wilson line state, which is a fluctuation of a combination of components $A_M (M = \mu, 5, 6)$. The motivation is two-fold: few quantum investigations exist for such compactification and the field $\varphi$ may play the role of a higgs field in realistic models, which is interesting for model building and the hierarchy problem.

We pay particular attention to the regularization of the quantum corrections. Indeed, the one-loop integrals are divergent and call for a UV regularization consistent with the symmetries of the theory. The regularization ensures that (the series of) these integrals are well-defined and any divergences of the result in the limit of removing the regulator dictate the form of the corresponding counterterm operators. Since effective theories are non-renormalizable, the counterterms may be higher dimensional operators. The offshell calculation is important and is actually necessary in order to capture the behaviour of the effective theory which is an expansion in powers of $(\partial/\Lambda)$ [27] where $\Lambda$ is a high scale (e.g. compactification scale). We use dimensional regularization (DR), since it respects all symmetries, in particular gauge symmetry. We then compare the UV behaviour of our result for the quantum correction in the presence of magnetic flux against similar results in orbifold compactifications without flux such as effective theory on $T_2/Z_2$, etc.

2 Magnetic compactification on a torus

We begin our study with the relevant part of the action. Consider first the action of a 6D $\mathcal{N}=1$ vector superfield and hypermultiplet compactified to 4D on a torus $T_2$ in the presence of magnetic flux. This can be described in 4D $\mathcal{N}=1$ superfields language [28]. For the details of this compactification we refer the reader to [10, 11, 12, 13]. For the vector superfield

$$S_v = \int d^6x \left\{ \int d^4\theta \left[ \partial V - V \Phi u + \sqrt{2} V (\partial^\theta \Phi + \text{h.c.}) \right] + \frac{1}{4} \int d^2\sigma W^\alpha W_\alpha + \text{h.c.} \right\}$$

where $\partial \equiv \partial_5 - i \partial_6$. Only zero-modes of $V$ (hereafter $V_0$), of gauge kinetic field-strength $W$ ($W_0$) and of the superfield $\Phi$ ($\Phi_0$) are relevant below. We have $\Phi_0|_{\theta=\bar{\sigma}=0} = 1/\sqrt{2}(A_6 + iA_5) + \varphi$, where $\varphi$ defines a complex continuous Wilson line state on $T_2$.

We also need a 6D $\mathcal{N}=1$ hypermultiplet of chiral superfields $Q, \bar{Q}$ of charges $\pm q_0$.
\[
S_h = \int d^6x \left\{ \int d^4\theta \left[ \tilde{Q}^\dagger e^{2q_0 g V} Q + \tilde{Q}^\dagger e^{-2q_0 g V} \tilde{Q} \right] + \left[ \int d^2\theta \left( \partial + \sqrt{2} g q_0 \Phi \right) Q + \text{h.c.} \right] \right\}
\]

(2)

with \(g\) the gauge coupling. One must integrate \(S_h, S_v\) over \(T_2\) in the presence of magnetic flux \([10]\), but a set of basis functions is required. First, we use a symmetric gauge choice with \(A_5 = -(1/2)f x_6\) and \(A_6 = (1/2)f x_5\) (\(f=\text{constant}\)), satisfying a constant field strength \(F_{56} = \partial_5 A_6 - \partial_6 A_5 = f\). Its flux through \(T_2\) closed surface is then quantised \(q_0 g/(2\pi) \int_{T_2} F_{56} dx_5 dx_6 = q_0 g f A/(2\pi) \in \mathbb{Z}\), \((A\) is the area of the torus). The Kaluza-Klein (KK) spectrum of the charged fields will then resemble that of Landau levels \([10, 11, 12, 15]\).

To find the basis set of functions, notice that covariant derivatives \(D_k = \partial_k + iq_0 g A_k\) \((k = 5, 6)\) satisfy \([i D_5, i D_6] = -i q_0 g f\). Assuming \(f < 0\), one can construct a 1D harmonic oscillator Hamiltonian \(H = \frac{p^2}{2m} + 1/2 m \omega^2 x^2\) of \(p \sim i D_6\) and \(x \sim i D_5\), \(m = 1/2, \omega = 2\). Its eigenfunctions define the basis set of functions \(\psi_{n,j}\) \([10, 11, 12]\). The ladder operators are \(a = (1/\sqrt{\alpha}) (i D_5 - D_6), a^\dagger = 1/\sqrt{\alpha} (i D_5 + D_6)\) with \([a, a^\dagger] = 1\), so \(H = \alpha (a^\dagger a + 1/2)\) and

\[
\alpha = -2q_0 g f = \frac{4\pi N}{A} > 0, \quad (N \in \mathbb{Z}_+) \quad (3)
\]

The basis functions are \(\psi_{n,j} = (a^\dagger)^n/\sqrt{n!} \psi_{0,j}\), where \(n\) refers to the Landau level and \(j\) reflects the \(N\)-fold degeneracy. These are orthonormal on \(T_2\), and \(a^\dagger \psi_{n,j} = \sqrt{n + 1} \psi_{n,j}\), with \(\psi_{0,j}\) as zero mode: \(\psi_{0,j} = 0\). Then \(\partial + \sqrt{2} q_0 g \Phi_0 = -i\sqrt{\alpha} a^\dagger + \sqrt{2} q_0 g \varphi\), which is used in \(S_h\), together with an expansion of superfields in the basis functions \(\psi_{n,j}(x_m)\):

\[
Q(x_M, \theta, \bar{\theta}) = \sum_{n,j} Q_{n,j}(x_M, \theta, \bar{\theta}) \psi_{n,j}(x_m), \quad M = \mu, 5, 6. \quad (4)
\]

A similar expansion exists for \(\tilde{Q}(x_M, \theta, \bar{\theta})\) in this basis, with coefficients \(\tilde{Q}_{n,j}(x_M, \theta, \bar{\theta})\). One finds the relevant part of the 4D action \([10]\)

\[
S \supset \int d^4x \left\{ \int d^4\theta \left[ \varphi^\dagger \varphi + \sum_{n,j} Q_{n,j}^\dagger e^{2q_0 g V_0} Q_{n,j} + \sum_{n,j} \tilde{Q}_{n,j}^\dagger e^{-2q_0 g V_0} \tilde{Q}_{n,j} + 2f V_0 \right] + \int d^2\theta \left[ \frac{1}{4} W_0^a W_{0,a} - i \sum_{n,j} \sqrt{\alpha(n+1)} \tilde{Q}_{n+1,j} Q_{n,j} + \sum_{n,j} \sqrt{2} q_0 g \tilde{Q}_{n,j} \varphi Q_{n,j} \right] + \text{h.c.} \right\}
\]

(5)

where we kept only the zero modes of the gauge kinetic term and of Wilson line scalar \(\varphi\). After eliminating the auxiliary fields one identifies the scalar fields mass: \(m_{Q_{n,j}}^2 = m_{\tilde{Q}_{n,j}}^2 = \alpha(n + 1/2)\); for fermions their mass can be read from the last line of the above

Another way to see the quantisation condition is the following. We can make a gauge choice near \(x_5 = 0\) and \(x_5 = 2\pi R_5\). Region I \((-\pi R_5 < x_5 < \pi R_5)\): \(A_5 = 0, A_6 = f x_5\), and Region II \((\pi R_5 < x_5 < 2\pi R_5)\): \(A_5 = 0, A_6 = f(x_5 - 2\pi R_5)\). Then, two gauge potentials are connected by a gauge transformation in the overlapping region: \(A_{1\mu} = A_{1\mu} = -2\pi f R_5 \partial A_5\), with \(\Lambda = -2\pi f R_5 x_6\). As a result, the wavefunctions of charged fields, \(\phi\), are connected in this overlapping region as \(\phi_{1\mu} = e^{-iq_0 g f 2\pi R_5 x_6} \phi_{1\mu}\). Then, single-valuedness of wavefunctions along the \(x_6\) direction requires \(q_0 g f (2\pi) R_5 R_6 = N\) with \(N\) integer. The periodicity along the \(x_6\) direction is guaranteed by the same quantisation condition.
equation: \( m^2_{\Psi_{n,j}} = \alpha(n + 1) \) for a Dirac fermion composed of two Weyl spinors as in \( \Psi_{n,j} \equiv (\tilde{\chi}_{n+1,j}, \chi_{n,j})^T \). The (onshell-SUSY) couplings of these fields, in components, are:

\[
L = -i\sqrt{2} q_0 g \sum_{n,j} \sqrt{\alpha (n + 1)} \varphi [\tilde{Q}^\dagger_{n+1,j} \tilde{Q}_{n,j} - Q^\dagger_{n,j} Q_{n+1,j}] - \sqrt{2} q_0 g \sum_{n,j} \varphi \tilde{\chi}_{n,j} \chi_{n,j} + \text{h.c.}
\]

where the sums are over \( n \geq 0 \); \( \tilde{\chi} (\chi) \) are the Weyl spinors of \( \tilde{Q} (Q) \) superfields. With this information we can investigate the quantum corrections to the mass of the scalar field \( \varphi \).

### 3 One-loop corrections to Wilson line

With the above action, we compute the one-loop corrections to the Wilson line scalar, shown in fig. 1 for non-vanishing external 4-momentum \( q \). This allows us to investigate their UV behaviour under scaling of the momentum. Since the integrals are divergent, we use the DR scheme, in order to find the poles and identify their corresponding counterterms. This regularization preserves all symmetries of the theory. After performing a Wick rotation to the Euclidean space and with the DR subtraction scale \( \mu \) introduced to ensure dimensionless coupling \( (g) \) in \( d = 4 - 2\epsilon \) dimensions, we find for the bosonic contribution

\[
\delta m^2_b(q^2) = 2q_0^2 g^2 N \mu^{2\epsilon} \sum_{n \geq 0} \int \frac{d^d k}{(2\pi)^d} \frac{2k^2 + \alpha}{[(q + k)^2 + \alpha(n + 1/2)][k^2 + \alpha(n + 3/2)]}. \tag{7}
\]

For the fermionic part

\[
\delta m^2_f(q^2) = -2q_0^2 g^2 N \mu^{2\epsilon} \sum_{n \geq 0} \int \frac{d^d k}{(2\pi)^d} \frac{2k(q + k)}{[(q + k)^2 + \alpha n][k^2 + \alpha(n + 1)]}. \tag{8}
\]

Performing the integrals in the DR scheme (see the Appendix) gives

\[
\delta m^2_b(q^2) = K_0 (4\pi \mu^2/\alpha)^\epsilon \int_0^1 dx \left( 2q_0^2 x^2 + \alpha \right) \Gamma[\epsilon] \zeta[\epsilon, \rho_1] + d \alpha \Gamma[-1 + \epsilon] \zeta[-1 + \epsilon, \rho_1] \]

\[
\delta m^2_f(q^2) = -K_0 (4\pi \mu^2/\alpha)^\epsilon \int_0^1 dx \left[ 2q_0^2 x(x - 1) \Gamma[\epsilon] \zeta[\epsilon, \rho_2] + d \alpha \Gamma[-1 + \epsilon] \zeta[-1 + \epsilon, \rho_2] \right] \tag{9}
\]

with the notation

\[
K_0 \equiv \frac{2q_0^2 g^2 N}{(4\pi)^2}, \quad \rho_2 = \rho_1 - \frac{1}{2} = (1 - x) \left( 1 + x \frac{q_0^2}{\alpha} \right) > 0 \tag{10}
\]

\(^3\)Unlike in 6D orbifolds, in the present case only one KK sum is present, which would apparently make the result less UV divergent. This is however misleading because in the present case the (masses)\(^2\) under the sum are linear rather than quadratic in the level \( (n) \), thus there is no UV improvement in this sense.
where we introduced the Hurwitz zeta function \( \zeta(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s} \). The above bosonic and fermionic contributions have poles from Gamma functions, \( \Gamma[\epsilon] \) and \( \Gamma[-1+\epsilon] \). One could proceed in eqs.(9) to Taylor expand the zeta functions for small \( \epsilon \) and isolate the poles from the finite part, however, one cannot then integrate the resulting terms involving \( (d/dz \zeta[z, \rho])_{z=-1} \) since for this derivative only asymptotic expansions are known \([29, 30]\). To avoid this, we integrate by parts the second term in both \( \delta m_{b,f}^{2} \) and use

\[
\frac{\partial \zeta[s, \rho]}{\partial \rho} = -s \zeta[s+1, \rho]
\]

This gives

\[
\delta m_{b}^{2}(q^{2}) = K_{0} \left( \frac{4\pi\mu^{2}}{\alpha} \right)^{\epsilon} \left[ d\alpha \Gamma[\epsilon - 1] \zeta[-1+\epsilon, 1/2] + \Gamma[\epsilon] \int_{0}^{1} dx \zeta[\epsilon, \rho_{1}] f_{1}(x) \right] \]

\[
\delta m_{f}^{2}(q^{2}) = -K_{0} \left( \frac{4\pi\mu^{2}}{\alpha} \right)^{\epsilon} \left[ d\alpha \Gamma[\epsilon - 1] \zeta[-1+\epsilon, 0] + \Gamma[\epsilon] \int_{0}^{1} dx \zeta[\epsilon, \rho_{2}] f_{2}(x) \right] \]

Here \( f_{1}(x) = 2q^{2} x^{2} + \alpha + x d\alpha \rho_{1}^{4}(x) \) and \( f_{2}(x) = 2q^{2} x(x-1) + x d\alpha \rho_{2}^{4}(x) \) with a notation \( \rho_{j}(x) = (d/dx)\rho_{j}(x), \) \( j = 1, 2 \). Further

\[
\Gamma[\epsilon] = \frac{1}{\epsilon} - \gamma_{E} + O(\epsilon)
\]

\[
\zeta[\epsilon, \rho_{j}] = \zeta[0, \rho_{j}] + \epsilon \zeta^{(1,0)}[0, \rho_{j}] + O(\epsilon^{2}), \quad j = 1, 2.
\]

\[
\left( \frac{4\pi\mu^{2}}{\alpha} \right)^{\epsilon} = 1 + \epsilon \ln \frac{4\pi\mu^{2}}{\alpha} + O(\epsilon^{2})
\]

with the Euler constant \( \gamma_{E} \approx 0.577216 \). We then find

\[
\delta m_{b}^{2}(q^{2}) = K_{0} \left[ -\frac{q^{4}}{30\alpha} \left( \frac{1}{\epsilon} + \ln \frac{4\pi\mu^{2}e^{-\gamma_{E}}}{\alpha} \right) - \frac{\alpha}{12} \ln \frac{e^{3}G^{24}}{4} - \frac{q^{4}}{30\alpha} + H_{1}(q) \right] + O(\epsilon)
\]

\[
\delta m_{f}^{2}(q^{2}) = -K_{0} \left[ \frac{4q^{4}}{30\alpha} \left( \frac{1}{\epsilon} + \ln \frac{4\pi\mu^{2}e^{-\gamma_{E}}}{\alpha} \right) + \alpha \ln G^{4} - \frac{q^{4}}{30\alpha} + H_{2}(q) \right] + O(\epsilon)
\]

\(^{4}\)One has \( \zeta[0, \rho] = 1/2 - \rho, \zeta^{(1,0)}[0, \rho] = \ln \Gamma[\rho] - \ln \sqrt{2\pi}, \) and \( \zeta^{(1,0)}[-1, 1/2] = -\frac{1}{2}\zeta^{(1,0)}[-1, 0] - \frac{1}{2\pi}\ln 2 \).
with $G = 1.28243$ the Glaisher constant. Above we introduced the functions $H_1, H_2$

$$H_1(q) = \int_0^1 dx \left(2 q^2 x^2 + \alpha + 4 x x \rho'_1(x)\right) \ln \frac{\Gamma[\rho_1(x)]}{\sqrt{2\pi}}$$

$$= \frac{\alpha}{12} \ln \frac{G^{24} \epsilon^3}{4} + \frac{q^2}{6} - \frac{9 \zeta[3]}{8\pi^2} q^2 + c_b \frac{q^4}{\alpha} + \mathcal{O}((q^2/\alpha)^3)$$

(14)

with $c_b = -109/720 - (1/15) \ln 2 + \ln G - 14 \zeta[-3] \approx -0.02414$, and

$$H_2(q) = \int_0^1 dx \left(2 q^2 x (x - 1) + 4 x x \rho'_2(x)\right) \ln \frac{\Gamma[\rho_2(x)]}{\sqrt{2\pi}}$$

$$= -\alpha \ln G^4 + \frac{3 \zeta[3]}{2\pi^2} q^2 + c_f \frac{q^4}{\alpha} + \mathcal{O}((q^2/\alpha)^3)$$

(15)

and $c_f = 11/45 + 16 \zeta[-3] \approx 0.3305$. Therefore, up to irrelevant $\mathcal{O}(\epsilon)$ terms

$$\delta m_b^2(q^2) = K_0 \left[ -\frac{q^4}{30\alpha \epsilon} + \mathcal{O}(q^2/\alpha) \right]$$

$$\delta m_f^2(q^2) = K_0 \left[ -\frac{4q^4}{30\alpha \epsilon} + \mathcal{O}(q^2/\alpha) \right]$$

(16)

The sum of bosonic and fermionic contributions $\delta m^2(q^2) = \delta m_b^2(q^2) + \delta m_f^2(q^2)$, is found in general from eq.(13), but for small momenta $q^2 \ll \alpha$ it simplifies

$$\delta m^2(q^2) = K_0 \left[ -\frac{q^4}{60\alpha \epsilon} + \mathcal{O}(q^2/\alpha) \right]$$

$$= K_0 \left[ -\frac{q^4}{60\alpha \epsilon} + \frac{4\pi^2 \mu^2 e^{-\gamma_0}}{\alpha} - \frac{21 \zeta[3]}{8\pi^2} q^2 + \mathcal{O}(q^6/\alpha^3) \right].$$

(17)

with $\gamma_0 = \gamma_E + 6(c_b - c_f)$. A pole is present in the two-point Green function, reflecting the UV divergences of the theory and the limits $q^2 \to 0$ and $\epsilon \to 0$ do not commute, which shows the importance of this calculation. A finite quantum correction ($\propto q^2$) is also present.

4 \ Counterterms and symmetries

Eq.(17) shows that a counterterm is needed to cancel the pole $q^4/\epsilon$. The counterterm involves the 2-point self-energy, so it has the form $L_{c.t.} = -K_0/(6\alpha) \varphi^1 \phi^2 \varphi$. In superfield language, this operator has the form (in a new dimensionless coupling):

$$L = \frac{\varphi}{\alpha} \int d^4 \theta \varphi^1 \varphi^2 \varphi = -\frac{\lambda}{\alpha} \varphi^1 \phi^2 \varphi + \cdots$$

(18)

\footnote{Glaisher constant is given by $\ln G = 1/12 - \zeta[-1]$, with $\zeta[x]$ the Riemann zeta function.}

\footnote{The poles in eqs.14 are identical to those obtained if we Taylor expanded the expressions in eqs.10 about $\epsilon = 0$, and used $\zeta[0, \rho] = 1/2 - \rho$, and $\zeta[-1, \rho] = -1/2 (\rho^2 - \rho + 1/6)$ [39].}

\footnote{This is a genuine 6D divergence.}
where we used the same notation for the superfield and its scalar component.

This operator respects all symmetries of the theory and its presence is a reminder that our theory, although supersymmetric, is nevertheless non-renormalizable. Indeed, such theories are an expansion in powers \((\partial/\Lambda)^n\) \[27\], so such counterterms are expected; here \(\Lambda\) is the scale of new physics (from a 4D perspective), in this case \(\Lambda^2 \sim \alpha \sim 1/A\). Higher loops will generate more operators of this type. This operator, often overlooked in similar quantum calculations due to technical difficulties, is not specific to compactification with fluxes - it was also seen in 5D and 6D orbifold models at one-loop \[25, 26\]. The counterterm modifies the dispersion relations (the poles of the propagator) of the scalar \(\varphi\), which acquires a new solution, ghost-like, due to the higher order derivative \[31\]. Eqs.\((17), (18)\) show the propagator of \(\varphi\) has new pole at

\[
m^2_{\text{pole}} = \frac{\alpha}{\Lambda} \left[ 1 + \frac{21\zeta[3]}{8\pi^2} K_0 \right]
\]

This mass state is of the order of the compactification scale \(\sqrt{\alpha} \sim 1/\sqrt{A}\) and corresponds to the ghost degree of freedom. Note that the effective theory approach is reliable for a large torus area/radii (or small flux \(\sqrt{\alpha} \sim 1/A\)) but then also operator (18) is enhanced!

In applications it is useful to replace this operator by an equivalent polynomial one \[32\]: this is done by a non-linear field-redefinition or, equivalently, by using the equation of motion (in superfields) for \(\varphi\): 

\[-1/4\mathcal{D}^2\varphi^\dagger + \sqrt{2g_0}g \sum_{n,j} \hat{Q}_{n,j} Q_{n,j} = O(A),\]

where we used eq.\((5)\). This is used back in the action and effectively integrates the ghost \((\mathcal{D}^2\varphi^\dagger)\) but leaves \(\varphi\) in the action; then operator (18) becomes (with 

\[-16\varphi^\dagger \square \varphi = \varphi^\dagger \mathcal{D}^2 \mathcal{D}^2 \varphi\]

\[\mathcal{L} \propto \frac{\lambda}{\alpha} \int d^4\theta q_0^2 g^2 \left| \sum_{n,j} \hat{Q}_{n,j} Q_{n,j} \right|^2 \]

This is a dimension-six polynomial effective operator, equivalent to \(\mathcal{L}\) of (18) and brings many non-renormalizable operators in the action, suppressed by \(\sqrt{\alpha} \sim 1/\sqrt{A}\).

Having identified the counterterm operator, we can now formally set \(q^2 = 0\) in the one-loop correction \(\delta m^2_{b,f}(q^2)\) of eqs.\((13)\) and by using the exact relations

\[H_1(0) = (\alpha/12) \ln(G^{24} e^3/4), \quad H_2(0) = -\alpha \ln G^4\]

we find from eqs.\((13)\)

\[\delta m^2_b(0) = 0, \quad \delta m^2_f(0) = 0, \quad \Rightarrow \delta m^2(0) = \delta m^2_b(0) + \delta m^2_f(0) = 0.\]

Therefore the bosonic and fermionic contributions vanish separately for \(q^2 \to 0\), as conjectured in [10]. This indicates that at one-loop \(\varphi\) is a flat direction of the corresponding potential which has a vanishing curvature: \(\delta m^2(q^2 = 0) = 0\), as we showed. Beyond one-loop, any quantum calculation must include the one-loop counterterm of eq.\((18)\).

\*One cannot take \(\alpha \to 0\) since the flux is quantised.
Let us comment briefly on the result of eq.(22). In compactifications without flux the Wilson line $\varphi$ changes the boundary conditions of the charged fields, giving a continuous shift of the KK levels masses [9]. Then $\varphi$ acquires at one-loop a potential and a nonzero correction $\delta m^2(q^2 = 0)$ [33]. By contrast, in our compactification with flux, $\varphi = \varphi_1 + i\varphi_2$ only shifts [12] the argument $z = (x_5, x_6)$ of the wavefunction of the KK modes by an amount $\propto (\varphi_1/f, \varphi_2/f)$ and so the Wilson line does not enter in the mass formulae of the KK modes (and of the potential). This explains why the momentum-independent correction $\delta m^2(0)$ vanished at one-loop, with $\varphi$ a flat direction. This appears as a consequence of the continuous (classical) translation symmetry of $T_2$ which can “shift away” $\varphi$, so the KK spectrum (and the potential) does not depend on it.

The initial continuous translation symmetry of $T_2$ is broken however at the quantum level by non-local Wilson loops [10]. To see this, we put together the solution for the background gauge potential in the symmetric gauge and the constant Wilson line $\varphi = \varphi_1 + i\varphi_2$ in the following form [14]. $A_5 = -\frac{1}{2} fx_6 + \varphi_2$ and $A_6 = \frac{1}{2} fx_5 + \varphi_1$. Now, the Wilson loops integrals are: $w_1(x_6) = \exp[i q_0 g \int_0^{a_5} dx_5 A_5] = \exp[i q_0 g (-f x_6/2 + \varphi_2)a_5]$ and $w_2(x_5) = \exp[i q_0 g (f x_5/2 + \varphi_1)a_6]$. Here, $k, l, N \in \mathbb{Z}$ are integers and we used the flux quantisation condition $f a_5 a_6 q_0 = 2\pi N$ (see Section 2). As a result, the continuous translation symmetry of $T_2$ is broken by non-local Wilson loops to a discrete (accidental) translation symmetry $x_5 \rightarrow x_5 + a_5(2k/N)$, $x_6 \rightarrow x_6 + a_6(2l/N)$ [13, 34]. With this continuous symmetry broken, one must investigate at higher orders if the one-loop flat direction of $\varphi$ can be maintained.

To complete our discussion, let us also examine what happens if the sum over the modes in the calculation of the quantum corrections to $\delta m^2(q^2)$ is truncated to a fixed number of levels. Truncating the summation to $0 \leq n \leq n_0 - 1$ for bosons ($n_0$ levels) and to $0 \leq n \leq n_0' - 1$ for fermions ($n_0'$ levels) we find from eqs.[7, 8], after some algebra:

$$
\delta m^2_b(q^2) = -\frac{1}{\epsilon} K_0 \alpha n_0 (2n_0 + 1) + O(\epsilon^0)
$$

$$
\delta m^2_f(q^2) = \frac{1}{\epsilon} K_0 \alpha n_0' (2n_0' + q^2/\alpha) + O(\epsilon^0)
$$

---

4Let us show this for the KK zero modes. The Wilson line changes the equation for the zero mode: $a \psi_0 = 0$: $(i\partial + \frac{1}{2} q_0 g [f(z - z_0)\varphi^2])\psi_0 = 0$, with $z = x_5 + ix_6$. Then, the solution for the zero mode becomes $\psi_0 = h(z) \exp[-\frac{1}{2} q_0 g [f(z - z_0)/(z - z_0)]$, where $z_0 \equiv -2\sqrt{2} i\varphi^2/|f|$ and $h(z)$ is a holomorphic function. Therefore, a constant Wilson line only shifts $z$ by $z_0$, but the number of zero modes (equivalent to the number of the possible centers within the fundamental domain on a torus) remains fixed by the quantisation condition as it is for a vanishing Wilson line. The same conclusion can be drawn in an asymmetric gauge for the background, such as $A_5 = 0$ and $A_6 = f x_5$.

10In orbifolds (no flux) this translation symmetry is broken explicitly by the orbifold fixed points.

11The gauge $A_5 = -fx_6/2, A_6 = fx_5/2$ is not invariant under translations $x_j \rightarrow x_j + \delta_j$ ($j = 5, 6$), but a gauge transformation $\hat{A} \rightarrow \hat{A} - f/2\sqrt{2}(\delta_5 x_6 - \delta_6 x_5)$ with $\hat{A} = (A_5, A_6)$ restores the translation symmetry.

12This is done by writing the “truncated” sum as a difference of two infinite towers/sums, bringing in eq.[10] a difference of Hurwitz zeta functions for each zeta function present there, e.g. for bosons: $\zeta[\epsilon, \rho_1] \rightarrow \zeta[\epsilon, \rho_1] - \zeta[\epsilon, n_0 + \rho_1]$ and similar for fermions with $\rho_1 \rightarrow \rho_2, n_0 \rightarrow n_0'$. Similar for $\zeta[\epsilon - 1, \rho_2]$. 


Their sum becomes, for $n_0 = n'_0$ (by supersymmetry)

$$\delta m^2(q^2) = - \frac{1}{\epsilon} K_0 \alpha n_0 (1 - q^2/\alpha) + O(\epsilon^0)$$

This shows that a truncation of the towers to a same finite level would bring in the action a wavefunction renormalization for the superfield $\varphi$ (due to the term $\propto q^2/\epsilon$) familiar in softly broken supersymmetry and also a momentum-independent quadratic divergence $\propto \alpha/\epsilon$ (due to broken supersymmetry), but no higher dimensional counterterm is present. The theory is exactly 4-dimensional and renormalizable. Summing instead the whole tower, as we did, changes these two divergences into a worse “quartic” divergence $\propto q^4/(\alpha \epsilon)$ discussed earlier; this demanded instead a higher dimensional counterterm operator $(L)$ specific to non-renormalizable theories ($n_0, n'_0$ being now infinite).

A situation similar to that above is expected for the quantum corrections to the gauge coupling in this theory, when the 6D Lorentz invariance “promotes” counterterm [18] for the Wilson line to $F^{MN} \square F_{MN}$ which also contains $F^{\mu\nu} \square F_{\mu\nu}$. This is similar to 6D orbifolds without flux where such a higher dimensional counterterm ($A \int d^2 \theta \text{Tr} W^\alpha \square W_\alpha + \text{h.c.}$, in superfield notation) is generated [26] and is actually the reason for the so-called “power-like” running near the compactification scale.

5 Conclusions

Compactifications of effective theories in the presence of magnetic flux are interesting for model building since they provide supersymmetry breaking and chiral fermions. However, very few quantum calculations exist in such cases and this motivated our study. We examined the one-loop offshell correction to the two-point Green function of the Wilson line self-energy, in 6D N=1 Abelian gauge theories compactified on $T_2$ with magnetic flux ($\propto \alpha$). The offshell calculation is important and necessary in order to capture the usual effective theory expansion in powers of $\partial/\Lambda$; (from a 4D view $\Lambda \sim 1/\sqrt{A} \sim \sqrt{\alpha}$, $A=$torus area).

Since the one-loop momentum integrals are UV divergent, a regularization is needed. We used the DR scheme which preserves all the symmetries of the initial 6D gauge theory. The result shows that in the limit of removing the regulator, the two-point Green function has a pole which dictates the form of the counterterm. This is a higher dimensional (derivative) operator that was also seen in orbifold compactifications without flux. One consequence of this counterterm is that a ghost state is present of (mass)$^2 \propto \alpha$. We showed that such operator is equivalent to an operator of the same dimension (six) that is actually polynomial (quartic) in the charged superfields and is obtained by integrating out (decoupling) the ghost state. This operator is enhanced by a larger compactification area (when effective theory is applicable) and a reminder that effective theories are non-renormalizable.

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13 This situation is worse than the case of eq.16 in the first paper in [25] of ordinary orbifolds (no flux) where only usual $q^2/\epsilon$ poles i.e. wavefunction renormalization existed for a “truncated” tower summation.
After identifying the counterterm for the one-loop offshell self-energy, one may also consider the momentum independent mass correction \( \delta m^2(q^2 = 0) \), which is the curvature of the corresponding one-loop potential. Unlike in orbifold compactifications (without flux), this mass correction vanishes at one-loop and the Wilson line corresponds to a flat direction. The reason for this is a translation symmetry in internal dimensions which is broken however at the quantum level by non-local Wilson loops. It is worth investigating this issue beyond the one-loop order considered here.

**Appendix**

In the text we used the following integrals in Euclidean space

\[
I_1 \equiv \int \frac{d^dp}{(2\pi)^d} \frac{p_\mu}{((p+q)^2 + m_2^2)(p^2 + m_1^2)} = \frac{-q_\mu}{(4\pi)^{d/2}} \int_0^1 dx \ x \left[ 2 - d/2 \right] \left[ L(x, q^2, m_{1,2}) \right]^{\frac{d}{2} - 2}
\]

\[
I_2 \equiv \int \frac{d^dp}{(2\pi)^d} \frac{1}{((p+q)^2 + m_2^2)(p^2 + m_1^2)} = \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \ x \left[ 2 - d/2 \right] \left[ L(x, q^2, m_{1,2}) \right]^{\frac{d}{2} - 2}
\]

\[
I_3 \equiv \int \frac{d^dp}{(2\pi)^d} \frac{p_\mu p_\nu}{((p+q)^2 + m_2^2)(p^2 + m_1^2)} = \frac{1}{(4\pi)^{d/2}} \int_0^1 dx \ x \left[ 1 - d/2 \right] \left[ L(x, q^2, m_{1,2}) \right]^{\frac{d}{2} - 1}
\]

\[
+ \frac{1}{(4\pi)^{d/2}} \ q_\mu q_\nu \int_0^1 dx \ x^2 \left[ 2 - d/2 \right] \left[ L(x, q^2, m_{1,2}) \right]^{\frac{d}{2} - 2}
\]

where

\[
L(x, q^2, m_{1,2}) \equiv x (1 - x) q^2 + x m_2^2 + (1 - x) m_1^2
\]

and \( \sum_\mu \delta_{\mu\mu} = d, \ (d = 4 - 2 \epsilon) \).

**Acknowledgements:** The work of HML is supported in part by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2016R1A2B4008759).

**References**

[1] M. Berkooz, M. R. Douglas and R. G. Leigh, “Branes intersecting at angles,” Nucl. Phys. B 480 (1996) 265 [hep-th/9606139].

[2] C. Angelantonj, I. Antoniadis, E. Dudas and A. Sagnotti, “Type I strings on magnetized orbifolds and brane transmutation,” Phys. Lett. B 489 (2000) 223 [hep-th/0007090].

[3] R. Blumenhagen, L. Goerlich, B. Kors and D. Lust, “Noncommutative compactifications of type I strings on tori with magnetic background flux,” JHEP 0010 (2000) 006 [hep-th/0007024].
[4] P. Anastasopoulos, I. Antoniadis, K. Benakli, M. D. Goodsell and A. Vichi, “One-loop adjoint masses for non-supersymmetric intersecting branes,” JHEP 1108 (2011) 120 [arXiv:1105.0591 [hep-th]].

[5] G. Aldazabal, S. Franco, L. E. Ibanez, R. Rabadan and A. M. Uranga, “Intersecting brane worlds,” JHEP 0102 (2001) 047 [hep-ph/0011132].

[6] D. Cremades, L. E. Ibanez and F. Marchesano, “Computing Yukawa couplings from magnetized extra dimensions,” JHEP 0405 (2004) 079 [hep-th/0404229].

[7] R. Blumenhagen, B. Kors, D. Lust and S. Stieberger, “Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes,” Phys. Rept. 445 (2007) 1 [hep-th/0610327].

[8] C. Angelantonj and A. Sagnotti, “Open strings,” Phys. Rept. 371 (2002) 1 Erratum: [Phys. Rept. 376 (2003) no.6, 407] [hep-th/0204089].

[9] C. Bachas, “A Way to break supersymmetry,” [hep-th/9503030]

[10] W. Buchmuller, M. Dierigl, E. Dudas and J. Schweizer, “Effective field theory for magnetic compactifications,” [arXiv:1611.03798v2 [hep-th]].

[11] M. Ishida, K. Nishiwaki and Y. Tatsuta, “Brane-localized masses in magnetic compactifications,” [arXiv:1702.08226 [hep-th]].

[12] Y. Hamada and T. Kobayashi, “Massive Modes in Magnetized Brane Models,” Prog. Theor. Phys. 128 (2012) 903 [arXiv:1207.6867 [hep-th]].

[13] M. H. Al-Hashimi and U.-J. Wiese, “Discrete Accidental Symmetry for a Particle in a Constant Magnetic Field on a Torus,” Annals Phys. 324 (2009) 343 [arXiv:0807.0630 [quant-ph]].

[14] W. Buchmuller, M. Dierigl, F. Ruehle and J. Schweizer, “de Sitter vacua and supersymmetry breaking in six-dimensional flux compactifications,” Phys. Rev. D 94 (2016) no.2, 025025 [arXiv:1606.05653 [hep-th]].

[15] L. D. Landau and E. M. Lifshitz, Quantum Mechanics (Non-relativistic theory), Third Edition, Betterworth-Heiniemann (1977), p458.

[16] W. Buchmuller and J. Schweizer, “Flavour mixings in flux compactifications,” [arXiv:1701.06935 [hep-ph]].

[17] T. H. Abe, Y. Fujiimoto, T. Kobayashi, T. Miura, K. Nishiwaki and M. Sakamoto, “$Z_N$ twisted orbifold models with magnetic flux,” JHEP 1401 (2014) 065 [arXiv:1309.4925].

[18] T. H. Abe, Y. Fujiimoto, T. Kobayashi, T. Miura, K. Nishiwaki and M. Sakamoto, “Operator analysis of physical states on magnetized $T^2/Z_N$ orbifolds,” Nucl. Phys. B 890 (2014) 442 [arXiv:1409.5421 [hep-th]].
[19] T. h. Abe, Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki, M. Sakamoto and Y. Tatsuta, “Classification of three-generation models on magnetized orbifolds,” Nucl. Phys. B 894 (2015) 374 [arXiv:1501.02787 [hep-ph]].

[20] Y. Fujimoto, T. Kobayashi, K. Nishiwaki, M. Sakamoto and Y. Tatsuta, “Comprehensive analysis of Yukawa hierarchies on $T^2/Z_N$ with magnetic fluxes,” Phys. Rev. D 94 (2016) no.3, 035031 [arXiv:1605.00140 [hep-ph]].

[21] T. Kobayashi, K. Nishiwaki and Y. Tatsuta, “CP-violating phase on magnetized toroidal orbifolds,” arXiv:1609.08608 [hep-th].

[22] W. Buchmuller, M. Dierigl, F. Ruehle and J. Schweizer, “Split symmetries,” Phys. Lett. B 750 (2015) 615 [arXiv:1507.00819 [hep-th]].

[23] H. Abe, T. Kobayashi, K. Sumita and Y. Tatsuta, “Supersymmetric models on magnetized orbifolds with flux-induced Fayet-Iliopoulos terms,” Phys. Rev. D 95 (2017) no.1, 015005 [arXiv:1610.07730 [hep-ph]].

[24] Y. Fujimoto, T. Kobayashi, T. Miura, K. Nishiwaki and M. Sakamoto, “Shifted orbifold models with magnetic flux,” Phys. Rev. D 87 (2013) no.8, 086001 [arXiv:1302.5768].

[25] D. Ghilencea, “Higher derivative operators as loop counterterms in one-dimensional field theory orbifolds,” JHEP 0503 (2005) 009 [hep-ph/0409214]; D. Ghilencea and H. M. Lee, “Higher derivative operators from transmission of supersymmetry breaking on $S_1/Z_2$, ” JHEP 0509 (2005) 024 [hep-ph/0505187]. “Higher derivative operators from Scherk-Schwarz supersymmetry breaking on $T^2/Z_2$,” JHEP 0512 (2005) 039 [hep-ph/0508221].

[26] S. Groot Nibbelink, M. Hillenbach, “Quantum corrections to non-Abelian SUSY theories on orbifolds,” Nucl. Phys. B 748 (2006) 60 [hep-th/0602155]; “Renormalization of supersymmetric gauge theories on orbifolds: Brane gauge couplings and higher derivative operators,” Phys. Lett. B 616 (2005) 125 [hep-th/0503153]. D. Ghilencea, H. M. Lee, K. Schmidt-Hoberg, “Higher derivatives and brane-localised kinetic terms in gauge theories on orbifolds,” JHEP 0608 (2006) 009 [hep-ph/06041215]. D. Ghilencea, “Compact dimensions and their radiative mixing,” Phys. Rev. D 70 (2004) 045018 [hep-ph/0311264].

[27] H. Georgi, “Effective field theory,” Ann. Rev. Nucl. Part. Sci. 43 (1993) 209.

[28] N. Marcus, A. Sagnotti and W. Siegel, “Ten-dimensional Supersymmetric Yang-Mills Theory in Terms of Four-dimensional Superfields,” Nucl. Phys. B 224 (1983) 159. N. Arkani-Hamed, T. Gregoire and J. G. Wacker, “Higher dimensional supersymmetry in 4-D superspace,” JHEP 0203 (2002) 055 [hep-th/0101233].

[29] E. Elizalde, “Ten physical applications of spectral zeta functions” 2nd ed, Berlin, Springer, 2012. ISBN-13: 978-3642294044.
[30] I. Gradshteyn, I. Ryzhik, “Table of integrals, series and products”, 7th edition, Ed. A. Jeffrey, D. Zwillinger, Academic Press, Elsevier 2007. ISBN-13: 978-0-12-373637-6.

[31] See e.g. Section 2.1 in: I. Antoniadis, E. Dudas and D. Ghilencea, “Living with ghosts and their radiative corrections,” Nucl. Phys. B 767 (2007) 29 [hep-th/0608094].

[32] I. Antoniadis, E. Dudas and D. M. Ghilencea, “Supersymmetric Models with Higher Dimensional Operators,” JHEP 0803 (2008) 045 [arXiv:0708.0383 [hep-th]]. E. Dudas and D. M. Ghilencea, “Effective operators in SUSY, superfield constraints and searches for a UV completion,” JHEP 1506 (2015) 124 [arXiv:1503.08319 [hep-th]].

[33] See for example D. M. Ghilencea, D. Hoover, C. P. Burgess and F. Quevedo, “Casimir energies for 6D supergravities compactified on T(2)/Z(N) with Wilson lines,” JHEP 0509 (2005) 050 [hep-th/0506164].

[34] E. Onofri, “Landau levels on a torus,” Int. J. Theor. Phys. 40 (2001) 537 [quant-ph/0007055].