Entanglement enhances performance in microscopic quantum fridges

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Understanding the thermodynamics of quantum systems is of fundamental importance, from both theoretical and experimental perspectives. A growing interest has been recently given to small self-contained quantum thermal machines, the functioning of which requires no external source of work or control, but only incoherent interactions with thermal baths. The simplicity of such machines makes them an ideal test-bed for exploring quantum thermodynamics. So far, however, the importance of quantum effects in these machines has remained elusive. Here we show that entanglement, the paradigmatic quantum effect, plays a fundamental role in small self-contained quantum refrigerators, as it can enhance cooling and energy transport—except notably when the efficiency is close to the Carnot limit. Hence a truly quantum refrigerator can outperform a classical one. Furthermore, the amount of entanglement alone quantifies the enhancement in cooling. More generally, our work shows that entanglement opens new possibilities in thermodynamics.

The study of quantum thermal machines has a long history, from the thermodynamic analysis of lasers [1–3], to considerable work on quantum cycles and the second law [4–17]. Recently, models of small self-contained quantum thermal machines [18–22] have attracted growing attention. The key feature of such machines is that they function without any external source of work or control. Only incoherent interaction with thermal baths are required. Interestingly, there exist no fundamental limit on the size of such machines [18], nor on their efficiency [19]. Their main interest resides in their simplicity, which allows us to explore novel ideas in quantum thermodynamics.

An important question which has not been addressed so far is whether quantum effects play any significant role in small self-contained thermal machines. Indeed, although these machines are described within the formalism of quantum mechanics, it is not immediately clear to what extent their working is inherently quantum. One can give an heuristic account of the functioning of the machine in classical terms.

Here, our aim is to establish the importance of quantum effects in self-contained quantum thermal machines. Our main focus will be on the concept of entanglement, often considered as the defining feature of quantum theory. Hence, if entanglement turns out to play an important role in self-contained quantum thermal machines, this would make it clear that the working of such machines is truly quantum mechanical. Moreover, it would then raise the question of whether entanglement can enhance the performance of such machines. Below, we address these questions focusing our attention on the model of the smallest possible self-contained quantum refrigerator [18, 19]. We will first show that in the regime of high efficiency, that is machines operating with efficiency close to Carnot limit, the machine does not feature any entanglement. Hence, in a sense, entanglement appears to be detrimental as far as efficiency is concerned since an entangled state cannot get close to Carnot efficiency. Next, moving away from the high efficiency regime, we show that there exist regimes featuring entanglement. In fact, a wide variety of types of entanglement can be found in our system—including genuine multipartite entanglement—depending upon the external conditions. Finally, and most importantly, we show that this entanglement is useful, as it enhances cooling and energy transport. Specifically, given an object to cool and a set of resources (for instance fixing the temperatures of the heat baths), we show that a refrigerator featuring entanglement can outperform a ‘classical’ refrigerator (i.e. which features no entanglement), as it allows one to cool the object to lower temperatures. Moreover, we demonstrate that the improvement grows monotonically with entanglement measures, strongly suggesting of a functional relationship.

QUANTUM FRIDGE MODEL

We start by briefly reviewing the model of the smallest quantum refrigerator of Ref. [18, 19], which
we will focus on throughout this work. Let us consider three qubits, which in the absence of interaction have vanishing ground state energies and excited state energies $E_j$ ($j = 1, 2, 3$). The free Hamiltonian of the refrigerator is thus given by

$$H_0 = E_1\Pi_1 + E_2\Pi_2 + E_3\Pi_3$$

(1)

where $|1\rangle_j$ is the exited state of qubit $j$, (i.e. energy eigenstate at energy $E_j$), and $\Pi_j = |1\rangle_j\langle 1|$ denotes the projector onto that excited state. We also fix a relationship between the excited state energies of the free Hamiltonian and place an interaction between the qubits. This interaction takes the form

$$H_{int} = g \langle 010\rangle \langle 101| + |101\rangle \langle 010|.$$  

(2)

We require that this interaction Hamiltonian couples only states within a degenerate subspace of the free Hamiltonian, such that the coupling constant $g$ can be taken to be arbitrarily small while still producing changes in the steady state behaviour of the refrigerator. In this regime, where $g \ll E_j$, the eigenvalues and eigenstates remain governed by $H_0$. The above requirements thus impose that $E_2 = E_1 + E_3$ so that the states $|010\rangle$ and $|101\rangle$, connected via the interaction Hamiltonian, become degenerate in energy.

Finally, each qubit is taken to be in contact with a separate thermal reservoir. The temperatures of the reservoirs are denoted by $T_C$ (cold), $T_R$ (room), and $T_H$ (hot), for qubits 1, 2 and 3 respectively. The thermal contact between each qubit and bath is governed by Lindbladian dissipative dynamics, which we model here using a simple reset model, the justification of which we shall comment on briefly. In this model, with probability $p_i dt$ per time $dt$, qubit $i$ is reset to the thermal state $\tau_i$, at the temperature of its bath, while for all other times it evolves unitarily according to the combined Hamiltonian $H_0 + H_{int}$. That is, in this model thermalization events are taken to be rare but strong events. It is straightforward to derive the equation of motion for the refrigerator using this model of dissipation [18], which is given by the following Master equation

$$\frac{\partial \rho}{\partial t} = -i[H_0 + H_{int}, \rho] + \sum_i p_i(\tau_i \otimes Tr_i(\rho) - \rho)$$

(3)

where $\tau_i = r_i |0\rangle \langle 0| + (1 - r_i) |1\rangle \langle 1|$ with $r_i = 1/(1 + e^{-E_i/T_i})$. In general one would expect there to be additional terms in equation (3), corresponding to dissipative dynamics on qubit $j$ originating from the combination of the interaction Hamiltonian and the dissipative dynamics on qubit $i \neq j$. In other words, one may expect each qubit to be effectively in contact with all three baths due to the interaction Hamiltonian [22, 23]. However, when focusing on the regime where $p_i \approx g \ll E_i$, these additional effects, whose strength is approximately $g p_i \ll g$ can be safely neglected. This setup is depicted schematically in Fig. 1.

Here our focus is on the stationary state (i.e. long term behaviour) of the refrigerator, $\rho_S$, which satisfies $\dot{\rho}_S = 0$ i.e.

$$i[H_0 + H_{int}, \rho_S] = \sum_i p_i(\tau_i \otimes Tr_i(\rho_S) - \rho_S).$$

(4)

As shown in [19], this equation can be solved analytically for all values of the parameters. The solution takes the form

$$\rho_S = \tau_1 \tau_2 \tau_3 + \gamma \sigma$$

(5)

where $\gamma$ is a dimensionless parameter depending upon all parameters of the model (namely $p_i$, $g$, $E_i$, and temperatures $T_{C,R,H}$), and $\sigma$ is a traceless matrix with a single off-diagonal term (see [19] for details). The important property of the solution is that it can be shown that the refrigerator cools qubit 1 whenever $\gamma > 0$. In this case, one finds that qubit 1 is in a stationary state that is diagonal, with corresponding temperature $T^S < T_C$. Moreover, the efficiency of the refrigerator tends to the Carnot limit in the limit $\gamma \to 0$. 

FIG. 1. Schematic diagram of the quantum refrigerator. The fridge contains three qubits (inside the yellow circle), each in weak thermal contact (wiggly lines) with a bath at a different temperature. The qubits interact via the weak interaction Hamiltonian $H_{int}$, which couples the two degenerate levels $|010\rangle$ and $|101\rangle$, depicted by the arrows. The lower qubit (purple) is the object to be cooled. At equilibrium, it reaches a temperature $T_S < T_C$. The other two qubits (red and blue) are the machine qubits, connected to heat baths at temperatures $T_R$ and $T_H$. 

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(5)
AROUND THE CARNOT POINT

Let us first discuss the properties of \( \rho_S \) for those refrigerators which are operating close to the Carnot efficiency—which from hereon we shall refer to as refrigerators around the Carnot point. From inspection of Eq. (5), it is clear that for \( \gamma = 0, \rho_S \) is a fully separable state, as it is nothing other than the direct product of thermal state for each qubit. Hence it can immediately be concluded that no entanglement is present at the Carnot point. More interestingly, this statement remains true for a small region within the set of all refrigerators which are highly efficient function without entanglement. Hence it can be concluded that no entanglement is present at the Carnot point. Thus all refrigerators which exhibit these forms of entanglement are given in the Appendix B.

To see this let us first rewrite \( \rho_S \) in the following form

\[
\rho_S = w|GHZ\rangle\langle GHZ| + (1 - w)\sigma_{\text{diag}} \tag{6}
\]

where \(|GHZ\rangle = (|101\rangle + |110\rangle)/\sqrt{2}\) is a tripartite entangled state (of the Greenberger-Horne-Zeilinger form), and \(\sigma_{\text{diag}}\) is a diagonal density matrix, hence corresponding to a fully separable state. While there is no unique notion of entanglement in multipartite systems, it turns out that the entanglement of states of the form (6) can be conveniently characterized.

We first note that in the vicinity of any Carnot point, the state \( \rho_S \) has full rank and off-diagonal terms which are small compared to diagonal ones. Hence, in this regime, the state can be decomposed as

\[
\rho_S = (1 - \epsilon)\sigma_{\text{diag}} + \epsilon\rho(p), \quad \textnormal{where} \quad \sigma_{\text{diag}} \quad \textnormal{is a diagonal separable state and} \quad \rho(p) = p|GHZ\rangle\langle GHZ| + (1 - p)\frac{1}{3}I.
\]

Since \( \rho(p) \) is fully separable for \( p \leq \frac{3}{11} \), it follows that \( \rho_S \) is fully separable in the vicinity of any Carnot point. In fact, it can even be shown that any \( \rho_S \) at the Carnot point has a small ball of fully separable state in the full Hilbert space around it—see Appendix A for more details.

ENTANGLED REGIME

Next we ask whether there exists regimes in which entanglement is present in \( \rho_S \). At this point it is useful to recall that entanglement can appear under several forms in a state of 3 qubits. Indeed, there can be bipartite entanglement along a given bipartition (e.g. qubit 1 versus qubits 2 and 3), or genuine tripartite entanglement. In order to detect entanglement, our main tool will be a class of entanglement witnesses developed in [25], which allow one to fully characterize the entanglement of states of the form \( \rho_S \).

Moreover, these witnesses also provide a meaningful entropy based measure of multipartite entanglement [29] and necessary and sufficient conditions for biseparability for our system [27]. Formally, these witnesses are given by inequalities of the form

\[
W_S(\rho) = 2\left(\prod_{i=1}^{n} \left| \rho_{i,j} \right| \sum_{k=S} \sqrt{\rho_{k,k}} - k \omega_S \right) \leq 0 \tag{7}
\]

where \( \rho_{i,j} \) denotes elements of the density matrix and the set \( S \) depends on the partition and type of entanglement one is interested in. When inequality (7) is violated, its LHS gives the concurrence [28] of \( \rho_S \).

Moving away from the Carnot point we find, by sweeping though the parameter space numerically, that there exists regimes where entanglement is present. In fact, most types of entanglement can be found. We find regimes where there is entanglement (i) along only a single bipartition of the system, (ii) on all three bipartitions at the same time, and (iii) on all three bipartitions at the same time, and (iii) genuine tripartite entanglement, the strongest form of multipartite entanglement. Table 1 in the appendix gives further details and the specific choice of parameters which exhibit these forms of entanglement are given in the Appendix B.

ENTANGLEMENT ENHANCES COOLING

In the remainder of the paper, we investigate the usefulness of the entanglement that we have just uncovered in the fridge. We will see that, strikingly, entanglement can in fact enhance the performance of the refrigerator. For this we consider the task of cooling a qubit with given energy \( E_1 \), immersed in a bath at a given temperature \( T_C \) with fixed coupling \( p_1 \). As a source of free energy, we have at our disposal two heat baths, at temperatures \( T_R \) and \( T_H \) (again assuming that \( T_C < T_R < T_H \)). The challenge is then to adjust the remaining parameters in order to minimize the temperature of the qubit in its stationary state. The free parameters are the energy of the hot qubit \( E_3 \), the thermalization coefficients for the machine qubits \( p_2 \) and \( p_3 \), and the interaction strength \( g \). These parameters are not completely free, but all constrained. We require that \( g \ll \frac{E_1}{2}, p_j \ll \frac{E_1}{2}, p_j g \ll g \) and \( p_j g \ll p_i \). These constraints can be enforced by choosing a cutoff for the \( p_j \) and \( g \). We observe that all of our conclusions
FIG. 2. Entanglement can enhance cooling. (a) Density plot depicting the relative advantage, \( \zeta = \frac{T_C - T_S}{T_C - T^*_S} \), in cooling a qubit at \( T_C = 1K \), as a function of the available bath temperatures (\( T_R \) and \( T_H \)), for optimal refrigerators. Entanglement provides an advantage (\( \zeta > 1 \)) for all points above the red dashed line. In the dark blue region, below the red dashed line, the optimal fridge is separable on all bipartitions (\( \zeta = 1 \)). For \( T_R \) close to \( T_C \) and sufficiently high \( T_H \), entanglement is a widely present feature of the optimal quantum refrigerator. (b) Slices through the density plot for three different values of \( T_H \), showing in more detail the relative advantage in cooling. The parameters used here are \( p_C = 10^{-5}, E_1 = 1J, \) and parameter bounds \( p_R, p_{PH} \leq 10^{-4} \).

below remain valid independent of the precise choice of the cutoff. The only change is that the strength of the effect becomes weaker as we make the constraints stronger, as is intuitively expected. We comment further on this at the end of this section.

We first perform the optimization without additional constraints. That is, among all possible refrigerators, we look for the one achieving the best cooling, i.e. the smallest value of \( T_S \). Next, we repeat this optimization, but now adding the constraint that no entanglement is present in the fridge. More precisely, we find the optimal cooling (now denoted \( T^*_S \)), imposing that the stationary state \( \rho_S \) satisfies all the entanglement witness inequalities \( (7) \) (and their relevant symmetries), hence ensuring separability across every bipartition.

The results are presented in Fig. (2). We observe that the cold qubit can be cooled to lower temperatures when no restrictions are placed, compared to the case when the system is constrained to be separable. In the regime where \( T_R \ll T_H \), entanglement provides a significant enhancement in cooling, which is quantified by the ratio

\[
\zeta = \frac{T_C - T_S}{T_C - T^*_S}.
\]

Notably, although in general we were able to find entanglement across any bipartition, here, when restricting only to optimal machines, we find that the entanglement is almost always preferentially between the room qubit and the other two. Only if \( T_R \approx T_C \) and \( T_R \ll T_H \) does entanglement between the other two bipartitions also appear. An intriguing aspect of this behaviour is that the entanglement therefore appears to be preferentially between the partitions energy in: energy out, since in the stationary state it is the room qubit which is heated up, and the cold and hot qubits which are cooled down. Hence, it seems that quantum coherence (here entanglement) enhances energy transport, a phenomenon that has received considerable attention in the context of photosynthetic complexes \[33\].

It is important to note that the cooling enhancement observed here is much larger than the uncertainty in the result (which is \( O(gp) \)) and hence represents a genuine effect, and not a mere consequence of our approximate dynamics. Not unexpectedly, the optimal
cooling is obtained by maximizing the interaction coupling as well as the thermalization rates. Here these parameters must be constrained in the optimization in order to remain within the regime of validity of the master equation (5). It would thus be interesting to consider more general master equations (e.g. the one studied in [22]), accounting for more sophisticated thermalization processes. In this case, one could describe machines working at much higher rates, in which entanglement may become even more beneficial. We leave it for future research to explore this direction.

Finally, we investigate the link between the amount of entanglement on the bipartition \( R|CH \) (as measured by the concurrence \( C \)) and the relative cooling enhancement \( \zeta \). Remarkably, \( \zeta \) appears to be solely determined by the concurrence (see Fig. 3), and hence does not depend on the temperatures of the heat baths \( T_R \) and \( T_H \). The results strongly suggest a functional relationship between the relative cooling enhancement and concurrence. Indeed, the monotonic relationship observed shows that the more entanglement that is present, the larger the advantage one gains for cooling a system.

**CONCLUSIONS**

We have demonstrated that entanglement plays a fundamental role in the smallest self-contained quantum refrigerator. Entanglement is a feature of a wide range of operation regimes, with the notable exception of the Carnot point and its vicinity. Crucially, this entanglement is not merely a byproduct, but its presence is beneficial: the refrigerator is able to cool to lower temperatures when it becomes entangled. Moreover, our results show that the extra quantum performance is directly related to measures of entanglement.

One important question is to what extent the results obtained here tell us about entanglement and its usefulness in general thermal machines. In [21] it was argued that any thermal machine which is operating close to the Carnot efficiency is necessarily functioning in the same manner as the smallest thermal machines studied here. This is precisely the regime where no entanglement is present, and thus, based upon the above argument, we can conclude that close to the Carnot efficiency no thermal machine is entangled (at least regarding those degrees of freedom directly relevant to the cooling process). What would be more interesting however is to go in the other direction, and make statements about the regime where entanglement is beneficial. At the present such general results appear to be beyond our reach, but any such statements in this direction would represent significant progress.

Another fundamental question is whether entanglement can enhance the performance of machines producing work. The model discussed here, when analysed appropriately (in particular by replacing the cooled qubit by a ‘weight’ which can store work), can also function as a small work-producing heat engine [21]. Hence an analysis of the presence and role of entanglement in this system will also be of particular interest, especially given the recent results [34-37] where the role of entanglement in work extraction from quantum systems is explored.

To conclude, we believe that the present results provide strong evidence that entanglement plays a fundamental role in thermodynamic processes. Clearly, turning these preliminary results into a general and quantitative understanding of the exact role played by entanglement in thermodynamics is highly desirable, highlighting the necessity of pursuing this line of research further.

**Note added.** While finishing this work, we became aware of the work of Correa and colleagues [22] dis-
cussing the effect of quantum discord in the small self-contained quantum refrigerator.

Acknowledgments. We thank J. Reid for contributions in early stages of this work. We acknowledge financial support from the Swiss National Science Foundation (grant PP00P2_138917), the FP7-MarieCurie grant ‘Quacocos’, the ERC (Advanced Grant NLST), and the Templeton Foundation.

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APPENDIX A

In this appendix we show that around any Carnot point (i.e. any any machine which is operating at the Carnot efficiency) there is a ball of states which are fully biseparable (i.e. separable on any bipartition). It is also shown that in some circumstances, there is a ball of fully separable states.

More precisely, the first statement that we would like to prove is that there exists an $\epsilon > 0$, such that for all states $\sigma$ which satisfy $\|\rho_\mathcal{S} - \sigma\|_1 \leq \epsilon$ are biseparable, where $\rho_\mathcal{S}$ is a stationary state of the refrigerator corresponding to a machine at Carnot efficiency.

The first important point is that all Carnot points have the form $\rho_\mathcal{S} = \tau_x \otimes \tau_y \otimes \tau_z$, where each $\tau_i$ is a thermal state at at strictly positive temperature $T_i > 0$ (since if we consider the cold bath to be at absolute zero there is no cooling to be achieved). As such, it is immediately clear that $\rho_\mathcal{S}$ is a full rank fully separable state (in fact it is a direct product state). The fact that $\rho_\mathcal{S}$ is full rank guarantees that there is a ball of states around it (i.e. it is strictly in the interior of the set of quantum states). This ball however may contain entangled states, and so we must show that this ball contains within it a small ball of separable states.

Any states $\sigma$ which are diagonal in the same basis as $\rho_\mathcal{S}$ are clearly also fully separable, and therefore we shall focus on states which are not diagonal. Let us consider first states $\sigma$ which are still within the interior of the set of quantum states but contain only a single pair of off-diagonal elements, i.e. states of the form

$$\sigma = \rho_D + \epsilon_{xy} |x\rangle\langle y| + \epsilon_{yx}^* |y\rangle\langle x|$$

where $\rho_D$ is diagonal in the same basis as $\rho_\mathcal{S}$, and $x = (x_C, x_R, x_H)$ collectively specifies the state for the three qubits in the energy eigenbasis (and analogously for
TABLE I. **Regimes of the quantum refrigerator featuring entanglement.** Here we give the full list of parameters required to exhibit the different entanglement regimes present in the quantum refrigerator. The first row gives a point where there is genuine multipartite entanglement between all three qubits, whilst the second and third rows are points where there is only bipartite entanglement along a single bipartition. The last row shows a regime with bipartite entanglement across every bipartition but without genuine tripartite entanglement.

| $C_{CRH}$ | $C_{R|CH}$ | $C_{H|CR}$ | $C_{CRH}$ | $T_C$ | $T_R$ | $T_H$ | $E_1$ | $E_3$ | $g$ | $p_1$ | $p_2$ | $p_3$ |
|-----------|------------|------------|------------|-------|-------|-------|-------|-------|-----|-------|-------|-------|
| 0.003 | 0.004 | 0.004 | 0.003 | 1.0 | 1.1 | $1.0 \times 10^4$ | 2.0 | 300 | $1.0 \times 10^{-4}$ | 1.0 | $1.0 \times 10^{-5}$ | 1.0 | $1.0 \times 10^{-5}$ |
| 0 | 0 | 0.00002 | 0 | 1.0 | 2.0 | $1.0 \times 10^4$ | 4.4 | 379 | $3.4 \times 10^{-4}$ | 1.3 | $1.0 \times 10^{-5}$ | 1.6 | $1.0 \times 10^{-4}$ |
| 0 | 0.00005 | 0 | 0 | 1.0 | 42.6 | $1.0 \times 10^4$ | 4.4 | 421 | $2.8 \times 10^{-4}$ | 1.3 | $1.0 \times 10^{-5}$ | 8.3 | $1.0 \times 10^{-4}$ |
| 0.0008 | 0.005 | 0.004 | 0 | 1.0 | 1.1 | $1.0 \times 10^4$ | 2.0 | 300 | $1.0 \times 10^{-4}$ | 1.0 | $1.0 \times 10^{-5}$ | 2.0 | $1.0 \times 10^{-4}$ |