Trigonometric S-matrices, Affine Toda Solitons and Supersymmetry

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ABSTRACT

Using $U_q(a^{(1)}_n)$– and $U_q(a^{(2)}_{2n})$–invariant \( R \)-matrices we construct exact \( S \)-matrices in two-dimensional space–time. These are conjectured to describe the scattering of solitons in affine Toda field theories. In order to find the spectrum of soliton bound states we examine the pole structure of these \( S \)-matrices in detail. We also construct the \( S \)-matrices for all scattering processes involving scalar bound states. In the last part of this paper we discuss the connection of these \( S \)-matrices with minimal \( N = 1 \) and \( N = 2 \) supersymmetric \( S \)-matrices. In particular we comment on the folding from \( N = 2 \) to \( N = 1 \) theories.

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1 Introduction

During the last twenty years, a great deal of effort has been expended on the study of integrable two-dimensional field theories. Unlike their higher dimensional counterparts, these theories can often be solved exactly without the need to resort to perturbative methods. It is for this reason that two-dimensional field theories can provide important insights which may lead towards a better understanding of non-perturbative aspects of general quantum field theories. Apart from their role as models for higher dimensional theories, two-dimensional theories also have some important applications in string theory. In particular, the study of supersymmetric (and superconformal) theories is closely related to the study of world sheet supersymmetries in superstring theories.

This paper deals with the construction of exact $S$-matrices for a certain class of relativistic quantum field theories defined in $(1 + 1)$-dimensional space-time, namely theories displaying a quantum affine symmetry. We also briefly discuss the general structure of $S$-matrices for supersymmetric quantum field theories and point out some interesting connections between the trigonometric $S$-matrices and supersymmetric $S$-matrices. The construction of the $S$-matrices in this paper follows closely the construction in the series of papers [1, 2, 3, 4]. A more detailed exposition of some of the material presented here can also be found in [5].

The layout of this paper is as follows. In the first chapter we provide a brief introduction to some of the main results of the study of affine Toda field theories (ATFTs). A review of the subject of trigonometric $R$-matrices and exact $S$-matrices for theories with quantum affine symmetries is given in chapter 2. Chapter 3 deals with the construction of exact $S$-matrices using $U_q(a_n^{(1)})$ invariant $R$-matrices. We examine the pole structure of these $S$-matrices and determine scattering amplitudes for the scattering of bound states. We also summarise the evidence for the conjecture that these $S$-matrices describe the scattering of solitons in $a_n^{(1)}$ ATFTs. In chapter 4 we repeat this construction for the case of $U_q(a_n^{(2)})$ invariant $R$-matrices and $a_n^{(2)}$ affine Toda solitons. Chapter 5 deals with a slightly separate subject, namely the minimal $S$-matrices for two-dimensional $N = 1$ and $N = 2$ supersymmetric models. We find, however, that the $S$-matrix scalar factors in these supersymmetric cases are related to those in chapters 3 and 4 at one particular value of the coupling constant. We also comment briefly on a possible “folding” from $N = 2$ to $N = 1$ theories as discussed recently by Moriconi and Schoutens [6].
1.1 Affine Toda field theories

Affine Toda field theories (ATFTs) are a family of (classically) integrable field theories in (1+1)–dimensional Minkowski space. They are defined by their Lagrangian density

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi^a) (\partial_\mu \phi^a) - \frac{m^2}{\bar{\beta}^2} \sum_{j=0}^{r} n_j e^{\bar{\beta} \alpha_j \phi}, \]  

(1.1)

in which \( \phi = (\phi^1, \ldots, \phi^r) \) is a \( r \)-dimensional scalar field, the \( \alpha_i \)'s are a set of \( r + 1 \) \( r \)-dimensional vectors and \( m \) and \( \bar{\beta} \) are mass and coupling constant parameters. \( \{\alpha_1, \ldots, \alpha_r\} \) form the root system of a semi–simple classical Lie algebra \( g \) of rank \( r \), and \( \alpha_0 \) is chosen to be the extended root, such that \( \{\alpha_0, \alpha_1, \ldots, \alpha_r\} \) form the root system of the affine Lie algebra \( \hat{g} \) (the longest root is taken to have length \( \sqrt{2} \), except in the case of a twisted Lie algebra \( \hat{g}^{(k)} \) in which we take the longest root to have length \( \sqrt{2k} \)). The numbers \( n_j \) are the Kac marks of the affine algebra \( \hat{g} \) and we have chosen \( n_0 = 1 \). Thus, there is one ATFT associated with each simple affine Lie algebra \( \hat{g} \).

Among the ATFTs one has to distinguish between the two fundamentally different cases, the so–called real and imaginary ATFTs, which are distinguished by the coupling constant \( \bar{\beta} \) being either a real or purely imaginary number.

In the case of a real coupling constant \( \bar{\beta} \) the Lagrangian describes a unitary field theory containing \( r \) scalar particles. The classical masses and three point couplings for all real ATFTs were computed in \cite{7} and it was found that the particle masses form the right Perron–Frobenius eigenvector of the Cartan matrix of the underlying algebra. This fact suggested that each particle can be associated with one of the nodes of the Dynkin diagram.

It is generally believed that the integrability of these theories is preserved after quantisation and that ATFTs possess exact factorised \( S \)-matrices. The masses of the particles in real ATFTs are non–degenerate and the \( S \)-matrices are therefore diagonal. By using the axioms of analyticity, unitarity, crossing symmetry and the bootstrap, exact \( S \)-matrices for the simply–laced ATFTs (i.e. based on the a, d and e series of affine Lie algebras) have been constructed by Braden et al. in \cite{7} and independently by Christe and Mussardo in \cite{8}. The \( S \)-matrices for the non–simply–laced and twisted algebras, in which the poles corresponding to bound states require an additional coupling constant dependence, were found some time later by Delius et al. in \cite{9}. One of the most intriguing features of these \( S \)-matrices is the fact that they exhibit a strong–weak coupling duality. This means that if we change the coupling constant in the \( S \)-matrices for the \( \hat{g} \) ATFT in the following way

\[ \bar{\beta} \mapsto 4\pi \beta, \]  

(1.2)
then we recover the $S$-matrices of the ATFT associated with the dual algebra $\hat{g}^\vee$. (The dual of an affine Lie algebra is obtained by exchanging long and short roots.)

1.2 Affine Toda solitons

Let us now take the coupling constant in (1.1) to be purely imaginary, i.e. $\tilde{\beta} = i\beta$ with $\beta$ being a real number. This seemingly small change in the Lagrangian has some very significant implications for these theories. First of all, apart from the $a_1^{(1)}$ theory, which is the well known Sine–Gordon theory, the imaginary ATFTs are in general non–unitary.

The most striking feature of imaginary ATFTs, however, is the fact that they display an infinitely degenerate vacuum and therefore admit classical soliton solutions. By using Hirota’s method for the solution of nonlinear differential equations, Hollowood was able in [10] to construct explicit expressions for one–soliton and multi–soliton solutions in $a_n^{(1)}$ ATFTs. Some time later Olive et al. [11] used a more algebraic construction in order to obtain the general solution to the classical equations of motion for ATFTs based on all affine algebras. Despite the complex form of the Hamiltonian, the solitons turn out to have real and positive energies and masses. As pointed out by Olive et al., this fact suggests that there might be a unitary theory embedded in the (generally non–unitary) imaginary ATFTs. Remarkably, the masses of the solitons were also found to be proportional to the particle masses in the dual theory$^b$.

Apart from the elementary solitons, there are also bound states of solitons which can be regarded as two solitons oscillating around a fixed point. Whereas in Sine–Gordon theory only bound states with zero topological charge (the ‘breathers’) occur, in most other ATFTs there are also bound states with non–zero topological charges (the ‘breathing solitons’ or ‘excited solitons’). In [12] Harder et al. obtained classical bound state solutions for $a_n^{(1)}$ ATFTs by changing the real velocity into an imaginary velocity in the expressions of the two–soliton solutions. They found scalar bound states, which are the analogues to the Sine–Gordon breathers, and also bound states transforming under the $2a$th (for $a = 1, 2, \ldots \leq (n + 1)/2$) and $(2a - n - 1)$th (for $(n + 1)/2 < a < n$) fundamental representations of $a_n^{(1)}$. While in the classical theory a continuous spectrum of bound states exists, in the quantum theory we expect the spectrum to be quantised and a finite number of discrete bound states to emerge.

$^b$Note that by dual theory we mean in this case the theory obtained after the exchange of the underlying simple Lie algebra $g$ with its dual $g^\vee$, whereas the weak–strong coupling duality of the $S$-matrices relates the affine algebra $\hat{g}$ with its affine dual $\hat{g}^\vee$. 
Unlike in the case of real coupling constant the quantum theory of the imaginary ATFTs is not very well established. This is mainly due to the fact that the Lagrangian in this case defines, in general, a non–unitary field theory. Despite this problem, attempts have been made to construct exact $S$-matrices for the scattering of quantum solitons in ATFTs. These attempts rest primarily on the assumption of a quantum affine symmetry, which has been established in \[13, 14\] for certain restrictions of imaginary ATFTs. As we will review in the following chapter such a quantum affine symmetry permits the use of trigonometric $R$-matrices as scattering matrices. Hollowood was the first to achieve the explicit construction of exact $S$-matrices for $a_n^{(1)}$ affine Toda solitons by using $U_q(a_n^{(1)})$ invariant $R$-matrices. Following the realisation of the crucial importance of $R$-matrix gradations in \[15, 16\] this construction was extended to the cases of $d_{n+1}^{(2)}$ and $b_n^{(1)}$ ATFTs in \[3, 4\]. In two very recent papers \[17\] the same construction was used for the two exceptional cases of $d_4^{(3)}$ and $g_2^{(1)}$ ATFTs. In the present paper we extend Hollowood’s construction for the $a_n^{(1)}$ case to the scattering of bound states of solitons and we construct the $S$-matrices for the case of the twisted algebra $a_{2n}^{(2)}$.

2 Exact $S$-matrices from trigonometric $R$-matrices

In this section we provide a short introduction to the subject of $R$-matrices of quantised universal enveloping algebras (QUEAs) of affine Lie algebras. These $R$-matrices are trigonometric solutions of the quantum Yang–Baxter equation. Although they were not originally introduced for this purpose, trigonometric $R$-matrices turn out to be exactly the right objects to use as $S$-matrices for theories displaying a so–called quantum affine symmetry. Following Delius \[16\] (see that paper for details) we give a general scheme of how to build exact soliton $S$-matrices in terms of these trigonometric $R$-matrices.

2.1 $R$-matrices and the Yang Baxter Equation

Associated with each QUEA $U_q(\hat{g})$ there exists a universal $R$-matrix $\mathcal{R} \in U_q(\hat{g}) \otimes U_q(\hat{g})$, which satisfies the following identities:

\[
(\Delta \otimes I)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (I \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (\forall a \in U_q(\hat{g})) \quad (2.1)
\]

\[
\overline{\Delta}(a)\mathcal{R} = \mathcal{R}\Delta(a) \quad (\forall a \in U_q(\hat{g})) \quad (2.2)
\]
in which $\Delta$ is the (Hopf algebra) coproduct $\Delta : U_q(\hat{g}) \to U_q(\hat{g}) \otimes U_q(\hat{g})$, $\Box$ is the opposite coproduct defined by $\Box = T \circ \Delta$ and $T$ denotes the transposition map, i.e. $T(a_1 \otimes a_2) = a_2 \otimes a_1$ ($\forall a_1, a_2 \in U_q(\hat{g})$). If we write $R = \sum_n R^{(1)}_n \otimes R^{(2)}_n$, then $R_{ij} \equiv \sum_n 1 \otimes \ldots \otimes R^{(1)}_n \otimes 1 \otimes \ldots \otimes R^{(2)}_n \otimes \ldots \otimes 1$, in which $R^{(1)}_n$ and $R^{(2)}_n$ appear in the $i$th and $j$th position. It is also possible to deduce the following identities, which later will be used in connection with the crossing symmetry of $S$-matrices:

$$(\epsilon \otimes I)R = (I \otimes \epsilon)R = 1$$

and

$$(S \otimes I)R = R^{-1}, \quad (I \otimes S)R^{-1} = R,$$

in which $I$, $\epsilon$ and $S$ denote the identity, counit and antipode in $U_q(\hat{g})$, respectively. For our purpose the most important property of universal $R$-matrices is the fact that they satisfy the quantum Yang-Baxter equation (YBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$  \hfill (2.5)

Instead of the universal $R$-matrices, here we require expressions of $R$-matrices which depend on a spectral parameter and on some finite dimensional representations of $U_q(\hat{g})$. In order to achieve this we first define an automorphism $D^s_x : U_q(\hat{g}) \to U_q(\hat{g})$ which acts on the Chevalley generators $e_i, f_i$ and $h_i$ of $U_q(\hat{g})$ in the following way:

$$D^s_x(e_i) = x^{s_i}e_i, \quad D^s_x(f_i) = x^{-s_i}f_i, \quad D^s_x(h_i) = h_i,$$

in which $x$ is a complex number and the $s_i$ ($i=0,1,\ldots,n$) are a set of real numbers which determine the gradation $s$ of the $R$-matrix. The most commonly used gradations are the homogeneous gradation, in which $s_i = \delta_{i0}$, and the principal gradation, in which $s_i = 1$, ($\forall i = 0,1,\ldots,n$). Thus we can define a $R$-matrix depending on a spectral parameter $x$ and on the gradation $s$:

$$R^{(s)}(x) \equiv (D^s_x \otimes I)R.$$  \hfill (2.7)

Now let $V_i$ and $V_j$ be two finite dimensional $U_q(\hat{g})$-modules and $\pi_{i(j)} : U_q(\hat{g}) \to \text{End}(V_{i(j)})$ the associated representations, and we define

$$\tilde{R}^{(s)}_{i,j}(x) \equiv \sigma_{ij}(\pi_i \otimes \pi_j)R^{(s)}(x),$$

in which $\sigma_{ij}$ is a permutation operator $\sigma_{ij}(v \otimes w) = w \otimes v$ ($\forall v \in V_i, w \in V_j$). Thus $\tilde{R}^{(s)}_{i,j}(x)$ acts as an intertwiner on the spaces $V_i$ and $V_j$:

$$\tilde{R}^{(s)}_{i,j}(x) : V_i \otimes V_j \to V_j \otimes V_i.$$  \hfill (2.9)
Note that $R$-matrices in different gradations can be related to each other through a gauge transformation, i.e. given two gradations $s^1$ and $s^2$ we can always find a transformation $\tau_{ij}$: $V_i \otimes V_j \rightarrow V_i \otimes V_j$, such that $\tilde{R}_{i,j}^{(s^1)}(x) = \tau_{ji} \tilde{R}_{i,j}^{(s^2)}(x) \tau_{ij}^{-1}$. In the remainder of this paper we will mainly use $R$-matrices in the homogeneous gradation $s^h$. For the sake of simplicity we therefore omit the gradation label on $R$-matrices in the homogeneous gradation, i.e. $\tilde{R}_{i,j}(x) \equiv \tilde{R}_{i,j}^{(s^h)}(x)$.

For any three finite dimensional representations the associated YBE now takes the form

$$\left[\tilde{R}_{j,k}(x) \otimes I_j\right] \left[I_j \otimes \tilde{R}_{i,k}(xy)\right] \left[\tilde{R}_{i,j}(y) \otimes I_k\right] =$$

$$= \left[I_k \otimes \tilde{R}_{i,j}(y)\right] \left[\tilde{R}_{i,k}(xy) \otimes I_j\right] \left[I_i \otimes \tilde{R}_{j,k}(x)\right],$$

in which $I_l$ denotes the identity on $V_l$ and therefore both sides of equation (2.10) map $V_i \otimes V_j \otimes V_k$ into $V_k \otimes V_j \otimes V_i$. Furthermore, let us assume that we can write the tensor product of two modules $V_i$ and $V_j$ as a direct sum of irreducible modules in the following way

$$V_i \otimes V_j = \bigoplus_k V_k.$$  

(2.11)

It can then be shown that the general solution to equation (2.10) can be written as a spectral decomposition

$$\tilde{R}_{i,j}(x) = \sum_k \rho_k(x) \tilde{P}^{ij}_k,$$

(2.12)

in which $\tilde{P}^{ij}_k: V_i \otimes V_j \rightarrow V_k \subset V_j \otimes V_i$ denotes the intertwining projector onto the irreducible module $V_k$ and the $\rho_k(x)$ are scalar factors. Accordingly, it is also immediately evident that these $R$-matrices satisfy the following normalisation condition:

$$\tilde{R}_{a,b}(x) \tilde{R}_{b,a}(x^{-1}) = I_b \otimes I_a.$$  

(2.13)

Using a method known as the tensor product graph method it is possible to find explicit expressions of trigonometric $R$-matrices in the form (2.12). At this stage, however, this method is only applicable to the case of multiplicity–free tensor products, i.e. those cases in which each $V_k$ appears only once at the right hand side of (2.11).

### 2.2 Exact $S$-matrices with quantum affine symmetries

Let us assume that we have a relativistic quantum field theory in $(1 + 1)$-dimensional space–time. Following Delius \[16\], we say this theory has a $U_q(\hat{g})$ quantum affine symmetry if the following two conditions hold:
1) The theory has quantum conserved charges $H_i$, $E_i$, $F_i$, for $(i = 1, 2, ..., n)$, which satisfy the same commutation relations as the Chevalley generators of $U_q(\hat{g})$.

2) These conserved charges possess definite Lorentz spin. If we denote the infinitesimal Lorentz generator by $D$, the conserved charges transform under $D$ in the following way

$$D(E_i) = s_i E_i, \quad D(F_i) = -s_i F_i, \quad D(H_i) = 0,$$  \hspace{1cm} (2.14)

in which $s_i$ and $-s_i$ are the Lorentz spins of the conserved charges $E_i$ and $F_i$ respectively.

In other words, we can find conserved charges which act as the generators of an $U_q(\hat{g})$ charge algebra, in which $\hat{g}$ is a rank $n$ affine Lie algebra. And one finds that the Lorentz spin of these conserved charges plays the role of the derivation in $U_q(\hat{g})$. Thus we obtain that the gradation of the charge algebra is determined by the Lorentz spins $s_i$ of the conserved charges. We will therefore call this gradation the spin gradation.

As Delius pointed out in [16], the existence of a quantum affine symmetry also implies the complete integrability of the theory. Thus we assume that we are dealing with an integrable theory which possesses factorised $S$-matrices, which means that all the information of any multi-particle scattering process is contained in the two-particle $S$-matrices. Let us further assume that we can arrange the quantum states in the theory into mass degenerate multiplets which are the modules of some finite dimensional irreducible representations of $U_q(\hat{g})$. Let us denote these multiplets by $V_1, V_2, \ldots, V_n$. Thus the two–particle $S$-matrices must act as intertwiners on these representation spaces

$$S_{a,b}(\theta) : V_a \otimes V_b \longrightarrow V_b \otimes V_a,$$  \hspace{1cm} (2.15)

and depend only on the rapidity difference $\theta$ of the incoming particles. These $S$-matrices are severely constrained by the so–called axioms of analytic $S$-matrix theory, which we briefly review in the following:

(i) Factorisation equation:

$$[S_{b,c}(\theta_1) \otimes I_a] [I_b \otimes S_{a,c}(\theta_1 + \theta_2)] [S_{a,b}(\theta_2) \otimes I_c] =$$

$$= [I_c \otimes S_{a,b}(\theta_2)] [S_{a,c}(\theta_1 + \theta_2) \otimes I_b] [I_a \otimes S_{b,c}(\theta_1)],$$  \hspace{1cm} (2.16)

in which $\theta_1 \equiv \theta_b - \theta_c$ and $\theta_2 \equiv \theta_a - \theta_b$. The factorisation equation is a result of the existence of higher spin conserved charges in an integrable theory.
(ii) Unitarity:
The unitary condition can be expressed as

\[ S_{a,b}(\theta)S_{b,a}(-\theta) = I_b \otimes I_a . \]  

(2.17)

(iii) Crossing Symmetry:
The matrix elements \( S_{a,b}(\theta) \) must be symmetric under the transformation \( \theta \mapsto i\pi - \theta \), such that

\[ S_{a,b}(\theta) = (I_b \otimes C_a)[\sigma_{ab}S_{b,a}(i\pi - \theta)] \sigma_{ab}(I_a \otimes C_b) , \]

in which \( C_a : V_a \rightarrow V_a \) is the charge conjugation operator which maps the particles in \( V_a \) to their charge conjugated partners in \( V_a \) and \( t_1 (t_2) \) means transposition in the first (second) space.

(vi) Analyticity and Bootstrap:
\( S_{a,b}(\theta) \) is a meromorphic function of \( \theta \) with the only singularities on the physical strip \( (0 \leq \text{Im}\theta \leq \pi) \) at \( \text{Re}\theta = 0 \). Simple poles in the physical strip correspond to bound states in the direct or crossed channel. If \( S_{a,b}(\theta) \) exhibits a simple pole \( \theta_{ab}^c \) on the physical strip corresponding to a bound state in \( V_c \subset V_a \otimes V_b \) in the direct channel, then the mass of this particle in \( V_c \) is given by the formula:

\[ m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos(\text{Im}\theta_{ab}^c) . \]

(2.19)

In this case there must also be poles \( \theta_{ca}^b \) in \( S_{c,a}(\theta) \) and \( \theta_{ab}^c \) in \( S_{b,a}(\theta) \), such that \( \theta_{ab}^c + \theta_{ca}^b + \theta_{ba}^c = 2\pi i \).

The bootstrap equations express the fact that there is no difference whether the scattering process with any particle in, say, \( V_d \) occurs before or after the fusion of particles \( a \) and \( b \) into particle \( c \):

\[ S_{d,c}(\theta) = [I_b \otimes S_{d,a}(\theta - (i\pi - \theta_{ca}^b))] [S_{d,b}(\theta + (i\pi - \theta_{ba}^c)) \otimes I_a] , \]

(2.20)

in which both sides are restricted to \( V_d \otimes V_c \subset V_d \otimes V_a \otimes V_b \).

If we now compare these axioms with the properties of the \( R \)-matrices as defined in (2.8), we find that the \( R \)-matrices, which are also intertwining maps on the representation spaces, already satisfy most of the \( S \)-matrix axioms. The most important of these axioms is the factorisation equation, which is the same as the YBE with additive spectral parameter \( \theta \). However, the \( R \)-matrix satisfies the YBE (2.10) with multiplicative spectral parameter \( x \). This difference is easily circumvented if we choose \( x(\theta) = \exp(\theta) \). The \( R \)-matrix normalisation (2.13) now becomes

\[ I_b \otimes I_a = \tilde{R}_{a,b}(x) \tilde{R}_{b,a}(x^{-1}) = \tilde{R}_{a,b}(x(\theta)) \tilde{R}_{b,a}(x(-\theta)) , \]

(2.21)
which is identical to the unitarity condition \((2.17)\) of the \(S\)-matrix. Furthermore, the equations \((2.1)\) for the \(R\)-matrices appear to be analogous to the bootstrap equations \((2.20)\). The only condition which is not directly satisfied by the \(R\)-matrices is the crossing symmetry condition. However, it was also demonstrated in \cite{14} that the crossing symmetry condition can be satisfied by choosing an overall scalar factor (although care must be taken that this scalar factor does not violate the unitarity condition). This ‘crossing property’ of the \(R\)-matrices is essentially implied by equations \((2.4)\).

We therefore make the following general ansatz for the two-particle \(S\)-matrix of a relativistic field theory displaying an \(U_q(\hat{g})\) quantum affine symmetry:

\[
S_{a,b}(\theta) = F_{a,b}(\theta) \tilde{R}^{(s)}_{a,b}(x),
\]

in which \(F_{a,b}(\theta)\) is an overall scalar factor, \(\tilde{R}^{(s)}_{a,b}(x)\) is the \(U_q(\hat{g})\) invariant \(R\)-matrix in the spin gradation, and the spectral parameter is given by \(x = e^\theta\). The scalar factor \(F_{a,b}\) is determined by the requirements of crossing symmetry and unitarity.

In order to use this general ansatz for specific models we need to find an expression for the gauge transformation which transforms the homogeneous gradation \(R\)-matrices into \(R\)-matrices in the ‘physical’ spin gradation. This is necessary since we only know exact expressions of the \(U_q(\hat{g})\) invariant \(R\)-matrices in the homogeneous gradation. This has first been done by Bracken et al. in \cite{13}, in which the following formula was found

\[
\mathcal{R}^{(s)}(x) = \left(x^{\nu\xi} \otimes I\right) \mathcal{R}^{(h)}(x^\nu) \left(x^{\nu\xi} \otimes I\right)^{-1}, \tag{2.23}
\]

in which \(\xi\) is some linear combination of the conserved charges \(H_1, H_2, \ldots, H_n\). In particular it was found that the change of gradation shifts the spectral parameter \(x\) to \(x^\nu\), and \(\nu\) is given by \(\nu = \sum_{i=0}^{n} n_is_i\), in which the \(n_i\)'s are the Kac marks of the affine symmetry algebra and the \(s_i\)'s are the Lorentz spins of the conserved charges. Hence, if we write the spectral parameter in the spin gradation \(R\)-matrix as \(x = e^\theta\) we obtain the spectral parameter of the homogeneous gradation \(R\)-matrix as

\[
x = \exp \left(\sum_{i=0}^{n} n_is_i\theta\right). \tag{2.24}
\]

(For details of this derivation see \cite{13,16} or \cite{5}).

The only remaining problem concerns the role of the deformation parameter \(q\) in \(U_q(\hat{g})\). Heretofore \(q\) has been an arbitrary complex number (although not a root of unity). Clearly, if we choose \(R\)-matrices as \(S\)-matrices the deformation parameter \(q\), like the spectral parameter \(x\),
must somehow be determined by physical quantities. In the example of ATFTs in the following section we will see that $q$ is related to the coupling constant in the theory.

### 2.3 $S$-matrices for affine Toda solitons

The question of whether there is a quantum affine symmetry present in ATFTs with imaginary coupling constant has not yet been answered satisfactorily. However, some strong evidence exists suggesting that an ATFT with imaginary coupling constant based on a rank $n$ affine Lie algebra $\hat{g}$ displays a $U_q(\hat{g}^\vee)$ quantum affine symmetry, in which $\hat{g}^\vee$ denotes the dual algebra of $\hat{g}$. This evidence emerges from the work by Bernard and LeClair [13] which was later followed up by Felder and LeClair [14]. These papers discuss restrictions of imaginary ATFTs which can be regarded as certain integrable perturbations of conformal field theories. This allows the construction of non–local conserved charges, which can be shown to satisfy the commutation relations of a quantum affine algebra. We will not go into any detail regarding the construction of these non–local charges. What is important for the construction of $S$-matrices is the fact that the construction of these non-local charges gives us the explicit form of the Lorentz spins and the dependence of the deformation parameter $q$ on the coupling constant $\beta$. It was found that that the Lorentz spins of the conserved charges in imaginary ATFTs have the form

$$s_i = \frac{8\pi}{\beta^2 \alpha_i^2} - 1, \quad (\forall i = 0, 1, \ldots, n). \tag{2.25}$$

Note in particular that for the case of simply-laced algebras, in which all simple roots have the same length, the Lorentz spins are all equal and the $R$-matrix in the spin gradation therefore is equal to the $R$-matrix in the principal gradation. Using (2.25) it is possible to derive the explicit $\theta$ dependence of the spectral parameter $x(\theta)$ in the homogeneous gradation $R$-matrix from equation (2.24) in which the $n_i$’s are now the Kac marks of the dual algebra $\hat{g}^\vee$. In [3] we found the result

$$x(\theta) = \exp\left(\frac{4\pi h}{\beta^2} - h^\vee\right) \theta = \exp(h\lambda \theta), \tag{2.26}$$

in which we have introduced the coupling constant dependent function \[c\]

$$\lambda \equiv \frac{4\pi}{\beta^2} - \frac{h^\vee}{h}. \tag{2.27}$$

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\[c\]This function $\lambda(\beta)$ is the analogue of the $B(\beta)$ used in the $S$-matrices for real coupling ATFTs [7]. Note, however, that $\lambda(\beta)$ is computed exactly using the quantum affine symmetry algebra, whereas $B(\beta)$ was only conjectured and checked to low orders in perturbation theory.
In the remainder of this paper we always assume that $\beta$ takes values such that $\lambda$ is positive.

The other important information which we extract from the paper by Felder and LeClair is the form of the deformation parameter $q$. It is possible to deduce that

$$q = \exp\left(\frac{4\pi^2 i}{\beta^2}\right).$$

(2.28)

In particular we notice that in the case of self-dual algebras $q = -\exp(i\pi \lambda)$, from which it emerges that $S$-matrix poles at $x = q^k$ (for integer $k$) do not depend on the coupling constant $\beta$. This reflects the fundamental property that the mass ratios of the solitons in the self-dual cases remain constant under renormalisation [18].

From the study of the properties of classical soliton solutions of ATFTs we also know that the topological charges of the solitons lie in the weight spaces of the fundamental representations of $g$. Together with the assumption of a quantum affine symmetry this observation leads us to expect that the quantum affine Toda solitons can be grouped into mass degenerate multiplets $V_1, V_2, \ldots, V_n$ corresponding to the $n$ fundamental representations of $U_q(\hat{g}^\vee)$, in which $n$ is the rank of $g$. Thus a two particle $S$-matrix describing the scattering of affine Toda solitons should be of the form (2.22) in which the $R$-matrix gradation is determined by (2.23).

3 The $a^{(1)}_n$ invariant $S$-matrices

In this section the general scheme for constructing exact $S$-matrices from trigonometric $R$-matrices will be applied to the $R$-matrices associated with the fundamental representations of $U_q(a^{(1)}_n)$. The construction of the soliton–soliton $S$-matrices in section 2.1 and 2.2 essentially reviews Hollowood’s construction of the $S$-matrices for the scattering of $a^{(1)}_n$ affine Toda solitons in [19, 1], the only new result being an explanation of the inclusion of the minimal Toda factor. We then study the pole structure of these $S$-matrices and compute the scattering amplitudes for the scattering of bound states.

3.1 $R$-matrix fusion and crossing properties

We may write the $U_q(a^{(1)}_n)$ invariant $R$–matrices in the following form:

$$\hat{R}_{a,b}(x) = \sum_{c=0}^{\min(b,n+1-a)} \prod_{i=1}^{c} (2i + a - b)^x \hat{P}_{\lambda_{a+c}+\lambda_{b-c}},$$

(3.1)
in which we have labelled the projectors $\hat{P}$ by the highest weights of the associated irreducible modules. $\lambda_1, \lambda_2, ..., \lambda_n$ are the highest weights of the fundamental modules $V_1, V_2, ..., V_n$ and we set $\lambda_0 = \lambda_{n+1} = 0$. We have also used the bracket notation

$$\langle a \rangle \equiv \frac{1 - xq^a}{x - q^a}.$$  \hspace{1cm} (3.2)

The $R$-matrix (3.1) was given in this form in [20] and also in [21].

For our purpose, the first important feature of these $R$-matrices is their fusion properties. We are always free to rescale $R$-matrices by a scalar factor and we have chosen our $R$-matrix normalisation such that the projector $\hat{P}_{\lambda_a + \lambda_b}$ always has coefficient 1. However, this form of the $R$-matrix is not the one preserved by fusion. Let us denote the $R$-matrix preserved by fusion by $\hat{R}_{a,b}'(x)$, which means

$$\hat{R}_{a,b}'(x) = \prod_{j=1}^{a} \prod_{k=1}^{b} \left[ \hat{R}_{1,1}(xq^{-2a-2b+2j+2k}) \right]_{a+1-j,a+k},$$  \hspace{1cm} (3.3)

in which the equation acts on $V_a \otimes V_b \subset V_{\lambda_a+b}$ and $[ ]_{j,k}$ indicates that the $R$-matrix is taken to act on the $j$th and $k$th $V_1$'s. The product is taken in order of increasing $j$ and $k$. To make this more clear we could also write this in the form

$$\hat{R}_{a,b}'(x) = \left[ I_b \otimes \hat{R}_{a,c}(xq^b) \right] \left[ \hat{R}_{a,b}'(xq^{-c}) \otimes I_c \right],$$

$$\hat{R}_{a+b,c}(x) = \left[ \hat{R}_{a,c}(xq^b) \otimes I_b \right] \left[ I_a \otimes \hat{R}_{b,c}(xq^{-a}) \right],$$

in which the first equation is restricted to $V_a \otimes V_{b+c}$ and the second to $V_{a+b} \otimes V_c$. Since the totally antisymmetric representation (i.e. $V_{\lambda_{a+b}}$) is obtained by fusing totally antisymmetric representations only, the $R$-matrices $\hat{R}_{a,b}'$ must be those in which the projector $\hat{P}_{\lambda_{a+b}}$ has prefactor 1 (or in the case of $a + b \geq n + 1$ those in which $\hat{P}_{\lambda_{a+b}}$ would have prefactor 1, if the tensor product graph was not truncated). Therefore the $R$-matrices $\hat{R}_{a,b}'$ are related to $\hat{R}_{a,b}$ in the following way:

$$\hat{R}_{a,b}(x) = \prod_{k=1}^{b} \langle a - b + 2k \rangle \hat{R}_{a,b}'(x).$$  \hspace{1cm} (3.4)

Let us define a scalar factor $k_{a,b}(x)$ such that

$$k_{a,b}(x)\hat{R}_{a,b}(x) \equiv \prod_{j=1}^{a} \prod_{k=1}^{b} \left[ \hat{R}_{1,1}(xq^{-2a-2b+2j+2k}) \right]_{a+1-j,a+k}.$$  

Using (3.3) and (3.4) we can compute $k_{a,b}(x)$ explicitly and obtain

$$k_{a,b}(x) = (-1)^a \prod_{k=1}^{b} \langle a + b - 2k \rangle.$$
Following (2.26) and (2.28) we now set
\[ x = e^{2i\pi \mu}, \quad q = e^{i\pi \lambda}, \]
(3.5)
in which \( \mu \) and \( \lambda \) are related to the rapidity and coupling constant:
\[ \mu = -\frac{h\lambda}{2\pi}, \quad \lambda = \frac{4\pi}{\beta^2} - 1. \]
(3.6)
And \( h \) denotes the Coxeter number of the underlying affine Lie algebra, which for the case of \( a_n^{(1)} \) is given by
\[ h = n + 1. \]
(3.7)
In terms of these variables we can write
\[ k_{a,b}(x(\mu)) = (-1)^{a+b} \prod_{k=1}^{b} \frac{\sin\left(\pi(\mu + \frac{1}{2}(a + b - 2k))\right)}{\sin\left(\pi(\mu - \frac{1}{2}(a + b - 2k))\right)} . \]
(3.8)
In order to proceed also we need the following crossing formula for the \( R \)-matrix:
\[ \tilde{R}_{1,1}(x) = \frac{\sin(-\pi \mu)}{\sin(\pi(\mu - \lambda))} \tilde{R}_{1,1}^{(cross)} \tilde{R}_{a,b}^{-1}(x^{-1} q^{n+1}) , \]
(3.9)
in which the ‘crossed’ \( R \)-matrix is given by
\[ \tilde{R}_{1,1}^{(cross)}(x) = (C_n \otimes I_1) \left[ \sigma_{1n} \tilde{R}_{n,1}(x) \right] \tau_1 \sigma_{1n} (I_1 \otimes C_1) . \]
(3.10)
The formula (3.9) has been derived by Hollowood in [19].

3.2 The \( S \)-matrix scalar factor

In order to use the above \( R \)-matrices as scattering matrices we must multiply them by an overall scalar factor which ensures unitarity and crossing symmetry. We therefore make the following ansatz:
\[ S_{a,b}(\theta) = F_{a,b}(\mu) k_{a,b}(\mu) \tau \tilde{R}_{a,b}(x) \tau^{-1} , \]
(3.11)
in which \( \tilde{R}_{a,b}(x) \) and \( k_{a,b}(\mu) \) are as given above and \( \tau \) denotes the gauge transformation from the homogeneous to the principal gradation. The overall scalar factor \( F_{a,b}(\mu) \) will be constructed in the following.

\(^d\)Note, however, that we follow [4, 5] and choose \( q \) to differ from (2.28) by a minus sign. This has no effect on the \( S \)-matrix pole structure and agrees with the conventions in [1].

\(^e\)Recall that the physically relevant spin gradation is equal to the principal gradation in the case of self–dual algebras.
We have chosen the definition of $x$ and $q$ in such a way that the poles in the $S$-matrix, which correspond to the fusion of two solitons of type $a$ and $b$ into a soliton of type $a + b$, are at $x = q^a b$. Therefore, $S_{a,b}(\theta)$ satisfies the same fusion formula as $\tilde{R}_{a,b}$ and we therefore must have

$$F_{a,b}(\mu) = \prod_{j=1}^{a} \prod_{k=1}^{b} F_{1,1}(\mu + \frac{\lambda}{2}(a + b - 2j - 2k + 2)) .$$

(3.12)

Thus all $F_{a,b}(\mu)$ can be computed from the lowest factor $F_{1,1}(\mu)$ via this fusion formula, and it is therefore sufficient to determine the factor $F_{1,1}(\mu)$. Putting the ansatz (3.11) (with $a = b = 1$) into equation (2.17) and using the $R$-matrix normalisation (2.13), we obtain

$$F_{1,1}(\mu)F_{1,1}(-\mu) = 1 ,$$

(3.13)

since $k_{1,1}(\mu) = 1$. In order to obtain the condition imposed on $F_{1,1}(\mu)$ by the requirement of $S$-matrix crossing symmetry, we use the second line of equation (2.18) with $a = b = 1$ and the crossing property (3.9) of the $R$-matrix. We then obtain

$$F_{1,1}(\mu)(-1)^n \frac{\sin(\pi \mu)}{\sin(\pi(\mu - n\lambda))} = \prod_{k=1}^{n} F_{1,1}(-\mu + k\lambda) .$$

(3.15)

Combining (3.13) and (3.15) we finally arrive at

$$\prod_{k=0}^{n} F_{1,1}(-\mu + k\lambda) = (-1)^n \frac{\sin(\pi \mu)}{\sin(\pi(\mu - n\lambda))} .$$

(3.16)

One solution to this equation has been found by Hollowood in [1]. There is, however, an infinite number of different solutions. We are only interested in the solution with a minimum number of poles and zeros on the physical strip ($0 \leq \mu \leq \frac{n+1}{2}\lambda$). For later convenience let us remove the overall minus sign from the right hand side of (3.16) and write

$$\prod_{k=0}^{n} -F_{1,1}(-\mu + k\lambda) = \frac{\sin(-\pi \mu)}{\sin(\pi(\mu - n\lambda))} .$$

(3.17)

Hence, we wish to solve an equation of the general form

$$\prod_{k=0}^{n} F(-\mu + k\lambda) = C(\mu) ,$$

(3.18)

\footnote{Note that the equation (B.1) in appendix B of [1] is the same as equation (3.16) if $\mu$ is not a negative integer.}
for a given function $C(\mu)$. One can immediately write down two (formal) solutions to this equation

$$
F_1(\mu) = \prod_{j=1}^{\infty} \frac{C(-\mu + jh\lambda - \lambda)}{C(-\mu + jh\lambda)},
$$

(3.19)

$$
F_2(\mu) = \prod_{j=1}^{\infty} \frac{C(-\mu - (j-1)h\lambda)}{C(-\mu - (j-1)h\lambda - \lambda)}.
$$

(3.20)

It is easy to see that these functions solve equation (3.18) if $h = n+1$, which is the Coxeter number of $a_{n}^{(1)}$. Of course these solutions only make sense if we can prove the convergence of the infinite products.

If we simply solve equation (3.18) with $C(\mu)$ equal to the right hand side of (3.17) we will find that solutions $F_1$ and $F_2$ both have an infinite number of poles in the physical strip. We can, however, find a scalar factor with only a finite number of poles, if we combine the two solutions in a certain way. In order to do this let us write the right hand side of equation (3.17) in terms of Gamma functions

$$
\frac{\sin(-\pi \mu)}{\sin(\pi (\mu - n\lambda))} = \frac{\Gamma(\mu - n\lambda)\Gamma(1 - \mu + n\lambda)}{\Gamma(-\mu)\Gamma(1 + \mu)}.
$$

(3.21)

We notice that we would at most obtain a finite number of zeros and poles on the physical strip if in the infinite product only terms of the form $\Gamma(\pm \mu + jh\lambda \pm \ldots)$ appeared and no terms with a minus sign in front of $jh\lambda$ appeared. (Recall that $\lambda > 0$.) It is then simple to construct a solution with this property, if we define

$$
C_1(\mu) \equiv \frac{\Gamma(\mu - n\lambda)}{\Gamma(1 + \mu)} \quad \text{and} \quad C_2(\mu) \equiv \frac{\Gamma(1 - \mu + n\lambda)}{\Gamma(-\mu)}.
$$

(3.22)

Now we use (3.19) for $C_1$ and (3.20) for $C_2$ and we obtain the desired solution to equation (3.17) as the product of these two solutions:

$$
F^{(A)}(\mu) = \prod_{j=1}^{\infty} \frac{\Gamma(\mu + jh\lambda - n\lambda)\Gamma(\mu + jh\lambda - \lambda + 1)}{\Gamma(-\mu + jh\lambda - n\lambda)\Gamma(-\mu + jh\lambda - \lambda + 1)} 
\times \frac{\Gamma(\mu + jh\lambda - n\lambda - \lambda + 1)\Gamma(\mu + jh\lambda)}{\Gamma(\mu + jh\lambda - n\lambda - \lambda + 1)\Gamma(\mu + jh\lambda)}.
$$

(3.23)

This is the solution Hollowood found in [1]9. However, with our method we are able additionally to obtain a very similar but different solution by simply rewriting the right hand side of (3.17)

9The different prefactor in front of the infinite product is due to the different R-matrix normalisation.
in the following trivial way
\[
\frac{\sin(-\pi \mu)}{\sin(\pi(\mu - n \lambda))} = \frac{\sin(\pi \mu)}{\sin(\pi(-\mu + n \lambda))} = \frac{\Gamma(-\mu + n \lambda)\Gamma(1 + \mu - n \lambda)}{\Gamma(\mu)\Gamma(1 - \mu)}.
\] (3.24)

Using the same procedure as above (now with \( C_1 = \frac{\Gamma(\mu + n \lambda)}{\Gamma(\mu)} \) and \( C_2 = \frac{\Gamma(-\mu + n \lambda)}{\Gamma(-\mu)} \)) we obtain another solution to (3.17) in the form
\[
F^{(B)}(\mu) = -\prod_{j=1}^{\infty} \frac{\Gamma(\mu + jh\lambda - n \lambda + 1)\Gamma(\mu + jh\lambda - \lambda)}{\Gamma(-\mu + jh\lambda - n \lambda + 1)\Gamma(-\mu + jh\lambda - \lambda)} \times \frac{\Gamma(-\mu + jh\lambda - n \lambda - \lambda + 1)\Gamma(-\mu + jh\lambda)}{\Gamma(\mu + jh\lambda - n \lambda - \lambda + 1)\Gamma(\mu + jh\lambda)}.
\] (3.25)

Thus we have obtained two slightly different solutions. In [1] Hollowood found that the factor \( F^{(A)}(\mu) \) does not quite give the correct pole structure. It was thus necessary to include an additional CDD–factor in the \( S \)-matrix conjecture. This CDD-factor was equal to the minimal \( a_n \) Toda \( S \)-matrix. For \( a = b = 1 \) this minimal Toda \( S \)-matrix is simply
\[
S^{(\text{min})}_{1,1}(\mu) = \frac{\sin(\frac{\mu + \lambda}{h\lambda} \mu)}{\sin(\frac{\mu - \lambda}{h\lambda} \mu)}.
\] (3.26)

Using the expansion of the sine function in terms of an infinite product we can rewrite this in the following form:
\[
S^{(\text{min})}_{1,1}(\mu) = \frac{\mu + \lambda}{\mu - \lambda} \prod_{j=1}^{\infty} \frac{(jh\lambda)^2 - (\mu + \lambda)^2}{(jh\lambda)^2 - (\mu - \lambda)^2} = -\prod_{j=1}^{\infty} \frac{(\mu + jh\lambda - n \lambda)(-\mu + jh\lambda - \lambda)}{(\mu + jh\lambda - \lambda)(-\mu + jh\lambda - n \lambda)}.
\]

Using the fundamental property of the Gamma function we can then see that
\[
F^{(B)}(\mu) = S^{(\text{min})}_{1,1}(\mu)F^{(A)}(\mu).
\] (3.27)

If we now carefully examine the poles and zeros of \( F^{(A)}(\mu) \) and \( F^{(B)}(\mu) \) on the physical strip \( (0 \leq \mu \leq \frac{n+1}{2}\lambda) \), we find that both functions displays simple poles at \( \mu = m \) (for all \( m = 1, 2, \ldots \leq \frac{n+1}{2}\lambda \)) and simple zeros at \( \mu = \lambda + m \) (for all \( m = 1, 2, \ldots \leq \frac{n-1}{2}\lambda \)). \( F^{(A)}(\mu) \), however, additionally displays a zero at \( \mu = \lambda \). Thus the scalar function originally found in [1] is not minimal and the additional pole in the included CDD–factor only serves to cancel this additional zero. This derivation explains the inclusion of the minimal Toda \( S \)-matrix in [1]. There are in fact two (almost) minimal solutions to equation (3.17) and they are distinguished by exactly the minimal \( a_n \) Toda \( S \)-matrix.

Therefore we choose the solution \( F^{(B)}(\mu) \) as our scalar factor for \( S_{1,1}(\theta) \). We define \( F_{1,1}(\mu) \equiv F^{(B)}(\mu) \) and all scalar factors for the higher \( S \)-matrices can then be determined by the fusion
After a somewhat lengthy but straightforward calculation, we find the following expression for the general scalar factor $F_{a,b}(\mu)$:

$$F_{a,b}(\mu) = (-1)^{a+b} \prod_{k=1}^{b-1} \frac{\sin(\pi + \frac{\lambda}{2}((-a-b)+2k+1))}{\sin (\pi (\mu + \frac{a+b}{2})(-a+b)-2k))} \prod_{j=1}^{\infty} \frac{\Gamma (\mu + jh\lambda - \frac{1}{2}(2n+2-a-b)+1) \Gamma (\mu + jh\lambda - \frac{1}{2}(a+b))}{\Gamma (-\mu + jh\lambda - \frac{1}{2}(2n+2-a-b)+1) \Gamma (-\mu + jh\lambda - \frac{1}{2}(a+b))}$$

$$\times \frac{\Gamma (-\mu + jh\lambda - \frac{1}{2}(2n+2-a-b)+1) \Gamma (-\mu + jh\lambda - \frac{1}{2}(a+b))}{\Gamma (\mu + jh\lambda - \frac{1}{2}(2n+2-a-b)+1) \Gamma (\mu + jh\lambda - \frac{1}{2}(a+b))}.$$

(3.28)

As mentioned earlier, it is of course important to show that these infinite products converge at least within the physical strip. This can be done by rewriting the infinite products of Gamma functions in terms of integrals over hyperbolic functions. This has been shown in detail in [5] and for the case of the $a_n^{(1)}$ scalar factor it was found that

$$F_{1,1}(\mu) = \exp \left( \int_0^\infty dt \frac{dt}{2} \sinh (\mu t) \mathcal{I}(t) \right),$$

in which

$$\mathcal{I}(t) \equiv \frac{\sinh(\frac{1}{2}t) \sinh((\frac{h}{2} - \frac{1}{2})t)}{\sinh(\frac{1}{2}t) \sinh((\frac{h}{2} + \frac{1}{2})t)}.$$

(3.30)

A similar form of the $S$-matrix scalar factors for the cases of simply laced algebras has recently been constructed in a very interesting paper [22], which uses a so-called regularised quantum dilogarithm. This form has the advantage that the $S$-matrices can be compared easily with the time delays in the classical scattering of affine Toda solitons. Although these kinds of integral representations provide a more compact way of writing the overall scalar factors, for our purposes it is advantageous to use infinite products of Gamma functions, since in this case the study of the pole structure is greatly simplified.

The $S$-matrix (3.11) with the scalar factor (3.28) is exactly that suggested in [1]. If we take the underlying algebra to be $a_1^{(1)}$ then this $S$-matrix is the well known $S$-matrix for the Sine-Gordon solitons [23]. In [2] this $S$-matrix was used to construct the $S$-matrices for the scattering of bound states in $a_2^{(1)}$ ATFT. In following section we extend this to the general case of $a_n^{(1)}$. 


3.3 The bound states

The $S$-matrices describe the quantum scattering of a set of quantum states (which will be referred to as solitons) grouped into mass degenerate multiplets $V_a$ (for $a = 1, 2, ..., n$) corresponding to the $n$ fundamental representations of $U_q(a_n^{(1)})$. Let us denote the solitons in a multiplet $V_a$ by $A^{(a)}$. Although in the following we will always speak of a ‘soliton’ $A^{(a)}$ we should point out that $A^{(a)}$ actually denotes an entire multiplet $V_a$ of solitons transforming under the $a$th fundamental representation. The single solitons should therefore carry an additional multiplet label, i.e. $A^{(a)}_{(j)}$ in which $j = 1, 2, ..., \dim V_a$. However, in order to construct bound state $S$-matrices we will not need to distinguish different solitons of the same multiplet and we will therefore suppress the multiplet label in the following.

The quantum masses of these solitons are determined by the pole structure of the $S$-matrices. We find that the fusion of two solitons $A^{(a)}$ and $A^{(b)}$ into a soliton $A^{(a+b)}$ corresponds to the simple pole in the $S$-matrix at $\mu = \frac{a+b}{2} \lambda$, which is at $x = q^{a+b}$. According to the mass formula (2.19) the quantum masses of the elementary solitons must have the form

$$M_a = 2mC \sin \left( \frac{\pi a}{n+1} \right), \quad (a = 1, 2, \ldots, n),$$

(3.31)

with an unspecified overall factor $C$. As we will discuss at the end of this chapter, these masses are proportional to the masses of the classical solitons in $a_n^{(1)}$ ATFTs. We can also see that the solitons in the multiplets $V_a$ and $V_{n+1-a}$ have the same mass. They are charge conjugate to each other and transform into each other under time reversal. We therefore introduce the following notation for a ‘charge conjugate’ soliton:

$$\bar{A}^{(a)}(\theta) \equiv A^{(n+1-a)}(\theta).$$

3.3.1 Breather bound states

It has already been shown that in Sine–Gordon theory as well as in the $a_2^{(1)}$ ATFT two solitons of conjugate type can fuse into so-called breather bound states. These are bound states with zero topological charge, which means that they transform under the singlet representation. The poles in the $S$-matrix at which two solitons fuse into a breather bound state are distinguished by the fact that the $S$-matrix projects onto the module of the singlet representation at these poles. Thus, if we denote possible breather poles by $\theta_p$, then we must have

$$S_{a,b}(\theta_p) \sim \tilde{P}_0.$$ 

(3.32)
Now let us look at possible scalar bound states in the \( S \)-matrices (3.11). We first notice that this last equation can only be true if \( a = \overline{b} \) (in which \( \overline{b} \equiv n + 1 - b \)) since only in this case does the projector \( \tilde{P}_0 \) appear in the spectral decomposition of the \( a_n^{(1)} \) \( R \)-matrix (3.1). Therefore, breathers must be bound states of two solitons of conjugate type, say \( b \) and \( \overline{b} \). From the spectral decomposition of the corresponding \( R \)-matrix we can see that possible breather poles must be contained in the factor \( \langle n + 1 \rangle \), since this is the only factor which appears in front of the projector \( \tilde{P}_0 \) and in front of no other projector. \( \langle n + 1 \rangle \) becomes singular at \( x = q_n^{n+1} \) and therefore

\[
\tilde{R}_{\overline{b}b}(q^{n+1}) \sim \tilde{P}_0.
\]

From the definitions (3.6) we see that \( x = q^{n+1} \) corresponds to

\[
\mu = \frac{n + 1}{2} \lambda + m , \quad (\forall m \in \mathbb{Z}) , \tag{3.33}
\]

of which only the poles with \( m = 0, -1, -2, \ldots \geq -\frac{n+1}{2} \lambda \) lie in the physical strip. Of course we also have to examine possible pole–zero cancellations from the overall scalar factor \( F_{\overline{b},b}(\mu)k_{\overline{b},b}(\mu) \). Using (3.8) and (3.28) we find that no physical strip poles of the form (3.33) appear in this factor, but we do find a simple zero at \( \mu = \frac{n+1}{2} \lambda \), which cancels the corresponding pole for \( m = 0 \) in (3.33). We are therefore left with the following simple poles in the \( S \)-matrix:

\[
\theta_p = i\pi \left( 1 - \frac{2p}{(n + 1)\lambda} \right) ,
\]

or in terms of \( \mu \)

\[
\mu_p \equiv \mu(\theta_p) = \frac{n + 1}{2} \lambda - p , \tag{3.34}
\]

in which \( p = 1, 2, \ldots \leq \frac{n+1}{2} \lambda \). We conjecture that these poles correspond to the fusion of two solitons into bound states as depicted in figure 1. These bound states transform under the singlet representation and we will call them breathers. Let us denote the breather bound states of two solitons of type \( b \) and \( \overline{b} \) by the symbols \( B_p^{(b)}(\theta) \). We can also define the notion of a ‘conjugate breather’, in which the order of the incoming solitons of type \( b \) and \( \overline{b} \) is reversed, and we introduce the notation \( \overline{B}_p^{(b)}(\theta) \equiv B_p^{(n+1-b)}(\theta) \). The breathers carry an additional quantum number \( p \), the so–called excitation number, which takes integer values \( p = 1, 2, \ldots \leq \frac{n+1}{2} \lambda \). Thus the number of breather states in the spectrum decreases with increasing coupling constant \( \beta \) and for \( \lambda < \frac{2}{n+1} \) (i.e. \( \beta^2 > 4\pi \frac{n+1}{n+3} \)) all breather states disappear from the spectrum.
From the mass relation (2.19) we can determine the quantum masses of the breathers

\[ m_{B_p^{(b)}} = 2M_b \sin \left( \frac{\pi p}{(n+1)\lambda} \right), \quad (b = 1, 2, \ldots, n), \]  

in which \( M_b \) is the mass of the \( b \)-th elementary soliton given by (3.31).

### 3.3.2 Excited solitons

By analogy with the study in [2] we expect the theory to contain some sort of excited solitons which are soliton bound states with non-zero topological charge. From the classical examination of the \( a_n^{(1)} \) bound states in [12], we expect bound states of two solitons of the same species to exist. Let us therefore look at the \( R \)-matrix associated to the tensor product \( V_a \otimes V_a \) for some \( a \in \{1, 2, \ldots, n\} \). From (3.33) we get

\[ \tilde{R}_{a,a}(x) = \sum_{c=0}^{\min(a,n+1-a)} \prod_{i=1}^{\min(a,n+1-a)} (2i) \tilde{P}_{\lambda_a+c+c} = \tilde{P}_{2\lambda_a} + \ldots + \prod_{i=1}^{\min(a,n+1-a)} (2i) \tilde{P}_{\lambda_{\min(2a,2a-n-1)}}. \]

We expect the excited solitons to transform under fundamental representations and we therefore have to consider the poles in the factor \( \langle 2 \min(a, n+1-a) \rangle \) at which the \( R \)-matrix projects onto the fundamental module \( V_{\min(2a,2a-n-1)} \). After a careful study of the corresponding poles and zeros in the overall scalar factor \( F_{a,a}(\mu)k_{a,a}(\mu) \) we find all poles at which the soliton \( S \)-matrix projects onto \( V_{\min(2a,2a-n-1)} \). We have to distinguish the following three cases:

1) \( a < \frac{n+1}{2} \)

The excited soliton poles are contained in the factor \( \langle 2a \rangle \) which has poles at \( x = q^{2a} \), and the overall scalar factor \( F_{a,a}(\mu)k_{a,a}(\mu) \) displays simple zeros on the physical strip at \( \mu = a\lambda + m \) for \( m = 1, 2, \ldots \). Therefore, we end up with the following simple poles on the physical strip:

\[ \mu = a\lambda - p, \quad (\text{for } p = 0, 1, 2, \ldots \leq a\lambda), \]  

(3.36)
which correspond to a fusion process $A^{(a)} + A^{(a)} \rightarrow A_{p}^{(2a)}$.

ii) $a > \frac{n+1}{2}$

Here the excited soliton poles are contained in $\langle 2(n + 1 - a) \rangle$ which displays simple poles at $\mu = (n + 1 - a)\lambda + m$ for integer $m$. However we find corresponding zeros in the scalar factor for all positive $m$ and we thus end up with the following simple poles on the physical strip:

$$\mu = (n + 1 - a)\lambda - p,$$

(for $p = 0, 1, 2, \ldots \leq (n + 1 - a)\lambda$),

which correspond to the fusion process $A^{(a)} + A^{(a)} \rightarrow A_{p}^{(2a-n-1)}$.

iii) $a = \frac{n+1}{2}$

In this case we have $\tilde{R}_{a,a}(\theta) = \tilde{R}_{a,a}(\theta)$ and the excited solitons are nothing other than the breathers $B_{p}^{(a)}$ which were studied above.

Thus we have found bound states of two elementary solitons of the same species which transform under the fundamental representations $\pi_{2a \text{mod} h}$. We have denoted these excited solitons by $A_{p}^{(2a \text{mod} h)}$, in which $p$ is the excitation number taking values $p = 1, 2, \ldots \leq \min(a, n + 1 - a)\lambda$. Thus, as in the case of the breather bound states, the number of excited solitons in the spectrum of the theory is restricted by the coupling constant. The quantum masses of these excited solitons are

$$m_{A_{p}^{(2a)}} = 2M_{a} \cos \left( \frac{\pi}{n + 1} (a - \frac{p}{\lambda}) \right),$$

(for $a = 1, 2, \ldots < \frac{n+1}{2}$ and $p = 0, 1, 2, \ldots \leq a\lambda$),

$$m_{A_{p}^{(2a-n-1)}} = 2M_{a} \cos \left( \frac{\pi}{n + 1} (a + \frac{p}{\lambda}) \right),$$

(for $\frac{n+1}{2} < a < n + 1$ and $p = 0, 1, 2, \ldots \leq (n + 1 - a)\lambda$).

We find that the excited solitons with the lowest mass, i.e. $A_{0}^{(2a \text{mod} h)}$, are indeed just the elementary solitons $A^{(2a \text{mod} h)}$ themselves.

Before we discuss any of the other simple and multiple poles in the soliton S-matrices we will first construct the scattering amplitudes for the scattering of breathers and excited solitons in the following subsection.

### 3.4 The bound state S-matrices

Now we are able to compute the S-matrices for the scattering of breather bound states by using the bootstrap equations. We start with the S-matrix for the scattering of a breather of type
Using the crossing symmetry of the $S$-matrix we can write

$$S_{A^{(a)}B^{(b)}_p}(\theta) \left( I_a \otimes P_0 \right) = P_0 \otimes I_a \cdot I_b \otimes S_{a,b}^{\text{cross}}(\theta + \frac{1}{2} \theta_p) \cdot S_{a,b}(\theta - \frac{1}{2} \theta_p) \otimes I^- \ , \quad (3.38)$$

in which $P_0$ denotes the projector from $V_b \otimes V_b^*$ onto the singlet space, such that both sides of equation (3.38) map $V_a \otimes V_b \otimes V_b^*$ into $V_a \otimes V_b$. This bootstrap equation is illustrated in figure 2, in which as usually time is meant to run upwards.

![Figure 2: Bootstrap for $S_{A^{(a)}B^{(b)}_p}$](image)

Using the crossing symmetry of the $S$-matrix we can write

$$S_{A^{(a)}B^{(b)}_p}(\theta) \left( I_a \otimes P_0 \right) = P_0 \otimes I_a \cdot I_b \otimes S_{a,b}^{\text{cross}}(i \pi - (\theta + \frac{1}{2} \theta_p)) \cdot S_{a,b}(\theta - \frac{1}{2} \theta_p) \otimes I^- \ ,$$

$$= \mathcal{F}(\mu) \left[ P_0 \otimes I_a \cdot I_b \otimes \hat{R}_{a,b}^{\text{cross}}(x^{-1} q^{(n+1)/2}) \cdot \hat{R}_{a,b}(x q^{(n+1)/2}) \otimes I_b \right] \ ,$$

$$= \mathcal{F}(\mu) \left( I_a \otimes P_0 \right) . \quad (3.39)$$

The last step in this calculation is non-trivial and uses the $R$-matrix unitarity (a proof of this can be found in the appendix of [4]). Thus the $S$-matrix $S_{A^{(a)}B^{(b)}_p}(\theta)$ is simply a scalar factor equal to $\mathcal{F}(\mu)$ which is given by

$$\mathcal{F}(\mu) = F_{b,a}(\mu(i \pi - \theta - \frac{1}{2} \theta_p))F_{a,b}(\mu(\theta - \frac{1}{2} \theta_p))k_{b,a}(\mu(i \pi - \theta - \frac{1}{2} \theta_p))k_{a,b}(\mu(\theta - \frac{1}{2} \theta_p)) . \quad (3.40)$$

We can compute this explicitly and we find a surprisingly simple expression for $S_{A^{(a)}B^{(b)}_p}(\theta)$ in terms of a finite product of sine functions:

$$S_{A^{(a)}B^{(b)}_p}(\theta) = \prod_{l=1}^{P-1} \left[ \frac{x}{2} (a + b - \frac{n+1}{2}) - \frac{p}{2} + l \right] \left[ -\frac{x}{2} (a + b + \frac{n+1}{2}) - \frac{p}{2} + l \right] \prod_{k=1}^{b} \left[ \frac{x}{2} (a + b - \frac{n+1}{2} - 2k + 2) + \frac{p}{2} \right] \left[ \frac{x}{2} (b - a - \frac{n+1}{2} - 2k + 2) + \frac{p}{2} \right] \ , \quad (3.41)$$
Finally, we also compute the following bracket notation:
\[ [y] \equiv \sin\left(\frac{\pi}{4\lambda}(\mu + y)\right), \quad (3.42) \]
and for later use we also introduce
\[ (y) \equiv \frac{[y]}{[-y]} . \quad (3.43) \]

It can be easily checked that these \( S \)-matrix elements are themselves crossing symmetric, i.e. \( S_{A(a)B_p(b)}(i\pi - \theta) = S_{A(a)B_p(b)}(\theta) \) and therefore satisfy the following required symmetry conditions:
\[ S_{A(a)B_p(b)}(\theta) = S_{A'(a)B_p'(b)}(\theta) = S_{A''(a)B_p''(b)}(\theta) = S_{B_p'(a)A(a)}(\theta) , \]
and since \( S_{A(a)B_p(b)} = S_{A(a)bB_p} \), we also have
\[ S_{B_p(a)A(a)}(\theta) = S_{B_p(b)A(a)}(\theta) = S_{A(a)B_p(b)}(\theta) = S_{A(a)B_p'(b)}(\theta). \]

Using these identities we can apply the bootstrap method again in order to obtain the breather-breather \( S \)-matrices. If we replace the soliton \( A(a) \) in figure 2 with a breather \( B_r(a) \), we obtain the following bootstrap equation:
\[
S_{B_r(a)B_p(b)}(\theta) = S_{A(a)B_p(b)}(\theta + \frac{1}{2}\theta_r)S_{A'(a)B_p'(b)}(\theta - \frac{1}{2}\theta_r) \\
= \prod_{l=1}^{p-1} \left( \frac{\lambda}{2}(b - a) + \frac{r - p}{2} + l \right) \left( \frac{\lambda}{2}(a - b) + \frac{r - p}{2} + l \right) \\
\times \left( \frac{\lambda}{2}(a + b) - \frac{p + r}{2} + l \right) \left( \frac{\lambda}{2}(a + b) - \frac{p + r}{2} + l \right) \\
\times \prod_{k=1}^{b} \left( \frac{\lambda}{2}(b - a - 2k + 2) + \frac{p + r}{2} \right) \left( \frac{\lambda}{2}(a + b - 2k) + \frac{r - p}{2} \right) \\
\times \left( \frac{\lambda}{2}(a + b - 2k + 2) + \frac{p - r}{2} \right) \left( \frac{\lambda}{2}(b - a - 2k) - \frac{p + r}{2} \right). \quad (3.44) \]

Finally, we also compute the \( S \)-matrix for the scattering of an excited soliton with a breather:
\[
S_{A_r(2a)B_p(b)}(\theta) = \prod_{l=1}^{p-1} \left[ \frac{\lambda}{2}(2a + b - n + \frac{1}{2}) - \frac{p + r}{2} + l \right] \left[ -\frac{\lambda}{2}(2a + b + n + \frac{1}{2}) + \frac{r - p}{2} + l \right] \\
\times \left[ -\frac{\lambda}{2}(b + n + \frac{1}{2}) - \frac{p + r}{2} + l \right] \left[ \frac{\lambda}{2}(b - n - \frac{1}{2}) + \frac{r - p}{2} + l \right] \\
\times \left[ \frac{\lambda}{2}(b - n - \frac{1}{2}) - \frac{p + r}{2} + l \right] \left[ -\frac{\lambda}{2}(b + n + \frac{1}{2}) + \frac{r - p}{2} + l \right] \\
\times \prod_{k=1}^{b} \left[ \frac{\lambda}{2}(2a + b - n + \frac{1}{2} - 2k + 2) + \frac{p + r}{2} \right] \left[ \frac{\lambda}{2}(b - 2a - n + \frac{1}{2} - 2k + 2) + \frac{r - p}{2} \right] \\
\times \left[ \frac{\lambda}{2}(b - n + \frac{1}{2} - 2k + 2) + \frac{p + r}{2} \right] \left[ \frac{\lambda}{2}(b - n - \frac{1}{2} - 2k + 2) + \frac{p + r}{2} \right] . \quad (3.45) \]
The same procedure could in principle be applied again in order to construct the $S$-matrices for the scattering of two excited solitons. This $S$-matrix, however, would not be just a scalar function but, like $S_{a,b}(\theta)$, an intertwiner on the tensor product of the two corresponding modules. This construction remains beyond the scope of this paper.

3.5 The spectrum

We conjecture that the entire spectrum of states of a quantum field theory, the on-shell information of which is provided by the above $S$-matrices, consists of fundamental solitons and two kinds of bound states, the breathers and the excited solitons as described above. This is obviously a bold assertion given the fact that the $S$-matrices contain a vast number of so far unexplained poles. However, it was already discovered in previous work that a very large number of poles in trigonometric $S$-matrices can be explained by higher order diagrams, many of which involve a generalised Coleman-Thun mechanism (see [27, 28, 3] for details).

Let us, as an example, consider the pole at which the $S$-matrix projects onto the module of the fundamental representation $\pi_{\lambda_{a+b}}$. For the sake of simplicity let us assume that $a \geq b$ and $b \leq n + 1 - a$. The $R$-matrix in this case is then given by

$$\hat{R}_{a,b}(x) = \prod_{c=0}^{b} \prod_{i=1}^{c} \langle 2i + a - b \rangle \hat{P}_{\lambda_{a+b+c}} \cdot \hat{P}_{\lambda_{a+b-c}}.$$  \hspace{1cm} (3.46)

From this formula we can see that at the poles in the factor $\langle a + b \rangle$ the $R$-matrix projects onto $V_{a+b}$. The factor $\langle a + b \rangle$ has simple poles at $x = q^{a+b}$ which correspond to $\mu = \frac{\lambda}{2}(a + b) + m$ (for $m \in \mathbb{Z}$). If we carefully study the corresponding zeros and poles in $F_{a,b}(\mu)k_{a,b}(\mu)$ we find that $S_{a,b}(\mu)$ displays simple poles on the physical strip at

$$\mu = \frac{\lambda}{2}(a + b) - p, \hspace{1cm} \text{for} \hspace{0.5cm} p = 0, 1, \ldots \leq \frac{\lambda}{2}(a + b).$$  \hspace{1cm} (3.47)

As mentioned earlier, the pole with $p = 0$ among these corresponds to the fusion process $A^{(a)} + A^{(b)} \rightarrow A^{(a+b)}$. The poles with $p > 0$ do not correspond to a fusion process into a bound state but they correspond to the crossed box process depicted in figure 3.
If the incoming solitons in this diagram have a rapidity difference equal to the poles (3.47) then the scattering process in the center of the diagram occurs at exactly $\mu = \frac{a+b}{4} \lambda + \frac{a}{2} \lambda + \frac{b}{2}$. From formula (3.41) we can see that the $S$-matrix element $S_{A^{(a-b)}B^{(b)}}$ has a simple zero at this value of rapidity difference. This zero reduces the expected double pole to the single poles (3.47). Thus we have shown that the poles (3.47) do not correspond to new bound states, but can be explained by a generalised Coleman-Thun mechanism. We also note that the process in figure 3 can only exist for the case of $a \neq b$, because of the occurrence of the soliton $A^{(a-b)}$. This is consistent with the fact that only in the case $a = b$ do the poles (3.47) correspond to the fusion into an excited soliton.

We have examined a large number of simple and multiple poles and found that all of these poles can be explained in a similar fashion in terms of the conjectured spectrum of solitons, breathers and excited solitons$^h$. We believe that this will prove true for all poles in the soliton and bound state $S$-matrices and that the bootstrap closes on this conjectured spectrum. Thus, unlike previously expected, we believe that no bound states transforming under non-fundamental or reducible representations exist.

3.6 The connection with affine Toda solitons

As already mentioned in the introduction we conjecture that the above constructed exact $S$-matrices describe the scattering of solitons and their bound states in $a_n^{(1)}$ affine Toda field theories with purely imaginary coupling constant. In this section we will summarise briefly

$^h$For a detailed discussion of a large number of poles in the case of the $a_2^{(1)}$ theory see 2.
some of the results which support this view.

The reason we chose trigonometric $R$-matrices as the basic building blocks for the soliton $S$-matrices was the fact that imaginary ATFTs are believed to display a quantum affine symmetry related to the affine dual algebra and that the solitons transform under fundamental representations of $U_q(\hat{g}^\vee)$. Strong evidence for this quantum affine symmetry stems from the work by Bernard and LeClair as well as Felder and LeClair. In our $S$-matrix construction we have therefore used the values of the Lorentz spins of the non-local conserved charges as derived in [14].

Possibly the strongest connection of the trigonometric $S$-matrices with ATFTs at this stage is the breather–particle identification. By analogy with all previously considered cases, we expect the $S$-matrices for the lowest breathers to coincide with the $S$-matrices for the real coupling $a_n^{(1)}$ ATFTs. Using formula (3.44) for $p = r = 1$ we find

$$S_{B_1^{(a)}B_1^{(b)}}(\theta) = \prod_{k=1}^{b} \left( \frac{\lambda}{2} (b-a-2k) - 1 \right) \left( \frac{\lambda}{2} (a+b-2k+2) \right) \left( \frac{\lambda}{2} (a+b-2k) \right) \left( \frac{\lambda}{2} (b-a-2k) + 1 \right).$$

(3.48)

The $S$-matrix for the fundamental quantum particles in the real theory on the other hand was given in [7] in the following form:

$$S_{ab}^{(r)}(\theta) = \prod_{a-b+1}^{a+b-1} \left\{ y \right\}_r,$$

(3.49)

in which

$$\left\{ y \right\}_r \equiv \frac{(y + 1)_r (y - 1)_r}{(y + 1 - B)_r (y - 1 + B)_r}, \quad \left( y \right)_r \equiv \frac{\sin\left(\frac{\theta}{2\pi} + \frac{\pi y}{2(n+1)}\right)}{\sin\left(\frac{\theta}{2\pi} - \frac{\pi y}{2(n+1)}\right)},$$

(3.50)

and it was conjectured (and shown up to order $\beta^4$ in perturbation theory in [24]) that $B(\beta) = \frac{1}{2\pi} \frac{\beta^2}{1+\beta^2/4\pi}$. If we analytically continue $\beta \to i\beta$ we thus obtain

$$B(\beta) \to -\frac{2}{\lambda}, \quad \text{and} \quad \left( y \right)_r \to \left( \frac{\lambda}{2} y \right).$$

(3.51)

Therefore, we find

$$S_{ab}^{(r)}(\theta) \to S_{B_1^{(a)}B_1^{(b)}}(\theta),$$

(3.52)

thus establishing the lowest breather–particle identification for $a_n^{(1)}$ ATFTs. (This has also been demonstrated in [22].)
Further evidence for the S-matrix conjecture for affine Toda solitons comes from the comparison of our results with some of the results of the study of classical soliton solutions in ATFTs. First of all, the trigonometric $S$-matrices (without any additional CDD factor included) display a pole structure which is consistent with the classical soliton mass ratios. The classical masses of the $a_n^{(1)}$ solitons in the $a$th multiplet are

$$M_{a}^{(cl)} = 2m \sin \left( \frac{\pi a}{n+1} \right), \quad (a = 1, 2, \ldots, n),$$

which are the same as the quantum soliton masses (3.31). (Recall that the soliton mass ratios in theories based on self–dual algebras are expected to remain constant when the theory is quantised [18].)

In [12] bound states of classical affine Toda solitons were constructed and it was found that the classical bound state spectrum coincides with the spectrum of breathers and excited solitons conjectured in the previous section. Note also that the conjecture of the spectrum implies that only bound states of two solitons with equal mass exist. This also agrees with results in classical ATFTs.

There are of course still some serious problems regarding the existence of a quantised version of imaginary ATFTs, which include the fact that the Lagrangian defines an in general non–unitary theory or that there are not enough classical solitons to fill out the representation spaces [25]. However, we believe that the results of the construction of soliton and bound state $S$-matrices in this and a series of other papers further support the conjecture that there are some unitary theories embedded in the imaginary ATFTs. This concludes our discussion of the $a_n^{(1)}$ ATFTs and their $S$-matrices. In the following section we repeat the above construction for the case of the twisted algebra $a_{2n}^{(2)}$.

4 The $a_{2n}^{(2)}$ invariant $S$-matrices

The $a_{2n}^{(2)}$ affine Lie algebras play a somewhat special role in the sense that they are the only non–simply–laced but nevertheless self–dual affine Lie algebras. In this case we therefore will find some similarities with the $a_{n+1}^{(2)}$ invariant $S$-matrices from [3] as well as with those from the preceding chapter. We find that most of the computations used in [3,4] can be used analogously for the case of $a_{2n}^{(2)}$ and we refer the reader to these papers for details.
The construction of the $a_{2n}^{(2)}$ S-matrices was made possible by the extension of the tensor product graph method to the case of twisted algebras in [26], which allowed the construction of $U_q(a_{2n}^{(2)})$-invariant $R$-matrices. These were given in the following form:

$$\tilde{R}_{a,b}(x) = \sum_{c=0}^{a} \sum_{d=0}^{c} \prod_{i=c}^{a-1} (a + b - 2i) - \prod_{j=1}^{c-d} (n - a - b + 2j) + \tilde{P}_\lambda \tau^{-1} \tilde{R}_{a,b}(x),$$ \hspace{1cm} (4.1)

in which $a, b = 1, 2, \ldots, n$, $a \geq b$ and a slight generalisation of the $<,>$ bracket notation has been used: $\langle a \rangle_\pm \equiv \frac{1 \pm q^a}{q \mp q^a}$. Since $a_{2n}^{(2)}$ is a self–dual algebra, the connections of the $R$-matrix parameters with the $S$-matrix parameters are again given by (3.5) and (3.6) but now with the $a_{2n}^{(2)}$ Coxeter number 

$$h = 2n + 1.$$ 

### 4.1 The soliton $S$-matrices

We make the by now familiar ansatz for a set of $a_{2n}^{(2)}$ invariant soliton $S$-matrices acting on the representation spaces of the fundamental $U_q(a_{2n}^{(2)})$ representations:

$$S_{a,b}(\theta) = F_{a,b}(\mu(\theta)) k_{a,b}(\mu(\theta)) \tau \tilde{R}_{a,b}(x(\theta)) \tau^{-1}. \hspace{1cm} (4.2)$$

The fusion factor $k_{a,b}(\mu)$ is again given by (3.8) and $\tau$ denotes the gauge transformation from the homogeneous to the principal gradation. In order to find the overall scalar factor $F_{a,b}(\mu)$ we first need the $a_{2n}^{(2)}$ $R$-matrix crossing property. This is completely analogous to the derivation in [3] and was done explicitly in the appendix of [3] where it was found that

$$c_{1,1}(i\pi - \theta) \tilde{R}_{1,1}^{(\text{cross})}(x(i\pi - \theta)) = c_{1,1}(\theta) \tilde{R}_{1,1}(x(\theta)),$$ \hspace{1cm} (4.3)

in which

$$c_{1,1}(\theta) = \sin(\pi(\mu - \lambda)) \sin(\pi(\mu - (n + \frac{1}{2})\lambda)).$$

Following exactly the same method as employed in [3] we find the following scalar factor in terms of Gamma functions:

\footnote{Note, that some care must be taken in this case regarding the fact that for the sake of convenience we have again chosen $q$ to differ from (2.28) by a minus sign.}
\[
F_{1,1}(\mu) = \prod_{j=1}^{\infty} \frac{\Gamma(\mu + jh\lambda - \lambda)\Gamma(\mu + jh\lambda - 2n\lambda + 1)}{\Gamma(-\mu + jh\lambda - \lambda)\Gamma(-\mu + jh\lambda - 2n\lambda + 1)}
\]
\[
\times \frac{\Gamma(\mu + jh\lambda - (n + \frac{1}{2})\lambda)\Gamma(\mu + jh\lambda - (n + \frac{1}{2})\lambda + 1)}{\Gamma(-\mu + jh\lambda - (n + \frac{1}{2})\lambda)\Gamma(-\mu + jh\lambda + 1)}
\]
\[
\times \frac{\Gamma(-\mu + jh\lambda - (2n + 1)\lambda)\Gamma(\mu + jh\lambda + 1)}{\Gamma(-\mu + jh\lambda - (n + \frac{1}{2})\lambda)\Gamma(-\mu + jh\lambda - (n - \frac{1}{2})\lambda + 1)}
\]
\[
\times \frac{\Gamma(-\mu + jh\lambda - (n + \frac{3}{2})\lambda)\Gamma(-\mu + jh\lambda - (n - \frac{3}{2})\lambda + 1)}{\Gamma(-\mu + jh\lambda - (n + \frac{3}{2})\lambda)\Gamma(-\mu + jh\lambda - (n - \frac{3}{2})\lambda + 1)}, \quad (4.4)
\]

and the higher scalar factors are again given by the fusion formula

\[
F_{a,b}(\mu) = \prod_{j=1}^{a} \prod_{k=1}^{b} F_{1,1}(\mu + \frac{\lambda}{2}(2j + 2k - a - b - 2)). \quad (4.5)
\]

As in the previous chapter we can also rewrite this scalar factor in terms of an integral and obtain

\[
F_{1,1}(\mu) = \exp\left(\int_{0}^{\infty} \frac{dt}{t} \frac{\sinh(\mu t)\mathcal{I}(t)}{2\sinh(\frac{\lambda}{2}t) \cosh((\frac{n}{2} - \frac{1}{4})\lambda t)}\right), \quad (4.6)
\]

in which

\[
\mathcal{I}(t) = \frac{\sinh(\frac{\lambda}{2}t) \cosh((\frac{n}{2} - \frac{1}{4})\lambda t)}{\sinh(\frac{\lambda}{2}t) \cosh((\frac{n}{2} + \frac{1}{4})\lambda t)} \quad (4.7)
\]

From the pole structure of the \(S\)-matrix we can then derive the quantum mass ratios of the solitons. They are found to be

\[
M_{a} = -C8\sqrt{2} \frac{hm}{\beta_{2}} \sin\left(\frac{a\pi}{K}\right), \quad (4.8)
\]

in which \(C\) is some (unknown) scale factor. As expected these match the soliton mass ratios of classical \(a_{2n}^{(2)}\) ATFTs.

### 4.2 The bound states

In analogy to the previous cases, the \(a_{2n}^{(2)}\) \(S\)-matrices contain simple poles corresponding to scalar bound states, the so-called breathers, as well as bound states transforming under fundamental representations, the so-called excited solitons. Following the same procedure as before, we find the bound state poles in the soliton \(S\)-matrices, as the poles at which the \(S\)-matrices project onto the corresponding fundamental modules. The results of this analysis are the following:
i) The breathers \( B_p^{(a)} \) are bound states of two solitons of the same species. They correspond to the following simple poles in \( S_{a,a}(\theta) \):

\[
\mu = (n + \frac{1}{2})\lambda - p , \quad \text{for } a = 1, 2, \ldots , n , \text{ and } p = 1, 2, \ldots \leq (n + \frac{1}{2})\lambda . \tag{4.9}
\]

From the position of these poles we can deduce the quantum masses of these breathers:

\[
m_{B_p^{(a)}} = 2M_a \sin \left( \frac{p\pi}{h\lambda} \right) . \tag{4.10}
\]

ii) The other type of bound states correspond to the following poles in \( S_{a,a}(\theta) \):

\[
\mu = a\lambda - p , \quad \text{for } a = 1, 2, \ldots < \frac{n + 1}{2} , \text{ and } p = 1, 2, \ldots \leq a\lambda , \tag{4.11}
\]

at which the \( S \)-matrix projects onto the module \( V_{\lambda_{2n}} \). These are the excited solitons \( A_p^{(2a)} \), which only exist for even representations. Their quantum masses are

\[
m_{A_p^{(2a)}} = 2M_a \cos \left( \frac{\pi}{h}(a - \frac{p}{\lambda}) \right) . \tag{4.12}
\]

Using these fusion poles in the bootstrap equations we can derive the \( S \)-matrices for the scattering of bound states. This is again completely analogous to the previously constructed cases and we therefore only list the results here. We can express the \( S \)-matrices in terms of a block notation:

\[
\begin{pmatrix} y \\ \end{pmatrix} = \begin{pmatrix} (n + \frac{1}{2})\lambda - y \\ \lambda + 1 + y \\ (n + \frac{1}{2})\lambda - 1 - y \end{pmatrix} , \tag{4.13}
\]

in which the \( \left( \right) \) notation was defined in (3.43). Using these notations we find the \( S \)-matrices for the following scattering processes:

**soliton–breather scattering:**

\[
S_{A^{(a)}} B_{p}^{(b)} (\theta) = \prod_{k=1}^{b} \prod_{l=1}^{p} \left\{ \frac{\lambda}{2} (a + b + n + \frac{1}{2} - 2k) + \frac{p - r}{2} - l \right\} , \tag{4.14}
\]

**breather–breather scattering:**

\[
S_{B_{p}^{(a)}} B_{p}^{(b)} (\theta) = \prod_{k=1}^{b} \prod_{l=1}^{p} \left\{ \frac{\lambda}{2} (b - a - 2k) + \frac{p - r}{2} - l \right\} \left\{ \frac{\lambda}{2} (a + b - 2k) + \frac{p + r}{2} - l \right\} , \tag{4.15}
\]

**excited soliton–breather scattering:**

\[
S_{A^{(2a)}} B_{p}^{(b)} (\theta) = \prod_{k=1}^{b} \prod_{l=1}^{p} \left\{ \frac{\lambda}{2} (2a + b + n + \frac{1}{2} - 2k) + \frac{p - r}{2} - l \right\} \left\{ \frac{\lambda}{2} (b + n + \frac{1}{2} - 2k) + \frac{p + r}{2} - l \right\} , \tag{4.16}
\]

We have not attempted to construct explicit expressions of scattering amplitudes for the scattering of two excited solitons with each other.
4.3 Connection with affine Toda solitons

In analogy to the previous chapter, we conjecture that these \( S \)-matrices describe the scattering of quantum solitons in \( a_{2n}^{(2)} \) ATFTs. All the arguments for this conjecture mentioned in section 3.6 hold equally in this case. We also can again compare the lowest breather \( S \)-matrices with the \( S \)-matrix for the quantum particles. The real \( a_{2n}^{(2)} \) affine Toda \( S \)-matrices were given in [28] in the following form:

\[
S_{a,b}^{(r)}(\theta) = \prod_{2 | a - b | + 2}^{2a+2b-2} \left\{ \frac{p}{r} \right\}_r \left\{ 4n + 2 - p \right\}_r, \quad (a, b = 1, 2, \ldots, n) \tag{4.17}
\]

in which

\[
\left\{ x \right\}_r \equiv \frac{(x - 2)_r (x + 2)_r}{(x - 2 + 2B)_r (x + 2 - 2B)_r}, \quad \left( x \right)_r \equiv \frac{\sin \left( \frac{\theta}{2} + \frac{x}{4} \right)}{\sin \left( \frac{\theta}{2} - \frac{x}{4} \right)}. \tag{4.18}
\]

The coupling constant dependent function \( B(\beta) \) in this case was conjectured to be of the form \( B = \frac{2\beta^2}{4\pi + \beta^2} \). Without loss of generality let us assume \( a \geq b \). Then we can rewrite the expression for \( S_{a,b}^{(r)} \) as

\[
S_{a,b}^{(r)}(\theta) = \prod_{a - b + 1}^{a+b-1} \left\{ 2p \right\}_r \left\{ 4n + 2 - 2p \right\}_r = \prod_{k=1}^{b} \left\{ 4k - 2 + 2a - 2b \right\}_r \left\{ 4n + 4 - 4k - 2a + 2b \right\}_r.
\]

If we analytically continue \( \beta \rightarrow i\beta \) we find that we have to make the following replacements:

\[
\left( y \right)_r \longrightarrow \left( \frac{\lambda}{4} y \right), \quad B \longrightarrow -\frac{2}{\lambda}, \tag{4.19}
\]

which lead to the expected result that

\[
S_{ab}^{(r)}(\theta) \longrightarrow S_{B_1^{(a)}B_1^{(b)}}(\theta). \tag{4.20}
\]

Thus we have established the identification of the lowest \( a_{2n}^{(2)} \) breathers with the \( a_{2n}^{(2)} \) affine Toda quantum particles.

Using the same arguments as in the previous chapter we conjecture that the full spectrum of \( a_{2n}^{(2)} \) ATFTs consists of the elementary solitons and the above defined bound states. We expect that all other simple and higher order poles in the \( S \)-matrices can again be explained in terms of higher order diagrams using only these states.

Here we additionally can compare our result with another result obtained in a different context. In [29] Smirnov discussed minimal conformal models perturbed by the \( \phi_{1,2} \) operator.
This $\phi_{1,2}$-perturbed minimal model is exactly the $a_2^{(2)}$ ATFT with imaginary coupling constant. In the same way as we have done here, Smirnov used an $a_2^{(2)}$-invariant $R$-matrix to construct the $S$-matrix for this model. Additionally, he explicitly computes the $S$-matrix for the first breathers, which after adjusting the different notation is indeed equal to equation (4.15) for $n = 1$. The overall $S$-matrix scalar factor in [29] was given in terms of an exponential of an integral over hyperbolic functions, which can be shown to be equal to (4.6). This agreement with the results obtained by Smirnov gives further support for the conjectured connection of these $S$-matrices with the scattering of affine Toda solitons.

In the remainder of this article we leave the subject of affine Toda field theories and concentrate on some intriguing features of the above constructed overall scalar factor in connection with minimal supersymmetric $S$-matrices.

5 Minimal supersymmetric $S$-matrices

Exact $N = 1$ and $N = 2$ supersymmetric $S$-matrices in two-dimensional integrable models have been studied by many different authors during the last few years (see for instance [30, 31, 32, 33, 34]). Recently some of the work on $N = 1$ theories has been extended further in [6] and [35]. Apart from the usual constraints of unitarity, crossing symmetry and the bootstrap, supersymmetric $S$-matrices additionally are required to commute with the generators of the supersymmetry algebra. One of the common features of integrable supersymmetric $S$-matrices in two dimensions appears to be that they can be written as a product of a supersymmetric part and a bosonic part:

$$S_{a,b}(\theta) = S_{a,b}^{(bos)}(\theta) \otimes S_{a,b}^{(SUSY)}(\theta),$$

in which the supersymmetric part seems to be universal for a large class of different models. The entire pole structure and therefore the spectrum of bound states is determined exclusively by the bosonic part of the $S$-matrix. Here we only study the basic (minimal) supersymmetric $S$-matrices without considering a specific model. In the following we will look at two examples of sets of minimal $N = 2$ and $N = 1$ supersymmetric $S$-matrices.
5.1 Minimal $N = 2$ supersymmetric $S$-matrices

$N = 2$ supersymmetric $S$-matrices were constructed by Fendley and Intriligator in [33]. Without going into any detail regarding the model they considered, we simply state their result for the $S$-matrix. Following the second reference in [33] we consider a theory containing $2n$ different $N = 2$ SUSY multiplets denoted by $\{ |u_a(\theta)\rangle, |d_a(\theta)\rangle \}$ ($a = 1, 2, \ldots, 2n$), which have masses

$$m_a = m \sin \left( \frac{a\pi}{2n+1} \right), \quad (5.2)$$

in which $m$ denotes some model specific mass parameter. These states have rational fermion numbers ($u_a$ has fermion number $\frac{a}{2n+1}$ and $d_a$ has fermion number $-(1 - \frac{a}{2n+1})$) and $u_a$ and $d_{2n+1-a}$ are charge conjugate to each other.

Writing the incoming states in a column left of the $S$-matrix and outgoing states in a row above, the basic form of the $S$-matrix appears as follows:

$$
\begin{pmatrix}
  u_a u_b & d_b u_a & u_b d_a & d_b d_a \\
  u_a d_b & 0 & B_{a,b}(\mu) & C_{a,b}(\mu) \\
  d_a u_b & 0 & C_{a,b}(\mu) & \tilde{B}_{a,b}(\mu) \\
  d_a d_b & 0 & 0 & 0
\end{pmatrix}
= S_{a,b}^{(N=2)}(\theta). \quad (5.3)
$$

This form is dictated by integrability and the requirement of fermion number conservation. As usual, the $S$-matrix depends on the rapidity difference of the incoming particles and we have used the abbreviation $\mu = \frac{\theta}{2\pi}$. The $S$-matrix elements themselves are determined (up to an overall scalar factor) by the requirement that the $S$-matrix must commute with the generators of the $N = 2$ supersymmetry algebra. The $S$-matrix elements can be written in the following form:

$$
A_{a,b}(\mu) = (-)^{a+b} G_{a,b}^{(N=2)}(\mu),
$$

$$
\tilde{A}_{a,b}(\mu) = (-)^{a+b-1} \left( -\frac{1}{2\hbar} (a - b) \right) G_{a,b}^{(N=2)}(\mu),
$$

$$
B_{a,b}(\mu) = (-)^{a+b} \frac{1}{2\hbar} (b - a) G_{a,b}^{(N=2)}(\mu),
$$

$$
\tilde{B}_{a,b}(\mu) = (-)^{a+b} \frac{1}{2\hbar} (a - b) G_{a,b}^{(N=2)}(\mu),
$$

$$
C_{a,b}(\mu) = (-)^{a+b} e^{\frac{a\pi}{\hbar} (b-a)} \frac{\sin(\frac{a\pi}{\hbar}) \sin(\frac{b\pi}{\hbar})}{\frac{1}{2\hbar} (a + b)} G_{a,b}^{(N=2)}(\mu),
$$
\[ C_{a,b}(\mu) = (-)^{a+b} e^{\frac{i\pi}{2}a-b} \frac{\sqrt{\sin(a\pi h)\sin(b\pi h)}}{[2/n(a+b)]} G_{a,b}^{(N=2)}(\mu). \] (5.4)

Here we have used the bracket notation

\[ [y] = \sin(\pi(\mu + y)) \quad \text{and} \quad (y) = \frac{[y]}{[-y]}, \] (5.5)

which is the same as (3.42,3.43) for \( \lambda = \frac{1}{h} \). (Note that the abbreviation \( \mu = \frac{\theta}{2\pi} \) is also equal to that used in chapter 3 for \( \lambda = \frac{1}{h} \).) The overall scalar factors \( G_{a,b}^{(N=2)}(\mu) \) are determined by \( S \)-matrix unitarity, crossing symmetry and the bootstrap. They have been given in [33] in terms of an infinity product of Gamma functions:

\[ G_{1,1}^{(N=2)}(\mu) = \prod_{j=1}^{\infty} \frac{\Gamma(\mu+j+\frac{1}{2})\Gamma(\mu+j-\frac{1}{2})\Gamma(-\mu+j-1)\Gamma(-\mu+j+1)}{\Gamma(-\mu+j+\frac{1}{2})\Gamma(-\mu+j-\frac{1}{2})\Gamma(\mu+j-1)\Gamma(\mu+j+1)}, \] (5.6)

and

\[ G_{a,b}^{(N=2)}(\mu) = \prod_{j=1}^{a} \prod_{k=1}^{b} G_{1,1}^{(N=2)}(\mu + \frac{1}{2(2n+1)}(a+b-2j-2k+2)). \] (5.7)

Note, however, that (5.3) is not yet a consistent \( S \)-matrix in itself, since it contains no poles. In order to close the bootstrap one has to include an overall CDD factor. The simplest possible choice in this case is the minimal \( a_{2n} \)-Toda \( S \)-matrix given by

\[ S_{a,b}^{(min)} = (a + b)\left(a + b - 2\right)^2 \left(a + b - 4\right)^2 \ldots \left|a - b\right|. \] (5.8)

This CDD factor was included in the \( S \)-matrix in [33] and ensures that the following fusing processes are allowed:

\[ u_a + u_b \rightarrow u_{a+b} , \quad \text{and} \quad d_a + d_b \rightarrow d_{2(2n+1) - (a+b)}. \]

Without the Toda factor, the basic \( S \)-matrix (5.3) satisfies the associated bootstrap equations passively in the sense of [33].

In order to find a connection between these \( S \)-matrices and those of the preceding chapters, we first observe that the number of particle multiplets and the particle mass ratios (5.2) are the same as those in the \( a_{2n}^{(1)} \) ATFTs. It is known from the work in [36] that some intriguing connection exists between ATFTs at one particular value of the coupling constant and \( N = 2 \) supersymmetric scattering theories. This special value of the coupling constant is \( \beta^2 = 4\pi\frac{h}{n+1} \), which in the case of \( a_{2n}^{(1)} \) ATFTs is \( 4\pi\frac{2n+1}{2n+2} \). In terms of the coupling constant dependent function \( \lambda \) this is equivalent to

\[ \lambda = \frac{1}{h}. \] (5.9)
Furthermore, it is well known that the $N = 2$ supersymmetry algebra can be regarded as a special case of a quantum affine algebra at one particular value of the deformation parameter $q$ (see [37]). This value also corresponds to (5.9), if we consider the relationship (3.5) of the deformation parameter with the Toda coupling constant.

This suggests the possibility of comparing the overall scalar factor from chapter 3 (at $\lambda = \frac{1}{n}$) with the $N = 2$ scalar factor (5.7), and we indeed find that they are identical:

$$G_{a,b}^{(N=2)}(\mu) = F_{a,b}(\mu)|_{\lambda=\frac{1}{n}},$$

in which $F_{a,b}(\mu)$ is given by (3.28) with $n$ replaced by $2n$. This demonstrates a further aspect of the relationship between two–dimensional $N = 2$ supersymmetric scattering theories and imaginary ATFTs.

In this paper we restrict ourselves to the discussion of this relationship on the level of the scalar factors. It would, however, be very interesting to examine this relationship directly on the level of $R$-matrices. This would also require the study of $R$-matrices of quantum affine superalgebras. The $N = 2$ $S$-matrices (5.3), for instance, are known to be related to the $R$-matrices of the superalgebra $U_q(sl(1|1)^{(1)})$ (for further details see [38]).

5.2 Minimal $N = 1$ supersymmetric $S$-matrices

$N = 1$ supersymmetric $S$-matrices for integrable theories have first been studied in a general framework by Schoutens in [30]. This work has also recently been extended in [3] and in [35]. Here we consider the $S$-matrices for the scattering of multiplets of bosons and fermions and we therefore mainly follow the notation for the particle $S$-matrices of [35].

Let us assume we have a $N = 1$ supersymmetric integrable field theory containing $n$ multiplets of bosons and fermions, denoted by $\{|\phi_a\rangle, |\psi_a\rangle\}$ ($a = 1, 2, \ldots, n$). These particles have masses

$$m_a = m \sin(\frac{a\pi}{2n+1}).$$

The paper [35] actually considered a slightly more general case, in which $2n + 1$ was replaced by an arbitrary function $H$. Since our main purpose here is to find a connection with the above $N = 2$ $S$-matrices, we will restrict ourselves to the case where $H = 2n + 1$.

The minimal $N = 1$ $S$-matrices, which are again determined by the requirement that they must commute with the generators of the $N = 1$ superalgebra, can be written in the following
The overall scalar factors can again be written as a product of Gamma functions, which were given explicitly in [35]. For later convenience we slightly rewrite the lowest of these factors in the following form:

\[
\begin{pmatrix}
\phi_a \phi_b & \psi_b \psi_a & \phi_b \psi_a & \psi_b \phi_a \\
\phi_a \psi_b & 0 & B_{a,b}(\mu) & D_{a,b}(\mu) \\
\psi_a \phi_b & 0 & C_{a,b}(\mu) & \tilde{B}_{a,b}(\mu) \\
\psi_a \psi_b & D_{a,b}(\mu) & 0 & 0
\end{pmatrix} = S_{a,b}^{(N=1)}(\theta). 
\tag{5.12}
\]

The individual scattering amplitudes in this matrix have been given as:

\[
\begin{align*}
A_{a,b}(\mu) &= \left( 1 + \frac{2 \sin(\frac{a+b}{2h} \pi) \cos(\frac{a-b}{2h} \pi)}{\sin(\frac{\theta}{h})} \right) G_{a,b}^{(N=1)}(\mu), \\
\tilde{A}_{a,b}(\mu) &= \left( -1 + \frac{2 \sin(\frac{a+b}{2h} \pi) \cos(\frac{a-b}{2h} \pi)}{\sin(\frac{\theta}{h})} \right) G_{a,b}^{(N=1)}(\mu), \\
B_{a,b}(\mu) &= \left( 1 - \frac{2 \sin(\frac{a-b}{2h} \pi) \cos(\frac{a+b}{2h} \pi)}{\sin(\frac{\theta}{h})} \right) G_{a,b}^{(N=1)}(\mu), \\
\tilde{B}_{a,b}(\mu) &= \left( 1 + \frac{2 \sin(\frac{a-b}{2h} \pi) \cos(\frac{a+b}{2h} \pi)}{\sin(\frac{\theta}{h})} \right) G_{a,b}^{(N=1)}(\mu), \\
C_{a,b}(\mu) &= \frac{\sqrt{\sin(\frac{\theta}{h})}}{\sin(\frac{\theta}{h})} G_{a,b}^{(N=1)}(\mu), \\
D_{a,b}(\mu) &= \frac{\sqrt{\sin(\frac{\theta}{h})}}{\cos(\frac{\theta}{h})} G_{a,b}^{(N=1)}(\mu). 
\end{align*}
\tag{5.13}
\]

The overall scalar factors can again be written as a product of Gamma functions, which were given explicitly in [35]. For later convenience we slightly rewrite the lowest of these factors in the following form:

\[
G_{1,1}^{(N=1)}(\mu) = \frac{\Gamma(\mu + j + \frac{1}{2})\Gamma(\mu + j - \frac{1}{2})}{\Gamma(-\mu + j + \frac{1}{2})\Gamma(-\mu + j - \frac{1}{2})} \frac{\Gamma(-\mu + j - 1)\Gamma(-\mu + j + 1)}{\Gamma(\mu + j - 1)\Gamma(\mu + j + 1)} \times \frac{\Gamma(\mu + j - \frac{1}{2})\Gamma(\mu + j + \frac{1}{2})}{\Gamma(-\mu + j - \frac{1}{2})\Gamma(-\mu + j + \frac{1}{2})} \frac{\Gamma(-\mu + j + \frac{1}{2})\Gamma(\mu + j + \frac{1}{2})}{\Gamma(\mu + j - \frac{1}{2})\Gamma(\mu + j + \frac{1}{2})} ,
\tag{5.14}
\]

in which we have again used the bracket notation (5.3) and \( h = 2n + 1 \). All higher scalar factors can be obtained from the following fusion formula (which can easily be derived from equation (3.17) in [35]):

\[
K_{a,b}(\mu)G_{a,b}^{(N=1)}(\mu) = \prod_{j=1}^{a} \prod_{k=1}^{b} G_{1,1}^{(N=1)}(\mu + \frac{1}{2h}(a + b - 2j - 2k + 2)) ,
\tag{5.15}
\]
in which
\[ K_{a,b}(\mu) = \left[ \frac{1}{2\hbar}(a + b) \right] \left[ \frac{1}{2\hbar}(b - a) + \frac{1}{2} \right] \prod_{k=1}^{b} \left[ \frac{1}{2\hbar}(b - a - 2k + 2) \right] \left[ \frac{1}{2\hbar}(a + b - 2k + 1) \right]. \] (5.16)

By analogy with the case of the \( N = 2 \) S-matrices, we first notice that the number of multiplets and the mass ratios in the theory are now identical to those of the \( a_{2n}^{(2)} \) ATFTs. We thus try to compare the above factor \( G_{a,b}^{(N=1)}(\mu) \) with the \( a_{2n}^{(2)} \) scalar factor from chapter 4. First we find that the factor \( F_{1,1}(\mu) \) as given in (4.4) with \( \lambda = \frac{1}{\hbar} \) is identical to the infinite product part in (5.14). We thus have
\[ G_{1,1}^{(N=1)}(\mu) = \left[ 0 \right] \left[ \frac{1}{\hbar} + \frac{1}{2} \right] F_{1,1}(\mu) \big|_{\lambda = \frac{1}{\hbar}}. \] (5.17)

Using the above fusion formula we can easily extend this to the case of general \( a \) and \( b \) and obtain
\[ G_{a,b}^{(N=1)}(\mu) = \Lambda_{a,b}(\mu) F_{a,b}(\mu), \] (5.18)
in which
\[ \Lambda_{a,b}(\mu) = (-)^b \left[ \frac{1}{2\hbar}(a + b) \right] \left[ \frac{1}{2\hbar}(b - a) + \frac{1}{2} \right] \prod_{k=1}^{b} \left( \frac{1}{2\hbar}(a - b + 2k) + \frac{1}{2} \right). \] (5.19)

Thus, we have established a connection between the S-matrices constructed from \( U_q(a_{2n}^{(2)}) \) invariant R-matrices and the minimal \( N = 1 \) supersymmetric particle S-matrices.

As mentioned before, it would be interesting to explore this relationship further and examine the underlying connection between the \( N = 1 \) superalgebra and the quantum algebra \( U_q(a_{2n}^{(2)}) \). It would be interesting to explore whether similar connections can be obtained for the trigonometric S-matrices based on other non–simply laced or twisted algebras. We hope to address these issues in a future publication. In the following section we will proceed by establishing a relationship between the \( N = 2 \) and the \( N = 1 \) S-matrix scalar factors among each other.

5.3 Folding from \( N = 2 \) to \( N = 1 \)

In [39] Melzer conjectured a connection between the \( N = 2 \) and \( N = 1 \) scattering theories via a “folding” of their TBA systems (see also [40]). In [3] Moriconi and Schoutens completed the Thermodynamic Bethe Ansatz for these \( N = 1 \) supersymmetric theories and were able to prove Melzer’s conjecture. This folding procedure takes the following form:

If \( \Phi_{a,b} \) denotes the kernel of the TBA system of the \( N = 2 \) scattering theories discussed in section 5.1, then one can define a folded kernel by
\[ \Phi_{a,b}^{folded} = \Phi_{a,b} + \Phi_{a,2n+1-b}, \quad (\text{for } a, b = 1, 2, \ldots, n). \] (5.20)
It turns out that $\Phi_{a,b}^{folded}$ is exactly the kernel of the TBA system of the $N = 1$ theories as discussed in section 5.2. This immediately raises the question, whether a similar folding procedure can be established directly on the level of $S$-matrices. As a small step towards the answer of this question, here we will show that the overall scalar factors satisfy a similar folding identity.

First note that the kernel of a TBA system is related to the logarithm of the $S$-matrix and therefore the sum in (5.20) should become a product in terms of $S$-matrices. We then use the explicit expressions for the overall scalar factors as given above and obtain the following identity

$$G_{1,1}^{(N=2)}(\mu)G_{1,2n}^{(N=2)}(\mu) = -\frac{[\frac{1}{1}]}{[0]}G_{1,1}^{(N=1)}(\mu) ,$$

(5.21)

Using the fusion relations and the connections with the affine Toda scalar factors at $\lambda = \frac{1}{n}$ we obtain after a somewhat tedious but straightforward computation

$$G_{a,b}^{(N=2)}(\mu)G_{a,2n+1-b}^{(N=2)}(\mu) = -\frac{[\frac{1}{2n}(a+b)][\frac{1}{2n}(b-a)+\frac{1}{2}]}{[0][\frac{1}{2}]} G_{a,b}^{(N=1)}(\mu) .$$

(5.22)

These relations between the $N = 2$ and $N = 1$ scalar factors provide the first step towards establishing Melzer’s folding from $N = 2$ to $N = 1$ theories directly on the level of $S$-matrices. Although the above folding relation for the scalar factors is encouraging, it is still very hard to see how a similar folding relation could be possible for the $S$-matrices. Such a relation could obviously not be just a product of two $S$-matrices due to the non–diagonality of the matrices.

It is also hard to see how the states in the theories could be related, given the fact that the $N = 2$ theory in [33] was formulated in terms of fractionally charged states, whereas the $N = 1$ $S$-matrices describes the scattering of bosons and fermions. There are certainly no obvious connections between the $S$-matrix elements (5.4) and (5.13).

As shown in the preceding sections the overall scalar factors of the supersymmetric $S$-matrices are related to those of the $a_{2n}^{(1)}$ and $a_{2n}^{(2)}$ invariant $S$-matrices. These two algebras have the same Coxeter number $h = 2n+1$ and the solitons in the related ATFTs have the same mass ratios. The main difference between the two theories is the fact that the $a_{2n}^{(1)}$ theory contains $2n$ particle multiplets which occur in mass–degenerate pairs ($m_a = m_{2n+1-a}$), whereas the $a_{2n}^{(2)}$ theory contains only $n$ (non–degenerate) multiplets. Since each multiplet in ATFT can be associated with a spot in the corresponding Dynkin diagram, it seems plausible that any $S$-matrix folding should somehow be related to the folding of Dynkin diagrams. We therefore expect the folding from $N = 2$ to $N = 1$ theories to be related to the folding of the Dynkin diagram of $a_{2n}^{(1)}$ to that of $a_{2n}^{(2)}$. This folding of the Dynkin diagrams was described in [41]. However, a more detailed discussion of this will have to wait for a future publication.
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