GLOBAL WELL-POSEDNESS FOR A NLS-KDV SYSTEM ON $\mathbb{T}$

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Abstract. We prove that the Cauchy problem of the Schrödinger - Korteweg - deVries (NLS-KdV) system on $\mathbb{T}$ is globally well-posed for initial data $(u_0, v_0)$ below the energy space $H^1 \times H^1$. More precisely, we show that the non-resonant NLS-KdV is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 11/13$ and the resonant NLS-KdV is globally well-posed for initial data $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $s > 8/9$. The idea of the proof of this theorem is to apply the I-method of Colliander, Keel, Staffilani, Takaoka and Tao in order to improve the results of Arbieto, Corcho and Matheus concerning the global well-posedness of the NLS-KdV on $\mathbb{T}$ in the energy space $H^1 \times H^1$.

1. Introduction

We consider the Cauchy problem of the Schrödinger-Korteweg-deVries system

$$\begin{cases}
  i \partial_t u + \partial_x^2 u = \alpha uv + \beta |u|^2 u, \\
  \partial_t v + \partial_x^2 v + \frac{1}{2} \partial_x (v^2) = \gamma \partial_x (|u|^2), \\
  u(x, 0) = u_0(x), \quad v(x, 0) = v_0(x), \quad t \in \mathbb{R}.
\end{cases}$$

The Schrödinger-Korteweg-deVries (NLS-KdV) system naturally appears in fluid mechanics and plasma physics as a model of interaction between a short-wave $u = u(x,t)$ and a long-wave $v = v(x,t)$.

In this paper we are interested in global solutions of the NLS-KdV system for rough initial data. Before stating our main results, let us recall some of the recent theorems of local and global well-posedness theory of the Cauchy problem (1.1).

For continuous spatial variable (i.e., $x \in \mathbb{R}$), Corcho and Linares [5] recently proved that the NLS-KdV system is locally well-posed for initial data $(u_0, v_0) \in H^k(\mathbb{R}) \times H^s(\mathbb{R})$ with $k \geq 0$, $s > -3/4$ and

- $k - 1 \leq s \leq 2k - 1/2$ if $k \leq 1/2$,
- $k - 1 \leq s < k + 1/2$ if $k > 1/2$.

Furthermore, they were able to prove the global well-posedness of the NLS-KdV system in the energy $H^1 \times H^1$ using three conserved quantities discovered by M. Tsutsumi [7], whenever $\alpha \gamma > 0$.

Also, Pecher [6] improved this global well-posedness result by an application of the I-method of Colliander, Keel, Staffilani, Takaoka and Tao (for instance, see [3]) combined with some refined bilinear estimates. In particular, Pecher proved that, if $\alpha \gamma > 0$, the NLS-KdV
system is globally well-posed for initial data \((u_0, v_0) \in H^s \times H^s\) with \(s > 3/5\) in the resonant case \(\beta = 0\) and \(s > 2/3\) in the non-resonant case \(\beta \neq 0\).

On the other hand, in the periodic setting (i.e., \(x \in \mathbb{T}\)), Arbieto, Corcho and Matheus \[1\] proved the local well-posedness of the NLS-KdV system for initial data \((u_0, v_0) \in H^k \times H^s\) with \(0 \leq s \leq 4k - 1\) and \(-1/2 \leq k - s \leq 3/2\). Also, using the same three conserved quantities discovered by M. Tsutsumi, one obtains the global well-posedness of NLS-KdV on \(\mathbb{T}\) in the energy space \(H^1 \times H^1\) whenever \(\alpha \gamma > 0\).

Motivated by this scenario, we combine the new bilinear estimates of Arbieto, Corcho and Matheus \[1\] with the I-method of Tao and his collaborators to prove the following result

**Theorem 1.1.** The NLS-KdV system \((1.1)\) on \(\mathbb{T}\) is globally well-posed for initial data \((u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})\) with \(s > 11/13\) in the non-resonant case \(\beta \neq 0\) and \(s > 8/9\) in the resonant case \(\beta = 0\), whenever \(\alpha \gamma > 0\).

The paper is organized as follows. In the section 2, we discuss the preliminaries for the proof of the theorem 1.1: Bourgain spaces and its properties, linear estimates, standard estimates for the non-linear terms \(|u|^2u\) and \(\partial_x(|u|^2)\), the bilinear estimates of Arbieto, Corcho and Matheus \[1\] for the coupling terms \(uv\) and \(\partial_x(|u|^2)\), the I-operator and its properties. In the section 3, we apply the results of the section 2 to get a variant of the local well-posedness result of \[1\]. In the section 4, we recall some conserved quantities of \((1.1)\) and its modification by the introduction of the I-operator; moreover, we prove that two of these modified energies are almost conserved. Finally, in the section 5, we combine the almost conservation results in section 4 with the local well-posedness result in section 3 to conclude the proof of the theorem 1.1.

## 2. Preliminaries

A successful procedure to solve some dispersive equations (such as the nonlinear Schrödinger and KdV equations) is to use the Picard’s fixed point method in the following spaces:

\[
\|f\|_{X^{b,k}} := \left(\int \sum_{n \in \mathbb{Z}} \langle n \rangle^{2k} \langle \tau + n^2 \rangle^{2b} |\hat{f}(n, \tau)|^2 d\tau\right)^{1/2}
\]

\[
= \|U(-t)f\|_{H^k_b} \quad \text{and} \quad \|g\|_{Y^{b,s}} := \left(\int \sum_{n \in \mathbb{Z}} \langle n \rangle^{2s} \langle n^3 \rangle^{2b} |\hat{g}(n, \tau)|^2 d\tau\right)^{1/2}
\]

\[
= \|V(-t)f\|_{H^s_b}
\]

where \(\langle \cdot \rangle := 1 + |\cdot|\), \(U(t) = e^{it\partial_x^2}\) and \(V(t) = e^{-it\partial_x^2}\). These spaces are called Bourgain spaces. Also, we introduce the restriction in time norms

\[
\|f\|_{X^{k,b}(I)} := \inf_{f|_I = f} \|\hat{f}\|_{X^{k,b}} \quad \text{and} \quad \|g\|_{Y^{s,b}(I)} := \inf_{\tilde{g}|_I = g} \|\tilde{g}\|_{Y^{s,b}}
\]

where \(I\) is a time interval.
The interaction of the Picard method has been based around the spaces $Y_{s,1/2}$. Because we are interested in the continuity of the flow associated to $\mathcal{F}$ and the $Y_{s,1/2}$ norm do not control the $L^\infty_t H^s_x$ norm, we modify the Bourgain spaces as follows:

$$
\|u\|_{X^k} := \|u\|_{X^{k,1/2}} + \|\langle n \rangle^k \hat{u}(n, \tau)\|_{L^2_t L^1_x} \quad \text{and} \quad \|v\|_{Y^s} := \|v\|_{Y^{s,1/2}} + \|\langle n \rangle^s \hat{v}(n, \tau)\|_{L^2_t L^1_x}
$$

and, given a time interval $I$, we consider the restriction in time of the $X^k$ and $Y^s$ norms

$$
\|u\|_{X^k(I)} := \inf_{\tilde{u}} \|\tilde{u}\|_{X^k} \quad \text{and} \quad \|v\|_{Y^s(I)} := \inf_{\tilde{v}} \|\tilde{v}\|_{Y^s}
$$

Furthermore, the mapping properties of $U(t)$ and $V(t)$ naturally leads one to consider the companion spaces

$$
\|u\|_{Z^k} := \|u\|_{X^{k,-1/2}} + \|\langle n \rangle^{-k} \hat{u}(n, \tau)\|_{L^2_t L^1_x} \quad \text{and} \quad \|v\|_{W^s} := \|v\|_{Y^{s,-1/2}} + \|\langle n \rangle^{-s} \hat{v}(n, \tau)\|_{L^2_t L^1_x}
$$

In the sequel, $\psi$ denotes a non-negative smooth bump function supported on $[-2, 2]$ with $\psi = 1$ on $[-1, 1]$ and $\psi_\delta(t) := \psi(t/\delta)$ for any $\delta > 0$.

Next, we recall some properties of the Bourgain spaces:

**Lemma 2.1.** $X^{0.3/8}([0,1]), Y^{0.1/3}([0,1]) \subset L^1(T \times [0,1])$. More precisely,

$$
\|\psi(t) f\|_{L^1_{xt}} \lesssim \|f\|_{X^{0.3/8}} \quad \text{and} \quad \|\psi(t) g\|_{L^1_{xt}} \lesssim \|g\|_{Y^{0.1/3}}.
$$

**Proof.** See [2].

Another basic property of these spaces are their stability under time localization:

**Lemma 2.2.** Let $X^{s,b}_{\tau=h(\xi)} := \{f : \langle \tau - h(\xi)\rangle^b \langle \xi \rangle^s |f(\tau, \xi)| \in L^2\}$. Then,

$$
\|\psi(t) f\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{\psi,b} \|f\|_{X^{s,b}_{\tau=h(\xi)}}
$$

for any $s, b \in \mathbb{R}$. Moreover, if $-1/2 < b' \leq b < 1/2$, then for any $0 < T < 1$, we have

$$
\|\psi_T(t) f\|_{X^{s,b'}_{\tau=h(\xi)}} \lesssim_{\psi,b', T} T^{b-b'} \|f\|_{X^{s,b}_{\tau=h(\xi)}}.
$$

**Proof.** First of all, note that $\langle \tau - \tau_0 - h(\xi)\rangle^b \lesssim_{b} \langle \tau_0 \rangle^{|b|} \langle \tau - h(\xi)\rangle^b$, from which we obtain

$$
\|e^{it\tau_0} f\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{b} \langle \tau_0 \rangle^{|b|} \|f\|_{X^{s,b}_{\tau=h(\xi)}}.
$$

Using that $\psi(t) = \int \hat{\psi}(\tau_0) e^{it\tau_0} d\tau_0$, we conclude

$$
\|\psi(t) f\|_{X^{s,b}_{\tau=h(\xi)}} \lesssim_{b} \left( \int |\hat{\psi}(\tau_0)| \langle \tau_0 \rangle^{|b|} \right) \|f\|_{X^{s,b}_{\tau=h(\xi)}}.
$$

Since $\psi$ is smooth with compact support, the first estimate follows.

Next we prove the second estimate. By conjugation we may assume $s = 0$ and, by composition it suffices to treat the cases $0 \leq b' \leq b$ or $\leq b' \leq b \leq 0$. By duality, we may take $0 \leq b' \leq b$. Finally, by interpolation with the trivial case $b' = b$, we may consider $b' = 0$. This reduces matters to show that

$$
\|\psi_T(t) f\|_{L^2} \lesssim_{\psi,b} T^{b} \|f\|_{X^{0,b}_{\tau=h(\xi)}}.
$$
Lemma 2.5. Let \( f \) be a function such that \( \langle \tau - h(\xi) \rangle \geq 1/T \) and \( \langle \tau - h(\xi) \rangle \leq 1/T \), we see that in the former case we’ll have

\[
\| f \|_{X^{b,0}_{\tau - h(\xi)}} \leq T^b \| f \|_{X^{0,b}_{\tau - h(\xi)}}
\]

and the desired estimate follows because the multiplication by \( \psi \) is a bounded operation in Bourgain’s spaces. In the latter case, by Plancherel and Cauchy-Schwarz

\[
\| f(t) \|_{L^2} \lesssim \| \widehat{f(t)}(\xi) \|_{L^1_\xi} \lesssim \left\| \int_{\langle \tau - h(\xi) \rangle \leq 1/T} |\widehat{f}(\tau, \xi)| d\tau \right\|_{L^2_\xi} \\
\lesssim b T^{b-1/2} \left\| \int (\tau - h(\xi))^{2b} |\widehat{f}(\tau, \xi)|^2 d\tau \right\|^{1/2}_{L^2_\xi} \leq T^{b-1/2} \| f \|_{X^{b,0}_{\tau - h(\xi)}}.
\]

Integrating this against \( \psi_T \) concludes the proof of the lemma. \( \square \)

Also, we have the following duality relationship between \( X^k \) (resp., \( Y^s \)) and \( Z^k \) (resp., \( W^s \)):

Lemma 2.3. We have

\[
\left| \int \chi_{[0,1]}(t) f(x,t) g(x,t) dx dt \right| \lesssim \| f \|_{X^s} \| g \|_{Z^{-s}}
\]

and

\[
\left| \int \chi_{[0,1]}(t) f(x,t) g(x,t) dx dt \right| \lesssim \| f \|_{Y^s} \| g \|_{W^{-s}}
\]

for any \( s \) and any \( f, g \) on \( \mathbb{T} \times \mathbb{R} \).

Proof. See [1], p. 182–183] (note that, although this result is stated only for the spaces \( Y^s \) and \( W^s \), the same proof adapts for the spaces \( X^k \) and \( Z^k \)). \( \square \)

Now, we recall some linear estimates related to the semigroups \( U(t) \) and \( V(t) \):

Lemma 2.4 (Linear estimates). It holds

- \( \| \psi(t)U(t)u_0 \|_{X^k} \lesssim \| u_0 \|_{H^k} \) and \( \| \psi(t)V(t)v_0 \|_{W^s} \lesssim \| v_0 \|_{H^s} \);
- \( \| \psi_T(t) \int_0^t U(t-t')F(t') dt' \|_{X^k} \lesssim \| F \|_{Z^k} \) and \( \| \psi_T(t) \int_0^t V(t-t')G(t') dt' \|_{Y^s} \lesssim \| G \|_{W^s} \).

Proof. See [2], [3] or [11]. \( \square \)

Furthermore, we have the following well-known multi-linear estimates for the cubic term \( |u|^2 u \) of the nonlinear Schrödinger equation and the nonlinear term \( \partial_x (v^2) \) of the KdV equation:

Lemma 2.5. \( \| uvw \|_{Z^k} \lesssim \| u \|_{X^{k,\frac{1}{3}}} \| v \|_{X^{k,\frac{1}{3}}} \| w \|_{X^{k,\frac{1}{3}}} \) for any \( k \geq 0 \).

Proof. See [2] and [11]. \( \square \)

Lemma 2.6. \( \| \partial_x (v_1 v_2) \|_{W^s} \lesssim \| v_1 \|_{Y^{s,1}} \| v_2 \|_{Y^{s,1}} \| v_2 \|_{Y^{s,1}} + \| v_1 \|_{Y^{s,1}} \| v_2 \|_{Y^{s,1}} \| v_2 \|_{Y^{s,1}} \) for any \( s \geq -1/2 \), if \( v_1 = v_1(x,t) \) and \( v_2 = v_2(x,t) \) are \( x \)-periodic functions having zero \( x \)-mean for all \( t \).

Proof. See [2], [3] and [11]. \( \square \)
Next, we revisit the bilinear estimates of mixed Schrödinger-Airy type of Arbieto, Corcho and Matheus for the coupling terms $uv$ and $\partial_x|u|^2$ of the NLS-KdV system.

**Lemma 2.7.** $\|uv\|_{Z^k} \lesssim \|u\|_{X^k} \cdot \|v\|_{Y^s}$ whenever $s \geq 0$ and $k - s \leq 3/2$.

**Lemma 2.8.** $\|\partial_x(u_1 \overline{u_2})\|_{W^s} \lesssim \|u_1\|_{X^{k,3/8}} \cdot \|u_2\|_{X^{k,3/8}}$ whenever $1 + s \leq 4k$ and $k - s \geq -1/2$.

**Remark 2.1.** Although the lemmas 2.7 and 2.8 are not stated as above in [1], it is not hard to obtain them from the calculations of Arbieto, Corcho and Matheus.

Finally, we introduce the I-operator: let $m(\xi)$ be a smooth non-negative symbol on $\mathbb{R}$ which equals 1 for $|\xi| \leq 1$ and equals $|\xi|^{-1}$ for $|\xi| \geq 2$. For any $N \geq 1$ and $\alpha \in \mathbb{R}$, denote by $I_N^\alpha$ the spatial Fourier multiplier

$$I_N^\alpha f(\xi) = m\left(\frac{\xi}{N}\right) \alpha \hat{f}(\xi).$$

For latter use, we recall the following general interpolation lemma:

**Lemma 2.9.** (Lemma 12.1 of [4]). Let $\alpha_0 > 0$ and $n \geq 1$. Suppose $Z, X_1, X_2, \ldots, X_n$ are translation-invariant Banach spaces and $T$ is a translation invariant $n$-linear operator such that

$$\|I_N^\alpha T(u_1, \ldots, u_n)\|_Z \lesssim \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j},$$

for all $u_1, \ldots, u_n$ and $0 \leq \alpha \leq \alpha_0$. Then,

$$\|I_N^\alpha T(u_1, \ldots, u_n)\|_Z \lesssim \prod_{j=1}^n \|I_N^\alpha u_j\|_{X_j}$$

for all $u_1, \ldots, u_n$, $0 \leq \alpha \leq \alpha_0$ and $N \geq 1$. Here the implicit constant is independent of $N$.

After these preliminaries, we can proceed to the next section where a variant of the local well-posedness of Arbieto, Corcho and Matheus is obtained. In the sequel we take $N \gg 1$ a large integer and denote by $I$ the operator $I = I_N^{1-s}$ for a given $s \in \mathbb{R}$.

### 3. A Variant Local Well-posedness Result

This section is devoted to the proof of the following proposition:

**Proposition 3.1.** For any $(u_0, v_0) \in H^s(\mathbb{T}) \times H^s(\mathbb{T})$ with $\int_T v_0 = 0$ and $s \geq 1/3$, the periodic NLS-KdV system (1.1) has a unique local-in-time solution on the time interval $[0, \delta]$ for some $\delta \leq 1$ and

$$\delta \sim \begin{cases} (\|I u_0\|_{X^1} + \|I v_0\|_{Y^1})^{-\frac{4\beta}{3}}, & \text{if } \beta \neq 0, \\ (\|I u_0\|_{X^1} + \|I v_0\|_{Y^1})^{-3}, & \text{if } \beta = 0. \end{cases}$$

Moreover, we have $\|Iu\|_{X^1} + \|Iv\|_{Y^1} \lesssim \|Iu_0\|_{X^1} + \|Iv_0\|_{Y^1}$. 


Proof. We apply the I-operator to the NLS-KdV system (1.1) so that
\[\begin{align*}
  i u_t + u_{xx} &= \alpha I(uv) + \beta I(|u|^2 u), \\
  v_t + v_{xxx} + I(vv_x) &= \gamma I(|u|^2)_x, \\
  u(0) &= u_0, \quad v(0) = v_0.
\end{align*}\]

To solve this equation, we seek for some fixed point of the integral maps
\[\begin{align*}
  \Phi_1(Iu, Iv) &= U(t)Iu_0 - i \int_0^t U(t - t')\{\alpha I(u(t')v(t')) + \beta I(|u(t')|^2 u(t'))\}dt', \\
  \Phi_2(Iu, Iv) &= V(t)Iv_0 - \int_0^t V(t - t')\{I(v(t')v_x(t')) - \gamma I(|u(t')|^2)_x\}dt'.
\end{align*}\]

The interpolation lemma [2,3] applied to the linear and multilinear estimates in the lemmas 2.4, 2.5, 2.6, 2.7 and 2.8 yields, in view of the lemma 2.2,

\[\begin{align*}
  \|\Phi_1(Iu, Iv)\|_{X_1} &\lesssim \|Iu_0\|_{H^1} + \alpha \delta^\frac{8}{3} - \|Iu\|_{X^1} \|Iv\|_{Y^1} + \beta \delta^\frac{8}{3} - \|Iu\|_{X^1}^3, \\
  \|\Phi_2(Iu, Iv)\|_{Y^1} &\lesssim \|Iv_0\|_{H^1} + \delta^\frac{8}{3} - \|Iv\|_{Y^1} + \gamma \delta^\frac{8}{3} - \|Iu\|_{X^1}^2.
\end{align*}\]

In particular, these integral maps are contractions provided that \(\beta \delta^\frac{8}{3} - (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1})^2 \ll 1\) and \(\delta^\frac{8}{3} - (\|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}) \ll 1\). This completes the proof of the proposition 3.1 \(\square\)

4. Modified energies

Consider the following three quantities:

\[\begin{align*}
  M(u) &= \|u\|_{L^2}, \\
  L(u, v) &= \alpha \|v\|_{L^2}^2 + 2\gamma \int \Re(u\overline{v})dx, \\
  E(u, v) &= \alpha \gamma \int v|u|^2 dx + \gamma \|u_x\|_{L^2}^2 + \frac{\alpha}{2} \|v_x\|_{L^2}^2 - \frac{\alpha}{6} \int v^3 dx + \frac{\beta \gamma}{2} \int |u|^4 dx.
\end{align*}\]

In the sequel, we suppose \(\alpha \gamma > 0\). Note that

\[\begin{align*}
  |L(u, v)| &\lesssim \|v\|_{L^2}^3 + M\|u_x\|_{L^2} \\
  \|v\|_{L^2}^2 &\lesssim |L| + M\|u_x\|_{L^2}.
\end{align*}\]

Also, the Gagliardo-Nirenberg and Young inequalities implies

\[\begin{align*}
  \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 &\lesssim |E| + |L|^\frac{5}{2} + M^8 + 1 \quad \text{and} \\
  |E| &\lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + |L|^\frac{5}{2} + M^8 + 1.
\end{align*}\]
In particular, combining the bounds (4.4) and (4.7),

\[(4.8) \quad |E| \lesssim \|u_x\|_{L^2}^2 + \|v_x\|_{L^2}^2 + \|v\|_{L^2}^{10} + M^{10} + 1.\]

Moreover, from the bounds (4.5) and (4.6),

\[(4.9) \quad \|v\|_{L^2}^2 \lesssim |L| + M|E|^{1/2} + M^6 + 1\]

and hence

\[(4.10) \quad \|u\|_{H^1}^2 + \|v\|_{H^1}^2 \lesssim |E| + |L|^{5/3} + M^8 + 1\]

\[
\frac{d}{dt} L(Iu, Iv) = 2\alpha \int Iv(IvIv_x - Ivv_x)dx + 2\alpha\gamma \int Iv(|u|^2 - |Iu|^2)x dx + 4\alpha\gamma |\Pi_x(IuIv - Ivuv)|dx + 4\beta\gamma |\Pi_x(IvIu - Ivuv)|dx
\]

\[(4.11) \quad =: \sum_{j=1}^{4} L_j.\]

\[
\frac{d}{dt} E(Iu, Iv) = \alpha \int (Ivv_x - Ivv_x)Iv_x dx + \frac{\alpha}{2} \int (Iv)^2(IvIv_x - Ivv_x)dx + 2\beta\gamma \int (I(|u|^2u)_x - (Iu^2I\Pi)_x)\Pi_x dx
\]

\[
+ \alpha \int |Iu|^2(IvIv_x - Ivv_x)dx + \alpha \int (|Iu|^2 - |Iu|^2)IvIv_x dx + \alpha \int Ivv_x(Iu^2 - Iu^2)IvIv_x dx + 2\alpha\gamma \int Iu(IuIv - Iuvv)_x dx
\]

\[
+ \alpha^2 \int (I(|u|^2) - |Iu|^2)_x |Iu|^2 dx + 2\alpha^2\gamma |IvIu(IuIv - Iuvv)|dx + 2\beta^2\gamma |Iu(Iu^2I\Pi - (Iu)^2I\Pi)|dx
\]

\[-2\alpha\beta\gamma \int IvIu(I(|u|^2\Pi) - Iu(I\Pi))^2 dx - 2\alpha\beta\gamma \int (Iu)^2I\Pi(Iuvv) - I\Pi Ivy)|dx
\]

\[
=: \sum_{j=1}^{12} E_j.
\]

4.1. Estimates for the modified L-functional.
\textbf{Proposition 4.1.} Let \((u, v)\) be a solution of (1.2) on the time interval \([0, \delta]\). Then, for any \(N \geq 1\) and \(s > 1/2\),

\begin{equation}
|L(Iu(\delta), Iv(\delta)) - L(Iu(0), Iv(0))| \lesssim N^{-1+\delta\frac{3}{24}}(\|Iu\|_{X^{1,1/2}} + \|Iv\|_{Y^{1,1/2}})^3 + N^{-2+\delta\frac{3}{24}}\|Iu\|^4_{X^{1,1/2}}.
\end{equation}

\textit{Proof.} Integrating (4.11) with respect to \(t \in [0, \delta]\), it follows that we have to bound the (integral over \([0, \delta]\) of the) four terms on the right hand side. To simplify the computations, we assume that the Fourier transform of the functions are non-negative and we ignore the appearance of complex conjugates (since they are irrelevant in our subsequent arguments).

Also, we make a dyadic decomposition of the frequencies \(|n_i| \sim N_j\) in many places. In particular, it will be important to get extra factors \(N_j^{0-}\) everywhere in order to sum the dyadic blocks.

We begin with the estimate of \(\int_0^\delta L_1\). It is sufficient to show that

\begin{equation}
\int_0^\delta \sum_{n_1+n_2+n_3=0} \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \hat{v}_1(n_1,t)\hat{v}_2(n_2,t)\hat{v}_3(n_3,t) \lesssim N^{-1+\delta\frac{3}{24}} \prod_{j=1}^3 \|v_j\|_{Y^{1,1/2}}
\end{equation}

- \(|n_1| \ll |n_2| \sim |n_3|, |n_2| \gtrsim N\). In this case, note that

\[
\left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \|\nabla m(n_2)/m(n_2)\| \lesssim \frac{N_1}{N_2}, \text{ if } |n_1| \leq N, \text{ and } \left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left( \frac{N_1}{N} \right)^{1/2} \text{, if } |n_1| \geq N.
\]

Hence, using the lemmas 2.1 and 2.2 we obtain

\[
| \int_0^\delta L_1 | \lesssim \frac{N_1}{N_2} \|v_1\|_{L^4} \|v_2\|_{L^4} \|v_3\|_{L^2} \lesssim N^{-2+\delta\frac{3}{24}} N_\max^{-3} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}
\]

if \(|n_1| \leq N\), and

\[
| \int_0^\delta L_1 | \lesssim \left( \frac{N_1}{N} \right)^{1/2} \frac{1}{N_1 N_3} \delta\frac{3}{24} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}} \lesssim N^{-2+\delta\frac{3}{24}} N_\max^{-3} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}.
\]

- \(|n_2| \ll |n_1| \sim |n_3|, |n_1| \gtrsim N\). This case is similar to the previous one.
- \(|n_1| \sim |n_2| \gtrsim N\). The multiplier is bounded by

\[
\left| \frac{m(n_1+n_2)-m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left( \frac{N_1}{N} \right)^{1-}.
\]

In particular, using the lemmas 2.1 and 2.2

\[
| \int_0^\delta L_1 | \lesssim \left( \frac{N_1}{N} \right)^{1-} \|v_1\|_{L^2} \|v_2\|_{L^4} \|v_3\|_{L^4} \lesssim N^{-1+\delta\frac{3}{24}} N_\max^{-3} \prod_{j=1}^3 \|v_i\|_{Y^{1,1/2}}.
\]
Now, we estimate \( \int_0^\delta L_2 \). Our task is to prove that

\[
\int_0^\delta \sum_{n_1+n_2+n_3=0} \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |n_1 + n_2| |\tilde{u}_1(n_1,t)\tilde{u}_2(n_2,t)\tilde{v}_3(n_3,t)| \lesssim N^{-1+\delta_{14}} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}
\]

(4.15)

\[
\int_0^\delta L_2 \lesssim \left( \frac{N_2}{N} \right)^{1-} \frac{1}{N_1N_2} \delta_{24} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}
\]

Thus, using \( L_2^2 L_4^4 L_4^4 \) Hölder inequality and the lemmas 2.1 and 2.2.

- \(|n_2| \ll |n_1| \sim |n_3| \gg N\). This case is similar to the previous one.

- \(|n_1| \sim |n_2| \gg N\). Estimating the multiplier by

\[
\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left( \frac{N_2}{N} \right)^{1-}
\]

we conclude

\[
\int_0^\delta L_2 \lesssim \left( \frac{N_2}{N} \right)^{1-} \frac{1}{N_1N_2} \delta_{24} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{Y^{1,1/2}}
\]

Next, let us compute \( \int_0^\delta L_3 \). We claim that

\[
\int_0^\delta \sum_{n_1+n_2+n_3=0} \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |\tilde{u}_1(n_1,t)\tilde{u}_2(n_2,t)\tilde{v}_3(n_3,t)| \lesssim N^{-2+\delta_{14}} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}} \|v_3\|_{X^{1,1/2}}
\]

(4.16)

- \(|n_2| \ll |n_1| \sim |n_3|, \ |n_1| \gg N\). The multiplier is bounded by

\[
\left\{ \begin{array}{ll}
\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \|\nabla m(n_1)\| \lesssim \frac{N_1}{N}, \quad \text{if } |n_2| \leq N, \text{ and}
\end{array} \right.
\]

\[
\left\{ \begin{array}{ll}
\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left( \frac{N}{N_1} \right)^{1/2}, \quad \text{if } |n_2| \geq N.
\end{array} \right.
\]

So, it is not hard to see that

\[
\int_0^\delta L_3 \lesssim N^{-2+\delta_{14}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}
\]

- \(|n_1| \ll |n_2| \sim |n_3|, \ |n_2| \gg N\). This case is completely similar to the previous one.
Finally, it remains to estimate the contribution of $\int_0^\delta L_3$. It suffices to see that
\[
\int_0^\delta \sum_{n_1+n_2+n_3+n_4=0} \frac{|m(n_1+n_2+n_3) - m(n_1)m(n_2)m(n_3)|}{m(n_1)m(n_2)m(n_3)} |n_4| \prod_{j=1}^4 u_j(n_j,t) \lesssim N^{-2+\delta^{3/2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}
\]
\[\tag{4.17}
\]

- $N_1, N_2, N_3 \gtrsim N$. Since the multiplier verifies

$$\left| \frac{m(n_1+n_2+n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \right| \lesssim \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\delta^{3/2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

the application of $L_4^4 L_4^4 L_4^4 L_4^4$ Hölder inequality and the lemmas \[2.1, 2.2\] yields

$$\int_0^\delta L_4 \lesssim \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\delta^{3/2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_2 \gtrsim N$ and $N_3, N_4 \ll N_1, N_2$. Here the multiplier is bounded by \( \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} \). Hence,

$$\int_0^\delta L_4 \lesssim \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\delta^{3/2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_1 \gtrsim N$ and $N_2, N_3 \ll N_1, N_4$. In this case we have the following estimates for the multiplier

$$\left| \frac{m(n_1+n_2+n_3) - m(n_1)m(n_2)m(n_3)}{m(n_1)m(n_2)m(n_3)} \right| \lesssim \begin{cases} 
|\nabla m(n_1)(n_2+n_3)| / m(n_1) & \text{if } N_2, N_3 \leq N \\
\left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} & \text{if } N_2 \geq N, \\
\left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} & \text{if } N_3 \geq N.
\end{cases}$$

Therefore, it is not hard to see that, in any of the situations $N_2, N_3 \leq N$, $N_2 \geq N$ or $N_3 \geq N$, we have

$$\int_0^\delta L_4 \lesssim N^{-2+\delta^{3/2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$

- $N_1 \sim N_2 \sim N_4 \gtrsim N$ and $N_3 \ll N_1, N_2, N_4$. Here we have the following bound

$$\int_0^\delta L_4 \lesssim \left( \frac{N_1}{N} \right)^{\frac{1}{2}} \left( \frac{N_2}{N} \right)^{\frac{1}{2}} \left( \frac{N_3}{N} \right)^{\frac{1}{2}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.$$
At this point, clearly the bounds \(4.11\), \(4.13\), \(4.16\) and \(4.17\) concludes the proof of the proposition \(4.1\).

### 4.2. Estimates for the modified E-functional.

**Proposition 4.2.** Let \((u, v)\) be a solution of \(4.1)\) on the time interval \([0, \delta]\) such that \(\int_T v = 0\). Then, for any \(N \geq 1\), \(s > 1/2\),

\[
|E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))| \lesssim \\
N^{-1+\delta_s^+} \left( N^{-\frac{2}{3} + \delta_s^+} + N^{-\frac{2}{3} + \delta_s^+} \right) (\|Iu\|_{X_1} + \|Iv\|_{Y_1})^3 + \\
N^{-1+\delta_s^+} (\|Iu\|_{X_1} + \|Iv\|_{Y_1}) + N^{-2+\delta_s^+} (\|Iu\|_{X_1}^2 + \|Iv\|_{Y_1}^2).
\]

**Proof.** Again we integrate \(4.14\) with respect to \(t \in [0, \delta]\), decompose the frequencies into dyadic blocks, etc., so that our objective is to bound the (integral over \([0, \delta]\) of the) \(E_j\) for each \(j = 1, \ldots, 12\).

For the expression \(\int_0^\delta E_1\), apply the lemma \(2.3\). We obtain

\[
\left| \int_0^\delta E_1 \right| \lesssim \|v_{xx}\|_{Y_1} \|v_Iv - v_Iv_x - I(vv_x)\|_{Y_1} \lesssim \|v_Iv\|_{Y_1} \|v_Iv_x - I(vv_x)\|_{Y_1}.
\]

Writing the definition of the norm \(W_1\), it suffices to prove the bound

\[
\left\| \frac{\langle n_3 \rangle}{\langle n_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \tilde{v}_1(n_1, \tau_1) n_2 \tilde{v}_2(n_2, \tau_2) \right\|_{L_{n_3, r_3}^3} + \\
\left\| \frac{\langle n_3 \rangle}{\langle n_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \tilde{v}_1(n_1, \tau_1) n_2 \tilde{v}_2(n_2, \tau_2) \right\|_{L_{n_3, L_{r_3}}^2} \lesssim \\
N^{-1+\delta_s^+} \|v_1\|_{Y_1,1/2} \|v_2\|_{Y_1,1/2}.
\]

Recall that the dispersion relation \(\sum_{j=1}^3 \tau_j - n_3^3 = -3n_1n_2n_3\) implies that, since \(n_1n_2n_3 \neq 0\), if we put \(L_j := \langle \tau_j - n_3^3 \rangle\) and \(L_{\max} = \max\{L_j; j = 1, 2, 3\}\), then \(L_{\max} \gtrsim \langle n_1 \rangle \langle n_2 \rangle \langle n_3 \rangle\).

- \(\|n_2\| \sim |n_3| \gtrsim N\), \(|n_1| \ll |n_2|\). The multiplier is bounded by

\[
\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \begin{cases} \frac{N_1}{N_2}, & \text{if } |n_1| \leq N, \\ \left( \frac{N_1}{N_2} \right)^{\frac{1}{2}}, & \text{if } |n_1| \geq N. \end{cases}
\]

Thus, if \(|\tau_3 - n_3^3| = L_{\max}\), we have

\[
\langle n_3 \rangle \left\| \frac{\langle n_3 \rangle}{\langle n_3 - n_3^3 \rangle^{\frac{1}{2}}} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \tilde{v}_1(n_1, \tau_1) n_2 \tilde{v}_2(n_2, \tau_2) \right\|_{L_{n_3, r_3}^3} \lesssim \\
\left( \frac{N_1}{N_2} \right)^{\frac{1}{2}} \left( \frac{N_1}{N_2} \right) \frac{1}{\langle n_1n_2n_3 \rangle^{\frac{1}{2}}} \|v_1\|_{L_{r_3}^3} \|v_2\|_{L_{r_3}^3} \lesssim N^{-\frac{1}{2} + \delta_s^+} \sum_{n_1} \|v_1\|_{Y_1,1/2} \|v_2\|_{Y_1,1/2}, \text{ if } |n_1| \leq N,
\]

\[
\left( \frac{N_1}{N_2} \right)^{\frac{1}{2}} \left( \frac{N_1}{N_2} \right) \frac{1}{\langle n_1n_2n_3 \rangle^{\frac{1}{2}}} \|v_1\|_{L_{r_3}^3} \|v_2\|_{L_{r_3}^3} \lesssim N^{-1+\delta_s^+} \sum_{n_1} \|v_1\|_{Y_1,1/2} \|v_2\|_{Y_1,1/2}, \text{ if } |n_1| \geq N.
\]
and

\[
\left\| \frac{(n_3)}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{v}_1(n_1, \tau_1) n_2 \hat{v}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3} L^1_{\tau_3}} \\
\leq \left\{ \frac{N_1}{N_2} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \right\} \delta_1^- \|v_1\|_{Y^{1, \frac{1}{2}}_T} \|v_2\|_{Y^{1, \frac{1}{2}}_T} \lesssim N^{-1 + \frac{3}{2} - \frac{5}{2} - N_{\max}^0} \|v_1\|_{Y^{1, 1/2}_T} \|v_2\|_{Y^{1, 1/2}_T}, \quad \text{if } |n_1| \leq N,
\]

\[
\left\{ \frac{N_1}{N_2} \frac{N_3}{N_1 (N_1 N_2 N_3)^{\frac{1}{2}}} \right\} \delta_1^- \|v_1\|_{Y^{1, \frac{1}{2}}_T} \|v_2\|_{Y^{1, \frac{1}{2}}_T} \lesssim N^{-1 + \frac{3}{2} - \frac{5}{2} - N_{\max}^0} \|v_1\|_{Y^{1, 1/2}_T} \|v_2\|_{Y^{1, 1/2}_T}, \quad \text{if } |n_1| \geq N.
\]

If either $|\tau_1 - n_1^3| = L_{\max}$ or $|\tau_2 - n_2^3| = L_{\max}$, we have

\[
\left\| \frac{(n_3)}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{v}_1(n_1, \tau_1) n_2 \hat{v}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3} L^1_{\tau_3}} \\
\leq \left\{ \frac{N_1}{N_2} \frac{N_3}{(N_1 N_2 N_3)^{\frac{1}{2}}} \right\} \delta_1^- \|v_1\|_{Y^{1, \frac{1}{2}}_T} \|v_2\|_{Y^{1, \frac{1}{2}}_T} \lesssim N^{-1 + \frac{3}{2} - \frac{5}{2} - N_{\max}^0} \|v_1\|_{Y^{1, 1/2}_T} \|v_2\|_{Y^{1, 1/2}_T}, \quad \text{if } |n_1| \leq N,
\]

\[
\left\{ \frac{N_1}{N_2} \frac{N_3}{N_1 (N_1 N_2 N_3)^{\frac{1}{2}}} \right\} \delta_1^- \|v_1\|_{Y^{1, \frac{1}{2}}_T} \|v_2\|_{Y^{1, \frac{1}{2}}_T} \lesssim N^{-1 + \frac{3}{2} - \frac{5}{2} - N_{\max}^0} \|v_1\|_{Y^{1, 1/2}_T} \|v_2\|_{Y^{1, 1/2}_T}, \quad \text{if } |n_1| \geq N.
\]

\[
|n_1| \sim |n_2| \gtrsim N. \quad \text{Estimating the multiplier by}
\]

\[
\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left( \frac{N_1}{N} \right)^{1-}
\]

we have that, if $|\tau_3 - n_3^3| = L_{\max}$,

\[
\left\| \frac{(n_3)}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{v}_1(n_1, \tau_1) n_2 \hat{v}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3} L^1_{\tau_3}} + \\
\left\| \frac{(n_3)}{\langle \tau_3 - n_3^3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{v}_1(n_1, \tau_1) n_2 \hat{v}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3} L^1_{\tau_3}} \\
\lesssim \left\{ \frac{N_1}{N} \right\}^{1-} \frac{N_3}{N_1 (N_1 N_2 N_3)^{\frac{1}{2}}} + \left\{ \frac{N_1}{N} \right\}^{1-} \frac{N_3}{N_1 (N_1 N_2 N_3)^{\frac{1}{2}}} \|v_1\|_{Y^{1, \frac{1}{2}}_T} \|v_2\|_{Y^{1, \frac{1}{2}}_T}
\lesssim N^{-1 + \frac{3}{2} - \frac{5}{2} - N_{\max}^0} \|v_1\|_{Y^{1, 1/2}_T} \|v_2\|_{Y^{1, 1/2}_T}
\]
and, if either $|\tau_1 - n_3|^2 = L_{\text{max}}$ or $|\tau_2 - n_3|^2 = L_{\text{max}},$

$$\left\| \frac{\langle n_3 \rangle}{\langle r_3 - n_3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{v}_1(n_1, \tau_1) n_2 \hat{v}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3} L^1_{r_3}} +$$

$$\left\| \frac{\langle n_3 \rangle}{\langle r_3 - n_3 \rangle} \int \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{v}_1(n_1, \tau_1) n_2 \hat{v}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3} L^1_{r_3}}$$

$$\lesssim \left\{ \left( \frac{N_1}{N} \right)^{1-} \frac{N_3}{(N_1 N_2 N_3)^{1/2}} + \left( \frac{N_1}{N} \right)^{1-} \frac{N_3}{(N_1 N_2 N_3)^{1/2}} \right\} \|v_1\|_{Y_{1,1/2}}^4 \|v_2\|_{Y_{1,1/2}}^4 \lesssim N^{-\frac{3}{2} + \delta \frac{3}{N} - N_{\text{max}}^0} \|v_1\|_{Y_{1,1/2}}^4 \|v_2\|_{Y_{1,1/2}}^4.$$
we obtain
\[
\int_0^\delta E_3 \lesssim \left( \frac{N_1}{N} \right)^{\frac{3}{7}} \frac{N_4}{N_1 N_2 N_3} \delta^{\frac{1}{7}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\frac{1}{2}} N_3^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.
\]

- Exactly two frequencies are \( \gtrsim N \). We consider the most difficult case \(|n_4| \gtrsim N\), \(|n_1| \sim |n_4| \) and \(|n_2|, |n_3| \ll |n_1|, |n_4|\). The multiplier is estimated by

\[
\frac{m(n_1 n_2 n_3) - m(n_1) m(n_2) m(n_3)}{m(n_1) m(n_2) m(n_3)} \lesssim \begin{cases} \left( \frac{N_3}{N} \right)^{\frac{1}{7}}, & \text{if } |n_2| \geq N, \\ \left( \frac{N_3}{N} \right)^{\frac{1}{7}}, & \text{if } |n_3| \geq N, \\ \frac{N_3}{N_1}, & \text{if } |n_2|, |n_3| \leq N. \end{cases}
\]

Thus,
\[
\int_0^\delta E_3 \lesssim N^{-1+\frac{1}{7}} N_3^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.
\]

- Exactly three frequencies are \( \gtrsim N \). The most difficult case is \(|n_1| \sim |n_2| \sim |n_4| \gtrsim N\) and \(|n_3| \ll |n_1|, |n_2|, |n_4|\). Here the multiplier is bounded by

\[
\frac{m(n_1 n_2 n_3) - m(n_1) m(n_2) m(n_3)}{m(n_1) m(n_2) m(n_3)} \lesssim \left( \frac{N_1 N_2}{N} \right)^{\frac{1}{7}} \left( \frac{N_3}{N} \right)^{\frac{1}{7}}.
\]

Hence,
\[
\int_0^\delta E_3 \lesssim \left( \frac{N_1 N_2}{N} \right)^{\frac{1}{7}} \left( \frac{N_3}{N} \right)^{\frac{1}{7}} N_3^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+\frac{1}{7}} N_3^{0-} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}}.
\]

The contribution of \( \int_0^\delta E_4 \) is controlled if we are able to show that
\[
\int_0^\delta \sum \frac{m(n_1 + n_2) - m(n_1) m(n_2)}{m(n_1) m(n_2)} \hat{v}_1(n_1, t) |n_2| \hat{v}_2(n_2, t) \hat{v}_3(n_3, t) \hat{v}_4(n_4, t) \lesssim N^{-1+\frac{1}{7}} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}.
\] (4.22)

We crudely bound the multiplier by
\[
|\frac{m(n_1 + n_2) - m(n_1) m(n_2)}{m(n_1) m(n_2)}| \lesssim \left( \frac{N_{\text{max}}}{N} \right)^{1-}.
\]

The most difficult case is \(|n_2| \geq N\). We have two possibilities:

- Exactly two frequencies are \( \gtrsim N \). We can assume \( N_3 \ll N_2 \). In particular,

\[
\int_0^\delta E_4 \lesssim \left( \frac{N_{\text{max}}}{N} \right)^{1-} \frac{\delta^{\frac{1}{7}}}{N_1 N_3 N_1} \prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}} \lesssim \frac{N^{-1+\frac{1}{7}} N_{\text{max}}^{0-}}{\prod_{j=1}^4 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}.}
\]
• At least three frequencies are $\gtrsim N$. In this case, 
\[ \int_0^\delta E_4 \lesssim N^{-2+\delta^2} - N_\alpha^0 \prod_{j=1}^2 \|u_j\|_{X^{1,\frac{3}{2}}} \|v_j\|_{Y^{1,\frac{3}{2}}} \cdot \]

The expression $\int_0^\delta E_5$ is controlled if we are able to prove 
\[ \int_0^\delta \sum m(n_1+n_2) - m(n_1)m(n_2) \frac{\hat{u}_1(n_1,t)\hat{u}_2(n_2,t)}{m(n_1)m(n_2)} \hat{v}_3(n_3,t) |n_4| \hat{v}_4(n_4,t) \lesssim \]
\[ N^{-1+\delta^2} - \prod_{j=1}^2 \|u_j\|_{X^{1,1/2}} \|v_j\|_{Y^{1,1/2}}. \]

This follows directly from the previous analysis for (4.22).

For the term $\int_0^\delta E_6$, we apply the lemma 2.3 to obtain 
\[ \int_0^\delta E_6 \lesssim \|(Iv)_{xx}\|_{Y^{-1}} \|(|u|^2 - I(|u|^2))_x\|_{W^1} \lesssim \|Iv\|_{Y^1} \|(|u|^2 - I(|u|^2))_x\|_{W^1}. \]

So, the definition of the $W^1$ norm means that we have to prove 
\[ \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3 \rangle^{\frac{3}{2}}} \langle n_3 \rangle \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1,\tau_1)\hat{u}_2(n_2,\tau_2) \right\|_{L^2_{\tau_1,L^1_{\tau_3}}} + \]
\[ \left\| \frac{\langle n_3 \rangle}{\langle \tau_3 - n_3 \rangle^{\frac{3}{2}}} \langle n_3 \rangle \int \sum \frac{m(n_1+n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1,\tau_1)\hat{u}_2(n_2,\tau_2) \right\|_{L^2_{\tau_1,L^1_{\tau_3}}} \lesssim \]
\[ \left\{ N^{-\frac{3}{2}+\delta^2} + N^{-\frac{2}{3}\delta^2} \right\} \|u_1\|_{X^{1,1/2}} \|u_2\|_{X^{1,1/2}}. \]

Note that $\sum \tau_j = 0$ and $\sum n_j = 0$. In particular, we obtain the dispersion relation 
\[ \tau_3 - n_3^2 + \tau_2 + n_2^2 + \tau_1 + n_1^2 = -n_3^2 + n_1^2 + n_2^2. \]

• $|n_1| \gtrsim N$, $|n_2| \ll |n_1|$. Denoting by $L_1 := |\tau_1 + n_1^2|$, $L_2 := |\tau_2 + n_2^2|$ and $L_3 := |\tau_3 - n_3^2|$, the dispersion relation says that in the present situation $L_{\max} := \max\{L_j\} \gtrsim N_3^2$. Since the multiplier is bounded by 
\[ \frac{|m(n_1+n_2) - m(n_1)m(n_2)|}{m(n_1)m(n_2)} \lesssim \left\{ \frac{\sum m(n_1)n_2}{m(n_1)} \right\} \lesssim \frac{N^2}{N_1^3}, \text{ if } |n_2| \leq N, \]
\[ \left( \frac{N^2}{N} \right)^{\frac{3}{2}}, \text{ if } |n_2| \geq N, \]
we deduce that
\[
\begin{align*}
\left\| \frac{\langle n_3 \rangle}{(\tau_3 - n_3^3)^{1/2}} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1, \tau_1) \hat{u}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3, \tau_3}} \\
\left\| \frac{\langle n_3 \rangle}{(\tau_3 - n_3^3)^{1/2}} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1, \tau_1) \hat{u}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3, \tau_3}} \\
\frac{N_3^2}{N_3^2} & \left\| \frac{\langle n_3 \rangle}{N_1 N_1} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| u_1 \right\|_{X_1,1/2} \left\| u_2 \right\|_{Y_1,1/2} \lesssim N^{-\frac{2}{3}} + \delta^\frac{1}{6} - N_{\max}^{0-} \left\| u_1 \right\|_{X_1,1/2} \left\| u_2 \right\|_{X_1,1/2}.
\end{align*}
\]

- \(|n_1| \sim |n_2| \gtrsim N_1, |n_3|^3 \gg |n_2|^2\). In the present case the multiplier is bounded by \(\left( \frac{N_3}{N} \right)^{1/2}\) and the dispersion relation says that \(L_{\max} \gtrsim N_3^3\). Thus,
\[
\begin{align*}
\left\| \frac{\langle n_3 \rangle}{(\tau_3 - n_3^3)^{1/2}} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1, \tau_1) \hat{u}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3, \tau_3}} \\
\left\| \frac{\langle n_3 \rangle}{(\tau_3 - n_3^3)^{1/2}} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1, \tau_1) \hat{u}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3, \tau_3}} \\
\frac{N_3^2}{N_3^2} \left( \frac{N_1}{N} \right)^{1/2} & \left\| \frac{\langle n_3 \rangle}{N_1 N_1} \right\|_{L^2_{\tau_3, \tau_3}} \left\| u_1 \right\|_{X_1,1/2} \left\| u_2 \right\|_{X_1,1/2} \lesssim N^{-\frac{2}{3}} + \delta^\frac{1}{6} - N_{\max}^{0-} \left\| u_1 \right\|_{X_1,1/2} \left\| u_2 \right\|_{X_1,1/2},
\end{align*}
\]

- \(|n_1| \sim |n_2| \gtrsim N_1, |n_3|^3 \lesssim |n_2|^2\). Here the dispersion relation does not give useful information about \(L_{\max}\). Since the multiplier is estimated by \(\left( \frac{N_3}{N} \right)^{1/2}\), we obtain the crude bound
\[
\begin{align*}
\left\| \frac{\langle n_3 \rangle}{(\tau_3 - n_3^3)^{1/2}} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1, \tau_1) \hat{u}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3, \tau_3}} \\
\left\| \frac{\langle n_3 \rangle}{(\tau_3 - n_3^3)^{1/2}} \right\|_{L^2_{\tau_3, \tau_3}} & \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \hat{u}_1(n_1, \tau_1) \hat{u}_2(n_2, \tau_2) \right\|_{L^2_{\tau_3, \tau_3}} \\
\frac{N_3^2}{N_3^2} \left( \frac{N_2}{N} \right)^{1/2} & \left\| \frac{\langle n_3 \rangle}{N_1 N_1} \right\|_{L^2_{\tau_3, \tau_3}} \left\| u_1 \right\|_{X_1,1/2} \left\| u_2 \right\|_{X_1,1/2} \lesssim N^{-\frac{2}{3}} + \delta^\frac{1}{6} - N_{\max}^{0-} \left\| u_1 \right\|_{X_1,1/2} \left\| u_2 \right\|_{X_1,1/2}.
\end{align*}
\]

Next, the desired bound related to \(\int_0^\delta E_7\) follows from
\[
\int_0^\delta \left\| \sum \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right\|_1 n_1 + n_2 \| \hat{u}_1(n_1, t) \hat{u}_2(n_2, t) \| \| \hat{u}_3(n_3, t) \| \lesssim N^{-1/2} \delta^{\frac{1}{6}} \| u_1 \|_{X_1,1/2} \| v_2 \|_{Y_1,1/2} \| u_3 \|_{X_1,1/2}
\]
\[ \int_0^\delta E_7 \lesssim \frac{1}{N^{1/2}} \int_0^\delta \sum |n_1 + n_2|^2 \hat{v}_2(n_2, t)|n_3| \hat{u}_3(n_3, t) \lesssim N^{-1} \delta^{\frac{10}{24}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}. \]

- \( |n_1| \ll |n_2| \gtrsim N \). The multiplier is \( \lesssim (|n_2|/N)^{1/2} \) so that

\[
\int_0^\delta E_7 \lesssim \frac{1}{N^{1/2}} \int_0^\delta \sum |n_1 + n_2|^2 \hat{v}_2(n_2, t)|n_3| \hat{u}_3(n_3, t) \lesssim N^{-1} \delta^{\frac{10}{24}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.
\]

- \( |n_1| \sim |n_2| \gtrsim N \). The multiplier is \( \lesssim |n_2|/N \). Hence,

\[
\int_0^\delta E_7 \lesssim N^{-1} \delta^{\frac{10}{24}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}}.
\]

- \( |n_1| \gtrsim N, |n_2| \leq N \). The multiplier is again \( \lesssim N_2/N \), so that it can be estimated as above.

Now we turn to the term \( \int_0^\delta E_8 \). The objective is to show that

\[
\int_0^\delta \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| |n_1 + n_2| \prod \hat{v}_j(n_j, t) \lesssim N^{-1+\delta^{\frac{1}{2}}} \prod \|u_j\|_{X^{1,1/2}}.
\]

- At least three frequencies are \( \gtrsim N \). We can assume \( |n_1| \geq |n_2| \). The multiplier is bounded by \( N_{\max}/N \) so that

\[
\int_0^\delta E_8 \lesssim \frac{N_{\max}}{N} \delta^{\frac{1}{2}} \prod \|u_j\|_{X^{1,1/2}} \lesssim N^{-2+\delta^{\frac{1}{2}}} \prod \|u_j\|_{X^{1,1/2}}.
\]

- Exactly two frequencies are \( \gtrsim N \). Without loss of generality, we suppose \( |n_1| \sim |n_2| \gtrsim N \) and \( |n_3|, |n_4| \ll N \). Since the multiplier satisfies

\[
\left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \lesssim \left( \frac{N_{\max}}{N} \right)^{1-}
\]

we get the bound

\[
\int_0^\delta E_8 \lesssim \left( \frac{N_{\max}}{N} \right)^{1-} \delta^{\frac{1}{2}} \prod \|u_j\|_{X^{1,1/2}} \lesssim N^{-1+\delta^{\frac{1}{2}}} \prod \|u_j\|_{X^{1,1/2}}.
\]

The contribution of \( \int_0^\delta E_9 \) is estimated if we prove that

\[
\int_0^\delta \left| \frac{m(n_1 + n_2) - m(n_1)m(n_2)}{m(n_1)m(n_2)} \right| \hat{u}_1(n_1, t) \hat{v}_2(n_2, t) \hat{u}_3(n_3, t) \hat{v}_4(n_4, t) \lesssim N^{-2+\delta^{\frac{1}{2}}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}.
\]
This follows since at least two frequencies are $\gtrsim N$ and the multiplier is always bounded by $(N_{\max}/N)^{1-}$, so that

$$\int_0^\delta E_0 \lesssim \left( \frac{N_{\max}}{N} \right)^{1-} \|u_1\|_{L^4} \|v_2\|_{L^4} \|u_3\|_{L^4} \|v_4\|_{L^4} \lesssim$$

$$(N_{\max})^{1-} \frac{\delta^{1/2} + \delta^{1/3}}{N_1 N_2 N_3 N_4} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}} \lesssim$$

$$N^{-2+\delta^{1/2}} \|u_1\|_{X^{1,1/2}} \|v_2\|_{Y^{1,1/2}} \|u_3\|_{X^{1,1/2}} \|v_4\|_{Y^{1,1/2}}.$$

Now, we treat the term $\int_0^\delta E_{10}$. It is sufficient to prove

$$\int_0^\delta \sum \frac{|m(n_4 + n_5 + n_6) - m(n_4)m(n_5)m(n_6)|}{m(n_4)m(n_5)m(n_6)} \prod_{j=1}^6 \tilde{u}_j(n_j, t) \lesssim$$

$$N^{-2+\delta^{1/2}} \prod_{j=1}^6 \|u_j\|_{X^1} \quad (4.28)$$

However, this follows easily from the facts that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are $\gtrsim N$, say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$, the Strichartz bound $X^{0,3/8} \subset L^4$ and the inclusion $X^{1/2+} \subset L^{27}_x$. Indeed, if we combine these informations, it is not hard to get

$$\int_0^\delta E_{10} \lesssim \left( \frac{N_{\max}}{N} \right)^{3/2} \frac{1}{N_{i_1} N_{i_2} N_{i_3} N_{i_4}} \delta^{1/2} \frac{1}{(N_{i_5} N_{i_6})^{1/2}} \prod_{j=1}^6 \|u_j\|_{X^1} \lesssim N^{-2+\delta^{1/2}} N_{\max}^{1-} \prod_{j=1}^6 \|u_j\|_{X^1}$$

For the expression $\int_0^\delta E_{11}$, we use again that the multiplier is bounded by $(N_{\max}/N)^{3/2}$, at least two frequencies are $\gtrsim N$ (say $|n_{i_1}| \geq |n_{i_2}| \gtrsim N$), the Strichartz bounds in lemma 2.1 and the inclusions $X^{1/2+}, Y^{1/2+} \subset L^{27}_x$ to obtain

$$\int_0^\delta \sum \frac{|m(n_1 + n_2 + n_3) - m(n_1)m(n_2)m(n_3)|}{m(n_1)m(n_2)m(n_3)} \prod_{j=1}^4 \tilde{u}_j(n_j, t) \tilde{v}_5(n_5, t) \lesssim$$

$$(N_{\max})^{3/2} \frac{1}{N_{i_1} N_{i_2} N_{i_3} N_{i_4}} \delta^{1/2} \frac{1}{N_{i_5}^{1/2}} \prod_{j=1}^4 \|u_j\|_{X^1} \|v_5\|_{Y^1} \lesssim$$

$$N^{-2+\delta^{1/2}} \prod_{j=1}^4 \|u_j\|_{X^1} \|v_5\|_{Y^1} \quad (4.29)$$

The analysis of $\int_0^\delta E_{12}$ is similar to the $\int_0^\delta E_{11}$. This completes the proof of the proposition 4.12.

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1. This inclusion is an easy consequence of Sobolev embedding.
5. Global well-posedness below the energy space

In this section we combine the variant local well-posedness result in proposition 3.1 with the two almost conservation results in the propositions 4.1 and 4.2 to prove the theorem 1.1.

**Remark 5.1.** Note that the spatial mean \( \int_T v(t,x)dx \) is preserved during the evolution \( \{u(t), v(t)\} \). Thus, we can assume that the initial data \( v_0 \) has zero-mean, since otherwise we make the change \( w = v - \int_T v_0 dx \) at the expense of two harmless linear terms (namely, \( u \int_T v_0 dx \) and \( \partial_x v \int_T v_0 \)).

The definition of the I-operator implies that the initial data satisfies \( \|Iu_0\|_{H^1} + \|Iv_0\|_{H^1}^2 \lesssim N^{2(1-s)} \) and \( \|Iu_0\|_{L^2}^2 + \|Iv_0\|_{L^2}^2 \lesssim 1 \). By the estimates (4.1) and (4.8), we get that \( |L(Iu_0, Iv_0)| \lesssim N^{1-s} \) and \( |E(Iu_0, Iv_0)| \lesssim N^{2(1-s)} \).

Also, any bound for \( L(Iu, Iv) \) and \( E(Iu, Iv) \) of the form \( |L(Iu, Iv)| \lesssim N^{1-s} \) and \( |E(Iu, Iv)| \lesssim N^{2(1-s)} \) implies that \( \|Iu\|_{L^2}^2 \lesssim M, \|Iv\|_{L^2}^2 \lesssim N^{1-s} \) and \( \|Iu\|_{H^1}^2 + \|Iv\|_{H^1}^2 \lesssim N^{2(1-s)} \).

Given a time \( T \), if we can uniformly bound the \( H^1 \)-norms of the solution at times \( t = \delta, t = 2\delta \), etc., the local existence result in proposition 3.1 says that the solution can be extended up to any time interval where such a uniform bound holds. On the other hand, given a time \( T \), if we can interact \( T^{-1} \) times the local existence result, the solution exists in the time interval \([0, T]\). So, in view of the propositions 4.1 and 4.2, it suffices to show

\[
(N^{-1+\delta^{12}\delta^{-3}N^{3(1-s)}} + N^{-2+\delta^{12}N^4(1-s)})T\delta^{-1} \lesssim N^{1-s}
\]

and

\[
\left\{ (N^{-1+\delta^{12}\delta^{-3}N^{3(1-s)}} + N^{-2+\delta^{12}N^4(1-s)})N^{3(1-s)} + N^{-1+\delta^{12}N^4(1-s)} + N^{-2+\delta^{12}N^6(1-s)} \right\} \frac{T}{\delta} \lesssim N^{2(1-s)}
\]

At this point, we recall that the proposition 3.1 says that \( \delta \sim N^{-\frac{16}{17}(1-s)} \) if \( \beta \neq 0 \) and \( \delta \sim N^{-\frac{8}{17}(1-s)} \) if \( \beta = 0 \). Hence,

- \( \beta \neq 0 \). The condition 5.1 holds for

\[
-1 + \frac{5}{24}\frac{16}{3}(1-s) + 3(1-s) < (1-s), \text{ i.e. } s > 19/28
\]

and

\[
-2 + \frac{1}{2}\frac{16}{3}(1-s) + 4(1-s) < (1-s), \text{ i.e. } s > 11/17;
\]

Similarly, the condition 5.2 is satisfied if

\[
-1 + \frac{5}{6}\frac{16}{3}(1-s) + 3(1-s) < 2(1-s), \text{ i.e. } s > 40/49;
\]

\[
-2 + \frac{5}{6}\frac{16}{3}(1-s) + 3(1-s) < 2(1-s), \text{ i.e. } s > 11/13;
\]

\[
-3 + \frac{7}{8}\frac{16}{3}(1-s) + 3(1-s) < 2(1-s), \text{ i.e. } s > 25/34;
\]
\[-1 + \frac{116}{2^3} (1 - s) + 4(1 - s) < 2(1 - s), \text{ i.e. } s > 11/14\]

and

\[-2 + \frac{116}{2^3} (1 - s) + 6(1 - s) < 2(1 - s), \text{ i.e. } s > 7/10.\]

Thus, we conclude that the non-resonant NLS-KdV system is globally well-posed for any $s > 11/13$.

- $\beta = 0$. The condition (5.1) is fulfilled when

\[-1 + \frac{5}{24} 8(1 - s) + 3(1 - s) < (1 - s), \text{ i.e. } s > 8/11\]

and

\[-2 + \frac{1}{2} 8(1 - s) + 4(1 - s) < (1 - s), \text{ i.e. } s > 5/7;\]

Similarly, the condition (5.2) is verified for

\[-1 + \frac{5}{6} 8(1 - s) + 3(1 - s) < 2(1 - s), \text{ i.e. } s > 20/23;\]

\[-\frac{2}{3} + \frac{5}{6} 8(1 - s) + 3(1 - s) < 2(1 - s), \text{ i.e. } s > 8/9;\]

\[-\frac{3}{2} + \frac{7}{8} 8(1 - s) + 3(1 - s) < 2(1 - s), \text{ i.e. } s > 13/16;\]

\[-1 + \frac{1}{2} 8(1 - s) + 4(1 - s) < 2(1 - s), \text{ i.e. } s > 5/6\]

and

\[-2 + \frac{1}{2} 8(1 - s) + 6(1 - s) < 2(1 - s), \text{ i.e. } s > 3/4.\]

Hence, we obtain that the resonant NLS-KdV system is globally well-posed for any $s > 8/9$.

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