On the Migdal-Watson approach to FSI effects in meson production in $NN$ collisions

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Abstract

The influence of the nucleon-nucleon final state interaction (FSI) on properties of the meson production amplitude near threshold is discussed. For the nucleon-nucleon interaction a simple Yamaguchi potential as well as realistic potential models are considered. It is shown that FSI effects cannot be factorized from the production amplitude. The absolute magnitude of FSI effects depends on the momentum transfer (or on the mass of the produced meson) and hence is not universal. Only in the case of the production of rather heavy mesons like $\eta'$ or $\phi$ FSI effects become universal. The Jost function approach to FSI effects is critically examined.

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I. INTRODUCTION

Already in the 1950’s K. Watson [1] and A. Migdal [2] have shown that the energy dependence for meson production reactions $NN \rightarrow NNx$ near threshold is predominantly determined by the strong $NN$ interaction in the final state. Their arguments have been used for justifying a rather simple treatment of effects from the final state interaction (FSI) (see, e.g., Refs. [3–6]). It consists in simply multiplying the basic production amplitude with the on-shell $NN$ $T$-matrix, i.e.

$$M = -N A_{\text{prod}}^m \frac{e^{i\delta} \sin \delta}{ka_{NN}},$$

where $\delta = \delta(k)$ is the $NN$ phase shift, $a_{NN}$ the $NN$ scattering length, $A_{\text{prod}}^m$ the on-shell meson production amplitude and $N$ a normalization factor. This expression suggests that the FSI effect is universal, i.e. does not depend on the specific meson emitted.

Recently, some aspects of FSI effects in the reaction $NN \rightarrow NNx$ were investigated by Hanhart and Nakayama [7] and Niskanen [8]. In particular, these authors pointed out that the evaluation of the total reaction amplitude by just multiplying the production amplitude by the on-shell $NN$ $T$-matrix is not acceptable for obtaining quantitative predictions. In the present paper we want to study those FSI effects in more detail. Specifically, we want to shed some light on the validity of the multiplication prescription Eq. (1). We examine the influence of the $NN$ FSI on the absolute value of the reaction amplitude by employing realistic models of the $NN$ interaction. Furthermore we investigate the dependence of the FSI effects on the mass of the produced meson. For that purpose we will vary the mass of $x$ and adopt values corresponding to those of the $\pi$, $\eta$, and $\eta'$ mesons.

In general the total amplitude for the reaction $pp \rightarrow ppx$ can be determined from the DWBA expression

$$M = A_{\text{prod}}^m + A_{\text{prod}}^{\text{off}} G_0 T_{NN},$$

where the term on the very right side implies an integration over the off-shell production amplitude and the off-shell $NN$ $T$-matrix. Eq. (2) corresponds to the sum of the two diagrams shown in Fig. 1. Meson production in $NN$ collisions requires a large momentum transfer between the initial and final nucleons which is typically of the order of $\sqrt{mm_x}$, where $m$ is the nucleon mass and $m_x$ the mass of produced meson. Thus the range of the production interaction will be much smaller than the characteristic range of the $NN$ interaction in the final state. Goldberger and Watson argued that in such a case the meson can be considered to be produced practically from a point like region so that the production amplitude can be factored out of the integral [9], i.e.

$$M = A_{\text{prod}}^m + A_{\text{prod}}^{\text{off}} G_0 T_{NN} \approx A_{\text{prod}}^m [1 + G_0 T_{NN}] = A_{\text{prod}}^m \psi_k^{(-)*}(0).$$

Here $\psi_k^{(-)}(\vec{r})$ is the (suitably normalized) $NN$ wave function in the continuum [9], where $\psi_k^{(-)}(0)$ is related to the Jost function $J$ via $\psi_k^{(-)*}(0) = J^{-1}(-k)$ [9].

Clearly also in this case one arrives at results where the FSI effects are reduced to a mere multiplicative factor $|\psi_k^{(-)*}(0)|^2$ (commonly referred to as enhancement factor).
The prescription Eq. (3) has been utilized by several authors \[10–14\] in their studies of meson production. Its validity has been examined by an explicit calculation of the loop diagram in Ref. [10] employing an OBE model for the production amplitude. However, one has to keep in mind that this investigation is based on a simple separable Yamaguchi potential for the \(NN\) FSI. It is well-known that the off-shell behavior of the \(T\)-matrix for the Yamaguchi potential is rather different from the one resulting from realistic models of the \(NN\) force. This can be seen from Fig. 2 where we compare the off-shell properties of the Paris [15] and (one version of) the Bonn [16] \(NN\) models with the one of the Yamaguchi potential for the \(^1S_0\) partial wave. The most striking difference is definitely the zero crossing of the \(T\)-matrix that occurs for realistic potential models at off-shell momenta \(q \approx 350\) MeV/c whereas the one of the Yamaguchi potential never changes sign. As we will show below, this specific feature has a strong and important influence on the result for the FSI effects.

The paper is structured in the following way. In Sect. 2 we present our formalism. We specify the meson production amplitude that we use in the present investigation and we give the explicit expression for the loop diagram of Fig. 1b. In Sect. 3 we present and discuss our results. Specifically, we show calculations for the effects of the FSI considering different \(NN\) models and the production of mesons with different masses. Furthermore we take a look at the energy dependence of the FSI effects and examine the validity of some commonly used approximations. The paper ends with a short summary.

II. LOOP DIAGRAM CALCULUS

For the calculation of the loop diagram of Fig. 1b with off-shell amplitudes of realistic \(NN\) interactions we need to specify a model for the production amplitude. We assume that it has the form

\[
A_{\text{prod}} = \frac{g \cdot A_{\mu N \rightarrow xN}}{t - m_{\mu}^2}, \tag{4}
\]

which corresponds to the exchange of a scalar meson \(\mu\) of mass \(m_{\mu}\) in the \(t\)-channel followed by the production of a meson \(x\) in a rescattering process. The corresponding diagram is shown in Fig. 3a. The coupling \(g\) at the \(NN\mu\) vertex and the amplitude \(A_{\mu N \rightarrow xN}\) are assumed to be constants. For \(m_{\mu}\) we take the value of the pion mass, i.e. \(m_{\mu} = 135\) MeV. Furthermore, for simplicity reasons, we use non-relativistic kinematics for the intermediate nucleons. The total reaction amplitude for this production model is then given by the sum of the two diagrams of Fig. 3b, i.e.

\[
\mathcal{M} = -\frac{mg}{E} \frac{A_{\mu N \rightarrow xN}}{[\overline{k} - \overline{P}/2 + \frac{m_{\mu}E}{E}\overline{\tau}]^2 + \lambda^2} \Psi(\overline{k}), \tag{5}
\]

where \(E = \sqrt{m^2 + p^2}\), \(\lambda^2 = \frac{m_{\mu}^2}{E}m_{\mu}^2 + \frac{m_{\mu}^2}{E}\tau^2\), with \(\tau = E - m\). \(\Psi(\overline{k})\) is given by the expression

\[
\Psi(\overline{k}) = 1 - \frac{m\pi[\overline{k} - \overline{P}/2 + \frac{m_{\mu}E}{E}\overline{\tau}]^2 + \lambda^2}{\int_0^\infty dq \ln\left[\frac{(q + r)^2 + \lambda^2}{(q - r)^2 + \lambda^2}\right] T_{NN}(q, k) ln[(q + r)^2 + \lambda^2],} \tag{6}
\]
where $r = |\frac{m}{E}\vec{p} - \vec{P}/2|$. $T_{NN}(q,k)$ is the NN half-off-shell T-matrix in the $^1S_0$ partial wave. The function $F_{NN}(k) = |\Psi(k)|^2$ can be considered as a generalization of the FSI enhancement factor $|\psi(-)^*\kappa(0)|^2$ that follows from the factorization assumption Eq. (3). We would like to emphasize, however, that (contrary to $\psi(-)^*\kappa(0)$ in Eq. (3)) $\Psi(k)$ does contain also information on the production mechanism and not only on the NN FSI.

In the actual calculations we want to include the Coulomb interaction between the outgoing protons. Therefore we have to replace the $NN$ half-off-shell $T$-matrix in Eq. (6) by the quantity $T_{cs}^{ee}NN$, i.e. the Coulomb-distorted hadronic $T$-matrix. This quantity is obtained by the prescription introduced in Ref. [17], namely via

$$T_{NN}(q,k) = C(\gamma_q) T_{NN}(q,k) T_{NN}(k,k) T_{cs}^{ee}NN(k,k),$$

where $k$ and $q$ denote the on-shell and off-shell momentum, respectively. $T_{NN}(k,k)$ and $T_{NN}(q,k)$ are the on-shell and half-off-shell $T$-matrices for the strong interaction alone. The Coulomb penetration factor $C$ is given by

$$C^2(\gamma_q) = \frac{2\pi\gamma_k}{e^{2\pi\gamma_k} - 1}; \quad \gamma_k = \frac{m}{\alpha k},$$

with $\alpha$ the fine-structure constant. Furthermore, the first term on the l.h.s. of Eq. (8) (the “1”) has to be replaced by $C(\gamma_k)$.

### III. DISCUSSION

First we want to discuss the dependence of $\Psi(k)$ on the mass of the produced meson. For that purpose we start out from a somewhat simpler expression for $\Psi$ which follows from Eq. (3) for the kinematics at the production threshold:

$$\Psi(k) = C(\gamma_k) - \frac{m\pi[mm_x + m^2_\mu]}{\sqrt{mm_x + m^2_x/4}} \int_0^{\infty} dq \frac{T_{NN}^{ee}(q,k)}{[q^2 - k^2 - i0]} \ln \left( \frac{(q + \tilde{r})^2 + \bar{\lambda}^2}{(q - \tilde{r})^2 + \lambda^2} \right),$$

where

$$\tilde{r} = \frac{m}{m + m_x/2} \sqrt{mm_x + m^2_x/4}, \quad \bar{\lambda}^2 = \frac{m^2}{(m + m_x/2)^2} + \frac{m}{m + m_x/2} m^2_\mu.$$

For the production of a light meson, $m_x \ll m$, we get $\tilde{r} \approx \sqrt{mm_x}$, $\bar{\lambda}^2 = m^2_\mu + m^2_x/4$, so that there is a dependence of the integral on the r.h.s. of Eq. (8) on $m_x$. In the case of a heavy meson, $m_x \gg m$, it follows that $\tilde{r} \approx m$, $\bar{\lambda}^2 \approx m^2$ and consequently $\Psi(k)$ does not depend on the mass of the emitted meson $x$. In other words, we expect that FSI effects are getting universal for the production of heavy mesons via an OBE-type production mechanism Eq. (4).

Let us now come to the results for the FSI factor $F_{NN} = |\Psi(k)|^2$. In Fig. 4 we show calculations for different $NN$ models and for some typical masses of the emitted meson $x$. It can be seen from those figures that the magnitudes of $F_{NN}$ resulting for the Bonn and the
Paris potentials are fairly similar whereas the one for the separable Yamaguchi potential is quite different. (Note that different scales are used for each $NN$ model!) This result can be understood qualitatively from the features of the corresponding off-shell T-matrices shown in Fig. 2. The T-matrices for the Bonn and Paris potentials are very similar. In particular, for both models there is a change of sign at an off-shell momentum of $q \approx 350$ MeV/c. Because of this change of sign cancellations occur in the integral for $\Psi(k)$ (cf. Eq. 3). The off-shell T-matrix of the Yamaguchi potential does not change sign. Therefore, no such cancellations take place in the integration and as a consequence the FSI factor $F_{NN}$ is significantly larger than the ones for the realistic interaction models, cf. Fig. 4.

There is also a striking difference in the results with regard to the mass of the emitted meson. For the Paris and Bonn potentials the FSI factor decreases with increasing mass of the produced meson. However, for the Yamaguchi potential we observe the opposite effect. Here $F_{NN}(k)$ becomes larger with the mass of the produced meson. These features can again be understood in terms of the $NN$ off-shell properties. However, now the off-shell behavior of the production amplitude, that enters into the integral ($\Psi$) as well, becomes also relevant. With increasing mass of the produced meson the required momentum transfer $t$ increases as well and, accordingly, the production mechanism gets more and more short-ranged. As a consequence the production amplitude remains a constant over a larger (off-shell) momentum range, as can be seen in Fig. 3. This feature enhances the cancellation effects for the Bonn and Paris $NN$ T-matrices, discussed above, and therefore leads to a reduction of $F_{NN}$ for larger meson masses. In case of the Yamaguchi potential no such cancellations can occur and therefore the FSI factor turns out to be almost independent of the mass of the produced meson.

Nevertheless, we see that also for realistic $NN$ potentials the FSI factors become more and more similar with increasing mass of the produced meson, i.e. for high momentum transfers. This is expected. It simply reflects the universality of FSI effects for the production of heavy mesons as discussed above. We would like to emphasize that the universality of FSI effects at large $t$ should set in not only for the particular production amplitude used in the present investigation (cf. Eq. (4)), but is expected to occur in general. Actually, we examined the behavior of $F_{NN}$ for the OBE-type production amplitude Eq. (4) with inclusion of form factors of monopole and dipole type at the $NN\mu$ vertex. Corresponding numerical calculations clearly indicate that the qualitative behaviour of the FSI factors remains basically unchanged.

However, it is important to realize that the actual values for the FSI factors do, of course, depend on the particular production amplitude. Thus, the results presented in Figs. 3 are by not means absolute predictions that can be taken from this paper and used blindly for FSI corrections in any other study of meson production. Rather our results demonstrate that FSI effects have to be calculated always explicitly, utilizing the respective production amplitudes and a proper $NN$ off-shell T-matrix, if one wants to obtain reliable quantitative predictions. (In this context we also would like to draw attention to the requirement of a consistent treatment of both the $NN$ scattering and production amplitudes as discussed in the Appendix of Ref. [18].) Specifically, this means that the apparent universality of the FSI effects for large meson masses does not imply that one can use the prescription applied in the studies [10,12,14], i.e. take the production amplitude out of the integral in Eq. (9). Even though the factor $\Psi(k)$ becomes independent of the mass of the produced meson its
actual magnitude is still determined by the off-shell properties of the NN T-matrix as well as by the production amplitude. In order to demonstrate this we show also results based on the factorization assumption Eq. (3) (dash-dotted curves). In this case the FSI factor is simply given by the enhancement factor $F_{NN}(k) = |J(k)|^{-2}$. It is really startling how strongly the results for the Yamaguchi potential and for realistic NN interaction models differ. For the former potential the enhancement factor based on the Jost function is larger than the FSI factor obtained from Eq. (2) whereas for the latter models it turns out to be much smaller than the DWBA values. Clearly, these results suggest that it is rather questionable to use the Jost function of some arbitrary potentials for the evaluation of FSI effects in meson-production reactions [10,12–14].

Finally a remark on the differences between the results for realistic NN models. Obviously these are small (about 20%) for the pion-production case. But for the $\eta'$ meson the Paris result is almost a factor 2 larger then the one for the Bonn model. This is not too surprising because for the production of heavier mesons a larger momentum transfer is required and therefore the features of the NN interaction at shorter distances (or larger off-shell momenta) become more and more important in the actual calculations. As we can see in Fig. 2 there are fairly large differences in the off-shell properties of these two models for large off-shell momenta. Note that such a sensitivity to the off-shell behaviour of realistic NN models at large off-shell momenta has been also seen, e.g., in proton-proton bremsstrahlung producing hard photons [19].

By all the variations we see in the magnitudes of the FSI factors presented in Fig. 4 we would like to point out that their energy dependence is basically the same for all the different potentials and for the different meson masses. If we normalize them to the same value for small $k$ (e. g., at the peak of $F_{NN}$ at $k \approx 20$ MeV/c) all curves would lie essentially on top of each other. This means that the energy dependence of the FSI factors is really primarily determined by the on-shell NN T-matrix. Consequently, the on-shell prescription Eq. (2) is indeed a fairly good approximation, at least for energies near the production threshold. In order to demonstrate this let us compare one of the curves based on the Paris potential with the result corresponding to Eq. (2) (normalized to the Paris curve at $k \approx 20$ MeV/c), cf. Fig. 6. None-the-less, we do observe an increasing difference between these two curves for $k \geq 50$ MeV/c, which corresponds to excess (cms) energies $Q \geq 3$ MeV. For $k = 100$ MeV/c ($Q \approx 10$ MeV) the curves differ already by a factor of around 2. It is interesting to see that the Jost function approach (Eq. (3)) deviates even more strongly from the correct results than the on-shell approximation in the energy range $k \leq 100$ MeV/c.

Let us now investigate the origin of those discrepancies in more detail. For that purpose we re-write the amplitude $\mathcal{M}$ Eq. (2) in the form given in Ref. [7]

$$\mathcal{M} = -A_{prod}^{on} e^{ik} \sin \delta \cdot \left[ P(k) - a_{pp} k \cot \delta \right]$$

$$= -A_{prod}^{on} e^{ik} \sin \delta \cdot \left[ P(k) + 1 - 1/2a_{pp}r_0k^2 + O(k^4) + ... \right]$$

(10)

Here $P(k)$ is proportional to the principal value of the loop integral (cf. Fig. 3b) and contains the information on the off-shell behavior of both $A_{prod}$ and $T_{NN}$. (Note that we have neglected corrections coming from the Coulomb interaction in Eq. (10) for simplicity
reasons. Those terms don’t play a role anymore at the energies were the discrepancies discussed above occur.)

Evidently corrections to the simple on-shell prescription Eq. (1) come from the energy dependence of the function $P(k)$ as well as from the cot $\delta$ term. Actual calculations with the $NN$ potentials utilized in the present study revealed that the value of $P(k)$ at $k = 0$ is positive and about 3 to 5 units large which makes it to be the dominant piece of the terms in the bracket of Eq. (10). Furthermore, $P(k)$ is slowly decreasing with $k$. The $k^2$ term is slowly increasing with $k$ (Note that $a_{pp}$ is negative for the $1S_0$ partial wave!) so that there is a compensation in the energy dependence of the terms in the brackets of Eq. (10). This circumstance is certainly partly responsible for the fact that Eq. (1) works relatively well.

As already mentioned above, the value at $P(k = 0)$ is positive and fairly large (cf. also the comments in Ref. [7]). Note that in order to get Eq. (1) with the normalization $N$ set to one, as chosen in Refs. [5,6], we have to assume that $P(k) \equiv 0$ and omit all the terms proportional to $k^2$, $k^4$, etc., in the square brackets on the r.h.s. of Eq. (10). Therefore, this particular normalization can be only obtained under very specific conditions, cf. the discussion in the Appendix of Ref. [18].

IV. SUMMARY

In the present paper we have studied some aspects of effects from the final state interaction in the meson-production reaction $NN \rightarrow NNx$ near threshold. Specifically, we have demonstrated that the nucleon-nucleon FSI cannot be factorized from the production amplitude if one wants to obtain reliable quantitative predictions. This conclusion confirms the arguments given in the paper [7]. Furthermore, we have demonstrated that the absolute value of the FSI factor depends on the momentum transfer, i.e. on the mass of the produced meson. It is not universal! Only for large momentum transfers, i.e. for the production of heavy mesons, the FSI factor is getting independent of the mass of the produced meson. Finally, we have shown that the use of the Jost function of some arbitrary potentials for the evaluation of FSI effects is rather questionable and may lead to a considerable over-estimation of those FSI effects.

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REFERENCES

[1] K. M. Watson, Phys. Rev. 88, 1163 (1952).
[2] A. B. Migdal, Sov. Phys. JETP 1, 2 (1955).
[3] A. Moalem, E. Gedalin, L. Razdolskaja, and Z. Shorer, Nucl. Phys. A589, 649 (1995).
[4] E. Gedalin, A. Moalem, and L. Razdolskaja, Nucl. Phys. A634, 368 (1998).
[5] V. Bernard, N. Kaiser, and Ulf-G. Meißner, Eur. Phys. J., A 4, 259 (1999).
[6] N. Kaiser, Phys. Rev. C 60, 057001 (1999).
[7] C. Hanhart and K. Nakayama, Phys. Lett. B 454, 176 (1999).
[8] J. A. Niskanen, Phys. Lett. B 456, 107 (1999).
[9] M. L. Goldberger and K. M. Watson, Collision Theory (Wiley, New York 1964), chapter 9.3.
[10] B. L. Druzhinin, A. E. Kudryavtsev, and V. E. Tarasov, Z. Phys. A359, 205 (1997).
[11] G. Fäl dt and C. Wilkin, Phys. Lett. B 382, 209 (1996).
[12] R. Shyam and U. Mosel, Phys. Lett. B 426, 1 (1998).
[13] A. Sibirtsev and W. Cassing, nucl-th/9904046, Eur. Phys. J. A 2, 333 (1998).
[14] A. I. Titov, B. Kämpfer, and B. L. Reznik, Eur. Phys. J. A 7, 543 (2000).
[15] M. Lacombe et al., Phys. Rev. C 21, 861 (1980).
[16] J. Haidenbauer, K. Holinde, and M.B. Johnson, Phys. Rev. C 48, 2190 (1993).
[17] C. Hanhart, J. Haidenbauer, A. Reuber, C. Schütz, and J. Speth, Phys. Lett. B 358, 21 (1995).
[18] Cf. the Appendix of C. Hanhart and K. Nakayama, nucl-th/9809059.
[19] V. Hermann, K. Nakayama, O. Scholten, and H. Arellano, Nucl. Phys. A582, 568 (1995).
FIG. 1. Diagrammatic representation of the DWBA expression Eq. (2). $A$ is the elementary meson-production amplitude and $T$ the $NN$ T-matrix.

FIG. 2. Real part of the $NN\,^1S_0$ T-matrix as a function of the off-shell momentum $q$ calculated at the fixed on-shell momentum $k = 10$ MeV/c. The solid, long-dashed, and short-dashed lines are the results for the Paris \cite{15}, Bonn \cite{16}, and Yamaguchi potentials, respectively.
FIG. 3. Contributions to the total reaction amplitude $\mathcal{M}$: (a) Born term $A$; (b) loop diagram including the final state interaction.
FIG. 4. The FSI factor $F_{pp} = | \Psi(k) |^2$ (cf. Eq. (4)) for the Paris (a), Bonn (b), and the Yamaguchi (c) potentials and the production of the $\pi$ (solid curve), $\eta$ (dotted curve), and $\eta'$ (dashed curve) mesons. The dash-dotted lines are the results based on the factorization assumption Eq. (3), i.e. $F_{pp}(k) = | J(k) |^{-2}$. Note that different scales are used for each $NN$ model!
FIG. 5. Ratio $A_{\text{off}}/A_{\text{on}}$ of the production amplitude as a function of the off-shell momentum $q$ calculated at the fixed on-shell momentum $k = 0$ MeV/c. The solid, dotted and dashed curves correspond to the production of the $\pi$, $\eta$, and $\eta'$ mesons, respectively.
FIG. 6. The FSI factor $F_{pp} = |\Psi(k)|^2$ for the Paris $NN$ potential and for pion production (solid line) in comparison to results based on the approximations Eqs. (1) (dashed line) and (3) (dotted line). The latter two curves are normalized to the one of the Paris potential at the peak ($k \approx 20$ MeV/c).