PROBABILISTIC LIMIT THEOREMS FOR THE COEFFICIENTS OF A CLASS OF ROOT-UNITARY POLYNOMIALS

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Abstract. Sequences of discrete random variables are studied whose probability generating functions have roots located on the unit circle in the complex plane. Bounds on the cumulants of all orders for a class of such random variables are established, leading to Berry-Esseen bounds, moderate deviation results, concentration inequalities and mod-Gaussian convergence. A variety of examples is discussed in detail that naturally fit into this context.

1. Introduction

Polynomials with non-negative coefficients are closely related to bounded \( \mathbb{N}_0 \)-valued random variables, where \( \mathbb{N}_0 := \{0, 1, 2, \ldots\} \), even by a one-to-one correspondence if the sum of the coefficients is assumed to be equal to 1. Indeed, the probability generating function of any bounded \( \mathbb{N}_0 \)-valued random variable is a polynomial with non-negative coefficients which sum up to 1, and any such polynomial may be interpreted as the probability generating function of a random variable taking finitely many values all of which are non-negative integers.

It is therefore natural to search for connections between the location of the zeros of the polynomials and the asymptotic distributional properties of the corresponding random variables. For instance, if the polynomials only have real roots and thus can be written as the product of linear factors, the respective random variables can be represented as sums of independent Bernoulli variables in distribution. As a consequence, a central limit theorem is satisfied if and only if the variance tends to infinity. For a survey of this by now classical approach for proving probabilistic limit theorems, which was originally introduced by L.H. Harper [15], we refer to the survey article of J. Pitman [27].

In this note, we are interested in strong probabilistic limit theorems for the coefficients of polynomials all of whose roots lie only on the unit circle in the complex plane. Polynomials of this kind have been called root-unitary in the literature, and we adapt this name in our work as well. H.-K. Hwang and V. Zacharovas [17] proved (among other results) that in typical cases, the limiting behaviour of sequences of the corresponding random variables \( X_N \) is already determined by the limit of the fourth moment of their standardizations \( X_N^* = (X_N - \mathbb{E}X_N) / \sqrt{\text{var}(X_N)} \). For example, they proved the remarkable fact that \( X_N^* \) converges to the standard Gaussian distribution if and only if \( \mathbb{E}(X_N^*)^4 \to 3 \). This phenomenon can be regarded as an instant of what is known as a fourth-moment phenomenon, which asserts the remarkable fact that a central limit theorem for a sequence of random variables...
is implied by the convergence of the fourth moment to 3, the fourth moment of a standard Gaussian distribution.

The arguments in [17] involve cumulants up to order four, but the authors also provide formulas for cumulants \( \kappa_j(X_N^*) \) of any order \( j \geq 1 \) for the random variables \( X_N^* \). It is therefore natural to relate their results to the approach put forward by L. Saulis and V.A. Statulevičius [30] (see also the recent survey [11]). Here, a cumulant bound of the form

\[
|\kappa_j(X_N^*)| \leq \frac{j!}{\Delta_N^{j-2}} \quad (j \geq 3),
\]

known as the Statulevičius condition, gives rise to a larger number of strong limit theorems involving the quantity \( \Delta_N \). In addition to central limit theorems, this includes Berry-Esseen bounds, moderate deviation principles and concentration inequalities. Moreover, the Statulevičius condition can be used to establish mod-Gaussian convergence, a fairly new concept with far-reaching probabilistic consequences, which was introduced only within the last decade. For details about the role of cumulant bounds in mod-Gaussian convergence we refer to [13, Chapter 5]. In particular, mod-Gaussian convergence leads to sharper versions of some of the bounds which can be derived by the methods established in [30]. We remark that for the coefficients of real-rooted polynomials, mod-Gaussian convergence is a consequence of [13, Theorem 8.1].

In this note, we pursue this line of research further and establish a Statulevičius condition for a class of root-unitary polynomials which is introduced in Section 2. It turns out that the polynomials we consider appear in many applications like inversions in classical random permutations or in more general Coxeter groups, Gaussian polynomials, \( q \)-Catalan numbers or descending plane partitions. A detailed discussion of our general result applied to these combinatorial random structures is the content of Section 3. The proofs of our main result can be found in Section 4 at the end of this paper.

2. Main results

Throughout this note, we shall consider real-valued polynomials with non-negative coefficients which admit a representation of the form

\[
P_N(z) = \prod_{1 \leq j \leq N} \frac{1 - z^{b_j}}{1 - z^{a_j}} \quad (z \in \mathbb{R})
\]

for some natural number \( N \) and exponents \( a_j, b_j \in \mathbb{N}, j = 1, \ldots, N \). We assume that \( b_j \geq a_j \) for all \( j = 1, \ldots, N \). The degree of \( P_N \) is denoted by \( n \) and is related to the parameter \( N \) by \( n = \sum_{1 \leq j \leq N} (b_j - a_j) \).

Polynomials of type (1) have been discussed in [17, Section 4], where they form an important subclass of root-unitary polynomials with many applications, and previously also in [8, Section 3] in the more specific context of \( q \)-Catalan numbers. We emphasize that we use a different notation as in [17], who index the polynomials \( [1] \) by the degree \( n \). However, since in all of our applications \( N \) is the parameter which naturally appears in the definition of the polynomials and which tends to infinity in our convergence results, we have altered the notation accordingly.
As already sketched in the introduction, the polynomials (1) give rise to \( \mathbb{N}_0 \)-valued random variables \( X_N \) defined by their probability generating functions:

\[
E(z^{X_N}) = \frac{P_N(z)}{P_N(1)} \quad (z \in \mathbb{R}).
\]

As opposed to the (probability) generating function (2), we refer to \( P_N(z) \) as the generating polynomial of \( X_N \). We note that \( P_N(1) = \prod_{1 \leq j \leq N} b_j / a_j \), which follows readily from the fact that \( (1 - z^k) = (1 - z)(1 + z + \ldots + z^{k-1}) \) for \( k \geq 1 \).

The cumulants \( \kappa_{m,N} := \kappa_m(X_N) \) of \( X_N \) for \( m \in \mathbb{N} \) play a crucial role in this note. In the framework of the present paper they can be defined as the coefficients in the following power series expansion of the moment generating function:

\[
E(e^{sX_N}) = \exp \left( \sum_{m=1}^{\infty} \frac{\kappa_{m,N}}{m!} s^m \right).
\]

For the polynomials (1) and any \( m \geq 1 \), \( \kappa_{m,N} \) has the representation

\[
\kappa_{m,N} = \frac{B_m}{m} \sum_{1 \leq j \leq N} (b_j^m - a_j^m),
\]

where \( B_m \) denotes the \( m \)-th Bernoulli number with \( B_1 = \frac{1}{2} \), see Section 4.1 for the proof. In particular, since \( B_{2m+1} = 0 \) for any \( m \geq 1 \), all cumulants of odd order \( \geq 3 \) vanish identically, and when establishing cumulant bounds it is therefore sufficient to consider the cumulants of even orders \( 2m \). In the following, we write \( \sigma_N^2 := \text{var}(X_N) \) for the variance of \( X_N \) and put

\[
X_N^* := \frac{X_N - \mathbb{E}X_N}{\sigma_N}.
\]

Our main result, whose proof will be given in Section 4.2, provides bounds for the cumulants of \( X_N^* \), which we denote by \( \kappa_{m,N}^* := \kappa_m(X_N^*) \) for brevity.

**Theorem 2.1.** Let \( P_N(z) \) be a polynomial as in (1) and \( X_N \) a random variable defined as in (2). Then, for \( m \geq 2 \), the \( 2m \)-th cumulant of \( X_N^* \) satisfies

\[
|\kappa_{2m,N}^*| \leq \frac{(2m)!}{\Delta_N^{m-2}}
\]

with

\[
\Delta_N = \sqrt{\frac{7}{30}} \pi^2 \min \left\{ \frac{\sigma_N}{M_N}, \frac{1}{\kappa_{4,N}^* |^{1/2}} \right\},
\]

where \( M_N = \max_{1 \leq j \leq N} b_j \).

Theorem 2.1 establishes the Statulevičius condition (4) for the class of random variables defined via (1). Note that (4) essentially relates all cumulants of higher order to the fourth cumulant and the quantity \( \sigma_N / M_N \). In this sense, our result mirrors and extends the work of [17], whose central limit theorems are solely based on controlling fourth moments (or cumulants) as discussed in the introduction.

The Statulevičius condition (4) opens the door to a variety of remarkable probabilistic implications, of which we will discuss a few exemplarily and refer to the monograph [30] as well as the survey article [11] for further details. First, recall that by [17, Theorem 4.1], for a sequence of random variables \( X_N \) as in (2), the normalized sequence \( (X_N^*)_{N \in \mathbb{N}} \) follows a central limit theorem if and only if \( \kappa_{4,N}^* \to 0 \).
as $N \to \infty$. Under slightly stronger assumptions, we get a Berry-Esseen bound for the speed of convergence in this central limit theorem.

**Corollary 2.2.** Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of random variables defined as in [2]. Then, for a standard Gaussian random variable $Z$ and $\Delta_N$ as in [5],

$$\sup_{x \in \mathbb{R}} |P(X_N^* \geq x) - P(Z \geq x)| \leq \frac{C}{\Delta_N},$$

where $C > 0$ is an absolute constant. In particular, if $M_N/\sigma_N \to 0$ and $\kappa_{4,N}^* \to 0$ as $N \to \infty$, the sequence $(X_N^*)_{N \in \mathbb{N}}$ satisfies a central limit theorem with speed of convergence as given in [6].

This immediately follows combining the cumulant bound [4] with [11] Theorem 2.4. According to [30] Corollary 2.1, the constant in (6) can be chosen as $C = 32\sqrt{2}$.

Compared to the central limit theorem [17] Theorem 4.1, we additionally require that $M_N/\sigma_N \to 0$ as $N \to \infty$. A similar quantity also appears in [16] Lemma 4, which yields Berry-Esseen type bounds for sequences of random variables whose probability generating functions factorize into polynomials with non-negative coefficients. In particular, assuming all the factors $(1 - z^{b_j})/(1 - z^{a_j})$ in (1) to be polynomials with non-negative coefficients, by [16] Lemma 4 the Berry-Esseen bound (3) holds with $\Delta_N = \sigma_N/M'_N$, where $M'_N := \max_{1 \leq j \leq N} (b_j - a_j)$. Note that in all examples discussed in Section 3, the asymptotic behaviour of $M_N$, $\kappa_{4,N}^*$ and $M'_N$ (whenever applicable) coincides. On the other hand, some of the examples we treat do not fall into the class of random variables considered in [16].

We would like to point out that M. Michelen and J. Sahasrabudhe [23, 24] proved central limit theorems for the coefficients of general polynomials together with Berry-Esseen bounds on the speed of convergence in terms of the variance and the quantity $\min_{\zeta} |1 - \zeta|$ with the minimum running over all roots. Non-Gaussian limits especially for sequences of integer-valued random variables which are related to certain classes of integer partitions have been studied in [32].

The cumulant bound [4] also leads to moderate deviation results, see [11] Theorem 3.1. To formulate them, for two sequences of real numbers $(a_N)_{N \in \mathbb{N}}$ and $(b_N)_{N \in \mathbb{N}}$ we write $a_N = o(b_N)$, provided that $a_N/b_N \to 0$ as $N \to \infty$.

**Corollary 2.3.** Let $(X_N)_{N \in \mathbb{N}}$ be a sequence of random variables where for each $N \geq 1$, $X_N$ is defined as in [2], and assume $M_N/\sigma_N \to 0$ and $\kappa_{4,N}^* \to 0$ as $N \to \infty$. Let $(a_N)_{N \in \mathbb{N}}$ be a sequence of real numbers with $a_N \to \infty$ and $a_N = o(\Delta_N)$, then

$$\lim_{N \to \infty} \frac{1}{a_N^2} \log P(X_N^*/a_N \in (x, \infty)) = -\frac{x^2}{2} \quad (x \geq 0).$$

In fact, even a full moderate deviation principle (MDP) is valid under the assumptions of Corollary 2.3, see [10] Theorem 1.1. Formally, the definition of an MDP is similar to that of a large deviation principle, but with the rescaling ranging between that of a law of large numbers and that of a distributional (typically central) limit theorem. For a detailed explanation see, for example, [9] Chapter 3.7. In our case, the sequence $(X_N^*/a_N)_{N \in \mathbb{N}}$ satisfies an MDP with speed $a_N^2$ and rate function $I(x) = x^2/2$, meaning that

$$\liminf_{N \to \infty} \frac{1}{a_N^2} \log P(X_N^*/a_N \in A) \geq -\inf_{x \in A} \frac{x^2}{2},$$
\[
\limsup_{N \to \infty} \frac{1}{a_N^2} \log P \left( X_N^*/a_N \in A \right) \leq - \inf_{x \in \bar{A}} \frac{x^2}{2}
\]
for any Borel set \( A \subset \mathbb{R} \), where \( A^o \) and \( \bar{A} \) denote the interior and the closure of \( A \), respectively. This includes (7) as a special case.

Theorem 2.1 also implies mod-Gaussian convergence for the sequence of random variables \( X_N \). We recall from [13, 22] that a sequence of random variables \( X_N \) converges in the mod-Gaussian sense with parameters \( t_N \to \infty \) and limiting function \( \Psi \), provided that
\[
\lim_{N \to \infty} E \left[ e^{zX_N} e^{-t_N \frac{z^2}{2}} \right] = \Psi(z)
\]
locally uniformly on \( C \), where \( \Psi \) is a non-degenerate analytic function. The next result establishes mod-Gaussian convergence of the sequence of random variables \( X_N \) under the assumption that the minimum in (5) is attained for the second term, possibly multiplied by some absolute constant (i.e., even if the first term might actually dominate it still has to be of the same order as the second one). In the examples we treat in Section 3 this assumption will always be satisfied.

Corollary 2.4. Let \( (X_N)_{N \in \mathbb{N}} \) be a sequence of random variables defined as in (2), and suppose that (4) holds with \( \Delta_N = \frac{\gamma}{|\kappa_{4,N}^2|^{1/2}} \) for some absolute constant \( \gamma > 0 \) and that \( \kappa_{4,N}^2 \to 0 \) as \( N \to \infty \). Then the sequence of random variables \( \sqrt{\Delta_N} X_N^* \) converges in the mod-Gaussian sense with parameters \( \Delta_N \) and limiting function \( e^{Lz^4} \) with \( L = \gamma^{-2} \).

Proof. This is a direct consequence of the result in Section 5.1 in [13]. Indeed, condition (5.1) there is implied by the Statulevičius condition (4) with \( C = 1 \), \( \alpha_N = \Delta_N^2 \) and \( \beta_N = \sigma_N / \Delta_N \). We also have \( v = 4 \), since the third cumulant of \( X_N \) vanishes. Moreover, condition (5.2) is satisfied with \( \sigma^2 = 1 \) and \( L = \gamma^{-2} \) by our assumption on \( \Delta_N \).

The last result directly gives further implications as Cramér-Petrov type large deviations and precise deviations (see [13, Chapter 5.2]). The latter means an estimate for probabilities of the type \( P \left( X_N^* \in (x, \infty) \right) \) on a non-logarithmic scale for suitable values of \( x \), which are allowed to depend on the parameter \( N \) and which involve the limiting function \( \Psi \). Since such results are rather technical to formulate, we refrain from presenting them. Instead, we mention the following concentration inequality, which is a consequence of [30, Lemma 2.4] and the corresponding corollary therein (by setting \( H = 2 \)). For a more detailed explanation, see the discussion around [11, Theorem 2.5].

Corollary 2.5. For \( N \geq 1 \), let \( X_N \) be a random variable as defined in (2). Then, there exists an absolute constant \( C > 0 \) such that for all \( x \geq 0 \),
\[
P (X_N^* \geq x) \leq C \exp \left( - \frac{1}{4} \min \left\{ x^2/2, x\Delta_N \right\} \right).
\]

3. Applications

Throughout the literature there exists a variety of examples of random variables whose generating functions are of the form (1). In the following, we discuss a few of them. In particular, we consider the number of inversions in classical random permutations and inversions in the classical Coxeter groups of type \( A_N \), \( B_N \) and
We also discuss Gaussian polynomials, (generalized) $q$-Catalan numbers and descending plane partitions.

In all these examples, we explicitly calculate the parameter $\Delta_N$ from (5), so that all the probabilistic consequences mentioned in Section 2 follow. In the case of Sections 3.1, 3.3, 3.4 and 3.5 this extends the central limit theorems shown in [17, Section 4], though typically there is a larger amount of literature and previous results which we discuss in each of the subsections.

For further examples, we refer to [17, Section 4]. Moreover, the online database FindStat [28] provides a variety of combinatorial statistics, of which the following have a generating polynomial in the form (1): the number of even inversions of a permutation [St000538], the number of even and odd descents in a permutation [St001115, St001114], the number of strict 3-descents of a permutation [St001520] and the number of inversions of the second or third entry of a permutation [St001557, St001556]. The generating functions of these statistics are all closely related to the polynomial studied in Section 3.1 and the related probabilistic limit theorems directly follow from the result presented there.

3.1. **Number of inversions in a classical random permutation.** Let $X_N$ be the number of inversions in a uniform random permutation on $N \geq 2$ elements. The generating polynomial of $X_N$ is of the form (1) with $b_j = j$ and $a_j = 1$, that is,

$$ P_N(z) = \prod_{1 \leq j \leq N} \frac{1 - z^j}{1 - z}, $$

see e.g. [29, Chapter 1.3.1] or [17, Section 4.2.1]. It is listed as [St000018] in [28], where it can also be found in a larger number of further situations such as [St000004], [St000156], [St000794]. In particular, the polynomial $P_N$ has degree $n = \sum_{j=1}^{N} (j - 1) = \binom{N}{2}$. Following (3), the $m$-th cumulant of the random variable $X_N$ with generating polynomial $P_N$ is given by

$$ \kappa_{m,N} = \frac{B_m}{m} \sum_{1 \leq j \leq N} (j^m - 1). $$

Straightforward calculations yield

$$ \sigma_N^2 = \kappa_{2,N} = \frac{1}{72} (2N^3 + 3N^2 - 5N), $$

$$ \kappa_{4,N} = -\frac{1}{3600} (6N^5 + 15N^4 + 10N^3 - 31N), $$

and we clearly have $M_N = \max_{1 \leq j \leq N} j = N$. This leads to

$$ \frac{\sigma_N}{M_N} = \frac{(2N^3 + 3N^2 - 5N)^{1/2}}{6\sqrt{2N}}. $$

It is not hard to check that $|\kappa_{4,N}|/\sigma_N^2 \leq N^2 = M_N^2$ for any $N \geq 2$ and since $|\kappa_{4,N}^*| = |\kappa_{4,N}|/\sigma_N$, we obtain

$$ \min \left\{ \frac{\sigma_N}{M_N}, \frac{1}{|\kappa_{4,N}^*|^{1/2}} \right\} = \frac{\sigma_N}{M_N}. $$

In conclusion, this leads to following bound for the parameter $\Delta_N$ from (5):

$$ \Delta_N = \pi^2 \sqrt{\frac{7}{30}} \frac{\sigma_N}{M_N} = \frac{\pi^2}{12} \sqrt{\frac{7}{15}} \frac{(2N^3 + 3N^2 - 5N)^{1/2}}{N} > \frac{1}{6} \sqrt{\frac{7}{30}} \pi^2 \sqrt{N}. $$
In particular, by Corollary 2.2 the random variables $X_N$ satisfy a central limit theorem with speed of convergence of order $N^{-1/2}$. Central limit theorems for the number of inversions in a uniform random permutation are very classical and well-known. We refer to [12, Section 5 (a)], [5, Example 5.5], [29, Theorem 3.1] or [20, Theorem 2]. A Berry-Esseen bound of order $N^{-1/2}$ has been shown in [16, Example 5]. In addition, our approach also yields a moderate deviation principle (Corollary 2.3), a concentration inequality (Corollary 2.5) as well as mod-Gaussian convergence (Corollary 2.4). Figure 1 shows the number of inversions in a sample of 350,000 random permutations of $N = 50$ elements.

3.2. General Coxeter Groups of Types $A_N$, $B_N$ and $D_N$. Inversions in a random permutation as discussed in the previous section are a special type of the more general theory of inversions in finite Coxeter groups. For a detailed introduction to Coxeter groups we refer to [7, Chapter 1]. For simplicity, we only consider the following three classical types, although our techniques also apply to the more general cases discussed in [18].

The Coxeter group of type $A_N$ corresponds to the permutations on the symmetric group on $\{1, \ldots, N+1\}$. The Coxeter group of type $B_N$ can be realized as the group of signed permutations, that is, the group of all bijections $\sigma$ on $\{\pm 1, \ldots, \pm N\}$, such that $\sigma(-i) = -\sigma(i)$, see [7, Chapter 8.1]. Following the one-line notation of [18, Section 2], we write $\sigma = [\sigma(1), \ldots, \sigma(N)]$, with $\sigma(i) \in \{\pm 1, \ldots, \pm N\}$ and $\{|\sigma(1)|, \ldots, |\sigma(n)|\} = \{1, \ldots, N\}$. The Coxeter group of type $D_N$ can then be realized as a subgroup of $B_N$ with an even number of negative entries in the one-line notation, in other words,

$$D_N = \{\sigma \in B_N : \sigma(1)\sigma(2)\cdots\sigma(N) > 0\},$$

see [7, Chapter 8.2]. Note that the groups $A_N$, $B_N$ and $D_N$ all have rank $N$.

For any of these three groups, certain elements may be identified as inversions, generalizing the notion of inversions in classical permutations. Indeed, according to the notation in [18, Section 2], we set

$$\text{inv}^+ (\sigma) := \{1 \leq i < j \leq N : \sigma(i) > \sigma(j)\},$$

$$\text{inv}^- (\sigma) := \{1 \leq i < j \leq N : -\sigma(i) > \sigma(j)\},$$

$$\text{inv}^\circ (\sigma) := \{1 \leq i \leq N : \sigma(i) < 0\}.$$
Then, the inversions in an element \( \sigma \in W_N, W \in \{A, B, D\} \), are given by
\[
\begin{align*}
\text{inv}_{A_N}(\sigma) &= \text{inv}^+(\sigma), \\
\text{inv}_{B_N}(\sigma) &= \text{inv}^+(\sigma) \cup \text{inv}^-(\sigma) \cup \text{inv}^0(\sigma), \\
\text{inv}_{D_N}(\sigma) &= \text{inv}^+(\sigma) \cup \text{inv}^-(\sigma).
\end{align*}
\]
We also refer to [21, Section 2] for a discussion of more general \( d \)-inversions in the classical types of Coxeter groups.

The number of elements in \( W_N, W \in \{A, B, D\} \), with exactly \( k \in \{0, 1, \ldots, N\} \) inversions are called the \( W_N \)-Mahonian numbers. If we define a random variable \( X_N := X(W_N) \) as the number of \( W_N \)-inversions of an element of \( W_N \) chosen uniformly at random, the distribution of \( X_N \) is called the \( W_N \)-Mahonian distribution.

The generating polynomial for \( X_N \) is given by
\[
P_N(z) = \prod_{1 \leq j \leq N} \frac{1 - z^{d_j}}{1 - z} \quad (z \in \mathbb{R}),
\]
where \( d_1, \ldots, d_N \) are the degrees of \( W_N \), cf. [7, Theorem 7.1.5] and note that the exponents \( e_j \) of Coxeter groups therein are related to the degrees by \( d_j = e_j + 1 \). The degrees of the three types of Coxeter groups we consider are summarized in the following table.

| Type | Degrees |
|------|---------|
| \( A_N \) | \( d_1 = 2, d_2 = 3, \ldots, d_N = N + 1 \) |
| \( B_N \) | \( d_1 = 2, d_2 = 4, \ldots, d_N = 2N \) |
| \( D_N \) | \( d_1 = 2, d_2 = 4, \ldots, d_{N-1} = 2N - 2, d_N = N \) |

In particular, the polynomials (8) are of the form (1) with \( b_j = d_j \), so that \( M_N = \max_{1 \leq j \leq N} d_j \), and \( a_j = 1 \) for \( j \in \{1, \ldots, N\} \) and have degree \( n = \sum_{1 \leq j \leq N} (d_j - 1) \). The cumulants of the corresponding random variables \( X_N \) admit the representation (3). For instance, the variance of \( X_N \) is given by
\[
\sigma_N^2 = \frac{1}{12} \sum_{1 \leq j \leq N} (d_j^2 - 1).
\]
For each of the three possible choices for \( W \) we consider, the respective results for \( \sigma_N^2, M_N \) and \( \kappa^+_{1,N} \) are summarized in the following table.

| Type | \( \sigma_N^2 \) | \( M_N \) | \( |\kappa^+_{1,N}| \) |
|------|----------------|----------|----------------|
| \( A_N \) | \( \frac{1}{12} \left( 2N^3 + 9N^2 + 7N \right) \) | \( N + 1 \) | \( \frac{36}{25} \left( 6N^2 + 45N^4 + 130N^3 + 180N^2 + 89N \right) \) |
| \( B_N \) | \( \frac{1}{36} \left( 4N^3 + 6N^2 - N \right) \) | \( 2N \) | \( \frac{18}{25} \left( 48N^3 + 120N^2 + 80N^3 - 23N \right) \) |
| \( D_N \) | \( \frac{1}{36} \left( 4N^3 - 3N^2 - N \right) \) | \( 2N - 2 \) | \( \frac{18}{25} \left( 48N^3 - 105N^2 + 80N^2 - 23 \right) \) |

Given these values, one can check analogously to Section 3.1 that for any \( N \geq 2 \),
\[
\min \left\{ \frac{\sigma_N}{M_N}, \left| \frac{\kappa^+_{1,N}}{M_N} \right|^{1/2} \right\} = \frac{\sigma_N}{M_N}
\]
for $A_N$, $B_N$ and $D_N$. This leads to expressions for $\Delta N$, all of which can be bounded from below by

$$\Delta N > \frac{1}{6} \sqrt{\frac{7}{30}} \pi \sqrt{N} \quad (N \geq 2).$$

Hence, for these three types of Coxeter groups, we observe a central limit theorem for $X_N$ with speed of convergence of order $N^{-1/2}$. This agrees with the results found in [21, Theorem 2.8, Corollary 2.9], where generalized inversions in finite Weyl groups are studied. Also, we have a moderate deviation principle as in Corollary 2.3 and a concentration inequality as given in Corollary 2.5. In addition we have mod-Gaussian convergence in Corollary 2.4 and note in this context that in Remark 6.9 in [18] the authors already mention without proof and without giving details a possible mod-Gaussian convergence for the sequence of random variables $X_N$.

### 3.3. Gaussian polynomials

Let $p(N, \ell, j)$ denote the number of partitions of an integer $j$ into at most $\ell$ summands, each of which is less than or equal to $N$ (in particular, $p(N, \ell, N\ell) = 1$ and $p(N, \ell, j) = 0$ if $j > N\ell$). According to [3, Theorem 3.1], the corresponding generating polynomial $G(N, \ell; z)$ is given by

$$G(N, \ell; z) = \sum_{0 \leq j \leq N\ell} p(N, \ell, j) z^j = \prod_{1 \leq j \leq N} \frac{1 - z^{j+\ell}}{1 - z^j} \quad (z \in \mathbb{R}).$$

These polynomials were historically first studied by Gauss and hence are called Gaussian polynomials.

The right hand side of (9) matches (1) with $b_j = \ell + j$ and $a_j = j$, and thus $G(N, \ell; z)$ is a polynomial of degree $n = \sum_{1 \leq j \leq N} \ell = N\ell$ (note the additional dependence on the parameter $\ell$). By [3], the cumulants of the corresponding random variables $X_{N,\ell}$ are given by

$$\kappa_{m,N,\ell} = \frac{B_m}{m} \sum_{1 \leq j \leq N} ((j + \ell)^m - j^m)$$

$$= \frac{B_m}{m} \left(-H_N^{(-m)} - \zeta(-m, \ell + N + 1) + \zeta(-m, \ell + 1)\right),$$

where $H_N^{(-m)} = \sum_{0 \leq k \leq N} k^m$ is the $N$-th harmonic number of order $-m$ and $\zeta(-m, a)$ denotes the Hurwitz zeta function, which for any $a \in \mathbb{R}$ is given by

$$\zeta(-m, a) = -\frac{B_{m+1}(a)}{m+1},$$

with $B_m(\cdot)$ denoting the $m$-th Bernoulli polynomial. In case of the variance, this leads to

$$\sigma^2_{N,\ell} = \frac{1}{12} \left(-\frac{N^3}{3} - \frac{N^2}{2} - \frac{N}{6} - \zeta(-2, \ell + N + 1) + \zeta(-2, \ell + 1)\right)$$

$$= \frac{1}{12} \left(-\frac{N^3}{3} - \frac{N^2}{2} - \frac{N}{6} + \frac{B_3(\ell + N + 1)}{3} - \frac{B_3(\ell + 1)}{3}\right).$$

The Bernoulli polynomial $B_3(a)$ is given by $B_3(a) = a^3 - \frac{3}{2}a^2 + \frac{1}{2}a$, see for example [1, Chapter 23] for details. Thus,

$$\sigma^2_{N,\ell} = \frac{1}{12} (\ell^2 N + \ell N + \ell N^2).$$
In the same way, using that \( B_5(a) = a^5 - \frac{5}{2} a^4 + \frac{5}{3} a^3 - \frac{1}{6} a \), we find
\[
|\kappa_{4,N,\ell}| = \frac{1}{120} \left( -\frac{N^5}{5} - \frac{N^4}{2} - \frac{N^3}{3} + \frac{N}{30} + \frac{B_5(\ell + N + 1)}{5} - \frac{B_5(\ell + 1)}{5} \right)
\]
\[
= \frac{1}{120} \left( \ell^4 N + 2 \ell^3 N^2 + 2 \ell^2 N^3 + 3 \ell^2 N^2 + \ell^2 N + \ell N^4 + 2 \ell N^3 + \ell N^2 \right),
\]
which leads to
\[
|\kappa_{4,N,\ell}^*| = \frac{6}{5} \left( \frac{1}{N} + \frac{1}{\ell} - \frac{1}{\ell + N + 1} \right),
\]
after straightforward simplifications. Further, \( M_{N,\ell} = \max_{1 \leq j \leq N} (j + \ell) = N + \ell \), so that
\[
\frac{M_{N,\ell}^2}{\sigma_{N,\ell}^2} = \frac{12 (\ell + N)^2}{\ell N (\ell + N + 1)}.
\]
One may readily check that
\[
|\kappa_{4,N,\ell}^*| \leq \frac{M_{N,\ell}^2}{\sigma_{N,\ell}^2},
\]
which directly leads to
\[
\Delta_{N,\ell} = \sqrt{\frac{7}{30} \pi^2 \frac{\sigma_{N,\ell}}{M_{N,\ell}}} = \sqrt{\frac{7}{360} \pi^2 \frac{\ell N (\ell + N + 1)}{(\ell + N)}},
\]
with \( \Delta_{N,\ell} \) as in [4], taking into account the additional dependence on \( \ell \). As we take both \( N \to \infty \) and \( \ell \to \infty \), we have \( \sigma_{N,\ell}/M_{N,\ell} \to \infty \). In particular, Corollary 2.2 yields a central limit theorem for the random variables \( X_{N,\ell} \) as previously shown in [19, Section 4] and [34, Section 3]. Moreover, by [6], the speed of convergence is bounded by
\[
\sup_{x \in \mathbb{R}} |\mathbb{P}(X_{N,\ell}^* \geq x) - \mathbb{P}(Z \geq x)| \leq C \left( N^{-1/2} + \ell^{-1/2} \right),
\]
where \( C > 0 \) is some absolute constant and \( Z \) is a standard random variable. It appears that this Berry-Esseen bound, as also the other results which follow from Section 2 (in particular, mod-Gaussian convergence), seem new. In particular, note that for Gaussian polynomials the generating function does not factorize as required in [16, Lemma 4], so that the latter result cannot be applied in this situation in order to deduce a Berry-Esseen bound for the coefficients of a Gaussian polynomial.

3.4. \( q \)-Catalan numbers. There are several ways of generalizing the usual Catalan numbers \( C_N = \frac{1}{N+1} \binom{2N}{N} \) to \( q \)-Catalan numbers. One of them reads
\[
C_N(q) = \frac{1}{[N+1]_q} \binom{2N}{N}_q,
\]
where \([N]_q := 1 + q + \cdots + q^{N-1}\) and
\[
\binom{N}{k}_q := \frac{[N]_q!}{[k]_q! [N-k]_q!},
\]
with \([N]_q! := [N]_q[N-1]_q\cdots[1]_q\). For a broader view on \(q\)-Catalan numbers (including various generalizations like the one discussed in the next subsection) we refer the reader to [14]. In the sequel, we assume \(N \geq 2\). It is easy to check that

\[
C_N(q) = \prod_{2 \leq j \leq N} \frac{1 - q^{N+j}}{1 - q^j},
\]

which shows that \(q\)-Catalan numbers are closely related to the Gaussian polynomials [9]. Indeed, \(C_N(q) = G(N, N; q) / [N+1]_q\). In particular, \(C_N(q)\) is a polynomial as in (1) with \(b_1 = 1\), \(a_1 = 1\), \(b_j = N + j\) and \(a_j = j\) for \(2 \leq j \leq N\). Hence, the \(m\)-th cumulant of the corresponding random variable \(X_N\) is given by

\[
\kappa_{m,N} = \frac{B_m}{m} \sum_{2 \leq j \leq N} \left((N+j)^m - j^m\right).
\]

It follows that

\[
\sigma_N^2 = \frac{1}{6} \left(N^3 - N\right), \quad \kappa_{4,N} = -\frac{1}{60} \left(3N^5 + 3N^4 - N^3 - 3N^2 - 2N\right),
\]

implying that

\[
\kappa^*_{4,N} = -\frac{3}{5} \left(\frac{3N^2 + 3N + 2}{N^3 - N}\right).
\]

Further, since \(M_N = \max_{1 \leq j \leq N} b_j = 2N\), we have

\[
\frac{M_N^2}{\sigma_N^2} = \frac{24N}{N^2 - 1}.
\]

One can verify that \(M_N^2 / \sigma_N^2 \geq \left|\kappa^*_{4,N}\right|\), and therefore,

\[
\Delta_N = \sqrt{\frac{7}{30} \pi^2} \frac{\sigma_N}{M_N} = \frac{1}{12} \sqrt{\frac{7}{30} \pi^2} \sqrt{\frac{N^2 - 1}{N}}.
\]

Hence, Corollary 2.2 yields a central limit theorem for the sequence \((X_N^*)_{N \in \mathbb{N}}\) as previously shown in [8, Corollary 3.3], see also [6, Theorem 3.1]. Using Corollary 2.2 the speed of convergence in this central limit theorem is bounded by

\[
\sup_{x \in \mathbb{R}} \left|\mathbb{P}(X_N^* \geq x) - \mathbb{P}(Z \geq x)\right| \leq C N^{-1/2},
\]

where \(C > 0\) is some absolute constant and \(Z\) a standard normal random variable. This answers the question for a Berry-Esseen bound for \(q\)-Catalan numbers which was raised in [35, Remark 4.2]. Note that moreover, Corollary 2.3 recovers the moderate deviation principle established in [35, Theorem 2.1], while the remaining results which follow from Section 2 seem new.

3.5. **Generalized \(q\)-Catalan numbers.** Generalizing the usual Catalan numbers \(C_N\) in another way, one may define for integers \(k \geq 1\) the \(k\)-Catalan numbers by

\[
C_{N,k} = \frac{1}{(k-1)N + 1} \binom{kN}{N},
\]

see [31]. Given the \(q\)-Catalan numbers, it is natural to define

\[
C_{N,k}(q) := \frac{1}{(N + 1)_{[N]_q}} \left[\frac{kN}{N}\right]_{[N]_q},
\]
which yields
$$\gamma$$
and first introduced by [2]) is an array of non-negative integers
$$\text{Descending Plane Partitions.}$$

Section 2 follow readily.

are called the parts of the DPP, as in Figure 2a, such that the following conditions
$$\text{Noting that } \Delta$$
In particular, for any choice of $$N,k$$

One can show also in this case that
$$M$$
for the variance and
$$\kappa$$
for the fourth cumulant. This leads to
$$\kappa$$
Further, $$M_{N,k} = \max_{1 \leq j \leq N} b_j = kN$$ and hence,
$$\frac{M_{N,k}^2}{\sigma_{N,k}^2} = \frac{12k^2N}{(k^2 - k)N^2 + (-k^2 + 3k - 2)N^2 + (2 - 2k)N}.$$

One can show also in this case that $$M_{N,k}^2/\sigma_{N,k}^2 \geq |\kappa_{4,N,k}^*|$$, so that
$$\Delta_{N,k} = \sqrt{\frac{7}{30}} \pi^2 \frac{\sigma_{N,k}}{M_{N,k}} = \pi^2 \sqrt{\frac{7}{10} \sqrt{(k - 1)(N - 1)N(kN + 2)}} / 6kN.$$

In particular, for any choice of $$k > 1$$ by Corollary 2.2, the sequence $$(X_N^N)_{N \in \mathbb{N}}$$
noting that $$\Delta_{N,k}$$ can always be chosen of order $$N^{-1/2}$$, all the other results from
Section 2 follow readily.

3.6. Descending Plane Partitions. A descending plane partition (DPP for short
and first introduced by [2]) is an array of non-negative integers $$\gamma_{i,j}$$ ($$i \leq j$$), which
are called the parts of the DPP, as in Figure 2a such that the following conditions hold:

(D1) The values of the parts are decreasing in each row from left to right and
strictly decreasing in each column from top to bottom. In particular, $$\gamma_{i,i}$$
is the largest part of the $$i$$-th row and the $$i$$-th column.

(D2) The entry $$\gamma_{i,j}$$ is strictly greater than the number of parts in the $$i$$-th row
and less or equal to the number of parts in the $$(i - 1)$$-th row.

We refer, for example, to [33 Section 1] or [25 Section 1] for a detailed introduction.
A descending plane partition is said to be of order $$N$$, if its largest part is at most
$$N$$. For instance, there are two DPPs of order $$N = 2$$ and seven DPPs of order
$$N = 3$$. Note that $$\emptyset$$ always counts as a DPP (the empty one). If $$\gamma_{i,j} \leq j - i$$, $$\gamma_{i,j}$$
A general descending plane partition

\begin{align*}
\gamma_{1,1} \gamma_{1,2} \gamma_{1,3} \ldots \gamma_{1,\lambda_1} \\
\gamma_{2,2} \gamma_{2,3} \ldots \gamma_{2,\lambda_2} \\
\ldots \\
\gamma_{\ell,\ell} \gamma_{\ell,\lambda_\ell}
\end{align*}

\text{(a) A general descending plane partition}

\begin{align*}
12 & \quad 12 & \quad 9 & \quad 8 & \quad 5 & \quad 1 \\
7 & \quad 7 & \quad 6 & \quad 5 & \quad 4 \\
5 & \quad 5 & \quad 3 & \quad 3 \\
3 & \quad 2
\end{align*}

\text{(b) A descending plane partition of order } N = 12 \text{ where special parts are marked in bold}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Descending plane partitions}
\end{figure}

is called a special part of the DPP. See Figure 2b for a DPP of order \( N = 12 \) (and thus also of any order \( N > 12 \)), where the specials parts are marked in bold.

Writing \( S := \sum_{i,j} \gamma_{i,j} \) for the sum of the parts, a DPP of order \( N \) can be regarded as a partition of \( S \) whose largest part is at most \( N \), see especially [33, Lemma 5] which translates (D1) and (D2) into conditions on the partition for DPPs with no special parts. Descending plane partitions moreover admit connections to other combinatorical structures. For instance, in [4] a bijection between DPPs of order \( N \) with no special parts and certain permutations of \( N \) elements is established. There is also a relation of DPPs to alternating sign matrices as studied in [26] and [25].

According to [33, Corollary 6], the generating polynomial for DPPs of order \( N \) with no special parts is given by

\begin{equation}
\prod_{1 \leq j \leq N} [j]_{q^j} = \prod_{1 \leq j \leq N} \frac{1 - q^{j^2}}{1 - q^j},
\end{equation}

where we recall the notation \([j]_q = 1 + q + q^2 + \cdots + q^{j-1}\) from Section 3.4. This generating polynomial also appears in the context of the inversion index of a permutation of \( N \) elements as shown in [33] using permutations matrices see also [St000616] in [28].

The polynomial (10) matches (1) with \( b_j = j^2 \) and \( a_j = j \), hence the cumulants are given by

\[
\kappa_{m,N} = \frac{B_m}{m} \sum_{1 \leq j \leq N} (j^{2m} - j^m).
\]

This directly leads to

\[
\sigma_N^2 = \frac{1}{120} \left( 2N^5 + 5N^3 - 5N^2 - 2N \right),
\]

and

\[
\kappa_{4,N} = -\frac{1}{2160} \left( 2N^8 + 9N^6 + 12N^7 - 12N^5 - 9N^4 - 2N^3 \right).
\]

Standardizing the fourth cumulant then yields

\[
\kappa_{4,N}^* = -\frac{20(N^2 + N)}{6N^3 + 9N^2 - 9N - 6}.
\]

Having \( M_N = N^2 \), one can show that \( |\kappa_{4,N}^*| \leq M_N^2/\sigma_N^2 \) and hence,

\[
\Delta_N = \pi^2 \sqrt{\frac{7}{60}} \left( \frac{2N^5 + 5N^3 - 5N^2 - 2N}{N^2} \right)^{1/2} > 30\pi^2 \sqrt{\frac{7}{2}} \sqrt{N},
\]

where the inequality holds for any \( N \geq 2 \). Therefore, Corollary 2.2 provides a central limit theorem for the number of DPPs with largest part at most \( N \) with speed of convergence of order \( N^{-1/2} \). See Figure 3 for a plot of 350,000 samples.
with \( N = 50 \). The Gaussian shape is clearly visible. Further, Corollary 2.3 provides a moderate deviation principle and Corollary 2.5 a concentration inequality. Moreover, mod-Gaussian convergence follows from Corollary 2.4. All these results seem new.

4. Proofs

4.1. Proof of Cumulant Representation. For the reader’s convenience and since the formula in [17] needs a correction we derive here the cumulant representation (3) based on the methods of the proof of [8, Theorem 2.3]. First note that the moment generating function of the random variable \( X_N \) is given by

\[
m_{X_N}(t) := E(e^{tX_N}) = \frac{P_N(e^t)}{P_N(1)} = \prod_{1 \leq j \leq N} \frac{1 - e^{tb_j}}{1 - e^{ta_j}} \cdot \frac{a_j}{b_j},
\]

recalling that \( P_N(1) = \prod_{1 \leq j \leq N} b_j/a_j \) (see the discussion following (2)). Using

\[
2 \sinh(x) = e^x - e^{-x}
\]

then leads to

\[
m_{X_N}(t) = \prod_{1 \leq j \leq N} e^{t(b_j-a_j)/2} \frac{\sinh(tb_j/2)/(b_j/2)}{\sinh(ta_j/2)/(a_j/2)}
\]

\[
= \exp \left( \frac{t}{2} \sum_{1 \leq j \leq N} (b_j - a_j) \right) \cdot \prod_{1 \leq j \leq N} \frac{\sinh(tb_j/2)/(b_j/2)}{\sinh(ta_j/2)/(a_j/2)}
\]

and hence,

\[
\ln(m_{X_N}(t)) = \frac{t}{2} \sum_{1 \leq j \leq N} (b_j - a_j)
\]

\[
+ \sum_{1 \leq j \leq N} \left( \ln \left( \frac{\sinh(tb_j/2)}{b_j/2} \right) - \ln \left( \frac{\sinh(ta_j/2)}{a_j/2} \right) \right).
\]

By [29, Chapter 1.3.1],

\[
\ln \left( \frac{\sinh(x/2)}{x/2} \right) = \sum_{m \geq 1} B_{2m} \frac{x^{2m}}{2m(2m)!},
\]

where \( B_{2m} \) are the Bernoulli numbers.
with $B_m$ being the Bernoulli numbers. This yields
\[
\ln m X_N(t) = \frac{t}{2} \sum_{1 \leq j \leq N} (b_j - a_j) + \sum_{m \geq 1} B_{2m} \frac{t^{2m}}{2m(2m)!} \sum_{1 \leq j \leq N} (b_j^m - a_j^m).
\]
By definition, the cumulants of $X_N$ are given as the coefficients of
\[
m X_N(t) = \exp \left( \sum_{m \geq 1} \frac{\kappa_m}{m!} t^m \right),
\]
which by comparison leads to
\[
\kappa_{2m,N} = \frac{B_{2m}}{2m} \sum_{1 \leq j \leq N} (b_j^m - a_j^m)
\]
and $\kappa_{2m+1,N} = 0$. Moreover,
\[
\kappa_{1,N} = \frac{1}{2} \sum_{1 \leq j \leq N} (b_j - a_j) = B_1 \sum_{1 \leq j \leq N} (b_j - a_j)
\]
for the choice of $B_1 = \frac{1}{2}$ (the sign of $B_1$ is a matter of convention). In particular, we see that $\kappa_{1,N} = \mathbb{E}[X_N] = n/2$ in accordance with [17, Lemma 2.2 & Equation (20)].

4.2. Proof of Theorem 2.1. The core of the proof of Theorem 2.1 is to relate the higher-order cumulants to the fourth one. For this the following simple lemma will turn out to be useful.

Lemma 4.1. For any two real numbers $b \geq a \geq 0$ and any integer $m \geq 2$, it holds that
\[
(11) \quad b^{2m} - a^{2m} \leq (b^4 - a^4) 2^m - 2 b^{2m-4}.
\]
Proof. First note, that if $a = b$, (11) is trivial. In the case $b > a$, the statement can be proven by induction over $m$. The case $m = 2$ is clear and we are left to show that if (11) holds for some $m \in \mathbb{N}$, then also
\[
b^{2m+2} - a^{2m+2} \leq (b^4 - a^4) 2^{m-1} b^{2m-2}.
\]
Since $b \geq a$, we obtain
\[
\frac{b^{2m+2} - a^{2m+2}}{b^{2m} - a^{2m}} = b^2 + \frac{a^{2m} b^2 - a^{2m+2}}{b^{2m} - a^{2m}} \leq b^2 + \frac{a^2 b^{2m} - a^{2m+2}}{b^{2m} - a^{2m}} = b^2 + a^2,
\]
which yields
\[
b^{2m+2} - a^{2m+2} \leq (b^4 - a^4)(b^2 + a^2) \leq (b^{2m} - a^{2m}) 2b^2 \leq (b^4 - a^4) 2^{m-1} b^{2m-2},
\]
where the last step follows by induction. \qed

Proof of Theorem 2.1. We start by observing that the bound (11) trivially holds for $m = 2$. If $m > 2$, the cumulant representation (3), the identity $\kappa_{2m,N} = \kappa_{2m,N}/\sigma_N^{2m}$ and our assumption that $b_j \geq a_j$ for all $j \in \{1, \ldots, N\}$ give
\[
|\kappa_{2m,N}^*| = \frac{|B_{2m}|}{2m\sigma_N^{2m}} \sum_{1 \leq j \leq N} (b_j^{2m} - a_j^{2m}).
\]
An application of Lemma 4.1 then leads to
\[ |\kappa_{2m,N}^*| \leq \frac{|B_{2m}|}{2m\sigma_N^2} \sum_{1 \leq j \leq N} 2^{m-2} b_j 2^{m-4} (b_j^4 - a_j^4) \]
\[ \leq \frac{2^{m-3}}{m} |B_{2m}| \frac{M_N^{2m-4}}{\sigma_N^{2m-4}} \sum_{1 \leq j \leq N} (b_j^4 - a_j^4), \]
where we recall that \( M_N = \max_{1 \leq j \leq N} b_j \) as stated in the theorem. Again by (3), the fourth cumulant can be written as
\[ \kappa_{4,N} = \frac{1}{120} \sum_{1 \leq j \leq N} (b_j^4 - a_j^4). \]
This yields
\[ |\kappa_{2m,N}^*| \leq \frac{120 \cdot 2^{m-3}}{m} |B_{2m}\kappa_{4,N}^*| \left( \frac{M_N}{\sigma_N} \right)^{2m-4} \]
\[ = \frac{15 \cdot 2^m}{m} |B_{2m}\kappa_{4,N}^*| \left( \frac{M_N}{\sigma_N} \right)^{2m-4}. \]
To bound the Bernoulli numbers \( B_{2m} \) we use [1 Equation 3.1.15], which says that
\[ |B_{2m}| \leq \frac{2(2m)!}{(2\pi)^{2m}} \frac{1}{1 - 2^{1-2m}} \quad (m \geq 1). \]
As a consequence,
\[ |\kappa_{2m,N}^*| \leq \frac{15 \cdot 2^m}{m} 2(2m)! \left( \frac{2^{1-2m}}{2\pi} \right)^{2m} \frac{1}{1 - 2^{1-2m}} \kappa_{4,N}^* \left( \frac{M_N}{\sigma_N} \right)^{2m-4} \]
\[ = (2m)! \frac{15}{2^{m-3} m \pi^{2m}} \left( \frac{2^{2m}}{4m - 2} \right) \kappa_{4,N}^* \left( \frac{M_N}{\sigma_N} \right)^{2m-4} \]
\[ = (2m)! \frac{15 \cdot 2^{m+1}}{m \pi^{2m} (4m - 2)} \kappa_{4,N}^* \left( \frac{M_N}{\sigma_N} \right)^{2m-4}. \]
The sequence \((c_m)_{m \geq 2}\) defined by \( c_m := \frac{15 \cdot 2^{m+1}}{m \pi^{2m} (4m - 2)} \) is strictly decreasing, since
\[ c_m - c_{m+1} = 15 \pi^{-2m} 2^{m+1} \left( \frac{1}{m(4m - 2)} - \frac{1}{\pi^2 (2^{m+1} - 1)(m+1)} \right) > 0. \]
To see this, note that for \( m \geq 2 \),
\[ \pi^2 (2^{m+1} - 1)(m+1) = m(4^m + 1) \pi^2 + (2^{m+1} - 1) \pi^2 > m(4^m - 2), \]
and therefore,
\[ \frac{1}{m(4^m - 2)} > \frac{1}{(m+1)(2^{m+1} - 1) \pi^2}. \]
In particular, this implies that \( \max_{m \geq 2} c_m = c_2 = 30/(7 \pi^4) \). Thus,
\[ |\kappa_{2m,N}^*| \leq (2m)! \frac{30}{\pi^4} \kappa_{4,N}^* \left( \frac{M_N}{\sigma_N} \right)^{2m-4} \]
\[ \leq (2m)! \frac{30}{\pi^4} \min\left\{ \frac{\sigma_N}{M_N}, \left[ \kappa_{4,N}^* \right]^{1/2} \right\}^{2m-2}. \]
So, choosing
\[
\Delta_N = \max_{m \geq 2} \left\{ \frac{2m - 2}{\sqrt{\frac{2\pi^4}{30}}} \right\} \min \left\{ \frac{\sigma_N}{M_N}, \frac{1}{|\kappa^*_N|^{1/2}} \right\}
\]
we arrive at
\[
|\kappa^*_N| \leq \frac{(2m)!}{\Delta_N^{2m-2}}.
\]
This completes the proof. □

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