Instability of particle inertial migration in shear flow

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In a shear flow particles migrate to their equilibrium positions in the microchannel. Here we demonstrate theoretically that if particles are inertial, this equilibrium can become unstable due to the Saffman lift force. We derive an expression for the critical Stokes number that determines the onset of instable equilibrium. We also present results of lattice Boltzmann simulations for spherical particles and prolate spheroids to validate the analysis. Our work provides a simple explanation of several unusual phenomena observed in earlier experiments and computer simulations, but never interpreted before in terms of the instable equilibrium.

I. INTRODUCTION

In shear flows particles experience an inertial lift force which induces their migration across streamlines. This effect was first discovered for neutrally buoyant particles in tubes and is currently widely employed to separate particles in microfluidic devices. In an unbounded shear flow, the Saffman lift force emerges when a slip velocity (i.e. a difference between particle velocity and fluid velocity at the particle center) becomes finite. The disturbance of the flow in this case is caused by a streamwise drag force and momentum released into the fluid. In the channel (wall-bounded) flows, another type of the lift force emerges for a freely rotating and translating particle. Namely, a neutrally buoyant lift force, which is due to the curvature of the undisturbed velocity profile or wall effects.

The inertial migration of particles is traditionally considered as a quasi-steady process. This implies that the particle inertia is neglected, so that hydrodynamic forces (the drag, the neutrally buoyant lift, the Dean force) as well as any external forces are balanced, i.e. \( \mathbf{F}(x_p, \mathbf{V}, \mathbf{U}) = 0 \), where \( x_p \) and \( \mathbf{V} \) are the particle position and velocity, correspondingly, and \( \mathbf{U} \) is the fluid velocity. Besides, it is commonly considered that the Saffman lift emerges only under forces acting in the streamwise direction (e.g. non-neutrally buoyant particles under gravity in vertical channels). Such a quasi-steady approach allows one to infer the particle velocity \( \mathbf{V}(x) \) by using the lift and drag coefficients. Since the drag coefficient is positive-definite, the behavior of the particle is controlled by the variation of the lift force across the channel: the zeros of the lift force correspond to the particle equilibrium positions and its gradients define their stability. In computer simulations, the lift force on a particle at different positions can be measured independently and is often used to predict the particle behavior. By contrast, direct experimental measurements of the lift force are impossible. This force is usually calculated from the measured migration velocity.\(^a\)

The particle inertia is characterized by the Stokes number \( \text{St} = 2 \rho_p \text{Re}_G/(9 \rho) \), where \( \rho_p \) and \( \rho \) are the particle and fluid densities, \( \text{Re}_G = Ga^2/\nu \) is the particle Reynolds number defined using the particle radius \( a \), shear rate \( G \) and kinematic viscosity of the fluid \( \nu \). Clearly, at sufficiently large \( \text{Re}_G \) the Stokes number can become finite even for neutrally buoyant particles with \( \rho_p/\rho = 1 \). Consequently, when such particles migrate across the streamlines, they accelerate by the fluid. In this case, the momentum exchange between the fluid and the particle generates the Saffman lift force, and the quasi-steady approach is no longer applicable.

Despite this obvious fact, the correctness of the quasi-steady approach at \( \text{Re}_G > 1 \) is still not under dispute, and migration phenomena are commonly analyzed in terms of the dependence of the lift force on the particle position, its zeros and bifurcations.\(^7,9\) As one example, numerous experiments with neutrally buoyant particles in circular tubes found that at high channel Reynolds numbers the Segre-Silberberg equilibrium position shifts towards the wall, but some particles migrate towards the center to form an inner annulus.\(^10–12\) However, their interpretation implies that in the long run all particles will focus at the zeros of the lift curve, although note that there have been some suggestions that the inner annulus is a second “true” equilibrium position or represents only a transient configuration.\(^11\) Another example refers to the computer simulations of the inertial behavior of spheroids in shear flow that is currently a subject of active research.\(^13–16\) It is well known that at large \( \text{Re}_G \) and \( \text{St} \) prolate spheroids undergo a series of transitions between different rotational regimes. All

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The equations of particle motion in dimensional variables read

\[ m \frac{dV'}{dt'} = F', \quad (1) \]
\[ \frac{dx_p'}{dt'} = V', \quad (2) \]

where \( m = 4/3 \rho_p a^3 \) is the particle mass, \( V' = (V'_x, V'_y) \) is the particle translational velocity, and \( F' = (F'_x, F'_y) = F'_d + F'_l + F_l^{sa} \) is the hydrodynamic force. Here, \( F'_d \) is the quasi-steady drag force and \( F'_l \) is an unsteady force due to particle acceleration which includes the Basset and added-mass forces\(^{12}\). The Saffman lift force \( F_l^{sa} = F_l^{sa} e_y \) is proportional to the momentum released by the particle into shear flow. Note that we do not include the equation for the rotational velocity since it has little effect on the sphere’s dynamics in shear flows\(^{12}\).

A steady-state solution of Eqs. (1) and (2) in the absence of external force is

\[ V'_0 = 0, \quad x'_{p0} = 0. \quad (3) \]

To distinguish between the stable and unstable equilibrium states we employ the linear stability analysis. Let us consider how the particle at the equilibrium position reacts to a small disturbance in the initial velocity. For small (slip Reynolds number) \( Re_V = (V'_y - y_p)/\nu \) Eqs. (1) and (2) can be linearized by expanding \( V' \). The drag force then takes the form \( F'_d = -6\pi \mu a f_s (V'_y - y_p) \), \( F'_dy = -6\pi \mu a f_s V'_y \), where \( f_s (Re_G) \) and \( f_y (Re_G) \) are the correction factors accounting for the effect of fluid inertia at finite \( Re_G \). At small \( Re_G \) they converge to unity.

It is now convenient to introduce dimensionless variables by scaling the velocities by \( G a \), the coordinates by \( a \), the time by \( G^{-1} \), and the forces by \( 6\pi \mu a^2 G \). The linearized equations can then be formulated as

\[ St \frac{dV_x}{dt} = -f_x (V_x - y_p) + F_{lx}, \quad (4) \]
\[ St \frac{dV_y}{dt} = -f_y V_y + F_{l}^{sa} + F_{ly}, \quad (5) \]
\[ St \frac{dy}{dt} = V_y, \quad (6) \]

where \( St \) is the Stokes number,

\[ St = \frac{2\rho_p G a^2}{9 \mu} = \frac{2\rho_p}{9 \mu} Re_G. \quad (7) \]

The origin of instability can be understood as follows. Assume a small disturbance of the particle from its equilibrium position leading to a positive transverse velocity \( V'_y \) (see Fig. (a)). The emerging negative transverse drag \( F_{dy} = -f_y V'_y < 0 \) should tend to stop the particle. However, since the particle enters the region of a larger fluid velocity, it lags behind the fluid due to its inertia. The particle that is subject to a drag force, \( F_{dx} = -f_x V_x > 0 \), where \( V_x = V_x - y_p < 0 \) is the slip velocity, begins to accelerate in the streamwise direction. This, in turn, should induce a positive Saffman lift force \( F_l^{sa} > 0 \) which may exceed the transverse drag by leading to further acceleration of the particle in the transverse direction. We refer this situation to as unstable.

By contrast, when the equilibrium is neutral, which implies that small disturbances neither grow nor disappear, \( V'_y \) and \( V_x \) are constant (see Fig. (b)). The acceleration of a particle is also constant and equal to the fluid...
acceleration along its trajectory
\[
\frac{dV_y}{dt} = V_y \frac{dU}{dy} e_x = V_y e_x = \text{const.} \tag{8}
\]

It follows from Eq. (8) that if \(dV_y/dt = 0\), the unsteady force \(F_y\) and the inertia term in the left-hand side vanish. The condition of a neutral equilibrium can then be formulated as
\[
F^{Sa}_t - f_y V_y = 0. \tag{9}
\]

To apply the criterion (9) it is necessary to calculate the lift force \(F^{Sa}_t\) for the neutral equilibrium, which is not straightforward. In the classical Saffman theory, the lift force is proportional to (constant) \(V_x\). In our case the situation is different since the particle accelerates in the streamwise direction, and the lift force is related to the particle acceleration given by Eq. (A4) (see Appendix A for a derivation) as
\[
F^{Sa}_t = C_l St V_y \quad \text{as} \quad Re_V \ll Re_G \ll 1. \tag{10}
\]

Here \(C^{Sa}_l = 0.343Re_G^{1/2}\) is the Saffman lift coefficient, which characterizes the lift-to-drag ratio. One can expect that the lift force at finite \(Re_G\) is described by (10), i.e.
\[
F^{Sa}_t = C_l St V_y \quad \text{as} \quad Re_V \ll 1, \tag{11}
\]

but \(C_l \neq C^{Sa}_l\) and its dependence on \(Re_G\) has to be calculated.

By balancing the lift (Eq. (11)) and the transverse drag (Eq. (5)) forces, we can find the critical Stokes number
\[
St_{cr} = \frac{f_y}{C_l}, \tag{12}
\]

which determines the onset of the unstable equilibrium. For \(St < St_{cr}\) the equilibrium is stable, but when \(St > St_{cr}\), the lift force becomes larger than the transverse drag, \(F^{Sa}_t > f_y V_y\), and the unstable regime develops. Note that at small \(Re_G\) the value of \(C_l\) can be found using (14), and the correction factor is \(f_y = 1\). Therefore, in this limiting case \(St_{cr} = 2.92 Re_G^{1/2}\).

Eq. (7) can be used to reformulate Eq. (12) as
\[
\left \frac{9f_y}{2 Re_G C_l} \right \ = \ \frac{\rho_p}{\rho}, \tag{13}
\]

This equation can be seen as an implicit condition on the critical Reynolds number. Thus, when the Saffman theory is valid, using Eq. (A4) one can obtain from (13) that
\[
Re_{cr} = 5.56 \left ( \frac{\rho_p}{\rho} \right )^{-2/3} \ll 1 \quad \text{as} \quad \rho_p/\rho \gg 1. \tag{14}
\]

It indicates that at a large density ratio, e.g. for aerosol particles the critical Reynolds numbers is small. Say, for water droplets in air (\(\rho_p/\rho \approx 800\)) the instability is expected at \(Re_G > Re_{cr} \approx 0.065\).

For smaller density ratios, including \(\rho_p/\rho = 1\) (neutrally buoyant particles), the instability should also occur, but at finite \(Re_G\). To generalize the scaling equation (14) to the case of finite \(Re_G\) we have to calculate \(f_y(Re_G)\) and \(C_l(Re_G)\). Moreover, in practice we normally deal with a wall-bounded flow, termed the Couette flow, where these coefficients depend on the channel thickness.

### III. PARTICLE MIGRATION UNDER A TRANSVERSE FORCE

The theoretical model described above corresponds to an idealized situation of migration in an unbounded shear flow and without external forces acting. In this Section we consider the channel flow, where the interactions with the walls should be taken into account. Besides, the particle may also experience an extra transverse force \(F_{ex} e_y\) induced by external fields (gravitational, electric, magnetic) or an additional hydrodynamic force, such as, for example, the Dean force (in curved channels) or the so-called neutrally buoyant lift force \(F^{nb}_l\).

The force \(F^{nb}_l\) is usually evaluated numerically, assuming that the particle is free to rotate and move in the \(x\)-direction, but is fixed in the transverse direction. Then, inertial migration of a particle to its equilibrium position is simulated by balancing the lift force \(F^{nb}_l\) and the transverse drag, i.e., the Saffman lift force \(F^{Sa}_l\) is neglected although it can be significant at finite \(Re_G\).

Let us now generalize our analysis of instability to the case of a finite transverse force \(F_{ex}(y_p)\) which depends only on the particle position \(y_p\), but not on its migration velocity \(V_y\). We make an additional assumption that the Reynolds number \(Re_V\) is small and the force changes slowly during particle migration, i.e. the characteristic migration time \(H/|V_y|\) is large compared to the hydrodynamic time scale \(G^{-1}\). Here \(H\) is the characteristic length scale for the change of \(F_{ex}\) (that is usually the channel width). This is justified, i.e. the ratio of the two time scales is large, provided particles are small
\[
\frac{GH}{|V_y|} = \frac{Re_G H}{Re_V a} \gg 1 \quad \text{as} \quad a/H \ll 1, \text{ Re}_V \ll \text{Re}_G.
\]

In this case the migration is quasi-steady so that the acceleration term in Eq. (5) can be ignored as in the neutral stability regime. We further assume that the Saffman lift \(F^{Sa}_l\) is controlled by streamwise acceleration of the particle due to its transverse motion, i.e., given by (10). For such a situation Eq. (5) for the transverse momentum can be rewritten as
\[
0 = -f_y (y_p) V_y + C_l (y_p) St (y_p) V_y + F_{ex} (y_p), \tag{15}
\]

where the Stokes number is based on a local shear rate \(G(y_p)\), and the coefficients \(f_y\) and \(C_l\) should depend on the location of the particle to account for hydrodynamic interactions with the walls.
Eq. (15) allows one to obtain the migration velocity under a slowly varying transverse force $F_{ex}$:

$$V_y = \frac{F_{ex}}{f_y - C_l \text{St}}.$$  

(16)

We recall that the migration velocity is usually determined by balancing $F_{ex}$ and $F_{dy} = f_y V_y$, so that $V_{y0} = F_{ex}/f_y$. Since for neutrally buoyant particles it is traditionally assumed that $f_y = 1$, the migration velocity is simply $V_{y0} = F_{ex}/f_y$. Our results, however, show that due to the effect of the Saffman lift force $V_y$ significantly deviates from $V_{y0}$, especially when the denominator in (16) is small.

If we consider a small perturbation $\delta V_y$ of the quasi-steady velocity (16), one can find that transverse motion becomes unstable when $C_l (y_p) \text{St} (y_p) - f_y (y_p) > 0$. Therefore, we recover the stability criterion (12) for the force-free case, but now it involves functions of $y_p$. This suggests that the motion can be unstable only in some parts of the channel.

Eq. (16) for the migration velocity can be rewritten as

$$V_y = \frac{F_{ex}}{f_y (1 - \text{Re}_G/\text{Re}_C)},$$  

(17)

which includes $\text{Re}_C$. The last equation should be used for interpreting experimental data on migration velocity. One can also conclude that the application of a steady state model to calculate the lift force from data obtained at finite $\text{Re}_G$ can strongly overestimate the result, and would also lead to incorrect scaling relationships.

IV. SIMULATION METHOD

To simulate the flow we use a 3D implementation of the lattice Boltzmann method (LBM) with a 19 velocity, single relaxation time scheme and the Batnagar Gross Krook (BGK) collision operator. Particles are discretized on the fluid lattice and implemented as moving no-slip boundaries following Ladd(24). The relaxation time of the BGK collision operator is fixed to unity leading to a kinematic viscosity of $\nu = 1/6$. Here and below the variables are given in simulation units. In addition, we set the fluid density $\rho = 1$. Further implementation details are provided in our previous publications (21,25,26).

The size of the computational domain in most simulations is $(N_x, N_y, N_z) = (200, 161, 100)$. We used spherical particles with radius $a = 8$, which provides $H/a \approx 20$, and prolate spheroids with equatorial radius $a = 4$ and polar radius $b = 8$. To generate a shear flow we implement impermeable no-slip walls moving with opposite velocities ($V_w$ at the top wall and $-V_w$ at the bottom wall) and impose periodic boundary conditions in the other two directions. The generated shear rate in simulation units is $G = 2V_w/(N_y - 1)$.

To search the unstable regimes for the particle equilibrium position at the channel mid-plane we vary $\text{Re}_G$ in the range from 0.25 to 2 and the particle density $\rho_p$ in the range from 15 to 200 to obtain different values of the Stokes number $\text{St}$. We set initial rotational velocity $\omega_z = -G/2$ and translational velocity $V = (0.1Ga, 0, 0)$, and fix the transverse coordinate $y_p = 0$ during $3 \times 10^4$ time steps, waiting for the system to equilibrate. Then, we release the particle and track its position for $\sim 10^5$ time steps. If the transverse coordinate $y_p$ grows with time exponentially the equilibrium is deemed unstable.

V. RESULTS AND DISCUSSION

It is of considerable interest to compare LBM simulation data with our analytical theory and to determine the regimes of validity of the theoretical results. Here we present results of our simulations together with specific calculations using theoretical expressions.

A. Lift-to-drag ratio and transverse drag

According to Eq. (12) the critical Stokes number that should give us the instability onset depends on the ratio of $C_l$ and $f_y$. We, therefore, begin with the investigation of these parameters.

To obtain the dependence of $C_l$ on $\text{Re}_G$ we put the particle at the mid-plane of the channel and then move it in the $x$-direction with a velocity $V_x$ that corresponds to fixed $\text{Re}_V = 0.05$. By setting different $V_w$ we vary $\text{Re}_G$ from 0.1 to 2. After equilibration, i.e. when the rotational velocity of the particle becomes stationary, we measure the lift $(F_{ex}^{Sa})$ and drag $(F_{dx})$ forces on the particle and average them over $10^4$ time steps. Then the lift-to-drag ratio is calculated as $C_l = F_{ex}^{Sa}/F_{dx}$. We remark that in these simulations we use our standard box that gives $H/a = 20$, but we have verified that the results for $C_l$ do not change if we set $H/a = 40$.

The computed lift-to-drag ratio $C_l = F_{ex}^{Sa}/F_{dx}$ as a function of $\text{Re}_G$ is shown in Fig. 2. We see that on
increasing Re\textsubscript{G} the lift-to-drag ratio increases quickly when Re\textsubscript{G} \ll 1 and then shows a weak nonlinear growth. The simulation data are compared with calculations from Eq. (A4), which is the Saffman formula derived for small Re\textsubscript{G}. It can be seen that the Saffman formula fits well the simulation data obtained at Re\textsubscript{G} \ll 1, but strongly over-estimates results at larger Re\textsubscript{G}. Also included in Fig. 2 are calculations made using

\begin{equation}
C_l = 0.343Re_G^{1/2} - 0.106Re_G \quad (18)
\end{equation}

obtained by fitting our data in the range Re\textsubscript{G} \leq 2. The first term here coincides with the Saffman lift C\textsubscript{Saff} \textsubscript{a}, and the second, linear in Re\textsubscript{G}, term is associated with a correction for finite Re\textsubscript{G}.

We now turn to the correction factor to the transverse drag \( f_y \), which depends not only on Re\textsubscript{G}, but also on \( a/H \) and \( y_p \) due to the wall effect. To obtain \( f_y = F_y/V_y \) as a function of \( y_p \) we place a particle at \( y = 0.4 \), apply a small vertical force \( F_y \), and then measure \( V_y \) along the trajectory. In these simulations we fix Re\textsubscript{G} = 0, i.e. perform measurements in a stagnant fluid since at finite Re\textsubscript{G} it is difficult to distinguish between the transverse drag and the lift force arising when the particle moves in the \( y \)-direction. We stress, however, that the effect of \( H/a \) on \( f_y \) is stronger than that of Re\textsubscript{G}. Consequently, the qualitative features of the \( f_y \) curves at finite Re\textsubscript{G} are the same, and the quantitative difference from the case of Re\textsubscript{G} = 0 should be insignificant. Fig. 3 shows \( f_y \) plotted as a function of particle position \( y_p \), multiplied by \( a/H = 0.05 \). It has been earlier proposed that a sensible approximation for \( f_y \) in the case of the channel can be simply a superposition of single-wall contributions

\begin{equation}
f_y = 1 + \frac{1}{H/2a + 1} + \frac{1}{H/2a - 1 - y_p}. \quad (19)
\end{equation}

The calculations from Eq. (19) are also shown in Fig. 3 and we see that the fit is quite good. The function \( f_y(y_p) \) takes its minimum value (of ca. 1.22 with our parameters) at the mid-plane, \( y_p = 0 \). Note that this exceeds \( f_y = 1 \) corresponding to the Stokes drag in an unbounded flow. On approaching the walls \( f_y \) increases, which implies that the critical Stokes number St\textsubscript{cr} given by Eq. (12) also grows. Consequently, inertial migration in the near-wall region can remain stable even when the stability condition is violated in the central part of the channel.

B. Instability for spherical particles in Couette flow

Next we examine the dependence of St\textsubscript{cr} on Re\textsubscript{G} and \( \rho_p/\rho \). In these simulations particles are released at the mid-plane of the channel with a small initial velocity in the \( x \)-direction.

Figure 4 show the time dependence of particle trajectories and transverse velocities obtained at Re\textsubscript{G} = 0.5. These simulations are made using \( \rho_p/\rho \) from 50 to 80, which corresponds to St varying from 5.6 to 8.9. It can be seen that the particle with St = 5.6 remains at the mid-plane, but those with larger St accelerate in the \( y \)-direction, demonstrating the instability of the mid-plane equilibrium (Fig. 3(a)). In turn, the velocity \( V_y \) reduces for the particle with St = 5.6, but augments exponentially with time if St is larger (Fig. 3(b)). The simulation data show that at large Stokes numbers the transverse velocity grows with their value.

If similar analysis is made to a variety of simulations performed at different Re\textsubscript{G} and \( \rho_p/\rho \), we can find St\textsubscript{cr} that determines an onset of instability depending on these parameters. Fig. 5 summarizes the simulation re-
squares correspond to simulation data for spheroids.

particles discernibly deviate from their equilibrium positions, tend to oscillate of their motion. The particles then accelerate towards the opposite wall, etc. In other words, we observe the oscillations of particles between channel walls instead of immobilization due to wall-bounded flow the local St depends on St. We see in Fig. 6 that the particles of larger St accelerate faster and oscillate with a smaller period, but larger amplitude. It is well seen that the extrema of y_p become less pronounced and of smaller absolute value on decreasing St – St_{cr}. Clearly, the oscillations would disappear at St = St_{cr} and smaller. These observations are, of course, very different from expected for an unbounded shear flow, where the particle would accelerate until the lift force (which reduces with Re_G) becomes equal to the transverse drag. However, for our wall-bounded flow the local St_{cr} depends on y_p. Besides, in addition to the drag and the Saffman lift forces, the neutrally buoyant lift force F_{nb}(y_p) (the force F_{nb} in Eq. (14)) is acting on the particles. The latter does not depend on V_y and is caused by inertial hydrodynamic interactions with the walls. Note that although this force is traditionally termed neutrally buoyant, it would be the same for particles of any density. The local St_{cr} increases with the absolute value of y_p, i.e. on approaching the wall, since C_l decreases near the wall, but f_y is much larger near the wall than in the central part of the channel (see Fig. 5). As a result, the particles retard near the wall and V_y tends to zero, so does the Saffman lift force given by (11). One can speculate that particles commence the movement towards an opposite wall instead of immobilization due to F_{nb} that is directed away from the wall. They are pushed back to the mid-plane, but since the equilibrium there is unstable, continue to migrate until approach the wall.

Similar oscillatory trajectories have been found in simulations of neutrally buoyant and heavy particles migrating in pressure-driven flows, but no attempt has been made to connect these results to the equilibrium instability. In Fig. 7 the data and the theoretical cal-

Figure 5. Stability diagram for spherical particles in the Couette flow. Filled circles correspond to the onset of instability obtained in simulations. The solid line is calculated from Eq. (12) using C_l given by Eq. (15) and f_y = 1.22. Open squares correspond to simulation data for spheroids.

Figure 6. Evolution of transverse positions in Couette flow for the particles with Re_G = 1 and St = 7.4 (solid), 7.9 (dashed) and 9 (dashed-dotted). Dotted lines indicate contact with the walls.

Figure 7. The same as in Fig. 5 but plotted in the (\rho_p/\rho, Re_G) plane. Dashed line is calculated from Eq. (14). The data inferred from earlier results for a pressure-driven flows are shown by filled triangles and diamonds (flat-parallel channels), the star (a circular tube), and the square (aerosol particles in a square channel). The earlier data for steady-state trajectories are marked by the open triangle and diamond.

results (black circles) obtained for Re_G from 0.25 to 2 and several density ratios \rho_p/\rho in the range from 15 to 200 in the (Re_G, St) plane. The solid line is calculated from Eq. (12). The calculations are made using f_y = 1.22 and C_l given by Eq. (18). An overall conclusion from this plot is that finite Re_G dramatically reduce the value of St_{cr}. We also conclude that the theory reproduces well the qualitative features of the neutral stability curve, although there is some quantitative discrepancy. The discrepancy is always in the direction of smaller St_{cr} than obtained in simulations, which is likely due to underestimated (obtained for Re_G = 0) f_y used in theoretical calculations.

We now fix Re_G = 1 and monitor the time evolution of y_p at several supercritical St. The results are plotted in Fig. 6. It can be seen that after some interval of time particles discernibly deviate from their equilibrium position y_p = 0, and that they move with the acceleration towards the wall indicating unstable equilibrium. However, they slow down in the near-wall region, and, without making contact with the wall, reverse the direction

\[ y_p = \frac{y}{H} \]

\[ t = \frac{t}{H/V_y} \]

\[ V_y = \frac{V}{H} \]

\[ St = \frac{y_p}{V} \]

\[ Re_G = \frac{Re}{\rho G} \]
calculation are reproduced from Fig. 5 but plotted in the $(\rho_p/\rho, \text{Re}_G)$ plane and in a log-log scale. They are compared with the above mentioned simulation data obtained for pressure-driven flows and with another calculation, made from Eq.(14). It can be seen that Eq. (12) provides quite good fit of our simulation data, but Eq.(14) underestimates $\text{Re}_{Gc}$. We also conclude that earlier data for heavy particles in oscillatory regimes (filled symbols) always either fall into the instability region of a diagram or coincide with its onset. However, the data for steady-state trajectories (open symbols) fall into the stability region. Finally, we remark that the neutrally buoyant particles in an unstable equilibrium correspond to $\text{Re}_G \simeq 20$, which is close to the values at which the Segre-Silberberg equilibrium position disappears, leaving only the inner annulus. Therefore, one can speculate that the particle inertia and the Saffman lift force may be important for interpreting this phenomenon too.

C. Translational instability for spheroids

Our theory and above simulation results refer to spherical particles. Here we report some simulation data showing that our model could be suitable for spheroid particles too.

We investigate prolate spheroids of a polar radius $b$ and an equatorial radius $a$. In all simulations we use $H/b = 20$ and aspect ratio $b/a = 2$. We define

$$\text{Re}_G = \frac{Gb^2}{\nu}, \quad \text{St} = \frac{2\rho_pGa^2}{9\mu} = \frac{2\rho_p}{9\rho} \left(\frac{a}{b}\right)^2 \text{Re}_G.$$  \hspace{1cm} (20)

and fix $\text{Re}_G = 0.5$ and $\text{St} = 7.8$. The spheroid is initially located at the mid-plane with some small inclination relative to $x$–axis and move with $V_z$. The time evolution of the symmetry vector $\mathbf{n} = (n_x, n_y, n_z)$, which characterizes the particle orientation in the channel, is illustrated in Fig. 8(a). We see that after some time a stable tumbling motion is established. This observation is in agreement with prior work. Simultaneously, the spheroid migrates in the $y$–direction with a growing with time velocity as seen in Fig. 8(b). This is exactly what we have observed for spheres (cf. Fig. 4(a)). We have performed additional simulations using several $\text{Re}_G$ and $\text{St}$. The results are included in Fig. 5 and indicate that the onset of instability for our spheroid particles is very close to that for the spheres.

VI. CONCLUSION

We have demonstrated theoretically that migration of inertial spherical particles in a shear flow becomes unstable, thanks to the Saffman lift force. It is shown that when their Stokes number exceeds the critical value, inertial particles migrate with an exponential acceleration. Lattice Boltzmann simulations of the critical Stokes numbers generally validate our analysis. Simulations also show that our simple theoretical model is also applied for prolate spheroids, and that the lift-induced instability of spheroid motion occurs approximately at the same Stokes numbers as for spheres.

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Appendix A: Calculation of the lift force on a migrating particle

In this Appendix, we derive the formula for the Saffman lift force on a particle migrating in the neutral-stability regime.

In his pioneering work Saffman calculated a lift force on a sphere moving in unbounded shear flow $U = ye_x$ with a constant velocity $V = V_x e_x$ parallel to the flow, using the method of matched asymptotic expansions. In the inertial coordinate system associated with the particle, $X = (X, Y, Z) = x - x_p$, the unperturbed flow reads

$$U - V = (Y - V_x)e_x,$$  
(A1)

where $V_x = V - y_p$ is particle slip velocity.

The problem was solved in a strong shear limit, when the shear-based and slip-based particle Reynolds numbers satisfy the condition

$$Re_Y \ll Re_G^{1/2} \ll 1. \quad \text{(A2)}$$

Condition (A2) means that the linear flow dominates over the slip velocity in the outer region of the flow where $Y \sim Re_G^{-1/2}$. Therefore, far from the particle the unperturbed flow is $U - V \simeq Ye_x$ and the disturbance induced by the particle velocity $u$ is governed by the Oseen-like equations,

$$Re_G \left(Y \frac{\partial u}{\partial X} + u_x e_x\right) + \nabla p - \nabla^2 u = 6\pi F_p \delta (X). \quad \text{(A3)}$$

Here, the terms in the brackets are the Oseen-like inertial terms, $\delta (X)$ is the delta-function, so that the particle effect is approximated by the point force $F_p$ exerted by the particle on the fluid. For a particle moving with constant slip velocity $V_x$ this force is equal and opposite to the drag on the particle, $F_p = -F_{dx} e_x$. Therefore, the lift force is proportional to the drag, and the ratio of the two forces is

$$C_l^{Sa} = \frac{F_p}{F_{dx}} = 0.343 Re_G^{1/2} \quad \text{for} \quad Re_G \ll 1. \quad \text{(A4)}$$

Equations (A3) and (A4) are usually written in terms of the slip velocity $V_s$, since $F_{dx} = -V_s$ for the steady case at $Re_Y$, $Re_G \ll 1$.

In our case the situation is different, since the particle accelerates in the streamwise direction and migrates in the transverse direction. We consider the flow using a non-inertial coordinate system translating with the particle, $X = (X, Y, Z)$. The unperturbed flow around the particle then reads

$$U - V = Ye_x - (V_x - y_p) e_x - V_y e_y. \quad \text{(A5)}$$

For the neutral-stability regime, the streamwise component of the force $F_p$ can be found by using Eqs. (4), (5), while the transverse forces are balanced, and hence $F_p = -StV_y e_y$. Since the force is constant the disturbance flow is steady. Assuming that $Re_Y, Re_G$ satisfy the condition (A2), we can neglect the last two terms in (A9) in the outer region. Therefore, the disturbance velocity $u$ in our case is governed by the momentum equation similar to Eq. (A3), with the drag force $F_{dx}$ replaced by $StV_y$. The lift force on the particle in the neutral stability regime is then given by

$$F_l^{Sa} = C_l^{Sa} StV_y \quad \text{as} \quad Re_G \ll 1. \quad \text{(A6)}$$
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