Transience, Recurrence and the Speed of a Random Walk in a Site-Based Feedback Environment

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Abstract
We study a random walk on $\mathbb{Z}$ which evolves in a dynamic environment determined by its own trajectory. Sites flip back and forth between two modes, $p$ and $q$. $R$ consecutive right jumps from a site in the $q$-mode are required to switch it to the $p$-mode, and $L$ consecutive left jumps from a site in the $p$-mode are required to switch it to the $q$-mode. From a site in the $p$-mode the walk jumps right with probability $p$ and left with probability $1 - p$, while from a site in the $q$-mode these probabilities are $q$ and $1 - q$.

1 Introduction and Statement of Results

In this paper we introduce a process we call a site-based feedback random walk on $\mathbb{Z}$. The process $(X_n)_{n\geq 0}$ is a nearest neighbor random walk governed by four parameters: $p, q \in (0, 1)$ and $R, L \in \mathbb{N}$. An informal description is as follows.

Initially each site $x \in \mathbb{Z}$ is set to either the $p$-mode or the $q$-mode. From a site in the $p$-mode the walk jumps right with probability $p$ and left with probability $1 - p$, whereas from a site in the $q$-mode these probabilities are $q$ and $1 - q$, respectively. Also, a site $x$ switches from the $q$-mode to the $p$-mode after the walk jumps right from $x$ on $R$ consecutive visits to $x$, and a site $x$ switches from the $p$-mode to the $q$-mode after the walk jumps left from $x$ on $L$ consecutive visits to $x$.

In light of this description, we say the random walk $(X_n)$ has positive feedback if $q < p$ and negative feedback if $q > p$. Of course, if $q = p$ the situation is trivial; we just have a simple random walk of bias $p$.

We now give the formal description and set some notation.

- $\Lambda = \{(p, 0), \ldots, (p, L - 1), (q, 0), \ldots, (q, R - 1)\}$ is the set of single site configurations. A typical configuration is denoted $\lambda = (r, i)$, where $r \in \{p, q\}$ is the mode and $i$ is the number of charges in favor of the alternative mode.

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For each identical at each \( M \) consider periodic cookies with period \( M \). Instead of having a cookie only the first \( M \) times, whenever the process visits that site again, it behaves like an ordinary simple, symmetric random walk, jumping left or right with equal probability. This site-based feedback random walk is motivated by so-called cookie random walks and shares certain fundamental properties of two outgrowths of the basic cookie random walk. A basic cookie random walk on \( \mathbb{Z} \) is defined as follows. Let \( M \geq 1 \) be a positive integer. At each site \( x \in \mathbb{Z} \), place a pile of \( M \) “cookies” with values \( \omega(x,k) \in [0,1], k = 1, \ldots, M \). For \( k \leq M \), the \( k \)-th time the process reaches site \( x \), it eats the \( k \)-th cookie at that site, whose value is \( \omega(x,k) \), and this empowers the process to jump to the right with probability \( 1 - \omega(x,k) \) and to the left with probability \( 1 - \omega(x,k) \). After the site \( x \) has been visited \( M \) times, whenever the process visits that site again, it behaves like an ordinary simple, symmetric random walk, jumping left or right with equal probability. Cookie random walks were first introduced by Benjamini and Wilson [1]; see the survey paper of Kosygina and Zerner [2] for more on cookie random walks and a bibliography.

We now describe two outgrowths of the basic cookie random walk. Kozma, Orenshtein, and Shinkar [3] recently considered a periodic cookie random walk. Instead of having a cookie only the first \( M \) times the process is at a given site, consider periodic cookies with period \( M \), and assume that these cookies are identical at each \( x \in \mathbb{Z} \). Thus, one defines \( \omega(k), k \in \mathbb{N} \), with \( \omega(k+M) = \omega(k) \). For each \( x \in \mathbb{Z} \), the \( k \)-th time the process is at \( x \) it jumps right or left with probabilities \( \omega(k) \) and \( 1 - \omega(k) \) respectively. In particular, the process never reverts to a simple, symmetric random walk at any site. Another outgrowth of the basic cookie random walk is the “have your cookie and eat it” random walk.
Now there is only one cookie at each site; call it $\omega(x) , x \in \mathbb{Z}$, and assume $\omega(x) > 1/2$. When the process first reaches $x$, it jumps right with probability $\omega(x)$ and left with probability $1 - \omega(x)$. For each site $x$, as long as the process continues to jump to the right from $x$, it continues to use this right-biased cookie; but after the first time the process jumps to the left from $x$, the cookie at $x$ is removed. From then on, whenever the process is at $x$, it behaves like a simple, symmetric random walk, jumping left or right with equal probability.

The site-based feedback random walk has something in common with each of the two above processes. In particular, as with the periodic cookie process, the site-based feedback process never reverts to the simple, symmetric random walk at any site, and as with the “have your cookie and eat it” process, the site-based feedback process’ current jump mechanism at a site depends upon the direction in which the process has jumped in the past from that site.

In this paper we study the transience/recurrence properties of the site-based feedback random walk, and in the transient case we study the speed of the process. For this process, some new features occur that were not present in other cookie random walk models. In particular, the initial environment influences the above properties in an interesting way.

Before stating the results, we need to introduce a bit more notation and terminology. Let $P_{\omega,k}$ denote the probability measure for the random walk started at $X_0 = k$ in the initial environment $\omega$, and let $P_{\omega} = P_{\omega,0}$. Also, let $E_{\omega}$ and $E_{\omega,k}$ denote, respectively, expectations with respect to the measures $P_{\omega}$ and $P_{\omega,k}$. Finally, for $x \in \mathbb{Z}$, let $N_x$ be the total number of visits to site $x$:

\[ N_x = |\{ n \geq 0 : X_n = x \}|. \] (1)

We say that the random walk path $(X_n)$ is:

- **recurrent** if $N_0 = \infty$.
- **right transient**, or **transient to $+\infty$**, if $\lim_{n \to \infty} X_n = +\infty$, and **left transient**, or **transient to $-\infty$**, if $\lim_{n \to \infty} X_n = -\infty$.
- **ballistic** if $\lim \inf_{n \to \infty} |X_n|/n > 0$.

Our first theorem gives the cutoff point for left/right transience.

**Theorem 1.** Define $0 < \alpha < 1$ by

\[ \alpha = \frac{p \cdot [(1-q)q^R(1-(1-p)L)] + q \cdot [(p(1-p)L(1-q^R))]}{[(1-q)q^R(1-(1-p)L)] + [p(1-p)L(1-q^R)]}. \] (2)

- If $\alpha > 1/2$ then the random walk $(X_n)$ is $P_{\omega}$ a.s. right transient, for any initial environment $\omega$.
- If $\alpha < 1/2$ then the random walk $(X_n)$ is $P_{\omega}$ a.s. left transient, for any initial environment $\omega$. 

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We will call the vector \((p, q, R, L)\) the parameter quadruple for the random walk \((X_n)\). In light of Theorem 1, we say that the parameter quadruple \((p, q, R, L)\) is critical if \(\alpha(p, q, R, L) = 1/2\), and noncritical otherwise. Our next theorem shows that in the noncritical case, the random walk is not just transient, but in fact ballistic.

**Theorem 2.** If \(\alpha(p, q, R, L) \neq 1/2\), then there exists a \(\beta = \beta(p, q, R, L) > 0\) such that, for any initial environment \(\omega\),

\[
\liminf_{n \to \infty} \frac{|X_n|}{n} \geq \beta, \ P_\omega \text{ a.s.} \tag{3}
\]

Moreover, if \(\alpha > 1/2\) (\(\alpha < 1/2\)) and the initial environment \(\omega(x)\) is constant for \(x \geq m\) \((x \leq -m)\) then \(E_\omega(N_x)\) is also constant for \(x \geq m\) \((x \leq -m)\), and denoting this common value by \(\gamma\),

\[
\lim_{n \to \infty} \frac{|X_n|}{n} = \frac{1}{\gamma}, \ P_\omega \text{ a.s.} \tag{4}
\]

Here, \(m\) can be any nonnegative integer.

**Remark.** We note that the speed, when it exists, clearly depends on the initial environment \(\omega\).

The following proposition characterizes some properties of the fundamental function \(\alpha\). We choose to analyze \(\alpha\) as a function of \(p\) for fixed \(R, L, q\); of course, a similar analysis also works to analyze \(\alpha\) as a function of \(q\) for fixed \(R, L, p\).

**Proposition 1.** Let \(R, L, q\) be fixed and consider \(\alpha\) as a function of \(p\), \(\alpha(p) \equiv \alpha(p, q, R, L)\).

(i) If \(q = 1/2\), then

\[
\alpha(1/2) = 1/2, \ \alpha(p) < 1/2 \ for \ p < 1/2, \ \alpha(p) > 1/2 \ for \ p > 1/2.
\]

(ii) If \(q < 1/2\), then there exists a unique critical point \(p_0 = p_0(q, R, L) \in (1/2, 1)\) such that

\[
\alpha(p_0) = 1/2, \ \alpha(p) < 1/2 \ for \ p < p_0, \ \alpha(p) > 1/2 \ for \ p > p_0. \tag{5}
\]

(iii) For \(q < 1/2\) the critical point \(p_0\) from (ii) satisfies

\[
q + p_0(q, R, L) < 1, \ if \ R < L;
q + p_0(q, R, L) > 1, \ if \ R > L. \tag{6}
\]

Also, for any fixed \(R\) and \(L\), \(p_0(q, R, L)\) is a decreasing function of \(q\), for \(q \in (0, 1/2)\).
(iv) If \( q < 1/2 \) and \( L = 1 \), then

\[
p_0 = \frac{1 - 2q + q^{R+1}}{1 - 2q + q^R}.
\]

If \( q > 1/2 \) and \( L = 1 \), then (5) still holds with

\[
p_0 = \frac{1 - 2q + q^{R+1}}{1 - 2q + q^R}
\]

as long as \( 1 - 2q + q^{R+1} > 0 \). However, if \( 1 - 2q + q^{R+1} \leq 0 \), then \( \alpha(p) > 1/2 \), for all \( p \in (0,1) \).

(v) If \( q < 1/2 \) and \( L = R \), then \( p_0 = 1 - q \). If \( q > 1/2 \) and \( L = R \), then \( 1 - q \) is still a critical point (i.e. \( \alpha(1 - q) = 1/2 \)), but it is not always unique.

(vi) For any \( R, L, q \), \( \lim_{p \to 1} \alpha(p) = 1 \). In particular, \( \alpha > 1/2 \) for all sufficiently large \( p \).

**Remark 1.** Part (v) shows that \( \alpha(p) \) is not always a monotonic function of \( p \), and, in fact, often it is not. Consequently, increasing \( p \) (with fixed \( q, R, L \)) may sometimes change the process from the right transient regime to the left transient regime. However, this phenomena can only occur when \( q > 1/2 \), by part (ii), in which case the process has negative feedback at all critical points. An illustrative plot is given below in Figure 1.

![Figure 1: Plot of \( \alpha(p) \) with \( L = 10, R = 10, q = 0.75 \). As \( p \) increases from 0 to 1 the parameter quadruple \( (p, q, R, L) \) passes from right transient \((\alpha > 1/2)\), to left transient \((\alpha < 1/2)\), and back to right transient.](image-url)
Remark 2. As noted before the proposition, we could have considered $\alpha$ as a function of $q$ for fixed $p, R, L$. We note, in particular, that in the case that $p > 1/2$, there exists a unique critical point $q_0 = q_0(p, R, L) \in (0, 1/2)$, and when in addition, $R = 1$, one has

$$q_0 = \frac{p(1-p)^L}{2p - 1 + (1-p)^L}.$$  \hspace{1cm} (8)

Moreover, if $R = 1$ and $p \leq 1/2$, then there is still a unique critical point $q_0$ given by (8) as long as $2p - 1 + (1-p)^L > 0$. However, if $2p - 1 + (1-p)^L \leq 0$, then $\alpha < 1/2$, for all $q \in (0, 1)$.

In general for cookie-type random walks, it is very difficult to obtain an explicit formula for the speed in the ballistic regime. However, the additional level of interaction between the random walker and the environment in the site-based feedback case makes a calculation of the speed possible in some situations. Before moving on to the critical case, we present two results that give an exact characterization of the limiting speed with $R$ or $L$ equal to 1, in certain initial environments. We assume that $\alpha > 1/2$, but analogous results for $\alpha < 1/2$ are easily inferred by symmetry considerations. Specifically, if $\alpha(p, q, R, L) < 1/2$ then $\alpha(1-q, 1-p, L, R) > 1/2$, and the speed to $-\infty$ with parameters $p, q, R, L$ in an initial environment $\omega$ is the same as the speed to $+\infty$ with parameters $1-q, 1-p, L, R$ in an initial environment $\omega'$ defined by $\omega'(x) = \omega(-x)^{\ast}, x \in \mathbb{Z}$, where $(q, i)^{\ast} = (1-q, i)$ and $(p, i)^{\ast} = (1-p, i)$.

Theorem 3. Let $L = 1$ and $\alpha > 1/2$. If $\omega(x) = (q, 0)$ in a neighborhood of $+\infty$, then

$$\lim_{n \to \infty} \frac{X_n}{n} = \frac{1 - t_{\ast}}{1 + t_{\ast}}, \quad \mathbb{P}_\omega \text{ a.s.} \hspace{1cm} (9)$$

where $t_{\ast}$ is the unique root of the polynomial

$$P(t) = (1-q) + (pq - p - 1)t + (p + q)t^2 - pq^3 - (p-q)q^R(t^R - t^{R+1}) \hspace{1cm} (10)$$

in the interval $(1-q, 1)$.

Theorem 4. Let $R = 1$ and $\alpha > 1/2$. Assume that the limiting right density of $(p, i)$ sites $d_i = \lim_{n \to \infty} \frac{1}{n} |\{0 \leq x \leq n - 1 : \omega(x) = (p, i)\}|$ exists, for each $0 \leq i \leq L - 1$, and let $d_L = 1 - \sum_{i=0}^{L-1} d_i$ denote the limiting right density of $(q, 0)$ sites. Then,

$$\lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{\sum_{i=0}^{L-1} a_id_i}, \quad \mathbb{P}_\omega \text{ a.s.}, \hspace{1cm} (11)$$

where

$$a_0 = \frac{1 + (p/q - 1)(1-p)^L}{(2p - 1) - (p/q - 1)(1-p)^L}, \hspace{1cm} (12)$$

and, for $1 \leq i \leq L$,

$$a_i = \frac{1 + (p/q - 1)(1-p)^{L-i}}{p} + \left(\frac{(1-p) + (p/q - 1)(1-p)^{L-i}}{p}\right) a_0. \hspace{1cm} (13)$$
We now turn to the critical case, $\alpha = 1/2$. Here, there are two possibilities: positive feedback with $q < 1/2 < p$ or negative feedback with $p < 1/2 < q$. In the case of positive feedback the situation is somewhat simpler, but in both cases the analysis is more delicate than before, and the transience/recurrence of the random walk often depends heavily on the initial environment.

We begin with a result which states that in the case of positive feedback, there always exist initial environments for which the random walk is recurrent. In the case of negative feedback, we expect there are also initial environments for which the random walk is recurrent. However, there are some technical issues that must still be resolved.

**Theorem 5.** If $\alpha(p, q, R, L) = 1/2$ and $q < p$, then there exist initial environments $\omega$ for which the random walk $(X_n)$ is a.s. recurrent. In particular, if $\omega(x) = (q, 0)$ for $x$ in a neighborhood of $+\infty$ and $\omega(x) = (p, 0)$ for $x$ in a neighborhood of $-\infty$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. recurrent.

Our remaining results concern the situation that either $L$ or $R$ is 1. In this case, we can give an essentially complete description of when the random walk is recurrent, right transient, or left transient. However, for technical reasons, we will need to assume in many instances that the initial environment $\omega$ is constant in a neighborhood of $+\infty$, a neighborhood of $-\infty$, or both. Our first result indicates that, when $R$ or $L$ is equal to 1, only one of the two directions is possible for transience.

**Theorem 6.** Assume $\alpha = 1/2$.

- If $R = 1$ and $q < p$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. not transient to $+\infty$, for any initial environment $\omega$.
- If $R = 1$ and $p < q$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. not transient to $+\infty$, for any environment $\omega$ which is constant in a neighborhood of $+\infty$.
- If $L = 1$ and $q < p$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. not transient to $-\infty$, for any initial environment $\omega$.
- If $L = 1$ and $p < q$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. not transient to $-\infty$, for any environment $\omega$ which is constant in a neighborhood of $-\infty$.

The following corollary is an immediate consequence of this theorem and part (ii) of Lemma 1 in Section 2.2.

**Corollary 1.** Let $R = L = 1$ and $p = 1 - q$, so $\alpha = 1/2$.

- If $q < p$ then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. recurrent, for any initial environment $\omega$.
- If $p < q$ then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. recurrent, for any initial environment $\omega$ which is constant in neighborhoods of $+\infty$ and $-\infty$. 
Our next theorem gives specific conditions to determine if the random walk is recurrent or transient to $+\infty$ in the case $L = 1$ and $R > 1$ (which, by Theorem 6, are the only possibilities). By symmetry considerations, if $R = 1$, instead of $L = 1$, then the result obtained for $L = 1$ will hold with the roles of $q, p, R$ and $\pm \infty$ replaced by $1 - p, 1 - q, L$ and $\mp \infty$ respectively.

**Theorem 7.** Assume that $L = 1$, $R \geq 2$, and $\alpha = 1/2$. Thus, by (7), $p = p_0 = \frac{1 - 2q + q^{R+1}}{1 - 2q + q^R}$. In the case of negative feedback, $p < q$, assume also that the initial environment $\omega$ is constant in a neighborhood of $-\infty$.

(i) If $\omega(x) = (q, 0)$ in a neighborhood of $+\infty$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. recurrent.

(ii) If $\omega(x) = (q, i)$ in a neighborhood of $+\infty$, $1 \leq i \leq R - 1$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. recurrent if $P_{R,i}(q) \geq 0$, and $\mathbb{P}_\omega$ a.s. transient to $+\infty$ if $P_{R,i}(q) < 0$,

where

$$P_{R,i}(q) = (2R - 1)q^{R+2} - (3R + 1)q^{R+1} + (R + 1)q^R - 2q^{R+2-i} + 3q^{R+1-i} - q^{R-i} + (1 - 2q)^2. \quad (14)$$

(iii) If $\omega(x) = (p, 0)$ in a neighborhood of $+\infty$, then the random walk $(X_n)$ is $\mathbb{P}_\omega$ a.s. recurrent if $P_{R,R}(q) \geq 0$, and $\mathbb{P}_\omega$ a.s. transient to $+\infty$ if $P_{R,R}(q) < 0$,

where

$$P_{R,R}(q) = (2R - 1)q^{R+2} - (3R + 1)q^{R+1} + (R + 1)q^R + 2q^2 - q. \quad (15)$$

Moreover, for each $R \geq 2$ there exists a unique root $q_\ast(R) \in (0, 1/2)$ of the polynomial $P_{R,R}(q)$, $P_{R,R}(q) > 0$ for $q > q_\ast(R)$, $P_{R,R}(q) < 0$ for $q < q_\ast(R)$, and $\lim_{R \to \infty} q_\ast(R) = 1/2$.

**Remark 1.** In the case of positive feedback, $p > q$, one can also determine between right transience and recurrence for some environments that are not constant in a neighborhood of $+\infty$, using the comparison lemma given in Section 2.3. In particular, $(p, 0)$ is the most favorable environment for right transience in the positive feedback case, so if the random walk is not right transient with the $(p, 0)$ environment in a neighborhood of $+\infty$, then it is not right transient for any initial environment.

**Remark 2.** $P_{R,R}(q)$ is, of course, the same polynomial one obtains by substituting $i = R$ into the definition of $P_{R,i}(q)$.
Remark 3. For any $1 \leq i \leq R$, $P_{R,i}(q)$ has a double root at 1. $P_{R,R}(q)$ also has a single root at 0 and factors as $P_{R,R}(q) = q(1-q)^2 \tilde{P}_{R,R}(q)$ where

$$\tilde{P}_{R,R}(q) = -1 + \sum_{j=1}^{R-3} jq^{j+1} + (2R-1)q^{R-1}. \quad (16)$$

Here, the sum is defined to be 0, for $R = 2, 3$.

Remark 4. Using (16) one finds that $q_*(2) = 1/3$ and $q_*(3) = 1/\sqrt{3} \approx 0.477$.

Using a combination of analytical techniques and computer generated plots one also finds the following behavior for $P_{R,i}$, $1 \leq i \leq R - 1$. For $R = 2, 3, 4$, $P_{R,i}(q) \geq 0$ for all $1 \leq i \leq R - 1$ and $q \in (0,1)$. For $R = 5, 6$, $P_{R,i}(q) \geq 0$ for all $1 \leq i \leq R - 2$ and $q \in (0,1)$. However, $P_{5,4}(q) < 0$ if (and only if) $q \in (a,b)$, where $a \approx .410$ and $b \approx .473$, and $P_{6,5}(q) < 0$ if (and only if) $q \in (a,b)$, where $a \approx .391$ and $b \approx .490$. For $i = 5, 6$ there are ranges of $q$ for which $P_{R,i} < 0$.

Our final result characterizes asymptotic properties of the function $P_{R,i}(q)$ in the limit of large $R$, for two different cases of $i = i_R$. In the first case, $R - i_R$ grows to infinity, so the process must jump right many consecutive times from a given site to switch it to the $p$-mode, starting in the $(q, i_R)$ initial environment. In the second case, $i_R = R - k$, for a fixed $k$, so the process need jump right only $k$ consecutive times from any site to switch it to the $p$-mode, starting in the $(q, i_R)$ initial environment. The proof of both cases is straightforward, and is left to the reader.

Proposition 2.

(i) If $(R - i_R) \to \infty$ as $R \to \infty$ then, for any fixed $q \in (0,1)$, $P_{R,i_R}(q) > 0$ for all sufficiently large $R$.

(ii) Let $i_R = R - k$, and define

$$Q_k(q) = (1-2q)^2 - q^k + 3q^{k+1} - 2q^{k+2}. \quad (17)$$

If $Q_k(q) > 0$ then $P_{R,i_R}(q) > 0$ for sufficiently large $R$, and if $Q_k(q) < 0$ then $P_{R,i_R}(q) < 0$ for sufficiently large $R$.

Remark The polynomial $Q_k(q)$ factors as $(2q-1)(2q-1+q^k-q^{k+1})$. Using this representation it is not hard to verify the following facts: $Q_k(q) > 0$ for $q > 1/2$, and there exists an $a_k \in (1/2, 3 - \sqrt{5}, 1/2) \approx (.382, 1/2)$ such that $Q_k(q) > 0$ for $q \in (0, a_k)$ and $Q_k(q) < 0$ for $q \in (a_k, 1/2)$. Furthermore, $a_k$ is increasing in $k$ and $\lim_{k \to \infty} a_k = 1/2$.

The remainder of the paper is organized as follows. In Section 2, we introduce some important constructions that will be central to our proofs and establish a number of simple lemmas. In Section 3 we prove Theorems 1-4 concerning the behavior of the random walk $(X_n)$ in the noncritical case. In Section 4 we prove Theorems 5-7 concerning the behavior of the random walk in the critical case. Finally, in Section 5 we prove Proposition 1 which characterizes properties of the important function $\alpha$. 
2 Preliminaries

In this section we introduce a basic framework for proving the theorems stated above and establish a number of useful lemmas. Section 2.1 gives constructions of the single site Markov chains \((Y_n^x)_{n \in \mathbb{N}}\) and the right jumps Markov chain \((Z_n^x)_{x \geq 0}\), which will be the primary tools used in the proofs of Theorems 1.2 and 7. Section 2.2 gives three simple lemmas that will be used in a number of places. The first two concern conditions for transience, and the other relates hitting times to speed. Finally, Section 2.3 gives an important lemma comparing the possibility of transience in different environments.

2.1 Auxiliary Markov Chains

2.1.1 The Single Site Markov Chains \((Y_n^x)_{n \in \mathbb{N}}\)

Let \(M\) be the stochastic transition matrix on the set of single site configurations \(\Lambda\), with nonzero entries defined as follows:

\[
M_{i,j} = p, M_{(p,i)\rightarrow (p,i+1)} = 1 - p, \quad \text{for } 0 \leq i \leq L - 2.
\]

\[
M_{i,j} = p, M_{(p,L-1)\rightarrow (q,0)} = 1 - p.
\]

\[
M_{i,j} = 1 - q, M_{(q,i)\rightarrow (q,i+1)} = q, \quad \text{for } 0 \leq i \leq R - 2.
\]

\[
M_{i,j} = 1 - q, M_{(q,R-1)\rightarrow (p,0)} = q.
\]

For \(x \in \mathbb{Z}\), let \((Y_n^x)_{n \in \mathbb{N}}\) be the Markov chain with state space \(\Lambda\), transition matrix \(M\), and initial state \(\omega(x)\). We refer to the chain \((Y_n^x)_{n \in \mathbb{N}}\) as the single site Markov chain at \(x\). It is the Markovian sequence of configurations at site \(x\) that would occur if \(x\) were to be visited infinitely often. That is,

\[
\mathbb{P}(C_{n+1}^x = \lambda^l | C_n^x = \lambda, N_x \geq n + 1) = M_{\lambda \lambda^l}, \quad \lambda, \lambda^l \in \Lambda
\]

where \(N_x\) is the total number of visits to site \(x\), as above, and \(C_n^x\) is the configuration at site \(x\) immediately after the \(n\)-th visit.

The extended single site chain at \(x\), \((\hat{Y}_n^x)_{n \in \mathbb{N}} = (Y_n^x, J_n^x)_{n \in \mathbb{N}}\), is the Markov chain whose states are pairs \((\lambda, j)\), where \(\lambda \in \Lambda\) denotes the current configuration at site \(x\) and \(j \in \{1, -1\}\) represents the next jump from \(x\) (1 for right, -1 for left). The state space of this chain is \(\hat{\Lambda} = \Lambda \times \{1, -1\}\) and the transition matrix \(\hat{M}\) is defined by:

\[
\hat{M}_{(i,j),(p,0)} = p, \hat{M}_{(i,j),(p,0),-1} = 1 - p, \quad \text{for } 0 \leq i \leq L - 1.
\]

\[
\hat{M}_{(i,j),(p,i+1)} = p, \hat{M}_{(i,j),(p,i+1),-1} = 1 - p, \quad \text{for } 0 \leq i \leq L - 2.
\]

\[
\hat{M}_{(i,j),(p,L-1)} = q, \hat{M}_{(p,L-1),(q,0),-1} = 1 - q.
\]

\[
\hat{M}_{(q,i),(q,0)} = q, \hat{M}_{(q,i),(q,0),-1} = 1 - q, \quad \text{for } 0 \leq i \leq R - 1.
\]

\[
\hat{M}_{(q,i),(q,i+1)} = q, \hat{M}_{(q,i),(q,i+1),-1} = 1 - q, \quad \text{for } 0 \leq i \leq R - 2.
\]

\[
\hat{M}_{(q,R-1),(p,0)} = p, \hat{M}_{(q,R-1),(p,0),-1} = 1 - p.
\]

The initial state \(\hat{Y}_1^x\) for the chain has the following distribution:
• If \( \omega(x) = (p, i) \), for some \( 0 \leq i \leq L - 1 \), then
  \[
P(\hat{Y}_1^x = ((p, i), 1)) = p, \quad P(\hat{Y}_1^x = ((p, i), -1)) = 1 - p.
\]  \hspace{1cm} (18)

• If \( \omega(x) = (q, i) \), for some \( 0 \leq i \leq R - 1 \), then
  \[
P(\hat{Y}_1^x = ((q, i), 1)) = q, \quad P(\hat{Y}_1^x = ((q, i), -1)) = 1 - q.
\]  \hspace{1cm} (19)

By construction, the sequence of site configurations \((Y^x_n)\) obtained by projection from this extended Markov chain state sequence \((\hat{Y}^x_n)\) with transition matrix \(\hat{M}\) and initial state distributed according to (18) and (19) has the same law as above, when defined directly by the transition matrix \(M\) with initial state \(\omega(x)\).

**Coupling to the Random Walk \((X_n)\)**

For a given initial position \(x_0\) and initial environment \(\omega = \{\omega(x)\}_{x \in \mathbb{Z}}\) one can construct the random walk \((X_n)_{n \geq 0}\) according to the following two step procedure, similar to that given in [5] for cookie random walks.

1. Run the extended single site Markov chains \((\hat{Y}^x_n)_{n \in \mathbb{N}}\) at each site \(x\) independently.

2. Walk deterministically from the initial point \(x_0\) according to the corresponding “jump pattern” \(\{J^x_k\}_{n \in \mathbb{N}, x \in \mathbb{Z}}\). That is, upon the the \(k_{th}\) visit to site \(x\), the walk jumps right if \(J^x_k = 1\) and left if \(J^x_k = -1\). Formally, we have
   \[
   \begin{align*}
   &X_0 = x_0, \\
   &\text{For } n \geq 0, \\
   &X_{n+1} = X_n + J^x_{K_n} \quad \text{where} \quad K_n = |\{0 \leq m \leq n : X_m = X_n\}|.
   \end{align*}
   \]

By definition of the extended single site chains, the random walk \((X_n)_{n \geq 0}\) constructed by this two step procedure will have the correct law, and in the sequel we always assume our random walk \((X_n)\) to be defined in this fashion. We also denote by \(P_\omega\) the probability measure for the extended single site chains, run independently at each site \(x\), with initial environment \(\omega = \{\omega(x)\}_{x \in \mathbb{Z}}\). This is a slight abuse of notation since the probability measure \(P_\omega = P_{\omega,0}\) introduced in Section [1] also specifies the initial position of the random walk as \(X_0 = 0\). However, things should be clear from the context.

**Stationary Distribution**

Since \(\Lambda\) is finite and \(M\) is an irreducible transition matrix, there exists a unique stationary probability distribution \(\pi\) on \(\Lambda\) satisfying \(\pi = \pi M\). Solving
the linear system \( \{ \pi = \pi M, \sum_{\lambda \in \Lambda} \pi_{\lambda} = 1 \} \), one obtains the following explicit form for \( \pi \) (see Appendix A.1):

\[
\pi(p, i) = p \left( 1 - q \right) \left( 1 - q R \right) \cdot \left( 1 - p \right)^i, \quad 0 \leq i \leq L - 1.
\]

\[
\pi(q, i) = p \left( 1 - q \right) \left( 1 - q R \right) \cdot \left( 1 - p \right)^i, \quad 0 \leq i \leq R - 1.
\]

(20)

In particular,

\[
\pi_p \equiv \sum_{i=0}^{L-1} \pi(p, i) = \frac{\left( 1 - q \right) q^R \left( 1 - \left( 1 - p \right)^L \right)}{(1 - q)q^R(1 - (1 - p)L) + p(1 - p)L(1 - q^R)} \cdot (1 - p)^i, \quad 0 \leq i \leq L - 1.
\]

\[
\pi_q \equiv \sum_{i=0}^{R-1} \pi(q, i) = \frac{p(1 - q)(1 - p)^L}{(1 - q)q^R(1 - (1 - p)L) + p(1 - p)L(1 - q^R)} \cdot q^i.
\]

(21)

So, defining \( \phi : \hat{\Lambda} \to \{0, 1\} \) by \( \phi(\lambda, j) = 1 \{ j = 1 \} \), we have

\[
E_{\hat{\pi}}(\phi) = p \cdot \pi_p + q \cdot \pi_q = \alpha,
\]

(22)

where \( \hat{\pi} \) is the stationary distribution for the transition matrix \( \hat{M} \), and \( \alpha \in (0, 1) \) is as in Theorem 1. It follows, by the ergodic theorem for finite-state Markov chains, that the limiting fraction of right jumps in the sequence \( (J^x_n)_{n \in \mathbb{N}} \) is equal to \( \alpha \) a.s., for each site \( x \).

### 2.1.2 The Right Jumps Markov Chain \( (Z_x)_{x \geq 0} \)

The right jumps Markov chain \( (Z_x)_{x \geq 0} \) is defined as follows:

- \( Z_0 = 1 \).
- For \( x \geq 1 \),

\[
Z_x = \Theta_x - Z_{x-1} \quad \text{where} \quad \Theta_x = \inf \left\{ n \geq 0 : \sum_{m=1}^{n} \mathbf{1}(J^x_m = -1) = Z_{x-1} \right\}.
\]

(23)

That is, \( \Theta_x \) is the first time that there are \( Z_{x-1} \) left jumps in the sequence \( (J^x_n)_{n \in \mathbb{N}} \), and \( Z_x = \Theta_x - Z_{x-1} \) is the total number of right jumps in the sequence \( (J^x_n)_{n \in \mathbb{N}} \) before there are \( Z_{x-1} \) left jumps.

For an initial environment \( \omega \), we denote the probability measure for the right jumps chain \( (Z_x) \) also by \( P_\omega \). This is simply the projection of the measure \( P_\omega \) for the extended single site Markov chains, of which the right jumps chain is a deterministic function.
Relation to the Random Walk \((X_n)\)

We denote by \(T_x\) the first hitting time of site \(x\),
\[
T_x = \inf\{n \geq 0 : X_n = x\}, x \in \mathbb{Z}.
\]
Also, we say that a jump pattern \(\{J^x_n\}_{n \in \mathbb{N}, x \in \mathbb{Z}}\) is non-degenerate if
\[
|\{n : J^x_{n+1} \neq J^x_n\}| = \infty, \text{ for each } x \in \mathbb{Z}.
\]
Clearly, for any initial environment \(\omega\), the corresponding jump pattern \(\{J^x_n\}_{n \in \mathbb{N}, x \in \mathbb{Z}}\) is non-degenerate \(P_\omega\) a.s. The following important proposition relating transience/recurrence of the random walk \((X_n)\) to survival of the Markov chain \((Z_x)\) is shown in [5].

**Proposition 3.** If \(X_0 = 1\) and \(\{J^x_n\}_{n \in \mathbb{N}, x \in \mathbb{Z}}\) is non-degenerate, then
\[
T_0 = \infty \text{ if and only if } Z_x > 0, \text{ for all } x > 0.
\]

Moreover, if \(T_0 < \infty\) then, for each \(x \in \mathbb{N}\), \(Z_x\) is equal to the number of right jumps of the process \((X_n)\) from site \(x\) before hitting 0.

### 2.2 Basic Lemmas

For \(n \geq 0\) we denote by \(A^+_n\) the event that the random walk steps right at time \(n\) and never returns to its time-\(n\) location, and by \(A^-_n\) the event that the random walks steps left at time \(n\) and never returns:

\[
A^+_n = \{X_m > X_n, \forall m > n\} \text{ and } A^-_n = \{X_m < X_n, \forall m > n\}.
\]

The following simple facts will be needed in several instances below. A proof is provided in Appendix B.

**Lemma 1.** For any initial environment \(\omega\):

(i) \(P_\omega(A^+_n) > 0\) if and only if \(P_\omega(X_n \to \infty) > 0\), and
\[
P_\omega(A^-_n) > 0 \text{ if and only if } P_\omega(X_n \to -\infty) > 0.
\]

(ii) \(P_\omega(X_n \to \infty) = P_\omega(\lim \inf_{n \to \infty} X_n > -\infty)\), and
\[
P_\omega(X_n \to -\infty) = P_\omega(\lim \sup_{n \to \infty} X_n < \infty).
\]

(iii) \(P_\omega(X_n \to \infty) = 1\) if \(P_\omega(X_n \to \infty) > 0\) and \(P_\omega(X_n \to -\infty) = 0\).
\[
P_\omega(X_n \to -\infty) = 1 \text{ if } P_\omega(X_n \to -\infty) > 0 \text{ and } P_\omega(X_n \to \infty) = 0.
\]

Combining Proposition 3 and (i) also gives the following useful lemma.

**Lemma 2.** For any initial environment \(\omega\),
\[
P_\omega(A^+_n) = P_\omega(X_1 = 1) \cdot P_\omega(Z_z > 0, \forall x > 0).
\]

Consequently, \(P_\omega(X_n \to \infty) > 0\) if and only if \(P_\omega(Z_z > 0, \forall x > 0) > 0\).
Proof. Fix any initial environment \( \omega \), and let \( \omega' \) denote the environment at time 1 induced by jumping right from \( X_0 = 0 \) starting in \( \omega \):

\[
\{ \omega_0 = \omega, X_0 = 0, X_1 = 1 \} \implies \omega_1 = \omega'.
\]

Since \( \omega(x) = \omega'(x) \), for all \( x > 0 \), the distribution of the random variables \( (J^n_x)_{n,x>0} \), is the same in the two environments \( \omega \) and \( \omega' \). Thus,

\[
P_\omega(Z_x > 0, \forall x > 0) = P_{\omega'}(Z_x > 0, \forall x > 0).
\]

So,

\[
P_\omega(A^+_0) = P_\omega,0(X_n > 0, \forall n > 0)
\]

\[
= P_\omega,0(X_1 = 1) \cdot P_{\omega,1}(X_n > 0, \forall n > 1 | X_1 = 1)
\]

\[
= P_{\omega,0}(X_1 = 1) \cdot P_{\omega',1}(X_n > 0, \forall n > 0)
\]

\[
= P_{\omega,0}(X_1 = 1) \cdot P_{\omega'}(Z_x > 0, \forall x > 0)
\]

This proves (26), and the “consequently” part of the proposition follows immediately from part (i) of Lemma 1. Step (*) follows from Proposition 3.

For the proofs of Theorems 2 and 4 we will need the following lemma relating hitting times to speed. The same result is shown in [6, Lemma 2.1.17], for the case \( C < \infty \) without the a priori assumption that \( X_n \to \infty \). It is easy to see that with this assumption the claim also holds in the case \( C = \infty \).

**Lemma 3.** If \( \lim_{n \to \infty} X_n = \infty \) and \( \lim_{x \to \infty} T_x/x = C \in (0, \infty] \), then

\[
\lim_{n \to \infty} X_n/n = 1/C.
\]

We note that, although stated in [6, Lemma 2.1.17] in the context of random walks in random environment, the proof is entirely non-probabilistic and holds for any nearest neighbor walk trajectory \( (X_0, X_1, \ldots) \) such that \( X_n \to \infty \) and \( \lim_{x \to \infty} T_x/x = C \).

### 2.3 Comparison of Environments

Let \( \prec \) be the ordering on the set of single site configurations \( \Lambda \) defined by

\[
(q, 0) \prec \ldots \prec (q, R - 1) \prec (p, L - 1) \prec \ldots \prec (p, 0).
\]

We write \( \lambda \preceq \bar{\lambda} \) if \( \lambda \prec \bar{\lambda} \) or \( \lambda = \bar{\lambda} \), and \( \omega \preceq \bar{\omega} \) if \( \omega(x) \preceq \bar{\omega}(x) \), for all \( x \in \mathbb{Z} \).

In this case, we also say that the environment \( \bar{\omega} \) *dominates* the environment \( \omega \). The following lemma relating the possibility of right transience in different environments will be important for the analysis of transience and recurrence in the critical case \( \alpha = 1/2 \).

---

1. In the proof we have used the explicit notation \( P_{\omega,0} \), rather than simply \( P_\omega \), for the random walk variables \( X_n, n \geq 0 \), to emphasize that the initial position \( X_0 = 0 \) plays a role in their distribution. By contrast, \( P_\omega, P_{\omega'} \) are used for the distribution of the right jumps Markov chain \( (Z_x)_{x \geq 0} \), where the initial position of the random walk plays no role.
Lemma 4. If \( q < p \) and \( \omega \leq \bar{\omega} \), then \( \mathbb{P}_\omega(A_0^+) \leq \mathbb{P}_\bar{\omega}(A_0^+) \). In particular, by Lemma 7, if \( q < p, \omega \leq \bar{\omega} \), and \( \mathbb{P}_\omega(X_n \to \infty) = 0 \), then \( \mathbb{P}_\bar{\omega}(X_n \to \infty) = 0 \).

For the proof it will be convenient to introduce the following definitions.

- The threshold function \( f : \Lambda \times [0, 1] \to \{1, -1\} \) is defined by
  \[
  f(\lambda, u) = 1\{u \leq p\} - 1\{u > p\}, \quad \text{if } \lambda \in \Lambda_p
  \]
  \[
  f(\lambda, u) = 1\{u \leq q\} - 1\{u > q\}, \quad \text{if } \lambda \in \Lambda_q.
  \]

- The transition function \( g : \Lambda \times \{1, -1\} \to \Lambda \) is defined by
  \[
  g(\lambda, j) = \lambda' \iff \{Y_n^x = \lambda, J_n^x = j\} \implies Y_{n+1}^x = \lambda.'
  \]

That is, \( g(\lambda, j) \) is the (deterministic) next configuration at site \( x \) if the walk jumps in direction \( j \) from site \( x \) when \( x \) is in configuration \( \lambda \).

Proof of Lemma. For \( x \in \mathbb{Z} \), let \( (Y_n^x, J_n^x)_{n \in \mathbb{N}} \) and \( (\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}} \) denote, respectively, the state sequences of the extended single site Markov chains at \( x \) for the environments \( \omega \) and \( \bar{\omega} \). Also, let \( (U_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}} \) be i.i.d. uniform([0,1]) random variables. For each \( x \), we will use the i.i.d. sequence \( (U_n^x)_{n \in \mathbb{N}} \) to couple the state sequences \( (Y_n^x, J_n^x)_{n \in \mathbb{N}} \) and \( (\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}} \) in such a way that \( J_n^x \leq \tilde{J}_n^x \), for all \( n \).

By independence, this coupling at each individual site \( x \) passes to a coupling of the entire joint processes \( (Y_n^x, J_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}} \) and \( (\tilde{Y}_n^x, \tilde{J}_n^x)_{x \in \mathbb{Z}, n \in \mathbb{N}} \), with the correct law. This final larger coupling will be used to show that \( \mathbb{P}_\omega(A_0^+) \leq \mathbb{P}_\bar{\omega}(A_0^+) \).

Step 1: The Coupling
For a fixed site \( x \), we construct the sequences \( (Y_n^x, J_n^x)_{n \in \mathbb{N}} \) and \( (\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}} \) inductively from the i.i.d. random variables \( (U_n^x)_{n \in \mathbb{N}} \) as follows.

- \( Y_1^x = \omega(x) \) and \( \tilde{Y}_1^x = \bar{\omega}(x) \).

- For \( n \geq 1 \),
  \[
  J_n^x = f(Y_n^x, U_n), \quad Y_{n+1}^x = g(Y_n^x, J_n^x) \quad \text{and} \quad \tilde{J}_n^x = f(\tilde{Y}_n^x, U_n), \quad \tilde{Y}_{n+1}^x = g(\tilde{Y}_n^x, \tilde{J}_n^x).
  \]

Clearly, the sequences \( (Y_n^x, J_n^x)_{n \in \mathbb{N}} \) and \( (\tilde{Y}_n^x, \tilde{J}_n^x)_{n \in \mathbb{N}} \) each have the appropriate marginal laws under this coupling. Moreover, by considering the various possible cases for \( Y_n^x, Y_n^x \in \Lambda \) and possible ranges for \( U_n \in [0,1] \) one finds that, since \( q < p \), whatever the value of \( U_n \) is:

\[
Y_n^x \leq \tilde{Y}_n^x \implies J_n^x \leq \tilde{J}_n^x \quad \text{and} \quad Y_{n+1}^x \leq \tilde{Y}_{n+1}^x.
\]

Since \( Y_1^x = \omega(x) \leq \bar{\omega}(x) = \tilde{Y}_1^x \) it follows, by induction, that

\[
J_n^x \leq \tilde{J}_n^x, \quad \text{for all } n.
\]

Step 2: Relation to the probability of \( A_0^+ \)
Let \( (Z_x)_{x \geq 0} \) and \( (\tilde{Z}_x)_{x \geq 0} \) denote, respectively, the right jumps Markov chains.
constructed from the jump patterns \((J^x_n)_{x \in \mathbb{Z}, n \in \mathbb{N}}\) and \((\tilde{J}^x_n)_{x \in \mathbb{Z}, n \in \mathbb{N}}\) according to (23). Also, for \(x, k \geq 0\) define \(\Theta_{x,k}\) and \(\tilde{\Theta}_{x,k}\) by

\[
\Theta_{x,k} = \inf \left\{ n : \sum_{m=1}^{n} 1 \{ J^x_m = -1 \} = k \right\}, \quad \tilde{\Theta}_{x,k} = \inf \left\{ n : \sum_{m=1}^{n} 1 \{ \tilde{J}^x_m = -1 \} = k \right\}.
\]

If \(Z_{x-1} \leq \tilde{Z}_{x-1}\), then applying the definition (23) gives

\[
Z_x = \sum_{m=1}^{\Theta_{x,\tilde{Z}_{x-1}}} 1 \{ J^x_m = 1 \} \leq \sum_{m=1}^{\tilde{\Theta}_{x,\tilde{Z}_{x-1}}} 1 \{ J^x_m = 1 \} \leq \sum_{m=1}^{\tilde{\Theta}_{x,\tilde{Z}_{x-1}}} 1 \{ \tilde{J}^x_m = 1 \} = \tilde{Z}_x.
\]

Here, (a) follows from (27), which implies \(\Theta_{x,k} \leq \tilde{\Theta}_{x,k}\) for any \(k\), and (b) follows directly from (27). Since \(Z_0 = \tilde{Z}_0 = 1\), it follows, by induction, that

\[
Z_x \leq \tilde{Z}_x, \text{ for all } x \in \mathbb{Z}.
\] (28)

Now, since (27) and (28) both hold with probability 1, under our coupling, it follows from Lemma 2 that

\[
\mathbb{P}_\omega(A^+_0) = \mathbb{P}_\omega(X_1 = 1) \cdot \mathbb{P}_\omega(Z_x > 0, \forall x > 0) = \mathbb{P}_\omega(J^0_1 = 1) \cdot \mathbb{P}_\omega(Z_x > 0, \forall x > 0) \leq \mathbb{P}_\tilde{\omega}(J^0_1 = 1) \cdot \mathbb{P}_\tilde{\omega}(Z_x > 0, \forall x > 0) = \mathbb{P}_\tilde{\omega}(A^+_0).
\]

3 The Noncritical Case

Here we analyze the behavior of the random walk \((X_n)\) for \(\alpha \neq 1/2\), proving Theorems 1-4. We begin in Section 3.1 with a key lemma for the survival probability of the right jumps Markov chain \((Z_x)\), from which we derive a number of useful corollaries. Using these results, Theorem 1 on the cutoff for right/left transience and Theorem 2 on ballisticity of the random walk are then proved in Sections 3.2 and 3.3. Theorems 3 and 4 on the exact speed of the random walk in certain special cases are proved afterward in Sections 3.4 and 3.5.

Throughout we use the following notation:

- \(T^{(i)}_x\), \(x \in \mathbb{Z}\) and \(i \in \mathbb{N}\), is the \(i\)-th hitting time of site \(x\).

\[
T^{(1)}_x = T_x \quad \text{and} \quad T^{(i+1)}_x = \inf \{ n > T^{(i)}_x : X_n = x \}, \quad (29)
\]

with the convention \(T^{(j)}_x = \infty\), for all \(j > i\), if \(T^{(i)}_x = \infty\).
• $N_x$ is the total number of visits to site $x$, as in (1), and $N^y_x$ is the number of visits to site $x$ up to time $T_y$.

\[ N_x = |\{ n \geq 0 : X_n = x \} | , \ x \in \mathbb{Z} . \]
\[ N^y_x = |\{ 0 \leq n \leq T_y : X_n = x \} | , \ x,y \in \mathbb{Z} . \]

• $R_x$ is the total number of right jumps from site $x$, and $L_x$ is the total number of left jumps from site $x$.

\[ R_x = |\{ n \geq 0 : X_n = x , X_{n+1} = x+1 \} | , \ x \in \mathbb{Z} . \]
\[ L_x = |\{ n \geq 0 : X_n = x , X_{n+1} = x-1 \} | , \ x \in \mathbb{Z} . \]

• $B_x$ is the farthest distance the random walk ever steps backward from site $x$ after hitting $x$ for the first time.

\[ B_x = \sup \{ k \geq 0 : \exists n \geq T_x \text{ with } X_n = x-k \} , \ x \in \mathbb{Z} . \]

In the case $T_x = \infty$, $B_x \equiv 0$.

• $A^*_n$, given by (25), is the event that $B_x \leq \epsilon x$, for all sufficiently large $x$.

\[ A^*_n = \{ \exists N \in \mathbb{N} \text{ s.t. } B_x \leq \epsilon x , \forall x \geq N \} . \]

3.1 Survival of Right Jumps Markov Chain ($Z_x$)

**Lemma 5.** If $\alpha = \alpha(p,q,R,L) > 1/2$, then there exists some $\beta = \beta(p,q,R,L) > 0$ such that, for any initial environment $\omega$,

\[ \mathbb{P}_\omega(Z_x > 0 , \forall x > 0 ) \geq \beta . \]

**Proof.** Fix $p,q,R,L$ such that $\alpha > 1/2$ and any initial environment $\omega$. Define $0 < \epsilon < 1/4$ by the relation $\alpha = 1/2 + 2\epsilon$, and for $\lambda = (\lambda,j) \in \hat{\Lambda}$, let $\phi(\hat{\lambda}) = \mathbb{1}\{j = 1\}$.

By [22] we have $\alpha = \mathbb{E}_{\hat{\pi}}(\phi)$, where $\hat{\pi}$ is the stationary distribution for the extended single site transition matrix $\hat{M}$. So, by standard large deviation bounds for finite-state Markov chains, there exist some $0 < a < 1$ and $n_0 \in \mathbb{N}$ such that for any initial state $\hat{\lambda} \in \hat{\Lambda}$ the Markov chain $(\hat{Y}_n)$ with transition matrix $\hat{M}$ satisfies

\[ \mathbb{P}_{\hat{\lambda}} \left( \frac{1}{n} \sum_{m=1}^{n} \phi(\hat{Y}_m) \leq 1/2 + \epsilon \right) = \mathbb{P}_{\hat{\lambda}} \left( \frac{1}{n} \sum_{m=1}^{n} \phi(\hat{Y}_m) \leq \mathbb{E}_{\hat{\pi}}(\phi) - \epsilon \right) \leq a^n , n \geq n_0 . \]
Using this estimate we obtain the following important inequality:

\[
\mathbb{P}_\omega(Z_x \leq n(1/2 + \epsilon)/(1/2 - \epsilon) \mid Z_{x-1} = n) \\
= \mathbb{P}_\omega(\Theta_x \leq n/(1/2 - \epsilon) \mid Z_{x-1} = n) \\
= \mathbb{P}_\omega\left(\exists n \leq m \leq n/(1/2 - \epsilon) : \sum_{i=1}^{m} (1 - \phi(Y_i^x)) = n\right) \\
= \mathbb{P}_\omega\left(\exists n \leq m \leq n/(1/2 - \epsilon) : \frac{1}{m} \sum_{i=1}^{m} \phi(Y_i^x) = \frac{m-n}{m}\right) \\
\leq \mathbb{P}_\omega\left(\exists n \leq m \leq n/(1/2 - \epsilon) : \frac{1}{m} \sum_{i=1}^{m} \phi(Y_i^x) \leq 1/2 + \epsilon\right) \\
\leq \mathbb{P}_\omega\left(\exists m \geq n : \frac{1}{m} \sum_{i=1}^{m} \phi(Y_i^x) \leq 1/2 + \epsilon\right) \\
\leq \sum_{m=n}^{\infty} a^m = \frac{a^n}{1-a}, \text{ for all } n \geq n_0 \text{ and } x \in \mathbb{N}.
\]

(31)

Now, define \( b > 1 \) by \( b = \frac{1/2 + \epsilon}{1/2 - \epsilon} \), and take \( n_1 \geq n_0 \) sufficiently large that \( \frac{a^{n_1}}{1-a} < 1 \). Thus, \( \frac{a^{n_1(b^{x-1})}}{1-a} < 1, \forall x \in \mathbb{N} \). Applying the inequality (31) gives,

\[
\mathbb{P}_\omega(Z_x > 0, \forall x > 0) \\
\geq \mathbb{P}_\omega(Z_x \geq n_1 b^x, \forall x > 0) \\
= \mathbb{P}_\omega(Z_1 \geq n_1 b) \cdot \prod_{x=2}^{\infty} \mathbb{P}_\omega(Z_x \geq n_1 b^x \mid Z_1 \geq n_1 b, ..., Z_{x-1} \geq n_1 b^{x-1}) \\
\geq \mathbb{P}_\omega(Z_1 \geq n_1 b) \cdot \prod_{x=2}^{\infty} \mathbb{P}_\omega(Z_x \geq n_1 b^x \mid Z_{x-1} = \lceil n_1 b^{x-1} \rceil) \\
\geq (\min\{p, q\})^{[n_1 b]} \cdot \prod_{x=2}^{\infty} \left( 1 - \frac{a^{[n_1 b^{x-1}]} - 1}{1-a} \right) \equiv \beta.
\]

Note that \( \sum_{x=2}^{\infty} \frac{a^{[n_1 b^{x-1}]}}{1-a} < \infty \), so \( \prod_{x=2}^{\infty} \left( 1 - \frac{a^{[n_1 b^{x-1}]} - 1}{1-a} \right) > 0 \). \( \square \)

**Corollary 2.** If \( \alpha = \alpha(p, q, R, L) > 1/2 \) then there exists some \( \beta = \beta(p, q, R, L) > 0 \), such that for any initial environment \( \omega \) and random walk path \((x_0, x_1, ..., x_n)\),

\[
\mathbb{P}_{\omega,x_0}(A^+_n \mid X_0 = x_0, ..., X_n = x_n) \geq \beta.
\]

(32)

**Proof.** Since the claimed bound is uniform in the initial environment \( \omega \), it suffices to consider the case \( x_0 = n = 0 \). By Lemma 5 there exists some \( \beta' > 0 \) such that \( \mathbb{P}_\omega(Z_x > 0, \forall x > 0) \geq \beta' \), for any initial environment \( \omega \). Thus, by Lemma 2

\[
\mathbb{P}_\omega(A^+_n) \geq \min\{p, q\} \cdot \beta' \equiv \beta
\]

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for any initial environment $\omega$. □

**Corollary 3.** If $\alpha > 1/2$ then, for any initial environment $\omega$ and site $x \geq 0$,

$$\mathbb{P}_\omega(N_x \geq k) \leq (1 - \beta)^{k-1}, \quad \text{for all } k \geq 1$$

where $\beta > 0$ is the constant in Corollary 2.

**Proof.** Let $A_x^{(i)}$ be the set of all random walk paths $(x_0, x_1, \ldots, x_n)$, of any length $n$, which end in an $i$-th hitting time of site $x$. That is, $\{X_0 = x_0, X_1 = x_1, \ldots, X_n = x_n\} \implies T_x^{(i)} = n$. For brevity we denote $(X_0, \ldots, X_n)$ as $X^n$ and $(x_0, \ldots, x_n)$ as $x^n$. By Corollary 2, for any $i \geq 1$, we have

$$\mathbb{P}_\omega(T_x^{(i+1)} < \infty|T_x^{(i)} < \infty) = \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n|T_x^{(i)} < \infty) \cdot \mathbb{P}_\omega(T_x^{(i+1)} < \infty|T_x^{(i)} < \infty, X_0^n = x_0^n)$$

$$= \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n|T_x^{(i)} < \infty) \cdot \mathbb{P}_\omega(T_x^{(i+1)} < \infty|X_0^n = x_0^n)$$

$$\leq \sum_{x_0^n \in A_x^{(i)}} \mathbb{P}_\omega(X_0^n = x_0^n|T_x^{(i)} < \infty) \cdot \mathbb{P}_\omega((A^n)^c|X_0^n = x_0^n)$$

$$\leq (1 - \beta).$$

Hence, for each $k \geq 1$,

$$\mathbb{P}_\omega(N_x \geq k) = \mathbb{P}_\omega(T_x^{(1)} < \infty) \cdot \prod_{i=1}^{k-1} \mathbb{P}_\omega(T_x^{(i+1)} < \infty|T_x^{(i)} < \infty) \leq (1 - \beta)^{k-1}. \quad \square$$

**Corollary 4.** If $\alpha > 1/2$ then, for any initial environment $\omega$ and site $x \geq 0$,

$$\mathbb{P}_\omega(B_x \geq k) \leq (1 - \beta)^k, \quad \text{for all } k \geq 1$$

where $\beta > 0$ is the constant in Corollary 2. In particular, by the Borel-Cantelli lemma,

$$\mathbb{P}_\omega(B_x) = 1, \quad \text{for each } 0 < \epsilon < 1.$$

**Proof.** The proof is similar to that of Corollary 3. For $x \in \mathbb{Z}$, let $\tau_x^{(0)}$ be the first hitting time of site $x$, and let $\tau_x^{(i)}$, $i \in \mathbb{N}$, be the first time greater than $\tau_x^{(i-1)}$ at which the walk steps backward from its position $x - (i - 1)$ at time $\tau_x^{(i-1)}$. That is, $\tau_x^{(0)} = T_x$, and for $i \geq 1$,

$$\tau_x^{(i)} = \inf\{n > \tau_x^{(i-1)}: X_n < X_{\tau_x^{(i-1)}}\}$$

$$= \inf\{n > T_x: X_n = x - i\}$$

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with the convention $\tau_x^{(j)} = \infty$, for all $j > i$, if $\tau_x^{(i)} = \infty$. Also, let $A_x^{(i)}$ be the set of all random walk paths $(x_0, x_1, ..., x_n)$, of any length $n$, which end in an $i$-th “back step time” from site $x$. That is, \{$X_0 = x_0, X_1 = x_1, ..., X_n = x_n$\} $\implies$ \$x_x^{(i)} = n \$. As above, we denote $(X_0, ..., X_n)$ as $X_n^0$ and $(x_0, ..., x_n)$ as $x_n^0$. By Corollary 2, for any $i \geq 0$, we have

$$P_\omega(\tau_x^{(i+1)} < \infty | \tau_x^{(i)} < \infty) = \sum_{x_n^0 \in A_x^{(i)}} P_\omega(\tau_x^{(i)} < \infty | \tau_x^{(i+1)} < \infty, X_n^0 = x_n^0).$$

So, for each $k \geq 1$,

$$P_\omega(B_\infty \geq k) = P_\omega(\tau_x^{(0)} < \infty) \cdot \prod_{i=0}^{k-1} P_\omega(\tau_x^{(i+1)} < \infty | \tau_x^{(i)} < \infty) \leq (1 - \beta)^k.$$

3.2 Proof of Theorem 1

If $\alpha > 1/2$ then Corollary 4 implies that $B_\infty$ is $P_\omega$ a.s. finite, for any initial environment $\omega$. Thus, by part (ii) of Lemma 1, for $\alpha > 1/2$ we must have $X_n \to \infty$, $P_\omega$ a.s., for any initial environment $\omega$. It follows by symmetry that, for $\alpha < 1/2$ and any initial environment $\omega$, $X_n \to -\infty$, $P_\omega$ a.s.

3.3 Proof of Theorem 2

For the proof of Theorem 2 we will assume that $\alpha > 1/2$, the case $\alpha < 1/2$ follows by symmetry considerations. The primary ingredients for the proof are Corollaries 3 and 4 above, and Lemmas 6 and 7 given below. Lemma 6 is a simple consequence of Theorem 1. Lemma 7 shows that, when $\alpha > 1/2$, the sequence $(N_x)$ obeys a strong law of large numbers. The proof of this fact is somewhat lengthy and is deferred to Appendix C.

Lemma 6. Assume that $\alpha > 1/2$ and $X_0 = 0$.

(i) For all $x \geq 0$, the random variables $N_x, L_x, R_x$ are each independent of the environment to the left of site $x$ when site $x$ is first reached:

$$N_x, L_x, R_x \perp \{\omega_{\tau_x}(y), y < x\}.$$
(ii) \( N_x^y \) and \( N_y \) are independent, for all \( 0 \leq x < y \).

(iii) If, for some \( y \geq 0 \), \( \omega(x) \) is constant for \( x \geq y \), then \( N_x \) and \( N_y \) have the same distribution for all \( x \geq y \). Similarly, if \( \omega(x) = \omega(y) \) for all \( x \geq y \), then \( R_x \) and \( R_y \) have the same distribution for all \( x \geq y \), and \( L_x \) and \( L_y \) have the same distribution for all \( x \geq y \).

Proof. Since \( \alpha > 1/2 \) and \( X_0 = 0 \), Theorem 1 shows that \( T_x \) is a.s. finite, for each \( x \geq 0 \), and that regardless of the environment to the left of site \( x \) at time \( T_x \), the walk returns to site \( x \) with probability 1 each time it steps left from \( x \). This implies (i). Now, (ii) and (iii) follow easily since (i) shows that the distribution of \( N_x \), \( L_x \), and \( R_x \) are each entirely determined by the values of \( \omega_{T_x}(y), y \geq x \), which are the same as the original values \( \omega(y), y \geq x \).

Lemma 7. If \( \alpha > 1/2 \) then, for any initial environment \( \omega \),

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{x=1}^{n} (N_x - E_{\omega}(N_x)) = 0, \quad P_\omega \text{ a.s.}
\]

Proof of Theorem 2, Equation (3), with \( \alpha > 1/2 \).

Let \( \alpha > 1/2 \) and fix any initial environment \( \omega \). Also, let \( \beta > 0 \) be the constant defined in Corollary 2. We will show that:

(i) \( \limsup_{x \to \infty} \frac{1}{x} \sum_{y=1}^{x} N_y \leq 1/\beta, \quad P_\omega \text{ a.s.} \)

(ii) \( \limsup_{x \to \infty} T_x/x \leq \limsup_{x \to \infty} \frac{1}{x} \sum_{y=1}^{x} N_y, \quad P_\omega \text{ a.s.} \)

(iii) \( \liminf_{n \to \infty} X_n/n \geq \left( \limsup_{x \to \infty} T_x/x \right)^{-1}, \quad P_\omega \text{ a.s.} \)

The result (3) follows directly from these three facts.

Proof of (i): This is immediate from Lemma 7 and Corollary 3.

Proof of (ii): Since \( \alpha > 1/2 \), \( X_n \to \infty \) \( P_\omega \text{ a.s.} \). So, \( \sum_{x \leq 0} N_x \) is \( P_\omega \text{ a.s.} \) finite. Thus, \( P_\omega \text{ a.s.} \) we have

\[
\limsup_{x \to \infty} \frac{T_x}{x} = \limsup_{x \to \infty} \frac{1}{x} \sum_{y=-\infty}^{x} N_y \leq \limsup_{x \to \infty} \frac{1}{x} \sum_{y=-\infty}^{x} N_y = \limsup_{x \to \infty} \frac{1}{x} \sum_{y=1}^{x} N_y.
\]

Proof of (iii): For \( 0 < \epsilon < 1 \), let \( \mathcal{B}_{\epsilon}' = \mathcal{B}_\epsilon \cap \{ T_x < \infty, \forall x > 0 \} \), where \( \mathcal{B}_\epsilon \) is defined by (30). On the event \( \mathcal{B}_{\epsilon}' \), for all sufficiently large \( x \) and \( T_z \leq n < T_{x+1} \), we have

\[
\frac{X_n}{n} \geq \frac{x - \epsilon x}{T_{x+1}} = (1-\epsilon) \frac{x+1}{T_{x+1}} - \frac{1-\epsilon}{T_{x+1}}.
\]

So,

\[
\liminf_{n \to \infty} \frac{X_n}{n} \geq \liminf_{x \to \infty} \left( (1-\epsilon) \frac{x+1}{T_{x+1}} - \frac{1-\epsilon}{T_{x+1}} \right) = (1-\epsilon) \cdot \left( \limsup_{x \to \infty} T_x/x \right)^{-1}.
\]
The result follows since $P_\omega(B'_\epsilon) = 1$, for each $\epsilon > 0$, due to Corollary 4 and the fact that the random walk $(X_n)$ is a.s. right transient with $\alpha > 1/2$.

**Proof of Theorem 2**. Equation (4), with $\alpha > 1/2$. By assumption $\omega(x)$ is constant for $x \geq m$, so Lemma 6 implies $N_x$ and $N_{m}$ are equal in law, for all $x \geq m$, under $P_\omega$. Thus,

$$E_\omega(N_x) = E_\omega(N_m) \equiv \gamma, \text{ for all } x \geq m.$$  \hspace{1cm} (34)

To show that $\lim_{n \to \infty} X_n/n = 1/\gamma$, $P_\omega$ a.s., note first that (34) and Lemma 7 imply that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{y=1}^{x} N_y = \gamma, \text{ } P_\omega \text{ a.s.}$$  \hspace{1cm} (35)

So, by point (ii) above, we have

$$\limsup_{x \to \infty} T_x/x \leq \gamma, \text{ } P_\omega \text{ a.s.}$$  \hspace{1cm} (36)

On the other hand, on the event $B'_\epsilon = B_\epsilon \cap \{T_x < \infty, \forall x > 0\}$, we have

$$\liminf_{x \to \infty} \frac{T_x}{x} = \liminf_{x \to \infty} \frac{1}{x} \sum_{y=-\infty}^{x} N_y \geq \liminf_{x \to \infty} \frac{1}{x} \sum_{y=1}^{\lfloor(1-\epsilon)x\rfloor} N_y = \liminf_{x \to \infty} \frac{1}{x} \sum_{y=1}^{\lfloor(1-\epsilon)x\rfloor} N_y.$$  \hspace{1cm} (37)

Since $P_\omega(B'_\epsilon) = 1$, for each $\epsilon > 0$, and the RHS of (37) is equal to $(1-\epsilon)\gamma$, $P_\omega$ a.s., by (35), this implies

$$\liminf_{x \to \infty} \frac{T_x}{x} \geq \gamma, \text{ } P_\omega \text{ a.s.}$$  \hspace{1cm} (38)

Together, (36) and (38) imply $\lim_{x \to \infty} T_x/x = \gamma, \text{ } P_\omega \text{ a.s.}$, so the result follows from Lemma 8.  \hspace{1cm} \square

### 3.4 Proof of Theorem 3

The proof of Theorem 3 is based on the speed formula given in Theorem 2 and uses the assumptions on $L$ and $\omega$ to obtain a more explicit expression for $\gamma$.

**Proof of Theorem 3**. We will prove the theorem under the assumption $\omega(x) = (q,0)$, for all $x \geq 0$. The case $\omega(x) = (q,0)$ in a neighborhood of $+\infty$ follows immediately from this. The main observation is that since $L = 1$ and the random walk starts from $X_0 = 0$ in an environment $\omega$ satisfying $\omega(x) = (q,0)$, for all $x \geq 0$, we have

$$\omega_n(x) = (q,0), \text{ for each } n \geq 0 \text{ and } x > X_n.$$  

That is, the environment to the right of the current position of the random walk always consists entirely of sites in the $(q,0)$ configuration. Consequently, when
the walk jumps right the environment both at its current position and to its
right consists entirely of sites in the \((q,0)\) configuration:

\[
\{X_{n-1} = x-1 \text{ and } X_n = x\} \implies \omega_n(y) = (q,0), \text{ for all } y \geq x. \tag{39}
\]

Using this fact we will show that:

(i) \(\gamma \equiv E_{\omega}(N_0) = \frac{1+\eta}{1-\eta}\), where \(\eta \equiv P_{\omega}(T_{-1} < \infty)\).

(ii) \(\eta\) satisfies \(P(\eta) = 0\), where \(P\) is as in (10).

Also, using direct calculus arguments we will show that:

(iii) The polynomial \(P(t)\) has a unique real root in the interval \((0,1)\).

Clearly, \(\eta > P_{\omega'}(T_{-1} = 1) = 1 - q\), and by Corollary 2 we know \(\eta < 1\). Thus, the theorem follows from points (i)-(iii) and Theorem 2.

Proof of (i): Since \(\alpha > 1/2\) the random walk returns to site 0 with probability 1 every time it steps left from 0, and by (39), applied in the case \(x = 0\), we know that at each time \(n\) when the random walk returns to site 0 after stepping left on its last visit, we have \(\omega_n(x) = \omega_0(x) = (q,0)\), for all \(x \geq 0\). Therefore, since \(P_{\omega'}(T_{-1} < \infty)\) does not depend on the values of \(\omega'(x), x < 0\), it follows that \(L_0\) is a geometric random variable with distribution

\[P_{\omega}(L_0 = k) = \eta^k(1-\eta), \quad k \geq 0.\]

Hence, by Lemma 6

\[E_{\omega}(N_0) = E_{\omega}(R_0 + L_0) \overset{(\ast)}{=} \left[ E_{\omega}(L_1) + 1 \right] + E_{\omega}(L_0) = 2E_{\omega}(L_0) + 1 = \frac{1+\eta}{1-\eta}. \]

Step (*) follows from the fact that \(R_0 = L_1 + 1\) a.s., since the random walk is a.s. transient to \(+\infty\).

Proof of (ii): For \(i \geq 0\), let \(A_i\) be the event that the random walk steps right from site 0 and eventually returns \(i\) times without ever jumping left from 0, and let \(A_i'\) be the event that the random walk steps right from site 0 and eventually returns \(i\) times without stepping left from 0, but then does step left on its next visit:

\[
A_i = \{N_0 \geq i + 1, T_{-1} > T_0^{(i+1)}\},
\]

\[
A_i' = \{N_0 \geq i + 1, T_{-1} = T_0^{(i+1)} + 1\}.
\]

Clearly, \(P_{\omega}(A_0) = 1\). We claim also that:

\[
P_{\omega}(A_i'|A_i) = \begin{cases} 
(1-q), & \text{for } 0 \leq i \leq R - 1 \\
(1-p), & \text{for } i \geq R
\end{cases} \tag{40}
\]
and

\[ \mathbb{P}_\omega(A_{i+1} | A_i) = \begin{cases} q \eta , & \text{for } 0 \leq i \leq R - 1 \\ p \eta , & \text{for } i \geq R. \end{cases} \]  

(41)

To see (40), note that after jumping right from site 0 and returning \( i \) times in a row, site 0 will be in configuration \((q, i)\), for \( 0 \leq i \leq R - 1 \), and in configuration \((p, 0)\) for \( i \geq R \). Thus, for \( 0 \leq i \leq R - 1 \), we have

\[ \mathbb{P}_\omega(A_i | A_i') = \mathbb{P}_\omega \left( X_{T_0^{(i+1)}+1} = -1 \middle| \omega_{T_0^{(i+1)}}(0) = (q, i) \right) = (1 - q) \]

and, for \( i \geq R \), we have

\[ \mathbb{P}_\omega(A_i | A_i') = \mathbb{P}_\omega \left( X_{T_0^{(i+1)}+1} = -1 \middle| \omega_{T_0^{(i+1)}}(0) = (p, 0) \right) = (1 - p). \]

Now, (41) follows from (40) and the following calculation which is valid for all \( i \geq 0 \):

\[ \mathbb{P}_\omega(A_i | A_{i+1}) = \mathbb{P}_\omega(X_{T_0^{(i+1)}+1} = 1 | A_i) \cdot \mathbb{P}_\omega(T_0^{(i+2)} < \infty | A_i, X_{T_0^{(i+1)}+1} = 1) = \mathbb{P}_\omega((A_i')^c | A_i) \cdot \eta. \]

The second equality above follows from (39), which implies that on the event \( \{X_{T_0^{(i+1)}+1} = 1\} \), all sites \( x \geq 1 \) are in the \((q, 0)\) configuration at time \( T_0^{(i+1)} + 1 \).

Now, from (40) and (41), along with the fact \( \mathbb{P}_\omega(A_0) = 1 \), we conclude that

\[ \mathbb{P}_\omega(A_i') = \left( \prod_{j=1}^{i} \mathbb{P}_\omega(A_j | A_{j-1}) \right) \cdot \mathbb{P}_\omega(A_i' | A_i) = \begin{cases} (q \eta)^i (1 - q) , & \text{for } 0 \leq i \leq R - 1 \\ (q \eta)^R (p \eta)^{i-R} (1 - p) , & \text{for } i \geq R. \end{cases} \]

So,

\[ \eta = \mathbb{P}_\omega(T_{-1} < \infty) = \sum_{i=0}^{\infty} \mathbb{P}_\omega(A_i') = (1 - q) \frac{1 - (q \eta)^R}{1 - q \eta} + (1 - p) \frac{(q \eta)^R}{1 - p \eta}. \]

For \( 0 < \eta < 1 \), this condition is equivalent to \( P(\eta) = 0 \).

Proof of (iii): From point (ii) above we know that \( \eta \in (0, 1) \) is a root of the polynomial \( P(t) \). We now show that there cannot be any other roots in \((0, 1)\). Observe that \( t = 1 \) is a root of \( P \) and that \( P \) factors as

\[ P(t) = (1 - t) \left( 1 - q + (pq - p - q)t + pqt^2 - (p - q)q R t R \right) \equiv (1 - t)Q(t). \]
Thus, we need to show that the only root of $Q$ in $(0, 1)$ is $\eta$. Observe that $\frac{1}{q} > 1$ is a root of $Q$. For $R \geq 3$, we have

$$Q''(t) = 2pq - (p - q)q^R R(R - 1)t^{R-2}. $$

So, if $R \geq 3$ and $q \geq p$ then $Q$ is convex and can have at most two real roots. Also, if $R \in \{1, 2\}$ then $Q$ is quadratic and, thus, has at most two real roots. In either case, this completes the proof.

Now assume that $R \geq 3$ and $q < p$. In this case, $Q''$ has one real root. Thus, $Q'$ can have no more than two real roots. Let $t^+$ denote the largest root of $Q$ in $(0,1)$. We will show below that $Q(1) < 0$. Using this along with the facts that $Q(t^+) = Q(\frac{1}{q}) = 0$ and $Q(t) < 0$ for sufficiently large $t$, it follows that $Q'$ has two roots in $(t^+, \infty)$. If there were another root $t^- \in (0, 1)$ of $Q$, then $Q'$ would have to have a root in $(t^-, t^+)$, but this is impossible since $Q'$ cannot have more than two real roots. It remains to show that $Q(1) < 0$.

We have

$$Q(1) = 1 - 2q + q^{R+1} - p(1 - 2q + q^R). \quad (42)$$

Since we are assuming that $\alpha > \frac{1}{2}$, it follows from Proposition 1 that $p > \frac{1 - 2q + q^{R+1}}{1 - 2q + q^R} \equiv p_0$, if $1 - 2q + q^{R+1} > 0$. On the other hand, if $1 - 2q + q^{R+1} \leq 0$, then $p \in (0, 1)$ is unrestricted. In the former case, it follows from (42) that for any $q$, $Q(1) < 1 - 2q + q^{R+1} - p_0(1 - 2q + q^R) = 0$. In the latter case, it follows from (42) that

$$Q(1) < \max\{1 - 2q + q^{R+1}, 1 - 2q + q^{R+1} - (1 - 2q + q^R)\}$$

$$= \max\{1 - 2q + q^{R+1}, q^{R+1} - q^R\} \leq 0.$$

\[\square\]

### 3.5 Proof of Theorem 4

Unlike the proof of Theorem 3 for the speed with $L = 1$, the proof of Theorem 4 for the speed with $R = 1$ does not rely on the implicit characterization of the speed given by Theorem 2 in terms of $\gamma$. Instead, the proof is based on a direct method for estimating the hitting times $T_x$ for large $x$.

**Proof of Theorem 4** For $0 \leq i \leq L - 1$, we define $a_i$ to be the expected hitting time of site 1, starting from site 0, in an initial environment with all sites $x < 0$ in the $(p, 0)$ configuration and site 0 in the $(p, i)$ configuration. Also, we define $a_{L}$ to be the expected hitting time of site 1, starting from site 0, in an initial environment with all sites $x < 0$ in the $(p, 0)$ configuration and site 0 in the $(q, 0)$ configuration.

$$a_i = \mathbb{E}_{\omega(i)}(T_1), \quad 0 \leq i \leq L,$$
where the environments $\omega^{(i)}$ satisfy:

\[ \omega^{(i)}(x) = (p,0) , \ x < 0 \quad \text{and} \quad 0 \leq i \leq L. \]

\[ \omega^{(i)}(0) = (p,i) , \ 0 \leq i \leq L - 1. \]

\[ \omega^{(L)}(0) = (q,0). \]

The proof proceeds in two steps. First we set up a linear system of equations for the $a_i$’s, which can be solved to obtain the desired speed formula in the case that the initial environment $\omega$ satisfies $\omega(x) = (p,0)$, for all $x < 0$. Then, using this result, we show that the same speed formula holds in the general case.

**Case (1):** $\omega(x) = (p,0)$, for all $x < 0$.

Since $\alpha > 1/2$, $T_x$ is a.s. finite for each $x > 0$, and we define $\Delta_x$, $x \geq 0$, by

\[ \Delta_x = T_{x+1} - T_x. \]

The key observation is that because $R = 1$ and the random walk starts at $X_0 = 0$ in an environment $\omega$ satisfying $\omega(x) = (p,0)$, for all $x < 0$, we have

\[ \omega_n(x) = (p,0), \ \text{for each} \ n \geq 0 \ \text{and} \ x < X_n. \]

That is, the environment to the left of the current position of the random walk always consists entirely of sites in the $(p,0)$ configuration. Applying this fact at the random time $T_x$ it follows that, for each $x > 0$, $\Delta_x$ is independent of $\Delta_0, \ldots, \Delta_{x-1}$ and has distribution:

\[
\begin{align*}
\mathbb{P}_\omega(\Delta_x = k) &= \mathbb{P}_{\omega^{(i)}}(T_1 = k) , \text{ if } \omega(x) = (p,i) , 0 \leq i \leq L - 1. \\
\mathbb{P}_\omega(\Delta_x = k) &= \mathbb{P}_{\omega^{(q)}}(T_1 = k) , \text{ if } \omega(x) = (q,0).
\end{align*}
\]

Thus, defining

\[
A_x^i = \{ 0 \leq y \leq x - 1 : \omega(x) = (p,i) \} , 0 \leq i \leq L - 1 \\
A_x^L = \{ 0 \leq y \leq x - 1 : \omega(x) = (q,0) \}
\]

and applying the strong law of large numbers for the i.i.d. random variables $\{\Delta_y : \omega(y) = (p,i)\}$ and $\{\Delta_y : \omega(y) = (q,0)\}$ we have that $\mathbb{P}_\omega$ a.s.

\[
\lim_{x \to \infty} \frac{T_x}{x} = \lim_{x \to \infty} \frac{1}{x} \sum_{i=0}^{L} \sum_{y \in A_x^i} \Delta_y = \lim_{x \to \infty} \frac{1}{x} \sum_{i=0}^{L} \frac{|A_x^i|}{x} \sum_{y \in A_x^i} \Delta_y = \frac{L}{\sum_{i=0}^{L} d_ia_i} \tag{43}
\]

So, by Lemma 3

\[
\lim_{n \to \infty} \frac{X_n}{n} = \frac{1}{\sum_{i=0}^{L} d_ia_i} , \ \mathbb{P}_\omega \text{ a.s.} \tag{44}
\]

\[ \text{2} \text{Of course, in order to apply the strong law to conclude that } \lim_{x \to \infty} \frac{1}{|A_x^i|} \sum_{y \in A_x^i} \Delta_y = a_i, \text{ we need } |A_x^i| \to \infty. \text{ However, if } |A_x^i| \not\to \infty, \text{ for some } i, \text{ then } d_i = 0. \text{ So,} \lim_{x \to \infty} \frac{1}{x} \sum_{y \in A_x^i} \Delta_y = 0 = d_ia_i, \text{ and } (43) \text{ still holds.} \]
Now, by conditioning on the first step of the walk it is easy to see that the following relations between the \(a_i\)’s hold:

\[
a_i = p \cdot 1 + (1 - p) \cdot (1 + a_0 + a_{i+1}), \quad 0 \leq i \leq L - 1.
\]

\[
a_L = q \cdot 1 + (1 - q) \cdot (1 + a_0 + a_L).
\]

(45)

One possible solution to the system (45) is \(a_0 = a_1 = \ldots = a_L = \infty\). However, by (44), this implies \(X_n/n \to 0, \mathbb{P}_\omega \) a.s., which contradicts Theorem 2. Also, if \(a_j = \infty\), for any \(j\), then to satisfy (45) we must have \(a_i = \infty\), for all \(i\), which, as just shown, cannot happen. Over the real numbers the system (45) has a unique solution given by (12) and (13). This is shown in Appendix A.2.

Case (2): General Case

Fix any initial environment \(\omega\) such that the limiting right densities \(d_i\) exist, and let \(s = 1/(\sum_{i=0}^L d_i a_i)\). Also, for an arbitrary environment \(\omega'\), let \(\tau\) denote the last hitting time of site 0 (which is a.s. finite by Theorem 1).

We observe that:

1. For any \(\omega'\),

\[
\mathbb{P}_{\omega'}(\tau = 0) = \mathbb{P}_{\omega''}(\tau = 0) > 0,
\]

by Corollary 2, where \(\omega''\) is the environment defined by

\[
\omega''(x) = \omega'(x), \quad x \geq 0 \quad \text{and} \quad \omega''(x) = (p, 0), \quad x < 0.
\]

(46)

2. For any environment \(\omega'\) with \(\mathbb{P}_\omega(\omega_{\tau} = \omega') > 0\), we have

\[
\mathbb{P}_\omega(X_n/n \to s | \omega_{\tau} = \omega') = \mathbb{P}_\omega(X_n/n \to s | \tau = 0) = \mathbb{P}_{\omega''}(X_n/n \to s | \tau = 0),
\]

where \(\omega''\) is defined by (46).

3. For any environment \(\omega'\) with \(\mathbb{P}_\omega(\omega_{\tau} = \omega') > 0\), \(\omega'(x) = \omega(x)\), for all but finitely many \(x\). So, the limiting right densities \(d_i\) of states in each configuration for the environment \(\omega'\) are the same as the limiting right densities \(d_i\) for the initial environment \(\omega\).

It follows from these three observations and the result for Case (1) that, for any environment \(\omega'\) with \(\mathbb{P}_\omega(\omega_{\tau} = \omega') > 0\),

\[
\mathbb{P}_\omega(X_n/n \to s | \omega_{\tau} = \omega') = \mathbb{P}_{\omega''}(X_n/n \to s | \tau = 0) = \mathbb{P}_{\omega''}(X_n/n \to s) = 1.
\]

Hence, \(X_n/n \to s, \mathbb{P}_\omega\) a.s.

\[\square\]
4 The Critical Case

Here we analyze the transience/recurrence properties of the random walk \((X_n)\) in the critical case \(\alpha = 1/2\), proving Theorems 5-7. We begin in Section 4.1 with some lemmas for transience/recurrence of Markov chains on \(\mathbb{N}_0\). Then, in Section 4.2 we establish a framework relating the right jumps Markov chain \((Z_x)\) to the setup of these lemmas. Using this framework, Theorem 5 is proved in Section 4.3 and Theorem 7 in Section 4.4. Finally, Theorem 6 is proved in Section 4.5 using other methods.

4.1 Transience and Recurrence for Markov Chains on \(\mathbb{N}_0\)

In Lemmas 8 and 9 below we assume that \((Z_x)_{x \geq 0}\) is an irreducible, aperiodic Markov chain on \(\mathbb{N}_0 = \{0, 1, 2, \ldots\}\) with step distribution \(U(n)\). That is, \(P(Z_{x+1} = m | Z_x = n) = P(U(n) = m), \ n, m \geq 0.\)

Also, we denote by \(P_k\) the probability measure for this chain started from initial state \(k\). Finally, we say that the step distribution \(U(n)\) is well concentrated if \(\mu \equiv \lim_{n \to \infty} E(U(n))/n\) exists and there exists constants \(C, c > 0\) and \(N \in \mathbb{N}\) such that:

\[
P(|U(n) - \mu_n| > tn) \leq Ce^{-ctn}, \quad \text{for} \quad t \geq 1 \quad \text{and} \quad n \geq N.
\]

In this case, we define also the quantities \(\rho(n), \nu(n), \theta(n)\) by

\[
\rho(n) = E(U(n) - \mu n), \quad \nu(n) = E((U(n) - \mu n)^2)/n, \quad \theta(n) = 2\rho(n)/\nu(n).
\]

Lemma 8. If \(E(U(n)) \leq n, \ \text{for all sufficiently large} \ n, \ \text{then}

\[
P_k(Z_x > 0, \forall x \geq 0) = 0, \ \text{for any} \ k \geq 1.
\]

Lemma 9. If the step distribution \(U(n)\) is well concentrated then the following hold for any initial state \(k \geq 1.\)

(i) If \(\mu < 1\), then \(P_k(Z_x > 0, \forall x \geq 0) = 0.\)

(ii) If \(\mu > 1\), then \(P_k(Z_x > 0, \forall x \geq 0) > 0.\)

(iii) If \(\mu = 1, \ \liminf_{n \to \infty} \nu(n) > 0, \ \text{and} \ \theta(n) > 1 + \frac{1}{\ln(n)} - \frac{a(n)}{n^{\frac{1}{2}}}, \ \text{for sufficiently large} \ n, \ \text{for some function} \ a(n) \to \infty, \ \text{then} \ P_k(Z_x > 0, \forall x \geq 0) = 0.\)

(iv) If \(\mu = 1, \ \liminf_{n \to \infty} \nu(n) > 0, \ \text{and} \ \theta(n) < 1 + \frac{2}{\ln(n)} + \frac{a(n)}{n^{\frac{1}{2}}}, \ \text{for sufficiently large} \ n, \ \text{for some function} \ a(n) \to \infty, \ \text{then} \ P_k(Z_x > 0, \forall x \geq 0) > 0.\)

Lemma 8 follows immediately from part (1) of Theorem A.1 in [3] by taking the Lyapunov function \(V(x) = x,\) and Lemma 9 is essentially Theorem 1.3 in...
The following observations will be useful for applying the lemmas in several instances below.

(I) The claims in both lemmas do not depend at all on the transition probabilities from state 0. Thus, if 0 is an absorbing state for the Markov chain \((Z_x)\), but one can redefine the transition probabilities from state 0 to make the chain irreducible and aperiodic then the lemmas can still be applied. In this case, we say that the chain \((Z_x)\) is irreducible and aperiodic with the exception of 0.

(II) If \(\xi_1, \xi_2, \ldots\) are i.i.d. random variables with exponential tails and \(\chi\) is another random variable with exponential tail, independent of the \(\xi_i\), then standard large deviation estimates show that (47) and (48) hold with \(\mu = \mathbb{E}(\xi_i)\) if \(U(n) = \chi + \sum_{i=1}^{n-1} \xi_i\).

4.2 Step Distribution of the Right Jumps Markov chain

By definition (23) for the right jumps Markov chain \((Z_x)_{x \geq 0}\),

\[
P(Z_{x+1} = m | Z_x = n) = \mathbb{P}(U(n, x+1) = m), \quad n, m, x \geq 0
\]

where \(U(n, x)\) is the (random) number of right jumps in the sequence \((J^x_k)_{k \in \mathbb{N}}\) before the time of the \(n\)-th left jump:

\[
U(n, x) = \inf \left\{ \ell \geq 0 : \sum_{k=1}^{\ell} \mathbb{1}\{J^x_k = -1\} = n \right\} - n. \tag{50}
\]

If the initial environment \(\omega(x)\) is constant for all \(x \geq 0\), then the distribution of the jump sequence \((J^x_k)_{k \in \mathbb{N}}\) is the same for all \(x \geq 0\), so the distribution of \(U(n, x)\) is also the same for all \(x \geq 0\). In this case, the right jumps chain \((Z_x)_{x \geq 0}\) is time-homogeneous (where \(x\) is the time variable) with step distribution \(\bar{U}(n) = U(n, x)\). It is also irreducible and aperiodic with the exception of state 0.

For the remainder of this section we assume \(\omega(x) = \omega(0)\), for all \(x \geq 0\). For our analysis of the step distribution \(U(n)\) we fix an arbitrary site \(x\) and decompose \(U(n) = U(n, x)\) as

\[
U(n) = \sum_{j=1}^{n} \Gamma_j, \tag{51}
\]

[3] There are two small differences. First, in Theorem 1.3 of [3] instead of (47) and (48) the following somewhat stronger concentration condition for \(U(n)\) is assumed: There exist \(c > 0\) and \(N \in \mathbb{N}\) such that

\[
\mathbb{P}(|U(n) - \mu n| > tn) \leq 2e^{-ct^2n}, \text{ for all } t > 0 \text{ and } n \geq N. \tag{49}
\]

Second, there is no assumption that \(\lim \inf_{n \to \infty} \nu(n) > 0\) for cases (iii) and (iv).

The concentration condition (49) is used in [3] only to bound the error terms in certain Taylor series expansions. These estimates remain valid if (47) and (48) hold instead, so there is no issue with using the weaker concentration condition. However, the proof of cases (iii) and (iv) given in [3] actually works as stated only if \(\lim \inf_{n \to \infty} \nu(n) > 0\), so we require this condition also in our statement.
where $\Gamma_j$ is the number of right jumps in the sequence $(J^x_k)_{k \in \mathbb{N}}$ between the $(j-1)$-th and $j$-th left jumps. That is, $\Gamma_j = k_j - k_{j-1} - 1$, where $k_0 = 0$ and, for $j \geq 1$, $k_j = \inf \{ k > k_{j-1} : J^x_k = -1 \}$.

We think of the $(\Gamma_j)_{j=1}^n$ as the values obtained in $n$ “sessions,” and denote by $\omega^{(j)}(x)$ the configuration at site $x$ at the beginning of the $j$-th session. That is, $\omega^{(j)}(x) = Y^x_{k_{j-1}+1}$ is the configuration at site $x$ immediately after the $(j-1)$-th left jump in the sequence $(J^x_k)_{k \in \mathbb{N}}$. It is straightforward to see that conditioned on $\omega^{(j)}(x)$, $\Gamma_j$ is independent of $\Gamma_1, \ldots, \Gamma_{j-1}$ and has the following distribution:

\[
\begin{align*}
\Gamma_j &\sim S_i, \text{ if } \omega^{(j)}(x) = (q, i), \text{ for some } 0 \leq i \leq R - 1, \\
\Gamma_j &\sim S_R, \text{ if } \omega^{(j)}(x) = (p, i), \text{ for some } 0 \leq i \leq L - 1,
\end{align*}
\]

(52)

where $S_0, \ldots, S_R$ are random variables with law

\[
\mathbb{P}(S_i = k) = \begin{cases} q^k (1 - q), & 0 \leq k \leq R - i - 1; \\ q^{R-i} p^{k-(R-i)} (1 - p), & k \geq R - i. \end{cases}
\]

(53)

In particular, $S_R$ is a standard geometric random variable with parameter $1 - p$.

Now, the configuration $\omega^{(j+1)}(x)$ at the beginning of the next session is determined entirely by $\omega^{(j)}(x)$ and $\Gamma_j$. More precisely, $\omega^{(j+1)}(x)$ is the (deterministic) configuration obtained by jumping right $\Gamma_j$ times from site $x$, starting in configuration $\omega^{(j)}(x)$, and then jumping left once:

If $\omega^{(j)}(x) = (q, i), 0 \leq i \leq R - 1$, then $\omega^{(j+1)}(x) = \begin{cases} (p, 1), & \text{if } \Gamma_j \geq R - i; \\ (q, 0), & \text{if } \Gamma_j < R - i. \end{cases}$

(54)

If $\omega^{(j)}(x) = (p, i), 0 \leq i \leq L - 2$, then $\omega^{(j+1)}(x) = \begin{cases} (p, 1), & \text{if } \Gamma_j \geq 1; \\ (p, i + 1), & \text{if } \Gamma_j = 0. \end{cases}$

This, of course, is assuming that $L \geq 2$. If $L = 1$ then the configuration $\omega^{(j)}(x)$ at the beginning of each of the right jumps sessions after the first is always $(q, 0)$, since the configuration at site $x$ immediately after a left jump is $(q, 0)$.

From (52)-(54) it follows that, for any $L \geq 2$, the sequence of configurations $\{\omega^{(j)}(x)\}_{j=1}^n$ is a Markov chain with (initial state $\omega(x)$) and transition matrix $\hat{A}$ given by

\[
\begin{align*}
\hat{A}_{(q,i),(q,0)} &= 1 - q^{R-i}, \quad \hat{A}_{(q,i),(p,1)} = q^{R-i} \quad \text{for } 0 \leq i \leq R - 1; \\
\hat{A}_{(p,i),(p,1)} &= p, \quad \hat{A}_{(p,i),(p,i+1)} = 1 - p \quad \text{for } 0 \leq i \leq L - 2; \\
\hat{A}_{(p,L-1),(p,1)} &= p, \quad \hat{A}_{(p,L-1),(q,0)} = 1 - p.
\end{align*}
\]

In the case $L = 1$, $\{\omega^{(j)}(x)\}_{j=1}^n$ is still a Markov chain, but it is degenerate. The transition matrix $\hat{A}$ has $\hat{A}_{\lambda,(q,0)} = 1$, for all $\lambda \in \Lambda$. 

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In either case, the transition matrix $\hat{A}$ is indecomposable, and the $L$ states $\{(q,0), (p,1), \ldots, (p, L-1)\}$ constitute a closed, irreducible set of states. We denote by $A$ the corresponding transition matrix obtained from $\hat{A}$ by restricting to these $L$ states, and by $\psi$ the unique invariant measure for $A$ (for $L = 1$, $A = \psi = 1$). Also, we denote by $e_{(p,i)}$ the unit $L$-vector with a 1 in the position of state $(p,i)$, and by $c_{(q,0)}$ the unit $L$-vector with a 1 in the position of $(q,0)$. Finally, we denote by $E$ denote the $L$-vector with components, $E_{(q,0)} = E(S_0)$ and $E_{(p,i)} = E(S_R)$, for $i \in \{1, \ldots, L-1\}$. For the proofs of Theorems 5 and 7 below we will need the following simple lemma.

**Lemma 10.** If $\alpha = 1/2$, then
\[
\lim_{n \to \infty} \frac{\mathbb{E}(U(n))}{n} = \langle \psi, E \rangle = 1,
\]
where $\langle \cdot, \cdot \rangle$ denotes the inner product of two $L$-vectors.

**Proof.** Since $U(n) = \sum_{j=1}^n \Gamma_j$, if follows from the Markov chain representation described above and the ergodic theorem for finite-state Markov chains that
\[
\lim_{n \to \infty} \frac{\mathbb{E}(U(n))}{n} = \lim_{j \to \infty} \mathbb{E}(\Gamma_j) = \langle \psi, E \rangle \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n \Gamma_j = \langle \psi, E \rangle, \ a.s.
\]
Thus,
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^m \mathbbm{1}\{J_k^x = 1\} = \lim_{n \to \infty} \left( \frac{\sum_{j=1}^n \Gamma_j}{n + \sum_{j=1}^n \Gamma_j} \right) = \langle \psi, E \rangle \left/ \mathbb{1} + \langle \psi, E \rangle \right., \ a.s.
\]
On the other hand, as noted at the end of Section 2.1.1
\[
\lim_{m \to \infty} \frac{1}{m} \sum_{k=1}^m \mathbbm{1}\{J_k^x = 1\} = \alpha, \ a.s.
\]
Since $\alpha = 1/2$, it follows that $\langle \psi, E \rangle = 1$.

**4.3 Proof of Theorem 5**

Throughout this section, $U(n)$ and $\Gamma_j$ are defined as in the previous section whenever the initial environment $\omega$ is constant for $x \geq 0$. Also, we use the notation $\Gamma_j^{(\lambda)}$, when necessary, to specifically denote the random variable $\Gamma_j$ with initial environment $\omega(x) = \lambda$, $x \geq 0$. Finally, we use the ordering $\prec$ on $\Lambda$ as in Section 2.3

For any state $\lambda \in \{(q,0), (p,1), \ldots, (p, L-1)\}$, the Markov chain representation described above shows that
\[
\mathbb{E}(\Gamma_j^{(\lambda)}) = \sum_{\lambda'} \mathbb{P}(\omega^{(j)}(x) = \lambda' | \omega^{(1)}(x) = \lambda) \cdot \mathbb{E}(\Gamma_j | \omega^{(j)}(x) = \lambda') = \langle e_{\lambda} A^{j-1}, E \rangle.
\]
A coupling argument very similar to the one used in Section 2.3 to show Lemma 4 gives the following.

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Lemma 11. If \( q < p \) and \( \lambda \prec \lambda' \), then \( \Gamma_j^{(\lambda)} \) is stochastically dominated by \( \Gamma_j^{(\lambda')} \). In particular, if \( q < p \) then for any \( \lambda \in \{(q,0), (p,1), \ldots, (p,L - 1)\} \), we have
\[
\langle e_{(q,0)}A_j^{j-1}, E \rangle = \mathbb{E}(\Gamma_j^{(q,0)}) \leq \mathbb{E}(\Gamma_j^{(\lambda)}) = \langle e_{\lambda}A_j^{j-1}, E \rangle.
\]

Proof of Theorem 7. We will show that, \( P_\omega(Z_x > 0, \forall x > 0) = 0 \), if \( \omega(x) = (q,0) \), for all \( x \geq 0 \), which implies also that
\[
P_\omega(X_n \rightarrow \infty) = 0, \text{ if } \omega(x) = (q,0) \text{ in a neighborhood of } +\infty.
\]
Symmetry consideration then imply
\[
P_\omega(X_n \rightarrow -\infty) = 0, \text{ if } \omega(x) = (p,0) \text{ in a neighborhood of } -\infty.
\]
Hence, by part (ii) of Lemma 1, the random walk must be \( P_\omega \) a.s. recurrent for any initial environment \( \omega \) that is equal to \( (q,0) \) in a neighborhood of \( +\infty \) and equal to \( (p,0) \) in a neighborhood of \( -\infty \).

For the remainder of the proof we assume that the initial environment \( \omega \) satisfies \( \omega(x) = (q,0) \), for all \( x \geq 0 \). To show (55), we observe that Lemmas 10 and 11 imply,
\[
\mathbb{E}(\Gamma_j) = \langle e_{(q,0)}A_j^{j-1}, E \rangle \leq \langle \psi A_j^{j-1}, E \rangle = \langle \psi, E \rangle = 1,
\]
for each \( j \in \mathbb{N} \). Thus, for each \( n \in \mathbb{N} \), we have
\[
\mathbb{E}(U(n)) = \sum_{j=1}^{n} \mathbb{E}(\Gamma_j) \leq n.
\]
Since the right jumps Markov chain \( (Z_x)_{x \geq 0} \) with step distribution \( U(n) = \sum_{j=1}^{n} \Gamma_j \) is irreducible and aperiodic with the exception of state 0, it follows by Lemma 8 and observation (I) in Section 4.1 that \( P_\omega(Z_x > 0, \forall x > 0) = 0 \).

4.4 Proof of Theorem 7

For the proof of Theorem 7 we again use the framework for analyzing the step distribution of the right jumps Markov chain introduced in Section 4.2. Also, for notational convenience we define
\[
\lambda_0 = (q,0), \ldots, \lambda_{R-1} = (q, R-1), \lambda_R = (p,0).
\]
As discussed above, in the case \( L = 1 \) the transition matrix \( \hat{A} \) is degenerate and \( \omega^{(j)}(x) = (q,0) \), for all \( j \geq 2 \) (independent of the values of the \( \Gamma_j \)'s). Also, with \( L = 1 \), \( \psi \) is simply the length-1 vector 1 and \( E \) is simply the length-1 vector \( \mathbb{E}(S_0) \). The following facts are immediate from this.
1. If $L = 1$ and $\omega(x) = \lambda_i$, $x \geq 0$, then

\[ \Gamma_1, \Gamma_2, \ldots \text{ are independent with } \Gamma_1 \sim S_i \text{ and } \Gamma_j \sim S_0, \ j \geq 2. \] (56)

2. If $L = 1$ and $\alpha = 1/2$ then, by Lemma 10,

\[ \mathbb{E}(S_0) = \langle \psi, E \rangle = 1. \] (57)

Using these facts we now prove Theorem 7.

**Proof of Theorem 7.** By assumption $\alpha = 1/2$ and $L = 1$, and the initial environment is a constant in a neighborhood of $-\infty$ in the case of negative feedback, $p < q$. Thus, by Theorem 6, the probability of the random walk $(X_n)$ being transient to $-\infty$ is equal to $0$. So, by Lemma 1, the probability of being transient to $+\infty$ is either 0 or 1, and if it is 0, the process is recurrent. Moreover, without loss of generality, we may clearly assume that the initial environment $\omega$ is constant for all $x \geq 0$ (rather than only in a neighborhood of $+\infty$). Further, by Lemma 2, there is 0 probability of transience to $+\infty$ for an initial environment $\omega$ if and only if the hitting time of state 0 for the right jumps Markov chain $(Z_x)$ is a.s. finite.

In light of this discussion, it suffices to show the following to establish the transience/recurrence claims in the theorem:

- If $\omega(x) = \lambda_0$ for all $x \geq 0$, then $\mathbb{P}_\omega(Z_x > 0, \forall x > 0) = 0$.
- If $\omega(x) = \lambda_i$ for all $x \geq 0$, 1 $\leq i \leq R$, then $\mathbb{P}_\omega(Z_x > 0, \forall x > 0) = 0$ if and only if $P_{R,i}(q) \geq 0$. (58)

For the remainder of the proof we assume that

\[ \omega(x) = \lambda_i, \ x \geq 0, \] (59)

for some $0 \leq i \leq R$. Since the random variables $S_i, 0 \leq i \leq R$, have exponential tails, it follows from (56) and point (II) in Section 4.1 that the step distribution $U(n) = \sum_{j=1}^{n} \Gamma_j$ of the right jumps Markov chain $(Z_x)$ is well concentrated. Thus, by point (I) in Section 4.1, we can apply Lemma 9 to determine if the right jumps chain $(Z_x)$ has a positive probability of survival.

Since $\alpha = 1/2$, Lemma 10 gives $\mu \equiv \lim_{n \to \infty} \frac{\mathbb{E}(U(n))}{n} = 1$. Thus, to apply Lemma 9 we need to analyze the quantity

\[ \theta(n) = \frac{2\rho(n)}{\nu(n)}, \text{ where } \rho(n) = \mathbb{E}(U(n) - n) \text{ and } \nu(n) = \frac{\mathbb{E}[(U(n) - n)^2]}{n}. \]

We begin with $\rho(n)$. Since $L = 1$ and $\alpha = 1/2$, $\mathbb{E}(S_0) = 1$, by (57). Thus, by (56),

\[ \rho(n) = \mathbb{E}(S_i) + (n-1)\mathbb{E}(S_0) - n = \mathbb{E}(S_i) - 1. \] (60)

\[ \text{Theorem 6 is not proved till later in Section 4.5, but the proof is independent of the proof of this theorem.} \]
A direct computation yields

\[
\mathbb{E}(S_i) = \sum_{k=0}^{R-i-1} k \cdot q^k (1 - q) + \sum_{k=R-i}^{\infty} k \cdot q^{R-i} p^{k-(R-i)} (1 - p)
\]

\[
= \frac{1}{1-q} \left[ -(1-q)q^{R-i}(R-i) + (1-q^{R-i})q \right] + \frac{1}{1-p} q^{R-i} \left[ (1-p)p^{R-i}(R-i) + p^{R-i+1} \right].
\]

Since \(L = 1\) and \(\alpha = 1/2\), Proposition 1 implies

\[
\rho(n) = \frac{1-2q + q^{i+1}}{q^i(1-q)}.
\]

Thus, by (60),

\[
\rho(n) = \frac{1-2q - q^i + 2q^{i+1}}{q^i(1-q)}.
\]

Since \(\rho(n)\) is independent of \(n\), from now on we will just write \(\rho\).

We now turn to the calculation of \(\nu(n)\). Using the independence of the random variables \(\Gamma_1, \ldots, \Gamma_n\) we have

\[
\mathbb{E} \left[ (U(n) - n)^2 \right] = \mathbb{E} \left[ \left( \sum_{j=1}^{n} \Gamma_j - (n - 1 + \mathbb{E}(S_i)) - (1 - \mathbb{E}(S_i)) \right)^2 \right] = \sigma^2(S_i) + (n-1)\sigma^2(S_0) + (1-\mathbb{E}(S_i))^2 = n[\mathbb{E}(S_0^2) - 1] + O(1).
\]

A tedious computation gives

\[
\mathbb{E}(S_0^2) = \sum_{k=0}^{R-1} k^2 \cdot q^k (1 - q) + \sum_{k=R}^{\infty} k^2 \cdot q^R p^{k-R} (1 - p)
\]

\[
= \frac{q + q^2 - q^R \left( R^2 - (2R^2 - 2R - 1)q + (R - 1)^2 q^2 \right)}{(1-q)^2} + \frac{q^R \left( R^2 - (2R^2 - 2R - 1)p + (R - 1)^2 p^2 \right)}{(1-p)^2}.
\]

Substituting for \(p\) from (61) and doing a lot of algebra, one eventually finds that

\[
\mathbb{E}(S_0^2) = \frac{1}{q^R(1-q)^2} \left[ 2 - 8q + 8q^2 + (2R+1)q^R + (2-6R)q^{R+1} + (4R-5)q^{R+2} \right].
\]
Finally, from (63) and (64), we have
\[
\nu(n) = \frac{\mathbb{E}[(U(n) - n)^2]}{n} = \mathbb{E}(S_0^2) - 1 + O\left(\frac{1}{n}\right) =
\]
\[
\frac{1}{q^R(1-q)^2} \left[ 2 - 8q + 8q^2 + 2Rq^R + (4 - 6R)q^{R+1} + (4R - 6)q^{R+2} \right] + O\left(\frac{1}{n}\right).
\]
(65)

Since \(\mathbb{E}(S_0) = 1\) and \(S_0\) is non-degenerate, we know \(\mathbb{E}(S_0^2) > 1\). Thus, it follows from this calculation that \(\lim_{n \to \infty} \nu(n) > 0\), which is also required to apply Lemma 9 in the case \(\mu = 1\).

Now, combining (62) and (65) gives
\[
\theta(n) = \frac{q^{R-i}(1-q)(1-2q - q^i + 2q^{i+1})}{1 - 4q + 4q^2 + Rq^R + (2 - 3R)q^{R+1} + (2R - 3)q^{R+2}} + O\left(\frac{1}{n}\right) =
\]
\[
\frac{q^{R-i} - 3q^{R-i+1} + 2q^{R-i+2} - q^R + 3q^{R+1} - 2q^{R+2}}{1 - 4q + 4q^2 + Rq^R + (2 - 3R)q^{R+1} + (2R - 3)q^{R+2}} + O\left(\frac{1}{n}\right).
\]
(66)

Thus, for the initial environment \(\omega\) satisfying (59), Lemma 9 shows that \(\mathbb{P}_\omega(Z_x > 0, \forall x > 0) = 0\) if and only if
\[
\frac{q^{R-i} - 3q^{R-i+1} + 2q^{R-i+2} - q^R + 3q^{R+1} - 2q^{R+2}}{1 - 4q + 4q^2 + Rq^R + (2 - 3R)q^{R+1} + (2R - 3)q^{R+2}} \leq 1.
\]

For \(1 \leq i \leq R\), this inequality is equivalent to \(P_{R,i}(q) \geq 0\), and in the case \(i = 0\) the left hand side is 0, so the inequality always holds. Therefore, (58) is satisfied.

It remains only to show the claims concerning the polynomial \(P_{R,R}(q)\). For these we will use the factored representation \(P_{R,R}(q) = q(1-q)^2 \tilde{P}_{R,R}(q)\), where \(\tilde{P}_{R,R}(q) = -1 + \sum_{j=1}^{R-1} jq^j + (2R - 1)q^{R-1}\), as in (16). Since, \(P_{R,R}(q)\) and \(\tilde{P}_{R,R}(q)\) have the same sign for all \(q \in (0,1)\), it suffices to prove the claims for the polynomial \(\tilde{P}_{R,R}(q)\).

Now, clearly, \(\tilde{P}_{R,R}\) is increasing, and \(\tilde{P}_{R,R}(0) = -1\). For \(R \geq 4\), one can rewrite \(\tilde{P}_{R,R}\) as \(\tilde{P}_{R,R}(q) = -1 + \left(\frac{q}{1-q}\right)^2 (1 - (R - 2)q^{R-3} + (R - 3)q^{R-2}) + (2R - 1)q^{R-1}\). Using this, we find that \(\tilde{P}_{R,R}(\frac{1}{2}) = (\frac{1}{2})^{R-1}\), for all \(R \geq 2\). Consequently, \(\tilde{P}_{R,R}\) has a unique root \(q_*(R) \in (0,\frac{1}{2})\), with \(\tilde{P}_{R,R}(q) < 0\) for \(q < q_*(R)\) and \(\tilde{P}_{R,R}(q) > 0\) for \(q > q_*(R)\). Furthermore,
\[
\tilde{P}_{R+1,R+1}(q) - \tilde{P}_{R,R}(q) = (2R+1)q^R - (R+1)q^{R-1} =
\]
\[
q^{R-1} \left[(2R+1)q - (R+1)\right] \leq -\frac{1}{2}q^{R-1} < 0, \text{ for } q \in [0,\frac{1}{2}].
\]
(67)

So, \(q_*(R)\) is increasing in \(R\). Also, we have \(\tilde{P}_{\infty,\infty}(q) \equiv \lim_{R \to \infty} \tilde{P}_{R,R}(q) = \frac{2q-1}{(1-q)^2}\).

Since the root of \(\tilde{P}_{\infty,\infty}\) is at \(q = \frac{1}{2}\), it follows that \(\lim_{R \to \infty} q_*(R) = \frac{1}{2}\). □
4.5 Proof of Theorem 6

Thus far the proofs of transience/recurrence for the random walk \((X_n)\) have centered around an analysis of the right jumps Markov chain \((Z_x)\). For the proof of Theorem 6 we will need to construct another auxiliary process called the left jumps Markov chain.

Consider the random walk \((X_n)_{n \geq 0}\) started from \(X_0 = 0\) and restricted to \(\mathbb{N} \cup \{0, -1\}\) by the following modification of its transition mechanism: when the walker is at a site \(x \geq 0\), it behaves as before, but at the site \(-1\) it jumps right with probability one. Denote the modified random walk by \((\tilde{X}_n)_{n \geq 0}\). Note that the modified random walk can be defined in terms of the extended single site Markov chains, \((\tilde{Y}_n^x)_{n \in \mathbb{N}} = (Y_n^x, J_n^x)_{n \in \mathbb{N}}, x \geq 0\), along with an appropriately defined deterministic single site mechanism at \(x = -1\). Fix \(N \in \mathbb{Z}^+\) and let \(T_N = \inf\{n \geq 0 : \tilde{X}_n = N\}\) denote the first time the modified random walk hits \(N\). Note that \(T_N\) is almost surely finite. We define a process \((\tilde{W}^N_x)_{x = 0}^N\) by setting \(\tilde{W}^N_x\) equal to the number of times the modified walk \((\tilde{X}_n)\) jumps left from site \(x\) before time \(T_N\). That is,

\[
\tilde{W}^N_x = |\{n \leq T_N - 1 : \tilde{X}_n = x, \tilde{X}_{n+1} = x - 1\}|
\]

We will refer to this process \((\tilde{W}^N_x)_{x = 0}^N\) as the left jumps \(N\)-chain. It can also be defined directly in terms of the jump sequences \((J_k^x)_{k \in \mathbb{N}}\), \(0 \leq x \leq N\):

\[
\tilde{W}^N_x = \Theta^{(N)}_x = \Theta^{(N)}_{x+1} + 1, \quad x \in \{N-1, N-2, \ldots, 0\},
\]

where

\[
\Theta^{(N)}_x = \inf \left\{ n \geq 1 : \sum_{k=1}^{n} \mathbb{1}\{J_k^x = 1\} = \tilde{W}^N_{x+1} + 1 \right\}.
\]

That is, \(\tilde{W}^N_x\) is the number of left jumps in the jump sequence \((J_k^x)_{k \in \mathbb{N}}\) before the \((\tilde{W}^N_{x+1} + 1)\)-th right jump. In particular, \(\tilde{W}^N_x\) is independent of \(\tilde{W}^N_{x+2}, \tilde{W}^N_{x+3}, \ldots, \tilde{W}^N_N\) conditioned on \(\tilde{W}^N_{x+1}\), so the sequence \((\tilde{W}^N_0, \tilde{W}^N_1, \ldots, \tilde{W}^N_N)\) is Markovian. The distribution of the jump sequence \((J_k^x)_{k \in \mathbb{N}}\) is the same for all \(x \geq 0\), if the initial environment \(\omega\) is constant for all \(x \geq 0\). So, in this case, the transition probabilities

\[
\mathbb{P}(\tilde{W}^N_x = \ell | \tilde{W}^N_{x+1} = m) = \mathbb{P} \left( \inf \left\{ n \geq 1 : \sum_{k=1}^{n} \mathbb{1}\{J_k^x = 1\} = m + 1 \right\} - (m + 1) = \ell \right)
\]

are independent of \(N\) and \(x \in \{0, \ldots, N - 2\}\), and we may define a single time-homogeneous Markov chain \((W_n)_{n=0}^\infty\) such that \((\tilde{W}^N_0, \tilde{W}^N_1, \ldots, \tilde{W}^N_N)\) has the same distribution as \((W_0, W_1, \ldots, W_N)\), for all \(N\).

We call \((W_n)_{n=0}^\infty\) the left jumps Markov chain. The following proposition characterizes the transience or recurrence of the original random walk \((X_n)\) in terms of the positive recurrence or non-positive recurrence of the left jumps Markov chain.
Proposition 4. If \( X_0 = 0 \) and the initial environment \( \omega(x) \) is constant for \( x \geq 0 \), then the random walk \( (X_n) \) has positive probability of being transient to \( +\infty \) if and only if the left jumps Markov chain \( (W_n) \) is positive recurrent.

Proof. Arguments exactly like the proof of part (iii) of Lemma 1 show that the modified random walk \( (\tilde{X}_n) \) either has probability 1 of being transient to \( +\infty \) or probability 1 of being recurrent, and clearly the former occurs if and only if the original random walk \( (X_n) \) has a positive probability of being transient to \( +\infty \). Thus, it suffices to show that the left jumps Markov chain \( (W_n) \) is positive recurrent if and only if the modified random walk \( (\tilde{X}_n) \) is transient to \( +\infty \).

Now, by construction of the left jumps Markov chain \( (W_n) \), we know \( W_N \) and \( \tilde{W}_0^{(N)} \) have the same distribution for each \( N > 0 \), where \( \tilde{W}_0^{(N)} \) is the number of jumps of the modified random walk \( (\tilde{X}_n) \) from 0 to \( -1 \) before it first reaches \( N \). Thus, the distribution of \( W_N \) is stochastically increasing, and it converges to a limiting finite distribution if and only if the modified random walk \( (\tilde{X}_n) \) is transient. On the other hand, since \( (W_n)_{n \geq 0} \) is a (time-homogeneous) irreducible, aperiodic, Markov chain, the distribution of \( W_N \) converges to a finite limiting distribution if and only if this chain is positive recurrent.

We now use Proposition 4 to prove Theorem 6.

Proof of Theorem 6. By symmetry it suffices to treat the case \( R = 1 \). In the statement of the theorem, it is assumed that the environment is constant in a neighborhood of \( +\infty \) in the negative feedback case. For the proof, we will make this assumption even in the positive feedback case. This causes no problem because in the positive feedback case if we can prove that the probability of transience to \( +\infty \) is 0 for any constant environment then, by Lemma 4, it is also true for any non-constant initial environment. Without loss of generality, we may assume also that the initial environment is constant for all \( x \geq 0 \).

In this case, by Proposition 4 it suffices to show the left jumps Markov chain \( (W_n) \) is not positive recurrent. By construction of the left jumps chain we have

\[
P(W_n = m|W_{n-1} = m) = P(\tilde{W}^{(N)}_x = m|\tilde{W}^{(N)}_{x+1} = m),
\]

where the right hand side is independent of \( N \) and \( x \in \{0, ..., N-2\} \) (due to the assumption on the initial environment). Now, if we condition on \( W^{(N+1)}_x = m \), it follows from (68) and (69) that \( \tilde{W}^{(N)}_x \) is equal to the number of left jumps in the jump sequence \( (J^x_k)_{k \in \mathbb{N}} \) before the time of the \((m+1)\)-th right jump.

Similarly to the analysis of the right jumps chain, we decompose \( \tilde{W}^{(N)}_x \) as

\[
\tilde{W}^{(N)}_x = \sum_{j=1}^{m+1} V_j,
\]

where \( V_j \) is the number of left jumps in the sequence \( (J^x_k)_{k \in \mathbb{N}} \) between the \((j-1)\)-th and \( j \)-th right jumps. Since \( R = 1 \), the configuration at site \( x \) is always \((p, 0)\) immediately after a right jump from site \( x \). So, the starting configuration for
each of the “left jump sessions” after the first one is \((p, 0)\), independent of the number of left jumps in all previous sessions. It follows that the random variables \(V_1, ..., V_{m+1}\) are independent and \(V_2, ..., V_{m+1}\) are i.i.d. with common distribution \(V\), which is the distribution of the number of left jumps from site \(x\) before the first right jump, starting in the \((p, 0)\) configuration:

\[
\mathbb{P}(V = k) = \begin{cases} 
(1 - p)^{kp}, & k = 0, \ldots, L - 1; \\
(1 - p)^{L - k}(1 - q)^{k - L}q, & k \geq L.
\end{cases}
\] (70)

(This is analogous to the situation \(L = 1\) for the right jumps Markov chain, where \(U(m) = \sum_{j=1}^{m} \Gamma_j\) with \(\Gamma_1, ..., \Gamma_m\) independent and \(\Gamma_2, ..., \Gamma_m\) i.i.d.)

We now show that since \(\alpha = \frac{1}{2}\), \(\mathbb{E}(V) = 1\). After a somewhat messy calculation and some algebraic simplification, one finds that

\[
\mathbb{E}(V) = \frac{(1 - p)q + (1 - p)^L(p - q)}{pq}.
\]

From this it follows that \(\mathbb{E}(V) = 1\) if and only if \(q = q_0 = \frac{p(1 - p)^L}{2p - 1 + (1 - p)^L}\) by Remark 2 after Proposition 1. So, we conclude that \(\mathbb{E}(V) = 1\).

We have now shown that

\[
\mathbb{P}(W_n = m|W_{n-1} = m) = \mathbb{P}\left(\sum_{j=1}^{m+1} V_j = m\right),
\]

where \(V_1, ..., V_{m+1}\) are independent and \(V_2, ..., V_{m+1}\) are i.i.d. with mean 1. So, the Markov chain \((W_n)\) has the transition probabilities of a critical branching process with immigration. The immigration term \(V_1\) depends on the initial environment, but is always nonnegative and not identically zero with finite mean. Also, clearly \(\mathbb{E}(V^2) < \infty\), so the branching terms have finite variance. It thus follows from [7] that \(\bar{W}_n\) converges in probability to a certain nonzero limiting distribution, which implies the Markov chain \((W_n)_{n \geq 0}\) cannot be positive recurrent.

\[\square\]

5 Analysis of \(\alpha\)

In this section we prove Proposition 1 which characterizes some properties of the important quantity

\[
\alpha = p \cdot \pi_p + q \cdot \pi_q
\] (71)

that determines the direction of transience for our random walk (away from borderline critical case). We recall from [21] that

\[
\pi_p = \frac{(1 - q)q^R(1 - (1 - p)^L)}{(1 - q)q^R(1 - (1 - p)^L) + p(1 - p)^L(1 - q^R)}
\]

\[
\pi_q = \frac{p(1 - p)^L(1 - q^R)}{(1 - q)q^R(1 - (1 - p)^L) + p(1 - p)^L(1 - q^R)}.
\]

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The various pieces of the proposition will be proved separately, but we begin first with two useful observations.

(I) For any fixed \( q, R, L \) the quantity

\[
\frac{\pi_p}{\pi_q} = \frac{(1-q)q^R}{1-q^R} \cdot \frac{1-(1-p)L}{p(1-p)L}
\]

satisfies \( \lim_{p \to 1} \left( \frac{\pi_p}{\pi_q} \right) = \infty \). Since \( \pi_p + \pi_q = 1 \), this implies \( \lim_{p \to 1} \pi_p = 1 \).

(II) For any fixed \( q, R, L \) the quantity \( \frac{\pi_p}{\pi_q} \) satisfies

\[
\frac{d}{dp} \left( \frac{\pi_p}{\pi_q} \right) = \frac{(1-q)q^R}{1-q^R} \cdot \frac{p(L+1)+(1-p)L^+1-1}{p^2(1-p)L^+1} > 0, \quad \forall p \in (0, 1).
\]

Since \( \pi_p + \pi_q = 1 \), this implies \( d \pi_p / dp > 0, \forall p \in (0, 1) \). So,

\[
\frac{d}{dp}(\alpha) = \frac{d}{dp}(p \cdot \pi_p + q \cdot \pi_q) = \frac{d}{dp}(p \cdot \pi_p + q \cdot (1-\pi_p))
\]

\[
= \pi_p + (p-q) \cdot \frac{d}{dp}(\pi_p) > 0, \quad \forall p \geq q.
\]

Proof of (vi): By (I), \( \lim_{p \to 1} \alpha = \lim_{p \to 1}(p \cdot \pi_p + q \cdot \pi_q) = 1 \cdot 1 + q \cdot 0 = 1 \).

Proof of (i): This is immediate from (71) since \( \pi_p + \pi_q = 1 \) and \( \pi_p, \pi_q > 0 \), for any \( p, q \).

Proof of (ii): If \( q < 1/2 \), then \( \alpha < 1/2 \) for all \( p \leq 1/2 \), by (I). But, by (II) and (vi), we also know that \( \alpha(p) \) is monotonically increasing on the interval \( [1/2, 1] \subset [q, 1) \), with \( \lim_{p \to 1} \alpha(p) = 1 \). Thus, the claim follows by continuity of \( \alpha(p) \).

Proof of (iii): Plugging \( p = 1 - q \) into (2) and simplifying one finds that

\[
\alpha(1-q) < 1/2 \iff q^R(1/2 - q) - q^L(1/2 - q) < 0, \quad \text{and}
\]

\[
\alpha(1-q) > 1/2 \iff q^R(1/2 - q) - q^L(1/2 - q) > 0.
\]

Thus, for \( q < 1/2 \) and \( R > L \), \( \alpha(1-q) < 1/2 \), which implies \( p_0 > 1 - q \). While, for \( q < 1/2 \) and \( R < L \), \( \alpha(1-q) > 1/2 \), which implies \( p_0 < 1 - q \). This proves (6).

Now, by (II) and symmetry considerations, for any fixed \( R, L, p \) we know that \( d/dq(\alpha) > 0 \) for \( q \leq p \). Thus, for any \( 0 < q < q' < 1/2 \), we have

\[
\alpha(p_0(q, R, L), q', R, L) > \alpha(p_0(q, R, L), q, R, L) = 1/2,
\]

which implies \( p_0(q', R, L) < p_0(q, R, L) \). So, \( p_0 \) is a decreasing function of \( q \), for \( q \in (0, 1/2) \).
Proof of (iv): Plugging \( L = 1 \) into (2) and simplifying one finds that
\[
\alpha = \frac{1}{2} \iff p(1 - 2q + q^R) = 1 - 2q + q^{R+1}
\]
and, similarly,
\[
\alpha < \frac{1}{2} \iff p(1 - 2q + q^R) < 1 - 2q + q^{R+1},
\]
\[
\alpha > \frac{1}{2} \iff p(1 - 2q + q^R) > 1 - 2q + q^{R+1}.
\]
(iv) follows by considering separately the two cases \( 1 - 2q + q^{R+1} > 0 \) and \( 1 - 2q + q^{R+1} \leq 0 \).

Proof of (v): If \( L = R \), then plugging in \( p = 1 - q \) into (2) gives \( \alpha = \frac{1}{2} \). So, by (ii), if \( q < \frac{1}{2} \) then \( p_0 = 1 - q \) is the unique critical point. On the other hand, for any \( q > \frac{1}{2} \), if \( L = R \) is sufficiently large then there exists another critical point \( p'_0 > 1 - q \). This follows from (vi), continuity of \( \alpha \), and the following claim.

Claim: For any fixed \( q > \frac{1}{2} \), if \( L = R \) is sufficiently large then \( \left. \frac{d}{dp} (\alpha) \right|_{p=1-q} < 0 \).

Proof: Computing \( \frac{d}{dp} (\alpha) \) directly from (2) and then substituting \( L = R \) and \( p = 1 - q \), one finds, after some lengthy simplifications, that the condition \( \left. \frac{d}{dp} (\alpha) \right|_{p=1-q} < 0 \) is equivalent to the condition
\[
R(1 - q)(1 - 2q) + q(1 - q^R) < 0.
\]
For fixed \( q > \frac{1}{2} \), this condition is satisfied for all sufficiently large \( R \). \( \square \)

A Solution of Linear Systems

A.1 Stationary Distribution of Single Site Markov Chains

Here we solve the linear system \( \{ \pi = \pi M, \sum \pi_\lambda = 1 \} \) for the stationary distribution \( \pi \) of the single site Markov chain transition matrix \( M \). In expanded form this system becomes
\[
\pi_{(p,i)} = (1 - p) \cdot \pi_{(p,i-1)}, \quad 1 \leq i \leq L - 1 \tag{74}
\]
\[
\pi_{(p,0)} = p \cdot \pi_p + q \cdot \pi_{(q,R-1)} \tag{75}
\]
\[
\pi_{(q,i)} = q \cdot \pi_{(q,i-1)}, \quad 1 \leq i \leq R - 1 \tag{76}
\]
\[
\pi_{(q,0)} = (1 - q) \cdot \pi_q + (1 - p) \cdot \pi_{(p,L-1)} \tag{77}
\]
\[
\pi_p + \pi_q = 1, \tag{78}
\]
where \( \pi_p = \sum_{i=0}^{L-1} \pi_{(p,i)} \) and \( \pi_q = \sum_{i=0}^{R-1} \pi_{(q,i)} \). Applying (74) and (76) repeatedly gives
\[
\pi_{(p,i)} = (1 - p)^i \cdot \pi_{(p,0)}, \quad 0 \leq i \leq L - 1; \tag{79}
\]
\[
\pi_{(q,i)} = q^i \cdot \pi_{(q,0)}, \quad 0 \leq i \leq R - 1. \tag{80}
\]
Hence, 
\[
\pi_p = \sum_{i=0}^{L-1} (1-p)^i \cdot \pi(p,0) = \frac{1 - (1-p)^L}{p} \cdot \pi(p,0), \quad (81)
\]
\[
\pi_q = \sum_{i=0}^{R-1} q^i \cdot \pi(q,0) = \frac{1 - q^R}{1-q} \cdot \pi(q,0). \quad (82)
\]

Plugging (80) and (81) into (75) gives
\[
\pi(p,0) = p \cdot \left( \frac{1 - (1-p)^L}{p} \cdot \pi(p,0) \right) + q \cdot \left( q^{R-1} \cdot \pi(q,0) \right),
\]
which implies
\[
\pi(p,0) = \pi(q,0) \cdot \frac{q^R}{(1-p)^L}. \quad (83)
\]

But, by (78), (81), and (82), we also have
\[
\frac{1 - (1-p)^L}{p} \cdot \pi(p,0) + \frac{1 - q^R}{1-q} \cdot \pi(q,0) = 1
\]
or, equivalently,
\[
\pi(p,0) = \left( 1 - \pi(q,0) \frac{1 - q^R}{1-q} \right) \cdot \frac{p}{1 - (1-p)^L}. \quad (84)
\]

Equating the right hand sides of (83) and (84) and solving for \(\pi(q,0)\) gives
\[
\pi(q,0) = \frac{p(1-q)(1-p)^L}{(1-q)q^R(1-(1-p)^L) + p(1-p)^L(1-q^R)}.
\]

Substituting this value of \(\pi(q,0)\) into (83) gives an explicit expression for \(\pi(p,0)\), and the values of \(\pi(q,i), 1 \leq i \leq R-1\), and \(\pi(p,i), 1 \leq i \leq L-1\), are then easily found by substituting the expressions for \(\pi(p,0)\) and \(\pi(q,0)\) in (79) and (80), giving (83).

\section*{A.2 Expected Hitting Times with \(R = 1\)}

Here we solve the linear system (45) for the expected hitting times \(a_i, 0 \leq i \leq L\). As shown in the proof of Theorem 4 using soft methods, these expected hitting times must all be finite.

For simplicity of notation we define \(b_i = a_{L-i}, 0 \leq i \leq L\). Rearranging slightly the system (45) then becomes
\[
b_{i+1} = 1 + (1-p)(a_0 + b_i), \quad 0 \leq i \leq L-1 \quad (85)
\]
\[
b_0 = \frac{1}{q} + \left( \frac{1-q}{q} \right) a_0. \quad (86)
\]
Thus, for each \(0 \leq i \leq L\), we have

\[
b_i = u_i + v_i \cdot a_0
\]

where the sequences \((u_i)_{i=0}^L\) and \((v_i)_{i=0}^L\) are defined recursively by

\[
u_0 = \frac{1}{q} \quad \text{and} \quad u_{i+1} = 1 + (1-p)u_i, \quad 0 \leq i \leq L - 1,
\]

\[
v_0 = \frac{1-q}{q} \quad \text{and} \quad v_{i+1} = (1-p)(1 + v_i), \quad 0 \leq i \leq L - 1.
\]

By induction on \(i\), we find that, for each \(1 \leq i \leq L\),

\[
\begin{align*}
\frac{1}{p}u_i &= \frac{(1-p)^i}{q} + \sum_{j=0}^{i-1} (1-p)^j = 1 + \frac{(p/q - 1)(1-p)^i}{p}, \\
\frac{1}{p}v_i &= \frac{(1-p)^i}{q} + \sum_{j=1}^{i-1} (1-p)^j = 1 - p + \frac{(p/q - 1)(1-p)^i}{p}.
\end{align*}
\]

Substituting, first for the \(b_i\)’s and then for the \(a_i\)’s with \(a_i = b_{L-i}\), one obtains (12) and (13).

### B Proof of Lemma 1

Here we prove Lemma 1 from Section 2.2. The three parts are proved separately. In each case, we prove only the first of the two statements, since the second follows by symmetry. The following notation will be used for the proofs.

- \(T_x^{(i)}\) is the \(i\)-th hitting time of site \(x\):
  \[T_x^{(i)} = T_x \quad \text{and} \quad T_x^{(i+1)} = \inf\{n > T_x^{(i)} : X_n = x\},\]
  with the convention \(T_x^{(j)} = \infty\), for all \(j > i\), if \(T_x^{(i)} = \infty\).

- \(m_i = \sup\{X_n : n \leq T_0^{(i)}\}\) is the maximum position of the random walk up to the \(i\)-th hitting time of site 0.

- For an initial environment \(\omega\) and path \(\zeta = (x_0, \ldots, x_n)\), \(\omega^{(\zeta)}\) is the environment induced at time \(n\) by following the path \(\zeta\) starting in \(\omega\):
  \[
  \{\omega_0 = \omega, X_0 = x_0, \ldots, X_n = x_n\} \implies \omega_n = \omega^{(\zeta)}.
  \]

**Proof of (ii):** Clearly, \(P_\omega(X_n \to \infty) \leq P_\omega(\lim\inf_{n \to \infty} X_n > -\infty)\). To show the reverse inequality also holds observe that, for any \(k \in \mathbb{Z}\), \(P_\omega(\lim\inf_{n \to \infty} X_n = k) = 0\). Thus,

\[
P_\omega \left(\lim\inf_{n \to \infty} X_n > -\infty, X_n \neq \infty\right) = P_\omega \left(-\infty < \lim\inf_{n \to \infty} X_n < \infty\right) = 0.
\]
Proof of (i): By (ii), \( \mathbb{P}_\omega(X_n \to \infty) \geq \mathbb{P}_\omega(A_0^+) \). Thus, \( \mathbb{P}_\omega(X_n \to \infty) > 0 \), if \( \mathbb{P}_\omega(A_0^+) > 0 \).

On the other hand, if \( \mathbb{P}_\omega(X_n \to \infty) > 0 \) then there exists some finite path \( \zeta = (x_0, \ldots, x_n) \), such that \( x_0 = 0, x_n = 2 \), and

\[
\mathbb{P}_\omega(X_m > 1, \forall m \geq n | X_0 = x_0, \ldots, X_n = x_n) > 0.
\]

We construct from \( \zeta = (x_0, \ldots, x_n) \) the reduced path \( \tilde{\zeta} = (\tilde{x}_0, \ldots, \tilde{x}_n) \) by setting \( \tilde{x}_0 = x_0 = 0 \), and then removing from the tail \( (x_1, \ldots, x_n) \) all steps before the first hitting time of site 1 and all steps in any leftward excursions from site 1. For example,

if \( \zeta = (0, -1, 0, 1, 2, 1, 0, 1, 2, 1, 0, -1, -2, -1, 0, 1, 2, 3, 2) \),

then \( \tilde{\zeta} = (0, 1, 2, 1, 2, 1, 2, 3, 2) \)

(where we denote the removed steps in bold for visual clarity). By construction, \( \omega^{(\tilde{\zeta})}(x) = \omega^{(\zeta)}(x) \), for all \( x \geq 2 \). So, \( \mathbb{P}_\omega(X_m > 1, \forall m \geq n | (X_0, \ldots, X_n) = \tilde{\zeta} = \mathbb{P}_\omega(X_m > 1, \forall m \geq n | (X_0, \ldots, X_n) = \zeta) > 0 \). Thus,

\[
\mathbb{P}_\omega(A_0^+) \geq \mathbb{P}_\omega((X_0, \ldots, X_n) = \tilde{\zeta}) \cdot \mathbb{P}_\omega(X_m > 1, \forall m \geq n | (X_0, \ldots, X_n) = \zeta) > 0.
\]

Proof of (iii): Since we assume \( \mathbb{P}_\omega(X_n \to -\infty) = 0 \), it follows from (ii) that (a) \( T_x \) is \( \mathbb{P}_\omega \) a.s. finite for each \( x \geq 0 \), and (b) every time the random walk steps left from a site \( x \) it will eventually return with probability 1. Now (b) implies that the probability that the walk is transient to \( +\infty \), after first hitting a site \( x \geq 0 \), is independent of the trajectory taken to get to \( x \). That is, \( \mathbb{P}_\omega(X_n \to \infty | (X_0, \ldots, X_n) = \zeta) = \mathbb{P}_\omega(X_n \to \infty | T_x < \infty) \), for any \( x \geq 0 \) and path \( \zeta = (x_0, \ldots, x_n) \) such that \( x_0 = 0, x_n = x \), and \( x_m < x \) for \( m < n \). Combining this last observation with (a) shows that

\[
\mathbb{P}_\omega(X_n \to \infty | T_0^{(i)} < \infty, m_i = x) = \mathbb{P}_\omega(X_n \to \infty | T_0^{(i)} < \infty, m_i = x, T_{x+1} < \infty) = \mathbb{P}_\omega(X_n \to \infty | T_{x+1} < \infty) = \mathbb{P}_\omega(X_n \to \infty), \text{ for all } x \geq 0 \text{ and } i \geq 1.
\]

So, \( \mathbb{P}_\omega(X_n \to \infty | T_0^{(i)} < \infty) = \mathbb{P}_\omega(X_n \to \infty) \), for all \( i \geq 1 \). Thus, by (ii),

\[
\mathbb{P}_\omega(X_n \not\to \infty) = \mathbb{P}_\omega(X_n \not\to \infty | T_0^{(i)} < \infty) = \prod_{j=i}^{\infty} \mathbb{P}_\omega(T_0^{(j+1)} < \infty | T_0^{(j)} < \infty), \forall i \geq 1.
\]

Since the LHS is independent of \( i \), the product on the RHS is constant for \( i \geq 1 \). Thus, there are two possibilities: either the product is 0 (for all \( i \geq 1 \)) or \( \mathbb{P}_\omega(T_0^{(j+1)} < \infty | T_0^{(j)} < \infty) = 1 \), for all \( j \geq 1 \). In the latter case, \( \mathbb{P}_\omega(X_n \not\to \infty) = 1 \), which contradicts the assumption that \( \mathbb{P}_\omega(X_n \not\to \infty) > 0 \). In the former case, \( \mathbb{P}_\omega(X_n \to \infty) = 1 \), as required. \( \square \)
C Proof of Lemma 7

The following strong law for sums of dependent random variables is a special case of [8, Theorem 1] with $w_i = 1$ and $W_i = i$.

**Theorem 8.** Let $(\xi_i)_{i \in \mathbb{N}}$ be a sequence of nonnegative random variables satisfying:

1. $\sup_i E(\xi_i) < \infty$.
2. $E(\xi_i^2) < \infty$, for each $i$.
3. $\sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{1}{j^2} \cdot \text{Cov}^+(\xi_i, \xi_j) < \infty$.

Then

$$\frac{1}{n} \sum_{i=1}^{n} (\xi_i - E(\xi_i)) \xrightarrow{a.s.} 0, \text{ as } n \to \infty.$$ 

Using this theorem we will prove Lemma 7. Throughout our proof the initial environment $\omega$ is fixed, and all random variables are distributed according to the measure $P_\omega$, which we will abbreviate simply as $P$. Also, $\beta > 0$ is the constant given in Corollary 2.

**Proof of Lemma 7.** By Corollary 3, $E(N_x) \leq \frac{1}{\beta}$ and $E(N_x^2) \leq \frac{2 - \beta^2}{\beta^2}$, for each $x \in \mathbb{N}$. (89)

Thus, by Theorem 8 it suffices to show that

$$\sum_{y=1}^{\infty} \sum_{x=1}^{y} \frac{1}{y^2} \text{Cov}^+(N_x, N_y) < \infty.$$ 

Since $N_x$ and $N_y$ are nonnegative integer valued random variables, $\text{Cov}(N_x, N_y)$ can be represented as the following absolutely convergent double sum:

$$\text{Cov}(N_x, N_y) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left( \mathbb{P}(N_x \geq k, N_y \geq j) - \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j) \right). \quad (90)$$

To bound this sum we will need the following two estimates for the differences $D_{k,j} \equiv \mathbb{P}(N_x \geq k, N_y \geq j) - \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j)$:

For any $1 \leq x < y$ and $k, j \in \mathbb{N}$, $D_{k,j} \leq (1 - \beta)^{\max\{j,k\} - 1}. \quad (91)$

For any $1 \leq x < y$ and $k, j \in \mathbb{N}$, $D_{k,j} \leq (1 - \beta)^{y-x}. \quad (92)$

(91) follows from Corollary 3

$$D_{k,j} \equiv \mathbb{P}(N_x \geq k, N_y \geq j) - \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j)$$

$$\leq \mathbb{P}(N_x \geq k, N_y \geq j) \leq \min\{\mathbb{P}(N_x \geq k), \mathbb{P}(N_y \geq j)\} \leq (1 - \beta)^{\max\{j,k\} - 1}.$$
To see (92) recall that \(N^y_x\) and \(N_y\) are independent for all \(1 \leq x < y\), by Lemma 6. Thus, for any \(1 \leq x < y\), we have

\[
\mathbb{P}(N_x \geq k, N_y \geq j) = \mathbb{P}(N^y_x \geq k, N_y \geq j) + \mathbb{P}(N^y_x < k, N_x \geq k, N_y \geq j)
\]

\[
\leq \mathbb{P}(N_x \geq k) \mathbb{P}(N_y \geq j) + \mathbb{P}(N^y_x < k, N_x \geq k, N_y \geq j)
\]

by Corollary 4.

Now, for given \(1 \leq x < y\), let \(n = y - x\) and let \(N = \lfloor (1 - \beta)^{-n/4} \rfloor\). Breaking the (absolutely convergent) double sum in (90) into pieces and applying Fubini’s Theorem gives

\[
\text{Cov}(N_x, N_y) = \sum_{j=1}^{n} \sum_{k=1}^{N} D_{k,j} + \sum_{j=1}^{n} \sum_{k=N+1}^{\infty} D_{k,j} + \sum_{j=N+1}^{\infty} \sum_{k=1}^{N} D_{k,j} + \sum_{j=N+1}^{\infty} \sum_{k=N+1}^{\infty} D_{k,j}.
\]

By (92), the first term on the RHS of this equation is bounded above by \(N^2(1 - \beta)^n\). Similarly, by (91):

- The second term is bounded by \(N \cdot \sum_{j=N+1}^{\infty} (1 - \beta)^{k-1} = N(1 - \beta)^N / \beta\).
- The third term is bounded by \(N \cdot \sum_{k=N+1}^{\infty} (1 - \beta)^{j-1} = N(1 - \beta)^N / \beta^2\).
- The fourth term is bounded by \(\sum_{k=N+1}^{\infty} \sum_{j=k}^{\infty} (1 - \beta)^{k-1} = (1 - \beta)^N / \beta^2\).
- The fifth term is bounded by \(\sum_{j=N+1}^{\infty} \sum_{k=N+1}^{\infty} (1 - \beta)^{k-1} = (1 - \beta)^N / \beta^2\).

The upper bound on the first term is at most \((1 - \beta)^{n/2}\), and the same is also true for the upper bounds on each of the other 4 terms for all sufficiently \(n\), since \(N\) grows exponentially in \(n\). Thus, there exists some \(n_0 \in \mathbb{N}\) such that

\[
\text{Cov}(N_x, N_y) \leq 5(1 - \beta)^{n/2}, \text{ whenever } y - x = n \geq n_0.
\]

But, for any \(1 \leq x \leq y\) such that \(y - x = n < n_0\) we also have

\[
\text{Cov}(N_x, N_y) \leq \mathbb{E}(N_x^2)^{1/2} \cdot \mathbb{E}(N_y^2)^{1/2} \leq \frac{2 - \beta}{\beta^2} \leq \left( \frac{2 - \beta}{\beta^2(1 - \beta)^{n_0-1}} \right) (1 - \beta)^n
\]

by (89). Thus, for all \(1 \leq x \leq y\),

\[
\text{Cov}(N_x, N_y) \leq C(1 - \beta)^{n/2}, \text{ where } C = \max \left\{ 5, \frac{2 - \beta}{\beta^2(1 - \beta)^{n_0-1}} \right\} \text{ and } n = y - x.
\]

So,

\[
\sum_{y=1}^{\infty} \sum_{x=1}^{y} \frac{1}{y^2} \text{Cov}^+(N_x, N_y) \leq \sum_{y=1}^{\infty} \sum_{x=1}^{y} \frac{1}{y^2} \cdot C(1 - \beta)^{(y-x)/2} < \infty.
\]
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