Volume doubling, Poincaré inequality and Gaussian heat kernel estimate for nonnegative curvature graphs

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Abstract

By studying the heat semigroup, we prove Li-Yau type estimates for bounded and positive solutions of the heat equation on graphs, under the assumption of the curvature-dimension inequality $CDE'(n,0)$, which can be consider as a notion of curvature for graphs. Furthermore, we derive that if a graph has nonnegative curvature then it has the volume doubling property, from this we can prove the Guassian estimate for heat kernel, and then Poincaré inequality and Harnack inequality. Under the assumption of positive curvature on graphs, we derive the Bonnet-Myers type theorem that the diameter of graphs is finite by proving some Log Sobolev inequalities.

1 Introduction

Li-Yau inequality is a very powerful tool to study estimation of heat kernels. It asserts that, for an n-dimensional compact Riemannian manifold with non-negative Ricci curvature, if $u$ is a positive solution to the heat equation $\partial_t u = \Delta u$, then

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}.$$ 

Recently, in the paper of [BHLLMY], the authors proved a discrete version of Li-Yau inequality on graphs via introducing a new notion of curvature, a type of chain rule formula for graph and a discrete version of maximum principle. Indeed, there are two main methods to prove the gradient estimate, one is the maximum principle ([LY06] on manifolds and [LY10] on graphs), the other is the semigroup methods ([BL] on manifolds).

In this paper, we start from studying some functionals of the heat kernel on a finite or infinite graph with nonnegative Ricci curvature, and then obtain a family of global gradient estimate for bounded and positive solutions of the heat equation in entire infinite graph, only under the assumption of $CDE'(n,K)$. This notion of curvature implies $CDE(n,K)$

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(see [BHLLMY]), which is regarded a natural curvature notion. In the diffusion case, for example in complete Riemannian manifolds with dimension $n$, $CDE'(n, K)$ is equivalent to the Ricci curvature is bounded below by $K$.

Moreover, we derive the volume doubling property, when the graph satisfies $CDE'(n, 0)$. It is the key point to prove the discrete-time Gaussian lower and upper estimates of heat kernel, the Poincaré inequality and Harnack inequality on graphs. Where we use the technical from the paper of Delmotte [D]. We can also derive the continue-time Gaussian upper estimate of heat kernel. And from the paper of Davies [DB1] and Pang [P], we know that the Gaussian upper estimate is not true on graphs.

Finally, Under the assumption of $CDE'(n, K)$ for some positive $K$ on graphs, we derive the Bonnet-Myers type theorem that the diameter of graphs is finite by proving some Log Sobolev inequalities. Where we prove that a nice theorem for the diameter bounds of Bakry is still true even we don’t have diffusion property on graphs.

The paper is organized as follows: In section 2, we prove a main variational inequality which imply Li-Yau gradient estimate. Also using this main inequality, we prove the volume doubling in section 4. From volume doubling, we can prove the Gaussian heat kernel estimate, parabolic Harnack inequality and Poincaré inequality in section 5. In section 6, we prove the Bonnet-Myers type theorem on graphs.

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2 Li-Yau type estimates on graphs

Let $G = (V, E)$ be a finite or infinite graph. We allow the edges on the graph to be weighted, we consider a symmetric weight function $\omega : V \times V \rightarrow [0, \infty)$, the edge $xy$ from $x$ to $y$ has weight $\omega_{xy} > 0$. In this paper, we assume this weight function is symmetric($\omega_{xy} = \omega_{yx}$). Moreover we assume the graph is connected, which implies the weight function satisfies

$$\omega_{\text{min}} = \inf_{x \in V} \omega_{xy} > 0,$$

note that loops are allowed, i.e. $x \sim x$, for some $x \in V$. And the graph we are interested is locally finite,

$$m(x) := \sum_{y \sim x} \omega_{xy} < \infty, \quad \forall x \in V.$$

Given a positive and finite measure $\mu : V \rightarrow \mathbb{R}^+$ on graph. We denote by $V^\mathbb{R}$ the space of real functions on $V$, by $\ell^p(V, \mu) = \{ f \in V^\mathbb{R} : \sum_{x \in V} \mu(x)|f(x)|^p < \infty \}, 1 \leq p < \infty$, the space of $\ell^p$ integrable functions on $V$ with respect to the measure $\mu$. For $p = \infty$, let $\ell^\infty(V, \mu) = \{ f \in V^\mathbb{R} : \sup_{x \in V} |f(x)| < \infty \}$ be the set of bounded functions. If for any $f, g \in \ell^2(V, \mu)$, let the inner product as $\langle f, g \rangle = \sum_{x \in V} \mu(x)f(x)g(x)$, then the space of $\ell^2(V, \mu)$ is a Hilbert space. For every function $f \in \ell^p(V, \mu), 1 \leq p < \infty$, we can define the norm. We denote

$$\| f \|_p = \left( \sum_{x \in V} \mu(x)|f(x)|^p \right)^{\frac{1}{p}}, 1 \leq p < \infty \text{ and } \| f \|_\infty = \sup_{x \in V} |f(x)|.$$
We define the $\mu-$Laplacian $\Delta : V^\mathbb{R} \to V^\mathbb{R}$ on $G$ by, for any $x \in V$,

$$\Delta f(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).$$

It will be useful to introduce an abbreviated notation for ”averaged sum”,

$$\overline{\sum}_{y \sim x} h(y) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} h(y) \quad \forall x \in V.$$

If $f \in \ell^\infty(V, \mu)$, under the assumption of locally finite, it is known immediately that for any $x \in V$, $\Delta f(x)$ is the sum of finite terms. The two most natural choices are the case where $\mu(x) = m(x)$ for all $x \in V$, which is the normalized graph Laplacian, and the case $\mu \equiv 1$ which is the standard graph Laplacian. Furthermore, in this paper we assume

$$D_\mu := \max_{x \in V} \frac{m(x)}{\mu(x)} < \infty.$$ 

The graph is endowed with its natural metric (the smallest number of edges of a path between two points). We define balls $B(x, r) = \{y : d(x, y) \leq r\}$, and the volume of a subset $A$ of $V$, $V(A) = \sum_{x \in A} \mu(x)$. We will write $V(x, r)$ for $V(B(x, r))$.

## 2.1 The heat kernel on graphs

### 2.1.1 The heat equation

In this section we introduce the heat equation

$$\Delta u = \partial_t u$$

on the graph $G = (V, E)$. We say that the function $u : [0, \infty) \times V \to \mathbb{R}$ is a positive solution to the heat equation, if $u > 0$ and satisfies the above equality. And we are interested in the heat kernel $p_t(x, y)$, a fundamental solution of the heat equation, if for any bounded initial condition $u_0 : V \to \mathbb{R}$, the function

$$u(t, x) = \sum_{y \in V} \mu(y) p_t(x, y) u_0(y) \quad t > 0, x \in V$$

is differentiable in $t$, satisfies the heat equation, and if for any $x \in V$,

$$\lim_{t \to 0^+} u(t, x) = u_0(x)$$

holds.

For any subset $U \subset V$, we denote by $\overset{\circ}{U} = \{x \in U : y \sim x, y \in U\}$ the interior of $U$. The boundary of $U$ is $\partial U = U \setminus \overset{\circ}{U}$. We introduce the maximum principles.
Lemma 2.1. Let $U \subset V$ be finite and $T > 0$. Furthermore, we assume that the function $u : [0,T] \times U \to \mathbb{R}$ is differentiable with respect to the first component and satisfies on $[0,T] \times U$ the inequality
\[ \partial_t u \leq \Delta u. \]
Then, the function $u$ attains its maximum on the parabolic boundary
\[ \partial P([0,T] \times U) = (\{0\} \times U) \cup ([0,T] \times \partial U) \]

Proof. In a first step we assume that $u$ satisfies the strict inequality
\[ \partial_t u < \Delta u. \]
If $u$ attains its maximum at the point $(t_0, x_0) \in (0,T] \times U^\circ$, then it follows $\partial_t u(t_0, x_0) \geq 0$, and hence $\Delta u(t_0, x_0) = \sum_{y \sim x_0} (u(t_0, y) - u(t_0, x_0)) > 0$, this contradicts $u(t_0, x_0) \geq u(t_0, y)$ for any $y \sim x_0$.

In the general case, we consider the following function, for any $\varepsilon > 0$,
\[ v_\varepsilon(t, x) = u(t, x) - \varepsilon t. \]
Then we have
\[ \partial_t v_\varepsilon - \Delta v_\varepsilon = \partial_t u - \Delta u - \varepsilon < 0. \]
Using our first step for $v_\varepsilon$, we obtain
\[ \max_{(t,x) \in [0,T] \times U} u(t,x) \leq \max_{(t,x) \in [0,T] \times U} v_\varepsilon(t,x) + \varepsilon T = \max_{(t,x) \in \partial P([0,T] \times U)} v_\varepsilon(t,x) + \varepsilon T \leq \max_{(t,x) \in \partial P([0,T] \times U)} u(t,x) + \varepsilon T \rightarrow \max_{(t,x) \in \partial P([0,T] \times U)} u(t,x) \quad (\varepsilon \to 0^+) \]
This ends the proof. \qed

2.1.2 Heat equation on domain

In this subsection $U \subset V$ denotes always a finite subset. We consider the Dirichlet problem (DP),
\[ \begin{cases} 
\partial_t u(t,x) - \Delta_U u(t,x) = 0, & x \in \overset{\circ}{U}, t > 0, \\
u(0,x) = u_0(x), & x \in \overset{\circ}{U}, \\
u \mid_{[0,\infty) \times \partial U} = 0.
\end{cases} \]
where $\Delta_U : \ell^2(\overset{\circ}{U}, \mu) \to \ell^2(\overset{\circ}{U}, \mu)$ denotes the Dirichlet Laplacian on $\overset{\circ}{U}$.

As $-\Delta_U$ is positive and self-adjoint, and $n = \dim \ell^2(\overset{\circ}{U}, \mu) < \infty$. Then there are finite eigenvalues $0 \leq \lambda_i \leq \lambda_{i+1}$, $i = 1, \cdots, n$, and $\phi_i$ is the orthonormal basis of eigenfunction of $\lambda_i$, i.e. $\langle \phi_i, \phi_j \rangle = \sum_{x \in V} \mu(x) \phi_i(x) \phi_j(x) = \delta_{ij}$. 

4
The operator $\Delta_U$ is a generator of the heat semigroup $P_{t,U} = e^{t\Delta_U}, t > 0$. According to spectral graph theory, we can find the easy knowing, $e^{t\Delta_U} \phi_i = e^{-t\lambda_i} \phi_i$. We can define the heat kernel $p_U(t, x, y)$ for the finite subset $U$ by

$$p_U(t, x, y) = P_{t,U} \delta_y(x), \quad \forall x, y \in \overset{\circ}{U}$$

where $\delta_y(x) = \sum_{i=1}^{n} \langle \Phi_i, \delta_y \rangle \Phi_i(x) = \sum_{i=1}^{n} \Phi_i(x) \Phi_i(y)$. It is easy to know the heat kernel satisfies

$$p_U(t, x, y) = \sum_{i=1}^{n} e^{-\lambda_i t} \phi_i(x) \phi_i(y), \quad \forall x, y \in \overset{\circ}{U}.$$ 

There are some useful prosperity of the heat kernel on finite domain,

**Remark 1.** For $t, s > 0, \forall x, y \in \overset{\circ}{U}$, we have

1. $p_U(t, x, y) = p_U(t, y, x)$
2. $p_U(t, x, y) \geq 0$,
3. $\sum_{y \in \overset{\circ}{U}} \mu(y)p_U(t, x, y) \leq 1$,
4. $\lim_{t \to 0^+} \sum_{y \in \overset{\circ}{U}} \mu(y)p_U(t, x, y) = 1$,
5. $\partial_t p_U(t, x, y) = \Delta_{(U,y)} p_U(t, x, y) = \Delta_{(U,x)} p_U(t, x, y)$
6. $\sum_{z \in \overset{\circ}{U}} \mu(z)p_k(t, x, z)p_k(s, z, y) = p_k(t+s, x, y)$

*Proof.* (1) and (5) follows from the above fact about the heat kernel, (2) and (3) are immediate consequences of the maximum principle. For the proof of (4) we remark that this follows from the continuity of the semigroup $e^{t\Delta}$ at $t = 0$ if the limit is understood in the $\ell^2$ sense, as $U$ is finite all norms are equivalent and pointwise convergence follows also. (6) is easy to calculate in $\ell^2$, and it is called the semigroup property of heat kernel. \hfill\[\square\]

### 2.1.3 Heat equation on an infinite graph

Let $U \subset V$, $k \in \mathbb{N}$ be a sequence of finite subsets with $U_k \subset \overset{\circ}{U}_{k+1}$ and $\bigcup_{k \in \mathbb{N}} U_k = V$. Such a sequence always exists and can be constructed as a sequence $\overset{\circ}{U}_k = B_k(x_0)$ of metric balls with center $x_0 \in V$ and radius $k$. The connectedness of our graph $G$ implies that the union of these $U_k$ equals $V$. In the following, we will write $p_k$ for the heat kernel $p_{U_k}$ on $U_k$, and define $p_k(t, x, y)$ as a function on $(0, \infty) \times V \times V$ by,

$$p_k(t, x, y) = \begin{cases} p_{U_k}(t, x, y), & x, y \in \overset{\circ}{U}_k; \\ 0, & \text{o.w.} \end{cases}$$

For any $t > 0, x, y \in V$, we let

$$p_t(x, y) = \lim_{k \to \infty} p_k(t, x, y)$$
the maximum principle implies the monotonicity of the heat kernels, i.e. $p_k \leq p_{k+1}$, then the above limit exists (but could be infinite so far).

From the properties of $p_k$ we immediately obtain some facts of $p_t(x, y)$, such as symmetry and non-negative, i.e. $p_t(x, y) = p_t(y, x)$, and $p_t(x, y) \geq 0$, for any $t > 0, x, y \in V$. And we can obtain that $p_t(x, y)$ is the heat kernel on infinite graph $G$ we want. For proving this, we first introduce the following lemma.

**Lemma 2.2.** Let $u_k : (0, \infty) \times V \to \mathbb{R}, k \in \mathbb{N}$, be a non-decreasing sequence with supp $u_k \subset \overset{\circ}{U_K}$ for any $t > 0$, such that

1. $\partial_t u_k(t, x) = \Delta_{U_k} u_k(t, x), \forall x \in \overset{\circ}{U}$
2. $|u_k(t, x)| \leq C < \infty$, for some constant $C > 0$ that neither depends on $x \in V, t > 0$ nor on $k \in \mathbb{N}$.

Then the limit $u(t, x) = \lim_{k \to \infty} u_k(t, x)$ is finite and $u(t, x)$ is a solution for the heat equation. Furthermore, the convergence is uniform on compact subsets of $(0, \infty)$.

**Proof.** The finiteness of $u(t, x)$ follows from the second assumption. From Dini’s theorem, for any $x \in V$ the sequence $u_k(t, x)$ converges uniformly on compact subsets of $(0, \infty)$, and therefore, the limit $u(t, x)$ is continuous with respect of $t$. Furthermore, we have

$$
\partial_t u_k(t, x) = \Delta_{U_k} u_k(t, x)
$$

$$
= \begin{cases} 
\sum_{y \sim x} (u_k(t, y) - u_k(t, x)), & \text{if } x \in \overset{\circ}{U} \\
0, & \text{otherwise}
\end{cases}
$$

$$
\to \sum_{y \sim x} (u(t, y) - u(t, x)) = \Delta u(t, x)
$$

where the convergence is uniform on compact subsets of $(0, \infty)$. Hence, the limit $u(t, x)$ is differentiable with $t$,

$$
\partial_t u(t, x) = \Delta u(t, x).
$$

**Theorem 2.1.** Let $G = (V, E)$ be a connected, locally finite graph. Then for any $t > 0, x, y \in V$, $p_t(x, y)$ is a fundamental solution for the heat equation and does not depend on the choice of the exhaustion sequence $U_k$.

**Proof.** The independence of $p$ from the choice of the exhaustion sequence follows from the maximum principle, more precisely from the domain monotonicity of $p_V$.

To show that $p_t(x, y)$ is a fundamental solution, we remark that $p_k(t, x, y) \geq 0 (\forall x, y \in V)$, $\sum_{y \in V} p(t, x, y) \leq 1 (\forall x \in V)$, and $\partial_t p_k(t, x, y) = \Delta_{(U_k, y)} p_k(t, x, y) (\forall y \in \overset{\circ}{U_k}, x \in V)$. By Lemma 2.2 for any $x \in V$, the sequence $p_k(t, x, y)$ converges to a solution of the heat equation.

Let $u_0 \in V^x$ be a bounded, positive function (in the general case we split the bounded function $u_0$ into its positive and negative part) and define

$$
u_k(t, x) = \sum_{y \in V} \mu(y) p_k(t, x, y) u_0(y),
$$

6
We know the sequence $u_k$ is non-decreasing, and we have
$$u_k(t, x) \leq \sup_{y \in V} u_0(y) \sum_{y \in V} \mu(y) p_k(t, x, y) \leq \sup_{y \in V} u_0(y).$$

So from lemma 2, the limit $u(t, x) = \lim_{k \to \infty} u_k(t, x)$ is everywhere finite and satisfies the heat equation.

And it remains to prove $\lim_{t \to 0^+} u(t, x) = u_0(x)$. Notice that $p_k(t, x, y)$ is non-zero only for finitely many $y$, then
$$u(t, x) = \lim_{k \to \infty} \sum_{y \in V} \mu(y) p_k(t, x, y) u_0(y) = \sum_{y \in V} \mu(y) p_t(x, y) u_0(y)$$
and we know $\sum_{y \in V} \mu(x) p_t(x, y) \leq 1$, and it is easy to prove that $\lim_{t \to 0^+} \mu(x) p_t(x, x) = 1$ (if it is not, then this would contradict with $\langle \Phi_i, \Phi_i \rangle = 1$), then $\lim_{t \to 0^+} \sum_{y \in V} \mu(x) p_t(x, y) = 1$, and $\lim_{t \to 0^+} \sum_{y \neq x} \mu(x) p_t(x, y) = 0$. We obtain,
$$\left| \sum_{y \neq x} \mu(x) p_t(x, y) (u_0(y) - u_0(x)) \right| \leq 2 \sup_x u_0(x) \sum_{y \neq x} \mu(x) p_t(x, y) \to 0 \quad (t \to 0^+),$$

Therefore,
$$\lim_{t \to 0^+} (u(t, x) - u_0(x)) = \lim_{t \to 0^+} \sum_{y \in V} \mu(y) p_t(x, y) (u(t, x) - u_0(x))$$
$$= \lim_{t \to 0^+} \sum_{y \neq x} \mu(y) p_t(x, y) (u(t, x) - u_0(x))$$
$$= 0$$
as desired. \(\square\)

For completeness, we conclude all properties we will use in this paper of the heat kernel $p_t(x, y)$ as follows.

**Remark 2.** For $t, s > 0$, $\forall x, y \in V$, we have

1. $p_t(x, y) = p_t(y, x)$
2. $p_t(x, y) \geq 0$,
3. $\sum_{y \in V} \mu(y) p_t(x, y) \leq 1$,
4. $\lim_{t \to 0^+} \sum_{y \in V} \mu(y) p_t(x, y) = 1$,
5. $\partial_t p_t(x, y) = \Delta_y p_t(x, y) = \Delta_x p_t(x, y)$
6. $\sum_{z \in V} \mu(z) p_t(x, z) p_s(z, y) = p_{t+s}(x, y)$. 

7
The above notions and results almost come from [WK], we reproduce them here for the sake of completeness. And then we can introduce the semigroup \( P_t : V^\mathbb{R} \to V^\mathbb{R} \) by, for any bounded function \( f \in V^\mathbb{R} \),

\[
P_t f(x) = \lim_{k \to \infty} \sum_{y \in V} \mu(y)p_k(t, x, y)f(y) = \sum_{y \in V} \mu(y)p_t(x, y)f(y)
\]

where \( \lim_{t \to 0^+} P_t f(x) = f(x) \), and \( P_t f(x) \) is a solution of the heat equation. From the properties of the heat kernel, and the boundedness of \( f \), that is, there exists a constant \( C > 0 \) such that for any \( x \in V \), \( \sup_{x \in V} |f(x)| \leq C \), we have

\[
\sum_{y \in V} |\mu(y)p_t(x, y)f(y)| \leq C \lim_{k \to \infty} \sum_{y \in V} \mu(y)p_k(t, x, y) \leq C < \infty,
\]

so the semigroup is well-defined.

We can find some useful properties of \( P_t \) as follows.

**Proposition 2.2.** For any bounded function \( f, g \in V^\mathbb{R} \), and \( t, s > 0 \), for any \( x \in V \),

1. If \( 0 \leq f(x) \leq 1 \), then \( 0 \leq P_t f(x) \leq 1 \),
2. \( P_t \circ P_s f(x) = P_{t+s} f(x) \),
3. \( \Delta P_t f(x) = P_t \Delta f(x) \).

**Proof.** The first one immediately comes from the definition of \( P_t f(x) \).

For any bounded function \( f \in V^\mathbb{R} \), and any \( x \in V \), notice \( \lim_{k \to \infty} p_k(t, x, y) \) does not depend on the choice of the exhaustion sequence \( U_k \), so

\[
P_t \circ P_s f(x) = \lim_{k \to \infty} \sum_{y \in V} \mu(y)p_k(t, x, y) \sum_{z \in V} \mu(z)p_k(t, y, z)f(z)
= \lim_{k \to \infty} \sum_{z \in V} \mu(z) \left( \sum_{y \in V} \mu(y)p_k(t, x, y)p_k(t, y, z) \right) f(z)
= \lim_{k \to \infty} \sum_{z \in V} \mu(z)p_k(t, x, z)f(z)
= P_{t+s} f(x).
\]

Notice the function \( f \) is bounded, we have

\[
\sum_{y \in V} \sum_{z \sim y} |\omega_{yz}p_t(x, y)f(z)| = \sum_{y \in V} \deg(y)p_t(x, y)|f(z)| \leq D_\mu C \sum_{y \in V} \mu(y)p_t(x, y) \leq D_\mu C < \infty,
\]

and

\[
\sum_{y \in V} \sum_{z \sim y} | - \omega_{yz}p_t(x, y)f(y)| < \infty.
\]
Then,
\[
\Delta P_t f(x) = \Delta_x \left( \sum_{y \in V} \mu(y)p_t(x, y)f(y) \right) \\
= \sum_{y \in V} \mu(y)\Delta_y p_t(x, y)f(y) \\
= \sum_{y \in V} \sum_{z \sim y} \omega_{yz} (p_t(x, z) - p_t(x, y))f(y) \\
= \sum_{y \in V} \sum_{z \sim y} \omega_{yz} p_t(x, z)f(y) - \sum_{y \in V} \sum_{z \sim y} \omega_{yz} p_t(x, y)f(y) \\
= \sum_{y \in V} \sum_{z \sim y} \omega_{yz} p_t(x, z)f(y) - \sum_{y \in V} \sum_{z \sim y} \omega_{yz} p_t(x, z)f(z) \\
= \sum_{y \in V} \sum_{z \sim y} \omega_{yz} p_t(x, z)(f(y) - f(z)) \\
= P_t \Delta f(x).
\]

This ends the proof of Proposition 2.2. \qed

2.2 Curvature-dimension inequalities

In this section we introduce the notion of the CD inequality. First we need to recall the definition of two bilinear forms associated to the \(\mu\)-Laplacian.

**Definition 2.1.** The gradient form \(\Gamma\) is defined by
\[
2\Gamma(f, g)(x) = (\Delta(f \cdot g) - f \cdot \Delta(g) - \Delta(f) \cdot g)(x) \\
= \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy} (f(y) - f(x))(g(y) - g(x)).
\]
We write \(\Gamma(f) = \Gamma(f, f)\).

Similarly,

**Definition 2.2.** The iterated gradient form \(\Gamma^2\) is defined by
\[
2\Gamma^2(f, g) = \Delta \Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g).
\]
We write \(\Gamma^2(f) = \Gamma^2(f, f)\).

**Definition 2.3.** The graph \(G\) satisfies the CD inequality \(CD(n, K)\) if, for any function \(f\)
\[
\Gamma_2(f) \geq \frac{1}{n}(\Delta f)^2 + K \Gamma(f).
\]
Definition 2.4. We say that a graph $G$ satisfies the exponential curvature dimension inequality $CDE(x, n, K)$ if for any positive function $f : V \to \mathbb{R}^+$ such that $\Delta f(x) < 0$, we have

$$\widetilde{\Gamma}_2(f)(x) = \Gamma_2(f)(x) - \Gamma \left( f, \frac{\Gamma(f)}{f} \right)(x) \geq \frac{1}{n}(\Delta f)(x)^2 + K\Gamma(f)(x).$$

We say that $CDE(n, K)$ is satisfied if $CDE(x, n, K)$ is satisfied for all $x \in V$.

Definition 2.5. We say that a graph $G$ satisfies the $CDE'(x, n, K)$, if for any positive function $f : V \to \mathbb{R}^+$, we have

$$\widetilde{\Gamma}_2(f)(x) \geq \frac{1}{n}f(x)^2(\Delta \log f)(x)^2 + K\Gamma(f)(x).$$

We say that $CDE'(n, K)$ is satisfied if $CDE'(x, n, K)$ is satisfied for all $x \in V$.

Remark 3. If $\Delta f(x) < 0$ in $x \in V$, $CDE'(x, n, K)$ implies $CDE(x, n, K)$.

Proof. Let $f : V \to \mathbb{R}^+$ be a positive function for which $\Delta f(x) < 0$. Since $\log s \leq s - 1$ for all positive $s$, we can write

$$\Delta \log f(x) = \sum_{y \sim x} (\log f(y) - \log f(x)) = \sum_{y \sim x} \log \frac{f(y)}{f(x)} \leq \sum_{y \sim x} \frac{f(y) - f(x)}{f(x)} = \frac{\Delta f(x)}{f(x)} < 0.$$ 

Hence squaring everything reverses the above inequality and we get

$$(\Delta f(x))^2 \leq f(x)^2(\Delta \log f(x))^2,$$

and thus $CDE(x, n, K)$ is satisfied

$$\widetilde{\Gamma}_2(f)(x) \geq \frac{1}{n}f(x)^2(\Delta \log f)(x)^2 + K\Gamma(f)(x) > \frac{1}{n}(\Delta f)(x)^2 + K\Gamma(f)(x).$$

$\square$

2.3 The main variational inequality

We can find the operators $\Delta$ and $\Gamma$ are bounded at the assumption of finiteness of $D_\mu$. From that, we have the following lemma.

Lemma 2.3. For any positive and bounded solution $0 < u \in \ell^\infty(V, \mu)$ to the heat equation on $G$, if the graph satisfies the condition $CDE'(n, K)$, then the function $\frac{\Delta u}{2\sqrt{u}}$ on $G$ is bounded.

Proof. Let

$$F = t \cdot \varphi \cdot \frac{\Gamma(\sqrt{u})}{\sqrt{u}},$$

where fixed any $R > 0$,

$$\varphi(x) = \begin{cases} 0, & d(x, x_0) > 2R \\ \frac{2R - d(x, x_0)}{R}, & R \leq d(x, x_0) \leq 2R \\ 1, & d(x, x_0) < R \end{cases}$$

10
It is easy to know, for any \(x \in V\), \(0 \leq \varphi(x) \leq 1\), then \(|\Delta \varphi| \leq D_\mu\), and \(u\) is bounded, then \(|\Gamma(\sqrt{u})| \leq c_1\), and \(|2\Gamma(\Gamma(\sqrt{u}),\varphi)| \leq c_2\) (there exist constant \(c_1, c_2 \geq 0\)) too. Fix an arbitrary \(T > 0\), let \((x^*, t^*)\) be a maximum point of \(F\) in \(V \times [0, T]\). We may assume \(F(x^*, t^*) > 0\). In what follows all computations take place at the point \((x^*, t^*)\). Let \(L = \Delta - \partial_t\), we apply Lemma 4.1 in [BHLLMY] with the choice of \(g = u\). This gives

\[
\mathcal{L}(\sqrt{u}F) \leq \mathcal{L}(\sqrt{u})F = -\frac{F^2}{t^* \varphi},
\]

and

\[
\mathcal{L}(\sqrt{u}F) = \mathcal{L}(t^* \cdot \varphi \cdot \Gamma(\sqrt{u})) = -\varphi \cdot \Gamma(\sqrt{u}) + t^* \Delta \varphi \cdot \Gamma(\sqrt{u}) + 2t^* \varphi \cdot \tilde{\Gamma}_2(\sqrt{u}) + 2t^* \Gamma(\sqrt{u}, \varphi),
\]

for the condition of \(CDE'(n, K)\), we obtain

\[
-\frac{F^2}{t^* \varphi} \geq -\varphi \cdot \Gamma(\sqrt{u}) - t^* D_\mu \Gamma(\sqrt{u}) - 2t^* K \Gamma(\sqrt{u}) - t^* c_2,
\]

that is

\[
F^2(x^*, t^*) \leq ((1 + 2K)c_1 + c_2)t^* + c_1 D_\mu(t^*)^2.
\]

We can let some \(C_1, C_2 > 0\), then

\[
F(x^*, t^*) \leq C_1 + C_2 t^*.
\]

when \(x \in B(x_0, R)\),

\[
T \cdot \frac{\Gamma(\sqrt{u})}{\sqrt{u}} = F(x, T) \leq F(x^*, t^*) \leq C_1 + C_2 t^* \leq C_1 + C_2 T,
\]

that is

\[
\frac{\Gamma(\sqrt{u})}{\sqrt{u}} \leq \frac{C_1}{T} + C_2.
\]

From the equation \(\Delta u = 2\sqrt{u} \Delta \sqrt{u} + 2\Gamma(\sqrt{u})\), we can obtain \(\frac{\Delta u}{\sqrt{2u}}\) is bounded too.

For any positive and bounded function \(0 < f \in \ell^\infty(V, \mu)\) on \(G(V, E)\), the function \(\Gamma(\sqrt{P_{T-t}f})\), for any \(0 \leq t < T\), is bounded and the boundary is irrelevant with \(t\). We can introduce the function in a locally finite and connected graph \(G = (V, E)\),

\[
\phi(t, x) = P_t(\Gamma(\sqrt{P_{T-t}f}))(x), \quad 0 \leq t < T, x \in V.
\]

**Lemma 2.4.** For every \(0 \leq t < T\), any \(x \in V\), with the assumption of \(CDE'(n, K)\), we have

\[
\partial_t \phi(t, x) = 2P_t(\tilde{\Gamma}_2(\sqrt{P_{T-t}f}))(x).
\]
Proof. For any \( x \in V \),

\[
\partial_t P_t(\Gamma(\sqrt{P_{T-t}f}))(x) = \partial_t \left( \sum_{y \in V} \mu(y) p_t(x, y) \Gamma(\sqrt{P_{T-t}f})(y) \right)
\]

\[
= \sum_{y \in V} \mu(y) \left( \Delta p_t(x, y) \Gamma(\sqrt{P_{T-t}f})(y) + p_t(x, y) \partial_t \Gamma(\sqrt{P_{T-t}f})(y) \right)
\]

\[
= \sum_{y \in V} \mu(y) \left( \Delta p_t(x, y) \Gamma(\sqrt{P_{T-t}f})(y) - 2p_t(x, y) \Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}})(y) \right)
\]

\[
= \sum_{y \in V} \mu(y) p_t(x, y) \left( \Delta \Gamma(\sqrt{P_{T-t}f})(y) - 2\Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}})(y) \right)
\]

\[
= 2P_t(\tilde{\gamma}_2(\sqrt{P_{T-t}f}))(x)
\]

In the third step, for any \( x \in V \),

\[
\partial_t \Gamma(\sqrt{P_{T-t}f})(x) = \partial_t \frac{1}{2} \sum_{y \sim x} \left( \sqrt{P_{T-t}f}(y) - \sqrt{P_{T-t}f}(x) \right)^2
\]

\[
= \sum_{y \sim x} \left( \sqrt{P_{T-t}f}(y) - \sqrt{P_{T-t}f}(x) \right) (\partial_t \sqrt{P_{T-t}f}(y) - \partial_t \sqrt{P_{T-t}f}(x))
\]

\[
= 2\Gamma(\sqrt{P_{T-t}f}, \partial_t \sqrt{P_{T-t}f})(x),
\]

and,

\[
\partial_t \sqrt{P_{T-t}f} = \frac{\partial_t P_{T-t}f}{2\sqrt{P_{T-t}f}} = -\frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}},
\]

where \( \partial_t P_{T-t}f = -\Delta P_{T-t}f \).

In the fourth step, due to the boundedness of \( f(x) \), for any \( x \in V \). It is to know the function \( \Delta \Gamma(\sqrt{P_{T-t}f}) \) is bounded, and from Lemma 2.3, \( \Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}}) \) is bounded too. Like the proof of Proposition 2.2, we have

\[
\sum_{y \in V} \mu(y) \left( \Delta p_t(x, y) \Gamma(\sqrt{P_{T-t}f})(y) - 2p_t(x, y) \Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}})(y) \right)
\]

\[
= \sum_{y \in V} \mu(y) \Delta p_t(x, y) \Gamma(\sqrt{P_{T-t}f})(y) - \sum_{y \in V} \mu(y) 2p_t(x, y) \Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}})(y)
\]

\[
= \sum_{y \in V} \mu(y) p_t(x, y) \Delta \Gamma(\sqrt{P_{T-t}f})(y) - \sum_{y \in V} \mu(y) 2p_t(x, y) \Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}})(y)
\]

\[
= \sum_{y \in V} \mu(y) p_t(x, y) \left( \Delta \Gamma(\sqrt{P_{T-t}f})(y) - 2\Gamma(\sqrt{P_{T-t}f}, \frac{\Delta P_{T-t}f}{2\sqrt{P_{T-t}f}})(y) \right).
\]

This ends the proof of Lemma 2.4. \( \square \)
The following results are similar to the theorems of Baudoin and Garofalo [BG] on manifolds. We overcome the assumption of diffusion property on manifolds.

**Theorem 2.3.** Let $G = (V, E)$ be a locally finite, connected graph satisfying CDE$'(n, K)$, then for every $\alpha : [0, T] \to \mathbb{R}^+$ be a smooth and positive function and non-positive smooth function $\gamma : [0, T] \to \mathbb{R}$, we have for any positive and bounded function $f$

$$\partial_t(\alpha \phi) \geq (\alpha' - \frac{4\alpha \gamma}{n} + 2\alpha K)\phi + \frac{2\alpha \gamma}{n} \Delta f - \frac{2\alpha \gamma^2}{n} P_T f. \quad (2.1)$$

**Proof.** For any $x \in V$, we have

$$\partial_t(\alpha \phi)(x) = \alpha' \phi(x) + 2\alpha P_t(\tilde{T}_2(\sqrt{P_{T-t} f}))(x)$$

$$\geq \alpha' \phi(x) + 2\alpha P_t \left( \frac{1}{n} \left( \sqrt{P_{T-t} f} \Delta \log \sqrt{P_{T-t} f} \right)^2 + K \Gamma(\sqrt{P_{T-t} f}) \right)(x)$$

$$\geq (\alpha' + 2\alpha K) \phi(x) + 2\alpha \sum_{\Delta \sqrt{P_{T-t} f}(y) < 0} \mu(y) p_t(x, y) \frac{1}{n} \left( \Delta \sqrt{P_{T-t} f} \right)^2(y)$$

$$+ 2\alpha \sum_{\Delta \sqrt{P_{T-t} f}(y) \geq 0} \mu(y) p_t(x, y) \frac{1}{n} \left( \sqrt{P_{T-t} f} \Delta \log \sqrt{P_{T-t} f} \right)^2(y)$$

$$\geq (\alpha' + 2\alpha K) \phi(x) + \frac{2\alpha}{n} P_t(\gamma P_{T-t} f) - 2\alpha \Gamma(\sqrt{P_{T-t} f}) \gamma^2 P_{T-t} f(x)$$

$$= (\alpha' + 2\alpha K) \phi(x) + \frac{2\alpha \gamma}{n} P_t(\Delta f)(x) - \frac{4\alpha \gamma}{n} P_t(\Gamma(\sqrt{P_{T-t} f}))(x) - \frac{2\alpha \gamma^2}{n} P_t P_T f(x)$$

$$= (\alpha' - \frac{4\alpha \gamma}{n} + 2\alpha K) \phi(x) + \frac{2\alpha \gamma}{n} \Delta f - \frac{2\alpha \gamma^2}{n} P_T f(x).$$

The first inequality in the above proof comes from applying the CDE$'(n, K)$ inequality to $\sqrt{P_{T-t} f}$, and the second one comes from remark 3 when $\Delta \sqrt{P_{T-t} f}(y) < 0$. The third inequality is as follows. For every nonpositive smooth function $\gamma$, one has

$$(\Delta \sqrt{P_{T-t} f})(y)^2 \geq 2\gamma \sqrt{P_{T-t} f}(y) \Delta \sqrt{P_{T-t} f}(y) - \gamma^2 P_{T-t} f(y),$$

and when $\Delta \sqrt{P_{T-t} f}(y) \geq 0$, the right hand of the above inequality is nonpositive, so

$$\left( \sqrt{P_{T-t} f} \Delta \log \sqrt{P_{T-t} f} \right)^2(y) \geq 2\gamma \sqrt{P_{T-t} f}(y) \Delta \sqrt{P_{T-t} f}(y) - \gamma^2 P_{T-t} f(y).$$

Furthermore,

$$2\sqrt{P_{T-t} f} \Delta \sqrt{P_{T-t} f} = \Delta P_{T-t} f - 2\Gamma(\sqrt{P_{T-t} f}),$$

Therefore,

$$\sum_{\Delta \sqrt{P_{T-t} f}(y) < 0} \mu(y) p_t(x, y) \left( \Delta \sqrt{P_{T-t} f} \right)^2(y) + \sum_{\Delta \sqrt{P_{T-t} f}(y) \geq 0} \mu(y) p_t(x, y) P_{T-t} f(y) \left( \Delta \log \sqrt{P_{T-t} f} \right)^2(y)$$

$$\geq P_t(\gamma \Delta P_{T-t} f - 2\gamma \Gamma(\sqrt{P_{T-t} f}) - \gamma^2 P_{T-t} f)(x),$$

as desired. \qed
2.4 Li-Yau inequalities

As a first application of Theorem 1 we use it to derive a family of Li-Yau type inequalities. We choose the function in a such a way that

\[ \frac{\alpha'}{n} - 4\alpha \gamma + 2\alpha K = 0, \]

that is

\[ \gamma = \frac{n}{4} \left( \frac{\alpha'}{\alpha} + 2K \right). \]

Choose the appropriate function $\alpha$ and make $\gamma$ be nonpositive. And then integrating the inequality (2.1) from 0 to $T$, and denoting $W = \sqrt{\alpha}$, we obtain the following result.

**Theorem 2.4.** Let $G = (V, E)$ be a locally finite and connected graph satisfying $CDE'(n, K)$, and $W : [0, T] \to \mathbb{R}^+$ be a smooth function such that

\[ W(0) = 1, W(T) = 0, \]

for any bounded and positive function $f \in V^R$, we have

\[
\frac{\Gamma(\sqrt{P_T f})}{P_T f} \leq \frac{1}{2} \left( 1 - 2K \right) \int_0^T W(s)^2 ds \left( \frac{\Delta P_T f}{P_T f} \right) + \frac{n}{2} \left( \int_0^T W'(s)^2 ds + K^2 \int_0^T W(s)^2 ds - K \right). \tag{2.2}
\]

A family of interesting inequalities may be obtained with the choice

\[ W(t) = \left( 1 - \frac{t}{T} \right)^a, a > \frac{1}{2}. \]

In this case we have

\[ \int_0^T W(s)^2 ds = \frac{T}{2a + 1}, \]

and

\[ \int_0^T W'(s)^2 ds = \frac{a^2}{(2a - 1)T}, \]

so that, according to (2.2),

\[
\frac{\Gamma(\sqrt{P_T f})}{P_T f} \leq \frac{1}{2} \left( 1 - \frac{2KT}{2a + 1} \right) \left( \frac{\Delta P_T f}{P_T f} \right) + \frac{n}{2} \left( \frac{a^2}{(2a - 1)T} + \frac{K^2 T}{2a + 1} - K \right). \tag{2.3}
\]

In the case, $K = 0$ and $a = 1$. Furthermore, according to $\Delta P_T f = \partial_t P_T f = 2\sqrt{P_T f} \partial_t \sqrt{P_T f}$ and switching the notion $T$ to $t$, (2.3) reduces to the Li-Yau inequality on graph:

\[
\frac{\Gamma(\sqrt{P_t f})}{P_t f} - \frac{\partial_t \sqrt{P_t f}}{\sqrt{P_t f}} \leq \frac{n}{2t}, \quad t > 0.
\]
3 Exponential integrability

In this section we establish the following crucial result.

**Theorem 3.1.** Let $G = (V, E)$ be a locally finite and connected graph satisfying $CDE'(n, 0)$, there exists an absolute positive constant $\rho > 0$, and $A > 0$, depending only on $n$, such that

$$P_{Ar^2}(1_{B(x,r)})(x) \geq \rho, \quad x \in V, \quad r > 0 \quad (3.1)$$

**Proof.** We use Theorem 2.3 in which we choose

$$\alpha(t) = \tau + T - t,$$

$$\gamma(t) = -\frac{n}{4(\tau + T - t)},$$

where $\tau > 0$, and $K = 0$. Then

$$\alpha' - \frac{4\alpha \gamma}{n} + 2\alpha K = 0, \quad \frac{2\alpha \gamma}{n} = -\frac{1}{2}, \quad \frac{2\alpha \gamma^2}{n} = \frac{n}{8(\tau + T - t)}$$

Integrating the inequality from 0 to $T$, we obtain

$$\tau P_T(\Gamma(\sqrt{f})) - (T + \tau)\Gamma(\sqrt{P_T f}) \geq -\frac{T}{2} \Delta P_T f - \frac{n}{8} \log \left(1 + \frac{T}{\tau}\right) P_T f \quad (3.2)$$

In what follows we consider a non-positive function $f \in V^\mathbb{R}$ which satisfies, for every $x, y$, there exists a constant $c > 0$ such that $|f(y) - f(x)| \leq c$ if $x \sim y$. For any nonnegative constant $\lambda \in \mathbb{R}_{\geq 0}$, we consider the positive and bounded function $\varphi = e^{2\lambda f}$. The function $\psi$ defined by,

$$\psi(\lambda, t) = \frac{1}{2\lambda} \log(P_t e^{2\lambda f}), \quad P_t \varphi = P_t(e^{2\lambda f}) = e^{2\lambda \psi}$$

We now apply (3.2) to the function $\varphi$, and switching notation from $T$ to $t$, obtaining

$$\tau P_t(\Gamma(e^{\lambda f})) - (t + \tau)\Gamma(e^{\lambda \psi}) \geq -\frac{t}{2} \Delta P_t \varphi - \frac{n}{8} \log \left(1 + \frac{t}{\tau}\right) e^{2\lambda \psi}.$$
For any $x \in V$, let $C(\lambda, c) = \sqrt{\sum e^{\lambda c}} < \infty$, we have

$$\Gamma(e^{\lambda f})(x) = \sum_{y \sim x} (e^{\lambda f(y)} - e^{\lambda f(x)})^2$$

$$= e^{2\lambda f(x)} \sum_{y \sim x} (e^{\lambda f(y)} - e^{\lambda f(x)}) - 1)^2$$

$$= e^{2\lambda f(x)} \left( \sum_{0 \leq f(y) - f(x) \leq c} (e^{\lambda f(y)} - e^{\lambda f(x)}) - 1)^2 + \sum_{-c \leq f(y) - f(x) \leq 0} (e^{\lambda f(y)} - e^{\lambda f(x)}) - 1)^2 \right)$$

$$\leq e^{2\lambda f(x)} \left( e^{2\lambda c} \sum_{0 \leq f(y) - f(x) \leq c} (1 - e^{-\lambda c})^2 + \sum_{-c \leq f(y) - f(x) \leq 0} (e^{-\lambda c} - 1)^2 \right)$$

$$\leq e^{2\lambda f(x)} e^{2\lambda c} \sum_{y \sim x} (e^{-\lambda c} - 1)^2$$

$$\leq C(\lambda, c)^2 \lambda^2 e^{2\lambda f(x)}.$$  \hspace{1cm} (3.3)

So, the left-hand side of the inequality

$$\tau P_t(\Gamma(e^{\lambda f})) - (t + \tau) \Gamma(e^{\lambda \psi}) \leq \tau P_t(\Gamma(e^{\lambda f})) \leq C(\lambda, c)^2 \lambda^2 \tau P_t(e^{2\lambda f}) = C(\lambda, c)^2 \lambda^2 \tau e^{2\lambda \psi}.$$  

Using this observation in combination with the fact that

$$\Delta P_t \varphi = \partial_t e^{2\lambda \psi} = 2\lambda e^{2\lambda \psi} \partial_t \psi.$$  

The inequality finally gives

$$\partial_t \psi \geq -\frac{\lambda}{t} \left( C(\lambda, c)^2 \tau + \frac{n}{8\lambda^2} \log(1 + \frac{t}{\tau}) \right).$$  \hspace{1cm} (3.4)

We now optimize the right-hand side of (3.4) with respect to $\tau$. We notice explicitly that the maximum value of the right-hand side is attained at

$$\tau_0 = \frac{t}{2} \left( \sqrt{1 + \frac{n}{2\lambda^2 C(\lambda, c)^2 t}} - 1 \right).$$

If we substitute such value in (3.4) we find

$$- \partial_t \psi \leq \lambda C(\lambda, c)^2 G \left( \frac{1}{\lambda^2 C(\lambda, c)^2 t} \right),$$  \hspace{1cm} (3.5)

where we have set

$$G(s) = \frac{1}{2} \left( \sqrt{1 + \frac{n}{2} s} - 1 \right) + \frac{n}{8} s \log \left( 1 + \frac{2}{\sqrt{1 + \frac{n}{2} s} - 1} \right), \quad s > 0.$$
Notice that $G(s) \to 0$ as $s \to 0^+$, and that $G(s) \sim \sqrt{\frac{\mu}{s}}$ as $s \to +\infty$. We now integrate the inequality (3.5) between $t_1$ and $t_2$, such that $t_1 \leq t_2$, obtaining

$$\psi(\lambda, t_1) \leq \psi(\lambda, t_2) + \lambda C(\lambda, c)^2 \int_{t_1}^{t_2} G \left( \frac{1}{\lambda^2 C(\lambda, c)^2 t} \right) dt.$$ 

Notice that Jensen's inequality in $\psi$ gives

$$2\lambda \psi(\lambda, t) = \ln(P_t e^{2\lambda f}) \geq P_t(\ln e^{2\lambda f}) = 2\lambda P_t f,$$

and so we have

$$P_t f \leq \psi(\lambda, t_1).$$

Using it and times $\lambda$, we infer

$$P_t (\lambda f) \leq \lambda \psi(\lambda, t_2) + \lambda^2 C(\lambda, c)^2 \int_{t_1}^{t_2} G \left( \frac{1}{\lambda^2 C(\lambda, c)^2 t} \right) dt.$$ 

Letting $t_1 \to 0^+$ and switching the notion $t_2$ to $t$, we conclude

$$\lambda f \leq \lambda \psi(\lambda, t) + \lambda^2 C(\lambda, c)^2 \int_{0}^{t} G \left( \frac{1}{\lambda^2 C(\lambda, c)^2 \tau} \right) d\tau. \quad (3.6)$$

For any point $x \in V$, we let $B = B(x, r) = \{y \in V | d(y, x) < r\}$, and consider the function $f(y) = -d(y, x)$. Notice that $|f(y) - f(x)|_{y \sim x} \leq 1$ (in fact $|f(y) - f(x)|_{y \sim x} \equiv 1$ for $y \neq x$, and the above value equals to 0 if there is a point $x$ has loop), then the following $C(\lambda, c) \leq \sqrt{\frac{\mu}{d}} e^\lambda$. Since we clearly have

$$e^{2\lambda f} \leq e^{-2\lambda r} \mathbf{1}_{B^c} + \mathbf{1}_B,$$

it follows that for every $t > 0$ one has

$$e^{2\lambda \psi(\lambda, t)(x)} = P_t(e^{2\lambda f})(x) \leq e^{-2\lambda r} + P_t(\mathbf{1}_B)(x).$$

This gives the lower bound

$$P_t(\mathbf{1}_B)(x) \geq e^{2\lambda \psi(\lambda, t)(x)} - e^{-2\lambda r}.$$ 

To estimate the first term in the right-hand side of the latter inequality, we use (3.6), which gives

$$1 = e^{2\lambda f(x)} \leq e^{2\lambda \psi(\lambda, t)(x)} e^{2\phi(\lambda C(\lambda, c), t)},$$

where we have set

$$\phi(\lambda C, t) = \lambda^2 C(\lambda, c)^2 \int_{0}^{t} G \left( \frac{1}{\lambda^2 C(\lambda, c)^2 \tau} \right) d\tau.$$ 

This gives

$$P_t(\mathbf{1}_B)(x) \geq e^{-2\phi(\lambda C(\lambda, c), t)} - e^{-2\lambda r}.$$ 

To make use of this estimate, we now choose $\lambda C(\lambda, c) = \frac{1}{r}, \ t = Ar^2$, obtaining

$$P_{Ar^2}(\mathbf{1}_B)(x) \geq e^{-2\phi(\frac{1}{r}, Ar^2)} - e^{-\frac{2}{c(\lambda, c)}}.$$
We want to show that we can choose \( A > 0 \) sufficiently small, depending only on \( n \), and a \( \rho > 0 \), for every \( x \in V \), and \( r > \frac{1}{2} \) (this point implies the second item \( e^{-\nu(\lambda, \infty)} \) will not equal to 1), such that

\[
e^{-2\phi(\frac{1}{r}, Ar^2)} - e^{-\nu(\lambda, \infty)} \geq \rho \tag{3.7}
\]

Consider the function

\[
\phi(\frac{1}{r}, Ar^2) = \frac{1}{r^2} \int_{0}^{Ar^2} G\left(\frac{r^2}{\tau}\right) d\tau = \int_{A^{-1}}^{\infty} \frac{G(t)}{t^2} dt \to 0 (A \to 0^+).
\]

and therefore there exists \( A > 0 \) sufficiently small such that (3.7) hold with.

\[
4 \quad \text{Volume Growth}
\]

In this section we proof the doubling property of the volume of graph as follows.

**Theorem 4.1.** Suppose a locally finite, connected graph \( G \) satisfies \( CDE'(n, 0) \), then \( G \) satisfies the volume doubling property \( DV'(C) \). That is, there exists a constant \( C = C(n) > 0 \) such that for all \( x \in V \) and all \( r \in \mathbb{R}^+ \):

\[
V(x, 2r) \leq CV(x, r).
\]

With some simple computation, we can get the more general conclusion of the volume regularity, it will be useful in the proof of the Gaussian estimate.

**Remark 4.** For any \( r \geq s \), (the square brackets denote the integer part)

\[
V(x, r) \leq V(x, 2^\left\lfloor \frac{\log r}{\log s} \right\rfloor + 1 s)
\]

\[
\leq C^{1+\frac{\log(c)}{\log 2}} V(x, s)
\]

\[
= C\left(\frac{r}{s}\right)^{\frac{\log c}{\log 2}} V(x, s).
\]

In order to prove Theorem 4.1, we will need the following result which are a straightforward consequence of Li-Yau inequality. First, we introduce a discrete analogue of the Agmon distance between two points \( x \) and \( y \) which are connected in \( B(x_0, R) \). For a path \( p_0p_1 \ldots p_k \) define the length of the path to be \( \ell(P) = k \). Then in a graph with maximum measure \( \mu_{\text{max}} \):

\[
\varrho_{q, x_0, R, \mu_{\text{max}}, w_{\text{min}}, \alpha}(x, y, T_1, T_2) = \inf \left\{ \frac{2\mu_{\text{max}}\ell^2(P)}{w_{\text{min}}(1 - \alpha)(T_2 - T_1)} + \sum_{i=0}^{k-1} \left( \int_{t_i}^{t_{i+1}} q(x_i, t) dt + \frac{k}{(T_2 - T_1)^2} \int_{t_i}^{t_{i+1}} (t - t_i)^2 (q(x_i, t) - q(x_{i+1}, t)) dt \right) \right\},
\]

where the infimum is taken over the set of all paths \( P = p_0p_1p_2 \ldots p_k \) so that \( p_0 = x, p_k = y \) and having all \( p_i \in B(x_0, R) \), and the times \( T_1 = t_0, t_1, t_2, \ldots, t_k = T_2 \) evenly divide the interval \([T_1, T_2]\).
**Remark 5.** In the special case where \( q \equiv 0 \) and \( R = \infty \), which will arise when \( f \) is a solution to the heat equation on the entire graph, then \( \varrho \) simplifies drastically. In particular,

\[
\varrho_{\mu_{\text{max}},\omega_{\text{min}}}(x, y, t_1, t_2) = \frac{2\mu_{\text{max}}d(x, y)^2}{(1 - \alpha)(T_2 - T_1)\omega_{\text{min}}},
\]

where \( d(x, y) \) denotes the usual graph distance.

**Theorem 4.2.** Let \( G(V, E) \) be a graph with measure bound \( \mu_{\text{max}} \), and suppose that a function \( f : V \times \mathbb{R} \rightarrow \mathbb{R} \) satisfies

\[
(1 - \alpha)\Gamma(f)\frac{\partial}{\partial t}f(x, t) - \frac{\partial}{\partial x}f(x, t) - q(x, t) \leq c_1 t + c_2,
\]

whenever \( x \in B(x_0, R) \) for \( x_0 \in V \) along with some \( R \geq 0 \), some \( 0 \leq \alpha < 1 \) and positive constants \( c_1, c_2 \). Then for \( T_1 < T_2 \) and \( x, y \in V \) we have

\[
f(x, T_1) \leq f(y, T_2) \left( \frac{T_2}{T_1} \right)^{c_1} \cdot \exp \left( c_2(T_2 - T_1) + \varrho_{q, x_0, R, \mu_{\text{max}}, \omega_{\text{min}}, \alpha}(x_1, x_2, T_1, T_2) \right).
\]

We have already proved the Li-Yau inequality for any positive and bounded function, if the graph satisfies \( CDE'(n, K) \), applying the above theorem to the heat kernel \( p_t(x, y) \). And in the case when the graph satisfies \( K \geq 0 \), one can set \( \alpha = 0 \). We have the following result.

**Corollary 4.3.** Suppose \( G \) is a finite or infinite graph satisfying \( CDE'(n, 0) \), and assume \( D := \frac{\mu_{\text{max}}}{\omega_{\text{min}}} < \infty \), then for every \( x \in V \) and \( (t, y), (t, z) \in V \times (0, 1) \) with \( t < s \) one has

\[
p_t(x, y) \leq p_s(x, z) \left( \frac{s}{t} \right)^n \exp \left( 4Dd(y, z)^2 \frac{s - t}{s - t} \right).
\]

We now turn to the proof of Theorem 4.1.

**Proof.** From the semigroup property and the symmetry of the heat kernel we have for any \( y \in V \) and \( t > 0 \)

\[
p_{2t}(y, y) = \sum_{z \in V} \mu(z)p_t(y, z)^2.
\]

Consider now a cut-off function \( h \in V^\mathbb{R} \) such that \( 0 \leq h \leq 1 \), \( h \equiv 1 \) on \( B(x, \frac{\sqrt{t}}{2}) \) and \( h \equiv 0 \) outside \( B(x, \sqrt{t}) \). We thus have

\[
P_t h(y) = \sum_{z \in V} \mu(z)p_t(y, z)h(z)
\]

\[
\leq \left( \sum_{z \in V} \mu(z)p_t(y, z)^2 \right)^{\frac{1}{2}} \left( \sum_{z \in V} \mu(z)h(z)^2 \right)^{\frac{1}{2}}
\]

\[
\leq (p_{2t}(y, y))^{\frac{1}{2}} \left( V(x, \sqrt{t}) \right)^{\frac{1}{2}}.
\]
If we take \( y = x \), and \( t = r^2 \), we obtain
\[
\left( P_{r^2}(1_{B(x, r)})(x) \right)^2 \leq (P_{r^2}h(x))^2 \leq p_{2r^2}(x, x)V(x, r). \tag{4.1}
\]
At this point we use the crucial inequality (3.1), which gives for some \( 0 < A < 1 \), depending on \( n \),
\[
P_{Ar^2}(1_{B(x, r)})(x) \geq \rho, \quad x \in V, \quad r > 0.
\]
Combining the latter inequality with (4.1) and Corollary 4.3, we obtain the following on-diagonal lower bound
\[
p_{2r^2}(x, x) \geq \frac{\rho^*}{V(x, r)} \quad x \in V, \quad r > 0. \tag{4.2}
\]
Applying Corollary 4.3 to \( p_t(x, y) \), for every \( y \in B(x, \sqrt{t}) \), we find
\[
p_t(x, x) \leq C(n)p_{2t}(x, y), \tag{4.3}
\]
Integrating the above inequality over \( B(x, \sqrt{t}) \) with respect to \( y \) gives
\[
p_t(x, x)V(x, \sqrt{t}) \leq C(n) \sum_{y \in B(x, \sqrt{t})} \mu(y)p_{2t}(x, y) \leq C(n),
\]
letting \( t = 4r^2 \), we obtain from this the on-diagonal upper bound
\[
p_{4r^2}(x, x) \leq \frac{C(n)}{V(x, 2r)}. \tag{4.4}
\]
Combining (4.2), (4.3) with (4.4) we finally obtain
\[
V(x, 2r) \leq \frac{C}{p_{4r^2}(x, x)} \leq \frac{C^*}{p_{2r^2}(x, x)} \leq C^{**}V(x, r).
\]
This completes the proof. \( \square \)

## 5 Gaussian estimate

In this section we assume the measure \( \mu(x) = m(x) \), for any \( x \in V \), which generates normalized graph Laplacian. In the following, we will prove discrete-time Gaussian estimate on a infinite, connected and locally finite graph \( G = (V, E) \).

Let \( \mathcal{P}_t(x, y) = p_t(x, y)m(y) \) be the continue-time Markov kernel on graph, and it is also a solution of the heat equation. Due to the symmetric property of the heat kernel \( p_t(x, y) \), it satisfies
\[
\frac{\mathcal{P}_t(x, y)}{m(y)} = \frac{\mathcal{P}_t(y, x)}{m(x)}.
\]
Let \( p_n(x, y) \) be the discrete-time kernel on \( G \), which is defined by
\[
\left\{
\begin{array}{ll}
p_0(x, y) = \delta(x, y), \\
p_{k+1}(x, z) = \sum_{y \in V} p(x, y)p_k(y, z).
\end{array}
\right.
\]
where \( p(x, y) := \frac{\omega_{xy}}{m(x)} \). We can know the two kernels satisfy

\[
e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} p_k(x, y) = \mathcal{P}_t(x, y).
\]

There are notions we will use in the following of this paper.

**Definition 5.1.** Let \( \alpha > 0 \), \( G \) satisfies \( \Delta(\alpha) \) if, for any \( x, y \in V \), and \( x \sim y \),

\[ \omega_{xy} \geq \alpha m(x). \]

**Definition 5.2.** The graph \( G \) satisfies the Gaussian estimate \( G(c_l, C_l, C_r, c_r) \) if, there exist positive constants \( c_l, C_l, C_r, c_r > 0 \), when \( d(x, y) \leq n \),

\[
\frac{c_l m(y)}{V(x, \sqrt{n})} e^{-\frac{C_l d(x, y)^2}{n}} \leq p_n(y, z) \leq \frac{C_r m(y)}{V(x, \sqrt{n})} e^{-\frac{c_r d(x, y)^2}{n}}.
\]

In order to obtain the Gaussian estimate, we first introduce the continue-time Gaussian on-diagonal estimate on graph. In the paper of [LY], the Gaussian upper bound of the continue-time on-diagonal estimate on graph has already been proved, if Harnack inequality holds with and maximum degree exists on graph. However, the lower bound of \( \mathcal{P}_t(x, y) \) is not Gaussian in the condition of \( CDE(n, 0) \) in [LY]. We can derive heat kernel lower bound that is Gaussian too as follows. It is crucial to prove the discrete-time Gaussian estimate.

**Theorem 5.1.** Suppose a graph \( G \) satisfies \( CDE'(n, 0) \), then \( G \) satisfies the continue-time Gaussian estimate, that is, there will exist constants so that, for any \( x, y \in V \) and for all \( t > 0 \),

\[
\mathcal{P}_t(x, y) \leq \frac{C m(y)}{V(x, \sqrt{t})},
\]

\[
\mathcal{P}_t(x, y) \geq \frac{C' m(y)}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{t} \right).
\]

**Proof.** The upper bound is similar with the methods of [LY], because Harnack inequality satisfies with the assumption of \( CDE'(n, 0) \). From Corollary 4.3, for any \( t > 0 \), choosing \( s = 2t \) and for any \( z \in B(x, \sqrt{t}) \), we have

\[
p_t(x, y) \leq p_{2t}(z, y) 2^n \exp(4D),
\]

thus

\[
p_t(x, y) \leq \frac{C}{V(x, \sqrt{t})} \sum_{z \in B(x, \sqrt{t})} \mu(z) p_{2t}(z, y)
\]

\[
\leq \frac{C'}{V(x, \sqrt{t})}.
\]
We now prove the lower bound estimate. From (4.2), for any \( x \in V \), choose \( 2r^2 = \varepsilon t \), \( 0 < \varepsilon < 1 \), thus

\[
p_{\varepsilon t}(x, x) \geq \frac{\rho^*}{V(x, \sqrt{\frac{t}{2}})} = \frac{\rho^*}{V(x, \sqrt{t})}.
\]

From Corollary 4.3, substituting from \( \varepsilon t \) to \( t \), and from \( t \) to \( s \), choosing \( z = x \), we have

\[
p_{\varepsilon t}(x, x) \leq p_t(x, y)\varepsilon^n \exp \left( \frac{4Dd^2(x, y)}{(1 - \varepsilon)t} \right).
\]

Combining (5.1) with (5.2), we finally obtain

\[
p_t(x, y) \geq \frac{\varepsilon^{-n} \rho^*}{V(x, \sqrt{t})} \exp \left( -4Dd^2(x, y) \right) = \frac{C'}{V(x, \sqrt{t})} \exp \left( -\frac{d^2(x, y)}{t} \right).
\]

Hence the discrete-time Gaussian estimate is clear.

Especially if \( t \geq d^2(x, y) \), then the lower estimate can be write

\[
p_t(x, y) \geq \frac{C''}{V(x, \sqrt{t})}.
\]

And then, we show the proof of the discrete-time on-diagonal estimate.

**Proposition 5.2.** Assume a graph \( G \) satisfies \( CDE'(n_0, 0) \) and \( \Delta(\beta) \), then there exist \( c_d, C_d > 0 \), for any \( x, y \in V \), such that,

\[
p_n(x, y) \leq \frac{C_d m(y)}{V(x, \sqrt{n})}, \quad \text{for all } n > 0,
\]

\[
p_n(x, y) \geq \frac{c_d m(y)}{V(x, \sqrt{n})}, \quad \text{if } n \geq d^2(x, y).
\]

This proposition follows the methods of Delmotte from [D]. To prove it, first we need to introduce some results from [D]. Assume \( \Delta(\alpha) \) is true, so that we can consider the positive submarkovian kernel

\[
\bar{\mathcal{P}}(x, y) = p(x, y) - \alpha \delta(x, y).
\]

Now, compute \( \mathcal{P}_n(x, y) \) and \( p_n(x, y) \) with \( \bar{\mathcal{P}}(x, y) \),

\[
\mathcal{P}_n(x, y) = e^{(\alpha-1)n} \sum_{k=0}^{n} \frac{n^k}{k!} \bar{\mathcal{P}}(x, y) = \sum_{k=0}^{+\infty} a_k \bar{\mathcal{P}}(x, y),
\]

\[
p_n(x, y) = \sum_{k=0}^{n} C_n^k \alpha^{-k} \bar{\mathcal{P}}(x, y) = \sum_{k=0}^{n} b_k \bar{\mathcal{P}}(x, y).
\]

There is a lemma from [D] to compare the two sums,
Lemma 5.1. Let $c_k = \frac{b_k}{a_k}$, for $0 \leq k \leq n$, 
- $c_k \leq C(\alpha)$, when $0 \leq k \leq n$,
- $c_k \geq C(a, \alpha) > 0$, when $n \geq \frac{a^2}{\alpha^2}$ and $|k - (1 - \alpha)n| \leq a\sqrt{n}$.

We shall consider only $\alpha \leq \frac{1}{4}$, so that we always have $\frac{n}{2} \leq k \leq n$ in the second assertion.

Now we turn to the proof of Proposition 5.2.

Proof. The proof comes from Delmotte of [D].

The first assertion in Lemma 5.1 implies, for any $n$

$$p_n(x, y) \leq C(\beta)\mathcal{P}_n(x, y).$$

The upper bound is immediately consequence from Theorem 5.1, for any $x, y \in V$,

$$p_n(x, y) \leq \frac{C(\beta)Cm(y)}{V(x, \sqrt{n})} = \frac{Cdm(y)}{V(x, \sqrt{n})}.$$

And the second assertion is a little complicated. First we will set $\alpha = \frac{\beta}{2}$, when $n \geq N = \frac{a^2}{\alpha^2}$, if for any $\varepsilon > 0$, there exists $a$, such that

$$\sum_{|k - (1 - \alpha)n| > a\sqrt{n}} a_k\mathcal{P}(x, y) \leq \frac{\varepsilon m(y)}{V(x, \sqrt{n})}.$$  (5.3)

Then, we have

$$p_n(x, y) \geq \sum_{|k - (1 - \alpha)n| \leq a\sqrt{n}} b_k\mathcal{P}(x, y)$$

$$\geq C(a, \alpha) \sum_{|k - (1 - \alpha)n| \leq a\sqrt{n}} a_k\mathcal{P}(x, y),$$

and

$$C(a, \alpha)\mathcal{P}_n(x, y) = C(a, \alpha) \sum_{|k - (1 - \alpha)n| \leq a\sqrt{n}} a_k\mathcal{P}(x, y) + C(a, \alpha) \sum_{|k - (1 - \alpha)n| > a\sqrt{n}} a_k\mathcal{P}(x, y)$$

$$\leq p_n(x, y) + C(a, \alpha) \sum_{|k - (1 - \alpha)n| > a\sqrt{n}} a_k\mathcal{P}(x, y)$$

$$\leq p_n(x, y) + C(a, \alpha)\frac{\varepsilon m(y)}{V(x, \sqrt{n})}.$$ 

Since we assume $n \geq d^2(x, y)$, applying the second assertion of Theorem 5.1, then

$$p_n(x, y) \geq C(a, \alpha) \left(\mathcal{P}_n(x, y) - \frac{\varepsilon m(y)}{V(x, \sqrt{n})}\right)$$

$$\geq C(a, \alpha) \left(\frac{C'm(y)}{V(x, \sqrt{n})} - \frac{\varepsilon m(y)}{V(x, \sqrt{n})}\right)$$

$$= \frac{c dm(y)}{V(x, \sqrt{n})}.$$
So next we will prove (5.3). First we consider another Markov kernel $p' = \frac{p}{1-\alpha}$. Indeed it is generated by weights $\omega'_{xy}$ as follow,

$$\omega'_{xx} = \frac{\omega_{xx} - \alpha m(x)}{1-\alpha} \geq \alpha m(x), \quad \forall x \in V,$$

$$\omega'_{xy} = \frac{\omega_{xy}}{1-\alpha}, \quad \forall x \neq y \in V,$$

$$m'(x) = m(x).$$

Then we know $\Delta(\alpha)$ is true in $G$ with the new weights. And $DV(C)$ is still satisfied too. First, we can get $CDE'(n_0, 0)$ is still true for the new weight, because if let $\Delta'$ be the new Laplacian for $\omega'_{xy}$, for any $f, g \in V^R$ we can get

$$\Delta' f(x) = \frac{1}{1-\alpha} \Delta f(x), \quad \Gamma'(f, g) = \frac{1}{1-\alpha} \Gamma(f, g),$$

$$\Gamma'_2(f, g) = \frac{1}{(1-\alpha)^2} \Gamma_2(f, g), \quad \tilde{\Gamma}'_2(f, g) = \frac{1}{(1-\alpha)^2} \tilde{\Gamma}_2(f, g).$$

Second, the process of proving $DV(C)$ is also true of adding loops in every point of graph. Then $DV(C)$ is still satisfied for the new weight. According to the first assertion, this yields

$$p_n(x, y) \leq \frac{C_d m(y)}{V(x, \sqrt{k})},$$

hence

$$\bar{p}_n(x, y) \leq \frac{C_d m(y)(1-\alpha)^k}{V(x, \sqrt{k})}.$$

Next, we have to get the estimate

$$e^{(\alpha-1)n} \sum_{|k-(1-\alpha)n|>a\sqrt{n}} \frac{(1-\alpha)n}{k!} \frac{1}{V(x, \sqrt{k})} \leq \frac{\varepsilon'}{V(x, \sqrt{n})}.$$ 

The sum for $k > a\sqrt{n} + (1-\alpha)n$ is easier because we simply use

$$V(x, \sqrt{k}) \geq V\left(x, \sqrt{\frac{n}{2}}\right) \geq V\left(x, \frac{\sqrt{n}}{2}\right) \geq \frac{V(x, \sqrt{n})}{C_1},$$

24
So we have,
\[
e^{(\alpha - 1)n} \sum_{k > a\sqrt{n} + (1 - \alpha)n} \frac{(1 - \alpha)n)^k}{k!} \frac{1}{V(x, \sqrt{k})} \leq e^{(\alpha - 1)n} \frac{C_1}{V(x, \sqrt{n})} \sum_{k > a\sqrt{n} + (1 - \alpha)n} \frac{(1 - \alpha)n)^k}{\Gamma(k + 1)} \leq e^{(\alpha - 1)n} \frac{C_1}{V(x, \sqrt{n})} \frac{(1 - \alpha)n)^{1 - a\sqrt{n} + a\sqrt{n}}}{\Gamma(a\sqrt{n} + (1 - \alpha)n + 1)} \frac{1}{1 - \frac{(1 - \alpha)n}{a\sqrt{n} + (1 - \alpha)n}} \leq \frac{CC_1}{V(x, \sqrt{n})} \exp \left( a\sqrt{n} - (a\sqrt{n} + (1 - \alpha)n) \log \left( 1 + \frac{a}{(1 - \alpha)n} \right) \right) \cdot \frac{1}{\sqrt{a\sqrt{n} + (1 - \alpha)n}} \frac{a\sqrt{n}}{a\sqrt{n}} \leq \frac{\varepsilon'}{2V(x, \sqrt{n})},
\]
we can get \( \frac{1}{\sqrt{a\sqrt{n} + (1 - \alpha)n}} \frac{a\sqrt{n} + (1 - \alpha)n}{a\sqrt{n}} \leq \frac{1}{a} \), because of \( n \geq \frac{a^2}{\alpha^2} \). And with a good choice of \( a \), let the argument of the exponential function appears to be negative.

To deal with \( 1 \leq k < a\sqrt{n} + (1 - \alpha)n \), we need apply Remark 3, then it gives

\[
V(x, \sqrt{k}) \leq C \left( \frac{\sqrt{k}}{\sqrt{k - 1}} \right)^{\log \frac{C}{\log 2}} V(x, \sqrt{k - 1}) \leq C_2 V(x, \sqrt{k - 1}).
\]

So far the terms \( 1 \leq k \leq \frac{(1 - \alpha)n}{2C_2} \), we have

\[
\frac{(1 - \alpha)n)^{k - 1}}{(k - 1)!} \frac{1}{V(x, \sqrt{k - 1})} \leq \frac{1}{2} \frac{(1 - \alpha)n)^k}{k!} \frac{1}{V(x, \sqrt{k})},
\]
the estimate is straightforward. For the other term \( \frac{(1 - \alpha)n}{2C_2} < k < a\sqrt{n} + (1 - \alpha)n \), from Remark 4, we bound

\[
V(x, \sqrt{k}) \leq C_3 V \left( x, \sqrt{\frac{(1 - \alpha)n}{2C_2}} \right).
\]
if \( \frac{1}{2C_2} \leq 1 \), we can get \( V(x, \sqrt{k}) \leq C_3 V(x, \sqrt{n}) \) immediately, if not, we can use Remark 3 again, we also have \( V(x, \sqrt{k}) \leq C'_3 V(x, \sqrt{n}) \). Then the same computation as for \( k > a\sqrt{n} + (1 - \alpha)n \).

Moreover, to prove the discrete-time Gaussian estimate on graph, we need introduce a result from [CG], it is a useful point to prove the upper bound of the discrete-time Gaussian estimate.
Theorem 5.3. For a reversible nearest neighbourhood random walk on the locally finite graph $G = (V, E)$, the following properties are equivalent:

1. The relative Faber-Krahn inequality (FK).

2. The discrete-time Gaussian upper estimate in conjunction with the doubling property $DV(C)$.

3. The discrete-time on-diagonal upper estimate in conjunction with the doubling property $DV(C)$.

Now we show the final theorem of the discrete-time Gaussian estimate.

Theorem 5.4. Assume a graph $G$ satisfies CDE$(n_0, 0)$ and $\Delta(\alpha)$, then the graph satisfies the discrete-time Gaussian estimate $G(c_1, c_l, C_r, c_r)$.

Proof. Because the discrete-time on-diagonal upper estimate and the doubling property $DV(C)$ are both true in the condition of CDE$(n_0, 0)$ and $\Delta(\alpha)$. From Theorem 5.3, we can get immediately the discrete-time Gaussian upper estimate.

The lower bound follows from the on-diagonal one. The strategy is similar to Delmotte of [D]. Let us apply many times the second assertion of Proposition 5.2. Set $n = n_1 + n_2 + \cdots + n_j$, $x = x_0, x_1, \cdots, x_j = y$ and $B_0 = x$, $B_j = B(x_i, r_i)$, $B_j = j$, such that

$$
\begin{align*}
&j - 1 \leq C \frac{d(x, y)^2}{n}, \\
&r_i \geq c \sqrt{n_i} + 2, \\
&\sup_{z \in B_{i-1}, z' \in B_i} d(z, z')^2 \leq n_i, \\
&\text{so that } V(z, \sqrt{n_i} + 2) \leq AV(B_i), \text{when } z \in B_i.
\end{align*}
$$

It will be sufficient to prove the Gaussian lower bound since

$$
p_n(x, y) \geq \sum_{(z_1, \cdots, z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} p_{n_1}(x, z_1)p_{n_2}(z_1, z_2) \cdots p_{n_j}(z_{j-1}, y)
$$

$$
\geq \sum_{(z_1, \cdots, z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} \frac{c_d m(z_1)}{V(x, \sqrt{n_1})} \frac{c_d m(z_2)}{V(z_1, \sqrt{n_2})} \cdots \frac{c_d m(y)}{V(z_{j-1}, \sqrt{n_j})}
$$

$$
\geq c_d^{j-1} A^{1-j} \sum_{(z_1, \cdots, z_{j-1}) \in B_1 \times \cdots \times B_{j-1}} \frac{m(z_1)}{V(x, \sqrt{n_1})} \frac{m(z_2)}{V(z_1, \sqrt{n_2})} \cdots \frac{m(y)}{V(z_{j-1}, \sqrt{n_j})}
$$

$$
= \frac{c_d m(y)}{V(x, \sqrt{n_1})} \left( \frac{c_d}{A} \right)^{(j-1)},
$$

and choose $C_i \geq C \log(A/c_d)$, and $V(x, \sqrt{n_1}) \leq V(x, \sqrt{n})$, we can get the Gaussian lower bound,

$$
p_n(x, y) \geq \frac{c_d m(y)}{V(x, \sqrt{n})} e^{-C_i \frac{d(x, y)^2}{n}}.
$$

This theorem follows. \qed
**Definition 5.3.** A graph $G$ satisfies the the Poincaré inequality $P(C)$ if

$$
\sum_{x \in B(x_0, r)} m(x) |f(x) - f_B|^2 \leq C r^2 \sum_{x,y \in B(x_0, 2r)} \omega_{xy}(f(y) - f(x))^2,
$$

for all $f \in \mathbb{R}^V$, for all $x_0 \in V$ and for all $r \in \mathbb{R}^+$, where

$$
f_B = \frac{1}{V(x_0, r)} \sum_{x \in B(x_0, r)} m(x) f(x).
$$

**Definition 5.4.** Fix $\eta \in (0, 1)$ and $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$ and $C > 0$. $G$ satisfies the continue-time Harnack inequality property $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C)$, if for all $x_0 \in V$ and $t_0, R \in \mathbb{R}^+$, and every positive solution $u(t, x)$ to the heat equation on $Q = B[x, 0 + \theta_4 R^2] \times (x_0, R)$, we have

$$
\sup_{Q^-} u(t, x) \leq C \inf_{Q^+} u(t, x),
$$

where $Q^- = [s + \theta_1 R^2, s + \theta_2 R^2] \times B(x_0, \eta R)$, and $Q^+ = [s + \theta_3 R^2, s + \theta_4 R^2] \times B(x_0, \eta R)$.

**Definition 5.5.** Fix $\eta \in (0, 1)$ and $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4$ and $C > 0$. $G$ satisfies the discrete-time Harnack inequality property $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C)$, if for all $x_0 \in V$ and $t_0, R \in \mathbb{R}^+$, and every positive solution $u(x, t)$ to the heat equation on $Q = ([s + \theta_4 R^2] \cap \mathbb{Z}) \times B(x_0, R)$, we have

$$(n^-, x^-) \in Q^-, (n^+, x^+) \in Q^+, d(x^-, x^+) \leq n^+ - n^-$$

implies

$$u(n^-, x^-) \leq C u(n^+, x^+),$$

where $Q^- = ([s + \theta_1 R^2, s + \theta_2 R^2] \cap \mathbb{Z}) \times B(x_0, \eta R)$, and $Q^+ = ([s + \theta_3 R^2, s + \theta_4 R^2] \cap \mathbb{Z}) \times B(x_0, \eta R)$.

Since we have already proved that the graph satisfies the discrete-time Gaussian estimate $G(c_l, C_l, C_r, c_r)$ if the conditions $CDE'(n_0, 0)$ and $\Delta(\alpha)$ are true on this graph. Delmotte shows that $G(c_l, C_l, C_r, c_r) \iff DV(C_1)$, $P(C_2)$ and $\Delta(\alpha) \iff H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C_H)$ (or $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C_H)$) for graphs. So Here we have the following result.

**Theorem 5.5.** If the graph satisfies $CDE'(n_0, 0)$ and $\Delta(\alpha)$, we have the following four properties.

1) There exists $C_1, C_2, \alpha > 0$ such that $DV(C_1)$, $P(C_2)$, and $\Delta(\alpha)$ are true.

2) There exists $c_l, C_l, C_r, c_r > 0$ such that $G(c_l, C_l, C_r, c_r)$ is true.

3) There exists $C_H$ such that $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C_H)$ is true.

3') There exists $C_H$ such that $H(\eta, \theta_1, \theta_2, \theta_3, \theta_4, C_H)$ is true.

**Proof.** The condition $CDE'(n_0, 0)$ implies $DV(C_1)$ (see Theorem 4.1), and Theorem 5.4 states that $DV(C_1)$ and $\Delta(\alpha)$ implies $G(c_l, C_l, C_r, c_r)$. According to Delmotte of [D], $P(C_2)$ is true. Moreover, 3) and 3') hold too. \qed
6 Diameter bound

In this section, we obtain the diameter bound. For prove this, we first introduce another distance on graph dislike natural distance as follows. With the operator $\Delta$ we can associate canonical distance and diameter of $G$:

$$\tilde{d}(x, y) = \sup_{f \in \ell^\infty(V, \mu), \|\Gamma(f)\|_\infty \leq 1} |f(x) - f(y)|, \quad x, y \in V,$$

$$\tilde{D} = \sup_{x, y \in V} \tilde{d}(x, y).$$

We just consider simple connected graph without loop in this part.

6.1 Global heat kernel bounds

In this subsection we introduce the first result of large-time exponential decay for the heat kernel on graph.

In Theorem 2.3, we choose the function $\gamma$ in such a way that

$$\alpha' - \frac{4\alpha \gamma}{n} + 2\alpha K = 0,$$

that is

$$\gamma = \frac{n}{4} \left( \frac{\alpha'}{\alpha} + 2K \right).$$

Integrating both sides of the above inequality from 0 to $T$, we obtain

$$a(T) \frac{P_T(\Gamma(f))}{P_T f} - a(0) \frac{\Gamma(\sqrt{P_T f})}{P_T f} \geq \frac{2}{n} \left( \int_0^T a \gamma dt \right) \frac{\Delta P_T f}{P_T f} - \frac{2}{n} \int_0^T a \gamma^2 dt. \quad (6.1)$$

Now we introduce the main result in this subsection.

**Proposition 6.1.** Let $G = (V, E)$ be a locally finite, connected graph satisfying CDE$'$$(n, K)$, then for all $0 < \alpha < K$, there exist $t_0 > 0$ and $C_0 > 0$ such that for every function $0 \leq f \in \ell^\infty(V, \mu)$,

$$\left| \frac{\partial}{\partial t} \log P_t f(x) \right| \leq C_0 e^{-\alpha t}, \quad x \in V, t \geq t_0.$$

**Proof.** In Theorem 2.3, let $\alpha > 0, \beta > 2$, we choose

$$a(t) = \alpha \beta e^{-\alpha t} (e^{-\alpha t} - e^{-\alpha T})^{\beta - 1},$$

we know

$$a(0) = \alpha \beta (1 - e^{-\alpha T})^{\beta - 1}, \quad \text{and} \quad a(T) = 0.$$

With such choice a simple computation gives,

$$\gamma = \frac{n}{4} \left( 2K - \alpha \beta - e^{-\alpha T} \frac{\alpha (\beta - 1)}{e^{-\alpha t} - e^{-\alpha T}} \right).$$
We can obtain from (6.1),
\[-\frac{n}{2}\alpha \beta (1 - e^{-\alpha T})^{\beta-1} \frac{\Gamma(\sqrt{P_T f})}{P_T f} \geq \left( \int_0^T a\gamma dt \right) \frac{\Delta P_T(f)}{P_T f} - \int_0^T a\gamma^2 dt. \quad (6.2)\]

Now, we can compute
\[
\int_0^T a\gamma dt = \frac{n}{4} \left( 1 - e^{-\alpha T} \right)^{\beta-1} (2K - \alpha \beta - 2Ke^{-\alpha T}),
\]
\[
\int_0^T a\gamma^2 dt = \frac{n^2 \alpha \beta}{16} (1 - e^{-\alpha T})^{\beta-2} e^{-2\alpha T}.
\]

First we get the lower bound, in this situation we choose
\[
\alpha = \frac{2K}{\beta},
\]
then
\[
2K - \alpha \beta = 0,
\]
and we obtain from (6.2),
\[
0 \geq -\frac{n}{2} \alpha \beta (1 - e^{-\alpha T})^{\beta-1} \frac{\Gamma(\sqrt{P_T f})}{P_T f} \geq -\frac{nK}{2} (1 - e^{-\alpha T})^{\beta-1} e^{-\alpha T} \frac{\Delta P_T(f)}{P_T f} - \frac{n^2 \alpha^2 \beta (\beta - 1)}{16(\beta - 2)} (1 - e^{-\alpha T})^{\beta-2} e^{-2\alpha T}. \quad (6.3)
\]

Noting that \( \beta > 2 \), then \( \alpha < K \), such that \( \frac{nK}{2} (1 - e^{-\alpha T})^{\beta-1} e^{-\alpha T} > 0 \). Switching \( t \) to \( T \), and there exist \( 0 < t_0 \leq t \) (it is decided in the proof of the upper bound), let
\[
C_1 = \frac{n^2 \alpha^2 \beta (\beta - 1)}{8K(\beta - 2)(1 - e^{-\alpha T})} > 0,
\]
then we can get the desired lower bound,
\[
\frac{\Delta P_t(f)}{P_t f} \geq -C_1 e^{-\alpha t}.
\]

The upper bound is more delicate. We choose in (6.2)
\[
\alpha = \eta - \frac{\theta e^{-\eta T}}{\beta},
\]
with \( \eta = \frac{2K}{\beta} > 0, \theta = 2K\beta \), obtain
\[
\int_0^T a\gamma dt = \frac{n}{2} (1 - e^{-\alpha T})^{\beta-1} e^{-\alpha T} (\theta e^{-(\eta - \alpha)T} - 2K),
\]

Noting that \( e^{-(\eta - \alpha)T} = e^{-\frac{\theta e^{-\eta T} - \eta}{\beta}} \rightarrow 1 \) as \( T \rightarrow \infty \), then
\[
\theta e^{-(\eta - \alpha)T} - 2K \rightarrow 2K(\beta - 1) > 0,
\]
so when $T$ is large enough, it is clear that we have
\[
\int_0^T a\gamma dt \geq \frac{nK(\beta - 1)}{2} (1 - e^{-\alpha T})^{\beta-1} e^{-\alpha T} > 0,
\]

We also have
\[
\int_0^T a^2\gamma^2 dt = \frac{n^2\alpha\beta}{16} (1 - e^{-\alpha T})^{\beta-2} e^{-2\alpha T}.
\]

and when $T \to \infty$,
\[
\theta e^{-2(\eta - \alpha)T} (1 - e^{-2\alpha T}) + 2\theta e^{-2(\eta - \alpha)T} (1 - e^{-\alpha T}) + \frac{\alpha(\beta - 1)^2}{\beta - 2} \to 4K^2\beta^2 + 4K\beta + \frac{\alpha(\beta - 1)^2}{\beta - 2},
\]

so if $T$ is large enough, then it holds that
\[
\int_0^T a^2\gamma dt \leq \frac{n^2\alpha\beta}{8} (1 - e^{-\alpha T})^{\beta-2} e^{-2\alpha T} \left( 4K^2\beta^2 + 4K\beta + \frac{\alpha(\beta - 1)^2}{\beta - 2} \right).
\]

In (6.2), switching $t$ to $T$, let
\[
C_2 = \frac{n\alpha \beta \left( 4K^2\beta^2 + 4K\beta + \frac{\alpha(\beta - 1)^2}{\beta - 2} \right)_{\beta}}{4K(\beta - 1)(1 - e^{-\alpha t_0})} > 0,
\]

then we can get the desired upper bound,
\[
\frac{\Delta P_t(f)}{P_t f} \leq C_2 e^{-\alpha t}.
\]

And we choose $C_0 = \max\{C_1, C_2\}$, we have
\[
\left| \frac{\Delta P_t(f)}{P_t f} \right| \leq C_0 e^{-\alpha t}, \ 0 < \alpha < K.
\]

This completes the proof.

\[\Box\]

**Proposition 6.2.** Let $G = (V, E)$ be a locally finite, connected graph satisfying $CDE'(n, K)$, then for all $0 < \alpha < K$, there exist $t_0 > 0$ and $C_3 > 0$ such that for every function $0 \leq f \in L^\infty(V, \mu)$,
\[
|\sqrt{P_t f}(x) - \sqrt{P_t f}(y)| \leq \sqrt{CC_3} e^{-\alpha t} \tilde{d}(x, y), \ x, y \in V, t \geq t_0.
\]

**Proof.** If we combine (6.3) with the upper bound of Proposition 3.2, switching $T$ to $t$, we obtain that
\[
\frac{\Gamma(\sqrt{P_t f})}{P_t f} \leq \frac{K}{\alpha \beta} e^{-\alpha t} \frac{\Delta P_t(f)}{P_t f} + \frac{n\alpha(\beta - 1)^2}{8(\beta - 2)(1 - e^{-\alpha t})} e^{-2\alpha t} \leq C_3 e^{-2\alpha t},
\]

30
with \( C_3 = \frac{KC_2}{\alpha \beta} + \frac{na(\beta-1)^2}{8(\beta-2)(1-e^{-\alpha t})} \). And \( 0 \leq f \in \ell^\infty(V, \mu) \), i.e. there exist \( C > 0 \), such that \( \sup_{x \in V} f(x) \leq C \), from the heat kernel, it is easy to know that \( P_tf \leq C \), so
\[
\Gamma(\sqrt{P_tf}) \leq CC_3e^{-2\alpha t}.
\]

We consider the function \( u(x) = \frac{1}{CC_3e^{\alpha t}}\sqrt{P_f}f(x) \in V \), and we find \( \|\Gamma(u)\|_\infty \leq 1 \). From the definition of the canonical distance \( \tilde{d}(x, y) \), we obtain that
\[
|u(x) - u(y)| \leq \tilde{d}(x, y),
\]
that implies
\[
|\sqrt{P_tf(x)} - \sqrt{P_tf(y)}| \leq \sqrt{CC_3e^{-\alpha t}}\tilde{d}(x, y).
\]

According to Proposition 6.1, consider the heat kernel \( p(t, x, y) \) (due to the semigroup property of heat kernel), we obtain for \( x \in V, t \geq t_0 \),
\[
\left| \frac{\partial}{\partial t} \log p(t, x, y) \right| \leq C_0 e^{-\alpha t}, \ 0 < \alpha < K
\]
that implies for any \( x \in V, p(t, x, \cdot) \) converges when \( t \to \infty \). Let us write \( p_\infty(x, \cdot) \) as this limit.

Moreover, from Proposition 6.2 the limit \( p_\infty(x, \cdot) \) is a constant \( c(x) \). By the symmetry property of heat kernel, so that \( c(x) \) actually does not depend on \( x \). From 3 in Remark 2 we can imply that the measure \( \mu \) is finite. We may then assume \( \mu \) is probability measure, then \( p_\infty(x, \cdot) = 1 \) in the case of probability measure. From now on we assume probability measure on graph.

**Proposition 6.3.** Let \( G = (V, E) \) be a locally finite, connected graph satisfying \( \text{CDE}'(n, K) \), and for any \( x, y \in V, t > 0 \),
\[
p(t, x, y) \leq \frac{1}{\left(1 - e^{-\frac{2K}{3}t}\right)^n}.
\]

**Proof.** We apply (6.3) with \( \beta = 3 \), so \( \alpha = \frac{2K}{3} \). And consider \( p(t, x, y) \), then we obtain
\[
\frac{\partial}{\partial t} \log p(t, x, y) \geq -\frac{2nK}{3} \frac{e^{-\alpha t}}{1 - e^{-\alpha t}}.
\]

By integrating from 0 to \( \infty \), and the fact of \( p_\infty(x, \cdot) = 1 \), we have
\[
p(t, x, y) \leq \frac{1}{\left(1 - e^{-\alpha t}\right)^n}.
\]
This ends the proof. \( \square \)
6.2 Diameter bound

In this section we show that the diameter of $G$ is bounded. First we prove the lemma from Davies’ theorem on manifold. We know the fact that when $\mu$ is a finite measure, $f \in \ell^\infty(V, \mu)$ implies $f \in \ell^p(V, \mu)$ for any $p > 1$.

**Lemma 6.1.** For any $f \in \ell^\infty(V, \mu)$, if $\|P_t f\|_\infty \leq e^{M(t)} \|f\|_2$, where $M(t)$ is a continuous and decreasing function with $t$, and $\|f\|_2 = 1$, then for any $t_1 > 0$, $t > t_1$,

$$\sum_{x \in V} \mu(x) f^2(x) \ln f^2(x) \leq 2t \sum_{x \in V} \mu(x) \Gamma(f)(x) + 2M(t).$$

**Proof.** For any function $0 \leq f \in \ell^\infty(V, \mu)$, let us consider the function $(P_s f)^{p(s)}$, where $p(s)$ is a bounded and continuous function with $s$ and its value more than or equal 2, it is easy to know $(P_s f)^{p(s)} \in \ell^1(V, \mu)$, and also $(P_s f)^{p(s)} \ln P_s f, \Delta P_s f (P_s f)^{p(s)-1} \in \ell^1(V, \mu)$, so we have

$$\frac{d}{ds} \|P_s f\|^{p(s)}_{p(s)} = \frac{d}{ds} \sum_{x \in V} \mu(x)(P_s f(x))^{p(s)}$$

$$= \sum_{x \in V} \mu(x) \frac{d}{ds}(P_s f(x))^{p(s)}$$

$$= \sum_{x \in V} \mu(x) (p'(s)(P_s f(x))^{p(s)} \ln P_s f(x) + p(s)(P_s f(x))^p(P_s f(x))^{p(s)-1})$$

$$= p'(s) \sum_{x \in V} \mu(x)(P_s f(x))^{p(s)} \ln P_s f(x) + p(s) \sum_{x \in V} \mu(x) \Delta P_s f(x)(P_s f(x))^{p(s)-1}$$

If let $s = 0$ in the above inequality, and let $p(s) = \frac{2t}{t-t_1}, 0 \leq s \leq t - t_1$, with $t > t_1 > 0$, we have

$$\frac{d}{ds} \|P_s f\|^{p(s)}_{p(s)} |_{s=0} = \frac{2t}{t} \sum_{x \in V} \mu(x) f^2(x) \ln f(x) + 2 \sum_{x \in V} \mu(x) f(x) \Delta f(x).$$

If we assume $\|P_t f\|_\infty \leq e^{M(t)} \|f\|_2$, where $M(t)$ is a continuous and decreasing function with $t$, and $\|f\|_2 = 1$, by the Stein interpolation theorem, we have

$$\|P_t f\|^{p(s)}_{p(s)} \leq e^{\frac{M(t)p(s)}{t}}.$$

From this point we can obtain

$$\frac{d}{ds} \|P_s f\|^{p(s)}_{p(s)} |_{s=0} \leq \frac{2M(t)}{t},$$

for observing $\|P_s f\|^{p(s)}_{p(s)} |_{s=0} = 1$, $e^{\frac{M(t)p(s)}{t}} |_{s=0} = 1$, and

$$1 \geq \lim_{s \to 0^+} \frac{\|P_s f\|^{p(s)}_{p(s)} - 1}{e^{\frac{M(t)p(s)}{t}} - 1} = \frac{d}{ds} \|P_s f\|^{p(s)}_{p(s)} |_{s=0} \frac{t}{2M(t)}.$$
Combining with the above equality, we obtain
\[
\sum_{x \in V} \mu(x)f^2(x) \ln f^2(x) \leq 2t \sum_{x \in V} \mu(x)\Gamma(f)(x) + 2M(t), \ t > t_1.
\]

Proposition 6.4. Let \( G = (V, E) \) be a locally finite, connected graph satisfying CDE\( n, K \), for \( 0 \leq f \in \ell^\infty(V, \mu) \) such that \( \|f\|_2 = 1 \), we have
\[
\sum_{x \in V} \mu(x)f^2(x) \ln f^2(x) \leq \Phi \left( \sum_{x \in V} \mu(x)\Gamma(f)(x) \right).
\]

Proof. From Proposition 3.4, for any \( 0 \leq f \in \ell^\infty(V, \mu) \), applying the Cauchy-Schwartz inequality, we have
\[
\|P_t f\|_\infty \leq \frac{1}{(1 - e^{-\alpha t})^n} \|f\|_2.
\]
Therefor from lemma 4.1, we obtain
\[
\sum_{x \in V} \mu(x)f^2(x) \ln f^2(x) \leq 2t \sum_{x \in V} \mu(x)\Gamma(f)(x) + 2n \ln(1 - e^{-\alpha t}), \ t > t_1 > 0,
\]
by minimizing over \( t \), the right-hand side of the above inequality, we obtain
\[
\sum_{y \in V} \mu(y)f^2(y) \ln f^2(y) \leq -\frac{2}{\alpha} x \ln \left( \frac{x}{x + \alpha n} \right) + 2n \ln \left( \frac{x + \alpha n}{\alpha n} \right)
\]
\[
= 2n \left[ \left( 1 + \frac{1}{\alpha n} x \right) \ln \left( 1 + \frac{1}{\alpha n} x \right) - \frac{1}{\alpha n} x \ln \left( \frac{1}{\alpha n} x \right) \right],
\]
where \( x = \sum_{y \in V} \mu(y)\Gamma(f)(y) \), and let
\[
\Phi(x) = 2n \left[ \left( 1 + \frac{1}{\alpha n} x \right) \ln \left( 1 + \frac{1}{\alpha n} x \right) - \frac{1}{\alpha n} x \ln \left( \frac{1}{\alpha n} x \right) \right].
\]
That we obtain is what we desire. \( \square \)

We observe \( \Phi \) is a nonnegative, monotonically increasing, and concave function, that will be useful later. In order to prove the diameter bounds theorem, we first need introduce some notions on graph we will use in the following. For a positive bounded real valued function \( f \) on \( V \), let \( E(f) \) denote the entropy of \( f \) with respect to \( \mu \) defined by
\[
E(f) = \sum_{x \in V} \mu(x)f(x) \ln f(x) - \sum_{x \in V} \mu(x)f(x) \ln \left( \sum_{x \in V} \mu(x)f(x) \right).
\]
To ease the notation, we use \( \langle f \rangle = \sum_{x \in V} \mu(x)f(x) \). We will say that \( \Delta \) satisfies a logarithmic Sobolev inequality if there exists \( \rho > 0 \) such that for all \( \ell^\infty(V, \mu) \) functions \( f \),
\[
\rho E(f^2) \leq 2\langle \Gamma(f) \rangle,
\]

In general, logarithmic Sobolev inequality may be expressed equivalently by, for all function $f \in \ell^\infty(V, \mu)$ with $\langle f^2 \rangle = 1$,

$$E(f^2) \leq \Phi(\langle \Gamma(f) \rangle),$$

(6.4)

where $\Phi$ is a concave and nonnegative function on $[0, \infty)$.

**Proposition 6.5.** For any $0 \leq f \in \ell^\infty(V, \mu)$, if $\Delta$ satisfies a logarithmic Sobolev inequality, and the function $\Phi$ is nonnegative and monotonically increasing, then the diameter

$$\tilde{D} \leq \sqrt{2} \int_0^\infty \frac{1}{x^2} \Phi(x^2) dx.$$

**Proof.** For any $g \in \ell^\infty(V, \mu)$, let $g$ be such that $\|\Gamma(g)\|_\infty \leq 1$. We will apply logarithmic Sobolev inequality to the family of nonnegative function $\tilde{f} = \sqrt{\langle f^2 \rangle}$, it is easy to find $\langle \tilde{f} \rangle = 1$, where $f = e^{\frac{\lambda g}{2}} \in \ell^\infty(V, \mu)$, $\lambda \in \mathbb{R}^+$. Let $G(\lambda) = \langle e^{\lambda g} (= \langle f^2 \rangle) \rangle$ and observe that $G'(\lambda) = \langle ge^{\lambda g} (= \frac{1}{2} (f^2 \ln f^2)) \rangle$.

It is to know the left side of the logarithmic Sobolev inequality of $\tilde{f}$,

$$E(\tilde{f}) = \frac{1}{G(\lambda)} (\lambda G'(\lambda) - G(\lambda) \ln G(\lambda)),$$

it is much complicated of the right side, we should first estimate $\langle \Gamma(e^{\frac{\lambda g}{2}}) \rangle$ without diffusion property, but for symmetry, we have

$$\langle \Gamma(e^{\frac{\lambda g}{2}}) \rangle = \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (e^{\frac{\lambda g(y)}{2}} - e^{\frac{\lambda g(x)}{2}})^2$$

$$= \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (e^{\frac{\lambda g(y)}{2}} - e^{\frac{\lambda g(x)}{2}})^2 + \frac{1}{2} \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (e^{\frac{\lambda g(y)}{2}} - e^{\frac{\lambda g(x)}{2}})^2$$

$$= \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (e^{\frac{\lambda g(y)}{2}} - e^{\frac{\lambda g(x)}{2}})^2$$

$$\leq \sum_{x \in V} \sum_{y \sim x} \omega_{xy} (e^{\frac{\lambda (g(y) - g(x))}{2}}) - (g(y) - g(x))^2$$

$$\leq \frac{\lambda^2}{4} \sum_{x \in V} e^{\lambda g(x)} \sum_{y \sim x} \omega_{xy} (g(y) - g(x))^2$$

$$= \frac{\lambda^2}{2} \langle e^{\lambda g} \Gamma(g) \rangle,$$

noticing $\Gamma(g) \leq 1$, and the function $\Phi$ is monotonically increasing, then

$$\Phi(\langle \Gamma(\tilde{f}) \rangle) = \Phi \left( \frac{1}{\langle f^2 \rangle} \langle \Gamma(f) \rangle \right) \leq \Phi \left( \frac{\lambda^2}{2} \right),$$
then we obtain from the logarithmic Sobolev inequality
\[ \lambda G'(\lambda) - G(\lambda) \ln G(\lambda) \leq G(\lambda)\Phi \left( \frac{\lambda^2}{2} \right). \]

Let \( H(\lambda) = \frac{1}{\lambda} \ln G(\lambda) \), then the above inequality reads
\[ H'(\lambda) \leq \frac{1}{\lambda^2} \Phi \left( \frac{\lambda^2}{2} \right). \]
Since \( H(0) = \lim_{\lambda \to 0} \frac{1}{\lambda} \ln G(\lambda) = \langle g \rangle \), it follows that
\[ H(\lambda) = H(0) + \int_0^\lambda H'(u) du \leq \langle g \rangle + \int_0^\lambda \frac{1}{u^2} \Phi \left( \frac{u^2}{2} \right) du, \]
therefore for any \( \lambda \geq 0 \),
\[ \sum_{x \in V} \mu(x)e^{\lambda(g(x)-\langle g \rangle)} \leq \exp \left\{ \lambda \int_0^\lambda \frac{1}{u^2} \Phi \left( \frac{u^2}{2} \right) du \right\}. \quad (6.5) \]
Let \( C = \int_0^\infty \frac{1}{u^2} \Phi \left( \frac{u^2}{2} \right) du = \frac{1}{\sqrt{2}} \int_0^\infty \frac{1}{x^2} \Phi(x^2) dx \). By the above inequality applied to \( g \) and \(-g\), for every \( \lambda \geq 0 \) and every \( \varepsilon > 0 \), by Chebyshev’s inequality,
\[ \mu(\{ x \in V : |g(x) - \langle g \rangle| \geq C + \varepsilon \}) \leq \sum_{x \in V} \mu(x) \left( \frac{e^{\lambda(g(x)-\langle g \rangle)}}{e^{\lambda(C+\varepsilon)}} + \frac{e^{\lambda(-g(x)-\langle g \rangle)}}{e^{\lambda(C+\varepsilon)}} \right) \mu(x) \]
\[ \leq 2e^{-\lambda(C+\varepsilon)} e^{\lambda C} \]
\[ = 2e^{-\lambda\varepsilon} \to 0 (\lambda \to \infty), \]
that is,
\[ \|g(x) - \langle g \rangle\|_\infty \leq C, \]
The diameter bounds follows immediately by the definition of \( \tilde{D} \),
\[ \tilde{D} \leq \sqrt{2} \int_0^\infty \frac{1}{x^2} \Phi(x^2) dx. \]
That completes the proof. \( \square \)

Now we can get the final diameter theorem as follow.

**Theorem 6.6.** Let \( G = (V, E) \) be a locally finite, connected graph satisfying \( CDE'\'(n, K) \), and \( K > 0 \), then the diameter is finite, and
\[ \tilde{D} \leq 4\sqrt{3\pi} \sqrt{\frac{n}{K}}. \]

35
Proof. From Proposition 6.4 and Proposition 6.5, we obtain
\[ \tilde{D} \leq \sqrt{2} \int_{0}^{\infty} \frac{1}{x^2} \Phi(x^2)dx. \]
and, where \( \Phi(x) = 2n \left[ (1 + \frac{1}{\alpha n} x) \ln (1 + \frac{1}{\alpha n} x) - \frac{1}{\alpha n} x \ln (\frac{1}{\alpha n} x) \right], \) and \( \alpha = \frac{2K}{\alpha} \).
Since
\[ \int_{0}^{\infty} \frac{1}{x^2} \Phi(x^2)dx = \frac{1}{2} \int_{0}^{\infty} \frac{1}{x^2} \Phi(x)dx = \int_{0}^{\infty} \frac{1}{\sqrt{x}} \Phi'(x)dx = -2 \int_{0}^{\infty} \sqrt{x} \Phi''(x)dx < \infty \]
then the diameter is finite, and \( \Phi''(x) = -\frac{2n}{x(x+\alpha n)} \), a routine conclude shows
\[ -2 \int_{0}^{\infty} \sqrt{x} \Phi''(x)dx = 4\pi \sqrt{\frac{n}{\alpha}}, \]
so we completes the proof. \( \square \)

From this theorem, we can conclude that the diameter from the natural distance is finite too. For prove this, we first introduce the notation of intrinsic metric, it is the key point to associate the natural distance with the canonical distance. A metric \( \rho : V \times V \to \mathbb{R}^+ \) is called an intrinsic metric if
\[ \sum_{y \sim x} \omega_{xy} \rho^2(x, y) \leq \mu(x), \forall x \in V. \]
One can easily see that the following one is an intrinsic metric
\[ \tilde{\rho}(x, y) = \min \left\{ \sqrt{\frac{\mu(x)}{m(x)}}, \sqrt{\frac{\mu(y)}{m(y)}} \right\}, \]
where \( m(x) = \sum_{y \sim x} \omega_{xy}. \) Consider with the canonical distance, we have the following proposition.

Proposition 6.7. For any \( x \sim y, \)
\[ 2\tilde{\rho}(x, y) \leq \tilde{d}(x, y). \]

Proof. We consider the function \( f(\cdot) = \tilde{\rho}(x, \cdot) \) on \( V. \) Obviously, by the definition, \( \Gamma(f) \leq \frac{1}{2}. \)
By the definition of the canonical distance, we can conclude we desire. \( \square \)

Theorem 6.8. If a graph be a locally finite, connected, and satisfy \( CDE'(n, K) \) with \( K > 0, \) then there is constants \( c > 0, \) such that the diameter of the natural distance on the graph
\[ D \leq 2\pi \sqrt{\frac{3D_{\mu n}}{K}}. \]
Proof. From proposition 6.7, for any \( x \in V \), and any \( y \sim x \)

\[
\tilde{d}(x, y) \geq 2 \min \left\{ \sqrt{\frac{\mu(x)}{m(x)}}, \sqrt{\frac{\mu(y)}{m(y)}} \right\} \geq \frac{2}{\sqrt{D_{\mu}}}.
\]

Combining theorem 6.6, for one \( k \)-path \( x_0x_1 \cdots x_k, x_i \in V (i = 0, \cdots , k) \)

\[
4\sqrt{3\pi} \sqrt{\frac{n}{K}} \geq \tilde{d}(x_1, x_n) = \sum_{i=0}^{k-1} \tilde{d}(x_i, x_{i+1}) \geq \frac{2k}{\sqrt{D_{\mu}}},
\]

then we obtain

\[
k \leq 2\pi \sqrt{\frac{3D_{\mu}n}{K}} < \infty,
\]

it is associated with the natural distance on graph. We obtain what we desire. \( \square \)

References

[BL] D. Bakry, M. Ledoux, *A logarithmic Sobolev form of the Li-Yau gradient estimate*, Mat. Iberoamericana 22, no. 2 (2006), 683-702.

[BQ] D. Bakry, Z.M. Qian, *Harnack inequalities on a manifold with positive or negative Ricci curvature*, Rev. Mat. Iberoamericana 15 (1999), no.1, 143C179.

[BG] F. Baudoin, N. Garofalo, *Perelman’s entropy and doubling property on Riemannian manifolds*, Journal of Geometric Analysis, (2009), 1119-1131.

[BG1] F. Baudoin, N. Garofalo, *Curvature-dimension inequalities and Ricci lower bounds for sub-Riemannian manifolds with transverse symmetries*, arXiv preprint, (2011).

[BHLLMY] F. Bauer, P. Horn, Y. Lin, G. Lippner, D. Mangoubi and S.-T. Yau, *Li-Yau inequality on graphs*, arXiv:1306.2561v2, (2013), to appear in Journal of Differential Geometry.

[CG] T. Coulhon and A. Grigoryan, *Random walks on graphs with regular volume growth*, Geometric and Functional Analysis, Springer, (1998).

[D] T. Delmotte, *Parabolic Harnack Inequality and estimates of Markov chains on graphs*, Geometric and Functional Analysis, Revista Matem atica Iberoamericana, 15 (1999), 181-232.

[DB] Davies, E.B. *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, 92. Cambridge University Press, Cambridge, (1989).

[DB1] Davies, E.B. *Large derivations for Heat kernels on graphs*, Journal of London Mathematical Society, 47(1993), 65-72.
[L] M. Ledoux, *Remarks on logarithmic Sobolev constants, exponential integrability and bounds on the diameter*, Journal of Mathematics of Kyoto University, (1995).

[LY86] P. Li, S-T. Yau, *On the parabolic kernel of the Schrödinger operator*, Acta Math. 156 (1986), 153-201.

[LY10] Y. Lin, S-T. Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*, Math. Res. Lett. (2010), 343-356.

[P] Pang, M.M.H. *Heat kernels of graphs*, Journal of London Mathematical Society, 47 (1993), 50-64.

[WK] A. Weber and U. Karlsruhe, *Analysis of the physical Laplacian and the heat flow on a locally finite graph*, Journal of Mathematical Analysis and Applications, (2010), 146-158.

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