ON WEAK-STAR CONVERGENCE IN PRODUCT HARDY SPACES ON SPACES OF HOMOGENEOUS TYPE

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Abstract. A classical theorem of Jones and Journé on weak-star convergence in the Hardy space $H^1$ was generalised to the multiparameter setting by Pipher and Treil. We prove the analogous result when the underlying space is a product space of homogeneous type. The main tools we use for this setting are from recent work in papers by Chen, Li and Ward and by Han, Li and Ward.

1. Introduction

In this paper we extend to the setting of product Hardy spaces $H^1$ on spaces of homogeneous type the result that almost-everywhere convergence of a sequence of uniformly bounded $H^1$ functions implies weak-star convergence. See [PT] for the history of this result and its connections with commutators, singular integral operators, Riesz transforms, BMO, div-curl lemmas, and the theory of compensated compactness in partial differential equations.

Our main result is the following.

Theorem 1.1. Suppose that a sequence of functions $\{f_k\} \subset H^1(X_1 \times \cdots \times X_n)$ satisfies $\|f_k\|_{H^1} \leq 1$ for all $k$ and $f_k(x) \to f(x)$ for $\mu$-almost every $x \in X_1 \times \cdots \times X_n$. Then $f \in H^1(X_1 \times \cdots \times X_n)$, $\|f\|_{H^1} \leq 1$, and for all $\phi \in \text{VMO}(X_1 \times \cdots \times X_n)$,

$$\int_{X_1 \times \cdots \times X_n} f_k(x) \phi(x) \, d\mu(x) \longrightarrow \int_{X_1 \times \cdots \times X_n} f(x) \phi(x) \, d\mu(x).$$

To extend the Calderón–Zygmund singular integral operator theory to a more general setting, in the early 1970s Coifman and Weiss introduced spaces of homogeneous type. As Meyer remarked in his preface to [DH], "One is amazed by the dramatic changes that occurred in analysis during the twentieth century. . . . After many improvements, mostly achieved by the Calderón–Zygmund school, the action takes place today on spaces of homogeneous type. No group structure is available, the Fourier transform is missing, but a version of harmonic analysis is still present. Indeed the geometry is conducting the analysis." We say that $(X, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss if $d$ is a quasi-metric on $X$ and $\mu$ is a nonzero measure.
satisfying the doubling condition. To be more precise, let us begin by recalling these spaces. A quasi-metric \( d \) on a set \( X \) is a function \( d : X \times X \to [0, \infty) \) satisfying

1. \( d(x, y) = d(y, x) \geq 0 \) for all \( x, y \in X \);
2. \( d(x, y) = 0 \) if and only if \( x = y \); and
3. the quasi-triangle inequality holds: there exists a constant \( A_0 \in [1, \infty) \) such that for all \( x, y \) and \( z \in X \),

\[
d(x, y) \leq A_0[d(x, z) + d(z, y)].
\]

We define the quasi-metric ball by \( B(x, r) := \{ y \in X : d(x, y) < r \} \) for \( x \in X \) and \( r > 0 \). Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open. In this paper, we assume that

4. given a neighborhood \( N \) of a point \( x \) there is an \( \epsilon > 0 \) such that the sphere \( \{ y \in X : d(x, y) \leq \epsilon \} \) with center at \( x \) is contained in \( N \); and
5. the sphere \( \{ y \in X : d(x, y) \leq r \} \) is measurable, and the measure \( \mu(\{ y \in X : d(x, y) \leq r \}) \) is a continuous function of \( r \) for each \( x \).

We say that a nonzero measure \( \mu \) satisfies the doubling condition if there is a constant \( C_\mu \) such that for all \( x \in X \) and all \( r > 0 \),

\[
\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r)) < \infty.
\]

As noted by the reviewer of [PT] in Mathematical Reviews, since \( H^1 \) is not reflexive, the fact that \( H^1 \) is the dual of VMO does not lead to a functional analytic proof of this result using known methods.

The paper is organised as follows. In Section 2 we present some background about spaces of homogeneous type. In Section 3 we prove the one-parameter version of our result, and in Section 4 we prove the product version.

2. Preliminaries

We recall the ingredients and tools that we will use below to prove Theorem 1.1, namely systems of dyadic cubes, the orthonormal basis and wavelet expansion of Auscher and Hytönen [AH], the spaces of test functions and of distributions, the definitions from [HLW] (using these spaces) of \( H^1 \), BMO and VMO on product spaces of homogeneous type, and the duality relations between them. See [HLW] for a full account of this material.

2.1. Systems of dyadic cubes in a doubling quasi-metric space. Let \( X \) be a set equipped with a quasi-metric \( d \) and a doubling measure \( \mu \); in particular, \( (X, d, \mu) \) is a space of homogeneous type. As shown in [HK], building on [Chr], there exists a dyadic decomposition for this space \( X \). There exist positive absolute constants \( c_1, C_1 \) and \( 0 < \delta < 1 \) such that we can construct a set
of points \( \{x^k_{\alpha}\}_{k,\alpha} \) and families of sets \( \{Q^k_{\alpha}\}_{k,\alpha} \) in \( X \) satisfying the following properties:

\[
\begin{align*}
(2.1) & \quad \text{if } \ell \leq k, \text{ then either } Q^k_{\alpha} \subset Q^\ell_{\beta} \text{ or } Q^k_{\alpha} \cap Q^\ell_{\beta} = \emptyset; \\
(2.2) & \quad \text{for every } k \in \mathbb{Z} \text{ and } \alpha \neq \beta, Q^k_{\alpha} \cap Q^k_{\beta} = \emptyset; \\
(2.3) & \quad \text{for every } k \in \mathbb{Z}, X = \bigcup_{\alpha} Q^k_{\alpha}; \\
(2.4) & \quad B(x^k_{\alpha}, 1 \cdot \delta^k) \subset Q^k_{\alpha} \subset B(x^k_{\alpha}, C_1 \delta^k); \\
(2.5) & \quad \text{if } \ell \leq k \text{ and } Q^k_{\alpha} \subset Q^\ell_{\beta}, \text{ then } B(x^k_{\alpha}, C_1 \delta^k) \subset B(x^\ell_{\beta}, C_1 \delta^\ell).
\end{align*}
\]

Here for each \( k \in \mathbb{Z}, \alpha \) runs over an appropriate index set. We call the set \( Q^k_{\alpha} \) a dyadic cube and \( x^k_{\alpha} \) the center of the cube. Also, \( k \) is called the level of this cube. We denote the collection of dyadic cubes at level \( k \) by \( \mathcal{D}^k \), and the collection of all dyadic cubes by \( \mathcal{D} \). When \( Q^k_{\alpha} \subset Q^{k-1}_{\beta} \), we say \( Q^k_{\alpha} \) is a child of \( Q^{k-1}_{\beta} \) and \( Q^{k-1}_{\beta} \) is the parent of \( Q^k_{\alpha} \). Because \( X \) is a space of homogeneous type, there is a uniform constant \( N \) such that each cube \( Q \in \mathcal{D} \) has at most \( N \) children.

### 2.2. Orthonormal basis and wavelet expansion

We recall the orthonormal basis and wavelet expansion of \( L^2(X) \) due to Auscher and Hytönen \[AH\]. To state their result, we first recall the set of reference dyadic points \( x^k_{\alpha} \) as follows. Let \( \delta \) be a fixed small positive parameter (for example, as pointed out in Section 2.2 of \[AH\], it suffices to take \( \delta \leq 10^{-3} A_0^{-10} \)). For \( k = 0 \), let \( \mathcal{X}^0 := \{x^0_{\alpha}\}_\alpha \) be a maximal collection of 1-separated points in \( X \). Inductively, for \( k \in \mathbb{Z}_+ \), let \( \mathcal{X}^k := \{x^k_{\alpha}\}_\alpha \supseteq \mathcal{X}^{k-1} \) and \( \mathcal{X}^{-k} := \{x^{-k}_{\alpha}\}_\alpha \subseteq \mathcal{X}^{-(k-1)} \) be \( \delta^k \)- and \( \delta^{-k} \)-separated collections in \( \mathcal{X}^{k-1} \) and \( \mathcal{X}^{-(k-1)} \), respectively.

Lemma 2.1 in \[AH\] shows that, for all \( k \in \mathbb{Z} \) and \( x \in X \), the reference dyadic points satisfy

\[
(2.6) \quad d(x^k_{\alpha}, x^\ell_{\beta}) \geq \delta^k (\alpha \neq \beta), \quad d(x, \mathcal{X}^k) = \min_{\alpha} d(x, x^k_{\alpha}) < 2 A_0 \delta^k.
\]

Now let \( c_0 := 1, C_0 := 2 A_0 \) and \( \delta \leq 10^{-3} A_0^{-10} \). Then there exists a set of half-open dyadic cubes \( \{Q^k_{\alpha}\}_{k \in \mathbb{Z}, \alpha \in \mathcal{X}^k} \) associated with the reference dyadic points \( \{x^k_{\alpha}\}_{k \in \mathbb{Z}, \alpha \in \mathcal{X}^k} \). We consider the reference dyadic point \( x^k_{\alpha} \) as the center of the dyadic cube \( Q^k_{\alpha} \). We also identify with \( \mathcal{X}^k \) the set of indices \( \alpha \) corresponding to \( x^k_{\alpha} \in \mathcal{X}^k \).

Note that \( \mathcal{X}^k \subseteq \mathcal{X}^{k+1} \) for \( k \in \mathbb{Z} \), so that every \( x^k_{\alpha} \) is also a point of the form \( x^{k+1}_{\beta} \), and thus of all the finer levels. We denote \( \mathcal{Y}^k := \mathcal{X}^{k+1}\setminus \mathcal{X}^k \), and relabel the points \( \{x^k_{\alpha}\}_\alpha \) that belong to \( \mathcal{Y}^k \) as \( \{y^k_{\alpha}\}_\alpha \).

**Theorem 2.1** \[AH\] Theorem 7.1. Let \( (X, d, \mu) \) be a space of homogeneous type with quasi-triangle constant \( A_0 \), and let \( a := (1 + 2 \log_2 A_0)^{-1} \). There exists an orthonormal basis \( \psi^k_{\alpha}, k \in \mathbb{Z}, y^k_{\alpha} \in \mathcal{Y}^k \), of \( L^2(X) \), having exponential decay

\[
|\psi^k_{\alpha}(x)| \leq \frac{C}{\sqrt{\mu(B(y^k_{\alpha}, \delta^k))}} \exp(-\nu(\delta^{-k}d(y^k_{\alpha}, x)^a)),
\]

Holder-regularity

\[
|\psi^k_{\alpha}(x) - \psi^k_{\beta}(y)| \leq \frac{C}{\sqrt{\mu(B(y^k_{\alpha}, \delta^k))}} \left(\frac{d(x, y)}{\delta^k}\right) \eta \exp(-\nu(\delta^{-k}d(y^k_{\alpha}, x)^a))
\]
for some \( \eta \in (0, 1] \) and for \( d(x, y) \leq \delta^k \), and the cancellation property

\[
\int_X \psi^k_\alpha(x) \, d\mu(x) = 0, \quad k \in \mathbb{Z}, \quad y^k_\alpha \in \mathcal{Y}^k.
\]

Moreover,

\[
f(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{Y}^k} (f, \psi^k_\alpha) \psi^k_\alpha(x)
\]

in the sense of \( L^2(X) \).

Here \( \delta \) is a fixed small parameter, say \( \delta \leq \frac{1}{1000} A_0^{-10} \), and \( \nu > 0 \) and \( C < \infty \) are two constants that are independent of \( k, \alpha, x \) and \( y^k_\alpha \). In what follows, we also refer to the functions \( \psi^k_\alpha \) as wavelets.

### 2.3. Spaces of test functions and distributions.

We refer the reader to [HLW], Definitions 3.9 and 3.10 and the surrounding discussion, for the definitions of the space \( \hat{G} \) of product test functions and its dual space \( (\hat{G})' \) of product distributions on the product space \( X_1 \times X_2 \). In [HLW], \( \hat{G} \) is denoted by \( \hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \) and \( (\hat{G})' \) is denoted by \( \hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2)' \), where the \( \beta_i \) and \( \gamma_i \) are parameters that quantify the size and smoothness of the test functions, and \( \beta_i \in (0, \eta_i) \) where \( \eta_i \) is the regularity exponent from Theorem 2.1. (In fact, in [HLW] the theory is developed for \( \beta_i \in (0, \eta_i] \), but for simplicity here we only use \( \beta_i \in (0, \eta_i) \) since that is all we need.) We note that the one-parameter scaled Auscher–Hytönen wavelets \( \psi^k_\alpha(x)/\sqrt{\mu(B(y^k_\alpha, \delta^k))} \) are test functions, and that their tensor products \( \psi^{k_1}_{\alpha_1}(x)\psi^{k_2}_{\alpha_2}(y) \left( \frac{1}{\mu(B(y^{k_1}_{\alpha_1}, \delta^{k_1}_{\alpha_1}))} \right) \frac{1}{\mu(B(y^{k_2}_{\alpha_2}, \delta^{k_2}_{\alpha_2}))} \right)^{-1/2} \) are product test functions in \( \hat{G} \), for all \( \beta_i \in (0, \eta_i] \) and all \( \gamma_i > 0 \), for \( i = 1, 2 \). These facts follow from the theory in [HLW], specifically Definition 3.1 and the discussion after it, Theorem 3.3, and Definitions 3.9 and 3.10 and the discussion between them.

We have the following version of the reproducing formula in the product setting \( X_1 \times X_2 \).

**Theorem 2.2** ([HLW]). The reproducing formula

\[
f(x_1, x_2) = \sum_{k_1} \sum_{\alpha_1 \in \mathcal{Y}^{k_1}} \sum_{k_2} \sum_{\alpha_2 \in \mathcal{Y}^{k_2}} (f, \psi^{k_1}_{\alpha_1} \psi^{k_2}_{\alpha_2}) \psi^{k_1}_{\alpha_1}(x_1) \psi^{k_2}_{\alpha_2}(x_2)
\]

holds in both \( \hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2) \) and \( (\hat{G}(\beta_1, \beta_2; \gamma_1, \gamma_2))' \), with \( 0 < \beta_i < \eta_i \) and \( \gamma_i < \eta_i \) for \( i = 1, 2 \).

We recall from [HLW] the definitions of the Hardy space \( H^1(X_1 \times X_2) \), the bounded mean oscillation space \( \text{BMO}(X_1 \times X_2) \), and the vanishing mean oscillation space \( \text{VMO}(X_1 \times X_2) \).

**Definition 2.3** ([HLW]). The product Hardy space \( H^1 \) is defined by

\[
H^1(X_1 \times X_2) := \left\{ f \in (\hat{G})' : S(f) \in L^1(X_1 \times X_2) \right\},
\]

where \( S(f) \) is the product Littlewood–Paley square function defined as

\[
S(f)(x_1, x_2) := \left\{ \sum_{k_1} \sum_{\alpha_1 \in \mathcal{Y}^{k_1}} \sum_{k_2} \sum_{\alpha_2 \in \mathcal{Y}^{k_2}} \left| (\psi^{k_1}_{\alpha_1} \psi^{k_2}_{\alpha_2}, f) \overline{\chi}^{k_1}_{\alpha_1}(x_1) \overline{\chi}^{k_2}_{\alpha_2}(x_2) \right|^2 \right\}^{1/2},
\]

where \( \overline{\chi}^{k_i}_{\alpha_i}(x_i) := \chi^{k_i}_{\alpha_i}(x_i) \mu_i(Q^{k_i}_{\alpha_i})^{-1/2} \) and \( \chi^{k_i}_{\alpha_i}(x_i) \) is the indicator function of the dyadic cube \( Q^{k_i}_{\alpha_i} \) for \( i = 1, 2 \).
For \( f \in H^1(X_1 \times X_2) \), we define \( \|f\|_{H^1(X_1 \times X_2)} := \|S(f)\|_{L^1(X_1 \times X_2)} \).

**Definition 2.4** ([HLW]). We define the *product BMO space* as

\[
\text{BMO}(X_1 \times X_2) := \{ f \in (\mathring{G})' : C_1(f) < L^\infty \},
\]

with \( C_1(f) \) defined as follows:

\[
C_1(f) := \sup_{\Omega} \left\{ \frac{1}{\mu(\Omega)} \sum_{k_1, k_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{Y}^k, R = Q_{\alpha_1}^k \times Q_{\alpha_2}^k \subset \Omega} \left| \langle \psi_{\alpha_1}^{k_1}, \psi_{\alpha_2}^{k_2}, f \rangle \right|^2 \right\}^{1/2},
\]

where \( \Omega \) runs over all open sets in \( X_1 \times X_2 \) with finite measures.

Now we introduce the following

**Definition 2.5** ([HLW]). We define the *product vanishing mean oscillation space* \( \text{VMO}(X_1 \times X_2) \) as the subspace of \( \text{BMO}(X_1 \times X_2) \) consisting of those \( f \in \text{BMO}(X_1 \times X_2) \) satisfying the three properties

\text{(a)} \quad \lim_{\delta \to 0} \sup_{\Omega : \mu(\Omega) < \delta} \left\{ \frac{1}{\mu(\Omega)} \sum_{k_1, k_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{Y}^k, R = Q_{\alpha_1}^k \times Q_{\alpha_2}^k \subset \Omega} \left| \langle \psi_{\alpha_1}^{k_1}, \psi_{\alpha_2}^{k_2}, f \rangle \right|^2 \right\}^{1/2} = 0;

\text{(b)} \quad \lim_{N \to \infty} \sup_{\Omega : \text{diam}(\Omega) > N} \left\{ \frac{1}{\mu(\Omega)} \sum_{k_1, k_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{Y}^k, R = Q_{\alpha_1}^k \times Q_{\alpha_2}^k \subset \Omega} \left| \langle \psi_{\alpha_1}^{k_1}, \psi_{\alpha_2}^{k_2}, f \rangle \right|^2 \right\}^{1/2} = 0;

\text{(c)} \quad \lim_{N \to \infty} \sup_{\Omega \subseteq (B(x_1, N) \times B(x_2, N))^c} \left\{ \frac{1}{\mu(\Omega)} \sum_{k_1, k_2 \in Z, \alpha_1, \alpha_2 \in \mathbb{Y}^k, R = Q_{\alpha_1}^k \times Q_{\alpha_2}^k \subset \Omega} \left| \langle \psi_{\alpha_1}^{k_1}, \psi_{\alpha_2}^{k_2}, f \rangle \right|^2 \right\}^{1/2} = 0, \text{ where } x_1 \text{ and } x_2 \text{ are any fixed points in } X_1 \text{ and } X_2, \text{ respectively.}

**Theorem 2.6** ([HLW]). The following duality results hold:

\[
(H^1(X_1 \times X_2))' = \text{BMO}(X_1 \times X_2),
\]

\[
(\text{VMO}(X_1 \times X_2))' = H^1(X_1 \times X_2).
\]

### 3. Proof of Theorem 1.1 for One Parameter

To prove Theorem 1.1, paralleling the Euclidean one-parameter case, we will make use of several properties of the \( A_p \) classes on spaces of homogeneous type. These properties are collected in Lemma 3.1, Theorem 3.2, and Lemmas 3.3-3.5 below.

Let \((X, d, \mu)\) be a space of homogeneous type. A nonnegative locally integrable function \( \omega : X \to \mathbb{R} \) is said to belong to \( A_p(X) \), \( 1 < p < \infty \), if

\[
\sup_B \left( \frac{1}{\mu(B)} \int_B \omega(x) \, d\mu(x) \right) \left( \frac{1}{\mu(B)} \int_B \omega(x)^{-\frac{1}{p-1}} \, d\mu(x) \right)^{p-1} < \infty,
\]
and \( \omega \) is said to belong to \( A_1(X) \) if

\[
\sup_B \left( \frac{1}{\mu(B)} \int_B \omega(x) \, d\mu(x) \right) \left( \text{ess sup} \frac{\omega^{-1}(x)}{x \in B} \right) < \infty.
\]

**Lemma 3.1** ([C2] Lemma 4). Let \( \omega \in A_p, 1 \leq p < \infty \). There exists a constant \( C > 0 \) such that, for any subset \( E \) of \( B \),

\[
\left( \frac{\mu(E)}{\mu(B)} \right)^p \leq C \frac{\int_E \omega(x) \, d\mu(x)}{\int_B \omega(x) \, d\mu(x)}.
\]

The centered Hardy–Littlewood maximal operator \( M \) with respect to the measure \( \mu \) is defined by

\[
M f(x) := \sup_{r > 0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, d\mu(y).
\]

**Theorem 3.2** ([C2] Theorem 3). If \( \omega \in A_1 \), then \( M \) is of \( \omega \)-weak type \((1,1)\) with respect to \( \mu \); that is, there exists a constant \( C > 0 \) such that, for all \( \lambda > 0 \) and all \( f \in L^1_\omega(d\mu) \),

\[
\int_{\{x \in X : M f(x) > \lambda \}} \omega(x) \, d\mu(x) \leq C \frac{1}{\lambda} \int_X |f(x)| \omega(x) \, d\mu(x).
\]

Similarly, the uncentered Hardy–Littlewood maximal operator \( \tilde{M} \) with respect to the measure \( \mu \) is defined by

\[
\tilde{M} f(x) := \sup_{x \in B} \frac{1}{\mu(B)} \int_B |f(y)| \, d\mu(y).
\]

**Lemma 3.3.** The weight \( \omega \in A_1 \) if and only if there is a constant \( C > 0 \) such that

\[
M \omega(x) \leq C \omega(x) \quad \mu\text{-almost everywhere } x \in X.
\]

**Proof.** Suppose that there is a constant \( C > 0 \) such that \( \tilde{M} \omega(x) \leq C \omega(x) \) \( \mu\)-almost everywhere. Since \( \tilde{M} \omega(x) \) is equivalent to \( M \omega(x) \), it is clear that

\[
\frac{1}{\mu(B)} \int_B \omega(y) \, d\mu(y) \leq C \omega(x) \quad \mu\text{-almost everywhere } x \in B.
\]

Hence \( \omega \in A_1 \). Conversely, Theorem 3.2 shows that there exists \( C > 0 \) such that, for any \( \lambda > 0 \) and \( f \in L^1_\omega \),

\[
\int_{\{x \in X : \tilde{M} f(x) > \lambda \}} \omega(x) \, d\mu(x) \leq C \frac{1}{\lambda} \int_X |f(x)| \omega(x) \, d\mu(x).
\]

Suppose \( x \in B_1 \subset B_2 \). Let \( f = \chi_{B_1} \) and \( z \in B_2 \). Then

\[
\tilde{M} f(z) \geq \frac{1}{\mu(B_2)} \int_{B_2} f(y) \, d\mu(y) = \frac{\mu(B_1)}{\mu(B_2)}.
\]

The above inequality shows that \( B_2 \subset \{ x : \tilde{M} f(x) \geq \mu(B_1)/\mu(B_2) \} \). Hence,

\[
\int_{B_2} \omega(x) \, d\mu(x) \leq \int_{\{x : \tilde{M} f(x) \geq \mu(B_1)/\mu(B_2) \}} \omega(x) \, d\mu(x)
\]

\[
\leq C \frac{\mu(B_2)}{\mu(B_1)} \int_{B_1} \omega(x) \, d\mu(x).
\]

By Lebesgue’s differentiation theorem, the Lemma follows. \( \square \)
We will need the following generalization to spaces of homogeneous type of one direction of a well-known result of Coifman and Rochberg \[\text{[CR]}\].

**Lemma 3.4.** Let \( f \in L^1_{\text{loc}}(X) \) such that \( Mf(x) < \infty \) \( \mu \)-almost everywhere. Then \( (Mf)^\delta \in A_1 \) for \( 0 \leq \delta < 1 \).

**Proof.** By Lemma 3.3, it suffice to show that there exists a constant \( C \) such that, for any \( B \) and \( \mu \)-almost every \( x \in B \),
\[
\frac{1}{\mu(B)} \int_B (\widetilde{Mf})^\delta \, d\mu \leq C (\widetilde{Mf}(x))^\delta.
\]

Fix \( B = B(x_0, t_0) \) and decompose \( f \) as \( f = f_1 + f_2 \), where \( f_1 = f \chi_{2B} \) and \( f_2 = f \chi_{(2B)^c} \) with \( 2B = B(x_0, 2t_0) \). Then \( \widetilde{Mf}(y) \leq \widetilde{Mf}_1(y) + \widetilde{Mf}_2(y) \) and
\[
(\widetilde{Mf}(y))^\delta \leq (\widetilde{Mf}_1(y))^\delta + (\widetilde{Mf}_2(y))^\delta \quad \text{for} \quad 0 \leq \delta < 1.
\]

Since \( \widetilde{M} \) is weak \((1,1)\) with respect to the measure \( \mu \), Kolmogorov’s inequality shows that
\[
\frac{1}{\mu(B)} \int_B (\widetilde{Mf}_1(y))^\delta \, d\mu \leq \frac{C}{\mu(B)} \mu(B)^{1-\delta} ||f_1||_{L^1}^\delta \leq C \left( \frac{1}{\mu(B)} \int_{2B} f \, d\mu \right)^\delta \leq C (\widetilde{Mf}(x))^\delta.
\]

Now we estimate \( \widetilde{Mf}_2 \). Given \( y \in B \), for any \( B(y_0, R) \) that contains \( y \), we have \( B \subset B(y_0, A_0^4 \max\{t_0, R\}) \). If \( R < t_0 \), we have \( B(y_0, t_0) \cap B(x_0, t_0) \neq \emptyset \) and hence \( B(y_0, t_0) \subset B(x_0, A_0^4 t_0) \) which gives \( B(y_0, A_0^4 R) \subset B(x_0, 2t_0) \). Then the inequality \( \int_{B(y_0, R)} |f_2| \, d\mu > 0 \) implies \( R > \frac{t_0}{2A_0} \) that concludes \( B \subset B(y_0, 2A_0^4 R) \) when \( R < t_0 \). It is clear that \( B \subset B(y_0, 2A_0^4 R) \) when \( R \geq t_0 \). Thus,
\[
\frac{1}{\mu(B(y_0, R))} \int_{B(y_0, R)} |f_2| \leq \frac{C}{\mu(B(y_0, 2A_0^4 R))} \int_{B(y_0, 2A_0^4 R)} |f_2| \, d\mu \leq C \widetilde{Mf}(x),
\]
so that \( \widetilde{Mf}_2(y) \leq C \widetilde{Mf}(x) \) for all \( y \in B \). Therefore,
\[
\frac{1}{\mu(B)} \int_B (\widetilde{Mf}_2(y))^\delta \, d\mu(y) \leq C (\widetilde{Mf}(x))^\delta.
\]
This completes the proof. \( \square \)

**Lemma 3.5.** If \( \omega \in A_2(X) \), then \( \log \omega \in \text{BMO}(X) \).

We omit the proof of Lemma 3.5 which echoes the Euclidean version (see for example \[\text{[D]}\]).

We are ready to show the main result in the one-parameter case. We follow the proof in \[\text{[JJ]}\].

**Proof of Theorem 1.1** for one parameter. Since \( H^1(X) \) is a subspace of \( L^1(X) \), it follows from Fatou’s lemma that \( f \in L^1(X) \). To show (1.1) for all \( \phi \in \text{VMO}(X) \), by density it suffices to consider \( \phi \in \tilde{G}(\beta, \gamma) \). Fix \( \delta \in (0, \frac{1}{2A_0^4}) \) and pick \( \eta > 0 \) such that \( \eta \exp(\delta^{-1}) \leq \delta C_{\mu}^{\log_2 \delta} \) and \( \int_E |f\, d\mu \leq \delta \) whenever \( \mu(E) \leq C \eta \exp(\delta^{-1}) \). Now choose \( k \) large enough so that
\[
\mu(E_k) := \mu(\{x \in X : |f_k(x) - f(x)| > \eta\}) \leq \eta.
\]

We construct a bump function \( \tau(x) \) on \( X \), as follows. Define
\[
\tau(x) := \max \{0, 1 + \delta \log(M\chi_{E_k})(x)\}.
\]
It is clear that $0 \leq \tau(x) \leq 1$ and $\tau \equiv 1$ $\mu$-almost everywhere on $E_k$. Also, $\|\tau\|_{\text{BMO}(X)} \leq 2\delta \log(MX_{E_k})^{1/2}\|\mu\|_{\text{BMO}(X)} \leq C\delta$ due to Lemmas 3.4 and 3.5 By the weak $(1, 1)$ estimate for $M$ with respect to $\mu$, 

$$\mu(\text{supp}(\tau)) \leq C\mu(E_k) \exp(\delta^{-1}) \leq C\eta \exp(\delta^{-1}).$$

Consequently, 

$$\int_{\text{supp}(\tau)} |f| \, d\mu \leq \delta.$$ 

We now write 

$$\left| \int_X (f - f_k)\phi \, d\mu \right| \leq \left| \int_X (f - f_k)\phi(1 - \tau) \, d\mu \right| + \left| \int_X (f - f_k)\phi \tau \, d\mu \right| \leq \eta \|\phi\|_{L^1(d\mu)} + \int_{\text{supp}(\tau)} |f| \, d\mu + \int_X f_k\phi \tau \, d\mu \leq \delta + \delta + \left| \int_X f_k\phi \tau \, d\mu \right|.$$ 

The proof of (1.1) will therefore be established provided we verify 

$$\|\phi \tau\|_{\text{BMO}(X)} \leq C\delta.$$ 

Suppose $B = B(y_0, r_0)$ with $r_0 < \delta$. The Hölder regularity of $\phi$ gives 

$$\frac{1}{\mu(B)} \int_B |\phi \tau - (\phi \tau)_B| \, d\mu \leq \frac{2}{\mu(B)} \int_B |\phi \tau - \phi B \tau_B| \, d\mu \leq \frac{2}{\mu(B)} \int_B |\phi \tau - \phi B \tau| \, d\mu + \frac{2|\phi_B|}{\mu(B)} \int_B |\tau - \tau_B| \, d\mu \leq C\delta^3 + 2\|\phi\|_{L^\infty} \|\tau\|_{\text{BMO}(X)} \leq C(\delta^3 + \delta).$$ 

For $r_0 > \delta$ and $B(y_0, \delta) \cap B(x_0, \delta^{-1}) = \emptyset$, the size condition of $\phi$ yields 

$$\frac{1}{\mu(B)} \int_B |\phi \tau - (\phi \tau)_B| \, d\mu \leq \frac{2}{\mu(B)} \int_B |\phi \tau| \, d\mu \leq C\delta^\gamma.$$ 

For $r_0 > \delta$ and $B(y_0, \delta) \cap B(x_0, \delta^{-1}) \neq \emptyset$, we obtain $B(y_0, \delta^{-1}) \subset B(x_0, A_0\delta^{-1})$ and hence $\mu(B(x_0, \delta^{-1})) \leq \mu(B(y_0, A_0\delta^{-1})).$ The doubling condition shows that 

$$\mu(B(y_0, A_0\delta^{-1})) \leq C^{\log_2(A_0\delta^{-2})} \mu(B(y_0, \delta)).$$ 

Thus, 

$$\frac{1}{\mu(B)} \leq C^{\log_2(A_0\delta^{-2})} \mu(B(y_0, A_0\delta^{-1})) \leq C^{\log_2(A_0\delta^{-2})} \mu(B(x_0, \delta^{-1})) \leq \frac{C^{\log_2(A_0\delta^{-2})}}{V_1(x_0)},$$ 

and then 

$$\frac{1}{\mu(B)} \int_B |\phi \tau - (\phi \tau)_B| \, d\mu \leq \frac{2}{\mu(B)} \int_B |\phi \tau| \, d\mu \leq \frac{2C^{\log_2(A_0\delta^{-2})}}{V_1(x_0)} \mu(\text{supp}(\tau)) \leq \frac{2C^{\log_2(A_0\delta^{-2})}}{V_1(x_0)} \eta \exp(\delta^{-1}) \leq C\delta.$$
Therefore,

\[
\frac{1}{\mu(B)} \int_B |\phi \tau - (\phi \tau)_B| \, d\mu \leq C\delta.
\]

and (3.1) follows. By weak-star compactness of the ball in \( H^1 \), there exists a \( g \in H^1 \) with \( \|g\|_{H^1} \leq 1 \) and a subsequence \( \{f_{k_\ell}\}_{\ell \in \mathbb{N}} \) such that \( \{f_{k_\ell}\}_{\ell \in \mathbb{N}} \) weak-star converges to \( g \). By (2.1), we have \( \int f \phi = \int g \phi \) for all \( \phi \in \dot{G}(\beta, \gamma) \), and hence \( f = g \in H^1 \).

\[\Box\]

4. Proof of Theorem 1.1 in the product case

We begin by recalling several key tools we will use to pass from the product Euclidean setting to the setting of product spaces of homogeneous type. These tools are the random dyadic lattices, the dyadic product BMO space, the averaging theorem relating the dyadic and continuous product BMO spaces, several properties of product \( \text{bmo} \) ("little BMO"), and the construction of a product bump function \( \tau(x_1, x_2) \) on \( X_1 \times X_2 \). Then we prove Theorem 1.1 for product spaces of homogeneous type.

In [HK] Theorem 5.1 Hytönen and Kairema constructed random dyadic lattices on spaces of homogeneous type, extending an earlier result of Nazarov, Treil and Volberg [NTV]. Specifically, there exists a probability space \((\Omega, \mathbb{P})\) such that for each \( \omega \in \Omega \) there is an associated dyadic lattice \( \mathcal{D}(\omega) = \{Q^k_\alpha(\omega)\}_{k, \alpha} \) related to dyadic points \( \{x^k_\alpha(\omega)\}_{k, \alpha} \), with the properties (2.1)–(2.5) above, and the following smallness property holds: there exist absolute constants \( C, \eta > 0 \) such that

\[
\mathbb{P}\left\{ \{\omega \in \Omega : x, x^* \text{ are not in the same cube } Q \in \mathcal{D}^k(\omega)\} \right\} \leq C \left( \frac{d(x, x^*)}{\delta^k} \right)^{\eta}
\]

for all \( x, x^* \in X \), where \( \mathcal{D}^k(\omega) \) is the set of all dyadic cubes at level \( k \) in \( \mathcal{D}(\omega) \).

Fix \( \omega \in \Omega \). For a cube \( Q \in \mathcal{D}(\omega) \), let \( \text{ch}(Q) \) denote the set of all children of \( Q \in \mathcal{D}(\omega) \). From (2.1) and (2.2), we know that \( Q = \bigcup_{I \in \text{ch}(Q)} I \). For a cube \( Q \in \mathcal{D}(\omega) \), define the averaging operator \( E^\omega_Q \) by

\[
E^\omega_Q f = E^{\mathcal{D}(\omega)} Q f := \left( \int_Q f \, d\mu \right) \chi_Q,
\]

where as usual \( \int_Q f \, d\mu = \mu(Q)^{-1} \int_Q f \, d\mu \) and \( \chi_Q \) is the characteristic function of \( Q \). (We reserve the more usual name of expectation operator for the expectation \( \mathbb{E}_\omega \) over random dyadic lattices, defined below.) Define the difference operator \( \Delta^\omega_Q \) by

\[
\Delta^\omega_Q f = \Delta^{\mathcal{D}(\omega)} Q f := \left( \sum_{J \in \text{ch}(Q)} E^\omega_J f \right) - E^\omega_Q f.
\]

For convenience, we sometimes write \( E_Q \) and \( \Delta_Q \) instead of \( E^\omega_Q \) and \( \Delta^\omega_Q \). Note that for every \( x \in X \), at each level \( k \) there exists exactly one cube \( Q^k(x) \in \mathcal{D}^k(\omega) \) such that \( x \in Q^k(x) \). So for each \( k \in \mathbb{Z} \) we can define

\[
E_k f(x) := \sum_{\alpha} E^\omega_{Q^k_\alpha} f(x) = E_{Q^k(x)} f(x) \quad \text{and} \quad \Delta_k f(x) := \sum_{\alpha} \Delta^\omega_{Q^k_\alpha} f(x) = E_{k+1} f(x) - E_k f(x).
\]
For \( j = 1, 2 \), let \((\Omega_j, \mathbb{P}_j)\) be a probability space for \((X_j, \rho_j, \mu_j)\) such that for each \( \omega_j \in \Omega_j \) there is an associated dyadic lattice \( \mathcal{D}_j(\omega_j) \) satisfying properties (1)–(6). We define the dyadic product \( \text{BMO}(X_1 \times X_2) \) space via the difference operator. Let \( \Delta^\omega := \Delta^\omega_{1,1} \cup \Delta^\omega_{2,2} \) where \( \Delta^\omega_{1,1} \in \mathcal{D}_1(\omega_1) \) and \( \Delta^\omega_{2,2} \in \mathcal{D}_2(\omega_2) \). Let \( R^\omega \) denote the rectangle \( Q^\omega_{1,1} \times Q^\omega_{2,2} \).

**Definition 4.1.** Let \( f^\omega(x) = f^{(\omega_1, \omega_2)}(x_1, x_2) \) be a locally integrable function on \( X_1 \times X_2 \). We say that \( f^\omega \) belongs to the **dyadic product bounded mean oscillation space** \( \text{BMO}^{\omega_1, \omega_2} := \text{BMO}_{\mathcal{D}_1(\omega_1) \times \mathcal{D}_2(\omega_2)}(X_1 \times X_2) \) if there exists a constant \( C > 0 \) such that for every open set \( A \subset X_1 \times X_2 \),

\[
\frac{1}{\mu(A)} \sum_{R^\omega \subset A} \int_{X} |\Delta^\omega f^\omega|^2 \, d\mu \leq C^2.
\]

We define the dyadic product \( \text{BMO} \) norm \( \| f^\omega \|_{\text{BMO}^{\omega_1, \omega_2}} \) of the function \( f^\omega \) to be the infimum of \( C \) such that the inequality above holds.

**Theorem 4.2 ([CLW]).** Let \( (X_1, d_1, \mu_1) \) and \( (X_2, d_2, \mu_2) \) be spaces of homogeneous type. For \( j = 1, 2 \), let \((\Omega_j, \mathbb{P}_j)\) be a probability space, and \( \{\mathcal{D}(\omega_j)\}_{\omega_j \in \Omega_j} \) a collection of random dyadic lattices on \( X_j \), such that properties (1)–(6) hold. Let \( \{f^\omega\} \), \( \omega := (\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 \), be a family of functions with \( f^\omega \in \text{BMO}_{\mathcal{D}_1(\omega_1) \times \mathcal{D}_2(\omega_2)}(X_1 \times X_2) \) for each \( \omega \in \Omega_1 \times \Omega_2 \), such that

1. \( \omega \mapsto f^\omega \) is measurable, and
2. \( \| f^\omega \|_{\text{BMO}_{\mathcal{D}_1(\omega_1) \times \mathcal{D}_2(\omega_2)}(X_1 \times X_2)} \leq C_d \) for some constant \( C_d \) independent of \( \omega \).

Then the function \( f \) defined by the expectation

\[
f(x) := \mathbb{E}_\omega f^\omega(x)
\]

belongs to \( \text{BMO}(X_1 \times X_2) \), and \( \| \mathbb{E}_\omega f^\omega \|_{\text{BMO}(X_1 \times X_2)} \leq C C_d \).

**Definition 4.3.** A real-valued function \( f \in L^1_{\text{loc}}(X_1 \times X_2) \) is in the space \( \text{bmo}(X_1 \times X_2) \) (called “little \( \text{BMO} \)” in the literature) if its \( \text{bmo} \) norm is finite:

\[
\| f \|_{\text{bmo}(X_1 \times X_2)} := \sup_R \int_R |f(x_1, x_2) - f_R| \, d\mu_1(x_1) \, d\mu_2(x_2) < \infty.
\]

**Lemma 4.4.** If \( f \) and \( g \) belong to \( \text{bmo} \), then \( \max\{f, g\} \in \text{bmo} \).

**Lemma 4.5.** Suppose \( \Omega \) is an open set in \( X_1 \times X_2 \) with finite measure. Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be dyadic cubes in \( X_1 \) and \( X_2 \), respectively. Then

\[
\sum_{R=Q_1 \times Q_2 \in \mathcal{D}_1 \times \mathcal{D}_2, R \subset \Omega} \| \Delta_{Q_1 \times Q_2} f \|_2^2 \leq \int_{X_1} \sum_{Q_2 \in \mathcal{D}_2(\omega_2)} \| \Delta_{Q_2} f(x_1, \cdot) \|_2^2 \, d\mu_1(x_1).
\]

**Proof.** Let \( \tilde{f}(x_1, \cdot) := \sum_{Q_2 \in \mathcal{D}_2(\omega_2)} \Delta_{Q_2} f(x_1, \cdot) \) for \( x_1 \in X_1 \). Then \( \Delta_{Q_2} f(x_1, \cdot) = \Delta_{Q_2} \tilde{f}(x_1, \cdot) \). Since \( \Delta_{Q_1 \times Q_2} = \Delta_{Q_2} \otimes \Delta_{Q_2} \), we get that

\[
\Delta_{Q_1 \times Q_2} f = \Delta_{Q_1 \times Q_2} \tilde{f},
\]

and so

\[
\sum \| \Delta_{Q_1 \times Q_2} f \|_2^2 = \sum \| \Delta_{Q_1 \times Q_2} \tilde{f} \|_2^2.
\]
\[ \leq \|f\|_2^2 = \int_{X_1} \left\| \sum_{Q_2 \in D_2(\omega_2)} \Delta f(x_1, \cdot) \right\|_2^2 d\mu_1(x_1) \]

\[ = \int_{X_1} \sum_{Q_2 \in D_2(\omega_2)} \|\Delta f(x_1, \cdot)\|_2^2 d\mu_1(x_1). \]

Lemma 4.6. Suppose \( \phi \in \tilde{G}(\beta_1, \beta_2, \gamma_1, \gamma_2) \) and \( b \) is a bounded function with \( \|b\|_\infty \leq 1 \). Then, for all \( \alpha \in (0, 1) \), for each open \( \Omega \subset X_1 \times X_2 \), and for each rectangle \( R = Q_1 \times Q_2 \in D_1(\omega_1) \otimes D_2(\omega_2) \), we have

\[ \sum_{D \subset \Omega, \text{diam}(D) \leq \alpha} \|\Delta_R(\phi b)\|_2^2 \leq C(\|b\|_{bmo} + \alpha) \mu(\Omega). \]  

Proof. The proof is by iteration. For one parameter, it suffices to prove (4.3) for \( \Omega = Q_0 \), where \( Q_0 \) is a dyadic cube in \( X_1 \). Without loss of generality we may assume that \( \text{diam} Q_0 \leq \alpha \). Then

\[ \sum_{Q \subset Q_0} \|\Delta_Q(\phi b)\|_2^2 = \int_{Q_0} |\phi b(x) - (\phi b)_{Q_0}|^2 d\mu(x) \]

\[ \leq 2 \int_{Q_0} |\phi b(x) - (\phi)_{Q_0} b_{Q_0}|^2 d\mu(x) \]

\[ \leq 2 \int_{Q_0} |\phi b(x) - (\phi)_{Q_0} b_{Q_0}|^2 d\mu(x) + 2 \int_{Q_0} |(\phi)_{Q_0} b_{Q_0} - (\phi)_{Q_0} b_{Q_0}|^2 d\mu(x) \]

\[ \leq C(\|b\|_{bmo} + \alpha) \mu(\Omega) \]

by inequality (3.2). Applying Lemma 4.5 we obtain

\[ \sum_{Q_1 \in D_1(\omega_1), Q_2 \in D_2(\omega_2)} \|\Delta_{Q_1 \times Q_2} f\|^2 \leq \int_{X_1} \sum_{Q_2 \in D_2(\omega_2)} \|\Delta_{Q_1} f(x_1, \cdot)\|^2 d\mu_1(x_1) \]

\[ + \int_{X_2} \sum_{Q_1 \in D_1(\omega_1)} \|\Delta_{Q_2} f(\cdot, x_2)\|^2 d\mu_2(x_2) \]

\[ \leq C(\|b\|_{bmo}^2 + \alpha) \int_{X_1} \mu_2(\{x_2 : (x_1, x_2) \in \Omega\}) d\mu_1(x_1) \]

\[ + C(\|b\|_{bmo}^2 + \alpha) \int_{X_2} \mu_1(\{x_1 : (x_1, x_2) \in \Omega\}) d\mu_2(x_2) \]

\[ \leq 2C(\|b\|_{bmo}^2 + \alpha) \mu(\Omega). \]

Next we construct a bump function \( \tau(x_1, x_2) \) in the product setting.

Lemma 4.7. Let \( E \) be a subset of \( X_1 \times X_2 \) with finite measure, and let \( \delta \in (0, 1) \) be a given parameter. Then there exists a function \( \tau \in bmo \) such that \( \tau \equiv 1 \) on \( E \), \( \|\tau\|_{bmo} < C_{11} \delta \), and \( \mu(\text{supp} \tau) < C_{2} \delta^{2/3} \mu(E) \), where \( C_1 \) and \( C_2 \) are some absolute constants.

Proof. Let \( M_\delta \) be the strong maximal function, in which the averages are taken over arbitrary rectangles in \( X_1 \times X_2 \). A weight \( w \) is in \( A_1(X_1 \times X_2) \) if there exists a constant \( C \) such that \( M_\delta w(x) \leq Cw(x) \) for \( \mu \)-almost every \( x \in X_1 \times X_2 \).
We define the following $A_1$ weight, with $M_s^{(k)}$ denoting the $k$-fold iteration of the strong maximal function:

$$m(x_1, x_2) = K^{-1} \sum_{k=0}^{\infty} c^k M_s^{(k)} \chi_E(x_1, x_2),$$

where $K = \sum_k c^k$ and $c > 0$ is chosen to insure the convergence of the series. Then $\|m\|_2 \leq C\|\chi_E\|_2 = C\mu(E)^{1/2}$. Observe that $m = 1$ $\mu$-almost everywhere on $E$, and $m \leq 1$ $\mu$-almost everywhere outside $E$.

Define the function

$$\tau(x_1, x_2) := \max\{0, 1 + \delta \log m(x_1, x_2)\}.$$

The function $\tau$ is in $bmo$, and satisfies $\tau = 1$ $\mu$-almost everywhere on $E$. By Lemma 4.4 and the fact that $\log w \in bmo$ for every $A_1$ weight $w$, which is proved exactly as in the one-parameter Euclidean setting, we have $\|\tau\|_{bmo} \leq C\delta$.

The estimate for the size of the support of $\tau$ follows from Tchebychev’s theorem and the estimate $\|m\|_2 \leq C\mu(E)^{1/2}$.

We are ready to prove our main result for product spaces of homogeneous type. We follow the lines of the product Euclidean proof from [PT].

**Proof of Theorem 1.1 in the product case.** First note that $\mathcal{G}(\beta_1, \beta_2, \gamma_1, \gamma_2)$ is dense in $VMO(X_1 \times X_2)$. To prove Theorem 1.1 it suffices to show (1.1) for all $\phi \in \mathcal{G}(\beta_1, \beta_2, \gamma_1, \gamma_2)$.

Next, note that as shown in [HPFW], $H^1(X_1 \times X_2)$ is a subspace of $L^1(X_1 \times X_2)$. Thus, since $f_n \to f$ a.e. and $\|f_n\|_{H^1(X_1 \times X_2)} \leq 1$, by Fatou’s lemma we have that $f \in L^1(X_1 \times X_2)$ with $\|f\|_{L^1(X_1 \times X_2)} \leq 1$.

Fix $\delta \in (0, \frac{1}{2\log_2 \mu(E)})$ and pick $\eta > 0$ such that $\eta \exp(2/\delta) \leq \delta C_\mu^{\log_2 \delta} \chi_E$ and $\int |f| \, d\mu \leq \delta$ whenever $\mu(E) \leq C_2 \eta \exp(2/\delta)$, where $C_2$ is as in Lemma 4.7. Now choose $K_0$ large enough such that when $k > K_0$,

$$\mu(E_k) := \mu\{(x_1, x_2) \in X_1 \times X_2 : |f_k(x_1, x_2) - f(x_1, x_2)| > \eta\} \leq \eta.$$

Define

$$\tau(x_1, x_2) = \max\{0, 1 + \delta \log m(x_1, x_2)\},$$

where $m(x_1, x_2) = K^{-1} \sum_{k=0}^{\infty} c^k M_s^{(k)} \chi_{E_k}(x_1, x_2)$ as defined in Lemma 4.7. It is clear that $0 \leq \tau(x_1, x_2) \leq 1$ and $\tau = 1$ $\mu$-almost everywhere on $E_k$. By Lemma 4.7 we have $\tau \in bmo$ with $\|\tau\|_{bmo} \leq C_2 \delta$ and

$$\int_{\text{supp}(\tau)} |f| \, d\mu \leq \delta.$$

For every $k > K_0$, we now write

$$\int_{X_1 \times X_2} (f - f_k) \phi \, d\mu = \int_{X_1 \times X_2} (f - f_k) \phi (1 - \tau) \, d\mu + \int_{X_1 \times X_2} (f - f_k) \phi \tau \, d\mu.$$
Note that $\tau = 1$ $\mu$-almost everywhere on $E_k$. In the complement of $E_k$ we have $|f - f_k| < \eta$. Thus the first interval on the right-hand side of the above equality is bounded by $\eta \|\phi\|_{L^1(X_1 \times X_2)}$, which is in turn less than $\delta$ if $\eta$ is sufficiently small. Further, the second interval is bounded by
\[
\int_{\supp(\tau)} |f\phi| \, d\mu + \int_{X_1 \times X_2} f_k \phi \tau \, d\mu 
\leq \delta + \int_{X_1 \times X_2} f_k \phi \tau \, d\mu.
\]

The proof of (4.4) will therefore be established provided we verify
\[
(4.4) \quad \|\phi \tau\|_{\text{BMO}(X_1 \times X_2)} \leq C\delta.
\]

We will now verify (4.4) by first proving that the dyadic BMO norm of $\phi \tau$ has the required estimate, and then by using Theorem 4.2.

For every arbitrary open set $A \subset X_1 \times X_2$ with finite measure and $x \in A$, there exists a constant $r(x) < \frac{\delta}{3A_0}$ such that $B(x, r(x)) \subset A$ and then
\[
A = \bigcup_{x \in A} B(x, r(x)).
\]

By [C1] Lemma 3], there exists a countable subfamily of disjoint spheres $B(x_i, r(x_i))$ such that each sphere $B(x, r(x)), x \in A$ is contained in $B(x_i, 3A_0 r(x_i))$ for some $i \in \mathbb{N}$. Hence,
\[
\int_A |\phi \tau|^2 \, d\mu \leq \sum_{i=1}^{\infty} \int_{B(x_i, 3A_0 r(x_i))} |\phi \tau|^2 \, d\mu.
\]

Since $3A_0 r(x_i) < \delta$, we use Lemma [L4] to get
\[
\int_{B(x_i, 3A_0 r(x_i))} |\phi \tau|^2 \, d\mu = \sum_{R \subset B(x_i, 3A_0 r(x_i))} \|\Delta_R(\phi b)\|^2 \leq C(\|\tau\|_{\text{bmo}} + \delta) \mu(B(x_i, 3A_0 r(x_i))).
\]

Therefore,
\[
\int_A |\phi \tau|^2 \, d\mu(x) \leq \sum_{i=1}^{\infty} \int_{B(x_i, 3A_0 r(x_i))} |\phi \tau|^2 \, d\mu(x) \leq C\delta \sum_{i=1}^{\infty} \mu(B(x_i, 3A_0 r(x_i))).
\]

Since $\mu(B(x_i, 3A_0 r(x_i))) \leq C\mu(B(x_i, r(x_i)))$ and $\{B(x_i, r(x_i))\}_{i \in \mathbb{N}}$ are disjoint, we have
\[
\int_A |\phi \tau|^2 \, d\mu(x) \leq \sum_{i=1}^{\infty} \mu(B(x_i, 3A_0 r(x_i))) \leq C\mu(A).
\]

This completes the proof of (4.4). \qed

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