Let $\mu$ be a positive finite measure on the unit circle. The Dirichlet type space $D(\mu)$, associated to $\mu$, consists of holomorphic functions on the unit disc whose derivatives are square integrable when weighted against the Poisson integral of $\mu$. First, we give an estimate of the norm of the reproducing kernel $k_{\mu}$ of $D(\mu)$. Next, we study the notion of $\mu$-capacity associated to $D(\mu)$, in the sense of Beurling–Deny. Namely, we give an estimate of $\mu$-capacity of arcs in terms of the norm of $k_{\mu}$. We also provide a new condition on closed sets to be $\mu$-polar. Note that in the particular case where $\mu$ is the Lebesgue measure, this condition coincides with Carleson’s condition [4].

Our method is based on sharp estimates of norms of some outer test functions which allow us to transfer these problems to an estimate of the reproducing kernel of an appropriate weighted Sobolev space.

1. Introduction

Let $H^2$ denote the classical Hardy space of analytic functions on the unit disc $D$ having square summable Taylor coefficients at the origin. Every function $f \in H^2$ has non-tangential limits almost everywhere on the unit circle $T = \partial D$. We denote by $f(\zeta)$ the non-tangential limit of $f$ at $\zeta \in T$ if it exists.

Let $\mu$ be a positive finite measure on $T$, the Dirichlet type space $D(\mu)$ is the set of analytic functions $f \in H^2$, such that

$$D_\mu(f) := \frac{1}{2\pi} \int_T \int_T \frac{|f(\zeta) - f(\xi)|^2}{|\zeta - \xi|^2} |d\zeta|d\mu(\xi) < \infty,$$

The space $D(\mu)$ is endowed with the norm

$$\|f\|_{D(\mu)}^2 := \|f\|_{H^2}^2 + D_\mu(f).$$

If $d\mu(e^{it}) = 0$, then $D(\mu) = H^2$ and if $d\mu(e^{it}) = dm(t) = dt/2\pi$, the normalized arc measure on $T$, then $D(\mu)$ is the classical Dirichlet space $D$.

These spaces were introduced by Richter by examining the 2-isometries on the Hilbert spaces. A bounded operator $T$ in a Hilbert space $H$ is called 2-isometry if $T^*T - 2TT^* - I = 0$, is said to be cyclic if there exists $x \in H$ such that $\text{span}\{T^n x, n \geq 0\}$ is dense in $H$ and is called analytic if $\bigcap_{n \geq 0} T^n H = \{0\}$. In [1], Richter proved that every cyclic, analytic 2-isometry can be represented as a multiplication by $z$ on a Dirichlet type space.
\( D(\mu) \) for some positive finite measure \( \mu \). As consequence [11, 12] Richter gave an analogue of Beurling’s theorem for the Dirichlet space.

1.1. Reproducing kernels. The reproducing kernel plays an important role in the study of Hilbert spaces of analytic functions. In particular, it allows to determine the rate of growth of functions near the boundary and its tangential behavior; their properties are closely related to embedding theorems, sampling and interpolation sets, and other topics.

Let \( P[\mu] \) be the Poisson integral of the positive finite measure \( \mu \) on \( T \)

\[
P[\mu](z) = \int_T \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad z \in \mathbb{D}.
\]

In the following theorem, we provide an asymptotic estimate of the reproducing kernel \( k^{\mu} \) of \( D(\mu) \) on the diagonal.

**Theorem 1.** Let \( \mu \) be a finite positive measure on \( T \). We have

\[
k^{\mu}(z, z) \asymp 1 + \int_0^{|z|} \frac{dr}{(1 - r) P[\mu](r z/|z|) + (1 - r)^2},
\]

where the implied constants are absolute.

Let us recall that Shimorin [15] proved that all Dirichlet type spaces have complete Nevanlinna–Pick reproducing kernels. As an important consequence (see [2, §9.4] and [14, Theorem 1]), each sequence \( Z = \{z_n\} \subset \mathbb{D} \) satisfying Shapiro–Shields condition \( \sum_{z \in Z} 1/k^{\mu}(z, z) < \infty \) is a zero set of \( D(\mu) \). Theorem 1 allows to give examples of zero sets of \( D(\mu) \).

To prove the lower estimates of \( k^{\mu}(z, z) \), we establish a sharp norm estimates of some outer functions which peak near \( z \). This allows us to transfer our problem to an estimation of the norm of the kernel of an appropriate weighted Sobolev space. In fact and roughly speaking, Theorem 1 says that \( k^{\mu}(z, z) \asymp K_\varphi(1 - |z|, 1 - |z|) \) where \( K_\varphi \) is the reproducing kernel of the weighted Sobolev space defined by

\[
W^2(\varphi) := \{ f \in C((0, 2\pi)) : f(x) = f(1) + \int_0^{2\pi} g(t) dt, \ g \in L^2((0, 2\pi), \varphi dt) \},
\]

where \( \varphi(t) = tP[|\mu|((1 - t)z/|z|) + t^2. \)

1.2. Capacity. The Dirichlet type space is closely related to some notions of potential theory. Let \( D^h(\mu) \) be the harmonic version of \( D(\mu) \) given by

\[
D^h(\mu) := \{ f \in L^2(T) : \|f\|^2_{\mu} := \|f\|^2_{L^2(T)} + D_\mu(f) < \infty \}.
\]

\( D^h(\mu) \) is a Dirichlet space in the sense of Beurling–Deny [3]. Some aspects of the potential theory associated to the general Dirichlet spaces were studied in several papers (see for instance [8]). In this paper we will focus on the notion of capacity. We recall at first, the
definition of capacity in the sense of Beurling–Deny. Let $U$ be an open subset of the unit circle. The $c_\mu$-capacity of $U$ is defined by

$$c_\mu(U) := \inf \{ \|u\|_\mu^2 : u \in D^h(\mu), \ u \geq 0 \text{ and } u \geq 1 \text{ a.e. on } U \}.$$  \hspace{1cm} (1)

As usual we define the $c_\mu$-capacity of any subset $F \subset \mathbb{T}$ by

$$c_\mu(F) = \inf \{ c_\mu(U) : U \text{ open}, \ F \subset U \}.$$  

Since the $L^2$ norm dominates the Dirichlet–type norm, it is completely obvious that sets having $c_\mu$–capacity 0 have Lebesgue measure 0. We say that a property holds $c_\mu$–quasi-everywhere ($c_\mu$–q.e.) if it holds everywhere outside a set of $c_\mu$–capacity 0. So, $c_\mu$–q.e. implies a.e.. A closed set of capacity zero will be called, throughout this paper, $\mu$–polar set. If $d\mu(e^{it}) = dt/2\pi$, the normalized arc measure on $\mathbb{T}$, then $D(\mu)$ is the classical Dirichlet space $D$ and $c_\mu$ is comparable to the logarithmic capacity, see [10, Theorem 14] and [1, Theorem 2.5.5].

Our first result on $\mu$-capacity gives an estimate of capacity of arcs in terms of the kernel. More precisely we have:

**Theorem 2.** Let $I \subset \mathbb{T}$ be the arc of length $|I| = 1 - \rho$ with the midpoint at $\zeta \in \mathbb{T}$. Then

$$c_\mu(I) \lesssim \frac{1}{k_\mu(\rho\zeta, \rho\zeta)},$$

where the implied constants are absolute.

As a consequence,

$$c_\mu(\{e^{i\theta}\}) = 0 \iff \int_0^1 \frac{dr}{(1 - r)P[\mu](re^{i\theta}) + (1 - r)^2} = \infty.$$ \hspace{1cm} (2)

In the sequel we will suppose that $E$ is a closed set which has Lebesgue measure zero and $\mu$ is a finite positive measure on $\mathbb{T}$. Now our goal is to give sufficient condition on $E$ to be $\mu$–polar. Let us introduce the local modulus of continuity of $\mu$ on $E$ which will play a crucial role in this paper. It is defined by

$$\rho_{\mu,E}(t) := \sup \{ \mu(\zeta e^{-it}, \zeta e^{it}) : \zeta \in E \}.$$ \hspace{1cm} (3)

Note that $\rho_{\mu,T} = \rho_\mu$ is the classical modulus of continuity of $\mu$. Let us write

$$E_t := \{ \zeta \in \mathbb{T} : d(\zeta, E) \leq t \},$$

where $d$ denotes the distance with respect to arc-length, and denote by $|E_t|$ the Lebesgue measure of $E_t$. We can express the function $|E_t|$ in terms of

$$N_E(t) := 2 \sum_j 1_{\{|I_j| > 2t\}},$$

where $(I_j)$ are the components of $\mathbb{T}\setminus E$, as follows

$$\int_0^t N_E(s)ds = |E_t|.$$
In Theorem 5.4, we give sufficient conditions on a closed subset \( E \), in terms of \( \rho_{\mu,E} \) and \( N_E \), to be \( \mu \)-polar. To illustrate this theorem we give here some of its corollaries.

(i) If

\[
\int_0^\pi \frac{dt}{\int_0^t (\rho_{\mu}(s)N_E(s)/s)ds} = +\infty,
\]

then \( c_{\mu}(E) = 0 \).

This result can be considered as an extension of Carleson’s Theorem \([4, \text{section IV, Theorem } 2]\). In fact if \( \mu = m \) is the Lebesgue measure then \( \rho_{\mu,E}(t) = t \) and \( c_{\mu} \approx c \) (\( c \) is the logarithmic capacity). We obtain Carleson’s theorem which says that if \( \int_0^\pi dt/|E_t| = \infty \), then \( c(E) = 0 \).

(ii) Suppose that \( \rho_{\mu,E}(t) = O(t^\alpha) \) with \( 1 \leq \alpha < 2 \). If

\[
\int_0^\pi \frac{dt}{t^{\alpha-1}|E_t|} = +\infty,
\]

then \( c_{\mu}(E) = 0 \).

(iii) If \( \rho_{\mu,E}(t) = O(t^\alpha) \) with \( \alpha > 2 \), we have \( c_{\mu}(E) = 0 \).

Note also that if \( t^\alpha = O(\rho_{\mu,E}(1)(t)) \) with \( \alpha < 1 \), then by \([2]\), \( c_{\mu}(\{1\}) > 0 \).

The proof of Theorem 5.4 uses an idea analogous to the proof of Theorem \([3]\). However, our test functions must peak on the whole set \( E \) and the desired weighted Sobolev spaces will depend on \( \mu \) and \( E \). In fact, we prove that there is no bounded point evaluation at 0 for \( W^2(\varphi) \) (where \( \varphi \) depends on \( \rho_{\mu,E} \) and \( N_E \)), then \( c_{\mu}(E) = 0 \). Note finally, that there is no bounded point evaluation at 0 for \( W^2(\varphi) \) if and only if \( \lim_{t \to 0^+} K_{\varphi}(t,t) = \infty \).

The plan of the paper is the following. In the next section we recall two formulas of the Dirichlet type norm; we also give punctual estimates of some outer functions. In Section 3 we give norm estimates of our test functions. In Section 4 we give diagonal asymptotic estimates of reproducing kernel. In section 5, we prove the announced results on capacity.

Throughout the paper, we use the following notations:

- \( A \lesssim B \) means that there is an absolute constant \( C \) such that \( A \leq CB \).
- \( A \asymp B \) if both \( A \lesssim B \) and \( B \lesssim A \).
- \( C(x_1, \ldots, x_n) \) denote a constant which depends only on variables \( x_1, \ldots, x_n \).
2. Preliminaries

2.1. Norm formulas. In this subsection we recall some results about norm formulas in Dirichlet type spaces which will be used in what follows.

For a finite positive measure \( \mu \) on \( \mathbb{T} \), the harmonic Dirichlet space \( \mathcal{D}^h(\mu) \) consists of functions \( f \in L^2(\mathbb{T}) \) such that

\[
\mathcal{D}_\mu(f) := \int_{\mathbb{T}} \mathcal{D}_\xi(f) d\mu(\xi) < \infty,
\]

where \( \mathcal{D}_\xi(f) \) is the local Dirichlet integral of \( f \) at \( \xi \in \mathbb{T} \) given by

\[
\mathcal{D}_\xi(f) := \int_{\mathbb{T}} \frac{|f(e^{it}) - f(\xi)|^2}{|e^{it} - \xi|^2} dt.
\]

The Douglas’ formula, see [6, Theorem 7.1.3], expresses the Dirichlet integral of a function \( f \) in terms of the Poisson transform of \( \mu \), namely

\[
\mathcal{D}_\mu(f) = \int_{\mathbb{D}} |\nabla P[f]|^2 P[\mu] dA, \quad f \in \mathcal{D}^h(\mu),
\]

where \( dA(z) = dx dy / \pi \) stands for the normalized area measure in \( \mathbb{D} \). In particular, if \( f \in \mathcal{D}(\mu) = \mathcal{D}^h(\mu) \cap H^2 \), Douglas’ Formula becomes

\[
\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P[\mu](z) dA(z) < \infty.
\]

Another useful formula, due to Richter and Sundberg [13, Theorem 3.1], gives the local Dirichlet integral of function \( f \) in terms of their zeros sequence, their singular measure and the modulus of their radial limit. We will need, throughout this paper, the Richter–Sundberg formula mainly for outer functions. Recall that outer functions are given by

\[
f(z) = \exp \int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z} \log \varphi(\zeta) \frac{|d\zeta|}{2\pi}, \quad (z \in \mathbb{D}),
\]

where \( \varphi \) is a positive function such that \( \log \varphi \in L^1(\mathbb{T}) \). Note that the radial limit of \( f \), noted also by \( f \), exists a.e. and \( |f| = \varphi \) a.e. on \( \mathbb{T} \).

Let \( f \in H^2 \) be an outer function such that \( f(\zeta) \) exists at \( \zeta \in \mathbb{T} \). We have

\[
\mathcal{D}_\zeta(f) = \int_{\mathbb{T}} \frac{|f(\lambda)|^2 - |f(\zeta)|^2 - 2|f(\zeta)|^2 \log |f(\lambda)/f(\zeta)| |d\lambda|}{|\lambda - \zeta|^2}.
\]  

2.2. Punctual estimates of test functions. The result obtained in this subsection will be used in the proof of the lower estimate of the kernel.
Lemma 2.1. Let $1/2 < r = 1 - a < 1$ and let $I_k = [e^{ia_k}, e^{ia_{k+1}}]$ with $a_0 = 0$, $a_k = 2^k a$ ($k \geq 1$). Let $N$ be the integer such that $2^N a \leq \pi < 2^{N+1} a$, then

$$\sum_{k=0}^{N-1} (k + 1) \varpi(r, I_k, \mathbb{D}) \asymp 1,$$

where $\varpi(r, I_k, \mathbb{D})$ denotes the harmonic measure of $I_k$ at $r$ in $\mathbb{D}$.

Proof. Without loss of generality, we may suppose that $2^N a = \pi$. Note that

$$\sum_{k=0}^{N-1} (k + 1) \varpi(r, I_k, \mathbb{D}) \geq \sum_{k=0}^{N-1} \varpi(r, I_k, \mathbb{D}) \asymp 1.$$

For the reverse inequality, let $g(z) = \log 1/|1 - rz|$. Since $g$ is harmonic in the neighbourhood of $\mathbb{D}$,

$$g(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|1 - rz|^2} \log \frac{1}{|1 - re^{i\theta}|} d\theta.$$

So,

$$g(r) = \log \frac{1}{1 - r^2} = \sum_{k=0}^{N-1} \frac{1}{\pi} \int_{a_k}^{a_{k+1}} \frac{1 - r^2}{|1 - re^{-i\theta}|^2} \log \frac{1}{|1 - re^{i\theta}|} d\theta.$$

(5)

For $k = 0, \ldots, N-1$, and $\theta \in (a_k, a_{k+1})$, we have

$$\frac{1}{|1 - re^{i\theta}|} \asymp \frac{1}{2^k} \frac{1}{1 - r}.$$

By (5), we get

$$\log \frac{1}{1 - r^2} = \log \frac{1}{1 - r} - \log 2 \sum_{k=0}^{N-1} k \varpi(r, I_k, \mathbb{D}) + O(1)$$

and our result follows. \hfill \square

Let $w : (0, \pi) \to (0, +\infty)$ be a continuous positive function such that $\log w \in L^1(\mathbb{T})$. As before $f_w$ denote the outer function satisfying

$$|f_w(e^{it})| = w(|t|) \quad \text{a.e on} \ (-\pi, \pi).$$

(6)

Proposition 2.2. Let $w : (-\pi, \pi) \to (0, +\infty)$ be an even continuous positive decreasing function such that $\log w \in L^1(\mathbb{T})$. Suppose that $w(x) \leq 2w(2x)$. Let $f_w$ be an outer function given by (6). Then

$$w(1 - r) \lesssim |f_w(r)|, \quad 0 \leq r < 1.$$
Proof. Let \( a \), \( I_k \) and \( N \) as in lemma 2.1 and suppose that \( a_N = \pi \). We have

\[
|f_w(r)| = \exp \left\{ \sum_{k=0}^{N-1} \frac{1}{\pi} \int_{I_k} \frac{1 - r^2}{|1 - re^{i\theta}|^2} \log w(\theta) d\theta \right\}
\]

\[
\geq \exp \left\{ \sum_{k=0}^{N-1} \log \omega(2^{k+1}a) \frac{1}{\pi} \int_{I_k} \frac{1 - r^2}{|1 - re^{i\theta}|^2} d\theta \right\}
\]

\[
\geq \exp \left\{ \sum_{k=0}^{N-1} (\log \omega(a) - (k+1) \log 2) \frac{1}{\pi} \int_{I_k} \frac{1 - r^2}{|1 - re^{i\theta}|^2} d\theta \right\}
\]

\[
\geq \exp \left\{ \log w(a) \sum_{k=0}^{N-1} \mathcal{W}(r, I_k, D) - \log 2 \sum_{k=0}^{N-1} (k+1) \mathcal{W}(r, I_k, D) \right\}.
\]

We obtain from Lemma 2.1 that \( w(1 - r) \lesssim |f_w(r)| \). The case \( a_N < \pi \) can be treated in the same way by taking into account the interval \([e^{iaN}, e^{i\pi}]\).

2.3. Regularization lemma. Let \( \mu \) be a positive finite measure on \( \mathbb{T} \), we set \( d\mu(s) = d\mu(e^{is}) \). Denote by

\[
\hat{\mu}(s) = \mu([e^{-i\pi}, e^{i\pi}]) \quad (0 \leq s \leq \pi) \quad \text{and} \quad \hat{\mu}(s) = \hat{\mu}(\pi) \quad (s > \pi).
\]

Let

\[
F_\mu(x) = \int_{-\pi}^{\pi} \frac{x^2}{x^2 + s^2} d\mu(s) \quad x > 0.
\]

(7)

Note that \( F_\mu \) is increasing and \( F_\mu(x)/x^2 \) is decreasing. We extend \( F_\mu \) at the origin by \( F_\mu(0) = F_\mu(0^+) \). In the following lemma we collect some elementary properties of \( F_\mu \) which will be used in the sequel.

Lemma 2.3. Let \( \nu \) be a positive finite measure on \( \mathbb{T} \). We have the following

1. \( F_\nu(x) \simeq xP[\nu](1 - x), \) for \( x > 0, \)

2. \( \hat{\nu}(x) \lesssim F_\nu(x) \) for \( x \geq 0, \)

3. \( \int_{x \leq |s| \leq \pi} \frac{d\nu(s)}{s^2} \lesssim \frac{F_\nu(x)}{x^2}, \) for \( x > 0, \)

4. If \( h \) is a positive monotone function on \( (0, \pi) \). Then

\[
\int_{-a}^{a} h(|x|) d\nu(x) \leq \int_{0}^{a} \frac{h(x)}{x} \hat{\nu}(2x) dx,
\]

5. If \( \hat{\nu}(2x) \leq c\hat{\nu}(x) \) for some constant \( c < 4, \) then \( F_\nu(x) \lesssim \frac{\hat{\nu}(x)}{4 - c}. \)
Proof. (1), (2) and (3) are obvious. To prove (4) suppose that $h$ is a decreasing function. Clearly if $\nu(\{0\}) > 0$, then (4) is obvious. So, suppose that $\nu(\{0\}) = 0$. We have
\[
\int_{-a}^{a} h(|x|)\nu(x) = \sum_{n \geq 0} \int_{2^{-n}a}^{2^{-(n+1)}a} h(x)(\nu(x) + \nu(-x)) \leq \sum_{n \geq 0} h(2^{-n}a)\tilde{\nu}(2^{-n}a) \leq \int_{0}^{a} \frac{h(x)}{x}\tilde{\nu}(2x)dx.
\]

Analogue argument works if $h$ is increasing.

Finally to prove (5), suppose that $\tilde{\nu}(2x) \leq c\tilde{\nu}(x)$ with $c < 4$. We have $\tilde{\nu}(2^{n+1}x) \leq c^{n}\tilde{\nu}(x)$ and
\[
\int_{|t| \geq x} \frac{\nu(t)}{t^2} dt = \sum_{n \geq 0, 2^nx \leq |t| \leq 2^{n+1}x} \frac{\nu(t)}{t^2} \leq \sum_{n \geq 0} \frac{\tilde{\nu}(2^{n+1}x)}{2^{2n}x^2} \leq \frac{4c}{4-c} \frac{\tilde{\nu}(x)}{x^2}.
\]
So
\[
F_{\nu}(x) \leq \tilde{\nu}(x) + \int_{|\theta| > x} \frac{x^2}{x^2 + \theta^2}d\nu(\theta) \leq \tilde{\nu}(x) + x^2 \int_{\theta > x} \frac{d\nu(\theta)}{\theta^2} \leq \frac{\tilde{\nu}(x)}{4 - c}.
\]

3. Norm estimate of test functions

3.1. Norm estimate of analytic test functions. The purpose of this subsection is to give estimate of norms of some outer functions which play an important role in what follows.

The following lemma is the first step to prove Theorem 3.2.

Lemma 3.1. Let $w : [0, \pi] \to (0, +\infty)$ be a $C^1$ decreasing convex function such that $w(x) \leq 2w(2x)$. Suppose that $x^2|w'(x)|$ is increasing and let $f_w$ be the outer function given by (6). Then
\[
D_{\mu}(f_w) \lesssim J_1 + J_2 + J_3,
\]
where
\[
J_1 := \int_{x=0}^{\pi} \int_{y=0}^{x} |w'(y)|w(y)\frac{|w'(x)|}{w(x)} \hat{\mu}(y) dx dy,
\]
\[
J_2 := \int_{s=-\pi}^{\pi} w'(s)^2 s d\mu(s),
\]
\[
J_3 := \int_{x=0}^{\pi} \int_{y=x}^{\pi} x \cdot |w'(y)|w(y)\frac{|w'(x)|}{w(x)} \left( \int_{y \leq |s| \leq \pi} \frac{d\mu(s)}{s^2} \right) dx dy.
\]
Proof. Without loss of generality, we may assume that \(d\mu(s) = d\mu(-s)\). By Richter-Sundberg formula (4) we have

\[
\mathcal{D}_{e^{is}}(f_w) = \frac{8}{2\pi} \int_{t=0}^{\pi} \int_{x=s}^{t} \int_{y=s}^{x} w'(y)w(y) \frac{w'(x)}{w(x)} dy dx dt \frac{d\mu(s)}{|e^{is} - e^{it}|^2}.
\]

So,

\[
\mathcal{D}_\mu(f_w) = \frac{16}{2\pi} \int_{t=0}^{\pi} \int_{x=s}^{t} \int_{y=s}^{x} w'(y)w(y) \frac{w'(x)}{w(x)} dy dx dt \frac{d\mu(s)}{|e^{is} - e^{it}|^2}
\]

\[
= \int_{t=0}^{\pi} \int_{s=0}^{2s} \cdots + \int_{t=0}^{s/2} \int_{s=0}^{2s} \cdots + \int_{t=0}^{s/2} \int_{s=0}^{\pi} \cdots \quad \text{(11)}
\]

To complete the proof we will estimate each term separately.

If \(2s \leq t \leq \pi\), we have \(|t - s| \geq t/2\),

\[
\mathcal{I}_1 \lesssim \int_{t=0}^{\pi} \int_{x=0}^{t} \int_{y=0}^{x} |w'(y)|w(y) \frac{|w'(x)|}{w(x)} dy dx dt \frac{d\mu(s)}{t^2}
\]

\[
\lesssim \int_{t=0}^{\pi} \int_{x=0}^{t} \int_{y=0}^{x} |w'(y)|w(y) \frac{|w'(x)|}{w(x)} \left( \int_{|s|}^{\pi} d\mu(s) \right) dt dy dx
\]

Since \(w(2x) \asymp w(x)\) and \(w'(2x) \asymp w'(x)\), we have

\[
\mathcal{I}_2 \lesssim \int_{t=0}^{\pi} |w'(s)|^2 s d\mu(s).
\]

If \(0 \leq t \leq s/2\), then \(|t - s| \geq s/2\),

\[
\mathcal{I}_3 \lesssim \int_{t=0}^{s} \int_{x=0}^{t} \int_{y=0}^{x} |w'(y)|w(y) \frac{|w'(x)|}{w(x)} dy dx \frac{d\mu(s)}{s^2} dt
\]

\[
\lesssim \int_{t=0}^{s} \int_{x=0}^{t} \int_{y=0}^{x} |w'(y)|w(y) \frac{|w'(x)|}{w(x)} \left( \int_{y \leq |s| \leq \pi} \frac{d\mu(s)}{s^2} dt \right)
\]

\[
\lesssim \int_{x=0}^{s} \int_{y=0}^{x} |w'(y)|w(y) \frac{|w'(x)|}{w(x)} \left( \int_{y \leq |s| \leq \pi} \frac{d\mu(s)}{s^2} \right) x dx dy.
\]

\[\square\]

Theorem 3.2. Let \(w : [0, \pi] \to (0, +\infty)\) be a \(C^1\) decreasing convex function such that \(w(x) \leq 2w(2x)\). Suppose that \(x^2|w'(x)|\) is increasing and let \(f_w\) be the outer function given by (9). Then

\[
\mathcal{D}_\mu(f_w) \lesssim \|F_\mu w'\|_\infty \|w\|_\infty,
\]

where \(F_\mu\) is given by (7).
Proof. By (3) of Lemma 2.3 we have $\hat{\mu} \leq F_\mu$. Now Lemma 3.1 gives

$$\mathcal{J}_3 \lesssim \int_{x=0}^{\pi} \int_{y=x}^{\pi} |w'(y)|w(y) |w'(x)| \frac{F_\mu(y)}{y^2} xdydx$$

$$\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi} \int_{y=x}^{\pi} |w'(x)| \frac{w(y)x}{w(x)y^2} dydx$$

$$\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi} \int_{y=x}^{\pi} |w'(x)| \frac{x}{y^2} dydx$$

$$\lesssim \|F_\mu w'\|_\infty \|w\|_\infty.$$  

Note that $x|w'(x)| \lesssim w(x)$ for all $x \in [0, \pi]$. Indeed, since $w(2x) \asymp w(x)$ and $|w'(2x)| \asymp |w'(x)|$, it suffices to prove the inequality for $x \in [0, \pi/2]$. We have

$$w(x) \geq \int_{x}^{\pi} t^2 |w'(t)| \frac{dt}{t^2} \geq x^2 |w'(x)| \left( \frac{1}{x} - \frac{1}{\pi} \right) \geq \frac{x}{2} |w'(x)|.$$  

So, again by Lemma 2.3 we get

$$\mathcal{J}_1 \lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi} \int_{y=0}^{x} |w'(x)| \frac{w(y)}{w(x)x} dydx$$

$$\lesssim \|F_\mu w'\|_\infty \int_{x=0}^{\pi} \int_{y=x}^{\pi} \int_{u=y}^{\pi} |w'(x)||w'(u)| \frac{dydx}{w(x)x}$$

$$+ \|F_\mu w'\|_\infty \int_{x=0}^{\pi} \int_{y=0}^{x} |w'(x)| \frac{w(\pi)}{w(x)x} dydx$$

$$= \mathcal{J}_{12} + \mathcal{J}_{22}.$$  

We have

$$\mathcal{J}_{12} \lesssim \|F_\mu w'\|_\infty \int_{u=0}^{\pi} \int_{x=0}^{\pi} \int_{y=0}^{\min(u,x)} |w'(x)||w'(u)| \frac{dx dy du}{w(x)x}$$

$$\lesssim \|F_\mu w'\|_\infty \left( \int_{u=0}^{\pi} \int_{x=0}^{\pi} \frac{|w'(x)||w'(u)|}{w(x)} dx du + \int_{u=0}^{\pi} \int_{x=u}^{\pi} \frac{u|w'(x)||w'(u)|}{xw(x)} dx du \right)$$

$$\lesssim \|F_\mu w'\|_\infty \left( \int_{x=0}^{\pi} |w'(x)| dx + \int_{u=0}^{\pi} \int_{x=u}^{\pi} \frac{u|w'(x)||w'(u)|}{xw(x)} dx du \right)$$

$$\lesssim \|F_\mu w'\|_\infty \left( \|w\|_\infty + \int_{u=0}^{\pi} \int_{x=u}^{\pi} \frac{u|w'(u)|}{x^2} dx du \right)$$

$$\lesssim \|F_\mu w'\|_\infty \|w\|_\infty.$$ 

Since $w$ is decreasing, we get

$$\mathcal{J}_{22} \leq \|F_\mu w'\|_\infty \int_{x=0}^{\pi} \int_{y=0}^{x} |w'(x)| \frac{dx}{x} dy$$

$$\leq \|F_\mu w'\|_\infty \|w\|_\infty.$$
Finally, applying (4) of Lemma 2.3 with \( d\nu(s) = sd\mu(s) \), we have \( \hat{\nu}(t) \leq t\hat{\mu}(t) \) and
\[
J_2 \lesssim \int_0^\pi \frac{w'(t)^2}{t} \hat{\nu}(2t)dt \lesssim \|F_{\mu}w'||\infty \|w\|\infty.
\]

\[\square\]

3.2. Norm estimate of test functions in \( D^h(\mu) \). Our goal here is to give an estimate of the norm of some distance functions in \( D^h(\mu) \) (For analytic distance functions see [7]). The result of this subsection will be used in the proof of Theorem 5.4.

Let \( E \) be a closed subset of \( \mathbb{T} \), \( \mu \) be a positive finite measure and denote by \( \rho_{\mu,E} \) the local modulus of continuity of \( \mu \) on \( E \) given by (3). Note that \( \rho_{\mu,E}(\{1\}) = \hat{\mu}(t) \). Recall that \( N_E(t) := 2\sum_j 1_{|I_j|>2t} \), where \( (I_j) \) are the components of \( \mathbb{T}\setminus E \).

**Lemma 3.3.** Let \( \Omega : (0,\pi] \to \mathbb{R}^+ \) be a positive decreasing function, then
\[
\int_{\mathbb{T}} \Omega(\text{dist}(\zeta,E))d\mu(\zeta) \lesssim \int_0^1 \Omega(t)\rho_{\mu,E}(t) t N_E(t)dt.
\]

**Proof.** Write \( \mathbb{T}\setminus E = \cup_n I_n \), where \( (I_n) = (e^{i\alpha_n},e^{i\beta_n}) \) are the components of \( \mathbb{T}\setminus E \). Let \( d\mu_n(t) = d\mu(t+\alpha_n) + d\mu(\beta_n-t) \), By Lemma 2.3
\[
\int_{I_n} \Omega(\text{dist}(\zeta,E))d\mu(\zeta) \sim \int_0^{|I_n|/2} \Omega(t)\hat{\mu}_n(t) t N_E(t)dt
\]
\[
\lesssim \int_0^{|I_n|/2} \Omega(t)\rho_{\mu,E}(t) t N_E(t)dt.
\]

Summing over all \( I_n \), we get
\[
\int_{\mathbb{T}} \Omega(\text{dist}(\zeta,E))d\mu(\zeta) \lesssim \int_0^1 \Omega(t)\rho_{\mu,E}(t) t N_E(t)dt,
\]
and the proof is complete. \[\square\]

**Lemma 3.4.** Let \( E \) be a closed subset of \( \mathbb{T} \). Let \( w \) be a convex decreasing function and let \( \Omega(\zeta) = w(d(\zeta,E)) \). Then
\[
\mathcal{D}_\mu(\Omega) \lesssim \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3,
\]
where
\[
\mathcal{I}_1 := \int_0^\pi \int_0^\pi \frac{(w(t) - w(t+s))^2}{s^2} \rho_{\mu,E}(t) t N_E(t)dsdt,
\]
\[
\mathcal{I}_2 = \int_0^\pi w'(t)^2 \rho_{\mu,E}(2t) N_E(t)dt,
\]
\[
\mathcal{I}_3 := \int_{t=0}^\pi \int_{s=t}^\pi \frac{(w(t) - w(t+s))^2}{s^2} \rho_{\mu,E}(s) s N_E(t)dsdt.
\]
Proof. Set $\delta = d(\zeta, E)$ and $\delta' = d(\zeta', E)$. By Lemma 3.3 we have

$$J_1 = \frac{1}{2\pi} \int_T \int_{|\zeta| \leq \delta} \frac{[w(\delta) - w(\delta')]^2}{|\zeta - \zeta'|^2} d\zeta |d\mu(\zeta')|$$

$$\leq \int_T \int_{|\zeta| \leq \delta} \frac{[w(\delta) - w(\delta') + |\zeta - \zeta'|]^2}{|\zeta - \zeta'|^2} d\zeta |d\mu(\zeta')|$$

$$\leq \int_T \int_{|\zeta| \leq \delta} \frac{[w(\delta) - w(\delta') + |\zeta - \zeta'|]^2}{|\zeta - \zeta'|^2} d\zeta |d\mu(\zeta')|$$

and

$$J_2 = \frac{1}{2\pi} \int_T \int_{|\zeta| \leq \delta} \frac{[w(\delta) - w(\delta')]^2}{|\zeta - \zeta'|^2} d\zeta |d\mu(\zeta')|$$

Clearly we have

$$J_{21} \lesssim \int_T w'(\delta) \rho_{\mu,E}(2\delta) |d\zeta| \lesssim \int_0^\pi w'(t)^2 \rho_{\mu,E}(2t) N_E(t) dt.$$ 

With the same calculation, as in Lemma 3.3, we have

$$J_{22} \lesssim \int_{t=0}^\pi \int_{s=0}^\pi \frac{(w(t) - w(t + s))^2}{s^2} \rho_{\mu,E}(s) N_E(t) ds dt.$$ 

Since $D_\mu(w) = J_1 + J_2$, we get our result. \qed

For a positive increasing function $\psi$ such that $\psi(0) = 0$, we set

$$M_{\psi,E}(s) = \max \left( \int_0^s \frac{\psi(t)}{t} N_E(t) dt, \frac{\psi(s)}{s} |E_s| \right),$$

(8)where $E_t = \{\zeta \in E : \text{dist}(\zeta, E) \leq t\}$. If $\psi$ is concave, then $\psi(x)/x$ is decreasing and

$$\frac{\psi(s)}{s} |E_s| \leq \int_0^s N_E(t) dt \leq \int_0^s \frac{\psi(t)}{t} N_E(t) dt = M_{\psi,E}(s).$$

And if $\psi$ is convex then $\psi(x)/x$ is increasing, so

$$\int_0^s \frac{\psi(t)}{t} N_E(t) dt \leq \int_0^s \frac{\psi(s)}{s} N_E(t) dt = \frac{\psi(s)}{s} |E_s| = M_{\psi,E}(s).$$

The function $\psi$ is called $\alpha$–admissible if $\psi$ is concave or convex and $\psi(s)/s^\alpha$ is decreasing for some $\alpha > 0$. Now we can state the main result of this subsection
**Theorem 3.5.** Let $E$ be a closed subset of $\mathbb{T}$. Let $w$ be a convex decreasing function and let $\Omega(\zeta) = w(d(\zeta, E))$. Suppose that there exists an $\alpha$–admissible function $\psi$, with $\alpha < 2$, such that $\rho_{\mu,E}(s) \leq \psi(s)$. Then

$$D_\mu(\Omega) \leq C(\alpha)\|w'M_{\psi,E}\|_{\infty}\|w\|_{\infty}.$$ 

**Proof.** We apply Lemma 3.4. An analogue calculation, as in the proof of Theorem 3.2, gives

$$I_1 + I_2 \lesssim \|w'M_{\rho_{\mu,E,E}}\|_{\infty}\|w\|_{\infty}.$$

Now we consider the integral $I_3$. We have

$$I_3 = 2\int_{t=0}^{\pi} \int_{s=t}^{\pi} \int_{u=t}^{t+s} \int_{v=t}^{v+s} w'(u)w'(v)\frac{\rho_{\mu,E}(s)}{s^3} N_E(t)dvudsdt \lesssim \int_{t=0}^{\pi} \int_{s=t}^{\pi} \int_{u=t}^{2s} \int_{v=t}^{2s} |w'(u)||w'(v)|\frac{\rho_{\mu,E}(s)}{s^3} N_E(t)dvudsdt \lesssim \int_{v=0}^{\pi} |w'(v)|\frac{\psi(v)}{v^2} \int_{u=0}^{v} |w'(u)||E_u|dudv \lesssim \int_{v=0}^{\pi} |w'(v)|\frac{1}{v^{2-\alpha}} \int_{u=0}^{v} |w'(u)|\frac{\psi(u)}{u^\alpha} |E_u|dudv \leq C(\alpha) \sup_u \left(|w'(u)|\frac{\psi(u)}{u}|E_u|\right) \int_{v=0}^{\pi} |w'(v)|\frac{1}{v^{2-\alpha}} dv \leq C(\alpha) \sup_u \left(|w'(u)|\frac{\psi(u)}{u}|E_u|\right) \|w\|_{\infty}.$$ 

**Corollary 3.6.** Let $E$ be a closed subset of $\mathbb{T}$. Let $w$ be a convex decreasing function and let $\Omega(\zeta) = w(d(\zeta, E))$. Suppose that there exists an $\alpha$–admissible function $\psi$, with $\alpha < 2$, such that $\rho_{\mu,E}(s) \leq \psi(s)$. Then

1. $D_\mu(\Omega) \leq C(\alpha) \sup_{t \geq 0} |w'(t)| \int_t^{\pi} t^{\alpha-1} N_E(t)dt \|w\|_{\infty}$, if $0 < \alpha \leq 1$,
2. $D_\mu(\Omega) \leq C(\alpha) \sup_{t \geq 0} |w'(t)t^{\alpha-1}| |E_t| \|w\|_{\infty}$, if $1 \leq \alpha < 2$,
3. $D_\mu(\Omega) \leq C(\alpha) \sup_{t \geq 0} |w'(t)| \log t |E_t| \|w\|_{\infty}$, if $\alpha = 2$,
4. $D_\mu(\Omega) \leq C(\alpha, h) \sup_{t \geq 0} |w'(t)h(t)| |E_t| \|w\|_{\infty}$, if $\alpha > 2$, where $h$ is a positive increasing function such that $h(0) = 0$ and $\int_0^\infty ds/h(s) < \infty$.

**Proof.** (1) and (2) are direct consequences of Theorem 3.5. Now we prove (3). The proof of Theorem 3.5 gives

$$I_3 \lesssim \int_{v=0}^{\pi} |w'(v)| \log \frac{1}{v} \int_{u=0}^{v} |w'(u)||E_u|dudv,$$

and we get our estimate.

Finally we prove (4). Since $|E_t| \to 0$, then there exists a positive increasing function $h,$
\( h(0) = 0 \), such that \( \int_0^\infty ds/h(s) < \infty \). Again, by the proof of Theorem 3.5 we have
\[
\mathcal{I}_3 \lesssim C(\alpha) \int_{v=0}^{\pi} |w'(v)| \int_{u=0}^{\nu} \frac{h(u)}{h(v)} |w'(u)||E_u|dudv,
\]
which gives the desired estimate. \( \square \)

4. Kernel Estimate

In this section we will prove Theorem 1. The reproducing kernel \( k^\mu \) of \( D(\mu) \) is defined by,
\[
f(z) = \langle f, k^\mu(\cdot,z) \rangle, \quad f \in D(\mu), \quad z \in \mathbb{D}.
\]
So,
\[
k^\mu(z,z) = \sup\{ |f(z)|^2 : f \in D(\mu), \|f\|_\mu^2 \leq 1 \}. \tag{9}
\]
It follows obviously that for |\( z \)| \( \leq 1/2 \) we have
\[
k^\mu(z,z) \approx 1 + \int_0^{|z|} \frac{dr}{(1-r)P[\mu](rp/|z|) + (1-r)^2}.
\]
By Littlewood–Paley identity, we have
\[
\|f\|^2_\mu = \|f\|^2_{H^2} + D_\mu(f) = |f(0)|^2 + \int_\mathbb{D} |f'(w)|^2 [|| \log |w|| + P[\mu](w)]dA(w) \leq |f(0)|^2 + \int_\mathbb{D} |f'(w)|^2 [(1-|w|) + P[\mu](w)]dA(w).
\]
Let \( f \in D(\mu) \setminus \{0\} \) and let \( f = I_f O_f \) be the inner–outer factorization of \( f \). Then \( O_f \in D(\mu) \) and \( D_\mu(O_f) \leq D_\mu(f) \) (see \[13\]). Thus by (9), we get
\[
k^\mu(z,z) = \sup\{ |f(z)|^2 : f \in D(\mu) \text{ outer function and } \|f\|_\mu \leq 1 \}. \tag{10}
\]
This observation will be useful in the proof of the lower estimate.

4.1. Proof of the upper estimate. Let \( z = \rho \in [1/2, 1) \). By Lemma 2.3 it suffices to prove that
\[
k^\mu(\rho, \rho) \lesssim 1 + \int_{1-\rho}^{1} \frac{dx}{F_\mu(x) + x^2}.
\]
Let \( f \in D(\mu) \), since \( f \in H^2 \),
\[
|f(iy)| \leq \frac{\|f\|_{H^2}}{\sqrt{1-y}} \leq \sqrt{2}\|f\|_{H^2}, \quad 0 < y < 1/2.
\]
So, for \( 0 < y < 1/2 \), we have
\[
|f(\rho + i(1-\rho)y)| = |f(iy)| + \int_0^{\rho} f'(t + i(1-t)y)dt \lesssim \int_0^{\rho} |f'(t + i(1-t)y)|dt + \|f\|_{H^2}.
\]
let $\Delta$ be the triangle with vertices $-i/2$, 1, $i/2$ and let $\Delta_\rho = \{x + iy \in \Delta : 0 \leq x \leq \rho \}$. By change of variables $w = u + iv = t + i(1 - t)y$, we get

$$\frac{1}{1 - \rho} \int_{-1/2}^{1/2} |f(\rho + iy)|dy = \int_{-1/2}^{1/2} |f(\rho + i(1 - \rho)y)|dy$$

$$\lesssim \int_0^\rho \int_{-1/2}^{1/2} |f'(t + i(1 - t)y)|dydt + \|f\|_{H^2}$$

$$\lesssim \int_{\Delta_\rho} |f'(w)| \frac{dudv}{1 - u} + \|f\|_{H^2}$$

$$\lesssim D_\mu(f)^{1/2} \left[ \int_{\Delta_\rho} \frac{dA(w)}{(1 - u)^2 + (1 - u)P[\mu](u)} \right]^{1/2} + \|f\|_{H^2}$$

$$\lesssim D_\mu(f)^{1/2} \left[ \int_0^\rho \frac{du}{(1 - u)^2 + (1 - u)P[\mu](u)} \right]^{1/2} + \|f\|_{H^2}$$

$$\lesssim D_\mu(f)^{1/2} \left[ \int_{1 - \rho}^1 \frac{dx}{F_\mu(x) + x^2} \right]^{1/2} + \|f\|_{H^2}. \quad (11)$$

Denote by $D(\lambda, r)$ the disc of radius $r$ centered at $\lambda$. Since $D(\rho, (1 - \rho)/4) \subset \{z = x + iy : |x - \rho| \leq (1 - \rho)/4 \text{ and } |y| \leq (1 - x)/4\}$, by (11) and the subharmonicity of $|f|$ we obtain

$$|f(\rho)| \lesssim \frac{1}{(1 - \rho)^2} \int_{x=\rho}^{\rho+\frac{1-x}{4}} \left( \int_{y=\frac{1-x}{2}}^{\frac{1-x}{2}} |f(x + iy)|dy \right)dx$$

$$\lesssim D_\mu(f)^{1/2} \left[ \int_{\frac{1}{4}(1 - \rho)}^1 \frac{dx}{F_\mu(x) + x^2} \right]^{1/2} + \|f\|_{H^2} \quad (12)$$

Now from (9), we get

$$k^\mu(\rho, \rho) \leq 1 + \int_{1 - \rho}^1 \frac{dx}{F_\mu(x) + x^2} \leq 1 + \int_0^{|z|} \frac{dr}{(1 - r)P[\mu](rz/|z|) + (1 - r)^2}.$$
As usual $g$ will be denoted by $f'$. Equipped with the following norm
\[ \|f\|_{W^\infty(\varphi)} = \|f'\|_{\infty}, \]
$W^\infty(\varphi)$ is a Banach space. It becomes a topological Banach algebra, if and only if,
\[ \int_0^{2\pi} dx \varphi(x) < \infty. \]
We say that $f$ is regular, and write $f \in R$, if $f$ is a $C^1$ convex decreasing function on $[0, 2\pi]$, satisfying $f(2t) \leq 2f(t)$ and $t^2|f'(t)|$ is increasing.

Our goal is to estimate
\[ \gamma_{\varphi}(a) = \sup \{ f^2(a) : f \in W^\infty(\varphi) \cap R, \|f\|_{W^\infty(\varphi)} \|f\|_{\infty} + \|f\|^2_2 \leq 1 \}. \]
First we will examine the hilbertian case
\[ W^2(\varphi) = \{ f \text{ of the form (13) : } \|f\|^2_{W^2(\varphi)} = \|f\|^2_2 + \int_0^{2\pi} |f'(t)|^2 \varphi(t) dt < \infty \}. \]
We also need the following subspace of $W^2(\varphi)$
\[ W^2_0(\varphi) = \{ f \text{ of the form (13) : } f(2\pi) = 0, \|f\|^2_{W^2_0(\varphi)} = \int_0^{2\pi} |f'(t)|^2 \varphi(t) dt < \infty \}. \]

Clearly, evaluations at points of $]0, 2\pi]$ define continuous linear functionals on $W^2(\varphi)$ and on $W^2_0(\varphi)$. Let $K_\varphi, L_\varphi$ be the reproducing kernels of $W^2(\varphi)$ and of $W^2_0(\varphi)$ respectively. One can give, with an elementary calculation, the expression of the reproducing kernel $L_\varphi$ of $W^2_0(\varphi)$. Indeed we have
\[ L_\varphi(t, s) = \begin{cases} \int_t^{2\pi} \frac{dx}{\varphi(x)} & t \geq s, \\ L_\varphi(s, s) & t \leq s. \end{cases} \]
The estimates of $K_\varphi$, on the diagonal is given by
\[ K_\varphi(a, a) \asymp 1 + L_\varphi(a, a) = 1 + \int_a^{2\pi} \frac{dx}{\varphi(x)}. \]
It means that
\[ \sup \{ f(a)^2 : f \in W^2(\varphi), \|f\|^2_{W^2(\varphi)} \leq 1 \} \asymp 1 + \int_a^{2\pi} \frac{dx}{\varphi(x)}. \]
The following proposition will be used several times in what follows.

**Proposition 4.1.** Suppose that $t^2/\varphi(t)$ is increasing and $\varphi(t) \geq t^2$. Let $a < 1/2$, we have
\[ \gamma_{\varphi}(a) \asymp K_\varphi(a, a). \]
Proof. Since \( \|f\|_{W^2(\varphi)} \leq \|f\|_{W^\infty(\varphi)}\|f\|_\infty \), then \( \gamma_\varphi(a) \leq K_\varphi(a,a) \).

Conversely, let \( f = f_0 + 1 \), where

\[
f_0(x) = \int_{2\pi \frac{x+a}{2\pi+a}}^{2\pi} \frac{ds}{\varphi(s)}, \quad 0 < x \leq 2\pi.
\]

Clearly \( f \in \mathcal{R} \). Since \( \varphi(2t) \asymp \varphi(t) \), we have

\[
\|f_0\|_2^2 = \int_0^{2\pi} f_0(t)^2 dt \\
\asymp \int_0^{\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\varphi(u)\varphi(v)} dudvdt \\
\asymp \int_0^{a} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\varphi(u)\varphi(v)} dudvdt + \int_0^{\pi} \int_0^{2\pi} \int_0^{2\pi} \frac{1}{\varphi(u)\varphi(v)} dudvdt. \\
\asymp \int_a^{2\pi} \int_0^{2\pi} \frac{a}{\varphi(u)v^2} dudv + \int_0^{2\pi} \int_0^{2\pi} \frac{u}{\varphi(u)v^2} dudv. \\
\asymp \|f_0\|_\infty.
\]

Then we obtain

\[
K_\varphi(a,a) \lesssim \frac{f(a)^2}{\|f\|_{W^\infty(\varphi)}\|f\|_\infty + \|f\|_2^2} \leq \gamma_\varphi(a).
\]

And the proof is complete. \( \square \)

4.2.2. Proof of the lower estimates. Let \( z = r \in ]1/2, 1[ \) and let \( w \in \mathcal{R} \). We consider the outer function \( f_w \) given by

\[
|f_w(e^{it})| = w(|t|), \quad \text{ a.e on } [-\pi, \pi].
\]

Let \( \varphi(x) = F_\mu(x) + x^2 \). By (10), Proposition 2.2 and Theorem 3.2 we have

\[
k_\mu(r, r) \geq \frac{|f_w(r)|^2}{\|f_w\|_{D(\mu)}^2} \\
\geq \frac{w^2(1-r)}{\|w\|_{W^\infty(F_\mu)}\|w\|_\infty + \|w\|_2^2} \\
\geq \frac{w^2(1-r)}{\|w\|_{W^\infty(\varphi)}\|w\|_\infty + \|w\|_2^2}.
\]

This implies that

\[
k_\mu(r, r) \gtrsim \gamma_\varphi(1-r).
\]

By Proposition 4.1, we obtain the result.
5. Capacity

Let $\mu$ be a positive finite measure on $\mathbb{T}$ and let $c_\mu$ be the capacity given by (1)

The capacity $c_\mu$ is a Choquet capacity [4, 9] and so for every borelian set of $\mathbb{T}$ we have

$$c_\mu(E) = \sup\{c_\mu(K) : K \text{ compact }, K \subset E\}.$$  

Note that $c_\mu$ satisfies the weak-type inequality. Namely:

$$c_\mu(\{\zeta \in \mathbb{T} : |f(\zeta)| \geq t \text{ $c_\mu$-q.e.}\}) \leq \frac{\|f\|^2_{\mathcal{D}^h(\mu)}}{t^2}, \quad f \in \mathcal{D}^h(\mu).$$

As a consequence of this inequality we have the following properties.

**Proposition 5.1.** The following properties are satisfied

- Let $E \subset \mathbb{T}$ be a Borel set and let $\mathcal{M}_\mu(E) := \{f \in \mathcal{D}(\mu) : g|E = 0 \text{ $c_\mu$-q.e.}\}$. Then the set $\mathcal{M}_\mu(E)$ is closed in $\mathcal{D}(\mu)$.

- If $f \in \mathcal{D}(\mu)$ is cyclic for $\mathcal{D}(\mu)$ then $f$ is outer function and $c_\mu(Z_\mathbb{T}(f)) = 0$.

- Every function $f \in \mathcal{D}(\mu)$ has non-tangential limits $c_\mu$-q.e on $\mathbb{T}$, more precisely

  The radial limit $\lim_{r \to 1^-} f(r\zeta)$ exists and is finite for every $f \in \mathcal{D}^h(\mu)$ if and only if $c_\mu(\zeta) > 0$.

**Proof.** See [5, 9]. \(\square\)

Now we will give the proof of the estimate of the capacity of arcs.

**Proof Theorem 2** Suppose that $\zeta = 1$. Let $w \in \mathcal{R}$ and let $f_w$ be the outer function satisfying

$$|f_w(e^{it})| = w(|t|), \quad \text{a.e on } [-\pi, \pi].$$

It’s clear that $w(|I|) \lesssim w(x)$ for $|x| \leq 2|I|$. We have

$$\gamma_\varphi(|I|) \lesssim 1/c_\mu(I).$$

By proposition 4.1 we obtain

$$c_\mu(I) \lesssim 1/k_\mu(\rho, \rho), \quad (\rho = 1 - |I|).$$

For the reverse inequality note that

$$c_\mu(I) = \inf\{\|f\|^2_\mu : f \in C^1, 0 \leq f \leq 1 \text{ and } f = 1 \text{ on } I\}.$$

Consider the function $u \in \mathcal{D}^h(\mu)$ such that $0 \leq u \leq 1$ and $u|_I = 1$. Hence $u \in \mathcal{D}^h(\mu) \cap C^1$. We have $P[u](1 - |I|) \asymp 1$. Let $\rho = 1 - |I|$, a similar argument, as in the proof of [12], gives

$$P[u](\rho) \lesssim \mathcal{D}_\mu(u)^{1/2} \sqrt{k_\mu(\rho, \rho)} + \|u\|_{L^2(\mathbb{T})},$$

So $k_\mu(\rho, \rho)^{-1/2} \lesssim \mathcal{D}_\mu(u)^{1/2}$, and

$$c_\mu(I) \geq \frac{1}{k_\mu(\rho, \rho)}.$$  \(\square\)
As an immediate consequence, we obtain

**Corollary 5.2.** Let \( \lambda \in \mathbb{T} \).

\[
c_\mu(\{\lambda\}) = 0 \iff \int_0^1 \frac{dx}{(1-x)P[\mu](x\lambda) + (1-x)^2} = \infty.
\]

Now we will give some sufficient conditions on a closed subset of \( \mathbb{T} \) to be \( \mu \)-polar. Let \( E \) be a closed subset of \( \mathbb{T} \). We define \( n_\varepsilon(E) \), the \( \varepsilon \)-covering number of \( E \), to be the smallest number of the closed arcs of length \( 2\varepsilon \) that cover \( E \). Note that

\[
\varepsilon n_\varepsilon(E) \leq |E\varepsilon| \leq 4\varepsilon n_\varepsilon(E), \quad 0 < \varepsilon \leq \pi.
\]

Let

\[
k_\mu(r) = \inf\{k^\mu(r\zeta, r\zeta) : \zeta \in \text{supp} \mu\}.
\]

It’s easy to see that \( k_\mu \) is unbounded if and only if, for each \( \zeta \in \mathbb{T} \), we have \( c_\mu(\zeta) = 0 \). In this case one can prove easily, by the sub-additivity property of capacity, that

**Corollary 5.3.** Let \( E \) be a closed subset of \( \mathbb{T} \) such that \( \lim_{r \to 1^-} k_\mu(r) = \infty \). If \( n_\varepsilon(E) = o(k_\mu(1-\varepsilon)) \), \( \varepsilon \to 0 \),

then \( c_\mu(E) = 0 \).

Recall that for a positive increasing function \( \psi \) on \((0, 2\pi)\) such that \( \psi(0) = 0 \), we set

\[
M_{\psi,E}(s) = \max \left( \int_0^s \frac{\psi(t)}{t} N_E(t) dt, \frac{\psi(s)}{s} |E_s| \right), \quad s \in (0, 2\pi)
\]

Now we can state the main result of this section.

**Theorem 5.4.** Let \( E \) be closed subset of \( \mathbb{T} \) such that \( \rho_{\mu,E} \leq \psi \) where \( \psi \) is \( \alpha \)-admissible for some \( \alpha < 2 \). If

\[
\int_0^\pi \frac{dt}{M_{\psi,E}(t)} = +\infty,
\]

then \( c_\mu(E) = 0 \).

**Proof.** Note that \( M_{\psi,E} \) is given by (5), so \( M_{\psi,E}(s) \) is increasing. Let \( a > 0 \). By the definition of capacity and using Theorem 3.5 and Proposition 4.1 we have

\[
K_{M_{\psi,E}}(a, a) \asymp \gamma_{M_{\psi,E}}^2(a) \lesssim \frac{1}{c_\mu(E)}.
\]

When \( a \) goes to zero, we get

\[
\int_0^\pi \frac{dt}{M_{\psi,E}(t)} \lesssim \frac{1}{c_\mu(E)}.
\]

And the proof is complete. \( \square \)
Remarks. Now we give some examples:

(i) Let $E$ be a closed subset of $\mathbb{T}$ and let $\widetilde{M}_{\mu,E}(t) = \int_0^t (\rho_{\mu}(s)N_E(s)/s)ds$. If
\[
\int_0^\pi \frac{dt}{\widetilde{M}_{\mu,E}(t)} = \infty
\]
then $c_\mu(E) = 0$.

(ii) If $\mu = m$ is the Lebesgue measure, then $c_\mu$ is comparable to the logarithmic capacity and $\widetilde{M}_{\mu,E}(s) = |E_s|$. And theorem [5.3] says that if $\int_0^\infty dt/|E_t|$ diverges then $c(E) = 0$. This result is due to Carleson [4, Theorem 2, p.30].

(iii) Let $K$ be a closed subset of $\mathbb{T}$ such that $\rho_{\mu,K}(s) = O(s^{1+\beta})$ for some $0 < \beta$. If $\beta < 1$, then every subset of $K$ with Hausdorff dimension less than $\beta$ is $\mu$-polar. If $\beta > 1$, then every subset $E$ of $K$ with $|E| = 0$ is $\mu$-polar. The following measures $d\mu(\zeta) = d(\zeta,K)^\beta dm(\zeta)$ provide such examples.

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