NEW INEQUALITIES FOR THE GENERALIZED KARCHER MEAN

MOHAMMED SABABHEH, HAMID REZA MORADI AND ZAHRA HEYDARBAYGI

Abstract. Recently, Pálfia introduced a generalized Karcher mean as a solution of an operator equation. In this article, we present several relations for this new mean. In particular, we investigate the behavior of this generalized mean when filtered through positive linear maps, thus its information monotonicity is revealed, and operator monotone function.

1. Introduction

Let $\mathcal{M}_n$ be the set of all $n \times n$ matrices over the complex number field $\mathbb{C}$ and $I$ stands for the identity matrix. For Hermitian matrices $A, B$ we write $A \geq B$ or $B \leq A$ to mean that $A - B$ is positive semidefinite. In particular, $A \geq 0$ indicates that $A$ is positive semidefinite. If $A$ is positive definite, that is, positive semidefinite and invertible, we write $A > 0$.

For a positive probability vector $w = (w_1, \ldots, w_n)$ and positive definite matrices $A = (A_1, \ldots, A_n)$, the Karcher mean $\Lambda(w; A)$ is the unique positive solution of

$$\sum_{i=1}^{n} w_i \log \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) = 0. \tag{1.1}$$

We call (1.1) the Karcher equation (see [11]).

In [8], Lim and Pálfia introduced the notion of matrix power mean of positive definite matrices of some fixed dimension. The matrix power mean $P_t (w; A)$ is defined by the unique positive definite solution of the following non-linear equation:

$$X = \sum_{i=1}^{n} w_i (X^t_{#} A_i), \quad t \in (0, 1] \tag{1.2}$$

where $A_{#} B = A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^t A^{1/2}$ is the $t$-weighted geometric mean of $A$ and $B$. For $t \in [-1, 0)$, it is defined by $P_t (w; A) = P_{-t} (w; A^{-1})^{-1}$, where $A^{-1} = (A_1^{-1}, \ldots, A_n^{-1})$. As $t \to 0$, the power mean $P_t$ coincides with the Karcher mean $\Lambda$.

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Pálfia [13] generalized the operator equation (1.2) to the following form

(1.3) \[ \sum_{i=1}^{n} w_i g \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) = 0 \]

where \( w \) is a probability vector and \( g \) is an operator monotone function on \((0, \infty)\) with \( g(1) = 0 \) and \( g'(1) = 1 \).

Of course, the Karcher and the power means can be obtained by setting \( g(x) = \log x \) and \( g(x) = \frac{x^t - 1}{t} \) in (1.3), respectively. In what follows \( \sigma_g(w; A) \) denotes the solution \( X \) of (1.3).

Let \( \mathcal{M} \) denote the set of all operator monotone functions on \((0, \infty)\), and let

\[ \mathcal{L} = \{ g \in \mathcal{M} \mid g(1) = 0 \text{ and } g'(1) = 1 \} \]

Very recently, Yamazaki [14, Lemma 4] showed the following order among \( \sigma_g(w; A) \), weighted harmonic and arithmetic means

(1.4) \[ \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \leq \sigma_g(w; A) \leq \sum_{i=1}^{n} w_i A_i. \]

The proof of (1.4) is based on the observation that when \( g \in \mathcal{L} \), it follows that \( 1 - x^{-1} \leq g(x) \leq x - 1 \).

In the same paper, the following extension of Ando-Hiai inequality has been shown: Let \( g \in \mathcal{L} \), \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of positive definite matrices and \( w = (w_1, \ldots, w_n) \) be a weight vector. Then the implication

(1.5) \[ \sigma_g(w; A) \leq I \Rightarrow \sigma_{g_p}(w; A^p) \leq I \]

holds for all \( p \geq 1 \), where \( g_p(x) = p g \left( x^{\frac{1}{p}} \right) \).

Throughout this paper we assume that \( g \in \mathcal{L} \). Our target in this article is to present generalizations and counterparts of (1.4) and (1.5) via Kantorovich constant \( K(h, 2) = \frac{(h+1)^2}{4h} \). In applications, we give an analogous result of [7] for \( n \)-tuple of positive definite matrices. Our results extend the results appearing in [6] to the context of the solution of the generalized Karcher equation (GKE) and present new generalizations that reflect the behavior of these means under positive linear maps and operator monotone functions.

Further, we present a natural extension of the inequality [5]

(1.6) \[ \langle (A_x^v B)x, x \rangle \leq \langle Ax, x \rangle^v \langle Bx, x \rangle \]

valid for any vector \( x \in \mathcal{H} \), \( 0 \leq v \leq 1 \) and positive definite matrices \( A, B \). Many other results generalizing the action of operator monotone functions on two matrices will be presented too.
2. REVERSES OF (1.4) AND THEIR REFINEMENTS

In this section we present the reversed versions of (1.4) first, then we prove refinements using the well known Kantorovich inequality and its refinement.

**Proposition 2.1.** Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be an \( n \)-tuple of positive definite matrices with \( m \leq A_i \leq M (i = 1, \ldots, n) \) for some scalars \( 0 < m < M \) and \( w = (w_1, \ldots, w_n) \) be a weight vector. Then

\[
\sum_{i=1}^{n} w_i A_i \leq K(h, 2) \sigma_g(w; \mathcal{A}),
\]

and

\[
\sigma_g(w; \mathcal{A}) \leq K(h, 2) \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1}.
\]

**Proof.** The celebrated Kantorovich inequality asserts that, for a positive matrix \( A \) satisfying \( 0 < m \leq A \leq M \) for any normalized postive linear map \( \Phi \), it follows from the above inequality

\[
\Phi(A) \leq K \left( \frac{1}{h}, 2 \right) \Phi(A^{-1})^{-1} = K(h, 2) \Phi(A^{-1})^{-1}.
\]

Letting \( \Psi(\mathcal{A}) = \sum_{i=1}^{n} w_i A_i \) in (2.3), we obtain

\[
\sum_{i=1}^{n} w_i A_i = \Psi(\mathcal{A}) \\
\leq K(h, 2) \Psi(\mathcal{A}^{-1})^{-1} \\
= K(h, 2) \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \\
\leq K(h, 2) \sigma_g(w; \mathcal{A})
\]

where we have used the LHS of (1.4) to obtain the last inequality. This proves (2.1).

The inequality (2.2) follows from RHS of (1.4) and the inequality (2.4). □

Next, we use the improvement of the Kantorovich inequality to deduce refinements of both inequalities (2.1) and (2.2).

**Proposition 2.2.** Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be a n-tuple of positive definite matrices with \( m \leq A_i \leq M (i = 1, \ldots, n) \) for some scalars \( 0 < m < M \), and \( w = (w_1, \ldots, w_n) \) be a weight vector. Then

\[
\sum_{i=1}^{n} w_i A_i \leq \sum_{i=1}^{n} w_i M_{m^{-1}A_i^{-1}, M^{-1}}^m A_i^{-1} \leq K(h, 2) \sigma_g(w; \mathcal{A}),
\]
and

$$\sigma_g (w; A_i) \leq \sum_{i=1}^{n} w_i M^{\frac{m^{-1}A_i^{-1} - 1}{m^{-1} - M^{-1}}} M^{\frac{A_i^{-1} - M^{-1}}{m^{-1} - M^{-1}}} \leq K(h, 2) \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1},$$

where $K(h, 2) = \frac{(h+1)^2}{4h}$ and $h = M/m$.

**Proof.** On account of [12, Theorem 1.1] if $\Phi$ is a normalized positive linear map and $m \leq A_i \leq M$ for some scalars $0 < m < M$, then

$$\Phi (A^{-1}) \leq \Phi \left( M^{\frac{A-M}{M-m}} M^{\frac{m-A}{m-M}} \right) \leq K(h, 2) \Phi (A)^{-1}. \tag{2.5}$$

It follows from the inequality (2.5) that

$$\Phi (A) \leq \Phi \left( M^{\frac{m^{-1}A^{-1} - 1}{m^{-1} - M^{-1}}} M^{\frac{A^{-1} - M^{-1}}{m^{-1} - M^{-1}}} \right) \leq K \left( \frac{1}{h}, 2 \right) \Phi (A^{-1})^{-1} = K(h, 2) \Phi (A^{-1})^{-1}. \tag{2.6}$$

Now, if we apply the same argument presented in the proof of Proposition 2.1, we reach the desired results. We omit the details. \qed

In the following, we complement the inequality (1.5).

**Theorem 2.1.** Let $A = (A_1, \ldots, A_n)$ be a $n$-tuple of positive definite matrices with $m \leq A_i \leq M$ ($i = 1, \ldots, n$) for some scalars $0 < m < M$ and $w = (w_1, \ldots, w_n)$ be a weight vector. Then for all $p \geq 1$ and every unitarily invariant norm $||| \cdot |||$, 

$$|||\sigma_g (w; A) p ||| \leq K(m, M, p) K(h, 2)^p |||\sigma_g (w; A) p |||,$$

where

$$K(m, M, p) = \frac{m M^p - M m^p}{(p-1)(M-m)} \left( \frac{p-1}{p} \frac{M^p - m^p}{m M^p - M m^p} \right)^p.$$

In particular, if $\sigma_g (w; A) \leq I$, we have

$$\sigma_g (w; A) p \leq K(m, M, p) K(h, 2)^p. \tag{2.7}$$

**Proof.** We have

$$\sigma_g (w; A) p \leq \sum_{i=1}^{n} w_i A_i^p \tag{2.8}$$

$$\leq K(m, M, p) \left( \sum_{i=1}^{n} w_i A_i \right)^p \leq K(m, M, p) K(h, 2)^p U \sigma_g (w; A) p U^*$$

where the first inequality is due to RHS of (1.4) and the fact that $m \leq A_i \leq M$ implies $m^p \leq A_i^p \leq M^p$ ($p > 0$), the second one is due to [9, Remark 4.14], and the last inequality follows from (2.1) and the fact that for two positive definite matrices $X, Y$ with $X \leq Y$ there exists a unitary matrix $U$, such that $X^p \leq U Y^p U^*$ ($p > 0$).
One can infer from the above discussion
\[ \| \| \sigma_g (w; \mathbb{A}^p) \| \| \leq K (m, M, p) K(h, 2)^p \| \| \sigma_g (w; \mathbb{A})^p \| \|, \]
for all \( p \geq 1 \). Consequently,
\[ \sigma_g (w; \mathbb{A}^p) \leq \| \sigma_g (w; \mathbb{A})^p \| \leq K (m, M, p) K(h, 2)^p \| \sigma_g (w; \mathbb{A})^p \|. \]
The assumption \( \sigma_g (w; \mathbb{A}) \leq I \), implies \((2.7)\).

3. INEQUALITIES INVOLVING POSITIVE LINEAR MAPS

In this section we present several relations that describe the behavior of the solution of the GKE under positive linear maps. This study is usually referred to as information monotonicity. The following lemma is needed to prove our results.

**Lemma 3.1.** \([14]\) Let \( \mathbb{A} = (A_1, \ldots, A_n) \) be an \( n \)-tuple of positive definite matrices and \( w = (w_1, \ldots, w_n) \) be a weight vector.

(i) \( \sum_{i=1}^{n} w_i g (A_i) \geq 0 \) implies \( \sigma_g (w; \mathbb{A}) \geq I \).

(ii) \( \sigma_g (w; X^* \mathbb{A} X) = X^* \sigma_g (w; \mathbb{A}) X \) for all invertible \( X \).

Our first result in this direction is the study of information monotonicity of \( \sigma_g \). This result extends the corresponding result of \([6]\), where the power mean \( P_t \) was studied.

**Theorem 3.1.** Let \( \mathbb{A} = (A_1, \ldots, A_n) \) be a \( n \)-tuple of positive definite matrices with \( m \leq A_i \leq M \) \((i = 1, \ldots, n)\) for some scalars \( 0 < m < M \) and \( w = (w_1, \ldots, w_n) \) be a weight vector. Then, for the normalized positive linear map \( \Phi \),
\[ \Phi (\sigma_g (w; \mathbb{A})) \leq \sigma_g (w; \Phi (\mathbb{A})) \leq K (h, 2) \Phi (\sigma_g (w; \mathbb{A})). \]

**Proof.** For the first inequality, let \( X = \sigma_g (w; \mathbb{A}) \). Then
\[ 0 = \sum_{i=1}^{n} w_i g (X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}}) \Rightarrow 0 = \sum_{i=1}^{n} w_i (X \sigma_g A_i), \]
where \( X \sigma_g A_i = X^{\frac{1}{2}} g \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) X^{\frac{1}{2}} \). Taking \( \Phi \), we obtain
\[ 0 = \sum_{i=1}^{n} w_i \Phi (X \sigma_g A_i) \leq \sum_{i=1}^{n} w_i \Phi (X) \sigma_g \Phi (A_i) \]
where we used the well known Ando's inequality. Whence
\[ 0 \leq \sum_{i=1}^{n} w_i g \left( \Phi(X)^{-\frac{1}{2}} \Phi (A_i) \Phi(X)^{-\frac{1}{2}} \right). \]
Applying Lemma 3.1, we get
\[ \sum_{i=1}^{n} w_i g \left( \Phi(X)^{-\frac{1}{2}} \Phi(A_i) \Phi(X)^{-\frac{1}{2}} \right) \geq I \Rightarrow \Phi(X)^{-\frac{1}{2}} \sigma_g(w; \Phi(\mathcal{A})) \Phi(X)^{-\frac{1}{2}} \geq I. \]
That is
\[ \sigma_g(w; \Phi(\mathcal{A})) \geq \Phi(X) \]
which is equivalent to
\[ \sigma_g(w; \Phi(\mathcal{A})) \geq \Phi(\sigma_g(w; \mathcal{A})). \]
This proves the first desired inequality. For the second inequality, noting the RHS of (1.4) and the inequality (2.1), respectively, we obtain
\[ \sigma_g(w; \Phi(\mathcal{A})) \leq \sum_{i=1}^{n} w_i \Phi(A_i) \]
\[ = \Phi \left( \sum_{i=1}^{n} w_i A_i \right) \]
\[ \leq \Phi(\Phi(X)^{-\frac{1}{2}} \sigma_g(w; \mathcal{A})) \]
\[ = \Phi(\Phi(X)^{-\frac{1}{2}} \Phi(\sigma_g(w; \mathcal{A}))). \]
This completes the proof of the second inequality. □

Now we show how the inequalities in Proposition 2.2 could be squared.

**Theorem 3.2.** Let \( \mathcal{A} = (A_1, \ldots, A_n) \) be an \( n \)-tuple of positive definite matrices with \( m \leq A_i \leq M \) for some scalars \( 0 < m < M \) and \( w = (w_1, \ldots, w_n) \) be a weight vector. If \( \Phi : \mathcal{M}_n \rightarrow \mathcal{M}_p \) is a normalized positive linear map, then for any \( p \geq 2 \),
\[
\text{Proof.} \quad (3.1) \quad \Phi \left( \sum_{i=1}^{n} w_i M \frac{m^{-1} - A^{-1}}{m^{-1} - M^{-1}} \frac{A^{-1} - M^{-1}}{m^{-1} - M^{-1}} \right)^p \leq \left( \frac{(m + M)^2}{4^p M m} \right)^p \Phi(\sigma_g(w; \mathcal{A}))^p,
\]
and
\[
\text{Proof.} \quad (3.2) \quad \Phi(\sigma_g(w; \mathcal{A}))^p \leq \left( \frac{(m + M)^2}{4^p M m} \right)^p \Phi \left( \sum_{i=1}^{n} w_i M \frac{m^{-1} - A^{-1}}{m^{-1} - M^{-1}} \frac{A^{-1} - M^{-1}}{m^{-1} - M^{-1}} \right)^p.
\]
whenever \( m \leq A \leq M \). Then (3.4) implies

\[
(3.5) \quad \Phi \left( \sum_{i=1}^{n} w_i A_i^{-1} \right) + M^{-1} m^{-1} \Phi \left( \sum_{i=1}^{n} w_i M^{m-1} m^{-1} \right) \leq M^{-1} + m^{-1}
\]

where \( m \leq A_i \leq M \) \((i = 1, \ldots, n)\). Now, we can write

\[
M^{-\frac{p}{2}} m^{-\frac{p}{2}} \left\| \Phi \left( \sum_{i=1}^{n} w_i M^{m-1} m^{-1} \right)^{\frac{p}{2}} \Phi (w; \mathcal{A})^{-\frac{p}{2}} \right\|^2 \leq \frac{1}{4} \left( M^{-1} m^{-1} \Phi \left( \sum_{i=1}^{n} w_i M^{m-1} m^{-1} \right) + \Phi (w; \mathcal{A})^{-1} \right) \left\| \Phi \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^p \right\|^2 \leq \frac{1}{4} \left( M^{-1} m^{-1} \Phi \left( \sum_{i=1}^{n} w_i M^{m-1} m^{-1} \right) + \Phi (w; \mathcal{A})^{-1} \right) \left\| \Phi \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^p \right\|^2 \leq \frac{1}{4} \left( M^{-1} + m^{-1} \right)^p \quad \text{(by (3.5))}.
\]

Thus, we have shown

\[
\left\| \Phi \left( \sum_{i=1}^{n} w_i M^{m-1} m^{-1} \right)^{\frac{p}{2}} \Phi (w; \mathcal{A})^{-\frac{p}{2}} \right\| \leq \frac{(M^{-1} + m^{-1})^p}{4M^{-\frac{p}{2}} m^{-\frac{p}{2}}},
\]

which is equivalent to the desired inequality (3.1).

Now to prove (3.2), we proceed similarly noting that for \( t \in [m, M] \), we have

\[
(3.6) \quad t + mM \left( M^{-1} \right)^{m^{-1} t^{-1} (m^{-1})^{m^{-1} t^{-1}}} \leq t + mM t^{-1} \leq m + M.
\]

Then
\[
M^{\frac{p}{2}} m^\frac{p}{2} \left\| \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} A_i^{-1} - m^{-1} - A_i^{-1} M^{-1}} \right)^{-\frac{p}{2}} \Phi (\sigma_g (w; A))^\frac{p}{2} \right\| \\
\leq \frac{1}{4} \left\| M m \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} A_i^{-1} - m^{-1} - A_i^{-1} M^{-1}} \right)^{-1} \Phi (\sigma_g (w; A)) \right\|^p \\
\leq \frac{1}{4} \left\| M m \Phi \left( \sum_{i=1}^{n} w_i (M^{-1})^{m^{-1} M^{-1}} \right)^p \Phi \left( \sum_{i=1}^{n} w_i A_i \right) \right\|^p \\
\leq \frac{(M + m)^p}{4},
\]

where we have used (1.4) and the fact that the function \( f(t) = t^{-1} \) is operator convex to obtain (3.7), then (3.6) to obtain the last inequality.

\[\square\]

As a complementary result to Theorem 3.2 we have:

**Proposition 3.1.** Let all assumptions as in Theorem 3.2. Then

\[
\Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} A_i^{-1} - m^{-1} - A_i^{-1} M^{-1}} \right)^p \leq \left( \frac{(m + M)^2}{4^{\frac{p}{2}} m M} \right)^p \sigma_g (w; \Phi (A))^p,
\]

and

\[
\sigma_g (w; \Phi (A))^p \leq \left( \frac{(m + M)^2}{4^{\frac{p}{2}} m M} \right)^p \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} A_i^{-1} - m^{-1} - A_i^{-1} M^{-1}} \right)^p.
\]
Proof. We prove the first inequality. Notice that
\[
M^{-\frac{p}{2}} m^{-\frac{p}{2}} \left\| \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} - A^{-1}_i} m^{A^{-1}_i - M^{-1}} \right) \right\|_{\frac{p}{2}} \leq \frac{1}{4} \left\| M^{-\frac{p}{2}} m^{-\frac{p}{2}} \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} - A^{-1}_i} m^{A^{-1}_i - M^{-1}} \right) + \sigma_g(w; \Phi(A))^{-\frac{p}{2}} \right\|^{2} \\
\leq \frac{1}{4} \left\| M^{-1} m^{-1} \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} - A^{-1}_i} m^{A^{-1}_i - M^{-1}} \right) + \sigma_g(w; \Phi(A))^{-1} \right\|^{p} \\
\leq \frac{1}{4} \left\| M^{-1} m^{-1} \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} - A^{-1}_i} m^{A^{-1}_i - M^{-1}} \right) \right\|^{p} + \frac{1}{4} \left\| \sigma_g(w; \Phi(A))^{-1} \right\|^{p} \tag{by LHS of (2.8)} \\
\leq \frac{1}{4} \left\| M^{-1} m^{-1} \Phi \left( \sum_{i=1}^{n} w_i M^{m^{-1} - A^{-1}_i} m^{A^{-1}_i - M^{-1}} \right) \right\|^{p} + \frac{1}{4} \left\| \Phi \left( \sum_{i=1}^{n} w_i A^{-1}_i \right) \right\|^{p} \tag{by [4, Theorem 2.3.6]} \\
\leq \frac{(M^{-1} + m^{-1})^p}{4} \tag{by (3.5)}. 
\]
Then the desired inequality follows immediately. The second inequality follows similarly. □

Following the same steps as Theorem 3.2 and Proposition 3.1, we obtain the following squared versions of Theorem 3.1.

Corollary 3.1. Let \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of positive definite matrices with \( m \leq A_i \leq M \) \( (i = 1, \ldots, n) \) for some scalars \( 0 < m < M \) and \( w = (w_1, \ldots, w_n) \) be a weight vector. Then, for the normalized positive linear map \( \Phi \) and \( p \geq 2 \),

\[
\sigma_g(w; \Phi(A))^{p} \leq \left( \frac{(M + m)^2}{4m M} \right)^{p} \Phi(\sigma_g(w; A))^{p}.
\]

Related to positive linear maps, the inequality (1.6) was shown in [5] as a main tool to prove a reversed version of the inequality \( \Phi(A_i A) \leq \Phi(A) \Phi(A_i \Phi(B)) \). Our next result is the natural extension of (1.6) to the context of the solution of the GKE.

Theorem 3.3. Let \( A = (A_1, \ldots, A_n) \) be an \( n \)-tuple of positive definite matrices and let \( w = (w_1, \ldots, w_n) \) be a weight vector. Then, for any \( x \in \mathbb{C}^n \),

\[
\langle \sigma_g(w; A)x, x \rangle \leq \sigma_g(w; \langle A x, x \rangle).
\]

Proof. If \( x = 0 \), the result holds trivially, hence we may assume \( x \neq 0 \). Let \( X = \sigma_g(w; A) \). Then

\[
\sum w_i g \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) = 0 \Rightarrow \sum w_i X^{\frac{1}{2}} g \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) X^{\frac{1}{2}} = 0.
\]
Therefore, if $x \in \mathbb{C}^n$ is any nonzero vector, so that $Xx \neq 0$, then

$$0 = \sum w_i \left\langle X^{\frac{1}{2}} g \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) X^{\frac{1}{2}} x, x \right\rangle = \left\| X^{\frac{1}{2}} x \right\|^2 \sum w_i \left\langle g \left( X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \right) \frac{X^{\frac{1}{2}} x}{\left\| X^{\frac{1}{2}} x \right\|}, \frac{X^{\frac{1}{2}} x}{\left\| X^{\frac{1}{2}} x \right\|} \right\rangle \leq \left\| X^{\frac{1}{2}} x \right\|^2 \sum w_i g \left( \left\langle X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} \frac{X^{\frac{1}{2}} x}{\left\| X^{\frac{1}{2}} x \right\|}, \frac{X^{\frac{1}{2}} x}{\left\| X^{\frac{1}{2}} x \right\|} \right\rangle \right) \leq \left\| X^{\frac{1}{2}} x \right\|^2 g \left( \sum w_i \left\langle A_i x, x \right\rangle \right).$$

Applying Lemma 3.1, we infer that

$$\sigma_g \left( w; \frac{\left\langle A x, x \right\rangle}{\left\| X^{\frac{1}{2}} x \right\|^2} \right) \geq 1 \Rightarrow \left\| X^{\frac{1}{2}} x \right\|^2 \sigma_g (w; \left\langle A x, x \right\rangle) \geq 1,$$

which implies

$$\left\langle X^{\frac{1}{2}} x, X^{\frac{1}{2}} x \right\rangle \leq \sigma_g (w; \left\langle A x, x \right\rangle) \Rightarrow \left\langle X x, x \right\rangle \leq \sigma_g (w; \left\langle A x, x \right\rangle).$$

That is

$$\left\langle \sigma_g (w; A) x, x \right\rangle \leq \sigma_g (w; \left\langle A x, x \right\rangle).$$

\[\square\]

4. Inequalities for operator monotone functions

Given an operator monotone function $f$, we discuss the relation between $\sigma_g (w; f (A))$ and $f \left( \sigma_g (w; A) \right)$ in the following theorem.

**Theorem 4.1.** Let $A = (A_1, \ldots, A_n)$ be a n-tuple of positive definite matrices with $m \leq A_i \leq M (i = 1, \ldots, n)$ for some scalars $0 < m < M$ and $w = (w_1, \ldots, w_n)$ be a weight vector.

(I) If $f$ is an operator monotone function, then

$$\sigma_g (w; f (A)) \leq K (h, 2) f \left( \sigma_g (w; A) \right).$$

(II) If $f$ is an operator monotone decreasing function, then

$$f \left( \sigma_g (w; A) \right) \leq K (h, 2) \sigma_g (w; f (A)).$$
Proof. Compute

\[ \sigma_g(w; f(A_i)) \leq \sum_{i=1}^{n} w_i f(A_i) \quad \text{(by (1.4))} \]

\[ \leq f \left( \sum_{i=1}^{n} w_i A_i \right) \quad \text{(f being operator concave)} \]

\[ \leq f \left( K(h, 2) \sigma_g(w; A_i) \right) \quad \text{(by (2.1))} \]

\[ \leq K(h, 2) f \left( \sigma_g(w; A_i) \right) \]

where, to obtain the last inequality, we have used the fact that if \( f(t) \) is operator monotone and \( \alpha \geq 1 \), then \( f(\alpha t) \leq \alpha f(t) \). This proves the first inequality.

For the second inequality, notice first that when \( f \) is operator monotone decreasing, we have \( \frac{1}{\alpha} f(t) \leq f(\alpha t) \) for \( \alpha \geq 1 \). Then

\[ \frac{1}{K(h, 2)} f \left( \sigma_g(w; A_i) \right) \leq f \left( K(h, 2) \sigma_g(w; A_i) \right) \]

\[ \leq f \left( \sum_{i=1}^{n} w_i A_i \right) \quad \text{(by (2.1))} \]

\[ \leq \left( \sum_{i=1}^{n} w_i f(A_i)^{-1} \right)^{-1} \]

\[ \leq \sigma_g(w; f(A_i)) \quad \text{(by (1.4))} \]

where we have used [1, Remark 2.7] to obtain the third inequality. \( \square \)

As a counterpart of (4.1), we have the following reversed version. The notation \( \nabla_n A \) will be used for the arithmetic mean \( \frac{1}{n}(A_1 + \cdots + A_n) \).

**Proposition 4.1.** Let \( \mathbb{A} = (A_1, \ldots, A_n) \) be a \( n \)-tuple of positive definite matrices with \( m \leq A_i \leq M \) \((i = 1, \ldots, n)\) for some scalars \( 0 < m < M \) and \( w = (w_1, \ldots, w_n) \) a weight vector. If \( f \) is an operator monotone function, then

\[ f \left( \sigma_g(w; A) \right) \leq K(h, 2) \sigma_g(w; f(A)) + nw_{\text{max}} f \left( \nabla_n A - \nabla_{nf}(A) \right), \]

where \( w_{\text{max}} = \max_{1 \leq i \leq n} w_i \) and \( h = \frac{f(M)}{f(m)} \).
Proof. Noting operator concavity and monotonicity of \( f \), we have
\[
f(\sigma_g(w; A)) \leq f\left(\sum_{i=1}^{n}w_iA_i\right)
\leq \sum_{i=1}^{n}w_i f(A_i) + nw_{\max} (f(\nabla_n A) - \nabla_n f(\hat{A}))
\leq K(h, 2) \sigma_g(w; f(\hat{A})) + nw_{\max} (f(\nabla_n A) - \nabla_n f(\hat{A}))
\]
where, to obtain the second inequality, we have used an argument similar to that used in [10]. \( \square \)

Notice that the function \( f(t) = t^{-1} \) is operator monotone decreasing. Therefore, Inequality (4.2) implies
\[
\sigma_g(w; A)^{-1} \leq K(h, 2) \sigma_g(w; A)^{-1}.
\]

In the following result, we present a counterpart of this inequality.
For this, notice that when \( g(x) = x - 1 \), we have \( \sigma_g(w; \hat{A}) = \sum_{i=1}^{n} w_i A_i \), while we obtain \( \sigma_g(w, A) = (\sum_{i=1}^{n} A_i^{-1})^{-1} \) when \( g(x) = 1 - x^{-1} \). Therefore, letting \( g(x) = x - 1 \) in (2.2), we have
\[
\sum_{i=1}^{n} w_i A_i \leq K(h, 2) \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1}.
\]

**Proposition 4.2.** Let all the assumptions of Theorem 4.1 hold. Then
\[
\sigma_g(w; \hat{A})^{-1} \leq K(h, 2) \sigma_g(w; A)^{-1}.
\]

**Proof.** Notice that
\[
\sigma_g(w; \hat{A})^{-1} \leq \sum_{i=1}^{n} w_i A_i^{-1} \quad \text{(by (1.4))}
\leq K(h, 2) \left( \sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \quad \text{(by (4.4))}
\leq K(h, 2) \sigma_g(w; \hat{A})^{-1} \quad \text{(by (1.4))}.
\]
\( \square \)

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References

[1] T. Ando and F. Hiai, Operator log-convex functions and operator means, Math Ann., 350(3) (2011), 611–630.
[2] T. Ando and X. Zhan, Norm inequalities related to operator monotone functions, Math Ann., 315(4) (1999), 771–780.
[3] R. Bhatia and F. Kittaneh, Notes on matrix arithmetic-geometric mean inequalities, Linear Algebra Appl., 308 (2000), 203–211.
[4] R. Bhatia, Positive definite matrices, Princeton University Press, 2007.
[5] J.-C. Bourin, E.-Y. Lee, M. Fujii and Y. Seo, A matrix reverse Hölder inequality, Linear Algebra Appl., 431 (2009), 2154–2159.
[6] M. Delhghani, M. Kian and Y. Seo, Matrix power means and the information monotonicity, Linear Algebra Appl., 521 (2017), 57–69.
[7] I.H. Güümüş, H.R. Moradi and M. Sababheh, More accurate operator means inequalities, J. Math. Anal. Appl. (2018), https://doi.org/10.1016/j.jmaa.2018.05.003.
[8] Y. Lim and M. Pálfia, The matrix power means and the Karcher mean, J. Funct. Anal., 262 (2012), 1498–1514.
[9] J. Mićić, J. Pečarić, Y. Seo and M. Tominaga, Inequalities for positive linear maps on Hermitian matrices, Math. Inequal. Appl., 3 (2000), 559–591.
[10] F. Mitroi, About the precision in Jensen-Steffensen inequality, An. Univ. Craiova Ser. Mat. Inform., 37 (2010), 73–84.
[11] M. Moakher, A differential geometric approach to the geometric mean of symmetric positive definite matrices, SIAM J. Matrix Anal. Appl., 26 (2005), 735–747.
[12] H.R. Moradi, I.H. Güümüş and Z. Heydarbeygi, A glimpse at the operator Kantorovich inequality, Linear Multilinear Algebra. (2018), https://doi.org/10.1080/03081087.2018.1441799.
[13] M. Pálfia, Operator means of probability measures and generalized Karcher equations, Adv. Math., 289 (2016), 951–1007.
[14] T. Yamazaki, Generalized Karcher equation, relative operator entropy and the Ando-Hiai inequality, Preprint 2018, arXiv:1802.06200 [math.FA].

(M. Sababheh) Department of Basic Sciences, Princess Sumaya University for Technology, Amman, Jordan.
E-mail address: sababheh@yahoo.com, sababheh@psut.edu.jo

(H.R. Moradi) Young Researchers and Elite Club, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail address: hrmoradi@mshdiau.ac.ir

(Z. Heydarbeygi) Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran.
E-mail address: zheydarbeygi@yahoo.com