On the Derived Category of the Cayley Grassmannian

L. A. Guseva*

Received August 9, 2022; in final form, August 9, 2022; accepted August 20, 2022

Keywords: Lefschetz collection, derived category, Cayley Grassmannian, quadric bundle.

We construct a full exceptional collection in the bounded derived category of coherent sheaves on the Cayley Grassmannian \( \text{CG} \). The geometric properties of the Cayley Grassmannian were studied in [1]. In [2], the small quantum cohomology of the Cayley Grassmannian was calculated; in particular, it was proved to be semisimple. Thus, the main result of the present paper confirms Dubrovin’s conjecture that the semisimplicity of small quantum cohomology implies the existence of a full exceptional collection in the case of the Cayley Grassmannian.

It is known that the existence of a full exceptional collection does not imply the semisimplicity of small quantum cohomology. In [3], Dubrovin’s conjecture was refined for Fano varieties with one-dimensional Picard group; namely, it was conjectured that the semisimplicity of the small quantum cohomology of a variety implies the existence of a full Lefschetz collection whose residual category is generated by completely orthogonal exceptional objects (the definition of a Lefschetz collection and its residual category is given below; see Definition 3). In the present paper, we prove that this conjecture is also true for the Cayley Grassmannian.

We fix an algebraically closed field \( k \) of characteristic 0. The Cayley Grassmannian \( \text{CG} \) is defined as follows. Consider the Grassmannian \( \text{Gr}(3, V) \) parametrizing the 3-dimensional subspaces in a 7-dimensional vector space \( V \). We denote the tautological vector bundles on \( \text{Gr}(3, V) \) of ranks 3 and 4 by \( \mathcal{U} \subset V \otimes \mathcal{O} \) and \( \mathcal{U}^\perp \subset V^\vee \otimes \mathcal{O} \), respectively. By \( \mathcal{O}(1) \), the Plücker line bundle. By definition, there are the following exact sequences of bundles on \( \text{Gr}(3, V) \):

\[
0 \to \mathcal{U} \to V \otimes \mathcal{O} \to \mathcal{O} \to 0, \quad (1)
\]

\[
0 \to \mathcal{U}^\perp \to V^\vee \otimes \mathcal{O} \to \mathcal{U}^\vee \to 0. \quad (2)
\]

By the Borel–Bott–Weil theorem, \( H^0(\text{Gr}(3, V), \mathcal{U}^\perp(1)) \simeq \Lambda^4 V^\vee \). Let us fix a general global section of the bundle \( \mathcal{U}^\perp(1) \), i.e., a generic 4-form \( \lambda \in \Lambda^4 V^\vee \). The Cayley Grassmannian \( \text{CG} \) is defined as the zero locus of a global section \( \lambda \in H^0(\text{Gr}(3, V), \mathcal{U}^\perp(1)) \). In other words, \( \text{CG} \) parametrizes the 3-dimensional vector subspaces \( U \subset V \) such that \( \lambda(u_1, u_2, u_3, -) = 0 \) for all \( u_1, u_2, u_3 \in U \). It readily follows from the definition that \( \text{CG} \) is a smooth Fano variety of dimension 8 and that the canonical class \( \omega_{\text{CG}} \) of the Cayley Grassmannian is isomorphic to \( \mathcal{O}(-4) \).

Let us recall the definition of a full exceptional collection in a \( k \)-linear triangulated category \( \mathcal{T} \).

**Definition 1.** An object \( E \) in \( \mathcal{T} \) is said to be exceptional if \( \text{Ext}^i(E, E) = 0 \) for all \( i > 0 \).

**Definition 2.** A sequence of objects \( E_1, \ldots, E_m \) in \( \mathcal{T} \) is called an exceptional collection if all \( E_i \) are exceptional and \( \text{Ext}^i(E_i, E_j) = 0 \) for all \( i > j \). A collection \( (E_1, E_2, \ldots, E_m) \) is said to be full if the minimal triangulated subcategory in \( \mathcal{T} \) containing \( (E_1, E_2, \ldots, E_m) \) coincides with \( \mathcal{T} \).

*E-mail: lguseva@hse.ru*
Given an exceptional object $E \in \mathcal{T}$, by $\mathbb{L}_E$ we denote the functor of left mutation through $E$, which takes each object $G \in \mathcal{T}$ to
\[
\mathbb{L}_E(G) := \text{Cone}(\text{Ext}^\bullet(E, G) \otimes E \to G),
\]
where the morphism is the evaluation morphism. If $(E, E')$ is an exceptional pair, then so is the pair $(\mathbb{L}_E(E'), E)$; moreover, $(\mathbb{L}_E(E'), E)$ and $(E, E')$ generate the same subcategory in $\mathcal{T}$ (see [4]). Given an exceptional collection $(E_1, \ldots, E_m)$ of any length, we define the left mutation through the subcategory $\langle E_1, \ldots, E_m \rangle$ as the composition
\[
\mathbb{L}_{(E_1,\ldots,E_m)} = \mathbb{L}_{E_1} \circ \cdots \circ \mathbb{L}_{E_m}.
\]

Proposition 1 [1]. The left mutation functor through an exceptional collection induces the equivalence
\[
\langle E_1, \ldots, E_m \rangle \xrightarrow{\mathbb{L}_{(E_1,\ldots,E_m)}} \langle E_1, \ldots, E_m \rangle^\perp.
\]
The mutation of a full exceptional collection is a full exceptional collection.

A full exceptional collection in the bounded derived category $\mathbb{D}^b(\mathcal{CG})$ of coherent sheaves on the Cayley Grassmannian $\mathcal{CG}$, which we are going to describe, is Lefschetz. Below we give the definition of a Lefschetz collection and its residual category in the derived category of a smooth projective variety $X$.

Definition 3 [3]. Let $\mathcal{O}_X(1)$ be an ample line bundle on $X$.

1. A Lefschetz collection in $\mathbb{D}^b(X)$ with respect to $\mathcal{O}_X(1)$ is an exceptional collection in $\mathbb{D}^b(X)$ consisting of several blocks of the form
\[
\begin{align*}
&\text{block 1} \quad \frac{E_1, E_2, \ldots, E_{\vartheta_0}, E_1(1), E_2(1), \ldots, E_{\vartheta_1}(1), \ldots, E_1(i-1), E_2(i-1), \ldots, E_{\vartheta_{i-1}}(i-1),}{E_1, E_2, \ldots, E_{\vartheta_{i-1}}, E_1(1), E_2(1), \ldots, E_{\vartheta_{i-1}}(1), \ldots,}, \\
&\text{block 2} \quad \frac{E_1(i-1), E_2(i-1), \ldots, E_{\vartheta_{i-1}}(i-1),}{E_1(i-1), E_2(i-1), \ldots, E_{\vartheta_{i-1}}(i-1),}
\end{align*}
\]
where $\vartheta = (\vartheta_0 \geq \vartheta_1 \geq \cdots \geq \vartheta_{i-1} > 0)$ is a nonincreasing sequence of positive integers.

2. The rectangular part of a Lefschetz collection is the subset
\[
\begin{align*}
&\text{block 1} \quad \frac{E_1, E_2, \ldots, E_{\vartheta_{i-1}}, E_1(1), E_2(1), \ldots, E_{\vartheta_{i-1}}(1), \ldots,}{E_1, E_2, \ldots, E_{\vartheta_{i-1}}, E_1(1), E_2(1), \ldots, E_{\vartheta_{i-1}}(1), \ldots,}, \\
&\text{block 2} \quad \frac{E_1(i-1), E_2(i-1), \ldots, E_{\vartheta_{i-1}}(i-1),}{E_1(i-1), E_2(i-1), \ldots, E_{\vartheta_{i-1}}(i-1),}
\end{align*}
\]

3. The subcategory in $\mathbb{D}^b(X)$ orthogonal to the rectangular part of a Lefschetz collection is called the residual category:
\[
\text{Res} := \langle E_1, E_2, \ldots, E_{\vartheta_{i-1}}, E_1(1), E_2(1), \ldots, E_{\vartheta_{i-1}}(1), \ldots, \\
&\quad E_1(i-1), E_2(i-1), \ldots, E_{\vartheta_{i-1}}(i-1) \rangle^\perp.
\]

The collection in $\mathbb{D}^b(\mathcal{CG})$ constructed above consists of four blocks. The common part of these four blocks comprises three bundles: $(\mathcal{O}, \mathcal{W}^\vee, \Lambda^2 \mathcal{W}^\vee)$. We set
\[
\mathfrak{E}(i) := (\mathcal{O}(i), \mathcal{W}^\vee(i), \Lambda^2 \mathcal{W}^\vee(i)).
\]
To describe the remaining part of the collection, we must define two additional vector bundles on $\mathcal{CG}$. The first one is defined as
\[
\Sigma^{2,1} \mathcal{W}^\vee := (\mathcal{W}^\vee \otimes \Lambda^2 \mathcal{W}^\vee)/\Lambda^3 \mathcal{W}^\vee.
\]
To describe the second one, we need a lemma.

Lemma 1. On the Cayley Grassmannian $\mathcal{CG}$, the 4-form $\lambda$ determines an embedding
\[
i_\lambda : \Lambda^2 \mathcal{W} \hookrightarrow \Lambda^2 \mathcal{W}^\perp
\]
of vector bundles.
In particular, we can define the quotient bundle $\Lambda^2\mathcal{U}^\perp / \Lambda^2\mathcal{U}$ on $\mathbf{CG}$. In the exceptional collection, we use the dual bundle 
\[
\mathcal{R} := (\Lambda^2\mathcal{U}^\perp / \Lambda^2\mathcal{U})^\vee.
\] (10)
Thus, by the definition of $\mathcal{R}$, there is an exact sequence
\[
0 \to \mathcal{R} \to \Lambda^2\mathcal{Q} \to \Lambda^2\mathcal{Q}^\vee \to 0
\] (11)
on $\mathbf{CG}$. The main result of this paper is the following theorem.

**Theorem 1.** The collection
\[
(\mathcal{E}, \mathcal{R}, \Sigma^2 \mathcal{U}^\vee; \mathcal{E}(1), \mathcal{R}(1); \mathcal{E}(2); \mathcal{E}(3))
\] (12)
of 15 vector bundles on the Cayley Grassmannian $\mathbf{CG}$ is a full Lefschetz collection with respect to $\mathcal{O}(1)$.

Using the Koszul resolution
\[
0 \to \mathcal{O}(−3) \to \Lambda^3\mathcal{Q}(−3) \to \Lambda^2\mathcal{Q}(−2) \to \mathcal{Q}(−1) \to \mathcal{O} \to \mathcal{O}_\mathbf{CG} \to 0
\] (13)
for the structure sheaf $\mathcal{O}_\mathbf{CG}$ (here $\mathbf{CG} \hookrightarrow \text{Gr}(3, V)$ is an embedding of $\mathbf{CG}$ in $\text{Gr}(3, V)$), we reduce proving that the collection (14) is exceptional to calculating the cohomology on $\text{Gr}(3, V)$, which can be done by using the Borel–Bott–Weil theorem.

The idea of the proof that the collection (14) is full is as follows.

It is a little more convenient to prove the fullness of the collection
\[
(\mathcal{U}, \mathcal{E}, \mathcal{R}, \Sigma^2 \mathcal{U}^\vee, \mathcal{E}(1), \mathcal{R}(1), \mathcal{E}(2), \mathcal{E}(3)), \mathcal{U}^\vee(3),
\] (14)
which is obtained from (12) by deleting the last object $\Lambda^2\mathcal{U}^\vee(3)$ and inserting $\mathcal{U} \simeq \Lambda^2\mathcal{U}^\vee(3) \otimes \omega_{\mathbf{CG}}$ at the beginning. By Theorem 4.1 in [4], the fullness of the collection (12) is equivalent to that of (14).

It can be shown that $\mathbf{CG}$ is covered by the family of subvarieties $\mathbf{CG}_f \hookrightarrow \mathbf{CG}$ being the zeros of sufficiently general global sections $f \in H^0(\mathbf{CG}, \mathcal{U}^\vee)$. It is easily seen that the subvarieties $\mathbf{CG}_f$ are isomorphic to a smooth hyperplane section of the isotropic Grassmannian $\text{IGr}(3, 6)$; thus, Theorem 2.3 of [5] implies the existence of a full exceptional collection in $D^b(\mathbf{CG}_f)$. A standard argument in [6] reduces the problem to checking the following relations for five vector bundles:
\[
S^2\mathcal{U}^\vee(m) \in \mathcal{D}, \quad m = 0, 1, 2, \quad \Sigma^2 \mathcal{U}^\vee(1), \Sigma^2 \mathcal{U}^\vee(2) \in \mathcal{D},
\] (15)
where $\mathcal{D} \subset D^b(\mathbf{CG})$ is the subcategory generated by (14). More specifically, using (15), the Koszul resolution of the sheaf $i_f^*\mathcal{O}_{\mathbf{CG}}$, on $\mathbf{CG}$, and an exceptional collection on $\mathbf{CG}_f$, we can show that the inverse image $i_f^*F$ of any object $F \in \mathcal{D}^\perp$ equals 0. This implies $F = 0$, whence we obtain $\mathcal{D}^\perp = 0$ and $\mathcal{D} = D^b(\mathbf{CG})$.

The proof of (15) is the most important part of the proof that the collection (14) is full. To prove (15), we use the operation of gluing together two quadric bundles with isomorphic cokernels. It is described as follows.

Recall that a quadric bundle $Q \to S$ over a scheme $S$ is a proper morphism which can be represented as the composition $Q \hookrightarrow \mathbb{P}_S(\mathcal{F}) \to S$, where $\mathcal{F}$ is a vector bundle on $S$ and $Q \hookrightarrow \mathbb{P}_S(\mathcal{F})$ is a divisorial embedding of relative degree 2 over $S$. A quadric bundle is determined by a self-dual morphism
\[
\mathcal{F} \overset{f}{\to} \mathcal{F}^\vee \otimes \mathcal{L},
\] (16)
where $\mathcal{L}$ is a linear bundle. A quadric bundle $Q \to S$ can be associated with the coherent sheaf $\text{Coker}(f)$, which is called the cokernel of the quadric bundle. The precise description of the gluing operation which we need is as follows.
Proposition 2. There exists a bijection between the set of triple isomorphism classes
\[ \{(f_1, f_2, g) \mid f_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_1^\vee \otimes \mathcal{L}, f_2 : \mathcal{F}_2 \rightarrow \mathcal{F}_2^\vee \otimes \mathcal{L} g : \text{Coker}(f_1) \cong \text{Coker}(f_2) \}, \]
where \( f_1 \) and \( f_2 \) are self-dual morphisms, and the set of pair isomorphism classes
\[ \{(f, \epsilon) \mid f : \mathcal{F} \cong \mathcal{F}^\vee \otimes \mathcal{L}, \epsilon : 0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_2 \otimes \mathcal{L} \rightarrow 0 \}, \]
where \( f \) is a self-dual morphism.

Now we outline the derivation of relations (15) from Proposition 2.

First, we sketch the proof that \( \Sigma^{2,1} \mathcal{U}^\vee(1), \Sigma^{2,1} \mathcal{U}^\vee(2) \in \mathcal{D} \). Using the results in [1], we can construct quadric bundles with isomorphic cokernels in the projectivizations of the vector bundles \( \mathbb{P}_{\mathcal{CG}}(\mathcal{U}^\perp \otimes \Lambda^2 \mathcal{U}^\vee) \) and \( \mathbb{P}_{\mathcal{CG}}(\mathcal{U}^\vee \otimes \mathcal{O}) \). Thus, by Proposition 2, there exists an extension
\[ 0 \rightarrow \mathcal{U}^\perp \otimes \Lambda^2 \mathcal{U}^\vee \rightarrow \mathcal{E}_{16} \rightarrow \Lambda^2 \mathcal{U}^\vee \oplus \mathcal{O}(1) \rightarrow 0 \]
(17)
such that the bundle \( \mathcal{E}_{16} \) is self-dual, i.e., \( \mathcal{E}_{16} \cong \mathcal{E}_{16}^\vee(1) \). It is easy to see that \( \mathcal{E}_{16} \) is the kernel of the evaluation morphism
\[ \text{ev} : \text{Hom}_{\mathcal{CG}}(\Lambda^2 \mathcal{U}^\vee, \Sigma^{2,1} \mathcal{U}^\vee) \otimes \Lambda^2 \mathcal{U}^\vee \rightarrow \Sigma^{2,1} \mathcal{U}^\vee \]
on \( \mathcal{CG} \) and that \( \text{Hom}_{\mathcal{CG}}(\Lambda^2 \mathcal{U}^\vee, \Sigma^{2,1} \mathcal{U}^\vee) \cong \mathcal{V} \oplus \mathcal{k} \). Using the self-duality of \( \mathcal{E}_{16} \), we obtain a self-dual exact sequence
\[ 0 \rightarrow \Sigma^{2,1} \mathcal{U}^\vee(-1) \xrightarrow{\text{ev}^\vee} (\mathcal{V} \oplus \mathcal{k}) \otimes \mathcal{U}^\vee \rightarrow (\mathcal{V} \oplus \mathcal{k}) \otimes \Lambda^2 \mathcal{U}^\vee \xrightarrow{\text{ev}} \Sigma^{2,1} \mathcal{U}^\vee \rightarrow 0 \]
(18)
on \( \mathcal{CG} \) (here \( \text{ev}^\vee \) is the morphism dual to \( \text{ev} \)). The existence of such a sequence readily implies \( \Sigma^{2,1} \mathcal{U}^\vee(1), \Sigma^{2,1} \mathcal{U}^\vee(2) \in \mathcal{D} \).

The membership relations \( \mathcal{S}^{2,1} \mathcal{U}^\vee(1), \mathcal{S}^{2,1} \mathcal{U}^\vee(2) \in \mathcal{D} \) follow from standard exact sequences on \( \mathcal{CG} \). To be more precise, the sequence (11) gives \( \Lambda^2 \mathcal{D}, \Lambda^2 \mathcal{D}(1) \in \mathcal{D} \), and the required relation is obtained by using the (dual) Koszul complex
\[ 0 \rightarrow \Lambda^2 \mathcal{D}(n - 1) \rightarrow \Lambda^2 \mathcal{V}^\vee \otimes \mathcal{O}(n) \rightarrow \mathcal{V} \otimes \mathcal{U}^\vee(1) \rightarrow \mathcal{S}^{2,1} \mathcal{U}^\vee \rightarrow 0 \]
(19)
for \( n = 1, 2 \).

To prove the relation \( \mathcal{S}^{2,1} \mathcal{U}^\vee \in \mathcal{D} \), we again need Proposition 2. Using the results in [1], we can construct quadric bundles with isomorphic cokernels in the projectivizations \( \mathbb{P}_{\mathcal{CG}}(S^2 \mathcal{U}) \) and \( \mathbb{P}_{\mathcal{CG}}(\mathcal{D}) \) of vector bundles. Thus, by Proposition 2, there exists an extension
\[ 0 \rightarrow S^2 \mathcal{U} \rightarrow \mathcal{E}_{10} \rightarrow \mathcal{U}^\perp \rightarrow 0 \]
(20)
such that the bundle \( \mathcal{E}_{10} \) is self-dual, i.e., \( \mathcal{E}_{10} \cong \mathcal{E}_{10}^\vee \). From the (twisted) Koszul complex
\[ 0 \rightarrow S^2 \mathcal{U}(1) \rightarrow S^2 \mathcal{V}^\vee \otimes \mathcal{O}(1) \rightarrow \Lambda^2 \mathcal{D}(1) \rightarrow 0, \]
(21)
we see that \( S^2 \mathcal{U}(1) \in \mathcal{D} \). From the exact sequence (20) and the exact sequence (2) twisted by \( \mathcal{O}(1) \), we obtain \( \mathcal{E}_{10}(1) \in \mathcal{D} \), and self-duality implies \( \mathcal{E}_{10}^\vee \in \mathcal{D} \). The exact sequence
\[ 0 \rightarrow \mathcal{D} \rightarrow \mathcal{E}_{10}^\vee \rightarrow S^2 \mathcal{U}^\vee \rightarrow 0 \]
(22)
dual to (20) and the sequence (21) give the desired relation \( S^2 \mathcal{U}^\vee \in \mathcal{D} \).

Now we prove that the residual category (12) is generated by completely orthogonal objects. Thereby, we show that the conjecture in [3] is true for \( \mathcal{CG} \).

Theorem 2. The residual category \( \text{Res} \) of the Lefschetz collection (12) is generated by three completely orthogonal objects; namely,
\[ \text{Res} = \langle L_{L}, \Sigma^{2,1} \mathcal{U}^\vee(-1), \mathcal{D}(-1) \rangle. \]
The idea of the proof of this theorem is as follows. By definition,

\[ \text{Res} := \langle \mathcal{E}, \mathcal{E}(1), \mathcal{E}(2), \mathcal{E}(3) \rangle^\perp. \] (24)

Thus, \( \text{Res} \) is generated by the set \( \langle \mathcal{L}_{\mathcal{E}} \mathcal{R}, \mathcal{L}_{\mathcal{E}} \Sigma^{2,1} \mathcal{U}^\vee, \mathcal{L}_{\mathcal{E}} \mathcal{E}(1) (\mathcal{R}(1)) \rangle \).

The isomorphism \( \mathcal{L}_{\mathcal{E}} \Sigma^{2,1} \mathcal{U}^\vee \cong \Sigma^{2,1} \mathcal{U}^\vee (-1)[2] \) follows from the exact sequence (18). Using the same exact sequences as in the proof of \( S^2 \mathcal{U}^\vee \in \mathcal{D} \), we prove that \( \mathcal{R}(-1) \in \langle \mathcal{E}, \mathcal{E}(1), \mathcal{R}(1) \rangle \). The fact that the collection (12) is exceptional gives the isomorphism \( \mathcal{L}_{\mathcal{E}} \mathcal{E}(1) (\mathcal{R}(1)) \cong \mathcal{R}(-1) \) (up to shift). This proves (23).

Let us prove the complete orthogonality of the objects generating \( \text{Res} \). Recall (see Theorem 2.8 in [3]) that the residual category \( \text{Res} \) for the Lefschetz collection (12) has the self-equivalence

\[ \tau(-) := \mathcal{L}_{\mathcal{E}} (- \otimes \mathcal{O}(1)). \] (25)

By definition, \( \tau(\mathcal{R}(-1)) = \mathcal{L}_{\mathcal{E}} \mathcal{R} \). It follows from these relations that (up to shift)

\[ \tau(\Sigma^{2,1} \mathcal{U}^\vee (-1)) = \Sigma^{2,1} \mathcal{U}^\vee (-1), \quad \tau(\mathcal{L}_{\mathcal{E}} \mathcal{R}) = \mathcal{R}(-1). \] (26)

The semiorthogonality of the three objects generating \( \text{Res} \) is obvious. Complete orthogonality follows from \( \tau \) being a self-equivalence.

**FUNDING**

This work was supported by the Basic Research Program, National Research University Higher School of Economics.

**REFERENCES**

1. L. Manivel, J. Algebra **503**, 277 (2018).
2. V. Benedetti and L. Manivel, Int. J. Math. **31**(3) (2020).
3. A. Kuznetsov and M. Smirnov, Proc. London Math. Soc. **120**(5), 617 (2020).
4. A. I. Bondal, Izv. Math. **34**(1), 23 (1990).
5. A. Samokhin, C. R. Math. Acad. Sci. Paris **340**(12), 889 (2005).
6. A. Kuznetsov, Proc. London Math. Soc. **97**(1), 155 (2008).