Structure-Dependent Electromagnetic
Finite-Size Effects

Matteo Di Carlo\textsuperscript{a}, Maxwell T. Hansen\textsuperscript{a}, Nils Hermansson-Truedsson\textsuperscript{b},\textsuperscript{\dagger} Antonin Portelli\textsuperscript{a}

\textsuperscript{a}Higgs Centre for Theoretical Physics, School of Physics and Astronomy, University of Edinburgh, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom

\textsuperscript{b} Albert Einstein Center for Fundamental Physics, Institute for Theoretical Physics, Universität Bern, Sidlerstrasse 5, CH–3012 Bern, Switzerland

Abstract

We present a model-independent and relativistic approach to analytically derive electromagnetic finite-size effects beyond the point-like approximation. The key element is the use of electromagnetic Ward identities to constrain vertex functions, and structure-dependence appears via physical form-factors and their derivatives. We apply our general method to study the leading finite-size structure-dependence in the pseudoscalar mass (at order $1/L^3$) as well as in the leptonic decay amplitudes of pions and kaons (at order $1/L^2$). Knowledge of the latter is essential for Standard Model precision tests in the flavour physics sector from lattice simulations.

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1 Introduction

Lattice quantum chromodynamics (QCD) allows for systematically improvable Standard Model (SM) precision tests from numerical simulations performed in a finite-volume (FV), discretised Euclidean spacetime. In order to reach (sub-)percent precision in lattice predictions, also strong and electromagnetic isospin breaking corrections have to be included. The latter are encoded via quantum electrodynamics (QED), but the inclusion of QED in a FV spacetime is complicated because of Gauss’ law [1]. This problem is related to zero-momentum modes of photons and the absence of a QED mass-gap. Several prescriptions of how to include QED in a finite volume have been formulated and the one used here is QED_L where the spatial zero-modes are removed on each time-slice. The long-range nature of QED in addition enhances the FV effects (FVEs), which typically leads to power-law FVEs that are larger than the exponentially suppressed ones for single-particle matrix elements in QCD alone.

The FVEs for a QCD+QED process depend on properties of the involved particles, including masses and charges, but also structure-dependent quantities such as electromagnetic form-factors and their derivatives. In order to analytically capture the finite-volume scaling fully, one cannot neglect hadron structure, and in the following we develop a relativistic and model-independent method to go beyond the point-like approximation at order e^2 in QED_L.

We consider a space-time with periodic spatial extents L but with infinite time-extent. To exemplify the method, we first consider the pseudoscalar mass in Sec. 2, and then proceed to leptonic decays in Sec. 3. The discussion is based on the results in Ref. [2], and the reader is referred there for further technical details.

2 Pseudoscalar Mass

To study the finite-size scaling in the mass m_P(L) of a charged hadronic spin-0 particle P, we first define the full QCD+QED infinite-volume two-point Euclidean correlation function

\[ C_2^\infty(p) = \int d^4x \langle 0 | T[\phi(x)\phi^\dagger(0)] | 0 \rangle e^{-ipx}. \]  

(1)

Here φ is an interpolating operator coupling to P, and p = (p_0, \mathbf{p}) is the momentum. We denote the finite-volume counterpart of this correlator \( C_2^L(p) \), but for the moment only consider \( C_2^\infty(p) \). This can be diagrammatically represented as

\[ C_2^\infty(p) = \begin{array}{c} \phi \end{array} \begin{array}{c} \phi \end{array} = Z_P \cdot D(p) \cdot Z_P, \quad D(p) = \frac{Z(p^2)}{p^2 + m_P^2}, \quad Z_P = \langle 0 | \phi(0) | P, p \rangle, \]

(2)

where the double-line represents the QCD+QED propagator D(p), the φ-blob is the overlap between φ and P and \( Z(p^2) = 1 + O(p^2 + m_P^2) \) is the residue of the propagator. Expanding \( C_2^\infty(p) \) in (2) around \( e = 0 \) yields

\[ \begin{array}{c} \phi \end{array} \begin{array}{c} \phi \end{array} = \phi_0 \phi_0 + \phi_0 \phi_0 + \phi_0 \phi_0 + O(e^4), \]

(3)
where quantities with subscript 0 are evaluated in QCD alone. The grey blob is the Compton scattering kernel defined via

\[
C_{\mu\nu}(p, k, q) = \int d^4x \, d^4y \, d^4z \, e^{i(px + iy + iz)} \left\{ 0 \, T[\phi(0)J_\mu(x)J_\nu(y)\phi^\dagger(z)] \right\}_{0} / Z_{2p}^2 D_0(p)D_0(p + k + q). \tag{4}
\]

Here \( k \) and \( q \) are incoming photon momenta and \( J_\mu(x) \) is the electromagnetic current. Note that the unphysical dependence on the arbitrary interpolating operator \( \phi \) must cancel for any physical quantity, and when the external legs in \( C_{\mu\nu}(p, k, q) \) go on-shell the kernel is nothing but the physical forward Compton scattering amplitude. Using (3) the electromagnetic mass-shift of the meson is readily obtained in terms of an integral over the photon loop-momentum \( k \). One may follow an equivalent procedure for the finite-volume correlation function \( C_{2L}^T(p) \), where the integral over the spatial momentum \( k \) is replaced by a sum. The leading electromagnetic FVEs in the mass, \( \Delta m_P^2(L) \), are thus given by the sum-integral difference

\[
\Delta m_P^2(L) = -\frac{e^2}{2} \lim_{p^2 \to -m_P^2} \left( \frac{1}{L^3} \sum' k - \int \frac{d^3k}{(2\pi)^3} \sum_{k} \left\{ \frac{C_{\mu\nu}(p, k, -k)}{k^2} \right\}_{k=0} \right), \tag{5}
\]

where the rest-frame \( p = 0 \) was chosen for convenience and the primed sum indicates the omission of the photon zero-mode \( k = 0 \) in QED\(_L\). The analytical dependence on \( 1/L \) including structure-dependence can now be obtained from this formula through a soft-photon expansion of the integrand, i.e. an expansion order by order in \( |k| \) which is directly related to the expansion in \( 1/L \) via \( |k| = 2\pi|n|/L \) where \( n \) is a vector of integers. The first step is to decompose \( C_{\mu\nu}(p, k, q) \) into two irreducible electromagnetic vertex functions \( \Gamma_1 \) and \( \Gamma_2 \) according to

\[
\begin{align*}
\begin{array}{c}
\Gamma_1 \quad \Gamma_1 \quad \Gamma_2 \\
C \quad C \quad C
\end{array}
\end{align*}
\]

(6)

The vertex functions depend in general on the structure of the particle, as can be seen from e.g. the form-factor decomposition

\[
\Gamma_1 = \Gamma_1(p, k) = (2p + k)\mu F(k^2, (p + k)^2, p^2) + k_\mu G(k^2, (p + k)^2, p^2), \tag{7}
\]

where \( F(k^2, (p + k)^2, p^2) \) and \( G(k^2, (p + k)^2, p^2) \) are structure-dependent electromagnetic form-factors depending on three virtualities. This means that \( F \) and \( G \) contain off-shell effects, but we stress that these non-physical quantities always cancel in the FVEs. The cancellation occurs since the vertex functions \( \Gamma_{1,2} \) are related to each other and the propagator \( D_0(p) \) via Ward identities. An example of an off-shell relation is \( F(0, p^2, -m_{2P0}^2) = Z_0(p^2)^{-1} \). The derivatives of \( Z_0(p^2) \) are already known in the literature as \( \delta D^{(n)}(0) \) [3] and \( z_0 \) [4], but these could in principle be set to zero as they always cancel in the final results. The Ward identities further yield \( G \) as a function of \( F \). The form-factor \( F \) also contains physical information, and for our purposes it suffices to know that \( F^{(1,0,0)}(0, -m_{2P0}^2, -m_{2P0}^2) = F^0(0) = -\langle r_P^2 \rangle /6 \), where \( \langle r_P^2 \rangle \) is the physical electromagnetic charge radius of \( P \) which is well-known experimentally [5].

Using our definitions of the vertex functions in \( C_{\mu\nu}(p, k, q) \) in (5) we obtain the FVEs

\[
\Delta m_P^2(L) = e^2 m_P^2 \left\{ \frac{c_2}{4\pi^2 m_P L} + \frac{c_1}{2\pi (m_P L)^2} + \frac{\langle r_P^2 \rangle}{3 m_P L^3} + \frac{C}{(m_P L)^3} + O \left[ \frac{1}{(m_P L)^4} \right] \right\}, \tag{8}
\]

2
where the $c_j$ are finite-volume coefficients specific to QED$_L$, arising from the sum-integral difference in [5]. These are discussed in detail in Ref. [2]. Here we see the charge radius $\langle r^2_p \rangle$ appearing at order $1/L^3$ and its coefficient agrees with that derived within non-relativistic scalar QED [6]. However, there is an additional structure-dependent term $C$ related to the branch-cut of the forward, on-shell Compton amplitude. This contribution can be found, in other forms, also in Refs. [3, 7], and only arises because of the QED$_L$ prescription with the subtracted zero-mode. Its value is currently unknown but one can show $C > 0$ [2], meaning that it cannot cancel the charge radius contribution. Note that all unphysical off-shell contributions from the form-factors $F$ and $G$ have vanished.

### 3 Leptonic Decays

Leptonic decay rates of light mesons are of the form $P^- \to \ell^- \bar{\nu}_\ell$, where $P$ is a pion or kaon, $\ell$ a lepton and $\nu_\ell$ the corresponding neutrino. These are important for the extraction of the Cabibbo-Kobayashi-Maskawa matrix elements $|V_{us}|$ and $|V_{ud}|$ [8,9]. The leading virtual electromagnetic correction to this process yields an infrared (IR) divergent decay rate $\Gamma_0$. One must therefore add the real radiative decay rate $\Gamma_1(\Delta E)$ for $P^- \to \ell^- \bar{\nu}_\ell \gamma$, where the photon has energy below $\Delta E$, to cancel the IR-divergence in $\Gamma_0$. The IR-finite inclusive decay rate is thus $\Gamma(\ell^- \nu_\ell(\gamma))$, and following the lattice procedure first laid out in Ref. [8] we may write

$$\Gamma_0 + \Gamma_1(\Delta E) = \lim_{L \to \infty} \Gamma_0^\text{uni}(L) + \Gamma_1(L, \Delta E).$$

Here, Ref. [8] chose to add and subtract the universal finite-volume decay rate $\Gamma_0^\text{uni}(L)$, calculated in point-like scalar QED in Ref. [4], to cancel separately the IR-divergences in $\Gamma_0$ and $\Gamma_1$. In the following we are interested in only the first term in brackets. The subtracted term $\Gamma_0^\text{uni}(L)$ cancels the FVEs in $\Gamma_0^\text{uni}(L)$ through order $1/L$, and hence $\Gamma_0^\text{uni}(L) - \Gamma_0^\text{uni}(L) \sim \mathcal{O}(1/L^2)$. Structure-dependence enters at order $1/L^2$. With the goal of systematically improving the finite-volume scaling order by order including structure-dependence, we replace the universal contribution by

$$\Gamma_0^\text{uni}(L) \to \Gamma_0^{(n)}(L) = \Gamma_0^\text{uni}(L) + \sum_{j=2}^n \Delta \Gamma_0^{(j)}(L),$$

where $n \geq 2$ and $\Delta \Gamma_0^{(j)}(L)$ contains the FVEs at order $1/L^j$. This means that the finite-volume residual instead scales as $\Gamma_0^\text{uni}(L) - \Gamma_0^{(n)}(L) \sim \mathcal{O}(1/L^{n+1})$. We may parametrise $\Gamma_0^{(n)}(L)$ in terms of a finite-volume function $Y^{(n)}(L)$ according to

$$\Gamma_0^{(n)}(L) = \Gamma_0^\text{free} \left[ 1 + 2 \frac{\alpha}{4\pi} Y^{(n)}(L) \right] + \mathcal{O} \left( \frac{1}{L^{n+1}} \right),$$

where $\Gamma_0^\text{free}$ is the tree-level decay rate.

Since we are interested in the leading structure-dependent contribution we consider $Y^{(2)}(L)$. In order to derive it, we define the QCD+QED correlation function

$$C_W^{s}(p, p_{\nu}) = \int d^4z \ e^{ipz} \langle \ell^-, p_{\ell}, r; \nu_\ell, p_{\nu_\ell}, s | T[\mathcal{O}_W(0)\phi^\dagger(z)] | 0 \rangle,$$

$$3$$
where \( p_\ell = (p_\ell^0, \mathbf{p}_\ell) \) is the momentum of the on-shell lepton of mass \( m_\ell \), \( p_\nu_\ell = (p_\nu_\ell^0, \mathbf{p}_\nu_\ell) \) is the momentum of the massless neutrino and \( \mathcal{O}_W(0) \) is the four-fermion operator of the decay in question. We may diagrammatically represent this in a similar way as for the mass according to

\[
C^{rs}_W(p, p_\ell) = \begin{array}{c}
\phi \\
\overline{\mathcal{M}} \\
\phi_0 \\
\overline{\mathcal{M}_0} \\
\overline{W}
\end{array} = \begin{array}{c}
\phi \\
\overline{\mathcal{M}} \\
\phi_0 \\
\overline{\mathcal{M}_0} \\
\overline{W}
\end{array} + \begin{array}{c}
\phi \\
\overline{\mathcal{M}} \\
\phi_0 \\
\overline{\mathcal{M}_0} \\
\overline{W}
\end{array}.
\]

(13)

The grey blob containing \( W \) is of order \( e^2 \) and can be separated, just like the Compton amplitude, into several irreducible vertex functions. The exact definitions of these vertex functions are quite involved and can be found in Ref. [2], but several comments can be made. First of all, the vertex functions are related to various structure-dependent form-factors containing both on-shell and off-shell information. Again, the off-shellness must cancel. The vertex functions also contain physical structure-dependent information (similar to how \( \Gamma_1 \) depends on the charge radius) and for \( Y^{(2)}(L) \) this is the axial-vector form-factor \( F_A(-m_\ell^2) = F_A^P \) from the real radiative decay \( P^- \rightarrow \ell^- \bar{\nu}_\ell \gamma \).

By performing the amputation on the external meson leg in (12) to obtain the matrix element needed for the decay rate in (11), one finds the finite-volume function \( Y^{(2)}(L) \) to be

\[
Y^{(2)}(L) = \frac{3}{4} + 4 \log \left( \frac{m_\ell}{m_W} \right) + 2 \log \left( \frac{m_W L}{4\pi} \right) + \frac{c_3 - 2(c_3(v_\ell) - B_1(v_\ell))}{2\pi} - 2 A_1(v_\ell) \left[ \log \left( \frac{m_P L}{2\pi} \right) + \log \left( \frac{m_\ell L}{2\pi} \right) - 1 \right] - \frac{1}{m_P L} \left[ \frac{(1 + r_\ell^2)^2 c_2 - 4 r_\ell^2 c_2(v_\ell)}{1 - r_\ell^4} \right] + \frac{1}{(m_P L)^2} \left[ \frac{E_A^P}{f_P} \frac{4\pi m_P [(1 + r_\ell^2)^2 c_1 - 4 r_\ell^2 c_1(v_\ell)]}{1 - r_\ell^4} + \frac{8\pi [(1 + r_\ell^2) c_1 - 2 c_1(v_\ell)]}{(1 - r_\ell^2)} \right].
\]

(14)

Here, \( r_\ell = m_\ell/m_P, v_\ell = p_\ell/E_\ell \) the lepton velocity in terms of the energy \( E_\ell \), and \( m_W \) the W-boson mass. Also, \( c_k, A_1(v_\ell), B_1(v_\ell) \) and \( c_j(v_\ell) \) are finite-volume coefficients defined in Ref. [2]. Note that no unphysical quantities appear. At order \( 1/L^2 \), there is one structure-dependent contribution proportional to \( F_A^P \) and the other term is purely point-like. This result is in perfect agreement with Ref. [4] for the universal terms up to \( \mathcal{O}(1/L) \), which we derived in a completely different approach. The numerical impact of the \( 1/L^2 \)-corrections is studied in Ref. [2].

4 Conclusions

We have presented a relativistic and model-independent method to derive electromagnetic FVEs beyond the point-like approximation. We are currently working to obtain the leading FVEs for semi-leptonic kaon decays, relevant for future precision tests in the SM flavour physics sector.

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