Asymptotic series for the splitting of separatrices near a Hamiltonian bifurcation

V. Gelfreich and N. Brännström*

Mathematics Institute
University of Warwick
V.Gelfreich@warwick.ac.uk
N.L.A.Brannstrom@warwick.ac.uk

June 14, 2008

Abstract

This is a proof of an asymptotic formula which describes exponentially small splitting of separatrices in a generic analytic family of area-preserving maps near a Hamiltonian saddle-centre bifurcation. As a particular case and in combination with an earlier work on a Stokes constant for the Hénon map (Gelfreich, Sauzin (2001)), it implies exponentially small transversality of separatrices in the area-preserving Hénon family when the multiplicator of the fixed point is close to one.

Contents

1 Introduction 3
2 Overview of the proof 7
   2.1 Standard scaling and limit flow 6
   2.2 Formal interpolation and Formal separatrix 10
   2.3 Separatrices for close-to-identity maps 12
   2.4 Parametrisation the separatrices 13

*This work is partially supported by the EPSRC grant EP/C000595/1.
2.5 Extension towards the singularity ............................. 14
2.6 Complex matching ........................................... 15
2.7 The stable manifold .......................................... 18
2.8 Upper bounds for the splitting of separatrices ......... 19
2.9 The flow box .................................................. 20
2.10 Splitting function ............................................. 23
2.11 Asymptotic expansion for the homoclinic invariant ........ 27

3 Normal form for the bifurcation .............................. 30
3.1 Formal series and quasi-homogeneous polynomials .......... 30
3.2 Formal interpolation .......................................... 32
3.3 Simplification of the interpolating Hamiltonian ........... 36

4 Formal separatrix ............................................... 38
4.1 Auxiliary functions ........................................... 38
4.2 Formal separatrix of the flow ............................... 39
4.3 Re-expansion near the singularity .......................... 48

5 Close to identity maps .......................................... 49
5.1 Fixed points and their multipliers ........................... 50
5.2 Formal interpolation by a flow .............................. 51
5.3 Approximation of the local separatrix ...................... 53
5.4 Extension lemma .............................................. 56
5.5 Application of the extension lemma ......................... 61
There has been a substantial progress in the study of exponentially small splitting of separatrices (see for example [4, 2, 19, 21, 22]). In most cases exponentially small asymptotic expansions are very sensitive to the form of the system. In the case of local bifurcations with an integrable normal form some generically valid formulae have been obtained [18, 11, 12]. In degenerate cases the splitting of separatrices may be studied by the Melnikov method [2, 4] but application of this method to exponentially small phenomena add substantial restrictions onto systems under consideration (see e.g. survey [13]). Since the pioneering work by Lazutkin, it is accepted that in the case when Melnikov method fails the asymptotic formulae contain a special pre-factor which comes from a parameter-independent problem and can be interpreted as a Stokes constant (see e.g. [16]). For the class of system considered in the present paper, that problem is considered separately in [14].

We note that transversality of separatrices plays an important role in some applications ([6, 7]).

1 Introduction

Let $f_\epsilon$ be an analytic family of area-preserving maps defined in a neighbourhood of the origin. We assume that when $\epsilon = 0$ the origin is a parabolic fixed point of the map. This means that $f_0(0) = 0$ and one is a double eigenvalue of the Jacobian $Df_0(0)$. These eigenvalues are called multipliers of the fixed point. In the generic case the Jacobian is not diagonalisable. Then we may assume that

$$Df_0(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

as this form can be achieved by a linear area-preserving change of variables. Since the multipliers are equal to one the fixed point is not structurally stable and can be destroyed by a small perturbation of $f_0$. We will consider the non-degenerate bifurcation. This means that two leading coefficients in a normal form of $f_\epsilon$ do not vanish: one of them is a leading non-linear term of $f_0$ and the second one guarantees that $f_\epsilon$ is a “generic unfolding” of $f_0$.

Let us state these two assumptions in a more precise way. We will see that the bifurcation can be described by an integrable normal form. We will provide a detailed description of the normal form later but at the moment we note that the map $f_\epsilon$ can be approximated by a time one map of an autonomous Hamiltonian flow. The leading order of the Hamiltonian has the form

$$h_0 = \frac{y^2}{2} + \frac{ax^3}{3} - bx\epsilon.$$
In this paper we assume

\[ a, b > 0. \tag{1.3} \]

The positivity assumption is not more restrictive than \( ab \neq 0 \). Indeed, the sign of \( a \) and \( b \) can be changed by substitutions \( x \mapsto -x \) and \( \varepsilon \mapsto -\varepsilon \). Using assumption (1.3) we ensure that the phenomena we study in this paper are on the side of positive \( \varepsilon \).

Near the bifurcation the map \( f_\varepsilon \) is rather accurately approximated by the time one map \( \Phi^1_{h_\varepsilon} \) generated by the Hamiltonian (1.2). When \( \varepsilon \) changes sign an equilibrium of \( h_\varepsilon \) bifurcates following the scenario sketched in Figure 1. For \( \varepsilon > 0 \), separatrices of the saddle point form a loop around the elliptic fixed point.

Under these assumptions \( f_\varepsilon \) with \( \varepsilon > 0 \) has two fixed points which collide at the origin when \( \varepsilon = 0 \) and then disappear. One of the fixed points is elliptic (centre) and another one is hyperbolic (saddle). For this reason the bifurcation is sometimes called a Hamiltonian saddle-centre bifurcation.

The bifurcation of the map is very similar to the bifurcation of the leading order normal form (1.2). Unlike the normal form, we do not expect a generic map to be integrable and therefore the separatrices of the map may split. We will derive an asymptotic formula which describes the splitting of the separatrices. This formula implies that in a generic family of maps the separatrices intersect transversally for all sufficiently small \( \varepsilon \).

We note that the area preserving Hénon map satisfies assumptions of our theorem.

The leading order of the asymptotic expansion was described in [11, 12]. The goal of this paper is to present a complete proof of a refined version of this asymptotic formula. The proof is based on the programme of [10] and is a development of the original idea proposed by Lazutkin in [17].

Let us state the main result of the paper. First we note that many of the series involved in our proof involve powers of \( \varepsilon^{1/4} \). It is therefore convenient to introduce a new small parameter \( \delta = \varepsilon^{1/4} \). We will occasionally use both
of them indicating the type of a power series expansion for the corresponding function.

In this paper we will study separatrices of a family of hyperbolic fixed points:

$$p_\delta = f_\varepsilon(p_\delta)$$

such that $p_0 = 0$. Let $\lambda_\delta \geq 1$ be the largest of the two multipliers of $f_\varepsilon$ at $p_\delta$. We will prove that $p_\delta$ and $\lambda_\delta$ depend analytically on $\delta$. Moreover, $\log \lambda_\delta$ is of the same order of smallness as $\delta$ itself. We will see that

$$\log \lambda_\delta = \sqrt[4]{4ab} \delta + O(\delta^3). \quad (1.4)$$

It is convenient to represents stable and unstable separatrices $W^\pm(p_\delta)$ using solutions of the following finite-difference equation (for a more detailed discussion see the review [13])

$$\psi^\pm(t + \log \lambda_\delta) = f_\varepsilon \circ \psi^\pm(t). \quad (1.5)$$

We have suppressed explicit dependence of $\psi^\pm$ on $\delta$ to shorten the notation. An application of $f_\varepsilon$ increases the parameter $t$ by $\log \lambda_\delta > 0$. Therefore the conditions

$$\lim_{t \to -\infty} \psi^-(t) = p_\delta \quad \text{and} \quad \lim_{t \to +\infty} \psi^+(t) = p_\delta$$

imply that $\psi^-$ represents the unstable separatrix and $\psi^+$ represents the stable one. It is easy to see that these conditions do not defined the functions $\psi^\pm$ uniquely.

The most convenient parameterization is obtained using analytic linearizations of restrictions of $f_\varepsilon$ onto $W^-(p_\delta)$ and $W^+(p_\delta)$ respectively. These parametrisations can be represented in the form $\psi^-(t) = \Psi^-(e^t)$ and $\psi^+(t) = \Psi^+(e^{-t})$ where $\Psi^\pm(z)$ are analytic at 0 and $\Psi^\pm(0) = p_\delta$. This choice reduces the freedom in $\psi^\pm$ to a translation $t \mapsto t + t_0$. Our proof of their existence does not use the linearization but relies on a contraction mapping argument which has the advantage of providing an accurate approximation for the functions.

From the technical point of view a substantial part of the present paper is dedicated to a detailed study of the analytic continuation for $\psi^\pm$.

Let $\gamma \in W^-(p_\delta) \cap W^+(p_\delta)$ be an homoclinic point. Then

$$\gamma = \psi^+(t_s) = \psi^-(t_u)$$

for some $t_u$ and $t_s$. The vectors $\dot{\psi}^-(t_u)$ and $\dot{\psi}^+(t_s)$ are tangent respectively to $W^-(p_\delta)$ and $W^+(p_\delta)$ at $\gamma$. The area of the parallelogram generated by these tangent vectors is called the homoclinic invariant of $\gamma$:

$$\omega(\gamma) = \Omega \left( \dot{\psi}^-(t_s), \dot{\psi}^+(t_u) \right),$$
where $\Omega$ stands for the standard area form. The homoclinic invariant has some advantages over alternative measures for the separatrices splitting (see the review [13] for a discussion).

**Theorem 1.1 (Main Theorem).** There are constants $a_k \in \mathbb{R}$ such that the homoclinic invariant of each of the two primary homoclinic orbits has the form

$$\omega \simeq \pm \frac{2\pi}{\log \lambda_2} e^{-\frac{2\pi^2}{\log \lambda_2^2}} \sum_{k \geq 0} a_k \delta^{2k}.$$  \hfill (1.6)

This theorem implies that for a generic family the separatrices intersect transversally. Indeed, we note that in (1.6) the pre-factor is exponentially small compared to $\delta$ due to the smallness of $\log \lambda_2$ estimated in (1.4). Nevertheless for small $\delta > 0$ the theorem implies transversality of the homoclinic points provided $a_0 \neq 0$. We note that $a_0$ describes the splitting of complex parabolic invariant manifolds for function $f_0$ and does not vanish generically (on an open dense subset). In [15] it was proved that $a_0 \neq 0$ for the Hénon map. Finally, numerical evaluation of $a_0$ for a given $f_0$ is a relatively easy task.

The difficulty of this theorem is due to the exponential smallness of the separatrices splitting which is to be detected on the background of much larger effects.

The proof consists of the following main steps:

1. We show that $f_\varepsilon$ can be formally represented as a time-one map of a Hamiltonian flow and provide a normal form theory for the flow.

2. We develop a general theory of close-to-identity maps which provides sufficient condition for the existence of saddle points and their separatrices. Moreover, we prove that the separatrices of the map can be quite accurately approximated by separatrices of a flow which approximately interpolates the map.

3. We show that In the complexified time domain the separatrix of the interpolating flow is close to the separatrix of $f_\varepsilon$ in a set which has a non-empty intersection with a $\delta$-neighbourhood of $\pm i\pi$.

4. We show that in the latter region an approximation based on the separatrices of the parabolic fixed point of $f_0$ provides an even more accurate approximation. This approximation distinguishes between the stable and unstable separatrices of $f_\varepsilon$. 

6
5. We show that flow-box time coordinates can be introduced.

6. Flow-box coordinates are used to get an asymptotic expansion for the splitting of separatrices.

The rest of the paper is dedicated to proving the main theorem and is structured in the following way. Section 2 contains a detailed overview of the proof leaving technical estimates to corresponding sections. Section 3 describes the construction of the normal form for the bifurcation up an arbitrary order. Section 5 develops a general theory of close to identity maps, which can be of independent interest since it covers a much wider class of maps than that studied in other parts of this paper. In Section 4 we provide a detailed description of a formal solution of the separatrix equation (1.5).

We will prove that $f_\varepsilon$ can be analytically interpolated by an autonomous analytic flow and establish properties for the corresponding energy-time coordinates. Note that this property does not imply integrability of $f_\varepsilon$ since the domain in question is not invariant. Several sections are included to provide details on approximation of $\psi^-(t)$ on various subsets of the complex plane.

# 2 Overview of the proof

In this section we will describe the main steps of the proof postponing proofs of technical statements to subsequent sections. Before proceeding to the proof we need to introduce some definitions.

In this paper we will use power series in $x, y$ and $\varepsilon$. The arguments are easier when terms of a similar magnitude are grouped together. We consider $x$ to be of order 2, $y$ to be of order 3 and $\varepsilon$ of order 4. We say that a polynomial $Q_p$ is quasi-homogeneous of order $p$ if it satisfies the identity

$$Q_p(\lambda^2 x, \lambda^3 y, \lambda^4 \varepsilon) = \lambda^p Q_p(x, y, \varepsilon).$$  \hspace{1cm} (2.1)

Any power series can be represented as a sum of quasi-homogeneous polynomials.

In order to shorten the notation, we define

$$\hat{O}_n := O \left( \left( \max\{ |x|^{1/2}, |y|^{1/3}, \varepsilon^{1/4} \} \right)^n \right)$$

and write $\hat{O}_{n,n+1}$ for a vector with components $(\hat{O}_n, \hat{O}_{n+1})$. It is easy to see that $Q_p = \hat{O}_p$. 

Figure 2: Overview of the proof: Domains of approximation
2.1 Standard scaling and limit flow

The change of the variables

\[ x = \delta^2 X, \quad y = \delta^3 Y, \quad \varepsilon = \delta^4 \]  

is called the standard scaling. In the new variables the map \( f_\varepsilon \) takes the form of a close to identity map

\[ F_\delta = \text{id} + \delta G_\delta. \]  

We note that \( G_0 \) is called the limit flow. Its time-\( \delta \) map \( \Phi_{\delta}^t = \text{id} + \delta G_0 + O(\delta^2) \) and therefore is \( O(\delta^2) \) close to \( F_\delta \).

Taking into account that \( f_0(0) = 0 \) and the equality (1.1) we obtain

\[ G_0 = \begin{pmatrix} Y \\ -aX^2 + b \end{pmatrix} \]

for some constants \( a, b \). The same constants enter the non-degeneracy assumption (1.3). Indeed, the limit flow is Hamiltonian with the Hamilton function

\[ H_0 = \frac{Y^2}{2} + \frac{aX^3}{3} - bX \]

which coincides, after reverting the standard scaling and scaling time, with (1.2). The dynamics of the limit flow is defined by the system of Hamiltonian equations:

\[ \dot{X} = Y, \quad \dot{Y} = -aX^2 + b. \]

The limit flow has a saddle equilibrium at \((-\sqrt{\frac{b}{a}}, 0)\). At this point the linearised vector-field has a positive eigenvalue \( \mu_0 = \sqrt{4ab} \). The separatrix of the limit flow can be found explicitly by integrating the Hamiltonian equations:

\[ X_0(t) = -\sqrt{\frac{b}{a}} + \frac{3\sqrt{ab}}{\cosh^2 \frac{\mu_0 t}{2}}, \quad Y_0(t) = -3\sqrt{ab} \mu_0 \frac{\sinh \frac{\mu_0 t}{2}}{\cosh^3 \frac{\mu_0 t}{2}}. \]

A substitution shows that these functions satisfy the Hamiltonian system. Obviously they converge to the saddle equilibrium as \( t \to \pm \infty \).

It is convenient to scale time and define the separatrix solution of the limit flow by

\[ \varphi_0(t) = (X_0, Y_0)(t/\mu_0). \]  

This function satisfies the differential equation

\[ \dot{\varphi}_0 = \mu_0^{-1} J \nabla H_0(\varphi_0). \]
We will consider the separatrix solution for complex values of $t$. We note that $\varphi_0$ is analytic for all complex $t$ except for second order poles at $i\pi(2k + 1)$ with $k \in \mathbb{Z}$. In particular this solution is analytic in the strip $|\text{Im}(t)| < \pi$ and has poles at the points $t_* = \pm \pi i$ on its boundary. These singularities play the central role in our analysis.

### 2.2 Formal interpolation and Formal separatrix

We use $h^n = \sum_{p=0}^{n+5} h_p(x, y, \varepsilon)$ to denote a partial sum of a formal series

$$h_\varepsilon = \sum_{p \geq 6} h_p(x, y, \varepsilon) \quad (2.5)$$

where $h_p(x, y, \varepsilon)$ are quasi-homogeneous polynomials of order $p$. In Section 3.2 we will prove that there exists a unique formal series such that for each $n \in \mathbb{N}$

$$f_\varepsilon = \Phi^1_{h^n} + \dot{\mathcal{O}}_{n+3,n+4}, \quad (2.6)$$

where $\Phi^1_{h^n}$ is the time one map with hamiltonian $h^n$. The leading order of the series is given by (1.2). It is a quasi-homogeneous polynomial of order 6 and coincides, up to scaling of the space and time variables, with the Hamiltonian of the limit flow.

We say that the series $h_\varepsilon$ formally interpolates the map $f_\varepsilon$. We should not expect the series $h_\varepsilon$ to converge.

**Theorem 2.1** (Formal separatrix). If $h_\varepsilon$ is a formal Hamiltonian of the form (2.5) and its leading order satisfies the non-degeneracy condition (1.3), then there is a unique formal series

$$\mu(\delta) = \sum_{k \geq 0} \mu_k \delta^{2k+1}$$

such that the equation

$$\mu(\delta) \dot{X} = J \nabla h_\varepsilon(X) \quad (2.7)$$

has a formal solution of the form

$$X(t, \varepsilon) = \left( \sum_{p \geq 1} \delta^{2p} X^1_p + \dot{\eta}_0 \sum_{p \geq 1} \delta^{2p+1} X^2_{p-1} \right)$$

$$+ \dot{\eta}_0 \sum_{p \geq 1} \delta^{2p+1} Y^2_{p-1} + \sum_{p \geq 2} \delta^{2p} Y^1_p \quad (2.8)$$

where $X^1_p, Y^1_p, X^2_p, Y^2_p$ are polynomials of order $p$ in $\eta_0 = \cosh^{-2} \frac{t}{2}$.
The proof of this theorem is placed in Section 4. We remind that $\delta = \varepsilon^{1/4}$.

We say that $X$ is a formal separatrix. The series $h_\varepsilon$ may diverge and consequently there is no reason to expect that the formal separatrix converges. On the other hand we will see that its partial sums provide a rather accurate approximation for the separatrices of the map $f_\varepsilon$.

Let us consider the domain

$$T_0 = \{ t \in \mathbb{C} : \text{Re}(t) \leq r_1 \text{ and } \varphi_0(t - s) \in D, \forall s \geq 0 \}$$

(2.9)

where $r_1 > 0$ is an arbitrary constant and $D$ is a bounded domain. Later we will assume it to be a sufficiently large ball centred around the origin. Of course, some constants in future estimates will depend on the choice of $r_1$ and $D$.

**Lemma 2.2.** For any $n \in \mathbb{N}$, the Hamiltonian $h^n_\delta$ has a saddle equilibrium with a Lyapunov exponent $\mu_{n,\delta} > 0$ in a neighbourhood of the origin. Moreover, there exists a separatrix solution $\varphi^n_\delta$ of the equation

$$\varphi_\delta^n = \mu_{n,\delta}^{-1} J \nabla h^n_\delta |_{\varphi^n_\delta}$$

(2.10)

such that uniformly for $t \in T_0$

$$\varphi^n_\delta(t) = X^n(t, \delta) + O(\delta^n),$$

where $X^n$ denotes the sum of the first $n$ orders in $\delta$ of the formal series (2.8).

**Proof.** The proof is straightforward: this lemma is a statement about polynomial vector fields and after the standard scaling the saddle of the equation is non-degenerate.

Let $X$ be the formal separatrix given by (2.8). We will study it close to the singularity by substituting $t = i \pi + \tau \log \lambda_\delta$ and expanding $X^{1,2}_k, Y^{1,2}_k$ into their Laurent series. We also re-expand $\log \lambda_\delta = \mu(\delta)$. The result is a formal series of the form

$$X = \left( \sum_{m \geq 0} \delta^{2m} \tau^{2m-2} \sum_{k \geq 0} x_{mk} \tau^{-k}, \sum_{m \geq 0} \delta^{2m} \tau^{2m-3} \sum_{k \geq 0} y_{mk} \tau^{-k} \right),$$

(2.11)

where $x_{mk}$ and $y_{mk}$ are real coefficients. See Section 4.3 for a derivation of (2.11). It is convenient to introduce formal power series in $\tau$ by setting

$$\hat{\psi}_m = \left( \tau^{2m-2} \sum_{k \geq 0} x_{mk} \tau^{-k}, \tau^{2m-3} \sum_{k \geq 0} y_{mk} \tau^{-k} \right)$$

(2.12)
for $m \geq 0$. Then
\[ X = \sum_{m \geq 0} \delta^{2m} \hat{\psi}_m(\tau). \quad (2.13) \]

We will use these formal series in the complex matching procedure described in Section 2.6.

### 2.3 Separatrices for close-to-identity maps

Let us consider a family of close to identity maps (2.3). For the purpose of this section it is not necessary to assume that the map is obtained as a result of the standard scaling.

The implicit function theorem implies that if the limit flow $G_0$ has a non-degenerate equilibrium then $F_\delta$ has an analytic family of fixed points which tend to the equilibrium when $\delta \to 0$. If the equilibrium is a saddle then the fixed point of $F_\delta$ is also a saddle (see Section 5.1 for formal statements and proofs).

The separatrix of $F_\delta$ is close to the separatrix of the limit flow (see [9]). We will use a more accurate approximation provided by a separatrix of an (approximately) interpolating flow. Let us state it more formally.

Let $D \subset \mathbb{C}^2$ be a bounded domain and assume the origin is inside $D$.

**Theorem 2.3** (Approximation theorem). Let $F_\delta$ be an analytic family of area preserving maps such that the limit flow has a saddle equilibrium at the origin. Let $H_\delta^n$ be a Hamiltonian function such that
\[ F_\delta = \Phi_{H_\delta^n} + O(\delta^{n+1}) \quad (2.14) \]
on $D$ and $\phi_\delta^n$ be a separatrix solution of the equation
\[ \dot{\phi}_\delta^n = \mu_{n, \delta} \nabla H_\delta^n \]
such that
\[ \phi_\delta^n(t) = \varphi_0(t) + O(\delta) \]
for all $t \in T_0$. Then there exists a parametrisation $\Psi^-$ of the local unstable separatrix of the map $F_\delta$ which satisfies the equation
\[ \Psi^-(t + \log \lambda_\delta) = F_\delta(\Psi^-(t)) \quad (2.15) \]
and
\[ \Psi^-(t) = \phi_\delta^n(t) + O(\delta^n) \quad (2.16) \]
uniformly on the set $T_0$. 
The proof of this theorem consists of two steps. First the theorem is proved for the local separatrix which corresponds to a half-plane $\text{Re} t < -r_0$ as described in Section 5.3. Then an extension lemma (Lemma 5.3) stated and proved in Section 5.4 is applied.

For completeness, we also provide a proof of the existence for the Hamiltonian $H^n_\delta$ in Section 5.2.

### 2.4 Parametrisation the separatrices

We note that the existence of a parameterisation which satisfies equation (1.5) follows from the general theory [13]. The parametrisation is not unique and is defined up to a translation $t \mapsto t + t_0(\varepsilon)$. The next theorem states that there is a parameterisation which is close to the separatrix of the interpolating flow.

Let $\varphi^n_\delta$ be defined by Lemma 2.2.

**Theorem 2.4 (Existence and Local approximation).** Equation (1.5) has a solution

$$\psi^-(t) = \varphi^n_\delta(t) + O_{n+2,n+3}(\delta)$$  \hspace{1cm} (2.17)

uniformly for $t \in T_0$.

**Proof.** The theory of the previous section applies to the map $F_\delta$ obtained after the standard scaling of the map $f_\varepsilon$. Let $D$ be a large ball of a fixed radius. Since the map $f_\varepsilon$ is defined in an $\varepsilon$ independent neighbourhood of the origin, the domain of the scaled map $F_\delta$ contains the ball $D$ provided $\varepsilon$ is sufficiently small.

The limit flow separatrix $\varphi_0$ is given explicitly by (2.4) and has poles at $i\pi(2k + 1)$ for each $k \in \mathbb{Z}$. Therefore the set $T_0$ takes the shape similar to the one shown on Figure 3: it is a half plane $\text{Re} t < r_1$ without “shadows” of small disks centred around the singular points of $\varphi_0$.

The interpolating Hamiltonian is obtained from (2.5) after the change of variables (2.2). Although the change of variables is not symplectic it has a constant Jacobian, and therefore the Hamiltonian flow of $h^n_\delta$ is transformed into a Hamiltonian flow with the Hamiltonian function

$$H^n_\delta = \sum_{k=1}^{n} \delta^k h_{5+k}(X, Y, 1).$$  \hspace{1cm} (2.18)

Indeed, under the standard scaling the symplectic form becomes $dx \wedge dy = \delta^n dX \wedge dY$, to work with the standard symplectic form in the scaled variables we scale the Hamiltonian by $\delta^{-5}$. The upper bound (2.14), which is necessary to apply Theorems 2.3 follows from (2.6) and (2.2).
To complete the proof we reverse the standard scaling which transforms the error term $O(\delta^n)$ of equation (2.16) into $O_{n+2,n+3}(\delta)$.

### 2.5 Extension towards the singularity

The theory presented in the previous subsection provides a rather accurate approximation for the invariant manifolds. However, it is not sufficient to distinguish between the stable and unstable separatrices of the map $F_\delta$. Instead of further improving the accuracy, we will study the separatrices for values of $t$ closer to the singular points of the limit flow separatrix. In this region the splitting of the separatrices is larger and consequently easier to detect. This extension is sensitive to the form of the map and cannot be performed with the same level of generality as the estimates of Section 2.3.

So from now on we restrict our consideration to our original map $f_\varepsilon$.

Let us fix two small positive constants $c_1$ and $\beta$, and consider a domain $\mathcal{T}_1(\varepsilon)$ shown on Figure 4 (left), which also depends on a constant $c_2 > 0$. The constant $c_2$ is to be chosen sufficiently large to ensure that $\varphi_0$, the separatrix of the limit flow written in the unscaled variables, does not leave a small $\varepsilon$-independent ball centred around the origin.

It is convenient to work with a time parameter centred at the singularity and to scale the time step to one, hence we let

$$\tau = \frac{t - i\pi}{\log \lambda_\delta}.$$ (2.19)

The domain $\mathcal{T}_1(\varepsilon)$ expressed in the terms of $\tau$ is shown on Figure 4 (right).

**Theorem 2.5.** Let $\psi^-$ be defined by Theorem 2.4. There is $c_2 > 0$ such that

$$\psi^-(\tau) = \varphi_\delta^n(\tau) + O_{n,n+1}(\tau^{-1})$$ (2.20)
uniformly for all $\tau \in T_1(\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$.

We note that $t = i\pi + \tau \log \lambda_\delta$ and we have overloaded notation by writing $\psi^-(\tau)$ instead of $\psi^-(i\pi + \tau \log \lambda_\delta) = \psi^-(t)$.

The estimate \[ \text{(2.20)} \] is uniform which means that the constant in the term $O_{n,n+1}(\tau^{-1})$ is independent from both $\varepsilon$ and $\tau$. It is important because the size of $T_1(\varepsilon)$ increases as $\varepsilon \to 0$ and includes $\tau$ up to the order of $\delta^{-1} = \varepsilon^{-1/4}$.

We note that in Theorem \[ \text{2.5} \] the error of the approximation is of order $O(\delta^n)$ on the left boundary of $T_1(\varepsilon)$ but it is just of order $O(1)$ on the boundary near the central circle.

Therefore the theorem suggests that the distance between the separatrix of the map and separatrix of the interpolating flow may gradually increase as the parameter $t$ comes closer to $i\pi$.

### 2.6 Complex matching

In this section we construct another approximation for $\psi^-$ which provides higher accuracy in the central part of $T_1(\varepsilon)$. Let us start by looking for a formal solution

\[ \hat{\psi}(\tau) = \sum_{k \geq 0} \delta^{2k} \psi_k(\tau) \quad \text{(2.21)} \]

of the separatrix equation

\[ \hat{\psi}(\tau + 1) = f_\varepsilon(\hat{\psi}(\tau)). \quad \text{(2.22)} \]

We assume that the new time $\tau$ is related to the original $t$ by \[ \text{(2.19)}. \] We formally substitute the series \[ \text{(2.21)} \] into the equation \[ \text{(2.22)} \] and expand the right hand side in powers of $\delta^2$. Collecting terms of equal order in $\delta$ we get at the leading order

\[ \psi_0(\tau + 1) = f_0(\psi_0(\tau)), \quad \text{(2.23)} \]
and the following system of equations for all other orders:

\[
\begin{align*}
\psi_1(\tau + 1) &= D f_0(\psi_0(\tau))\psi_1(\tau), \\
\psi_2(\tau + 1) &= D f_0(\psi_0(\tau))\psi_2(\tau) + R_2(\psi_0, \psi_1), \\
&\vdots
\end{align*}
\]

In general, the equation corresponding to \( \delta^{2m} \) with \( m \geq 2 \) can be written as

\[
\psi_m(\tau + 1) = D f_0(\psi_0(\tau))\psi_m(\tau) + R_m(\psi_0, \psi_1, \ldots, \psi_{m-1}),
\]

(2.24)

where \( R_m \) is a polynomial in \((\psi_1, \ldots, \psi_{m-1})\). We stress that the dependence on \( \psi_0 \) is not necessarily polynomial. For example,

\[
R_2 = \psi_1^t D^2 f_0(\psi_0)\psi_1 + f_1(\psi_0)
\]

where \( t \) is used to denote the transposition and \( f_1 \) is a coefficient of the Taylor series \( f_ε = \sum_{k \geq 0} ε^k f_k \).

An analytic solution \( \psi_m^- \) of the equation (2.24) can be selected by the following asymptotic condition:

\[
\psi_m^-(\tau) \approx \hat{\psi}_m(\tau) \quad \text{as } \tau \to \infty \text{ in the sector } \beta_0 < \arg \tau < 2\pi - \beta_0. \quad (2.25)
\]

The formal series \( \hat{\psi}_m(\tau) \) comes from the re-expansion of the formal separatrix given by (2.13). In fact, \( \hat{\psi}_m \) is a formal solution of equation (2.24).

The method of fixing a solution in one region by initial or boundary conditions which come from a neighbouring region is known as “complex matching”.

We will now continue with the approximation inside the matching zone, we call this domain \( T_2(ε) \):

\[
T_2(ε) = \{ τ ∈ T_1(ε) : \text{Re } τ > -δ^{1/2}, -δ^{1/2} < \text{Im } τ < δ^{1/2} \}.
\]

**Theorem 2.6 (Second approximation theorem).**

\[
\psi^-(\tau) = \sum_{k=0}^{n-1} δ^{2k} \psi_k^-(\tau) + O(δ^n)
\]

uniformly in \( T_2(ε) \).

We note that this theorem implies the following.
Figure 5: Complex matching.

Figure 6: Complex matching.
Corollary 2.7. There is a parametrisation of the unstable separatrix such that
\[ \psi^- (\tau) = \sum_{k=0}^{n-1} \delta^{2k} \psi_k^- (\tau) + O((\delta \tau)^{2n}). \] (2.26)
uniformly in $\mathcal{T}_2 (\varepsilon)$.

Proof. The asymptotic boundary conditions implies $\psi_k^- (\tau) = O(\tau^{2k})$. Then
the theorem with $n$ replaced by $2n$ implies
\[ \psi^- (\tau) = \sum_{k=0}^{2n-1} \delta^{2k} \psi_k^- (\tau) + O(\delta^{2n}) = \sum_{k=0}^{n-1} \delta^{2k} \psi_k^- (\tau) + O((\delta \tau)^{2n}). \]
\[ \square \]

2.7 The stable manifold

So far we have only considered the unstable separatrix. In order to estimate
the splitting of separatrices we also need to study the stable one. We note
that the stable manifold is an unstable one for the inverse map.

Therefore the study of the stable separatrix is analogous to the study of
the unstable one but with time reversed. The transformation $t \mapsto -t$ swaps
the upper and lower half-planes of $\mathbb{C}$ and it is convenient to use real-analytic
symmetry to swap these back. Let $\mathcal{T}_0^+, \mathcal{T}_1^+ (\varepsilon), \mathcal{T}_2^+ (\varepsilon) \subset \mathbb{C}$ denote domains
obtained by reflecting $\mathcal{T}_0, \mathcal{T}_1 (\varepsilon), \mathcal{T}_2 (\varepsilon)$ in the imaginary axis respectively. The
limit flow and approximately interpolating flows are integrable and their
stable and unstable separatrices coincide. Therefore the same separatrix
solution arise and the same formal expansion (2.13) is used for the complex
matching of the stable separatrix. However, the matching is done on the
right hand side of the singularity with asymptotic boundary conditions as
$\tau \to +\infty$.

Similarly to Theorem 2.4 we conclude that for every $n \in \mathbb{N}$ there is a
solution $\psi^+$ of the equation (1.5) such that
\[ \psi^+ (t) = \varphi^{2n} (t) + O_{2n+2,2n+3} (\delta) \] (2.27)
uniformly for $t \in \mathcal{T}_0^+$ (see Figure 7). Note that we directly started with $n$
replaced by $2n$. For this solution, instead of (2.20) we get
\[ \psi^+ (t) = \varphi^{2n} (t) + O_{2n,2n+1} (\tau^{-1}) \] (2.28)
uniformly in $\mathcal{T}_1^+ (\varepsilon)$ and $\varepsilon \in (0, \varepsilon_0)$ and instead of Corollary 2.7 we get
Corollary 2.8. Uniformly in $\mathcal{T}_2^+(\varepsilon)$

$$\psi^+(\tau) = \sum_{k=0}^{n-1} \delta^{2k} \psi_k^+(\tau) + O(\delta^2 \tau^2)$$  \hspace{1cm} (2.29)

where $\psi_k^+(\tau)$ is the solution of the asymptotic boundary condition problem (2.23) or (2.24) subject to the asymptotic boundary conditions

$$\psi_k^+(\tau) \asymp \hat{\psi}_k(\tau) \quad \text{as} \quad \tau \to \infty \quad \text{in the sector} \quad -\pi + \beta_0 < \arg \tau < \pi - \beta_0.$$

We note that in general $\psi_0^+ \neq \psi_0^-$. \hspace{1cm} \cite{11, 14}.

2.8 Upper bounds for the splitting of separatrices

We assume that both $\psi^-$ and $\psi^+$ are chosen to be closed to $\varphi_\delta^{2n}$. Then comparing the estimates (2.17) and (2.27)

$$\psi^-(t) - \psi^+(t) = O(\delta^2 \tau^{2n})$$

uniformly in $\mathcal{T}_0 \cap \mathcal{T}_0^+$. Then we use the estimates (2.28) (with $n$ replaced by $2n$) and (2.28) to conclude

$$\psi^-(t) - \psi^+(t) = O_{2n,2n+1}(\tau^{-1})$$

uniformly in $\mathcal{T}_1(\varepsilon) \cap \mathcal{T}_1^+(\varepsilon)$. Taking into account the definitions of the domains we see that the union of validity domains of these two estimates includes the rectangle defined by the inequalities $|\text{Im} t| < \pi - \delta^{1/2}$ and $|\text{Re} t| < \delta^{1/2}$ and for these values of $t$

$$\psi^+(t) - \psi^-(t) = O(\tau^{-n}) = O(\delta^n).$$  \hspace{1cm} (2.30)
In the innermost region $T_2(\varepsilon)$, closer to the singularity at $i\pi$, we use a different argument. In this region the estimate has to be close to optimal: we will use the upper bound to show that the square of the distance between the stable and unstable separatrices is negligible compared to the leading term of the splitting (on a line with $\text{Im} \tau = -\sigma \log \delta$).

The upper bound for the splitting near the singularity is obtained in the following way: by Corollaries 2.7 and 2.8

$$
\psi^+(\tau) - \psi^-(\tau) = \sum_{k=0}^{n-1} \delta^{2k} (\psi^+_k(\tau) - \psi^-_k(\tau)) + O((\delta \tau)^{2n})
$$

(2.31)

uniformly in $T_2(\varepsilon) \cap T_2^+(\varepsilon)$. We use that

$$
\psi^+_k(\tau) - \psi^-_k(\tau) = O(\tau^{m_k} e^{-2\pi i \tau})
$$

(2.32)

provided $-\pi + \beta < \text{arg} \tau < -\beta$ and $\text{Im} \tau < -c_2$. Consequently

$$
\psi^+(\tau) - \psi^-(\tau) = O(\tau^4 e^{-2\pi i \tau}) + O((\delta \tau)^{2n})
$$

(2.33)

provided $-c_2 > \text{Im} \tau > -\delta^{-1/2}$ and $-\pi + \beta < \text{arg} \tau < -\beta$.

### 2.9 The flow box

In order to provide a quantitative description for the difference between $\psi^+$ and $\psi^-$ we use the method based on flow box coordinates. We will prove a theorem which essentially says that the restriction of the map $f_\varepsilon$ on a non-invariant subset can be analytically interpolated by an autonomous Hamiltonian flow with one degree of freedom. The subset in question contains large (but again non-invariant) segments of the stable and unstable separatrices. Simultaneously we introduce “energy-time” coordinates $(T,E)$ for the interpolating flow.

Consider a domain in $\mathbb{C}^2$ defined by

$$
\mathcal{U}_\delta = T \times E = \left\{ |\text{Re}(T)| < 2, |\text{Im}(T)| < \frac{\pi}{\log \lambda_\delta} - c_3 \right\} \times \left\{ |E| < E_0(\delta) \right\}.
$$

(2.34)

**Theorem 2.9** (The flow box theorem). If $c_3$ is sufficiently large, there exists a symplectic diffeomorphism $S : \mathcal{U}_\delta \to \mathcal{V}_\delta := S(\mathcal{U}_\delta)$ such that

1. $f_\varepsilon|_{\mathcal{V}_\delta} = \Phi^1_E$, where $E : \mathcal{V}_\delta \to \mathbb{C}$ is the second component of the map $S^{-1}$.
2. $S(T,0) = \psi^-(T \log \lambda_\delta)$ (Normalisation).
3. \( \|S^{-1}\|_{C^2} \) is bounded uniformly for \( \varepsilon \in (0, \varepsilon_0) \).

It is convenient to consider \( S^{-1} \) as a symplectic coordinate map defined on \( V_\delta \). In coordinates \((T,E)\), the Hamiltonian equations with the Hamiltonian function \( E \) take the form

\[
T' = 1, \quad E' = 0
\]

and can be easily integrated. The corresponding time-one map has the form \((T,E) \mapsto (T + 1, E)\). It coincides with \( S^{-1} \circ \Phi^{1}_{E} \circ S \). In the original coordinates the map \( \Phi^{1}_{E} = f_\varepsilon \). Consequently the first property of \( S \) means that \( S \) conjugates \( f_\varepsilon \) and the translation:

\[
(T + 1, E) = S^{-1} \circ f_\varepsilon \circ S(T, E). \tag{2.35}
\]

Let us use \((T(x, y), E(x, y))\) for the components of \( S^{-1} \) and \((x(T, E), y(T, E))\) for the components of \( S \). Equation (2.35) implies

\[
T \circ f_\varepsilon (x, y) = T(x, y) + 1, \quad E \circ f_\varepsilon (x, y) = E(x, y). \tag{2.36, 2.37}
\]
By the inverse function theorem, the Jacobian matrix $DS^{-1} = (DS)^{-1}$. In coordinate form this equality implies that

$$
\begin{pmatrix}
\frac{\partial T}{\partial x} & \frac{\partial T}{\partial y} \\
\frac{\partial E}{\partial x} & \frac{\partial E}{\partial y}
\end{pmatrix} = \left(\begin{pmatrix}
\frac{\partial x}{\partial T} & \frac{\partial x}{\partial E} \\
\frac{\partial y}{\partial T} & \frac{\partial y}{\partial E}
\end{pmatrix}\right)^{-1} = \left(\begin{pmatrix}
\frac{\partial y}{\partial E} & -\frac{\partial x}{\partial E} \\
-\frac{\partial y}{\partial T} & \frac{\partial x}{\partial T}
\end{pmatrix}\right),
$$

where we used that the Jacobian matrix has the unit determinant to get the second equality. The second row gives the following Hamiltonian equations:

$$
\frac{\partial y}{\partial T} = -\frac{\partial E}{\partial x} \quad \text{and} \quad \frac{\partial x}{\partial T} = \frac{\partial E}{\partial y}.
$$

Substituting $E = 0$ into the equations, using the normalisation condition in the form

$$
(x(T, 0), y(T, 0)) = \psi^{-}(T \log \lambda_{s}) = (\psi_{1}^{-}(T \log \lambda_{s}), \psi_{2}^{-}(T \log \lambda_{s}))
$$

and switching to derivation with respect to $t$ (where $t = T \log \lambda_{s}$) we get

$$
\frac{\partial E}{\partial x}\bigg|_{\psi^{-}} = -\log \lambda_{s} \dot{\psi}_{2}^{-},
$$

$$
\frac{\partial E}{\partial y}\bigg|_{\psi^{-}} = \log \lambda_{s} \dot{\psi}_{1}^{-}.
$$

The last equation has the form of a Hamiltonian system with the Hamiltonian $E(x, y)$ after a time rescaling. For future reference we write this identity using the vector notation:

$$
\nabla E(\psi^{-}) = -\log \lambda_{s} J \dot{\psi}^{-}
$$

where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the standard symplectic matrix.

**Remark 2.10.** The following facts are not directly used in the proof. It is easy to see that equation (2.35) alone does not define $S$ uniquely. Indeed, consider a substitution $(T, E) \mapsto (\tilde{T}, \tilde{E})$ of the form

$$
(\tilde{T}, \tilde{E}) = (T + a(T, E), b(T, E))
$$

where $a$ and $b$ are 1-periodic in $T$. A map $\tilde{S}$ obtained as a result of this substitution also satisfy (2.35). Although the uniqueness is not important for our study, we note that the substitutions of this form exhaustively describe the freedom in the definition of $S$.

The normalisation property of Theorem 2.9 does not eliminate this freedom completely but only fixes $S$ on the line $E = 0$ in a way which makes comparison between the stable and unstable separatrices more convenient.
2.10 Splitting function

Let us define the splitting function by

\[ \Theta(T) = E \circ \psi^+(T \log \lambda). \] (2.39)

We can use \( \Theta \) to

1. Find the homoclinic points.
2. Find the homoclinic invariant \( \omega \).
3. Calculate the lobe areas.
4. Calculate the distances between \( \psi^+ \) and \( \psi^- \).

Since \( E = 0 \) corresponds to the unstable separatrix, zeroes of the splitting functions correspond to homoclinic points, i.e., if \( \Theta(T_1) = 0 \) for some \( T_1 \) then \( \psi^+(t_1) \in W^- \) for \( t_1 = T_1 \log \lambda \). Therefore there is \( t_2 \) such that \( \psi^+(t_1) = \psi^-(t_2) \).

Let us consider the derivative of the splitting function with respect to \( T \) at \( T = T_1 \):

\[ \frac{d\Theta}{dT}(T_1) = \nabla E^t|_{\psi^+(t_1)} \frac{d\psi^+}{dT}(T_1 \log \lambda) = \log \lambda \nabla E^t|_{\psi^-(t_2)} \dot{\psi}^+(t_1), \]

where we used the equality \( \psi^+(t_1) = \psi^-(t_2) \). We use (2.38) to eliminate the gradient of \( E \):

\[ \frac{d\Theta}{dT}(T_1) = -\log^2 \lambda (J \dot{\psi}^-)^t \dot{\psi}^+ = \log^2 \lambda (\dot{\psi}^-)^t J \dot{\psi}^+ \]

since \( J^t = -J \). Taking into account the definition of the symplectic form we get

\[ \frac{d\Theta}{dT}(T_1) = -\log^2 \lambda \Omega(\dot{\psi}^-, \dot{\psi}^+). \]

We remind ourselves that the homoclinic invariant, also known as the Lazutkin invariant, is defined by

\[ \omega = \Omega(\dot{\psi}^+, \dot{\psi}^-). \]

Therefore we have established that the homoclinic invariant is proportional to the gradient of the splitting function

\[ \omega = \frac{1}{\log^2 \lambda} \Theta'(T_1) \] (2.40)

\(^1\)The domain of this function is to be studied.
at its zero, \( \Theta(T_1) = 0 \).

Next, we note that \( \Theta(T) \) is periodic. Indeed using the definition (2.39), equation (1.5) with \( t = T \log \lambda \) and the equality (2.37) we obtain

\[
\Theta(T + 1) = E \circ \psi^+((T + 1) \log \lambda) = E \circ f_e \circ \psi^+(T \log \lambda) \\
= E \circ \psi^+(T \log \lambda) = \Theta(T).
\]

Hence we can write the Fourier series

\[
\Theta(T) = \sum_{k \in \mathbb{Z}} \Theta_k e^{2\pi ikT}.
\]

**Lemma 2.11.** Let \( c_1 > 0 \) be fixed and let \( \rho = \pi - c_1 \). If \( \delta \) is sufficiently small, the domain of the function \( \Theta \) contains the strip

\[
\{ T : |\text{Im } T| \leq \rho/\log \lambda \}
\]

Moreover, for all \( s \in \mathbb{R} \)

\[
\Theta(s) = \Theta_1 e^{2\pi is} + \Theta_1^* e^{-2\pi is} + O(e^{-4\pi \rho/\log \lambda}) \tag{2.41}
\]

where \( \Theta_1 \equiv \Theta_1(\delta) \) is a first Fourier coefficient of \( \Theta \). The asymptotic formula (2.41) can be differentiated with respect to \( s \), the error term does not change its order.

**Proof.** We note that on the strip the absolute value of \( \Theta(T) \) does not exceed \( E_0(\delta) \) defined in (2.34). Writing the classical integrals for Fourier coefficients of a periodic function and shifting the integration path to the boundary of the analyticity strip, we derive

\[
|\Theta_k| \leq E_0(\delta) e^{-|k|\frac{2\pi \rho}{\log \lambda}}. \tag{2.42}
\]

Since \( \log \lambda \delta \) is of the order of \( \delta \) all Fourier coefficients are exponentially small (the zero order coefficient being an exception). Moreover, the coefficients with \( |k| \geq 2 \) are negligible compared to the size of the separatrix splitting we intend to detect.

Now we show that \( \Theta_0 \) is also negligible. We define an auxiliary function \( g(s) = T \circ \psi^+(s \log \lambda) \) which describes the \( T \) component of the stable separatrix in the flow box coordinates. Using the definition (2.39), equation (1.5) with \( t = T \log \lambda \) and the equality (2.36) we obtain

\[
g(s + 1) = T \circ \psi^+((s + 1) \log \lambda) = T \circ f_e \circ \psi^+(s \log \lambda) \\
= T \circ \psi^+(s \log \lambda) + 1 = g(s) + 1.
\]
Consequently \( g' \) is periodic and its mean value is 1
\[
\int_{T_1}^{T_1+1} (1 - g'(s)) ds = 0 \quad (2.43)
\]
for any \( T_1 \) from its domain.

Let \( T_1 \) be a zero of the function \( \Theta \). Then \( p_1 = \psi^+(t_1) = \psi^-(t_2) \) is an homoclinic point. The segments of the stable and unstable separatrices with their ends at \( p_1 \) and \( f_\epsilon(p_1) \) form a closed loop on the plane. The total algebraic area of this loop vanishes. Since \( S \) is symplectic, this area can be evaluated in the flow box coordinates. The segment of the unstable separatrix is a straight line which connects the points \((T_2, 0)\) and \((T_2 + 1, 0)\) and therefore the area is given by the integral
\[
\int_{T_1}^{T_1+1} \Theta(s) g'(s) ds = \int_{T_1}^{T_1+1} \Theta(s) dg(s) = 0. \quad (2.44)
\]

Now we can come back to estimates for the zero order Fourier coefficient of the function \( \Theta \) which is given by the integral
\[
\Theta_0 = \int_{T_1}^{T_1+1} \Theta(s) ds.
\]

Taking into account (2.44) we rewrite it
\[
\Theta_0 = \int_{T_1}^{T_1+1} \Theta(s)(1 - g'(s)) ds.
\]
Then using equation (2.43) and the fact that $\Theta_0$ is constant we get:

$$\Theta_0 = \int_{T_1}^{T_1+1} (\Theta(s) - \Theta_0)(1 - g'(s))ds .$$

Under the integral we see the product of two functions, each one is periodic and of zero mean. The Fourier series arguments show that

$$\Theta(s) - \Theta_0 = O(e^{-2\rho/\log \lambda_3}) \quad \text{and} \quad 1 - g'(s) = O(e^{-2\rho/\log \lambda_3}) \quad (2.45)$$

since $s$ is real. Consequently, $\Theta_0$ is extremely small:

$$\Theta_0 = O(e^{-4\rho/\log \lambda_3}),$$

as the exponent in the right hand side is almost double the expected leading order of the separatrices splitting.

In particular, combining the last upper bound with (2.42) we conclude that on the real axis

$$|\Theta(s)| \leq \sum_{k \in \mathbb{Z}} |\Theta_k| = O(e^{-2\rho/\log \lambda_3}) . \quad (2.46)$$

Moreover, this function is almost sinusoidal since if we keep apart the two largest Fourier coefficients the sum of all others is very small and for $s \in \mathbb{R}$ we get (2.41).

Of course the Fourier coefficients $\Theta_1$ and $\Theta_{-1} = \Theta_1^*$ depend on $\delta$ and are exponentially small due to the upper bound (2.42) with $|k| = 1$. Nevertheless the the error term is much smaller than the upper bound for $\Theta_{\pm 1}$. In the next section we will construct an asymptotic expansion for $\Theta_{-1}$ and conclude that generically it really dominates the error.

Finally we show how to use the splitting function to estimate the lobe area between the stable and unstable manifolds. Suppose $T_1$ and $T_2$ are two zeroes of $\Theta$. Then the lobe area is represented by the following integral:

$$\int_{T_1}^{T_2} \Theta(s)g'(s)ds = \int_{T_1}^{T_2} \Theta(s)ds + \int_{T_1}^{T_2} \Theta(s)(g'(s) - 1)ds .$$

Taking into account the upper bound (2.45) and (2.46) for the functions under the second integral we conclude that the lobe area equals to

$$\int_{T_1}^{T_2} \Theta(s)ds + O(e^{-4\rho/\log \lambda_3}) .$$
2.11 Asymptotic expansion for the homoclinic invariant

Consider a line in the complex plane of the variable $T$ defined by

$$\ell(\delta) = \{ \text{Im } T = \rho_1(\delta), \ |\text{Re } T| \leq 1 \},$$

where

$$\rho_1(\delta) = \frac{\pi}{\log \lambda_{\delta}} - \sigma \log \delta^{-1}$$

and $\sigma = \frac{n}{\pi}$ is a positive constant. We fix the following relation between the parameters: $t = T \log \lambda_{\delta}$ and

$$\tau = \frac{t - i\pi}{\log \lambda_{\delta}} = T - \frac{i\pi}{\log \lambda_{\delta}}.$$  \hfill (2.47)

We estimate the splitting function defined in (2.39) by expanding the energy $E$ in Taylor series centred at a point $\psi^-$ of the unstable manifold. Using $E \circ \psi^- = 0$, equation (2.38) for the gradient of $E$ and the uniform boundedness of the second derivatives of $E$ we find that

$$\Theta(T) = \log \lambda_{\delta} \det \left( \frac{d\psi}{dt}; \psi^- - \psi^+ \right) + O \left( |\psi^- - \psi^+|^2 \right).$$

The functions in the right hand side are evaluated at a point $\tau$ with $\text{Im } \tau = -\sigma \log \delta^{-1}$. Using (2.31) with $n$ replaced by $2n$ gives us the following estimate for the difference of $\psi^\pm$

$$\psi^+ - \psi^- = \sum_{k=0}^{2n-1} \delta^{2k} (\psi_k^+ - \psi_k^-) + O(\delta^{4n} \log^4 \delta^{-1}).$$
We note that on $\ell(\delta)$
\[ \psi_k^+ - \psi_k^- = O(\delta^{2n} \log^{m_k} \delta^{-1}) \] (2.48)
by the upper bound (2.32), hence it follows that
\[ \psi^+ - \psi^- = \sum_{k=0}^{n-1} \delta^{2k}(\psi_k^+ - \psi_k^-) + O(\delta^{4n} \log^m \delta^{-1}) \]
and
\[ \psi^+ - \psi^- = O(\delta^{2n} \log^m \delta^{-1}), \]
where $m = \max\{4n, m_0, \ldots, m_{2n-1}\}$. On a sufficiently large neighbourhood of $\ell(\delta)$ we have $|\psi^-(\tau)| < 1$, therefore the Cauchy estimate implies $\left|\frac{d\psi^-}{d\tau}\right| < 1$ and consequently
\[ \Theta(T) = \det\left(\frac{d\psi^-}{d\tau}, \psi^- - \psi^+\right) + O(|\psi^- - \psi^+|^2) \]
\[ = \det\left(\frac{d\psi^-}{d\tau}, \sum_{k=0}^{n-1} \delta^{2k}(\psi_k^+ - \psi_k^-)\right) + O(\delta^{4n} \log^{2m} \delta^{-1}). \]

Now, since $\psi^-$ is analytic the series (2.26) can be differentiated term-wise, so using (2.48) we obtain
\[ \Theta(T) = \det\left(\sum_{k=0}^{n-1} \delta^{2n} \frac{d\psi_k^-}{d\tau}, \sum_{k=0}^{n-1} \delta^{2k}(\psi_k^+ - \psi_k^-)\right) + O(\delta^{4n} \log^{\tilde{m}} \delta^{-1}), \]
where $\tilde{m} = \max\{2m, (2n - 1)m_0\}$. We use that the determinant is bi-linear to rewrite this estimate in the following form
\[ \Theta(T) = \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \delta^{2(k_1+k_2)} \det\left(\frac{d\psi_k^-}{d\tau}, \psi_k^+ - \psi_k^-\right) + O(\delta^{4n} \log^{\tilde{m}} \delta^{-1}). \] (2.49)

Next we estimate the derivative of $\psi_k^-$. Since it is an analytic function in a small domain $D$ we can use the Cauchy estimate to estimate it using $\psi_k^-$ itself. Using that $\psi_k^-$ is asymptotic to, see (2.25), the series given by (2.12) yields
\[ \left|\frac{d\psi_k^-}{d\tau}\right| = O(\tau^{m_{k_1}}) = O(\log^{m_{k_1}} \delta^{-1}) \]
which together with the estimate (2.48) gives
\[ \det\left(\frac{d\psi_k^-}{d\tau}, \psi_k^- - \psi_k^+\right) = O(\delta^{2n} \log^{m_{k_1}+m_{k_2}} \delta^{-1}). \]
The last estimate shows that terms in (2.49) with \( k_1 + k_2 \geq n \) are not larger than the error term. We collect terms with \((k_1 + k_2) < n\) to get

\[
\Theta(T) = \sum_{k=0}^{n-1} \delta^{2k} \sum_{k_1+k_2=k \atop k_1,k_2 \geq 0} \det \left( \frac{d\psi_{k_1}^-}{d\tau}, \psi_{k_2}^- - \psi_{k_2}^+ \right) + O(\delta^{4n} \log \tilde{m} \delta^{-1}), \tag{2.50}
\]

where \( \tilde{m} = \max_{k_1+k_2=n} \{ \tilde{m}, m_{k_1} + m_{k_2} \} \).

Let us introduce the auxiliary functions

\[
\theta_k(\tau) = \sum_{k_1+k_2=k} \det \left( \frac{d\psi_{k_1}}{d\tau} (\tau); \psi_{k_2}^- (\tau) - \psi_{k_2}^+ (\tau) \right). \tag{2.51}
\]

We note that these functions are independent from the parameter \( \delta \). The next lemma gives a more accurate estimate for these functions.

**Lemma 2.12.** For every integer \( k \geq 0 \) there exist \( \omega_k \in \mathbb{C} \) and \( N_k \in \mathbb{N} \) such that

\[
\theta_k(\tau) = \omega_k e^{-2\pi i \tau} + O(\delta^{4n} \log N_k \delta^{-1}), \tag{2.52}
\]

where \( \tau \in \{ -\pi + \beta_1 < \arg \tau < -\beta_1, \Im \tau < c \} \).

According to the lemma we have

\[
\theta_k(\tau) = \omega_k e^{-2\pi i \tau} + O(\delta^{4n} \log N_k \delta^{-1}) \tag{2.53}
\]

on the line \( \ell(\delta) \). Substituting this and (2.51) into equation (2.50), we obtain

\[
\Theta(T) = \sum_{k=0}^{n-1} \delta^{2k} \omega_k e^{-2\pi i \tau} + O(\delta^{4n} \log M \delta^{-1}),
\]

where \( M = \max\{m, N_0\} \). Now we can evaluate the first Fourier coefficients of \( \Theta \):

\[
\Theta_{-1} = \int_{i\rho_1(\delta)}^{1+i\rho_1(\delta)} e^{2\pi i T} \Theta(T) dT
\]

\[
= e^{-\frac{\pi^2}{\log x} \log \delta^{-1}} \int_{-i\sigma \log \delta^{-1}}^{1-i\sigma \log \delta^{-1}} e^{2\pi i \tau} \left( \sum_{k=0}^{n-1} \delta^{2k} \omega_k e^{-2\pi i \tau} + O(\delta^{4n} \log M \delta^{-1}) \right) d\tau,
\]

where the exponential factor comes from the change of the variables (2.47).

Expanding the parenthesis and taking into account that \( |e^{2\pi i \tau}| = \delta^{-2n} \) on the integration path we get

\[
\Theta_{-1} = \left( \sum_{k=0}^{n-1} \delta^{2k} \omega_k + O(\delta^{2n} \log M \delta^{-1}) \right) e^{-\frac{\pi^2}{\log x}}.
\]
With this bound on the first Fourier coefficient we are now ready to estimate the size of $\Theta(t)$ for $t$ real. In particular, if $s \in \mathbb{R}$ then (2.41) with $\rho = \frac{3}{4}\pi$ implies

$$
\Theta(s) = \Theta_{-1}e^{-2\pi is} + \Theta^*_{-1}e^{2\pi is} + O\left(e^{-\frac{2\pi^2}{\log \lambda}}\right)
$$

$$
= \sum_{k=0}^{n-1} \delta^{2k} \left(\omega_k e^{-2\pi is} + \omega^*_k e^{2\pi is}\right) e^{-\frac{2\pi^2}{\log \lambda}} + O\left(\delta^{2n} e^{-\frac{2\pi^2}{\log \lambda}}\right).
$$

There are two cases to consider. If $\omega_k$ vanish for all $k$, this formula simply gives an upper bound for the derivative of $\Theta$ and, as a result, for the homoclinic invariant of any primary homoclinic trajectory. If, on the other hand, some $\omega_k \neq 0$ the leading term is larger than the error. In this case it is more convenient to rewrite the formula in the form

$$
\Theta(s) = \sum_{k=0}^{n-1} a_k \delta^{2k} \cos\left(2\pi s + \sum_{k=0}^{n-1} \varphi_k \delta^{2k}\right) e^{-\frac{2\pi^2}{\log \lambda}} + O\left(\delta^{2n} e^{-\frac{2\pi^2}{\log \lambda}}\right),
$$

where $a_0 = |\omega_0|$ and at least one of the amplitudes does not vanish. The implicit function theorem then implies that the function $\Theta$ has exactly two zeroes per period. The derivative of $\Theta$ at the zeroes is $\pm 2\pi e^{-\frac{2\pi^2}{\log \lambda}} \sum_{k=0}^{n-1} a_k \delta^{2k}$. Therefore there are exactly two primary homoclinic orbits and the relation (2.40) implies our main result:

$$
\omega = \pm \frac{2\pi}{\log \lambda^2} \sum_{k=0}^{n-1} a_k \delta^{2k} e^{-\frac{2\pi^2}{\log \lambda}} + O\left(\delta^{2n} e^{-\frac{2\pi^2}{\log \lambda}}\right).
$$

3 Normal form for the bifurcation

3.1 Formal series and quasi-homogeneous polynomials

In this section we will mainly be interested in transformations given in the form of formal power series. We will consider the series in powers of the space variables $(x,y)$ combined with expansions in the parameter $\varepsilon$. The series have the form

$$
g(x, y, \varepsilon) = \sum_{k,l,m} c_{k,l,m} x^k y^l \varepsilon^m.
$$

A formal series is treated as a collection of coefficients. Formal series form an infinite dimensional vector space. Addition, multiplication, integration and
differentiation are defined in a way compatible with the common definition on the subset of convergent series.

The series involve several variables and it is convenient to group terms which are “of the same order”. A usual choice is to consider $x$ and $y$ to be of the same order. But for purpose of this paper it is much more convenient to assume

- $x$ is of order 2;
- $y$ is of order 3;
- $\varepsilon$ is of order 4.

Then a monomial $x^k y^l \varepsilon^m$ is considered to be of order $2k + 3l + 4m$. We can write

$$g(x, y, \varepsilon) = \sum_{p \geq 0} g_p(x, y, \varepsilon)$$

where

$$g_p(x, y, \varepsilon) = \sum_{2k+3l+4m=p} c_{klm} x^k y^l \varepsilon^m$$

is a quasi-homogeneous polynomial of order $p$. We stress that this notation does not refer to a resummation of the divergent series but simply indicates the order in which the coefficients are to be treated.

In order to give a rigorous background for manipulation with formal series we define a metric on the space of formal series $\mathcal{F}$. Let $g$ and $\tilde{g}$ be two formal series and let $p$ denote the lowest (quasi-homogeneous) order of $g - \tilde{g}$. If $g \neq \tilde{g}$ then $p$ is finite and we let

$$d(g, \tilde{g}) = 2^{-p},$$

otherwise we assume

$$d(g, g) = 0.$$

It is a straightforward to check that $(\mathcal{F}, d)$ is a complete metric space. Moreover polynomials are dense in $(\mathcal{F}, d)$. Hence we can define formal convergence and formal continuity on the space of formal series. In particular an operator is formally continuous if each coefficient of a series in its image is a function of a finite number of coefficients of a series in its argument.

Let $\chi$ and $g$ be two formal power series. We note that any of the series involved in next definitions may diverge. The linear operator defined by the formula

$$L_\chi g = \{g, \chi\}$$

is called the Lie derivative generated by $\chi$. We note that if $\chi$ starts from an order $p$ and $g$ starts with an order $q$, then the series $L_\chi g$ starts with the
order $p + q - 5$ as the Poisson bracket involves differentiation with respect to $x$ and $y$. If $p \geq 6$ the lowest order in $L_\chi g$ is at least $q + 1$.

We define the exponent of $L_\chi$ by

$$\exp(L_\chi)g = \sum_{k \geq 0} \frac{1}{k!} L_\chi^k g,$$

where $L_\chi^k$ stands for the operator $L_\chi$ applied $k$ times. The lowest order in the series $L_\chi^k g$ is at least $q + k$ and consequently every coefficient of $\exp(L_\chi)g$ depends only on a finite number of coefficients of $\chi$ and $g$.

If the lowest order of $\chi$ is at least 6 the series (3.2) converges with respect to the metric $d$, i.e., each coefficient of the result is function of finite number of coefficients of $\chi$ and $g$.

We consider consider the formal series

$$\Phi^1_\chi = (\exp(L_\chi)x, \exp(L_\chi)y).$$

We say that $\Phi^1_\chi$ is a Lie series generated by the formal Hamiltonian $\chi$. If $\chi$ is a polynomial the series converge on a poly-disk and coincide with a map which shifts points along trajectories of the Hamiltonian system with Hamiltonian function $\chi$. For this reason sometimes we will call this series a time one map of the Hamiltonian system $\chi$.

We note that it is easy to construct the formal series for the inverse map:

$$\Phi^{-1}_\chi = (\exp(-L_\chi)x, \exp(-L_\chi)y).$$

Then $\Phi^1_\chi \circ \Phi^{-1}_\chi(x, y) = (x, y)$. We also note that

$$g \circ \Phi^1_\chi = \exp(L_\chi)g.$$

These formulae are well known to be valid for convergent series and can be extended onto $\mathcal{H}$ due to the density property.

### 3.2 Formal interpolation

We study the following family of area-preserving maps

$$F_{\varepsilon} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y + f(x, y, \varepsilon) \\ y + g(x, y, \varepsilon) \end{pmatrix},$$

and the Taylor series of $f(x, y, 0)$ and $g(x, y, 0)$ do not contain constant and linear terms. Since $F_{\varepsilon}$ is area-preserving we have

$$\det DF_{\varepsilon} = \left| \begin{array}{cc} 1 + \partial_x f & 1 + \partial_y f \\ \partial_x g & 1 + \partial_y g \end{array} \right| = 1.$$
which is equivalent to
\[ \partial_x f + \partial_y g + \{f, g\} - \partial_x g \equiv 0. \quad (3.3) \]

Next theorem states that \( F_\varepsilon \) can be formally interpolating by an autonomous Hamiltonian flow.

**Theorem 3.1.** Let \( F_\varepsilon \) be family of area-preserving maps such that
\[ DF_0(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]
then there exists a unique (up to adding a formal series in powers \( \varepsilon \) only) formal Hamiltonian \( h_\varepsilon \) such that
\[ \Phi_{h_\varepsilon}^1 = F_\varepsilon. \quad (3.4) \]

**Proof.** The Taylor series for \( F_\varepsilon : (x, y) \mapsto (x_1, y_1) \) is written as a sum of quasi-homogeneous polynomials:
\[
\begin{align*}
  x_1 &= x + y + \sum_{p \geq 4} f_p(x, y, \varepsilon), \\
  y_1 &= y + \sum_{p \geq 4} g_p(x, y, \varepsilon),
\end{align*}
\]
where \( f_p \) and \( g_p \) are quasi-homogeneous polynomials:
\[
\begin{align*}
  f_p(x, y, \varepsilon) &= \sum_{2k+3l+4m=p} f_{klm} x^k y^l \varepsilon^m, \\
  g_p(x, y, \varepsilon) &= \sum_{2k+3l+4m=p} g_{klm} x^k y^l \varepsilon^m.
\end{align*}
\]
Similarly we look for \( h_\varepsilon \) in the form of a sum of quasi-homogeneous polynomials using the same quasi-homogeneous ordering for terms:
\[ h_\varepsilon(x, y) = \sum_{p \geq 6} h_p(x, y, \varepsilon) \quad \text{where} \quad h_p(x, y, \varepsilon) = \sum_{2k+3l+4m=p} h_{klm} x^k y^l \varepsilon^m. \]
The time-1 map of this Hamiltonian is given by the Lie series
\[ \Phi_{h_\varepsilon}^1(x, y) = \begin{pmatrix} x + L_{h_\varepsilon} x + \sum_{k \geq 2} \frac{1}{k!} L_{h_\varepsilon}^k x \\ y + L_{h_\varepsilon} y + \sum_{k \geq 2} \frac{1}{k!} L_{h_\varepsilon}^k y \end{pmatrix}, \]
where the operator \( L_{h_\varepsilon} \) is defined by
\[ L_{h_\varepsilon}(\varphi) = \{\varphi, h_\varepsilon\} = \frac{\partial \varphi}{\partial x} \frac{\partial h_\varepsilon}{\partial x} - \frac{\partial \varphi}{\partial y} \frac{\partial h_\varepsilon}{\partial y}. \]
Now we use induction to show that there is a formal Hamiltonian \( h_\varepsilon \) such that \( \Phi^1_{h_\varepsilon} = F_\varepsilon \) at all orders. This is an equality between two formal series, i.e., all coefficients of two series coincide. We note the first component of the series starts with the quasi-homogeneous order 2 and the second one starts with 3. Therefore it is convenient to consider an order \( p \) in the first component simultaneously with the order \( p + 1 \) in the second one. In this situation we say that we consider a term of the order \((p, p + 1)\).

Equality (3.4) is equivalent to the following infinite system

\[
L_{h_{p+3}}x + \sum_{k\geq 2} \frac{1}{k!} [L^k_{h_\varepsilon} x]_p = \begin{cases} f_p & \text{if } p \geq 4 \\ y & \text{if } p = 3 \end{cases} \tag{3.6}
\]

\[
L_{h_{p+3}}y + \sum_{k\geq 2} \frac{1}{k!} [L^k_{h_\varepsilon} y]_p = g_{p+1} \tag{3.7}
\]

where \([\cdot]_p\) is used to denote terms of the quasi-homogeneous order \( p \) of a formal series. We note that

\[
\text{order}(L_{h_k} x) = k - 3 \quad \text{and} \quad \text{order}(L_{h_k} y) = k - 2.
\]

First we check that the equations can be solved for \( p = 3 \). It is easy to see that

\[
L_{h_6} x = \frac{\partial h_6}{\partial y},
\]

\[
L_{h_6} y = -\frac{\partial h_6}{\partial x}.
\]

Then the equations (3.6), (3.7) take the form

\[
\frac{\partial h_6}{\partial y} = y, \tag{3.8}
\]

\[-\frac{\partial h_6}{\partial x} = g_4(x, y, \varepsilon).\]

Since \( g_4 \) is a quasihomogeneous polynomial of order 4 it is independent from \( y \) which is of order 3. Consequently

\[
g_4(x, y, \varepsilon) = g_4(x, \varepsilon).
\]

Then system (3.8) has a unique solution of the form

\[
h_6 = \frac{y^2}{2} + G_6(x, \varepsilon),
\]
where $G_6$ is a quasi-homogeneous polynomial of order 6 in $x$ and $\varepsilon$.

Let us now proceed with the induction step. Assume we have $h_6, \ldots, h_{p+2}$ and want to compute $h_{p+3}$ for $p > 3$. To make the combinatorics easier we introduce the notation

$$L_s \varphi = \{ \varphi, h_{s+5} \}.$$ 

Then

$$L_{h_\varepsilon} \varphi = \{ \varphi, h_{\varepsilon} \} = \sum_{p \geq 6} \{ \varphi, h_p \} = \sum_{s \geq 1} L_s \varphi.$$ 

Note that $L_s$ maps a quasi-homogeneous polynomial of order $j$ into a quasi-homogeneous polynomial of order $j + s$. So we have

$$L_k^{\varphi_j} = \left( \sum_{s \geq 1} L_s \right)^k \varphi_j = \sum_{s_1 + s_2 + \ldots + s_k \geq k, s_1, s_2, \ldots, s_k \geq 1} L_{s_1} \ldots L_{s_k} \varphi_j.$$ 

Now let us consider the components of $\Phi_{h_\varepsilon}^1$ at order $(p, p+1)$, the first component is

$$L_{h_\varepsilon} x + \sum_{k \geq 2} \frac{1}{k!} L_{h_\varepsilon}^k x = L_{p-2} x + \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + s_2 + \ldots + s_k = p-2} L_{s_1} \ldots L_{s_k} x.$$ 

and the second component

$$L_{h_\varepsilon} y + \sum_{k \geq 2} \frac{1}{k!} L_{h_\varepsilon}^k y = L_{p-2} y + \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + s_2 + \ldots + s_k = p-2} L_{s_1} \ldots L_{s_k} y.$$ 

We want

$$\begin{pmatrix} 1^{st \, comp} \\ 2^{nd \, comp} \end{pmatrix} = \begin{pmatrix} f_p \\ g_{p+1} \end{pmatrix}.$$ 

We have

$$L_{p-2} x = \{ x, h_{p+3} \},$$

$$L_{p-2} y = \{ y, h_{p+3} \}.$$ 

Thus

$$\begin{pmatrix} \frac{\partial h_{p+3}}{\partial y} \\ \frac{\partial h_{p+3}}{\partial x} \end{pmatrix} = \begin{pmatrix} f_p - \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + s_2 + \ldots + s_k = p-2} L_{s_1} \ldots L_{s_k} x \\ g_{p+1} - \sum_{k=2}^{p-2} \frac{1}{k!} \sum_{s_1 + s_2 + \ldots + s_k = p-2} L_{s_1} \ldots L_{s_k} y \end{pmatrix}.$$ 

(3.9)
The Hamiltonian \( h_{p+3} \) is defined from this equation uniquely up to a function of \( \varepsilon \) if and only if the right hand side is divergence free. The last condition follows from the area-preservation property due to the following lemma.

**Lemma 3.2.** If expansions of two area-preserving maps coincide up to the order \( (p-1, p) \) then the divergences of the terms of the order \( (p, p+1) \) are equal.

**Proof.** We use the area-preserving property (3.3) and collect the terms of order \( p-2 \)

\[
\partial_x f_p + \partial_y g_{p+1} - \partial_x g_p + \sum_{k,l \geq 3} \{f_k, g_l\} = 0.
\]

Consequently,

\[
\partial_x f_p + \partial_y g_{p+1} = \partial_x g_p - \sum_{k+l=p+3, k,l \geq 4} \{f_k, g_l\} = \partial_x \tilde{f}_p + \partial_y \tilde{g}_{p+1}.
\]

We finish the proof by the following observation. Let \( \tilde{h} = \sum_{k=0}^{p+3} h_k \). For any choice of \( h_{p+3} \) the maps \( \Phi_1^1 \) and \( F_\varepsilon \) satisfy the assumptions of the previous lemma therefore their orders \( (p, p+1) \) have the same divergence. That is the solvability condition for the equation (3.9).

\[\square\]

### 3.3 Simplification of the interpolating Hamiltonian

The Hamiltonian \( h_\varepsilon \) can be simplified with the help of a formal canonical change of variables.

**Theorem 3.3.** If \( h(x, y, \varepsilon) = \sum_{p \geq 6} h_p(x, y, \varepsilon) \) such that the leading order has the form

\[
h_6(x, y, \varepsilon) = \frac{y^2}{2} + a \frac{x^3}{3} - b \varepsilon x
\]

then there exists a formal canonical substitution such that the Hamiltonian takes the form

\[
\tilde{h}(x, y, \varepsilon) = \frac{y^2}{2} + \sum_{p \geq 6} u_p(x, \varepsilon).
\]
Proof. We prove the theorem by induction. The order 6 is already in the desired form, just let \( u_6 = ax^3/3 - bx \). Then suppose we transformed the Hamiltonian to the desired form up to the order \( p \) with \( p \geq 6 \). For any quasi-homogeneous polynomial \( \chi_p \) of order \( p \) we have

\[
\tilde{h} = e^{L_{\chi_p}}h = h + \{h, \chi_p\} + \hat{O}_{2p-10+6}
\]

\[
= h + \{h_6, \chi_p\} + \hat{O}_{p+2}
\]

We see that this change does not affect terms of order less or equal \( p \). Collecting all terms of order \( p + 1 \) we get

\[
\tilde{h}_{p+1} = h_{p+1} + \{h_6, \chi_p\}.
\]

(3.11)

In order to complete the proof we show that there is a polynomial \( \chi_p \) such that \( \tilde{h}_{p+1} \) is independent from \( y \). The situation is different for \( p \) even and \( p \) odd. We will show that if \( p \) is even then for any \( h_{p+1} \) there is \( \chi_p \) such that

\[
h_{p+1} + \{h_6, \chi_p\} = 0.
\]

(3.12)

The polynomial \( \chi_p \) is defined up to addition of an arbitrary polynomial of \( h_6 \) and \( \varepsilon \). On the other hand, if \( p \) is odd then there are two unique polynomials \( u_{p+1} = u_{p+1}(x, \varepsilon) \) and \( \chi_{p+1} = \chi_{p+1}(x, y, \varepsilon) \) such that

\[
h_{p+1} + \{h_6, \chi_p\} = u_{p+1}.
\]

(3.13)

In order to prove these statements we note that since \( y \) is counted as a term of order three and \( x, \varepsilon \) as terms of order two and four respectively, a quasihomogeneous polynomial of even order contains only even powers of \( y \) and one of an odd order contains only odd powers. Let \( p + 1 = 3q + r \) with \( r \in \{0, 1, 2\} \). The largest possible power of \( y \) in \( h_{p+1} \) does not exceed \( q \) and we can write

\[
h_{p+1} = \sum_{l=0}^{q/2} y^{q-2l}s_{r+6l}(x, \varepsilon) \quad \text{and} \quad \chi_p = \sum_{l=0}^{(q-1)/2} y^{q-2l-1}\sigma_{r+6l+2}(x, \varepsilon),
\]

where \( s_j \) and \( \sigma_j \) are quasi-homogeneous polynomials of \( x \) and \( \varepsilon \). Taking into account

\[
\{h_6, \chi_p\} = \frac{\partial u_6}{\partial x} \frac{\partial \chi_p}{\partial y} - ay \frac{\partial \chi_p}{\partial x}
\]

and collecting terms in \( h_{p+1} + \{h_6, \chi_p\} \) which have of the same order in \( y \) (excluding the \( y \)-independent terms) we get

\[
-a \frac{\partial \sigma_{r+2}}{\partial x} + s_r = 0,
\]

\[
-a \frac{\partial \sigma_{r+6l+2}}{\partial x} + (q - 2l + 1) \frac{\partial u_6}{\partial x} \sigma_{r+6l-4} + s_{r+6l} = 0.
\]
The first equation defines $\sigma_{r+2}$. If $r \neq 2$ the solution is unique, otherwise a constant times $\varepsilon$ can be added. Then $\sigma_{r+6l+2}$ are to be chosen recursively for $1 \leq l \leq q/2$ to satisfy the second equation.

The theorem follows by induction in $p$. \hfill \Box

4 Formal separatrix

In this section we will describe properties of an asymptotic expansion for the separatrices providing a proof for the statements of Section 2.2.

4.1 Auxiliary functions

In this subsection we describe a useful class of functions. A function belongs to this class if

1. it is periodic with the imaginary period $2\pi i$.
2. it is analytic on the entire $\mathbb{C}$ except for poles at $\pi i(2k+1)$, $k \in \mathbb{Z}$.
3. it vanishes as $\text{Re} t \to \pm \infty$.

Any function which satisfies these three assumptions is a polynomial, without a constant term, of two “base” functions\footnote{We note that $\tanh \frac{t}{2}$ is $2\pi i$ periodic and analytic on $\mathbb{C}$ except for simple poles at $i\pi(2k+1)$ with $k \in \mathbb{Z}$. Suppose a function $\varphi(t)$ satisfies properties 1 and 2 and is bounded as $\text{Re} t \to \pm \infty$. Comparing Laurent expansions around any of the poles, we can construct a polynomial $p$ such that $\varphi(t) - p(\tanh \frac{t}{2})$ has no singularities and, consequently, is constant. The condition 3 adds a restriction on coefficients of the polynomial ($p(1) = p(-1) = 0$).}

$$
\eta_0 = \frac{1}{\cosh^2 \frac{t}{2}} \quad \text{and} \quad \eta_1 = -\frac{\sinh \frac{t}{2}}{\cosh^3 \frac{t}{2}}. \quad (4.1)
$$

Equivalently, the function can be written in the form $P(\eta_0) + \eta_1 Q(\eta_0)$ where $P, Q$ are polynomials of $\eta_0$ only and $P(0) = 0$. Indeed, a straightforward substitution shows that $\eta_0$, $\eta_1$ satisfy the differential equation

$$
\begin{align*}
\dot{\eta}_0 &= \eta_1, \\
\dot{\eta}_1 &= \eta_0 - \frac{3}{2} \eta_0^2.
\end{align*} \quad (4.2)
$$

This equation is Hamiltonian with the Hamiltonian function

$$
H = \frac{1}{2} \left( \eta_1^2 - \eta_0^2 + \eta_0^3 \right).
$$
The Hamiltonian is constant along solutions of the differential equation. Since both \( \eta_0 \) and \( \eta_1 \) vanish as \( \text{Re} \to \infty \), we get the identity

\[
\eta_1^2 - \eta_0^2 + \eta_0^3 = 0. \tag{4.3}
\]

This last equality and (4.2) imply that

\[
\dot{\eta}_0^2 = \eta_1^2 = \eta_0^2 - \eta_0^3. \tag{4.4}
\]

Using this identity we can exclude all even powers of \( \eta_1 \) and reduce all odd powers to the power one. These identities will be used to simplify formulae involved in the construction of the formal separatrix.

### 4.2 Formal separatrix of the flow

In this section we prove Theorem 2.1. We remind that \( \delta = \varepsilon^{1/4} \). By Theorem 3.3 there is a formal canonical change of variables which transforms the Hamiltonian \( h_\varepsilon \) to the simpler form

\[
H_\varepsilon(x, y) = \frac{y^2}{2} + U(x, \varepsilon), \tag{4.5}
\]

where

\[
U(x, \varepsilon) = \sum_{m \geq 0, k \geq 1, k + 2m \geq 3} u_{km} x^k \varepsilon^m. \tag{4.6}
\]

After this change equation (2.7), which we for convenience repeat

\[
\mu(\delta) \dot{X} = J \nabla H_\varepsilon(X), \tag{4.7}
\]

takes the form

\[
\begin{align*}
\mu(\varepsilon) \dot{x} &= y, \\
\mu(\varepsilon) \dot{y} &= -U'(x, \varepsilon),
\end{align*} \tag{4.8}
\]

where \( U' \equiv \frac{\partial U}{\partial x} \). This system can be conveniently rewritten in an equivalent form as a scalar equation of second order:

\[
(\mu(\varepsilon))^2 \ddot{x} = -U'(x, \varepsilon).
\]

In order to simplify notation we introduce the new auxiliary series

\[
\beta(\varepsilon) = \frac{1}{2}(\mu(\varepsilon))^2.
\]

Since \( \mu \) was an odd series in \( \delta \), the series \( \beta \) is even. The equation takes the form

\[
2\beta(\varepsilon) \ddot{x} = -U'(x, \varepsilon)
\]
Multiplying with \( \dot{x} \) and integrating with respect to \( t \) we obtain

\[
\beta(\varepsilon) \dot{x}^2 = -U(x, \varepsilon) + c(\varepsilon),
\]

(4.9)

where \( c(\varepsilon) \) is a constant of integration. Now we use the following ansatz for solving this equation:

\[
x(t, \varepsilon) = \sum_{k \geq 1} \delta^{2k} x_k(t),
\]

\[
\beta(\varepsilon) = \sum_{k \geq 1} a_k \delta^{2k},
\]

and

\[
c(\varepsilon) = \sum_{k \geq 3} c_k \delta^{2k}.
\]

Substituting these series into equation (4.9) we get

\[
\left( \sum_{k \geq 1} a_k \delta^{2k} \right) \left( \sum_{k \geq 1} \delta^{2k} \dot{x}_k(t) \right)^2 + \sum_{k+2m \geq 3} u_{km} \left( \sum_{j \geq 1} \delta^{2j} x_j \right)^k \delta^{4m} = \sum_{k \geq 3} c_k \delta^{2k}.
\]

(4.10)

We recall that this equation is considered in the class of formal series which assumes that two formal series are equal if and only if their coefficients coincide. We will use induction to show that the equation can be satisfied at every order in \( \delta^2 \).

We note that in (4.10) the least order of \( \delta \) is 6. Collecting all terms of this order we get the following equation

\[
a_1 \dot{x}_1^2 + u_{30} x_1^3 + u_{11} x_1 = c_3.
\]

(4.11)

This equation looks very similar to (4.4) but has different coefficients in front of its terms. This suggests \( x_1 \) to be of the form

\[
x_1 = b_0 + b_1 \eta_0.
\]

(4.12)

where \( b_0, b_1 \in \mathbb{R} \) are to be determined. We insert \( x_1 \) into (4.11) which gives

\[
a_1 (b_1 \dot{\eta}_0)^2 + u_{30} (b_0 + b_1 \eta_0)^3 + u_{11} (b_0 + b_1 \eta_0) = c_3.
\]

Using equation (4.4) to replace \( \dot{\eta}_0^2 \) and expanding the parentheses we rewrite this expression as follows

\[
c_3 = a_1 b_1^2 (\eta_0^2 - \eta_0^3) + u_{30} b_0^3 + u_{30} 3b_0^2 b_1 \eta_0 + u_{30} 3b_0 b_1^2 \eta_0^2 + u_{30} b_1^3 \eta_0^3 + u_{11} b_0 + u_{11} b_1 \eta_0.
\]

(4.13)
We now collect terms in (4.13) of equal powers in $\eta_0$ and obtain the following system of equations

\[
\begin{aligned}
-a_1 b_1^2 + u_{30} b_1^3 &= 0, \\
a_1 b_1^2 + u_{30} 3b_0 b_1^2 &= 0, \\
3u_{30} b_0^2 b_1 + u_{11} b_1 &= 0, \\
c_3 &= u_{11} b_0 + u_{30} b_0^3.
\end{aligned}
\] (4.14)

The last equation defines $c_3$. The first three lines imply

\[
\begin{aligned}
b_0 &= \pm \sqrt{-\frac{u_{11}}{3u_{30}}}, \\
b_1 &= -3b_0, \\
a_1 &= -3u_{30} b_0.
\end{aligned}
\] (4.15)

We make the following two remarks. Note that, to the leading order, $b_0$ is the $x$ coordinate of the equilibrium point and $c_3$ is its energy.

The sign of $b_0$ should be chosen to ensure $a_1 > 0$ since the latter coefficient is the first term of the expansion of $\beta(\varepsilon)$ which is a square of a real series.

We have seen that equation (4.10) has a solution at the leading order $\delta^6$, and continue by induction. We explain the first step of the induction in more detail in order to illustrate the method used in the general step.

Let us collect the terms of order $\delta^8$ which gives the equation

\[
2a_1 \dot{x}_1 \dot{x}_2 + a_2 \dot{x}_1^2 + u_{11} x_2 + u_{21} x_1^2 + 3u_{30} x_1^2 x_2 + u_{40} x_1^4 = c_4.
\] (4.16)

We note that this is a linear non-homogeneous equation in $x_2$. Therefore it defines $x_2$ uniquely up to addition of a solution of the corresponding homogeneous equation.

Using the form of $x_1$ given in equation (4.12) and the relationship (4.4) we rearrange this equation as

\[
c_4 = 2a_1 \dot{x}_1 \dot{x}_2 + (3u_{30} x_1^2 + u_{11}) x_2 + a_2 b_1^2 (\eta_0^2 - \eta_0^3) + u_{21} (b_0 + b_1 \eta_0)^2 + u_{40} (b_0 + b_1 \eta_0)^4.
\] (4.17)

Before solving this equation for each power of $\eta_0$ separately we make some observations which will simplify our analysis. First we note that the second line of (4.17) is a polynomial in $\eta_0$ of order 4. The first line of (4.17) tells us that $x_2$ should have a pole of order 4, hence we assume that $x_2$ takes the following form

\[
x_2 = b_{20} + b_{21} \eta_0 + b_{22} \eta_0^2.
\]
Now let us simplify the first line of (4.17) beginning with the second term

\[
(3u_{30}x_1^2 + u_{11}) = 3u_{30}(b_0 + b_1\eta_0)^2 + u_{11} \\
= 3u_{30}b_0^2 + 6u_{30}b_0b_1\eta_0 + 3u_{30}b_1^2\eta_0^2 + u_{11}.
\]

Using (4.15) we conclude that

\[
(3u_{30}x_1^2 + u_{11}) = 3u_{30}b_1b_0(2\eta_0 - 3\eta_0^2) \quad (4.18)
\]

To ease the notation we let

\[
A = 3u_{30}b_1b_0, \quad (4.19)
\]

and we notice that \( A \neq 0 \) by virtue of (4.15). Next we simplify the first term in the first line of (4.17)

\[
\dot{x}_1\dot{x}_2 = (b_{21} + 2b_{22}\eta_0)\dot{\eta}_0b_1\dot{\eta}_0 \\
= b_1(b_{21} + 2b_{22}\eta_0)(\eta_0^2 - \eta_0^3),
\]

where in the last step we used (4.4). Then equation (4.17) reads

\[
c_4 = 2a_1b_1(b_{21} + 2b_{22}\eta_0)(\eta_0^2 - \eta_0^3) \\
+ A(2\eta_0 - 3\eta_0^2)(b_{20} + b_{21}\eta_0 + b_{22}\eta_0^2) \\
+ a_2b_1^2(\eta_0^2 - \eta_0^3) + u_{21}(b_0 + b_1\eta_0)^2 + u_{40}(b_0 + b_1\eta_0)^4. \quad (4.20)
\]

Let us now solve (4.20) for \( x_2, c_4 \) and \( a_2 \). The strategy is to solve (4.20) in each power of \( \eta_0 \) separately in the following order

1. \( c_4 \) is defined from the order 0 in \( \eta_0 \).
2. \( b_{20} \) is defined from the order 1 in \( \eta_0 \).
3. \( b_{22} \) is defined from the order 4 in \( \eta_0 \).
4. \( b_{21} \) and \( a_2 \) solve a system of equations obtained from the orders 2 and 3 in \( \eta_0 \).

At order 0 in \( \eta_0 \) we get

\[
c_4 = u_{21}b_0^2 + u_{40}b_0^4, \quad (4.21)
\]

which defines the coefficient \( c_4 \). Next we proceed with the terms of order 1 in \( \eta_0 \)

\[
2Ab_{20} + u_{21}2b_0b_1 + u_{40}4b_0^3b_1 = 0,
\]

42
which gives
\[ b_{20} = -\frac{u_{21}2b_0b_1 + u_{40}4b_0^3b_1}{2A}. \] (4.22)

Since \( A \) is nonzero equation (4.22) defines the coefficient \( b_{20} \) uniquely. At order 4 in \( \eta_0 \) we have
\[-(4a_1b_1 + 3A)b_{22} + u_{40}b_4^4 = 0,\]
hence
\[ b_{22} = \frac{u_{40}b_4^4}{4a_1b_1 + 3A}. \] (4.23)

The denominator in (4.23) is non-zero since, using (4.15) and (4.19),
\[ 4a_1b_1 + 3A = -12u_{30}b_0b_1 + 9u_{30}b_0b_1 = 9u_{30}b_0^2, \]
and \( u_{30} \neq 0 \) by assumption and \( b_0 \neq 0 \) by (4.15). Therefore (4.23) determines \( b_{22} \) uniquely. The terms of order 3 in \( \eta_0 \) are
\[-a_2b_1^2 + 4u_{40}b_0b_1^3 + (-2a_1b_1 - 3A)b_{21} + 4a_1b_1b_{22} + 2Ab_{22} = 0,\]
and the terms of order 2 in \( \eta_0 \) are
\[-3Ab_{20} + 2a_1b_1b_{21} + 2Ab_{21} + a_2b_1^2 + u_{21}b_1^2 + u_{40}6b_0^2b_1^2 = 0.\]

These two terms define the following system of equations in \( b_{21} \) and \( a_2 \)
\[ \begin{pmatrix} 2a_1b_1 + 2A & b_1^2 \\ -2a_1b_1 - 3A & -b_1^2 \end{pmatrix} \begin{pmatrix} b_{21} \\ a_2 \end{pmatrix} = \begin{pmatrix} 3Ab_{20} - u_{21}b_1^2 - u_{40}6b_0^2b_1^2 \\ -4a_1b_1b_{22} - 2Ab_{22} - u_{40}4b_0b_1^2 \end{pmatrix}. \] (4.24)

Call the matrix in the left-hand side \( M \). This system of equations has a unique solution since
\[ \det M = Ab_1^2 = -81u_{30}b_0^4 \neq 0. \]

Therefore \( b_{21} \) and \( a_2 \) is the unique solution of (4.24). We have now shown that \( x_2, a_2 \) and \( c_4 \) are uniquely defined.

Now let us continue with the inductive step at an arbitrary order of \( \delta^2 \). For a general \( n \) the terms corresponding to \( \delta^{2n} \) are
\[ \sum_{k+l+m=n} a_k\dot{x}_l\dot{x}_m + \sum_{k+2m=3} u_{km} \sum_{j_1+\ldots+j_k=n-2m} x_{j_1} \ldots x_{j_k} = c_n. \]

This equation has the form
\[ 2a_1\dot{x}_1\dot{x}_n + (3u_{30}x_1^2 + u_{11})x_n + \tilde{P}(x_1, \ldots, x_{n-1}) + a_n\dot{x}_1^2 = c_n, \] (4.25)
where $\tilde{P}(x_1, \ldots, x_{n-1})$ is a polynomial. We note that $\tilde{P}$ is a (known) polynomial of order $n+2$ in $\eta_0$ and that all coefficients are unique since our induction assumption is that $x_1, \ldots, x_{n-1}$ are unique. We will show by induction that there exist a unique $a_n$ and a unique

$$x_n = \sum_{m=0}^{n} b_{nm} \eta_0^m. \quad (4.26)$$

That is, we will show that $a_n$ and all $b_{nm}$ are uniquely defined.

Using equality (4.18) which states that

$$(3u_{30}x_1^2 + u_{11}) = A(2\eta_0 - 3\eta_0^2),$$

where $A = 3u_{30}b_1b_0 \neq 0$ (see equation (4.19)) and that $x_n$ has the form (4.20) we simplify (4.25) as follows

$$c_n = 2a_1b_1(\eta_0^2 - \eta_0^3) \left(b_{n1} + 2b_{n2}\eta_0 + \ldots + nb_{nn}\eta_0^{n-1}\right)$$

$$+ A(2\eta_0 - 3\eta_0^2) (b_{n0} + \ldots + b_{nn}\eta_0^n)$$

$$+ a_n b_1^2 (\eta_0^2 - \eta_0^3) + \tilde{P}(x_1, \ldots, x_{n-1}). \quad (4.27)$$

We will now solve this equation for each power in $\eta_0$ separately. The strategy is the following

1. The power 0 will define $c_n$ uniquely.

2. The power 1 will define $b_{n0}$ uniquely.

3. The power $n+2$ will define $b_{nn}$ uniquely.

4. The powers $j \in \{4, \ldots, n+1\}$ (starting from $j = n+1$ and proceeding in decreasing order) will define the coefficients $b_{n,j-2}$ uniquely.

5. Finally, at powers 2 and 3 we will obtain a system of equations in $b_{n1}$ and $a_n$ which we show has a unique solution.

In what follows we use $\left[\tilde{P}\right]_k$ to denote the terms in the polynomial $\tilde{P}(x_1, \ldots, x_{n-1})$ which are of order $k$ in $\eta_0$. Let us now start solving (4.27) at order 0 in $\eta_0$. At this order we have

$$c_n = \left[\tilde{P}\right]_0,$$

which defines $c_n$ uniquely. Next, at order 1 in $\eta_0$ we have

$$2Ab_{n0} + \left[\tilde{P}\right]_1 = 0,$$

where $A = 3u_{30}b_1b_0 \neq 0$ (see equation (4.19)).
thus
\[ b_{n0} = -\left[ \tilde{P} \right]_{n+2} \frac{1}{2A}. \]

Since \( A \neq 0 \) this equation defines \( b_{n0} \) uniquely.

At order \( n + 2 \) in \( \eta_0 \) we get
\[-2a_1nb_1b_{nn} - 3Ab_{nn} + \left[ \tilde{P} \right]_{n+2} = 0,\]
thus
\[ b_{nn} = \frac{-\left[ \tilde{P} \right]_{n+2}}{2na_1b_1 + 3A}. \]

By equation (4.15) \( a_1b_1 = A \), thus the denominator is equal to \( A(2n + 3) \) which is non-zero, hence \( b_{nn} \) is defined uniquely.

Now we continue with the orders \( j \in \{4, \ldots, n+1\} \). At order \( j \) we solve the equation
\[ b_{n,j-2}(2a_1b_1(j - 2) + 3A) - b_{n,j-1}(2a_1b_1(j - 1) + 2A) - \left[ \tilde{P} \right]_j = 0,\]
thus
\[ b_{n,j-2} = \frac{b_{n,j-1}(2a_1b_1(j - 1) + 2A) + \left[ \tilde{P} \right]_j}{2a_1b_1(j - 2) + 3A}. \]

Since we are proceeding in decreasing \( j \) the coefficient \( b_{n,j-1} \) in the right-hand side has already been determined. The denominator
\[ 2a_1b_1(j - 2) + 3A = A(2(j - 2) + 3) \]
is non-zero since \( j - 2 \) is positive for all \( j \in \{4, \ldots, n+1\} \). Therefore the coefficients \( b_{n,j-2} \) are uniquely determined for \( j \in \{4, \ldots, n+1\} \).

Finally we have to solve for the remaining coefficients \( a_n \) and \( b_{n1} \). They are obtained from the equations at order 2 and 3 which are given by
\[ 2a_1b_1b_{n1} + 2Ab_{n1} - 3Ab_{n0} + a_nb_1^2 + \left[ \tilde{P} \right]_2 = 0 \]
and
\[ 4a_1b_1b_{n2} - 2a_1b_1b_{n1} + 2Ab_{n2} - 3Ab_{n1} - a_nb_1^2 + \left[ \tilde{P} \right]_3 = 0, \]
respectively. We observe that this system of equations may be written as
\[
\begin{pmatrix}
  2a_1b_1 + 2A & b_1^2 \\
-2a_1b_1 - 3A & -b_1^2
\end{pmatrix}
\begin{pmatrix}
  b_{n1} \\
  a_n
\end{pmatrix}
= \begin{pmatrix}
  3Ab_{n0} - \left[ \tilde{P} \right]_2 \\
-4a_1b_1b_{n2} - 2Ab_{n2} - \left[ \tilde{P} \right]_3
\end{pmatrix}.
\]

(4.28)
The determinant of the matrix in the left-hand side is \( Ab_1^2 = -81u_{30}b_0^4 \) which is non-zero, hence \( b_{n1} \) and \( a_n \) is the unique solution of (4.28).

We have now shown that \( b_{n0}, \ldots, b_{nm} \) and \( a_n \) and \( c_n \) are uniquely defined, hence \( x_n \) is a unique solution of (4.25).

It follows by induction that \( x_m \) is uniquely defined for all \( m \).

Now we can reconstruct the second component of the solution of system (4.8). Since

\[
x = \sum_{k \geq 1} \delta^{2k} x_k(t)
\]

the second component of the solution is restored by

\[
y = \mu(\varepsilon) \hat{x} = \left( 2 \sum_{k \geq 1} a_k \delta^{2k} \right)^{\frac{1}{2}} \sum_{k \geq 1} \delta^{2k} \hat{x}_k(t).
\]

Since \( x_k(t) \) is a polynomial in \( \eta_0 \) of order \( k \) we conclude that the solution of the system (4.8) can be written in the form

\[
\begin{cases}
  x = \sum_{k \geq 1} \delta^{2k} x_k(t), \\
y = \eta_1(t) \sum_{k \geq 1} \delta^{2k+1} y_{k-1}(t),
\end{cases}
\]

(4.29)

where \( y_{k-1} \) is a polynomial in \( \eta_0 \) of order \( k - 1 \).

The solution (4.29) is a formal solution of the system of equations (4.8). Equation (4.8) was obtained from (2.7) by a change of variables

\[
x = \phi_{\chi_e}^{-1}(X)
\]

(4.30)

where \( \chi_e = \sum_{p \geq 6} \chi_p \) where \( \chi_p \) is a quasi-homogeneous polynomial of order \( p \).

To finish the proof of the theorem we have to invert this change of variables and consider the structure of the resulting formal series \( X \). As the change of variables is close-to-identity (in the quasi-homogeneous sense) it is easy to invert (4.30). For our purposes it is sufficient to consider the change in its most general form, hence

\[
X = \phi_{\chi_e}^1(\mathbf{x}) = \left( \begin{array}{c}
x + \sum_{2k+3l+4m \geq 3} d_{klm} x^k y^l \delta^{4m} \\
y + \sum_{2k+3l+4m \geq 4} f_{klm} x^k y^l \delta^{4m}
\end{array} \right),
\]

(4.31)

where \( d_{klm}, f_{klm} \) are coefficients and \((x, y)\) are given by (4.29).
Let us now consider the structure of $X$, that is, let us consider the sums in the right hand side of the previous equation. Obviously $\delta^{4m}$ is an even power of $\delta$. We also note that any power of the series $x$ is a series of the same type as $x$ itself, i.e. a series in even powers of $\delta$. This is not the case with $y$. An even power of the series $y$ gives a series in even powers of $\delta$ while an odd power of $y$ gives a series in odd powers of $\delta$. Hence $x^k y^l \delta^{4m}$ is a series in even powers of $\delta$ if $l$ is even and in odd powers if $l$ is odd.

Furthermore, since $(x,y)$ are series of polynomials in $\eta_0(t)$ and $\dot{\eta}_0(t)$ any power of them are series of polynomials as well. Hence we conclude that $X$ is a power series in both odd and even powers of $\delta$ and that each power of $\delta$ is multiplied by a polynomial in $\eta_0(t)$ and $\dot{\eta}_0(t)$.

A more careful study reveals even more about the structure. Consider (4.29) again. We see that taking any power of $x$ gives a series of terms of the type

$$\delta^{2k} P_k(t),$$

where $P_k$ is a quasi-homogeneous polynomial in $\eta_0$ of order $k$. When taking powers of $y$ we note that

$$\eta_1^k = \eta_0^k = \begin{cases} \frac{(\eta_0^2 - \eta_0^3)^{k/2}}{\eta_0(\eta_0^2 - \eta_0^3)^{(k-1)/2}}, & \text{if } k \text{ is even} \\ \frac{\dot{\eta}_0(\eta_0^2 - \eta_0^3)^{(k-1)/2}}{\eta_0^2}, & \text{if } k \text{ is odd} \end{cases}.$$

Bearing this in mind an even power of $y$ gives a series with terms of the type

$$\delta^{2k} Q_{\leq k}(t),$$

where $Q_{\leq k}(t)$ is a homogeneous polynomial in $\eta_0(t)$ of order at most $k$. Similarly, taking an odd power of $y$ gives a series with terms of the type

$$\delta^{2k+1} Q_{\leq k-1}(t),$$

where $Q_{\leq k-1}(t)$ is a polynomial in $\eta_0(t)$ of order at most $k - 1$. Based on these observations we conclude that the form of the formal separatrix $X$ is

$$X(t, \varepsilon) = \left( \sum_{p \geq 1} \delta^{2p} x_p^1 + \eta_0(t) \sum_{p \geq 1} \delta^{2p+1} x_{p-1}^2, \dot{\eta}_0(t) \sum_{p \geq 1} \delta^{2p+1} y_{p-1}^2 + \sum_{p \geq 2} \delta^{2p} y_p^1 \right), \tag{4.32}$$

where $x_p^1, x_p^2, y_p^1, y_p^2$ denotes polynomials in $\eta_0(t)$ of order $p$. This finishes the proof of the theorem.
4.3 Re-expansion near the singularity

In this section we will derive equation (2.11), i.e. we will expand the formal separatrix \( X \) in its Laurent series around the singularity \( t = i\pi \). We do this by first expanding the two ”base” functions \( \eta_0(t) \) and \( \dot{\eta}_0(t) \) into their Laurent series around the singularity. The Laurent series of \( \eta_0(t) \) at \( t = i\pi \) is given by

\[
\eta_0(t - i\pi) = -\frac{4}{t^2} \left( 1 + \sum_{k=1}^{\infty} c_k t^{2k} \right),
\]

where \( c_k \in \mathbb{R} \ \forall k \in \mathbb{N} \). The distance to the next singularity is \( 2\pi \), which is then the radius of convergence for this series. Secondly we note that the Laurent expansion of \( \eta_1(t) \) at \( t = i\pi \) is given by

\[
\eta_1(t - i\pi) = \frac{8}{t^3} \left( 1 + \sum_{k=1}^{\infty} d_k t^{2k} \right),
\]

where \( d_k \in \mathbb{R} \ \forall k \in \mathbb{N} \) and it converges for \( |t| < 2\pi \).

We are now ready to determine the Laurent expansion of (4.33). Let us consider its first component which we write as

\[
[X]_{n-1} = \sum_{k=2}^{n-1} \delta^k \psi_1^k(t) \quad (4.33)
\]

where \( \psi_1^k(t) \) is the term corresponding to \( \delta^k \) in the formal separatrix (4.32). \( \psi_1^k \) has a pole of order \( k \). Using the Laurent expansions of \( \eta_0(t) \) and \( \eta_1(t) \) respectively, the Laurent expansion of each \( \psi_1^k(t) \) is given by

\[
\psi_1^k(t) = \frac{1}{(t - i\pi)^k} \sum_{j=0}^{\infty} (t - i\pi)^{2j} \psi_{kj},
\]

where \( \psi_{kj} \) are the coefficients of the Laurent expansion of \( \psi_1^k(t) \). Next we substitute \( \tau = \frac{t - i\pi}{\log \lambda} \), where we re-expand \( \log \lambda = \delta \sum_{l\geq 0} \lambda_l \delta^{2l} \), this gives

\[
\psi_1^k(t) = \frac{1}{\tau^k \delta^k (\sum_{l\geq 0} \lambda_l \delta^{2l})^k} \sum_{j=0}^{\infty} \tau^{2j} \delta^{2j} (\sum_{l\geq 0} \lambda_l \delta^{2l})^{2j} \psi_{kj}.
\]

Inserting this expression into (4.33) we note that \( \delta^{-k} \) and \( \delta^k \) cancel each other and we obtain an expansion in even powers of \( \delta \)

\[
\sum_{k=2}^{n-1} \delta^k \psi_1^k(t) = \sum_{k=2}^{n-1} \frac{1}{\tau^k} \sum_{j=0}^{\infty} \tau^{2j} \delta^{2j} (\sum_{l\geq 0} \lambda_l \delta^{2l})^{2j-k} \psi_{kj}.
\]
Since $\lambda(\delta)$ is even, analytic at $\delta = 0$ and $\lambda(0) = 1$ we use that $\lambda^m$ expands as
\[
\lambda^m = \left( \sum_{l=0}^{\infty} \lambda_l \delta^{2l} \right)^m = \sum_{l=0}^{\infty} \lambda_{m,l} \delta^{2l}
\]
to obtain
\[
\sum_{k=2}^{n-1} \delta^k \psi_k^1(t) = \sum_{k=2}^{n-1} \frac{1}{\tau_k} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \psi_{kj} \tau^{2j} \delta^{2(l+j)} \lambda_{2j-k,l}.
\]
We will now change the summation indices in two steps to simplify this expression. First we let $m = j + l$ which allow us to write the sums as
\[
\sum_{k=2}^{n-1} \delta^k \psi_k^1(t) = \sum_{k=2}^{n-1} \sum_{m \geq 0} \delta^{2m} \sum_{j+l=m} \psi_{kj} \tau^{2j-k} \lambda_{2j-k,l}.
\]
Next we let $p = 2j - k$ which gives us the desired result
\[
\sum_{k=2}^{n-1} \delta^k \psi_k^1(t) = \sum_{m \geq 0} \delta^{2m} \sum_{j+l=m} \sum_{2j-k=p} \tau^p \psi_{kj} \lambda_{p,l}.
\]
\[
= \sum_{m \geq 0} \delta^{2m} \sum_{p=-n+1} \tau^p \tilde{\psi}_{m,p}^{(1)}.
\]
The derivation of the Laurent expansion of the second component of (4.32) is analogous. The result is
\[
[X]_{n-1} = \sum_{k=2}^{n-1} \delta^k \psi_k(t) = \begin{pmatrix}
\sum_{m \geq 0} \delta^{2m} \sum_{p=-n+1} \tau^p \tilde{\psi}_{m,p}^{(1)} \\
\sum_{m \geq 0} \delta^{2m} \sum_{p=-n+1} \tau^p \tilde{\psi}_{m,p}^{(2)}
\end{pmatrix}.
\]
This finishes the derivation of the Laurent expansion of the formal separatrix around the singularity $t = i\pi$.

5 Close to identity maps

In this section study an analytic family of close to identity maps of the form
\[
F_\delta(x) = x + \delta G_\delta(x)
\]
where $\delta$ is a small parameter. Expanding a solution of the differential equation $\dot{x} = G_0(x)$ into Taylor series in time, we can easily check that $F_\delta$ is approximated by a map which shifts a point along a trajectory of the differential equation by time $\delta$:

$$F_\delta = \Phi_0^\delta + O(\delta^2).$$

We say that $G_0$ generate the limit flow associated with the map $F_\delta$.

If $F_\delta$ is a family of area-preserving maps then the limit flow is divergence free and has a (possibly local) Hamiltonian function $H_0$, i.e.,

$$G_0 = J \nabla H_0$$

where $J$ is the standard two dimensional symplectic matrix.

### 5.1 Fixed points and their multipliers

**Theorem 5.1.** If $p_0$ is a hyperbolic saddle of the limit flow $G_0$ then there is a unique smooth family $p_\delta$ of fixed points of $F_\delta$ which converges to $p_0$ as $\delta \to 0$. Moreover, the fixed point depends analytically on $\delta$ and is hyperbolic with multiplier $\lambda_\delta$. The multiplier depends analytically on $\delta$ and

$$\lambda_\delta = 1 + \delta \mu_0 + O(\delta^2),$$

where $\mu_0$ is an eigenvalue of $DG_0(p_0)$.

**Proof.** The theorem easily follows from the implicit function theorem. Indeed, by the definition of a fixed point we have

$$p_\delta = F_\delta(p_\delta),$$

which is equivalent to

$$G_\delta(p_\delta) = 0$$

due to the form of the map (5.1). Since $F_\delta$ is an analytic function, $G_\delta$ is analytic. Since $p_0$ is an equilibrium of the limit flow we have

$$G_0(p_0) = 0.$$

Since $p_0$ is a saddle equilibrium of $G_0$ one of the eigenvalues of $DG_0(p_0)$ is positive and the other one is negative. Therefore

$$\det DG_0(p_0) < 0,$$
and in particular \( \det DG_0(p_0) \neq 0 \). Then by the implicit function theorem there exists an analytic solution of \( (5.2) \) for all \( \delta \) such that \( |\delta| < \delta_0 \).

Let us now consider the eigenvalues and eigenvectors of the fixed point. Let

\[
A_\delta := DF_\delta(p_\delta) = \text{id} + \delta DG_\delta(p_\delta).
\]

The eigenvalues of \( DG_0(p_0) \) are simple, hence eigenvalues and normalised eigenvectors of \( DG_\delta(p_\delta) \) are analytic in \( \delta \). The matrices \( DG_\delta(p_\delta) \) and \( A_\delta \) have the same eigenvectors. Moreover, if \( \mu(\delta) \) is an eigenvalue of \( DG_\delta(p_\delta) \), then \( \lambda_\delta = 1 + \delta \mu(\delta) \) is an eigenvalue of \( A_\delta \).

5.2 Formal interpolation by a flow

The results of this subsection are of independent interest. We show that the map \( F_\delta \) can be formally interpolated by an autonomous Hamiltonian flow.

**Proposition 5.2.** Let \( F_\delta \) be an analytic family of area-preserving maps \( (5.1) \) defined on a simply connected domain \( D \subset \mathbb{R}^2 \) (or \( D \subset \mathbb{C}^2 \)), then there is a formal Hamiltonian \( H_\delta \) with coefficients analytical on \( D \), which formally interpolates \( F_\delta \). The Hamiltonian \( H_\delta \) is defined uniquely up to addition of a formal series in \( \delta \) only.

**Proof.** For a function \( F_\delta \) let us denote by \( [F_\delta]_n \) the \( n \)-th-order coefficient of the Taylor series in \( \delta \). In coordinates we write

\[
[F_\delta]_n = (f_n(x,y), g_n(x,y)).
\]

We construct the formal series

\[
H_\delta(x,y,\delta) = \sum_{k \geq 0} \delta^k H_k(x,y),
\]

such that for every \( n \)

\[
[\Phi^\delta_{H_n}]_\ell = [F_\delta]_\ell \quad \text{for all } 1 \leq \ell \leq n \tag{5.3}
\]

where \( \Phi^\delta_{H_n} \) is the Hamiltonian flow generated by the Hamiltonian

\[
H_n^\delta(x,y) = \sum_{k=0}^{n-1} \delta^k H_k(x,y). \tag{5.4}
\]

It is convenient to write the Hamiltonian flow generated by a Hamiltonian \( H^\delta \) using Lie series

\[
\Phi^\delta_{H_n}(x) = x + \sum_{n=1}^{\infty} \frac{\delta^n L^\delta_{H_n}(x)}{n!},
\]

51
where $L_{H_0}$ is the Lie derivative generated by the Hamiltonian $H_0^n$. The Lie derivative is linear in $H^n_\delta$. Then taking into account (5.4) we get

$$
\left[ \Phi_{H_\delta}^\ell \right] = \sum_{m=1}^\ell \frac{1}{m!} \sum_{n_1+n_2+...+n_m = \ell-m} L_{H_1} \ldots L_{H_{nm}}(x) .
$$

(5.5)

We construct $H_n$ inductively. Let $n = 0$. Then (5.3) takes the form

$$
\left[ \Phi_{H_\delta}^1 \right] = \left[ F_\delta \right]^1 .
$$

Taking into account (5.5) we rewrite it in the form

$$
L_{H_0}(x) = G_0 .
$$

Taking into account that in coordinates $x = (x,y)$ and

$$
L_{H_0}(x) = \left( \frac{\partial H_0}{\partial y}, -\frac{\partial H_0}{\partial x} \right)
$$

we see that (5.3) is satisfied for $n = 0$ provided $H_0$ is the Hamiltonian of the limit flow.

We continue by induction. Suppose $H_0, \ldots, H_{n-1}$ are chosen in such a way that the condition (5.3) is satisfied for some $n \geq 0$. We show that $H_n$ can be chosen in such a way that (5.3) is satisfied for $n$ replaced by $n+1$.

We note that (5.5) implies that

$$
\left[ \Phi_{H_\delta}^{n+1} \right] = \left[ \Phi_{H_\delta}^n \right] + L_{H_n}(x) .
$$

Then the induction assumption implies that

$$
\left[ \Phi_{H_\delta}^{n+1} \right] = \left[ F_\delta \right]^{n+1} .
$$

For $\ell = n + 1$ the equality is achieved provided $H_n$ satisfies the equation

$$
L_{H_n}(x) = \left[ \Phi_{H_\delta}^{n+1} \right] - \left[ F_\delta \right]^{n+1} .
$$

(5.6)

This equation has a solution in $D$ provided the right hand side is divergence free. The solution is defined uniquely up to addition of a constant.
In order to check that the right hand side has indeed zero divergence, we make a simple observation. Let $F_\delta$ and $\tilde{F}_\delta$ be two analytic families of close-to-identity area preserving maps such that $[F_\delta]_\ell = [\tilde{F}_\delta]_\ell$ for all $0 \leq \ell \leq n$. Then
\[
\text{div}[F_\delta - \tilde{F}_\delta]_{n+1} = 0. \tag{5.7}
\]
Indeed, let us write the maps in the coordinates:
\[
F_\delta(x, y) = \left( x + \sum_{k \geq 1} \delta^k a_k(x, y), y + \sum_{k \geq 1} \delta^k b_k(x, y) \right)
\]
and a similar formula for $\tilde{F}_\delta$. The preservation of the area is equivalent to the system
\[
\partial_x a_1 + \partial_y b_1 = 0,
\partial_x a_{n+1} + \partial_y b_{n+1} = -\sum_{\ell=1}^{n} \{ a_\ell, b_{n+1-\ell} \}, \quad n \geq 1.
\]
The system was obtained by expanding the Jacobian of $F_\delta$ in power series of $\delta$ and comparing the result with 1. Since $a_\ell = \tilde{a}_\ell$ and $b_\ell = \tilde{b}_\ell$ for $1 \leq \ell \leq n$, we immediately see that
\[
\partial_x a_{n+1} + \partial_y b_{n+1} = \partial_x \tilde{a}_{n+1} + \partial_y \tilde{b}_{n+1} \tag{5.8}
\]
which is equivalent to (5.7).
Setting $\tilde{F}_\delta = \Phi^\delta_{H_0}$ and applying the last statement we see that the right hand side of (5.6) has zero divergence and consequently $H_n$ is defined by the equation. The theorem follows by induction.

5.3 Approximation of the local separatrix

In this section we prove that the local separatrix of the map $F_\delta$ is close to a local separatrix of any Hamiltonian flow which is $O(\delta^{n+1})$ close to $F_\delta$. The approximation is of order $O(\delta^n)$. We note that the existence of a local separatrix near a hyperbolic fixed point of the map and a hyperbolic equilibrium of the flow follows from the Hartman-Grobman theorem.

The proof consists of two steps. First we transform the fixed point of the map to the origin and diagonalise its linear part by a linear symplectic change of variables. We also transform the Hamiltonian flow into a similar form.

\footnote{The proof does not really uses the fact that $F$ is area-preserving and the flow is Hamiltonian.}
Then we prove the theorem for the transformed map and the transformed flow. Then we come back to the original variables.

Theorem 5.1 implies the map $F_\delta$ has a fixed point $p_\delta$ which is analytic in $\delta$. Consider the following change of variables

$$S : x \mapsto p_\delta + B_\delta x,$$

where $B_\delta$ is a symplectic matrix which diagonalises $DF_\delta(p_\delta)$. This change of variables transforms $F_\delta$ to

$$\tilde{F}_\delta = S^{-1} \circ F_\delta \circ S.$$

The change of variables is chosen to ensure that the fixed point is moved to the origin

$$\tilde{F}_\delta(0) = 0,$$

and the Jacobian of the map at the origin is diagonal:

$$DF_\delta(0) = A_\delta := \text{diag}(\lambda_\delta, \lambda_\delta^{-1}).$$

Note that the multipliers of the fixed point are left unchanged under any smooth change of variables. After the change of coordinates the unstable separatrix is parametrised by the function

$$\tilde{\psi}_\delta^\leftarrow = S \circ \tilde{\psi}_\delta^\leftarrow$$

which satisfies the equation

$$\tilde{\psi}_\delta^\leftarrow(t + \log \lambda_\delta) = \tilde{F}_\delta(\tilde{\psi}_\delta^\leftarrow(t)).$$ (5.9)

We note that the vector field generated by $H^n_\delta$ has an equilibrium $q_\delta$ which is close to $p_\delta$ but does not necessarily coincide with it. In general the linear part of the flow is not exactly diagonal after the substitution $S$. So we introduce another change of variables

$$\hat{S} : x \mapsto q_\delta + \hat{B}_\delta x,$$

where $\hat{B}_\delta$ is a symplectic matrix that diagonalises $J\nabla^2 H^n_\delta(q_\delta)$. Using the implicit function theorem and smooth dependence of eigenvectors in a way similar to Section 5.1 we conclude that

$$p_\delta = q_\delta + O(\delta^n),$$ (5.10)

$$B_\delta = \hat{B}_\delta + O(\delta^n).$$
Then we define a new Hamiltonian function
\[
\tilde{H}_\delta^n = H_\delta^n \circ \hat{S}.
\]
Let \(\tilde{\varphi}_\delta^n\) denote the separatrix solution of the differential equation
\[
\mu_{n,\delta} \tilde{\varphi}_\delta^n = J \nabla \tilde{H}_\delta^n \bigg|_{\tilde{\varphi}_\delta^n}.
\] (5.11)
The original separatrix solution can be easily restored by the formula
\[
\varphi_\delta^n = \hat{S} \circ \tilde{\varphi}_\delta^n.
\]
Let us write \(\tilde{\Phi}_\delta := \Phi_{\log \lambda_\delta}^{\mu_{n,\delta}^{-1} H_\delta^n}\) to shorten the notation. By the definition of the Hamiltonian flow,
\[
\tilde{\varphi}_\delta^n(t + \log \lambda_\delta) = \tilde{\Phi}_\delta(\tilde{\varphi}_\delta^n(t)).
\] (5.12)
It is easy to see that
\[
\tilde{\Phi}_\delta = \tilde{F}_\delta + O(\delta^{n+1}),
\] (5.13)
which will play a crucial role in the proof.

Now we define
\[
\xi(t) := \tilde{\varphi}_\delta^n(t) - \tilde{\varphi}_\delta^n(t).
\] (5.14)
It follows from the definition of \(\xi(t)\) and equations (5.12) and (5.9) that
\[
\xi(t + \log \lambda_\delta) = \tilde{F}_\delta(\tilde{\varphi}_\delta^n(t) + \xi(t)) - \tilde{\Phi}_\delta(\tilde{\varphi}_\delta^n(t)).
\] (5.15)
We are proving that there exists a \(\xi(t)\) such that
\[
\xi(t) = O(\delta^n e^{2t}).
\] (5.16)
First we rewrite the equation for \(\xi\) in the form
\[
\xi(t + \log \lambda_\delta) - A_\delta \xi(t) = R(t, \xi(t)),
\] (5.17)
where
\[
R(t, \xi(t)) := \tilde{F}_\delta(\tilde{\varphi}_\delta^n(t) + \xi(t)) - \tilde{\Phi}_\delta(\tilde{\varphi}_\delta^n(t)) - A_\delta \xi(t).
\]
Expanding $\tilde{F}_\delta$ in Taylor series around $\tilde{\varphi}_n(t)$ we get
\[
R(t, \xi(t)) = R_0(t) + R_1(t)\xi(t) + R_2(\xi(t)),
\]
(5.18)
where
\[
R_0(t) = \tilde{F}_\delta(\tilde{\varphi}_n(t)) - \tilde{\Phi}_\delta(\tilde{\varphi}_n(t)),
\]
(5.19)
\[
R_1(t) = D\tilde{F}_\delta(\tilde{\varphi}_n(t)) - A_\delta,
\]
(5.20)
\[
R_2(\xi(t)) = O(\|\xi(t)\|^2),
\]
(5.21)
where the upper bound for $R_2(\xi(t))$ is a standard estimate for the remainder of the Taylor expansion. Next we show that
\[
R_0(t) = O(\delta^n e^{2t})
\]
(5.21)
and
\[
R_1(t) = O(\delta e^t).
\]
(5.22)
We begin with (5.22). We note that since $\tilde{F}_\delta$ is close to identity we have $\tilde{F}_\delta = \text{id} + \delta \tilde{G}_\delta$ and consequently $D\tilde{F}_\delta = \text{id} + \delta D\tilde{G}_\delta$ and in particular $A_\delta = D\tilde{F}_\delta(0) = \text{id} + \delta D\tilde{G}_\delta(0)$. Therefore
\[
D\tilde{F}_\delta(x) - A_\delta = \delta \left(D\tilde{G}_\delta(x) - D\tilde{G}_\delta(0)\right) = O(\delta|x|).
\]
Substituting $\tilde{\varphi}_n(t)$ instead of $x$ and using the upper bound $\tilde{\varphi}_n(t) = O(e^t)$ as $\text{Re}(t) \to -\infty$, we see the upper bound (5.22) follows from the definition (5.19).

The derivation of the upper bound for $R_0(t)$ is a bit more sophisticated. First we note that
\[
\tilde{F}_\delta(0) = \tilde{\Phi}_\delta(0) = 0 \quad \text{and} \quad D\tilde{F}_\delta(0) = D\tilde{\Phi}_\delta(0) = A_\delta.
\]
Consequently
\[
\tilde{F}_\delta(x) - \tilde{\Phi}_\delta(x) = O(|x|^2).
\]
Substituting $\tilde{\varphi}_n(t)$ instead of $x$ and taking into account the definition of $R_2$ we conclude that the components of $\frac{R_0(t)}{(e^t)^2}$ are bounded analytic function on $\text{Re}(t) \leq r_0$. Moreover $\tilde{\varphi}_n(t)$ is $2\pi i$ periodic and analytic, therefore each component of the vector $R_0(t)$ is $2\pi i$ periodic and analytic in $t$. Consequently the maximum of $\frac{R_0(t)}{(e^t)^2}$ is achieved at the line $\text{Re}(t) = r_0$. Using (5.13) we get for $\text{Re } t < r_0$
\[
\left|e^{-2t}R_0(t)\right| \leq \sup_{\text{Re}(t)=r_0} \frac{\left|\tilde{F}_\delta(\tilde{\varphi}_n(t)) - \tilde{\Phi}_\delta(\tilde{\varphi}_n(t))\right|}{|e^{2t}|} \leq c_2\delta^{n+1}
\]
\footnote{The $2\pi i$ periodicity allows a substitution $z = e^t$ and an application of a maximum modulus principle to the result.}
which is equivalent to the desired estimate (5.21).

Now we observe that if the following sum converges

\[ \xi(t) = \sum_{k \geq 1} A_k^{t} f(t) \]

then it solves the difference equation

\[ \xi(t + \log \lambda) - A_\delta \xi(t) = f(t) . \]

Indeed, substituting the series into the left hand side of the equation we get

\[ \xi(t + \log \lambda) - A_\delta \xi = \sum_{k \geq 1} A_k^{t} f(t - (k - 1) \log \lambda) \]

\[ - \sum_{k \geq 1} A_k^{t} f(t - k \log \lambda) = f(t). \]

Thus instead of solving (5.17) we solve the “integral equation”

\[ \xi(t) = \sum_{k=1}^{\infty} A_k^{t} R(t - k \log \lambda, \xi(t - k \log \lambda)), \quad (5.23) \]

where \( R(t, \xi(t)) \) is given by (5.18). We will show that we can solve (5.23) by applying contraction mapping arguments in a ball in the Banach space of functions which are analytic, \( 2\pi i \) periodic in a half plane \( \text{Re}(t) < r_0 \) and have bounded norm

\[ \|\xi\| = \sup_{\text{Re}(t) < r_0} |e^{-2t} \xi(t)| \]

where \(| \cdot |\) stands for the Euclidean norm in \( C^2 \). Let

\[ \mathcal{B} = \{ \xi : \|\xi\| < K \delta^n \} \quad (5.24) \]

where \( K > 0 \) and \( r_0 \in \mathbb{R} \) will be chosen later in the proof. We note that if \( \xi \in \mathcal{B} \) then for all \( \text{Re}(t) < r_0 \)

\[ |\xi(t)| \leq \|\xi\| e^{2\text{Re}(t)} \]

\[ < K \delta^n e^{2\text{Re}(t)} . \]

We will show that there exist \( K \) and \( r_0 \) such that the ball \( \mathcal{B} \) is invariant under the operator in the right-hand side of (5.23) and that the operator is a contraction operator on the ball \( \mathcal{B} \). Let us denote the operator in the right-hand side of (5.23) by \( \mathcal{F} \). We begin by showing that the ball \( \mathcal{B} \) is invariant under \( \mathcal{F} \), i.e. \( \mathcal{F}(\xi) \in \mathcal{B} \) for any \( \xi \in \mathcal{B} \). Let \( \xi \in \mathcal{B} \) then

\[ \mathcal{F}(\xi) = \sum_{k=1}^{\infty} A_k^{t} (R_0 + R_1 \xi + R_2(\xi)) \]

\[ |t - k \log \lambda| . \quad (5.25) \]
Using the upper bounds for $R_0(t), R_1(t)$ and $R_2(\xi(t))$ from equations (5.21), (5.22) and (5.20) respectively we obtain

\[
\begin{align*}
|R_0|_{t-k \log \lambda_\delta} &\leq c_1 \delta^{n+1} e^{2 \text{Re}(t)} \lambda_\delta^{-2k}, \\
|R_1 \xi|_{t-k \log \lambda_\delta} &\leq c_2 \delta e^{3 \text{Re}(t)} \lambda_\delta^{-2k} \|\xi\|, \\
|R_2(\xi)|_{t-k \log \lambda_\delta} &= c_3 \delta^{2n} K^2 e^{4 \text{Re}(t)} \lambda_\delta^{-4k},
\end{align*}
\]

where $c_1, c_2, c_3$ are constants. Since the norm of $A_\delta$ does not exceed $\lambda_\delta$, we can use the above estimates to get an upper bound for the left hand side of (5.26):

\[
\|F(\xi)\| \leq c_1 c_3 \delta^{n} e^{2 r_0} + c_2 c_4 \delta^{n} e^{3 r_0} K \lambda_\delta^{-1} + c_3 c_4 \delta^{-2} \delta^{2n-1} K^2 e^{4 r_0},
\]

where $c_4 = \sup_{|\delta| < \delta_0} \frac{\lambda_\delta^2}{1 - 1 - \lambda_\delta}$ is finite since $\frac{1}{1 - 1 - \lambda_\delta} = O(\delta^{-1})$. To show that there exists a $K, r_0$ such that $\|F(\xi)\| \leq \delta^n K$ for all $\delta < \delta_0$ we start by choosing $K = 4c_1 c_4 e^{2r_0}$. Then by choosing $r_0 = \frac{1}{2} \log \frac{\lambda_\delta}{4c_2 c_3}$ we obtain that the second term in the right-hand side is bounded by

\[
c_2 c_4 \delta^{n} e^{3 r_0} K \lambda_\delta^{-1} < \frac{1}{4} \delta^n K.
\]

Finally letting $\delta_0 = \sqrt{-\frac{c_2}{4c_1 c_3}}$, we see that the last term in the right-hand side of (5.26)

\[
c_3 c_4 \lambda_\delta^{-2} \delta^{2n-1} K^2 e^{4 r_0} \leq \frac{1}{4} \delta^n K
\]

for all $\delta < \delta_0$. These choices yield

\[
\|F(\xi)\| \leq \frac{3}{4} \delta^n K,
\]

hence $F(\xi) \in B$.

Next we show that $F$ is a contraction. Let $\xi_1, \xi_2 \in B$, and consider

\[
|R(t, \xi_1(t)) - R(t, \xi_2(t))| = |R_1(t) (\xi_1(t) - \xi_2(t)) + R_2(t, \xi_1(t)) - R_2(t, \xi_2(t))| \\
\leq (c_2 \delta e^{\text{Re}(t)} + c_3 \|R_2\|_{C^2} K \delta^n e^{2 \text{Re}(t)}) |\xi_1(t) - \xi_2(t)|.
\]

Then we have

\[
\|F(\xi_1) - F(\xi_2)\| \leq \sum_{k=1}^{\infty} \lambda_\delta^{1+k} \sup_{\text{Re}(t) < r_0} \left| (R(\xi_1) - R(\xi_2)) |_{t-k \log \lambda_\delta} e^{2 \text{Re}(t)} \right|.
\]

58
Taking into account the previously obtained upper bound for $R_1$ and $R_2$ we get

$$\|F(\xi_1) - F(\xi_2)\| \leq c_5 \left( c_2 e^{r_0} + c_3 \|R_2\|_{C^2} K e^{4r_0 \delta^n \lambda_\delta^{-1}} \right) \|\xi_1 - \xi_2\|,$$

where $c_2, c_3$ and $c_5$ are constants. By choosing $r_0$ sufficiently small the contraction constant

$$\alpha = c_5 \left( c_2 e^{r_0} + c_3 \|R_2\|_{C^2} K e^{4r_0 \delta^n \lambda_\delta^{-1}} \right) < \frac{1}{2}.$$

It now follows by the contraction mapping principle that there exists a unique solution $\xi \in B$ which satisfies equation (5.16).

Now we recall that

$$\psi - \delta(t) = S \circ \tilde{\psi}_\delta$$

and

$$\varphi^n_\delta = \tilde{S} \circ \tilde{\varphi}^n_\delta$$

Taking the difference we get the estimate:

$$\psi(t) - \varphi^n_\delta(t) = S \circ \tilde{\psi}_\delta(t) - \tilde{S} \circ \tilde{\varphi}^n_\delta(t) = S \circ \tilde{\psi}_\delta(t) - S \circ \tilde{\varphi}^n_\delta(t) + O(\delta^n)$$

$$= S \left( \tilde{\psi}_\delta(t) - \tilde{\varphi}^n_\delta(t) \right) + O(\delta^n) = O(\delta^n).$$

We have proved that there is $r_0 \in \mathbb{R}$ and a separatrix solution of (1.5) such that

$$\psi(t) = \varphi^n_\delta(t) + O(\delta^n) \quad (5.27)$$

for all $t$ with $\Re t < r_0$.

### 5.4 Extension lemma

In the previous section we have seen that the local unstable separatrix of the flow and the map approximates each other well for the half-plane $\Re(t) < r_0$ for some $r_0 \in \mathbb{R}$. In this section we show that we can extend the domain in which the separatrices stay close to each other.

Consider two close to identity maps

$$F = \text{id} + \delta G,$$

$$\tilde{F} = \text{id} + \delta \tilde{G}, \quad (5.28)$$

The following lemma gives a closeness result for the difference of their first $O(\delta^{-1})$ iterates.

59
Lemma 5.3 (Extension Lemma). Suppose two close to identity maps $F_\delta$ and $\tilde{F}_\delta$ are both defined on a $\delta$-independent convex neighbourhood of a set $\bar{D} \subset \mathbb{C}^2$ and on this neighbourhood $\tilde{F}_\delta = F_\delta + O(\delta^{n+1})$. Then for any $r > 0$ and any $T > 0$ there are two constants $\delta_0 > 0$ and $\tilde{K}$ such that for any two points $z_0, \tilde{z}_0 \in D$ such that $z_k := F_\delta^k(z_0) \in D$ for all $k$, $0 \leq k \leq T\delta^{-1}$, the inequality $|\tilde{z}_0 - z_0| \leq r\delta^n$ implies that

$$|\tilde{z}_k - z_k| \leq K\delta^n \quad \text{for all} \ 0 \leq k \leq T\delta^{-1}$$

where $\tilde{z}_k = \tilde{F}_\delta^k(\tilde{z}_0)$.

To prove this lemma we will need the following Gronwall type inequality.

Lemma 5.4. Let $b \geq 0$ and $(a_n)$ be an increasing sequence of positive numbers. If $z_n$ is a sequence such that $z_n \leq a_n + b \sum_{k=1}^{n-1} z_k$, then $z_n \leq a_n (1 + b)^{n-1}$.

Proof. By induction. The statement is true for $n = 1$. Assume that the statement is true for all $n \leq p$, i.e. $z_n \leq a_n (1 + b)^{n-1}$. Then for $n = p + 1$ we have

$$z_{p+1} \leq a_{p+1} + b \sum_{k=0}^{p} z_k \leq a_{p+1} + b \sum_{k=1}^{p} a_k (1 + b)^{k-1}$$

$$\leq a_{p+1} \left( 1 + b \sum_{k=1}^{p} (1 + b)^{k-1} \right) = a_{p+1} \left( 1 + b \frac{1 - (1 + b)^p}{1 - (1 + b)} \right)$$

$$= a_{p+1} (1 + b)^p$$

By induction the statement is true for all $n \in \mathbb{N}$. \qed

Proof of Lemma 5.3. Let us estimate the difference of the $N$th iterates of the two maps (5.28). It is easy to see that

$$z_N = z_0 + \delta \sum_{k=0}^{N-1} G(z_k) \quad \text{and} \quad \tilde{z}_N = \tilde{z}_0 + \delta \sum_{k=0}^{N-1} \tilde{G}(\tilde{z}_k).$$

Let $\tilde{G} - G = \delta^n R$, where $R$ is bounded due to the assumptions of the lemma. Taking the difference and applying the triangle inequality we get

$$|\tilde{z}_N - z_N| = |\tilde{z}_0 - z_0| + \delta \sum_{k=0}^{N-1} |\tilde{G}(\tilde{z}_k) - G(z_k)|$$

$$\leq |\tilde{z}_0 - z_0| + \delta \sum_{k=0}^{N-1} |G(\tilde{z}_k) - G(z_k)| + \delta^{n+1} \sum_{k=0}^{N-1} |R(\tilde{z}_k)|.$$
Since the domain of $G$ is convex the map is Lipschitz
\[ |G(\tilde{z}_k) - G(z_k)| \leq L|\tilde{z}_k - z_k|, \]
where $L = \|DG\|$. Moreover $R$ is bounded and we conclude
\[ |\tilde{z}_N - z_N| \leq |\tilde{z}_0 - z_0| + \delta L \sum_{k=0}^{N-1} |\tilde{z}_k - z_k| + 2\delta^N N \|R\|. \]

We recall that $b := |\tilde{z}_0 - z_0| \leq r\delta^n$. The sequence $a_N := \delta^n r + \delta^{n+1} N \|R\|$ is increasing and we apply the Gronwall type inequality of Lemma 5.4 to obtain
\[ |\tilde{z}_N - z_N| \leq \left( \delta^n r + \delta^{n+1} N \|R\| \right) (1 + \delta L)^{N-1}. \]

Then we use the textbook inequality
\[ (1 + \delta L)^{\frac{1}{\delta L}} \leq e \]
as an upper bound for the last parenthesis in the right-hand side and get
\[ |\tilde{z}_N - z_N| \leq \delta^n (r + \delta N \|R\|) e^{\delta L(N-1)}. \]

Then for all $0 \leq N \leq \frac{T}{\delta}$ we have
\[ |\tilde{z}_N - z_N| \leq \delta^n (r + T \|R\|) e^{LT}. \]

Let $K = (r + T \|R\|) \exp(LT)$ to complete the proof.

5.5 Application of the extension lemma

We will now use Lemma 5.3 to extend the domain in which the local separatrix of the flow approximates the local separatrix of the map. We extend the approximation to the domain defined by
\[ \mathcal{T}_0 = \{ t \in \mathbb{C} : \varphi_0(t - s) \in \mathcal{D} \quad \forall s \geq 0, \quad \text{Re}(t) \leq T \} , \quad (5.29) \]
where $\varphi_0$ is the separatrix of the limit flow $\mu_0 \dot{\varphi}_0 = G_0(\varphi_0)$. Note that $\mathcal{T}_0$ is independent of $\delta$.

Corollary 5.5. Let $\psi_\delta(t)$ and $\varphi_\delta^n(t)$ be defined as in Theorem 2.3 and let $T \in \mathbb{R}$ be fixed. Then $\psi_\delta(t)$ is defined on $\mathcal{T}_0$ and
\[ \psi_\delta(t) = \varphi_\delta^n(t) + O(\delta^n) \]
uniformly in $\mathcal{T}_0$. 

61
Remark 5.6. Uniformly in $T_0$ means that

$$|\psi_\delta^-(t) - \varphi_\delta^n(t)| \leq K(D,T)\delta^n.$$  

Proof. By definition

$$\psi_\delta^-(t + \log \lambda_\delta) = F_\delta(\psi_\delta^-)$$

and

$$\dot{\varphi}_\delta^n(t) = \mu_{n,\delta}^{-1}J\nabla H^n_\delta(\varphi_\delta^n(t)),$$

which implies that

$$\varphi_\delta^n(t + \log \lambda_\delta) = \Phi_{1 + O(\delta^n)}(\varphi^n_\delta(t)).$$

Theorem 2.3 also states that

$$F_\delta = \Phi_{1 + O(\delta^n)} + O(\delta^{n+1}),$$

hence

$$\Phi_{1 + O(\delta^n)}(\varphi^n_\delta(t)) = \Phi_{1 + O(\delta^n)} + O(\delta^{n+1}) = F_\delta + O(\delta^{n+1}).$$ \hfill (5.30)

Let us denote this map by $\tilde{F}_\delta$. Then we have

$$\varphi_\delta^n(t + \log \lambda_\delta) = \tilde{F}_\delta(\varphi_\delta^n(t)).$$

It now follows that $F_\delta$ and $\tilde{F}_\delta$ are two close to identity maps such that $\tilde{F}_\delta = F_\delta + O(\delta^{n+1})$. Define

$$z_k := F_\delta^k(\psi_\delta^-(t_0))$$

and

$$z_k^* := \tilde{F}_\delta^k(\varphi_\delta^n(t_0)).$$

Since $\text{Re}(t_0) \leq r_0$ it follows from Theorem 2.3 that

$$|z_0 - z_0^*| \leq O(\delta^n).$$

To estimate $|\psi_\delta^- (t) - \varphi_\delta^n(t)|$ we do \left\lfloor \frac{\text{Re}(t) - r_0}{\log \lambda_\delta} \right\rfloor$ iterations of the maps $F_\delta, \tilde{F}_\delta$, hence the number of iterates is bounded by $\frac{L(T,r_0)}{\delta}$ where $L$ is a constant. Finally we note that by the definition of $T_0$ it follows that $z_k^*$ belongs to a $\delta$-neighbourhood of $D$ for all $0 \leq k \leq \frac{L(T,r_0)}{\delta}$. The corollary now follows from Lemma 5.3. \hfill $\square$
References

[1] Arnold, V.I., Kozlov, V.V., Neishtadt, A.I., Mathematical aspects of classical and celestial mechanics. Dynamical systems. III. Third edition. Encyclopaedia of Mathematical Sciences, 3. Springer-Verlag, Berlin, 2006.

[2] Baldomá, I., Seara, T. M., Breakdown of heteroclinic orbits for some analytic unfoldings of the Hopf-zero singularity. J. Nonlinear Sci. 16 (2006), no. 6, 543–582.

[3] Champneys, A. R. Codimension-one persistence beyond all orders of homoclinic orbits to singular saddle centres in reversible systems. Nonlinearity 14 (2001), no. 1, 87–112.

[4] Delshams, A., Ramrez-Ros, R., Exponentially small splitting of separatrices for perturbed integrable standard-like maps. J. Nonlinear Sci. 8 (1998), no. 3, 317–352.

[5] Deprit, Andr´e Canonical transformations depending on a small parameter. Celestial Mech. 1 1969/1970 12–30.

[6] Duarte, P., Abundance of elliptic isles at conservative bifurcations. Dynam. Stability Systems 14 (1999), no. 4, 339–356.

[7] Duarte, P., Persistent homoclinic tangencies for conservative maps near the identity. Ergodic Theory Dynam. Systems 20 (2000), no. 2, 393–438.

[8] Dumortier, F., Rodrigues, P.R., Roussarie, R., Germs of diffeomorphisms in the plane. Lecture Notes in Mathematics, 902. Springer-Verlag, Berlin-New York, 1981. 197 pp.

[9] Fontich, E.; Simó, C. The splitting of separatrices for analytic diffeomorphisms. Ergodic Theory Dynam. Systems 10 (1990), no. 2, 295–318.

[10] Gelfreich V., A proof of the exponentially small transversality of the separatrices for the standard map, Comm. Math. Phys. 201/1, (1999) 155–216

[11] Gelfreich V., Splitting of a small separatrix loop near the saddle-center bifurcation in area-preserving maps, Physica D 136, no.3–4, (2000) 266–279

63
[12] Gelfreich V., *Near strongly resonant periodic orbits in a Hamiltonian system*. Proc. Nat. Acad. Sci. USA, vol. 99, no. 22, (2002) 13975–13979.

[13] Gelfreich V., Lazutkin V., *Splitting of Separatrices: perturbation theory and exponential smallness*, Russian Math. Surveys Vol. 56, no. 3, (2001) 499–558.

[14] Gelfreich V., Naudot N. (2008) Analytic invariants associated with a parabolic fixed point in C2 Ergodic Theory and Dynamics Systems (accepted for publication).

[15] Gelfreich V., Sauzin D. *Borel summation and splitting of separatrices for the Henon map*, Annales l’Institut Fourier, vol. 51, fasc. 2, (2001) p.513–567.

[16] Hakim, V., Mallick, K., Exponentially small splitting of separatrices, matching in the complex plane and Borel summation. Nonlinearity 6 (1993), no. 1, 57–70.

[17] Lazutkin, V.F., Splitting of separatrices for the Chirikov standard map. J. Math. Sci., New York 128, No. 2, 2687-2705 (2005) (translation from a Russian preprint (1984))

[18] Lombardi, E., Oscillatory integrals and phenomena beyond all algebraic orders with applications to homoclinic orbits in reversible systems. Lecture Notes in Mathematics. 1741. Berlin: Springer, 412 p (2000).

[19] Ramírez-Ros, R., Exponentially small separatrix splittings and almost invisible homoclinic bifurcations in some billiard tables. Phys. D 210 (2005), no. 3-4, 149–179.

[20] Simó, C., Broer, H., Roussarie, R., A numerical survey on the Takens-Bogdanov bifurcation for diffeomorphisms. European Conference on Iteration Theory (Batschuns, 1989), 320–334, World Sci. Publ., River Edge, NJ, 1991.

[21] Treschev, D. Width of stochastic layers in near-integrable two-dimensional symplectic maps. Phys. D 116 (1998), no. 1-2, 21–43.

[22] Treschev, D., Splitting of separatrices for a pendulum with rapidly oscillating suspension point. Russian J. Math. Phys. 5 (1997), no. 1, 63–98 (1998).