Crossing and Antisolitons in Affine Toda Theories

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Abstract
Affine Toda theory is a relativistic integrable theory in two dimensions possessing solutions describing a number of different species of solitons when the coupling is chosen to be imaginary. These nevertheless carry real energy and momentum. To each species of soliton there has to correspond an antisoliton species. There are two different ways of realising the antisoliton whose equivalence is shown to follow from a surprising identity satisfied within the underlying affine Kac-Moody group. This is the classical analogue of the crossing property of analytic S-matrix theory. Since a complex parameter related to the coordinate of the soliton is inverted, this identity implies a sort of modular transformation property of the soliton solution. The results simplify calculations of explicit soliton solutions.
1. Introduction

As integrable field theories in two dimensions, the affine Toda theories have long attracted particular attention [1], [2], [3], [4], [5], [6], [7]. There were two main reasons: (1) their relativistic invariance and (2) their natural association with affine Kac-Moody algebras. These algebras (which we shall take to be untwisted and simple laced for definiteness and simplicity) play a key rôle in the integrability which is manifested by the occurrence of an infinite number of local conservation laws. Correspondingly, there exists a hierarchy of integrable systems amongst which the affine Toda theory is distinguished by its Poincaré invariance.

This significance was reinforced by Zamolodchikov’ observation that the Lorentz symmetry is a subgroup of a much large, conformal symmetry realised in a critical limit of the theory [8], [9]. The conservation laws are seen as the non-chiral relics of the conformal (or larger) chiral symmetry when it is broken to its Poincaré subgroup. Affine Toda theory provides a very good illustration of this [10].

When the affine Toda couplings are made imaginary, the theory acquires classical soliton solutions [11], [12], [13] whose energy and momentum are topological in the sense of being surface terms and, as a consequence, real despite the complex nature of the equations [14], [15], [16]. If \( \mathfrak{g} \) denotes the associated untwisted affine Kac-Moody algebra, with the Lie algebra \( \mathfrak{g} \) simple and of rank \( r \), the number of soliton species equals \( r \), [15], the same as the number of species of particles created by the quantum Toda fields. Furthermore, there are resemblances between the solitons and quantum excitation particles such as the mass formulae and the rules governing trilinear couplings [17], [18], [19], [20], [21], [22].
There is an extra discrete symmetry of a relativistic theory which implies that to every species $i$ of particles there exist a corresponding species $\bar{i}$ of antiparticles to $i$. For the particles created by the quantum Toda fields there is a very specific construction of $\bar{i}$ from $i$ [23]. But for solitons there is a second, apparently different construction, in which the soliton solution is run backwards in space. The main result here is to find and demonstrate a surprising identity in terms of the generators of $\hat{g}$ which guarantees the equivalence of these two presentations of the antisoliton. For reasons to be explained, this identity can be thought as the classical analogue of the crossing property familiar in analytic S-matrix theory [24]. The identity explains various regularities in the explicit soliton solutions and can simplify their calculations.

In section 2 we recap the soliton formalism and how it is advantageous to reformulate $\hat{g}$ in terms of a new basis consisting of its principal Heisenberg subalgebra and the remaining generators which are the Laurent coefficients of the $r$ “fields” $\hat{F}^1(z), \hat{F}^2(z), \ldots, \hat{F}^r(z)$ which ad-diagonalise the Heisenberg subalgebra and plays the rôle of creating the $r$ species of solitons [15].

It has been shown that in a highest weight representation of $\hat{g}$ at level $x$ the powers of $\hat{F}^i(z)$ higher than $x$ vanish [25]. In section 3 we show that the highest non-vanishing power, $(\hat{F}^i(z))^x$, takes the form of a vertex operator obtained by exponentiating the Heisenberg subalgebra. This generalises a result previously known at level 1. In section 4, this vertex operator is used to relate the power $k(\leq x)$ of $\hat{F}^i(z)$ to the power $(x-k)$ of $\hat{F}^\bar{i}(\bar{z})$ where the variable $\bar{z}$ is obtained from $z$ by the standard crossing procedure in a sense to be explained.
This is the surprising identity and constitutes our main result. Since it can be put in a form involving exponentials of Kac-Moody generators (including the central extension) it can be viewed as a group theoretic relation.

Since the single soliton solution is expressed in terms of expectation values of powers of $\hat{F}^i(z)$ with respect to highest weight states, the ideas mentioned relate powers $k$ and $x - k$ in a very simple way. Since powers up to the integer part of $(x + 1)/2$ only need to be considered, rather than $x$, calculations are consequently simplified.

2. Preliminaries

Corresponding to the $(r + 1)$ simple roots of the affine untwisted Kac-Moody algebra $\hat{g}$, we denote the $3(r + 1)$ Chevalley generators as $e_i, f_i$ and $h_i$ ($i = 0, 1, ..., r$) in the usual notation [26], [15]. In the principal grade, which counts the height of the roots by the operator

$$d' = -hL_0 + T_0^3,$$

these have grades 1, −1 and 0 respectively. $h$ is the Coxeter number of $g$, $T_0^3$ the intersection of its principal so(3) subalgebra and its Cartan subalgebra and $L_0$ the Virasoro generator.

The principal Heisenberg subalgebra of $\hat{g}$ plays an important role in the affine Toda theories, [6], just as it does in other integrable theories considered earlier, [27] and the final chapter of [26]. It is infinite dimensional, possessing commutation relations

$$[\hat{E}_M, \hat{E}_N] = M x \delta_{M+N,0},$$

(2.2)
where the suffix denotes the principal grade and only takes values equal to the exponents of \( g \) modulo \( h \), as far as the Heisenberg subalgebra is concerned. \( x \) denotes the integer level. The generators which ad-diagonalise the Heisenberg subalgebra can be written as \( \hat{F}(\alpha, z) \) where \( \alpha \) is a root of \( g \) and \( z \) is a complex number

\[
\left[ \hat{E}_M, \hat{F}(\alpha, z) \right] = \alpha \cdot q([M])z^M \hat{F}(\alpha, z)
\]

(2.3)

\( q(\nu) \) is the eigenvector of the Coxeter element \( \sigma \) corresponding to the exponent \( \nu \) of \( g \),

\[
\sigma(q(\nu)) = e^{\frac{2\pi i \nu}{h}}q(\nu).
\]

(2.4)

It follows, with an appropriate normalization choice such as

\[
\langle \Lambda_0 | \hat{F}(\alpha, z) | \Lambda_0 \rangle = 1,
\]

(2.5)

that

\[
\hat{F}(\sigma(\alpha), z) = \hat{F}(\alpha, ze^{-\frac{2\pi i}{h}}).
\]

(2.6)

Here it is sensible to choose a standard representative of each of the \( r \) orbits of the \( hr \) roots under action of \( \sigma \). A convenient choice is, \cite{21}, \cite{23},

\[
\gamma_i = c(i)\alpha_i,
\]

(2.7)

where \( \alpha_i \) is a simple root of \( g \) and \( c(i) = \pm 1 \) according as the vertex \( i \) of the Dynkin diagram is coloured black or white. Then, the Laurent coefficient of the \( r \) quantities \( \hat{F}(\gamma_i, z) \) taken with the \( \hat{E}_M \), span \( \hat{g} \). We further define
\[ \hat{F}^i(\eta) = \hat{F} \left( \gamma_i, ie^{\eta e^{-\frac{i(1+c(1))}{2\hbar}}} \right), \]  

(2.8)
since this will create a soliton of species \( i \) with rapidity \( \eta \). In fact, with this choice of \( z \),

\[ [\hat{E}_{\pm 1}, \hat{F}^i(\eta)] = -|\gamma_i \cdot q(1)|e^{\pm \eta \hat{F}^i(\eta)}. \]  

(2.9)
To see this interpretation, we consider the general soliton solution of the affine Toda theory associated with \( g \)

\[ e^{-\beta \lambda_j} \Phi = \frac{<\Lambda_j|e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} |\Lambda_j>}{<\Lambda_0|e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} |\Lambda_0>n_{m_j}} \]  

(2.10)
where \( |\Lambda_j> \), \((j = 0, 1, \ldots, r)\), denote the highest weight states of the \( r \) fundamental representations of \( \hat{g} \). When \( j \neq 0 \) they have \( g \) weights \( \lambda_j \) and levels \( m_j \) where \( \frac{\psi}{\psi^x} = \sum_{j=1}^{r} m_j \frac{\alpha_j}{\alpha'_j} \).

\( |\Lambda_0> \) has \( g \) weight zero and unit level and can be thought of as the vacuum. \( x^\pm \) are the light-cone coordinates \((t \pm x)/\sqrt{2}\) and the coupling constant \( \beta \) is understood to be imaginary.

The Kac-Moody group element \( g(0) \) contains the soliton data and consists of a product of factors \( \exp Q_k \hat{F}^{i_k}(\eta_k) \), one for each soliton \( i_k \). \( Q_k \) is a complex number upon which the coordinate of the \( k \)'th soliton depends.

The time development operators \( \exp(-\mu \hat{E}_{\pm 1} x^\pm) \) can be eliminated using the fact that \( \hat{E}_N \) annihilates \( |\Lambda_j> \) for \( N > 0 \) and the commutation relations (2.2) and (2.3). The result is that \( g(0) \) is replaced by a space-time dependent \( g(t) \) in which each \( \hat{F}^i(\eta) \) is multiplied by the factor

\[ W_i = \exp (\mu |\gamma_i \cdot q(1)|(x^+ e^{\eta} - x^- e^{-\eta})). \]  

(2.11)
The choice \( z = ie^\eta exp - \left\{ \frac{i\pi(1+c(i))}{2\hbar} \right\} \) in (2.8) ensured that \( W_i \) is real and that when \( \eta = 0 \) the time, \( t \), cancels out of \( W_i \) so that the soliton \( i \) is then stationary. More generally, we see that \( \eta \) is its rapidity. The energy and momentum of the general solution of this type has been evaluated and found to be appropriate to this interpretation and independent of the complex numbers \( Q_k \). [15]

Now we shall consider how to formulate an antisoliton. If we replace the rapidity \( \eta \to \eta + i\pi \) in \( W_i \), (2.10), we see that the sign of \( x \) and \( t \) is reversed. This means that the solution runs backwards in space, reversing the boundary condition and describing the antisoliton to species \( i \). But for the particles which are the quantum excitations of the Toda fields there is a precise way of obtaining the antispecies \( \bar{i} \) of \( i \). This is given by the orbit of \( \sigma \) containing \(-\gamma_i\) rather than \( \gamma_i \). Denoting its representative element (2.6), \( \gamma_{\bar{i}} \), we conclude that the antisoliton of rapidity \( \eta \) can be equally well be created by \( \hat{F}^i(\eta + i\pi) \) and \( \hat{F}^{\bar{i}}(\eta) \). These quantities are certainly not equal, but we shall find a surprising, non-linear relation between them that will mean that the antisoliton solutions created by them are indeed the same after the parameters \( Q_k \) are readjusted.

Notice that the prescription that \( i \to \bar{i} \) can be achieved by an analytic continuation \( \eta \to \eta + i\pi \), is a familiar property of the scattering matrices in the quantum theory of the particle excitations of the Toda fields where it is known as “crossing”. Thus it will be possible to consider our identity as a classical version of “crossing” which can be formulated purely in terms of the affine Kac-Moody group.

Notice also that since \( \pm i\mu \hat{E}_{\pm 1} \) are displacement operators along the two light cones, their commutation relations (2.1) with the principal grade, \( d' \), show that, in this soliton
formalism, \(d'\) has the physical interpretation of being the Lorentz boost.

3. The Highest Non-Vanishing Power of \(\hat{F}^i(\eta)\) as a Vertex Operator

The exponential factor \(\exp Q \hat{F}^i(\eta)\) responsible for creating a soliton of species \(i\) and rapidity \(\eta\) can be expanded as a power series which terminates as a consequence of the vanishing of those powers of \(\hat{F}^i(\eta)\) greater than the level \(x\) of the representation considered. (This statement is slightly more complicated if \(g\) is not simply laced). Consequently, each expression (2.10) for the soliton solutions comprises a ratio of polynomials in the \(W_i\), (2.11), of the same order. This has been proven, [25], starting with the known result that, at level \(x = 1\), \(\hat{F}^i(\eta)\) can be represented as a vertex operator involving exponentials of elements of the Heisenberg subalgebra (2.2) [28]. We shall now extend this result by showing that, at level \(x\), the \(x'\)th power of \(\hat{F}^i(\eta)\) is again such a vertex operator. Thus the highest non-vanishing power of \(\hat{F}^i(\eta)\) has a particularly simple structure. This is important as it controls the asymptotic behaviour of the soliton solutions.

We recall that in the fundamental representation with highest weight \(\Lambda_j\) and level 1 (so \(m_j = 1\),

\[
\hat{F}^i(\eta) = e^{-2\pi i \lambda_i \cdot \lambda_j} Y^i Z^i, \tag{3.1a}
\]

where

\[
Y^i = \exp \sum_{N > 0} \gamma_i \cdot q([N]) z^N \hat{E}_{-N}, \quad Z^i = \exp \sum_{N > 0} -\gamma_i \cdot q([N])^* z^{-N} \hat{E}_N, \tag{3.1b}
\]

and we have used the result of section 8 of [25] that
\[ < \Lambda_j \vert \hat{F}^i(\eta) \vert \Lambda_j > = e^{-2\pi i \lambda_i \cdot \lambda_j} < \Lambda_0 \vert \hat{F}^i(\eta) \vert \Lambda_0 > \]

and the normalization choice (2.5). The phase factor occurring here and consequently in (3.1a) has its origin in the isomorphism between the centre of the universal covering group of \( g \) and \( W_0 \), the subgroup of the Weyl group which relates the vacuum to the level one representations of \( \hat{g} \).

Now let us consider another fundamental representation with weight \( \Lambda_k \), now of level \( m_k \) greater than 1. The corresponding highest weight state \( |\Lambda_k > \) can be represented as a state occurring in the \( m_k \)-fold tensor product with weights \( \Lambda_j(1), ..., \Lambda_j(m_k) \) which must satisfy

\[ e^{-2\pi i \lambda_i \cdot \lambda_k} = e^{-2\pi i \lambda_i \cdot \sum_{p=1}^{m_k} \lambda_j(p)}. \] (3.2)

An example at level 2 was furnished by (6.8) of [25] but the detailed construction of the state will, fortunately, be immaterial for the results that follow. We label the spaces 1, 2, ..., \( m_k \), respectively. So

\[ \hat{F}^i(\eta) = \hat{F}^i(\eta)(1) + \hat{F}^i(\eta)(2) + \cdots + \hat{F}^i(\eta)(m_k), \] (3.3)

and similarly for the Heisenberg subalgebra,

\[ \hat{E}_M = \hat{E}_{M(1)} + \hat{E}_{M(2)} + \cdots + \hat{E}_{M(m_k)}. \] (3.4)

In this notation the bracketed subscript denotes which of the spaces it is upon which the operator acts non-trivially. Thus the terms in (3.3) mutually commute and each have
vanishing square as each is given by the vertex operator construction (3.1). Hence

\[
\frac{(\hat{F}^i(\eta))^{m_k}}{m_k!} = \hat{F}^i(\eta)(1)\hat{F}^i(\eta)(2)\cdots\hat{F}^i(\eta)(m_k)
\]

\[
= e^{-2\pi i\lambda_i \cdot \lambda_k Y^i_{(1)} \cdots Y^i_{(m_k)}} Z^i_{(1)} \cdots Z^i_{(m_k)}
\]

using (3.2). Finally, using (3.1b) and (3.4),

\[
\frac{(\hat{F}^i(\eta))^{m_k}}{m_k!} = e^{-2\pi i\lambda_i \cdot \lambda_k Y^i Z^i}.
\]

(3.6)

This is the new vertex operator construction announced earlier. Of course (3.1a) is now seen to be the special case of (3.6) when \(m_k = 1\). More generally, given an irreducible representation with highest weight \(\Lambda\), at level \(x\),

\[
\frac{(\hat{F}^i(\eta))^x}{x!} = e^{-2\pi i\lambda_i \cdot \lambda_k Y^i Z^i}.
\]

(3.7)

In the numerator of the single soliton solution we see that the coefficient of the highest power of \(QW_i\) is given by the expectation value of (3.6) with respect to the weight state \(|\Lambda_k\rangle\). Since the state is annihilated by all the step operators for positive roots of \(\hat{g}\), that is, by all the generators of positive principal grade, it is certainly annihilated by those elements of the Heisenberg subalgebra (3.4) with positive suffix. Hence \(Y^i\) yields unity, and

\[
< \Lambda_k | (\hat{F}^i(\eta))^{m_k} | \Lambda_k > = e^{-2\pi i\lambda_i \cdot \lambda_k},
\]

(3.8)

a surprisingly simple result, holding for all \(k = 0,1,2,\ldots r\) and \(i = 1,2,\ldots r\). Noting that (3.8) is independent of \(z\) (or \(\eta\)). This is a special case of a more general result: the
expectation value of any power of $\hat{F}^i(\eta)$ is independent of $\eta$. As we noted, the operator $d'$, (2.1), plays the role of the Lorentz boost for solitons. More explicitly, from the definitions,

$$[d', \hat{F}^i(\eta)] = \frac{d}{d\eta} \hat{F}^i(\eta).$$

Hence

$$[d', (\hat{F}^i(\eta))^a] = \frac{d}{d\eta} (\hat{F}^i(\eta))^a.$$\[10\]

The expectation value of this vanishes as $d'$ has equal, real eigenvalues to each side. Thus $< \Lambda_k| (\hat{F}^i(\eta))^a | \Lambda_k >$ is independent of the rapidity $\eta$ and so is Lorentz invariant. Using (3.8), the solution for the single soliton created by $g(0) = exp Q \hat{F}^i(\eta)$ in (2.10) has the form

$$e^{-\beta \lambda_k \cdot \Phi} = 1 + \cdots + e^{-2\pi i \lambda_i \cdot \lambda_k (QW_i)^{m_k}} \quad (3.9)$$

where the dots indicates powers of $QW_i$ intermediate between 0 and $m_k$. These intermediate powers, whose coefficients we have not calculated, do not affect the asymptotic limit $x \to \pm \infty$, which are equivalent to $W_i \to \infty$ or $W_i \to 0$. Thus

$$e^{-\beta \lambda_k \cdot \Phi} = \left\{ \begin{array}{ll}
e^{-2\pi i \lambda_i \cdot \lambda_k} & x \to \infty \\
1 & x \to -\infty \end{array} \right.$$\[10\]

In particular, this shows that, asymptotically, $\Phi$ does approach one of the “degenerate vacua”, so satisfying the boundary conditions expected of a soliton solution.

This result can readily be extended to solutions with any number of solitons. The operator product of two vertex operators (3.6) is
\[
\frac{(\hat{F}^i(\eta))^{m_k}}{m_k!} \frac{(\hat{F}^i(\eta'))^{m_k}}{m_k!} = [X_{i,i'}(z, z')]^{m_k} : \frac{(\hat{F}^i(\eta))^{m_k}}{m_k!} \frac{(\hat{F}^i(\eta'))^{m_k}}{m_k!} : \]

where \(|z| > |z'|\) i.e. \(\eta' > \eta\), where the notation of section 5 of [25] for the complex numbers \(X_{ij}\) is used.

Hence the two soliton solution created by \(\hat{F}^i(\eta)\) and \(\hat{F}^i(\eta')\) takes the form

\[
e^{-\beta \lambda_k \cdot \Phi} = 1 + \cdots + e^{-2\pi i (\lambda_i + \lambda_{i'} - \lambda_k)(QW_i Q'W_i')^{m_k} X_{i,i'}(z, z')}^{m_k}
\]

with asymptotic limits \(\exp[-2\pi i (\lambda_i + \lambda_{i'}) \cdot \lambda_k]\) and 1. Again this confirms that the solution interpolates degenerate vacua. This argument is readily extended to more solitons using the work of section 5 of [25].

4. The Crossing Relations for Solitons

This identity will arise naturally in extending the result (3.7) to lower powers of \(\hat{F}^i(\eta)\). Consider a representation of level \(x\) with highest weight \(\Lambda\), constructed again as occurring in the decomposition of tensor product of \(x\) fundamental representations of level 1. Consider a power \(a\) of \(\hat{F}^i(\eta)\) less then \(x\):

\[
\frac{(\hat{F}^i(\eta))^{a}}{a!} = \hat{F}^i(\eta)_{(1)} \hat{F}^i(\eta)_{(2)} \cdots \hat{F}^i(\eta)_{(a)} + \cdots ,
\]

where the dots indicates extra terms of a similar structure obtained by assigning the “\(a\)” quantities \(\hat{F}^i(\eta)\) separately to the \(x\) different species. There are consequently \(\binom{x}{a}\) terms on the right hand of (4.1) in all. Similarly, if we consider the power \(x - a\):

\[
\frac{(\hat{F}^i(\eta))^{x-a}}{(x-a)!} = \hat{F}^i(\eta)_{(a+1)} \hat{F}^i(\eta)_{(a+2)} \cdots \hat{F}^i(\eta)_{(x)} + \cdots ,
\]
The first terms on the right hand side of (4.1) and (4.2) multiplied give \((\hat{F}^i(\eta))^x/x!\) are thus naturally complementary. There is a similar pairing between any term on the right hand side of equation (4.1) and one on the right hand side of (4.2). Thus the respective number of terms \(\binom{x}{a}\) and \(\binom{x}{x-a}\) must be equal as indeed they are. We shall now relate the first two terms exhibited above so that a similar relation will follow for all the \(\binom{x}{a}\) complementary pairs

\[
\hat{F}^i (\eta)_{(1)} \hat{F}^i (\eta)_{(2)} \cdots \hat{F}^i (\eta)_{(a)} = e^{-2\pi i \lambda_i \cdot \sum_{p=1}^{a} \lambda_j(p)} Y^i_1(\eta) \cdots Y^i_a(\eta) Z^i_1(\eta) \cdots Z^i_a(\eta)
\]

\[
= e^{-2\pi i \lambda_i \cdot \sum_{p=a+1}^{x} \lambda_j(p)} Y^i_{a+1}(\eta)^{-1} \cdots Y^i_x(\eta)^{-1} Z^i_{a+1}(\eta)^{-1} \cdots Z^i_x(\eta)^{-1} Z^i(\eta).
\]

(4.3)

Notice that the factors outside the curly bracket are independent of the subset of \(x\) chosen since \(Y^i(\eta)\) and \(Z^i(\eta)\) are expressed by exponentiating (3.4). We shall relate the terms inside the curly brackets to the first term on the right hand side of (4.2) for a suitable new choice of species and rapidity.

Consider a typical factor in the curly brackets in (4.3) associated with one of the species whose tensor product is taken, with label \(p\), say, namely

\[
e^{2\pi i \lambda_i \cdot \lambda_j(p)} (Y^i(\eta)_{(p)})^{-1} (Z^i(\eta)_{(p)})^{-1}.
\]

We shall temporarily drop the label \(p\) and verify the following remarkable identity for this factor:

\[
e^{2\pi i \lambda_i \cdot \lambda_j} (Y^i(\eta))^{-1} (Z^i(\eta))^{-1} = \hat{F}^i(\eta + i\pi).
\]

(4.4)

Thus, the contents of the curly brackets in (4.3) can be written

\[
\hat{F}^i_{(a+1)}(\eta + i\pi) \hat{F}^i_{(a+2)}(\eta + i\pi) \cdots \hat{F}^i_{(x)}(\eta + i\pi).
\]

(4.5)
To prove (4.4) note that, by (3.1),

\[
(Y^i(\eta))^{-1} = \exp - \sum_{N>0} \frac{\gamma_i \cdot q([N]) z^N \hat{E}_{-N}}{N},
\]

(4.6a)

where, as in (2.8),

\[
z = ie^{\eta}e^{-\frac{i\pi(1+c(i))}{2h}}.
\]

(4.6b)

Now, $-\gamma_i$ lies on the orbit of $\sigma$ whose representative element is defined as $\gamma_{\bar{i}}$. As observed in the study of the quantum particle excitations, $\bar{i}$ is interpreted as the antispecies of $i$ [23]. Precisely [29]

\[
-\gamma_i = \sigma \frac{b}{2} - \frac{c(i)-c(\bar{i})}{4} \gamma_{\bar{i}}.
\]

(4.7)

Recalling (2.4) we see that

\[
Y^i(\eta)^{-1} = \exp \sum_{n>0} \gamma_{\bar{i}} \cdot q([N]) \left[ze^{-\frac{2\pi i}{\hbar} \left(\frac{b}{2} - \frac{c(i)-c(\bar{i})}{4}\right)}\right]^N \hat{E}_{-N}.
\]

Now, by (4.6b), the argument of the square brackets is equal to

\[
-ie^{\eta}e^{-\frac{i\pi(1+c(i))}{2h}}e^{-\pi i}e^{i\pi \frac{c(i)-c(\bar{i})}{2h}} = ie^{\eta + i\pi}e^{-\frac{i\pi(1+c(\bar{i}))}{2h}} = -\bar{z}.
\]

where $\bar{z}$ differs from $z$ defined by (4.6b) only in the species $\bar{i}$ replacing its antispecies $i$. Hence, $Y^i(\eta)^{-1} = Y^{\bar{i}}(\eta + i\pi)$ and similarly for $Z^i(\eta)^{-1}$. Finally, in order to prove (4.4) we need to show

\[
e^{2\pi i \lambda_{\bar{i}} \lambda_j} = e^{-2\pi i \lambda_{\bar{i}} \lambda_j}
\]

(4.8)

and this relies on the fact that the map $i \rightarrow \bar{i}$ is a symmetry $\tau$, say, of the ordinary Dynkin diagram of $g$. Hence this can be lifted to a linear automorphism of the root system, also called $\tau$, which is either outer or equal to the identity, such that $\alpha_{\bar{i}} = \tau(\alpha_i)$. Comparing
with (4.7), it follows that \( \tau = -\sigma_0 \), where \( \sigma_0 \) can be calculated but is known as that special element of the Weyl group of \( \mathbf{g} \) which maps the positive Weyl chamber into its negative. As a consequence,

\[
\lambda_i = \tau(\lambda_i) = -\sigma_0(\lambda_i).
\]

(4.8) follows from this as \( \sigma_0(\lambda_i) \) differ from \( \lambda_i \) by an element of the root lattice. Thus, (4.4) is proven and, using (4.3) and (4.5), we conclude

\[
\frac{(\hat{F}^i(\eta))^a}{a!} = e^{-2\pi i \lambda_i \cdot \lambda} Y^i(\eta) \frac{(\hat{F}^{\bar{i}}(\eta + i\pi))^{x-a}}{(x-a)!} Z^i(\eta).
\]  

(4.9)

If we take expectation values of (4.9) with respect to the highest weight state \( |\Lambda> \), the dependence on the rapidity drops out as we explained, leaving

\[
\text{\frac{<\Lambda|(F^i)^a|\Lambda>}{a!}} = e^{-2\pi i \lambda_i \cdot \lambda} \text{\frac{<\Lambda|(F^{\bar{i}})^{x-a}|\Lambda>}{(x-a)!}}.
\]  

(4.10)

This relates coefficients in the single soliton solution. As it relates high powers of \( \hat{F}^i(\eta) \) to low powers, it simplifies calculations of these solutions.

Since \( Y^i(\eta) \) and \( Z^i(\eta) \) appearing in (4.9) are independent of the integer \( a \), we can deduce the following relation between group elements:

\[
e^{Q\hat{F}^i(\eta)} = e^{-2\pi \lambda_i \cdot \lambda} Q^x Y^i(\eta) e^{\frac{i}{2} Q^{\bar{i}}(\eta + i\pi)} Z^i(\eta).
\]  

(4.11)

This is our main result, being the surprising, nonlinear relation between \( \hat{F}^i(\eta) \) and \( \hat{F}^{\bar{i}}(\eta + i\pi) \). We shall argue that it expresses the “crossing property” of a soliton solution into an antisoliton. This is clearer if we rewrite it as

\[
e^{Q\hat{F}^{\bar{i}}(\eta)} = e^{-2\pi \lambda_i \cdot \lambda} Q^x Y^{\bar{i}}(\eta) e^{\frac{1}{2} Q^{\bar{i}}(\eta + i\pi)} Z^{\bar{i}}(\eta).
\]  

(4.12)
This relates the analytic continuation in rapidity, \( \eta \rightarrow \eta + i\pi \), of the group element 
\( e^{rac{1}{\hbar} \hat{F}^{i}(\eta)} \) creating a soliton of species \( i \), to the group element \( e^{Q\hat{F}^{\bar{i}}(\eta)} \) creating an antisoliton of species \( \bar{i} \). The remaining factors \( Y^{\bar{i}}(\eta), Z^{\bar{i}}(\eta) \) can be thought of as “crossing matrices”. The reason for saying this is that in a local quantum field theory, the S-matrix describing the scattering of particles obeys a “crossing property”, more precisely “the substitution rule for crossed processes” [24]. In two dimensions, the S-matrix for the elastic scattering of an antiparticle on a target is obtained by the analytic continuation \( \eta \rightarrow \eta + i\pi \) of the S-matrix for the elastic scattering of the particle on the same target. If the particle has internal degrees of freedom, then an additional crossing matrix multiplies the S-matrix.

The equation (4.11) is obviously a very similar relation. One difference is that the crossing property just described, holds in the quantum theory as the S-matrix elements are probability amplitudes. The solitons are the only sort of particles possible in a classical theory such as we consider. Thus, (4.12) is actually the classical analogue of the crossing property and is remarkable because it is formulated purely in terms of the underlying Kac-Moody group. What is attractive about this is that, hitherto, the crossing property of the affine Toda S-matrix has been incorporated in a rather ad-hoc way. Relation (4.12) suggests the possibility of a more systematic understanding.

We now justify our interpretation by using the crossing identity (4.12) inside the expression for the multisoliton solution in order to see explicitly how analytic continuation through \( i\pi \) in the rapidity of one soliton of species \( i \) yields the multisoliton solution in which the soliton of species \( i \) is replaced by its antisoliton with species \( \bar{i} \). The rapidities are unchanged but the complex numbers \( Q_{1}, Q_{2}, \ldots \) associated with each soliton undergo
a transformation which will be determined.

Consider the \( n \) soliton solution created by the group element

\[
g(0) = e^{Q_1 \hat{F}_{i1}(\eta_1)} \cdots e^{Q_{k-1} \hat{F}_{i(k-1)}(\eta_{k-1})} e^{Q_k \hat{F}_{i}(\eta)} e^{Q_{k+1} \hat{F}_{i(k+1)}(\eta_{k+1})} \cdots e^{Q_n \hat{F}_{in}(\eta_n)}.\]

The \( k \)th factor creates the antisoliton of species \( i \) to be crossed into an soliton of species \( i \) and is labelled in the simple way indicated. Inserting the crossing identity (4.12)

\[
g(0) = e^{\frac{-2\pi i \lambda - \lambda Q}{\hat{E}_1} \hat{Y}(\eta) \tilde{g}(0) \hat{Z}(\eta)}
\]

where

\[
\tilde{g}(0) = e^{\hat{Q}_1 \hat{F}_{i1}(\eta_1)} \cdots e^{\hat{Q}_{k-1} \hat{F}_{i(k-1)}(\eta_{k-1})} e^{\hat{Q}_k \hat{F}_{i}(\eta+i\pi)} e^{\hat{Q}_{k+1} \hat{F}_{i(k+1)}(\eta_{k+1})} \cdots e^{\hat{Q}_n \hat{F}_{in}(\eta_n)}.
\]

is similar to \( g(0) \) but with the \( k' \)th factor now of the form \( e^{\hat{Q}_k \hat{F}_{i}(\eta+i\pi)} \), being the analytic continuation of the factor for the creation of a soliton of species \( i \). The quantities \( Q \) have all changed:

\[
\begin{cases}
\hat{Q} = 1/Q \\
\hat{Q}_r = Q_r X_{i,ij}, \quad r \neq k.
\end{cases}
\]

(4.13)

We have used the normal ordering relations

\[
\hat{Z}(z) \hat{F}(z_j) = \hat{F}(z_j) \hat{Z}(z) X_{ij}(z, z_j)
\]

\[
\hat{F}(z_j) \hat{Y}(z) = \hat{Y}(z) \hat{F}(z_j) X_{ij}(z, z_j)
\]

where \( X_{ij}(z, \zeta) \) is defined in section 5 of [25]. Finally we use

\[
e^{-\mu \hat{E}_1 x^+ \hat{Y}(\eta) \tilde{g}(0) \hat{Z}(\eta)} e^{-\mu \hat{E}_{-1} x^-} = (W_i(\eta))^{-x} \hat{Y}(\eta) e^{-\mu \hat{E}_1 x^+ \tilde{g}(0)} e^{-\mu \hat{E}_{-1} x^-} \hat{Z}(\eta)
\]

which follows from (2.2) and the definitions. Putting all this together

\[
< \Lambda_k | e^{-\mu \hat{E}_1 x^+} g(0) e^{-\mu \hat{E}_{-1} x^-} | \Lambda_k >
\]

\[
e^{-2\pi i \lambda - \lambda Q} \left( \frac{Q}{W_i} \right)^{m_k} < \Lambda_k | e^{-\mu \hat{E}_1 x^+ \tilde{g}(0)} e^{-\mu \hat{E}_{-1} x^-} | \Lambda_k >.
\]

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When we insert this into (2.10), the factors $(Q/W)^z$ cancel leaving the solution created by $g(0)$ equalling that created by $\tilde{g}(0)$, apart from a factor $e^{-2\pi i \lambda^\tau \lambda_k}$ which merely indicates that the vacua are interpolated in reverse order. This then demonstrates that a soliton solution crosses into another solution with one soliton replaced by the antispecies under the analytic continuation $\eta \to \eta + i\pi$. This procedure is accompanied by the transformation (4.13) on the $Q$'s. Since they describe not only the coordinate of the soliton, but also its internal degrees of freedom, this is to be viewed as the effect of the crossing matrix. It is intriguing that for the soliton which is crossed, $Q \to 1/Q$ which reminiscent of a modular transformation.

Because relativistically invariant soliton theories play a preferred rôle within their integrable hierarchies according to the ideas of Zamolodchikov [9], and because the concept of antiparticles and crossing are special to relativistic theories, one might anticipate interesting structure to emerge. What we have shown is that in affine Toda theories the crossing relation is embodied in the remarkable identity (4.12) which is purely a statement about the underlying affine Kac-Moody algebra and a highest weight representation of arbitrary level. As we already know that the “fusing” rules for trilinear couplings are embodied in the operator product expansion of the $\hat{F}^z(\eta)$, [25], it would appear that, for affine Toda solitons, there is the possibility of a purely group theoretical understanding of the principles of S-matrix theory.

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