The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms

Nariya Kawazumi* and Yusuke Kuno**

Department of Mathematical Sciences, University of Tokyo
3-8-1 Komaba Meguro-ku, Tokyo 153-8914 JAPAN
e-mail: kawazumi@ms.u-tokyo.ac.jp

Department of Mathematics, Tsuda College,
2-1-1, Tsuda-Machi, Kodaira-shi, Tokyo 187-8577 JAPAN
e-mail: kunotti@tsuda.ac.jp

Abstract. We survey a geometric approach to the Johnson homomorphisms using the Goldman-Turaev Lie bialgebra.

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1 Introduction

The purpose of this chapter is to survey a new aspect of topological studies of Riemann surfaces via infinite dimensional Lie algebras. More concretely, we discuss a geometric approach to the Torelli-Johnson-Morita theory using an infinite dimensional Lie algebra called the Goldman Lie algebra, which comes from global structure of surfaces.

The Torelli-Johnson-Morita theory, initiated by Johnson [23, 24] and elaborated later by Morita [57], is a place where infinite dimensional Lie algebras appear in the study of the mapping class groups and the Torelli groups. In this chapter, surfaces are always assumed to be differentiable. Let \( \Sigma = \Sigma_{g,1} \) be a compact oriented surface of genus \( g > 0 \) with one boundary component, and \( \mathcal{M}_{g,1} \) the mapping class group of \( \Sigma \) relative to the boundary. The Torelli group \( \mathcal{I}_{g,1} \) is a normal subgroup of \( \mathcal{M}_{g,1} \) consisting of mapping classes acting trivially on the homology of \( \Sigma \). There are many motivations from various
fields of mathematics for studying this group, but still we are very far from full understanding of it. To study $\mathcal{I}_{g,1}$ we consider a central filtration of $\mathcal{I}_{g,1}$ called the Johnson filtration, which is defined by the action of mapping classes on the lower central series of the fundamental group of the surface. A central object of the theory is an injective, graded Lie algebra homomorphism

$$\tau : \bigoplus_{k=1}^{\infty} \text{gr}^k(\mathcal{I}_{g,1}) \to \bigoplus_{k=1}^{\infty} h_{g,1}^Z(k).$$  

(1.1)

The $k$-th component $\tau_k : \text{gr}^k(\mathcal{I}_{g,1}) \to h_{g,1}^Z(k)$ is called the $k$-th Johnson homomorphism. Here the target is an infinite dimensional Lie algebra called the Lie algebra of symplectic derivations of type “Lie” in the sense of Kontsevich [40]. It is purely algebraically defined and was introduced by Kontsevich [40] and Morita [53] [54] independently. The image of $\tau$ is called the Johnson image. Characterization of it is a hard but very important problem in the study of $\mathcal{I}_{g,1}$. Morally, the problem asks what the Lie algebra of the Torelli group is.

An infinite dimensional Lie algebra related to an oriented surface $S$ also arises in the following way. Let $\hat{\pi}(S)$ be the set of homotopy classes of oriented loops on $S$. Motivated by the study of the symplectic structure of the moduli space of flat bundles over a surface, Goldman [17] introduced a Lie bracket called the Goldman bracket on the free $\mathbb{Z}$-module $\mathbb{Z}\hat{\pi}(S)$ with basis the set $\hat{\pi}(S)$. The definition of the Lie bracket involves the intersections of two loops and this Lie algebra is called the Goldman Lie algebra. Later Turaev [79] found a Lie cobracket called the Turaev cobracket on the quotient Lie algebra $\mathbb{Z}\hat{\pi}'(S) = \mathbb{Z}\hat{\pi}(S)/\mathbb{Z}1$, where 1 is the class of a constant loop, and showed that $\mathbb{Z}\hat{\pi}'(S)$ has a structure of a Lie bialgebra. The definition of the Lie cobracket involves the self-intersections of a loop. This Lie bialgebra is called the Goldman-Turaev Lie bialgebra and will be our central object of consideration. In this chapter we consider over the rationals $\mathbb{Q}$ and work with $\mathbb{Q}\hat{\pi}(S) = \mathbb{Z}\hat{\pi}(S) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Our primary goal is to show that the Goldman Lie algebra appears naturally in the Torelli-Johnson-Morita theory. This relation is first found in [31] and further developed in [33] [34]. Here we explain the main idea briefly. To start with, the mapping class group $\mathcal{M}_{g,1}$ acts on the fundamental group of $\Sigma$ by automorphisms, where we take a base point on the boundary of $\Sigma$. This is a point of view illustrated in the Dehn-Nielsen theorem. A basic observation is that the Goldman Lie algebra $\mathbb{Q}\hat{\pi}(\Sigma)$ also acts on the fundamental group, but this time the action is by derivations. As for the relationship between Lie algebras and Lie groups, derivations and automorphisms are related by the exponential map. Here we come to a technical but inevitable point; since we work with the exponential map, we have to consider completions of objects and care about convergence. In any case we can construct suitable completions of the Goldman Lie algebra and the fundamental group, and we have
the exponential map from a subset of the completion of $\hat{\mathbb{Q}}\pi(\Sigma)$ to the automorphism group of the completion of the fundamental group. After that we introduce a Lie subalgebra $L^+(\Sigma)$ of the completion of $\hat{\mathbb{Q}}\pi(\Sigma)$, and show that the automorphisms of the completion of the fundamental group of $\Sigma$ induced by elements of $I_{g,1}$ are in the image of the derivations coming from $L^+(\Sigma)$ by the exponential map. Taking the logarithm, we obtain an injective group homomorphism

$$\tau: I_{g,1} \rightarrow L^+(\Sigma), \quad (1.2)$$

where the group structure of the target is described by the Hausdorff series. We call (1.2) the geometric Johnson homomorphism, since taking the graded quotients of it we can recover (1.1). Actually, $\tau$ is essentially the same as Massuyeau’s total Johnson map [47]. However, our construction is free from any choice and is more intrinsic. A practical advantage of our construction is that it can be applied to other compact surfaces with more than one boundary component.

The Turaev cobracket induces a map $\delta$ from the target of the geometric Johnson homomorphism $\tau$. By the fact that any diffeomorphism of a surface preserves the self-intersections of loops on the surface, we show that $\delta \circ \tau = 0$. This gives a non-trivial geometric constraint on the Johnson image. Indeed, we show that all the Morita traces [57], which are obstructions of the surjectivity of (1.1) found first, can be derived from $\delta$. We hope that some of other known constraints on the Johnson image also can admit interpretations from our geometric context.

This survey is organized as follows. In §2, we give an overview of the construction of the Johnson homomorphisms and known results about the Johnson image. We also discuss how to extend the Johnson homomorphisms to the Torelli group or to the whole mapping class group. The main body of this chapter is from §3 to §7. We always consider the mapping class groups relative to the boundary; all the diffeomorphisms and the isotopies that we consider are required to fix the boundary pointwise. Therefore, when the surface $S$ has more than one boundary component, it is natural to consider that the mapping class group acts on the fundamental groupoid of the surface with base points chosen from each boundary component, instead of the fundamental group. In §3 we provide some languages to deal with such a situation. In §4 we give the definition of the Goldman-Turaev Lie bialgebra, and explain how it interacts with the homotopy set of based paths on the surface. In particular, we show that $\hat{\mathbb{Q}}\pi(S)$ acts on the fundamental groupoid of $S$ by derivations. We also discuss other operations to curves on surfaces. In §5 we investigate Dehn twists from our point of view in detail. We show that the action of a Dehn twist on the completion of the fundamental groupoid has the canonical logarithm, and specify it as an element of the completion of the Goldman Lie algebra. This was first observed in [31], and leads us to introducing a “generalized...
Dehn twist”, which is an automorphism of the completion of the fundamental groupoid associated to a loop on the surface which is not necessarily simple. Generalized Dehn twists are first introduced in [41] and are further studied in [33] [34]. Massuyeau and Turaev [49] also study them from a slight different point of view. In §5.2 and §5.3 we present basic properties of generalized Dehn twists. In §5.2 and §5.3 we define the geometric Johnson homomorphism. We first treat the case \( S = \Sigma \) in §6, then the general case in §7. In the general case, we obtain an injective group homomorphism

\[
\tau : \mathcal{T}^L(S) \to L^+(S).
\] (1.3)

Here \( \mathcal{T}^L(S) \) is the “largest” Torelli group in the sense of Putman [68], and \( L^+(S) \) is a Lie subalgebra of the completion of \( \hat{\mathbb{Q}} \hat{\pi}(S) \). Note that when \( S \) has more than one boundary component, there are natural choices for the Torelli group, see [65]. Recently Church [7] constructed the first Johnson homomorphism for all kinds of Putman’s Torelli groups. We do not know any relation between Church’s construction and ours. In §6 and §7 we also give an algebraic description of the Goldman bracket. When \( S = \Sigma \), this means that the completion of \( \hat{\mathbb{Q}} \hat{\pi}(S) \) is isomorphic to (the degree completion of an enhancement of) the Lie algebra of symplectic derivations of type “associative” in the sense of Kontsevich [40]. Note that the tensorial description of the Goldman bracket is also obtained by Massuyeau and Turaev [49] [50] by a different approach. In the case \( S = \Sigma \), we also mention a partial result about a tensorial description of the Turaev cobracket based on a result of [49]. In §8 we discuss other related topics.

Finally, we make a remark that is less relevant to the main part of the text but is still worth mentioning. As for an infinite dimensional Lie algebra coming from local structures of surfaces, there is an observation due to Kontsevich [39] and Beilinson, Manin and Schechtman [3]. They discovered that the Lie algebra of germs of meromorphic vector fields at the origin of \( \mathbb{C} \), i.e., a complex analytic version of the Lie algebra \( \text{Vect}(S^1) \), acts on the moduli space of compact Riemann surfaces with local coordinates in an infinitesimally transitive way. It enables us to regard the Lie algebra as the Lie algebra of the stable mapping class group. This idea has been well-understood for the last few decades. For example, from this fact, we can derive some topological information on the stable cohomology of the mapping class group of a surface. For details, see [1], [26], [27] and [28]. Compared with this idea, our approach, which will be presented in this chapter, suggests us a quite new interaction between the mapping class group and an infinite dimensional Lie algebra coming from the global nature of surfaces.
2 Classical Torelli-Johnson-Morita theory

We describe the Torelli-Johnson-Morita theory for once bordered surface and its recent developments. This theory, initiated by Johnson [23] [24] and elaborated later by Morita [57], studies a certain filtration of the mapping class group and a graded Lie algebra associated to it. The treatment here is brief and limited. In particular, we confine ourselves to compact surfaces with one boundary component. For more details and other aspects, we refer to the chapters of Habiro and Massuyeau [18], Morita [61], Sakasai [70] and Satoh [73].

2.1 Lower central series and the higher Torelli groups

Let $\Sigma = \Sigma_{g,1}$ be a compact connected oriented surface of genus $g > 0$ with one boundary component, and $\mathcal{M}_{g,1}$ the mapping class group of $\Sigma$ relative to the boundary, i.e., the group of diffeomorphisms of $\Sigma$ fixing the boundary $\partial \Sigma$ pointwise, modulo isotopies fixing $\partial \Sigma$ pointwise. Taking a base point $*$ on $\partial \Sigma$, we denote $\pi = \pi_1(\Sigma, *)$. The group $\mathcal{M}_{g,1}$ acts naturally on $\pi$. Let $\Gamma_k = \Gamma_k(\pi)$, $k \geq 1$, be the lower central series of $\pi$, i.e., a series of normal subgroups of $\pi$ successively defined by $\Gamma_1 = \pi$ and $\Gamma_k = [\Gamma_{k-1}, \pi]$ for $k \geq 2$. The intersection $\bigcap_{k=1}^{\infty} \Gamma_k$ is trivial since $\pi$ is a free group. Since $\Gamma_k$ is characteristic, $\mathcal{M}_{g,1}$ acts naturally on the $k$-th nilpotent quotient $N_k = N_k(\pi) = \pi/\Gamma_{k+1}$.

For $k \geq 1$, the $k$-th Torelli group is defined as $\mathcal{M}_{g,1}(k) = \{ \varphi \in \mathcal{M}_{g,1} | \varphi \text{ acts trivially on } N_k \}$. Then we obtain a decreasing filtration $\{ \mathcal{M}_{g,1}(k) \}_{k=1}^{\infty}$ of normal subgroups of $\mathcal{M}_{g,1}$ called the Johnson filtration. The first term $\mathcal{M}_{g,1}(1)$ is nothing but the Torelli group $T_{g,1}$ since $N_1 = \pi/\pi$ is canonically isomorphic to the first homology group $H_1 = H_1(\Sigma; \mathbb{Z})$. The second term $\mathcal{M}_{g,1}(2)$ is known as the Johnson kernel $K_{g,1}$, which is by definition the kernel of the first Johnson homomorphism $\tau_1$ (see §2.2). Due to a deep result by Johnson [25], $K_{g,1}$ is equal to the group generated by Dehn twists along separating simple closed curves on $\Sigma$.

It is known that the filtration $\{ \mathcal{M}_{g,1}(k) \}_{k=1}^{\infty}$ is central, i.e.,

$$[\mathcal{M}_{g,1}(k), \mathcal{M}_{g,1}(\ell)] \subset \mathcal{M}_{g,1}(k + \ell) \quad \text{for } k, \ell \geq 1$$

(see [55] Corollary 3.3). Thus commutator product induces a structure of a graded Lie algebra on the graded module $\bigoplus_{k=1}^{\infty} \text{gr}^k(I_{g,1})$, where $\text{gr}^k(I_{g,1}) = \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)$. On the other hand, the intersection $\bigcap_{k=1}^{\infty} \mathcal{M}_{g,1}(k)$ is trivial since $\bigcap_{k=1}^{\infty} \Gamma_k = \{1\}$. We can regard the quotient groups $\mathcal{M}_{g,1}/\mathcal{M}_{g,1}(k)$ and $I_{g,1}/I_{g,1}(k)$ as approximations of the whole group $\mathcal{M}_{g,1}$ and the Torelli group $I_{g,1}$. From this point of view it is important to understand $\text{gr}^k(I_{g,1})$.
for a specific $k$ or the whole graded Lie algebra $\bigoplus_{k=1}^{\infty} \text{gr}^k(I_{g,1})$. The Johnson homomorphisms are key tool to study them.

### 2.2 The Johnson homomorphisms and their images

We briefly recall the definition of the Johnson homomorphisms. Let us fix $k \geq 1$ and consider a $\mathcal{M}_{g,1}$-equivariant exact sequence $0 \to \Gamma_{k+1}/\Gamma_{k+2} \to N_{k+1} \to N_k \to 1$. Since $\pi$ is free, the quotient $\Gamma_k/\Gamma_{k+1}$ is canonically isomorphic to $\mathcal{L}_Z(k)$, the degree $k$-part of the free Lie algebra generated by $N_1 = H_Z$ (see e.g., [46, 75]). Thus the exact sequence becomes a central extension

$$0 \to \mathcal{L}_Z(k + 1) \to N_{k+1} \to N_k \to 1. \tag{2.2}$$

Take $\varphi \in \mathcal{M}_{g,1}(k)$. Since $\varphi$ acts trivially on $N_k$, for any $x \in \pi$ the image of $\varphi(x)x^{-1}$ in $N_{k+1}$ is actually an element of $\mathcal{L}_Z(k + 1)$ in view of $(2.2)$. Then we obtain a mapping $\pi \to \mathcal{L}_Z(k + 1)$, $x \mapsto [\varphi(x)x^{-1}]$. One can show that this mapping is a homomorphism, thus induces a homomorphism $\tau_k(\varphi) : H_Z \to \mathcal{L}_Z(k + 1)$. The mapping $\tau_k : \mathcal{M}_{g,1}(k) \to \text{Hom}(H_Z, \mathcal{L}_Z(k + 1))$, $\varphi \mapsto \tau_k(\varphi)$ is in fact a homomorphism, and is called the $k$-th Johnson homomorphism. It was introduced by Johnson [23, 24]. Note that using the intersection form $\cdot : H_Z \times H_Z \to \mathbb{Z}$ on the surface, we can identify $H_Z$ and its dual $H_Z^* = \text{Hom}(H_Z, \mathbb{Z})$ by $H_Z \to H_Z^*$, $X \mapsto (Y \mapsto (Y \cdot X))$, where $X, Y \in H_Z$. This induces an isomorphism

$$\text{Hom}(H_Z, \mathcal{L}_Z(k + 1)) = H_Z^* \otimes \mathcal{L}_Z(k + 1) \cong H_Z \otimes \mathcal{L}_Z(k + 1),$$

through which we can also write $\tau_k$ as

$$\tau_k : \mathcal{M}_{g,1}(k) \to H_Z \otimes \mathcal{L}_Z(k + 1). \tag{2.3}$$

One can easily see that the kernel of $\tau_k$ is $\mathcal{M}_{g,1}(k + 1)$, hence $\tau_k$ induces an injective group homomorphism

$$\tau_k : \text{gr}^k(I_{g,1}) \hookrightarrow H_Z \otimes \mathcal{L}_Z(k + 1) \tag{2.4}$$

(using the same letter $\tau_k$). In particular the graded quotient $\text{gr}^k(I_{g,1})$ is isomorphic to $\text{Im}(\tau_k)$.

**Remark 2.1.** Let $Sp(H_Z)$ be the group of $\mathbb{Z}$-linear automorphisms of $H_Z$ preserving the intersection form. Fixing a symplectic basis of $H_Z$, we have an isomorphism $Sp(H_Z) \cong Sp(2g; \mathbb{Z})$. The group $Sp(H_Z)$ acts on both the domain and the target of $(2.3)$. First of all for each $k \geq 1$ the group $\mathcal{M}_{g,1}$ acts on $\mathcal{M}_{g,1}(k)$ by conjugation, hence on $\text{gr}^k(I_{g,1})$. From $(2.1)$ we see that the subgroup $\mathcal{M}_{g,1}(1) = I_{g,1}$ acts trivially on $\text{gr}^k(I_{g,1})$. Since we have an exact sequence $1 \to I_{g,1} \to \mathcal{M}_{g,1} \to Sp(H_Z) \to 1$, the action of $Sp(H_Z)$ on the domain is induced. The action of $Sp(H_Z)$ on the target is naturally induced by the action of $Sp(H_Z)$ on $H_Z$. Then one can see that the map $(2.4)$
is $Sp(H_Z)$-equivariant. This point of view is particularly important when we study $\tau_k \otimes \mathbb{Q}$, since we can apply representation theory of $Sp(2g; \mathbb{Q})$.

Johnson [23] proved $\tau_1(I_{g,1}) = \Lambda^3_H \mathbb{Z} \subseteq H_2 \otimes \mathcal{L}_Z(2)$. Morita [57] found that the target of $\tau_k$ can be smaller and the collection $\{\tau_k\}_{k=1}^{\infty}$ constitutes a graded Lie algebra homomorphism. He introduced a submodule $h^Z_{g,1}(k) \subset H_2 \otimes \mathcal{L}_Z(k + 1)$ defined by

$$
h^Z_{g,1}(k) = \text{Ker}(\{ , \} : H_2 \otimes \mathcal{L}_Z(k + 1) \to \mathcal{L}_Z(k + 2)).$$

When $k = 1$, we have $h^Z_{g,1}(1) = \Lambda^3_H \mathbb{Z}$. Let $\mathcal{L}_Z = \bigoplus_{k=1}^{\infty} \mathcal{L}_Z(k)$ be the free Lie algebra generated by $H_2$. Any element of $h^Z_{g,1}(k)$ can be considered as a symplectic derivation of $\mathcal{L}_Z$ as follows. For $u \in h^Z_{g,1}(k)$, we define a $\mathbb{Z}$-linear map $D_u : H_2 = \mathcal{L}_Z(1) \to \mathcal{L}_Z(k + 1)$ by $D_u(X) = C_{12}(X \otimes u)$, where $C_{12} : H^\otimes k+3 \to H^\otimes k+1$, $X_1 \otimes X_2 \otimes X_3 \otimes \cdots \otimes X_{k+3} \mapsto (X_1 \cdot X_2)X_3 \otimes \cdots \otimes X_{k+3}$ is the contraction of the first and the second factor by the intersection form. Then we can extend $D_u$ uniquely to a derivation $D_u : \mathcal{L}_Z \to \mathcal{L}_Z$ (using the same letter), that is, a $\mathbb{Z}$-linear map satisfying the Leibniz rule $D_u([v, w]) = \{D_u(v), w\} + [v, D_u(w)]$ for any $v, w \in \mathcal{L}_Z$. The derivation $D_u$ is of degree $k$ in the sense that $D_u(\mathcal{L}_Z(\ell)) \subset \mathcal{L}_Z(k + \ell)$ for any $\ell \geq 1$, and is symplectic in the sense that $D_u(\omega) = 0$, where $\omega \in \mathcal{L}_Z(2) = \Lambda^2 H_2 \subset H^2 \mathbb{Z}$ is the tensor called the symplectic form, corresponding to $-1_H \in \text{Hom}(H_2, H_2) = H^2 \mathbb{Z} \otimes H_2 = H_2 \otimes H_2$. Note that if $\{A_i, B_i\}_{i=1}^g \subset H_2$ is a symplectic basis, then $D_u = \sum_{i=1}^g A_i \otimes B_i - B_i \otimes A_i$, cf. [6, 2]. The correspondence $u \mapsto D_u$ is injective. On the other hand any symplectic derivation of $\mathcal{L}_Z$ of degree $k$ can be written as the form $D_u$ for some $u \in h^Z_{g,1}(k)$. Thus we can identify $h^Z_{g,1}(k)$ with the $\mathbb{Z}$-module of symplectic derivations of $\mathcal{L}_Z$ of degree $k$. Then the graded module $\bigoplus_{k=1}^{\infty} h^Z_{g,1}(k)$ is the $\mathbb{Z}$-module of symplectic derivations of $\mathcal{L}_Z$ and naturally has a structure of a graded Lie algebra. We will discuss more details of the Lie algebra of symplectic derivations in [6, 2].

**Theorem 2.2** (Morita [57]).

(1) The image of $\{\tau_k\}_{k=1}^{\infty}$ is contained in $h^Z_{g,1}(k)$.

(2) The maps $\{\tau_k\}_{k=1}^{\infty}$ induce an injective homomorphism of graded Lie algebras

$$
\tau : \bigoplus_{k=1}^{\infty} \text{gr}^k(I_{g,1}) \to \bigoplus_{k=1}^{\infty} h^Z_{g,1}(k).
$$

By the result of Morita, we can write $\tau_k$ as

$$
\tau_k : \mathcal{M}_{g,1}(k) \to h^Z_{g,1}(k).
$$

In this chapter, we understand the $k$-th Johnson homomorphism on the $k$-th Torelli group to be [23, 5].
As posed in Morita [61], characterization of \( \bigoplus_{k=1}^{\infty} \text{gr}^k(I_g,1) \) as a Lie subalgebra of \( \bigoplus_{k=1}^{\infty} b_{g,1}^Z(k) \) is one of big and basic problems in the Torelli-Johnson-Morita theory. Actually, \( \tau_k \) is not surjective in general, which was observed first by Morita [57]. One often considers the problem over \( \mathbb{Q} \) to make use of representation theory of \( Sp(2g;\mathbb{Q}) \), (see Remark 2.1), but still it is very hard. In this chapter we call the subalgebra \( \bigoplus_{k=1}^{\infty} \text{gr}^k(I_g,1) \) tensored by the rationals \( \mathbb{Q} \) the Johnson image. In his monumental paper [20], Hain gave an explicit presentation of the Malcev completion of the Torelli group \( I_g,1 \) when \( g \geq 6 \). In particular, from the presentation together with his other result, Proposition 7.1 in [19], the Johnson image is generated by the first degree component \( \text{gr}^1(I_g,1) \otimes \mathbb{Q} = \Lambda^3 H_Z \otimes \mathbb{Q} \). In other words, the comprehension of the Johnson image is completely determined by Hain. Hence what we want is a complete system of the defining equations of the Johnson image in the Lie algebra \( \bigoplus_{k=1}^{\infty} hZ_g,1(k) \otimes \mathbb{Q} \). Such an equation is called a Johnson cokernel or an obstruction of the surjectivity of the Johnson homomorphism.

First of all, Morita [57] first found an obstruction for the surjectivity of \( \tau_k \). Let \( S^k H_Z \) be the \( k \)-th symmetric power of \( H_Z \), \( C_1: H_Z^{\otimes k+2} \rightarrow H_Z^{\otimes k} \) the contraction of the first and the second factor, and \( s: H_Z^{\otimes k} \rightarrow S^k H_Z \) the natural projection. Let \( \text{Tr}_k: b_{g,1}^Z(k) \rightarrow S^k H_Z \) be a \( \mathbb{Z} \)-linear map defined by

\[
\text{Tr}_k: b_{g,1}^Z(k) \subset H_Z \otimes L_Z(k+1) \subset H_Z^{\otimes k+2} \xrightarrow{C_1} H_Z^{\otimes k} \xrightarrow{s} S^k H_Z.
\]

The map \( \text{Tr}_k \) is called the \( k \)-th Morita trace.

**Theorem 2.3** (Morita [57]).

1. If \( k \geq 2 \), we have \( \text{Tr}_k \circ \tau_k = 0: \text{gr}^k(I_g,1) \rightarrow S^k H_Z \).
2. If \( k \) is odd, then \( \text{Tr}_k \) is non-trivial. In fact, \( \text{Tr}_k \otimes \mathbb{Q} \) is surjective.
3. If \( k \) is even, then \( \text{Tr}_k = 0 \) on the whole \( b_{g,1}^Z(k) \).

In the original definition [57] the map \( \text{Tr} \) is defined as a map \( b_{g,1}^Z(k-1) \rightarrow S^{k-1} H_Z \), while we follow the grading in [59], p.376. In §6.3 we will give a topological interpretation of the Morita traces by the Turaev cobracket [34].

In a natural way the absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) of the rational number field \( \mathbb{Q} \) acts on the arithmetic fundamental group of a pointed algebraic curve defined over the rationals \( \mathbb{Q} \), which is a group extension of the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) by the (geometric) fundamental group of the curve. This induces an image of the Galois group in the Lie algebra \( \bigoplus_{k=1}^{\infty} b_{g,1}^Z(k) \otimes \mathbb{Q} \), which is called the Galois image. The origin of this construction is in Grothendieck, Ihara and Deligne. For its precise description, see [64] and references therein. The relation between the Johnson image and the Galois image has been studied by T. Oda, H. Nakamura, M. Matsumoto and others. For example, H. Nakamura [64] introduced some explicit Johnson cokernels coming from the Galois image. Such Johnson cokernels are called Galois obstructions.
In his study of the IA-automorphism group of a free group, Satoh \cite{Satoh-1999} \cite{Satoh-2000} discovered a refinement of the Morita traces. Let $F_n$ be a free group of rank $n \geq 2$, and $H\mathbb{Z}$ its abelianization as in \cite{Hain-1998}. We denote by $H\mathbb{Z}^*$ its dual $\text{Hom}_{\mathbb{Z}}(H\mathbb{Z}, \mathbb{Z})$, and by $L_2(k)$ the degree $k$ part of the free Lie algebra generated by $H\mathbb{Z}$. The cyclic group of degree $k$ acts on the tensor space $H\mathbb{Z}^k$ by cyclic permutation of the components. Following Satoh, we denote by $C_n(k)$ the coinvariants of the action, i.e.,

$$C_n(k) := H\mathbb{Z}^k/\langle X_1 \otimes X_2 \otimes \cdots \otimes X_k - X_2 \otimes X_3 \otimes \cdots \otimes X_k \otimes X_1; X_i \in H\mathbb{Z} \rangle.$$ 

The Satoh trace $\widehat{\text{Tr}}_k : H\mathbb{Z}^* \otimes L_2(k+1) \to C_n(k)$ is defined to be the composite of the inclusion $H\mathbb{Z}^k \otimes L_2(k+1) \hookrightarrow H\mathbb{Z}^k \otimes H\mathbb{Z}^{(k+1)}$, the contraction map $C_{12} : H\mathbb{Z} \otimes H\mathbb{Z}^{(k+1)} \to H\mathbb{Z}^k$, $f \otimes X_2 \otimes X_3 \otimes \cdots \otimes X_{k+2} \mapsto f(X_2)X_3 \otimes \cdots \otimes X_{k+2}$, $(f \in H\mathbb{Z}^k, X_i \in H\mathbb{Z})$, and the quotient map $H\mathbb{Z}^k \to C_n(k)$. Satoh \cite{Satoh-1999} \cite{Satoh-2000} proved that the images of the lower central series of the IA-automorphism group under the Johnson homomorphisms stably coincide with the kernels of the Satoh traces $\widehat{\text{Tr}}_k$ up to torsion. For details, see his own chapter \cite{Satoh-2002}.

The fundamental group $\pi_1(\Sigma_{g,1}, \ast)$ is free of rank $2g$, so that we can consider the Satoh traces $\widehat{\text{Tr}}_k$ on the Lie algebra $\bigoplus_{k=1}^{\infty} b_{g,1}^*(k)$. Then the contraction map $C_{12}$ is exactly the same as the map $C_{12}$ under the Poincaré duality. From Satoh’s result \cite{Satoh-1999} together with Hain’s result \cite{Hain-1998}, we have $\widehat{\text{Tr}}_k \circ \tau_k = 0$ on $\mathcal{M}_{g,1}(k)$ for any $k \geq 2$. Hence $\widehat{\text{Tr}}_k$ is a refinement of the Morita trace $\text{Tr}_k$.

Enomoto and Satoh \cite{Enomoto-Satoh-2002} carried out some explicit computation of $\widehat{\text{Tr}}_k$‘s on $\bigoplus_{k=1}^{\infty} b_{g,1}^*(k) \otimes \mathbb{Q}$, to prove that they have many non-trivial components of the Johnson cokernels other than the Morita traces. Thus the restriction of $\widehat{\text{Tr}}_k$ to $\bigoplus_{k=1}^{\infty} b_{g,1}^*(k)$ is called the Enomoto-Satoh trace.

### 2.3 Extensions of the Johnson homomorphisms

From Theorem 2.2 (2) by Morita, the totality of the Johnson homomorphisms (tensored by the rationals $\mathbb{Q}$)

$$\tau : \bigoplus_{k=1}^{\infty} \text{gr}^k(\mathcal{I}_{g,1}) \otimes \mathbb{Q} \to \bigoplus_{k=1}^{\infty} b_{g,1}^*(k) \otimes \mathbb{Q}$$

is an injective homomorphism of graded Lie algebras. Hence the Johnson image $\tau(\bigoplus_{k=1}^{\infty} \text{gr}^k(\mathcal{I}_{g,1}) \otimes \mathbb{Q})$ can be regarded as the “Lie algebra” of the Torelli group $\mathcal{I}_{g,1}$. But the map $\tau$ is not defined on the Torelli group itself, but on the graded quotients. So it is desirable to find a lift of $\tau$, or equivalently, an extension of $\tau$ to the Torelli group or to the whole mapping class group $\mathcal{M}_{g,1}$.

As will be stated below, there are various ways to construct extensions of the Johnson homomorphisms. The diversity of constructions comes from that of realizations of the Malcev completion of the free group $\pi = \pi_1(\Sigma_{g,1}, \ast)$. 


The first result on this problem was given by Morita \cite{56,58} through an explicit construction of the automorphism group of the group $N_{k}$, a truncated Malcev completion. Here it should be remarked that the abelianization $\mathcal{M}_{g,1}^{\text{abel}}$ is trivial ($g \geq 3$) or finite ($g = 2$). Hence there exists no non-trivial homomorphism from $\mathcal{M}_{g,1}$ to any rational vector space if $g \geq 2$. In \cite{56} Morita gave an extension as a crossed homomorphism $\tilde{k}: \mathcal{M}_{g,1} \to \text{gr}^{1}(I_{g,1}) \otimes \mathbb{Q} = \Lambda^{3}H_{Z} \otimes \mathbb{Q}$ of the first Johnson homomorphism $\tau_{1}$. More precisely, he proved that there is a unique cohomology class $2\tilde{k} \in H^{1}(\mathcal{M}_{g,1}; \Lambda^{3}H_{Z})$ whose restriction to $I_{g,1}$ is twice the first Johnson homomorphism $2\tau_{1}$. Here $\Lambda^{3}H_{Z}$ is a non-trivial $\mathcal{M}_{g,1}$-module in an obvious way. Let $\rho_{0}: \mathcal{M}_{g,1} \to \text{Sp}(H_{Z})$ be the natural action of $\mathcal{M}_{g,1}$ on the first homology group $H_{Z}$. The crossed homomorphism $\tilde{k}$ defines a group homomorphism
\[
\rho_{1}: \mathcal{M}_{g,1} \to (\frac{1}{2}\Lambda^{3}H_{Z}) \rtimes \text{Sp}(H_{Z}),
\]
which induces a homomorphism of the cohomology groups
\[
\hat{k}^{*}: H^{*}(\frac{1}{2}\Lambda^{3}H_{Z}; \mathbb{Q})^{\text{Sp}(H_{Z})} \to H^{*}(\frac{1}{2}\Lambda^{3}H_{Z} \rtimes \text{Sp}(H_{Z}); \mathbb{Q}) \xrightarrow{\rho_{1}^{*}} H^{*}(\mathcal{M}_{g,1}; \mathbb{Q}).
\]

**Theorem 2.4** (Kawazumi-Morita \cite{37}). *The image Image($\hat{k}^{*}$) equals the subalgebra of $H^{*}(\mathcal{M}_{g,1}; \mathbb{Q})$ generated by the Morita-Mumford classes $e_{i} = (-1)^{i+1}\kappa_{i}$, $i \geq 1$.*

We remark that the theorem holds also for the unstable range. So it is not covered by the Madsen-Weiss theorem \cite{45}. The original proof of Theorem 2.4 is obtained by interpreting the extended first Johnson homomorphism $\tilde{k}$ as the $(0,3)$-twisted Morita-Mumford class $m_{0,3}$ \cite{28}.

As for the second Johnson homomorphism $\tau_{2}$, Morita \cite{58} constructed a group homomorphism
\[
\rho_{2}: \mathcal{M}_{g,1} \to (\frac{1}{24}\mathbb{b}^{2}_{g,1}(2))^\times (\frac{1}{2}\mathbb{b}^{2}_{g,1}(1)) \rtimes \text{Sp}(M_{Z})
\]
extending the homomorphisms $\rho_{1}$ and $\tau_{2}$, where $^\times$ means some central extension of $\frac{1}{24}\mathbb{b}^{2}_{g,1}(1)$ by $\frac{1}{24}\mathbb{b}^{2}_{g,1}(2)$. From the Madsen-Weiss theorem, all the rational cohomology classes coming from $\rho_{2}$ in the stable range are generated by the Morita-Mumford classes.

From Hain’s theorem \cite{20} stated above follows the existence of an extension of the $k$-th Johnson homomorphism to the whole $\mathcal{M}_{g,1}$ for any $k \geq 1$. On the other hand, Kawazumi \cite{29} gave an explicit recipe for constructing extensions of the totality of the Johnson homomorphisms from a generalized Magnus expansion of a free group. For any $k \geq 1$, Kitano \cite{38} described the $k$-th Johnson homomorphism in terms of the standard Magnus expansion of the free group $\pi = \pi_{1}(\Sigma_{g,1}, *)$ associated to a symplectic generating system. Moreover Perron \cite{67} constructed an extension of the $k$-th Johnson
homomorphism $\tau_k$ for any $k \geq 1$ in terms of the standard Magnus expansion. In general, consider a free group $F_n$ of rank $n \geq 2$ with a generating system \{x_1, x_2, \ldots, x_n\}. Let $H$ be the abelianization of $F_n$ tensored by the rationals $\mathbb{Q}$, $H := F_n^{\text{abel}} \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\widehat{T} = \widehat{T}(H)$ the completed tensor algebra generated by $H$, $\widehat{T} = \widehat{T}(H) := \prod_{p=0}^{\infty} H^{\otimes p}$, which has a decreasing filtration $\{\widehat{T}_m\}_{m=1}^{\infty}$ of two-sided ideals defined by $\widehat{T}_m := \prod_{p=m}^{\infty} H^{\otimes p}$. The set $1 + \widehat{T}_1$ is a subgroup of the multiplicative group of the algebra $\widehat{T}$. The standard Magnus expansion of $F_n$ associated to $\{x_1, x_2, \ldots, x_n\}$ is the group homomorphism $\text{std} : F_n \to 1 + \widehat{T}_1$ defined by $\text{std}(x_i) := 1 + [x_i]$, $1 \leq i \leq n$. Here $[\gamma] := (\gamma \mod [F_n, F_n]) \otimes 1 \in H = F_n^{\text{abel}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is the homology class of $\gamma \in F_n$. On the other hand, Bourbaki [25] developed a basic theory of group homomorphisms $F_n \to 1 + \widehat{T}_1$. See also [73]. So we define the notion of a (generalized) Magnus expansion of the free group $F_n$ by the minimum conditions for describing the Johnson homomorphisms.

**Definition 2.5** ([29]). A map $\theta : F_n \to \widehat{T}$ is a ($\mathbb{Q}$-valued) Magnus expansion of the free group $F_n$ if it is a group homomorphism of $F_n$ into $1 + \widehat{T}_1$ and satisfies the condition $\theta([\gamma]) \equiv 1 + [\gamma] \pmod{\widehat{T}_2}$ for any $\gamma \in F_n$.

The standard Magnus expansion $\text{std}$ is a Magnus expansion in this definition. Let $\mathbb{Q}F_n$ be the rational group ring of the group $F_n$, and $\widehat{\mathbb{Q}F_n}$ its completion

$$\widehat{\mathbb{Q}F_n} := \lim_{m \to \infty} \mathbb{Q}F_n/(IF_n)^m.$$ 

Here $IF_n$ is the augmentation ideal, or equivalently, the kernel of the augmentation map $\text{aug} : \mathbb{Q}F_n \to \mathbb{Q}$, $\sum_{\gamma \in F_n} a_{\gamma} \gamma \mapsto \sum_{\gamma \in F_n} a_{\gamma}$. The algebra $\widehat{\mathbb{Q}F_n}$ has a natural decreasing filtration $\{F_m \mathbb{Q}F_n\}_{m=1}^{\infty}$ defined by $F_m \mathbb{Q}F_n := \text{Ker}(\mathbb{Q}F_n \to \mathbb{Q}F_n/(IF_n)^m)$.

Fix an arbitrary Magnus expansion $\theta$ of the free group $F_n$. Then its $\mathbb{Q}$-linear extension $\theta : \mathbb{Q}F_n \to \widehat{T}$, $\sum_{\gamma} a_{\gamma} \gamma \mapsto \sum_{\gamma} a_{\gamma} \theta(\gamma)$, induces an algebra isomorphism

$$\theta : \widehat{\mathbb{Q}F_n} \xrightarrow{\cong} \widehat{T}$$

such that $\theta(F_m \widehat{\mathbb{Q}F_n}) = \widehat{T}_m$ for any $m \geq 1$. See, for example, [29] Theorem 1.3.

For any automorphism $\varphi \in \text{Aut}(F_n)$ of the group $F_n$, we define an automorphism $T^\theta(\varphi)$ of the algebra $\widehat{T}$ by $T^\theta(\varphi) := \theta \circ \varphi \circ \theta^{-1} : \widehat{T} \xrightarrow{\cong} \widehat{\mathbb{Q}F_n} \cong \widehat{\mathbb{Q}F_n} \xrightarrow{\cong} \widehat{T}$, which satisfies $T^\theta(\varphi)(\widehat{T}_m) = \widehat{T}_m$ for any $m \geq 1$. Denote by $\text{Aut}(\widehat{T})$ the group of all automorphisms $U$ of the algebra $\widehat{T}$ satisfying the condition $U(\widehat{T}_m) = \widehat{T}_m$ for any $m \geq 1$. Since the completion map $\mathbb{Q}F_n \to \widehat{\mathbb{Q}F_n}$ is injective, the group
homomorphism

\[ T^\theta : \text{Aut}(F_n) \to \hat{\text{Aut}}(\hat{T}), \quad \varphi \mapsto T^\theta(\varphi), \]

is injective. All the Johnson homomorphisms come from the homomorphism \( T^\theta \). So we call \( T^\theta \) the total Johnson map of the automorphism group \( \text{Aut}(F_n) \).

There are at least two ways to extract an extension of the \( k \)-th Johnson homomorphism \( \tau_k \) from the map \( T^\theta \). One way was prepared for the group cohomology of \( \text{Aut}(F_n) \), and the other is suitable for the Mal’cev completion of the group \( F_n \).

First we explain the original Johnson map introduced in \([29]\). Let \( IA(\hat{T}) \) be the kernel of the natural action of \( \text{Aut}(\hat{T}) \) on the space \( \hat{T}_1/\hat{T}_2 = H \). Then the restriction to the subspace \( H \subset \hat{T} \) induces a linear isomorphism

\[ IA(\hat{T}) \cong \text{Hom}(H, \hat{T}_2) = \prod_{k=1}^\infty \text{Hom}(H, H^{\otimes (k+1)}), \]

by which we identify these linear spaces. For any \( \varphi \in \text{Aut}(F_n) \) the induced map \( |\varphi| \) on \( H = F_n^{\text{abel}} \otimes \mathbb{Q} \) acts on the algebra \( \hat{T} \) in an obvious way, so that we may regard the composite \( T^\theta(\varphi) \circ |\varphi|^{-1} \) as an element of \( IA(\hat{T}) = \prod_{k=1}^\infty \text{Hom}(H, H^{\otimes (k+1)}) \). We define the \( k \)-th Johnson map \( \tau^\theta_k : \text{Aut}(F_n) \to \text{Hom}(H, H^{\otimes (k+1)}) \), \( k \geq 1 \), by

\[ T^\theta(\varphi) \circ |\varphi|^{-1} = (\tau^\theta_k(\varphi))_{k=1}^\infty \in IA(\hat{T}) = \prod_{k=1}^\infty \text{Hom}(H, H^{\otimes (k+1)}). \]

The maps \( \tau^\theta_k \)'s are no longer group homomorphisms. Instead they satisfy an infinite sequence of coboundary equations. For example, we have

\[ -d\tau^\theta_1(\varphi) = 0 \in C^2(\text{Aut}(F_n); \text{Hom}(H, H^{\otimes 2})), \]
\[ -d\tau^\theta_2(\varphi) = (\tau^\theta_1 \otimes 1_H + 1_H \otimes \tau^\theta_1) \cup \tau^\theta_1 \in C^2(\text{Aut}(F_n); \text{Hom}(H, H^{\otimes 3})). \] (2.7)

Here \( C^*(\text{Aut}(F_n); M) \) is the normalized cochain complex of the group \( \text{Aut}(F_n) \) with values in an \( \text{Aut}(F_n) \)-module \( M \), \( d \) the coboundary operator, and \( \cup \) the Alexander-Whitney cup product. From the equation \([64]\) we obtain a straightforward proof of Theorem \([24]\). Let \( IA_n \) be the kernel of the natural action of \( \text{Aut}(F_n) \) on the abelianization \( F_n^{\text{abel}} \), which is called the IA-automorphism group, and an analogue of the Torelli group. Then we have an injective group homomorphism \( T^\theta : IA_n \to IA(\hat{T}) \). In the case \( n = 2g \) and \( F_n = \pi = \pi_1(\Sigma_{g,1}, *) \), the restriction \( \tau^\theta_k \big|_{M_{g,1}(k)} \) equals the (original) \( k \)-th Johnson homomorphism \( \tau_k \). In other words, the graded quotient of the restriction \( T^\theta \big|_{\Sigma_{g,1}} \) equals the totality of the (original) Johnson homomorphisms

\[ \text{gr}(T^\theta \big|_{\Sigma_{g,1}}) = \tau : \bigoplus_{k=1}^\infty \text{gr}^k(\Sigma_{g,1}) \to \bigoplus_{k=1}^\infty \text{Hom}(H, H^{\otimes (k+1)}). \] (2.8)
For details, see [29].

Next we discuss Massuyeau’s total Johnson map [47]

\[ \tau^\theta : IA_n \to \text{Hom}(H, L^+(\hat{T})). \]

We need some generalities on a complete Hopf algebra to explain the definition of the target. The completed group ring \( \hat{\mathbb{Z}}F_n \) and the completed tensor algebra \( \hat{T} = \hat{T}(H) \) are complete Hopf algebras, whose coproducts \( \Delta \) are given by \( \Delta(\gamma) = \gamma \otimes \gamma \in \hat{\mathbb{Z}}F_n \otimes \hat{\mathbb{Z}}F_n \) for \( \gamma \in F_n \), and by \( \Delta(X) = X \otimes 1 + 1 \otimes X \in \hat{T} \otimes \hat{T} \) for \( X \in H \), respectively. We denote by \( \text{Gr}(R) \) the set of all group-like elements in a complete Hopf algebra \( R \), or equivalently \( \text{Gr}(R) := \{ r \in R \setminus \{0\} ; \Delta(r) = r \otimes r \in R \otimes R \} \), which is a subgroup of the multiplicative group of the algebra \( R \). The group \( \text{Gr}(\hat{\mathbb{Z}}F_n) \) is, by definition, the Mal’cev completion of the group \( F_n \). Similarly we denote by \( \mathcal{L}(R) := \{ u \in R ; \Delta(u) = u \otimes 1 + 1 \otimes u \} \) the set of all primitive elements, which is a Lie subalgebra of the associative algebra \( R \).

The Lie algebra \( \mathcal{L}(\hat{T}) \) equals the degree completion of \( \bigoplus_{k=1}^\infty \mathcal{L}_k \otimes \mathbb{Q} \). As is known [60], the exponential \( \exp : \mathcal{L}(R) \to \text{Gr}(R) \), \( \exp(u) := \sum_{k=1}^\infty (1/k!)u^k \), and the logarithm \( \log : \text{Gr}(R) \to \mathcal{L}(R) \), \( \log(r) := \sum_{k=1}^\infty ((-1)^{k-1}/k)(r-1)^k \), are the inverses of each other.

Let \( IA_\Delta(\hat{T}) \) be the stabilizer of the coproduct \( \Delta \) in the group \( IA(\hat{T}) \), and \( L^+(\hat{T}) \) the degree completion of the Lie algebra \( \bigoplus_{k=2}^\infty \mathcal{L}_k \otimes \mathbb{Q} \). Then the Lie algebra consisting of continuous derivations of \( \hat{T} \) which stabilize the coproduct \( \Delta \) and vanish on the quotient \( H = \hat{T}_1/\hat{T}_2 \) is naturally identified with \( \text{Hom}(H, L^+(\hat{T})) \), which is the target of the map \( \tau^\theta \). The exponential \( \exp : \text{Hom}(H, L^+(\hat{T})) \to IA_\Delta(\hat{T}) \), \( \exp(D) := \sum_{k=1}^\infty (1/k!)D^k \), and the logarithm \( \log : IA_\Delta(\hat{T}) \to \text{Hom}(H, L^+(\hat{T})) \), \( \log(U) := \sum_{k=1}^\infty ((-1)^{k-1}/k)(U-1)^k \), are the inverses of each other. Massuyeau [47] introduced the notion of a group-like expansion of the group \( F_n \).

**Definition 2.6** (Massuyeau [47]). A Magnus expansion \( \theta : F_n \to \hat{T} \) of the free group \( F_n \) is group-like if \( \theta(F_n) \subset \text{Gr}(\hat{T}) \), or equivalently \( \Delta(\theta(\gamma)) = \theta(\gamma) \otimes \theta(\gamma) \) for any \( \gamma \in F_n \).

Fix a group-like expansion \( \theta \). Then the isomorphism \( \theta : \hat{\mathbb{Z}}F_n \cong \hat{T} \) preserves the coproduct, so that the Mal’cev completion of \( F_n \) is isomorphic to the group of the group-like elements of \( \hat{T} \) through \( \theta \), and we have \( T^\theta(IA_n) \subset IA_\Delta(\hat{T}) \). Massuyeau introduced the composite

\[ \tau^\theta := \log \circ T^\theta : IA_n \to \text{Hom}(H, L^+(\hat{T})), \quad \varphi \mapsto \sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}(T^\theta(\varphi) - 1)^k|_H. \]

(2.9)

which we call Massuyeau’s total Johnson map. From (2.8) the graded quotient of \( \tau^\theta \) equals the totality of the (original) Johnson homomorphisms (for
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In the case \( n = 2g \) and \( F_n = \pi = \pi_1(\Sigma_{g,1}, *) \), it is desirable that \( \tau^\theta(T_g,1) \subset \mathfrak{t}_g^+ := \prod_{k=1}^\infty \mathfrak{h}^\mathbb{Z}_g(k) \otimes \mathbb{Q} \). \( \mathfrak{t}_g^+ \) is the Lie algebra of \( \pi_1(\Sigma_g, *) \). A symplectic expansion introduced by Massuyeau \[47\] makes it possible as will be stated in \( \S 6 \). Our purpose is to re-construct the map \( \tau^\theta \) in a geometric context with no use of Magnus expansions.

We conclude this subsection by reviewing some other approaches to extending the Johnson homomorphisms or their enlargement to the whole mapping class group or some wider objects. The fatgraph decompositions of the surface \( \Sigma_{g,1} \) define the Ptolemy groupoid of \( \Sigma_{g,1} \), which includes the mapping class group \( \mathcal{M}_{g,1} \). Morita and Penner \[62\] introduced an explicit 1-cocycle on the Ptolemy groupoid representing the extended first Johnson homomorphism \( \tilde{k} \). Bene, Kawazumi and Penner \[4\] discovered a canonical way to associate a group-like expansion to any bordered trivalent fatgraph with one tail. Unfortunately it is not symplectic. But the 1-cocycle in \[62\] is the first term of the difference of two group-like expansions associated to two fatgraphs adjacent by one Whitehead move. Contracting the coefficients \( \Lambda^3 H \to H \) by the intersection form on the homology group \( H = H_1(\Sigma_{g,1}; \mathbb{Q}) \), we have the Earle class \( k \in H^1(\mathcal{M}_{g,1}; H) \). Kuno, Penner and Turaev \[43\] introduced an explicit 1-cocycle on the Ptolemy groupoid representing the Earle class, which is simpler than the contraction of the Morita-Penner cocycle. On the other hand, in \[57\], Morita introduced a refinement of the \( k \)-th Johnson homomorphism \( \mathcal{M}_{g,1}(k) \to H_3(N_k) \) for any \( k \geq 1 \). Here \( H_3(N_k) \) is the third homology group of the nilpotent group \( N_k \) in \( \mathcal{M}_{g,1} \). Massuyeau \[18\] discovered a canonical way to attach a 3-chain of the group \( \pi \) modulo the boundaries to each marked trivalent fatgraph, which is an extension of Morita’s refinement of the Johnson homomorphisms to the Ptolemy groupoid. It is unknown that the cocycle representing the Johnson homomorphisms induced from Massuyeau’s and that in \[4\] coincide with each other or not. As was proved by Massuyeau \[47\] Theorem 4.4, Morita’s refinement is equivalent to the sum \( \bigoplus_{j=k-1}^{2k-1} \tau_j \). In \[47\], he gave an extension of all of Morita’s refinements to the monoid of homology cylinders, which includes the mapping class group \( \mathcal{M}_{g,1} \). See the chapter by Habiro and Massuyeau \[18\]. M. Day \[18\] realized truncations of the Malcev completion of the group \( \pi \) in the framework of general Lie theory of nilpotent groups \[65\] to present two geometric ways to extend Morita’s refinements to the whole mapping class group \( \mathcal{M}_{g,1} \). It is also unknown whether one of them coincides with any of what we have stated above or not.

### 3 Dehn-Nielsen embedding

In study of the mapping class group of a surface, it is often useful to consider its action on curves on the surface. In the classical case, the surface is \( \Sigma = \Sigma_{g,1} \)
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as in \(2\) and the mapping class group \(\mathcal{M}_{g,1}\) acts on \(\pi = \pi_1(\Sigma,*)\). This action induces an injective group homomorphism

\[
\text{DN}: \mathcal{M}_{g,1} \to \text{Aut}(\pi),
\]

whose image is characterized as the automorphisms of \(\pi\) preserving the boundary loop of \(\Sigma\). This is the Dehn-Nielsen theorem.

In this section we work with general oriented surfaces and consider an analogue of (3.1) for their mapping class groups. Instead of the fundamental group as in the case \(\Sigma = \Sigma_{g,1}\), we consider the fundamental groupoid of the surface with suitably chosen base points and the action of the mapping class group on it.

### 3.1 Groupoids and their completions

We begin by some general discussions about groupoids.

Let us recall a classical construction for a group \(G\) (see \([69]\)). The group ring \(\mathbb{Q}G\) is a \(\mathbb{Q}\)-vector space with basis the set \(G\). Extending \(\mathbb{Q}\)-bilinearly the product of \(G\), it is a \(\mathbb{Q}\)-algebra. Also it is a Hopf algebra with respect to the coproduct \(\Delta: \mathbb{Q}G \to \mathbb{Q}G \otimes \mathbb{Q}G\), \(G \ni g \mapsto g \otimes g\) and the antipode \(\iota: \mathbb{Q}G \to \mathbb{Q}G\), \(G \ni g \mapsto g^{-1}\). The augmentation ideal \(IG\) is the kernel of the \(\mathbb{Q}\)-algebra homomorphism \(\mathbb{Q}G \to \mathbb{Q}, G \ni g \mapsto 1\). The powers \((IG)^n, n \geq 0\), are two sided ideals of \(\mathbb{Q}G\). We denote by \(\hat{\mathbb{Q}}G\) the projective limit \(\lim_{\leftarrow n} \mathbb{Q}G/(IG)^n\).

The product, the coproduct, and the antipode of \(\hat{\mathbb{Q}}G\) are induced by those of \(\mathbb{Q}G\). It is called the completed group ring of \(G\), and is naturally a complete Hopf algebra with respect to the filtration \(F_n \hat{\mathbb{Q}}G = \text{Ker}(\hat{\mathbb{Q}}G \to \mathbb{Q}G/(IG)^n), n \geq 0\).

The set of group-like elements \(\hat{G} = \{g \in \hat{\mathbb{Q}}G; \Delta(g) = g \otimes g, g \neq 0\}\) is a group with respect to the product of \(\hat{\mathbb{Q}}G\), and is called the Malcev completion of \(G\). We have a canonical group homomorphism \(G \to \hat{G}\). This map is not injective in general. Note that in \([2,3]\) we have already seen the above construction for a free group.

Let us consider an analogous construction for groupoids. Let \(\mathcal{G}\) be a groupoid such that the set of objects is \(\text{Ob}(\mathcal{G})\) and the set of morphisms from \(p_0 \in \text{Ob}(\mathcal{G})\) to \(p_1 \in \text{Ob}(\mathcal{G})\) is \(\mathcal{G}(p_0,p_1)\). First we consider the “group ring” for \(\mathcal{G}\). Let \(\mathbb{Q}\mathcal{G}\) be the following small category. The set of objects of \(\mathbb{Q}\mathcal{G}\) is the same as that of \(\mathcal{G}\), i.e., \(\text{Ob}(\mathcal{G})\). The set of morphisms from \(p_0 \in \text{Ob}(\mathcal{G})\) to \(p_1 \in \text{Ob}(\mathcal{G})\) is \(\mathbb{Q}\mathcal{G}(p_0,p_1)\), the \(\mathbb{Q}\)-vector space with basis the set \(\mathcal{G}(p_0,p_1)\). By an obvious manner the product of morphisms in \(\mathbb{Q}\mathcal{G}\) is induced from that in \(\mathcal{G}\).

For any \(p_0, p_1, p_2 \in \text{Ob}(\mathcal{G})\) the product \(\mathbb{Q}\mathcal{G}(p_0,p_1) \times \mathbb{Q}\mathcal{G}(p_1,p_2) \to \mathbb{Q}\mathcal{G}(p_0,p_2)\) is \(\mathbb{Q}\)-bilinear. We define the coproduct, the antipode, and the augmentation of
\( \mathbb{Q} G \) as the collections
\[
\{ \Delta_{p_0,p_1} : \mathbb{Q} G(p_0,p_1) \to \mathbb{Q} G(p_0,p_1) \otimes \mathbb{Q} G(p_0,p_1) \}_{p_0,p_1 \in \text{Ob}(G)},
\{ \iota_{p_0,p_1} : \mathbb{Q} G(p_0,p_1) \to \mathbb{Q} G(p_1,p_0) \}_{p_0,p_1 \in \text{Ob}(G)},
\text{ and}
\{ \text{aug}_{p_0,p_1} : \mathbb{Q} G(p_0,p_1) \to \mathbb{Q} \}_{p_0,p_1 \in \text{Ob}(G)},
\]
of \( \mathbb{Q} \)-linear maps respectively, where \( \Delta_{p_0,p_1} \) is defined by \( \Delta_{p_0,p_1}(\ell) = \ell \otimes \ell \)
for \( \ell \in \text{Hom}(p_0,p_1) \), \( \iota_{p_0,p_1} \) is induced by taking the inverse of morphisms in \( G \),
and \( \text{aug}_{p_0,p_1} \) is defined by \( \text{aug}_{p_0,p_1}(\ell) = 1 \) for \( \ell \in G(p_0,p_1) \). For \( \ell \in G(p_0,p_1) \),
we denote \( \overline{\ell} := \iota_{p_0,p_1}(\ell) \). If there is no fear of confusion, we simply write \( \Delta, \iota, \) and \( \text{aug} \) instead of \( \Delta_{p_0,p_1}, \iota_{p_0,p_1}, \) and \( \text{aug}_{p_0,p_1} \), respectively. Clearly \( \iota \) is a contravariant functor from \( \mathbb{Q} G \) to itself. Let \( \mathbb{Q} G \otimes \mathbb{Q} G \) be the following small category. The set of objects of \( \mathbb{Q} G \otimes \mathbb{Q} G \) is \( \text{Ob}(G) \), the set of morphisms \( p_0 \in \text{Ob}(G) \) to \( p_1 \in \text{Ob}(G) \) is \( \mathbb{Q} G(p_0,p_1) \otimes \mathbb{Q} G(p_0,p_1) \), and the product of morphisms in \( \mathbb{Q} G \otimes \mathbb{Q} G \) is the tensor product of morphisms in \( \mathbb{Q} G \). We call \( \mathbb{Q} G \otimes \mathbb{Q} G \) the tensor product. Then we can regard the coproduct \( \Delta \) as a covariant functor from \( \mathbb{Q} G \) to \( \mathbb{Q} G \otimes \mathbb{Q} G \).

We next consider a concept corresponding to the augmentation ideal \( IG \) and its powers. Notice that for any \( p \in \text{Ob}(G) \) the set \( G_p = G(p,p) \) is a group. Let \( p_0, p_1 \in \text{Ob}(G) \) and \( n \geq 0 \). If there is no morphism from \( p_0 \) to \( p_1 \), i.e., \( G(p_0,p_1) = \emptyset \), we set \( F_n \mathbb{Q} G(p_0,p_1) = 0 \). Otherwise, taking a morphism \( \ell \in G(p_0,p_1) \) we set \( F_n \mathbb{Q} G(p_0,p_1) = (IG_p)^n \ell. \) Here \( IG_{p_0} \) is the augmentation ideal of the group \( G_{p_0} \). We understand that \( F_n \mathbb{Q} G(p_0,p_1) = \mathbb{Q} G(p_0,p_1) \) for \( n < 0 \).

**Proposition 3.1.** (1) The subspace \( F_n \mathbb{Q} G(p_0,p_1) \) is independent of the choice of \( \ell \), and \( \{ F_n \mathbb{Q} G(p_0,p_1) \}_{n \geq 0} \) is a decreasing filtration of \( \mathbb{Q} G(p_0,p_1) \). The augmentation induces an isomorphism \( \mathbb{Q} G(p_0,p_1)/F_1 \mathbb{Q} G(p_0,p_1) \cong \mathbb{Q} \).

(2) For any \( p_0, p_1, p_2 \in \text{Ob}(G) \) and \( n_1, n_2 \geq 0 \), we have
\[
F_{n_1} \mathbb{Q} G(p_0,p_1) : F_{n_2} \mathbb{Q} G(p_1,p_2) \subset F_{n_1+n_2} \mathbb{Q} G(p_0,p_2).
\]

(3) For any \( p_0, p_1 \in \text{Ob}(G) \) and \( n \geq 0 \), we have
\[
\Delta F_n \mathbb{Q} G(p_0,p_1) \subset \sum_{n_1+n_2=n} F_{n_1} \mathbb{Q} G(p_0,p_1) \otimes F_{n_2} \mathbb{Q} G(p_0,p_1),
\]
\[
\iota F_n \mathbb{Q} G(p_0,p_1) \subset F_n \mathbb{Q} G(p_1,p_0).
\]

Clearly \( F_n \mathbb{Q} G(p,p) = (IG_p)^n \) for any \( p \in \text{Ob}(G) \) and \( n \geq 0 \), and \( F_1 \mathbb{Q} G(p_0,p_1) = \text{Ker}(\text{aug}) \) for any \( p_0, p_1 \in \text{Ob}(G) \).

Now we construct a completion of \( \mathbb{Q} G \). Let \( \hat{\mathbb{Q}} G \) be the following small category. The set of objects of \( \hat{\mathbb{Q}} G \) is \( \text{Ob}(G) \). For \( p_0, p_1 \in \text{Ob}(G) \) we set
\[
\hat{\mathbb{Q}} G(p_0,p_1) := \varprojlim_n \mathbb{Q} G(p_0,p_1)/F_n \mathbb{Q} G(p_0,p_1),
\]
and define the set of morphisms from \( p_0 \) to \( p_1 \) to be \( \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \). By Proposition \( 3.1(2) \), the product of morphisms in \( \hat{\mathcal{Q}}\mathcal{G} \) is induced from that in \( \mathcal{Q}\mathcal{G} \). Also, by Proposition \( 3.1(1)(3) \) the coproduct, the antipode, and the augmentation of \( \hat{\mathcal{Q}}\mathcal{G} \) are induced naturally. We shall use the same letters \( \Delta, \iota, \varepsilon \) for them. For example the coproduct of \( \hat{\mathcal{Q}}\mathcal{G} \) is the collection of maps \( \Delta = \Delta_{p_0, p_1}, p_0, p_1 \in \text{Ob}(\mathcal{G}) \), where \( \Delta \) is a map from \( \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \) to the completed tensor product

\[
\hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \otimes \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) = \lim_n (\mathcal{Q}\mathcal{G}(p_0, p_1) \otimes \mathcal{Q}\mathcal{G}(p_0, p_1))/\sum_{n_1+n_2=n} F_{n_1} \mathcal{Q}\mathcal{G}(p_0, p_1) \otimes F_{n_2} \mathcal{Q}\mathcal{G}(p_0, p_1).
\]

Again, introducing a small category \( \hat{\mathcal{Q}}\mathcal{G} \otimes \hat{\mathcal{Q}}\mathcal{G} \) by an obvious manner, we can regard \( \Delta \) as a covariant functor from \( \mathcal{Q}\mathcal{G} \) to \( \hat{\mathcal{Q}}\mathcal{G} \otimes \hat{\mathcal{Q}}\mathcal{G} \).

We call \( \hat{\mathcal{Q}}\mathcal{G} \) the completion of \( \mathcal{Q}\mathcal{G} \). We define a filtration of \( \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \) by

\[
F_n \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) := \text{Ker}(\hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \rightarrow \mathcal{Q}\mathcal{G}(p_0, p_1)/F_n \mathcal{Q}\mathcal{G}(p_0, p_1)), \quad \text{for } n \geq 0.
\]

There is a canonical isomorphism \( \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \cong \lim_n \mathcal{Q}\mathcal{G}(p_0, p_1)/F_n \mathcal{Q}\mathcal{G}(p_0, p_1) \). This filtration enjoys a property similar to Proposition \( 3.1 \) and endows \( \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \) with a topology. We shall say \( u \in \hat{\mathcal{Q}}\mathcal{G}(p_0, p_1) \) is group-like if \( \Delta(u) = u \otimes u \) and \( u \neq 0 \). Note that the set of group like elements of \( \hat{\mathcal{Q}}\mathcal{G} \) is closed under the product of morphisms and the antipode, and any group like element of \( \hat{\mathcal{Q}}\mathcal{G} \) is an isomorphism. Thus the set of group like elements of \( \hat{\mathcal{Q}}\mathcal{G} \) constitutes a subcategory \( \text{Gr}(\hat{\mathcal{Q}}\mathcal{G}) \) of \( \hat{\mathcal{Q}}\mathcal{G} \) and is in fact a groupoid. There is a natural homomorphism of groupoids from \( \mathcal{G} \) to \( \text{Gr}(\hat{\mathcal{Q}}\mathcal{G}) \). We call \( \text{Gr}(\hat{\mathcal{Q}}\mathcal{G}) \) the Malcev completion of the groupoid \( \mathcal{G} \).

We end this subsection by recording the following fact which will be used later. Let \( n \geq 1 \) and \( p_0, p_1, \ldots, p_n \in \text{Ob}(\mathcal{G}) \), and assume \( \mathcal{G}(p_{i-1}, p_i) \neq \emptyset \) for \( 1 \leq i \leq n \). Then the multiplication \( \otimes_{i=1}^n F_1 \mathcal{Q}\mathcal{G}(p_{i-1}, p_i) \rightarrow F_n \mathcal{Q}\mathcal{G}(p_0, p_n) \) is surjective, and the sum of the multiplication and the inclusion \( \bigoplus_{i=1}^n F_1 \mathcal{Q}\mathcal{G}(p_{i-1}, p_i) \rightarrow F_{n+1} \mathcal{Q}\mathcal{G}(p_0, p_n) \rightarrow F_n \mathcal{Q}\mathcal{G}(p_0, p_n) \) is surjective.

### 3.2 Derivations and their exponentials

Let \( \mathcal{G} \) be a groupoid. Recall that a derivation of an associative \( \mathbb{Q} \)-algebra \( A \) is a \( \mathbb{Q} \)-endomorphism \( D: A \rightarrow A \) satisfying the Leibniz rule \( D(ab) = (Da)b + a(Db) \) for any \( a, b \in A \). We generalize this notion to \( \mathcal{Q}\mathcal{G} \) and \( \hat{\mathcal{Q}}\mathcal{G} \). We define a derivation of \( \mathcal{Q}\mathcal{G} \) to be a collection \( D = \{D_{p_0, p_1}\}_{p_0, p_1 \in \text{Ob}(\mathcal{G})} \) of \( \mathbb{Q} \)-endomorphisms \( D_{p_0, p_1}: \mathcal{Q}\mathcal{G}(p_0, p_1) \rightarrow \mathcal{Q}\mathcal{G}(p_0, p_1) \) satisfying the Leibniz rule in the sense that

\[
D_{p_0, p_2}(uv) = (D_{p_0, p_1}u)v + u(D_{p_1, p_2}v).
\]
for any $p_0, p_1, p_2 \in \text{Ob}(\mathcal{G})$, $u \in \mathbb{Q}G(p_0, p_1)$ and $v \in \mathbb{Q}G(p_1, p_2)$. To simplify the notation we often write $D$ instead of $D_{p_0, p_1}$. The derivations of $\mathbb{Q}G$ form a Lie algebra $\text{Der}(\mathbb{Q}G)$ with the Lie bracket $[D_1, D_2] = D_1D_2 - D_2D_1$, $D_1, D_2 \in \text{Der}(\mathbb{Q}G)$. Similarly, we define a derivation of $\mathcal{G}$ to be a collection of continuous $\mathbb{Q}$-endomorphisms of $\mathcal{G}(p_0, p_1)$, $p_0, p_1 \in \text{Ob}(\mathcal{G})$, satisfying the Leibniz rule in the same sense as before. We denote by $\text{Der}(\mathcal{G})$ the set of derivations of $\mathcal{G}$. This is a Lie algebra by an obvious manner. For later use we introduce a filtration of $\text{Der}(\mathcal{G})$. For $n \in \mathbb{Z}$, we define $F_n \text{Der}(\mathcal{G})$ to be the set of $D \in \text{Der}(\mathcal{G})$ such that

$$D(F_l\mathcal{G}(p_0, p_1)) \subset F_{\ell+n}\mathcal{G}(p_0, p_1)$$

for any $p_0, p_1 \in \text{Ob}(\mathcal{G})$ and $l \geq 0$. We say that a derivation $D \in \text{Der}(\mathcal{G})$ stabilizes the coproduct if $\Delta D = (D \otimes 1 + 1 \otimes D)\Delta : \mathcal{G}(p_0, p_1) \to \mathcal{G}(p_0, p_1)$ for any $p_0, p_1 \in \text{Ob}(G)$. The derivations of $\mathcal{G}$ stabilizing the coproduct form a Lie subalgebra $\text{Der}_\Delta(\mathcal{G})$ of $\text{Der}(\mathcal{G})$.

We show that a derivation of $\mathcal{G}$ naturally induces a derivation of $\mathcal{G}$. Let $D \in \text{Der}(\mathcal{G})$. We claim that for any $p_0, p_1 \in \text{Ob}(\mathcal{G})$ and $n \geq 0$ we have

$$D(F_n\mathcal{G}(p_0, p_1)) \subset F_{n-1}\mathcal{G}(p_0, p_1).$$

To prove this, we may assume that $\mathcal{G}(p_0, p_1) \neq \emptyset$. By the remark at the end of Lemma 1.3.1 there exist $u_1, \ldots, u_n \in F_1\mathcal{G}(p_0, p_0)$ and $u_n \in F_1\mathcal{G}(p_0, p_1)$ such that $u = u_1 \cdots u_{n-1}u_n$. Then

$$D(u_1 \cdots u_n) = \sum_{i=1}^{n} u_1 \cdots u_{i-1}(Du_i)u_{i+1} \cdots u_n \in F_{n-1}\mathcal{G}(p_0, p_1),$$

as desired. This shows that $D = D_{p_0, p_1}$ induces a continuous $\mathbb{Q}$-endomorphism of $\mathcal{G}(p_0, p_1)$, and there is a natural Lie algebra homomorphism $\text{Der}(\mathcal{G}) \to \text{Der}(\mathcal{G})$.

We next discuss the exponential of derivations. Recall that if $A$ is an associative $\mathbb{Q}$-algebra and $D$ is a derivation of $A$, then the formal power series $\exp(D) = \sum_{n=0}^{\infty} (1/n!)D^n$ is a $\mathbb{Q}$-algebra automorphism of $A$, provided it converges. To prove this note that for any $a, b \in A$ and $n \geq 0$, we have

$$D^n(ab) = \sum_{n_1, n_2 \geq 0, \ n_1 + n_2 = n} \frac{n!}{n_1!n_2!} D^{n_1}(a)D^{n_2}(b)$$

by the Leibniz rule. Now let us consider the exponential of derivations of $\mathcal{G}$.

**Lemma 3.2 (3.3 Lemma 1.3.2).** Suppose $D \in \text{Der}(\mathcal{G})$ satisfies the following conditions.

1. For any $p_0, p_1 \in \text{Ob}(\mathcal{G})$, $n \geq 0$, we have $D(F_n\mathcal{G}(p_0, p_1)) \subset F_n\mathcal{G}(p_0, p_1)$. 

(2) For any \( p_0, p_1 \in \text{Ob}(\mathcal{G}) \), we have \( D(p_0, p_1) \subset F_1 \widehat{\mathcal{G}}(p_0, p_1) \).

(3) For any \( p_0, p_1 \in \text{Ob}(\mathcal{G}) \), there exists \( \nu > 0 \) such that \( D^\nu(F_1 \widehat{\mathcal{G}}(p_0, p_1)) \subset F_2 \widehat{\mathcal{G}}(p_0, p_1) \).

Then for any \( p_0, p_1 \in \text{Ob}(\mathcal{G}) \) the series \( \exp(D) = \sum_{n=0}^\infty (1/n!) D^n \) converges and is a \( \mathbb{Q} \)-linear homeomorphism of \( \widehat{\mathcal{G}}(p_0, p_1) \). Moreover, if \( D' \in \text{Der}(\widehat{\mathcal{G}}) \) satisfies the above conditions and \( \exp(D) = \exp(D') \), then we have \( D = D' \).

Assume that \( D \in \text{Der}(\widehat{\mathcal{G}}) \) satisfies the assumption of Lemma 3.2. It is a formal consequence of the Leibniz rule that \( \exp(D)(uv) = (\exp(D)u)(\exp(D)v) \) for any \( p_0, p_1, p_2 \in \text{Ob}(\mathcal{G}) \), \( u \in \widehat{\mathcal{G}}(p_0, p_1) \), and \( v \in \widehat{\mathcal{G}}(p_1, p_2) \). Thus \( \exp(D) \) is an automorphism of the small category \( \widehat{\mathcal{G}} \) acting on the set of objects as the identity. Moreover, by the condition (2) of the assumption of Lemma 3.2 the automorphism \( \exp(D) \) preserves the augmentation in the sense that \( \text{aug} \circ \exp(D) = \text{aug} : \widehat{\mathcal{G}}(p_0, p_1) \to \mathbb{Q} \) for any \( p_0, p_1 \in \text{Ob}(\mathcal{G}) \).

### 3.3 Fundamental groupoid

Let us consider the construction in §3.1 for surfaces. Let \( S \) be an oriented surface. For \( p_0, p_1 \in S \), let

\[ \Pi S(p_0, p_1) = \{([0, 1], [0, 1], (S, p_0, p_1)) \} \]

be the homotopy set of paths from \( p_0 \) to \( p_1 \). Throughout this chapter, we often ignore the distinction between a path and its homotopy class.

Let \( E \) be a non-empty closed subset of \( S \), which is the disjoint union of finitely many simple closed curves and finitely many points. We denote by \( \Pi S|_E \) the fundamental groupoid of \( S \) based at \( E \). Namely, the set of objects of \( \Pi S|_E \) is \( E \), and the set of morphisms from \( p_0 \in E \) to \( p_1 \in E \) is \( \Pi S(p_0, p_1) \). The product of morphisms is induced by conjunction of paths. For \( p_0, p_1 \in E \) we denote \( \widehat{\Pi S}(p_0, p_1) \) and \( \widehat{\Pi S}|_E(p_0, p_1) \) instead of \( \Pi S|_E(p_0, p_1) \) and \( \Pi S|_E(p_0, p_1) \), respectively.

### 3.4 Dehn-Nielsen homomorphism

Let \( S \) and \( E \) be as in §3.3. We define the mapping class group of the pair \((S, E)\), denoted by \( \mathcal{M}(S, E) \), as the group of diffeomorphisms of \( S \) fixing \( E \cup \partial S \) pointwise, modulo isotopies fixing \( E \cup \partial S \) pointwise. If \( E \subset \partial S \), we denote \( \mathcal{M}(S, \partial S) \) or \( \mathcal{M}(S) \) instead of \( \mathcal{M}(S, E) \). Unless otherwise stated we ignore the distinction between a diffeomorphism and its mapping class in \( \mathcal{M}(S, E) \).

The mapping class group \( \mathcal{M}(S, E) \) acts naturally on the groupoid \( \Pi S|_E \). Let \( \text{Aut}(\Pi S|_E) \) be the group of automorphisms of the groupoid \( \Pi S|_E \) acting...
on the set of objects as the identity. If \( \varphi \) is a diffeomorphism fixing \( E \cup \partial S \) pointwise, then for any \( p_0, p_1 \in E \) and any path \( \ell \) from \( p_0 \) to \( p_1 \) the path \( \varphi(\ell) \) is from \( p_0 \) to \( p_1 \). Moreover the homotopy class \( \varphi(\ell) \in \text{II}S(p_0, p_1) \) depends only on the isotopy class of \( \varphi \) and the homotopy class of \( \ell \). In this way (the mapping class of) \( \varphi \) induces an automorphism of \( \text{II}S|_E \), giving a group homomorphism

\[
\text{DN}: \mathcal{M}(S, E) \to \text{Aut}(\text{II}S|_E).
\]

We call it the Dehn-Nielsen homomorphism.

We are interested in the case that \( \text{DN} \) is injective. We say \( S \) is of finite type, if \( S \) is a compact oriented surface, or a surface obtained from a compact oriented surface by removing finitely many points in the interior.

**Theorem 3.3.** Suppose \( S \) is of finite type and any component of \( S \) has the non-empty boundary, \( E \subset \partial S \), and any connected component of \( \partial S \) has an element of \( E \). Then the homomorphism \( \text{DN}: \mathcal{M}(S, \partial S) \to \text{Aut}(\text{II}S|_E) \) is injective.

To prove Theorem 3.3, we argue as follows. Let \( \varphi \in \mathcal{M}(S, \partial S) \) and suppose \( \text{DN}(\varphi) = 1 \). Take a system of proper arcs in \( S \) such that the surface obtained from \( S \) by cutting along these arcs is the union of disks and punctured disks. Since \( \text{DN}(\varphi) = 1 \), we may assume that \( \varphi \) is identity on these arcs. Finally we deform \( \varphi \) out side of these arcs to the identity to conclude that \( \varphi = 1 \). For more detail, see the proof of Theorem 3.1.1 in [33].

Let \( \text{Aut}(\overline{\text{II}S}|_E) \) be the group of automorphisms of the small category \( \overline{\text{II}S}|_E \) acting on the set of objects as the identity and on the set of morphisms \( \overline{\text{II}S}|_E \)-linearly. Further, let \( \overline{\text{II}S}|_E \) be the completion of \( \text{II}S|_E \) introduced in \[3.1\]. We introduce the group \( \text{Aut}(\overline{\text{II}S}|_E) \) by the same manner as for \( \text{II}S|_E \) except for considering only the automorphisms acting on the set of morphisms continuously. Then we have natural group homomorphisms \( \text{Aut}(\text{II}S|_E) \to \text{Aut}(\overline{\text{II}S}|_E) \) and \( \text{Aut}(\text{II}S|_E) \to \text{Aut}(\overline{\text{II}S}|_E) \). By post-composing them to \( \text{DN} \), we get a group homomorphism

\[
\overline{\text{DN}}: \mathcal{M}(S, E) \to \text{Aut}(\overline{\text{II}S}|_E)
\]

which we call the completed Dehn-Nielsen homomorphism.

For technical reasons and topological considerations, we introduce a subgroup of \( \text{Aut}(\overline{\text{II}S}|_E) \) in which the homomorphism \( \overline{\text{DN}} \) takes value.

**Definition 3.4.** Define the group \( A(S, E) \) as the subgroup of \( \text{Aut}(\overline{\text{II}S}|_E) \) consisting of automorphisms \( U \) satisfying the following conditions.

1. If \( \gamma \in \text{II}S(p_0, p_1) \) is represented by a path included in \( E \), then \( U(\gamma) = \gamma \).
2. We have \( \text{aug} \circ U = \text{aug}: \overline{\text{II}S}(p_0, p_1) \to \mathbb{Q} \) for any \( p_0, p_1 \in E \).
(3) We have $\Delta U = (U \otimes U) \Delta: \widehat{\text{QII}}S(p_0, p_1) \to \widehat{\text{QII}}S(p_0, p_1) \otimes \widehat{\text{QII}}S(p_0, p_1)$ for any $p_0, p_1 \in E$.

By (3), any $U \in A(S, E)$ preserves the group-like elements of $\widehat{\text{QII}}|E|$. For any $\varphi \in \mathcal{M}(S, E)$, the element $\widehat{\text{DN}}(\varphi)$ satisfies the three conditions above.

Note that $\widehat{\text{DN}}(\varphi)$ satisfies (1) since $\varphi$ fixes $E \cup \partial S$ pointwise. Thus we can write

$$\widehat{\text{DN}}: \mathcal{M}(S, E) \to A(S, E).$$

If $S$ and $E$ satisfy the assumption of Theorem 3.3, the fundamental group of each component of $S$ is a finitely generated free group. Then for any $p \in E$ the natural map $\pi_1(S, p) \to \hat{\pi}_1(S, p)$ is injective, since $\int_{n=1}^{\infty} I\pi_1(S, p)^n = 0$ (see [46]). It follows that for any $p_0, p_1 \in E$, the natural map $\text{II}S(p_0, p_1) \to \hat{\text{QII}}S(p_0, p_1)$ is also injective.

**Corollary 3.5.** If $S$ and $E$ satisfy the assumption of Theorem 3.3, the completed Dehn-Nielsen homomorphism $\hat{\text{DN}}$ is injective.

### 3.5 Cut and paste arguments

Notice that the construction of $\hat{\text{QG}}$ and $\hat{\text{G}}$ for a groupoid $\mathcal{G}$ is functorial.

If $S$ is a subsurface of an oriented surface $S'$, and $E \subset S$ and $E' \subset S'$ are closed subsets as in (3.3) such that $E \subset E'$, then the inclusion map $S \hookrightarrow S'$ induces a groupoid homomorphism from $\text{II}S|E|$ to $\text{II}S'|E'$, called the inclusion homomorphism. In this subsection we study certain kind of cut and paste arguments associated to the inclusion homomorphism.

First we show the easier half of the van Kampen theorem for $\text{II}S|E$. Let $S$ and $E$ be as in (3.3) and let $S_1$ and $S_2$ be closed subsurfaces of $S$ such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2$ is a disjoint union of finitely many simple closed curves on $S$. We further assume that for $i = 1, 2$, the set $E_i := S_i \cap E$ is a disjoint union of finitely many simple closed curves and finitely many points, and any connected component of $S_1 \cap S_2$ has an element of $E$. We denote $C_i := \text{II}S_i|E_i$, $i = 1, 2$.

We claim that $\text{II}S_i|E_i$ is “generated by $C_i$ and $C_j$”. To formulate this claim we prepare some notations. For $p_0, p_1 \in E$, we denote by $\hat{\text{E}}(p_0, p_1)$ the set of finite sequences of points in $E$, $\lambda = (q_0, q_1, \ldots, q_n) \in E^{n+1}$, $n \geq 0$, such that

1. We have $q_0 = p_0$ and $q_n = p_1$.
2. For $1 \leq j \leq n$, either $\{q_{j-1}, q_j\} \subset S_1$ or $\{q_{j-1}, q_j\} \subset S_2$.

Further let be $\hat{\text{E}}(p_0, p_1)$ the set of pairs $(\lambda, \mu)$, $\lambda = (q_0, q_1, \ldots, q_n) \in \hat{\text{E}}(p_0, p_1)$, $\mu = (\mu_0, \ldots, \mu_n) \in \{1, 2\}^n$ such that $\{q_{j-1}, q_j\} \subset S_{\mu_j}$ for any $1 \leq j \leq n$. For $(\lambda, \mu) \in \hat{\text{E}}(p_0, p_1)$, we set $\hat{\text{QC}}(\lambda, \mu) := \hat{\text{Q}}C_{\mu_1}(q_{j-1}, q_j)$. Then the multiplication map $\hat{\text{QC}}(\lambda, \mu) \to \hat{\text{QII}}S(p_0, p_1)$ is defined.
Proposition 3.6 (the easier half of the van Kampen theorem, [33]). Keep the notations as above. For any \( p_0, p_1 \in E \) the multiplication map

\[
\bigotimes_{(\lambda, \mu) \in \mathcal{E}(p_0, p_1)} \mathbb{Q}C(\lambda, \mu) \to \mathbb{Q}\Pi S(p_0, p_1)
\]

is surjective.

We next consider the forgetful homomorphisms. Let \( S \) and \( E \) be as in §3.3, and we assume that \( S \) is of finite type and has the non-empty boundary. If \( C \) is a simple closed curve on \( \text{Int}(S) \setminus E \), we can consider the forgetful homomorphism \( M(S, E \cup C) \to M(S, E) \), and the kernel of this is generated by push maps along simple closed curves on \( \text{Int}(S) \setminus (E \cup C) \) parallel to \( C \). This can be proved by a standard argument found in e.g., [15] §3.6. We shall give a corresponding result for \( A(S, E) \).

Let \( C_i \subset \text{Int}(S) \setminus E \), \( 1 \leq i \leq n \), be disjoint simple closed curves not null-homotopic in \( S \). Set \( E_1 := \bigcup_{i=1}^n C_i \). The inclusion homomorphism \( \Pi S \mid E \to \Pi S \mid E \cup E_1 \), naturally induces the forgetful homomorphism \( \phi : A(S, E \cup E_1) \to A(S, E) \). We study the kernel of \( \phi \). For definiteness, we fix an orientation of each curve \( C_i \), \( 1 \leq i \leq n \). For \( 1 \leq i \leq n \) and \( p \in C_i \), we denote by \( \eta_{i,p} \) the loop \( C_i \) based at \( p \). We can regard that \( \eta_{i,p} \in \pi_1(S, p) \). Then for a rational number \( a \in \mathbb{Q} \), we can define

\[
\eta_{i,p}^a := \exp(a \log \eta_{i,p}) \in \hat{\mathbb{Q}} \pi_1(S, p).
\]

Proposition 3.7 ([33]). Keep the notations as above. Let \( U \in A(S, E \cup E_1) \) and suppose \( \phi(U) = 1 \in A(S, E) \). Then there exist rational numbers \( a_i = a_i^U \in \mathbb{Q} \), \( 1 \leq i \leq n \), such that for any \( p_0, p_1 \in E \cup E_1 \) and \( v \in \hat{\mathbb{Q}} \Pi S(p_0, p_1) \), we have

\[
Uv = \begin{cases} 
    v, & \text{if } p_0, p_1 \in E, \\
    \eta_{i_0, p_0}^a v, & \text{if } p_0 \in C_{i_0}, p_1 \in E, \\
    v(\eta_{i_1, p_1})^{-1}, & \text{if } p_0 \in E, p_1 \in C_{i_1}, \\
    \eta_{i_0, p_0} v(\eta_{i_1, p_1})^{-1}, & \text{if } p_0 \in C_{i_0}, p_1 \in C_{i_1}.
\end{cases}
\]

Morally, this proposition says that the kernel of \( \phi \) is generated by “rational push maps” along \( C_i \).

4 Operations to curves on surfaces

Let \( S \) be an oriented surface. In this section we consider several operations to (the homotopy classes of) curves on \( S \). Here a curve on \( S \) means a loop or a
path on $S$. These operations are first defined for curves in general position, then shown to be homotopy invariant.

The quite natural but important property of these operations is that they are equivariant with the action of the mapping class group. In later sections we will see applications of this fact. Rather technical but worth mentioning is that these operations are compatible with filtrations on the $\mathbb{Q}$-vector spaces based on curves coming from the augmentation ideal of $\mathbb{Q}\pi_1(S)$ (see §3). This point will be explained in §4.5.

We say that a curve on $S$ is generic if it is an immersion and its self intersections consist of transverse double points. Likewise, we say that finitely many curves on $S$ are in general position if each of the curves is generic and their intersections consist of transverse double points. We often identify a generic curve, which is a map to $S$, with its image, which is a subset of $S$.

For simplicity, we will consider over the rationals $\mathbb{Q}$. However, all the constructions in this section works well over the integers $\mathbb{Z}$ as well as over any commutative ring with unit.

### 4.1 Goldman-Turaev Lie bialgebra

Let $\hat{\pi}(S) = [S^1, S]$ be the homotopy set of oriented free loops on $S$. For $p \in S$ we denote by $| |: \pi_1(S) = \pi_1(S, p) \to \hat{\pi}(S)$ the map obtained by forgetting the base point of a based loop. If $S$ is connected, $| |$ is surjective. Let $\mathbb{Q}\hat{\pi}(S)$ be the $\mathbb{Q}$-vector space with basis the set $\hat{\pi}(S)$. The map $| |$ extends $\mathbb{Q}$-linearly to $| |: \mathbb{Q}\pi_1(S) \to \mathbb{Q}\hat{\pi}(S)$.

Let us recall the definition of the Goldman bracket. We use the intersection of two generic oriented loops on $S$. Let $\alpha$ and $\beta$ be oriented loops on $S$ in general position. For each $p \in \alpha \cap \beta$, let $\varepsilon(p; \alpha, \beta) \in \{\pm 1\}$ be the local intersection number of $\alpha$ and $\beta$ at $p$. Also let $\alpha_p$ be the loop $\alpha$ based at $p$ and define $\beta_p$ similarly. Then the conjunction $\alpha_p \beta_p \in \pi_1(S, p)$, and $| \alpha_p \beta_p | \in \hat{\pi}(S)$ are defined. The Goldman bracket \[ [\alpha, \beta] := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) | \alpha_p \beta_p | \in \mathbb{Q}\hat{\pi}(S). \] (4.1)

**Theorem 4.1** (Goldman \[17\]). The Goldman bracket \[ [\, , \]: \mathbb{Q}\hat{\pi}(S) \otimes \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}\hat{\pi}(S) \].

We call $\mathbb{Q}\hat{\pi}(S)$ the Goldman Lie algebra of $S$. Goldman introduced this Lie algebra along the study of the Poisson bracket of two trace functions on the moduli space of flat $G$-bundles $\text{Hom}(\pi_1(S), G)/G$, where $G$ is a Lie group satisfying very general conditions. The proof of Theorem 4.1 goes as follows.

(1) To prove that the Goldman bracket is well-defined, it suffices to check that $[\alpha, \beta]$ is unchanged under the three local moves in Figure 1. For,
every pair of free loops on $S$ is homotopic to a generic pair of free loops, and if two generic pairs of free loops on $S$ are homotopic to each other, then they are related by a sequence of the three moves. For another proof using twisted homology, see [31] Proposition 3.4.3.

(2) To prove that the Goldman bracket is a Lie bracket, one needs to check that it is skew-symmetric and satisfies the Jacobi identity. The skew-symmetry is clear from (4.1) since $|\alpha_p\beta_p| = |\beta_p\alpha_p|$ and $\varepsilon(p;\alpha,\beta) = -\varepsilon(p;\beta,\alpha)$ for $p \in \alpha \cap \beta$. To prove the Jacobi identity, take three free loops $\alpha, \beta, \gamma$ in general position. Then one can directly check $[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0$ using (4.1).

In this section we will see statements similar to Theorem 4.1, e.g., Theorems 4.2, 4.3, 4.5, and 4.8. They can be proved by the same method as above. Note that if $S = \coprod_{\lambda} S_{\lambda}$ is the decomposition of $S$ into connected components, then $\mathbb{Q}\hat{\pi}(S) = \bigoplus_{\lambda} \mathbb{Q}\hat{\pi}(S_{\lambda})$ as Lie algebras.

Next let us recall the definition of the Turaev cobracket. We use the self-intersection of a generic oriented loop on $S$. For simplicity and a direct sum decomposition given in the last sentence of the proceeding paragraph we assume that $S$ is connected. We denote by $1 \in \hat{\pi}(S)$ the class of a constant loop. The $\mathbb{Q}$-linear subspace $\mathbb{Q}1$ is an ideal of $\mathbb{Q}\hat{\pi}(S)$. We denote by $\mathbb{Q}\hat{\pi}'(S)$ the quo-
tient Lie algebra $\mathbb{Q}\hat{\pi}(S)/\mathbb{Q}1$, and let $\varpi: \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}\hat{\pi}'(S)$ be the projection. We write $t := \varpi \circ t: \mathbb{Q}\pi_1(S) \to \mathbb{Q}\hat{\pi}'(S)$.

Let $\alpha: S^1 \to S$ be a generic oriented loop. Set $D = D_\alpha := \{(t_1, t_2) \in S^1 \times S^1; t_1 \neq t_2, \alpha(t_1) = \alpha(t_2)\}$. For $(t_1, t_2) \in D$, let $\alpha_{t_1, t_2}$ (resp. $\alpha_{t_2, t_1}$) be the restriction of $\alpha$ to the interval $[t_1, t_2]$ (resp. $[t_2, t_1]$) $\subset S^1$ (they are indeed loops since $\alpha(t_1) = \alpha(t_2)$). Also, let $\varepsilon(\alpha(t_1), \alpha(t_2)) \in \{\pm 1\}$ be the local intersection number of the velocity vectors $\dot{\alpha}(t_i) \in T_{\alpha(t_i)} S, i = 1, 2$. The Turaev cobracket [79] of $\alpha$ is

$$\delta(\alpha) := \sum_{(t_1, t_2) \in D} \varepsilon(\alpha(t_1), \alpha(t_2))|\alpha_{t_1, t_2}|' \otimes |\alpha_{t_2, t_1}|' \in \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\hat{\pi}'(S). \quad (4.2)$$

**Theorem 4.2** (Turaev [79], the involutivity is due to Chas [6]). The Turaev cobracket (4.2) induces a Lie cobracket $\delta: \mathbb{Q}\hat{\pi}'(S) \to \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\hat{\pi}'(S)$. Moreover, the $\mathbb{Q}$-vector space $\mathbb{Q}\hat{\pi}'(S)$ is an involutive Lie bialgebra with respect to the Goldman bracket and the Turaev cobracket.

To be more precise we have the following.

1. The space $\mathbb{Q}\hat{\pi}'(S)$ is a Lie algebra with respect to the Goldman bracket.
2. The space $\mathbb{Q}\hat{\pi}'(S)$ is a Lie coalgebra with respect to the Turaev cobracket.
3. We have $\delta(u, v) = \sigma(u)(\delta v) - \sigma(v)(\delta u)$ for any $u, v \in \mathbb{Q}\hat{\pi}'(S)$. Here $\sigma(u)(v \otimes w) = [u, v] \otimes w + v \otimes [u, w]$ for $u, v, w \in \mathbb{Q}\hat{\pi}'(S)$.
4. We have $[\, , ] \circ \delta = 0: \mathbb{Q}\hat{\pi}'(S) \to \mathbb{Q}\hat{\pi}'(S)$.

The condition (3) is called the compatibility, and (4) is called the involutivity. We call $\mathbb{Q}\hat{\pi}'(S)$ the Goldman-Turaev Lie bialgebra. He introduced this Lie bialgebra along the study of a skein quantization of Poisson algebras of loops on surfaces, and he showed that some skein bialgebra of links in $S \times [0, 1]$ quantizes the Goldman-Turaev Lie bialgebra ([79] Theorem 10.1).

### 4.2 The action of free loops on based paths

We introduce an operation denoted by $\sigma$, using the intersection of an oriented loop and a based path in general position. Let $S$ and $E$ be as in [3.3]. Put $S^* = S \setminus (E \setminus \partial S)$. Note that $S^* = S$ if $E \subset \partial S$. We show that the Goldman Lie algebra $\mathbb{Q}\hat{\pi}(S^*)$ acts on $\Pi S|_E$ by derivations.

Take two points $*_0, *_1 \in E$ which are not necessarily distinct. Let $\alpha$ be an oriented loop on $S^*$ and $\beta: [0, 1] \to S$ a path from $*_0$ to $*_1$, and assume that they are in general position. For $p \in \alpha \cap \beta$, let $\varepsilon(p; \alpha, \beta)$ be the local intersection number as before. Also let $\beta_{*_0p}$ be the path from $*_0$ to $p$ traversing $\beta$, and define $\beta_{p*_1}$ similarly. Then the conjunction $\beta_{*_0p} \alpha_p \beta_{p*_1} \in \Pi S(*_0, *_1)$ is defined.
Set

$$\sigma(\alpha \otimes \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \beta_{s_0} \alpha_p \beta_{p^*} \in \text{QILS}(\ast_0, \ast_1).$$  (4.3)

**Theorem 4.3**: The formula (4.3) induces a $\mathbb{Q}$-linear map $\sigma: \mathbb{Q}\hat{\pi}(S^*) \otimes \text{QILS}(\ast_0, \ast_1) \to \text{QILS}(\ast_0, \ast_1)$. Moreover, with respect to $\sigma$ and the Goldman bracket, the vector space $\text{QILS}(\ast_0, \ast_1)$ is a left $\mathbb{Q}\hat{\pi}(S^*)$-module.

Recall that $1 \in \hat{\pi}(S^*)$ denotes the class of a constant loop. We have $\sigma(1 \otimes v) = 0$ for any $v \in \text{QILS}(\ast_0, \ast_1)$. Thus $\sigma$ naturally induces a map $\mathbb{Q}\hat{\pi}(S^*) \otimes \text{QILS}(\ast_0, \ast_1) \to \text{QILS}(\ast_0, \ast_1)$, which we denote by the same letter $\sigma$. For $u \in \mathbb{Q}\hat{\pi}(S^*)$ and $m \in \text{QILS}(\ast_0, \ast_1)$ we often write $\sigma(u)m$ or $um$ for short instead of $\sigma(u \otimes m)$. That $\text{QILS}(\ast_0, \ast_1)$ is a left $\mathbb{Q}\hat{\pi}(S^*)$-module means

$$[u, v]m = u(vm) - v(um)$$

for $u, v \in \mathbb{Q}\hat{\pi}(S^*)$ and $m \in \text{QILS}(\ast_0, \ast_1)$.

If we consider not only a single pair $(\ast_0, \ast_1)$ but also all the ordered pairs of elements of $E$, we obtain a derivation of $\text{QILS}|_E$. First notice that the operation $\sigma$ satisfies the Leibniz rule in the following sense. For any $\ast_0, \ast_1, \ast_2 \in E$ and $\alpha \in \mathbb{Q}\hat{\pi}(S^*)$, $\beta_1 \in \text{QILS}(\ast_0, \ast_1)$, $\beta_2 \in \text{QILS}(\ast_1, \ast_2)$, we have

$$\sigma(\alpha)(\beta_1 \beta_2) = (\sigma(\alpha)\beta_1)\beta_2 + \beta_1(\sigma(\alpha)\beta_2).$$  (4.4)

This shows that for any $\alpha \in \mathbb{Q}\hat{\pi}(S^*)$ the collection $\sigma(\alpha) = \sigma(\alpha)_{\ast_0, \ast_1, \ast_0, \ast_1} \in E$, determines a derivation of $\text{QILS}|_E$ in the sense of (3.2). Thus we get a $\mathbb{Q}$-linear map

$$\sigma: \mathbb{Q}\hat{\pi}(S^*) \to \text{Der}(\text{QILS}|_E),$$  (4.5)

and by the second sentence of Theorem 4.3, this is a Lie algebra homomorphism. As a special case, if $E = \{\ast\}$ is a singleton with $\ast \in \partial S$, the group ring $\mathbb{Q}\pi_1(S, \ast)$ is a $\mathbb{Q}\hat{\pi}(S)$-module and we have a Lie algebra homomorphism $\sigma: \mathbb{Q}\hat{\pi}(S) \to \text{Der}(\mathbb{Q}\pi_1(S, \ast))$. Note that for any $u \in \mathbb{Q}\hat{\pi}(S)$ and $v \in \mathbb{Q}\pi_1(S, \ast)$ we have

$$[u, v] = |\sigma(u \otimes v)|.$$  (4.6)

### 4.3 Intersection of based paths

Take points $\ast_1, \ast_2, \ast_3, \ast_4$ on the boundary of $S$. We define a $\mathbb{Q}$-linear map

$$\kappa: \text{QILS}(\ast_1, \ast_2) \otimes \text{QILS}(\ast_3, \ast_4) \to \text{QILS}(\ast_1, \ast_4) \otimes \text{QILS}(\ast_3, \ast_2),$$

using the intersection of two based paths in general position. Then we show that this is closely related to an operation called the homotopy intersection form by Massuyeau and Turaev.
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First we discuss the most generic case. Namely, we assume \( \{ *, _1, _2 \} \cap \{ *, _3, _4 \} = \emptyset \). Let \( x: [0, 1] \to S \) be a path from \(*, _1 \) to \(*, _2 \) and \( y: [0, 1] \to S \) a path from \(*, _3 \) to \(*, _4 \), and assume that they are in general position. Set

\[
\kappa(x, y) = \sum_{p \in x \cap y} \varepsilon(p; x, y)(x_{*1}p_{y3}) \otimes (y_{*3}p_{*4})
\]

for this case, we move the points \(*, _1 \) slightly along the negatively oriented boundary of \( S \) to achieve \( \{ *, _1 \} \cap \{ *, _3 \} = \emptyset \), then apply the formula (4.7). For more precise explanation we use an example, which is the most extreme. Namely,
let us consider the case \( *_1 = *_2 = *_3 = *_4 \). Take a base point \( * \in \partial S \) and pick an orientation preserving embedding \( \nu: [0, 1] \to \partial S \) such that \( \nu(1) = * \). Set \( \nu(0) = \bullet \). See Figure 2. Then we have three isomorphisms \( \pi_1(S, *) \cong \pi_1(S, \bullet) \), \( x \mapsto \nu x \pi, \pi_1(S, *) \cong \Pi S(\bullet, *), x \mapsto \nu x \), and \( \pi_1(S, *) \cong \Pi S(*, \bullet), x \mapsto \nu x \). Now we define

\[
\kappa: \pi(S, *) \otimes \pi(S, *) \to \pi(S, *) \otimes \pi(S, *)
\]

so that the diagram

\[
\begin{array}{ccc}
\pi(S, *) \otimes \pi(S, *) & \xrightarrow{\kappa} & \pi(S, *) \otimes \pi(S, *) \\
\cong & & \cong \\
\pi(S, \bullet) \otimes \pi(S, *) & \xrightarrow{\kappa} & \Pi S(*, \bullet) \otimes \Pi S(*, \bullet)
\end{array}
\]

commutes. Here the vertical maps are via the above isomorphisms, and the bottom horizontal arrow is the map already defined. To write down \( \kappa \) in (4.10) explicitly, let \( \alpha \) be a loop based at \( \bullet \), and \( \beta \) a loop based at \( * \) and assume that they are in general position. By the isomorphism \( \pi_1(S, *) \cong \pi_1(S, \bullet) \) given by \( \nu \), we regard that \( \alpha \) represents an element of \( \pi_1(S, *) \). Then

\[
\kappa(\alpha, \beta) := - \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) (\nu p \alpha \nu p \beta) \otimes (\beta \nu p \alpha \nu p \beta \nu).
\]

Note this \( \kappa \) satisfies (4.8) (4.9) for any four points \( *_1, *_2, *_3, *_4 \in \partial S \), which are not necessarily distinct, we can define the operation \( \kappa \). Since we use only the most extreme case (4.10), we omit the detail of the construction.

Post-composing \(-1 \otimes \text{aug}: \pi(S, *) \otimes \pi(S, *) \to \pi(S, *) \otimes \pi(S, *) \) to (4.10), we obtain a \( \mathbb{Q} \)-linear map

\[
\eta: \pi(S, *) \otimes \pi(S, *) \to \pi(S, *).
\]

By (4.11), an explicit formula for \( \eta \) is given by

\[
\eta(\alpha, \beta) := \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) (\nu p \alpha \nu p \beta) \in \pi(S, *),
\]

where notations are the same as in the preceding paragraph. The map \( \eta \) is introduced by Massuyeau and Turaev [49], and is called the homotopy intersection form. It is actually a modification of the operation \( \lambda: \pi_1(S, *) \times \pi_1(S, *) \to \pi_1(S, *) \) introduced by Papakyriakopoulos [66] and Turaev [78] independently. The relationship between \( \lambda \) and \( \eta \) is given by \( \lambda(\alpha, \beta) = \eta(\alpha, \beta) \eta^{-1} \) for \( \alpha, \beta \in \pi_1(S, *) \). By (4.8) (4.9), we have the following, which is essentially due to Papakyriakopoulos [66] and Turaev [78].
Proposition 4.5 ([49]). The homotopy intersection form satisfies the following identities:

\[
\eta(\alpha_1 \alpha_2, \beta) = \eta(\alpha_1, \beta)\text{aug}(\alpha_2) + \alpha_1 \eta(\alpha_2, \beta), \\
\eta(\alpha, \beta_1 \beta_2) = \eta(\alpha, \beta_1)\beta_2 + \text{aug}(\beta_1)\eta(\alpha, \beta_2),
\]

where \(\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 \in \pi_1(S, *). \) Here \(\text{aug} : \pi_1(S, *) \rightarrow \mathbb{Q} \) is the augmentation map.

In [49], a bilinear pairing on the group ring satisfying (4.13) is called a Fox pairing. In their theory, given a Fox pairing one can consider its derived form. Actually the derived form \(\eta\) turns out to be \(\sigma\). Let \(u, v \in \mathbb{Q}\pi_1(S, *)\). The element \(v\) is uniquely written as \(v = \sum_{x \in \pi_1} c_x x\) where \(c_x \in \mathbb{Q}\). We denote \(u'' = \sum_{x \in \pi_1} c_x x^{-1}ux\). We define a \(\mathbb{Q}\)-linear map \(\sigma^{\eta} : \pi_1(S, *) \otimes \pi_1(S, *) \rightarrow \mathbb{Q}\pi_1(S, *)\) by setting \(\sigma^{\eta}(x \otimes y) = y(x^{v(x,y)})\) for \(x, y \in \pi_1(S, *)\) and extending \(\pi\)-linearly to \(\pi_1(S, *) \otimes \pi_1(S, *)\). In [49], \(\sigma^{\eta}\) is called the derived form of \(\eta\). From (4.13), we have

\[
\begin{align*}
\sigma^{\eta}(u, uv) &= \sigma^{\eta}(u, v)w + v\sigma^{\eta}(u, w), \\
\sigma^{\eta}(uv, w) &= \sigma^{\eta}(vu, w)
\end{align*}
\]

for \(u, v, w \in \mathbb{Q}\pi_1(S, *)\).

Lemma 4.6 (Masuyae-Turaev [49]). The composition of \(1 \otimes 1 : \pi_1(S, *) \otimes \pi_1(S, *) \rightarrow \mathbb{Q}\hat{\pi}(S) \otimes \pi_1(S, *)\) and \(\sigma : \mathbb{Q}\hat{\pi}(S) \otimes \pi_1(S, *) \rightarrow \mathbb{Q}\pi_1(S, *)\) coincides with the map \(\sigma^{\eta}\).

We end this subsection by a remark that one can recover \(\kappa\) from \(\eta\).

Proposition 4.7. Let \(\kappa\) be the map in (4.10). We have

\[
\kappa = -(1 \otimes m)(1 \otimes 1 \otimes m)P_{2431}(1 \otimes ((1 \otimes \iota)\Delta \eta) \otimes 1)(\Delta \otimes \Delta).
\]

Here, \(1\) is the identity map, \(\Delta, \iota\), and \(m\) are the coproduct, the antipode, and the product of the group ring \(\mathbb{Q}\pi_1(S, *)\), and \(P_{2431} : \pi_1(S, *) \otimes \pi_1(S, *) \rightarrow \mathbb{Q}\pi_1(S, *)\) is the \(\mathbb{Q}\)-linear map given by \(P_{2431}(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_2 \otimes x_4 \otimes x_3 \otimes x_1\).

4.4 Self intersections

Take two points \(*_0, *_1\) on the boundary of \(S\). We define a \(\mathbb{Q}\)-linear map

\[
\mu : \text{II}S(*_0, *_1) \rightarrow \pi_1(S, *_0) \otimes \mathbb{Q}\hat{\pi}(S),
\]

using the self intersections of a generic path from \(*_0\) to \(*_1\). Then we mention a certain product formula for \(\mu\) and a relationship with the Turaev cobracket.

First we consider the general case \(*_0 \neq *_1\). Let \(\gamma : [0, 1] \rightarrow S\) be a generic path from \(*_0\) to \(*_1\). We denote by \(\Gamma \subset S\) the set of double points of \(\gamma\). For
Theorem 4.8. The \( \mathbb{Q} \)-vector space \( \text{QIIS}(\ast_0, \ast_1) \) is an involutive right \( \hat{\mathbb{Q}}'(S) \)-bimodule with respect to \( \sigma \) and \( \mu \).

To be more precise we have the following.

1. The space \( \text{QIIS}(\ast_0, \ast_1) \) is a left \( \hat{\mathbb{Q}}'(S) \)-module with respect to \( \sigma \) (see Theorem 4.3).

2. The space \( \text{QIIS}(\ast_0, \ast_1) \) is a right \( \hat{\mathbb{Q}}'(S) \)-comodule with respect to \( \mu \). That is, the diagram

\[
\begin{array}{ccc}
\text{QIIS}(\ast_0, \ast_1) & \xrightarrow{\mu} & \text{QIIS}(\ast_0, \ast_1) \otimes \hat{\mathbb{Q}}'(S) \\
\downarrow & & \downarrow 1 \otimes \delta \\
\text{QIIS}(\ast_0, \ast_1) \otimes \hat{\mathbb{Q}}'(S) & \xrightarrow{(1 \otimes (1 - T))(\mu \otimes 1)} & \text{QIIS}(\ast_0, \ast_1) \otimes \hat{\mathbb{Q}}'(S) \otimes \hat{\mathbb{Q}}'(S)
\end{array}
\]

commutes. Here \( \delta \) is the Turaev cobracket and \( T : \hat{\mathbb{Q}}'(S) \otimes \hat{\mathbb{Q}}'(S) \to \hat{\mathbb{Q}}'(S) \otimes \hat{\mathbb{Q}}'(S), u \otimes v \mapsto v \otimes u \) is the switch map.

3. The operations \( \sigma \) and \( \mu \) satisfy the compatibility in the sense that

\[
\sigma(u) \mu(m) - \mu(\sigma(u)m) - (\sigma \otimes 1)(1 \otimes \delta)(m \otimes u) = 0
\]

for \( u \in \hat{\mathbb{Q}}'(S), m \in \text{QIIS}(\ast_0, \ast_1) \). Here \( \overline{\sigma} : \text{QIIS}(\ast_0, \ast_1) \otimes \hat{\mathbb{Q}}'(S) \to \text{QIIS}(\ast_0, \ast_1) \) is given by \( \overline{\sigma}(m \otimes u) = -\sigma(u \otimes m) \), and \( \sigma(u) \mu(m) = (\sigma \otimes 1)(u \otimes \mu(m)) + (1 \otimes \text{ad}(u))\mu(m) \).
The operation \( \mu \) is introduced in \cite{KawazumiKuno1}, and inspired by Turaev’s self intersection \( \mu = \mu^T: \pi_1(S, \ast) \to \mathbb{Z}\pi_1(S, \ast) \) in \cite{Turaev1} §1.4. Indeed, for any \( \gamma \in \pi_1(S, \ast) \) we have \( \mu^T(\gamma)\gamma = -(1 \otimes \varepsilon)\mu(\gamma) \), where \( \varepsilon: \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q} \) is the \( \mathbb{Q} \)-linear map given by \( \varepsilon(\omega(\alpha)) = 1 \) for \( \alpha \in \hat{\pi}(S) \setminus \{1\} \).

We end this subsection by stating two results about \( \mu \). The first one is a certain product formula, and the second one is a relation with the Turaev cobracket.

**Lemma 4.9.** For any \( \ast_1, \ast_2, \ast_3 \in E \) and \( u \in \mathbb{Q}\text{IIS}(\ast_1, \ast_2) \), \( v \in \mathbb{Q}\text{IIS}(\ast_2, \ast_3) \), we have
\[
\mu(uv) = \mu(u)(v \otimes 1) + (u \otimes 1)\mu(v) + (1 \otimes | \cdot \rangle)\kappa(u, v).
\]
Here \( \mu(u)(v \otimes 1) \) is the image of \( \mu(u) \otimes v \) by the map \( \mathbb{Q}\pi_1(S, \ast) \otimes \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\pi_1(S, \ast) \to \mathbb{Q}\pi_1(S, \ast) \otimes \mathbb{Q}\hat{\pi}'(S), \ a \otimes b \otimes c \mapsto ac \otimes b, \) etc.

As a corollary, for any \( n \geq 2 \) and \( \ast_0, \ldots, \ast_n \in \partial S, \ u_i \in \text{IIS}(\ast_{i-1}, \ast_i) \), \( 1 \leq i \leq n \), we have
\[
\mu(u_1 \cdots u_n) = \sum_{i=1}^{n} ((u_1 \cdots u_{i-1}) \otimes 1)\mu(u_i)((u_{i+1} \cdots u_n) \otimes 1)
+ \sum_{i<j} ((u_1 \cdots u_{i-1}) \otimes 1)K_{i,j}((u_{j+1} \cdots u_n) \otimes 1),
\]
where \( K_{i,j} = (1 \otimes | \cdot \rangle)\kappa(u_i, u_j)(1 \otimes (u_{i+1} \cdots u_{j-1})) \).

**Proposition 4.10.** The following diagram is commutative:
\[
\begin{array}{ccc}
\mathbb{Q}\pi_1(S, \ast) & \xrightarrow{\mu} & \mathbb{Q}\pi_1(S, \ast) \otimes \mathbb{Q}\hat{\pi}'(S) \\
\downarrow & & \downarrow (1-T)(| \cdot \rangle \otimes 1) \\
\mathbb{Q}\hat{\pi}'(S) & \xrightarrow{\delta} & \mathbb{Q}\hat{\pi}'(S) \otimes \mathbb{Q}\hat{\pi}'(S)
\end{array}
\]

### 4.5 Completions of the operations

We shall see that the operations we have considered extends naturally to completions.

First of all let us introduce a filtration of the vector space \( \mathbb{Q}\hat{\pi}(S) \) and its completion. Recall from \cite{KawazumiKuno1} the map \( | \cdot |: \mathbb{Q}\pi_1(S) \to \mathbb{Q}\hat{\pi}(S) \). Note that the...
constant loop 1 is always in the kernel of the homomorphism \([33](\mathfrak{g}) \). For \( n \geq 0 \), set
\[
\mathcal{Q}\pi(S)(n) := \mathcal{Q}1 + (I\pi_1(S))^n
\]
and \( \mathcal{Q}\pi'(S)(n) := \mathcal{Q}(\mathcal{Q}\pi(S)(n)) \). We define the \( \mathcal{Q} \)-vector space \( \mathcal{Q}\pi(S) \) by
\[
\mathcal{Q}\pi(S) := \lim_{n} \mathcal{Q}\pi(S)/\mathcal{Q}\pi(S)(n) \cong \lim_{n} \mathcal{Q}\pi'(S)/\mathcal{Q}\pi'(S)(n),
\]
and introduce its filtration by
\[
\mathcal{Q}\pi(S)(n) := \text{Ker}(\mathcal{Q}\pi(S) \rightarrow \mathcal{Q}\pi(S)/\mathcal{Q}\pi(S)(n)), \quad n \geq 0.
\]
The map \( | \cdot | \) naturally induces a \( \mathcal{Q} \)-linear map \( | \cdot | : \mathcal{Q}\pi_1(S) \rightarrow \mathcal{Q}\pi(S) \). We understand that if \( n < 0 \), \( \mathcal{Q}\pi(S)(n) = \mathcal{Q}\pi(S) \).

**Proposition 4.11 (33, 51).** Let \( m, n \) be integers \( \geq 0 \).

1. Let \( S \) and \( E \) be as in (33). For any \( *_0, *_1 \in E \) we have
   \[
   \sigma(\mathcal{Q}\pi(S^*)(m) \otimes F_n \mathcal{Q}\Pi S(*_0, *_1)) \subset F_{m+n-2} \mathcal{Q}\Pi S(*_0, *_1).
   \]

2. For any \( *_0, *_1 \in \partial S \) we have
   \[
   \mu(F_n \mathcal{Q}\Pi S(*_0, *_1)) \subset \sum_{p+q=n-2} F_p \mathcal{Q}\Pi S(*_0, *_1) \otimes \mathcal{Q}\pi'(S)(q).
   \]

We remark that (2) follows from (4.10). As an immediate consequence, we see that \( \sigma \) and \( \mu \) extends to completions:
\[
\sigma : \widehat{\mathcal{Q}\pi(S^*)} \otimes \mathcal{Q}\Pi S(*_0, *_1) \rightarrow \mathcal{Q}\Pi S(*_0, *_1),
\]
\[
\mu : \mathcal{Q}\Pi S(*_0, *_1) \rightarrow \mathcal{Q}\Pi S(*_0, *_1) \otimes \mathcal{Q}\pi(S), \quad (4.17)
\]
Here \( \widehat{\otimes} \) means the complete tensor product. From (4.9) and Proposition 4.10 we have the following corollary to Proposition 4.11

**Corollary 4.12.**

1. For \( u \in \mathcal{Q}\pi(S)(m) \) and \( v \in \mathcal{Q}\pi(S)(n) \), we have \([u, v] \in \mathcal{Q}\pi(S)(m+n-2)\). In particular, the Goldman bracket naturally induces a complete Lie bracket \([\cdot, \cdot] : \widehat{\mathcal{Q}\pi(S)} \otimes \widehat{\mathcal{Q}\pi(S)} \rightarrow \widehat{\mathcal{Q}\pi(S)} \).

2. If \( u \in \mathcal{Q}\pi'(S)(n) \), then \( \delta(u) \in \sum_{p+q=n-2} \mathcal{Q}\pi'(S)(p) \otimes \mathcal{Q}\pi'(S)(q) \). In particular, the Turaev cobracket naturally induces a complete Lie cobracket \( \delta : \mathcal{Q}\pi'(S) \rightarrow \mathcal{Q}\pi(S) \otimes \mathcal{Q}\pi(S) \).

It is easy to see that the completed Lie bracket and cobracket on \( \mathcal{Q}\pi(S) \) inherit the compatibility and the involutivity from those on \( \mathcal{Q}\pi'(S) \). We call \( \mathcal{Q}\pi(S) \) the completed Goldman-Turaev Lie bialgebra. Also if \( *_0, *_1 \in \partial S \), the
vector space \( \hat{\mathbb{Q}}\hat{\Pi}S(\ast_0, \ast_1) \) is a complete \( \hat{\mathbb{Q}}\hat{\pi}(S) \)-bimodule with respect to the completed operations \( 4.17 \).

Let \( S \) be a compact connected oriented surface with non-empty boundary, and \( E \subset \partial S \) a finite subset consisting of one point from each component of the boundary \( \partial S \). Then we have \( S^* = S \), so that the homomorphism \( 4.13 \) induces a Lie algebra homomorphism \( \sigma : \hat{\mathbb{Q}}\hat{\pi}(S) \to \text{Der}(\hat{\mathbb{Q}}\hat{\Pi}S|_E) \). We denote by \( \text{Der}_\partial(\hat{\mathbb{Q}}\hat{\Pi}S|_E) \) the Lie subalgebra consisting of continuous derivations on \( \hat{\mathbb{Q}}\hat{\Pi}S|_E \) annihilating all based loops inside the boundary \( \partial S \). Clearly it includes the image \( \sigma(\hat{\mathbb{Q}}\hat{\pi}(S)) \). The following is an infinitesimal version of the Dehn-Nielsen theorem in §3.

**Theorem 4.13.** Let \( S \) and \( E \) be as above. Then the Lie algebra homomorphism 
\[
\sigma : \hat{\mathbb{Q}}\hat{\pi}(S) \to \text{Der}_\partial(\hat{\mathbb{Q}}\hat{\Pi}S|_E)
\]
is an isomorphism.

The injectivity is proved in [33]. The proof of the surjectivity, which follows from a tensorial description of \( \hat{\mathbb{Q}}\hat{\pi}(S) \), will appear in our forthcoming paper [35]. In §7.2 we will give an outline of the proof.

Finally we consider \( \kappa \) and \( \eta \). From Lemma 4.4, for any integers \( m, n \geq 0 \), and points \( \ast_i \in \partial S, 1 \leq i \leq 4 \), we have
\[
\kappa(F_m\hat{\mathbb{Q}}\hat{\Pi}S(\ast_1, \ast_2) \otimes F_n\hat{\mathbb{Q}}\hat{\Pi}S(\ast_3, \ast_4)) \\
\subseteq \bigoplus_{p+q=m+n-2} F_p\hat{\mathbb{Q}}\hat{\Pi}S(\ast_1, \ast_4) \otimes F_q\hat{\mathbb{Q}}\hat{\Pi}S(\ast_3, \ast_2).
\]
We conclude that \( \kappa \) extends naturally to completions:
\[
\kappa : \hat{\mathbb{Q}}\hat{\Pi}S(\ast_1, \ast_2) \hat{\otimes} \hat{\mathbb{Q}}\hat{\Pi}S(\ast_3, \ast_4) \to \hat{\mathbb{Q}}\hat{\Pi}S(\ast_1, \ast_4) \hat{\otimes} \hat{\mathbb{Q}}\hat{\Pi}S(\ast_3, \ast_2),
\]
and by \( \eta = (-1 \otimes \text{aug})\kappa \) so does \( \eta \):
\[
\eta : \hat{\mathbb{Q}}\hat{\pi}_1(S, *) \hat{\otimes} \hat{\mathbb{Q}}\hat{\pi}_1(S, *) \to \hat{\mathbb{Q}}\hat{\pi}_1(S, *).
\]

We end this section with a couple of remarks.

**Remark 4.14.**

1. In later sections we consider the logarithms on the completed group ring of the fundamental group of the surface, which is defined by a formal power series with coefficients in \( \mathbb{Q} \). Thus we have to work with coefficients in a commutative ring including \( \mathbb{Q} \). For simplicity we confine ourselves to the case of \( \mathbb{Q} \).

2. To define \( \kappa \) for the degenerate case \( \{\ast_1, \ast_2\} \cap \{\ast_3, \ast_4\} \neq \emptyset \), we move the points \( \ast_1, \ast_2 \) slightly along the negatively oriented boundary. However
this is not a unique way. Our aim is to achieve \( \{*1, *2\} \cap \{*3, *4\} = \emptyset \) by moving the endpoints of paths we consider. We may move the points \(*1, *2\) slightly along the \textit{positively} oriented boundary, etc., and we obtain a similar but different operation. A similar matter occurs for the definition of \(\mu\). It is possible and might be desirable to develop this point in full generalities, but here we avoid it for simplicity. Our convention, in particular the choice of \(\ast\) and \(\bullet\) in Figure 2, follows that of Massuyeau and Turaev [49].

5 Dehn twists

Let \(S\) be an oriented surface and \(C \subset S \setminus \partial S\) a simple closed curve. The right handed Dehn twist along \(C\), denoted by \(t_C\), is a diffeomorphism of the surface as in Figure 3. By definition, a Dehn twist is \textit{local} in the sense that the support of \(t_C\) is contained in a regular neighborhood of \(C\). Dehn twists play a fundamental role in study of the mapping class group from combinatorial group theory. For example, they give a generating set for the group, cf. Dehn [10], Lickorish [44] and Humphries [22], and a finite presentation of the group can be given in terms of Dehn twists. The explicit presentation given first was Wajnryb [80] based on a result of Hatcher-Thurston [21], see also Matsumoto [52] and Gervais [16].

In this section, we introduce an invariant of unoriented closed curves on \(S\), and using this invariant we give a formula for the image of \(t_C\) by the completed Dehn-Nielsen homomorphism \((3.3)\). The formula naturally leads us to introducing the \textit{generalized Dehn twist} along an unoriented loop which are not necessarily simple. This generalization takes value in \(A(S, E)\), a group introduced in \((3.4)\). We can ask whether a generalized Dehn twist comes from a diffeomorphism of the surface. We partially give a negative answer to this question.
5.1 The logarithms of Dehn twists

Let $S$ and $E$ be as in §3.3. Recall that $S^* = S \setminus (E \setminus \partial S)$. The homotopy set $\hat{\pi}(S^*) = [S^1, S^*]$ has an involution which maps each $\gamma \in \hat{\pi}(S^*)$ to $\overline{\gamma}$, the loop $\gamma$ with the reversed orientation. An unoriented loop on $S^*$ means an element of the quotient set $\hat{\pi}(S^*)/(\gamma \sim \overline{\gamma})$. We say that a (based) oriented loop $x$ on $S^*$ represents an unoriented loop $\gamma$ on $S^*$ if a suitable lift of $\gamma$ to $[S^1, S^1]$ equals (the homotopy class of) $x$. As in [14] we often identify an unoriented loop on $S^*$ with its image. Likewise we use the word “generic” for unoriented loops with the same meaning as before.

Let us consider the formal power series $L(t) := (1/2)(\log t)^2 \in \mathbb{Q}[[t - 1]]$, where $\log t = \sum_{n=1}^{\infty}((-1)^{n-1}/n)(t - 1)^n$.

**Definition 5.1.** Let $\gamma$ be an unoriented loop on $S^* \setminus \partial S$. Take a base point $q \in S^*$ on the connected component of $S^*$ containing $\gamma$, and a based oriented loop $x \in \pi_1(S^*, q)$ representing $\gamma$. Set

$$L(\gamma) := |L(x)| \in \hat{\pi}(S^*)(2).$$

Here $| \cdot | : \mathbb{Q} \pi_1(S^*, q) \to \hat{\pi}(S^*)$ is the map introduced in §1.5. Since $L(t) = (1/2)(t - 1)^2 + (\text{higher term})$, we have $L(x) \in F_2 \mathbb{Q} \pi_1(S^*, q)$ and $|L(x)| \in \hat{\pi}(S^*)(2)$. The element $L(\gamma)$ does not depend on the choice of $q$ and $x$, because the operation $| \cdot |$ is conjugate invariant and $L(t) = L(t^{-1})$.

We show that the invariant $L(\gamma)$ gives rise to an element of $A(S, E)$. Recall from §3.2 the Lie algebra $\text{Der}(\mathbb{Q}I\!I\!S|_E)$ and its Lie subalgebra $\text{Der}_{\Delta}(\mathbb{Q}I\!I\!S|_E)$. By Proposition 4.11 (1) (see also 4.17), the Lie algebra homomorphism (4.5) induces a Lie algebra homomorphism

$$\sigma: \hat{\pi}(S^*) \to \text{Der}(\mathbb{Q}I\!I\!S|_E). \quad (5.1)$$

We claim that for any unoriented loop $\gamma$ on $S^* \setminus \partial S$, the derivation $\sigma(L(\gamma))$ belongs to $\text{Der}_{\Delta}(\mathbb{Q}I\!I\!S|_E)$. To see this, take an oriented loop $\alpha$ representing $\gamma$. Let $\ast_0, \ast_1 \in E$ and take a path $\beta$ from $\ast_0$ to $\ast_1$, and assume that $\alpha$ and $\beta$ are in general position. It is sufficient to show that $\Delta \sigma(L(\gamma))\beta = (\sigma(L(\gamma))) \otimes 1 + 1 \otimes \sigma(L(\gamma)) \Delta \beta$. For $n \geq 0$, we have $\sigma(\alpha^n)\beta = \sum_{p \in \alpha \cap \beta} \beta_{\ast_0 p} \alpha_p f'(\alpha_p) \beta_{p \ast_1}$, thus for any formal power series $f(t) \in \mathbb{Q}[[t - 1]]$ we have

$$\sigma(f(\alpha))\beta = \sum_{p \in \alpha \cap \beta} \beta_{\ast_0 p} \alpha_p f'(\alpha_p) \beta_{p \ast_1}.$$

Here $f'(t)$ is the derivative of $f(t)$. In particular, since $L'(t) = (\log t)/t$,

$$\sigma(L(\gamma))\beta = \sum_{p \in \alpha \cap \beta} \varepsilon(p; \alpha, \beta) \beta_{\ast_0 p} (\log \alpha_p) \beta_{p \ast_1}.$$
On the other hand, $\Delta(\log \alpha_p) = \log \alpha_p \otimes 1 + 1 \otimes \log \alpha_p \in \mathbb{Q}[\mathbb{S}^* \times \mathbb{S}^*]$. Thus

$$\Delta \sigma(L(\gamma)) \beta = \sum_{p \in \alpha \cap \beta} \epsilon(p; \alpha, \beta)(\beta_{*,p} \otimes \beta_{*,p})(\log \alpha_p \otimes 1 + 1 \otimes \log \alpha_p)(\beta_{p,1} \otimes \beta_{p,1})$$

$$= (\sigma(L(\gamma)) \otimes \beta + \beta \otimes (\sigma(L(\gamma))) \Delta \beta,$$

as was to be shown.

**Lemma 5.2** (33). The derivation $\sigma(L(\gamma)) \in \text{Der}(\mathcal{QIIS}|_E)$ satisfies the three assumptions of Lemma 3.2.

For the proof of this Lemma we refer to (33) Lemma 5.1.1. We remark that for any $*_0, *_1 \in E$ we can take $\nu = 2$ in the assumption (3), and this corresponds to the following fact. Using the intersection form $(\cdot) : H_1(S^*) \otimes H_1(S^*, \partial S^*) \to \mathbb{Q}$ on the surface, we can assign any path $\gamma \in \text{IIS}(*_0, *_1)$ a skew-symmetric, the square of the Dehn twist along a core curve $C$ of $\gamma$. The condition (1) is automatic (see the end of 3.2), and (2) follows from $\sigma(\alpha) \beta = 0$. The condition (2) is automatic (see the end of 3.2), and (3) follows from $\sigma(L(\gamma)) \in \text{Der}_\Delta(\mathcal{QIIS}|_E)$.

Now we have finished the main construction in this section. The reason we are interested in $\exp(\sigma(L(\gamma)))$ comes from the following result.

**Theorem 5.3** (33). Let $S$ and $E$ be as in (33) and $C$ a simple closed curve on $S^* \setminus \partial S$, where $S^* = S \setminus (E \setminus \partial S)$. Then we have

$$\hat{\Delta}N(t_C) = \exp(\sigma(L(C))) \in A(S, E).$$

Using a toy model, we illustrate how the formula in Theorem 5.3 comes up. Let $S$ be an annulus and $E = \{p_0, p_1\}$ as in Figure 4. We consider the Dehn twist along a core $C$ of $S$. Take curves $x$ and $y$ as in Figure 4. Then $x$ is a representative of $C$. For $n \geq 0$ we have $\sigma([x^n])x = 0$ and $\sigma([y^n])y = nx^ny$. Thus for any formal power series $f(t) \in \mathbb{Q}[t - 1]$ we have $\sigma([f(x)])x = 0$ and $\sigma([f(x)])y = x f'(x)y$. In particular, since $tL'(t) = \log t$ we have $\sigma(L(C))x = 0$ and $\sigma(L(C))y = (\log x)y$. By the Leibniz rule (4.4) we obtain $\exp(\sigma(L(C)))x = x$ and $\exp(\sigma(L(C)))y = xy$. On the other hand, clearly $t_C(x) = x$ and $t_C(y) = xy$. Since $x$ and $y$ generate the fundamental groupoid $\text{IIS}|_E$, we obtain $\exp(\sigma(L(C))) = \hat{\Delta}N(t_C)$. In fact, this is an essential part of the proof; Theorem 5.3 for general $S$ is proved by the theorem for an annulus and Proposition 3.6.
As an immediate consequence of Theorem 5.3, for any $n > 0$ the Dehn-Nielsen image $\hat{\DN}(t_C)$ has a canonical $n$-th root: $\hat{\DN}(t_C)^{1/n} = \exp((1/n)\sigma(L(C)))$. Also, for any $a \in \mathbb{Q}$ we can consider the rational Dehn twist $(t_C)^a := \exp(a\sigma(L(C))) \in A(S,E)$. Suppose a generic path $\ell$ from $p_0 \in E$ to $p_1 \in E$ intersects $C$ transversely and $\ell \cap C = \{p\}$. We orient $C$ so that $\varepsilon(p; C, \ell) = +1$ and define $\eta \in \pi_1(S, p)$ to be a oriented loop $C$ based at $p$. Setting $\eta^a = \exp(a \log \eta) \in \hat{\mathbb{Q}}\pi_1(S, p)$ as in §3.3 we have
\begin{equation}
(t_C)^a \ell = \ell_{p_0 p} \eta^a \ell_{p_1 p}.
\end{equation}

**Remark 5.4.** Theorem 5.3 was originally proved in [31] for a compact surface with one boundary component. The formulation and proof in [31] use a symplectic expansion of the fundamental group of the surface. See also §6 and §8. Massuyeau and Turaev [49] proved a similar result from another point of view. See also Remark 5.6.

### 5.2 Generalized Dehn twists

Let $S$ and $E$ be as in §3.3. Motivated by Theorem 5.3 we introduce the following generalization of Dehn twists.

**Definition 5.5 ([33]).** Let $\gamma$ be an unoriented loop on $S^* \setminus \partial S$. We define the *generalized Dehn twist* along $\gamma$ to be $t_\gamma := \exp(\sigma(L(\gamma))) \in A(S,E)$.

**Remark 5.6.** Generalized Dehn twists were introduced first for a compact surface with one boundary component [41], based on a main result of [31]. Then the definition for any oriented surfaces and their fundamental groupoids was given as above. In [49] Massuyeau and Turaev gave a similar construction. They introduced the notion of a *Fox pairing* on the group ring of a group, which generalizes the homotopy intersection form in [4.3] and worked in a more general framework. Massuyeau and Turaev further discussed generalized Dehn twists for closed surfaces.
We state two properties of generalized Dehn twists. First of all, if $\gamma$ is an unoriented loop on $S^* \setminus \partial S$ and $n \geq 0$ is an integer, we can consider the $n$-th power of $\gamma$, denoted by $\gamma^n$. Then

$$t_\gamma^n = (t_\gamma)^{n^2}.$$  \hspace{1cm} (5.3)

This follows from $L(t^n) = n^2 L(t)$. Secondly, we have the following.

**Proposition 5.7.** For any $\ast_i \in E \cap \partial S$, $1 \leq i \leq 4$, and $u \in \hat{\Pi}S(\ast_1, \ast_2)$, $v \in \hat{\Pi}S(\ast_3, \ast_4)$, we have

$$\kappa(t_\gamma(u) \hat{\otimes} t_\gamma(v)) = (t_\gamma \hat{\otimes} t_\gamma)\kappa(u \hat{\otimes} v).$$

Namely, $t_\gamma$ preserves the intersections of two paths on $S$. This is first proved by Massuyeau and Turaev for the homotopy intersection form $\eta$, see [49] Lemma 8.2. The proof of Proposition 5.7 follows the same line as the proof of [49] Lemma 8.2 by Massuyeau and Turaev. Indeed, we first prove that

$$\kappa(\sigma(x)u, v) + \kappa(u, \sigma(x)v) = \sigma(x)\kappa(u, v)$$  \hspace{1cm} (5.4)

for any $u \in \PiS(\ast_1, \ast_2)$, $v \in \PiS(\ast_3, \ast_4)$, and $x \in \hat{\pi}(S)$ (see [49] Lemma 7.4). Here, $\sigma(x)(a \otimes b) = (\sigma(x)a) \otimes b + a \otimes (\sigma(x)b)$ for $a \in \PiS(\ast_1, \ast_4)$, $b \in \PiS(\ast_2, \ast_3)$. The equation (5.4) naturally induces an equality on completions.

Putting $x = L(\gamma) \in \hat{\pi}(S^*)$, we compute

$$(t_\gamma \hat{\otimes} t_\gamma)\kappa(u \hat{\otimes} v) = (e^{D} \hat{\otimes} e^{D})\kappa(u \hat{\otimes} v) = \sum_{r \geq 0} \frac{1}{r!} D^r \kappa(u \hat{\otimes} v))$$

$$= \sum_{r \geq 0} \frac{1}{r!} \sum_{i=0}^{r} \binom{r}{i} \kappa(D^i u \hat{\otimes} D^{r-i} v) = \sum_{i,j \geq 0} \frac{1}{i! j!} \kappa(D^i u \hat{\otimes} D^j v) = \kappa(e^{D} u \hat{\otimes} e^{D} v),$$

where $D = \sigma(L(\gamma))$. This proves Proposition 5.7.

We say that $t_\gamma$ is **realizable as a diffeomorphism**, or **realizable** in short, if $t_\gamma$ is in the image of the completed Dehn-Nielsen homomorphism (3.3). For example, if $C$ is a simple closed curve on $S^* \setminus \partial S$ and $n \geq 0$ is an integer, $t_{C^n}$ is realizable by Theorem 5.3 and (5.3).

**Question.** For which unoriented loop $\gamma$ on $S^* \setminus \partial S$, is $t_\gamma$ realizable as a diffeomorphism?

To study this question, we confine ourselves mainly to the case that $\hat{DN}$ is injective. Then if $t_\gamma$ is realizable, there exists uniquely up to isotopy an orientation diffeomorphism $\varphi$ of $S$ fixing $E \cup \partial S$ pointwise such that $\hat{DN}(\varphi) = t_\gamma$. We call $\varphi$ a **representative** for $t_\gamma$. Generalized Dehn twists are local in the following sense.
Theorem 5.8 ([44], [33]). Suppose $S$ and $E$ satisfy the assumption of Theorem 3.3. Let $\gamma$ be an unoriented loop on $S \setminus \partial S$, and suppose $t_\gamma$ is realizable as a diffeomorphism. Then there is a representative of $t_\gamma$ whose support lies in a regular neighborhood of $\gamma$.

Here the support of a diffeomorphism $\varphi: S \to S$ is the closure of the set $\{x \in S; \varphi(x) \neq x\}$.

5.3 Criterion using the self intersection

We give an application of the operation $\mu$ in §4.4 to generalized Dehn twists. Assume that $S$ and $E$ satisfy the assumption of Theorem 3.3. Let $\gamma$ be an unoriented loop on $S \setminus \partial S$. Suppose $t_\gamma$ is realizable as a diffeomorphism and $\varphi$ is a representative of $t_\gamma$. Let $*_0, *_1 \in E \subset \partial S$ be distinct points and $\ell$ a simple path from $*_0$ to $*_1$. Since $\mu$ maps simple paths to zero and any diffeomorphism preserves the simplicity of paths, for any $n \geq 0$ we have $\mu(\varphi^n(\ell)) = \mu(\exp(n\sigma(L(\gamma)))) = 0$. Therefore, we must have $\mu(\sigma(L(\gamma))\ell) = 0 \in \widehat{\mathbb{Q}}\widehat{\pi}(S)$. (5.5)

This observation is useful to detect the non-realizability of generalized Dehn twists. Using our cut and paste arguments in §3.5, let us consider this in a more refined form.

Let $N \subset S \setminus \partial S$ be a connected compact subsurface which is a neighborhood of $\gamma$. First of all we give the following variant of Proposition 3.7. Let $N \subset S^* \setminus \partial S$ be a connected compact subsurface with non-empty boundary, which is not diffeomorphic to the disk. Assume that the inclusion homomorphism of fundamental groups $\pi_1(N) \to \pi_1(S)$ is injective. We number the components of $\partial N$ as $\partial N = \bigsqcup_{i=1}^n C_i$. For rational numbers $a_i, 1 \leq i \leq n$, we set $F(a_1, \ldots, a_n) := \sum_{i=1}^n a_i L(C_i) \in \widehat{\mathbb{Q}}\widehat{\pi}(N)$. Note that since $C_i$ are disjoint, the derivations $L(C_i)$ commute with each other.

Proposition 5.9 ([33]). Keep the assumptions as above. Let $U \in A(N, \partial N)$ and $\tilde{U} \in A(S, E \cup \partial N)$ and assume $\tilde{U}(i(u)) = iU(u)$ for any $p_0, p_1 \in \partial N$ and $u \in \widehat{\mathbb{Q}}\widehat{\pi}(p_0, p_1)$. Here $i: \Pi N|_{\partial N} \to \Pi S|_{E \cup \partial N}$ is the inclusion homomorphism. Further assume $\phi(\tilde{U}) = 1$, where $\phi: A(S, E \cup \partial N) \to A(S, E)$ is the forgetful homomorphism. Then there exist rational numbers $a_i = a_i^U \in \mathbb{Q}$, $1 \leq i \leq n$, such that $U = \exp(\sigma(F(a_1, \ldots, a_n)))$.

Morally, this proposition says such $U$ is a product of rational Dehn twists: $U = \prod_{i=1}^n (t_{C_i})^{a_i}$. From the observation (5.5), Theorem 5.8 and Proposition 5.9 we can prove the following theorem.
Figure 5. the Pochhammer contour

\textbf{Theorem 5.10} \cite{34}. \textit{Keep the notations as above and suppose that the inclusion homomorphism} \( \pi_1(N) \rightarrow \pi_1(S) \) \textit{is injective. If the generalized Dehn twist} \( t_\gamma \) \textit{is realizable as a diffeomorphism, we have}

\[ \mu(\sigma(L(\gamma))\ell) = 0 \in \hat{\Pi}N(*_{0},*_{1}) \hat{\otimes} \hat{\pi}(N) \]

\textit{for any distinct points} \( *_{0},*_{1} \in \partial N \) \textit{and any simple path} \( \ell \in \Pi N(*_{0},*_{1}) \).

The following theorem provides many examples of unoriented loops \( \gamma \) such that \( t_\gamma \) is not realizable as a diffeomorphism. The proof is by taking \( N \) to be a regular neighborhood of \( \gamma \) and \( \ell \) a simple path intersecting \( \gamma \) transversely in a single point, and showing that \( \mu(\sigma(L(\gamma))\ell) \neq 0 \).

\textbf{Theorem 5.11} \cite{34}. \textit{Let} \( \gamma \subset S \setminus \partial S \) \textit{be a generic non-simple loop and assume that the inclusion homomorphism} \( \pi_1(N(\gamma)) \rightarrow \pi_1(S) \) \textit{is injective, where} \( N(\gamma) \) \textit{is a closed regular neighborhood of} \( \gamma \). \textit{Then the generalized Dehn twist} \( t_\gamma \) \textit{is not realizable as a diffeomorphism.}

For example, the generalized Dehn twist along a figure eight is not realizable as a diffeomorphism. This is first proved in \cite{41} \cite{33} by a rather ad hoc way. Here we say that an oriented generic loop \( \gamma \) is a \textit{figure eight} if \( \gamma \) has a single self intersection and the inclusion homomorphism \( \pi_1(N(\gamma)) \rightarrow \pi_1(S) \) is injective. For another example, suppose that \( \gamma \) is generic and non-simple, and the inclusion map \( N(\gamma) \hookrightarrow S \) is a homotopy equivalence. Then \( t_\gamma \) is not realizable as a diffeomorphism. This shows that \textit{locally}, a generalized Dehn twist is never realized as a diffeomorphism. On the other hand, let \( \gamma \) be an unoriented loop shown in Figure 5. Since \( \pi_1(N(\gamma)) \rightarrow \pi_1(S) \) is not injective, Theorem 5.11 cannot be applied. However, if \( N \) is a neighborhood of \( \gamma \) diffeomorphic to a pair of pants as shown in Figure 5, and \( \pi_1(N) \rightarrow \pi_1(S) \) is injective, then we can apply Theorem 5.10 to conclude that \( t_\gamma \) is not realizable as a diffeomorphism.

We end this section with a conjecture about the characterization of simple closed curves.
Conjecture. Suppose $t_{\gamma}$ is realizable as a diffeomorphism. Then $\gamma$ is homotopic to a power of a simple closed curve.

6 Classical theory revisited

In this section we reconsider the classical Torelli-Johnson-Morita theory for the surface $\Sigma = \Sigma_{g,1}$, $g > 0$, in [2]. Let $\Gamma_g^+ := \prod_{k=1}^{\infty} \mathbb{H}_{g,1}(k) \otimes \mathbb{Q}$ be the degree completion of the target of the Johnson homomorphisms tensored by the rationals $\mathbb{Q}$. Massuyeau's total Johnson map $\tau^\theta$ associated to a symplectic expansion $\theta$ of the group $\pi = \pi_1(\Sigma, \ast)$, introduced by Massuyeau [47], is an embedding of the Torelli group $I_{g,1}$ into the pro-nilpotent Lie algebra $L_g^+$. We introduce a Lie subalgebra $L_g^+(\Sigma)$ of the completed Goldman Lie algebra $\hat{Q}\hat{\pi}(\Sigma)$, and decompose the map $\tau^\theta$ into a natural embedding of the group $I_{g,1}$ into the pro-nilpotent Lie algebra $L_g^+(\Sigma)$ and an isomorphism of Lie algebras $\lambda^\theta: L_g^+(\Sigma) \rightarrow L_g^+$ induced by a tensorial description of $\hat{Q}\hat{\pi}(\Sigma)$ through the symplectic expansion $\theta$. Then we can give an algebraic description of the geometric constraint on the Johnson image in §6.3. In particular, we show that the Morita traces are recovered from this geometric constraint. In the next section §7 we generalize some of these results to an arbitrary compact oriented surface with non-empty boundary.

6.1 Symplectic expansions

We begin by considering a tensorial description of the completed group ring $\hat{Q}\pi$, where $\pi = \pi_1(\Sigma, \ast)$. As in §1.3, let $H = H_1(\Sigma; \mathbb{Q}) = H^1 \otimes_{\mathbb{Z}} \mathbb{Q}$ be the first rational homology group of the surface $\Sigma$, and let $T := \prod_{n=0}^{\infty} H^\otimes_n$ be the completed tensor algebra generated by $H$. The algebra $T$ has the decreasing filtration given by $T_p := \prod_{n \geq p} H^\otimes_n$, $p \geq 0$, and is a complete Hopf algebra with the coproduct $\Delta$ given by $\Delta(X) = X \otimes 1 + 1 \otimes X$ for $X \in H$, and the antipode $\iota$ given by $\iota(X) = -X$ for $X \in H$. Note that the subset $1 + T_1$ is a subgroup of the multiplicative group of the algebra $\hat{T}$. For the rest of this chapter we omit the symbol $\otimes$ for the multiplication in the algebra $\hat{T}$. For example, we denote $\omega = \sum_{i=1}^n A_i B_i - B_i A_i \in H^\otimes_2 \subset H^\otimes_2 \subset \hat{T}$ for the symplectic form (see [22]). Massuyeau [17] introduced the notion of a symplectic expansion, which is a group like expansion of the free group $\pi = \pi_1(\Sigma, \ast) \cong F_{2g}$ satisfying the symplectic condition (4) stated below. Let $\zeta \in \pi$ be the based loop parallel to the negatively oriented boundary of $\Sigma$.

Definition 6.1 (Massuyeau [17]). A map $\theta: \pi \rightarrow 1 + T_1$ is called a symplectic expansion of $\pi$ if
(1) The map \( \theta : \pi \to 1 + \hat{T}_1 \) is a group homomorphism.

(2) For any \( x \in \pi \) we have \( \theta(x) \equiv 1 + [x] \mod \hat{T}_2 \).

(3) For any \( x \in \pi \) the element \( \theta(x) \) is group-like, i.e., \( \Delta \theta(x) = \theta(x) \hat{\otimes} \theta(x) \).

(4) We have \( \theta(\zeta) = \exp(\omega) = \sum_{n=0}^{\infty} (1/n!) \omega^\otimes n \).

Let \( \hat{L} \subset \hat{T} \) be the completed free Lie algebra generated by \( H \). In other words, \( \hat{L} \) is the set of primitive elements of the complete Hopf algebra \( \hat{T} \), \( \hat{L} = L(\hat{T}) \). For a map \( \theta \) satisfying the condition (1)(2), we denote \( \ell : \hat{\theta} \).

Let \( \psi \) be the completed free Lie algebra generated by \( H \). In other words, \( \hat{L} \) is the set of primitive elements of the complete Hopf algebra \( \hat{T} \), \( \hat{L} = L(\hat{T}) \). For a map \( \theta \) satisfying the condition (3), then \( \ell : \) takes values in \( \hat{L} \). In general, we have \( \ell(\zeta) = \omega + (\text{higher term}) \). The condition (4) is equivalent to \( \ell(\zeta) = \omega \), i.e., all the higher terms vanish, which we call the symplectic condition. This seems a quite severe condition, however, symplectic expansions do exist and there are infinitely many. The first example (with real coefficients) was given by Kawazumi [30] via iterated integrals and the Green operator, and Massuyeau [47] gave the second example using the LMO functor. There is also a purely combinatorial construction by Kuno [42].

Fix a symplectic expansion \( \theta \). Then \( \theta \) induces a filter-preserving isomorphism \( \theta : \hat{Q} \pi \to \hat{T} \) of complete Hopf algebras. Moreover, from the condition (4) we have an isomorphism \( \theta : (\hat{Q}, \hat{Q}(\zeta)) \to (\hat{T}, \hat{Q}[\omega]) \) of pairs of complete Hopf algebras. Here \( \langle \zeta \rangle \) is the infinite cyclic group generated by \( \zeta \) and \( Q[\omega] \) is a formal power series ring generated by a primitive element \( \omega \).

6.2 The Lie algebra of symplectic derivations

We recall the definition of the Lie algebra of symplectic derivations. First we make a couple of remarks about the intersection form on the surface \( \Sigma \). The first homology group \( H \) is equipped with a skew symmetric non-degenerate bilinear form \( (\cdot, \cdot) : H \otimes H \to \mathbb{Q} \) called the intersection form. We identify \( H \) and its dual \( H^* \) by \( H \cong H^* \), \( X \mapsto (Y \mapsto (Y \cdot X)) \).

A symplectic basis of \( H \) is a subset \( \{A_i, B_i\}_{i=1}^{g} \subset H \) satisfying \( (A_i \cdot B_j) = \delta_{ij} \), \( (A_i \cdot A_j) = (B_i \cdot B_j) = 0 \). Then the symplectic form \( \omega \) is given by \( \omega = \sum_{i=1}^{g} A_i B_i - B_i A_i \).

By definition, the Lie algebra of symplectic derivations, denoted by \( \mathfrak{a}^\omega_{g} \) or \( \text{Der}_\omega(\hat{T}) \), is the Lie algebra of continuous derivations on the algebra \( \hat{T} \) annihilating \( \omega \). Since the algebra \( \hat{T} \) is generated by the degree 1 part as a complete algebra, the restriction

\[
\mathfrak{a}^\omega_{g} \to \text{Hom}(H, \hat{T}) = H^* \otimes \hat{T} = H \otimes \hat{T} = \hat{T}_1, \quad D \mapsto D|_H
\]
is injective. It is easy to show that the image coincides with \( \text{Ker}(\cdot, \cdot): H \otimes \hat{T} \to \hat{T}_1 \). In other words, \( a^-_g \) is identified with the space of cyclically invariant tensors. If we define a homogeneous \( \mathbb{Q} \)-linear map \( N: \hat{T} \to \hat{T} \) by \( N(X_1 \cdots X_m) = \sum_{i=1}^m X_i \cdots X_m X_1 \cdots X_{i-1} \), for \( m \geq 1, X_1, \ldots, X_m \in H \), and \( N|_{H \otimes 0} = 0 \), then we have an identification

\[
a^-_g = \prod_{m=1}^{\infty} N(H^{\otimes m}). \tag{6.3}
\]

By a straightforward computation, we have the following.

**Lemma 6.2.** Under the identification (6.3), the Lie bracket of \( a^-_g \) is given by

\[
\begin{align*}
[N(X_1 \cdots X_m), N(Y_1 \cdots Y_n)] &= -\sum_{i,j} (X_i \cdot Y_j) N(X_{i+1} \cdots X_m X_1 \cdots X_{i-1} Y_{j+1} \cdots Y_n Y_1 \cdots Y_{j-1}),
\end{align*}
\]

where \( m, n \geq 1 \) and \( X_i, Y_j \in H \).

The Lie subalgebra \( a_g := N(\hat{T}_2) \) is nothing but (the completion of) Kontsevich’s “associative” \( a\). \[40\]

Let \( l_g \) be the Lie subalgebra of \( a^-_g \) consisting of derivations \( D \) stabilizing the coproduct of \( \hat{T} \) in the sense that \( \Delta D = (D \otimes 1 + 1 \otimes D) \Delta \). This condition is equivalent to \( D(H) \subset \hat{\mathcal{L}} \), thus the restriction of (6.4) to \( l_g \) gives an identification \( l_g = \text{Ker}(\cdot, \cdot): H \otimes \hat{\mathcal{L}} \to \hat{\mathcal{L}} \). Moreover, we have \( \text{Ker}(\cdot, \cdot): H \otimes \hat{\mathcal{L}} \to \hat{\mathcal{L}} = N(\hat{\mathcal{L}} \otimes \hat{\mathcal{L}}) \) (see \[31\] Lemma 2.7.2). The Lie algebra \( l_g \) is the degree completion of Kontsevich’s “Lie” \( \ell_g \) \[40\]. It should be remarked that the Lie algebra \( \ell_g \) was introduced earlier by Morita \[53\] \[54\] as a target of the Johnson homomorphisms. Let \( l^+_g \) be the ideal of \( l_g \) consisting of derivations \( D \) such that \( D(H) \subset \hat{\mathcal{L}} \cap \hat{T}_2 \). In fact, the Lie algebra \( l^+_g \) is nothing but the completion of the Lie algebra \( \bigoplus_{k=1}^{\infty} \mathfrak{b}_{g,1}^Z(k) \otimes \mathbb{Q} \) (see \[42\]).

6.3 Algebraic interpretation of the Goldman bracket

Let \( \theta \) be a symplectic expansion of \( \pi \). Consider the \( \mathbb{Q} \)-linear map \( \mathbb{Q}\pi \to N(\hat{T}_1) = a^-_g \), \( \pi \ni x \mapsto N\theta(x) \). Since \( N(uv) = N(vu) \) for any \( u, v \in \hat{T} \), this map factors through to a map \( \lambda_\theta: \mathcal{Q}\hat{\pi}(\Sigma) \to a^-_g \). Namely, if \( x \in \pi \), then \( \lambda_\theta([x]) = N\theta(x) \).

Using a symplectic expansion, we can relate the Goldman Lie algebra with the Lie algebra of symplectic derivations.

**Theorem 6.3 (\[31\]).** (1) The map

\[
-\lambda_\theta: \mathcal{Q}\hat{\pi}(\Sigma) \to N(\hat{T}_1) = a^-_g
\]
is a filter preserving Lie algebra homomorphism. The kernel is spanned by the class of a constant loop 1 and the image is dense with respect to the $\hat{T}_1$-adic topology.

(2) The following diagram commutes:

\[
\begin{array}{ccc}
\mathbb{Q}\hat{\pi}(\Sigma) \times \mathbb{Q}\pi & \longrightarrow & \mathbb{Q}\pi \\
-\lambda_0 \times \theta & \downarrow & \theta \\
\mathbb{a}_0^+ \times \hat{T} & \longrightarrow & \hat{T}
\end{array}
\]

Here the bottom horizontal arrow is the action by derivations.

Note that the minus sign comes from our convention about the isomorphism \((6.1)\). From Theorem 6.3 (1) the map $\lambda_0$ induces a filtered Lie algebra isomorphism $\mathbb{Q}\hat{\pi}(\Sigma) \cong \mathbb{a}_0^-$. As was shown in \([42]\), Theorem 6.3 holds also for any Magnus expansion satisfying the symplectic condition (4) in Definition \([6.1]\) By definition, we have $\mathbb{a}_0^- = \{ D \in \text{Der}(\hat{T}); \ D(\omega) = 0 \} = \{ D \in \text{Der}(\hat{T}); \ D(e^\omega) = 0 \}$. Hence, from Theorem 6.3 we have a filtration-preserving isomorphism $\sigma: \mathbb{Q}\hat{\pi}(\Sigma) \cong \text{Der}_\partial(\mathbb{Q}\pi)$.

This is Theorem 4.13 for \((S, E) = (\Sigma, \{*\})\).

Now consider the Torelli group $I_{g,1}$. For any $\varphi \in I_{g,1}$, the logarithm of $\hat{\text{DN}}(\varphi)$ converges as an element of $F_1 \text{Der}(\mathbb{Q}\pi)$, since $\hat{\text{DN}}(\varphi)((I\pi)^m) \subset (I\pi)^{m+1}$ for any $m \geq 1$. From the fact $\varphi(\zeta) = \zeta \in \pi$ follows $\log \hat{\text{DN}}(\varphi) \in \text{Der}_\partial(\mathbb{Q}\pi)$. Hence we define the geometric Johnson homomorphism by

\[
\tau := \sigma^{-1} \circ \log \circ \hat{\text{DN}}: I_{g,1} \to \mathbb{Q}\hat{\pi}(\Sigma)(3), \quad \varphi \mapsto \sigma^{-1}(\log \hat{\text{DN}}(\varphi)).
\]

We remark $\hat{\text{DN}}(\varphi) = e^{\sigma(\tau(\varphi))}$ for any $\varphi \in I_{g,1}$. Hence, if $\varphi$ is the right handed Dehn twist along a separating simple closed curve $C \subset \Sigma$, then we have $\tau(t_C) = L(C)$ by Theorem 6.3.

Recall that the action of the group $M_{g,1}$ on $\mathbb{Q}\pi$ preserves the coproduct $\Delta$. Hence $\tau(I_{g,1})$ is included in the stabilizer of $\Delta$, which we denote

\[
L(\Sigma) := \{ u \in \mathbb{Q}\hat{\pi}(\Sigma): (\sigma(u) \hat{\otimes} \sigma(u)) \Delta = \Delta \sigma(u) \}, \quad \text{and}
\]

\[
L^+(\Sigma) := L(\Sigma) \cap \mathbb{Q}\hat{\pi}(\Sigma)(3).
\]

The Lie algebra $L^+(\Sigma)$ is pro-nilpotent, so that the Hausdorff series define a natural group structure on it. Hence the geometric Johnson homomorphism

\[
\tau: I_{g,1} \to L^+(\Sigma)
\]

is an injective group homomorphism.

On the other hand, recall that $t_g = \prod_{k=0}^\infty h_{g,1}(k) \otimes \mathbb{Q}$ and $t_g^+ = \prod_{k=1}^\infty h_{g,1}(k) \otimes \mathbb{Q}$ are exactly the stabilizer of $\Delta$ in $\mathbb{a}_0^+$ and $\mathbb{a}_0^+ := N(\hat{T}_3)$, respectively. Since
\( \theta : \hat{\mathbb{Q}} \pi \xrightarrow{\sim} \hat{T} \) is a filtration-preserving isomorphism of complete Hopf algebras, the isomorphism \(-\lambda_\theta \) induces isomorphisms \(-\lambda_\theta : L^+(\Sigma) \xrightarrow{\sim} \ell_g^+ \) and \(-\lambda_\theta : L(\Sigma) \xrightarrow{\sim} \ell_g \). From the construction, Massuyeau’s total Johnson map \( \tau^\theta \) is decomposed as

\[
\tau^\theta = -\lambda_\theta \circ \tau : \mathcal{I}_{g,1} \to L^+(\Sigma) \xrightarrow{\sim} \ell_g^+.
\]

In particular, the graded quotient of the geometric Johnson homomorphism \( \tau \) is the totality of the (original) Johnson homomorphisms.

Our geometric re-construction of the Johnson homomorphisms leads us to finding a geometric constraint of the Johnson image. We look at the map \( \mu \) as in [49]. It is clear that the action of any \( \varphi \in \mathcal{I}_{g,1} \) preserves the map \( \mu : \hat{\mathbb{Q}} \pi \to \hat{\mathbb{Q}} \pi \otimes \hat{\mathbb{Q}} \pi (\Sigma) \). Hence \( \mu(e^{\operatorname{na}(\tau(\varphi)))}v) = (e^{\operatorname{na}(\tau(\varphi)))} \otimes e^{\operatorname{na}(\tau(\varphi)))})\mu(v) \) for any \( n \in \mathbb{Z} \) and \( v \in \hat{\mathbb{Q}} \pi \). Taking the linear terms in \( n \), we have \( \mu(\sigma(\tau(\varphi))) = \tau(\varphi)\mu(v) \).

This means \( (\nabla \otimes 1)(v \otimes \delta(\tau(\varphi))) = 0 \) from Theorem 6.3 (3), and \( \delta(\tau(\varphi)) = 0 \) from the isomorphism [49]. Hence we obtain the following theorem.

**Theorem 6.4** ([34]).

\[
\delta \circ \tau = 0 : \mathcal{I}_{g,1} \to L^+(\Sigma) \to \hat{\mathbb{Q}} \pi (\Sigma) \otimes \hat{\mathbb{Q}} \pi (\Sigma).
\]

### 6.4 The Turaev cobracket and the Morita trace

From Theorem 6.3 the space \( a_\pi^g \) has a structure of a complete Lie bialgebra with a (\( \theta \)-dependent) Lie cobracket \( \delta^\theta := ((-\lambda_\theta) \otimes (-\lambda_\theta)) \circ \delta \circ (-\lambda_\theta)^{-1} \). In this subsection we shall study the Laurent expansion of \( \delta^\theta \). The key ingredients is a tensorial description of the homotopy intersection form (see [34]) due to Massuyeau and Turaev [49]. As in [49] we see that \( \eta \) extends to \( \eta : \hat{\mathbb{Q}} \pi \otimes \hat{\mathbb{Q}} \pi \to \hat{\mathbb{Q}} \pi \). Let \( \varepsilon : \hat{T} \to \hat{T}/\hat{T}_1 \equiv \mathbb{Q} \) be the augmentation map. Define a \( \mathbb{Q} \)-bilinear map \( \bullet : \hat{T} \otimes \hat{T}_1 \to \hat{T} \) by

\[
X_1 \cdots X_m \bullet Y_1 \cdots Y_n := (X_m \cdot Y_1)X_1 \cdots X_{m-1}Y_2 \cdots Y_n \in H^\otimes m+n-2
\]

for any \( m, n \geq 1 \), and \( X_i, Y_j \in H \). Here \( (X_m \cdot Y_1) \in \mathbb{Q} \) is the intersection pairing of \( X_m \) and \( Y_1 \in H \). A \( \mathbb{Q} \)-linear map \( \rho : \hat{T} \otimes \hat{T} \to \hat{T} \) is defined by

\[
\rho(a \bullet b) = \rho(a, b) := (a - \varepsilon(a)) \bullet (b - \varepsilon(b)) + (a - \varepsilon(a))s(\omega)(b - \varepsilon(b)) \quad (6.5)
\]

for any \( a \) and \( b \) in \( \hat{T} \), where \( s(z) \) is the formal power series

\[
s(z) = \frac{1}{e^{-z} - 1} + \frac{1}{z} = -\frac{1}{2} - \sum_{k \geq 1} \frac{B_{2k}}{(2k)!} z^{2k-1} = -\frac{1}{2} \frac{z}{12} + \frac{z^3}{720} - \frac{z^5}{30240} + \cdots.
\]

Here \( B_{2k} \)'s are the Bernoulli numbers.
Theorem 6.5 (\[49\]). Let \( \theta : \pi \to \hat{T} \) be a symplectic expansion. Then the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{Q}_\pi \otimes \mathcal{Q}_\pi & \rightarrow & \mathcal{Q}_\pi \\
\theta \otimes \theta & \downarrow & \theta \\
\hat{T} \otimes \hat{T} & \rightarrow & \hat{T}.
\end{array}
\]

Let us fix a symplectic expansion \( \theta \). We define \( \mathbb{Q} \)-linear maps \( \kappa^\theta : \hat{T} \otimes \hat{T} \to \hat{T} \otimes \hat{T} \) and \( \mu^\theta : \hat{T} \to \hat{T} \otimes \mathfrak{a}_g^- \) so that the diagrams
\[
\begin{array}{ccc}
\mathcal{Q}_\pi \otimes \mathcal{Q}_\pi & \rightarrow & \mathcal{Q}_\pi \otimes \mathcal{Q}_\pi \\
\theta \otimes \theta & \downarrow & \theta \\
\hat{T} \otimes \hat{T} & \rightarrow & \kappa^\theta \\
\hat{T} \otimes \hat{T} & \rightarrow & \mu^\theta \otimes \mathfrak{a}_g^-.
\end{array}
\]
commute. From Proposition 1.7 and Theorem 6.5, the map \( \kappa = \kappa^\theta \) does not depend on the choice of \( \theta \). Explicitly, for \( X, Y \in H \) we have
\[
\kappa(X \otimes Y) = -(1 \otimes 1)((1 \otimes \epsilon)\Delta \rho(X, Y))(1 \otimes 1)
\]
\[
= -(X \cdot Y)(1 \otimes 1) - (1 \otimes \epsilon)\Delta(X \omega Y).
\]
\[\text{(6.6)}\]

On the other hand, the map \( \mu^\theta \) depends on the choice of \( \theta \). By Proposition 4.16 for any \( m \geq 0 \) and \( X_i \in H, 1 \leq i \leq m \), we have
\[
\mu^\theta(X_1 \cdots X_m)
\]
\[
= (1 \otimes (-N)) \sum_{i < j} (X_1 \cdots X_{i-1} \otimes 1) \kappa(X_i, X_j)(X_{j+1} \cdots X_m \otimes X_{i+1} \cdots X_{j-1})
\]
\[
+ \sum_{i=1}^m (X_1 \cdots X_{i-1} \otimes 1) \mu^\theta(X_i)(X_{i+1} \cdots X_m \otimes 1).
\]
\[\text{(6.7)}\]

We consider the Laurent expansion of \( \mu^\theta \). We denote by \( \mu^\theta_{(k)} \) the degree \( k \) part of \( \mu^\theta \). In other words, we have
\[
\mu^\theta(X_1 \cdots X_m) = \sum_{k=-\infty}^{\infty} \mu^\theta_{(k)}(X_1 \cdots X_m), \quad \mu^\theta_{(k)}(X_1 \cdots X_m) \in H^{\otimes (m+k)}
\]
for \( X_i \in H, 1 \leq i \leq m \). We define the homogeneous \( \mathbb{Q} \)-linear map \( \mu^\text{alg} : \hat{T} \to \hat{T} \otimes \mathfrak{a}_g^- \) by
\[
\mu^\text{alg}(X_1 \cdots X_m) = \sum_{i < j} (X_i \cdot X_j)X_1 \cdots X_{i-1} X_{j+1} \cdots X_m \otimes N(X_{i+1} \cdots X_{j-1})
\]
for $m \geq 0$ and $X_i \in H$, $1 \leq i \leq m$. Looking at \[ \text{Eq. 6.7} \] and \[ \text{Eq. 6.10} \] in detail, we have the following. As is announced in \[ \text{Eq. 5.10} \] Remark 7.4.3, this result is obtained independently by Massuyeau and Turaev \[ \text{Eq. 51} \].

**Theorem 6.6** \[ \text{Eq. 32} \] \[ \text{Eq. 51} \]. For any $u \in \hat{T}$ we have

1. $\mu_{(k)}^0(u) = 0$ for $k \leq -3$ and $k = -1$,
2. $\mu_{(-2)}^0(u) = \mu_{(1)}^{\text{alg}}(u)$, and
3. $\mu_{(0)}^0(u) = (-1/2)(1 \otimes N(u))$.

Therefore, for any $u \in \hat{T}$ we can write

$$
\mu^0(u) = \mu_{(1)}^{\text{alg}}(u) - \frac{1}{2}(1 \otimes N(u)) + \mu_{(1)}^{\text{alg}}(u) + \mu_{(2)}^{\text{alg}}(u) + \cdots.
$$

In general, the higher terms $\mu_{(k)}^0(u)$, $k \geq 1$, do depend on the choice of $\theta$.

Now we consider the Lie cobracket $\delta^\theta$. We denote by $\delta_{(k)}^\theta$ the degree $k$ part of $\delta^\theta$, i.e.,

$$
\delta^\theta(N(X_1 \cdots X_m)) = \sum_{k=-\infty}^{\infty} \delta_{(k)}^\theta(N(X_1 \cdots X_m)),
$$

and $\delta_{(k)}^\theta(N(X_1 \cdots X_m)) \in N(H \otimes (m+k))$ for $X_i \in H$, $1 \leq i \leq m$. We define the homogeneous $\mathbb{Q}$-linear map $\delta_{\text{alg}}: a^-_g \to a^-_g \otimes a^-_g$ by

$$
\delta_{\text{alg}}(N(X_1 \cdots X_m)) = -\sum_{i<j}(X_i \cdot X_j) \left\{ N(X_{i+1} \cdots X_{j-1}) \otimes N(X_{i+1} \cdots X_m X_1 \cdots X_{i-1}) \right\}
$$

for $m \geq 1$ and $X_i \in H$, $1 \leq i \leq m$. We call $\delta_{\text{alg}}$ Schedler’s cobracket since it was introduced by Schedler \[ \text{Eq. 74} \]. By Proposition \[ \text{Eq. 4.11} \] and Theorem \[ \text{Eq. 6.10} \] for any $u \in a^-_g$ we have

$$
\delta^\theta(u) = \delta_{\text{alg}}(u) + \delta_{(1)}^\theta(u) + \delta_{(2)}^\theta(u) + \cdots,
$$

and in general the higher terms $\delta_{(k)}^\theta(u)$, $k \geq 1$, depend on the choice of $\theta$. As a corollary of Theorem \[ \text{Eq. 6.4} \] we have

$$
\delta_{\text{alg}} \circ \tau = 0: \bigoplus_{k=1}^{\infty} \text{gr}^k(I_{g,1}) \to \bigoplus_{k=1}^{\infty} \text{b}^g_{g,1}(k) \to a^-_g \otimes a^-_g.
$$

Finally we show that Schedler’s cobracket $\delta_{\text{alg}}$ restricted to $I_{g,1}^+$ recovers the Morita traces of all degrees $\text{Tr}_k: (I_{g,1}^+(k+2)) := \text{b}^g_{g,1}(k) \otimes \mathbb{Q} \to S^kH$, $k \geq 3$ (see \[ \text{Eq. 2.2} \]). Here $S^kH$ is the $k$-th symmetric power of $H$. Let $p_1: a^-_g =
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\[ \prod_{m=1}^{\infty} N(H^{\otimes m}) \to N(H^{\otimes 1}) = H \] be the first projection, \( i : a_{g}^{\sim} = \prod_{m=1}^{\infty} N(H^{\otimes m}) \hookrightarrow \prod_{m=1}^{\infty} H^{\otimes m} = \hat{T} \) the inclusion map, and \( \varpi : \hat{T} \to \hat{\text{Sym}}(H) := \prod_{m=0}^{\infty} S^{m}(H) \) the natural projection. We define

\[ \mathfrak{s} := \varpi \circ (p_{1} \otimes i) : a_{g}^{\sim} \otimes a_{g}^{\sim} \to H \otimes \hat{T} = \hat{T} \to \hat{\text{Sym}}(H). \]

By some straightforward computation we obtain the following.

**Theorem 6.7** (**[34]**). For any \( k \geq 3 \), we have

\[ \mathfrak{s} \circ \delta_{\text{alg}}|_{(1_{g}^{+})_{(k+2)}} = (-k) \times \text{Tr}_{k} : (1_{g}^{+})_{(k+2)} = h_{g,1}^{\mathbb{Z}}(k) \otimes \mathbb{Q} \to S^{k}H. \]

Thus all the Morita traces are derived from the geometric fact that any diffeomorphism preserves the self-intersection of any curve on the surface. Very recently Enomoto [12] proved that the Enomoto-Satoh traces [13] are inside of Schedler’s cobracket \( \delta_{\text{alg}} \). But we do not know whether they are inside of the Turaev cobracket \( \delta^{0} \) itself or not.

### 7 Compact surfaces with non-empty boundary

Let \( \Sigma_{g,r} \) be a compact connected oriented surface of genus \( g \) with \( r \) boundary components. Now we generalize some of the results in [40] to such a surface with \( r > 0 \). Throughout this section we work over the rationals \( \mathbb{Q} \). In particular, we denote by \( H_{*}(X, A) \) the rational homology group \( H_{*}(X, A; \mathbb{Q}) \) for any pair of spaces \((X, A)\). Let \( S = \Sigma_{g,n+1} \) for some \( g \) and \( n \geq 0 \). Note that the interior of the surface \( S \) has a complete hyperbolic structure. We number the components of the boundary \( \partial S = \bigsqcup_{j=0}^{n} \partial_{j} S \). Choose one point \( *_{j} \) from each \( \partial_{j} S \) to form a finite set \( E := \{ *_{j} \}_{j=0}^{n} \subset \partial S \). Gluing \((n+1)\) copies of the 2-disks on the surface \( S \) along the boundary, we obtain a closed surface \( \overline{S} \equiv \Sigma_{g} \).

In [71] we construct an analogous Lie algebra associated to the surface \( S \) for the degree completion \( a_{g}^{\sim} \) of an enhancement of Kontsevich’s associative \( a_{g}^{\sim} \), where we need an additional data on the inclusion homomorphism \( H_{1}(S) \to H_{1}(\overline{S}) \). In [72] we introduce a Magnus expansion of the groupoid \( \Pi_{S}|_{E} \) satisfying some boundary condition, which induces an isomorphism of Lie algebras from the completed Goldman Lie algebra onto the Lie algebra constructed in [71]. As a consequence of the isomorphism, we obtain Theorem 4.13. The geometric Johnson homomorphism on the largest Torelli group of \( S \) in the sense of Putman [68] is defined to be an embedding of the Torelli group into some pro-nilpotent Lie subalgebra of \( \hat{\mathbb{Q}}\hat{\pi}(S) \) in [73]. Its image is included in the kernel of the Turaev cobracket in a similar way to \( \Sigma_{g,1} \).
7.1 The “associative” Lie algebra for a compact surface

We use similar notation to that in [33] and [31]. Let $H$ be a finite-dimensional $\mathbb{Q}$-vector space. Then let $\hat{T}(H) := \prod_{m=0}^{\infty} H^\otimes m$ be the completed tensor algebra over $H$, and $N = N^H: \hat{T}(H) \to \hat{T}(H)$ the cyclic symmetrizer or the cyclicizer defined by $N|_{H^\otimes 0} := 0$ and $N(X_1 \cdots X_m) := \sum_{i=1}^{m} X_i \cdots X_m X_1 \cdots X_{i-1}$ for $m \geq 1$ and $X_i \in H$. As in [30] we omit the symbol $\otimes$ if it means the product in the algebra $\hat{T}(H)$. The filtered $\mathbb{Q}$-vector space $N(\hat{T}(H)_1)$ is an analogue of the Lie algebra $a^g_y$, the degree completion of an enhancement of Kontsevich’s “associative” Lie algebra $a_g$. The algebra $\hat{T}(H)$ is filtered by the two-sided ideals $\hat{T}(H)_p := \prod_{m \geq p} H^\otimes m$, $p \geq 1$, and constitutes a complete Hopf algebra whose coproduct $\Delta: \hat{T}(H) \to \hat{T}(H) \hat{T}(H)$ is given by $\Delta(X) = X \hat{\otimes} 1 + 1 \hat{\otimes} X$ for any $X \in H$.

Let $S$ be a surface as above. Then the fundamental group $\pi_1(S)$ is a free group of finite rank, where we may choose any point in $S$. The reason why we introduce a Lie algebra structure on the filtered $\mathbb{Q}$-vector space $N(\hat{T}(H_1(S)))$ comes from the following proposition.

**Proposition 7.1** ([33] Corollary 4.3.5.). For any group-like expansion $\theta: \pi_1(S) \to \hat{T}(H_1(S))$, the map $N\theta: \hat{Q}\pi(S) \to N(\hat{T}(H_1(S)))$, given by $(N\theta)(|x|) := N(\theta(x))$ for any $x \in \pi_1(S)$, is injective, and induces an isomorphism of filtered $\mathbb{Q}$-vector space

$$N\theta: \hat{Q}\pi(S) \overset{\cong}{\to} N(\hat{T}(H_1(S))).$$

**Proof.** We simply write $\pi = \pi_1(S)$ and $\hat{\pi} = \hat{T}(S)$. Let $Q\pi^c$ and $\hat{Q}\pi^c$ be the group ring of the group $\pi$ and its completion, respectively, which we regard as left $\mathbb{Q}\pi$-modules by conjugation of the group $\pi$. Since the interior of $S$ has a complete hyperbolic structure, we have a natural injective map $\lambda: \hat{Q}\pi/\mathbb{Q}1 \to H_1(\pi; Q\pi^c)$ introduced in Proposition 3.4.3 [31]. The completion map $Q\pi^c \to \hat{Q}\pi^c$ is injective, so that the induced map $H_1(\pi; Q\pi^c) \to H_1(\pi; \hat{Q}\pi^c)$ is also injective, since $\pi$ is free and so $H_2(\pi; \hat{Q}\pi^c / Q\pi^c) = 0$. The group-like expansion $\theta$ induces an isomorphism $H_1(\pi; \hat{Q}\pi^c) \overset{\cong}{\to} N(\hat{T}(H_1(S)))$ in the context of twisted homology of (complete) Hopf algebras. By straightforward computation in Lemma 5.3.2 [31] using the group-like condition of $\theta$, we can prove that the composite of these maps equals the map $N\theta$. In particular, $N\theta: \hat{Q}\pi(S) \to N(\hat{T}(H_1(S)))$ is injective. Clearly the image of $N\theta$ is dense in $N(\hat{T}(H_1(S)))$, and $N(\hat{T}(H_1(S)))$ is complete with respect to the filtration $\{N(\hat{T}(H_1(S)))_m\}_{m=1}^{\infty}$. As was proved in Lemma 4.3.3 [33], $(N\theta)^{-1}(N(\hat{T}(H_1(S)))_m) = |\mathbb{Q}1 + I\pi^m|$ for any $m \geq 1$. This proves the rest of the assertions of the proposition. \qed
Through the isomorphism $N \theta$ we can consider a Lie algebra structure on $N(\tilde{T}(H_1(S)))$. In this subsection, we will give a candidate for such a structure. As will be stated in the next subsection, the map $-N \theta$ is an isomorphism of Lie algebras for some expansion $\theta$.

Recall $S \cong \Sigma_{g,n+1}$. $\partial S = \bigsqcup_{i=0}^n \partial_i S$ and $\overline{S}$ is a closed surface of genus $g$ obtained from capping the boundary components of $S$ by $(n+1)$ disks. The first homology group $H_1(\overline{S})$ is a symplectic vector space of dimension $2g$. We denote by $\overline{\omega} \in H_1(\overline{S}) \otimes \mathbb{R}$ the symplectic form on it. If $\{A_i, B_i\}_{i=1}^n \subset H_1(\overline{S})$ is a symplectic basis, then we have $\overline{\omega} = \sum_{i=1}^n A_i B_i - B_i A_i$. Let $C_j \in H_1(S)$ be the homology class of a boundary loop on $\partial_j S$ in the positive direction. Consider the inclusion map $i: S \hookrightarrow \overline{S}$. In the homology exact sequence

$$H_2(\overline{S}, S) \xrightarrow{\partial_n} H_1(S) \xrightarrow{i_*} H_1(\overline{S}) \rightarrow 0,$$

the set $\{C_j\}_{j=1}^n$ is a basis of the image $\text{Im}(\partial_n)$. To define our Lie bracket on $N(\tilde{T}(H_1(S)))$, we need to choose a section of the surjection $i_*: H_1(S) \rightarrow H_1(\overline{S})$. We denote by $\text{Sect}(i_*)$ the set of all sections of the surjection $i_*$. Any complex structure of the surface $\overline{S}$ defines a canonical $\mathbb{R}$-valued section of the surjection $i_*: H_1(S; \mathbb{R}) \rightarrow H_1(\overline{S}; \mathbb{R})$ induced by the normalized Abelian integrals of the third kind if we regard the interior of $S$ as a punctured Riemann surface.

Fix a section $s \in \text{Sect}(i_*)$. We denote $A_i^s := s(A_i), B_i^s := s(B_i) \in H_1(S)$ and $\omega_s := s^\otimes 2(\overline{\omega}) = \sum_{i=1}^g A_i^s B_i^s - B_i^s A_i^s \in H_1(S) \otimes \mathbb{R}$. The set $\{A_i^s, B_i^s\}_{i=1}^n \sqcup \{C_j\}_{j=1}^n$ is a basis of $H_1(S)$. Let $u = \sum_{i=1}^g A_i^s u_i^s + \sum_{i=1}^n B_i^s u_i^s + \sum_{j=1}^n C_j v_j^0$ and $v = \sum_{i=1}^g A_i^s v_i^s + \sum_{i=1}^g B_i^s v_i^s + \sum_{j=1}^n C_j v_j^0$ be elements of $N(\tilde{T}(H_1(S)))_1$, where $u_i^s, u_i^s, v_i^s, v_i^s, v_j^0 \in \tilde{T}(H_1(S))$. Then a bracket $[u, v] = [u, v]_s$ of $u$ and $v$ is defined by

$$[u, v]_s := -N \left( \sum_{i=1}^g u_i^s v_i^s - u_i^s v_i^s + \sum_{j=1}^n C_j (u_j^0 v_j^0 - v_j^0 u_j^0) \right) \in N(\tilde{T}(H_1(S)))_1.$$

It is easy to prove that the bracket does not depend on the choice of the symplectic basis $\{A_i, B_i\}_{i=1}^g$ and satisfies the Jacobi identity. We denote by $N(\tilde{T}_1)_s = N(\tilde{T}(H_1(S)))_1$ the Lie algebra $N(\tilde{T}(H_1(S)))_1$ equipped with the Lie bracket $[\cdot, \cdot]$. As Massuyeau and Turaev [50, 51] point out, this Lie algebra structure is related to quiver theory. If $g = 0$ or $n = 0$, then the set $\text{Sect}(i_*)$ is a singleton, and $N(\tilde{T}_1)_s$ is just the Lie algebra of special derivations of the algebra $\tilde{T}$ or that of symplectic derivations, $\mathfrak{a}_\omega$, respectively.

For any compact connected oriented surface $S$ the map $-N \theta$ is a Lie algebra isomorphism if an expansion $\theta$ satisfies some condition, which will be formulated in the next subsection.
7.2 A tensorial description of the Goldman Lie algebra

Let \((S, E)\) be as above. We begin this subsection by introducing the notion of a Magnus expansion of the groupoid \(\Pi S|_E\). In this subsection we simply write \(H = H_1(S), \hat{T} = \hat{T}(H_1(S))\) and so on. If \(M\) is a monoid, then we denote by \(M_E\) the small category such that the set of objects is \(E\), and the set of morphisms from \(*_a\) to \(*_b\), \(0 \leq a, b \leq n\), is defined by \((M_E)(*_a, *_b) = M\). The composite and the unit in the monoid \(M\) induce the composite and the units in the category \(M_E\). For example, we consider the small category \(\hat{T}_E\) and the groupoid \((1 + \hat{T}_1)_E\) over the set \(E\).

**Definition 7.2.** A homomorphism \(\theta : \Pi S|_E \to (1 + \hat{T}_1)_E\) of groupoids over the set \(E\) is a Magnus expansion of the pair \((S, E)\), if the restriction to \(\pi_1(S, *_a)\), \(\theta : \pi_1(S, *_a) \to 1 + \hat{T}_1\), is a Magnus expansion in Definition 2.5 for any \(a\), \(0 \leq a \leq n\).

It is easy to show that a homomorphism \(\theta \colon \Pi S|_E \to (1 + \hat{T}_1)_E\) is a Magnus expansion if \(\theta \colon \pi_1(S, *_b) \to 1 + \hat{T}_1\) is a Magnus expansion of the free group \(\pi_1(S, *_a)\) for some \(b\), \(0 \leq b \leq n\). Any Magnus expansion \(\theta\) induces an isomorphism \(\theta : \Pi S|_E \cong T_E\). Hence, for any two Magnus expansions \(\theta\) and \(\theta'\), there exists a unique derivation \(\hat{u}_0 \in F_1 \text{Der}(\hat{T}_E)\) such that \(\theta' = (\exp \hat{u}_0) \circ \theta : \Pi S|_E \to (1 + \hat{T}_1)_E\). Here \(F_1 \text{Der}(\hat{T}_E)\) is the Lie subalgebra of all derivations \(D\) increasing the filtration on \(\hat{T}_E\) strictly (see \(\S\ 3.2\)). A group-like expansion is defined to be a Magnus expansion whose target is reduced to \(\text{Gr}(\hat{T})_E\), where \(\text{Gr}(\hat{T})\) is the set of group-like elements in the complete Hopf algebra \(\hat{T}\).

Let \(s \in \text{Sect}(i_*)\) be a section of the surjection \(i_*\) as in \(\S\ 7.1\). Then we introduce some boundary condition on a Magnus expansion \(\theta\) with respect to the section \(s\). Let \(\xi_j \in \pi_1(S, *_j), 0 \leq j \leq n\), be a based boundary loop along \(\partial_j S\) in the positive direction. We define the boundary condition with respect to the section \(s\), which we denote by \((\sharp_s)\), by

\[
\theta(\xi_j) = \begin{cases} 
\exp(-\omega_s + C_0) = \exp(-\omega_s - \sum_{j=1}^n C_j), & \text{if } j = 0, \\
\exp(C_j), & \text{if } 1 \leq j \leq n.
\end{cases}
\]

(7.3)

A group-like expansion satisfying the condition \((\sharp_s)\) is a generalization of a symplectic expansion. If we fix a complex structure on the surface \(\overline{S}\) and regard the interior of \(S\) as a punctured Riemann surface, then we can construct a canonical \(\mathbb{R}\)-valued group-like expansion satisfying the condition \((\sharp_s)\) with respect to the canonical section stated above by a similar way to \(\S\ 30\).

Now we can state the following theorem.
The Goldman-Turaev Lie bialgebra and the Johnson homomorphisms

**Theorem 7.3** ([35][51]). If a Magnus expansion \( \theta \) satisfies the condition \((\sharp_s)\), then the map

\[-N\theta : \widehat{Q}(S) \to N(\widehat{T}_1)_s\]

is a Lie algebra isomorphism.

Recall the natural action of \( \widehat{Q}(S) \) on the completion \( \widehat{Q}IS|_E \) and the Lie algebra homomorphism

\[\sigma : \widehat{Q}(S) \to \text{Der}_\theta(\widehat{Q}IS|_E)\]

stated in §4.3. Theorem 4.13 asserts that the map \( \sigma \) is an isomorphism. Here we remark that \( \sigma \) does not preserve the filtrations if \( S \neq \Sigma_{g,1} \). This seems to be related to the diversity of the Torelli groups [68]. A homomorphism similar to \( \sigma \) is constructed for the Lie algebra \( N(\widehat{T}_1)_s \) and the small category \( \widehat{T}_E \). We begin by defining a derivation on \( \widehat{T}_E = (\widehat{T}_E)(\ast_0, \ast_0) \).

Let \( u = \sum_{i=1}^g A_i u_i' + \sum_{i=1}^g B_i u_i'' + \sum_{j=1}^n C_j u_j^0 \) be an element of \( N(\widehat{T}_1)_s \). Then a continuous derivation \( \sigma^0_s(u) \) of the algebra \( \widehat{T} \) is defined by

\[ \sigma^0_s(u)(A_i) := u_i'' \quad \sigma^0_s(u)(B_i) := -u_i'' \quad \text{and} \quad \sigma^0_s(u)(C_j) := u_j^0C_j - C_j u_j^0 \]

for \( 1 \leq i \leq g \) and \( 1 \leq j \leq n \). We define \( u^0_0 := 0 \) for our convenience. Then a continuous derivation \( \sigma_s(u) \) of the small category \( \widehat{T}_E \) is defined by

\[ \sigma_s(u)(v) := -u_0^0v + \sigma^0_s(u)(v) + vu_0^0 \quad (7.4) \]

for any \( v \in (\widehat{T}_E)(\ast_a, \ast_b) = \widehat{T} \), \( 0 \leq a, b \leq n \). Similarly we denote by \( \text{Der}_\theta(\widehat{T}_E) \) the Lie algebra of continuous derivations annihilating \(-\omega_s + C_0 \in (\widehat{T}_E)(\ast_0, \ast_0)\) and \( C_j \in (\widehat{T}_E)(\ast_j, \ast_j), 1 \leq j \leq n \). Then we have a Lie algebra homomorphism

\[ \sigma_s : N(\widehat{T}_1)_s \to \text{Der}_\theta(\widehat{T}_E) \]

By some straightforward computation, we can prove that \( \sigma_s \) is an isomorphism.

**Theorem 7.4** ([35][51]). If \( \theta \) is a Magnus expansion of the pair \((S, E)\) satisfying the condition \((\sharp_s)\), then the diagram

\[
\begin{array}{ccc}
\widehat{Q}(S) & \xrightarrow{\sigma} & \text{Der}_\theta(\widehat{Q}IS|_E) \\
-N\theta & \downarrow & \theta \\
N(\widehat{T}_1)_s & \xrightarrow{\sigma_s} & \text{Der}_\theta(\widehat{T}_E)
\end{array}
\]

commutes.

To prove Theorems 7.3 and 7.4 we consider a group-like expansion \( \bar{\theta} \) obtained by gluing a symplectic expansion of \( \Sigma_{g,1} \) and a special expansion of
Hence we obtain a natural embedding \( \tau: \Pi \Sigma_{0,n+2} |_{E} \to (1 + \hat{T}(H_1(\Sigma_{0,n+2}))_1)_E \) is called a special expansion if it satisfies \( \theta(\xi_j) = e^{C_j} \) for any \( j \geq 0 \). By a similar argument to that in \([31]\) on relative twisted homology we deduce Theorems 7.3 and 7.4 for the expansion \( \theta \). As a corollary of them, we obtain Theorem 4.13, from which Theorems 7.3 and 7.4 for any expansion with the condition \((s)\) follow.

Independently Massuyeau and Turaev \([51]\) give a proof of Theorems 7.3 and 7.4 in the context of quiver theory. See also \([50]\).

### 7.3 The geometric Johnson homomorphism

Let \((S, E)\) be as above, and \( \mathcal{M}(S) = \mathcal{M}(S, \partial S) \) the mapping class group of the surface \( S \) fixing the boundary \( \partial S \) pointwise (see \([33]\)). The largest Torelli group \( \mathcal{I}^L(S) \) in the sense of Putman \([68]\) is defined to be the kernel of the natural action on the quotient \( H_1(S)/\left(\sum_{j=0}^{n} \pi \hat{C}_j\right) \). In this subsection we introduce a pro-nilpotent Lie subalgebra \( L^+(S) \) of the completed Goldman Lie algebra \( \hat{Q}\pi(S) \), and construct a natural embedding \( \tau: \mathcal{I}^L(S) \hookrightarrow L^+(S) \), the geometric Johnson homomorphism for the surface \( S \), using the isomorphism

\[
\sigma: \hat{Q}\pi(S) \cong \text{Der}_\partial(\hat{\mathcal{I}}IS|_E)
\]

in Theorem 4.13.

By Theorem 3.3 the Dehn-Nielsen map \( \hat{\mathcal{D}}\hat{N}: \mathcal{M}(S) \to \text{Aut}(\hat{\mathcal{I}}IS|_E) \) is injective. If \( \varphi \in \mathcal{I}^L(S) \), then, by some straightforward argument, we have the logarithm \( \log \hat{\mathcal{D}}\hat{N}(\varphi) = \sum_{n=1}^{\infty}((-1)^{n-1}/n)(\hat{\mathcal{D}}\hat{N}(\varphi) - 1)^n \) as an element of \( \text{Der}_\partial(\hat{\mathcal{I}}IS|_E) \). From Theorem 4.13 we can define

\[
\tau(\varphi) := \sigma^{-1}\left(\log \hat{\mathcal{D}}\hat{N}(\varphi)\right) \in \hat{Q}\pi(S).
\]

Hence we obtain a natural embedding \( \tau: \mathcal{I}^L(S) \hookrightarrow \hat{Q}\pi(S) \).

We use Putman’s capping argument \([68]\) to define Lie subalgebras \( \hat{Q}\pi(S)(2\frac{1}{2}) \) and \( L^+(S) \). Let \( g_j \geq 1 \) be a positive integer for \( 1 \leq j \leq n \). We glue \( \Sigma_{g_j,1} \) on \( S \) along \( \partial S \) for each \( j \geq 1 \) to obtain a compact connected oriented surface \( \bar{S} := S \cup_{\partial S \setminus \partial \bar{S}} \bigcup_{j=1}^{n} \Sigma_{g_j,1} \) of genus \( g + \sum_{j=1}^{n} g_j \) with one boundary component.

We denote by \( \iota: S \hookrightarrow \bar{S} \) the inclusion map. Then the kernel of the induced homomorphism \( \iota_*: \hat{Q}\pi(S) \to \hat{Q}\pi(\bar{S}) \) is spanned by the set \( \{ |\log \xi_j| \}_{j=1}^{n} \) \([33]\) Lemma 6.2.3 (2). Choose a section \( s \in \text{Sect}(\iota_*). \) We define a weight \( w_\ast \) on the algebra \( \hat{T} \) by \( w_\ast(A_i^\ast) = w_\ast(B_i^\ast) = 1 \) and \( w_\ast(C_j^\ast) = 2 \) for \( 1 \leq i \leq g \) and \( 1 \leq j \leq n \). The following is the key lemma to define \( \hat{Q}\pi(S)(2\frac{1}{2}) \) and \( L^+(S) \).
Theorem 7.6. We do not know any relation between Church’s construction and constructed the first Johnson homomorphism for all kinds of Putman’s Torelli groups \([68]\). In fact, when \(L\) as a subgroup of \(\hat{T}_1\) have defined in §6.3 and [33] [34] is a proper subspace of \(L^{+}(S)\) in [33] [34] is a proper subspace of \(L^{+}(S)\). Recently Church \[7\] constructed the first Johnson homomorphism for all kinds of Putman’s Torelli groups [68]. We do not know any relation between Church’s construction and ours.

From the same argument as in Theorem 6.4 follows

\[ \delta \circ \tau = 0: \mathcal{I}^{L}(S) \rightarrow L^{+}(S) \rightarrow \widehat{\pi}(S) \widehat{\pi}(S). \]
8 Other topics and applications

In this section, we discuss other related topics. In §8.1, continuing the discussion in §6, we study the action of a Dehn twist on the fundamental groupoid of the surface. In §8.2, we describe the Lie algebra of chord diagrams, which comes from the Sp-invariants of the Lie algebra of symplectic derivations. An idea of using chord diagrams for description of the Sp-invariant tensors in $H^\otimes m$ goes back to a classical result due to Weyl [81]. In §8.3, we make remarks on the center of the Goldman Lie algebra and show a result on a surface of infinite genus. In §8.4, we make a review on the homological Goldman Lie algebra, which is a quotient of the Goldman Lie algebra. It is easier to handle than the Goldman Lie algebra, but still we have not reached a full understanding of it. We especially focus on its homological properties.

8.1 The action of Dehn twists on the nilpotent quotients

Let $S$ and $E$ be as in §3.3. Fix an integer $k \geq 1$. For $p_0, p_1 \in E$, we define

$$N_kQIS(p_0, p_1) : = F_1QIS(p_0, p_1)/F_{k+1}QIS(p_0, p_1)$$

$$\cong F_1\hat{QIS}(p_0, p_1)/F_{k+1}\hat{QIS}(p_0, p_1).$$

The mapping class group $\mathcal{M}(S, E)$ acts on the $\mathbb{Q}$-vector space $N_kQIS(p_0, p_1)$ through the Dehn-Nielsen homomorphism (§2.4). In this subsection we mention a few results about this action.

Notice that generalized Dehn twists induce $\mathbb{Q}$-linear automorphisms of $N_kQIS(p_0, p_1)$. Let $\gamma \subset S^* \setminus \partial S$ be an unoriented loop. Since $L(\gamma) \in \hat{\pi}(S^*)\langle 2 \rangle$, we have $\sigma(L(\gamma))u \in F_{k+1}\hat{QIS}(p_0, p_1)$ for any $u \in F_{k+1}\hat{QIS}(p_0, p_1)$, cf. Proposition 4.11. This implies that $\sigma(L(\gamma))$ induces a $\mathbb{Q}$-linear endomorphism of $N_kQIS(p_0, p_1)$, and $t_\gamma = \exp(\sigma(L(\gamma)))$ induces a $\mathbb{Q}$-linear automorphism of $N_kQIS(p_0, p_1)$. Fix $q \in S^*$ and let $\gamma_1$ and $\gamma_2$ be unoriented loops on $S^*$ represented by loops $x_1$ and $x_2$ based at $q$, respectively. As in §2.4 let $N_k\pi_1(S^*, q) = \pi_1(S^*, q)/\Gamma_{k+1}(\pi_1(S^*, q))$ be the k-th nilpotent quotient of $\pi_1(S^*, q)$.

**Proposition 8.1.** If $x_1 = x_2 \in N_k\pi_1(S^*, q)$, then $t_{\gamma_1} = t_{\gamma_2}$ on $N_kQIS(p_0, p_1)$. Moreover, if the homology classes $[x_1], [x_2] \in H_1(S^*; \mathbb{Z})$ are zero, then $t_{\gamma_1} = t_{\gamma_2}$ on $N_k\pi_1QIS(p_0, p_1)$.

**Proof.** For simplicity we denote $\pi_1 = \pi_1(S^*, q)$. Since $x_1 = x_2 \in N_k(\pi_1)$ there exists some $a \in \Gamma_{k+1}(\pi_1)$ such that $x_1 = x_2a$. Then $a - 1 \in (I\pi_1)^{k+1}$. Since $(x_1 - 1) - (x_2 - 1) = x_2(a - 1) \in (I\pi_1)^{k+1}$, $(1/2)(\log x_1)^2 -(1/2)(\log x_2)^2 \in F_{k+1}\hat{Q}\pi_1$. Therefore, $L(\gamma_1) - L(\gamma_2) \in \hat{\pi}(S^*)\langle k + 2 \rangle$. By Proposition 4.11, $\sigma(L(\gamma_1)) = \sigma(L(\gamma_2))$ on $N_kQIS(p_0, p_1)$. 


The condition \([x_1] = [x_2] = 0 \in H_1(S^*; \mathbb{Z})\) means that \(x_1 - 1\) and \(x_2 - 1\) are elements of \((I\pi^*)^2\). Hence \((1/2)(\log x_1)^2 - (1/2)(\log x_2)^2 \in F_{k+3}\mathbb{Q}\pi^*\) and \(L(\gamma_1) = L(\gamma_2) \in \mathbb{Q}\pi^*(S^*)(k + 3)\). This proves the latter part.

Suppose \(S\) and \(E\) satisfy the assumption of Theorem 5.3 and \(p_0 = p_1\). The map \(\pi_1(S, p_0) \to N_k\mathbb{Q}\pi(S, p_0) = I\pi_1(S, p_0)/(I\pi_1(S, p_0))^{k+1}, x \mapsto x - 1\) induces a \(M(S, E)\)-equivariant injection \(N_k(\pi_1(S, p_0)) \to N_k\mathbb{Q}\pi(S, p_0)\).

**Corollary 8.2.** Keep the assumptions as above and choose \(q \in S\). Let \(C_1\) and \(C_2\) be simple closed curves on \(S\), and let \(x_1\) and \(x_1\) be loops based at \(q\) representing \(C_1\) and \(C_2\). If \(x_1 = x_2 \in N_k(\pi_1(S, p_0))\), then the action of the Dehn twists \(t_{C_1}\) and \(t_{C_2}\) on \(N_k(\pi_1(S, p_0))\) coincide. Moreover, if \(C_1\) and \(C_2\) are separating, the the action of \(t_{C_1}\) and \(t_{C_2}\) on \(N_{k+1}(\pi_1(S, p_0))\) coincide.

This is first proved for the classical case \(S = \Sigma_{g,1}\) in [31] Theorem 1.1.2. We do not know whether this corollary can be proved without using the results in [31].

Now we consider the case the surface is \(\Sigma = \Sigma_{g,1}\) as in [31] and [32]. We shall give a partial result about formulas for the Johnson maps of a Dehn twist, which is closely related to the action of a Dehn twist on the nilpotent quotients of the fundamental group \(\pi = \pi_1(\Sigma, \ast)\). Fix a symplectic expansion \(\theta\) of \(\pi\), cf. Definition 6.1. Recall from [31] the total Johnson map \(T^\theta\) and the \(k\)-th Johnson map \(\tau_k^\theta\) associated to \(\theta\). Note that the action of \(\varphi \in \mathcal{M}_{g,1}\) on the \(k\)-th nilpotent quotient \(N_k = N_k(\pi)\) is determined by \(|\varphi|\) and \(\tau_k^\theta(\varphi)\), \(1 \leq i \leq k - 1\).

Recall from [31] that any symplectic expansion \(\theta\) induces a filtered Lie algebra isomorphism \(-\lambda_\theta: \mathbb{Q}\pi \to \mathfrak{a}_g^-\). Let \(\gamma\) be an unoriented free loop on \(\Sigma\). We set \(L^\theta(\gamma) := \lambda_\theta(L(\gamma)) = (1/2)N(\ell^\theta(x)\ell^\theta(x)) \in \mathfrak{l}_g\), where \(x \in \pi\) is a representative of \(\gamma\). By Theorems 5.3 and 6.3 if \(C\) is a simple closed curve on \(\Sigma\), then

\[T^\theta(t_C) = \exp(-L^\theta(C)) \in \text{Aut}(\mathcal{T}).\]  

(8.1)

Let \(L_k^\theta\) be the degree \(k\) part of \(L^\theta\). For example, \(L_0^\theta(C) = [C][C] \in H^{\otimes 2}\), where \([C] \in H\) is the homology class of \(C\) with a fixed orientation. For \(X \in H\) we have \(L_2^\theta(C)X = (X \cdot [C])[C]\), thus \((L_2^\theta(C))^2\chi = 0\). From [31], computing modulo \(\mathcal{T}_2\) we obtain

\[|t_C|X = X - L_2^\theta(C)X = X - (X \cdot [C])[C], \quad \text{for } X \in H.\]

This is the classical transection formula. Computing modulo higher tensors, we obtain explicit formulas for \(\tau_k^\theta(t_C)\).

**Theorem 8.3** (31). Let \(\theta\) be a symplectic expansion and \(C\) a non-separating simple closed curve on \(\Sigma\). For simplicity we denote \(L_k = L_k^\theta(C)\). Then we have
Let $L_{c} \leq k$ for some standard diagram as in Figure 6. If $i_{\omega}$ idea is to make the symplectic form description of $\text{Sp}^{3}$, which come from the Lie bracket on the $\text{Sp}^{j}$ the space of $\text{Sp}^{j}$ diagrams introduced in [32], which are derived from the Lie bracket on the $\text{Sp}^{j}$. As in [84], let $\Lambda^{j}$ explicit formulas for $\tau^{\theta}$ invariants of the Lie algebras Der($\text{Sp}$) and Der($\text{Sp}$) such that $\pi^{t}$ is another labeled linear chord diagram such that $\pi^{t}$ is a representative of $\pi^{t}$, then $\pi^{t}$ be a positive integer. A $\pi^{t}$ be the $\pi^{t}$-linear space spanned by the labeled linear chord diagram of $\pi^{t}$, and Der($\text{Sp}$) and Der($\text{Sp}$) are Lie subalgebras of Der($\text{Sp}$) and Der($\text{Sp}$), respectively.

We draw a picture of a labeled linear chord diagram as in Figure 6. If $i_{\omega}$ for any $1 \leq k \leq m$, we say the label of $C$ is standard. If $C'$ is another labeled linear chord diagram such that $C' = \{(i_{1}, j_{1}), \ldots, (i_{m}, j_{m})\}$ for some $1 \leq k \leq m$, we say $C'$ is obtained from $C$ by a single label change. Let $\mathcal{LC}_{m}$ be the $\mathcal{Q}$-linear space spanned by the labeled linear chord diagram

1. $\tau^{0}_{1}(t_{C}) = -L_{3}$,
2. $\tau^{0}_{2}(t_{C}) = -L_{4} + \frac{1}{2}[L_{2}, L_{4}] + \frac{1}{2}(L_{4})^{2}$.

Note that if $x \in \pi$ is a representative of $C$, then $L_{c}^{0}(C) = [x] \wedge \ell_{0}^{0}(x) \in \Lambda^{3}H \subset H^{\otimes 3}$ (see [31] Lemma 6.4.1). At the present stage we do not know explicit formulas for $\tau^{0}_{k}(t_{C})$, $k \geq 3$ and $C$ non-separating. If $C$ is separating, the formula for $\tau^{0}_{k}$ becomes simple since $L_{2}^{0}(C) = 0$. See [31] Theorem 6.3.1.

8.2 Lie algebras based on chord diagrams

As in [31] let $H = H_{1}(\Sigma; \mathbb{Q})$ be the first rational homology group of the surface $\Sigma = \Sigma_{g,1}$ and $\text{Sp} = \text{Sp}(H) \cong \text{Sp}(2g; \mathbb{Q})$. A classical result of Weyl [81] is that the space of $\text{Sp}$-invariant tensors in $H^{\otimes m}$ is generated by chord diagrams. The idea is to make the symplectic form $\omega$ correspond to a labeled chord. This description of $\text{Sp}$-invariant tensors has been used in several works such as [30, 37, 59].

In this subsection we review Lie algebra structures on the spaces of chord diagrams introduced in [32], which come from the Lie bracket on the $\text{Sp}$-invariants of the Lie algebras Der($T$) and Der$_{\omega}(T)$. Here $T = \bigoplus_{m=0}^{\infty} H^{\otimes m}$ is the tensor algebra generated by $H$, Der($T$) is the Lie algebra of derivations of $T$, and Der$_{\omega}(T)$ is the Lie subalgebra of Der($T$) consisting of derivations annihilating $\omega$. Note that the degree completion of Der$_{\omega}(T)$ is the Lie algebra $\mathfrak{a}_{g}$ in [62]. The symplectic group $\text{Sp}$ acts naturally on Der($T$), and this action preserves Der$_{\omega}(T)$. As in [62] we can identify Der($T$) with $\bigoplus_{m=1}^{\infty} H^{\otimes m}$ by the restriction

$$\text{Der}(T) \xrightarrow{\cong} \text{Hom}(H, T) = H^{*} \otimes T \cong H \otimes T = \bigoplus_{m=1}^{\infty} H^{\otimes m}, \quad D \mapsto D|_{H}.$$ 

Also we have Der$_{\omega}(T) = \bigoplus_{m=1}^{\infty} N(H^{\otimes m})$. The action of $\text{Sp}$ coincides with the diagonal action on the tensor spaces $H^{\otimes m}$. The $\text{Sp}$-invariant parts Der($T$)$^{\text{Sp}}$ and Der$_{\omega}(T)^{\text{Sp}}$ are Lie subalgebras of Der($T$) and Der$_{\omega}(T)$, respectively.

Let $m$ be a positive integer. A labeled linear chord diagram of $m$ chords is a set of $m$ ordered pairs $C = \{(i_{1}, j_{1}), (i_{2}, j_{2}), \ldots, (i_{m}, j_{m})\}$ satisfying $\{i_{1}, \ldots, i_{m}, j_{1}, \ldots, j_{m}\} = \{1, 2, \ldots, 2m\}$. We draw a picture of a labeled linear chord diagram as in Figure 6. If $i_{k} < j_{k}$ for any $1 \leq k \leq m$, we say the label of $C$ is standard. If $C'$ is another labeled linear chord diagram such that $C' = \{(i_{1}, j_{1}), \ldots, (i_{m}, j_{m})\}$ for some $1 \leq k \leq m$, we say $C'$ is obtained from $C$ by a single label change. Let $\mathcal{LC}_{m}$ be the $\mathcal{Q}$-linear space spanned by the labeled linear chord diagram

$$\mathcal{LC}_{m} = \text{Span}_{\mathcal{Q}}(\{i_{1}, j_{1}, \ldots, i_{k-1}, j_{k-1}, j_{k}, i_{k}, i_{k+1}, j_{k+1}, \ldots, (i_{m}, j_{m})\})$$

for some $1 \leq k \leq m$, we say $C'$ is obtained from $C$ by a single label change.
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Figure 6. $C = \{(1,2),(3,5),(4,6)\}$

of $m$ chords modulo the $\mathbb{Q}$-linear subspace generated by the set

$$\{C + C'; C' \text{ is obtained from } C \text{ by a single label change}\}.$$  

Note that $\mathcal{L}C_m$ is $(2m - 1)!!$ dimensional, and the set of linear chord diagrams with standard label is a basis for $\mathcal{L}C_m$.

The symmetric group $\mathfrak{S}_{2m}$ acts naturally on the tensor space $H^\otimes 2m$. For a labeled linear chord diagram $C$, we define

$$a(C) := \begin{pmatrix} 1 & 2 & \cdots & 2m - 1 & 2m \\ i_1 & j_1 & \cdots & i_m & j_m \end{pmatrix} (\omega^\otimes m) \in H^\otimes 2m.$$  

This is an $Sp$-invariant tensor. Since $a(C') = -a(C)$ if $C'$ is obtained from $C$ by a single label change, the correspondence $a$ induces a $\mathbb{Q}$-linear map

$$a: \mathcal{L}C_m \to (H^\otimes 2m)^{Sp}, \quad C \mapsto a(C).$$

The following proposition is due to Weyl [81] except for “only if” part of (3) which is due to Morita [59].

**Proposition 8.4.**  

1. If $n$ is odd, the space of $Sp$-invariant tensors $(H^\otimes n)^{Sp}$ is zero.  

2. The map $a: \mathcal{L}C_m \to (H^\otimes 2m)^{Sp}$ is surjective for any $m \geq 1$.  

3. The map $a: \mathcal{L}C_m \to (H^\otimes 2m)^{Sp}$ is an isomorphism if and only if $m \geq g$.

Set $\mathcal{LL} := \bigoplus_{m=1}^{\infty} \mathcal{L}C_m$. From Proposition 8.4 the map

$$a: \mathcal{LL} \to \bigoplus_{m=1}^{\infty} (H^\otimes 2m)^{Sp} = \text{Der}(T)^{Sp} \quad (8.2)$$

is a stable isomorphism, and we can introduce a Lie bracket on $\mathcal{LL}$ so that $\mathcal{LL}$ is a Lie algebra homomorphism.

To describe the Lie bracket on $\mathcal{LL}$, we define the *amalgamation* of two linear chord diagrams. Let $C$ and $C'$ be linear chord diagrams with standard label of $m$ and $l$ chords, respectively. For $2 \leq t \leq 2l$, we define the $t$-th amalgamation $C \ast_t C'$ as a linear chord diagram with standard label as follows. We first cut $C'$ at the $t$-th vertex and $C$ at the first vertex, insert $C$ into the $t$-th hole of the cut $C'$, then connect the first vertex of $C$ to the $t$-th vertex of $C'$. See Figure 7. The amalgamation $C \ast_t C'$ is the linear chord diagram of the result
Figure 7. the \( t \)-th amalgamation \( C \ast_t C' \)

with standard label. Then the bracket \( [C, C'] \in \mathcal{LC}_{m+t-1} \) is given by

\[
[C, C'] = -\sum_{t=2}^{2l} C \ast_t C' + \sum_{s=2}^{2m} C' \ast_s C.
\] (8.3)

We next consider the Lie algebra \( \text{Der}_\omega(T)^{Sp} \). Let \( \nu = \nu \in \mathfrak{S}_{2m} \) be the cyclic permutation

\[
\nu = \begin{pmatrix} 1 & 2 & 3 & \cdots & 2m \\ 2m & 1 & 2 & \cdots & 2m-1 \end{pmatrix}.
\]

For a labeled linear chord diagram \( C = \{(i_1, j_1), (i_2, j_2), \ldots, (i_m, j_m)\} \) and \( s \in \mathbb{Z} \), we define

\[
\nu^s(C) := \{(\nu^s(i_1), \nu^s(j_1)), (\nu^s(i_2), \nu^s(j_2)), \ldots, (\nu^s(i_m), \nu^s(j_m))\}.
\]

The cyclic group of order \( 2m \), generated by \( \nu \), acts on the \( \mathbb{Q} \)-vector space \( \mathcal{LC}_m \). Let \( \mathcal{C}_m \subset \mathcal{LC}_m \) be the \( \mathbb{Z}_{2m} \)-invarints under this action. If \( m = 1 \) and \( C = \{(1, 2)\} \), then \( \nu(C) = -C \in \mathcal{LC}_1 \). This means that \( \mathcal{C}_1 = 0 \). The \( \mathbb{Q} \)-linear space \( \mathcal{C}_m \) is generated by labeled \textit{circular} chord diagrams. More precisely, the space \( \mathcal{C}_m \) is generated by element of the form \( N(C) = \sum_{s=0}^{2m-1} \nu^s(C) \), where \( C \) is a labeled linear chord diagram of \( m \) chords. We draw a picture of \( N(C) \) as in Figure 8. Here the picture is a labeled circular chord diagram obtained as the \textit{"closing"} of the picture of \( C = \{(1, 2), (3, 5), (4, 6)\} \) in Figure 6. We call \( \mathcal{C}_m \) the \textit{space of oriented circular chord diagram} of \( m \) chords. The direct sum \( \mathcal{C} = \bigoplus_{m=2}^{\infty} \mathcal{C}_m \) is a Lie subalgebra of \( \mathcal{LC} \). Since the tensor \( a(N(C)) \) is cyclically invariant, \( \nu \) induces a Lie algebra homomorphism

\[
a : \mathcal{C} \rightarrow \bigoplus_{m=2}^{\infty}(N(H^\otimes 2m))^{Sp} = \text{Der}_\omega(T)^{Sp}.
\]

The Lie bracket on \( \mathcal{C} \) is given as follows. Let \( D \) and \( D' \) be labeled circular chord diagrams of \( m \) and \( m' \) chords, respectively. For vertices \( p \) of \( D \) and \( q \)
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Figure 8. A labeled circular chord diagram

\[ \mathcal{D}(D, p, D', q) \]

Figure 9. \( \mathcal{D}(D, p, D', q) \)

of \( D' \), we define the labeled circular chord diagram \( \mathcal{D}(D, p, D', q) \) as the result of a certain surgery at \( p \) and \( q \) illustrated in Figure 9. Here the label of the chord connecting \( \overline{p} \) and \( \overline{q} \) are determined by the rule in Figure 10. Then the bracket \( [D, D'] \in C^m_{m'+1} \) is given by

\[
[D, D'] = \sum_{(p, q)} \mathcal{D}(D, p, D', q),
\]

where the sum is taken over all pairs of the vertices of \( D \) and \( D' \).

The structure of graded Lie algebras \( \mathcal{L} \mathcal{C} \) and \( \mathcal{C} \) are not fully understood. The Lie algebra \( \mathcal{L} \mathcal{C} \) has the trivial center and its homology \( H_*(\mathcal{L} \mathcal{C}) \) is the same as the homology of the circle \( S^1 \). However, the homology of the Lie subalgebra \( \mathcal{L} \mathcal{C}^1 := \bigoplus_{m=2}^{\infty} \mathcal{L} \mathcal{C}_m \) is highly non-trivial, and so is the homol-

Figure 10. The label of the chord \( \overline{pq} \)
ogy of \( \mathcal{C} \). In [32], the center of \( \mathcal{C} \) was computed. For an integer \( m \geq 2 \) let 
\[ C_m = \{ (1, 2), (3, 4), \ldots, (2m - 1, 2m) \} \] 
and set \( \Omega_m = N(C_m) \in C_m \). Note that 
\[ a(\Omega_m) = N(\omega \otimes^m) \] 

**Theorem 8.5** ([32]). The center of the Lie algebra \( \mathcal{C} \) is spanned by \( \Omega_m, m \geq 2 \):
\[ Z(\mathcal{C}) = \bigoplus_{m=2}^{\infty} \mathbb{Q}\Omega_m. \]

### 8.3 The center of the Goldman Lie algebra

Let \( \mathfrak{g} \) be a Lie algebra. The center of \( \mathfrak{g} \), denoted by \( Z(\mathfrak{g}) \), is the set of \( X \in \mathfrak{g} \) such that \([X, Y] = 0\) for any \( Y \in \mathfrak{g} \). It is a fundamental problem to compute the center of \( \mathfrak{g} \).

Let \( S \) be an oriented surface. It is clear from the definition of the Goldman bracket that if \( \xi \) is a loop parallel to a boundary component of \( S \), \( \xi \) and its powers \( \xi^n, n \in \mathbb{Z} \), are in the center \( Z(\mathbb{Q}\hat{\pi}(S)) \). The question is whether these elements span \( Z(\mathbb{Q}\hat{\pi}(S)) \). Goldman gave a partial result in this direction.

**Theorem 8.6** (Goldman [17], Theorem 5.17). Let \( \alpha, \beta \in \hat{\pi}(S) \) and assume that \( \alpha \) is represented by a simple closed curve. Then \([\alpha, \beta] = 0 \) in \( \mathbb{Q}\hat{\pi}(S) \) if and only if \( \alpha \) and \( \beta \) are freely homotopic to disjoint curves.

For example, we see that if \( S \) is compact then \( \hat{\pi}(S) \cap Z(\mathbb{Q}\hat{\pi}(S)) \) is the set of loops parallel to some boundary component of \( S \) and their powers. To see this, we take a system of simple closed curves that fills \( S \). This means that each component of the complement of these curves is a disk or an annulus whose boundary contains some boundary component of \( S \). If \( \beta \in \hat{\pi}(S) \cap Z(\mathbb{Q}\hat{\pi}(S)) \), by Theorem 8.6 one can assume that \( \beta \) is disjoint from each member of the filling curves. Therefore, \( \beta \) is homotopic to a point or a power of a boundary loop.

Whether the set \( \hat{\pi}(S) \cap Z(\mathbb{Q}\hat{\pi}(S)) \) spans the center \( Z(\mathbb{Q}\hat{\pi}(S)) \) or not is an open question. If \( S \) is closed, this is affirmative. The following result was conjectured by Chas and Sullivan.

**Theorem 8.7** (Etingof [14]). If \( S \) is closed, the center \( Z(\mathbb{Q}\hat{\pi}(S)) \) is spanned by the constant loop \( 1 \in \hat{\pi}(S) \).

The proof of Etingof uses symplectic geometry of the moduli space of flat \( GL_N(\mathbb{C}) \)-bundles over the surface \( S \).

As a bi-product of the proof of Therem 8.6, we obtain a partial result on the center \( Z(\mathbb{Q}\hat{\pi}(\Sigma_{g,1})) \) The idea is to use the relation between \( \mathbb{Q}\hat{\pi}(\Sigma_{g,1}) \) and
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Let \( \zeta \in \pi_1(\Sigma_{g,1}) \) be the boundary loop as in §6.3 and the fact that any element of \( Z(\mathfrak{a}_g^-) \) must be an \( Sp \)-invariant tensor since it commutes with the degree two part \( N(H^{\otimes 2}) \cong \mathfrak{sp}(H) \).

Theorem 8.8 ([32]). For a positive integer \( g \), set \( m(g) := \lfloor (g-1)/4 \rfloor + 1 \). Here \( \lfloor x \rfloor \) is the greatest integer less than or equal to \( x \). For any \( u \in Z(\hat{\pi}(\Sigma_{g,1})) \), there exists a polynomial \( f(\zeta) \in \mathbb{Q}[\zeta] \) such that
\[
u \equiv |f(\zeta)| \pmod{\hat{\pi}(\Sigma_{g,1})(2m(g))}.
\]

Let \( \Sigma_{\infty,1} \) be the inductive limit of the embeddings \( \Sigma_{g,1} \hookrightarrow \Sigma_{g+1,1} \), \( g > 0 \), obtained by gluing \( \Sigma_{1,2} \) on \( \Sigma_{g,1} \) along the boundary. Based on Theorem 8.8, we can determine the center of \( Z(\hat{\pi}(\Sigma_{\infty,1})) \).

Theorem 8.9 ([32]). The center \( Z(\hat{\pi}(\Sigma_{\infty,1})) \) is spanned by the constant loop \( 1 \in \hat{\pi}(\Sigma_{\infty,1}) \).

8.4 Homological Goldman Lie algebra

As was mentioned in §4.1, the Goldman bracket comes from the Poisson bracket of two trace functions on the moduli space of flat \( G \)-bundles over the surface, \( \text{Hom}(\pi_1(S), G)/G \). Goldman [17] §3 already showed the explicit formula for the Poisson bracket depends heavily on the choice of a Lie group \( G \). This fact led him to introducing some variants of the (original) Goldman Lie algebra. Later Andersen, Mattes and Reshetikhin [2] unified diversity of the Poisson structures into the Poisson algebra of chord diagrams on a surface. It would be very interesting if some phenomena analogous to what was stated in §4 and §6.3 could be found for this Poisson algebra.

In this subsection we discuss a variant which appears for an abelian \( G \), and some relation to the first and the second homology groups of the original one. Results on the second homology group are due to Toda [77]. If \( G \) is abelian, the Poisson action of \( \hat{\pi}(S) \) on \( \text{Hom}(\pi_1(S), G)/G \) factors through the group ring of the integral homology group \( H_\mathbb{Z} = H_1(S; \mathbb{Z}) \), \( \mathbb{Z}H_\mathbb{Z} \), which we call the homological Goldman Lie algebra of the surface \( S \). We denote by \( [X] \in \mathbb{Z}H_\mathbb{Z} \) the basis element corresponding to \( X \in H_\mathbb{Z} \). Then the Lie bracket on \( \mathbb{Z}H_\mathbb{Z} \) is given by
\[
[[X], [Y]] = (X \cdot Y)[X + Y] \in \mathbb{Z}H_\mathbb{Z}
\]
(8.5)
for any \( X, Y \in H_\mathbb{Z} \), cf. [17] §5.10. Here \( (X \cdot Y) \) is the intersection number of \( X \) and \( Y \). More generally, if \( H \) is an additive group with a bi-additive alternating pairing \( (\cdot, \cdot): H \times H \to \mathbb{Z} \), then the formula (8.5) defines a structure of a Lie algebra on the group ring \( \mathbb{Z}H \), which we also call the homological Goldman Lie algebra associated to the alternating pairing \( (\cdot, \cdot) \). We remark that the pairing
( · ) is not necessarily non-degenerate. In the last part of this subsection, we will present an outline of Toda’s works [76] [77] on the algebraic structure of the homological Goldman Lie algebra in this general setting.

In §6.3 we constructed a Lie algebra homomorphism of $\hat{\pi}(\Sigma_{g,1})$ into the Lie algebra of symplectic derivations. A similar homomorphism for $\mathbb{R}H_Z$ was already given in [17] §5.10. In the first half of this subsection, until Corollary 8.15 we suppose $S = \Sigma_{g,1}$, $g \geq 1$. Note that $H_1(\Sigma_{g,1};\mathbb{Z}) = H_1(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}^{2g}$, and that the intersection pairing is non-degenerate. Here we give a slightly modified version of Goldman’s homomorphism. The $2g$-dimensional torus $T_{2g} := H_1(\Sigma_{g,1};\mathbb{R}/\mathbb{Z})$ has a natural symplectic form $\omega \in \Omega^2(T_{2g})$ derived from the intersection form on the surface $\Sigma_{g,1}$. Hence the Poisson bracket makes $C^\infty(T_{2g}) = C^\infty(T^{2g};\mathbb{C})$ a complex Lie algebra. We define a linear map $\rho: CH_Z \to C^\infty(T_{2g})$ by

$$\rho([X])(Z) := -\frac{1}{4\pi^2} e^{2\pi \sqrt{-1}(X \cdot Z)}$$

for any $X \in H_Z$ and $Z \in T_{2g}$. It is easy to check that $\rho$ is a Lie algebra homomorphism.

On the other hand, for any closed $2g$-dimensional symplectic manifold $(M, \omega)$ the linear map $\varphi^M: C^\infty(M) \to \mathbb{C}$ given by

$$\varphi^M(f) := \int_M f \omega^g \in \mathbb{C}$$

induces a linear map on the abelianization $C^\infty(M)^{ab}$ of the Poisson Lie algebra $C^\infty(M)$. In fact, we have $\varphi^M([f, h]) = \int_M H_f(h) \omega^g = \int_M \mathcal{L}_{H_f}(h) \omega^g = 0$ for any $f$ and $h \in C^\infty(M)$. Here $H_f \in X(M)$ is the Hamiltonian vector field associated to $f$. In our situation, we have

$$(\varphi^{T_{2g}} \circ \rho)([X]) = \begin{cases} 0, & \text{if } X \neq 0, \\ -\frac{g!}{4\pi^2}, & \text{if } X = 0. \end{cases} \quad (8.6)$$

This induces a non-trivial element of the first cohomology group of the Lie algebra $CH_Z$, $H^1(CH_Z)$.

From (8.6) we have

$$\frac{16\pi^2}{g!} \int_{T_{2g}} \rho([X])\rho([Y]) \omega^g = \begin{cases} 0, & \text{if } X \neq Y, \\ 1, & \text{if } X = Y. \end{cases}$$

Hence the homomorphism $\rho: CH_Z \to C^\infty(T_{2g})$ is injective. We can use $\rho$ to compute the center of $CH_Z$ in a similar way to that in §8.3.

**Proposition 8.10.** The center of $CH_Z$, $Z(CH_Z)$, is spanned by $[0]$. 
Proof. Let \( \{A_i, B_i\}_{i=1}^g \subset H_Z \) be a symplectic basis, and \((x_i, y_i)\) the global coordinates of \(T^{2g}\) corresponding to the basis. We have \( \omega = \sum_{i=1}^g dx_i \wedge dy_i \), \( \rho([A_i]) = \frac{1}{4 \pi^2} e^{2 \pi \sqrt{-1} y_i} \) and \( \rho([B_i]) = -\frac{1}{4 \pi^2} e^{-2 \pi \sqrt{-1} x_i} \). Suppose \( u \in Z(CH_Z) \). Then \( 0 = \rho([A_i, u]) = -\frac{1}{4 \pi^2} \left\{ e^{2 \pi \sqrt{-1} y_i}, \rho(u) \right\} = \frac{1}{4 \pi^2} \left( \frac{\partial}{\partial y_i} e^{2 \pi \sqrt{-1} u} \right) \left( \frac{\partial}{\partial x_i} \rho(u) \right) \), and so \( \frac{\partial}{\partial y_i}(\rho(u)) = 0 \). Similarly \( \frac{\partial}{\partial x_i}(\rho(u)) = 0 \). Hence \( \rho(u) \in C^\infty(T^{2g}) \) is a constant function \( \in \mathbb{C} \). Since \( \rho \) is injective, we obtain \( u \in \mathbb{C}[0] \). Clearly we have \( \{0\} \subset Z(CH_Z) \). This proves the proposition.

As will be stated in Theorem \(8.19\), Toda \(76\) classifies the ideals of the homological Goldman Lie algebra over the rationals \( \mathbb{Q} \) in the most general setting. This proposition follows also from his result.

Next we discuss the abelianization, i.e., the first homology group of the Goldman Lie algebra \( Z\pi_1(S_{g,1}) \). We begin by computing the abelianization of the homological Goldman Lie algebra \( ZH_Z \). Let \( \{A_i, B_i\}_{i=1}^g \subset H_Z \) be a symplectic basis. We define \( \nu(X) \in \mathbb{Z}_{>0} \) for \( X \in H_Z \setminus \{0\} \) by \( \nu(X) = \gcd\{a_i, b_i; 1 \leq i \leq g\} \) where \( X = \sum a_i A_i + b_i B_i \). In other words, \( \nu(X)^{-1}X \) is in \( H_Z \), and primitive. We define \( \nu(0) := 0 \) for \( 0 \in H_Z \).

Lemma 8.11.

\[
[ZH_Z, ZH_Z] = \bigoplus_{X \in H_Z \setminus \{0\}} \mathbb{Z}\nu(X)[X].
\]

Proof. We have \( (X \cdot Y) = ((X + Y) \cdot Y) \) for any \( X \) and \( Y \in H_Z \). Hence \( [[X], Y] \in Z\nu(X + Y)[X + Y] \). Conversely, for any \( X \in H_Z \setminus \{0\} \), there exists \( Y \in H_Z \) such that \( (X \cdot Y) = \nu(X) \). Then we have \( [[X - Y], Y] = \nu(X)[X] \). This proves the lemma.

Corollary 8.12.

\[
ZH_Z^{\text{abel}} = \bigoplus_{X \in H_Z} (\mathbb{Z}/\nu(X)).
\]

In particular, the Lie algebra \(ZH_Z\) is not finitely generated, while \( QH_Z^{\text{abel}} = Q \).

The result \( QH_Z^{\text{abel}} = Q \) follows also from Toda’s Theorem \(8.19\) \(76\). Furthermore we have

Theorem 8.13 \(83\). There is a subset \( S \) of \( H_Z \) such that \( \{[X]; X \in S\} \) generates \( QH_Z \) as a Lie algebra and \( 2g + 2 \). In particular, the Lie algebra \( QH_Z \) is finitely generated. Moreover, if \( S \) is a subset of \( H \) and \( \{[X]; X \in S\} \) generates \( QH_Z \) as a Lie algebra, we have \( 2g + 2 \).

Since we have a natural surjection of Lie algebras \( Z\pi_1(S_{g,1}) \to ZH_Z \), we obtain the following from Corollary 8.12.
Corollary 8.14. The Goldman Lie algebra $\mathbb{Z}\hat{\pi}(\Sigma_{g,1})$ is not finitely generated.

The following question arises from this Corollary.

**Question.** Is the abelianization $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})^\text{abel}$ finite dimensional, or not? Furthermore, is the (rational) Goldman Lie algebra $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})$ finitely generated, or not?

This question is open. As was explained in §6.3, we have a Lie algebra homomorphism of $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})$ into the Lie algebra of symplectic derivations $a_{g}^{-}\mathfrak{g}$. Recently Morita, Sakasai and Suzuki [63] proved that a stable part of the abelianization of the Lie algebra $\mathfrak{a}_{g}$ is finite dimensional. But the Lie algebra homomorphism does not fit to the homomorphisms $\varphi\{1\}$ and $\varphi[\pi,\pi]$ stated below. They do not induce maps on $a_{g}^{-}\mathfrak{g}$. On the other hand, from Toda’s classification of the ideals, Theorem 8.19 [76], $\mathbb{Q}H_{1}(\Sigma_{g,1};\mathbb{Z})^{\text{abel}}$ is infinite-dimensional if $r \geq 2$.

The abelianization $\mathbb{Q}H_{1}^{\text{abel}}$ is spanned by $[0]$. The map $\varphi[\pi,\pi]: \mathbb{Q}H_{1} \to \mathbb{Q}$ defined by

$$\varphi[\pi,\pi](\{X\}) = \begin{cases} 0, & \text{if } X \neq 0, \\ 1, & \text{if } X = 0, \end{cases}$$

(8.7)

is proportional to the map $\varphi^{\mathfrak{r}^{2}\mathfrak{g}} \circ \rho$ in (8.4). Hence it induces an isomorphism $\varphi[\pi,\pi]: \mathbb{Q}H_{1}^{\text{abel}} \cong \mathbb{Q}$. Since we have a natural surjection of Lie algebras $\mathbb{Q}\hat{\pi}(\Sigma_{g,1}) \to \mathbb{Q}H_{1}$, the map $\varphi[\pi,\pi]: \mathbb{Q}\hat{\pi}(\Sigma_{g,1})^{\text{abel}} \to \mathbb{Q}H_{1}^{\text{abel}} \to \mathbb{Q}$ is nontrivial. Moreover, as will be shown later in Corollary 8.17 the map $\varphi\{1\}: \mathbb{Q}\hat{\pi}(\Sigma_{g,1}) \to \mathbb{Q}$ defined in (8.8) descends to $\mathbb{Q}\hat{\pi}(\Sigma_{g,1})^{\text{abel}}$. Since $\varphi\{1\}$ and $\varphi[\pi,\pi]$ are linearly independent, we obtain

Corollary 8.15.

$$\dim_{\mathbb{Q}} \mathbb{Q}\hat{\pi}(\Sigma_{g,1})^{\text{abel}} \geq 2.$$  

Now let $S$ be any connected oriented surface. Choose a base point $* \in S$, and denote $\pi := \pi_{1}(S,*)$. Consider a normal subgroup $\Gamma \subset \pi$. We denote by $\overline{N} = \overline{N}_{\Gamma}$ the set of conjugacy classes in the quotient $N = N_{\Gamma} := \pi/\Gamma$. We have a natural surjection $\varpi = \varpi_{\Gamma}: \mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}\overline{N}$.

Now the following question seems to be natural.

**Question.** Does the Goldman bracket descend to $\mathbb{Q}\overline{N}_{\Gamma}$? Or, equivalently, is the subspace $\text{Ker}(\varpi_{\Gamma})$ an ideal of $\mathbb{Q}\hat{\pi}(S)$?

Clearly the answer is yes if $\Gamma = \{1\}$ or $\Gamma = [\pi,\pi]$. Remark that $N[\pi,\pi] = \overline{N}_{[\pi,\pi]} = H_{1}$. The answer is no if $S = \Sigma_{g,1}$, $g \geq 2$ and $\Gamma = [\pi,\pi]$. To see this
choose a Goldman-Turaev Lie bialgebra and the Johnson homomorphisms

choose a symplectic generator system \( \{ \alpha_i, \beta_i \}_{i=1}^g \subset \pi \). By a straightforward computation, we obtain
\[
\langle [\alpha_1], [[\beta_1, \alpha_2^2] \alpha_2^{-1}] \alpha_1^{-1} \rangle - |\alpha_1^{-1}| = \langle [1 - [\alpha_2, \alpha_1]) (1 - [\alpha_1, \alpha_2]) \rangle \in \mathbb{Q}\mathcal{N}_{[\pi, [\pi, \pi]]},
\]
which is not zero. This implies that the Goldman bracket does not descend to \( \mathbb{Q}\mathcal{N}_{[\pi, [\pi, \pi]]} \).

The question is open for the other normal subgroups. It is closely related to the former question. We introduce a map \( \varphi_{\Gamma} : \mathbb{Q} \hat{\pi}(S) \to \mathbb{Q} \) by
\[
\varphi_{\Gamma}([x]) = \begin{cases} 
0, & \text{if } x \in \pi \setminus \Gamma, \\
1, & \text{if } x \in \Gamma.
\end{cases}
\]
(8.8)

It is well-defined since \( \Gamma \) is a normal subgroup of \( \pi \).

**Lemma 8.16.** If the Goldman bracket descends to \( \mathbb{Q}\hat{\pi}(S) \), then

\[
\varphi_{\Gamma}([\mathbb{Q}\hat{\pi}(S), \mathbb{Q}\hat{\pi}(S)]) = 0.
\]

In other words, \( \varphi_{\Gamma} \) induces a nonzero element of \( H^1(\mathbb{Q}\hat{\pi}(S)) \).

**Proof.** Assume \( \varphi_{\Gamma}([\alpha, \beta]) \neq 0 \) for some \( \alpha \) and \( \beta \in \hat{\pi}(S) \), from which we will deduce a contradiction. We may assume \( \alpha \) is in \( \pi_1(S, \ast) \), and \( \alpha \prod_{i=1}^g \beta \) a generic immersion. Then we have \( \varphi_{\Gamma}([\alpha, \beta]) = 1 \) for some \( \gamma \). If we write \( \gamma := \alpha_{\ast p_0} \alpha_{p_0}^{-1} \), then \( \gamma \in \pi_1(S, \ast) \). Since the Goldman bracket descends to \( \hat{\pi}_{\Gamma} \), we have
\[
0 \neq \varphi_{\Gamma}([\alpha, \beta]) = \varphi_{\Gamma}([\varpi_{\Gamma} \alpha, \varpi_{\Gamma} \beta]) = \varphi_{\Gamma}([\varpi_{\Gamma} \alpha, \varpi_{\Gamma} \alpha^{-1}]) = \varphi_{\Gamma}([\alpha, \alpha^{-1}]).
\]

On the other hand, let \( \alpha^{-1} \) be represented by a generic immersion such that \( \alpha \cup \alpha^{-1} \) bounds a narrow annulus, as in [17], p.295. Let \( p \) be a double point of the loop \( \alpha \). It divides the loop \( \alpha \) into two based loops \( \alpha_1 \) and \( \alpha_2 \) with base point \( p \) as in Figure 11. The two intersection points derived from \( p \) contribute \( \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \) and \( \alpha_2 \alpha_1 \alpha_2^{-1} \alpha_1^{-1} \), respectively, with the opposite sign. Then \( \alpha_1 \alpha_2 \alpha_1^{-1} \alpha_2^{-1} \in \Gamma \) is equivalent to \( \alpha_2 \alpha_1 \alpha_2^{-1} \alpha_1^{-1} \in \Gamma \). This implies that the contributions of the two points cancel, namely, \( \varphi_{\Gamma}([\alpha, \alpha^{-1}]) = 0 \). This contradicts what we proved above, and proves the lemma.

In the case \( \Gamma = \{1\} \), we have

**Corollary 8.17** ([17] Proposition 5.9). If we write \( \hat{\pi}'(S) := \hat{\pi}(S) \setminus \{1\} \), then
\[
[\mathbb{Q}\hat{\pi}(S), \mathbb{Q}\hat{\pi}(S)] \subset \mathbb{Q}\hat{\pi}'(S).
\]

Goldman’s original proof [17] pp.294–294 is not correct. For details, see [31] Remark 3.1.2.
Figure 11. loops $\alpha_1$ and $\alpha_2$

**Corollary 8.18.** If $S = \Sigma_g$ or $\Sigma_{g,1}$, $g \geq 1$, then the Goldman bracket does not descend to $\mathbb{Q} \tilde{\mathcal{N}}_\Gamma$ for any normal subgroup $\Gamma$ with $[\pi, \pi] \subsetneq \Gamma \subsetneq \pi$.

**Proof.** This follows from $\mathbb{Q}H^\text{abel} \cong \mathbb{Q}$, Lemma 8.12.

We conclude this chapter by giving an outline of Toda’s works [76] [77] on the algebraic structure of the rational homological Goldman Lie algebra for any additive group $H$ equipped with a bi-additive alternating pairing $(\cdot): H \times H \to \mathbb{Z}$. The pairing induces the map $\mu: H \to \text{Hom}(H, \mathbb{Z})$ defined by $\mu(x)(y) = (x \cdot y)$ for any $x$ and $y \in H$. In Toda’s results, the set $\text{Ker}(\mu)$ and its complement subset $H \setminus \text{Ker}(\mu)$ play some important roles.

Toda classified all the ideals of the rational homological Goldman Lie algebra $\mathbb{Q}H$ as follows. For any $x \in H$, we define $T(x): \mathbb{Q}H \to \mathbb{Q}H$ by $T(x)(Y) := [X + Y]$ for any $Y \in H$.

**Theorem 8.19** ([76]). For any ideal $\mathfrak{h}$ in $\mathbb{Q}H$, there exists a unique pair $(V_0, V)$ such that

1. $V_0$ and $V$ are subspaces of the linear span of $\text{Ker}(\mu)$,
2. For any $Z \in \text{Ker}(\mu)$ we have $T(Z)(V) \subset V$, and
3. $\mathfrak{h} = V_0 \oplus \sum_{X \in H \setminus \text{Ker}(\mu)} T(X)(V)$.

If $\mu = 0$, we define $V = 0$. Conversely, a subset $\mathfrak{h} \subset \mathbb{Q}H$ satisfying the conditions (1),(2) and (3) is an ideal of $\mathbb{Q}H$.

As a corollary, the center of $\mathbb{Q}H$ equals the $\mathbb{Q}$-linear span of $\text{Ker}(\mu)$, and it is isomorphic to the abelianization of the Lie algebra $\mathbb{Q}H$.

**Corollary 8.20** ([11]). If $(\cdot)$ is non-degenerate, then any ideal of $\mathbb{Q}H$ equals one of the followings

$$0, \mathbb{Q}[0], \mathbb{Q}H \text{ and } \mathbb{Q}(H \setminus \{0\}).$$

This corollary was already obtained by Doković and Zhao [11]. Moreover they asserted that their results covered the degenerate cases. But it was based on the non-correct claim on p.154, l.-10 that the quotient $\mathbb{Q}H/\mathbb{Q}\text{Ker}(\mu)$ would
be isomorphic to the rational homological Goldman Lie algebra associated to the quotient \(H/\text{Ker}(\mu)\).

In [77] he computed the second homology group of \(\mathbb{Q}H\) in the general setting. Let \(\mathbb{Q}H^{(1)}\) be the derived ideal of \(\mathbb{Q}H\), which equals \(\mathbb{Q}(H \setminus \text{Ker}(\mu))\) by Theorem 8.19.

**Theorem 8.21** ([77] Theorem 1). If the pairing \((\cdot \cdot)\) is non-zero, then we have natural isomorphisms
\[
H_2(\mathbb{Q}H) \cong (\wedge^2 \mathbb{Q}\text{Ker}(\mu)) \oplus H_2(\mathbb{Q}H^{(1)}),
\]
\[
H_2(\mathbb{Q}H^{(1)}) \cong \bigoplus_{z \in \text{Ker}(\mu)} \mathbb{Q} \otimes (H/\mathbb{Z}z).
\]

**Corollary 8.22** ([11]). If the pairing \((\cdot \cdot)\) is non-degenerate, then we have \(H_2(\mathbb{Q}H) = H \otimes \mathbb{Q}\).

In [77] Theorem 13, Toda proved that the third homology group of \(\mathbb{Q}H\) is non-trivial if the pairing is non-zero.

Finally we go back to the (original) Goldman Lie algebra. Let \(S\) be a compact connected oriented surface. Then Toda [77] proved

**Theorem 8.23** ([77] Theorem 11). The composite of the map induced by the natural surjection \(\mathbb{Q}\hat{\pi}(S) \to \mathbb{Q}H_1(S; \mathbb{Z})\) and the projection in the decomposition (8.9)
\[
H_2(\mathbb{Q}\hat{\pi}(S)) \to H_2(\mathbb{Q}H_1(S; \mathbb{Z})) \to H_2(\mathbb{Q}H_1(S; \mathbb{Z})^{(1)}))
\]

is surjective.

To prove the theorem, he constructed some explicit abelian 2-cycles in the Goldman Lie algebra \(\mathbb{Q}\hat{\pi}(S)\).

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