We derive the slow-roll conditions for a non-minimally coupled scalar field (extended quintessence) during the radiation/matter dominated era extending our previous results for thawing quintessence. We find that the ratio $\ddot{\phi}/3H\dot{\phi}$ becomes constant but negative, in sharp contrast to the ratio for the minimally coupled scalar field. We also find that the functional form of the equation of state of the scalar field asymptotically approaches that of the minimally coupled thawing quintessence.

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I. INTRODUCTION

There is strong evidence that the Universe is dominated by dark energy, and the current cosmological observations seem to be consistent with $\Lambda$CDM. The equations of state of dark energy, $w$, is close to $-1$ within 10% or less. This implies that even if a scalar field (dubbed "quintessence" [1]) plays the role of dark energy, it should roll down its potential slowly because its kinetic energy density should be much smaller than its potential. In this situation, as in the case of inflation, it is useful to derive the slow-roll conditions for quintessence because the dynamics of the scalar field can be discussed only by simple conditions without having to solve its equation of motion directly. Quintessence models are classified according to their motion [2]: In "thawing" models [3–5] the scalar fields hardly move in the past and begin to roll down the potential recently, while in "freezing" models the scalar fields move in the opposite ways and gradually slow down the motion [6–8]. We will consider the slow-roll conditions for thawing models since there are several particle physics models for them. For example, massive scalar fields (like axions or moduli) before their oscillations move like thawing models. Moreover, polynomial potentials beyond the Planck scale field value can be naturally realized by F-term [9] and D-term [10] in supergravity and in superstring [11], and axion-type potentials are obtained by instanton effects [12].

In our former study, one of us (TC) derived the slow-roll conditions for a scalar field minimally coupled to gravity [5]. It is found that for thawing models the acceleration term, $\ddot{\phi}$, is never negligible compared with the Hubble friction term, $3H\dot{\phi}$, if the Universe is dominated by radiation/matter. Moreover the ratio, $\ddot{\phi}/3H\dot{\phi}$, becomes constant during the radiation/matter dominated epoch [5]:

$$\frac{\ddot{\phi}}{3 H \dot{\phi}} = \frac{1 + w_B}{2},$$

where $w_B$ is the equation of state of radiation/matter. So it is intriguing to examine to what extent the relation Eq. [5] holds universally. In [13], one of us (TC) with Dutta and Scherrer studied the slow-roll conditions for k-essence [14] and again found that the relation Eq. [5] persists for slow-roll k-essence in the radiation/matter dominated era since the k-essence Lagrangian can be Taylor-expanded for small kinetic energy if it is analytical and it reduces to that of canonical scalar field by field redefinition.

In this paper, we further study the slow-roll conditions for a scalar field non-minimally coupled to gravity (called extended quintessence [15]). We will find again that the ratio becomes constant but that its value is negative:

$$\frac{\ddot{\phi}}{3 H \dot{\phi}} = \frac{w_B - 1}{2},$$

being in sharp contrast to the minimally coupled scalar field case. Therefore, the ratio makes the non-minimally coupled scalar field distinguishable from the minimally coupled scalar field even for very small coupling constant $\xi$. 

* Present address: Department of Physics, Tokyo Institute of Technology, Tokyo 152-8551, Japan
The slow-roll extended quintessence can provide a dynamical solution to the coincidence problem: why dark energy dominates recently, not in the past \[16\]. Also if the scalar field has a non-minimal coupling during inflation, such a non-minimal coupling may provide a dynamical solution to the fine-tuning of the initial conditions of the scalar field. Note that the situation is not limited to quintessence but is applied to the case when the scalar fields which are subdominant components in the universe move slowly. Axions, curvaton, and moduli before the oscillation can be such fields.

The paper is organized as follows: In Sec. 2, we derive the slow-roll conditions for non-minimally coupled scalar field during the radiation/matter dominated epoch and discuss the dynamics of the scalar field. In Sec. 3, we derive analytic solutions for the scalar field during the matter dominated era to examine the slow-roll behavior of the scalar field. Sec. 4 is devoted to summary.

II. EXTENDED THAWING QUINTESSENCE

We consider the cosmological dynamics described by the action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R - F(\phi) R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] + S_m. \tag{2}
\]

Here \(\kappa^2 \equiv 8\pi G_{\text{bare}}\) is the bare gravitational constant, \(F(\phi)\) is the non-minimal coupling and \(S_m\) denotes the action of matter (radiation and nonrelativistic particle). We note that since matter is universally coupled to \(g_{\mu\nu}\) in the action Eq. (2), this “Jordan frame metric” defines the lengths and times actually measured by laboratory rods and clocks. All experimental data will thus have their usual interpretation in this frame.

The equations of motion in a flat FRW universe model are

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) + 6F'(\phi) \left( \dot{H} + 2\dot{H}^2 \right) = 0, \tag{3}
\]

\[
3H^2 = \kappa^2 \left( \rho_B + \frac{1}{2} \dot{\phi}^2 + U \right) =: \kappa^2 \rho_{\text{tot}}, \tag{4}
\]

\[
2\dot{H} = -\kappa^2 \left( \rho_B + p_B + \rho_\phi + \dot{\phi}^2/2 - V - 2\dot{F} - 4H\dot{F} - 2F(2\dot{H} + 3H^2) \right) \tag{5}
\]

\[
U := V + 6H \left( \dot{F} + HF \right), \tag{6}
\]

where \(\dot{} = d/d\phi\), \(\rho_B\) and \(p_B\) denote the background (radiation and matter) energy density and pressure, respectively, and \(w_B = p_B/\rho_B\) is the equation of state of radiation and matter.

A. Slow-roll Conditions

We derive the slow-roll conditions for extended (thawing) quintessence during the matter/radiation dominated epoch. Then Eq. (3) becomes

\[
\ddot{\phi} + 3H \dot{\phi} + V_{\text{eff}}' = 0, \tag{7}
\]

\[
V_{\text{eff}}' \equiv V' + 3F' H^2(1 - 3w_B). \tag{8}
\]

By “slow-roll”, we mean that the movement of \(\phi\) during one Hubble time is much smaller than \(\phi\). On the other hand, the condition that the kinetic energy density of the scalar field is much smaller than the potential \(U\) (Eq. (11)) in the energy density of the scalar field \(\rho_\phi\) (Eq. (11))

\[
\frac{1}{2} \dot{\phi}^2 \ll U, \tag{9}
\]

implies that

\[
\dot{\phi}^2 H^{-2} \ll \kappa^{-2} \ll \phi^2, \tag{10}
\]

from \(U \ll \rho_{\text{tot}} \simeq \kappa^{-2} H^2\) if \(\kappa \phi \gtrsim 1\). Hence we regard Eq. (9) as the slow-roll condition.
FIG. 1: $K = \frac{\dot{\phi}^2}{2}$, $V$, $V_{\text{eff}}$, and $U$ (upper figure) and $\kappa \phi$ and $\kappa \dot{\phi}/H$ (lower figure) are shown as a function of $a$ for a massive scalar field model with $F = \frac{1}{2} \xi \phi^2$ with $\xi = 10^{-2}$.

Note that, since the term $|3F'H^2(1 - 3w_B)|$ is much larger than $|V'|$ in $V'_{\text{eff}}$ during the matter or the radiation dominated era, the dynamics of $\phi$ is governed by the term $3F'H^2(1 - 3w_B)$, which is of the same order as $U'$. Thus, different from the minimal coupling case ($\xi = 0$), the kinetic energy density of the scalar field is much larger than $V$

$$\frac{1}{2} \dot{\phi}^2 \gg V,$$

during the slow-roll in the matter or the radiation dominated era. As shown in Fig. 1 during the matter or the radiation dominated era, it is realized that $V \ll \frac{1}{2} \dot{\phi}^2 \ll U$ and also $|\dot{\phi}H^{-1}| \ll \phi$, which guarantees the slow-roll of the scalar field $\phi$.

On the other hand, in the case that the scalar field dominates the energy density of the universe, $U$ almost reduces to $V$ so that the condition Eq. (9) coincides with the slow-roll condition defined in Refs. [17], in which the slow-roll conditions are discussed in the context of the inflation with a non-minimally coupled scalar field.

Unlike the case of inflation, $H$ is not determined by the potential alone, but by the matter/radiation along with the scalar field energy density so that the Hubble friction is not effective and hence $\ddot{\phi}$ is not necessarily small compared with $3H\dot{\phi}$ in Eq. (9) and cannot be neglected.

Now we develop the consistent set of the slow-roll conditions. Following [5, 18], we consider the ratio,

$$\beta = \frac{\ddot{\phi}}{3H\dot{\phi}},$$

For slow-roll (thawing) models, we first assume that $\beta$ is an $O(1)$ approximately constant quantity not equal to $-1$ in the sense $|\beta| \ll H|\beta|$, and the consistency of the assumption will be checked later. In terms of $\beta$, using Eq. (10), $\dot{\phi}$
is rewritten as

$$
\dot{\phi} = - \frac{V'_\text{eff}}{3(1 + \beta)H},
$$

(13)

and the condition Eq. [5] gives the first one of the slow-roll conditions

$$
\epsilon := \frac{V''_\text{eff}}{6H^2U} \ll 1,
$$

(14)

where we have omitted $1 + \beta$ since it is an $\mathcal{O}(1)$ quantity and introduced the factor of $1/6$ in $\epsilon$ so that $\epsilon$ coincides with the inflationary slow-roll parameter, $\epsilon = \frac{1}{2} \left( \frac{V'}{\kappa V} \right)^2$, if the scalar field dominates the expansion: $H^2 \simeq \kappa^2 V/3$ and $U \simeq V_{\text{eff}} \simeq V$.

Similar to the case of inflation, the consistency of Eq. (12) and Eq. (7) should give the second slow-roll condition. In fact, from the time derivative of Eq. (13)

$$
\ddot{\phi} = - \frac{\ddot{H}}{H} \dot{\phi} - \frac{V''}{3(1 + \beta)H} \dot{\phi} - \frac{F''H(1 - 3w_B)}{1 + \beta} \dot{\phi} + \frac{3F' H^2(1 - 3w_B)}{1 + \beta} - \frac{\dot{\beta}}{1 + \beta} \dot{\phi},
$$

(15)

where we have used $(H^2(1 - 3w_B)) \simeq -3H^3(1 - 3w_B)$. On the other hand, from Eq. (12) and Eq. (13), $\ddot{\phi} = 3\beta H \dot{\phi} = -\beta V'_{\text{eff}}/(1 + \beta)$, and so we obtain

$$
\beta = \frac{3}{3H} \frac{\ddot{\phi}}{\dot{\phi}} \simeq - \frac{\ddot{H}}{3H^2} - \frac{V''}{9(1 + \beta)H^2} - \frac{F''(1 - 3w_B)}{3(1 + \beta)} - \frac{V''_{\text{eff}}}{V'_{\text{eff}}},
$$

(16)

$$
= \frac{w_B - 1}{9(1 + \beta)} - \frac{V''}{9(1 + \beta)H^2} - \frac{F''(1 - 3w_B)}{3(1 + \beta)} + \frac{V'}{V'_{\text{eff}}},
$$

where we have used $3F' H^2(1 - 3w_B) = V'_{\text{eff}} - V'$ and $|\beta| \ll H|\beta|$. While the left-hand-side of Eq. (16) is assumed to be an almost time-independent quantity, the terms other than the first in the right-hand-side are time-dependent quantities in general. Therefore the assumption is consistent if they are negligible:

$$
\eta := \frac{V''}{3H^2}; \quad |\eta| \ll 1 \quad \text{and} \quad |F''(1 - 3w_B)| \ll 1 \quad \text{and} \quad \left| \frac{V'}{V'_{\text{eff}}} \right| \ll 1,
$$

(17)

so that $\beta$ becomes

$$
\beta = \frac{w_B - 1}{2}.
$$

(18)

$\beta$ given by Eq. (18) is consistently an $\mathcal{O}(1)$ constant not equal to $-1$. Here the factor $1/3$ is introduced in $\eta$ so that $\eta$ coincides with the inflationary slow-roll parameter, $\eta = \frac{V''}{V'}$, if $H^2 \simeq \kappa^2 V/3$. The conditions in Eq. (17) are quintessence counterparts of the inflationary slow-roll condition $\frac{\ddot{V}}{V} \ll 1$.

Eq. (14) and Eq. (17) constitute the slow-roll conditions for extended quintessence during matter/radiation epoch. $\beta$ (Eq. (18)) is negative and is quite different from that for a minimally coupled scalar field (Eq. (1)) which is positive. Therefore, this can be a discriminating probe of the non-minimal coupling of the scalar field. Although it may be difficult to determine the thawing dynamics from distance measurements [13, 19], the ratio $\beta$ may be determined by measuring the time variation of the fine structure constant $\alpha$ if $\phi$ induces such a variation [20] and $\alpha$ depends linearly on $\phi$.

In Fig. 2 the evolution of $\beta$ is shown for a massive scalar field ($V = \frac{1}{2} m^2 \phi^2$) with a non-minimal coupling $F = \frac{1}{2} \xi \phi^2$ with $\xi = 10^{-2}$. The evolution of $\beta$ agrees nicely with Eq. (18).

---

1 The exception is the case of $F'' = \text{const}$. In this case $F''$ needs not to be small. For example, if $F = \frac{1}{2} \xi \phi^2$, then $\beta$ satisfies $\beta = -\frac{1}{3}$ during the radiation era and $\beta = -\frac{1}{3} \sqrt{\frac{4\xi \phi^2}{(1 + \beta)}}$, so that $\beta = -\frac{1}{3} \sqrt{\frac{4\xi \phi^2}{1 + \beta}}$ during the matter era.
FIG. 2: $\beta$ as a function of $a$ for a massive scalar field model with $F = \frac{1}{2} \xi \phi^2$ with $\xi = 10^{-2}$. The dotted lines are $\beta = -\frac{1}{3}, -\frac{1}{2}$, respectively.

B. Tracking without Tracking

In the following we examine the cosmological dynamics of the extended quintessence using the slow-roll conditions Eq. (14) and Eq. (17). We shall first show that the equation of state of the slow-roll extended quintessence, $w_\phi = p_\phi/\rho_\phi$, is the same as the background equation of state, $w_B$, although the scalar field moves slowly [16].

Consider a slowly rolling scalar field non-minimally coupled to gravity which satisfies the above slow-roll conditions. If the universe is dominated by radiation/matter, the energy density of the scalar field then becomes

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V + 6H \left( \dot{F} + HF \right) \simeq 6H^2 F, \quad (19)$$

and the pressure becomes

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V - 2\ddot{F} - 4HF - 2\dot{F}(2\dot{H} + 3H^2) \simeq 6w_B H^2 F. \quad (20)$$

Since $F(\phi)$ is almost constant, this implies that the equation of state of the scalar field $w_\phi = p_\phi/\rho_\phi$ becomes $w_B$ and behaves as background fluid although $\phi$ itself moves hardly (See Fig. 3). We dubbed this behavior as “tracking without tracking”\textsuperscript{2} [16]. We should emphasize that this behavior is independent of the details of the shape of a

\textsuperscript{2} It is no surprise that the equation of state of tracking without tracking state is the same as $w_B$ because in this case the Lagrangian
potential as long as the slow-roll conditions, Eq. (14) and Eq. (17), are satisfied. “Tracking without tracking” is rather kinematical tracker inherent in a wide class of extended quintessence.

Next, we shall consider a scenario based on extended quintessence which solves the coincidence problem: why dark energy becomes dominant now? In order to “solve” the problem dynamically, the dark energy density should scale in the same way as the radiation density during the radiation dominated epoch; otherwise it is nothing but introducing a fine-tuning to account for the coincidence from the very beginning, and it is no surprise that there is some epoch when the two energy components coincide. On the other hand, however, during the matter dominated epoch, dark energy should not track matter; otherwise dark energy cannot dominate.

The slow-roll extended quintessence tracks the background matter during the radiation dominated epoch (tracking without tracking) but it can begin to move during the matter dominated epoch due to the violation of one (or some) of the slow-roll conditions so that $w_\phi$ deviates from $w_B$ and the scalar field eventually dominates the universe. In Fig. 3, the energy density of the scalar field is shown.

It is to be noted that one of the slow-roll conditions (the second one in Eq. (17)) depends on the equation of state of background matter $w_B$ and are automatically satisfied during the radiation dominated epoch $w_B \approx 1/3$. Hence, the non-minimal coupling with $F'' \approx O(1)$ naturally realizes the scenario.

density of the scalar field is simply $-F(\phi)R$. Therefore the energy momentum tensor of the scalar field is the same as the Einstein tensor which is dominated by the background matter.
C. Dynamical Solution of Fine-tuning Problems

We note that in order to solve the dark energy "why now problem", it is not sufficient to explain the miniscule energy scale of dark energy, but the solution should explain why the dark energy becomes dominant after the matter dominated epoch. For example, particle physics models of quintessence axion have been constructed which realize the tiny mass scale ($\simeq H_0 \approx 10^{-33}$ eV) via instanton effects. However, the dynamics of the quintessence axion field is dependent on the initial condition, and if the field is initially near the minimum of the potential, the scalar field soon oscillates around the minimum and can never dominate the universe. Therefore, in the context of quintessence axion, the why now problem is replaced with the initial condition problem: why the scalar field started near the top of the potential?

A non-minimal coupling can alleviate the fine-tuning. Consider the situation where the global PQ-like symmetry is broken (to $Z_2$) during inflation and quintessence axion acquires a non-minimal coupling of the form

$$F(\phi) = \xi f^2 \cos(\phi/f).$$

(21)

Then during inflation, due to the large curvature $R \simeq 12H^2$, the scalar field is dynamically tuned toward the minimum of $F(\phi)$. Depending on the sign of $\xi$, the axion is thus dynamically tuned toward the maximum/minimum of $\cos(\phi/f)$: $\phi \to \pi f$ for $\xi > 0$; $\phi \to 0$ for $\xi < 0$. Therefore a $\xi > 0$ case can be used for a dynamical solution of the fine-tuning problem of quintessence axion. Note that if the minimum of $F(\phi)$ coincides with the local maximum of $V(\phi)$, it can hardly start rolling down and almost behaves like the cosmological constant because the quantum fluctuations are significantly suppressed due to the large positive effective masses squared.

III. ANALYTIC SOLUTIONS

In this section, we analytically investigate the dynamics of $\phi$ and its equation of state $w_\phi(a) = p_\phi/\rho_\phi$, as done in Refs. [4, 5, 13]. We consider the case that the non-minimal coupling is given by $F(\phi) = \frac{1}{2} \xi \phi^2$. In this case, from Eqs. (22-23), the equation of motion and the equation of state are given by

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) + 6\xi \phi \left( \dot{H} + 2H^2 \right) = 0,$$

(22)

$$w_\phi = p_\phi/\rho_\phi,$$

(23)

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V + 3\xi H \left( 2\phi \dot{\phi} + H \phi^2 \right),$$

(24)

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V - 2\xi(\dot{\phi}^2 + \phi \ddot{\phi}) - 4\xi H \dot{\phi}^2 - \xi \phi^2 (2\dot{H} + 3H^2).$$

(25)

First of all, we change the variable in order to eliminate the first derivative in Eq. (22),

$$u = (\phi - \phi_i)a^{3/2},$$

(26)

where $\phi_i$ is an arbitrary constant and is set to the initial value later. Then, the equation of motion becomes

$$\ddot{u} + \left[ -\frac{3}{2} \left( \dot{H} + \frac{3}{2} H^2 \right) + 6\xi \left( \dot{H} + 2H^2 \right) \right] u + \left[ V' + 6\xi \phi_i \left( \dot{H} + 2H^2 \right) \right] a^2 = 0.$$

(27)

Since we are interested in the slow-roll motion of the quintessence field $\phi$, we may expand the potential $V(\phi)$ around the initial value $\phi_i$ up to the quadratic order [3],

$$V(\phi) = V(\phi_i) + V'(\phi_i)(\phi - \phi_i) + \frac{1}{2} V''(\phi_i)(\phi - \phi_i)^2.$$

(28)

Since the present equation of state of $\phi$ should be almost $-1$, we also assume that the scale factor $a(t)$ is well approximated by that in the $\Lambda$CDM model, which is given by

$$a(t) = \left( \frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}} \right)^{1/3} \sinh^{2/3}(t/t_\Lambda),$$

(29)

where $\Omega_{\phi 0}$ is the present density parameter of quintessence $\phi$, the scale factor $a$ is normalized to $a = 1$ at present, and $t_\Lambda$ is defined as

$$t_\Lambda = \frac{2}{\sqrt{3\kappa^2 V(\phi_i)}}.$$

(30)
Then, the equation motion can be written as
\[
\ddot{u} + \left[ V''(\phi_i) - \frac{3}{4} \kappa^2 V(\phi_i) + \xi \kappa^2 V(\phi_i) \left\{ 3 + \coth^2 \left( \frac{t}{t_\Lambda} \right) \right\} \right] u \\
+ \left[ V'(\phi_i) + \xi \kappa^2 \phi_i V(\phi_i) \left\{ 3 + \coth^2 \left( \frac{t}{t_\Lambda} \right) \right\} \right] \left\{ 1 - \frac{\Omega_{\phi 0}}{\Omega_{\phi 0}} \right\}^{1/2} \sinh \left( \frac{t}{t_\Lambda} \right) = 0.
\]

Unfortunately, unlike the case of the minimal coupling $\xi = 0$, we cannot solve this equation analytically for the whole range of the cosmic time. Instead, we consider the two extreme regions.

- **Region I** ($t \ll t_\Lambda$):

  The first one is the region with $t \ll t_\Lambda$. In this region, the equation of motion reduces to

\[
\ddot{u} + \frac{4}{3} \kappa^2 u + \frac{4}{3} \left( \frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}} \right)^{1/2} \xi \phi_i \frac{1}{t_\Lambda} = 0,
\]

whose solution is given by

\[
\phi(t) = \frac{1}{A} \left[ \left( \frac{1 + A}{2} \phi_i + t_i(\dot{\phi}_i) \right) \left( \frac{t}{t_i} \right)^{-\frac{1+A}{2}} + \left( -\frac{1 + A}{2} \phi_i - t_i(\dot{\phi}_i) \right) \left( \frac{t}{t_i} \right)^{-\frac{1-A}{2}} \right]
\]

with $A = \sqrt{1 - \frac{4}{3} \xi}$. This solution satisfies $\phi = \phi_i$ and $\dot{\phi} = (\dot{\phi}_i)$ at $t = t_i$.

As was explicitly shown for the general coupling $F(\phi)$ in the subsection II.B in this region, the equation of state of $\phi$ is almost equal to the equation of state of the background matter, that is, $w_\phi \simeq w_B$.

- **Region II** ($t \gtrsim t_\Lambda$):

  Next, we consider the region with $t \gtrsim t_\Lambda$. Then, the equation of motion reduces to

\[
\ddot{u} - k^2 u + \left[ V'(\phi_i) + 4 \xi \kappa^2 \phi_i V(\phi_i) \right] \left( \frac{1 - \Omega_{\phi 0}}{\Omega_{\phi 0}} \right)^{1/2} \sinh \left( \frac{t}{t_\Lambda} \right) = 0
\]

with $k \equiv \sqrt{\left( \frac{q}{4} - 4 \xi \right) \kappa^2 V(\phi_i) - V''(\phi_i)}$.

Apart from the coefficients, this equation coincides with that in the minimal coupling $\Phi$. Then, the solution for $K \equiv kt_\Lambda \neq 1$ is given by

\[
\phi(t) - \phi_i = \frac{\sinh(t_i/t_\Lambda)}{kt_\Lambda \sinh(t/t_\Lambda)} \left[ \sinh(kt) \cosh(kt_i) \left\{ \frac{V'(\phi_i) + 4 \xi \kappa^2 \phi_i V(\phi_i)}{V''(\phi_i)} \left( \coth \left( \frac{t_i}{t_\Lambda} \right) - kt_\Lambda \tanh(kt_i) \right) + t_\Lambda \phi_i \right\} \\
- \cosh(kt) \sinh(kt_i) \left\{ \frac{V'(\phi_i) + 4 \xi \kappa^2 \phi_i V(\phi_i)}{V''(\phi_i)} \left( \coth \left( \frac{t_i}{t_\Lambda} \right) - kt_\Lambda \coth(kt_i) \right) + t_\Lambda \phi_i \right\} \right] \\
- \frac{V'(\phi_i) + 4 \xi \kappa^2 \phi_i V(\phi_i)}{V''(\phi_i)}.
\]

As far as $a(t_\Lambda) \ll a(t)$ and $a(t_i) \ll a(t)$, this solution may be approximated by that with $t_i = 0 = t_\Lambda$,

\[
\phi(t) = \phi_i + \frac{V'(\phi_i) + 4 \xi \kappa^2 \phi_i V(\phi_i)}{V''(\phi_i)} \left[ \frac{\sinh(kt) \cosh \left( \frac{t}{t_\Lambda} \right)}{kt_\Lambda \sinh \left( \frac{t}{t_\Lambda} \right)} - 1 \right].
\]

For $t \gtrsim t_\Lambda$, the dynamics of the field $\phi$ with $\xi \ll 1$ is determined by the potential term $V$ so that $\rho_\phi \simeq V(\phi_i)$.
and \( \rho_\phi + p_\phi = \dot{\phi}^2 - 2\xi(\ddot{\phi} + \dot{\phi}^2) + 2\xi H \dot{\phi}^2 - 2\xi H \dot{\phi}^2 \simeq \dot{\phi}^2 \). Then, the equation of state \( w_\phi \) is given by

\[
1 + w_\phi \simeq \frac{\dot{\phi}^2}{V(\phi)}
\]

\[
= \frac{3\kappa^2}{4} \cosh^2 \left( \frac{t}{t_\Lambda} \right) \left( \frac{V'(\phi) + 4\xi \kappa^2 \phi V(\phi)}{\kappa A V''(\phi)} \right)^2 \left[ \frac{k \omega \cosh(kt) \sinh \left( \frac{t}{t_\Lambda} \right) - \sinh(kt) \cosh \left( \frac{t}{t_\Lambda} \right) }{\sinh^2 \left( \frac{t}{t_\Lambda} \right)} \right]^2
\]

\[
= (1 + w_{\phi_0}) a^{3(K-1)} \left[ \frac{(K - F(a)(F(a) + 1)^K + (K + F(a))(F(a) - 1)^K)}{(K - \Omega_{\phi_0}^{-1/2})(\Omega_{\phi_0}^{-1/2} + 1)^K + (K + \Omega_{\phi_0}^{-1/2})(\Omega_{\phi_0}^{-1/2} - 1)^K} \right]^2,
\]

(37)

where \( w_{\phi_0} \) is the present equation of state of the field \( \phi \), \( F(a) = \sqrt{1 + (\Omega_{\phi_0}^{-1} - 1)a^{-3}} \) and \( K = k t_\Lambda = \sqrt{1 - \frac{16\xi}{3} - \frac{4 V''(\phi)}{3 \kappa^2 V(\phi)}} \). We have normalized the expression to \( w_{\phi_0} \) in Eq. (37). This expression completely coincides with that in the minimal coupling [4, 5, 13]. However, it is noted that this expression applies only for \( t \gtrsim t_\Lambda \) and that the definition of \( K \) is different but, when \( \xi \) approaches 0, reduces to that of the minimally coupled scalar field.

Finally, the solution for \( K = k t_\Lambda = 1 \) is given by

\[
\dot{\phi}(t) = \dot{\phi}_i + \frac{2}{3} \left[ \frac{V'(\phi_i)}{\kappa^2 V(\phi_i)} + 4 \xi \dot{\phi}_i \right] \left( 1 - \frac{kt}{\tanh(kt)} \right),
\]

(38)

which yields the equation of state,

\[
1 + w_\phi = \frac{3}{\kappa^2} \left( \frac{V'(\phi) + 4\xi \kappa^2 \phi V(\phi)}{\kappa^2 V(\phi)} \right)^2 \left( \frac{\sinh(kt) \cosh(kt) - kt}{\sinh^2(kt)} \right)^2
\]

\[
= (1 + w_{\phi_0}) \left[ \frac{F(a) - 1 - \Omega_{\phi_0}^{-1/2}}{\Omega_{\phi_0}^{-1/2} - 1 - \Omega_{\phi_0}^{-1/2}} \ln \left\{ \sqrt{\frac{\Omega_{\phi_0} a^{-3}}{1 - \Omega_{\phi_0} a^{-3}}} (1 + F(a)) \right\} \right]^2.
\]

(39)

In Fig. 3 \( w_\phi \) is shown as a function of \( a \). We find that apart from the slight offset \( w_\phi \) approaches the asymptotic solution given by Eq. (37). This, together with [13], makes the functional form of \( w_\phi(a) \) derived in [4, 5] even more useful. It is noted, however, that the asymptotic solution is actually a transient solution since the scalar field would oscillate around the minimum of \( V \) in the future and \( w_\phi \) would tend to 0.

**IV. SUMMARY**

We have derived the slow-roll conditions for non-minimally coupled scalar field during the radiation/matter dominated epoch by extending the previous results for a minimally coupled scalar field [4] and for non-minimally coupled inflaton(s) [13]. We have also derived the slow-roll equation of motion of the scalar field and found that the ratio \( \dot{\phi}/3H \dot{\phi} \) becomes constant but negative, in sharp contrast to the result for the minimally coupled scalar field. This ratio can be a discriminating probe of the non-minimal coupling of the scalar field.

We have presented two applications of the slow-roll extended quintessence: a dynamical solution to the coincidence problem ("tracking without tracking") and a dynamical solution to the fine-tuning problem of quintessence axion.

We have solved the equation of motion for two limiting cases and found that for \( t \gtrsim t_\Lambda \) the functional form of the equation of state of the scalar field coincides with that of the minimally coupled thawing quintessence derived in [4, 5]. While this strengthens the universality of the functional form of \( w_{\phi}(a) \) derived in [4, 5], this also implies "the attraction toward minimality": the scalar field dynamics reduces to that of the minimally coupled scalar field. It would be interesting to investigate whether this property holds more generally.

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FIG. 4: $w_\phi$ as a function of $a$. The solid line is the numerical solution, while the dotted line is the asymptotic solution Eq. (37). Computations were performed at YITP at Kyoto University.

[1] R. R. Caldwell, R. Dave and P. J. Steinhardt, Phys. Rev. Lett. 80, 1582 (1998).
[2] R. R. Caldwell and E. V. Linder, Phys. Rev. Lett. 95, 141301 (2005) arXiv:astro-ph/0505494.
[3] J. A. Frieman, C. T. Hill, A. Stebbins and I. Waga, Phys. Rev. Lett. 75, 2077 (1995) arXiv:astro-ph/9505060.
[4] S. Dutta and R. J. Scherrer, Phys. Rev. D 78, 123525 (2008) [arXiv:0809.4411 [astro-ph]].
[5] T. Chiba, Phys. Rev. D 79, 083517 (2009) [Erratum-ibid. D 80, 109902 (2009)] arXiv:0902.4037 [astro-ph.CO].
[6] B. Ratra and P. J. E. Peebles, Phys. Rev. D 37, 3406 (1988).
[7] J. A. Frieman, C. T. Hill, A. Stebbins and I. Waga, Phys. Rev. Lett. 75, 2077 (1995) arXiv:astro-ph/9505060.
[8] S. Dutta and R. J. Scherrer, Phys. Rev. D 78, 123525 (2008) [arXiv:0809.4411 [astro-ph]].
[14] T. Chiba, T. Okabe and M. Yamaguchi, Phys. Rev. D 62, 023511 (2000) [arXiv:astro-ph/9912463]; C. Armendariz-Picon, V. F. Mukhanov and P. J. Steinhardt, Phys. Rev. Lett. 85, 4438 (2000) [arXiv:astro-ph/0004134].

[15] J. P. Uzan, Phys. Rev. D 59, 123510 (1999) [arXiv:gr-qc/9903004]; L. Amendola, Phys. Rev. D 60, 043501 (1999) [arXiv:gr-qc/9903094]; T. Chiba, Phys. Rev. D 60, 083508 (1999) [arXiv:gr-qc/9903094]; F. Perrotta, C. Baccigalupi and S. Matarrese, Phys. Rev. D 61, 023507 (1999) [arXiv:astro-ph/9906066]; O. Bertolami and P. J. Martins, Phys. Rev. D 61, 064007 (2000) [arXiv:gr-qc/9910056]; B. Boisseau, G. Esposito-Farese, D. Polarski and A. A. Starobinsky, Phys. Rev. Lett. 85, 2236 (2000) [arXiv:gr-qc/0001066]; G. Esposito-Farese and D. Polarski, Phys. Rev. D 63, 063504 (2001) [arXiv:gr-qc/0009034].

[16] T. Chiba, Phys. Rev. D 64, 103503 (2001) [arXiv:astro-ph/0106550].

[17] T. Chiba and M. Yamaguchi, JCAP 0810, 021 (2008) [arXiv:0807.4965 [astro-ph]]; T. Chiba and M. Yamaguchi, JCAP 0901, 019 (2009) [arXiv:0810.5387 [astro-ph]].

[18] R. Crittenden, E. Majerotto and F. Piazza, Phys. Rev. Lett. 98, 251301 (2007) [arXiv:astro-ph/0702003].

[19] S. Sen, A. A. Sen and M. Sami, Phys. Lett. B 686, 1 (2010) [arXiv:0907.2814 [astro-ph.CO]].

[20] P. P. Avelino, C. J. A. Martins, N. J. Nunes and K. A. Olive, Phys. Rev. D 74, 083508 (2006) [arXiv:astro-ph/0605690].