Abstract. In this paper we study the $C^*$-convex set of unital entanglement breaking (EB-)maps on matrix algebras. General properties and an abstract characterization of $C^*$-extreme points are discussed. By establishing a Radon-Nikodym type theorem for a class of EB-maps we give a complete description of the $C^*$-extreme points. It is shown that a unital EB-map $\Phi : M_{d_1} \to M_{d_2}$ is $C^*$-extreme if and only if it has Choi-rank equal to $d_2$. Finally, as a direct consequence of the Holevo form of EB-maps, we derive a noncommutative analogue of the Krein-Milman theorem for $C^*$-convexity of the set of unital EB-maps.

1. Introduction

Completely positive (CP-)maps between matrix algebras that are entanglement breaking (EB) plays an important role in quantum information theory. The class of (unital) CP-maps and its subclass of (unital) EB-maps form ($C^*$-)convex sets, and hence one would like to identify its extreme points. Though the linear extreme points of the convex set of unital CP-maps are not well-understood objects, its quantum analogue, called $C^*$-extreme points, have been explored to a great extent (see [FaMo97, FaZh98, Gre09, BBK21, BhKu21]). A complete description of $C^*$-extreme points of the convex set of unital CP-maps on matrix algebras is known ([FaMo97, FaZh98]). But, except for some necessary or sufficient conditions ([Gre09, BBK21, BhKu21]), a complete description of the structure of $C^*$-extreme points is unknown in general, for example, when the underlying Hilbert space is infinite-dimensional.

Though various characterizations and structure theorems for EB-maps between matrix algebras are known ([HSR03, Hol98]), the structure of linear extreme points of the convex set of unital or trace-preserving EB-maps is not well-understood. In [HSR03], the authors considered a class of trace-preserving EB-maps, called extreme classical-quantum (CQ) maps, and showed that such maps are linear extreme points of the convex set of trace-preserving EB-maps; but they are not all in general ([HSR03 Counterexample]). (Note that a CP-map $\Phi : M_{d_1} \to M_{d_2}$ is unital if and only if the CP-map $\Phi^* : M_{d_2} \to M_{d_1}$ is trace-preserving, where $\Phi^*$ is the adjoint of $\Phi$ with respect to the Hilbert-Schmidt inner product given by $\langle A, B \rangle := \text{tr}(A^*B)$.) In this article, we introduce and study $C^*$-extreme points of the convex set of unital EB-maps.
We organize the paper as follows. Section 2 recalls some definitions and basic results and fix some notations. In Section 3, we observe that some of the basic properties of $C^*$-extreme points of unital CP-maps will also hold in the case of unital EB-maps. In particular, every $C^*$-extreme point of unital EB-maps is a linear extreme point (Proposition 3.3). In the case of unital CP-maps, the Radon Nikodym theorem for CP-maps ([Arv69]) provides the basic framework for determining its $C^*$-extreme points. Keeping this in mind, in Section 4, we prove a Radon Nikodym type theorem (Theorem 4.2) for a particular class of EB-maps. In Section 5, using this result, we give an abstract characterization (Theorem 5.1) and further prove a structure theorem (Theorem 5.3) for $C^*$-extreme points of unital EB-maps. It is seen that whether an EB-map is $C^*$-extreme or not can be determined by knowing its Choi-rank. The only $C^*$-extreme points of unital EB-maps are direct sum of pure states, and this establishes that (Remark 5.6) adjoints of the class of extreme CQ-maps discussed in [HSR03] coincides with the $C^*$-extreme points. Finally, a Krein-Milman type theorem is proved for the $C^*$-convexity of the convex cone of unital EB-maps.

2. Preliminaries

Throughout, $d, d_1, d_2 \in \mathbb{N}$, and $\{e_i\}_{i=1}^d \subseteq \mathbb{C}^d$ denote the standard orthonormal basis. We let $M_{d_1 \times d_2}$ denote the space of all complex matrices of size $d_1 \times d_2$, and $M_d^+ = \{A \in M_d : A \geq 0\}$ denote the cone of all positive semidefinite matrices in $M_d := M_{d \times d}$. We write $A = \begin{bmatrix} a_{ij} \end{bmatrix} \in M_{d_1 \times d_2}$ to denote that $A$ is a complex matrix of size $d_1 \times d_2$ with $(i, j)^{th}$ entry $a_{ij} \in \mathbb{C}$. Given $A = \begin{bmatrix} A_{ij} \end{bmatrix} \in M_d$, and $B = \begin{bmatrix} b_{ij} \end{bmatrix} \in M_d$, we define $A \otimes B := \begin{bmatrix} a_{ij} b_{kl} \end{bmatrix}$, and thereby identify $M_{d_1} \otimes M_{d_2} = M_{d_1} (M_{d_2}) = M_{d_1 d_2}$. A matrix $X \in (M_{d_1} \otimes M_{d_2})^+$ is said to be separable if $X = \sum_{i=1}^{d_1} A_i \otimes B_i$ for some $A_i \in M_{d_1}^+$ and $B_i \in M_{d_2}^+$; otherwise called entangled. If $X \in (M_{d_1} \otimes M_{d_2})^+$ is separable, then the optimal ensemble cardinality ([DTT00]) or length ([CDD13]) of $X$ is the minimum number $\ell(X)$ of rank-one positive operators $A_i, B_i$ required to write $X = \sum A_i \otimes B_i$. Clearly $\text{rank}(X) \leq \ell(X)$; strict inequality can also happen ([DTT00]).

A linear map $\Phi : M_{d_1} \to M_{d_2}$ is said to be a completely positive (CP) map if for every $k \geq 1$ the map $id_k \otimes \Phi : M_k \otimes M_{d_1} \to M_k \otimes M_{d_2}$ satisfies $(id_k \otimes \Phi)(M_k \otimes M_{d_1})^+ \subseteq (M_k \otimes M_{d_2})^+$, where $id_k : M_k \to M_k$ is the identity map. A linear map $\Phi : M_{d_1} \to M_{d_2}$ is a CP-map if and only if the Choi-matrix ([Cho75]),

$$C_\Phi := \sum_{i, j=1}^{d_1} E_{i j} \otimes \Phi(E_{i j}) = [\Phi(E_{i j})] \in (M_{d_1} \otimes M_{d_2}) = M_{d_1} (M_{d_2})$$

is positive semidefinite, where $E_{i j} := |e_i\rangle \langle e_j| \in M_{d_i}$ for all $1 \leq i, j \leq d_1$. This condition is also equivalent to saying that $\Phi$ has a Kraus decomposition ([Kra71]), i.e.,

$$\Phi = \sum_{i=1}^{n} \text{Ad}_{V_i}$$

for some $V_i \in M_{d_1 \times d_2}$ called Kraus operators, where $\text{Ad}_V (X) := V^* XV$ for all $X \in M_{d_1}$. There is no uniqueness in Kraus decomposition. But, if $\Phi = \sum_{i=1}^{n} \text{Ad}_{V_i} = \sum_{i=1}^{m} \text{Ad}_{W_i}$, are two
Kraus decomposition, then it can be shown that \( \text{span}\{V_i : 1 \leq i \leq n\} = \text{span}\{W_i : 1 \leq i \leq m\} \). The minimum number of linearly independent Kraus’ operators required to represent \( \Phi \) by a Kraus decomposition is known as the \textit{Choi-rank} of \( \Phi \).

We let \( \text{UCP}(d_1, d_2) \) denote the space of all unital CP-maps from \( M_{d_1} \) into \( M_{d_2} \). Elements of \( \text{UCP}(d, 1) \) are called states. Given any \( \Phi \in \text{UCP}(d_1, d_2) \), due to Stinespring ([Sti55]), there exists a triple \((\pi, V, \mathcal{K})\), called Stinespring’s dilation, consisting of a Hilbert space \( \mathcal{K} \), a representation \( \pi \) of \( M_{d_1} \) in the algebra \( \mathcal{B}(\mathcal{K}) \) of all bounded linear maps on \( \mathcal{K} \), and an isometry \( V : \mathcal{C}^{d_2} \to \mathcal{K} \) such that \( \Phi = Ad_V \circ \pi; \) the triple is said to be minimal if \( \text{span}\{\pi(M_{d_1})V\mathcal{C}^{d_2}\} = \mathcal{K} \), and such a triple is unique up to unitary equivalence. If \( \Psi : M_{d_1} \to M_{d_2} \) is a CP-map such that the difference \( \Phi - \Psi \) is also a CP-map, then we write \( \Psi \leq_{CP} \Phi \). In such cases, due to [Arv69, Theorem 1.4.2], there exists a positive contraction \( T \) in the commutant \( \pi(M_{d_1})' \subseteq \mathcal{B}(\mathcal{K}) \) of \( \pi(M_{d_1}) \) such that \( \Psi(X) = V^* T \pi(X) V \) for all \( X \in M_{d_1} \). (The conclusion holds even if \((\pi, V, \mathcal{K})\) is not minimal.)

A completely positive map \( \Phi : M_{d_1} \to M_{d_2} \) is said to be

- **irreducible** if \( \text{range}(\Phi)' = \mathcal{C}I_{d_2} \); equivalently \( \text{range}(\Phi) \) has only trivial invariant subspaces;
- **pure** if, whenever \( \Psi : M_{d_1} \to M_{d_2} \) is a CP-map such that \( \Psi \leq_{CP} \Phi \), then \( \Psi = \lambda \Phi \) for some scalar \( \lambda \geq 0 \).

There are different notions of irreducibility for CP-maps. The definition presented here is motivated from the representation theory of \( C^* \)-algebras. It can be seen easily that a state \( \phi : M_d \to \mathcal{C} \) is pure if and only if \( \phi(X) = \langle u, Xu \rangle = \text{tr}(X|u\rangle\langle u|) \) for some unit vector \( u \in \mathcal{C}^d \), where given \( x \in \mathcal{C}^{d_1} \), \( y \in \mathcal{C}^{d_2} \), the matrix \( |x\rangle \langle y| \in M_{d_1 \times d_2} \) defines the linear map \( z \mapsto \langle y, z|x \rangle \) from \( \mathcal{C}^{d_2} \) to \( \mathcal{C}^{d_1} \).

Suppose \( \Phi_i \in \text{UCP}(d_1, d_2) \) and \( T_i \in M_{d_2} \), \( 1 \leq i \leq n \) are such that \( \sum_{i=1}^n T_i^* T_i = I \). Then the sum \( \sum_{i=1}^n \text{Ad}_{T_i^*} \Phi_i \) is called a \( C^* \)-convex combination, which is said to be proper if all the \( T_i \)'s are invertible. A linear map \( \Phi \) in \( \text{UCP}(d_1, d_2) \) is said to be a

- **linear extreme point** of \( \text{UCP}(d_1, d_2) \) if, whenever \( \Phi = \sum_{i=1}^n t_i \Phi_i \), where \( \Phi_i \in \text{UCP}(d_1, d_2) \) and \( t_i \in (0, 1) \) with \( \sum_{i=1}^n t_i = 1 \), then \( \Phi_i = \Phi \) for all \( 1 \leq i \leq n \);
- **\( C^* \)-extreme point** of \( \text{UCP}(d_1, d_2) \) if, whenever \( \Phi \) is a proper \( C^* \)-convex combination, say \( \Phi = \sum_{i=1}^n \text{Ad}_{T_i^*} \Phi_i \), then each \( \Phi_i \) is unitarily equivalent to \( \Phi \), i.e., there exist unitaries \( U_i \in M_{d_2} \) such that \( \Phi_i = \text{Ad}_{U_i} \Phi \) for all \( 1 \leq i \leq n \).

A linear map \( \Phi : M_{d_1} \to M_{d_2} \) is said to be a \textit{PPT-map} if both \( \Phi \) and \( \text{Tr} \circ \Phi \) are CP-maps (equivalently, both \( C_\Phi \) and \( (\text{id}_{d_1} \otimes \text{Tr})(C_\Phi) \) are elements of \( (M_{d_1} \otimes M_{d_2})^+ \)), where \( T : M_{d_2} \to M_{d_2} \) denotes the transpose map. A completely positive map \( \Phi : M_{d_1} \to M_{d_2} \) is said to be an \textit{entanglement breaking (EB)-map} if for every \( k \geq 1 \) and for every \( X \in (M_k \otimes M_{d_1})^+ \) the matrix \( (\text{id}_k \otimes \Phi)(X) \in (M_k \otimes M_{d_2})^+ \) is separable.

**Theorem 2.1** ([Hol98, HSR03]). Given a CP-map \( \Phi : M_{d_1} \to M_{d_2} \) the following conditions are equivalent:

(i) \( \Phi \) is an EB-map.
(ii) \( \Phi \in (\mathcal{M}_{d_1} \otimes \mathcal{M}_{d_2})^+ \) is separable.

(iii) There exist rank-one operators \( V_i \in \mathcal{M}_{d_1} \times d_2 \), \( 1 \leq i \leq n \), such that \( \Phi = \sum_{i=1}^{n} Ad V_i \).

(iv) (Holevo form:) There exist \( F_i \in \mathcal{M}_{d_1}^+ \) and \( R_i \in \mathcal{M}_{d_2}^+ \) such that \( \Phi(X) = \sum_{i=1}^{n} \text{tr}(XF_i)R_i \) for all \( X \in \mathcal{M}_{d_1} \). (Here ‘tr’ denotes the trace map.)

Note that EB-maps are necessarily PPT-maps. The converse is true when \( d_1d_2 \leq 6 \), but not true in general ([HHH96] [Hor97]).

We will denote the convex set of all unital EB-maps from \( \mathcal{M}_{d_1} \) to \( \mathcal{M}_{d_2} \) by \( \text{UEB}(d_1, d_2) \). Given \( \Phi \in \text{UEB}(d_1, d_2) \) its entanglement breaking rank, which we denote by EB-rank\( (\Phi) \), is defined ([PPPR20]) as the minimum number of rank-one Kraus operators required to represent \( \Phi \) as in Theorem 2.1 (iii). It is known ([Hor97] [HSR03]) that:

\[ d_2 \leq \text{Choi-rank}(\Phi) \leq \text{EB-rank}(\Phi) \leq (d_1d_2)^2. \]

Note that Choi-rank\( (\Phi) = \text{rank}(C_\Phi) \) and EB-rank\( (\Phi) = \ell(C_\Phi) \).

3. Basic properties of \( C^* \)-extreme points

Following [FaMo97] we define \( C^* \)-extreme points of \( \text{UEB}(d_1, d_2) \) as follows.

**Definition 3.1.** A linear map \( \Phi \in \text{UEB}(d_1, d_2) \) is said to be a

(i) linear-extreme point of \( \text{UEB}(d_1, d_2) \) if, whenever \( \Phi = \sum_{i=1}^{n} t_i \Phi_i \), where \( \Phi_i \in \text{UEB}(d_1, d_2) \) and \( t_i \in (0, 1) \) with \( \sum_{i=1}^{n} t_i = 1 \), then \( \Phi_i = \Phi \) for all \( 1 \leq i \leq n \);

(ii) \( C^* \)-extreme point of \( \text{UEB}(d_1, d_2) \) if, whenever \( \Phi \) is written as a proper \( C^* \)-convex combination, say

\[ \Phi = \sum_{i=1}^{n} Ad T_i \circ \Phi_i, \]

where \( T_i \in \mathcal{M}_{d_2} \) are invertible with \( \sum_{i=1}^{n} T_i^* T_i = I_{d_2} \) and \( \Phi_i \in \text{UEB}(d_1, d_2) \), then there exist unitaries \( U_i \in \mathcal{M}_{d_2} \) such that \( \Phi_i = Ad U_i \circ \Phi \) for all \( 1 \leq i \leq n \).

Observe that \( \text{UEB}(d_1, d_2) \) is a \( C^* \)-convex set in the sense it is closed under \( C^* \)-convex combinations. We denote the set of linear-extreme points and the set of \( C^* \)-extreme points of \( \text{UEB}(d_1, d_2) \) by \( \text{UEB}_{\text{lex}}(d_1, d_2) \) and \( \text{UEB}_{\text{C*ext}}(d_1, d_2) \), respectively. Observe that if \( \Phi, \Psi \in \text{UEB}(d_1, d_2) \) are unitarily equivalent (and we write \( \Phi \cong \Psi \)) and if \( \Phi \) is a \( C^* \)-extreme point, then \( \Psi \) is so.

The following two propositions are analogue of [Zho98] Proposition 2.1.2 and [FaMo97] Proposition 1.1, respectively, in the context of unital EB-maps. Though proofs are similar to that for unital CP-maps we add them here for the sake of completeness.

**Proposition 3.2.** Given \( \Phi \in \text{UEB}(d_1, d_2) \) the following conditions are equivalent:

(i) \( \Phi \) is a \( C^* \)-extreme point of \( \text{UEB}(d_1, d_2) \).

(ii) If \( \Phi = \sum_{i=1}^{2} Ad T_i \circ \Phi_i \), where \( \Phi_i \in \text{UEB}(d_1, d_2) \) and \( T_i \in \mathcal{M}_{d_2} \) are invertible with \( \sum_{i=1}^{2} T_i^* T_i = I_{d_2} \), then \( \Phi_i \)'s are unitarily equivalent to \( \Phi \).
Proof. We only prove the nontrivial part (ii) ⇒ (i). To show that whenever \( \Phi = \sum_{i=1}^{n} \text{Ad}_{T_i} \circ \Phi_i \) is a proper \( C^* \)-convex combination with \( \Phi_i \in \text{UEB}(d_1, d_2) \), then \( \Phi_i \cong \Phi \) for all \( 1 \leq i \leq n \). We prove by induction on \( n \). By assumption the result is true for \( n = 2 \). Assume that the result is true for \( m \geq 2 \). Now, suppose

\[
\Phi = \sum_{i=1}^{m+1} \text{Ad}_{T_i} \circ \Phi_i = \sum_{i=1}^{m} \text{Ad}_{T_i} \circ \Phi_i + \text{Ad}_{T_{m+1}} \circ \Phi_{m+1},
\]

where \( T_i \in M_{d_2} \) invertible with \( \sum_{i=1}^{m+1} T_i^* T_i = I_{d_2} \) and \( \Phi_i \in \text{UEB}(d_1, d_2) \). Let \( S \) be the positive square root of \( \sum_{i=1}^{m} T_i^* T_i \). Then \( S \) is invertible and \( S^* S + T_{m+1}^* T_{m+1} = I_{d_2} \). Let \( \Psi = \sum_{i=1}^{m} \text{Ad}_{T_i, S^{-1}} \circ \Phi_i \); then \( \Psi \in \text{UEB}(d_1, d_2) \) is such that

\[
\Phi = \text{Ad}_S \circ \Psi + \text{Ad}_{T_{m+1}} \circ \Phi_{m+1}.
\]

Hence, by assumption (ii), \( \Psi \cong \Phi \cong \Phi_{m+1} \). Now, as \( \Psi \) is a proper \( C^* \)-convex combination, by induction hypothesis, we conclude that \( \Phi_i \cong \Phi \) for all \( 1 \leq i \leq m \). This completes the proof. \( \square \)

Note 3.3. We can easily verify that, like the above, it is enough to consider a convex combination of two maps in the definition of linear extreme points.

The following result (\cite{AGG02} Theorem 3.5(a)) on fixed points of CP-maps seems to be well-known.

Lemma 3.4. Let \( \Phi : M_d \to M_d \) be a unital, trace-preserving CP-map given by

\[
\Phi = \sum_{i=1}^{m} \text{Ad}_{T_i},
\]

for some \( T_1, T_2, \ldots, T_m \) in \( M_d, m \in \mathbb{N} \). Then for \( A \in M_d \), \( \Phi(A) = A \) if and only if \( AT_i = T_i A \) for all \( 1 \leq i \leq m \).

Proposition 3.5. \( \text{UEB}_{C^*-ext}(d_1, d_2) \subseteq \text{UEB}_{ext}(d_1, d_2) \).

Proof. Let \( \Phi \in \text{UEB}_{C^*-ext}(d_1, d_2) \). Suppose \( \Phi = \sum_{i=1}^{2} t_i \Phi_i \), where \( \Phi_i \in \text{UEB}(d_1, d_2) \) and \( t_i \in (0, 1) \) such that \( \sum_{i=1}^{2} t_i = 1 \). Since \( \Phi \) is a \( C^* \)-extreme point there exist unitaries \( U_i \in M_{d_2} \) such that \( \Phi_i = \text{Ad}_{U_i} \circ \Phi \), and thus

\[
\Phi = \sum_{i=1}^{2} t_i \text{Ad}_{U_i} \circ \Phi = \sum_{i=1}^{2} \text{Ad}_{T_i} \circ \Phi,
\]

where \( T_i = \sqrt{t_i} U_i, 1 \leq i \leq 2 \). Consider the unital trace-preserving CP-map \( \Psi = \sum_{i=1}^{2} \text{Ad}_{T_i} \) on \( M_{d_2} \). Let \( T \in \Phi(M_{d_1}) \). Note that \( \Psi(T) = T \) and hence, from the above Lemma, it follows that \( T_i \) commutes with \( T \). But \( T_i \) is scalar multiple of \( U_i \), so that \( U_i \) commutes with \( T \). Since \( T \) is arbitrary it follows that \( U_i \)’s commutes with range of \( \Phi \). Hence

\[
\Phi_i(X) = \text{Ad}_{U_i} \circ \Phi(X) = U_i^* \Phi(X) U_i = \Phi(X)
\]

for all \( X \in M_{d_1}, 1 \leq i \leq 2 \), and concludes that \( \Phi \in \text{UEB}_{ext}(d_1, d_2) \). \( \square \)
4. Radon Nikodym type theorem for EB-maps

The well-known Radon-Nikodym theorem tells us that a measure absolutely continuous with respect to a given measure can be recovered using a positive function called the Radon-Nikodym derivative. W. Arveson showed that CP-maps dominated by a given CP-map can be described through positive contractions in the commutant of the Stinespring representation. In particular we get the following: Suppose \( \Phi = \sum_{i=1}^{n} \text{Ad}V_i \in \text{UCP}(d_1, d_2) \). If \( \Psi : M_{d_1} \to M_{d_2} \) is a CP-map such that \( \Psi \leq_{CP} \Phi \), then due to [Arv69] there exists a positive contraction \( T = [t_{ij}] \in M_n \) such that \( \Psi(X) = \sum_{i,j=1}^{n} t_{ij}V^*_i XV_j \). Further, writing \( T = S^*S \) for some \( S = [s_{ij}] \in M_n \) we get \( t_{ij} = \sum_{k=1}^{n} s_{kj}V_j \) for all \( 1 \leq i, j \leq n \). Thus, \( \Psi = \sum_{k=1}^{n} \text{Ad}_{L_k} \), where \( L_k = \sum_{j=1}^{n} s_{kj}V_j \) for every \( 1 \leq k \leq n \).

**Notation.** Given \( x = \{x_i : 1 \leq i \leq n\} \subseteq \mathbb{C}^{d_1} \) and \( y = \{y_i : 1 \leq i \leq n\} \subseteq \mathbb{C}^{d_2} \) we define the EB-map \( \mathcal{E}_{x,y} : M_{d_1} \to M_{d_2} \) by

\[
\mathcal{E}_{x,y}(X) := \sum_i \mathcal{E}_{x_i,y_i},
\]

where \( \mathcal{E}_{u,v}(X) := \langle u, Xu \rangle |v\rangle \langle v| \) for all \( u \in \mathbb{C}^{d_1}, v \in \mathbb{C}^{d_2} \) and \( X \in M_{d_1} \).

**Lemma 4.1.** Let \( \Phi \in \text{UEB}(d_1, d_2) \) be such that

\[
\Phi(X) = \sum_{i=1}^{n} (u_i, X u_i) P_i
\]

for all \( X \in M_{d_1} \), where \( u_i \in \mathbb{C}^{d_1} \) are unit vectors such that \( \{u_i, u_j\} \) is linearly independent whenever \( i \neq j \), and \( P_i \in M_{d_2} \) are mutually orthogonal projections such that \( \sum_{i=1}^{n} P_i = I_{d_2} \).

If \( x \in \mathbb{C}^{d_1} \) and \( y \in \mathbb{C}^{d_2} \) are non-zero vectors such that

\[
\mathcal{E}_{x,y} \leq_{CP} \Phi,
\]

then \( x \in \text{Cu}_j \) and \( y \in \text{range}(P_j) \) for some \( 1 \leq j \leq n \). In particular,

\[
\mathcal{E}_{x,y}(X) = \langle u_j, X u_j \rangle R_j
\]

for all \( X \in M_{d_1} \) and for some positive contraction \( R_j \in M_{d_2} \) with \( R_jP_k = P_kR_j = \delta_{jk}R_j \) for all \( 1 \leq k \leq n \).

**Proof.** For every \( 1 \leq k \leq n \) let \( P_k = \sum_{i=1}^{r_k} |v_i^k \rangle \langle v_i^k| \), where \( \{v_i^k : 1 \leq i \leq r_k\} \) is an orthonormal basis for range\( (P_k) \). Then

\[
\Phi(X) = \sum_{k=1}^{n} \sum_{i=1}^{r_k} |v_i^k \rangle \langle u_k|X|u_k \rangle |v_i^k \rangle \langle v_i^k| \quad \forall X \in M_{d_1},
\]

so that \( \{|u_k \rangle \langle v_i^k| : 1 \leq i \leq r_k, 1 \leq k \leq n\} \) is a set of Kraus operators for \( \Phi \). As \( \mathcal{E}_{x,y} \) is a CP-map with Kraus operator \( |x \rangle \langle y| \) and \( \mathcal{E}_{x,y} \leq_{CP} \Phi \), we have

\[
|x \rangle \langle y| \in \text{span}\{ |u_k \rangle \langle v_i^k| : 1 \leq i \leq r_k, 1 \leq k \leq n\}.
\]
Suppose $\lambda_{k,i} \in \mathbb{C}$ are such that
\[ |x\rangle\langle y| = \sum_{k=1}^{n} \sum_{i=1}^{r_k} \lambda_{k,i} |u_k\rangle\langle v_i^k|. \]

Since $\{v_i^k : 1 \leq i \leq r_k, 1 \leq k \leq n\} \subseteq \mathbb{C}^{d_2}$ is an orthonormal basis and $0 \neq y \in \mathbb{C}^{d_2}$, applying the operator $|x\rangle\langle y|$ on these basis vectors we get
\[ x(y, v_i^k) = \lambda_{k,i} u_k \]
for all $k, i$. Therefore if $\langle y, v_i^k \rangle \neq 0$ then $\lambda_{k,i} \neq 0$ and $x \in \mathbb{C}u_k$. Since $u_k$'s are pairwise linearly independent it follows that there is a unique $1 \leq j \leq n$, such that $x \in \mathbb{C}u_j$ and $\langle v_i^k, y \rangle = 0$ for all $1 \leq i \leq r_k$ and for all $k \neq j$. So $y = P_j(y) \in \text{range}(P_j)$ and
\[ E_{x,y}(X) = \langle u_j, Xu_j \rangle R_j \quad \forall X \in M_{d_2}, \]
where $R_j = \langle x, x \rangle |y\rangle\langle y|$. Note that $R_j = E_{x,y}(I) \leq \Phi(I) = I$ and $R_j P_k = P_k R_j = \delta_{jk} R_j$ for all $1 \leq k \leq n$. □

Suppose $\Phi, \Psi : M_{d_1} \to M_{d_2}$ are two EB-maps. Then we write $\Psi \leq_{EB} \Phi$ whenever the difference $\Phi - \Psi$ is also an EB-map.

**Theorem 4.2** (Radon-Nikodym type theorem). Let $\Phi \in UEB(d_1, d_2)$ be such that
\[ \Phi(X) = \sum_{i=1}^{n} \phi_i(X) P_i \]
for all $X \in M_{d_1}$, where $\phi_i : M_{d_1} \to \mathbb{C}$ are distinct pure states and $P_i \in M_{d_2}$ are mutually orthogonal projections such that $\sum_{i=1}^{n} P_i = I_{d_2}$. Given an EB-map $\Psi : M_{d_1} \to M_{d_2}$ the following conditions are equivalent:

(i) $\Psi \leq_{EB} \Phi$.

(ii) $\Psi \leq_{CP} \Phi$.

(iii) There exist positive contractions $R_i \in M_{d_2}$ with $P_i R_j = R_j P_i = \delta_{ij} R_j, 1 \leq i, j \leq n$ such that
\[ \Psi(X) = \sum_{i=1}^{n} \phi_i(X) R_i \]
for all $X \in M_{d_1}$.

(iv) There exists a positive contraction $R \in \text{range}(\Phi)' \subseteq M_{d_2}$ such that
\[ \Psi(X) = \text{Ad}_{\sqrt{R}} \circ \Phi(X) = \Phi(X) R \]
for all $X \in M_{d_1}$.

**Proof.** (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) Choose unit vectors $u_i \in \mathbb{C}^{d_1}$ such that $\phi_i(X) = \langle u_i, Xu_i \rangle$ for all $X \in M_{d_1}$. Due to Holevo form $\Psi = \sum_{i=1}^{m} E_{x_i, y_i}$ for some non-zero vectors $x_i \in \mathbb{C}^{d_1}$ and $y_i \in \mathbb{C}^{d_2}$. Since $E_{x_i, y_i} \leq_{CP} \Phi$ for all $1 \leq i \leq m$, from Lemma 4.1 there exist $1 \leq j_i \leq n$ and a positive contraction $R_i \in M_{d_2}$ with $R_i P_k = P_k R_i = \delta_{k,j_i} R_i$ such that $E_{x_i, y_i}(X) = \langle u_{j_i}, Xu_{j_i} \rangle R_i$ for all
$X \in M_{d_1}$. Hence

$$\Psi(X) = \sum_{i=1}^{m} \phi_{j_i}(X) \tilde{R}_i = \sum_{k=1}^{n} \phi_k(X) R_k \quad \forall X \in M_{d_1},$$

with $R_k = \sum \tilde{R}_i$, where the summation is taken over all $i$ for which $\phi_{j_i} = \phi_k$; if there is no such $j_i$'s, then take $R_k = 0$. Observe that $R_k$ has the required properties.

(iii) $\Rightarrow$ (iv) Suppose $\Psi(X) = \sum_{i=1}^{n} \phi_i(X) R_i$ for some positive contractions $R_i \in M_{d_2}$ satisfying $P_i R_i = R_i P_i = \delta_{ij} R_j$ for all $1 \leq i, j \leq n$. Then $R := \sum_i R_i = \Psi(I) \in M_{d_2}$, is a positive contraction such that $R \in \text{range}(\Phi')$ and $\Psi = \text{Ad}_{\sqrt{\Phi}} \circ \Phi = \Phi(\cdot) R$.

(iv) $\Rightarrow$ (i) Follows since $\Phi - \Psi = \text{Ad}_{\sqrt{1-R}} \circ \Phi$ is EB. This completes the proof. \hfill \Box

**Example 4.3.** Define $\Phi, \Psi : M_2 \to M_2$ by $\Phi(X) = \phi(X) I$, where $\phi(X) = \frac{x_{11} + x_{22}}{2}$. Consider the map $\Psi : M_2 \to M_2$ by

$$\Phi(X) = \begin{bmatrix} \frac{x_{11} + x_{22}}{2} & 0 \\ 0 & \frac{x_{11} + x_{22}}{2} \end{bmatrix} \quad \text{and} \quad \Psi(X) = \begin{bmatrix} \frac{x_{11} + x_{22}}{4} & \frac{x_{11}}{4} \\ \frac{x_{11}}{4} & \frac{x_{11} + x_{22}}{4} \end{bmatrix}$$

for all $X = [x_{ij}] \in M_2$. Being PPT-maps $\Phi, \Psi$ and $\Phi - \Psi$ are EB-maps. Thus, $\Psi \leq \text{EB} \Phi$. But statements (iii) and (iv) of the above theorem do not hold here. This does not contradict the theorem as $\phi$ here is not pure.

**Example 4.4.** In general, $\Psi \leq \text{CP} \Phi$ does not imply that $\Psi \leq \text{EB} \Phi$. For example, consider the map $\Phi : M_d \to M_d$ given by

$$\Phi(X) = \frac{\text{tr}(X) I + cX}{d + c}$$

where $c \in (0, 1]$. By [Sto13, Corollary 7.5.5] we have $\Phi \in \text{UEB}(d)$. Observe that the EB-map $\Psi(X) = \frac{\text{tr}(X) I}{d + c}$ is such that $\Phi - \Psi$ is CP but not EB.

5. Structure of $C^*$-extreme points

In this section, we give a complete description of the structure of $C^*$-extreme points of $\text{UEB}(d_1, d_2)$. To this end, first, we prove an abstract characterization of $C^*$-extreme points using standard techniques.

**Theorem 5.1.** Given $\Phi \in \text{UEB}(d_1, d_2)$ the following conditions are equivalent:

(i) $\Phi \in \text{UEB}_{C^*} \text{-cxt}(d_1, d_2)$.

(ii) If $\Psi : M_{d_1} \to M_{d_2}$ is any EB-map with $\Psi \leq \text{EB} \Phi$ and $\Psi(I)$ is invertible, then there exists an invertible matrix $Z \in M_{d_2}$ such that $\Psi = \text{Ad}_Z \circ \Phi$.

**Proof.** (i) $\Rightarrow$ (ii) Let $\Psi : M_{d_1} \to M_{d_2}$ be an EB-map with $\Psi \leq \text{EB} \Phi$ and $\Psi(I)$ be invertible. Let $(\pi, V, \mathcal{K})$ be the minimal Stinespring dilation of $\Phi$. (Note that here $\mathcal{K}$ can chosen to be finite dimensional.) Since $\Psi \leq \text{CP} \Phi$ there exists a positive contraction $T \in (\pi(M_{d_1}))' \subseteq \mathcal{B}(\mathcal{K})$ such that

$$\Psi(X) = V^*T \pi(X)V = \text{Ad}_{T^*} \circ \pi(X) \quad \forall X \in M_{d_1}.$$


Fix a scalar $\lambda \in (0,1)$. Then $\lambda V^*TV = \lambda \Psi(I)$ and $I - \lambda V^*TV$ are invertible positive contractions in $M_{d_2}$, and hence $T_1 = (\lambda V^*TV)^\dagger$ and $T_2 = (I - \lambda V^*TV)^\dagger$ are also invertible matrices with $T_1^*T_1 + T_2^*T_2 = I$. Define $\Phi_i : M_{d_1} \to M_{d_2}$ by

$$
\Phi_1 := \lambda \Ad_{T_1^{-1}} \circ \Psi \quad \text{and} \quad \Phi_2 := \Ad_{T_2^{-1}} \circ (\Phi - \lambda \Psi)
$$

Since $\Psi$ and $\Phi - \Psi$ are EB-maps we observe that $\Phi_1$ and $\Phi_2$ are unital EB-maps such that

$$
\Ad_{T_1} \circ \Phi_1 + \Ad_{T_2} \circ \Phi_2 = \Phi.
$$

As $\Phi$ is a $C^*$-extreme point there exists a unitary $W \in M_{d_2}$ such that $\Phi = \Ad_W \circ \Phi_1$. Let $V_1 = \sqrt{\lambda^2 + 1}V T_1^{-1}W \in B(\mathbb{C}^{d_2}, K)$. Note that

$$
\Ad_{V_1} \circ \pi = \lambda \Ad_W \circ \Ad_{T_1^{-1}} \circ \Ad_{T_2^{-1}} \circ \pi = \lambda \Ad_W \circ \Ad_{T_1^{-1}} \circ \Psi = \Ad_W \circ \Phi_1 = \Phi
$$

and $V_1^*V_1 = \Phi(I) = I$. Set $\mathcal{H} = \overline{\text{span}}\{ \pi(X)V_1z : z \in \mathbb{C}^{d_2}, X \in M_{d_2} \} \subseteq K$. Then $(\pi|_{\mathcal{H}}, V_1, \mathcal{H})$ is the minimal Stinespring dilation of $\Phi$, and hence there exists a unitary $U : \mathcal{H} \to K$ such that

$$
UV_1 = V, \quad \Ad_U \circ \pi = \pi|_{\mathcal{H}}.
$$

Now, since $K$ is finite dimensional, both $\mathcal{H}$ and $K$ having same dimension implies that $\mathcal{H} = K$. Therefore, we have $U \pi(X) = \pi(X)U$ for all $X \in M_{d_1}$, and

$$
V = UV_1 = \sqrt{\lambda^2 + 1}UT_2^*VT_1^{-1}W = UT_1^*VZ^{-1},
$$

where $Z = \frac{1}{\sqrt{\lambda}} W^*T_1 \in M_{d_2}$. Clearly $Z$ is invertible, and

$$
\Ad_Z \circ \Phi(X) = Z^*V^*\pi(X)(UU^*)VZ = (U^*VZ)^*\pi(X)(U^*VZ) = (T_2^*V)^*\pi(X)(T_2^*V) = V^*T_1\pi(X)V = \Psi(X)
$$

for all $X \in M_{d_1}$.

$$(iii) \Rightarrow (i)$$

Assume that $\Phi = \sum_{i=1}^n \Ad_{T_i} \circ \Phi_i$, where $\Phi_i$ are unital EB-maps and $T_i \in M_{d_2}$ are invertible with $\sum_{i=1}^n T_i^*T_i = I$. Fix $1 \leq i \leq n$. Then $\Psi := \Ad_{T_i} \circ \Phi_i$ is an EB-map with $\Psi(I) = T_i^*T_i$ is invertible and $\Phi - \Psi = \sum_{j \neq i} \Ad_{T_j} \circ \Phi_j$ is EB. Hence, by assumption, there exists an invertible matrix $Z \in M_{d_2}$ such that $\Psi = \Ad_Z \circ \Phi$. Note that

$$
(ZT_i^{-1})^*ZT_i^{-1} = (T_i^*)^{-1}\Psi(I)T_i^{-1} = (T_i^*)^{-1}T_i^*T_iT_i^{-1} = I,
$$

so that $U := ZT_i^{-1} \in M_{d_2}$ is a unitary. Further,

$$
\Ad_U \circ \Phi = \Ad_{T_i^{-1}} \circ \Ad_Z \circ \Phi = \Ad_{T_i^{-1}} \circ \Psi = \Phi_i,
$$

that is, $\Phi_i \equiv \Phi$. Since $1 \leq i \leq n$ is arbitrary we conclude that $\Phi \in \text{UEB}_{C^*_{-ext}}(d_1, d_2)$. \qed
Example 5.2. To illustrate Theorem 5.1 consider the map \( \Phi : \mathcal{M}_3 \to \mathcal{M}_3 \) defined by

\[
\Phi(X) = \begin{bmatrix} x_{11} & 0 & 0 \\ 0 & x_{11} & 0 \\ 0 & 0 & x_{33} \end{bmatrix} = \text{tr}(XE_{11})(E_{11} + E_{22}) + \text{tr}(XE_{33})E_{33}
\]

for all \( X = [x_{ij}] \in \mathcal{M}_3 \). We will prove in Theorem 5.3 below that \( \Phi \in \text{UEB}_{\text{ext}}(3) \). Now, let \( t_{33} \in (0,1) \) and \( [t_{11}, t_{22}] \in \mathcal{M}_2 \) be an invertible positive contraction. Define \( \Psi : \mathcal{M}_3 \to \mathcal{M}_3 \) by

\[
\Psi(X) = \begin{bmatrix} t_{11}x_{11} & t_{12}x_{11} & 0 \\ t_{21}x_{11} & t_{22}x_{11} & 0 \\ 0 & 0 & t_{33}x_{33} \end{bmatrix} = \text{tr}(XE_{11}) \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} + \text{tr}(XE_{33}) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & t_{33} \end{bmatrix}.
\]

Then \( \Psi \) is an EB-map such that \( \Psi(I) \) is invertible and \( \Psi \leq \text{EB} \Phi \). Note that \( \Psi = \text{Ad}_\sqrt{\psi(I)} \circ \Phi \).

Theorem 5.3. Given \( \Phi \in \text{UEB}(d_1, d_2) \) the following conditions are equivalent:

(i) \( \Phi \in \text{UEB}_{\text{ext}}(d_1, d_2) \).

(ii) There exists a Choi-Kraus decomposition, \( \Phi = \sum_{i=1}^{d_1} \text{Ad}V_i \), with rank-one operators \( V_i \in \mathbb{M}_{d_1 \times d_2} \) such that given any \( k < d_2 \) and \( \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, d_2\} \), the matrix \( \sum_{j=1}^{k} V_{i_j}^*V_{i_j} \) is not invertible.

(iii) \( \text{EB-rank}(\Phi) = d_2 \).

(iv) \( \text{Choi-rank}(\Phi) = d_2 \).

(v) There exist (distinct) pure states \( \phi_i : \mathbb{M}_{d_1} \to \mathbb{C} \) and mutually orthogonal projections \( P_i \in \mathbb{M}_{d_2} \) with \( \sum_{i=1}^{n} P_i = I \) such that

\[
\Phi(\cdot) = \sum_{i=1}^{n} \phi_i(\cdot)P_i,
\]

where \( n \leq d_2 \).

(vi) There exist unit vectors \( \{u_i : 1 \leq i \leq d_2\} \subseteq \mathbb{C}^{d_1} \) and an orthonormal basis \( \{v_i : 1 \leq i \leq d_2\} \subseteq \mathbb{C}^{d_2} \) such that

\[
\Phi(X) = \sum_{i=1}^{d_2} |u_i, X|v_i\rangle\langle v_i |
\]

for all \( X \in \mathbb{M}_{d_1} \).

(vii) There exist pure states \( \phi_i : \mathbb{M}_{d_1} \to \mathbb{C} \) such that \( \Phi \cong \bigoplus_{i=1}^{d_2} \phi_i \).

Proof. (i) \( \Rightarrow \) (ii) Let \( r = \text{EB-rank}(\Phi) \) and \( \Phi = \sum_{i=1}^{r} \text{Ad}V_i \), where \( V_i = |u_i\rangle\langle v_i| \in \mathbb{M}_{d_1 \times d_2} \) with \( u_i \in \mathbb{C}^{d_1} \) and \( v_i \in \mathbb{C}^{d_2} \) for all \( 1 \leq i \leq r \). Note that \( \sum_{j=1}^{r} V_{i_j}^*V_{i_j} = \Phi(I) = I \) is invertible.

Step 1: We show that given any \( k < r \) and \( \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, r\} \), the matrix \( \sum_{j=1}^{k} V_{i_j}^*V_{i_j} \) is not invertible. Suppose not. Then consider the map \( \Psi := \sum_{j=1}^{k} \text{Ad}V_{i_j} \). Clearly \( \Psi \) is an EB-map and \( \Psi \leq \text{EB} \Phi \). Since \( \Psi(I) = \sum_{j=1}^{k} V_{i_j}^*V_{i_j} \) is invertible, by Theorem 5.1 there exists an
invertible matrix $Z \in M_{d_2}$ such that $Ψ = Ad_{Z} \circ Φ$, therefore

$$Φ(X) = Ad_{Z^{-1}} \circ Ψ(X) = \sum_{j=1}^{k} (V_{j} Z^{-1})^* X (V_{j} Z^{-1}).$$

This contradicts the minimality of $r$ as $V_{i} Z^{-1}$’s are rank-one operators.

Step 2: We show that $r = d_2$. With out of generality assume that $\{v_1, v_2, \ldots, v_q\}$ is the maximal linearly independent set in $\{v_i : 1 \leq i \leq r\}$. Then, since $I = Φ(I) = \sum_{i=1}^{r} ||v_i||^2 |v_i⟩⟨v_i|$, we have

$$\text{span}\{v_i : 1 \leq i \leq q\} = \text{span}\{v_i : 1 \leq i \leq r\} = \mathbb{C}^{d_2}$$

so that $q = d_2$. But, as $V_j^* V_j \leq \sum_{i=1}^{d_2} V_i^* V_i$, from [Dou66], we have $v_j \in \text{range}(V_j^* V_j) \subseteq \text{range}(\sum_{i=1}^{d_2} V_i^* V_i)$ for all $1 \leq j \leq q$, so that

$$\text{range}(\sum_{i=1}^{q} V_i^* V_i) = \text{span}\{v_i : 1 \leq i \leq q\} = \mathbb{C}^{d_2}.$$

Thus, $\sum_{i=1}^{q} V_i^* V_i$ is invertible, hence from Step 1, it follows that $r = q = d_2$.

$(ii) \Rightarrow (iii)$ Follows as $d_2 \leq \text{Choi-rank}(Φ) \leq \text{EB-rank}(Φ) \leq d_2$, where the last inequality follows from the assumption.

$(iii) \iff (iv)$ Clearly, if $\text{EB-rank}(Φ) = d_2$, then $\text{Choi-rank}(Φ) = d_2$. Conversely assume that $\text{Choi-rank}(Φ) = d_2$. Since $Φ^* : M_{d_2} \rightarrow M_{d_1}$ is trace preserving EB-map we have $(id \otimes \text{tr})(C_{Φ^*}) = I_{d_2}$. Thus both $C_{Φ^*}$ and $((id \otimes \text{tr})(C_{Φ^*}))$ has rank $d_2$. Hence, from [HSR03, Lemma 8], we have $d_2 \geq \text{EB-rank}(Φ) \geq \text{Choi-rank}(Φ) = d_2$.

$(iii) \Rightarrow (v)$ Suppose $Φ = \sum_{i=1}^{d_2} Ad_{V_i}$ with $V_i = |u_i⟩⟨v_i|$, where $u_i \in \mathbb{C}^{d_1}$ and $v_i \in \mathbb{C}^{d_2}$. With out loss of generality assume that $u_i$’as are unit vectors. Then

$$I = \sum_{i=1}^{d_2} V_i^* V_i = \left( [w_{k_i} u_{l_i}] \right)_{k,l=1}^{d_2},$$

where $w_j$’s are the columns of the adjoint of $V := \begin{bmatrix} v_1 & v_2 & \cdots & v_{d_2} \end{bmatrix} \in M_{d_2}$, so that $V$ is a unitary. Thus,

$$Φ(X) = \sum_{i=1}^{d_2} V_i^* X V_i = \sum_{i=1}^{d_2} (u_i X u_i)|v_i⟩⟨v_i| = \sum_{i=1}^{d_2} \phi_i(X) P_i$$

for all $X \in M_{d_1}$, where $\phi_i(X) = ⟨u_i, X u_i⟩$ are pure states and $P_i = |v_i⟩⟨v_i| \in M_{d_2}$ are mutually orthogonal projections such that $\sum_i P_i = I$. Now, whenever $\phi_i = \phi_j$ for some $i \neq j$, then we replace $\phi_i(P_i + P_j)$ by $ϕ_i(P_i + P_j)$ in the above sum and thereby assume that all $ϕ_i$’s are distinct.

$(v) \Rightarrow (i)$ Suppose there exist distinct pure states $ϕ_i : M_{d_1} \rightarrow \mathbb{C}$ and mutually orthogonal projections $P_i \in M_{d_2}$ with $\sum_{i=1}^{n} P_i = I$ such that

$$Φ(X) = \sum_{i=1}^{n} ϕ_i(X) P_i, \quad \forall X \in M_{d_1}.$$
Let $\Psi : M_{d_1} \to M_{d_2}$ be an EB-map with $\Psi \leq_{EB} \Phi$ and $\Psi(I)$ invertible. By Theorem 4.2 there exists a positive contraction $R \in \text{range}(\Phi)'$ such that $\Psi = \text{Ad}\sqrt{R} \circ \Phi$. Note that $R = \Psi(I)$, which is invertible. Hence $Z := \sqrt{R}$ is an invertible positive contraction such that $\Psi = \text{Ad}_{Z} \circ \Phi$. From Theorem 5.4 it follows that $\Phi \in \text{UEB}_{C^*-ext}(d_1, d_2)$.

(v) $\iff$ (vi) $\iff$ (vii) Follows from basic linear algebra. This completes the proof. □

**Example 5.4.** Define $\Phi : M_2 \to M_2$ by $\Phi(X) = \phi(X)I$, where $\phi(X) = \frac{x_{11} + x_{22}}{2}$ for all $X = [x_{ij}] \in M_2$. Note that $\Phi_1, \Phi_2 : M_2 \to M_2$ defined by

$$
\Phi_1(X) = \begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} \end{bmatrix} \quad \text{and} \quad \Phi_2(X) = \begin{bmatrix} x_{22} & 0 \\ 0 & x_{11} \end{bmatrix}
$$

are unital EB-maps such that $\Phi = \frac{1}{2}\Phi_1 + \frac{1}{2}\Phi_2$, hence $\Phi \notin \text{UEB}_{C^*-ext}(2)$. Though $\phi$ is a state it is not pure. Thus, the condition ‘pure’ cannot be dropped in Theorem 5.3.

**Note 5.5.** If $\Phi \in \text{UEB}_{C^*-ext}(d_1, d_2)$, then from Theorem 5.4(chori-rank$(\Phi)$) = EB-rank$(\Phi)$. But the converse is not true in general. For example, consider the map $\Phi \in \text{UEB}(d)$ given by

$$
\Phi(X) = \frac{\text{tr}(X)}{d} I = \frac{1}{d} \sum_{i,j=1}^{d} \text{Ad}_{E_{ij}}(X).
$$

Note that Choi-rank$(\Phi)$ = EB-rank$(\Phi) = d^2$. But, from Theorem 5.3(iii), $\Phi \notin \text{UEB}_{C^*-ext}(d)$.

**Remark 5.6.** Given a set $u = \{u_i\}_{i=1}^{d} \subseteq \mathbb{C}^d$ of unit vectors and an orthonormal basis $v = \{v_i\}_{i=1}^{d} \subseteq \mathbb{C}^d$ the map $E_{u,v}$ is called extreme CQ-channel in [HSR03]. It is shown in [HSR03 Theorem 5] that $E_{u,v}$'s are linear extreme points of the convex set of trace-preserving EB-maps; in other words, $E_{u,v}^* = E_{u,v} \in \text{UEB}_{ext}(d)$. But, Theorem 5.3(vi) says that every element of $\text{UEB}_{C^*-ext}(d)$ is of the form $E_{u,v}$ for some $u,v$. Thus, [HSR03 Theorem 5] combined with Theorem 5.3 leads to:

(i) Let $\Phi \in \text{UEB}_{C^*-ext}(d)$. Then $\Phi \in \text{UCP}_{ext}(d)$ if and only if $\Phi = E_{u,v}$ with $\langle u_j, u_k \rangle \neq 0$ for all $j, k$.

(ii) $\text{UEB}(d) \cap \text{UCP}_{ext}(d) \subseteq \text{UCP}_{C^*-ext}(d)$.

(iii) If $d = 2$, then $\text{UEB}_{ext}(2) = \text{UEB}_{C^*-ext}(2)$.

(iv) If $d \geq 3$, then there are linear extreme points in $\text{UEB}(d)$ which are not C*-extreme points.

**Example 5.7.** Consider the unital trace-preserving EB-map $\Phi : M_3 \to M_3$ defined by

$$
\Phi(X) = \frac{3}{4} \sum_{i=1}^{4} \langle v_i, X v_i \rangle |v_i\rangle \langle v_i| = \frac{1}{3} \begin{bmatrix} \text{tr}(X) & x_{12} + x_{21} & x_{13} + x_{31} \\ x_{21} + x_{12} & \text{tr}(X) & x_{23} + x_{32} \\ x_{31} + x_{13} & x_{32} + x_{23} & \text{tr}(X) \end{bmatrix}
$$

for all $X = [x_{ij}] \in M_3$, where

$$
v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \quad v_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \quad v_4 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.
$$
It is shown in [HSR03, Counterexample] that $\Phi = \Phi^* \in \text{UEB}_{ext}(3)$ but not an extreme CQ, i.e., $\Phi \notin \text{UEB}_{C^*-ext}(3)$. Note that Choi-rank$(\Phi) = 4$.

**Note 5.8.** Let $\Phi \in \text{UEB}(d_1, d_2) \subset \text{UCP}(d_1, d_2)$. Clearly,

(i) $\Phi \in \text{UCP}_{C^*-ext}(d_1, d_2)$ implies that $\Phi \in \text{UEB}_{C^*-ext}(d_1, d_2)$.

(ii) $\Phi \in \text{UCP}_{ext}(d_1, d_2)$ implies that $\Phi \in \text{UEB}_{ext}(d_1, d_2)$.

But the converse of the above statements may not hold. See the example below.

**Example 5.9.** Consider the map $\Phi : M_2 \to M_2$ defined by

$$\Phi(X) := \begin{bmatrix} x_{11} & 0 \\ 0 & x_{22} \end{bmatrix} = \sum_{i=1}^{2} (e_i, X e_i)|e_i)(e_i|$$

for all $X = [x_{ij}] \in M_2$. From the above theorem $\Phi \in \text{UEB}_{C^*-ext}(2) = \text{UEB}_{ext}(2)$. But, it is observed in [FaMo97] that $\Phi \notin \text{UCP}_{ext}(2) \supset \text{UEB}_{C^*-ext}(2)$. In fact, $\Phi$ is a convex combination of unital CP-maps:

$$\Phi(X) = \frac{1}{2} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -x_{12} & -x_{11} \\ x_{21} & x_{22} \end{bmatrix} = \frac{1}{2}(\text{id}(X) + \text{Ad}_V(X))$$

where $V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in M_2$. Thus $\text{UEB}(2) \cap \text{UCP}_{C^*-ext}(2) \subsetneq \text{UEB}_{C^*-ext}(2)$ and $\text{UEB}(2) \cap \text{UCP}_{ext}(2) \subseteq \text{UEB}_{ext}(2)$.

**Corollary 5.10.** Let $\Phi \in \text{UEB}_{C^*-ext}(d_1, d_2)$. If $\Phi$ irreducible, then $d_2 = 1$ and consequently $\Phi$ is a pure state on $M_{d_1}$.

**Proof.** Since $\Phi \in \text{UEB}_{C^*-ext}(d_1, d_2)$, by Theorem [5.3 iii], there exist unit vectors $\{u_i : 1 \leq i \leq d_2\} \subseteq C^{d_1}$ and an orthonormal basis $\{v_i : 1 \leq i \leq d_2\} \subseteq C^{d_2}$ such that

$$\Phi(X) = \sum_{i=1}^{d_2} \langle u_i, X u_i | v_i \rangle |v_i\rangle.$$  \hfill (5.1)

for all $X \in M_{d_1}$. If $d_2 > 1$, then let $A = \sum_{k=1}^{d_2} k|v_k\rangle\langle v_k| \in M_{d_2}$. Then for all $X \in M_{d_1}$,

$$A\Phi(X) = \sum_{k=1}^{d_2} k \langle u_k, X u_k | v_k \rangle |v_k\rangle = \Phi(X)A,$$

so that $A \in \text{range}(\Phi)^{\perp}$. But $\Phi \notin C(I)$, which is contradiction to the fact that $\Phi$ is irreducible. Hence $d_2 = 1$, consequently from (5.1), we have $\Phi : M_{d_1} \to C$ and is a pure state. $\Box$

Next we prove a quantum version of the classical Krein-Milman theorem, which asserts that every nonempty convex compact subset of a locally convex topological vector space is closure of the convex hull of its extreme points. To prove a quantum analogue of this result for $\text{UEB}(d_1, d_2)$ we define the $C^*$-convex hull of a subset $S \subseteq \text{UEB}(d_1, d_2)$ as

$$\left\{ \sum_{i=1}^{n} \text{Ad}_{T_i} \circ \Phi_i : \Phi_i \in S, T_i \in M_{d_2}, \text{ with } \sum_{i=1}^{n} T_i^* T_i = I_{d_2} \right\}.$$
Theorem 5.11 (Krein-Milman type theorem). UEB\((d_1, d_2)\) is the \(C^*\)-convex hull of its \(C^*\)-extreme points.

Proof. Let \(\Phi \in \text{UEB}(d_1, d_2)\) with Holevo form \(\Phi(X) = \sum_{i=1}^{n} \text{tr}(XF_i)R_i\) for all \(X \in M_{d_1}\).

Without loss of generality assume that \(F_i = |u_i\rangle\langle u_i|\) and \(R_i = \lambda_i |v_i\rangle\langle v_i|\), where \(u_i \in \mathbb{C}^{d_1}\) and \(v_i \in \mathbb{C}^{d_2}\) are unit vectors, and \(\lambda_i \in (0, \infty)\) for all \(1 \leq i \leq n\). Then, for all \(X \in M_{d_1}\),

\[
\Phi(X) = \sum_{i=1}^{n} \lambda_i \langle u_i, Xu_i \rangle |v_i\rangle\langle v_i| = \sum_{i=1}^{n} \text{Ad}_{T_i} \circ \Phi_i(X),
\]

where \(\Phi_i = \langle u_i, (\cdot)u_i \rangle I\) and \(T_i = \sqrt{\lambda_i} |v_i\rangle\langle v_i| \in M_{d_2}\) are such that \(\sum_{i=1}^{n} T_i^* T_i = \Phi(I) = I\). By Theorem 5.3, each \(\Phi_i\) is a \(C^*\)-extreme point. This completes the proof. \(\Box\)

6. Conclusion

Our main aim was to characterize the \(C^*\)-extreme points of the \(C^*\)-convex set of unital EB-maps between matrix algebras. To this end, we recalled the analogous results for the \(C^*\)-convex set of unital CP-maps in the finite-dimensional setup. In Propositions 3.2, 3.5 we showed that some of the basic properties of \(C^*\)-extreme points of unital CP-maps hold for unital EB-maps also. So one may think that EB-maps that are \(C^*\)-extreme points of unital CP-maps form the complete set of \(C^*\)-extreme points of unital EB-maps. But, in Example 5.9 we showed that this is not the case. This motivates us to study \(C^*\)-extreme points of EB-maps independently. First, with the help of the technical Lemma 4.1 we proved a Radon-Nikodym type theorem (Theorem 4.2) for a particular class of EB-maps. As in the case of CP-maps, this theorem acts as the building block for characterizing \(C^*\)-extreme points of EB-maps and we arrived at Theorem 5.1. Finally, in Theorem 5.3 we completely characterize \(C^*\)-extreme points of UEB\((d_1, d_2)\). Below we highlight two important characterizations:

- Theorem 5.3 (iv) shows that a unital EB-map is a \(C^*\)-extreme point if and only if its Choi-rank equals \(d_2\). This is a very useful result as typically it is much easier to compute the Choi-rank compared to the EB-rank. No such result exists for \(C^*\)-extreme points of general unital CP-maps.
- Theorem 5.3 (vii) proves that \(C^*\)-extreme EB-maps are just direct sums of pure states.

We conclude with an analogue of classical Krein-Milman theorem (Theorem 5.11) for EB maps under \(C^*\)-convexity.

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