On the spectrum of a waveguide with periodic cracks

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Abstract

The spectral problem on a periodic domain with cracks is studied. An asymptotic form of dispersion relations is calculated under the assumption that the opening of the cracks is small.

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Let $\Omega$ be a connected smooth domain satisfying the following conditions:

- $\Omega$ is invariant under the shift $(x, y) \mapsto (x + 1, y)$,
- the domains $\Omega_n := \Omega \cap ([n, n + 1] \times \mathbb{R})$, $n \in \mathbb{Z}$, are bounded,
- the points $A_n := (n, 0)$ are interior points of $\Omega$.

Assume that the vertical lines $L_n = (n, \mathbb{R})$, $n \in \mathbb{Z}$, are non-tangent to the boundary of $\Omega$ and denote $\Gamma_n := L_n \cap \partial \Omega \setminus B_\varepsilon(A_n)$, where $B_\varepsilon(A_n)$ is the ball of radius $\varepsilon$ centered at $A_n$. See figure 1.

Consider in $L^2(\Omega)$ the operator $H_\varepsilon$ which is the Laplacian with the Neumann boundary conditions on $\partial \Omega$ and on all the curves $\Gamma_n$. We are studying some spectral properties of $H_\varepsilon$ as $\varepsilon$ tends to $0$.

The operator $H_\varepsilon$ can be viewed as the Hamiltonian of elementary cells $\Omega_n$ coupled by small windows. For positive $\varepsilon$ one has a periodic system with a band spectrum, while at $\varepsilon = 0$ the system is decoupled. There are a number of papers concerning the spectral asymptotics for waveguides with windows or cracks in various settings, see e.g. [3–5, 7–9]. In particular, the paper [9] studied a situation which is close to ours, more precisely, the case of the Dirichlet boundary conditions on a similar periodic structure with particular symmetries (straight strip with symmetric cuts), and the asymptotics of the lowest dispersion relation was calculated. All the papers cited use the method of matching of asymptotic expansions [6], which is quite sensible to the geometric properties. We give here another proof based on the analysis of operator pencils involving boundary integrals and inspired by the recent monograph [1]. This allows one to obtain the asymptotics of the spectrum in rather general situations (the assumption made above can be relaxed in various directions, we have just chosen a basic situation in order to keep simple notations). Integral operator pencils of a similar type were used by Pierre Duclos with co-authors as a part of the skeleton method for the study of multi-particle problems [2].

* Dedicated to the memory of Pierre Duclos.
By periodicity, the spectrum of $H_\varepsilon$ can be studied with the help of the Bloch theory. For $\theta \in [0, 2\pi]$ consider the values $E_n(\theta; \varepsilon)$ for which there exists a non-zero function $\Psi_n(x, y; \theta, \varepsilon)$ satisfying the equation $-\Delta \Psi_n(x, y; \theta, \varepsilon) = E_n(\theta; \varepsilon) \Psi_n(x, y; \theta, \varepsilon)$, the above boundary conditions, and the Bloch quasiperiodicity condition $\Psi_n(x + 1, y; \theta, \varepsilon) = e^{i\theta} \Psi_n(x, y; \theta, \varepsilon)$; here, $\theta \in [0, 2\pi]$ is a real number called quasimomentum. By a suitable reordering one can assume that the functions $\theta \mapsto E_n(\theta, \varepsilon)$ are continuous and $2\pi$ periodic; these functions are called dispersion relations of $H_\varepsilon$. The range of a dispersion relation is called a spectral band of $H_\varepsilon$, and the spectrum of $H_\varepsilon$ is the union of the spectral bands.

We are interested in the asymptotic form of the dispersion relations for small $\varepsilon$. By the standard variational arguments, see e.g. proposition 1.3 in [9] or proposition 9.8 in [3], each function $E_n(\cdot, \varepsilon)$ converges to a constant function equal to an eigenvalue of the decoupled system, i.e. to an eigenvalue of the Neumann Laplacian on $\Omega_0$. Our main result is the following theorem.

**Theorem 1.** Let $E$ be a simple eigenvalue of the Neumann Laplacian in $\Omega_0$ and $u$ be the corresponding normalized eigenfunction. As $\varepsilon \to 0$, the dispersion relation of $H_\varepsilon$ near $E$ is of the form

$$E(\theta, \varepsilon) = E + \frac{2\pi}{|\log \varepsilon|} |u(A_0) - e^{i\theta} u(A_1)|^2 + O\left(\frac{\log^2 \varepsilon}{|\log \varepsilon|}\right)$$

uniformly in $\theta \in [0, 2\pi]$.

We note that $E$ can be *any* simple eigenvalue, and not just the bottom one.

The rest of the paper is devoted to the proof of theorem 1.

Let $G_\nu(r, r'; z)$ be the Green’s function of the Neumann Laplacian $N_\nu$ in $\Omega_\nu$, i.e. the integral kernel of the resolvent $(N_\nu - z)^{-1}$. Recall that, if $z$ is not an eigenvalue of $N_\nu$, then the boundary value problem

$$(-\Delta - z)f = 0 \quad \text{in} \quad \Omega_\nu, \quad \frac{\partial f}{\partial v} |_{\partial \Omega_\nu} = g,$$

where $v$ is the exterior normal vector, has the unique solution

$$f(r) = \int_{\partial \Omega_\nu} G_\nu(r, r'; z) g(r') \, dl(r').$$
Consider a Bloch solution $\Psi$ of $H_\epsilon$ corresponding a quasimomentum $\theta$ and to an eigenvalue $z = E(\theta)$. Introduce the functions

$$f_n(y) := \frac{\partial \Psi(x, y)}{\partial x} \bigg|_{x=-\epsilon}, \quad y \in (-\epsilon, \epsilon).$$

Denote by $\Psi_n$ the restriction of $\Psi$ to $\Omega_n$. One has, obviously,

$$\Psi_n(x, y) = -\int_{-\epsilon}^{\epsilon} G_n(x, y; n, y'; z)f_n(y') \, dy' + \int_{-\epsilon}^{\epsilon} G_n(x, y; n + 1, y'; z)f_{n+1}(y') \, dy'.$$

The Bloch condition for $\Psi$ implies $f_{n+1}(y') = e^{i\theta} f_n(y')$. The continuity of $\Psi$ at $x = n$ takes the form $\Psi_n(n, y) = \Psi_{n-1}(n, y)$, or, using the above integral representation

$$e^{i\theta} \int_{-\epsilon}^{\epsilon} G_n(n, y; n + 1, y'; z)f_n(y') \, dy' - \int_{-\epsilon}^{\epsilon} G_n(n, y; n, y'; z)f_n(y') \, dy' = \int_{-\epsilon}^{\epsilon} G_{n-1}(n, y; n - 1, y'; z)f_n(y') \, dy'.$$

Using the obvious identity $G_n(x, y; x', y'; z) = G_0(x - n, y; x' - n, y'; z)$ one arrives at a single integral equation

$$K_{\theta,\epsilon}(z)f_{\theta,\epsilon}(y', z) = G_0(0, 0; 0, y'; z) + G_0(1, 1; 1, y'; z) - e^{i\theta} G_0(0, y; 1, y'; z) - e^{-i\theta} G_0(1, y; 0, y'; z).$$

In order to determine the dispersion relations of $H_\epsilon$ one needs to find the values of $z$ for which equation (1) has non-zero solutions, i.e. the nonlinear eigenvalues of $K_{\theta,\epsilon}$.

As noted above, integral equations of this type were studied in detail in the recent monograph [1], and here we add some details needed for the treatment of the periodic problem. First of all, by lemma 5.6 in [1], for each fixed $\theta$, equation (1) has a unique solution in a neighborhood of $E$ if $\epsilon$ is small enough.

Like in [1] introduce the Hilbert space

$$X_\epsilon := \left\{ \psi : \|\psi\|_{X_\epsilon}^2 := \int_{-\epsilon}^{\epsilon} \sqrt{\epsilon^2 - y^2} |\psi(y)|^2 \, dy < +\infty \right\},$$

and the linear space

$$Y_\epsilon := \{ \psi \in C[-\epsilon, \epsilon] : \psi' \in X_\epsilon \}.$$

By the trace theorems, the functions $f_{\theta,\epsilon}$ belong to both $X_\epsilon$ and $Y_\epsilon$.

We emphasize that the symbol $\langle \cdot, \cdot \rangle$ will always denote the usual scalar product in $L^2[-\epsilon, \epsilon]$, i.e.

$$\langle f, g \rangle := \int_{-\epsilon}^{\epsilon} f(y)g(y) \, dy.$$

Using lemma 5.1 in [1] and the spectral representation of the Green’s function $G_0$ one can decompose $K_{\theta,\epsilon}$ as follows:

$$K_{\theta,\epsilon}(z) := -\frac{1}{2\pi} L_\epsilon + \frac{M_{\theta,\epsilon}}{E - z} + R_{\theta,\epsilon}(z),$$

where $L_\epsilon$ is the integral operator:

$$L_\epsilon f(x) = \int_{-\epsilon}^{\epsilon} \log |x - y| f(y) \, dy,$$
$M_{\theta, \epsilon}$ is a rank one operator:

$$M_{\theta, \epsilon} f(x) = u_{\theta, \epsilon}(x) \int_{-\epsilon}^{\epsilon} u_{\theta, \epsilon}(y) f(y) \, dy,$$

$$u_{\theta, \epsilon}(y) := u(0, y) - e^{i\theta} u(1, y),$$

and $R_{\theta, \epsilon}(z)$ is an integral operator:

$$R_{\theta, \epsilon}(z)f(x) = \int_{-\epsilon}^{\epsilon} R_{\theta, \epsilon}(x, y; z) f(y) \, dy,$$

where the kernel $R_{\theta, \epsilon}(x, y; z)$ is holomorphic in $(z, \theta)$ and of H"older class $C^{1+\alpha}$ with respect to $(x, y)$; here, $\alpha \in (0, 1)$ is arbitrary. Note that the functions $u_{\theta, \epsilon}$ are in both $X_{\epsilon}$ and $Y_{\epsilon}$.

By lemma 5.2 in [1], the operator $L_{\epsilon} : X_{\epsilon} \to Y_{\epsilon}$ is a bijection for $\epsilon$ small enough, and, by lemma 5.4 in [1], one has, uniformly in $\theta \in [0, 2\pi]$ and $z$ in a neighborhood of $E$, the norm estimate

$$\|L_{\epsilon}^{-1} R_{\theta, \epsilon}(z)\|_{L(X_{\epsilon}, X_{\epsilon})} = O\left(\frac{1}{\log \epsilon}\right);$$

(3)

here and below $L(X_{\epsilon}, X_{\epsilon})$ is the Banach space of bounded linear operators on $X_{\epsilon}$. We will need the following estimates:

**Proposition 2.** As $\epsilon \to 0$ one has, uniformly in $\theta$,

$$\langle L_{\epsilon}^{-1} u_{\theta, \epsilon}, u_{\theta, \epsilon} \rangle = \frac{|u_{\theta, \epsilon}(0)|^2}{\log \epsilon} + O\left(\frac{1}{|\log \epsilon|^2}\right).$$

(4)

Furthermore, there exists $A > 0$ with the following property: if $B$ is a bounded operator in $X_{\epsilon}$ then, as $\epsilon \to 0$, one has

$$\left|\langle BL_{\epsilon}^{-1} u_{\theta, \epsilon}, u_{\theta, \epsilon} \rangle\right| \leq A \|B\|_{L(X_{\epsilon}, X_{\epsilon})} \left|\log \epsilon\right|.$$  

(5)

**Proof.** As in lemma 5.4 of [1] we use an explicit form for the inverse of $L_{\epsilon}$ and some properties of the finite Hilbert transform. We have

$$L_{\epsilon}^{-1} u_{\theta, \epsilon}(x) = \frac{1}{\pi^2} \sqrt{\epsilon^2 - x^2} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{\epsilon^2 - y^2} u'_{\theta, \epsilon}(y)}{x - y} \, dy$$

$$+ \frac{a(u_{\theta, \epsilon})}{\pi \log \frac{2}{\sqrt{\epsilon^2 - x^2}}} =: I_1 + I_2,$$

where

$$a(u_{\theta, \epsilon}) = u_{\theta, \epsilon}(x) - L_{\epsilon} v_{\theta, \epsilon}(x)$$

(the choice of $x$ is arbitrary) with

$$v_{\theta, \epsilon}(x) = \frac{1}{\pi^2} \sqrt{\epsilon^2 - x^2} \int_{-\epsilon}^{\epsilon} \frac{\sqrt{\epsilon^2 - y^2} u'_{\theta, \epsilon}(y)}{x - y} \, dy;$$

here and below all the integrals are understood in the sense of the Cauchy principal value.

We will use the following well-known estimate: for any $\alpha \in (0, 1)$ there exists $C > 0$ such that for any $\varphi \in C^\alpha[-1, 1]$ and $x \in [-1, 1]$ there holds

$$\left|\int_{-1}^{1} \frac{\varphi(y)}{x - y} \, dy\right| \leq C \|\varphi\|_{C^\alpha[-1, 1]}.$$
One then has
\[
\|I_1\|_{X_1} = \frac{1}{\pi} \sqrt{\int \frac{1}{\sqrt{\varepsilon^2 - x^2}} \left( \int \frac{\sqrt{\varepsilon^2 - y^2} u_\theta,\varepsilon'(y)}{x-y} \, dy \right)^2 \, dx}
\]
\[
\leq \frac{1}{\pi} \sqrt{\int \frac{1}{\sqrt{\varepsilon^2 - x^2}} \left( \int \frac{\sqrt{\varepsilon^2 - y^2} u_\theta,\varepsilon'(y)}{x-y} \, dy \right) \| \cdot \|_{L^\infty([-\varepsilon,\varepsilon])}}
\]
\[
= \frac{\varepsilon}{\pi^{1/2}} \left\| \int_{-1}^{1} \frac{1}{\sqrt{1 - y^2}} u_\theta,\varepsilon'(y) \, dy \right\|_{L^\infty([-1,1])}
\]
\[
\leq \frac{C \varepsilon}{\pi^{1/2}} \left\| \sqrt{1 - y^2} u_\theta,\varepsilon'(y) \right\|_{C^1([-1,1]^\prime)}
\]
and one obtains \(\|I_1\|_{X_1} = O(\varepsilon)\) uniformly in \(\theta\). This implies \(\langle I_1, u_\theta,\varepsilon \rangle = O(\varepsilon)\) as well. Hence it is sufficient to study the asymptotics of \(\langle I_2, u_\theta,\varepsilon \rangle\). We have
\[
\langle I_2, u_\theta,\varepsilon \rangle = \frac{1}{\pi \log |\varepsilon/2|} \left( \int_{-\varepsilon}^{\varepsilon} \frac{|u_\theta,\varepsilon'(x)|^2}{\varepsilon^2 - x^2} \, dx - \int_{-\varepsilon}^{\varepsilon} \frac{u_\theta,\varepsilon(x)}{\sqrt{\varepsilon^2 - y^2}} \, dy \right)
\]
\[
\frac{1}{\pi \log |\varepsilon/2|} (K_1 - K_2).
\]
There holds
\[
K_1 = \int_{-\varepsilon}^{\varepsilon} \frac{|u_\theta,\varepsilon(x)|^2}{\varepsilon^2 - x^2} \, dx = \int_{-1}^{1} \frac{|u_\theta,\varepsilon(\varepsilon x)|^2}{\varepsilon^2 - x^2} \, dx
\]
\[
= |u_\theta,\varepsilon(0)|^2 \int_{-1}^{1} \frac{dx}{\varepsilon^2 - x^2} + \int_{-1}^{1} \frac{|u_\theta,\varepsilon(\varepsilon x)|^2 - |u_\theta,\varepsilon(0)|^2}{\varepsilon^2 - x^2} \, dx
\]
\[
= \pi |u_\theta,\varepsilon(0)|^2 + O(\varepsilon),
\]
where the remainder term is uniform in \(\theta\).

To estimate \(K_2\) let us note first that, by the same arguments as in (6),
\[
\int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - y^2}}{x-y} u_\theta,\varepsilon'(y) \, dy =: K_3(\varepsilon) = O(\varepsilon)
\]
uniformly in \(x\) and \(\theta\). Therefore,
\[
\sup_{x \in [-\varepsilon, \varepsilon]} |L_\varepsilon v_{\theta,\varepsilon}(x)| = \frac{1}{\pi \varepsilon} \sup_{x \in [-\varepsilon, \varepsilon]} \left| \int \frac{\log |x - y| \cdot K_3(\varepsilon) \cdot \frac{1}{\sqrt{\varepsilon^2 - y^2}} \, dy}{\varepsilon^2 - x^2} \right|
\]
\[
\leq \frac{1}{\pi \varepsilon} \sup_{y \in [-\varepsilon, \varepsilon]} |K_3(\varepsilon)| \sup_{x \in [-\varepsilon, \varepsilon]} \left| \int \frac{\log |x - y|}{\sqrt{\varepsilon^2 - y^2}} \, dy \right|
\]
\[
= \frac{1}{\pi} \sup_{y \in [-\varepsilon, \varepsilon]} |K_3(\varepsilon)| \log \frac{\varepsilon}{2} = O(\varepsilon \log \varepsilon)
\]
uniformly in \(\theta\). Finally,
\[
|K_2| \leq \sup_{x \in [-\varepsilon, \varepsilon]} |L_\varepsilon v_{\theta,\varepsilon}(x)| \int \frac{|u_\theta,\varepsilon(y)|}{\sqrt{\varepsilon^2 - y^2}} \, dy = O(\varepsilon \log \varepsilon).
\]
We obtain
\[
\langle L_\varepsilon^{-1} u_\theta,\varepsilon, u_\theta,\varepsilon \rangle = \langle I_1, u_\theta,\varepsilon \rangle + \langle I_2, u_\theta,\varepsilon \rangle = (I_1, u_\theta,\varepsilon) + \frac{K_1}{\pi \log \frac{\varepsilon}{2}} - \frac{K_2}{\pi \log \frac{\varepsilon}{2}}
\]
\[
= O(\varepsilon) + \frac{\pi |u_\theta,\varepsilon(0)|^2}{\pi \log \frac{\varepsilon}{2}} + O(\varepsilon \log \varepsilon) = \frac{|u_\theta,\varepsilon(0)|^2}{\log \varepsilon} + O\left(\frac{1}{|\log \varepsilon|^2}\right),
\]
which proves (4).
To show (5) we use first the Cauchy–Schwartz inequality for $X_\varepsilon$,

$$\left| \left[ B L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon} \right] \right| = \left| \int_{-\varepsilon}^{\varepsilon} \frac{\sqrt{\varepsilon^2 - x^2}}{\sqrt{\varepsilon^2 - x^2}} \cdot B L_\varepsilon^{-1} u_{\theta,\varepsilon} (x) \cdot u_{\theta,\varepsilon} (x) \, dx \right|$$

$$\leq \| B \|_{L(X_\varepsilon, X_\varepsilon)} \cdot \| L_\varepsilon^{-1} u_{\theta,\varepsilon} \|_{X_\varepsilon} \cdot \sqrt{\int_{-\varepsilon}^{\varepsilon} \frac{|u_{\theta,\varepsilon} (x)|^2}{\sqrt{\varepsilon^2 - x^2}} \, dx}$$

$$\leq a \| B \|_{L(X_\varepsilon, X_\varepsilon)} \cdot \| L_\varepsilon^{-1} u_{\theta,\varepsilon} \|_{X_\varepsilon}, \quad a > 0,$$

and the previous estimates show that $\| L_\varepsilon^{-1} u_{\theta,\varepsilon} \|_{X_\varepsilon} = O(1/\log \varepsilon)$. Proposition 2 is proved.

Let us conclude the proof of theorem 1. Assume that $K_{\theta,\varepsilon} (z) f_{\theta,\varepsilon} (z) = 0$ with $f_{\theta,\varepsilon} \neq 0$ and $z = E(\theta, \varepsilon)$ sufficiently close to $E$. Using the invertibility of $L_\varepsilon$ and the decomposition (2) one obtains

$$\left( 1 - 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon} \right) f_{\theta,\varepsilon} = \frac{2\pi}{E - z} \langle u_{\theta,\varepsilon}, f_{\theta,\varepsilon} \rangle L_\varepsilon^{-1} u_{\theta,\varepsilon}.$$

The operator $1 - 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon}$ is invertible for $\varepsilon$ small enough; therefore,

$$f_{\theta,\varepsilon} = \frac{2\pi}{E - z} \left( 1 - 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon} \right)^{-1} \langle u_{\theta,\varepsilon}, f_{\theta,\varepsilon} \rangle L_\varepsilon^{-1} u_{\theta,\varepsilon},$$

which implies $\langle u_{\theta,\varepsilon}, f_{\theta,\varepsilon} \rangle \neq 0$ (otherwise one would get $f_{\theta,\varepsilon} \equiv 0$). Taking the scalar product $\langle \cdot, \cdot \rangle$ of the both parts with $u_{\theta,\varepsilon}$ one arrives at

$$z = E(\theta, \varepsilon) = E - 2\pi \left| \left( 1 - 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon} \right)^{-1} L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon} \right|.$$

Using (3), let us fix $B > 0$ such that

$$\| L_\varepsilon^{-1} R_{\theta,\varepsilon} (z) \|_{L(X_\varepsilon, X_\varepsilon)} \leq \frac{B}{| \log \varepsilon |} \quad (7)$$

for small $\varepsilon$. As the map $f \mapsto \langle f, u_{\theta,\varepsilon} \rangle$ is continuous in the $X_\varepsilon$-norm, we have, for sufficiently small $\varepsilon$ such that $\frac{2\pi B}{| \log \varepsilon |} < 1$,

$$\left| \left( 1 - 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon} \right)^{-1} L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon} \right|$$

$$= \langle L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon} \rangle + \sum_{k \geq 1} \left| \left( 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon} \right)^k L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon} \right| =: D_1 + D_2.$$

In view of the estimate (4) for $D_1$, it is sufficient to show that $D_2 = O(1/| \log \varepsilon |^2)$. By (3) and (5),

$$\left| \left( 2\pi L_\varepsilon^{-1} R_{\theta,\varepsilon} \right)^k L_\varepsilon^{-1} u_{\theta,\varepsilon}, u_{\theta,\varepsilon} \right| \leq \frac{A}{| \log \varepsilon |} \left( \frac{2\pi B}{| \log \varepsilon |} \right)^k,$$

hence

$$|D_2| \leq \frac{A}{| \log \varepsilon |} \sum_{k \geq 1} \left( \frac{B}{| \log \varepsilon |} \right)^k = \frac{A}{| \log \varepsilon |} \cdot \frac{2\pi B}{1 - 2\pi B} = O(1/| \log \varepsilon |^2),$$

which finishes the proof of theorem 1.
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