On the interplay among hypergeometric functions, complete elliptic integrals, and Fourier–Legendre expansions

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Abstract

Motivated by our previous work on hypergeometric functions and the parbelos constant, we perform a deeper investigation on the interplay among generalized complete elliptic integrals, Fourier–Legendre (FL) series expansions, and \( \mathcal{P}_FQ \) series. We produce new hypergeometric transformations and closed-form evaluations for new series involving harmonic numbers, through the use of the integration method outlined as follows: Letting \( K \) denote the complete elliptic integral of the first kind, for a suitable function \( g \) we evaluate integrals such as

\[
\int_0^1 K(\sqrt{x}) g(x) \, dx
\]

in two different ways: (1) by expanding \( K \) as a Maclaurin series, perhaps after a transformation or a change of variable, and then integrating term-by-term; and (2) by expanding \( g \) as a shifted FL series, and then integrating term-by-term. Equating the expressions produced by these two approaches often gives us new closed-form evaluations, as in the formulas involving Catalan’s constant \( G \)

\[
\sum_{n=0}^{\infty} \left( \frac{2n}{n} \right)^2 \frac{H_{n+\frac{1}{4}} - H_{n-\frac{1}{4}}}{16^n} = \frac{\Gamma^4 \left( \frac{1}{4} \right)}{8\pi^2} - \frac{4G}{\pi},
\]

\[
\sum_{m,n \geq 0} \frac{\left( \frac{2m}{n} \right)^2 \left( \frac{2n}{m} \right)^2}{16^m+n(m+n+1)(2m+3)} = \frac{7\zeta(3) - 4G}{\pi^2}.
\]

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1. Introduction

In the history of mathematical analysis, there are many strategies for computing infinite series in symbolic form and it remains a very active area of research. In our present article, we introduce a variety of new results on the closed-form evaluation of hypergeometric series and harmonic summations through the use of new techniques that are mainly based on the use of complete elliptic integrals and the theory of Fourier–Legendre (FL) expansions.

Inspired in part by our previous work on the evaluation of a \(3F_2(1)\) series related to the parbelos constant \(\mathcal{P}\), which, in turn, came about through the discovery [4] of an integration technique for evaluating series involving squared central binomial coefficients and harmonic numbers in terms of \(\frac{1}{\pi}\), in this article we apply a related integration method to determine new identities for hypergeometric expressions, as well as new evaluations for binomial-harmonic series.

We recall that a hypergeometric series is an infinite series \(\sum_{i=0}^{\infty} c_i\) such that there exist polynomials \(P\) and \(Q\) satisfying

\[
\frac{c_{i+1}}{c_i} = \frac{P(i)}{Q(i)}
\]

for each \(i \in \mathbb{N}_0\). If \(P\) and \(Q\) can be written as \(P(i) = (i + a_1)(i + a_2) \cdots (i + a_p)\) and \(Q(i) = (i + b_1)(i + b_2) \cdots (i + b_q)(i + 1)\), a generalized hypergeometric function is an infinite series of the form \(\sum_{i=0}^{\infty} c_i x^i\), and is denoted by

\[
\sum_{i=0}^{\infty} c_i x^i = {}_pF_q\left[ \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{array} \mid x \right].
\]

The complete elliptic integrals of the first and second kinds, which are respectively denoted with \(K\) and \(E\), may be defined as

\[
K(k) = \frac{\pi}{2} \cdot {}_2F_1\left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2} \\ 1 \end{array} \mid k^2 \right].
\]
\[ E(k) = \frac{\pi}{2} \cdot 2F_1 \left[ \begin{array}{c} \frac{1}{2}, -\frac{1}{2} \\ 1 \end{array} \middle| k^2 \right]. \]

We are interested in evaluating integrals such as
\[ \int_0^1 K(\sqrt{x}) g(x) \, dx \quad (2) \]
for a suitable function \( g \), by expanding \( K \) as a Maclaurin series, perhaps after a manipulation of the expression \( K \), and integrating term-by-term. However, by replacing \( g(x) \) with its shifted Fourier–Legendre series expansion, integrating term-by-term and equating the two resulting series, we often obtain new closed-form evaluations. We illustrate this idea with the example described in Section 2 below that is taken from [6], after a preliminary discussion concerning the basics of FL theory.

Legendre functions of order \( n \) are solutions to Legendre’s differential equation
\[(1 - x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + n(n + 1)y = 0 \]
for \( n > 0 \) and \( |x| < 1 \). For \( n \in \mathbb{N}_0 \), Legendre polynomials \( P_n(x) \) are examples of Legendre functions, and may be defined via the Rodrigues formula
\[ P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n. \quad (3) \]

The following equivalent definition for \( P_n(x) \) will also be used:
\[ P_n(x) = \frac{1}{2^n} \sum_{k=0}^n \binom{n}{k}^2 (x - 1)^{n-k} (x + 1)^k. \quad (4) \]

The Legendre polynomials form an orthogonal family on \((-1, 1)\), with
\[ \int_{-1}^1 P_n(x)P_m(x) \, dx = \frac{2}{2n + 1} \delta_{m,n}, \]
which gives us the Fourier–Legendre series for a suitable function \( g \):
\[ g(x) = \sum_{n=0}^{\infty} \left[ \frac{2n + 1}{2} \int_{-1}^1 g(t)P_n(t) \, dt \right] P_n(x). \quad (5) \]

Letting shifted Legendre polynomials be denoted as \( \tilde{P}_n(x) = P_n(2x - 1) \), polynomials of this form are orthogonal on \([0, 1]\), with
\[ \int_0^1 \tilde{P}_m(x)\tilde{P}_n(x) \, dx = \frac{1}{2n + 1} \delta_{m,n}. \]
By analogy with the expansion from [5], for a reasonably well-behaved function $f$ on $(0,1)$, this function may be expressed in terms of shifted Legendre polynomials by writing

$$f(x) = \sum_{m=0}^{\infty} c_m P_m(2x - 1).$$

We can determine the scalar coefficient $c_m$ in a natural way using the orthogonality of the family of shifted Legendre polynomials. In particular, we see that if we integrate both sides of

$$\tilde{P}_m(x) f(x) = \sum_{n=0}^{\infty} c_n \tilde{P}_m(x) \tilde{P}_n(x)$$

over $[0,1]$, we get

$$c_m = (2m + 1) \int_0^1 P_m(2x - 1) f(x) \, dx.$$

Brafman’s formula states that

$$\sum_{n=0}^{\infty} \frac{(s)_n (1-s)_n}{(n!)^2} P_n(x) z^n = 2 F_1 \left[ \begin{array}{c} s, 1-s \\ 1 \end{array} \right]_{\alpha} 2 F_1 \left[ \begin{array}{c} s, 1-s \\ 1 \end{array} \right]_{\beta},$$

letting $\alpha = \frac{1-\rho-z}{2}$, $\beta = \frac{1+\rho-z}{2}$, and $\rho = \sqrt{1 - 2xz + z^2}$ [2]. The canonical generating function for Legendre polynomials is [12, 13]

$$\frac{1}{\sqrt{1 - 2xz + z^2}} = \sum_{n=0}^{\infty} P_n(x) z^n.$$  

(7)

This gives the following result (see [9] and [13]) which we exploit heavily:

$$K\left(\sqrt{k}\right) = \sum_{n \geq 0} \frac{2}{2n+1} P_n(2k - 1).$$

(8)

If we make use of the standard moment formula

$$\int_0^1 x^i P_n(2x - 1) \, dx = \frac{(i!)^2}{(i-n)! (i+n+1)!}$$

for shifted Legendre polynomials, then it is not difficult to see why (8) holds.

Namely, from the Maclaurin series for $K$, we have that

$$K\left(\sqrt{x}\right) P_n(2x - 1) = \frac{\pi}{2} \sum_{i \in \mathbb{N}_0} \left(\frac{1}{16}\right)^i \binom{2i}{i}^2 x^i P_n(2x - 1),$$

(8)
and by rewriting the right-hand side as
\[
\frac{\pi}{2} \sum_{i \in \mathbb{N}_0} \left( \frac{(2i)!}{i!} \right)^2 \frac{1}{16^i(i-n)!(i+n+1)!} = \frac{2(\sin(\pi n) + 1)}{(2n + 1)^2}
\]
using the moments for the family \( \{P_n(2x-1)\}_{n \in \mathbb{N}_0} \), we obtain the desired result.

2. A motivating example

Following the integration method from [4], since
\[
\int_0^1 x^{4n} \ln(1-x^2) \sqrt{1-x^2} \frac{dx}{\sqrt{1-x^2}} = -\frac{\sqrt{\pi} \Gamma(2n + \frac{1}{2}) (H_{2n} + 2 \ln(2))}{2\Gamma(2n + 1)},
\]
and since
\[
\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n} \left( \frac{1}{2} \right) \ln(1-x^2)}{\sqrt{1-x^2}} = \sqrt{x^2 + 1} \ln(1-x^2)
\]
we have that the series
\[
\sum_{n=0}^{\infty} \frac{(2n)!}{n!} \frac{(4n)!}{(2n)!} H_{2n} \frac{1}{64^n(2n-1)}
\]
may be evaluated in terms of the following \( 3\text{F}_2(1) \) series, which was conjectured by the first author to be equal to the parbelos constant \( \frac{\sqrt{2} + \ln(1+\sqrt{2})}{\pi} \): \( 3\text{F}_2 \left[ -\frac{1}{2}, \frac{1}{4}, \frac{3}{4} \right| \frac{1}{2}, 1 \right] \).

This conjecture is proved in a variety of different ways in [4], where the “palindromic” formula
\[
3\text{F}_2 \left[ \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \right| \frac{1}{2}, 1 \right] = \frac{8}{\pi} \tan^{-1} \tan \frac{\pi}{8}
\]
is introduced and is used in one proof through an application of Fourier–Legendre theory that heavily makes use of the complete elliptic integral \( K \).

Many aspects of our present article are directly inspired by the FL-based proof presented in [6], so it is worthwhile to review it. Adopting notation from [2], we let \( g(x) = \frac{1}{(2-x)^{3/2}} \); the main integral under investigation is then
\[
\int_0^1 K(\sqrt{x}) \frac{1}{(2-x)^{3/2}} \frac{dx}{(2-x)^{3/2}}.
\]
Letting \((x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\) denote the Pochhammer symbol, by writing the series in (10) as

\[ _3F_2 \left[ \begin{array}{l} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ \frac{3}{2} \end{array} \right] = \sum_{n=0}^{\infty} \frac{(1)_n (1/2)_n (3/2)_n}{(1)_n (3/2)_n n!} \]

and by noting that

\[ \sum_{n=0}^{\infty} \frac{(2n)(4n)}{(2n+1)64^n} \]

we thus find that

\[ _3F_2 \left[ \begin{array}{l} \frac{1}{4}, \frac{1}{2}, \frac{3}{4} \\ \frac{3}{2} \end{array} \right] = \frac{2\sqrt{2}}{\pi} \int_0^1 \frac{K(\sqrt{r})}{(2-r)^{3/2}} \, dr. \]  

(12)

Manipulating (7) so as to obtain the generating function for shifted Legendre polynomials, we find that

\[ \frac{1}{(2-x)^{3/2}} = \sum_{n \in \mathbb{N}_0} (2n+1)\sqrt{2} \left( \sqrt{2} - 1 \right)^{2n+1} \tilde{P}_n(x), \]

so that

\[ \int_{x \in [0,1]} \tilde{P}_m(x) \frac{1}{(2-x)^{3/2}} \, dx = \sqrt{2} \left( \sqrt{2} - 1 \right)^{2m+1} \]

for all \(m \in \mathbb{N}_0\), from the orthogonality relations for shifted FL polynomials. So, from the important expansion in (8), we have that the integral in (11) equals

\[ \int_0^1 \sum_{n \geq 2n+1} \frac{2}{2n+1} \tilde{P}_n(x) \frac{1}{(2-x)^{3/2}} \, dx = \sum_{n \geq 2n+1} \frac{2}{2n+1} \sqrt{2} \left( \sqrt{2} - 1 \right)^{2n+1} \]

\[ = \sqrt{2} \ln \left( 1 + \sqrt{2} \right). \]

This, together with equation (12), gives us the desired evaluation in (10).

It is natural to consider variants and generalizations of this proof technique, and this serves as something of a basis for our article.
3. Summary of main results

In Section 5, we show how manipulations related to the fundamental formula \((8)\) together with symbolic evaluations of definite integrals of the form given in (2) for a function \(g(x)\) with a “reasonable” or “manageable” shifted Fourier–Legendre series may be used to construct new results on hypergeometric series.

Generalized harmonic functions are of the form

\[ H^{(b)}_a = \zeta(b) - \zeta(b, a + 1), \tag{13} \]

where \(\zeta(b)\) denotes the Riemann zeta function evaluated at \(b\), and

\[ \zeta(b, a + 1) = \sum_{i \in \mathbb{N}_0} \frac{1}{(i + a + 1)^b} \]

denotes the Hurwitz zeta function with parameters \(b\) and \(a + 1\). In the case \(b = 1\), we often omit the superscript on the left-hand side of (13). We also adopt the standard convention whereby \(H^{(0)}_a = a\). If we let \(c_n, c'_n, \ldots, c^{(m)}_n\) satisfy the hypergeometric condition in (1), then we define a twisted hypergeometric series to be one of the form

\[ \sum_{n=0}^{\infty} \left( c_n H^{(\gamma)}_{\alpha n + \beta} + c'_n H^{(\gamma')}_{\alpha' n + \beta'} + \cdots + c^{(m)}_n H^{(\gamma^{(m)})}_{\alpha^{(m)} n + \beta^{(m)}} \right). \]

The evaluation of such series using our main technique is of central importance in this paper.

In Section 5.1, we use this technique to prove that

\[ \frac{\pi}{4} = \frac{3F_2}{3F_2} \left[ \begin{array}{c} -\eta, \frac{1}{2}, 1 \\ \frac{3}{2}, 2 + \eta \end{array} \right| -1 \right] \tag{14} \]

for \(\eta > -1\), and we obtain the following identity on the moments of the elliptic-type function \(E(\sqrt{x})\):

\[ \int_0^1 E(\sqrt{x}) x^n \, dx = \frac{\pi}{2(1 + \eta)} \cdot 3F_2 \left[ \begin{array}{c} -\frac{1}{2}, \frac{1}{2}, 1 + \eta \\ 1, 2 + \eta \end{array} \right| 1 \right] \]
\[
= \frac{4}{3(1 + \eta)} \cdot _3F_2 \left[ \begin{array}{c}
\frac{1}{2}, 1, -\eta \\
\frac{3}{2}, 2 + \eta
\end{array} \right] - 1.
\]

In Section 5.2, we prove the equality
\[
\sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{H_n - H_{n-1/2}}{64^n} = \frac{\pi}{\sqrt{2}} - \frac{2\sqrt{2}}{\pi} \ln^2 \left( \sqrt{2} + 1 \right)
\]
and we also provide a closed-form evaluation for the series
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \frac{H_n^{1/2} - nH_n}{64^n}.
\]

We offer a new FL-based proof of the \( \frac{1}{\pi} \) formula
\[
\sum_{n=1}^{\infty} \frac{\binom{2n}{n}^2 H_n}{16^n(2n - 1)} = \frac{8 \ln(2) - 4}{\pi}
\]
introduced in [4], along with a new proof of the formula
\[
\sum_{m,n \geq 0} \binom{n}{16} m \frac{\binom{2m}{m}^2 H_n}{(n + 1)(m + n + 2)} = \frac{48 + 32(\ln(2) - 2) \ln(2) - 4\pi}{3}
\]
given in [4]. By applying a moment formula that is used to prove the identity in (14), we prove the new double series result
\[
\sum_{n,m \in \mathbb{N}_0} \frac{(n + 1)! n! H_n}{(2m + 1)(n - m + 1)! (m + n + 2)!} = 12 - \frac{\pi^2}{3} + 8 \ln^2(2) - 16 \ln(2).
\]

Inspired in part by an integration method given in [4] on the evaluation of series containing factors of the form \( H_n^2 + H_n^{(2)} \), we offer a new proof of the formula
\[
\sum_{n=0}^{\infty} \binom{1/16}{n} \frac{\binom{2n}{n}^2 H_n^2 + H_n^{(2)}}{n + 1} = \frac{64 \ln^2(2)}{\pi} - \frac{8\pi}{3}
\]
using the machinery of Fourier–Legendre expansions. We also offer a stronger version of this result by showing how our main technique can be used to evaluate
\[
\sum_{n=0}^{\infty} \binom{1/16}{n} \frac{\binom{2n}{n}^2 H_n^{(2)}}{n + 1} = \frac{32G}{\pi} + \frac{2\pi}{3} - 16 \ln(2)
\]
which also gives us a symbolic form for its companion
\[
\sum_{n=0}^{\infty} \binom{1/16}{n} \frac{\binom{2n}{n}^2 H_n^2}{n + 1}.
\]
We also prove the formula
\[ \sum_{n=0}^{\infty} \frac{(2n)^2 H_{2n}}{16^n(2n-1)} = \frac{6 \ln(2) - 2}{\pi}, \]
which extends our proof of (15). In Section 5.2, we prove the equality
\[ \sum_{n=0}^{\infty} \frac{(2n)^2 H_{n+\frac{1}{2}} - H_{n-\frac{1}{2}}}{16^n} = \frac{\Gamma^4(\frac{1}{4})}{8\pi^2} - \frac{4G}{\pi}, \]
and offer a new proof of the equation
\[ \sum_{n=1}^{\infty} \frac{(2n)^2 H_{2n}}{16^n(2n-1)^2} = \frac{4G + 6 - 12 \ln(2)}{\pi} \]
introduced in [7]. Using FL theory, we prove the formula
\[ \sum_{m,n\geq 0} \frac{(2m)^2 (2n)^2}{16^m+n+1}(2m+3) = \frac{7\zeta(3) - 4G}{\pi^2}, \]
strongly motivating further explorations on our main techniques.

Recall that the polylogarithm function \( \text{Li}_n(z) \) is defined so that
\[ \text{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n} \]
for \( |z| \leq 1 \), and in the case \( n = 2 \) we obtain the dilogarithm mapping. In Section 5.4 we use our integration methods to explore connections between generalized hypergeometric functions and polylogarithmic functions, providing an evaluation of
\[ \sum_{n\geq 1} \frac{(4n)(2n)}{n64^n} = \frac{3}{16} \cdot 4F3 \left[ \frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{7}{4} \left| \frac{2}{\pi} \right] 1 \right] \]
in terms of a dilogarithmic expression, with a similar evaluation being given for
\[ \sum_{n=0}^{\infty} \frac{(2n)^2 (2n+1)}{16^n(n+1)^4}. \]

In Section 6, we prove a variety of new results on generalized complete elliptic integrals of the form
\[ \mathcal{J}(x) = \int_0^{\pi/2} \left( \sqrt{1 - x \sin^2 \theta} \right)^3 d\theta. \]
Using the moments of this function, we prove that the identity

\[
\frac{15\pi}{32} = \frac{\genfrac{3}{3}{2}{\begin{array}{c}
-\frac{3}{2}, 1, -\eta \\
\frac{7}{2}, 2 + \eta
\end{array}}{-1}}{\genfrac{3}{3}{2}{\begin{array}{c}
-\frac{3}{2}, \frac{7}{2}, 1 + \eta \\
1, 2 + \eta
\end{array}}} - 1
\]

holds for \( \eta > -1 \). A generalization \( J_m(x) \) of \( J(x) \) is introduced in Section 6, and we introduce formulas for evaluating the moments of \( J_m(x) \).

4. Related mathematical literature

Some of our main results are the construction of new formulas for \( \frac{1}{\pi} \) using Fourier–Legendre expansions. So, it is natural to consider FL-based techniques that have previously been applied to derive new infinite series formulas for \( \frac{1}{\pi} \). One of Ramanujan’s most famous formulas for \( \frac{1}{\pi} \) is

\[
\frac{2}{\pi} = \sum_{n=0}^{\infty} \left( -\frac{1}{64} \right)^n (4n + 1) \left( \frac{2n}{n} \right)^3,
\]

which was proved by Bauer in 1859 in [1] using a Fourier–Legendre expansion. Levrie applied Bauer’s classical FL-based method to functions of the form \((\sqrt{1 - a^2x^2})^{2k-1}\) to derive new infinite sums for \( \frac{1}{\pi} \), such as the Ramanujan-like equation [16]

\[
\frac{8}{9\pi} = \sum_{n=0}^{\infty} \left( -\frac{1}{64} \right)^n \frac{(4n + 1)\left( \frac{2n}{n} \right)^3}{(n + 1)(n + 2)(2n - 3)(2n - 1)}.
\]

Much of the subject matter in Wan’s dissertation [20] is closely related to some of our main techniques. One of our key methods is the manipulation of generating functions for Legendre polynomials to construct new rational approximations for \( \frac{1}{\pi} \), as is the case with [20]. The section in [20] on Legendre polynomials and series for \( \frac{1}{\pi} \) makes use of Brafman’s formula [6], proving many series for \( \frac{1}{\pi} \) typically involving summands with irrational powers.

Wan also explored the use of Legendre polynomials to construct new series for \( \frac{1}{\pi} \) in [21] and new results in this area were also introduced by Chan, Wan, and
Zudilin in [8]. Brafman’s formula is also applied in [21] to produce new results on \( \frac{1}{\pi} \) series, whereas the FL-based methods in our present article mainly make use of Fourier–Legendre expansions for elliptic-type expressions such as \( K(\sqrt{x}) \).

New series for \( \frac{1}{\pi} \) are given in [18] through a generalization of Bailey’s identity for generating functions given by componentwise products of Apéry-type sequences and the sequence of Legendre polynomials. The construction of hypergeometric series identities using expansions in terms of Legendre polynomials has practical applications in mathematical physics [14] and related areas; a variety of binomial sum identities given in terms of generalized hypergeometric functions are proved in [11] through the use of the family \( \{ P_n(x) : n \in \mathbb{N}_0 \} \).

5. Applications of FL expansions

From the ordinary generating function for the binomial sequence

\[
\left( \binom{4n}{2n} : n \in \mathbb{N}_0 \right),
\]

we have that

\[
\sum_{n \geq 0} \binom{4n}{2n} \frac{x^{2n}}{16^n} = \frac{1}{2} \left( \frac{1}{\sqrt{1 + x}} + \frac{1}{\sqrt{1 - x}} \right). \tag{16}
\]

From (16), together with Wallis’ formula

\[
\binom{2n}{n} = \frac{2}{\pi} \int_0^\pi 2^{2n} \sin^{2n}(t) \, dt, \tag{17}
\]

we may compute the generating function for the sequence \( (\binom{4n}{2n}\binom{2n}{n})_{n \geq 0} \):

\[
\sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} y^n 64^n = 2 F_1 \left[ \begin{array}{c} 1, 3 \\ 1 \end{array} \right | y \\
= \frac{1}{\pi} \int_0^{\pi/2} \left( \frac{1}{\sqrt{1 - \sqrt{y} \sin \theta}} + \frac{1}{\sqrt{1 + \sqrt{y} \sin \theta}} \right) d\theta \\
= \frac{2}{\pi \sqrt{1 + \sqrt{y}}} K \left( \frac{2\sqrt{y}}{1 + \sqrt{y}} \right). 
\]

The identity (16) alone is powerful enough to lead us to the explicit computation of many hypergeometric \( _3F_2 \) functions with quarter-integer parameters.
Through the use of (17), it is not difficult to evaluate series of the form
\[ \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^{n}(n+m)^{m}} \quad m \in \mathbb{N}^{+}, \eta \in \{1, 2\} \]
so as to obtain closed-form evaluations for series as in
\[ \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^{n}(n+m_{1})^{m_{1}} \cdots (n+m_{k})^{m_{k}}} \quad m_{i} \in \mathbb{N}^{+}, \eta_{j} \in \{1, 2\} \]
via partial fraction decomposition, not to mention the case whereby \( m \in \frac{1}{2} + \mathbb{N} \) and \( \eta = 1 \); in this case we can also apply Wallis’ identity to obtain closed-form evaluations. For example, the following equalities hold:
\[ \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^{n}(2n+1)} = \frac{4}{\pi} \ln(1+\sqrt{2}), \]
\[ \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^{n}(2n+3)} = \frac{4\sqrt{2}}{15\pi} + \frac{16}{15\pi} \ln(1+\sqrt{2}), \]
\[ \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^{n}(2n+5)} = \frac{68\sqrt{2}}{315\pi} + \frac{64}{105\pi} \ln(1+\sqrt{2}). \]

As Zhou shows in [22], the Legendre functions \( P_{-1/2} \) and \( P_{-1/4} \) are associated with \( K(\sqrt{x}) \) and \( \frac{1}{\sqrt{1+y}} K \left( \sqrt{\frac{2y}{1+y}} \right) \), which in turn are associated with the weights \( \binom{2n}{n}^{2} \) and \( \binom{4n}{2n} \binom{2n}{n} \). From the Rodrigues formula, the computation of the FL expansion of \( \frac{1}{\sqrt{1+y}} K \left( \sqrt{\frac{2y}{1+y}} \right) \) turns out to be an exercise in fractional calculus. On the other hand,
\[ \int_{0}^{1} \frac{y^\eta}{\sqrt{1+y}} K \left( \sqrt{\frac{2y}{1+y}} \right) dy = \frac{\pi}{2} \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^{n}(2n+\eta+1)} \]
\[ = \frac{\pi}{2(1+\eta)} \cdot _{3}F_{2} \left[ \begin{array}{c} \frac{1}{2} \frac{1+\eta}{2} \\
1, \frac{3+\eta}{2} \end{array} \bigg| 1 \right]. \]

As illustrated in the following example, the main technique in our paper as formulated in the Introduction can also be applied to obtain hypergeometric transformations related to series involving binomial products; we later find that the series in (18) arises in our evaluation of a new binomial-harmonic summation.

**Example 5.1.** From the Rodrigues formula \( (3) \), we have that:
\[ \int_{0}^{1} \frac{P_{n}(2x-1)}{\sqrt{x(1-x)}} dx = \frac{\pi ((-1)^{n} + 1) \binom{n}{\frac{1}{2}}^{2}}{2^{2n+1}}. \]
So, from the fundamental formula for expressing $K\left(\sqrt{k}\right)$ as a shifted FL expansion, we obtain the following:

$$
\int_0^1 \frac{K(\sqrt{x})}{\sqrt{1-x}} \, dx = \int_0^1 \sum_{n \geq 0} \frac{2}{2n+1} \frac{P_n(2x-1)}{\sqrt{(1-x)x}} \, dx
= \sum_{n \geq 0} \frac{2}{2n+1} \frac{\pi \left((-1)^n + 1\right)}{2^{2n+1}} \left(\frac{2}{\pi}\right)^2 \, dx.
$$

So, the above expressions are equal to

$$
2\pi \sum_{n=0}^{\infty} \frac{(2n)^2}{16^n(4n+1)}, \quad (18)
$$

which Mathematica 11 can evaluate directly. On the other hand, by using the Maclaurin series for $K$, we see that

$$
\int_0^1 \frac{K(\sqrt{x})}{\sqrt{1-x}} \, dx = \frac{\pi^2}{2} \sum_{n=0}^{\infty} \left(\frac{1}{64}\right)^n \left(\frac{2n}{n}\right)^3
$$

so that the evaluation as \(\frac{\Gamma(\frac{1}{4}\frac{1}{\sqrt{2}})}{8\pi}\) follows immediately from the evaluation in terms of $K$ of the generating function for cubed central binomial coefficients. Variants of the approach outlined above allow us to compute a wide array of \(pF_q\) series with quarter-integer parameters.

From the equalities

$$
K(\sqrt{x}) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x \sin^2 \theta}} = \frac{\pi}{2} \sum_{n \geq 0} \binom{2n}{n} x^n \frac{16^n}{n},
$$

we see that series of the form

$$
\sum_{n \geq 0} \frac{(2n)^2}{16^n(n+a)^2}
$$

and summations such as

$$
\sum_{n \geq 0} \frac{(2n)^2 H_n}{16^n(n+a)^2}
$$

are closely related to the moments of $K(\sqrt{x})$ and $K(\sqrt{x}) \ln(1-x)$, as we later explore.
5.1. On the moments of elliptic-type integrals

The results given in the present subsection are inspired in part by those in [19] on the moments of products of complete elliptic integrals. We remark that the contiguity relations for hypergeometric functions can be applied to obtain similar results.

**Lemma 5.2.** For $m$ such that $\Re(m) > -1$ we have the equation

$$
\int_0^1 K(\sqrt{x}) x^m \, dx = 2 \sum_{i=0}^{\infty} \frac{(m!)^2}{(2i+1)(m-i)!(m+i+1)!}.
$$

**(19)**

**Proof.** Through the use of the Rodrigues formula, it is not difficult to see that the identity

$$
\int_0^1 P_i(2x-1)x^m \, dx = \frac{\Gamma^2(m+1)}{\Gamma(m-i+1)\Gamma(m+i+2)}
$$

holds. So, we have that

$$
K(\sqrt{x}) x^m = \sum_{i \in \mathbb{N}_0} \frac{(2i+1)(m!)^2}{(m-i)!(m+i+1)!} K(\sqrt{x}) P_i(2x-1),
$$

and the desired result follows by integrating both sides over $[0, 1]$ and applying the fundamental formula (8).

**Theorem 5.3.** For $\eta > -1$, we have that

$$
\frac{\pi}{4} = \frac{3F_2\left[\begin{array}{c}
-\eta, \frac{1}{2}, 1 \\
\frac{3}{2}, 2 + \eta
\end{array} \right]}{3F_2\left[\begin{array}{c}
\frac{3}{2}, 1 + \eta \\
1, 2 + \eta
\end{array} \right]}.
$$

**Proof.** The right-hand side of formula (19) may be evaluated as

$$
\frac{2}{m+1} \cdot 3F_2\left[\begin{array}{c}
\frac{1}{2}, 1, -m \\
\frac{3}{2}, m + 2
\end{array} \right] - 1.
$$

Using the Maclaurin series for $K$ in the integrand in (19), we see that the definite integral in Lemma 5.2 must be equal to

$$
\frac{\pi}{2} \sum_{i \in \mathbb{N}_0} \left(\frac{1}{16}\right)^i \frac{(2i)^2}{i + m + 1}.
$$
which, in turn, is equal to
\[
\frac{\pi}{2m + 2} \, _3F_2\left[\frac{1}{2}, m + 1 \mid \frac{1}{2}, 1, m + 2 \mid 1 \right], \quad (20)
\]
giving us the desired result. \(\square\)

Similar methods may be applied to the complete elliptic integral of the second kind, using the FL expansion formula
\[
E(\sqrt{x}) = 4 \sum_{n \geq 0} P_n (2x - 1) \frac{-1}{(2n - 1)(2n + 1)(2n + 3)}, \quad (21)
\]
and this leads us to the following result.

**Theorem 5.4.** For \(\eta > -1\), we have the equalities
\[
\int_0^1 E(\sqrt{x}) \, x^n \, dx = \frac{\pi}{2(1 + \eta)} \cdot _3F_2\left[-\frac{1}{2}, \frac{1}{2}, 1 + \eta \mid 1, 2 + \eta \right] \\
= \frac{4}{3(1 + \eta)} \cdot _3F_2\left[-\frac{1}{2}, 1, -\eta \mid \frac{5}{2}, 2 + \eta \right] - 1 \right].
\]

**Proof.** The first equality follows from the Taylor series expansion for \(E(x)\). Recalling the FL expansion
\[
x^n = \sum_{n \geq 0} P_n (2x - 1)(2n + 1)(-1)^n \frac{\Gamma(n - \eta)\Gamma(1 + \eta)}{\Gamma(-\eta)\Gamma(n + 2 + \eta)},
\]
we immediately see that
\[
\int_0^1 E(\sqrt{x}) x^n \, dx = \frac{4}{3(1 + \eta)} \cdot _3F_2\left[-\frac{1}{2}, 1, -\eta \mid \frac{5}{2}, 2 + \eta \right] - 1 \right],
\]
as desired. \(\square\)

**5.2. New results involving series containing harmonic numbers and central binomial coefficients**

Inspired in part by the results introduced in [3, 4, 5, 7], we evaluate new binomial series for \(\frac{1}{x}\) involving harmonic numbers, but through the use of Fourier-Legendre theory.
Theorem 5.5. The following equality holds:

\[
\sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{H_n - H_{n-1/2}}{64^n} = \frac{\pi}{\sqrt{2}} - \frac{2\sqrt{2}}{\pi} \ln^2 \left( \sqrt{2} + 1 \right).
\]

Proof. From the FL series

\[
\frac{1}{\sqrt{2} - x} = \sum_{k \geq 0} 2(\sqrt{2} - 1)^{2k+1} P_k(2x - 1)
\]

for the elementary function \( \frac{1}{\sqrt{2} - x} \), we have that

\[
\int_0^1 \frac{\mathbf{K}(\sqrt{x})}{\sqrt{2} - x} \, dx = 4 \sum_{n \geq 0} \frac{(\sqrt{2} - 1)^{2n+1}}{(2n + 1)^2}.
\]

As noted above, the dilogarithm function \( \text{Li}_2(z) \) is defined so that

\[
\text{Li}_2(z) = \sum_{k \in \mathbb{N}} \frac{z^k}{k^2}, \tag{22}
\]

whereas the Rogers dilogarithm function \( L(x) \) is

\[
L(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} + \frac{1}{2} \ln(x) \ln(1 - x).
\]

By bisecting the series in (22), we see that

\[
\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n + 1)^2} = \text{Li}_2(x) - \frac{1}{4} \text{Li}_2 \left( x^2 \right).
\]

Writing \( \alpha \) in place of \( \sqrt{2} - 1 \), we have that

\[
\sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{(2n + 1)^2} = \text{Li}_2(\alpha) - \frac{1}{4} \text{Li}_2 \left( \alpha^2 \right). \tag{23}
\]

The evaluation \( 4L(\alpha) - L \left( \alpha^2 \right) = \frac{\pi^2}{4} \) is known [15], which shows that

\[
\text{Li}_2(\alpha) - \frac{\text{Li}_2 \left( \alpha^2 \right)}{4} = \frac{\pi^2}{16} - \frac{1}{4} \ln^2 \left( 1 + \sqrt{2} \right).
\]

So, we have that

\[
\int_0^1 \frac{\mathbf{K}(\sqrt{x})}{\sqrt{2} - x} \, dx = \frac{\pi^2}{4} - \ln^2 \left( \sqrt{2} - 1 \right),
\]
and by enforcing the substitution $x = \frac{2y}{1+y}$, we obtain the equality

$$\int_0^1 K \left( \sqrt{\frac{2y}{1+y}} \right) \frac{\sqrt{2}}{(1+y)^{3/2}} dy = \frac{\pi}{\sqrt{2}} \sum_{n \geq 0} \binom{4n}{2n} \frac{(2n)}{64^n} \int_0^1 \frac{y^{2n}}{1+y} dy.$$ 

Now, we claim that the identity

$$\int_0^1 \frac{y^{2n}}{1+y} dy = \frac{H_n - H_{n-1}}{2}$$

holds for $n \in \mathbb{N}_0$. Let $H_n = \frac{H_n - H_{n-1}}{2}$ for $n \in \mathbb{N}_0$, and from (9) we see that the recursion

$$H_{n+1} - H_n = -\frac{1}{2(n+1)(2n+1)}$$

holds, with $H_0 = \ln(2)$ as the base case. We see that $\int_0^1 \frac{y^{2n}}{1+y} dy = \ln(2)$, and we also have the equalities

$$\int_0^1 \frac{y^{2(n+1)}}{1+y} dy - \int_0^1 \frac{y^{2n}}{1+y} dy = \int_0^1 \frac{y^{2(n+1)} - y^{2n}}{1+y} dy = \int_0^1 (y-1)y^{2n} dy = -\frac{1}{2(n+1)(2n+1)}.$$ 

By induction, the desired result follows.  \hfill \Box 

**Remark 5.6.** In general, the problem of evaluating a series containing central binomial coefficients and harmonic-type expressions is difficult. However, as explored in [4] and [7], there are sometimes ways of obtaining such evaluations through elementary manipulations of generating functions. For example, consider the following variant of the series in Theorem 5.5:

$$\sum_{n=0}^{\infty} \binom{2n}{n} \left( \frac{H_n - H_{n-1/2}}{n+1} \right)^2 \left( \frac{1}{16} \right)^n. \quad (24)$$

In this particular case, if we observe that

$$\sum_{n=0}^{\infty} \binom{2n}{n} \frac{(H_n - H_{\frac{1}{2}+n}) x^n}{16^n} = \frac{2 \ln(4-x)}{\sqrt{4-x}} - 2 F_1^{(0,0,1,0)} \left[ \frac{1}{2}, 1 \mid \frac{x}{4} \right]$$

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and make use of the identity
\[
\frac{1}{2\pi} \int_0^4 x^n \sqrt{\frac{1-x}{x}} \, dx = \frac{(2n)}{n+1},
\]
(25)
it is not difficult to see that the series in (24) equals \(4 - \frac{8}{\pi}\). However, through
the use of our main technique, we can also obtain new evaluations that cannot
be evaluated in this way, as well as new proofs of known results.

The study of the relationships between definite integrals of the form from
(2) and generalized hypergeometric functions produces many new results, as
further illustrated below.

**Theorem 5.7.** The harmonic-binomial series
\[
\sum_{n=0}^{\infty} \binom{2n}{n} \binom{4n}{2n} \left\{ \frac{1}{n} + nH_{n-\frac{1}{2}} - nH_n \right\}
\]
is equal to
\[
\frac{3\pi}{16\sqrt{2}} + \frac{1}{4\sqrt{2\pi}} \left( 2 + 2\sqrt{2} \ln(1 + \sqrt{2}) - 3\ln^2(1 + \sqrt{2}) \right).
\]

**Proof.** A shifted FL expansion of \(\sqrt{2-x}\) is given by
\[
\sqrt{2-x} = \sum_{n=0}^{\infty} \frac{2}{(1-2n)(2n+3)} + \int_1^2 \frac{dv}{(\sqrt{v} + \sqrt{v-1})^{2n+1}} P_n(2x-1).
\]
(26)
Evaluating the integral in (26) using the substitution \(v \mapsto \cosh^2 \theta\), we find that
\[
\sqrt{2-x} = \sum_{n=0}^{\infty} \frac{2(\sqrt{2} - 1)^{2n+1}(3 + \sqrt{2} + 2\sqrt{2}n)}{(1-2n)(2n+3)} P_n(2x-1),
\]
and we thus obtain the formula
\[
\int_0^1 K(\sqrt{2} - x) \, dx = 2 + \sqrt{2} \ln(\sqrt{2} - 1) + 12 \sum_{n=0}^{\infty} \frac{(\sqrt{2} - 1)^{2n+1}}{(1-2n)(2n+1)^2(2n+3)}.
\]
Using partial fraction decomposition, we obtain the evaluation
\[
\int_0^1 K(\sqrt{2} - x) \, dx = \frac{8 + 3\pi^2 + 8\sqrt{2} \ln(1 + \sqrt{2}) - 12\ln^2(1 + \sqrt{2})}{16}.
\]
If we substitute \(x \mapsto \frac{2y}{1+y}\), we see that
\[
\int_0^1 K \left( \sqrt{x} \right) \sqrt{2-x} \, dx = 2\sqrt{2} \int_0^1 K \left( \sqrt{\frac{2y}{1+y}} \right) \left( \frac{1}{1+y} \right)^{5/2} \, dy.
\]
(27)
Since
\[ K \left( \sqrt{\frac{2y}{1+y}} \right) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(2^n)(4n)\sqrt{1+y}}{64^n}, \]
we see that the integral on the right-hand side in (27) may be rewritten as
\[ \pi \sqrt{2} \sum_{n \geq 0} \binom{4n}{2n} \binom{2n}{n} \frac{1}{64^n} \int_0^1 y^{2n} \frac{1}{(1+y)^2} dy, \]
and we thus obtain the desired closed form.

The evaluation in Theorem 5.8 below illustrates the power of the integration method from [4], where it is used to give a one-line proof of (28). Formula (28) is also proved through a complicated manipulation of the generating function for \( \binom{2n}{n} \cdot H_n : n \in \mathbb{N}_0 \). Through an application of integration by parts with respect to the known integral formula
\[ \int_0^1 \ln(1-x)K \left( \sqrt{x} \right) dx = 8 \ln(2) - 8, \]
we are able to use the resulting formula
\[ \int_0^1 \frac{1 - E \left( \sqrt{x} \right)}{1-x} dx = 2 - 4 \ln(2) \]
to prove (28), by applying the Maclaurin series for \( E \left( \sqrt{x} \right) \). The following proof complements the final proof of (28) from [4], and highlights the connections between harmonic-binomial series for \( \frac{1}{\pi} \) and Fourier–Legendre theory.

**Theorem 5.8.** The following result holds [4]:
\[ \sum_{n=1}^{\infty} \frac{(2n)^2 H_n}{16^n(2n-1)} = \frac{8 \ln(2) - 4}{\pi}. \] (28)

**Proof.** From the Fourier–Legendre series identity
\[ E(\sqrt{x}) = 4 \sum_{n \geq 0} P_n(2x-1) \frac{-1}{(2n-1)(2n+1)(2n+3)}, \]
given in (21), together with the identity
\[ \int_0^1 \ln(1-x)P_n(2x-1) \, dx = -\frac{1}{n(n+1)}, \]
we have...
we find that
\[
\int_0^1 E(\sqrt{x}) \ln(1-x) \, dx = -\frac{4}{3} + \sum_{n \geq 1} \frac{4 \left( \frac{1}{n} + \frac{1}{n+1} \right)}{(2n-1)(2n+1)^2(2n+3)},
\] (29)
with the infintie series in (29) being evaluated as \( \frac{4}{3}(12 \ln 2 - 11) \). Using the Maclaurin series for \( E(\sqrt{x}) \), we get
\[
\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2 H_{n+1}}{(n+1)(2n-1)} = \frac{96 \ln(2) - 88}{9\pi},
\]
and the desired result follows using partial fraction decomposition.

As a variation on the main lemma from [4], an integration method in [4] allows us to construct new series for \( \frac{1}{\pi} \) involving factors of the form \( H_n^2 + H_n^{(2)} \). For example, the formula
\[
\sum_{n=1}^{\infty} \frac{(2n)^2}{16^n(2n-1)^2} \left( H_n^2 + H_n^{(2)} \right) = \frac{64 + 64 \ln^2(2) - 96 \ln(2)}{\pi} - \frac{8\pi}{3},
\] (30)
was proved in [4] following this method. As noted in [4], using the identity
\[
\int_0^1 x^{n-1} \ln^2(1-x) \, dx = (H_n)^2 + H_n^{(2)},
\]
we find that the formula in (30) is equivalent to the evaluation
\[
\int_0^1 K(\sqrt{x}) \ln^2(1-x) \, dx = 48 - \frac{4\pi^2}{3} + 32(\ln(2) - 2) \ln(2).\] (31)
As also noted in [4], from (30) and (31), together with the identity
\[
\sum_{n=0}^{\infty} \frac{x^{n+1}H_n}{n+1} = \frac{1}{2} \ln^2(1-x)
\] (32)
we obtain the double series formula given in Theorem 5.10 below, which we prove using the machinery of FL expansions, thereby furthering the main “thesis” of our article concerning the connections among the evaluation of twisted and non-twisted hypergeometric series, that of integrals involving elliptic-type integrals, and Fourier–Legendre theory.
Lemma 5.9. For \( n \in \mathbb{N} \) we have the identity

\[
\int_0^1 \ln^2(1-x) P_n(2x-1) \, dx = \frac{4n+2}{n^2(n+1)^2} + \frac{4H_n-1}{n(n+1)}.
\] (33)

Proof. From (4), we see that the equality in (33) is equivalent to

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left( (H_{n+1} - H_{n-k})^2 + H_{n+1}^{(2)} - H_{n-k}^{(2)} \right)
\] (34)

being equal to \( \frac{4n+2}{n^2(n+1)^2} + \frac{4H_n-1}{n(n+1)} \). Expanding the summand in (34) and simplifying using the binomial theorem, we obtain the sum

\[
\sum_{k=0}^{n} (-1)^{n-k+1} \left( 2 \binom{n}{k} H_{n+1} H_{n-k} - \binom{n}{k} H_{n-k}^2 + \binom{n}{k} H_{n-k}^{(2)} \right).
\]

The following basic binomial-harmonic identities are known to hold:

\[
\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} H_{n-k} = \frac{(-1)^{n+1}}{n},
\]

\[
\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{n-k}^2 = \frac{H_n}{n} - \frac{2}{n^2},
\]

\[
\sum_{k=0}^{n} (-1)^{n-k+1} \binom{n}{k} H_{n-k}^{(2)} = \frac{1 + nH_n-1}{n^2}.
\]

From them, together with the above expansion of (34), we see that the desired result holds.

Theorem 5.10. The following equality holds [4]:

\[
\sum_{m,n \geq 0} \left( \frac{1}{16} \right)^m \frac{(2m)!^2 H_n}{(n+1)(m+n+2)} = \frac{48 + 32 \ln(2) - 2 \ln(2)}{\pi} - \frac{4\pi}{3}.
\]

Proof. By Lemma 5.9 we have that

\[
\int_0^1 K \sqrt{x} \ln^2(1-x) \, dx = 1 + \sum_{n \geq 1} \frac{1}{n^2(n+1)^2} + 2 \sum_{n \geq 1} \frac{H_{n-1}}{n(n+1)(2n+1)},
\] (35)

and we thus find that an application of partial fraction decomposition shows that (31) holds, giving us a new proof of it. Rewriting the integral in (35) as

\[
\int_0^1 K \sqrt{x} \sum_{n=0}^{\infty} \frac{2x^{n+1} H_n}{n+1} \, dx,
\]

and from the hypergeometric evaluation in (20) for the moments indicated in Lemma 5.2, we obtain the desired result.

\[ \square \]
On the other hand, if we use the alternative moment formula in Lemma 5.2, by mimicking the above proof, we obtain the following evaluation.

**Theorem 5.11.** The following equality holds:

\[
12 - \frac{\pi^2}{3} + 8 \ln^2(2) - 16 \ln(2) = \sum_{n,m \in \mathbb{N}_0} \frac{(n+1)! n! H_n}{(2m+1)(n-m+1)! (m+n+2)!}.
\]

**Proof.** This follows almost immediately from Lemma 5.2 and formulas (31) and (32), in the manner suggested above. \[\square\]

The following evaluation may also be obtained using the latter integration method from [4], but the following FL proof illustrates the connections between this evaluation and recent results from [3].

**Theorem 5.12.** The following equality holds:

\[
\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \binom{2n}{n}^2 \frac{H_n^2 + H_n^{(2)}}{n+1} = \frac{64 \ln^2(2)}{\pi} - \frac{8\pi}{3}.
\]

**Proof.** Expanding the left-hand side of (35), we find that

\[
\frac{4}{3} (36 - \pi^2 - 48 \ln(2) + 24 \ln^2(2)) = \frac{\pi}{2} \sum_{n \geq 0} \binom{2n}{n}^2 \frac{H_{n+1}^2 + H_{n+1}^{(2)}}{16^n(n+1)}.
\]

By expanding the numerator in the summand, and by evaluating the generalized hypergeometric expression

\[
_{4}F_{3}\left[ \begin{matrix} 
\frac{1}{2}, \frac{1}{2}, 1, 1 \\
2, 2, 2
\end{matrix} \bigg| 1 \right] = \frac{16(-2G + 3 + \pi(\ln(2) - 1))}{\pi}
\]

using the standard integral formula (25) for the Catalan number \( C_n = \binom{2n}{n+1} \), and by making use of the recently-discovered formula

\[
\sum_{n=0}^{\infty} \frac{(2n)!^2 H_n}{16^n(n+1)^2} = 16 + \frac{32G - 64 \ln(2)}{\pi} - 16 \ln(2)
\]

introduced in [3], we obtain the desired result. \[\square\]
Through the use of our main technique, as applied to the shifted FL expansion for \( \ln(1 - x) \ln(x) \), we obtain the following stronger version of Theorem 5.12, which also allows us to evaluate

\[
\sum_{n \in \mathbb{N}_0} \left( \frac{1}{16} \right)^n \left( \frac{2n}{n} \right)^2 \frac{H_n^2}{n + 1}
\]

in closed form. The proof depends in a non-trivial way on a recent evaluation provided in [3].

**Theorem 5.13.** The following equality holds:

\[
\sum_{n=0}^{\infty} \left( \frac{1}{16} \right)^n \left( \frac{2n}{n} \right)^2 \frac{H_n^{(2)}}{n + 1} = 32G \frac{2}{\pi} + 2\pi 3 - 16 \ln(2).
\]

**Proof.** We find that the equality

\[
- \frac{((-1)^m + 1) (2m + 1)}{m^2(m + 1)^2} = (2m + 1) \int_0^1 P(m, 2x - 1) \ln(1 - x) \ln(x) \, dx
\]

holds for \( m \in \mathbb{N} \). By expressing the integral in (2) in the two different ways our main strategy is based upon, letting \( g(x) = \ln(1 - x) \ln(x) \) in (2), we obtain the equality

\[
\sum_{n=0}^{\infty} \left( \frac{2n}{n} \right)^2 \left[ (1 + (1 + n)H_{1+n} - (1 + n)^2 \psi(1)(1 + n) \right] 16^n(n + 1)^3 = 96 - 64 \ln(2) - 16,
\]

and from the recent evaluation [3]

\[
\sum_{n \in \mathbb{N}} \left( \frac{2n}{n} \right)^2 H_n 16^n(n + 1)^2 = 16 + \frac{32G - 64 \ln(2)}{\pi} - 16 \ln(2),
\]

we obtain the desired result. \( \square \)

The integration result in the following lemma is especially useful in the construction of harmonic-binomial series. We may obtain similar results for integrals involving expressions such as \( \ln(1 + \sqrt{1 - x}) \), \( \ln(1 + \sqrt{x}) \), \( \ln(1 - \sqrt{1 - x}) \), etc.

**Lemma 5.14.** The following identity holds for \( n \in \mathbb{N} \):

\[
\int_0^1 \ln \left( 1 - \sqrt{x} \right) P_n(2x - 1) \, dx = \frac{(-1)^n - 4n - 2}{2n(n + 1)(2n + 1)}.
\]
Proof. It seems natural to make use of the moment formula from (9), together with a Maclaurin expansion of \( \ln(1 - \sqrt{x}) \). Indeed, since

\[
\sum_{i=0}^{\infty} \left( \frac{1}{1+i} \right) P_n(2x - 1) = \ln(1 - \sqrt{x}) P_n(2x - 1),
\]

from (9) we see that

\[
\int_0^1 \ln \left( 1 - \sqrt{x} \right) P_n(2x - 1) \, dx = -\sum_{i=0}^{\infty} \frac{((i+1)/2)!^2}{(i+1)(i+1/n-n+1/n+1)}.
\]

Rewrite the series as

\[
-\sum_{i=0}^{\infty} \frac{\Gamma^2 \left( \frac{i+3}{2} \right)}{(i+1) \Gamma \left( \frac{n}{2} - n + \frac{n}{2} \right) \Gamma \left( \frac{n}{2} + n + \frac{n}{2} \right)}.
\]

Bisect this resulting series so as to produce the symbolic form

\[
\frac{-2n + \cos(\pi n) - 1}{4n^3 + 6n^2 + 2n} - \frac{\pi n(n+1) - \sin(\pi n)}{2\pi n^2(n+1)^2},
\]

which is the same as the right-hand side of (36) for the desired integer parameters.

The formula in Lemma 5.14 can be used in conjunction with the identity

\[
\int_0^1 x^n \ln(1 - \sqrt{x}) \, dx = -\frac{H_{2n+2}}{n+1},
\]

to produce some new results. For example, since

\[
\int_0^1 K(\sqrt{x}) \ln(1 - \sqrt{x}) \, dx = -\frac{\pi}{2} \sum_{n \geq 1} \frac{(2n)^2}{16^n(n+1)^2} H_{2n+2},
\]

by rewriting the integral in (37) as

\[
-3 + \sum_{n \geq 1} \left( \frac{-1}{2} \right)^n \left( \frac{1}{n} + \frac{1}{n+1} \right) - \left( \frac{1}{n} + \frac{1}{n+1} \right) \left( \frac{2}{(2n+1)^2} \right),
\]

through an application of partial fraction decomposition we obtain a new proof of the known series formula

\[
\sum_{n=0}^{\infty} \frac{(2n)^2}{16^n(n+1)} H_{2n} = 2 + \frac{4 - 12 \ln(2)}{\pi}
\]

which Mathematica 11 is able to verify. However, if we apply the same approach with respect to the definite integral obtained by replacing \( K \) in the integrand in (37) with the complete elliptic integral of the second kind, we obtain an evaluation of a new binomial-harmonic series.
**Theorem 5.15.** We have the evaluation

\[
\sum_{n=0}^{\infty} \frac{{(2n)}^2 H_{2n}}{16^n(2n - 1)} = \frac{6 \ln(2) - 2}{\pi}.
\]

**Proof.** Rewrite the integral

\[
\int_0^1 E(\sqrt{x}) \cdot \ln(1 - \sqrt{x}) \, dx
\]

using the Maclaurin series for \(E\), so that the integral is equal to

\[
\frac{\pi}{2} \sum_{i=0}^{\infty} \frac{\left(\frac{1}{16}\right)^i \left(\frac{2}{i} \right)^2 H_{2i+2}}{(2i - 1)(i + 1)}.
\]

The theorem then follows naturally from Lemma 5.14, as the integral in (38) may also be written as

\[
-2 + \sum_{n \geq 1} \left[\frac{(-1)^n}{2} \left(\frac{1}{n} - \frac{1}{n + 1}\right) - \left(\frac{1}{n} + \frac{1}{n + 1}\right)\right] \frac{4}{(1 - 2n)(2n + 1)^2(2n + 3)}.
\]

Using partial fraction decomposition, we obtain the desired result. \(\square\)

**Theorem 5.16.** The following equality holds:

\[
\sum_{n=0}^{\infty} \frac{(2n)^2 H_{n+\frac{1}{2}} - H_{n-\frac{1}{2}}}{16^n} = \frac{\Gamma^4\left(\frac{1}{4}\right)}{8\pi^4} - \frac{4G}{\pi}.
\]

**Proof.** By recalling two standard (equivalent) definitions

\[
G = \frac{1}{2} \int_0^{\pi/2} \frac{\theta \, d\theta}{\sin \theta}, \quad G = \sum_{n \geq 0} \frac{(-1)^n}{(2n + 1)^2}
\]

of Catalan’s constant, we may prove (by exploiting the FL expansions of \(K(\sqrt{x})\) and \(\frac{1}{\sqrt{x}}\), together with the substitution \(\theta \mapsto \arcsin \sqrt{t}\)) the following identity:

\[
4G = \int_0^1 \frac{\arcsin \sqrt{t}}{\sqrt{t}} \cdot \frac{1}{\sqrt{t(1-t)}} \, dt = \int_0^1 \frac{K(\sqrt{x})}{\sqrt{x}} \, dx.
\]

From the FL expansion

\[
\frac{\arcsin \sqrt{x}}{\sqrt{x}} = \sum_{n \geq 0} P_n(2x - 1) \left[\frac{2}{2n + 1} - \int_0^1 \frac{4t^{2n+2}}{1 + t^2} \, dt\right]
\]

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together with the evaluation for (18), we find that
\[
4G = 2\pi \sum_{n \geq 0} \frac{(2n)^2}{16^n(4n + 1)} - 4\pi \int_0^1 \sum_{n \geq 0} \frac{(2n)^2}{16^n} x^{4n+2} \frac{dx}{1 + x^2}
\]
\[
\frac{1}{\pi^2} \Gamma^2\left(\frac{1}{4}\right)
\]
and that
\[
G = \int_0^1 \frac{1 - x^2}{1 + x^2} K(x^2) \, dx = \frac{1}{4} \int_0^1 \frac{1 - \sqrt{x}}{1 + \sqrt{x}} x^{-3/4} K(\sqrt{x}) \, dx
\]
\[
= \frac{\pi}{4} \sum_{n \geq 0} \frac{(2n)^2}{16^n} \left( \frac{2}{4n + 1} + H_{n-1/4} - H_{n+1/4} \right)
\]
as desired. \(\Box\)

Given the FL expansion of \(x K(\sqrt{x})\), and introducing the notation
\[
W(m) = \frac{1}{2m - 1} + \frac{2}{2m + 1} + \frac{1}{2m + 3} + \frac{1}{(2m - 1)^2} - \frac{1}{(2m + 3)^2}
\]
for the sake of brevity, we may compute the following integral:
\[
\int_0^1 \frac{x K(\sqrt{x})}{\sqrt{1 - x}} \, dx = \sum_{m \geq 0} \frac{W(m)}{2m + 1}
\]
\[
= \frac{3\pi^2}{8} = 2\pi \sum_{n \geq 0} \frac{(2n)(n + 1)}{4^n(2n + 1)(2n + 3)}
\]
\[
= \frac{\pi}{2} \sum_{n \geq 0} \frac{(2n)}{4^n(2n + 1)} + \frac{\pi}{2} \sum_{n \geq 0} \frac{(2n)}{4^n(2n + 3)}
\]

Using the sequence \((W(m) : m \in \mathbb{N}_0)\), we offer a new proof of the following evaluation introduced in [17].

**Theorem 5.17.** The following equality holds [2]:
\[
\sum_{n=1}^{\infty} \frac{(2n)^2}{16^n(2n - 1)^2} H_{2n} = \frac{4G + 6 - 12 \ln(2)}{\pi}.
\]

**Proof.** Letting \(W\) be as given above, we find that
\[
\int_0^1 x K(\sqrt{x}) \ln(1 - \sqrt{x}) \, dx
\]
\[
= \frac{5}{3} + \frac{1}{2} \sum_{m \geq 1} \frac{W(m)}{2m + 1} \left[ (-1)^m \left( \frac{1}{m} \frac{1}{m + 1} \right) - \left( \frac{1}{m} + \frac{1}{m + 1} \right) \right].
\]

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Evaluate the right-hand side as
\[
\frac{10}{3} \ln(2) - \frac{7}{3} G - \frac{151}{54}.
\]
Evaluate the above integral as
\[
-\frac{\pi}{2} \sum_{n \geq 0} \frac{(2n)^2}{16^n(n + 2)} H_{2n+4}.
\]
It is not difficult to see that it follows that
\[
\sum_{n=1}^{\infty} \left( \frac{1}{16} \right)^n \frac{(2n)^2}{n+2} H_{2n} = \frac{92 + 24\pi - 180 \ln(2)}{27\pi}.
\]
Through a re-indexing argument, it is seen that
\[
\frac{12G + 8 - 6\ln(2)}{9\pi} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{(2n)^2}{16^n(2n - 1)^2} H_{2n} + \frac{5}{9} \sum_{n=1}^{\infty} \frac{(2n)^2}{16^n(2n - 1)}.
\]
and the desired result follows from Theorem 5.15.

5.3. Double hypergeometric series

We prove a transformation formula related to the main results from the preceding sections, and we apply this formula to arrive at a new evaluation for a double hypergeometric series.

**Lemma 5.18.** If \( g \in L^1(0,1) \), then the definite integral
\[
\int_0^1 E(\sqrt{x})g(x) \, dx
\]
is equal to
\[
\int_0^1 \left[ \frac{\pi}{2} - \left( \frac{\pi}{2} - 1 \right) \sqrt{x} \right] g(x) \, dx + \frac{1}{3} \int \int_{(0,1)^2} \left( K(\sqrt{x}) - \frac{\pi}{2} \right) g(xz^{2/3}) \, dz \, dx.
\]

**Proof.** Write \( \tilde{g}(x) = \int_0^x z^2 g(z^2) \, dz \). We note that
\[
\frac{\tilde{g}(x)}{x^{2}} = \frac{1}{x^{2}} \int_0^{\sqrt{x}} z^2 g(z^2) \, dz = \int_0^1 z^2 g(xz^2) \, dz.
\]
Using the Maclaurin series expansion for the complete elliptic integral of the second kind, we find that
\[
\int_0^1 E(\sqrt{x})g(x) \, dx = \frac{\pi}{2} \int_0^1 g(x) \, dx - \pi \int_0^1 \sum_{n \geq 1} \frac{(2n)^2}{16^n(2n - 1)} x^{2n-1} g(x^2) \, dx.
\]
Using integration by parts, we may rewrite the right-hand side as

\[ \frac{\pi}{2} \int_0^1 g(x) \, dx - \pi \left[ \sum_{n \geq 1} \frac{(2n)^2}{16^n (2n-1)} \hat{g}(x) \right]_0^1 + \pi \int_0^1 \sum_{n \geq 1} \frac{(2n)^2}{16^n} \cdot \frac{\hat{g}(x)}{x^2} \, dx. \]

Rewriting this as

\[ \frac{\pi}{2} \int_0^1 g(x) \, dx - (\pi - 2) \int_0^1 z^2 g(z^2) \, dz + \int_0^1 \left( K(\sqrt{x}) - \frac{\pi}{2} \right) \frac{\hat{g}(\sqrt{x})}{\sqrt{x}} \, dx, \quad (39) \]

we see that (39) is equal to

\[ \int_0^1 \left[ \frac{\pi}{2} - \left( \frac{\pi}{2} - 1 \right) \sqrt{x} \right] g(x) \, dx + \int_0^1 \left( K(\sqrt{x}) - \frac{\pi}{2} \right) \int_0^1 z^2 g(z^2) \, dz \, dx 
= \int_0^1 \left[ \frac{\pi}{2} - \left( \frac{\pi}{2} - 1 \right) \sqrt{x} \right] g(x) \, dx + \frac{1}{2} \int_{(0,1)^2} \left( K(\sqrt{x}) - \frac{\pi}{2} \right) \sqrt{z} g(xz) \, dz \, dx \]

and this gives us the desired result. \( \square \)

**Theorem 5.19.** The following formula holds:

\[ \sum_{m,n \geq 0} \frac{(2m)^2 (2n)^2}{16^m(n+m+1)(2m+3)} = \frac{7 \zeta(3) - 4G}{\pi^2}. \]

**Proof.** We begin with the following:

\[ S = \sum_{m,n \geq 0} \frac{(2n)^2}{16^m(n+m+1)(2m+3)} \]

\[ = \int_{(0,1)^2} \sum_{m,n \geq 0} \frac{(2n)^2}{16^m} \frac{(2m)^2}{x^{n+m} y^{2m+2}} \, dx \, dy. \]

Due to the Maclaurin series of \( K(x) \) the above expressions are equal to:

\[ \frac{4}{\pi^2} \int_{(0,1)^2} y^2 K(\sqrt{x}) K(\sqrt{xy}) \, dx \, dy. \]

By integrating with respect to \( dy \) first we have

\[ S = \frac{2}{3\pi} \int_0^1 K(\sqrt{x}) \cdot _3F_2 \left( \begin{array}{c} 1, 1, 1 \\ \frac{3}{2}, \frac{3}{2} \end{array} \middle| x \right) \, dx, \]

with

\[ g(x) = \sum_{n \geq 0} \left[ \frac{1}{4^n} \left( \frac{2n}{n} \right)^2 \right] x^n \frac{3x^n}{2n+3}. \]
Lemma 5.18 gives us that
\[
\int_0^1 \int_0^1 \left( K(\sqrt{x}) - \frac{\pi}{2} \right) y^2 \frac{g(xy^2)}{\pi^2} \, dx \, dy
\]
\[
= \int_0^1 E(\sqrt{x}) g(x) \, dx - \int_0^1 \sqrt{x} g(x) \, dx - \frac{\pi}{2} \int_0^1 (1 - \sqrt{x}) g(x) \, dx.
\]

By considering the instance \( g(x) = K(\sqrt{x}) \) we have
\[
\int_0^1 \int_0^1 \left( K(\sqrt{x}) - \frac{\pi}{2} \right) y^2 K(\sqrt{xy}) \, dx \, dy
\]
\[
= \frac{2 + 7\zeta(3)}{4} - \frac{1 + 2G}{2} - \frac{\pi(3 - 2G)}{4}.
\]

The above evaluation is obtained through the FL-expansions of \( K(\sqrt{x}) \), \( E(\sqrt{x}) \), \( 135 \) and \( \sqrt{x} \).

In order to complete the evaluation of \( S \) it is enough to compute
\[
\int_0^1 \int_0^1 y^2 K(\sqrt{xy}) \, dx \, dy = \frac{\pi}{6} \int_0^1 g(x) \, dx
\]
\[
= \frac{\pi}{2} \sum_{n \geq 0} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{2n + 3} \frac{3}{(2n + 3)(n + 1)},
\]
but this is straightforward by partial fraction decomposition, since
\[
\frac{\pi}{2} \sum_{n \geq 0} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{2n + 3} = \int_0^1 x^2 K(x) \, dx
\]
\[
= \frac{1}{2} \int_0^1 \sqrt{x} K(\sqrt{x}) \, dx
\]
\[
= \frac{1 + 2G}{4},
\]
and since
\[
\frac{\pi}{2} \sum_{n \geq 0} \left[ \frac{1}{4^n} \binom{2n}{n} \right]^2 \frac{1}{n + 1} = \int_0^1 K(\sqrt{x}) \, dx = 2. \quad \square
\]

We have that
\[
\int_{-1}^1 x P_N(x) P_L(x) \, dx = \begin{cases} 
\frac{2L+2}{(2L+1)(2L+3)} & \text{if } N = L + 1 \\
\frac{2L}{(2L-1)(2L+1)} & \text{if } N = L - 1,
\end{cases}
\]
and hence
\[
\int_0^1 x P_N(2x - 1) P_L(2x - 1) \, dx = \begin{cases} 
\frac{2L+2}{4(2L+1)(2L+3)} & \text{if } N = L + 1 \\
\frac{1}{2(2L+1)} & \text{if } N = L \\
\frac{2L}{4(2L-1)(2L+1)} & \text{if } N = L - 1.
\end{cases}
\]
so the previously computed FL expansions allow us an explicit evaluation of any integral of the form \( \int_0^1 xK(\sqrt{x})g(x)\,dx \) or \( \int_0^1 xE(\sqrt{x})g(x)\,dx \), with \( g(x) \) being a function with a previously computed FL expansion.

5.4. Connections between generalized hypergeometric functions and polylogarithms

The FL-based techniques explored in our article are powerful enough to find symbolic expressions for some \( _4F_3 \) functions with quarter-integer parameters, as shown in the proof of the following theorem. Our manipulations of Fourier–Legendre expansions often allow us to express \( _pF_q \) series in terms of the dilogarithm, expanding upon results given in [10].

**Theorem 5.20.** The hypergeometric series

\[
\sum_{n \geq 1} \frac{(4n)(2n)}{n\,64^n} \bigg[ \frac{2K(\sqrt{\frac{2\sqrt{2}+1}{1+y}})}{\sqrt{1+y}} - 1 \bigg] = 3 \cdot \frac{1}{16} \cdot _4F_3 \left[ \left. \begin{array}{c} 1, 1, 1, \frac{7}{4} \\ 2, 2, 2 \end{array} \right| 1 \right]
\]  

(40)

is equal to

\[
6 \ln 2 - 2 \ln(1 + \sqrt{2}) - \frac{16}{15} \ln \text{Li}_2((\sqrt{2} - 1)i)
\]

Proof.

Since

\[
\sum_{n \geq 1} \frac{(4n)(2n)}{n\,64^n} = 2 \int_0^1 \bigg[ \frac{2K(\sqrt{\frac{2\sqrt{2}+1}{1+y}})}{\sqrt{1+y}} - 1 \bigg] \frac{dy}{y}
\]

we find that

\[
\sum_{n \geq 1} \frac{(4n)(2n)}{n\,64^n} = \frac{\pi}{\sqrt{2}} + 6 \ln 2 - \frac{2\sqrt{2}}{\pi} \ln^2(1 + \sqrt{2}) - \frac{16}{15} G - 2 \frac{\pi}{\sqrt{2}} \int_0^1 \frac{K(\sqrt{2})\,dx}{1 + \sqrt{1 - \frac{2}{x}}}.
\]

(41)

So, the whole problem of evaluating the series in (40) boils down to finding the FL expansion of \( \frac{1}{1 + \sqrt{1 - \frac{2}{x}}} \):

\[
\frac{1}{1 + \sqrt{1 - \frac{2}{x}}} = \sum_{n \geq 0} P_n(2x - 1) \cdot 2 \int_1^{+\infty} \frac{dv}{(v + 1)^{3/2} (\sqrt{v} + \sqrt{v + 1})^{2n+1}}
\]

\[
= \sum_{n \geq 0} P_n(2x - 1) \cdot 4 \int_0^{(\sqrt{2} - 1)^2} \frac{1-x}{(1+x)^2x^n} \,dx.
\]
Manipulate the above integral so as to obtain the following:

$$\sum_{n \geq 0} P_n(2x - 1) \cdot 8 \left[ \frac{1}{2\sqrt{2}} (\sqrt{2} - 1)^{2n+1} - \int_0^{\sqrt{2}-1} (2n + 1)x^{2n+1} \frac{dx}{1 + x^2} \right].$$

From the FL expansion for $K(\sqrt{x})$, we have that

$$f(x) = \sum_{n \geq 0} P_n(2x - 1)c_n \implies \int_0^1 K(\sqrt{x}) f(x) \, dx = 2 \sum_{n \geq 0} \frac{c_n}{(2n + 1)^2},$$

letting $c_n$ denote a scalar coefficient for $n \in \mathbb{N}_0$. By taking $f(x) = \frac{1}{1 + \sqrt{1 - x^2}}$ and considering the previous line, we find that the definite integral in (41) equals

$$4\sqrt{2} \sum_{n \geq 0} \frac{(\sqrt{2} - 1)^{2n+1}}{(2n + 1)^2} - 16 \int_0^{\sqrt{2}-1} \frac{\arctanh x}{1 + x^2} \, dx. \quad (42)$$

We again encounter the series from (23), and we thus find that the expression in (42) must be equal to

$$\frac{\pi^2}{2\sqrt{2}} - \sqrt{2}\ln^2(1 + \sqrt{2}) - 16 \int_0^{\sqrt{2}-1} \frac{\arctanh x}{1 + x^2} \, dx.$$

The above integral may also be computed through the machinery of dilogarithms, yielding the following evaluation of (42):

$$\frac{\pi^2}{2\sqrt{2}} - \sqrt{2}\ln^2(1 + \sqrt{2}) - 8G + \pi \ln(1 + \sqrt{2}) + 8\text{Im}Li_2 \left[ (\sqrt{2} - 1)i \right].$$

We thus obtain the desired result. \hfill \Box

**Theorem 5.21.** The following equality holds:

$$\sum_{n=0}^{\infty} \frac{(\sqrt{n})^2(2n + 1)}{16^n(n + 1)^2} = 16 - 6\pi^2 - 32 \ln(2) + 24\ln^2(2) + \frac{64G - 32 + 256 \cdot \text{Im} \left( \text{Li}_3 \left( \frac{1+i}{2} \right) \right)}{\pi}.$$
Proof. We have that

\[
\sum_{n \geq 0} \frac{(2n)^2(2n+1)}{16^n(n+1)^4} = -\frac{4}{\pi} \int_0^1 \frac{d}{dx} E(\sqrt{x}) \ln^2(x) \, dx,
\]

and we may evaluate the right-hand side as

\[
-\frac{4}{\pi} \left[ (2 - \pi) + \sum_{n \geq 1} (-1)^n \left( 2 \left( \frac{2n+1}{n(n+1)} \right)^2 + 4 \frac{2n+1}{n(n+1)} H_{n-1} \right) \right] - \frac{1}{2n+1} + \int_0^1 \frac{2x^{2n+2}}{1+x^2} \, dx.\]

It is seen that the above expression is equal to

\[
= -\frac{4}{\pi} \left[ -4\pi + 8 - 16G + \frac{11}{12} \pi^3 + (8\pi - 16G) \ln 2 + \pi \ln^2 2 
- 32 \text{Im} \text{Li}_3(\frac{1+i}{2}) - 8 \int_0^1 \frac{\ln^2(1+x^2)}{1+x^2} \, dx \right].
\]

The desired result follows. \qed

6. Generalized complete elliptic integrals

The study of the moments of \(K(\sqrt{x})\) and \(E(\sqrt{x})\) provides us with some new results on generalized complete elliptic integrals. While the following definition is not new per se \[17\], the notation introduced below will be convenient.

Definition 6.1. We define the generalized complete elliptic integral \(\mathcal{J}\) over \([0, 1)\) as

\[
\mathcal{J}(x) = \int_{\pi/2}^\pi \left( \sqrt{1 - x \sin^2 \theta} \right)^3 d\theta
= \frac{x-1}{3} K(\sqrt{x}) + \frac{4-2x}{3} E(\sqrt{x}).
\]

Inspired by the main results in the preceding sections, we are interested in:

- the computation of the Taylor series for \(\mathcal{J}\) at the origin;
- the computation of the FL expansion for \(\mathcal{J}\); and
- the computation of its moments \(\int_0^1 \mathcal{J}(x) x^n \, dx\).
It is not difficult to see that $J(x)$ may be expanded as

$$
\frac{3\pi}{2} \sum_{n \geq 0} \frac{(2n)^2}{16^n(1 - 2n)(3 - 2n)} x^n = \frac{\pi}{2} \cdot \left[ {}_2F_1 \left( \begin{array}{c} -\frac{3}{2}, \frac{1}{2} \\ 1 \end{array} \right) \right], \tag{43}
$$

and it is seen that the following equality holds:

$$
J(x) = 48 \sum_{n \geq 0} \frac{P_n(2x - 1)}{(2n + 5)(2n + 3)(2n + 1)(2n - 1)(2n - 3)}.
$$

By multiplying the right-hand side of (43) by $x^n$ and performing a termwise integration we immediately have

$$
\int_0^1 J(x)x^\eta \, dx = \frac{\pi}{2(1 + \eta)} \cdot \left[ {}_3F_2 \left( \begin{array}{c} -\frac{3}{2}, \frac{1}{2}, 1 + \eta \\ 1, 2 + \eta \end{array} \right) \right].
$$

From (29) it follows that the $\eta^{th}$ moment of $J(x)$ is equal to

$$
48 \sum_{n \geq 0} (-1)^n \frac{\Gamma(n - \eta)\Gamma(1 + \eta)}{\Gamma(-\eta)\Gamma(n + 2 + \eta)(2n + 5)(2n + 3)(2n + 1)(2n - 1)(2n - 3)}.
$$

So, we find for $\eta > -1$ that

$$
\int_0^1 J(x)x^\eta \, dx = \frac{\pi}{2(1 + \eta)} \cdot \left[ {}_3F_2 \left( \begin{array}{c} -\frac{3}{2}, \frac{1}{2}, 1 + \eta \\ 1, 2 + \eta \end{array} \right) \right] = \frac{16}{15(\eta + 1)} \cdot \left[ {}_3F_2 \left( \begin{array}{c} -\frac{3}{2}, 1, -\eta \\ \frac{7}{2}, 2 + \eta \end{array} \right) \right].
$$

**Corollary 6.2.** The following equality holds for $\eta > -1$:

$$
\frac{15\pi}{32} = \left[ {}_3F_2 \left( \begin{array}{c} -\frac{3}{2}, 1, -\eta \\ \frac{7}{2}, 2 + \eta \end{array} \right) \right] = \left[ {}_3F_2 \left( \begin{array}{c} -\frac{3}{2}, 1, 1 + \eta \\ 1, 2 + \eta \end{array} \right) \right].
$$

**Proof.** This follows from the expansions of $\int_0^1 J(x)x^n \, dx$ given above.

**Definition 6.3.** The generalized complete elliptic integral $J_m$ is given for all $x \in [0, 1)$ by

$$
J_m(x) = \int_0^{\pi/2} (1 - x \sin^2 \theta)^{m-1/2} \, d\theta.
$$
We observe that $J_0(x) = K(\sqrt{x})$ and that $J_1(x) = E(\sqrt{x})$. Also, we have that $J_2(x) = J(x)$.

By considering the FL expansions of $J_0, J_1, J_2$ and by performing induction on $m$, we obtain many by-products.

**Theorem 6.4.** For any $\eta > -1$, the moment of $J_m(x)$ of order $\eta$ satisfies all of the properties

$$
\int_0^1 J_m(x) x^n dx = \frac{2 \cdot 4^m}{(2m)(2m+1)} \cdot 3F2 \left[ \begin{array}{c}
-\eta, 1, m + 1 \\
\frac{3}{2}, \frac{3}{2} + m
\end{array} \right | 1
$$

$$
= \frac{\pi}{2(1+\eta)} \cdot 3F2 \left[ \begin{array}{c}
\frac{1}{2} - m, \frac{1}{2}, 1 + \eta \\
1, 2 + \eta
\end{array} \right | 1
$$

$$
= \frac{2 \cdot 4^m}{(2m)(2m+1)(\eta+1)} \cdot 3F2 \left[ \begin{array}{c}
\frac{1}{2} - m, 1, -\eta \\
\frac{3}{2} + m, 2 + \eta
\end{array} \right | -1
$$

*Proof.* This follows by termwise integration of a function of the form $x^n \cdot 2F1$, and by exploiting the “trigonometric definition” of $J_m(x)$ along with Fubini’s Theorem, and by using FL expansions.

**Corollary 6.5.** The following equality holds for $m \geq 0$ and $\eta > -1$:

$$
\pi = \frac{4^{m+1}}{(2m)(2m+1)} \cdot 3F2 \left[ \begin{array}{c}
\frac{3}{2} - m, 1, -\eta \\
\frac{3}{2} + m, 2 + \eta
\end{array} \right | -1
$$

*Proof.* This follows immediately from Theorem 6.4.

By applying parameter derivatives to the equality (44), we often end up with results on harmonic summations for $\frac{\pi}{6}$. For example, if we apply the operator $\frac{\partial}{\partial m} \big|_{n=0}$ to both sides of (44) and then let the parameter $\eta$ approach a natural number $j$, we obtain an explicit evaluation of the infinite series

$$
\sum_{i=0}^{\infty} \frac{(2i)^2 (2H_{2i} - H_i)}{16^i (i+j+1)} = \frac{8(j!)^2}{\pi} \sum_{i=0}^{j} \frac{(2i+1) \left( H_{i+\frac{1}{2}} + \ln(2) \right) + 1}{(2i+1)^2 (j-i)! (i+j+1)!}
$$
It seems worthwhile to consider the evaluation of integrals of the form
\[
\int_0^1 x^a (1 - x)^b J_m(x)^2 \, dx \in \mathbb{Q} + \zeta(3) \mathbb{Q}
\]
for \(m, a, b \in \mathbb{N}\). For instance, in the case \(m = 0\) and \(n = 2\), from the FL expansion of \(x(1-x)K(\sqrt{x})\)
\[
x(1-x)K(\sqrt{x}) = \sum_{n \geq 0} \frac{8(9 - 4n - 4n^2)}{(2n + 1)(4n^2 + 4n - 15)^2} P_n(2x - 1)
\]
we have:
\[
\int_0^1 x^2 (1-x)^2 K^2(\sqrt{x}) \, dx = \frac{7}{24} (251 \zeta(3) - 18).
\]
The following formulas can also be obtained from our FL-based technique:
\[
\int_0^1 x K^2(\sqrt{x}) \, dx = \frac{1}{4} (7 \zeta(3) + 2),
\]
\[
\int_0^1 x^2 K^2(\sqrt{x}) \, dx = \frac{1}{64} (77 \zeta(3) + 34).
\]
Let us state in an explicit way the structure of the Taylor series and the FL expansions of our generalized complete elliptic integrals, together with their special values at \(x = \frac{1}{2}\). The following results may be proved by considering the information about \(J_0, J_1, J_2\) collected so far and by applying induction on \(m \in \mathbb{N}\):
\[
J_m(x) = \frac{\pi}{2} \cdot \frac{2}{2F_{1}} \left[ \begin{array}{c} \frac{-2m+1}{2} \frac{1}{2} \\ 1 \end{array} \right] x
\]
\[
= \frac{\pi}{2} (2m + 1)!! \sum_{n \geq 0} \frac{(2n)^2 x^n}{16^n (1 - 2n) \cdots (2m + 1 - 2n)}
\]
\[
= 2(2m)! \sum_{n \geq 0} \frac{P_n(2x - 1)}{(2m + 1 + 2n)(2m - 1 + 2n) \cdots (2m - 1 - 2n)},
\]
\[
J_m \left( \frac{1}{2} \right) = 2(2m)! \sum_{n \geq 0} \frac{(2n)^2 (-1)^n}{4^n (2m + 1 + 4n)(2m - 1 + 4n) \cdots (2m - 1 - 4n)}
\]
\[
= \frac{2 \cdot 4^m}{(2m)(2m + 1)} \cdot \frac{1}{3F_{2}} \left[ \begin{array}{c} 1-2m \frac{3-2m}{4} \frac{1}{2} \\ 2m+3 \frac{2m+3}{4} \end{array} \right] - 1
\]
\[
= \int_0^{\pi/2} (1 - \frac{1}{2} \sin^2 \theta)^{2m-1} d\theta
\]
\[
\begin{align*}
&= \frac{1}{2} \int_0^1 \left(1 - \frac{u}{2}\right)^{\frac{2m-1}{2}} \frac{du}{\sqrt{u(1-u)}} \\
&= \frac{\Gamma(1+2m)}{\Gamma^2(1)} \int_0^1 (1+u^2)^{2m} \frac{du}{\sqrt{1-u^4}} \\
&= e_m E \left( \frac{1}{\sqrt{2}} \right) + k_m K \left( \frac{1}{\sqrt{2}} \right), \quad e_m, k_m \in \mathbb{Q}.
\end{align*}
\]

A generalization of Legendre’s relation for the $J_m$ functions has already been proved by Shingo Takeuchi in [17]. By FL expansions, the computation of the integrals $\int_0^1 J_m(x)^2 dx$ boils down to the computation of the partial fraction decomposition of

\[
q_n^{(m)} = \frac{1}{(2n+2m+1)^2} \cdots \frac{1}{(2n+1)^2} \cdot \frac{1}{(2n+1)^3} \cdots \frac{1}{(2n-2m+1)^2},
\]

which, regarded as a meromorphic function of the $n$ variable, has a triple pole at $n = -\frac{1}{2}$ and double poles at $n \in \{-m - \frac{1}{2}, \ldots, -\frac{3}{2}, \frac{1}{2}, \ldots, m - \frac{1}{2}\}$. We have

\[
2(2m)! \frac{(-1)^m (2n-2m-1)!!}{(2n+2m+1)!!} = 2 \frac{\Gamma(2m)}{4m} \sum_{k=0}^{2m} \binom{2m}{k} \frac{(-1)^k}{2n+2m+1-2k}
\]

and by squaring both sides, multiplying them by $\frac{1}{2n+1}$ and applying $\sum_{n \geq 0}(\ldots)$, we obtain an evaluation of $\int_0^1 J_m(x)^2 dx$ in terms of Apéry’s constant $\zeta(3)$. By the same argument, the integral $\int_0^1 J_a(x)J_b(x) dx$ has a similar structure.

### 7. Conclusion

We have investigated the relationships between classical hypergeometric sums and twisted hypergeometric series, FL series, and the $K$ and $E$ mappings. Our article may lead to new areas of research into generalizing our main theorems. In particular, from the results presented in Section 6, we are especially interested in generalizing the $J_m$ transformation as much as possible in a meaningful way, through a suitable modification of our methods.
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