Intermittency at critical transitions and aging dynamics at edge of chaos

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Abstract

We recall that, at both the intermittency transitions and at the Feigenbaum attractor in unimodal maps of non-linearity of order \( \zeta > 1 \), the dynamics rigorously obeys the Tsallis statistics. We account for the \( q \)-indices and the generalized Lyapunov coefficients \( \lambda_q \) that characterize the universality classes of the pitchfork and tangent bifurcations. We identify the Mori singularities in the Lyapunov spectrum at the edge of chaos with the appearance of a special value for the entropic index \( q \). The physical area of the Tsallis statistics is further probed by considering the dynamics near criticality and glass formation in thermal systems. In both cases a close connection is made with states in unimodal maps with vanishing Lyapunov coefficients.

Key words: Intermittency, criticality, glassy dynamics, ergodicity breakdown, edge of chaos, external noise, nonextensive statistics

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1 Introduction

Ever since the proposition, in 1988, by Tsallis [1], [2] of a nonextensive generalization of the canonical formalism of statistical mechanics a spirited discussion [3], [4] has grown on the foundations of this branch of physics. But, only
until recently [5]-[12] has there appeared firm evidence about the relevance of the generalized formalism for specific model system situations. These studies present rigorous results on the dynamics associated to critical attractors in prototypical nonlinear one-dimensional maps, such as those at the pitchfork and tangent bifurcations and at the accumulation point of the former, the so-called onset of chaos [13]. As these are very familiar and well understood features of these maps it is of interest to see how previous knowledge fits in with the new perspective. Also, clear-cut calculations may help clarify the physical reasons (believed in these examples to be a breakdown in the chain of increasing randomness from non-ergodicity to completely developed chaoticity) for the departure from the Boltzmann-Gibbs (BG) statistics and the competence of the non-extensive generalization. Here we review briefly specific results associated to the aforementioned states in unimodal maps which are all characterized by vanishing Lyapunov coefficients. We also recall how the dynamics at the tangent bifurcation appears to be related to that at thermal critical states. In addition, time evolution at the noise-perturbed onset of chaos is seen to be closely analogous to the glassy dynamics observed in supercooled molecular liquids.

2 Dynamics at the pitchfork and tangent bifurcations

As pointed out in Refs. [5], [6], [9] the long-known [13] exact geometric or static solution of the Feigenbaum renormalization group (RG) equations for the tangent bifurcation in unimodal maps of nonlinearity $\zeta > 1$ also describes the dynamics of iterates at such state. A straightforward extension of this approach applies also to the pitchfork bifurcations. We recall that the period-doubling and intermittency transitions are based on the pitchfork and the tangent bifurcations, respectively, and that at these critical states the ordinary Lyapunov coefficient $\lambda_1$ vanishes. The sensitivity to initial conditions $\xi_t$ was determined analytically and its relation with the rate of entropy production examined [9]. The fixed-point expressions have the specific form that corresponds to the temporal evolution suggested by the nonextensive formalism. These studies contain the derivation of their $q$-generalized Lyapunov coefficients $\lambda_q$ and the description of the different possible types of
sensitivity $\xi_t$.

By considering as starting point the $\zeta$-logistic map $f_\mu(x) = 1 - \mu |x|^\zeta$, $\zeta > 1$, $-1 \leq x \leq 1$, it is found that for both infinite sets of pitchfork and tangent bifurcations $\xi_t$, defined as $\xi_t \equiv \lim_{\Delta x_0 \to 0} (\Delta x_t / \Delta x_0)$ (where $\Delta x_0$ is the length of the initial interval and $\Delta x_t$ its length at time $t$), has the form suggested by Tsallis,

$$\xi_t(x_0) = \exp_q[\lambda_q(x_0) t] \equiv [1 - (q - 1)\lambda_q(x_0) t]^{-\frac{1}{q-1}},$$

that yields the customary exponential $\xi_t$ with Lyapunov coefficient $\lambda_1(x_0)$ when $q \to 1$. In Eq. (1) $q$ is the entropic index and $\lambda_q$ is the $q$-generalized Lyapunov coefficient; $\exp_q(x) \equiv [1 - (q - 1)x]^{-1/(q-1)}$ is the $q$-exponential function. The pitchfork and the left-hand side of the tangent bifurcations display weak insensitivity to initial condition, while the right-hand side of the tangent bifurcations presents a ‘super-strong’ (faster than exponential) sensitivity to initial conditions [6].

For the transition to periodicity of order $n$ the composition $f_\mu^{(n)}$ is first considered. In the neighborhood of one of the $n$ points tangent to the line with unit slope one obtains $f^{(n)}(x) = x + u|x|^2 + o(|x|^2)$, where $u > 0$ is the expansion coefficient. The general result obtained is $q = 2 - z^{-1}$ and $\lambda_q(x_0) = zux_0^{z-1}$[6], [9]. At the tangent bifurcations one has $f_\mu^{(n)}(x) = x + ux^2 + o(x^2)$, $u > 0$, and from $z = 2$ one gets $q = 3/2$. For the pitchfork bifurcations one has instead $f_\mu^{(n)}(x) = x + ux^3 + o(x^3)$, because $d^2 f_\mu^{(2k)}(dx^2 = 0$ at these transitions, and $u < 0$ is now the coefficient associated to $d^3 f_\mu^{(2k)}(dx^3 < 0$. In this case we have $z = 3$ in $q = 2 - z^{-1}$ and one obtains $q = 5/3$. Notably, these specific results for the index $q$ are valid for all $\zeta > 1$ and therefore define the existence of only two universality classes for unimodal maps, one for the tangent and the other one for the pitchfork bifurcations [6]. See Figs. 2 and 3 in Ref. [6].

Notice that our treatment of the tangent bifurcation differs from other studies of intermittency transitions [14] in that there is no feed back mechanism of iterates into the origin of $f^{(n)}(x)$ or of its associated fixed-point map. Here impeded or incomplete mixing in phase space (a small interval neighborhood around $x = 0$) arises from the special 'tangency' shape of the map at the pitchfork and tangent transitions that produces monotonic trajectories. This has the effect of confining or expelling trajectories causing anomalous phase-space sampling, in contrast to the thorough coverage in generic states.
with $\lambda_1 > 0$. By construction the dynamics at the intermittency transitions, describe a purely nonextensive regime.

3 Link between intermittent dynamics and dynamics at thermal criticality

An unanticipated relationship has been shown by Contoyiannis et al. [15]-[18] to exist between the intermittent dynamics of nonlinear maps at a tangent bifurcation and the equilibrium dynamics of fluctuations at an ordinary thermal critical point. With the aim of obtaining the properties of clusters or domains of the order parameter at criticality, a Landau-Ginzburg-Wilson (LGW) approach was carried out such that the dominant contributions to the partition function arise from a singularity (similar to an instanton) located in the space outside the cluster [15], [16]. Then [17], [18], a nonlinear map for the average order parameter was constructed whose dynamics reproduce the averages of the cluster critical properties. This map has as a main feature the tangent bifurcation and as a result time evolution is intermittent.

The starting point is the partition function of the $d$-dimensional system at criticality,

$$Z = \int D[\phi] \exp(-\Gamma_c[\phi]),$$

where

$$\Gamma_c[\phi] = g_1 \int_\Omega dV \left[ \frac{1}{2}(\nabla \phi)^2 + g_2 |\phi|^{\delta+1} \right]$$

is the critical LGW free energy of a system of $d$-dimensional volume $\Omega$, $\phi$ is the order parameter (e.g. magnetization per unit volume) and $\delta$ is the critical isotherm exponent. By considering the space-averaged magnetization $\Phi = \int_\Omega \phi(x) dV$, the statistical weight

$$\rho(\Phi) = \exp(-\Gamma_c[\Phi])/Z,$$

where $\Gamma_c[\Phi] \sim g_1 g_2 \Phi^{\delta+1}$ and $Z = \int d\Phi \exp(-\Gamma_c[\Phi])$, was shown to be the invariant density of a statistically equivalent one-dimensional map. The functional form of this map was obtained as the solution of an inverse Frobenius-Perron problem [17]. For small values of $\Phi$ the map has the form

$$\Phi_{n+1} = \Phi_n + u\Phi_n^{\delta+1} + \epsilon,$$
where the amplitude \( u \) depends on \( g_1, g_2 \) and \( \delta \), and the shift parameter \( \epsilon \sim R^{-d} \). Eq. (5) can be recognized as that describing the intermittency route to chaos in the vicinity of a tangent bifurcation [13]. The complete form of the map displays a superexponentially decreasing region that takes back the iterate close to the origin in one step. Thus the parameters of the thermal system determine the dynamics of the map. Averages made of order-parameter critical configurations are equivalent to iteration time averages along the trajectories of the map close to the tangent bifurcation. The mean number of iterations in the laminar region was seen to be related to the mean magnetization within a critical cluster of radius \( R \). There is a corresponding power law dependence of the duration of the laminar region on the shift parameter \( \epsilon \) of the map [17]. For \( \epsilon > 0 \) the (small) Lyapunov coefficient is simply related to the critical exponent \( \delta \) [16].

As the size of subsystems or domains of the critical system is allowed to become infinitely large the Lyapunov coefficient vanishes and the duration of the laminar episodes of intermittency diverge. Without the feedback to the origin feature in the map we recover the conditions described in the previous section. See Ref. [10] for more details.

4 Dynamics inside the Feigenbaum attractor

The dynamics at the chaos threshold, also referred to as the Feigenbaum attractor, of the \( \zeta \)-logistic map at \( \mu_c \) has been analyzed recently [7], [8]. By taking as initial condition \( x_0 = 0 \) we found that the resulting orbit consists of trajectories made of intertwined power laws that asymptotically reproduce the entire period-doubling cascade that occurs for \( \mu < \mu_c \). This orbit captures the properties of the so-called ‘superstable’ orbits at \( \mu_n < \mu_c, n = 1, 2, ... \) [13]. Here again the Lyapunov coefficient \( \lambda_1 \) vanishes and in its place there appears a spectrum of \( q \)-Lyapunov coefficients \( \lambda_q^{(k)} \). This spectrum was originally studied in Refs. [20], [21] and our interest has been to examine the relationship of its properties with the Tsallis statistics. We found that the sensitivity to initial conditions has precisely the form of a \( q \)-exponential, of which we determine the \( q \)-index and the associated \( \lambda_q^{(k)} \). The appearance of a specific value for the \( q \) index (and actually also that for its conjugate value \( Q = 2 - q \)) turns out to be due to the occurrence of Mori’s ‘\( q \) transitions’ [21] between ‘local attractor structures’ at \( \mu_c \). Furthermore, we have also shown
that the dynamical and entropic properties at \( \mu_c \) are naturally linked through the nonextensive expressions for the sensitivity to initial conditions \( \xi_t \) and for the entropy \( S_q \) in the rate of entropy production \( K_q^{(k)} \). We have corroborated analytically [7], the equality \( \lambda_q^{(k)} = K_q^{(k)} \) given by the nonextensive statistical mechanics. Our results support the validity of the \( q \)-generalized Pesin identity at critical points of unimodal maps.

Thus, the absolute values for the positions \( x_\tau \) of the trajectory with \( x_{t=0} = 0 \) at time-shifted \( \tau = t + 1 \) have a structure consisting of subsequences with a common power-law decay of the form \( \tau^{-1/\alpha} \) with \( q = 1 - \ln 2 / (\zeta - 1) \ln \alpha(\zeta) \), where \( \alpha(\zeta) \) is the Feigenbaum universal constant that measures the period-doubling amplification of iterate positions [6]. That is, the Feigenbaum attractor can be decomposed into position subsequences generated by the time subsequences \( \tau = (2k + 1)2^n \), each obtained by proceeding through \( n = 0, 1, 2, ... \) for a fixed value of \( k = 0, 1, 2, ... \). See Fig. 1. The \( k = 0 \) subsequence can be written as \( x_t = \exp_{-q}(\lambda_q^{(0)} t) \) with \( \lambda_q^{(0)} = (\zeta - 1) \ln \alpha(\zeta) / 2 \). These properties follow from the use of \( \zeta > 0 \) in the scaling relation [6]

\[
x_\tau \equiv \left| g^{(\tau)}(x_0) \right| = \tau^{1/q} \left| g(\tau^{1/q} x_0) \right|.
\]

The sensitivity associated to trajectories with other starting points \( x_0 \neq 0 \) within the attractor can be determined similarly with the use of the time subsequences \( \tau = (2k + 1)2^n \). One obtains \( \lambda_q^{(k)} = (\zeta - 1) \ln \alpha(\zeta) / (2k + 1) \ln 2 > 0, \ k = 0, 1, 2, ... \), the positive branch of the Lyapunov spectrum, when the trajectories start at the most crowded \( (x_{\tau=0} = 1) \) and finish at the most sparse \( (x_{\tau=2^n} = 0) \) region of the attractor. By inverting the situation we obtain \( \lambda^{(k)}_{q-2} = -2(\zeta - 1) \ln \alpha / (2k + 1) \ln 2 < 0, \ k = 0, 1, 2, ... \), the negative branch of \( \lambda^{(k)}_q \), i.e. starting at the most sparse \( (x_{\tau=0} = 0) \) and finishing at the most crowded \( (x_{\tau=2^n+1} = 0) \) region of the attractor. Notice that \( Q = 2 - q \) as \( \exp_{Q}(y) = 1 / \exp_{q}(-y) \). For the case \( \zeta = 2 \) see Refs. [7] and [8], for general \( \zeta > 1 \) see Refs. [11] and [19] where also a different and more direct derivation is used. So, when considering these two dominant families of orbits all the \( q \)-Lyapunov coefficients appear associated to only two specific values of the Tsallis index, \( q = 2 - q \).

As a function of the running variable \( -\infty < q < \infty \) the \( \lambda^{(k)}_q \) coefficients become a function \( \lambda(q) \) with two steps located at \( q = q = 1 \mp \ln 2 / (\zeta - 1) \ln \alpha(\zeta) \). In this manner contact can be established with the formalism developed by Mori and coworkers and the \( q \) phase transition obtained in Ref.
Figure 1: Absolute values of positions in logarithmic scales of the first 1000 iterations $\tau$ for a trajectory of the logistic map at the onset of chaos $\mu_c(0)$ with initial condition $x_{in} = 0$. The numbers correspond to iteration times. The power-law decay of the time subsequences described in the text can be clearly appreciated.

[21]. The step function for $\lambda(q)$ can be integrated to obtain the spectrum $\phi(q)$ ($\lambda(q) \equiv d\phi/d\lambda(q)$) and its Legendre transform $\psi(\lambda)$ ($\equiv \phi - (1 - q)\lambda$), the dynamic counterparts of the Renyi dimensions $D_q$ and the spectrum $f(\alpha)$ that characterize the geometry of the attractor. The constant slopes in the spectrum $\psi(\lambda)$ represent the Mori’s $q$ transitions for this attractor and the value $1 - q$ coincides with that of the slope previously detected [20], [21]. See Fig. 2. Details appear in Ref. [19].

Ensembles of trajectories with starting points close to $x_{\tau=0} = 1$ expand in such a way that a uniform distribution of initial conditions remains uniform for all later times $t \leq T$ where $T$ marks the crossover to an asymptotic regime. As a consequence of this we established [8] the identity of the rate of entropy production $K_q^{(k)}$ with $\lambda_q^{(k)}$. The $q$-generalized rate of entropy production $K_q$ is defined via $K_q t = S_q(t) - S_q(0)$, $t$ large, where

$$S_q \equiv \sum_i p_i \ln_q \left( \frac{1}{p_i} \right) = \frac{1 - \sum_i W^q i_q}{q - 1}$$

is the Tsallis entropy, and where $\ln_q y \equiv (y^{1-q} - 1)/(1 - q)$ is the inverse of $\exp_q(y)$. See Figs. 2 and 3 in Ref. [8].
Consider now the logistic map $\zeta = 2$ in the presence of additive noise

$$x_{t+1} = f_\mu(x_t) = 1 - \mu x_t^2 + \chi_t \sigma, \quad -1 \leq x_t \leq 1, \quad 0 \leq \mu \leq 2,$$

where $\chi_t$ is Gaussian-distributed with average $\langle \chi_t \chi_{t'} \rangle = \delta_{t,t'}$, and $\sigma$ measures the noise intensity. We recall briefly the known properties of this problem [13], [22]. Except for a set of zero measure, all the trajectories with $\mu_c(\sigma = 0)$ and initial condition $-1 \leq x_0 \leq 1$ fall into the attractor with fractal dimension $d_f = 0.5338\ldots$. These trajectories represent nonergodic states, since as $t \to \infty$ only a Cantor set of positions is accessible out of the total phase space $-1 \leq x \leq 1$. For $\sigma > 0$ the noise fluctuations wipe the sharp features of the periodic attractors as these widen into bands similar to those in the chaotic attractors, nevertheless there remains a well-defined transition to chaos at $\mu_c(\sigma)$ where the Lyapunov exponent $\lambda_1$ changes sign. The period doubling of bands ends at a finite value $2^N(\sigma)$ as the edge of chaos transition is approached and then decreases at the other side of the transition. This effect displays scaling features and is referred to as the bifurcation gap [13], [22]. When $\sigma > 0$ the trajectories visit sequentially a set of $2^n$ disjoint bands or segments leading to a cycle, but the behavior inside each band is completely chaotic. These trajectories represent ergodic states as the accessible positions have a fractal dimension equal to the dimension of phase space. Thus the removal of the noise $\sigma \to 0$ leads to an ergodic to nonergodic transition in the map.
In the presence of noise (σ small) one obtains instead of Eq. (6) [12]

\[ x_\tau = \tau^{-1/1-q} g(\tau^{1/1-q}x) + \chi \sigma \tau^{1/1-r} G_\Lambda(\tau^{1/1-q}x), \]  

(9)

where \( G_\Lambda(x) \) is the first order perturbation eigenfunction, and where \( r = 1 - \ln 2 / \ln \kappa \approx 0.6332 \). Use of \( x_0 = 0 \) yields \( x_\tau = \tau^{-1/1-q} \left| 1 + \chi \sigma \tau^{1/1-r} \right| \) or \( x_t = \exp_{2-q}(-\lambda_q t) [1 + \chi \sigma \exp_r(\lambda_q t)] \) where \( t = \tau - 1 \) and \( \lambda_r = \ln \kappa / \ln 2 \).

At each noise level \( \sigma \) there is a 'crossover' or 'relaxation' time \( t_x = \tau_x - 1 \) when the fluctuations start suppressing the fine structure of the orbits with \( x_0 = 0 \). This time is given by \( \tau_x = \sigma^{r-1} \), the time when the fluctuation term in the perturbation expression for \( x_\tau \) becomes unbounded by \( \sigma \), i.e. \( x_{\tau_x} = \tau_x^{-1/1-q} [1 + \chi] \). There are two regimes for time evolution at \( \mu_c(\sigma) \).

When \( \tau < \tau_x \) the fluctuations are smaller than the distances between neighboring subsequence positions of the \( \sigma = 0 \) orbit at \( \mu_c(0) \), and the iterate positions with \( \sigma > 0 \) fall within small non overlapping bands each around the \( \sigma = 0 \) position for that \( \tau \). Time evolution follows a subsequence pattern analogous to that in the noiseless case. When \( \tau \approx \tau_x \) the width of the noise-generated band reached at time \( \tau_x = 2^N \) matches the distance between adjacent positions where \( N \approx -\ln \sigma / \ln \kappa \), and this implies a cutoff in the progress along the position subsequences. At longer times \( \tau > \tau_x \) the orbits no longer follow the detailed period-doubling structure of the attractor. The iterates now trail through increasingly chaotic trajectories as bands merge with time. This is the dynamical image - observed along the time evolution for the orbits of a single state \( \mu_c(\sigma) \) - of the static bifurcation gap originally described in terms of the variation of the control parameter \( \mu \) [22], [23], [24].

The plateau structure of relaxation and the crossover time \( t_x \) can be clearly observed in Fig. 1b in Ref. [25] where \( <x_t^2> - <x_t>^2 \) is shown for several values of \( \sigma \).

5 Parallels with dynamics near glass formation

We recall the main dynamical properties displayed by supercooled liquids on approach to glass formation. One is the growth of a plateau and for that reason a two-step process of relaxation in the time evolution of two-time correlations [26]. This consists of a primary power-law decay in time difference
\( \Delta t \) (so-called \( \beta \) relaxation) that leads into the plateau, the duration \( t_x \) of which diverges also as a power law of the difference \( T - T_g \) as the temperature \( T \) decreases to a glass temperature \( T_g \). After \( t_x \) there is a secondary power law decay (so-called \( \alpha \) relaxation) away from the plateau [26]. A second important (nonequilibrium) dynamic property of glasses is the loss of time translation invariance observed for \( T \) below \( T_g \), a characteristic known as aging [27]. The time fall off of relaxation functions and correlations display a scaling dependence on the ratio \( t/t_w \) where \( t_w \) is a waiting time. A third notable property is that the experimentally observed relaxation behavior of supercooled liquids is effectively described, via reasonable heat capacity assumptions [26], by the so-called Adam-Gibbs equation,

\[
t_x = A \exp\left(\frac{B}{TS_c}\right),
\]

where \( t_x \) is the relaxation time at \( T \), and the configurational entropy \( S_c \) is related to the number of minima of the fluid’s potential energy surface [26]. We compare the dynamic properties at the edge of chaos described in the previous section with those known for the process of vitrification of a liquid as \( T \to T_g \).

At noise level \( \sigma \) the orbits visit points within the set of \( 2^N \) bands and, as explained in Ref. [12], this takes place in time in the same way that period doubling and band merging proceeds in the presence of a bifurcation gap when the control parameter is run through the interval \( 0 \leq \mu \leq 2 \). Namely, the trajectories starting at \( x_0 = 0 \) duplicate the number of visited bands at times \( \tau = 2^n \), \( n = 1, ..., N \), the bifurcation gap is reached at \( \tau_x = 2^N \), after which the orbits fall within bands that merge by pairs at times \( \tau = 2^{N+n} \), \( n = 1, ..., N \). The sensitivity to initial conditions grows as \( \xi_t = \exp_q(\lambda_q t) \) \((q = 1 - \ln 2 / \ln \alpha < 1)\) for \( t < t_x \), but for \( t > t_x \) the fluctuations dominate and \( \xi_t \) grows exponentially as the trajectory has become chaotic \((q = 1)\) [12]. This behavior was interpreted [12] to be the dynamical system analog of the \( \alpha \) relaxation in supercooled fluids. The plateau duration \( t_x \to \infty \) as \( \sigma \to 0 \). Additionally, trajectories with initial conditions \( x_0 \) not belonging to the attractor exhibit an initial relaxation process towards the plateau as the orbit approaches the attractor. This is the map analog of the \( \beta \) relaxation in supercooled liquids.

The entropy \( S_c(\mu_c(\sigma)) \) associated to the distribution of iterate positions (configurations) within the set of \( 2^N \) bands was determined in Ref. [12]. This entropy has the form \( S_c(\mu_c(\sigma)) = 2^N \sigma s \), since each of the \( 2^N \) bands contributes with an entropy \( \sigma s \), where \( s = - \int_{-1}^{1} p(\chi) \ln p(\chi) d\chi \) and where \( p(\chi) \) is the distribution for the noise random variable. Given that \( 2^N = 1 + t_x \)
and $\sigma = (1 + t_x)^{-1/1-r}$, one has $S_c(\mu_c, t_x)/s = (1 + t_x)^{-r/1-r}$ or, conversely,

$$t_x = (s/S_c)^{(1-r)/r}. \quad (10)$$

Since $t_x \simeq \sigma^{r-1}, \ r - 1 \simeq -0.3668$ and $(1 - r)/r \simeq 0.5792$ then $t_x \to \infty$ and $S_c \to 0$ as $\sigma \to 0$, i.e. the relaxation time diverges as the 'landscape' entropy vanishes. We interpret this relationship between $t_x$ and the entropy $S_c$ to be the dynamical system analog of the Adam-Gibbs formula for a supercooled liquid. Notice that Eq.(10) is a power law in $S_c^{-1}$ while for structural glasses it is an exponential in $S_c^{-1}$ [26]. This difference is significant as it indicates how the superposition of molecular structure and dynamics upon the bare ergodicity breakdown phenomenon built in the map modifies the vitrification properties.

The aging scaling property of the trajectories $x_t$ at $\mu_c(\sigma)$ was examined in Ref. [12]. The case $\sigma = 0$ is readily understood because this property is actually built into the position subsequences $x_\tau = \left| g^{(\tau)}(0) \right|, \ \tau = (2k + 1)2^n, \ k, n = 0, 1, \ldots$ referred to above. These subsequences can be employed for the description of trajectories that are at first held at a given attractor position for a waiting period of time $t_w$ and then released to the normal iterative procedure. For illustrative purposes we select the holding positions to be any of those for a waiting time $t_w = 2k + 1, \ k = 0, 1, \ldots$ and notice that for the $x_{in} = 0$ orbit these positions are visited at odd iteration times. The lower-bound positions for these trajectories are given by those of the subsequences at times $(2k + 1)2^n$. See Fig. 1. Writing $\tau$ as $\tau = t_w + t$ we have that $t/t_w = 2^n - 1$ and $x_{t+t_w} = g^{(t_w)}(0)g^{(t/t_w)}(0)$ or

$$x_{t+t_w} = g^{(t_w)}(0) \exp_q(-\lambda_q t/t_w). \quad (11)$$

This fully developed aging property is gradually modified when noise is turned on. The presence of a bifurcation gap limits its range of validity to total times $t_w + t < t_x(\sigma)$ and so progressively disappears as $\sigma$ is increased.

### 6 Concluding remarks

The implications of joining the results about intermittency described in Sections 2 and 3 are apparent. In the critical clusters of infinite size $R \to \infty$ the dynamics of fluctuations obeys the nonextensive statistics. This is expressed
via the time series of the average order parameter $\Phi_n$, i.e. trajectories $\Phi_n$ with close initial values separate in a superexponential fashion according to Eq. (1) with $q = (2\delta + 1)/(\delta + 1) > 1$ and with a $q$-Lyapunov coefficient $\lambda_q$ determined by the system parameter values $\delta, g_1, g_2$ [10].

Also, as described in Sections 4 and 5, the dynamics of noise-perturbed logistic maps at the chaos threshold exhibits the most prominent features of glassy dynamics in supercooled liquids. The existence of this analogy cannot be considered accidental since the limit of vanishing noise amplitude $\sigma \to 0$ (the counterpart of the limit $T - T_g \to 0$ in the supercooled liquid) entails loss of ergodicity. Here we have shown that this nonergodic state corresponds to the limiting state, $\sigma \to 0$, $t_x \to \infty$, for a family of small $\sigma$ noisy states with glassy properties, that are noticeably described for $t < t_x$ via the $q$-exponentials of the nonextensive formalism [12].

What is the significance of the connections we have reviewed? Are there other connections between critical phenomena and transitions to chaos? Are all critical states - infinite correlation length with vanishing Lyapunov coefficients - outside BG statistics? Where, and in that case why, does Tsallis statistics apply? Is ergodicity failure the basic playground for applicability of generalized statistics? We can mention some noticeable limitations in the examples discussed. In the case of intermittency we have focused only on a single laminar episode (or scape from a position very close to tangency), though this can be of very large duration. In the case of dynamics at the edge of chaos we have payed attention only to the dominant features of the multifractal attractor (the most sparse $x = 0$ and most crowded $x = 1$ regions) as starting and final orbit positions.

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