A MIN-MAX PRINCIPLE FOR NON-DIFFERENTIABLE FUNCTIONS WITH A WEAK COMPACTNESS CONDITION

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(Dedicated to Professor Mario Marino on the occasion of his sixty-fifth birthday)

Abstract. A general critical point result established by Ghoussoub is extended to the case of locally Lipschitz continuous functions satisfying a weak Palais-Smale hypothesis, which includes the so-called non-smooth Cerami condition. Some special cases are then pointed out.

1. Introduction. A trend in today’s literature on critical point theory is the attempt to weaken, in a fruitful way, the key assumptions of the famous Ambrosetti-Rabinowitz’s Mountain Pass Theorem, namely

(a) the Mountain Pass geometry,
(b) the Palais-Smale compactness condition, and
(c) the regularity of the involved functional.

These questions have by now been widely investigated, and excellent monographs are already available. For instance, [10, 20] contain meaningful generalizations of (a)–(b), while [17, 18, 8] mainly deal with (c). The book [12] represents a general reference on the subject.

In this paper we first extend the min-max principle of Ghoussoub [9] to locally Lipschitz continuous functions satisfying a weak Palais-Smale hypothesis, which includes both the usual one [2, Definition 2] and the non-smooth Cerami condition [13, p. 248]; see Theorem 3.1 below. Thus, since (a) is a very special case of the situation treated in [9], assumptions (a)–(c) are weakened here. From a technical point of view, Theorem 3.1 is achieved by adapting Ghoussoub’s approach to our non-smooth framework and exploiting Ekeland’s Variational Principle with a suitable metric of geodesic type. When the functional is $C^1$ while the compactness condition is that of Cerami, this idea basically goes back to Ekeland [6, p. 138]. Two simple but meaningful consequences of Theorem 3.1 are then pointed out, i.e., Theorems 3.2 and 3.3. The first of them provides a more refined version of several recent critical point results, as, for instance, Theorem 1.bis and Corollary 2 in [9], Corollary 1 and Theorem 2 of [19] (vide also [18, Section 2.1]), Theorem 6 in [13],

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besides Corollaries 9 and 10 at p. 145 of [6]. A special mention deserves the paper [14], where a general deformation lemma and a subsequent min-max principle for locally Lipschitz continuous functions are obtained under (a) and a weak Palais-
Smale hypothesis similar to but different from the one adopted here. It is also
worth noting that non-differentiable variants of structure or multiplicity results, as
well as applications to elliptic hemivariational inequalities, might be drawn from
Theorem 3.3; see for instance [1].

Let us finally observe that Ghoussoub’s result has already been extended by the
authors in [15] to the case of functionals which are the sum of a locally Lipschitz
continuous term and of a convex, proper, lower semi-continuous function, namely
the so-called Motreanu-Panagiotopoulos’ non-smooth framework [17]. However, the
compactness condition employed there reduces to the standard one [2, Definition 2]
in the locally Lipschitz continuous setting and hence it is more restrictive than that
taken on below.

2. Basic definitions and auxiliary results. Let \((X, \|\cdot\|)\) be a real Banach space.
If \(U\) is a nonempty subset of \(X\), \(x \in X\), and \(r > 0\), we define \(B(x, r) := \{z \in X : \|z - x\| < r\}\) as well as
\[
d(x, U) := \inf_{z \in U} \|x - z\|.
\]
Given \(x, z \in X\), the symbol \([x, z]\) indicates the line segment joining \(x\) to \(z\), i.e.,
\([x, z] := \{x + t(z - x) : t \in [0, 1]\}\), and \(d(x, z) := \|x - z\|\). We denote by \(X^*\) the
dual space of \(X\), while \(\langle \cdot, \cdot \rangle\) stands for the duality pairing between \(X\) and \(X^*\).
A function \(f : X \to \mathbb{R}\) is called locally Lipschitz continuous when to every \(x \in X\)
there correspond a neighborhood \(V_x\) of \(x\), besides a constant \(L_x \geq 0\), such that
\[
|f(z) - f(w)| \leq L_x \|z - w\| \quad \forall z, w \in V_x.
\]
If \(x, z \in X\), we write \(f^0(x; z)\) for the generalized directional derivative of \(f\) at
the point \(x\) along the direction \(z\), namely
\[
f^0(x; z) := \limsup_{w \to x, t \to 0^+} \frac{f(w + tz) - f(w)}{t}.
\]
One evidently has \(f^0(x; tz) = tf^0(x; z)\) for every \(t \geq 0\). It is known [3, Proposition
2.1.1] that \(f^0\) turns out to be upper semi-continuous on \(X \times X\). The generalized
gradient of the function \(f\) in \(x\), denoted by \(\partial f(x)\), is the set
\[
\partial f(x) := \{x^* \in X^* : \langle x^*, z \rangle \leq f^0(x; z) \ \forall z \in X\}.
\]
Proposition 2.1.2 of [3] ensures that \(\partial f(x)\) is nonempty, convex, in addition to weak*
compact, and that
\[
f^0(x; z) = \max\\{\langle x^*, z \rangle : x^* \in \partial f(x)\}.
\]
Hence, it makes sense to write
\[
m_f(x) := \min\\{\|x^*\|_{X^*} : x^* \in \partial f(x)\}.
\]
We say that \(x \in X\) is a critical point of \(f\) when \(0 \in \partial f(x)\), namely \(f^0(x; z) \geq 0\)
for all \(z \in X\). Given a real number \(c\), write
\[
K_c(f) := \{x \in X : f(x) = c, \ x \text{ is a critical point of } f\}.
\]
In this setting, the classical Palais-Smale compactness hypothesis at the level \(c \in \mathbb{R}\)
for \(C^1\) functions becomes (cf. [2, Definition 2]):
(PS)$_c$ Every sequence $\{x_n\} \subseteq X$ such that $f(x_n) \to c$ and $m_f(x_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence.

Now, let $h: [0, +\infty[ \to [0, +\infty[$ be a continuous function enjoying the following property:

$$
\int_0^{+\infty} \frac{1}{1 + h(\xi)} \, d\xi = +\infty. 
$$

(1)

We say that $f$ satisfies a weak Palais-Smale condition at the level $c \in \mathbb{R}$ when for some $h$ as above one has:

(PS)$_h^c$ Every sequence $\{x_n\} \subseteq X$ such that

$$
\lim_{n \to +\infty} f(x_n) = c \quad \text{and} \quad \lim_{n \to +\infty} (1 + h(\|x_n\|)) m_f(x_n) = 0 
$$

possesses a convergent subsequence.

**Remark 1.** If $h(\xi) \equiv 0$ then (PS)$_h^c$ reduces to (PS)$_c$. Setting $h(\xi) := \xi, \xi \in [0, +\infty[$, we obtain a non-smooth version, previously introduced in [13], of the so-called Cerami compactness assumption.

**Remark 2.** Under the further request that $h$ be nondecreasing in $[0, +\infty[$, condition (PS)$_h^c$ is already known. We refer to [22] for $C^1$ functions and to [16] for a much more general situation. A similar but different hypothesis has recently been exploited in the paper [14].

A weaker form of (PS)$_h^c$ is the one below, where $U$ denotes a nonempty closed subset of $X$. For $U := X$ it coincides with (PS)$_h^c$.

(PS)$_h^c$ Every sequence $\{x_n\} \subseteq X$ such that $d(x_n, U) \to 0$ as $n \to +\infty$ and (2) holds true possesses a convergent subsequence.

Next, let $\alpha: [0, +\infty[ \to [0, 1]$ be a continuous function such that $\int_0^{+\infty} \alpha(\xi) d\xi = +\infty$ and let $c \in \mathbb{R}$. We say that $f$ fulfills condition (C)$_\alpha^c$ if:

(j1) Every bounded sequence $\{x_n\} \subseteq X$ complying with $f(x_n) \to c$ and $m_f(x_n) \to 0$ as $n \to +\infty$ possesses a convergent subsequence.

(j2) There exist $r, \sigma > 0$ such that

$$
m_f(x) \geq \alpha(\|x\|) \quad \forall x \in f^{-1}([c - \sigma, c + \sigma]) \setminus B(0, r). 
$$

This condition is patterned after that of [7]; see also [12, Section 13.1]. An elementary argument provides the following

**Proposition 1.** The function $f$ satisfies (PS)$_h^c$ if and only if it fulfills (C)$_\alpha^c$.

Given $x, z \in X$, we denote by $\mathcal{P}(x, z)$ the family of all piecewise $C^1$ paths $p: [0, 1] \to X$ such that $p(0) = x$ and $p(1) = z$. Moreover, put

$$
l_h(p) := \int_0^1 \frac{\|p'(t)\|}{1 + h(\|p(t)\|)} \, dt, \quad p \in \mathcal{P}(x, z),
$$

as well as

$$
\delta_h(x, z) := \inf \{l_h(p) : p \in \mathcal{P}(x, z)\}. 
$$

(3)

For $h(\xi) := \xi, \xi \in [0, +\infty[$, the function $\delta_h: X \times X \to \mathbb{R}$ defined by (3) coincides with the geodesic distance introduced in [6, p. 138]. Exploiting (1) and the arguments of [6, p. 138] (cf. besides [4, Section 4], where a more general situation is treated) yields the following basic properties of $\delta_h$.

**Property (p$_1$)** $\delta_h(x, z) \leq \|x - z\|$ for all $x, z \in X$. 


(p₂) If $U$ is a nonempty bounded subset of $X$ then there exists a constant $c_U > 0$ such that

$$\delta_h(x, z) \geq c_U \|x - z\| \quad \forall x, z \in U.$$ 

(p₃) $\delta_h$ turns out to be a distance on $X$ and the metric topology derived from $\delta_h$ coincides with the norm topology.

(p₄) $\delta_h$-bounded and norm-bounded sets in $X$ are the same.

Through (p₁), (p₂), and (p₄) one easily verifies that the metric space $(X, \delta_h)$ is complete.

Finally, the following version [5, pp. 444, 456] of the famous variational principle of Ekeland will be employed.

**Theorem 2.1.** Let $(Z, \delta)$ be a complete metric space and let $\Phi : Z \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous, bounded below function. Then to each $\varepsilon, r > 0$ and $\varpi \in Z$ satisfying $\Phi(\varpi) \leq \inf_{z \in Z} \Phi(z) + \varepsilon$ there corresponds a point $z_0 \in Z$ such that

$$\Phi(z_0) \leq \Phi(\varpi), \quad \delta(z_0, \varpi) \leq 1/r, \quad \Phi(z) - \Phi(z_0) \geq -\varepsilon \rho(z, z_0) \quad \forall z \in X.$$

3. **Existence of critical points.** Let $B$ be a nonempty closed subset of $X$ and let $\mathcal{F}$ be a class of nonempty compact sets in $X$. According to [9, Definition 1], we say that $\mathcal{F}$ is a homotopy-stable family with extended boundary $B$ when for every $A \in \mathcal{F}$, $\eta \in C^0([0, 1] \times X, X)$ such that $\eta(t, x) = x$ on $\{(0) \times X\} \cup \{(0, 0) \times B\}$ one has $\eta(1) \times A) \in \mathcal{F}$. Some meaningful situations are special cases of this notion. For instance, if $Q$ denotes a compact set in $X$, $Q_0$ is a non-empty closed subset of $Q$, $\gamma_0 \in C^0(Q_0, X)$,

$$\Gamma := \{ \gamma \in C^0(Q, X) : \gamma|_{Q_0} = \gamma_0 \},$$

and $\mathcal{F} := \{ \gamma(Q) : \gamma \in \Gamma \}$, then $\mathcal{F}$ enjoys the above-mentioned property with $B := \gamma_0(Q_0)$. In particular, it holds true when $Q$ indicates a compact topological manifold in $X$ having a nonempty boundary $Q_0$ while $\gamma_0 := id|_{Q_0}$.

The following assumptions will be posited in the sequel.

(a₁) $f : X \to \mathbb{R}$ is a locally Lipschitz continuous function.

(a₂) $\mathcal{F}$ denotes a homotopy-stable family with extended boundary $B$.

(a₃) There exists a nonempty closed subset $F$ of $X$ such that

$$(A \cap F) \setminus B \neq \emptyset \quad \forall A \in \mathcal{F}$$

and, moreover,

$$\sup_{x \in B} f(x) \leq \inf_{x \in F} f(x).$$

(a₄) $h : [0, +\infty] \to [0, +\infty]$ is a continuous function fulfilling (i), while $\delta_h$ indicates the metric defined by (3).

Set, as usual,

$$c := \inf_{A \in F} \max_{x \in A} f(x).$$

Thanks to (5) one has

$$\inf_{x \in F} f(x) \leq c.$$ 

**Theorem 3.1.** Let (a₁)–(a₄) be satisfied. Then to every sequence $\{A_n\} \subseteq \mathcal{F}$ such that $\lim_{n \to +\infty} \max_{x \in A_n} f(x) = c$ there corresponds a sequence $\{x_n\} \subseteq X \setminus B$ having the following properties:

(i₁) $\lim_{n \to +\infty} f(x_n) = c$.

(i₂) $(1 + h(\|x_n\|))(f^h(x_n; z) \geq -\epsilon_n \|z\| \text{ for all } n \in \mathbb{N}, z \in X, \text{ where } \epsilon_n \to 0^+.$
The function

Moreover, set

is complete. Define, for every

A simple computation then shows that

Since

we denote by

which obviously provides a sequence \( \{ x_n \} \subseteq X \setminus B \) enjoying properties (i1)–(i4). To shorten notation, write

where

We shall prove the existence of a point \( x_\varepsilon \in X \setminus B \) such that

(10)
\[
(1 + h(\| x_\varepsilon \|)) f^0(x_\varepsilon ; z) \geq -3\varepsilon \| z \| \quad \forall z \in X ,
\]

(11)
\[
\delta_h(x_\varepsilon , F) \leq \frac{3\varepsilon}{2} ,
\]

(12)
\[
\delta_h(x_\varepsilon , A_\varepsilon) \leq \frac{\varepsilon}{2} ,
\]

which obviously provides a sequence \( \{ x_n \} \subseteq X \setminus B \) enjoying properties (i1)–(i4). To shorten notation, write

(9)
\[
c - \frac{\varepsilon^2}{8} \leq f(x_\varepsilon) \leq c + \frac{5\varepsilon^2}{4} ,
\]

(7)
\[
\inf_{x \in F} f(x) = c .
\]

Pick an \( \varepsilon > 0 \) and choose \( A_\varepsilon \in \mathcal{F} \) fulfilling

(8)
\[
c \leq \max_{x \in A_\varepsilon} f(x) < c + \frac{\varepsilon^2}{8} .
\]

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Proof. On account of (6), we first consider the case

(13)
\[
\eta(A'_\varepsilon) \in \mathcal{F} \quad \forall \eta \in \mathcal{L} .
\]

A simple computation then shows that \( \mathcal{L} \), equipped with the uniform distance

is complete. Define, for every \( x \in X \),

(10)
\[
\rho(\eta_1, \eta_2) := \sup_{(t,x) \in [0,1] \times X} \delta_h(\eta_1(t,x), \eta_2(t,x)) , \quad \eta_1, \eta_2 \in \mathcal{L} ,
\]

Moreover set

(11)
\[
f_1(x) := \max \{ 0, \varepsilon^2 - \varepsilon \delta_h(x, F) \} , \quad f_2(x) := \min \{ \varepsilon^2/8 , \varepsilon \delta_h(x, (A_\varepsilon \setminus F) \cup B) \} ,
\]

\[
g(x) := f(x) + f_1(x) + f_2(x) .
\]

Moreover, set

(12)
\[
I(\eta) := \max_{z \in \eta(A'_\varepsilon) \cap F} g(z) , \quad \eta \in \mathcal{L} .
\]

The function \( I : \mathcal{L} \to \mathbb{R} \) is evidently lower semi-continuous. Gathering (13) and (4) together yields

(13)
\[
(\eta(A'_\varepsilon) \cap F) \setminus B \neq \emptyset \quad \forall \eta \in \mathcal{L} .
\]

Consequently, on account of (7),

(14)
\[
\inf_{\eta \in \mathcal{L}} I(\eta) \geq c + \varepsilon^2 .
\]
Let \( \bar{\eta}(t, x) := x \) for all \((t, x) \in [0, 1] \times X\). Since, by (8) and (14),
\[
I(\bar{\eta}) \leq \max_{x \in A} f(x) + \epsilon^2 + \frac{e^2}{8} < c + \epsilon^2 + \frac{e^2}{4} \leq \inf_{\eta \in \mathcal{L}} I(\eta) + \frac{e^2}{4},
\]
(15)
Theorem 2.1 can be applied. Hence, there exists an \( \eta_0 \in \mathcal{L} \) such that
\[
I(\eta_0) \leq I(\bar{\eta}),
\]
(16)
\[
\rho(\eta_0, \bar{\eta}) \leq \frac{\epsilon}{2},
\]
(17)
\[
I(\eta) \geq I(\eta_0) - \frac{\epsilon}{2} \rho(\eta, \eta_0) \quad \forall \eta \in \mathcal{L}. \tag{18}
\]
Define
\[
C := \left\{ w \in \eta_0(A') : g(w) = I(\eta_0) = \max_{z \in \eta_0(A')} g(z) \right\}.
\]
Obviously, the set \( C \) turns out nonempty and compact. We claim that for some \( z_0 \in (\eta_0(A'_z) \cap F) \setminus B \) one has
\[
f(z_0) = \max_{z \in \eta_0(A') \cap F} f(z).
\]
(19)
In fact, let \( \hat{z} \in \eta_0(A'_z) \cap F \) satisfy \( f(\hat{z}) = \max_{z \in \eta_0(A') \cap F} f(z) \). If \( \hat{z} \notin B \), then the assertion is true with \( z_0 := \hat{z} \). Otherwise, exploiting (4), pick any \( z_0 \in (\eta_0(A'_z) \cap F) \setminus B \). Thanks to (5) we obtain
\[
\max_{z \in \eta_0(A') \cap F} f(z) = f(\hat{z}) \leq \sup_{x \in B} f(x) \leq \inf_{x \in F} f(x) \leq f(z_0) \leq \max_{\eta \in \eta_0(A') \cap F} f(z),
\]
and (19) holds. The point \( z_0 \) does not lie in \((A_x \setminus F_x) \cup B\). So, \( f_2(z_0) > 0 \). Because of (7) this implies
\[
\max_{z \in \eta_0(A')} g(z) \geq f(z_0) + f_1(z_0) + f_2(z_0) \geq c + \epsilon^2 + f_2(z_0) > c + \epsilon^2. \tag{20}
\]
Through (8), (5), and (7) we then achieve
\[
\sup_{x \in A_x \setminus F_x} g(x) \leq \max_{x \in A_x} f(x) < c + \frac{\epsilon^2}{8}, \tag{21}
\]
\[
\sup_{x \in B \setminus F_x} g(x) \leq \sup_{x \in B} f(x) + \epsilon^2 \leq \inf_{x \in F} f(x) + \epsilon^2 \leq c + \epsilon^2, \tag{22}
\]
\[
\sup_{x \in B \setminus F_x} g(x) \leq \sup_{z \in \eta_0(A') \cap F} f(z) \leq \sup_{x \in B} f(x) \leq \inf_{x \in F} f(x) = c. \tag{23}
\]
Let \( B' := (A_x \setminus F_x) \cup B \). From (20)–(23) it results in
\[
\sup_{x \in B'} g(x) \leq c + \epsilon^2 < \max_{z \in \eta_0(A')} g(z),
\]
which clearly forces
\[
B' \cap C = \emptyset. \tag{24}
\]
We shall next prove that:
\[
\text{There is a point } x_\epsilon \in C \text{ fulfilling } (1 + h(||x_\epsilon||)) f^0(x_\epsilon; z) \geq -3\epsilon ||z|| \quad \forall z \in X. \tag{25}
\]
If (25) were false then for each \( x \in C \) one could find a \( v_x \in X \) such that
\[
(1 + h(||x||)) f^0(x; v_x) < -3\epsilon ||v_x||. \tag{26}
\]
Without loss of generality, suppose \( ||v_x|| = 1 \). Since the function
\[
(w, z) \mapsto (1 + h(||w||)) f^0(z; v_x) + 3\epsilon, \quad (w, z) \in X \times X,
\]
turns out upper semi-continuous in \( X \times X \) and less than zero at \((x,x)\), there exists an \( r_x > 0 \) such that
\[
(1 + h(\|w\|))f^0(z; v_x) < -3\epsilon \quad \forall w, z \in B(x, r_x).
\]
Let \( r_x \) be small enough to have also \( f \) Lipschitz continuous on \( B(x, r_x) \),
\[
B' \cap B(x, r_x) = \emptyset
\]
by (24), as well as
\[
h(\|w\|) < h(\|x\|) + 1, \quad w \in B(x, r_x).
\]
Put \( V_x := B(x, r_x) \). The family \( \mathcal{B} := \{ V_x : x \in C \} \) represents an open covering of \( C \).
Since this set is compact, \( \mathcal{B} \) possesses a finite sub-covering \( \{ V_{x_j} : j = 1, 2, \ldots, m \} \) to which we can associate a continuous partition of unity \( \{ \chi_j : j = 1, 2, \ldots, m \} \).
For simplicity of notation, write \( v_j := v_{x_j} \), \( r_j := r_{x_j} \), besides \( V_j := V_{x_j} \). Moreover, define, for every \( x \in X \),
\[
v(x) := \begin{cases} \sum_{j=1}^m \chi_j(x)v_j & \text{if } x \in \bigcup_{j=1}^m V_j, \\ 0 & \text{otherwise}, \end{cases}
u(x) := (1 + h(\|x\|))v(x).
\]
Observe that \( \|v(x)\| \leq 1 \) for all \( x \in X \) because \( \|v_j\| = 1 \), \( j = 1, 2, \ldots, m \). Hence, the function \( u : X \to X \) is continuous,
\[
\|u(x)\| \leq 1 + h(\|x\|) \quad \forall x \in X,
\]
and, on account of (28),
\[
u(x) = 0 \quad \text{in } B'.
\]
Through (29)–(30) we then obtain
\[
\|u(x)\| \leq 2 + M, \quad x \in X,
\]
where \( M := \max_{x \in C} h(\|x\|) \). Now, set, provided \( n \in \mathbb{N} \),
\[
\eta_n(t, x) := \eta_0(t, x) + \frac{t}{n}u(\eta_0(t, x)) \quad \forall (t, x) \in [0, 1] \times X.
\]
From \( \eta_0 \in \mathcal{L} \), (31), (32), and (17) it follows \( \eta_n \in \mathcal{L} \). Further,
\[
\rho(\eta_n, \eta_0) = \sup_{(t,x) \in [0,1] \times X} \delta_n(\eta_n(t, x), \eta_0(t, x))
\leq \frac{1}{n} \sup_{(t,x) \in [0,1] \times X} \int_0^1 \frac{\|u(\eta_0(t, x))\|}{1 + h(\|\eta_0(t, x) + \frac{t\tau}{n} u(\eta_0(t, x))\|)} d\tau.
\]
This inequality, combined with (18), leads to
\[
I(\eta_n) - I(\eta_0) \geq -\frac{\epsilon}{2n} \sup_{(t,x) \in [0,1] \times X} \int_0^1 \frac{\|u(\eta_0(t, x))\|}{1 + h(\|\eta_0(t, x) + \frac{t\tau}{n} u(\eta_0(t, x))\|)} d\tau \quad \forall n \in \mathbb{N}.
\]
Letting \( n \to +\infty \) and using (30) we thus get
\[
\liminf_{n \to +\infty} \frac{I(\eta_n) - I(\eta_0)}{1/n} \geq -\frac{\epsilon}{2} > -\epsilon.
\]
Pick \( x_n \in A_\epsilon \) such that \( I(\eta_n) = g(\eta_n(1, x_n)) \), \( n \in \mathbb{N} \). By the compactness of \( A_\epsilon \) one has, along a subsequence when necessary, \( x_n \to x_0 \) for some \( x_0 \in A_\epsilon \) as well as \( \eta_0(1, x_0) \in C \), since
\[
g(\eta_0(1, x)) = \lim_{n \to +\infty} g(\eta_n(1, x)) \leq \lim_{n \to +\infty} g(\eta_n(1, x_n)) = g(\eta_0(1, x_0)) \quad \forall x \in A_\epsilon.
\]
If $J_0 := \{ j : \eta_0(1, x_0) \in V_j \}$ then
\[
f_0^0(\eta_0(1, x_0); u(\eta_0(1, x_0))) \leq \sum_{j=1}^{m} \chi_j(\eta_0(1, x_0))(1 + h(\|\eta_0(1, x_0)\|))f_0^0(\eta_0(1, x_0); v_j)
= \sum_{j \in J_0} \chi_j(\eta_0(1, x_0))(1 + h(\|\eta_0(1, x_0)\|))f_0^0(\eta_0(1, x_0); v_j)
\leq -3\epsilon \sum_{j \in J_0} \chi_j(\eta_0(1, x_0)) = -3\epsilon
\]
thanks to (27) besides Proposition 2.1.1 in [3]. Thus, likewise before, we can find an $r_0 > 0$ such that
\[
f_0^0(w; u(z)) < -3\epsilon \quad \forall w, z \in B(\eta_0(1, x_0), r_0).
\]
In view of [18, Theorem 2.3.7], for any $n \in \mathbb{N}$ sufficiently large there exist $z_n \in [\eta_0(1, x_n), \eta_n(1, x_n)]$, $z_n^* \in \partial f(z_n)$ satisfying
\[
f(\eta_n(1, x_n)) - f(\eta_0(1, x_n)) = \frac{1}{n} \langle z_n^*, u(\eta_0(1, x_n)) \rangle \leq \frac{1}{n}f_0^0(z_n; u(\eta_0(1, x_n))).
\]
It is evident that
\[
f_i(\eta_0(1, x_n)) - f_i(\eta_0(1, x_n)) \leq \epsilon \delta_\eta(\eta_0(1, x_n), \eta_0(1, x_n))
\leq \frac{\epsilon}{n} \int_0^1 \frac{\|u(\eta_0(1, x_n))\|}{1 + h(\|\eta_0(1, x_n) + \frac{\tau}{n}u(\eta_0(1, x_n))\|)} d\tau, \quad i = 1, 2, \quad n \in \mathbb{N}.
\]
Gathering (35) and (36) together yields
\[
I(\eta_n) - I(\eta_0) \leq g(\eta_0(1, x_n)) - g(\eta_0(1, x_n))
\leq \frac{1}{n} \left( f_0^0(z_n; u(\eta_0(1, x_n))) + 2\epsilon \int_0^1 \frac{\|u(\eta_0(1, x_n))\|}{1 + h(\|\eta_0(1, x_n) + \frac{\tau}{n}u(\eta_0(1, x_n))\|)} d\tau \right).
\]
Since $z_n, \eta_0(1, x_n) \in B(\eta_0(1, x_0), r_0)$ for $n$ large enough, due to (34) besides (30) we actually have
\[
\limsup_{n \to +\infty} \frac{I(\eta_n) - I(\eta_0)}{1/n} \leq -\epsilon,
\]
against inequality (33). Therefore, Claim (25) holds true.

Choose any $x_\epsilon$ in $C$ complying with (25). By (24) one has $x_\epsilon \notin B$, i.e., (10) is fulfilled. Observe that $x_\epsilon = \eta_0(1, \hat{x})$ for some $\hat{x} \in A_\epsilon$ and that $\hat{x} \in F_\epsilon$, because otherwise $x_\epsilon = \hat{x} \in A_\epsilon \setminus F_\epsilon \subseteq B'$, which would contradict (24). Consequently, on account of (17),
\[
\delta_\eta(x_\epsilon, F) \leq \delta_\eta(\eta_0(1, \hat{x}), \eta(1, \hat{x})) + \delta_\eta(\hat{x}, F) \leq \frac{3\epsilon}{2},
\]
\[
\delta_\eta(x_\epsilon, A_\epsilon) \leq \delta_\eta(\eta_0(1, \hat{x}), \eta(1, \hat{x})) \leq \frac{\epsilon}{2}.
\]
This shows (11) and (12). Thus, the conclusion is achieved once we verify (9).

Bearing in mind the choice of $x_\epsilon$, (16), (8), besides the properties of $f_1$ and $f_2$, one achieves
\[
f(x_\epsilon) \leq g(x_\epsilon) = I(\eta_0) \leq I(\tilde{\eta}) = \max_{x \in A_\epsilon} g(x) < c + \frac{\epsilon^2}{8} + \epsilon^2 + \frac{\epsilon^2}{8} = c + \frac{5\epsilon^2}{4}.
\]
Through (14) we then get
\[ f(x_\varepsilon) = I(\eta_0) - f_1(x_\varepsilon) - f_2(x_\varepsilon) \geq \inf_{\eta \in \mathcal{L}} f(\eta) - \varepsilon^2 - \frac{\varepsilon^2}{8} \geq c - \frac{\varepsilon^2}{8}. \]
Combining the above inequalities yields (9) and completes the proof when (7) holds.
Let us now come to the case
\[ \inf_{x \in \mathcal{F}} f(x) < c. \]
(37)
Pick an \( \varepsilon > 0 \) and choose \( A_\varepsilon \) in \( \mathcal{F} \) fulfilling
\[ c \leq \max_{x \in A_\varepsilon} f(x) < c + \frac{\varepsilon^2}{4}. \]
(38)
We shall seek a point \( x_\varepsilon \in X \setminus B \) such that
\[ c \leq f(x_\varepsilon) < c + \frac{\varepsilon^2}{4}, \]
(39)
\[ (1 + h(||x_\varepsilon||))f^0(x_\varepsilon; z) \geq -3\varepsilon||z|| \quad \forall z \in X, \]
(40)
\[ \delta_h(x_\varepsilon, A_\varepsilon) \leq \frac{\varepsilon}{2}, \]
(41)
which obviously provides a sequence \( \{x_n\} \subseteq X \setminus B \) enjoying properties (i_1), (i_2), and (i_4). Denote by \( \mathcal{L} \) the space of all \( \eta \in C^0([0, 1] \times X, X) \) such that
\[ \eta(t, x) = x \quad \forall (t, x) \in ([0] \times X) \cup ([0, 1] \times B), \quad \sup_{(t, x) \in [0, 1] \times X} \delta_h(\eta(t, x), x) < +\infty. \]
One clearly has \( \eta(A'_\varepsilon) \in \mathcal{F} \) for every \( \eta \in \mathcal{L} \). Moreover \( \mathcal{L} \), when equipped with the uniform distance \( \rho \), is complete. If
\[ I(\eta) := \max_{z \in \eta(A'_\varepsilon)} f(z), \quad \eta \in \mathcal{L}, \]
then the function \( I : \mathcal{L} \to \mathbb{R} \) is evidently lower semi-continuous, bounded from below by \( c \), and, due to (38), satisfies (15). Through Theorem 2.1 we thus get an \( \eta_0 \) in \( \mathcal{L} \) complying with (16)–(18). Define
\[ C := \left\{ w \in \eta_0(A'_\varepsilon) : f(w) = I(\eta_0) = \max_{z \in \eta_0(A'_\varepsilon)} f(z) \right\}. \]
The set \( C \) turns out to be nonempty, compact, and such that
\[ f(x) \geq \inf_{\eta \in \mathcal{L}} I(\eta) \geq c \quad \forall x \in C. \]
Exploiting (5), (37), and the above inequality yields
\[ \sup_{x \in B} f(x) \leq \inf_{x \in \mathcal{F}} f(x) < c \leq \min_{x \in \mathcal{C}} f(x), \]
namely \( B \cap C = \emptyset \). Now, the same reasoning adopted before gives a point \( x_\varepsilon \) in \( C \) fulfilling (40). Since \( x_\varepsilon \) belongs to \( C \), from (16) and (38) we immediately infer (39). Indeed,
\[ c \leq \min_{x \in C} f(x) \leq f(x_\varepsilon) = I(\eta_0) \leq I(\tilde{\eta}) = \max_{x \in \mathcal{A}_\varepsilon} f(x) < c + \frac{\varepsilon^2}{4}. \]
Finally, (41) can be achieved exactly as in the preceding case.

**Remark 3.** This proof is patterned after that of [15, Theorem 3.1]. The main difference lies in showing Claim (25), which represents a key point of the argument.
The next result is a straightforward, although meaningful, consequence of Theorem 3.1. It extends Theorem 1.bis and Corollary 2 in [9], besides Corollaries 9 and 10 at p. 145 of [6], to locally Lipschitz continuous functions $f$ fulfilling a weak Palais-Smale compactness condition.

**Theorem 3.2.** Let $(a_1)-(a_4)$ be satisfied. Suppose that either $(PS)_b^c$ holds or $F$ is bounded and $(PS)_{b,c}^c$ holds, according to whether $\inf_{x \in F} f(x) < c$ or $\inf_{x \in F} f(x) = c$. Then $K_c(f) \neq \emptyset$. If, moreover, $\inf_{x \in F} f(x) = c$, then $K_c(f) \cap F \neq \emptyset$.

**Proof.** Theorem 3.1 provides a sequence $\{x_n\} \subseteq X \setminus B$ with properties (i$_1$)-(i$_4$). Observe that (i$_2$) actually means

$$\lim_{n \to +\infty} (1 + h(\|x_n\|))m_f(x_n) = 0.$$  

(42)

In fact, by [21, Lemma 1.3], for any $n \in \mathbb{N}$ there exists a $z_n^* \in X^*$ such that $\|z_n^*\|_{X^*} \leq 1$ and

$$\epsilon_n^{-1}(1 + h(\|x_n\|))f^0(x_n; z) \geq \langle z_n^*, z \rangle \quad \forall z \in X.$$  

Hence,

$$\epsilon_n(1 + h(\|x_n\|))^{-1}z_n^* \in \partial f(x_n),$$

which leads to

$$(1 + h(\|x_n\|))m_f(x_n) \leq \epsilon_n\|z_n^*\| \leq \epsilon_n, \quad n \in \mathbb{N}.$$  

Now, (42) is an immediate consequence of $\epsilon_n \to 0^+$. Bearing in mind (p$_1$)-(p$_2$), for bounded $F$ one has $\delta_h(x_n, F) \to 0$ if and only if $d(x_n, F) \to 0$. By the weak Palais-Smale condition we may thus assume that $x_n \to x$ in $X$, where a subsequence is considered when necessary. At this point, the conclusion comes from (i$_1$)-(i$_3$). In fact, (i$_2$) and the upper semi-continuity of $f^0$ yield $f^0(x; z) \geq 0$ for all $z \in X$, namely $x \in K_c(f)$. \hfill $\Box$

Making suitable choices of $\mathcal{F}$, $B$, and $F$, more refined versions of several recent results can easily be obtained through Theorem 3.2. By way of example, we find the result below, which includes Theorem (1.bis) in [11], Corollary 1 and Theorem 2 of [19] (vide also [18, Section 2.1]), as well as Theorem 6 in [13]. Keep the same notation introduced at the beginning of this section.

**Theorem 3.3.** Let $(a_1)$ and $(a_4)$ be satisfied. Suppose that:

(a$_5$) There exists a closed subset $F$ of $X$ complying with $(\gamma(Q) \cap F) \setminus \gamma_0(Q_0) \neq \emptyset$ for all $\gamma \in \Gamma$ and, moreover, $\sup_{z \in Q_0} f(\gamma_0(z)) \leq \inf_{x \in F} f(x).

(a$_6$) Setting $c := \inf_{\gamma \in \Gamma} \max_{x \in Q} f(\gamma(x))$, either $(PS)_c^b$ holds or $F$ is bounded and $(PS)_{b,c}^c$ holds, according to whether $\inf_{x \in F} f(x) < c$ or $\inf_{x \in F} f(x) = c$.

Then the conclusion of Theorem 3.2 is true.

**Remark 4.** Taking into account Proposition 1, the weak Palais-Smale condition $(PS)^h_c$, that appears in Theorems 3.2 and 3.3, can be replaced by assumption $(C)^c_c$.

**Remark 5.** The above result reveals useful to get non-smooth variants of structure or multiplicity results for $C^1$ functions where, instead of $(PS)_c$, a weak Palais-Smale condition is requested; see [1].
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