Dynamics of Toroidal Spiral Strings around Five-dimensional Black Holes

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We examine the separability of the Nambu-Goto equation for test strings in a shape of toroidal spiral in a five-dimensional Kerr-AdS black hole. In particular, for a ‘Hopf loop’ string which is a special class of the toroidal spiral strings, we show the complete separation of variables occurs in two cases, Kerr background and Kerr-AdS background with equal angular momenta. We also obtain the dynamical solution for the Hopf loop around a black hole and for the general toroidal spiral in Minkowski background.

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I. INTRODUCTION

Recently, much attention has been focused on the study of higher-dimensional spacetimes in relation to ideas of AdS/CFT correspondence [1, 2, 3] and braneworld [4, 5, 6, 7]. In the context of the topics, higher-dimensional black hole spacetimes would play crucial roles. In the braneworld scenarios, since matter fields are confined on the four-dimensional membrane while the gravity propagates in all dimensions, the observation of phenomena related to the gravity is the only way to see the existence of extra-dimensions. For this reason, physical properties in the strong gravitational field around higher-dimensional black holes should be illuminated to explore the higher-dimensional spacetime.

A typical exact solution of a higher-dimensional rotating black hole is a generalization of the Kerr metric found by Myers and Perry [8], and the solution was extended to the Kerr-AdS black hole solution [9, 10]. Furthermore, it has been found that there are several black holes which have a variety of the horizon topology such as black strings and black rings (see [11] for a review). It has been recognized that the higher-dimensional black holes possess richer properties than the...
four-dimensional black holes.

Though there are differences in properties of black holes in four-dimensions and in higher dimensions, it is known that the Kerr-AdS black holes in general dimensions have a remarkable common property, that is, separation of variables in the geodesic Hamilton-Jacobi equation. This is because the Kerr-AdS black hole metrics in general dimensions admit rank-2 Killing tensor fields.

A geodesic particle plays an important role as a probe of black hole spacetime because it gives us information of the geometry around the black hole. In addition to a particle, a string is regarded as a probe of geometry of a higher-dimensional black hole. It has been also found that separation of variables in the Hamilton-Jacobi equation for the stationary strings occurs in the Kerr-AdS metric.

In this paper we study dynamics of a Nambu-Goto string in five-dimensional spacetimes. We consider a special string that we call a toroidal spiral string, which has a spiral shape along a circle on a time slice. The toroidal spiral string is a cohomogeneity-one string associated with the Killing vector field which generates a combination of two commutable rotational isometry. Here we discuss dynamics of the toroidal spiral string in the Kerr-AdS background, and examine the separability of the Hamilton-Jacobi equation for the string.

In particular, we study ‘Hopf loop’ strings, which are in a special class of the toroidal spirals. We consider five-dimensional spacetimes of which time slices are foliated by the $S^3$. It is well known that the $S^3$ is a Hopf fibration, i.e., a twisted $S^1$ bundle over $S^2$. We call a closed string which lies along the Hopf fiber a Hopf loop. We show that separation of variables occurs for the Hopf loop in the Kerr metric with two independent rotations, and in the Kerr-AdS metric with the equal rotations in five-dimensions. We also analyze dynamics of the Hopf loop in the latter black hole metrics, and general toroidal spirals in five-dimensional flat metric, where the separation of variables occurs.

This paper is organized as follows. In Sec. II, we briefly review the formalism of a cohomogeneity-one string. In Sec. III, we introduce the toroidal spiral strings in the five-dimensional Kerr-AdS black hole spacetime, and study the separability of the Hamilton-Jacobi equation corresponding to the Nambu-Goto equation for the string. In Sec. IV, We show that complete separation of variables occurs for the Hopf loop strings in two cases of the black hole geometry and discuss dynamical properties of the Hopf loops in the five-dimensional black holes. We obtain solutions of general toroidal spirals explicitly in five-dimensional Minkowski background in Sec. V. Finally, we summarize results in Sec. VI. We use the sign convention $-++++$ for the metric, and units in
which $c = G = 1$.

II. COHOMOGENEITY-ONE STRING MOTION

A test string motion is described by a two-dimensional worldsheet $\Sigma$ in a target spacetime $(\mathcal{M}, g_{MN})$. The embedding of $\Sigma$ in $\mathcal{M}$ is determined by the parametric equation

$$y^M = y^M(\zeta^a),$$

where $y^M$ are coordinates of $\mathcal{M}$ and $\zeta^a(\zeta^0 = \tau, \zeta^1 = \sigma)$ are coordinates of $\Sigma$. We assume that the dynamics of the string is governed by the Nambu-Goto action

$$S_{NG} = -\mu \int_{\Sigma} d^2\zeta \sqrt{-\gamma},$$

where parameter $\mu$ denotes the string tension and $\gamma$ is the determinant of the induced metric on $\Sigma$ given by

$$\gamma_{ab} = g_{MN} \frac{\partial y^M}{\partial \zeta^a} \frac{\partial y^N}{\partial \zeta^b}.$$  

Let us suppose that the background spacetime metric $g_{MN}$ possesses Killing vector fields. If one of the Killing vector fields, say $\xi$, is tangent to $\Sigma$, we call the string a cohomogeneity-one string associated with the Killing vector field $\xi$ \cite{21, 22}. A stationary string is one of the cohomogeneity-one string. The Killing vector field associated with the stationary string is timelike on $\Sigma$.

The group action of isometry generated by $\xi$ defines Killing orbits in $\mathcal{M}$. One can consider a set $\mathcal{O}$ of the Killing orbits as a quotient space of $\mathcal{M}$, on which the metric is naturally introduced by the projection tensor with respect to $\xi$,

$$h_{MN} = g_{MN} - \xi_M \xi_N / F,$$  

where $F = \xi_M \xi_M$. The action for the cohomogeneity-one string associated with $\xi$ is reduced to the geodesic action for a curve $C$ in $\mathcal{O}$ with respect to the metric $Fh_{\mu\nu}$ \cite{21, 22}

$$S = -\mu \int_C \sqrt{-Fh_{\mu\nu} dx^\mu dx^\nu};$$

where $x^\mu$ are coordinates on $\mathcal{O}$. Therefore, the problem for finding solution of cohomogeneity-one strings associated with Killing vector field $\xi$ reduces to the problem for solving geodesic equations in the $(\dim \mathcal{M} - 1)$-dimensional space $(\mathcal{O}, Fh_{\mu\nu})$. Signature of $Fh_{\mu\nu}$ is Euclidean for a timelike $\xi$, and is Lorentzian for a spacelike $\xi$. There are a lot of works dealing with the cohomogeneity-one strings in a variety of context \cite{21, 22, 23, 24, 25}. 
III. TOROIDAL SPIRAL STRINGS IN FIVE-DIMENSIONAL BLACK HOLES

We consider now the motion of a cohomogeneity-one string in a five-dimensional rotating black hole with a cosmological constant. The corresponding metric of these five-dimensional Kerr-AdS black holes in the Boyer-Lindquist type coordinates [10] is

\[ ds^2 = -\frac{\Delta_\theta \Xi_r dt^2}{\Xi_a \Xi_b} + \frac{2M}{\Sigma} \left( \frac{\Delta_\theta dt}{\Xi_a \Xi_b} - \nu \right)^2 + \frac{\Sigma dr^2}{\Delta_r} + \frac{\Sigma d\theta^2}{\Delta_\theta} + \frac{r^2 + a^2 \sin^2 \theta d\Phi^2 + \frac{r^2 + b^2 \cos^2 \theta d\Psi^2}{\Xi_b}}{\Xi_a}, \]

with

\[ \Xi_a = 1 - a^2 \lambda^2, \quad \Xi_b = 1 - b^2 \lambda^2, \quad \Xi_r = 1 + \lambda^2 r^2, \]

\[ \Delta_r = \frac{(r^2 + a^2)(r^2 + b^2)(1 + \lambda^2 r^2)}{r^2} - 2M, \quad \Delta_\theta = 1 - a^2 \lambda^2 \cos^2 \theta - b^2 \lambda^2 \sin^2 \theta, \]

\[ \nu = a \sin^2 \theta \frac{d\Phi}{\Xi_a} + b \cos^2 \theta \frac{d\Psi}{\Xi_b}, \quad \Sigma = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \]

where \( M \) is the mass parameter, and \( a \) and \( b \) are two independent rotation parameters. The parameter \( \lambda \) is connected with the cosmological constant \( \Lambda \) as \( \lambda^2 = -\Lambda/6 \). Note that the metric possesses three commutable Killing vector fields \( \partial_t, \partial_\Phi \) and \( \partial_\Psi \).

Now, let us concentrate on a special cohomogeneity-one string. We suppose that a worldsheet of a string is tangent to the Killing vector field

\[ \xi = \partial_\Phi + \alpha \partial_\Psi, \]

where \( \alpha \) is a constant. Here we name the string as a toroidal spiral string since the string has a toroidal spiral shape on a snapshot. The constant \( \alpha \) is interpreted as a ratio of winding number in the directions of two azimuthal angles \( \Psi \) and \( \Phi \). If \( \alpha \) takes a rational number, the toroidal spiral string is closed.

We make the coordinate transformation \((\Phi, \Psi) \rightarrow (\phi, \psi)\), which is defined as

\[ \phi = \Phi, \quad \psi = \Psi - \alpha \Phi, \]

so that \( \xi = \partial_\phi \). Hence the quotient space \( O \) with respect to \( \xi \) is covered by the coordinate system \((t, r, \theta, \psi)\). The motion of the toroidal spiral string is regarded as a geodesic motion in \( O \) with the metric \( F h_{\mu\nu} \). The action for the geodesics is regarded as a geodesic motion in \( O \) with the metric \( F h_{\mu\nu} \). The action for the geodesics is equivalent to the action

\[ S = -\mu \int d\tau \left( \frac{1}{2N} F h_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{N}{2} \right), \]
where $N$, an arbitrary function of $\tau$, denotes a Lagrange multiplier which is related to the reparametrization invariance of the curve.

In order to study the geodesics, it is convenient to use the Hamilton-Jacobi method. The corresponding Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial \tau} + \frac{N}{2} \frac{h_{\mu\nu}}{F} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = 0,$$

where $S$ denotes Hamilton’s principal function and $F$ is given by

$$F = \frac{2M}{\Sigma} \left( \frac{a \sin^2 \theta}{\Xi_a} + \frac{b \cos^2 \theta}{\Xi_b} \right)^2 + \frac{(r^2 + a^2) \sin^2 \theta}{\Xi_a} + \alpha \frac{(r^2 + b^2) \cos^2 \theta}{\Xi_b}. \quad (13)$$

The contravariant components of the projection metric (4) are given by

$$h_{tt} = -\frac{\Sigma \Xi_a \Xi_b (r^2 + a^2)(r^2 + b^2) + 2MX}{r^2 \Sigma \Delta_\theta \Delta_r}, \quad (15)$$

$$h_{\psi\psi} = \frac{1}{r^2 \Sigma \Delta_r \cos^2 \theta \sin^2 \theta} \left[ F \Xi_a \Xi_b \Sigma - 2M \Delta_\theta \left[ \frac{(r^2 + a^2) \sin^2 \theta}{\Xi_a} + \alpha \frac{(r^2 + b^2) \cos^2 \theta}{\Xi_b} \right] \right], \quad (16)$$

$$h_{t\psi} = \frac{2MY}{r^2 \Sigma \Delta_r}, \quad h_{rr} = \frac{\Delta_r}{\Sigma}, \quad h_{\theta\theta} = \frac{\Delta_\theta}{\Sigma}, \quad (17)$$

where

$$X = a^2 \Xi_b (r^2 + b^2) \sin^2 \theta + b^2 \Xi_a (r^2 + a^2) \cos^2 \theta, \quad Y = -b(r^2 + a^2) + \alpha a(r^2 + b^2). \quad (18)$$

Since the metric $F h_{\mu\nu}$ admits the two Killing vector fields $\partial_t$ and $\partial_\psi$, the momenta $p_t$ and $p_\psi$ take constant values, say $-E$ and $L$, respectively. Since $t$ and $\psi$ are cyclic coordinates, we can obtain a complete solution of the equation (13) of the form

$$S = \frac{1}{2} \mu^2 \chi - Et + L\psi + S_{r\theta}, \quad (19)$$

where $S_{r\theta}$ is a function of $r$ and $\theta$, and $\chi$ is function of parameter $\tau$ such that $\dot{\chi} = N$. Then, the system reduces to a particle system in two-dimensions described by $S_{r\theta}(r, \theta)$. Here, we assume complete separation of variables of $S_{r\theta}$ in the form

$$S_{r\theta}(r, \theta) = S_r(r) + S_\theta(\theta), \quad (20)$$

where $S_r$ and $S_\theta$ are functions of $r$ and $\theta$, respectively.

Substituting the expression for Hamilton’s principal function into the Hamilton-Jacobi equation,
we can write down it in the form
\[
\left( \frac{r^2 + a^2}{r^2} \right)(r^2 + b^2) \Xi_a \Xi_b - 2M(r^2 + a^2b^2\lambda^2) \right) \frac{2(1 - a^2\lambda^2)(1 - b^2\lambda^2)}{\lambda^2(a^2 + b^2) + \lambda^2(a^2 - b^2) \cos 2\theta - 2} E^2 \lambda^2
\]
\[+ 4abM \alpha \Xi_r + (\alpha^2 - 1)(b^2 - a^2)r^2 \Xi_r + a^2b^2\Xi_r \left[ (\alpha^2 - 1)(1 + \lambda^2(a^2 - b^2)) - 2\alpha^2 \right] \]
\[+ a^2(1 + \alpha^2r^2\lambda^2)(a^2\Xi_r - 2M) + b^2(\alpha^2 + \alpha^2r^2\lambda^2)(b^2\Xi_r - 2M) + \left( \frac{\alpha^2 \Xi_a}{\sin^2 \theta} + \frac{\Xi_b}{\cos^2 \theta} \right) L^2 \]
\[+ 4MY \frac{E^2}{r^2} \Delta_r \left( \frac{dS_r}{dr} \right)^2 + \Delta_\theta \left( \frac{dS_\theta}{d\theta} \right)^2 = -\mu^2 F \Sigma. \] (21)

We find that the separability of variables \( r \) and \( \theta \) in this equation crucially depends only on the term in the right-hand-side. The explicit form of the term is given by
\[
\mu^2 F \Sigma = \mu^2 \left( \frac{r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta}{\Xi_b} \right) \left( \frac{r^2 + a^2 \cos^2 \theta}{\Xi_a} \right)
\]
\[+ 2M \mu^2 \left( \frac{\alpha b \cos^2 \theta}{\Xi_b} + \frac{\alpha \sin^2 \theta}{\Xi_a} \right)^2. \] (22)

The complete separation of variables in the Hamilton-Jacobi equation depends on the parameter \( \alpha \) which describes winding ratio of the string, and parameters of the background geometry. In a particular case of toroidal spirals with \( \alpha^2 = 1 \), the separation of variables occurs in two cases of the black hole geometry. We see this fact in the following section.

### IV. HOPF LOOPS

In this section, we concentrate on the case of \( \alpha^2 = 1 \). As is shown below, the toroidal spiral string of this case lies along a fiber of the Hopf fibration of \( S^3 \), then we call the string with \( \alpha^2 = 1 \) Hopf loop string.

Let us suppose a timeslice of the spacetime with the metric (20). The time slice is foliated by \( r = \text{const.} \) surfaces, which have the topology of \( S^3 \), with the metric
\[
dS^2_{S^3} = g_{\theta \Phi} d\theta^2 + g_{\Phi \Psi} d\Phi^2 + g_{\Psi \Psi} d\Psi^2 + 2g_{\Phi \Psi} d\Phi d\Psi
\]
\[= \Sigma \Delta_\theta \left( r^2 + a^2 \right) \Xi_a \Xi_b \left( \frac{r^2 + a^2}{\Xi_a} + \frac{2Ma^2}{\Sigma \Xi_a^2} \sin^2 \theta \right) \sin^2 \theta d\phi^2
\]
\[+ \left( r^2 + b^2 \right) \Xi_b \Xi_b \left( \frac{2Mb^2}{\Sigma \Xi_b^2} \cos^2 \theta \right) \cos^2 \theta d\psi^2 + \frac{4Ma b}{\Sigma \Xi_a \Xi_b} \sin^2 \theta \cos^2 \theta d\Phi d\Psi. \] (23)

The metric describes \( S^3 \) that is deformed from the round metric by the continuous parameters \( M, a, b \) and \( \lambda \). The metric (23) is rewritten as
\[
dS^2_{S^3} = \frac{1}{4} \left[ g_{\theta \Phi} d\theta_E^2 + \frac{4}{F} (g_{\Phi \Phi} g_{\Psi \Psi} - g_{\Phi \Psi}^2) d\phi_E^2 \right] + \frac{F}{4} d\psi_E + \frac{1}{F} (g_{\Psi \Phi} - g_{\Phi \Phi}) d\phi_E, \] (24)
where
\[ \theta = \frac{\theta_E}{2}, \quad \Phi = \frac{1}{2}(\psi_E - \phi_E), \quad \Psi = \frac{1}{2}(\psi_E + \phi_E), \quad (25) \]

and
\[ F = g_{\Phi\Phi} + g_{\Psi\Psi} + 2g_{\Phi\Psi}. \quad (26) \]

The metric (24) provides the scheme of the Hopf fibration of \( S^3 \), where the projection along the integral curves of the Killing vector field \( \partial_{\psi_E} \) defines a map from \( S^3 \) to \( S^2 \). The first term in the metric (24) is the metric on the \( S^2 \) base space, and the second term is the metric on the \( S^1 \) fiber of the Hopf fibration of \( S^3 \). If \( a = b \), the base space is round \( S^2 \), otherwise the base space \( S^2 \) is deformed by the rotation in the direction of \( \phi_E \).

On each timeslice, the toroidal spiral with \( \alpha = 1 \), which is associated with the Killing vector field \( \xi = \partial_\Phi + \partial_\Psi \), lies on a fiber of the Hopf fibration of \( S^3 \) because \( \xi = 2\partial_{\psi_E} \), i.e., the tangent vector field of the string generates the \( S^1 \) fiber of \( S^3 \). Therefore, we refer the toroidal spiral string with \( \alpha = 1 \) as a Hopf loop. The toroidal spiral string with \( \alpha = -1 \) is also the Hopf loop which is obtained by the coordinate reflection \( \Psi \mapsto -\Psi \).

When we consider the Hopf loop around the black hole, we see that the Hamilton-Jacobi equation written in the Boyer-Lindquist coordinates can be separable for two cases: (i) the vanishing cosmological constant \( i.e., \) the background is a Kerr black hole, (ii) the black hole with non-zero cosmological constant and two equal angular momenta. In what follows we discuss the separability in the two cases separately.

### A. Five-dimensional Kerr background

For a Hopf loop in the Kerr background, \( i.e., \) vanishing cosmological constant, the equation (21) is separated into two independent equations of \( r \) and \( \theta \) because \( \mu^2 F \Sigma \) consists of three terms as
\[ \mu^2 F \Sigma = \mu^2(r^2 + a^2)(r^2 + b^2) + \mu^2(a^2 - b^2)\cos^2 \theta \sin^2 \theta + 2M\mu^2(a\sin^2 \theta + b\cos^2 \theta)^2, \quad (27) \]
where the first term depends only on \( r \), and the second and third on \( \theta \). Therefore the separated equations are given by
\[
\Delta_r \left( \frac{dS_r}{dr} \right)^2 - \frac{(r^2 + a^2)(r^2 + b^2)(r^2 + 2M) - 2Mr^4E^2 + [(a^2 - b^2)^2 - 2M(a - b)^2]L^2}{(r^2 + a^2)(r^2 + b^2) - 2Mr^2} \frac{r^2 \Delta_r}{\mu^2} \\
+ 4M[-b(r^2 + a^2) + L(r^2 + b^2)]EL \frac{r^2 \Delta_r}{\mu^2} + \mu^2(r^2 + a^2)(r^2 + b^2) = -K, \quad (28)
\]
and
\[
\Delta_\theta \left( \frac{dS_\theta}{d\theta} \right)^2 - (a^2 \cos^2 \theta + b^2 \sin^2 \theta) E^2 + \left( \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \right) L^2 \\
+ \mu^2 \left[ 2M(a \sin^2 \theta + b \cos^2 \theta)^2 + (a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta \right] = K,
\]
where \( K \) is a separation constant. The complete separability implies that the metric \( Fh_{\mu\nu} \) admits a Killing tensor field \( K^{\mu\nu} \) which gives the quadratic constant of motion \( K = K^{\mu\nu} p_\mu p_\nu \). Using equation (29), one obtains the irreducible Killing tensor field
\[
K^{\mu\nu} = \Delta_\theta \delta^\mu_\theta \delta^\nu_\theta - (a^2 \cos^2 \theta + b^2 \sin^2 \theta) \delta^\mu_t \delta^\nu_t + \left( \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \right) \delta^\mu_\psi \delta^\nu_\psi \\
- \left[ 2M(a \sin^2 \theta + b \cos^2 \theta)^2 + (a^2 - b^2)^2 \sin^2 \theta \cos^2 \theta \right] h^{\mu\nu}.
\]
(30)

Therefore the motion of the Hopf loop in Kerr background can be integrable.

**B. Five-dimensional Kerr-AdS with equal angular momenta**

For a Hopf loop with \( \alpha = \pm 1 \) in Kerr-AdS background with \( a = \pm b \), the expression (21) is also separated into two independent equations of \( r \) and \( \theta \) because \( \mu^2 F_{\Sigma} \) is simply the function of \( r \) in the form
\[
\mu^2 F_{\Sigma} = \frac{\mu^2}{\Xi_a} \left[ \Xi_a(r^2 + a^2)^2 + 2Ma^2 \right].
\]
(31)

Then, we obtain that
\[
\Delta_r \left( \frac{dS_r}{dr} \right)^2 = \frac{\Xi_a(r^2 + a^2)^3 + 2Ma^2(r^2 + a^2)}{r^2 \Delta_r} E^2 + \frac{\mu^2[\Xi_a(r^2 + a^2)^2 + 2Ma^2]}{\Xi_a^2} = -K,
\]
(32)
and
\[
(1 - a^2 \lambda^2) \left[ \left( \frac{dS_\theta}{d\theta} \right)^2 + \left( \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \right) L^2 \right] = K.
\]
(33)

Thus, the Hopf loop motions in the background of the Kerr-AdS black hole with two equal angular momenta are completely integrable. By using the above expression, one can read the form of the Killing tensor field
\[
K^{\mu\nu} = (1 - a^2 \lambda^2) \left[ \delta^\mu_\theta \delta^\nu_\theta + \left( \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} \right) \delta^\mu_\psi \delta^\nu_\psi \right].
\]
(34)

Here, let us see the metric \( Fh_{\mu\nu} \) on the quotient space \( \mathcal{O} \). As is mentioned above, the metric of the base space in (21) becomes round \( S^2 \) in the case \( a = b \), and \( F \), the norm of \( \xi \), is a function of \( r \).
Then, the metric $F_{\mu\nu}$ admits SO(3) isometry group. Therefore, we can restrict our attention to study geodesics confined in the equatorial plane, i.e., $\theta_E = 2\theta = \pi/2$, without loss of generality\(^1\).

Thus, setting
\[
\frac{dS_\theta}{d\theta} = 0 \quad \text{and} \quad \theta = \pi/4
\]
in (33), we have
\[
K = 4L^2\Xi_a.
\]

Then the Hamilton-Jacobi equation (32) becomes
\[
\Delta_r \left( \frac{dS_r}{dr} \right)^2 - \frac{a^2 \Delta_a}{r^2 \Delta_r} (r^2 + a^2) E^2 + \frac{\mu^2 a^2 \Delta_a}{\Xi_a^2} + 4L^2\Xi_a = 0,
\]
where
\[
a^2 \Delta_a := (r^2 + a^2) \Xi_a + 2Ma^2.
\]

We can obtain Hamilton’s principal function in the form
\[
S = \frac{1}{2} \mu^2 \chi - Et + L\psi + \sigma_r \int dr \sqrt{\Theta_r},
\]
where $\Theta_r$ is given by
\[
\Theta_r = \frac{1}{\Delta_r} \left[ \frac{a^2 \Delta_a}{r^2 \Delta_r} (r^2 + a^2) E^2 - \frac{\mu^2 a^2 \Delta_a}{\Xi_a^2} - 4L^2\Xi_a \right],
\]
and $\sigma_r = \pm 1$. By setting the derivatives of $S$ with respect to $\mu, E$ and $L$ equal to zero, we get a solution for the Hamilton-Jacobi equation in the following form
\[
\chi = \sigma_r \int dr \frac{a^2 \Delta_a}{\Delta_r \Xi_a \sqrt{\Theta_r}},
\]
\[
t = \sigma_r \int dr \frac{a^2 \Delta_a (r^2 + a^2) E}{r^2 \Delta^2_r \sqrt{\Theta_r}},
\]
\[
\psi = \sigma_r \int dr \frac{4L\Xi_a}{\Delta_r \sqrt{\Theta_r}}.
\]

By differentiating these equations with respect to $\tau$, one obtains the first order differential equations
\[
\frac{1}{N} \dot{\chi} = \sigma_r \frac{\Delta_r \Xi_a^2 \sqrt{\Theta_r}},
\]
\[
\frac{1}{N} \dot{t} = \frac{(r^2 + a^2) \Xi_a^2 E}{r^2 \Delta_r},
\]
\[
\frac{1}{N} \dot{\psi} = \frac{4L\Xi_a^3}{a^2 \Delta_a}.
\]

\(^1\) Since the symmetry of the quotient space $(O, F_{\mu\nu})$ are enhanced in the case $a = b$, the Killing tensor field (34) is reducible.
If we choose the Lagrange multiplier $N$ as

$$N^2 = \frac{a^2 \Delta \alpha \Delta r^2}{\Xi \alpha (r^2 + a^2)},$$

(47)

the radial motion is determined by

$$\dot{r}^2 + V_{\text{eff}} = E^2,$$

(48)

where the effective potential $V_{\text{eff}}$ is given by

$$V_{\text{eff}} = \frac{\mu^2 r^2 \Delta r}{(r^2 + a^2) \Xi \alpha} + \frac{4r^2 \Delta r \Xi \alpha L^2}{a^2 \Delta \alpha (r^2 + a^2)},$$

(49)

The effective potential shapes are given in FIG. 1.

FIG. 1: The effective potential for radial motion of a Hopf loop in the five-dimensional Kerr-AdS metric with equal angular momenta. The mass parameter is chosen as $2M = 1$. The parameters in the panels are the following.

(A) $\Lambda = 0$, $a = 1/4$; (i) $L^2 = 1$, (ii) $L^2 = 6$, (iii) $L^2 = 20$, (iv) $L^2 = 50$.

(B) $\Lambda = -1/100$, $a = 1/4$; (i) $L^2 = 1$, (ii) $L^2 = 6.9$, (iii) $L^2 = 15$, (iv) $L^2 = 30$.

(C) $\Lambda = 1/100$, $a = 1/4$; (i) $L^2 = 1$, (ii) $L^2 = 7$, (iii) $L^2 = 20$, (iv) $L^2 = 140$.

(D) $\Lambda = 0$, $a = 1/2$; (i) $L^2 = 1$, (ii) $L^2 = 5.7$, (iii) $L^2 = 20$, (iv) $L^2 = 40$. 
FIG.1 shows that the radial motion of the Hopf loop is classified into two types, bounded or unbounded. The existence of bounded orbits for the Hopf loop is analogous to the case of a geodesic particle around a four-dimensional black hole. We note that there is no bounded orbit around the five-dimensional Kerr black hole [13]. Stationary Hopf loop solution exists at the local minimum of \( V_{\text{eff}} \). By the effect of the gravitational force, there exists a critical radius for each black hole such that no stable Hopf loop inside the radius, namely, the innermost stable orbit exists. In addition, in the case of \( \Lambda > 0 \), Hopf loops can grow up to infinite radius by the de Sitter expansion, and the outermost stable orbit exists. In the panel (D) of FIG.1 we see that the radius of innermost stable orbit does not touch the degenerate horizon of the extremal black hole. This property is different from the case of a geodesic particle in four-dimensional extremal black hole.

V. DYNAMICS OF TOROIDAL SPIRAL STRINGS IN FIVE-DIMENSIONAL MINKOWSKI BACKGROUND

In this section, we consider general toroidal spiral strings in Minkowski background. Stationary solutions of toroidal spirals in five-dimensional flat spacetime are studied in Ref.[19, 20]. Here, we investigate the dynamical motions of the toroidal spirals. Already we know that the Hamilton-Jacobi equation is not separable for \( \alpha^2 \neq 1 \) in the \((r, \theta)\) coordinates even in the case of Minkowski background. However, we find that the Hamilton-Jacobi equation is separable for general \( \alpha \) in the Minkowski background by using the new coordinates defined by

\[
\rho = r \sin \theta, \quad \zeta = r \cos \theta.
\] (50)

Furthermore, \((\rho, \zeta)\) coordinates make the physical description clear. The Minkowski metric in the \((\rho, \zeta)\) coordinates becomes

\[
d s^2 = -dt^2 + d\rho^2 + \rho^2 d\Phi^2 + d\zeta^2 + \zeta^2 d\Psi^2.
\] (51)

Two couples of coordinates \((\rho, \Phi)\) and \((\zeta, \Psi)\) cover two independent flat planes.

Now we can apply the Hamilton-Jacobi method to the dynamical system in the coordinates (50). We transform the angular coordinates \((\Phi, \Psi)\) into \((\phi, \psi)\) by (11). Then in the coordinates \((t, \rho, \zeta, \psi)\) on the quotient space \(O\) with respect to \(\xi\) in (10), the contravariant components of the effective metric are given by

\[
\frac{h^{\mu\nu}}{F} = \frac{1}{F} (-\delta_t^t \delta_t^t + \delta_r^t \delta_r^t + \delta_{\zeta}^t \delta_{\zeta}^t) + \frac{1}{\rho^2 \zeta^2} \delta_{\psi}^\mu \delta_{\psi}^\nu,
\] (52)
where $F = \rho^2 + \alpha^2 \zeta^2$. In the same way as in the previous section, Hamilton’s principal function is assumed to be

$$S = \frac{1}{2} \mu^2 \chi - Et + L \psi + S_\rho + S_\zeta,$$  \hfill (53)

where $S_\rho$ and $S_\zeta$ are functions of $\rho$ and $\zeta$, respectively. Substituting this expression into the Hamilton-Jacobi equation, one obtains

$$- E^2 + \left( \frac{1}{\zeta^2} + \frac{\alpha^2}{\rho^2} \right) L^2 + \left( \frac{dS_\rho}{d\rho} \right)^2 + \left( \frac{dS_\zeta}{d\zeta} \right)^2 + \mu^2 (\rho^2 + \alpha^2 \zeta^2) = 0.$$  \hfill (54)

For all $\alpha$, the separability of the above equation is manifest. We solve the Hamilton-Jacobi equation (54) for the toroidal spiral strings in Minkowski background. Introducing a separation constant $K$, we have

$$\left( \frac{dS_\rho}{d\rho} \right)^2 + \frac{\mu^2}{\rho^2} \rho^2 + \frac{\alpha^2 L^2}{\rho^2} - \frac{E^2}{2} = K,$$  \hfill (55)

$$\left( \frac{dS_\zeta}{d\zeta} \right)^2 + \mu^2 \alpha^2 \zeta^2 + \frac{L^2}{\zeta^2} - \frac{E^2}{2} = -K.$$  \hfill (56)

Hamilton’s principal function is obtained as

$$S = \frac{1}{2} \mu^2 \chi - Et + L \psi + \sigma_\rho \int d\rho \sqrt{\Theta_\rho} + \sigma_\zeta \int d\zeta \sqrt{\Theta_\zeta}.$$  \hfill (57)

where

$$\Theta_\rho = K - \frac{\mu^2}{2} \rho^2 - \frac{\alpha^2 L^2}{\rho^2} + \frac{E^2}{2},$$  \hfill (58)

$$\Theta_\zeta = -K - \mu^2 \alpha^2 \zeta^2 - \frac{L^2}{\zeta^2} + \frac{E^2}{2},$$  \hfill (59)

and $\sigma_\rho$ and $\sigma_\zeta$ are sign functions. By differentiating $S$ with respect to $\mu$, $E$, $L$ and $K$, we obtain the solution of the Hamilton-Jacobi equations (54) as

$$\chi = \sigma_\rho \int d\rho \frac{\rho^2}{\sqrt{\Theta_\rho}} + \sigma_\zeta \int d\zeta \frac{\alpha^2 \zeta^2}{\sqrt{\Theta_\zeta}},$$  \hfill (60)

$$t = \sigma_\rho \int d\rho \frac{-E}{\sqrt{\Theta_\rho}},$$  \hfill (61)

$$\psi = L \left[ \sigma_\rho \int d\rho \frac{\alpha^2}{\rho^2 \sqrt{\Theta_\rho}} + \sigma_\zeta \int \frac{d\zeta}{\zeta^2 \sqrt{\Theta_\zeta}} \right],$$  \hfill (62)

$$\sigma_\rho \int \frac{d\rho}{\sqrt{\Theta_\rho}} = \sigma_\zeta \int \frac{d\zeta}{\sqrt{\Theta_\zeta}}.$$  \hfill (63)

Note that in the Schwarzschild background, the separation of variables $\rho$ and $\zeta$ does not occur even for Hopf loops.
Often, it is convenient to rewrite these equations in the form of the first-order differential equations

\[
\begin{align*}
\frac{1}{N} \dot{t} & = \frac{E}{F}, \\
\frac{1}{N} \dot{\psi} & = \frac{L}{\rho^2 \zeta^2}, \\
\frac{1}{N} \dot{\rho} & = \frac{1}{F} \sigma \rho \sqrt{\Theta}, \\
\frac{1}{N} \dot{\zeta} & = \frac{1}{F} \sigma \zeta \sqrt{\Theta},
\end{align*}
\]  

(64) (65) (66) (67)

where overdot stand for differentiation with respect to \( \tau \).

If we choose the Lagrange multiplier \( N \) as

\[
N = F = \rho^2 + \alpha^2 \zeta^2,
\]

(68)

then (66) and (67) become

\[
\begin{align*}
\dot{\rho}^2 + \rho^2 + \frac{\alpha^2 L^2}{\rho^2} - \left( \frac{E^2}{2} + K \right) & = 0, \\
\dot{\zeta}^2 + \alpha \zeta^2 + \frac{L^2}{\zeta^2} - \left( \frac{E^2}{2} - K \right) & = 0.
\end{align*}
\]

(69) (70)

At this point, we put \( \mu = 1 \) since it can be absorbed by the parameters \( K, L, \) and \( E \).

We can obtain the general solution explicitly in the form

\[
\begin{align*}
\rho^2 & = \frac{\rho_{\text{max}}^2 - \rho_{\text{min}}^2}{2} \cos(2\tau + \delta_\rho) + \frac{\rho_{\text{max}}^2 + \rho_{\text{min}}^2}{2}, \\
\zeta^2 & = \frac{\zeta_{\text{max}}^2 - \zeta_{\text{min}}^2}{2} \cos(2\alpha \tau + \delta_\zeta) + \frac{\zeta_{\text{max}}^2 + \zeta_{\text{min}}^2}{2},
\end{align*}
\]

(71) (72)

where the constants \( \delta_\rho \) and \( \delta_\zeta \) are initial phases of \( \rho \) and \( \zeta \), respectively. It is clear that \( \rho_{\text{min}}^2 \leq \rho^2 \leq \rho_{\text{max}}^2 \) and \( \zeta_{\text{min}}^2 \leq \zeta^2 \leq \zeta_{\text{max}}^2 \) where the maximum and minimum values of \( \rho \) and \( \zeta \) are given by

\[
\begin{align*}
\rho_{\text{max}}^2 & = \frac{1}{4} \left[ E^2 + 2K + \sqrt{(E^2 + 2K + 4\alpha L)(E^2 + 2K - 4\alpha L)} \right], \\
\rho_{\text{min}}^2 & = \frac{1}{4} \left[ E^2 + 2K - \sqrt{(E^2 + 2K + 4\alpha L)(E^2 + 2K - 4\alpha L)} \right], \\
\zeta_{\text{max}}^2 & = \frac{1}{4\alpha^2} \left[ E^2 - 2K + \sqrt{(E^2 - 2K + 4\alpha L)(E^2 - 2K - 4\alpha L)} \right], \\
\zeta_{\text{min}}^2 & = \frac{1}{4\alpha^2} \left[ E^2 - 2K - \sqrt{(E^2 - 2K + 4\alpha L)(E^2 - 2K - 4\alpha L)} \right].
\end{align*}
\]

(73) (74) (75) (76)

In the case of \( \rho_{\text{max}} = \rho_{\text{min}} \) and \( \zeta_{\text{max}} = \zeta_{\text{min}} \), i.e., \( K = 0, E^2 = 4\alpha L \), we have the stationary solution \( \rho = \rho_0 = \text{const.} \) and \( \zeta = \zeta_0 = \text{const.} \) where

\[
\rho_0 = \sqrt{\alpha L}, \quad \zeta_0 = \sqrt{\frac{L}{\alpha}}.
\]

(77)
Properties of these solutions were discussed in Ref. [19, 20].

The general solution (71) and (72) describe harmonic oscillations of $\rho^2$ and $\zeta^2$ in time $t = E\tau$. The frequency of $\rho^2$ and $\zeta^2$ are given by $2/E$ and $2\alpha/E$ then they draw a Lissajous figure in the $\rho^2$-$\zeta^2$ plane. If $\alpha$ is a rational number, the toroidal spiral string is closed on a snapshot, the Lissajous figure becomes a closed curve i.e., the motion of the toroidal spiral string become periodic. On the other hand an open toroidal spiral string, irrational $\alpha$, has ergode-like motion.

VI. CONCLUSION

In this paper, we have studied the separability of the Nambu-Goto equations for toroidal spiral strings, whose worldsheet is tangent to a Killing vector field, say $\xi$, that is a linear combination of two commutable rotational Killing vector fields in the five-dimensional Kerr-AdS black holes. Dynamics of the toroidal spiral string associated with $\xi$ is determined by geodesics in the quotient space with respect to $\xi$ with the effective metric which is given by the projection tensor weighted by the norm of $\xi$. We have used the Hamilton-Jacobi method to solve geodesic equations, and have shown that the Hamilton-Jacobi equation in the Boyer-Lindquist coordinates admits the separation of variables for the Hopf loop strings, in the two black hole spacetimes: the Kerr background and the Kerr-AdS background with equal angular momenta. The Hopf loop string is a special case of the toroidal spiral strings which lie along a fiber of the Hopf fibration of $S^3$ which foliates spacial sections of the target spacetime. The separability is owing to the existence of a rank-2 Killing tensor field on the quotient space with the effective metric.

We have demonstrated the dynamical properties of the Hopf loop strings in the Kerr-AdS background with equal angular momenta. Because of high symmetry of the metric, the dynamics is determined by radial motion of geodesic particle in the quotient space. By using the effective potential for the radial motion, we have shown that there exist bounded orbits and unbounded orbits in the quotient space, and there exist the innermost or outermost stable orbit. The motions of Hopf loops are driven by three forces: tension of string, the centrifugal force, and force of gravity [20], and the orbits are determined by the competition of these forces. The stationary solutions are achieved by the balance of these forces. The existence of the bounded orbits of the Hopf loops around five-dimensional black holes makes us recall the geodesic particles around a four-dimensional black hole.

We have also shown that the Hamilton-Jacobi equation is completely separable for the general toroidal spiral strings in the five-dimensional Minkowski background if we choose appropriate
coordinates. We have presented the general dynamical solution of the toroidal spiral strings. The stationary toroidal spirals have been given in the Ref.[19,20], and generalization in less symmetric case is discussed in Ref.[19]. We have presented another generalization to dynamical toroidal spirals in this paper.

It would be easy to replace the five-dimensional black holes with other target spacetimes which admit two commutable rotational Killing vector fields. Toroidal spiral strings in black rings are interesting targets of next study.

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