Asymptotic Distribution of Unit Root Tests Base on ESTAR Model with Flexible Fourier Form

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Authors’ contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In this paper, the asymptotic distribution of Fourier ESTAR model (FKSS) proposed by [1], which was not given in the original paper are derived. Result shows that the asymptotic distributions are functions of brownian motion, only depends on K and free from nuisance parameters.

Keywords: Structural break; Nonlinear unit root tests; Flexible Fourier form.

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1 Introduction

In the recent years, a large body of time series literatures that use Fourier approximation of unknown functional forms have emerged (see [2],[3] and [4]). Moreover, [5] propose Fourier approximation which is sufficient to approximate a wide range of functional forms, since the advantage of the Fourier approach to capture the behavior of a deterministic function of unknown form works better than dummy variable method proposed by [6] irrespective of the breaks are instantaneous or smooth.

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After that, unit root test based on nonlinear deterministic component has been widely concerned.\[7\] adopt the Lagrange multiplier methodology by [8] and develop a unit root test using Fourier form approximation. Similarly, [9] develop a unit root with a Fourier function in the deterministic term in a Dickey fuller type regression frame work. Furthermore, [10] develop the generalize least square unit root test proposed by [11] to allow for a Fourier approximation to the unknown deterministic component.

Considering ESTAR model with Fourier form that capture nonlinear adjustment and structural breaks well, [1] develop a new tests procedure for unit roots base on ESTAR model with flexible Fourier form. The main constraint is that [1] failed to give the asymptotic distribution of their proposed test. However, our main concern in this paper is to derive the asymptotic distribution of Fourier-ESTAR model proposed by [1]. The corresponding asymptotic distribution are given in next part. Result show that our derived asymptotic distributions will provides a foundation for the derivation of complex model with Fourier function, which is uncorrelated with nuisance parameter.

2 Asymptotic Properties of the Test Statistics

According to [5], a single-frequency Component of a Fourier approximation can mimic a wide variety of breaks and other types of non-linearity we begin our analysis with a Data generating process containing only one frequency.

\[ y_t = \alpha_0 + \alpha_1 t + \alpha_2 \sin \left( \frac{2\pi kt}{T} \right) + \alpha_3 \cos \left( \frac{2\pi kt}{T} \right) + \nu_t, \quad (2.1) \]

\[ \nu_t = \rho \nu_{t-1} + \gamma \nu_{t-1} (1 - \exp(-\theta \nu_{t-1}^2)) + \epsilon_t. \quad (2.2) \]

where \( \epsilon_t \sim iid(0, \sigma^2) \), k represent a particular frequency and T is the sample Size

In this study, two step testing procedure are used to derive the asymptotic distribution of the Fourier-ESTAR model.

Remark 2.1: The deterministic kernel considered in equation(2.1) includes a linear time trend, but we may also consider the case where only a constant and a Fourier terms are considered; i.e., the case where \( \alpha_1 = 0 \) in equation(2.1). This will be referred to demeaned case in what follows, while the more general case \( \alpha_1 \neq 0 \) will be termed the detrended case.

In the first step, we obtain the demeaned and detrended series of equation(1) as follows:

Demeaned Case, \( \alpha_1 = 0 \):

\[ \tilde{v}_t = y_t - \hat{\alpha}_0 - \hat{\alpha}_2 \sin \left( \frac{2\pi kt}{T} \right) - \hat{\alpha}_3 \cos \left( \frac{2\pi kt}{T} \right). \]

This can be re-written as:

\[ \tilde{v}_t = y_t - Z_t(\hat{\theta}). \quad (2.3) \]
where $\theta = (\alpha_0, \alpha_1, \alpha_3)'$, $\hat{\theta}$ is the OLS estimate of $\theta$ and $Z_t = (1, \sin(\frac{2\pi kt}{T}), \cos(\frac{2\pi kt}{T}))'$

Detrended Case, $\alpha_1 \neq 0$

$$\tilde{v}_t = y_t - \hat{\alpha}_0 - \hat{\alpha}_1 t - \hat{\alpha}_2 \sin(\frac{2\pi kt}{T}) - \hat{\alpha}_3 \cos(\frac{2\pi kt}{T}),$$

This can be re-written as:

$$\tilde{v}_t = y_t - Z_t'(\hat{\theta}). \quad (2.4)$$

where $\theta = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)'$, $\hat{\theta}$ is the OLS estimate of $\theta$ and $Z_t = (1, t, \sin(\frac{2\pi kt}{T}), \cos(\frac{2\pi kt}{T}))'$

In the second step (section (2.1)) we adopt a Fourier base unit root test (KSS test) with demeaned or detrended and run the ESTAR-type unit root test.

### 2.1 KSS test

To construct a Fourier base unit root test we use classical KSS unit root. The ESTAR model in equation (2.2) will be reparametrized with $\tilde{v}_t$ instead of $v_t$ as follows:

$$\tilde{v}_t = \rho \tilde{v}_{t-1} + \gamma \tilde{v}_{t-1}(1 - \exp(-\theta \tilde{v}_{t-1}^2)) + \epsilon_t. \quad (2.5)$$

equation (2.5) can be written as:

$$\Delta \tilde{v}_t = \phi \tilde{v}_{t-1} + \gamma \tilde{v}_{t-1}(1 - \exp(-\theta \tilde{v}_{t-1}^2)) + \epsilon_t. \quad (2.6)$$

Where $\phi = \rho - 1$

Following the practice in the literature (e.g. [12] in the context of TAR models and [13] in the context of ESTAR models), we impose $\phi = 0$ in equation (2.6), implying that $\tilde{v}_t$ follows a unit root process in the middle regime. we can write equation (2.6) as :

$$\Delta \tilde{v}_t = \gamma \tilde{v}_{t-1}(1 - \exp(-\theta \tilde{v}_{t-1}^2)) + \epsilon_t. \quad (2.7)$$

equation (2.7) implies that $\tilde{v}_t$ follows either a unit root or globally stationary, we consider testing the null hypothesis that $\tilde{v}_t$ follow a unit root process given by $\gamma = 0$ or $\theta = 0$, against the alternative that $\tilde{v}_t$ is nonlinear and globally stationary, i.e $\theta = 0$ with $-2 < \gamma < 0$.

Obviously, testing the null hypothesis $H_0 : \theta = 0$ in equation (2.7) directly is not feasible, since $\gamma$ is not identified under the null hypothesis. See for example [14]. A popular approach to avoid the presence of nuisance parameters under the null hypothesis is to use a Taylor approximation of the smooth transition function $G(\tilde{v}_{t-1}; \theta) = 1 - \exp(-\theta \tilde{v}_{t-1}^2)$ around $\theta = 0$ see [15]. An application of a first-order Taylor approximation to the ESTAR model leads to the auxiliary equation below:

$$\Delta \tilde{v}_t = \delta \tilde{v}_{t-1}^3 + \epsilon_t. \quad (2.8)$$

The unit root hypothesis is set up by estimating equation (2.8) with OLS and testing the null $H_0 : \delta = 0$ against the alternative $H_1 : \delta < 0$ using t-statistics define as:

$$t_{kss}^i = \frac{\hat{\delta}}{s.e(\hat{\delta})}. \quad (2.9)$$

where $\hat{\delta}$ is the OLS estimate of $\delta$ in equation (2.8), $s.e(\hat{\delta})$ is the corresponding standard error, and $i = (\mu, \tau)$ for demeaned and detrended cases respectively.
To obtain the asymptotic distribution of the Fourier ESTAR model defined in equation (2.9), the following results are needed.

**Proposition 2.1.**

\[\begin{align*}
  i & \quad \frac{1}{T^{3/2}} \sum_{k=1}^{T} y_t \Rightarrow \sigma \int_{0}^{1} W(r) dr = \sigma f_1 \\
  ii & \quad \frac{1}{T^{3/2}} \sum_{k=1}^{T} t y_t \Rightarrow \sigma \int_{0}^{1} r W(r) dr = \sigma f_2 \\
  iii & \quad \frac{1}{T^{3/2}} \sum_{k=1}^{T} \sin\left(\frac{2\pi k t}{T}\right) y_t \Rightarrow \sigma \int_{0}^{1} \sin(2\pi kr) W(r) dr = \sigma f_3 \\
  iv & \quad \frac{1}{T^{3/2}} \sum_{k=1}^{T} \cos\left(\frac{2\pi k t}{T}\right) y_t \Rightarrow \sigma \int_{0}^{1} \cos(2\pi kr) W(r) dr = \sigma f_4 \\
  v & \quad \frac{1}{T} \sum_{k=1}^{T} \sin\left(\frac{2\pi k t}{T}\right) \Rightarrow 0 \\
  vi & \quad \frac{1}{T} \sum_{k=1}^{T} \cos\left(\frac{2\pi k t}{T}\right) \Rightarrow 0 \\
  vii & \quad \frac{1}{T} \sum_{k=1}^{T} \sin^2\left(\frac{2\pi k t}{T}\right) \Rightarrow 1/2 \\
  viii & \quad \frac{1}{T} \sum_{k=1}^{T} \cos^2\left(\frac{2\pi k t}{T}\right) \Rightarrow 1/2 \\
  ix & \quad \frac{1}{T^2} \sum_{k=1}^{T} t \sin\left(\frac{2\pi k t}{T}\right) \Rightarrow -\frac{1}{(2\pi k)} \\
  x & \quad \frac{1}{T^2} \sum_{k=1}^{T} t \cos\left(\frac{2\pi k t}{T}\right) \Rightarrow 0 \\
  xi & \quad \frac{1}{T} \sum_{k=1}^{T} \cos\left(\frac{2\pi k t}{T}\right) \sin\left(\frac{2\pi k t}{T}\right) \Rightarrow 0
\end{align*}\]

**Theorem 2.1.** Under the null hypothesis the t-statistics defined in equation (2.9), for the demeaned case has the following asymptotic distribution:

\[\begin{align*}
t_{\mu}^{F-kss} & \Rightarrow \frac{1}{\sqrt{T}} \frac{1}{\left\{ \frac{1}{0} W_{\mu}(k, r)^3 dr \right\}^{1/2}} W_{\mu}(k, r)^3 dw(r)
\end{align*}\]

where \(W_{\mu}(k, r)\) is demeaned Brownian motion defined on \(r \in (0, 1)\)
Theorem 2.2. Under the null hypothesis the t-statistics defined in equation (2.9), for the detrended case has the following asymptotic distribution:

\[ t_{F-kss} \Rightarrow 1 \int_{0}^{1} \frac{W_r(k, r)^3 dw(r)}{(\int_{0}^{1} W_r(k, r)^3 dr)^{1/2}} \]

where \( W_r(k, r) \) is detrended Brownian motion defined on \( r \in (0, 1) \)

The asymptotic distribution of the test-statistics will only depend on \( K \) and free from nuisance parameter.

Proof. See the Appendix.

3 Conclusions

[1] Focus on the potential effect that structural breaks and non-linear mean reversion have on tests of the Purchasing Power Parity (PPP) hypothesis. They present tests that, far from considering these two features separately, model both breaks and non-linear adjustment jointly, but the main constraints is that [1] do not give asymptotic distributions of there test. This article extend the work of [1] by providing the asymptotic distributions of Fourier-ESTAR model which is not available in the original paper.

Competing Interests

Authors have declared that no competing interests exist.

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Appendix

Proof of Proposition:

From Proposition (2.1) under null hypothesis $\Delta y_t = \epsilon_t$

i. $\frac{1}{T^{3/2}} \sum_{k=1}^{T} y_t \implies \sigma \int_{0}^{1} W(r) dr = \sigma f_1$

ii. $\frac{1}{T^{3/2}} \sum_{k=1}^{T} t y_t \implies \sigma \int_{0}^{1} r W(r) dr = \sigma f_2$

The result of (i) and (ii) are standard, (see [16]). from the continuous mapping theorem , we have:

iii. $\frac{1}{T^{3/2}} \sum_{k=1}^{T} \sin\left(\frac{2\pi k t}{T}\right)t y_t \implies \sigma \int_{0}^{1} \sin(2\pi kr)W(r) dr = \sigma f_3$

iv. $\frac{1}{T^{3/2}} \sum_{k=1}^{T} \cos\left(\frac{2\pi k t}{T}\right)t y_t \implies \sigma \int_{0}^{1} \cos(2\pi kr)W(r) dr = \sigma f_4$

v. $\frac{1}{T} \sum_{k=1}^{T} \sin\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} \sin(2\pi kr) dr = \frac{1}{2\pi k} (1 - \cos(2\pi kr)) = 0$

vi. $\frac{1}{T} \sum_{k=1}^{T} \cos\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} \cos(2\pi kr) dr = \frac{\sin(2\pi k)}{2\pi k} = 0$

vii. $\frac{1}{T} \sum_{k=1}^{T} \sin^2\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} \sin^2(2\pi kr) dr = \frac{1}{2} \int_{0}^{1} (1 - \cos(4\pi kr)) dr = \frac{1}{2} \frac{\sin 4\pi k}{4\pi k} = 1/2$

viii. $\frac{1}{T} \sum_{k=1}^{T} \cos^2\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} \cos^2(2\pi kr) dr = \frac{1}{2} \int_{0}^{1} (1 - \sin^2(4\pi kr)) dr = \frac{1}{2} + \frac{\sin 4\pi k}{4\pi k} = 1/2$

ix. $\frac{1}{T^2} \sum_{k=1}^{T} t \sin\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} t \sin(2\pi kr) dr = \frac{\sin(2\pi k)}{(2\pi k)^2} - \frac{\cos(2\pi k)}{(2\pi k)} = -\frac{1}{2\pi k}$

x. $\frac{1}{T^2} \sum_{k=1}^{T} t \cos\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} t \cos(2\pi kr) dr = \frac{\cos(2\pi k) - 1}{(2\pi k)^2} + \frac{\sin(2\pi k)}{(2\pi k)} = 0$

xi. $\frac{1}{T} \sum_{k=1}^{T} \cos\left(\frac{2\pi k t}{T}\right) \sin\left(\frac{2\pi k t}{T}\right) y_t \implies \int_{0}^{1} \cos(2\pi kr) \sin(2\pi kr) dr = \frac{1 - \cos(4\pi k)}{8\pi k} = 0$

Proof of Theorem 2.1:

Proof. Consider the level of stationarity with Fourier Function. for the demeaned case i.e $\alpha_1 = 0$ in equation(2.1), by using the OLS residual define in equation(2.6) with $Z_t = (1, \sin\left(\frac{2\pi k t}{T}\right), \cos\left(\frac{2\pi k t}{T}\right))$. define as follows:
\[ \tilde{v}_t = y_t - Z_t'(\hat{\theta}) \]

where \( \theta = (\alpha_0, \alpha_2, \alpha_3)' \) and \( \hat{\theta} \) is the OLS estimate of \( \theta \), and \( \Delta y_t = \epsilon_t \). Let \( z = (z_1, \ldots, z_t)' \) and \( y = (y_1, \ldots, y_t)' \) also we define the scaling parameter as \( \Upsilon = \text{diag}[\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}] \) then we have:

\[
\Upsilon(\hat{\theta}) = \Upsilon - \frac{1}{\Upsilon} Z_t'(\hat{\theta}) \frac{1}{\Upsilon} - \frac{1}{\Upsilon} Z_t'Y
\]

from the above equation we show the asymptotic distribution of the demeaned case as follows:

\[
\frac{1}{\sqrt{T}} \tilde{v}_{[T]} = \frac{1}{\sqrt{T}} y_{[T]} - \frac{1}{\sqrt{T}} Z_{[T]}'(\hat{\theta}) \to W(r)
\]

\[
\frac{1}{\sqrt{T}} \tilde{u}_{[T]} = \frac{1}{\sqrt{T}} \sum_{t=1}^{[T]} u_{[T]} \to \sigma W(r)
\]

\[
[\Upsilon^{-1}(Z'Z)^{-1}Y^{-1}]^{-1} = \begin{bmatrix}
\frac{T}{T} & \frac{1}{T} \sum_{k=1}^{T} \sin(\frac{2\pi kt}{T}) & \frac{1}{T} \sum_{k=1}^{T} \cos(\frac{2\pi kt}{T})
\frac{1}{T} \sum_{k=1}^{T} \sin^2(\frac{2\pi kt}{T}) & \frac{1}{T} \sum_{k=1}^{T} \sin(\frac{2\pi kt}{T}) \cos(\frac{2\pi kt}{T}) & \frac{1}{T} \sum_{k=1}^{T} \cos^2(\frac{2\pi kt}{T})
\end{bmatrix}^{-1}
\]

by using the proposition define earlier we have:

\[
[\Upsilon^{-1}(Z'Z)^{-1}]^{-1} \to \begin{bmatrix}
1 & 0 & 0
\frac{1}{2} & 0 & 0
\frac{1}{2} & 0 & 2
\end{bmatrix}^{-1} = \begin{bmatrix}
1 & 0 & 0
\frac{1}{2} & 0 & 0
\frac{1}{2} & 0 & 2
\end{bmatrix}
\]

Also

\[
T^{-1} \Upsilon^{-1} Z'Y = \begin{bmatrix}
\frac{1}{T^{1/2}} \sum_{k=1}^{T} y_t
\frac{1}{T^{1/2}} \sum_{k=1}^{T} \sin(\frac{2\pi kt}{T}) y_t
\frac{1}{T^{1/2}} \sum_{k=1}^{T} \cos(\frac{2\pi kt}{T}) y_t
\end{bmatrix} \to \begin{bmatrix}
\sigma f_1
\sigma f_3
\sigma f_4
\end{bmatrix}
\]
Then we can write that:

\[
\frac{1}{T}Z'_{[T]} \Upsilon^{-1} [\Upsilon^{-1} (Z'Z) \Upsilon^{-1}]^{-1} \Upsilon^{-1} Z'Y \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \sigma f_1 \\ \sigma f_3 \\ \sigma f_4 \end{bmatrix}
\]

\[
\rightarrow \begin{bmatrix} \sigma f_1 + 2 \sin(2\pi kr) \sigma f_3 + 2 \cos(2\pi kr) \sigma f_4 \end{bmatrix}
\]

by combining all the result we obtain the demeaned brownian motion as:

\[
\frac{1}{\sqrt{T}} \tilde{v}_t \rightarrow \sigma W(r) - f_1 - 2 \sin(2\pi kr) f_3 - 2 \cos(2\pi kr) f_4 \equiv \sigma W_\mu(kr)
\]

\[
\frac{1}{\sigma \sqrt{T}} \tilde{v}_t \rightarrow W_\mu(kr)
\]

using above results, under the null we can obtain that:

\[
t^*_\mu \rightarrow \frac{1}{\int_0^T W(\mu,k,r)^3 dw(r)} \int_0^T W_\mu(k,r)^3 dw(r) \right)^{1/2}
\]

**Proof of Theorem 2.2:**

*Proof.* Consider the level of sattionarity with Fourier Function. for the detrended case i.e \( \alpha_1 \neq 0 \) in equation (2.1), by using the OLS residual define in equation (2.7) with \( Z_t = (1, t, \sin(2\pi kt), \cos(2\pi kt)) \).

define as follows:

\[
\tilde{v}_t = y_t - Z'_t(\hat{\theta})
\]

where \( \theta = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)' \) \( \text{and } \hat{\theta} \text{ is the OLS estimate of } \theta \text{ and } \Delta y_t = \epsilon_t \). Let \( z = (z_1, ..., z_t)' \) \( \text{and } y = (y_1, ..., y_t)' \) 
also we define the scalling parameter as \( \Upsilon = diag \left[ \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}}, \frac{1}{\sqrt{T}} \right] \) then we have:

\[
\Upsilon(\hat{\theta}) = [\Upsilon^{-1} (Z'Z) \Upsilon^{-1}]^{-1} \Upsilon^{-1} Z'Y
\]

from the above equation we show the asymptotic distributon of the detrended case as follows:

\[
\frac{1}{\sqrt{T}} \tilde{v}_t \rightarrow \frac{1}{\sqrt{T}} Z'_{[T]} [\Upsilon^{-1} (Z'Z) \Upsilon^{-1}]^{-1} \Upsilon^{-1} Z'Y
\]

following the same procedure discused earlier and according to functional central limit theorem we have terms as follows:

\[
\frac{1}{\sqrt{T}} y_{[T]} \rightarrow \frac{1}{\sqrt{T}} \sum_{t=1}^{[T]} y_t \rightarrow \sigma W(r)
\]
by using the proposition define earlier we have:

\[
[Y^{-1}(Z'Z)Y^{-1}]^{-1} = \begin{bmatrix}
T^{-1} & \frac{1}{T^2} \sum_{k=1}^{T} t & \frac{1}{T} \sum_{k=1}^{T} \sin\left(\frac{2\pi kt}{T}\right) & \frac{1}{T} \sum_{k=1}^{T} \cos\left(\frac{2\pi kt}{T}\right) \\
\frac{1}{T^2} \sum_{k=1}^{T} t^2 & \frac{1}{T} \sum_{k=1}^{T} t \sin\left(\frac{2\pi kt}{T}\right) & \frac{1}{T} \sum_{k=1}^{T} t \cos\left(\frac{2\pi kt}{T}\right) \\
\frac{1}{T} \sum_{k=1}^{T} \sin^2\left(\frac{2\pi kt}{T}\right) & \frac{1}{T} \sum_{k=1}^{T} \sin\left(\frac{2\pi kt}{T}\right) \cos\left(\frac{2\pi kt}{T}\right) \\
\frac{1}{T} \sum_{k=1}^{T} \cos^2\left(\frac{2\pi kt}{T}\right)
\end{bmatrix}^{-1}
\]

Also

\[
T^{-1}Y^{-1}Z'Y = \begin{bmatrix}
\frac{1}{T} \sum_{k=1}^{T} y_k \\
\frac{1}{T} \sum_{k=1}^{T} t y_k \\
\frac{1}{T} \sum_{k=1}^{T} \sin\left(\frac{2\pi kt}{T}\right) y_k \\
\frac{1}{T} \sum_{k=1}^{T} \cos\left(\frac{2\pi kt}{T}\right) y_k
\end{bmatrix} \rightarrow \begin{bmatrix}
\sigma f_1 \\
\sigma f_2 \\
\sigma f_3 \\
\sigma f_4
\end{bmatrix}
\]
Then we can write that:

\[
\frac{1}{T} Z^r_{\{T_r\}} Y^{-1} [Y^{-1} (Z^r Z) Y^{-1}]^{-1} Y^{-1} Z^r Y \rightarrow \left[ \begin{array}{c} 1 \\ r \\ \sin(2\pi k r) \\ \cos(2\pi k r) \end{array} \right]
\]

by combining all the results we obtain the detrended brownian motion as:

\[
\frac{1}{\sqrt{T}} \tilde{v}_{\{T_r\}} = \frac{1}{\sqrt{T}} y_{\{T_r\}} - \frac{1}{\sqrt{T}} \tilde{v}^r_{\{T_r\}}(\hat{\theta})
\]

\[
\frac{1}{\sqrt{T}} \tilde{v}_{\{T_r\}} \longrightarrow \sigma(W(r) - \sigma \left[ \begin{array}{c} (a_{11} f_1 + a_{12} f_2 + a_{13} f_3) + (a_{21} f_1 + a_{22} f_2 + a_{23} f_3) r \\ + (a_{31} f_1 + a_{32} f_2 + a_{22} f_3) \sin(2\pi k r) + a_{44} f_4 \cos(2\pi k r) \end{array} \right]) \equiv W_r(k r)
\]

using above results, under the null we can obtain that:

\[
t_{F-k^*} \rightarrow \frac{1}{\sqrt{1 \int_0^T W_k^r(k r)^3 dr}}.
\]

\[
\frac{1}{\sigma \sqrt{T}} \tilde{v}_{\{T_r\}} \longrightarrow W_r(k r)
\]

\[
\frac{1}{\int_0^T W_k^r(k r)^6 dr}^{1/2}
\]

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