ON THE MAXIMAL DIRECTIONAL HILBERT TRANSFORM
IN THREE DIMENSIONS

FRANCESCO DI PLINIO AND IOANNIS PARISSIS

ABSTRACT. We establish the sharp growth rate, in terms of cardinality, of the $L^p$ norms of the maximal Hilbert transform $H_\Omega$ along finite subsets of a finite order lacunary set of directions $\Omega \subset \mathbb{R}^3$, answering a question of Parcet and Rogers in dimension $n = 3$. Our result is the first sharp estimate for maximal directional singular integrals in dimensions greater than 2.

The proof relies on a representation of the maximal directional Hilbert transform in terms of a model maximal operator associated to compositions of two-dimensional angular multipliers, as well as on the usage of weighted norm inequalities, and their extrapolation, in the directional setting.

1. INTRODUCTION

Let $n \geq 2$. The Hilbert transform along a direction $\omega \in S^{n-1}$ acts on Schwartz functions on $\mathbb{R}^n$ by the principal value integral

$$H_\omega f(x) := \text{p.v.} \int_{\mathbb{R}} f(x + t\omega) \frac{dt}{t}, \quad x \in \mathbb{R}^n.$$ 

If $\Omega \subset S^{n-1}$, we may define the corresponding maximal directional Hilbert transform

$$H_\Omega f := \sup_{\omega \in \Omega} |H_\omega f|.$$ 

The main result of this paper is the following sharp estimate in the three-dimensional case.

**Theorem 1.1.** Let $\Omega \subset S^2$ be a finite order lacunary set [28]. Then for all $1 < p < \infty$

$$\sup_{O \subset \Omega} \|H_O\|_{L^p(\mathbb{R}^3) \to L^p(\mathbb{R}^3)} \leq C \sqrt{\log N}.$$ 

The positive constant $C$ may depend on $1 < p < \infty$ and on the lacunary order of $\Omega$ only.

We stress that the supremum in Theorem (1.2) is taken over all subsets $O$ having finite cardinality $N$ of a given finite order lacunary set $\Omega$, which may be infinite. Theorem 1.1 is in fact the Lebesgue measure case of a more general sharp weighted norm inequality which is a natural byproduct of our proof techniques, and is detailed in Corollary 1 for the interested
reader. In [22] Laba, Marinelli, and Pramanik have extended to dimensions $n \geq 2$ the lower bound (due to Karagulyan [19] in the case $n = 2$)

$$\inf_{\#\Omega = N} \|H_{\Omega}\|_{L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)} = c_n \sqrt{\log N},$$

(1.3)

where the infimum is taken over all sets $\Omega \subset S^{n-1}$ of finite cardinality $N$. A comparison with the upper bound of Theorem 1.1 and interpolation reveals that the dependence on the cardinality of the set of directions in our theorem is sharp for all $1 < p < \infty$. In fact, [22] proves the analogue of (1.3) for all $1 < p < \infty$.

1.2. Maximal and singular integrals along sets of directions. The study of cardinality-free, or sharp bounds for the companion directional maximal operator to (1.1)

$$M_{\Omega}f(x) = \sup_{\omega \in \Omega} M_{\omega}f(x), \quad M_{\omega}f(x) := \sup_{t > 0} \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} |f(x + t\omega)|dt, \quad x \in \mathbb{R}^n,$$

(1.4)

is a classical subject in real and harmonic analysis, with deep connections to multiplier theorems, Radon transforms, and the Kakeya problem, to name a few. The seminal article by Nagel, Stein and Wainger [26] contains a proof that the projection on $S^{n-1}$ of the set of directions

$$\Omega_{\lambda} := \{ (\lambda^{k\alpha}, \ldots, \lambda^{k\alpha}) : k \geq 1 \}, \quad 0 < \lambda < 1,$$

gives rise to a bounded maximal operator $M_{\Omega_{\lambda}}$ in any dimension $n \geq 2$. Besides providing the first higher dimensional example of such a set of directions, the article [26] contains the important novelty of treating the geometric maximal operator $M_{\Omega}$ through Fourier analytic tools. This allowed the authors to break the barrier $p = 2$ that was present in previous work of Córdoba and Fefferman [7], and Strömberg [31], where the authors used mostly geometric arguments.

Sjögren and Sjölin [29] proved that, in dimension $n = 2$, a sufficient condition for the $L^p$-boundedness of $M_{\Omega}$ for some (equivalently for all) $1 < p < \infty$ is that $\Omega$ is a lacunary set of finite order; loosely speaking, in dimension $n = 2$, a lacunary set $\Omega'$ of order $L$ is obtained from a lacunary set $\Omega$ of order $L - 1$ by inserting within each gap between two consecutive elements $a, b \in \Omega$ two subsequences of suitably rotated copies of $\Omega^1$ having $a, b$ as limit points.

Bateman [2] subsequently showed that (up to finite unions) finite order lacunarity of $\Omega$ is necessary in order for $M_{\Omega}$ to admit nontrivial $L^p$-bounds when $\Omega$ is an infinite set. While the counterexample by Bateman is highly nontrivial and employs a probabilistic construction based upon tree percolation, it is rather easy to see that the $L^p$ norm of $M_{\Omega}$ must depend on $N$ if $\Omega$ is, say, the set of $N$-th roots of unity. In fact, the sharp dependence

$$\|M_{\Omega}\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \sim \log N$$

(1.5)

for sets of this type was proved by Strömberg [31]; in [20], the structural restriction on $\Omega$ was lifted and the upper bound in (1.5) was shown to hold for all finite $\Omega \subset S^1$ with cardinality $N$. Further results concerning maximal operators along directions coming from sets with intermediate Hausdorff and fractal dimension can be found in [15, 17].

We already reviewed that in all dimensions $n \geq 2$, [26] provides us with an example of a lacunary set of directions $\Omega_{\lambda}$ for which $M_{\Omega_{\lambda}}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Another
significant higher dimensional example is given in the article of Carbery, [5], where the author considers the projection on $S^{n-1}$ of the infinite set

$$\Omega^{n-1} := \{(2^{k_1}, \ldots, 2^{k_n}) : (k_1, \ldots, k_n) \in \mathbb{Z}\},$$

and proves that $M_{\Omega^{n-1}}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. In dimension $n = 2$ the set $\Omega^1$ is the paradigmatic example of lacunary subset of $S^1$. By the same token, the Carbery set $\Omega^{n-1}$ can be considered as the canonical example of a higher order, higher dimensional lacunary set. More precisely (cf. Definition 2.2) $\Omega^{n-1}$ is a lacunary set of order $n - 1$ with exactly one direction in each cell of the dissection.

In dimensions $n \geq 3$, a general sufficient characterization of those infinite $\Omega \subset S^{n-1}$ giving rise to bounded directional maximal operators, subsuming those of [5, 26], was recently established by Parcet and Rogers [28] via an almost-orthogonality principle for the $L^p$ norms of $M_\Omega$, resembling in spirit that of [1] by Alfonseca, Soria and Vargas in the two-dimensional case. This principle leads naturally to the notion of a lacunary subset of $S^{n-1}$ when $n \geq 3$, which is a sufficient condition for nontrivial $L^p$-bounds of (1.4). Again loosely speaking, $\Omega \subset S^{n-1}$ is lacunary of order 1 if there exists a choice of orthonormal basis— in the language of [28], a dissection of the sphere $S^{n-1}$— such that for all pairs of coordinate vectors $e_j, e_k$ the projection of $\Omega$ on the linear span of $e_j, e_k$ is a two-dimensional lacunary set; higher order lacunary sets are defined inductively in the natural way. The authors of [28] also provide a necessary condition which is slightly less restrictive than finite order lacunarity; we send to their article for a precise definition.

As (1.3) shows, if $\Omega \subset S^{n-1}$ is infinite, $H_\Omega$ is necessarily unbounded. Therefore, the question of sharp quantitative bounds for $H_\Omega$ in terms of the (finite) cardinality of the set $\Omega$ arises as a natural substitute of uniform bounds. In dimension $n = 2$, several sharp or near-sharp results of this type have been obtained by Demeter [8], Demeter and the first author [9], and the authors [11]. We choose to send to these references for detailed statements and just mention the quantitative bounds which are the closest precursors of our Theorem 1.1. To begin with, the two-dimensional analogue of (1.2), with the same $O(\sqrt{\log N})$ quantitative dependence, was proved by the authors in [11]. The methods of [11] are essentially relying on the fact that (lacunary) directions in $n = 2$ can be naturally ordered, and that this order yields a telescopic representation of $H_\Omega$ as a maximal partial sum of Fourier restrictions to disjoint (lacunary) cones: see also [8, 19]. These methods do not extend to dimensions three and higher, where no ordering is possible in general. On the other hand, in [28, Corollary 4.1], following the ideas of [9, Theorem 1], the authors derive the quantitative estimate

$$\sup_{\Omega \subseteq \Omega \subset \mathbb{R}^n} \|H_\Omega\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C \log N \|M_\Omega\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)}$$

at the root of which lies Hunt’s exponential good-$\lambda$ comparison principle between maximal and singular integrals [18]. Coupling (1.6) with the main result of [28] yields that the norms of the maximal Hilbert transform over finite subsets of a given finite order lacunary set in any dimension grow at most logarithmically with the cardinality of the subset. When $n = 3$, Theorem 1.1 improves this result to the sharp $O(\sqrt{\log N})$ quantitative dependence, answering the question posed by Parcet and Rogers in [28, Section 4]. The estimate of Theorem 1.1 appears to be the first sharp quantitative estimate for directional singular integrals in dimension $n \geq 3$. 

\[ \text{MAXIMAL DIRECTIONAL HILBERT TRANSFORM} \]
1.3. Techniques of proof. The key observation leading to the Parcet-Rogers theorem [28] is that the Fourier support of the single scale distribution

$$f \mapsto \int_{\mathbb{R}} f(x - t\omega)\psi(t) \, dt,$$

where $\psi$ is a Schwartz function on $\mathbb{R}$, is covered by a union of two dimensional wedges $\Psi_{\sigma,\omega}$ over pairs $\sigma$ of coordinate directions, provided a suitable smooth $n$-dimensional average of $f$ at the same scale is subtracted off; the latter piece is controlled by the strong maximal function of $f$. While these wedges heavily overlap with respect to $\sigma$, see e.g. Figure 3.2, the authors use the inclusion-exclusion principle to reduce to a square function estimate for compositions of two-dimensional multipliers adapted to the wedges $\Psi_{\sigma,\omega}$. The fact that this square function is a bounded operator on $L^p$ follows from the bounded overlap, for fixed $\sigma$, as $\omega$ ranges over a lacunary set $\Omega$, of the associated wedges $\Psi_{\sigma,\omega}$. The proof of our Theorem 1.1 is also based on a representation of the directional Hilbert transform $H_\omega$ involving two-dimensional wedge multipliers, which splits $H_\omega$ into an inner and an outer part: cf. Lemma 3.2.

The inner part, which is supported on the union of the wedges $\Psi_{\sigma,\omega}$, is amenable to a square function treatment; however, additional difficulties are encountered in comparison to [28] as $H_\Omega$ is not a positive operator and does not obey a trivial $L^\infty$-estimate. We circumvent this difficulty by aiming for the stronger $L^2$-weighted norm inequality and relying on extrapolation theory for suitable weights in the natural directional $A_2$ classes. This requires extending the maximal inequality of [28] to the weighted setting; while this extension does not require substantial additional efforts we wrote out the proofs in detail for future reference. As we previously remarked, it also has the pleasant effect of giving a much more general weighted version of Theorem 1.1: see Corollary 1, Section 5.

Unlike the single scale operator, the outer part of the decomposition is nontrivial, and is actually the one introducing the dependence on the cardinality of the set of directions. It is a signed sum of $2^n$ terms which are compositions of two-dimensional angular multipliers; in general we cannot do better than estimating the maximal operator associated to each summand. The key observation of our analysis at this point is that these compositions can be bounded pointwise by (compositions of) strong maximal operators, upon pre-composition with at most $\lfloor \frac{n}{2} \rfloor$ directional Littlewood-Paley projections; see Lemma 3.3 and Remark 6.4. An application of at most $\lfloor \frac{n}{2} \rfloor$ Chang-Wilson-Wolff decouplings, see Proposition 5.2, then reduces the maximal estimate to a square function estimate upon loss of $\lfloor \frac{n}{2} \rfloor$ factors of order $\sqrt{\log N}$. This is enough to obtain the sharp result for $n = 2, 3$ (and, less interestingly, recover (1.6) when $n = 4, 5$), hinting on the other hand that this approach is not feasible in general dimensions.

In fact, perhaps surprisingly, we show with a counterexample that this growth rate, worse than that of $H_\Omega$ whenever $n \geq 6$, is actually achieved by the maximal operator associated to the outer parts. This phenomenon displays how the model operator of Lemma 3.2, based on the combinatorics of two-dimensional wedges, is not subtle enough to completely capture the cancellation present in $H_\Omega$.

1.4. Relation to the Hilbert transform along vector fields. In addition to their intrinsic interest, Theorem 1.1 and predecessors may be seen as building blocks towards the resolution of the following question, apocryphally attributed to E. Stein and often referred to as the vector
**field problem:** if \( v : \mathbb{R}^n \to S^{n-1} \) is a vector field with Lipschitz constant equal to 1 and pointing within a small neighborhood of \((1/\sqrt{n}, \ldots, 1/\sqrt{n})\), prove or disprove that the truncated directional Hilbert transform along \( v \)

\[
H_v f(x) = \text{p.v.} \int_{|t|<\epsilon_0} f(x - tv(x)) \frac{dt}{t}
\]

for \( \epsilon_0 > 0 \) small enough, is a bounded operator from \( L^2(\mathbb{R}^n) \) into \( L^{2,\infty}(\mathbb{R}^n) \). The partial progress in dimension \( n = 2 \), beginning with the work of Lacey and Li [23, 25] and continued in e.g. [3, 14, 10] by several authors, rests upon using the Lipschitz property to achieve decoupling of the full maximal operator into a Littlewood-Paley square function similar in spirit to the one appearing in (5.6). The estimation of a single Littlewood-Paley piece in the vector field case is more difficult than the pointwise estimate available to us in Lemma 3.3 and involves, in dimension \( n = 2 \), time-frequency analysis of roughly the same parametric complexity as of that appearing in the Lacey-Thiele proof of Carleson’s theorem [24]. Lemma 3.3 in this context may be interpreted as a single tree estimate (cf. [24,25]), showing that the annular estimate for \( n = 3 \) might display the same essential complexity as the \( n = 2 \) case.

1.5. **Plan of the article.** In the forthcoming Section 2, we set up the notation for the remainder of the article and provide the precise definition of finite order lacunary sets in \( \mathbb{R}^n \). Section 3 contains the reduction of \( H_\Omega \) to the above mentioned model operators, Lemma 3.2 as well as their single tree estimate of Lemma 3.3. In Section 4, after the necessary setup for directional weighted classes, we prove a weighted version of the Parcet-Rogers maximal estimate in Theorem 4.6 which, together with the extrapolation techniques of Lemma 4.3, is relied upon in the proof of our main result. Theorem 1.1 is derived in Section 5 as the Lebesgue measure case of a more general sharp weighted estimate, Corollary 1. This corollary in turn descends from Theorem 5.1, a \( L^2 \)-weighted almost-orthogonality principle for \( H_\Omega \) in the vein of [1, 28]. The final Section 6 contains the above mentioned sharp counterexamples for the model operator of Lemma 3.2 in dimension 4 and higher: the main result of this section is the lower bound of Theorem 6.3.

**Acknowledgments.** The authors are deeply grateful to Sara Maloni for fruitful discussions on the subject of completion of a lacunary set. We would also like to thank Maria J. Carro for helpful discussion related to weighted norm inequalities for directional operators. We are indebted to Keith Rogers for an expert reading and insightful comments that helped us improve the presentation. Finally, we would like to thank the anonymous referees for providing helpful comments and references.

2. **Lacunary sets of directions: definitions and notation**

In this section, we give a rigorous definition of finite order lacunary sets which will be used throughout the article. In essence, our definition is the same as the one given by Parcet and Rogers in [28].

2.1. **Lacunary sets of directions of finite order.** For convenience we keep most of the notational conventions of [28]. Throughout the paper we work in \( \mathbb{R}^n \) and consider sets of directions \( \Omega \subset S^{n-1} \). We allow the possibility that \( \text{span}(\Omega) = \mathbb{R}^d \) for some non-negative integer \( d \leq n \) and write \( \Sigma(d) := \{(j,k) : 1 \leq j < k \leq d\} \); we will drop the dependence on \( d \).
and just write $\Sigma$ when there is no ambiguity. We typically denote the members of $\Omega$ as $\omega$ and the members of $\Sigma$ as $\sigma = (j,k)$. Note that $|\Sigma(d)| = d(d - 1)/2$.

With the roles of $n, d,$ and $\Omega$ as above we assume that for each $\sigma = (j,k) \in \Sigma(d)$ we are given a sequence $\{\theta_{\sigma,\ell} : \ell \in \mathbb{Z}\}$ with the property that there exists $\lambda_{\sigma} \in (0, 1)$ such that
\begin{equation}
\theta_{\sigma,\ell+1} \leq \lambda_{\sigma} \theta_{\sigma,\ell}, \quad \theta_{\sigma,0} = \theta_0, \quad \forall \sigma.
\end{equation}
(2.1)
Here we set $\lambda := \max_{\sigma \in \Sigma} \lambda_{\sigma}$ and throughout the paper we will fix a numerical value of $\lambda \in (0, 1)$ and we will adopt the convention that all sequences $\theta_{\sigma,\lambda}$ have lacunarity constants uniformly bounded by the same number $\lambda$. A choice of orthonormal basis (ONB) of span($\Omega$) = $\mathbb{R}^d$
(2.2)
$\mathcal{B} := \{e_j : j = 1, \ldots, d\}$
and of lacunary sequences $\{\theta_{\sigma,\ell}\}$ as above induces for each $\sigma \in \Sigma(d)$ a partition of the sphere $S^{d-1}$ into sectors $S_{\sigma,\ell}$:
(2.3)
$S^{d-1} = \bigcup_{\ell \in \mathbb{Z}} S_{\sigma,\ell}, \quad S_{\sigma,\ell} = S_{(j,k)\ell} := \left\{ \omega \in S^{d-1} : \theta_{\sigma,\ell+1} \leq \left| \frac{\omega \cdot e_k}{|\omega \cdot e_j|} \right| < \theta_{\sigma,\ell} \right\}.$
We will henceforth write $\omega_j := \omega \cdot e_j$ for $1 \leq j \leq d$ once the coordinate system is clear from context. The partition above is completed by adding the set $S_{\sigma,\infty} = S_{(j,k)\infty} := S^{d-1} \cap (e_j^+ \cup e_k^+)$. We henceforth write $\mathbb{Z}^* := \mathbb{Z} \cup \{\infty\}$. Now such a partition of the sphere immediately gives a partition of $\Omega$ by setting
(2.4)
$\Omega_{\sigma,\ell} := \Omega \cap S_{\sigma,\ell}, \quad \sigma \in \Sigma, \quad \ell \in \mathbb{Z}^*.$
The family of $\binom{d}{2} = d(d - 1)/2$ partitions indexed by $\sigma \in \Sigma(d)$,
\[ \Omega = \bigcup_{\ell \in \mathbb{Z}} \Omega_{\sigma,\ell}, \]
will be called a lacunary dissection of $\Omega$, with parameters an ONB $\mathcal{B}$ as in (2.2) and a choice of sequences $\{\theta_{\sigma,\ell}\}$ as in (2.1). Note that $\{\{S_{\sigma,\ell}\}_{\ell \in \mathbb{Z}^*} : \sigma \in \Sigma(d)\}$ is a lacunary dissection of $S^{d-1}$.

We will refer to sets of the type $S_{\sigma,\ell}$ and $\Omega_{\sigma,\ell}$ as sectors of the lacunary dissection. We will also work with the partition of $\Omega$ into disjoint cells induced by a dissection, namely intersections of sectors $\Omega_{\sigma,\ell}$. More precisely, let $\mathcal{B}$ be a choice of ONB as in (2.2). Given $\ell = \{\ell_{\sigma} : \sigma \in \Sigma(d)\} \in \mathbb{Z}^\Sigma$ we define the $\ell$-cell of the dissection corresponding to $\mathcal{B}$ as
\[ S_{\ell} := \bigcap_{\sigma \in \Sigma} S_{\sigma,\ell_{\sigma}}, \quad \Omega_{\ell} := \bigcap_{\sigma \in \Sigma} \Omega_{\sigma,\ell_{\sigma}}. \]
Observe that this provides the finer partition of $S^{d-1}$ and $\Omega$, respectively, into cells
\[ S^{d-1} = \bigcup_{\ell \in \mathbb{Z}^\Sigma} S_{\ell}, \quad \Omega = \bigcup_{\ell \in \mathbb{Z}^\Sigma} \Omega_{\ell}. \]

The following definition, which is the principal assumption in our main results, was given in [28, p. 1537].

**Definition 2.2 (Lacunary set).** Let $\Omega \subset S^{n-1}$ be a set of directions with span($\Omega$) = $\mathbb{R}^d$. Then

- $\Omega$ is a lacunary set of order 0 if it consists of a single direction;
If $L$ is a positive integer, then $\Omega$ is lacunary of order $L$ if there exists an ONB $\mathcal{B}$ as in (2.2) and a choice of sequences $\{\theta_{\sigma,\ell}\}$ as in (2.1) with the property that for each $\sigma \in \Sigma(d)$ and each $\ell \in \mathbb{Z}^*$ the sector $\Omega_{\sigma,\ell}$ in (2.4) is a lacunary set of order $L - 1$.

A set $\Omega$ will be called *lacunary* if it is a finite union of lacunary sets of finite order.

For example, $\Omega$ is 1-lacunary if there exists a dissection such that, for each $\sigma \in \Sigma(d)$ and $\ell \in \mathbb{N}$ the set $\Omega_{\sigma,\ell}$ contains at most one direction.

**Remark 2.3.** Let $\Omega$ be a lacunary set of directions and $\beta \in (0, 1)$. Then $\Omega$ is a lacunary set of directions with respect to dissections given by the sequence $\theta_{\sigma,\ell} := \beta^\ell$. This is automatic if $\beta \geq \lambda$ while in the case $\beta < \lambda$ it follows easily by suitably splitting the set $\Omega$ into $O(\log \beta / \min_{\sigma} \log \lambda_{\sigma})$ congruence classes. Unless explicitly mentioned otherwise, all lacunary sets in this paper are given with respect to the sequence $\theta_{\sigma,\ell} = 2^{-\ell}$, $\ell \in \mathbb{Z}$.

As our choice of sequences $\{\theta_{\sigma,\ell}\}$ is universal, prescribing a lacunary dissection amounts to fixing an orthonormal basis $\mathcal{B}$ as in (2.2). It is also clear that for all proofs in this paper it suffices to consider the case that $\Omega$ is contained in the open positive $2^d$-tant of the sphere $S^{d-1} := S^d \cap \mathbb{R}^d_+$. While there are different coordinate systems involved in the definition of a lacunary set $\Omega$, by splitting any lacunary set into finitely many pieces we can assume this property for all dissections that come into play. Furthermore, by standard approximation arguments (e.g. monotone convergence) we can assume that $\Omega$ has empty intersection with all coordinate hyperplanes. These conventions allow us to only consider sectors $S_{\sigma,\ell}, \Omega_{\sigma,\ell}$ with $\ell \in \mathbb{Z}$ instead of $\ell \in \mathbb{Z}^*$.

On the other hand, and in contrast with the previous conventions concerning the proofs, in the statements of our theorems we always assume that the set $\Omega$ is closed. Furthermore, the basis vectors of any dissection used in the definition of a lacunary set of any order are assumed to be contained in the set. We adopt these conventions throughout the paper without further mention.

**Remark 2.4.** Although it is necessary to distinguish the case $\text{span} \Omega = \mathbb{R}^d$ with $d < n$ in the definitions, in the proofs of our estimates we will argue with $d = n$ without explicit mention; by Fubini’s theorem, this is without loss of generality.

### 3. Model operators

For $\omega \in S^{n-1}$ (re)define the directional Hilbert transform on $\mathbb{R}^n$

$$H_\omega f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) \text{sign}(\xi \cdot \omega) e^{ix \cdot \xi} d\xi.$$  

In this section we set up a representation formula for (3.1). The central result is Lemma 3.2 below. Before the statement we need to introduce some additional notation and auxiliary functions. For $\ell \in \mathbb{Z}$ and $\gamma > 0$ we consider the two-dimensional wedges

$$\Psi_{\sigma,\ell,\gamma} := \left\{ \dot{\xi} \in \mathbb{R}^n \setminus (e_{\sigma(2)})^\perp : \frac{2^{-\ell+1}}{\gamma} \leq \frac{\dot{\xi}_{\sigma(1)}}{\dot{\xi}_{\sigma(2)}} < \gamma 2^{-\ell} \right\}.$$
We are interested in the particular cases \( \gamma \in \{ n, n + 1 \} \) for which we use the special notations
\[
\Psi_{\sigma, \ell, n} =: \Psi_{\sigma, \ell}, \quad \Psi_{\sigma, \ell, n+1} =: \widetilde{\Psi}_{\sigma, \ell}.
\]
Furthermore, let \( \phi^+, \phi^- : \mathbb{R} \to [0, 1] \) be smooth functions satisfying
\[
\phi^+(x) := \begin{cases} 
0, & x < -(n + 1), \\
1, & x > -n,
\end{cases} \quad \phi^-(x) := \begin{cases} 
1, & x < \frac{-1}{2n}, \\
0, & x > \frac{-1}{2(n+1)}. 
\end{cases}
\]
We now use the functions \( \phi^+, \phi^- \) in order to define the essentially two-dimensional angular Fourier multiplier operators
\[
\widehat{K}^\pm_{\sigma, \ell, \sigma}(\xi) = \kappa^\pm_{\sigma, \ell, \sigma}(\xi_{\sigma(1)}, \xi_{\sigma(2)}) := \phi^\pm\left(2\ell_{\sigma} \frac{\xi_{\sigma(1)}}{\xi_{\sigma(2)}}\right),
\]
\[
\widehat{K}^\circ_{\sigma, \ell, \sigma}(\xi) = \kappa^\circ_{\sigma, \ell, \sigma}(\xi) := \kappa^+_{\sigma, \ell, \sigma}(\xi)\kappa^-_{\sigma, \ell, \sigma}(\xi),
\]
and their compositions
\[
K^\varepsilon_{U, \ell} := \prod_{\sigma \in U} K^\varepsilon_{\sigma, \ell, \sigma}, \quad \emptyset \subset U \subseteq \Sigma, \quad \varepsilon \in \{\pm, \circ\}^U;
\]
when \( \varepsilon = \circ \) for all \( \sigma \in U \) we simply write \( K_{U, \ell} \) in place of \( K^\circ_{U, \ell} \).

**Remark 3.1.** Let \( \varepsilon \in \{\pm, \circ\} \). We record the support conditions (see Figure 3.1)
\[
(\nabla_{\xi} \kappa^\varepsilon_{\sigma, \ell, \sigma})1_{\Psi_{\sigma, \ell}, \sigma} = (\nabla_{\xi} \kappa^\circ_{\sigma, \ell, \sigma})1_{\mathbb{R}^n \setminus \widetilde{\Psi}_{\sigma, \ell}, \sigma} = 0, \quad \kappa^\varepsilon_{\sigma, \ell, \sigma}1_{\Psi_{\sigma, \ell}, \sigma} = 1, \quad \kappa^\circ_{\sigma, \ell, \sigma}1_{\mathbb{R}^n \setminus \widetilde{\Psi}_{\sigma, \ell}, \sigma} = 0.
\]
Moreover we have the derivative estimates
\[
\sup_{|\alpha| \leq 10n} \sup_{\xi \in \mathbb{R}^n} |\xi_{\sigma(1)}|^{\alpha_1} |\xi_{\sigma(2)}|^{\alpha_2} |\partial_{\xi_{\sigma(1)}}^{\alpha_1} \partial_{\xi_{\sigma(2)}}^{\alpha_2} \kappa^\varepsilon_{\sigma, \ell, \sigma}(\xi)| \leq 1, \quad |\alpha| = \alpha_1 + \alpha_2.
\]
We will also use below that if \( \xi \notin \Psi_{\sigma, \ell, \sigma} \), then \( \kappa^\varepsilon_{\sigma, \ell, \sigma} \) is constant in a neighborhood of \( \xi \).

**Figure 3.1.** The Fourier support of the multipliers \( K^\circ_{(1, 2), \sigma}, K^-_{(1, 2), \sigma}, \) and \( K^+_{(1, 2), \sigma} \).
Lemma 3.2. Suppose $\omega \in S_\ell$, the cell of $S^{n-1}$ with lacunary parameters $\ell = \{\ell_\sigma : \sigma \in \Sigma\}$. Then we have the pointwise bound

$$|H_\omega f| \lesssim |f| + \sup_{\varnothing \subset U \subset \Sigma} |H_\omega K_{U,\ell} f| + \sup_{\varepsilon \in \{+,-\}} \sup_{\varnothing \subset U \subset \Sigma} |K_{U,\ell}^\varepsilon f|.$$  

Proof. As

$$\text{Id} = \left[ \sum_{\varnothing \subset U \subset \Sigma} (-1)^{nU+1} K_{U,\ell} \right] + \left[ \prod_{\sigma \in \Sigma} (\text{Id} - K_{\sigma,\ell_\sigma}) \right]$$

we write

(3.7) \hspace{1cm} H_\omega f = \left[ \sum_{\varnothing \subset U \subset \Sigma} (-1)^{nU+1} H_\omega K_{U,\ell} f \right] + T f

where $T$ is the Fourier multiplier with symbol

(3.8) \hspace{1cm} m(\xi) = \widehat{T}(\xi) = \text{sign}(\omega \cdot \xi) \prod_{\sigma \in \Sigma} (1 - \kappa_{\sigma,\ell_\sigma}^\omega (\xi)).

We have to treat the term $T$. First of all, we check that

(3.9) \hspace{1cm} C_\omega := \left\{ \xi \in \mathbb{R}^n : |\xi \cdot \omega| < \frac{1}{n} \max_{1 \leq j \leq n} |\omega_j \xi_j| \right\} \subset D_\ell := \bigcup_{\sigma \in \Sigma} \Psi_{\sigma,\ell_\sigma}.

This is essentially depicted in Figure 3.2 and is a sharpening of the argument in [28, Proof of Theorem A]. We prove (3.9) by showing that $\mathbb{R}^n \setminus D_\ell \subseteq \mathbb{R}^n \setminus C_\omega$. To that end let $\bar{\xi} \in \mathbb{R}^n \setminus D_\ell$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3_2.png}
\caption{Suppose $\omega$ belongs to the cell $S_\ell$. The red line is the intersection with the sphere $S^2$ of the singularity $\xi \cdot \omega = 0$ of $H_\omega$. The blue and yellow wedges are respectively $\Psi(1,2),\ell_{(1,2)}$ and $\Psi(2,3),\ell_{(2,3)}$ from (3.2). As in the depicted octant $\xi_1$ and $\xi_3$ have the same sign, $\Psi(1,3),\ell_{(1,3)}$ is not visualized.}
\end{figure}
Writing \( \eta_j := \omega_j \xi_j \) and remembering the convention \( \omega_j > 0 \) for all \( j \) we then have that
\[
- \frac{\eta_{\sigma(1)}}{\eta_{\sigma(2)}} \notin \left[ \frac{1}{n}, n \right] \quad \forall \sigma \in \Sigma.
\]
Choose \( j_* \) such that \( |\eta_{j_*}| = \max_{1 \leq j \leq n} |\eta_j| \). Now we note that if \( \eta_j \eta_{j_*} \geq 0 \) for all \( j \in \{1, \ldots, n\} \setminus \{j_*\} \) then \( \xi \notin C_\omega \) so we are done. Otherwise we define \( k_* \) by means of \( |\eta_{k_*}| := \max_{j \in \{1, \ldots, n\} \setminus \{j_*\}} |\eta_j| \); as \( |\eta_{j_*}| \geq n|\eta_{k_*}| \) we end up with
\[
|\xi \cdot \omega| = \left| \sum_{1 \leq j \leq n} \eta_j \right| \geq |\eta_{j_*}| - (n - 1)|\eta_{k_*}| \geq \frac{|\eta_{j_*}|}{n} = \max_{1 \leq j \leq n} |\omega_j \xi_j|
\]
which is the claim (3.9). Noting that
\[
\text{supp } m = \mathbb{R}^n \setminus \bigcup_{\sigma \in \Sigma} \Psi_{\sigma, f, \omega} = \mathbb{R}^n \setminus D_k
\]
this claim tells us that \( \text{supp } m \cap C_\omega = \emptyset \) whence if \( \xi \in \text{supp } m \) the signum of \((\omega \cdot \xi)\) is constant in a neighborhood of \( \xi \). Now using the easy to verify fact that \((1 - \phi^+ \phi^-) = (1 - \phi^+) + (1 - \phi^-)\) and the two summand are supported in disjoint intervals we can rewrite (3.8) as
\[
m(\xi) = \sum_{\varepsilon \in \{-, +\}}^{\Sigma} \text{sign}(\omega \cdot \xi) \kappa_{\varepsilon}(\xi), \quad \kappa_{\varepsilon}(\xi) := \prod_{\sigma \in \Sigma} \left( 1 - \kappa_{\varepsilon, \sigma, f, \omega} \right),
\]
for \( \varepsilon = \{\varepsilon_\sigma : \sigma \in \Sigma\} \). As \( \text{supp } \kappa_{\varepsilon} \) is a connected set not intersecting \( C_\omega \) we conclude that \( \text{sign}(\omega \cdot \xi) \) is constant on \( \text{supp } \kappa_{\varepsilon} \). Therefore if \( T_\varepsilon \) is the Fourier multiplier with symbol \( \kappa_{\varepsilon} \)
\[
(3.10) \quad |T f| \leq \sum_{\varepsilon \in \{-, +\}}^{\Sigma} |T_\varepsilon f|.
\]
Now we observe that the symbol of \( \text{Id} - T_\varepsilon \) is equal to
\[
1 - \prod_{\sigma \in \Sigma} \left( 1 - \kappa_{\varepsilon, \sigma, f, \omega} \right) = \sum_{\emptyset \subseteq U \subseteq \Sigma} (-1)^{|U| + 1} \prod_{\sigma \in U} \kappa_{\varepsilon, \sigma, f, \omega},
\]
and putting together the last display with (3.7) and (3.10) we achieve the pointwise estimate claimed in the Lemma.

In the next lemma we prove an annular estimate for the multiplier operators of (3.4). To do so we will need to precompose these operators with suitable Littlewood-Paley projections which we now define. Let \( p, q \) be smooth functions on \( \mathbb{R} \) with
\[
\text{supp } p \subset \{ \xi \in \mathbb{R} : \frac{1}{2} < |\xi| < 2 \}, \quad \sum_{t \in \mathbb{Z}} p(2^{-t} \xi) = 1, \quad \xi \neq 0,
\]
\[
\text{supp } q \subset \{ \xi \in \mathbb{R} : \frac{1}{4} < |\xi| < 4 \}, \quad q = 1 \text{ on } \{ \xi \in \mathbb{R} : \frac{1}{2} < |\xi| < 2 \}.
\]
Now for \( u \in \{1, \ldots, n\} \) we define the Fourier multiplier operators on \( \mathbb{R}^n \)
\[
\widehat{P_u^q} f(\xi) := \widehat{f}(\xi) p(2^{-u} \xi \cdot e_u), \quad \widehat{Q_u^q} f(\xi) := \widehat{f}(\xi) q(2^{-u} \xi \cdot e_u).
\]
Thus \( \{P_u^q\}_u \) is a one-dimensional Littlewood-Paley decomposition, acting on the \( u \)-th variable only, and being the identity with respect to all other frequency variables. Here and in the rest of the paper we write \( M_s \) for the strong maximal function and \( M_s^2 := M_s \circ M_s \).
**Lemma 3.3.** Let $\text{supp} \hat{f} \subset Q$ where $Q$ is any of the $2^3$ octants of $\mathbb{R}^3$. Let $\emptyset \subset U \subset \Sigma$, $\varepsilon \in \{+, -\}^U$. There is a choice $\nu = \nu(U, \varepsilon, Q) \in \{1, \ldots, n\}$ such that the pointwise estimate

$$|K_{U, \ell}^\nu (P_t^\nu f)(x)| \lesssim M_5^\nu (P_t^\nu f)(x), \quad x \in \mathbb{R}^3,$$

holds uniformly over all $t \in \mathbb{R}$.

**Proof.** As $\varepsilon$, $\ell$ are fixed throughout the proof, and in order to avoid proliferation of indices, we shall write below

$$\kappa_\ell = \kappa, \quad K_\ell = K, \quad K_U = K_U,$$

when these parameters are unimportant. As we are working with the strong maximal function, by rescaling on the sphere we may assume $\ell = 0$ for all $\sigma \in \Sigma$; this is just for convenience of notation as we shall see. We divide the proof to different cases according to the cardinality of the set $U \subset \Sigma$.

**Case #U = 1.** In this case there exists $\sigma \in \Sigma(3)$ such that $U = \{\sigma\}$, $K_U = K$, and we may choose either $\nu = \sigma(1)$ or $\nu = \sigma(2)$. The choice does not depend on the quadrant $Q$. To fix ideas, we work with $\sigma = (1, 2)$ and choose $\nu = 1$. By the observation (3.5) of Remark 3.1, we know that $\kappa_\sigma$ is constant in a neighborhood of $\xi$ unless $\xi \in \overline{\Sigma}_0$, in which case $|\xi_1| \sim |\xi_2|$. Therefore if $\xi \in \overline{\Sigma}_0$, and $2^{t-2} < |\xi_1| < 2^{t+2}$, there holds $2^t \sim |\xi_1| \sim |\xi_2|$ and

$$|\partial_\xi^0 \kappa_\sigma \partial_\xi^2 \kappa_\sigma (\xi)| \lesssim |\xi_1|^{-\alpha_1} |\xi_2|^{-\alpha_2} \lesssim 2^{-\alpha}, \quad \alpha = \alpha_1 + \alpha_2.$$

Using the above inequality for $\alpha = 0, \ldots, 10 \cdot 3$, it follows that

$$\Phi_\sigma(x_{\sigma(1)}, x_{\sigma(2)}) := \int_{\mathbb{R}^2} \kappa_\sigma(\xi_1, \xi_2) q(2^{-t} \xi_1) e^{i(x_{\sigma(1)} \xi_1 + x_{\sigma(2)} \xi_2)} \, d\xi_1 d\xi_2$$

satisfies

$$|\Phi_\sigma(x_{\sigma(1)}, x_{\sigma(2)})| \lesssim 2^{2t} \left(1 + 2^t |x_{\sigma(1)}| + 2^t |x_{\sigma(2)}|\right)^{-3(3+1)}.$$

We now write $f_t = P_t^\nu f$. Denoting convolution in the variables $\sigma(1), \sigma(2)$ by $*_{\sigma}$ we have that

$$K_{\sigma} f_t = (K_{\sigma} Q_t^\nu)(f_t) = \Phi_\sigma *_{\sigma} f_t.$$

Hence using (3.12) we see that

$$|K_U f_t(x)| \leq \int_{\mathbb{R}^2} |f_t(x_1 - y_1, x_2 - y_2, x_3)| |\Phi_\sigma(y_1, y_2)| \, dy \lesssim M_5(f_t)(x)$$

as claimed.

**Case #U = 2.** In this case $U = \{\sigma, \tau\}$ for some $\sigma, \tau \in \Sigma(3)$ and necessarily $\sigma, \tau$ must have a common component. We choose $\nu$ to be this common component. This choice also does not depend on the quadrant $Q$. To fix ideas $\sigma = (1, 2), \tau = (1, 3)$ and we choose $\nu = 1$. Note that in this case $K_U = K_{(1,2)} K_{(1,3)}$. With the same notation of (3.11) from the previous case we have the equality

$$K_U f_t = (K_{(1,2)} Q_t^\nu) \circ (K_{(2,3)} Q_t^\nu)(f_t) = \Phi_{(1,2)} *_{(1,2)} \Phi_{(1,3)} *_{(1,3)} f_t.$$
so using (3.12) again we see that
\[
|K_U f_t(x)| \leq \int_{\mathbb{R}^2 \times \mathbb{R}^2} |f_t(x_1 - y_1 - z_1, x_2 - y_2, x_3 - z_3)||\Phi_{(1,2)}(y_1, y_2)||\Phi_{(1,3)}(z_1, z_3)| \, dy \, dz
\]
\[
\leq M^2_2(f_t)(x)
\]
as claimed.

Case \#U = 3. We show that this case reduces to the preceding ones, with choice of \(\nu\) depending on the quadrant \(Q\). Let
\[Q_\sigma = \{ \xi \in \mathbb{R}^3 : \xi_{(1)}\xi_{(2)} \geq 0 \}.\]
Notice that the constraints on the supports of \(\phi^\pm\) imply that
\[
\kappa^{-}_\sigma, e_{\ell} 1_{Q_\sigma} \equiv 0, \quad \kappa^{+}_\sigma, e_{\ell} 1_{Q_\sigma} \equiv 1, \quad \forall \sigma \in \Sigma.
\]
As for each of the 8 quadrants \(Q\) of \(\mathbb{R}^3\) there exists (at least one) \(\sigma_Q \in \Sigma\) such that \(Q \subset Q_{\sigma_Q}\), we see that
\[
K^e_{U,\ell} 1_Q = \begin{cases} 
0, & \text{if } \exists \sigma \in U \text{ with } \varepsilon_\sigma = -, \\
K^e_{U,\ell \setminus \sigma_Q} 1_{Q_\sigma}, & \text{otherwise}.
\end{cases}
\]
As \(#\{U \setminus \{\sigma_Q\}\} = 2\) for each quadrant \(Q\) the proof follows by the cases \(#U \in \{1, 2\}\) considered above. \(\square\)

4. Weighted norm inequalities for directional maximal operators

We dedicate this section to the discussion of weighted norm inequalities for the maximal directional operator. These will serve as a tool for the proof of Theorem 1.1; in fact, they will be used to prove a weighted almost orthogonality principle that subsumes both Theorem 1.1 and its weighted analogue, which will be stated at the end of this section. However, we do think they are also of independent interest.

The weighted theory of the directional maximal operator has been studied, at least in the two-dimensional case, in [13], for the case of 1-lacunary sets of directions. Here we recall all the basic definitions and tools, and then proceed to prove weighted norm inequalities for the directional maximal function \(M_\Omega\) associated to a finite order lacunary set \(\Omega \subset \mathbb{S}^{n-1}\). In essence, the main result of this section, Theorem 4.6, is a weighted generalization of the main result of [28] by Parcet and Rogers.

4.1. Directional \(A_p\) weights. We begin by defining the appropriate directional \(A_p\) classes. The easiest way to define the appropriate class is to ask for non-negative, locally integrable functions \(w\) (we will refer to such functions as weights) such that for all nice functions \(f\) we have
\[
\|M_\Omega f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}, \quad \|f\|_{L^p(w)} := \left( \int |f|^p w \right)^{1/p}, \quad 1 < p < \infty,
\]
where \(\Omega\) is a set of directions such that \(M_\Omega\) is bounded on \(L^p(\mathbb{R}^n)\). Without explicit mention, we work under the purely qualitative assumptions that all weights appearing below will be continuous and nonvanishing functions on \(\mathbb{R}^n\); this assumption may be removed via a standard approximation procedure which we omit. We will very soon specialize to sets \(\Omega\) which are lacunary of finite order so we encourage the reader to keep this example in mind. Note that
for smooth functions $f$ we have $M_{\Omega} f = M_{\Omega} f$. We can then assume that $\Omega$ is closed when deriving necessary conditions for $w$.

For $\omega \in \Omega$, $x \in \mathbb{R}^n$ and $\eta > 0$ we then define segments and corresponding one-dimensional averages of $f \in C(\mathbb{R}^d)$ as follows

$$I(x, \eta, \omega) := \{x + t \omega : |t| < \eta\}, \quad \langle f \rangle_{I(x, \eta, \omega)} := \frac{1}{2\eta} \int_{-\eta}^{\eta} f(x + t \omega) \, dt.$$ We set $I_\Omega := \{I(x, \eta, \omega) : x \in \mathbb{R}^d, \eta > 0, \omega \in \Omega\}$ and for $p \in (1, \infty)$ we adopt the usual notation for the dual weight $\sigma := w^{\frac{1}{p^*}}$. Now the $L^p(\mathbb{R}^n)$-boundedness of $M_{\Omega}$ clearly implies the boundedness of $M_\omega$ on $L^p(I(x, \omega))$, where $I(x, \omega) := \{x + t \omega : t \in \mathbb{R}\}$, uniformly in $x \in \mathbb{R}^n$ and $\omega \in \Omega$. Now testing this one-dimensional boundedness property $M_\omega$ for some fixed $p \in (1, \infty)$ against functions of the form $\sigma 1_{I(x, \eta, \omega)}$ shows the necessity of the directional $A_p$ condition

$$[w]_{A_p^\omega} := \sup_{I \in I_\Omega} \int_I w \left( \int_I \sigma \right)^{p-1} < \infty;$$

here we remember that we have made the qualitative assumption that $w$ is a continuous non-vanishing function.

Note that if we write $w(x) = w(x \cdot \omega, x \cdot \omega^\perp)$, the previous condition means that for almost every $x \in \mathbb{R}^n$ and $\omega \in \Omega$, the one-dimensional weight $\nu_{x,\omega}(s) := w(s, x \cdot \omega^\perp)$, $s \in \mathbb{R}$, is in $A_p(\mathbb{R})$, with uniformly bounded $A_p$ constant:

$$\sup_{x \in \mathbb{R}^n, \omega \in \Omega} \nu_{x,\omega} = [w]_{A_p^\omega} < \infty.$$ We complete the set of definitions by defining $A_1^\Omega$ to be the class of weights $w$ such that

$$[w]_{A_1^\omega} := \sup_{x \in \mathbb{R}^n} \frac{M_{\Omega} w(x)}{w(x)} < \infty.$$ A well known class of Muckenhoupt weights is produced be considering $\Omega = \{e_1, \ldots, e_n\}$; then $A_p^\omega$ is just the class $A_p^*$ of strong or $n$-parameter Muckenhoupt weights. We also note that an obvious corollary of one dimensional theory is that

$$\|M_\omega\|_{L^p(w) \to L^p(w)} \leq [w]_{A_p^\omega}^{\frac{1}{p-1}}, \quad \omega \in \Omega,$$

and the implicit constant is independent of $w$ and $\omega$. We refer to [4] for the sharp one-dimensional weighted bound for $M_\omega$.

4.2. **Extrapolation for $A_p^\omega$ weights.** Having established the appropriate $A_p^\omega$ classes, we now proceed to proving one of the most useful properties of weighted norm inequalities, that of extrapolation.

We begin by noting that, as in the case of classical $A_p$ weights, it is easy to create $A_p^\omega$-weights by using the Rubio de Francia method and factorization; see [12, Lemmata 2.1,2.2]. We omit the proofs which are essentially identical to the one-directional case.
**Lemma 4.3.** Let $w \in A^\Omega_p$. For a nonnegative function $g \in L^p(w)$ we define

$$E_g := \sum_{k=0}^{\infty} \frac{M_{\Omega}^{(k)} g}{2^k \|M_{\Omega}\|_{L^p(w)}^k}$$

Then $E_g$ satisfies the following properties

(i) $g \leq E_g$.

(ii) For every $g \in L^p(w)$ we have $\|E_g\|_{L^p(w)} \leq 2\|g\|_{L^p(w)}$.

(iii) If $\|M_{\Omega}\|_{L^p(w)\rightarrow L^p(w)} < \infty$ then $E_g$ is an $A^\Omega_1$ weight with constant

$$[E_g]_{A^\Omega_1} \leq 2\|M_{\Omega}\|_{L^p(w)\rightarrow L^p(w)}.$$ 

Furthermore for all exponents $1 \leq p < \infty$, $1 < p_0 < \infty$ and weights $u, w$ there holds

$$\left\{ \begin{array}{ll}
[wu^{p-p_0}]_{A^\Omega_p} \leq [w]_{A^\Omega_p}[u]_{A^\Omega_1}, & p \leq p_0 \\
[w^{p-p_0} u^{-p}]_{A^\Omega_p} \leq [w]_{A^\Omega_1}^{p-1} [w]_{A^\Omega_p}^{p-p_0}, & p > p_0.
\end{array} \right.$$ 

We now provide the basic extrapolation result for $A^\Omega_p$ weights which will be our main tool for passing from $L^2(w)$-estimates to $L^p(w)$-estimates for all $p \in (1, \infty)$. This result and its proof are completely analogous to [12, Theorem 3.1], making use of Lemma 4.3 as the analogous of [12, Lemmata 2.1, 2.2].

**Lemma 4.4.** Let $\Omega \subset S^{n-1}$ be a set of directions such that for all $1 < p < \infty$ and for all $w \in A^\Omega_p$ we have the weighted boundedness property $M_{\Omega} : L^p(w) \rightarrow L^p(w)$. Assume that for some family of pairs of nonnegative functions, $(f, g)$, for some $p_0 \in [1, \infty]$ and for all $w \in A_{p_0}$ we have

$$\left( \int_{\mathbb{R}^n} g^{p_0} w \right)^{\frac{1}{p_0}} \leq C \mathcal{P}(\|w\|_{A^\Omega_{p_0}}) \left( \int_{\mathbb{R}^n} f^{p_0} w \right)^{\frac{1}{p_0}},$$

where $\mathcal{P} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing function and $C > 0$ does not depend on $w$ or the pairs $(f, g)$. Then for all $1 < p < \infty$ we have

$$\left( \int_{\mathbb{R}^n} g^{\frac{1}{p}} w \right)^{\frac{1}{p}} \leq C \mathcal{K}(w) \left( \int_{\mathbb{R}^n} f^{\frac{1}{p}} w \right)^{\frac{1}{p}},$$

with

$$\mathcal{K}(w) := \begin{cases} \mathcal{P}(\|w\|_{A^\Omega_p}(2\|M_{\Omega}\|_{L^p(w)})^{p_0-p}), & p < p_0, \\ \mathcal{P}(\|w\|_{A^\Omega_{p_0}}^{p-1}(2\|M_{\Omega}\|_{L^p(w)})^{p_0-p_0}), & p > p_0. \end{cases}$$

### 4.5. Weighted inequalities for the lacunary directional maximal operator.

In this subsection, we consider directional maximal operators associated to lacunary sets of order $L$.

According to the previous discussion, the condition $w \in A^\Omega_p$ is necessary for the boundedness property $M_{\Omega} : L^p(w) \rightarrow L^p(w)$. In this paragraph we also show the sufficiency of condition $A^\Omega_p$, thus giving a characterization of the $A^\Omega_p$ class in terms of $M_{\Omega}$.

**Theorem 4.6.** Let $\Omega \subset S^{n-1}$ be a lacunary set of directions of order $L$, where $L$ is a positive integer, and $w$ be a weight. For every $1 < p < \infty$, the following are equivalent.
For all \( f \in L^p(w) \) we have \( \|M\Omega f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \), with implicit constant depending on \( w \), the dimension, and the lacunarity constants of \( \Omega \).

Furthermore, if \( w \in A_\Omega^\alpha \) then we have the estimate \( \|M\Omega\Omega f\|_{L^p(w) \to L^p(w)} \lesssim \|w\|_{A_\Omega^\alpha}^\delta \) for some exponent \( \delta = \delta(p, n) > 0 \) and implicit constant independent of \( w \).

**Remark 4.7.** We note here that in dimension \( n = 2 \) and for \( L = 1 \) this theorem was known and contained in [13, Theorem 4].

**4.8. Proof of Theorem 4.6.** Recall from §2 that a set \( \Omega \subset S^{n-1} \) is called lacunary of order \( L \), where \( L \geq 1 \) is a positive integer, if there exists a dissection as in (2.3) such that for each \( \sigma \in \Sigma(n) \) and \( \ell \in \mathbb{N} \), the sets \( \Omega_{\sigma, \ell} \) are lacunary of order \( L - 1 \). As mentioned before, cf. Remark 2.3, we assume that \( \Omega \) is closed and that the axes \( \{e_1, \ldots, e_n\} \) of the dissection of order \( L \) are contained in \( \Omega \). Then, the inclusion \( A_\Omega^\alpha \subset A_\sigma^\alpha \) holds, the latter being the class of strong \( A_p \) weights with respect to these coordinate axes. In consequence, the strong maximal function \( M_s \) is automatically bounded on \( L^p(w) \) for \( w \in A_\Omega^\alpha \).

As in the proof of [27, Theorem A] we rely on the covering of the singularity hyperplane \( \xi \cdot \omega = 0 \) by finitely overlapping unions of two dimensional wedges \( \{\Psi_{\sigma, \ell} : \sigma \in \Sigma\} \) defined in (3.2), where \( \ell = (\ell_\sigma : \sigma \in \Sigma) \) is the unique index in \( \mathbb{Z}^\Sigma \) such that \( \omega \) belongs to the cell \( \Omega_{\ell} \). The core of the proof is contained in the following two lemmata which are weighted versions of the corresponding results from [27].

The first result we need is a weighted analogue of [27, Lemma 1.1]. Note that it does not require the lacunarity assumption on \( \Omega \) and the weight class needed is just the usual class of strong Muckenhoupt weights \( A_\sigma^\alpha \).

**Lemma 4.9.** Let \( p > 1 \) and \( w \in A_\sigma^\alpha \) be a weight. There holds

\[
\|M\Omega f\|_{L^p(w)} \lesssim \|w\|_{A_\sigma^\alpha}^{-\frac{n}{p-1}} \sup_{\sigma \neq U \subseteq \Sigma} \|M_{\Omega_{\ell}}K_{U, \ell} f\|_{L^p(w)},
\]

with the implicit constant depending upon dimension and \( p \).

**Proof.** The proof follows from the arguments in the proof of [27, Lemma 1.1]. Indeed one just needs to note that the corresponding unweighted estimate in [27] is proved via the use of pointwise estimates, which of course are independent of the underlying measure, and the boundedness of the strong maximal function \( M_s f \) on \( L^p(\mathbb{R}^n) \). The latter fact is replaced by the observation that \( M_s \) maps \( L^p(w) \) to itself whenever \( w \in A_\sigma^\alpha \), and satisfies the quantitative norm estimate

\[
\|M_s\|_{L^p(w) \to L^p(w)} \lesssim \|w\|_{A_\sigma^\alpha}^{-\frac{n}{p-1}}.
\]

Here again we use the one-dimensional sharp weighted estimate for the Hardy-Littlewood maximal operator from [4].

The second result is a weighted square function estimate for the angular multipliers \( K_{U, \ell} \) associated to a lacunary dissection of the sphere.
Lemma 4.10. Let $1 < p < \infty$ and $\Sigma$ corresponding to a given dissection of the sphere. Then for all $w \in A_p^*$ we have
\[
\sup_{U \subseteq \Sigma} \left\| \left( \sum_{\ell \in \mathbb{Z}^d} \left| K_{U,\ell} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \leq [w]_{A_p^*}^\beta \|f\|_{L^p(w)}.
\]
The implicit constant depends upon dimension $n$, $p$, and $\beta > 0$ depends on $p$ and $n$.

Proof. As in the proof of [27, Lemma 1.2] we note that it will be enough to prove the $L^p(w)$-boundedness of the randomized map $T$ given as
\[
f \mapsto \left( \sum_{\ell \in \mathbb{Z}^d} \chi_{\ell} \prod_{\sigma \in U} |\hat{K}_{\sigma,\ell} f|^\gamma \right)^{\frac{1}{\gamma}} =: (m\hat{f})^\gamma,
\]
uniformly over choices of signs $\{\chi_{\ell}\}_{\ell \in \mathbb{Z}^d}$. The unweighted $L^2(\mathbb{R}^n)$-boundedness of this map follows simply by Plancherel and the finite overlap property of the supports $\{\overline{U}_{\sigma,\ell}: \ell \in \mathbb{N}\}$, which shows that $m \in L^\infty$, uniformly over choices of signs. For $L^p(w)$-bounds, we need an $A_p^*$-weighted version of the standard Marcinkiewicz multiplier theorem. This can be found for example in [21, Theorem 3] so the proof of the lemma reduces to checking a number of conditions on averaged derivatives of $m$. In fact these conditions are identical to the hypothesis of the unweighted Marcinkiewicz multiplier theorem, as can be found for example in [30, p. 109] and can be verified by using estimates (3.6) for each single multiplier $\hat{K}_{\sigma,\ell}$. An inspection of the proof, which relies on the weighted vector valued boundedness of frequency projections on rectangles, and the weighted multiparameter Littlewood-Paley inequalities, shows that there exists a constant $\beta$ depending on $n$ and $p$ such that $\|T\|_{L^p(w) \rightarrow L^p(w)} \leq [w]_{A_p^*}^\beta$. \qed

We now give the conclusion of the proof of Theorem 4.6.

Conclusion of the proof of Theorem 4.6. The key step is the estimate
\[
\|M\Omega\|_{L^p(w) \rightarrow L^p(w)} \leq [w]_{A_p^*}^\delta \sup_{\sigma \in \Sigma} \sup_{\ell \in \mathbb{Z}} \|M_{\Omega_{\sigma,\ell}}\|_{L^p(w) \rightarrow L^p(w)}
\]
for some exponent $\delta > 0$ depending on the dimension $n$ and on $p$. Indeed, if $L = 1$, each sector $\Omega_{\sigma,\ell}$ contains at most one direction, whence using the well-known weighted maximal inequality for each such direction and the obvious inequality $[w]_{A_p^*} \leq [w]_{A_p^*}$ for $\omega \in \Omega$,
\[
\sup_{\sigma \in \Sigma} \sup_{\ell \in \mathbb{Z}} \|M_{\Omega_{\sigma,\ell}}\|_{L^p(w) \rightarrow L^p(w)} \leq \sup_{\omega \in \Omega} \|M_{\Omega}(\omega)\|_{L^p(w) \rightarrow L^p(w)}
\]
Coupling the latter display with (4.1) yields the claimed estimate in Theorem 4.6. We now proceed by induction and derive the $L$-lacunary case assuming the $L - 1$ holds true. Estimate (4.1) and the inductive assumption read
\[
\|M\Omega\|_{L^p(w) \rightarrow L^p(w)} \leq [w]_{A_p^*}^\delta \sup_{\sigma \in \Sigma} \sup_{\ell \in \mathbb{Z}} \|M_{\Omega_{\sigma,\ell}}\|_{L^p(w) \rightarrow L^p(w)} \leq [w]_{A_p^*}^\delta \sup_{\sigma \in \Sigma} \sup_{\ell \in \mathbb{Z}} \|M_{\Omega_{\sigma,\ell}}\|^{\delta(L-1)}_{A_p^*} \leq [w]_{A_p^*}^{\delta L}
\]
where in the last inequality we have used the obvious fact that $\sup_{\sigma,\ell}[w]_{A_p^*} \leq [w]_{A_p^*}$ and $[w]_{A_p^*} \leq [w]_{A_p^*}$. This completes the proof of the theorem up to showing estimate (4.1) holds true. \qed
Proof of (4.1). We first perform the proof in the case $p \geq 2$. Let us for a moment fix a $U \subseteq \Sigma$ and write $\mathbb{Z}^2 = \mathbb{Z}^U \otimes \mathbb{Z}^{\Sigma \setminus U}$ so that given $\ell = \{\ell_\sigma\}_{\sigma \in \Sigma}$ we decompose $\ell = \tau \times t$ with $\tau = \{\tau_\sigma : \sigma \in U\}$. Replacing the supremum by an $\ell^p$ function gives

$$\sup_{\ell \in \mathbb{Z}^2} \| M_{\Omega, \ell} f_\tau \|_{L^p(w)} \leq \sup_{\sigma \in \Sigma} \| M_{\Omega, \sigma, \ell} \|_{L^p(w) \to L^p(w)} \left( \sum_{\tau \in Z^U} |f_\tau|^p \right)^{\frac{1}{p}} \| f_\tau \|_{L^2(w)}.$$  

As $p \geq 2$ estimate (4.2) implies

$$\sup_{\ell \in \mathbb{Z}^2} \| M_{\Omega, \ell} f_\tau \|_{L^p(w)} \leq \sup_{\sigma \in \Sigma} \| M_{\Omega, \sigma, \ell} \|_{L^p(w) \to L^p(w)} \left( \sum_{\tau \in Z^S} |f_\tau|^2 \right)^{\frac{1}{2}} \| f_\tau \|_{L^2(w)}.$$  

Now (4.1) follows by taking $f_\tau := K_{U, \tau}$ where $\tau = \{\tau_\sigma : \sigma \in U\}$ and bounding the right hand side in the last display, from above, by Lemma 4.10, and the left hand side of the last display, from below, by Lemma 4.9. For $1 < p < 2$ we note that, by monotone convergence, it suffices to show the estimate for every finite subset of $\Omega$, which we still call $\Omega$. Then $\| M_\Omega \|_{L^p(w) \to L^p(w)} < \infty$, $w \in A^\Omega_p$, so we can interpolate between the estimates

$$\sup_{\ell \in \mathbb{Z}^2} \| M_{\Omega, \ell} f_\tau \|_{L^p(w)} \leq \| M_{\Omega, \ell} \|_{L^p(w) \to L^p(w)} \left( \sum_{\tau \in Z^S} |f_\tau|^2 \right)^{\frac{1}{2}} \| f_\tau \|_{L^2(w)}$$

and (4.2) to conclude

$$\sup_{\ell \in \mathbb{Z}^2} \| M_{\Omega, \ell} f_\tau \|_{L^p(w)} \leq \left( \sup_{\sigma \in \Sigma} \| M_{\Omega, \sigma, \ell} \|_{L^p(w) \to L^p(w)} \right)^{\frac{p}{2}} \left( \sum_{\tau \in Z^S} |f_\tau|^2 \right)^{\frac{1}{2}} \| f_\tau \|_{L^2(w)}.$$  

Taking again $f_\tau = K_{U, \tau}$ an application of Lemmata 4.10 and 4.9 yields

$$\| M_\Omega \|_{L^p(w) \to L^p(w)} \leq \left( \sup_{\sigma \in \Sigma} \| M_{\Omega, \sigma, \ell} \|_{L^p(w) \to L^p(w)} \right)^{\frac{p}{2}} \| f_\tau \|_{L^p(w)}$$

with $\gamma = \beta + n/(p - 1)$. As we have assumed that $\| M_\Omega \|_{L^p(w) \to L^p(w)} < \infty$ we may rearrange and complete the proof of the theorem.

5. An almost orthogonality principle for the maximal Hilbert transform

We now prove an almost orthogonality principle for the maximal Hilbert transform of a set $\Omega \subset S^2$. In the statements below it is convenient to write for all nonnegative integers $N$, weights $w$ on $\mathbb{R}^3$, and $\Omega \subset S^2$

$$\Theta_N(\Omega, w) := \sup_{\emptyset \subset \Omega \subset \mathbb{R}^3} \| H_\Omega \|_{L^2(\mathbb{R}^3; w) \to L^2(\mathbb{R}^3; w)}.$$  

Theorem 5.1. There exist $C, \gamma \geq 1$ such that the following holds. Let $N$ be a positive integer, $\mathcal{B}$ be a choice of ONB, $\Omega \subset S^2$ a set of directions containing $\mathcal{B}$ and $w \in A^\Omega_2$. Then

$$\Theta_N(\Omega, w) \leq C[w]^{\gamma} \left[ \sqrt{\log N} + \sup_{\sigma \in \Sigma} \Theta_N(\Omega, \sigma, w) \right]$$

where the lacunary dissection is taken with respect to $\mathcal{B}$ as in (2.3).
By iterative application of the almost orthogonality principle, and extrapolation, we obtain the following corollary, of which Theorem 1.1 is the particular case \( w = 1 \).

**Corollary 1.** Let \( 1 < p < \infty \) and \( L \geq 0 \). There exists constants \( C = C_{p,L}, \gamma = \gamma_{p,L} \) such that for any \( \Omega \subset S^2 \) lacunary set of order \( L \) and \( w \in A^\Omega_p \)

\[
\sup_{O \subset \Omega \atop \#O \leq N} \| H_O \|_{L^p(\mathbb{R}^3; w)} \leq C \| w \|^\gamma_{A^\Omega_p} \sqrt{\log N}.
\]

The proof of Theorem 5.1 rests upon the results of the previous sections, as well as on the proposition below, a weighted version of the Chang-Wilson-Wolff principle, which we state and prove before the main argument. In the statement of the proposition below we remember that \( A^\ast_p = A^\Omega_p \) with \( \Omega \) being the canonical basis of \( \mathbb{R}^n \), namely \( \Omega = \{e_1, \ldots, e_n\} \). We also use the standard notation \( A^\ast_\infty := \cup_{p>1} A^\ast_p \).

**Proposition 5.2.** Let \( \{K_1, \ldots, K_N\} \) be Fourier multiplier operators on \( \mathbb{R}^n \) with uniform bound

\[
\sup_{1 \leq j \leq N} \| K_j \|_{L^2(w) \to L^2(w)} \leq [w]^\alpha_{A^\ast_p}
\]

for some \( \alpha > 0 \). Let \( \{P^j_t\}_{t \in \mathbb{Z}} \) be a smooth Littlewood-Paley decomposition acting on the \( v \)-th frequency variable, where \( 1 \leq v \leq n \). For \( w \in A^\ast_p \) and \( 1 < p < \infty \) we then have

\[
\left\| \sup_{1 \leq j \leq N} |K_j f| \right\|_{L^p(w)} \leq [w]^\gamma_{A^\ast_p} \| f \|_{L^p(w)} + (\log(N + 1))^{\frac{1}{2}} \left( \sum_{t \in \mathbb{Z}} \sup_{1 \leq j \leq N} |K_j P^j_t f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)}
\]

for some exponent \( \gamma = \gamma(\alpha, p, n) \) and implicit constant depending on \( \alpha, p, n \).

**Proof:** To simplify the notation we work with \( v = 1 \) and set

\[
K^\ast f := \sup_{1 \leq j \leq N} |K_j f|, \quad DK^\ast := \left( \sum_{t \in \mathbb{Z}} \sup_{1 \leq j \leq N} |K_j P^j_t f|^2 \right)^{\frac{1}{2}}.
\]

Let \( \{D_j : j \in \mathbb{Z}\} \) be the standard dyadic filtration on \( \mathbb{R} \), \( E_j \) be the associated sequence of conditional expectations, and \( \Delta f \) denote the associated martingale square function. Let \( E^1_j f := E_j g \otimes h(x_2, \ldots, x_n) \) by \( E^1_j f := E_j g \otimes h \) and denote by \( \Delta^1 f \) the associated martingale square functions. The Chang-Wilson-Wolff inequality [6] tells us that if \( w \) is an \( A^\ast_\infty \)-weight then

\[
w \left( \{ x \in \mathbb{R}^n : |g(x) - E^1_0 g(x)| > 2\lambda, |\Delta^1 f(x)| \leq \gamma \lambda \} \right) \leq A \exp \left( - \frac{b}{[w]_{A^\ast_\infty}^{(1)} \gamma^2} \right) w \left( \{ x \in \mathbb{R}^n : |M_{e_1} g(x)| > \lambda \} \right),
\]

where \( A, b \) are absolute positive constants and

\[
[w]_{A^\ast_\infty}^{(1)} := \sup_{x \in \mathbb{R}^n} [w(x + \cdot e_1)]_{A^\ast_\infty},
\]

here \([\cdot]_{A^\ast_\infty} \) denotes the Wilson \( A^\ast_\infty \) constant of a weight on the real line, see [32]. The inequality (5.1) for \( n > 1 \) is in fact obtained from the one dimensional version of [6] and Fubini. As
\([w]_{A^+_p} \leq [w]_{A^*_p}\) and proceeding exactly as in the proof of [10, Corollary 1.14] we can use the above inequality to reach

\[
\|K^* f\|_{L^p(w)} \leq \|M_{\mathbf{e}_1} f\|_{L^p(w)} + \sup_{1 \leq /f_i \leq N} \|M_{\mathbf{e}_i} K_{/f_i} f\|_{L^p(w)} + \sqrt{\log(N + 1)} \||M_{\mathbf{e}_i}||_{L^p(w)} \|DK^* f\|_{L^p(w)}
\]

(5.2)

where \(\mathbf{e}_i f := (M_{\mathbf{e}_i} |f|^r)^{1/r}\), and \(r > 1\) can be chosen arbitrarily close to \(1\); the implicit constant depends on \(p, r\), and polynomially on \([w]_{A^*_p}\). Since our weight \(w \in A^*_p\), \(M_{\mathbf{e}_i}\) is a bounded operator on \(L^p(w)\). Furthermore, using the reverse Hölder property for \(A^*_\infty\) weights, see e.g. [16, Theorem 1.4], we actually have the openness property

\[ [w]_{A^*_p} \leq 2[w]_{A^*_p}, \quad r \leq \frac{p}{p - c([w]_{A^*_\infty})^{-1}}, \]

where the positive constant \(c = c(p, n) \leq 1\) can be explicitly computed. Therefore \(\mathbf{e}_i f\) is also a bounded operator on \(L^p(w)\) provided \(r\) is chosen small enough to comply with the restriction in the last display. Making use of these \(L^p(w)\)-bounds in (5.2) finally yields the proposition. \(\Box\)

5.3. Proof of Theorem 5.1. In this proof the implicit constants occurring in the inequalities as well as the exponent \(\gamma\) are meant to be absolute and are allowed to vary without explicit mention. Let \(\Omega \subset S^2\) and an ONB \(\mathcal{B} \subset \Omega\) be given. Fix a subset \(O \subset \Omega\) with \(#O = N\). Of course the set of addresses of the cells whose intersection with \(O\) is nonempty, in symbols \(\mathcal{S}_O := \{\ell \in \mathbb{Z}^2 : O \cap B_{\ell} \neq \emptyset\}\), has cardinality at most \(N\). We use the pointwise estimate of Lemma 3.2 for each \(\omega \in O\) to obtain that

\[
|H_{O f}| \leq |f| + \sup_{\omega \subset U \subset \omega} \sup_{\ell \in \mathcal{L}_O} |H_{O \cap \ell} K_{U, \ell} f| + \sup_{\omega \subset U \subset \omega} \sup_{\ell \in \mathcal{L}_O} \sup_{\sigma \in \{+,-\}^U} |K_{U, \ell}^\sigma f|.
\]

(5.3)

We may ignore the first summand on the right hand side. We bound the norm of the second summand on the right hand side by a constant multiple of

\[
\sup_{\omega \subset U \subset \omega} \left\| \left( \sum_{\ell \in \mathcal{L}_O} \left| H_{O \cap \ell} K_{U, \ell} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w)} \leq B \sup_{\sigma \in \{+,-\}^U} \left\| \left( \sum_{\ell \in \mathcal{L}} \left| K_{U, \ell} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w)}
\]

(5.4)

where in the last step we have used the weighted estimate of Lemma 4.10, and we have also used the easy estimate

\[
B := \sup_{\|g\|_{L^2(w)} = 1} \left\| \left( \sum_{\ell \in \mathcal{L}} \left| H_{O \cap \ell} g \right|^2 \right)^{\frac{1}{2}} \right\|_{L^2(w)} \leq \sup_{\sigma \in \{+,-\}^U} \Theta_N(\Omega_{\sigma, \ell}, w).
\]

The third summand in (5.3) is treated in the next Proposition. In fact, coupling the bounds (5.4) above, and (5.5) below, with the pointwise estimate (5.3), and noticing that \([w]_{A^*_2} \leq [w]_{A^*}\) since the coordinate basis vectors are contained in \(\Omega\), completes the proof of Theorem 5.1.

Proposition 5.4. Let \(\mathcal{L}\) be a finite subset of \(\mathbb{Z}^3\). Then

\[
\sup_{\omega \subset U \subset \omega} \sup_{\ell \in \mathcal{L}} \left\| \sup_{\sigma \in \{+,-\}^U} |K_{U, \ell}^\sigma f| \right\|_{L^2(w)} \leq [w]_{A^*_2} \sqrt{\log(|\mathcal{L}| + 1)} \|f\|_{L^2(w)}.
\]

(5.5)
Proof. Fix $U \subseteq \Sigma, \varepsilon \in \{+,-\}^U$ throughout the proof. By means of compositions of Hilbert transforms along the coordinate directions we may decompose

$$f = \sum_Q f_Q, \quad \|f_Q\|_{L^2(w)} \leq [w]_{A^2}^{3} \|f\|_{L^2(w)},$$

where each $f_Q$ has frequency support in one of the octants $Q$ of $\mathbb{R}^3$. By virtue of the norm estimate of the above display, we may fix one of these octants $Q$ and prove (5.5) for functions $f$ whose frequency support is contained in $Q$, which we do here onwards. Now we remember that by Lemma 4.10 the multiplier operators $\{K^\varepsilon_{U,\ell} : \ell \in \mathbb{L}\}$ satisfy weighted $L^2$ bounds with weighted operator norms bounded polynomially in $[w]_{A^2}$, uniformly in $U$ and $\ell$. This allows us to use Proposition 5.2 on the $N = \#\mathbb{L}$ Fourier multiplier operators $\{K^\varepsilon_{U,\ell} : \ell \in \mathbb{L}\}$, to get that

$$\left\| \sup_{\ell \in \mathbb{L}} |K^\varepsilon_{U,\ell} f| \right\|_{L^2(w)} \leq [w]_{A^2}^{3} \|f\|_{L^2(w)} + \sqrt{\log(N + 1)} \left( \sum_{\ell \in \mathbb{L}} |K^\varepsilon_{U,\ell} P_\ell f|^2 \right)^{\frac{1}{2}} \|f\|_{L^2(w)}$$

for any $v \in \{1, 2, 3\}$. We make the choice $v = v(U, \varepsilon, Q) \in \{1, 2, 3\}$ according to Lemma 3.3, so that based on supp $\hat{f} \subset Q$

$$|K^\varepsilon_{U,\ell} (P_\ell f)(x)| \leq M_3^2(P_\ell f)(x).$$

Combining the last two inequalities followed by weighted Fefferman-Stein and Littlewood-Paley estimates

$$\left\| \sup_{\ell \in \mathbb{L}} |K^\varepsilon_{U,\ell} f| \right\|_{L^2(w)} \leq [w]_{A^2}^{3} \|f\|_{L^2(w)} + \sqrt{\log(N + 1)} \left( \sum_{\ell \in \mathbb{L}} M_3^2(P_\ell f)^2 \right)^{\frac{1}{2}} \|f\|_{L^2(w)}$$

which is the claimed (5.5).

6. Quantitative counterexamples for the model operator

In this section, we show that sharp higher dimensional $(n \geq 4)$ analogues of Theorem 1.1 cannot be attacked by means of the model operators of Section 3, which are essentially compositions of smooth two-dimensional lacunary cutoffs. To wit, we show that the maximal operators

$$\sup_{\ell \in \mathbb{L}} \left| \prod_{\sigma \in \Sigma} (1 - K^\varepsilon_{\sigma,\ell,\sigma}) f \right| \quad \sup_{\ell \in \mathbb{L}} |K^\varepsilon_{U,\ell} f|,$$

intervening in the decomposition of the maximal Hilbert transform induced by Lemma 3.2, have operator norms which grow at order $(\log \#\mathbb{L})^{\frac{3}{2} + \frac{1}{4}}$. For $n \geq 4$, this is unfavorable compared to the maximal Hilbert transform over finite subsets $O$ of a (finite order) lacunary set $\Omega$, whose operator norm is of order at most $\log(\#O)$; see [28, Corollary 4.1]. Our counterexamples are obtained by careful tensoring of the lower bound for the two-dimensional case $\Sigma = \{(1,2)\}$ which in turn descends from the main theorem of [19].
We use the notation of Section 3 and in particular of (3.3). However in this section it will be more convenient to use the equivalent (up to identity) definition
\[ H_\omega f(x) := \int_{\mathbb{R}^n} \hat{f}(\xi) 1_{(0,\infty)}(\xi \cdot \omega) e^{ix \cdot \xi} \, d\xi. \]

6.1. **A lower bound in \( n = 2 \).** The lower bound for \( p = 2 \) of Karagulyan [19] combined with the upper bound for all \( 1 < p < \infty \) of [9, 11] tells us that for all \( L \geq 0 \) and \( 1 < p < \infty \) there exists \( c_{p,L} > 0 \) such that the following holds: Whenever \( \Omega \subset S^1 \) is a lacunary set of order \( L \) and \( O \subset \Omega \) is finite there exists a Schwartz function \( f_O \) with

\[ \|f_O\|_{L^p(\mathbb{R}^2)} = 1, \quad \|H_O f_O\|_{L^p(\mathbb{R}^2)} \geq c_{p,L} \sqrt{\log \#O}. \]  

Let now \( \Omega \) be a lacunary set of order 1 with \( \Omega \subset \{ \omega \in S^1 : \omega_1, \omega_2 > 0 \} \). We can take \( f_O \) to be frequency supported in the quadrants \( \{ \xi \in \mathbb{R}^2 : \xi_1 \xi_2 < 0 \} \) as \( H_O \) acts trivially on the remaining frequency plane. By a symmetry argument we can actually take

\[ \text{supp } \hat{f}_O \subset Q_{(1,2)} := \{ \xi \in \mathbb{R}^2 : \xi_1 > 0, \xi_2 \leq 0 \}. \]

Rewriting (3.7) in this particular case we see that if \( \omega \in \Omega \cap S_{(1,2),\ell(\omega)} \) and \( \text{supp } \hat{f} \subset Q_{(1,2)} \) then

\[ \hat{H}_\omega f(\xi) = \mathcal{F}(H_\omega K^\circ_{(1,2),\ell(\omega)} f)(\xi) + 1_{(0,\infty)}(\xi \cdot \omega)(1 - \kappa^\circ_{(1,2),\ell(\omega)}(\xi)) \hat{f}(\xi) \]
\[ = \mathcal{F}(H_\omega K^\circ_{(1,2),\ell(\omega)} f)(\xi) + (1 - \kappa^+_{(1,2),\ell(\omega)}(\xi)) \hat{f}(\xi). \]

We notice that, for some absolute constant \( C_p \)

\[ \left\| \left( \sum_{\omega \in \Omega} |H_\omega K^\circ_{(1,2),\ell(\omega)} f_O|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^2)} \leq C_p \|f_O\|_{L^p(\mathbb{R}^2)} = C_p; \]

this \( L^p \)-boundedness is most easily seen by proving the weighted \( L^2 \)-bound as in Section 5 first. Comparing this last display with (6.1) we obtain that

\[ \| \sup_{\omega \in \Omega} |K^+_{(1,2),\ell(\omega)} f_O| \|_{L^p(\mathbb{R}^2)} \geq c_p \sqrt{\log \#O} \]

provided \( \#O \) is large enough, with \( c_p = c_{p,1}/2 \). The arguments of Section 3 and symmetry considerations finally show that there exist positive absolute constants \( c_p, C_p \) such that for \( \varepsilon \in \{+, \circ, -\} \) and all finite index sets \( \mathbb{L} \subset \mathbb{Z} \) we have

\[ c_p \leq \frac{1}{\sqrt{\log \#\mathbb{L}}} \left\| f \mapsto \sup_{\ell \in \mathbb{L}} |K_{(1,2),\ell}^\varepsilon f| \right\|_{L^p(\mathbb{R}^2)} \leq C_p. \]

**Remark 6.2.** Just like the maximal Hilbert transform, the maximal operators defined in (6.2) are invariant under dilation and reflection through the frequency origin, and act trivially on functions supported outside \( \pm Q_{(1,2)} \). For any fixed \( \mathbb{L} \subset \mathbb{Z} \) with \( \#\mathbb{L} = N \), using the lower bound in (6.2), the reflection symmetry and an approximation argument we may find \( M > 0 \) and a Schwartz function \( f_\mathbb{L} \) with

\[ \text{supp } \hat{f}_\mathbb{L} \subset Q_{(1,2)} \cap A^{(1,2)}(2^{-M}, 2^M), \quad \| \sup_{\ell \in \mathbb{L}} |K_{(1,2),\ell}^\varepsilon f_\mathbb{L}| \|_{L^p(\mathbb{R}^2)} \geq C_p \sqrt{\log N} \|f_\mathbb{L}\|_{L^p(\mathbb{R}^2)}. \]
where
\[ A^{(1,2)}(a, b) := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : a < \sqrt{\xi_1^2 + \xi_2^2} < b \right\}. \]

Given any \( s \in \mathbb{R} \), the dilation invariance can then be used to find \( f_{s,L} \) with the same properties as \( f_L \) in (6.3) but \( \text{supp} f_{s,L} \subset Q_{(1,2)} \cap A^{(1,2)}(2^s, 2^{s+M}) \).

The next result is the anticipated counterexample to estimate (5.5) in dimensions 4 and higher.

**Theorem 6.3.** Let \( n \geq 2 \) be the dimension of the ambient space. Then

\[
\inf_{\varepsilon \in \{+, -\}^2} \sup_{L \subset \mathbb{Z}^n, \varepsilon L \subseteq \Omega} \sup_{\phi \subseteq \Omega \subseteq \Sigma} \left\| f \mapsto \sup_{\ell \in L} \left\| K_{U, \ell}^\varepsilon f \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \right\| \geq C_p \left( \sqrt{\log N} \right)^{\frac{n}{2}}.
\]

**Proof.** It suffices to prove the statement for even \( n = 2d \) and for \( N > 10d \), say. By symmetry considerations we may argue in the case where \( \varepsilon = (+, \ldots, +) \). Let \( \mathbb{L} \) be the set of \( N^d \) indices such that \( S_\ell \cap \Omega \neq \emptyset \), where \( \Omega \) is the set of vectors on \( S^{2d-1} \) obtained by normalizing the vectors \( (x_1, \ldots, x_n) \) with components

\[ x_{2k-1} = 2^{-2kN}, \quad x_{2k} = 2^{-2kN-m_k}, \quad m_k \in \{1, \ldots, N\}, \quad k = 1, \ldots, d. \]

In practice \( \ell = \ell(m_1, \ldots, m_d) \in \mathbb{L} \) is completely determined by the \( 2d - 1 \) conditions

\[ \ell_{(2k-1,2k)} = m_k, \quad k = 1, \ldots, d; \quad \ell_{(2k-1,2k+1)} = 2N, \quad k = 1, \ldots, d-1. \]

As

\[ 1 - \prod_{\sigma \in \Sigma} \left( 1 - \kappa_{\sigma, \ell, \sigma}^+ \right) = \sum_{\phi \subseteq U \subseteq \Omega} (-1)^{\# U + 1} \prod_{\sigma \in \Sigma} \kappa_{\sigma, \ell, \sigma}^+, \]

estimate (6.4) will follow if we prove that

\[
\left\| f \mapsto \sup_{\ell \in L} \left\| \prod_{\sigma \in \Sigma} (\text{Id} - K_{\sigma, \ell, \sigma}^+) f \right\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \right\| \geq C_p \left( \sqrt{\log N} \right)^{d},
\]

where product denotes composition. Now for each \( k = 1, \ldots, d \) define the function of two variables \( f_k = f_k(x_{2k-1}, x_{2k}) \) given by \( f_{s,L} \) in Remark 6.2 with the pair \( (2k-1, 2k) \) in the place of \((1, 2)\), with \( L = \{1, \ldots, N\} \), and with \( s \) chosen so that

\[ \text{supp} f_k \subset Q_{(2k-1,2k)} \cap A^{(2k-1,2k)}(2^{-3kM}, 2^{-(3k-1)M}). \]

Here

\[ Q_{(2k-1,2k)} := \{ \xi \in \mathbb{R}^2 : \xi_{2k-1} > 0, \xi_{2k} < 0 \}. \]

We now define

\[ f(x) := \prod_{k=1}^d f_k(x_{2k-1}, x_{2k}). \]

The point of this choice is that if \( \sigma = (\sigma(1), \sigma(2)) \) is such that \( \sigma(1), \sigma(2) \) have the same parity then \( \xi_{\sigma(1)}, \xi_{\sigma(2)} \) have the same sign on the frequency support of \( f \), so that \( (\text{Id} - K_{\sigma, \ell, \sigma}^+) f = f \).

Also, unless \( \sigma = (2k-1, 2k) \) for some \( k = 1, \ldots, d \), there holds

\[ \text{supp} \hat{f} \subset \left\{ \xi \in \mathbb{R}^n : \frac{|\xi_{\sigma(1)}|}{|\xi_{\sigma(2)}|} \geq 2^{3M} \right\}, \]

\[ (\text{Id} - K_{\sigma, \ell, \sigma}^+) = \text{Id} \quad \text{on the cone} \quad |\xi_{\sigma(1)}| > (2d + 1)2^{-\ell} |\xi_{\sigma(2)}|. \]
which is a larger cone than the one where \( \hat{f} \) is supported, as \( \ell_\sigma \geq N \) in this case. Summarizing we may delete from the composition in (6.5) all the \( \sigma \) which are not of the form \( \sigma = (2k - 1, 2k) \), and we have for all \( m = 1, \ldots, N \) that

\[
\prod_{\sigma \in \Sigma} \left( \text{Id} - K^+_{\sigma, (\ell(m_1, \ldots, m_d)_\sigma)} \right) f = \prod_{k=1}^d \left( \text{Id} - K^+_{(2k-1, 2k), m_k} \right) f_k,
\]

with the caveat that the product sign on the left hand side denotes composition while the product sign on the right hand side denotes pointwise product. Therefore using (6.3) for the lower bound in the third line

\[
\left\| \sup_{\ell \in \mathcal{L}} \left\| \prod_{\sigma \in \Sigma} (\text{Id} - K^+_{\sigma, (\ell(m_1, \ldots, m_d)_\sigma)}) f \right\|_{L^p(\mathbb{R}^n)} \right\|
\]

\[
= \left\| \prod_{k=1}^d \left( \text{Id} - K^+_{(2k-1, 2k), m_k} \right) f_k \right\|_{L^p(\mathbb{R}^n)}
\]

\[
= \prod_{k=1}^d \sup_{m_k \in \{1, \ldots, N\}} \left\| \left( \text{Id} - K^+_{(2k-1, 2k), m_k} \right) f_k \right\|_{L^p(\mathbb{R}^n)}
\]

\[
\geq c_p^d (\log N) \frac{d}{2} \prod_{k=1}^d \| f_k \|_{L^p(\mathbb{R}^n)} = c_p^d (\log N) \frac{d}{2} \| f \|_{L^p(\mathbb{R}^n)}.
\]

This proves (6.5) and thus completes the proof of the theorem. \( \Box \)

**Remark 6.4.** This remark shows that the counterexample of Theorem 6.3 is sharp. We say that \( U \subset \Sigma(n) \) has no odd cycles if it does not contain tuples of pairs which are images under permutation of \( \{1, \ldots, n\} \) of the tuple of pairs

\[
\{(1, 2), (2, 3), \ldots, (k - 1, k), (1, k)\}
\]

with \( k \) odd. In the case that \( U \) has odd cycles, in each given quadrant of \( \mathbb{R}^n \) at least one of the multipliers \( K_{\sigma, \ell}^\varepsilon \) is trivial for both \( \varepsilon = \pm \); we can thus reduce to the case that \( U \) has no odd cycles. This case is treated below.

Suppose that \( \{v_1, \ldots, v_j\} \subset \{1, \ldots, n\} \) are such that for all \( \sigma \in U \) there exists \( j \) such that \( v_j \in \sigma \); in this case \( \{v_1, \ldots, v_j\} \) is called spanning set of \( U \). Notice that for every \( U \subset \Sigma \) we may find a spanning set with \( s \leq \lfloor n/2 \rfloor \). Arguing in similar fashion as in the proof of Lemma 3.3 we may obtain the pointwise estimate

\[
|K_{U, \ell}(P_{t_1}^{v_1} \circ \cdots \circ P_{t_s}^{v_s} f)(x)| \lesssim M_2^n(P_{t_1}^{v_1} \circ \cdots \circ P_{t_s}^{v_s} f)(x), \quad x \in \mathbb{R}^n,
\]

uniformly over all \( t_1, \ldots, t_s \in \mathbb{R} \). Now, we may use an \( s \)-parametric version of the Chang-Wilson-Wolff inequality to reduce estimates for the maximal operator associated to the multipliers \( K_{U, \ell}^\varepsilon \) over \( \ell \in \mathcal{L} \) to an \( s \)-fold Littlewood-Paley square function estimate involving the left hand side of (6.6) with a loss of \((\log \# \mathcal{L})^{\frac{d}{2}}\). An application of the bound (6.6) as in Proposition
5.4 will thus lead to the estimate
\[
\sup_{\varepsilon \in \{+,-\}} \left\| \sup_{U \in \mathcal{L}} |K^\varepsilon_{U,L}f| \right\|_{L^p(\mathbb{R}^n)} \leq (\log |\mathcal{L}|)^{\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^n)},
\]
which, together with the previously made observation that \( s \) may be taken \( \leq [n/2] \) shows the sharpness of Theorem 6.3; in general the worst case is \( U = \{(1,2),(3,4),\ldots,(2[n/2]-1,2[n/2]),\} \). We leave the details to the interested reader.

References

[1] A. Alfonseca, F. Soria, and A. Vargas, *A remark on maximal operators along directions in \( \mathbb{R}^2 \)*, Math. Res. Lett. **10** (2003), no. 1, 41–49. MR1960122 ↑3, 5
[2] M. Bateman, *Kakeya sets and directional maximal operators in the plane*, Duke Math. J. **147** (2009), no. 1, 55–77. MR2494456 ↑2
[3] M. Bateman and C. Thiele, *\( L^p \) estimates for the Hilbert transforms along a one-variable vector field*, Anal. PDE **6** (2013), no. 7, 1577–1600. MR3148061 ↑5
[4] S. M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 253–272. MR1124164 ↑13, 15
[5] A. Carbery, *Differentiation in lacunary directions and an extension of the Marcinkiewicz multiplier theorem*, Ann. Inst. Fourier (Grenoble) **38** (1988), no. 1, 157–168. MR949003 ↑3
[6] S.-Y. A. Chang, J. M. Wilson, and T. H. Wolff, *Some weighted norm inequalities concerning the Schrödinger operators*, Comment. Math. Helv. **60** (1985), no. 2, 217–246. MR800004 ↑18
[7] A. Córdoba and R. Fefferman, *On the equivalence between the boundedness of certain classes of maximal and multiplier operators in Fourier analysis*, Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 2, 423–425. MR0433117 ↑2
[8] C. Demeter, *Singular integrals along \( N \) directions in \( \mathbb{R}^2 \)*, Proc. Amer. Math. Soc. **138** (2010), no. 12, 4433–4442. MR2680067 ↑3
[9] C. Demeter and F. Di Plinio, *Logarithmic \( L^p \) bounds for maximal directional singular integrals in the plane*, J. Geom. Anal. **24** (2014), no. 1, 375–416. MR3145928 ↑3, 21
[10] F. Di Plinio, S. Guo, C. Thiele, and P. Zorin-Kranich, *Square functions for bi-lipschitz maps and directional operators* (201706), available at 1706.07111. ↑5, 19
[11] F. Di Plinio and I. Parissis, *A sharp estimate for the Hilbert transform along finite order lacunary sets of directions*, Israel J. Math., to appear (2017), available at 1704.02918. ↑3, 21
[12] J. Duoandikoetxea, *Extrapolation of weights revisited: new proofs and sharp bounds*, J. Funct. Anal. **260** (2011), no. 6, 1886–1901. MR2754896 ↑13, 14
[13] J. Duoandikoetxea and A. Moyua, *Weighted inequalities for square and maximal functions in the plane*, Studia Math. **102** (1992), no. 1, 39–47. MR1164631 ↑12, 15
[14] S. Guo, *Hilbert transform along measurable vector fields constant on Lipschitz curves: \( L^2 \) boundedness*, Anal. PDE **8** (2015), no. 5, 1263–1288. MR3393679 ↑5
[15] P. Hagelstein, *Maximal operators associated to sets of directions of Hausdorff and Minkowski dimension zero*, Recent advances in harmonic analysis and applications, 2013, pp. 131–138. MR3066883 ↑2
[16] P. Hagelstein and I. Parissis, *Weighted Solyanik estimates for the strong maximal function*, Publ. Mat. **62** (2018), no. 1, 133–159. ↑19
[17] K. E. Hare, *Maximal operators and Cantor sets*, Canad. Math. Bull. **43** (2000), no. 3, 330–342. MR1776061 ↑2
[18] R. A. Hunt, *An estimate of the conjugate function*, Studia Math. **44** (1972), 371–377. Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, IV. MR0338667 ↑3
[19] G. A. Karagulyan, *On unboundedness of maximal operators for directional Hilbert transforms*, Proc. Amer. Math. Soc. **135** (2007), no. 10, 3133–3141. MR2322743 ↑2, 3, 20, 21
[20] N. H. Katz, *Maximal operators over arbitrary sets of directions*, Duke Math. J. **97** (1999), no. 1, 67–79. MR1681088 ↑2
[21] D. S. Kurtz, *Littlewood-Paley and multiplier theorems on weighted \( L^p \) spaces*, Trans. Amer. Math. Soc. **259** (1980), no. 1, 235–254. MR561835 ↑16
[22] I. Laba, A. Marinelli, and M. Pramanik, *On the maximal directional Hilbert transform*, preprint arXiv:1707.01061 (July 2017), available at 1707.01061.

[23] M. Lacey and X. Li, *On a conjecture of E. M. Stein on the Hilbert transform on vector fields*, Mem. Amer. Math. Soc. **205** (2010), no. 965, viii+72. MR2654385

[24] M. Lacey and C. Thiele, *A proof of boundedness of the Carleson operator*, Math. Res. Lett. **7** (2000), no. 4, 361–370. MR1783613

[25] M. T. Lacey and X. Li, *Maximal theorems for the directional Hilbert transform on the plane*, Trans. Amer. Math. Soc. **358** (2006), no. 9, 4099–4117. MR2219012

[26] A. Nagel, E. M. Stein, and S. Wainger, *Differentiation in lacunary directions*, Proc. Nat. Acad. Sci. U.S.A. **75** (1978), no. 3, 1060–1062. MR0466470

[27] J. Parcet and K. M. Rogers, *Differentiation of integrals in higher dimensions*, Proc. Natl. Acad. Sci. USA **110** (2013), no. 13, 4941–4944. MR3047650

[28] J. Parcet and K. M. Rogers, *Directional maximal operators and lacunarity in higher dimensions*, Amer. J. Math. **137** (2015), no. 6, 1535–1557. MR3432267

[29] P. Sjögren and P. Sjölin, *Littlewood-Paley decompositions and Fourier multipliers with singularities on certain sets*, Ann. Inst. Fourier (Grenoble) **31** (1981), no. 1, vii, 157–175. MR613033

[30] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095

[31] J.-O. Strömberg, *Maximal functions associated to rectangles with uniformly distributed directions*, Ann. Math. (2) **107** (1978), no. 2, 399–402. MR0481883

[32] M. Wilson, *Weighted Littlewood-Paley theory and exponential-square integrability*, Lecture Notes in Mathematics, vol. 1924, Springer, Berlin, 2008. MR2359017

Department of Mathematics, University of Virginia, Box 400137, Charlottesville, VA 22904, USA
E-mail address: francesco.diplinio@virginia.edu

Departamento de Matemáticas, Universidad del País Vasco, APTDO. 644, 48080 Bilbao, Spain and IKERBASQUE, Basque Foundation for Science, Bilbao, Spain
E-mail address: ioannis.parissis@ehu.es