Coloring n-String Tangles

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ABSTRACT
This expository paper describes how the knot invariant Fox coloring can be applied to tangles.

1. Introduction
This expository paper describes how the knot invariant Fox coloring [3,8] can be applied to tangles. An n-string tangle T is a 3-ball with n strings properly embedded in the 3-ball. The boundary of the 3-ball and the 2n endpoints of the n-strings on the boundary of the 3-ball are not allowed to move. Tangles were first used by John H. Conway to tabulate knots [10]. Following the presentation in [7], we will describe coloring via systems of linear equations so that only an introductory background in linear algebra will be needed. Fox coloring is related to many beautiful areas in topology. Our interest in this method of coloring links and tangles is to make this paper accessible to non-mathematicians as this method is used computationally to solve tangle equations arising from protein-DNA interactions [2]. Also, this approach can make open problems in this area accessible to undergraduates. For example, the results of [11,6] can be proved using only this linear algebra definition of Fox coloring combined with a neat trick of Przytycki [8].

We will begin with a brief review on coloring knots/links in section 2. In this section we will provide examples, but no proofs. For proofs see [7]. Most of the proofs for knots/links are also similar to those for tangles given in section 3. In section 4, we extend the coloring definition to tangles containing a finite number of circles. In section 5, we give some formulas for determining these invariants for 3-string braids and 2-string rational tangles. In sections 7 and 8, we discuss embedding tangles in knots. We make some concluding remarks in section 9.

2. Coloring knots and links
An m-coloring of a diagram of a knot or link or tangle is a function C : {arcs of a diagram} \rightarrow \mathbb{Z}_m where the elements of \mathbb{Z}_m = \{0, 1, \cdots, m-1\} are called colors and where at each crossing the following relation holds: if x is the color corresponding to the overarc and y and z are the two colors corresponding to the two underarcs, then y + z - 2x = 0 mod m (Fig. 1A). If the coloring function is the constant map (i.e., all the arcs are assigned the same value or color), then the coloring is said to be trivial. A link is said to be m-colorable if there exists a non-trivial m-coloring. Coloring mod 3 can easily distinguish a trefoil from an unknot. Any projection of a trefoil can be colored non-trivially mod 3 while any projection of an unknot can only be trivially colored. See Fig. 1B. We explain how to determine if a knot or link is m-colorable below using an example.

2.1. Example: A figure-eight knot is 5-colorable
Let us color a particular projection of 4_{1} (also called the figure-eight knot) in figure 2. Color each arc of this diagram of 4_{1} using x_1, x_2, x_3, x_4. Note that every knot K with k crossings has exactly k arcs. Then, this particular diagram of 4_{1} has four arcs since it has four crossings in this projection. The crossings are described by the equations shown in figure 2. Writing these equations in matrix form, we obtain Eqn. 2.1
Let \( M_{4_1} \) be the 4 \times 4 coefficient matrix in Eqn 2.1. In order to transform this matrix, \( M_{4_1} \), into echelon form, \( EF(M_{4_1}) \), we will only use the following elementary row operations:

a) Exchange two rows (row \( i \) \( \leftrightarrow \) row \( j \))

b) Add a multiple of one row to a different row (row \( i \) \( \rightarrow \) row \( i \) + \( t \cdot \) row \( j \) where \( i \neq j, t \in \mathbb{Z} \))

c) Multiply a row by -1 (row \( i \) \( \rightarrow \) -row \( i \))

\[
\begin{pmatrix}
-2 & 1 & 1 & 0 \\
1 & 0 & -2 & 1 \\
0 & -2 & 1 & 1 \\
1 & 1 & 0 & -2
\end{pmatrix}
\times
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}
\mod m
\] (2.1)

For those familiar with group presentations, we are forming a finitely generated abelian group where the arcs correspond to generators while the crossing equations give relations among these generators. Given a knot/link, tangle \( K \), \( M_K \) is the presentation matrix corresponding to this group. The allowed row operations allow us to simplify the relations without changing the group.
Each entry of the last row of $EF(M_{4_1})$ (Eqn. 2.2) is zero. The last row of the echelon form of a coloring matrix will always be a row of all zeros. This is because all knots and links can be colored trivially; i.e., given any $a \in \mathbb{Z}_m$, $(x_1, x_2, \cdots, x_k) = (a, a, \cdots, a)$ will always be a solution to the system of coloring equations.

From $EF(M_{4_1})$ (Eqn. 2.2), we see that if $m = 5$, the mod 5 solutions to the system of equations in Eqn. 2.1 are $(x_1, x_2, x_3, x_4) = (2a - b, 3a - 2b, a, b), a, b \in \mathbb{Z}_5$. For example if we let $a = 1, b = 0$, then we have the non-trivial 5-coloring, $(x_1, x_2, x_3, x_4) = (2, 3, 1, 0)$. Thus $4_1$ is 5-colorable. If $m$ is a multiple of 5, $m = 5r$ for some $r \in \mathbb{Z}$, then $(x_1, x_2, x_3, x_4) = (2r, 3r, 1r, 0r)$ will be a solution to $M_{4_1}x = 0 \mod 5r$. Hence $4_1$ is also $m$-colorable if $m$ is a multiple of 5. If $m$ is not a multiple of 5, then $4_1$ is only non-trivially colored mod $m$.

Note that we did not use a scaling operation $row_i \rightarrow t \cdot row_j$, $t \in \mathbb{Z} \{-\pm 1\}$, as part of the three row operations above to convert $M_{4_1}$ to echelon form, $EF(M_{4_1})$. In the example above, notice that scaling by $t = \frac{1}{5}$ was not done on the third row of $EF(M_{4_1})$. Had we scaled by $t = \frac{1}{5}$, we would lose the information that the linear system in equation 2.2 has a nontrivial mod 5 solution. Also, had we scaled the third row by 3, we would have been led to the false conclusion that $4_1$ is 3-colorable which it is not.

There are other invariants that can be gleaned from this matrix method of presenting a link. Since all knots/links have $m$ trivial $m$-colorings, the determinant of the coloring matrix is always zero. However, the absolute value of the determinant of the matrix obtained after removing one row and one column, $d(L)$, is an invariant. For example, $d(4_1) = 5$. A link $L$ is $m$-colorable if and only if $gcd(m, d(L)) > 1$. The coloring matrix with one row and one column removed is the same as the Alexander matrix when $t = -1$. Hence this determinant is actually the Alexander polynomial evaluated at -1. For more information on the Alexander matrix/polynomial see [7].

3. Coloring of n-string Tangles

We can similarly color n-string tangles. One of the invariants coming from coloring a tangle will depend on a chosen ordering of the $2n$ endpoints of the $n$ strings. We will call the arcs which have one endpoint on the boundary of the 3-ball endpoint arcs. We will fix a particular ordering for the endpoint arcs. For example, for a 2-string tangle, we will label the endpoint arcs in a clockwise manner starting with labeling the top left arc $x_1$ as in Fig. 3. The arcs which are not endpoint arcs will be called interior arcs.

\[ x_6 + x_7 - 2x_1 = 0 \]
\[ x_1 + x_5 - 2x_6 = 0 \]
\[ x_4 + x_6 - 2x_5 = 0 \]
\[ x_2 + x_8 - 2x_7 = 0 \]
\[ x_3 + x_7 - 2x_8 = 0 \]
\[ x_4 + x_8 - 2x_2 = 0 \]

Fig. 3. Coloring A 2-string Tangle

Note that from Fig. 3, at each crossing we form an equation just as in the knot or link case. This equation represents a row in the matrix we are going to form out of this colored tangle. Each arc represents a column of this matrix. A matrix which is row equivalent to a matrix which comes from coloring a tangle $T$ will be called a coloring matrix of $T$. In the 2-string tangle example in Fig. 3, there are six crossings and eight arcs, thereby giving a $6 \times (6 + 2)$ coloring matrix with entries of zeroes, ones and negative twos. This is one of the differences between knots and tangles. We end up having a non-square matrix when coloring a tangle. Normally if an $n$-string tangle has $k$ crossings, it will have $k + n$ arcs, and hence its coloring matrix will have $k$ rows and
$k + n$ columns. See the note just before the proof of Theorem 3.2 for an example of an $n$ string tangle with $k$ crossings for which we choose a coloring matrix which is not $k \times (k + n)$.

Based on the labeling of the given 2-string tangle in Fig. 3, we get the system of linear equations, $(M_T)x = 0$ in Eqn. 3.3. Notice that we put the endpoint arcs unknowns, $x_1, x_2, x_3, x_4$, as the four rightmost columns of matrix $M_T$.

\[
\begin{pmatrix}
0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -2 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & -2 & 0
\end{pmatrix}
\begin{pmatrix}
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\] (3.3)

After performing the allowed elementary row operations, we obtain an echelon form of $M_T$. Recall that scaling a row is not allowed. An echelon form, $EF(M_T)$ is:

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\
0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & -2 & 5 & -3
\end{pmatrix}
\] (3.4)

Since we are dealing with $n = 2$ strings, we are interested in the lower right hand corner $2 \times 4$ submatrix of the echelon form of the matrix $M_T$. We show below in general that for an $n$-string tangle, this lower right-hand corner $n \times 2(n)$ submatrix is an invariant up to the allowed elementary row operations. To make it an invariant, we define the standard echelon form of a matrix.

Let $EF(M_T) = \{a_{ij}\}_{1 \leq i \leq k, 1 \leq j \leq (k+n)}$ be an echelon form of a matrix of size $k \times (k + n)$. A leading entry of $EF(M_T)$ is the first nonzero entry of a row of $EF(M_T)$. A matrix $M_T$ is in standard echelon form if the following three properties hold:

(i) It is in echelon form.
(ii) Its leading entries are positive.
(iii) If $a_{ij}$ is a leading entry of the $i$th row, then $0 \leq a_{\lambda j} \leq a_{ij} - 1, 1 \leq \lambda < i$, i.e., all entries above a leading entry are non-negative and less than that leading entry.

**Lemma 3.1.** Let $SF(M_T)$ be the standard echelon form of a matrix $M_T$. Then $SF(M_T)$ is unique.

**Proof.** Similar to showing reduced echelon form is unique. \qed

From Eqn. 3.4, we obtain the standard echelon form, $SF(M_T)$, based on the three criteria listed above:
Theorem 3.2. Suppose we have chosen a fixed ordering of the endpoint arcs. The following are invariants of an \( n \)-string tangle \( T \):

1. \( d_U(T) = \) absolute value of the the determinant of the upper left hand corner \( (k - n) \times (k - n) \) submatrix of \( T \).
2. \( M_l(T) = \) the \( n \times 2n \) lower right hand corner submatrix of \( SF(M_T) \).

For the tangle in Fig. 3.3, \( d_U(T) = 3 \) and \( M_l(T) = \begin{pmatrix} 1 & -4 & 2 \\ 0 & 2 & -5 & 3 \end{pmatrix} \).

Note: It is possible that one or more strings of an \( n \)-string tangle does not pass under any string. Such a string will project to a single arc. See, for example, the tangle in Fig. 4. To calculate the invariant \( M_l(T) \) for an \( n \)-string tangle, \( T \), we need to have \( 2n \) distinct variables corresponding to endpoint arcs. Hence if any string projects to a single arc, we will doubly label this arc with two variables, \( x_i \) and \( x_j \) (depending on the ordering of the endpoint arcs) and add the equation, \( x_i - x_j = 0 \). For example, the matrix in Eqn. 3.6 is a coloring matrix for the one crossing tangle in Fig. 4. Additionally, doubly labeling any arc and adding an equation(s) equating the variables corresponding to this doubly labeled arc does not affect the invariants listed in Theorem 3.2.

\[
SF(M_T) = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & -2 & 0 \\
0 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 1 & -2 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 3 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -4 & 2 \\
0 & 0 & 0 & 0 & 2 & -5 & 3 & -4
\end{pmatrix}
\]

(3.5)

Fig. 4. Arcs can be doubly labeled.

\[
\begin{pmatrix}
1 & 0 & 1 & -2 \\
0 & 1 & 0 & -1
\end{pmatrix}
\]

(3.6)

Proof. [Theorem 3.2]

Two coloring matrices of a tangle diagram may differ with respect to how the interior arcs are labeled. One can convert between \( k \times (k + n) \) coloring matrices for the same tangle diagram which differ with respect to interior arc labeling by performing column operations on the first \( k - 2n \) columns. Such column operations only affect the sign of the determinant and do not affect the lower right \( n \times 2n \) matrix since no column operations are performed on the last \( 2n \) columns. Similarly doubly labeling an arc has no affect on \( d_U(T) \) and \( M_l(T) \).

Our allowed row operations can only change the sign of the determinant. Also, no matter how the allowed row operations are performed, \( SF(M_T) \) is unique.
Hence $d_U(T)$ and $M_l(T)$ are invariants of a given tangle diagram. Thus we only need to check if they are the same for two different tangle diagrams corresponding to the same tangle. Hence we only need to check if they are preserved under Reidemeister moves (Fig. 5).

A Reidemeister move can be thought of in terms of modifying a subtangle within a tangle. For example, an RI move can be thought of as replacing a 1-string subtangle containing no crossings with a 1-string subtangle containing exactly one crossing (or vice versa). An RI move results in the addition (or removal) of a crossing and the creation (or deletion via joining) of a new arc. This results in the addition (or removal) of one equation, $x - y = 0$, and one variable. The new equation can be used to eliminate the new variable from all other equations. Since $x = y$, the RI move does not affect the color of the endpoint arc(s) of the RI subtangle (Fig. 5, top). Hence after eliminating the new variable from equations resulting from crossings outside of the RI subtangle, the only difference between the coloring matrices is the addition (or removal) of a row and column. Hence since the endpoint colors are not affected, $M_l(T)$ is unchanged by an RI move. As the leading entry of the added (or deleted) row is 1, the determinant, $d_U(T)$, is unchanged by an RI move.

Similarly an RII move consists of modifying a 2-string subtangle (Fig. 5, middle). In this case, an RII move results in the creation (or deletion) of two new crossings and two new arcs. Since the endpoint arc colors of the RII subtangle are not affected by an RII move, we can again remove the new subtangle endpoint arc variable from any equation resulting from crossings outside of the RII subtangle so that these equations are identical both before and after the RII move. Hence $M_l(T)$ is unchanged by an RII move. Also since the leading entries of the added (or deleted) rows are 1, the determinant, $d_U(T)$, is unchanged by an RII move.

Similarly, the endpoint arc colors of the 3-string RIII subtangle are not affected by an RIII move (Fig. 5, bottom). Hence $M_l(T)$ is unchanged by an RIII move. The equation corresponding to the interior arc of the
RIII subtangle is affected by an RIII move, but as an interior arc of the RIII subtangle, this does not affect $M_l(T)$. Since the leading entry of the row corresponding to this interior arc of the RIII subtangle is 1, the determinant, $d_{U_l}(T)$, is also unchanged by an RIII move.

Thus $M_l(T)$ and $d_{U_l}(T)$ are not affected by any of the Reidemeister moves and hence are tangle invariants.

4. Other definitions of tangle coloring

We defined an n-string tangle as 3-ball containing n properly embedded arcs. Sometimes one would also like to allow a finite number of circles to be embedded within the 3-ball. We can also apply coloring to these tangles. In this case we not only label all arcs in the tangle diagram (including those from both strings and circles), but we also label any circular component in the tangle diagram. We also add a row of all zeros for each such closed circular component in the tangle diagram. For example, the 2-string tangle in Fig. 6 contains two circles. One of these projects to a closed circular component (labeled $x_5$) while the other projects to single arc (labeled $x_6$) in this tangle diagram. The former results in Eqn. 3: $0 = 0$ while the later is involved in two equations, Eqn. 4. Other definitions of tangle coloring

In order to obtain the most information from the coloring equations, we will not use $SF(M_T)$ in this case. We can obtain an echelon form, but we will then place the n rows with a leading entry corresponding to an endpoint arc as the last n rows even below rows of all zero’s. Hence after obtaining an echelon form, all rows of all zeros should be moved above the last n rows. Thus we obtain the matrix in Eqn. 4.7B. Thus $d_u(T) = 0$

and $M_l(T) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

![Fig. 6. A tangle containing two arcs and two circular components](image)

Jozef Pryztycki determined a relationship among endpoint arcs which all tangles must satisfy:

**Theorem 4.1.** [8] If $T$ is an n-string tangle (possibly containing a finite number of circles), then $M_l(T)$ is row equivalent to a matrix where the first row consists of alternating 1’s and -1’s.

We will illustrate his theorem and proof with an example. The tangle in Fig. 7A contains an unknotted, unlinked circle. Note for the tangle diagram in Fig. 7A, we have the following relationship among the endpoint arcs: Eqn 1 - Eqn 2 + Eqn 3 - Eqn 4 = 0. Hence $(x_1+x_5-2x_2)-(x_2+x_6-2x_3)+(x_3+x_7-2x_5)-(x_4+x_7-2x_5) = x_1 - x_2 + x_3 - x_4 = 0$. Since both the tangles in Fig. 7 are the same, the coloring equations corresponding to the tangle diagram in Fig. 7B also satisfies the endpoint arcs relationship, $x_1 - x_2 + x_3 - x_4 = 0$. Removing the circle $x_5$ from the tangle diagram in Fig. 7B corresponds to removing the column corresponding to $x_5$ (containing all zeros) as well as the row containing all zeros. The remaining equations are unchanged. Hence, $M_l(T)$ is not affected and we still have the relationship $x_1 - x_2 + x_3 - x_4 = 0$ for the tangle diagram without
the circle $x_5$. Note that we can add an unknotted, unlinked circular component to any $n$-string tangle to determine the endpoint arcs relationship, $x_1 - x_2 + \ldots + x_{n-1} - x_n = 0$. Since removing the unknotted, unlinked component does not affect $M(T)$, we can see that the coloring equations of any $n$-string tangles must satisfy this relationship \[8\] thus giving us Theorem 4.1.

$$
\begin{align*}
\text{Eqn 1: } & x_1 + x_5 - 2x_3 = 0 \\
\text{Eqn 2: } & x_2 + x_5 - 2x_3 = 0 \\
\text{Eqn 4: } & x_4 + x_7 - 2x_5 = 0 \\
\text{Eqn 3: } & x_3 + x_7 - 2x_5 = 0
\end{align*}
$$

Fig. 7.

$$
\begin{align*}
\begin{pmatrix}
-2 & 1 & 0 & 1 & 0 & 0 \\
-2 & 1 & 0 & 0 & 1 & 0 \\
-2 & 0 & 1 & 0 & 0 & 1 \\
-2 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} & \sim \\
\begin{pmatrix}
-2 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{align*}
$$

B.)

$$
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix} & \sim \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\end{align*}
$$

(4.8)

Hence instead of using $SF(M_T)$, we will sometimes use a matrix row equivalent to $M(T)$ where the first row consists of alternating 1’s and -1’s.

5. 3-string braid and 2-string rational tangle coloring formulas

Let us start with a formal definition of an $n$-string braid. An $n$-string braid is the union $B = b_1 \cup b_2 \cup \ldots \cup b_n$ of $n$ strings $b_i (i = 1, 2, \ldots , n)$ in the cylinder $D^2 \times [0, 1]$ such that for each $t \in [0, 1]$, $B$ intersects the 2-disk $D^2 \times \{t\}$ transversely in $n$ distinct interior points of $D^2 \times \{t\}$ with the 2$n$ endpoints fixed. An example of a 3-string braid is given in Fig. 8.

$$
\begin{align*}
\begin{pmatrix}
X_1 & \ldots & X_4 \\
X_2 & \ldots & X_5 \\
X_3 & \ldots & X_6
\end{pmatrix}
\end{align*}
$$

Fig. 8. A 3-string braid

In order to calculate coloring formulas for 3-string braids, we will use the Euler bracket function, $E[c_1, \ldots, c_h]$ which equals the sum of products of the $x_i$’s where zero or more disjoint pairs of consecutive $x_i$’s are omitted [9]. For example, $E[c_1, c_2] = c_1 c_2 + 1$, $E[c_1, c_2, c_3] = c_1 c_2 c_3 + c_1 + c_3$, $E[c_1, c_2, c_3, c_4] = c_1 c_2 c_3 c_4 + c_1 c_2 + c_1 c_4 + c_3 c_4 + 1$. If $h = 0$, then $E[\] = 1$. Two useful formulas involving the Euler bracket are $E[c_1, \ldots, c_h] = E[c_h, \ldots, c_1]$ and $E[c_1, \ldots, c_h] = c_1 E[c_2, \ldots, c_h] + E[c_3, \ldots, c_h]$ [9]. The following theorem is an unpublished result of Arun Ponnusamy and D.
Theorem 5.1. If \( B \) is an \( n \)-string braid, then \( d_U(B) = 1 \). Furthermore, for a 3-string braid, \( B = \sigma_1 c_1 \sigma_2 c_2 \sigma_1^{-1} \cdots \sigma_2^{-1} c_1 \), h odd, if the endpoint arcs have been ordered as in Fig. 8, then \( M_l(B) \) is row equivalent to the matrix in Eqn. 5.9.

\[
\begin{pmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & 0 & E[1, c_1, \ldots, c_h] - 1 \\
0 & 0 & 1 & E[c_2, \ldots, c_h] - 1 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

(5.9)

Proof. Induction on \( h \).

A rational 2-string tangle is a tangle which can be formed from a 3-string braid by connecting the endpoint arcs \( x_2 \) and \( x_3 \) (compare Figs. 8, 9). For other definitions of rational tangle, see for example [6]. The 3-string braid \( B = \sigma_1 c_1 \sigma_2 c_2 \sigma_1^{-1} \cdots \sigma_2^{-1} c_1 \), \( h \) odd, forms the 2-string tangle \( \langle c_1, \ldots, c_h \rangle \). 2-string tangles are uniquely identified by the continued fraction \( \frac{p}{q} = \frac{E[c_1, \ldots, c_h]}{E[c_1, \ldots, c_h - 1]} = c_h + \frac{1}{c_h - 1 + \frac{1}{\ldots}} \).

Thus a coloring matrix for \( T \) can be obtained from a coloring matrix for \( B \) by adding the equation \( x_2 = x_3 \) and switching the columns corresponding to old endpoint arcs of \( B \), \( x_1 \) and \( x_3 \), so that the 2-string tangle endpoint arcs are the last four columns of the coloring matrix (see Eqn. 5.10). Hence if \( T \) is a rational 2-string tangle, then \( d_U(T) = 1 \). We can determine \( M_l(T) \) by putting the matrix in Eqn. 5.10 into echelon form. Hence we have Theorem 5.2.

\[
\begin{pmatrix}
1 & -1 & 1 & -1 \\
0 & 1 & 0 & E[1, c_1, \ldots, c_h] - 1 \\
1 & 0 & 0 & E[c_2, \ldots, c_h] - 1 \\
1 & -1 & 0 & 0
\end{pmatrix}
\]

(5.10)

Theorem 5.2. For a rational 2-string tangle \( T = \frac{p}{q} \), \( d_U(T) = 1 \) and \( M_l(T) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & -p & -p - q & q \end{pmatrix} \)

Thus, as also noted by [5], coloring classifies rational tangles.

6. Numerator and Denominator Closure of 2-string tangles

There are a number of operations which can be performed on tangles in order to obtain knots or links. In this section we will look at the operations of numerator and denominator closures of 2-string tangles.
Suppose $T$ is a 2-string tangle with coloring matrix in Eqn. 6.11.

$$M_T = \begin{pmatrix} A_{(k-2)\times(k-2)} & B_{(k-2)\times4} \\ 0_{2\times(k-2)} & 1 \\ 0 & a & b & c \end{pmatrix}$$

(6.11)

### 6.1. Numerator Closure

The numerator closure of a tangle is formed by connecting the endpoint arcs $x_1$ and $x_2$ as well as $x_3$ and $x_4$ as shown in Fig. 10.

![Numerator Closure](image)

Fig. 10. Numerator Closure, $N(T)$

Thus to obtain a coloring matrix for the knot or link $N(T)$, we can add the equations $x_1 - x_2 = 0$ and $x_3 - x_4 = 0$ to a coloring matrix of the tangle $T$ (Eqn. 6.12). Hence the determinant of the numerator closure of $T$, $d(N(T)) = |a|d_U(T)$. For example $d(N(pq)) = |p|$.

$$\begin{pmatrix} A_{(k-2)\times(k-2)} & B_{(k-2)\times4} \\ 0_{2\times(k-2)} & 1 \\ 0 & a & b & c \end{pmatrix} \sim \begin{pmatrix} A_{(k-2)\times(k-2)} & B_{(k-2)\times4} \\ 0_{4\times(k-2)} & 1 \\ 0 & a & b & c \end{pmatrix}$$

(6.12)

### 6.2. Denominator Closure

The denominator closure of a tangle is formed by connecting the endpoint arcs $x_1$ and $x_4$ as well as $x_2$ and $x_3$ as shown in Fig. 11.

![Denominator Closure](image)

Fig. 11. Denominator Closure, $D(T)$

Thus to obtain a coloring matrix for the knot or link $D(T)$, we can add the equations $x_1 - x_3 = 0$ and $x_2 - x_4 = 0$ to a coloring matrix of the tangle $T$ (Eqn. 6.13). Hence $d(D(T)) = |a + b|d_U(T)$, For example $d(D(pq)) = |q|$. 
March 30, 2022 14:49 coloring

D

E

mirror image. For example the trefoil knot, \( N \)

Since \( d \)

is embedded in \( K \). In the last section, we

embedded tangles into knots/links via numerator \( N(T) \) and denominator \( D(T) \) closures. One can also embed

tangle into a knot/link via much more complicated operations. Krebes \[6\] proved that if a tangle, \( T \), is a subtangle of a knot or link, \( K \), then \( gcd(d(N(T), D(T))) \) divides \( d(K) \). A short proof of this result is also given in \[10, 11\]. We will also provide a short proof of this result. A similar technique to that presented here was used in \[8\] to prove several related results.

Recall that if \( M_1(T) \) is row equivalent to \( \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & a & b & c \end{pmatrix} \), then \( d(N(T)) = |a|d_U(T) \) and \( d(D(T)) = |b + a|d_U(T) \). Hence \( gcd(d(N(T), D(T))) = gcd(|a|d_U(T), |a + b|d_U(T)) = d_U(T)gcd(a, b) \).

Let \( g = gcd(a, b) \). Since we started out with a matrix where the sum of the entries in a row is 0, \( a + b + c = 0 \).

Thus \( g \) also divides \( c \). Let \( a' = \frac{a}{g}, b' = \frac{b}{g}, c' = \frac{c}{g} \). Suppose \( T \) is embedded in a knot \( K \). Then the matrix \( M_1 \) in equation 7.14 is a coloring matrix of \( K \) where the upper left \( k \times (k + 2) \) submatrix is a coloring matrix of \( T \). The matrix \( M_2 \) in equation 7.14 is obtained from the matrix \( M_1 \) by dividing a row by \( g \). Hence \( det(M_1) = gdet(M_2) \). Since \( d_U(T) = det(A) \) and \( det(A) \) divides \( det(M_2) \), \( d_U(T)gcd(a, b) \) divides \( det(M_1) \). Thus \( gcd(d(N(T), D(T))) \) divides \( d(K) \).

\[
M_1 = \begin{pmatrix}
A_{(k-2)\times(k-2)} & B_{(k-2)\times4} \\
0_{2\times(k-2)} & 1 -1 1 & -1 \\
0_{c\times(k-2)} & D_{c\times4} & E_{c\times(c-2)}
\end{pmatrix}
\]

\[
M_2 = \begin{pmatrix}
A_{(k-2)\times(k-2)} & B_{(k-2)\times4} \\
0_{2\times(k-2)} & 1 -1 1 & -1 \\
0_{c\times(k-2)} & D_{c\times4} & E_{c\times(c-2)}
\end{pmatrix}
\]

7. Embedding Tangles in Knots/Links

If a tangle, \( T \), is a subtangle of a knot/link/tangle, \( K \), we say that \( T \) is embedded in \( K \). We will also provide a short proof of this result. A similar technique to that presented here was used in \[8\] to prove several related results.

Recall that if \( M_1(T) \) is row equivalent to \( \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & a & b & c \end{pmatrix} \), then \( d(N(T)) = |a|d_U(T) \) and \( d(D(T)) = |b + a|d_U(T) \). Hence \( gcd(d(N(T), D(T))) = gcd(|a|d_U(T), |a + b|d_U(T)) = d_U(T)gcd(a, b) \).

Let \( g = gcd(a, b) \). Since we started out with a matrix where the sum of the entries in a row is 0, \( a + b + c = 0 \).

Thus \( g \) also divides \( c \). Let \( a' = \frac{a}{g}, b' = \frac{b}{g}, c' = \frac{c}{g} \). Suppose \( T \) is embedded in a knot \( K \). Then the matrix \( M_1 \) in equation 7.14 is a coloring matrix of \( K \) where the upper left \( k \times (k + 2) \) submatrix is a coloring matrix of \( T \). The matrix \( M_2 \) in equation 7.14 is obtained from the matrix \( M_1 \) by dividing a row by \( g \). Hence \( det(M_1) = gdet(M_2) \). Since \( d_U(T) = det(A) \) and \( det(A) \) divides \( det(M_2) \), \( d_U(T)gcd(a, b) \) divides \( det(M_1) \). Thus \( gcd(d(N(T), D(T))) \) divides \( d(K) \).

8. How good of a tangle invariant is colorability?

Recall the \( d_U(B) = 1 \) for all braids \( B \). For the unbraided, \( U \), shown on the left in Fig. 12, \( M_1(U) \) is given in Eqn. 8.15. This invariant is the same for the braid shown on the right-side of Fig. 12. Thus the coloring invariants are the same for these two braids. Hence, coloring cannot distinguish the unbraided from all other braids.

\[
M_1(U) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & -1 & 0 & 0
\end{pmatrix}
\]

8.1. Coloring Can Distinguish Between A Tangle And Its Mirror Image

Coloring mod \( m \) cannot distinguish between a knot and its mirror image. If a knot is \( m \)-colorable, then so is its mirror image. For example the trefoil knot, \( N(\frac{3}{7}) \), and its mirror image, \( N(-\frac{3}{7}) \), are both 3-colorable. However, coloring can distinguish between the rational tangle \( \frac{2}{5} \) and its mirror image \( \frac{3}{7} \): \( M_1(\frac{2}{5}) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -4 & 1 \end{pmatrix} \)
while $M_1\left(\frac{3}{1}\right) = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 3 & -2 & 1 \end{pmatrix}$.

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