Mixed-state certification of quantum capacities for noisy communication channels

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We extend a recent method to detect lower bounds to the quantum capacity of quantum communication channels by considering realistic scenarios with general input probe states and arbitrary detection procedures at the output. Realistic certification relies on a new bound for the coherent information of a quantum channel that can be applied with arbitrary bipartite mixed input states and generalized output measurements.
The quantum capacity represents a central quantitative notion in quantum information science [1–4]. However, in general, its computation is a hard task, since it requires a regularisation procedure over an infinite number of channel uses, and it is therefore by itself not directly accessible experimentally. Its analytical value is known mainly for some channels that have the property of degradability [5–7], since regularisation is not needed in this case.

In many practical situations a complete knowledge of the kind of noise present along the channel is not available, and sometimes noise can be completely unknown. It is then important to develop efficient means to establish whether in these situations the channel can still be profitably employed for information transmission. A standard method to establish this relies on quantum process tomography [8–15], where a complete reconstruction of the completely positive map describing the action of the channel can be achieved, and therefore all its communication properties can be estimated. This, however, is a demanding procedure in terms of the number of different measurement settings needed, since it scales as $d^4$ for a finite $d$-dimensional quantum system.

When one is not interested in reconstructing the complete form of the noise affecting the channel but only in detecting its quantum capacity, which is a very specific feature, a novel and less demanding procedure in terms of needed resources (measurements) has been presented in Ref. [19]. In the same spirit as it is done, for example, in entanglement detection [20], parameter estimation [21], and detection of entanglement-breaking property [22] or non-Markovianity [23] of quantum channels, the method of Ref. [19] allows one to experimentally detect lower bounds to the quantum capacity by means of a number of local measurements that scales as $d^2$. The method can be applied to generally unknown noisy channels, and has been proved to be very efficient for many examples of single qubit channels, for generalized Pauli channels in arbitrary dimension [19], and for two-qubit memory Pauli and amplitude damping channels [24]. The first experimental demonstration has been also recently shown in Ref. [25], based on a quantum optical implementation for various forms of noisy single-qubit channels, proving the feasibility and efficiency of the method.

In the original proposal of Ref. [19] we considered a pure maximally entangled input state as a probe, which was used to sample the channel and reconstruct the probabilities for output measurements over orthogonal projectors. In this paper we extend our certification method by considering the case of a generally mixed bipartite input state and probabilities pertaining to generalized measurements (i.e. POVM’s). Clearly, such a generalization makes the method more flexible \textit{per se}, and also allows one to compare theoretical predictions with experiments, where pure states and perfect measurements always represent an idealization. In fact, already in Ref. [25], a specific treatment of the experimental data was in order, since the ideal input maximally entangled state was realistically replaced with a Werner state, because of unavoidable imperfections in the procedures of quantum-state preparation.

Let us denote the action of a generic quantum memoryless channel on a single system as $\mathcal{E}$ and define $\mathcal{E}_N = \mathcal{E}^\otimes N$, where $N$ represents the number of channel uses. The quantum capacity $Q$ measured in qubits per channel use is defined as [1–4]

$$Q = \lim_{N \to \infty} \frac{Q_N}{N},$$

where $Q_N = \max_{\rho} I_c(\rho, \mathcal{E}_N)$, and $I_c(\rho, \mathcal{E}_N)$ denotes the coherent information [26]

$$I_c(\rho, \mathcal{E}_N) = S[\mathcal{E}_N(\rho)] - S_c(\rho, \mathcal{E}_N).$$

In Eq. [2], $S(\rho) = -\text{Tr}[\rho \log_2 \rho]$ is the von Neumann entropy, and $S_c(\rho, \mathcal{E})$ represents the entropy exchange [27], i.e. $S_c(\rho, \mathcal{E}) = S[(I_R \otimes \mathcal{E})(|\Psi_\rho\rangle\langle\Psi_\rho|)]$, where $|\Psi_\rho\rangle$ is any purification of $\rho$ by means of a reference quantum system $R$, namely $\rho = \text{Tr}_R[|\Psi_\rho\rangle\langle\Psi_\rho|]$.

Let us consider an arbitrary bipartite input mixed state $\sigma$ for the tensor-product Hilbert space of reference and system, and write it as a convex decomposition (not necessarily the spectral one) of pure states, namely

$$\sigma = \sum_i a_i |A_i\rangle\langle A_i|,$$

with $a_i \geq 0$ and $\sum_i a_i = 1$. The double-ket notation $|A\rangle$ introduced in Eq. [3] is useful to remind one of the isomorphism between bipartite vectors $|A\rangle = \sum_{n,m} A_{nm} |n\rangle\langle m|$ and linear operators $A = \sum_{n,m} A_{nm} |n\rangle\langle m|$, along with the identities [28]

$$\langle A | B \rangle = \text{Tr}[A^\dagger B],$$

and

$$A \otimes B |C\rangle = | ACB\tau \rangle,$$

where $\tau$ denotes transposition on a fixed basis. The partial trace of $\sigma$ over the reference is the system mixed state

$$\rho = \text{Tr}_R[\sigma] = \left(\sum_i a_i A_i^\dagger A_i\right)^\tau.$$
Since any purification of a mixed state $\rho$ of the system can be written as $|V\sqrt{\rho^r}\rangle$ with arbitrary unitary $V$, then the entropy exchange for $\rho$ is given by

$$S_e(\rho, \mathcal{E}) = S[(\mathcal{I}_R \otimes \mathcal{E})(|V\sqrt{\rho^r}\rangle\langle V\sqrt{\rho^r}|)].$$

(7)

By writing the following spectral decomposition

$$(\mathcal{I}_R \otimes \mathcal{E})(|V\sqrt{\rho^r}\rangle\langle V\sqrt{\rho^r}|) = \sum_j s_j |\phi_j\rangle\langle \phi_j|,$$

(8)

one has

$$S_e(\rho, \mathcal{E}) = - \sum_j s_j \log_2 s_j .$$

(9)

For full-rank $\rho = \text{Tr}_R[\sigma]$, the probability $p_i$ for input mixed state $\sigma$ as in Eq. (3) and measurement outcome $i$ pertaining to the element $\Pi_i$ of an arbitrary POVM for the tensor product of reference and system is given by

$$p_i = \text{Tr}[\sigma \Pi_i] = \sum_l a_l \text{Tr}[(\mathcal{I}_R \otimes \mathcal{E})(|I\rangle\langle I|)(A_l^\dagger \otimes I)\Pi_i(A_l \otimes I)]$$

$$= \sum_l a_l \text{Tr} \left[ (\mathcal{I}_R \otimes \mathcal{E})(|V\sqrt{\rho^r}\rangle\langle V\sqrt{\rho^r}|) \left( V\frac{1}{\sqrt{\rho^r}} A_l^\dagger \otimes I \right) \Pi_i \left( A_l \frac{1}{\sqrt{\rho^r}} V^\dagger \otimes I \right) \right]$$

$$= \sum_j s_j |\phi_j\rangle \sum_l a_l \left( V\frac{1}{\sqrt{\rho^r}} A_l^\dagger \otimes I \right) \Pi_i \left( A_l \frac{1}{\sqrt{\rho^r}} V^\dagger \otimes I \right) |\phi_j\rangle$$

$$\equiv \sum_j s_j p(i|j) ,$$

(10)

where in the last line we introduced the conditional probability

$$p(i|j) = |\langle \phi_j | \sum_l a_l \left( V\frac{1}{\sqrt{\rho^r}} A_l^\dagger \otimes I \right) \Pi_i \left( A_l \frac{1}{\sqrt{\rho^r}} V^\dagger \otimes I \right) |\phi_j\rangle| .$$

(11)

By denoting the Shannon entropy for the vector of the probabilities $\{p_i\}$ as $H(\bar{p}) = - \sum_i p_i \log_2 p_i$, then one has

$$S_e(\rho, \mathcal{E}) - H(\bar{p}) = \sum_i p_i \log_2 p_i - \sum_j s_j \log_2 s_j$$

$$= \sum_{i,j} s_j p(i|j) \log_2 \frac{p_i}{s_j} \leq \log_2 \left( \sum_{i,j} s_j p(i|j) \frac{p_i}{s_j} \right)$$

$$= \log_2 \bar{r} \cdot \bar{p} \leq \log_2 \bar{t} \cdot \bar{p} ,$$

(12)

where we used Jensen’s inequality in the second line, and defined the vectors $\bar{r}$ and $\bar{t}$ with components $r_i = \sum_j p(i|j)$ and

$$t_i = \text{Tr} \left[ \left( \sum_l a_l A_l \frac{1}{\rho^r} A_l^\dagger \otimes I \right) \Pi_i \right] \geq r_i ,$$

(13)

respectively. Notice that the elements of $\bar{t}$ are independent of the unknown channel and can be evaluated from the explicit form of the input state and the output measurement. For a mixed state $\rho$ in Eq. (6) which is not full-rank, since $\text{ker} \rho^r = \cap_l A_l$ and hence $\text{ker} \rho^r \subset \text{ker} A_l$ for all $l$, it is easy to see that Eq. (13) is replaced with

$$t_i = \text{Tr} \left[ \left( \sum_l a_l A_l (\rho^r)^+ A_l^\dagger \otimes I \right) \Pi_i \right] ,$$

(14)

where $M^+$ denotes the Moore-Penrose pseudo inverse of $M$. Eq. (12) provides the following bound for the entropy exchange

$$S_e(\rho, \mathcal{E}) \leq H(\bar{p}) + \log_2 \bar{t} \cdot \bar{p} ,$$

(15)
where $p_i$ represents the probability of measurement outcome in Eq. (10) for the mixed input state (3), and the components $t_i$ are given in Eq. (14). From Eqs. (1), (2), and (15) it follows that for any $\rho$ and $\vec{p}$ one has the following chain of bounds

$$Q \geq Q_1 \geq I_c(\rho, E_1) \geq S[\mathcal{E}(\rho)] - H(\vec{p}) - \log_2 \vec{t} \cdot \vec{p} = Q_{DET}.$$  

(16)

The lower bound $Q_{DET}$ to the quantum capacity of any unknown channel can then be easily accessed without requiring full process tomography of the quantum channel, by means of the following procedure: i) prepare a bipartite state $\sigma$ and send it through the unknown channel $I_R \otimes \mathcal{E}$, where $\mathcal{E}$ acts on one of the two subsystems; ii) measure suitable local observables on the joint output state to estimate $\vec{p}$ and $S[\mathcal{E}(\rho)]$ in order to compute $Q_{DET}$; $iii)$ after performing the measurements, the detected bound $Q_{DET}$ can be further optimized over all probability vectors that can be obtained from the used measurement setting. In fact, for a fixed measurement setting, one can infer different vectors of probabilities pertaining to different POVM’s $\{\Pi_c\}$. This last step is achieved by performing ordinary classical processing of the measurement outcomes. The bound for quantum capacity certification in Eq. (16) generalizes the result of Ref. [19], where only a maximally entangled pure input state and set of orthogonal projectors were considered (for which $\log_2 \vec{t} \cdot \vec{p} = 0$).

Differently from a complete process tomography, we do not need to measure a complete set of observables and, moreover, the bound is directly obtained from the measured expectations, without need of linear inversion and/or maximum likelihood technique. Notice also that, like in quantum process tomography assisted by an ancilla, entanglement is not mandatory, since the bipartite input state $\sigma$ just has to be faithful [11] [30], namely such that the output state $(I_R \otimes \mathcal{E})(\sigma)$ is in one-to-one correspondence to the map $\mathcal{E}$.

Finally, we remark that the detectable bound (16) also gives a lower bound to the private information $P$, since $P \geq Q_1$ [31], and to the entanglement-assisted classical capacity $C_E$, since $C_E = \max_{\rho} [S(\rho) + I_c(\rho, E_1)]$, and then clearly $C_E \geq S(\rho) + Q_{DET}$.

A number of particular cases for Eq. (14) can be inspected as follows:

i) If $\Pi_i$ is a projector on a maximally entangled state, namely $\Pi_i = \frac{1}{d} |U_i\rangle \langle U_i|$, with $U_i$ unitary operator, then

$$t_i = \frac{\text{rank} \rho}{d}.$$  

ii) If the bipartite input state $\sigma$ is pure with maximal Schmidt number, namely $\sigma = |A\rangle \langle A|$, with $A$ invertible, then $t_i = \text{Tr}[\Pi_i]$.

iii) If the reduced input state is $\rho = \frac{I}{d}$, then $t_i = d \text{Tr}[S[\sigma] \text{Tr}[\Pi_i]]$.

iv) If the input state $\sigma$ is diagonal on maximally entangled states, namely $A_i = \frac{U_i}{\sqrt{d}}$, where $\{U_i\}$ is an operator basis of unitary operators, then $t_i = \text{Tr}[\Pi_i]$.

v) Finally, when $t_i = \text{Tr}[\Pi_i] = k$, $k$ being a constant for all $i$ (and then necessarily $k = d^2/N$ for a POVM with $N$ elements), one has $\log_2 \vec{t} \cdot \vec{p} = \log_2 k$.

Notice also that the relation $\sum_i t_i = d \text{rank} \rho$ always holds.

In the following we provide two examples of quantum capacity detection for specific bipartite mixed input states $\sigma$ and channels $\mathcal{E}$, comparing the results with those of the original procedure of Ref. [19], where only a maximally entangled input state was considered.

**Example 1.** Let us consider a Pauli channel in dimension $d$

$$\mathcal{E}(\rho) = \sum_{m,n=0}^{d-1} p_{m,n} U_{mn} \rho U_{mn}^\dagger,$$  

(17)

where $U_{mn}$ represents the unitary operator $U_{mn} = \sum_{k=0}^{d-1} e^{2\pi i km} |k\rangle \langle (k+n) \mod d|$, and $\sum_{m,n=0}^{d-1} p_{m,n} = 1$, along with a bipartite mixed input state diagonal on the generalized Bell basis, namely

$$\sigma = \frac{1}{d} \sum_{m,n=0}^{d-1} p_{m,n} |U_{mn}\rangle \langle U_{mn}|,$$  

(18)

with $q_{m,n} \geq 0$, and $\sum_{m,n=0}^{d-1} q_{m,n} = 1$.

Since the generalized Bell projectors can be written as follows [28] [32]

$$\Pi_{mn} = \frac{1}{d} |U_{mn}\rangle \langle U_{mn}| = \frac{1}{d^2} \sum_{p,q=0}^{d-1} e^{2\pi i (np + mq)} U_{pq} \otimes U_{pq}^\dagger,$$  

(19)
with * denoting complex conjugation, then a set of local measurements on the eigenstates of $U_{mn} \otimes U_{mn}^*$ allows one to estimate $Q_{\text{DET}}$ in Eq. (16), and one has

$$Q \geq Q_{\text{DET}} = \log_2 d - H(p') ,$$

(20)

where $p'$ is the $d^2$-dimensional vector of probabilities pertaining to the generalized Bell projectors (19), whose components are given by

$$p'_{m,n} = \sum_{l,s=0}^{d-1} p_{l,s} q_{m-l,n+s} .$$

(21)

For a pure maximally entangled state, e.g. $\sigma = \frac{1}{d} |I \rangle \langle I |$, one recover $p'_{m,n} = p_{m,n}$ for all $m, n$. In Ref. 25 one can find some experimental results for the qubit case and a number of different channels (dephasing, depolarizing, and Pauli), with a mixed input state of the specific form

$$\sigma = \frac{4F - 1}{3} |\Phi^+ \rangle \langle \Phi^+ | + \frac{1 - F}{3} I \otimes I ,$$

(22)

namely a maximally entangled state $|\Phi^+ \rangle$ affected by isotropic noise that reduces the fidelity from 1 to $F = (\Phi^+ | \sigma | \Phi^+ )$. For the special case of a depolarizing channel [i.e. $p_{0,0} = 1 - p$ and $p_{m,n} = \frac{F}{d^2 - 1}$ for $(m, n) \neq (0, 0)$ in Eq. (17)], by using an ideal maximally entangled input state $\frac{1}{\sqrt{d}} |I \rangle$, the detectable bound coincides with the hashing bound (19), namely

$$Q_{\text{DET}} = \log_2 d - H_2(p') - p' \log_2 (d^2 - 1) .$$

(23)

Let us consider now an isotropic input state for dimension $d$

$$\sigma = \frac{d^2 F - 1}{d^2 - 1} |I \rangle \langle I | + \frac{1 - F}{d^2 - 1} I \otimes I ,$$

(24)

where $F$ represents the fidelity of $\sigma$ with the ideal maximally entangled state $\frac{1}{\sqrt{d}} |I \rangle$, Notice that such a choice corresponds to Eq. (18), with $q_{0,0} = F$ and $q_{m,n} = \frac{F}{d^2 - 1}$ for $(m, n) \neq (0, 0)$. By applying Eq. (21) one obtains

$$p'_{0,0} = (1 - p)F + \frac{p(1 - F)}{d^2 - 1} ,$$

$$p'_{m,n} = \frac{1}{d^2 - 1}(1 - p'_{0,0}) \quad \text{for} \quad (m, n) \neq (0, 0).$$

(25)

Hence, the detected quantum capacity in Eq. (23) is replaced with

$$Q_{\text{DET}} = \log_2 d - H_2(p') - p' \log_2 (d^2 - 1) ,$$

(26)

where $p' = 1 - p'_{0,0} = \frac{d^2[1 - F(1 - p)] + F - p - 1}{d^2 - 1}$.

(27)

In Fig. 1 we plot the detectable quantum capacity for the qubit depolarizing channel for different values of the fidelity $F$ of the input state (24). Clearly, for decreasing values of $F$, the certification of quantum capacity deteriorates. In fact, the ideal hashing bound (23) approaches zero for $F \gtrsim 0.1892$ [39]. This threshold value of $p$ for certifying positive quantum capacity decreases when using noisy input states. We notice that for any value of $p$, if $F \lesssim 0.818$ no quantum-capacity certification is obtained since one has $Q_{\text{DET}} < 0$.

**Example 2.** Erasure channel with erasure probability $p$ in dimension $d$, namely

$$E(\rho) = (1 - p)\rho \oplus p|e\rangle \langle e| \text{ Tr}[\rho] ,$$

(28)

where $|e\rangle$ denotes the erasure flag which is orthogonal to the system Hilbert space. Since it is a degradable channel [5], its quantum capacity coincides with the one-shot single-letter quantum capacity $Q_1$, and one has

$$Q = Q_1 = (1 - 2p) \log_2 d ,$$

(29)
for $p \leq \frac{1}{2}$, and $Q = 0$ for $p \geq \frac{1}{2}$. Let us consider a bipartite mixed input state as in Eq. (24). The bipartite output is given by

$$\quad (\mathcal{I}_R \otimes \mathcal{E}) \sigma = (1 - p) \left[ \frac{d^2 F - 1}{d^2 - 1} |I\rangle \langle I| + \frac{1 - F}{d^2 - 1} I \otimes I \right] + p \left[ \frac{I_R}{d} \otimes |e\rangle \langle e| \right]. \quad (30)$$

A basis constructed by the union of the projectors on $|i\rangle \otimes |e\rangle$ (with $i = 0, 1, \cdots, d - 1$) and Bell projectors (where one of them corresponds to $\frac{1}{\sqrt{d}} |I\rangle \langle I|$) gives a vector of probability $\vec{p}$ with $d$ elements equal to $p/d$ [corresponding to $|i\rangle \otimes |e\rangle$], one element equal to $(1 - p)F$ [corresponding to $\frac{1}{\sqrt{d}} |I\rangle \langle I|$], and $d^2 - 1$ elements equal to $(1 - p)(1 - F)/(d^2 - 1)$ [corresponding to projectors on maximally entangled states orthogonal to $\frac{1}{\sqrt{d}} |I\rangle$].

We then have

$$H(\vec{p}) = H_2(p) + (1 - p)H_2(F) + p \log_2 d + (1 - p)(1 - F) \log_2 (d^2 - 1), \quad (31)$$

where $H_2(p) \equiv -p \log_2 p - (1 - p) \log_2 (1 - p)$ denotes the binary Shannon entropy. The von Neumann entropy of the reduced output state $\mathcal{E} \left( \frac{1}{d} \right) = (1 - p) \frac{1}{d} \otimes p |e\rangle \langle e|$ is given by $S \left( \mathcal{E} \left( \frac{1}{d} \right) \right) = H_2(p) + (1 - p) \log_2 d$. Then, it follows that the detectable bound $Q_{DET}$ for the quantum capacity of the erasure channel is given by

$$Q \geq Q_{DET} \equiv (1 - 2p) \log_2 d - (1 - p)[H_2(F) + (1 - F) \log_2 (d^2 - 1)]. \quad (32)$$

Notice that for perfect fidelity $F = 1$, one achieves $Q = Q_{DET}$.

In Fig. 2 we compare the detectable quantum capacity for the qubit erasure channel for different values of fidelity $F$ with the exact quantum capacity. We also remark that for any value of erasure probability $p$, if $F \lesssim 0.811$ no quantum-capacity certification is obtained since one has $Q_{DET} < 0$.  

**FIG. 1.** Detectable quantum capacity for the qubit depolarizing channel versus $p$ for mixed input state as in Eq. (24) with $d = 2$ and fidelity $F = 0.98$ (dashed), $F = 0.95$ (dotted), $F = 0.9$ (dot-dashed), along with the hashing bound (solid line) achieved for $F = 1$.

**FIG. 2.** Detectable quantum capacity for the qubit erasure channel versus erasure probability $p$ for mixed input state as in Eq. (24) with $d = 2$ and fidelity $F = 0.98$ (dashed), $F = 0.95$ (dotted), $F = 0.9$ (dot-dashed), along with the exact quantum capacity (solid line) achieved for $F = 1$. 


In conclusion, we have extended our recent method to detect lower bounds to capacities of quantum communication channels (specifically, to the quantum capacity, the private capacity, and the entanglement-assisted classical capacity). This capacity certification does not require any \textit{a priori} knowledge about the quantum channel and relies on a number of measurement settings that scales as $d^2$, thus much more favorably than complete process tomography. We think that the presented more general approach can be relevant for the realistic scenario where experimental imperfections are taken into account. In this way one can theoretically predict and evaluate the robustness of quantum-capacity witnessing with respect to input noisy states and output generalized measurements.

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