FLAT PLUMBING BASKET, SELF-LINKING NUMBER AND THURSTON-BENNEQUIN NUMBER

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Abstract. A flat plumbing basket is a surface consisting a disk and finitely many bands which are contained in distinct pages of the trivial open book decomposition of $S^3$. In this paper, we construct a Legendrian link from a flat plumbing basket, and we describe a relation among the self-linking number, the Thurston-Bennequin number and the flat plumbing basket number of the link. As a corollary, we determine the flat plumbing basket numbers of torus links.

Comments: This is a previous version of [12]. In [12], we correct and generalize the construction of the Legendrian link $L_F$ associated with a flat plumbing basket $F$ given in Section 3. Moreover, we expand and improve the results in this paper. We also give a front projection of $L_F$, which answers Question 6.2.

1. Introduction

It is well known that for any oriented link $L$ in the 3-sphere $S^3$, there is some oriented compact surface whose boundary is the link $L$. Such a surface is called a Seifert surface of $L$. Furutachi-Hirasawa-Kobayashi [6] introduced a concept of positions of a Seifert surface, which is called a “flat plumbing basket”.

A Seifert surface is a flat plumbing basket if it is obtained from a disk by plumbing some unknotted and untwisted annuli so that the gluing regions are in the disk. Flat plumbing baskets can be expressed in terms of the trivial open book decomposition $O$ of $S^3$. Namely, a flat plumbing basket consists a page $D_0$ of $O$ and finitely many bands which are contained in distinct pages of $O$.

We say that an oriented link $L$ admits a flat plumbing basket presentation $F$ if $F$ is a flat plumbing basket and its boundary is $L$.

Theorem 1.1 ([6]). Any oriented link admits a flat plumbing basket presentation.

On flat plumbing baskets, there are some related works (for example, see [3, 6, 9, 11, 13, 14, 15]). In particular, in [13], the concept of the flat plumbing basket number of a link is introduced. The flat plumbing basket number $f_{pbk}(L)$ of an oriented link $L$ is the minimal number of bands to obtain a flat plumbing basket presentation of the link. Hirose-Nakashima [9] gave a lower bound for $f_{pbk}(K)$ of a knot $K$ as follows.

Theorem 1.2 ([9, Theorem 1.3]). Let $K$ be a non-trivial knot, and $g(K)$ be the minimal genus of the Seifert surface (i.e. three genus) of $K$. For the Alexander polynomial $\Delta_K(t)$ of $K$, let $a$ be the coefficient of the term of highest degree, and

$$\deg \Delta_K(t) := (\text{the highest degree of } \Delta_K(t)) - (\text{the lowest degree of } \Delta_K(t)).$$

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Then \( fpbk(K) \) is evaluated as follows.

1. If \( a = \pm 1 \), then \( fpbk(K) \geq 2g(K) + 2 \).
2. If \( a \neq \pm 1 \), then \( fpbk(K) \geq \max\{2g(K) + 2, \deg \Delta_K(t) + 4\} \).

As mentioned above, flat plumbing baskets can be expressed in terms of an open book decomposition. On the other hand, by Thurston-Winkelnkemper’s work \[23\], we can construct a contact structure of \( S^3 \) from an open book decomposition. Hence, it seems that there are some relations between flat plumbing baskets and contact topology.

In this paper, we construct a Legendrian link \( L_F \) from a flat plumbing basket \( F \) (see Section 3). By using this Legendrian link, we give two inequalities on flat plumbing basket numbers as follows.

**Theorem 1.3.** Let \( L \) be a non-trivial oriented link in \( S^3 \). Then we have

\[
\max\{-\bar{sl}(L), -\bar{sl}(\overline{L})\} - 1 \leq fpbk(L),
\]

where \( \overline{L} \) is the mirror image of \( L \) and \( \bar{sl}(L) \) is the maximal self-linking number of \( L \) (for detail see Section 4.1).

**Theorem 1.4.** Let \( L \) be a non-trivial oriented link in \( S^3 \). Suppose that \( L \) has no split component which is isotopic to the unknot. Then, we obtain

\[
\max\{-\bar{tb}(L), -\bar{tb}(\overline{L})\} + 2 \leq 2fpbk(L),
\]

where \( \bar{tb}(L) \) is the maximal Thurston-Bennequin number of \( L \). The equality holds if and only if \( L \) is a non-trivial alternating torus link \( T_{2,n} \) for some \( |n| \geq 2 \).

As a corollary, we determine \( fpbk(T_{p,q}) \) for the \((p, q)\)-torus link \( T_{p,q} \) (see Corollary 4.2). Moreover, by using Theorems 1.3 and 1.4 we improve \[9\], Table 1 (see Table 1).

This paper is organized as follows: In Section 2, we recall the definitions of open book decompositions, flat plumbing baskets and contact structures. In Section 3 we construct a Legendrian link from a flat plumbing basket. Moreover, we describe a relation among the maximal self-linking numbers, the maximal Thurston-Bennequin numbers and the flat plumbing basket numbers. In Section 4 we prove Theorem 1.3 (Theorem 4.1). In Section 5 we prove Theorem 1.4. In Section 6 we give further observations.

2. **Preliminary**

2.1. **Trivial open book decomposition and flat plumbing basket.** Let \( M \) be an oriented closed 3-manifold. Suppose that for a link \( L \) in \( M \), there is a fiber projection \( \pi : M \setminus L \to S^1 \) such that each fiber is the interior of a Seifert surface of \( L \). Then, \((L, \pi)\) is called an open book decomposition of \( M \). The closure of each fiber is called a page, and \( L \) is called the binding.

Let \( U \) be the unknot in \( S^3 \). The knot complement \( S^3 \setminus U \) is a product \( \text{Int}(D^2) \times S^1 \). Hence, there exists a fiber projection \( \pi : S^3 \setminus U \to S^1 \) whose fibers \( \pi^{-1}(D_\theta) \) are open disks for \( 0 \leq \theta < 2\pi \). Put \( D_\theta := \pi^{-1}(D_\theta) \). An orientation of \( U \) induces an orientation of each fiber \( D_\theta \) and a positive direction of the fibration \( \{D_\theta\}_{0 \leq \theta < 2\pi} \) (see Figure 1). This fibration is called the trivial open book decomposition of \( S^3 \), denoted by \( \mathcal{O} \). Then, the unknot is the binding, and each \( D_\theta \) is a page of \( \mathcal{O} \). Throughout this paper, we consider the trivial open book decomposition.
A Seifert surface $F$ is a flat plumbing basket if there are finitely many bands $B_1, \ldots, B_n$ and $0 < \theta_1 < \cdots < \theta_n < 2\pi$ such that $F = D_0 \cup B_1 \cup \cdots \cup B_n$, each band $B_i$ is contained in $D_{\theta_i}$ and $B_i \cap U$ consists of two arcs. We call the subscript $i$ of $B_i$ the label of the band. A flat plumbing basket $F$ is a flat plumbing basket presentation of an oriented link $L$ if $\partial F$ is ambient isotopic to $L$ (for example, see Figure 2).

For a flat plumbing basket $F$, by recording the labels of the bands as one travels along $U = \partial D_0$, one obtains a cyclic word $W_F$ in $\{1, \ldots, n\}$ such that each letter appears exactly twice, where $n$ is the number of the bands of $F$. We call $W_F$ the flat basket code for $F$.

We define the flat plumbing basket number $fpbk(L)$ of an oriented link $L$ to be the minimal number of bands to obtain a flat plumbing basket presentation of $L$ from $D_0$. Namely,

$$fpbk(L) := \min\{b_1(F) \mid F \text{ is a flat plumbing basket presentation of } L\},$$

where $b_1(F)$ is the first betti number of $L$. We remark that $fpbk(L) \in 2\mathbb{Z} + \vert L \vert - 1$, where $\vert L \vert$ is the number of the components of $L$, and $fpbk$ is preserved under taking mirror image.
2.2. **Trivial open book decomposition and standard contact structure.** In this section, we recall the relation between open book decompositions and contact structures. For detail, for example, see [19].

Let \( M \) be an oriented closed 3-manifold. A 1-form \( \alpha \in \Omega^1(M) \) is a contact form if \( \alpha \wedge d\alpha \) is nowhere 0. A 2-dimensional distribution \( \xi \subset TM \) is a contact structure if there is a contact form \( \alpha \) such that \( \xi = \text{Ker} \alpha \). A contact structure \( \xi \) on \( M \) is supported by an open book decomposition \((L, \pi)\) if \( \xi \) can be represented by a contact form \( \alpha \) such that the binding is transverse to \( \xi \), \( d\alpha \) is a volume form on every page and the orientation of binding induced by \( \alpha \) agrees with the orientation of the pages.

It is known that for any open book decomposition of \( M \), we can construct a contact structure on \( M \) supported by the open book decomposition, by Thurston-Winkelnkemper’s construction [23]. By Giroux’s work [8], such a contact structure is unique up to contact isotopy. In particular, the trivial open book decomposition \( \mathcal{O} \) of \( S^3 \) supports the standard contact structure \( \xi_{\text{std}} \) of \( S^3 \). See Figure 3.

For a surface \( \Sigma \) in \( S^3 \), we consider \( \xi_{\text{std}} \cap T\Sigma \). For generic \( \Sigma \), this intersection is a line field except at finitely many points where \( \Sigma \) is tangent to \( \xi_{\text{std}} \). By integrating \( \xi_{\text{std}} \cap T\Sigma \), we obtain a foliation of \( \Sigma \) with singularities. Such a foliation is called the characteristic foliation of \( \Sigma \) in \( \xi_{\text{std}} \). In the next section, Section 3 we will construct a Legendrian link \( \mathcal{L}_F \) from a flat plumbing basket \( F \), which is isotopic to the boundary \( \partial F \). Then, it is convenient to see the characteristic foliation of the boundary of a tubular neighborhood of \( U \) and each fiber \( D_\theta \) (see Figures 4 and 5).

![Figure 3. The standard contact structure \( \xi_{\text{std}} \) on \( S^3 \)](image-url)
Figure 4. Characteristic foliation of $D_0$. This characteristic foliation has one singular point. The characteristic foliations of other fibers $D_0$ are similar to that of $D_0$. In particular there is one singular point.

$r_1 < r_2 \Rightarrow \phi_{r_1} < \phi_{r_2}$

Figure 5. Characteristic foliation of the boundary of a tubular neighborhood of the binding $U$. If we increase the radius of the tube, then the angle $\phi_r$ between the foliation and the meridian also increases.

characteristic foliation of $D_0$. See Figure 6. Then, there is a small perturbation (isotopy) $f_t: S^3 \to S^3$ such that $f_0 = \text{id}$ and $f_1(\partial B_i \setminus N(U))$ is Legendrian (by Legendrian realization principle [10]). After this operation, $\partial B_i \setminus N(U)$ can be regarded as Legendrian arcs.

As mentioned above, we see that $\partial F \cap N(U)$ is not Legendrian. In order to construct a Legendrian link from $\partial F$, we replace each component $\beta$ of $\partial F \cap N(U)$ by an arc $\alpha$ depicted in Figure 7. Each $\alpha$ is in a leaf of the characteristic foliation of $\partial(N(U))$, connects the two points of $\partial \beta$ and satisfies $|\alpha \cap D_0| = 1$. We can take such an $\alpha$ by adjusting the radius of $N(U)$ locally.

By this replacement (and smoothening the curve at the points of $\partial F \cap \partial(N(U))$ in $\xi_{std}$), we obtain a Legendrian link from $\partial F$. We denote the Legendrian link by $\mathcal{L}_F$, and called the Legendrian link associated with $F$. For example, see Figure 8.
The classical invariants, the Thurston Bennequin number $tb$ and the rotation number $\text{rot}$, of $\mathcal{L}_F$ can be computed as follows.

![Figure 6](image)

**Figure 6.** Thinning and moving a band $B_i$ so that the core of $B_i$ is on two leaves of the characteristic foliation of $D_{\theta_i}$.

![Figure 7](image)

**Figure 7.** In order to construct a Legendrian link $\mathcal{L}_F$ from $\partial F$, we replace each component $\beta$ of $\partial F \cap N(U)$ by an arc $\alpha$. The arc $\alpha$ is contained in a leaf of the characteristic foliation of $\partial(N(U))$ (see Figure 4), connects the two points of $\partial \beta$ and satisfies $|\alpha \cap D_0| = 1$. By adjusting the radius of $N(U)$ locally, we can take such an $\alpha$.

![Figure 8](image)

**Figure 8.** An example of $\mathcal{L}_F$. This is isotopic to the negative trefoil knot. Precisely, the radius of $N(U)$ is non-uniform.
Lemma 3.1. Let $F$ be a flat plumbing basket with $b_1(F) > 0$ and $\mathcal{L}_F$ be the Legendrian link associated with $F$. Then, we obtain

$$tb(\mathcal{L}_F) = -2b_1(F),$$

where $tb(\mathcal{L}_F)$ is the Thurston-Bennequin number of $\mathcal{L}_F$.

Proof. Let $\mathcal{L}_F^{\pm}$ be a Legendrian link obtained by pushing of $\mathcal{L}_F$ in the direction of a nonzero vector field transverse to $\xi_{std}$. Then the Thurston-Bennequin number $tb(\mathcal{L}_F)$ is computed by

$$tb(\mathcal{L}_F) = lk(\mathcal{L}_F, \mathcal{L}_F^{\pm}),$$

where $lk(\mathcal{L}_F, \mathcal{L}_F^{\pm})$ is the linking number between $\mathcal{L}_F$ and $\mathcal{L}_F^{\pm}$. By Figure 9 we see that each band of $F$ contributes $-2$ to the linking number. Note that the crossings between distinct bands do not contribute the linking number. Hence, we finish the proof. □

Figure 9. Near a band of a flat plumbing basket. Each band contributes $-4/2 = -2$ to the linking number $lk(\mathcal{L}_F, \mathcal{L}_F^{\pm})$.

Lemma 3.2. Let $F$ be a flat plumbing basket with $b_1(F) > 0$ and $\mathcal{L}_F$ be the Legendrian link associated with $F$. Suppose that $\mathcal{L}_F$ has the orientation which agrees with the orientation of the binding $U$. Then, we obtain

$$rot(\mathcal{L}_F) = -b_1(F) + 1,$$

where $rot(\mathcal{L}_F)$ is the rotation number of $\mathcal{L}_F$.

Proof. The rotation number $rot(\mathcal{L})$ of an oriented Legendrian link $\mathcal{L}$ is the winding number of $T\mathcal{L}$ with respect to a trivialization of $\xi_{std}$ along $\mathcal{L}$. Let $W_F = (i_1, i_2, \ldots, i_{2n})$ be the flat basket code of $F$, where $n = b_1(F)$. For convenience, define $i_{2n+1} = i_1$. Let $\alpha$ be the arc in $\mathcal{L}_F$ which connects $B_{i_k}$ and $B_{i_{k+1}}$, as in Figure 10. Define $\zeta_k$ be the angle corresponding to the arc in $U$ which connects $B_{i_k}$ and $B_{i_{k+1}}$. Then, as in Figure 10 the arc $\alpha$ contributes

$$-2\pi + (\theta_{i_{k+1}} - \theta_{i_k}) + \frac{\pi}{2} \times 2 + \zeta_k$$

to $rot(\mathcal{L}_F)$. Moreover, when we go across a band $B_i$ twice along $\mathcal{L}_F$, the winding number increases $\xi_i$, where $\xi_i$ is the angle corresponding to the two arcs in $B_i \cap U$.
Then, we obtain
\[
2\pi \times \text{rot}(\mathcal{L}_F) = \sum_{k=1}^{2n} (-2\pi + (\theta_{ik+1} - \theta_{ik}) + \frac{\pi}{2} \times 2 + \zeta_k) + \sum_{i=1}^{n} \xi_i
\]
\[
= -2n\pi + \sum_{k=1}^{2n} \zeta_k + \sum_{i=1}^{n} \xi_i
\]
\[
= -2n\pi + 2\pi.
\]
Hence, we have \(\text{rot}(\mathcal{L}_F) = -n + 1 = -b_1(F) + 1\). □

4. Maximal self-linking number and flat plumbing basket

Let \(l\) be an oriented link in \(S^3\). Then, \(l\) is a transverse link in \(\xi_{std}\) if it is positively transverse to \(\xi_{std}\). The self-linking number \(\text{sl}(l)\) of \(l\) is defined as the linking number \(\text{lk}(l, l')\) of \(l\) and \(l'\), where \(l'\) is a push-off of \(l\) obtained by a non-zero vector field in \(\xi_{std}\). It is known that for any Legendrian link \(\mathcal{L}\), by pushing \(\mathcal{L}\) in a sufficiently

\[
\xi_i = \xi_i^{(1)} + \xi_i^{(2)}
\]

\(\text{Figure 10.}\)

\(\text{Figure 11.}\)
On the other hand, by [6, Theorem 2.4], we see that

\[ s_l(l_F) = \text{tb}(\mathcal{L}) + |\text{rot}(\mathcal{L})|. \]

For example, see \[7, 19\].

Let \( F \) be a flat plumbing basket with \( b_1(F) > 0 \) and \( \partial F = L \). Let \( \mathcal{L}_F \) be the Legendrian link associated with \( F \). By Lemmas \[8,11\] and \[8,22\] we can construct a transverse link \( l_F \) such that \( l_F \) is isotopic to \( L \) and

\[ s_l(l_F) = \text{tb}(\mathcal{L}_F) + |\text{rot}(\mathcal{L}_F)| = -b_1(F) - 1 \in 2\mathbb{Z} + |L|. \]

Hence, we obtain the following.

**Theorem 4.1** (Theorem \[1.3\]). Let \( L \) be a non-trivial oriented link in \( \mathbb{S}^3 \). Define the maximal self-linking number \( s_l(L) \) of \( L \) as

\[ s_l(L) := \max\{\text{tb}(\mathcal{L}) + |\text{rot}(\mathcal{L})| \mid \mathcal{L} \text{ is a Legendrian link in } \xi_{\text{std}} \text{ and isotopic to } L\}. \]

Then we have

\[ \max\{-s_l(L), -s_l(L)\} - 1 \leq \text{fpbk}(L) = \text{fpbk}(\mathcal{L}), \]

where \( \mathcal{L} \) is the mirror image of \( L \).

**Corollary 4.2.** For any \( p \geq q > 1 \), we have

\[ pq - p + q - 1 = -s_l(T_{p,q}) - 1 = \text{fpbk}(T_{p,q}), \]

where \( T_{p,q} \) is the positive \((p, q)\)-torus link and \( T_{p,q} = \overline{T_{p,q}} \).

**Proof.** By Morton-Franks-Williams (MFW) inequality \[8,16\], we have

\[ -2b(L) \leq s_l(L) + s_l(\mathcal{L}) \leq -\text{breadth}_v P_L(v, z) - 2, \]

where \( b(L) \) is the braid index of \( L \) and \( P_L(v, z) \) is the HOMFLYPT polynomial. Franks and Williams \[5\] proved that for any torus link, MFW inequality is sharp, that is \( 2q = \text{breadth}_v P_{T_{p,q}}(v, z) + 2 \). Moreover, it is known that \( s_l(T_{p,q}) = pq - p - q \) \((\ref{1})\). Hence, we have

\[ -s_l(T_{p,q}) \geq s_l(T_{p,q}) + \text{breadth}_v P_{T_{p,q}}(v, z) + 2 = pq - p + q. \]

On the other hand, by \[8,24\] Theorem 2.4, we see that \( T_{p,q} \) has a flat plumbing basket presentation \( F \) with \( b_1(F) = pq - p + q - 1 \). By Theorem \[1.1\] we have

\[ -s_l(T_{p,q}) \leq \text{fpbk}(T_{p,q}) + 1 \leq pq - p + q, \]

and we finish the proof. \( \square \)

**Corollary 4.3.** Let \( L \) be an oriented link with \( s_l(L) = -\chi(L) \), where \( \chi(L) \) is the maximal Euler characteristic of \( L \) (for example, if \( L \) is strongly quasipositive, \( L \) satisfies this condition). Then we have

\[ 1 - \chi(L) + \text{breadth}_v P_L(v, z) \leq \text{fpbk}(L). \]

**Proof.** By MFW inequality and Theorem \[1.1\] we have

\[ s_l(L) + \text{breadth}_v P_L(v, z) + 2 \leq -s_l(\mathcal{L}) \leq \text{fpbk}(L) + 1. \]

By the assumption, we finish the proof. \( \square \)

**Corollary 4.4.** Let \( L \) be an oriented link. Then, we have

\[ \max\text{deg}_v P_L(v, z) \leq -s_l(L) - 1 \leq \text{fpbk}(L). \]
Proof. The first inequality is the HOMFLYPT bound on the self-linking number, which follows from MFW inequality. The second follows from Theorem 4.1. □

Corollary 4.5. Let $K_m$ be the $m$-twist knot (Figure 12). Then, for any $k \geq 0$, we have $2k \leq fpbk(K_{2k})$ and $2k + 4 \leq fpbk(K_{2k+1})$.

Proof. By [3, Theorem 1.2], we see that

- $\max\{-s_l(K_{2k}), -s_l(K_{2k+1})\} \geq 2k + 1$,
- $\max\{-s_l(K_{2k+1}), -s_l(K_{2k+1})\} \geq 2k + 5$.

By Theorem 4.1, we finish the proof. □

Remark 4.6. Mikami Hirasawa showed that $fpbk(K_{2k+1}) \leq 2k + 4$ for $k \geq 0$, and $fpbk(K_{2k}) \leq 2k$ for $k \geq 3$ in his forthcoming paper. Hence, by Corollary 4.5, we have

- $fpbk(K_{2k+1}) = 2k + 4$ for $k \geq 0$,
- $fpbk(K_{2k}) = 2k$ for $k \geq 3$,
- $fpbk(K_2) = 4$ and $fpbk(K_4) = 6$.

![Figure 12. The m-twist knot $K_m$](image)

Example 4.7. By Corollary 4.2, we have

$$fpbk(3_1) = 4 = \max_{v,z} P_{3_1}(v, z).$$

Moreover, since the right-hand side is additive and $fpbk$ is subadditive under connected sum, by Corollary 4.4, we have

$$fpbk(2_n 3_1) = 4n.$$

5. **Maximal Thurston-Bennequin number and flat plumbing basket**

Let $L$ be an oriented link in $S^3$. The **maximal Thurston-Bennequin number** $\overline{tb}(L)$ of $L$ is the maximal number of Thurston-Bennequin numbers of Legendrian links in $\xi_{std}$ which are isotopic to $L$. In this section, we compare $tb(L_F)$ with $\overline{tb}(L)$, where $L = \partial F$.

Let $\alpha$ be an arc used in the construction of $L_F$. Suppose that $\alpha$ goes round the meridian of $\partial(N(U))$. See the left picture of Figure 13. If we increase the radius of $N(U)$, the angle between the characteristic foliation and the meridian also increases as in Figure 5. Hence, by increasing the radius of $N(U)$ locally, we can obtain a new Legendrian arc $\alpha'$ instead of $\alpha$ (see Figure 13). Then, we can construct a new Legendrian link by replacing $\alpha$ in $L_F$ with $\alpha'$ as in Figure 13. We call this operation a **shortcut** for $L_F$. By considering the Lagrangian projection, we see that a shortcut corresponds to a (de)stabilization in a Lagrangian projection of $L_F$. 
Lemma 5.1. Let $F$ be a flat plumbing basket. Let $L'_F$ be a Legendrian link obtained from $L_F$ by taking one shortcut. Then, we obtain $tb(L'_F) = tb(L_F) + 1$.

Proof. See Figure 14.

Figure 13. Shortcut operation. In this picture, $j > i$, that is, $\theta_j > \theta_i$ (cf. Figure 7). In this case, we can take a shortcut. Namely we can replace $\alpha$ with $\alpha'$.

Figure 14. One shortcut contributes $(+1 - (-1))/2 = +1$ to the Thurston-Bennequin number.

Lemma 5.2. Let $F$ be a flat plumbing basket with $b_1(F) = n \geq 2$. Suppose that $\partial F$ has no split component which is isotopic to the unknot. Then, we can take at least two shortcuts for $L_F$.

Proof. Let $W_F$ be the flat basket code of $F$. Because $\partial F$ has no split component which is isotopic to the unknot, $W_F$ has no subcode $(n, n)$. Since the cyclic word $W_F$ has exactly two $n$, there are two subcodes $(n, i_1)$ and $(n, i_2)$ of $W_F$ for some $i_1, i_2 \in \{1, \ldots, n-1\}$. At the corresponding places in $F$ to subcodes $(n, i_1)$ and $(n, i_2)$, we can take shortcuts for $L_F$ since $n > i_1$ and $n > i_2$.

By Lemmas 5.1, 5.1 and 5.2, we obtain the following,

Lemma 5.3. Let $L$ be a non-trivial oriented link. Suppose that $L$ has no split component which is isotopic to the unknot. Then, we obtain

$$-tb(L) + 2 \leq 2fpbk(L).$$
Proof. Let $F$ be a flat plumbing basket presentation of $L$ with $b_1(F) = fpbk(L)$. By Lemma 3.1, we have $tb(L_F) = -2b_1(F)$. By Lemmas 5.1 and 5.2, we obtain $tb(L_F) \leq -\overline{tb}(L) - 2$. Hence, we have $-\overline{tb}(L) + 2 \leq 2b_1(F)$. This implies $-\overline{tb}(L) + 2 \leq 2fpbk(L)$.

Example 5.4. Let $F$ be the flat plumbing basket depicted in Figure 2 which presents the negative trefoil knot. The Legendrian link $L_F$ associated with $F$ is as in Figure 8. Its Thurston-Bennequin number $tb(L_F)$ is $-8$. We can take 2 shortcuts for $tb(L_F)$. Hence, $tb(L_F) + 2 \leq \overline{tb}(3_1)$. It is known that $\overline{tb}(3_1) = -6$. We see that the Legendrian link obtained by taking 2 shortcuts for $L_F$ attains the maximal Thurston-Bennequin number of $3_1$. Moreover, we obtain

$$8 = -\overline{tb}(3_1) + 2 \leq 2fpbk(3_1) \leq 2 \times 4 = 8.$$ Hence, we have $fpbk(3_1) = 4$. This coincides the result of Example 4.7.

On the equality of Lemma 5.3, we obtain the following.

Lemma 5.5. Let $L$ be a non-trivial oriented link. Suppose that $L$ has no split component which is isotopic to the unknot. Then, $L$ satisfies

$$-\overline{tb}(L) + 2 = 2fpbk(L)$$

if and only if $L$ is a negative alternating torus link $T_{2,n}$ for some $n \geq 2$.

Proof. Suppose that $L$ satisfies $-\overline{tb}(L) + 2 = 2fpbk(L)$. Let $F$ be a flat plumbing basket presentation of $L$ with $b_1(F) = fpbk(L)$. Then, by Lemma 3.1, we have $tb(L_F) = -2b_1(F) = -2fpbk(L) = -\overline{tb}(L) - 2$. By Lemmas 5.1 and 5.2, we can take exactly two shortcuts for $L_F$. Such a flat plumbing basket is depicted in Figure 15 and its boundary is a negative alternating torus link $\overline{T_{2,n}}$ for some $n \geq 2$.

Conversely, suppose that $L = \overline{T_{2,n}}$ for some $n \geq 2$. It is known that $\overline{tb}(T_{2,n}) = -2n$ (for example, see [17]). Moreover, by Figure 15, $T_{2,n}$ has a flat plumbing basket presentation $F$ with $b_1(F) = n + 1$. Hence, we obtain $2n + 2 = -\overline{tb}(L) + 2 = 2fpbk(L) \leq 2(n + 1)$. This implies the equality. 

\[ \]
Proof of Theorem 1.4. Note that $fpbk(L) = fpbk(\bar{L})$, where $\bar{L}$ is the mirror image. Hence, Theorem 1.4 follows from Lemmas 5.3 and 5.5. □

Corollary 5.6. Let $L$ be a non-trivial oriented link. Suppose that $L$ has no split component which is isotopic to the unknot. If $L$ is not an alternating torus link, we obtain

$$\max\{-tb(L), -tb(\bar{L})\} + 3 \leq 2fpbk(L).$$

Question 5.7. When does the equality of Corollary 5.6 hold? Classify such links.

For example, by Table 1, $8_{21}$ satisfies

$$\max\{-tb(8_{21}), -tb(\bar{8}_{21})\} + 3 = 2fpbk(8_{21}).$$

6. Further discussion

6.1. Negativity of links and flat plumbing basket number. It is known that the maximal self-linking number $sl(L)$ and the maximal Thurston-Bennequin number $tb(L)$ of a positive link $L$ hold the equality of the Bennequin inequality [1], that is, $sl(L) = tb(L) = -\chi(L)$ (for example, see [22]). In this sense, $sl$ and $tb$ of a positive link are large. On the other hand, as corollaries of MFW inequality and the Rasmussen bound on the maximal Thurston-Bennequin number [20, 21], we obtain

$$sl(L) + sl(\bar{L}) \leq -\text{breadth}_e P_L(v, z) - 2,$$

$$tb(L) + tb(\bar{L}) \leq -2.$$

Hence, we see that the maximal self-linking number and the maximal Thurston-Bennequin number of a negative link are small. By this observation, it seems that Theorem 1.3 (or Corollary 4.4) and Theorem 1.4 (or Lemma 5.3) are effective for negative links. In particular, in Table 1 we see that $-sl(L) - 1 = fpbk(L)$ for negative links $L$ with up to 9 crossings.

Question 6.1. For any negative link $L$, does the following hold?

$$-sl(L) - 1 = fpbk(L).$$

6.2. Non-sharpness of Theorem 1.4. For any $M > 0$, there is a link $L$ such that

$$2fpbk(L) - (\max\{-tb(L), -tb(\bar{L})\} + 2) > M.$$ 

In fact, we can construct such a link as follows. Let $K = 8_9$. Then, $g(8_9) = 3$ and $\max\{-tb(8_9), -tb(\bar{8}_9)\} = 5$. Let $K_n$ be the connected sum of $n$ copies of $8_9$, where we take the mirror image so that $tb(8_9) = -5$. It is known that $tb(K_n + K') = tb(K) + tb(K') + 1$ for any knots $K$ and $K'$ [24]. Hence, we obtain

$$g(K_n) = 3n,$$

$$tb(K_n) = ntb(8_9) + (n - 1) = -5n + n - 1 = -4n - 1.$$

By Theorem 1.2, we see that $6n + 2 \leq fpbk(K_n)$. Hence, we have

$$2fpbk(L) - (\max\{-tb(L), -tb(\bar{L})\} + 2) \geq 2(6n + 2) - (4n + 1 + 2) = 8n + 1.$$
6.3. Front and Lagrangian projections. Let $F$ be a flat plumbing basket. If we can draw a front (or Lagrangian) projection of $L_F$, we may use front projections of Legendrian links to study flat plumbing baskets.

Question 6.2. Find a method to draw a front projection of the Legendrian link $L_F$ associated with a flat plumbing basket $F$.

An answer to Question 6.2 is given in [12].

6.4. Non-trivial open book decomposition and flat plumbing basket. In this paper, we only consider the trivial open book decomposition $O$ of $S^3$. However, we can define flat plumbing baskets $F$ and the Legendrian knot $L_F$ associated with $F$ in non-trivial open book decompositions of $S^3$ naturally.

Question 6.3. Consider non-trivial open book decompositions of $S^3$ (or general 3-manifold $M$), and give analogies of Theorems 1.3 and 1.4.

6.5. Tabulation. We improve [9, Table 1] as in Table 1, where we use Theorems 1.2, 1.3 and 1.4. In Table 1, $-tb(K)$ means $\max\{-tb(K), -\overline{tb}(K)\}$ and $-sl(K)$ means $\max\{-sl(K), -\overline{sl}(K)\}$. We refer to [18, Proposition 1.6] for $sl(K)$ and [2] for $\overline{tb}(K)$. Note that $fpbk(K) \in 2\mathbb{Z}$ for a knot $K$.

In this table, the asterisks * are improved points of [9, Table 1]. The double asterisks ** are given by [3] and Mikami Hirasawa. In fact, Mikami Hirasawa taught the author that a flat basket code for 8_1 is (1, 2, 4, 3, 6, 1, 4, 2, 5, 3) and a flat basket code for 9_4 is (1, 2, 5, 6, 1, 4, 3, 5, 6, 2, 4, 3). The daggers † mean that Theorem 1.3 or 1.4 detects $fpbk(K)$.

For example, 8_15 has a flat plumbing basket presentation $F$ with $b_1(F) = 10$. On the other hand, it is known that $\max\{-\overline{sl}(8_{15}), -\overline{sl}(8_{15})\} = 11$ (see [13, Proposition 1.6]). Hence, by Theorem 1.3 we have $fpbk(8_{15}) = 10$.

For another example, the knot 9_45 has a flat plumbing basket presentation $F$ with $b_1(F) = 8$. On the other hand, it is known that $\max\{-\overline{tb}(9_{45}), -\overline{tb}(9_{45})\} = 10$ (for example, see [2]). Moreover, 9_45 is a non-torus knot. Hence, by Theorem 1.3 we have $10 + 3 = 13 \leq 2fpbk(9_{45}) \leq 16$. Since $fpbk(9_{45}) \in 2\mathbb{Z}$, we have $fpbk(9_{45}) = 8$.

Question 6.4. Determine $fpbk(K)$ for $K = 9_{25}, 9_{34}, 9_{39}, 9_{40}, 9_{41}$ and 9_43.

Note that $fpbk$ is subadditive under connected sum of knots. However, in general, it is not additive. For example, Hirose-Nakashima [9, Remark 1.4(b)] proved that $fpbk(3_1) = fpbk(3_1) = 4$ but $fpbk(3_1\#3_1) = 6$. Nao Kobayashi (Imoto) proved that $fpbk(6_1) = fpbk(6_1) = 6$ but $fpbk(6_1\#6_1) = 8$ in [13, Proposition 5.4].

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| $K$ | $-tb(K)$ | $-sl(K)$ | $fpbk(K)$ | $K$ | $-tb(K)$ | $-sl(K)$ | $fpbk(K)$ |
|-----|---------|----------|-----------|-----|---------|----------|-----------|
| $3_1$ | 6†      | 5†       | 4         | $9_8$ | 8       | 7         | 8         |
| $4_1$ | 3 3     | 4        | $9_g$     | 16†   | 11†     | 10        |
| $5_1$ | 10†     | 7†       | 6         | $9_{10}$ | 14†     | 11†       | 10†       |
| $5_2$ | 8†      | 7†       | 6         | $9_{11}$ | 12†     | 9†        | 8         |
| $6_1$ | 5 5 6   | 9 8      | $9_{12}$  | 10†   | 9†      | 8         |
| $6_2$ | 7†      | 5 6      | $9_{13}$  | 14†   | 11†     | 10†       |
| $6_3$ | 4 3 6   |          | $9_{14}$  | 7 7 8 | 8       |
| $7_1$ | 14†     | 9†       | 8         | $9_{15}$ | 10†     | 9†        | 8         |
| $7_2$ | 10†     | 9† 8*    | $9_{16}$  | 16†   | 11†     | 10         |
| $7_3$ | 12†     | 9† 8     | $9_{17}$  | 8 5 8 | 8       |
| $7_4$ | 10†     | 9† 8*    | $9_{18}$  | 14†   | 11†     | 10†        |
| $7_5$ | 12†     | 9† 8     | $9_{19}$  | 6 5 8 | 8       |
| $7_6$ | 8†      | 7† 6     | $9_{20}$  | 12†   | 9†      | 8         |
| $7_7$ | 4 5 6   |          | $9_{21}$  | 10†   | 9†      | 8         |
| $8_1$ | 7†      | 7† 6**   | $9_{22}$  | 8 5 8 | 8       |
| $8_2$ | 11†     | 7 8      | $9_{23}$  | 14†   | 11†     | 10†        |
| $8_3$ | 5 5 6   |          | $9_{24}$  | 6 5 8 | 8       |
| $8_4$ | 7 5 8   |          | $9_{25}$  | 10 9  | 8–10    |
| $8_5$ | 11†     | 7 8      | $9_{26}$  | 9 7 8 | 8       |
| $8_6$ | 9 7 8   |          | $9_{27}$  | 6 5 8 | 8       |
| $8_7$ | 8 5 8   |          | $9_{28}$  | 9 7 8 | 8       |
| $8_8$ | 6 5 8   |          | $9_{29}$  | 8 5 8 | 8       |
| $8_9$ | 5 3 8   |          | $9_{30}$  | 6 5 8 | 8       |
| $8_{10}$ | 8 5 8 | $9_{31}$ | 9 7 8 | 8 |
| $8_{11}$ | 9 7 8 | $9_{32}$ | 9 7 8 | 8 |
| $8_{12}$ | 5 5 6 | $9_{33}$ | 6 5 8 | 8 |
| $8_{13}$ | 6 5 8 | $9_{34}$ | 6 5 8 | 8–12 |
| $8_{14}$ | 9 7 8 | $9_{35}$ | 12 11†  | 10† |
| $8_{15}$ | 13 11† | 10* | $9_{36}$ | 12† | 9† | 8 |
| $8_{16}$ | 8 5 8 | $9_{37}$ | 6 5 8 | 8 |
| $8_{17}$ | 5 3 8 | $9_{38}$ | 14† | 11† | 10* |
| $8_{18}$ | 5 3 8 | $9_{39}$ | 10 9 | 8–10 |
| $8_{19}$ | 12 11† | 10* | $9_{40}$ | 9 7 | 8–12 |
| $8_{20}$ | 6† 5 6 | $9_{41}$ | 7 7 | 8–10 |
| $8_{21}$ | 9† 7† 6 | $9_{42}$ | 5 5 | 6 |
| $9_1$ | 18†     | 11†      | 10        | $9_{43}$ | 10 9  | 8–10    |
| $9_2$ | 12 11†  | 10*      | $9_{44}$  | 6† 5  | 6** |
| $9_3$ | 16†    | 11† 10   | $9_{45}$  | 10†   | 9†      | 8*        |
| $9_4$ | 14†    | 11† 10*  | $9_{46}$  | 7† 7  | 6       |
| $9_5$ | 12 11†  | 10*      | $9_{47}$  | 7 7 8 | 8       |
| $9_6$ | 16†    | 11† 10   | $9_{48}$  | 8† 7  | 6       |
| $9_7$ | 14†    | 11† 10*  | $9_{49}$  | 12 11† | 10†    |

*Table 1.* Table of flat plumbing basket numbers $fpbk(K)$ for prime knots $K$ with up to 9 crossings. For the notations, see Section 6.5.
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