Relativistic and Non-Relativistic Quantum Brownian Motion in an Anisotropic Dissipative Medium

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Abstract Using a minimal-coupling-scheme we investigate the quantum Brownian motion of a particle in an anisotropic-dissipative-medium under the influence of an arbitrary potential in both relativistic and non-relativistic regimes. A general quantum Langevin equation is derived and explicit expressions for quantum-noise and dynamical variables of the system are obtained. The equations of motion for the canonical variables are solved explicitly and an expression for the radiation-reaction of a charged particle in the presence of a dissipative-medium is obtained. Some examples are given to elucidate the applicability of this approach.

Keywords Anisotropic dissipative medium · Langevin-equation · Coupling tensor · Radiation reaction · Dirac equation · Cherenkov radiation

1 Introduction

In the present paper we introduce a scheme for investigating the quantum dynamics of a particle embedded in an anisotropic dissipative medium, under the influence of an arbitrary potential. This problem is fundamental to many fields of physics: statistical mechanics, chemical physics [1–9], condensed matter [10–12], quantum optics [13–19], quantum information and atomic physics [20]. Our purpose is to show that our approach is a macroscopic description that can be applied in a general way from the classical to the relativistic domain and is consistent with physics requirements, in particular causality and fluctuation-dissipation theorem.
For investigating the quantum mechanical description of a dissipative system, there are usually two approaches: the first approach is based on the assumption that the damping of the system is caused by an irreversible transfer of its energy to the reservoir due to the coupling of the system with the reservoir [21–23]. Therefore, the loss of energy is phenomenologically described in terms of a frictional force. In addition, the system also is subject to a fluctuating or noise force and that dissipation and fluctuations are related. The second approach is essentially a rigorous one in which the effect of dissipation is introduced by ingeniously construction a suitable Lagrangian or Hamiltonian for the system [24, 25]. Historically, the first Hamiltonian was introduced by Caldirola [26] and rederived independently by Kanai [27] and afterward by several others [28, 29]. They employed a time dependent mass in such a way that a friction term appears in the corresponding equation of motion. There are significant difficulties in the quantum mechanical solution of the Caldirola-Kanai Hamiltonian, for example quantizing in this way violates the uncertainty relation or canonical commutation rules and the former vanishes as time tends to infinity [30–33].

These common approaches to quantum dissipative systems can generally be divided into two classes. Working in Schrodinger picture and Heisenberg picture, that the dynamics is described in terms of generalized master equations [34–36] and Langevin equations [37, 38], respectively. Although, the effects of environmental degrees of freedom on the system can be investigated with the method of Feynman-Vernon influence functional by integrating out environment variables within the context of the closed-time-path formalism [39–46]. The more complicated interaction by considering nonlinear couplings of the particle with the reservoir has also been studied in [47, 48]. Furthermore, the relativistic Brownian motion has also been discussed in [49–51].

The main purpose of the present work is to develop a canonical theory of Brownian motion to extract the classical, the nonrelativistic and the relativistic quantum Langevin equation consistent with fluctuation-dissipation theorem. The result is a relation between noise correlations and susceptibility in frequency domain where the proportionality constant depends on temperature. To achieve this goal, we first introduce an appropriate Lagrangian that includes the dissipative effect in a consistent form and then generalize this Lagrangian to charged particles in presence of the electromagnetic fields. This prepares not only the grounds to survey the radiation reaction but extracts a Dirac equation for a relativistic particle. Our approach suggests the simplest way in which the dissipation and the fluctuation effects can emerge from the classical to the relativistic quantum theory. On this base, it is enough using a minimal coupling scheme to obtain a suitable Hamiltonian and also the motion equations to describe the dissipative system.

The layout of the paper is as follows: In Section 2, a Lagrangian for the total system is proposed and a classical treatment of the damped system is investigated. In Section 3, we use the Lagrangian introduced in the Section 2 to canonically quantize the system and obtain the corresponding Langevin equation. Subsequently in Section 4, as a simple application, we calculate the spontaneous decay rate of an initially excited two-level atom embedded in an anisotropic dissipative medium then this formalism is generalized to describe the radiation reaction of a charged particle in this medium. In Section 4, a modified Lagrangian is introduced to describe the relativistic effects of charged particles embedded in an anisotropic dissipative medium. We then apply our relativistic Langevin equation for a Dirac particle in an anisotropic dissipative medium to calculate the Cherenkov radiation that is emitted by such a medium when a relativistic charged particle moves through it. Finally, conclusions are given in Section 5.
2 Classical Dynamics

Classical and quantum description of a dissipative system under influence of a potential \( V \) can be accomplished by modeling it in interaction with a heat bath. We assume that the heat bath consists of a continuum of three dimensional harmonic oscillators labeled by a continuous parameter \( \omega \). This kind of heat bath could be a model for an elastic solid, a dissipative medium and electromagnetic field that known as the Hopfield model \([52–56]\). This model can be also used to describe the quantization of electromagnetic field in presence of an amplifying magnetodielectric medium \([57]\) where the electric and magnetic properties of the medium are modeled by two independent sets of harmonic oscillators. In order to examine the classical and the quantum treatment of a linear dissipative system, we start the following classical Lagrangian for the total system

\[
L(t) = L_\text{m} + L_\text{s} + L_\text{int}.
\]  

The first term \( L_\text{m} \) is the Lagrangian of the medium which is a continuum of three dimensional harmonic oscillators defined by

\[
L_\text{m} = \frac{1}{2} \int_0^\infty d\omega \left[ \dot{X}^2(\omega, t) - \omega^2 X^2(\omega, t) \right],
\]

where \( X(\omega, t) \) is the dynamical variable of the medium. The second term \( L_\text{s} \) is the Lagrangian of the main system with mass \( m \), position \( q \) and potential \( V(q) \)

\[
L_\text{s} = \frac{1}{2} m \dot{q}^2(t) - V(q),
\]

and \( L_\text{int} \) is the interaction term defined by

\[
L_\text{int} = \int_0^\infty d\omega f_{ij}(\omega) \dot{q}_i(t) X_j(\omega, t),
\]

where summation should be done over the repeated indices and \( f_{ij}(\omega) \) is the coupling tensor which for an isotropic medium is written as \( f(\omega) \delta_{ij} \). We can simply obtain the classical equations of motion from the Euler-Lagrange equations

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{q}_i(t)} \right) - \frac{\delta L}{\delta q_i(t)} = 0 \quad i = 1, 2, 3
\]

\[
m \ddot{q}(t) + \nabla_q V(q) = -\dot{R}(t),
\]

and

\[
\frac{d}{dt} \left( \frac{\delta L}{\delta \dot{X}_i(\omega, t)} \right) - \frac{\delta L}{\delta X_i(\omega, t)} = 0 \quad i = 1, 2, 3
\]

\[\ddot{X}_i(\omega, t) + \omega^2 X_i(\omega, t) = \dot{q}_j(t) f_{ji}(\omega),\]

where the components of the field \( R \) is defined by

\[
R_i = \int_0^\infty d\omega f_{ij}(\omega) X_j(\omega).
\]

The formal solution of the field equation (6) is

\[
X_i(\omega, t) = \dot{X}_i(\omega, 0) \frac{\sin \omega t}{\omega} + X_i(\omega, 0) \cos \omega t + \int_0^t dt' \frac{\sin \omega (t - t')}{\omega} f_{ji}(\omega) \dot{q}_j(t')
\]

where the first term is the inhomogeneous solution of (6) and the second term is the homogeneous one. We will show later, the homogeneous solution after quantization becomes a
noise operator. Now by substituting (8) in the integrand of (7), we find that the field $R$ is as follows

$$R_i(t) = \int_0^\infty dt' \chi_{ij}(t-t')\dot{q}_j(t') + R_i^N(t),$$

(9)

where $\chi_{ij}$ is the causal susceptibility tensor of the medium and in terms of the coupling tensor $f_{ij}$ can be written as

$$\chi_{ij}(t) = \int_0^\infty d\omega \frac{\sin \omega t}{\omega} f_{il} f_{jl}(\omega) \Theta(t),$$

(10)

where $\Theta(t)$ is the Theta function. The second term in (9) is a noise function which in terms of the coupling tensor $f_{ij}$ are obtained as

$$R_i^N(t) = \int_0^\infty d\omega f_{ij}(\omega) \left( \dot{X}_j(\omega, 0) \frac{\sin \omega t}{\omega} + X_j(\omega, 0) \cos \omega t \right).$$

(11)

It is easily shown that (10) is the origin of the significant Kramers-Kronig relations

$$\text{Re}[\chi_{ij}(\omega)] = P \int_{-\infty}^\infty \frac{dv}{\pi} \frac{\text{Im}[\chi_{ij}(v)]}{v - \omega},$$

$$\text{Im}[\chi_{ij}(\omega)] = -P \int_{-\infty}^\infty \frac{dv}{\pi} \frac{\text{Re}[\chi_{ij}(v)]}{v - \omega}. $$

(12)

Here, the symbol $P$ denotes the Cauchy principal value and

$$\chi_{ij}(\omega) = \int_{-\infty}^\infty dt \chi_{ij}(t)e^{i\omega t}. $$

(13)

Note that, for a definite susceptibility tensor $\chi_{ij}$ the coupling tensor $f_{ij}$ can not be determined uniquely. In fact if $f_{ij}$ is a solution then for any arbitrary unitary matrix $U$, $fU$ is also a solution. But this freedom does not affects the physical observables. Therefore, we may take the coupling tensor to be symmetric, i.e, $f_{il} f_{jl} = f^2_{ij}$, and invert the relation (13) for a specified susceptibility tensor $\chi_{ij}$. Then, the corresponding coupling tensor up to a unitary freedom is as follows:

$$f_{ij}(\omega) = \sqrt{\frac{2\omega}{\pi}} \text{Im} \chi_{ij}(\omega). $$

(14)

Now by substituting (9) into (5), the classical Langevin equation are obtained as

$$m\ddot{q}_i(t) + \int_0^t dt' \chi_{ij}(t-t')\dot{q}_j(t') + \frac{\partial V(q)}{\partial q_i} = \xi^N_i(t), $$

(15)

where $\xi^N(t) = -\dot{R}_i^N(t)$. Here, the frictional force is a linear functional of the history of the velocity $\dot{q}$ of the particle and the stochastic force $\xi^N(t)$ which is a result of the dissipation character of the medium and obeys the Gaussian statistics [10, 13].

### 3 Non-relativistic Quantum Dynamics

The Lagrangian (1) can now be used to obtain the canonical conjugate variables that correspond to the dynamical variables $X$ and $q$ respectively as

$$Q_i(\omega, t) = \frac{\delta L}{\delta \dot{X}_i(\omega, t)} = \dot{X}_i(\omega, t), $$

(16)

$$p_i(t) = \frac{\delta L}{\delta \dot{q}_i} = m\dot{q}_i + R_i(t). $$
The fields can be canonically quantized in a standard fashion by demanding equal-time commutation relations among the variables and their conjugates

\[
[q_i(t), p_j(t)] = i\hbar \delta_{ij},
\]

\[
[X_i(\omega, t), Q_j(\omega', t)] = i\hbar \delta_{ij} \delta(\omega - \omega'),
\]

with all other equal-time commutators being zero. Using the Lagrangian (1) and the expression for the canonical conjugate variables in (16), we obtain the Hamiltonian of the total system

\[
H = \frac{(p - R(t))^2}{2m} + V(q) + \frac{1}{2} \int_0^\infty d\omega (Q^2(\omega, t) + \omega^2 X^2(\omega, t)).
\]

It is seen that this has the same form as a charged particle Hamiltonian that interacts with the electromagnetic field through a minimal coupling provided that the vector field \( A \) be taken as the vector field \( R \). This result justify a minimal coupling scheme by making the substitution \( p \to (p - R(t)) \) to introduce the dissipation effect of the medium [58]. To facilitate calculations, let us introduce the following annihilation operator

\[
b_i(\omega, t) = \sqrt{\frac{1}{2\hbar\omega}}[\omega X_i(\omega, t) + iQ_i(\omega, t)],
\]

from equal-time commutation relations (17)–(18), we find

\[
[b_i(\omega, t), b^\dagger_j(\omega', t)] = \delta_{ij} \delta(\omega - \omega').
\]

Inverting the equation (20), we can write the canonical conjugate variables \( X_i(\omega, t) \) and \( Q_i(\omega, t) \) in terms of the creation and annihilation operators \( b_i^\dagger \) and \( b_i \) as

\[
X_i(\omega, t) = \sqrt{\frac{\hbar}{2\omega}}\left(b_i(\omega, t) + b_i^\dagger(\omega, t)\right),
\]

\[
Q_i(\omega, t) = i\sqrt{\frac{\hbar\omega}{2}}\left(b_i^\dagger(\omega, t) - b_i(\omega, t)\right).
\]

Using these relations, we can obtain the Hamiltonian of the total system in terms of the creation and annihilation operators of the medium

\[
H = \frac{(p - R(t))^2}{2m} + V(q) + H_m,
\]

where

\[
R_i(t) = \int_0^\infty d\omega \sqrt{\frac{\hbar}{2\omega}} f_{ij}(\omega) \left[b_j(\omega, t) + b_j^\dagger(\omega, t)\right],
\]

and

\[
H_m =: \int d\omega \hbar\omega \ b_i^\dagger(\omega, t) b_i(\omega, t) :
\]

is the Hamiltonian of the medium in normal ordering form. In the Heisenberg picture, by using commutation relations (17), (18) and the total Hamiltonian (19), the equations of motion for the canonical variables \( X \) and \( Q \) are obtained as

\[
\dot{X}_i(\omega, t) = \frac{i}{\hbar}[H, X_i(\omega, t)] = Q_i(\omega, t),
\]

\[
\dot{Q}_i(\omega, t) = \frac{i}{\hbar}[H, Q_i(\omega, t)] = -\omega^2 X_i(\omega, t) + \dot{q}_j(t) f_{ji}(\omega).
\]
It can be easily shown that the combination of these equations leads to the same classical equation (6) with the formal solution (8). In a similar way, using Heisenberg equation for the conjugate dynamical variables $\dot{\mathbf{q}}(t)$ and $\mathbf{p}(t)$, we find

$$\dot{\mathbf{q}}(t) = \frac{i}{\hbar} [H, \mathbf{q}(t)] = \frac{(\mathbf{p} - \mathbf{R}(t))}{m},$$

(28)

$$\dot{\mathbf{p}}(t) = \frac{i}{\hbar} [H, \mathbf{p}(t)] = -\nabla V(\mathbf{q}).$$

(29)

Combination of these recent equations also lead to the same classical equation of motion (5). In addition, by substituting the solution (8) in the latter equation, the quantum analogous of the Langevin equation (15) are obtained as

$$m\ddot{\chi}_{ij}(t) + \int_0^t dt' \dot{\chi}_{ij}(t - t')\dot{\chi}_{ij}(t') + \frac{\partial V(\mathbf{q})}{\partial q_i} = \xi^N_i(t),$$

(30)

where the susceptibility $\chi_{ij}$ has been already defined in (10) and $\xi^N_i(t)$ is a fluctuating force induced by the medium and in terms of the introduced operator (20) are written as

$$\xi^N_i(t) = t \int_0^\infty d\omega \sqrt{\frac{\hbar\omega}{2}} f_{ij}(\omega) (b_j(\omega, 0)e^{-i\omega t} - \text{h.c.}).$$

(31)

The (30) is the quantum Langevin equation, wherein the explicit form of the noise is known. The quantum Langevin equation can be considered as the basis of the macroscopic description of a quantum system coupled to an environment or a heat bath. This equation contains a memory tensor $\chi_{ij}(t)$ and a noise or fluctuating force $\xi^N_i(t)$. If the medium is kept in thermal equilibrium at temperature $T$, by using (31) the force noise correlations [10] can be found as

$$\zeta_{ij}(t - t') \equiv \frac{1}{2} \left[ \xi^N_i(t)\xi^N_j(t') + \xi^N_i(t')\xi^N_j(t) \right]$$

$$= \int_0^\infty d\omega \frac{\hbar\omega}{\pi} \text{Im}[\chi_{ij}(\omega)] \cos \omega(t - t') \coth\left(\frac{\hbar\omega}{2k_B T}\right).$$

(32)

From this recent relation, the power spectrum of the noise force is obtained as

$$\zeta_{ij}(\omega) \equiv \int_{-\infty}^{\infty} dt \zeta_{ij}(t) \cos \omega t = \hbar\omega \coth\frac{\hbar\omega}{2k_B T} \text{Im}[\chi_{ij}(\omega)],$$

(33)

which is a version of the quantum mechanical Fluctuation-dissipation theorem [1, 10, 59–62]. It should be noted that the above equation can be used to find a special susceptibility tensor in order to have a predetermined correlation function, for example a white noise. The limit $\hbar \rightarrow 0$ clearly gives a smooth transition to the classical Langevin equation, in the sense that in this limit all commutators vanish and the equation of motion becomes an equation for c-numbers driven by a noise term. Since $\lim_{\hbar \rightarrow 0} \hbar\omega \coth(\hbar\omega/2k_B T) = 2k_B T$, this noise term possesses a flat spectrum [10, 13].

In the case of a time-local friction proportional to the velocity that is usually called Ohmic, i.e, $\chi_{ij}(t) = \gamma_0 \delta_{ij}(t)$, we find that $f_{ij}(\omega) = \sqrt{\frac{\gamma_0}{\pi}} \delta_{ij}$ and the force noise correlations (32) are simplified as

$$\zeta_{ij}(t - t') = \gamma k_B T \frac{d}{dt} \coth\left[\frac{\pi k_B T(t - t')}{\hbar}\right]\delta_{ij},$$

(34)

which is consistent with the results have been reported in [10, 13, 59–62]. In the remainder of the paper we consider four examples to show the applicability of this scheme.
3.1 Free Particle

The simplest model for a dissipative Brownian particle is described by a free particle. We assume here, the particle with mass $m$ is moving in an anisotropic dissipative medium. This model can be applied to the problem of Cherenkov radiation in presence of a polarizable medium [63, 64]. We set $V(\mathbf{q}) = 0$ in the equation (30) and find

$$m \ddot{q}_i(t) + \int_0^t dt' \dot{\chi}_{ij}(t-t')\dot{q}_j(t') = \xi_i(t).$$  \hspace{1cm} (35)

For any function $q(t)$ the forward- and the backward-Laplace-transform of $q(t)$ are respectively defined by

$$\tilde{q}^f(s) = \int_0^\infty dt e^{-st} q(t),$$
$$\tilde{q}^b(s) = \int_0^\infty dt e^{-st} q(-t).$$

Depending on the initial conditions which are usually set at $t = 0$, there is no need to use backward Laplace transform but here we do not restrict ourselves and the reader can choose the plus sign in what follows to recover this situation. Using these definitions, we take the Laplace transform of the both sides of the equation (35) as

$$\Lambda_{ij}(s)\tilde{q}^f_j(s) = \frac{1}{s} \Lambda_{ij}q_j(0) \pm m\dot{q}_i(0) + \int_0^\infty d\omega \frac{\hbar}{2\omega} \sqrt{f_{ij}(\omega)} \left( \frac{b_j(\omega)}{s \pm i\omega} - h.c. \right),$$

where $\Lambda_{ij}(s) = (ms^2\delta_{ij} + s^2\dot{\chi}_{ij}(s))$ and the upper(lower) sign corresponds to $q^f_j(s)$ ($q^b_j(s)$). After some simple but elaborate calculations one finds

$$q_i(t) = q_i(0) \pm \eta_{ij}(t) p_j(0) + \int_0^\infty d\omega \frac{\hbar}{2\omega} \left( Z^*_{ij}(\omega, t) b_j(\omega, 0) + h.c. \right),$$

where now the upper(lower) sign corresponds to $t > 0$ ($t < 0$) and the functions $\eta_{ij}(t)$ and $Z^*_{ij}(\omega, \pm t)$ are defined by

$$\eta_{ij}(t) = L^{-1}[\Lambda_{ij}^{-1}(s)],$$
$$Z^*_{ij}(\omega, \pm t) = L^{-1} \left[ \Lambda_{ik}^{-1}(s) \frac{\pm s}{s \pm i\omega} \right] f_{kj}(\omega),$$

in which, $L^{-1}[f(s)]$ denotes the inverse Laplace transform of function $f(s)$ and $\Lambda^{-1}(s)$ is the inverse of the matrix $\Lambda(s)$.

Diffusion of a Brownian particle is characterized by the long-time behavior of the mean-square displacement, therefore we compute the mean square distance traveled by the free particle in a time interval starting at the time $t'$ and ending at the time $t$

$$\left\langle [\mathbf{q}(t) - \mathbf{q}(t')]^2 \right\rangle = \sum_{i=1}^3 (\eta_{ij}(t) - \eta_{ik}(t'))(p_j(0)p_k(0))$$
$$+ \int_0^\infty d\omega \frac{\hbar\omega}{2} \coth \frac{\hbar\omega}{2KT} \left\{ [Z^*_{ij}(\omega, t) - Z^*_{ij}(\omega, t')] [Z^*_{ij}(\omega, t) - Z^*_{ij}(\omega, t')] \right\},$$

\hspace{1cm} (40)
which in the case of the Ohmic damping is reduced to

\[
\left\langle (q(t) - q(t'))^2 \right\rangle = \int_0^\infty d\omega \frac{\hbar \gamma}{\pi} \frac{\coth(\hbar \omega/2KT)}{\gamma^2 + m^2 \omega^2} \left( e^{-\omega t} - e^{-\omega t'} - e^{-\frac{\gamma}{m} t} + e^{-\frac{\gamma}{m} t'} \right) \left( e^{\omega t} - e^{\omega t'} - e^{-\frac{\gamma}{m} t} + e^{-\frac{\gamma}{m} t'} \right).
\]

(41)

This integral contains transient terms, which appears to allow the particle to absorb a large amount of energy from medium, however after this transient, for sufficiently large times, diffusion of the particle is given by a formula which is known in classical stochastic theory. In large-time-limit we find

\[
\left\langle (q(t) - q(t'))^2 \right\rangle \approx \frac{\hbar \gamma}{\pi} \int_0^\infty d\omega \frac{\coth(\hbar \omega/2KT)}{\gamma^2 + m^2 \omega^2} \frac{4 \sin^2 \frac{\omega}{2}(t - t')}{\omega^2},
\]

(42)

from which the following asymptotic formula can be found [10, 13, 40].

\[
\left\langle (q(t) - q(t'))^2 \right\rangle \approx \frac{4KT}{\gamma} (t - t').
\]

(43)

3.2 Harmonic Oscillator

The problem of a damped harmonic quantum mechanical oscillator has been studied intensively because of its universal relevance [65–68]. Indeed, it describes a quantum electromagnetic field propagating in a linear dielectric medium [69], a particle interacting with a quantum field in dipole approximation [70]. In addition to these quantum optical applications, this problem is used in nuclear physics [71] and quantum chemistry [1, 72]. For a harmonic oscillator with mass \( m \) and frequency \( \omega_0 \), we set \( V(q) = \frac{1}{2} m \omega_0^2 q^2 \) in the equation (30) and find

\[
m \ddot{q}_i(t) + \int_0^t dt' \dot{\chi}_{ij}(t - t') \dot{q}_j(t') + m \omega_0^2 q_i(t) = \xi^N_i(t),
\]

(44)

with the following solution

\[
q_i(t) = \alpha_{ij}(t) q_j(0) \pm \eta'_{ij}(t) p_j(0) + \int_0^\infty d\omega \sqrt{\frac{\hbar}{2\omega}} \left( Z^\pm_{ij}(\omega, \pm t) b_{j}(\omega, 0) + h.c. \right),
\]

(45)

in which the upper (lower) sign again corresponds to \( t > 0 \) (\( t < 0 \)). The functions \( \alpha_{ij}(t), \eta'_{ij}(t) \) and \( Z^\pm_{ij}(\omega, \pm t) \) are defined by

\[
\alpha_{ij}(t) = L^{-1} \left[s \Lambda'^{\perp s}_{ik}(s)(m \delta_{kj} + \chi_{kj}(s)) \right],
\]

(46)

\[
\eta'_{ij}(t) = L^{-1} \left[ \Lambda'^{\perp s}_{ij}(s) \right],
\]

(47)

\[
Z^\pm_{ij}(\omega, \pm t) = L^{-1} \left[ \Lambda'^{\perp s}_{ik}(s) \pm s \right] f_{kj}(\omega),
\]

(48)

with

\[
\Lambda'^{\perp s}_{ij}(s) = ms^2 \delta_{ij} + s^2 \chi_{ij}(s) + m \omega_0^2.
\]

(49)

During the last decade huge advances in laser cooling and trapping experimental techniques have made it possible to confine harmonically a single ion and cool it down to very low temperatures where purely quantum manifestations begin to play an important role [73, 74]. The reference [75, 76] reports recently experimental results on how to couple a properly
engineered reservoir with a quantum oscillator. These experiments aim at measuring the
decoherence of a quantum superposition of coherent states and Fock states due to the
presence of the reservoir [68]. Therefore, in order to survey the theory bases of this kind of issue,
we calculate the transition probabilities of an initially excited harmonic oscillator embedded
in an anisotropic dissipative medium. For this purpose we write the Hamiltonian (23) as
\[ H = H_0 + H', \]
where
\[ H_0 = \frac{p^2}{2m} + \frac{1}{2} m \omega_0^2 q^2 + H_m \]
\[ = \hbar \omega_0 \left( \alpha_{j} a_{j} + \frac{3}{2} \right) + H_m, \]
with
\[ H' = -\frac{p \cdot R}{m} + \frac{R^2}{2m}. \]
The term \( R^2 \) can be ignored for a sufficiently weak coupling and the dominant term in this
case is \( \frac{p \cdot R}{m} \). Using the rotating-wave approximation [73–76], \( H' \) in the interaction picture
can be written as
\[ H'_I = \frac{i}{\hbar} \sqrt{\frac{\hbar \omega_0}{2m}} \int_0^\infty d\omega f_{ij}(\omega) \left[ a_i(0) b_j^+(\omega, 0) e^{-t(\omega_0 - \omega)t} - a_j^+(0) b_j(\omega, 0) e^{t(\omega_0 - \omega)t} \right]. \]
In order to obtain the transition-probabilities, we find the density operator to using the time-
dependent perturbation theory. The time-evolution of the density-operator in the interaction-
picture can be obtained from [77–80]
\[ \rho_I(t) = U_I(t) \rho_I(0) U_I(t), \]
where \( U_I(t) \), up to the first order time-dependent perturbation is
\[ U_I(t) = 1 - \frac{t}{\hbar} \int_0^\infty dt' H_I(t') \]
\[ = 1 + \sqrt{\frac{\hbar \omega_0}{2m}} \int_0^\infty d\omega f_{ij}(\omega) \left( a_i(0) b_j^+(\omega, 0) e^{-t(\omega_0 - \omega)t} - a_j^+(0) b_j(\omega, 0) e^{t(\omega_0 - \omega)t} \right) \frac{\sin \left( \frac{(\omega_0 - \omega)t}{2} \right)}{\left( \frac{(\omega_0 - \omega)t}{2} \right)}. \]
Let the initial density-operator be \( \rho_I(0) = |n_j\rangle \omega_0 \omega_0 \langle n_j| \otimes |0\rangle_{m} \langle 0| \) where \( |0\rangle_{m} \) is the
vacuum state of the dissipative medium and \( |n_j\rangle_{\omega_0} \) an excited state of the harmonic oscillator
for the mode \( j \), \( (j = 1, 2, 3) \). Then, by substituting \( U_I(t) \) from (55) in (54) and tracing
out the medium degrees of freedom, the transition probabilities \( |n_j\rangle_{\omega_0} \rightarrow |n_j \pm 1\rangle_{\omega_0} \) are
obtained as
\[ \Gamma_{n_j \rightarrow n_j+1} = \text{Tr}_m \left[ C_1 \right] = 0, \]
\[ \Gamma_{n_j \rightarrow n_j-1} = \text{Tr}_m \left[ C_2 \right] \]
\[ = \frac{\omega_0 n_j}{m \pi \hbar} \int_0^\infty d\omega \omega \sin \left( \frac{\omega_0 \omega}{2} t \right) \frac{\sin \left( \frac{\omega \omega_0}{2} t \right)}{\left( \frac{\omega_0 - \omega}{2} t \right)^2}, \]
in which \( \text{Tr}_m(\text{Tr}_m) \) denotes that the trace is taken over the degrees of freedom of the medium (harmonic oscillator). In the large-time-limit, \( \frac{\sin^2 \left( \frac{\omega - \omega_0}{2} \right) t}{\left( \frac{\omega - \omega_0}{2} \right)^2} \approx 2\pi t \delta(\omega - \omega_0) \), therefore

\[
\Gamma_{n_j \rightarrow n_j-1} = \frac{2\omega_0^2 n_j t}{m\hbar} \text{Im}[\chi_{jj}(\omega_0)].
\]

In a similar way, when the medium has a Maxwell-Boltzmann distribution, i.e., \( \rho_m = e^{-\frac{H_m}{\hbar T}} \) and \( \rho_I(0) = |n_j\rangle_{\omega_0} \langle n_j| \otimes \rho_m \), then the transition-probabilities \( |n_j\rangle_{\omega_0} \rightarrow \text{Tr}_m[e^{iB_T}] |n_j \pm 1\rangle_{\omega_0} \), in the large-time-limit, can be obtained as

\[
\Gamma_{n_j \rightarrow n_j+1} = \frac{2\omega_0^2 (n_j + 1) \tilde{n}(\omega, T) t}{m\hbar} \text{Im}[\chi_{jj}(\omega_0)],
\]

\[
\Gamma_{n_j \rightarrow n_j-1} = \frac{2\omega_0^2 n_j (\tilde{n}(\omega, T) + 1) t}{m\hbar} \text{Im}[\chi_{jj}(\omega_0)],
\]

where \( \tilde{n}(\omega, T) = \left[ \exp\left(\frac{\hbar\omega_0}{k_B T}\right) - 1 \right]^{-1} \) is the mean number of thermal photons at frequency \( \omega \) and temperature \( T \). It is clear from above equations, when the medium is held at zero temperature then \( \Gamma_{n_j \rightarrow n_j+1} = 0 \), and as expected the energy flows only from the system to the medium until the system falls in its ground state and remains in this state forever.

### 3.3 Spontaneous Emission of an Excited Two-Level Atom

In this section we calculate the decay constant and level shift for an initially excited atom embedded in an anisotropic medium. Theoretical treatments often simplify matters even more by assuming that the atoms only have two levels and is localized, which experimentally can only be achieved by rather complicated optical pumping techniques [79]. Indeed it describes systems composed of atoms interacting with a few modes of the electromagnetic field which has been studied both theoretically and experimentally [13, 78–80]. For this purpose, let us consider a one-electron atom and write the Hamiltonian (23) as

\[
H = H_0 + H',
\]

where

\[
H_0 = H_{\text{Atom}} + H_m,
\]

and

\[
H' = q \cdot \dot{\hat{R}}(t),
\]

with

\[
H_{\text{Atom}} = \frac{\mathbf{p}^2}{2m} + V(q).
\]

For a two-level atom with upper state \( |2\rangle \), lower state \( |1\rangle \) and transition frequency \( \omega_0 \), the Hamiltonian (61) can be written as

\[
H = \hbar \omega_0 \sigma^+ \sigma + (q_{12} \sigma + q_{12}^* \sigma^+) \cdot \dot{\hat{R}}(t) + H_m
\]

where \( \sigma = |1\rangle\langle 2| \) and \( \sigma^+ = |2\rangle\langle 1| \) are Pauli operators of the two-level atom and \( q_{12} = \langle 1| q |2\rangle \) is its transition dipole momentum. To study the spontaneous decay of an initially
excited atom we may look for the wave function of total system as follows
\[ |\psi(t)\rangle = c(t) |2\rangle |0\rangle_m + \sum_{i=1}^{3} \int_{0}^{\infty} d\omega M_i(\omega, t) |1\rangle |1_i(\omega)\rangle_m \]  
(66)
where \(|0\rangle_m\) and \(|1_i(\omega)\rangle_m\) are the vacuum state and the excited state of the medium with a single photon with frequency \(\omega\) and polarization \(i\), respectively. The coefficients \(c(t)\) and \(M_i(\omega, t)\) are to be specified by the Schrödinger equation
\[ i\hbar \frac{\partial |\psi(t)\rangle}{\partial t} = H |\psi(t)\rangle, \]  
(67)
with the initial conditions \(c(0) = 1, M_i(\omega, 0) = 0\). According to the relations shown in Appendix, the spontaneous decay of an initially excited atom are given by
\[ c(t) = e^{-\Gamma t} e^{-i(\Delta + \omega_0)t}, \]  
(68)
where
\[ \Gamma = \frac{\omega_0^2}{\hbar} \left[ q_{12,i}^* \text{Im}[\chi_{ij}(\omega_0)] q_{12,j} \right], \]  
(69)
and
\[ \Delta = P \int_{0}^{\infty} d\omega \frac{q_{12,i}^* \text{Im}[\chi_{ij}(\omega)] q_{12,j}}{\pi \hbar (\omega_0 - \omega)}, \]  
(70)
are the decay constant and the level shift due to the presence of the dissipative medium, respectively. The symbol \(P\) denotes the Cauchy principal value.

In the case of an isotropic susceptibility such as the Lorentz model, i.e. \(\chi_{ij}(t) = \beta e^{-\gamma t} \sin \nu t \delta_{ij}\) which \(\beta\) and \(\nu\) are a positive constants and \(\gamma\) is a damping coefficient, we find that \(f^2(\omega) = \frac{\beta \omega}{2\pi \nu} \left[ \frac{\gamma}{\nu^2 + (\nu + \omega)^2} - \frac{\gamma}{\nu^2 + (\nu - \omega)^2} \right]\). This recent coupling function in nondissipative limit, \(\gamma \to 0\), are reduced to \(f^2(\omega) = \frac{\beta \omega}{\nu} \delta(\nu - \omega)\). Accordingly, by inserting it into (69) and (70), we obtain
\[ \Gamma = \frac{\pi \beta \omega_0^2 |q_{12}|^2}{2\hbar \nu} \delta(\nu - \omega_0), \]  
(71)
\[ \Delta = \frac{\beta \nu |q_{12}|^2}{2\hbar (\omega_0 - \nu)}, \]  
(72)
which is consistent with the results have been reported in [78–80].

3.4 Radiation Reaction

A charge particle in accelerated motion radiates electromagnetic waves, and as a result it experiences a friction-like force, i.e. radiation reaction. Consider a charged particle embedded in an anisotropic-dissipative-medium interacting with its own field and also the quantum vacuum field. The total Lagrangian that describe this system is a generalized version of the Lagrangian (1) by adding the Lagrangian electromagnetic field together with its interaction with the charged particle. Therefore, the total Lagrangian in Coulomb-gauge can be written as
\[ L = \frac{1}{2} m \dot{q}^2(t) + \frac{1}{2} \int_{0}^{\infty} d\omega [\dot{X}^2(\omega, t) - \omega^2 X^2(\omega, t)] + \frac{1}{2} \int d^3 r \left[ \varepsilon_0 \dot{A}^2 - \frac{(\nabla \times A)^2}{\mu_0} \right] - \frac{1}{2} \int d^3 r \rho(r) \phi(r, t) + \dot{q} \cdot \mathbf{R}(t). \]  
(73)
By using Lagrangian (73), the corresponding canonically conjugate variables can be found

\[ -\varepsilon_0 E_i^\perp (\mathbf{r}, t) = \frac{\delta L}{\delta \dot{A}_i (\mathbf{r}, t)} = \varepsilon_0 \dot{A}_i (\mathbf{r}, t), \]  

\[ p_i(t) = \frac{\delta L}{\delta \dot{q}_i} = m \dot{q}_i + R_i(t) + e A_i(\mathbf{r}, t). \]  

The electromagnetic field can be canonically quantized by imposing the following equal-time commutation relation

\[ [A_i(\mathbf{r}, t), -\varepsilon_0 E_j^\perp (\mathbf{r}', t)] = i \hbar \delta_{ij} \delta^\perp (\mathbf{r} - \mathbf{r}'). \]  

where \( \delta^\perp (\mathbf{r} - \mathbf{r}') \) is the transverse delta function. To facilitate the calculations, let us introduce new annihilation-operators

\[ a_\lambda (\mathbf{k}, t) = \sqrt{\frac{c|\mathbf{k}|\varepsilon_0}{2\hbar}} \left( e^{i\mathbf{k} \cdot \mathbf{r}} a_\lambda (\mathbf{k}, t) - e^{-i\mathbf{k} \cdot \mathbf{r}} a_\lambda^\dagger (\mathbf{k}, t) \right), \]  

where \( A_\lambda (\mathbf{k}, t) \) and \( E_\lambda^\perp (\mathbf{k}, t) \) are the cartesian components of the spatial Fourier transformation of \( \mathbf{A}(\mathbf{r}, t) \) and \( \mathbf{E}^\perp (\mathbf{r}, t) \). From equal-time commutation relation (76), the following equal-time commutation relations are obtained

\[ \left[ a_\lambda (\mathbf{k}, t), a_\lambda^\dagger (\mathbf{k}', t) \right] = \delta_{\lambda,\lambda'} \delta (\mathbf{k} - \mathbf{k'}). \]  

Inverting (77) and taking the inverse Fourier transform, we obtain the electromagnetic field operators in terms of the creation and annihilation operators in the real-space as

\[ \mathbf{A}(\mathbf{r}, t) = \sum_{\lambda=1}^2 \int d^3k \sqrt{\frac{\hbar}{2(2\pi)^3\varepsilon_0 c|\mathbf{k}|}} \left( a_\lambda^\dagger (\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{r}} + a_\lambda (\mathbf{k}, t) e^{-i\mathbf{k} \cdot \mathbf{r}} \right) \mathbf{e}_\lambda (\mathbf{k}), \]

\[ \mathbf{E}^\perp (\mathbf{r}, t) = -i \sum_{\lambda=1}^2 \int d^3k \sqrt{\frac{\hbar c|\mathbf{k}|}{2(2\pi)^3\varepsilon_0}} \left( a_\lambda^\dagger (\mathbf{k}, t) e^{-i\mathbf{k} \cdot \mathbf{r}} - a_\lambda (\mathbf{k}, t) e^{i\mathbf{k} \cdot \mathbf{r}} \right) \mathbf{e}_\lambda (\mathbf{k}), \]

where \( \mathbf{e}_\lambda (\mathbf{k}), (\lambda = 1, 2) \) are orthonormal polarization vectors. Using Lagrangian (73) and the expressions for the canonical conjugate variables in (17) and (21), we find Hamiltonian of the total system as

\[ H = \frac{(\mathbf{p} - \mathbf{R}(t) - e\mathbf{A}(\mathbf{q}, t))^2}{2m} + \frac{1}{2} \int d^3r \rho(\mathbf{r}) \phi(\mathbf{r}, t) + H_F + H_m, \]

where

\[ H_F =: \sum_{\lambda=1}^2 \int d^3k \frac{\hbar c|\mathbf{k}|}{2} a_\lambda^\dagger (\mathbf{k}, t) a_\lambda (\mathbf{k}, t) : \]

is the Hamiltonian of the electromagnetic field in the normal ordering form and \( H_m \) is the Hamiltonian of the heat bath which is already defined in (25).

In the Heisenberg picture, the equations of motion for canonical variables \( \mathbf{X}_\omega \) and \( \mathbf{Q}_\omega \) are the same equations (26) and (27) with the formal solution (29) and likewise the \( \mathbf{R} \) field is defined by (30). In a similar way, by combining the Heisenberg equations for the conjugate dynamical variables \( \dot{\mathbf{q}}(t) \) and \( \dot{\mathbf{p}}(t) \) we find

\[ m \ddot{q}_i(t) + \int_0^t dt' \chi_{ij}(t - t') \dot{q}_j(t') + e \frac{\delta \phi(\mathbf{q})}{\delta q_i} = e E_i(\mathbf{q}, t) \]

\[ + e \varepsilon_{ijk} \dot{q}_j(t) B_k(\mathbf{q}, t) + \xi^N_i(t), \]
where the $\epsilon_{ijk}$ are the components of the Levi-Civita pseudotensor and the susceptibility tensor $\chi_{ij}$ and the noise operator $\xi_i^N(t)$ are the same equations as previously defined by (10) and (31). Consider now a single electron with binding potential energy $V(\mathbf{r}) = e\phi(\mathbf{r})$. Suppose that the distances over which the bound electron can move in this potential are small compared with the wavelength of any field with the electron undergoes a significant interaction. Therefore, it is convenient to make the electric dipole approximation in which spatial variation of $\mathbf{A}$ is ignored. Using this note, the Heisenberg equation for the operator $a_\lambda(\mathbf{k}, t)$ is found from the Hamiltonian (79) to be

$$\dot{a}_\lambda(\mathbf{k}, t) = -i\omega_\lambda a_\lambda(\mathbf{k}, t) + ie \frac{\hat{q}(t) \cdot e_\lambda(\mathbf{k})}{\sqrt{2(2\pi)^3 \varepsilon_0 \hbar \omega_\lambda}},$$

with the following formal solution

$$a_\lambda(\mathbf{k}, t) = e^{-i\omega_\lambda t} a_\lambda(\mathbf{k}, 0) + \frac{te}{\sqrt{2(2\pi)^3 \varepsilon_0 \hbar \omega_\lambda}} \int_0^t dt' e^{-i\omega_\lambda(t-t')} e_\lambda(\mathbf{k}) \cdot \hat{q}(t'),$$

where $\omega_\lambda = c k$. Now, by inserting $a_\lambda(\mathbf{k}, t)$ into the right-hand side of (81), we obtain

$$m\ddot{q}_i(t) + \int_0^t dt' \chi_{ij}(t-t')\dot{q}_j(t') + e \frac{\partial \phi(\mathbf{q})}{\partial q_i} = e E_{0,i}(t) + e E_{RR,i}(t) + \xi_i^N(t).$$

where $E_{0,i}(t)$ and $E_{RR,i}(t)$ are, respectively, the components of the vacuum field and the radiation reaction field and define as follow:

$$E_{0,i}(t) = t \sqrt{\frac{\hbar}{2(2\pi)^3 \varepsilon_0}} \sum_{\lambda=1}^2 \int_0^\infty d^3k \sqrt{\omega_\lambda} \left( a_\lambda(\mathbf{k}, 0) e^{-i\omega_\lambda t} - a_\lambda^\dagger(\mathbf{k}, 0) e^{i\omega_\lambda t} \right) e_\lambda(\mathbf{k}),$$

$$E_{RR}(t) = -\frac{e}{(2\pi)^3 \varepsilon_0} \sum_{\lambda=1}^2 \int_0^t dt' \cos \omega_\lambda(t-t') \int d^3k e_\lambda(\mathbf{k}) \cdot \hat{q}(t')$$

$$= \tau \ddot{q}(t) - \frac{\delta m}{e} \bar{q}(t).$$

Here, $\tau = \frac{e^2}{6\pi c^2}$ and $\delta m = \frac{e^2}{3\pi^2 \varepsilon_0 c^3} \int_0^\infty d\omega$ [80]. The mass $m$ in (84) is the mass of a bare particle which does not interact with electromagnetic field. It is fictitious, since the interaction cannot be turned off. The experimental mass of the particle must include the present interaction with field. Therefore, $\delta m$ is effectively a contribution to the mass and arises from the action of its own field, namely, from the radiation reaction.

4 Dissipative Dirac Field

The description of particles used in the preceding sections is valid only when the particles are moving at velocities small compared to the velocity of light. In this section, we generalize the preceding formalism to describe relativistic particles embedded in an anisotropic-dissipative-medium. For this purpose, we use the following Lagrangian for the Dirac field under influence of a potential $V$ and its interaction with an external electromagnetic field and the dissipative medium [64, 80]

$$L = L_m + L_s + L_{int},$$

(87)
where
\[
L_s = \frac{i\hbar c}{2} \int d^3 x \left[ \sum_{\mu=0}^{3} \left( \bar{\psi}(x, t) \gamma^\mu \frac{\partial \psi(x, t)}{\partial x^\mu} - \frac{\partial \bar{\psi}(x, t)}{\partial x^\mu} \gamma^\mu \psi(x, t) \right) - (mc^2 + V(x)) \bar{\psi} \psi \right] ,
\]
\(\tag{88}\)

and
\[
L_{\text{int}} = e \int d^3 x \left[ c \bar{\psi}(x, t) \gamma^j \psi(x, t) A^j(x, t) - \bar{\psi}(x, t) \gamma^0 \psi(x, t) \varphi(x, t) \right] + c \int_0^\infty d\omega \int d^3 x f_{ij}(\omega) \bar{\psi}(x, t) \gamma^i \psi(x, t) X^j(\omega) .
\]
\(\tag{89}\)

Here, \(\gamma^\mu, (\mu = 0, \ldots, 3)\), are the Dirac matrices with \(\gamma^0 \equiv \beta, \gamma^j \equiv \beta\alpha^j\) and \(\bar{\psi} \equiv \psi^\dagger\beta\). In a standard representation we have
\[
\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} ,
\]
\(\tag{90}\)

where \(\sigma_j, (j = 1, 2, 3)\), are Pauli spin matrices and \(I\) is the unit matrix. The Lagrangian (87) is the relativistic generalization of the nonrelativistic Lagrangian (73) in the Coulomb gauge. The part of the Lagrangian attributable to the transverse field, involving the energy density of electromagnetic field, has the same form as (73), as does the part associated with the medium and the part corresponded to the instantaneous Coulomb interactions among the charged particles. (In the nonrelativistic Lagrangian the Coulomb interactions are written explicitly in the form appropriate for classical point particles.) Here, the medium Lagrangian (3) stays the same, since the medium is stationary on the one hand. On the other hand, we often deal with a non-relativistic medium, although the relativistic version of the Lagrangian (3) can be simply considered \([82]\). The main difference between the relativistic and nonrelativistic Lagrangian lied in the treatment of the particle kinetic term (88) and it’s interaction terms (89). We proceed along the lines of the preceding section and define the canonical conjugate variable of the Dirac particle as
\[
\frac{\partial L}{\partial \dot{\psi}} = \frac{i\hbar}{2} \psi^\dagger .
\]
\(\tag{91}\)

The Dirac field is quantized by imposing anti-commutation relations among the field components
\[
\{ \psi_\alpha(x, t), \psi^\dagger_\beta(x', t) \} = \delta_{\alpha\beta} \delta(x - x') ,
\]
\[
\{ \psi_\alpha(x, t), \psi_\beta(x', t) \} = 0 .
\]
\(\tag{92}\)

The Hamiltonian of the total system can also be find as
\[
H = c \int d^3 x \psi^\dagger(x, t) \left[ \alpha \cdot (p - R(t) - eA(x, t)) + (mc^2 + V(x)) \beta \right] \psi(x, t)
\]
\[
+ \frac{1}{2} \int_0^\infty d\omega \left( Q^2(\omega, t) + \omega^2 X^2(\omega, t) \right) + \frac{1}{8\pi\epsilon_0} \int d^3 x \int d^3 x' \frac{\rho(x, t) \rho(x', t)}{|x - x'|} .
\]
\(\tag{93}\)

where \(p = -i\hbar \nabla\), the field \(R(t)\) is defined by (7) and \(\rho(x, t) = e\psi^\dagger(x, t)\psi(x, t)\) is the charge density. In the Heisenberg picture, by using commutation relations (92) and the total
Hamiltonian (93), the relativistic analogues of the motion equation (6) is obtained as
\[ \ddot{X}_i(\omega, t) + \omega^2 X_i(\omega, t) = f_{ji}(\omega)J_j(t), \] (94)
with the solution
\[ X_i(\omega, t) = Q_i(\omega, 0) \sin \omega t + X_i(\omega, 0) \cos \omega t + \int_0^t dt' \frac{\sin \omega (t - t')}{\omega} f_{ji}(\omega)J_j(t'), \] (95)
where
\[ J(t) = c \int d^3x \psi^\dagger(x, t) \alpha \psi(x, t). \] (96)

Now by substituting \( X_i(\omega, t) \) from (95) in (7) we find
\[ R_i(t) = \int_0^\infty dt \chi_{ij}(t - t')J_j(t') + R_i^N(t), \] (97)
where \( \chi_{ij} \) and \( R_i^N(t) \) are the causal susceptibility tensor of the medium and the noise operator that previously defined by (10) and (11), respectively. If we apply the Heisenberg equation to the Dirac field \( \psi(x, t) \) and make use of the anticommutation relations (92), then the Dirac equation in the presence of a dissipative medium are found as
\[ i\hbar \dot{\psi}(x, t) = \left[ c \alpha \cdot \pi + e \phi(x, t) + (mc^2 + V(x))\beta \right] \psi(x, t), \] (98)
in which \( \pi \equiv (p - R(t) - eA(x, t)) \), and
\[ \phi(x, t) = \frac{e}{4\pi \epsilon_0} \int d^3x' \frac{\psi(x', t) \psi(x', t)}{|x - x'|}. \] (99)
Substitution (97) in (98), a relativistic Langevin equation for a Dirac particle in an anisotropic-dissipative-medium can be obtained as
\[ i\hbar \dot{\psi}(x, t) + c \alpha \cdot (i\hbar \nabla + eA(x, t))\psi(x, t) + \int_0^\infty dt' \alpha_i \chi_{ij}(t - t')J_j(t') \psi(x, t)
- [e\phi(x, t) + (mc^2 + V(x))\beta] \psi(x, t) = -\alpha \cdot R^N(t) \psi(x, t). \] (100)

What we have here is a case of the fluctuation-dissipation relation. Generally speaking, if a system is coupled to a dissipative medium that can take energy from the system in an effectively irreversible way, then the medium must also cause fluctuations. The fluctuations and the dissipation effects go hand in hand, we cannot have one without the other. In (100), the effects of the dissipation and the fluctuation appear as two contributions \( \chi_{ij}(t - t') \) and \( R^N(t) \) from the vector field \( R \). Thus, our approach suggests the simplest way in which the dissipation and the fluctuation effects can emerge from the classical to the relativistic quantum regime. In fact, based on a minimal coupling scheme, we can introduce the dissipation and the fluctuation effects of the medium by writing \( p^2 \) equivalently as \( (p - R)^2 \) in the Heisenberg equation for a free brownian particle, or making the replacement \( (p - R)^2 \to (p - R - eA)^2 \) in presence of the electromagnetic field. This approach to the derivation of the motion equation is simpler than that proceeding from other approach [49–51] and provides a consistent and rigorous basis for the introduction of the dissipation and the fluctuation in a Hamiltonian formalism and motion equations from the classical to the relativistic domain.
Let us examine the nonrelativistic limit of the Dirac equation (98). We introduce the Dirac field $\psi(x, t)$ as

$$\psi = \left( \begin{array}{c} \tilde{\eta} \\ \tilde{\xi} \end{array} \right),$$  

where $\tilde{\eta}$ and $\tilde{\xi}$ are each two-component column vectors, so that by applying (98) we have

$$i\hbar \frac{\partial}{\partial t} \left( \begin{array}{c} \tilde{\eta} \\ \tilde{\xi} \end{array} \right) = c\sigma \cdot \pi \left( \begin{array}{c} \tilde{\xi} \\ \tilde{\eta} \end{array} \right) + e\phi(x, t) \left( \begin{array}{c} \tilde{\eta} \\ \tilde{\xi} \end{array} \right) + (mc^2 + V(x)) \left( \begin{array}{c} \tilde{\eta} \\ -\tilde{\xi} \end{array} \right).$$  

In the nonrelativistic limit the energy $mc^2$ is large compared with any kinetic or potential energy, and this suggests writing

$$\left( \begin{array}{c} \tilde{\eta} \\ \tilde{\xi} \end{array} \right) = e^{-i\frac{mc^2}{\hbar}t} \left( \begin{array}{c} \eta \\ \xi \end{array} \right),$$  

in which assuming that $\eta$ and $\xi$ slowly varying compared with $e^{-i\frac{mc^2}{\hbar}t}$ in a nonrelativistic approximation. Therefore, (102) becomes

$$i\hbar \frac{\partial}{\partial t} \left( \begin{array}{c} \eta \\ \xi \end{array} \right) = c\sigma \cdot \pi \left( \begin{array}{c} \xi \\ \eta \end{array} \right) + e\phi(x, t) \left( \begin{array}{c} \eta \\ \xi \end{array} \right) - 2mc^2 \left( \begin{array}{c} 0 \\ \xi \end{array} \right) + V(x) \left( \begin{array}{c} \eta \\ -\xi \end{array} \right),$$  

and in the nonrelativistic limit the second of the two indicated equations is replaced by

$$\xi \approx \frac{\sigma \cdot \pi}{2mc} \eta.$$  

Then the equation for $\eta$ becomes

$$i\hbar \frac{\partial \eta}{\partial t} \approx \left[ \frac{(\sigma \cdot \pi)^2}{2m} + e\phi(x, t) + V(x) \right] \eta.$$  

This result can be cast in a more familiar form by using the general identity $(\sigma \cdot C)(\sigma \cdot D) = C \cdot D + i\sigma \cdot (C \times D)$ [80], and we can write (106) for a spinless particle as the nonrelativistic equation

$$i\hbar \frac{\partial \eta}{\partial t} = \left[ \frac{(\mathbf{p} - \mathbf{R}(t) - e\mathbf{A}(x, t))^2}{2m} + e\phi(x, t) + V(x) \right] \eta.$$  

Comparison above obtained result with (98) suggests the interpretation of $\alpha$ as the operator corresponding to the particle’s velocity, i.e., $c \int d^3x \psi^\dagger(x, t) \alpha \psi(x, t) = \dot{q}(t)$. This interpretation is strengthened by the Heisenberg equation $\dot{q}(t) = \frac{i}{\hbar} [H, q(t)] = c \int d^3x \psi^\dagger(x, t) \alpha \psi(x, t).$  

Let us expand the Dirac field $\psi(x, t)$ in term of the eigenfunctions of the free Dirac equation in the absence of dissipative medium

$$\psi(x, t) = \frac{1}{(2\pi)^{3/2}} \sum_{\mu=1}^{4} \int d^3q c_\mu(q, t) \psi_\mu(q),$$  

where $\psi_\mu(q) = u_\mu(q)e^{i\mathbf{q} \cdot \mathbf{x}}$ and $u_\mu(q)$ are four-component spinors of the Dirac equation with corresponding eigenvalues $E_q = \pm\sqrt{\hbar^2c^2q^2 + m^2c^4}$ and normalization.
\( u_\mu^\dagger(q)u_\nu(q) = \delta_{\mu\nu} \) [63]. Here, the operator \( c_\mu(q, t) \) annihilates a particle with momentum \( h\mathbf{q} \). Substitution (109) in (93), the Hamiltonian of the total system can be written as

\[
H = \sum_{\mu=1}^{4} \int d^3 \mathbf{q} E_\mu c_\mu^\dagger(q, t)c_\mu(q, t) + H_m + H_F + H_{\text{int}},
\]

(110)

where

\[
H_{\text{int}} = -c \sum_{\mu, \mu'=1}^{4} \int_0^\infty d\omega \int d^3 \mathbf{q} \sqrt{\frac{h}{2\omega}} \left[ u_\mu^\dagger(q)\alpha_i f_{ij}(\omega) u_{\mu'}(q) \right] \left[ c_{\mu'}^\dagger(q)c_{\mu'}(q) \right] \beta_j(\omega) + \text{h.c.}
\]

\[
-\epsilon \sum_{\lambda=1}^{2} \sum_{\mu, \mu'=1}^{4} \int d^3 \mathbf{k} \int d^3 \mathbf{q} \sqrt{\frac{hc}{2(2\pi)^3e_0|k|}} \left[ u_\mu^\dagger(k + q)\alpha \cdot e_\lambda(k) u_\mu(q) \right]
\]

\[\times c_{\mu'}^\dagger(k + q)c_{\mu}(q)\alpha_\lambda(k) + \text{h.c.},\]

(111)

and \( H_m \) and \( H_F \) are defined in (25) and (80), respectively. The Heisenberg equation for the operator \( a_\lambda(k, t) \) is found from Hamiltonian (110) as

\[
\dot{a}_\lambda(k) = \frac{i}{\hbar}[H, a_\lambda(k)] = -i\omega_k a_\lambda(k)
\]

\[+\frac{tec}{\sqrt{2(2\pi)^3h\epsilon_0\omega_k}} \sum_{\mu, \mu'} \int d^3 \mathbf{q} \left( u_\mu^\dagger(q)\alpha \cdot e_\lambda(k) u_{\mu'}(k + q) \right) c_{\mu'}^\dagger(q)c_{\mu'}(k + q),\]

(112)

with the formal solution

\[
a_\lambda(k, t) = e^{-i\omega_k t}a_\lambda(k, 0)
\]

\[+\frac{tec}{\sqrt{2(2\pi)^3h\epsilon_0\omega_k}} \sum_{\mu, \mu'} \int d^3 \mathbf{q} C_{\lambda, \mu, \mu'}(k, q)
\]

\[\times \int_0^t dt' e^{-i\omega_k(t-t')} c_{\mu'}^\dagger(q', t')c_{\mu'}(k + q', t')\]

\[= a_{0, \lambda}(k, t) + a_{RR, \lambda}(k, t),\]

(113)

where \( C_{\lambda, \mu, \mu'}(k, q) \equiv u_\mu^\dagger(q)\alpha \cdot e_\lambda(k) u_{\mu'}(k + q) \). Now, by substitution (113) into (100) a relativistic-Langevin-equation is obtained that describes a relativistic moving particle through an anisotropic-dissipative-medium in presence of the electromagnetic field

\[
\dot{\psi}(x, t) + c \alpha \cdot (i\hbar \nabla + eA_0(x, t))\psi(x, t) - [e\phi(x, t) + (mc^2 + V(x))\beta]\psi(x, t)
\]

\[+ \int_0^\infty dt' \alpha_i \chi_{ij}(t - t') J_j(t') \psi(x, t) = -\alpha \cdot (R^N(t) + ecA_{RR}(x, t))\psi(x, t),\]

(114)

where

\[
A_0(r, t) = \sum_{\lambda=1}^{2} \int d^3 k \sqrt{\frac{\hbar}{2(2\pi)^3\epsilon_0\omega_k}} \left( a_{0, \lambda}(k, t)e^{ikr} + \text{h.c.} \right) e_\lambda(k),
\]

(115)

and

\[
A_{RR}(r, t) = \sum_{\lambda=1}^{2} \int d^3 k \sqrt{\frac{\hbar}{2(2\pi)^3\epsilon_0\omega_k}} \left( a_{RR, \lambda}(k, t)e^{ikr} + \text{h.c.} \right) e_\lambda(k),
\]

(116)
are the vacuum and the radiation reaction contributions, respectively. It is seen that, the coupling of the Dirac field to the electromagnetic field has a dissipative component, in the form of radiation reaction, and a fluctuation component, in the form of the vacuum field. Given the existence of radiation reaction, the vacuum field must also exist in order to satisfy the fluctuation-dissipation relation [80].

The vacuum field and the radiation-reaction field can be calculated from the time derivative of (115) and (116). It is interesting to compare these latter fields with the result of the nonrelativistic quantum mechanic (85) and (86). At first glance, we can easily find that the vacuum field contributions are the same on the one hand. On the other hand, in nonrelativistic theory, the “velocity” is
\[ c \int d^3 x \psi^\dagger(x, t) \alpha \psi(x, t) \rightarrow \dot{q}(t) \]
and therefore, by using (113) and (109) we have

\[
\begin{align*}
  c \, e_\lambda(k) & \cdot \sum_{\mu, \mu'} \int d^3 q u^\dagger_\mu(q) \alpha u_{\mu'}(k + q) c^\dagger_\mu(q, t') c_{\mu'}(k + q, t') \\
  &= c \, e_\lambda(k) \cdot \int d^3 x \psi^\dagger(x, t) \alpha \psi(x, t) \rightarrow e_\lambda(k) \cdot \dot{q}(t). 
\end{align*}
\]

Substitution (117) into the time derivative of (116), we indeed revert to the nonrelativistic expression (86) for the radiation reaction field.

4.1 Relativistic Quantum Theory of Cherenkov Radiation

In order to illustrate the applicability of our approach, we attempt to treat the relativistic theory of Cherenkov radiation in the presence of an anisotropic polarizable medium. Cherenkov radiation is the radiation with continuous spectrum that emitted by the medium due to the motion of a charged particle moving through the medium with a velocity exceeding the phase velocity of light in it. We consider a charge particle with mass \( m \) and electric charge \( e \) uniformly moving in the anisotropic polarizable medium which is described by the Hamiltonian (110) with the field operator \( R \) that now plays the role of the polarization density of the medium. The Hamiltonian operator of the total system (110), i.e. the electromagnetic field, the polarizable medium and the particle, in the large-time limit when the medium and electromagnetic field tend to an equilibrium state can be rewritten as follows [64]

\[
H = H_0 + H_{\text{int}},
\]

where

\[
H_0 = H_{\text{ele}} + H_F,
\]

with

\[
H_{\text{ele}} = \sum_{j=1}^{4} \int d^3 k \int d^3 q E_q \, c^\dagger_{\mu j}(q, t) c_{\mu j}(q, t),
\]

\[ H_F =: \int d\omega \sqrt{\frac{\hbar \omega}{2(2\pi)^3 \epsilon_0}} \left\{ u^\dagger_\mu(q) \alpha_i \times G_{ij}(k, \omega) f_{ij}(\omega) u_{\mu'}(q - k) c^\dagger_{\mu'}(q) c_{\mu'}(q - k) b_i(k, \omega, 0) e^{-i\omega t} - h.c. \right\}.
\]

\[
(122)
\]
Here, the different components of the Green tensor \(G_{ij}(k, \omega)\) in (122) satisfy the following set of algebraic equations
\[
H_{ij}(k, \omega)G_{ij}(k, \omega) = \delta_{ij},
\]
where
\[
H_{ij}(k, \omega) = \left( \mu_0^{-1}k^2c^2 - \omega^2\epsilon_{ij}(\omega) \right),
\]
in which the permittivity tensor \(\epsilon_{ij}(\omega)\) is related to the susceptibility (13) as \(\epsilon_{ij}(\omega) = 1 + \chi_{ij}(\omega)\). It is seen from (123), looked upon either as a matrix or as an operator equation, that the response tensor \(G_{ij}(k, \omega)\) is the inverse of \(H_{ij}(k, \omega)\). The solution of it is obtained most easily by evaluating the inverse of \(H_{ij}(k, \omega)\) with the help of dyadic analysis [83].

The unperturbed Hamiltonian \(H_0 = H_{ele} + H_F\) has the eigenstate \(|ele + rad\rangle = |ele\rangle \otimes |rad\rangle\) which are the direct product of the eigenstates of \(H_{ele}\) and \(H_F\). We apply the perturbation theory up to the first order to treat the transition probability per unit time for a free Dirac particle of momentum \(h \mathbf{q}\) to emit a photon of momentum \(h \mathbf{k}\) and energy \(\hbar \omega\), thereby changing its momentum to \(h(\mathbf{q} - \mathbf{k})\)
\[
\Gamma_{\mathbf{q} \rightarrow \mathbf{q} - \mathbf{k}} = \frac{2\pi}{\hbar} | \langle 1_k | \langle \mathbf{q} - \mathbf{k} | H_{\text{int}} | \mathbf{q} \rangle | 0 \rangle|^2 	imes \delta \left( \sqrt{\hbar^2c^2q^2 + m^2c^4} - \sqrt{\hbar^2c^2|\mathbf{q} - \mathbf{k}|^2 + m^2c^4} - \hbar \omega \right),
\]
where the states \(|0\rangle\) and \(|1_k\rangle\) present the vacuum state of the electromagnetic field and the excited state of the electromagnetic field with a single photon with wave vector \(\mathbf{k}\) and frequency \(\omega\), respectively. The argument of the Dirac \(\delta\) function displays the conservation of energy. The radiation intensity in a form of the Cherenkov radiation is obtained by multiplying (125) by \(\hbar \omega\) and integrating over \(\mathbf{k}\) and \(\omega\). We find that
\[
\frac{dW}{dt} = \frac{1}{2} \sum_{\lambda=1}^{2} \sum_{\mu, \mu' = 1}^{2} \int d^3k \int_0^\infty d\omega \hbar \omega \Gamma_{\mathbf{q} \rightarrow \mathbf{q} - \mathbf{k}}.
\]
Here, the sum is taken over the final spin states of the particle with positive energy, \((\mu = 1, 2)\), as well as an average over the initial spin states. In order to calculate above equation we need to evaluate the following sum
\[
S = \frac{1}{2} \sum_{\lambda=1}^{2} \sum_{\mu, \mu' = 1}^{2} |u^\dagger_{\mu}(\mathbf{q}) \alpha \cdot G(\mathbf{k}, \omega) \cdot f(\omega) u_{\mu'}(\mathbf{q} - \mathbf{k})|^2.
\]
For this purpose, we introduce the annihilation operator [64]
\[
\Lambda(\mathbf{q}) = \frac{c\alpha \cdot \mathbf{q} + \beta mc^2 + |E_{\mathbf{q}}|}{2 |E_{\mathbf{q}}|}.
\]
It is straightforward with the help of (128) to show that
\[
S = \frac{1}{8} \text{Tr}[\alpha \cdot G(\mathbf{k}, \omega) \cdot f(\omega)] \Lambda(\mathbf{q} - \mathbf{k}) (\alpha \cdot G(\mathbf{k}, \omega) \cdot f(\omega))^\dagger \Lambda(\mathbf{q})]
\]
\[
= \nu \text{Im} \left[ G_{ij}^+(\mathbf{k}, \omega) \right] v_j + \frac{1}{2} \left( 1 - \sqrt{1 - v^2/c^2} (1 - v^2/c^2) - \frac{\mathbf{v} \cdot \mathbf{v}'}{c^2} \right),
\]
in which \(\mathbf{v} = \hbar^2 c^2 E_{\mathbf{q}} / E_{\mathbf{q}}\) and \(\mathbf{v}' = \hbar^2 c^2 E_{\mathbf{q}'} / E_{\mathbf{q}'}\) are the velocities before an after the emission of the photon, respectively. Notice that in writing (129), we have used the tensor identity
\[
\frac{\omega^2}{c^2} G(\mathbf{k}, \omega) \cdot \text{Im} [\chi(\omega)] \cdot G^+(\mathbf{k}, \omega) = \text{Im}[G(\mathbf{k}, \omega)],
\]
\[
\text{Im} [\chi(\omega)] = \frac{1}{2} \left[ \chi_{\mu\mu'}(\omega) - \delta_{\mu\mu'} \chi_{\mu\mu}(\omega) \right].
\]
as well as just the transverse part of the Green tensor are considered in calculation, since the transverse contribution yields the radiative effect, while the longitudinal contribution provides purely nonradiative effects. On the other hand, the second term in (129) has no contribution to the radiation intensity (126), since the wavelength of the electron is much smaller than the optical wavelength. Using these properties one can easily show that

\[
\frac{dW}{dt} = \frac{e^2}{4\pi^3\epsilon_0} \int_0^{+\infty} d^3k \int_0^{+\infty} d\omega \omega v_j \text{Im}\left[ G_{ij}^+(k, \omega) \right] v_i
\]

\[
\times \delta \left( \mathbf{v} \cdot \mathbf{k} - \omega \left[ 1 + \frac{\hbar \omega}{2mc^2} \left( \frac{k^2c^2}{\omega^2} - 1 \right) \sqrt{1 - \frac{v^2}{c^2}} \right] \right). \tag{131}
\]

which is a main result of this section. Now, it is interesting to compare this recent relation with the result of the relativistic Cherenkov radiation in presence of isotropic medium [64]. In the latter, the transverse Green tensor (123) is given by

\[
G_{ij}^+(k, \omega) = \frac{\delta_{ij} - k_i k_j / k^2}{\mu_0^{-1} k^2 c^2 - \omega^2 \epsilon(\omega)}. \tag{132}
\]

Let \( \theta \) be the angle between \( \mathbf{q} \) and \( \mathbf{k} \), according to the argument of the Dirac \( \delta \) function (125), the photon is emitted at an angle to the path of the particle as

\[
\cos \theta = \frac{\omega}{vk} \left( 1 + \frac{\hbar \omega}{2mc^2} \left( \frac{k^2c^2}{\omega^2} - 1 \right) \sqrt{1 - \frac{v^2}{c^2}} \right). \tag{133}
\]

Substitution of (132) and (133) into (131) and integration over the polar angle \( \theta \) and the azimuthal angle \( \varphi \), yields that in the presence of isotropic medium

\[
\frac{dW}{dt} = \frac{e^2 v}{2\pi^2 \epsilon_0} \int_0^{+\infty} dk k \int_0^{+\infty} d\omega \omega \left( 1 - \frac{\omega^2}{v^2 k^2} \left[ 1 + \frac{\hbar \omega}{2mc^2} \left( \frac{k^2c^2}{\omega^2} - 1 \right) \sqrt{1 - \frac{v^2}{c^2}} \right]^2 \right)
\]

\[
\times \text{Im} \left( \frac{1}{\mu_0^{-1} k^2 c^2 - \omega^2 \epsilon(\omega)} \right), \tag{134}
\]

which is consistent with the results reported in [64].

5 Conclusion

In this paper, the classical, the non-relativistic and the relativistic quantum dynamics of a Brownian particle under the influence of an arbitrary potential in anisotropic medium is canonically investigated. In this formalism the dissipative medium is modeled by collections of harmonic oscillators. A fully canonical quantization of the dynamical variables that model the dissipative medium is demonstrated and, as a direct consequence of it the susceptibility tensor of the dissipative medium in terms of the coupling functions are calculated. In Heisenberg picture, a non-relativistic and relativistic quantum Langevin-equation is derived and explicit expressions for quantum noise and dynamical variables of the system are obtained. It is shown how radiation reaction in this theory can be expressed in terms of the source operators, then the radiated spectrum of the medium as the Cherenkov radiation is calculated. We finally introduced a minimal coupling scheme to enter the dissipation and
the fluctuation of the medium in the Hamiltonian formalism and the motion equations from the classical to the relativistic regime.

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**Appendix**

In this appendix we evaluate the time-dependent coefficient $c(t)$ in (66) for the spontaneous decay of an initially excited atom embedded in anisotropic dissipative medium. By substituting $|\psi(t)\rangle$ from (66) into (67) and using the expansions (7), (22) and (25), we find the following coupled differential equations

$$
\dot{c}(t) = -i\omega_0 c(t) - \int_0^\infty d\omega \sqrt{\frac{\hbar \omega}{2}} q_{12i}^* f_{ij}(\omega) M_j(\omega, t),
$$

(135)

$$
\dot{M}_i(\omega, t) = -i\omega M_i(\omega, t) + \sqrt{\frac{\omega}{2\hbar^3}} q_{12j} f_{ji}(\omega) c(t).
$$

(136)

We can solve these coupled differential equations by using Laplace transformation technique. Let $\tilde{c}(s)$ denotes the Laplace transform of $c(t)$. Taking the Laplace transform of (135), combining them and using the relations (33), we find

$$
s \tilde{c}(s) = c(0) - i\omega_0 \tilde{c}(s) + \frac{1}{\hbar} \left[q_{12i}^* \tilde{G}_{ij}(is) q_{12j}\right] \tilde{c}(s),
$$

(137)

where

$$
\tilde{G}_{ij}(is) = \int_0^\infty d\omega \frac{\omega^2}{\pi (is - \omega)} \text{Im} \chi_{ij}(\omega).
$$

(138)

The tensor $\tilde{G}_{ij}(is)$ gives the spontaneous-emission and frequency-shift of the atom due to the presence of dissipative medium. From definition (138) it is obvious that $\tilde{G}_{ij}(is)$ is an analytic tensor in the upper half-plane $Re(s) > 0$, therefore

$$
\dot{c}(t) = -i\omega_0 c(t) + \int_0^t K(t - t') c(t') dt',
$$

(139)

where

$$
K(t - t') = \frac{1}{2\pi \hbar} \int_{-\infty}^{+\infty} du e^{-i(u(t - t'))} \left[q_{12i}^* \tilde{G}_{ij}(u + i\tau) q_{12j}\right].
$$

(140)

Here we use the Markov’s approximation [80] and replace $c(t')$ in (139) by

$$
c(t') = c(t) e^{i\omega_0(t-t')}.
$$

(141)

By lengthy but straightforward calculations and using Kramers-Kroning relations we deduce

$$
\dot{c}(t) = -i\omega_0 c(t) - (\Gamma + i\Delta) c(t)
$$

(142)

where

$$
\Gamma = -\frac{1}{\hbar} \left[q_{12i}^* \text{Im}[\tilde{G}_{ij}(\omega_0 + i0^+) q_{12j}]\right] = \frac{\omega_0^2}{\hbar} \left[q_{12i}^* \text{Im}[\chi_{ij}(\omega_0)] q_{12j}\right],
$$

(143)

and

$$
\Delta = -\frac{1}{\hbar} \left[q_{12i}^* \text{Re}[\tilde{G}_{ij}(\omega_0 + i0^+)] q_{12j}\right] = P \int_0^\infty d\omega \frac{q_{12i}^* \text{Im}[\chi_{ij}(\omega)] q_{12j}}{\pi \hbar(\omega_0 - \omega)}.
$$

(144)
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