A thermodynamic counterpart of the Axelrod model of social influence

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Abstract

We propose a thermodynamic version of the Axelrod model of social influence. In one-dimensional lattices, the thermodynamic model becomes a Potts model of several coupled chains with a site (agent) interaction that increases with the site matching traits. We analytically calculate thermodynamic and critical properties for a one-dimensional system and show that an order-disorder phase transition only occurs at $T = 0$ independent of the number of cultural traits $q$ and features $F$ of the agents. We find that the parameter $q$ does not induce any transition or anomaly in the thermodynamic model, as it does in the standard social model that violates detailed balance. The one-dimensional thermodynamic Axelrod model belongs to the same universality class of the Ising and Potts models notwithstanding the increase of the internal dimension of the local degree of freedom (agent).

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I. INTRODUCTION

Statistical mechanics has helped us to understand macroscopic collective phenomena of a system in terms of its microscopic constituents and their interactions. Sociophysics is born under the idea of transferring this premise to social phenomena (formation of public opinions, actions of social groups, epidemic spreads, etc.), motivated by the fact that in many cases those phenomena observed in complex networks are analogous to those found in systems in thermal equilibrium [1].

The Axelrod model is an example where a local (microscopic) rule is used for the study of global social behaviors [2]. This model studies, in particular, the dissemination of cultures among interacting individuals or agents in which (a) the more culturally similar the agents, the greater the chance of interaction between them, and (b) interaction increases similarity between individuals. More explicitly, the model considers that an agent located at the \( i \)th site of a lattice is defined by a set of \( F \) cultural features (e.g., religion, sports, politics, etc.) represented by a vector \( \sigma_i = (\sigma_{i1}, \sigma_{i2}, ..., \sigma_{iF}) \). Each feature \( \sigma_{ik} \) can take integer values in the interval \([1, q]\), where \( q \) defines the cultural traits allowed per feature and measures the cultural variability in the system. There are \( q^F \) possible cultural states.

The model’s dynamics is as follows: (1) Choose randomly two nearest neighbor agents \( i \) and \( j \), then (2) calculate the number of shared features between the agents \( \ell_{ij} = \sum_k \delta_{\sigma_{ik}, \sigma_{jk}} \). If \( 0 < \ell_{ij} < F \), then (3) pick up randomly a feature \( k \) such that \( \sigma_{ik} \neq \sigma_{jk} \) and with probability \( \ell_{ij}/F \) set \( \sigma_{ik} = \sigma_{jk} \). These time steps are iterated and the dynamics stops when a frozen state is reached; i.e., either \( \ell_{ij} = 0 \) or \( \ell_{ij} = F, \forall i, j \). A cluster is a set of connected agents with the same state. Monocultural or ordered phases are composed of a cluster of the size of the system where \( \ell_{ij} = F, \forall i, j \). Multicultural or disordered phases consist of two or more clusters.

One of the main features of this model is a change of behavior, at a value \( q_c \), from a monocultural state, where all agents share the same cultural features, to a multicultural state, where individuals mostly have their own features, as the cultural diversity \( q \) increases [3]. This change can be characterized by an order parameter \( \phi \) that is usually defined as the average size of the largest cultural cluster \( C_{\text{max}} \) normalized by the total number of agents \( N \) in the system; \( \phi = C_{\text{max}}/N \). In the monocultural (ordered) state \( \phi \rightarrow 1 \) and in the multicultural (disordered) state \( \phi \rightarrow 0 \).
This change of behavior in the Axelrod model is often associated to phase transitions that occur in some physical systems [3–5]. However, it is known that true phase transitions occur only in the thermodynamic limit. The suggested ”transitions” when noise is added to the Axelrod model, depend on finite size effects and the critical values $q_c$ disappear in the thermodynamic limit as $q_c \propto 1/N$ [5, 14, 15]. On the other hand, social systems are believed to be out of equilibrium. The association of the phenomena in the Axelrod model with physical systems is thus purely figurative. Nevertheless, concepts like temperature, critical phase transition, critical point, applied magnetic field, among others, have previously been used in studies of the Axelrod model [5, 8]. Here, with the aim of finding a possible thermodynamic role of the parameters $F$, $q$, and $q_c$, we introduce a thermodynamic model that captures the main characteristics of the nonequilibrium Axelrod model.

For our model, hereafter called *thermodynamic Axelrod*, we use the interaction rule or number of shared features $\ell_{ij}$ of the Axelrod model to construct a new Hamiltonian distinguishable from the $F$ parallel one-dimensional (1D) Potts models in that the interaction strength between agents increases with $\ell_{ij}$. This feature of the interaction precipitates ordering preempting fluctuations. In the thermodynamic Axelrod model $F$ is related to the coupling energy of the system and $q$ has the same meaning as in the Potts model.

The paper is structured as follows: In Sect. II we demonstrate that the standard Axelrod model does not satisfy the detailed balance condition. In Sect. III we present the mathematical framework for computing the physical properties of our model, which are actually determined in Sect. IV. We compare the original and thermodynamic Axelrod models in Sect. V and give some conclusions in Sect. VI.

II. AXELROD MODEL: OUT OF EQUILIBRIUM

Before we introduce the thermodynamic version of the Axelrod model, here we demonstrate that the original version does not satisfy equilibrium conditions by showing that it does not obey detailed balance.

Let a link between two sites $i$ and $j$ be of type $n$ if they share $n$ components $(\ell_{ij} = n)$, $P_n$ be the probability that the system is in a state with links of type $n$, and $W_{nm}$ be the transition probability per unit time from a state with type-$n$ to one with type-$m$ links. $W_{nm}$ being time independent. Since in the dynamics of the Axelrod model a feature $k$ is changed
with probability $\ell_{ij}/F$ to make two sites have one more component in common ($\sigma_{ik} = \sigma_{jk}$), we have

$$W_{nm} = \begin{cases} \frac{n}{F} & \text{for } m = n + 1 \\ 0 & \text{otherwise} \end{cases}. \quad (1)$$

The master equation

$$\frac{dP_n}{dt} = \sum_m [W_{nm}P_m(t) - W_{mn}P_n(t)]$$

controls the time evolution and determines the equilibrium states of the system \cite{9}. As is well known, an equilibrium state is characterized by a probability $P_n$ independent of time $t$. Thus,

$$\sum_m [W_{nm}P_m - W_{mn}P_n] = 0 \quad \text{for any } n.$$

The detailed balance relation implies that

$$W_{nm}P_m = W_{mn}P_n \quad \forall \ n, m. \quad (2)$$

In the Axelrod model Eq. (2) cannot be satisfied since one side or the other is always zero according to Eq. (1). This feature of the model introduces some very strong constraints into both the evolution and the time-independent states of human interactions that emphasize nonequilibrium; i.e. i) completely different individuals do not interact, ii) individuals "conform" once they have modified their cultural profile, and iii) individuals that are alike, that interact, increase their similarity at interaction. These rules yield frozen final states with no further evolution or fluctuation.

The thermodynamic model we propose relaxes all of the previous constraints, while preserving similarity increasing interactions and introducing fluctuations controlled by a temperature parameter. These conditions permit arriving at a dynamical equilibrium state in the regular sense of statistical mechanics. On the other hand, we are interested in evaluating whether behaviors reported in nonequilibrium networks models survive a thermodynamic driven scenario.
III. NATURE OF THERMAL SOCIETY OF AGENTS

Two basic questions come to mind when proposing a thermodynamic model for society, namely: what is the meaning of temperature and detailed balance? In the Axelrod model each agent is described as a vector whose $F$ components can take $q$ values that reflect the variety of postures an individual can have on a particular scope of action in society. Temperature may be thought as related to an energy scale that competes with the regular interactions between individuals; a high temperature renders their interactions moot, while a low temperature leads to a domination of the individual interactions and to the system settling into a minimum energy state. In this sense, in a society at high temperatures individuals would have fluctuating positions with a small relation to those of their circle of interaction, whereas at low temperatures individuals would pay much attention to their circle of interaction and tend to lower posture differences. In terms of a statistical ensemble approach, a group of individuals of the society would be subjected to a temperature reservoir that competes with individual interactions.

What sets this temperature scale might be any motive for confusion, for speculation or for uncertainty that could disrupt bonds between individuals. This would be consistent with regarding temperature as a parameter coupled to entropy, in the sense of the First Law of Thermodynamics. In social terms, as the temperature increases the agents communicate less effectively, their coupling decreases, they are less convincing in a sense to their neighbours who choose or preserve their own positions without regard to their peers. This has entropic benefits since there is an increasing amount of ways people can disagree. On the other hand, when the system approaches to the critical point each individual takes the states of its neighborhood and each neighborhood takes the state of its around neighborhood and in this way all individuals remain correlated with the rest of individuals of the system.

Thermal equilibrium is a situation where some Helmholtz-type free energy is minimal, and energy fluctuations exist subject to the condition of detailed balance. From Eq. (2) one can see that detailed balance dictates that an individual in a highly probable state $n$ should balance with an individual in a less probable state $m$. It is reasonable to expect that it is easier for the individual in state $m$ to adjust itself to the mainstream rather than the other way around. This can be seen as peer pressure or pressure to conform to the norm. This is an interesting perspective in the sense that varying temperature can yield
thresholds for phase changes and set tipping points for collective behavior. On the other hand, a single temperature may not be set for different scopes of action, since a religious posture is certainly less fluctuating than a political posture. In our model this may be taken into account by weighting the interaction factor $J_{ij}$ (see below) that depends on the cultural-featured $F$-vector. We will not address this last point in the present work.

**IV. THERMODYNAMIC AXELROD MODEL**

To mimic the interaction rule of the Axelrod model, in which the interaction probability is proportional to the number of shared features, the Hamiltonian for the thermodynamic model is defined as

$$\mathcal{H} = -\sum_{k=1}^{F} \sum_{ij} \left( J_{ij} \delta(\sigma_{ik}, \sigma_{jk}) + \frac{\mu H}{2} \left[ \delta(\sigma_{ik}, H_k) + \delta(\sigma_{jk}, H_k) \right] \right),$$

with the interaction factor

$$J_{ij} = \sum_{n=1}^{F} J \delta(\sigma_{in}, \sigma_{jn}).$$

Here, $H_k$ works as an applied magnetic field that can point in one of the Potts-model-like directions $k$, can take values $1, \ldots, q$, and has an energy weight proportional to the magnitude $H$. $\sigma_{ik} = 1, \ldots, q$ specifies each of the $F$ variables $(\sigma_{i1}, \sigma_{i2}, \ldots, \sigma_{iF})$ of the agent $\sigma_i$ at the $i$th lattice site. $N$ is the size of the system. The second term in the Hamiltonian is symmetrized for convenience.

This Hamiltonian is evidently inspired on the Potts model, with the significant distinction that the interaction factor $J_{ij}$ always depends on the global state of the $F$-vector and not on the state of the particular Potts variable. This is a somewhat rare type of Hamiltonian interaction, in a sense similar to a non-linear sigma model where a vector interaction occurs subject to normalization of the interacting vectors [10]. Our model on a 1D lattice is like an $F$-layer coupled Potts model; it is a quasi one-dimensional system so that some signatures of the two-dimensional Potts model are expected as crossovers.

We consider a 1D chain of $N$ sites occupied by $q^F$-valued agents and use the transfer matrix method to compute the relevant physical properties. For periodic boundary conditions $\sigma_{(N+1)k} = \sigma_{1k}$, the partition function corresponding to the above Hamiltonian can be expressed as
\[
Z = \sum_{\sigma_1} \sum_{\sigma_2} \cdots \sum_{\sigma_N} \prod_{i=1}^{N} \exp \left[ \sum_{k=1}^{F} \left( \beta J_{i(i+1)} \delta(\sigma_{ik}, \sigma_{(i+1)k}) + \frac{\beta \mu H}{2} \left[ \delta(\sigma_{ik}, H_k) + \delta(\sigma_{(i+1)k}, H_k) \right] \right) \right] = Tr[W^N].
\] (5)

Here, \( \beta = 1/k_B T \) and we introduced a \( q^F \times q^F \) transfer matrix \( W \) with elements

\[
\langle \sigma_i | W | \sigma_{i+1} \rangle = \exp \left[ \sum_{k=1}^{F} \left( \beta J_{i(i+1)} \delta(\sigma_{ik}, \sigma_{(i+1)k}) + \frac{\beta \mu H}{2} \left[ \delta(\sigma_{ik}, H_k) + \delta(\sigma_{(i+1)k}, H_k) \right] \right) \right].
\] (6)

The eigenvalues \( \lambda_j \) of the transfer matrix are determined from the solution of the secular equation \( \text{Det}|W - \lambda E| = 0 \). Then, the partition function can be written as

\[
Z = \lambda_1^N + \lambda_2^N + \cdots + \lambda_{q^F}^N = \lambda_{\text{max}}^N \left( 1 + \frac{\lambda_1^N}{\lambda_{\text{max}}^N} + \cdots + \frac{\lambda_{q^F}^N}{\lambda_{\text{max}}^N} \right),
\] (7)

where \( \lambda_{\text{max}} \) is the largest eigenvalue. In the thermodynamic limit (\( N \to \infty \)), \( Z \approx \lambda_{\text{max}}^N \).

Then, a standard procedure [11] can be followed to obtain thermodynamic and critical properties.

V. CASE \( F = 2, \ q = 2 \)

Here, we present explicitly the simplest non-trivial case \( F = 2, \ q = 2 \), as an explicit illustration. However the analytical calculations were performed for higher values of \( F \) and \( q \). The transfer matrix takes the form

\[
W = \begin{pmatrix}
    e^{2\beta \mu H + 4\beta J} & e^{\frac{3}{2}\beta \mu H + \beta J} & e^{\frac{3}{2}\beta \mu H + \beta J} & e^{\beta \mu H} \\
    e^{\frac{3}{2}\beta \mu H + \beta J} & e^{\beta \mu H + 4\beta J} & e^{\beta \mu H} & e^{\frac{1}{2}\beta \mu H + \beta J} \\
    e^{\frac{3}{2}\beta \mu H + \beta J} & e^{\beta \mu H} & e^{\beta \mu H + 4\beta J} & e^{\frac{1}{2}\beta \mu H + \beta J} \\
    e^{\beta \mu H} & e^{\frac{1}{2}\beta \mu H + \beta J} & e^{\frac{1}{2}\beta \mu H + \beta J} & e^{4\beta J}
\end{pmatrix}.
\] (8)

The largest eigenvalue for this matrix is
\[ \lambda_{\text{max}} = \frac{1}{3} \left( e^{4\beta J + \beta \mu H} + e^{4\beta J + 2\beta \mu H} + e^{4\beta J} + e^{\beta \mu H} \right) - \frac{2^{1/3} a}{3 \left[ b + \sqrt{4\alpha^3 + b^2} \right]^{1/3}} \]

\[
+ \frac{1}{3} \left[ b + \sqrt{4\alpha^3 + b^2} \right]^{1/3},
\]

where

\[ a = -6e^{2\beta J + \beta \mu H} + e^{4\beta J + \beta \mu H} + e^{8\beta J + \beta \mu H} - 2e^{4\beta J + 2\beta \mu H} - 6e^{2\beta J + 3\beta \mu H} \\
+ e^{4\beta J + 3\beta \mu H} + e^{8\beta J + 3\beta \mu H} - e^{8\beta J + 4\beta \mu H} - e^{8\beta J} - 4e^{2\beta \mu H} \]

\[ b = 18e^{6\beta J + \beta \mu H} - 3e^{8\beta J + \beta \mu H} - 3e^{12\beta J + \beta \mu H} + 18e^{2\beta J + 2\beta \mu H} + 6e^{4\beta J + 2\beta \mu H} \\
+ 18e^{6\beta J + 2\beta \mu H} - 6e^{8\beta J + 2\beta \mu H} - 6e^{12\beta J + 2\beta \mu H} + 108e^{2\beta J + 3\beta \mu H} - 12e^{4\beta J + 3\beta \mu H} \\
- 72e^{6\beta J + 3\beta \mu H} + 18e^{8\beta J + 3\beta \mu H} + 14e^{12\beta J + 3\beta \mu H} + 18e^{2\beta J + 4\beta \mu H} + 6e^{4\beta J + 4\beta \mu H} \\
+ 18e^{6\beta J + 4\beta \mu H} - 6e^{8\beta J + 4\beta \mu H} - 6e^{12\beta J + 4\beta \mu H} + 18e^{6\beta J + 5\beta \mu H} - 3e^{8\beta J + 5\beta \mu H} \\
- 3e^{12\beta J + 5\beta \mu H} + 2e^{12\beta J + 6\beta \mu H} + 2e^{12\beta J} - 16e^{3\beta \mu H}. \]

The free energy, magnetization, magnetic susceptibility, and specific heat are given in terms of \( \lambda_{\text{max}} \):

\[ F = -Nk_B T \ln \lambda_{\text{max}}, \] (10)

\[ M = -\frac{\partial F}{\partial H} = \frac{Nk_B T}{\lambda_{\text{max}}} \frac{\partial \lambda_{\text{max}}}{\partial H}, \] (11)

\[ \chi = \frac{\partial M}{\partial H} = \frac{\partial}{\partial H} \left( \frac{Nk_B T}{\lambda_{\text{max}}} \frac{\partial \lambda_{\text{max}}}{\partial H} \right), \] (12)

\[ C = -T \frac{\partial^2 F}{\partial T^2} = 2 \frac{Nk_B T}{\lambda_{\text{max}}} \frac{\partial \lambda_{\text{max}}}{\partial T} + Nk_B T^2 \frac{1}{\lambda_{\text{max}}} \frac{\partial \lambda_{\text{max}}}{\partial T}. \] (13)

Figures 1-3 show 3D plots of the magnetization, magnetic susceptibility, and specific heat in terms of \( k_B T/J \) and \( \mu H/k_B T \). All these properties behave qualitatively as they do in the Ising and Potts models in 1D lattices, no new qualitatively striking behavior is obvious.
The magnetization (Fig. 1) goes to 0 as $H \to 0$ at any finite $T$. At $T = 0$ the magnetization saturates to its maximum value for any $H$. This implies a spontaneous transition to an ordered state only at $T = 0$. As $T$ becomes finite, the magnetization saturates to its maximum only at large $H$. The susceptibility (Fig. 2) diverges as $T \to 0$ and $H \to 0$. For zero field the specific heat goes through a maximum before vanishing smoothly as $T = 0$ (Fig. 3). Thus, the differences in internal symmetry and interaction in the thermodynamic Axelrod model with respect to the Ising model do not seem to be relevant. We go back this point later in the discussion on the critical exponents.
In terms of the thermal society of agents, the above results imply that the fluctuating positions will be always so large as for the individuals to conform spontaneously to a particular posture, unless there exist a highly persuasive agent or mass media. On the other hand, the specific heat can be regarded as the ability of the society to reject the increase of spontaneous changes of posture. This ability decreases as the occurrence of these changes goes down and disappears for zero occurrence.

A. Spatial correlations

We now calculate the correlation function

\[ G(i, i + j) = \langle \sigma_i \sigma_{i+j} \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle \]  

(14)

using the transfer matrix method [12]. The two-agent correlation function is given by

\[ \langle \sigma_i \sigma_{i+j} \rangle = \frac{1}{Z} \text{tr}[AW^j AW^{N-j}] , \]  

(15)

where \( A = \sum_{\sigma_i} |\sigma_i\rangle \sigma_i \langle \sigma_i| \). In the original space for \( F = 2 \) and \( q = 2 \), the state matrix

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 4
\end{pmatrix}.
\]  

(16)

Following a standard procedure, we evaluate Eq. (15) in a basis where \( W \) is diagonal. For \( H = 0 \), the unitary matrix that diagonalizes \( W \) is

\[
P = \begin{pmatrix}
-1 & 0 & 1 & 1 \\
0 & -1 & -1 & 1 \\
0 & 1 & -1 & 1 \\
1 & 0 & 1 & 1
\end{pmatrix}.
\]  

(17)
The evaluation yields

\[
\langle \sigma_i \sigma_{i+j} \rangle = \frac{1}{Z} \text{tr} \left[ \begin{pmatrix}
\frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} \\
0 & 2 & 0 & 0 \\
\frac{3}{4} & 0 & \frac{9}{4} & \frac{1}{4} \\
0 & \frac{3}{4} & 0 & \frac{9}{4} \\
\end{pmatrix} \begin{pmatrix}
\lambda_1^j & 0 & 0 & 0 \\
0 & \lambda_2^j & 0 & 0 \\
0 & 0 & \lambda_3^j & 0 \\
0 & 0 & 0 & \lambda_4^j \\
\end{pmatrix} \begin{pmatrix}
\frac{5}{2} & 0 & \frac{3}{2} & \frac{3}{2} \\
0 & 2 & 0 & 0 \\
\frac{3}{4} & 0 & \frac{9}{4} & \frac{1}{4} \\
0 & \frac{3}{4} & 0 & \frac{9}{4} \\
\end{pmatrix} \right] 
\times \begin{pmatrix}
\lambda_{1}^{N-j} & 0 & 0 & 0 \\
0 & \lambda_{2}^{N-j} & 0 & 0 \\
0 & 0 & \lambda_{3}^{N-j} & 0 \\
0 & 0 & 0 & \lambda_{4}^{N-j} \\
\end{pmatrix} \right] 
= \frac{81}{16} + \frac{9}{8} \left( \frac{\lambda_1}{\lambda_4} \right)^j.
\]

(18)

Here, \( \lambda_4 \) and \( \lambda_1 \) are the largest and the second largest eigenvalues, respectively, and the evaluation was performed in the thermodynamic limit.

The average of the agent at site \( i \), \( \langle \sigma_i \rangle \), is evaluated in the same manner:

\[
\langle \sigma_i \rangle = \frac{1}{Z} \text{tr}[AW^N] = \frac{9}{4}.
\]

(19)

Then,

\[
G(i, i + j) = \frac{9}{8} \left( \frac{\lambda_1}{\lambda_4} \right)^j \approx e^{-j \ln(\lambda_4/\lambda_1)} \equiv e^{-j/\xi}.
\]

(20)

The correlation function has the same form as in the Ising and Potts models \[12, 13\], with the correlation length

\[
\xi = \frac{1}{\ln (\lambda_4/\lambda_1)}.
\]

(21)

Since \( \lambda_1 = -1 + e^{4J/k_B T} \) and \( \lambda_4 = 1 + 2e^{4J/k_B T} + e^{4J/k_B T} \), the correlation length becomes

\[
\xi = \frac{1}{\ln \left((1 + 2e^{4J/k_B T} + e^{4J/k_B T}) / (-1 + e^{4J/k_B T})\right)}.
\]

(22)

As in Ising and Potts models, for \( H = 0 \) the two largest eigenvalues become degenerate at \( T = 0 \), leading to a divergence of the correlation length and, therefore, to a zero-temperature phase transition.

In social terms, the correlation length measures the distance at which there are relations between agents beyond their own mean values. This is a causal or influence relationship in the sense that changing the opinions in one place generates an influence that causes change up to the correlation length. This influence operates through local interactions.
general, correlation lengths are linked to a return force that limits their growth. In terms of social influence the existence of a correlation length invokes a limit to the propagation of influence, some return force avoids the propagation of fluctuations beyond a certain distance. When no return force is present fluctuations run wild and influence runs over the whole society. The return force is a measure of the cost of producing the fluctuation. If there is no cost, fluctuations diverge and a change of opinion in one place generates a fluctuation that propagates throughout the system.

**B. Critical exponents**

To get the critical behavior of the thermodynamic properties one needs to evaluate them near the transition temperature $T_c$. Usually, for this purpose $T$ is replaced by the parameter $t = (T - T_c)/T_c$. However, for 1D models the transition occurs at $T_c = 0$ with exponential singularities [11]. In this case, the regular parameter $t$ as defined is inappropriate. A different critical point approach parameter $t = e^{-\Delta/k_B T}$ [11] is required in the 1D models to convert the exponential singularities in $T$ into power-law singularities in $t$. The constant $\Delta$ has been thus far taken arbitrarily, which leads to ambiguous critical exponents.

Here, we propose that $\Delta$ be given by half the energy difference between the ground state and the first excited state of the system. In this way $\Delta$ is no longer arbitrary. This convention correctly unifies, independent of the interaction strength, the Ising and Potts exponents. For the 1D Ising model the energy difference between the ground state (say, all spins aligned up) and the first excited state (one spin aligned opposite to the others) is $4J$; then, for this case $\Delta = 2J$. This agrees with the choice of $p = 2$ in $t = e^{-pJ/k_B T}$ to bring together the exponents of the 1D discrete-symmetry models.

For the Potts and thermodynamic Axelrod models, the energy of the ground state is $-JNF^2$ and of the first excited state is $-J [(N - 2)F^2 + 2(F - 1)^2]$. Then,

$$\Delta = J(2F - 1).$$

(23)

In the case of the Potts model, $F = 1$ and $\Delta = J$. This value of $\Delta$ yields critical exponents of the Potts model that agree with those of the Ising model [13]. For the thermodynamic Axelrod model we are analyzing, $F = 2$ and $\Delta = 3J$.

We are now prepared to estimate the critical exponents of our model for the case $F = 2$. 


and \( q = 2 \). We define \( h = \mu H/k_B T \) and have \( t = e^{-3J/k_B T} \). For \( H = 0 \) and \( t \to 0 \) the singular part of the free energy, Eq. (10), for a zero temperature transition [11] becomes

\[
f(t) = \frac{F + 4NJ}{Nk_B T} \sim t .
\]

(24)

Since \( f \sim t^{2-\alpha} \), \( \alpha = 1 \). In the same limits, the magnetization, Eq. (11), is

\[
m(t) = \frac{M}{N\mu} \sim 1 .
\]

(25)

This means from \( m \sim t^\beta \) that \( \beta = 0 \). Now, for \( t = 0 \) and \( H \to 0 \), the magnetization becomes

\[
m(h) = \frac{M}{N\mu} \sim 1 .
\]

(26)

Since \( m \sim h^{1/\delta} \), the exponent \( \delta \to \infty \). The low-field susceptibility is obtained from Eq. (12)

\[
\chi_0(t) = \frac{\chi k_B T}{N\mu^2} \sim t^{-1} .
\]

(27)

The susceptibility should go as \( t^{-\gamma} \), then \( \gamma = 1 \). The specific heat, Eq. (13), for \( H = 0 \) and \( t \to 0 \) is

\[
c(t) = \frac{C k_B T^2}{N.J^2} \sim t .
\]

(28)

Then, from \( C \sim t^\alpha \) one gets \( \alpha = 1 \). Finally, from Eq. (22) the correlation length

\[
\xi(t) \sim t^{-1} .
\]

(29)

The correlation length goes as \( t^{-\nu} \); then, \( \nu = 1 \).

In conclusion, the differences in the internal space dimensionality (\( F = 2 \)) and interaction strengths do not alter the Ising universality class of our model.

VI. THERMODYNAMIC AND SOCIAL AXELROD MODELS: A COMPARISON

The original Axelrod model in the absence of external field displays a crossover from a disordered to an ordered state at certain value \( q_c \) that depends on dimensionality [3, 4]. Moreover, the way this crossover takes place depends on the cultural feature \( F \); for \( F > 2 \) it is abrupt and for \( F = 2 \) it is smooth [3]. For \( F = 2 \) the crossover occurs for \( q < 5 \) [4]. As shown above, our thermodynamic Axelrod model in 1D has a continuous phase transition to an ordered state at \( T = 0 \), just as expected for discrete symmetry systems and as is found in the Ising and Potts models. To see if there is any correspondence between the trait \( q \) and
feature $F$ variables in the two Axelrod models; that is, if $q$ and $F$ have any effect on the thermodynamic model that can be mapped into the social model, we carried out analytical calculations for different $F$ and $q$, sweeping the latter across the region were the crossover happens in the social model.

Figure 4 depicts the analytical results for the temperature dependence of the order parameter for $F = 2$ and $q = 2, 3, 4, 5$. For comparison purpose only, we also show curves for $F = 3$ and $q = 2, 3$. A monotonic and smooth dependence on $q$ is observed for $F = 2$, indicating no signal of crossover or phase transition at all. For both values of $F$, the order parameter behaviors are similar to those seen in the Potts model [13]. We note that for the analyzed values of $F$ as $q$ increases the transition becomes gradually sharper, which could mistakenly suggest a change in the transition type as a function of $q$. We know that for 1D systems this is not allowed and that the observed effect is due to numerical limitations as $H \rightarrow 0$. The data of Fig. 4 were obtained for $H = 0.005$.

The results suggest that thermodynamically the 1D Axelrod hamiltonian does not exhibit a phase transition as a function of the variable $q$. The transition depends only on temperature and in one-dimension it takes place in $T=0$. It is interesting that in the non-equilibrium model with noise according to Klem et. al [14,15], if one equates temperature to noise, for any finite noise intensity the system is "polarized" for large systems, in the sense that it breaks up into finite islands. This is equivalent to our zero magnetization phase for any finite temperature.

VII. SUMMARY

We presented a thermodynamic counterpart of the Axelrod model of social influence. For the thermodynamic version we calculated thermodynamic and critical properties in 1D and showed that an order-disorder phase transition occurs at $T = 0$, independent of the cultural trait $q$ and feature $F$ variables. A relevant result is that for our thermodynamic model the parameter $q$ does not induce any transition or anomaly in the system, as it does in the social model. We showed that the thermodynamic Axelrod model belongs to the same universality class of the Ising and Potts models.
FIG. 4: Magnetization or order parameter of the thermodynamic Axelrod model. The group of curves to the left correspond to $F = 2$ and the one to the right to $F = 3$. No anomaly is observed as $q$ varies from 2 to 5 for the $F = 2$ case. In the original Axelrod model a crossover from an ordered to a disordered state is seen below $q = 5$ in the one-dimensional case [4].

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