1 Introduction

A harmonic sum for the simple index, \( s \in \mathbb{N}_+ \), is defined by the sum \( H_s(N) := 1 + \frac{1}{2^s} + \ldots + \frac{1}{N^s} \). We know that the limit \( \lim_{N \to \infty} H_s(N) \) is also finite whenever \( s > 1 \) and one calls this limit the zeta number. For example

\[
\lim_{N \to \infty} H_2(N) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2).
\]

These definitions are also extended to a set of multi-index called multiple harmonic sums and polyzetas (or multiple zeta values), respectively. For each composition of positive integers \( s = (s_1, \ldots, s_r), s_1 > 1, r, N \in \mathbb{N}_+ \),

\[
H_s(N) := \sum_{N \geq n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}, \tag{1}
\]

\[
\zeta(s) := \sum_{n_1 > \ldots > n_r > 0} \frac{1}{n_1^{s_1} \ldots n_r^{s_r}}. \tag{2}
\]

Example 1.

\[
H_{2,1}(N) = \sum_{N \geq n_1 \geq n_2 \geq 1} \frac{1}{n_1^{2}n_2} = \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{1}{2^2} \cdot N^2(N - 1) + \frac{1}{3^2} \cdot N^2(N - 2) + \ldots + \frac{1}{N^2 \cdot 1} \sum_{n_1, n_2 \geq 1} \frac{1}{n_1^{2}n_2} = \zeta(2, 1).
\]

Furthermore, this structure also has an other infinity form, called multiple polylogarithms, such a

\[
\lim_{N \to \infty} H_s(N) = \lim_{N \to \infty} \sum_{n=1}^{N} \frac{1}{n^s} = \zeta(2).
\]

They all have famous relations in limits by Abel’s theorem:

\[
\lim_{N \to \infty} H_s(N) = \lim_{z \to 1} \zeta(s) = \zeta(s), \quad \forall s_1 > 1. \tag{4}
\]

Fortunately, the multiple harmonic sums are compatible with the algebra of the stuffle product; whereas, the multiple polylogarithms are compatible with the algebra of the shuffle product when they are observed in the forms of iterated Chen integrals. As a consequence of these results, the polyzetas are compatible with both of the structures.

In this paper, we briefly review a general result about Hopf algebras (in Section 2), of the quasi-shuffle product and the concatenation product, constructed on a space of formal polynomials freely generated by some alphabet. They admit transcendence bases (see [1]) on which the special functions can be expressed as non-commutative generating series in respect of Hausdorff group\(^1\):

\[
H(N) = \prod_{l \in \mathcal{L}_{Z} \setminus \{y_1\}} \exp(H_{\Sigma_i}(N)\Pi_i),
\]

\[
\mathcal{L}_{\omega} = \prod_{l \in \mathcal{L}_{Z} \setminus \{y_1\}} \exp(\zeta(\Sigma_i)\Pi_i),
\]

\[
L(z) = \prod_{l \in \mathcal{L}_{Z} \setminus X} \exp(L_{S_i}(z)\Pi_i),
\]

\[
\mathcal{L}_{\omega} = \prod_{l \in \mathcal{L}_{Z} \setminus X} \exp(\zeta(S_i)\Pi_i).
\]

Thanks to relations among the non-commutative generating series \( L(z) \xrightarrow{z \to 1} \mathcal{L}_{\omega}, H(N) \xrightarrow{N \to \infty} \mathcal{L}_{\omega} \) in

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the equivalent algebraic structures, we establish relations
and reduce representations in the forms of polynomial
relations and asymptotic expansions indexed by the transcendence bases.

2 Quasi-shuffle algebra with the deformation $q$

Let’s denote $Y := \{y_k \mid k \in \mathbb{N}_q\}$ an alphabet totally
ordered by $y_1 > y_2 > \cdots$. A word is a finite sequence
of letters and $Y^*$ denotes the set of all words including
the empty word, denoted by $1_Y$. This set is a free
monoid and $1_Y$ is a neutral element. We call each linear
combination, over the field $Q$ of words in $Y^*$
(a formal polynomial and $Q(Y)$ denotes the set of all
polynomials. This set equipped with the concatenation
product follows a free algebra with unit $1_Y$. A Lyndon word
is a nonempty word that is smaller than all its nontrivial proper right factors and $\mathcal{L}(Y)$
denotes the set of all Lyndon words in $Y^*$.

For any $q$ belonging to any field containing the
field of rational number, the $q$–shuffle product, denoted
by $\omega_q$, is defined by recurrent formula as follows:
$$u \omega_q y = 1_Y \cdot y, \quad y_k u \omega_q y_k v = y_k (u \omega_q y_k v) + y_k (y_k u \omega_q v) + q y_k (y_k u \omega_q v).$$

Example 2.

$$y_2 \omega_q y_3 y_1 = y_2 (1_Y \cdot y_2 \omega_q y_3 y_1) + y_3 (y_2 \omega_q y_1) + q y_2 (1_Y \cdot y_2 \omega_q y_1) = y_2 y_3 y_1 + y_2 y_3 y_1 + y_3 y_2 y_2 + q (y_3 y_2 y_1).$$

This product is exactly the shuffle product (denoted by $\mathcal{L}(Y)$ for $q = 0$ and the shuffle product (denoted by $\omega_q$) for $q = 1$ product is commutative and
associative hence, $(A(Y), \omega_q, 1_Y)$ is a commutative, associative algebra with unit, where $A := Q[q]$ is the
field extension of $Q$ containing $q$. Here, we still use the notation $\omega_q$ as a morphism
$$\omega_q : A(Y) \otimes A(Y) \longrightarrow A(Y) \quad \otimes v \quad \mapsto u \omega_q v.$$

We denote $\Delta_\omega$ and $\Delta_{conc}$ as the dual laws of the
$q$–shuffle product and the concatenation product, respectively; this means that for all $w$ in $Y^*$,
$$\Delta_\omega (w) = \sum_{u,v \in Y^*} \langle \Delta_\omega (w) \mid u \otimes v \rangle u \otimes v = \sum_{u,v \in Y^*} \langle w \mid u \omega_q v \rangle u \otimes v, \quad \Delta_{conc} (w) = \sum_{u,v \in Y^*} \langle \Delta_{conc} (w) \mid u \otimes v \rangle u \otimes v = \sum_{u,v \in Y^*} \langle w \mid u \omega_q v \rangle u \otimes v. \quad (7)$$

We proved (in paper [1]) that the coproduct $\Delta_\omega$ is compatible with the concatenation product. This
means that $\Delta_\omega (uv) = \Delta_\omega (u) \Delta_\omega (v)$ whereas $\Delta_{conc}$ is compatible with the $q$–shuffle product, that means $\Delta_{conc} (u \omega_q v) = \Delta_{conc} (u) \omega_q \Delta_{conc} (v)$. An important point to note here is the weight of the word
$w = y_{s_1} \cdots y_{s_i}$ to be (and denoted by) $(w) = s_1 + \cdots + s_i$. Due to these definitions we can see that $\Delta_\omega (w)$ is the polynomial of weights in weight $(w)$ and $u \omega_q v$ is the polynomial of weights in weight $(u) + (v)$. Consequently, they all form the two algebraic structures in duality as follows:

Proposition 1 ([1]). $(A(Y), conc, 1_Y, \Delta_\omega, \epsilon, S^{conc})$ and $(A(Y), \omega_q, 1_Y, \Delta_{conc}, \epsilon, S^{\omega_q})$ are the graded Hopf algebras in duality.

On the other hand, we proved that the algebraic morphism defined on letters by
$$\pi_1(y_k) = y_k + \sum_{i \geq 2} \frac{(-q)^{i-1}}{i} \sum_{s_1 + \cdots + s_i = k} y_{s_1} \cdots y_{s_i} \quad (8)$$
to be an isomorphism between the two algebraic algebras $\mathcal{H}_\omega = (A(Y), conc, 1_Y, \Delta_\omega, \epsilon, S^\omega)$ and $\mathcal{H}_{\omega_q} = (A(Y), conc, 1_Y, \Delta_{\omega_q}, \epsilon, S^{\omega_q})$. Therefore, each letter is a primitive element of $\mathcal{H}_\omega$ and follows its image $\pi_1(y_k)$ to be primitive in $\mathcal{H}_{\omega_q}$. This result helps us to construct a linear basis for the space of the Lie algebra generated by primitive elements. We denote here by $\{\Pi_i\}_{i \in \mathcal{L}(Y)}$ the Poincaré–Birkhoff Witt basis (PBW-basis for short), and it is computed according to the recurrent formula [1]:

$$\begin{align*}
\Pi_{y_k} & = \pi_1(y_k) \quad \text{for } y_k \in Y, \\
\Pi_l & = [\Pi_{l_1}, \Pi_{l_2}] \quad \text{for } l \in \mathcal{L}(Y) \setminus Y, \\
\Pi_w & = \Pi_{l_1} \cdots \Pi_{l_k} \quad \text{for } w = l_{i_1} \cdots l_{i_k},
\end{align*} \quad (9)$$

where $(l_1, l_2)$ is the standard factorization of $l$, $w = l_{i_1} \cdots l_{i_k}, i_1 > \cdots > i_k, i_1, \ldots, i_k \in \mathcal{L}(Y)$.

Example 3.

$$\begin{align*}
\Pi_{y_1} & = y_1, \quad \Pi_{y_2} = y_2 - \frac{q}{2} y_1^2, \\
\Pi_{y_1 y_2} & = y_1 y_2 - y_1^2 y_2, \\
\Pi_{y_1 y_2 y_1} & = y_1 y_2 y_1 - y_2 y_1 y_2 y_1 - y_2 y_1 y_2^2 - \frac{q}{2} y_1^2 y_2 y_1^2 - \frac{q}{2} y_2^2 y_1^2 + \frac{q}{2} y_1^2 y_2^2 + \frac{q}{2} y_1 y_2 y_1 y_2^2 + \frac{q}{2} y_1 y_2^2 y_1 y_2 + \frac{q}{2} y_2^2 y_1 y_2.
\end{align*}$$

On the other hand, we also established a formula for the dual basis, denoted by $(\Sigma u)_{u \in Y^*}$, by

$$= \sum_{u, v \in Y^*} \langle w \mid u \omega_q v \rangle u \otimes v. \quad (7)$$
the recurrent formula [1]:

\[
\begin{align*}
\Sigma_{y_1} &= y_1, \\
\Sigma_{y_1} &= \sum_{i=1}^{n} y_{i,1}\Sigma_{y_1} \cdots \Sigma_{y_1}, \\
\Sigma_{y_1} &= \sum_{\lambda \geq 1} \sum_{i_1, \ldots, i_n} y_{i_1, \ldots, i_n} \Sigma_{y_1}, \\
\Sigma_{y_1} &= \frac{\Sigma_{y_1} \Sigma_{y_1} \cdots \Sigma_{y_1}}{1 - \lambda}.
\end{align*}
\]

(10)

Example 4.

\[
\begin{align*}
\Sigma_{y_1} &= y_1, \\
\Sigma_{y_1} &= y_{1,1}y_{1,2} + \sum_{i=2}^{n} y_{i,1}y_{i,2}, \\
\Sigma_{y_1} &= \sum_{i, j, k, l} y_{i,1} y_{i,2} \cdots y_{l,1} y_{l,2} y_{l,3}.
\end{align*}
\]

This basis reduces a transcendence basis, \{\Sigma_{y_1}\}_{y \in \mathbb{Y}_\mathbb{D}}, of the algebra \(\mathcal{A}((Y))\). Furthermore, we can express the diagonal series \(\mathbb{D}_Y := \sum_{w \in \mathbb{Y}_\mathbb{D}} w \otimes w\), an element in the algebra \(\mathcal{A}((Y)) \otimes \mathcal{A}((Y))\) of the \(q\)-shuffle product on the left of the tensor and the concatenation product on the right.

Proposition 2 ([1]).

\[
\begin{align*}
\mathbb{D}_Y &= \sum_{w \in \mathbb{Y}_\mathbb{D}} \Sigma_w \otimes \Pi_w \\
&= \sum_{\lambda \geq 1} \sum_{i_1, \ldots, i_n} \Sigma_{\Pi_{i_1} \cdots \Pi_{i_k}} \otimes \Pi_{i_1} \cdots \Pi_{i_k} \\
&= \prod_{\lambda \in \mathbb{L}\mathbb{Y}_\mathbb{D}} \exp(\Sigma_\lambda \otimes \Pi_\lambda),
\end{align*}
\]

(11)

where the last product takes Lyndon words in decreasing order.

3 Representation of special functions on transcendence bases

3.1 Representation of multiple polylogarithms

We now consider the above algebra in the case of the alphabet \(X = \{x_0, x_1\}\), totally ordered by \(x_0 < x_1\), with the shuffle product (it means \(q = 0\)). At that time, the couple of bases in duality \([2]\) is denoted by \(\{P_w \}_{w \in X^*}\), the PBW-basis, and \(\{S_w \}_{w \in X^*}\), Schützenberger basis. It follows from (11)\(^6\)

\[
\begin{align*}
\mathbb{D}_X &= \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w \\
&= \prod_{\lambda \in \mathbb{L}\mathbb{X}} \exp(S_{\lambda} \otimes P_{\lambda}).
\end{align*}
\]

We have seen at (3) that a multiple polylogarithms is determined for each multi-index \(s = (s_1, \ldots, s_r)\). In this section, we use encoding that each

\(^6\)Note that, \(x_0, x_1\) are respectively the smallest and the largest Lyndon words in \(X^*\).

\(^7\)This morphism isn’t change on the tensor product \((\mathbb{Q}(X), \otimes, 1_{X^*}) \otimes (\mathbb{Q}(X), \text{conc}, 1_{X^*})\) but in the subalgebra \(\mathbb{L}_{\mathbb{Q}(X)}\).

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Proposition 3 ([3]). i) For all curve $z_0 \leadsto z$ in $\Omega$, one has $L(z) = S_{z_0 \leadsto z}(z_0)$.

ii) In the special case of curve $1 - t$, one has

$$L(x_0, x_1 | 1 - t) = L(-x_1, -x_0 | t)Z_{\perp}.$$  \hspace{1cm} (17)

We now use an automorphism of $Q(X)$ of the concatenation product, denoted by $\sigma$, verified $\sigma(x_0) = -x_1, \sigma(x_1) = -x_0$. Note that, for all words $w \in X^*, P_w$ and $S_w$ are homogeneous polynomial of weight $|w|$, the length of $w$. Furthermore, $Q(X)$ is a graded space admitting two graded bases $\{P_w\}_{w \in X^*}$ and $\{S_w\}_{w \in X^*}$. We can see more precisely by the following diagram illustrating a matrix representation of $\sigma$ in a subspace of weight $n_0$, denoted by $X_n := \text{span}\{u_1^{(n)}, \ldots, u_{2^n}^{(n)}\}$, where $\{u_1^{(n)}, \ldots, u_{2^n}^{(n)}\}$ is the set of all words of length in this case $n_0$. $E(n)$ denotes the matrix representation of $\sigma$ with respect to this basis: for all $1 \leq i, j \leq 2^n$,

$$E(n) := \langle \sigma(P_{u_i^{(n)}}) | P_{u_j^{(n)}} \rangle = \langle \sigma(S_{u_i^{(n)}}) | S_{u_j^{(n)}} \rangle.$$  \hspace{1cm} (18)

$$(X_n, \{P_{u_i^{(n)}}\}_{1 \leq i \leq 2^n}) \xrightarrow{\text{duality}} (X_n, \{P_{u_i^{(n)}}\}_{1 \leq i \leq 2^n})$$

$$(X_n, \{S_{u_i^{(n)}}\}_{1 \leq i \leq 2^n}) \xrightarrow{\text{duality}} (X_n, \{S_{u_i^{(n)}}\}_{1 \leq i \leq 2^n}).$$

Proposition 4. Let $L(z)$ be the non-commutative generating series of multiple polylogarithms, we have

$$\sigma[L(z)] = \sum_{w \in X^*} Li_{S_{w}}(z)\sigma(P_w) = \sum_{w \in X^*} Li_{S_{w}}(z)P_w.$$  \hspace{1cm} (19)

Proof:

$$\sum_{w \in X^*} Li_{S_{w}}(z)\sigma(P_w) = \sum_{n \geq 0} \sum_{i=1}^{2^n} Li_{S_{u_i^{(n)}}}(z)\sigma(P_{u_i^{(n)}})$$

$$= \sum_{n \geq 0} \sum_{i=1}^{2^n} Li_{S_{u_i^{(n)}}}(z) \sum_{j=1}^{2^n} E(n)_{ij} P_{u_j^{(n)}}$$

$$= \sum_{n \geq 0} \sum_{i=1}^{2^n} Li_{S_{u_i^{(n)}}}(z) \sum_{j=1}^{2^n} E(n)_{ij} P_{S_{u_j^{(n)}}}$$

$$= \sum_{n \geq 0} \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} Li_{S_{u_i^{(n)}}}(z) E(n)_{ij} S_{u_j^{(n)}}$$

$$= \sum_{n \geq 0} \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} Li_{S_{u_i^{(n)}}}(z) S_{u_j^{(n)}}$$

$$= \sum_{n \geq 0} \sum_{i=1}^{2^n} Li_{S_{u_i^{(n)}}}(z) P_{u_j^{(n)}}$$

$$= \sum_{w \in X^*} Li_{S_{w}}(z)P_w.$$

For this reason, we can rewrite relation (17) as follows:

$$\sum_{w \in X^*} Li_{S_{w}}(z)P_w = \sum_{w \in X^*} Li_{S_{w}}(1 - z)P_w Z_{\perp}.$$  \hspace{1cm} (20)

From this formula, by identifying local coordinates, we get relations among the multiple polylogarithms indexed by basis $\{S_i\}_{i \in L_{\text{proj}}X}$. The following example are computed by our program running under Maple.

Example 5.

$$Li_{S_{w_0}}(z) = \log(z), \quad Li_{S_{w_1}}(z) = -\log(1 - z),$$

$$Li_{S_{w_2}}(z) = -\log(z) \log(1 - z)$$

$$- Li_{S_{w_3}}(1 - z) + \zeta(S_{w_0}) + \frac{1}{2} \log(1 - z)^2 \log(z),$$

$$Li_{S_{w_4}}(z) = \frac{1}{2} \log(1 - z)^2 \log(z) + \log(1 - z) Li_{S_{w_0}}(1 - z)$$

$$- Li_{S_{w_5}}(1 - z)$$

$$+ \zeta(S_{w_0}^{(2)}) + \log(z) \zeta(S_{w_0}) \zeta(S_{w_0}).$$

3.2 Representation of multiple harmonic sums

We have seen at (1) that a multiple harmonic sum is determined for each multi-index $s = (s_1, \ldots, s_r) \in N^r_*$. Similar to the idea of the previous subsection, these compositions of positive integers, $s = (s_1, \ldots, s_r) \in Y^r$, are encoded by the words $w = y_{s_1} \cdots y_{s_r}$. Thus, the multiple harmonic sums can be rewritten as

$$H_w(N) := \sum_{N \geq n_1 \geq \cdots \geq n_r \geq 1} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}.$$  \hspace{1cm} (21)

Note that, for each composition $s = (s_1, s_2, \ldots, s_r)$, we have the reducing expression

$$H_y(N) = \sum_{n_1 = r}^{N} \frac{H(y_{s_2} \cdots y_{s_r})(n_1 - 1)}{n_1}.$$  \hspace{1cm} (22)

by the reason

$$H_y(N) = \sum_{n_1 = r}^{N} \frac{1}{n_1^{s_1} \cdots n_r^{s_r}}$$

$$= \sum_{n_1 = r}^{N} \frac{1}{n_1^{s_1} n_1 - 1 \cdots n_r^{s_r} n_r - 1}$$

$$= \sum_{n_1 = r}^{N} \frac{H(y_{s_2} \cdots y_{s_r})(n_1 - 1)}{n_1^{s_1} \cdots n_r^{s_r}}.$$  \hspace{1cm} (23)

This allows us to prove, by induction, that multiple harmonic sums are compatible with the shuffle product [7]. It means that for all words $w_1, w_2 \in Y^r$, we have

$$H_{w_1}(N)H_{w_2}(N) = H_{w_1 \shuffle w_2}(N).$$

Proposition 5. The mapping $w \mapsto H_w$ is the isomorphism between $(Q(Y), \shuffle, 1_Y)$ and the algebra of multiple harmonic sums with the standard product, denoted by $(H_{\text{R}}, \cdot, 1)$. 
Because the set of the Lyndon words freely generates the algebra of the quasi-shuffle product [8], it follows the isomorphism $\mathcal{H}_n \cong \mathcal{Q}[H_i, i \in \mathcal{L}_{yn}]$. Moreover, by using the expression of diagonal series $\mathcal{D}_Y$ (see (11)), we can factorize the non-commutative generating series of multiple harmonic sums $H := \sum_{w \in Y^*} \mathcal{H}_w w$ as follows

**Proposition 6.**

$$H = \prod_{Y \in \mathcal{L}_{yn} Y} \exp(\mathcal{H}_{y_1})$$ \hspace{1cm} (24)

$$= \exp H_{y_1} \prod_{Y \in \mathcal{L}_{yn} \setminus \{y_1\}} \exp(\mathcal{H}_{y_1}).$$ \hspace{1cm} (25)

The original generating series of multiple harmonic sums forms a multiple polylogarithms deformed the factor $\frac{1}{1-z}$, namely for all multiindices $s = (s_1, s_2, \ldots, s_r)$,

$$\sum_{n \geq 0} H_s(n) z^n = \frac{\text{Li}_s(z)}{1-z}.$$ \hspace{1cm} (26)

Indeed,

$$\frac{\text{Li}_s(z)}{1-z} = \sum_{n \geq 0} z^n \sum_{n_1 \geq 1, \ldots, n_r \geq 1} z^{n_1 \ldots n_r}$$

$$= \sum_{n \geq 0} \sum_{n_1 \geq 1, \ldots, n_r \geq 1} z^{n_1 \ldots n_r}$$

$$= \sum_{n \geq 0} H_s(n) z^n.$$

Here we accept that $H_s(n) = 0$ for any $n < r$. In other words, $H_s(n)$ is the coefficient of $z^n$ in the Taylor development of $\frac{\text{Li}_s(z)}{1-z}$ in the system $\{z^n | n \in \mathbb{N}\}$. By the way, according to the representations of multiple polylogarithms (in the above subsection) we obtain relations or asymptotic expansions of multiple harmonic sums.

### 3.2.1 Generating series of multiple harmonic sums on the alphabet $X$

For any word $w \in X^*$, we denote $G^{X}_s (z) := \frac{\text{Li}_s(z)}{1-z}$ and $G^{X}(z) := \sum_{w \in X^*} G^{X}_w (z)$. By the way, using formula (20), we have the following expressions:

$$G^{X}(1-z) = \frac{1-z}{z} \sigma(G^X(z))Z_{\perp}$$

$$\sum_{w \in X^*} G^{X}_w (z) = \frac{1-z}{z} \prod_{Y \in \mathcal{L}_{yn} X} \exp(\text{Li}_{y_1}(z) \sigma(P_i))Z_{\perp}.$$

**Example 6.** According to equality (27), we reduce the following relations by identifying local coordinates$^9$:

$$G^{X}_{s_{\sigma}^{-1}}(1-z) = \frac{\log(1-z)}{z}.$$
\[
\begin{align*}
&= \sum_{N \geq 0} \left( \sum_{k \geq 0} H_{\Theta_2} (N) y_1^k \right) z^N \\
&= \sum_{N \geq 0} \exp \left( - \sum_{k \geq 1} H_{\Theta_0} (N) \frac{(-y_1)^k}{k} \right) z^N.
\end{align*}
\]
Consequently,
\[
H = \exp (H_{y_1} y_1) \prod_{l \in \mathcal{L}_{\Phi Y \setminus \{y_1\}}} \exp (H_{\Sigma_2} \Pi_l)
= [z^N] G^Y (z)
\lim_{N \to \infty} \exp \left( - \sum_{k \geq 1} H_{\Theta_0} (N) \frac{(-y_1)^k}{k} \right) \pi_Y Z_{\mathbb{L}}.
\]
**Example 8.** According to expression (28), we reduce the following relations by identifying local coordinates:
\[
\begin{align*}
H_{\Sigma_2} (N) &= \ln (N) + \gamma + 1/2 N^{-1} - 1/12 N^{-2} \\
&\quad + \frac{120}{N^4} + O(N^{-5})
\end{align*}
\]
\[
\begin{align*}
H_{\Sigma_2} (N) &= -N^{-1} + 1/2 N^{-2} - 1/6 N^{-3} \\
&\quad + \frac{1}{N^2} + \zeta (\Sigma_4)
\end{align*}
\]
\[
\begin{align*}
H_{\Theta_2} (N) &= 1/2 (\ln (N) + \gamma)^2 + \frac{1}{N} \ln (N) + 1/2 \gamma \\
&\quad - \ln (N) - 1/12 \gamma + 1/8 - 1/24 N^{-3} \\
&\quad + \frac{120}{N^4} + O(N^{-5})
\end{align*}
\]
\[
\begin{align*}
H_{\Theta_2} (N) &= -1/2 N^{-2} + 1/2 N^{-3} - 1/4 \frac{1}{N^4} \\
&\quad + \zeta (\Sigma_4) + O(N^{-5})
\end{align*}
\]
\[
\begin{align*}
H_{\Sigma_2 \mathbb{L}} (N) &= 1/2 \zeta (\Sigma_4) + \frac{1 + \ln (N) + \gamma}{N} \\
&\quad - 1/2 - 1/2 \gamma - 1/2 \ln (N) \\
&\quad + \frac{7}{18} + 1/6 \ln (N) + 1/6 \gamma) \ln (N) - \frac{5}{24} \frac{1}{N^4} \\
&\quad + O(N^{-5}).
\end{align*}
\]
\[\text{3.3 \ Representations of polyzetas}\]
As we see the definition of polyzetas at (2), these convergent series are also compatible with the stuff product like multiple harmonic sums. Using the expression in Proposition 6, we set
\[
Z_{\mathbb{L}} := \prod_{l \in \mathcal{L}_{\Phi Y \setminus \{y_1\}}} \exp \left( \zeta (\Sigma_l) \Pi_l \right).
\]
On the other hand, we conclude from (16) that polyzetas are also obtained by letting \( z \to 1 \) in multiple polylogarithms. Due to the isomorphism in the algebraic structures, we establish a bridge equation between the generating series \( Z_{\mathbb{L}} \) and \( Z_{\mathbb{L}} \) as follows.

**Proposition 7 ([9]).** We have a bridge equation between the two spaces \( C \langle X \rangle \) and \( C \langle Y \rangle \):
\[
Z_{\mathbb{L}} = B' (y_1) \pi_Y (Z_{\mathbb{L}}),
\]
where \( B' (y_1) = \exp \left( \sum_{k \geq 2} \frac{(-1)^{k-1} \zeta (k)}{k} y_1^k \right) \).

Let \( Z_{\mathbb{L}} \) be the \( Q \)-vector space generated by polyzetas of weight \( n \). Using this formula, we can rewrite the two sides on the same transcendence basis and then reduce the relations among polyzetas by identifying the local coordinates. On the one hand, by expressing the right hand side of (29) on the basis \( \{ \Sigma_l \} \in \mathcal{L}_{\Phi Y} \), we can identify coefficients on this basis [9, 10].

**Example 9.** Relations of polyzetas in terms of irreducible elements indexed by the basis \( \{ \Sigma_l \} \in \mathcal{L}_{\Phi Y} \):

- **Weight 3:** \( \zeta (\Sigma_{y_2 y_1}) = \frac{2}{5} \zeta (\Sigma_{y_3}) \),
- **Weight 4:** \( \zeta (\Sigma_{y_3}) = \frac{5}{2} \zeta (\Sigma_{y_2}) \), \( \zeta (\Sigma_{y_3 y_3}) = \frac{11}{6} \zeta (\Sigma_{y_3}) \),
- **Weight 5:** \( \zeta (\Sigma_{y_3 y_3 y_3}) = 3 \zeta (\Sigma_{y_3}) \zeta (\Sigma_{y_2}) - 5 \zeta (\Sigma_{y_3}) \),
- **Weight 6:**
  \[
  \begin{align*}
  \zeta (\Sigma_{y_4}) &= \frac{8}{35} \zeta (\Sigma_{y_3})^3, \\
  \zeta (\Sigma_{y_4 y_3}) &= \zeta (\Sigma_{y_3})^2 - \frac{4}{21} \zeta (\Sigma_{y_3})^3, \\
  \zeta (\Sigma_{y_3 y_3}) &= \frac{2}{7} \zeta (\Sigma_{y_3})^3 - \frac{3}{2} \zeta (\Sigma_{y_2})^2, \\
  \zeta (\Sigma_{y_4 y_3 y_3}) &= -\frac{17}{30} \zeta (\Sigma_{y_3})^3 + \frac{9}{4} \zeta (\Sigma_{y_2})^2, \\
  \zeta (\Sigma_{y_5}) &= 3 \zeta (\Sigma_{y_4}) - \frac{9}{10} \zeta (\Sigma_{y_3})^3, \\
  \zeta (\Sigma_{y_4 y_3 y_2}) &= \frac{3}{10} \zeta (\Sigma_{y_3})^3 - \frac{3}{4} \zeta (\Sigma_{y_3})^2, \\
  \zeta (\Sigma_{y_5}) &= \frac{11}{63} \zeta (\Sigma_{y_3})^3 - \frac{1}{4} \zeta (\Sigma_{y_3})^2.
  \end{align*}
\]
\[
\zeta(\Sigma_{\gamma y_1}) = \frac{1}{27} \zeta(\Sigma_{\gamma y})^3, \\
\zeta(\Sigma_{\gamma y_2}) = \frac{17}{50} \zeta(\Sigma_{\gamma y})^3 + \frac{3}{16} \zeta(\Sigma_{\gamma y})^2.
\]

Weight 7:
\[
\zeta(\Sigma_{\gamma y_1,y_2}) = \frac{35}{2} \zeta(\Sigma_{y y}) - 10 \zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}), \\
\zeta(\Sigma_{\gamma y_1,y_2}) = 5 \zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}) - \frac{21}{2} \zeta(\Sigma_{y y}) + 4 \frac{5}{3} \zeta(\Sigma_{y y})^2 \zeta(\Sigma_{y y}), \\
\zeta(\Sigma_{\gamma y_1,y_3}) = -\zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}) - \frac{2}{5} \zeta(\Sigma_{y y})^2 \zeta(\Sigma_{y y}) + 7 \frac{2}{3} \zeta(\Sigma_{y y}), \\
\zeta(\Sigma_{\gamma y_2,y_3}) = \frac{23}{7} \zeta(\Sigma_{y y})^2 \zeta(\Sigma_{y y}) - \frac{217}{48} \zeta(\Sigma_{y y}), \\
\zeta(\Sigma_{\gamma y_3,y_3}) = \frac{7}{24} \zeta(\Sigma_{y y}), \\
\zeta(\Sigma_{\gamma y_1,y_3}) = \frac{1}{10} \zeta(\Sigma_{y y})^2 \zeta(\Sigma_{y y}) + \frac{7}{48} \zeta(\Sigma_{y y}).
\]

Weight 8:
\[
\zeta(\Sigma_{\gamma y}) = \frac{24}{175} \zeta(\Sigma_{y y})^4, \\
\zeta(\Sigma_{\gamma y_1,y_2}) = \frac{126}{25} \zeta(\Sigma_{y y})^4 - 720 \zeta(\Sigma_{y y}^2 y), \\
\zeta(\Sigma_{\gamma y_1,y_2}) = -\frac{282}{125} \zeta(\Sigma_{y y})^4 + 2 \zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}) + \frac{288}{125} \zeta(\Sigma_{y y}^2 y), \\
\zeta(\Sigma_{\gamma y_1,y_3}) = \frac{6}{25} \zeta(\Sigma_{y y})^4 - \zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}), \\
\zeta(\Sigma_{\gamma y_2,y_3}) = \frac{9}{2} \zeta(\Sigma_{y y})^2 \zeta(\Sigma_{y y}) + \frac{4}{3} \frac{2}{5} \zeta(\Sigma_{y y})^4 - 15 \zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}) + 1440 \zeta(\Sigma_{y y}^2 y), \\
\zeta(\Sigma_{\gamma y_3,y_3}) = \frac{6}{25} \zeta(\Sigma_{y y})^4 - \zeta(\Sigma_{y y}) \zeta(\Sigma_{y y}),
\]

On the other hand, we use the inverse of \(\pi_Y\), denoted by \(\pi_X\), to express equality (29) on the basis \(\{S_i\} \in \mathcal{E}_{\mathcal{Y},X}\). It means that \(\pi_X\) is a morphism defined on the word by \(\pi_X(y_1, \ldots, y_n) = x_n^{\pi_X} - x_1 \cdot \ldots \cdot x_n^{\pi_X} - x_1\), and applying to the two sides of (29) we have
\[
\pi_X(Z_{\mathcal{W}}) = B'(x_1) Z_{\mathcal{Y}}. \tag{30}
\]

Hence, we can represent the left hand side of this bridge equation on the basis \(\{S_i\} \in \mathcal{E}_{\mathcal{Y},X}\) and then identifying local coordinates to reduce polynomial relations among polyzetas [9, 10].

**Example 10.** Relations of polyzetas in terms of irreducible elements indexed by the basis \(\{S_i\} \in \mathcal{E}_{\mathcal{Y},X}\):

Weight 3:
\[
\zeta(S_{\gamma x_1}) = \zeta(S_{\gamma x_1}^2),
\]

Weight 4:
\[
\zeta(S_{\gamma x_1}) = \frac{2}{5} \zeta(S_{\gamma x_1})^2, \\
\zeta(S_{\gamma x_2}) = \frac{1}{10} \zeta(S_{\gamma x_1})^2.
\]

Weight 5:
\[
\zeta(S_{\gamma x_1}) = -\zeta(S_{\gamma x_1}) \zeta(S_{\gamma x_1}) + 2 \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = -\zeta(S_{\gamma x_1}) + \frac{3}{2} \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = -\zeta(S_{\gamma x_1}) \zeta(S_{\gamma x_1}) + 2 \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = \frac{1}{2} \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = \zeta(S_{\gamma x_1}).
\]

Weight 6:
\[
\zeta(S_{\gamma x_1}) = \frac{8}{35} \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = \frac{6}{35} \zeta(S_{\gamma x_1})^3 - \frac{1}{2} \zeta(S_{\gamma x_1})^2, \\
\zeta(S_{\gamma x_1}) = \frac{4}{210} \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = \frac{23}{70} \zeta(S_{\gamma x_1})^3 - \zeta(S_{\gamma x_1})^2, \\
\zeta(S_{\gamma x_1}) = \frac{2}{105} \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = \frac{8}{35} \zeta(S_{\gamma x_1})^3 - \zeta(S_{\gamma x_1})^2, \\
\zeta(S_{\gamma x_1}) = \frac{8}{85} \zeta(S_{\gamma x_1})^3.
\]

Weight 7:
\[
\zeta(S_{\gamma x_1}) = -\zeta(S_{\gamma x_1}) \zeta(S_{\gamma x_1}) + \frac{1}{2} \zeta(S_{\gamma x_1})^2 - \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = -\zeta(S_{\gamma x_1}) + 3 \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = \zeta(S_{\gamma x_1})^2 - \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = \frac{1}{10} \zeta(S_{\gamma x_1})^2 - \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = \frac{1}{105} \zeta(S_{\gamma x_1})^3 + \frac{19}{16} \zeta(S_{\gamma x_1})^4, \\
\zeta(S_{\gamma x_1}) = \zeta(S_{\gamma x_1}).
\]

Weight 8:
\[
\zeta(S_{\gamma x_1}) = \frac{24}{175} \zeta(S_{\gamma x_1})^4, \\
\zeta(S_{\gamma x_1}) = \frac{24}{30625} \zeta(S_{\gamma x_1})^4 + 3 \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = \frac{6}{35} \zeta(S_{\gamma x_1})^4 - \zeta(S_{\gamma x_1})^3, \\
\zeta(S_{\gamma x_1}) = \frac{2}{30625} \zeta(S_{\gamma x_1})^4 + 149753 \zeta(S_{\gamma x_1}), \\
\zeta(S_{\gamma x_1}) = \frac{2}{105} \zeta(S_{\gamma x_1})^2 - \zeta(S_{\gamma x_1})^3,
\]

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\[ \zeta(S_{\Sigma x_1}) = -\frac{11}{4} \zeta(S_{\Sigma x_1}) \zeta(S_{\Sigma x_1}) + \frac{1}{2} \zeta(S_{\Sigma x_1})^2 \zeta(S_{\Sigma x_1}) + \frac{61}{175} \zeta(S_{\Sigma x_1})^4, \ldots \]

The above examples, one again, show us that each polyzetas only has a linear representation of polyzetas of the same weight. Hence, we can list the elements of linear bases of \( Z \) corresponding to the bases \( \{ \Sigma \} \in \mathcal{L}(Y) \) and \( \{ S \} \in \mathcal{L}(X) \) that confirm the Zagier’s dimension conjecture \(^{10}\) (see [4]). Moreover, we can reduce algebraic bases (the normal product) from these representations. We show here two lists of irreducible elements up to weight 12 (see more [11]):

1. For the basis \( \{ \Sigma \} \in \mathcal{L}(Y) \): \( \zeta(\Sigma y_2), \zeta(\Sigma y_3), \zeta(\Sigma y_4), \zeta(\Sigma y_5), \zeta(\Sigma y_6), \zeta(\Sigma y_7), \zeta(\Sigma y_8), \zeta(\Sigma y_9), \zeta(\Sigma y_{10}), \zeta(\Sigma y_{11}), \zeta(\Sigma y_{12}), \zeta(\Sigma y_{13}), \zeta(\Sigma y_{14}). \)

2. For the basis \( \{ S \} \in \mathcal{L}(X) \): \( \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}), \zeta(S_{x_0 x_1}). \)

4 Conclusion

We represented special functions (multiple harmonic sums, polyzetas, and multiple polylogarithms) in forms of non-commutative generating series indexed by transcendence bases of quasi-shuffle algebras. By identifying the local coordinates of the Hausdorff graphs, in shuffle and stuffle Hopf algebras, we can reduce polynomial relations or asymptotic expansions of these special functions indexed by the bases.

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