Eigenvalues of the hermitian Wilson-Dirac operator and chiral properties of the domain-wall fermion

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Chiral properties of QCD formulated with the domain-wall fermion (DWQCD) are studied using the anomalous quark mass \( m_{5q} \) and the spectrum of the 4-dimensional Wilson-Dirac operator. Numerical simulations are made with the standard plaquette gauge action and a renormalization-group improved gauge action. Results are reported on the density of zero eigenvalue obtained with the accumulation method, and a comparison is made with the results for \( m_{5q} \).

1. Introduction

Formulation of chiral fermions on the lattice has been one of long-standing problems in lattice field theories. Several years ago, the domain-wall fermion (DWF) formalism \[ \mathbb{D} \], which is a Wilson fermion in \( D + 1 \) dimensions with Dirichlet boundary condition in the extra dimension, has been proposed as a new formulation of lattice chiral fermion. In the limit of large extra dimension size, \( N_s \to \infty \), the spectrum of free domain-wall fermion contains massless modes at the edges in the extra dimension.

While the massless modes are shown to be stable in perturbation theory \[ \mathbb{P} \], their existence may be spoiled non-perturbatively in the presence of dynamical gauge fields. We studied this issue through an anomalous quark mass \( m_{5q} \) in Ref. \[ \mathbb{R} \]. This quantity measures the magnitude of chiral symmetry breaking with the domain-wall QCD (DWQCD).

In this article we make a status report of our attempt to understand the results on the \( N_s \)-dependence of \( m_{5q} \) obtained in Ref. \[ \mathbb{R} \] through measurements of the eigenvalue distribution of the 4-dimensional Wilson-Dirac operator.

2. Chiral Properties of DWQCD

We define the anomalous quark mass by \[ \mathbb{K} \]

\[ m_{5q} = \lim_{t \to \infty} \frac{\sum_x \langle J_{5q}^a(t,x)P^b(0,0) \rangle}{\sum_x \langle P^a(t,x)P^b(0,0) \rangle}. \quad (1) \]

This quantity measures the chiral symmetry breaking effect in the axial Ward-Takahashi identity:

\[ \sum_{\mu} \langle \nabla_{\mu} A^a_{\mu}(x)P^b(0) \rangle = 2m_f \langle P^a(x)P^b(0) \rangle \]
the plaquette action, and (iv) from data in the range of \( N_s \) we explore, \( m_{5q} \) seems to remain non-zero in the limit \( N_s \to \infty \), in all cases except at \( \beta = 2.6 \) for the RG-improved action. If confirmed with studies at larger values of \( N_s \), the last point means that DWQCD realizes chiral symmetry at \( a^{-1} \approx 2 \text{ GeV} \) only for the case of the RG-improved action.

3. Eigenvalues of the hermitian Wilson-Dirac operator and chiral property

Chiral symmetry of DWQCD can be studied also through the transfer matrix in the direction of the extra dimension\([7,8]\). When the transfer matrix has a unit eigenvalue, chiral symmetry is not realized in DWQCD because the left and right chiral modes on the two edges in the extra dimension couple with each other.

A unit eigenvalue of the transfer matrix is in one-to-one correspondence with a zero eigenvalue of the hermitian Wilson-Dirac operator defined by

\[
H_W(m_0) = \gamma_5 D_W(-m_0) ,
\]

which is much easier to calculate. Here, \( D_W(-m_0) \) is the four dimensional Wilson-Dirac kernel with a bare mass \(-m_0\). Therefore, a failure of exponential decay of \( m_{5q} \) would result if \( H_W \) develops a zero eigenvalue.

We calculate eigenvalues of \( H_W^2 \) by the Lanczos method using 50–100 configurations at several values of coupling in the range \( a^{-1} \approx 1–2 \text{ GeV} \) using both plaquette and RG-improved actions. The results from the Lanczos method are checked by the Ritz functional method for \( H_W \). We also study the dependence on the lattice size. The maximum lattice at \( a^{-1} \approx 1 \text{ GeV} \) is \( 12^4 \) for both actions, while the one at \( a^{-1} \approx 2 \text{ GeV} \) is \( 24^4 \) for the RG-improved action and \( 16^3 \times 32 \) for the plaquette action.

3.1. Eigenvalue distributions

In Figs. 2 and 3 we plot Monte Carlo time histories for the six lowest eigenvalues of \( H_W^2 \) for the plaquette and RG-improved actions. In each figure the left panel shows results for \( a^{-1} \approx 1 \text{ GeV} \) and the right panel for \( a^{-1} \approx 2 \text{ GeV} \). The lattice size at \( a^{-1} \approx 2 \text{ GeV} \) is the same as in the previous
work of $m_{5q}$ shown in Fig. 1. Open squares plot the minimum eigenvalue $\lambda_{\text{min}}^2$ and filled diamonds are the five higher eigenvalues.

There is a clear trend that the minimum eigenvalues become larger for smaller lattice spacings. Another interesting point is that the RG-improved action gives larger values of $\lambda_{\text{min}}^2$ than the plaquette action, which indicates that the RG-improved action has a better chiral behavior. These trends are parallel to the features we noted for $m_{5q}$ in Sec. 2.

### 3.2. Spectral density

The spectral density of $H_W$ is defined by

$$\rho(\lambda) = \lim_{V \to \infty} \frac{1}{3 \cdot 4 \cdot V} \sum_{\lambda'} \delta(\lambda' - \lambda),$$

where the summation is over the eigenvalues of $H_W$. We are interested in the density of zero eigenvalues, $\rho(0)$, since we expect this quantity to be related to the existence of unit eigenvalue of the transfer matrix. To calculate this quantity, we adopt the accumulation method proposed in [9], which is based on the relation

$$A(\lambda) \equiv \int_{-\lambda}^{\lambda} d\lambda' \rho(\lambda') = \frac{1}{3 \cdot 4 \cdot V} \sum_{|\lambda'| \leq \lambda} 1$$

where $\tilde{\rho}(\lambda^2)$ is the spectral density function for $H_W^2$. We note that, for the small-$\lambda$ expansion of $A(\lambda)$ in (5), analyticity of $\rho(\lambda)$ at the origin is assumed.

In Fig. 4, we show typical examples of the accumulation $A(\lambda)$ from the eigenvalue distribution of $H_W^2$ for the case of the RG-improved action. Results for $\rho(0)$ obtained by a linear fitting following (5), normalized by the string tension, are summarized in Fig. 5.

Our results for $\rho(0)$ for the plaquette action are consistent with the previous data by Edwards et al. [9]. Results for the RG-improved action show a similar $\beta$ dependence. A significant difference is that the RG-improved action leads to much smaller values of $\rho(0)$ than the plaquette action, roughly by an order of magnitude.

### 4. Discussions

We have applied the accumulation method to estimate the spectral density at zero eigenvalue of the hermitian Wilson-Dirac operator, $\rho(0)$. We found that this method leads to non-zero values of $\rho(0)$ at $a^{-1} \simeq 1$–2 GeV for both the plaquette and RG-improved actions.

At $a^{-1} \simeq 1$ GeV, the non-zero result for $\rho(0)$

$$\rho(0) \simeq 2\rho(0) + O(\lambda^2),$$

where $\tilde{\rho}(\lambda^2)$ is the spectral density function for $H_W^2$. We note that, for the small-$\lambda$ expansion of $A(\lambda)$ in (5), analyticity of $\rho(\lambda)$ at the origin is assumed.

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is consistent with the finite $m_{5q}$ in the large $N_s$ limit observed in with both the plaquette and RG-improved actions. At $a^{-1} \simeq 2$ GeV, while a consistency also holds with the plaquette action, there is an apparent contradiction for the case of the RG-improved action since $m_{5q}$ seems to decay exponentially with $N_s$ for this case.

In the accumulation data shown in Fig. 4 we observe that results are very noisy at $\beta = 2.6$ (2 GeV). Since the fit results for $\rho(0)$ fluctuates with volume, it is difficult to determine the size dependence. Therefore simulations with larger lattices are needed to check if the slope remains non-vanishing toward infinite volume. Another point to examine is if the analyticity assumption for $\rho(\lambda)$ at the origin is justified if there is a spectral gap. Further studies are required to clarify these points.

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