Cross-scale energy transfer of chaotic oscillator chain in stiffness-dominated range

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Abstract The investigations on the dynamics of hierarchical oscillator chains provide basic theories for the fields of acoustic metamaterial and passive guidance of energy flow. In the present study, the characteristics of stiffness-dominated energy transfer of a reduced cross-scale oscillator chain that generates chaotic vibrations are studied. The semi-analytical solutions are obtained by the complexification-averaging method and the least-square method for thoroughly searching the branches of responses. Then, the energy transfer patterns of the oscillator chain are analyzed based on Runge–Kutta method. Moreover, chaotic responses are determined using the displacements and Lyapunov exponents of the system. The average energy and normalized energy flux are defined to evaluate the energy distribution in each oscillator and the energy transfer between oscillators, respectively. The semi-analytical results show that the oscillator chain produces complex unstable branches of response around the resonance frequency of the linear oscillator. It is found that the detached higher branches of response of the chain are similar to that of a system consisting of one linear oscillator and one cubic oscillator. When chaotic responses occur, the energy transfer patterns that reflected by average energy and normalized energy flux are similar in the same branch of response for different excitation frequencies. However, energy transfer patterns in different branches of response are quite different even at the same excitation frequency, and the higher branch generates the lower energy transfer. Furthermore, analyses of the input energy and the scale between cubic oscillators demonstrate that...

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variations of the two parameters slightly affect the normalized energy flux.

**Keywords**  Energy transfer · Cross-scale · Chaotic response · Stiffness-dominated range

1 Introduction

Energy transfer such as energy transfer between vortices in turbulent flows, energy transfer between molecules through heat conduction, and solar energy transfer to plants through photosynthesis occurs widely in nature. With the development and maturity of the nonlinear theory, nonlinear elements have been added into various linear systems to improve their performances [1, 2], and construct the so-called artificial nonlinear systems. Accordingly, the construction of various nonlinear paths has become a research hotspot to achieve the expected energy transfer in mechanical and acoustic systems [3, 4].

Usually, turbulent flows are considered as a typical example of the cross-scale energy transfer between vortices of different scales where large and unstable vortices break up into several smaller ones, thereby irreversibly transferring energy from larger scales to small scales. Based on the idea of decreasing-scale binomial tree network, an equivalent decreasing-scale oscillator chain has been used to simulate the energy transfer in turbulent flows. Through parameter adjustment, energy transfer between oscillators in the chain is in accordance with the well-known Kolmogorov law [5]. This study is helpful to find some general rules of energy transfer between oscillators and has the potential to provide a new angle for the study of fluid turbulence. The models that comprise linear and weakly nonlinear oscillator chains, respectively, were both studied [6, 7]. To explore the similarities of energy transfer between chaotic mechanical system and fluid turbulence, a model consisting of a linear oscillator and $N$ purely cubic oscillators was proposed, and the irreversible energy transfer from stiffness-dominated larger scales to damping-dominated smaller scales was analyzed [8].

Reviewing the literature indicates that predictable designs for nonlinear targeted energy transfer motivated by fluid turbulence have been investigated for more than two decades [9]. Nonlinear energy sink (NES) is a representative application of the principle of targeted energy transfer, where the energy transfer path constructed by purely cubic nonlinearity, and has been widely studied so far [10–12]. To improve the efficiency of energy transfer, several components have been considered to modify the standard NES, thereby forming several novel vibration absorption devices such as lever-type [13], limited-type [14, 15], encapsulated-type [16], collision enhanced-type [17], inertial-type [18], and stiffness descending-type [19]. Recently, studies about NESs have merged with energy harvesting, which is another hot research topic that concentrates on the energy transfer between mechanical and electrical sources of energy [20, 21]. In view of the bright prospects of NESs, investigations about the energy transfer between vibratory systems and NESs are expected to be continued for a long time.

In the past few years, the nonreciprocity of energy transfer has attracted special attention, which has become a challenging issue in the fields of acoustic and mechanical metamaterials and can be considered as a modified irreversible energy transfer. The nonreciprocity of energy transfer was analyzed for a reduced two-degree-of-freedom model with purely nonlinear and asymmetric oscillators, demonstrating that the transient resonance energy capture is an important phenomenon that should be considered to realize nonreciprocity [22]. Accordingly, the linear and purely nonlinear oscillators were utilized to construct decreasing-scale oscillator chains with different structures [23] and periodic oscillator chains with several identical cells to achieve global nonreciprocity [24]. Subsequently, the direction control of energy transfer in an oscillator network was achieved by using purely nonlinear components, which expands the application of nonreciprocity [25].

Currently, most investigations in this area are mainly concentrated on the vibration absorption of cubic oscillators. In this regard, NESs with two and multiple degrees-of-freedom, which can be seemed as reduced oscillator chains, have been used to suppress vibrations of impulsively and harmonically excited linear primary systems [26–29], aeroelastic instability of a rigid wing with structural nonlinearities [30], and responses of a six-story structure subjected to a shock-type loading [31]. Considering the limitations of the vibration absorber structure, equal-weight cubic oscillators are usually used (at least the weight of the oscillators are close). Accordingly, the energy transfer
between such cubic oscillators is an equi-scale phenomenon.

In the present study, nonlinear energy transfer in a cross-scale chain with a linear oscillator and three cubic oscillators is analyzed when the stiffness-dominated chaotic responses are excited. The energy transfer patterns in the oscillator chain are explored using the Runge–Kutta, complexification-averaging, and least-square methods. It is intended to study the characteristics of energy transfer of the chain with different steady-state responses by analyzing the average energy and normalized energy flux. The main objective of this study is to find and investigate the differences and similarities of different intense energy transfers produced in the cross-scale oscillator chain.

2 Reduced model of the stiffness-dominated range

Figure 1 shows a nonlinear hierarchical oscillator chain consisting of a harmonically excited linear oscillator and $N$ purely cubic nonlinear oscillators. $M$, $K$, and $\mu$ denote mass, linear stiffness, and viscous damping coefficients of the primary linear oscillator, respectively. Moreover, $m_n$, $k_n$ and $\lambda_n$ ($n = 1, \ldots, N$) denote the mass, cubic stiffness, and viscous damping coefficients of the cubic oscillators, respectively. $x_{n+1}$, $n = 0, 1, \ldots, N$ reflects the displacements of $N + 1$ oscillators of the system. The linear oscillator is excited by a harmonic excitation represented by $f \cos(\Omega t)$, where $f$ and $\Omega$ are the excitation amplitude and frequency, respectively.

The governing equations of the oscillator chain can be expressed as follows [8]:

$$ M\ddot{x}_1 + \mu \dot{x}_1 + Kx_1 + \lambda_1 (\dot{x}_1 - \dot{x}_2) + k_1 (x_1 - x_2)^3 = f \cos(\Omega t) 
$$ (1a)

$$ m_n \ddot{x}_{n+1} + \lambda_n (\dot{x}_{n+1} - \dot{x}_n) + k_n (x_{n+1} - x_n)^3 + \lambda_{n+1} (\dot{x}_{n+1} - \dot{x}_{n+2}) + k_{n+1} (x_{n+1} - x_{n+2})^3 = 0, \quad n = 1, 2, \ldots, N - 1
$$ (1b)

$$ m_N \ddot{x}_{N+1} + \lambda_N (\dot{x}_{N+1} - \dot{x}_N) + k_N (x_{N+1} - x_N)^3 = 0
$$ (1c)

Introducing the following normalized variables and parameters

$$ e_n = \frac{m_n}{M}, \tilde{f} = \sqrt{\frac{K}{M}}, \tilde{\mu} = \frac{\mu}{\sqrt{KM}}, \tilde{\lambda} = \frac{\lambda}{\sqrt{KM}}, \tilde{k}_n = \frac{k_n}{K}, \tilde{\Omega} = \sqrt{\frac{M}{K}}. \quad (2) $$

Equation (1) can be rewritten in the following non-dimensional form:

$$ \ddot{x}_1 + \tilde{\mu} \dot{x}_1 + x_1 + \tilde{\lambda}_1 (\dot{x}_1 - \dot{x}_2) + \tilde{k}_1 (x_1 - x_2)^3 = \tilde{f} \cos(\tilde{\Omega} t) 
$$ (3a)

$$ e_n \ddot{x}_{n+1} + \tilde{\lambda}_n (\dot{x}_{n+1} - \dot{x}_n) + \tilde{k}_n (x_{n+1} - x_n)^3 + \tilde{\lambda}_{n+1} (\dot{x}_{n+1} - \dot{x}_{n+2}) + \tilde{k}_{n+1} (x_{n+1} - x_{n+2})^3 = 0, \quad n = 1, 2, \ldots, N - 1
$$ (3b)

$$ e_N \ddot{x}_{N+1} + \tilde{\lambda}_N (\dot{x}_{N+1} - \dot{x}_N) + \tilde{k}_N (x_{N+1} - x_N)^3 = 0 \quad (3c)$$

For the convenience of writing, the tildes super-script will be dropped in the subsequent analysis. The hierarchy of scales can be achieved by decreasing the system parameters from the left oscillator to the right one according to the following scaling laws:

$$ e_{n-1} = 8, \quad k_{n-1} = 30, \quad \lambda_{n-1} = 2, \quad n = 2, 3, \ldots, N \quad (4) $$

Moreover, several parameters of the system are chosen as $\mu = 0.01$, $e_1 = 0.05$, $k_1 = 0.5$ and $\lambda_1 = 0.001$.

Figure 2 shows the average energy of each oscillator in different chains. The average steady-state energy in the linear ($E_{LO}$) and nonlinear oscillators ($E_n$) can be expressed as follows:

![Fig. 1 Mechanical model of a cross-scale oscillator chain](image-url)
\[ \begin{align*}
E_{LO} &= \frac{1}{T} \int_{t_0}^{t_0+T} \left( \frac{1}{2} (x_1^2 + x_1^2) + \frac{1}{8} k_1 (x_1 - x_2)^4 \right) dt, \\
E_n &= \frac{1}{T} \int_{t_0}^{t_0+T} \left( \frac{1}{2} \frac{x_n^2}{\epsilon_n} + \frac{1}{8} k_n (x_n - x_{n+1})^4 + \frac{1}{8} k_{n+1} (x_{n+1} - x_{n+2})^4 \right) dt,
\end{align*} \]

where \( t_0 = 2000 \) and \( T = 1000 \), indicating that a long enough initial time is removed to reach steady-state vibrations.

It is shown that the energy in the stiffness-dominated (large-scale) range is much larger than that in the damping-dominated (small-scale) range. The \( x \) and \( y \) axes both use logarithmic coordinates in Fig. 2, the mass scales are defined as \( 1/\epsilon_n, n = 1, 2, \ldots, N \), and the mass scale corresponding to the linear oscillator is normalized to unity. The energies in chains with three, six, and nine cubic oscillators are compared. It is found that when the input energy is at a low level (excitation amplitude is small), it hardly transfers to small scales. However, when the input energy reaches a certain level, the energy level in cubic oscillators of different chains enhances dramatically. It is also observed that the three lines in each sub-figures almost overlapped with each other, and as the input energy increases the differences between the three lines vary slightly. Moreover, it should be noted

![Fig. 2](image-url) Distribution of average energy of the chain with different numbers of oscillators, \( \Omega = 1 \) and (a) \( f = 0.002 \), (b) \( f = 0.006 \), (c) \( f = 0.01 \), and (d) \( f = 0.014 \)
that the black and red lines only have four and seven points, respectively. In other words, the 3COs (cubic oscillators) system uses four mass scales and 6COs system uses seven mass scales. Figures 2(b)–2(d) reveal that the blue and red lines all have two regions, and the lines in one region decrease according to a power law. The lines in the other region are present as broken lines, which are close to straight lines and have a relatively small angle with the x-axis.

Figures 3 and 4 show the distribution of the coupling force $F_n$ versus the relative displacements and velocities for oscillators in a chain with six cubic oscillators, respectively. In this way, the influence of cubic stiffness nonlinearities and the linear viscous dissipative forces on the energy transfer process can be investigated. The coupling force can be calculated using the following expression:

$$ F_n = k_n(x_n - x_{n+1})^3 + \lambda_n(\dot{x}_n - \dot{x}_{n+1}), \quad n = 1, 2, \ldots, N. $$

In Figs. 3(a) and 3(b), the relations between the coupling force and relative displacement are almost cubic, but the relations between the coupling force and relative velocity are irregular, as shown in Figs. 4(a) and 4(b). Elastic force is related to displacement and dissipative force is related to velocity. It is concluded that the dynamics of the first two cubic oscillators are dominated by nonlinear stiffness. Similarity, in Figs. 4(e) and 4(f), the relations between the coupling force and relative velocity are linear, and thus, the last two oscillators are dominated by linear damping. Moreover, the stiffness has more influence than the damping on the dynamics of the third cubic oscillator. Therefore, the third cubic oscillator is also deemed stiffness-dominated.

Based on Figs. 2, 3, and 4, a reduced mechanical model of Fig. 1 (i.e., $N = 3$) can be established to study the energy transfer and dynamics in the stiffness-dominated range of the cross-scale oscillator chain. Another advantage of using the reduced model is that the analytical procedure can be significantly simplified. When $N = 3$, Eq. (3) can be rewritten in the form below:

$$ \ddot{x}_1 + \mu \dot{x}_1 + x_1 + \lambda_1(\dot{x}_1 - \dot{x}_2) + k_1(x_1 - x_2)^3 = f \cos(\Omega t), \quad (7a) $$

$$ \ddot{x}_3 + \lambda_3(\dot{x}_3 - \dot{x}_4) + k_3(x_3 - x_4)^3 = 0, \quad (7b) $$

$$ \ddot{x}_2 + \lambda_2(\dot{x}_2 - \dot{x}_3) + k_2(x_2 - x_3)^3 = 0, \quad (7c) $$

$$ \ddot{x}_4 + \lambda_3(\dot{x}_4 - \dot{x}_3) + k_3(x_4 - x_3)^3 = 0. \quad (7d) $$

### 3 Energy transfer in oscillator chain

In this section, the complexification-averaging method is used to obtain the first-order equation and analyze the distribution of the system response in frequency bands. The derivation procedure is presented in Appendix.

In order to obtain the steady-state response of the system, we set $a_n = 0$, $b_n = 0$, $n = 1, 2, 3, 4$, the sources of $a_n$ and $b_n$ are also reported in Appendix. Furthermore, the amplitudes of the oscillators can be calculated by $A_n = \frac{\sqrt{a_n^2 + b_n^2}}{\Omega}$.

To verify the accuracy of the calculation, the results of the complexification-averaging method are compared with that of the Runge–Kutta method. To this end, the steady-state response equations of the system are solved by using a code based on least-square method and Eq. (7) is solved directly by using Runge–Kutta method, respectively. It should be indicated that these equations are both solved in the MATLAB software environment. In Fig. 5, the semi-analytical results are obtained from the complexification-averaging method and least-square method, and the numerical results are obtained from Runge–Kutta method. Figure 5 reveals that the results in the stable region of Fig. 5(a) and in the lower stable branch of Fig. 5(b) obtained by the two ways are consistent. It is worth noting that the values of the blue circles are the maximal values of the displacement in the selected time window. Therefore, there are some deviations in unstable regions and the stable higher branch. More specifically, when the excitation amplitude is large enough (e.g., $f = 0.014$),
the stability analysis between the semi-analytical and numerical methods is unsatisfactory. The reduced model is expected to generate complex dynamic behaviors because it still has three strong nonlinearities. However, when only the frequency responses of the reduced model are compared with those of a system consisting of one linear oscillator and one cubic oscillator [32, 33], multiple unstable branches arise around the resonance frequency and two unstable branches are extended to the frequency band beyond the resonance frequency for a large excitation amplitude. Figure 6 shows the frequency responses of the cubic oscillators for \( f = 0.006 \). Similar to the linear oscillator, it is observed that unstable branches of the first cubic oscillator are connected. However, unstable branches of the second and third oscillators are detached. In the present study, the semi-analytical method is used to find the branches of the studied system and further to guide the investigations on the differences in the energy transfer between these branches.

Fig. 3 Relations between coupling force and relative displacement of the chain with six oscillators for \( \Omega = 1 \) and \( f = 0.006 \)
Figure 7 shows the system displacements for $\Omega = 1$ and $f = 0.006$. It is found that the system always has chaotic dynamics and its maximal response amplitude only changes slightly independent of the initial conditions. Accordingly, only a waveform with zero initial conditions is presented. Although the system has three unstable semi-analytical solutions whose values are quite different for this excitation, there is no significant difference between the responses with different initial conditions obtained by Runge–Kutta method. Figure 8 reveals that the responses of the system with $\Omega = 0.98$ are periodic. The vibrations are unstable at the beginning of the response (also called the transient response) and energy transfers to small scales. However, the vibrations become stable rapidly and energy mainly localizes in the linear oscillator.

To further prove the chaotic state of the system, Lyapunov exponents are obtained where the responses in the last 1000 of 3000 (non-dimensional time) are used. The system (7) is transferred into state-space...
form involving eight variables, where four variables are related to the displacement and the other four variables reflect the velocity. Consequently, eight Lyapunov exponents are obtained, as shown in Fig. 9. The results reveal that when \( \Omega = 1 \), five Lyapunov exponents are positive while all Lyapunov exponents are negative for \( \Omega = 0.98 \). Figure 10 illustrates the distribution of the average energy and the normalized energy flux which can be applied to evaluate the energy transfer between the oscillators. The input energy flux \( (E_{in}) \) and normalized energy flux \( (\dot{E}_n) \) between neighboring oscillators can be expressed as follows:

\[
E_{in} = \int_{t_0}^{t_0+T} f \cos(\Omega t) \ddot{x}_1 \, dt \tag{8a}
\]

\[
\dot{E}_n = \frac{1}{E_{in}v_n} \int_{t_0}^{t_0+T} k_n(x_n - x_{n+1})^3 \ddot{x}_{n+1} \, dt, \quad n = 1, 2, ..., N \tag{8b}
\]
Fig. 7 Displacements of a the linear oscillator, b the first cubic oscillator, c the second cubic oscillator, and d the third cubic oscillator for $\Omega = 1$ and $f = 0.006$ with zero initial conditions.

Fig. 8 Displacements of a the linear oscillator, b the first cubic oscillator, c the second cubic oscillator, and d the third cubic oscillator for $\Omega = 0.98$ and $f = 0.006$ with zero initial conditions.
Figure 10(a) shows that as chaotic responses occur, energy distributes in the entire chain, while it is concentrated on the linear oscillator when periodic responses generate. It is found that the average energy of the smaller oscillator is lower than that of the larger oscillator even if the chain produces chaotic responses. This phenomenon is attributed to the hierarchy of scales (i.e., the smaller oscillator has smaller mass and stiffness). However, the black straight line with a positive slope in Fig. 10(b) demonstrates that the energy transfer between the smaller oscillators is more intense than that between larger oscillators in a chaotic state. As expected, the energy transfer between smaller oscillators is weak in a periodic state.

Figure 11 shows the frequency responses of the cubic oscillators for $f = 0.014$. It is observed that the maximal unstable solution in the resonance band is close to solutions of the stable higher branch, which is quite different from the frequency response of the linear oscillator in Fig. 5(b). Figures 12, 13, and 14 show three typical responses. The responses in Fig. 12 are obtained in the resonance band. The responses in Figs. 13 and 14 are obtained for the same excitation frequency which is out of the resonance band, and different initial conditions. Considering the Lyapunov exponents in Fig. 15, it is concluded that the three responses are chaotic. The responses obtained from the Runge–Kutta method in the chain with random initial conditions are chaotic, while the semi-analytical solutions for this excitation frequency are stable. It is noted that the results of stability analysis are consistent obtained by numerical and semi-analytical methods for relatively small excitation amplitudes. However, there exists some deviations between them for large excitation amplitudes. Even so, the semi-analytical results still can be used to assist the search of the higher branches in the numerical study. Interestingly, the two significantly different responses obtained by Runge–Kutta method under the same excitation frequency are both chaotic.

In this section, the characteristics of the cross-scale energy transfer in the higher branch are compared with those in the lower branch. Figure 16(a) reveals that the energy distributions in the chain for $\Omega = 1$ and $\Omega = 0.95$ with zero initial conditions are slightly different and the distribution rates in the four oscillators are
**Fig. 11** Frequency responses of a the first cubic oscillator; b the second cubic oscillator; c the third cubic oscillator for $f = 0.014$

**Fig. 12** Displacements of a the linear oscillator, b the first cubic oscillator, c the second cubic oscillator, d the third cubic oscillator with $\Omega = 1, f = 0.014$ and zero initial conditions
Fig. 13 Displacements of a the linear oscillator, b the first cubic oscillator, c the second cubic oscillator, d the third cubic oscillator with $\Omega = 0.95$, $f = 0.014$ and zero initial conditions.

Fig. 14 Displacements of a the linear oscillator, b the first cubic oscillator, c the second cubic oscillator, d the third cubic oscillator with $\Omega = 0.95$, $f = 0.014$ and nonzero initial conditions (0.049, −0.635, 0.011, 0.301, 0.449, −0.049, −0.792, −0.257).
consistent (black and red lines have the similar shape). Moreover, it is observed that the average energy in the chain for the higher branch of $\Omega = 0.95$ is much larger than those in the other cases. The same result can be achieved from the waveforms. Furthermore, the energy fluxes in the two cases with zero initial conditions are almost identical, while the energy flux in the case with nonzero initial condition is much lower than that in the former two cases, shown in Fig. 16(b). Accordingly, it is concluded that high levels of energy do not necessarily lead to a high energy transfer.

Then, it is necessary to investigate the influence of the excitation amplitude on the cross-scale energy transfer of the oscillator chain with chaotic responses. In this regard, the results of two case studies are shown in Figs. 17 and 18. It is observed that as the excitation increases, the distributed energy in the three cubic oscillators monotonically increases in the resonance band (e.g., $\Omega = 1$) except a special case (e.g., $f = 0.016$). Moreover, the energy level of four oscillators increases slightly as the system generates a higher branch. This phenomenon may be attributed to difficulties in increasing the amplitude for a higher...
branch. The initial conditions for obtaining high branches in Fig. 18 are listed in Table 1. It is worth noting that the normalized energy fluxes almost remain constant in the two cases, indicating that the variations of the energy transfer between oscillators have a linear correlation with the increase in the input energy.

Figures 19 and 20 show the influences of the scale (mass ratio) between cubic oscillators on the cross-scale energy transfer and the initial conditions are listed in Table 2. The oscillator chains produce chaotic responses. It is found that the energy level distributed in the cubic oscillators decreases with the increase in the mass ratio in both excitation conditions. Moreover, the distribution of normalized energy fluxes has the same slope and almost overlaps with each other in Figs. 19(b) and 20(b). Recalling Fig. 16(b), the normalized energy flux of the oscillator chain has a significant difference as it produces two different vibration states. The responses in the lower and higher branches are deemed as two vibration states. However, as the chain vibrates in one state, the normalized energy flux is almost independent of the variations of the input energy and the scale between cubic oscillators.

**4 Conclusions**

In the present study, the cross-scale nonlinear energy transfer of a harmonically excited chain with a linear oscillator and three purely cubic oscillators as the chaotic responses occur is investigated. The performed analyses reveal that the chaotic dynamics of the three cubic oscillators are dominated by the strongly nonlinear stiffness in the oscillator chain. Accordingly, the complexityification-averaging method and the least-square method are used to analyze the branches of response in the studied frequency band and to assist the search of the higher branches in the numerical study. The time histories of displacement and Lyapunov exponents of the system are used to determine the occurrence of chaotic responses. Moreover, the average energy and normalized energy flux are defined to evaluate the energy transfer in the oscillator chain.

The obtained results from the semi-analytical method are in good agreement with numerical results in terms of amplitude, but there are some deviations in lower- and higher branches in terms of stability. The semi-analytical results show that the two additional small-scale cubic oscillators lead to complex branches of response around the resonance frequency of the linear oscillator in the lower branch. Moreover, the numerical results show that the oscillator chain can generate steady-state chaotic responses with entirely different amplitudes under the same excitation frequency.

Based on the obtained average energy and normalized energy flux of the oscillator chain with chaotic dynamics, it is concluded that the energy transfer patterns are similar in the same branch of response but quite different in different branches. It is noted that the dynamics in the higher branch are along with a lower energy transfer than that in the lower branch. Meanwhile, the normalized energy flux is almost

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**Fig. 17** Large-to-small scale nonlinear energy transfer with $\Omega = 1$, zero initial conditions, and different forces. **a** average energies, **b** normalized energy fluxes.
**Fig. 18** Large-to-small scale nonlinear energy transfer with $\Omega = 0.95$, nonzero initial conditions and different forces.

- **a** average energies,
- **b** normalized energy fluxes.

**Table 1** Initial conditions for obtaining high branches in Fig. 18

| $f$   | Initial conditions |
|-------|--------------------|
| 0.014 | 0.569 -0.522 -0.765 -0.211 -0.105 0.489 -0.091 -0.255 |
| 0.016 | -0.693 -0.636 -0.203 -0.147 0.181 -0.533 -0.106 0.762 |
| 0.018 | 0.513 -0.268 0.118 -0.321 -0.038 0.311 -0.785 0.182 |
| 0.02  | 0.099 -0.165 0.492 0.209 0.283 -0.582 -0.308 0.423 |

**Fig. 19** Distributions of the **a** average energy and **b** normalized energy flux against the mass scale for different mass ratios with $\Omega = 1, f = 0.014$ and zero initial conditions.

**Fig. 20** Distributions of the **a** average energy and **b** normalized energy flux against the mass scale for different mass ratios with $\Omega = 0.95, f = 0.014$ and nonzero initial conditions.
dependent of the variations of the input energy and the scale between cubic oscillators.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Data availability** All data included in this study are available from the corresponding author on reasonable request.

**Appendix**

To obtain the first-order equation of the oscillator chain by the complexification-averaging method, the following transformations are introduced:

Let \( u_1 = x_1 \), \( u_k = x_{k-1} - x_k \), \( k = 2, 3, 4 \), and introduce the following transformations:

\[
\dot{u}_k + i\Omega u_k = \hat{\Delta} e^{i\alpha t}, \quad k = 1, ..., 4
\] (9)

Subsequently, these expressions can be obtained:

\[
u_k = \frac{\hat{\Delta} e^{i\alpha t} - \bar{\alpha} e^{-i\alpha t}}{2i\Omega}, \quad \dot{u}_k = \frac{\hat{\Delta} e^{i\alpha t} + \bar{\alpha} e^{-i\alpha t}}{2},
\]

\[
\ddot{u}_k = \frac{\hat{\Delta} e^{i\alpha t} + i\Omega e^{i\alpha t} \hat{\Delta} e^{-i\alpha t} + \bar{\alpha} e^{-i\alpha t} - i\Omega e^{-i\alpha t} \bar{\alpha} e^{-i\alpha t}}{2}.
\] (10)

Substituting expression (10) into Eq. (7) and retaining the slow flow parts results in the following expressions:

\[
\dot{\Omega}_1 + i\Omega \dot{\Omega}_1 + \mu \dot{\Omega}_1 + \frac{\hat{\Delta}_1}{i\Omega} + \lambda_1 \dot{\Omega}_2 - \frac{3k_1 \hat{\Delta}^2 \Omega_2}{4\Omega^2} = f, \quad (11a)
\]

\[
e_1 \left( \dot{\Omega}_1 + i\Omega \dot{\Omega}_1 - \hat{\Delta}_2 - i\Omega \dot{\Omega}_2 \right) - \lambda_1 \dot{\Omega}_2 + \frac{3k_1 \hat{\Delta}^2 \Omega_2}{4\Omega^2} + \lambda_2 \dot{\Omega}_3 - \frac{3k_2 \hat{\Delta}^2 \Omega_3}{4\Omega^2} = 0,
\] (11b)

\[
e_2 \left( \dot{\Omega}_1 + i\Omega \dot{\Omega}_1 - \hat{\Delta}_2 - i\Omega \dot{\Omega}_2 - \hat{\Delta}_3 - i\Omega \dot{\Omega}_3 \right) - \lambda_2 \dot{\Omega}_3 + \frac{3k_2 \hat{\Delta}^2 \Omega_3}{4\Omega^3} + \lambda_3 \dot{\Omega}_4 - \frac{3k_3 \hat{\Delta}^2 \Omega_4}{4\Omega^3} = 0,
\] (11c)

\[
e_3 \left( \dot{\Omega}_1 + i\Omega \dot{\Omega}_1 - \hat{\Delta}_2 - i\Omega \dot{\Omega}_2 - \hat{\Delta}_3 - i\Omega \dot{\Omega}_3 - \hat{\Delta}_4 - i\Omega \dot{\Omega}_4 \right) - \lambda_3 \dot{\Omega}_4 + \frac{3k_3 \hat{\Delta}^2 \Omega_4}{4\Omega^4} = 0.
\] (11d)

Introducing \( \dot{\Omega}_k = a_k + ib_k, k = 1, ..., 4 \) into Equation (11) yields:

\[
\hat{\Delta}_1 + i\Omega \hat{\Delta}_1 + \mu \hat{\Delta}_1 + \frac{\hat{\Delta}_1}{i\Omega} + \lambda_1 \hat{\Delta}_2 - \frac{3k_1 \hat{\Delta}^2 \hat{\Delta}_2}{4\Omega^2} + \lambda_2 \hat{\Delta}_3 - \frac{3k_2 \hat{\Delta}^2 \hat{\Delta}_3}{4\Omega^2} = f,
\] (12a)

\[
e_1 \left[ \hat{\Delta}_1 + i\Omega \hat{\Delta}_1 + \mu \hat{\Delta}_1 + \frac{\hat{\Delta}_1}{i\Omega} + \lambda_1 \hat{\Delta}_2 - \frac{3k_1 \hat{\Delta}^2 \hat{\Delta}_2}{4\Omega^2} + \lambda_2 \hat{\Delta}_3 - \frac{3k_2 \hat{\Delta}^2 \hat{\Delta}_3}{4\Omega^2} \right] - \lambda_1 \hat{\Delta}_2 + \frac{3k_1 (a_2^2 + b_2^2) (a_2 + ib_2)}{4\Omega}
\]

\[
+ \lambda_2 (a_3 + ib_3) - \frac{3k_2 (a_3^2 + b_3^2) (a_3 + ib_3)}{4\Omega} = 0.
\] (12b)
\[ e_2 \left[ a_i + ib_i + i\Omega (a_1 + ib_1) - a_2 - i\hat{b}_2 - i\Omega (a_2 + ib_2) \right] \\
- a_3 - i\hat{b}_3 - i\Omega (a_3 + ib_3) - \lambda_2 (a_3 + ib_3) \\
+ \frac{3i\kappa_2 (a_3^2 + b_3^2)(a_3 + ib_3)}{4\Omega^3} + \lambda_3 (a_4 + ib_4) \\
- \frac{3i\kappa_3 (a_4^2 + b_4^2)(a_4 + ib_4)}{4\Omega^3}, \] 

\[ (12c) \]

\[ e_3 \left[ a_i + ib_i + i\Omega (a_1 + ib_1) - a_2 - i\hat{b}_2 \\
- i\Omega (a_2 + ib_2) - a_3 - i\hat{b}_3 - i\Omega (a_3 + ib_3) \\
- a_4 - i\hat{b}_4 - i\Omega (a_4 + ib_4) \right] \\
- \lambda_3 (a_4 + ib_4) + \frac{3i\kappa_3 (a_4^2 + b_4^2)(a_4 + ib_4)}{4\Omega^3} = 0. \] 

\[ (12d) \]

Then, the real and imaginary parts of Equation (12) can be separated in the form below:

\[ \dot{a}_1 - \Omega b_1 + \mu a_1 + \frac{b_1}{\Omega} + \lambda_1 a_2 + \frac{3k_1 (a_2^2 + b_2^2)b_2}{4\Omega^3} = f, \] 

\[ (13a) \]

\[ \dot{b}_1 + \Omega a_1 + \mu b_1 - \frac{a_1}{\Omega} + \lambda_1 b_2 - \frac{3k_1 (a_2^2 + b_2^2)a_2}{4\Omega^3} = 0, \] 

\[ (13b) \]

\[ e_1 \left[ \dot{a}_1 - \Omega b_1 - a_2 + \Omega b_2 \right] - \lambda_1 a_2 - \frac{3k_1 (a_2^2 + b_2^2)b_2}{4\Omega^3} + \lambda_2 a_3 + \frac{3k_2 (a_3^2 + b_3^2)b_3}{4\Omega^3} = 0, \] 

\[ (13c) \]

\[ e_1 \left[ \dot{b}_1 + \Omega a_1 - \dot{b}_2 - \Omega a_2 \right] - \lambda_1 b_2 + \frac{3k_1 (a_2^2 + b_2^2)a_2}{4\Omega^3} + \lambda_2 b_3 - \frac{3k_2 (a_3^2 + b_3^2)a_3}{4\Omega^3} = 0, \] 

\[ (13d) \]

\[ e_2 \left[ \dot{a}_1 - \Omega b_1 - a_2 + \Omega b_2 - a_3 + \Omega b_3 \right] - \lambda_2 a_3 \\
- \frac{3k_2 (a_3^2 + b_3^2)b_3}{4\Omega^3} + \lambda_3 a_4 + \frac{3k_3 (a_4^2 + b_4^2)b_4}{4\Omega^3} = 0, \] 

\[ (13e) \]
\[ a_3 = \frac{\lambda_1 a_2}{\epsilon_1} + \frac{3k_1(a_2^2 + b_2^2)b_2}{4\Omega^3\epsilon_1} - \frac{\lambda_2 a_3}{\epsilon_1} - \frac{3k_2(a_1^2 + b_1^2)b_3}{4\Omega^3\epsilon_1} + \Omega b_3 - \frac{\lambda_3 a_3}{\epsilon_2} + \frac{3k_3(a_2^2 + b_2^2)a_4}{4\Omega^3\epsilon_2} \]

\[ b_3 = \frac{\lambda_1 b_2}{\epsilon_1} + \frac{3k_1(a_2^2 + b_2^2)a_2}{4\Omega^3\epsilon_1} - \frac{\lambda_2 b_3}{\epsilon_1} + \frac{3k_3(a_2^2 + b_2^2)a_3}{4\Omega^3\epsilon_2} - \frac{\Omega a_3}{\epsilon_2} + \frac{\lambda_3 b_4}{\epsilon_2} + \frac{3k_3(a_2^2 + b_2^2)a_4}{4\Omega^3\epsilon_2} \]

\[ a_4 = \frac{\lambda_2 a_3}{\epsilon_2} - \frac{\lambda_3 a_4}{\epsilon_2} + \frac{3k_2(a_1^2 + b_1^2)b_3}{4\Omega^3\epsilon_2} - \frac{3k_3(a_2^2 + b_2^2)b_4}{4\Omega^3\epsilon_2} + \Omega a_4 - \frac{\lambda_3 a_4}{\epsilon_3} - \frac{3k_3(a_2^2 + b_2^2)a_4}{4\Omega^3\epsilon_3} \]

\[ b_4 = \frac{\lambda_2 b_3}{\epsilon_2} - \frac{\lambda_3 b_4}{\epsilon_2} + \frac{3k_2(a_1^2 + b_1^2)a_3}{4\Omega^3\epsilon_2} - \frac{3k_3(a_2^2 + b_2^2)a_4}{4\Omega^3\epsilon_2} - \Omega a_4 - \frac{\lambda_3 b_4}{\epsilon_3} + \frac{3k_3(a_2^2 + b_2^2)a_4}{4\Omega^3\epsilon_3} \]

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