PRIMAL AND DUAL PREDICTION-CORRECTION METHODS FOR
TIME-VARYING CONVEX OPTIMIZATION *

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Abstract. We propose a unified framework for time-varying convex optimization based on the prediction-correction paradigm, both in the primal and dual spaces. In this framework, a continuously varying optimization problem is sampled at fixed intervals, and each problem is approximately solved with a primal or dual correction step. The solution method is warm-started with the output of a prediction step, which solves an approximation of a future problem using past information. Prediction approaches are studied and compared under different sets of assumptions. Examples of algorithms covered by this framework are time-varying versions of the gradient method, splitting methods, and the celebrated alternating direction method of multipliers (ADMM).

Key words. Time-varying, online, prediction-correction, convex optimization, operator theory

AMS subject classifications. 68W27, 65K10, 90C25, 90C30, 47N10

1. Introduction. Continuously varying optimization programs have appeared as a natural extension of time-invariant ones when the cost function, the constraints, or both, depend on a time parameter and change continuously in time. This setting captures relevant problems in the data streaming era, see e.g. [12, 48].

We focus here on two complementary time-varying convex programs; the first is of the form,

\[ \begin{align*}
\text{P}_1(t) : \quad & \mathbf{x}^*(t) = \arg \min_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x};t) + g(\mathbf{x};t) \\
\text{P}_2(t) : \quad & \mathbf{x}^*(t), \mathbf{y}^*(t) = \arg \min_{\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^m} f(\mathbf{x};t) + h(\mathbf{y};t), \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{c},
\end{align*} \]

where \( t \in \mathbb{R}_+ \) is non-negative, continuous, and is used to index time; \( f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \) is a smooth strongly convex function uniformly in time; \( g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\} \) is a closed convex and proper function. The second problem is linearly constrained and of the form,

\[ \begin{align*}
\text{P}_2(t) : \quad & \mathbf{x}^*(t), \mathbf{y}^*(t) = \arg \min_{\mathbf{x} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^m} f(\mathbf{x};t) + h(\mathbf{y};t), \quad \text{s.t.} \quad \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{y} = \mathbf{c},
\end{align*} \]

where, in addition, \( h : \mathbb{R}^m \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\} \) is a closed convex and proper function, matrices \( \mathbf{A} \in \mathbb{R}^{p \times n} \) and \( \mathbf{B} \in \mathbb{R}^{p \times m} \), and vector \( \mathbf{c} \in \mathbb{R}^p \). Solving any of the two problems means determining, at each time \( t \), the optimizers \( \mathbf{x}^*(t) \) or \( \mathbf{x}^*(t), \mathbf{y}^*(t) \), and therefore, computing the optimizers’ trajectory (i.e., the optimizers’ evolution in time), up to some arbitrary but fixed accuracy. We notice here that problems \( \text{P}_1(t), \text{P}_2(t) \) are not available a priori, but they are revealed as time evolves; e.g., problem \( \text{P}_1(t') \) will be revealed at \( t = t' \) and known for all \( t \geq t' \). A practical example of such situation is when \( f(\mathbf{x};t) = ||\mathbf{x} - \mathbf{b}(t)||^2 \) and \( \mathbf{b}(t) \) are streaming data that are revealed in time, as \( t \) evolves; see [12, 48] for other examples. In this context, we are interested in modeling how the problems \( \text{P}_1(t), \text{P}_2(t) \) evolve in time.

We will look at primal and dual first-order methods. Problem \( \text{P}_1(t) \) is the time-varying version of a composite optimization problem (i.e., of the form \( f + g \)) and we will consider primal first-order methods. Note here that \( g \) could be the indicator
function of a closed convex set, thereby enabling modeling constrained optimization problems varying with time. Problem (1.2) is the time-varying version of the alternating direction method of multipliers (ADMM) setting, and we will consider dual first-order methods. The idea is to present in a unified way a broad class of time-varying optimization algorithms that can tackle instances of problems (1.1)-(1.2). Further note that the two problems could be transformed into each other, if one so wishes, but we prefer to treat them separately to encompass both primal and dual methods.

The focus is on discrete-time settings as in [54, 15, 47]. In this context, we will use sampling arguments to reinterpret Equations (1.1) and (1.2) as a sequence of time-invariant problems. In particular, focusing here only on (1.1) for simplicity, upon sampling the objective functions \( f(x; t) + g(x; t) \) at time instants \( t_k, k = 0, 1, 2, \ldots \), where the sampling period \( T_s := t_k - t_{k-1} \) can be chosen arbitrarily small, one can solve the sequence of time-invariant problems

\[
P_1(t_k) : \quad x^*(t_k) = \arg \min_{x \in \mathbb{R}^n} f(x; t_k) + g(x; t_k), \quad k \in \mathbb{N}.
\]

By decreasing \( T_s \), an arbitrary accuracy may be achieved when approximating problem (1.1) with (1.3). In this context, we will hereafter assume that \( T_s \) is a small constant and \( T_s < 1 \). However, solving (1.3) for each sampling time \( t_k \) may not be computationally affordable in many application domains, even for moderate-size problems. We therefore consider here approximating the discretized optimizers’ trajectory \( \{x^*(t_k)\}_{k \in \mathbb{N}} \) by using first-order methods. In particular, we will focus on prediction-correction methods [54, 15, 47, 33]. This methodology arises from non-stationary optimization [31, 35], parametric programming [39, 22, 54, 15, 24, 27], and continuation methods in numerical mathematics [2].

This paper extends the current state-of-the-art methods, e.g., [47, 46], by offering the following contributions.

1. We provide a framework for prediction-correction methods in both primal space and dual space for time-varying optimization with time-varying cost and constraints. In doing so we re-obtain as special cases (for static constraints) existing prediction-correction time-varying algorithms, such as the ones based on (projected) gradient method [47], proximal point, forward-backward splitting, Peaceman-Rachford splitting [4], as well as the ones based on dual ascent [46]. Our framework allows us to derive new prediction-correction time-varying algorithms based on the method of multipliers, dual forward-backward splitting, and ADMM, and to generalize many of the methods appeared in the literature.

2. We unify the convergence results of the above mentioned methods with prediction-agnostic meta-results and improve some key inequalities to obtain sharper convergence bounds. In particular, in contrast with current literature, (i) the convergence and error bounds for prediction-correction methods do reduce to the case of correction-only methods, when the number of prediction steps is zero; (ii) full rankness of the matrix \( A \) is only required when needed; (iii) we are able to include time-varying convex constraints, both as indicator functions and in a regularized primal-dual fashion.

3. We look at different prediction strategies, in addition to the more used Taylor-expansion prediction, and in particular we show how an extrapolation-based strategy can offer global and tighter convergence, notably when the curvature of the problem does not change in time, which is relevant in online learning with data streams. This alternative strategy exploits only the knowledge of past costs to define the prediction, and does not require the computation of derivatives, which
instead are needed in Taylor-expansions. This is key for computational-light algorithms. Moreover, the accuracy of the prediction can be tuned by choosing the number of past functions used in the scheme.

The paper therefore presents a complete and comprehensive framework for primal and dual prediction-correction methods for a large class of time-varying optimization problems, and different choices of prediction functions.

Applications. To further motivate the need for a solid time-varying algorithmic framework, we cite here two significant applications, while the reader is referred to [48] for more. The first is the smart grid, where the high penetration of renewables and increasing frequency of operation, along with the size of the grid, make key energy optimization problems more and more difficult to be solved within the timeframe that is allocated for them. This has motivated the broadening research in optimal power flow pursuit [11] and related [23, 19]. The second is robotics and controls, where one would like to solve time-varying problems at high frequency to determine trajectories and control laws of mobile robots, that respond to external dynamic inputs. This has spurred keen interest in time-varying approaches to model predictive control [54, 3, 33], among others.

Organization. In section 2, we introduce the necessary background. In sections 3 to 5, we present the proposed prediction-correction framework, we analyze different prediction strategies, and give convergence guarantees. Section 6 describes the dual version of the proposed framework. Section 7 concludes with some numerical results.

2. Mathematical Background. In this section, we establish the notation used throughout the paper, and we review some useful definitions and results.

2.1. Notation. Vectors are written as $x \in \mathbb{R}^n$ and matrices as $A \in \mathbb{R}^{n \times n}$. We denote by $\lambda_M(A)$ and $\lambda_m(A)$ the largest and smallest eigenvalues of a square matrix $A \in \mathbb{R}^{n \times n}$. We use $\| \cdot \|$ to denote the Euclidean norm in the vector space, as well as the respective induced norms for matrices and tensors. In particular, given a matrix $A \in \mathbb{R}^{p \times n}$, we have $\| A \| = \sigma_M(A) = \sqrt{\lambda_M(A^\top A)}$, where $\sigma_M$ denotes the largest singular value. The gradient of a differentiable function $f(x; t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ with respect to $x$ at the point $(x, t)$ is denoted as $\nabla_x f(x; t) \in \mathbb{R}^n$, the partial derivative of the same function with respect to (w.r.t.) $t$ at $(x, t)$ is written as $\nabla_t f(x; t) \in \mathbb{R}$. Similarly, the notation $\nabla_{xx} f(x; t) \in \mathbb{R}^{n \times n}$ denotes the Hessian of $f(x; t)$ w.r.t. $x$ at $(x, t)$, whereas $\nabla_x f(x; t) = \nabla_{tt} f(x; t) \in \mathbb{R}^n$ denotes the partial derivative of the gradient of $f(x; t)$ w.r.t. the time $t$ at $(x, t)$. Higher order tensors $\nabla_{x \cdots x} f(x; t)$ are defined in a similar way. We indicate the inner product of vectors belonging to $\mathbb{R}^n$ as $\langle v, u \rangle := v^\top u$, for all $v \in \mathbb{R}^n, u \in \mathbb{R}^n$, where $(\cdot)^\top$ means transpose. We denote by $\circ$ the composition operation. We use $\{x^\ell\}_{\ell \in \mathbb{N}}$ to indicate sequences of vectors indexed by non-negative integers, for which we define linear convergence as follows (see [36] for details).

**Definition 2.1 (Linear convergence).** Let $\{x^\ell\}_{\ell \in \mathbb{N}}$ and $\{y^\ell\}_{\ell \in \mathbb{N}}$ be sequences in $\mathbb{R}^n$, and consider the points $x^*, y^* \in \mathbb{R}^n$. We say that $\{x^\ell\}_{\ell \in \mathbb{N}}$ converges Q-linearly to $x^*$ if there exists $\lambda \in (0, 1)$ such that: $\| x^{\ell+1} - x^* \| \leq \lambda \| x^\ell - x^* \|$, $\forall \ell \in \mathbb{N}$.

We say that $\{y^\ell\}_{\ell \in \mathbb{N}}$ converges R-linearly to $y^*$ if there exists a Q-linearly convergent sequence $\{x^\ell\}_{\ell \in \mathbb{N}}$ and $C > 0$ such that: $\| y^\ell - y^* \| \leq C \| x^\ell - x^* \|$, $\forall \ell \in \mathbb{N}$.

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1 E.g., the tensor $\nabla_{x,x} f(x; t) \in \mathbb{R}^{n \times n \times n \times n}$ indicates the third derivative of $f(x; t)$ w.r.t. $x$ at $(x, t)$, the matrix $\nabla_{u,x} f(x; t) = \nabla_{x,x} f(x; t) \in \mathbb{R}^{n \times n}$ indicates the time derivative of the Hessian of $f(x; t)$ at $(x, t)$, the vector $\nabla_{t,x} f(x; t) \in \mathbb{R}^n$ indicates the second derivative in time of the gradient of $f(x; t)$ at $(x, t)$. 
2.2. Convex analysis. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is \( \mu \)-strongly convex, \( \mu > 0 \), iff \( f(x) - \frac{\mu}{2} \|x\|^2 \) is convex. It is said to be \( L \)-smooth iff \( \nabla f(x) \) is \( L \)-Lipschitz continuous or, equivalently, iff \( f(x) - \frac{L}{2} \|x\|^2 \) is concave. We denote by \( S_{\mu,L}(\mathbb{R}^n) \) the class of twice differentiable, \( \mu \)-strongly convex, and \( L \)-smooth functions, and \( \kappa := L/\mu \) will denote the condition number of such functions. An extended real line function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is closed if its epigraph \( \text{epi}(f) = \{ (x,a) \in \mathbb{R}^{n+1} | x \in \text{dom}(f), f(x) \leq a \} \) is closed. It is proper if it does not attain \(-\infty\). We denote by \( \Gamma_0(\mathbb{R}^n) \) the class of closed, convex and proper functions \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \). Notice that functions in \( \Gamma_0(\mathbb{R}^n) \) need not be smooth. Given a function \( f \in \Gamma_0(\mathbb{R}^n) \), we define its convex conjugate as the function \( f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) such that \( f^*(w) = \sup_{x \in \mathbb{R}^n} \{ \langle w, x \rangle - f(x) \} \). The convex conjugate of a function \( f \in \Gamma_0(\mathbb{R}^n) \) belongs to \( \Gamma_0(\mathbb{R}^n) \) as well, and if \( f \in S_{\mu,L}(\mathbb{R}^n) \) then \( f^* \in S_{1/L,1/\mu}(\mathbb{R}^n) \), [41, Chapter 12.H].

The subdifferential of a convex function \( f \in \Gamma_0(\mathbb{R}^n) \) is defined as the set-valued operator \( \partial f : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) such that: \( x \rightharpoonup \{ z \in \mathbb{R}^n \mid \forall y \in \mathbb{R}^n : \langle y - x, z \rangle + f(x) \leq f(y) \} \), and we denote by \( \nabla f(x) \in \partial f(x) \) the subgradients. The subdifferential of a convex function is monotone, that is, for any \( x,y \in \mathbb{R}^n \): \( 0 \leq \langle x - y, u - v \rangle \) where \( u \in \partial g(x), v \in \partial g(y) \).

2.3. Operator theory. We briefly review some notions and results in operator theory, and we refer to [42, 6] for a thorough treatment.

**Definition 2.2.** An operator \( T : \mathbb{R}^n \to \mathbb{R}^n \) is:

- \( \lambda \)-Lipschitz, with \( \lambda > 0 \), iff \( \|T x - T y\| \leq \lambda \|x - y\| \) for any two \( x,y \in \mathbb{R}^n \); it is non-expansive iff \( \lambda \in (0,1] \) and \( \lambda \)-contractive iff \( \lambda \in (0,1) \);

- \( \beta \)-strongly monotone, with \( \beta > 0 \), iff \( \beta \|x - y\|^2 \leq \langle x - y, T x - T y \rangle \), for any two \( x,y \in \mathbb{R}^n \).

By using the Cauchy-Schwarz inequality, a \( \beta \)-strongly monotone operator can be shown to satisfy:

\[
\beta \|x - y\| \leq \|T x - T y\|, \quad \forall x,y \in \mathbb{R}^n.
\]

**Definition 2.3.** Let \( f \in \Gamma_0(\mathbb{R}^n) \) and let \( \rho > 0 \). The proximal operator of \( f \) with penalty parameter \( \rho \) is defined as: \( \text{prox}_{\rho f}(x) = \arg\min_{y \in \mathbb{R}^n} \{ f(y) + \frac{\rho}{2} \|y - x\|^2 \} \). The corresponding reflective operator is \( \text{refl}_{\rho f}(x) = 2 \text{prox}_{\rho f}(x) - x \).

**Definition 2.4.** Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be an operator, a point \( x^* \in \mathbb{R}^n \) is a fixed point for \( T \) iff \( x^* = Tx^* \).

By the Banach-Picard theorem [6, Theorem 1.51], contractive operators have a unique fixed point.

2.4. Operator theory for convex optimization. Operator theory can be employed to solve convex optimization problems; the main idea is to translate a minimization problem into the problem of finding the fixed points of a suitable operator.

Let \( f \in S_{\mu,L}(\mathbb{R}^n) \) and \( g \in \Gamma_0(\mathbb{R}^n) \) and consider the optimization problem

\[
(2.2) \quad x^* = \arg\min_{x \in \mathbb{R}^n} \{ f(x) + g(x) \}.
\]

Let \( T : \mathbb{R}^p \to \mathbb{R}^p \) and \( X : \mathbb{R}^p \to \mathbb{R}^n \) be two operators. Let \( T \) be \( \lambda \)-contractive and such that its fixed point \( z^* \) yields the solution to (2.2) through the operator \( x^* = Xz^* \). Let the operator \( X \) be \( \chi \)-Lipschitz.

We employ then the Banach-Picard fixed point algorithm, defined as the update:

\[
(2.3) \quad z^{\ell+1} = Tz^{\ell}, \quad \ell \in \mathbb{N}.
\]
By the contractiveness of $\mathcal{T}$, the Q-linear convergence to the fixed point is guaranteed [6, Theorem 1.51]:

$$
\|z^{\ell+1} - z^*\| \leq \lambda \|z^\ell - z^*\| \leq \lambda^{\ell+1} \|z^0 - z^*\|,
$$

as well as R-linear convergence of $\{x^\ell\}_{\ell \in \mathbb{N}}$ obtained through $x^\ell = \mathcal{X} z^\ell$ as

$$
\|x^{\ell+1} - x^*\| \leq \chi \lambda^{\ell+1} \|z^0 - z^*\|.
$$

The following lemma further characterizes the convergence in terms of $x$.

**Lemma 2.5.** Let $\mathcal{X}$ be $\beta$-strongly monotone. Then, convergence of the sequence $\{x^\ell\}_{\ell \in \mathbb{N}}$ with $x^\ell = \mathcal{X} z^\ell$ is characterized by the following inequalities:

$$
\|x^\ell - x^*\| \leq \zeta(\ell) \|x^0 - x^*\|, \quad \|x^\ell - x^0\| \leq \xi(\ell) \|x^0 - x^*\|
$$

where

$$
\zeta(\ell) := \begin{cases} 1, & \text{for } \ell = 0, \\ \frac{\beta}{\lambda} \lambda^\ell, & \text{otherwise} \end{cases} \quad \text{and} \quad \xi(\ell) := \begin{cases} 0, & \text{for } \ell = 0, \\ 1 + \frac{\beta}{\lambda} \lambda^\ell, & \text{otherwise} \end{cases}.
$$

**Proof.** In the case in which $\ell = 0$, then we have $\|x^0 - x^*\| = \|x^0 - x^0\|$ and $\|x^\ell - x^0\| = \|x^0 - x^0\| = 0$, which give the first cases in the definitions of $\zeta(\ell)$ and $\xi(\ell)$. If $\ell > 0$, then by $\beta$-strong monotonicity of $\mathcal{X}$, Eq. (2.1), we have

$$
\|x^\ell - z^*\| \leq \frac{1}{\beta} \|x^\ell - x^0\| \leq \frac{1}{\beta} \|x^0 - x^*\|.
$$

Combining (2.5) with (2.8) then yields the second case in the definition of $\zeta(\ell)$. Moreover, using the triangle inequality we have:

$$
\|x^\ell - x^0\| \leq \|x^\ell - z^*\| + \|z^* - x^0\|,
$$

and the second case in the definition of $\xi(\ell)$ follows by (2.5) and (2.8).

Examples of $\mathcal{T}$ and $\mathcal{X}$ operators for primal methods (i.e., gradient method, proximal point algorithm forward-backward splitting (FBS), and Peaceman-Rachford splitting) as well for dual methods (i.e., dual ascent, the method of multipliers, dual FBS, and the Alternating direction method of multipliers (ADMM)) are provided in Appendix A, together with their contraction rates.

We now give a formal definition of operator theoretical solver, which will be needed for our developments.

**Definition 2.6 (Operator theoretical solver).** Let $\mathcal{T} : \mathbb{R}^p \to \mathbb{R}^p$ and $\mathcal{X} : \mathbb{R}^p \to \mathbb{R}^n$ be two operators, respectively, $\lambda$-contractive for $\mathcal{T}$ and $\lambda$-Lipschitz and $\beta$-strongly monotone for $\mathcal{X}$, such that the solution $x^*$ of (2.2) can be computed as $x^* = \mathcal{X} z^*$ with $z^*$ being the fixed point of $\mathcal{T}$. Suppose that a recursive method, e.g. the Banach-Picard in (2.3), is available to compute the fixed point $z^*$. Then we call this recursive method an operator theoretical solver for problem (2.2), and call each recursive update of the method a step of the solver. We also use the short-hand notation $\mathcal{O}(\lambda, \chi, \beta)$ to indicate such a solver, for which the contraction rates in Lemma 2.5 are valid.

### 3. Prediction-Correction Framework.

We start by describing in this section the proposed prediction-correction framework, referring to problem (1.3):

$$
x_k^* = \arg \min_{x \in \mathbb{R}^n} \{f(x; t_k) + g(x; t_k)\}, \quad k \in \mathbb{N}
$$

where hereafter $x_k^* := x^*(t_k)$. As said, problem (3.1) can model a wide range of both constrained and unconstrained optimization problems, in which a smooth term $f$ is
(possibly) summed to a non-smooth term \( g \). For example, we may have that \( g \) is the indicator function of a constraint set, or a non-smooth function promoting some structural properties (such as an \( \ell_1 \) norm enforcing sparsity). We remark that \( g \) may actually be a static function, or that it may be absent. In some cases, \( g \) could be itself a composite function \( g' + g'' \), which yields a 3-composite problem [13].

### 3.1. Assumptions.

**Assumption 3.1.** (i) The cost function \( f: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \) belongs to \( \mathcal{S}_{\mu,L}(\mathbb{R}^n) \) uniformly in \( t \). (ii) The function \( g: \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\} \) either belongs to \( \mathcal{I}_0(\mathbb{R}^n) \) uniformly in \( t \), or \( g(\cdot; t_k) \equiv 0 \) for any \( k \in \mathbb{N} \). (iii) The solution to (3.1) is finite for any \( k \in \mathbb{N} \).

Assumption 3.1(ii) guarantees that problem (3.1) is strongly convex and has a unique solution for each time instance. Uniqueness of the solution implies that the solution trajectory is also unique. **Assumption 3.1(iii)** guarantees that the solution does not diverge over time, excluding for example the case \( f(x; t) = (x - t)^2 \).

**Assumption 3.2.** The gradient of function \( f \) has bounded time derivative, that is, there exists \( C_0 > 0 \) such that \( \|\nabla f(x; t)\| \leq C_0 \) for any \( x \in \mathbb{R}^n \), \( t \in \mathbb{R}_+ \).

**Assumption 3.3.** The cost function \( f \) is (at least) three times differentiable, and there exist \( C_1, C_2, C_3 > 0 \) such that, for any \( x \in \mathbb{R}^n \), \( t \in \mathbb{R}_+ \):

\[
\|\nabla^3 f(x; t)\| \leq C_1, \quad \|\nabla^2 f(x; t)\| \leq C_2, \quad \|\nabla f(x; t)\| \leq C_3.
\]

By imposing Assumption 3.2 we ensure that the solution trajectory is Lipschitz in time, as we will see, and therefore prediction-type methods would work well.

Assumption 3.3 on the bound of the third derivative tensor \( \nabla^3 f(x; t) \) is typically required for the analysis of Newton-type algorithms. Even though Assumption 3.3 is not strictly needed to prove convergence of the proposed algorithm (as we will show), it will be useful to derive more refined results. Assumption 3.3 (or similar ones) is also typical in time-varying optimization literature, see e.g. [15, 49, 33].

**Assumption 3.4.** There exist a finite constant \( D_0 \geq 0 \) such that, for any \( k \in \mathbb{N} \), it holds that \( \|\nabla g(x^*_k; t_k)\| - \|\nabla g(x^*_k; t_k)\| \leq D_0 \), with \( \nabla g(x^*_k; t_j) \in \partial g(x^*_k; t_j) \), \( j = k, k+1 \).

First of all, Assumption 3.4 is verified with \( D_0 = 0 \) if \( g \) is static. For a time-varying, non-smooth \( g \), this assumption may be easy to verify (e.g. for an \( \ell_1 \) norm) or it may be more restrictive (e.g. for indicator functions). What it imposes in the latter case is that the sampling period is small enough such that there is a “feasibility overlap”, in the sense that \( x_k^* \) is not only feasible at time \( t_k \) but also at time \( t_{k+1} \). This is a consequence of the fact that the subdifferentials of \( g_k \) and \( g_{k+1} \) need to be defined at \( x_k^* \). In other words, this assumption may become restrictive when the subdifferential of the non-smooth cost has a time-varying domain, in which case we require that there be sufficient overlap in subsequent domains.

### 3.2. Framework.

Suppose that an operator theoretical solver for problem (3.1) is available. The prediction-correction scheme is characterized by the following two steps:

- **Prediction:** at time \( t_k \), we approximate the as yet unobserved costs \( f(x; t_{k+1}) \), \( g(x; t_{k+1}) \) using the past observations; let \( \hat{f}_{k+1}(x) \), \( \hat{g}_{k+1}(x) \) be such approximations, then we solve the problem

\[
\hat{x}_{k+1}^* = \arg \min_{x \in \mathbb{R}^n} \left\{ \hat{f}_{k+1}(x) + \hat{g}_{k+1}(x) \right\}
\]


with initial condition \( x_k \), which yields the prediction \( \hat{x}_{k+1}^* \). In practice, it is possible to compute only an approximation of \( \hat{x}_{k+1}^* \), denoted by \( \hat{x}_{k+1} \), by applying \( N_P \) steps of the solver.

- **Correction:** when, at time \( t_{k+1} \), the costs \( f_{k+1}(x) := f(x; t_{k+1}) \), \( g_{k+1}(x) := g(x; t_{k+1}) \) are made available, we can correct the prediction computed at the previous step by solving:

\[
(3.3) \quad x_{k+1}^* = \arg\min_{x \in \mathbb{R}^n} \{ f_{k+1}(x) + g_{k+1}(x) \}
\]

with initial condition equal to \( \hat{x}_{k+1} \). We will denote by \( x_{k+1} \) the (possibly approximate) correction computed by applying \( N_C \) steps of the solver.

The prediction step relies on approximating the future cost functions using the information available up to time \( t_k \). In general, we allow for any choice of predicted function such that the following assumption is verified.

**Assumption 3.5.** (i) The predicted functions \( \hat{f}_{k+1} \) and \( \hat{g}_{k+1} \) belong to \( S_{\mu,L}(\mathbb{R}^n) \) and \( \Gamma_0(\mathbb{R}^n) \), respectively, and the solution to the prediction problem (3.2) is finite. In addition, (ii) for the solution of the prediction problem \( \hat{x}_{k+1}^* \), one has that \( \| \tilde{\nabla}_x g_{k+1}(\hat{x}_{k+1}^*) - \tilde{\nabla}_x \hat{g}_{k+1}(\hat{x}_{k+1}) \| \leq D_0 \), with \( \tilde{\nabla}_x g_{k+1}(\hat{x}_{k+1}^*) \in \partial g_{k+1}(\hat{x}_{k+1}^*) \), and \( \tilde{\nabla}_x \hat{g}_{k+1}(\hat{x}_{k+1}^*) \in \partial \hat{g}_{k+1}(\hat{x}_{k+1}^*) \).

Assumption 3.5 therefore imposes that the predicted problem be in the same class as the actual problem, although the respective minimizers may of course differ. Thus the same solver can be applied to both the correction and prediction steps, and its convergence properties are the same. Assumption 3.5(ii) also imposes that we have a bound on the approximation quality of the non-smooth part, which we have put as \( D_0 \), since it enters in the proofs in the same way of Assumption 3.4. We remark that in general Assumption 3.4 and Assumption 3.5(ii) may feature two different, unrelated bounds to the difference of subgradients, which we omit for simplicity.

Section 4 will present different possible prediction methods, both for the smooth and non-smooth terms in the cost of problem (3.1).

Figure 1 depicts the flow of the prediction-correction scheme, in which information observed up to time \( t_k \) is used to compute the prediction \( \hat{x}_{k+1} \). In turn, the prediction serves as a warm-starting condition for the correction problem, characterized by the cost observed at time \( t_{k+1} \).

**Algorithm 3.1** reports the proposed prediction-correction scheme. The parameters that need to be chosen are the prediction and correction horizons \( N_P \), \( N_C \), the operator theoretical solver \( O(\lambda, \beta, \chi) \) and its parameters (e.g. the step-size), see Appendix A, and a prediction method. At time \( t_k \), the algorithm generates the prediction function and applies \( N_P \) steps of the solver to it, using as a warm-starting condition the solution to the last correction problem. Afterwards, the actual problem at time \( t_{k+1} \) is sampled and the algorithm applies \( N_C \) steps of the solver to it, using the prediction as an initial condition.

3.3. **Solvers.** As described above, the proposed framework requires that an operator theoretical solver for the prediction and correction steps is available. In par-
Algorithm 3.1 Primal prediction-correction method template

Require: $x_0$, horizons $N_P$ and $N_C$, specify an operator theoretical solver $O(\lambda, \chi, \beta)$ (e.g. gradient, proximal point, FBS, or PRS) and its parameters, and a prediction method.
1: for $k = 0, 1, \ldots$ do
   // time $t_k$ (prediction)
2: Generate the function $f_{k+1}(x)$ and $g_{k+1}(x)$ using the available information up to time $t_k$
3: Apply $N_P$ steps of the solver to problem (3.2) with initial condition $x_k$
   // time $t_{k+1}$ (correction)
4: Observe the cost functions $f_{k+1}(x)$ and $g_{k+1}(x)$
5: Apply $N_C$ steps of the solver to problem (3.3) with initial condition the prediction $\hat{x}_{k+1}$
6: Set $x_{k+1}$ equal to the last iterate of the method
7: end for

In particular, there are $\lambda$-contractive operators $\hat{T}_{k+1}, \hat{\mathcal{R}}_{k+1} : \mathbb{R}^p \to \mathbb{R}^p$ with fixed points $\hat{z}_{k+1}^*, z_{k+1}^*$, and $\chi$-Lipschitz, $\beta$-strongly monotone operators $\hat{X}_{k+1}, X_{k+1} : \mathbb{R}^p \to \mathbb{R}^n$, such that

\[ \hat{x}_{k+1}^* = \hat{X}_{k+1} z_{k+1}^* \quad \text{and} \quad x_{k+1}^* = X_{k+1} z_{k+1}^*. \]

For simplicity, we assume that the convergence rate of the prediction and correction solvers are the same, and we denote them by $O(\lambda, \chi, \beta)$. Therefore the contraction functions $\zeta$ and $\xi$ in Lemma 2.5 are the same in both cases. As reviewed in subsection 2.4, the contractiveness and strong monotonicity are consequences of the strong convexity and smoothness of the problem at hand, and thus they will hold throughout the remainder of this paper.

There is a broad range of solvers that can be used within the proposed framework, depending on the structure of problem (3.1). For example, if $g \equiv 0$, then gradient method and proximal point algorithms are suitable solvers, while if $g \neq 0$ then forward-backward$^2$ and Peaceman-Rachford splitting can be used. If the cost functions $f$ and $g$ are separable because they represent the sum of decentralized costs, then distributed algorithms can be employed as solvers, see e.g. [1, 51]. More details are available in Appendix A.

Remark 3.6. Assuming that the prediction and correction solvers have the same properties does not imply that they are the same solver. Indeed, assume we had two solvers with $O'(\lambda', \chi', \beta')$ and $O''(\lambda', \chi'', \beta'')$, then to satisfy the assumption we simply use $O(\max\{\lambda', \lambda''\}, \max\{\chi', \chi''\}, \min\{\beta', \beta''\})$. The possibility of employing different solvers may be useful when, due to the chosen prediction method, the prediction problem is more efficiently solved with a different operator than the correction problem.

3.4. On constrained online optimization. The proposed framework can also handle (explicit) constrained optimization problems in the form

\[ \min_{x \in \mathbb{R}^n} f(x; t) \quad \text{s.t.} \quad c(x; t) \leq 0, \]

where $f(\cdot; t) \in S_{\mu, \ell}(\mathbb{R}^n)$ and the constraint $c(\cdot; t) : \mathbb{R}^n \to \mathbb{R}^m$ is convex and differentiable w.r.t. $x$.

Define the Lagrangian of the problem as $\mathcal{L}(x, w; t) = f(x; t) + \langle w, c(x; t) \rangle - \epsilon \|w\|_2^2 / 2$, with the constraint $w \geq 0$, and where we added a dual regularization term with $0 < \epsilon \ll 1$. The Tikhonov regularization ensures that the Lagrangian is strongly

\(^2\)Also called proximal gradient method.
concave in $w$, and does not significantly alter the solutions if the weight $\epsilon$ is very small. Indeed, one may argue that since we are interested in an online setup, a small additional error due to regularization does not significantly impact the asymptotic behavior, see [25, 5, 53].

Since $L$ is strongly convex-strongly concave and differentiable in $x$ and $w$, to find a saddle point we can then apply a projected gradient descent-ascent on the Lagrangian, in which we project $w$ onto $\mathbb{R}^m_+$. For sufficiently small stepsize $\alpha$, the resulting primal-dual gradient descent-ascent is a contractive solver in $[x^\top, w^\top]^\top$ and fits in the proposed framework.

Alternative approaches to designing primal-dual algorithms for time-varying optimization have been proposed for example in [54, 26, 29, 10, 9, 55, 43], mainly as correction-only or prediction-only methods. Closer in spirit to our methodology are [54, 26], both however in the non-convex domain and with results around the optimizer trajectory. In [54], the authors propose a correction-only method based on a quadratic approximation around the optimizer trajectory (see their Eq. (5.3)), while in [26], a prediction-correction is proposed, also employing quadratic approximations around the solution trajectory (see their Eqs (3.1)-(3.2)).

Finally, some works analyze optimization problems that have time-varying cost functions but static constraints (possibly non-smooth ones), for example [28, 47, 52, 7]. The proposed framework naturally includes this class of problems, in which case Assumption 3.4 becomes trivially satisfied with $D_0 = 0$.

4. Prediction Methods. This section will explore different approaches to prediction, which, depending on the properties of the cost, yield different degrees of approximation accuracy. The first, described in subsection 4.1, is the most general, and it can be applied also to non-smooth costs. The following ones, described in subsections 4.2 and 4.3, can instead be applied only to smooth costs, but achieve better precision.

The accuracy of the proposed prediction methods will be quantified in terms of the bound $\|\hat{x}^*_k - x^*_k\|$, where we remind that $\hat{x}^*_k$ and $x^*_k$ represent the optimal solutions to the prediction and correction problems, respectively.

4.1. One-step-back prediction. The most straightforward prediction method is the choice

$$f_{k+1}(x) = f_k(x), \quad g_{k+1}(x) = g_k(x)$$

which simply employs the last observed cost as a prediction of the next. This choice is suitable also for non-smooth costs, for example indicator functions, and does not require any knowledge of how the cost varies over time, except for the assumption that the costs at two subsequent times are sufficiently similar. Indeed, in the following, we assume that the (generally) non-smooth costs $g$ are always predicted with a one-step-back prediction. We remark that using this strategy guarantees that the predicted problem inherits the same properties of the correction one.

This one-step-back strategy is implicitly employed in online learning, in which the decision $x_{k+1}$ to be applied at time $t_{k+1}$ is computed solving the problem observed at time $t_k$. After the decision has been computed, the next problem is made available and the solver incurs a regret.

Consider problem (3.1), and suppose that we employ a one-step-back prediction for both terms in the cost function. It follows that the prediction error satisfies $\|\hat{x}^*_{k+1} - x^*_{k+1}\| = \|x^*_k - x^*_{k+1}\|$, which can be bounded using the following lemma (which will be instrumental to derive the subsequent results as well).
Lemma 4.1. Let Assumptions 3.1, 3.2 and 3.4 hold, then the distance between the optimizers of problems (3.1) at \( t_k \) and \( t_{k+1} \) is bounded by:

\[
\|\hat{x}_{k+1}^* - \hat{x}_k^*\| \leq (C_0 T_s + D_0)/\mu.
\]

Proof. See Appendix B.2. \( \square \)

4.2. Taylor expansion-based prediction. We now turn our attention to the smooth term \( f \) in problem (3.1), and present a prediction method under Assumptions 3.1 and 3.2.

The idea is to use the 2nd order Taylor expansion of \( f(x; t) \) centered in \((x; t_k)\) as a prediction. In particular, we choose:

\[
\begin{align*}
\hat{f}_{k+1}(x) &= f_k(x_k) + \langle \nabla f_k(x_k), x - x_k \rangle + T_k \nabla^2 f_k(x_k) \left( \frac{T^2}{2} \nabla^2 f_k(x_k) \right) + T_k \nabla \nabla f_k(x_k) + \frac{1}{2}(x - x_k)^\top \nabla^2 f_k(x_k)(x - x_k).
\end{align*}
\]

This is a quadratic approximation of the cost, with Hessian \( \nabla^2 f_k(x_k) \), which implies that the prediction verifies \( \hat{g}_{k+1}(x) = g_k(x) \), then Assumption 3.5(i) holds.

We remark that, although the Taylor-based prediction requires the computation of second order derivatives, it does so only once, at the beginning of each prediction phase. Moreover, we use this second order information to construct a quadratic approximation of the cost, but we do not require the inversion of the Hessian. This is different from second order methods, proposed in a time-varying context in e.g. [15, 49], which evaluate the Hessian and compute its inverse at each step.

Lemma 4.2. Let Assumptions 3.1, 3.2, 3.4 and 3.5(ii) hold. Using the Taylor-based prediction (4.3) for \( f \) and the one-step-back prediction (4.1) for \( g \) then yields the following error:

\[
\|\hat{x}_{k+1}^* - x_{k+1}^*\| \leq \frac{2L}{\mu} \|x_k - x_k^*\| + 2 \frac{C_0 T_s + D_0}{\mu} \left( 1 + \frac{L}{\mu} \right).
\]

Moreover, if Assumption 3.3 holds as well, then a time quadratic bound can be derived:

\[
\|\hat{x}_{k+1}^* - x_{k+1}^*\| \leq \frac{C_4}{2\mu} \|x_k - x_k^*\|^2 + C_4 \|x_k - x_k^*\| + C_5 T_s^2/2 + C_6 D_0,
\]

where \( C_4 := T_s(C_0 C_1/\mu^2 + C_2/\mu) + C_1 D_0/\mu^2, C_5 := C_0^2 C_1/\mu^3 + 2 C_0 C_2/\mu^2 + C_3/\mu, \) and \( C_6 := T_s(C_0 C_1/\mu + C_2)/\mu^2 + (1 + C_1 D_0/2\mu^2)/\mu. \)

Proof. See Appendix B.3. \( \square \)
Remark 4.3. The Taylor expansion-based prediction of (4.3) yields a quadratic function, and this fact suggests that we may apply different solvers for the prediction and correction steps, cf. Remark 3.6. Consider for example a composite problem which can be solved by either FBS or PRS, with the PRS requiring the computation of the proximal of the smooth term \( f_k \) (or \( \hat{f}_{k+1} \) for the prediction). Since the prediction is quadratic, there is a closed form for its proximal and we can apply PRS, which can achieve faster convergence than FBS. On the other hand, the correction problem may be highly non-linear and we may not be able to compute the proximal of \( f_k \), thus using FBS would be best.

4.2.1. Approximate time-derivative. In order to compute the Taylor approximation (4.3), knowledge of the derivative \( \nabla_t x f_k \) is required. However, in general it may not be available in closed form, and approximations need to be combined with the proposed scheme.

A simple and effective choice is the backward finite difference [37]:

\[
\hat{\nabla}_t x f_k(x) = \frac{(\nabla_x f_k(x) - \nabla_x f_{k-1}(x))}{T_s}.
\]

The following result shows how Lemma 4.2 is adapted when one employs approximate time-derivatives.

**Lemma 4.4.** Let Assumptions 3.1, 3.2, 3.4 and 3.5(ii) hold. Using the Taylor expansion-based prediction for \( f \) with approximate \( \hat{\nabla}_t x f_k \), and the one-step-back prediction for \( g \) then yields the same error bound (4.3). Moreover, if Assumption 3.3 holds as well, then the time quadratic bound (4.5) holds with \( C'_5 = C_5 + C_3/\mu \) substituting \( C_5 \).

**Proof.** See Appendix B.4.

4.3. Extrapolation-based prediction. We now look at extrapolation-based prediction. First of all we briefly review a numerical technique for polynomial interpolation [37], which we then leverage to design a novel prediction strategy.

4.3.1. Polynomial interpolation. Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a function that we want to interpolate from the pairs \( \{(t_i, \varphi_i)\}^I_{i=1} \) where \( \varphi_i := \varphi(t_i) \), and with \( t_i \neq t_j \) for any \( i \neq j \). The interpolated function is then defined as [37, Theorem 8.1]:

\[
\hat{\varphi}(t) = \sum_{i=1}^I \varphi_i \ell_i(t), \quad \text{with} \quad \ell_i(t) = \prod_{1 \leq j \leq I, j \neq i} \frac{t - t_j}{t_i - t_j}.
\]

The interpolation error can be characterized by [37, Theorem 8.2]:

\[
\varphi(t) - \hat{\varphi}(t) = \frac{\varphi^{(I)}(v)}{I!} \omega_I(t) \quad \text{with} \quad \omega_I(t) = \prod_{i=1}^I (t - t_i)
\]

and where \( v \) is a scalar in the smallest interval that contains \( t \) and \( \{t_i\}^I_{i=1} \).

Since in the following we are interested in evaluating the interpolated function at a point \( t \) that lies outside the interval \([t_1, t_I]\), we will refer to the resulting function as extrapolation.

4.3.2. Extrapolation-based prediction. Let us now apply the polynomial interpolation technique (4.7) to the function \( f(x; t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \) w.r.t. the scalar variable \( t \in \mathbb{R}_+ \). In particular, we compute the predicted function \( \hat{f}_{k+1} \) from the set
of past functions \( \{ f_i(x) \}_{i=1}^{k-1} \). Since the sampling times are multiples of \( T_n \), it is easy to see that the coefficients in (4.7) become:

\[
\ell_i := \ell_i(t_{k+1}) = (-1)^{i-1} \binom{I}{i},
\]

and the prediction is thus given by

\[
\hat{f}_{k+1}(x) = \sum_{i=1}^{I} \ell_i f_{k+1-i}(x), \quad \forall x \in \mathbb{R}^n.
\]

In general, however, the predicted cost \( \hat{f}_{k+1} \) may not be strongly convex – as a matter of fact, it can even fail to be convex; in the following we assume that \( C_2 = 0 \) in Assumption 3.3, which implies \( \| \nabla_{xx} f(x; t) \| = 0 \), and thus that \( f_{k+1} \in \mathcal{S}_{\mu, L}(\mathbb{R}^n) \) (and since we will take \( g_{k+1}(x) = g_k(x) \), then Assumption 3.5(ii) holds).

Indeed, if \( C_2 = 0 \) then \( \nabla_{xx} f(x; t) = \nabla_{xx} f(x) \) for any \( (x; t) \in \mathbb{R}^n \times \mathbb{R}_+ \), and the Hessian of \( \hat{f}_{k+1} \) is \( \nabla_{xx} \hat{f}_{k+1}(x) = (\sum_{i=1}^{I} \ell_i) \nabla_{xx} f(x) \). Using the fact that \( \sum_{i=1}^{I} \ell_i = 1 \), we see that \( \nabla_{xx} \hat{f}_{k+1}(x) = \nabla_{xx} f(x) \), and so \( \hat{f}_{k+1} \) inherits the same strong convexity and smoothness properties of the cost. By imposing \( C_2 = 0 \), we are here excluding functions whose curvature change over time, but we are not restricting ourselves to quadratic functions alone. We further note that problems in which the curvature does not change are ubiquitous in signal processing and machine learning with data streams, see [12], for instance when considering online regularized least-squares with cost \( \| Ax - b(t) \|^2 + \lambda \| x \|_1 \).

With this choice of prediction function, the error \( \| \hat{x}_{k+1}^* - x_{k+1}^* \| \) can be bounded as follows.

**Lemma 4.5.** Let Assumptions 3.1 and 3.3 as well as Assumption 3.5(ii) hold and let \( C_2 = 0 \). Choose the extrapolation order \( I \in \mathbb{N} \), \( I \geq 2 \), and assume there exists \( C(I) > 0 \) such that:

\[
\left\| \frac{\partial(I)}{\partial(I)} \nabla_x f(x_{k+1}^*; \tau) \right\| \leq C(I), \quad \tau \in [t_{k+1-I}, t_{k+1}].
\]

Using the extrapolation-based prediction (4.9) of order \( I \) for \( f \) and the one-step-back prediction (4.1) for \( g \) then yields the following error:

\[
\| \hat{x}_{k+1}^* - x_{k+1}^* \| \leq (C(I)T_s^I + D_0)/\mu.
\]

**Proof.** See Appendix B.5.

**Prediction comparison.** We conclude with a comparison of the prediction methods proposed in this section in terms of the information and the assumptions that they require, summarized in Table 1. We remark that the extrapolation approach does not require access to the derivatives (neither first nor second order) of the cost, which is instead necessary for the Taylor expansion. This is a favorable aspect of extrapolation in applications where computing derivatives is very expensive. Moreover, the prediction accuracy can be tuned by choosing the extrapolation order \( I \), with higher orders leading to better accuracy (recalling that \( T_s < 1 \) in (4.11)).

**5. Convergence Analysis.** We are now ready to present our main convergence results. We will start with meta-results that are prediction-agnostic, meaning that could be applied to any prediction strategies once their error bound is available; then we specify these meta-results for the prediction mechanism we have defined in the previous sections, along with their error bounds.
Table 1: Comparison of the different prediction methods.

| Method                  | Assumptions         | Information                                      |
|-------------------------|---------------------|-------------------------------------------------|
| One-step-back           | As. 3.2 and 3.4     | \(f_k, g_k\)                                    |
| Taylor expansion        | As. 3.1, 3.2, 3.4, 3.5(ii), optionally As. 3.3 | \(f_k (\nabla_x x f_k, \nabla_t x f_k); g_k\) |
| Extrapolation           | As. 3.1, 3.3 with \(C_2 = 0, 3.5(ii), (4.10)\) | \(\{f_{k+1-i}\}_{i=1}^1; g_k\)                  |

5.1. General convergence results. In this section, we present some general convergence results which are independent of the particular prediction method that is employed. The main result is as follows.

**Proposition 5.1 (General error bound).** Let Assumptions 3.1 and 3.5 hold, and let \(\sigma_k, \tau_k \in (0, +\infty)\) be such that for any \(k \in \mathbb{N}\):

\[
\|x_{k+1}^* - x_k^*\| \leq \sigma_k \quad \text{and} \quad \|\hat{x}_{k+1}^* - x_{k+1}^*\| \leq \tau_k.
\]

Choose the prediction and correction horizons \(N_C\) and \(N_P\) such that \(\zeta(N_C)\zeta(N_P) < 1\). Then the error incurred by a prediction-correction method that uses the solver \(O(\lambda, \chi, \beta)\) is upper bounded by:

\[
\|x_{k+1} - x_{k+1}^*\| \leq \zeta(N_C) \left(\zeta(N_P) \|x_k - x_k^*\| + \zeta(N_P)\sigma_k + \xi(N_P)\tau_k\right),
\]

with functions \(\zeta\) and \(\xi\) defined in Lemma 2.5.

**Proof.** See Appendix C.1.

Iterating (5.2) in Proposition 5.1 yields the bound

\[
\|x_k - x_k^*\| \leq \left(\zeta(N_C)\zeta(N_P)\right)^k \|x_0 - x_0^*\| + \sum_{j=0}^{k-1} \left(\zeta(N_C)\zeta(N_P)\right)^{k-j-1} \zeta(N_C) \left(\zeta(N_P)\sigma_j + \xi(N_P)\tau_j\right).
\]

Taking the limit of (5.3) for \(k \to \infty\), we can then see that the first term converges to zero (and so, asymptotically the initial condition does not influence the error). However, the second term in (5.3) does not converge to zero, which implies that there may be an asymptotical tracking error.

In particular, let \(\sigma := \sup_{k \in \mathbb{N}} \sigma_k\) and \(\tau := \sup_{k \in \mathbb{N}} \tau_k\), and using the facts \(\sigma_k \leq \sigma\) and \(\tau_k \leq \tau\) in (5.3) yields

\[
\limsup_{k \to \infty} \|x_k - x_k^*\| = \frac{\zeta(N_C)}{1 - \zeta(N_C)\zeta(N_P)} \left(\zeta(N_P)\sigma + \xi(N_P)\tau\right)
\]

which gives an upper bound to this asymptotic tracking error.

5.1.1. Prediction- and correction-only convergence. In this section, we analyze the convergence of the prediction- and correction-only methods as a particular case of the proposed framework.

First of all, we consider the *correction-only* method, which is characterized by \(N_P = 0\) and \(N_C > 0\). Convergence of this method with some particular solvers has been already studied in the literature, see *e.g.* [12] for a survey, and the following result extends the convergence analysis to a broader range of solvers.
Consider the problem (3.1) and let Assumptions 3.1, 3.2 and 3.4 hold. Consider the correction-only method derived from Algorithm 3.1 imposing $N_P = 0$, and choose $N_C > 0$ such that

$$\zeta(N_C) < 1.$$  

Then the trajectory $\{x_k\}_{k \in \mathbb{N}}$ generated by the correction-only algorithm converges $Q$-linearly with rate $\zeta(N_C)$ to a neighborhood of the optimal trajectory $\{x_k^*\}_{k \in \mathbb{N}}$, whose radius is bounded as

$$\limsup_{k \to \infty} \|x_k - x_k^*\| = \frac{\zeta(N_C)}{1 - \zeta(N_C)} (C_0 T_s + D_0) / \mu.$$  

Proof. Choosing $N_P = 0$ in (5.2) we get $\|x_{k+1} - x_{k+1}^*\| \leq \zeta(N_C) (\|x_k - x_k^*\| + \sigma_k)$. Combining the bound with Lemma 4.1 and iterating yields the result.

We consider now the prediction-only method, which is characterized by $N_C = 0$ and $N_P > 0$, and by using a one-step-back prediction for the cost functions. This is the framework usually employed in the context of online learning, see e.g. [44].

Consider the problem (3.1) and let Assumptions 3.1, 3.2 and 3.4 hold. Consider the prediction-only method derived from Algorithm 3.1 imposing $N_C = 0$, using a one-step-back prediction for the costs, and choosing $N_P > 0$ such that

$$\zeta(N_P) < 1.$$  

Then the trajectory $\{x_k\}_{k \in \mathbb{N}}$ generated by the prediction-only algorithm converges $Q$-linearly with rate $\zeta(N_C)$ to a neighborhood of the optimal trajectory $\{x_k^*\}_{k \in \mathbb{N}}$, whose radius is bounded as

$$\limsup_{k \to \infty} \|x_k - x_k^*\| = \frac{1}{1 - \zeta(N_P)} (C_0 T_s + D_0) / \mu.$$  

Proof. By the choice of $N_C = 0$ we have $\|x_{k+1} - x_{k+1}^*\| = \|\hat{x}_{k+1} - \hat{x}_{k+1}^*\| = \|\hat{x}_{k+1} - x_{k+1}^*\| + \tau_k$ where we used the triangle inequality. Since $\hat{x}_{k+1} = x_k$, and $\hat{x}_{k+1}$ is derived applying $N_P$ steps of the solver to $x_k$, then we have $\|\hat{x}_{k+1} - x_{k+1}^*\| = \|\hat{x}_{k+1} - x_{k+1}^*\| \leq \zeta(N_P) \|x_k - x_k^*\|$. The thesis follows by combining this fact with Lemma 4.1.

Remark 5.4. There is a subtle but important difference between the prediction- and correction-only methods. In the prediction-only case, the problem sampled at time $t_k$ is used to compute the approximate solution – or decision – that will be applied at time $t_{k+1}$. When a new problem is revealed (possibly, by an adversarial agent) at time $t_{k+1}$, the computed decision incurs a regret. On the other hand, in the correction-only case the problem sampled at time $t_k$ is used to compute the approximate solution that will be applied at time $t_k$ itself. Observing the bounds for the two schemes (cf. Theorems 5.2 and 5.3):

$$\|x_{k+1} - x_{k+1}^*\| \leq \zeta(N_P) \|x_k - x_k^*\| + (C_0 T_s + D_0) / \mu \quad \text{(prediction-only)}$$

$$\|x_{k+1} - x_{k+1}^*\| \leq \zeta(N_C) (\|x_k - x_k^*\| + (C_0 T_s + D_0) / \mu) \quad \text{(correction-only)}$$

we can see that for the correction-only case, the constant term $(C_0 T_s + D_0) / \mu$ is weighted by $\zeta(N_C)$, while this is not the case for the prediction-only case. This is a consequence of the fact that the prediction-only algorithm cannot mitigate the effect of $x_{k+1}^*$ being different from $x_k^*$, because the decision $x_{k+1}$ is taken without knowledge of the problem at time $t_{k+1}$. On the other hand, the correction-only algorithm is working towards the solution of the sampled problem, which means that the distance of consecutive optima has still a role, but it can be reduced.
5.1.2. Eventually static problem. An interesting scenario for which the tracking error can be shown to converge to zero is that of an “eventually static” problem, in the sense that asymptotically the problem stops varying.

**Corollary 5.5** (Eventually static problem). Let Assumptions 3.1 and 3.5 hold, and assume that \( \lim_{k \to \infty} \sigma_k = 0 \), \( \lim_{k \to \infty} \tau_k = 0 \). Then, the error incurred by a prediction-correction method that uses the solver \( \mathcal{O}(\lambda, \chi, \beta) \) converges to zero:

\[
\limsup_{k \to \infty} \| x_k - x_k^* \| = 0.
\]

**Proof.** Follows from the recursion (5.3) and Lemma 3.1(a) in [38]. 

While this convergence result is interesting, usually time-varying problems are not eventually static, and therefore in the rest of the paper we will consider time-varying problems for which Lemma 4.1 holds.

5.2. Convergence with Taylor expansion-based prediction. In this section we present some convergence results for the proposed framework when a Taylor expansion-based prediction is used for \( f \) and a one-step-back prediction is used for \( g \).

**Theorem 5.6.** Consider the problem (3.1). Consider also a prediction-correction method as defined in Algorithm 3.1 with a Taylor expansion-based prediction strategy as (4.3) for \( f \) and a one-step-back strategy for \( g \). Let Assumptions 3.1, 3.2 and 3.4 as well as Assumption 3.5(ii) hold. Consider the operator theoretic solver \( \mathcal{O}(\lambda, \beta, \chi) \) to solve both the prediction and correction problems with contraction rates \( \zeta \) and \( \xi \) given in Lemma 2.5. Choose the prediction and correction horizons \( N_p \) and \( N_C \) such that

\[
(5.7) \quad \zeta(N_C)[\zeta(N_P) + 2\kappa \xi(N_P)] < 1.
\]

Then the trajectory \( \{x_k\}_{k \in \mathbb{N}} \) generated by the prediction-correction algorithm converges Q-linearly to a neighborhood of the optimal trajectory \( \{x_k^*\}_{k \in \mathbb{N}} \), whose radius is upper bounded as

\[
(5.8) \quad \limsup_{k \to \infty} \| x_k - x_k^* \| = \zeta(N_C) C_0 T_s + D_0 \frac{\zeta(N_P) + 2(1 + \kappa)\xi(N_P)}{1 - \zeta(N_C)[\zeta(N_P) + 2\kappa \xi(N_P)]}.
\]

**Proof.** See Appendix C.2. 

Theorem 5.6 implies that the trajectory converges to a neighborhood of the optimal trajectory that depends linearly on \( T_s \). This result is in accordance with the observation that the smaller the sampling time, the better the solutions to the sampled problems (3.3) track the underlying optimal trajectory.

If Assumption 3.3 holds as well, then we can refine these results as follows.

**Theorem 5.7.** Consider the same conditions and algorithmic choices of Theorem 5.6. In addition, let Assumption 3.3 hold. Select the prediction and correction horizons \( N_p \) and \( N_C \), and \( \gamma \in (0, 1) \), such that

\[
(5.9) \quad \gamma > \zeta(N_C)\xi(N_P).
\]

Then there exist an upper bound \( \bar{T}_s \) for the sampling time and a convergence region \( R \) such that if \( T_s < \bar{T}_s \) and the algorithm is initialized with \( \| x_0 - x_0^* \| < R \), then

\[
(5.10) \quad \limsup_{k \to \infty} \| x_k - x_k^* \| = \zeta(N_C) \frac{C_0 T_s + D_0}{1 - \gamma} + \xi(N_P) \left( \frac{C_5 T_s^2}{2} + C_6 D_0 \right).
\]

\[3\text{Recall that } \kappa := L/\mu \text{ denotes the condition number of } f.\]
We can characterize the bounds for the sampling time and the convergence region with

\[
\hat{T}_s := \gamma - \zeta(N_C)(\zeta(N_P) + \xi(N_P)C_1D_0/\mu^2) \left( \frac{C_0C_1}{\mu^2} + \frac{C_2}{\mu} \right)^{-1},
\]

\[
R := \frac{2\mu}{C_1} \left( \frac{C_0C_1}{\mu^2} + \frac{C_2}{\mu} \right) (T_s - T_s).
\]

Proof. See Appendix C.3.

Theorem 5.7 shows that, with better knowledge of how the cost function \( f(x; t) \) varies over time, we can identify a tighter neighborhood of the optimal trajectory to which the proposed algorithm is guaranteed to converge. Indeed, the bound in Theorem 5.7 depends on the square of \( T_s \), while the bound in Theorem 5.6 depends linearly on \( T_s \). However, this result holds only provided that the algorithm is initialized in a neighborhood of the optimal trajectory with radius \( R \). Since \( R \) depends on \( T_s - T_s \), the theorem implies that the larger the sampling time, the closer to the optimum we need to start in order to compute a good approximation of the solution \( \{x_k\}_{k\in\mathbb{N}} \). This local convergence result is due to the quadratic norm term in (4.15) as it would happen in Newton’s method.

Remark 5.8. The results Theorems 5.6 and 5.7 reduce to the convergence result for correction-only time-varying algorithms [12] when \( N_P = 0 \), and present therefore tighter bounds than previous works [49, 47, 4], in the sense that the error bound does not present spurious terms depending on the prediction even though we set \( N_P = 0 \).

Remark 5.9. As observed in subsection 4.2.1, in order to compute the Taylor expansion of \( f \) we need some derivatives w.r.t. time, and in particular \( \nabla_{tx} f \). When they are not known in closed form, they can be approximated using backward finite differences. As a consequence of Lemma 4.4 then, we can see that Theorem 5.6 still holds, while Theorem 5.7 holds with \( C_5 = C_5 + C_3/\mu \) substituting \( C_5 \).

5.3. Convergence with extrapolation-based prediction. Finally, we present a convergence result when the extrapolation-based prediction (4.9) is employed.

**Theorem 5.10.** Consider Problem (3.1). Consider a prediction-correction method as defined in Algorithm 3.1 with the extrapolation-based prediction strategy (4.9) of order \( I \in \mathbb{N}, I \geq 2 \), for \( f \) and a one-step-back strategy for \( g \). Let Assumptions 3.1 to 3.4 as well as Assumption 3.5(ii) hold, set \( C_2 = 0 \) and let (4.10) hold for some \( C(I) > 0 \). Consider the operator theoretic solver \( O(\lambda, \beta, \chi) \) to solve both the prediction and correction problems with contraction rates \( \zeta \) and \( \xi \) given in Lemma 2.5. Choose the prediction and correction horizons \( N_P \) and \( N_C \) such that

\[
\zeta(N_C)\zeta(N_P) < 1.
\]

Then the trajectory \( \{x_k\}_{k\in\mathbb{N}} \) generated by the prediction-correction algorithm converges Q-linearly with rate \( \zeta(N_C)\zeta(N_P) \) to a neighborhood of the optimal trajectory \( \{x_k^*\}_{k\in\mathbb{N}} \), whose radius is upper bounded as

\[
\limsup_{k \to \infty} \left\| x_k - x_k^* \right\| = \frac{C_5}{\mu} \frac{T_s}{1 - \zeta(N_C)\zeta(N_P)} + C(I)\xi(N_P)T_s^I + D_0(\zeta(N_P) + \xi(N_P))]
\]

Proof. See Appendix C.4.

Remark 5.11. Notice that if the operator \( \mathcal{X} \) converting between \( z \) and the primal variable \( x \) is the identity (which is the case e.g. for gradient and proximal gradient methods – see Appendix A.1), then \( \zeta(N_C)\zeta(N_P) < 1 \) is automatically satisfied whenever at least one of \( N_C \) or \( N_P \) is non-zero.
We conclude presenting in Table 2 a comparison of the convergence results proposed in this section, with the required assumptions and the asymptotic error that they achieve highlighting their dependence on $N_P, N_C$ and the sampling period $T_s$. In addition, to further discuss how the different methods behave in different regimes, we also explicitly compute the asymptotic radius for a few notable cases in Table 3. As one can see, the use of one method or the other is case dependent. In particular, (in the considered setting) for the case of poor prediction ($\bar{\zeta}(N_P) \approx 1$), then correction-only appear to be the best, while in good prediction regimes Taylor-based prediction and Extrapolation-based prediction achieve lower error bounds and they are alternatively the best in different situations.

Table 2: Summary of the different convergence results.

| Method         | Result Assumptions | Convergence Conditions |
|---------------|--------------------|-----------------------|
| Correction-only | Th. 5.2 $3.1, 3.2, 3.4, N_P = 0$ | $\zeta(N_C) < 1$ $\Rightarrow$ Eq. (5.5) |
| Prediction-only | Th. 5.3 $3.1, 3.2, 3.4, N_C = 0$ | $\zeta(N_P) < 1$ $\Rightarrow$ Eq. (5.6) |
| Taylor-expansion | Th. 5.6 $3.1, 3.2, 3.4, 3.5(i)$ | $\zeta(N_C)|\zeta(N_P) + 2\zeta(N_P)| < 1$ $\Rightarrow$ Eq. (5.8) |
|                | Th. 5.7 $3.1, 3.2, 3.3, 3.4, 3.5(ii)$ | $\gamma > \zeta(N_C)\zeta(N_P), T_s < T_\nu, \|x_0 - x^*_0\| < R$ $\Rightarrow$ Eq. (5.10) (local) |
| Extrapolation   | Th. 5.10 $3.1, 3.2, 3.3, 3.4, 3.5(iii), C_2 = 0$ | $\zeta(N_C)\zeta(N_P) < 1$ $\Rightarrow$ Eq. (5.11) |

Table 3: Comparison of asymptotic error bounds with $D_0 = 0, C_2 = 0, \mu = 1, \gamma \approx \zeta(N_C)\zeta(N_P)$. Let $q = \frac{\zeta(N_C)}{1 - \zeta(N_C)}$ and $C_7 = C_0^2C_1 + C_3$ for convenience.

| Regime          | Sub-case | C-only | P-only | Taylor | Extrap. |
|----------------|----------|--------|--------|--------|---------|
| Exact pred.: $N_P \to \infty$ | $C_1 = 0$ | $qC_0T_s$ | $C_0T_s$ | $\zeta(N_C)\frac{C_0T_s^2}{2}$ | $\zeta(N_C)C_0T_s^2$ |
|                 | $C_1 > 0$ | $qC_0T_s$ | $C_0T_s$ | $\zeta(N_C)\frac{C_0T_s^2}{2}$ | $\zeta(N_C)C_0T_s^2$ |
| Poor pred.: $\zeta(N_P) \approx 1$ | $C_1 = 0$ | $qC_0T_s$ | $+\infty$ | $q(C_0T_s + C_3T_s^2)$ | $q(C_0T_s + 2C_3T_s^2)$ |
|                 | $C_1 > 0$ | $qC_0T_s$ | $+\infty$ | $q(C_0T_s + C_3T_s^2)$ | $q(C_0T_s + 2C_3T_s^2)$ |

Remark 5.12 (Prediction of $g$). As discussed above, in the proposed strategies we always apply a one-step-back prediction for the cost $g$. This choice is dictated by the fact that we assume $g$ to be closed, convex and proper, and hence potentially non-smooth. Indeed, the generality of this assumption – which allows to treat a wide class of problems – has the drawback of reducing the predictive power at our disposal. We remark that if $g$ were to be smooth, then it would be possible to set $f \mapsto f + g$ and $g \mapsto 0$, opening up the possibility of predicting $g$ with a Taylor expansion or extrapolation strategy. Alternatively, specific knowledge of $g$’s evolution over time could be exploited to design a better prediction strategy. However, for the sake of generality, in this paper we do not consider specific problem instances.
6. Dual Prediction-Correction Methods. We now propose a dual version to the prediction-correction framework, that allows us to solve linearly constrained time-varying problems. We present convergence results and a dual Taylor expansion-based prediction strategy.

6.1. Problem formulation. We are interested in solving the following time-varying convex optimization problem with linear constraints, cf. (1.2):

\begin{equation}
(x^*(t), y^*(t) = \arg \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} f(x; t) + b(y; t), \text{ s.t. } Ax + By = c,
\end{equation}

where $A \in \mathbb{R}^{p \times n}$, $B \in \mathbb{R}^{p \times m}$ and $c \in \mathbb{R}^p$. The following assumption will hold throughout this section, and we will further use Assumptions 3.2 and 3.3 for $f$.

Assumption 6.1. (i) The cost function $f$ belongs to $\mathcal{S}_{\mu,L}(\mathbb{R}^n)$ uniformly in time and satisfies Assumption 3.2. (ii) The cost $h$ either belongs to $I_0(\mathbb{R}^m)$ uniformly in time or $h(t) = 0$ with $B = 0$. (iii) The matrix $A \in \mathbb{R}^{p \times n}$ is full row rank and the vector $c$ can be written as the sum of two vectors $c' \in \text{im}(A)$ and $c'' \in \text{im}(B)$.4

The Fenchel dual of (6.1) is

\begin{equation}
w^*(t) = \arg \min_{w \in \mathbb{R}^p} \{d^f(w; t) + d^h(w; t)\}
\end{equation}

where $d^f(w; t) = f^*(A^\top w; t) - \langle w, c \rangle$ and $d^h(w; t) = h^*(B^\top w; t)$. Problem (6.2) conforms to the class of problems that can be solved with the prediction-correction splitting methods of section 3. Indeed, by Assumption 6.1 we can see that $d^f \in \mathcal{S}_{\tilde{\mu},L}(\mathbb{R}^p)$, with $\tilde{\mu} := \lambda_m(AA^\top)/L$ and $\tilde{L} := \lambda_M(AA^\top)/\mu$, and $d^h \in I_0(\mathbb{R}^p)$, uniformly in time, as proved in Lemma A.1. In the following, we denote $\tilde{\kappa} := \tilde{L}/\tilde{\mu}$.

Remark 6.2. The assumption that $A$ be full row rank is necessary to guarantee that $d^f \in \mathcal{S}_{\tilde{\mu},L}(\mathbb{R}^p)$ (cf. Lemma A.1). However, when problem (6.4) reduces to $\min_x f(x)$ s.t. $Ax = c$, this assumption can be relaxed. In this case we are able to prove that the dual function $d^f$ is strongly convex in the subspace of the image of $A$, i.e., $\text{im}(A)$. Therefore, if $\text{im}(A)$ is an invariant set for the trajectory generated by the solver, the solver is contractive and the convergence analysis of this paper applies to show linear convergence. The solvers dual ascent and method of multipliers indeed satisfy these conditions, see [46] and Appendix A.3 for more details. Moreover, even when the full problem (6.1) is considered, there are some solvers that – exploiting a particular structure of $A$ – do achieve linear convergence. In particular, some decentralized ADMM formulations have been shown to converge linearly [45, 30].

We now characterize the derivatives of $d^f$ with respect to both $w$ and $t$.

Proposition 6.3. Under Assumption 6.1, let

\begin{equation}
\bar{x}(w, t) := \arg \min_{x} \{f(x; t) - \langle A^\top w, x \rangle \}.
\end{equation}

The derivatives of the dual function $d^f$ are

\begin{align*}
\nabla_w d^f(w; t) &= A\bar{x}(w, t) - c, \quad \nabla_{ww} d^f(w; t) = A\nabla_{xx} f(\bar{x}(w, t); t)^{-1} A^\top, \\
\nabla_{tt} d^f(w; t) &= \nabla_{ww} d^f(w; t) = -A\nabla_{xx} f(\bar{x}(w, t); t)^{-1} \nabla_{tt} f(\bar{x}(w, t); t).
\end{align*}

4This assumption ensures that the problem does indeed have a solution; otherwise, it would not be possible to satisfy the linear constraints.
Appendix D.1

section 3

and with initial condition

reports the pseudo-code for the dual prediction-correction scheme

sampled at times

6.1

Assumption 3.2

(6.1)

6.1

hold for the primal problem

(6.1)

Assumptions

we are

holds for the dual problem as well.

and using the sub-multiplicativity of the norm we have

where the last inequality holds by Assumptions 3.2 and 6.1.

6.2. Convergence analysis.

Applying the same approach of section 3, we are interested in solving

(6.1) from the last iterate

of the solver to it, using as initial condition the prediction just computed.

In the following we present first a convergence result that holds independently of

the particular prediction strategy that is chosen, and then we particularize this result

when a Taylor expansion-based prediction is used.

Algorithm 6.1 reports the pseudo-code for the dual prediction-correction scheme

analyzed in this section. After specifying the prediction and correction horizons

and using the available information up to time \( t_k \), a prediction of the costs

and

is generated, and the algorithm applies \( N_p + 1 \) steps of the solver to the resulting

problem. When, at time \( t_{k+1} \), the actual problem is observed, then the algorithm

applies \( N_C \) of the solver to it, using as initial condition the prediction just computed.

Algorithm 6.1 Dual prediction-correction methods

Require: \( x_0 \), horizons \( N_p \) and \( N_C \), and specify an operator theoretical solver \( O(\lambda, \beta, \chi) \) (e.g. dual ascent,

MM, dual FBS, or ADMM) and its parameters, and a prediction method.

1: for \( k = 0, 1, \ldots \) do

// time \( t_k \) (prediction)

2: Generate the functions \( \hat{d}_k^f(x) \) and \( \hat{d}_k^h(y) \) (prediction)

3: Apply \( N_p \) steps of the dual solver to problem (6.10) with initial condition \( w_k \)

// time \( t_{k+1} \) (correction)

4: Observe the cost functions \( d_{k+1}^f(x) \) and \( d_{k+1}^h(y) \)

5: Apply \( N_C \) steps of the solution method to problem (6.5) with initial condition the prediction

\( \hat{w}_{k+1} \)

6: Derive \( x_{k+1}, y_{k+1} \) from the last iterate \( w_{k+1} \) of the dual solver

7: end for
6.2.1. General convergence analysis.

**Proposition 6.5** (General dual error bound). Let Assumption 6.1 hold, and let \( \sigma_k, \tau_k \in (0, +\infty) \) be such that for any \( k \in \mathbb{N} \):

\[
\|w^*_k - w^*_{k+1}\| \leq \sigma_k \quad \text{and} \quad \| \hat{w}^*_k - w^*_{k+1}\| \leq \tau_k.
\]

The error incurred by a prediction-correction method using the dual solver \( O(\lambda, \chi, \beta) \) is upper bounded in the dual space by:

\[
\| \hat{w}^*_{k+1} - w^*_{k+1}\| \leq \zeta(N_C) \left( \zeta(N_P)\|w_k - w^*_k\| + \zeta(N_P)\sigma_k + \zeta(N_P)\tau_k \right),
\]

with functions \( \zeta \) and \( \tilde{\zeta} \) are defined as in Lemma 2.5 for \( d^f \in S_{\mu, \ell}(\mathbb{R}^p) \); and in the primal space by:

\[
\begin{align*}
\|x^*_{k+1} - x^*_{k+1}\| & \leq (\|A\|/\mu) \|w^*_{k+1} - w^*_{k+1}\|, \\
\|B(y^*_{k+1} - y^*_{k+1})\| & \leq \|B\| \left( 1/\rho + \|A\|^2/\mu \right) \|w^*_{k+1} - w^*_{k+1}\|.
\end{align*}
\]

**Proof.** The bound (5.2) is a consequence of Proposition 5.1 when applied to the dual problem (6.5), while (6.8) is a consequence of Lemma A.2.

Therefore Proposition 6.5 shows that, since the dual problem has a composite structure with the smooth component being strongly convex, then we can use the results of section 5 to prove convergence in the dual space. Moreover, we can combine the dual convergence with the properties of (6.4) to spell out the convergence in the primal space.

6.2.2. Taylor expansion-based prediction. In the following, we apply a dual version of the Taylor expansion-based prediction strategy of subsection 4.2 to solve (6.5). In particular, we define the approximation \( \hat{d}^f_{k+1}(w) \) of the function \( d^f_{k+1}(w) \) through the second order Taylor expansion of \( d^f_k(w) \) around \((w_k; t_k)\):

\[
\begin{align*}
\hat{d}^f_{k+1}(x) &= d^f_k(w_k) + \langle \nabla w d^f_k(w_k), w - w_k \rangle + T_w \nabla d^f_k(w_k) + \frac{T^2}{2} \nabla_{tt} d^f_k(w_k) \\
& \quad + T_s \langle \nabla_{tw} d^f_k(w_k), w - w_k \rangle + \frac{1}{2} (w - w_k)^\top \nabla_{ww} d^f_k(w_k)(w - w_k)
\end{align*}
\]

alongside with the approximation \( \hat{d}^h_{k+1}(w) = d^h_k(w) \).

We make the following additional assumption.

**Assumption 6.6.** The cost \( h \) is such that \( d^h \) satisfies Assumptions 3.4 and 3.5(ii) with \( D_0 \), w.r.t. the dual optimal trajectory and the dual optimal prediction.

The corresponding dual prediction problem

\[
\hat{w}^*_{k+1} = \arg\min_{w \in \mathbb{R}^p} \left\{ \hat{d}^f_{k+1}(w) + \hat{d}^h_{k+1}(w) \right\}
\]

satisfies the following Proposition.

**Proposition 6.7.** The dual prediction problem (6.10) corresponds to the primal prediction problem

\[
\begin{align*}
\hat{x}^*_{k+1}, \hat{y}^*_{k+1} &= \arg\min_{x \in \mathbb{R}^p, y \in \mathbb{R}^m} \left\{ \hat{f}_{k+1}(x) + \hat{h}_{k+1}(y) \right\} \quad \text{s.t.} \quad Ax + By = c.
\end{align*}
\]
where \( h_{k+1}(y) = h_k(y) \) and, denoting \( \hat{x}_k = \hat{x}(w_k, t_k) \), we define:

\[
\hat{f}_{k+1}(x) = \hat{a}_k + \langle \nabla_x f_k(\hat{x}_k), x - \hat{x}_k \rangle + \frac{1}{2} (x - \hat{x}_k)^\top \nabla_{xx} f_k(\hat{x}_k)(x - \hat{x}_k)
\]

(6.12)

with \( \hat{a}_k = d_k^f(w_k) + T_s \nabla_t d_k^f(w_k) + (T_s^2/2) \nabla_{tt} d_k^f(w_k) \), being a constant independent of \( x \) at time \( t_k \).

**Proof.** See Appendix D.2. \[\Box\]

**Remark 6.8.** The dual prediction-correction method designed above is based on a Taylor expansion of the dual cost \( d^f \) which, as shown in Proposition 6.7, yields a quadratic approximation of the primal problem. But this is not the only approach to dual prediction that could fit the convergence result of Proposition 6.5. In particular, we could think of employing a **primal** prediction strategy for (6.4), which would yield a corresponding prediction for the dual problem. However, in order to exploit the results presented in section 4, in this section we focus on the dual Taylor expansion-based prediction. Future research will look into different prediction methods for (6.4).

The following result characterizes the convergence in terms of the primal and dual variables.

**Corollary 6.9.** Consider the problem (6.4). Consider the prediction-correction method defined in Algorithm 6.1, and assume that the dual Taylor expansion-based prediction strategy is employed. Let \( O(\lambda, \beta, \chi) \) be a suitable dual solver with contraction rates \( \zeta \) and \( \xi \) given in Lemma 2.5 for \( d^f \in S_{h,L}(\mathbb{R}^p) \). Let Assumptions 6.1 and 6.6 hold.

Choose the prediction and correction horizons such that

\[
\zeta(N_C)[\zeta(N_P) + 2\kappa \zeta(N_P)] < 1.
\]

Then the dual trajectory \( \{w_k\}_{k \in \mathbb{N}} \) generated by the dual prediction-correction method converges to a neighborhood of the optimal trajectory \( \{w_k^*\}_{k \in \mathbb{N}} \), whose radius is upper bounded as

\[
\limsup_{k \to \infty} \|w_k - w_k^*\| = \zeta(N_C) \frac{\zeta_0 T_s + \bar{D}_0}{\mu} \frac{[\zeta(N_P) + 2(1 + \kappa)\zeta(N_P)]}{1 - \zeta(N_C)[\zeta(N_P) + 2\kappa \zeta(N_P)]}.
\]

Moreover, the primal trajectories \( \{x_k\}_{k \in \mathbb{N}}, \{y_k\}_{k \in \mathbb{N}} \) converge to a neighborhood of the optimal trajectories \( \{x_k^*\}_{k \in \mathbb{N}}, \{y_k^*\}_{k \in \mathbb{N}} \), whose radii are upper bounded as

\[
\limsup_{k \to \infty} \|x_k - x_k^*\| = (\|A\|/\mu) \limsup_{k \to \infty} \|w_k - w_k^*\|,
\]

\[
\limsup_{k \to \infty} \|B(y_k - y_k^*)\| = \|B\| (\|A\|^2/\mu + 1/\rho) \limsup_{k \to \infty} \|w_k - w_k^*\|.
\]

**Proof.** This is a straightforward consequence of Theorem 5.6 for the dual convergence, and Lemma A.2 for the primal convergence (cf. the proof of Proposition 6.5). \[\Box\]

Corollary 6.9 states that a dual prediction-correction algorithm with the specified choice of Taylor-based prediction function converges to a neighborhood of the optimizers trajectory, whose radius is dictated by the sampling period. The results of Corollary 6.9 for dual methods is on par with the results for primal methods in Theorem 5.6. In this context, a similar result for dual methods, mimicking the one for primal methods in Theorem 5.7, can be derived almost directly and not presented here for
the sake of space. Finally, we refer the interested reader to Appendix A.2.1 for how to derive the dual convergence rates of the proposed approach, with an example for ADMM.

What remains still open are results for the extrapolation-based prediction in the dual setting, which seems not so straightforward, since ensuring the extrapolation-based prediction function to be convex would require both $C_1 = C_2 = 0$, in the dual case. We have left this as future research.

7. Numerical Results. In this section we briefly present some numerical results that showcase the different performance of the proposed prediction strategies.

We consider the composite problem (cf. \cite{49}):

\begin{equation}
(7.1) \quad f(x; t) = \|x - b(t)\|^2 / 2 + \epsilon \log(1 + \exp((1_n, x))) \quad \text{and} \quad g(x) = \nu \|x\|_1
\end{equation}

with $n = 20$, and where $b(t) \in \mathbb{R}^n$ is a signal with sinusoidal components (with angular velocity $\omega = 0.02\pi$ and randomly generated phases), $\epsilon = 0.75$, $\nu = 0.5$. The function $f$ has $\mu = 1$, $L = 1 + \epsilon N/4$, $C_0 = \omega$, while $g$ has $D_0 = 0$. The prediction and correction problems are solved using the forward-backward splitting (FBS, see Appendix A.1) with a step-size of $\rho = 2/(L + \mu)$, and $N_P = [5, 20, 40]$, $N_C = 5$.

In Table 4, we report the asymptotic tracking error for different strategies, and for three different sampling times. Note that the prediction-only approach employs a one-step-back strategy (and hence corresponds to the algorithm proposed in \cite{17} – see also \cite{18, 14}]), and that we use extrapolation-based predictions with order $I = 2$ – which gives $f_{k+1} = 2f_k - f_{k-1}$ – and order $I = 3$ – which yields $f_{k+1} = 3f_k - 3f_{k-1} + f_{k-2}$. We can see that in general the methods that employ prediction and correction achieve better results, and that the larger the sampling time is, the larger the asymptotic error. Also we observe how extrapolation with $I = 3$ can boost performance, especially when $N_P$ is large, with respect to Taylor.

Appendix A. Examples of solvers. We review some examples of operator theoretical methods. See Table 5 for a comparison.

A.1. Primal solvers. Consider the optimization problem (2.2), in this section we review widely used algorithms to solve it in case (i) $g(\cdot) = 0$ or (ii) $g \in \mathcal{L}_b(\mathbb{R}^n)$.

Case (i). In this case, the solution $x^\ast$ can be computed as the fixed point of the following operators:

- **Gradient method (or forward step):** $T = I - \rho \nabla_x f$, which is contractive if $\rho < 2/L$; the Banach-Picard iteration is then: $x^{\ell+1} = x^\ell - \rho \nabla_x f(x^\ell), \quad \ell \in \mathbb{N}$.
- **Proximal point algorithm (PPA) (or backward step):** $T = \text{prox}_{\rho f}$, which is contractive for any $\rho > 0$; the Banach-Picard becomes: $x^{\ell+1} = \text{prox}_{\rho f}(x^\ell), \quad \ell \in \mathbb{N}$.

Case (ii). In case the cost of (2.2) is composite, then we can apply the following splitting algorithms:

- **Forward-backward splitting (FBS) (or proximal gradient method):** we choose $T = \text{prox}_{\rho g} \circ (I - \rho \nabla_x f)$, which is contractive for $\rho < 2/L$; the fixed point of $T$ coincides with $x^\ast$ and the Banach-Picard update can be split in the following updates:

\begin{equation}
(A.1) \quad y^\ell = x^\ell - \rho \nabla_x f(x^\ell), \quad x^{\ell+1} = \text{prox}_{\rho g}(y^\ell), \quad \ell \in \mathbb{N}.
\end{equation}

- **Peaceman-Rachford splitting (PRS):** we choose $T = \text{refl}_{\rho g} \circ \text{refl}_{\rho f}$, which is contractive for any $\rho > 0$; the Banach-Picard iteration becomes, for $\ell \in \mathbb{N}$:

\begin{equation}
(A.2) \quad x^\ell = \text{prox}_{\rho f}(z^\ast), \quad y^\ell = \text{prox}_{\rho g}(2x^\ell - z^\ell), \quad z^{\ell+1} = z^\ell + (y^\ell - x^\ell)
\end{equation}

and, from the fixed point $z^\ast$ of $T$ we compute the solution $x^\ast$ through $X = \text{prox}_{\rho f}$.
Table 4: Comparison of asymptotic errors observed in the numerical simulations for (7.1).

| Method          | $T_s = 0.2$ | $T_s = 0.002$ |
|-----------------|-------------|---------------|
|                 | $N_p = 5$   | $N_p = 20$    | $N_p = 40$ |
| Prediction-only | $3.98 \times 10^{-2}$ | $3.35 \times 10^{-3}$ | $3.31 \times 10^{-3}$ |
| Correction-only | $4.70 \times 10^{-3}$ | $4.70 \times 10^{-3}$ | $4.70 \times 10^{-3}$ |
| Taylor          | $4.96 \times 10^{-4}$ | $3.11 \times 10^{-4}$ | $3.11 \times 10^{-5}$ |
| Extrapolation $I = 2$ | $4.98 \times 10^{-4}$ | $6.22 \times 10^{-5}$ | $6.22 \times 10^{-5}$ |
| Extrapolation $I = 3$ | $4.95 \times 10^{-4}$ | $1.91 \times 10^{-5}$ | $6.54 \times 10^{-7}$ |

| Method          | $T_s = 0.002$ |
|-----------------|---------------|
|                 | $N_p = 5$   | $N_p = 20$    | $N_p = 40$ |
| Prediction-only | $3.98 \times 10^{-3}$ | $3.35 \times 10^{-3}$ | $3.31 \times 10^{-3}$ |
| Correction-only | $4.70 \times 10^{-4}$ | $4.70 \times 10^{-4}$ | $4.70 \times 10^{-4}$ |
| Taylor          | $4.96 \times 10^{-5}$ | $3.17 \times 10^{-7}$ | $3.11 \times 10^{-7}$ |
| Extrapolation $I = 2$ | $4.96 \times 10^{-5}$ | $6.27 \times 10^{-7}$ | $6.22 \times 10^{-7}$ |
| Extrapolation $I = 3$ | $4.96 \times 10^{-5}$ | $7.96 \times 10^{-8}$ | $6.41 \times 10^{-10}$ |

Table 5: Possible operator theoretical solvers, for the primal and dual problem. For dual problems: convergence is guaranteed for the dual variable $w$, and R-linear convergence is shown for the primal variables. In the dual case, the gradient method becomes to dual ascent, the PPA to the method of multipliers, FBS to dual FBS, and PRS to ADMM. Moreover, $f$, $g$, $L$ and $\mu$ are substituted by $d^f$, $d^g$, $\bar{L}$ and $\bar{\mu}$, respectively.

| Method: $\mathcal{T}$ | Gradient | PPA | FBS | PRS |
|------------------------|----------|-----|-----|-----|
| $\mathcal{T}$          | $I - \rho \nabla f$ | $\text{prox}_{\rho f}$ | $\text{refl}_{\rho g} \circ \text{refl}_{\rho f}$ | $\text{refl}_{\rho g} \circ \text{refl}_{\rho f}$ |
| $\mathcal{X}$          | $I$      | $I$ | $I$ | $\text{prox}_{\rho f}$ |

Convergence Cond.$s$
- $\rho \in (0, 2/L)$
- $\rho > 0$
- $\rho \in (0, 2/L)$
- $\rho > 0$

| $\lambda$ | $\max\{|1 - \rho L|, |1 - \rho \mu|\}$ | $1/(1 + \rho \mu)$ | $\max\{|1 - \rho L|, |1 - \rho \mu|\}$ | $\max\left\{\frac{|1 - \rho L|}{1 + \rho \mu}, \frac{|1 - \rho \mu|}{1 + \rho \mu}\right\}$ |
|------------|------------------|-------------------|------------------|------------------|
| $\chi$     | $1$               | $1$               | $1$               | $1/(1 + \rho \mu)$ |
| $\beta$    | $1$               | $1$               | $1$               | $1/(1 + \rho L)$  |

References: [50] [40] [50] [21]

A.1.1. Other solvers. We mention, without going into details, some other solvers that fit into the proposed prediction-correction method.
Decentralized primal-dual solvers. Consider a set of $N$ agents with the task of cooperatively solving the following problem:

$$
\min_{x_i, \; i=1,\ldots,N} \sum_{i=1}^{N} f_i(x_i) + g_i(x_i) \quad \text{s.t.} \quad x_i = x_j \forall i,j \in 1,\ldots,N,
$$

where the local costs $f_i \in \mathcal{S}_{\mu,L}$ and the $g_i \in \Gamma_0$ are privately stored by each agent. The consensus constraints $x_i = x_j \forall i,j \in 1,\ldots,N$ enforce consistency between the local states $x_i$ of the agents.

Different solvers have been proposed for this class of decentralized problems, both with $g_i(\cdot) \equiv 0$ and $g_i(\cdot) \not\equiv 0$. These solvers are mainly based on primal-dual schemes which sequentially update the primal and dual variables, and on the idea of gradient tracking, that is, the use of a scheme that allows to estimate global descent directions (for $\sum_{i=1}^{N} f_i(x)$) locally.

We refer the interested reader to the papers [1, 51] – and references therein – which propose two alternative unifying frameworks for this class of primal-dual solvers (which can be specialized to the case $g_i(\cdot) \equiv 0$).

3-splitting methods. A different class of solvers that fits into the proposed method is that of “3-splitting” methods designed to solve (2.2) when we have $g(x) = g^\prime(x) + g''(x)$. These methods exploit the further separability of the non-smooth cost $g$ to develop tailored solvers which, under certain conditions, can be proved to converge linearly. We refer the interested reader to [13, 34].

### A.2. Dual solvers.

We consider now linearly constrained optimization problems\(^5\) of the form

\begin{equation}
(A.3) \quad \mathbf{x}^*, \mathbf{y}^* = \arg \min_{\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^m} \{f(\mathbf{x}) + h(\mathbf{y})\} \quad \text{s.t.} \quad A\mathbf{x} + B\mathbf{y} = \mathbf{c}\n\end{equation}

in which $f \in \mathcal{S}_{\mu,L}(\mathbb{R}^n)$, $A \in \mathbb{R}^{p \times n}$ full row rank, and either (i) $h(\cdot) \equiv 0$ with $B = 0$ or (ii) $h \in \Gamma_0(\mathbb{R}^m)$. In order to solve (A.3), we compute its Fenchel dual [21]

\begin{equation}
(A.4) \quad \mathbf{w}^* = \arg \min_{\mathbf{w} \in \mathbb{R}^p} \{d^f(\mathbf{w}) + d^h(\mathbf{w})\}
\end{equation}

where $d^f(\mathbf{w}) = f^*(A^\top \mathbf{w}) - \langle \mathbf{w}, \mathbf{c} \rangle$ and $d^h(\mathbf{w}) = h^*(B^\top \mathbf{w})$. The following lemma shows that (A.4) inherits the properties of (A.3), and therefore we can apply the algorithms reviewed in Appendix A.1 to solve it\(^6\).

**Lemma A.1.** Assuming that $f \in \mathcal{S}_{\mu,L}(\mathbb{R}^n)$ and $A$ is full row rank, then the dual function $d^f : \mathbb{R}^p \times \mathbb{R}_+ \to \mathbb{R}$ belongs to $\mathcal{S}_{\bar{\mu},\bar{L}}(\mathbb{R}^p)$, with $\bar{\mu} = \lambda_m(AA^\top)/L$ and $\bar{L} = \lambda_M(AA^\top)/\mu$. Moreover, if $h \in \Gamma_0(\mathbb{R}^m)$ then $d^h : \mathbb{R}^p \to [+,\infty]$ belongs to $\Gamma_0(\mathbb{R}^p)$, otherwise if $h(\cdot) \equiv 0$ and $B = 0$ we have $d^h(\cdot) \equiv 0$.

**Proof.** The first fact follows from [21, Prop. 4]; the second from [6, Cor. 13.38].\(\square\)

In the following we review the dual algorithms derived by applying the methods of Appendix A.1 to (A.4), and with Lemma A.2 we provide a convergence result.

---

\(^5\)We remark that (2.2) can also model constrained problems when $g$ is chosen as the indicator function of a convex set, but this is usually done only when projecting over said convex set is computationally easy to do.

\(^6\)We remark that the primal and dual solutions are related through the update equations of the following dual solvers, in that $(x^*, y^*, w^*)$ must be a fixed point of these equations.
Case (i). The primal solution \( x^* \) can be computed via the following algorithms:

- **Dual ascent**: applying the gradient method to (A.4) with \( \rho < 2/L \) yields:
  \[
  \begin{align*}
  x^\ell &= \arg \min_{x \in \mathbb{R}^n} \{ f(x) - \langle u^\ell, Ax \rangle \}, \quad w^{\ell+1} = w^\ell - \rho(Ax^\ell - c), \quad \ell \in \mathbb{N}.
  \end{align*}
  \]

- **Method of multipliers (MM)**: applying the proximal point algorithm to (A.4) yields:
  \[
  \begin{align*}
  x^\ell &= \arg \min_{x \in \mathbb{R}^n} \{ f(x) - \langle w^\ell, Ax \rangle + \frac{\rho}{2} \| Ax - c \|^2 \}, \\
  w^{\ell+1} &= w^\ell - \rho(Ax^\ell - c), \quad \ell \in \mathbb{N}.
  \end{align*}
  \]

Case (ii). The primal solutions \( x^* \) and \( y^* \) are computed with the algorithms:

- **Dual FBS**: applying the forward-backward splitting to (A.4) with \( \rho < 2/L \) gives:
  \[
  \begin{align*}
  x^\ell &= \arg \min_{x \in \mathbb{R}^n} \{ f(x) - \langle w^\ell, Ax \rangle \}, \quad u^\ell = w^\ell - \rho(Ax^\ell - c), \\
  y^\ell &= \arg \min_{y \in \mathbb{R}^m} \{ h(y) - \langle u^\ell, By \rangle + \frac{\rho}{2} \| By \|^2 \}, \quad w^{\ell+1} = u^\ell - \rho By^\ell, \quad \ell \in \mathbb{N}.
  \end{align*}
  \]

- **Alternating direction method of multipliers (ADMM)**: for any \( \rho > 0 \), applying the Peaceman-Rachford splitting to (A.4) yields:
  \[
  \begin{align*}
  x^\ell &= \arg \min_{x \in \mathbb{R}^n} \{ f(x) - \langle z^\ell, Ax \rangle + \frac{\rho}{2} \| Ax - c \|^2 \}, \quad w^\ell = z^\ell - \rho(Ax^\ell - c), \\
  y^\ell &= \arg \min_{y \in \mathbb{R}^m} \{ h(y) - \langle 2w^\ell - z^\ell, By \rangle + \frac{\rho}{2} \| By \|^2 \}, \quad u^\ell = 2w^\ell - z^\ell - \rho By^\ell, \quad z^{\ell+1} = z^\ell + 2(u^\ell - w^\ell), \quad \ell \in \mathbb{N}.
  \end{align*}
  \]

**Lemma A.2.** Consider problem (A.3) with \( f \in \mathcal{S}_{\mu,L}(\mathbb{R}^n) \) and \( A \) full row rank, and either \( h \in \mathcal{I}_0(\mathbb{R}^m) \), or \( h(\cdot) \equiv 0 \) with \( B = 0 \).

The dual variable converges linearly to the solution \( w^* \) of the Fenchel dual (A.4), and the primal variables converge R-linearly; indeed the following inequalities hold:

\[
\begin{align*}
\| x^\ell - x^* \| &\leq \| A \|/\mu \| w^\ell - w^* \|, \\
\| B(y^\ell - y^*) \| &\leq \| B \| (1/\rho + \| A \|^2/\mu) \| w^\ell - w^* \| \quad \text{(in case (ii)).}
\end{align*}
\]

**Proof.** Recall that by the \( \mu \)-strong convexity of \( f \), the following holds \cite[p. 72]{8}:

\[
\mu \| x - x^* \|^2 \leq \langle \nabla_x f(x) - \nabla_x f(x^*), x - x^* \rangle;
\]

moreover, the KKT conditions for the Problem (A.3) are:

\[
\nabla_x f(x^*) = A^\top w^*, \quad \partial g(y^*) \ni B^\top w^*, \quad Ax^* + By^* = c.
\]

**Convergence in \( x \)** Imposing the first order optimality condition to the \( x \) updates of dual ascent and dual FBS yields:

\[
\nabla_x f(x^\ell) = A^\top w^\ell.
\]

The same equation can be derived for the ADMM by substituting \( z^\ell = w^\ell + \rho(Ax^\ell - c) \) into the optimality condition of the \( x \) update.
For the method of multipliers instead the optimality condition is
\begin{equation}
\nabla_x f(x^\ell) - A^T \omega^\ell + \rho A^T (Ax^\ell - c) = 0.
\end{equation}

Subtracting the first KKT condition \((A.11)\) to \((A.12)\) yields \(\nabla_x f(x^\ell) - \nabla_x f(x^*) = A^T (w^\ell - w^*)\), which, substituted into the strong convexity inequality \((A.10)\) gives:
\begin{equation}
\mu \|x^\ell - x^*\|^2 \leq \langle A^T (w^\ell - w^*), x^\ell - x^* \rangle \leq \|A\| \|w^\ell - w^*\| \|x^\ell - x^*\|
\end{equation}
where the second inequality holds by Cauchy-Schwarz. Simplifying and rearranging \((A.14)\) proves the thesis for dual ascent, dual FBS and ADMM.

For the method of multipliers, summing the first and third KKT conditions \((A.11)\) (the latter with \(B = 0\)) to \((A.13)\) gives \(\nabla_x f(x^\ell) - \nabla_x f(x^*) = A^T (w^\ell - w^*) - \rho A^T (Ax^\ell - x^*)\), and substituting it into \((A.10)\), then
\[\mu \|x^\ell - x^*\|^2 \leq \langle A^T (w^\ell - w^*), x^\ell - x^* \rangle - \rho \|A(x^\ell - x^*)\|^2 \leq \|A\| \|w^\ell - w^*\| \|x^\ell - x^*\|\]
and the thesis follows.

**Convergence in y** We consider now the case in which \(h \in \mathcal{I}_0(\mathbb{R}^m)\) and prove R-linear convergence of \(\{By^\ell\}_{\ell \in \mathbb{N}}\) to \(By^*\) for dual FBS and ADMM.

For the dual FBS, combine the optimality condition of the \(y\) update with \(\omega^\ell = w^\ell - \rho (Ax^\ell - c)\); and for the ADMM, combine the optimality condition of the \(y\) update with \(2\omega^\ell - z^\ell = w^\ell - \rho (Ax^\ell - c)\); this yields
\begin{equation}
\partial h(y^\ell) - B^T \omega^\ell + \rho B^T (Ax^\ell + By^\ell - c) \ni 0.
\end{equation}

Summing the second and third KKT conditions \((A.11)\) to \((A.15)\) yields
\begin{equation}
\partial h(y^\ell) - \partial h(y^*) - B^T (w^\ell - w^*) + \rho B^T [A(x^\ell - x^*) + B(y^\ell - y^*)] \ni 0.
\end{equation}
By the monotonicity of the subdifferential, for any choice of subgradients \(\nabla h(y^\ell) \in \partial h(y^\ell)\) and \(\nabla h(y^*) \in \partial h(y^*)\) we have: \(0 \leq \langle \nabla h(y^\ell) - \nabla h(y^*), y^\ell - y^* \rangle\); thus substituting \((A.16)\) yields:
\begin{equation}
\rho \|B(y^\ell - y^*)\|^2 \leq \langle B^T (w^\ell - w^*) - \rho B^T A(x^\ell - x^*), y^\ell - y^* \rangle - \|B\| \|y^\ell - y^*\| \|w^\ell - w^*\| + \rho \|A\| \|x^\ell - x^*\|\]
\end{equation}
where the second inequality follows by Cauchy-Schwarz. Finally, using \((A.9a)\) into \((A.17)\), simplifying and rearranging, the thesis is proved.

**A.2.1. Dual convergence rate.** As an example of the convergence rate corresponding to these dual solvers, we report in the following the case of ADMM. Using the facts that \(\mathrm{cf. Lemma A.1}\) \(\bar{\mu} := \lambda_m(\mathbf{A}^\top \mathbf{A})/\bar{L}\) and \(\bar{L} := \lambda_M(\mathbf{A}^\top \mathbf{A})/\mu\) we have \(\bar{\kappa} = (\bar{L}/\mu)\lambda_M(\mathbf{A}^\top \mathbf{A})/\lambda_m(\mathbf{A}^\top \mathbf{A})\), and \(\mathrm{cf. Lemma 2.5}\)
\[\zeta(\ell) := \begin{cases} 1, & \text{for } \ell = 0, \\ \frac{1}{2} \lambda^\ell, & \text{otherwise} \end{cases} \quad \text{and} \quad \xi(\ell) := \begin{cases} 0, & \text{for } \ell = 0, \\ 1 + \frac{1}{2} \lambda^\ell, & \text{otherwise} \end{cases}\]
where \(\mathrm{cf. Table 5}\)
\[\frac{1}{\beta} = 1 + \rho \bar{L} \quad \text{and} \quad \lambda = \max \left\{ \frac{1 - \rho \bar{L}}{1 + \rho \bar{L}}, \frac{1 - \rho \bar{\mu}}{1 + \rho \bar{\mu}} \right\}.\]

All the convergence rates of the other solvers can be derived similarly using the results reported in Table 5.
A.3. Rank deficient $A$. As mentioned in Remark 6.2, there are two scenarios in which requiring $A$ to be full row rank is actually not necessary.

A.3.1. Decentralized ADMM. One of them is the case of decentralized problems, in which $A$ encodes consistency (consensus) constraints between the agents cooperating toward the solution of the problem. In this case, $A$ has a very specific structure which can be leveraged to derive a linear convergence result for ADMM:

$$\|z^{\ell+1} - z^*\| \leq \zeta \|z^\ell - z^*\| \quad \text{and} \quad \|x^{\ell} - x^*\| \leq \chi \|z^{\ell} - z^*\|.$$ 

for appropriate definitions of $\zeta$ and $\chi$, and where $\|\cdot\|$ may represent a generic norm, see e.g. [45, Theorem 1] or [30, Theorem 3].

As a consequence, these decentralized ADMM formulations fit into the proposed framework and we can prove linear convergence using the results presented in section 5.

A.3.2. Reduced problem. The second scenario in which linear convergence can be proved despite $A$ being rank deficient is when we solve the reduced form of (A.3):

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{s.t.} \quad Ax = c$$

where $f \in S_{\mu,L}(\mathbb{R}^n)$ and $c \in \text{im}(A)$. In this case, it is possible to prove that $d^\ell(w) = f^*(A^\ell w) - (w, c)$ is $\mu$-strongly convex and $L$-smooth in the subspace $\text{im}(A)$, see [46, Theorem 1]. As a consequence, the operators $I - \rho \nabla d^\ell(\cdot)$ and $\text{prox}_{\rho d^\ell} -$ which characterize the dual ascent and the method of multipliers – are contractive in $\text{im}(A)$ with contraction constants $\max\{1 - \rho L, 1 - \rho \mu\}$ and $1/(1 + \rho \mu)$, respectively (cf. Table 5). Moreover, the dual variable is updated in both algorithms through the equation $w^{\ell+1} = w^\ell - \rho(Ax^\ell - c)$, which guarantees that, if $w^\ell \in \text{im}(A)$, then so $w^{\ell+1} \in \text{im}(A)$ (since recall that $c \in \text{im}(A)$). By induction, if $w^0 \in \text{im}(A)$, it follows that $\text{im}(A)$ is an invariant space for the dual trajectory $\{w^\ell\}_{\ell \in \mathbb{N}}$ generated by either dual ascent or method of multipliers. This implies that the dual variable always belongs to the subspace in which $d^\ell$ is strongly convex and smooth, and thus in which linear convergence can be guaranteed.

Appendix B. Proofs of Section 4.

B.1. A preliminary result.

**Theorem B.1.** Let $f \in S_{\mu,L}(\mathbb{R}^n)$ and $g \in \Gamma_0(\mathbb{R}^n)$, then the solution mapping $S(p) = \{ y \mid \nabla_x f(y) + \partial g(y) \ni p \}$ of the parameterized generalized equation $\nabla_x f(y) + \partial g(y) \ni p$ is single-valued and $\mu^{-1}$-Lipschitz continuous.

**Proof.** The proof follows from [32, Theorem 1], and it is reworked here for completeness and matching notation. The single-valuedness of the solution mapping follows from the strong convexity of $f$, which implies that, for a fixed $p$, $S(p) = \arg \min_y \{ f(y) + g(y) - \langle p, y \rangle \}$ has a unique solution. In order to prove the Lipschitz continuity of the solution mapping, we need to prove that for any $p, q \in \mathbb{R}^n$ it holds $\|S(p) - S(q)\| \leq \mu^{-1}\|p - q\|$. Let $y = S(p)$ and $w = S(q)$, by definition of the solution mapping it holds $p - \nabla_x f(y) \in \partial g(y)$ and $q - \nabla_x f(w) \in \partial g(w)$. Using the definition of subdifferential, the following inequalities are thus verified:

$$\langle z - y, p - \nabla_x f(y) \rangle \leq g(z) - g(y), \quad \langle z - w, q - \nabla_x f(w) \rangle \leq g(w) - g(y),$$

for any choice of $z \in \mathbb{R}^n$. In particular, choosing $z = w$ in the first of (B.1) and $z = y$ in the second of (B.1), and, multiplying it by $-1$, yields $\langle w - y, p - \nabla_x f(y) \rangle \leq g(w) - g(y) \leq \langle w - y, q - \nabla_x f(w) \rangle$, which implies

$$\langle w - y, \nabla_x f(w) - \nabla_x f(y) \rangle \leq \langle w - y, q - p \rangle.$$
Finally, since $\nabla_x f$ is the gradient of a $\mu$-strongly convex function, it holds
\[
(x - y, \nabla_x f(x) - \nabla_x f(y)) \geq \mu \|x - y\|^2,
\]
for any $x, y \in \mathbb{R}^n$. Thus combining (B.2) and (B.3) yields
\[
\mu \|w - y\|^2 \leq \langle w - y, \nabla_x f(w) - \nabla_x f(y) \rangle \leq \|w - y\| \|q - p\|.
\]

B.2. Proof of Lemma 4.1. The following proof is an extension of [16, Theorem 2F.10] when $\nabla_x f$ exists everywhere. First, we define the auxiliary functions: $\psi(y) = \nabla_x f_k(x) + \partial g_k(y)$ and $\psi(y) = \nabla_x f_k(y) - \nabla_x f_{k+1}(y)$ and, by the fact that $\nabla_x f_k(x_k^*) + \partial g_k(x_k^*) \ni 0$ and $\nabla_x f_{k+1}(x_k^*) + \partial g_{k+1}(x_{k+1}^*) \ni 0$, we have $(\psi + \psi)(x_k^*) \ni \psi(x_{k+1}^*)$. Moreover, we can see that
\[
(\psi + \psi)(x_k^*) = \partial g_{k+1}(x_k^*) - \partial g_k(x_k^*) \ni \nabla g_{k+1}(x_k^*) - \nabla g_k(x_k^*).
\]

We define now the function $F(y) = (\psi + \psi)(y)$ and consider the parametric generalized equation $F(y) + p \ni 0$. Under Assumptions 3.1 and 3.2, Theorem B.1 implies that the solution mapping $p \mapsto y(p)$ for this generalized equation is everywhere single valued and Lipschitz continuous with constant $\mu^{-1}$, i.e., $\|y(p) - y(p')\| \leq \|p - p'\|/\mu$. Therefore, setting $p = \nabla g_{k+1}(x_k^*) - \nabla g_k(x_k^*)$ and $p' = -\psi(x_{k+1}^*)$, implies
\[
\|x_k^* - x_k^*\| \leq \|\psi(x_{k+1}^*) - \nabla g_{k+1}(x_k^*) - \nabla g_k(x_k^*)\|/\mu \leq (C_0T_0 + D_0)/\mu.
\]

B.3. Proof of Lemma 4.2. In this section we bound, using Theorem B.1, the error that derives from approximating (3.3) at time $t_{k+1}$ with (3.2). Let us first introduce the following functions:

\[
\begin{align*}
\Psi(y) &= \nabla_x f_{k+1}(x_k^*) + \nabla_x f_{k+1}(x_{k+1}^*)(y - x_{k+1}^*) + \partial g_{k+1}(y), \\
\psi(y) &= \nabla_x f_{k+1}(y) - \left[\nabla_x f_{k+1}(x_{k+1}^*) + \nabla_x f_{k+1}(x_{k+1}^*)(y - x_{k+1}^*)\right].
\end{align*}
\]

Solving (3.3) at time $t_{k+1}$ and (3.2) is equivalent to solving the generalized equations $\nabla_x f_{k+1}(x) + \partial g_{k+1}(x) \ni 0$ and $\nabla_x f_{k+1}(x) + \partial g_k(x) \ni 0$, respectively. As a consequence, it holds that $(\psi + \psi)(x_{k+1}^*) \ni \psi(x_{k+1}^*)$, and that
\[
(\psi + \psi)(x_{k+1}^*) = \nabla_x f_{k+1}(x_{k+1}^*) + \partial g_{k+1}(x_{k+1}^*) = \nabla_x f_{k+1}(x_{k+1}^*) + \partial g_{k+1}(x_{k+1}^*) - \partial g_k(x_{k+1}^*) \ni \nabla g_{k+1}(x_{k+1}^*) - \nabla g_k(x_{k+1}^*)
\]

Consider now the parametrized generalized equation $(\psi + \psi)(x) \ni p$ and denote by $S(p)$ its solution mapping. Using the above characterizations of the correction and prediction problems' solutions, we have $S(\nabla g_{k+1}(x_{k+1}^*) - \nabla g_k(x_{k+1}^*)) = x_{k+1}$ and $S(\psi(x_{k+1}^*)) = x_{k+1}$. Applying Theorem B.1 thus implies that the approximation error satisfies:
\[
\|x_{k+1} - x_{k+1}\| \leq \mu^{-1} \|\psi(x_{k+1}^*) - (\nabla g_{k+1}(x_{k+1}^*) - \nabla g_k(x_{k+1}^*))\| \leq \mu^{-1} (\|\psi(x_{k+1}^*)\| + D_0)
\]

(B.4)
where for the second inequality we used the triangle inequality and Assumption 3.5(ii).

We can then bound the term $\|\psi(x_{k+1}^*)\|$ following the steps detailed in [47, Appendix B] under Assumptions 3.1 and 3.2, which yields

\[
(B.5) \quad \|\psi(x_{k+1}^*)\| \leq 2L \|x_k - x_k^*\| + 2L \|x_{k+1}^* - x_k^*\| + 2C_0 T_s.
\]

Combining (B.5) with Lemma 4.1 and substituting into (B.4) then yields:

\[
\|\hat{x}_{k+1}^* - x_{k+1}^*\| \leq 2\kappa \|x_k - x_k^*\| + 2(1 + \kappa)C_0 T_s / \mu + (1 + 2\kappa)D_0 / \mu.
\]

The thesis follows by using the fact that $1 + 2\kappa < 2(1 + \kappa)$.

Moreover, if Assumption 3.3 holds as well, then we have the bound

\[
(B.6) \quad \|\psi(x_{k+1}^*)\| \leq (C_1 / 2) \|x_{k+1}^* - x_k\|^2 + T_s C_2 \|x_{k+1}^* - x_k\| + T_s^2 C_3 / 2
\]

and using $\|x_{k+1}^* - x_k\| \leq \|x_{k+1}^* - x_k^*\| + \|x_k - x_k^*\|$ along with Lemma 4.1 yields the bound:

\[
(B.7) \quad \|\hat{x}_{k+1}^* - x_{k+1}^*\| \leq \frac{C_1}{2\mu} \|x_k - x_k^*\|^2 + C_4 \|x_k - x_k^*\| + C_5 T_s^2 / 2 + C_6 D_0,
\]

where $C_4 := T_s (C_0 C_1 / \mu^2 + C_2 / \mu) + C_1 D_0 / \mu^2$, $C_5 := C_0^2 C_1 / \mu^3 + 2C_0 C_2 / \mu^2 + C_3 / \mu$, and $C_6 := T_s (C_0 C_1 / \mu + C_2) / \mu^2 + (1 + C_1 D_0 / 2\mu^2) / \mu$.

**B.4. Proof of Lemma 4.4.** We can observe that, by the mean value theorem, for any $\tau \in [t_{k-1}, t_k]$:

\[
\left\|\nabla_{tx} f(x; t) - (1 / T_s) \nabla_{tx} f(x; t_k) + \nabla_{tx} f(x; \tau) - \nabla_{tx} f(x; t_k)\right\| \leq C_0.
\]

Since the bound derived in Lemma 4.2 depends on $C_0$, then the result applies also when we use approximate time-derivatives.

Moreover, if Assumption 3.3 holds, then following [49, Eq. (100)] we can see that $\|\nabla_{tx} f(x; t_k) - \nabla_{tx} f(x; t_k)\| \leq T_s C_4 / 2$ and therefore using approximate time-derivatives does indeed introduce an additional error. This means that we need to substitute $\psi(y)$ with $\hat{\psi}(y) = \psi(y) - T_s \nabla_{tx} f(x; t_k) + T_s \nabla_{tx} f(x; t_k)$, in the proof of Lemma 4.2. As a consequence, the distance between the solutions to the prediction and correction problems is bounded by

\[
\|\hat{x}_{k+1} - x_{k+1}^*\| \leq \frac{C_1}{2\mu} \|x_k - x_k^*\|^2 + C_4 \|x_k - x_k^*\| + C_5 T_s^2 / 2 + C_6,
\]

where the first two terms are the same of the exact case, and $C_5 = C_5 + C_3 / \mu$.

**B.5. Proof of Lemma 4.5.** Define the functions $\Psi(y) = \nabla_{x} f_k(x_k + 1)(y) + \partial g_{k+1} (y)$ and $\hat{\psi}(y) = \nabla_{x} \hat{f}_{k+1}(x_{k+1}) - \nabla_{x} f_{k+1}(y)$; by the optimality conditions of the correction and prediction problems we have that $(\Psi + \hat{\psi})(x_{k+1}^*) \supseteq \psi(x_{k+1}^*), \text{ and } (\Psi + \hat{\psi})(\hat{x}_{k+1}^*) \supseteq \nabla g_{k+1}(\hat{x}_{k+1}^*) - \nabla g_k(\hat{x}_{k+1}^*)$. Then, applying Theorem B.1 to the parametrized generalized equation $(\Psi + \hat{\psi})(y) \equiv p$ we have the following bound

\[
\|\hat{x}_{k+1} - x_{k+1}^*\| \leq \|\psi(x_{k+1}) - \nabla g_{k+1}(\hat{x}_{k+1}) - \nabla g_k(\hat{x}_{k+1})\| / \mu \leq (\|\psi(x_{k+1})\| + D_0) / \mu.
\]
where the second inequality follows by triangle inequality and Assumption 3.5(iii). By the interpolation error formula (4.8), we have the bound:

\[
\|\psi(x_{k+1}^*)\| = \left\| \nabla x f_{k+1}(x_{k+1}^*) - \nabla x f_k(x_{k+1}) \right\| \\
\leq \left\| \frac{1}{T^l} \frac{\partial f(l)}{\partial t} \nabla x f(x_{k+1}; \tau) \omega_l(t_{k+1}) \right\| \leq C(T)^l
\]

where \( \tau \in [t_{k-1}, t_{k+1}] \), and we used the facts that \( \omega_l(t_{k+1}) = \prod_{i=1}^{l}(t_{k+1} - t_{k+1-i}) = IT^l \) (cf. (4.8)) and (4.10) to derive the last inequality.

**Appendix C. Proofs of Section 5.**

**C.1. Proof of Proposition 5.1.** First of all, the prediction and correction problems have finite solutions, therefore we know that there exist \( \sigma_k, \tau_k \in (0, +\infty) \) such that (5.1) holds. Moreover, since we apply only \( N_P \) and \( N_C \) steps during prediction and correction, respectively, then by Lemma 2.5 the following hold:

\[
\begin{align*}
(C.1a) \quad & \|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| \leq \zeta(N_P) \|x_k - \bar{x}_{k+1}^*\| \\
(C.1b) \quad & \|x_{k+1} - \hat{x}_{k+1}^*\| \leq \zeta(N_C) \|\hat{x}_{k+1} - \bar{x}_{k+1}^*\|.
\end{align*}
\]

The goal now is to bound the prediction error \( \|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| \). If \( N_P = 0 \) then no prediction steps are applied, and thus, using the triangle inequality, we can write:

\[
\|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| = \|x_k - \bar{x}_{k+1}^*\| \leq \|x_k - x_k^*\| + \|x_k^* - \bar{x}_{k+1}^*\| \leq \|x_k - x_k^*\| + \sigma_k
\]

where we used the facts [cf. Lemma 2.5] that \( \zeta(N_P) = 1 \) and \( \zeta(N_P) = 0 \) if \( N_P = 0 \) to derive the last equality.

Consider now the case of \( N_P > 0 \). By the triangle inequality, the early termination inequality (C.1a), and (5.1), the following chain of inequalities holds:

\[
\begin{align*}
\|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| & \leq \|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| + \|\hat{x}_{k+1} - x_k^*\| \leq \zeta(N_P) \|x_k - x_k^*\| + \tau_k \\
& \leq \zeta(N_P) \left( \|x_k - x_k^*\| + \|x_k^* - x_k^*\| + \|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| \right) + \tau_k \\
& \leq \zeta(N_P) \|x_k - x_k^*\| + \zeta(N_P)\sigma_k + \zeta(N_P)\tau_k
\end{align*}
\]

where the last equality follows by the fact [cf. Lemma 2.5] that \( \zeta(N_P) = 1 + \zeta(N_P) \) if \( N_P > 0 \). Therefore for any \( N_P \geq 0 \) we can bound the prediction error as

\[
\begin{align*}
\|\hat{x}_{k+1} - \bar{x}_{k+1}^*\| & \leq \zeta(N_P) \|x_k - x_k^*\| + \zeta(N_P)\sigma_k + \zeta(N_P)\tau_k
\end{align*}
\]

and combining (C.2) with (C.1b) for the correction step yields (5.2). The error thus converges provided that \( \zeta(N_C)\zeta(N_P) < 1 \).

**C.2. Proof of Theorem 5.6.** By Lemmas 4.1 and 4.2 we know that \( \sigma_k = (C_0 T_s + D_0)/\mu \) and \( \tau_k = 2\kappa \|x_k - x_k^*\| + 2(C_0 T_s + D_0)(1 + \kappa)/\mu \), and by Proposition 5.1 we know that

\[
\|x_{k+1} - x_{k+1}^*\| \leq \zeta(N_C) \left( \zeta(N_P) \|x_k - x_k^*\| + \zeta(N_P)\sigma_k + \zeta(N_P)\tau_k \right).
\]
Combining these facts then yields the following error bound:

(C.3) \[
\|x_{k+1} - x^*_k\| \leq \\
\zeta(N_C) \left( \frac{\zeta(N_P) + 2\kappa \xi(N_P)}{\mu} \|x_k - x^*_k\| + \frac{C_0 T_s + D_0}{\mu} \right) \left( \zeta(N_P) + 2(1 + \kappa) \xi(N_P) \right).
\]

The convergence is thus proved if (5.7) is satisfied.

Indeed, recursively applying (C.3) we get the following bound:

(C.4) \[
\|x_k - x^*_k\| \leq \eta_1^{k} \|x_0 - x^*_0\| + \sum_{\ell=0}^{k-1} \eta_1^{k-\ell} \eta_0 = \eta_1^{k} \|x_0 - x^*_0\| + \frac{1 - \eta_1^{k}}{1 - \eta_1} \eta_0.
\]

where we defined:

\[
\eta_1 = \zeta(N_C) \left( \frac{\zeta(N_P) + 2\kappa \xi(N_P)}{\mu} \right) \|x_0 - x^*_0\| \quad \text{and} \quad \eta_0 := \zeta(N_C) \left( \frac{C_0 T_s + D_0}{\mu} \right) \left( \zeta(N_P) + 2(1 + \kappa) \xi(N_P) \right).
\]

If \( \eta_1 \in (0, 1) \) we have \( \limsup_{k \to \infty} \|x_k - x^*_k\| = \frac{1}{1 - \eta_1} \eta_0 \) and the thesis follows.

**C.3. Proof of Theorem 5.7.** By Lemma 4.2, under Assumption 3.3 we have:

\[
\|\tilde{x}_{k+1} - x^*_k\| \leq \frac{C_1}{2\mu} \|x_k - x^*_k\|^2 + C_4 \|x_k - x^*_k\| + C_5 T_s^2/2 + C_6 D_0 = \tau_k,
\]

and, by Lemma 4.1, we know that \( \sigma_k = (C_0 T_s + D_0)/\mu \). Using these facts into

\[
\|x_{k+1} - x^*_k\| \leq \zeta(N_C) \left( \zeta(N_P) \|x_k - x^*_k\| + \zeta(N_P) \sigma_k + \zeta(N_P) \tau_k \right)
\]

yields the following error bound:

(C.5) \[
\|x_{k+1} - x^*_k\| \leq \eta_2 \|x_k - x^*_k\|^2 + \eta_1 \|x_k - x^*_k\| + \eta_0, \quad \text{where:}
\]

\[
\eta_2 = \zeta(N_C) \xi(N_P) \frac{C_1}{2\mu}, \quad \eta_1 = \zeta(N_C) \left( \zeta(N_P) + \xi(N_P) C_4 \right)
\]

\[
\eta_0 = \zeta(N_C) \left[ \zeta(N_P) \left( \frac{C_0 T_s + D_0}{\mu} + \xi(N_P) \left( \frac{T_s^2}{2} C_5 + C_6 D_0 \right) \right) \right].
\]

Therefore, the error converges provided that there exists \( \gamma \in (0, 1) \) such that

(C.6) \[
\eta_2 \|x_k - x^*_k\|^2 + \eta_1 \|x_k - x^*_k\| + \eta_0 \leq \gamma \|x_k - x^*_k\| + \eta_0
\]

Condition (C.6) is satisfied if \( N_P \) and \( N_C \) are chosen such \( \eta_1 < \gamma \) and if the initial condition guarantees that \( \|x_0 - x^*_0\| \leq \frac{2 - \eta_1}{\eta_2} =: R \). In particular, we can see that \( \eta_1 < \gamma \) holds if \( \gamma > \zeta(N_C) \left( \zeta(N_P) + \xi(N_P) C_1 D_0/\mu^2 \right) \) and

\[
T_s < \frac{\gamma - \zeta(N_C) \left( \zeta(N_P) + \xi(N_P) C_1 D_0/\mu^2 \right) \left( \frac{C_0 C_1}{\mu^2} + \frac{C_2}{\mu} \right)^{-1}}{\zeta(N_C) \xi(N_P)} =: \bar{T}_s
\]

Suppose now that \( N_P, N_C, \) and \( \gamma \) are chosen such that \( \gamma > \zeta(N_C) \left( \zeta(N_P) + \xi(N_P) C_1 D_0/\mu^2 \right), T_s < \bar{T}_s, \) and \( \|x_0 - x^*_0\| < R \). Hence, the norm \( \|x_{k+1} - x^*_k\| \leq \gamma \|x_k - x^*_k\| + \eta_0 \frac{1 - \gamma^k}{1 - \gamma} \) and the thesis is proved as \( k \to +\infty \).
C.4. Proof of Theorem 5.10. By Proposition 5.1 we have
\[ \|x_{k+1} - x^*_k\| \leq \zeta(N_C) \left( \zeta(N_P) \|x_k - x^*_k\| + \zeta(N_P) \sigma_k + \zeta(N_P) \tau_k \right), \]
where, by Lemmas 4.1 and 4.5 we can bound \( \sigma_k \leq (C_0T_s + D_0)/\mu \) and \( \tau_k \leq (C(I)T_s^I + D_0)/\mu \) and the overall error becomes:
\[ \|x_{k+1} - x^*_k\| \leq \zeta(N_C) \zeta(N_P) \|x_k - x^*_k\| + \\
+ \left( \zeta(N_C)/\mu \right) \left[ (C_0 T_s + \zeta(N_P) C(I) T_s^I + D_0 (\zeta(N_P) + \zeta(N_P))) \right]. \]
Choosing \( N_P \) and \( N_C \) such that \( \zeta(N_C) \zeta(N_P) \in (0, 1) \) the thesis follows. \( \square \)

Appendix D. Proofs of Section 6.

D.1. Proof of Proposition 6.3. The expression for the gradient can be derived using the properties of the convex conjugate and the chain rule [20]. First, notice that \( \bar{x}(w, t) \) is the unique solution to the equation \( \psi(x; w, t) := \nabla_x f(x; t) - A^\top w = 0 \), where \( \psi(x; w, t) \) is differentiable in \( x \) with \( \nabla_x \psi(x; w, t) = \nabla_x f(x; t) \) non-singular. Now, set \( \bar{x} = \bar{x}(w, t) \) for ease of read. Fixing \( t \), we can apply the implicit function theorem [16, Theorem 1.B.1] w.r.t. \( w \), which yields
\[ \frac{\partial \bar{x}}{\partial w} = -\nabla_x \psi(\bar{x}; w, t)^{-1} \nabla_w \psi(\bar{x}; w, t) = \nabla_x f(\bar{x}; t)^{-1} A^\top. \]
Similarly, fixing \( w \) and applying [16, Theorem 1.B.1] w.r.t. \( t \) gives
\[ \frac{\partial \bar{x}}{\partial t} = -[\nabla_x \psi(\bar{x}; w, t)]^{-1} \nabla_t \psi(\bar{x}; w, t) = -\nabla_x f(\bar{x}; t)^{-1} \nabla_{tx} f(\bar{x}; t) \]
thus proving the proposition. Notice that applying the same procedure, we can easily show that \( \nabla_w d^f(w; t) = -\nabla_x f(\bar{x}; t)^{-1} \nabla_{tx} f(\bar{x}; t) = \nabla_{tw} d^f(w; t) \).

D.2. Proof of Proposition 6.7. First of all, the choice \( \hat{h}_{k+1}(w) = d^h_k(w) \) clearly implies that \( \hat{h}_{k+1}(y) = h_k(y) \) by the definition of the dual function \( d^h \).

Now, using the formulae for the derivatives of \( d^f \) reported in Proposition 6.3 and the fact that \( A^\top w_k = \nabla_x f_k(\bar{x}_k) \) we can explicitly write
\[ \begin{aligned}
&\hat{d}^f_{k+1}(w) = \hat{d}_k + \langle A\bar{x}_k - c, w - u_k \rangle - T_s \langle \nabla_{xx} f_k(\bar{x}_k) \nabla_{tx} f_k(\bar{x}_k), A^\top w - \nabla f_k(\bar{x}_k) \rangle \\
&+ \frac{1}{2} (A^\top w - \nabla f_k(\bar{x}_k))^\top \nabla_{xx} f_k(\bar{x}_k) (A^\top w - \nabla f_k(\bar{x}_k)) \\
&\text{(D.1)}
\end{aligned} \]

Our aim thus is to show that \( \hat{d}^f_{k+1}(w) = \hat{f}^*_k(A^\top w) - \langle w, c \rangle \) with \( \hat{f}_{k+1} \) defined through (6.12). By the definition of convex conjugate for continuous functions:
\[ \hat{f}^*_k(u) = \max_{x \in \mathbb{R}^n} \left\{ \langle u, x \rangle - \hat{f}_{k+1}(x) \right\}, \]
where \( \max_{x \in \mathbb{R}^n} \left\{ \langle u, x \rangle - \hat{f}_{k+1}(x) \right\} = \bar{x}_k + \nabla_{xx} f_k(\bar{x}_k)^{-1} [u - \nabla f_k(\bar{x}_k)] - T_s \nabla_{tx} f_k(\bar{x}_k) \). Finally, substituting \( u = A^\top w \) and rearranging the thesis follows.

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