Intermediate Assouad-like dimensions

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Abstract. We introduce and study bi-Lipschitz-invariant dimensions that range between the box and Assouad dimensions. The quasi-Assouad dimensions and \( \theta \)-spectrum are other special examples of these intermediate dimensions. These dimensions are localized, like Assouad dimensions, but vary in the depth of scale which is considered, thus they provide very refined geometric information.

We investigate the relationship between these and the familiar dimensions. We construct a Cantor set with a non-trivial interval of dimensions, the endpoints of this interval being given by the quasi-Assouad and Assouad dimensions of the set. We study continuity-like properties of the dimensions. In contrast with the Assouad-type dimensions, we see that decreasing sets in \( \mathbb{R} \) with decreasing gaps need not have dimension 0 or 1.

Formulas are given for the dimensions of Cantor-like sets and these are used in some of our constructions. We also show that, as is the case for Hausdorff and Assouad dimensions, the Cantor set and the decreasing set have the extreme dimensions among all compact sets in \( \mathbb{R} \) whose complementary set consists of open intervals of the same lengths.

1. Introduction

Over the years, many notions of dimension have been introduced to help understand the geometry of (often ‘small’) subsets of metric spaces, such as subsets of \( \mathbb{R}^n \) of Lebesgue measure zero. Hausdorff, box and packing dimensions are well known examples of such notions. More recently, the Assouad dimensions, which quantify the ‘thickest’ or ‘thinnest’ part of the space, were introduced by Assouad in \cite{Assouad} and Larman in \cite{Larman} and have been extensively studied within the fractal geometry community; see for example, \cite{Barlow, BBS, DD, GLS, MD, MDL} and the references cited therein.

The Assouad dimensions can roughly be thought of as local refinements of the box-counting dimensions where one takes the worst local behaviour. For the upper box dimension one considers the minimal number of balls of radius \( r \) that are required to cover the entire space \( E \), say \( N_r(E) \), and computes the infimal exponent \( s \) such that \( N_r(E) \leq r^{-s} \) as \( r \to 0 \). For the upper Assouad dimension (denoted \( \dim_A E \)) one determines, instead, the infimal \( s \) such that \( N_r(B(x, R) \cap E) \leq (R/r)^s \).
for all \( r \leq R \) and all centres \( x \in E \). The lower Assouad dimension, denoted \( \dim_L E \), is a similar local variation of the lower box dimension. The related quasi-Assouad dimensions, denoted \( \dim_{qA} E \) and \( \dim_{qL} E \) and introduced in [4,23], have also been much studied; c.f., [5,9,13,17]. These are moderated versions of the Assouad dimensions, requiring only that the bounds hold for \( r \leq R^p \) where the exponent \( p \) decreases to 1.

For all compact sets \( E \) we have the relationships

\[
\dim_L E \leq \dim_{qL} E \leq \dim_H E \leq \dim_B E \leq \dim_{qA} E \leq \dim_A E,
\]

where \( \dim_H, \dim_B, \dim_{A} \) denote the Hausdorff, lower and upper box dimensions respectively. See Proposition 2.12 for the proof that \( \dim_{qL} E \leq \dim_H E \) and \([6,8,23]\) otherwise. The set \( E \) is doubling if and only if \( \dim_A E < \infty \); [18]. For instance, if \( E \subseteq \mathbb{R}^n \), then \( \dim_A E \leq n \). The set \( E \) is uniformly perfect if and only if \( \dim_L E > 0 \); [19]. These dimensions can all be different, reflecting different geometric properties of the space, thus it is natural to ask about more refined, ‘in-between’ dimensions.

In [7], Falconer, Fraser and Kempton introduced a continuum of dimensions that are intermediate between the Hausdorff and box dimensions by restricting the size of the allowable covers. In a different direction, Fraser and Yu observed in [11] that a rich dimension theory can be developed by considering decreasing continuous functions \( F(x) \leq x \), choosing \( r = F(R) \) (or \( r \leq F(R) \) in the modified case) and studying the corresponding Assouad-like dimensions. In [9,11,12], Fraser with various coauthors studied the special case of \( F(x) = x^{1/\theta} \) for (fixed) \( \theta \in (0,1) \). This gives rise to a family of dimensions, known as the upper (or lower) \( \theta \)-spectrum, which lie between the upper (or lower) box and quasi-upper (resp., lower) Assouad dimensions. As \( \theta \to 1 \), the \( \theta \)-spectrum tends (by definition) to the quasi-Assouad dimension. As \( \theta \to 0 \), the upper \( \theta \)-spectrum tends to the upper box dimension, but the analogous statement need not be true for the lower \( \theta \)-spectrum, even when the lower box dimension of the set is positive; [5].

Motivated by the work of Fraser, in this paper we consider the very general class of functions \( F(R) = R^{1+\Phi(R)} \), which we require only to decrease to 0 as \( R \to 0 \), and the upper and lower \( \Phi \)-dimensions, denoted \( \overline{\dim}_{\Phi} \), \( \underline{\dim}_{\Phi} \) respectively, where we consider \( r \leq R^{1+\Phi(R)} \). (See Subsection 2.1 for definitions.) When \( \Phi = 0 \) we recover the Assouad dimension and when \( \Phi = 1/\theta - 1 \), this gives the modified \( \theta \)-spectrum. Moreover, it can be shown that if \( \Phi \to \infty \) (and \( \dim_B E > 0 \)) the upper (or lower) \( \Phi \)-dimension is the upper (resp., lower) box dimension. Provided \( \Phi \to 0 \), the \( \Phi \)-dimensions will be intermediate between the quasi-Assouad and Assouad dimensions. Indeed, in Proposition 2.11 we show that for any set \( E \) there are functions \( \Phi_1, \Phi_2 \) such that the \( \Phi_i \)-dimensions of \( E \) are the quasi-Assouad dimensions. These intermediate dimensions preserve bi-Lipschitz equivalence and provide more detailed and precise information about the geometric scaling structure of the set.

**Main Results**

The primary purpose of this paper is to study the basic properties of these intermediate dimensions. A natural question to ask is how the \( \Phi \)-dimensions compare, both with each other and to the familiar dimensions. Of course, they are naturally ordered: if \( \Phi_1 \leq \Phi_2 \), then \( \overline{\dim}_{\Phi_1}(E) \geq \overline{\dim}_{\Phi_2}(E) \) and vice versa for the lower \( \Phi \)-dimensions.

\[1\] This is the only inequality requiring \( E \) to be closed.
Here are some of our main theoretical results on this question.

(i) If \( \Phi(x) \leq c/|\log x| \) then the \( \Phi \)-dimension coincides with the Assouad dimension for all sets \( E \). (Proposition 2.6)

(ii) If \( \Phi_1/\Phi_2(x) \to 1 \) as \( x \to 0 \), then the two dimensions coincide for all sets \( E \). (Proposition 2.7)

(iii) If \( \Phi_1 \) is bounded above away from \( \Phi_2 \), then there will be a set \( E \) with different \( \Phi_1 \) and \( \Phi_2 \)-dimensions. (Theorem 3.6)

(iv) Given any family of decreasing dimension functions, \( \{\Phi_p\}_{p \in (0,1)} \), with \( \Phi_p \gg \Phi_q \) for \( p > q \), and given any decreasing, continuous function \( d : (0,1) \to [a,b] \subseteq (0,1) \), there is a set \( E \subseteq \mathbb{R} \) with \( \dim_{\Phi_p}(E) = d(p) \) for all \( p \). The analogous result holds for the lower dimensions. (Theorem 3.7)

Hence there are subsets of \( \mathbb{R} \) with a full (non-trivial) interval of dimensions whose endpoints are given by the quasi-Assouad and Assouad dimensions. (Corollary 3.9)

In \([5, 12]\), it is shown that the \( \theta \)-spectrum is continuous in \( \theta \). More generally, the following is true.

(v) If \( \Phi_t(x) = g(t)\Phi(x) \) and \( g \) is continuous, then \( \dim_{\Phi_t}(E) \to \dim_{\Phi_{t_0}}(E) \) if \( t \to t_0 \neq 0 \) and similarly for the lower dimensions. This need not be true when \( t_0 = 0 \). (Propositions 2.13 and 3.4)

This suggests that it may be difficult to find a one-parameter family of continuous dimension functions that interpolate precisely between the quasi-Assouad and Assouad dimensions.

In \([14]\) it was shown that if \( E = \{x_n\}_n \subseteq \mathbb{R} \) is a decreasing sequence with decreasing gaps, then the Assouad dimension is either 0 or 1. Likewise, the quasi-Assouad dimension is 0 if \( \dim_{B}E = 0 \) and 1 otherwise. In contrast, in Example 2.18 we construct a decreasing set \( E \) with decreasing gaps and a dimension function \( \Phi \to 0 \) with \( \dim_{qA}E = 0 < \dim_{B}E < \dim_{A}E = 1 \).

Many of our examples are Cantor-like sets. For such sets there are known formulas for the Hausdorff, box and Assouad dimensions. Similar formulas are given in Theorem 3.3 for the \( \Phi \)-dimensions and these are used in some of our constructions, such as in exhibiting sets with different values for various \( \Phi \)-dimensions.

Every compact subset \( E \subseteq [0,1] \) of Lebesgue measure zero can be viewed as a rearrangement of a Cantor-like set in the sense that the complement of \( E \) in \([0,1]\) consists of countably many disjoint open intervals whose lengths are the same as the lengths of the complementary intervals of a Cantor set. Besicovitch and Taylor in \([3]\) proved that if \( C \) is a Cantor set, then the interval \([0,\dim_{H}C]\) is precisely the set of Hausdorff dimensions of ‘rearrangements’ of \( C \). This was later extended to the Assouad dimensions in \([14]\) where it was found that under natural assumptions on the decay of the gap sizes, the set of attainable dimensions was \([0,\dim_{L}C]\) and \([\dim_{A}C,1]\) for the lower and upper Assouad dimensions respectively, with 0 and 1 being the Assouad dimensions of the decreasing rearrangement.

In Section 4 we study the analogous problem for the \( \Phi \)-dimensions. We again show that under natural assumptions, the \( \Phi \)-dimensions of any rearranged set lie between the dimensions of the Cantor set and the decreasing rearrangement (which might be \( < 1 \) in the case of the upper \( \Phi \)-dimension) and for any \( \Phi \to p \) with \( 0 \leq p \leq \infty \) (including the quasi-Assouad dimensions), this full range of values can be attained. While some of these results can be proven by methods analogous to the Assouad dimension case, for others a different approach is required.
2. Basic Properties of the $\Phi$ dimensions

2.1. Definitions. Given a metric space $X$, we denote the ball centred at $e \in X$ with radius $R$ by $B(e, R)$. For a bounded set $E \subseteq X$, the notation $N_r(E)$ will mean the least number of balls of radius $r$ that cover $E$.

**Definition 2.1.** By a dimension function, we mean a map $\Phi : (0, 1) \to \mathbb{R}^+$ that has the property that $R^{1+\Phi(R)}$ decreases as $R$ decreases to 0.

Of course, $R^{1+\Phi(R)} \leq R$, so $R^{1+\Phi(R)} \to 0$ as $R \to 0$ for any dimension function $\Phi$. Special examples of dimension functions include the constant functions $\Phi$.

**Definition 2.2.** Let $\Phi$ be a dimension function and $X$ a metric space. The **upper** and **lower** $\Phi$-dimensions of $E \subseteq X$ are given by

$$\overline{\dim}_\Phi E = \inf \left\{ \alpha : (\exists c_1, c_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < c_1) \sup_{e \in E} N_r(B(e, R) \cap E) \leq c_2 \left( \frac{R}{r} \right)^\alpha \right\}$$

and

$$\underline{\dim}_\Phi E = \sup \left\{ \alpha : (\exists c_1, c_2 > 0)(\forall 0 < r \leq R^{1+\Phi(R)} \leq R < c_1) \inf_{e \in E} N_r(B(e, R) \cap E) \geq c_2 \left( \frac{R}{r} \right)^\alpha \right\}.$$

The **upper Assouad** and **lower Assouad dimensions** of $E$, denoted $\dim_A E$ and $\dim_L E$ respectively, are the special cases of the upper and lower $\Phi$-dimensions with $\Phi = 0$.

If we let $\Phi_\delta(x) = \delta$ and put $1/\theta = 1+\delta$, then the upper and lower $\Phi_\delta$ dimensions are the **upper and lower Assouad spectrum**, $\overline{\dim}_A^\delta E$ and $\underline{\dim}_A^\delta E$, studied in [5], [9] and [11]. (To be precise, the upper and lower Assouad spectrum defined in [5] and [11] only requires consideration of $r = R^{1/\theta}$, however it is shown in [9] that if we denote this dimension by $\overline{\dim}_A^{1/\theta} E$, then $\overline{\dim}_A^\delta E = \sup_{\psi \leq \theta} \overline{\dim}_A^{\psi} E$, and similarly for the lower Assouad spectrum; see [5] or [17].)

The **upper quasi-Assouad** and **lower quasi-Assouad dimensions**, denoted $\dim_{qA} E$ and $\dim_{qL} E$, are defined as the limit as $\delta \to 0$ of the upper and lower $\Phi_\delta$ dimensions, respectively. In Proposition 2.11 we will see that the quasi-Assouad dimensions can also be attained as $\Phi$-dimensions, but the choice of $\Phi$ depends on the set $E$. We refer the reader to the references cited in the Introduction of this paper for further information on the (quasi-)Assouad dimensions.

We will always assume the underlying metric space $X$ is **doubling**, which means that there is a doubling constant $M \geq 1$ so that for any $R > 0$, each ball of radius $R$ can be covered with at most $M$ balls of radius $R/2$. This condition is equivalent to saying the space has bounded upper Assouad dimension. For example, if $E \subseteq \mathbb{R}^d$, then $\dim_A E \leq d$.

As a set and its closure have the same $\Phi$-dimensions, unless we say otherwise we will assume all sets are compact.

**Remark 2.3.** As with box dimension, in the definitions of the $\Phi$-dimensions the covering number $N_r(B(e, R) \cap E)$ could be replaced by the packing number...
$$P_r(B(e,R) \cap E),$$ the maximum number of disjoint balls of radius $r$, centred in $B(e,R) \cap E$. This is because, in any doubling metric space, packing and covering numbers are comparable in the sense that there is a constant $c > 0$ such that for any $F \subseteq X$, $\frac{1}{c} N_{2r}(F) \leq P_r(F) \leq cN_{r/2}(F)$.

### 2.2. Relationships between dimensions

We begin by recalling the relationships between Assouad-like dimensions and various classical dimensions. We denote by $\dim_H$, $\dim_B$ and $\underline{\dim}_B$ the Hausdorff, lower box and upper box dimensions respectively, which satisfy

$$\dim_H E \leq \underline{\dim}_B E \leq \underline{\dim}_B E \leq \dim_B E \leq \dim_A E.$$

We refer to [4] for the definitions and basic properties of these dimensions.

Clearly we have the following relationships:

$$\dim_L E \leq \dim_{qL} E \leq \underline{\dim}_B E \leq \dim_B E \leq \dim_{qA} E \leq \dim_A E$$

and

$$\dim_L E \leq \underline{\dim}_\Phi E \leq \underline{\dim}_{\Phi} E \leq \dim_B E \leq \dim_A E.$$

Since $\underline{\dim}_B E \geq \dim_H E$, we obviously have

$$\dim_{qA} E \geq \dim_H E.$$

In [22] it was shown that if $E$ is closed, then $\dim_L E \leq \dim_H E$. Using similar arguments one can prove that

$$\dim_{qL} E \leq \dim_H E;$$

see Proposition 2.12. This inequality is not true in general: for the set of rational numbers $\mathbb{Q}$, $\dim_H \mathbb{Q} = 0$, but $\dim_{qL} \mathbb{Q} = 1$.

Obviously, if $\Phi \leq \Psi$, then

$$\underline{\dim}_\Phi E \leq \underline{\dim}_\Psi E \quad \text{and} \quad \underline{\dim}_\Phi E \leq \underline{\dim}_\Psi E.$$

Consequently, if $\Phi(x) \to 0$ as $x \to 0$, then the $\Phi$-dimensions give a range of dimensions between the Assouad and quasi-Assouad type dimensions:

$$\dim_L E \leq \underline{\dim}_\Phi E \leq \dim_{qL} E \leq \dim_{qA} E \leq \underline{\dim}_\Phi E \leq \dim_A E.$$

**Remark 2.4.** We remark that in Section 3 of this paper many examples are constructed which demonstrate strictness in these inequalities. These are based on the formulas given in Theorem 3.3 for the $\Phi$-dimensions of Cantor sets. The reader can also refer to [13, Ex. 16] or [23, Ex. 1.18] for similar constructions illustrating the strictness of the relationship between the quasi-Assouad and Assouad dimensions.

The aim of this subsection is to give more detailed information than the above inequalities. First, we consider the relationship between the box and $\Phi$-dimensions.

**Proposition 2.5.** (i) For any dimension function $\Phi$,

$$\underline{\dim}_\Phi E \leq \dim_B E \leq \underline{\dim}_B E \leq \underline{\dim}_\Phi E.$$

(ii) If $\Phi(x) \to \infty$, then $\underline{\dim}_\Phi E = \underline{\dim}_B E$. If, in addition, $\dim_{\Phi} E > 0$, then $\underline{\dim}_\Phi E = \dim_B E$. 


PROOF. (i) Suppose $b = \dim_B E$ and $b - \varepsilon = \dim_Φ E$ for some $\varepsilon > 0$. Given small $r$, choose $R$ such that $r = R^{β(R)}$ where $β(R) = \max(1 + Φ(R), 4)$. It is easy to see that

$$N_r(E) \leq \sup_{r} N_r(B(e, R) \cap E) \cdot N_R(E).$$

If $R$ is small enough, then for some constant $c$,

$$N_r(E) \leq c \left( \frac{R}{r} \right)^{b-\varepsilon/2} R^{-β(R)(b-\varepsilon/2+3\varepsilon/(2β(R))} \leq cr^{-(b-\varepsilon/8)}.$$ 

This implies $\dim_B E \leq b - \varepsilon/8$, which is a contradiction.

The arguments are similar to show the lower box dimension is an upper bound for the lower $Φ$-dimensions, but using packing numbers instead of covering numbers, both for the lower $Φ$-dimensions and the lower box dimension.

(ii) If $Φ → \infty$ as $x → 0$, then $Φ(x) > p$ for any $p$, provided $x$ is sufficiently small. Thus the monotonicity of the $Φ$-dimensions implies that if $θ = (1+p)^{-1}$ and $Ψ(x) = p$, then $\dim_B E \leq \dim_Φ E \leq \dim_Φ E = \dim_A E$. The result for the upper dimension then follows since the upper Assouad spectrum converges to $\dim_B E$ as $p → \infty$, as a consequence of [9] Theorem 2.1] and [11] Proposition 3.1].

For the lower dimension case, suppose that $\dim_Φ E = d > 0$ and let $0 < \epsilon < d/2$ be given. Since $Φ(x) → \infty$, we have

$$\frac{R^{d-\epsilon}}{r} \geq \frac{R^{d-\epsilon}}{R^{1+Φ(R)}} \to \infty \text{ implies } \left( \frac{R}{r} \right)^{d-\epsilon} \to \infty$$

as $R → 0$, provided $r ≤ R^{1+Φ(R)}$. We note that for $r ≤ R^{1+Φ(R)} ≤ \text{diam}(E)$,

$$N_r(E) \geq N_r(B(e, R) \cap E) \geq C \left( \frac{R}{r} \right)^{d-\epsilon} \geq C r^{2\epsilon-d}$$

and thus $\dim_B E \geq d - 2\epsilon$ for all such $\epsilon$. This implies that $\dim_B E \geq d$. Since we know that $\dim_Φ E \leq \dim_0 E$, we obtain equality. \hfill \Box

It is natural to ask when two dimension functions give rise to the same dimensions for all sets $E$. Here are two positive results.

PROPOSITION 2.6. If there is some constant $c$ such that $Φ(x) ≤ c/|\log x|$, then $\dim_L E = \dim_Φ E$ and $\dim_A E = \dim_Φ E$. 

PROOF. This is simply due to the fact that if $Φ(x) ≤ c/|\log x|$, then $R^{Φ(R)}$ is bounded above and below from 0. \hfill \Box

PROPOSITION 2.7. Suppose $Φ_1, Φ_2$ are dimension functions and $Φ_1(x)/Φ_2(x) → 1$ as $x → 0$. Then $\dim_Φ E = \dim_Φ E$ and $\dim_Φ E = \dim_Φ E$ for all sets $E$.

From this, we immediately deduce the following.

COROLLARY 2.8. If $Φ(x) → p$ as $x → 0$ for some $0 < p < \infty$, then for $θ = (1+p)^{-1}$,

$$\dim_Φ E = \dim_Φ E \text{ and } \dim_Φ E = \dim_Φ E.$$

The proof of the proposition is an easy consequence of the estimates in the Lemma below, which will also be useful for proving continuity results in Proposition 2.13.
Lemma 2.9. Let $\Phi$ and $\Psi$ be dimension functions and assume that for some $\epsilon > 0$ we have $|\Phi(x)/\Psi(x) - 1| \leq \epsilon$ for all $x$ small.

(i) For any $e \in E$ and $0 < r \leq R^{1 + \Psi(R)}$,
\begin{equation}
N_e(B(e, R) \cap E) \leq C \left( \frac{R}{r} \right)^{(\dim_{\Phi,E} + \epsilon)(1+\epsilon)}
\end{equation}
whenever $R \leq \rho_\epsilon$, with $\rho_\epsilon$ the corresponding constant $c_1$ from the definition of $\dim_{\Phi,E}$ (with $\alpha = \dim_{\Phi,E} + \epsilon$).

(ii) Analogously, for any $e \in E$ and $0 < r \leq R^{1 + \Phi(R)} \leq R \leq \rho_\epsilon$,
\begin{equation}
N_e(B(e, R) \cap E) \geq c \left( \frac{R}{r} \right)^{\max(\dim_{\Phi,E} - \epsilon, \dim_{\Psi,E} + \epsilon)(1+\epsilon)}
\end{equation}
where $M$ is the doubling constant of the space.

Proof. (i) Let $d = \dim_{\Phi,E}$ and pick $0 < r \leq R^{1 + \Psi(R)} < R \leq \rho_\epsilon$. If $r \leq R^{1 + \Phi(R)}$, then (2.1) follows by the definition of $\dim_{\Phi,E}$. Otherwise, $R^{1 + \Phi(R)} < r \leq R^{1 + \Psi(R)}$, hence $\Psi(R) < \Phi(R)$ and therefore $0 < \Phi(R) - \Psi(R) \leq \epsilon\Psi(R)$. Consequently,
\begin{align*}
N_e(B(e, R) \cap E) \leq N_{R^{1 + \Phi(R)}}(B(e, R) \cap E) &\leq C \left( \frac{R}{R^{1 + \Phi(R)}} \right)^{d + \epsilon} \\
&\leq C \left( \frac{R}{r} \right)^{d + \epsilon} \left( \frac{1}{R} \right)^{(\Phi(R) - \Psi(R))(d + \epsilon)} \\
&\leq C \left( \frac{R}{r} \right)^{d + \epsilon} \left( \frac{1}{R} \right)^{(\Psi(R))(d + \epsilon)} \leq C \left( \frac{R}{r} \right)^{(d + \epsilon)(1+\epsilon)}.
\end{align*}

(ii) Now let $d = \dim_{\Phi,E}$ and pick $0 < r \leq R^{1 + \Psi(R)} < R \leq \rho_\epsilon$. Again, if $r \leq R^{1 + \Phi(R)}$, then (2.2) follows by the definition of $\dim_{\Phi,E}$. Otherwise, $R^{1 + \Psi(R)} < r \leq R^{1 + \Phi(R)}$, hence $\Phi(R) < \Psi(R)$ so $\Phi(R)(1 - \epsilon) \leq \Phi(R)$. Then,
\begin{align*}
N_e(B(e, R) \cap E) &\geq N_{R^{1 + \Phi(R)}}(B(e, R) \cap E) \geq N_{R^{1 + (1-\epsilon)\Psi(R)}}(B(e, R) \cap E).
\end{align*}
But $R^{1 + (1-\epsilon)\Psi(R)} = R^{1 + \Psi(R)} R^{-\epsilon\Psi(R)} = R^{1 + \Psi(R)} 2^T$ with $T = \epsilon\Psi(R) \log_2 R^{-1}$, and since the space is doubling with constant $M$, iterating this definition we have that each ball of radius $R^{1 + (1-\epsilon)\Psi(R)}$ can be covered by at most $M^T$ balls of radius $R^{1 + \Psi(R)}$. Therefore,
\begin{align*}
N_{R^{1 + (1-\epsilon)\Psi(R)}}(B(e, R) \cap E) &\geq M^{-T} N_{R^{1 + \Psi(R)}}(B(e, R) \cap E) \\
&\geq cM^{-T} \left( \frac{R}{R^{1 + \Psi(R)}} \right)^{d - \epsilon} = cR^{\epsilon\Psi(R)} \log_2 M \left( \frac{1}{R^{\Phi(R)}} \right)^{d - \epsilon} \\
&= c \left( \frac{1}{R^{\Psi(R)}} \right)^{d - \epsilon(1 + \log_2 M)} \geq c \left( \frac{R}{r} \right)^{d - \epsilon(1 + \log_2 M)}.
\end{align*}
Here the last inequality holds since $R^{1 + \Psi(R)} < r$. \qed

Proof of Proposition 2.7. The assumption $\Phi_1(x)/\Phi_2(x) \to 1$ as $x \to 0$ ensures that for all $\epsilon > 0$ there is some $\rho'_\epsilon$ such that $|\Phi_1(x)/\Phi_2(x) - 1| \leq \epsilon$ for $R \leq \rho'_\epsilon$. Thus (2.1) holds for all $R \leq \min(\rho_\epsilon, \rho'_\epsilon)$ and that implies $\dim_{\Phi,E} \leq (\dim_{\Phi,E} + \epsilon)(1 + \epsilon)$.
for any \( \epsilon > 0 \), so that \( \dim_{\Phi_2} E \leq \dim_{\Phi_1} E \). Since the roles of \( \Phi_1 \) and \( \Phi_2 \) can be interchanged because the condition is symmetric, the opposite inequality also holds.

The statement for the lower dimensions follows in the same way. Observe that by symmetry there is no need to consider separately the hypothetical case when one of the dimensions is zero. \( \square \)

Later, in Theorem 3.9 we will see that if \( \Phi_1 \) is bounded above away from \( \Phi_2 \), then the functions give rise to different dimensions.

**Remark 2.10.** To summarize, the \( \Phi \)-dimensions coincide with the Assouad dimension if \( \Phi \) tends to 0 very quickly. If \( \Phi(x) \) is monotonic and does not tend to 0, then the \( \Phi \)-dimensions coincide with either the box dimensions or \( \theta \)-spectrum. The \( \Phi \)-dimensions lie between the quasi-Assouad and Assouad dimensions if and only of \( \Phi(R) \to 0 \) as \( R \to 0 \). This is the case we will be primarily interested in.

In fact, the quasi-Assouad dimensions can also be understood as special cases of \( \Phi \)-dimensions, but the functions \( \Phi \) need to be tailored for the specific set.

**Proposition 2.11.** For any \( E \subseteq X \), there are dimension functions \( \Phi_1, \Phi_2 \) (depending on \( E \)), which tend to 0 and satisfying \( \dim_{\Phi_1} E = \dim_{\Phi_2} E \).

**Proof.** Since our choice of dimension functions will satisfy \( \Phi_i \to 0 \), it will automatically be true that \( \dim_{\Phi_i} E \geq \dim_{\Phi_j} E \) and \( \dim_{\Phi_1} E \leq \dim_{\Phi_2} E \). Thus we need only check the opposite inequalities.

First, consider the quasi-Assouad dimension. Put \( d = \dim_{\Phi_i} E \). By definition, for each \( n \in \mathbb{N} \), there are \( \delta_n \downarrow 0 \) and \( \rho_n \downarrow 0 \) such that for all \( r \leq R^{1+\delta_n} \leq R \leq \rho_n \), and \( e \in E \), we have

\[
N_e(B(e, R) \cap E) \leq \left( \frac{R}{r} \right)^{d+1/n}.
\]

Put \( n_1 = 1 \) and inductively define a subsequence \( \{n_j\} \) so that \( \rho_{n+1}^{1+\delta_{n-1}} \geq \rho_{n+1}^{1+\delta_{n-1}} \). For notational convenience, we will put \( p_j = \rho_{n_j} \) and \( \varepsilon_j = \delta_{n_j} \). Since \( \varepsilon_{j-1} > \varepsilon_j \), we have \( p_{j+1}^{1+\varepsilon_{j-1}} \leq p_{j+1}^{1+\varepsilon_j} \), hence there is some \( q_j \in \{p_{j+1}, p_j\} \) with \( q_j^{1+\varepsilon_{j-1}} = p_{j+1}^{1+\varepsilon_j} \).

We are now ready to define \( \Phi_1 \). For \( R \in (q_j, p_j] \) we put \( \Phi_1(R) = \varepsilon_{j-1} \), while for \( R \in (p_{j+1}, q_j] \) we define \( \Phi_1(R) \) by the rule that \( R^{1+\Phi_1(R)} = p_{j+1}^{1+\varepsilon_j} \). Observe that in either case, \( \Phi_1(R) \geq \varepsilon_j \). It is straightforward to verify that the function \( R^{1+\Phi_1(R)} \) decreases to 0 as \( R \) decreases to 0, and therefore \( \Phi_1 \) is a dimension function.

If \( R \in (p_{j+1}, q_j] \), then \( R^{1+\varepsilon_{j-1}} \leq q_j^{1+\varepsilon_{j-1}} = R^{1+\Phi_1(R)} \) and thus \( \Phi_1(R) \leq \varepsilon_{j-1} \). This shows that \( \Phi_1 \) tends to 0.

We will check that \( \dim_{\Phi_1} E \leq d + 1/n \) for any given \( N \). To do this, choose any \( R \leq p_N, r \leq R^{1+\Phi_2(R)} \) and \( e \in E \). Find \( k \geq N \) so \( R \in (p_{k+1}, p_k] \) so that \( \Phi_1(R) \geq \varepsilon_k \). Thus \( r \leq R^{1+\varepsilon_k} = R^{1+\delta_{n_k}} \) and consequently, since \( n_k \leq n_N \), \( N_e(B(e, R) \cap E) \leq \left( \frac{R}{r} \right)^{d+1/n_k} \leq \left( \frac{R}{r} \right)^{d+1/n_N} \), completing the verification.

The argument for the quasi-lower Assouad dimension is the same, building \( \Phi_2 \) using the fact that for each \( n \in \mathbb{N} \), there are \( \delta_n \downarrow 0 \) and \( \rho_n \downarrow 0 \) such that for all \( r \leq R^{1+\delta_n} \leq R \leq \rho_n \) and \( e \in E \), we have

\[
N_e(B(e, R) \cap E) \geq \left( \frac{R}{r} \right)^{d+1/n}.
\]
and the proposition follows.

As was shown in [11], we have \( \dim_{\mathcal{A}}^0 E \to \dim_{\mathcal{B}} E \) as \( \theta \to 0 \). The same statement is false if we replace upper by lower dimensions, since the lower box dimension need not be attained as \( \theta \to 0 \) (see [5]). Furthermore, it is also known that the lower spectrum is not uniformly bounded above by the Hausdorff dimension. (This follows from the results in [5] and [12].) However, we have the following relationship between the lower spectrum and Hausdorff dimensions that does not seem to have been previously observed.

**Proposition 2.12.** If \( E \) is a closed subset of \( X \), then \( \dim_{\mathcal{A}}^0 E \leq \frac{1}{\theta} \dim_{\mathcal{H}} E \) for any \( \theta \in (0, 1) \). In particular, \( \dim_{\mathcal{AP}} E \leq \dim_{\mathcal{H}} E \).

**Proof.** Our proof is based on the method of proof of [22, Theorem 6]. Fix \( \theta \in (0, 1) \) and recall that \( \dim_{\mathcal{A}} E = \dim_{\mathcal{B}} E \) for \( \delta = 1/\theta - 1 \). If \( \dim_{\mathcal{A}} E = 0 \) there is nothing to prove, so assume \( \alpha < \dim_{\mathcal{A}} E \) for some \( \alpha > 0 \). We will show \( \dim_{\mathcal{H}} E \geq \alpha/(1 + \delta) \). This will prove the proposition.

The doubling property ensures that \( P_{2r}(F) \geq cP_r(F) \) for any \( F \subseteq X \), where the positive constant \( c \) depends only on the doubling constant of the space. Also, we pick \( \rho_\delta > 0 \) such that for any \( e \in E \) and any \( r \leq 2^{1+\delta} \leq R \leq \rho_\delta \),

\[
P_r(B(e, R) \cap E) \geq c^{-1}(R/r)^\alpha.
\]

In particular, \( P_{2R^{1+\delta}}(B(e, R) \cap E) \geq R^{-\delta \alpha} \).

Fix \( e \in E \) and \( R_1 \leq \rho_\delta \). There are \( e_1, \ldots, e_{R_1^{-\delta \alpha}} \) points in \( E \cap B(e, R_1) \) such that the balls \( B(e_j, 2R_1^{1+\delta}) \) are disjoint for \( j = 1, \ldots, R_1^{\delta \alpha} \). Now let \( R_2 = R_1^{1+\delta} \) and notice that

\[
B(e_j, 2R_2) \subseteq B(e_j, 2R_1^{1+\delta}) \subseteq B(e, 2R_1)
\]

(as we can take \( 2R_1^{1+\delta} < 1 \)). We let \( C_0 = B(e, 2R_3) \), \( C_1 = \bigcup_{j=1}^{R_1^{\delta \alpha}} B(e_j, 2R_2) \) and refer to the balls \( B(e_j, 2R_2) \) as the Cantor balls of level 1.

Repeating this procedure, we see that for each \( j \),

\[
P_{2R_2^{1+\delta}}(B(e_j, R_2) \cap E) \geq R_2^{-\delta \alpha}
\]

so there are \( e_{j_1}, \ldots, e_{j_{R_2^{-\delta \alpha}}} \in B(e_j, R_2) \cap E \), such that \( \{B(e_{j_k}, 2R_2^{1+\delta})\}_{k=1}^{R_2^{\delta \alpha}} \) are pairwise disjoint. Put \( R_3 = 2R_2^{1+\delta} \). Furthermore,

\[
B(e_{j_k}, 2R_3) \subseteq B(e_{j_k}, R_2 + 2R_3) \subseteq B(e_j, 2R_2) \subseteq C_1,
\]

so all these balls are disjoint. Let

\[
C_2 = \bigcup_{k=1}^{R_2^{-\delta \alpha}} \bigcup_{j=1}^{R_1^{-\delta \alpha}} B(e_{j_k}, 2R_3)
\]

and call these the Cantor balls of level 2.

Inductively, given disjoint balls \( B(e_{j_1}, \ldots, j_{k-1}, 2R_k) \), we find points \( e_{j_1}, \ldots, j_{k-1}, l \) belonging to \( B(e_{j_1}, \ldots, j_{k-1}, R_k) \cap E \) for \( l = 1, \ldots, R_k^{-\delta \alpha} \), such that \( \{B(e_{j_1}, \ldots, j_{k-1}, l, 2R_k^{1+\delta})\}_l \) are disjoint. Put \( R_{k+1} = R_k^{1+\delta} \) and let

\[
C_k = \bigcup_{j \in \{1, \ldots, R_k^{-\delta \alpha}\}} B(e_{j_1}, \ldots, j_k, 2R_{k+1}) \subseteq C_{k-1}.
\]
Let \( C = \bigcap_{k=1}^{\infty} C_k \). As each element of \( C \) is a limit point of the centre of the Cantor balls and \( E \) is closed, then \( C \subseteq E \). We will use the mass distribution principle to check \( \dim_H C \geq \alpha/(1+\delta) \); see [6] Proposition 2.1.

Let \( \mu \) be the probability measure that assigns equal mass on the Cantor balls of each level, i.e., each ball in \( C_k \) gets measure \( M^{-1} = R_1^\alpha((1+\delta)^k-1) \). We want to show that there is some constant \( A = A(\alpha, E) \) such that \( \mu(U) \leq A(\diam(U))^{\frac{\theta\alpha}{1+\delta}} \) for all Borel sets \( U \).

Without loss of generality we assume \( U = B(y, r) \), where \( 2R_{k+1} < r \leq 2R_k \), \( y \in E \). Any ball of radius \( 2R_k \) that intersects \( U \) will have its centre in \( B(y, 4R_k) \).

Since \( X \) is doubling, there is a constant \( A_1 \) such that \( P_{2R_k}(B(y, 4R_k)) \leq A_1 \) for all \( y \in E \) and all \( k \). As Cantor balls at level \( k-1 \) are disjoint, of radius \( 2R_k \) and centred in \( E \), at most \( A_1 \) of such balls can intersect \( U \). Thus \( U \) intersects at most \( A_1 R_k^{-\delta} \) level \( k \) Cantor balls, so

\[
\mu(U) \leq A_1 R_k^{-\delta} \cdot R_1^{\alpha((1+\delta)^k-1)} = \frac{A_1}{R_1^{\alpha}} R_1^{\alpha(1+\delta)^{k-1}} \leq A(\diam(U))^{\frac{\theta\alpha}{1+\delta}}.
\]

Hence the mass distribution principle implies \( \theta\alpha = \alpha/(1+\delta) \leq \dim_H C \leq \dim_H E \), completing the proof.

### 2.3. Other basic properties

The \( \Phi \)-dimensions have some natural continuity properties. For example, we have the following.

**Proposition 2.13.** Assume \( g : (0, 1) \to \mathbb{R}^+ \) is continuous at \( t_0 \) and \( g(t_0) \neq 0 \). Suppose \( \Phi \) is a dimension function and put \( \Phi_t(x) = g(t)\Phi(x) \). For any set \( E \), \( \dim_{\Phi_t} E \to \dim_{\Phi_{t_0}} E \) as \( t \to t_0 \). The same statement holds for the lower dimensions.

**Proof.** Given \( \epsilon > 0 \) choose \( \delta > 0 \) such that

\[
\max(|g(t_0)/g(t) - 1|, |g(t)/g(t_0) - 1|) \leq \epsilon
\]

for any \( |t - t_0| < \delta \). For each such \( t \), apply Lemma 2.9 to \( \Phi_t \) and \( \Phi_{t_0} \).

In Proposition 3.3 we will see that this convergence need not hold if \( g(t_0) = 0 \).

**Remark 2.14.** In order to apply Lemma 2.9 to show continuity, it was necessary that the convergence of \( \Phi_t(x)/\Phi_{t_0}(x) \) was uniform in \( x \). For instance, Lemma 3.3 cannot be applied to families such as \( \Phi_t(x) = |\log x|^{-t} \) with \( t \in (0, 1] \). We do not know if there is a one-parameter family of dimension functions which range continuously from the quasi-Assouad to the Assouad dimensions.

Unlike the case for the upper \( \Phi \)-dimension (see Proposition 2.3(ii)), it is not true in general that \( \varliminf E = \varliminf B E \) if \( \Phi(R) \to \infty \) since \( \varliminf E = 0 \) whenever \( E \) has an isolated point. This also means that we need not have \( \varliminf E \leq \varliminf F \) if \( E \subseteq F \), as is easily seen to be true for the upper \( \Phi \)-dimension.

**Proposition 2.15.** (i) If \( E \subseteq F \), then \( \varliminf E \leq \varliminf F \). Indeed, for all \( E, F \subseteq X \),

\[
\varliminf (E \cup F) = \max (\varliminf E, \varliminf F).
\]
(ii) For all $E, F \subseteq X$,
$$\min(\dim_\Phi E, \dim_\Phi F) \leq \dim_\Phi(E \cup F) \leq \max(\dim_\Phi E, \dim_\Phi F).$$

Proof. (i) The fact that $\dim_\Phi(E \cup F) \geq \max(\dim_\Phi E, \dim_\Phi F)$ follows easily from the observation that if $c \in E \cup F$, say $c \in E$, then
$$N_r(B(e, R) \cap (E \cup F)) \geq N_r(B(e, R) \cap E).$$

To see the reverse inequality, first note that
$$N_r(B(e, R) \cap (E \cup F)) \leq N_r(B(e, R) \cap E) + N_r(B(e, R) \cap F).$$

Fix $\varepsilon > 0$ and assume $e \in E$ and $r \leq R^{1+\Phi(R)}$ for small $R$. Then there is a constant $c$ such that $N_r(B(e, R) \cap E) \leq c \left(\frac{R}{r}\right)^{d_1+\varepsilon}$ where $d_1 = \dim_\Phi E$. If $B(e, R) \cap F$ is empty, then trivially
$$N_r(B(e, R) \cap (E \cup F)) \leq c \left(\frac{R}{r}\right)^{d_1+\varepsilon}.$$

Otherwise, there is some $y \in B(e, R) \cap F$, and as $B(e, R) \cap F \subseteq B(y, 2R) \cap F$ we have
$$N_r(B(e, R) \cap F) \leq N_r(B(y, 2R) \cap F) \leq c \left(\frac{R}{r}\right)^{d_2+\varepsilon}$$
for $d_2 = \dim_\Phi F$. In either case, there is a constant $C$ such that
$$N_r(B(e, R) \cap (E \cup F)) \leq C \left(\frac{R}{r}\right)^{d+\varepsilon}$$
for $d = \max(\dim_\Phi E, \dim_\Phi F)$ for all $e \in E \cup F$ and $r \leq R^{1+\Phi(R)}$, proving that $\dim_\Phi(E \cup F) \leq d$.

(ii) The lower bound is obvious. For the upper bound, choose $e_j$ and $r_j \leq R^{1+\Phi(R_j)}$ such that $N_{r_j}(B(e_j, R_j) \cap E) \leq c_1 \left(\frac{R_j}{r_j}\right)^{d_1+\varepsilon}$ where $d_1 = \dim_\Phi E$ and $\varepsilon > 0$ is fixed. Choosing $y_j \in B(e_j, R_j) \cap F$ (if this set is non-empty), we have
$$N_{r_j}(B(e_j, R_j) \cap (E \cup F)) \leq N_{r_j}(B(e_j, R_j) \cap E) + N_{r_j}(B(y_j, 2R_j) \cap F) \leq c_1 \left(\frac{R_j}{r_j}\right)^{d_1+\varepsilon} + c_2 \left(\frac{R_j}{r_j}\right)^{d_2+\varepsilon} \leq C \left(\frac{R_j}{r_j}\right)^{d+\varepsilon}$$
where $d_2 = \dim_\Phi F$ and $d = \max(d_1, d_2)$. □

Proposition 2.16. Upper and lower $\Phi$-dimensions are preserved under bi-Lipschitz maps.

Proof. The preservation of $\Phi$-dimensions under bi-Lipschitz invariance is a standard argument based upon the relationship between balls in the bi-Lipschitz spaces.

We finish this subsection with an example.

Example 2.17. It is shown in [13, Theorem 1] that if $E$ is a suitably controlled Moran construction in $\mathbb{R}^d$, such as a self-conformal set, then, regardless of any separation conditions, $\dim_\Phi A E = \dim_\Phi E$. A similar proof can be given to show that if $\Phi$ is any dimension function satisfying $R^{\Phi(R)} \to 0$ (equivalently, $|\log x| \Phi(x) \to \infty$ as $x \to 0$) the same result holds. In contrast, it is known that for self-similar sets $E \subseteq \mathbb{R}$ failing the weak separation condition, $\dim_\Phi A E = 1$ (see [10]).
2.4. Dimensions of Decreasing Sequences. In \cite{13} and \cite{14} it was shown that if $E = \{x_n\} \subseteq \mathbb{R}^+$ is a decreasing sequence with the sequence of ‘gaps’, \{x_n - x_{n+1}\}, also decreasing, then both the Assouad and quasi-Assouad dimensions of $E$ are either 0 or 1. The Assouad dimension of such a set is 0 if and only if the sequence of gaps is lacunary. Likewise, the quasi-Assouad dimension is 0 if and only if $\dim_B E = 0$.

This dichotomy fails for the $\Phi$-dimensions. Indeed, if we choose $\Phi(x) = \delta > 0$ for all $x$, it follows from \cite{9} and \cite{11} Theorem 6.2, that

$$\overline{\dim}_\Phi E = \min\{(1 + \delta)\overline{\dim}_B E/\delta, 1\}.$$ 

Therefore, $\overline{\dim}_\Phi E \in (0, 1)$ if $0 < \overline{\dim}_B E < \delta/(1 + \delta)$. However, in this case the $\Phi$-dimension is bounded above by the quasi-Assouad dimension, and necessarily $\dim_{qA} E = \dim_A E = 1$.

More interestingly, the dichotomy fails also for $\Phi$-dimensions that lie between the quasi-Assouad and Assouad dimensions. More precisely, we have the following.

**Example 2.18.** There is a decreasing set $E$, with decreasing gaps, and a dimension function $\Phi \to 0$, with $\dim_{qA} E = 0$ and $\dim_A E = 1$, but with $0 < \overline{\dim}_\Phi E < 1$.

**Construction.** Let $E = \{x_n\}_{n=1}^\infty$ where $x_n = n^{-\log n}$. Define $\Phi$ by the rule $x_n^{1+\Phi(x_n)} = 2(4n)^{-(1+\log 4n)} \log(4n)$ and extend $\Phi$ to $\mathbb{R}$ by setting $\Phi(x) = \Phi(x_n)$ if $x \in (x_{n+1}, x_n)$. We will verify that $E$ and $\Phi$ have the stated properties. Of course, if $\Phi(x) \to 0$ as $x \to 0$ then we must have $\dim_{qA} E \leq \overline{\dim}_\Phi E \leq \dim_A E$ and thus the properties $\dim_{qA} E = 0$ and $\dim_A E = 1$ will follow once we have shown that $\Phi$ tends to 0 and $0 < \overline{\dim}_\Phi E < 1$.

The fact that $E$ is decreasing follows from the fact that the function $f(z) = z^{-\log z}$ has negative derivative. Similarly, $x^{1+\Phi(x)}$ can be seen to be decreasing by checking the function $g(z) = z^{-(1+\log z)} \log z$ has negative derivative for large $z$ and thus $\Phi$ is a dimension function. One can directly calculate $\Phi$ and see that $\Phi(x_n) \sim 1/\log n$. That shows $\Phi(x) \to 0$ as $x \to 0$.

From the derivative of the function $h(x) = x^{-\log x} - (x + 1)^{-\log(x + 1)}$ one can also confirm that the sequence $\{x_n - x_{n+1}\}$ is decreasing. Moreover, an application of the mean value theorem shows that $x_n - x_{n+1} = -f'(\xi_n)$ for some $\xi_n \in [n, n+1]$ and thus

$$2(n + 1)^{-\log(n+1)} \log(n + 1)/(n + 1) \leq x_n - x_{n+1} \leq 2n^{-\log n} \log n/n.$$ 

This shows that if we take $R = x_k$, then if

$$r = R^{1+\Phi(R)} = 2(4k)^{-(1+\log 4k)} \log(4k)$$

we have

$$x_{4k} - x_{4k+1} \leq r \leq x_{4k-1} - x_{4k}.$$ 

Because the gaps are decreasing in length, $x_i - x_{i+1} \geq r$ whenever $i = k, \ldots, 4k - 1$ and $x_i = x_{i+1} \leq r$ whenever $i \geq 4k$. Consequently,

$$N_{r/2} \left( B(0, R) \cap E \right) = 3k + \frac{x_{4k}}{r} = 3k + \frac{(4k)^{-\log 4k} k}{2(4k)^{-\log 4k} \log 4k}$$

and hence for large enough $k$,

$$3k \leq N_{r/2} \left( B(0, R) \cap E \right) \leq 4k.$$
Since 
\[ \frac{R}{r} = \frac{k^{1+2\log 4}}{2\log 4}k^{1+\log 4}, \]
we deduce that \( \operatorname{dim}_4 E \geq 1/(1 + 2\log 4). \)

A similar statement holds for any \( r \) with \( x_{4k} - x_{4k+1} \leq r \leq x_{4k-1} - x_{4k} \). More generally, if \( x_{4n} - x_{4n+1} \leq r \leq x_{4n-1} - x_{4n} \), where \( n = Lk + j \), with \( 0 \leq j < k \) and \( L \geq 4 \), then
\[ N_{r/2}(B(0, R) \cap E) = n - k + \frac{x_n}{r} \sim n - k + \frac{n}{\log n} \leq 2n, \]
and for some \( c > 0 \),
\[ \frac{R}{r} \geq c \frac{n^{1+\log n}}{k\log k \log n}. \]
Thus, for \( t = (1 + \log 3)^{-1} \), we get
\[ \left( \frac{R}{r} \right)^t \geq n \left( c \frac{n^{(\log n - \log 3)}}{k\log k \log n} \right)^t \geq n \left( c \frac{L^{\log Lk - \log 3} k^{\log L - \log 3}}{\log((L+1)k)} \right)^t \]
and the last quotient is bounded away from 0.

If \( R \in (x_{k+1}, x_k) \), then since \( \Phi(R) = \Phi(x_k) \) we make a similar argument. Finally, we note that if \( z > 0 \), then the decreasingness of the gaps means
\[ N_{r/2}(B(z, R) \cap E) \leq N_{r/2}(B(0, R)). \]
Thus \( 0 < 1/(1 + 2\log 4) \leq \operatorname{dim}_4 E \leq 1/(1 + \log 3) < 1. \)

\begin{remark}
It would be interesting to characterize for which dimension functions \( \Phi \) the 0, 1 dichotomy holds.
\end{remark}

3. Examples of \( \Phi \)-Dimensions

In this section we will construct various examples. These will show the sharpness of some of the basic properties, such as Propositions 2.7 and 2.13, as well as illustrating their distinctness. In particular, we will give an example of a set with specified values for a countable family of \( \Phi \)-dimensions and whose set of all dimensions between quasi-Assouad and Assouad is an interval.

In all these examples, the set \( E \) will be a Cantor set. We begin this section by determining a formula for the \( \Phi \)-dimension of Cantor sets. It will be convenient to make use of the following notation.

\begin{notation}
We write \( f \sim g \) if there are positive constants \( c_1, c_2 \) such that \( c_1 f \leq g \leq c_2 f \). The symbols \( \gtrsim \) and \( \lessapprox \) are defined similarly.
\end{notation}

3.1. \( \Phi \)-Dimensions of Cantor sets. Given a decreasing, summable sequence, \( a = \{a_j\} \) with (without loss of generality) \( \sum_j a_j = 1 \), by the \textbf{Cantor set associated with} \( a \), denoted by \( C_a \), we mean the compact subset of \([0, 1]\) constructed as follows: In the first step, we remove from \([0, 1]\) an open interval of length \( a_1 \), resulting in two closed intervals \( I_1^1 \) and \( I_1^2 \). Having constructed the \( k \)-th step, we obtain the closed intervals \( I_1^k, \ldots, I_{2^k}^k \) contained in \([0, 1]\). The intervals \( I_j^k, j = 1, \ldots, 2^k \), are called the Cantor intervals of step \( k \). The next step consists in removing from
each $I^k_i$ an open interval of length $a_{2^k+i-1}$, obtaining the closed intervals $I^{k+1}_{2^k-1}$ and $I^{k+1}_{2^k}$. We define

$$C_a := \bigcap_{k \geq 1} \bigcup_{j=1}^{2^k} I^k_j.$$ 

This construction uniquely determines the set because the lengths of the removed intervals on each side of a given gap are known. The classical middle-third Cantor set is the Cantor set associated with the sequence $\{a_i\}$ where $a_i = 3^{-n}$ if $2^{n-1} \leq i \leq 2^n - 1$. All associated Cantor sets are uncountable, compact, totally disconnected and, in fact, are all homeomorphic.

If we put

$$s_n = 2^{-n} \sum_{j \geq 2^n} a_j,$$

then $s_n$ is the average length of the Cantor intervals of step $n$. The decreasing property of the sequence $\{a_j\}$ ensures that all the intervals of step $n$ have lengths satisfying

$$s_{n+1} \leq \text{length}(I^n_j) \leq s_{n-1}$$

and that $s_n \geq a_{2^n - 1}$. Of course, always $s_{n+1} \leq s_n/2$.

When the gap sizes $a_{2^n} = \cdots = a_{2^n - 1}$ for all $n$, the intervals at step $n$ all have the same length (namely $s_n$), and the Cantor set is sometimes called a central Cantor set. The classical middle-third Cantor set is such an example. In this case, the ratio $s_{j+1}/s_j$ is referred to as the ratio of dissection at step (or level) $j$.

We will assume the sequence $\{a_j\}$ is **doubling**, meaning there is a constant $\kappa$ such that $a_n \leq \kappa a_{2n}$ for all $n$. This ensures that

$$s = \inf s_{n+1}/s_n > 0$$

since

$$s_n \leq \frac{1}{2^n} \left( \sum_{j \geq 2^n} a_j + a_{2^n} 2^n \right) \leq 2s_{n+1} + \kappa^2 a_{2^n+2} \leq (2 + \kappa^2)s_{n+1}.$$

Thus under the doubling assumption we have $s_j \sim s_{j+1}$.

For Cantor sets, it is helpful to understand the comparison $r \leq R^{1+\Phi(R)}$ in terms of the sequence $(s_n)$. For this we introduce the following notation.

**Notation 2.** Given a dimension function $\Phi(x)$ and a doubling, decreasing, summable sequence $a = (a_j)$, define the associated **depth function** $\phi : \mathbb{N} \to \mathbb{N}$ by the rule that $\phi(n)$ is the minimal integer $j$ such that $s_{n+j} \leq s_{n}^{1+\Phi(s_n)}$. In other words, $\phi(n)$ is the minimal integer with $s_{n+\phi(n)}/s_n \leq s_{n}^{\Phi(s_n)}$.

If $\phi$ is bounded, then the sequence $\{\phi_n(s_n)\}$ is bounded above and below from 0. The decreasingness of the function $R^{1+\Phi(R)}$ implies that if $s_n \leq R \leq s_{n-1}$, then

$$\tau s_{n}^{\phi(s_n)} \leq R^{\phi(R)} \leq \frac{1}{\tau} s_{n-1}^{\phi(s_{n-1})}.$$ 

Hence if $\phi$ is bounded, then Proposition 2.1 implies the upper (or lower) $\Phi$-dimension coincides with the upper (resp., lower) Assouad dimension.
A very useful observation for constructing examples is to note that if \( E \) is any Cantor set with \( \tau = \inf s_{j+1}/s_j \) and \( \rho = \sup s_{j+1}/s_j \), and \( \Phi/\phi \) is a dimension/depth function pair for \( E \), then we have

\[
(3.1) \quad \frac{(\phi(n) - 1) \log \rho}{n} \leq \Phi(s_n) \leq \frac{\phi(n) \log \tau}{n}.
\]

This is because the doubling property ensures \( \tau^\phi(n) \leq s_{n+\phi(n)}/s_n \leq \Phi(s_n) \leq \tau^{\phi(n)} \) and \( \rho(\phi(n)-1) \geq s_{n+\phi(n)-1}/s_n \geq s_n^{\phi(n)} \).

If, in addition, \( \phi(n) \geq 2 \) (as is typically the case in interesting examples), then we see that \( \phi(n) \) is comparable to \( n\Phi(s_n) \) with constants depending only on \( \tau, \rho \) as we have

\[
(3.2) \quad \frac{\phi(n) \log \rho}{n} \leq \Phi(s_n) \leq \frac{\phi(n) \log \tau}{n}.
\]

**Remark 3.1.** Notice that if we are given an increasing function \( \phi : \mathbb{N} \to \mathbb{N} \), and a Cantor set \( C_\alpha \), we can define a function \( \Phi \) by the rule \( R^{1+\Phi(R)} = s_{n+\phi(n)} \) if \( R \in (s_{n+1}, s_n] \). If \( R_1 \leq R_2 \) with \( R_1 \in (s_{n+1}, s_n] \) and \( R_2 \in (s_{k+1}, s_k] \), then \( n \geq k \), so \( \phi(n) \geq \phi(k) \) and hence \( s_{n+\phi(n)} \leq s_{k+\phi(k)} \). Consequently, \( R^{1+\Phi(R_1)} = s_{n+\phi(n)} \leq R^{1+\Phi(R_2)} \). Furthermore, \( R^{1+\Phi(R)} = s_{n+\phi(n)} \to 0 \) as \( n \to \infty \) and hence as \( R \to 0 \). Thus \( \Phi \) is a dimension function with associated depth function \( \phi \).

**Corollary 3.2.** (i) If \( \phi(n)/n \to \infty \), then \( \dim_\Phi C_\alpha = \dim_B C_\alpha \).

(ii) The quasi-Assouad dimensions are obtained by taking \( \phi(n) = \delta n \) and letting \( \delta \to 0 \).

**Proof.** These follow from the fact that \( \phi(n)/n \sim \Phi(s_n) \). \( \square \)

It is easy to see that \( \dim_B C_\alpha > 0 \) for any doubling sequence \( a \), hence the upper and lower \( \Phi \)-dimensions of \( C_\alpha \) coincide with the upper and lower box dimension (respectively) if \( \Phi(R) \to \infty \) as \( R \to 0 \). More generally, we have the following formulas for the \( \Phi \)-dimensions of Cantor sets.

**Theorem 3.3.** Let \( a \) be a decreasing, summable, doubling sequence and \( C_\alpha \) the associated Cantor set. The upper and lower \( \Phi \)-dimensions of \( C_\alpha \) are given by

\[
(3.3) \quad \dim_\Phi C_\alpha = \inf \left\{ \beta : (\exists k_0, a_0 > 0) \ (\forall k \geq k_0, n \geq \phi(k)) \ \left( \frac{s_k}{s_{k+n}} \right)^\beta \geq a_02^n \right\}
\]

and

\[
(3.4) \quad \dim_\Phi C_\alpha = \sup \left\{ \beta : (\exists k_0, a_0 > 0) \ (\forall k \geq k_0, n \geq \phi(k)) \ \left( \frac{s_k}{s_{k+n}} \right)^\beta \leq a_02^n \right\}.
\]

We refer the reader to [14] for the formulas for the Assouad dimensions and to [5] and [22] for the quasi-Assouad dimensions.

**Proof.** The proof is similar to that given in [14] for the Assouad dimensions, but we include it here for completeness. Let \( d = \dim_\Phi C_\alpha \) and \( \alpha \) equal the right hand side in (3.3). Given \( \epsilon > 0 \), there are positive constants \( c_1, c_2 \) (depending on \( \epsilon \)) such that if \( 0 < r \leq R^{1+\Phi(R)} \leq R < c_1 \), then

\[
N_r(B(e, R) \cap C_\alpha) \leq c_2 \left( \frac{R}{r} \right)^{d+\epsilon}
\]

for all \( e \in C_\alpha \).
Pick $k_0$ so large that $s_k < c_1$ for all $k \geq k_0$. Let $k \geq k_0$ and $n \geq \phi(k)$. Put $R = s_k$, $r = s_{k+n}$ and $e$ to be the endpoint of a Cantor interval $I$ of step $k + 1$. Since $R > \text{length}(I)$, $I \subseteq B(e, R)$. Since $s_{k+n}$ is dominated by the length of any Cantor interval of step $k + n - 1$, the left endpoints of those intervals that are contained in $I$ are more than $r$ apart. Consequently,

$$N_r(B(e, R) \cap C_a) \geq \frac{1}{2} 2^{k+n-1-(k+1)} \gtrsim 2^n.$$  

But, as

$$r \leq s_{k+\phi(k)} \leq 2 s_k^{\Phi(s_k)} = R^{1+\Phi(R)} \leq R = s_k < c_1,$$

we know that

$$N_r(B(e, R) \cap C_a) \leq c_2 \left( \frac{R}{r} \right)^{d+\epsilon} = c_2 \left( \frac{s_k}{s_{k+n}} \right)^{d+\epsilon}.$$  

Putting these inequalities together, it follows that for a suitable constant $c_0$ we have

$$\left( \frac{s_k}{s_{k+n}} \right)^{d+\epsilon} \geq c_0 2^n$$

and that implies $\alpha \leq d$.

On the other hand, given $\epsilon > 0$ choose $c_0$ and $k_0$ such that for all $k \geq k_0$ and $n \geq \phi(k)$, we have

$$\left( \frac{s_k}{s_{k+n}} \right)^{\alpha+\epsilon} \geq c_0 2^n,$$

as per (5.3). Put $c_1 = s_{k_0}$ and assume $r \leq R^{1+\Phi(R)} \leq R \leq c_1$ and $e \in C_a$. Choose $k, m$ such that $s_{k+1} \leq R < s_k$ and $s_m \leq r < s_{m-1}$. As $r \leq R$, $n := m - k \geq 1$.

Further, the relations $s_{k+n} \leq r \leq R^{1+\Phi(R)} \leq s_k^{1+\Phi(s_k)} \leq 2 s_k^{\Phi(k)-1}$, which hold by the decreasingness of $R^{1+\Phi(R)}$ as $R \downarrow 0$ and the definition of $\phi$, imply $n \geq \phi(k) - 1$ and therefore $(s_k/s_{k+n})^{\alpha+\epsilon} \geq c_0 2^{n+1}$.

The size of $R$ ensures that $B(e, R)$ cannot intersect five Cantor intervals of step $k - 1$ in $C_a$ for otherwise it would have to contain at least one of level $k - 2$ and that would have length at least $s_{k-1} > 2 s_k \geq \text{diam}(B(e, R))$. The (closed) balls of radius $r$ centred at the left endpoints of the step $m + 1$ intervals contained in these step $k - 1$ intervals cover $B(e, R) \cap C_a$, and since $s_j \sim s_{j+1}$ we have

$$N_r(B(e, R) \cap C_a) \leq 4 \cdot 2^{m+1-(k-1)} \lesssim \left( \frac{s_k}{s_{k+n}} \right)^{\alpha+\epsilon} \lesssim \left( \frac{R}{r} \right)^{\alpha+\epsilon},$$

which proves $d \leq \alpha$.

The proof of the formula for the lower $\Phi$-dimension of Cantor sets is similar. Here are the main ideas.

Let $d = \dim_a C_a$ and $\alpha$ equal the right hand side in (5.3). Given $\epsilon > 0$, there are positive constants $c_1, c_2$ (depending on $\epsilon$) such that if $0 < r \leq R^{1+\Phi(R)} \leq R < c_1$, then

$$N_r(B(e, R) \cap C_a) \geq c_2 \left( \frac{R}{r} \right)^{-d-\epsilon}$$

for all $e \in C_a$.

Pick $k_0$ so large that $s_k < c_1$ for all $k \geq k_0$. Let $k \geq k_0$ and $n \geq \phi(k)$. Put $R = s_k$, $r = s_{k+n}$. As above, $B(e, R) \cap C_a$ cannot intersect five Cantor intervals of level
$k - 1$ and hence can be covered by the level $k + n + 1$ Cantor intervals contained in the at most four Cantor intervals of level $k - 1$ that $B(e, R) \cap C_a$ intersects. Thus
\[
\left( \frac{s_k}{s_{k+n}} \right)^{d-\varepsilon} \sim c_2 \left( \frac{R}{r} \right)^{d-\varepsilon} \leq N_r(B(e, R) \cap C_a) \leq 4 \cdot 2^{k+n+1-(k-1)} \approx 2^n
\]
and that shows $\alpha \geq d$.

Conversely, given $\varepsilon > 0$ choose $c_0$ and $k_0$ such that for all $k \geq k_0$ and $n \geq \phi(k)$, we have
\[
\left( \frac{s_k}{s_{k+n}} \right)^{\alpha-\varepsilon} \leq c_0 2^n
\]
as per (3.4). Put $c_1 = s_{k_0}$ and assume $r \leq R^{1+\Phi(R)} \leq c_1$ and $e \in C_a$. Choose $k, m$ such that $s_{k+1} \leq R < s_k$ and $s_m \leq r < s_{m-1}$. As $n = m - k \geq \phi(k) - 1$, we have $(s_k/s_{k+n+1})^{\alpha-\varepsilon} \leq c_0 2^{n+1}$. Similar to the first step in the argument above, one can see that
\[
N_r(B(e, R) \cap C_a) \geq \frac{1}{2} 2^{k+n-1-(k+2)} \approx 2^n
\]
and that implies $d \geq \alpha$. \qed

### 3.2. Basic Properties Revisited.
With the formulas for the $\Phi$-dimensions of Cantor sets, it is easy to give examples of sets with any specified $\Phi$-dimension in $(0, 1)$. The key idea is that if $E$ is a central Cantor set with ratios of dissection $r_k$ at step $k$, and there is an increasing sequence of integers $\{n_j\}$ (possibly even very sparse) such that $r_k = \rho$ for all $k = n_j + 1,...,n_j + \phi(n_j)$ and $r_k = \tau \leq \rho$ otherwise, then $\dim_{\Phi} E = \log 2/|\log \rho|$, where $\Phi/\phi$ is a dimension/depth function pair. A similar idea can be applied for the lower $\Phi$-dimension.

In this subsection we will use this principle to obtain (partial) converses to Propositions 2.7 and 2.13. First, we will show that the continuity properties described in Proposition 2.13 can fail when $g(t_0) = 0$.

**Proposition 3.4.** Suppose $\phi$ is an increasing depth function tending to infinity, but with $\phi(n)/n \to 0$ as $n \to \infty$. There is a central Cantor set $E$ such that if $\Phi$ is the dimension function with associated depth function $\phi$, and $\Phi_t = t\Phi$, then $\dim_{\Phi_t} E \in [\dim_{\Phi} E, \lim_{t \to 0} \dim_{\Phi_t} E]$, but $\lim_{t \to 0} \dim_{\Phi_t} E < \dim_{\Phi} E$.

**Proof.** Choose $A, B > 0$ such that if $E$ is a Cantor set with inf $s_{j+1}/s_j \geq 1/27$ and $\psi$ is the depth function associated with a dimension function $\Psi$ on $E$, then according to (3.1).

\[
A \frac{\psi(n)}{n} - 1 \leq \Psi(s_n) \leq B \frac{\psi(n)}{n} \text{ for all } n.
\]

In particular, this holds for the depth function $\phi$ and any associated dimension function $\Phi$, and also for the depth function $\bar{\phi}_k$ associated with $t\Phi$. Without loss of generality we can assume $\phi(n) \geq 2$ for all $n$ and therefore for all $k \in \mathbb{N}$
\[
A \frac{\phi(n)}{2kn} \leq \frac{1}{k} \Phi(s_n) = \Phi_{1/k}(s_n) \leq B \frac{\phi_{1/k}(n)}{n}.
\]
That shows that for each \( k \) there is some \( N_k \) such that if \( n \geq N_k \), then \( \phi_{1/k}(n) \geq 2 \). Thus we also have \( A_0 1/k(n)/(2n) \leq \Phi_{1/k}(s_n) \) for all \( n \geq N_k \) and therefore with the constant \( C = A/(2B) \) (and any such Cantor set \( E \)) we have

\[
\frac{C}{k} \phi(n) \leq \phi_{1/k}(n) \leq \frac{1}{Ck} \phi(n) \text{ for all } k \text{ and } n \geq N_k.
\]

To construct the Cantor set \( E \), we will first choose an integer-valued function \( f(n) \to \infty \) with \( f(n)/\phi(n) \to 0 \) as \( n \to \infty \). Then choose an increasing sequence of integers \( \{n_k\} \) with \( n_k \geq \max(2N_k, 8n_{k-1}) \) and satisfying

\[
f(n_k) \leq \min\left(\frac{n_k}{8}, \frac{C}{2k} (\phi(n_k) - f(n_k))\right).
\]

The Cantor set will be defined by setting the ratios of dissection to be \( 1/3 \) on steps \( n_j + 1, \ldots, n_j + f(n_j) \) for all \( j = 1, 2, \ldots \) and equal to \( 1/27 \) on all other levels. Certainly, \( \dim_A E = \log 2/\log 3 \).

Let \( r_i \) denote the ratio of dissection at step \( i \). Our choice of \( n_j \) ensures that if \( n \in \{n_j - f(n_j) + 1, \ldots, n_j + f(n_j)\} \) and \( m \geq 2f(n_j) \), then at least as many \( n \) are equal to \( 1/3 \) for \( i \) ranging over \( \{n+1, \ldots, n+m\} \). Hence the geometric mean of these ratios is at most \( 1/9 \). The same conclusion clearly also holds if \( n \notin \{n_j - f(n_j) + 1, \ldots, n_j + f(n_j)\} \).

In order to bound \( \dim_{\Phi_{1/k}} E \) we use formula (3.3), noting first that it suffices to consider \( (s_n/s_{n+m})^{1/m} \) where \( n \geq n_k \) and \( m \geq \phi_{1/k}(n) \). If \( n \in \{n_j - f(n_j) + 1, \ldots, n_j + f(n_j)\} \) for some \( j \geq k \), then as \( \phi \) is increasing and \( n \geq N_k \),

\[
m \geq \phi_{1/k}(n) \geq \frac{C}{k} \phi(n) \geq \frac{C}{k} \phi(n_j - f(n_j)) \geq \frac{C}{j} \phi(n_j - f(n_j)) > 2f(n_j).
\]

By our previous remark, \( (s_n/m/s_{n+m})^{1/m} \leq 1/9 \). The same bound clearly holds if \( n \geq n_k \) does not belong to any such interval. Consequently, \( \Phi_{1/k} \) implies \( \dim_{\Phi_{1/k}} E \leq \log 2/\log 9 \). By monotonicity, \( \dim_{\Phi_{1/k}} E \leq \log 2/\log 9 \) for all \( t > 0 \).

**Remark 3.5.** We remark that a similar argument could be used to prove that there is a central Cantor set \( E \) and dimension function \( \Phi \) so that \( \lim_{t \to 0} \dim_{\Phi} E = \dim_{\Phi} E \). One could also similarly arrange for \( \dim_{\Phi} E \in [\dim_{A} E, \dim_{A} E] \), but \( \lim_{t \to \infty} \dim_{\Phi} E \in [\dim_{A} E, \dim_{A} E] \) and likewise for the quasi-lower dimension.

We will use a similar technique to obtain a partial converse to Proposition 2.7.

**Theorem 3.6.** Suppose \( \Phi_1, \Phi_2 \) are dimension functions decreasing to 0 as \( x \to 0 \) with \( |\log x| \Phi_2(x) \to \infty \) as \( x \to 0 \). Assume there is some \( \eta > 0 \) such that \( \Phi_1(x) \geq (1 + \eta)\Phi_2(x) \) for all \( x \) sufficiently small. Then there is a Cantor set \( E \) such that \( \dim_{\Phi_1} E < \dim_{\Phi_2} E \) and a Cantor set \( E' \) with \( \dim_{\Phi_1} E' > \dim_{\Phi_2} E' \).

**Proof.** We will give the proof for the upper \( \Phi \)-dimension. The lower \( \Phi \)-dimension case is similar.

The monotonicity properties of the \( \Phi \)-dimensions implies that \( \dim_{\Phi_1} E \leq \dim_{\Phi_2} E \) for all sets \( E \). It is the strictness of the inequality that we need to verify for an appropriate choice of \( E \).

The strategy of the proof will be to build a central Cantor set by inductively specifying the ratios of dissection at each level. For most levels, the ratio will be a fixed small number, say \( \tau \). However, we will specify the ratios to be a fixed number \( \rho > \tau \) on the levels \( n_j + 1, \ldots, n_j + \phi_2(n_j) \), where \( \phi_2 \) is the depth function.
associated with $\Phi_2$ and the Cantor set, and $\{n_j\}$ is a sparse set. By consideration of $(s_{n_1} + \phi_2(n_j)/s_{n_1})^{1/\phi_2(n_j)}$ (hence, the geometric mean of the ratios at levels $n_1 + 1, \ldots, n_j + \phi_2(n_j)$) and the formula for the $\Phi$-dimensions of Cantor sets from (3.3), we have $\dim_{K,E} = \log 2/|\log \rho|$. However, these depths will be too shallow to give the $\Phi_1$-dimension and consequently we will be able to conclude that $\dim_{K,E} < \log 2/|\log \rho|$.

One complication with this strategy is that the depth functions depend on the construction of the Cantor set. However, our construction of the Cantor set depends (at least, to some extent) on the depth functions. Fortunately, we do have enough control on the depth functions to overcome this complication. We address this issue first.

Fix small $\varepsilon > 0$ such that

\[
\left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^2 (1 + \eta) \geq (1 + \eta/2).
\]

Choose $0 < \tau < \rho < 1/2$ with $|\log \tau / \log \rho| \leq 1 + \varepsilon$. It follows from (3.1) that if $E$ is any Cantor set with all ratios between $\tau$ and $\rho$, and $\Phi/\phi$ any dimension/depth function pair associated with $E$, then

\[
(3.5) \quad \frac{(\phi(n) - 1) \log \rho}{n} \leq \Phi(s_n) \leq \frac{\phi(n) \log \tau}{n} \leq (1 + \varepsilon) \frac{\phi(n)}{n}.
\]

By assumption, given any $C > 0$, there is some $x_0 = x_0(C)$ such that if $x \leq x_0$, then $|\log x| \Phi_i(x) \geq C$. Choose $n_0$ such that $\tau^{n_0} \leq x_0$. Since the functions $\Phi_i$ are decreasing as $x \to 0$, it follows that if $n \geq n_0$ and $E$ is a Cantor set with all ratios of dissection at least $\tau$, then $\Phi_i(s_n) \geq \Phi_i(\tau^n) \geq C/|\log \tau^n|$ and hence $n\Phi_i(s_n) \geq (1 + \varepsilon)/\varepsilon$ if we take a suitable choice for $C$, depending on $\varepsilon$ and $\tau$. Coupled with the right hand side of (3.5), this shows that for all $n \geq n_0$,

\[
\phi_i(n) \geq \frac{n\Phi_i(s_n)}{1 + \varepsilon} \geq 1/\varepsilon
\]

and hence $\phi_i(n) - 1 \geq (1 - \varepsilon)\phi_i(n)$ for $i = 1, 2$. Consequently, using the left hand side of (3.5) we also have

\[
\frac{\phi_i(n)(1 - \varepsilon)}{n(1 + \varepsilon)} \leq \Phi_i(s_n) \text{ for all } n \geq n_0.
\]

As $\Phi_i \downarrow 0$, this further ensures that there exists $n_0'$ such that

\[
\phi_i(n) \leq \varepsilon n \text{ for } n \geq n_0'.
\]

We remind the reader that having fixed $\varepsilon, \tau, \rho$, these inequalities and the choices of $n_0, n_0'$ depend only $\Phi_1$ and $\Phi_2$ for any choice of Cantor set, provided the ratios of dissection are chosen from $[\tau, \rho]$. As we will see, these relationships give us enough control on the depth functions.

Now let $n_1 \geq \max(8n_0, 8n_0')$ and choose $n_{j+1} \geq 16n_j$. We will inductively define a central Cantor set by specifying the ratios of dissection $r_k$ at each level $k$. To begin, we put $r_k = \tau$ for $k = 1, \ldots, n_1$. Thus $s_{n_1} = \tau^{n_1}$. Define $L_1$ to be the least integer with $\rho^{L_1} \leq s_{n_1}(\tau^{n_1})$ and let $r_k = \rho$ for $k = n_1 + 1, \ldots, n_1 + L_1$. Notice that this construction means $L_1 = \phi_2(n_1) \leq \varepsilon n_1$, thus $n_1 + L_1 < n_2$. We put $r_k = \tau$ for $k = n_1 + L_1 + 1, \ldots, n_2$.

Now we proceed inductively. We assume $L_1, \ldots, L_{j-1}$ have been chosen in the same fashion and we have put $r_k = \rho$ if $k = n_i + 1, \ldots, n_i + L_i$ for $i = 1, \ldots, j-1$, then

\[
L_j = \phi_2(n_j) \leq \varepsilon n_j.
\]

Thus $n_j + L_j < n_{j+1}$. We put $r_k = \tau$ for $k = n_j + L_j + 1, \ldots, n_{j+1}$.
and \( r_k = \tau \) otherwise on \( \{1, ..., n_j \} \). Thus \( s_{n_j} \) is determined. Define \( L_j \) to be the least integer satisfying \( \rho^{L_j} \leq \Phi_2(s_{n_j}) \). We will put \( r_k = \rho \) if \( k = n_j + 1, ..., n_j + L_j \) and \( r_k = \tau \) on \( \{n_j + L_j + 1, ..., n_{j+1}\} \). Again \( L_j = \phi_2(n_j) \). This completes the construction of \( E \).

The fact that the ratios equal \( \rho \) on the consecutive levels \( n_j + 1, ..., n_j + \phi_2(n_j) \) for all \( j \) and are equal to \( \tau \) otherwise, certainly means \( \dim E = \log 2/|\log \rho| \).

Since \( \phi_1(n_j) \leq \varepsilon n_j \) and \( \Phi_1 \) is decreasing, the choice of \( \varepsilon \) gives that for each \( j \) and \( n \in \{n_j - L_j, ..., n_j\} \),

\[
\phi_1(n) \geq \frac{n}{1 + \varepsilon} \Phi_1(s_n) \geq \frac{n_j - L_j}{1 + \varepsilon} \Phi_1(s_{n_j})
\]

(3.6)

\[
\geq \left( \frac{1 - \varepsilon}{1 + \varepsilon} \right)^2 \left( 1 + \varepsilon \right) \Phi_2(n_j) \geq (1 + \varepsilon/2) \Phi_2(n_j).
\]

Since the sequence \( \{s_n^{1+\Phi_2(s_n)}\} \) is decreasing (for any dimension function \( \Phi \)), for any \( n, m \geq 1 \) and associated depth function \( \phi \) we have

\[ s_{n+m+\phi(n+m)} \leq s_{n+m}^{1+\Phi(s_n)} < s_{n+\phi(n)} \]

by the definition of \( \phi \). That means \( n + m + \phi(n) > n + \phi(n) - 1 \), and as these are integers this implies, in particular, that for \( i = 1, 2 \),

(3.7)

\[ n_j + m + n_i(n_j + m) \geq n_j + \phi_i(n_j) \]

for all \( m \geq 1 \). As (3.6) holds for \( n = n_j \), this gives

\[
n_j + m + \phi_1(n_j + m) - (n_j + \phi_2(n_j)) \geq n_j + \phi_1(n_j) - (n_j + \phi_2(n_j)) \geq (\eta/2) \phi_2(n_j).
\]

Since \( \phi_i(n) \leq \eta n \) for all \( n \geq n_0 \) and \( L_j = \phi_2(n_j) \), we also know that

\[
n_j + \phi_2(n_j) + \max_{n \in \{n_j - L_j, n_j + L_j\}} \phi_1(n) \leq n_j + \varepsilon n_j + \varepsilon(n_j + L_j)
\]

\[
\leq (1 + \varepsilon)^2 n_j \leq n_j + 1/4 < (n_j + 1 - \phi_2(n_j + 1))/2.
\]

In particular, this guarantees that if \( n \in \{n_j - L_j + 1, ..., n_j + L_j\} \) and \( m = \phi_1(n) \), then \( n + m < (n_j + 1 - \phi_2(n_j + 1))/2. \) Together with (3.8), it follows that for such \( n \) there are at least \((\eta/2) \phi_2(n_j)\) ratios equal to \( \tau \) and at most \( \phi_2(n_j) = L_j \) ratios equal to \( \rho \) on the levels \( n + 1, ..., n + m \). Hence the geometric mean of these ratios is dominated by

\[
\left( \rho^{L_j + \eta L_j/2} \right)^{1/(1+\eta/2) L_j} = \rho^{1/(1+\eta/2) L_j + n_j/(2+\eta)} := \sigma < \rho.
\]

If \( m \geq \phi_1(n) \), the choice of ratios ensures that there could only be an even greater proportion of the ratios on the levels \( n + 1, ..., n + m \) having value \( \tau \). Thus we can conclude that the geometric mean of the ratios from the levels \( n + 1, ..., n + m \) is also dominated by \( \sigma \) whenever \( m \geq \phi_1(n) \) and \( n \in \{n_j - L_j + 1, ..., n_j + L_j\} \).

If \( n \notin \{n_j - L_j + 1, ..., n_j + L_j\} \) for any \( j \), then it is obvious from the construction that there are at least as many ratios equal to \( \tau \) as equal to \( \rho \) on the levels \( n + 1, ..., n + m \) (for any \( m \geq 1 \)) and hence the geometric mean is even smaller.
We deduce that
\[
\overline{\dim}_{\Phi_1} E \leq \frac{\log 2}{\log \sigma} < \frac{\log 2}{\log \rho} = \overline{\dim}_{\Phi_2} E,
\]
which concludes the proof.

A modification of this argument would allow us to show that given \(0 < a < b < 1/2\) there is an example of a Cantor set \(E\) for which
\[
\overline{\dim}_{\Phi_1} E = \frac{\log 2}{\log a} < \frac{\log 2}{\log b} = \overline{\dim}_{\Phi_2} E.
\]
To do this, we will choose \(0 < c < a\). Then, instead of assigning ratio \(\rho\) on the levels \(n_j + 1, \ldots, n_j + \phi_2(n_j)\) and \(\tau\) otherwise, we will put ratios \(b\) on levels \(n_{2j} + 1, \ldots, n_{2j} + \phi_2(n_{2j})\), ratios \(a\) on levels \(n_{2j+1} + 1, \ldots, n_{2j+1} + \phi_1(n_{2j+1})\) and ratio \(c\) elsewhere. The choice of sequence \(\{n_j\}\) may need to be even more sparse to ensure that \(\phi_1(n_j)\) is enough larger than \(\phi_2(n_j)\) to guarantee that the geometric mean of ratios from any \(\phi_1(n)\) consecutive levels beginning at \(n\) is at most \(a\). The fact that the ratios at levels \(n_{2j+1} + 1, \ldots, n_{2j+1} + \phi_1(n_{2j+1})\) are equal to \(a\) implies that \(\overline{\dim}_{\Phi_1} E = \frac{\log 2}{\log a}\). From their values on levels \(n_{2j} + 1, \ldots, n_{2j} + \phi_2(n_{2j})\) one can deduce that \(\overline{\dim}_{\Phi_2} E = \frac{\log 2}{\log b}\). The details are left for the reader.

A further modification of the argument would also enable us to construct a (single) Cantor set \(E\) with both \(\overline{\dim}_{\Phi_1} E < \overline{\dim}_{\Phi_2} E\) and \(\overline{\dim}_{\Phi_1} E > \overline{\dim}_{\Phi_2} E\).

3.3. Continuum of \(\Phi\)-Dimensions. In the next result we use the method described in the previous remark to show that we can construct a Cantor set with countably many specified values for \(\Phi\)-dimensions. Furthermore, there is a Cantor set with a continuum of \(\Phi\)-dimensions between the quasi-Assouad and Assouad dimensions.

**Theorem 3.7.** Assume that for each \(p \in (0, 1)\), \(\Phi_p\) are dimension functions decreasing to 0 as \(x \to 0\) and satisfying \(\log x \cdot \Phi_p(x) \to \infty\) as \(x \to 0\). Assume, also, that
\[
\Phi_p(x)/\Phi_q(x) \to \infty \text{ as } x \to 0 \text{ whenever } p > q.
\]
Choose any \(0 < \alpha < \beta < 1\) and suppose \(d : (0, 1) \to [\alpha, \beta]\) is monotonically decreasing and continuous. Then there is a central Cantor set \(E\) with
\[
\overline{\dim}_{\Phi_1} E = d(p) \text{ for each } p \in (0, 1).
\]
The analogous result holds for the lower \(\Phi\)-dimensions.

**Remark 3.8.** The functions \(\Phi_p(x) = |\log x|^{p-1}\) are an example of a class of such functions.

**Proof.** We will actually construct a central Cantor set \(E\) with the property that if \(f : (0, 1) \cap \mathbb{Q} \to [a, b]\) is monotonically decreasing, then \(\overline{\dim}_{\Phi_1} E = \log 2/|\log f(p)|\) for every rational \(p \in (0, 1)\). To obtain the theorem, put \(a = 2^{-1/\alpha}\), \(b = 2^{-1/\beta}\) and define the decreasing continuous function \(f : (0, 1) \to [a, b]\) by \(f(x) = 2^{-1/d(x)}\). The proof follows directly from this using the monotonicity of the functions \(p \to \overline{\dim}_{\Phi_p} E\) and the fact that the function \(d\) of the theorem is assumed to be continuous and decreasing.
As in the proof of the previous theorem our strategy will be to inductively define the ratios of dissection of the Cantor set. These ratios will lie in \([a^2, b]\) and so by (3.10), with \(c = \log b/2\log a\) we have

\[ c(\phi(n) - 1) \leq n\Phi(s_n) \leq \frac{1}{c}\phi(n) \text{ for all } n, \]

for any depth/dimension function pair \(\phi/\Phi\) associated with such a Cantor set.

Since \(|\log x|\Phi_p(x) \to \infty\) for each \(p\), there is a choice of \(I_p \in \mathbb{N}\) such that if \(n \geq I_p\) and \(x \leq a^{2^n}\), then \(\Phi_p(x) \geq C/|\log x|\) for \(C = 4|\log a|/c\). Consequently, as \(s_n \geq a^{2^n}\), we will have \(\phi_p(n) \geq cn\Phi_p(a^{2^n}) \geq 2\) for all \(n \geq I_p\), (whatever the choice of \(E\), as long as the ratios lie between \(a^2\) and \(b\)). Thus with \(A = 2/c\) and \(B = c\),

\[ Bn\Phi_p(s_n) \leq \phi_p(n) \leq An\Phi_p(s_n) \text{ for all } n \geq I_p. \]

As \(\Phi_p\) decreases to 0, there is also an index \(J_p \in \mathbb{N}\) such that

\[ \Phi_p(b^n) \leq 1/(8A) \text{ for all } n \geq J_p. \]

As in the proof of Theorem 3.22, we will pick a sparse sequence \(\{N_j\}\) and assign ratios \(a^2\) except on the levels \(\{N_j + 1, ..., N_j + \phi_{r_j}(N_j)\}\) where the ratios will be \(f(r_j)\). Each \(p\) must occur as an \(r_j\) infinitely often so that we will have \(\dim_{\phi_p} E \geq \log 2/|\log f(p)|\). The numbers \(N_j\) will need to be sufficiently sparse so that if \(q > p\), this length of levels (where the ratio exceeds \(f(q)\)) is too short to influence the \(\dim_{\phi_E}\) calculation.

To begin, we list \((0, 1) \cap \mathbb{Q}\) as \(\{r_i\}_{i=1}^\infty\) where each rational number is repeated infinitely often in \(\{r_i\}\). To start the construction of \(E\), pick \(N_1 \geq \max(I_{r_1}, 8J_{r_1})\). We will set the ratios of dissection to be \(a^2\) on the levels \(\{1, ..., N_1\}\). Choose the minimal integer \(L_1\) such that \(f(r_1)L_1 \leq \phi_{r_1}(N_{1+1})\) and put \(M_1 = 4(N_1 + L_1)\). Set the ratios equal to \(f(r_1)\) on the levels \(\{N_1 + 1, ..., N_1 + L_1\}\) and \(a^2\) on the levels \(\{N_1 + L_1 + 1, ..., M_1\}\).

Notice that \(L_1 = \phi_{r_1}(N_1)\) and the choice of \(N_1\) ensures that

\[ L_1 \leq AN_1\Phi_{r_1}(s_{N_1}) \leq AN_1\Phi_{r_1}(b^{N_1}) \leq N_1/8 \]

by (3.9) and (3.10).

We proceed inductively and suppose we have chosen \(N_i, L_i, M_i\) for \(i = 1, ..., j-1\), (with the properties described below) and have specified that the ratios of dissection on levels \(\{1, ..., M_{j-1}\}\) should be \(a^2\) except on the levels \(\{N_i + 1, ..., N_i + L_i\}\), for \(i = 1, ..., j-1\), when they will be \(f(r_i)\).

Now pick \(N_j\) large enough to satisfy the following conditions:

(i) \(N_j \geq 8\max(I_{r_j}, J_{r_j}, M_{j-1})\) and

(ii) If \(i < j\) and \(r_i > r_j\), then

\[ \Phi_{r_i}(s_{M_{j-1}}a^{2(N_j-M_{j-1})}) \geq \frac{8A}{B}\Phi_{r_j}(s_{M_{j-1}}a^{2(N_j-M_{j-1})}), \]

which can be done since \(\Phi_{r_i}(x)/\Phi_{r_j}(x) \to \infty\) as \(x \to 0\).

We will assign ratio \(a^2\) on levels \(\{M_{j-1} + 1, ..., N_j\}\), so \(s_{M_{j-1}}a^{2(N_j-M_{j-1})} = s_{N_j}\), and that means (ii) actually says

\[ \Phi_{r_i}(s_{N_j}) \geq \frac{8A}{B}\Phi_{r_j}(s_{N_j}) \text{ whenever } i < j \text{ and } r_i > r_j. \]
Choose the minimal integer $L_j$ such that $f(r_j)^{L_j} \leq S_{N_j}^{\Phi_{r_j} (N_j)}$, put $M_j = 4(N_j + L_j)$ and assign the ratios on levels \{ $N_j + 1$, ..., $N_j + L_j$ \} to be $f(r_j)$ and the ratios on the levels \{ $N_j + L_j + 1$, ..., $M_j$ \} to be $a^2$.

Note that $L_j = \phi_{r_j} (N_j)$ and property (i) in the definition of $N_j$, together with \[ (3.9) \] and \[ (3.10) \] ensures $L_j \leq N_j / 8$. In particular, $N_j - L_j \geq 7/8 N_j \geq 7 M_{j-1}$ and $N_j + L_j = M_j / 4$.

This completes the construction of $E$. We now need to verify that we obtain the desired value for each $\dim \Phi_q E$. We can easily see that $\dim \Phi_q E \geq \log 2 / |\log f(q)|$ by noting that

$$
\left( \frac{s_{N_j} + \phi_{r_j} (N_j)}{s_{N_j}} \right)^{1/\phi_{r_j} (N_j)} = f(r_j)
$$

for the infinitely many choices of $r_j = q$. So we only need to prove the other inequality.

Assume the first occurrence of $q$ in $\{ r_i \}$ is with $i = j_0$. It will be enough to show that $(s_{k+m}/s_k)^{1/m} \leq f(q)$ whenever $k \geq N_{j_0}$ and $m \geq \phi_q (k)$. In other words, we want to prove that the geometric mean of the ratios $r_{k+1}, ..., r_{k+m}$ is at most $f(q)$ for all $m \geq \phi_q (k)$ and $k \geq N_{j_0}$. A key point to observe is that the geometric mean of any collection of ratios where there are at least as many ratios equal to $a^2$ as otherwise, is at most $a \leq f(q)$ for any $q$.

Given $k \geq N_{j_0}$, choose $j \geq j_0$ such that $k \in \{ M_{j-1} + 1, ..., M_j \} := B_j$. If either $k \leq N_j - L_j$ or $k > N_j + L_j$, then this is, in fact, the situation with respect to the ratios $r_{k+1}, ..., r_{k+m}$ (regardless of the size of $m$), so the geometric mean is suitably small.

Thus we can assume $k \in \{ N_j - L_j + 1, N_j + L_j \}$. If $r_j \geq q = r_{j_0}$, then $f(r_j) \leq f(q)$ and hence all ratios from $B_j$ are at most $f(q)$. In this case it is clear that the geometric mean of the collection $r_{k+1}, ..., r_J$, where $J = \min(k + m, M_j)$, is at most $f(q)$. If $k + m > M_j$, then the set of ratios $\{ r_{M_j+1}, ..., r_{k+m} \}$ contains more ratios equal to $a^2$ than otherwise, so its geometric mean is even at most $a$ and thus the geometric mean of the full collection $\{ r_{k+1}, ..., r_{k+m} \}$ is at most $f(q)$.

The last case to consider is that for this choice of $j$ (which we remind the reader is $\geq j_0$), we have $r_j < q = r_{j_0}$ and therefore $f(r_j) > f(q)$. From \[ (3.11) \], we note that

$$
\Phi_q (s_{N_j}) = \Phi_{r_{j_0}} (s_{N_j}) \geq \frac{8A}{B} \Phi_{r_{j_0}}(s_{N_j}).
$$

The remaining arguments are now similar to the proof of Theorem \[ 3.6 \] Recall that $k \in \{ N_j - L_j + 1, N_j + L_j \}$. If $N_j - L_j < k \leq N_j$, then $k \geq I_{r_j}$, so

$$
\phi_q (k) \geq B k \Phi_q (s_k) \geq B (N_j - L_j) \Phi_q (s_{N_j}) \geq 7 A N_j \Phi_{r_j} (s_{N_j}) \geq 7 \phi_{r_j} (N_j) = 7 L_j.
$$

The fact that $N_j \geq 8 J_q$ also guarantees that $\phi_q (k) \leq A k \Phi_q (s_k) \leq k / 8$, so $k + \phi_q (k) < M_j / 2$. Thus the collection $\{ r_{k+1}, ..., r_{k+\phi_q (k)} \}$ contains at most $L_J$ terms of ratio $f(r_j)$ and at least $6 L_j$ terms of ratio $a^2$, and therefore has geometric mean at most $a$.

If, instead $N_j < k \leq N_j + L_j$, then as in the proof of Theorem \[ 3.10 \] (see \[ 3.7 \])

$$
k + \phi_q (k) - (N_j + L_j) \geq \phi_q (N_j) - \phi_{r_j} (N_j) \geq 6 L_j
$$
where the final inequality comes from applying (3.12) with \( k = N_j \). Again,
\[
k + \phi_q(k) \leq 9k/8 < M_j/2
\]
and thus again we deduce that the geometric mean of \( \{r_{k+1}, \ldots, r_{k+\phi_q(k)}\} \) is at most \( a \).

For either choice of \( k \), if \( m > \phi_q(k) \), then since \( N_j + L_j < k + \phi_q(k) \leq M_j/2 \) the collection of ratios \( \{r_{k+\phi_q(k)+1}, \ldots, r_{k+m}\} \) has more that value \( a^2 \) than otherwise, and hence has geometric mean at most \( a \), as well. Thus, again, we conclude \( (s_k/s_{k+m})^{1/m} \leq a \) in this (final) case.

This completes the proof. \( \square \)

**Corollary 3.9.** Given \( 0 < \alpha < \beta < 1 \), there is a set \( E \subseteq [0,1] \) such that
\[
\{\overline{\dim}_q E \colon \Phi(x) \to 0 \text{ as } x \to 0\} = [\alpha, \beta] = [\dim_{q_A} E, \dim_A E].
\]

**Proof.** Let \( D(E) = \{\overline{\dim}_q E \colon \Phi(x) \to 0\} \). For \( p \in (0,1) \),
\[
\{\overline{\dim}_{q_p} E \colon \Phi_p(x) = |\log x|^{p-1} \} \subseteq D(E) \subseteq [\dim_{q_A} E, \dim_A E],
\]
so it will be sufficient to construct a set \( E \) with \( \{\overline{\dim}_{q_p} E \colon \Phi_p(x) = |\log x|^{p-1} \} = (\alpha, \beta) \) and \( \dim_{q_A} E = \alpha, \dim_A E = \beta \). The previous theorem would permit us to construct such a set satisfying the first property and would also have \( \dim_A E = \beta \). However, its quasi-Assouad dimension is \( a^2 \), so we need to modify the construction slightly.

We can do this by requiring the sequence \( \{N_j\} \) to grow so rapidly that in addition to the requirements from before, we can also have \( K_j \) much greater than \( M_j \) and \( N_{j+1} \) much greater than \( 2K_j \). On the levels \( K_j + 1, \ldots, 2K_j \) we will set the ratios to equal to \( a = 2^{-1/\alpha} \) (rather than \( a^2 \)). One can see that \( \dim_{q_A} E = \log 2/|\log a| = \alpha \) by considering the terms \( s_{K_j}/s_{2K_j} \). The sparseness of the \( \{K_j\} \) will ensure that the other dimensions are not affected by this change. We leave the technical details to the reader. \( \square \)

**4. \( \Phi \)-Dimensions of Complementary sets in \( \mathbb{R} \)**

**4.1. Bounds for \( \Phi \)-dimensions of complementary sets.**

**4.1.1. Complementary sets.** Every closed subset of the interval \([0,1]\) of Lebesgue measure zero is of the form \( E = [0,1] \setminus \bigcup_{j=1}^{\infty} U_j \) where \( \{U_j\} \) is a disjoint family of open subintervals of \([0,1]\) whose lengths sum to one. We will let \( a = \{a_j\}_{j=1}^{\infty} \) where \( a_j \) is the length of \( U_j \). Of course, \( \sum a_j = 1 \) and without loss of generality we can assume \( a_{j+1} \leq a_j \). We will denote by \( C_a \) the collection of all such closed sets \( E \). These are called the **complementary sets of** \( a \).

One example of a complementary set is the Cantor set associated with \( a \), denoted \( C_a \). Another is the countable set, \( D_a \), called the decreasing rearrangement, defined as
\[
D_a = \{\sum_{i \geq k} a_i\}_{k=1}^{\infty} = \{1, 1-a_1, 1-a_1-a_2, \ldots\}.
\]

As is well known, all complementary sets of a given sequence \( a \) have the same upper and lower box dimensions [6] Section 3.2], but, of course, this need not be true for other dimensions. For instance, the Hausdorff dimension of the decreasing rearrangement is 0, but this need not be true for the Cantor set. In [3], Besicovitch and Taylor proved that the Cantor set \( C_a \) had the maximum Hausdorff dimension of any set in \( C_a \). Further, they showed given any \( s \leq \dim_H C_a \) there is some set...
$E \in \mathcal{C}_a$ with $\dim H E = s$. The same result was shown to be true with the Hausdorff dimension replaced by the packing dimension in [16]. In [14], it was shown that the Cantor set and the decreasing set also have the extremal Assouad dimensions (under natural assumptions on the gap sequence $a$). But unlike the situation for Hausdorff, packing and lower Assouad dimensions, $\dim A C_a$ is minimal among the sets in $\mathcal{C}_a$ and $\dim A D_a$ is maximal (and equals 1 for such $a$). Again, it was shown that the full range of possible dimensions is attained, namely $\{\dim L E : E \in \mathcal{C}_a\} = [0, \dim L C]$ and $\{\dim A E : E \in \mathcal{C}_a\} = [\dim A C, 1]$.

In this section, we will prove analogous results for the $\Phi$-dimensions, although some proofs are necessarily quite different.

4.1.2. Decreasing rearrangement. We first prove that the decreasing rearrangement is always one of the extreme values of the $\Phi$-dimension over the class $\mathcal{C}_a$. This requires a proof in the case of the upper $\Phi$-dimension as this dimension need not be 1, c.f., Example [2, 18]. To begin, we first point out the following elementary result which essentially can be found in [6].

**Lemma 4.1.** Suppose $F, G$ are two compact sets in $\mathcal{C}_a$ for some decreasing, summable sequence $a = (a_n)$. For any $r > 0$,

$$\frac{1}{16} \leq \frac{N_r(F)}{N_r(G)} \leq 16.$$

**Proof.** Let $F_r$ be the $r$-dilation of $F$. As noted in [6 Sec 3.2], the Lebesgue measure of $F_r$, denoted $|F_r|$, depends only on the lengths $a_n$, and not their rearrangement. Thus $|F_r| = |G_r|$. If $F$ can be covered by $N_r(F)$ intervals of radius $r$ (length $2r$), then $F_r$ can be covered by the concentric intervals with radius $2r$. Thus the Lebesgue measure of $F_r$ is at most $4rN_r(F)$. On the other hand, suppose $\{U_i\}$ is a collection of $N_r(F)$ intervals from a $2r$-covering of $F$, ordered from left to right. Then $\{U_{4i}\}$ is a disjoint collection of intervals of radius $r$ separated by at least $2r$. Moving each interval less than $r$ to the left or right suitably, we obtain a collection of $N_r(F)/4$ disjoint intervals of radius $r$, with centers in $F$.

The above observations show

$$\frac{|F_r|}{4r} \leq \frac{N_r(F)}{N_r(G)} \leq \frac{4|F_r|}{r},$$

and the conclusion of the lemma follows from these inequalities. \qed

**Proposition 4.2.** If $a$ is any decreasing, summable sequence, then $\underline{\dim}_\Phi E \leq \underline{\dim}_\Phi D_a$ and $\underline{\dim}_\Phi E \geq \underline{\dim}_\Phi D_a = 0$ for all $E \in \mathcal{C}_a$.

**Proof.** As $D_a$ has isolated points, $\underline{\dim}_\Phi D_a = 0$ for all dimensions functions $\Phi$ and hence is the minimal lower $\Phi$-dimension.

To prove that $\overline{\dim}_\Phi D_a$ is the maximal upper $\Phi$-dimension we will simply show that

$$N_r(E \bigcap B(e, R)) \leq 64N_r(D_a \bigcap B(0, R))$$

for all $e \in E$ and $r \leq R$.

To see this, let $\{a_j\}$ be the set of gap lengths that are completely contained in $E \bigcap B(e, R)$ and let $R' = \sum a_j$. Then $N_r(E \bigcap B(e, R)) = N_r(E \bigcap I)$ for a suitable interval $I \subseteq B(e, R)$ of length $R' \leq 2R$. Let $E'$ denote the set formed by removing from $[0, R']$ the gaps of lengths $(a_j)$ in decreasing order (from right to left). By Lemma [4.1] $N_r(E \bigcap I) \leq 16N_r(E' \bigcap [0, R'])$. 

Choose \( n \) such that \( a_n \leq 2r < a_{n-1} \) and suppose that \( R' \in (\sum_{i=n+1}^{\infty} a_i, \sum_{i=m}^{\infty} a_i] \). (We will say that \( R' \) belongs to gap \( a_m \) in the set \( D_a \).) If \( n \leq m \), then all gaps in the construction of \( D_a \) intersecting \([0, R']\) have length at most \( 2r \). Thus 
\[
N_r(D_a \cap [0, R']) = \left\lceil \frac{R'}{2r} \right\rceil
\]
is the maximum possible value and hence it dominates 
\[
N_r(E' \cap [0, R']) = \left\lceil \frac{A}{2r} \right\rceil + s.
\]

So assume \( n > m \) and let \( A = \sum_{i=n}^{\infty} a_i < R' \). As \( a_j \leq 2r \) for \( j \geq n \) and \( a_j > 2r \) for \( j \leq n-1 \),
\[
N_r(D_a \cap [0, R']) \geq N_r(D_a \cap [0, A]) + N_r(D_a \cap [A + a_{n-1}, R']) \geq \left\lceil \frac{A}{2r} \right\rceil + \max(n - m - 2, 0)
\]
(where \([A + a_{n-1}, R']\) is empty if \( R' < A + a_{n-1} \)).

Assume that the number \( A \) belongs to gap \( a_j \), in the set \( E' \), so 
\[
N_r(E' \cap [0, R']) \leq \left\lceil \frac{A}{2r} \right\rceil + s.
\]

Since \( \sum_{i=n}^{\infty} a_i = A \geq \sum_{i=s+1}^{\infty} a_{j_i} \), it follows that \( j_s \leq n-1 \) and consequently, 
\[a_{j_{s-k}} \geq a_{n-1-k}\] for all \( k = 0, ..., s-1 \). Thus if \( n-s+1 \leq m \), then 
\[
R' - A > a_{j_{s-1}} + a_{j_{s-2}} + \cdots + a_{j_2} + a_{j_1} \geq a_{n-2} + a_{n-3} + \cdots + a_{n-s+1} + a_{n-s} \geq a_{n-2} + a_{n-3} + \cdots + a_{n-s+1} + a_{n-1} \geq R' - A.
\]

This contradiction proves \( s \leq n - m \). Thus 
\[
N_r(D_a \cap [0, R']) \geq \left\lceil \frac{A}{2r} \right\rceil + \max(s - 2, 0),
\]
from which it is easy to check that 
\[
N_r(D_a \cap [0, R]) \geq N_r(D_a \cap [0, R']) \geq \frac{1}{4} \left( \left\lceil \frac{A}{2r} \right\rceil + s \right) \geq \frac{1}{4} N_r(E' \cap [0, R']) \geq \frac{1}{64} N_r(E \cap B(c, R)),
\]
and the proposition follows.

4.1.3. Cantor Sets. We now focus our attention on decreasing, summable sequences \( a \) with the property that there are constants \( \tau \) and \( \lambda \) with
\[
0 < \tau \leq s_{j+1}/s_j \leq \lambda < 1/2.
\]
(As before, \( s_j = 2^{-j} \sum_{i \geq 2^j} a_i \).) We will call a sequence \( \{a_j\} \) with this property level comparable. Of course, the doubling assumption automatically gives the left hand inequality and central Cantor sets have the level comparable property precisely when their ratios of dissection are bounded away from 0 and 1/2.

The level comparable assumption is very useful as it ensures that \( s_k \sim a_{2^k} \) since \( s_k \geq a_{2^{k+1}} \geq a_{2^k} \) and
\[
(1 - 2\lambda)s_k \leq s_k - 2s_{k+1} \leq a_{2^k}.
\]

For level comparable sequences, the Cantor set has the other extreme value for the \( \Phi \)-dimensions.

**Theorem 4.3.** If \( a = \{a_j\} \) is a level comparable sequence, then for all \( E \in C_a \) we have 
\[
\dim_\Phi E \geq \dim_\Phi C_a \text{ and } \dim_\Phi E \leq \dim_\Phi C_a.
\]
PROOF. We begin with the upper $Φ$-dimension. Observe that if $φ$ is bounded, then the upper $Φ$-dimension is the Assouad dimension and the result is already known in that case, see [14] Thm. 3.5. So assume otherwise. Some modifications to the proof of Theorem 3.5 in [14] are required.

Let $d = \dim_Φ C_α$. From the formula for the upper $Φ$-dimension of $C_α$, Theorem 4.3 we know there must exist $κ_0, c_0$ and indices $k ≥ k_0$ and $n ≥ φ(k)$ such that

$$c_0 2^n ≥ \left(\frac{s_k}{s_{k+n}}\right)^{d-ε} ≥ 2^{2nε} \left(\frac{s_k}{s_{k+n}}\right)^{d-2ε},$$

where the latter inequality holds because $s_k/s_{k+1} ≥ 2$ for all $k$.

We will refer to the complementary gaps of lengths $a_{2k-1},...,a_{2k-1}$ as the gaps of level $k$.

Remove from $[0, \sum a_j]$ the complementary gaps of levels $1,...,k$ to obtain the set $J_1 \cup \cdots \cup J_{M_k} \cup \{\text{singletons}\}$ where $J_i$ are non-trivial, closed, disjoint intervals, $M_k ≤ 2^k$ and $\sum |J_i| = 2^k s_k$. Let $b_i$ denote the number of gaps of step $k+n$ contained in $J_i$ and put $r = a_{2k+n}/2$. If we let $x_i$ be an endpoint of $J_i$, then as the gaps of step $k+n$ are at least $2r$ in length, $N_r(B(x_i, |J_i|) \cap E) ≥ b_i$. Since $\sum_i b_i = 2^{k+n}$,

$$\sum_i N_r(B(x_i, |J_i|) \cap E) ≥ 2^{k+n} ≥ 2^{2k nε} \left(\frac{s_k}{s_{k+n}}\right)^{d-2ε}.$$  

Let

$$I = \{i \in \{1,...,M_k\} : |J_i| ≤ s_k\}.$$  

If there is some index $i \in I$ with $N_r(B(x_i, |J_i|) \cap E) ≥ \left(\frac{s_k}{s_{k+n}}\right)^{d-2ε}$, then

$$N_r(B(x_i, s_k) \cap E) ≥ N_r(B(x_i, |J_i|) \cap E) ≥ \left(\frac{s_k}{s_{k+n}}\right)^{d-2ε}. \tag{4.3}$$

Otherwise,

$$\sum_{i \notin I} N_r(B(x_i, |J_i|) \cap E) = \sum_i N_r(B(x_i, |J_i|) \cap E) - \sum_{i \in I} N_r(B(x_i, |J_i|) \cap E) \geq 2^{k+n} - |I| \max_{i \in I} N_r(B(x_i, |J_i|) \cap E) \geq 2^{k+n} - |I| \max_{i \in I} N_r(B(x_i, |J_i|) \cap E) \geq 2^{k+n} - |I| \frac{2^{k+n}}{2} \left(\frac{s_k}{s_{k+n}}\right)^{d-2ε} \left(\frac{2^{nε}}{c_0} - 1\right).$$

Since $n ≥ φ(k) → \infty$, we can assume $2^{nε}/c_0 - 1 ≥ 2^{2nε}$. Recall that $\sum_i |J_i| = 2^k s_k$, thus

$$\sum_{i \notin I} N_r(B(x_i, |J_i|) \cap E) ≥ 2^k \left(\frac{s_k}{s_{k+n}}\right)^{d-2ε} 2^{2nε} ≥ 2^{k+2nε} \left(\frac{2^{-k} \sum_{i \notin I} |J_i|}{a_{2k+n}}\right)^{d-2ε}.$$
An application of Holder’s inequality gives
\[ \sum_{i \notin I} N_r(B(x_i, |J_i|) \cap E) \gtrsim 2^{k \log_2 \left( \frac{2^k}{a_{2^{k+n}}} \right)} \cdot \sum_{i \in I} |J_i|^{-d-2\varepsilon} |I^c|^{-(1-d+2\varepsilon)} \]
\[ \gtrsim 2^{\varepsilon} \left( \frac{2^k}{|I^c|} \right)^{1-d+2\varepsilon} \cdot \sum_{i \notin I} |J_i|^{-d-2\varepsilon} \frac{1}{\varepsilon} \sum_{i \notin I} |J_i|^{-d-2\varepsilon} \]
with the final inequality arising because \(|I^c| \leq M_k \leq 2^k\) and \(d \leq 1\). It follows that in this case there must be some choice of \(i \notin I\) such that
\[ N_r(B(x_i, |J_i|) \cap E) \gtrsim 2^{\varepsilon} \left( \frac{|J_i|}{r} \right)^{d-2\varepsilon} \]
By definition, \(i \notin I\) implies \(|J_i| \geq s_k\) and thus \(|J_i|^1+\Phi(|J_i|) \geq \frac{1}{s_k}+\Phi(s_k) \geq s_{k+n} \sim r\).

As either (4.3) or (4.4) must hold, we deduce that \(\text{dim}_\Phi E \geq d - 2\varepsilon\) and that gives the desired result.

The proof for the lower \(\Phi\)-dimension is a straightforward modification of Theorem 4.1 of (14).

Combining Proposition 4.2 and Theorem 4.3 gives the following statement.

**Corollary 4.4.** If \(a\) is any level comparable sequence, then for all \(E \in C_a\) we have \(\text{dim}_\Phi E \in \left[ \text{dim}_\Phi C_a, \text{dim}_\Phi D_a \right]\) and \(\text{dim}_\Phi E \in [0, \text{dim}_\Phi C_a]\). In particular, these statements are true for the quasi-Assouad dimensions.

**4.2. An interval of \(\Phi\)-dimensions for complementary sets.** In (14) it was shown that if \(a\) is any level comparable sequence, then for every \(c \in [0, \text{dim}_L C_a]\) and \(d \in [\text{dim}_A C_a, 1]\) there are sets \(E_c, E_d \in C_a\) with \(\text{dim}_L E_c = c\) and \(\text{dim}_A E_d = d\)\(\footnote{Actually, the assumption that \(a\) is doubling suffices for the upper Assouad dimension.}\)

These results continue to be true for the quasi-Assouad and \(\Phi\)-dimensions when \(\Phi \to p\), with \(p \in [0, \infty]\). For the lower \(\Phi\)-dimensions essentially the same proof as given in (14) for the lower Assouad dimension works. We give a brief sketch of the main idea at the beginning of the proof of Theorem 4.5.

For the upper \(\Phi\)-dimension, note that the case \(p = \infty\) is trivial since we recover the upper box dimension and all complementary sets of a given sequence have the same upper box dimension. Different proofs are required for the cases \(\Phi \to p\) for \(p = 0\) or \(p > 0\), and these are necessarily different from the proof given for the Assouad dimension in (14) as the set constructed there only exhibits large local ‘thickness’ on scales \(r\) that are nearly as large as \(R\), and hence are not suitable for use in obtaining these other dimensions.

**Theorem 4.5.** Suppose \(a\) is a level comparable sequence and \(\Phi\) is a dimension function with \(\Phi(R) \to p\), for some \(p \in [0, \infty]\). Then for every \(c \in [0, \text{dim}_\Phi C_a]\) and \(d \in [\text{dim}_\Phi C_a, \text{dim}_\Phi D_a]\), there are sets \(E_c, E_d \in C_a\) with \(\text{dim}_\Phi E_c = c\) and \(\text{dim}_\Phi E_d = d\). A similar statement holds with \(\text{dim}_\Phi C_a\) replaced by \(\text{dim}_\Phi C_a\) and \(\text{dim}_\Phi E_d = d\). [13]

**Remark 4.6.** We remind the reader that for any doubling sequence \(a\) (and hence any level comparable sequence) and any dimension function \(\Phi \to 0\), we have \(\text{dim}_B D_a > 0\) and thus \(\text{dim}_\Phi D_a \geq \text{dim}_\Phi^\alpha D_a = 1\) by [13].

Combining this result with Theorem 4.3 gives the following.
Corollary 4.7. Suppose \( a \) is any level comparable sequence and \( \Phi \) is a dimension function with \( \Phi(R) \to p \). Then

\[
\overline{\dim}_a E : E \in C_a = [\overline{\dim}_a C_a, \overline{\dim}_a D_a]
\]

and

\[
\underline{\dim}_a E : E \in C_a = [\underline{\dim}_a D_a, \underline{\dim}_a C_a] = [0, \underline{\dim}_a C_a].
\]

Proof of Theorem 4.5. For the lower dimension case, the same proof given in [14] Theorem 4.3 for the lower Assouad dimension, with the obvious modifications, works for the lower \( \Phi \)-dimensions and the lower quasi-Assouad dimension.

A sketch of the proof is that for \( 0 < \alpha < \overline{\dim}_a C_a \), it is possible to find a subsequence of \( a \) whose Cantor rearrangement is an \( \alpha \)-Ahlfors regular set and such that the Cantor rearrangement of the remaining gaps has lower \( \Phi \)-dimension equal to \( \overline{\dim}_a C_a \). This gives a complementary set \( E \) with \( \overline{\dim}_a E = \alpha \).

For the upper dimension case, we will first give the proof for the case \( \Phi \to p \) for some \( 0 < p < \infty \), where we can take advantage of an explicit formula for the \( \Phi \)-dimension of the decreasing rearrangement. The harder case, \( p = 0 \), is left to the end.

Case \( \Phi \to p \).

We have by Corollary 2.8 that \( \overline{\dim}_a E = \overline{\dim}_A E \) for all \( E \), where \( \theta = (1 + p)^{-1} \).

Observe that

\[
(4.5) \quad \overline{\dim}_A D_a = \min \left( \frac{\overline{\dim}_B D_a}{1 - \theta}, 1 \right) = T
\]

which follows from [11] Theorem 6.2] and [9] Theorem 2.1]. Given any \( 0 < s < T \), we use the above formula to construct a subsequence \( b \) of \( a \) such that if \( \tilde{a} \) is the subsequence obtained after removing \( b \) from \( a \), then \( \overline{\dim}_A D_{\tilde{a}} = s \) and \( \overline{\dim}_A C_{\tilde{a}} = \overline{\dim}_A C_a \).

Thus \( E = D_b \cup C_{\tilde{a}} \subseteq C_a \) and by the union property, \( \overline{\dim}_A E = \max(s, \overline{\dim}_A C_a) \), which proves the statement.

Let \( d := \overline{\dim}_B D_a \). By [27] Section 3.4 we have

\[
d = \lim \sup_{n \to \infty} \frac{\log n}{-\log a_n} = \lim_{k \to \infty} \frac{\log n_k}{-\log a_{n_k}},
\]

where \( \{n_k\} \) is chosen to be a sparse sequence, say \( n_{k+1} \geq 2^{n_k} \). This allows us, for \( 0 < B < 1 \), to define the subsequence \( b \) by

\[
b_m = \begin{cases} a_{n_k}, & \lceil m^{1/B} \rceil \leq n_k < \lceil (m + 1)^{1/B} \rceil \\ a_{\lceil m^{1/B} \rceil}, & \text{otherwise} \end{cases}
\]

Note that for the integers \( m_k \) where \( b_{m_k} = a_{n_k} \) we have \( m_k \sim n_k^{1/B} \), so

\[
\lim_{k} \frac{\log m_k}{-\log b_m} = \lim_{k} \frac{\log n_k^{1/B}}{-\log a_{n_k}} = Bd.
\]

Moreover, for \( \epsilon > 0 \) we have \( \log n/( -\log a_n) < d + \epsilon \) for all \( n \) large enough, so for large \( m \), with \( m \neq m_k \),

\[
\frac{\log m}{-\log b_m} = \frac{B \log m^{1/B}}{-\log a_{\lceil m^{1/B} \rceil}} \leq B(d + 2\epsilon).
\]

Therefore, \( \overline{\dim}_B D_b = Bd \), and by (4.5) we can choose \( B \) so that \( \overline{\dim}_A D_b = s \).
Finally, note that \( \lfloor (m+1)^{1/B} \rfloor - \lfloor m^{1/B} \rfloor \to \infty \) as \( m \) increases. As the original sequence was doubling, this ensures that the sequence \( \tilde{a} \) consisting of the remaining gaps is comparable to the original sequence \( a \). In consequence, \( \dim_A C_{\tilde{a}} = \dim_A C_a \).

**Case \( \Phi \to 0 \).**

We will give the proof for the quasi-Assouad dimension. It will be clear that the same arguments will work for the upper \( \Phi \)-dimension with \( \Phi \to 0 \). Our proof is constructive. The set \( E = E_d \subseteq C_a \) will again have the form \( E = A \cup B \), with \( \dim_q A \) equal to the desired in-between value \( d \) and \( \dim_q B = \dim_q C_a \). The union property for the quasi-Assouad dimension will ensure that \( E \) has the desired quasi-Assouad dimension.

If \( b = \{b_j\} \) is the sequence with \( b_{2^j+1} = a_{2^j} \) for \( t = 0, \ldots, 2^j - 1 \), then \( a, b \) are comparable sequences and if \( E \) is the set formed with some rearrangement of \( a \) and \( F \) is the corresponding rearrangement of \( b \), then \( E \) and \( F \) are bi-Lipschitz equivalent. So without loss of generality we will assume \( a \) is constant along diadic blocks. Moreover, a level comparable sequence \( \{a_j\} \) has the property that there are constants \( u, v \) such that

\[
1 > u \geq \frac{a_{2^j}}{a_{2^j-1}} \geq v > 0 \text{ for all } j.
\]

If \( \dim_q C_a = 1 \), there is nothing to do. So assume \( 1 > d > \dim_q C_a \), say \( d = \log 2/|\log \beta| \) where \( \beta < 1/2 \).

Temporarily fix \( M \). Given \( j \geq 1 \), choose the minimal index \( i_j \geq 1 \) such that \( a_{2M+i_j}/a_{2M} \leq \beta^j \) and choose the maximal integer \( e_j \geq 1 \) such that

\[
e_j \frac{a_{2M+i_j}}{a_{2M}} \leq \beta^j.
\]

The minimality of \( i_j \) ensures that

\[
e_j \frac{a_{2M+i_j}}{a_{2M}} \leq \beta^j < \frac{a_{2M+i_j+1}}{a_{2M}}
\]

which implies

\[
e_j \leq \frac{a_{2M+i_j+1}}{a_{2M+i_j}} \leq \frac{1}{v^j}.
\]

Similarly, the maximality of \( e_j \) means that

\[
(e_j + 1) \frac{a_{2M+i_j}}{a_{2M}} > \beta^j,
\]

so

\[
a_{2M+i_j} \geq \beta^j \frac{a_{2M}}{1+1/v} = c_1 a_{2M} \beta^j,
\]

where \( c_1 > 0 \) is independent of \( M \) and \( j \). Moreover, the fact that \( (v^j \leq a_{2M+i_j}/a_{2M} \leq u^j \), coupled with the definition of \( i_j \), implies

\[
c_3 j \leq \frac{\log \beta}{\log v} + 1 \leq i_j \leq \frac{\log \beta}{\log v} = c_2 j
\]

where we again note that \( c_2, c_3 \) are positive constants, independent of \( M \) and \( j \).

We now form a Cantor-tree like arrangement with blocks of gaps. The first block will consist of \( e_1 \) gaps of length \( a_{2M+i_j} \) placed adjacently. The blocks of level 2 will each consist of \( e_2 \) gaps of length \( a_{2M+i_2} \) placed adjacently and there will be two blocks of level 2, one to the left and the other to the right of the block of level 1. In general, there will be \( 2^{j-1} \) blocks of level \( j \), each consisting of \( e_j \) gaps of length.
\(a_{p^{M+i}}\) placed in a Cantor-like arrangement. If we do this for \(j = 1, \ldots, N\), we will call the resulting finite set \(X_{M,N}\). Note that the length of any block of level \(j\) in \(X_{M,N}\) is equal to \(c_j a_{2^{M+i}}\) and satisfies

\[
(4.6) \quad c_1 a_2 m j \leq e_j a_{2^{M+i}} \leq a_{2M} \beta^j.
\]

Hence the diameter of \(X_{M,N}\) is at least the length of block 1 which is \(\geq c_1 a_2 m\), and the diameter of \(X_{M,N}\) is at most

\[
(4.7) \quad \sum_{j=1}^{N} 2^{j-1} a_2 m \beta^j \leq a_{2M} \frac{\beta}{1-2\beta} = c_4 a_2 m \beta.
\]

Since \(i_j \geq c_2 j\), for each \(k\) the number of gaps of length \(a_{2m+k}\) that we will require is

\[
\sum_{j \in \{1, \ldots, N\} : j + k = k} e_j 2^{j-1} \leq \sum_{j=1}^{\infty} e_j 2^{j-1} \leq \frac{1}{v} 2^{k/c_2}.
\]

As \(j \in \{1, \ldots, N\}\) and \(i_j \leq c_3 j\), we have \(k \leq c_3 N\). Of course, for each \(k\) there are a total of \(2^{M+k}\) gaps of this size available in the sequence \(a\), so we have enough gaps, even twice as many as we need, provided

\[
\frac{1}{v} 2^{k/c_2} \leq 2^{M+k-1} \text{ for each } k = 1, \ldots, c_3 N.
\]

Hence there is some \(c_5 > 0\) (and independent of \(M\)) such that if \(N \leq c_5 M\), then there will be enough gaps to carry out this construction.

Lastly, we will select a rapidly growing sequence of integers \(\{M_k\}\) and let \(N_k = [c_5 M_k]\). We will set \(A_k = X_{M_k, N_k}\). We will want \(M_k+1\) to be much larger than \(M_k + c_3 N_k\), so that we will not use any gaps from the same diadic blocks in two different sets \(A_j\). Also, we will want to choose \(M_k\) increasing so rapidly that the diameter of \(A_{k+1}\) is at most \(1/2\) diameter of \(A_k\).

We will position the sets \(A_k\) adjacent to each other in decreasing order and let \(A = \bigcup_{k=1}^{\infty} A_k\). The gaps of the sequence \(\{a_j\}\) that were not used in the construction of the sets \(A_k\) will then be placed to form a Cantor set \(B\) to the left of \(A_1\). This completes the construction of the set \(E = A \cup B \in C_a\).

Since there are at least half the gaps \(a_j\) left in each diadic block, the decreasing sequence consisting of the remaining gaps is comparable to the original sequence. Hence \(\dim_{\mathbb{q}} A \cup B = \dim_{\mathbb{q}} A \leq c \leq d\). Thus, to see that the rearranged set \(A \cup B\) has quasi-Assouad dimension \(d\), it will be enough to prove \(\dim_{\mathbb{q}} A = d\).

First, we will check that \(\dim_{\mathbb{q}} A \geq d\). For this, consider \(R = \text{diameter of } A_k \sim a_2 M_k \beta^j\) (by \((4.6)\)), and 

\[
\frac{r}{R^{1+\delta}} \leq \frac{a_2 M_k \beta^N_k}{(a_2 M_k \beta^j)^{1+\delta}} = c \frac{\beta^N_k}{\beta^{1+\delta} a_2^{M_k}} \leq c \frac{\beta^c}{\beta^{1+\delta} v^\delta} \leq \frac{1}{M_k} < 1.
\]

If we let \(e \in A_k\), then \(N(e, B(e, R) \cap A_k) \geq 2^{N_k-1}\) since the blocks of level \(N_k\) are separated by at least \(r\), while \((R/r)^d \sim \beta^{-d N_k} = 2^{N_k}\). In order for there to be a constant \(C\) such that \(N(e, B(e, R) \cap A_k) \leq C(R/r)^t\) for all \(k\), we must have \(t \geq \log 2/|\log \beta| = d\). This shows \(\dim_{\mathbb{q}} A \cup A_k \geq d\).
The next step will be to show that there is a constant $C$, independent of $k$, such that

$$(4.8) \quad N_r(B(e, R) \cap A_k) \leq C \left( \frac{\min(\ell(A_k), R)}{r} \right)^d$$

for all $r < \min(\ell(A_k), R)$ and all $e \in \mathcal{A}$, where $\ell(Y)$ denotes the diameter of set $Y \subseteq \mathbb{R}$.

Assuming this, we complete the proof as follows: We can assume $R \leq \ell(A_1)/2$. Suppose $r < R$ and that

$$\ell(A_k + 1)/2 < R < \ell(A_k)/2.$$ 

Then $B(e, R)$ can intersect at most two (consecutive) sets $A_i$ for $i \leq k$, (say $i = m, m + 1$), as well as possibly $\bigcup_{i=k+1}^{\infty} A_i$. Assume

$$\ell(A_j) \leq r < \ell(A_{j-1})$$

where, of course, $j \geq k + 1$. Since $\sum_{i=j}^{\infty} \ell(A_i) \leq 2\ell(A_j)$, one ball of radius $r$ will cover $\bigcup_{i=k+1}^{\infty} A_i$. As $r \leq \ell(A_k) \leq \ell(A_{m+1})$, from (4.8) we have

$$N_r(B(e, R) \cap A) \leq N_r(B(e, R) \cap (A_m \cup A_{m+1})) + \sum_{i=k+1}^{\infty} N_r(B(e, R) \cap A_i)$$

$$\leq 2C \left( \frac{R}{r} \right)^d + \sum_{i=k+1}^{j-1} N_r(B(e, R) \cap A_i) + 1$$

(where the sum is empty if $j-1 < k+1$). Since $r < \ell(A_i)$ for $i = k + 1, \ldots, j - 1$, from (4.8) we again see that

$$N_r(B(e, R) \cap A) \leq 2C \left( \frac{R}{r} \right)^d + C \sum_{i=k+1}^{j-1} \left( \frac{\ell(A_i)}{r} \right)^d + 1$$

$$\leq C'' \left( \frac{R}{r} \right)^d + C'' \left( \frac{\ell(A_{k+1})}{r} \right)^d \leq C'' \left( \frac{R}{r} \right)^d.$$ 

That proves that even $\dim_A \mathcal{A} \leq d$ and hence $\dim_{\mathcal{A}}^q \mathcal{A} = d$.

So it only remains to prove (4.8). Choose $\gamma \leq 1$ such that the diameter of $A_k \geq \gamma a_{2M_k} \beta$ for all $k$. Temporarily fix $k$. Choose $R$ and $r < \min(\ell(A_k), R)$.

First, suppose there is some $j \in \mathbb{N}$ such that

$$\gamma a_{2M_k} \beta^{j+1}/4 < R \leq \gamma a_{2M_k} \beta^j/4$$

(in particular, $R < \ell(A_k)$). If $j > N_k$, then $2R$ is smaller than the smallest block in $A_k$ and thus $B(e, R)$ can intersect at most two blocks in $A_k$. As there are at most $1/v$ gaps in each block,

$$N_r(B(e, R) \cap A_k) \leq \frac{2}{v} \leq C \left( \frac{R}{r} \right)^d.$$ 

Hence assume $j \leq N_k$. Then $2R$ is less than the length of any block of level $\leq j$ and thus $B(e, R) \cap A_k$ can intersect at most two (consecutive) blocks of level $\leq j$, as well as the interval $I$ in-between (where an in-between interval could mean the interval between the left or right-most block of level $j$ and the endpoint of the
set \(A_k\). The points in \(A\) from the two blocks of level at most \(j\) can be covered by \(2/v\) balls of radius \(r\), hence
\[
N_r(B(e, R) \cap A_k) \leq \frac{2}{v} + N_r(B(e, R) \cap I).
\]
Notice that the interval \(I\) will contain (at most) \(2^{n-j}\) blocks of level \(n \geq j + 1\). Also, observe that the interval between two consecutive blocks of level \(n\) (should it exist in \(A_k\)) has length at most
\[
\sum_{i=n+1}^{\infty} 2^{-(n+1)}a_{2^m_k}^\beta r < a_{2^m_k}^\beta \frac{2^n}{1-2\beta}.
\]
Thus if
\[
a_{2^m_k}^\beta \frac{2^n}{1-2\beta} < r \leq a_{2^m_k}^\beta, \quad \text{for some } n \geq j + 1,
\]
then each such subinterval can be covered by one ball of radius \(r\). There are at most \(2^{n-j}\) such subintervals contained in \(I\). Additionally, the points in \(A\) from each of the blocks of levels \(j + 1, \ldots, n\) contained in \(I\) can be covered by \(1/v\) balls of radius \(r\) and there are \(\leq 2^{n-j}\) such blocks. So
\[
N_r(B(e, R) \cap I) \leq 2^{n-j} + 2^{n-j}/v \leq C'2^{n-j}
\]
where
\[
C' = \beta d < \frac{1}{d}.
\]
Thus for such \(r\) we certainly have
\[
N_r(B(e, R) \cap A_k) \leq \frac{2}{v} + C' \left(\frac{R}{r}\right)^d \leq C' \left(\frac{\min(\ell(A_k), R)}{r}\right)^d
\]
for a suitable constant \(C\) (recalling that \(R < \ell(A_k)\) and \(R/r \geq 1\)).

If (4.9) does not hold, we must have
\[
a_{2^m_k}^\beta < r \leq \frac{\gamma a_{2^m_k}^\beta}{4}.
\]
Then \(B(e, R)\) is covered by a bounded number (independent of \(j, k\)) of balls of radius \(r\) and that also suffices to prove
\[
N_r(B(e, R) \cap A_k) \leq C' \left(\frac{R}{r}\right)^d \leq C' \left(\frac{\min(\ell(A_k), R)}{r}\right)^d
\]
for these \(r\).

Otherwise, \(R > \gamma a_{2^m_k}^\beta/4\). If (still) \(R \leq \ell(A_k)\), then we argue similarly, taking as \(I\) the full set \(A_k\). Finally, suppose \(R > \ell(A_k)\). Then
\[
N_r(B(e, R) \cap A_k) \leq N_r(B(e', \ell(A_k)) \cap A_k)
\]
where \(e' \in A_k\). As \(r < \ell(A_k)\), the previous work shows
\[
N_r(B(e', \ell(A_k)) \cap A_k) \leq C' \left(\frac{\ell(A_k)}{r}\right)^d \leq C' \left(\frac{\min(\ell(A_k), R)}{r}\right)^d.
\]
This completes the proof of (4.8).

Finally, we remark that the same arguments show that if \(\Phi \to 0\), then for each \(d \in (\dim \Phi C_\alpha, 1)\) there is some \(E = A \cup B \in C_\alpha\) with \(\dim \Phi A = \dim \Phi A\), so that also \(\dim \Phi A = d\). Further, \(\dim \Phi B = \dim \Phi C_\alpha\) and thus \(\dim \Phi E = d\) by the union
result, Proposition 2.15. Since we have $1 = \dim_{qA} D_a = \dim_{\Phi} D_a$, the proof is complete when $\Phi \to 0$.

Remark 4.8. In [15] we study the $\Phi$-dimensions of random rearrangements and show that typically their almost sure dimensional behaviour agrees with either that of the Cantor set or the decreasing set, depending on how $\Phi$ compares with $|\log |\log x||/\log x|$.

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