The Schur multiplier of an \( n \)-Lie superalgebra

Hesam Safa

Department of Mathematics, Faculty of Basic Sciences, University of Bojnord, Bojnord, Iran

ABSTRACT
In the present paper, we study the notion of the Schur multiplier \( M(L) \) of an \( n \)-Lie superalgebra \( L = L_0 \oplus L_1 \) and prove that \( \dim M(L) \leq \sum_{i=0}^{n} \binom{m}{i} n(n-i,k) \), where \( \dim L_0 = m \), \( \dim L_1 = k \), \( n(0,k) = 1 \) and \( n(t,k) = \sum_{j=1}^{t} \binom{-1}{j} \binom{t}{j} \), for \( 1 \leq t \leq n \). Moreover, we obtain an upper bound for the dimension of \( M(L) \) in which \( L \) is a nilpotent \( n \)-Lie superalgebra with one-dimensional derived superalgebra. It is also provided several inequalities on \( \dim M(L) \) as well as an \( n \)-Lie superalgebra analogue of the converse of Schur’s theorem.

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1. Introduction

Lie superalgebras first appeared in a joint work of F. A. Berezin and G. I. Kac (Kats) [4] in 1970, as algebras generated by even (commuting) and odd (anticommuting) variables, though it was not them who introduced the name. In 1971, Victor G. Kac [18] independently published his first results on the classification of Lie superalgebras after meeting and discussion with a physicist named Stavraki who worked on algebras of field operators and knew the graded Lie algebras [20]. Since then, many investigations have been made on Lie superalgebras such that this concept has become a topic of interest in algebra and mathematical physics in the recent decades. Moreover, the notion of \( n \)-Lie superalgebras (also known as Filippov superalgebras) was introduced by Daletskii and Kushnirevitch [9] in 1996, as a natural generalization of \( n \)-Lie algebras [13].

The concept of the Schur multiplier of a group originated from works of Schur [36] on projective representations in 1904. Nearly a century later, Stitzinger and his PhD students Batten and Moneyhun introduced the Schur multiplier of a Lie algebra in [3, 22]. Recently, Nayak [23, 24] has generalized this notion to the Lie superalgebras and discussed some results on the structure of the Schur multiplier \( M(L) \) of a Lie superalgebra \( L \). She also provided several upper bounds for the dimension of \( M(L) \).

In 1994, Moneyhun [22] proved that \( \dim M(L) \leq \frac{1}{2} m(m-1) \), in which \( L \) is an \( m \)-dimensional Lie algebra. This is in fact the Lie algebra version of Green’s result [15] which shows that \( |M(G)| \leq p^{\frac{1}{2} m(m-1)} \), for any finite \( p \)-group \( G \) of order \( p^m \). In the context of \( n \)-Lie algebras [10], this bound is \( \dim M(L) \leq \binom{m}{n} \). In [24], Nayak proved that if \( L = L_0 \oplus L_1 \) is a Lie superalgebra of dimension \( (m|k) \) (i.e. \( \dim L_0 = m \), \( \dim L_1 = k \)), then \( \dim M(L) \leq \frac{1}{2} \left( (m+k)^2 + (k-m) \right) \) (see also [21]). In this paper, it is shown that for a finite-dimensional \( n \)-Lie superalgebra \( L \),
\[
\dim \mathcal{M}(L) \leq \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n-i,k),
\]
where \(\dim L = (m|k)\), \(\mathcal{L}(0,k) = 1\) and
\[
\mathcal{L}(t,k) = \sum_{j=1}^{t} \binom{t-1}{j-1} \binom{k}{j},
\]
for \(1 \leq t \leq n\). This bound generalizes all above bounds (see Remark 3.2).

Further in [27], Niroomand and Russo proved that if \(L\) is an \(m\)-dimensional nilpotent Lie algebra with \(\dim L^2 = 1\), then \(\dim \mathcal{M}(L) \leq \frac{1}{2}(m-1)(m-2) + 1\). Moreover, Nayak [24] generalized this bound to Lie superalgebras and proved that if \(L\) is a nilpotent Lie superalgebra of dimension \((m|k)\) and \(\dim L^2 = (1|0)\), then
\[
\dim \mathcal{M}(L) \leq \frac{1}{2}(m+k-1)(m+k-2) + k + 1.
\]

Finally, Eshrat et al. [12] showed that if \(L\) is a nilpotent \(n\)-Lie algebra of dimension \(m\) with \(\dim L^2 = 1\), then \(\dim \mathcal{M}(L) \leq \binom{m-1}{n} + n - 1\). In the paper, we show that if \(L\) is a finite-dimensional nilpotent \(n\)-Lie superalgebra with \(\dim L^2 = (1|0)\), then
\[
\dim \mathcal{M}(L) \leq \sum_{i=0}^{n} \left[ \binom{p-1}{i} \mathcal{L}(n-i,q) + \binom{m-p+1}{i} \mathcal{L}(n-i,k-q) \right] + \sum_{i=1}^{n-1} \sum_{j=0}^{i} \left( \binom{p-1}{j} q^{i-j} \sum_{j=0}^{n-i} \binom{m-p}{j} (k-q)^{n-i-j} \right) - 1,
\]
where \(\dim L = (m|k)\), \(\dim Z(L) = (p|q)\) and \(\mathcal{L}(t,k)\) is the function defined above. This upper bound simultaneously generalizes the above three bounds (see Remark 3.7). We also discuss some inequalities on the dimension of \(\mathcal{M}(L)\) as well as a result on the converse of Schur’s theorem in \(n\)-Lie superalgebras.

2. Preliminaries

Throughout the paper, \(n \geq 2\) is a fixed integer and all (super)algebras are considered over a fixed field of characteristic zero. This section is devoted to list some preliminaries on \(n\)-Lie superalgebras from [19, 37].

Let \(\mathbb{Z}_2 = \{0,1\}\) be a field. A \(\mathbb{Z}_2\)-graded vector space (or superspace) \(V\) is a direct sum of vector spaces \(V_0\) and \(V_1\), whose elements are called even and odd, respectively. Non-zero elements of \(V_0 \cup V_1\) are said to be homogeneous. For a homogeneous element \(v \in V_a\) with \(a \in \mathbb{Z}_2\), \(|v| = a\) is the degree of \(v\). In the sequel, when the notation \(|v|\) appears, it means that \(v\) is a homogeneous element. A vector subspace \(U\) of \(V\) is called \(\mathbb{Z}_2\)-graded vector subspace (or sub-superspace), if \(U = U_0 \oplus U_1\) where \(U_0 = U \cap V_0\) and \(U_1 = U \cap V_1\).

**Definition 2.1.** A \(\mathbb{Z}_2\)-graded vector space \(L = L_0 \oplus L_1\) is said to be an \(n\)-Lie superalgebra, if there exists an \(n\)-linear map \([-,-,-,…,-] : L × ⋯ × L → L\) such that

(i) \([x_1,\ldots,x_n] = \sum_{i=1}^{n} |x_i| \) (modulo 2),

(ii) \([x_1,\ldots,x_i,x_{i+1},\ldots,x_n] = (-1)^{|x_i||x_{i+1}|}[x_1,\ldots,x_{i+1},x_i,\ldots,x_n]\) (graded antisymmetric property), and
(iii) the following graded Filippov-Jacobi identity holds:

\[
[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]] = [[x_1, \ldots, x_{n-1}, y_1], y_2, \ldots, y_n] + \sum_{i=2}^{n} (-1)^{i+1} \binom{m-1}{i} [x_1, \ldots, y_{i-1}, [x_{i+1}, \ldots, x_{n-1}, y_1], y_{i+1}, \ldots, y_n].
\]

Note that this is actually the definition of an \( n \)-Lie superalgebra of zero parity (see [37, Definition 2.1], for the general case). Clearly \( n \)-Lie algebras and Lie superalgebras are particular cases of \( n \)-Lie superalgebras. In fact, the even part of an \( n \)-Lie superalgebra \( L \) is an \( n \)-Lie algebra, which means that if \( L_1 = 0 \), then \( L \) becomes an \( n \)-Lie algebra. A sub-superspace \( I \) of an \( n \)-Lie superalgebra \( L \) is said to be a sub-superalgebra (resp. graded ideal), if \([I, I, \ldots, I] \subseteq I \) (resp. \([I, L, \ldots, L] \subseteq I \)). Also, the center and commutator (or derived superalgebra) of \( L \) are \( Z(L) = \{z \in L | [z, L, \ldots, L] = 0 \} \) and \( L^2 = \langle [L, \ldots, L] \rangle \), respectively, which are graded ideals of \( L \). An \( n \)-Lie superalgebra \( L \) is said to be nilpotent of class \( c \), if \( L^{c+1} = 0 \) and \( L^c \neq 0 \), where \( L^1 = L \) and \( L^{i+1} = [L^i, L, \ldots, L] \), \( i \geq 1 \). Let \( L \) and \( K \) be two \( n \)-Lie superalgebras. A multilinear map \( f : L \rightarrow K \) is called a homomorphism of \( n \)-Lie superalgebras, if \( f(L_a) \subseteq K_a \) for every \( a \in \mathbb{Z}_2 \), and \( f([x_1, \ldots, x_n]) = [f(x_1), \ldots, f(x_n)] \) for every \( x_i \in L \) (see [24] for more details).

Let \( 0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0 \) be a free presentation of an \( n \)-Lie superalgebra \( L \). We define the Schur multiplier of \( L \) as

\[
\mathcal{M}(L) = \frac{R \cap F^2}{[R, F, \ldots, F]},
\]

which is an abelian \( n \)-Lie superalgebra, independent of the choice of the free presentation of \( L \). Clearly if \( n = 2 \), then this definition coincides with the notion of the Schur multiplier of a Lie superalgebra given in [24]. An exact sequence \( 0 \rightarrow M \rightarrow K \rightarrow L \rightarrow 0 \) of \( n \)-Lie superalgebras is called a central extension (resp. stem extension) of \( L \), if \( M \subset Z(K) \) (resp. \( M \subset Z(K) \cap K^2 \)). A stem cover is a stem extension in which \( M \cong \mathcal{M}(L) \). In this case, \( K \) is said to be a cover of \( L \) (see [31] for more information). Finally, throughout the paper when an \( n \)-Lie superalgebra \( L = L_0 \oplus L_1 \) is of dimension \( m + k \), in which \( \dim L_0 = m \) and \( \dim L_1 = k \), we write \( \dim L = (m|k) \).

In what follows, we provide some results on \( n \)-Lie superalgebras whose proofs are similar to the corresponding results on Lie superalgebras given in [23, 24]. Hence we omit the proofs (see also [6, 7] for the Leibniz case).

**Proposition 2.2.** Any finite-dimensional \( n \)-Lie superalgebra has at least one cover.

**Proposition 2.3.** Let \( 0 \rightarrow M \rightarrow K \rightarrow L \rightarrow 0 \) be a stem extension of a finite-dimensional \( n \)-Lie superalgebra \( L \). Then \( K \) is a homomorphic image of a cover of \( L \).

**Lemma 2.4.** Let \( L \) be a (finite-dimensional) \( n \)-Lie superalgebra and \( N \) be a graded ideal of \( L \). Then there exists a (finite-dimensional) \( n \)-Lie superalgebra \( K \) with a graded ideal \( M \) such that

(i) \( L^2 \cap N \cong K/M \),

(ii) \( M \cong \mathcal{M}(L) \),

(iii) \( \mathcal{M}(L/N) \) is a homomorphic image of \( K \),

(iv) if \( N \subset Z(L) \), then \( L^2 \cap N \) is a homomorphic image of \( \mathcal{M}(L/N) \).

**Proof.** Straightforward. See also [24, Lemma 3.5] for the usual case.

**Proposition 2.5.** Let \( 0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0 \) be a free presentation of an \( n \)-Lie superalgebra \( L \). Also, let \( N \) be a graded ideal of \( L \) and \( S \) a graded ideal of \( F \) such that \( N \cong S/R \). Then the following sequences are exact:
(i) \[ 0 \to \frac{\mathcal{M}_0}{[R,F,\ldots,F]} \to \mathcal{M}(L) \to \mathcal{M}(L/N) \to \frac{N\cap L^2}{[N,L,\ldots,L]} \to 0, \]
(ii) \[ \mathcal{M}(L) \to \mathcal{M}(L/N) \to \frac{N\cap L^2}{[N,L,\ldots,L]} \to \frac{N}{[N,L,\ldots,L]} \to \frac{L^2}{N\cap L^2} \to 0, \]
(iii) \[ N^{\otimes n-1} \frac{L}{L^2} \to \mathcal{M}(L) \to \mathcal{M}(L/N) \to N\cap L^2 \to 0, \] provided that \( N \) is a central graded ideal of \( L \).

**Proof.** Lie version of these exact sequences are well-known in the literature (see [6, 35]). The proof is a simple generalization of them. \( \square \)

The following corollary is an immediate consequence of the above proposition (see also [32, Lemma 2.1]).

**Corollary 2.6.** Under the notation of Proposition 2.5, if \( L \) is a finite-dimensional \( n \)-Lie superalgebra, then

(i) \( \mathcal{M}(L) \) is finite-dimensional,
(ii) \( \dim \mathcal{M}(L/N) \leq \dim \mathcal{M}(L) + \dim \frac{N\cap L^2}{[N,L,\ldots,L]} \),
(iii) \( \dim \mathcal{M}(L) + \dim (N \cap L^2) = \dim \mathcal{M}(L/N) + \dim [N,L,\ldots,L] + \dim \frac{R|K,F,\ldots,F]}{[R,F,\ldots,F]} \),
(iv) \( \dim \mathcal{M}(L) + \dim (N \cap L^2) = \dim \mathcal{M}(L/N) + \dim \frac{S,F,\ldots,F}{[K,F,\ldots,F]} \),
(v) \( \dim \mathcal{M}(L) + \dim (L^2) = \dim \frac{F^2}{[K,F,\ldots,F]} \),
(vi) if \( \mathcal{M}(L) = 0 \), then \( \mathcal{M}(L/N) \cong \frac{N\cap L^2}{[N,L,\ldots,L]} \),
(vii) \( \dim \mathcal{M}(L) + \dim (N \cap L^2) \leq \dim \mathcal{M}(L/N) + \dim (N^{\otimes n-1} \frac{L}{L^2}) \), provided that \( N \) is a central graded ideal of \( L \).

3. Upper bounds on the dimension of \( \mathcal{M}(L) \)

A well-known result of Schur [36] states that if the central factor of a group \( G \) is finite, then so is \( G' \), where \( G' \) is the commutator subgroup of \( G \). Over half a century after Schur, Wiegold [38] proved that if \( \lvert G/Z(G) \rvert = p^m \), then \( G' \) is a \( p \)-group of order at most \( p^{2m(m-1)} \). Also in Lie algebra, Moneyhun [22] showed that if \( L \) is a Lie algebra with \( \dim L/Z(L) = m \), then \( \dim L^2 \leq \frac{1}{2} m(m-1) \). Her argument is very simple. If \( \{ x_1, \ldots, x_m \} \) is a basis for \( L/Z(L) \), then \( L^2 \) can be trivially generated by \( \{ [x_i,x_j] \mid 1 \leq i < j \leq m \} \). Therefore, \( \dim L^2 \leq \binom{m}{2} \). Moreover, if \( L \) is an \( n \)-Lie algebra such that \( \dim L/Z(L) = m \), then one may similarly show that \( \dim L^2 \) is at most \( \binom{m}{n} \). Nayak [24] has recently proved that if \( L \) is a Lie superalgebra with \( \dim (L/Z(L)) = (m|k) \), then \( \dim L^2 \leq \frac{k}{2} ((m+k)^2 + (k-m)) \).

As the first result, we provide an upper bound for the dimension of the commutator of an \( n \)-Lie superalgebra with finite-dimensional central factor which generalizes all above bounds.

**Theorem 3.1.** Let \( L \) be an \( n \)-Lie superalgebra such that \( \dim (L/Z(L)) = (m|k) \). Then

\[ \dim L^2 \leq \sum_{i=0}^{n} \binom{m}{i} L(n-i,k), \]

where \( L(0,k) = 1 \) and \( L(t,k) = \sum_{i=1}^{t} \binom{i-1}{j-1} \binom{k}{j} \), for \( 1 \leq t \leq n \).

**Proof.** Let \( L = L_0 \oplus L_1 \) and \( \{ x_1, \ldots, x_m, y_1, \ldots, y_k \} \) be a basis for \( L/Z(L) \) where \( x_1, \ldots, x_m \in L_0 \) and \( y_1, \ldots, y_k \in L_1 \). The graded antisymmetric property of \( n \)-Lie superalgebras implies that the following set generates \( L^2 \):

\[ B = \{ [x_i, y_j, \ldots, y_j] \mid 0 \leq s, t \leq n, s + t = n, 1 \leq i_1 < \cdots < i_s \leq m, 1 \leq r_1 \leq \cdots \leq r_t \leq k \}. \]
First, let $t = 0$. Then $s = n$ and the number of commutators $[x_1, \ldots, x_n]$ in $B$ is clearly $\binom{m}{n}$. Now, let $t = n$ and $L_j$ be the number of commutators $[y_1, \ldots, y_n]$ in $B$ such that the set $\{y_1, \ldots, y_n\}$ contains exactly $j$ distinct elements $(1 \leq j \leq n)$. Therefore, $L_1 = k = \binom{n - 1}{j - 1}$, since we have these $k$ commutators:

$$[y_1, \ldots, y_1, \ldots, y_k]$$

Also $L_2 = \binom{n - 1}{1} \binom{k}{2}$ which is the number of commutators of the form:

$$[y_{r_1}, y_{r_2}, \ldots, y_{r_2}], \ [y_{r_1}, y_{r_2}, y_{r_3}, \ldots, y_{r_3}], \ [y_{r_1}, \ldots, y_{r_2}, y_{r_3}],$$

where $1 \leq r_1 < r_2 < k$. Using a similar combinatorial argument, one may check that $L_j = \binom{n - 1}{j - 1} \binom{k}{j}$. Now put $L_n(k) = L_1 + L_2 + \cdots + L_n = \sum_{j=1}^n \binom{n - 1}{j - 1} \binom{k}{j}$. Thus in case $t = n$, the number of commutators $[y_1, \ldots, y_n]$ in $B$ is $L(n, k)$. Applying a similar technique for all $1 \leq t \leq n$ and adding the case $t = 0$, we get

$$|B| = \binom{m}{0} L(n, k) + \binom{m}{1} L(n - 1, k) + \cdots + \binom{m}{n} L(1, k) + \binom{m}{n},$$

which completes the proof. \hfill \Box

**Remark 3.2.** Clearly if $L$ is a Lie algebra, i.e. $n = 2$ and $k = 0$, then since $L(2, 0) = L(1, 0) = 0$ and $L(0, 0) = 1$, our bound is

$$\dim L^2 \leq \sum_{i=0}^2 \binom{m}{i} L(2 - i, 0) = \binom{m}{2} L(0, 0) = \frac{1}{2} m(m - 1),$$

which is the Moneyhun’s bound [22]. Also if $L$ is an $n$-Lie algebra, i.e. $n \geq 2$ and $k = 0$, then our bound will be $\binom{m}{n}$ given in [10]. Finally if $L$ is a Lie superalgebra, i.e. $n = 2$ and $k \geq 1$, then our bound will be

$$\dim L^2 \leq \sum_{i=0}^2 \binom{m}{i} L(2 - i, k) = \binom{m}{0} L(2, k) + \binom{m}{1} L(1, k) + \binom{m}{2} L(0, k) = \sum_{j=1}^2 \binom{1}{j - 1} \binom{k}{j} + m \binom{k}{1} + \binom{m}{2} = k + \binom{k}{2} + mk + \binom{m}{2} = \frac{1}{2} ((m + k)^2 + (k - m)),$$

which is the Nayak’s bound [24].
Theorem 3.3. Let $L$ be an $n$-Lie superalgebra of dimension $(m|k)$. Then

$$\dim \mathcal{M}(L) \leq \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n-i,k),$$

where $\mathcal{L}(t,k)$ is the function defined in Theorem 3.1. In particular, the equality occurs if and only if $L$ is an abelian $n$-Lie superalgebra.

Proof. Regarding Proposition 2.2, let $0 \rightarrow M \rightarrow K \pi L \rightarrow 0$ be a stem cover of $L$, i.e. $M \subseteq Z(K) \cap K^2$ and $M \cong \mathcal{M}(L)$. Since $\dim(K/Z(K)) \leq \dim(K/M) = \dim L = (m|k)$, we get

$$\dim \mathcal{M}(L) = \dim M \leq \dim K^2 \leq \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n-i,k).$$

Now if the equality holds, then the above inequality implies that $M = K^2$ and hence $L = K/M$ is abelian.

Conversely, assume that $L$ is abelian, $0 \rightarrow M \rightarrow K \pi L \rightarrow 0$ is a stem extension of $L$, and $\{\bar{x}_1, \ldots, \bar{x}_m, \bar{y}_1, \ldots, \bar{y}_k\}$ is a basis for $L = L_0 \oplus L_1$, where $\pi(x_i) = \bar{x}_i \in L_0$ for $1 \leq i \leq m$, and $\pi(y_j) = \bar{y}_j \in L_1$ for $1 \leq j \leq k$. Then $x_i \in K_0$ and $y_j \in K_1$. Clearly $M = K^2 \subseteq Z(K)$ and also $B$ (given in Theorem 3.1) is contained in $\ker \pi = M$, since $L$ is abelian. Moreover $K$, as a vector superspace, can be generated by the set $\mathcal{N} \cup M$ where $\mathcal{N}$ is the sub-superspace of $K$ generated by the set $\{x_1, \ldots, x_m, y_1, \ldots, y_k\}$. Clearly $\mathcal{N} \cap M = 0$ and hence as vector superspaces, we have $K = \mathcal{N} \oplus M$ which will have the structure of an $n$-Lie superalgebra, since $M$ is central. Furthermore as $M = K^2 \subseteq Z(K)$, $B$ is actually a basis for $M$. Thus

$$\dim M = |B| = \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n-i,k).$$

On the other hand, $0 \rightarrow K^2 \rightarrow K \pi L \rightarrow 0$ is a stem extension of $L$ in which $K$ is of maximal dimension. Hence by Proposition 2.3, $K$ is a cover of $L$ and so $M \cong \mathcal{M}(L)$. This completes the proof. \(\square\)

Let $0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0$ be a free presentation of a Lie algebra $L$ and $F$ be the free Lie algebra on $m$ generators. The Witt’s Formula shows that

$$\dim F^d/F^{d+1} = \frac{1}{d} \sum_{r|d} \mu(r) m^{d/r} \equiv l_m(d),$$

(3.1)

where $\mu$ is the Möbius function, defined by $\mu(1) = 1$, $\mu(r) = 0$ if $r$ is divisible by a square, and $\mu(p_1 \ldots p_t) = (-1)^t$ if $p_1, \ldots, p_t$ are distinct prime numbers (see [28, 35]). Also in [8, Theorem 4.1], it is proved that if $L$ is a nilpotent Lie algebra of nilpotency class $c$ which is generated by $m$ elements, then

$$\dim \mathcal{M}(L) \leq \sum_{j=1}^{c} l_m(j+1) = \sum_{j=1}^{c} \left( \frac{1}{j} \sum_{i|j+1} \mu(i) m^{j+1} \right).$$

In [8, Examples 4.3, 4.4], it is provided explicit values for $\mu(i)$ in order to evaluate numerically the above upper bound on $\dim \mathcal{M}(L)$ and then to compare with $\dim \mathcal{M}(L) \leq \frac{1}{2} m(m-1)$. It is in fact hard to describe the behavior of the Möbius function from a general point of view and so (3.1) is not very helpful when we do not evaluate the coefficients $\mu(i)$ (see [28] for more details). Note that our upper bound in Theorem 3.3 is actually representing a similar bound as those
which are known for the Witt’s Formula, replacing the Möbius function with the combinatorial function $\mathcal{L}(t, k)$ which is very simple to work with.

**Corollary 3.4.** Let $L$ be an $n$-Lie superalgebra of dimension $(m|k)$. Then

$$\dim \mathcal{M}(L) \leq \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n - i, k) - \dim L^2.$$ 

**Proof.** Let $0 \to R \to F \to L \to 0$ be a free presentation of $L$. Clearly, dimension of the central factor of $F/[R, F, ..., F]$ is less than or equal to $\dim(F/R) = (m|k)$. Hence by Theorem 3.1, $\dim(F^2/[R, F, ..., F]) \leq \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n - i, k)$. Now using Corollary 2.6 (v), we have $\dim \mathcal{M}(L) + \dim L^2 = \dim(F^2/[R, F, ..., F])$, which completes the proof. \hfill \Box

A well-known problem in Lie algebra is the characterization of Lie algebras using $t(L) = \frac{1}{2}m(m - 1) - \dim \mathcal{M}(L)$ where $\dim L = m$. The characterization of nilpotent Lie algebras for $0 \leq t(L) \leq 8$ has been studied in [3, 16, 17]. Similarly, Green’s result [15] yielded a lot of interest on the classification of finite p-groups by $t(G)$, investigated by several authors (see [5, 11] for instance). Now, let $L$ be an $n$-Lie superalgebra. A natural question arises whether one could characterize $(m|k)$-dimensional $n$-Lie superalgebras by $t(n, L)$, where $t(n, L) = \sum_{i=0}^{n} \binom{m}{i} \mathcal{L}(n - i, k) - \dim \mathcal{M}(L)$. In [21] this question has been answered for $t(2, L) \leq 2$ (see also [25, 29]), and in [33] for the Schur multiplier of a pair of Lie superalgebras.

The following lemma is needed to prove the next main theorem.

**Lemma 3.5.** Let $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ be two finite-dimensional $n$-Lie superalgebras. Then

$$\dim \mathcal{M}(A \oplus B) = \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{i=1}^{n-1} \left[ \sum_{j=0}^{i} \binom{s_j}{j} \frac{t_0}{t_1}^{i-j} \right],$$

where $s_a = \dim(A_a/A^2 \cap A_a)$ and $t_a = \dim(B_a/B^2 \cap B_a)$, for $a \in \mathbb{Z}_2$.

**Proof.** Let $0 \to N \to H \to A \oplus B \to 0$ be a stem cover of $A \oplus B$. Then $\frac{H}{N} \cong A \oplus B$ and $\mathcal{M}(A \oplus B) \cong N \subseteq Z(H) \cap H^2$. Consider graded ideals $X$ and $Y$ of $H$ such that $\frac{X}{N} \cong A$ and $\frac{Y}{N} \cong B$. Then since $H = X + Y$, we have

$$H^2 = X^2 + Y^2 + \sum_{i=1}^{n-1} \left[ \frac{X, ..., X, Y, ..., Y}{(i \text{ times})} \right].$$

One can easily see that $N = (N \cap X^2) + (N \cap Y^2) + \sum_{i=1}^{n-1} [X, ..., X, Y, ..., Y]$, and hence by Lemma 2.4 (iv) we get

$$\dim \mathcal{M}(A \oplus B) = \dim N \leq \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{i=1}^{n-1} \dim [X, ..., X, Y, ..., Y].$$

Now, define

$$f: \frac{A}{A^2} \times \cdots \times \frac{A}{A^2} \times \frac{B}{B^2} \times \cdots \times \frac{B}{B^2} \to [X, ..., X, Y, ..., Y]$$

$$(\bar{a}_1, \ldots, \bar{a}_i, \bar{b}_1, \ldots, \bar{b}_{n-i}) \mapsto [x_{a_1}, \ldots, x_{a_i}, y_{b_1}, \ldots, y_{b_{n-i}}],$$

where $x_{a_i} + N \mapsto a_i$ in $\frac{X}{N} \cong A$, and $y_{b_i} + N \mapsto b_i$ in $\frac{Y}{N} \cong B$. One may check that $f$ is a surjective multilinear map. Clearly
\[ \frac{A}{A^2} = \frac{A_0}{A^2 \cap A_0} \oplus \frac{A_1}{A^2 \cap A_1} \]

and we have a similar equality for \( \frac{\mathfrak{g}}{\mathfrak{g}^2} \). Therefore

\[
\dim \left[ x, \ldots, x, y, \ldots, y \right] = \dim(\text{Im} \mathfrak{m}) \leq \sum_{j=0}^{n-1} \left( \sum_{i=0}^{j} \binom{s_0}{i} s_1^{i-j} \sum_{j=0}^{n-i} \binom{t_0}{j} t_1^{n-i-j} \right).
\]

Hence

\[
\dim \mathcal{M}(A \oplus B) \leq \dim \mathcal{M}(A) + \dim \mathcal{M}(B) + \sum_{i=1}^{n-1} \left[ \sum_{j=0}^{i} \binom{s_0}{j} s_1^{i-j} \sum_{j=0}^{n-i} \binom{t_0}{j} t_1^{n-i-j} \right].
\]

For the next theorem, we only need the above inequality, so we left the rest of proof which is an \( n \)-Lie superalgebra analogue of [3, Theorem 1] and [10, Theorem 3.6].

It is easy to show that if \( L \) is a Lie algebra, i.e. \( n = 2 \) and \( s_1 = t_1 = 0 \), then the above sigma is equal to \( \dim(A/A^2) \dim(B/B^2) \) given in [3], and if \( L \) is an \( n \)-Lie algebra, i.e. \( s_1 = t_1 = 0 \), then it will be

\[
\left( \dim(A/A^2) + \dim(B/B^2) \right) - \left( \dim(A/A^2) \right) - \left( \dim(B/B^2) \right),
\]
discussed in [10].

In Theorem 3.3, we determined the dimension of \( \mathcal{M}(L) \) when \( L \) is an abelian \( n \)-Lie superalgebra. In the next result, using Theorem 3.3 and Lemma 3.5 we give an upper bound for the dimension of \( \mathcal{M}(L) \) when \( L \) is (non-abelian) nilpotent with \( (1|0) \)-dimensional derived superalgebra.

**Theorem 3.6.** Let \( L \) be a nilpotent \( n \)-Lie superalgebra such that \( \dim L = (m|k) \), \( \dim Z(L) = (p|q) \) and \( \dim L^2 = (1|0) \). Then

\[
\dim \mathcal{M}(L) \leq \sum_{i=0}^{n} \left[ \binom{p-1}{i} \binom{m-p+1}{i} \binom{n-i}{q} \binom{n-i}{k-q} \right] + \sum_{i=1}^{n-1} \left[ \sum_{j=0}^{i} \binom{p}{j} q^{j} \sum_{j=0}^{n-i} \binom{m-p}{j} (k-q)^{n-i-j} \right] - 1,
\]

**Proof.** Since \( L \) is nilpotent with \( \dim L^2 = 1 \), we have \( L^3 \nsubseteq L^2 \) and so \( L^3 = 0 \). Hence \( L^2 \subseteq Z(L) \). So one may choose a complement \( A \) of \( L^2 \) in \( Z(L) \) and a complement \( B/L^2 \) of \( Z(L)/L^2 \) in \( L/L^2 \).

It is easy to see that \( L = B + Z(L) \) and \( L^2 = B^2 \). On the other hand if \( x \in L \), then \( x = b + a + l \), for some \( b \in B, a \in A \) and \( l \in L^2 \). Hence \( x \in A + B \). Moreover if \( x \in A \cap B \), then \( x + L^2 \in (Z(L)/L^2) \cap (B/L^2) = L^2 \) and thus \( x \in A \cap L^2 = 0 \). So \( A \cap B = 0 \) and \( L \cong A \oplus B \). Also since \( A \) is abelian we have \( \dim A/A^2 = \dim A = \dim Z(L) = (1|0) = (p-1|q) \), and since \( B^2 = L^2 \) we have \( \dim B/B^2 = \dim B/L^2 = \dim L - \dim Z(L) = (m-p|k-q) \) and \( \dim B = (m-p+1|k-q) \). Now one can use Theorem 3.3 and Lemma 3.5. Note that the above \((-1)\) in the theorem appears because of this fact that \( B \) is not abelian and hence \( \mathcal{M}(B) \) cannot reach the maximum dimension in Theorem 3.3. This completes the proof. \( \square \)

**Remark 3.7.** Here we show that the above bound extends all previous bounds in Lie algebras, Lie superalgebras and \( n \)-Lie algebras.
In [27], Niroomand and Russo proved that if $L$ is an $m$-dimensional nilpotent Lie algebra with $\dim L^2 = 1$, then
\[
\dim \mathcal{M}(L) \leq \frac{1}{2} (m - 1)(m - 2) + 1,
\] (3.2)
and the equality holds if and only if $L \cong H(1) \oplus A(3)$, where $H(1)$ denotes the Heisenberg Lie algebra of dimension 3 and $A(3)$ is the abelian Lie algebra of dimension $m - 3$. Now in Theorem 3.6, if $L$ is a Lie algebra (i.e. $n = 2$ and $k = q = 0$), then our bound will be
\[
\dim \mathcal{M}(L) \leq \sum_{i=0}^{2} \left[ \binom{p - 1}{i} \mathcal{L}(2 - i, 0) + \binom{m - p + 1}{i} \mathcal{L}(2 - i, 0) \right] + \left( \frac{p - 1}{2} \right) \left( \frac{m - p}{2} \right) - 1
\]
\[
= \left( \frac{p - 1}{2} \right) \mathcal{L}(0, 0) + \left( \frac{m - p + 1}{2} \right) \mathcal{L}(0, 0) + \left( \frac{p - 1}{1} \right) \left( \frac{m - p}{1} \right) - 1
\]
\[
= \frac{1}{2} (p - 1)(p - 2) + \frac{1}{2} (m - p + 1)(m - p) + (p - 1)(m - p) - 1
\]
\[
= \frac{1}{2} (m - 1)(m - 2) + (m - p) - 1.
\]
Note that if $L \cong H(1) \oplus A(n - 3)$, then $m - p = \dim L - \dim Z(L) = 2$ which yields bound (3.2).

Also, Nayak [24] recently generalized bound (3.2) for Lie superalgebras. She proved that if $L$ is a nilpotent Lie superalgebra of dimension $(m|k)$ and $\dim L^2 = (1|0)$, then
\[
\dim \mathcal{M}(L) \leq \frac{1}{2} (m + k - 1)(m + k - 2) + k + 1,
\]
and the equality occurs if and only if $L \cong H(1, 0) \oplus A(m - 3, k)$, where $H(1, 0)$ is the special Heisenberg Lie superalgebra of dimension $(3|0)$ and $A(m - 3, k)$ is the abelian Lie superalgebra of dimension $m + k - 3$. In Theorem 3.6 if $L$ is a Lie superalgebra (i.e. $n = 2$), then after a similar computation our bound turns into
\[
\dim \mathcal{M}(L) \leq \frac{1}{2} (m + k - 1)(m + k - 2) + k + (m + k - p - q) - 1,
\]
and in case $L \cong H(1, 0) \oplus A(m - 3, k)$ we have again $m + k - p - q = \dim L - \dim Z(L) = 2$ which gives the Nayak’s bound.

Finally, Eshrati et al. [12] showed that if $L$ is a nilpotent $n$-Lie algebra of dimension $m$ with $\dim L^2 = 1$, then
\[
\dim \mathcal{M}(L) \leq \binom{m - 1}{n} + n - 1,
\] (3.3)
and the equality holds if and only if $L \cong H(n, 1) \oplus A(m - n - 1)$, where $H(n, 1)$ is the special Heisenberg $n$-Lie algebra of dimension $n + 1$ and $A(m - n - 1)$ is the abelian $n$-Lie algebra of dimension of $m - n - 1$. Similar to the above cases, in Theorem 3.6 if $L$ is an $n$-Lie algebra (i.e. $k = q = 0$), then one may check that our bound will be
\[
\dim \mathcal{M}(L) \leq \binom{m - 1}{n} + \binom{m - p}{n - 1} - 1,
\]
and if $L \cong H(n, 1) \oplus A(m - n - 1)$, then $m - p = \dim L - \dim Z(L) = n$ which yields bound (3.3).

Note that the above bounds obtained in [12, 24, 27] are also valid when $\dim L^2 = r \geq 1$. In fact, their general cases are decreasing functions of $\dim L^2$. 
Problem 3.8. As it is discussed in the above remark, in cases of Lie algebras, Lie superalgebras and $n$-Lie algebras, using the notion of Heisenberg (super)algebras, one could reach the maximum dimension of $\mathcal{M}(L)$ when $\dim L^2 = 1$. R. Bai and Meng [2] introduced methods to construct Heisenberg $n$-Lie algebras which are actually one-dimensional central extensions of abelian $n$-Lie algebras. Moreover, W. Bai and Liu [1] studied the cohomology of Heisenberg Lie superalgebras. A Heisenberg superalgebra can be considered as a subalgebra of Lie superalgebras of (odd) contact vector fields, which is precisely the negative part with respect to a certain natural grading.

Now, how should we define the Heisenberg $n$-Lie superalgebra (specially when $n$ is odd and considering Definition 2.1 (i)) to reach the maximum dimension in Corollary 3.4 (with $\dim L^2 = 1$) and then in Theorem 3.6 (together with an abelian $n$-Lie superalgebra)?

In what follows, we provide some inequalities on the dimension of the Schur multiplier of a nilpotent $n$-Lie superalgebra (see [34] for Lie algebra case).

Proposition 3.9. Let $L$ be a finite-dimensional nilpotent $n$-Lie superalgebra of class $c$. Then for every $i \geq 2$

$$\dim \mathcal{M}(L) \leq \dim \mathcal{M}(L/L^i) + (\dim L^c) \left( \sum_{j=0}^{n-1} \binom{s}{j} t^{n-1-j} - 1 \right),$$

where $\dim (L/L^2) = (s|t)$.

Proof. Clearly if either $c = 1$ or $i > c$, then the result follows. So assume that $2 \leq i \leq c$ and use induction on $\dim L$. If $\dim L = (1|0)$, then $L$ is abelian. Let $\dim L = (0|1)$ and $\{x\} \subseteq L_1$ be a basis for $L = L_0 \oplus L_1$. Then if $n$ is even, by Definition 2.1 (i) we have $[x, ..., x] = 0$ and hence $L$ is abelian. If $n$ is odd and $[x, ..., x] = x$, then since $n - 1$ is even and $|x_1| + \cdots + |x_{n-1}| = (n - 1)|x| = 0$ modulo 2, the graded Filippov-Jacobi identity implies that $x = nx$ and so $x = 0$ which is impossible. Hence in this case $[x, ..., x] = 0$ and $L$ is abelian as well, and the inequality holds. Now suppose that $\dim L > 1$ and the result holds for any $n$-Lie superalgebra of dimension less than $\dim L$. Choose a one-dimensional graded ideal $N$ of $L$ contained in $L^c \subseteq Z(L)$. By Corollary 2.6 (vii) and the technique applied in Lemma 3.5, we get

$$\dim \mathcal{M}(L) \leq \dim \mathcal{M}(L/N) + \sum_{j=0}^{n-1} \binom{s}{j} t^{n-1-j} - 1.$$

Put for convenience $\lambda = \sum_{j=0}^{n-1} \binom{s}{j} t^{n-1-j} - 1$. Since $\dim \frac{L/N}{(L/N)^t} = \dim L/L^2$, using induction hypothesis we have

$$\dim \mathcal{M}(L) \leq \dim \mathcal{M}\left(\frac{L/N}{L'/N}\right) + \dim (L'/N) \lambda + \lambda$$

$$= \dim \mathcal{M}(L/L^1) + (\dim L^c - 1) \lambda + \lambda$$

$$= \dim \mathcal{M}(L/L^1) + (\dim L^c) \lambda. \quad \square$$

Proposition 3.10. Let $L$ be a finite-dimensional nilpotent $n$-Lie superalgebra of class $c \geq 2$. Then for every $2 \leq i \leq c$

$$\dim \mathcal{M}(L) + \dim \mathcal{M}(L/L^i) \leq (\dim L - 1) \sum_{j=0}^{n-1} \binom{s}{j} t^{n-1-j},$$

where $\dim (L/L^2) = (s|t)$. 

Proof. Note that since \( c \geq 2 \), \( L \) is not abelian and so \( \dim L > 1 \). Choose a one-dimensional graded ideal \( N \) of \( L \) contained in \( L^i \cap Z(L) \). Put \( \mu = \sum_{j=0}^{n-1} \binom{s}{j} t^{n-1} \). By the above proposition and induction on \( \dim L \), we get

\[
\dim \mathcal{M}(L) \leq \dim \mathcal{M}(L/N) + \mu \\
\leq (\dim(L/N) - 1)\mu - \dim \mathcal{M}(L/L^i) + \mu \\
= (\dim L - 2)\mu - \dim \mathcal{M}(L/L^i) + \mu \\
= (\dim L - 1)\mu - \dim \mathcal{M}(L/L^i).
\]

Next, we give a result on the dimension of the Schur multiplier of an \( n \)-Lie algebra, i.e. \( \dim L = (m|0). \)

**Definition 3.11.** ([14]). An \( m \)-dimensional \( n \)-Lie algebra \( L \) (\( m \geq n \)) is called filiform if \( \dim L^i = m - n - i + 2 \), \( i \geq 2 \).

Clearly a filiform \( n \)-Lie algebra is nilpotent of class \( m - n + 1 \). Since this nilpotency class is maximal, such \( n \)-Lie algebras are also called nilpotent of maximal class.

**Example 3.12.** Consider the \( n \)-Lie algebra \( L \) with a basis \( \{x_1, \ldots, x_m\} \) (\( m > n \)) and non-zero multiplications \([x_1, \ldots, x_{n-1}, x_i] = x_{i+1}, \) for \( n i \leq m - 1 \). Now, since

\[
[x_{1}, \ldots, x_{n-1}, x_{n}, x_{1}, \ldots, x_{n-1}, x_{n}] = 0
\]

we have \( L^{m-n+2} = 0 \) and \( L^{m-n+1} \neq 0 \). Therefore \( L \) is a filiform \( n \)-Lie algebra.

**Theorem 3.13.** Let \( L \) be a finite-dimensional filiform \( n \)-Lie algebra. Then

\[
\dim \mathcal{M}(L) = \dim \mathcal{M}(L/Z(L)) + t,
\]

where \(-1 \leq t \leq n - 1\).

**Proof.** Put \( \dim L = m \). If \( m = n \), then the result follows. Suppose that \( m > n \), then \( L \) is not abelian. Since \( L \) is filiform, we have \( L^{m-n+1} \subseteq Z(L) \) and \( \dim L^{m-n+1} = 1 \). Let \( 0 \neq x = [y, l_1, \ldots, l_{n-1}] \in \dim L^{m-n+1} \), for some \( y \in L^{m-n} \) and \( l_i \in L \). Then \( y \not\in Z(L) \) and hence \( L^{m-n+1} \subseteq Z(L) \subseteq L^{m-n} \).

Therefore, \( Z(L) = L^{m-n+1} \) as \( L \) is filiform, and so \( \dim(Z(L) \cap L^2) = 1 \). Also \( \dim(L/L^2) = n \). Now, by Corollary 2.6 (iv) we have

\[
\dim \mathcal{M}(L) = \dim \mathcal{M}(L/Z(L)) + \dim \frac{[S, F, \ldots, F]}{[R, F, \ldots, F]} - 1,
\]

where \( 0 \rightarrow R \rightarrow F \rightarrow L \rightarrow 0 \) is a free presentation of \( L \) and \( Z(L) \cong S/R \). Similar to Lemma 3.5, \( \phi : Z(L) \times \frac{L^i}{F} \times \cdots \times \frac{L^i}{F} \rightarrow \frac{[S, F, \ldots, F]}{[R, F, \ldots, F]} \) given by \( \phi(z, l_1, \ldots, l_{n-1}) = [s, f_1, \ldots, f_{n-1}] + [R, F, \ldots, F] \) where \( \pi(s) = z \) and \( \pi(f_i) = l_i \), is a well-defined epimorphism. Then \( \dim(Z(L)) = 1 \) and \( \dim(L/L^2) = n \) imply that \( \dim(\text{Im } \phi) \leq \binom{n}{n-1} = n \). Therefore,
dim \( M(L) \leq \dim M(L/Z(L)) + n - 1 \).

On the other hand, by Corollary 2.6 (ii)

\[
\dim M(L/Z(L)) \leq \dim M(L) + 1.
\]

Combining two inequalities gives the result. \( \square \)

Finally, we discuss a result concerning the converse of Schur’s theorem in the context of \( n \)-Lie superalgebras. In [30], it is shown that if \( L \) is an \( n \)-Lie algebra with finite-dimensional derived algebra and finitely generated central factor, then \( \dim(L/Z(L)) \leq \binom{k}{n-1} \dim L^2 \), where \( k = d(L/Z(L)) \) is the minimal number of generators of \( L/Z(L) \). This is actually an \( n \)-Lie algebra analogue of the group case given in [26].

In the following theorem, we prove it for \( n \)-Lie superalgebras. Recall from [37] that for \( x_1, ..., x_{n-1} \in L \), the map \( \text{ad}(x_1, ..., x_{n-1}) : L \to L \) given by \( \text{ad}(x_1, ..., x_{n-1})(x) = [x_1, ..., x_{n-1}, x] \) for all \( x \in L \), is a derivation which is called an inner derivation of the \( n \)-Lie superalgebra \( L \).

**Theorem 3.14.** Let \( L = L_0 \oplus L_1 \) be an \( n \)-Lie superalgebra whose derived superalgebra is finite-dimensional and \( L/Z(L) \) is finitely generated. Also, let

\[
\{ x_1 + Z(L), ..., x_s + Z(L), y_1 + Z(L), ..., y_t + Z(L) \}
\]

be a generating set of \( L/Z(L) \) such that \( x_i \in L_0 \) for \( 1 \leq i \leq s \) and \( y_j \in L_1 \) for \( 1 \leq j \leq t \). Then

\[
\dim \left( \frac{L}{Z(L)} \right) \leq \sum_{i=0}^{n-1} \binom{s}{i} t^{n-1-i} \dim L^2.
\]

**Proof.** Let \( A = \{ x_1, ..., x_s \} \) and \( B = \{ y_1, ..., y_t \} \). Put \( \text{ad}_{u, v_i} = \text{ad}(u_1, ..., u_i, v_1, ..., v_{n-1-i}) \) such that \( u_1, ..., u_i \) are \( i \) distinct elements of \( A \), and \( v_1, ..., v_{n-1-i} \) are \( n-1-i \) (not necessarily distinct) elements of \( B \). Now, define

\[
\psi : \frac{L}{Z(L)} \to L^2 \oplus \cdots \oplus L^2
\]

\[
x + Z(L) \mapsto (..., \text{ad}_{u, v_i}(x), ...),
\]

whose codomain is including \( \sum_{i=0}^{n-1} \binom{s}{i} t^{n-1-i} \) copies of \( L^2 \). Clearly \( \psi \) is an injective linear map and this completes the proof. \( \square \)

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**ORCID**

Hesam Safa http://orcid.org/0000-0002-5418-8104

**References**

[1] Bai, W., Liu, W. (2017). Cohomology of Heisenberg Lie superalgebras. *J. Math. Phys.* 58(2):021701. DOI: 10.1063/1.4975606.

[2] Bai, R., Meng, D. (2006). The central extension of \( n \)-Lie algebras. *Chin. Ann. Math.* 27(4):491–502.
Safa, H. (2021). Isoclinic extensions of Lie superalgebras.

Saeedi, F., Veisi, B. (2014). On Schur's theorem. Linear Multilinear Algebra 62(9):1139–1297. DOI: 10.1080/03081087.2013.809871.

Berezin, F. A., Kac, G. I. (1970). Lie groups with commuting and anticommuting parameters. Math. USSR Sb. 11(3):311–325. DOI: 10.1070/SM1970v011n03ABEH001137.

Berkovich, Y. G. (1991). On the order of the commutator subgroup and the Schur multiplier of a finite p-group. J. Algebra 144(2):269–272. DOI: 10.1016/0021-8693(91)90106-I.

Biyogmam, G. R., Casas, J. M. (2019). The c-nilpotent Schur Lie-multiplier of Leibniz algebras. J. Geom. Phys. 138:55–69.

Niroomand, P. (2010). The converse of Schur's theorem. J. Algebra 33(11):4205–4210. DOI: 10.1016/j.jalgebra.2010.04.035.

Nayak, S. (2012). On the Schur multiplier of a pair of Leibniz algebras. Linear Multilinear Algebra 60(2):193–201. DOI: 10.1080/03081087.2011.592711.
[37] Sun, B., Chen, L., Zhou, X. (2018). Double derivations of \( n \)-Lie superalgebras. *Algebra Colloq.* 25(01): 161–180. DOI: 10.1142/S1005386718000111.

[38] Wiegold, J. (1965). Multiplicators and groups with finite central factor groups. *Math. Z.* 89(4):345–347. DOI: 10.1007/BF0112166.