Two-parameter Sturm-Liouville problems

B. Chanane† and A. Boucherif
Department of Mathematics and Statistics, 
King Fahd University of Petroleum and Minerals
Dhahran 31261, Saudi Arabia.
chanane@kfupm.edu.sa

May 4, 2012

Abstract This paper deals with the computation of the eigenvalues of two-parameter Sturm-Liouville (SL) problems using the Regularized Sampling Method, a method which has been effective in computing the eigenvalues of broad classes of SL problems (Singula r, Non-Self-Adjoint, Non-Local, Impulsive,...). We have shown, in this work that it can tackle two-parameter SL problems with equal ease. An example was provided to illustrate the effectiveness of the method.

1 Introduction

In an interesting paper published in 1963, F. M. Arscott [2] showed that the method of separation of variables used in solving boundary value problems for Laplace’s equation leads to a two-parameter eigenvalue problem for ordinary differential equations with the auxiliary requirement that the solutions satisfy boundary conditions at several points. This has led to an extensive development of multiparameter spectral theory for linear operators (see for instance [3-4], [7-10], [12], [15], [30-33], [35], [37], [39-42]). In the paper [12], the authors give an overview results on two-parameter eigenvalue problems for second order linear differential equations. Several properties of corresponding eigencurves are given. In [15] the authors have obtained interesting geometric properties of the eigencurves (for instance transversal intersections is equivalent to simplicity of the eigenvalues in the sense of Chow and Hale). All the above works are concerned with the theoretical aspect of existence of eigenvalues. Also, several authors have dealt with the theoretical numerical analysis of two-parameter eigenvalue problems (see [5], [11], [13], [34], [36], [38] and the references therein). Eigenvalue problems have played a major role in the applied sciences. Consequently, the problem of computing eigenvalues of one-parameter problems has attracted many researchers (see for example [1, 6, 16, 17, 19-29] and the references therein).

Concerning the computations of eigenvalues of one-parameter Sturm-Liouville problems, the authors in [16] introduced a new method based on Shannon’s sampling theory. It uses the analytic properties of the boundary function. The method has been generalized to a class of singular problems of Bessel type [17], to more general boundary conditions, separable [19] and coupled [21], to random Sturm-Liouville problems [23] and to fourth order regular Sturm-Liouville problems [28]. The books by Atkinson [4], Chow and Hale [30], Faierman [33], Mc Ghee and Picard [37], Sleeman [39], Volkmer [42] and the long awaited monograph by Atkinson and Mingarelli [5] contains several results on eigenvalues of multiparameter Sturm-Liouville problems and the corresponding bifurcation problems. However, no attempt has been made to compute the eigenvalues of two-parameter Sturm-Liouville problems using the approach based on the Regularized Sampling Method introduced recently by the first author in [25] to compute the eigenvalues of general Sturm-Liouville problems and extended to the case of Singular [24], Non-Self-Adjoint [23], Non-Local [20], Impulsive SLPs [22, 21]. We shall consider, in this paper the computation of the eigenpairs of two-parameter Sturm-Liouville problems with three-point boundary conditions using the Regularized Sampling Method.

2 The Characteristic Function

Consider the two-parameter Sturm-Liouville problem

1Corresponding author
where \( w_1, w_2 \) are positive and in \( C^2[0,1] \) and \( q \in L[0,1] \) and \( c \in (0,1) \) some given constant.

By an eigenvalue of (2.1) we mean a value of the couple \((\mu_1, \mu_2)\) for which problem (2.1) has a non trivial solution. Conditions that insure the existence of eigenvalue are given in [3], [9], [10], [15], [30], [31], [40] and [41]. In fact, under fairly general conditions it has been shown (see [15], [30]) that there are smooth curves of eigenvalues (actually eigenpairs). Our objective is to effectively localize the eigencurves in the parameter \((\mu_1, \mu_2)\)-plane. We should point out that we have restricted our attention to Dirichlet boundary conditions in order to eliminate technical details that might obscure the ideas.

We shall associate to (2.1) the initial value problem
\[
\begin{aligned}
\begin{cases}
-y'' + qy = (\mu_1^2 w_1 + \mu_2^2 w_2)y, & 0 < x < 1 \\
y(0) = 0, & y(1) = 0
\end{cases}
\end{aligned}
\]
and deal first with the unperturbed case \((q = 0)\) then with the perturbed case \((q \neq 0)\).

### 2.1 The unperturbed case \((q = 0)\)

In this case, (2.2) reduces to
\[
\begin{aligned}
\begin{cases}
\varphi'' + w\varphi_1 = 0, & 0 < x < 1 \\
\varphi_1(0) = 0, & \varphi_1'(0) = 1
\end{cases}
\end{aligned}
\]
where \( w = \mu_1^2 w_1 + \mu_2^2 w_2 \).

**Theorem 1** The solution \( \varphi_1 \) of (2.3) is an entire function of \((\mu_1, \mu_2)\) \( \in C^2 \) for each fixed \( x \in (0,1] \) of order \((1,1)\) and type \((\sigma_1(x), \sigma_2(x))\) and satisfies the estimate,
\[
|\varphi_1(x)| \leq K_1 \exp\left[\sigma_1(x) |\mu_1| + \sigma_2(x) |\mu_2| \right], \quad (\mu_1, \mu_2) \in C^2
\]
for each fixed \( x \in (0,1] \) where, \( \sigma_i(x) = 2 \{x \int_0^x w_i(\xi) d\xi\}^{\frac{1}{2}} \) for \( i = 1, 2 \).

**Proof.** From (2.3) we get the integral equation
\[
\varphi_1(x) = x - \int_0^x (x - \xi) \left[ \mu_1^2 w_1(\xi) + \mu_2^2 w_2(\xi) \right] \varphi_1(\xi) d\xi
\]
Let
\[
\varphi_{1,n+1}(x) = \int_0^x (x - \xi) \left[ \mu_1^2 w_1(\xi) + \mu_2^2 w_2(\xi) \right] \varphi_{1,n}(\xi) d\xi, \quad n \geq 0
\]
We shall show, by induction on \( n \), that
\[
|\varphi_{1,n}(x)| \leq \frac{x}{n! (n+1)!} \left\{ x \int_0^x \left[ |\mu_1|^2 w_1(\xi) + |\mu_2|^2 w_2(\xi) \right] d\xi \right\}^n, \quad n \geq 0
\]
It is true for \( n = 0 \). Assume it is true for \( n \). We shall show that it is true for \( n + 1 \). Indeed, from (2.5), we have
\[
|\varphi_{1,n+1}(x)| \leq \int_0^x (x - \xi) \left[ |\mu_1|^2 w_1(\xi) + |\mu_2|^2 w_2(\xi) \right] \times \frac{\xi}{n! (n+1)!} \left\{ x \int_0^\xi \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right\}^n d\xi
\]
Using the fact that the expression \((x - \xi)\xi^{n+1}\) attains its maximum at \(\xi = \frac{r}{n+2} x\), we get

\[
|\varphi_{1,n+1}(x)| \leq \frac{x}{n!(n+1)!} \left( \frac{n+1}{n+2} \right)^{n+1} \left\{ \int_0^x \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right\}^{n+1} \\
\leq \frac{x}{(n+1)!(n+2)!} \left\{ \int_0^x \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right\}^{n+1} \tag{2.8}
\]

that is \((2.0)\) is true for \(n + 1\). Hence, it is true for all \(n \geq 0\).

Now, \(\varphi_1(x) = \sum_{n \geq 0} (-1)^n \varphi_{1,n}(x)\) and the series is absolutely and uniformly convergent since

\[
|\varphi_1(x)| = \left| \sum_{n \geq 0} (-1)^n \varphi_{1,n}(x) \right| \leq \sum_{n \geq 0} |\varphi_{1,n}(x)| \\
\leq x \sum_{n \geq 0} \frac{1}{n!(n+1)!} \left\{ x \int_0^x \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right\}^n \\
= x I_1 \left( 2 \int_0^x \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right) \times \\
\left\{ x \int_0^x \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right\}^{-\frac{1}{2}} \tag{2.9}
\]

where \(I_1\) is the modified Bessel function of the first kind order 1

\[I_1(z) = \sum_{n \geq 0} \frac{1}{n!(n+1)!} \left( \frac{z}{2} \right)^{2n+1}.\]

Using the fact that \(I_1(z) \sim \frac{\exp(z)}{\sqrt{2\pi z}}\) as \(z \to \infty\), we get

\[
|\varphi_1(x)| \leq K_1 \exp \left[ 2 \left\{ x \int_0^x \left[ |\mu_1|^2 w_1(\tau) + |\mu_2|^2 w_2(\tau) \right] d\tau \right\}^{\frac{1}{2}} \right] \\
\leq K_1 \exp \left[ \sigma_1(x) |\mu_1| + \sigma_2(x) |\mu_2| \right] \tag{2.10}
\]

Therefore, \(\varphi_1\) is an entire function of \((\mu_1,\mu_2) \in \mathbb{C}^2\), as a uniformly convergent series of entire functions, for each fixed \(x \in (0, 1]\), of order \((1, 1)\) and type \((\sigma_1(x), \sigma_2(x))\). This concludes the proof.

\[\blacksquare\]

We shall make use of the Liouville-Green’s transformation

\[
\begin{align*}
& \left\{ \begin{array}{l}
    t(x) = \int_0^x \sqrt{w(\xi)} d\xi \\
    z(t) = \{w(x)\}^\frac{1}{2} \varphi_1(x)
\end{array} \right.
\end{align*} \tag{2.11}
\]

to bring \((2.2)\) to the form

\[
\begin{align*}
& \left\{ \begin{array}{l}
    \frac{d^2z}{dt^2} + (1 + r(t)) z = 0 , \quad 0 < t < \int_0^1 \sqrt{w(\xi)} d\xi \\
    z(0) = 0 , \quad \frac{dz}{dt}(0) = \{w(0)\}^{-\frac{1}{4}}
\end{array} \right.
\end{align*} \tag{2.12}
\]

which can be written as an integral equation

\[
z(t) = \{w(0)\}^{-\frac{1}{4}} \sin t - \int_0^t \sin(t-\tau) r(\tau) z(\tau)d\tau \tag{2.13}
\]

where \(r(t) = \left[ \{w(x)\}^{-\frac{3}{4}} \frac{w(x)}{w(x)} \right]_{x=x(t)}^{-\frac{1}{4}}\).
Returning to the original variables, we deduce that \( \varphi_1 \) satisfies the integral equation

\[
\varphi_1(x) = \left\{ w(0)w(x) \right\}^{-\frac{1}{2}} \sin \left\{ \int_0^x \sqrt{w(\xi)} d\xi \right\} - \int_0^x \sin \left\{ \int_0^x \sqrt{w(\xi)} \psi(x, \vec{x}) \varphi_1(\vec{x}) d\vec{x} \right\} \tag{2.14}
\]

where

\[
\psi(x, \vec{x}) = \left\{ w(x) \right\}^{-\frac{1}{2}} \left\{ w(\vec{x}) \right\}^{-\frac{1}{2}} \left[ -\frac{1}{4} w''(\vec{x}) \left\{ w(\vec{x}) \right\}^{-\frac{1}{2}} + \frac{5}{16} \left\{ w'(\vec{x}) \right\}^2 \left\{ w(\vec{x}) \right\}^{-\frac{1}{2}} \right]\tag{2.15}
\]

We shall present next some estimates whose proofs are immediate and left to the reader.

**Lemma 2** The function \( \psi(x, \vec{x}) \) satisfies the estimate

\[
|\psi(x, \vec{x})| \sim \frac{K_2}{|\mu_1| + |\mu_2|}, \text{ as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2 \tag{2.16}
\]

**Lemma 3** The function \( \alpha \) defined by

\[
\alpha(x) = (\text{sinc} \left\{ \sigma_1(x) \mu_1 + \sigma_2(x) \mu_2 \right\})^m
\]

where \( \text{sinc}(z) = z^{-1} \sin z \) and \( m \) is a positive integer, is an entire function of \( (\mu_1, \mu_2) \in \mathbb{C}^2 \) for each fixed \( x \in (0, 1] \) of order \( (1, 1) \) and type \( (\sigma_1(x), \sigma_2(x)) \). Furthermore, \( \alpha \) satisfies the estimate

\[
|\alpha(1)| \sim \frac{K_3}{|\mu_1| + |\mu_2|}, \text{ as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

**Lemma 4** The function \( \varphi_1 \) satisfies the estimate

\[
|\varphi_1(1)| \sim \frac{K_4}{|\mu_1| + |\mu_2|}, \text{ as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2 \tag{2.17}
\]

Combining the above results, we obtain the following theorem,

**Theorem 5** The function \( \alpha \varphi_1 \) is an entire function of \( (\mu_1, \mu_2) \in \mathbb{C}^2 \) for each fixed \( x \in (0, 1] \) of order \( (1, 1) \) and type \( ((m+1)\sigma_1(x), (m+1)\sigma_2(x)) \) and satisfies the estimate

\[
|\alpha(\varphi_1)(x)| \sim \frac{K(x)}{|\mu_1| + |\mu_2|} \tag{m+1}, \text{ as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

where \( K \) depends on \( x \in (0, 1] \) but is independent of \( (\mu_1, \mu_2) \).

Let \( \mathcal{PW}_{\beta_1, \beta_2} \) denote the Paley-Wiener space,

\[
\mathcal{PW}_{\beta_1, \beta_2} = \left\{ h(z_1, z_2) \text{ entire} / \begin{array}{l} |h(z_1, z_2)| \leq C \exp \{ \beta_1 |z_1| + \beta_2 |z_2| \}, \\
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |h(z_1, z_2)|^2 dz_1 dz_2 < \infty \end{array} \right\}
\]

we have,

**Theorem 6** \( \alpha(\varphi_1)(x) \), as a function of \( (\mu_1, \mu_2) \) belongs to the Paley-Wiener space \( \mathcal{PW}_{\beta_1, \beta_2} \) where \( (\beta_1, \beta_2) = ((m+1)\sigma_1(x), (m+1)\sigma_2(x)) \) for each fixed \( x \in (0, 1] \).
2.2 The perturbed case \((q \neq 0)\)

Let \(\varphi_1, \varphi_2\) be two linearly independent solutions of \(\varphi'' + w\varphi = 0\) satisfying \(\varphi_1(0) = \varphi_2'(0) = 0\), \(\varphi_1'(0) = \varphi_2(0) = 1\) then the method of variation of parameters shows that (2.2) can be written as the integral equation

\[
y(x) = \varphi_1(x) + \int_0^x \Phi(x, \xi) q(\xi) y(\xi) d\xi
\]

where \(\Phi(x, \xi) = \varphi_1(\xi)\varphi_2(x) - \varphi_2(\xi)\varphi_1(x)\).

Here again, it is not hard to show that \(y(x)\) is an entire function of \((\mu_1, \mu_2)\) for each \(x \in (0, 1]\), of order \((1, 1)\) and type \((\sigma_1(x), \sigma_2(x))\). Multiplication by \(\alpha\) gives a function \(\alpha(x) y(x)\) of \((\mu_1, \mu_2)\) in a Paley-Wiener space \(PW_{\beta_1, \beta_2}\) for each \(x \in (0, 1]\). More specifically, we have the following, 

**Theorem 7** The function \(\tilde{y}\) defined by \(\tilde{y}(x) = \alpha(x)y(x)\), belongs to \(PW_{\beta_1, \beta_2}\) where \((\beta_1, \beta_2) = ((m + 1)\sigma_1(x), (m + 1)\sigma_2(x))\) as a function of \((\mu_1, \mu_2)\) in \(C\) for each \(x \in (0, 1]\), and satisfies the estimate,

\[
|\alpha(x)y(x)| \sim \frac{K(x)}{|\mu_1| + |\mu_2|} = K_5, \quad \text{as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

where \(K\) depends on \(x \in (0, 1]\) but is independent of \((\mu_1, \mu_2)\).

**Proof.** Since \(\Phi_{xx} + w\Phi = 0\), \(\Phi(t, t) = 0\) and \(\Phi_x(t, t) = 1\), we have,

\[
\Phi(x, t) = (w(t)w(x))^{-\frac{1}{4}} \sin \left( \int_t^x \sqrt{w(\xi)} d\xi \right) - \int_t^x \sin \left( \int_t^\xi \sqrt{w(\eta)} d\eta \right) \psi(\xi, \tau) \Phi(\tau, t) d\tau
\]

so that,

\[
|\Phi(x, t)| \sim \frac{K_4}{|\mu_1| + |\mu_2|} \leq K_5, \quad \text{as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

from which we get, after using Gronwall’s lemma on (2.18) and the estimate for \(\varphi_1\),

\[
|y(x)| \leq K_1 \exp \left( \sigma_1(x)|\mu_1| + \sigma_2(x)|\mu_2| \right) \exp \left( K_5 \int_0^x |q(t)| dt \right)
\]

\[
|y(1)| \leq K_7 \exp \left( \sigma_1(1)|\mu_1| + \sigma_2(1)|\mu_2| \right), \quad \text{as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

and

\[
|\alpha(1)y(1)| \leq K_8 \exp \left[ (m + 1)\sigma_1(1)|\mu_1| + (m + 1)\sigma_2(1)|\mu_2| \right], \quad \text{as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

Furthermore, we have,

\[
|\alpha(x)y(x)| \sim \frac{K(x)}{|\mu_1| + |\mu_2|} = K_5, \quad \text{as } |\mu_1| + |\mu_2| \to \infty, (\mu_1, \mu_2) \in \mathbb{R}^2.
\]

where \(K\) depends on \(x \in (0, 1]\) but is independent of \((\mu_1, \mu_2)\). \(\blacksquare\)

To summarize, in both cases, unperturbed and perturbed, the transform \(\tilde{y}(x; \mu_1, \mu_2)\) of the solution \(y(x; \mu_1, \mu_2)\) of (2.2) is in a Paley-Wiener space \(PW_{\beta_1, \beta_2}\) where \((\beta_1, \beta_2) = ((m + 1)\sigma_1(x), (m + 1)\sigma_2(x))\). Thus \(\tilde{y}(x; \mu_1, \mu_2)\) can be recovered at each \(x \in (0, 1]\) from its samples at the lattice points \((\mu_{1j}, \mu_{2k}) = (j\frac{\pi}{(m + 1)\sigma_1(x)}, k\frac{\pi}{(m + 1)\sigma_2(x)}), (j, k) \in \mathbb{Z}^2\) using the rectangular cardinal series \((45, 50, 51), (46, 51, 51), (47, 51, 51), (48, 51, 51), (49, 51, 51), (50, 51, 51), (51, 51, 51)\).
Theorem 8 Let \( f \in PW_{\beta_1, \beta_2} \) then

\[
f(\mu_1, \mu_2) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(\mu_{1j}, \mu_{2k}) \frac{\sin \beta_1(\mu_1 - \mu_{1j}) \sin \beta_2(\mu_2 - \mu_{2k})}{\beta_1(\mu_1 - \mu_{1j}) \beta_2(\mu_2 - \mu_{2k})}
\]

the convergence of the series being uniform and in \( L^2_{\mu_1, \mu_2}(\mathbb{R}^2) \), and \( \mu_{mn} = n\pi/\beta_m \), \( m = 1, 2, n \in \mathbb{Z} \).

Let \( \sigma_{11} = \sigma_1(1), \sigma_{21} = \sigma_2(1), \sigma_{12} = \sigma_1(c), \sigma_{22} = \sigma_2(c) \).
The eigenpairs are therefore \((\mu_1^2, \mu_2^2)\) where \((\mu_1, \mu_2)\) solve the nonlinear system

\[
\begin{align*}
B_1(\mu_1, \mu_2) &= 0 \\
B_2(\mu_1, \mu_2) &= 0
\end{align*}
\]

where,

\[
\begin{align*}
B_1(\mu_1, \mu_2) &\equiv \frac{1}{\alpha(1)} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{y}(1; \mu_{1j}, \mu_{2k}) \sin 2(m+1)\pi \sin 2(n+1)\pi \mu_{1j} \mu_{2k} \\
B_2(\mu_1, \mu_2) &\equiv \frac{1}{\alpha(c)} \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \tilde{y}(c; \mu_{1j}, \mu_{2k}) \sin 2(m+1)\pi \sin 2(n+1)\pi \mu_{1j} \mu_{2k}
\end{align*}
\]

3 A numerical example

We shall consider in this section the two-parameter Sturm-Liouville problem with three-point boundary conditions given by

\[
\begin{align*}
-\gamma''(x) &= (\mu_1^2 + \mu_2^2 x) \gamma(x), \quad 0 < x < 1 \\
\gamma(0) &= \gamma(0.7) = \gamma(1)
\end{align*}
\]

The general solution \( y \) of the first differential equation can be expressed in terms of \( \text{Ai} \) and \( \text{Bi} \) functions and their first derivatives as

\[
y(x; \mu_1, \mu_2) = \frac{(-1)^{2/3}}{\mu_2^{2/3}} \left( \text{Ai} \left( \sqrt{-\mu_1^2} \mu_2 \right) \text{Bi} \left( \sqrt{-\mu_1^2} \mu_2 \right) - \text{Ai} \left( \sqrt{-\mu_1^2} \mu_2 \right) \text{Bi} \left( \sqrt{-\mu_1^2} \mu_2 \right) \right)
\]

Thus the eigenpairs \((\mu_1^2, \mu_2^2)\) can be obtained from the solutions \((\mu_1, \mu_2)\) of the system

\[
\begin{align*}
\tilde{y}(1; \mu_1, \mu_2) &= 0 \\
\tilde{y}(c; \mu_1, \mu_2) &= 0
\end{align*}
\]

For numerical purposes we have truncated the associated series to \(|j|, |k| \leq N = 50\) and took \( m = 5 \) in the function \( \alpha \). Thus, the approximate eigenpairs are seen as solutions of the system

\[
\begin{align*}
\tilde{y}^{[N]}(1; \mu_1, \mu_2) &= 0 \\
\tilde{y}^{[N]}(c; \mu_1, \mu_2) &= 0
\end{align*}
\]

The next table shows the exact eigenpairs together with their approximations using the Regularized Sampling Method (RSM).
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
$\mu_1$ (exact) & $\mu_2$ (exact) & $\mu_1$ (RSM) & $\mu_2$ (RSM) \\
\hline
7.788149097670813 & 7.590822436578645 & 7.78814883048352 & 7.590823605399021 \\
4.1940560473411936 & 22.273796542861913 & 4.194057357011064 & 22.273795699572364 \\
15.597607295800684 & 15.597605904429917 & 15.597609731797321 & 15.597609731797321 \\
13.8761575469624 & 13.876157718920586 & 13.876157718920586 & 13.876157718920586 \\
23.4024224972004 & 22.744341450749697 & 23.40242640980619 & 22.744329536815076 \\
9.51130897213563 & 44.29647446214639 & 9.51130897213563 & 44.29647446214639 \\
39.31835017121106 & 39.31833152483279 & 39.31833152483279 & 39.31833152483279 \\
47.18632817417246 & 47.186345272524804 & 47.186345272524804 & 47.186345272524804 \\
46.81208951678993 & 46.81208951678993 & 46.81208951678993 & 46.81208951678993 \\
30.03437464767369 & 30.034392317076748 & 30.034392317076748 & 30.034392317076748 \\
\hline
\end{tabular}
\end{table}

4 Conclusion

In this paper, we have successfully computed the eigenpairs of two-parameter Sturm-Liouville problems using the regularized sampling method. A method which has been very efficient in computing the eigenvalues of broad classes of Sturm-Liouville problems (Singular, Non-Self-Adjoint, Non-Local, Impulsive,...). We have shown, in this work that it can tackle two-parameter SL problems with equal ease. An example was provided to illustrate the effectiveness of the method.

Acknowledgements The authors are grateful to KFUPM for its usual support.

References

[1] S. D. Algazin, 1995, Calculating the eigenvalues of ODE Comp. Maths. Math. Phys. 35, pp. 477-482.
[2] F. M. Arscott, 1963, Paraboloid coordinates and Laplace’s equation, Proc. Royal Soc. Edinburgh 66, pp. 129-139.
[3] F. M. Arscott, 1964, Two-parameter eigenvalue problems in differential equations, Proc. London Math. Soc 14, pp. 459-470.
[4] F. V. Atkinson, 1972, Multiparameter Eigenvalue Problems, Academic Press, New York
[5] F. V. Atkinson, Angelo B. Mingarelli, Multipararameter problems Sturm-Liouville Theory, CRC Press, 2011
[6] P. Bailey, 1981, The automatic solution of two-parameter Sturm-Liouville eigenvalue problems in ordinary differential equations, Appl. Math. Comput., 8, No. 4, 251-259.
[7] P. B. Bailey, W. N. Everitt and A. Zettl, 1991, Computing eigenvalues of singular Sturm-Liouville problems, Results in Math. Vol. 20, Birkhauser Verlag, Basel.
[8] P. Binding, 1984, Perturbation and bifurcation of nonsingular multiparametric eigenvalues, Nonlinear Analysis, 8, pp. 1-20.
[9] P. Binding, 1991, Sturm-Liouville theory via examples of two parameter eigencurves, Proc. 1990 Dundee Conference on Differential Equations, R. Jarvis and B. D. Sleeman, Eds. Pitman, Boston, 1991, pp.38-49.
[10] P. Binding and P. J. Browne, 1981, Spectral properties of two parameter eigenvalue problems, Proc. Roy. Soc. Edinburgh, 89A, pp. 157-173.
[11] P. Binding and P. J. Browne, 1989, Eigencurves for two parameter self-adjoint ordinary differential equations of even order, J. Differential Equations, 79, pp. 289-303.
[12] P. Binding, P. J. Browne and Ji, Xing Zhi, 1993, A Numerical Method using the Pruefer transformation for the calculation of eigenpairs of two-parameter Sturm-Liouville problems, IMA J. Numer. Anal., 13, No. 4, pp. 559-569.

[13] P. Binding, H. Volkmer, 1996, Eigencurves for two-parameter Sturm- Liouville equations, SIAM Review 38, pp. 27-48.

[14] E. K. Blum and A. R. Curtis, 1978, A convergent gradient method for matrix eigenvector-eigenvalues problems, Numer. Math. 31, 247-263.

[15] A. Boucherif, 1980, Nonlinear three-point boundary value problems, J. Math. Anal. Appl. 77, pp.577-600.

[16] A. Boucherif, N. Boukli-Hacene, 1999, Two-parameter eigenvalue problem with time-dependent three-point boundary conditions, Maghreb Math. Rev., Vol. 8, No. 1 & 2, 57-66.

[17] A. Boumenir, B. Chanane, 1996, Eigenvalues of Sturm-Liouville systems using sampling theory, Applicable Analysis, 62, pp.323-334.

[18] A. Boumenir, B. Chanane, 1999, Computing Eigenvalues of Sturm- Liouville Systems of Bessel Type, Proceedings of the Edinburgh Math Society, 42, pp. 257-265.

[19] P. J. Browne and B. D. Sleeman, 1993, A note on the characterization of eigencurves for certain two parameter eigenvalue problems in ordinary differential equations, Glasgow Math. J., 35 , No. 1, pp. 63-67.

[20] B. Chanane, 2009, Computing the eigenvalues of a nonlocal Sturm-Liouville problem, Mathematical and Computer Modelling, doi:10.1016/j.mcm.2008.10.021, (50) 225–232

[21] B.Chanane, 2007, Sturm-Liouville problems with impulse effects, Applied Math. and Computation, 190/1 pp. 610-626,

[22] B. Chanane, 2007, Eigenvalues of Sturm-Liouville problems with discontinuity inside a finite interval , Applied Mathematics and Computation, 188/2 pp. 1725-1732, appeared electronically (2006) doi:10.1016/j.amc.2006.11.082

[23] B. Chanane, 2007, Computing the spectrum of non self-adjoint Sturm-Liouville problems with parameter dependent boundary, J. Computational and Applied Mathematics, 206/1 pp. 229-237, appeared electronically (2006) doi:10.1016/j.cam.2006.06.014

[24] B. Chanane, 2007, Computing the eigenvalues of singular Sturm-Liouville problems using the regularized sampling method, Applied Mathematics and Computation , 184/2 pp. 972-978, appeared electronically (2006) doi:10.1016/j.amc.2006.05.187

[25] B. Chanane, Computation of the eigenvalues of Sturm–Liouville Problems with parameter dependent boundary conditions using the regularized sampling method, Math. Comput. 74 (252) (2005) 1793–1801, published electronically S 0025-5718(05)01717-5.

[26] B. Chanane, 1999, Computing Eigenvalues of Regular Sturm-Liouville Problems, Applied Math. Letters, Vol.12, pp. 119-125.

[27] B. Chanane, 1998, High Order Approximations of Eigenvalues of Regular Sturm-Liouville Problems, J. Math. Analysis and Applications, 226, pp.121-129.

[28] B. Chanane, 2001, High Order Approximations of the Eigenvalues of Sturm-Liouville Problems with Coupled Self-Adjoint Boundary Conditions, Applicable Analysis, Vol. 80, pp. 317-330 .

[29] B. Chanane, 2000, The Paley-Wiener-Levinson theorem and the computation of Sturm-Liouville eigenvalues:Irregular sampling, Applicable Analysis, Vol.75 (3-4), pp.261-266.

[30] B. Chanane, 2002, On a class of random Sturm-Liouville problems, Intern. J. Applied Math., Vol. 8, pp. 171-182 .
[31] B. Chanane, 1998, Eigenvalues of Sturm-Liouville problems using Fliess series, Applicable Analysis, Vol. 69 (3-4) pp.233-238.

[32] B. Chanane, 1998, Eigenvalues of Fourth Order Sturm-Liouville problems using Fliess Series, J. of Applied and Computational Math., Vol. 96, Iss. 2., pp. 91-97.

[33] B. Chanane, 1999, Eigenvalue Problems: A Control Theoretic Approach, in Dynamics of Continuous, Discrete and Impulsive Systems, 1-4, pp. 465-471.

[34] B. Chanane, 2002, Fliess series approach to the computation of the eigenvalues of fourth order Sturm-Liouville problems, Applied Math. Letters, Vol. 15, Iss. 4, pp. 459-463.

[35] B. Chanane, 2002, Eigenvalues of Regular Fourth Order Sturm-Liouville Problems using Sampling Theory, Approximation Theory X: Wavelets, Splines and Applications, Charles K. Chui, Larry L. Schumaker, and Joachim Stockler (eds.), pp. 155-166.

[36] S. N. Chow, J. K. Hale, 1982, Methods of Bifurcation Theory, Springer Verlag, New York.

[37] L. Collatz, 1968, Multiparameter eigenvalue problems in inner-product spaces, . Comput. System. Sci. 2, pp. 333-341.

[38] M. Faierman, 1972, Asymptotic formulae for the eigenvalues of a two-parameter ordinary differential equation of the second order, Trans. Math. Soc., 168, pp. 1-52.

[39] M. Faierman, 1991, Two-parameter eigenvalue problems in ordinary differential equations, Pitman Research Notes in Mathematics, Vol. 205, Longman, UK.

[40] L. Fox et al., 1972, The double eigenvalue problem, Topics in Numerical Analysis, Proc. Royal. Irish Acad. Conference on Numerical Analysis, New York.

[41] N. Gregus, F. Neumann, F. M. Arscott, 1971, Three-point boundary value problems in differential equations, J. London Math. Soc. 3, pp. 429-436.

[42] Ji, Xing Zhi, 1992, A two-dimensional bisection method for solving two-parameter eigenvalue problems, SIAM J. Matrix Anal. Appl., 13, No. 4, pp.1085-1093.

[43] D. McGhee and R. Picard, 1988, Cordes’Two-Parameter Spectral Representation Theory, Pitman Research Notes, Longman, UK.

[44] B. A. Hargrave, B. D. Sleeman, 1974, The numerical solution of two-parameter eigenvalue problems in ordinary differential equations with an application to the problem of diffraction by a plane angular sector, J. Inst. Math. Appl., 14, pp. 9-22.

[45] E. Parzen, 1956, A Simple Proof and Some Extensions of Sampling Theorems, Tech. Rep. 7, Stanford University, Stanford.

[46] B. D. Sleeman, 1978, Multiparameter Spectral Theory in Hilbert Space, Pitman Research Notes, Longman, UK.

[47] B. D. Sleeman, 1972, The two-parameter Sturm-Liouville problem for ordinary differential equations II, Proc. Amer. Math. Soc. 34, pp. 165-170.

[48] L. Turyn, 1980, Sturm-Liouville problems with several parameters, J. Differential Equations, 38, pp. 239-259.

[49] H. Volkmer, 1988, Multiparameter Eigenvalue Problems and Expansion Theorems, Lecture Notes in Math, 1356, Springer-Verlag, New York, Berlin.

[50] A. I. Zayed, 1993, Advances in Shannon’s Sampling Theory, CRC Press, BOCA Raton.

[51] A. I. Zayed, 1994, A Sampling Theorem for Signals Bandlimited to a General Domain in Several Dimensions, J. Math. Anal. App. 187, 196–211.