Planar Skyrmions at High and Low Density

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Abstract

The O(3) Skyrme system in two space dimensions admits topological soliton solutions. This paper studies the transition between the high-density crystalline phase of such solitons and the low-density phase where there are multi-Skyrmions localized in space. The details depend crucially on the choice of the potential function. Two such choices are investigated: in the first system, multi-Skyrmions at low density form a ring; while in the second, there is an explicit crystal solution, and the preferred low-density configurations are chunks of this crystal. The second system is a particularly good analogue of three-dimensional Skyrmions.

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1 Introduction

Topological solitons are of interest both mathematically and in many areas of physics (as models for particles, topological defects in condensed-matter physics, etc). A particularly important question is the nature of, and transition between, the high-density and low-density phases of such solitons. Ultimately, one would like to understand the thermodynamics of these systems; the starting-point for this is to investigate their zero-temperature behaviour. At its most basic, this amounts to considering a fixed number $N$ of solitons confined to a finite volume, and investigating the way in which static classical $N$-soliton solutions depend on the volume (or density). Alternatively, one may use the dimensionless ratio $(size\ of\ soliton)/(size\ of\ space)$ as a parameter.

The typical situation is as follows. At large density, there is a high degree of symmetry and uniformity, and in particular it may not possible to identify individual solitons; in Skyrme models, for example, one gets a periodic crystal-like structure of half-Skyrmions. At low densities, by contrast, solitons (or multi-solitons) become localized in space, and there is less symmetry. The localized multi-solitons may be ‘chunks’ of the high-density crystal, or may have a quite different (for example shell-like) shape — this depends on the details of the system.

The picture for the three-dimensional Skyrme model may be summarized as follows. Let $N$ denote the Skyrme number, and $E$ the normalized energy-per-Skyrmion (so the Bogomolny-Faddeev bound is $E \geq 1$). At high density, the ground state is a triply-periodic lattice of half-Skyrmions [1, 2, 3, 4, 5, 6], with energy $E = 1.038$ at its most favourable density. At low density (for example for an isolated multi-Skyrmion in $\mathbb{R}^3$), and for relatively low values of $N$ (in particular for $N \leq 22$), the minimal-energy static Skyrmions take the form of polyhedral shells [7]. Their normalized energy $E$ is a decreasing function of $N$, and it appears [8] that these polyhedra have energy $E \approx 1.06$ for large $N$. Since this is larger than the energy of the Skyrme crystal, one expects that there is a critical value $N_c$ such that for $N > N_c$, the minimal-energy Skyrmion resembles a chunk of the Skyrme crystal. Such a lattice chunk will have energy $E \approx 1.038 + kN^{-1/3}$, with the second term being a surface contribution. There have been attempts [9] to estimate the constant $k$, and hence to determine $N_c$, but these have not been definitive. There are also other possibilities for the shape of large-$N$ Skyrmions, for example a multi-shell structure [10].

The present paper deals with the two-dimensional $O(3)$ Skyrme system, which one may view as an analogue of the three-dimensional case, as well as being of interest in its own right. The 2-D Skyrme system contains, as a limiting case, the two-dimensional $O(3)$ sigma model; the sigma-model crystal, obtained by imposing periodic boundary conditions, has been studied both as a model quantum field theory
and in connection with the dynamics of (classical) solitons. But sigma-model solitons do not have a fixed size, and in particular tend to shrink and decay; so in that sense they are not true solitons. We shall consider only the case where both the Skyrme term and a potential term $V$ are present, and the soliton size is consequently fixed. The interest here is in the nature of the low-density and the high-density configurations, and the transition between them; as we shall see, the details of these depend crucially on the choice of potential $V$. Two different systems, corresponding to two different choices of $V$, will be investigated in detail.

## 2 Two-Dimensional Skyrmions

The two-dimensional Skyrme system involves a unit 3-vector field $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ defined on (2+1)-dimensional space-time. We are interested only in static configurations, so $\vec{\phi}$ depends on the spatial variables $x^j = (x^1, x^2)$, thought of as (local) coordinates on a two-dimensional space $S$. If $S$ is a compact surface, then $\vec{\phi}$ has a winding number $N$, which we think of as the number of Skyrmions. One special case is where space is the plane $\mathbb{R}^2$ with boundary condition $\vec{\phi} \to (0, 0, 1)$ as $r \to \infty$ — this corresponds (by conformal invariance) to the 2-sphere $S = S^2$. Another case is that of periodic boundary conditions $\vec{\phi}(x^1, x^2) = \vec{\phi}(x^1 + L, x^2) = \vec{\phi}(x^1, x^2 + L)$, which corresponds to the 2-torus $S = T^2$. In either of these cases, we may regard the spatial metric as being the flat (Euclidean) metric, and we shall do so in what follows.

The integer $N$ can be either positive or negative; without loss of generality, we shall assume $N$ to be positive. The energy density of a configuration $\vec{\phi}$ is

$$E = \frac{1}{\alpha} (\partial_j \vec{\phi}) \cdot (\partial_j \vec{\phi}) + \frac{1}{\alpha} \Omega^2 + \frac{1}{\alpha} V(\vec{\phi}),$$

where $\Omega = \vec{\phi} \cdot \partial_1 \vec{\phi} \times \partial_2 \vec{\phi}$, $V$ is some potential function, and $\alpha$ is a dimensionless constant. The length-scale in this system is determined by the ratio of the coefficients of the $\Omega^2$ and $V$ terms, and so is fixed by choosing these coefficients to be equal as in (1). Static multi-Skyrmion solutions are critical points of the (normalized) energy functional

$$E = \frac{1}{4\pi N} \int \mathcal{E} \, d^2x.$$  

There is a topological (Bogomolny) lower bound on the energy, namely

$$E \geq 1 + \frac{\alpha}{4\pi} \int_{S^2} \sqrt{V(\vec{\phi})} \, d\omega,$$

where $d\omega$ is the usual area element on the space $S^2$ of unit vectors (in other words, $\int d\omega = 4\pi$). Under certain circumstances, this bound can be saturated; the following
statement is adapted from [14], see also [15]. Write \( z = x^1 + ix^2 \), and let \( W = (\phi_1 + i\phi_2)/(1 - \phi_3) \) denote the stereographic projection of \( \vec{\phi} \). Let \( W(z) \) be a complex-analytic function satisfying a first-order ordinary differential equation of the form 
\[
dW/dz = F(W)
\]
for some function \( F \). Then the configuration \( W(z) \) saturates the bound (3), and hence is a static Skyrmion solution, provided that the potential \( V \) is given by
\[
V = \frac{16 |F(W)|^4}{(1 + |W|^2)^4}.
\]
(4)
The simplest example of this is where \( F \) is a constant; the corresponding system (on \( \mathbb{R}^2 \)) has been investigated in some detail, for example in [16, 17]. In this case, there is a repulsive force between Skyrmions, and consequently there are no static multi-Skyrmion solutions.

For any system of the form (1), the nature and shape of multi-Skyrmion solutions depend on the choice of \( V \), and many different choices have been considered [14, 18]. The following two sections will deal with two possible choices. They are both analogous to the three-dimensional Skyrme system, in that they allow both crystal-like and ring-like solutions (the latter being the counterpart of the polyhedral shells mentioned earlier).

To conclude this section, it is worth mentioning yet another case which has been extensively investigated, namely \( V(\vec{\phi}) = 1 - \phi_3 \) [19, 20]. In this case, the static multi-Skyrmion solutions on \( \mathbb{R}^2 \) appear to form a lattice-like structure [19, 18] — for example, for even values of \( N \) one gets a lattice of double-Skyrmions as the lowest-energy state. There are also other local minima of the energy, but there do not appear to be any ring-like solutions (except for \( N = 2 \)). The thermodynamics of this system has been studied numerically [21].

3 The System \( V = 1 - \phi_3^2 \)

In this section, the potential function \( V \) appearing in (1) is taken to be \( V(\vec{\phi}) = 1 - \phi_3^2 \). Let us discuss, first, the localization-delocalization transition for this system. A useful order parameter in this regard is the quantity
\[
\langle \phi_3 \rangle = \frac{1}{A} \int_S \phi_3 \, d^2x,
\]
where \( A \) is the area of the 2-space \( S \). At high density, there is a high degree of symmetry, and we expect to have \( \langle \phi_3 \rangle = 0 \); while at low density, the field will localize in space and we will have \( \langle \phi_3 \rangle \neq 0 \). The transition between these two phases occurs at some critical density \( \rho_c \). This transition was investigated by taking space to be a sphere \( (S = S^2) \) in [15]; in particular, this paper looked at \( \langle \phi_3 \rangle \), as a function
of $A$, for a single skyrmion on $S^2$. Using approximate analytic expressions for the Skyrmion indicated that the transition occurs at $A = 4\pi \sqrt{5}$, i.e., the critical density is

$$\rho_c = 1/(4\pi \sqrt{5}) = 0.036.$$  \hfill (5)

Numerical simulations gave results that supported this estimate.

Let us now compare this to the situation on the torus $T^2$. So we are looking for doubly-periodic solutions — periodic in both $x^1$ and $x^2$ with period $L$. If $\alpha = 0$ (i.e., for the $O(3)$ sigma-model), then solutions correspond to elliptic functions, and these have topological charge $N = 2$ in the unit cell. So we expect the minimal-energy crystal in the Skyrme case also to have an $N = 2$ cell. Such solutions can be found by numerical minimization of the energy functional. It turns out there are two different crystals, namely one with (approximate) symmetry $A$:

$\text{A1. } (x, y) \mapsto (-x, y), (\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, \phi_3),$  

$\text{A2. } (x, y) \mapsto (y, x), (\phi_1, \phi_2, \phi_3) \mapsto (-\phi_1, \phi_2, \phi_3),$  

$\text{A3. } (x, y) \mapsto (x + L/2, y), (\phi_1, \phi_2, \phi_3) \mapsto (\phi_3, -\phi_2, \phi_1);$

and one with symmetry $B$:

$\text{B1. } (x, y) \mapsto (-x, y), (\phi_1, \phi_2, \phi_3) \mapsto (\phi_1, -\phi_2, \phi_3),$  

$\text{B2. } (x, y) \mapsto (y, x), (\phi_1, \phi_2, \phi_3) \mapsto (-\phi_3, \phi_2, -\phi_1),$  

$\text{B3. } (x, y) \mapsto (x + L/2, y), (\phi_1, \phi_2, \phi_3) \mapsto (-\phi_1, -\phi_2, \phi_3).$

In the $\alpha = 0$ system, symmetries $A$ and $B$ are equivalent; but this degeneracy is broken by the potential term $V$. The situation is illustrated in Figure 1 which plots the normalized energy $E$ for the two solutions, and the quantity $\Psi = \langle \phi_3 \rangle_A$ for the type-$A$ solution, as functions of the periodicity $L$. The value of the parameter $\alpha$ used in obtaining these graphs is $\alpha = 0.1$. The Bogomolny bound $[3]$ for this system is $E \geq 1 + \pi \alpha/4 = 1.0785$. The lowest crystal energy is attained when $L = L_0 = 6.0$, i.e., at density $\rho = 0.056$; this solution has symmetry of type $B$, and normalized energy $E_{\text{crys}} = 1.088$. The type-$B$ solution exists for lower density, and is always delocalized, i.e., has $\langle \phi_3 \rangle = 0$.

At relatively high density, in fact for $L < L_1 = 7.0$, the solution with type-$A$ symmetry also has $\langle \phi_3 \rangle = 0$; but it is unstable, and its energy (which is not plotted in Figure 1) is higher than that of the type-$B$ solution. When perturbed, it decays to the type-$B$ solution. For $L > L_1$, the type-$A$ solution delocalizes ($\Psi \neq 0$) and becomes stable. (It no longer has the symmetry $A$ — in fact $A3$ is invalid — but we still refer to it as the type-$A$ solution.) This is illustrated in the right-hand diagram.
Figure 1: $E$ and $\Psi = \langle \phi_3 \rangle_A$ for the periodic $N = 2$ solutions.

of Figure 1 the numerical results indicate that

$$\lim_{L \to L_1^+} \Psi \approx 0.06,$$

but this limit is difficult to study owing to the instability which sets in below $L_1$. In the range $L_1 < L < L_2 = 7.35$, the energy of the type-$A$ solution is greater than that of type $B$ — in other words, there are two local minima of the energy, one of which (type $B$) is more symmetric than the other. For $L > L_2$, the minimal-energy $N = 2$ solution is the localized type-$A$ one.

We see, therefore, that there is a transition between a dense, highly-symmetric homogeneous phase and a low-density, less homogeneous phase. The latter appears at $L_1 = 7.0$, corresponding to a density $\rho = 2/7^2 = 0.041$, and it becomes the minimal-energy state at $L_2 = 7.35$, corresponding to the critical density

$$\rho_c = 2/7.35^2 = 0.037.$$  \hfill (6)

The close agreement between (6) (for $N = 2$ Skyrmions on $T^2$) and (5) (for $N = 1$ Skyrmions on $S^2$) is remarkable.

Let us now consider the zero-density limit $L \to \infty$. The only known static solutions on $\mathbb{R}^2$ are rotationally-symmetric rings \cite{22}. The energy of these goes like $E \approx \beta + \gamma/N^2$, where $\beta$ and $\gamma$ are constants. The significant fact here is that $\beta$ is lower than $E_{\text{crys}}$; this has been checked numerically for $\alpha \leq 1$. For example, for $\alpha = 0.1$, we get $\beta = 1.082$, compared with $E_{\text{crys}} = 1.088$. The implication of this is that, unlike the case for three-dimensional Skyrmions, isolated crystal chunks are not energetically favourable, even for large values of $N$. In fact, it seems likely that the
minimum-energy configuration on $\mathbb{R}^2$ is, for all values of $N$, a single $O(2)$-symmetric ring (other local minima may, however, exist). In this respect, the system is different from the three-dimensional Skyrme model.

4 The System $V = (1 - \phi_3^2)(1 - \phi_1^2)$

The motivation for the potential $V$ studied in this section comes from requiring that the Bogomolny bound be saturated by a doubly-periodic solution. As a consequence of this, the energy of the corresponding Skyrme crystal will be as low as it possibly can be; and so an isolated multi-Skyrmion will, for $N$ large enough, take the form of a crystal chunk rather than a ring-like configuration. In this respect, it will be a better analogue of the three-dimensional Skyrme system.

Our assumption, therefore, is that the system should admit a solution $W(z)$ which is an elliptic function. One particularly simple choice is to take $W(z) = 2\wp(z)$, where $\wp$ is the Weierstrass $p$-function with parameters $g_2 = 1$ and $g_3 = 0$. So the function $F$ appearing in (4) is given by $F(W)^2 = 2W(W^2 - 1)$; and the corresponding potential function is easily seen to be

$$V(\bar{\phi}) = 16 (1 - \phi_3^2)(1 - \phi_1^2).$$

(7)

For the rest of the section we shall use this $V$.

The fundamental cell of the lattice is a square with sides of length $L_0 = 2 K(0.5) = 3.708$, where $K(m)$ is the complete elliptic integral of the first kind with parameter $m$. This cell contains two units of topological charge (that is, the map from $T^2$ to $S^2$ has winding number $N = 2$); the distribution of charge- and energy-density indicates that it should be thought of as a square lattice of half-Skyrmions. It has the symmetry of type $A$ defined in the previous section. If we scale the solution to fit a lattice with edge-length $L$, then we get a solution with normalized energy

$$E(L) = 1 + 2.3729 \times \frac{\alpha}{2} \left( \frac{L}{L_0} + \frac{L_0}{L} \right).$$

(8)

If $L = L_0$, the solution saturates the lower bound, which in this case is $E \geq 1 + 2.3729 \alpha$.

For large $L$ (low density) one expects that this solution (a scaled Weierstrass function) will no longer be the minimal-energy doubly-periodic $N = 2$ solution. In fact, the minimal solution will correspond to an isolated $N = 2$ lump at some point on $T^2$ — and so will have $\langle \phi_3 \rangle \neq 0$. The result of a numerical investigation of this phenomenon is presented in Figure 2; it was obtained by finding local minima of the energy with $N = 2$ for a range of values of the periodicity $L$. The value of
\[ \alpha \text{ was taken to be } \alpha = 0.1. \] For \( L < 5.15 \), there is only one solution, namely the Weierstrass-type one with \( \langle \phi_3 \rangle = 0 \). Its energy grows fairly rapidly with \( L \), but it remains a stable solution (i.e., a local minimum), at least up to \( L = 20 \). For \( L \geq 5.15 \), another solution (local minimum) appears — a spatially-localized one with \( \langle \phi_3 \rangle \neq 0 \). Its energy grows less rapidly with \( L \), and for \( L > 5.73 \) it is the lowest-energy solution. In this sense, there is a phase transition at \( L_1 = 5.73 \), corresponding to the critical density \( \rho_c = 2/5.73^2 = 0.061 \). It would be interesting to study this transition in other ways, for example by examining the system on a sphere (as in \[23\] \[15\]) or by Monte-Carlo simulations at finite temperature (as in \[21\]).

Let us now look at the \( L \to \infty \) limit, and consider finite-\( N \) solutions on the infinite plane \( \mathbb{R}^2 \), with the boundary condition \( \phi_3 = 1 \) at spatial infinity. First, the system admits ring-like solutions. For these, we have \( \phi_3 = -1 \) at a single point \( O \), and \( \phi_3 = 0 \) on a deformed ring centred at \( O \). Because of the factor \((1 - \phi_1^2)\) in (7), this ring is not quite rotationally-symmetric about \( O \); its symmetry group is a subgroup of the rotation group \( \text{O}(2) \), namely the dihedral group \( D_{2N} \). In other words, these solutions resemble polygonal rings, analogous to the polygonal shells of the three-dimensional Skyrme system. A numerical investigation (for the case \( \alpha = 0.1 \)) reveals

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Energy \( E \) for the two \( N = 2 \) solutions.}
\end{figure}
that their normalized energy $E$ is very well fitted by the function

$$E = 1.262 + 0.227/N^2$$

(9)

for $N \geq 2$. For example, the energy of the $N = 8$ ring is $E = 1.266$; for comparison, the topological lower bound is $E = 1.2373$. This $N = 8$ ring is depicted in the left-hand plot of Figure 3; it is a stable static solution with $D_{16}$ symmetry, and is best thought of as a ring of sixteen half-Skyrmions.

![Figure 3: Contour plots of $\phi_2$ for two static $N = 8$ solutions of the system (7).](image)

The other localized solution illustrated in Figure 3 corresponds to a clump of the Skyrme crystal — in the right-hand plot, we see a $2 \times 2$ clump of crystal, again containing sixteen half-Skyrmions. The energy of such a clump is approximately equal to the (bulk) energy of the crystal plus a surface contribution. If the circumference of the clump has length $M L_0$, where $M$ is a positive integer, then one expects that the normalized energy will have the form

$$E \approx E_0 + M \delta E / N$$

(10)

where $E_0 = 1 + 2.373 \alpha$ and where $\delta E$ is the contribution from a single edge of lattice-cell. One may estimate $\delta E$ as follows (this is analogous to the discussion in [9]). Start with a single cell, with coordinate range $0 \leq x \leq L_0$, $0 \leq y \leq L_0$. Now stretch the cell in the positive $x$-direction, so that the $x$-range becomes $0 \leq x < \infty$. The boundary conditions are that the field $\vec{\phi}$ remains periodic in the $y$-direction, maintains its original lattice-configuration on the edge $x = 0$, and tends to its asymptotic value $(0,0,1)$ as $x \to \infty$. We can then relax this configuration, minimizing its energy numerically to obtain a minimum energy $E_1$. The deformed cell still contains two
units of Skyrme charge, so the resulting estimate for $\delta E$ is

$$\delta E \approx E_1 - 2E_0.$$  \hfill (11)

For example, if $\alpha = 0.1$, one gets $\delta E = 0.026$. Consequently, our estimate for the energy of a localized square lattice clump of charge $N$, with $\alpha = 0.1$, is

$$E \approx E_0 + 2\sqrt{2}\delta E/\sqrt{N} = 1.237 + 0.074/\sqrt{N};$$  \hfill (12)

but for low $N$ this will be an underestimate, since there are also corner effects to be considered. These corner effects presumably give an additional contribution to $E$ of the form $C/N$, where $C$ is a constant.

If we take a single lattice cell ($N = 2$), then it resembles a square array of four half-Skyrmions, but this amounts to the same thing as a ring of four half-Skyrmions — indeed, the $N = 2$ solution on $\mathbb{R}^2$ has a unique shape. For $N = 8$, however, there is the possibility of the two solutions depicted in Figure 3, and each of these is in fact realized, as a local minimum of the energy. The formulas (12) and (9) suggest that the energy of the $N = 8$ ring should be higher than that of the square clump, by an amount $\Delta E = 0.0027$. The numerical result is that $\Delta E = -0.0035$, which gives an estimate for the corner contribution which has to be added to (12), namely $0.05/N$.

For larger crystal chunks, the ring will be less favourable. For example, we expect, from (9) and the modified version of (12), that the minimal-energy $N = 18$ solution will correspond to a $6 \times 6$ square array of half-Skyrmions; and in particular, that the energy of the $N = 18$ ring will be higher than that of the $6 \times 6$ clump by an amount $\Delta E = 0.0057$. It should be emphasized, however, that there are likely to be many local minima, of which the square chunk and the ring are just two examples. All this is very similar to the situation for the three-dimensional Skyrme system.

In conclusion, this particular planar system, in addition to admitting an explicit crystal solution which saturates the topological lower bound on the energy, is a good model for three-dimensional Skyrmions.

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