A formula by $LDLT^T$ decomposition for the minimal type-I seesaw mechanism and conditions of $CP$ symmetry in an arbitrary basis

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In this paper, defining a formula by $LDLT^T$ decomposition for the minimal type-I seesaw mechanism, we obtain conditions of $CP$ symmetry for the neutrino mass matrix $m$ in an arbitrary basis. The conditions are found to be $\text{Re} \ (M_{22}a_i - M_{12}b_i) \text{Im} \ (M_{22}a_j - M_{12}b_j) = -\det M \text{Re} b_i \text{Im} b_j$ or $= -\det M \text{Im} b_i \text{Re} b_j$ for the Yukawa matrix $Y_{ij} = (a_j, b_j)$ and the right-handed neutrino mass matrix $M_{ij}$. In other words, the real or imaginary part of $b_i$ must be proportional to the real or imaginary part of the quantity $(M_{22}a_i - M_{12}b_i)$.

\textbf{I. INTRODUCTION}

$CP$ violation (CPV) in the neutrino oscillation has been strongly suggested by T2K \cite{1} and NO\nu A \cite{2}. For this reason, CPV in the lepton sector has been widely studied using the seesaw mechanism \cite{3,4}. In the analysis of seesaw relations, it is common to reduce parameters by diagonalization and/or phase redefinition \cite{5,6}. However, such a representation obscures symmetries and relationships of the original Lagrangian. Information of $CP$ phases is often lost in such parameterization, even if many of phases are non-physical. It would be somewhat important to investigate structures of $CP$ phases in the Lagrangian without redefinitions. In this paper, we analyze conditions of $CP$ symmetry for the light neutrino mass matrix $m$ defining a formula by $LDLT^T$ decomposition \cite{9} for the minimal type-I seesaw mechanism \cite{10,32}. As a result, conditions are obtained for Lagrangian parameters in an arbitrary basis. Furthermore, we discuss relationships between the obtained solution and generalized $CP$ symmetry (GCP) \cite{33,59}.

This paper is organized as follows. The next section gives a formula for the minimal type-I seesaw mechanism. In Sec. 3, we discuss conditions for $CP$ symmetry in an arbitrary basis. In
Sec. 4, some relationships between the $CP$-invariant conditions and GCP are discussed. The final section is devoted to a summary.

II. A FORMULA FOR THE MINIMAL TYPE-I SEESAW MECHANISM

In the beginning, we define a formula by $LDLT^T$ decomposition [3] for the minimal type-I seesaw mechanism [10–13]. By setting the vacuum expectation value of Higgs to one, the two mass matrices of neutrinos are defined as follows,

$$
Y = \begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3
\end{pmatrix}
\equiv
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
b_3
\end{pmatrix},
\quad
M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\equiv
\begin{pmatrix}
M_1 \\
M_2
\end{pmatrix}.
$$

(1)

Here, two-dimensional complex row vectors $(Y_i)_j \equiv (Y_{ij})$ and $(M_i)_j \equiv M_{ij}$ have mass dimension one. Let us consider a case where $M$ and its eigenvalues $M_i$ are hierarchical;

$$
|M_{22}| \gg |M_{12}|, |M_{11}|, \quad M_2 \simeq M_{22} + \frac{M_{12}^2}{M_{22}}, \quad M_1 \simeq \frac{\det M}{M_2}.
$$

(2)

As in the case of the type-I seesaw mechanism, we perform an approximate spectral decomposition for $M^{-1}$,

$$
M^{-1} = \frac{1}{\det M}
\begin{pmatrix}
M_{22} & -M_{12} \\
-M_{12} & M_{11}
\end{pmatrix}
= \frac{1}{\det M}
\begin{pmatrix}
M_{22} & -M_{12} \\
-M_{12} & \frac{M_{12}^2}{M_{22}}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 \\
0 & 1/M_{22}
\end{pmatrix}
= M^{(1)} + M^{(2)}.
$$

(3)

(4)

The eigenvalues$^1$ of the $M^{(1,2)}$ are $(M_{22}^2 + M_{12}^2)/M_{22}$ det $M$ and $M_{22}^{-1}$, and it corresponds to the first-order perturbation of the spectral decomposition.

The mass matrix of the light neutrinos $m$ is found to be

$$
m = Y(M^{(1)} + M^{(2)})Y^T \equiv m^{(1)} + m^{(2)}
$$

(5)

$$
= \frac{M_{22}}{\det M}
\begin{pmatrix}
\tilde{a}_1^2 & \tilde{a}_1\tilde{a}_2 & \tilde{a}_1\tilde{a}_3 \\
\tilde{a}_1\tilde{a}_2 & \tilde{a}_2^2 & \tilde{a}_2\tilde{a}_3 \\
\tilde{a}_1\tilde{a}_3 & \tilde{a}_2\tilde{a}_3 & \tilde{a}_3^2
\end{pmatrix}
+ \frac{1}{M_{22}}
\begin{pmatrix}
b_1^2 & b_1b_2 & b_1b_3 \\
b_1b_2 & b_2^2 & b_2b_3 \\
b_1b_3 & b_2b_3 & b_3^2
\end{pmatrix}.
$$

(6)

$^1$ The modulus of eigenvalues correspond to singular values of the $M^{(1,2)}$, $(|M_{22}|^2 + |M_{12}|^2)/|M_{22} \det M|$ and $|M_{22}|^{-1}$ in the limit of $M_{12}/M_{22} \rightarrow 0$. 

where
\[ \tilde{a}_i \equiv \frac{\text{det}(Y_i, M_2)}{M_{22}} = a_i - b_i \frac{M_{12}}{M_{22}}. \] (7)

This is equivalent to a formula for the type-I seesaw mechanism by setting \( M_{33} \to \infty \). Since no approximation is used obviously, this formula is valid in any basis. Therefore, it can be useful for various analyses, such as flavor symmetries, GCPs, and fine-tunings of the seesaw mechanism.

For a unitary matrix \( U \) diagonalizing \( m \) as \( U^T m U = m^{\text{diag}} \), the values of neutrino masses are expressed without radical symbols (\( \sqrt{\cdot} \));
\[ m_i = \frac{M_{22}}{|M|} (\tilde{a} \cdot u_i)^2 + \frac{1}{M_{22}} (b \cdot u_i). \] (8)

Here, \( |M| \equiv \text{det} M \), \( \tilde{a}_i \equiv \tilde{a}_i \), \( b_i \equiv b_i \), \( (u_i)_j \equiv U_{ji} \) are three-dimensional vectors, and \( (u \cdot v) \equiv \sum_{j=1}^{3} u_j v_j \) is the inner product without Hermitian conjugation.

A deformed Yukawa matrix \( \tilde{Y} \) is defined by
\[ \tilde{Y} \equiv \begin{pmatrix} \tilde{a} & b \end{pmatrix} \equiv (a - b \frac{M_{12}}{M_{22}}, b) = Y \begin{pmatrix} 1 & 0 \\ -\frac{M_{12}}{M_{22}} & 1 \end{pmatrix} \equiv Y L, \] (9)

where \( L \) is a lower unitriangular matrix
\[ L = \begin{pmatrix} 1 & 0 \\ -\frac{M_{12}}{M_{22}} & 1 \end{pmatrix}, \quad L^{-1} = \begin{pmatrix} 1 & 0 \\ \frac{M_{12}}{M_{22}} & 1 \end{pmatrix}, \] (10)

that has all the diagonal entries equal to one. The mass matrix of heavy neutrinos \( M \) is diagonalized by \( L \) as
\[ \tilde{M}^{-1} = L^{-1} M^{-1} (L^{-1})^T = \begin{pmatrix} \frac{M_{22}}{\text{det} M} & 0 \\ 0 & \frac{1}{M_{22}} \end{pmatrix}, \] (11)

and \( m = \tilde{Y} \tilde{M}^{-1} \tilde{Y}^T \) holds. This is called the \( LDL^T \) (or generalized Cholesky) decomposition of a symmetric matrix.

This formula can be regarded as an extension of the natural representation [8]. Since it is possible to reverse the diagonalization by \( L \) using only \( M_{12}/M_{22} \), one advantage is that the information of Lagrangian includes \( CP \) phases is treated without solving quadratic equations.
III. CONDITIONS FOR CP SYMMETRY IN AN ARBITRARY BASIS

In this section, using the formula (6), we investigate conditions for CP symmetry for the minimal type-I seesaw mechanism. First, Eq. (6) is rewritten as

$$m = \frac{M_{22}}{|M|} \tilde{a} \otimes \tilde{a}^T + \frac{1}{M_{22}} b \otimes b^T.$$  \hspace{1cm} (12)

By redefining phases of the right-handed neutrinos, we can choose a basis in which $M_{22}$ and $|M|$ are real-positive. It is easy to incorporate these overall phases into the final result by inverse redefinitions.

If the matrix $m$ satisfies the CP symmetry, each imaginary part must cancel in Eq. (12);

$$\text{Im} m_{ij} = \frac{M_{22}}{|M|} \text{Im}(\tilde{a}_i \tilde{a}_j) + \frac{1}{M_{22}} \text{Im}(b_i b_j) = 0.$$  \hspace{1cm} (13)

By separating the imaginary parts of products,

$$\frac{M_{22}}{|M|} (\text{Re} \tilde{a}_i \text{Im} \tilde{a}_j + \text{Im} \tilde{a}_i \text{Re} \tilde{a}_j) + \frac{1}{M_{22}} (\text{Re} b_i \text{Im} b_j + \text{Im} b_i \text{Re} b_j) = 0.$$  \hspace{1cm} (14)

Furthermore, by considering cross products with $\text{Im} b_i$ and $\text{Im} b_j$ for $i$ and $j$ components, the term containing $1/M_{22}$ becomes zero;

$$\frac{M_{22}}{|M|} [(\text{Im} b \times \text{Re} \tilde{a}) \otimes (\text{Im} \tilde{a} \times \text{Im} b)^T + (\text{Im} b \times \text{Im} \tilde{a}) \otimes (\text{Re} \tilde{a} \times \text{Im} b)^T] = 0 \otimes 0.$$  \hspace{1cm} (15)

This is an antisymmetric condition for a matrix. However, since the only antisymmetric matrix with a rank less than one is the zero matrix, we obtain

$$\text{Im} b \times \text{Re} \tilde{a} = 0 \quad \text{or} \quad \text{Im} \tilde{a} \times \text{Im} b = 0, \quad \text{Re} \tilde{a} \propto \text{Im} b \quad \text{or} \quad \text{Im} \tilde{a} \propto \text{Im} b.$$  \hspace{1cm} (16)

Similarly, $\text{Re} \tilde{a} \propto \text{Re} b$ or $\text{Im} \tilde{a} \propto \text{Re} b$ can be shown. As a result, the conditions that $m$ is CP-invariant are equivalent to

$$M_{22}^2 \text{Re} \tilde{a}_i \text{Im} \tilde{a}_j = -|M| \text{Re} b_i \text{Im} b_j \quad \text{or} \quad -|M| \text{Im} b_i \text{Re} b_j.$$  \hspace{1cm} (17)

Thus, the real and imaginary parts of $\tilde{a}_i$ are proportional to those of $b_i$, and their coefficients are determined by Eq. (17). At first glance, these conditions appear to give nine constraints. However, since a matrix $A \equiv \text{Re} \tilde{a}_i \text{Im} \tilde{a}_j$ with rank one satisfies $A_{ij}A_{jj} = A_{ij}A_{ji}$, it is necessary and sufficient to give $A_{11}$ and $A_{j1}$. Therefore, there are only five independent constraints.
By the conditions (17), we can express \( \text{Re} \tilde{a} \) and \( \text{Im} \tilde{a} \) for given \( \text{Re} b \) and \( \text{Im} b \):

\[
(M_{22} \text{Re} \tilde{a}, M_{22} \text{Im} \tilde{a}) = (r \sqrt{|M|} \text{Re} b, -\frac{1}{r} \sqrt{|M|} \text{Im} b) \quad \text{or} \quad (r' \sqrt{|M|} \text{Im} b, -\frac{1}{r'} \sqrt{|M|} \text{Re} b).
\]

(18)

Here, \( r \) and \( r' \) are real constants defined by non-zero \( \text{Im} \tilde{a}_i \) and \( \text{Re} b_j \) or \( \text{Im} b_j \),

\[
r = \frac{M_{22} \text{Re} \tilde{a}_j}{\sqrt{|M|} \text{Re} b_j} = \frac{\sqrt{|M|} \text{Im} b_i}{M_{22} \text{Im} \tilde{a}_i}, \quad r' = \frac{M_{22} \text{Re} \tilde{a}_i}{\sqrt{|M|} \text{Im} b_i} = -\frac{\sqrt{|M|} \text{Re} b_j}{M_{22} \text{Im} \tilde{a}_j}.
\]

(19)

The other solution corresponds to \(-i\tilde{a}^*\) for a given solution \( \tilde{a} \).

\( \text{Re} \tilde{a} = 0 \) or \( \text{Im} \tilde{a} = 0 \) are special cases where the denominators of \( r(t) \) or \( 1/r(t) \) cannot be defined and the magnitudes of \( \tilde{a}_i \) and \( b_i \) are unconstrained. In such cases, the elements of \( \tilde{Y} \) are real or pure imaginal;

\[
\tilde{Y} = (\tilde{a}_i, b_i) = \{(x_i, y_i), (x_i, iy_i), (ix_i, y_i), (ix_i, iy_i)\},
\]

(20)

with real parameters \( x_i, y_i \in \mathbb{R} \). In each case, \( \tilde{Y} \) has a (generalized) \( CP \) symmetry;

\[
\tilde{Y}^* = \{\tilde{Y}, \tilde{Y} \sigma_3, -\tilde{Y} \sigma_3, -\tilde{Y}\},
\]

(21)

where \( \sigma_3 \equiv \text{diag}(1, -1) \).

For example, if \( \tilde{Y} \) has the canonical \( CP \) and \( \text{Im} \tilde{a}_i = \text{Im} b_i = 0 \) holds, the relation \( a_i = \tilde{a}_i + M_{12} b_i \) leads to

\[
\text{Re} a_i = \text{Re} \tilde{a}_i + \frac{M_{12}}{M_{22}} \text{Re} b_i,
\]

(22)

\[
\text{Im} a_i = \frac{M_{12}}{M_{22}} \text{Re} b_i,
\]

(23)

for any \( i \). In other words, the imaginary part of \( a \) and the real part of \( b \) are required to be proportional. If \( \text{Im} M_{12} = 0, \text{Im} M = 0 \) holds since we chose \( \text{Im} M_{22} = \text{Im} |M| = 0 \) in this basis. It leads to \( \text{Im} a_i = 0 \) and \( \text{Im} Y = 0, \text{i.e.} \), the neutrino sectors \( M \) and \( Y \) are \( CP \)-invariant. However, Eq. (23) shows the existence of other nontrivial solutions. They can be regarded as a kind of alignment conditions that are required for the naturality of the seesaw mechanism [60].

For the remaining three cases, similar alignments holds between \( \text{Re} a_i, \text{Im} a_i \) and \( \text{Re} b_i, \text{Im} b_i \).
A. Understanding from the original raw formula

We can understand the result (17) without LDLT decomposition. By adjusting coefficients in Eq. (17) and rewriting \( \tilde{a} \) to \( a \),

\[
\text{Re} (M_{22} a_i - M_{12} b_i) \text{Im} (M_{22} a_j - M_{12} b_j) = -|M| \text{Re} b_i \text{Im} b_j \quad \text{or} \quad -|M| \text{Im} b_i \text{Re} b_j. \tag{24}
\]

With some transformation, we obtain

\[
\text{Im} [(a_i M_{22} - b_i M_{12})a_j] = -\text{Im} [(b_i M_{11} - a_i M_{12})b_j]. \tag{25}
\]

This is equivalent to the condition \( m = m^* \) in the original raw formula. From Eq. (1), the mass matrix \( m \) is written as

\[
m = YM^{-1}Y^T = \frac{1}{|M|} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix} \begin{pmatrix} -(M_2 \times Y_1)_z & -(M_2 \times Y_2)_z & -(M_2 \times Y_3)_z \\ (M_1 \times Y_1)_z & (M_1 \times Y_2)_z & (M_1 \times Y_3)_z \end{pmatrix}, \tag{26}
\]

\[
m_{ij} = \frac{1}{|M|} [-a_i(M_{21}b_j - M_{22}a_j) + b_i(M_{11}b_j - M_{12}a_j)], \tag{27}
\]

where \((M_i \times Y_j)_z \equiv M_{i1}Y_{j2} - M_{i2}Y_{j1}\). Since the CP-invariance requires cancellation of the imaginary parts between the two terms, Eq. (25) is obtained as the conditions.

IV. UNDERSTANDING FROM GENERALIZED CP SYMMETRY

The solution (18) can also be understood from GCP. Although some results in this section are well known, an analysis is performed for confirmation of the conditions. By defining \( X \equiv \tilde{Y} \sqrt{M}^{-1} \), \( m \) is written only in \( X \);

\[
X = \left( \sqrt{\frac{M_{22}}{|M|}} \tilde{a} \right) \left( \sqrt{\frac{1}{M_{22}}} b \right), \quad m = XX^T. \tag{28}
\]

In order for \( m \) to have CP symmetry, \( X = (u, v) \) must satisfy the following GCP with a complex orthogonal matrix \( O \);

\[
X^*O = \pm X = \pm(u, v). \tag{29}
\]

Since \( XO^* = \pm X^*O = X \) holds from a conjugation of Eq. (29), the matrix \( O \) satisfies \( O^* = O^{-1} = O^T \) and \( O = O^1 \). In other words, \( O \) is a Hermitian orthogonal matrix. The
situation is divided into two cases with det \( O = \pm 1 \);

\[
O = \begin{pmatrix}
\cosh x & -i \sinh x \\
i \sinh x & \cosh x
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
\cos x & \sin x \\
\sin x & -\cos x
\end{pmatrix},
\quad (30)
\]

where \( x \) is a real parameter.

When \( O \) is real, the GCP relation is

\[
X^*O = (u^*, v^*) \begin{pmatrix}
\cos x & \sin x \\
\sin x & -\cos x
\end{pmatrix} = (c_x u^* + s_x v^*, s_x u^* - c_x v^*) = \pm (u, v) = \pm X.
\quad (31)
\]

where \( c_x \equiv \cos x, s_x \equiv \sin x \). From this, \( u^* \) and \( u \) can be solved for \( v \) and \( v^* \);

\[
u^* = \frac{1}{s_x} (\pm v + c_x v^*), \quad u = \pm \frac{c_x}{s_x} (\pm v + c_x v^*) \pm s_x v^* = \frac{c_x v}{s_x} \pm \frac{1}{s_x} v^*.
\quad (32)
\]

By rewriting them with the real and imaginary parts of \( u \) and \( v \),

\[
\sqrt{\frac{M_{22}}{|M|}} \text{Re} \hat{a} = \text{Re} u = \frac{c_x \pm 1}{s_x} \text{Re} v = \{\cot \frac{x}{2}, -\tan \frac{x}{2}\} \sqrt{\frac{1}{M_{22}}} \text{Re} b,
\quad (33)
\]

\[
\sqrt{\frac{M_{22}}{|M|}} \text{Im} \hat{a} = \text{Im} u = \frac{c_x \mp 1}{s_x} \text{Im} v = \{-\tan \frac{x}{2}, \cot \frac{x}{2}\} \sqrt{\frac{1}{M_{22}}} \text{Im} b.
\quad (34)
\]

Obviously, these coefficients satisfy

\[
c_x \pm 1 \times \frac{c_x \mp 1}{s_x} = -1,
\quad (35)
\]

and it corresponds to the first solution of Eq. (18). The signs \( \pm \) come from the parity for GCP (29) and it interchanges \( r \) and \( 1/r \).

The other solution with \( |O| = 1 \) satisfies

\[
X^*O = (u^*, v^*) \begin{pmatrix}
\cosh x & -i \sinh x \\
i \sinh x & \cosh x
\end{pmatrix} = (ch_x u^* + ish_x v^*, -ish_x u^* + ch_x v^*) = \pm (u, v) = \pm X,
\quad (36)
\]

where \( ch_x \equiv \cosh x, sh_x \equiv \sinh x \). A similar calculation yields

\[
\sqrt{\frac{M_{22}}{|M|}} \text{Re} \hat{a} = \text{Re} u = \frac{-ch_x \mp 1}{sh_x} \text{Im} v = \frac{-ch_x \mp 1}{sh_x} \sqrt{\frac{1}{M_{22}}} \text{Im} b,
\quad (37)
\]

\[
\sqrt{\frac{M_{22}}{|M|}} \text{Im} \hat{a} = \text{Im} u = \frac{ch_x \mp 1}{sh_x} \text{Re} v = \frac{ch_x \mp 1}{sh_x} \sqrt{\frac{1}{M_{22}}} \text{Re} b.
\quad (38)
\]
Since the product of the two coefficients becomes unity,

\[
\frac{-\sinh x \pm 1}{\cosh x} \times \frac{\cosh x \mp 1}{\sinh x} = \frac{-\cosh^2 x + 1}{\sinh^2 x} = -1,
\]

it corresponds to the second solution of Eq. (18).

These solutions can also be understood by matrices. For \(|O| = +1\), we can write

\[
1 \pm OT = (1 \pm c_z)1_2 \pm is_z \sigma_2,
\]

where \(\sigma_i\) are the Pauli matrices. From \(\sigma_3^2 = 1\), a certain relation exists between \(1 \pm OT\);

\[
(1 \pm OT) \frac{1 \pm c_z}{s_z} = s_z1_2 \pm i(1 \mp c_z)\sigma_2 = \pm (1 \mp OT)i\sigma_2.
\]

From \(X^* = \pm XO^T\), it leads to a relation between the real and imaginary parts of \(X\),

\[
\text{Re } X = X \frac{1 \pm OT}{2} = \pm \frac{s_z}{1 \mp c_z} X \frac{1 \mp OT}{2} i\sigma_2 = \pm \frac{s_z}{1 \mp c_z} i \text{Im } X \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

In other words,

\[
\begin{pmatrix} \text{Re } \tilde{a} & \text{Re } b \\ -\text{Im } b & \text{Im } \tilde{a} \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\tilde{M}}{|M|}} \\ 0 \end{pmatrix} = \frac{sh_x}{ch_x + 1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{\tilde{M}}{|M|}} \\ 0 \end{pmatrix}.
\]

and it comes down to the solutions (37) and (38).

For the case \(|O| = -1\), since \(1 \pm OT\) are singular matrices with \(\det(1 \mp OT) = 0\), we consider \(1 \pm OT \sigma_3\) instead. This extra \(\sigma_3\) exchanges \(\text{Re } b\) and \(\text{Im } b\). For example, in the case of \(X^* = XO^T\),

\[
X(1 + OT \sigma_3) = X + X^* \sigma_3 = 2(\text{Re } \tilde{a}, i \text{Im } b)\sqrt{\tilde{M}^{-1}},
\]

\[
X(1 - OT \sigma_3) = X - X^* \sigma_3 = 2(i \text{Im } \tilde{a}, \text{Re } b)\sqrt{\tilde{M}^{-1}}.
\]

A similar relationship as Eq. (40) holds for \(1 \pm OT \sigma_3\),

\[
1 \pm OT \sigma_3 = 1_2(1 \pm c_x) \mp i\sigma_2 s_x.
\]

From this,

\[
(1 \pm OT \sigma_3) \frac{1 \pm c_x}{s_x} = 1_2 s_x \mp i\sigma_2 (1 \mp c_x) = \mp (1 \mp OT \sigma_3)i\sigma_2.
\]
For example, the solution of the upper sign leads to Eqs. (13) and (14),
\[
(\text{Re} \, \tilde{a}, i\text{Im} \, b) \sqrt{\tilde{M}^{-1}} = \frac{-s_x}{1 - c_x} (i\text{Im} \, \tilde{a}, \text{Re} \, b) \sqrt{\tilde{M}^{-1}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{s_x}{1 - c_x} (\sqrt{\frac{1}{M_{22}}} \text{Re} \, b, -i \sqrt{\frac{M_{22}}{|M|}} \text{Im} \, \tilde{a}) .
\]
(48)

The symmetric conditions (17) satisfied by these solutions can be displayed in matrices as well. When $|O| = -1$, the condition $M_{22}^2 \text{Re} \, \tilde{a}_i \text{Im} \, \tilde{a}_j = -|M| \text{Re} \, b_i \text{Im} \, b_j$ is obviously equivalent to $\text{Re} \, X \text{Im} \, X^T = 0$. This condition can be rewritten as
\[
\text{Re} \, X \text{Im} \, X^T = \frac{1}{4i} X (1 \pm O^T)(1 \mp O^T) X^T = \frac{1}{4i} X (O^T - O) X^T = 0 .
\]
(49)

Since $O^T = O$ holds, it is a solution.

On the other hand, when $|O| = +1$, a similar trick as Eqs. (11) and (12) leads to a condition
\[
\frac{1}{4i} X (1 + O^T \sigma_3)(1 - O^T \sigma_3) X^T = (\text{Re} \, \tilde{a} , i\text{Im} \, b) \tilde{M}^{-1} (i\text{Im} \, \tilde{a}, \text{Re} \, b) = 0 ,
\]
(50)
that is equivalent to $M_{22}^2 \text{Re} \, \tilde{a}_i \text{Im} \, \tilde{a}_j = -|M| \text{Im} \, b_i \text{Re} \, b_j$. This can be rewritten as
\[
(1 + O^T \sigma_3)(1 - O^T \sigma_3)^T = O^T \sigma_3 - \sigma_3 O = 0 .
\]
(51)
In this case, it is still a solution because $\sigma_3 O^T \sigma_3 = O$ holds.

By understanding the solution $X^* O = \pm X$ as a product of matrices, the form of $\tilde{Y}$ and $Y$ can be specified. First, we show $O_0 \equiv Q^\dagger Q$ and $O_1 \equiv Q^\dagger \sigma_3 Q$ with a orthogonal matrix $Q$ are Hermitian orthogonal matrices. Specifically,
\[
Q \equiv \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix} \quad \Rightarrow \quad O_0 = Q^\dagger Q = \begin{pmatrix} \cos(w - w^*) & -\sin(w - w^*) \\ \sin(w - w^*) & \cos(w - w^*) \end{pmatrix} ,
\]
(52)
\[
O_1 = Q^\dagger \sigma_3 Q = \begin{pmatrix} \cos(w + w^*) & -\sin(w + w^*) \\ -\sin(w + w^*) & \cos(w + w^*) \end{pmatrix} .
\]
(53)
Indeed $O_{0,1}$ is Hermitian with $\det O_{0,1} = \pm 1$, and they agree with Eq. (20) by suitable redefinitions. The solution of GCP (29) is transformed as follows.
\[
X^* Q^\dagger = \pm X Q^T \quad \text{or} \quad X^* Q^\dagger = \pm X Q^T \sigma_3 .
\]
(54)
Then $X Q^T$ is one of the four trivial solutions in Eq. (21);
\[
X Q^T = \{ X_\pm , X_\pm' \}, \quad \{ X_\pm^* , X_\pm'^* \} = \{ \pm X_\pm , \pm X_\pm' \sigma_3 \} .
\]
(55)
From this $X = X \pm Q$ or $X = X' \pm Q$ holds. They certainly satisfy the GCP (23):

$$(X \pm Q)^* = \pm X \pm Q^* = \pm (X \pm Q) Q^T Q = \pm (X \pm Q) Q_0$$ \hspace{1cm} (56)

$$(X' \pm Q)^* = \pm X' \pm Q^* = \pm (X' \pm Q) Q^T Q = \pm (X' \pm Q) Q_1^*$$ \hspace{1cm} (57)

Since $X = X^{(l)} \pm Q$ holds, the Yukawa matrix is reconstructed as

$$\hat{Y} = X \sqrt{M} = X^{(l)} \pm Q \sqrt{M}, \quad Y = \hat{Y} L^{-1} = X^{(l)} \pm Q \sqrt{ML^{-1}}.$$ \hspace{1cm} (58)

Recalling that $L^{-1} M^{-1} (L^T)^{-1} = \tilde{M}^{-1}$ from Eq. (11), we finally obtain

$$m = X^{(l)} \pm Q \sqrt{ML^{-1}} = X^{(l)} \pm Q \sqrt{\tilde{M} L^{-1}},$$ \hspace{1cm} (59)

and $m$ is $CP$-invariant. Although such results may be well known, the main results of this paper are the formula (6) and Eqs. (17)-(18), the conditions of $CP$-invariance for $m$ in an arbitrary basis.

**V. SUMMARY**

In this paper, defining a formula by $LDL^T$ decomposition for the minimal type-I seesaw mechanism, we obtain conditions of $CP$-invariance for the neutrino mass matrix $m$ in an arbitrary basis. The conditions are found to be $\text{Re} (M_{22} a_i - M_{12} b_i) \text{Im} (M_{22} a_j - M_{12} b_j) = - \det M \text{Re} b_i \text{Im} b_j$ or $= - \det M \text{Im} b_i \text{Re} b_j$ for the Yukawa matrix $Y_{ij} = (a_j, b_j)$ and the right-handed neutrino mass matrix $M_{ij}$. In other words, the real or imaginary part of $b_i$ must be proportional to the real or imaginary part of the quantity $(M_{22} a_i - M_{12} b_i)$.

We also discussed the relevance of the generalized $CP$ symmetry that exists in such a solution. As a result, Yukawa matrix $Y$ found to be restricted to the form $Y = X^{(l)} \pm Q \sqrt{ML^{-1}}$ with $CP$-covariant Yukawa matrices $X^{(l)}$, a complex orthogonal matrix $Q$, the diagonal right-handed neutrino mass matrix $\tilde{M}$ by the $LDL^T$ decomposition, and the unitriangular matrix $L$ at that time. This result can be extended to other seesaw mechanisms, GCPs, and $Z_2$ symmetries [83].
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