Orientations of 1-Factorizations and
the List Chromatic Index of Small Graphs

Uwe Schauz
Xi’an Jiaotong-Liverpool University, Suzhou 215123, China,
uwe.schauz@xjtlu.edu.cn

Abstract As starting point, we formulate a corollary to the Quantitative Combinatorial Nullstellensatz. This corollary does not require the consideration of any coefficients of polynomials, only evaluations of polynomial functions. In certain situations, our corollary is more directly applicable and more ready-to-go than the Combinatorial Nullstellensatz itself. It is also of interest from a numerical point of view. We use it to explain a well-known connection between the sign of 1-factorizations (edge colorings) and the List Edge Coloring Conjecture. For efficient calculations and a better understanding of the sign, we then introduce and characterize the sign of single 1-factors. We show that the product over all signs of all the 1-factors in a 1-factorization is the sign of that 1-factorization. Using this result in an algorithm, we attempt to prove the List Edge Coloring Conjecture for all graphs with up to 10 vertices. This leaves us with some exceptional cases that need to be attacked with other methods.

Keywords: combinatorial nullstellensatz, one-factorizations, edge colorings, list edge coloring conjecture, combinatorial algorithms

1 Introduction

Using the polynomial method, we prove the List Edge Coloring Conjecture\(^1\) for many small graphs \(G\). This means, if such a graph \(G\) can be edge colored with \(k\) colors (\(\chi'(G) \leq k\)), then it can also be edge colored if the color of each edge \(e\) has to be taken from an arbitrarily chosen individual list \(L_e\) of \(k\) colors (\(\chi'_\ell(G) \leq k\)). There are no restriction on the lists, apart from the given cardinality \(k\). So, in general, there are very many essentially different list assignments \(e \mapsto L_e\), and brute-force attempts to find one coloring from every system of lists are computationally impossible. A way out may be found in the Combinatorial Nullstellensatz, which seems to be one of our strongest tools. It can also be used for list coloring of the vertices of a graph (see [1]), but it becomes even more powerful if applied to edge colorings of regular graphs. Ellingham and Goddyn [3] used it to prove the List Edge Coloring Conjecture for regular planar graphs of class 1. As, by definition, the edges of a class 1

\(^1\) See [6, Section 12.20] for a discussion of the origins of this coloring conjecture.
graph \( G \) can be partitioned into \( \Delta(G) \) color classes, the regular class 1 graphs are precisely the 1-factorable graphs. 1-factorable graphs, as we call regular class 1 graphs from now on, are also the first target in the current paper, but our results have implications for other graphs as well. In our previous paper \[15\], we could already prove the List Edge Coloring Conjecture for infinitely many 1-factorable complete graphs. There, we used a group action in connection with the Combinatorial Nullstellensatz. Häggkvist and Jansson \[5\] could prove the conjecture for all complete graphs of class 2. Nobody, however, has a proof for \( K_{16} \), and 120 edges and 15 colors are completely out of reach for all known numeric methods, including the algorithms that we suggest here. That we cannot even prove the conjecture for all complete graphs shows how hard the problem is.

Before this background, it is surprising that Galvin could prove the conjecture for all bipartite graphs \[4\]. His proof does not use the Combinatorial Nullstellensatz, but the so-called kernel method. Other methods were also used by Kahn \[7\], who showed that the List Edge Coloring Conjecture holds asymptotically, in some sense. Moreover, most of the mentioned results can also be generalized to edge painting \[13,14\], an on-line version of list coloring that allows alterations of the lists during the coloration process.

This paper has three further sections, and an appendix containing our algorithm. In Section 2, we formulate a corollary to the Combinatorial Nullstellensatz that does not require the consideration of any coefficients of polynomials, only evaluations of polynomial functions. There, we also explain a well-known connections between the sum of the signs over all 1-factorizations (edge colorings) of a graph and the List Edge Coloring Conjecture. In Section 3, we then provide another characterization of the sign. We explain how this can be used to calculate the sum of the signs over all 1-factorizations more efficiently. In Section 4, we explain to which conclusions this approach and our computer experiments with graphs on up to 10 vertices led.

## 2 A Nullstellensatz for List Colorings

We start our investigations from the following coefficient formula \[12\]:

**Theorem 1 (Quantitative Combinatorial Nullstellensatz).**

Let \( L_1, L_2, \ldots, L_n \) be finite non-empty subsets of a field \( \mathbb{F} \), set \( L := L_1 \times L_2 \times \cdots \times L_n \) and define \( d := (d_1, d_2, \ldots, d_n) \) via \( d_j := |L_j| - 1 \). For polynomials

\[
P = \sum_{\delta \in \mathbb{N}^n} P_\delta x^\delta \in \mathbb{F}[x_1, \ldots, x_n]
\]

of total degree \( \deg(P) \leq d_1 + d_2 + \cdots + d_n \), we have

\[
P_d = \sum_{x \in L} N_L(x)^{-1} P(x),
\]

where \( N_L(x) = N_L(x_1, \ldots, x_n) := \prod_j N_{L_j}(x_j) \) with \( N_{L_j}(x_j) := \prod_{\xi \in L_j \setminus x_j} (x_j - \xi) \neq 0 \).

In particular, if \( \deg(P) \leq d_1 + d_2 + \cdots + d_n \) then

\[
P_d \neq 0 \quad \Rightarrow \quad \exists x \in L : P(x) \neq 0.
\]
The implication in the second part is known as Alon’s Combinatorial Nullstellensatz\cite{Alon99}. The coefficient $P_d$ seems to play a central role in the Combinatorial Nullstellensatz, but it is not really important in various applications. One may get a wrong impression form the fact that $P_d$ is assumed as non-zero in that implication. There are applications of the theorem if the total degree $\deg(P)$ is strictly smaller than $d_1 + d_2 + \cdots + d_n$, and thus $P_d = 0$. If $P_d = 0$, then it cannot be that only one summand in the sum in that theorem is non-zero, and this means that there cannot be only one solution to the problem that was modeled by $P$. So, if there exist a solution, say a trivial solution, than there must also be a second solution, a non-trivial solution. This is a very elegant line of reasoning, and it does not require us to look at the coefficient $P_d$ at all. It is enough to know that the total degree is smaller than $d_1 + d_2 + \cdots + d_n$ and that there is a single trivial solution. Beyond that, the theorem can also be used to prove the existence of solutions to problems that do not have a trivial solution, for example the existence of a list coloring of a graph. In these cases, looking at the “leading coefficient” $P_d$ appears to be unavoidable. However, to actually calculate $P_d$, usually, the best idea is to use the Quantitative Combinatorial Nullstellensatz again, just with changed lists $L_j$. In fact, the polynomial $P$ can be changed, too, as long as the “leading coefficient” is not altered. So, theoretically, we can calculate $P_d$ by applying the theorem to modified lists $\tilde{L}_j$ and a modified polynomial $\tilde{P}$. Afterwards, the theorem can then be applied a second time, to $P$ and the original lists $L_j$, in order to prove the existence of a certain object. In this process, the coefficient $P_d$ stands in the middle, playing a crucial role. The coefficient $P_d$, however, does not appear in the initial setting and also not in the final conclusion. Therefore, it must be possible to formulate a all-in-one ready-to-go corollary in which $P_d$ does not occur. In providing that corollary, we free the user from the need to understand what $P_d$ is. Of course, in its most general form, there are two polynomials $P$ and $\tilde{P}$, and two list systems $L$ and $\tilde{L}$, which make that corollary look more technical, but it avoids mentioning $P_d$ and should be easier to apply in many situations:

**Corollary 1.** For $j = 1, 2, \ldots, n$, let $L_j$ and $\tilde{L}_j$ be finite non-empty subsets of a field $F$ with $|L_j| = |\tilde{L}_j|$. Let $N_L$ and $N_{\tilde{L}}$ be the corresponding coefficient functions over the cartesian products $L$ and $\tilde{L}$ of these sets. If two polynomials $P, \tilde{P} \in F[x_1, \ldots, x_n]$ of total degree at most $|L_1| + |L_2| + \cdots + |L_n| - n$ have the same homogenous component of degree $|L_1| + |L_2| + \cdots + |L_n| - n$ (or at least $P_d = P_{\tilde{d}}$), then

$$\sum_{x \in L} N_L(x)^{-1} \tilde{P}(x) = \sum_{x \in L} N_L(x)^{-1} P(x)$$

and, in particular\footnote{Also\cite[Th. 4.5]{Alon99}: $\sum_{x \in L} N_L(x)^{-1} \tilde{P}(x) \neq 0 \implies \exists x \in L: P(x) \neq 0$.} $\sum_{x \in \tilde{L}} N_{\tilde{L}}(x)^{-1} \tilde{P}(x) \neq 0 \implies \exists x \in L: P(x) \neq 0$.\footnote{Also\cite[Th. 4.5]{Alon99}: $\sum_{x \in L} N_L(x)^{-1} \tilde{P}(x) \neq 0 \implies P$ is $([L_1], \ldots, [L_n])$-paintable.}
We want to use this corollary to verify the existence of list colorings of graphs. Therefore, we apply the corollary to the edge distance polynomials $P_G$ of graphs $G$. The edge distance polynomial of a multi-graph $G$ on vertices $v_1, v_2, \ldots, v_n$ is a polynomial in the variables $x_1, x_2, \ldots, x_n$, with one variable $x_i$ for each vertex $v_i$. It is defined as the product over all differences $x_i - x_j$ with $v_i v_j \in E(G)$ and $i < j$, where the factor $x_i - x_j$ occurs as many times in $P$ as the edge $v_i v_j$ occurs in the multi-set $E(G)$. It is also called the graph polynomial and was introduced in [10]. We may view it as a polynomial over any field $F$. If $P_G$ is non-zero at a point $(x_1, x_2, \ldots, x_n)$ then the assignment $v_i \mapsto x_i$ is a proper vertex coloring of $G$. If the colors $x_i$ are supposed to lie in certain lists $L_i$ then the point $(x_1, x_2, \ldots, x_m)$ just has to be taken from the Cartesian product $L_1 \times L_2 \times \cdots \times L_m$. Here, we simply need to assume that the sets $L_i$ lie in $F$, or in an extension field of $F$. This is no restriction, as one can easily embed the color lists (and their full union $\bigcup_i L_i$) into any big enough field $F$.

We might just take $F = \mathbb{Q}$. With this ideas our corollary leads to the following more special result:

**Corollary 2.** Let $G$ be a multi-graph on the vertices $v_1, v_2, \ldots, v_n$. To each edge $e$, between any vertices $v_i$ and $v_j$ with $i < j$, choose a label $a_e$ in a field $F$ (possible $a_e = 0$) and associate the monomial $x_i - x_j - a_e$ to the edge $e$. Let $P$ be the product over all these monomials. For $j = 1, 2, \ldots, n$, let $L_j$ be a finite non-empty subset of $F$, and define $\ell = (\ell_1, \ell_2, \ldots, \ell_n)$ via $\ell_j := |L_j|$. If $|E(G)| \leq \ell_1 + \ell_2 + \cdots + \ell_n - n$ then

$$\sum_{x \in L} N_L(x)^{-1} P(x) \neq 0 \implies G \text{ is } \ell\text{-list colorable and } \ell\text{-paintable.}$$

In applications, one will often choose the $a_e$ as zero and take the lists $L_j$ all equal, but there are also examples where more complicated choices succeeded, as for example in the proof of the last lemma in [15]. Things can be further simplified if we examine edge colorings. In that case, one has to consider the line graph $L(G)$ of $G$ and its edge distance polynomial $P_{L(G)}$. If $G$ is $k$-regular, then $L(G)$ is the edge disjoint union of $n$ complete graphs $K_k$, and $P_{L(G)}$ factors into $n$ factors accordingly. For each vertex $v \in V(G)$ there is one complete graph $K_k$ whose vertices are the edges $e \in E(v)$ incident with $v$.

The corresponding factor of $P_{L(G)}$ is the edge distance polynomial $P_{K_k}(x_e \mid e \in E(v))$ of that $K_k$. If the $k$-regular graph is of class 1, i.e. if its edges can be colored with $k$ colors, then, in the corresponding vertex colorings of $L(G)$, every color occurs one time at each vertex of that $K_k$. Therefore, by choosing equal lists, say all equal to $(k) := \{1, 2, \ldots, k\}$, the coefficients $N_L(x)^{-1}$ in the sum in the last corollary become all the same. More precisely, $N_L(x) = N_L(y)$ if $P_{L(G)}(x) \neq 0$ and $P_{L(G)}(y) \neq 0$. Moreover, $P_{L(G)}(x)$ assumes, up to the sign, the same value for every edge coloring $x : E(G) \to (k)$. So, in that sum, one basically only has to see which edge colorings contribute a positive sign and which ones a negative sign. This was already observed in [1]. It is easy to see that the definition of the sign given there depicts what we need, but we simplify that a bit. Basically, we only have to be able to say if two edge colorings have same
or opposite sign. If \( c : E \rightarrow (k) \) and \( c_0 : E \rightarrow (k) \) are proper edge colorings, then \( c|_{E(v)} \) and \( c_0|_{E(v)} \) are bijections form the set \( E(v) \) of edges at \( v \in V(G) \) to \((k)\), and we set

\[
\sgn_v(c,c_0) := \sgn((c_0|_{E(v)})^{-1} \circ c|_{E(v)}) \quad \text{and} \quad \sgn(c) := \prod_{v \in V(G)} \sgn_v(c,c_0)
\]

where \((c_0|_{E(v)})^{-1} \circ c|_{E(v)}\) is a permutation of \( E(v) \) and \( \sgn((c_0|_{E(v)})^{-1} \circ c|_{E(v)}) \) is its usual sign. We could have also defined \( \sgn_v(c,c_0) \) as the sign of the inverse permutation \((c|_{E(v)})^{-1} \circ c_0|_{E(v)}\), or as sign of the permutations \(c|_{E(v)} \circ (c_0|_{E(v)})^{-1}\) or \(c_0|_{E(v)} \circ (c|_{E(v)})^{-1}\) in \(S_k\). This is all the same. It is the right definition here, because the sign of a permutation \(\rho\) in \(S_k\) is exactly the sign of the edge distance polynomial \(P_{K_k}\) of \(K_k\) evaluated at \((\rho_1,\rho_2,\ldots,\rho_k)\).

\[
\sgn(\rho) = \frac{P_{K_k}(\rho_1,\rho_2,\ldots,\rho_k)}{|P_{K_k}(\rho_1,\rho_2,\ldots,\rho_k)|}.
\]

Hence, we only need to fix one edge coloring \(c_0 : E(G) \rightarrow (k)\) and then count how many colorings \(c : E(G) \rightarrow (k)\) are positive or negative with respect to that reference coloring. It is convenient to define an absolute sign \(\sgn(c)\) through

\[
\sgn(c) := \sgn(c,c_0)\sgn(c_0),
\]

where \(\sgn(c_0)\) is fixed given as either +1 or −1. In this section, however, it does not matter whether \(c_0\) is viewed as positive or negative, and we postpone the stipulation of \(\sgn(c_0)\) till later. With that, we arrive at \(\text{Corollary 3.9}\):

**Corollary 3.** Let \(G = (V,E)\) be a \(k\)-regular graph and let \(C(G)\) be the set of its proper edge colorings \(c : E \rightarrow (k)\). Then

\[
\sum_{c \in C(G)} \sgn(c) \neq 0 \implies G \text{ is } k\text{-list edge colorable and edge } k\text{-paintable.}
\]

Actually, we may assume that \(G\) has even many vertices, as 1-factors and \(k\)-edge colorings only exist if there are even many vertices. If we exchange two colors in an edge coloring \(c : E \rightarrow (k)\) of a \(k\)-regular graph \(G\), then all the factors \(\sgn(c)\) in \(\sgn(c)\) change, but sign \(\sgn(c)\) does not change. Therefore, it makes sense to define the sign of a 1-factorization. A 1-factorization \(F\) of \(G\) is a partition \(F = \{F_1,F_2,\ldots,F_k\}\) of the edge set \(E(G)\) into \(k\) 1-factors (perfect matchings). To every 1-factorization \(F\) there are \(k!\) edge colorings \(c\) with \(F\) as set of fibers \(c^{-1}\{\{\alpha\}\}\). All of them have the same sign, and we define

\[
\sgn(F) := \sgn(c).
\]

With that, the last corollary can be rewritten as follows:

**Corollary 4.** Let \(G = (V,E)\) be a \(k\)-regular graph and let \(OF(G)\) be the set of 1-factorizations of \(G\). Then

\[
\sum_{F \in OF(G)} \sgn(F) \neq 0 \implies G \text{ is } k\text{-list edge colorable and edge } k\text{-paintable.}
\]
3 Another Characterization of the Sign

In this section, $G$ denotes a $k$-regular graph on the vertices $v_1, v_2, \ldots, v_{2n}$, and $F = \{F_1, F_2, \ldots, F_k\}$ denotes a 1-factorization of $G$. We examine the sign $\text{sgn}(F)$ in more detail, starting from the following definition:

**Definition 1.** Let $F_1 = \{e_1, e_2, \ldots, e_n\}$ be a 1-factor of a $k$-regular graph $G$ on the vertices $v_1, v_2, \ldots, v_{2n}$. Let $1 \leq i_k < j_k \leq 2n$ be such that $e_k = v_{i_k}v_{j_k}$, for $k = 1, 2, \ldots, n$. We say that an edge $e_k \in F_1$ intersects another edge $e_\ell \in F_1$ if $i_k < i_\ell < j_k < j_\ell$ or $i_\ell < i_k < j_\ell < j_k$. We define

$$\text{int}(e_k, e_\ell) := \begin{cases} 1 & \text{if } e_k \text{ intersects } e_\ell, \\ 0 & \text{otherwise}, \end{cases}$$

and set

$$\text{int}(F_1) := \sum_{1 \leq k < \ell \leq n} \text{int}(e_k, e_\ell) \quad \text{and} \quad \text{sgn}(F_1) := (-1)^{\text{int}(F_1)}.$$  

If we position the $2n$ vertices consecutively around a cycle and draw the edges as straight lines, then an intersection is an actual intersection between lines. With this picture in mind, it is not hard to see that, if $\text{int}(v_iv_j, F_1)$ denotes the number of intersections of an edge $v_iv_j \in F_1$ with other edges in $F_1$, then

$$\text{int}(v_iv_j, F_1) \equiv j - i - 1 \pmod{2}. \quad (5)$$

This, however, does not help to determine the sign $\text{sgn}(F_1)$ of $F_1$, as

$$\sum_{e \in F_1} \text{int}(e, F_1) = 2 \text{int}(F_1), \quad (6)$$

with a 2 in front of $\text{int}(F_1)$. Counting the number of all intersections of each edge $e$ is not the right approach here. We may order $F_1$ to $\vec{F}_1 = (e_1, e_2, \ldots, e_n)$ and count only the intersections of an edge $e_k$ with the subsequent edges $e_\ell$, $\ell > k$. If $\text{int}(e_k, \vec{F}_1)$ denotes this number, then the corresponding sum yields the desired result,

$$\text{int}(F_1) = \sum_{e \in F_1} \text{int}(e, \vec{F}_1). \quad (7)$$

Hence,

$$\text{sgn}(F_1) = \prod_{e \in F_1} \text{sgn}(e, \vec{F}_1), \quad (8)$$

if we set

$$\text{sgn}(e, \vec{F}_1) := (-1)^{\text{int}(e, \vec{F}_1)}. \quad (9)$$

This formula may be used to calculate the sign of a 1-factor in algorithms that generate a 1-factor by successively adding single edges. And, there is also an analog to Formula (5). We may just count how many of the vertices $b$ that lie
Orientations of 1-Factorizations 7

between the two ends \( v_{ik} \) and \( v_{jk} \) of the edge \( e_k \) are not yet matched when we add \( e_k \) to the sequence \( (e_1, e_2, \ldots, e_{k-1}) \). So,

\[
\text{int}(v_{ik} v_{jk}, F_1) \equiv \left| \{ b \mid i_k < b < j_k, b \notin e_1 \cup e_2 \cup \cdots \cup e_{k-1} \} \right| \quad \text{(mod 2)} \quad (10)
\]

In our algorithm, we kept track of these unmatched \( b \) by using a doubly linked linear lists. From each unmatched vertex \( b \), we have at any time a link to the unmatched vertex before \( b \) and a link to the unmatched vertex after \( b \). Updating these links can then be done without shifting all subsequent vertices one place forward.

The next theorem shows that the signs of the 1-factors in a 1-factorization \( F \) can be used to calculate the sign of \( F \). This can then be used in algorithms that calculate the 1-factorizations of a graph by successively adding new 1-factors. The advantage is that the sign of a 1-factor that is added at a certain point has to be calculated only once, for all the 1-factorizations that are generate afterwards, by adding more 1-factors in all possible ways. It is clear that the formula in the next theorem does not really depend on the sign of the underlying reference coloring \( c_0 \), or the equivalent reference 1-factorization \( \{ c_0^{-1}([\alpha]) \mid \alpha \in (k) \} \).

But, to avoid additional minus signs in the theorem, we synchronize our different signs at this point, and define

\[
\text{sgn}(c_0) := \prod_{\alpha \in (k)} (-1)^{\text{int}(c_0^{-1}([\alpha]))} = (-1)^{\text{int}(c_0)} \in \{-1, +1\}, \quad (11)
\]

where

\[
\text{int}(c_0) := \sum_{\alpha \in (k)} \text{int}(c_0^{-1}([\alpha])) \quad (12)
\]

is the number of intersections between edges of equal color in \( c_0 \), if the vertices \( v_1, v_2, \ldots, v_{2n} \) are arranged consecutively on a cycle and the edges are drawn as straight lines. With this stipulation of the sign of the reference coloring \( c_0 \), we have the following theorem:

**Theorem 2.** Let \( G = (V, E) \) be a \( k \)-regular graph on the vertices \( v_1, v_2, \ldots, v_{2n} \), and let \( F = \{ F_1, F_2, \ldots, F_k \} \) be a 1-factorization of \( G \). Then

\[
\text{sgn}(F) = \prod_{i=1}^{k} \text{sgn}(F_i) .
\]

In other words, if \( c: E \rightarrow (k) \) is an edge coloring, then

\[
\text{sgn}(c) = (-1)^{\text{int}(c)} ,
\]

where \( \text{int}(c) \) is the number of intersections between edges of equal color, if the vertices \( v_1, v_2, \ldots, v_{2n} \) of \( G \) are arranged consecutively on a cycle and the edges are drawn as straight lines.
Proving this theorem is the main task of this section. We do this in a tropologic way, using Jordan’s Curve Theorem. From this theorem, we know that any two closed curves on the sphere have even many intersections with each other (and that even if they also have intersection points with them selves, which we just do not count). We also use the fact that the sign of a permutation \( \rho \in S_k \) is \(-1\) to the power of the number of inversions of \( \rho \). Here, a pair \((i_1, i_2) \in (k)^2\) with \(i_1 < i_2\) is an inversion of \( \rho \) if \( \rho(i_1) > \rho(i_2) \). We will use that this property can be characterized as intersection of strait lines in \( \mathbb{R}^2 \). Indeed, the pair \((i_1, i_2)\) is an inversion if and only if the line from \((i_1, h_1)\) to \((\rho(i_1), h_2)\) intersects with the line from \((i_2, h_1)\) to \((\rho(i_2), h_2)\), where \(h_1\) and \(h_2\) are any two different real numbers:

**Proof.** Let \( c_0 : E \to [k] \) be the reference coloring of \( G \), and let \( c : E \to [k] \) be another edge coloring. We have to show that \( \text{int}(c) \equiv \text{int}(c_0) \pmod{2} \) if and only if \( \text{sgn}(c, c_0) = 1 \). To compare the numbers of intersections in \( c \) and \( c_0 \), we draw the colored graph \((G, c)\) on top of a round cylinder, with the vertices in counter-clockwise order along the boundary of the upper disc. The colored graph \((G, c_0)\) is drawn on the bottom of the cylinder, in such a way that every vertex \( v_j \) of \((G, c)\) lies vertically above the corresponding vertex \( v_j \) of \((G, c_0)\). Now, we remove the vertex \( v_j \) in \((G, c)\) and \((G, c_0)\) and connect the open ends of the edges in \( E(v_j) \) on the top disk with those in the bottom disk. We connect edges of equal color by a line along the lateral surface of the cylinder. As to every color \( \alpha \in [k] \) there exists exactly one edge of color \( \alpha \) incident with \( v_j \) in \((G, c)\) and in \((G, c_0)\), this makes exactly one line of every color (for every \( j \in (2n) \)). To avoid that these \( k \) lines lie on top of each other, we assume that we have cut down the radius of the cylinder a bit, so that the edges in \( E(v_j) \) do not end in exactly the same point of the boundary of the upper, resp. lower, disc. Hence, we have \( 2n \) disjoint intervals \( I_j \) on the edge of each disc, corresponding to the \( 2n \) removed vertices \( v_j \). In each interval \( I_j \), on each disc, the edges of \( E(v_j) \) arrive in consecutive order, corresponding to the clockwise order of the edges in \( E(v_j) \) around \( v_j \). We may imagine the area between the upper interval \( I_j \) and the lower interval \( I_j \) as a rectangle with \( k \) straight but slanted lines crossing from the upper interval \( I_j \) to the lower interval \( I_j \). If a color \( \alpha \) occurs on, say, the \( 2^{nd} \) edge of \( E(v_j) \) in \( c \), and on, say, the \( 5^{th} \) edge of \( E(v_j) \) in \( c_0 \), then there is a line of color \( \alpha \) running from the \( 2^{nd} \) position in the upper interval \( I_j \) to the \( 5^{th} \) position in the lower interval \( I_j \).

**Claim:** \( \text{sgn}_{v_j}(c, c_0) \) is equal to \(-1\) to the power of the number of intersections between the \( k \) lines that run from the upper interval \( I_j \) to the lower interval \( I_j \). \( \text{sgn}(c, c_0) \) is equal to \(-1\) to the power of the number of intersections between all lines on the lateral surface of the cylinder.

We prove the first part of this claim by observing that every intersection corresponds to an inversion of the permutation \( \rho := (c_0|E(v_j))^{-1} \circ c|E(v_j) \) of \( E(v_j) \). We identify the clockwise ordered edges \( e_1, e_2, \ldots, e_k \) in \( E(v_j) \), and the position in \( I_j \) where they arrive, with the integers \( 1, 2, \ldots, k \) (in that order). With that identification, \( \rho \) is actually an element of \( S_k \), and the pair \((1, 2)\), for
instance, is an inversion of $\rho$ if and only if the lateral lines that start in position 1 and 2 of the upper interval $I_j$ intersect. Obviously, the colors of these two lines are $c(e_1)$ and $c(e_2)$, respectively. Inside $(G, c_0)$, these two colors occur at the edges $\rho(e_1) = (c_0|_{E(v_j)})^{-1}(c(e_1))$ and $\rho(e_2)$ of $E(v_j)$, respectively. So, position 1 and 2 in the upper interval $I_j$ are connected to position $\rho(e_1)$ and $\rho(e_2)$ in the lower interval $I_j$. Our two lines cross if and only if $\rho(e_1) > \rho(e_2)$, if and only if $(1, 2)$ is an inversion of $\rho$. The first part of our claim follows from that. It holds for each $j \in (2n)$, and that is just summed up in the second part.

From the claim, we see that $\text{sgn}(c, c_0) = 1$ if and only if the number of intersections between lines on the lateral surface of the cylinder is even. Note also that all these lateral intersections are intersections between lines of different color. Overall, on the whole cylinder, there are even many intersections between lines of different color. This follows from Jordan’s Curve Theorem, as all lines together form a system of monochromatically colored closed curves on the surface of the cylinder. Therefore, modulo 2, the number of intersections of differently colored edges in the upper disk is equal to that number in the lower disk if and only if $\text{sgn}(c, c_0) = 1$. Since the total number of intersections (that between differently and equally colored edges) is the same on both disks, this also means that $\text{int}(c) \equiv \text{int}(c_0) \pmod{2}$ if and only if $\text{sgn}(c, c_0) = 1$.

4 The List Chromatic Index of Small Graphs

Based on Corollary 4 and the results of the previous section, we have tried to determine the list chromatic index $\chi'_L(G)$ of all graphs on up to 10 vertices, in an attempt to prove the List Edge Coloring Conjecture for small graphs. We implemented the approach explained in the previous sections in SageMath [11], importing regular graphs from the webpage [8] described in [9]. With that we attacked all regular graphs on 4, 6, 8 or 10 vertices. The results are shown in the first paragraph of the following subsection. We tried than to draw conclusions about the list chromatic index of all graphs with up to 10 vertices. We did this by considering embeddings into regular graphs on even many vertices. Unfortunately, there are many exceptional cases and special circumstances. We report about these difficulties, and some ideas how to overcome them, in quite a view case distinctions. It was not possible to go through all the cases and to prove the List Edge Coloring Conjecture for all graphs on up to 10 vertices. If, however, someone wants to prove the List Edge Coloring Conjecture for just one particular small graph, he or she may find a way to do so within our case distinctions.

In the following case distinctions, the word graph stands for connected graph, and a regular graph $G$ is a zero-sum graph if the sum $\sum \text{sgn}(F)$ over all 1-factorizations $F \in \text{OF}(G)$ vanishes. We call a graph small if it has at most 10 vertices, and we call it even resp. odd if it has even resp. odd many vertices.
4.1 Small even graphs

Regular Graphs. By checking all small regular even graphs, we found only three graphs of class 2. The Petersen graph and the following two graphs:

Our main method does not apply to class 2 graphs. In these three cases, however, one can simply add a suitable 1-factor, and prove the List Edge Coloring Conjecture for the resulting graph of class 1. It is, in fact, possible to choose the 1-factor in a way that the extended graph is not a zero-sum graph. So, in the shown three cases, the List Edge Coloring Conjecture holds. Unfortunately, our method also failed in a number of other cases, where the sum \( \sum \text{sgn}(F) \) over all 1-factorizations \( F \in \text{OF}(G) \) simple was zero. The smallest zero-sum graph is \( K_{3,3} \), but this graph is bipartite. Hence, it meets the List Edge Coloring Conjecture by Galvin’s Theorem [4]. On 8 vertices, there are exactly three zero-sum graphs. The complement \( C_3 \cup C_5 \) of the disjoint union of a 3-cycle and a 5-cycle, and the following graph and its complement:

On 10 vertices there are 51 zero-sum graphs out of 164 regular class 1 graphs (1-factorable graphs). There are 5 zero-sum graphs of degree 3, 17 of degree 4, 18 of degree 5, 8 of degree 6, and 3 of degree 7. It seems that, in every small zero-sum graph, one can find a symmetry of order 2 that turns even edge coloring (\( \text{sgn} = +1 \)) into odd ones (\( \text{sgn} = -1 \)) and vice versa; which explains the vanishing sum. The most simple symmetry of this kind is given if two non-adjacent vertices of odd degree have the same neighbors, or if two adjacent vertices of even degree have the same neighbors. But, there are also more complicated cases. In the complement of the Petersen graph, for example, it is more difficult to understand how odd and even edge colorings are matched through a graph symmetry. Overall, it should be possible to proof the List Edge Coloring Conjecture for all found zero-sum graphs with other methods. Some well chosen case distinctions with respect to the color lists might suffice. This kind of reasoning, however, is usually quite tedious and depends very much on the structure of the graph.

Non-regular Graphs. If a regular graph \( G \) is of class 1 and meets the List Edge Coloring Conjecture, then every subgraph of same maximal degree still is of class 1 and still meets the List Edge Coloring Conjecture. With this argument, most non-regular small even graphs can be proven to be of class 1 and to meet the List Edge Coloring Conjecture. We just have to consider regular even extensions of same maximal degree. If an extension is still small, we may apply our findings.
about small regular even graphs. There are, however, three difficulties:

(i) Some small non-regular even graphs cannot be embedded into a regular graph by adding edges only, which would keep these graphs small. Several examples of this kind can be constructed from \( k \)-regular graphs \( (k \geq 3) \) that contain an induced path \( u-v-w \) by removing the edges \( uv \) and \( vw \), and inserting the edge \( uw \).

(ii) The three small regular even graphs of class 2 are not suitable as regular extensions in this line of reasoning. Some of their subgraphs are actually of class 2, and we can only conclude that these class 2 subgraphs meet the List Edge Coloring Conjecture.

(iii) There are still some open cases among the small regular even class 1 graphs, for which we have not yet proven the conjecture. Circumventing these cases is not always possible, as there may not be many different ways to add edges.

4.2 Small odd graphs

Class 2 Graphs (including all Regular Graphs). All regular graphs of odd order are of class 2, as no 1-factors exist. Moreover, if we start from an \( k \)-regular odd graph and remove less than \( k/2 \) edges, then the graph remains in class 2, because it is still overfull \( (|E| > \Delta \cdot \lfloor |V|/2 \rfloor ) \). All graphs that we obtain in this way have maximal degree \( k \), which is necessarily an even number, as the initial regular graph was odd. Odd class 2 graphs with odd maximal degree are not obtained in this way. But, they do exist. One example is \( K_8 \) with one edge subdivided by a new vertex, which is still overfull. To prove the List Edge Coloring Conjecture for this graph and for all class 2 graphs \( G \), however, we do not need to embed \( G \) into a regular class 2 graph of same maximal degree \( \Delta(G) \). To prove that a graph \( G \) (whether of class 2 or not) has list chromatic index \( \Delta(G) + 1 \), we may simply embed it into a class 1 graph whose maximal degree is \( \Delta(G) + 1 \). If the List Edge Coloring Conjecture was proven for that extension graph, then \( \chi'_l(G) \leq \Delta(G) + 1 \), and then the List Edge Coloring Conjecture holds for \( G \) if \( G \) is of class 2. We may also add vertices. In this way, most small odd graphs can be embedded into a suitable regular graph. As in the case of even non-regular graphs, however, there are three difficulties:

(i) Some small odd graphs cannot be embedded into a regular graph by adding only one vertex and some edges, which would keep these graphs small. One example of this kind is \( K_8 \) with one edge subdivided by a new vertex.

(ii) The three small regular even graphs of class 2 are not suitable as regular extensions in this line of reasoning and must be circumvented. Since the maximal degree can go up by one, however, there is a lot of flexibility. One can show that the three exceptions of class 2 are not needed as extension graphs. Still, circumventing them is an additional difficulty if one tries to draw general conclusions.

(iii) There are still some open cases among the small regular even class 1 graphs. If we try to embed a single small odd class 2 graph, it is often easy to circumvent the open cases. But, in general examinations, avoiding open cases is difficult.
Class 1 graphs. The majority of small odd graphs are of class 1 and, in particular, non-regular. For these graphs, embedding without increasing the maximal degree frequently works. One can try to add just one vertex and some additional edges. In this way, the results about small even regular graphs can be applied. As in the other case where we discussed embedding, there are three difficulties:

(i) Adding just one vertex, to stay within the small graphs, does not work if there are not enough vertices of sub-maximal degree to which the new vertex can be connected. In this regard, there are obviously more problematic cases as in the discussion of small odd non-regular graphs of class 2, where we could increase the maximal degree by one.

(ii) The three small regular even graphs of class 2 are not suitable as regular extensions in this line of reasoning. However, if we remove just one vertex from any of them, they remain in class 2. Hence, the three class 2 graphs do not appear as single-vertex extensions of class 1 graphs. And, if we need to add a vertex plus some edges, we may be able to circumvent these three graphs.

(iii) If we try to embed a single small odd class 1 graph, circumventing the open cases among the small regular even class 1 graphs is sometimes not possible.

5 Appendix

We implemented our algorithm in SageMath [11] as function weighted_sum(), using only commands available in the underlying programming language python. Equation (10) in Section 3 provides the foundation for the accumulation of the sign

$$\text{sgn}(\overrightarrow{e_k, F_1}) := (-1)^{\text{int}(\overrightarrow{F_1})}$$

of an edge $e_k$ with respect to a partial 1-factor $\overrightarrow{F_1} = (e_1, e_2, \ldots, e_{k-1})$ in the variable sgn. The list previous_Unmatched resp. next_Unmatched in weighted_sum() contains in the cell with number um the link to the unmatched vertex before um resp. after um, as explained after Equation (10). By default these lists are set to [-1..9] resp. [1..11], but they can also be entered as optional parameters of weighted_sum(). The last entry next_Unmatched[-1] of next_Unmatched usually points to the very first unmatched vertex. If it is greater or equal to the number of vertices, however, it means that a fresh bootstrapping needs to be initiated. This is done in the elif part of the initial bootstrapping mechanism in our recursive function. Here, we also force the first edge of vertex 0 to be in the first 1-factor, its second edge to be in the second 1-factor, etc. Hence, in a $k$-regular graph, from the $k!$ equivalent edge colorings that arise out of one edge coloring by permutation of colors, only one is counted. In our algorithm, we also do not take first the product over all signs of all edges in an 1-factorization, to afterward add up all the products that we get for the different 1-factorizations. Instead, based on the distributive law, we take the sum over partial 1-factorizations during the construction process, and then multiply these partial sums with the sign of the edge that extends all these partial 1-factorizations. This speeds up our algorithm.
As input a regular graph on even many vertices is required. The format has to be as in the example of $K_6$, shown below the definition of `weighted_sum()` in line 35. The lists of adjacencies of each vertex has to be in strictly increasing order, without showing predeceasing vertices. For graphs with more than 10 vertices, the lists $[-1..9]$ and $[1..11]$ in line 1, 2, 12 and 13 have to be extended. As output, we obtain the sum $\sum sgn(F)$ over all 1-factorizations $F$ of the graph, as it is needed in Corollary 4.

Algorithm

```python
def weighted_sum(Graph, previous_Unmatched = [-1..9], 
                next_Unmatched = [1..11]): # 2 optional param.
    # by default, start = next_Unmatched[-1] = 11 > len(Graph)
    # next_Unmatched[j] is the unmatched vertex after j
    # previous_Unmatched[j] is the unmatched vertex before j
    to_match = next_Unmatched[-1] # next_Unmatched[-1] is start
    if to_match < len(Graph): # 1-factor under construction
        neighbors = Graph[to_match]
        if len(Graph[0]) <> 0: # start next 1-factor
            to_match = 0 # 0 shall be matched first
            neighbors = [Graph[0][0]] # to avoid color permutations
            previous_Unmatched = [-1..9] # fresh bootstrapping
            next_Unmatched = [1..11]
        else: return 1 # 1-factorization complete, edgeless graph
        um = next_Unmatched[to_match]
        previous_Unmatched[um] = -1 # bypass to_match
        next_Unmatched[-1] = um # bypass to_match
        w_sum = 0 # subtotal of weighted_sum()
        sgn = 1 # initial sign of edge {to_match,nbr}
        for i in range(len(neighbors)):
            nbr = neighbors[i] # i-th neighbor of to_match
            while um < nbr: # um is bridged by {to_match,nbr}
                sgn = -sgn # bridged unmatched vertices flip sgn
                um = next_Unmatched[um]
            if um == nbr: # match to_match with nbr
                gr = [in for n in lst] for lst in Graph] # deepcopy
del gr[to_match][i] # remove edge {to_match,nbr}
p_um = [n for n in previous_Unmatched] # deepcopy
n_um = [n for n in next_Unmatched] # deepcopy
p_um[n_um[nbr]] = p_um[nbr] # bypass nbr
n_um[p_um[nbr]] = n_um[nbr] # bypass nbr
w_sum = w_sum + sgn * weighted_sum(gr,p_um,n_um)
return w_sum # output w_sum

graph = [[1,2,3,4,5],[2,3,4,5],[3,4,5],[4,5],[5],[[]] # K6
# 0 adjacent to 1,2,3,4,5; 1 adjacent to 2,3,4,5 (and 0); etc.
weighted_sum(graph) # the initial call of weighted_sum() # returns the sum of all signs of all 1-factorizations of graph
```
References

1. N. Alon: Restricted Colorings of Graphs. In: Surveys in combinatorics, 1993. London Math. Soc. Lecture Notes Ser. 187, Cambridge Univ. Press, Cambridge 1993, 1-33.
2. N. Alon: Combinatorial Nullstellensatz. Combin. Probab. Comput. 8, No. 1-2 (1999), 7-29.
3. M.N. Ellingham, L. Goddyn: List Edge Colourings of Some 1-Factorable Multi-graphs. Combinatorica 16 (1996), 343-352.
4. F. Galvin: The List Chromatic Index of a Bipartite Multigraph. J. Combin. Theory Ser. B 63 (1995), 153-158.
5. R. Häggkvist and J. Janssen: New Bounds on the List-Chromatic Index of the Complete Graph and Other Simple Graphs. Combin. Probab. Comput. 6 (1997), 295-313.
6. T.R. Jensen, B. Toft: Graph Coloring Problems. Wiley, New York 1995.
7. J. Kahn: Asymptotically Good List-Colorings. J. Comb. Theory, Ser. A 73(1) (1996), 1-59.
8. M. Meringer: Connected Regular Graphs. http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html
9. M. Meringer: Fast Generation of Regular Graphs and Construction of Cages. Journal of Graph Theory 30 (1999), 137-146.
10. J. Petersen: Die Theorie der regularen Graphen. Acta Math. 15 (1891), 193-220.
11. SageMath, the Sage Mathematics Software System (Version 7.4.1), The Sage Developers, 2017, http://www.sagemath.org
12. U. Schauz: Algebraically Solvable Problems: Describing Polynomials as Equivalent to Explicit Solutions. The Electronic Journal of Combinatorics 15 (2008), #R10.
13. U. Schauz: Mr. Paint and Mrs. Correct. The Electronic Journal of Combinatorics 15 (2008), #R145.
14. U. Schauz: A Paintability Version of the Combinatorial Nullstellensatz, and List Colorings of k-partite k-uniform Hypergraphs. The Electronic Journal of Combinatorics 17/1 (2010), #R176.
15. U. Schauz: Proof of the List Edge Coloring Conjecture for Complete Graphs of Prime Degree. The Electronic Journal of Combinatorics 21/3 (2014), #P3.43.