Guaranteed satisficing and finite regret: Analysis of a cognitive satisficing value function

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ABSTRACT

As reinforcement learning algorithms are being applied to increasingly complicated and realistic tasks, it is becoming increasingly difficult to solve such problems within a practical time frame. Hence, we focus on a satisficing strategy that looks for an action whose value is above the aspiration level (analogous to the break-even point), rather than the optimal action. In this paper, we introduce a simple mathematical model called risk-sensitive satisficing (RS) that implements a satisficing strategy by integrating risk-averse and risk-prone attitudes under the greedy policy. We apply the proposed model to the $K$-armed bandit problems, which constitute the most basic class of reinforcement learning tasks, and prove two propositions. The first is that RS is guaranteed to find an action whose value is above the aspiration level. The second is that the regret (expected loss) of RS is upper bounded by a finite value, given that the aspiration level is set to an “optimal level” so that satisficing implies optimizing. We confirm the results through numerical simulations and compare the performance of RS with that of other representative algorithms for the $K$-armed bandit problems.

Introduction

Reinforcement learning (RL), a framework for learning and control in which agents search for proper actions in an environment through trial and error, has witnessed rapid development in recent years, as evidenced by the super-human performances of deep Q-networks (DQN)\textsuperscript{1} in video game playing and AlphaGo\textsuperscript{2} in the game of Go. Moreover, the application range of RL extends not only to more complicated tasks on computers but also to the control of robots\textsuperscript{3} and unmanned aerial vehicles (UAVs)\textsuperscript{4} in the real world.

As RL algorithms are being applied to increasingly complicated and realistic tasks, the limits of sensors, processors, and actuators of agents are posing serious obstacles for conventional optimization algorithms. Simon proposed the notion of bounded rationality as the principle underlying agents’ behavior under resource limits\textsuperscript{5}. A bounded rational agent may appear to behave irrationally, but by considering the limits and constraints, the agent’s behavior can be understood as rational. Bounded rationality has attracted considerable attention in recent years. Computational rationality,\textsuperscript{6} which has been claimed to integrate the three fields of neuroscience (brain), cognitive science (mind), and artificial intelligence (machine),\textsuperscript{7} is an updated form of bounded rationality. Further, it has been proposed that abstraction and hierarchy, which have been considered to enable flexible and efficient cognition of humans,\textsuperscript{8} result from the above-mentioned limitations and are bounded rational\textsuperscript{9}.

The representative decision making policy in the theory of bounded rationality is satisficing\textsuperscript{10,11}. Satisficing agents do not keep searching for the optimal action; instead, they stop searching when an action whose quality is above a certain level (aspiration) is found. The satisficing strategy has not attracted much attention in reinforcement learning, except for a few studies\textsuperscript{12,13} (to be discussed later). In previous studies\textsuperscript{14,15}, one of the authors proposed a simple satisficing value function called risk-sensitive satisficing (RS) and empirically validated its effectiveness through numerical simulations of reinforcement learning tasks.

In this paper, we apply RS to the $K$-armed bandit problems, which constitute the most basic class of reinforcement learning tasks, and prove two propositions. First, we prove that RS is guaranteed to find a satisfactory action: if the RS agent chooses an action in each trial and the number of trials is sufficient, the agent can stably choose an action whose value is above the aspiration level. Second, we prove the finiteness of the regret of RS. In general, the performance of algorithms in the $K$-armed bandit problems is measured by how small their regret (expected loss) is. It is known that the regret increases at least in the logarithmic order with the number of trials\textsuperscript{16}. Therefore, the regret increases infinitely as the trials are repeated. However, we prove that if a small amount of information on the reward distributions is available so that the aspiration level is set to an “optimal level” (hence, satisficing entails optimizing), then the regret of RS is upper bounded by a finite value. We confirm
We introduce two models of satisficing at the levels of policy and value function. The policy model follows the standard RA standard definition of satisficing is to keep exploring until an action whose value is above the aspiration level with the highest value based on the accumulated knowledge (of reinforcement learning) as follows. If there exists at least one action whose mean reward is above the aspiration level (one step means one trial) ends is defined as follows.

Before introducing the model, we first define the difference between the mean reward and to then stop searching and keep choosing the action (exploit). Satisficing, unlike optimization, can reduce the search cost because it does not involve searching for all actions and deciding on the optimal action. This is formulated as a policy.

Models of Satisficing

Models of Satisficing

Methods

K-armed Bandit Problems

The K-armed bandit problems that we deal with in this paper are as follows. Let there be K actions \{a_1, a_2, ..., a_K\} that lead to a reward of 1 or 0 according to the reward probabilities \{p_1, p_2, ..., p_K\}, which are unknown to the agent. If the agent chooses action \(a_i\), it acquires a reward of 1 with probability \(p_i\) or a reward of 0 with probability \(1 - p_i\). The goal of the repetition of choice is maximization of the expected accumulated rewards, which is measured by minimization of regret (the expected cumulative loss). \(a^*_i\) denotes the action with the maximal reward probability (i.e., \(p_i = \max_j p_j\)). The regret when the \(n\)-th step (one step means one trial) ends is defined as follows.

\[
\text{regret}(n) = \sum_{i=1}^{K} (p_i - p_i^*) E[n_i(n)],
\]

where \(n_i(n)\) is the number of times action \(a_i\) is chosen from the first to the \(n\)-th step (simply written as \(n_i\) when the number of steps is not explicitly indicated) and \(E[\cdot]\) is the expectation. Regret represents the expected loss, i.e., “how inferior the cumulative expected reward from the actual chosen actions is to the cumulative expected reward when the optimal action continues to be chosen from the first step?” The smaller the regret, the better is the performance of the algorithms. The minimum value of the regret is zero when the optimal action has been chosen in all the steps. It has been proven that the regret increases at least in \(\Theta(\log n)\) with the number of steps \(n\).

As for action selection by the agent, the basic policy is to take the action with the highest value (the greedy method). The basic valuation of action \(a_i\) is based on its mean reward:

\[
E_i = n_i^0 / (n_i^1 + n_i^0),
\]

where \(n_i^0\) is the number of times \(a_i\) is chosen and the reward \(r\) is acquired. \(n_i\), i.e., the number of times the action \(a_i\) is chosen, satisfies \(n_i = n_i^1 + n_i^0\) and \(n = \sum_{i=1}^{K} n_i\). Under the greedy method with the mean reward valuation, if there is a non-optimal action \(a_i (i \neq i^*)\) that has a high value in early trials, there is a risk of \(a_i\) being chosen all along. Each of the other actions must be tried for an appropriate number of times so that the optimal action is found in a timely manner. Merely choosing the action with the highest value based on the accumulated knowledge (exploitation) does not suffice, and various actions must be tried (exploration). Various algorithms have been proposed to balance exploitation and exploration.

Models of Satisficing

We introduce two models of satisficing at the levels of policy and value function. The policy model follows the standard description of satisficing. The second model is the risk-sensitive value function that we analyze and test in this paper. The former is tested through simulations for comparison with the latter.

Policy Satisficing (PS) Model

A standard definition of satisficing is to keep exploring until an action whose value is above the aspiration level \(R\) is found and to then stop searching and keep choosing the action (exploit). Satisficing, unlike optimization, can reduce the search cost because it does not involve searching for all actions and deciding on the optimal action. This is formulated as a policy (of reinforcement learning) as follows. If there exists at least one action whose mean reward is above the aspiration level \(R\), exploitation (following the greedy method) is executed. Otherwise, when the mean reward of all the actions is below the aspiration level \(R\), an action is randomly chosen. We refer to this algorithm as policy satisficing (PS).

Risk-sensitive Satisficing (RS) Value Function

One of the authors has proposed a value function called risk-sensitive satisficing (RS) that realizes satisficing action selection behavior when operated under the greedy policy \(14, 15\) (see Supplementary Information for its relationship with other models). Before introducing the model, we first define the difference between the mean reward \(E_i\) of action \(a_i\) and the aspiration level \(R\):

\[
\delta_i = E_i - R.
\]

If there exists a positive \(\delta_i\), then the agent will choose such \(a_i\) and be satisfied; otherwise, it will be unsatisfied. RS is defined as follows:

\[
RS_i = n_i \delta_i = n_i (E_i - R).
\]
We perform theoretical analysis of the basic satisficing and optimizing properties of RS. All data are generated by numerical simulations and they have all been reported in the paper.

RS integrates two risk-sensitive satisficing behaviors. When unsatisfied, RS is risk-seeking, leading to optimistic exploration. If $\delta_i < 0$ for all $i$, then actions with smaller $n_i$ are prioritized. Let $R = 0.7$ and let there be two unsatisfactory actions $a_1$ and $a_2$ with $E_1 = 0.4 < E_2 = 0.6$ and $n_1 = 7, n_2 = 2$. Then, $RS_1 = -2.1 < RS_2 = -0.2$; hence, $a_2$ is chosen. This preference of a less tried action can be interpreted as the optimistic expectation of the action’s actual reward probability $p_i$, being set above $R$. There might be some $p_i > R$; however, thus far, $E_i < R$ for all the actions. In terms of looking for a satisfactory action, it is rational to try actions with smaller $n_i$. This accords with the motto “optimism in the face of uncertainty,” which is considered a general and rational exploration strategy in reinforcement learning. The UCB model described later implements this idea.

When satisfied, RS is risk-averse, performing pessimistic exploitation. If there is only one $a_i$ for which $\delta_i$ is positive, the agent will keep choosing it. If there are multiple actions with positive $\delta_i$, then the actions with larger $n_i$ are prioritized. Let $R = 0.3$, and let there be two satisfactory actions $a_1$ and $a_2$ with $E_1 = 0.4 < E_2 = 0.6$ and $n_1 = 7, n_2 = 2$ that are equivalent to the example above. Then, $RS_1 = 0.7 > RS_2 = 0.6$; hence, $a_1$ is chosen. In this case, a more tried action is preferred. This can be interpreted as the pessimistic expectation of the action’s actual reward probability $p_i$ being set below $R$. It is possible that $a_i$ is a spuriously satisfactory action with $E_i > R$; however, $p_i < R$. In terms of looking for a truly satisfactory action and avoiding spuriously satisfactory ones, it is rational to try actions with $E_i > R$ for a larger $n_i$.

**Setting of the Aspiration Level**

The aspiration level $R$ defines the boundary between satisfactory and unsatisfactory, analogous to the break-even point between gain and loss or the neutral reference outcome in prospect theory. It can be set according to the internal need for it or its knowledge of the environment. As an ecological example, let the agent be an animal, and let the rewards 1 and 0 represent the gain and loss or the neutral reference outcome in prospect theory. It can be set optimally as follows:

$$R = (p_{1st} + p_{2nd})/2. \tag{5}$$

It is known that the regret increases at least in $\mathcal{O}(\log n)$ with the number of steps $n$. This is the result of assuming no knowledge of the agent on the reward distribution. By relaxing this assumption and allowing $R$ to be set as in Eq. 5, it will be shown that the regret is upper bounded by a finite value as in Proposition 2 described later.

Data availability

All data are generated by numerical simulations and they have all been reported in the paper.

**Results**

**Analysis**

We perform theoretical analysis of the basic satisficing and optimizing properties of RS. First, in Proposition 1, we prove that RS can stably choose actions above the aspiration level after a sufficient number of steps. Second, in Proposition 2, we prove that the regret of RS is upper bounded when an optimal aspiration level is given and satisficing becomes optimizing.

**Guarantee of Satisficing**

In the proof of Proposition 1, we adopt symbols clearly indicating the step number ($s$) and the chosen action ($a_i$) as follows. Both of the following represent values after $s$ steps: the mean reward

$$E(a_i, s) = \frac{n_i^I(s)}{n_i(s)} \tag{6}$$

and the RS value

$$RS(a_i, s) = n_i(s) \cdot (E(a_i, s) - R). \tag{7}$$

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Proposition 1 (Theoretical Guarantee of Satisficing). Let \( p_i \) be the reward probability of action \( a_i \) \( (i = 1, 2, \ldots, K) \). Let \( A_U \) be the set of actions whose reward probability is not smaller than the aspiration level \( R \), and let \( A_L \) be the set of actions whose reward probability is smaller than \( R \). Let \( I_U = \{ i \mid p_i \geq R \} \), \( I_L = \{ i \mid p_i < R \} \) and \( A_U = \{ a_i \mid i \in I_U \} \), \( A_L = \{ a_i \mid i \in I_L \} \), where \( A_U \) is supposed to be a non-empty set. Then, the following holds for \( RS \).

After a sufficient number of steps, a satisfactory action \( a_i \) with \( p_i > R \) will be always chosen, and this state is stable.

In other words, by letting \( P(A) \) be the probability that event \( A \) will occur,

\[
P \left( \arg \max_{a_i} RS(a_i, s) \in A_U \right) = 1 \quad (s \to \infty). \tag{8}
\]

Subsequently, by \( N_j = \{ s \mid \arg \max_a RS(a, s) = a_j \} \), we denote the set of steps in which action \( a_j \) is chosen. Let \( \#N \) be the number of elements in set \( N \). First, we prove two claims.

Claim A.

\[
\forall i \in I_L, \ P(\#N_i = \infty \leftrightarrow RS(a_i, s) \rightarrow -\infty \quad (s \to \infty)) = 1. \tag{9}
\]

Proof. (Claim A) \((\Leftarrow)\) Suppose that \( i \in I_L \) and \( RS(a_i, s) \rightarrow -\infty \quad (s \to \infty) \). If \( \#N_i < \infty \), \( RS(a_i, s) \) is constant for \( s \) greater than or equal to some number. This is a contradiction; hence, we have \( \#N_i = \infty \). \((\Rightarrow)\) Suppose that \( i \in I_L \) and \( \#N_i = \infty \). By the law of large numbers, for any positive number \( \varepsilon \), there exists some \( S \) such that we have

\[
P \left( \left| E(a_i, s) - p_i \right| < (R - p_i) / 2 > 1 - \varepsilon \right. \quad \text{for any } s \in N_i \quad \text{greater than } S. \quad \text{Now, if } \left| E(a_i, s) - p_i \right| < (R - p_i) / 2, \text{we have}
\]

\[
RS(a_i, s) = n_i(s) \cdot (E(a_i, s) - R) \\
< n_i(s) \cdot \left( p_i + \frac{R - p_i}{2} \right) \\
= n_i(s) \cdot \frac{p_i - R}{2} < 0. \tag{10}
\]

As \( s \to \infty \), we have \( n_i(s) (p_i - R) / 2 \to -\infty \); hence, \( RS(a_i, s) \to -\infty \). Therefore, \( P(RS(a_i, s) \to -\infty \mid \#N_i = \infty) > 1 - \varepsilon \). Since \( \varepsilon \) is arbitrary, we obtain \( P(RS(a_i, s) \to -\infty \mid \#N_i = \infty) = 1. \)

Claim B.

\[
\exists i \in I_U, \ P(\#N_i = \infty) = 1. \tag{11}
\]

Proof. (Claim B) We assume that for any \( i \in I_U \), \( \#N_i < \infty \). Then, for any \( i \in I_U \), \( RS(a_i, s) \) is constant for \( s \) greater than or equal to some number. Furthermore, for some \( j \in I_L \), we have \( \#N_j = \infty \). Hence, by Claim A, we have

\[
P(\exists j \in I_L, \ RS(a_j, s) \to -\infty \mid \forall i \in I_U, \ \#N_i < \infty) = 1. \tag{12}
\]

However, the following statements contradict each other: (i) \( RS(a_j, s) \to -\infty \), (ii) \( \forall i \in I_U, \ RS(a_i, s) \) is const. for any \( s \) greater than or equal to some number. Hence, we obtain

\[
P(\exists j \in I_L, \ RS(a_j, s) \to -\infty \text{ and } \forall i \in I_U, \ \#N_i < \infty) = 0. \tag{13}
\]

Now, the following formula holds.

\[
P(\exists j \in I_L, \ RS(a_j, s) \to -\infty \text{ and } \forall i \in I_U, \ \#N_i < \infty) \\
= P(\exists j \in I_L, \ RS(a_j, s) \to -\infty \mid \forall i \in I_U, \ \#N_i < \infty) \cdot P(\forall i \in I_U, \ \#N_i < \infty). \tag{14}
\]

Therefore, we must have \( P(\forall i \in I_U, \ \#N_i < \infty) = 0. \)

Proposition 1 (again).

\[
P \left( \arg \max_{a_i} RS(a_i, s) \in A_U \right) = 1 \quad (s \to \infty). \tag{8}
\]
Proof. (Proposition 1) By Claim B, we have \( \exists k \in I_U, \#N_k = \infty \). By the law of large numbers, for any positive number \( \epsilon \), there exists some \( S \) such that we have \( P(\{|E(a_k, s) - p_k| < (p_k - R)/2\}) > 1 - \epsilon \) for any integer \( s \in N_k \) greater than \( S \). Now, if \( |E(a_k, s) - p_k| < (p_k - R)/2 \), we have

\[
RS(a_k, s) = n_k(s) \cdot (E(a_k, s) - R) \\
> n_k(s) \cdot \left(p_k + \frac{R - p_k}{2} - R\right) \\
= n_k(s) \cdot \frac{p_k - R}{2} \geq 0. \tag{15}
\]

Hence, we have \( P(\text{for sufficiently large } s, \ RS(a_k, s) > 0) > 1 - \epsilon \). Since \( \epsilon \) is arbitrary, we obtain \( P(\text{for sufficiently large } s, \ RS(a_k, s) > 0) = 1 \).

Here, we assume that there exists \( i \in I_U \) such that \( \#N_i = \infty \). Then, we may have \( RS(a_i, s) \to -\infty \) because \( RS(a_k, s) > 0 \) for any sufficiently large \( s \). However, \( \#N_i = \infty \) and \( \#N_i < \infty \) contradict each other, which means that the initial assumption must be false. Hence, for any \( i \in I_U, P(\#N_i < \infty) = 1 \) holds. Therefore, the results obtained are summarized as \( \exists k \in I_U, P(\#N_k = \infty) = 1 \) and \( \forall i \in I_U, P(\#N_i < \infty) = 1 \). From these results, the following follows immediately. \( P(\arg \max_{a_i} RS(a_i, s) \in A_U) = 1 \ (s \to \infty) \).

Theoretical Analysis of Regret

We prove that \( RS \) is upper bounded by a finite value when the level \( R \) is set to the optimal aspiration level.

Proposition 2 (Finiteness of Regret of \( RS \)). Let the highest reward probability of all the actions be \( p_1 \) and the second-highest reward probability be \( p_2 \). Further, we set \( R = (p_1 + p_2)/2 \) (an optimal aspiration level). Then, the following holds for \( RS \):

“There exists a monotonically increasing function \( f(s) \) for step number \( s \) such that regret \( s < f(s) \). Then, \( f(s) \to M \ (s \to \infty) \), where \( M \) is constant. Thus, regret \( s < M \)”.

We conceived the following proof by referring the papers\(^{20-22}\) on TOW (tug-of-war) dynamics model (hereinafter simply referred to as TOW). TOW is similar to \( RS \) (See Supplementary Information for the similarities and differences between \( RS \) and TOW). However, in their paper, the analysis of the finiteness of the regret by TOW was strictly limited to cases in which there are only two actions and the variances of the reward probabilities are equal. In the case of the bandit problems with the reward following the Bernoulli distributions, equal variance implies \( p_1 = p_2 \) or \( p_2 = 1 - p_1 \). (Let \( V_i \) be the variance of action \( a_i \).

\( V_1 = V_2 \Leftrightarrow p_1(1 - p_1) = p_2(1 - p_2) \Leftrightarrow (p_1 - p_2)(1 - p_1 + p_2) = 0 \Leftrightarrow p_1 = p_2 \) or \( p_2 = 1 - p_1 \).) Thus, the equal variance is a strong assumption. Here, we generalize the proof to prove finite regret with \( K \) arms (\( K \geq 2 \)) and without assuming equal variance.

Proof. (Proposition 2) Suppose that \( p_1 > p_2 > p_i \ (i \neq 1, 2) \). Let \( RS(a_i, s) = n_i(s) \cdot (E(a_i, s) - R) \ (i = 1, 2, \ldots, K) \). The expectation \( E \) and the variance \( V \) of \( RS(a_i, s) \) are \( E[RS(a_i, s)] = n_i(s)(p_i - R) \) and \( V[RS(a_i, s)] = n_i(s)\sigma_i^2 \), respectively, where \( \sigma_i^2 = p_i(1 - p_i) \).

Note that

\[
RS(a_i, s) = n_i(s) \cdot (E(a_i, s) - R) \\
= n_i^1(s) - n_i(s)R \\
= (X_{i,1} - R) + (X_{i,2} - R) + \cdots + (X_{i,n_i} - R) \tag{16}
\]

holds, where \( X_{i,j} = 1 \) or 0, indicating the reward when action \( a_i \) was chosen in the \( j \)-th time. Let \( \Delta RS_i(s) = RS(a_1, s) - RS(a_i, s) \ (i \neq 1) \). Then,

\[
E[\Delta RS_i(s)] = n_i(s)(p_1 - R) - n_i(s)(p_i - R) \\
= \{(p_1 - p_i)/2\}(n_i(s) + n_i(s)) \\
+ \{(p_i + p_1)/2 - R\}(n_i(s) - n_i(s)). \tag{17}
\]

\[
V[\Delta RS_i(s)] = n_i(s)\sigma_i^2 + n_i(s)\sigma_i^2. \tag{18}
\]

Since \( (p_1 + p_2)/2 = R \),

\[
E[\Delta RS_i(s)] = \{(p_1 - p_i)/2\}(n_i(s) + n_i(s)) \\
+ \{(p_i - p_2)/2\}(n_i(s) - n_i(s)). \tag{19}
\]
By Proposition 1, if the step number $s$ is sufficiently large, then $n_1(s) \to s$ with probability 1. Hence,

$$E[\Delta RS_i(s)] = \{ (p_1 - p_i)/2 \} s + \{ (p_i - p_2)/2 \} s$$

$$= \{ (p_1 - p_2)/2 \} s. \quad (20)$$

$$V[\Delta RS_i(s)] \leq (n_1(s) + n_2(s)) \sigma^2_{i,j} \leq s \sigma^2_{i,j},$$

where $\sigma_{i,j} = \max(\sigma_1, \sigma_i)$. \hfill (21)

By Eq. (16) and the central limit theorem, $\Delta RS_i(s)$ follows the normal distribution with expectation $E[\Delta RS_i(s)]$ and variance $V[\Delta RS_i(s)]$. The probability that $\Delta RS_i(s) < 0$ is $Q(E[\Delta RS_i(s)]/\sqrt{V[\Delta RS_i(s)]})$. Here, $Q(x)$ is the $Q$-function, which represents the tail distribution function of the standard normal distribution. Thus, $Q(x) = (1/\sqrt{2\pi}) \cdot \int_{-\infty}^x \exp(-t^2/2) dt$. Let $P(s = n + 1, I = i)$ be the probability that action $a_i$ is chosen in the $(n + 1)$-th step.

Then, $P(s = n + 1, I = i)$ is given by

$$P(s = n + 1, I = i) = P[RS(a_j, n) < RS(a_i, n) \ (\forall j \neq i)]$$

$$< P[\Delta RS_i(n) < 0]$$

$$= Q \left( \frac{E[\Delta RS_i(n)]}{\sqrt{V[\Delta RS_i(n)]}} \right)$$

$$\leq Q \left( \frac{(p_1 - p_2)/\sqrt{n}}{2\sigma_{i,j}} \right)$$

$$= Q(\phi_i \sqrt{n}), \quad (22)$$

where we set $\phi_i = (p_1 - p_2)/(2\sigma_{i,j})$.

By using the Chernoff bound $Q(x) \leq (1/2) \exp(-x^2/2)$, we evaluate the upper bound of the regret.

$$E[n_i(n)] = \sum_{t=0}^{n-1} P[s = t + 1, I = i]$$

$$< \sum_{t=0}^{n-1} Q(\phi_i \sqrt{t})$$

$$= \frac{1}{2} + \sum_{t=1}^{n-1} \frac{1}{2} \exp \left( -\frac{\phi_i^2}{2} t \right)$$

$$\leq \frac{1}{2} + \int_0^{n-1} \frac{1}{2} \exp \left( -\frac{\phi_i^2}{2} t \right) dt$$

$$= \frac{1}{2} + \frac{1}{\phi_i^2} \left( \exp \left( -\frac{\phi_i^2}{2} (n - 1) - 1 \right) \right)$$

$$\to \frac{1}{2} + \frac{1}{\phi_i^2} \ (n \to \infty). \quad (24)$$

Therefore,

$$\text{regret}(n) = \sum_{i=1}^{K} (p_1 - p_i) E[n_i(n)]$$

$$< \sum_{i=1}^{K} (p_1 - p_i) \left\{ \frac{1}{2} - \frac{1}{\phi_i^2} \left( \exp \left( -\frac{\phi_i^2}{2} (n - 1) - 1 \right) \right) \right\}$$

$$\to \sum_{i=1}^{K} (p_1 - p_i) \left( \frac{1}{2} + \frac{1}{\phi_i^2} \right) \ (n \to \infty). \quad (25)$$

This concludes the proof.

\hspace{1cm} $\Box$

**Empirical Verification**

We verify the proven properties through simulations. As in Proposition 2, $R = (p_1 + p_2)/2$, where $p_1 > p_2 > p_i$ ($i \neq 1, 2$). All the results below are the averaged results of 1,000 simulations. As an additional performance index, we consider *accuracy,*
which is the proportion of the simulations in which the algorithm chose the optimal action in each step. Thus, the accuracy in the t-th step is as follows.

\[ \text{accuracy} = \frac{\text{Number of times action } a_t \text{ with the highest reward probability } p_t \text{ is chosen in the } t\text{-th step}}{\text{Total number of simulations}}. \]

First, we test whether the difference in reward probabilities can be detected, even if the difference is small, when the optimal aspiration level is set for RS. We test it with \( K = 2 \) where \( (p_1, p_2) = (0.51, 0.49), (0.501, 0.499) \). The result is shown in Fig. 1. The dotted line at the top in Fig. 1 (b) represents the upper bound of the regret shown by Proposition 2. We see that the accuracy nearly reaches 1 after \( 10^6 \) steps, even if the difference is only 0.002 as in \((0.501, 0.499)\). Moreover, we see that the regret does not exceed the upper bound (Eq. (27)) calculated by Proposition 2.

Next, we conduct simulations to confirm the propositions with \( K = 10 \). The reward probability of each action is generated uniformly randomly from \([0, 1]\). The result is shown in Fig. 2. We can see that the accuracy converges to 1 and the regret does not exceed the upper bound (Eq. (27)) calculated by Proposition 2. Here, the calculated upper bound of the regret for \( K = 10 \) is considerably higher than the actual regret compared with the case of \( K = 2 \). As we evaluate the probability of choosing \( a_i \) only by comparing \( a_i \) with action \( a_1 \) having the highest reward probability as shown in Eq. (22) in the proof of Proposition 2, the probability of choosing \( a_i \) is increasingly overestimated as the number of actions increases.

**Comparison with Other Algorithms**

Here, we clarify the performance and properties of RS by comparing it with some representative algorithms for the \( K \)-armed bandit problems, namely UCB1-Tuned and \( \epsilon_n \)-greedy.

**UCB1-Tuned**

Upper confidence bound (UCB) is an algorithm based on the idea that the value of relatively less tried actions (more uncertain) is potentially high, similar to RS’s risk-seeking evaluation when unsatisfied. The regret of UCB is guaranteed to increase in the logarithmic order, which is the theoretical limit. We include the result of UCB1-Tuned (hereinafter referred to as UCB1T), which shows better performance compared to UCB1.

\[
\text{UCB1T}(a_t) = E_i + \frac{\sqrt{\ln n / n_i}}{4} \min \left\{ \frac{1}{4}, V_i(n_t) \right\},
\]

Here, \( V_i(n_t) = v_t + \sqrt{2 \ln n / n_t} \), and \( v_t \) is the variance of the reward from choosing action \( a_t \). Further, \( 1/4 \) is the upper bound of the variance of the random variable following the binomial distribution. In the algorithm, the action with the highest UCB1T value is chosen (the greedy method). The first term \( E_i \) of UCB1T, which is the mean reward, represents the already acquired knowledge (and its exploitation), whereas the second term, which decreases as action \( a_t \) is tried more, expresses the (un-)reliability of \( E_i \) (which leads to exploration). When \( n_t = 0 \), the second term cannot be calculated, but in the first \( K \) steps, each action is chosen once so that the value of the second term for all the actions is subsequently finite.
The agent chooses action $a_i$ to set the level $R$ between bandit problems and algorithms with optimal order. As their algorithm is an adaptation of the standard UCB and "suffice" (from which the word "satisfice" is formed) and presented general problem settings that include the standard probability (not always), when unsatisfied. Therefore, the performance of their model is lower than that of $PS$ similar to bandit problems when the rewards are Bernoulli distributed. They mainly analyzed the limiting behavior of the policy model framework that is the closest to ours is that of Bendor et al. on the heuristics of satisficing. Here, we introduce the existing satisficing models and briefly explain the difference between those models and $RS$.

**Existing Satisficing Models**

Here, we introduce the existing satisficing models and briefly explain the difference between those models and $RS$. First, the framework that is the closest to ours is that of Bendor et al. on the heuristics of satisficing, which analyzes the two-armed bandit problems when the rewards are Bernoulli distributed. They mainly analyzed the limiting behavior of the policy model similar to $PS$. Their model is different from $PS$ in that it gives a probability parameter of switching actions with a certain probability (not always), when unsatisfied. Therefore, the performance of their model is lower than that of $PS$.

The most recent and comprehensive study was conducted by Reverdy et al. They decomposed satisficing into “satisfy” and “suffice” (from which the word “satisfice” is formed) and presented general problem settings that include the standard bandit problems and algorithms with optimal order. As their algorithm is an adaptation of the standard UCB, the difference between $RS$ and their algorithm is similar to the difference between $RS$ and UCB as described above. Furthermore, their $ε_n$-greedy

To set the level $R$ such that satisficing implies optimization, it is necessary to have some point in the interval between the highest and second-highest reward probabilities, usually unknown to the agent. Thus, having such “optimal” $R$ is a type of “cheating”. However, when such information is available, it should be utilized well, and $RS$ does so.

Furthermore, there is another algorithm, namely $ε_n$-greedy, which requires similar information for optimal performance. In this algorithm, the probability of random action selection, $ε_n$, is gradually reduced by annealing so that the regret of $ε_n$-greedy is guaranteed to be of the logarithmic order. It starts with maximal exploration (random action selection) and then gradually shifts to more exploitation as the information of the environment gets accumulated. In $ε_n$-greedy, there are two parameters $c$ and $d$ that are set as $c > 0$ and $0 < d < 1$. When there are $K$ arms, the stepwise decreasing sequence $ε_n \in (0, 1]$, $n = 1, 2, \ldots$ is defined as follows:

$$ε_n = \min \left\{ 1, \frac{cK}{d^2n} \right\}. \quad (29)$$

The agent chooses action $a_i$ with the highest mean reward with probability $1 - ε_n$, and it chooses a random action with probability $ε_n$ for $n = 1, 2, \ldots$. Let $p_{1st}$ be the highest reward probability, and define $Δ_i = p_{1st} - p_i$. Then, the parameter $d$ needs to satisfy

$$0 < d \leq \min_{i \neq 1st} Δ_i. \quad (30)$$

Further, $\min Δ_i = p_{1st} - p_{2nd}$ needs to be known in advance. Thus, some information about the reward probabilities is required, as in the case of $RS$ with the optimal aspiration level. In addition, the performance of $ε_n$-greedy is sensitive to the value of the parameter $c > 0$, and it is difficult to find the optimal value of $c$.

On the other hand, determining the optimal aspiration level $R$ for $RS$ may be easier. It does not require a parameter like $c$, and $(p_{1st} + p_{2nd})/2$ is sufficient. More generally, it is sufficient to obtain the interval $[p_{2nd}, p_{1st}]$ or the value of any point within the interval.

### Figure 2

Simulations of $RS$ with $K = 10$, where the reward probabilities are each generated uniformly randomly from $[0, 1]$ in each simulation. (a) Plot of accuracy and (b) regret. The dotted line at the top shows the upper bound of the regret calculated by Proposition 2.

(a) Plot of accuracy and (b) regret. The dotted line at the top shows the upper bound of the regret calculated by Proposition 2.
We set
\[ d \]

We compare the performance of UCB1T, PS, and \( \varepsilon_n \)-greedy: whether the expected reward always exceeds the aspiration level or not; therefore, it becomes the same framework as that of the ordinary bandit problems. In such cases, the regret of their algorithm increases in the logarithmic order, which is the theoretical limit, and it does not become finite. On the other hand, RS can achieve the finite regret without changing the definition of regret. Therefore, the purposes and problem settings are different in our study and their study.

According to the above-mentioned discussion, it is difficult to compare our study with other satisficing algorithms for reinforcement learning proposed in previous studies because the purposes and frameworks are different. It is sufficient to compare our approach with PS and UCB1. Accordingly, the other algorithms will not be handled directly hereafter.

**Performance Comparison**

We compare the performance of UCB1T, PS, \( \varepsilon_n \)-greedy, and RS with \( K = 100 \) through numerical simulations. Furthermore, the reward probabilities are uniformly randomly selected from \([0, 1]\), and the average is over 1,000 simulations. As mentioned above, it is difficult to determine the parameter \( c \) of \( \varepsilon_n \)-greedy. In this simulation, the regret of \( \varepsilon_n \)-greedy in the 10,000-th step is taken as a reference. It is empirically found by a long parameter sweep such that the regret of \( \varepsilon_n \)-greedy in the 10,000-th step is minimized at around \( c = 1 \times 10^{-5} \). Hence, the results of \( c = 1 \times 10^{-6}, 1 \times 10^{-5}, 1 \times 10^{-4} \) are shown as comparison targets. We set \( d = p_{1st} - p_{2nd} \). As for RS and PS, we set the aspiration level \( R \) to an optimal level, \( R = (p_{1st} + p_{2nd})/2 \), so that we can evaluate the efficiency when satisficing implies optimization.

The results are shown in Fig. 3. As for accuracy, RS approaches 1 the fastest among these algorithms. As for regret, PS increases rapidly because it randomly chooses actions unless an action whose reward is above \( R \) is found. The regret of RS remains small (and bound finitely), whereas UCB1T and \( \varepsilon_n \)-greedy diverge at a logarithmic order. In summary, we can see that RS with the optimal aspiration level \( R \) shows better performance than UCB1T, PS, and \( \varepsilon_n \)-greedy.

**Analysis of the Expected Change in Value Functions**

Here, we qualitatively consider why RS with the optimal aspiration level \( R \) performs better than the other algorithms. Let us consider how the value of RS in the \( n \)-th step changes when action \( a_i \) is chosen in the \( (n+1) \)-th step. In the following RS formula,

\[
RS(a_i, n) = n_i(n) \cdot (E(a_i, n) - R) = n_i(n) \cdot (n_i(n)R),
\]

\[ RS(a_i, n) = n_i(n) \cdot (E(a_i, n) - R) = n_i(n) \cdot (n_i(n)R), \]
When action $a_i$ is chosen, $RS(a_i, n)$ changes with probability $p_i$ to

$$RS(a_i, n + 1) = n_i^1(n) + 1 - (n_i(n) + 1)R = RS(a_i, n) + 1 - R,$$

whereas it otherwise changes with probability $(1 - p_i)$ to

$$RS(a_i, n + 1) = n_i^1(n) - (n_i(n) + 1)R = RS(a_i, n) - R.$$

Let $\Delta RS(a_i, n) = RS(a_i, n + 1) - RS(a_i, n)$. Then, the expected value of the change, $E[\Delta RS(a_i, n)]$, is as follows:

$$E[\Delta RS(a_i, n)] = p_i(1 - R) + (1 - p_i)(-R) = p_i - R.$$

Thus, we see that the following relationships hold in any step:

- If $p_i > R$ then $E[\Delta RS(a_i, n)] > 0$,
- If $p_i < R$ then $E[\Delta RS(a_i, n)] < 0$.

Let $R$ be set to an optimal level. Then, relationship 35 means that once the optimal action $a_i$ is chosen, $RS(a_i)$ will keep increasing on average, and it will continue to be chosen. On the other hand, relationship 36 means that if a non-optimal action $a_i$ has the highest $RS$ value, and continues to be chosen for a while, then the value keeps decreasing on average. The value for other actions remains invariant. Therefore, at some point, another action than $a_i$ will start to be chosen. Further, note that the $RS$ value decreases at an average rate of $p_i - R$. Therefore, on average, the lower the reward probability of an action, the faster the action will stop being chosen, and another action will start being chosen.

To clarify the idiosyncrasies of $RS$, we carry out similar analyses for other value functions. First, let us analyze the mean reward. The value function is $Q(a_i, n) = n_i^1/n_i(= E_i)$. When action $a_i$ is chosen, $E[Q(a_i, n)]$ is given by

$$E[\Delta Q(a_i, n)] = p_i\left(\frac{n_i^1 + 1}{n_i + 1} - \frac{n_i^1}{n_i}\right) + (1 - p_i)\left(\frac{n_i^1}{n_i + 1} - \frac{n_i^1}{n_i}\right)$$

$$= \frac{p_i n_i - n_i^1}{(n_i + 1)n_i} = \frac{p_i - E_i}{n_i + 1},$$

whereas the values for other actions do not change. Further, $E[\Delta Q(a_i, n)]$ is positive if $p_i > E_i$ and negative if $p_i < E_i$, and both cases may occur regardless of the reward probability $p_i$, because $E_i$ is a variable, in contrast to the constant $R$ for $RS$. If action $a_i$ is chosen for a sufficient number of times, $p_i \approx E_i$ holds. Then, it leads to $E[\Delta Q(a_i, n)] \approx 0$, and $Q(a_i, n)$ remains nearly unchanged. This implies that there is a possibility that a non-highest action keeps to be chosen (trapped into a local optimum). Let us consider the simplest example where there are only two actions (with $p_1 > p_2$), and choosing the optimal action $a_1$ does not give much rewards, leading to $E_1 < E_2$. As $n_2$ increases, $E_2$ converges to $E_2 \approx p_2$, and the relationship of $E_1 < E_2$ becomes fixed because of $E_1 < p_2$. This leads to $p_2$ being chosen constantly. To avoid the local optima, $\varepsilon_n$-greedy prevents a non-highest action from being continuously chosen by randomly choosing actions with probability $\varepsilon_n$. With the mean reward, unlike $RS$, we cannot say that the smaller the reward probability of the action chosen once, the faster on average is the switching of the agent to choose another action.

Next, let us analyze UCB1, which is the simplest algorithm in the UCB family.

$$UCB1(a_i, n) = E_i + \sqrt{\frac{2\ln n}{n_i}}.$$

When action $a_i$ is chosen, the expected change in the UCB1 value is

$$E[\Delta UCB1(a_i, n)] = p_i\left\{\frac{n_i^1 + 1}{n_i + 1} + \sqrt{\frac{2\ln (n + 1)}{n_i + 1}} - \left(\frac{n_i^1}{n_i} + \sqrt{\frac{2\ln n}{n_i}}\right)\right\}$$

$$+ (1 - p_i)\left\{\frac{n_i^1}{n_i + 1} + \sqrt{\frac{2\ln (n + 1)}{n_i + 1}} - \left(\frac{n_i^1}{n_i} + \sqrt{\frac{2\ln n}{n_i}}\right)\right\}$$

$$= \frac{p_i n_i - n_i^1}{(n_i + 1)n_i} + \sqrt{\frac{2\ln (n + 1)}{n_i + 1}} - \sqrt{\frac{2\ln n}{n_i}}$$

$$= \frac{p_i - E_i}{n_i + 1} + \sqrt{\frac{2\ln (n + 1)}{n_i + 1}} - \sqrt{\frac{2\ln n}{n_i}},$$
whereas the expected change of non-chosen action \( a_j \) is as follows:

\[
E[\Delta \text{UCB1}(a_j,n)] = E_j + \sqrt{\frac{2\ln(n+1)}{n_j}} - \left( E_j + \sqrt{\frac{2\ln n}{n_j}} \right) \\
= \sqrt{\frac{2}{n_j}(\sqrt{\ln(n+1)} - \sqrt{\ln n})}. \tag{40}
\]

In Eq. (39), the first term is the same as that in Eq. (37). In Eq. (39), the second and third terms approach zero if action \( a_i \) continues to be chosen. Hence, if we consider only Eq. (39), there is a possibility that the non-highest action continues to be chosen, as with Eq. (37). However, in UCB1, the value function of non-chosen action \( a_j \) also changes, as in Eq. (40). Moreover, we can see that the value of the non-chosen action increases infinitely because of the second term of Eq. (38). As a result, a non-highest action does not continue to be chosen.

In Eq. (39), the first term is positive if \( p_i > E_i \) and negative if \( p_i < E_i \), and both cases may occur regardless of the reward probability \( p_i \) because \( E_i \) is a variable, as it is for \( Q \) above. On the other hand, the second term between the parentheses is negative if \( n \geq 3 \), which results from the fact that \( f(x) = (\ln x)/x \) monotonically decreases with \( x > e^2(>2) \). As a result, \( E[\Delta \text{UCB1}(a_i,n)] \) may be positive or negative, regardless of the reward probability. Therefore, UCB1 does not have the property of \( RS \) whereby the action with a lower reward probability will be switched from earlier.

Based on the analyses presented above, let us reconsider the form of \( RS_i = n_i\delta_i = n_i(E_i - R) \). Starting from the most basic value function of the mean reward, \( E_i, RS \) is formed through two operations, \((\cdot - R) \) and \( n_i(\cdot) \). If it is merely \( \delta_i \), the value function \( \delta_i \) works exactly as the original \( E_i \) under the greedy policy. On the other hand, if only \( n_i(\cdot) \) is applied, the value function is \( n_iE_i = n_i^1 \), and it is a special case of \( RS \) with \( R = 0 \) where any action is satisfactory. With \( n_iE_i \), the agent will continue to choose the first action that gives a reward of 1. By applying the two operations, we acquire the property of \( E[\Delta \text{RS}(a_i,n)] = p_i - R \), the constant change in the \( RS \) value, regardless of the step number \( n \). Therefore, the \( RS \) value of an unsatisfactory action (with the reward probability below the aspiration level) constantly decreases on average; as a result, the action will cease to be chosen at some point. Furthermore, we can say that the smaller the reward probability of the action chosen once, the faster on average is the switching of the \( RS \) agent to the choice of other actions. As shown above, UCB and \( \varepsilon_n \)-greedy have no such property. Therefore, this property is considered to be one of the reasons why the performance of \( RS \) using the optimal aspiration level is superior to that of other basic algorithms.

**Discussion**

In this paper, we introduced a simple model called \( RS \) that implements a satisficing strategy for the \( K \)-armed bandit problems, which constitute one of the most basic classes of reinforcement learning tasks. We proved two propositions. One is that \( RS \) is guaranteed to find a satisfactory action with the reward probability above the aspiration level. The other is that the regret (expected loss) of \( RS \) is upper bounded by a finite value when an optimal aspiration level (where satisficing implies optimizing) is given. Then, we confirmed the results through numerical simulations and compared the performance of \( RS \) with that of other representative algorithms for the \( K \)-armed bandit problems. In addition, we analyzed the property of \( RS \) relative to other algorithms and validated why \( RS \) has its own form.

Except in Proposition 1, we assumed that we can set the aspiration level \( R \) to an optimal level. As the optimal aspiration is not always available to the agent, a future research direction would be to develop an algorithm that can learn an optimal aspiration level \( R \) online. As a preliminary result, an algorithm that exploits the properties of \( RS \) has shown performance comparable to that of Thompson sampling\(^{25}\), although it has not been theoretically guaranteed thus far\(^{24}\).

There are many other advantages of \( RS \) besides those mentioned in this paper. For example, the satisficing behavior is scalable in the sense that its performance does not depend on the scale of the problems, such as the number of actions, but rather on the proportion of satisfactory actions, unlike optimization algorithms\(^{15}\). In addition, as \( RS \) is a simple value function without assumptions such as the family of reward probability distributions, it can be applied to other reinforcement learning tasks through some straightforward generalization. In fact, it has been shown that the generalized \( RS \) can conduct autonomous and efficient searches in a robotic motion learning task in which a robot learns to perform giant swings (acrobat)\(^{14}\).

One of the computational advantages of satisficing, compared to optimization, is that it can convert an optimization problem into a decision problem. With \( RS \), the guaranteed satisficing algorithm, and \( R \) at a certain level, we can efficiently determine whether there is an action whose value is above \( R \). The decision framework is especially useful when a certain level of reward, rather than the optimal level, is necessary. It also facilitates parallelization. For example, we can set the aspiration levels \( R_1, R_2, \ldots, R_N \) to \( N \) agents in ascending order, respectively, and make the agents execute a certain task in parallel. If the task succeeds at the level \( R_i \) and fails at the level \( R_{i+1} \), we can see that the optimal solution exists somewhere in \([R_i, R_{i+1}]\), and the interval may be incrementally narrowed down. This is somewhat close to human learning for solving a task. When trying
to solve a task, we usually do not randomly try and err in a purely bottom-up manner. Instead, we tend to adopt a top-down constraint in our trials, such as trying to run one mile in four minutes. Guaranteed satisficing may lead to reinforcement learning methods that solve tasks somewhat similarly to humans.

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**Author contributions statement**

T.T. and A.T. conceived the analyses. A.T. conducted the proofs and experiments. Both the authors wrote and reviewed the manuscript.

**Additional information**

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**Supplementary information**

In this supplementary material, two distinctive aspects of RS are discussed. First, we show that RS can be considered as a generalization of another model, S0. In the bandit setting is based on the premise that high performance can be achieved through competitive evaluation of actions. However, our generalization from S0 to RS shows that competitive evaluation appears only in the two-armed settings, and in general (in the K-armed settings), the fundamental is the risk-sensitive satisficing behavior. Second, we compare RS with the Tug-of-war (TOW) dynamics models, which was referred to in the proof of Proposition 2. TOW is based on the notion of conservation of physical quantities, and it leads to competitive evaluation. We show that, under certain conditions, RS has the same mathematical form as a part of the recent TOW dynamics models. In addition, S0 and TOW are both limited in terms of their application to the evaluation of only two actions (or two classes of actions). On the other hand, as materialized in RS, the notion of risk-sensitive satisficing enables generalization (to an arbitrary number of actions), simplification, conceptual clarity, and high performance in terms of satisficing, as suggested in the main text of the paper.
A. RS as a generalization of S0

First, we show that RS is a generalization of another value function S0, from the number of actions $K = 2$ to arbitrary $K \geq 2$ and from constant aspiration level 0.5 to variable $R \in [0, 1]$. RS discussed in this paper was formerly called reference satisficing\textsuperscript{14,15}. It was subsequently renamed as risk-sensitive satisficing to characterize it more specifically, and abbreviated invariantly as RS. RS contains S0 model in a special form, which was first introduced by Shinohara et al. as a causal reasoning model\textsuperscript{25}. The S0 model was later termed as the RS (rigidly symmetric) model\textsuperscript{26}, and was then used as a value function\textsuperscript{27} in the bandit problems. Subsequent studies applied S0 in the two-armed bandit problems, and the performance of S0 was found to be similar to that of LS\textsuperscript{27}, which is a more complicated model. An analysis of these behaviors from a satisficing perspective was first published in 2013\textsuperscript{28,29}. The aspiration level for satisficing was made variable in 2011\textsuperscript{30}. Subsequently, in 2012\textsuperscript{31}, its generalization from two to any arbitrary number of actions of the model was proposed. However, LS is much more complicated than RS, and the analysis was rather indirect. Hereafter, we show the equivalence of RS and S0 under certain conditions (for two actions with $R = 0.5$).

Let $A$ and $B$ be actions in a two-armed bandit problem. Let $a_{k}^{A}$ be the number of times the choice of action $X \in \{A, B\}$ has given reward 1, and let $a_{k}^{B}$ be the number of times the choice of action X has given reward 0 (no reward). Thus, the mean reward is $a_{k}^{A}/(a_{k}^{A} + a_{k}^{B})$. Here, S0 defines the values of actions $A$ and $B$ as follows:

$$S0(A) = \frac{a_{k}^{A} + a_{0}^{A}}{a_{k}^{A} + a_{0}^{A} + a_{k}^{B} + a_{0}^{B}}, \quad (41)$$

$$S0(B) = \frac{a_{k}^{B} + a_{0}^{B}}{a_{k}^{A} + a_{0}^{A} + a_{k}^{B} + a_{0}^{B}}. \quad (42)$$

These comparative evaluations identify both the obtaining of reward from action $A$ and not obtaining of reward from action $B$. Hence, $S0(B) = 1 - S0(A)$ holds. Because the denominator is common, the comparison of the two values eventually results in the selection of action $A$ if the following inequality holds; if the inequality does not hold, action $B$ is selected:

$$a_{k}^{A} + a_{0}^{A} > a_{k}^{B} + a_{0}^{B}. \quad (43)$$

From the above inequality, we can see that transitive law is established when adding action $C$. That is, let the S0 evaluation of $A$ in comparison with $B$ be represented as $S0_{AB}(A)$. If $S0_{AB}(A) < S0_{BA}(B)$ and $S0_{BC}(B) < S0_{CB}(C)$, then $S0_{AC}(A) < S0_{CA}(C)$. Thus, we see that the comparable number of actions is not necessarily $K = 2$. The inequality (43) can be expressed as

$$a_{k}^{A} - a_{0}^{A} > a_{k}^{B} - a_{0}^{B}. \quad (44)$$

Using the notations presented in this paper, $a_{k}^{A} = n_{X}E_{X}$ and $a_{k}^{B} = n_{X}(1 - E_{X})$ holds. Then,

$$n_{A}E_{A} - n_{A}(1 - E_{A}) > n_{B}E_{B} - n_{B}(1 - E_{B}) \quad (45)$$

$$\Leftrightarrow \quad n_{A}(2E_{A} - 1) > n_{B}(2E_{B} - 1) \quad (46)$$

$$\Leftrightarrow \quad n_{A}(E_{A} - 0.5) > n_{B}(E_{B} - 0.5). \quad (47)$$

It can be seen that both sides of inequality (47) are identical to the form of RS (equation (4) in the main article) with $R = 0.5$. Because the value of a set of arbitrary actions can be totally ordered thanks to the property of transitivity, it is only necessary to calculate the RS value for each action, independently of all the other actions, and choose the action with the maximum value.

B. Comparison of RS and TOW

We referred to the TOW dynamics model\textsuperscript{20–22} (hereafter simply referred to as TOW) in the proof of Proposition 2. Here, we compare RS and TOW, and describe the relative advantages of RS over TOW. There are many variations of TOW, starting from around 2010\textsuperscript{32}. Here, we focus on recent papers\textsuperscript{21,22} where the proposed model of TOW is the closest to that of RS. Let $X_{k,i}$ be a random variable, representing the reward obtained by the $i$-th choice of the action $k$. Something like the value of action $k$ in TOW can be expressed as

$$S_{k} = X_{k,1} + X_{k,2} + \cdots + X_{k,n_{k}} - Kn_{k}$$

$$= (X_{k,1} - K) + (X_{k,2} - K) + \cdots + (X_{k,n_{k}} - K), \quad (48)$$

where $K$ is a parameter.

Let $n_{k}$ be the number of time action $k$ is chosen, and $E_{k}$ be the average rewards obtained by choosing action $k$, such that $E_{k} = \sum_{i=1}^{n_{k}} X_{k,i}/n_{k}$. Although in the main text of the paper, the probability distributions of the rewards were assumed to be the
Bernoulli distributions, herein, the distribution does not necessarily have to be Bernoulli. The value function $RS_k$ of the action $k$ of $RS$ is equivalent to the following form, as given in the proof of Proposition 2:

$$RS_k = n_k (E_k - R)$$

$$= \sum_{i=1}^{n_k} X_{k,i} - n_k R$$

$$= (X_{k,1} - R) + (X_{k,2} - R) + \cdots + (X_{k,n_k} - R).$$

When parameter $K$ in (48) is interpreted as the aspiration level $R$ in equation (49), $RS$ has the same mathematical form as a part of the recent TOW dynamics models under certain conditions. Hence, the regret calculation of TOW can be applied to $RS$ as well, and the regret of $RS$ also is upper bounded like TOW. In this work, we relaxed the assumption of equal variance in the proof for TOW.

However, there exist certain differences between $RS$ and TOW. The primary difference is that they model totally different phenomena. $RS$ is modeled on how humans make decisions (satisficing), while taking into account the associated risks. Moreover, as explained in Supplementary information A, $RS$ is also a generalized model of $S_0$ model in causal reasoning. On the other hand, TOW is derived from physical laws like volume conservation. An advantage of $RS$ over TOW lies in its simplicity, clarity, and generalizability. As regards clarity, $RS$ is the product of “reliability of obtained information” and “degree of satisficing,” and the parameter $R$ is associated to “aspiration.” On the other hand, the interpretation of the parameter $K$ of TOW, which corresponds to $R$ in $RS$, is not necessarily clear. Therefore, through straightforward generalization of these two constituent concepts, $RS$ can be applied not only to the $K$-armed bandit problems (instead of two-armed) but also generally to reinforcement learning settings. 

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