Abstract. We give a geometric construction of the $\mathcal{W}_{1+\infty}$ vertex algebra as the infinitesimal form of a factorization structure on an adèlic Grassmannian. This gives a concise interpretation of the higher symmetries and Bäcklund-Darboux transformations for the KP hierarchy and its multicomponent extensions in terms of a version of “$\mathcal{W}_{1,\infty}$-geometry”: the geometry of $\mathcal{D}$-bundles on smooth curves, or equivalently torsion-free sheaves on cuspidal curves.

1. Introduction

There is a beautiful relationship between the conformal field theory of free fermions, the KP hierarchy, and the geometry of moduli spaces of curves and bundles. In particular, the deformation theory of curves and of line (or vector) bundles on them is realized algebraically by the Virasoro and Heisenberg (or Kac-Moody) algebra symmetries of the free fermion system.

The algebra $\mathcal{W}_{1,\infty}$ (a central extension of differential operators with Laurent coefficients) and its more complicated nonlinear reductions, the $\mathcal{W}_n$ algebras, themselves play a well-established role as algebras of “higher symmetries” in integrable systems and string theory. For example, $\mathcal{W}$-algebras arise as the additional symmetries of the KP and Toda hierarchies, and tau-functions satisfying $\mathcal{W}$-constraints arise as partition functions of matrix models and topological string theories (see e.g. [vM]). Moreover, the search for $\mathcal{W}$-geometry and $\mathcal{W}$-gravity generalizing the moduli of curves and topological gravity has also been the topic of an extensive literature (see for example [H, Po, LO]).

In the present paper, we establish a “global” geometric realization of these higher $\mathcal{W}_{1+\infty}$-symmetries. More precisely, we give an algebro-geometric description of a factorization space (Section 3) and prove (Theorem 4.2) that it naturally integrates the $\mathcal{W}_{1+\infty}$-symmetry of the free fermion system (in a sense described below). Our work thus gives a precise analog of earlier work on the Virasoro and Kac-Moody algebras, in which the role of the moduli of curves and bundles is played by the moduli of $\mathcal{D}$-bundles (projective modules over differential operators) on a smooth complex curve $X$, or equivalently by the moduli of torsion-free sheaves on cusp quotients of $X$.

Beilinson and Drinfeld [BD2] have introduced the notion of factorization space as a nonlinear (or integrated) geometric counterpart to the notion of a vertex algebra or chiral CFT, which captures the algebraic structure of Hecke correspondences (see [FB] for a review). The Beilinson-Drinfeld Grassmannian, a factorization space built out of the affine Grassmannian of a Lie group $G$, simultaneously encodes the infinitesimal (Kac-Moody) and global (Hecke) symmetries of bundles on curves and is perhaps the central object in the geometric Langlands correspondence [BD1]. The case of $G = GL_1$ encodes geometrically the vertex algebra of a free fermion, or in
the language of integrable systems the KP flows together with the vertex operator [DJKM].

In Section 3, for any smooth complex curve \(X\) we define an algebro-geometric variant \(\text{Gr}_D\) of the adèlic Grassmannian introduced by Wilson [W1, W2] in the study of the bispectral problem and the rational solutions of KP (see Remark 3.1 for a discussion of the relationship of \(\text{Gr}_D\) to Wilson’s construction). We then show that \(\text{Gr}_D\) is naturally a factorization space in which the \(GL_1\) Beilinson-Drinfeld Grassmannian embeds as a factorization subspace. This is a special case of a more general construction in Section 3 of an adèlic Grassmannian \(\text{Gr}_P\) associated to any \(D\)-bundle \(P\) on \(X\); the case \(P = D^n\) leads to multicomponent KP (and the \(\mathcal{W}_{1+\infty}(\mathfrak{gl}_n)\) vertex algebra).

We show in Section 4 that the adèlic Grassmannian realizes the \(\mathcal{W}_{1+\infty}\)-algebra as the algebra of infinitesimal symmetries of \(D\)-bundles:

**Theorem 1.1** (See Theorem 4.2). *For any \(c \in \mathbb{C}\), the chiral algebra \(\mathcal{W}_{1+\infty}(\mathfrak{gl}_n)\) on \(X\) at level \(c\) associated to the \(\mathcal{W}_{1+\infty}(\mathfrak{gl}_n)\)-vertex algebra is isomorphic to the chiral algebra \(\delta_\mathfrak{g}_n\) of level \(c\) delta functions along the unit section of the adèlic Grassmannian \(\text{Gr}_{D^n}\).*

As we then explain in Section 5, the factorization structure of our adèlic Grassmannian thus encodes both the infinitesimal symmetries and the Hecke modifications of \(D\)-bundles: that is, the \(\mathcal{W}_{1+\infty}\) vertex algebra and the Bäcklund-Darboux transformations [BHY1, BHY3] of the KP hierarchy (and, more generally, the \(\mathcal{W}_{1+\infty}(\mathfrak{gl}_n)\)-algebras and Bäcklund-Darboux transformations of the multicomponent KP hierarchies). In other words, the factorization structure on the adèlic Grassmannian unites the vertex operators, Bäcklund transformations and additional symmetries of the KP hierarchy [vM] in a single geometric structure. The embedding of the \(GL_n\) Beilinson-Drinfeld Grassmannian into the rank \(n\) adèlic Grassmannian globalizes the inclusion of the Kac-Moody algebra \(\widehat{\mathfrak{g}}_n\) into \(\mathcal{W}_{1+\infty}(\mathfrak{gl}_n)\). Section 5 also contains an extended explanation of the interaction between our picture on the one hand and the free fermion theory and the Krichever construction on the other, realizing \(\mathcal{W}_{1+\infty}\) orbits on the Sato Grassmannian, and thereby the Orlov-Schulman additional symmetries of the KP hierarchy, via moduli of \(D\)-bundles or cusp line bundles. We close with a discussion of the significance of the factorization space \(\text{Gr}_P\) for \(W\)-geometry.

For related studies of (and background on) the geometry of \(D\)-bundles we refer the reader to [BN1, BN2, BN3], which originated as an attempt to understand geometrically and generalize the relations between bispectrality, the adèlic Grassmannian, ideals in the Weyl algebra and Calogero-Moser spaces first uncovered and explored in work of Wilson and Berest-Wilson [W1, W2, BW1, BW2]. In [BN1], the Cannings-Holland description of \(D\)-modules on curves and their cuspidal quotients is revisited from a more algebro-geometric viewpoint (and the Morita equivalence of Smith-Stafford generalized to arbitrary dimension). In [BN2], two constructions of KP solutions from \(D\)-bundles are explained—one (which is most relevant to this paper) related to line bundles on cusp curves and the other related to Calogero-Moser systems. (This description of moduli spaces of \(D\)-bundles or ideals in \(D\) as Calogero-Moser spaces is established for arbitrary curves in [BN3].) As explained in [BN2], in genus zero the two constructions are evidently exchanged by the Fourier transform and thus, by [BW1], by Wilson’s bispectral involution [W1]. However, in higher genus the constructions are completely independent.
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2. $\mathcal{D}$-bundles

In this section we review the definition and features of $\mathcal{D}$-bundles on a smooth complex algebraic curve $X$, following ideas of Cannings-Holland [CH1, CH2] as they are explained in [BN1]. Let $\mathcal{D} = \mathcal{D}_X$ denote the sheaf of differential operators on $X$ (which can be viewed as functions on the quantization of the cotangent bundle $T^*X$).

Definition 2.1. A $\mathcal{D}$-bundle $M$ on $X$ is a locally projective coherent right $\mathcal{D}_X$-module.

Equivalently, a $\mathcal{D}$-bundle is a torsion-free coherent right $\mathcal{D}_X$-module (since the sheaf of algebras $\mathcal{D}_X$ locally has homological dimension one, [SS, Section 1.4(e)]). Moreover, $\mathcal{D}_X$ possess a skew field of fractions (see [SS, Section 2.3]), from which it follows that any $\mathcal{D}$-bundle has a well-defined rank. Thus $\mathcal{D}$-bundles can be viewed alternatively as vector bundles or as torsion-free sheaves on the quantized cotangent bundle of $X$.

Some obvious examples of $\mathcal{D}$-bundles are the locally free (or induced) rank $n$ $\mathcal{D}$-bundles, of the form $M = V \otimes \mathcal{D}$ for a rank $n$ vector bundle $V$ on $X$. In general, however, $\mathcal{D}$-bundles of rank 1 are not locally free (isomorphic to $\mathcal{D}$), but only generically free, behaving more like the ideal sheaves of a collection of points on an algebraic surface. Note that every right ideal in $\mathcal{D}_X$ is torsion-free, hence a $\mathcal{D}$-bundle of rank 1 (conversely any rank one $\mathcal{D}$-bundle may locally be embedded as a right ideal in $\mathcal{D}_X$).

For example, let $X = \mathbb{A}^1$, so that $\mathcal{D}_X$ has as its ring of global sections the Weyl algebra $\mathbb{C}[z, \partial]/(\partial z - z\partial = 1)$. The right ideal $M_0$ generated by $z^2$ and $1 - z\partial$ agrees with $\mathcal{D}$ outside of 0 but is not locally free near $z = 0$. Let

$$W(M_0) = \{\theta(f) | \theta \in M_0, f \in \mathbb{C}[z]\} \subset \mathbb{C}[z].$$

Then $W(M_0) = \mathbb{C}[z^2, z^3]$, the subring of $\mathbb{C}[z]$ generated by $z^2$ and $z^3$; this is isomorphic to the coordinate ring of the cuspidal cubic curve $y^2 = x^3$. Furthermore, $M_0 = \mathcal{D}(\mathbb{C}[z], W(M_0))$, the space of differential operators that take $\mathbb{C}[z]$ into $W(M_0)$.

2.1. Grassmannian parametrization. In general, to describe $\mathcal{D}$-modules it is convenient to extract linear algebra data using the de Rham (or Riemann-Hilbert) functor, which (on right $\mathcal{D}$-modules) takes $M$ to the sheaf of $\mathbb{C}$-vector spaces which is its quotient by all total derivatives, or formally

$$M \mapsto h(M) := M \otimes_{\mathcal{D}} \mathcal{O}_X.$$ 

Note that if $M = V \otimes \mathcal{D}$ is induced, we recover the underlying vector bundle $V = h(M)$ in this fashion (though only as a sheaf of $\mathbb{C}$-vector spaces rather than as an $\mathcal{O}_X$-module). In the example of $M_0$ given above, $h(M_0) = W(M_0)$. 

We can parametrize \(\mathcal{D}\)-bundles by choosing generic trivializations: given a \(\mathcal{D}\)-bundle \(M\) we can embed \(M\) into \(n = \text{rk} M\) copies of rational differential operators, \(M \hookrightarrow \mathcal{D}^n(K_X)\), so that \(M \otimes K_X = \mathcal{D}^n(K_X)\). The module \(M\) is then determined by the finite collection of points \(x_i\) at which \(M\) differs from \(\mathcal{D}_{X_i}^n\), and choices of subspaces of \(n\) copies of Laurent series \(K_{x_i}^\bullet\) at \(x_i\). This is succinctly explained in [BD2, Section 2.1]: \(\mathcal{D}\)-submodules of a \(\mathcal{D}\)-module \(N\) that are cosupported at a point \(x \in X\) are in canonical bijection (via the de Rham functor \(h\)) with subspaces of the stalk of the de Rham cohomology \(h(N)_x\) at \(x\) that are open in a natural topology. In our case, we obtain collections of linearly compact open subspaces (or \(e\)-lattices, in the terminology of [BD2]) of \(n\) copies of Laurent series \(K_{x_i}^\bullet\) at \(x_i\) with respect to the usual (Tate) topology, i.e., subspaces commensurable with \(n\) copies of Taylor series. These subspaces correspond to points of the “thin Sato Grassmannian” \(\text{Gr}(K_{x_i}^\bullet)\) (not to be confused with the complementary original or “thick” Sato Grassmannian, see Section 5).

### 2.2. Cusps

\(\mathcal{D}\)-bundles can alternatively be described by torsion-free sheaves on cuspidal curves. Namely, given a \(\mathcal{D}\)-module \(M\) with a generic trivialization (identification with \(\mathcal{D}^n\)), we can find a subring (or rather subsheaf of rings) \(\mathcal{O}_Y \subset \mathcal{O}_X\) whose left action on \(\mathcal{D}^n(K_X)\) (by right \(\mathcal{D}\)-automorphisms) preserves \(M\). Equivalently, the \(\mathbb{C}\)-subsheaf \(h(M) \subset h(\mathcal{D}(K_X))\) is preserved by a ring \(\mathcal{O}_Y \subset \mathcal{O}_X\), which differs from \(\mathcal{O}_X\) only at the finitely many singularities \(x_i\) of the embedding. Thus \(h(M)\) defines a torsion-free sheaf \(V_Y\) on the cuspidal curve \(Y = \text{Spec} \mathcal{O}_Y\), which has \(X \to Y\) as a bijective normalization.

In fact, as was shown by Smith and Stafford [SS] and generalized to arbitrary dimension in [BN1], passing from \(X\) to such a cuspidal quotients \(Y\) doesn’t change the category of \(\mathcal{D}\)-modules: the sheaves of rings \(\mathcal{D}_X\) and \(\mathcal{D}_Y\) are Morita equivalent. Thus, given a torsion-free sheaf \(V_Y\) on \(Y\) we can define a torsion-free \(\mathcal{D}_Y\)-module \(M_Y = V_Y \otimes \mathcal{D}_Y\) by induction, and transport it to obtain a \(\mathcal{D}\)-bundle \(M\) on \(X\). Moreover, this process of “cusp-induction” reverses the above procedure \(M \mapsto V_Y\).

As a result, we obtain a geometric reinterpretation of the linear algebra data classifying \(\mathcal{D}\)-bundles. Every \(\mathcal{D}\)-bundle arises by cusp-induction from a torsion-free sheaf on a cuspidal quotient \(X \to Y\), but \(Y\) is not unique. We can consider \(M\) as associated to smaller and smaller subsheaves \(\mathcal{O}_{Y'} \subset \mathcal{O}_Y \subset \mathcal{O}_X\), introducing deeper and deeper cusps in the curves \(X \to Y \to Y'\). The collection of \(\mathcal{D}\)-bundles on \(X\) is obtained as a direct limit over these deepening cusp curves of the collections of torsion-free sheaves. Under this limit the geometry of \(X\) “evaporates” and we are left with a reinterpretation of the linear algebra data above, characterizing a class of \(\mathbb{C}\)-sheaves on \(X\) which arise from torsion-free sheaves on some cusp quotient.

### 3. Factorization

In this section we study the factorization space structure on the symmetries of a \(\mathcal{D}\)-bundle. Factorization spaces were introduced in [BD2]; we refer the reader to [FB, Section 20] for a leisurely exposition. In Section 3.1, we establish a factorization space structure for \(\mathcal{D}\)-bundles on a curve. As we explain in Section 3.2, standard constructions then produce a chiral algebra which acts infinitesimally simply transitively near the unit section of our factorization space \(\text{Gr}_p\). In Section 3.3, we describe how to twist this chiral algebra by the determinant line bundle to obtain a family of chiral algebras at different levels.
3.1. The Factorization Space $\text{Gr}_P$. Let $X$ denote a smooth complex curve. We usually fix a $\mathcal{D}$-bundle $P$ in what follows.

Recall the definition of a factorization space or factorization monoid from [KV, Definition 2.2.1]. We will define a factorization space, the $P$-adèlic Grassmannian $\text{Gr}_P$ of $X$, as follows. For each finite set $I$ we define $\text{Gr}^I_P(S)$, for a scheme $S$, as a set over $X^I(S)$, the set of maps $S \xrightarrow{\phi} X^I$. Such a map $\phi$ defines a divisor $D \subset S \times X$. Then $\text{Gr}^I_P(S)$ is the set of pairs $(M, \iota)$ consisting of:

1. An $S$-flat, locally finitely presented right $\mathcal{D}_{X \times S}$-module $M$, such that the restriction $M|_{X \times \{s\}}$ to each fiber is torsion-free (equivalently, locally projective).
2. An isomorphism $\iota : M|_{X \times S \setminus D} \xrightarrow{\simeq} P|_{X \times S \setminus D}$ such that the composite $M \rightarrow M|_{X \times S \setminus D} \xrightarrow{\iota} P|_{X \times S \setminus D}$ is injective with $S$-flat cokernel.

Remark 3.1. As we have mentioned in the introduction, our adèlic Grassmannian is not the same as the one introduced by Wilson [W1, W2]. Indeed, our space differs from Wilson’s in two significant respects. One of these is that Wilson’s space is not the same as the one introduced by Wilson [W1, W2]. Indeed, our space differs in the definition from [KV]) are standard. Briefly, we need to see that the fiber $\text{Gr}^I_P$ over an $I$-tuple $x_I$ of (S-)points of $X$ depends only on the support of the tuple—not the multiplicities—and factorizes as a product for every disjoint union decomposition of the tuple. The first property is automatic for any functor defined in terms of data outside of $x_I$. The second follows from the fact that modifications at points $x_I$ are local, and so modifications away from disjoint points can be glued together (explicitly, this follows from the description of modifications by collections of points of Grassmannians, as in the previous section).
Choose an induced $D$-bundle $V \otimes D$ and containments $V(-E) \otimes D \subseteq P \subseteq V(E) \otimes D$ (which is possible by Lemma 3.2). Then $P$ is cuspidal as induced from a cusp curve $Y$ with homeomorphism $X \to Y$ that fails to be an isomorphism only at the support of $E$. Suppose $Y$ has a cusp located “under” a point $p$ in $X$. If $z$ is a uniformizer at $p$, then for $n$ sufficiently large, $z^n$ lies in $\mathcal{O}_Y$. It follows that we can make sense of $D$-module inclusions $\mathcal{O}(-np) \otimes P \subseteq P \subseteq \mathcal{O}(np) \otimes P$ (say, as subsheaves of $P|_{X \setminus E}$). Write $P(kD) = \mathcal{O}(kD) \otimes P$ when this makes sense. We then give $\text{Gr}_P$ the ind-structure coming from Lemma 3.2: we let $\text{Gr}_P^i(n)$ denote the subset of $\text{Gr}_P^i$ of pairs $(M, i)$ for which we have $P(-nD) \subseteq M \subseteq P(nD)$.

There are now many ways to see that $\text{Gr}_P^i(n)$ is a scheme of finite type. For example, applying the de Rham functor to $M/P(-nD) \subseteq P(nD)/P(-nD)$ and applying Theorem 5.7 of [BN1], we see that choosing $M$ is equivalent to choosing a sheaf of vector subspaces of $h(P(nD)/P(-nD))$. The functor of such choices is a closed subsheaf of the relative Grassmannian of subspaces of $H^0(h(P(nD)/P(-nD)))$ (this is essentially the Cannings-Holland picture of the adelic Grassmannian, as explained and used to great effect in [W2]).

It remains to show that $\text{Gr}_P^i$ is formally smooth and to describe the formally integrable connection over $X^i$. We first show that $\text{Gr}_P^i \to X^i$ is formally smooth.

Let $(M, i)$ be an object of $\text{Gr}_P^i(S)$ for a scheme $S$. We may assume that $P(-nD) \subseteq M \subseteq P(nD)$ and that $M$ is, in the terminology of [BN1], cuspidal from a family $\mathcal{F}$ of $\mathcal{O}_Y$-modules, where $Y$ is a family of cusp curves over $S$ given by a sheaf of sub-algebras $\mathcal{O}_Y \subseteq \mathcal{O}_{X \times S}$. We then have inclusions of $\mathcal{O}_Y$-modules $h(P(-nD)) \subseteq \mathcal{F} \subseteq h(P(nD))$.

Now, given a nilpotent thickening $S \subseteq S'$ and a divisor $D'$ on $S' \times X$ that is a nilpotent thickening of the divisor $D$ to which $(M, i)$ corresponds, we need to extend $(M, i)$ to a pair $(M', i')$ over $X \times S'$. To do this, we first extend $Y$ to a family $Y'$ of cusp curves over $S'$ “with cusps determined by $D'$,” i.e., cusps whose depths are determined by $D'$. It then suffices to deform the map $h(P(nD)) \to h(P(nD))/\mathcal{F}$: the kernel of the cusp-induction of the deformed map will give $M'$, and the inclusion in $h(P(nD')) \otimes D$ will give $i'$. But, by construction, as an $\mathcal{O}_Y$-module $h(P(nD))/\mathcal{F}$ is a direct sum of skyscrapers: more precisely, it is isomorphic to the direct image to $Y$ of $\oplus_{i \in I} \mathcal{O}_{p_i}$, where $p_i$ is the $i$th section of $X \times S \to S$ determined by the map $S \to X^i$. So, as our deformation of $h(P(nD))/\mathcal{F}$ we can take $\oplus_{i \in I} \mathcal{O}_{p_i}$ as an $\mathcal{O}_Y$-module, and then deformations of this sum of skyscrapers, as well as the map from $h(P(nD'))$, certainly exist. This proves the existence of $(M', i')$. Hence $\text{Gr}_P^i \to X^i$ is formally smooth. (Note that an alternative proof of this assertion uses the formally transitive action of a Lie algebra of matrix differential operators, as in the next section.)

Finally, we describe the formally integrable connection over $X^i$—note, however, that the existence of such a connection follows formally (as in [FB, 20.3.8]) from the existence of a unit for the factorization space. This follows [Ga, Section 5.2]. Namely, given an Artinian scheme $A$, a pair $(M, i)$ parametrized by a scheme $S$, and a map $S \times A \to X^i$ determining a divisor $D_A$, we pull $M$ back along the projection $X \times S \times A \to X \times S$. Since $X \times S \times A \setminus D_A = (X \times S \setminus D) \times A$, this pullback comes equipped with an isomorphism to $P$ over $X \times S \times A \setminus D_A$, as desired. This canonical lift of infinitesimal extensions gives our connection. 

Suppose $P = V \otimes D$ is an induced $D$-bundle. Let $\text{Gr}^i_P$ denote the Beilinson-Drinfeld Grassmannian associated to $V$: over a divisor $D \subset X$, this parametrizes
vector bundles $W$ equipped with an isomorphism $W|_{X \setminus D} \cong V|_{X \setminus D}$. Such an isomorphism induces a $D$-bundle isomorphism $W \otimes D|_{X \setminus D} \cong P|_{X \setminus D}$, thus giving an embedding of unital factorization spaces:

$$\text{Gr}^t_V \hookrightarrow \text{Gr}^t_P .$$

We note that the relative tangent space of $\text{Gr}^t_P \to X^I$ at $(M, i)$ is given by

$$T_{(M,i)}(\text{Gr}^t_P / X^I) = \text{Hom}_D(M, P(\infty \cdot D)/M)$$

by the same analysis as one uses for a Quot-scheme. Given a divisor $D \subset X$, let $\text{supp}(D)$ denote the support of the divisor $D$ and $\hat{X}_D$ the formal completion of $X$ along $D$. For a quasi-coherent sheaf $\mathcal{F}$, let $\mathcal{F}_\eta$ denote the direct sum of stalks at points of $\text{supp}(D)$.

**Lemma 3.4.** There is a natural surjection:

$$\text{End}(P(\infty \cdot D))_\eta \twoheadrightarrow T_{(M,i)}(\text{Gr}^t_D / X^I).$$

If, in addition, $P = D^n$, this yields a surjection:

$$H^0(D(\infty \cdot D)|_{\hat{X}_D}) \otimes \mathfrak{gl}_n \twoheadrightarrow T_{(M,i)}(\text{Gr}^t_D / X^I).$$

In particular, if $D$ consists of a single point $x$, we get a surjection

$$\mathcal{D}(\mathcal{K}_x) \otimes \mathfrak{gl}_n \twoheadrightarrow T_{(M,i)}(\text{Gr}^t_D(x) / X).$$

**Proof.** We use the short exact sequence:

$$0 \to M \to P(\infty \cdot D) \to P(\infty \cdot D)/M \to 0.$$ 

Since $M$ and $P(\infty \cdot D)$ are locally projective, the sheaf Ext group $\text{Ext}^1_D(P(\infty \cdot D), M)$ vanishes, and we get a surjective map of sheaves

$$\text{Hom}_D(P(\infty \cdot D), P(\infty \cdot D)) \to \text{Hom}(P(\infty \cdot D), P(\infty \cdot D)/M).$$

Since $\text{Hom}(P(\infty \cdot D), P(\infty \cdot D)/M)$ is supported along $\text{supp}(D)$, the conclusions follow. \hfill \Box

3.2. **Chiral Algebra.** The $D$-bundle $P$, equipped with its canonical isomorphism $id$ with $P$ over $X \setminus D$ for any $D$, defines a “unit” section $\delta^t : X^I \to \text{Gr}^t_P$, compatible with the factorization isomorphisms and preserved by the relative connection. Let $\delta_P$ denote the $\mathcal{O}$-push-forward of the right $D$-module unit $\omega_X$ on $\text{Gr}^t_X = \text{Gr}^{[1]}_P$, and $\delta^t_P$ the push-forward of unit $\omega_{X^I}$ from $\text{Gr}^t_P$. The sheaves $\delta^t_P$ are right $D$-modules on $X^I$, and we let $\ell \delta^t_P$ denote the corresponding left $D$-modules.

**Corollary 3.5.** The left $D$-modules $\ell \delta^t_P$ form a factorization algebra on $X$. In particular $\delta_P$ is a chiral algebra.

**Proof.** This exactly follows Section 5.3.1 of [Ga]. \hfill \Box

**Remark 3.6.** We will identify this chiral algebra with a $W$-algebra below in the case $P = D^n$. It would be interesting to investigate the structure of $\delta_P$ further in the case that $P$ is not locally free.

**Proposition 3.7.** Suppose $P = D^n$. The Lie algebra $\mathcal{D}(\mathcal{K}_x) \otimes \mathfrak{gl}_n$ acts continuously and formally transitively on the Grassmannian fiber $\text{Gr}^{[*]}_{D^n}(x)$, inducing an isomorphism $T_{D^n} \text{Gr}^{[*]}_{D^n}(x) = \mathcal{D}(\mathcal{K}_x)/\mathcal{D}(\mathcal{O}_x) \otimes \mathfrak{gl}_n$. 


Proof. In the Grassmannian description of the fiber $\text{Gr}_{D^n}(x) = \text{Gr}(K^n_x)$, the action of $D(K^n_x) \otimes \mathfrak{gl}_n$ comes from its defining action by continuous endomorphisms of $K^n_x$. The tangent space to the Grassmannian $\text{Gr}(K^n_x)$ at $O^n_x$ is identified with

$$\text{Hom}_{\text{cont}}(O^n_x, K^n_x/O^n_x) = D(K^n_x)/D(O_x) \otimes \mathfrak{gl}_n,$$

since a continuous homomorphism factors through a map $(O_x/m_x)^n \to (m_x^{-k}/O_x)^n$ for $k$ sufficiently large, and all such homomorphisms may be realized by differential operators (or alternatively since $\text{Hom}_{\text{cont}}(O^n_x, K^n_x/O^n_x) = \text{Hom}_D(D^n, K^n_x/O^n_x \otimes D)$, since a homomorphism on either side is automatically $O_Y$-linear for a deep enough cusp $Y$). The formal transitivity at other points follows similarly. \hfill \Box

It follows that $D(K^n_x) \otimes \mathfrak{gl}_n$ also acts on the space of delta functions $\delta_{D^n}(x)$—we denote this action by

$$\text{act}_x : (D(K^n_x) \otimes \mathfrak{gl}_n) \otimes \delta_{D^n}(x) \to \delta_{D^n}(x).$$

This action induces an isomorphism of $D(K^n_x) \otimes \mathfrak{gl}_n$-modules

$$(3.3) \quad \delta_{D^n}|_x \cong U(D(K^n_x) \otimes \mathfrak{gl}_n) \otimes U(D(O_x) \otimes \mathfrak{gl}_n) \mathcal{C}.$$

3.3. Levels. Let $M$ be a coherent $D_{X \times S/S}$-module on $X \times S \to S$. We define a line bundle $\det(M)$ on $S$ as follows. Consider the bounded complex $R\pi_{\*} \text{DR}(M) = \underset{\Delta_X}{\overset{L}{\otimes}} \mathcal{O}_X D_X$ of coherent $\mathcal{O}_S$-modules, and form its determinant line bundle

$$\det(M) = \det(R\pi_{\*}M \overset{L}{\otimes} \mathcal{O}).$$

Note that if $S$ is smooth and $M$ is in fact a right (absolute) $D_{X \times S}$-module, then $R\pi_{\*}M = R\pi_{\*}M \overset{L}{\otimes} \mathcal{O} \in \text{mod} - D_S$ is the right $D$-module push-forward of $M$.

Thus the determinant of de Rham cohomology line bundle $\det(M)$ is the $D$-module analogue of the determinant of cohomology line bundles on moduli stacks of bundles.

Suppose that $M$ is a cuspidal-induced $D$-module, so that $M \cong \overline{M} \otimes D_{Y \to X}$ for a cuspidal quotient $\pi : Y \to S$ of $X \times S \to S$ over $S$. Then we have $\text{DR} M = h(M) = \overline{M}$, and $\det(M)$ is the determinant of cohomology

$$\det(M) = \det(R\pi_{\*}\overline{M})$$

of the $\mathcal{O}$-module push-forward of $\overline{M}$ to $S$. It follows that the determinant line bundle, restricted to a fiber $\text{Gr}_P^{(x)}(x)$, is naturally identified with the “Plücker” determinant line bundle on the Grassmannian $\text{Gr}(K^n_x)$. The following proposition follows from the description of $\det$ on $\text{Gr}_P^{(x)} \mid_x$ as a tensor product of local factors at the points of $x$ (for an algebraic treatment of factorization in the closely related context of $c$-factors, see [BBE]):

**Proposition 3.8.** The line bundle $\det$ on $\text{Gr}_P$ has a natural factorization structure over the factorization structure of $\text{Gr}_P$.

We can now define a new chiral algebra $\delta_P^c$ for every $c \in \mathbb{C}$ as level $c$ delta functions along the unit section on $\text{Gr}_P$. More precisely, the factorization line bundle $\det$ has a lift unit $\det$ of the unit section of $\text{Gr}_P$: in other words, $\det$ is canonically trivialized along unit. As a consequence, there is a natural direct image functor from $D_{X^1}$-modules to $\det \otimes c$-twisted $D$-modules on $\text{Gr}_P$. It follows that
we may take $\omega_X$, push it forward to $Gr^I$ as a det$^c$-twisted $D$-module, and take its $O$-module direct image to $X^I$ to obtain a new chiral algebra, denoted by $\delta^c$.

(Alternatively we can describe $\delta^c$ as the $O$-module restriction of the sheaf of det$^c$-twisted differential operators to the unit section.)

4. $W_{1+\infty}$ and $W_{1+\infty}(\mathfrak{g}_n)$

In this section, we explain how to identify the chiral algebra $\delta^c$ or, more generally, $\delta^c_{\mathfrak{g}_n}$: it is the chiral algebra associated to the vertex algebra $W_{1+\infty}$ (respectively, $W_{1+\infty}(\mathfrak{g}_n)$) at level $c$. We refer to [K] for a discussion of $W_{1+\infty}$ and [vdL] for $W_{1+\infty}(\mathfrak{g}_n)$. We begin in Section 4.1 by reviewing the definition of the chiral algebras $W_{1+\infty}$ and, more generally, $W_{1+\infty}(\mathfrak{g}_n)$. In Section 4.2 we then identify them with $\delta^c$ and $\delta^c_{\mathfrak{g}_n}$.

4.1. Introducing $W_{1+\infty}$. We recall (from [BD2], 2.5 and [BS]) the construction of the Lie* algebra (or conformal algebra [K]) $\mathfrak{gl}(D) = \text{End}^* D$. As a right $D$-module, $\mathfrak{gl}(D)$ is the induced $D$-module $D \otimes D$. We consider the sheaf $D$ as a quasi-coherent $O$-module equipped with a Lie algebra structure given by differential morphisms. (It forms the Lie algebra of right $D$-module endomorphisms of $D$.) It follows that the induced $D$-module has a natural Lie* algebra structure, as explained in [BD2] or [FB, 19,1,7]. We let $W_{1+\infty} = U_{ch} \mathfrak{gl}(D)$ denote the universal enveloping chiral algebra of $\mathfrak{gl}(D)$. It follows that the fibers of $W_{1+\infty}$ are identified with the vacuum representation of $D(K_x)$,

\[
W_{1+\infty}(x) \cong UD(K_x) \otimes UD(O_x) C. 
\]

(4.1)

The Lie* algebra $\mathfrak{gl}(D)$ has a central extension $\widehat{\mathfrak{gl}}(D)$ which may be described as follows. Consider the short exact sequence

\[
0 \to O \otimes \omega_X \to j_*j^*O \otimes \omega_X \to D \to 0,
\]

where $D$ is considered as $O$-bimodule. Restriction to the diagonal defines a morphism $O \otimes \omega_X \to \Delta_* \omega_X$, and we take the push-out exact sequence (and push forward to $X$), obtaining a central extension

\[
0 \to \omega_X \to \widehat{D} \to D \to 0.
\]

This $\omega_X$-extension of $D$ corresponds to an $\omega_X$-central extension of the Lie* algebra $\mathfrak{gl}(D)$. It is also shown in [BD2, BS] that the action of $D(K_x)$ on $K_x$ gives rise to a dense embedding of $D(K_x)$ in $\mathfrak{gl}(K_x)$, the Tate endomorphisms of $K_x$, and that the Tate central extension of $\mathfrak{gl}(K_x)$ restricts to the above central extension $\widehat{D}(K_x)$. The following lemma is an immediate consequence:

Lemma 4.1. The action of $D(K_x)$ on $Gr(K_x)$ lifts to an action of the central extension $\widehat{D}(K_x)$ on det with level one.

We denote $W^c_{1+\infty}$ the corresponding chiral enveloping algebra at level $c$. Repeating the construction with $D \otimes \mathfrak{gl}_n$ in place of $D$ leads to the chiral algebra $W^c_{\infty}(\mathfrak{g}_n)$. 


4.2. Comparison of $\delta_D$ and $W_{1+\infty}$. In this section we identify the chiral algebra $\delta^n_D$, associated to the factorization space $Gr_{D^n}$ with the $W$-algebra $W_{\infty}(gl_n)$:

**Theorem 4.2.** There is a natural isomorphism of chiral algebras $W_{\infty}(gl_n) \cong \delta^n_D$. Moreover for any $c \in C$, this isomorphism lifts to an isomorphism of the chiral algebra $W_{\infty}(gl_n)$ at level $c$ with the chiral algebra $\delta^n_D$ of level $c$ delta functions on $Gr_{D^n}$.

We will explain two proofs of this theorem below. First, in Section 4.2.1, we explain that the theorem is a special case of a very general construction of chiral algebras from the unit section of a unital factorization space. Because this general construction seems to be a folk theorem that does not appear in the literature, we also explain in Section 4.2.2 a more concrete proof that follows closely the approach to Kac-Moody algebras explained in [Ga].

4.2.1. Chiral Hopf Algebras and Factorization Spaces. We begin with some generality. Let $\{S^I \to X^I\}_I$ be a unital factorization space. Recall that, in particular, each $S^I$ is a formally smooth ind-scheme of ind-finite type over $X^I$. Write $S = S^{(*)} \subset X$. Pulling back the tangent sheaf unit* $T_S$ along the unit section gives a $D_X$-module (using the flat connection on $S/X$) which we denote by $L$. The factorization structure on $S$ equips $L$ with a structure of Lie*-algebra on $X$. We assume that this Lie*-algebra is an ind-$D$-vector bundle—that is, that it is a colimit of $D$-vector bundles $L_j$ and the Lie*-algebra structure is compatible with this realization (in the standard sense that $L_j$ and $L_k$ multiply to $L_{j+k}$). It then follows from [BD2, Lemma 2.5.7] that the dual $D$-module $L^\vee$ is a pro-Lie*-coalgebra, [BD2, Section 2.5.7].

Write $\hat{S}$ for the formal completion of $S$ along the unit section; then $\hat{S}$ is again a unital factorization space over $X$. This space can be completely reconstructed from the Lie*-algebra $L$: this is essentially just the factorization analog of the fact that a smooth formal group scheme can be reconstructed from its Lie algebra. In the factorization setting, this works as follows. To the Lie*-algebra one can associate a chiral algebra, the chiral enveloping algebra $U^{ch}(L)$ of $L$ [BD2, Section 3.7]. The chiral enveloping algebra is naturally a cocommutative Hopf chiral algebra (defined in [BD2, Section 3.4.16]. The dual $D$-module $U^{ch}(L)^\vee$ is then a commutative pro-Hopf chiral algebra, and, by the discussion in [BD2, Section 3.4.17], its formal spectrum gives a factorization space $\hat{S}(L) = \{\hat{S}(L)^I \to X^I\}$. Under the hypothesis that $L$ is an ind-$D$-vector bundle, moreover, $\hat{S}(L)$ is isomorphic to $\hat{S}$, the formal completion of our original factorization space along the unit section.

As an immediate consequence, one has:

**Theorem 4.3.** Suppose that $\{S^I \to X^I\}_I$ is a unital factorization space and that the associated Lie*-algebra $L$ is an ind-$D$-vector bundle. Then the delta-function chiral algebra $\delta = r_\ast \text{unit}_\omega_X$ is isomorphic to $U^{ch}(L)$.

There is also an extension of the theorem to the twisted chiral algebra $\delta^\tau$ associated to a factorization line bundle $L^I$ on $S^I$. We leave it to the reader to formulate the analog of Theorem 4.3 in this case.

Theorem 4.2 is an immediate corollary once we identify $L$ with $gl(D^n)$; this is a consequence of Proposition 4.4.
4.2.2. Direct Proof of Theorem 4.2. In this section we give a somewhat different, more direct and concrete, identification of the chiral algebra $\delta_{\mathbb{P}^n}$ associated to the factorization space $\text{Gr}_{\mathbb{P}^n}$ with the $\mathcal{W}$-algebra $\mathcal{W}_{1+\infty}(\mathfrak{gl}_n)$. For simplicity of exposition, we restrict to the case $n = 1$ and $c = 0$, in other words to $\mathcal{W}_{1+\infty}$. The proof is modeled directly on the identification of the Kac-Moody vertex algebra with the factorization algebra of delta functions on the affine Grassmannian due to Beilinson and Drinfeld explained in [Ga]. The result is a formal consequence of the construction of an extension of the action of $\mathcal{D}(\mathcal{K}_x)$ on $\text{Gr}_\mathcal{D}(x)$ to a “factorization action” of the Lie algebra $\mathfrak{gl}(\mathcal{D})$ on the factorization space $\text{Gr}_\mathcal{D}$ (Proposition 4.4 below).

We assume that $X$ is affine in the construction, though the final result will not depend on this assumption. Recall ([BD2] 3.7, [Ga]) that to any right $\mathcal{D}$-module $L$ on $X$ we can assign a collection $\mathcal{L}^I$ of left $\mathcal{D}$-modules on $X$, with certain factorization isomorphisms. Consider the open subset $j^{(I)}: U \subset X^I \times X$ of pairs $\{(x_i), x\}$ where $x \neq x_i$ for any $i \in I$. We let $\mathcal{L}^I$ denote the relative de Rham cohomology sheaf $\mathcal{L}^I = p_{X^I}(\text{id}_{\mathcal{D}}h(j^{(I)}_*j^{(I)^*}\mathcal{O}_{X^I} \boxtimes L))$ on $X^I$, whose fiber at $\{x_i\}$ is the de Rham cohomology of $L$ on $X$ with poles allowed at the $x_i$. The sheaf inherits a left $\mathcal{D}$-module structure from that on $\mathcal{O}_{X^I}$. The sheaves $\mathcal{L}^I$ satisfy several good factorization properties with respect to maps of sets $J \rightarrow I$ ([BD2]), in particular for $J \rightarrow I$ we have canonical isomorphisms $\Delta^+_I \mathcal{L}^I \cong \mathcal{L}^I$. The sheaf $\mathcal{L}^I$ contains as a $\mathcal{D}$-submodule the de Rham cohomology of $L$ with no poles allowed, $\mathcal{T}^I = p_{X^I}(\mathcal{O}_{X^I} \boxtimes L)$. We define $L^I$ as the quotient sheaf, and note that this sheaf is local in $X$, in the sense that the fiber at $\{x_i\}$ only depends on $L$ in the neighborhood of the $x_i$. In particular

$$L^{(1)} = p_{X^I}(j_*j^*(\mathcal{O}_X \boxtimes L)/\mathcal{O}_X \boxtimes L) = L^I$$

is the left $\mathcal{D}$-module version of $L$. When $L$ is a Lie algebra, then the sheaf $\mathcal{L}^I$ is a Lie algebra in the tensor category of left $\mathcal{D}$-modules, and $\mathcal{T}^I$ is a Lie sub-algebra. The sheaf $U\mathcal{L}^I \otimes_{U\mathcal{T}^I} \mathcal{C}$ of induced (vacuum) representations then forms a factorization algebra, corresponding to the chiral enveloping algebra $U^{ch}(L)$ of $L$.

In our case, the Lie algebra $L = \mathfrak{gl}(\mathcal{D})$ is an induced $\mathcal{D}$-module, with de Rham cohomology $h(\mathfrak{gl}(\mathcal{D})) = \mathcal{D}$. It follows that the sheaves $\mathfrak{gl}(\mathcal{D})^{(I)}$ may be identified (as Lie algebras in left $\mathcal{D}$-modules) with the usual (sheaf or $\mathcal{O}$-module) push-forward

$$\mathfrak{gl}(\mathcal{D})^{(I)} = p_{X^I}(j_*^{(I)}j^{(I)^*}\mathcal{O}_{X^I} \boxtimes \mathcal{D}) = p_{X^I}(\mathcal{D}_X^X \boxtimes x^I(\ast x)),$$

i.e. with differential operators with arbitrary poles along the universal divisor on $X^I \times X$ over $X^I$. The subsheaf $\mathfrak{g}(\mathcal{D})^{(I)}$ then consists of global differential operators on $X$. The quotient sheaf has fiber at $x$ the space $\mathcal{D}(\mathcal{K}_x)/\mathcal{D}(\mathcal{O}_x)$ of delta-functions at $x$, and has fiber at $\{x_i\}$ the global sections of the $\mathcal{D}$-module on $X$ which consists of delta functions at each of the points $x_i$, without multiplicities.

We now describe the “factorization action” of $\mathfrak{gl}(\mathcal{D})$ on $\text{Gr}_\mathcal{D}$, which is a multipoint generalization of the formally transitive action of $\mathcal{D}(\mathcal{K}_x)$ on $\text{Gr}_\mathcal{D}(x)$ and resulting description $\left. T_\mathcal{D} \text{Gr}_\mathcal{D} \right|_x = \mathcal{D}(\mathcal{K}_x)/\mathcal{D}(\mathcal{O}_x)$ of the tangent space.

**Proposition 4.4.**

1. The Lie algebra $\mathfrak{g}(\mathcal{D})^{(I)}$ in left $\mathcal{D}$-modules over $X^I$ acts on $\text{Gr}_\mathcal{D}^{(I)}$, compatibly with connections and factorization structures, and with stabilizer at the unit section the Lie sub-algebra $\mathfrak{g}(\mathcal{D})^{(I)}$. 
(2) For \( x \in X \), the Lie algebra \( \tilde{\mathfrak{gl}}(D)(x) \) is dense in \( D(K_x) \), and its action on \( \text{Gr}_D(x) \) is the restriction of the formally transitive action of \( D(K_x) \).

Proof. Consider the sheaf of Lie algebras \( \tilde{\mathfrak{gl}}(D)^I = p_{X,*} D(\bullet \times \tilde{x}) \) of differential operators with poles along the universal divisor \( \tilde{x} \). It acts by infinitesimal automorphisms on the sheaf \( D(\bullet \times \tilde{x}) \), and hence on its functor of submodules, preserving the sub-functor of submodules cosupported on \( \tilde{x} \). This is our desired action on \( \text{Gr}_D^I \). The action is compatible with decompositions of the divisor and forgetting of multiplicities, hence with the factorization structure. The formal transitivity follows from Lemma 3.4.

We are now ready to prove Theorem 4.2.

**Proof of Theorem 4.2.** We begin by constructing a natural map

\[
\text{act} : j_*j^*(\mathfrak{gl}(D) \otimes \delta_D) \to \Delta_\delta_D
\]

satisfying

1. \( \mathfrak{gl}(D) \) acts on the chiral algebra \( \delta_D \) by derivations.
2. The resulting action on \( \delta_D(x) \) of the completed de Rham cohomology \( \tilde{h}_x(\mathfrak{gl}(D)) \) coincides with the action \( D(K_x) \otimes \delta_D(x) \to \delta_D(x) \) induced by the action \( \text{act}_x \) of \( D(K_x) \) on \( \text{Gr}_D(x) \).

To construct the map \( \text{act} \), it is equivalent to define the map obtained by taking de Rham cohomology along the first factor, that is a map \( (j_*j^*(D \boxtimes \mathcal{O}))(\mathcal{O}_x) \otimes \delta_D \to \Delta, \delta_D \).

This map is supported set-theoretically on the diagonal, and so is equivalent to a continuous \( D \)-module action of the completion \( p_{I,*}(j_*j^*(D \boxtimes \mathcal{O}))(\mathcal{O}_x) \) along the diagonal on \( \delta_D \). Note that

\[
\tilde{\mathfrak{gl}}(D)^X = p_{I,*}(j_*j^*(D \boxtimes \mathcal{O})) \subset p_{I,*}(j_*j^*(D \boxtimes \mathcal{O}))(\mathcal{O}_x)
\]

is dense since \( X \) is affine, and so it suffices to define a continuous \( D \)-module action of \( \tilde{\mathfrak{gl}}(D)^X \) on \( \delta_D \). Such an action is provided by Proposition 4.4. The statement that this action is by derivations of the chiral algebra \( \delta_D \) follows from the compatibility of the action with factorization. More precisely, the action of \( \tilde{\mathfrak{gl}}(D)^X \) on \( \delta_D \) is the restriction to the diagonal of the action of \( \tilde{\mathfrak{gl}}(D)^I \) on \( \delta_D^I \) for \( I = \{1,2\} \), and the chiral bracket on \( \delta_D \) is simply the gluing data for \( \delta_D^I \) along the diagonal, so that we have a commutative diagram

\[
\begin{array}{ccc}
\tilde{\mathfrak{gl}}(D)^{I,12} \otimes \delta_D^{I,12} & \xrightarrow{\Delta_\delta} & \Delta(\tilde{\mathfrak{gl}}(D)^X \otimes \delta_D) \\
\downarrow & & \downarrow \\
\tilde{\mathfrak{gl}}(D)^{I,12} \otimes \Delta_\delta & \xrightarrow{\Delta_\delta} & \Delta(\tilde{\mathfrak{gl}}(D)^X \otimes \delta_D)
\end{array}
\]

The compatibility with the one-point action follows from the compatibility of the action of \( \tilde{\mathfrak{gl}}(D)^X \).

Next we construct a map of \( \mathfrak{gl}(D) \) in \( \delta_D \) which on fibers is the embedding of the tangent space \( i^*_x \mathfrak{gl}(D) = D(K_x) / D(\mathcal{O}_x) \) into delta-functions on \( \text{Gr}_D(x) \). This is simply the image of the action map \( D(K_x) \cdot 1 \subset \delta_D(x) \) on the unit, so the families version is the map

\[
j_*j^*(\mathfrak{gl}(D) \boxtimes \omega_X) \xrightarrow{\text{act}(\text{unit})} \Delta_\delta_D
\]

given by the action \( \text{act} \) on the unit, which factors through a map \( \mathfrak{gl}(D) \to \delta_D \). It follows from the derivation property of \( \text{act} \) that this is a map of chiral modules over \( \mathfrak{gl}(D) \). Moreover we claim that the restriction of the chiral bracket of \( \delta_D \) to
$j_*j^*(\mathfrak{gl}(\mathcal{D}) \boxtimes \delta_\mathcal{D})$ is the same as the action map $\text{act}$. This follows from the derivation property of $\text{act}$ and the unit axiom, by comparing the two ways of multiplying $\mathfrak{gl}(\mathcal{D}) \boxtimes \omega_X \boxtimes \delta_\mathcal{D}$.

It now follows that the map $\mathfrak{gl}(\mathcal{D}) \rightarrow \delta_\mathcal{D}$ lifts to a homomorphism of chiral algebras and $\mathfrak{gl}(\mathcal{D})$-modules $\mathcal{W}_{1+\infty} \rightarrow \delta_\mathcal{D}$, which on fibers induces the isomorphism $\mathcal{W}_{1+\infty}(x) \cong \delta_\mathcal{D}(x)$ arising from the actions of $\mathcal{D}(\mathcal{K}_x)$, (3.3) and (4.1). The theorem at level zero follows. Finally, the theorem at arbitrary level follows from the statement concerning the lifting of $\mathcal{D}(\mathcal{K}_x)$ to the determinant line bundle on $\text{Gr}_\mathcal{D}(x)$ at level one.

5. $\mathcal{W}_{1+\infty}$-symmetry and integrable systems

In this section, we explain how the factorization space $\text{Gr}_\mathcal{D}$ unifies geometric features of the free fermion CFT, the KP hierarchy, and the geometry of moduli of curves and bundles. We begin by reviewing the geometry of the Krichever construction (Section 5.1). We then explain, from the point of view of the geometry of $\mathcal{D}$-bundles, the additional $\mathcal{W}_{1+\infty}$-symmetry of these systems (Section 5.2). The infinitesimal $\mathcal{W}_{1+\infty}$-symmetry is only part of the full symmetry encoded in the factorization space $\text{Gr}_\mathcal{D}$, however. In Section 5.3, we explain the relationship between global symmetries of bundles—the Hecke modifications—and of solitons—the Bäcklund-Darboux transformations. Finally, we explore (Section 5.4) the significance of $\text{Gr}_\mathcal{D}$ for the interesting and still-mysterious subject of $\mathcal{W}$-geometry.

5.1. The Krichever construction. Let us begin by briefly reviewing the Krichever construction of solutions of KP in the formalism of the Sato Grassmannian following [KNTY]—see also [DJM, SW, Mu].

Let $\mathcal{K} = \mathbb{C}((z))$ denote the field of Laurent series, and $\mathcal{O} = \mathbb{C}[[z]]$ the ring of Taylor series. The Sato Grassmannian $\text{GR}$ [S] parametrizes subspaces of $\mathcal{K}$ which are complementary to subspaces commensurable with $\mathcal{O}$ (see [AMP, BN2] for algebraic constructions of $\text{GR}$). Note that this Grassmannian, which is a scheme, is quite different from the ind-scheme $\text{Gr}(\mathcal{K})$, the thin Grassmannian, parametrizing subspaces commensurable with $\mathcal{O}$.

The Lie algebras $\mathcal{K}$ and $\text{Der} \mathcal{K} = \mathbb{C}((z)) \partial_z$ of Laurent series (with zero bracket) and Laurent vector fields act on $\text{GR}$ through their action (by multiplication and derivation) on $\mathcal{K}$ itself. The KP hierarchy appears as the infinitely many commuting flows coming from the action of $\mathcal{K}$ (or more precisely of its sub-algebra $\mathbb{C}[z^{-1}]$) on the Grassmannian (or more precisely on its big cell, in natural coordinates).

Given a smooth projective complex curve $(X, x)$ and a line bundle $\mathcal{L}$ on it, together with a formal coordinate and a formal trivialization of $\mathcal{L}$ at $x$, Krichever’s construction produces a point of the Sato Grassmannian, and hence a solution of the KP hierarchy which may be expressed using theta functions. Namely, we consider the space $\mathcal{L}(X \setminus x)$ of sections on the punctured curve, embedded in $\mathcal{K}$ using the coordinate and trivialization. Note that for this construction we only require $X$ to be smooth at $x$, and can consider on an equal footing rank one torsion-free sheaves on singular curves with a marked smooth point. The construction can also be described as specifying the conformal blocks (or correlation functions) for the theory of a free fermion or a free boson on $X$ twisted by $\mathcal{L}$ [KNTY, U, FB].

The KP flows (action of $\mathcal{K}$) vary the line bundle $\mathcal{L}$ via translation in the Picard group of $X$ (the positive half $\mathcal{O}$ merely changing the trivialization). This action is in fact infinitesimally transitive on the moduli space $\text{Pic}(X, x)$ of line bundles on
with a formal trivialization at $x$, thereby giving a formal uniformization of the Picard group $\text{Pic}(X)$. Likewise, the action of $\text{Der} \mathcal{K}$ (for fixed line bundle $\mathcal{L} = \mathcal{O}_X$) or the action of the semi-direct product $\mathcal{K} \rtimes \text{Der} \mathcal{K}$ deforms the pointed curve $(X, x)$ (with the Taylor vector fields $\text{Der} \mathcal{O} = \mathbb{C}[[z]] \partial_z$ merely changing the coordinate and moving the point along $X$). Again, these actions are infinitesimally transitive on the moduli space $\hat{\mathcal{M}}_{g,1}$ of pointed curves with formal coordinate or of the $\hat{\text{Pic}}(X, x)$-bundle $\hat{\mathcal{M}}_{g,1}$ of all Krichever data, and give formal uniformizations of the moduli spaces $\mathcal{M}_{g,1}$ and $\text{Pic}_{g,1}$: see [Ko, AKDP, BS, TUY], or [FB] for a detailed exposition. We can thus consider the moduli spaces $\hat{\mathcal{M}}_{g,1}$ and $\hat{\text{Pic}}_{g,1}$ as global (integrated) versions of homogeneous spaces for the corresponding Lie algebras. The same construction applies with $\mathcal{GR} = \mathcal{GR}(\mathcal{K})$ replaced by the (isomorphic) Grassmannian $\mathcal{GR}^n = \mathcal{GR}(\mathcal{K}^n)$, line bundles replaced by vector bundles and $\mathcal{K} = \mathfrak{gl}_1(\mathcal{K})$ replaced by the loop algebra $\mathfrak{gl}_n(\mathcal{K})$. The KP flows on $\mathcal{GR}$ are now replaced by the multicomponent KP flows on $\mathcal{GR}^n$, corresponding to maximal tori in $\mathfrak{gl}_n(\mathcal{K})$ [KvdL] (see [Pl] and [BN2] for more on the geometry of multicomponent KP).

The Sato Grassmannian carries a canonical line bundle, the determinant line bundle, which defines the Plücker embedding of $\mathcal{GR}$ into the projective space of the fermionic Fock space. This line bundle restricts to the corresponding determinant or theta line bundles on the moduli of curves and bundles. The actions of the Lie algebras $\mathcal{K}$, $\text{Der} \mathcal{K}$ and $\mathfrak{gl}_n(\mathcal{K})$ on the Grassmannian (and on the corresponding moduli spaces) extend canonically to the actions of the Heisenberg, Virasoro and Kac-Moody central extensions $\mathfrak{gl}_1$, $\text{Vir}$ and $\mathfrak{gl}_n$ respectively on the determinant line bundles.

5.2. $\mathcal{W}_{1+\infty}$-orbits of Krichever data. The free fermion theory, the Sato Grassmannian and the determinant line bundle all carry an important additional symmetry: the action of the Lie algebra $\mathcal{D}(\mathcal{K})$ of differential operators with Laurent coefficients and of its central extension $\mathcal{W}_{1+\infty}$.

From the point of view of the KP hierarchy, the action of $\mathcal{W}_{1+\infty}$ is given by the Orlov-Schulman additional symmetries, and is given explicitly on tau functions by the Adler-Shiota-van Moerbeke formula (see [vM] for a review). Note that this symmetry algebra contains both the Heisenberg and Virasoro algebras as zeroth-order and first-order operators. It is natural, therefore, to ask for a moduli space interpretation of (integrated) orbits of the $\mathcal{W}_{1+\infty}$ action extending the Heisenberg-Virasoro uniformization of moduli of curves and bundles.

Thus, given a pointed curve $(X, x)$ we consider the moduli functor $\widehat{\text{Bun}}_{\mathcal{D}}(X, x)$ of $\mathcal{D}$-line bundles on $X$ trivialized in the formal neighborhood of $x$. By the Cannings-Holland correspondence (as in Section 2) this functor is the direct limit of the moduli functors of rank one torsion-free sheaves on cuspidal quotients of $X$ equipped with trivializations near $x$ (in other words, we allow singularities at all points other than $x$). It is then easy to see that the Krichever embedding on cusp curves gives an embedding

$$\widehat{\text{Bun}}_{\mathcal{D}}(X, x) \hookrightarrow \mathcal{GR}.$$  

Explicitly, this assigns to a $\mathcal{D}$-line bundle $M$ trivialized near $x$ its de Rham cohomology on the punctured curve $h(M|_{X\setminus x}) \subset h(M|_{D^*})$, viewed as a subspace of its de Rham cohomology on the punctured disc (which is itself identified with $\mathbb{C}((z))$).
The determinant bundle on the Sato Grassmannian restricts to the usual determinant line for torsion-free sheaves, which, as we have explained in Section 3.3, is the determinant line of de Rham cohomology for $D$-bundles. Furthermore, we have:

**Proposition 5.1.** The $W_{1+\infty}$ flows on the Sato Grassmannian span the tangent space to the embedding $\text{Bun}_D(X, x) \to \text{GR}$ of the moduli of $D$-line bundles on $X$, or equivalently rank one sheaves on cusp quotients of $X$, trivialized at $x$.

**Proof.** Let $D$ and $D^\infty$ denote the disc and punctured disc at $x$. Suppose $M$ is a $D$-bundle equipped with a trivialization on $D$. First-order deformations of $M$ are given by $\text{Ext}^1_D(M, M)$; since $M$ is locally projective over $D$, we find that $\text{Ext}^1_D(M, M) = H^1(X, \text{End}_D(M))$. Using the long exact sequence

$$0 \to \text{End}_D(M) \to \text{End}_D(M)|_{X \setminus x} \to \text{End}_D(M)|_{X \setminus x}/\text{End}_D(M) \to 0$$

and the isomorphisms

$$\text{End}_D(M)|_{X \setminus x}/\text{End}_D(M) \cong \text{End}(M|_{D^\infty})/\text{End}(M|_D) \cong D(K)/D(0)$$

(first the one coming from the given trivialization of $M$ over $D$), we find that

$$H^1(X, \text{End}(M)) = \text{End}(M|_{X \setminus x})/\text{End}(M|_{D^\infty})/\text{End}(M|_{D^\infty}).$$

Similarly, the tangent space to $\text{Bun}_D(X, x)$ at a pair consisting of a $D$-bundle $M$ and trivialization on $D$ is $\text{End}(M|_{X \setminus x})/D(0)$.

In particular the canonical action of $D(K)$ on $\text{Bun}_D(X, x)$ given by changing the transition functions on the punctured disc is infinitesimally transitive. Moreover, the de Rham functor identifies the action of $D(K)$ on itself by right $D$-module automorphisms with its action on $K$ as differential operators. Thus the embedding $\text{Bun}_D(X, x) \to \text{GR}$ is equivariant for $D(K)$. \qed

**Remark 5.2 (Isomonodromy and moving the curve).** We now have two interpretations of the action of vector fields $\text{Der}(K)$ (or the Virasoro algebra) on Krichever data: on the one hand this action deforms the pointed curve $(X, x)$, but on the other hand it is tangent to the moduli $\text{Bun}_D(X, x)$ of $D$-line bundles on a fixed pointed curve $(X, x)$. This apparent discrepancy is accounted for by the isomonodromy connection on the moduli spaces $\text{Bun}_D(X, x)$ (see [BF] for a parallel discussion of isomonodromy and the Segal-Sugawara construction).

More precisely, as we vary $(X, x)$ infinitesimally, each $D$-module on $X$ has a unique extension to a $D$-module on the family, i.e., a unique isomonodromic deformation. This gives a canonical horizontal distribution on the family of moduli of $D$-bundles (independent of the trivialization near $x$) over the moduli $\mathcal{M}_{g,1}$ of pointed curves. In coordinates (i.e., in terms of the Virasoro and $W_{1+\infty}$ uniformizations) this distribution is given by the embedding $\text{Der}(K) \to D(K)$. Thus, the effect of moving the curve infinitesimally on the Krichever data in the Sato Grassmannian can be realized by torsion-free sheaves on cusp quotients of the fixed curve.

**5.3. Bäcklund-Darboux transformations and the adèlic Grassmannians.** So far we have discussed only the infinitesimal symmetries of the Sato Grassmannian and KP hierarchy. We next turn our attention to global symmetries, the Bäcklund-Darboux transformations or, equivalently, Hecke modifications.
The global symmetry transformations of soliton hierarchies are the Bäcklund-Darboux transformations, which in their classical form for KdV involve replacing a Lax operator \( L \) by \( L' \) when the two are conjugate by a differential operator \( P \), i.e., \( L = PQ \) and \( L' = QP \) for some differential operator \( Q \). More generally, Bäcklund-Darboux transformations (for equations of KdV type) are the residual symmetries coming from the loop group symmetries of the corresponding Grassmannians (see e.g. [Pa]).

On the other hand, the global symmetry transformations of moduli spaces of vector bundles or principal bundles on curves are the Hecke correspondences. Two bundles on a curve are related by a (simple) Hecke modification when we are given an isomorphism between their restriction to the complement of a point (this is also the origin of Tyurin coordinates on moduli of bundles). Since the Hecke correspondences are the residue of the loop group symmetries underlying moduli of bundles, it is not surprising that Bäcklund-Darboux transformations are expressed as Hecke modifications on the level of geometric solutions of soliton equations, though this connection does not appear to be very explicit in the literature (see however [LOZ]).

The Hecke modifications of line bundles are directly related to the vertex operator of [DJKM] (the bosonic realization of a free fermion, see [DJM, KNTY]), which is itself a “Darboux operator in disguise” [vM]. As is explained in detail in [FB, Ch.20], the fermion vertex algebra, and specifically the fermion vertex operator, is obtained as distributions on (i.e., as the group algebra of) \( \mathcal{K}^\times / \mathcal{O}^\times \) (whose \( \mathbb{C} \)-points are the integers \( \mathbb{Z} \)). This group acts on the moduli of line bundles on a pointed curve, with the generator taking a line bundle \( L \) to its Hecke transform \( L(x) \). The corresponding element of the group algebra (the delta function at the generator) gives the vertex operator.

While Hecke modifications of line bundles form a group, Hecke modifications for vector bundles or principal \( G \)-bundles do not (they correspond to cosets, or double cosets, of a loop group). The algebraic structure of composition of Hecke modifications is precisely captured by the notion of factorization space, and specifically the factorization space structure on the Beilinson-Drinfeld Grassmannian. This is an adèlic version of the affine Grassmannian \( G(\mathcal{K})/G(\mathcal{O}) \), which in the \( GL_1 \) case reduces to the geometry of the group \( \mathcal{K}^\times / \mathcal{O}^\times \) above. The infinitesimal version of this structure is the Kac-Moody vertex algebra, which in the \( GL_1 \) case reduces to the Heisenberg algebra (i.e. the free boson, or the KP flows), while the fermionic vertex operator itself generalizes to the chiral Hecke algebra of Beilinson-Drinfeld (see [FB] for a discussion).

Above we have described a factorization structure on the adèlic Grassmannian of any rank, and shown that its infinitesimal version is precisely the \( \mathcal{W}_{1+\infty} \) vertex algebra and its higher rank analogues \( \mathcal{W}_{1+\infty}(\mathfrak{sl}_n) \). On the other hand, the global structure of the adèlic Grassmannian captures the Bäcklund-Darboux transformations of the KP hierarchy. We have already seen that the adèlic Grassmannian contains as a factorization subspace the Beilinson-Drinfeld Grassmannian for \( GL_n \) (see (3.1)), so in particular in the rank one case contains the information of the fermionic vertex operator or simple Bäcklund-Darboux-Hecke transformations.

The Bäcklund-Darboux transformations for the full KP hierarchy were defined in [BHY1], where it is observed that the adèlic Grassmannian (as originally defined by Wilson, as a space of “conditions” [W1]) is precisely the space of all Bäcklund-Darboux transformations of the trivial solution. (See [HvdL] for a related study.
of Bäcklund-Darboux transformations in terms of actions on the Grassmannian.)
A point $V \subset C((z))$ in the Sato Grassmannian $GR = GR(C((z)))$ is defined in [BHY1] to be a Bäcklund-Darboux transformation of a point $W \subset C((z))$ if there are polynomials $f, g$ in $z^{-1}$ such that

$$f \cdot V \subset W \subset g^{-1} \cdot V.$$ 

In the case when $W$ corresponds to an algebro-geometric solution $(X, x, L)$ of some $n$-KdV hierarchy, i.e. $z^{-n}$ extends to a global function on $X \setminus x$ (for example the trivial solution $C[z^{-1}]$), this implies that $V$ differs from $W$ by finite-dimensional conditions at some divisor $D$ on $X$. It then follows that $V$ is defined by a torsion-free sheaf on a curve obtained by adding cusps to $X$ along $D$, and that this sheaf is identified with $L$ off $D$. Equivalently, $V$ is defined by a $D$-bundle on $X$ which is identified with $L \otimes D$ off $D$—i.e. a Hecke modification of the $D$-bundle $L \otimes D$. Conversely, all torsion-free sheaves on cusp quotients of $X$ (or $D$-bundles on $X$) define points $V$ which are Bäcklund-Darboux transforms of the given solution $W$. In other words, the adelic Grassmannian of $L \otimes D$ precisely parametrizes Bäcklund-Darboux transforms of the corresponding KP solution.

Thus, we obtain a geometric approach to some of the results of [BHY2, BHY3] relating $W_{1+\infty}$ bispectrality and the adelic Grassmannian. (For a discussion of bispectrality for solutions of KP and its multicomponent versions in terms of $D$-bundles see [BN2].) In particular, the fact that the action of $W_{1+\infty}$ preserves the spaces of Darboux transformations of various solutions, or that the corresponding tau functions form representations of $W_{1+\infty}$, are explained by the fact that the $W_{1+\infty}$ vertex algebra is the “Lie algebra” of the “group” of Bäcklund-Darboux transformations, namely the adelic Grassmannian of Hecke modifications of the corresponding $D$-bundle.

One advantage of the geometric approach is that it immediately extends to the multicomponent KP hierarchies (see [vdL] for a study of the $W_{1+\infty}$ symmetries of these hierarchies). Namely the $W_{1+\infty}(gl_n)$-action on algebro-geometric solutions exponentiates to the space of Bäcklund-Darboux-Hecke transforms of the corresponding $D$-bundles, and in the case of solutions coming from the affine line we obtain a bispectral involution. It would be interesting to find explicit descriptions of the corresponding bispectral solutions and the action of the $W_{1+\infty}$ symmetries on them.

5.4. $W$-geometry. Conformal field theory has uncovered a fascinating new $W$-geometry. This mathematically mysterious geometric structure is expected to bear the same relation to the nonlinear $W_n$ vertex algebras (the quantized symmetries of the $n$th KdV hierarchy) as the moduli of bundles and curves bear to the Kac-Moody and Virasoro (or $W_2$) algebras. This geometry is further expected to have deep connections to higher Teichmüller theory, isomonodromy, quantization of higher Hitchin hamiltonians and geometric Langlands (for a sample see [H, Po, LO]).

The present paper yields a precise mathematical framework for $W$-geometry in the limiting, linear case of $W_{1+\infty}$ (studied for example in [Po] and references therein), as we sketch below. Namely, we propose that $W_{1+\infty}$ geometry is the geometry of a noncommutative variety, the quantized cotangent bundles of a Riemann
surface $X$ (for general $X$). This is a natural consequence of the interpretation of the $\mathcal{W}_{1+\infty}$ Lie algebra as the quantization of the Poisson algebra of functions on the cotangent bundle of the punctured disc (or Hamiltonian vector fields on the cylinder). Note that the quantized algebra generalizes simultaneously the ring structure on functions and the Lie bracket on Hamiltonian vector fields, hence the corresponding deformation problem generalizes simultaneously the deformations of line bundles and of the underlying variety.

Namely we consider the moduli stack $\text{Bun}_D(X)$ of all $D$-bundles on $X$, equipped with the line bundle defined by the determinant of de Rham cohomology, as a substitute for the moduli of bundles or curves with $\mathcal{W}_{1+\infty}$ symmetry. We have shown that the choice of trivialization at a point $x$ defines a space $\hat{\text{Bun}}_D(X, x)$ (and line bundle) which is embedded in the Sato Grassmannian (and determinant bundle) and is uniformized by $\mathcal{W}_{1+\infty}$. (An important distinction from the moduli of bundles, parallel to the moduli of torsion-free sheaves on a singular curve, is that not all $D$-bundles are locally trivial at a given $x$, though they are all generically trivial.) The tangent space to $\text{Bun}_D(X)$ at the trivial $D$-bundle is given by $H^1(X, D)$, and in fact functions on the formal neighborhood of the trivial bundle are easily seen to be calculated as the conformal blocks of the $\mathcal{W}_{1+\infty}$ vertex algebra on $X$, as expected from $\mathcal{W}_{1+\infty}$ geometry. Analogous statements hold at an arbitrary $D$-bundle $M$ with tangent space given by $H^1(X, \text{End}(M))$ and formal functions calculated as conformal blocks of the chiral algebra of endomorphisms of $M$. (More generally, one can define $D$-modules on $\text{Bun}_D(X)$ from conformal blocks of any representation of $\mathcal{W}_{1+\infty}$.)

It would be very interesting to obtain a better geometric understanding of the stack $\text{Bun}_D(X)$ and its relation to the many conjectured aspects of $\mathcal{W}$-geometry. In particular, for a curve of genus larger than 1 this stack is the noncommutative counterpart of the stack of torsion-free sheaves on a Stein surface with no global functions, and so may be expected to have good geometric properties. (Note that the stacks that were studied in [BN2, BN3]—the Calogero-Moser spaces—are the moduli of filtered $D$-bundles, corresponding to a compactification of the noncommutative surface under consideration.) We also hope that this geometry might provide a hint to the mystery of $\mathcal{W}_n$-geometry.

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1In the following discussion we will suppress the distinctions between the different variants of the $\mathcal{W}_{1+\infty}$ algebra, which differ by the central extension or the Heisenberg sub-algebra, whose geometric interpretations are evident.
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