Exact relation of lattice and continuum parameters in three-dimensional SU(2)+Higgs theories

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Abstract

The essential features of the high-temperature electroweak phase transition are contained in a three-dimensional super-renormalizable effective field theory. We calculate the exact counterterms needed for lattice simulations of the SU(2)-part of this theory. Scalar fields in both fundamental and adjoint representations are included. The three-dimensional U(1)+Higgs theory is also discussed.

I. INTRODUCTION

Due to its possible effect on the baryon number of the Universe [1], the cosmological electroweak phase transition should be understood in quantitative detail. Unfortunately, even resummed perturbation theory [2–13] may not be accurate enough, since there are infrared problems in the “symmetric” high-temperature phase. This calls for analytic [14–20] or lattice [21–27] studies of the relevant non-perturbative features.

The study of the non-perturbative features can be simplified by combining perturbation theory and non-perturbative methods. Indeed, the momentum scale $p \gtrsim T$ can be integrated out perturbatively, resulting in an effective theory for length scales larger than $1/T$. This is called dimensional reduction [13,28–34]. The effective theory is essentially a three-dimensional (3D) super-renormalizable SU(2) gauge theory with fundamental and adjoint Higgs fields. The effective theory can be studied with analytic [14,20] and lattice [25,27] methods with less effort than the original four-dimensional theory.

This paper is related to lattice simulations of the effective 3D theory. The purpose is to express the bare parameters of the lattice action in terms of the renormalized parameters of the effective continuum theory. The continuum theory is regularized in the

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The relation between lattice and continuum is needed when results from lattice simulations are transformed into physical values of continuum observables. Due to the super-renormalizability of the effective theory, the relation between lattice and continuum can be found exactly with a two-loop calculation. The only bare parameters having different expressions in the two schemes are the masses of the scalar fields. Our method is to calculate the value of a physical gauge-independent observable in both schemes, and to compare the results. We chose the value of the effective potential at the minimum, apart from unphysical vacuum terms, as the physical observable.

The relation between lattice and continuum in three-dimensional SU(2)+Higgs theories has previously been determined in \cite{27}, partly by analytical calculations, and partly by lattice Monte Carlo simulations. In the present paper, we calculate the relation fully analytically. This should improve the accuracy of that part of the result which was in \cite{27} determined by lattice Monte Carlo simulations.

Let us note that our problem is analogous to the problem of relating the values of $\Lambda_{\text{QCD}}$ in $\overline{\text{MS}}$ and lattice regularization schemes in QCD \cite{35–37}. In that case, logarithmic terms arise already at one-loop level, and there are contributions from all orders of perturbation theory. In our case, logarithmic terms arise only at two-loop level, and the result is exact in the continuum limit. Hence a very high accuracy can be reached in relating the results of lattice simulations to continuum physics.

The theories to be discussed are the SU(2) and U(1) gauge theories with Higgs fields in fundamental and adjoint representations. The SU(2) theory with both fundamental and adjoint Higgs fields, or only with a fundamental Higgs field, is relevant for the cosmological electroweak phase transition \cite{12,31,32}. The SU(2) theory with just an adjoint Higgs field is relevant for studies of dimensional reduction of the pure SU(2) gauge theory \cite{38}. The U(1) gauge theory with a fundamental Higgs field could be relevant for numerical studies of superconductivity \cite{39}.

The plan of the paper is the following. In Sec. II, the problem of expressing the lattice parameters in terms of the continuum parameters is explained, and the solution to the problem is outlined. We follow closely \cite{27}. In Sec. III, some details of the calculation are clarified. The results are in Sec. IV, and the conclusions in Sec. V. In the body of the paper, we deal with the SU(2) + fundamental Higgs theory; results for the other theories are collected in the Appendix.

II. FORMULATION OF THE PROBLEM

The SU(2) + fundamental Higgs theory in three Euclidian dimensions is described by the Lagrangian

$$\mathcal{L} = \frac{1}{4} F_{ij}^{a} F_{ij}^{a} + \frac{1}{2} \text{Tr}(D_{i}\Phi)^{\dagger}(D_{i}\Phi) + \frac{1}{2} m_{B}^{2} \text{Tr}\Phi^{\dagger}\Phi + \frac{1}{4} \lambda_{3} \left(\text{Tr}\Phi^{\dagger}\Phi\right)^{2}. \quad (1)$$
Here $F_{ij}^a = \partial_i A_j^a - \partial_j A_i^a - g_3 \epsilon^{abc} A_i^b A_j^c$, $m_B^2 = m_3^2(\mu) + \delta m_3^2$, 

$$\Phi = \frac{1}{\sqrt{2}}(\phi_0 \mathbf{1} + i \phi_3 \tau^a),$$

(2)

and $D_i \Phi = (\partial_i + ig_3 \tau^a A_i^a/2) \Phi$. The $\tau^a$:s are the Pauli matrices, the coupling constants $\lambda_3$ and $g_3^2$ have the dimension GeV, and the indices take the values $a, i, j \in \{1, 2, 3\}$. The sign of $g_3$ in $F_{ij}^a$ and $D_i$ is chosen so as to be compatible with [40].

The theory defined by the Lagrangian of eq. (1) is super-renormalizable [41]. There are only two kinds of divergences, the mass divergence and the unphysical vacuum divergence. In the $\overline{\text{MS}}$-scheme, the exact mass counterterm needed to make the theory finite is [12]

$$\delta m_3^2 = -\frac{\bar{h}^2}{16\pi^2} \frac{\mu^{-4\epsilon}}{4\epsilon} \left( \frac{51}{16} g_3^4 + 9 \lambda_3 g_3^2 - 12 \lambda_3^2 \right).$$

(3)

Here $\bar{h}$ is the loop counting parameter. Due to eq. (3), the renormalized mass squared is of the form

$$m_3^2(\mu) = \frac{\bar{h}^2}{16\pi^2} \left( \frac{51}{16} g_3^4 + 9 \lambda_3 g_3^2 - 12 \lambda_3^2 \right) \log \frac{\Lambda}{\mu}.$$ 

(4)

In the context of the electroweak phase transition, $\log \Lambda$ is known to two-loop order in terms of the physical 4D parameters and the temperature [12].

The two-loop vacuum counterterm of the theory of eq. (I) may be chosen as

$$\delta V = -\frac{\bar{h}^2}{16\pi^2} \frac{\mu^{-4\epsilon}}{4\epsilon} 3 g_3^2 m_3^2(\mu).$$

(5)

This choice removes the $1/\epsilon$-part from the value of the effective potential $V(\varphi)$ at $\varphi = 0$. As a result of eq. (5), $V(0)$ has an unphysical $\mu$-dependence

$$\mu \frac{d}{d\mu} V(0) = \frac{\bar{h}^2}{16\pi^2} 3 g_3^2 m_3^2(\mu).$$

(6)

Of course, the difference between the values of the effective potential at any two distinct minima is free of any $\mu$-dependence.

Next, consider the theory of eq. (I) in lattice (L) regularization. That is, calculations are made on a lattice with lattice spacing $a$ and spatial extension $N$, and in the end the limit $a \to 0$, $N \to \infty$ is taken. For simplicity, in the actual momentum integrations we take the lattice to be in the limit $N \to \infty$ from the beginning. The lattice Lagrangian is

$$\mathcal{L}_L = \frac{4}{a^4 g_3^2} \sum_{i<j} \left[ 1 - \frac{1}{2} \text{Tr} P_{ij}(x) \right]$$

$$+ \frac{1}{a^2} \sum_i \left[ \text{Tr} \Phi^\dagger(x) \Phi(x) - \text{Tr} \Phi^\dagger(x) U_i(x) \Phi(x + i) \right]$$

$$+ \frac{1}{2} m_L^2 \text{Tr} \Phi^\dagger \Phi + \frac{1}{4} \lambda_3 \left( \text{Tr} \Phi^\dagger \Phi \right)^2,$$

(7)
where \( P_{ij}(x) = U_i(x)U_j(x+i)U_i^\dagger(x+j)U_j^\dagger(x) \),

\[
U_i(x) = \exp\left[\frac{i}{2}ag_3\tau^bA^b_i(x)\right],
\]

(8)

\( \Phi \) is as in eq. (2), and \( x + i \equiv x + ae \). The action \( S \) corresponding to the Lagrangian in eq. (7) is

\[
S = a^3\sum_x L_L,
\]

where \( x \) enumerates the lattice sites. The Lagrangian \( L_L \) is invariant under the transformations

\[
U_i(x) \rightarrow U'_i(x) = g(x)U_i(x)g^{-1}(x+i), \quad \Phi(x) \rightarrow \Phi'(x) = g(x)\Phi(x),
\]

(9)

where \( g(x) \in SU(2) \). The path integration over the fields \( A^b_i(x) \) is defined using the Haar measure (see, e.g., [40]), to guarantee the gauge invariance, and hence the renormalizability, of the theory.

In the L-scheme, the counterterms differ from those in the \( \overline{\text{MS}} \)-scheme. For instance, there can be a one-loop mass counterterm in the L-scheme, since the lattice spacing \( a \) provides an extra scale that can be combined with \( g_3^2 \), to make a quantity of the dimension of mass squared. In general,

\[
m^2_L = m^2_L(\mu) + \delta m^2_L(\hbar) + \delta m^2_L(\hbar^2).
\]

(10)

As indicated by the notation, we have chosen the finite renormalized mass squared to be exactly the same as in the \( \overline{\text{MS}} \)-scheme, eq. (4). The bare term \( m^2_L \) as a whole is of course independent of \( \mu \).

The purpose of the present paper is to express the parameter \( m^2_L \) in eq. (10) in terms of the continuum parameters \( m^2_L(\mu) \), \( g_3 \), \( \lambda_3 \), and the lattice spacing \( a \). The one-loop mass-counterterm \( \delta m^2_L(\hbar) \), and the terms proportional to \( \lambda_3g_3^2 \) and \( \lambda_3^2 \) in the two-loop mass-counterterm \( \delta m^2_L(\hbar^2) \), have already been calculated analytically [27]. The two-loop contribution proportional to \( g_3^4 \) has been computed with lattice Monte Carlo methods [27]. Below we calculate analytically even the contribution proportional to \( g_3^4 \).

The method of calculation is the following. We extract both from eq. (1) and eq. (7) a measurable gauge-independent physical quantity. Since both calculations must give the same result, \( m^2_L \) can be fixed. The simplest suitable quantity is the value of the effective potential at the minimum, \( V(\text{min}) \). To be more precise, \( V(\varphi) \) contains unphysical divergent vacuum terms, such as the one shown in eq. (5). However, apart from these, \( V(\text{min}) \) gives the equation of state, and is thus physical. It has been explicitly proved that \( V(\text{min}) \) is gauge-independent, when calculated consistently in powers of \( \hbar \) [12, 14].

There is another, equivalent, way of formulating the problem, without reference to the effective potential. Indeed, one can just calculate the value of the path integral in the broken minimum using the loop expansion. In other words, \( \phi_0 \) is shifted to the classical broken minimum, and then all the connected vacuum graphs are calculated. It turns out that this gives just \( V(\text{min}) \). To separate the vacuum terms, one should calculate the value of the path integral in the symmetric minimum, as well. The conceptual advantage of calculating
directly the path integral is that complications related to fixing the gauge when calculating the effective potential are avoided.

As a matter of fact, the problem is even simpler than calculating \( V(\text{min}) \). From renormalizability, one knows that any difference in the \( \phi \)-dependent parts of \( V(\phi) \) between two schemes could only appear in the \( \phi^2 \)-term. This can roughly be seen also with simple power-counting arguments. Indeed, any difference between two schemes arises from the UV-region. Hence the difference should be analytic in the parameters \( m_\phi^2 \) appearing in the propagators, which depend quadratically on the field \( \phi \). At one loop, the difference is then dimensionally of the form \( (1 + a^2 m_\phi^2 + a^4 m_\phi^4 + \ldots) a^{-3} \), and at two loops, of the form \( g_\phi^2 (1 + a^2 m_\phi^2 + a^4 m_\phi^4 + \ldots) a^{-2} \). Apart from vacuum terms, higher loops give contributions vanishing as \( a \to 0 \). Hence non-vanishing differences could only arise in the \( \phi^2 \)-terms and at two-loop order. From the equation

\[
\begin{align*}
V(\text{min}) = V_0(\phi_0) + h V_1(\phi_0) + h^2 \left[ V_2 - \frac{1}{2} \left( \frac{V_1'}{V_0''} \right)^2 \right]_{\phi=\phi_0},
\end{align*}
\]

where \( \phi_0 \) is the location of the classical broken minimum, it follows that the difference of the \( \phi^2 \)-terms of two schemes determines the difference of the values \( V(\text{min}) \). In short, the counterterms in eq. (10) can be fixed by requiring that the two schemes produce the same \( \phi^2 \)-terms.

Let us state the problem in one more disguise: the effective potential \( V(\phi) \) itself is gauge-dependent, but the difference of the effective potentials in the \( L \) and \( \overline{\text{MS}} \)-schemes is not so, since it determines the gauge-independent quantity \( m_L^2 \).

To conclude this Section, we note that \( V(\text{min}) \) is directly related to the measurable quantity \( \langle \frac{1}{2} \text{Tr}\Phi^\dagger\Phi \rangle \) on lattice, by

\[
\langle \frac{1}{2} \text{Tr}\Phi^\dagger\Phi \rangle = \frac{d V(\text{min})}{d m_L^2(\mu)}. \tag{12}
\]

In consequence, one can actually measure the parameter \( m_L^2 \) of eq. (10) on lattice, by comparing lattice data to continuum perturbative results in a region where perturbation theory works well \[27\]. For such a comparison, even the mass-dependent unphysical vacuum contributions of the type in eq. (5), but in the \( L \)-scheme, are needed, since they enter through the right-hand side of eq. (12). Hence, we will write down also the mass-dependent vacuum counterterms \[27\] below, although mass-independent vacuum terms are neglected.

III. DETAILS OF THE CALCULATION

A. Choice of gauge

Since we are calculating the gauge-independent quantity \( V(\text{min}) \), the gauge may be chosen at will. The simplest possibility is the \( R_\xi \)-gauge with \( \xi = 1 \). It is not suitable
for calculating the effective potential for arbitrary $\varphi$ (see, e.g., [12]), but when $V(\text{min})$ is extracted from $V(\varphi)$ consistently in powers of $\bar{h}$ using eq. (11), the $R_\xi$-gauge can be used [14]. Hence the difference between the two-loop contributions to the effective potential in the L- and $\overline{\text{MS}}$-schemes can be calculated in this gauge.

To be absolutely sure, one could also just calculate all the connected graphs in the classical broken minimum, since then no reference is made to the effective potential. This amounts to fixing

$$\varphi \equiv \sqrt{-m_3^2(\mu)/\lambda_3},$$  \hspace{1cm} (13)

where $\mu$ is chosen so that $\varphi$ is real, and adding all the reducible two-loop graphs to the irreducible ones contributing to the effective potential. We shall indicate below the differences in the intermediate stages of the two mentioned ways of organizing the calculation.

The unshifted Lagrangian needed at the two-loop level is obtained by expanding eq. (7) in powers of $A_a^i$, and by adding the gauge-fixing and the ghost term. The gauge fixing term is chosen as $L_\xi = F^a F_a^b / 2 \xi a^2$, where

$$F^a(x) = \sum_i [A_i^a(x) - A_i^b(x - i)] + \frac{1}{2} \xi g_3 \varphi \phi(x).$$  \hspace{1cm} (14)

Gauge fixing is compensated for by the Faddeev-Popov determinant $\text{det}[\partial F^a(x)/\partial \theta^b(y)]$, where the $\theta^b(y)$ parametrize the gauge transformations of eq. (9) as

$$g(x) = \exp[\frac{i}{2} g_3 \gamma^b \theta^b(x)].$$  \hspace{1cm} (15)

With these terms added, the unshifted Lagrangian is complete.

To get the shifted Lagrangian needed for calculating $V(\varphi)$, one replaces $\phi_0$ by $\phi_0 + \varphi$. The non-diagonal terms between $A_a^i$ and $\phi_0$ are cancelled due to eq. (14). If one is calculating the effective potential, all the linear terms are neglected. If one is calculating the value of the path integral in the broken minimum, the linear term $\varphi \phi_0 [m_3^2(\mu) + 3 \lambda_3 \varphi^2]$ vanishes due to eq. (13), but the counterterm $\delta m_3^2 \varphi \phi_0$ remains. This enters when reducible two-loop graphs of the type in Fig. 2 of [13] are calculated.

**B. Feynman rules**

From the shifted Lagrangian, one can read the Feynman rules of the theory. From now on, we take the gauge parameter equal to unity, $\xi = 1$. The masses of the shifted theory are

$$m_T^2 \equiv \frac{1}{4} g_3^2 \varphi^2, \quad m_1^2 \equiv m_3^2(\mu) + 3 \lambda_3 \varphi^2, \quad m_2^2 \equiv m_3^2(\mu) + \lambda_3 \varphi^2, \quad m_T^2 = m_2^2 + m_T^2.$$  \hspace{1cm} (16)

If $\varphi$ is chosen according to eq. (13), the Goldstone boson mass squared $m_2^2$ vanishes. However, it is useful to keep it in calculations even in this case, since this allows one to separate
the unphysical vacuum contributions. Indeed, the vacuum contributions are obtained by
calculating the value of the loop expansion in the symmetric phase, which means putting
\( \varphi \to 0 \) in all the expressions, i.e., calculating \( V(0) \).

To display the propagators and the vertices of the theory, we use the notations

\[
\tilde{p}_i = \frac{2}{a} \sin \frac{a}{2} p_i, \quad \tilde{p}^2 = \sum_i \tilde{p}_i^2, \quad p_i = \cos \frac{a}{2} p_i. \tag{17}
\]

The propagators are then

\[
\langle \phi_0(p) \phi_0(-p) \rangle = \frac{1}{p^2 + m_i^2}, \quad \langle \phi_a(p) \phi_b(-p) \rangle = \frac{\delta_{ab}}{p^2 + m_i^2},
\]

\[
\langle e^i(p) e^b(p) \rangle = -\frac{\delta^{ab}}{p^2 + m_T^2}, \quad \langle A_i^a(p) A^b_j(-p) \rangle = \frac{\delta_{ab}}{p^2 + m_T^2} \delta_{ij}. \tag{18}
\]

The vertices relevant for the two-loop calculation are as follows. From (40) one can read the
two-gluon vertex [eq. (14.39) with \( 1/4a^2 \to 1/6a \)], the three-gluon vertex [eq. (14.43)], the
four-gluon vertex [eq. (14.44) with \( (2/3)(\delta_{AB}\delta_{CD} + \ldots) \to (\delta_{AB}\delta_{CD} + \ldots) \) and \( d_{ABC} \to 0 \)], and the two gluon-ghost vertices [on pages 212 and 213, with the sign of the \( \bar{c}cAA \)-vertex changed]. The remaining part of the action reads

\[
S_\phi = \frac{1}{2} \delta m^2_L (\phi_0^2 + \phi_a \phi_a) + \delta m^2_L \varphi \phi_0
\]

\[
+ \frac{1}{4} \lambda_3 \left[ \phi_0^4 + 2 \phi_0^2 \phi_a \phi_a + \phi_a \phi_a \phi_b \phi_b \right]
+ \frac{1}{4} \lambda_3 \varphi \phi_0 \left[ \phi_0^2 + \phi_a \phi_a \right]
\]

\[
- \frac{1}{2} g_3 \delta(p + q + r)(p_i - r_i) \left[ \phi_0(p) A_i^a(q) \phi_a(r) + \frac{1}{2} \epsilon_{abc} \phi_a(p) A^b_j(q) \phi_c(r) \right]
\]

\[
+ \frac{1}{8} g_3^2 \delta(p + q + r + s) (r_i - s_i) A_i^a(p) A^b_j(q) \left[ \phi_0(r) \phi_0(s) + \phi_b(r) \phi_b(s) \right]
\]

\[
+ \frac{1}{4} g_3^2 \varphi \phi \phi_0 \phi_0 + \frac{1}{2} \epsilon_{abc} \phi_a \phi_b \phi_c,
\]

where due summations and integrations are implied. The tree-level part was not displayed,
and the linear counterterm \( \delta m^2_L \varphi \phi_0 \) is not needed for \( V(\varphi) \). The integration measure is

\[
\int dp \equiv \int_{-\pi/a}^{\pi/a} \frac{d^3p}{(2\pi)^3}, \tag{20}
\]

and \( \delta(p) \) is a shorthand for \( (2\pi)^3 \delta_P(p) \), where \( \delta_P(p) \) is periodic with period \( 2\pi/a \).

### C. Integrations

In the limit \( a \to 0 \), eq. (19) naturally reproduces the corresponding part of the action
of the theory in eq. (10), apart from counterterms. However, when individual graphs are
calculated with finite \( a \), and the limit \( a \to 0 \) is taken only after the integrations, results differ from those in the \( \overline{\text{MS}} \)-scheme. In this Section we work out the differences of the one- and two-loop contributions. Mass-independent vacuum terms of the form \( 1/a^3 \) and \( g_3^2/a^2 \) are neglected.

Let us start by calculating the one-loop counterterms. These are known from [27], but we repeat the calculation. In the limit \( a \to 0 \), the difference of the one-loop effective potentials of the L- and \( \overline{\text{MS}} \)-schemes is

\[
V_L(\varphi) - V_{\overline{\text{MS}}}(\varphi) = \frac{1}{2} \delta m_L^2(h) \varphi^2 + \frac{\hbar}{24\pi a} (6m_L^2 + m_1^2 + 3m_2^2),
\]

where

\[
\Sigma = \frac{1}{\pi^2} \int_{-\pi/2}^{\pi/2} d^3x \frac{1}{\sum_i \sin^2 x_i}.
\]

Eq. (21) can easily be calculated in a general gauge, and is seen to be gauge-independent. Apart from vacuum terms, the two schemes must give the same result, and hence the difference in eq. (21) must disappear. Using eq. (16) one then sees that

\[
\delta m_L^2(h) = -\left( \frac{3}{2} g_3^2 + 6\lambda_3 \right) \frac{\hbar \Sigma}{4\pi a}.
\]

The mass-dependent vacuum counterterm, needed to make the vacuum part of the right-hand side of eq. (21) disappear, is

\[
\delta V_L(h) = -2m_3^2 \frac{\hbar \Sigma}{4\pi a}.
\]

The two-loop graphs are naturally much more tedious than the one-loop graphs. For illustration, we calculate the most complicated of them in some detail. This is the graph (vvv) in Fig. 1. From the Feynman rules it follows that

\[
(vvv) = -\frac{1}{2} g_3^2 \int dp dq dr \delta(p + q + r) \frac{F(p, q, r)}{(\bar{p}^2 + m_L^2)(\bar{q}^2 + m_T^2)(\bar{r}^2 + m_T^2)},
\]

where

\[
F(p, q, r) = \sum_{i,j,k} \left[ \delta_{kj} p_j (\bar{r}_i - q_i) + \delta_{ik} q_k (\bar{p}_j - r_j) + \delta_{ji} r_j (\bar{q}_k - p_k) \right]^2.
\]

Using trigonometric identities, one can express \( F(p, q, r) \) in terms of products of the functions \( \bar{p}_i, \bar{q}_i, \) and \( \bar{r}_i \). Utilizing the symmetry of eq. (25) in exchanges of \( p, q, \) and \( r, \) one then gets

\[
F(p, q, r) \Rightarrow 3 \left( 3 - \frac{1}{4} a^2 \bar{r}^2 \right) \left( 2\bar{p}^2 + 2\bar{q}^2 - \bar{r}^2 - a^2 \sum_i \bar{p}_i^2 \bar{q}_i^2 \right)
- 3 \left( 3\bar{r}^2 - 2a^2 \sum_i \bar{p}_i^2 \bar{r}_i^2 \right) \left( \frac{1}{4} a^2 \sum_i \bar{r}_i^4 + \frac{1}{4} a^4 \sum_i \bar{p}_i^2 \bar{q}_i^2 \bar{r}_i^2 \right).
\]
Next, factors of $m_T^2$ are added and subtracted in eq. (24), so that terms of the form $\vec{p}^2 + m_T^2$ cancel against similar terms in the denominator of eq. (25). As a result, the following nine types of integrals remain:

1. There is the integral

$$ I_1 = g_3^2 m_T^2 \int dp \, dq \frac{1}{[\vec{p}^2 + m_T^2][\vec{q}^2 + m_T^2][(p + q)^2 + m_T^2]} \equiv g_3^2 m_T^2 H_a(m_T, m_T, m_T). \quad (28) $$

In $[27]$ $H_a(m_T, m_T, m_T)$ was parametrized as

$$ H_a(m_T, m_T, m_T) = \frac{1}{16\pi^2} \left[ \log \frac{2}{a m_T} + \frac{1}{2} + \zeta + O(a) \right], \quad (29) $$

and from $[12]$ it is known that

$$ H_a(m_T, m_T, m_T) = H_c(m_T, m_T, m_T) + \frac{1}{16\pi^2} \left[ \log \frac{6}{a \mu} + \zeta \right] + O(a), \quad (30) $$

where $H_c(m_T, m_T, m_T)$ is the finite part of the continuum limit of $H_a(m_T, m_T, m_T)$ in the $\overline{\text{MS}}$-scheme. The contributions of eq. (28) to the renormalized two-loop effective potential $V_2(\varphi)$ in the lattice and $\overline{\text{MS}}$-schemes hence differ by $(g_3^2 m_T^2/16\pi^2)[\log(6/a\mu) + \zeta]$. There is also a contribution of the form $g_3^2 m_T^2 a^2 H_a(m_T, m_T, m_T)$ from the diagram (vvv) to $V_2(\varphi)$, but by eq. (29) this vanishes in the continuum limit.

2. There is the integral

$$ I_2 = g_3^2 \int dp \, dq \frac{1}{(\vec{p}^2 + m_T^2)(\vec{q}^2 + m_T^2)} \equiv g_3^2 I_a(m_T), \quad (31) $$

where $I_a(m_T)$ is $[27]

$$ I_a(m_T) \equiv \int dp \frac{1}{p^2 + m_T^2} = \frac{1}{4\pi} \left[ \frac{\Sigma}{a} - m_T - \xi a m_T \right] + O(a^2). \quad (32) $$

Note that the $\xi$ appearing in eq. (32) has nothing to do with the $\xi$ in eq. (14); the latter has been fixed to unity. In the $\overline{\text{MS}}$-scheme $I_a(m_T)$ is replaced by $I_c(m_T) = -m_T/4\pi$, so that the difference between the contributions of the two schemes is

$$ I_a(m_T)I_a(m_T) - I_c(m_T)I_c(m_T) = \frac{\Sigma^2}{16\pi^2 a^2} - \frac{\Sigma}{8\pi^2 a} m_T - \frac{\Sigma}{8\pi^2} \xi m_T^2 + O(a). \quad (33) $$

3. The integral

$$ I_3 = g_3^2 m_T^2 a^2 \int dp \, dq \frac{1}{(\vec{p}^2 + m_T^2)(\vec{q}^2 + m_T^2)} \equiv g_3^2 m_T^2 a^2 I_a(m_T)I_a(m_T) \quad (34) $$

has no analogue in the $\overline{\text{MS}}$-scheme, but gives by eq. (33) the finite contribution $g_3^2 m_T^2 \Sigma^2/16\pi^2$ in the L-scheme.

4. The integral

$$ I_4 = g_3^2 a^2 \int dp \, dq \frac{1}{p^2 + m_T^2} = g_3^2 I_a(m_T) \frac{1}{a} \quad (35) $$
has no analogue in the \( \overline{\text{MS}} \)-scheme, but gives by eq. (32) the term \(-g_3^2(m_T a^{-1} + \xi m_T^2)/4\pi \) in the L-scheme.

5. The integral

\[
I_5 = g_3^2 a^4 \int dp dq \frac{\sum_i \tilde{p}_i^2 \tilde{q}_i^2}{(p^2 + m_T^2)(q^2 + m_T^2)} = \frac{1}{3} g_3^2 a^4 \int dp dq \frac{\tilde{p}^2 \tilde{q}^2}{(p^2 + m_T^2)(q^2 + m_T^2)} = \frac{g_3^2}{3a^2} - \frac{2}{3} g_3^2 m_T^2 a I_a(m_T) + O(a^2)
\]

has no analogue in the \( \overline{\text{MS}} \)-scheme, but gives by eq. (32) \(-g_3^2 m_T^2 \Sigma/6\pi \) in the L-scheme.

6. There is the integral

\[
I_6 = g_3^2 a^2 \int dp dq \frac{\sum_i \tilde{p}_i^2 (p_i + q_i)^2}{[\tilde{p}^2 + m_T^2][\tilde{q}^2 + m_T^2][(p + q)^2 + m_T^2]} = \frac{g_3^2 m_T^2}{16\pi^6} \int_{\pi/2}^{\pi} d^3x \sum_i \sin^2 x_i \sin^2 (x_i + y_i) \frac{\sum_i \sin^2 x_i \sin^2 (x_i + y_i)}{[\sum_i \sin^2 x_i + z^2][\sum_i \sin^2 y_i + z^2][\sum_i \sin^2 (x_i + y_i) + z^2]},
\]

where \( z^2 = a^2 m_T^2/4 \). The integral \( I_6 \) contains a linear \( 1/a \)-divergence, which can be separated by adding and subtracting \( g_3^2 \alpha a I_a(m_T)/a \), where

\[
\alpha = a^3 \int dp \frac{\sum_i \tilde{p}_i^4}{(\tilde{p}^2)^2} = \frac{1}{\pi^3} \int_{\pi/2}^{\pi/2} d^3x \sum_i \frac{\sin^4 x_i}{(\sum_i \sin^2 x_i)^2}.
\]

In the rest of the integral, writing the propagators in the form

\[
\frac{1}{\sum_i \sin^2 x_i + z^2} = -\frac{z^2}{(\sum_i \sin^2 x_i + z^2)(\sum_i \sin^2 x_i)} + \frac{1}{\sum_i \sin^2 x_i},
\]

allows one to separate the vacuum part \( g_3^2/a^2 \), the finite part \( g_3^2 m_T^2 \), and the part vanishing with \( a \). Neglecting the vacuum part, the result is

\[
I_6 = \frac{g_3^2 \alpha}{a} I_a(m_T) - \frac{g_3^2 m_T^2}{4\pi^2}(\delta + \rho) + O(a),
\]

where

\[
\delta = \frac{1}{2\pi^4} \int_{-\pi/2}^{\pi/2} d^3x \sum_i \frac{\sin^2 x_i \sin^2 (x_i + y_i)}{(\sum_i \sin^2 x_i)^2 (\sum_i \sin^2 x_i + y_i) \sum_i \sin^2 y_i},
\]

\[
\rho = \frac{1}{4\pi^4} \int_{-\pi/2}^{\pi/2} d^3x \sum_i \left\{ \frac{\sin^2 x_i \sin^2 (x_i + y_i)}{(\sum_i \sin^2 x_i)^2} - \frac{\sum_i \sin^4 x_i}{(\sum_i \sin^2 x_i)^2} \right\} \frac{1}{(\sum_i \sin^2 y_i)^2}.
\]

7. In the integral

\[
I_7 = g_3^2 m_T^2 a^4 \int dp dq \frac{\sum_i \tilde{p}_i^2 (p_i + q_i)^2}{[\tilde{p}^2 + m_T^2][\tilde{q}^2 + m_T^2][\tilde{p}^2 + \tilde{q}^2 + m_T^2]},
\]

the mass terms in the propagators give contributions of higher order in \( a \). Hence \( I_7 \) has in the limit \( a \to 0 \) the value \( I_7 = g_3^2 m_T^2 \kappa_1/\pi^2 \), where
\[
\kappa_1 = \frac{1}{4\pi^4} \int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{\sum_i \sin^2 x_i \sin^2 (x_i + y_i)}{\sum_i \sin^2 x_i \sum_i \sin^2 (x_i + y_i) \sum_i \sin^2 y_i}. \tag{44}
\]

8. The integral
\[
I_8 = g_3^2 a^2 \int dp \, dq \frac{\sum_i (p_i + q_i)^4}{[p^2 + m_f^2][(q^2 + m_f^2)][(p + q)^2 + m_f^2]} \tag{45}
\]
can be handled exactly as \( I_6 \). The result is
\[
I_8 = 2 \frac{g_3^2 \alpha}{a} I_a(m_T) - \frac{g_3^2 m_f^2}{4\pi^2} (\kappa_2 + \kappa_3) + \mathcal{O}(a), \tag{46}
\]
where
\[
\kappa_2 = \frac{1}{4\pi^4} \int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{\sum_i \sin^4 x_i}{(\sum_i \sin^2 x_i) \sum_i \sin^2 (x_i + y_i) \sum_i \sin^2 y_i}. \tag{47}
\]
\[
\kappa_3 = \frac{1}{2\pi^4} \int_{-\pi/2}^{\pi/2} d^3x d^3y \left\{ \frac{1}{\sum_i \sin^2 x_i \sum_i \sin^2 (x_i + y_i)} - \frac{1}{(\sum_i \sin^2 x_i)^2} \right\} \frac{\sum_i \sin^4 x_i}{(\sum_i \sin^2 y_i)^2}. \tag{48}
\]

9. The integral
\[
I_9 = g_3^2 a^4 \int dp \, dq \frac{\sum_i \tilde{p}_{i}^2 \tilde{q}_{i}^2 (p_i + q_i)^2}{[\tilde{p}^2 + m_f^2][\tilde{q}^2 + m_f^2][(\tilde{p} + \tilde{q})^2 + m_f^2]} \tag{49}
\]
can be simplified with the method of eq. (33), giving \( I_9 = -3g_3^2 m_f^2 \kappa_4/4\pi^2 \), where
\[
\kappa_4 = \frac{1}{\pi^4} \int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{\sum_i \sin^2 x_i \sin^2 (x_i + y_i) \sin^2 y_i}{(\sum_i \sin^2 x_i)^2 \sum_i \sin^2 (x_i + y_i) \sum_i \sin^2 y_i}. \tag{50}
\]

This completes the enumeration of the integrals that appear in the graph (vvv).

In addition to the continuum contributions taken into account when discussing the integrals \( I_1 \) and \( I_2 \) in eqs. (32) and (33), there are extra continuum contributions in the \( \overline{\text{MS}} \)-scheme. Namely, the graphs (vvv) and (vvs) contain a part where the trace of the metric tensor \( \delta_{ii} = 3 - 2\epsilon \) multiplies the function \( H_\epsilon \propto 1/\epsilon \). The finite contributions arising from the products of \( \epsilon \) and \( 1/\epsilon \) are
\[
\text{(vvv)} \Rightarrow - \frac{\hbar^2}{16\pi^2} \frac{9}{16} g_3^4 \varphi^2, \quad \text{(vvs)} \Rightarrow \frac{\hbar^2}{16\pi^2} \frac{3}{32} g_3^4 \varphi^2. \tag{51}
\]

Naturally, this kind of contributions do not arise on lattice.

When all the numerical factors are taken into account, the integrals \( I_1-I_9 \) and the extra continuum contributions in eq. (33) finally yield for the difference of the L and \( \overline{\text{MS}} \)-schemes from the graph (vvv) the value
\[
\text{(vvv)} \Rightarrow g_3^2 (18\Sigma - 3\pi - 9\pi \alpha) \frac{m_T}{a} + \frac{9}{4} g_3^4 \varphi^2 \left( \log \frac{6}{a\mu} + \zeta \right) + \frac{3}{8} g_3^4 \varphi^2 \left( \frac{3}{2} - \frac{5}{4} \Sigma^2 + \frac{\pi}{3} \Sigma - 4\delta - 4\rho + 4\kappa_1 - \kappa_2 - \kappa_3 - 3\kappa_4 \right). \tag{52}
\]

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Here a common factor $h^2/16\pi^2$ has been neglected. In addition, terms proportional to $\xi$ are not shown explicitly, since one can see from the above that a term of the from $\xi m^2$ is always accompanied with the term $m/a$. In the end, the $1/a$-terms will cancel, so that the $\xi$-terms also cancel.

To conclude this Section, we list the differences of the L and $\overline{\text{MS}}$-schemes to $V(\varphi)$ from the rest of the irreducible two-loop graphs, shown in Fig. [1]. Again the factor $h^2/16\pi^2$, and all terms proportional to $\xi$, are neglected. The graph (s) arises from the one-loop mass-counterterm in eq. (23), and (v) arises from the gluon-gluon vertex induced by the Haar measure. The graph (vv') arises from the $\varphi^2 A^4$-vertex in eq. (19):

(v) $\Rightarrow -3\pi g_3^2 \frac{m_T}{a}$

\begin{equation}
(s) \Rightarrow \frac{3}{4}(g_3^2 + 4\lambda_3)\Sigma \left(\frac{m_1}{a} + 3\frac{m_2}{a}\right) \end{equation}

\begin{equation}
(vv) \Rightarrow g_3^2(14\pi - 18\Sigma) \frac{m_T}{a} + \frac{7}{8} g_3 A^2 \Sigma^2
\end{equation}

\begin{equation}
(vg) \Rightarrow -2\pi g_3^2 \frac{m_T}{a} - \frac{1}{8} g_3 A^2 \Sigma^2
\end{equation}

\begin{equation}
(vs) \Rightarrow -\frac{3}{2} g_3^2 (3\Sigma - 2\pi) \frac{m_T}{a} - \frac{9}{8} g_3 A^2 \Sigma \left(\frac{m_1}{a} + 3\frac{m_2}{a}\right) + \frac{3}{4} g_3^2 m_3 A^2 + \frac{9}{64} (g_3^4 + 8\lambda_3 g_3^2) A^2 \Sigma^2
\end{equation}

\begin{equation}
(ss) \Rightarrow -3\lambda_3 \Sigma \left(\frac{m_1}{a} + 3\frac{m_2}{a}\right)
\end{equation}

\begin{equation}
(vv') \Rightarrow -\frac{15}{128} g_3^4 A^2 \Sigma^2
\end{equation}

\begin{equation}
(vgg) \Rightarrow 3 g_3^2 (\pi\alpha - \Sigma) \frac{m_T}{a} + \frac{3}{4} g_3 A^2 \Sigma (\delta + \rho) - \frac{3}{8} g_3 A^2 \left(\log \frac{6}{a\mu} + \zeta\right)
\end{equation}

\begin{equation}
(vvsv) \Rightarrow \frac{3}{64} g_3^4 A^2 (\Sigma^2 - 2) - \frac{9}{16} g_3 A^2 \left(\log \frac{6}{a\mu} + \zeta\right)
\end{equation}

\begin{equation}
(ggs) \Rightarrow -\frac{3}{32} g_3 A^2 \left(\log \frac{6}{a\mu} + \zeta\right)
\end{equation}

\begin{equation}
(vss) \Rightarrow 3 g_3^2 (\Sigma - \pi\alpha) \frac{m_T}{a} + \frac{3}{8} g_3 A^2 \Sigma \left(\frac{m_1}{a} + 3\frac{m_2}{a}\right) - \frac{3}{16} g_3 A^2 (3\delta + 4\rho) - \frac{9}{2} \lambda_3 g_3^2 A^2 \delta
\end{equation}

\begin{equation}
+ \frac{3}{8} (g_3^4 + 12\lambda_3 g_3^2) A^2 \left(\log \frac{6}{a\mu} + \zeta\right) + 3 g_3^2 m_3 A^2 \left(\log \frac{6}{a\mu} + \zeta - \delta\right)
\end{equation}

\begin{equation}
(sss) \Rightarrow -6\lambda_3^2 A^2 \left(\log \frac{6}{a\mu} + \zeta\right).
\end{equation}
IV. RESULTS AT TWO-LOOP LEVEL

We are now ready to sum together the differences of the L- and \( \overline{\text{MS}} \)-schemes from all the two-loop graphs. First, let us note that the reducible two-loop graphs of the type in Fig. 2 of [43], needed when calculating the value of the path integral in the broken minimum, give exactly the same result in the L- and \( \overline{\text{MS}} \)-schemes, and hence do not affect \( \delta m^2_L(h^2) \). Only the irreducible graphs of Fig. [43] are significant. Second, when the eqs. (52)–(64) are summed together, all the terms proportional to \( m_1/a \) and \( m'_2/a \) cancel. Apart from vacuum terms, this leaves the result

\[
V_L(\varphi) - V_{\overline{\text{MS}}}(\varphi) = \frac{1}{2} \delta m^2_L(h^2) \varphi^2 + \frac{h^2}{16\pi^2} 9\pi g^2(1 - \frac{\Sigma}{2\pi} - \alpha) \frac{m_T}{a} \\
+ \frac{h^2}{16\pi^2} \frac{\varphi^2}{2} \left[ \left( \frac{51}{16} g^4_0 + 9\lambda_3 g^2_0 - 12\lambda^2_3 \right) \left( \log \frac{6}{a\mu} + \zeta \right) + 9\lambda_3 g^2_0 \left( \frac{1}{4} \Sigma^2 - \delta \right) \right] \\
+ \frac{3}{4} g^4_0 \left[ \left( \frac{15}{16} \Sigma^2 + \frac{\pi}{3} \Sigma + \frac{5}{4} - \frac{7}{2} \delta - 4\rho + 4\kappa_1 - \kappa_2 - \kappa_3 - 3\kappa_4 \right) \right].
\]

(65)

From the identity

\[
0 = \int_{-\pi/2}^{\pi/2} d^3x \frac{d}{dx_1 \sin^2 x_1 + \sin^2 x_2 + \sin^2 x_3} \sin x_1 \cos x_1,
\]

(66)

it follows that \( \alpha = 1 - \Sigma/2\pi \), and hence the \( m_T/a \)-terms cancel. Since eq. (65) must vanish, we finally get

\[
\delta m^2_L(h^2) = -\frac{h^2}{16\pi^2} \left[ \left( \frac{51}{16} g^4_0 + 9\lambda_3 g^2_0 - 12\lambda^2_3 \right) \left( \log \frac{6}{a\mu} + \zeta \right) + 9\lambda_3 g^2_0 \left( \frac{1}{4} \Sigma^2 - \delta \right) \right] \\
+ \frac{3}{4} g^4_0 \left[ \left( \frac{15}{16} \Sigma^2 + \frac{\pi}{3} \Sigma + \frac{5}{4} - \frac{7}{2} \delta - 4\rho + 4\kappa_1 - \kappa_2 - \kappa_3 - 3\kappa_4 \right) \right].
\]

(67)

From eqs. (67) and (68), one can also read the mass-dependent two-loop vacuum counterterm needed on lattice, in order to make the renormalized mass-dependent vacuum parts of the effective potentials the same in the two schemes:

\[
\delta V_L(h^2) = -\frac{h^2}{16\pi^2} 3g^2_0(\mu) \left( \log \frac{6}{a\mu} + \zeta + \frac{\Sigma^2}{4} - \delta \right).
\]

(68)

Note that the \( \mu \)-dependence of eq. (68) reproduces that of eq. (3).

Eq. (67) contains eight pure numbers, \( \zeta, \Sigma, \delta, \rho, \kappa_1, \kappa_2, \kappa_3 \), and \( \kappa_4 \). The parameter \( \Sigma \), defined in eq. (22), is known analytically [27], and its numerical value is \( \Sigma \approx 3.176 \). From the identity

\[
0 = \int_{-\pi/2}^{\pi/2} d^3x d^3y \frac{d}{dx_1 \left[ \sum_i \sin^2 x_i + m^2 \right] \left[ \sum_i \sin^2 (x_i + y_i) + m^2 \right] \left[ \sum_i \sin^2 y_i + m^2 \right]} \sin x_1 \cos x_1,
\]

(69)

it follows that \( \kappa_2 = \Sigma^2/4 - \delta/2 - 1/4 \). This still leaves six parameters to be calculated numerically. In [27] the values \( \zeta \approx 0.09, \delta \approx 1.94, \) and \( \rho \approx -0.314 \) were given. We have
calculated that \( \kappa_1 \approx 0.958, \kappa_3 \approx 0.751, \) and \( \kappa_4 \approx 1.20. \) The accuracy of these numbers could probably be considerably improved with the techniques of [45] adapted to three dimensions, but we have not attempted to do so. Numerically we then have that

\[
\delta m_L^2(h^2) \approx -\frac{h^2}{16\pi^2} \left[ \left( \frac{51}{16} g_3^4 + 9 \lambda_3 g_3^2 - 12 \lambda_3^2 \right) \left( \log \frac{6}{a\mu} + 0.09 \right) + 5.0 g_3^4 + 5.2 \lambda_3 g_3^2 \right].
\]  

(70)

There are a few ways of checking parts of the analytic result in eq. (67). First, the cancellation of \( 1/a \)-divergences indicating the renormalizability of the theory is a non-trivial check, since such terms arise from most of the graphs. Second, from eqs. (4) and (67) one sees that the \( \mu \)-dependence cancels in eq. (10), as it should. Third, in [27] the mass counterterm in the L-scheme was determined by a combination of analytical and lattice Monte Carlo methods. The parts proportional to \( \lambda_3 g_3^2 \) and \( \lambda_3^2 g_3^2 \) in \( \delta m^2_L(h^2) \) were determined analytically, and they agree with eq. (67). The part proportional to \( g_3^4 \) was determined by lattice Monte Carlo methods; for the SU(2) + fundamental Higgs theory, the coefficient of \( g_3^4 \) was parametrized with the number \( \eta_0 = 2.12(7) \), and for the SU(2) + fundamental Higgs + adjoint Higgs theory, with the number \( \eta = 2.18(6) \). Eq. (67) implies for \( \eta_0 \) the value 2.01, and eq. (A9) for \( \eta \) the value 1.96. The systematical error in the determination of \( \eta \) in [27] is larger than in the determination of \( \eta_0 \), since for the former it was assumed that \( \delta m^2_D(h^2) \), defined as the sum of eqs. (A3) and (A11), is negligible. We conclude that the agreement between our analytical result and lattice Monte Carlo simulations is good.

V. CONCLUSIONS

In this paper, the exact relations between lattice and continuum regularization schemes in 3D super-renormalizable SU(2) and U(1) gauge theories with Higgs fields in fundamental and adjoint representations have been calculated. These relations are needed when results from lattice simulations are related to continuum observables. The general structure of the calculated mass counterterms is that, in addition to linear \( 1/a \)-terms and logarithmic \( \log a \)-terms, there are two-loop constant terms proportional to \( g_3^4 \) and \( \lambda_3 g_3^2 \). Here \( \lambda_3 \) denotes the self-coupling of the relevant scalar field. Numerically, the \( g_3^4 \)-terms are rather large. The \( g_3^4 \)-terms are especially significant for the SU(2) gauge theory with a Higgs field in the adjoint representation, since then the “dominant” logarithmic term \( g_3^4 \log a \) vanishes. The results obtained have significance for numerical simulations of gauge theories at high enough temperatures, so that the theories undergo dimensional reduction into an effective 3D theory. In particular, the results are important for numerical simulations of the cosmological electroweak phase transition.

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APPENDIX:

In this Appendix, results for mass counterterms in the lattice regularization scheme are presented for a number of SU(2) and U(1) gauge theories, with Higgs fields in fundamental and adjoint representations.

1. SU(2) + adjoint Higgs

The Lagrangian for the SU(2) + adjoint Higgs theory consists of the standard plaquette action for gauge bosons on the first row of eq. (7), supplemented by the terms

\[ \mathcal{L}_{A_0} = \frac{1}{g^2} \sum_i \left[ \frac{1}{2} \text{Tr} A_0(x) A_0(x) - \frac{1}{2} \text{Tr} A_0(x + i) U_i^{-1}(x) A_0(x) U_i(x) \right] + \frac{1}{2} m^2 D_a A^a_0 + \frac{1}{4} \lambda_A (A^a_0 A^a_0)^2. \]  

(A1)

Here \( A_0 = A^a_0 \tau^a \). The matrix \( A_0 \) transforms in gauge transformations as \( A_0(x) \rightarrow A'_0(x) = g(x) A_0(x) g^{-1}(x) \). The effective potential is calculated by shifting \( A_0 \) to be \( A_0 + \alpha \), and the gauge fixing condition is chosen in complete analogy with eq. (14). The \( A^i_3 \) - and \( c^i_3 \) -fields remain massless despite the shift, and the \( A^i_1 \), \( A^i_2 \), \( c^i_1 \), and \( c^i_2 \)-fields get the mass squared \( m^2 D = g^2 A^2 \). The mass squared of \( A_0^3 \) is \( m^2 D + 3\lambda_A \alpha^2 \), and that of \( A_0^1 \) and \( A_0^2 \) is \( m^2 D + \lambda_A \alpha^2 + g^2 \alpha^2 \).

The two-loop graphs to be calculated in the SU(2) + adjoint Higgs theory are the same as those in Fig. 1, with the \( \Phi \)-field replaced by the \( A_0 \)-field, and the vertices corrected appropriately. As in eq. (51), there are extra continuum contributions in the \( \overline{\text{MS}} \)-scheme from the diagrams (vvv) and (vvs). However, the absolute value of both contributions is \( 3 \frac{g^4}{3 \pi^2} \), and the signs are different, so that these terms cancel.

The one-loop mass-counterterm resulting from eq. (A1) is

\[ \delta m^2_D(h) = -(4g_3^2 + 5\lambda_A) \frac{\hbar \Sigma}{4\pi a}, \]  

(A2)

and the two-loop counterterm is

\[ \delta m^2_D(h^2) = -\frac{\hbar^2}{16\pi^2} \left[ \left( 20\lambda_A g_3^2 - 10\lambda_A^2 \right) \left( \log \frac{6}{a\mu} + \zeta \right) + 20\lambda_A g_3^2 \left( \frac{1}{4} \Sigma^2 - \delta \right) + 2g_3^4 \left( \frac{5}{4} \Sigma^2 + \frac{\pi}{3} \Sigma - 6\delta - 6\rho + 4\kappa_1 - \kappa_2 - 3\kappa_3 - 3\kappa_4 \right) \right]. \]  

(A3)

Using the numerical values given in Sec. [V], \( \delta m^2_D(h^2) \) has the approximate value

\[ \delta m^2_D(h^2) \approx -\frac{\hbar^2}{16\pi^2} \left[ \left( 20\lambda_A g_3^2 - 10\lambda_A^2 \right) \left( \log \frac{6}{a\mu} + 0.09 \right) + 8.7g_3^4 + 11.6\lambda_A g_3^2 \right]. \]  

(A4)

Note that the constant term proportional to \( g_3^4 \) is numerically rather large. If the coupling constant \( \lambda_A \) is very small, as is the case in the context of the high-temperature electroweak
theory, the $g_3^4$-term gives the dominant contribution in eq. (A4) for moderate $a$, since the coefficient of the logarithmic term is vanishing.

The vacuum counterterms, determined analogously to eqs. (24) and (68), are

$$\delta V_L(h) = -\frac{3}{2} m_D^2 \frac{\bar{h} \Sigma}{4\pi a}.$$  \hspace{1cm} (A5)

$$\delta V_L(h^2) = -\frac{\bar{h}^2}{16\pi^2} 6g_3^2 m_D^2 \left( \log \frac{6}{a\mu} + \zeta + \frac{\Sigma^2}{4} - \delta \right).$$  \hspace{1cm} (A6)

2. SU(2) + fundamental Higgs + adjoint Higgs

The SU(2) + fundamental Higgs + adjoint Higgs theory consists of the sum of eqs. (7) and (A1), together with the interaction term

$$L_i = \frac{1}{2} h_3 \text{Tr} \Phi^i \Phi A_0^a A_0^a.$$  \hspace{1cm} (A7)

The one-loop mass counterterm $\delta m_L^2(h)$ is corrected from the value of eq. (23) by the amount

$$\Delta[\delta m_L^2(h)] = -3h_3 \frac{\bar{h} \Sigma}{4\pi a},$$  \hspace{1cm} (A8)

and $\delta m_L^2(h^2)$ is corrected from eq. (67) by the amount

$$\Delta[\delta m_L^2(h^2)] = -\bar{h}^2 \left[ \left( -\frac{3}{4} g_3^4 + 12h_3g_3^2 - 6h_3^2 \right) \log \frac{6}{a\mu} + \zeta \\
+12h_3g_3^2 \left( \frac{\Sigma^2}{4} - \delta \right) - 3g_3^4 \rho \right].$$  \hspace{1cm} (A9)

The adjoint mass counterterm $\delta m_D^2(h)$ of eq. (A2) is corrected by

$$\Delta[\delta m_D^2(h)] = -4h_3 \frac{\bar{h} \Sigma}{4\pi a},$$  \hspace{1cm} (A10)

and $\delta m_D^2(h^2)$ of eq. (A3) is corrected by

$$\Delta[\delta m_D^2(h^2)] = -\bar{h}^2 \left[ \left( -g_3^4 + 6h_3g_3^2 - 8h_3^2 \right) \log \frac{6}{a\mu} + \zeta \\
+6h_3g_3^2 \left( \frac{\Sigma^2}{4} - \delta \right) - 4g_3^4 \rho \right].$$  \hspace{1cm} (A11)

There are no extra contributions of the type in eq. (51) from the coupling constant $h_3$.

The mass-dependent vacuum counterterm of the SU(2) + fundamental Higgs + adjoint Higgs theory is the sum of eqs. (24), (68), (A5), and (A6). Note that for the leading order approximation $h_3 = g_3^2/4$ of dimensional reduction, the logarithmic term in eq. (A11) vanishes.
3. U(1) + fundamental Higgs

The Lagrangian for the U(1) + fundamental Higgs theory is, in analogy with eq. (7),

\[
L = \frac{1}{a^4 e_3^2} \sum_{i<j} \left\{ 1 - \frac{1}{2} \left[ P_{ij}(x) + P^*_{ij}(x) \right] \right\} \\
+ \frac{2}{a^2} \sum_i \left\{ \Phi^*(x)\Phi(x) - \frac{1}{2} \left[ \Phi^*(x)U_i(x)\Phi(x+i) + c.c. \right] \right\} \\
+ m^2_\Phi \Phi^* \Phi + \lambda_3 \left( \Phi^* \Phi \right)^2,
\]

(A12)

where \(U_i(x) = \exp[iae_3 A_i(x)]\), \(P_{ij}(x) = U_i(x)U_j(x+i)U^*_i(x+j)U^*_j(x)\), and \(\Phi = (\phi_0 + i\phi_1)/\sqrt{2}\).

A one-loop calculation produces the mass-counterterm

\[
\delta m^2_L(h) = -(2e_3^2 + 4\lambda_3) \frac{h\Sigma}{4\pi a},
\]

(A13)

and a two-loop calculation yields

\[
\delta m^2_L(h^2) = -\frac{h^2}{16\pi^2} \left[ \left( -4e_3^4 + 8\lambda_3 e_3^2 - 8\lambda_3^2 \right) \left( \log \frac{6}{a\mu} + \zeta \right) + 8\lambda_3 e_3^2 \left( \frac{1}{4} \Sigma^2 - \delta \right) \\
+ e_3^4 \left( \frac{1}{4} \Sigma^2 + \frac{8\pi}{3} \Sigma - 1 - 2\delta - 4\rho \right) \right].
\]

(A14)

Numerically, eq. (A14) gives

\[
\delta m^2_L(h^2) \approx -\frac{h^2}{16\pi^2} \left[ \left( -4e_3^4 + 8\lambda_3 e_3^2 - 8\lambda_3^2 \right) \left( \log \frac{6}{a\mu} + 0.09 \right) + 25.5e_3^4 + 4.6\lambda_3 e_3^2 \right]
\]

(A15)

There is again one extra continuum contribution of the type in eq. (51), present in eq. (A14):

\[
(vv) \Rightarrow \frac{h^2}{16\pi^2} \frac{1}{2} e_3^4 \nu^2.
\]

(A16)

The vacuum counterterms of the U(1) + fundamental Higgs theory are

\[
\delta V_L(h) = -m^2_\Phi \frac{h\Sigma}{4\pi a},
\]

(A17)

\[
\delta V_L(h^2) = -\frac{h^2}{16\pi^2} 2e_3^2 m_3^2 \left( \log \frac{6}{a\mu} + \zeta + \frac{\Sigma^2}{4} - \delta \right).
\]

(A18)

4. U(1) + adjoint Higgs

The U(1) + adjoint Higgs theory is very simple, since the \(A_0\)- and \(A_1\)-fields do not interact. The results for this theory have been given in [27], but to fix the notation, we restate the results. The \(A_1\)-part of the theory is the first row of eq. (A12), and the \(A_0\)-part is
The mass counterterms are
\[ \delta m_D^2(h) = -3\lambda_A \frac{h \Sigma}{4\pi a}, \]  
\[ \delta m_D^2(h^2) = \frac{h^2}{16\pi^2} \left[ 6\lambda_A^2 \left( \log \frac{6}{a\mu} + \zeta \right) \right]. \]  
At one loop there is the vacuum counterterm
\[ \delta V_L(h) = -\frac{1}{2} m_D^2 \frac{h \Sigma}{4\pi a}, \]
but there is no such term at two-loop order. There are no continuum contributions of the type in eq. (51).

5. U(1) + fundamental Higgs + adjoint Higgs

The U(1) + fundamental Higgs + adjoint Higgs theory consists of the sum of eqs. (A12) and (A19), together with the interaction term
\[ \mathcal{L}_i = h_3 \Phi^* \Phi A_0^2. \]  
The one-loop mass counterterm \( \delta m_L^2(h) \) is corrected from the value of eq. (A13) by the amount
\[ \Delta[\delta m_L^2(h)] = -h_3 \frac{h \Sigma}{4\pi a}, \]
and the two-loop counterterm \( \delta m_L^2(h^2) \) is corrected from eq. (A14) by the amount
\[ \Delta[\delta m_L^2(h^2)] = \frac{h^2}{16\pi^2} \left[ 2h_3^2 \left( \log \frac{6}{a\mu} + \zeta \right) \right]. \]  
The adjoint mass counterterm \( \delta m_D^2(h) \) of eq. (A20) is corrected by
\[ \Delta[\delta m_D^2(h)] = -2h_3 \frac{h \Sigma}{4\pi a}, \]
and \( \delta m_D^2(h^2) \) of eq. (A21) is corrected by
\[ \Delta[\delta m_D^2(h^2)] = -\frac{h^2}{16\pi^2} \left[ \left( 4h_3 e_3^2 - 4h_3^3 \right) \left( \log \frac{6}{a\mu} + \zeta \right) + 4h_3 e_3^2 \left( \frac{\Sigma^2}{4} - \delta \right) \right]. \]  
There are no extra continuum contributions of the type in eq. (51) from eq. (A23). The mass-dependent vacuum counterterm of this theory is the sum of eqs. (A17), (A18), and (A22).
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FIG. 1. The irreducible two-loop graphs contributing to the two-loop effective potential in the L-scheme. Wiggly line is the vector propagator, dashed line is the scalar propagator, and double line is the ghost propagator.