CONVERGENCE PROOF FOR THE MULTIGRID METHOD OF THE NONLOCAL MODEL *
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Abstract. Recently, nonlocal models attract the wide interests of scientist. They mainly come from two applied scientific fields: peridynamics and anomalous diffusion. Even though the matrices of the algebraic equation corresponding the nonlocal models are usually Toeplitz (denote $a_0$ as the principal diagonal element, $a_1$ as the trailing diagonal element, etc). There are still some differences for the models in these two fields. For the model of anomalous diffusion, $a_0/a_1$ is uniformly bounded; most of the time, $a_0/a_1$ of the model for peridynamics is unbounded as the stepsize $h$ tends to zero. Based on the uniform boundedness of $a_0/a_1$, the convergence of the two-grid method is well established [Chan, Chang, and Sun, SIAM J. Sci. Comput., 19 (1998), pp. 516–529; Pang and Sun, J. Comput. Phys., 231 (2012), pp. 693–703; Chen, Wang, Cheng, and Deng, BIT, 54 (2014), pp. 623–647]. This paper provides the detailed proof of the convergence of the two-grid method for the nonlocal model of peridynamics. Some special cases of the full multigrid and the V-cycle multigrid are also discussed. The numerical experiments are performed to verify the convergence.

Key words. multigrid method, nonlocal model, Toeplitz matrices

AMS subject classifications. 65M55

1. Introduction. Ranging from characterizing peridynamics [25] to anomalous diffusion [20], the nonlocal models have been built in more and more scientific fields. The used nonlocal operators include nonlocal diffusion operators [15], fractional Laplacian operators [15], Riesz fractional derivative [19,34], and the Riesz tempered fractional derivative [10,11]. In the field of anomalous diffusion, the nonlocal operators are derived in both the time and space directions. For the peridynamics, the nonlocal operators are just applied in the space direction. Mathematically, the nonlocal operators corresponding to these two applied fields have close connections, being examined in [14,15]. The nonlocal operator mentioned in this paper has a finite range of nonlocal interactions measured by a horizon parameter $δ$ [16,25]. When $δ \to 0$, the nonlocal effect diminishes and the local or classical partial differential equation (PDE) models are recovered, if the latter are well-defined. For $δ > 0$, compared with classical PDE models, the complexities are introduced by the nonlocal interactions and the matrices of the resulting discrete systems are no longer sparse. In a series of recent studies [28,29], the robust discretizations of the nonlocal models have been well developed. Based on the fast Toeplitz solver, the direct solution method for the resulting algebraic equation is discussed in [30]. In this work, we focus on the efficiently iterative solvers, especially, providing the strict convergence proof for the algorithm.

As is well know that the structure and conditioning of the resulting coefficient matrix of the numerical scheme play a key role for the effectiveness of the linear solver. For the nonlocal models, the associated stiffness matrices tend to be dense, and its condition number depends on both the nonlocal interaction kernel and horizon parameter $δ$. When $δ$ is fixed, the condition number is bounded even the stepsize $h$ tends

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to zero [4, 55]. However, if $\delta$ depends on $h$, the condition number will tend to infinity as $h$ becomes smaller and smaller. So, it is interesting/necessary to understand the performance of the linear solver for the different types of the horizon $\delta$. In particular, finding the effective linear solver should be paid much more attention for the case that $\delta$ depends on $h$. This work focuses on the multigrid method (MGM) with uniform convergence rate for various types of the horizon $\delta$. MGM has often been shown to be the most efficient iterative method for numerically solving the PDEs [4, 18]. For the uniform convergence of the V-cycle MGM, one can refer to [8, 31, 32, 33] for the second order elliptic operator and [13] for the block tridiagonal matrix. For the multilevel matrix algebras like circulant, tau, Hartely, the V-cycle convergence is theoretically obtained by using some special interpolation operators [2, 3]. For works on the so-called full MGM, i.e., recursive application of the two-grid method (TGM) procedure, see, e.g., [6, 17, 24].

For nonlocal models, peridynamics and anomalous diffusion are two of the most successful applied fields. The common feature of the stiffness matrix of the resulting algebraic equation from the models is to have Toeplitz structure. For the Toeplitz matrix, we denote $a_0$ as the principal diagonal element, $a_1$ as the trailing diagonal element, etc. For the stiffness matrix of nonlocal model describing anomalous diffusion, $a_0/a_1$ is bounded; using this attribute, the uniform convergence of the TGM is theoretically obtained [6, 12, 21]. However, most of the time, $a_0/a_1$ is unbounded for the stiffness matrix of the peridynamic model. So, some new ideas must be introduced for proving the uniform convergence of the TGM for the nonlocal peridynamic model. This paper provides the detailed proof of the uniform convergence of TGM with unbounded $a_0/a_1$. Furthermore, the special cases of the full MGM and V-cycle MGM are also discussed. The performed numerical experiments show the effectiveness of the MGM.

The outline of this paper is as follows. In the next section, we discuss the recently introduced finite difference discretizations of the nonlocal operator. The MGM algorithms are introduced in Section 3. In Section 4, we study the uniform convergence estimates of the TGM for the nonlocal model. Convergence of the full MGM and V-cycle MGM in a special case is analyzed in Section 5. To show the effectiveness of the presented schemes, results of numerical experiments are reported in Section 6. Finally, we conclude the paper with some remarks.

2. Preliminaries: numerical scheme and multigrid method. Before delivering the detailed convergence proof of the TGM, in this section, we review and discuss the numerical discretization and multigrid method for the nonlocal model (2.3).

2.1. The nonlocal operator and discretization scheme. In this subsection, we introduce the discretization of the nonlocal operator proposed in [28] and make some discussions on the generating of the matrix elements and treating of the non-homogeneous boundaries. Let $\Omega$ be a finite bar in $\mathbb{R}$. Without loss of generality, we take $\Omega = (0, b)$, $b > 0$. For $u = u(x) : \Omega \to \mathbb{R}$, the nonlocal operator $L_\delta$ is defined by [28],

$$L_\delta u(x) = \int_{B_\delta(x)} (u(y) - u(x)) \gamma_\delta(|x - y|) dy \quad \forall x \in \Omega$$

with $B_\delta(x) = \{y \in \mathbb{R} : |y - x| < \delta\}$ denoting a neighborhood centered at $x$ of radius $\delta$, which is the horizon parameter and represents the size of nonlocality; the symmetric nonlocal kernel $\gamma_\delta(|x - y|) = 0$ if $y \notin B_\delta(x)$. 
The operator $L$ is used in both the time-dependent nonlocal volume-constrained diffusion problem \cite{15}

\[
\begin{align*}
-u_t - L_\delta u &= f_\delta &\text{on } \Omega, t > 0, \\
u(x, 0) &= u_0 &\text{on } \Omega \cup \Omega_I, \\
u &= g &\text{on } \Omega_T, t > 0,
\end{align*}
\]

for the function $u = u(x, t)$ and its steady-state counterpart

\[
\begin{align*}
-L_\delta u &= f_\delta &\text{on } \Omega, \\
u &= g &\text{on } \Omega_I,
\end{align*}
\]

where $u = g$ denotes a volumetric constraint imposed on a volume $\Omega_I$ that has a nonzero volume and is made to be disjoint from $\Omega$. For 1D case, we use $\Omega_I = (\delta, 0) \cup (b, b + \delta)$.

Let $\gamma_\delta$ be nonnegative and radial, i.e., $\gamma_\delta = \gamma_\delta(|y - x|) \geq 0$. As in \cite{28}, we can rewrite (2.1) as

\[
\begin{align*}
L_\delta u(x) &= \int_0^\delta (u(x + s) - 2u(x) + u(x - s))\gamma_\delta(s)ds,
\end{align*}
\]

which makes the nonlocal operator as a *continuum difference operator*, or rather an average of finite difference operators over a continuum scale $(0, \delta)$ \cite{28}. Assuming that $u(x)$ is regular enough, from (2.4) there exists

\[
L_\delta u(x) = Cu''(x) + O \left( \int_0^\delta s^4\gamma_\delta(s)ds \right),
\]

where, $C$, is assumed to be positive and independent of $\delta$, i.e.,

\[
0 < C = \int_0^\delta s^2\gamma_\delta(s)ds < \infty.
\]

Now, we introduce and discuss the discretization scheme of (2.3). Denote the ratio of the horizon $\delta$ and the mesh size $h$ as

\[
\begin{align*}
R = \frac{\delta}{h} > 0 &\quad \text{and} \quad r = \lfloor R \rfloor,
\end{align*}
\]

which plays an important role in nonlocal diffusion models. Here $\lfloor R \rfloor$ denotes the greatest integer that is less than or equal to $R$. And we will use $\lceil R \rceil$ to denote the least integer that is greater than or equal to $R$.

Let $\Omega = (0, b)$ with $\delta < b$ and the mesh points $x_i = ih, h = b/(N + 1), i \in \Omega_N = \{-r, \ldots, 0, 1, \ldots, N + 1\}$, where $r$ is defined by (2.5); and $u_i$ as the numerical approximation of $u(x_i)$ and $f_{\delta,i} = f_\delta(x_i)$. Denote $I_p = ((p - 1)h, ph)$ for $1 \leq p \leq r$, and $I_{r+1} = (rh, Rh) = (rh, \delta)$, and the piecewise linear basis function is

\[
\phi_p(x) = \begin{cases} 
\frac{x - x_{p-1}}{h} & x \in [x_{p-1}, x_p], \\
\frac{x_{p+1} - x}{h} & x \in [x_p, x_{p+1}] \quad \text{for } i \in \Omega_N, \\
0 & \text{otherwise}.
\end{cases}
\]
Eq. (2.1) can be rewritten as

\[ \mathcal{L}_\delta u(x) = \sum_{p=0}^{r+1} \int_0^s \frac{u(x+s) - 2u(x) + u(x-s)}{s} \phi_p(s) s \gamma_\delta(s) ds, \]  

(2.6)

and an asymptotically compatible discretization of the nonlocal operator \( \mathcal{L}_\delta \) has the following form

\[ \mathcal{L}_\delta^h u_i = \sum_{p=1}^{r} \frac{u_{i-p} - 2u_i + u_{i+p}}{ph} \int_{I_p \cup I_{p+1}} \phi_p(s) s \gamma_\delta(s) ds \]

\[ + \frac{u_{i-r-1} - 2u_i + u_{i+r+1}}{(r+1)h} \int_{I_{r+1}} \phi_{r+1}(s) s \gamma_\delta(s) ds. \]

(2.7)

Note that the above integral over \( I_{r+1} \) automatically vanishes when \( r = R \).

The discretization of (2.5) then has the following form

\[ -\mathcal{L}_\delta^h u_i = f_{\delta,i}, \quad i \in \{1, 2, \ldots, N\}, \quad u_i = g_i, \quad i \in \{-r, \ldots, 0\} \cup \{N+1, \ldots, N+r+1\} \]

with the following sketch that characterizes different variables:

\[
\begin{bmatrix}
\cdots & x_{-r} & \cdots & x_0 & \cdots & x_{r-1} & x_r & \cdots & x_{N-r} & \cdots & x_{N-1} & \cdots & x_{N+r-1} & \cdots
\end{bmatrix}:
\begin{array}{cccccc}
\text{boundary points} & \text{interface points} & \text{internal points} & \text{interface points} & \text{boundary points}
\end{array}
\]

\[
\begin{bmatrix}
\cdots & x_{-r} & \cdots & x_0 & \cdots & x_{r-1} & x_r & \cdots & x_{N-r} & \cdots & x_{N-1} & \cdots & x_{N+r-1} & \cdots
\end{bmatrix}:
\begin{array}{cccccc}
\text{boundary values} & \text{interface values} & \text{internal values} & \text{interface values} & \text{boundary values}
\end{array}
\]

For notational convenience, we let

\[ U_\delta^h = [u_1, u_2, \ldots, u_N]^T, \quad F_\delta^h = [f_{\delta,1}, f_{\delta,2}, \ldots, f_{\delta,N}]^T; \]

\[ F_{V,\delta}^h = [f_{V,1}, f_{V,2}, \ldots, f_{V,r+1}, 0, \ldots, 0, f_{V,N-r}, \ldots, f_{V,N}]^T. \]

Thus, the finite difference scheme (2.8) can be recast as

\[ A_\delta^h U_\delta^h = F_\delta^h + F_{V,\delta}^h, \]

(2.10)

where the stiffness matrix \( A_\delta^h \) is \( a_{i,j} \) has a banded structure given by

\[ a_{i,j} = a_{i,j}, \quad \text{for } |i_1 - j_1| = |i_2 - j_2| \leq r + 1, \text{ and } a_{i,j} = 0 \text{ otherwise.} \]

(2.11)

We denote \( a_k \) with \( k = |i - j| \). The auxiliary vector \( F_{V,\delta}^h \) can be determined by the following matrix form

\[ \begin{bmatrix}
f_{V,1}^r \\
f_{V,2}^r \\
\vdots \\
f_{V,r}^r \\
f_{V,r+1}^r
\end{bmatrix} = \begin{bmatrix}
g_0 & g_{-1} & \cdots & g_{1-r} & g_{-r} \\
0 & g_0 & \cdots & g_{1-r} & g_{-r} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & g_0 & g_{-r} \\
0 & \cdots & 0 & 0 & g_0
\end{bmatrix} \begin{bmatrix}
a_1 \\
a_2 \\
\vdots \\
a_r \\
a_{r+1}
\end{bmatrix}, \]  

(2.12)
and

\[
\begin{bmatrix}
    f_{V,N} \\
    f_{V,N-1} \\
    \vdots \\
    f_{V,N-r+1} \\
    f_{V,N-r}
\end{bmatrix} =
\begin{bmatrix}
    g_{N+1} & g_{N+2} & \cdots & g_{N+r} & g_{N+r+1} \\
    0 & g_{N+1} & \cdots & g_{N+r} & \vdots \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \vdots & \cdots & g_{N+1} & g_{N+2} \\
    0 & 0 & \cdots & 0 & g_{N+1}
\end{bmatrix}
\begin{bmatrix}
    a_1 \\
    a_2 \\
    \vdots \\
    a_r \\
    a_{r+1}
\end{bmatrix}.
\]

(2.13)

In the following, we focus on the special case where the kernel \(\gamma(s)\) is taken to be a constant, i.e., \(\gamma(s) = 3s^{-3}\). More general kernel types [1, 15, 28] can be similarly studied. The entries of the stiffness matrix \(A_b\) can be explicitly documented by

Case 1: \(R \leq 1\).

\[
a_{i-j} = a_{i,j} = \begin{cases} 
\frac{2}{h^2}, & j = i, \\
\frac{-1}{h^2}, & |j - i| = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

(2.14)

Case 2: \(R > 1\). Let \(|j - i| \geq 1\). Then

\[
a_{i-j} = a_{i,j} = \begin{cases} 
-2 \sum_{p=1}^{r+1} a_{i,p}, & j = i, \\
-\frac{3}{h^2R^3}, & p = 1 : r - 1, \\
-\frac{3r - 1 + (R - r)(r^2 + rR - 2R^2 + 3r + 3R)}{2h^2R^3r}, & p = r, \\
-\frac{(R - r)(2R^2 - rR - r^2)}{2h^2R^3(r + 1)}, & p = r + 1, \\
0, & \text{otherwise}.
\end{cases}
\]

(2.15)

If we insert \(R \leq 1\) into (2.15), which reduces to (2.14).

2.2. Multigrid method. Let the finest mesh points \(x_i = ih, \ h = b/(N + 1)\).

Define the multiple level of grids [13] [23]

\[
\mathcal{M}_m = \left\{ x_i^m = \frac{i}{2^m}b, \ i = 1 : N_m \right\} \quad \text{with} \quad N_m = 2^m - 1, \ m = 1 : J,
\]

(2.16)

where \(\mathcal{M}_m\) represents not only the grid with grid spacing \(h_m = 2^{(J-m)}h\), but also the space of vectors defined on that grid. The classical restriction operator \(I_{m-1}^m\) and prolongation operator \(I_{m-1}^m\) are, respectively, defined by

\[
\nu^{m-1} = I_{m-1}^m \nu^m \quad \text{with} \quad \nu_i^{m-1} = \frac{1}{4} (\nu_{2i-1}^m + 2\nu_{2i}^m + \nu_{2i+1}^m), \quad i = 1 : N_{m-1};
\]

(2.17)
and
\[ \nu^m = I_{m-1}^m \nu^{m-1} \quad \text{with} \quad I_{m-1}^m = 2 (I_{m}^{m-1})^T. \]

We use the coarse grid operators defined by the Galerkin approach \[23\], p. 455]
\[ A_{m-1} = I_{m}^{m-1} A_{m} I_{m}^{m-1}, \quad m = 1 : J; \]
and for all intermediate \((m, m-1)\) coarse grids we apply the correction operators \[22\], p. 87]
\[ T^m = I_{m} - I_{m-1}^m A_{m-1}^{m-1} = I_{m} - I_{m-1}^m P_{m-1} \]
with
\[ P_{m-1} = A_{m-1}^{m-1} I_{m-1}^m. \]

We choose the damped Jacobi iteration matrix by \[5\], p. 9]
\[ K_m = I - S_m A_m \quad \text{with} \quad S_m := S_m \omega = \omega D_m^{-1} \]
with a weighting factor \(\omega \in (0, 1/3]\), and \(D_m\) is the diagonal of \(A_m\).

A multigrid process can be regarded as defining a sequence of operators \(B_m : M_m \mapsto M_m\) which is an approximate inverses of \(A_m\) in the sense that \(||I - B_m A_m||\) is bounded away from one. We list the following V-cycle multigrid algorithm \[33\]:

**Algorithm 1** V-cycle Multigrid Algorithm: Define \(B_1 = A_1^{-1}\). Assume that \(B_{m-1} : M_{m-1} \mapsto M_{m-1}\) is defined. We shall now define \(B_m : M_m \mapsto M_m\) as an approximate iterative solver for the equation associated with \(A_m \nu^m = f_m\).

1: Pre-smooth: Let \(S_m \omega\) be defined by \[221\] and \(\nu_0^m = 0, l = 1 : m_1 \)
\[ \nu_l^m = \nu_{l-1}^m + S_m \omega_{pre} (f_m - A_m \nu_{l-1}^m). \]

2: Coarse grid correction: \(e^{m-1} \in M_{m-1}\) is the approximate solution of the residual equation \(A_{m-1} e = I_{m}^{m-1} (f_m - A_m \nu_{m_1}^m)\) by the iterator \(B_{m-1}\):
\[ e^{m-1} = B_{m-1} I_{m}^{m-1} (f_m - A_m \nu_{m_1}^m). \]

3: Post-smooth: \(\nu_{m_1+1}^m = \nu_{m_1}^m + I_{m-1}^m e^{m-1}\) and \(l = m_1 + 2 : m_1 + m_2\)
\[ \nu_l^m = \nu_{l-1}^m + S_m \omega_{post} (f_m - A_m \nu_{l-1}^m). \]

4: Define \(B_m f_m = \nu_{m_1+m_2}^m.\)

3. Convergence of TGM for nonlocal model. Now, we start to prove the convergence of the TGM for nonlocal model. First, we give some Lemmas that will be used.
Lemma 3.1. \[7\text{ p. 5}\] Given \( n \times n \) symmetric matrices \( P \) and \( Q \) and let \( P' \) be a principal submatrix of \( P \) of order \( n - 1 \). Then, for \( m = 1, 2, \ldots, n \),

\begin{align*}
\lambda_k(P) + \lambda_1(Q) - \lambda_k(Q) & \leq \lambda_k(P) + \lambda_1(Q - P), \\
\lambda_1(P) & \leq \lambda_1(P') \leq \lambda_2(P) \leq \lambda_2(P') \leq \cdots \leq \lambda_{n-1}(P') \leq \lambda_n(P), \\
\lambda_{\min}(P) = \lambda_1(P) & = \min_{x \neq 0} \frac{x^T P x}{x^T x}, \quad \lambda_{\max}(P) = \lambda_n(P) = \max_{x \neq 0} \frac{x^T P x}{x^T x}.
\end{align*}

Definition 3.2. \[7\text{ p. 13}\] Let \( n \times n \) Toeplitz matrix \( T_n \) be of the following form:

\[
T_n = \begin{bmatrix}
t_0 & t_{-1} & \cdots & t_{2-n} & t_{1-n} \\
t_1 & t_0 & t_{-1} & \cdots & t_{2-n} \\
\vdots & t_1 & t_0 & \cdots & \vdots \\
t_{n-2} & \cdots & \cdots & \cdots & t_{-1} \\
t_{n-1} & t_{n-2} & \cdots & t_1 & t_0
\end{bmatrix},
\]

i.e., \( t_{i,j} = t_{i-j} \) and \( T_n \) is constant along its diagonals. Assume that the diagonals \( \{t_k\}_{k=-n+1}^{n-1} \) are the Fourier coefficients of a function \( f \), i.e.,

\[
t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-i k x} dx.
\]

Then the function \( f \) is called the generating function of \( T_n \).

Lemma 3.3. \[7\text{ p. 13-15}\] (Grenander-Szegö theorem) Let \( T_n \) be given by above matrix with a generating function \( f \), where \( f \) is a \( 2\pi \)-periodic continuous real-valued functions defined on \( [-\pi, \pi] \). Let \( \lambda_{\min}(T_n) \) and \( \lambda_{\max}(T_n) \) denote the smallest and largest eigenvalues of \( T_n \), respectively. Then we have

\[
f_{\min} \leq \lambda_{\min}(T_n) \leq \lambda_{\max}(T_n) \leq f_{\max},
\]

where \( f_{\min} \) and \( f_{\max} \) is the minimum and maximum values of \( f(x) \), respectively. Moreover, if \( f_{\min} < f_{\max} \), then all eigenvalues of \( T_n \) satisfy

\[
f_{\min} < \lambda(T_n) < f_{\max},
\]

for all \( n > 0 \). In particular, if \( f_{\min} > 0 \), then \( T_n \) is positive definite.

Lemma 3.4. Let the discrete Laplacian-like operators \( \{L_j\}_{j=1}^{N-1} \) be defined by

\[
L_j = \begin{bmatrix}
2 & \cdots & -1 \\
\vdots & \ddots & \ddots \\
-1 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
-1 & \cdots & \cdots & 2
\end{bmatrix}_{N \times N}
\]

with \( 1 \leq j \leq N-1 \).

Then, the smallest eigenvalues of \( L_j \) satisfy

\[
\lambda_1(L_j) \geq 4 \sin^2 \left( \frac{\pi}{2 \left( \lfloor N/j \rfloor + 1 \right)} \right), \quad j = 1, 2, \ldots, N-1.
\]
Moreover, if \( N/j \) is an integer
\[
\lambda_k(L_j) = 4 \sin^2 \left( \frac{k\pi}{2(N/j + 1)} \right), \quad k = 1, 2, \ldots N/j.
\]

**Proof.** Let \( \nu_{k,1}^{j} = [\nu_{1}^{j}, \nu_{2}^{j}, \ldots, \nu_{N}^{j}]^T \) be the associated eigenvector with the tridiagonal matrix \( L_1 \). It is well known that its eigenvalues are given by [26, p. 702]
\[
\lambda_{k,1} = 4 \sin^2 \left( \frac{k\pi}{2(N + 1)} \right), \quad k = 1, 2, \ldots N.
\]

Define
\[
\nu_{i}^{k,j} = [0 \cdots 0, \nu_{i}^{j}, 0 \cdots 0, \nu_{i-1}^{j}, 0 \cdots 0, \nu_{i}^{j}, 0 \cdots 0]^T, \quad i = 1, 2, \ldots, j.
\]

Then, for the matrix \( L_j \) with dimension \( (jN) \times (jN) \), we have
\[
\{L_j\}_{(jN) \times (jN)} \nu_{i}^{k,j} = \lambda_{k,1} \nu_{i}^{k,j} = 4 \sin^2 \left( \frac{k\pi}{2(N + 1)} \right) \nu_{i}^{k,j}, \quad i = 1, 2, \ldots, j,
\]
leading to all eigenvalues with multiplicity \( j \) and eigenvectors of \( \{L_j\}_{(jN) \times (jN)} \). A dimension rescaling then shows that
\[
(3.5) \quad \lambda_k(\{L_j\}_{N \times N}) := \lambda_{k,j} = 4 \sin^2 \left( \frac{k\pi}{2(N/j + 1)} \right), \quad k = 1, 2, \ldots N/j
\]
if \( N/j \) is an integer.

If \( N/j \) is not an integer, we extend \( N \) to \( \tilde{N} \) such that \( \tilde{N}/j \) is an integer, i.e.,
\[
\tilde{N}/j := \lceil N/j \rceil = \frac{N + j - \text{mod}(N,j)}{j},
\]
where \( \text{mod}(N,j) \) means the remainder of division of \( N \) by \( j \).

From (3.5) and (3.2), we obtain
\[
\lambda_1(\{L_j\}_{N \times N}) = 4 \sin^2 \left( \frac{\pi}{2(N/j + 1)} \right) = 4 \sin^2 \left( \frac{\pi}{2([N/j] + 1)} \right),
\]
and
\[
\lambda_1(\{L_j\}_{N \times N}) \geq \lambda_1(\{L_j\}_{\tilde{N} \times \tilde{N}}) = 4 \sin^2 \left( \frac{\pi}{2([N/j] + 1)} \right),
\]
The proof is completed. \( \square \)

**Lemma 3.5.** Let the matrix \( A^h_\delta \) be defined by (2.11) and (2.15) on a finite bar \( \Omega = (0, b) \), \( b > 0 \). Let \( \delta = ch^\beta, \beta \geq 0, h \to 0 \) and \( c > 0 \). Then
\[
\lambda_{\min}(A^h_\delta) \geq \frac{1}{276^2}.
\]
Proof. According the definition of $L_j$ given in Lemma 3.4, we can recast with its elements defined by (2.15) as

\begin{equation}
A_h^\delta = -a_1 L_1 - a_2 L_2 \cdots - a_{r+1} L_{r+1} = - \sum_{j=1}^{r+1} a_j L_j,
\end{equation}

where $\{a_j\}$ are entries on different off-diagonals. We should check the following two cases: $r \leq 1$ and $r \geq 2$.

Case 1: $r \leq 1$. From (2.14) and (3.6), we obtain

$$A_h^\delta = -a_1 L_1,$$

which means that

$$\lambda_{\text{min}}(A_h^\delta) = -a_1 \lambda_{\text{min}}(L_1) = \frac{1}{h^2} \cdot 4 \sin^2 \left( \frac{\pi}{2(N+1)} \right) = \frac{1}{h^2} \cdot 4 \sin^2 \left( \frac{\pi h}{2b} \right) = \frac{\pi^2}{b^2} + O(h^2).$$

Case 2: $r \geq 2$. Using

$$\frac{2}{\pi} x \leq \sin(x) \leq x, \quad x \in \left[0, \frac{\pi}{2}\right],$$

we obtain

$$\lambda_{\text{min}}(A_h^\delta) \geq - \sum_{j=1}^{r+1} a_j \lambda_{\text{min}}(L_j) \geq \frac{1}{h^2} \cdot \frac{3}{R^3} \sum_{j=1}^{r-1} \frac{4 \sin^2 \left( \frac{\pi}{2([N/j]+1)} \right)}{\frac{\pi}{6N}} 
\begin{align*}
\geq & \frac{1}{h^2} \cdot \frac{3}{R^3} \sum_{j=1}^{r-1} \frac{4 \sin^2 \left( \frac{j \pi}{6N} \right)}{\frac{\pi}{6N}} 
\geq & \frac{4}{3b^2(r+1)^3} \cdot \frac{r+1}{2} \cdot \frac{r+1}{2} \cdot \frac{2(r+1)}{2} = \frac{1}{27b^2}.
\end{align*}

The proof is completed. \( \square \)

Remark 3.1. From (3.6), we know that the discrete nonlocal operator $A_h^\delta$ can be viewed as the superposition of discrete Laplacian-like operators $\{L_j\}_{j=1}^{r+1}$; and it reduces to the classical discrete Laplacian operator when $r = 0$ or 1.

Lemma 3.6. Let the matrix $A_h^\delta$ be defined by (2.11) and (2.15) on a finite bar $\Omega = (0, b)$, $b > 0$. Let $\delta = ch^\beta$, $\beta \geq 0$, $h \rightarrow 0$ and $c > 0$. Then, there exists the bound of the condition number

$$\text{cond}(A_h^\delta) = \frac{\lambda_{\text{max}}(A_h^\delta)}{\lambda_{\text{min}}(A_h^\delta)} \leq c_* \min\{\delta^{-2}, h^{-2}\},$$

where $c_*$ is a positive constant.

Proof. Case 1: $\beta > 1$. From (2.10), there exists

$$R = \frac{\delta}{h} = ch^{\beta-1} \leq c_0 \quad \text{with} \quad c_0 \quad \text{a constant.}$$
If $R \leq 1$, from (2.14) and (3.6), we obtain
\[ A_h^t = -a_1 L_1, \]
which means that
\[
\lambda_{\text{max}}(A_h^t) = -a_1 \lambda_{\text{max}}(L_1) = \frac{1}{h^2} \cdot 4 \sin^2 \left( \frac{N\pi}{2(N + 1)} \right) \leq \frac{4}{h^2}.
\]
If $1 < R \leq c_0$, according to (3.6), (3.1), (3.4) and (2.15), there exists
\[
\lambda_{\text{max}}(A_h^t) \leq -r + 1 \sum_{j=1}^{r+1} a_j \lambda_{\text{max}}(L_j) \leq \frac{1}{R^3} \sum_{j=1}^{r+1} 4 \leq \frac{12(R + 1)}{h^2} \leq \frac{12(c_0 + 1)}{h^2}.
\]

Case 2: $0 \leq \beta \leq 1$. Using (2.15), we obtain
\[ a_0 = -2 \sum_{m=1}^{r+1} a_{i,m} \leq 2 \sum_{m=1}^{r+1} \frac{3}{h^2 R^3} \leq \frac{12}{\delta^2}. \]

From the Gerschgorin theorem [27, p. 133], the eigenvalues $\lambda$ of the matrix $A_h^t$ satisfy
\[ \lambda_{\text{max}}(A_h^t) \leq 2a_0 \leq \frac{24}{\delta^2}. \]

From Lemma 3.5 and the discussions of Case 1 and Case 2, the desired result is obtained.

Since the matrix $A_h^t$ is symmetric positive definite, we can define the following inner products
\[
(u, v)_D = (Du, v), \quad (u, v)_A = (Au, v), \quad (u, v)_{AD^{-1}A} = (Au, Av)_{D^{-1}},
\]
where for convenience we have dropped the explicit notational dependence on $h$ and $\delta$ so that $A := A_f = A_h^t$ and $D$ is its diagonal. Here $(\cdot, \cdot)$ is the usual Euclidean inner product.

**Lemma 3.7.** [22, p. 84] Let $A_f$ be a symmetric positive definite. If $\eta \leq \omega(2 - \omega \eta_0)$ with $\eta_0 \geq \lambda_{\text{max}}(D^{-1}A_f)$, then the damped Jacobi iteration with relaxation parameter $0 < \omega < 2/\eta_0$ satisfies
\[ \| K_f \nu^J \|_{A_f}^2 \leq \| \nu^J \|_{A_f}^2 - \eta \| A_f \nu^J \|_{D^{-1}}^2, \quad \forall \nu^J \in M_f. \]

**Lemma 3.8.** [22, p. 89] Let $A_f$ be a symmetric positive definite matrix and $K_f$ satisfies (3.8) and
\[ \min_{\nu^J \in M_f, J} \| \nu^J - I_{J-1}^{f} \nu^{J-1} \|_{D_f}^2 \leq \kappa \| \nu^J \|_{A_f}^2 \quad \forall \nu^J \in M_f \]
with $\kappa > 0$ independent of $\nu^J$. Then, $\kappa \geq \gamma > 0$ and the convergence factor of TGM satisfies
\[ \| K_f T^J \|_{A_f} \leq \sqrt{1 - \eta/\kappa} \quad \forall \nu^J \in M_f. \]
We present the convergence results of TGM in Theorem 3.9 and Theorem 3.11. The first part of the proof of Theorem 3.9 follows the traditional idea [6, 12, 21], but just the convergence result for \( \beta \geq 1 \) is obtained; we use a different technique to prove the case \( \beta = 0 \). After using the new idea, the convergence results for \( \beta \geq 0 \) are got, being proposed in Theorem 3.11.

**Theorem 3.9.** Let \( A_J = A^h_I \) be defined by (2.10) and (2.15) on a finite bar \( \Omega \in (0, b) \), where \( \delta = ch^\beta, \beta \geq 0, h \to 0 \) and \( c > 0 \). Then \( K_J \) satisfies (3.8) and the convergence factor of the TGM satisfies

\[
||K_J T^J||_{A_J} \leq \begin{cases} 
\sqrt{1 - \eta/c_0} < 1 & \text{with } \beta \geq 1, c_0 = \max(1, 2c), \\
\sqrt{1 - \eta c^2/(648b^2)} < 1 & \text{with } \beta = 0,
\end{cases}
\]

where \( \eta \leq 2\omega(1 - \omega) \) with \( 0 < \omega < 1 \).

**Proof.** Since \( \lambda_{\max}(D^{-1}_J A_J) \leq \eta_0 \) with \( \eta_0 = 2 \), it leads to \( 0 < \omega < 2/\eta_0 = 1 \). From Lemma 3.7, we conclude that \( K_J \) satisfies (3.8) with \( \eta \leq 2\omega(1 - \omega) \). Let

\[
\nu^J = (\nu_1, \nu_2, \ldots, \nu_N)^T \in M_J, \quad \nu^{J-1} = (\nu_2, \nu_4, \ldots, \nu_{N-1})^T \in M_{J-1},
\]

and \( \nu_0 = \nu_{N+1} = 0 \) with \( N = 2^J - 1 \) in (2.16). From [6, 12, 21] and (3.6), we have

\[
||\nu^J - I_{J-1}^J \nu^{J-1}||^2_{D_J} \leq a_0 \sum_{i=1}^N (\nu_i^2 - \nu_i \nu_{i+1}),
\]

(3.10)

\[
\sum_{i=1}^N \nu_i^2 \geq \sum_{i=1}^N |\nu_i \nu_{i+1}|,
\]

(3.11)

and

\[
||\nu^J||^2_{A_J} = (\nu^J, A_J \nu^J) \geq (\nu^J, -a_1 L_1 \nu^J) = -2a_1 \sum_{i=1}^N (\nu_i^2 - \nu_i \nu_{i+1}),
\]

where \( a_0 \) and \( a_1 \) are given in (2.15). According to (3.10) and (3.11), there exists

\[
||\nu^J - I_{J-1}^J \nu^{J-1}||^2_{D_J} \leq \frac{a_0}{-2a_1} ||\nu^J||^2_{A_J}.
\]

(3.13)

Next we prove that (3.3) holds.

Case 1: \( \beta \geq 1 \). Using (2.6), there exists

\[
R = \frac{\delta}{h} = ch^{\beta-1} \leq c \quad \text{as} \quad h \leq 1.
\]

When \( R \leq 1 \), from (2.14), it leads to \( \kappa = \frac{\eta_0}{2a_1} = 1 \); then from Lemma 3.8

\[
||K_J T^J||_{A_J} \leq \sqrt{1 - \eta}.
\]

(3.14)

When \( 1 < R \leq c \), using (2.15) and (3.7), we have

\[
a_1 = -\frac{1}{h^2} \cdot \frac{3}{R^3} = -\frac{3}{\delta^3} h = -\frac{3}{c^3} h^{1-3\beta},
\]

(3.15)

\[
a_0 = -2 \sum_{m=1}^{r+1} a_{i,m} \leq \frac{12}{\delta^2} = \frac{12}{c^2} h^{-2\beta}.
\]
It leads to

\[ (3.16) \quad \kappa = \frac{a_0}{-2a_1} \leq 2c h^{\beta - 1}, \]

equation is\( \leq 2c \) as \( h \leq 1 \). Thus we obtain

\[ (3.17) \quad ||K_J T_J||_{A_J} \leq \sqrt{1 - \eta/(2c)} < 1. \]

Combining (3.14) and (3.17), it yields

\[ ||K_J T_J||_{A_J} \leq \sqrt{1 - \eta/c_0} < 1. \]

Case 2: \( \beta = 0 \), i.e., \( \delta = c \). Since \( \kappa \to \infty \) as \( \beta = 0 \) in the estimate (3.16), we need to look for an estimate of the other form. From (3.15), we obtain \( a_0 \leq 12 c^2 \).

Using (3.11), it yields

\[ (3.18) \quad ||\nu^J||^2 = \sum_{i=1}^{N} \nu_i^2 = \frac{1}{2} \sum_{i=1}^{N} \nu_i^2 + \frac{1}{2} \sum_{i=1}^{N} (\nu_i^2 - \nu_i \nu_{i+1}). \]

From Lemma 3.5, we have

\[ (3.19) \quad ||\nu^J||_{A_J}^2 = (A_J \nu^J, \nu^J) \geq \lambda_{\min}(A_J)||\nu^J||^2 \geq \frac{1}{27 b^2}||\nu^J||^2. \]

According to (3.10), (3.18), (3.19) and (3.15), we get

\[ ||\nu^J - I_{J-1}^J \nu^{J-1}||_{A_J}^2 \leq a_0 \sum_{i=1}^{N} (\nu_i^2 - \nu_i \nu_{i+1}) \leq 2a_0 ||\nu^J||^2 \leq 54 b^2 a_0 ||\nu^J||_{A_J}^2 \leq \kappa ||\nu^J||_{A_J}^2 \]

with \( \kappa = 54 b^2 a_0 \leq \frac{648 b^2}{c^2} \). Hence

\[ ||K_J T_J||_{A_J} \leq \sqrt{1 - \eta c^2/(648 b^2)} \]

with \( \beta = 0 \).

The proof is completed. 

In the works [6, 12, 21], the convergence factor of the two-grid method is uniformly bounded below one independent of \( h \) by estimating \( \kappa = \frac{a_0}{2a_1} < \infty \). Since \( \kappa = \frac{a_0}{2a_1} \to \infty \) as \( \beta \in [0, 1] \) in the estimate (3.16), next we need to use a different idea to prove the case: \( \beta \geq 1 \).

**Lemma 3.10.** Let \( A = \sum_{j=1}^{n} L_j \) and \( B = n L_1 \) with \( n \geq 1 \), where \( L_j \) are defined by (3.4). Then \( 2A - B \) is a positive definite matrix.

**Proof.** The generating functions of \( A \) and \( B \) are

\[ f_A(x) = 2n - 2 \sum_{k=1}^{n} \cos(kx) \quad \text{and} \quad f_B(x) = 2n(1 - \cos x), \]

respectively. Since \( f_A(x) \) and \( f_B(x) \) are the even function and \( 2\pi \)-periodic continuous real-valued functions defined on \([-\pi, \pi]\), we just need to consider on \([0, \pi]\). Moreover

\[ (3.20) \quad 2f_A(x) - f_B(x) = 4ng(x) \]
with
\begin{equation}
(3.21) \quad g(x) = \cos^2 \frac{x}{2} - \frac{1}{n} \sum_{k=1}^{n} \cos(kx), \quad x \in [0, \pi].
\end{equation}

Next we prove \( g(x) \geq 0 \). If \( x = 0 \), it yields \( g(x) = 0 \). Denote
\begin{equation}
(3.22) \quad \varphi_n(x) := \frac{1}{n} \sum_{k=1}^{n} \cos(kx) = \frac{\sin \frac{2n+1}{2} x - \sin \frac{x}{2}}{2n \sin \frac{x}{2}}, \quad x \in (0, \pi].
\end{equation}

Case 1: \( 0 < \frac{2n+1}{2} x \leq \pi \). We can rewrite (3.22) as
\begin{equation}
\varphi_n(x) = \frac{x}{2 \sin \frac{x}{2}} \phi(y) \quad \text{with} \quad \phi(y) = \frac{\sin y - \sin \frac{x}{2}}{y - \frac{x}{2}}, \quad y = \frac{2n+1}{2} x, \quad x \in \left( \frac{x}{2}, \pi \right].
\end{equation}

It is easy to prove that \( \phi(y) \) decreases with respect to \( y \), which implies
\begin{equation}
\varphi_n(x) \leq \varphi_{n-1}(x) \leq \cdots \leq \varphi_1(x) = \frac{\sin \frac{3}{2} x - \sin \frac{x}{2}}{2 \sin \frac{x}{2}} = \cos x < \cos^2 \frac{x}{2},
\end{equation}
i.e., \( g(x) > 0 \).

Case 2: \( \pi \leq \frac{2n+1}{2} x \leq 2 \pi + \frac{x}{2} \). Since \( \varphi_n(x) \leq 0 < \cos^2 \frac{x}{2} \), it yields \( g(x) > 0 \).

Case 3: \( \frac{2n+1}{2} x \geq 2 \pi + \frac{x}{2} \). Using (3.22), there exists
\begin{equation}
\varphi_n(x) \leq \frac{1 - \sin \frac{x}{2}}{2n \sin \frac{x}{2}} = \frac{\cos^2 \frac{x}{2}}{2n \sin \frac{x}{2} \left( 1 + \sin \frac{x}{2} \right)} < \cos^2 \frac{x}{2},
\end{equation}
since
\begin{equation}
2n \sin \frac{x}{2} \left( 1 + \sin \frac{x}{2} \right) \geq \frac{4 \pi \sin \frac{x}{2}}{x} \geq \frac{4 \pi}{x}, \quad \frac{2}{\pi} \cdot \frac{x}{2} = 4.
\end{equation}

According to the above equations and Lemma 3.3, the desired result is obtained. \( \Box \)

**Theorem 3.11.** Let \( A_J = A_J^0 \) be defined by (2.10) and (2.15) on a finite bar \( \Omega \in (0, b) \), where \( \delta = c h^\beta, \beta \geq 0, h \to 0 \) and \( c > 0 \). Then \( K_J \) satisfies (3.8) and the convergence factor of the TGM satisfies
\begin{equation}
||K_J T^J||_{A_J} \leq \sqrt{1 - \eta/6} < 1, \quad \beta \geq 0,
\end{equation}
where \( \eta \leq 2 \omega(1 - \omega) \) with \( 0 < \omega < 1 \).

**Proof.** By the proof of Theorem 3.9, we know that \( K_J \) satisfies (3.8) and
\begin{equation}
(3.23) \quad ||\nu^r - I_{J-1} \nu^{r-1}||^2 \leq \sum_{i=1}^{N} (\nu_i^r - \nu_{i+1}^r)^2 = \frac{1}{2} (L_1 \nu^r, \nu^r).
\end{equation}

Case 1: \( r \leq 1 \). From (2.14) and (3.6), we obtain
\begin{equation}
A_J = A_J^0 = -a_1 L_1,
\end{equation}
which means that
\begin{equation}
(3.24) \quad ||\nu^r||_{A_J}^2 = (A_J \nu^r, \nu^r) = (-a_1 L_1 \nu^r, \nu^r).
\end{equation}
According to (3.23) and (3.24), there exists

\[ \| \nu' - I_{j-1}\nu'^{-1} \|_{D_j}^2 \leq \frac{a_0}{2} (L_1\nu'^ J, \nu'^ J) = \frac{a_0}{2a_1} \| \nu' \|^2_{A_J} = \| \nu' \|^2_{A_J}. \]

Thus from Lemma 3.8 we have

\[ \| K_j T^j \|_{A_J} \leq \sqrt{1 - \eta} \quad \text{with} \quad r \leq 1. \]

Case 2: \( r \geq 2 \). According to (3.6), (2.15), Lemma 3.10 and (3.23), we obtain

\[ \| \nu' \|^2_{A_J} = (A_j \nu'^ J, \nu'^ J) \geq -a_1 \left( \sum_{j=1}^{r-1} L_j \nu'^ J, \nu'^ J \right) \geq -\frac{a_1(r-1)}{2} (L_1\nu'^ J, \nu'^ J) \]

\[ \geq -\frac{a_1(r+1)}{6} (L_1\nu'^ J, \nu'^ J) \geq \frac{a_0}{12} (L_1\nu'^ J, \nu'^ J) \geq \frac{a_0}{6} \| \nu' - I_{j-1}\nu'^{-1} \|_{D_j}^2 \]

\[ = \frac{1}{6} \| \nu' - I_{j-1}\nu'^{-1} \|_{D_j}^2. \]

Thus from Lemma 3.8 we have

\[ \| K_j T^j \|_{A_J} \leq \sqrt{1 - \eta/6} \quad \text{with} \quad r \geq 2. \]

The proof is completed. □

4. Convergence of the full MGM and V-cycle MGM with \( \delta = ch \). We extend the convergence results of TGM given in the above section to the full MGM and V-cycle MGM in case that \( \delta = ch \) with \( c \) being an appropriate natural number. First, we introduce some lemmas. We will use the notion of M-matrix, which is a positive definite matrix with positive entries on the diagonal and nonpositive off-diagonal entries. And another notion is called weakly diagonal dominant [5, p. 3], if the diagonal element of a matrix is at least as large as the sum of the off-diagonal elements in the same row or column.

**Lemma 4.1** ([13]). Let \( A^{(1)} = \{a_{i,j}^{(1)}\}_{i,j=1}^{\infty} \) with \( a_{i,j}^{(1)} = a_{i-j}^{(1)} \) be a symmetric Toeplitz matrix and \( A^{(k)} = L_H^{(k)} A^{k-1} L_H^{(k)} \) with \( L_H^{(k)} = 4I_H^{(k-1)} \) and \( L_H^{(k)} = (L_H^{(k)})^T \). Then \( A^{(k)} \) can be computed by (4.1). Here

\[
\begin{align*}
a_0^{(k)} &= (4C_k + 2^{k-1})a_0^{(1)} + \sum_{m=1}^{2^k-1} 2^{k-1-m} a_m^{(1)}; \\
a_1^{(k)} &= C_k a_0^{(1)} + \sum_{m=1}^{3^k-1} 3^{k-1-m} a_m^{(1)}; \\
a_j^{(k)} &= \sum_{m=(j-2^{k-1})}^{(j+2)2^{k-1}-1} jC_m a_m^{(1)} \quad \forall j \geq 2 \quad \forall k \geq 2
\end{align*}
\]

with \( C_k = 2^{k-2} \cdot \frac{2^{2k-2}-1}{3} \). And

\[ 0^{C_m^k} = \begin{cases} 8C_k - (m^2 - 1)(2^k - m) & \text{for} \quad m = 1: 2^{k-1}; \\
1/3(2^k - m - 1)(2^k - m)(2^k - m + 1) & \text{for} \quad m = 2^{k-1} : 2 \cdot 2^{k-1} - 1; \end{cases} \]
\[ 1^{C_m^k} = \begin{cases} 
2C_k + m^2 \cdot 2^{k-1} - \frac{2}{3}(m-1)m(m+1) & \text{for } m = 1 : 2^{k-1}; \\
2C_k + (2k - m)^2 \cdot 2^{k-1} - \frac{2}{3}(2k - m)(2k - m + 1) & \\
-\frac{1}{6}(m - 2^{k-1} - 1)(m - 2^{k-1})(m - 2^{k-1} + 1) & \text{for } m = 2^{k-1} : 2 \cdot 2^{k-1}; \\
\frac{1}{6}(3 \cdot 2^{k-1} - m - 1)(3 \cdot 2^{k-1} - m)(3 \cdot 2^{k-1} - m + 1) & \text{for } m = 2 \cdot 2^{k-1} : 3 \cdot 2^{k-1} - 1;
\end{cases} \]

and for \( j \geq 2, \)

\[ j^{C_m^k} = \begin{cases} 
\varphi_1 & \text{for } m = (j - 2)2^{k-1} : (j - 1)2^{k-1}; \\
\varphi_2 & \text{for } m = (j - 1)2^{k-1} : j2^{k-1}; \\
\varphi_3 & \text{for } m = j2^{k-1} : (j + 1)2^{k-1}; \\
\varphi_4 & \text{for } m = (j + 1)2^{k-1} : (j + 2)2^{k-1} - 1,
\end{cases} \]

where

\[ \varphi_1 = \frac{1}{6}(m - (j - 2)2^{k-1} - 1)(m - (j - 2)2^{k-1})(m - (j - 2)2^{k-1} + 1); \]

\[ \varphi_2 = 2C_k + (m - (j - 1)2^{k-1})^2 \cdot 2^{k-1} \\
-\frac{1}{6}(j2^{k-1} - m - 1)(j2^{k-1} - m)(j2^{k-1} - m + 1) \\
-\frac{2}{3}(m - (j - 1)2^{k-1} - 1)(m - (j - 1)2^{k-1})(m - (j - 1)2^{k-1} + 1); \]

\[ \varphi_3 = 2C_k + ((j + 1)2^{k-1} - m)^2 \cdot 2^{k-1} \\
-\frac{1}{6}(m - j2^{k-1} - 1)(m - j2^{k-1})(m - j2^{k-1} + 1) \\
-\frac{2}{3}((j + 1)2^{k-1} - m - 1)((j + 1)2^{k-1} - m)((j + 1)2^{k-1} - m + 1); \]

\[ \varphi_4 = \frac{1}{6}((j + 2)2^{k-1} - m - 1)((j + 2)2^{k-1} - m)((j + 2)2^{k-1} - m + 1). \]

**Lemma 4.2.** Let \( A^{(1)} = \{a_{i,j}^{(1)}\}_{i,j=1}^\infty \) with \( a_{i,j}^{(1)} = a_{|i-j|}^{(1)} \) be a weakly diagonally dominant symmetric Toeplitz M-matrix and \( D_{(k)} \) be the diagonal of the matrix \( A^{(k)} \), where \( A^{(k)} = I_{k-1}^{-1} A^{(k-1)} I_{k-1} \). Then

\[ 1 \leq \lambda_{\max} \left( D_{(k)}^{-1} A^{(k)} \right) < 3. \]

In particular,

\[ 1 \leq \lambda_{\max} \left( D_{(k)}^{-1} A^{(k)} \right) \leq 2 \text{ if } a_{1}^{(k)} \leq 0. \]
Proof. We take \( A^{(k)} = L^H_h A^{(k-1)} L^H_h \) with \( L^H_h = 4I_k^{k-1} \) and \( L^H_h = (L^H_h)^T \), and denote
\[
A^{(k)} = (a^{(k)}_{i,j})_{i,j=1}^{\infty} \quad \text{with} \quad a^{(k)}_{i,j} = a_{|i-j|}^{(k)}, \quad \forall k \geq 1.
\]

By mathematical induction, we prove the estimates

\[
2 \sum_{j=1}^{n} |a_{j}^{(s)}| \leq a_{0}^{(s)} \quad \text{if} \quad a_{1}^{(s)} \leq 0, \quad s \geq 2, n \geq 1;
\]

and

\[
2 \sum_{j=1}^{n} |a_{j}^{(s)}| \leq \frac{2C_s + 6C_s + 2s-1}{4C_s + 2s-1} a_{0}^{(s)} < 2a_{0}^{(s)} \quad \text{if} \quad a_{1}^{(s)} \geq 0.
\]

For \( s = 2 \) and \( a_{1}^{(2)} \leq 0 \). From (4.1) and (2.13) of [13], we get \( a_{j}^{(2)} \leq 0, j \geq 2 \) and
\[
2 \sum_{j=1}^{n} |a_{j}^{(2)}| = -2 \sum_{j=1}^{n} a_{j}^{(2)}
\]
\[
= -2a_{0}^{(1)} + 8 \sum_{j=1}^{2n-1} |2a_{j}^{(1)}| + 8a_{1}^{(1)} + 2a_{2}^{(1)} + 2a_{2n}^{(1)} + 8a_{2n+1}^{(1)} + 14a_{2n+2}^{(1)}
\]
\[
\leq 6a_{0}^{(1)} + 8a_{1}^{(1)} + 2a_{2}^{(1)} = a_{0}^{(2)} ,
\]
where we use the property of \( A^{(1)} \), being a weakly diagonally dominant M-matrix.

For \( s = 2 \) and \( a_{1}^{(2)} \geq 0 \). According to (4.1) and (2.13) of [13], there exists \( a_{j}^{(2)} \leq 0, j \geq 2 \) and
\[
2 \sum_{j=1}^{n} |a_{j}^{(2)}| = 2a_{1}^{(2)} - 2 \sum_{j=2}^{n} a_{j}^{(2)}
\]
\[
= 2a_{0}^{(1)} + 24a_{1}^{(1)} + 26a_{2}^{(1)} + 16a_{4}^{(1)} + 4a_{4}^{(1)}
\]
\[
+ 2a_{2n}^{(1)} + 8a_{2n+1}^{(1)} + 14a_{2n+2}^{(1)}
\]
\[
\leq 10 \left( 6a_{0}^{(1)} + 8a_{1}^{(1)} + 2a_{2}^{(1)} \right) = \frac{5}{3} a_{0}^{(2)} .
\]

Then (4.2) and (4.3) hold for \( s = 2 \). Suppose that (4.2) and (4.3) hold for \( s = 2, 3, \ldots, k-1 \). Next we prove (4.3) holds for \( s = k \).
Taking \( a^{(k)}_1 \geq 0 \) and using (4.1) and \( a^{(k)}_j \leq 0, \ j \geq 2 \), we have
\[
2 \sum_{j=1}^{n} |a^{(k)}_j| = 2a^{(k)}_1 - 2 \sum_{j=2}^{n} a^{(k)}_j
\]
\[
=2C_k a^{(k)}_0 + 2 \sum_{m=1}^{2^{k-1}-1} (1C_m^k - 2C_m^k) a^{(1)}_m + 2 \sum_{m=2^{k-1}}^{2 \cdot 2^{k-1} - 1} (1C_m^k - 2C_m^k - 3C_m^k) a^{(1)}_m
\]
\[
+ 2 \sum_{m=2^{k-1}}^{n-3} (1C_m^k - 2C_m^k - 3C_m^k - 4C_m^k) a^{(1)}_m
\]
\[
- 2 \sum_{j=2}^{n-3} \sum_{m=(j+1)2^{k-1}}^{n-3 (j+2)2^{k-1} - 1} (jC_m^k + j+1C_m^k + j+2C_m^k + j+3C_m^k) a^{(1)}_m
\]
\[
-n2^{k-1} \sum_{m=(n-1)2^{k-1}}^{(n+1)2^{k-1} - 1} (n-2C_m^k + n-1C_m^k + nC_m^k) a^{(1)}_m
\]
\[
- 2 \sum_{m=n2^{k-1}}^{(n+2)2^{k-1} - 1} (n-1C_m^k + nC_m^k) a^{(1)}_m - 2 \sum_{m=(n+1)2^{k-1}}^{nC_m^k} a^{(1)}_m
\]
\[
\leq \frac{2C_k + 6C_k + 2^{k-1}}{4C_k + 2^{k-1}} a^{(k)}_0 < 2a^{(k)}_0,
\]
where we use \( jC_m^k + j+1C_m^k + j+2C_m^k + j+3C_m^k = 6C_k + 2^{k-1} \) and \( 1C_m^k - 2C_m^k + 6C_k + 2^{k-1} \geq 0 \), \( m = 1 : 2^{k-1} \) and \( 1C_m^k - 2C_m^k - 3C_m^k + 6C_k + 2^{k-1} \geq 0 \), \( m = 2^{k-1} : 2 \cdot 2^{k-1} \) and the property of \( A^{(k)} \). Similarly, we can prove (4.2). Then we obtain
\[
1 \leq \lambda_{\text{max}} \left( D^{(k)} A^{(k)} \right) < 3, \ a^{(k)}_1 \geq 0 \text{ and } 1 \leq \lambda_{\text{max}} \left( D^{(k)}^{-1} A^{(k)} \right) \leq 2, \ a^{(k)}_1 \leq 0.
\]
The proof is completed. \( \Box \)

### 4.1. The operation count and storage requirement

We now discuss the computation count and the required storage for the MGM of nonlocal problem (2.10).

From (2.11), we know that the matrix \( A_h = A_h^T \) is a symmetric banded Toeplitz matrix with bandwidth \( r+1 \) (obviously less than \( N \)). Then, we only need to store the first column of \( A_h \), which have \( N \) parameters, instead of the full matrix \( A_h \) with \( N^2 \) entries. From Lemma 4.3, we know that \( \{ A_k \} \) is the symmetric Toeplitz matrix with the grid sizes \( \{ 2^{J-k} h \}_{k=1}^{J} \), i.e., \( \mathcal{M}_k \) requires \( 2^{J-k} N \) storage. Adding these terms together, we have
\[
\text{Storage} = \mathcal{O}(N) \cdot \left( 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{J-1}} \right) = \mathcal{O}(N).
\]

As for operation counts, the matrix-vector product associated with the matrix \( A_h \) is a discrete convolution. While the cost of a direct product is \( \mathcal{O}(rN) \), the cost of using the FFT would lead to \( \mathcal{O}(N \log(N)) \) [7]. Moreover, from (4.1), we know that the bandwidth of \( \{ A_k \} \) is not bigger than the bandwidth of \( A_h \). Hence, with the change of \( r \), we may adopt different strategies. Thus, the total per \( V \)-cycle MGM operation count is
\[
\mathcal{O} \left( \min\{rN, N \log N\} \right) \cdot \left( 1 + \frac{1}{2} + \ldots + \frac{1}{2^{J-1}} \right) = \begin{cases} \mathcal{O}(N), & r \text{ is bounded,} \\ \mathcal{O}(N \log(N)), & \text{in the worst case.} \end{cases}
\]
4.2. Convergence of the full MGM with $\delta = \text{ch}$. We further consider the convergence of the full MGM (recursive application of the TGM procedure). More precisely we show that the constants $\eta$ in Lemma 3.7 and $\kappa$ in Lemma 3.8 do not depend on the levels; this level independence is crucial for the convergence theory of the full MGM [6, 24]. In the following, we consider the simple algebraic systems, but these algebraic arguments are mostly motivated from analytic considerations.

**Lemma 4.3.** Let $A^{(1)} = c_1 L_1 + c_2 L_2 + c_3 L_3$, $c_i > 0$, $i = 1, 2, 3$ and $A^{(k)} = I_{k-1} A_{k-1}^{(1)} I_{k-1}$. Then

$$||K_k T_k||_{A_k} \leq \sqrt{1 - \eta/k} < 1 \quad \forall 1 \leq k \leq J,$$

where $A_k = A^{(J-k+1)}$, $\kappa = \max \left\{ 1 + \frac{c_2 + 4c_3}{c_1}, 3 \right\}$, and $\eta \leq 2\omega(1 - \omega)$ with $0 < \omega < 1$.

**Proof.** Let $A^{(k)} = L_h^H A^{(k-1)} L_h$ with $L_h^H = 4I_{k-1}^H$ and $L_h^H = (L_h^H)^T$, and $A^{(k)} = \{a_{i,j}^{(k)}\}_{i,j=1}^{\infty}$ with $a_{i,j}^{(k)} = a_{i-j,j}^{(k)}$, where

$$A^{(1)} = c_1 \cdot \text{diag} (-1, 2, -1) + c_2 \cdot \text{diag} (-1, 0, 2, 0, -1) + c_3 \cdot \text{diag} (-1, 0, 0, 2, 0, -1).$$

According to Lemma 4.1 and above equations, we obtain

$$A^{(k)} = c_1 \cdot \text{diag} \left( -\frac{d_1^{(k)}}{2}, \frac{d_1^{(k)}}{2} \right)$$

$$+ c_2 \cdot \text{diag} \left( -1, -\frac{d_2^{(k)}}{2}, \frac{d_2^{(k)}}{2}, -1 \right)$$

$$+ c_3 \cdot \text{diag} \left( -4, -\frac{d_3^{(k)}}{2}, \frac{d_3^{(k)}}{2}, -4 \right)$$

$$= \sigma_1^{(k)} L_1 + \sigma_2^{(k)} L_2,$$

where

$$\sigma_1^{(k)} = c_1 \frac{d_1^{(k)}}{2} + c_2 \frac{d_2^{(k)}}{2} - 2 + c_3 \frac{d_3^{(k)}}{2} - 8, \quad \sigma_2^{(k)} = c_2 + 4c_3, \quad 2 \leq k \leq J$$

with $d_1^{(k)} = 2^k$, $d_2^{(k)} = 2^{k+2} - 6$ and $d_3^{(k)} = 9 \cdot 2^k - 24$.

Using (3.16) and (4.4), there exists

$$\kappa = \frac{a_0^{(k)}}{-2a_1^{(k)}} = \frac{2 \left( \sigma_1^{(k)} + \sigma_2^{(k)} \right)}{2\sigma_1^{(k)}} = 1 + \frac{c_2 + 4c_3}{\sigma_1^{(k)}}$$

$$\leq 1 + \frac{c_2 + 4c_3}{\sigma_1^{(k)}} = 1 + \frac{c_2 + 4c_3}{2c_1 + 4c_2 + 2c_3} < 3, \quad 2 \leq k \leq J;$$

and

$$\kappa = \frac{a_0^{(1)}}{-2a_1^{(1)}} = \frac{2 (c_1 + c_2 + c_3)}{2c_1} = 1 + \frac{c_2 + 4c_3}{c_1}, \quad k = 1.$$
Combining the proof of Theorem 3.11, the desired results are obtained.

**Theorem 4.4.** Let \( A_J = A_h^\delta \) be defined by (2.10) and (2.15) on a finite bar \( \Omega \in (0, b) \), where \( \delta = Rh, R = 3, h \to 0 \). Then

\[
||K_k T^k||_{A_k} \leq \sqrt{1 - \eta/6} < 1, \quad 1 \leq k \leq J,
\]

where \( \eta \leq 2\omega(1 - \omega) \) with \( 0 < \omega < 1 \).

**Proof.** According to Lemma 4.3 and Theorem 3.11, there exists

\[
||K_k T^k||_{A_k} \leq \sqrt{1 - \eta/3} < 1, \quad 1 \leq k \leq J - 1,
\]

and

\[
||K_J T^J||_{A_J} \leq \sqrt{1 - \eta/6} < 1, \quad k = J.
\]

The proof is completed.

**4.3. Convergence of the V-cycle MGM with \( \delta = h \).** In the special case, the convergence of the V-cycle MGM can also be simply obtained. Firstly, we have the following lemma.

**Lemma 4.5 (13).** Let the symmetric positive definite matrix \( A_k \) satisfy

\[
\omega \lambda_{\text{max}}(A_k)(\nu^k, \nu^k) \leq (S_k \nu^k, \nu^k) \leq (A_k^{-1} \nu^k, \nu^k) \quad \forall \nu^k \in B_k,
\]

and

\[
\min_{\nu^{k-1} \in B_{k-1}} ||\nu^k - I_{k-1}^k \nu^{k-1}||^2_{A_k} \leq m_0 ||A_k \nu^k||^2_{D_{k-1}} \quad \forall \nu^k \in B_k
\]

with \( m_0 > 0 \) independent of \( \nu^k \). Then

\[
||I - B_k A_k||_{A_k} \leq \frac{m_0}{2l\omega + m_0} < 1 \quad \text{with} \quad 1 \leq k \leq K,
\]

where the operator \( B_k \) is defined by the V-cycle method in Algorithm 7 and \( l \) is the number of smoothing steps.

We know that \( A_h^\delta \) reduces to the second order elliptic operator when \( R \leq 1 \) in (2.10). From [13, 31, 32, 33], it is easy to check that (4.6) and (4.7) hold with \( m_0 = 1 \). Then we have the following results.

**Theorem 4.6.** Let \( A_J = A_h^\delta \) be defined by (2.10) and (2.15) on a finite bar \( \Omega \in (0, b) \), where \( \delta = Rh, R \leq 1, h \to 0 \). Then

\[
||I - B_k A_k||_{A_k} \leq \frac{1}{2l\omega + 1} < 1 \quad \text{with} \quad 1 \leq k \leq J, \quad \omega \in (0, 1/2],
\]

where the operator \( B_k \) is defined by the V-cycle method in Multigrid Algorithm 7 and \( l \) is the number of smoothing steps.

**Remark 4.1.** Using Lemma 4.2, we know that (4.6) holds for the general nonlocal models or fractional models [13, 21] with \( \omega \in (0, 1/3] \), but it is not easy to check the condition (4.7).
5. Numerical Results. We employ the V-cycle MGM described in Algorithm 1 to solve the steady-state nonlocal problem (2.3). The stopping criterion is taken as \(|r^{(i)}| < 10^{-8}\), where \(r^{(i)}\) is the residual vector after \(i\) iterations; and the number of iterations \((m_1, m_2) = (1, 2)\) and \((\omega_{\text{pre}}, \omega_{\text{post}}) = (1, 1/3)\). In all tables, \(N\) denotes the number of spatial grid points; and the numerical errors are measured by the \(l_\infty\) (maximum) norm, ‘Rate’ denotes the convergence orders. ‘CPU’ denotes the total CPU time in seconds (s) for solving the resulting discretized systems; and ‘Iter’ denotes the average number of iterations required to solve a general linear system \(A_h u_h = f_h\) at each time level.

All numerical experiments are programmed in Matlab, and the computations are carried out on a laptop with the configuration: Inter(R) Core (tm) i3 CPU 2.27 GHZ and 2 GB RAM and a Windows 7 operating system.

Example 5.1. Consider the steady-state nonlocal problem

\[-L_\delta u(x) = -12x^2 + 12bx - 2b^2 - \frac{6}{5}\delta^2\]

with a finite domain \(0 < x < b, b = 4\). The exact solution of the equation is \(u(x) = x^2(b-x)^2\), and the boundary conditions \(u = g\) on \(\Omega_T\).

Table 5.1: Using Galerkin approach \(A_{k-1} = I_{k-1}^k A_k I_{k-1}^k\) computed by \(KK\) to solve the resulting systems (2.10) with \(h = 4/N\).

| \(N\) | \(\delta = 1\) | Rate | Iter | CPU | \(\delta = \sqrt{h}\) | Rate | Iter | CPU |
|---|---|---|---|---|---|---|---|---|
| 2^{10} | 4.0638e-05 | 42 | 0.133 s | 3.0999e-05 | 56 | 0.175 s |
| 2^{11} | 1.6971e-05 | 200 | 0.195 s | 7.1645e-06 | 200 | 0.250 s |
| 2^{12} | 2.5561e-06 | 200 | 0.312 s | 1.9184e-06 | 200 | 0.430 s |
| 2^{13} | 6.3918e-07 | 200 | 0.384 s | 4.8200e-07 | 200 | 0.669 s |

Table 5.2: Using doubling the mesh size \(A_{k-1} = A_k^{2(k+1)h}\) to solve the resulting systems (2.10) with \(h = 4/N\).

| \(N\) | \(\delta = 1\) | Rate | Iter | CPU | \(\delta = \sqrt{h}\) | Rate | Iter | CPU |
|---|---|---|---|---|---|---|---|---|
| 2^{10} | 4.0589e-05 | 42 | 0.133 s | 3.0999e-05 | 56 | 0.175 s |
| 2^{11} | 1.6971e-05 | 200 | 0.195 s | 7.1645e-06 | 200 | 0.250 s |
| 2^{12} | 2.5561e-06 | 200 | 0.312 s | 1.9184e-06 | 200 | 0.430 s |
| 2^{13} | 6.3918e-07 | 200 | 0.384 s | 4.8200e-07 | 200 | 0.669 s |
We use two coarsening strategies: Galerkin approach and doubling the mesh size, respectively, to solve the resulting system (2.10). Tables 5.1 and 5.2 show that these two methods have almost the same error values and the numerically confirm that the numerical scheme has second-order accuracy and the computation cost is almost $O(N\log N)$ operations.

6. Conclusions. There are already some theoretical convergence results for using the multigrid method to solve the PDEs, the algebraic system of which has the Toeplitz structure. We notice that the proofs are mainly based on the boundedness of $a_0/a_1$, where $a_0$ and $a_1$ are, respectively, the principal diagonal element and the trailing diagonal element of the Toeplitz matrix. However, in the nonlocal system, most of the time the boundedness of $a_0/a_1$ does not hold again. In this work, we rewrite the corresponding symmetric Toeplitz matrix as a sum of a series of Laplacian-like matrices. Then based on the analysis of the Laplacian-like matrix, we present the strict proof of the uniform convergence of the TGM. And the convergence results of the full MGM and V-cycle MGM in a special case are also derived. For the framework of the uniform convergence of the V-cycle MGM, the condition (4.6) has been confirmed to hold for the class of weakly diagonally dominant symmetric Toeplitz M-matrices, in the future we will try to find the way to verify the condition (4.7).

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REFERENCES

[1] B. Aksoylu and Z. Unlu, Conditioning analysis of nonlocal integral operators in fractional sobolev spaces, SIAM J. Numer. Anal., 52 (2014), pp. 653–677.
[2] A. Aricò and M. Donatelli, A V-cycle multigrid for multilevel matrix algebras: proof of optimality, Numer. Math., 105 (2007), pp. 511–547.
[3] A. Aricò, M. Donatelli, and S. Serra-Capizzano, V-cycle optimal convergence for certain (multilevel) structured linear systems, SIAM J. Matrix Anal. Appl., 26 (2004), pp. 186–214.
[4] J. H. Bramble and J. E. Pasciak, New convergence estimates for multigrid algorithms, Math. Comp., 49 (1987), pp. 311–329.
[5] W. L. Briggs, V. E. Henson, and S. F. McCormick, A Multigrid Tutorial, SIAM, 2000.
[6] R. H. Chan, Q. S. Chang, and H. W. Sun, Multigrid method for ill-conditioned symmetric toepiltz systems, SIAM J. Sci. Comput., 19 (1998), pp. 516–529.
[7] R. H. Chan and X. Q. Jin, An Introduction to Iterative Toeplitz Solvers, SIAM, 2007.
[8] Q. S. Chang and R. Q. Jia, A refined convergence analysis of multigrid algorithms for elliptic equations, Appl. Anal., 13 (2015), pp. 255–290.
[9] M. H. Chen and W. H. Deng, Fourth order accurate scheme for the space fractional diffusion equations, SIAM J. Numer. Anal., 52 (2014), pp. 1418–1438.
[10] M. H. Chen and W. H. Deng, Discretized fractional substantial calculus, ESAIM: Math. Mod. Numer. Anal., 49 (2015), pp. 373-394.
[11] M. H. Chen and W. H. Deng, High order algorithms for the fractional substantial diffusion equation with truncated Levy flights, SIAM J. Sci. Comput., 37 (2015), pp. A890-A917.
[12] M. H. Chen, Y. T. Wang, X. Cheng, and W. H. Deng, Second-order LOD multigrid method for multidimensional Riesz fractional diffusion equation, BIT, 54 (2014), pp. 623-647.
[13] M. H. Chen and W. H. Deng, Multigrid method for symmetric Toepiltz block tridiagonal matrix: Convergence analysis & application, arXiv:1602.08226.
[14] O. Defterli, M. D’Elia, Q. Du, M. Gunzburger, R. Lehoucq, and M. Meerschaert, Fractional diffusion on bounded domains, FCAA, 18 (2015), pp. 342–360.
[15] Q. Du, M. Gunzburger, R. Lehoucq, and K. Zhou, Analysis and approximation of nonlocal diffusion problems with volume constraints, SIAM Rev., 56 (2012), pp. 676–696.
[16] Q. Du, M. Gunzburger, R. Lehoucq, and K. Zhou, A nonlocal vector calculus, nonlocal volume-constrained problems, and nonlocal balance laws, Math. Model. Methods Appl. Sci., 23 (2013), pp. 493–540.
[17] G. Fiorentino and S. Serra, Multigrid methods for symmetric positive definite block Toeplitz matrices with nonnegative generating functions, SIAM J. Sci. Comput., 17 (1996), pp. 1068–1081.

[18] W. Hackbusch, Multigrid Methods and Applications, Springer-Verlag, Berlin, 1985.

[19] Y. H. Huang and A. Oberman, Numerical methods for the fractional Laplacian: a finite difference-quadrature approach, SIAM J. Numer. Anal., 52 (2014), pp. 3056–3084.

[20] R. Metzler and J. Klafter, The random walks guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep., 339 (2000), pp. 1–77.

[21] H. Pang and H. Sun, Multigrid method for fractional diffusion equations, J. Comput. Phys., 231 (2012), pp. 693–703.

[22] J. Ruge and K. Stüben, Algebraic multigrid, in Multigrid Methods, Ed: S. McCormick, pp. 73-130, SIAM, 1987.

[23] Y. Saad, Iterative Methods for Sparse Linear Systems, SIAM, 2003.

[24] S. Serra-Capizzano, Convergence analysis of two-grid methods for elliptic Toeplitz and PDEs matrix-sequences, Numer. Math., 92 (2002), pp. 433–465.

[25] S. A. Silling, Reformulation of elasticity theory for discontinuities and long-range forces, J. Mech. Phys. Solids, 48 (2000), pp. 175–209.

[26] J. Stoer and R. Bulirsch, Introduction to Numerical Analysis, Springer, 2002.

[27] J. W. Thomas, Numerical Partial Differential Equations: Finite Difference Methods, Springer, 1995.

[28] X. C. Tian and Q. Du, Analysis and comparison of different approximations to nonlocal diffusion and linear peridynamic equations, SIAM J. Numer. Anal., 51 (2013), pp. 3458–3482.

[29] X. C. Tian and Q. Du, Asymptotically compatible schemes and applications to robust discretization of nonlocal models, SIAM J. Numerical Analysis, 52 (2014), pp. 1641–1665.

[30] H. Wang and H. Tian, A fast Galerkin method with efficient matrix assembly and storage for a peridynamic model, J. Comput. Phys., 231 (2012), pp. 7730–7738.

[31] J. Xu, Iterative methods by space decomposition and subspace correction, SIAM Review, 34 (1992), pp. 581–613.

[32] J. Xu and L. Zikatanov, The method of alternating projections and the method of subspace corrections in Hilbert space, J. Am. Math. Soc., 15 (2002), pp. 573–597.

[33] J. Xu, An introduction to multilevel methods, in: M. Ainsworth, J. Levesley, W. A. Light and M. Marletta (Eds.), Wavelets, Multilevel Methods and Elliptic PDEs, Leicester, 1996, Oxford University Press, New York, (1997), pp. 213–302.

[34] Q. Yang, I. Turner, F. Liu, and M. Ilić, Novel numerical methods for solving the time-space fractional diffusion equation in two dimensions, SIAM J. Sci. Comput., 33 (2011), pp. 1159–1180.

[35] K. Zhou and Q. Du, Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions, SIAM J. Numer. Anal., 48 (2010), pp. 1759–1780.