Tunable ohmic environment using Josephson junction chains

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We propose a scheme to implement a tunable, wide frequency-band dissipative environment using a double chain of Josephson junctions. The two parallel chains consist of identical SQUIDs, with magnetic-flux tunable inductance, coupled to each other at each node via a capacitance much larger than the junction capacitance. Thanks to this capacitive coupling, the system sustains electromagnetic modes with a wide frequency dispersion. The internal quality factor of the modes is maintained as high as possible, and the damping is introduced by a uniform coupling of the modes to a transmission line, itself connected to an amplification and readout circuit. For sufficiently long chains, containing several thousands of junctions, the resulting admittance is a smooth function versus frequency in the microwave domain, and its effective dissipation can be continuously monitored by recording the emitted radiation in the transmission line. We show that by varying in situ the SQUIDs’ inductance, the double chain can operate as tunable ohmic resistor in a frequency band spanning up to one GHz, with a resistance that can be swept through values comparable to the resistance quantum \(R_q = h/(4e^2) \approx 6.5 \) k\( \Omega \). We argue that the circuit complexity is within reach using current Josephson junction technology.

I. INTRODUCTION

Dissipation in radio-frequency (rf) superconducting quantum electronic circuits\(^{1,2}\) is usually detrimental, giving rise to quantum decoherence. However, this does not necessarily have to be the case. Remarkably, in the last decade, engineered dissipation\(^3\) played an increasingly prominent role in quantum states stabilization\(^4\) or even in error correction schemes\(^5\).

So far, low-impedance dissipative environments have dominated the scene, as they are ubiquitously present in rf circuits and can be tailored using standard micro-wave design strategies. Designing high impedance environments, with an impedance comparable to the resistance quantum \(R_q = h/(4e^2) \approx 6.5 \) k\( \Omega \), has proven more challenging. Recently, significant success has been achieved in the fabrication of low-loss high impedance environments in the form of superconducting\(^6\) or metamaterial\(^7\) chains. However, the implementation of wide frequency-band, high-impedance ohmic environments remains an unsolved problem.

Tunable, high-impedance ohmic environments are potentially interesting for several applications in the field of superconducting electronics.

For instance, quantum simulations of fundamental models to study dissipative quantum phase transitions require the exploration of extended regions in their phase diagrams\(^8,9\). In a single Josephson junction, dissipation leads to a phase transition with suppression of the quantum tunneling of the superconducting phase when the effective resistance shunting the junction is swept through the resistance quantum \(R_q\). The phase diagram of such a transition was experimentally explored using different shunting resistances\(^10,11\). In circuit QED, the ratio between the characteristic impedance \(Z_c\) of a one-dimensional microwave waveguide and the quantum resistance \(R_q\) plays the role of the effective fine-structure constant between the artificial atoms, viz. superconducting qubits, and the electromagnetic field\(^12\), namely \(\alpha_{eff} = (Z_c/Z_{vac})\alpha \sim Z_c/R_q\), with the \(Z_{vac}\) the impedance of the vacuum and \(\alpha \approx 1/137\).

Ultra-strong coupling regime in circuit QED has been achieved in experiments in resonant cavities\(^13\) and in open microwave waveguides\(^14\) using galvanic coupling, which is characterized by a dual scaling of the coupling strength in matter-radiation interaction\(^15\), e.g. \(\sim 1/\alpha_{eff}\). This regime was also obtained experimentally in the effective rotating frame of a driven qubit coupled to an LC resonator\(^16\) with a theoretical extension to an ensemble of resonators\(^17\). Hence, circuits QED designs offer another realization of the spin-boson model, a reference model in the theory of quantum dissipation. For instance, a recent experiment investigated transmons coupled to transmission lines with different coupling strength\(^18\). Another recent approach is based on the use of 1D arrays of Josephson junctions to design the resonant modes of the electromagnetic environment\(^19\). In these systems it is desirable to have the ability to controllably sweep the relevant parameter over a wide range, i.e. the strength of the dissipative interaction between the quantum system and its environment\(^20,21\). Varying in situ the resistance opens the route for addressing novel issues as, for instance, quenching in the dissipative phase transition by varying rapidly the external dissipation across the critical point.

This class of environments could also be an asset for quantum state preparation and stabilization\(^22\) and autonomous quantum error correction via bath engineering\(^23\). For example, in the context of coherent cat states preparation, tuning the dissipative strength and the characteristic impedance might provide a significant resource\(^24\).

In this work we analyse the possibility to realize a tunable high-impedance environment, ohmic in a wide-
Josephson junctions (JJ) are versatile circuit elements, with widespread use in quantum mesoscopic systems, thanks to their intrinsic low dissipation and amenable non-linearity. They are the building blocks of superconducting quantum bits (qubits)\cite{Loss01,Devoret15}, hybrid systems\cite{Golovach16}, or Josephson photonic circuits\cite{Bergman04,Wallraff04}. Josephson junction chains exhibit rich and interesting many-body physical properties\cite{Loss01}, which can be influenced relatively accurately by circuit design and fabrication parameters. They have constituted the platform of choice for the investigation of quantum fluctuations of the phase induced by charge interactions, i.e. quantum phase slips\cite{Johansson96}, or quantum fluctuations of the charge induced by Josephson tunneling\cite{Carmichael83,Weinstein85}.

In the phase regime, where the Josephson energy $E_J = \hbar I_c/(2e)$, with $I_c$ the junction critical current, dominates over the charging energy of the junction $E_C = e^2/(2C_J)$, with $C_J$ the junction capacitance, Josephson junction chains have already been investigated as custom-designed electromagnetic environments, implementing metamaterials\cite{Johnson13,McMillan15} or tunable non-linearity\cite{Johnson13,Wagner16}, or parametric amplifiers\cite{Beaupre15}. The success of many-junction devices in the phase regime ($E_J \gg E_C$), paves the way towards more complex architectures, such as the two coupled Josephson junction chains we propose in Fig. 1 to implement a tunable, high-impedance ohmic environment.

The PJJC device shown in Fig. 1 consists of two JJ chains capacitively coupled to each other at each node and inductively coupled to a stripline microwave transmission line. Each element is formed by a SQUID, with $E_J \gg E_C$, threaded by a magnetic flux $\Phi_B$ that allows tuning of the Josephson inductance $L_J = \Phi_B^2/[8\pi^2 E_J \cos(2\pi \Phi_B/\Phi_0)]$. The coupling capacitance between the chains $C_C$ are designed to be dominant compared to $C_J$ ($C_J \ll C_C$), which imposes a dense and linear dispersion relation over a wide frequency range. Dissipation is introduced via the inductive coupling (using $L_C$) of the chains to an on-chip microwave transmission line, which is itself connected to an amplification and read-out circuit with using 180° hybrid couplers to mode-match between the on-chip transmission line and the standard 50 Ω coaxial cable. This matching is important to avoid the formation of standing waves in the transmission line, which would result in a non-uniform coupling of the PJJC eigenmodes to the 50 Ω environment.

We show that for sufficiently long chains, with $N$ in the range of $10^3$, the resulting real part of the impedance at the input port of the PJJC is a smooth function versus frequency in a band of $\sim 1$ GHz, and its value $\simeq \sqrt{L_J/C_C}$ is tunable in-situ, straddling the resistance quantum. Additionally, owing to the fact that dissipation is introduced via coupling to a transmission line, one can continuously monitor the photons emitted by the device of interest, connected to the input port of the PJJC.

Notice that, in our proposal, Josephson junctions are only used as linear inductances and could be in principle replaced by geometric inductors. Nevertheless, Josephson inductors are very convenient for this proposal, as they offer three essential ingredients: a) an intrinsically lossless medium, b) an ultra-compact inductor, much larger than the geometric inductance of an equivalent size wire, and c) tunability via the Josephson effect, when implemented in the shape of a SQUID.

The paper is structured as follows. In Sec. II we discuss the admittance of a single JJ chain. We compare a phenomenological model for dissipation, based on an infinite number of dissipationless junctions, with models for finite size JJ chains, formed by $N$ dissipative junctions. In Sec. III we analyze the effective circuit of the PJJC in Fig. 1 and show its equivalence to a single chain formed by $N$ JJs. In Sec. IV we discuss the realistically achievable values for the admittance of the PJJC device, taking into account the limited range of experimentally feasible parameters. Finally, we draw our conclusions in Sec. V.
II. ADMITTANCE OF A SINGLE CHAIN FORMED BY N LUMPED ELEMENTS

In sec. II A we recall the emergence of an ohmic resistor in the mathematical limit of an infinite line formed by dissipationless JJs acting as linear inductances and capacitances. In sec. II B we demonstrate that a similar result can be obtained for finite chain lengths, N, if the JJ element of the chain is intrinsically dissipative. In a first example, assuming typical measured values for the intrinsic dissipation of the JJ, the real part of the resulting admittance can only become a smooth function vs. frequency for chain lengths of the order $N = 10^3$. In a second example, we engineer the dissipation, and we can obtain a smooth admittance vs. frequency for much shorter chains with $N \sim 10^3$. The later results will be directly applicable to the PJJC device (as shown in Sec. III).

A. Ohmic admittance of a dissipationless JJ chain in the thermodynamic limit

We consider the chain shown in Fig. 2 formed by a series of inductances $L_J$, in parallel with capacitances $C_J$, with $C_0$ connecting each node to the ground. Introducing the two admittances $Y_J(\omega) = i\omega C_J + 1/(i\omega L_J)$ and $Y_0(\omega) = i\omega C_0$, we write Kirchhoff’s laws for current conservation at nodes $n = 1, \ldots, N$, in terms of the voltages $v_n = V_n(\omega)$, in frequency domain

$$Y_J(\omega)(v_n - v_{n-1}) = Y_J(\omega)(v_{n-1} - v_{n-2}) + Y_0(\omega)v_n,$$

(1)

with the boundary condition $V_0 = v_0 = 0$. We consider a vector of dimension $N - 1$ composed of the voltage values at nodes $n = 1, \ldots, N - 1$. Then, Eq. (1) can be cast in the following tridiagonal matrix form

$$
\begin{pmatrix}
  a(\omega) & -1 & 0 & \ldots & 0 \\
  -1 & a(\omega) & -1 & \ldots & 0 \\
  0 & -1 & a(\omega) & \ldots & 0 \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
  \ldots & \ldots & 0 & -1 & a(\omega)
\end{pmatrix}
\begin{pmatrix}
  v_{n-1} \\
  v_{n-2} \\
  \vdots \\
  \vdots \\
  \vdots \\
  0
\end{pmatrix}
= 
\begin{pmatrix}
  v_N \\
  v_{N-1} \\
  \vdots \\
  \vdots \\
  \vdots \\
  0
\end{pmatrix}
$$

(2)

with $a(\omega) = 2 + Y_0(\omega)/Y_J(\omega)$.

The previous matrix has eigenvalues $\lambda_k(\omega) = 2\left[1 - \cos(\pi k/N)\right] + Y_0(\omega)/Y_J(\omega)$ and eigenvectors $e_k(n) = \sqrt{2/N}\sin(\pi k n/N)$ for $k \in [1, N - 1]$ defined on the restricted lattice $n \in [1, N - 1]$. The eigenvectors are orthonormal $\sum_{k=1}^{N-1} e_k(n)e_k^*(m) = \delta_{km}$ and satisfy the completeness relation $\sum_{k=1}^{N-1} e_k(n)e_k(m) = \delta_{nm}$. The matrix appearing in the left of Eq. (2) can be written as $\tilde{Y} = \tilde{U} D \tilde{U}^{-1}$ where $D$ is the diagonal matrix of the eigenvalues, and $\tilde{U}$ $(\tilde{U}^{-1})$ is a matrix whose $k$-row (-column) are the components of the eigenvector $k$. By writing the inverse of the matrix as $Y^{-1} = \tilde{U} D^{-1} \tilde{U}^{-1}$, one can express the voltage at node $N - 1$ as a function of the voltage at node $N$, namely

$$v_{N-1} = v_N \sum_{k=1}^{N-1} e_k^2(n)/\lambda_k(\omega),$$

which reads

$$v_{N-1} = v_N \frac{2}{N} \sum_{k=1}^{N-1} Y_k(\omega) \sin^2(\pi k/N),$$

(3)

The admittance of the chain is defined by the relation

$$I(\omega) = Y_J(\omega)(v_N - v_{N-1}) \equiv Y_{ch}(\omega)v_N.$$

(4)

Inserting Eq. (3) into Eq. (4), using the relation $\sin^2(x) = (1 - \cos(x))(1 + \cos(x))$, and the sum $\sum_{k=1}^{N-1} [1 + \cos(\pi k/N)] = N - 1$, the admittance can be expressed as

$$Y_{ch}(\omega) = \frac{Y_J(\omega)}{N} \left[1 + \sum_{k=1}^{N-1} \frac{Y_0(\omega) + 2Y_J(\omega)}{2} \left[1 - \cos(\pi k/N)\right] \right].$$

(5)

We can cast the admittance $Y_{ch}(\omega)$ of Eq. (5) as the sum of three terms

$$Y_{ch}(\omega) = \frac{1}{i\omega N L_J} + i\omega \tilde{C} + Y_{har}(\omega).$$

(6)

$Y_{ch}(\omega)$ is characterized by an effective inductance $N L_J$ at small frequency and an effective capacitance $\tilde{C}$ at large frequency, given by

$$\tilde{C} = \frac{C_J}{N} + \frac{C_0}{2N} \sum_{k=1}^{N-1} \frac{\cos^2(\pi k/2N)}{\sin^2(\pi k/2N)} + C_0/(4C_J).$$

(7)

This same result was obtained using a different method in previous work, with different boundary conditions. The third term $Y_{har}(\omega)$ in Eq. (6) is related to the electromagnetic eigenmodes of the chain, whose spectrum reads

$$\omega_k = \frac{2\omega_0 \sin(\pi k/(2N))}{\sqrt{1 + (4C_J/C_0) \sin^2(\pi k/(2N))}}.$$

(8)

We introduce the characteristic frequencies of the spectrum

$$\omega_0 = \frac{1}{\sqrt{L_J C_0}}, \quad \omega_J = \frac{1}{\sqrt{L_J C_J}}, \quad \omega_m = \max_k \{\omega_k\},$$

(9)

corresponding, respectively, to the frequency scale in the linear regime, the plasma frequency of the single JJ, and
the maximum frequency of the spectrum, given by \( \omega_m = 2\omega_0/\sqrt{1 + 4C_J/C_0} \), for \( N \gg 1 \). Using the eigenmodes spectrum, \( Y_{har}(\omega) \) can be written as

\[
Y_{har}(\omega) = \frac{\alpha_0}{N}\sum_{k=1}^{N-1} \left( \frac{1 - \frac{\omega^2}{\omega_k^2}}{\omega_k^2 - \omega^2 - 2i\varepsilon_\omega} \right), \tag{10}
\]

In Eq. (10), we added phenomenologically an imaginary part \( \varepsilon > 0 \) in the denominator, which yields a finite real part for the admittance. From Eq. (8) for the modes, and from their corresponding admittance in Eq. (10), in the limit of \( \varepsilon = 0 \), we can recover previous theoretical results.59,81

It is now interesting to discuss the behavior of \( Y_{har}(\omega) \) at small frequency, with \( \pi k/(2N) \ll 1 \), such that we can assume a linear spectrum \( \omega_k \approx \omega_0 \pi k/N \). We define \( K \) as the approximated fraction of modes in the linear part of the spectrum, a number that scales as \( K \propto N \). At low frequency, the numerator of Eq. (10) converges to one \( (\omega_k \ll \min(\omega_m, \omega_J)) \). Then, for \( \omega > 0 \) and provided that the imaginary part is much smaller than the lowest eigenfrequency \( \varepsilon \ll \omega_{k=1} \) (which is equivalent to the requirement that \( N\varepsilon = \text{const.} \) or \( K\varepsilon = \text{const.} \), one can approximate the real part of the admittance to a sum of Lorentzian functions

\[
\text{Re} [Y_{har}(\omega)] \approx \frac{1}{Z_0} \frac{\pi \omega_0}{N} \sum_{k=1}^{K} \frac{\omega}{\omega_k (\omega_k - \omega)^2 + \varepsilon^2}, \tag{11}
\]

with the characteristic impedance of the line

\[
Z_0 = \sqrt{L_J/C_0}. \tag{12}
\]

Considering the limit of infinite length of the chain \( N \to \infty \) (or equivalently \( K \to \infty \)) and keeping constant the product \( N\varepsilon \) (or \( K\varepsilon \)), the admittance of the chain, given by Eq. (11), converge to an ohmic behavior \( 1/Z_0 \).

### B. Ohmic admittance of a finite size dissipative JJ chain

In the following we use a different approach compared to the previous section to introduce dissipation in the JJ chain.

We review two types of dissipative 1D JJ chains, where the dissipation can either be intrinsically associated to all circuit elements (see Fig. 3 and section II B 1), or it can be added in a controlled manner, in parallel with the ground capacitance \( C_0 \), using a coupling inductor \( L_C \) (see Fig. 4 and section II B 2). In both cases, the chain is composed of the effective junction admittances \( Y_J(\omega) \), and the effective admittances to the ground \( Y_0(\omega) \). Applying the current conservation at each node, similarly to Eq. (1), we obtain

\[
Y_J(\omega) (v_n - v_{n-1}) = Y_J(\omega) (v_{n-1} - v_{n-2}) + Y_0(\omega) v_n, \tag{13}
\]

Then one can repeat identically the steps following Eq. (1) in the previous section to obtain the admittance

\[
Y_{ch}(\omega) = \frac{Y_J(\omega)}{N} \left(1 + \sum_{k=1}^{N-1} \frac{Y_J(\omega) Y_0(\omega) [1 + \cos(\pi k/N)]}{Y_0(\omega) + 2Y_J(\omega) [1 - \cos(\pi k/N)]} \right), \tag{14}
\]

In the next two subsections we apply this result to two hypothetic circuit implementations: 1) the case of intrinsic dissipation associated with any real superconducting circuit element, and 2) a particular implementation of engineered dissipation, using resistive on-chip thin-films. We refer to them as intrinsic dissipation and engineered dissipation, respectively.

#### 1. JJ chain with intrinsic dissipation

We introduce dissipation by considering the inductances and capacitances to be nonideal elements, indicated by the resistances \( R_J \) and \( R_0 \) in the circuit model of Fig. 3. \( R_J \) takes into account the finite dissipation in a single JJ, potentially associated to (nonequilibrium) quasiparticles above the superconducting gap\(83,84\) or other imperfections of the JJ dielectric barrier\(85\). Similarly, \( R_0 \) accounts for dielectric losses in \( C_0 \). Then we have

\[
Y_J(\omega) = i\omega C_J + 1/(i\omega L_J) + 1/R_J, \tag{15}
\]

\[
Y_0(\omega) = i\omega C_0 + 1/R_0. \tag{16}
\]

Focusing on the limit in which the two shunt resistances are much larger than the characteristic resistance of the chain, \( R_J, R_0 \gg Z_0 \), following a method analogous to the one used in the previous sections, one can find the following approximate expression for the 1D JJ chain admittance,

\[
Y_J(\omega) \approx \frac{2\omega}{N L_J} \tilde{A}_J(\omega) \left(1 - \frac{\omega^2}{\omega_J^2}\right) \sum_{k=1}^{N-1} \frac{1 - \frac{\omega^2}{\omega_k^2}}{\omega_k^2 - \omega^2 - i\omega \gamma_k}, \tag{17}
\]

with the spectrum \( \omega_k \) given by Eqs. (8) and (9). As expected, Eq. (17) and Eq. (10) have a similar form. The complex amplitude \( \tilde{A}_J(\omega) = 1 - i/(\omega R_0 C_0) \) in Eq. (17) reduces to \( \sim 1 \) at frequency \( \omega R_0 C_0 \gg 1 \), and the damp-

Figure 3. Circuit model for the JJ chain with intrinsic dissipation, with shunt resistances \( R_J \) in parallel with each junction, and \( R_0 \) in parallel with the capacitance to the ground.
impedance of the JJ chain $Z$, i.e. the spacing between the low-frequency modes is much smaller than the width of the individual peaks. Since in typical JJ chains $R_0 \sim 100$ $\text{M} \Omega$, with a characteristic impedance of the JJ chain $Z_0 \sim k\Omega$, from Eq. (19) we get a minimum required number of junctions $N \gtrsim 10^5$, a number that is difficult to achieve in experimental JJ devices.

In Fig. 4 we plot the calculated real part of the JJ chain admittance, following Eq. (17), for $N = 10^5$. The inset shows the same calculation for a short chain with $N = 25$, to evidence the discrete mode structure of the JJ chain admittance. For $N = 10^5$, the admittance at low frequency still shows large amplitude oscillations caused by the discreteness of the eigenmodes spectrum, pointing out that even longer chains are needed to achieve an ohmic behavior in JJ chains with intrinsic dissipation.

2. JJ chain with engineered dissipation

Hereafter we neglect the large intrinsic resistances $R_J$ and $R_0$ associated with the dissipative part of non-ideal capacitances and inductances. As shown in Fig. 5, we assume $Y_J(\omega)$ to be a pure imaginary admittance, whereas the element $Y_0(\omega)$ is constructed using an ideal capacitance $C_C$, in parallel with the series combination $R_C$ and $L_C$.

\[
Y_J(\omega) = i\omega C_J + 1/(i\omega L_J), \quad Y_0(\omega) = i\omega C_C + \frac{1}{R_C + i\omega L_C}.
\]

The inductance $L_C$ opens a gap in the spectrum and the eigenmodes are now given by

\[
\omega_k = \frac{2}{\sqrt{L_J}C_C} \left\{ \sin \left[ \frac{\pi k}{2N} \right] + L_J/(4L_C) \right\},
\]

with the maximum frequency of the spectrum given by $\omega_m = 2\sqrt{1/L_J + 1/(4L_C)}/(C_C + 4C_J)$, and the minimum frequency $\omega_c = 1/\sqrt{L_JC_C}$, for $N \gg \max[1, \pi C_J/C_C]$. It is also convenient to introduce the characteristic impedance

\[
Z_C = \sqrt{L_J/C_C}.
\]

Focusing on the frequency range containing the spectrum, $\omega_c < \omega < \omega_m$, and in the regime

\[
C_J \ll C_C, \quad L_J \ll L_C, \quad \frac{R_C}{Z_C} \sqrt{\frac{L_J}{L_C}} \ll 1,
\]

using the method of Sec. II A, we obtain an approximate expression for the real part of the admittance of the chain:

\[
Y_{JJ}(\omega) \simeq i\frac{2\omega}{NL_J} A(\omega) \left( 1 - \frac{\omega^2}{\omega_J^2} \right) \sum_{k=1}^{N-1} \frac{1 - \Omega_k^2 - \omega^2 - i \omega \eta_k(\omega)}{\Omega_k^2 - \omega^2 - i \omega \eta_k(\omega)}.
\]

The complex amplitude $A(\omega)$ and the functions $\eta_k(\omega)$ are now given by

\[
A(\omega) = \left( 1 - \frac{\omega^2}{\omega_J^2} - i R_C \frac{L_J + 4L_C}{1 - L_J/C_C} \right),
\]

\[
\eta_k(\omega) = \frac{R_C}{1 - C_J L_J/(C_C L_C)} \left( 1 - \frac{\Omega_k^2}{\omega^2} \right).
\]
III. ADMITTANCE OF THE DOUBLE CHAIN WITH ENGINEERED DISSIPATION

Following the design of a 1D JJ chain with engineered dissipation introduced in Fig. 3 in this section we discuss a similar proposal, the PJJC device shown in Fig. 1 where dissipation is not added via on-chip dissipative elements, like in sec. IIIB2 but rather by a uniform coupling to a microwave transmission line, which could also allow the continuous monitoring of the dissipated energy.

We analyze theoretically an equivalent circuit of the PJJC, as shown in Fig. 7 which captures one essential ingredient of the PJJC proposal, namely a uniform dissipation distributed along the nodes of the chain. The resulting PJJA impedance can be connected to a probe system, for example a flux qubit or transmon qubit, coupled via the inductance $L_J$. The microstrip transmission line, which is ideally reflectionless and matched to a standard coaxial cable (50 Ω), acts as a resistor $R_J$ at each node of the chain. Under the condition of local mirror reflection symmetry for the two chains, we shown that the PJJC is equivalent to a single chain connected directly to the ground via $C_J$, as shown in Fig. 3. Hereafter, we assume the relevant regime $C_J \gg C_J$ and neglect the junction capacitance $C_J$ to simplify the formulas, although the treatment can be extended to the case $C_J \neq 0$. Similarly, we consider the local ground capacitance of each island negligible, i.e. $C_0 \ll C_J$.

We quantize the circuit of Fig. 7 using the standard method to construct the Lagrangian and equations of motion for a quantum electromagnetic circuit formed by lumped elements. We use the phase nodes variables $\Phi_{n,s}$, with $n = 1, \ldots, N$ and $s = a, b$ for the two chains connected via the capacitances $C_J$. The index $n$ runs from $n = 1, \ldots, N - 1$ for the two chains, with the boundary condition $\Phi_0 = 0$. The index $n = N$ is for the probe system, the qubit, formally described by the node phases $\Phi_{N,a,\Phi_{N,b}}$. For $\Phi_{n,s}$, with $n = 1, \ldots, N - 1$, the dynamics of the system is ruled by the equations of motion

$$C_J \frac{d^2(\Phi_{n,a} - \Phi_{n,b})}{dt^2} = -\frac{2}{L_J} (2\Phi_{n,a} - \Phi_{n-1,a} - \Phi_{n+1,a}) - \int_{-\infty}^{+\infty} dt' 2\gamma(t-t') \frac{d\Phi_{n,a}}{dt},$$

and

$$C_J \frac{d^2(\Phi_{n,b} - \Phi_{n,a})}{dt^2} = -\frac{2}{L_J} (2\Phi_{n,b} - \Phi_{n-1,b} - \Phi_{n+1,b}) - \int_{-\infty}^{+\infty} dt' 2\gamma(t-t') \frac{d\Phi_{n,b}}{dt'},$$

with the external admittance

$$Y_c(t) = \theta(t) e^{-t/\tau_c}/L_J, \quad Y_c(\omega) = 1/(R_J + i\omega L_J),$$

where $\theta(t)$ is the theta function, and $\tau_c = L_J/R_J$. It is now convenient to introduce the phase differences

$$\phi_n = \Phi_{n,a} - \Phi_{n,b}.$$
Consider the system shown in Eq. (22), where the eigenfrequencies of the modes \( \Omega_k \) are given in appendix A. After some algebra (see appendix A for details), we can express the solution

\[
\theta_k(t) = -e_k(N-1) \int_{-\infty}^{+\infty} dt' \phi(t-t') \sum_{i=1}^{3} \frac{A^{(k)}_i}{z^{(k)}_i} e^{iz^{(k)}_i(t-t')} \frac{d\phi_N}{dt'} + \frac{e_k(N-1)}{\lambda_k} \phi_N(t),
\]

Finally, we consider the equation for the node associated to the probe system (the qubit, at node \( N = n \)) in terms of the phase difference \( \phi_N \). For simplicity, we set \( L_P = L_J \) and write

\[
\frac{d}{dt} \left( \frac{\partial L_q}{\partial \phi_N} \right) = \frac{\partial L_q}{\partial \phi_N} - \frac{1}{L_J} (\phi_N - \phi_N - 1) = \frac{1}{L_J} \sum_{k=1}^{N-1} e_k(N-1) \theta_k,
\]

with \( L_q \) the Lagrangian function of the phase difference of the qubit probe: its explicit form is not relevant for our analysis. Inserting the solution Eq. (39) into Eq. (10) we get the phase difference \( \phi_N \) of the qubit

\[
\frac{d}{dt} \left( \frac{\partial L_q}{\partial \phi_N} \right) = \frac{\partial L_q}{\partial \phi_N} - \phi_N - 1 + \frac{1}{L_J} \sum_{k=1}^{N-1} e_k(N-1) \theta_k, \tag{40}
\]

in which we used the property \( \sum_{k=1}^{N-1} \frac{1}{\lambda_k} = 1/N \), and we set the admittance of the double chain to

\[
Y_{JJ}(t) = \frac{\theta(t)}{L_J} \sum_{k=1}^{N-1} e_k(N-1) \sum_{i=1}^{3} \frac{A^{(k)}_i}{z^{(k)}_i} e^{iz^{(k)}_i(t-t')} \tag{42}
\]
IV. PJJC ADMITTANCE WITH EXPERIMENTALLY FEASIBLE PARAMETERS

In the PJJC, each Josephson inductance is tuned by the applied magnetic flux as \( L_J = L_J^{(0)}/\cos(f) \) with the reduced flux-bias defined as \( f = 2\pi\Phi_B/\Phi_0 \). The admittance of the single JJ shown in Fig. 8 is equivalent to the admittance of the single JJ with applied magnetic flux as \( \omega_c^{(0)} = \omega_c(0) \) at zero flux \( f = 0 \). The PJJC parameters are the following: \( N = 8000 \), \( C_J/C_C = 0.25 \), and at \( f = 0 \) the inductance ratio \( L_C/L_J^{(0)} = 10 \) and the resistance ratio \( R_C/Z_C^{(0)} = 0.025 \).

By using some algebraic relations of the root \( z_k^{(c)} \) (see appendix A for details), we derive the final expression for the admittance in Eq. (42) in frequency space

\[
Y_{J,J}(\omega) = \frac{2\omega}{NL_J} \left( 1 - \frac{\omega^2}{\omega_c^2} - \frac{i}{\omega \tau_c} \right) \sum_{k=1}^{N-1} \left( 1 + \frac{L_J}{Z_C^{(0)}} \right) \left( 1 - \frac{\Omega_k^2}{\omega^2} - i\omega \gamma_k(\omega) \right),
\]

with the damping functions

\[
\gamma_k(\omega) = \frac{1}{\tau_c} \left( 1 - \frac{\Omega_k^2 - \omega_c^2}{\omega^2} \right).
\]

Equations (43) and (44) represent the goal of this section: the admittance \( Y_{J,J}(\omega) \) corresponds exactly to the limit \( C_J/C_C \rightarrow 0 \) of the admittance \( Y_{J,J}(\omega) \) in Eq. (25) of a single chain with engineered interaction, for \( L_J \ll L_C \).

To summarize, we showed that the effective circuit shown in Fig. 7 (case \( C_J \ll C_C \)) for the PJJC device of Fig. 4 is equivalent to the admittance of the single JJ chain with engineered dissipation discussed in Sec. II.B. For vanishing junction capacitance \( C_J = 0 \). Therefore, in the following we will use Eq. (25), (26), and Eq. (27) to calculate the PJJC admittance for circuits with experimentally feasible parameters.

V. SUMMARY

We have demonstrated that the parallel Josephson junction chain device shown in Fig. 1 can implement a tunable ohmic environment, over a frequency-band of the order of GHz, with an effective resistance that can be tuned through the resistance quantum \( R_q = 6.5 \text{k}\Omega \). The PJJC can be connected to any two-terminal device under test, such as a superconducting qubit or a resonator, and its dissipation can be continuously monitored using a low-noise rf amplification chain.

The PJJC principle of operation can also be applied for constituent SQUIDs with different geometries, such as the ones proposed in Ref. [13,22], implementing even higher impedances and resulting in larger effective resistances. It is also worth mentioning that the rapid increase of the PJJC resistance at low frequencies (see Fig. 9) protects the device from low energy thermal excitations.
The effective resistance of the PJJC electromagnetic environment shown in Fig. 1 vs. frequency, for experimentally relevant circuit parameters. We chose $R_C = 50 \Omega$ and, at flux bias $f = 0$, the characteristic impedance $Z_C(0) = 4k\Omega$, with a plasma frequency $\omega_p = 15$ GHz. The other PJJC parameters are: $N = 8000, C_C/C_J = 4$ and the inductance ratio $L_C/L_J(0) = 10$ at $f = 0$. Notice that as we increase the flux bias $f$, the effective resistance of the environment increases to values above the resistance quantum. The value of $f$ can be increased beyond the $0.35\pi$ threshold shown in the figure, which will further increase the effective resistance of the PJJC device. However, the frequency range where the resistance can be considered ohmic will continue to decrease, while the chain will become increasingly non-linear.

We believe that the tunable, high-impedance ohmic environment implemented by the PJJC will be a useful instrument in the route towards quantum simulations of dissipative phase transitions, or the engineering of environments for autonomous quantum error correction schemes.

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Appendix A: Susceptibility for the $k$ harmonic modes

In this section, we list the main steps of the calculations leading to the main results Eq. (37) and Eq. (44), in Sec. III, for the admittance of the double shown in Fig. 7. The solution for the dynamics of the harmonic modes connected to the probe qubit, in frequency space is given by Eq. (37) with the susceptibility for the single eigenmode $k$ defined by Eq. (38) and with factors given by

\[
A_1^{(k)} = -\frac{\left( z_1^{(k)} - \frac{1}{\tau_c}\right)}{(z_1^{(k)} - z_2^{(k)}) (z_1^{(k)} - z_3^{(k)})}, \quad (A1)
\]

and similar definitions of $A_2^{(k)}, A_3^{(k)}$. The roots of the cubic satisfy Veta’s relations

\[
\begin{align*}
z_1^{(k)} + z_2^{(k)} + z_3^{(k)} &= i/\tau_c \\
\sum_{i=1}^{3} z_i^{(k)} \\
\end{align*} \quad (A2)
\]

\[
\begin{align*}
z_1^{(k)} z_2^{(k)} + z_2^{(k)} z_3^{(k)} + z_3^{(k)} z_1^{(k)} &= -\Omega_k^2 \\
\sum_{i=1}^{3} z_i^{(k)} (z_1^{(k)} - z_i^{(k)}) &= (i/\tau_c) \left( \omega_c^2 - \Omega_k^2 \right) \quad (A3)
\end{align*}
\]

We also have the sum rules

\[
\begin{align*}
\sum_{i=1}^{3} \frac{A_i^{(k)}}{z_i^{(k)}} &= \frac{1}{\omega_c^2 - \Omega_k^2} \quad (A5)
\end{align*}
\]

\[
\begin{align*}
\sum_{i=1}^{3} \frac{A_i^{(k)}}{z_i^{(k)} (z_i^{(k)} - z_1^{(k)})} &= 1 + \frac{\omega_3 + \omega_2 - \omega_1}{\omega_c^3 - \omega_1 \omega_2 - \omega_1 \omega_3} \quad (A6)
\end{align*}
\]

We consider the solution in time which reads

\[
\begin{align*}
\theta_k(t) &= \alpha_k \sum_{i=1}^{3} i A_i^{(k)} \int_{-\infty}^{t} e^{i \omega_1^{(k)} (t-t')} \phi_N(t') \\
&= \alpha_k \sum_{i=1}^{3} \frac{A_i^{(k)}}{z_i^{(k)}} \left[ \phi_N(t) + \int_{-\infty}^{t} e^{i \omega_1^{(k)} (t-t')} \frac{d\phi_N(t')}{dt} \right] \quad (A7)
\end{align*}
\]

in which we set $\alpha_k = \epsilon_k (N-1) = \sqrt{2/N} \sin[\pi k (N-1)/N]$ and we have used the fact that $\lim_{\Delta t \to -\infty} e^{\omega_1^{(k)} \Delta t} \phi_N(t + \Delta t) = 0$ since the roots have positive imaginary parts. The weighted sum over modes reduces to

\[
\begin{align*}
\sum_{k=1}^{N-1} \alpha_k \theta_k(t) &= \left( \sum_{k=1}^{N-1} \alpha_k^2 \sum_{i=1}^{3} \frac{A_i^{(k)}}{z_i^{(k)}} \right) \phi_N(t) \\
&+ \sum_{k=1}^{N-1} \alpha_k^2 \sum_{i=1}^{3} \int_{-\infty}^{t} e^{i \omega_1^{(k)} (t-t')} \phi_N(t') \\
&= N-1 \phi_N(t) + L_J \int_{-\infty}^{+\infty} dt' Y_{JJ}(t-t') \frac{d\phi_N(t')}{dt'} \quad (A8)
\end{align*}
\]

with the admittance of the chain given by

\[
Y_{JJ}(t) = \frac{\theta(t)}{L_J} \sum_{k=1}^{N-1} \alpha_k^2 \sum_{i=1}^{3} \frac{A_i^{(k)}}{z_i^{(k)}} e^{i \omega_1^{(k)} (t-t')} . \quad (A9)
\]

In frequency domain, the admittance reads

\[
Y_{JJ} = -\frac{i}{L_J} \sum_{k=1}^{N-1} \alpha_k^2 \sum_{i=1}^{3} \frac{A_i^{(k)}}{z_i^{(k)} (\omega - \omega_1^{(k)})} . \quad (A10)
\]
Using Veta’s relations Eqs. (A2, A3, A4) and the sum rules Eqs. (A5, A5), we can obtain the main results Eq. (43) and Eq. (44).
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