SIMPLE SEMIGROUP GRADED RINGS

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Abstract. We show that if $R$ is a, not necessarily unital, ring graded by a semigroup $G$ equipped with an idempotent $e$ such that $eGe$ is a hypercentral group and $R_e$ has a non-zero idempotent $f$, then $R$ is simple if and only if it is graded simple and the center of the corner subring $fR_{eGe}f$ is a field. This is a generalization of a result of E. Jespers’ on the simplicity of a unital ring graded by a hypercentral group. We apply our result to partial skew group rings and obtain necessary and sufficient conditions for the simplicity of a, not necessarily unital, partial skew group ring by a hypercentral group. Thereby, we generalize a very recent result of D. Gonçalves’. We also point out how E. Jespers’ result immediately implies a generalization of a simplicity result, recently obtained by A. Baraviera, W. Cortes and M. Soares, for crossed products by twisted partial actions.

1. Introduction

Suppose that $R$ is an associative, not necessarily unital, ring and $G$ is a semigroup, i.e. a non-empty set equipped with an associative binary operation $G 	imes G \ni (g, h) \mapsto gh \in G$. Recall that $R$ is called $G$-graded if there for each $g \in G$ is an additive subgroup $R_g$ of $R$ such that $R = \oplus_{g \in G} R_g$ and the inclusion $R_gR_h \subseteq R_{gh}$ holds for all $g, h \in G$.

The investigation of semigroup graded rings has been carried out by many authors, see e.g. [1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20]. For an excellent overview of the theory of semigroup graded rings, we refer the reader to A. V. Kelarev’s extensive book [28], and the references therein.

Since many ring constructions are special cases of semigroup graded rings, e.g. monomial rings, crossed products, skew polynomial rings, twisted semigroup rings, skew power series rings, edge and path algebras, generalized matrix rings, incidence algebras and category graded rings, the theory of semigroup graded rings can be applied to the study of other...
less general constructions, giving new results for several constructions simultaneously, and unifying theorems obtained earlier.

An important problem in the investigation of semigroup graded rings is to explore how properties of the whole ring $R$ are connected to properties of subrings $R_H = \oplus_{g \in H} R_g$ where $H$ runs over subsemigroups of $G$. Many results of this sort are known for finiteness conditions, nil and radical properties, semisimplicity, semiprimeness and semiprimitivity (see the references in [26] and [27]).

The aim of this article is to establish a similar result (see Theorem 2) for simple semigroup graded rings. Recall that an (two-sided) ideal $I$ of a $G$-graded ring $R$ is said to be a graded ideal (or $G$-graded ideal) if $I = \oplus_{g \in G} (I \cap R_g)$ holds. A $G$-graded ring $R$ is called graded simple (or $G$-graded simple) if $R$ and $\{0\}$ are its only graded ideals. Clearly, graded simplicity is a necessary condition for simplicity. In the case when $G$ is a hypercentral group, that is when every non-trivial factor group of $G$ has non-trivial center, E. Jespers [24] has determined precisely when unital $G$-graded rings are simple.

**Theorem 1** (E. Jespers [24]). If $G$ is a hypercentral group and $R$ is a $G$-graded unital ring, then $R$ is simple if and only if $R$ is graded simple and the center of $R$ is a field.

In this article, we generalize Theorem 1 to a class of (potentially non-unital) semigroup graded rings.

**Theorem 2.** If $R$ is a ring graded by a semigroup $G$ equipped with an idempotent $e$ such that $eGe$ is a hypercentral group and $Re$ has a non-zero idempotent $f$, then $R$ is simple if and only if it is graded simple and the center of the corner subring $fR_{eGe}f$ is a field.

Here we would like to make two remarks. First of all, note that the above result might at first glance seem more general than it actually is. In fact, graded simplicity of $R$ often forces the semigroup $G$ to be an inverse semigroup (see Remark 9). Secondly, up until recently, the authors of the present article (and presumably also the authors of e.g. [4], [10]) were unaware of the existence of E. Jespers’ simplicity results in [23] and [24], as well as A. D. Bell’s in [5]. The technique used to prove the simplicity criterion for skew group rings in [10] is slightly different from the one used in [23] and [24]. In fact, after minor adjustments, it can be used to give a more direct proof of the main result of [23].

This article is organized as follows.

In Section 2, we recall the relevant definitions concerning semigroup graded rings and prove Theorem 2.

In Section 3, we apply Theorem 2 to partial skew group rings. Partial group actions were introduced by R. Exel in the context of crossed product C*-algebras [15]. A decade later, the investigation of its algebraic counterpart (the partial skew group rings) began by [14]. We obtain necessary and sufficient conditions for the simplicity of, not necessarily unital, partial skew group rings by hypercentral groups. Thereby, we generalize a very recent result of D. Gonçalves’ [20]. At the end of the section, we also point out how E. Jespers’ result immediately implies a generalization of a simplicity result, recently obtained by A. Baraviera, W. Cortes and M. Soares, for crossed products by twisted partial actions.
2. Semigroup Graded Rings

At the end of this section, we show Theorem 2. To this end, we show a series of results concerning ideals in semigroup graded rings (see Lemma 3, Lemma 5, Proposition 6 and Proposition 7).

We begin by fixing the notation. Throughout this section, \( R \) denotes a ring graded by a semigroup \( G \). Take \( r \in R \). There are unique \( r_g \in R_g \), for \( g \in G \), such that all but finitely many of them are zero and \( r = \sum_{g \in G} r_g \). We let the support of \( r \), denoted by \( \text{Supp}(r) \), be the set of \( g \in G \) such that \( r_g \neq 0 \). The cardinality of \( \text{Supp}(r) \) is denoted by \( |\text{Supp}(r)| \).

The element \( r \) is called homogeneous if \( |\text{Supp}(r)| \leq 1 \). If \( r \in R_g \setminus \{0\} \), for some \( g \in G \), then we write \( \deg(r) = g \).

**Lemma 3.** Let \( G \) be a group and \( R \) a unital \( G \)-graded ring. If \( R \) is graded simple and \( I \) is a non-zero \( G/Z(G) \)-graded ideal of \( R \), then for each non-zero \( r \in I \), there is a non-zero \( r' \in I \cap R_{Z(G)} \cap Z(R) \) with \( |\text{Supp}(r')| \leq |\text{Supp}(r)| \). If, in addition, the ring \( Z(R) \) is a field, then \( R \) is \( G/Z(G) \)-simple.

**Proof.** This is Proposition 4 of [24]. We show this result by a different method. Suppose that \( r_g \neq 0 \) for some \( g \in G \). Since \( R \) is graded simple, we get that \( R r_g R = R \). In particular, we get that \( 1 = \sum_{i=1}^{n} s_i r_g t_i \) for some homogeneous \( s_i, t_i \in R \). Therefore, there is \( j \in \{1, \ldots, n\} \) such that \( s_j r_g t_j \neq 0 \) and \( \deg(s_j r_g t_j) = e \). By replacing \( r \) with \( s_j r_g t_j \) we can assume that \( r_e \neq 0 \). Since \( I \) is \( G/Z(G) \)-homogeneous, the \( Z(G) \)-degree part of \( r \) belongs to \( I \) and is non-zero (since \( r_e \neq 0 \) ). Therefore, we can assume that \( r \) is a non-zero element belonging to \( I \cap R_{Z(G)} \).

Now put \( J = \{ s_e \mid s \in R r R, \text{Supp}(s) \subseteq \text{Supp}(r) \} \). Then \( J \) is a non-zero ideal of \( R_e \) and hence we get that \( RJ R = R \). Thus there are \( s^{(1)}, \ldots, s^{(n)} \in R r R \) and \( v_i, w_i \in R \), for \( i \in \{1, \ldots, n\} \), such that \( 1 = \sum_{i=1}^{n} v_i s^{(i)} w_i \) and \( \text{Supp}(s^{(i)}) \subseteq \text{Supp}(r) \). This implies that \( \deg(v_i) \deg(w_i) = e \) for \( i \in \{1, \ldots, n\} \). Put \( s = \sum_{i=1}^{n} v_i s^{(i)} w_i \). Then \( s \in I \) and since \( \text{Supp}(s^{(i)}) \subseteq \text{Supp}(r) \subseteq R_{Z(G)} \), we get that \( \text{Supp}(s) \subseteq \bigcup_{i=1}^{n} \text{deg}(v_i) \text{Supp}(s^{(i)}) \subseteq \bigcup_{i=1}^{n} \text{deg}(v_i) \text{deg}(w_i) \text{Supp}(s^{(i)}) \subseteq \bigcup_{i=1}^{n} \text{Supp}(r) = \text{Supp}(r) \). Therefore, \( 1 = \sum_{i=1}^{n} v_i s^{(i)} w_i = s_e \in J \).

Now pick a non-zero element \( r' \in I \) with \( |\text{Supp}(r')| \) minimal. By the above, we can assume that \( r'_e = 1 \) and that \( r' \in I \cap R_{Z(G)} \). Take \( g \in G \) and \( t \in R_g \). Since \( r'_e = 1 \) and \( \text{Supp}(r') \subseteq Z(G) \), we get that \( \text{Supp}(r't - tr') \subseteq \text{Supp}(r') \). By the definition of \( r' \) we get that \( \text{Supp}(r't - tr') = \emptyset \) and hence that \( r't - tr' = 0 \). Therefore \( r' \in Z(R) \).

**Remark 4.** Recall that if \( G \) is a group with identity element \( e \), then the ascending central series of \( G \) is the sequence of subgroups \( Z_i(G) \), for non-negative integers \( i \), defined recursively by \( Z_0(G) = \{e\} \) and, given \( Z_i(G) \), for some non-negative integer \( i \), \( Z_{i+1}(G) \) is defined to be the set of \( g \in G \) such that for every \( h \in G \), the commutator \( [g, h] = ghg^{-1}h^{-1} \) belongs to \( Z_i(G) \). For infinite groups this process can be continued to infinite ordinal numbers by transfinite recursion. For a limit ordinal \( \Theta \), we define \( Z_\Theta(G) = \bigcup_{\lambda < \Theta} Z_\lambda(G) \). If \( G \) is hypercentral, then \( Z_\Theta(G) = G \) for some limit ordinal \( \Theta \). For the details concerning this construction, see [34].
Lemma 5. If $G$ is a hypercentral group and $R$ is a $G$-graded ring with the property that for each $i \in I$ the ring $R$ is $G/Z_i(G)$-simple, then $R$ is simple.

Proof. Take a non-zero ideal $J$ of $R$ and a non-zero $a \in J$. We show that $\langle a \rangle = R$. Since $\bigcup_i Z_i(G) = G$ and $\text{Supp}(a)$ is finite, we can conclude that there is some $i$ such that $\text{Supp}(a) \subseteq Z_i(G)$. Then $\langle a \rangle$ is a non-zero $G/Z_i(G)$-graded ideal of $R$. Since $R$ is $G/Z_i(G)$-simple, we get that $\langle a \rangle = R$, which shows that $J = R$. \qed

Proposition 6. Suppose that $G$ is a semigroup and $R$ is a ring graded by $G$. If $R$ is simple, $e \in G$ is an idempotent and $f \in R_e$ is a non-zero idempotent, then the center of $f R_{eGe} f$ is a field.

Proof. Take a non-zero $x$ in $Z(f R_{eGe} f)$. Since the ideal $RxR$ of $R$ is non-zero and $R$ is simple, we get that $RxR = R$. In particular, $f$ equals a finite sum of elements of the form $y_i x z_i$, for $i \in \{1, \ldots, n\}$, where each $y_i$ and each $z_i$ is homogeneous. Hence $f = f \cdot f \cdot f = \sum_{i=1}^{n} f y_i x z_i f$ so we may assume that $x_i, y_i \in f R_{eGe} f$ for all $i$. But since $x$ belongs to $Z(f R_{eGe} f)$ we get that $f = wx = xw$ for some $w \in f R_{eGe} f$. All that is left to show now is that $w \in Z(f R_{eGe} f)$. Take $v \in f R_{eGe} f$. Then, since $x$ commutes with $v$, we get that $wv = vw f = vw f = xw = f v w = v w$. \qed

Proposition 7. Suppose that $G$ is a semigroup and $R$ is a ring graded by $G$. If $R$ is graded simple and there is an idempotent $e \in G$ and a non-zero idempotent $f \in R_e$ such that the center of $f R_{eGe} f$ is a field and $e Ge$ is a hypercentral group, then $R$ is simple.

Proof. Put $H = e Ge$ and $S = f R_H f$. We claim that $S$ is simple. Assume for a moment that the claim holds. We show that $R$ is simple. Take a non-zero ideal $I$ of $R$ and a non-zero $x \in I$. Take $g \in G$ such that $x_g \neq 0$. Since $R$ is graded simple, we get that $f \in Rx_g R$. Hence, we get that $f = \sum_{i=1}^{n} y_i x_g z_i f$ for some homogeneous $y_i, z_i \in R$. Since $f = f^3$, we get that $f = \sum_{i=1}^{n} f y_i x_g z_i f$. From the fact that $f$ is non-zero it now follows that there is $j \in \{1, \ldots, n\}$ such that $f y_j x_g z_j f$ is non-zero. Now put $x' = f y_j x_g z_j f$. By the construction of $x'$ it follows that $x' \in I \cap S$. Since $H$ is a group it also follows that $x'$ is non-zero. We thus get that $I \cap S \neq \{0\}$. But since $S$ is simple, we get that $I \cap S = S$, or, equivalently, that $S \subseteq I$. This implies, in particular, that $f \in I$. Since $f$ is homogeneous and $R$ is graded simple, we thus get that $I \supseteq R f R = R$. Hence $I = R$.

Now we show the claim in the beginning of the proof i.e. that $S$ is simple. Let $Z_i(H)$, for $i \geq 0$, be the ascending central series of $H$ (see Remark 4). By induction over $i$ we now show that for each $i \in I$, the ring $S$ is $H/Z_i(H)$-simple.

First we show the base case: $i = 0$. Since $H/Z_0(H) = H/\{e\} = H$, we need to show that $S$ is $H$-simple. Suppose that $J$ is a non-zero $H$-graded ideal of $S$. Then $RJR$ is a non-zero $G$-graded ideal of $R$. Since $R$ is graded simple, we get that $RJR = R$. In particular, we get that $u \in RJR$, and hence, we that $u = u^3 \in uRJR u \subseteq S$. Since $u$ is a multiplicative identity of $S$, we get that $J = S$.

Now we show the induction step. Suppose that the statement is true for some $i$, i.e. that $S$ is $H/Z_i(H)$-simple. By Lemma 3, we get that $S$ is $\frac{H/Z_i(H)}{Z(H/Z_i(H))}$-simple. Since the
center of $H/Z_i(H)$ equals $Z_{i+1}(H)/Z_i(H)$ we get that $S$ is $H/Z_i(H)$-simple, i.e. that $S$ is $H/Z_{i+1}(H)$-simple and the induction step is complete.

By Lemma 5 we get that $S$ is simple. □

**Proof of Theorem 2.** This follows immediately from Propositions 6 and 7. □

The following result generalizes another result of E. Jespers’ [23, Corollary 6] to the semigroup graded situation. We will not use this result in the sequel, but we think that it is interesting in its own right.

**Corollary 8.** If $R$ is a ring graded by a semigroup $G$ equipped with an idempotent $e$ such that $eGe$ is a torsion-free abelian group and $Re$ has a non-zero idempotent $f$, then $R$ is simple if and only if it is graded simple and the center of the corner subring $fReGe f$ is contained in $fRRe f$.

**Proof.** Put $H = eGe$ and $S = fRRe f$.

First we show the "only if" direction. Suppose that $R$ is simple. By Theorem 2 we get that $R$ is graded simple and that $Z(S)$ is a field. Since $H$ is abelian, we get that $Z(S)$ is a graded additive subgroup of $S$. Seeking a contradiction, suppose that there is a non-identity $g \in H$ and a non-zero $r \in R_g \cap Z(S)$. Then $1 + r \in Z(S) \setminus \{0\}$ and hence $1 + r$ is invertible in $Z(S)$ with inverse $s \in Z(S) \setminus \{0\}$. But this leads to a contradiction, since $|\text{Supp}((1 + r)s)|$ is even, $|\text{Supp}(u)| = 1$ is odd and $(1 + r)s = u$.

Now we show the "if" direction. Suppose that $R$ is graded simple and that the center of $S$ is contained in $S_e$. By the proof of Proposition 7 (base case), we get that $S$ is graded simple. Take a non-zero element $v$ in $Z(S)$. Since $SvS$ is a graded ideal, we therefore get that $SvS = S$. Thus $v$ is invertible in $S$. But since $Z(S) \subseteq S_e$, we get that the inverse of $v$ belongs to $S_e$ also. Hence $Z(S)$ is a field and the claim follows from Theorem 2. □

**Remark 9.** By assuming that a $G$-graded ring $R$ is graded simple, one often imposes restrictions on the semigroup $G$. In fact, if we suppose that the following rather mild conditions are satisfied

(i) for each non-zero $g \in G$, there is $p, q \in G$ such that $R_p R_g R_q$ is non-zero, and

(ii) there is a non-zero idempotent $e$ in $G$,

then $G$ is an inverse semigroup. In fact, take non-zero $g, h \in G$. By (i), we get that $RR_g R$ is a non-zero graded ideal of $R$. Then, by graded simplicity of $R$, we get that $R_h \subseteq RR_g R$.

By (i) again, we get that $R_h$ is non-zero. Hence, there are $p, q \in G$ such that $R_p R_g R_q$ is non-zero and $pqg = h$. In other words, we get that $h \in GgG$ for all non-zero $g, h \in G$.

Thus, $G$ is simple. This, in combination with the existence of the non-zero idempotent $e$, implies that $G$ is an inverse semigroup (see Theorem 3 in [33]).

**Remark 10.** Suppose that $G$ is a small category, i.e. with the property that its morphisms form a set. Let the domain and codomain of a morphism $g$ in $G$ be denoted by $d(g)$ and $c(g)$ respectively. Note that every category $G$ can be viewed as a semigroup if we adjoin a zero element $\theta$ with the property that $g\theta = \theta g = \theta$, for $g \in G$, and $gh = \theta$ whenever $gh$ is undefined in the category, i.e. when $d(g) \neq c(h)$. As a consequence, category graded
rings (in the sense of [30]) can be viewed as semigroup graded rings. Hence, Theorem 2 is applicable in this situation as well.

3. Applications to Partial Skew Group Rings

In this section, we apply Theorem 2 to partial skew group rings. We generalize a recent result by D. Gonçalves [20] to partial skew group rings by hypercentral groups over rings with local units (see Theorem 17). At the end of this section, we also point out how E. Jespers’ result immediately implies a generalization of a simplicity result, recently obtained by A. Baraviera, W. Cortes and M. Soares, for crossed products by twisted partial actions (see Remark 18). First we recall the definition of a partial skew group ring.

**Definition 11.** Let $G$ be a group with neutral element $e$ and let $A$ be a ring. A partial action $\alpha$ of $G$ on $A$ is a collection of ideals $\{D_g\}_{g \in G}$ of $A$ and a collection of ring isomorphisms $\alpha_g : D_{g^{-1}} \to D_g$ such that for all $g, h \in G$ and every $x \in D_{h^{-1}} \cap D_{(gh)^{-1}}$, the following three relations hold:

(i) $\alpha_e = \text{id}_A$;
(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;
(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$.

The partial skew group ring $A \ast_\alpha G$, associated with the partial action above, is defined as the set of all finite formal sums $\sum_{g \in G} a_g \delta_g$, where for each $g \in G$, $a_g \in D_g$ and $\delta_g$ is a symbol. Addition is defined in the obvious way and multiplication is defined as the linear extension of the rule $(a_g \delta_g)(b_h \delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}$ for $g, h \in G$, $a_g \in D_g$ and $b_h \in D_h$.

It is easy to check that if we put $(A \ast_\alpha G)_g = D_g \delta_g$, for $g \in G$, then this defines a gradation on the ring $A \ast_\alpha G$. Clearly, each classical skew group ring (see e.g. [17]) is a partial skew group ring where $D_g = A$ for all $g \in G$.

**Definition 12.** Recall from [3] that a ring $R$ has local units if there exists a set $E$ of idempotents in $R$ such that, for every finite subset $X$ of $R$, there exists an $f \in E$ such that $X \subseteq fRf$. From this it follows that $x = f x = xf$ holds for each $x \in X$. In that case, we will refer to $E$ as a set of local units for $R$ and to $f$ as a local unit for the subset $X$.

**Remark 13.** Rings with local units occur widely in mathematics, often in algebra (e.g. von Neumann regular rings [3] Example 1 and Leavitt path algebras [21] Lemma 1.6]), functional analysis (e.g. algebras of complex valued functions with compact support) and category theory (e.g. [21] Remark 1) or [18]).

**Remark 14.** A partial skew group ring $A \ast_\alpha G$ need not in general be associative (see [14] Example 3.5]). However, if each $D_g$, for $g \in G$, has local units, then, in particular, each $D_g$, for $g \in G$, is an idempotent ring, i.e. $D_g^2 = D_g$, which by [14] Corollary 3.2], ensures that $A \ast_\alpha G$ is associative. In that case, the set $E \delta_e = \{f \delta_e \mid f \in E\}$ is a set of local units for $A \ast_\alpha G$, if $E$ is a set of local units for $A$.

**Definition 15.** If $A \ast_\alpha G$ is a partial skew group ring, then an ideal $I$ of $A$ is said to be $G$-invariant if $\alpha_g(I \cap D_{g^{-1}}) \subseteq I$ holds for each $g \in G$. If $A$ and $\{0\}$ are the only $G$-invariant ideals of $A$, then $A$ is said to be $G$-simple.
Lemma 16. If \( \alpha \) is a partial action of a group \( G \) on a ring \( A \) such that for each \( g \in G \), the ring \( D_g \) has local units, then \( A \ast \alpha G \) is graded simple if and only if \( A \) is \( G \)-simple.

Proof. We begin by showing the "only if" statement. Suppose that \( A \ast \alpha G \) is graded simple. Let \( I \) be a non-zero \( G \)-invariant ideal of \( A \). Define \( I \ast \alpha G \) to be the set of all finite sums of the form \( \sum_{g \in G} a_g \delta_g \), where \( a_g \in \mathcal{I} \cap D_g \), for \( g \in G \). Note that \( I \ast \alpha G \) is a non-zero two-sided graded ideal of \( A \ast \alpha G \). Hence, \( I \ast \alpha G = A \ast \alpha G \). In particular, \( A \delta_e \subseteq I \ast \alpha G \) which shows that \( I \subseteq A \subseteq I \). We conclude that \( I = A \). Thus, \( A \) is \( G \)-simple.

Now we show the "if" statement. Suppose that \( A \) is \( G \)-simple. Let \( J \) be a non-zero graded ideal of \( A \ast \alpha G \). We claim that \( J_e = J \cap A \) is a non-zero \( G \)-invariant ideal of \( A \). If we assume that the claim holds, then \( A = J_e = A \cap J \subseteq J \) from which it follows that \( J = A \ast \alpha G \). Now we show the claim. First we show that \( J_e \) is non-zero. Since \( J \) is non-zero, there is \( g \in G \) and a non-zero \( a_g \in D_g \) with \( a_g \delta_g \in J \). Let \( b_{g^{-1}} \in D_{g^{-1}} \) be a local unit for \( \alpha_{g^{-1}}(a_g) \). Then \( J \ni a_g \delta_g b_{g^{-1}} \delta_g^{-1} = a_g(\alpha_{g^{-1}}(a_g)b_{g^{-1}})\delta_g = a_g(\alpha_{g^{-1}}(a_g))\delta_g = a_g \delta_g \) which is non-zero. Now we show that \( J_e \) is \( G \)-invariant. Take \( g \in G \) and \( a \in J_e \cap D_{g^{-1}} \). Let \( c_g \in D_g \) be such that \( \alpha_{g^{-1}}(c_g) \) is a local unit for \( a \). Then \( \alpha_g(a)u_e = \alpha_g(\alpha_{g^{-1}}(c_g)a)u_e = c_g \delta_g a \delta_{g^{-1}} \in J \). \( \Box \)

The following result generalizes the recent result by D. Gonçalves [20, Theorem 2.5] from the case when \( G \) is abelian to the case when \( G \) is a hypercentral group. Moreover, our result shows that it is enough to consider the center of one corner.

Theorem 17. Suppose that \( \alpha \) is a partial action of a hypercentral group \( G \) on a ring \( A \) such that for each \( g \in G \), the ring \( D_g \) has local units. Let \( E \) denote a set of local units for \( A \). The following three assertions are equivalent:

(i) \( A \ast \alpha G \) is a simple ring;
(ii) \( A \) is \( G \)-simple and the center of the corner subring \( f \delta_e(A \ast \alpha G) f \delta_e \) is a field, for some non-zero \( f \in E \);
(iii) \( A \) is \( G \)-simple and the center of the corner subring \( f \delta_e(A \ast \alpha G) f \delta_e \) is a field, for each non-zero \( f \in E \).

Proof. This follows immediately from Lemma [16] and Theorem [2] \( \Box \)

Remark 18. Analogously to the way in which Lemma [16] is proven, one can show that the crossed product by a twisted partial action [12], denoted by \( A \ast \omega G \), is graded simple if and only if \( A \) is \( G \)-simple. Thereby, [4, Theorem 2.25] which was recently observed by A. Baraviera, W. Cortes and M. Soares, can immediately be retrieved as a corollary to [23, Theorem 5]. In fact, by E. Jespers’ result (Theorem [11], we note that [4, Theorem 2.25] holds for \( A \ast \omega G \) even when \( G \) is a hypercentral (not necessarily abelian) group.

Acknowledgements

The second author was partially supported by The Swedish Research Council (repatriation grant no. 2012-6113).
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