Phase transition in the assignment problem for random matrices

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Abstract. – We report an analytic and numerical study of a phase transition in a P problem (the assignment problem) that separates two phases whose representatives are the simple matching problem (an easy P problem) and the traveling salesman problem (a NP-complete problem). Like other phase transitions found in combinatoric problems (K-satisfiability, number partitioning) this can help to understand the nature of the difficulties in solving NP problems an to find more accurate algorithms for them.

In the theory of computational complexity, two paradigmatic problems representative of the classes \textit{NP}-complete and \textit{P} are the traveling salesman and the assignment problem. Traveling salesman problem (TSP) requires \(N\) points or cities, with distances \(d_{i,j}\) between them, for which we must find the closed tour of minimum length that visits each city once. In the assignment or bipartite matching problem (AP), we have \(N\) objects of two classes (say A and B) with distances \(d_{i,j}\) between each object \(i\) of the class A and the object \(j\) of the class B. We must assign at each object \(i\) of the class A one and only one object \(\sigma(i)\) of the class B in such a way that the total distance \(\sum_{i=1}^{N} d_{i,\sigma(i)}\) is a minimum, so we can write \(D(\text{AP})\) as:

\[
D(\text{AP}) = \text{Min}_{\sigma \in S_N} \left[ \sum_{i=1}^{N} d_{i,\sigma(i)} \right], \tag{1}
\]

being \(S_N\) the symmetric group. This problem can be seen in another way when the sets A and B are the same (for example cities), in this case \(D(\text{AP})\) is the path of minimum length that visits each city once and is composed of so many closed sub tours as needed. In other words, when \(A = B\) the bipartite matching can be seen as a problem of \(n\)-traveling salesmen where each salesman must start and end his tour in the same city and we can use as many salesmen as needed in order to minimize the total length of all salesmen tours. Although the configuration space for the AP problem has \(N!\) elements while that of the TSP has only \((N - 1)!\), the AP is in the class \(P\) and can be solved in times that grows like \(N^3\) with the number \(N\) of cities.

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There is another interesting limit for the AP, when we introduce the constrain that the number of tours must be \(N/2\) (for \(N\) even) or equivalently, that the \(N\) cities must be pairwise matched. This limit is called simple matching problem (SMP) and in this case the configuration space has \((N-1)!!\) elements and the problem remains in the class \(P\).

Depending on the characteristics of the distance matrix the AP could, in principle, interpolate between those situations which are near of the SM problem (that is, situations where the optimal solution is composed approximately of \(N/2\) cycles) and those which are near of the TSP problem (when the optimal solution is composed only of a few cycles). We shall see that the control parameter that governs the transition between both limits is the correlation between the distances \(d_{i,j}\) and \(d_{j,i}\) \([1]\), in such a way that for positive correlations the optimal solution for the AP is in the “SMP regime”, whereas for anti correlated distances is in the “TSP regime” \([2]\).

The aim of this work is to study the structure (with reference to the number of cycles) of the optimal solution for the bipartite matching with random distance matrices e.g. matrices whose elements \(d_{i,j}\) are random numbers with a distribution of probability \(\rho(d_{i,j})\). The problem of bipartite matching on random matrices has been studied for many years, but those studies focused on the length of the minimal path \(D(\text{AP})\). For example for random matrices whose probability distribution is \(\rho(d_{i,j}) = \exp(-d_{i,j})\) it was conjectured first by G. Parisi \([3]\) and then proved rigorously \([4]-[6]\) that the average length is \(D(\text{AP}) = \sum_{m=1}^{\infty} \frac{1}{m}\) with \(N\) the number of points that must be matched. For more general distributions the first terms of the expansion in \(1/N\) of \(D(\text{AP})\) are known (\([7]-[11]\)). As for the TSP on random matrices with \(\rho(0) = 1\), the averaged length of the minimal tour is \(D(\text{TSP}) = 2.041... + \mathcal{O}(1/N)\) and the \(1/N\) corrections can be computed in principle although “the computations become rather long” \([12]\).

We shall work with random distance matrices given by

\[
\begin{align*}
  d_{i,j} &= R_{ij} + \lambda R_{ji} & i \neq j \\
  d_{i,i} &= \infty,
\end{align*}
\]

where \(R_{ij}, i \neq j\) are independent random variables with uniform distribution in the interval \([0, 1]\) and \(\lambda \in [-1, 1]\). \(d_{i,i} = \infty\) is imposed in order to make every salesman to travel out of his home city i.e. we do not allow 1-cycles \([13]\). \(\lambda = 0\) corresponds to a random matrix with all entries uncorrelated, \(\lambda = 1\) is the symmetric case \(d_{i,j} = d_{j,i} = R_{ij} + R_{ji}\) and \(\lambda = -1\) the antisymmetric one \(d_{i,j} = R_{ij} - R_{ji} = -d_{j,i}\). For each value of \(\lambda\) we generate \(M\) different distance matrices (typically \(M\) varies between \(10^4\) and \(10^6\)) and we solve for each of them the assignment problem using the algorithm of R. Jonker and A. Volgenant \([14]\). Then we measure the mean value of the number of cycles \(\langle n_c \rangle\), which is plotted in Fig. \(\boxed{1}\).

There we can see two very different regimes for the behavior of \(\langle n_c \rangle\) as a function of \(\lambda\). When the correlations between \(d_{i,j}\) and \(d_{j,i}\) are negative \((-1 \leq \lambda < 0)\), \(\langle n_c \rangle\) is (almost) constant with \(\lambda\), has a logarithmic dependence with \(N\) i.e. the solution for the assignment problem is composed typically of a few cycles, close to the TSP problem. This situation contrasts with that of positively correlated \(d_{i,j}\) and \(d_{j,i}\) \((0 < \lambda \leq 1)\), where \(\langle n_c \rangle\) varies with a nearly linear dependence in \(\lambda\) and \(N\), reaching its maximum for \(\lambda = 1\) with an approximate value \(\langle n_c \rangle_{\lambda=1} \approx N/2\). At this point the solutions are very close to those of the SM problem in the sense that they are dominated by 2-cycles.

The cases \(\lambda = 0\) and \(-1\) can be analyzed explicitly. For \(\lambda = 0\) the distance matrix is completely random in the sense that off diagonal entries are equally distributed, independent random variables. Hence all the permutations have the same probability of being the optimal tour (except those which have some 1-cycle that are excluded) \([15]\) and studying the structure...
of cycles of the optimal solution is the same that to study the structure of cycles of the corresponding subset of the permutation group. The latter can be done with the help of the associated Stirling numbers of the first kind \( d_2(N, k) \) defined as the number of permutations of \( N \) elements having \( k \) cycles, all of which of length \( r \geq 2 \) \([16]-[18]\). Using the generating function of \( d_2(N, k) \) we can calculate exactly the expected value of \( n_c \) as:

\[
\langle n_c \rangle_{\lambda=0} = -\frac{d^N}{dx^N} \left( \frac{(log(1-x)+x)\exp(-x)}{1-x} \right)_{x=0} = H_N - 1 + \sum_{i=1}^{\infty} C_i N^{-i},
\]

where \( H_N = \sum_{m=1}^{N} (1/m) \) is the harmonic series and the first coefficients in the expansion in (3) are \( C_i = 1, -1/2, -1/6, 1/4, 8/15, 1/12, -85/42, -125/24, 13/90, 479/10, 5800/33 \) for \( i = 1, \ldots, 11 \). In order to check the code of our simulations, in Fig. 2 we plot the expected values of the mean number of cycles at \( \lambda = 0 \) \((\langle n_c \rangle_{\lambda=0})\) obtained in our simulations (dots) and compared with the theoretical results of (3) (continuous line).

Note that this result is based, only, in the fact that all distances are identically distributed independent random variables, and is independent of the actual probability distribution. That is, a change of the probability will affect, only, to the value of the length of the tour and not to the value of \( \langle n_c \rangle \). Should we use random numbers also for \( d_{i,j} \) (recall (2)) then 1-cycles would be allowed and the average number of cycles \( \langle n_c \rangle_{\lambda=0} \) is exactly the Harmonic Series \( H_N \). So for \( N \to \infty \) we have that \( \langle n_c \rangle_{\lambda=0} = \langle n_c \rangle_{\lambda=0}^0 - 1 \), i.e. for large values of \( N \), to allow or to forbid one-cycles only changes in one unit the mean value of \( n_c \).

The case \( \lambda = -1 \) can be studied in a similar way, although here we do not have an exact result but an extremely good approximation. The key observation is that the minimal tour tends to be made of the smallest distances available, then for anti correlated \( d_{i,j} \) and \( d_{j,i} \) it
Figure 2 – Theoretical (continuous line) and experimental (dots) mean value of the number of cycles as a function of $N$, for $\lambda = 0$. Again, error bars correspond to three standard errors of the mean.

is very unlikely that both enter in the minimal tour (because one of them will be positive and the other negative adding up to zero). The net effect is that the anti correlations tend to suppress the appearance of 2-cycles in the optimal tour and this effect will increase with increasing $N$. This has been verified in the numerical simulations, whose results are plotted in Fig. 3. There we represent the logarithm of the average number of 2-cycles in the optimal tour ($P_2$) as a function of $N$ for $\lambda = -1, -0.2$ and $-0.1$.

As can be seen in Fig. 3, the probability of having a two-cycle in the optimal solution decays exponentially with $N$ with a coefficient that depends on $\lambda$. For $\lambda = -1$ the points can be fitted to $\log[P_2] = 2.17653 - 0.941985N$, i.e. $P_2 \sim A\xi^{-N}$ with $\xi = 2.565...$, so when $N > 20$ we can neglect the 2-cycles and then all the distances that appears in the optimal tour will be uncorrelated. The problem is again reduced to the study of the subset of the permutation group, in this case without 1-cycles and 2-cycles. This can be done with the help of the associated Stirling numbers $d_3(N, k)$ and their generating function. Finally for $\lambda = -1$,
the value of \( \langle n_c \rangle \) can be computed (up to corrections of order \( \xi^{-N} \)) as:

\[
\langle n_c \rangle_{\lambda=-1} = - \left[ \frac{d^N}{dx^N} \left( \frac{\log(1-x)+p(x)\exp(-p(x))}{1-x} \right) \right]_{x=0} - \left[ \frac{d^N}{dx^N} \left( \frac{\exp(-p(x))}{1-x} \right) \right]_{x=0} = H_N - \frac{3}{2} + \sum_{i=1}^{\infty} C_i N^{-i},
\]

(4)

where \( p(x) = x + x^2/2 \), and the first coefficients in (4) are \( C_i = 2, -3/2, -5/6, 7/4, 106/15, 67/12, -2627/42, -8633/24, 47929/90, 31758/5, 1989059/33 \) for \( i = 1, \ldots, 11 \). In the inset of Fig. 1 we plot, together, the values of \( \langle n_c \rangle / \log(N) \) obtained from (4) and the experimental points, so we can see that the agreement is excellent. It should be noted that in the limit \( N \to \infty \) we obtain from (3) and (4) the relation

\[
\lim_{N \to \infty} (\langle n_c \rangle_{\lambda=0} - \langle n_c \rangle_{\lambda=-1}) = \frac{1}{2},
\]

(5)

which states that, for large \( N \), between \( \lambda = 0 \) and \( \lambda = -1 \) the value of \( \langle n_c \rangle \) decreases only 0.5 units. Actually we can show that

\[
\lim_{N \to \infty} (H_N - \langle n_c \rangle_\lambda) = \frac{3}{2} \quad \forall \lambda \in [-1, 0)
\]

(6)

which tells us that in the thermodynamic limit \( N \to \infty \) the expected value of the number of cycles is independent of \( \lambda \) in the range \( \lambda < 0 \). The theoretical explanation for this fact is again based in the exponential suppression of 2-cycles shown in Fig. 3, and therefore the equal probability of all permutations (without 1 or 2-cycles) for producing the optimal tour. This, in turn, implies that up to corrections of order \( O(\xi^{-N}) \) the results are independent of the probability distribution as it happened in the random (\( \lambda = 0 \)) case. In this sense the concrete distribution used for the entries of our random matrix (2) that except for \( \lambda = 0 \) is non uniform, has little effect on the results and the influence weakens more and more as the dimension \( N \) grows.

Figure 4 – Plot of the ratio between the number of 2-cycles (\( P_2 \)) and the total number of cycles versus \( \lambda \) for different values of the dimension.

The right half of figure 1, corresponding to positive values of \( \lambda \), is much more poorly understood. It is clear from the diagram that \( \langle n_c \rangle \) behaves linearly with \( \lambda \) with a slope close
to $N/2$. This linear behavior has corrections near $\lambda = 1$ that are visible, for instance, in the change of the slope in the curve for $N = 200$ at $\lambda = 0.8$. It is apparent that the cycles of the optimal solutions in this side are dominated by 2-cycles, and as one can see from Fig. 4

$$\lim_{N \to \infty} \frac{P_2}{\langle n_c \rangle} = 1 \quad \forall \lambda > 0.$$  

Actually the number of 2-cycles grows linearly with $N$ for all positive values of $\lambda$ while the rest, i.e., the total number of $k$-cycles with $k \neq 2$ grows logarithmically. As $\lambda$ approaches 0 from the positive side the linear behavior with $N$ in $P_2$ has a smaller slope and the average number of 2-cycles goes to a constant $1/2$ (for large $N$) in the $\lambda = 0$ limit.

In relation to this we can prove that in the symmetric point ($\lambda = 1$) the probability of having a 2$k$-cycle with $k > 1$ vanishes. For if the links in even positions of the cycle add up to a length smaller (greater) than those in odd positions then breaking the 2$k$-cycle into $k$ 2-cycles, using the even (odd) links, lowers the length of the tour. As a result instead of a 2$k$-cycle we would have $k$ 2-cycles in the optimal tour. This feature helps to understand the abundance of 2-cycles for positive values of $\lambda$ and the change of the slope near $\lambda = 1$.

The dimerized phase corresponding to $\lambda > 0$ contrasts with the polymerized one for $\lambda < 0$. In this region there are $N$-cycles in the optimal solution with a probability $P_N$ obtained from the corresponding generating function

$$P_N(\lambda < 0) = e^{3/2N} + O(\xi^{-N})$$  

while $N$-cycles are exponentially suppressed in the dimerized region. The intermediate case corresponds to $\lambda = 0$ with

$$P_N(\lambda = 0) = \frac{e}{N} + O(1/N!).$$  

The error in (7) is produced by the residual presence of 2-cycles as discussed after Fig. 3, while that of (8), much smaller, comes from the tail of the series for the exponential. This has been verified numerically, the results are shown in Fig. 5. A more detailed analysis of the scaling as well as the “critical” properties of the transition is in progress.
Notice that in the polymerized phase ($\lambda < 0$) there are a non-negligible probability of finding a solution of the TSP in polynomial time, actually after $N$ runnings of the problem one would solve in a time of the order $N \cdot N^3$ a number of TSP’s approximately equal to $e^{3/2} = 4.48..$.

In conclusion, we have found a phase transition in the random assignment problem that separates two regimes where, while remaining always solvable in polynomial time, the system approaches the traveling salesman problem and the simple matching problem respectively. This phase transition can be seen as complementary of that which appears in many NP-complete problems separating the easy instances from the hard ones ([2], [19]). In our case, the control parameter is the correlation between the distances $d_{i,j}$ and $d_{j,i}$ which in turn controls the existence, or not, of allowed 2-cycles in the optimal tour. When both distances are positively correlated, the 2-cycles are allowed and favored and in the limit $N \to \infty$, the $\langle n_c \rangle$ is dominated by the 2-cycles as seen in Fig. 4, consequently, the AP optimal solution for those instances is very far away of that of the TSP solution for the same matrix. On the contrary, when $d_{i,j}$ and $d_{j,i}$ are negatively correlated it is very unlikely that a 2-cycle enters in the optimal solution, consequently they are strongly suppressed (see Fig. 3 and 4) and the optimal tour is composed of a “few” cycles ($\approx \log(N)$) which means that the AP optimal tour for those distances is near the one of the TSP problem for the same matrix. This property can be useful as a starting point for designing improved algorithms for solving the traveling salesman problem.

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[1] Note that $d_{i,j}$ are not true distances, in a mathematical sense, in particular they do not need to be positive or symmetric.

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invariant the probability distribution for $d$ and transforms $\sigma$ into $\sigma'$. Then both permutations have equal probability of leading to the shortest path.

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