Intermittency in Dynamics of Two-Dimensional Vortex-like Defects

V. V. Lebedev

Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel; and Landau Institute for Theoretical Physics, RAS, Kosygina 2, Moscow, 117940, Russia.

e-mail: lwlebede@wicc.weizmann.ac.il and lebede@landau.ac.ru

(March 1, 2018)

We examine high-order dynamical correlations of defects (vortices, disclinations etc) in thin films starting from the Langevin equation for the defect motion. We demonstrate that dynamical correlation functions $F_{2n}$ of vorticity and disclinity behave as $F_{2n} \sim y^2/r^{4n}$ where $r$ is the characteristic scale and $y$ is the renormalized fugacity. As a consequence, below the Berezinskii-Kosterlitz-Thouless transition temperature $F_{2n}$ are characterized by anomalous scaling exponents. The behavior strongly differs from the normal law $F_{2n} \sim F_n^2$ occurring for simultaneous correlation functions, the non-simultaneous correlation functions appear to be much larger. The phenomenon resembles intermittency in turbulence.

PACS numbers: 68.60.-p 05.20.-y 05.40.-a 64.60.Ht

INTRODUCTION

It is well known that defects like quantum vortices, spin vortices, dislocations and disclinations play an essential role in physics of low-temperature phases of thin films. Berezinskii [1] and then Kosterlitz and Thouless [2] recognized that there is a class of phase transitions in 2$d$ systems related to the defects. The main idea of their approach is that in 2$d$ the defects can be treated as point objects interacting like charged particles. It is usually called Coulomb gas analogy. The low-temperature phase corresponds to a fluid constituted of bound uncharged defect-antidefect pairs, which is an insulator, whereas the high-temperature phase contains free charged particles and can be treated as plasma. Correspondingly, in the low-temperature phase the correlation length is infinite whereas in the high-temperature phase it is finite. A huge number of works is devoted to different aspects of the problem, see, e.g., the surveys [3,4]. The scheme proposed by Kosterlitz and Thouless can be applied to superfluid and hexatic films and planar 2$d$ magnetics. It admits a generalization for crystalline films, see Refs. [5,6]. There are also applications to superconductive materials, especially to high-$T_c$ superconductors, see, e.g., Ref. [7].

The dynamics of the films in the presence of the defects was considered in the papers [8,9,10]. In the works a complete set of equations is formulated describing both motion of the defects and hydrodynamic degrees of freedom. Then, to obtain macroscopic dynamic equations, an averaging over an intermediate scale was performed. At the procedure the “current density” related to the defects was substituted by an expression proportional to the average “electric field” and to gradients of the temperature and of the chemical potential. The resulting equations perfectly correspond to the problems solved in the works [11,12]. Unfortunately, at the procedure an information concerning high-order correlations of the defect motion is lost. That is the motivation for the present work where these high-order correlations are examined.

We start from the same “microscopical” equations of the defect dynamics as was accepted in Ref. [11]. Following the works we focus mainly on the case when the motion of the defects is determined by the Langevin equation describing an interplay between the Coulomb inter-action and the thermal noise. We believe that the approach is correct for hexatic films (membranes, Langmuir films, freely suspended films). The situation is a bit more complicated for the vortices in superfluid films because of the Magnus force. Nevertheless, the equation for the vortices is close to the Langevin equation, see Ref. [11]. Similar equations can be formulated for the dislocations in crystalline films, see Ref. [12], for the vortices in superconductors in some interval of scales, see, e.g., Ref. [13], and for the spin vortices in planar 2$d$ magnetics. We will not consider the last cases here, though our scheme is, generally, applicable to the systems. Treating non-simultaneous correlation functions related to the defects one should take into account creation and annihilation processes also. For the purpose we use the Doi technique [14] who demonstrated that dynamics of classical particles involved into chemical reactions can be examined in terms of the creation and annihilation operators, like in the quantum field theory.

We consider correlation functions $F_{2n}$ of the “charge density” $\rho$ (vorticity, disclinity etc) provided that the so-called renormalized fugacity $y$ is small. The inequality $y \ll 1$ is satisfied for large scales in the low-temperature phase and probably in some region of scales above $T_c$. In statics, the normal estimate $F_{2n} \sim F_n^2$ is valid at the condition. Surprisingly, the non-simultaneous high-order correlation functions $F_{2n}$ appear to be much larger than...
their normal estimate $F^0$. In the low-temperature phase the phenomenon reveals an anomalous scaling on large scales. The reason for such unusual behavior is that the main contribution to high-order non-simultaneous correlation functions is associated with rare single defect-antidefect pairs. The situation resembles the intermittency phenomenon in turbulence, see, e.g., Ref. [4]. It can also be compared with non-trivial tails of probability distribution functions in the physics of disordered materials, see, e.g., Refs. [15,16]. Some preliminary results were published in the paper [17].

Let us give a qualitative explanation of the phenomenon. To obtain a non-zero contribution to the correlation function $F_{2n}(t_1,\ldots,t_n;\mathbf{r}_1,\ldots,\mathbf{r}_n)$ one must consider trajectories of the particles passing through the points $\mathbf{r}_1,\ldots,\mathbf{r}_n$ at the time moments $t_1,\ldots,t_n$. The situation is illustrated in Fig. 1. The “single-pair” contribution has to be compared with a “normal” contribution associated with a number of defect-antidefect pairs. Though the normal contribution contains an additional large entropy factor it has also an additional small factor related to a small probability to observe a defect-antidefect pair with a separation larger than the core radius. As a result of the competition, the normal contribution appears to be smaller. To avoid a misunderstanding, let us stress that the arguments do not work for the simultaneous correlation functions. The reason is that trajectories of two defects cannot pass through $n > 2$ points simultaneously, see Fig. 2.

Our paper is organized as follows. In Section 1 we remind some basic facts concerning static properties of the 2d defects and their dynamics and then we shortly review the Doi technique [13] suitable for our problem. In Section 2 we develop a diagrammatic representation for dynamical objects and examine the two-particle conditional probability which is extensively exploited in the subsequent consideration. In Section 3 we demonstrate how renormalization of different parameters can be obtained in the framework of our dynamic approach. Actually, the renormalization is reduced to the well known static renormalization group equations. In Section 4 we consider correlation functions of the “charge density” and ground the properties announced above. In Section 5 we generalize our procedure for the case of superfluid films. In the Conclusion we discuss the main results of our work and their possible relations to other systems. Some calculations are placed into Appendices.

1. BASIC RELATIONS

Static properties of the system of the vortex-like defects in thin films can be described quite universally. The starting point of the description is the free energy associated with the defects

$$\mathcal{F} = -\sum_{i\neq j} T \beta n_i n_j \ln \left(\frac{|\mathbf{x}_i - \mathbf{x}_j|}{a}\right) + \sum_j \mu(n_j), \quad (1.1)$$

where the subscripts $i, j$ label defects, $\mathbf{x}_j$ are positions of the defects, $a$ is a cutoff parameter of the order of the size of the defect core, $n_i$ are integer numbers determining the “strength” of the defects, $\beta$ is a dimensionless $T$-dependent factor and $\mu$ is the energy associated with the core. The expression (1.1) is correct for quantum vortices in superfluid films, for disclinations in hexatic films, and for spin vortices in 2d planar magnets. For dislocations in crystalline films the expression (1.1) has to be slightly modified [8], but the main peculiarity of the free energy, the logarithmic dependence on the separation, remains the same.

The Gibbs distribution exp($-\mathcal{F}/T$) corresponding to the energy (1.1) can be treated as the partition function of two-dimensional point particles with charges $n_j$, $\beta$ playing a role of the “inverse temperature”. The parameter $\beta$ can be considered also as the Coulomb coupling constant. Basing on the electrostatic analogy one can introduce the “charge density”

$$\rho(r) = \sum_j n_j \delta (r - \mathbf{x}_j), \quad (1.2)$$

The quantity $\rho$ is vorticity for superfluid films and disclination for hexatic films. We will treat the case when defects are produced by thermal fluctuations. Since both creation and annihilation processes conserve the “charge” we should accept that the total charge is zero: $\sum_j n_j = 0$. It leads to the constraint

$$\int d^2 r \rho(r) = 0, \quad (1.3)$$

where the integration is performed over the total area of the specimen.

Below we assume that for $|n| > 1$ the core energy $\mu(n)$ is so large that such defects are hardly created. Then only defects with the charges $n_i = \pm 1$ should be taken into account. We will call the objects with the charges $n_i = 1$ defects and the objects with the charges $n_i = -1$ antidefects. Because of the constraint $\sum_j n_j = 0$ there can be simultaneously $N$ defects and $N$ antidefects in the system. Thus, the partition function of the system can be characterized via a set of probability distribution functions $P_{2N}$ depending on coordinates of $2N$ “particles”. In accordance with Eq. (1.1) the functions can be written as

$$P_{2N} (\mathbf{x}_1, \ldots, \mathbf{x}_{2N}) = Z^{-1} \left( \frac{y_0}{a^2} \right)^{2N} \exp \left\{ \sum_{i \neq j} \beta n_i n_j \ln \left( \frac{|\mathbf{x}_i - \mathbf{x}_j|}{a} \right) \right\}, \quad (1.4)$$

where $Z$ is the sum over states and the quantity $y_0 = \exp(-\mu/T)$ is usually called fugacity. The possibility to neglect charges with $|n| > 1$ implies that the fugacity is small.
The low-temperature (insulator) phase can be treated as a system constituted of bound defect-antidefect pairs. In the high-temperature (plasma) phase there are unbound charges which essentially influence the system on scales larger than the correlation length \( r_c \). We will treat the low-temperature phase and the region of scales between \( a \) and \( r_c \) in the high-temperature phase where one can neglect the role of the unbound charges and only the bound defect-antidefect pairs have to be taken into account. The presence of the pairs in the system leads to non-trivial “dielectric” properties of the medium. As a result the interaction between the charges is modified, the effect can be described in terms of a scale-dependent “dielectric constant” of the medium as is suggested in Ref. \[1\]. In other words, the effective coupling constant \( \beta \) becomes dependent on the separation between the charges.

The scale dependence of \( \beta \) can be described in the framework of the scheme proposed by Kosterlitz, \[18\]. Namely, the partition function of the system can be integrated over separations of the defect-antidefect pairs between the core size \( a \) and a scale \( r \). After the procedure, that can be interpreted as shifting the core radius \( a \to r \), the form of the probability distribution functions \[2\] is reproduced (with \( r \) instead of \( a \)), but the parameters \( \beta \) and \( y \) are renormalized. The \( r \)-dependence of \( \beta \) and \( y \) is determined by the following renormalization group equations found in Ref. \[18\]

\[
\frac{d\beta}{d \ln(r/a)} = -cy^2, \quad \frac{dy}{d \ln(r/a)} = (2 - \beta)y, \quad (1.5)
\]

where \( c \) is a numerical factor of order unity. The \( r \)-dependent function \( y \) is the renormalized fugacity. It determines a concentration of defects belonging to the bound pairs with separations of the order of \( r \), the concentration can be estimated as \( y/r^2 \). The renormalized value of \( \beta \) determines the dependence of the strength of the Coulomb interaction on the separation between the charges. In the low-temperature phase, the effective value of \( \beta \) tends to a constant on large scales. The asymptotic value of \( \beta \) larger than 2, the critical value \( \beta = 2 \) corresponds to the transition temperature. In the asymptotic region, where \( \beta \) can be treated as \( r \)-independent, the renormalized fugacity \( y \) remains \( r \)-dependent. Its asymptotic behavior can easily be extracted from Eq. (1.5):

\[
y \propto r^{2-\beta}. \quad (1.6)
\]

Thus, in the low-temperature phase \( y \) tends to zero as scale increases.

Let us turn to simultaneous correlation functions of the charge density \( \rho \) \[12\]. The odd correlation function are zero. Indeed, the system is symmetric under permuting defects and antidefects whereas the charge density \[12\] changes its sign at the permutation. The pair correlation function can be written as (see, e.g., Ref. \[13\])

\[
\langle \rho(r)\rho(0) \rangle \sim y^2(r)/r^4. \quad (1.7)
\]

A generalization of the relation (1.7) can be obtained (see Ref. \[20\]) which is

\[
\langle \rho(r_1)\ldots \rho(r_{2n}) \rangle \sim \frac{y^{2n}(r_*)}{r_*^{2n}} \sim \langle \rho(r_*)\rho(0) \rangle^n, \quad (1.8)
\]

where all separation \( |r_i - r_j| \) are assumed to be of the same order \( r_* \). In the large-scale limit where \( \beta \) is saturated we have

\[
\langle \rho(Xr_1)\ldots \rho(Xr_{2n}) \rangle = X^{-2\beta n} \langle \rho(r_1)\ldots \rho(r_{2n}) \rangle, \quad (1.9)
\]

where \( X \) is an arbitrary factor. The relation (1.2) shows that the simultaneous statistics of \( \rho \) has normal scaling, that is scaling exponents of the correlation functions of the order \( 2n \) are equal to \( n \) times the scaling exponent of the pair correlation function \( (1.7) \). We will demonstrate that the behavior of non-simultaneous correlation functions of the charge density is quite different.

**A. Dynamics**

To examine dynamical characteristics of the system we should formulate a dynamical equation for a defect motion. Following Ref. \[11\] we accept the following stochastic equation

\[
\frac{dx_i}{dt} = -\frac{D}{T} \frac{\partial F}{\partial x_i} + \xi_j, \quad (1.10)
\]

determining the trajectory of the \( j \)-th defect. Here \( F \) is the free energy \[1.3\], \( D \) is a diffusion coefficient, and \( \xi_j \) are Langevin forces with the correlation function

\[
\langle \xi_{i,a}(t_1)\xi_{j,b}(t_2) \rangle = 2\delta_{ij}\delta_{ab}\delta(t_1 - t_2). \quad (1.11)
\]

The diffusion coefficient \( D \) determines mobility of the defects. We believe that the equation (1.10) is applicable to the dynamics of disclinations in hexatic films like membranes, freely suspended films and Langmuir films. The equation for the vortices in superfluid films is a bit more complicated. It is written in Section 5 where the correlation functions of the vorticity are analyzed.

The equations (1.10,1.11) describe trajectories of separate defects. We should also take into account annihilation and creation processes. Remember that we neglect defects with \( |n_j| > 1 \). Next, processes where a number of defect-antidefect pairs are created at the same point are suppressed since probability of such events is small due to the energy associated with the cores of defects. Then we have to take into account the creation processes of single pairs solely, they are characterized by the creation rate \( R(r) \) which is a probability density for a defect-antidefect pair with the separation \( r \) to be created per unit time per unit area. The annihilation processes have to be characterized by the annihilation rate \( \bar{R}(r) \) which is a probability for a defect-antidefect pair to annihilate per unit time if the pair is separated by the distance \( r \). Really,
both \( \tilde{R}(r) \) and \( R(r) \) are nonzero only if \( r \) is of the order of the core size \( a \). Let us introduce the integrals

\[
\tilde{\lambda} = \int d^2r \, \tilde{R}(r) , \quad \lambda = \int d^2r \, R(r) .
\]  

(1.12)

Here, the creation constant \( \tilde{\lambda} \) is a probability for a defect-antidefect pair to be created per unit time per unit area and \( \lambda \) is a constant having the same dimensionality as the diffusion coefficient \( D \). Below, the diffusion coefficient \( D \) is put to unity by rescaling time. Then the annihilation rate is the second power of the defect concentration.

As in statics, in dynamics the system of defects can be described in terms of probability distribution functions. In our case, when the total charge of the system is zero, only even probability densities \( \mathcal{P}_{2N}(t, x_1, \ldots, x_N, z_1, \ldots, z_N) \) are non-zero, where \( x_j \) and \( z_j \) are positions of the defects and of the antidefects correspondingly. The total probability should be equal to unity which gives the normalization condition

\[
\sum_{N=0}^{\infty} \frac{1}{(N!)^2} \int d^2x_1 \ldots d^2x_N d^2z_1 \ldots d^2z_N \times \mathcal{P}_{2N}(t, x_1, \ldots, x_N, z_1, \ldots, z_N) = 1 .
\]  

(1.14)

Starting from the equation (1.10) and taking into account the creation and annihilation processes one can derive a system of master equations for the probability densities

\[
\frac{\partial \mathcal{P}_{2N}}{\partial t} = \sum_{j,k} \left\{ \frac{\partial}{\partial x_j} \left[ \frac{1}{T} \frac{\partial F}{\partial x_j} \mathcal{P}_{2N} \right] + \frac{\partial}{\partial z_j} \left[ \frac{1}{T} \frac{\partial F}{\partial z_j} \mathcal{P}_{2N} \right] \right\} + \sum_{j,k} \left[ \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial z_j^2} \right] \mathcal{P}_{2N}
\]

\[
- \sum_{j,k} R(x_j - z_k) \mathcal{P}_{2N} + \int d^2x \, d^2z \, R(x - z) \mathcal{P}_{2N+2}(x_1, \ldots, x_N, \bar{x}, z_1, \ldots, z_N, z)
\]

\[
+ \sum_{j,k} \tilde{R}(x_j - z_k) \mathcal{P}_{2N-2}(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N, z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_N) - \tilde{\lambda} A \mathcal{P}_{2N} ,
\]  

(1.15)

where \( A \) is the area of the film.

The Gibbs distribution (1.4) must be a solution of the master equations (1.15). The condition imposes the following constraint on the creation and the annihilation rates

\[
\tilde{R}(r) = \frac{y_0^2}{a^4} \left( \frac{a}{r} \right)^{2\beta} R(r) ,
\]  

(1.16)

where we imply \( r > a \). The constraint (1.10) can be treated as the manifestation of the equilibrium state of the thermal bath which in our case is related to short-scale fluctuations. Thus the constraint (1.10) has the same origin as Eq. (1.11). At deriving Eq. (1.16) we assumed that the separation \( |x - z| \) in the argument of the rate \( R \) or \( \tilde{R} \) is smaller than separations between \( x \) or \( z \) and other points. That is accounted for the small characteristic value of the separation which are of the order of the core radius \( a \). Note that for such \( r \) the factor \( y_0 (r/a)^{2-\beta} \) entering Eq. (1.16) can be treated as the renormalized value \( y \) of the fugacity as follows from Eq. (1.3).

Principal, the master equations (1.15) enable one to find conditional probability densities, related to different time moments. Consequently, starting from the equations one can examine non-simultaneous correlations in the system. However, there are terms in the master equations (associated with the creation and annihilation processes) mixing the probability densities \( \mathcal{P}_{2N} \) with different \( N \). That makes the master equations hardly useful, that is a motivation to look for some more suitable technique. Such a technique was developed by Doi, Ref. [3], we formulate it in the subsequent subsection.

B. Quantum Field Formulation

The Doi technique enables one to treat systems of classical particles where creation and annihilation processes occur. The main idea introduced by Doi is that correlation functions of different quantities characterizing the particles can be written in the form close to the one known in the quantum field theory. Of course there are some peculiarities related to the fact that for classical particles one should deal directly with probabilities whereas in the quantum field theory one starts from the scattering matrix. Nevertheless the Doi technique enables, say, to formulate a diagrammatic expansion with the conventional rules. The technique was originally developed to describe systems of molecules involved into chemical reactions. But it is definitely applicable also to the system of point defects.

The Doi technique is formulated in terms of the creation \( \psi \) and annihilation \( \bar{\psi} \) operators which satisfy the same commutation rules as ones for Bose-particles

\[
[\psi(r_1), \bar{\psi}(r_2)] = \delta(r_1 - r_2) ,
\]

\[
[\psi(r_1), \psi(r_2)] = [\psi(r_1), \bar{\psi}(r_2)] = 0 .
\]  

(1.17)

For our system of defects we should introduce annihilation and creation operators \( \psi_\pm \) and \( \bar{\psi}_\pm \) where the sub-
scripts $+$ and $-$ label fields related to the defects and to the antidects. The state of the system at a time moment $t$ can be written in terms of a “quantum state”

$$|t\rangle = \frac{1}{(N!)^2} \sum_{N=0}^{\infty} \int d^2 x_1 \ldots d^2 x_N d^2 z_1 \ldots d^2 z_N \mathcal{P}_{2N} \times \hat{\psi}_+(x_1) \ldots \hat{\psi}_+(x_N) \hat{\psi}_-(z_1) \ldots \hat{\psi}_-(z_N)|0\rangle,$$  

(1.18)

where $\mathcal{P}_{2N}$ are the probability densities introduced above and $|0\rangle$ designates the vacuum state: $\hat{\psi}_+(z)|0\rangle = 0$. In accordance with the expression (1.18) an evolution of the quantum state $|t\rangle$ is determined by the master equations. The evolution equation can be written as

$$\partial_t|t\rangle = -\mathcal{H}|t\rangle, \quad \text{and} \quad |t_2\rangle = \exp[(t_1 - t_2)\mathcal{H}]|t_1\rangle,$$  

(1.19)

where $\mathcal{H}$ is an operator expressed in terms of the fields $\psi_+$ and $\psi_-$. By analogy with the quantum field formulation it can be called the Hamiltonian operator or simply the Hamiltonian.

The total probability must be equal to unity, which leads to Eq. (1.14). The condition can be written in terms of the quantum state $|t\rangle$ as

$$\langle \text{sum} |t\rangle = 1,$$  

(1.20)

where

$$\langle \text{sum} = \langle 0 | \exp \left[ \int d^2 r \ (\psi_+ + \psi_-) \right].$$

Note the following identity

$$\langle \text{sum} | \hat{\psi}_+(r) \rangle = \langle \text{sum} \rangle,$$  

(1.21)

which can be easily checked using the commutation rules (1.17) and the equality $\langle 0 | \hat{\psi}_+ = 0$. Since the evolution must conserve the total probability $\langle \text{sum} \rangle$ the relations

$$\langle \text{sum} | \mathcal{H} = 0, \quad \text{and} \quad \langle \text{sum} | \exp(\tau \mathcal{H}) = \langle \text{sum} \rangle,$$  

(1.22)

have to be satisfied, where $\tau$ is an arbitrary parameter.

Quantities characterizing the system can be represented by corresponding operators, see Ref. [13]. Say, the operator of the charge density is

$$\hat{\rho} = \hat{\psi}_+ \psi_+ - \hat{\psi}_- \psi_-.$$  

(1.23)

If $\hat{A}$ is such an operator corresponding to a quantity $A$, then an average value of the quantity at a time moment $t$ can be expressed as

$$\langle A(t) \rangle = \langle \text{sum} | \hat{A}(t) |t\rangle.$$  

(1.24)

Note that

$$\langle \hat{\psi}_\pm \rangle = \langle \text{sum} | \hat{\psi}_\pm |t\rangle = 1,$$  

(1.25)

which is a consequence of Eqs. (1.21,1.21). The relation (1.23) shows that it is natural to shift the creation operators $\hat{\psi}_\pm$ introducing new variables (see Ref. [21])

$$\hat{\psi}_\pm = 1 + \hat{\psi}_\pm, \quad \langle \hat{\psi}_\pm \rangle = 0.$$  

(1.26)

Correlation functions of different quantities can be presented analogously to Eq. (1.22). For example, the pair correlation function of two quantities $A(t_1)$ and $B(t_2)$ (we assume $t_2 > t_1$) can be written as

$$\langle B(t_2) A(t_1) \rangle = \langle \text{sum} | \hat{B}(t_2) \hat{A}(t_1) |t_1\rangle.$$  

(1.27)

Using the relations (1.19,1.22) we can rewrite the expression (1.27) as

$$\langle B(t_2) A(t_1) \rangle = \langle \text{sum} | \hat{B}(t_2) \hat{A}(t_1) |\text{in}\rangle,$$  

(1.28)

where $\hat{A}(t)$, $\hat{B}(t)$ in Eq. (1.28) are operators in the Heisenberg representation

$$\hat{A}(t) = \exp[-(t_1 - t)\mathcal{H}] \hat{A} \exp[-(t - t_2)\mathcal{H}],$$  

(1.29)

satisfying the equation $\partial_t \hat{A}(t) = [\mathcal{H}, \hat{A}(t)]$. In Eq. (1.28) $|\text{in}\rangle$ is an initial state (realized at a time moment $t_0$) and $t_0$ is a “final” time, so that $t_0 > t_2 > t_1 > t_0$.

The expressions like (1.28) enable one to reformulate the problem of calculating correlation functions in terms of a functional integral, see Ref. [22]. Namely, we can write

$$\langle A_1(t_1) \ldots A_n(t_n) \rangle = \int D\hat{\psi}_\pm D\psi_\pm A_1 \ldots A_n$$

$$\times \exp\left\{ - \int_{-\infty}^{t_1} dt \ [\mathcal{H} + \int d^2 r \ \left( \hat{\psi}_+ \partial_t \hat{\psi}_+ + \hat{\psi}_- \partial_t \hat{\psi}_- \right) \right\}$$

$$+ \int d^2 r \ \left( \langle \hat{\psi}_+(t, r) \rangle + \hat{\psi}_-(t, r) \right),$$  

(1.30)

where $\hat{\psi}_\pm$, $\hat{\psi}_\pm$ are to be interpreted as functions of $t$ and $r$. We assume that $t_1 > t_1, \ldots, t_n$ in Eq. (1.30). Deriving the expression one has taken the limit $t_0 \to -\infty$ and assumed $|\text{in}\rangle = |0\rangle$. Because of the creation processes the vacuum has to be turned into a stationary state during the infinite time. To ensure convergence of the functional integral (1.30), the integration contour over the field $\hat{\psi}$ should go parallel to the imaginary axis. Note that the shift (1.20) kills the boundary term $\int d^2 r (\hat{\psi}_+ + \hat{\psi}_-)$; it is cancelled by a contribution originating from the derivatives $\partial_t \hat{\psi}_\pm$ after integrating over time. Then we come to a conventional representation of the correlation functions in terms of a functional integral

$$\langle A_1(t_1) \ldots A_n(t_n) \rangle = \int D\hat{\psi}_\pm D\psi_\pm \exp\left\{ - \int dt \ \mathcal{H} \right.$$"
2. DIAGRAMMATIC REPRESENTATION

Below, we apply the Doi technique to our particular problem. The explicit expression for the Hamiltonian determining the evolution of the defect system in accordance with Eq. (1.19) can be found from the master equations (1.15). Comparing the equations with Eqs. (1.18,1.19) we get

\[ \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_\beta. \]  

(2.1)

The explicit expressions for the terms entering Eq. (2.1) are

\[ \mathcal{H}_0 = \int d^2r \left( \nabla \hat{\psi}_+ \nabla \psi_+ + \nabla \hat{\psi}_- \nabla \psi_- \right) \]

(2.2)

\[ \mathcal{H}_R = -\int d^2r_1 d^2r_2 \left[ R(r_1 - r_2)(\psi_{+1} \hat{\psi}_{-2} - 1) \right. \]

\[ + R(r_1 - r_2)(\psi_{+1} \hat{\psi}_{-2} - \hat{\psi}_{+1} \hat{\psi}_{-2} \psi_{+1} \psi_{-2}) \]  

(2.3)

\[ \mathcal{H}_\beta = 2\beta \int d^2r_1 d^2r_2 \left( \nabla \hat{\psi}_{+1} \hat{\psi}_{-2} - \hat{\psi}_{+1} \nabla \hat{\psi}_{-2} \right) \]

\[ \times \frac{r_1 - r_2}{|r_1 - r_2|^2} \psi_{+1} \psi_{-2} \]

\[ - 2\beta \int d^2r_1 d^2r_2 \left( \nabla \hat{\psi}_{+1} \hat{\psi}_{+2} \frac{r_1 - r_2}{|r_1 - r_2|^2} \psi_{+1} \psi_{+2} \right. \]

\[ + \nabla \hat{\psi}_{-1} \hat{\psi}_{-2} \frac{r_1 - r_2}{|r_1 - r_2|^2} \psi_{-1} \psi_{-2} \]  

(2.4)

where \( \psi_{+1} = \psi_+(t,r_1) \) and so further. The diffusive contribution (2.2) is related to the Langevin forces, in Eq. (2.3) \( R \) is the annihilation rate and \( \hat{R} \) is the creation rate for the defect-antidefect pairs (the quantities were introduced in Section 1), and the term (2.4) describe the Coulomb interaction. Using the property (1.21), one can easily check the conditions (1.23) for all contributions (2.2, 2.4).

Performing the substitution (1.26) we can express the Hamiltonian (2.1) in terms of the fields \( \psi_\pm \). Note that the shift (1.26) kills terms of the second order proportional to \( \lambda, \lambda \) and generates additional third-order vertices. Of course one can work in both representations. It is more convenient for us to use Eqs. (1.30,2.1). We can easily convince ourselves that odd correlation functions of the charge density (1.23) are zero. Indeed, the exponent in Eq. (1.30) is invariant under permuting \( \psi_+ \leftrightarrow \psi_- \), \( \psi_+ \leftrightarrow \psi_- \), whereas the charge density changes its sign at the permutation. The constraint (1.3) shows that the symmetry is not spontaneously broken what could lead to non-zero odd correlation functions.

It follows from Eq. (1.23) that the commutator \([\mathcal{H}, \hat{\rho}]\) should be equal to \( -\nabla \hat{j} \) where \( \hat{j} \) is the current density operator. Calculating the commutator with Eqs. (1.23,2.1) we get

\[ \hat{j} = \nabla \hat{\psi}_+ \psi_+ - \hat{\psi}_+ \nabla \psi_+ - \nabla \hat{\psi}_- \psi_- + \hat{\psi}_- \nabla \psi_- \]

\[ - (\hat{\psi}_+ \psi_+ + \hat{\psi}_- \psi_-) \nabla \phi, \]  

(2.5)

where \( \phi \) is an “electrostatic potential”

\[ \phi(r) = -2\beta \int d^2x \ln \left( \frac{|r - x|}{\sigma} \right) \hat{\rho}(x). \]  

(2.6)

Principally, besides the “internal” potential (2.6) an “external” potential \( \phi_{\text{ext}} \) can be imposed onto the system, satisfying the equation \( \nabla^2 \phi_{\text{ext}} = 0 \). Then an “external force” should be added to the right-hand side of the equation (1.10). The force generates an “external” contribution to the Hamiltonian

\[ \mathcal{H}_{\text{ext}} = \int d^2r \left( \nabla \hat{\psi}_+ \psi_+ - \nabla \hat{\psi}_- \psi_- \right) \nabla \phi_{\text{ext}}. \]  

(2.7)

The expression (2.3) for the current density has also to be corrected by substituting \( \phi \rightarrow \phi + \phi_{\text{ext}} \). With the term (2.7) we can examine susceptibilities describing a response of the system to the external influence.

Substituting the expression (2.1) into (1.30) or (1.31) and expanding the exponent over \( \mathcal{H}_R \) and \( \mathcal{H}_\beta \) one can obtain a conventional perturbation series for calculating different correlation functions of \( \psi, \hat{\psi} \). The series is an expansion over \( R, \hat{R} \) and \( \beta \) in terms of the conventional diffusion propagators:

\[ G(t,r) = \langle \psi_+(t,r) \hat{\psi}_+(0,0) \rangle_0 \]

\[ = \langle \psi_-(t,r) \hat{\psi}_-(0,0) \rangle_0 = \frac{\theta(t)}{4\pi t} \exp \left( -\frac{r^2}{4t} \right), \]  

(2.8)

where \( \theta(t) \) is the step function. However, effects related to the Coulomb interaction and to the annihilation processes are not weak. Therefore one must take into account the Coulomb interaction and the annihilation processes exactly. By other words, at calculating the correlation functions one must consider the complete series over \( \beta \) and \( \hat{R} \). Fortunately, the the expansion over \( \hat{R} \) is equivalent to an expansion over the fugaicity \( y \) which is assumed to be a small parameter. Therefore we can take only principal terms in the expansion over \( \hat{R} \).

The perturbation expansion can be formulated as a diagrammatic series. We develop the diagrammatic technique starting from the representation (1.30), pushing the final time \( t_f \) to the far future. We depict the propagator (2.8) by a line directed from \( \hat{\psi} \) to \( \hat{\psi} \). The term with the creation rate \( \hat{R} \) in Eq. (2.3) generates vertices where two propagator lines start, the vertices correspond to the defect-antidefect creation processes. The Coulomb term in Eq. (2.1) generates two-point objects which we will designate by dashed lines, such line describes the Coulomb interaction of defects located in points connected by the line. And the term proportional to the annihilation rate \( R \) in Eq. (2.1) produces two types of
vertices. First, it produces vortices where two propagator lines finish, that corresponds to an annihilation process. Second, it produces fourth-order vertices which correspond to an effective interaction related to a finite probability for a defect-antidefect pair to annihilate, see Ref. [3]. A typical diagram block is presented in Fig. 3. The block is drawn in real \( r - t \) space-time. The curves constituted of the propagator lines can be interpreted as trajectories of defects and antidefects. Due to causality the particles always move forward in time. Note that the dashed lines corresponding to the Coulomb interaction are perpendicular to the \( t \)-axis since the interaction is simultaneous.

### A. Pair Conditional Probability

In the subsection we examine an auxiliary object which will be needed for us at intermediate stages of subsequent calculations. The object is the following correlation function

\[
M(t_2-t_1,r_1,r_2,r_3,r_4) = \langle \psi_+\!(t_2,r_1)\psi_-\!(t_2,r_2)\hat{\psi}_+\!(t_1,r_3)\hat{\psi}_-\!(t_1,r_4) \rangle. \tag{2.9}
\]

For a stationary case the average \( \langle \cdot \rangle \) depends on the difference \( t = t_2 - t_1 \) only. Due to causality \( M \) is equal to zero provided \( t < 0 \). The quantity \( M \) can be interpreted as a probability density to find a defect and an antidefect at the time moment \( t_2 \) in the points \( r_1 \) and \( r_2 \) provided they were located in the points \( r_3 \) and \( r_4 \) at the time moment \( t_1 \). It can be considered also as a two-particle matrix element of the evolution operator \( \exp[-(t_2-t_1)\mathcal{H}] \).

As we explained above, the perturbation series in terms of the creation rate \( \dot{R} \) is an expansion over a small parameter. Here we examine the principal contribution to the conditional probability \( \langle \cdot \rangle \) which is of the zero order over \( \dot{R} \). Then the average \( \langle \cdot \rangle \) can be represented as a series of diagrams of the type depicted in Fig. 3. One can interpret the picture as trajectories of a defect and of an antidefect which are driven by the Langevin forces, and are influenced the Coulomb interaction (dashed lines) and the effective interaction associated with the annihilation processes (point vertex). Note that in this approximation direct annihilation events do not contribute to the conditional probability \( \langle \cdot \rangle \) since they would lead to terminating the lines in the diagrams.

It is of crucial importance that both the Coulomb interaction and the effective interaction associated with the annihilation processes are local in time. Therefore all the diagrams representing the conditional probability \( \langle \cdot \rangle \) are ladder diagrams, like in Fig. 3. Summing up the ladder sequence we get an equation for \( M \) which can be written in the differential form

\[
\partial_t M = \left( \nabla_r^2 + \nabla_t^2 \right) M + 2\beta \left( \nabla_1 - \nabla_2 \right) \frac{r_1 - r_2}{|r_1 - r_2|^2} M - R(r_1 - r_2)M + \delta(t)\delta(r_1 - r_3)\delta(r_2 - r_4). \tag{2.10}
\]

Since \( M = 0 \) at \( t < 0 \) we conclude from Eq. (2.10) that at \( t \to +0 \)

\[
M(t, r_1, r_2, r_3, r_4) \to \delta(r_1 - r_3)\delta(r_2 - r_4). \tag{2.11}
\]

The total probability to find the defect-antidefect pair is determined by the integral \( \int d^2r_1 \, d^2r_2 M \). Let us calculate the time derivative of this integral substituting \( \partial_t M \) by the right-hand side of Eq. (2.10). Then the first two terms will give zero contributions (since they are total derivatives) and only the term with \( R \) will produce a non-zero (negative) contribution. It is quite natural since the Langevin forces and the Coulomb interaction cannot change the total probability whereas the annihilation processes diminish it. Note that all the terms in the right-hand side of Eq. (2.10) proportional to \( M \) have the same dimensionality. Therefore one could expect a simple scaling behavior when \( t \) scales as \( r^2 \). The subsequent calculations confirm the expectation.

In terms of the variables

\[
r = r_1 - r_2, \quad q = \frac{r_1 + r_2}{2}, \quad r_0 = r_3 - r_4, \tag{2.12}
\]

the equation (2.10) for \( M \) is rewritten as

\[
\partial_t M = \left( \frac{1}{2} \nabla_q^2 + 2\beta \mathcal{V} \rho \frac{r}{r^2} - R(r) \right) M + \delta(t)\delta(q - r_0)\delta(q - r_3/r_2 - r_4/2). \tag{2.13}
\]

We see that the differential operator in the right-hand side of the equation falls into two parts depending on \( q \) and \( r \) only and that the “source” is a product of \( \delta \)-functions of the same variables. Therefore the solution of the equation can be written in a multiplicative form

\[
M = \frac{1}{2\pi t} \exp \left\{ -\frac{(2q - r_3 - r_4)^2}{8t} \right\} S(t, r, r_0). \tag{2.14}
\]

the function \( S \) satisfies the following equation

\[
\partial_t S = 2\mathcal{V}^2 S + 4\beta \mathcal{V} \left( \frac{r}{r^2} S \right) - R(r)S + \delta(t)\delta(q - r_0). \tag{2.15}
\]

Note that

\[
\text{if } \quad t \to +0 \quad \text{then} \quad S \to \delta(q - r_0). \tag{2.16}
\]

The relation follows from Eq. (2.14) and causality (leading to \( S = 0 \) for negative \( t \)).

In accordance with Eq. (2.13), a motion of the mass center and the relative motion of the defects are separated. The motion of the mass center is purely diffusive whereas the relative motion is strongly influenced by the interaction. The function \( S \) can be treated as the probability density for the relative motion of the defect-antidefect pair. It is natural to expand the function into
the Fourier series over the angle $\varphi$ between the vectors $r$ and $r_0$:

$$S(t, r, r_0) = \sum_{-\infty}^{+\infty} S_m(t, r, r_0) \exp(im\varphi). \quad (2.16)$$

Motions corresponding to different angular harmonics are separated. In terms of the angular harmonics Eq. (2.14) is rewritten as

$$\frac{1}{2} \partial_t S_m = \left[ \partial_r^2 + (1 + 2\beta) \frac{1}{r} \partial_r - \frac{m^2}{r^2} \right] S_m - \frac{1}{2} R(r) S_m + \frac{1}{4\pi r_0} \delta(t)(r - r_0). \quad (2.17)$$

It is possible to get equations for $S$ analogous to Eqs. (2.14, 2.17) in terms of $r_0$. They have practically the same form as Eqs. (2.14, 2.17). The only difference is in the sign of $\beta$ which is opposite. That leads to the relation

$$S_m(t, r, r_0) = \left( \frac{r_0}{r} \right)^{2\beta} S_m(t, r_0, r). \quad (2.18)$$

Let us stress that the relation (2.18) is correct for an arbitrary function $R(r)$.

Consider a behavior of the angular harmonics $S_m(t, r, r_0)$ at small $r$. More precisely, we assume $t \gg a^2$ and examine the region $\sqrt{t} \gg r \gg a$. Then it is possible to use the equation (2.17) with the time derivative and the annihilation term neglected. As a result we get

$$S_m = C_{1,m} r^{-\beta} + C_{2,m} r^{-\nu} (r/a)^{-2\nu}, \quad (2.19)$$

$$\nu = \sqrt{\beta^2 + m^2}, \quad (2.20)$$

where $C_{1,m}, C_{2,m}$ are some factors dependent on $t$ and $r_0$. The ratio of the factors is determined by a concrete $r$-dependence of the annihilation rate $R$, one can assert only that $C_{1,m}$ and $C_{2,m}$ are of the same order. Therefore, if we consider the behavior of the function $S$ for $r \gg a$, then the second term in the right-hand side of Eq. (2.19) can be neglected. By other words, being interested in the scales $r \gg a$, we can solve the equation (2.17) neglecting the annihilation term and requiring a finite value of $S_m$ at $r \to 0$ instead.

The requirement can be treated as the boundary condition for $S_m$ at small $r$. The other boundary condition is that $S_m$ tend to zero at $r \to \infty$. The equations for $S_m$ with the boundary conditions are solved in Appendix A.1. We present here only the answer

$$S_m(t, r, r_0) = \frac{1}{8\pi t} \left( \frac{r_0}{r} \right)^{\beta} \exp \left( -\frac{r^2 + r_0^2}{8t} \right) I_{\nu} \left( \frac{r r_0}{4t} \right), \quad (2.21)$$

where $I$ is the modified Bessel function and $\nu$ is introduced by Eq. (2.20). Remind that the expression is correct provided $r, r_0, \sqrt{t} \gg a$. Extracting from Eq. (2.21) an asymptotics at small $r$ we get

$$C_{1,m} = \frac{r_0^\beta}{8\pi t (1 + \nu)} \left( \frac{r_0}{t} \right)^{\nu} \exp \left( -\frac{r_0^2}{8t} \right).$$

Note that the Coulomb term in Eq. (2.14) produces a probability flux to the origin. To find it we should integrate the equation (2.14) over a disk of a radius $a < r < \sqrt{t}$ centered at the origin and single out the contribution to $\partial_t \int d^2 r S$ associated with the Coulomb term. Then we find the flux $\lambda_C 1 \omega$ where

$$\lambda_C = \frac{8\pi \beta}. \quad (2.22)$$

One can treat the quantity (2.22) as the renormalized (“dressed”) value of the annihilation constant. Now we understand why the solution (2.21) (realized at $r \gg a$) is insensitive to a particular form of the annihilation rate. The probability for a defect-antidefect pair with the separation $r \gg a$ to annihilate is determined by the Coulomb attraction. And only the behavior of the probability density at $r \sim a$ is sensitive to the particular form of the annihilation rate $R(r)$: The coefficients $C_{1,m}$ in Eq. (2.19) are positive if $\nu < \lambda_C$ and are negative if $\nu > \lambda_C$.

Returning to the conditional probability (2.4) we obtain from Eqs. (2.13, 2.16, 2.22) the expression

$$M = \frac{1}{(4\pi t)^2} \left( \frac{r_0}{t} \right)^{\beta} \exp \left\{ -\frac{(r_1 + r_2 - r_3 - r_4)^2}{8t} \right\} \times \exp \left( -\frac{r_1^2 + r_0^2}{4t} \right) \sum_{m=-\infty}^{+\infty} \exp(i m \varphi) I_{\nu} \left( \frac{r r_0}{4t} \right), \quad (2.23)$$

where $\nu$ is introduced by Eq. (2.20). One can easily check that the expression (2.23) is reduced to Eq. (2.11) at $t \to +0$. It is possible to calculate an explicit expression for the total probability to find a defect and an antidefect in any points at a fixed time separation $t$, see Appendix A.3. The asymptotic behavior of the expression (2.23) at the condition $r r_0 / t \gg 1$ is examined in Appendix A.3, see Eq. (2.14). If both $r, r_0$ are much greater than $\sqrt{t}$, it can be written as

$$M \approx \frac{1}{(4\pi t)^2} \exp \left\{ -\frac{(r - r_0)^2}{8t} \right\} \times \exp \left\{ -\frac{(r_1 + r_2 - r_3 - r_4)^2}{8t} \right\}. \quad (2.24)$$

The answer is quite natural. The characteristic values of the separations $r_1 - r_3$ and of $r_2 - r_4$ are of the order of $\sqrt{t}$ and are consequently much smaller than $|r_1 - r_2|$ (or $|r_3 - r_4|$). Then $r \approx r_0$ and it is possible to neglect all terms, containing $|r_1 - r_2|$ in the denominators, in the equation (2.10). Thus we come to a purely diffusive equation leading to the asymptotic law (2.22).

The consideration presented above can be generalized for the case when an “external electrostatic potential” $\phi_{ext}$ is imposed onto the system. Its influence is described by the contribution (2.7) to the Hamiltonian. Performing
the procedure as above we get a modified equation for the correlation function (2.9)
\[ \frac{\partial}{\partial t_2} M = (\nabla_1^2 + \nabla_2^2) M + 4\beta \nabla_2 (\frac{r}{r^2} M) - R(r) M \\
+ \nabla \phi_{\text{ext}}(t_2, r_1) \nabla_1 M - \nabla \phi_{\text{ext}}(t_2, r_2) \nabla_2 M \\
+ \delta(t_2 - t_1) \delta(r_1 - r_3) \delta(r_2 - r_4). \tag{2.25} \]
The expression (2.25) shows that the gradient of the external potential has to be added to the gradient of the internal one. Of course, in the presence of the external field the correlation function (2.9) gives zero contribution to the integral over \( t_1 \) and \( t_2 \). Note that the operator in the right-hand side of Eq. (2.25) is the same as that for the Fokker-Plank equation formulated in Ref. [11] (excluding the Hamiltonian (2.1)).

### 3. RENORMALIZATION

In this section we are going to discuss effects related to high-order terms over the creation rate \( \dot{R} \). The effects are relevant only near the transition point where \( \beta \) is close to 2. Then the influence of small-scale defect-antidefect pairs on larger scales becomes essential. In the situation the most natural language is the renormalization group approach. One can formulate a renormalization group procedure in the spirit of Kosterlitz, Ref. [18]. We will single out blocks corresponding to small separations of the pairs and treat them as renormalized quantities entering the Hamiltonian (2.1).

#### A. Creation and annihilation rates

Sizes of the pairs are small near creation and near annihilation points. Here, we consider vicinities of the points. Then it is possible to neglect the interaction of the defect and of the antidefect with the environment. Thus we turn to the situation when only a single pair can be treated. If this is the case then one should analyze diagram blocks of the type drawn in Fig. 3. The left part of the figure corresponds to a vicinity of the creation occurring at a time moment \( t_1 \) and the right part of the figure corresponds to a vicinity of the annihilation occurring at a time moment \( t_4 \).

Consider processes occurring during a time interval \( \tau \) from the creation time \( t_1 \). One can separately treat a block corresponding to the time interval from \( t_1 \) till \( t_2 = t_1 + \tau \). For the purpose we use the well-known property of the propagators (2.8)
\[ G(s_3 - s_1, r) = \int d^2 x G(s_3 - s_2, r - x) \times G(s_2 - s_1, x), \tag{3.1} \]
where \( s_3 > s_2 > s_1 \). For each diagram we extract propagators \( G \) containing \( t_2 \) inside their time interval and represent the propagators like in Eq. (3.1) believing \( s_2 = t_2 \).

The procedure is reflected in Fig. 3 where the dotted line represents a plane \( t = t_2 \) in the \( r - t \) space-time and the integration in Eq. (3.3) corresponds to the integration in the plane. As a result, the block to the left of the plane is separated, it is characterized by the time separation \( \tau \) and by two points \( r_1 \) and \( r_2 \) lying in the plane, the points are intersections of the plane with the trajectories of the particles. The block has to be inserted into more complicated objects via a convolution over \( r_1 \) and \( r_2 \).

The same is true for the vicinity of the annihilation point also. Let us take a time moment \( t_4 \) separated by a time interval \( \tau \gg a^2 \) from an annihilation time \( t_4 \). Then it is possible to introduce the block which is a sum of the diagrams where the trajectories of the annihilating particles start from two given points \( r_3 \) and \( r_4 \) at \( t = t_3 \). The block has to be inserted into more complicated objects via a convolution over the points. In the vicinity of the annihilation point we can take into account the interaction of the annihilating defect-antidefect pair solely. That leads to the same ladder diagrams treated in Section 2. Therefore we can write an expression for the block without an additional analysis
\[ R_\tau(r_0) = \int d^2 r_1 d^2 r_2 R(r) M(\tau, r_1, r_2, r_3, r_4) \]
\[ = \int d^2 r S(\tau, r, r_0) R(r). \tag{3.2} \]

Here \( r \) and \( r_0 \) are defined by Eq. (2.12), \( M \) is the conditional probability (2.9), \( S \) is the conditional probability for the relative motion of the defects, see Eq. (2.14), it is the solution of the equation (2.14). The physical meaning of the quantity \( R_\tau(r_0) \) is a distribution of the annihilating particles over the separation \( r_0 \) between the particles at the time moment \( t_3 \). It is natural to name this distribution “dressed” annihilation rate since the quantity determines a probability for the particles to annihilate after the time interval \( \tau \). Note that all processes occurring on scales larger than \( \sqrt{\tau} \) are sensitive only to this dressed quantity.

Let us substitute into Eq. (3.2) the product \( RS \) expressed from Eq. (2.14). The terms with the total derivatives give zero contribution to the integral over \( r \) and we get
\[ R_\tau(r_0) = -\partial_\tau \int d^2 r S(\tau, r, r_0), \tag{3.3} \]
Since \( S(\tau) \) tends to zero at \( \tau \to +\infty \) and is zero for negative \( \tau \) we get from (2.15, 4.3)
\[ \int d\tau R_\tau(r_0) = 1. \tag{3.4} \]

The relation means that the total probability for a given pair to annihilate is equal to unity. As is seen from Eq. (2.19) at the condition \( \tau \gg a^2 \) the main contribution to
the integral in the right-hand side of Eq. (3.3) is associated with the region \( r \sim \sqrt{\tau} \) and therefore the contribution to the integral associated with the region \( r \sim a \) is negligible. Therefore we can use the expression (2.16) with Eq. (2.21). Substituting the expression (A3) into Eq. (3.3), we get a universal expression for the dressed quantity \( R_{r}(\tau_{0}) \) which is insensitive to the bare quantity \( R_{r}(\tau) \).

In Sec. 2 we established the renormalized value (2.22) of the annihilation constant \( \lambda \). This analysis concerned the fourth-order interaction term written in Eq. (2.1). Below we demonstrate that the renormalized coefficient at the second-order annihilation term has the same value, independent of the bare one. In accordance with Eq. (2.12), to find the renormalized value \( \lambda_{r} \) we should calculate the integral of \( R_{r}(\tau_{0}) \). As is demonstrated in Appendix A, at \( \tau \gg a^{2} \) the value of the integral is independent of \( \tau \) and coincides with the written value in Eq. (2.22), as one anticipated:

\[
\lambda_{r} = \int d^{2}r_{0} R_{r}(\tau_{0}) = 8\pi\beta . \tag{3.5}
\]

The phenomenon resembles the renormalization of the reaction rate due to diffusion, see Refs. [13,23].

Analogously, one can introduce the renormalized creation rate \( \tilde{R}_{c}(r) \) which is determined by the block describing the vicinity of the creation point (see Fig. 3). Summing up the same ladder sequence of the diagrams we get

\[
\tilde{R}_{c}(r) = \int d^{2}r_{3} d^{2}r_{4} \tilde{R}(r_{0}) M(\tau, r_{1}, r_{2}, r_{3}, r_{4})
\]

\[
= \int d^{2}r_{0} \tilde{R}(r_{0}) S(\tau, r, r_{0}) . \tag{3.6}
\]

Here \( r \) and \( r_{0} \) are defined by Eq. (2.12), \( M \) is the conditional probability (2.9), \( S \) is the conditional probability for the relative motion of the defects, see Eq. (2.13). Using the relations (1.16,2.18) we get from Eq. (3.6)

\[
\tilde{R}_{c}(r) = \frac{y_{0}}{a^{4}} \left( \frac{a}{r} \right)^{2\beta} R_{r}(r) . \tag{3.7}
\]

Thus we see that the relation (1.16) is reproduced for the renormalized quantities \( \bar{R}_{c} \) and \( \bar{R}_{r} \).

The renormalized creation rate \( \bar{R}_{c}(r) \) can be interpreted as a probability density to find a defect-antidefect pair with a space separation \( r \) provided the pair was born on time separated by \( \tau \) from the measurement. Let us calculate the total probability density \( \bar{\lambda}_{c} \) to find the defect-antidefect pair at a fixed time separation \( \tau \) regarding \( \tau \gg a^{2} \). The probability is determined by the integral of \( \bar{R}_{c}(r) \) over \( r \). We conclude from the expressions (2.21,3.3) that the integral is determined by the region \( r \sim \sqrt{\tau} \). Taking into account Eq. (3.5) we get

\[
\bar{\lambda}_{c} = \int d^{2}r \bar{R}_{c}(r) \sim \frac{y_{0}}{a^{4}} \left( \frac{a^{2}}{\tau} \right)^{\beta} . \tag{3.8}
\]

We see that due to annihilation of defects at collisions the total probability diminishes at increasing the time separation \( \tau \) as a power of \( \tau \). The property can be interpreted as follows: The majority of defect-antidefect pairs annihilate fast after their creation and only a minor part of the defects achieve a separation \( r \gg a \). The probability of such event is proportional to \( (r/a)^{-2\beta} \).

The results obtained in the subsection are correct if the variation of the coupling constant \( \beta \) on the scale interval \( a < r < \sqrt{\tau} \) is small. The existence of such interval is justified by the assumed small value of the fugacity \( y_{0} \). Near \( T_{c} \) variations of \( \beta \) on a wide region of scales can be relevant. Then the consideration needs a generalization made in the last subsection of this section.

B. Coulomb interaction and diffusion coefficient

Let us consider the renormalization of the Coulomb interaction related to small defect-antidefect pairs. It is known that the influence of such pairs can be described in terms of a contribution to the effective dielectric constant, see Ref. [6]. The picture is naturally generalized for the dynamics.

Before proceeding to calculations, it will be convenient for us to express the Coulomb part (2.4) of the Hamiltonian (2.1) in an alternative form. Namely, using the Hubbard-Stratonovich trick we rewrite the fourth-order term \( \mathcal{H}_{\beta} \) as a functional integral over auxiliary fields \( \sigma \) and \( \phi \)

\[
\exp \left( - \int dt \mathcal{H}_{\beta} \right)
\]

\[
= \int D\phi D\sigma \exp \left[ - \int dt \left( \mathcal{H}_{1} + \mathcal{H}_{2} \right) \right] , \tag{3.9}
\]

\[
\mathcal{H}_{1} = \int d^{2}r \left[ \left( \nabla \psi_{+} \psi_{+} - \nabla \psi_{-} \psi_{-} \right) \nabla \phi \right. 
\]

\[
\left. - \left( \dot{\psi}_{+} \psi_{+} - \dot{\psi}_{-} \psi_{-} \right) \sigma \right] , \tag{3.10}
\]

\[
\mathcal{H}_{2} = \frac{1}{4\pi\beta} \int d^{2}r \nabla \sigma \nabla \phi . \tag{3.11}
\]

The relation (3.3) can be easily checked using the bare expression

\[
\langle \nabla \phi(t_{1}, r_{1}) \sigma(t_{2}, r_{2}) \rangle_{0} = -2\beta \frac{r_{1} - r_{2}}{|r_{1} - r_{2}|^{2}} . \tag{3.12}
\]

following from Eq. (3.11). We see that the correlation function (3.12) corresponds to the dashed line on the diagrams. Note that the field \( \phi \) in the expressions is the electrostatic potential introduced by Eq. (2.6). Indeed, integrating over the field \( \sigma \) in Eq. (3.1) we get the Poisson equation

\[
\nabla^{2} \phi = -\frac{1}{4\pi\beta} \rho , \tag{3.13}
\]
leading to the expression (2.6).

Now we can work in terms of the sum $\mathcal{H}_0 + \mathcal{H}_R + \mathcal{H}_2$ where two first terms are introduced by Eqs. (2.2,2.3). Note that the sum $\mathcal{H}_0 + \mathcal{H}_2$ is invariant under the following infinitesimal transformation

$$
\delta \psi_+ = \alpha \psi_+ \quad \delta \psi_- = -\alpha \psi_- \quad \delta \psi_+ = -\alpha \psi_+ \\
\delta \phi = 2\alpha \quad \delta \sigma = \nabla^2 \alpha ,
$$

(3.14)

where $\alpha$ is a function of coordinates. If to substitute $R(r) \to \lambda \delta(r)$ and $\tilde{R}(r) \to \tilde{\delta}(r)$ into the expression (3.3) then it will be invariant under the transformation (3.14) also. Deviations from the symmetry related to $r$-dependencies of $R$ and $\tilde{R}$ are irrelevant. If the function $\alpha$ contains only terms linear and quadratic over $r$ then the contribution $\mathcal{H}_2$ (3.13) is invariant under the transformation (3.14) also. The symmetry leads to a number of the Ward identities. Particularly, they connect the renormalized triple vertices to the self-energy function of the propagator.

A typical diagram contributing to renormalization of the effective “dielectric constant” is drawn in Fig. 6. There we see a loop composed of the trajectories of a defect and of an antidefect which annihilate after their creation. There are also two “external” dashed lines corresponding to the interaction of the defect-antidefect pair with an environment. Besides the diagrams of the type drawn in Fig. 6 there are also diagrams with two external dashed lines attached to the same trajectory. We draw the external lines with arrows to remember that two sides of the dashed line representing the correlation function (3.12) are not equivalent. We imply that the dashed lines are directed from the field $\sigma$ to the field $\phi$.

As previously, we can dissect the diagram into parts which can be treated separately. Then the answer can be found as a convolution of the corresponding expressions. We perform the dissection along the planes in the $r - t$ space-time perpendicular to the $t$-axis and corresponding to the time moments $t_2$ and $t_3$ of the external Coulomb lines. In Fig. 7 the dissection is shown by the dotted lines. We see that the loop is divided into three parts.

The left part of the loop implying the integration over the time $t_1$ (see Fig. 6) corresponds to

$$
\int_0^\infty dt \int d^2 r_3 d^2 r_4 \tilde{R}(r_3 - r_4) M(r, x_1, x_2, r_3, r_4) \\
= \int_0^\infty dt \tilde{R}_s(\lvert x_1 - x_2 \rvert) .
$$

(3.15)

Substituting here Eq. (3.7) and using Eq. (3.4) we get

$$
\int_0^\infty dt \tilde{R}_s(r) = \frac{y^2}{a^4} \left( \frac{a}{r} \right)^{2\beta - 4} .
$$

(3.16)

The central part of the diagram depicted in Fig. 6 corresponds to the conditional probability (2.9) $M(t_3 - t_2, x_3, x_4, x_1, x_2)$. And the right part of the diagram in

Fig. 8 corresponds to the integral (3.4). The relation can be recognized as a manifestation of independence of all results of the final time $t_1$ in the relation (1.30). If we chose $t_1 = t_3$ then the right part of the diagram in Fig. 8 disappears and we should substitute 1 instead, in accordance with (3.4).

The structure of the diagram depicted in Fig. 6 shows that the block related to the defect-antidefect pair can be treated as a self-energy insertion to the line corresponding to the Coulomb interaction. Thus it is natural to expect that this insertion can be treated as a contribution to the “dielectric constant”, leading to a renormalization of the Coulomb constant $\beta$. The corresponding quantitative analysis is presented in Appendix B giving the following expression for the correction to $\beta$

$$
\Delta \beta \sim -y^2 \int d r \left( \frac{a}{r} \right)^{2\beta - 4} .
$$

(3.17)

The expression can be treated as an integral over the characteristic sizes of the defect-antidefect pairs.

One may try to find more complicated blocks contributing to a renormalization of the Coulomb coupling constant $\beta$. An example of such block is depicted in Fig. 6 where a number (three) “external” lines are attached to the loop corresponding to the trajectories of the defect-antidefect pair. One can easily check that the block depicted in Fig. 6 gives a correction to the Coulomb force which diminishes faster than $r^{-4}$ at increasing the distance $r$ between the interacting particles. Therefore the contribution is irrelevant. The same is true for more complicated diagrams of the same type.

We can also consider blocks which can be treated as contributions to the diffusion coefficient $D$ introduced by Eqs. (1.10,1.11). An example is depicted in Fig. 8 where the block between two dotted lines is a self-energy insertion to the propagator (2.3) which gives the renormalization of the diffusion coefficient. We will assume that the fields $\psi_\pm$ are corrected to keep the term (2.3) unchanged. Then the contribution (3.18) has to be extracted from the renormalization of the coefficient in front of the time derivatives.

To analyze the correction $\Delta D$ quantitatively one should know a three-particle conditional probability which is more complicated than the two-particle conditional probability (2.9). Fortunately, one can estimate the value of $\Delta D$ without detailed calculations. The point is that the dependence of $\Delta D$ on the cutoff $a$ can be produced only by regions near the creation or near the annihilation point (which are designated by ovals in Fig. 8). The regions can be analyzed in terms of the two-particle conditional probability (2.9) since only the interaction of the nearest “particles” is relevant there. We already know the answer: the region near the creation point produces the renormalized creation rate (3.3) whereas the region near the annihilation point produce no $a$-dependence. Then simple dimensional estimates give the answer similar to the expression (3.17).
\[ \Delta D \sim -y_0^2 \int \frac{dr}{r} \left( \frac{r}{a} \right)^{2\beta - 4}. \]  

(3.18)

Remind that the bare value of $D$ is assumed to be equal to unity.

Of course there are also blocks which can be interpreted as corrections to the triple vertices describing the interaction of the fields $\psi_\pm, \bar{\psi}_\pm$ with the Coulomb fields $\phi$ and $\sigma$. An example of such block can be imagined if to attach an external dashed line to a trajectory between the dotted lines in Fig. 3. However, in the renormalization scheme accepted the corrections are fully absorbed into the renormalization of the fields $\psi_\pm$. That is a consequence of the symmetry of the Hamiltonian under the transformation $R$. Namely, the cubic term (3.10) is unchanged provided the term (2.2) is unchanged.

### C. Summary

In the previous subsection we obtained the expressions for the corrections (3.17,3.18) of the Coulomb constant and of the diffusion coefficient in the main order over the fugacity $y_0$. Now we are going to discuss high-order corrections over $y_0$ which in statics lead to the renormalization group equations (1.5). It will be more convenient for us to proceed in spirit of the Kosterlitz renormalization group scheme. Namely, we see that the expressions (3.17,3.18) are written as integrals over the space variable $r$ which can be treated as the size of a defect-antidefect pair. We can first perform the integration over a restricted interval of the sizes, what gives slightly renormalized values of the coupling constants. Then we can repeat the integration. In the limit this multi-step procedure gives the renormalization group equations for the coupling constants, as Kosterlitz suggested. On the diagrammatic language the procedure means that we gradually substitute blocks corresponding to small separations between the particles by their effective values relative to larger scales. The procedure can also be considered as increasing an effective size of the defects $a \to r$. Then the renormalization of the coupling constants can be described in terms of the differential renormalization group equations.

At each step of the procedure we deal with correlation functions like (2.4). For an interval of scales where a variation of the Coulomb constant $\beta$ is small one can use for the function the expression (2.23) where now one should substitute the renormalized value of the Coulomb constant $\beta$. Analogously, the renormalized annihilation rate is determined by Eq. (3.3) where one should substitute the expression (2.21) with the renormalized value of the Coulomb constant $\beta$. Next, for the renormalized creation rate we should use the relation

\[ \bar{R}_{\tau_1} = \int d^2 r_0 S(\tau_1 - \tau_2, r, r_0) \bar{R}_{\tau_2}, \]

(3.19)

where $\tau_2 > \tau_1$. The relation (3.19) can be derived from Eq. (3.6) if to use the property of $S$ analogous to Eq. (3.1). For the renormalized creation rate the relation (3.19) is correct only if $\beta(r_1)$ weakly differs from $\beta(r_2)$ for characteristic values of the parameters $r_1 \sim \sqrt{r}$ and $r_2 \sim \sqrt{r}$. A relation analogous to Eq. (3.13) can be formulated for the annihilation rate $R$. The relation leads to the same expression (2.3) where $\beta$ is now scale-dependent. Therefore the renormalized quantity of the annihilation constant $\lambda_r$ flows together with $\beta$ in accordance with Eq. (2.22). That is accounted for the non-logarithmic character of the integrals leading to the relation (2.22).

Then we should define the renormalized fugacity in dynamics. For the purpose let us generalize the expression (3.16)

\[ \int_{0}^{\infty} d\tau \bar{R}_{\tau}(r) = \frac{y^2}{r^4}. \]

(3.20)

Then Eq. (3.16) is rewritten as

\[ y = \left( \frac{a}{r} \right)^{\beta - 2} y_0. \]

(3.21)

The relation can be treated as an elementary step of the renormalization group procedure which is described by Eq. (1.5) for $y$. Thus the renormalization group equation for $\beta$ in dynamics coincides with one in statics.

The expression (3.17) for the correction to the Coulomb constant $\beta$ can be considered as arising at the elementary step of the renormalization group procedure. The corresponding renormalization group equation can be found if to pass to the differential form and to substitute $y_0$ by the renormalized value $y$ in accordance with Eq. (3.21). Then we obtain the renormalization group equation coinciding with Eq. (1.3) for $\beta$. The expression (3.18) for the correction to the diffusion coefficient leads to the following renormalization group equation

\[ \frac{dD}{d \ln(r/a)} \sim -y_0^2, \]

(3.22)

analogous to the equation (1.5) for $\beta$. We conclude that the correction to $D$ is small due to the small value of the fugacity and is therefore irrelevant. To avoid a misunderstanding, remind that the variation of the Coulomb constant $\beta$ with increasing scale is also small. Nevertheless, as seen from Eq. (1.3), it is the difference $\beta - 2$ that enters the renormalization group equations and the variation of the difference can be essential.

### 4. CORRELATION FUNCTIONS

Here, we treat non-simultaneous correlation functions of the charge density $\rho$ (1.2)

\[ F_{2n}(t_1, \ldots, t_{2n}; r_1, \ldots, r_{2n}) = \langle \rho(t_1, r_1) \ldots \rho(t_{2n}, r_{2n}) \rangle. \]

(4.1)
Note an obvious consequence of the constraint (1.3)
\[
\int d^2 r_1 \, F_{2n} = 0 .
\]  
(4.2)

To examine the correlation functions (4.1) we use the representation (4.23). We will assume that all the diagrammatic blocks corresponding to small defect-antidefect pairs are already included into the renormalization of the corresponding coupling constants as discussed in Section 3. Therefore the fugacity \( y \) and the Coulomb constant \( \beta \) entering all subsequent expressions should be taken at the current scale. First we will examine contributions to the functions associated with a single defect-antidefect pair and then we will consider contributions related to a number of defect-antidefect pairs.

### A. Pair Correlation Function

We start with the pair correlation function
\[
F_2(t_2 - t_1, r_2 - r_1) = \langle \rho(t_2, r_2) \rho(t_1, r_1) \rangle ,
\]  
(4.3)

with \( t_2 > t_1 \). The average (4.3) can be calculated in accordance with the relation (4.30) where one should substitute the expression (4.23). We assume \( t_1 = 0 \).

The contribution to the average (4.3) related to a single defect-antidefect pair can be represented by a series of the diagrams with two lines constituted of the defect and antidefect propagators. The lines start from the creation point. A half of the diagrams have the structure depicted in Fig. 9. Here we omitted lines and vertices corresponding to the interaction of the defects (which is implied) and keep only trajectories of the defects. The trajectories should pass through the points \( r_1 \) and \( r_2 \) at the time moments \( t_1 \) and \( t_2 \) (the events are designated by black circles). An additional contribution to the average (4.3) is determined by similar diagrams where both events \( r_1 \) and \( r_2 \) belong to the same trajectory.

As above, we dissect the diagram into parts which can be treated separately. Let us make a cut along planes in \( r - t \) space-time corresponding to the time moments \( t_1 \) and \( t_2 \), they are shown in Fig. 9 by dotted lines. Intermediate points appearing in the convolution (4.1) are designated in Fig. 9 as \( r_3 \) and \( r_4 \). After that the diagram is divided into two parts separated by the dotted line. The part of the diagram to the right from the dotted line corresponds to the conditional probability \( M(2.9) \) and the part of the diagram to the left from the dotted line corresponds to the correlation function
\[
\Phi_2(r_1 - r_2) = \langle \psi_+ (t, r_1) \psi_- (t, r_2) \rangle .
\]  
(4.4)

The quantity (4.4) can be treated as the probability density to find a defect-antidefect pair with a given separation. Correspondingly, the integral \( \int d^2 r \, \Phi_2(r) \) determines the density of the defect-antidefect pairs. The correlation function (4.4) coincides with the integral in the left-hand side of Eq. (3.10). Hence
\[
\Phi_2(r) = y^2 / r^4 ,
\]  
(4.5)

where \( y \) is the renormalized fugacity. The equations (1.6, 4.3) show that asymptotically in the low-temperature phase
\[
\Phi_2(r) \propto r^{-2 \beta} .
\]  
(4.6)

Note that the same behavior (4.6) is observed up to a slowly varying factor in the whole region of scales.

The diagram depicted in Fig. 4 gives a convolution of \( \Phi_2 \) and \( M \). Adding the contribution corresponding to the case where both events \( r_1, r_2 \) and \( t_2, r_2 \) belong to the same trajectory we get the following expression for the pair correlation function (4.3)
\[
F_2(t, r_2 - r_1) = -2 \int d^2 r_3 \int d^2 r_4 \, \Phi_2(r_4 - r_1)
\]  
(4.7)

\[
\times [M(t, r_3, r_2, r_1, r_4) - M(t, r_2, r_3, r_1, r_4)] ,
\]  
(4.8)

where \( t > 0 \). At small times \( t \) we turn to the limit law (2.11). Substituting the expression into Eq. (4.7) we get
\[
F_2(t = 0, r_1 - r_2)
\]  
(4.9)

\[
= 2 \delta (r_1 - r_2) \int d^2 r \, \Phi_2(r) - 2 \Phi_2(r_1 - r_2) .
\]  
(4.10)

To justify Eq. (4.9) one should check that there are no divergences in the integral (4.7). It can be done directly using Eq. (2.23). The convergence at small separations \( r \) and \( r_0 \) is accounted for the behavior of the modified Bessel functions \( L_v (x) \propto x^v \) at small values of the argument. The convergence at large separations \( r_0 \) can be checked using the asymptotic law (2.24). If \( |t| \gg r^2 \) then
\[
F_2 \sim - \frac{y^2 (r)}{r^{4 - 2 \beta} |t|^\beta} .
\]  
(4.10)

The asymptotic law is established in Appendix A.

The behavior of the pair correlation function determined by the laws (4.3, 4.10) corresponds to a conventional critical dynamics (see, e.g., Ref. 21) with the dynamical critical index \( z = 2 \). However, as we will see below, the behavior of the high-order correlation functions is beyond the conventional scheme. Besides, the scaling law \( t \sim r^2 \) is true for the high-order correlation functions as well.
B. High-Order Correlation functions

Here we extend the procedure of the preceding subsection to the case of the high-order correlation functions \( F_{2n} \). We will assume that \( t_1 < t_2 < \ldots < t_{2n} \). Again, we examine the contribution to \( F_{2n} \) associated with a single defect-antidefect pair. Corresponding diagrams contain two trajectories starting anywhere and passing through the points \( r_1, \ldots, r_{2n} \) at the time moments \( t_1, \ldots, t_{2n} \). We will designate the trajectories of the defect and of the antidefect as \( x(t) \) and \( z(t) \).

Let us dissect the diagrams along planes in the \( r - t \) space-time corresponding to the time moments \( t_1, \ldots, t_{2n} \). Then the diagram is divided into a number of blocks, see Fig. 10. The left block in Fig. 10 corresponds to the object \( \Phi \) and all the other blocks correspond to the correlation function \( \Phi \). Again, using Eq. (3.1) we can write the contribution to the correlation function \( \Phi \) associated with a single defect-antidefect pair as the following convolution

\[
F_{2n}(t_1, \ldots, t_{2n}; r_1, \ldots, r_{2n}) = \prod_{j=1}^{2n} \int d^2x_j \int d^2z_j \Phi_2(x_1 - z_1) \left[ \delta(r_j - x_j) - \delta(r_j - z_j) \right] \times M(t_{j+1} - t_j, x_{j+1}, z_j, x_j, z_j). \tag{4.11}
\]

Here, one must replace the last factor \( M(t_{2n+1} - t_1) \) by unity. The relation (4.11) is a generalization of Eq. (7). Thus we got an expression for the correlation function which is a multiple integral of functions determined by explicit formulas. A recurrent procedure for calculating \( F_{2n} \) is suggested in Appendix C.

Of course the expression (4.7) for the pair correlation function is reproduced by Eq. (4.11). Note also the following expression

\[
\langle \rho(t, r_1) \rho(t, r_2) \rho(0, r_3) \rho(0, r_4) \rangle = 2 \left[ M(t, r_1, r_2, r_3, r_4) + M(t, r_1, r_2, r_3, r_4) \right] \Phi_2(r_3 - r_4)
- B(r_1 - r_2) \delta(r_3 - r_4) - B(r_3 - r_4) \delta(r_1 - r_2) + \int d^2r B(r) \delta(r_1 - r_2) \delta(r_3 - r_4), \tag{4.12}
\]

where

\[
B(r_3 - r_4) = 2 \int d^2r_1 [M(t, r_1, r_2, r_3, r_4) + M(t, r_1, r_2, r_4, r_3)] \Phi_2(r_3 - r_4).
\]

The formula can be found from Eq. (4.11) using the relation (2.11). Naturally, the expression (4.12) is symmetric under the permutation \( r_1, r_2 \leftrightarrow r_3, r_4 \), which can be checked using the expressions (2.23). The formula (4.12) is in agreement with the general property (2.12).

The recurrent procedure for calculating \( F_{2n} \) suggested in Appendix C shows that there are no divergences in the integrals at all steps of calculating \( F_{2n} \). This means that we can evaluate the correlation functions from naive dimension estimates. Namely, if all space separations among \( |r_i - r_j| \) are of the same order \( r \) and all time intervals are of the order \( \tau \), then

\[
F_{2n} \sim y^2(r) r^{-4n}. \tag{4.13}
\]

In the large-scale limit when \( \beta \) is saturated we have in accordance with Eq. (4.6)

\[
F_{2n} \propto r_s^{-4(n - 1) - 3\beta}. \tag{4.14}
\]

If some space separations among \( |r_i - r_j| \) and/or some time intervals differ strongly then one can formulate some simple rules following from Eqs. (2.23, 4.11). Let us give some examples. If one of the time intervals \( \tau \) is much larger than all values of the squared separations \( |r_i - r_j|^2 \) then the correlation function behaves like \( F_{2n} \propto \tau^{-\beta} \). It can be proved like it is done in Appendix A for the pair correlation function. For small \( \tau \) there appear contributions to \( F_{2n} \) short-correlated in space (on scales \( \sim \sqrt{\tau} \)). In the limit \( \tau \to 0 \) the contributions turn into \( \delta \)-functions, as it was for the pair correlation function, see Eq. (4.8), representing an autocorrelation of single defects. If the points \( r_j \) can be divided into two “clouds” with a separation \( r \) between the clouds much larger than their sizes (and all time intervals are much smaller than \( r^2 \) then the principal \( r \)-dependence of the correlation function \( F_{2n} \) is the same as in the function \( \Phi_2(r) \).

Remember that the charge density \( \rho \) is related to the curl of the gradient of the hexatic angle \( \varphi \) for hexatics and to the curl of the gradient of the order parameter phase for superfluid films. It is instructive to re-express our results in terms of the phase gradient circulations of \( \nabla \varphi \) over a closed loop \( C \)

\[
\Gamma(t, C) = \oint_C \varphi = 2\pi \int d^2r \rho(t, r), \tag{4.15}
\]

where the second integral is taken over the area inside the loop. Correlation functions of \( \Gamma \)'s can be rewritten as integrals of the correlation functions \( F_{2n} \). As an example, consider the following average

\[
\Psi_{2n} = \langle \Gamma(t, C) \Gamma(t + \tau, C) \rangle \Gamma(t + (2n - 1)\tau, C) \rangle. \tag{4.16}
\]

Suppose that the characteristic size of the loop \( r \) is large enough so that we can assume that \( \beta \) is saturated, and that \( \tau \approx r^2 \). Then the following scaling law is satisfied: if \( r \to Xr \) and \( \tau \to X^2 \tau \) then

\[
\Psi_{2n} \to X^{4 - 2\beta} \Psi_{2n}, \tag{4.17}
\]

where \( X \) is an arbitrary factor. The law (4.17) is a consequence of Eq. (4.14). It has two striking peculiarities.
First, it possesses a clear critical dependence. Second, it is independent of the order $n$.

The procedure described above can be generalized to include the external potential. Then one should take the solution of the equation (2.23) for the function (2.9) and the corresponding expression for the object (4.4).

C. Many-Pair Contributions

We have established the contributions to the charge density correlation functions $F_{2n}$ associated with a single defect-antidefect pair. Now we are going to discuss other contributions to the correlation functions related to an arbitrary number of defect-antidefect pairs. Correspondingly, we should take diagrams with a number of trajectories passing through the points $r_1, \ldots, r_{2n}$ at the time moments $t_1, \ldots, t_{2n}$. The picture illustrating the situation is drawn in Fig. 1 where black circles correspond to the arguments of $F_{2n}$: $(t_1, r_1), \ldots, (t_{2n}, r_{2n})$. There we omitted blocks related to short-living defect-antidefect pairs regarding that the blocks are already included into the renormalization of the Coulomb constant $\beta$.

As previously, we can dissect the diagrams along the planes in the $r - t$ space-time corresponding to the time moments $t_1, \ldots, t_{2n}$. Then any diagram will be divided into a number of strips, see Fig. 1. The part of the diagram within each strip can be treated as the corresponding matrix element of the evolution operator $\exp(-\int dt \mathcal{H})$. Then the contribution to $F_{2n}$ will be written like (4.11) as a convolution of the matrix elements. Generally, the matrix elements can be estimated like the function (2.9): each trajectory segment gives a factor which scales as $1/t$ and $t$ scales as $r^2$. But there are obvious exceptions from the rule. Namely, going back in time we will come to a moment where a defect-antidefect pair was created. Again, when we consider small separations between the defect and the antidefect (which is correct for time moments close to the creation time) we can take into account only the Coulomb interaction between the two created defects. The corresponding regions in Fig. 1 are inside the ovals. Each such region produces the factor $y^2$. Therefore generally $F_{2n} \propto y^{2k}$, where $k$ is the number of the pairs. Taking into account also a scale-dependent factor we get

$$F_{2n} \sim y^{2k}(r_*)^4 \sim 4n,$$  \hspace{1cm} (4.18)

where we assume that all space separations are of the order $r_*$ and all time intervals are of the order $r_*^2$.

The expression (4.18) is a generalization of Eq. (4.13). Comparing these two expressions we see that the ratio of the contribution (4.18) to the contribution (4.13) is the $(k - 1)$-th power of a dimensionless small parameter $y^2(r_*)$. Thus we conclude that the leading contribution to $F_{2n}$ is related to a single defect-antidefect pair, that corresponds to $k = 1$. Now we can explain the origin of the estimate (4.8) for the simultaneous correlation functions which obviously does not coincide with Eq. (4.13). The estimate (4.8) in terms of Eq. (4.18) corresponds to $k = n$. The reason is quite obvious: Two defects cannot pass simultaneously through $2n$ points and at least $k = n$ defect-antidefect pairs should be taken to obtain a non-zero contribution to the simultaneous correlation function $F_{2n}$. The situation is illustrated by Fig. 2. Note that the estimate (4.8) is not correct for the autocorrelation contributions proportional to $\delta$-functions, as written in Eq. (4.5).

Thus we have two different regimes: for simultaneous and for non-simultaneous correlation functions. Let us establish the boundary between the regimes. For the purpose we should consider small time intervals where the single-pair contribution is finite but small. The smallness is associated with diffusive exponents presented, e.g., in the expression (2.23). Therefore the characteristic time where the simultaneous regime passes into the non-simultaneous one can be estimated as

$$t \sim \frac{r^2}{|\ln[y(r)]|},$$  \hspace{1cm} (4.19)

where $r$ is a space separation corresponding to the small time interval. In the low-temperature phase on large scales (where $\beta$ is saturated) we have $|\ln y| \approx (\beta - 2) \ln(r/a)$.

5. SUPERFLUID FILMS

Let us consider superfluid films. The equation of motion for the vortices contains an additional term (Magnus force). Thus instead of Eq. (1.10) we should write (see Ref. 1)

$$\frac{dx_{j,\alpha}}{dt} = -\frac{D}{T} \left[ \frac{\partial F}{\partial x_{j,\alpha}} + n_j \gamma \epsilon_{\alpha\beta} \frac{\partial F}{\partial r_{\beta}} \right] + \xi_{j,\alpha},$$  \hspace{1cm} (5.1)

where $\gamma$ is a new dimensionless parameter. The equation (5.1) can be derived in the spirit of the procedure proposed by Hall and Vinen for the 3$d$ superfluid, see Ref. 29. Huber [24] argued that the same equation is correct for spin vortices in planar 2$d$ magnets.

For the superfluid films the “charge density” (1.2) is proportional to the vorticity curl $\omega$. To calculate correlation functions $F_{2n}$ (4.13) one can use the scheme developed in the previous sections. The only difference is that instead of the expression (2.23) for the conditional probability $M$ (2.9) one should use the solution of the equation

$$\partial_t M = \left( \frac{1}{2} \nabla^2 + 2 \nabla^2 + 4 \frac{\beta}{r^2} \nabla \nabla \right) M + 2 \gamma \beta \epsilon_{\alpha\beta} \frac{r_{\beta}}{r^2} \nabla \epsilon_{\alpha\beta} M - R(r) M + \delta(t) \delta(|r - r_0|) \left( \delta - \frac{r^2}{2} - \frac{r_0^2}{2} \right).$$  \hspace{1cm} (5.2)

The variables $r, r_0$ and $a$ are introduced by Eq. (2.12). Again, on scales $r \gg a$ one can omit the term with the
annihilation rate $R$ in Eq. (5.2) demanding a finite value of $M$ at $r \to 0$ instead.

Unfortunately, a cross term over $r$ and $\varrho$ appears in the operator in the right-hand side of Eq. (5.2). Thus one cannot obtain an explicit expression for $M$ of the type of Eq. (2.23). Nevertheless, this additional term has the same dimensionality as the other terms and does not change the scaling estimates $M \sim t^{-2}, t \sim r^2$ determining the function $M$. Moreover, the equation for the object

$$S(t, r, r_0) = \int d^2 \varrho M(t, r, \varrho, r_0, r_3/2 + r_4/2),$$

(5.3)

following from (5.2) is identical to Eq. (2.14). Therefore for the object (5.3) we have the same series (2.16) with the coefficients (2.21).

Looking through the derivation presented in Section 3 we see that just the function $S$ (7.3) enters all the relations. Therefore we can make the same assertions as previously. First, on large scales the annihilation coefficient $\lambda$ is equal to its universal value (2.22). Second, we can write the same expression (4.2) for the average (4.4). Third, in dynamics we get the same renormalization group equation (1.5) for $\beta$, see Appendix B. Fourth, the renormalization of the diffusion coefficient is irrelevant. And finally, one can assert that a renormalization of the parameter $\gamma$ introduced by Eq. (7.1) is determined by the equation

$$\frac{d \gamma}{d \ln(r/a)} \sim y^2,$$

analogous to (2.22). Therefore the renormalization of $\gamma$ is irrelevant. Again, the scheme can be generalized to include the “external potential”, which now is the average value of the superfluid velocity.

Next, we proceed to the correlation functions $F_{2n}$ (1.1). Formally they are determined by the same convolution (1.11) as previously. However, one should substitute there the solution of the equation (5.2). Therefore the concrete expressions for $F_{2n}$ will be different. Nevertheless, the estimates like (1.13,1.14,1.18) remains true because of the following reasons. First, due to the same dimensionality of all the terms in the right-hand side of Eq. (5.2) the function $M$ possesses the simple scaling properties noted above. Second, there are no divergences in the convolutions like (4.11) determining the objects. To prove the second property, we should analyze a behavior of $M$ at large and at small separations. In the case $r_0/t \gg 1$ the characteristic values of the separations $r_1 - r_3$ and of $r_3 - r_4$ are $\sim \sqrt{t}$ and are consequently much smaller than $|r_1 - r_2|$ (or $|r_3 - r_4|$). Then it is possible to neglect all terms containing $|r_1 - r_2|$ in denominators in the equation (5.2) and we come to a purely diffusive equation leading to the asymptotics (2.23). It is possible to establish that the small-scale of the conditional probability $M$ for the vortices coincides with that examined above. The properties ensure convergence of all intermediate integrals appearing at calculating the correlation functions of vorticity $F_{2n}$.

We have also the same scaling law (1.17) for the correlation function (1.19) of the integrals (1.15) which are now proportional to the circulations of the superfluid velocity. We conclude that all the scaling laws for the correlation functions of the vorticity and their asymptotic behavior remains the same as previously.

6. Discussion

The main result of our consideration is the expression (1.13) for high-order correlation functions of the “charge density” (4.1) which is disclivity for hexatic films and vorticity for superfluid films. We see from Eq. (1.13) that the high-order correlation functions are much larger than their normal estimates via the pair correlation function. Namely, in accordance with Eqs. (1.9,4.13) we have

$$F_{2n}/F_2 \sim y^{-2n+2} \gg 1,$$

(6.1)

where $y$ is the renormalized fugacity. The asymptotic behavior of the ratio at large scales is determined by the law (1.6). Though at developing our scheme we accepted that the defect-antidefect pairs constitute a dilute solution we hope that the scaling law (6.1) is universal. The ground for the hope is the renormalization group procedure (formulated in Ref. [8]) which shows that on large scales we come to an effectively dilute solution of the pairs. We believe that the most interesting fact to be compared with experiment or numerics is the scaling law (1.17) which is a consequence of Eq. (6.1).

The physics behind the inequality (1.13) is as follows. The main contribution to the correlation functions is associated with a single defect-antidefect pair. Though the contribution associated with a number of defect-antidefect pairs contains an additional huge entropy factor it has also an additional small factor associated with small probability to observe defect-antidefect pairs with separations larger than the core radius $a$. The smallness is accounted for the strong Coulomb attraction. The considered effect is a consequence of the competition of those two factors. The result of the competition manifests in the law (1.13) which gives the estimate for the contribution associated with $k$ defect-antidefect pairs. For simultaneous correlation functions nothing similar occurs and we have the conventional estimate (1.8). That is the reason why the effect cannot be observed in statics. The property is directly related to causality since a defect-antidefect pair cannot simultaneously pass through $2n$ points and at least $n$ defect-antidefect pairs is needed to get a non-zero contribution to the simultaneous correlation function $F_{2n}$, see Fig. 2. That explains the estimate (1.8). Thus we have two different regimes for simultaneous and non-simultaneous correlation functions. The characteristic boundary time separating those two regimes is written in Eq. (1.19).
The considered effect resembles intermittency in turbulence (see, e.g., Ref. [14]), leading to large \( r \)-dependent factors in the ratios like \( [6,3] \) in the velocity correlation functions of a turbulent flow. However, as is seen from Eq. [2.11], for the defects the large \( r \)-dependent factors are related to the ultraviolet cutoff parameter \( a \) whereas for intermittency in turbulence the large \( r \)-dependent factors are related to the infrared (pumping) scale. Our situation is thus closer to the inverse cascade (see Ref. [27]) realized on scales much larger than the pumping length. There are experimental data [29] concerning the inverse cascade in 2d hydrodynamics and analytical observations concerning the inverse cascade for a compressible fluid [24], which indicate the absence of the intermittency in the inverse cascades. Note that only simultaneous objects were examined in the works [23,29], and there is no intermittency in our simultaneous correlation functions. So, based on the analogy, one may think that for the inverse cascades non-simultaneous objects reveal some intermittency.

The consideration presented in our work is applicable to superfluid films. There exist also films and quasi-2d systems of different symmetry. Huber [24] argued that the same equation as for quantum vortices is correct for spin vortices in planar 2d magnets. We believe that our approach based on the equation (1.10) is correct for the dynamics of disclinations in hexatic films like membranes, freely suspended films and Langmuir films. Next, the above scheme seems to work also for dislocations in solid films. The system needs a special treatment since a modification should be introduced into the procedure. Maybe some features of the presented picture can be observed also in superconductive materials, especially in high-\( T_c \) superconductors. There are analytical and numerical indications that for a purely Langevin dynamics of the order parameter there are logarithmic corrections to the law (1.10), see Refs. [31,36]. We believe that the logarithms are destroyed if to switch on an interaction of the order parameter with other degrees of freedom. Nevertheless, it would be interesting to generalize our scheme including the logarithmic corrections.

**ACKNOWLEDGMENTS**

I am grateful to G. Falkovich, K. Gawedzki, I. Kolokolov, and M. Vergassola for useful discussions and to E. Balkovsky, A. Kashuba and S. Korshunov for valuable remarks. This research was supported in part by a grant of Israel Science Foundation, by a grant of Minerva Foundation and by the Landau-Weizmann Prize program.

**APPENDIX A:**

In this Appendix we present some calculations leading to the results presented in the main body.

1. Some relations for the pair conditional probability

Here, we deduce some relations concerning the correlation function (2.9). We examine the solution of the equations (2.17) with the annihilation term \( R \) omitted. Then we should accept a finite value of \( S_m \) at \( r \to 0 \) as is explained in the main text.

Performing the Fourier transform over \( t \) we get from Eq. (2.17)

\[
-\frac{i\omega}{2} S_m(\omega) = \left[ \partial_r^2 + \left( 1 + 2\beta \right) \frac{1}{r} \partial_r - \frac{m^2}{r^2} \right] S_m(\omega) + \frac{1}{4\pi r_0} \delta(r - r_0).
\]  

(A1)

The solution of the equation (A1) which tends to zero at \( r \to \infty \) and remains finite at \( r \to 0 \) can be written as

\[
\begin{align*}
S_m(\omega) &= \frac{1}{4\pi} \left( \frac{r_0}{r} \right)^{\beta} I_{\nu} \left( \sqrt{\frac{-i\omega}{2} r} \right) \\
\times &K_{\nu} \left( \sqrt{\frac{-i\omega}{2} r_0} \right) \quad \text{if } r < r_0, \\
S_m(\omega) &= \frac{1}{4\pi} \left( \frac{r_0}{r} \right)^{\beta} K_{\nu} \left( \sqrt{\frac{-i\omega}{2} r} \right) \\
\times &I_{\nu} \left( \sqrt{\frac{-i\omega}{2} r_0} \right) \quad \text{if } r > r_0,
\end{align*}
\]

(A2)

where \( \nu = \sqrt{\beta^2 + m^2} \) and we used the relation

\[ K_{\nu}(z) \partial_z I_{\nu}(z) - I_{\nu}(z) \partial_z K_{\nu}(z) = z^{-1}. \]

At the next step one performs the inverse Fourier transform. Deforming the integration contour to the negative imaginary semi-axis and using the relation 6.633.2 from Ref. [37] one gets Eq. (2.23).

Next, we present two integral relations for the conditional probability (2.9) in the above approximation. Let us calculate the total probability to find a defect and an antidefect in any points at a fixed time separation \( t \). Using the relation 6.631.1 from Ref. [37] we get

\[
\begin{align*}
\int d^2r_1 \, d^2r_2 \, M &= \int d^2r \, S(t, r, r_0) \\
&= \frac{1}{\Gamma(1 + \beta)} \left( \frac{r_0^2}{8t} \right)^{\beta} \exp \left( -\frac{r_0^2}{8t} \right) I_F \left( 1, \beta + 1; \frac{r_0^2}{8t} \right).
\end{align*}
\]

(A3)

We conclude from Eq. (A3) that the total probability diminishes with increasing time \( t \). It is accounted for annihilation of defects at collisions. Using the relation 6.611.4 written in Ref. [37] we find from (2.23)
\[
\int_0^\infty dt M = \frac{1}{2\pi i} \left( \frac{r_0}{r} \right)^\beta \sum_{m=-\infty}^{+\infty} \exp(im\varphi) \\
\times \frac{(2rr_0)^\nu}{\sqrt{\varsigma^2 - 4r_0^2r_0^2}} ,
\]

where \( \varsigma = (r_1 + r_2 - r_3 - r_4)^2 + r_2^2 + r_0^2 \) and \( \nu = \sqrt{\beta^2 + m^2} \). The expression determines a distribution of the defect and of the antidote over space provided the pair was created near the origin at any time.

Now we calculate the integral (3.3). Substituting there Eqs. (3.3, A.3) we get after integrating in part

\[
\lambda_\nu = -8\pi \int_0^\infty dz \left[ \frac{z^\beta e^{-z}}{\Gamma(1+\beta)} F_1(1, \beta + 1; z) - 1 \right] ,
\]

(5.3)

where \( z = r_0^2/8\pi \). One can easily check using the asymptotic expression for \( F_1(1, \beta + 1; z) \) at large \( z \) that the integral (3.5) converges. Let us now rewrite the expression (3.5) as

\[
\lambda_\nu = -8\pi \lim_{\alpha \to 0} \int_0^\infty dz \left[ \frac{z^\beta}{\Gamma(1+\beta)} \exp(-z - \alpha z) \\
\times F_1(1, \beta + 1; z) - \exp(-\alpha z) \right] .
\]

(5.5)

Now integrals of both contributions to the integrand can be found explicitly (see the relation 7.621.5 from Ref. 27). Passing then to the limit \( \alpha \to 0 \) we come to the answer (2.22).

2. Asymptotics of the Pair Conditional Probability

Here, we will be interested in the asymptotic behavior of the expression (2.22) where the parameter \( z = r_0^2/(4t) \) is large: \( z \gg 1 \). More precisely, we will examine the function

\[
T(z, \varphi) = \sum_{m=-\infty}^{+\infty} \exp(im\varphi) I_\nu(z) ,
\]

(7.2)

where \( \nu = \sqrt{\beta^2 + m^2} \). It is obvious that \( T(z, \varphi) = T(z, -\varphi) \). Below we will assume \( \varphi > 0 \).

First of all, we convert the sum over \( m \) in Eq. (7.2) into an integral using the identity

\[
\sum_{m=-\infty}^{+\infty} f(m) = \frac{1}{2} \int_+ dm \coth(-i\nu m) f(m) \\
- \frac{1}{2} \int_- dm \coth(-i\nu m) f(m) ,
\]

where the contour in the first integral goes above the real axis and the contour in the second integral goes below the real axis. One can easily check that if \( f(-m) \) is complex conjugated to \( f(m) \) then the second integral is equal to the complex conjugated of the first integral with the sign minus. Therefore we can write

\[
T(z, \varphi) = \text{Re} \int_+ dm \coth(-i\nu m) \exp(im\varphi) I_\nu(z) .
\]

(8.2)

Next, we are going to shift the integration contour in Eq. (8.2) into the upper \( m \)-semi-plane. Then we should separately consider two contributions to \( I_\nu \) which are determined by two terms in its integral representation

\[
I_\nu(x) = \frac{1}{\pi} \int_0^\pi d\vartheta \exp(x \cos \vartheta) \cos(\nu \vartheta) .
\]

(9.2)

Substituting the first contribution in the right-hand side of Eq. (9.2) we get

\[
T_1(z, \varphi) = \text{Re} \int_+ dm \coth(-i\nu m) \\
\times \int_0^\pi d\vartheta \frac{\exp(z \cos \vartheta) \exp(im\varphi) \cos(\nu \vartheta)}{\pi} .
\]

(10.2)

As we will see, the answer will be determined by a vicinity of a saddle point where \( -im \gg 1 \). There one can substitute \( \coth(-i\nu m) \) by unity and \( \nu \) by \( m \). Then one obtains from Eq. (10.2) the following saddle-point conditions

\[
\vartheta = \varphi , \quad m = iz \sin \vartheta .
\]

One can easily find from Eq. (10.2) in the saddle-point approximation

\[
T_1(z, \varphi) \approx \exp(z \cos \varphi) ,
\]

(11.2)

where we have taken into account an pre-exponent besides the exponent. The above scheme is obviously broken at small \( \varphi \) or at \( \varphi \) close to \( \pi \) since the saddle-point value of \( m \) tends to zero there. Thus the cases need a separate consideration.

Let us consider the case where \( \varphi \) is close to zero. Then the main contribution to Eq. (11.2) is gained from the region of integration over \( \vartheta \) near zero. There it is possible to expand the factor at \( z \) as \( \cos \vartheta \approx 1 - \vartheta^2/2 \). Then the integration over \( \varphi \) is performed explicitly. Returning also to the sum we get

\[
T_1(z, \varphi) \approx \frac{\exp z}{\sqrt{2\pi z}} \sum_{m=-\infty}^{+\infty} \exp \left( im\varphi - \frac{m^2}{2z} \right) ,
\]

where we substituted \( \exp(-\beta^2/2z) \) by unity. Since both \( \varphi \ll 1 \) and \( 1/z \ll 1 \) we can replace here summation by the integration over \( m \). After the substitution we get

\[
T_1(z, \varphi) \approx \exp \left( z - \frac{z\varphi^2}{2} \right) ,
\]

what reproduces Eq. (11.2) at small \( \varphi \). Analogously the case of \( \varphi \) close to \( \pi \) can be examined. Again, we reproduce Eq. (11.2).
Next, we examine the contribution to the function $T$ associated with the second term in (A9). For the purpose we return to the initial expression (A7). The second contribution can be written as

$$T_2(z, \varphi) = -\frac{1}{\pi} \int_0^\infty dt \sum_{m=-\infty}^{+\infty} \exp(im\varphi - \nu t) \times \sin(\pi \nu) \exp(-z \cosh t).$$

If the sum is dominated by first terms in the sum over $m$ then we have

$$|T_2(z, \varphi)| \lesssim \frac{1}{\sqrt{z}} \exp(-z), \quad (A12)$$

which can be obtained after substituting $\cosh t \approx 1 + t^2/2$ into the integral. Thus we should estimate the contribution determined by large $|m|$. Then

$$\sin(\pi \nu) \approx (-1)^m \frac{\pi \beta^2}{2|m|}.$$ 

Substituting the expression into the sum we get a contribution to $T_2(z, \varphi)$

$$\int_0^\infty dt \exp(-z \cosh t) \times \sum_{m>1} \frac{\beta^2}{|m|} \cos[(\pi - \varphi)m] \exp(-mt)$$

$$= \int_0^\infty dt \exp(-z \cosh t) \Phi \left[t^2 + (\pi - \varphi)^2\right],$$

where we introduced a function $\Phi(x)$ which behaves as $\Phi(x) \sim \ln x$ if $x \ll 1$ and $|\Phi| \ll 1$ if $x \gg 1$. Therefore the contribution determined by large $|m|$ reproduces the same estimate (A12). The estimate (A12) shows that $|T_2|$ is much smaller than Eq. (A11). We conclude that the expression (A11) determines the main contribution to the sum (A7).

Rewriting the answer (A11) for the sum (A7) in terms of the original variables we get

$$\sum_{m=-\infty}^{+\infty} \exp(im\varphi) I_\nu \left(\frac{r_0}{4t}\right) \approx \exp \left(\frac{rr_0}{4t}\right). \quad (A13)$$

Then the expression (A13) leads to

$$M \approx \frac{1}{(4\pi t)^2} \frac{r_0}{\beta} \exp \left\{ \frac{(r - r_0)^2}{8t} \right\} \times \exp \left\{ -\frac{(r_1 + r_2 - r_3 - r_4)^2}{8t} \right\}. \quad (A14)$$

3. Asymptotics of the pair correlation function

Let us consider the asymptotics of $F_2$ for times $t$ much larger than $r_1^2$. Then the characteristic values of both $r$ and $r_0$ in the integral (4.7) are much larger than $r_1$ and therefore we can neglect a dependence on $r_1$ there. Substituting the expressions (2.23,3.14) into Eq. (4.7) and then neglecting $r_1$ we get after integrating over the angle between $r$ and $r_0$

$$F_2 \approx -\frac{y^2}{r^4-2\beta t^2} \int_0^\infty \frac{dr}{r^{\beta-1}} \frac{dr_0}{r_0^{\beta-1}} \exp \left( \frac{r^2 + r_0^2}{4t} \right)$$

$$\times \sum_{m=-\infty}^{+\infty} I_{2m+1} \left(\frac{rr_0}{4t}\right) I_\nu \left(\frac{rr_0}{4t}\right),$$

where $\nu = \sqrt{\beta^2 + (2m+1)^2}$ and $r_0 = r_1 - r$, $r = r_3 - r_2$. The ratio $y^2/r^{4-2\beta}$ is a slowly varying function and therefore it is placed outside the integral. Note that the integral is nonzero due to $I_\nu > 0$. Performing here the integration over $r/r_0$ we get

$$F_2 \approx -\frac{y^2}{r^4-2\beta t^2} \int_0^\infty \frac{dy}{y^{\beta-1} K_0(2y)}$$

$$\times \sum_{m=-\infty}^{+\infty} I_{2m+1} (y) I_\nu (y),$$

where we used the relation 3.547.4 from Ref. [37]. The integral over $y$ converges at any $\nu$ obviously. Thus we should examine the convergence of the sum over $m$. At large $\nu$ we can use the asymptotics

$$I_\nu (x) \approx \frac{1}{\sqrt{2\pi} (x^2 + \nu^2)^{1/4}} \times \exp \left[ \sqrt{x^2 + \nu^2 + \nu \ln \frac{x}{\nu + \sqrt{x^2 + \nu^2}} \right], \quad (A15)$$

and the integral over $y$ is gained at large $y$, where the asymptotics $K_0(2y) \approx \sqrt{\pi}/4y \exp(-2y)$ works. Therefore

$$\int_0^\infty \frac{dy}{y^{\beta-1}} K_0(2y) I_{2m+1} (y) I_\nu (y)$$

$$\sim \frac{1}{\nu^{\beta-1/2}} \int_0^{\infty} \frac{dz}{z^{\beta-1/2} \sqrt{z^2 + 1}} \exp(2f),$$

$$f(z) = \sqrt{z^2 + 1} - z + \ln \frac{z}{1 + \sqrt{z^2 + 1}}.$$ 

Since the function $f(z)$ increases monotonically, the integral over $z$ is gained where $f \sim \nu^{-1}$ that is at $z \sim \nu$. That leads to an estimate

$$\int_0^\infty \frac{dy}{y^{\beta-1}} K_0(2y) I_{2m+1} (y) I_\nu (y) \sim \frac{1}{m^{2\beta-1}}.$$ 

The estimate means that the sum over $m$ converges at large $m$ since $\beta > 2$. Thus we come to the estimate (4.10).
APPENDIX B: CORRECTION TO THE COULOMB COUPLING CONSTANT

Here, we calculate the correction to the Coulomb coupling constant $\beta$ produced by small defect-antidefect pairs. We start from Eqs. (3.10,3.11). The correction can be found from the relation

$$\exp \left[ - \int dt \Delta H_2(\phi, \sigma) \right] = \exp \left\{ \int dt \frac{d^2 r}{r^2} \left[ -\nabla \psi_+ \psi_+ + \nabla \psi_- \psi_- \right] \nabla \phi \right. \left. + \left( \psi_+ \psi_+ - \psi_- \psi_- \right) \right\},$$

where the angular brackets mean averaging over the small defect-antidefect pairs and $\phi$, $\sigma$ in the exponent are to be treated as "external" fields. Really, one of the contributions to $\Delta H$ is depicted in Fig. 6 where the upper arrowed dashed line correspond to the field $\phi$ and the lower arrowed dashed line correspond to the field $\sigma$. The other contribution to $\Delta H$ is associated with a similar diagram where both "external" dashed line are attached to the same trajectory. It is explained in Sec. 3 what objects correspond to the different parts of the diagram. Here we present an analytical expression written in accordance with the two noted diagrams (see designations of the points in Fig. 6)

$$\Delta H_2 = 2 \int dt dt' d^2 x_1 d^2 x_2 d^2 x_3 d^2 x_4$$

$$\times \nabla_\alpha \phi(t_2, x_1) \sigma(t_3, x_4) - \sigma(t_3, x_3)$$

$$\times \nabla_\sigma \Phi_2(x_1 - x_2) M(t_3 - t_2, x_3, x_4, x_1, x_2),$$

where $\Phi_2$ and $M$ are defined by Eqs. (2.23,4.4). The explicit expressions for the functions are written in Eqs. (2.24,4.5).

We are interested in the situation when the "external" fields $\phi$ and $\sigma$ in Eq. (3.12) vary on scales much larger than a characteristic size of the defect-antidefect pair. Then one can substitute

$$\sigma(t_3, x_4) - \sigma(t_3, x_3) \rightarrow -\nabla \sigma(t_2, x_1)(x_{4_1} - x_{3_1}),$$

where we expanded the difference and then shift the argument of the factor $\nabla \sigma$. Substituting the expression into Eq. (3.12) and integrating over $\sigma = (x_3 + x_4)/2$ we get

$$\Delta H_2 = -2 \int dt \frac{d^2 r}{r^2} \nabla_\alpha \phi(t, r) \nabla \sigma(t, r) \nabla \phi$$

$$\times \int d\tau d^2 x d^2 r_0 x_\gamma \nabla_\alpha \Phi_2(r_0) S(\tau, x, r_0),$$

where $r_0 = x_1 - x_2$, $x = x_3 - x_4$, $t = t_2$, $\tau = t_3 - t_2$, $r = x_1$, and we substituted the expression (2.13). Next, the integral (3.13) is proportional to $R_{\alpha \beta}$ (due to averaging over angles). Taking into account also the expressions (3.10,3.11) we get

$$\Delta H_2 = 4\pi^2 \beta \int dt \frac{d^2 r}{r^2} \nabla \phi(t, r) \nabla \sigma(t, r)$$

$$\times \int d\tau d^2 x d^2 r_0 x^2 \Phi_2(r_0) S(\tau, x, r_0).$$

Comparing the expression with Eq. (3.11) we conclude that the integral in Eq. (B4) can be considered as a correction to the coefficient $1/(4\pi\beta)$. Thus we get

$$\Delta \beta = -16\pi^3 \beta^3 \int d\tau d^2 x d^2 r_0 x^2 \Phi_2(r_0) S(\tau, x, r_0).$$

Substituting here the expression (2.21) we obtain

$$\Delta \beta = -\frac{2\pi^2 \beta^3}{\nu} \int d\tau d^2 r_0 x^2 \Phi_2(r_0)$$

$$\times \left( \frac{r_0^2}{2x_0} \right) \beta$$

where $\nu = \sqrt{\beta^2 + 1}$ and we used the relation 6.623.3 from Ref. [37]. Now, remembering the expression (3.10), one can easily check that the integral over $r_0$ in Eq. (B6) converges both for small and large $r_0$. Therefore the integral is gained at $r_0 \sim x$. Thus, substituting Eq. (3.10), we get from Eq. (B4) the answer (3.17).

Now some words about generalization to the case of superfluid films discussed in Section 5. The new term in the right-hand side of Eq. (5.1) produces an additional contribution to the “Hamiltonian” of the system. Then the term (3.10) should be substituted by

$$H_4 = -\int d^2 r \left[ \hat{\psi}_+ \nabla_\alpha \psi_+ + \hat{\psi}_- \nabla_\alpha \psi_- \right] \nabla_\alpha \phi$$

$$+ \left( \psi_+ \nabla_\alpha \psi_+ + \psi_- \nabla_\alpha \psi_- \right) \gamma \epsilon_{\alpha \beta} \nabla_\beta \phi$$

$$+ \left( \hat{\psi}_+ \psi_+ - \hat{\psi}_- \psi_- \right).$$

The expression (3.13) means a modification of the triple vertices shown by the arrows in diagrams, see e.g. Fig. 6. Next, we can repeat the same steps as above starting from the Hamiltonian determined by Eq. (3.13). Then the intermediate expressions like Eqs. (2.23,4.4) will be modified. The terms which does not contain $\gamma$ are reduced to the conditional probability $S$ and give the same correction (B4) to $\beta$. The terms containing $\gamma$ do not produce relevant terms at all because of the identity

$$\int d^2 r \nabla_\sigma \epsilon_{\alpha \beta} \nabla_\beta \phi = 0.$$

Thus for the superfluid films we get the same answers (3.17).
APPENDIX C: RECURRENT PROCEDURE

We see from the expression (4.11) that the correlation function \( F_{2n} \) can be calculated step by step: first integrating over \( x_1, z_1 \), then over \( x_2, z_2 \) and so further. To formulate the procedure one should introduce auxiliary objects

\[
H_k(t_1, \ldots, t_k; r_1, \ldots, r_{k-1}; x_k, z_k) = \prod_{j=1}^{k-1} \int d^2x_j d^2z_j \left[ \delta(r_j - x_j) - \delta(r_j - z_j) \right] \times M(t_k - t_{k-1}, x_k, z_k, x_{k-1}, z_{k-1}) \ldots \times M(t_2 - t_1, x_2, z_2, x_1, z_1) \Phi_2(x_1 - z_1). \tag{C1}
\]

One can easily check that the functions \( H_k \) are symmetric under permuting \( x_k \) and \( z_k \) for odd \( k \) and are antisymmetric under the permutation for even \( k \). That depends on the number of factors containing differences of \( \delta \)-functions which are antisymmetric under permuting \( x_j \) and \( z_j \), whereas the factors \( M \) are symmetric under the permutation. The correlation function \( F_{2n} \) can be restored from \( H_k \):

\[
F_{2n} = 2 \int d^2 \varrho H_{2n}(t_1, r_1, \ldots, r_{2n-1}, r_{2n}; \varrho), \tag{C2}
\]

as follows from Eq. (4.11). Note that an attempt to calculate odd correlation functions \( F_{2n+1} \) from \( H_{2n+1} \) using the same scheme gives zero (as it should be) since \( H_{2n+1} \) is symmetric under permuting \( x_{2n+1} \) and \( z_{2n+1} \) whereas the difference \( \delta(r - x_{2n+1}) - \delta(r - z_{2n+1}) \) is antisymmetric under the permutation.

To obtain \( H_k \) we can use a recurrent scheme. First of all we conclude from Eq. (C1) that

\[
H_1(t, x, z) = \Phi_2(x - z). \tag{C3}
\]

Next, one can easily obtain from the definition (C1) the following recurrent relation

\[
H_{k+1}(t_1, \ldots, t_{k+1}; r_1, \ldots, r_k, x_{k+1}, z_{k+1}) = \int d^2 \varrho \int d^2 z_k \left[ \delta(r_k - x_k) - \delta(r_k - z_k) \right] \times M(t_{k+1} - t_k, x_{k+1}, z_{k+1}, x_k, z_k) \times H_k(t_1, \ldots, t_k; r_1, \ldots, r_{k-1}, x_k, z_k). \tag{C4}
\]

Taking into account the symmetry properties of \( H_k \) we can rewrite the relation (C4) as

\[
H_{2n+1}(t_1, \ldots, t_{2n+1}; r_1, \ldots, r_{2n}, x_{2n+1}, z_{2n+1}) = \int d^2 \varrho H_{2n}(t_1, \ldots, t_{2n-1}, r_1, \ldots, r_{2n-1}, r_{2n}, \varrho) \times \left[ M(t_{2n} - t_{2n-1}, x_{2n}, z_{2n}, r_{2n-1}) - M(t_{2n} - t_{2n-1}, x_{2n}, z_{2n}, r_{2n-1}, \varrho) \right], \tag{3.5}
\]

\[
H_{2n+1}(t_1, \ldots, t_{2n+1}; r_1, \ldots, r_{2n}, x_{2n+1}, z_{2n+1}) = \int d^2 \varrho H_{2n}(t_1, \ldots, t_{2n-1}, r_1, \ldots, r_{2n-1}, r_{2n}, \varrho) \times \left[ M(t_{2n+1} - t_{2n}, x_{2n+1}, z_{2n+1}, r_{2n}, \varrho) + M(t_{2n+1} - t_{2n}, x_{2n+1}, z_{2n+1}, r_{2n}, \varrho) \right]. \tag{3.6}
\]

Thus we can subsequently obtain the functions \( H_k \) and then get the correlation functions \( F_{2n} \) in accordance with Eq. (C2).

Now we are going to establish scaling properties of the correlation functions. As is seen from the recurrent relations (3.5) and Eq. (C2) the crucial point is convergence of integrals over \( \varrho \). There are no problems with finite \( \rho \) since one can conclude from Eq. (2.23) that \( M \) remains finite at any values of its space arguments. Indeed, it follows from Eq. (2.23) that the only dangerous case is \( r \to 0 \). Then the arguments of \( I_\nu \) also tend to zero. Substituting there the first terms of the expansion \( I_\nu(x) \approx (x/2)^\nu/\Gamma(1 + \nu) \) we conclude that the value of \( M \) remains finite at \( r \to 0 \). Therefore we should examine a behavior of the integrands in Eqs. (3.5) at large \( \varrho \). The behavior is dominated by the asymptotics (2.24).

First of all note that at large \( \varrho \)

\[
H_k(t_1, \ldots, t_k; r_1, \ldots, r_{k-1}, x_k, \varrho) \propto \varrho^{-2\beta}. \tag{3.7}
\]

The behavior (3.7) is characteristic of \( \Phi_2 \), that is characteristic of \( H_1 \), see Eqs. (C3). Then the behavior is reproduced at each step of the recurrent procedure (3.5). Indeed, in the first terms in Eqs. (3.5) \( M \approx 1 \) if \( \varrho \) is close to \( x \) and \( M \) exponentially decays at increasing \( \varrho - x \). Therefore at large \( x \) the integrals (3.5) over \( \varrho \) are gained near \( x \), that leads to the above assertion. The asymptotic behavior (3.7) ensures convergence of the integral (C2) determining the correlation functions.

Thus there are no divergences in the integrals at all steps of calculating \( F_{2n} \) in accordance with (3.5) and (C2).

[1] V. L. Berezinskii, ZhETF 59, 907 (1970) [Sov. Phys. JETP 32, 493 (1971)]; ZhETF 61, 1144 (1971) [Sov. Phys. JETP 34, 610 (1972)].
[2] J. M. Kosterlitz and D. J. Thouless, J. Phys. C5, L124 (1972); J. Phys. C6, 1181 (1973).
[3] J. M. Kosterlitz and D. J. Thouless, in Progress in Low Temperature Physics, ed. D. F. Brewer, v. VII B, p. 373 (North-Holland, Amsterdam, 1978).
FIG. 1. Trajectories passing through given points.
FIG. 2. Possible and impossible trajectories passing through four points at a given time moment.

FIG. 3. Typical diagram block.

FIG. 4. Typical diagram for the conditional probability $M$. 
FIG. 5. Vicinities of Creation and Annihilation Points

FIG. 6. Typical diagram contributing to renormalization of the effective “dielectric constant”.

FIG. 7. A more complicated diagram giving a correction to the Coulomb interaction.
FIG. 8. Illustration to the renormalization of the diffusion coefficient.

FIG. 9. Trajectories contributing to the pair correlation function.

FIG. 10. Trajectories passing through a number of points.

FIG. 11. A number of defect-antidefect pairs passing through given points.