Some ‘Converses’ to Intrinsic Linking Theorems

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Abstract

A low-dimensional version of our main result is the following ‘converse’ of the Conway–Gordon–Sachs Theorem on intrinsic linking of the graph $K_6$ in 3-space:

For any integer $z$ there are six points $1, 2, 3, 4, 5, 6$ in 3-space, of which every two $i, j$ are joined by a polygonal line $ij$, the interior of one polygonal line is disjoint with any other polygonal line, the linking number of any pair of disjoint 3-cycles except for \{123, 456\} is zero, and for the exceptional pair \{123, 456\} is $2z + 1$. We prove a higher-dimensional analogue, which is a ‘converse’ of a lemma by Segal–Spież.

Keywords Intrinsic linking · Linking number · Embedding · Almost embedding · Deleted product

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1 Introduction and Main Result

We start with a low-dimensional intrinsic linking result (Theorem 1.1), its higher-dimensional generalization (Lemma 1.3), and a low-dimensional analogue (Proposition 1.2) of our main result (Theorem 1.4).
Disjoint closed polygonal lines $L_1, L_2$ in $\mathbb{R}^3$ (or, more generally, disjoint self-intersecting $k$-sphere and $\ell$-sphere in $\mathbb{R}^{k+\ell+1}$) are linked modulo 2 if a general position singular cone over $L_1$ intersects $L_2$ at an odd number of points; see [12, §77] and [23, §4].

**Theorem 1.1** (Conway–Gordon–Sachs [1, 9]). For any piecewise linear (PL) embedding $K_6 \to \mathbb{R}^3$ there are two disjoint cycles in $K_6$ whose images are linked modulo 2.

For a survey on ‘intrinsic linking’ results see e.g. [20] and the references therein.

The linking number $\text{lk} \in \mathbb{Z}$ of disjoint oriented closed polygonal lines in $\mathbb{R}^3$ (or, more generally, of disjoint oriented self-intersecting $k$-sphere and $\ell$-sphere in $\mathbb{R}^{k+\ell+1}$), is defined in [12, §77] and [23, §4]. For non-oriented closed polygonal lines (or singular spheres) the absolute value $|\text{lk}|$ is well defined.

This paper is motivated by finding a gap [18, §3] in the proof that embeddability is undecidable in codimension $> 1$ [2]. Theorem 1.1 and its higher-dimensional generalization (Lemma 1.3) give cycles (or spheres) linked modulo 2, i.e., having odd linking number. The gap was in trying to improve those results to get linking number $\pm 1$, not just odd. Our main result (Theorem 1.4) shows that this is not possible.

The following ‘converse’ of Theorem 1.1 shows that the existence of two cycles with odd linking number cannot be replaced by the existence of two cycles with $\pm 1$ linking number.

**Proposition 1.2** (proved in §2). For any integer $z \geq 0$ there exists a PL embedding $K_6 \to \mathbb{R}^3$ such that

- the image of any 3-cycle is unknotted,
- for any disjoint 3-cycles in $K_6$ except one pair the linking number of their images is zero, and
- for the exceptional pair of disjoint 3-cycles we have $|\text{lk}| = 2z + 1$.

A complex is a collection of closed simplices (= faces) of some simplex. (We abbreviate ‘finite simplicial complex’ to ‘complex’.) A $k$-complex is a complex containing at most $k$-dimensional simplices. The body (or geometric realization) $|K|$ of a complex $K$ is the union of simplices of $K$. Thus continuous or piecewise-linear (PL) maps $|K| \to \mathbb{R}^d$ and continuous maps $|K| \to S^m$ are defined. Below we abbreviate $|K|$ to $K$; no confusion should arise.

A map $g : K \to \mathbb{R}^d$ of a complex $K$ is called an almost embedding if $g\alpha \cap g\beta = \emptyset$ for any two disjoint simplices $\alpha, \beta \subset K$.

**Lemma 1.3** [11, Lem. 1.4] For any integers $0 \leq \ell \leq k$ there is a complex $F_-$ of dimension $\max\{k, \ell + 1\}$ containing disjoint subcomplexes $\Sigma^k \cong S^k$ and $\Sigma^\ell \cong S^\ell$, PL embeddable into $\mathbb{R}^{k+\ell+1}$ and such that for any PL almost embedding $f : F_- \to \mathbb{R}^{k+\ell+1}$ the images $f\Sigma^k$ and $f\Sigma^\ell$ are linked modulo 2.

Let us define $F_-, \Sigma^k$, and $\Sigma^\ell$ of Lemma 1.3. For this, define a complex $F = F_{k,\ell}$ (this is $P(k, \ell)$ of [11]). Let $[n] := \{1, 2, \ldots, n\}$. The vertex set is $[k + \ell + 3] \cup \{0\}$.

The simplices are formed by all the simplices of dimension at most $k$ of $[k + \ell + 3]$,
and all the simplices of dimension at most \( \ell + 1 \) that contain 0. In other words,

\[
F_{k, \ell} := \left( [k + \ell + 3] \cup \{0\}, \left( \begin{array}{c} k + \ell + 3 \\ \leq k + 1 \end{array} \right) \cup \{0\} \cup \sigma : \sigma \in \left( \begin{array}{c} [k + \ell + 3] \\ \leq \ell + 1 \end{array} \right) \right).
\]

Here \( \binom{[n]}{\leq m} \) is the set of all subsets of \([n]\) having at most \(m\) elements.

**Comment** Observe that \( F_{1,0} \) is the non-planar graph \( K_5 \). More generally, \( F_{k,k-1} \) is the \( k \)-skeleton of the \((2k+2)\)-simplex, which is not embeddable into \( \mathbb{R}^{2k} \). We have \( F_{k,k} = \text{Conv} \, F_{k,k-1} \). The complex \( F_{k,\ell} \) is not embeddable into \( \mathbb{R}^{k+\ell+1} \) for \( 0 \leq \ell \leq k \) by Lemma 1.3 (see the following definition of \( \Sigma^{\ell}_{1}, \Sigma^{k}_{1}, \) and \( F_{-} \)).

Let \( \Delta^{\ell+1} \subset F \) be the \((\ell+1)\)-simplex with the vertex set \( \{0, 1, 2, \ldots, \ell + 1\} \) and \( \Sigma^{\ell} = \partial \Delta^{\ell+1} \). Let \( \Sigma^{k} \subset F \) be the boundary sphere of the \((k+1)\)-simplex with the vertex set \( \{\ell + 2, \ell + 3, \ldots, k + \ell + 3\} \). Finally, define

\[
F_{-} = F_{k,\ell,-} := F - \text{Int} \, \Delta^{\ell+1}.
\]

Our main result (Theorem 1.4) shows that in Lemma 1.3 the linking number being odd cannot be replaced by the linking number being \( \pm 1 \). For a PL almost embedding \( f : F_{-} \to \mathbb{R}^{k+\ell+1} \) we have \( f \Sigma^{k} \cap f \Sigma^{\ell} = \emptyset \), so denote

\[
|\text{lk} \, f| := |\text{lk} \, (f \Sigma^{k}, f \Sigma^{\ell})| \in \mathbb{Z}.
\]

**Theorem 1.4** (proved in §2). For any integers \( 1 \leq \ell \leq k \) and \( z \geq 0 \) there is a PL almost embedding \( f : F_{-} \to \mathbb{R}^{k+\ell+1} \) such that \( |\text{lk} \, f| = 2z + 1 \).

**Open Problems**

The paper [3] announces that the analogue for \((k, \ell) = (1, 0)\) of Theorem 1.4 is wrong. It would be interesting to know if the analogue for \( \ell = 0, k \geq 2 \) of Theorem 1.4 holds. It would be interesting to know if

- for any graph \( K \) there is a PL embedding \( K \to \mathbb{R}^{3} \) such that for any disjoint cycles in \( K \) the linking number of their images is different from \( \pm 1 \);
- for any integers \( 0 < \ell < k \) and a \( k \)-complex \( K \) there is a PL embedding (or almost embedding) \( K \to \mathbb{R}^{k+\ell+1} \) such that for any disjoint \( k \)-sphere and \( \ell \)-sphere in \( K \) the linking number of their images is different from \( \pm 1 \);
- for any integers \( 1 < \ell \leq k \) and a \( k \)-complex \( K \) there is a PL embedding (or almost embedding) \( K \to \mathbb{R}^{k+\ell} \) such that for any disjoint \( k \)-simplex and \( \ell \)-simplex in \( K \) the intersection of their images is not transversal.

A negative answer would be a natural integer-valued generalization of Theorem 1.1 and Lemma 1.3. It would be interesting to know if ‘almost’ can be deleted from Theorem 1.4. For \( k < 2\ell \) this follows by Theorem 1.4 and Proposition 3.2(a). For \( k \geq 2\ell \) this can perhaps be proved analogously to Theorem 1.4, using Lemma 3.1(b) and the method of [13, §2], see the survey [8, §10].
Problem 1.5 Take fixed $d, k$ such that $8 \leq d \leq (3k + 1)/2$. Is there an algorithm recognizing PL almost embeddability of $k$-complexes in $\mathbb{R}^d$?

Theorem 1.4 allows to deduce a negative answer to Problem 1.5 (cf. [2] and [18, §3]) for the ‘extreme’ case $2d = 3k + 1 = 6\ell + 4$, $\ell$ even, from [18, 22, Conjecture 3.11(b)], see details in [18, end of §3]. Analogously one deduces such an answer from [18, 22, Conjecture 3.11(a)]. Analogously one deduces undecidability for embeddings from a version of Theorem 1.4 without ‘almost’ and [18, 22, Conjecture 3.11(a)].

More Information on Intrinsic Linking

Remark 1.6 (a) Lemma 1.3 is an important step in the proof of the important results [7, 10, 11, 24] on incompleteness of the deleted product criterion for embeddability of $k$-complexes in $\mathbb{R}^d$ for $2d < 3k + 3$ (Proposition 3.2(a)), and on NP-hardness of recognition of (almost) embeddability of $k$-complexes in $\mathbb{R}^d$ for $2d < 3k + 3$.

(b) The proof of Lemma 1.3 in [11, §1] uses the cohomological Smith index; a simpler argument by application of [24, Lemma 6] is presented after Theorem 1.7. See also [4, Rem. 1.6, (c) and (d)].

Theorem 1.4 can be regarded as a ‘converse’ also to the following ‘intrinsic linking’ result, Theorem 1.7. This result is a non-trivial generalization of Theorem 1.1, of Remark 2.2(a), and of their analogues for $k$-complexes in $\mathbb{R}^{2k+1}$, as well as a simple generalization of Lemma 1.3 and of [24, Lem. 5].

Theorem 1.7 For any integers $0 \leq \ell \leq k$ there is a $k$-complex $F' = F'_{k,\ell}$ containing disjoint subcomplexes $\Sigma^k_j \cong S^k$ and $\Sigma^\ell_j \cong S^\ell$, $j \in \binom{[k+\ell+3]}{k+2}$, PL embeddable into $\mathbb{R}^{k+\ell+1}$ and such that for any PL almost embedding $f : F' \to \mathbb{R}^{k+\ell+1}$ the number of linked modulo 2 unordered pairs of the images $f\Sigma^k_j$ and $f\Sigma^\ell_j$ is odd.

The above-mentioned analogues for $k$-complexes in $\mathbb{R}^{2k+1}$ are obtained by taking in Theorem 1.7, $k = \ell$ and $F'$ the $k$-skeleton of the $(k+\ell+3)$-simplex. They are proved in [5, 25], see survey [17, 21, §4]. The index argument of [11, §1] (see Remark 1.6(b)) has a simple generalization to Theorem 1.7; thus the analogues are implicit in [11].

Sketch of a proof of Theorem 1.7 Take $F'$ to be the complex whose vertex set is $[k + \ell + 3] \cup \{0\}$, and whose simplices are formed by all the simplices of dimension at most $k$ of $[k + \ell + 3]$, and all the simplices of dimension at most $\ell$ that contain 0. For $j \in \binom{[k+\ell+3]}{k+2}$ let

- $\Sigma^k_j \subset F'$ be the boundary sphere of the $(k + 1)$-simplex with the vertex set $j$;
- $\Sigma^\ell_j \subset F'$ be the boundary sphere of the $(\ell + 1)$-simplex with the vertex set $\{0\} \cup ([k + \ell + 3] - j)$.

Now the proof is analogous to the following proof of Lemma 1.3, and so to [24, §3].

Proof of Lemma 1.3 For a complex $K$, a general position PL map $f : K \to \mathbb{R}^d$ and $\dim K < d$ define the van Kampen number $v(f) \in \mathbb{Z}_2$ to be the parity of the number

\[ \text{connected components of ~} f^{-1}(y) \\text{~for all ~} y \in \mathbb{R}^d. \]
of points \( x \in \mathbb{R}^d \) such that \( x \in f(\sigma) \cap f(\tau) \) for some disjoint simplices \( \sigma, \tau \in K \) with \( \dim \sigma + \dim \tau = d \). The lemma follows because \( v(f) = 1 \) for any general position PL map \( f : F \to \mathbb{R}^{k+\ell+1} \). For some \( f \) this is Lemma 3.1(a) below. Then for any \( f \) this holds by the following [24, Lem. 6] (verification of its assumptions is analogous to [24, Lem. 7]): Let \( d \) be an integer and \( K \) a finite complex such that for every pair \( \sigma, \tau \) of disjoint oriented cycles \( f \) that \( lk(\sigma, \tau) = 0 \). For any disjoint oriented cycles \( \sigma, \tau \in K \) with \( s + t = d - 1 \) the following two numbers have the same parity:
- the number of \((s+1)\)-simplices \( v \) containing \( \sigma \) and disjoint with \( \tau \);
- the number of \((t+1)\)-simplices \( \mu \) containing \( \tau \) and disjoint with \( \sigma \).

Then \( v(f) \) is independent of a general position PL map \( f : |K| \to \mathbb{R}^d \). \( \square \)

2 Proofs of Proposition 1.2 and Theorem 1.4

For any disjoint oriented cycles \( \sigma, \tau \) in \( K_6 \), put \( \text{lk}_f(\sigma, \tau) := \text{lk}(f(\sigma), f(\tau)) \in \mathbb{Z} \). Observe that \( \text{lk}_f(\sigma, \tau) = \text{lk}(\tau, \sigma) \), so assume that the argument of \( \text{lk}_f \) is an unordered pair.

For an oriented edge \( c \) of \( K_6 \) issuing out of vertex \( A \) and going to vertex \( B \), and a vertex \( C \neq c \) denote by \( cC \) the oriented cycle \( CA \cup c \cup BC \) in \( K_6 \).

**Lemma 2.1** Let \( a, b \) be disjoint oriented edges of \( K_6 \) and \( f : K_6 \to \mathbb{R}^3 \) a PL embedding such that any 3-cycle in \( f(K_6) \) is unknotted. Then there is a PL embedding \( g : K_6 \to \mathbb{R}^3 \) such that any 3-cycle in \( g(K_6) \) is unknotted, for the remaining vertices \( P, Q \) of \( K_6 \) we have

\[
\text{lk}_f(aP, bQ) - \text{lk}_g(aP, bQ) = \text{lk}_f(aQ, bP) - \text{lk}_g(aQ, bP) = +1
\]

and \( \text{lk}_f(\sigma, \tau) = \text{lk}_g(\sigma, \tau) \) for any other unordered pair \( \sigma, \tau \).

**Proof** Informally, we obtain \( g \) by turning \( f(a) \) around \( f(b) \) once. Let us present an accurate construction. Take a point \( O \in \mathbb{R}^3 \) and general position arcs \( Of(V) \) joining \( O \) to the images of the vertices of \( K_6 \). For an oriented edge \( c \) of \( K_6 \) issuing out of vertex \( A \) and going to vertex \( B \) denote by \( Of(c) \) the oriented cycle \( Of(A) \cup f(c) \cup f(B)O \). Take an embedded oriented 2-disc \( \delta \subset \mathbb{R}^3 \) such that \( \delta \cap f(K_6) \) is the union of
- a point \( f(b) \cap \delta \subset \text{Int} \delta \) of sign +1, and
- an arc \( f(a) \cap \partial \delta \) at which the orientations from \( f(a) \) and from \( \partial \delta \) are the opposite.

Define \( g : K_6 \to \mathbb{R}^3 \) by ‘pushing a finger along \( \delta \’, i.e., so that
- \( g(\text{f}^{-1}(\partial \delta)) = \text{Cl}(\partial \delta - f(a)) \),
- \( g = f \) outside \( \text{f}^{-1}(\partial \delta) \),
- \( \text{lk}(Of(a), Of(b)) - \text{lk}(Og(a), Of(b)) = +1 \), and
- \( \text{lk}(Of(a), Of(c)) = \text{lk}(Og(a), Of(c)) \) for any oriented edge \( c \notin \{a, b\} \).

Since any 3-cycle in \( f(K_6) \) is unknotted, \( f(K_6) \cap \delta \subset f(a \cup b) \) and no 3-cycle in \( K_6 \) containing \( a \) contains \( b \), any 3-cycle in \( g(K_6) \) is unknotted.

Observe that \( \text{lk}_f(\sigma, \tau) \) is equal to the sum of nine summands of the form \( \text{lk}(Of(d), Of(e)) \), where \( d \) and \( e \) are oriented edges of \( \sigma \) and \( \tau \). Hence

\[
\text{lk}_f(aP, bQ) - \text{lk}_g(aP, bQ) = \text{lk}(Of(a), Of(b)) - \text{lk}(Og(a), Of(b))
\]
The relation $\text{lk}_f(aQ, bP) - \text{lk}_g(aQ, bP) = +1$ follows by exchanging $P$ and $Q$. Analogously $\text{lk}_f(\sigma, \tau) = \text{lk}_g(\sigma, \tau)$ for any other unordered pair $[\sigma, \tau]$.

\textbf{Proof of Proposition 1.2} Denote the vertices of $K_6$ by 1, 2, \ldots, 6. It is known that there is a PL embedding $f: K_6 \to \mathbb{R}^3$ such that any 3-cycle in $f(K_6)$ is unknotted, $\text{lk}_f(123, 456) = +1$ and $\text{lk}_f(\sigma, \tau) = 0$ for any other unordered pair $\sigma, \tau$ of disjoint oriented cycles in $K_6$. Make the modification of Lemma 2.1 for $(aP, bQ) = (123, 456), (162, 435), (234, 561)$. We have that $\text{lk}_f(ijk, pqr) = \text{lk}_f(jki, pqr) = - \text{lk}_f(jik, pqr)$ whenever $[6] = \{i, j, k, p, q, r\}$. Hence the resulting change of the symmetric matrix $\text{lk}_f$ is

\[
(\{123, 456\} + \{126, 453\}) + (\{162, 345\} + \{165, 342\})
+ (\{561, 234\} + \{564, 231\}) = 2\{123, 456\}.
\]

Thus making the same modification $z$ times we obtain the required PL embedding. \qed

\textbf{Remark 2.2} (a) Theorem 1.1 was proved in the following stronger form: For any piecewise linear (PL) embedding $K_6 \to \mathbb{R}^3$ the number of linked modulo-2 unordered pairs of images of two disjoint cycles in $K_6$ is odd.

(b) Part (a) is implied by the following assertion (see proof in [4, Rem. 2.2(b)])[a]) and the known fact stated at the beginning of the proof of Proposition 1.2 (this is essentially the standard argument). For any two PL embeddings $f, g: K_6 \to \mathbb{R}^3$ the symmetric matrix $\text{lk}_f$ can be obtained from the symmetric matrix $\text{lk}_g$ by several transformations described in Lemma 2.1.

(c) Proposition 1.2 shows that there are no linear relations or congruences on numbers $\text{lk}_f(\sigma, \tau)$ except (a).

\textbf{Proof of Theorem 1.4} If a closed polygonal line $L \subset \mathbb{R}^3$ is unknotted then the fundamental group of the complement is $\mathbb{Z}$. Hence a closed polygonal line in $\mathbb{R}^3 \setminus L$ is null-homotopic if and only if it is null-homologous, i.e., if and only if it has zero linking number with $L$. So if two closed polygonal lines in $\mathbb{R}^3$ have zero linking number and the second of them is unknotted then the first of them spans a mapped 2-disk disjoint from the second one. Hence the inductive base $k = \ell = 1$ follows by Proposition 1.2.

Let us prove the inductive step. If $k > 1$, then either $k > \ell$ or $\ell > 1$. If $k > \ell$, observe that

$$F_{k, \ell} = F_{k-1, \ell} \cup \text{Con}(F_{k-1, \ell} \cap F_{k, \ell-1}) \cup \binom{k + \ell + 2}{k + 1},$$

where the vertex of the cone is $k + \ell + 3$. The same formula is correct with $F_{k, \ell}$, $F_{k-1, \ell}$ replaced by $F_{k, \ell-1}$, $F_{k-1, \ell-1}$. Since $k > \ell$, by the inductive hypothesis there is a PL almost embedding $f: F_{k-1, \ell-1} \to \mathbb{R}^{k+\ell}$ such that $|\text{lk}_f| = 2z + 1$. Extend it to a map $f': F_{k, \ell-1} \to \mathbb{R}^{k+\ell+1}$ as follows. Extend $f$ conically over the cone, with the vertex in

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1 We are grateful to Florian Frick for allowing us to present this proof based on an idea he suggested. Alternative (earlier) proofs are presented in §3 and in [4, §4–5].
the upper half-space of $\mathbb{R}^{k+\ell+1}$ w.r.t. $\mathbb{R}^{k+\ell}$. Map the $k$-faces of $\binom{k+\ell+2}{k+1}$ to the lower half-space of $\mathbb{R}^{k+\ell+1}$ w.r.t. $\mathbb{R}^{k+\ell}$. Since $k > \ell$, we have $2(k+1) > k+\ell+2$, so any two such $k$-faces intersect. Thus the extension $f'$ is a PL almost embedding. Since $f'\Sigma^k$ is the ‘suspension’ over $f \Sigma^{k-1}$ and $f' = f$ on $\Sigma^k$, we have $|\text{lk } f'| = |\text{lk } f| = 2z + 1$.

If $\ell > 1$, observe that

$$F_{k,\ell} = F_{k,\ell-1} \cup \text{Con}(F_{k,\ell-1} \cap F_{k-1,\ell}) \cup 0\ast \left(\binom{k+\ell+2}{\ell+1}\right),$$

where the vertex of the cone is $k+\ell+3$. The complex $F_{k,\ell-1}$ is obtained from the above union by deleting the $(\ell+1)$-simplex $0\ast[\ell+1]$ and adding the $\ell$-simplex $0\ast[\ell]$. Since $\ell > 1$, by the inductive hypothesis there is a PL almost embedding $f : F_{k,\ell-1,-} \to \mathbb{R}^{k+\ell}$ such that $|\text{lk } f| = 2z + 1$. Extend it to a map $f' : F_{k,\ell,-} \to \mathbb{R}^{k+\ell+1}$ as follows. Extend $f$ conically over the cone, with the vertex in the upper half-space of $\mathbb{R}^{k+\ell+1}$ w.r.t. $\mathbb{R}^{k+\ell}$. Map the $(\ell+1)$-faces of $0\ast\binom{k+\ell+2}{\ell+1}$ (except $0\ast[\ell+1]$) and the $\ell$-face to the lower half-space of $\mathbb{R}^{k+\ell+1}$ w.r.t. $\mathbb{R}^{k+\ell}$. Any two such $(\ell+1)$- or $\ell$-faces intersect at 0. Thus the extension $f'$ is a PL almost embedding. Since $f'\Sigma^\ell$ is the ‘suspension’ over $f \Sigma^{\ell-1}$ and $f' = f$ on $\Sigma^k$, we have $|\text{lk } f'| = |\text{lk } f| = 2z + 1$. \hfill \Box

3 Alternative Proof of Theorem 1.4

Let $\Delta^k \subset \Sigma^k$ be the $k$-simplex with the vertex set $\{\ell+3, \ell+4, \ldots, k+\ell+3\}$. (So that $\Delta^k \neq \Delta^{\ell+1}$ even when $k = \ell + 1$.)

Lemma 3.1 For any integers $0 \leq \ell \leq k$ there is

(a) a PL map $g : F \to \mathbb{R}^{k+\ell+1}$ whose self-intersection set consists of two points, one in $\text{Int } \Delta^{\ell+1}$ and the other in $\text{Int } \Delta^k$, so that the images of these interiors intersect transversally [11, Lem. 1.1].

(b) a PL embedding $f : F_- \to \mathbb{R}^{k+\ell+1}$ such that $|\text{lk } f| = 1$.

Part (b) follows from (a).

The simplicial deleted product of a complex $K$ is

$$K_\Delta^{\times 2} := \bigcup \{\sigma \times \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset\}.$$ 

For a complex $K$, a map $g : K \to \mathbb{R}^d$, and an equivariant subset $G \subset K^2$ such that $g(x) \neq g(y)$ for each $(x, y) \in G$ define an equivariant map

$$g^{\times 2}_\Delta : G \to S^{d-1} \quad \text{by} \quad g^{\times 2}_\Delta(x, y) := \frac{g(x) - g(y)}{|g(x) - g(y)|}.$$ 

If $g$ is an almost embedding, we assume that $G = K_\Delta^{\times 2}$. \footnote{The condition $\ell < k$ is present in [11, Lem. 1.1] but is not used in the proof.}
Proposition 3.2 Let \( d \) be an integer and \( K \) a \( k \)-complex such that either
(a) \( 2d \geq 3k + 3 \); or
(b) \( d \geq k + 2 \) and \( 2d - k - 3 \geq \dim \alpha + \dim \beta \) for any disjoint simplices \( \alpha, \beta \subset K \).

For any equivariant map \( \Phi : K^{\times 2} \to S^{d-1} \) there is a PL almost embedding \( f : K \to \mathbb{R}^d \) such that \( f^{\times 2} \) is equivariantly homotopic to \( \Phi \).

Part (a) is a celebrated result of Weber [26], see survey [15, 16, §5]. Part (b) works for \( 2d < 3k + 3 \) and is an easy corollary of the generalization [14, Disjunction Thm. 3.1]. Part (b) in some sense generalizes Theorem 1.4, see [4, Rem. 3.6(a)].

Proof of Proposition 3.2(b) Apply [14, Disjunction Thm. 3.1] to \( N = |K| \), \( T = K \), \( A = \emptyset \), \( E_1 = K^{\times 2} \), \( E_0 = \emptyset \), \( h_0 \) any PL map, and the given map \( \Phi \). Let \( f \) be the obtained map \( h_1 \). Then by [14, (3.1.1)] \( f \) is an almost embedding. By [14, (3.1.2)] \( f^{\times 2} \) is equivariantly homotopic to \( \Phi \).

Lemma 3.3 For any integers \( 0 < \ell < k \) and \( z \) there is an equivariant map \( \Phi : (N-1)^{\times 2} \to S^{k+\ell} \) such that \( \deg \Phi|_{\Sigma^k \times \Sigma^\ell} = 2z + 1 \).

Proof of Theorem 1.4 for \( 1 < \ell < k \) modulo Lemma 3.3 Theorem 1.4 for \( 1 < \ell < k \) follows from Lemma 3.3 and Proposition 3.2(b) because \( f^{\times 2} \) is homotopic to \( \Phi \) on \( \Sigma^k \times \Sigma^\ell \), so \( |\text{lk } f| = 2z + 1 \) [19].

The simplicial deleted join of a complex \( K \) is
\[
K^{\times 2}_\Delta := \bigcup \{ \sigma \ast \tau : \sigma, \tau \in K, \sigma \cap \tau = \emptyset \}.
\]

Lemma 3.4 For any integers \( 0 \leq \ell < k \) we have \( F^{\times 2}_\Delta \cong_{\mathbb{Z}_2} S^{k+\ell+2} \).

Proof A subset \( \sigma \subset \{0, 1, \ldots, k + \ell + 3\} \) is a face of \( F \) if and only if the complement \( \overline{\sigma} \) is not a face of \( F \). Indeed,
- if \( 0 \in \sigma \), then both claims are equivalent to \( |\sigma| \leq \ell + 2 \);
- if \( 0 \notin \sigma \), then both claims are equivalent to \( |\sigma| \leq k + 1 \).

This property (\( F \) is Alexander dual to itself) implies that \( F^{\times 2}_\Delta \cong_{\mathbb{Z}_2} S^{k+\ell+2} \) by a result of Bier [6, Def. 5.6.1 and Thm. 5.6.2].

Lemma 3.5 (a) Any two \((k + \ell + 1)\)-cells of \( F^{\times 2}_\Delta \) can be joined by a sequence of \((k + \ell + 1)\)-cells of \( F^{\times 2}_\Delta \) in which any two consecutive \((k + \ell + 1)\)-cells have a common \((k + \ell)\)-cell.
(b) Any \((k + \ell)\)-cell of \( F^{\times 2}_\Delta \) belongs to precisely two \((k + \ell + 1)\)-cells of \( F^{\times 2}_\Delta \).
(c) There is a collection of orientations on \((k + \ell + 1)\)-cells of \( F^{\times 2}_\Delta \) such that for any \((k + \ell)\)-cell of \( F^{\times 2}_\Delta \) the orientations on the two adjacent \((k + \ell + 1)\)-cells of \( F^{\times 2}_\Delta \) induce the opposite orientations on the \((k + \ell)\)-cell.
(d) For the orientations on \((k + \ell + 1)\)-cells of \( F^{\times 2}_\Delta \) given by (b) the exchange \( \pi(x, y) := (y, x) \) of the factors acts on the orientations as multiplication by \((-1)^{k+\ell} \).
Proof This follows by Lemma 3.4 because the formula $\sigma \times \tau \mapsto \sigma \ast \tau$ defines a 1–1 correspondence between $(p + 1)$-cells of $F^2_\Delta$ and $p$-cells of $F^2 \Delta$, $p > 0$, which respects adjacency and orientation. For part (d) we also need that the antipodal involution of $S^{k+\ell+2}$ multiplies the orientation by $(-1)^{k+\ell+1}$. □

Proof of Lemma 3.3 Denote by $F^2_\Delta$ the $(k + \ell)$-skeleton of $F^2_\Delta$. Take a map $g$ given by Lemma 3.1(a). Then $g^2_\Delta : F^2_\Delta \to S^{k+\ell}$ is defined. We have that $\deg g^2_\Delta |_{\Sigma_k \times \Sigma_\ell} = \pm 1$.

Informally, the lemma now follows because $(F_-)_\Delta$ is obtained from the connected pseudomanifold $F^2$ by deleting two codimension 0 submanifolds $\Delta^{\ell+1} \times \Sigma^k$ and $\Sigma^k \times \Delta^{\ell+1}$, which go one to the other under the exchange of factors.

Formally, we shall modify the map $g^2_\Delta$ as follows. For an integer $a$, an equivariant map $\Psi : F^2_\Delta \to S^{k+\ell}$ and oriented $(k + \ell)$-cell $V$ of $F$ denote by $\Psi_{V,a} : F^2_\Delta \to S^{k+\ell}$ any equivariant map obtained by the following construction (in fact, this construction produces a map $\Psi_{V,a}$ well-defined up to homotopy). Define $\Psi_{V,a} |_V$ to be the connected sum of $\Psi |_V$ and a map $S^{k+\ell} \to S^{k+\ell}$ of degree $a$. (In other words, define $\Psi_{V,a} |_V$ to be the composition $V \xrightarrow{c} V \cup S^{k+\ell} \xrightarrow{\Psi_{V,a}} S^{k+\ell}$, where $c$ is the contraction of certain $(k + \ell - 1)$-sphere in the interior of $V$ and $\bar{a}$ is a map of degree $a$.) Define $\Psi_{V,a}(x,y) := -\Psi_{V,a}(y,x)$ for $(y,x) \in V$. Define $\Psi_{V,a} = \Psi$ elsewhere. We write that $\Psi_{V,a}$ is obtained from $\Psi$ by the modification $(V,a)$.

For oriented manifolds $A$ and $B$ of the same dimension denote $[A : B] = 1$ if $B \subset A$ and their orientations coincide, $[A : B] = -1$ if $B \subset A$ and their orientations are the opposite, and $[A : B] = 0$ otherwise (i.e., if $B \not\subset A$).

Clearly, $\deg \Psi_{V,a} |_A = \deg \Psi |_A$ for any $(k + \ell + 1)$-cell $A$ disjoint from $V \cup \pi V$. For a $(k + \ell + 1)$-cell $U \supset V$ we have

$$\deg \Psi_{V,a} |_U - \deg \Psi |_U = [\partial U : V] a.$$ 

By Lemma 3.5(d) and since the antipodal involution of $S^{k+\ell}$ multiplies the orientation by $(-1)^{k+\ell+1}$, we have

$$\deg \Psi_{V,a} |_{\partial U} - \deg \Psi |_{\partial U} = -[\partial U : V] a.$$ 

By Lemma 3.5(a) there is a sequence $\Delta^k \times \Delta^{\ell+1} = U_0, U_1, \ldots, U_m = \Delta^{\ell+1} \times \Delta^k$ of $(k + \ell + 1)$-cells of $F^2_\Delta$ such that $V_i := U_{i-1} \cap U_i$ is a $(k + \ell)$-cell for each $i = 1, 2, \ldots, m$. Take the above orientations on the $U_i$ and orient the $V_i$ so that $[\partial U_i : V_i] = 1$ for each $i = 1, 2, \ldots, m$. Denote by $\Phi : F^2_\Delta \to S^{k+\ell}$ any equivariant map obtained from $g^2_\Delta$ by the modifications $(V_1, -z), \ldots, (V_m, -z)$. Clearly, $\deg \Phi |_{\partial A} = \deg g^2_\Delta |_{\partial A}$ for any $(k + \ell + 1)$-cell $A \not\in \{U_0, U_m\}$. Then $\deg \Phi |_{\partial U_0} - \deg g^2_\Delta |_{\partial U_0} = 2z$. We have $\deg g^2_\Delta |_{\Sigma_k \times \Sigma_\ell} = \pm \deg g |_{F_-}$. If this degree is $+1$, then we are done. If this degree is $-1$, then we make additionally the same construction replacing $-z$ by $-1$. □
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