Quasinormal modes of magnetized black hole

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We investigate charged, massive scalar field around static, spherically symmetric black hole immersed into an external asymptotically uniform magnetic field $B$. It is shown that for given multipole number $\ell$ there are $2\ell + 1$ numbers of modes due to the Zeeman effect appearing by an interaction of the external magnetic and charged scalar fields introducing an effective mass of the scalar field $\mu_{\ell m} = \sqrt{\mu^2 - mqB}$ where $m$ is the azimuthal number and $q$ is the charge coupling constant. We calculate threshold value of effective mass in which quasinormal modes are arbitrarily long lived and beyond that value quasinormal modes vanish. In the case of $mqB < 0$ quasinormal modes are longer lived with larger oscillation frequencies. Whenever, magnetic and massive scalar fields satisfies condition $\mu_{\ell m}^2 < 0$, an instability appears, i.e., if $qB > 0$ or $qB < 0$ there is an instability for the values of azimuthal number $m > \mu^2/qB$ or $m < \mu^2/qB$, respectively.

I. INTRODUCTION

According to “no-hair theorem” black hole does not possess its own magnetic field. In the pioneering papers [1, 2] Ginzburg and Ozernoy have shown that during the gravitational collapse the dipolar magnetic moment of the star will decay vice proportional with time and estimation has been made more accurate for multipolar magnetic fields by Price [3]. However, the external magnetic field around the black hole can be generated by an accretion disc, or a rotating ring of matter that falls into the black hole, or a companion in binary systems containing neutron star or magnetar with strong magnetic field. The black hole immersed in an asymptotically uniform external magnetic field has been first considered by Wald [4] and several physical scenarios have been considered later studied, e.g., in [5–18]. It is generally accepted that a magnetic field is considered one of main sources of the most energetic processes around supermassive black holes at the center of galaxies, playing the role of “feeder” of the supermassive black hole by trapping dust near the galaxy’s center [19].

A complete detailed analysis of interaction between black hole and magnetic field generated by the accretion disc or companion object (it can be neutron star or magnetar with strong magnetic field) is complicated problem which requires numerical magnetohydrodynamic (MHD) simulations [20]. However, approximative methods are also very useful to draw picture of this phenomenon, by considering stationary magnetized black hole solutions in general relativity as was done by Wald [4] and Ernst [21].

The study of an interaction between scalar and electromagnetic fields is very interesting topic from theoretical and observational point of view. A comprehensive physical aspects of the theory of black holes in an external electromagnetic field are reviewed in [22, 23]. In the papers [24, 25] propagation of scalar field in the background of strongly magnetized black hole (or Ernst spacetime) has been studied and later it is considered for the massive scalar field in [26] in $G^{1/2}c^{-2}Br \ll 1$. It is shown that in the presence of the strong magnetic field the quasinormal modes are longer lived and have larger oscillation frequencies in both massless and massive scalar fields [27]. However, such approximation is not accurate enough in the strong magnetic and gravitational fields when $G^{1/2}c^{-2}Br \sim 1$. Moreover, it was shown in [28] that the spacetime local curvature created by the magnetic field $B$ can be neglected and it can be considered as test magnetic field if strength of the magnetic field satisfies the condition

$$B \ll B_M = \frac{c^4}{G^{3/2}M_\odot} \left( \frac{M_\odot}{M} \right) \sim 10^{19} \left( \frac{M_\odot}{M} \right) G . \tag{1}$$

On the other hand, observations have shown that the surface magnetic field of magnetar that is considered the most magnetic object known in the Universe is in the order of $B \approx 10^{15}G$ [29, 30] which is very small to change the geometry. Therefore, the Ernst solution of the field equations has more academic interest than the astrophysical relevance.

In the present paper we study massive and charged scalar field in the spacetime of Schwarzschild black hole immersed into the uniform magnetic field assuming that the magnetic field satisfies the condition (1). The paper is organized as follows. In section II we briefly describe Klein-Gordon equation for the massive and charged field in the presence of the electromagnetic field in the space-
time of static, spherically symmetric black hole. In section III we present quasinormal modes calculated by using the Wentzel-Kramers-Brillouin (WKB) and Leaver’s methods. Finally, we present some concluding remarks in section IV. Throughout the paper we use the geometric system of units \( c = G = \hbar = 1 \) and spacelike signature \((- , + , + , \tau)\).

## II. BASIC EQUATIONS

In this section we present equations of motion for the massive charged scalar field around the black hole immersed in the uniform magnetic field and examine a contribution of the external magnetic field on the quasinormal modes of the test scalar field. In the spherical coordinates \((t, r, \theta, \phi)\), line element of the Schwarzschild black hole is written as

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 ,
\]

with metric function

\[
f(r) = 1 - \frac{2M}{r} ,
\]

where \(M\) is total mass of the black hole.

In the paper [4] the electromagnetic field configuration in the vicinity of black hole that is immersed in the external asymptotically uniform magnetic field has been studied. By following that method we find the vector potential \(A_\mu\) of the electromagnetic field as

\[
A_\mu = \frac{1}{2} Br^2 \sin^2 \theta (0, 0, 0, 1) ,
\]

where \(B\) is the strength of uniform magnetic field.

The general relativistic form of Klein-Gordon equation for the massive and charged scalar field \(\Phi\) in presence of the electromagnetic field is given by

\[
g^{\alpha \beta} (\nabla_\alpha - iq A_\alpha) (\nabla_\beta - iq A_\beta) \Phi - \mu^2 \Phi = 0 ,
\]

where \(\mu\) is mass of the scalar field, \(q\) is the charge coupling constant between the scalar and electromagnetic fields, \(\nabla_\alpha\) is the covariant derivative.

Obviously, it is very difficult to separate variables in equation (5), however we can use the following physically reasonable assumptions which facilitate the problem:

- Lorentz gauge for the vector potential \(\nabla_\alpha A^\alpha = 0\);
- In weak interaction limit one can ignore high order terms starting \(q^2 B^2 \rightarrow 0\).

Then, equation (5) can be expressed in the following simple form:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 f \frac{\partial \Phi}{\partial r} \right) - \left( \frac{L^2}{r^2} + \mu^2 - qB L_z \right) \Phi - \frac{1}{f} \frac{\partial^2 \Phi}{\partial t^2} = 0 ,
\]

where \(L^2, L_z\) are square of the total orbital angular momentum and \(z\) component of the orbital angular momentum operators that have the following forms:

\[
L^2 = - \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \Phi}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Phi}{\partial \phi^2} \right] ,
\]

\[
L_z = - i \frac{\partial}{\partial \phi} .
\]

One can see that equation (6) can be separated into radial and angular variables parts if the wave function is chosen as harmonically time dependent form as

\[
\Phi(t, r, \theta, \phi) = \frac{1}{2\pi} \int d\omega e^{-i\omega t} \sum L Y_{\ell m}(\theta, \phi) ,
\]

where \(\omega\) is the frequency of quasinormal modes of the massive charged scalar field, \(L Y_{\ell m}(r, \omega)\) is radial part of the wave function. The angular part of the wave function \(Y_{\ell m}(\theta, \phi)\) is the spherical harmonic function which satisfies the following relations:

\[
L^2 Y_{\ell m}(\theta, \phi) = \ell(\ell + 1) Y_{\ell m}(\theta, \phi) ,
\]

\[
L_z Y_{\ell m}(\theta, \phi) = m Y_{\ell m}(\theta, \phi) ,
\]

where \(\ell\) is the multipole number \(\ell = 0, 1, 2, ...\) and \(m\) is the azimuthal number with range \(m \leq |\ell|\). Substituting expression (9) into (6) one can write the following stationary Schrödinger-like wave equation for the radial function:

\[
\left( \frac{d^2}{dx^2} + \omega^2 - V_{\text{eff}}(r) \right) R_{\ell m}(r, \omega) = 0 ,
\]

with the effective potential

\[
V_{\text{eff}}(r) = f(r) \left[ \frac{\ell(\ell + 1)}{r^2} + \frac{f'(r)}{r} + \mu_{\text{eff}}^2 \right] ,
\]

where

\[
dx = \frac{dr}{f(r)} , \quad \mu_{\text{eff}}^2 = \mu^2 - m qB .
\]

From the form of the effective potential one can realize that in addition to the squared mass of the scalar field, \(\mu^2\), there is another term \(-mqB\). This is due to the well-known Zeeman effect, which is the shift of energy of charged particle, \(q\), in the magnetic field, \(B\), due to the interaction of magnetic and scalar fields through the charge with an azimuthal momentum \(m\). Let us analyze the symmetry features of the effective potential through effective mass. Here all the parameters \(m, q,\) and \(B\) can be both positive and negative. Therefore, there are the following physically reasonable combinations of signatures of these parameters in that effective potential via effective mass remains invariant:

- \(q \rightarrow -q, \quad B \rightarrow -B\);
- \(m \rightarrow -m, \quad B \rightarrow -B\).
As we have mentioned before that the azimuthal number \( m \) accepts \( 2\ell + 1 \) values in the region \( -\ell \leq m \leq \ell \). Taking this into account one can see that the effective mass also splits into \( 2\ell + 1 \) parts as schematically shown in Fig. 1 and consequently, possible \( 2\ell + 1 \) numbers of the effective potentials appear.

In similar way, we can analyze the effective potential in the presence of the external magnetic field. From the expression (13) we can see that for the given multipole number \( \ell \), we get \( 2\ell + 1 \) different shapes of the effective potential. For simplicity, here we set the multipole number \( \ell = 2 \) which means that \(-2 \leq m \leq 2\). In Fig. 2 the radial dependence of the effective potential is illustrated. The black curve corresponds to uncharged scalar field (\( q = 0 \)), and for the charged scalar field (\( q \neq 0 \)) this curve is split into five different ones as shown in Fig. 2. Moreover, one should note that the massive scalar field which satisfies the relation \( \mu^2 = mqB \) has the same dynamics as massless one. Note that for some positive values of azimuthal number, \( m \), an instability may exist depending on the value of mass of the scalar field, \( \mu \), further away from the black hole.

Last term of the effective potential in expression (13) can be considered as induced mass due to the interaction of the scalar and magnetic fields and it plays very important role in calculation of quasinormal modes of the black hole which will be considered in next section III. Since, as we have mentioned before that \( m \) can also accept the negative values \((-\ell < m < \ell)\), there are the following scenarios related with interaction of the scalar and magnetic fields which cause either an increase or decrease in the value of the effective mass:

- \( mqB > 0 \) at \( m > 0 \), \( qB > 0 \) or \( m < 0 \), \( qB < 0 \);
- \( mqB < 0 \) at \( m > 0 \), \( qB < 0 \) or \( m < 0 \), \( qB > 0 \).

In the papers [24–26] the quasinormal modes of uncharged scalar field in the Ernst (strongly magnetized black hole) spacetime were studied and it was shown that in the \( G^{1/2}c\rightarrow 2Br \ll 1 \) approximation, the scalar field propagating on the Ernst background is equivalent to the massive scalar perturbation propagating on the Schwarzschild background with effective mass \( \mu_{\text{eff}}^2 = \mu^2 + 4B^2m^2 \) which is always positive. Here additional term \( \sim 4B^2m^2 \) does not characterize interaction between the scalar and the electromagnetic fields but arises directly due the spacetime geometry of the magnetized black hole. However, such approximation is not accurate enough in the strong magnetic and gravitational fields when \( G^{1/2}c^{-2}Br \sim 1 \) [27].

One can see from Fig. 2 that at large distances \( r \to \infty \) the effective potential tends to square of the effective mass as

\[
V_{\text{eff}}(r \to \infty) \to \mu_{\text{eff}}^2 \equiv \mu^2 - mqB .
\]

It means that depending on the values of mass and charge of the scalar field, azimuthal number, and strength of the magnetic field, value of the squared effective mass can be negative also. In this case, one can observe an instability. Moreover, if the value of the effective mass is positive, there is some threshold value of the effective mass after that the effective potential loses its barrier-like form and quasinormal modes disappear. At this threshold value of the effective mass, there are arbitrarily long lived quasinormal modes, so called quasi-resonance modes [31–33]. Since finding of this threshold value analytically is impossible, we present some values of the threshold effective mass for different values of the multipole number in Fig. 3.

### III. NUMERICAL RESULTS

In this section we briefly present results of the numerical calculations of the quasinormal modes of the Schwarzschild black hole immersed in the external asymptotically uniform magnetic field. In order to solve the Schrödinger-like wave equation (12), we set the following boundary conditions: asymptotic behaviour of the
wave is purely incoming at the event horizon and purely outgoing at the spatial infinity as

\[ R(r) = \begin{cases} 
  e^{-i\omega x} & \text{at } x \to -\infty, \quad (r \to 2M), \\
  e^{i\lambda x} & \text{at } x \to \infty, \quad (r \to \infty),
\end{cases} \tag{16} \]

where \( \lambda = \sqrt{\omega^2 - \mu_{\text{eff}}^2} \).

### A. WKB method

For numerical calculations of quasinormal frequencies we use the WKB method that was applied for the first time for the calculations of quasinormal modes of black holes by Schutz and Will \cite{34} and after that the method was extended to higher orders by several authors \cite{35–37}. The sixth order WKB method for solving the Schrödinger-like wave equation governing the quasinormal modes of black holes implies the relation

\[ \frac{i(\omega^2 - V(r_0))}{\sqrt{-2V''(r_0)}} + \sum_{j=2}^{6} A_j = n + \frac{1}{2}, \tag{17} \]

where \( r_0 \) is the extreme of the potential that is corresponding to its maximum, a prime denotes the derivative with respect to tortoise coordinate \( x \), \( n \) is the label the overtone, \( A_j \) are the correction terms that can be found in \cite{34–37}. By using the sixth order WKB method, we obtain the frequencies of quasinormal modes of electrically charged scalar perturbations in the field of magnetized Schwarzschild black hole. In Tab. I the real and imaginary part of the frequencies of quasinormal modes for different values of the azimuthal \( m \) and the multipole \( \ell \) numbers and different values of the parameter \( qB \) are produced. From the Tab. I one can easily see that spectrum of the quasinormal modes splits into \( 2\ell + 1 \) parts due to the magnetic field in analog to the Zeeman effect. Note that here we have set the overtone number as \( n = 0 \). One can see from Tab. I that an increase in the value of effective mass increases the frequency of the real oscillations and decreases damping rate of the scalar field.

### B. Leaver’s method

In order to check and increase accuracy of the quasinormal frequencies, we adopt alternate method, i.e., the Leaver’s method which is continued fraction method (also known as Leaver’s method) developed in 1985 by Leaver \cite{38}. In this method one chooses a solution of Eq. (12) to be in the following form:

\[ R(r) = r^{i(\omega^2 + \chi^2)/2\chi} (r - 1)^{i\omega e^{i\chi r}} \sum_{n=0}^{\infty} a_n \left( \frac{r - 1}{r} \right)^n \tag{18} \]

Here Leaver’s unit \( 2M = 1 \) is adopted. The coefficients \( a_n \in \{a_0 = 1, a_1, a_2, \ldots \} \) satisfy the following three-term recurrence relation:

\[ \begin{align*}
  a_0 a_1 + b_0 a_0 &= 0, \\
  a_n a_{n+1} + b_n a_n + c_n a_{n-1} &= 0. 
\end{align*} \tag{19} \]

The recurrence coefficient \( \alpha_n, \beta_n \) and \( \gamma_n \) are given by

\[ \begin{align*}
  \alpha_n &= n^2 + (c_0 + 1)n + c_0, \\
  \beta_n &= -2n^2 + (c_1 + 2)n + c_3, \\
  \gamma_n &= n^2 + (c_2 - 3)n + c_4 - c_2 + 2, 
\end{align*} \tag{20} \]

with

\[ \begin{align*}
  c_0 &= 1 - i2\omega, \\
  c_1 &= -4 + i(4\omega + 3\chi) + \frac{i\omega^2}{\chi}, \\
  c_2 &= 3 - i(2\omega + \chi) - \frac{i\omega^2}{\chi}, \\
  c_3 &= \left( 1 + \frac{\omega}{\chi} \right) \left[ (\omega + \chi)^2 + \frac{i(\omega + 3\chi)}{2} \right] - \ell(\ell + 1) - 1, \\
  c_4 &= -\left[ i + \frac{(\omega + \chi)^2}{2\chi} \right]^2. 
\end{align*} \]

For the given multipole number \( \ell \) by using the boundary conditions (16) for the series (18) the frequency \( \omega \) is found by solving the following equation:

\[ \beta_{n-1} - \frac{\alpha_{n-1}\gamma_n}{\beta_{n-2} - \alpha_{n-2}\gamma_{n-2}/\beta_{n-3} - \alpha_{n-3}\gamma_{n-3}/\beta_{n-4} - \alpha_{n-4}\gamma_{n-4}/\beta_{n-5} - \alpha_{n-5}\gamma_{n-5}/...} = 0. \tag{21} \]

This equation is complicated and cannot be solved analytically. Therefore, we present only numerical results in Tab. II. By comparing numerical results given in Tabs. I...
TABLE I: The fundamental \((n = 0)\) quasinormal frequencies \((M\omega)\) of massive, charged scalar field in the magnetized Schwarzschild black hole calculated via the sixth order WKB method.

| \(\ell\) | \(m\) | \(\mu = 0, qB = 0\) | \(\mu = 0.1, qB = 0\) | \(\mu = 0.1, qB = 0.05\) | \(\mu = 0.1, qB = 0.1\) | \(\mu = 0.1, qB = 0.15\) |
|---|---|---|---|---|---|---|
| 0 | 0 | 0.1104 - 0.1008i | 0.1122 - 0.0927i | 0.1122 - 0.0927i | 0.1122 - 0.0927i | 0.1122 - 0.0927i |
| 1 | 0 | 0.2929 - 0.0977i | 0.2974 - 0.0955i | 0.2974 - 0.0955i | 0.2974 - 0.0955i | 0.2974 - 0.0955i |
|    | -1 | 0.3200 - 0.0809i | 0.3429 - 0.0654i | 0.3429 - 0.0654i | 0.3543 - 0.0441i | 0.3543 - 0.0441i |
| 2 | 0 | 0.4836 - 0.0967i | 0.4868 - 0.0957i | 0.4868 - 0.0957i | 0.4868 - 0.0957i | 0.4868 - 0.0957i |
|    | -1 | 0.5027 - 0.0902i | 0.5188 - 0.0845i | 0.5188 - 0.0845i | 0.5351 - 0.0787i | 0.5351 - 0.0787i |
|    | -2 | 0.5188 - 0.0845i | 0.5516 - 0.0726i | 0.5516 - 0.0726i | 0.5852 - 0.0598i | 0.5852 - 0.0598i |

and II that are derived by the WKB and Leaver’s methods, respectively, one can easily notice the significant difference in the quasinormal frequencies for negative values of the squared effective mass, \(\mu^2_{\text{eff}} < 0\). In the results evaluated by the WKB method given in Tab. I imaginary parts of the frequencies are always negative which are an indication of stability of the spacetime against such perturbations, while in Tab. II calculated by the Leaver’s method, in addition to the stable modes which are almost the same as the one calculated by the WKB method in Tab. I, unstable modes with positive imaginary frequency appear. Existence of both stable and unstable modes for the negative squared effective mass with the same multipole is strange. Therefore, one needs to exclude one of them. In order to determine whether the unstable modes are physically relevant or not, in the next subsection we study the temporal evolution of the perturbations.

C. Instability

One can see from the effective potential depicted in Fig. 2 that for the negative values of the effective mass, \(\mu^2_{\text{eff}} < 0\), the effective potential is negative in the region \(r \in (r_n, \infty)\). Although, an increase in the value of multipole number increases the height of the potential, it does not change depth of the negative part of the effective potential, whereas it pushes away this negative region, i.e., it increases \(r_n\). The depth of negativity of the effective potential is only characterized by effective mass. Despite negativity of the effective potential indicates an instability, it does not guarantee it [39]. Therefore, to
determine if there is the instability in this region, we
investigate the temporal evolution of the scalar pertur-
bation by following the time-domain integration method
proposed by Gundlach et al. [40]. The time-domain inte-
gration method involves the retarded $dv = dt - dx$ and
advanced $dv = dt + dx$ light-cone variables. Then, tak-
ing into account that the scalar field is harmonically time
dependent, the wave equation (12) is written in the form

$$-4 \frac{\partial^2 \Phi}{\partial u \partial v} = V_{\text{eff}}(r(u,v)) \Phi.$$  \hspace{1cm} (22)

This equation is solved numerically. To solve this equa-
tion we use the following discretization scheme:

$$\Phi_N = (\Phi_W + \Phi_E) \frac{16 - \Delta^2 V_S}{16 + \Delta^2 V_S} - \Phi_S + O(\Delta^4),$$  \hspace{1cm} (23)

where the indices $N, W, E$, and $S$ refer to grid-points
$N \equiv (u, v), \ W \equiv (u - \Delta, v), \ E \equiv (u, v - \Delta)$, and $S \equiv (u - \Delta, v - \Delta)$. $\Delta$ is a constant separating neighboring
points of the grid (for details, please see [41, 42]). The
initial data is specified on the two null surfaces $u = u_0$
and $v = v_0$. For calculations we assume that the initial
perturbation is Gaussian pulse centered around the point $v_c$
with width $\sigma$ as

$$\Phi(u = u_0, v) = A \exp \left( -\frac{(v - v_c)^2}{\sigma^2} \right).$$  \hspace{1cm} (24)

In Fig. 4, evolution of the charged, massive scalar pertur-
bation with time are illustrated for the mode $\ell = 2$ and
all possible five values of $m$. Due to the external magnetic
field quasinormal modes of the scalar field around black
hole live long in comparison with the quasinormal modes
without magnetic field. It is also shown that the effects
of the magnetic field is more profound in the final stage
of the quasinormal modes. One can see from Fig. 4 that
instability occurs for the negative values of the squared
effective mass as it was stated in [23, 43], in particular,
for the positive value of $qB$ parameter ($qB > 0$) and for
positive value of the azimuthal number $m > 0$, only if

$$\mu_{\text{eff}}^2 < 0.$$  

From the symmetric property of the effective potential
for the values of $m$ and $qB$ one can find instability in the case of $m < 0$ and $qB < 0$ too. Thus, the
negativity of effective potential leads the instability.

Moreover, as we have indicated at the end of the pre-
vious subsection, here we briefly discuss the reason of the
contradiction on the quasinormal modes derived by the
WKB and Leaver’s methods which are given in Tabs. I
and II, respectively. As it has been perfectly explained in
[23], in the case of the single peak effective potential, the
WKB method is built on the basis of matching of three
wave solutions at the turning points [44] and therefore,
it is more accurate only in the region close to maximum
of the effective potential. Therefore, the WKB method
can be applied only near to the maximum of the potential
and in our case it can be used for calculation of the stable
modes when the effective potential does not have negative
region. On the contrary, fortunately, the Leaver’s method
does not have this problem and can be applied for the whole region. Consequently, the Leaver’s method
provides not only stable but also unstable modes for the
given multipole moment $(\ell, m)$ as shown in Tab. II.
However, the stable modes produced by the Leaver’s method
corespond to the ones of the WKB method which does
not take into account the negative region of the effective
potential. Therefore, for the cases of negative squared
effective mass, stable modes should be considered as just
mathematical solution having no physical meaning. With
increase of the negativity of the effective mass, instability
also increases. Interestingly, this instability does not
depend on multipole number of the scalar field, being
characterized by only effective mass of the charged, mas-
sive scalar field.

**IV. CONCLUSION**

In this paper we have investigated the dynamics of
charged and massive scalar field in the vicinity of static
spherically symmetric black hole immersed into the exter-
nal asymptotically uniform magnetic field. Assuming the
scalar field as spherical symmetric and harmonically time
dependent as well as using the general relativistic form of
the Klein-Gordon equation we have obtained stationary
Schrödinger-like equation with effective potential which
possesses the squared effective mass, $\mu_{\text{eff}}^2 = \mu^2 - mqB$
produced by the Zeeman effect that represents a shift
of energy of charge of the scalar field $q$ in the magnetic
field $B$ due to the interaction of magnetic field with an
azimuthal momentum $m$. By investigating the effective
potential we have found early indication of the instabil-
ity that occurs for negative values of the squared effective
mass. Since azimuthal number satisfies the condition $|m| \leq \ell$, for given $\ell$ the squared effective mass of the
scalar field $\mu_{\text{eff}}^2 = \mu^2 - mqB$ and consequently, the quasinormal
modes are splitted into $2\ell + 1$ parts due to the magnetic Zeeman effect. Moreover, the threshold value of
the effective mass of the scalar field from which the effec-

![Fig. 4: Temporal evolution of the charged, massive scalar field around Schwarzschild black hole immersed in the external uniform magnetic field.](image-url)
tive potential loses its barrier-like form and quasinormal modes disappear has been found. Analysis of temporal evolution of the scalar field has shown that for the negative values of the squared effective mass, $\mu^2_{\text{eff}} < 0$, there is instability. However, we have shown that the WKB method is not able to show this instability as the negative part of the effective potential is located further away from the peak of the potential. Therefore, in order to show the instability we have used the Leaver’s method. It has been shown that due to the external magnetic field and only in the case when the effective mass is positive, quasinormal modes are becoming longer lived and have larger oscillation frequencies for the positive charged scalar field until the effective mass reaches the threshold value.

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