Nature of the Gravitational Field and its Legitimate Energy-Momentum Tensor

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Abstract

In this paper we show how a gravitational field generated by a given energy-momentum distribution (for all realistic cases) can be represented by distinct geometrical structures (Lorentzian, teleparallel and non null nonmetricity spacetimes) or that we even can dispense all those geometrical structures and simply represent the gravitational field as a field, in the Faraday’s sense, living in Minkowski spacetime. The explicit Lagrangian density for this theory is given and the field equations (which are a set of four Maxwell’s like equations) are shown to be equivalent to Einstein’s equations. We also analyze if the teleparallel formulation can give a mathematical meaning to “Einstein’s most happy thought”, i.e. the equivalence principle. Moreover we discuss the Hamiltonian formalism for our theory and its relation to one of the possible concepts for energy of the gravitational field which emerges from it and the concept of ADM energy. One of the main results of the paper is the identification in our theory of a legitimate energy-momentum tensor for the gravitational field expressible through a really nice formula.

1 Introduction

As well known in General Relativity (GR), a classical field theory of gravitation, each gravitational field generated by a given energy-momentum tensor is represented by a Lorentzian spacetime, which is a structure \((M, D, g, \tau_g, \tau)\) where

...
$M$ is a non compact (locally compact) 4-dimensional Hausdorff manifold, $g$ is a Lorentzian metric on $M$ and $D$ is its Levi-Civita connection. Moreover $M$ is supposed to be oriented by the volume form $\tau_g$ and the symbol ↑ means that the spacetime is time orientable. From the geometrical objects in the structure $\langle M, D, g, \tau_g, ↑ \rangle$ we can calculate the Riemann curvature tensor $R$ of $D$ and a nontrivial GR model is one in which $R \neq 0$. In that way textbooks often say that in GR spacetime is curved. Unfortunately many people mislead the curvature of a connection $D$ on $M$ with the fact that $M$ can eventually be a bent surface in an (pseudo)Euclidean space with a sufficient number of dimensions. This confusion leads to all sort of wishful thinking because many forget that GR does not fix the topology of $M$ that often must be put “by hand” when solving a problem, and thus think that they can bend spacetime if they have an appropriate kind of some exotic matter. Worse, the insistence in supposing that the gravitational field is geometry lead the majority of physicists to relegate the search for the real physical nature of the gravitational field as not important at all (see a nice discussion of this issue in [10]). What most textbooks with a few exceptions (see, e.g., the excellent book by Sachs and Wu [19]) forget to say and give a proof to their readers is that in the standard formulation of GR there are no genuine conservation laws of energy-momentum and angular momentum unless spacetime has some additional structure which is not present in a general Lorentzian spacetime [13]. Some textbooks e.g., [12] even claim that energy-momentum conservation for matter plus the gravitational fields is forbidden due the equivalence principle because the energy-momentum of the gravitational field must be non localizable. Only a few people tried to develop consistent theories where the gravitational field (at least from the classical point of view) is simple another field, which like the electromagnetic field lives in Minkowski spacetime (see a list of references in [8]). A field of that nature will be called, in what follows, a field in Faraday’s sense.

Here we want to recall that: (i) the representation of gravitational fields by Lorentzian spacetimes is not a necessary one, for indeed, there are some geometrical structures different from $\langle M, D, g, \tau_g, ↑ \rangle$ that can equivalently represent such a field; (ii) The gravitational field can also be nicely represented as a field living in a fixed background spacetime. The preferred one which seems to describe all realistic situations is, of course, Minkowski spacetime.

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1For details, please consult, e.g., [16, 19].

2Any manifold $M$, dim $M = n$ according to the Whitney theorem can be realized as a submanifold of $\mathbb{R}^m$, with $m = 2n$. However, if $M$ carries additional structure the number $m$ in general must be greater than $2n$. Indeed, it has been shown by Eddington [6] that if dim $M = 4$ and if $M$ carries a Lorentzian metric $g$, which moreover satisfies Einstein’s equations, then $M$ can be locally embedded in a (pseudo)euclidean space $\mathbb{R}^{1,9}$. Also, isometric embedding of a general Lorentzian spacetime would require a lot of extra dimensions [3]. Indeed, a compact Lorentzian manifold can be embedded isometrically in $\mathbb{R}^{5,46}$ and a non-compact one can be embedded isometrically in $\mathbb{R}^{5,87}$.

3We will not discuss here that most presentations of the equivalence principle are devoid from mathematical and physical sense. See, e.g., [21, 17].

4Of course, the true background spacetime may be eventually a more complicated one, since that manifold must represent the global topological structure of the universe, something that is not known at the time of this writing [20]. We do not study this possibility in this
\( \langle M \simeq \mathbb{R}^4, D, \eta, \tau, \uparrow \rangle \).

Concerning the possible alternative geometrical models, the particular case where the connection is **teleparallel** (i.e., it is metric compatible, has **null** Riemann curvature tensor and **non null** torsion tensor) will be briefly addressed below (for other possibilities see [14]). What we will show, is that starting with a thoughtful representation of the gravitational field in terms of gravitational potentials \( g^a \in \text{sec} T^* M \rightarrow \text{sec} \mathcal{C}(M, g) \), \( a = 0, 1, 2, 3 \) and postulating a convenient Lagrangian density for the gravitational potentials which does not use any connection there is a posteriori different ways of geometrically representing the gravitational field, such that the field equations in each representation result equivalent in a precise mathematical sense to Einstein’s field equations. Explicitly we mean by this statement the following: any realistic model of a gravitational field in GR where that field is represented by a Lorentzian space-time (with **null** Riemann curvature tensor and **null** torsion tensor which is also parallelizable, i.e. admits four **global** linearly independent vector fields) is equivalent to a teleparallel spacetime (i.e., a spacetime structure equipped with a metrical compatible teleparallel connection, which has **null** Riemann curvature tensor and **non null** torsion tensor)\(^5\). The teleparallel possibility follows almost directly from the results in Section 2 and a recent claim that it can give a mathematical representation to “Einstein most happy though” is discussed in Section 3.

With our teleparallel equivalent version of GR and equipped with the powerful Clifford bundle formalism [8, 16] we are able to identify in Section 4 a legitimate energy momentum tensor for the gravitational field expressible in a very short and elegant formula.

Besides this main result we think that another important feature of this paper is that our representation of the gravitational field by the global 1-form fields potentials \( \{ g^a \} \) living on a manifold \( M \) and coupled among themselves and with the matter fields in a specific way (see below) shows that we can **dispense** with the concept of a connection and a corresponding geometrical description for that field. The simplest case is when \( M \) is part of Minkowski spacetime structure, in which case the gravitational field is (like the electromagnetic field) a field in Faraday’s sense\(^6\). In section 5 we present the Hamiltonian formalism for our theory and discuss the relation of one possible energy concept\(^7\) naturally appearing in it and its relation to the concept of ADM energy. In Section 6 we present the conclusions.

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\(^5\)There are hundreds of papers (as e.g., [15]) on the subject, but none (to the best of our knowledge) as the one presented here.

\(^6\)In [18] we even show that when a Lorentzian spacetime structure \( \langle M, D, g, \tau, \uparrow \rangle \) representing a gravitational field in GR possess a Killing vector field \( \mathbf{A} \), then there are Maxwell like equations with well determined source term satisfied for \( F = dA \), \( A = g(\mathbf{A}, \mathbf{v}) \) equivalent to Einstein equation and more, there is a Navier-Stokes equation equivalent to the Maxwell (like) equations and Einstein equation.

\(^7\)This other possibility does not define in general a legitimate energy-momentum tensor for the gravitational field in GR, but defined in our theory in which the gravitational field is interpreted as a field in the sense of Faraday living in Minkowski spacetime.
2 Representation of the Gravitational Field

Suppose that a 4-dimensional $M$ manifold is parallelizable, thus admitting a set of four global linearly independent vector fields $\{e_a\} \in \sec TM$, $a = 0, 1, 2, 3$ such that $\{e_a\}$ is a basis for $TM$ and let $\{g^a\}, \eta^a \in \sec T^*M$ be the corresponding dual basis $(g^a(e_b) = \delta_b^a)$. Suppose also that not all the $g^a$ are closed, i.e., $d\eta^a \neq 0$, for at least some $a = 0, 1, 2, 3$. This will be necessary for the possible interpretations we have in mind for our theory. The 4-form field $g^0 \wedge g^1 \wedge g^2 \wedge g^3$ defines a (positive) orientation for $M$.

Now, the $\{g^a\}$ can be used to define a Lorentzian metric field in $M$ by defining $g \in \sec T^*_2M$ by $g := \eta^a b^a \otimes g^b$, with the matrix with entries $\eta_{ab}$ being the diagonal matrix $(1, -1, -1, -1)$. Then, according to $g$ the $\{e_a\}$ are orthonormal, i.e., $e_a \cdot e_b := g(e_a, e_b) = \eta_{ab}$.

Since the $e_0$ is a global time like vector field it follows that it defines a time orientation in $M$ which we denote by $\uparrow$. It follows that that the 4-tuple $(M, g, \gamma, \uparrow)$ is part of a structure defining a Lorentzian spacetime and can eventually serve as a substructure to model a gravitational field in GR.

For future use we also introduce $g \in \sec T^*_0M$ by $g := \eta^a b^a \otimes e_b$, and we write $g^a \cdot g^b := g(g^a, g^b) = \eta^{ab}$.

Due to the hypothesis that $d\eta^a \neq 0$ the commutator of vector fields $e_a, a = 0, 1, 2, 3$ will in general satisfy $[e_a, e_b] = c_{ab}^k e_k$ where the $c_{ab}^k$, the structure coefficients of the basis $\{e_a\}$, and we easily show that $d\eta^a = \frac{1}{2} \epsilon_{abc} g^b \wedge g^c$.

Next, we introduce two different metric compatible connections on $M$, namely $D$ (the Levi-Civita connection of $g$) and a teleparallel connection $\nabla$. Metric compatibility means that for both connections it is $Dg = 0, \nabla g = 0$. Now, we put

$$
D e_a e_b = \omega^c_{ab} e_c, \quad D f^b = - \omega^c_{ac} g^c, \quad \nabla e_a e_b = 0, \quad \nabla f^b = 0. \tag{1}
$$

The objects $\omega^c_{ab}$ are called the connection coefficients of the connection $D$ in the $\{e_a\}$ basis and the objects $\omega^{ac} := \omega^a_{bc}$ are called the connection 1-forms in the $\{e_a\}$ basis. The connection coefficients $\omega^b_{ac}$ of $\nabla$ and the connection 1-forms of $\nabla$ in the basis $\{e_a\}$ are null according to the second line of Eq. (1) and thus the basis $\{e_a\}$ is called teleparallel and the connection $\nabla$ defines an absolute parallelism on $M$. Of course, as it is well known the Riemann curvature tensor of the Levi-Civita connection $D$ of $g$, is in general non null in all points of $M$, but the torsion tensor of $D$ is zero in all points of $M$. On the other hand the Riemann curvature tensor of $\nabla$ is null in all points of $M$, whereas the torsion tensor of $\nabla$ is non null in all points of $M$.

We recall also that for a general connection, say $D$ on $M$ (not necessarily metric compatible) the torsion and curvature operations and the torsion and

\footnote{We recall that $\sec TM$ means section of the tangent bundle and $\sec T^*M$ means section of the cotangent bundle. Also $\sec T^*_rM$ means the bundle of tensors of type $(r, s)$ and $\sec \Lambda^s T^*M$ a section of the bundle of $s$-forms fields.}
curvature tensors of a given general connection, say $D$, are respectively the mappings:

$$\rho : \text{sec}TM \otimes TM \otimes TM \longrightarrow \text{sec}TM,$$

$$\rho(u, v, w) = D_u D_v w - D_v D_u w - D_{[u,v]} w,$$

$$\tau : \text{sec}TM \otimes TM \longrightarrow \text{sec}TM,$$

$$\tau(u, v) = D_\nu v - D_v u - [u, v]. \quad (2)$$

It is usual to write \cite{3} $\rho(u, v, w) = \rho(u, v)w$ and $\Theta(\alpha, u, v) = \alpha(\tau(u, v))$ and $R(w, \alpha, u, v) = \alpha(\rho(u, v)w)$, for every $u, v, w \in \text{sec}TM$ and $\alpha \in \text{sec}T^*M$. In particular we write $T_a^b := \Theta(g^a, e_b, e_c)$ and $R^b_{a cd} := R(e_a, g^b, e_c, e_d)$, and define the Ricci tensor by $\text{Ricci} := R_{ac}g^a \otimes g^c$ with $R_{ac} := R^b_{a cb} = R_{ca}$.

From now on we imagine $\bigwedge T^*M = \bigoplus_{r=0}^4 \bigwedge^r T^*M \hookrightarrow \mathcal{C}(M, g)$, where $\mathcal{C}(M, g)$ is the Clifford bundle of non-homogeneous, differential forms and use the conventions about the scalar product, left and right contractions, the Hodge star operator and the Hodge codifferential. operator in $\mathcal{C}(M, g)$ as defined in \cite{[reference]} \cite{8}.

Given that we introduced two different connections $D$ and $\nabla$ defined in the manifold $M$ we can write two different pairs of Cartan’s structure equations. Those pairs describe respectively the geometry of the structures $(M, D, g, \tau, \uparrow)$ and $(M, \nabla, g, \tau, \uparrow)$ which will be called respectively a Lorentzian spacetime and a teleparallel spacetime. In the case $(M, D, g, \tau, \uparrow)$ we write

$$\Theta^a := dg^a + \omega^a_b \wedge g^b = 0, \quad R^a_b := d\omega^a_b + \omega^a_c \wedge \omega^c_b,$$

where the $\Theta^a \in \text{sec} \bigwedge^2 T^*M \hookrightarrow \text{sec}\mathcal{C}(M, g)$, $a = 0, 1, 2, 3$ and the $R^a_b \in \text{sec} \bigwedge^2 T^*M \hookrightarrow \text{sec}\mathcal{C}(M, g)$, $a, b = 0, 1, 2, 3$ are respectively the torsion and the curvature 2-forms of $D$ with

$$\Theta^a = \frac{1}{2} R_{b c d}^a \wedge g^c \wedge g^d,$$

$$R^a_b = \frac{1}{2} R^a_{b c d} \wedge g^c \wedge g^d. \quad (3)$$

In the case of $(M, \nabla, g, \tau, \uparrow)$ since $\omega^a_b = 0$ we have

$$\mathcal{F}^a := dg^a + \omega^a_b \wedge g^b = dg^a, \quad \mathcal{R}^a_b := d\omega^a_b + \omega^a_c \wedge \omega^c_b = 0, \quad (4)$$

where the $\mathcal{F}^a \in \text{sec} \bigwedge^2 T^*M$, $a = 0, 1, 2, 3$ and the $\mathcal{R}^a_b \in \text{sec} \bigwedge^2 T^*M$, $a, b = 0, 1, 2, 3$ are respectively the torsion and the curvature 2-forms of $\nabla$ given by formulas analogous to the ones in Eq. (3).

We next postulate that the $\{g^a\}$ are the basic variables representing the gravitation field, and moreover postulate that the $\{g^a\}$ interacts with the matter fields through the following Lagrangian density $\mathcal{L}$

$$\mathcal{L} = \mathcal{L}_g + \mathcal{L}_m, \quad (5)$$

\footnote{We observe that the first term in Eq. (5) can be proved (see, e.g. \cite{10}) to be equivalent just to the Lagrangian density used by Einstein in \cite{7}.}
where \( L_m \) is the matter Lagrangian density

\[
L_g = -\frac{1}{2} d\gamma^a \wedge \star d\gamma_a + \frac{1}{2} \delta \gamma^a \wedge \star \delta \gamma_a + \frac{1}{4} (d\gamma^a \wedge \gamma_a) \wedge \star (d\gamma^b \wedge \gamma_b),
\]

(6)

The form of this Lagrangian is notable, the first term is Yang-Mills like, the second one is a kind of gauge fixing term and the third term is an auto-interaction term describing the interaction of the "vorticities" of the potentials (or if you prefer, the interaction between Chern-Simons terms \( dq^a \wedge \gamma_a \)). Before proceeding we observe that this Lagrangian is not invariant under arbitrary point dependent Lorentz rotations of the basic cotetrad fields. In fact, if \( \gamma^a \mapsto g^a = \Lambda_b^a \gamma^b = R \gamma^a \tilde{R} \) (where for each \( x \in M, \Lambda_b^a(x) \in L_+^1 \), the homogeneous and orthochronous Lorentz group and \( R(x) \in \text{Spin}_{1,3} \subset \mathbb{R}_{1,3} \)) we get that

\[
L'_g = -\frac{1}{2} d\gamma'^a \wedge \star d\gamma'_a + \frac{1}{2} \delta \gamma'^a \wedge \star \delta \gamma'_a + \frac{1}{4} (d\gamma'^a \wedge \gamma'_a) \wedge \star (d\gamma^b \wedge \gamma'_b),
\]

(7)

differs from \( L_g \) by an exact differential. So, the field equations derived by the variational principle results invariant under a change of gauge \[16\] and we can always choose a gauge such that \( \delta \gamma_a = 0 \).

Now, to derive the field equations directly from Eq.(6) using constrained variations of the \( \gamma^a \) (i.e., variations induced by point dependent Lorentz rotations) that do not change the metric field \( g \) is a good exercise in the Clifford calculus, whose details the interested reader may find \[11\] in Appendix E of \[8\]. The result is:

\[
d \star S_d + \star t_d = - \star T_d,
\]

(8)

where

\[
\star t_d := \frac{\partial L_g}{\partial dq^a} = \frac{1}{2} \left[ (\partial_d d\gamma_a) \wedge \star d\gamma_a - d\gamma^a \wedge (\partial_d \star d\gamma_a) \right] + \frac{1}{2} d(\partial_d \gamma^a \wedge \gamma_a) \wedge \star \gamma_a + \frac{1}{2} (d\gamma^a \wedge \gamma_a) \wedge \star \gamma_a + \frac{1}{2} d\gamma_d \wedge \star (d\gamma^a \wedge \gamma_a) - \frac{1}{4} d\gamma^a \wedge \gamma_a \wedge \left[ \partial_d \gamma^c \wedge \gamma_c \right] - \frac{1}{4} \left[ \partial_d \gamma^c \wedge (d\gamma^a \wedge \gamma_c) \right] \wedge \star (d\gamma^a \wedge \gamma_a),
\]

(9)

\[
\star S_d := \frac{\partial L_g}{\partial d\gamma^a} = - \star d\gamma_d - (\partial_d \gamma^a \wedge \gamma_a) \wedge \star \gamma_a + \frac{1}{2} d\gamma_d \wedge \star (d\gamma^a \wedge \gamma_a),
\]

(10)

and the\[12\]

\[
\star T_d := \frac{\partial L_m}{\partial dq^a} = - \star T_d
\]

(11)

10 A Lagrangian density equivalent to \( L_g \) appeared in \[23\].
11 See errata for reference \[8\] at http://www.ime.unicamp.br/~walrod/plasticwr2012
12 We suppose that \( L_m \) does not depend explicitly on the \( dq^a \).
are the energy-momentum 3-forms of the matter fields.\footnote{In reality, due the conventions used in this paper the true energy-momentum 3-forms are $^{\ast g}T_d = - ^{\ast g}t_d$.}

Recalling that from Eq.(4) it is $F_a := dg^a$, it is, of course, $dF_a = 0$ and the field equations (Eq.(8)) can be written as

$$d \ast g^a F_a d = - \ast g^a T_d d - \ast g^a h_d,$$

where

$$h_d = d \left[ (g_d \ast g^a) \wedge \ast g^a \ast g^a - \frac{1}{2} g_d \wedge \ast (F^a \wedge g^a) \right].$$

Recalling the definition of the Hodge coderivative operator acting on sections of $\wedge^r T^* M$ we can write Eq.(12) as

$$\delta g^a F_a d = - (T^d + t^d),$$

with the $t^d \in \text{sec} \wedge^1 T^* M$ given by

$$t^d := t^d + h^d,$$

which are legitimate energy-momentum 1-form fields for the gravitational field. Note that the total energy-momentum tensor of matter plus the gravitational field is trivially conserved in our theory, i.e.,

$$\delta g^a (T^d + t^d) = 0.$$ 

\textbf{Remark 1} Recalling Eq.(9) and Eq.(13) the formula for the $t^d$ in Eq.(15) cannot be, of course, the nice and short formula we promised to present in the introduction. However, it is equivalent to the nice formula as shown in Section 4.

Recall the similarity of the equations satisfied by the gravitational field to Maxwell equations. Indeed, in electromagnetic theory on a Lorentzian space-time we have only one potential $A \in \text{sec} \wedge^1 T^* M \hookrightarrow \text{sec} Cl(M, g)$ and the field equations are

$$dF = 0, \quad \delta F = - J,$$

where $F \in \text{sec} \wedge^2 T^* M \hookrightarrow \text{sec} Cl(M, g)$ is the electromagnetic field and $J \in \text{sec} \wedge^1 T^* M \hookrightarrow \text{sec} Cl(M, g)$ is the electric current. As well known the two equations in Eq.(17) can be written (if you do not mind in introducing the connection $D$ in the game) as a single equation using the Clifford bundle formalism \footnote{This will become evident in Section 4 were derive the nice formula for the $t^d$.}, namely

$$\partial F = J.$$
where we can write \( \mathcal{D} = d - \delta = g^a D_a \), where \( \mathcal{D} \) is the Dirac operator (acting on sections of \( \mathcal{C}(M, g) \)).

Now, if you feel uncomfortable in needing four distinct potentials \( g^a \) for describing the gravitational field you can put them together defining a vector valued differential form

\[
g = g^a \otimes e_a \in \text{sec} \bigwedge^1 \mathcal{T}^* M \otimes \mathcal{T} M \rightarrow \text{sec} \mathcal{C}(M, g) \otimes \bigwedge \mathcal{T} M
\]

and in this case the gravitational field equations are

\[
d F = 0, \quad \delta g F = -(\mathcal{T} + t),
\]

where \( F = F^a \otimes e_a, \mathcal{T} = T^a \otimes e_a, t = t^a \otimes e_a \). Again, if you do not mind in introducing the connection \( D \) in the game by considering the bundle \( \mathcal{C}(M, g) \otimes \mathcal{T} M \) we can write the two equations in Eq. (??) as a single equation, i.e.,

\[
\partial F = \mathcal{T} + t
\]

At this point you may be asking: which is the relation of the theory just presented with Einstein’s GR theory? The answer is that recalling that the connection 1-forms \( \omega^{cd} \) of \( D \) are given by

\[
\omega^{cd} = \frac{1}{2} \left[ g^d \cdot g^c - g^c \cdot g^d + g^c \cdot (g^d \cdot g_a) g_a \right]
\]

one can show (see [16] for details) that the Lagrangian density \( \mathcal{L}_g \) becomes

\[
\mathcal{L}_g = \mathcal{L}_{EH} + d(g^a \wedge \star g_a),
\]

where

\[
\mathcal{L}_{EH} = \frac{1}{2} \mathcal{R}^{cd} \wedge \star (g^c \wedge g^d)
\]

(with \( \mathcal{R}^{cd} \) given by Eq. [3]) is the Einstein-Hilbert Lagrangian density. This permits (with some algebra) to show that Eqs. [8] are indeed equivalent to the usual Einstein equations.

Before ending this section we recall that from Eq. [9] we can also define for our theory a meaningful energy-momentum for the gravitational plus matter fields. Indeed, using Stokes theorem for a ‘certain 3-dimensional volume’, say a ball \( B \) we immediately get

\[
P^a := \int_B \frac{1}{g} \left( \mathcal{T}^a + t^a \right) = -\int_{\partial B} \frac{1}{g} \mathcal{S}^a.
\]

### 3 A Comment on Einstein Most Happy Though

The exercises presented above indicate that a particular geometrical interpretation for the gravitational field is no more than an option among many ones.
Indeed, it is not necessary to introduce any connection $D$ or $\nabla$ on $M$ to have a perfectly well defined theory of the gravitational field whose field equations are (in a precise mathematical sense) equivalent to the Einstein field equations. Note that we have not given until now details on the \textit{global topology} of the world manifold $M$, except that since we admitted that $M$ carries four global (not all closed) 1-form fields $g^a$ which defines the object $g$, it follows that $(M, D, g, \tau, \nabla)$ is a \textit{spin manifold} $[9, 10]$, i.e., it admits spinor fields. This, of course, is necessary if the theory is to be useful in the real world since fundamental matter fields are spinor fields. The most simple spin manifold is clearly Minkowski spacetime which is represented by a structure $(M = \mathbb{R}^4, D, \eta, \tau, \nabla)$ where $D$ is the Levi-Civita connection of the Minkowski metric $\eta$. In that case it is possible to interpret the gravitational field as a $(1, 1)$-extensor field $h$ which is a field in the Faraday sense living in $(M, D, \eta, \tau, \nabla)$. The field $h$ is a kind of square of $g$ which has been called in $[8]$ the plastic distortion field of the Lorentz vacuum. In that theory the potentials $g^a = h(\gamma^a)$ where $\gamma^a = \delta^a_\mu dx^\mu$, with $\{x^\mu\}$ global naturally adapted coordinates (in Einstein-Lorentz-Poincaré gauge) to the \textit{inertial reference frame} $I = \partial/\partial x^\mu$ according to the structure $(M = \mathbb{R}^4, D, \eta, \tau, \nabla)$, i.e. $D I = 0$. In $[8]$ we give the dynamics and coupling of $h$ to the matter fields. At last we want to comment that, as well known, in Einstein’s GR one can easily distinguish in any \textit{real physical laboratory}, i.e., not one modelled by a time like worldline (despite some claims on the contrary) $[15]$ a true gravitational field from an acceleration field of a given reference frame in Minkowski spacetime. This is because in GR the \textit{mark} of a real gravitational field is the non null Riemann curvature tensor of $D$, and the Riemann curvature tensor of the Levi-Civita connection of $D$ (present in the definition of Minkowski spacetime) is null. However if we interpret a gravitational field as the torsion 2-forms on the structure $(M, \nabla, g, \tau, \nabla)$ viewed as a deformation of Minkowski spacetime then one can also interpret an acceleration field of an accelerated reference frame in Minkowski spacetime as generating an effective teleparallel spacetime $(M, \nabla, \eta, \tau, \nabla)$. This can be done as follows. Let $Z \in sec TU, U \subset M$ with $\eta(Z, Z) = 1$ an \textit{accelerated reference frame} on Minkowski spacetime. This means (see, e.g., $[10]$ for details) that $a = D_Z Z \neq 0$. Put $c_0 = Z$ and define an accelerated reference frame as \textit{non} trivial if $\delta^0 = \eta(c_0, \cdot)$ is not an exact differential. Next recall that in $U \subset M$ there always exist $[3]$ three other $\eta$-orthonormal vector fields $e_i$, $i = 1, 2, 3$ such that $\{e_a\}$ is an $\eta$-orthonormal basis for $TU$, i.e., $\eta = \eta^a_b \delta^a \otimes \delta^b$, where $\{\delta^a\}$ is the dual basis$^{[15]}$ of $\{e_a\}$. We then have, $\tilde{D}_c e_b = \omega^c_{ab} e_c, \tilde{D}_c \delta^b = -\omega^c_{ab} \delta^c$.

What remains in order to be possible to interpret an acceleration field as a kind of ‘gravitational field’ is to introduce on $M$ a $\eta$-metric compatible connection $\hat{\nabla}$ such that the $\{e_a\}$ is teleparallel according to it, i.e., $\hat{\nabla}_{e_a} e_b = 0, \hat{\nabla}_{e_a} \delta^b = 0$. Indeed, with this connection the structure $(M \simeq \mathbb{R}^4, \hat{\nabla}, \eta, \tau, \nabla)$ has null Riemann curvature tensor but a non null torsion tensor, whose components are related with the components of the acceleration $a$ and with the other coefficients $\omega_{ab} = (\eta^a_b \delta^c - \delta^a_c \eta^b_{\phantom{b}c})$.

$^{[15]}$In general we will also have that $d \delta^i \neq 0, i = 1, 2, 3$. 

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\( \omega_{ab} \) of the connection \( \tilde{\nabla} \), which describe the motion on Minkowski spacetime of a grid represented by the orthonormal frame \( \{ e_a \} \). Schücking [20] thinks that such a description of the gravitational field makes Einstein most happy though, i.e., the equivalence principle (understood as equivalence between acceleration and gravitational field) a legitimate mathematical idea. However, a true gravitational field must satisfy (at least with good approximation) Eq. (12), whereas there is no single reason for an acceleration field to satisfy that equation.

4 The Nice Formula for the Legitimate Energy-Momentum Tensor of the Gravitational Field

Taking into account that \( F^d = d\mathbf{g}^d = \partial \wedge \mathbf{g}^d \) we return to Eq. (13) and write it as

\[
\partial^2 \mathbf{g}^d = \mathbf{T}^d + \mathbf{t}^d,
\]

with \( \mathbf{t}^d = \mathbf{t}^d - d\delta \mathbf{g}^d \). Next we recall that in the Clifford bundle formalism the operator \( \partial^2 \) (the Hodge D’Alembertian) has two non equivalent decompositions, namely, for each \( M \in \text{sec} \bigwedge T^* M \rightarrow \text{sec} \mathcal{CL}(M, g) \) we have

\[
\partial^2 M = -(d\delta + \delta d)M
\]

\[
= \partial \wedge \partial M + \partial \cdot \partial M
\]

where \( \partial \wedge \partial \) is an extensorial operator called the Ricci operator and \( \partial \cdot \partial \) is the covariant D’Alembertian operator. We have

\[
\partial \wedge \partial \mathbf{g}^d = \mathbf{R}^d,
\]

where the \( \mathbf{R}^d = R^d_a \mathbf{g}^a \in \text{sec} \bigwedge^1 T^* M \rightarrow \text{sec} \mathcal{CL}(M, g) \) (with \( R^d_a \) the components of the Ricci tensor) are called the Ricci 1-form fields. Then we can write Eq. (26) as

\[
\partial \wedge \partial \mathbf{g}^d + \partial \cdot \partial \mathbf{g}^d = \mathbf{T}^d + \mathbf{t}^d,
\]

or

\[
\mathbf{R}^d + \partial \cdot \partial \mathbf{g}^d = \mathbf{T}^d + \mathbf{t}^d.
\]

Now, we recall that Einstein equation in components form is

\[
R^a_d - \frac{1}{2} \delta^a_d R = -\mathbf{T}^a_d = \mathbf{T}^a_d
\]

from where it follows immediately that

\[
\mathbf{R}^d - \frac{1}{2} \mathbf{R} \mathbf{g}^d = \mathbf{T}^d.
\]

Then

\[
\mathbf{R}^d + \partial \cdot \partial \mathbf{g}^d = \mathbf{T}^d + \frac{1}{2} \mathbf{R} \mathbf{g}^d + \partial \cdot \partial \mathbf{g}^d,
\]
and comparing Eq.\((29)\) with Eq.\((33)\) we get
\[
t^d = \frac{1}{2} R g^d + \partial \cdot \partial g^d, \tag{34}
\]
and
\[
t^d = \frac{1}{2} R g^d + \partial \cdot \partial g^d + d\delta g^d \tag{35}
\]
the nice formula promised and that clearly demonstrates that the objects \(t_{da}^\gamma \) are components of a legitimate gravitational energy-momentum tensor tensor field \(t = t_{da}^\gamma \otimes \tilde{g}^a \in \text{sec} T_0^2 M\). We observe moreover that
\[
t^{da} - t^{ad} = 2 \partial \cdot \partial g^d \otimes \tilde{g}^a, \tag{36}
\]
i.e., the energy-momentum tensor of the gravitational field is not symmetric. As shown in \([8]\) this is important in order to have a total angular momentum conservation law for the system consisting of the gravitational plus the matter fields. At least observe that \(t^d = t^d\) when the potentials are chosen in the Lorenz gauge.

5 Hamilton Formalism

If we define as usual the canonical momenta associated to the potentials \(\{g^a\}\) by \(p_a = \partial L_g / \partial \delta g^a = \ast S_a\) and suppose that this equation can be solved for the \(d\delta g^a\) as function of the \(p_a\) we can introduce a Legendre transformation with respect to the fields \(d\delta g^a\) by

\[
L : (g^\alpha, p_\alpha) \mapsto L(g^\alpha, p_\alpha) = d g^\alpha \wedge p_\alpha - L_g(g^\alpha, d g^\alpha(p_\alpha)) \tag{37}
\]
We write in what follows \(L_g(g^\alpha, p_\alpha) := L_g(g^\alpha, d g^\alpha(p_\alpha))\) and observe that defining\(^{16}\)

\[
\frac{\delta L_g(g^\alpha, p_\alpha)}{\delta g^\alpha} := -d p_\alpha - \frac{\partial L}{\partial g^\alpha}, \quad \frac{\delta L_g(g^\alpha, p_\alpha)}{\delta p_\alpha} := d g^\alpha - \frac{\partial L}{\partial p_\alpha}
\]
we can obtain (see details in \([8]\))
\[
\delta g^\alpha \wedge \frac{\delta L_g(g^\alpha, d g^\alpha)}{\delta g^\alpha} = \delta g^\alpha \wedge \left( \frac{\delta L_g(g^\alpha, p_\alpha)}{\delta g^\alpha} + \left( \frac{\delta L_g(g^\alpha, p_\alpha)}{\delta p_\alpha} \right) \right) \wedge \delta p_\alpha. \tag{38}
\]

To define the Hamiltonian form, we need something to act the role of time for our manifold, and we choose this ‘time’ to be given by the flow of an arbitrary timelike vector field \(Z \in \text{sec} T M\) such that \(g(Z, Z) = 1\). Moreover, we define

\(^{16}\)We use only constrained variations of the \(g^a\), which as already recalled in Section 2 do not change the metric field \(g\).
\[
Z = g(Z, \cdot) \in \sec \wedge^1 T^* M \hookrightarrow \mathcal{C}(g, M). \]
With this choice, the variation \(\delta\) is generated by the Lie derivative \(\mathcal{L}_Z\). Using Cartan’s ‘magical formula’, we have
\[
\delta \Omega_g = \mathcal{L}_Z \Omega_g = d(Z \lrcorner \Omega_g) + Z \lrcorner d \Omega_g = d(Z \lrcorner \Omega_g). \tag{39}
\]
and after some algebra we get
\[
d(Z \lrcorner \Omega_g) = d(\mathcal{L}_Z g^\alpha \wedge p_\alpha) + \mathcal{L}_Z g^\alpha \wedge \frac{\delta \Omega_g}{\delta g^\alpha} + \mathcal{L}_Z p_\alpha \wedge \left( \frac{\delta \Omega_g}{\delta p_\alpha} \right) \tag{40}
\]
and also
\[
d(\mathcal{L}_Z g^\alpha \wedge p_\alpha - Z \lrcorner \Omega_g) = \mathcal{L}_Z g^\alpha \wedge \frac{\delta \mathcal{L}_g}{\delta g^\alpha}. \tag{41}
\]
Now, we define the Hamiltonian 3-form by
\[
H(g^\alpha, p_\alpha) := \mathcal{L}_Z g^\alpha \wedge p_\alpha - Z \lrcorner \Omega_g. \tag{42}
\]
We immediately have taking into account Eq.\((41)\) that, when the field equations for the free gravitational field are satisfied (i.e., when the Euler-Lagrange functional is null, \(\delta \mathcal{L}_g / \delta g^\alpha = 0\)) that
\[
dH = 0. \tag{43}
\]
Thus \(H\) is a conserved Noether current. We next write
\[
H = Z^\alpha H_\alpha + dB \tag{44}
\]
We can show (details in \[8\]) that \(\mathcal{H}_\alpha = -\delta \mathcal{L}_g / \delta g^\alpha\) and \(B = Z^\alpha p_\alpha\) and now we investigate the meaning of the boundary term\([\mathfrak{B}]\) \(B\). Consider an arbitrary spacelike hypersurface \(\sigma\). Then, we define
\[
H = \int_\sigma (Z^\alpha \mathcal{H}_\alpha + dB) = \int_\sigma Z^\alpha \mathcal{H}_\alpha + \int_{\partial \sigma} B.
\]
If we recall that \(\mathcal{H}_\alpha = -\delta \mathcal{L}_g / \delta g^\alpha\) we see that the first term in the above equation is null when the field equations (for the free gravitational field) are satisfied and we are thus left with
\[
E = \int_{\partial \sigma} B, \tag{45}
\]
which is called the quasi local energy\([\mathfrak{22}]\).

Now, if \(\{e_\alpha\}\) is the dual basis of \(\{g^\alpha\}\) we have \(g^0(e_1) = 0, \; i = 1, 2, 3\) and if we take \(Z = e_0\) orthogonal to the hypersurface \(\sigma\), such that for each \(p \in \sigma, \; T_{\sigma p}\) is generated by \(\{e_1\}\) and we get recalling that \(p_\alpha = *S_\alpha\) that
\[
E = \int_{\partial \sigma} *S_0, \tag{46}
\]
\footnote{More details on possible choices of the boundary term for different physical situations may be found in \[11\].}
which we recognize as being the same conserved quantity as the one defined by Eq. (25).

The relation of the energy defined by Eq. (46) with the energy concept defined in ADM formalism \[1\] can be seen as follows \[25\]. Instead of choosing an arbitrary unit timelike vector field \( Z \), start with a global timelike vector field \( n \in \text{sec} T^*M \) such that \( n = g(n, \cdot) = N^2 dt \in \text{sec} \wedge^1 T^*M \hookrightarrow \mathcal{C}(M, g) \), with \( N : \mathbb{R} \ni I \rightarrow \mathbb{R} \), a positive function called the lapse function of \( M \). Then \( n \wedge dnn = 0 \) and according to Frobenius theorem, \( n \) induces a foliation of \( M \), i.e., topologically it is \( M = I \times \sigma_t \), where \( \sigma_t \) is a spacelike hypersurface with normal given by \( n \). Now, we can decompose any \( A \in \text{sec} \wedge^p T^*M \hookrightarrow \mathcal{C}(M, g) \) into a tangent component \( A_t \) to \( \sigma_t \) and an orthogonal component \( A_\perp \) to \( \sigma_t \) by

\[
A = A_t + A_\perp, \tag{47}
\]

where

\[
A_t := n_\cdot(dt \wedge A), \quad A_\perp := n_\cdot A. \tag{48}
\]

Introduce also the parallel component \( d \) of the differential operator \( d \) by:

\[
dA := n_\cdot(dt \wedge dA) \tag{49}
\]

from where it follows (taking into account Cartan’s magical formula) that

\[
dA = dt \wedge (\mathbf{L}_n A - dA_\perp) + dA_t. \tag{50}
\]

Call

\[
m := -g + n \otimes n = g^i \otimes \tilde{g}_i,
\]

(where \( n = n/N \)) the first fundamental form on \( \sigma_t \) and next introduce the Hodge dual operator associated to \( m \), acting on the (horizontal forms) forms \( A \) by

\[
\star_A := \star \left( n \wedge \frac{A}{N} \right). \tag{51}
\]

At this point, we come back to the Lagrangian density Eq. (12) and, proceeding like above, but now leaving \( \delta n^\alpha \) to be non null, we eventually arrive at the following Hamiltonian density

\[
\mathcal{H}(\tilde{g}^i, p_i) = \mathbf{L}_n \tilde{g}^i \wedge \star p_i - K_g, \tag{52}
\]

where

\[
g^i - \tilde{g}^i = dt \wedge (n_\cdot g^i) = n^i dt, \tag{53}
\]

and where \( K_g \) depends on \( (n, dn, g_\perp, \tilde{g}_i, \mathbf{L}_n g^i) \). We can show (after some tedious but straightforward algebra that \( \mathcal{H}(\tilde{g}^i, p_i) \) can be put into the form

\[
\mathcal{H} = n^i \mathcal{H}_i + dB', \tag{54}
\]

with as before \( \mathcal{H}_i = -\delta \mathcal{L}_g / \delta \tilde{g}^i = -\delta K_g / \delta n^i \) and

\[
B' = -N \tilde{g}_i \wedge \star d\tilde{g}^i \tag{55}
\]
Then, on shell, i.e., when the field equations are satisfied we get

\[ E' = - \int_{\partial\sigma_t} N g_0 \wedge d\theta^i \]  

which is exactly the ADM energy, as can be seen if we take into account that taking \( \partial\sigma_t \) as a twosphere at infinity, we have (using coordinates in the ELP gauge) \( g_i = h_{ij} dx^j \) and \( h_{ij}, N \to 1 \). Then

\[ g_i \wedge \star d\theta^i = h_{ij} \left( \frac{\partial h_{ij}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^j} \right) \star g^k \]  

and under the above conditions we have the ADM formula

\[ E' = \int_{\partial\sigma_t} \left( \frac{\partial h_{ik}}{\partial x^i} - \frac{\partial h_{ik}}{\partial x^k} \right) \star g^k \]  

which, as is well known, is positive definite\(^{18}\). If we choose \( n = g^0 \) it may happen that \( g^0 \wedge d\theta^0 \neq 0 \) and thus it does not determine a spacelike hypersurface \( \sigma_t \). However all algebraic calculations above up to Eq.(55) are valid (and of course, \( g^k = g^k \)). So, if we take a spacelike hypersurface \( \sigma \) such that at spatial infinity the \( e_i (g^k(e_i) = 0) \) are tangent to \( \sigma \), and \( e_0 \to \partial/\partial t \) is orthogonal to \( \sigma \), then we have \( E = E' \) since in this case \( -N g_0 \wedge \star d\theta^0 \to -g_0 \wedge \star (g^0 \wedge \star d\theta^0) \) which as can be easily verified (see Eq.(10)) is the asymptotic value of \( \star S^0 \) (taking into account that at spatial infinity \( d\theta^0 \to 0 \))

6 Conclusions

In this paper we recalled that a gravitational field generated by a given energy-momentum distribution can be represented by distinct geometrical structures and if we prefer, we can even dispense all those geometrical structures and simply represent the gravitational field as a field in the Faraday’s sense living in Minkowski spacetime. The explicit Lagrangian density for this theory has been given in a Maxwell like form and shown to be equivalent to Einstein’s equations in a precise mathematical sense. We identify a legitimate energy-momentum tensor for the gravitational field which can be expressed through a really nice formula, namely Eq.(34). We hope that our study clarifies the real difference between mathematical models and physical reality and leads people to think about the real physical nature of the gravitational field (and also of the electromagnetic field\(^{19}\)). We discuss also an Hamiltonian formalism for our theory and the concept of energy defined by Eq.(25) and the one given by the ADM formalism, which are shown to coincide

\(^{18}\)See a nice proof in [23].

\(^{19}\)As suggested, e.g., by the works of Laughlin [10] and Volikov [24]. Of course,, it may be necessary to explore also other ideas, like e.g., existence of branes in string theory. But this is a subject for another publication.
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