A semigroup of theta-curves in 3-manifolds

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ABSTRACT. We establish an existence and uniqueness theorem for prime decompositions of theta-curves in 3-manifolds.

1 Introduction

This paper is concerned with the existence and uniqueness of prime decompositions of theta-graphs in 3-manifolds. A theta-graph is a graph formed by two ordered vertices, called the leg and the head, and three oriented edges leading from the leg to the head and labeled with the symbols \{-, +, 0\} (different edges should have different labels.) Theta-graphs embedded in 3-manifolds are called theta-curves; their study is parallel to the study of knots (i.e., knotted circles) in 3-manifolds. More precisely, a theta-curve is a pair \((M, \Theta)\), where \(M\) is a compact connected oriented 3-manifold and \(\Theta\) is a theta-graph embedded in \(\text{Int}M\). By abuse of language, we will call \(\Theta\) a theta-curve in \(M\). For example, every 3-manifold \(M\) contains a unique (up to isotopy) flat theta-curve that lies in a 2-disc embedded into \(M\). The flat theta-curve in \(S^3\) is called the trivial theta-curve. Further examples of theta-curves can be obtained by tying knots on the edges of flat theta-curves (see below for details). The resulting theta-curves are said to be knot-like.

By homeomorphisms of theta-curves we mean homeomorphisms of pairs preserving orientation in the ambient 3-manifolds and the orientation and the labels of the edges of theta-curves.

The set of homeomorphism classes of theta-curves is denoted \(\mathcal{T}\). We define a vertex multiplication in \(\mathcal{T}\), see [Wo] for the case of theta-curves in \(S^3\). Given theta-curves \((M_i, \Theta_i)\) with \(i = 1, 2\), pick regular neighborhoods \(B_1 \subset M_1\) and \(B_2 \subset M_2\) of the head of \(\Theta_1\) and the leg of \(\Theta_2\), respectively. Glue \(M_1 \setminus \text{Int} B_1\) and \(M_2 \setminus \text{Int} B_2\) along an orientation-reversing homeomorphism \(\partial B_1 \to \partial B_2\) that carries the only intersection point of \(\partial B_1\) with the \(i\)-labeled edge of \(\Theta_1\) to the intersection point of \(\partial B_2\) with the \(i\)-labeled edge of \(\Theta_2\) for \(i \in \{-, 0, +\}\). The union \(\Theta\) of \(\Theta_1 \cap (M_1 \setminus \text{Int} B_1)\) and \(\Theta_2 \cap (M_2 \setminus \text{Int} B_2)\) is a theta-curve in \(M = M_1 \# M_2\). The theta-curve \((M, \Theta)\) is called the vertex product of \(\theta_1 = (M_1, \Theta_1), \theta_2 = (M_2, \Theta_2)\) and denoted \(\theta_1 \circ \theta_2\).

The vertex multiplication is associative and turns \(\mathcal{T}\) into a semigroup. The unit of \(\mathcal{T}\) is the trivial theta-curve. The semigroup \(\mathcal{T}\) is non-commutative but has a big center: it follows from the definitions that all knot-like theta-curves lie in the center of \(\mathcal{T}\). Note also that a product theta-curve \(\theta_1 \circ \theta_2\) is trivial if and only if both \(\theta_1\) and \(\theta_2\) are trivial, see [Mo, Wo].

We call a theta-curve prime if it is non-trivial and does not expand as a product of two non-trivial theta-curves. The following theorem is the main result of this paper.

**Theorem 1.** Let \(\theta = (M, \Theta)\) be a non-trivial theta-curve such that all 2-spheres in \(M\) are separating. Then:

1. \(\theta\) expands as a product \(\theta = \theta_1 \circ \theta_2 \circ \cdots \circ \theta_n\) for a finite sequence \(\theta_1, \ldots, \theta_n\) of prime theta-curves.

2. This expansion is unique up to relations of type \(\theta' \circ \theta'' = \theta'' \circ \theta'\), where \(\theta'\) or \(\theta''\) is knot-like.

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For \( M = S^3 \), this theorem is due to Motohashi \([M]\). A similar theorem for knots in 3-manifolds is also true; we obtain it at the end of the paper as a corollary of Theorem \([\text{Tu}]\)

A study of prime decompositions is a traditional area of 3-dimensional topology. We refer to \([\text{Mi}]\) and \([\text{Sh}]\) for prime decompositions of 3-manifolds and knots in \( S^3 \), to \([\text{Miy}]\) for prime decompositions of knots in 3-manifolds, and to \([\text{Pa}],\) \([\text{HM}]\) for prime decompositions of orbifolds and knotted graphs in 3-manifolds. Our interest in prime decompositions of theta-curves is due to a connection to so-called knotoids recently introduced by the second named author \([\text{Tu}]\).

Let us explain the reasons for our assumption on the 2-spheres in Theorem \([\text{Tu}]\). If \( M \) contains a non-separating 2-sphere \( S \), then there is a theta-curve \( \Theta \subset M \) meeting \( S \) transversely in one point of an edge of \( \Theta \). Using \( S \) it is easy to see that tying any local knot on this edge one obtains a theta-curve isotopic to \( \Theta \). This yields an infinite family of knot-like theta-curves that are factors of \((M, \Theta)\) and compromises the existence and uniqueness of prime decompositions of \((M, \Theta)\).

To prove Theorem \([\text{Tu}]\) we follow the general scheme introduced in \([\text{HM}]\). This scheme has been successfully applied to prove the existence and uniqueness of prime decompositions in many similar geometric situations.

## 2 Preliminaries on knots

**Definition 1.** A knot is a pair \((Q, K)\) where \( Q \) is a compact connected oriented 3-manifold and \( K \) is an oriented simple closed curve in \( \text{Int} \ Q \). Two knots \((Q, K), (Q', K')\) are equivalent, if there is a homeomorphism \((Q, K) \to (Q', K')\) preserving orientations of both \( Q \) and \( K \).

We call a knot \((Q, K)\) flat if \( K \) bounds an embedded disc in \( Q \). A knot \((Q, K)\) is trivial if \( Q = S^3 \) and \( K \) is flat. We emphasize that all knots in \( Q \neq S^3 \) are non-trivial. Denote by \( \mathcal{K} \) the set of all equivalence classes of knots. We equip \( \mathcal{K} \) with a binary operation \( \# \) (connected sum) as follows. Let \( k_i = (Q_i, K_i) \in \mathcal{K} \) for \( i = 1, 2 \). Choose a closed 3-ball \( B_i \subset Q_i \) such that \( l_i = B_i \cap K_i \) is an unknotted arc in \( B_i \). Let \( h : (B_1, l_1) \to (B_2, l_2) \) be a homeomorphism which reverses orientations of both the ball and the arc. Glue \( Q_1 \setminus \text{Int} \ B_1 \) and \( Q_2 \setminus \text{Int} \ B_2 \) along \( h|_{\partial B_1} : (\partial B_1, \partial l_1) \to (\partial B_2, \partial l_2) \). The resulting knot \((Q_1 \# Q_2, K_1 \# K_2)\) does not depend on the choice of \( B_1, B_2, \) and \( h \). This knot is called the connected sum of \( k_1, k_2 \) and denoted \( k_1 \# k_2 \).

The operation \( \# \) is commutative, associative, and has a neutral element represented by the trivial knot. A classical argument due to Fox \([\text{Fox}]\) shows that the knot \( k = k_1 \# k_2 \) is trivial if and only if both \( k_1 \) and \( k_2 \) are trivial. Namely, if \( k \) is trivial, then \( \#_{i=1}^\infty k_i = k \) and \( k_1 \# (\#_{i=2}^\infty k_i) = k_1 \). Therefore \( k_1 = k \) is trivial.

## 3 From knots to theta-curves

Given a knot \( k = (Q, K) \) and a label \( i \in \{-, 0, +\} \), we define a theta-curve in \( Q \) as follows. Pick a disc \( D \subset Q \) meeting \( K \) along an arc \( l' = D \cap K = \partial D \cap K \). The complementary arc \( l = K \setminus \text{Int} \ l' \) of \( K \) receives the label \( i \) and the orientation induced by that of \( K \), the arcs \( l' \) and \( l'' = \partial D \setminus l' \) receive the remaining labels. Then \( \Theta_K = l \cup l' \cup l'' \) is a theta-curve in \( Q \) (cf. Figure \([\text{Tu}]\)). The homeomorphism class of the theta-curve \((Q, \Theta_K)\) does not depend on the choice of \( D \) because any two such disks are isotopic. This class is denoted \( \tau_i(k) \).

It is easy to see from the definitions that the map \( K \to \mathcal{T}, k \mapsto \tau_i(k) \) is a semigroup homomorphism. This homomorphism is injective because its composition with the map \( \mathcal{T} \to K \) removing the \( j \)-labeled edge (for \( j \neq i \)) is the identity.
A theta-curve is knot-like if it lies in the image of one of $\tau_i$ for $i \in \{-, 0, +\}$. As was mentioned above, the knot-like theta-curves commute with all theta-curves, i.e., lie in the center of $\mathcal{T}$.

Given a knot $k = (Q, K)$, a theta-curve $\theta = (M, \Theta)$, and a label $i \in \{-, 0, +\}$, we define the knot insertion of $k$ into $\theta$ to be the theta-curve $\tau_i(k) \circ \theta = \theta \circ \tau_i(k)$. To construct this theta-curve geometrically, pick a 3-ball $B \subset M$ such that $B \cap \Theta$ is an unknotted arc in the $i$-labeled edge of $\Theta$. The theta-curve $\tau_i(k) \circ \theta = \theta \circ \tau_i(k)$ is obtained by cutting off $(B, B \cap \Theta)$ from $(M, \Theta)$ and coherent filling the resulting hole by $(Q, K)$. For $Q = S^3$, this is the standard tying of local knots on the edges of $\Theta$.

In analogy with $\tau_i$, we define a homomorphism $\tau: M \to \mathcal{T}$, where $M$ is the semigroup of compact connected oriented 3-manifolds with respect to connected summation. If $M \in M$, then $\tau(M)$ is a flat theta-curve inside. This suggests a notion of a manifold insertion. The insertion of $M \in M$ into a theta-curve $\theta = (Q, \Theta)$ yields the theta-curve $\tau(M) \circ \theta = \theta \circ \tau(M)$ obtained by replacing a ball in $Q \setminus \Theta$ by a copy of punctured $M$. The same theta-curve can be obtained by inserting a flat knot in $M$ into $\theta$.

### 4 Prime theta-curves and knots

**Lemma 1.** Let $k = (Q, K)$ be a knot, $i \in \{-, 0, +\}$, and $\tau_i(k) = (Q, \Theta_K)$ the corresponding theta-curve. Let $D$ be a disc in $Q$ such that $\partial D$ is the union of two edges of $\Theta_K$ with labels distinct from $i$. If $S \subset Q$ is a 2-sphere meeting each edge of $\Theta_K$ in one point, then there is a self-homeomorphism of $Q$ which keeps $\Theta_K$ fixed and carries $S$ to a 2-sphere $S'$ such that $S' \cap D$ is a single arc.

**Proof.** The set $S \cap D$ consists of an arc $\alpha$ joining two points of $S \cap \Theta_K$ and possibly of several circles. The innermost circle argument yields a disc $A \subset S$ such that $A \cap D = \partial A$. The circle $\partial A$ bounds a disc $A' \subset D$. Then the disc $(D \setminus A') \cup A$ is isotopic to a disc in $Q$ which spans the same edges of $\Theta_K$ and crosses $S$ along $\alpha$ and fewer circles. Continuing by induction, we obtain a spanning disc $D'$ such that $D' \cap S = \alpha$. There is a homeomorphism $h: Q \to Q$ that keeps $\Theta_K$ pointwise and carries $D'$ to $D$. Then $S' = h(S)$ is a required sphere.

**Lemma 2.** A knot $k = (Q, K)$ is prime if and only if the theta-curve $\tau_i(k) = (Q, \Theta_K)$ is prime for some (and hence for any) $i \in \{-, 0, +\}$. The 3-manifold $Q$ is prime if and only if the flat theta-curve $\tau(Q)$ is prime.

**Proof.** Recall that a knot (resp., a theta-curve) is prime if it is non-trivial and does not split as a connected sum (resp., a product) of two non-trivial knots (resp., theta-curves). Since $\tau_i$ is an
injective homomorphism, if \( \tau_i(k) \) is prime then so is \( k \). Suppose that that \( \tau_i(k) \) is not prime. Then there is a sphere \( S \subset Q \) meeting each edge of \( \Theta_K \) in one point and dividing \( (Q, \Theta_K) \) into two pieces \( (Q_j, Q_j \cap \Theta_K), j = 1, 2, \) not homeomorphic to a 3-ball with three radii. Denote by \( D \) a disc spanning the edges of \( \Theta_K \) with labels distinct from \( i \). By Lemma [1] we may assume that \( S \cap D \) is an arc dividing \( D \) into two subdiscs. We conclude that after deleting one of the edges spanned by \( D, i.e., after returning to \( k = (Q, K) \), the pieces \( (Q_j, Q_j \cap K) \) remain non-trivial, i.e., are not homeomorphic to a 3-ball with two radii. We conclude that \( k \) splits as a connected sum of two non-trivial knots.

The second claim of the lemma is obtained by applying the first claim to the flat knot \( K_0 \subset Q \). It is clear that \( Q \) is prime if and only if the knot \((Q, K_0)\) is prime. The latter holds if and only if the theta-curve \( \tau_i(K_0) = \tau(Q) \) is prime.

5 Spherical reductions

Spherical reductions are operations inverse to taking products of theta-curves and inserting knots. Denote by \( U \) the set of all pairs \((M, G)\), where \( M \) is a compact connected oriented 3-manifold and \( G \subset M \) is either a theta-graph, or a knot labeled by \( i \in \{-, 0, +\}, \) or the empty set. The pairs are considered up to homeomorphisms preserving all orientations and labels. In other words, \( U = \mathcal{T} \bigcup \mathcal{K}_- \bigcup \mathcal{K}_0 \bigcup \mathcal{K}_+ \bigcup \mathcal{M} \), where \( \mathcal{K}_i \) is the set of \( i \)-labeled knots.

**Definition 2.** Let \((M, G) \in U\). A separating sphere \( S \) in \( M \) is admissible if it is in general position with respect to \( G \) and \( S \cap G \) is either empty or consists of 2 or 3 points.

**Definition 3.** Given an admissible sphere \( S \in (M, G) \in U \), we cut \((M, G)\) along \( S \) and add cones over two copies of \((S, S \cap G)\) on the boundaries of the resulting two pieces of \((M, G)\). This gives two pairs \((M_j, G_j) \in U, j = 1, 2, \) where the orientations and labels of the edges of \( G_j \) are inherited from those of \( G \). We say that these pairs are obtained by spherical reduction of \((M, G)\) along \( S \).

The sphere \( S \) as above and the reduction along \( S \) are inessential if \( S \) bounds a 3-ball \( B \subset M \) such that \( B \cap G \) is either empty, or a proper unknotted arc, or consists of three radii of \( B \). The reduction along an inessential sphere produces a copy of \((M, G)\) and a trivial pair, which is either \((S^3, \emptyset)\), or a trivial knot, or a trivial theta-curve.

6 Mediator spheres

**Definition 4.** Let \( M \) be a 3-manifold and \( S_1, S_2, S_3 \) three mutually transversal spheres in \( M \). We call \( S_3 \) a sphere-mediator for \( S_1, S_2 \), if both numbers \( \#(S_3 \cap S_2) \) and \( \#(S_3 \cap S_1) \) are strictly smaller than \( \#(S_2 \cap S_1) \). Here \( \# \) denotes the number of circles.

**Lemma 3.** Let \((M, G) \in U \) and \( S_1, S_2 \) be admissible essential mutually transversal spheres in \( M \) such that \( S_1 \cap S_2 \neq \emptyset \). If all 2-spheres in \( M \) are separating, then there is an essential sphere-mediator \( S_3 \) for \( S_1, S_2 \).

**Proof.** Using an innermost circle argument, we can find two discs in \( S_1 \) intersecting \( S_2 \) solely along their boundaries. Since \( S_1 \) meets \( G \) in \( \leq 3 \) points, one of the discs, \( D \), meets \( G \) in \( \leq 1 \) point. The circle \( \partial D \) splits \( S_2 \) into two discs \( D' , D'' \) such that \( S_2 = D' \cup D \) and \( S'' = D'' \cup D \) are embedded spheres in \( M \). Since all spheres in \( M \) are separating, \( S' \) and \( S'' \) bound manifolds \( W', W'' \subset M \) respectively so that \( W' \cap W'' = D \) and \( \partial(W' \cup W'') = S_2 \). Let \( X \) be the closure of \( M \setminus (W' \cup W'') \), see Figure [2].
Figure 2: The spheres $S_1$ and $S_2$

Case 1: $D \cap G = \emptyset$. Since the intersection of $G$ with a separating sphere cannot consist of one point and $G$ meets $S_2$ in $\leq 3$ points, at least one of the spheres $S'$, $S''$, say $S'$, does not meet $G$. If $S' = \partial W'$ is essential, then pushing it slightly inside $W'$ we obtain a sphere-mediator for $S_1$, $S_2$. Indeed, the latter sphere is disjoint from $S_2$ and meets $S_1$ in fewer circles.

If $S'$ is inessential, then it bounds a 3-ball in $M$ disjoint from $G$. This ball is either $W'$ or $W'' \cup X$. The second option is impossible, since $W'' \cup X$ contains the essential sphere $S_2$. Hence $W'$ is a 3-ball, and we can use it to isotope $D'$ to the other side of $S_1$ and thus transform $S_2$ into a parallel copy of $S''$ disjoint from $S_2$. This copy of $S''$ is a sphere-mediator for $S_1$, $S_2$.

If $W'$ is inessential, then $W'$ is a 3-ball and $W' \cap G$ is an unknotted arc. As above, we can use $W'$ to isotope $D'$ to the other side of $S_1$ and thus transform $S_2$ into a sphere-mediator for $S_1$, $S_2$.

Case 2: $D \cap G$ is a one-point set. An argument as above shows that one of the spheres $S'$, $S''$, say $S'$, meets $G$ in two points. If $S'$ is essential, then after a small isotopy it can be taken as a sphere-mediator. If $S'$ is inessential, then $W'$ is a 3-ball and $W' \cap G$ is an unknotted arc. As above, we can use $W'$ to isotope $D'$ to the other side of $S_1$ and thus transform $S_2$ into a sphere-mediator for $S_1$, $S_2$.

7 Digression into theory of roots

Let $\Gamma$ be an oriented graph. The set of vertices of $\Gamma$ will be denoted $V(\Gamma)$. By a path in $\Gamma$ from a vertex $V$ to a vertex $W$ we mean a sequence of coherently oriented edges $\overrightarrow{VV_1}, \overrightarrow{V_1V_2}, \ldots, \overrightarrow{V_nW}$, where $V_1, \ldots, V_n \in V(\Gamma)$. A vertex $W$ of $\Gamma$ is a subordinate of a vertex $V$, if either $V = W$ or there is a path from $V$ to $W$ in $\Gamma$. A vertex $W$ is a root of $V$, if $W$ is a subordinate of $V$ and $W$ has no outgoing edges.

We say that $\Gamma$ has property (F) if for any vertex $V \in V(\Gamma)$ there is an integer $C \geq 0$ such that any path in $\Gamma$ starting at $V$ consists of no more than $C$ edges. It is obvious that if $\Gamma$ has property (F) then every vertex of $\Gamma$ has a root. To study the uniqueness of the root, we need the following notion.

**Definition 5.** Two edges $e$ and $d$ of $\Gamma$ are equivalent if there is a sequence of edges $e = e_1, e_2, \ldots, e_n = d$ of $\Gamma$ with the same initial vertex such that the terminal vertices of $e_i$ and $e_{i+1}$ have a common root for all $i = 1, \ldots, n - 1$.

We say that $\Gamma$ has property (EE) if any edges of $\Gamma$ with common initial vertex are equivalent. The following theorem is a version of the classical Diamond Lemma due to Newman [Ne].

**Theorem 2.** ([HM]) If $\Gamma$ has properties (F) and (EE), then every vertex of $\Gamma$ has a unique root.
Note that in [HM] the role of the property (F) is played by a property (CF) which says that there is a map \( c: V(\Gamma) \to \{0, 1, 2, \ldots\} \) such that \( c(V) > c(W) \) for every edge \( VW \) of \( \Gamma \). The property (F) implies (CF); an appropriate map \( c \) is defined as follows: for any vertex \( V \) of \( \Gamma \), \( c(V) \) is the maximal number of edges in a path in \( \Gamma \) starting at \( V \).

Recall from Section 5 the set \( U \) whose elements are (homeomorphism classes of) theta-curves, labeled knots, and 3-manifolds. We construct an oriented graph \( \Gamma \) as follows. A vertex of \( \Gamma \) is a finite sequence of elements of \( U \) (possibly with repetitions) considered up to the following transformations: (i) permutations that change the position of labeled knots and 3-manifolds in the sequence but keep the order of theta-curves; (ii) permutations of two consecutive terms of a sequence \( \theta', \theta'' \) allowed when both terms \( \theta', \theta'' \) are theta-curves and at least one of them is knot-like; (iii) insertion or deletion trivial theta-curves, trivial labeled knots, and copies of \( S^3 \).

We now define the edges of \( \Gamma \). Let a vertex \( V \) of \( \Gamma \) be represented by a sequence \( u_1, \ldots, u_n \in U \) and let \( i \in \{1, \ldots, n\} \). Suppose that \( (M_1, G_1), (M_2, G_2) \) are obtained from \( u_i = (M, G) \) by an essential spherical reduction along a sphere \( S \subset M \). If \( G \) is a theta-curve, we choose the numeration so that \( (M_1, G_1) \) contains the leg of \( G \) and \( (M_2, G_2) \) contains the head of \( G \). If \( G \) is a knot or an empty set, then the numeration is arbitrary. Let \( W \) be the vertex of \( \Gamma \) represented by the sequence \( u_1, \ldots, u_{i-1}, (M_1, G_1), (M_2, G_2), u_{i+1}, \ldots, u_n \). We say that \( W \) is obtained from \( V \) by essential spherical reduction along \( S \). Two vertices \( V, W \) of \( \Gamma \) are joined by an edge \( VW \) if \( W \) can be obtained from \( V \) in this way.

**Lemma 4.** \( \Gamma \) has property (F).

**Proof.** This is a special case of Lemma 6 of [HM]. \[\[\]

**Definition 6.** For each \( u \in U \), we define a subgraph \( \Gamma_u \) of \( \Gamma \) as follows. The vertices of \( \Gamma_u \) are all vertices of \( \Gamma \) subordinate to the vertex of \( \Gamma \) represented by the 1-term sequence \( u \). The edges of \( \Gamma_u \) are all the edges of \( \Gamma \) with both endpoints in \( \Gamma_u \).

**Lemma 5.** Let \( u = (M, G) \) be a theta-curve or a labeled knot such that all 2-spheres in \( M \) are separating. Then \( \Gamma_u \) has property (EE).

**Proof.** Let \( V = (u_1, \ldots, u_n) \) be a vertex of \( \Gamma_u \). Suppose that edges \( VW_1, VW_2 \) of \( \Gamma_u \) correspond to reductions along essential spheres \( S_1, S_2 \). These spheres lie in the ambient 3-manifolds of \( u_p, u_q \) for some \( p, q \in \{1, \ldots, n\} \). If \( p \neq q \), then \( S_1, S_2 \) survive the reduction along each other. Thus we may consider \( S_1 \) as a sphere in (a term of) \( W_2 \) and \( S_2 \) as a sphere in (a term of) \( W_1 \). Both these spheres are essential and the reductions of \( W_2 \) along \( S_1 \) and of \( W_1 \) along \( S_2 \) yield the same vertex, \( W \), of \( \Gamma_u \). Any root of \( W \) is a common root of \( W_1 \) and \( W_2 \), and therefore the edges \( VW_1, VW_2 \) are equivalent.

It remains to consider the case where both spheres \( S_1, S_2 \) lie in the ambient 3-manifold \( M_p \) of the same term \( u_p = (M_p, G_p) \) of \( V \). Note that \( M_p \) is a submanifold of \( M \) and therefore all 2-spheres in \( M_p \) are separating. We prove the equivalence of the edges \( VW_1, VW_2 \) by induction on the number \( m \) of circles in \( S_1 \cap S_2 \).

**Base of induction.** Let \( m = 0 \), i.e., \( S_1, S_2 \) are disjoint. Then each of these spheres survives the reduction along the other. Thus we may consider \( S_1 \) as a sphere in (a term of) \( W_2 \) and \( S_2 \) as a sphere in (a term of) \( W_1 \). Consider the vertices \( W_3, W_3' \) of \( \Gamma_u \) obtained by reducing \( W_2 \) along \( S_1 \) and \( W_1 \) along \( S_2 \). Let us prove that \( W_3 = W_3' \), see the diagram on the left-hand side of Figure 3. Assume first that each sphere \( S_1, S_2 \) meets \( G_p \) in three points. Then \( G_p \) is a theta-curve and the reductions of \( (M_p, G_p) \) along \( S_1, S_2 \) give three nontrivial theta-curves \( \theta_i = (Q_i, \Theta_i), 1 \leq i \leq 3 \), where \( \Theta_1 \) and \( \Theta_3 \) contain the leg and the head of \( G_p \), respectively. It follows that both \( W_3 \) and \( W_3' \) are obtained from \( V \) by replacing the term \( (M_p, G_p) \) with 3 terms
Figure 3: (A) Reductions along disjoint spheres. (B) The ordering of $\theta_i$ is natural

$\theta_1, \theta_2, \theta_3$, see Figure 3. The other cases where at least one of the spheres $S_1, S_2$ meets $G_p$ in 2 or 0 points are treated similarly.

We claim that any root $R$ of $W_3$ is a common root of $W_1$ and $W_2$. Indeed, if $S_1$ is essential in $W_2$, then $R$ is a root of $W_2$ by the definition of a root. If $S_1$ is inessential in $W_2$, the reduction along it results in adding to $W_2$ either $S^3$, or a trivial knot, or a trivial theta-curve. Then $W_2 = W_3$ by the definition of a vertex of $\Gamma$. Therefore $R$ is a root of $W_2$. Similarly, $R$ is a root of $W_1$. Therefore, the edges $\overrightarrow{VW_1}, \overrightarrow{VW_2}$ are equivalent.

Inductive step. Let $\#(S_1 \cap S_2) = m + 1$. It follows from Lemma 3 that there is an essential sphere-mediator $S_3$ such that it intersects $S_1$ and $S_2$ in a smaller number of circles. By the inductive assumption we know that the corresponding edge $\overrightarrow{VW_3}$ is equivalent to $\overrightarrow{VW_1}$ and $\overrightarrow{VW_2}$. It follows that $\overrightarrow{VW_1}$ and $\overrightarrow{VW_2}$ are also equivalent.

Corollary 1. Any vertex of $\Gamma_u$ has a unique root.

This follows from Theorem 2 and Lemmas 4 and 5.

8 Proof of Theorem 1

We apply to $\theta$ consecutive reductions along essential spheres meeting the corresponding theta-curves in three points. After $m \geq 1$ reductions we obtain a sequence of $m+1$ theta-curves. Since $\Gamma$ has property (F), for some $m$ there will be no essential spheres meeting the corresponding theta-curves in three points. This means that all the theta-curves obtained after $m$ reductions are prime. We obtain thus a sequence of prime theta-curves whose product is equal to $\theta$. This proves the first claim of the theorem.

We now prove the second claim. Consider an expansion of $\theta$ as a product of $n$ prime theta-curves $\theta_1, \ldots, \theta_n$. Let $W$ be the sequence $\theta_1, \ldots, \theta_n$. It may happen that the theta-curve $\theta_j = (Q_j, \Theta_j)$ admits an essential reduction along a sphere $S \subset Q_j$ meeting $\Theta_j$ in two points. These points have to lie on the same edge $e$ of $\Theta_j$ because otherwise the sphere $S$ would be non-separating. If $i \in \{-, 0, +\}$ is the label of $e$, then this spherical reduction produces a theta-curve $\theta'_j$ and a knot $k_j \in K_i$ such that $\theta_j = \theta'_j \circ \tau_i(k_j)$. Since $\theta_j$ is prime, $\theta'_j$ is trivial. We may conclude that $\theta_j = \tau_i(k_j)$ is knot-like, where $k$ is a prime knot by Lemma 2. Similarly, if $\theta_j = (Q_j, \Theta_j)$ admits an essential reduction along a sphere disjoint from $\Theta_j$, then $\theta_j = \tau(Q)$ is also knot-like, where $Q$ is a prime manifold.
Replacing in the sequence $W$ all knot-like $\theta_j$ by the corresponding knots $k_j$, we obtain a root of the vertex $\theta$ of $\Gamma$. By the uniqueness of the root (Corollary 1), the expansion $\theta = \prod_{j=1}^{n} \theta_j$ is unique up to the commutation relations of knot-like theta-curves with all the others.

9 Corollaries

**Theorem 3.** Let $k = (Q, K)$ be a non-trivial knot such that all 2-spheres in $Q$ are separating. Then $k$ expands as a connected sum $k = k_1 \# k_2 \# \ldots \# k_n$ of $n \geq 1$ prime knots. This expansion is unique up to permutations of $k_1, ..., k_n$.

**Proof.** Pick any $i \in \{-, 0, +\}$. The claim follows from Theorem 1, the injectivity of the semigroup homomorphism $\tau_i : K \to T$, and the fact that $k \in K$ is prime if and only if $\tau_i(k)$ is prime (Lemma 2).

A more general version of this theorem was proved by Miyazaki [Miy].

Let $U^o = T^o \bigcup K^o \bigcup K^o_0 \bigcup K^o_+ \bigcup M^o$ be a subset of $U$ consisting of the pairs $(M, G)$ such that all spheres in $M$ are separating. Any element of $U^o$ expands as a product (or connected sum) of prime elements of $U^o$. Let $\hat{T}^o$ be the subsemigroup of $T^o$ consisting of theta-curves having no knot-like factors. Similarly, denote by $\hat{K}^o$ the subsemigroup of $K^o$ consisting of knots having no 3-manifold summands. A knot $(Q, K) \in K^o$ lies in $\hat{K}^o$ if and only if $Q \setminus K$ is an irreducible 3-manifold. For $i \in \{-, 0, +\}$ we denote by $\hat{K}^o_i$ a copy of $\hat{K}^o$ formed by $i$-labeled knots.

**Theorem 4.** The following holds:

1. $M^o$ and $\hat{K}^o_0$ are free abelian semigroups freely generated by their prime elements.
2. $\hat{T}^o$ is a free semigroup freely generated by its prime elements.
3. $K^o = \hat{K}^o \times M^o$.
4. $T^o = \hat{T}^o \times C$, where $C = \hat{K}^o_- \times \hat{K}^o_0 \times \hat{K}^o_+ \times M^o$ is the center of $T^o$.

**Proof.** This follows from Theorems 1 and 3.

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