TREES WITH GIVEN STABILITY NUMBER AND MINIMUM NUMBER OF STABLE SETS

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Abstract. We study the structure of trees minimizing their number of stable sets for given order \( n \) and stability number \( \alpha \). Our main result is that the edges of a non-trivial extremal tree can be partitioned into \( n - \alpha \) stars, each of size \( \left\lceil \frac{n-1}{n-\alpha} \right\rceil \) or \( \left\lfloor \frac{n-1}{n-\alpha} \right\rfloor \), so that every vertex is included in at most two distinct stars, and the centers of these stars form a stable set of the tree.

1. Introduction

The number \( F(G) \) of stable sets (or independent sets) of a graph \( G \) was first considered by Prodinger and Tichy \[18\]. They called it the Fibonacci number of a graph, based on the observation that the number of stable sets in a path on \( n \) vertices is exactly the \( n + 2 \)-th Fibonacci number. The invariant \( F(G) \) is also known as the Merrifield-Simmons index or \( \sigma \)-index of the graph \( G \) \[14\].

There is a rich literature dealing with extremal questions regarding \( F(G) \), and the case where \( G \) is a tree received much attention. For instance, Heuberger and Wagner \[6\] characterized the \( n \)-vertex trees \( T \) with maximum degree \( \Delta \) maximizing \( F(T) \), for given \( n \) and \( \Delta \). Li, Zhao, and Gutman \[11\], and independently Pedersen and Vestergaard \[17\] determined the \( n \)-vertex trees \( T \) maximizing \( F(T) \) when the diameter is fixed. Yu and Lv \[21\] considered similarly the case of trees with a fixed number of leaves. The reader is referred to \[8, 12, 20\] and the references therein for other results of this kind on trees. Other classes of graphs have been considered as well; this includes unicyclic graphs \[10, 15, 16, 19\], bicyclic graphs \[3\], tricyclic graphs \[22\], quasi-trees \[9\], maximal outerplanar graphs \[1\], connected graphs \[2\], and bipartite \( d \)-regular graphs \[7\].

In this paper, we propose to revisit an old result of Prodinger and Tichy \[18\] which is well-known in this area: among all trees \( T \) on \( n \geq 2 \) vertices, the path \( P_n \) and the star \( K_{1,n-1} \) respectively minimizes and maximizes \( F(T) \). Observe that the stability number (the largest cardinality of a stable set) of \( P_n \) is \( \lceil n/2 \rceil \), while that of \( K_{1,n-1} \) is \( n - 1 \). Since the stability number \( \alpha \) of every tree on \( n \) vertices is between these two extreme values, one may wonder which are the trees \( T \) minimizing or maximizing \( F(T) \) for fixed \( \alpha \). In a previous contribution \[2\], a subset of the authors showed that the trees maximizing \( F(T) \) have a rather simple structure: they are exactly the trees that can be obtained from the Turán graph \[1\] \( T(n, \alpha) \) by adding \( \alpha - 1 \) edges.

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1 The Turán graph \( T(n, \alpha) \) is the \( n \)-vertex graph consisting of \( \alpha \) vertex-disjoint cliques which are as balanced as possible.
Figure 1. An extremal tree for $n = 18$ and $\alpha = 13$. White vertices are the centers of the stars.

Here, we focus on the trees minimizing $F(T)$ for fixed $\alpha$. As will become apparent in the next sections, the structure of the corresponding extremal trees is less straightforward than when maximizing $F(T)$. Our main result is that the edges of a (non-trivial) extremal tree can uniquely be partitioned into $n - \alpha$ stars, each of size $\left\lceil \frac{n-1}{n-\alpha} \right\rceil$ or $\left\lfloor \frac{n-1}{n-\alpha} \right\rfloor$, so that every vertex is included in at most two distinct stars, and the centers of these stars form a stable set of the tree. (See Figure 1 for an illustration.)

It follows that an extremal tree can be completely described by specifying which stars have a vertex in common. This is captured by the following labeled tree, which we call center-tree: the tree has $n - \alpha$ vertices (representing the stars); there is an edge between two vertices if the corresponding stars have a vertex in common, and each vertex is labeled by the size of the corresponding star. However, characterizing the center-trees of extremal trees appears to be challenging. We obtained such a characterization in a very restricted case, namely when $(n - 1) \mod (n - \alpha) \in \{0, n - \alpha - 2, n - \alpha - 1\}$, but the general case is far from understood and is left as an open problem.

The paper is organized as follows. In Section 2, we provide the necessary definitions and a few basic lemmas. Sections 3 and 4, which constitute the bulk of this paper, are devoted to the proof that extremal trees have a partition into stars as described above (cf. Theorem 14). Finally, in Section 5 we conclude with some remarks on the problem of characterizing the center-trees of extremal trees, and provide a solution for the particular case mentioned above.

2. Definitions and Preliminaries

This section is devoted to basic definitions and notations used throughout the text. Also, we introduce the notion of tree of stars, and prove some useful facts about these trees.

All graphs are assumed to be finite, simple, and undirected. We generally follow the terminology and notations of Diestel [4]. The neighborhood of a vertex $v$ of a graph $G$ is denoted $N_G(v)$; also, we write $N_G[v]$ for the set $N_G(v) \cup \{v\}$. The degree of $v$ is denoted by $\deg_G(v)$. (We often drop the subscript $G$ when the graph is clear from the context.) We simply write $|G|$ for the order $|V(G)|$ of $G$. A subset $S \subseteq V(G)$ of vertices of a graph $G$ is a stable set if the vertices in $S$ are pairwise non adjacent in $G$. The maximum cardinality of a stable set in $G$ is the stability number of $G$, and is denoted $\alpha(G)$.

Recall that $F(G)$ denotes the number of stable sets of $G$ (including the empty set). In this paper, we call this invariant the Fibonacci number of $G$, as in [18]. The following easy properties (see [5,11,18]) of the Fibonacci number are used throughout the paper:

Lemma 1. Let $G$ be a graph. If $G$ is not empty, then,

- $F(G) = F(G - v) + F(G - N[v])$ for every vertex $v$ of $G$, and
• $F(G) = \prod_{i=1}^{k} F(G_i)$ if $G$ is the disjoint union of $k$ graphs $G_1, G_2, \ldots, G_k$.

In particular,

• $F(G) > F(G - v)$ for every vertex $v$ of $G$, and
• $F(G) \leq 2F(G - v)$, with strict inequality if $\deg_G(v) \geq 1$.

A tree $T$ is extremal if $T$ has minimum Fibonacci number among all trees on $|T|$ vertices with stability number $\alpha(T)$. As mentioned in the introduction, the structure of these extremal trees is the main topic of this paper; in particular, we will see that non-trivial extremal trees are trees of stars that are balanced (see below for the definitions).

In this paper, a leaf of a tree is defined as a vertex of degree at most 1. Thus in particular if the tree has a unique vertex, then this vertex is considered to be a leaf. (Usually leaves are required to have degree exactly 1, but this definition will be more convenient for our purposes.). A tree of stars is defined inductively as follows:

• a single vertex is a tree of stars;
• if $T_1, T_2, \ldots, T_k$ ($k \geq 2$) are disjoint trees of stars and $v_i$ is a leaf of $T_i$ for $i = 1, 2, \ldots, k$, then the tree obtained from $T_1 \cup \cdots \cup T_k$ by adding a new vertex $w$ adjacent to every vertex $v_i$ is also a tree of stars.

From the inductive definition given above, it is easy to check that, in a tree of stars $T$, the distances between a fixed vertex $v \in V(T)$ and all the leaves of $T$ have the same parity. The vertex $v$ is a center of $T$ if these distances are odd. An example of a tree of stars is given in Figure 2.

![Figure 2. A tree of stars. The white vertices are the centers of the tree.](image)

We note that trees of stars can equivalently be defined as follows. First observe that the vertex set of a tree $T$ can uniquely be partitioned into two stable sets $A$ and $B$. Then $T$ is a tree of stars if and only if one of these two sets, say $A$, contains all leaves of $T$ and no vertex with degree at least 3 (thus all vertices in $A$ have degree at most 2). In that case, the centers of $T$ are exactly the vertices in $B$. (We leave it to the reader to check that this definition of trees of stars is indeed equivalent to the original one.)

A tree of stars is balanced if the degrees of any two of its centers differ by at most 1 (for instance, the tree of stars in Figure 1 is balanced while the one in Figure 2 is not).

Let $T$ be a tree of stars. The set of centers of $T$ is denoted by $C(T)$. Observe that a vertex of $T$ with degree at least 3 is always a center; thus, the set $V(T) - C(T)$ includes only vertices with degree at most 2. Also, notice that $(C(T), V(T) - C(T))$ is a partition of $V(T)$ into two stable sets. Finally, note that a path $P$ is a tree of stars if and only if $P$ is even.

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2 $P$ is odd (even) if $P$ has odd (respectively even) length, where the length is defined as the number of edges.
Lemma 2. Let \( T \) be a tree of stars. Then \( V(T) - C(T) \) is the unique maximum stable set of \( T \).

Proof. We use induction on \( |C(T)| \). If \( |C(T)| = 0 \), then \( T \) is a single vertex and the claim trivially holds. Suppose \( |C(T)| \geq 1 \), and let \( S \) be any maximum stable set of \( T \). Also, let \( v \) be a leaf of \( T \) and \( w \) be its unique neighbor. Let \( T_1, \ldots, T_k \) be the components of \( T - \{v, w\} \).

Notice that each \( T_i \) is a tree of stars and \( C(T) = \{w\} \cup C(T_1) \cup \cdots \cup C(T_k) \).

First, suppose that \( S \cap V(T_i) \neq V(T_i) - C(T_i) \) for some \( i \in \{1, \ldots, k\} \). Then, by induction, \(|S \cap V(T_i)| < |V(T_i) - C(T_i)|\), and the set

\[
S' := (S - (S \cap V(T_i))) \cup (V(T_i) - C(T_i))
\]

has cardinality larger than \( S \). Since \( S' \) cannot be a stable set of \( T \), it follows \( w \in S' \), and thus \( v \notin S' \). But then \((S' - \{w\}) \cup \{v\}\) is a stable set of \( T \), with cardinality equal to that of \( S' \), a contradiction.

Therefore, \( S \cap V(T_i) = V(T_i) - C(T_i) \) for every \( i \in \{1, \ldots, k\} \), which implies \( w \notin S \), and hence \( v \in S \). It follows that \( S = V(T) - C(T) \). \( \square \)

Suppose that \( T \) is a tree not isomorphic to a path and that \( T \) has a leaf \( w \) such that the tree \( T' \) obtained by adding a new vertex \( v \) adjacent to \( w \) is a tree of stars. Then \( T \) is said to be almost a tree of stars, and a vertex of \( T \) is considered to be a center of \( T \) if it is a center of \( T' \). Note that the requirement that \( T \) is not a path ensures that the leaf \( w \) is uniquely determined. Hence, the set of centers of \( T \) is well-defined. The vertex \( w \) is the unique center of \( T \) which is also a leaf, and is said to be the exposed center of \( T \). See Figure 3 for an illustration.

![Figure 3](image-url)  

**Figure 3.** A tree which is almost a tree of stars. Its exposed center is drawn in grey.

Let us observe that, if \( v \) is a leaf of a tree of stars \( T \) with \(|T| \geq 3 \), then \( T - v \) is exactly one of the following:

- a tree of stars;
- almost a tree of stars;
- an odd path.

This fact will be repeatedly used in our proofs.

3. Extremal Trees are Trees of Stars

The following theorem is a first step towards understanding the structure of extremal trees.

**Theorem 3.** Every extremal tree is either a tree of stars or an odd path.

The proof of Theorem 3 is based on the concept of rotating an edge in a tree \( T \), which consists in first removing some edge \( uv \) of \( T \), and then adding an edge \( uv' \), where \( v' \) is some
vertex of the component of $T - uv$ containing $v$. We will show that if $T$ is neither a tree of stars nor an odd path, then there exists an edge of $T$ that can be rotated in such a way that the resulting tree $T'$ satisfies both $\alpha(T') = \alpha(T)$ and $F(T') < F(T)$. To this aim, we introduce a few technical lemmas.

3.1. Trees of Stars and the Golden Ratio. The golden ratio $\phi := \frac{1 + \sqrt{5}}{2} \approx 1.618$ appears naturally when considering trees of stars, as illustrated by the following two lemmas.

**Lemma 4.** Let $T$ be a tree of stars and let $v$ be a leaf of $T$. Then $F(T) > \phi \cdot F(T - v)$. Moreover, this remains true if $T$ is almost a tree of stars, provided $v$ is not the exposed center.

**Proof.** The proof is by induction on $|T|$. The claim is true when $|T| = 1$, since then $F(T) = 2 > \phi \cdot F(T - v) = \phi$.

For the inductive step, assume $|T| > 1$. Let $w$ be the center adjacent to $v$ and $T_1, \ldots, T_k$ be the components of $T - \{v, w\}$ (thus $k \geq 1$). Let also $v_i$ be the leaf of $T_i$ that is adjacent to $w$ in $T$. Letting

$$\gamma := \prod_{i=1}^{k} \frac{F(T_i)}{F(T_i - v_i)},$$

we deduce

$$\frac{F(T)}{F(T - v)} = \frac{2\prod_{i=1}^{k} F(T_i)}{\prod_{i=1}^{k} F(T_i) + \prod_{i=1}^{k} F(T_i - v_i)} = 2 - \frac{1}{1 + \gamma}.$$  

If $T$ is a tree of stars, then $T_i$ is also a tree of stars and $F(T_i) > \phi \cdot F(T_i - v_i)$ by the induction hypothesis, for every $i \in \{1, \ldots, k\}$. Hence, $\gamma > \phi^k \geq \phi$.

If, on the other hand, $T$ is almost a tree of stars, then all trees $T_i$ are trees of stars except for exactly one, say $T_1$. Thus, by induction, $F(T_i) > \phi \cdot F(T_i - v_i)$ for every $i \in \{2, \ldots, k\}$. Moreover, $T_1$ is either almost a tree of stars or an odd path. In the first case, $v_1$ is not the exposed center of $T_1$, and thus $F(T_1) > \phi \cdot F(T_1 - v_1)$ by induction, which again implies $\gamma > \phi^k \geq \phi$. In the second case, we have $k \geq 2$ since $T$ is not a path, and

$$\gamma \geq \prod_{i=2}^{k} \frac{F(T_i)}{F(T_i - v_i)} > \phi^{k-1} \geq \phi,$$

by induction.

Therefore, $\gamma > \phi$ holds in all possible cases, and it follows

$$\frac{F(T)}{F(T - v)} = 2 - \frac{1}{1 + \gamma} > 2 - \frac{1}{1 + \phi} = \phi.$$  

(The last equality is derived using the fact $\phi^2 = \phi + 1$.)

A similar but opposite property holds for centers:

**Lemma 5.** Let $T$ be a tree of stars and let $w$ be a center of $T$. Then $F(T) < \phi \cdot F(T - w)$. This moreover remains true if $T$ is almost a tree of stars.

Note that, contrary to Lemma 4, here the vertex $w$ is allowed to be the exposed center of $T$ if $T$ is almost a tree of stars.
Proof of Lemma 3. Let $T_1, \ldots, T_k$ be the components of $T - w$. (Thus $k \geq 1$ and $k = 1$ if and only if $w$ is the exposed center of $T$.) We denote by $v_i$ the leaf of $T_i$ that is adjacent to $w$. Let

$$\gamma := \prod_{i=1}^{k} \frac{F(T_i)}{F(T_i - v_i)}.$$ 

We have

$$\frac{F(T)}{F(T - w)} = \frac{\prod_{i=1}^{k} F(T_i) + \prod_{i=1}^{k} F(T_i - v_i)}{\prod_{i=1}^{k} F(T_i)} = 1 + \frac{1}{\gamma}.$$ 

If $T$ is a tree of stars, then $T_i$ is a tree of stars for every $i \in \{1, \ldots, k\}$. The same is true if $T$ is almost a tree of stars and $w$ is the exposed center of $T$. Thus, in these two cases, we have $F(T_i) > \phi \cdot F(T_i - v_i)$ for every $i \in \{1, \ldots, k\}$ by Lemma 4, which implies $\gamma > \phi^k \geq \phi$.

Now, if $T$ is almost a tree of stars and $w$ is not the exposed center of $T$, then $k \geq 2$ and all trees $T_i$ are trees of stars except for exactly one, say $T_1$. Thus, $F(T_i) > \phi \cdot F(T_i - v_i)$ for every $i \in \{2, \ldots, k\}$ by Lemma 4. Since $F(T_1) \geq F(T_1 - v_1)$, this implies

$$\gamma \geq \prod_{i=2}^{k} \frac{F(T_i)}{F(T_i - v_i)} > \phi^{k-1} \geq \phi.$$ 

It follows that $\gamma > \phi$ always holds, implying

$$\frac{F(T)}{F(T - w)} = 1 + \frac{1}{\gamma} < 1 + \frac{1}{\phi} = \phi.$$ 

$\square$

3.2. Edge Rotations. Let $T$ be a tree, $xy$ be one of its edges, and $x'$ be a vertex distinct from $x$ in the component of $T - xy$ containing $x$. Then the pair $\rho := (yx, yx')$ is called a rotation, and we let $\rho(T) := (T - yx) + yx'$ denote the tree resulting from the rotation. The rotation $\rho$ is good if $\alpha(\rho(T)) = \alpha(T)$ and $F(\rho(T)) < F(T)$. (Thus, in order to prove that a tree is not extremal, it is enough to show that it admits a good rotation.)

When the rotation $\rho = (yx, yx')$ is clear from the context, we use the following notations.

**Notation 6.** Denote by $z$ ($z'$) the neighbor of $x$ (respectively $x'$) that is included in the unique $xx'$-path in $T$. (Observe that $z = x'$ and $z' = x$ if $xx' \in E(T)$; similarly, $z = z'$ if $x$ and $x'$ are at distance 2.) Let $x_1, \ldots, x_\ell$ be the neighbors of $x$ distinct from $z$ and $y$, and $x_1', \ldots, x_{\ell'}$ be the neighbors of $x'$ distinct from $z'$. These notations are illustrated in Figure 4.

Notice that $x$ and $x'$ have degree $\ell + 2$ and $\ell' + 1$, respectively.

Furthermore, we refer to the components of $T - \{x, x'\}$ as follows: If $v \in V(T) - \{x, x'\}$, then $T_v$ denotes the component of $T - \{x, x'\}$ that includes $v$. Also, it will be convenient to define $T_v$ the empty tree if $v \in \{x, x'\}$. (In particular, $T_z = T_{z'}$, and the latter tree is empty if and only $xx' \in E(T)$.) Finally, we define ten numbers corresponding to these components:

$$X := \prod_{i=1}^{\ell} F(T_{x_i}), \quad \bar{X} := \prod_{i=1}^{\ell} F(T_{x_i} - x_i),$$
$$X' := \prod_{i=1}^{\ell'} F(T_{x_i'}), \quad \bar{X}' := \prod_{i=1}^{\ell'} F(T_{x_i'} - x'_i),$$
$$Y := F(T_y), \quad \bar{Y} := F(T_y - y),$$
$$Z := F(T_z), \quad \bar{Z} := F(T_z - z),$$
$$Z_{z'} := F(T_z - z'), \quad \bar{Z}_{\{z,z'\}} := F(T_z - \{z, z'\}).$$
Since the Fibonacci number of an empty graph is 1, by convention we let $X = Y := 1$ if $\ell = 0$. Similarly, $X' = Y' := 1$ if $\ell' = 0$, and $Z = Z_z = Z_{z'} = Z_{\{z,z'\}} := 1$ if $x$ is adjacent to $x'$.

**Lemma 7.** With Notation 6, we have $F(\rho(T)) < F(T)$ if and only if $X'X'Z_{z'} > XX'Z_z$.

**Proof.** Let us compute $F(T)$ by applying twice Lemma 1 (first item) with vertices $x$ and $x'$. We obtain

$$F(T) = XY \left( X'Z + X'Z_{z'} \right) + XX' \left( X'Z_z + X'Z_{\{z,z'\}} \right).$$

Similarly,

$$F(\rho(T)) = X'Y \left( XZ + XZ_z \right) + XX' \left( XZ_{z'} + XZ_{\{z,z'\}} \right).$$

Hence, we get

$$F(T) - F(\rho(T)) = XX'Z_{z'} \left( Y - Y \right) + XX'Z_z \left( Y - Y \right),$$

$$= \left( XX'Z'_{z'} - XX'Z_z \right) \cdot \left( Y - Y \right).$$

Since $Y - Y > 0$, we deduce that $F(\rho(T)) < F(T)$ if and only if $XX'Z_{z'} > XX'Z_z$. \[\square\]

For the remainder of this section, let $T$ be a tree with a vertex $v$ of degree $k+1 \geq 3$ such that the components of $T - v$ can be denoted as $T', T_1, T_2, \ldots, T_k$ in such a way that, letting $v_i$ ($i \in \{1, \ldots, k\}$) be the neighbor of $v$ in $V(T_i)$ and letting $T_i^+$ be the tree obtained from $T_i$ by adding the vertex $v$ and the edge $vv_i$, the following holds:

- $T_1^+$ is a tree of stars, and
- for each $i \in \{2, \ldots, k\}$, at least one of the following two conditions holds:
  - (C1) $T_i$ is a tree of stars and $v_i$ is a leaf of $T_i$,
  - (C2) $T_i^+$ is a tree of stars.

Also, let $v'$ be the neighbor of $v$ in $V(T')$, and let $w$ be a leaf of $T_1$ distinct from the vertex $v_1$ (such a leaf exists since $|T_1^+| \geq 3$). Finally, let us emphasize that no assumption is made on the component $T'$, that is, $T'$ is an arbitrary non-empty tree.

The following lemma is a crucial tool in our proof of Theorem 3:

**Lemma 8.** Let $T$ be as above. Then there exists a good rotation in $T$. 

![Figure 4. A rotation $\rho = (yx, yx')$.](image)
In order to prove Lemma 8, we distinguish three cases:

- $T_1$ is a path and $(C1)$ holds for some $i \in \{2, \ldots, k\}$;
- $T_1$ is not a path and $(C1)$ holds for some $i \in \{2, \ldots, k\}$, and
- $(C2)$ holds for every $i \in \{2, \ldots, k\}$.

(Observe that $T$ falls in at least one of these cases.) We consider each of these cases separately; Lemma 8 is obtained by combining Lemmas 11, 12, and 13, which respectively address the first, second, and third case. The latter lemmas rely in turn on Lemmas 9 and 10 below, showing that some specific rotations do not change the stability number of $T$.

Let $\rho_1$ and $\rho_2$ denote the two rotations $\rho_1 := (v'v, v'w)$ and $\rho_2 := (vv_1, vw)$ (see Figure 5 for an illustration).

**Lemma 9.** Suppose that $(C1)$ holds for some $i \in \{2, \ldots, k\}$. Then $\alpha(T) = \alpha(\rho_2(T))$, and $\alpha(T) = \alpha(\rho_1(T))$ if $T_1$ is a path.

Proof. Assume without loss of generality that $(C1)$ holds for $i = 2$. We first show:

1. Each of $T$, $\rho_1(T)$, and $\rho_2(T)$ contains a maximum stable set not including $v$.

First, consider the tree $T$. Let $S$ be a maximum stable set of $T$. Suppose that $v \notin S$. Then $v_2 \notin S$. Since $T_2$ is a tree of stars, $T_2$ has a unique maximum stable set, namely $S_2 := V(T_2) - C(T_2)$ (see Lemma 2). In particular, $v_2 \in S_2$, implying $|S \cap V(T_2)| < |S_2|$. Let

$$S' := (S - (V(T_2) \cup \{v\})) \cup S_2.$$ 

Observe that $S'$ is also a stable set of $T$, and $|S'| \geq |S|$. It follows that $S'$ is a maximum stable set of $T$ with $v \notin S'$, as claimed. Finally, note that the above argument still holds if we replace the tree $T$ by $\rho_1(T)$ or $\rho_2(T)$, which completes the proof of (1).

By (1), there is a maximum stable set $S$ of $T$ such that $v \notin S$. The set $S$ is also a stable set of $\rho_2(T)$, implying $\alpha(\rho_2(T)) \geq \alpha(T)$. Conversely, there is a maximum stable set $S'$ of
\(\rho_2(T)\) with \(v \notin S'\). Since \(S'\) is a stable set of \(T\), this shows \(\alpha(\rho_2(T)) \leq \alpha(T)\), and hence \(\alpha(T) = \alpha(\rho_2(T))\).

Now, suppose that \(T_1\) is a path. Observe that \(T_1\) has odd length, since \(T_1^+\) is a tree of stars. By \([1]\), there is a maximum stable set \(S'\) of \(\rho_1(T)\) such that \(v \notin S'\). This set is also a stable set of \(T\), implying \(\alpha(T) \geq \alpha(\rho_1(T))\). Let \(S\) be a maximum stable set of \(T\). If \(v' \notin S\) or \(w \notin S\) then \(S\) is also a stable set of \(\rho_1(T)\), showing \(\alpha(T) \leq \alpha(\rho_1(T))\), and hence \(\alpha(T) = \alpha(\rho_1(T))\).

Assume thus \(v', w \in S\). Then \(v \notin S\). Since \(T_1\) is a path with an even number of vertices, \(V(T_1)\) can be uniquely partitioned into two maximum stable sets \(S_1\) and \(S_2\) of \(T_1\). Assuming without loss of generality \(w \in S_1\), we then have \(S \cap V(T_1) = S_1\), because otherwise we could modify \(S\) and find a larger stable set of \(T\). It follows that

\[
\tilde{S} := (S - S_1) \cup S_2
\]

is another stable set of \(T\) with \(|\tilde{S}| = |S|\) and \(w \notin \tilde{S}\), and we deduce that \(\alpha(T) = \alpha(\rho_1(T))\), as previously. □

**Lemma 10.** Suppose that \((\mathcal{C}_i^+)\) holds for every \(i \in \{2, \ldots, k\}\). Then \(\alpha(T) = \alpha(\rho_1(T))\).

**Proof.** First, we show that both \(T\) and \(\rho_1(T)\) contain a maximum stable set that does not include \(v'\). Let \(S\) be a maximum stable set of \(T\) (respectively, \(\rho_1(T)\)), and suppose that \(v' \in S\). Then, \(v \notin S\) (respectively, \(w \notin S\)). The tree \(T_i^+\) for \(i \in \{1, \ldots, k\}\) is a tree of stars, and hence contains a unique maximum stable set \(S_i^+\) by Lemma 2. (Note that \(v, w \in S_i^+\) and \(v \in S_i^+\) for every \(i \in \{2, \ldots, k\}\).) Since \(v \notin S\) or \(w \notin S\), it follows that \(|S \cap V(T_i^+)\) < \(|S_i^+|\). Also, \(|S \cap V(T_i^+)\) \leq \(|S_i^+|\) for every \(i \in \{2, \ldots, k\}\). Let

\[
S' := (S \cap (V(T') - \{v'\})) \cup S_1^+ \cup S_2^+ \cdots \cup S_k^+.
\]

The set \(S'\) is a stable set of \(T\) (respectively, \(\rho_1(T)\)), and \(|S'| \geq |S|\) by the previous observations, implying that \(S'\) is a maximum stable set with \(v' \notin S'\).

Now, observe that every stable set of \(T\) (respectively \(\rho_1(T)\)) that does not include \(v'\) is also a stable set of \(\rho_1(T)\) (respectively \(T\)). Since there exists a maximum stable set not including \(v'\) in each of these two trees, it follows \(\alpha(T) = \alpha(\rho_1(T))\). □

**Lemma 11.** Suppose that \(T_1\) is a path and that \((\mathcal{C}_i^+)\) holds for some \(i \in \{2, \ldots, k\}\). Then the rotation \(\rho_1\) is good for \(T\).

**Proof.** We have that \(\alpha(T) = \alpha(\rho_1(T))\) by Lemma 9 thus it remains to prove that \(F(\rho_1(T)) < F(T)\). Let \(x := v, x' := w, y := v'\), and consider Notation 6 with respect to the rotation \(\rho_1 = (v'v, v'w) = (yx, yx')\). We have \(X' = X = 1\); and \(Z_i = Z_i'\) since \(T_1\) is a path. Also, \(X < X\), because \(\ell = k - 1 > 0\). Therefore, \(XX'Z_i' > XX'Z_i\), and hence \(F(\rho_1(T)) < F(T)\) by Lemma 7. □

**Lemma 12.** Suppose that \(T_1\) is not a path and that \((\mathcal{C}_i^+)\) holds for some \(i \in \{2, \ldots, k\}\). Then the rotation \(\rho_2\) is good for \(T\).

**Proof.** Since \((\mathcal{C}_1^+)\) holds for some \(i \in \{2, \ldots, k\}\), Lemma 9 implies that \(\alpha(T) = \alpha(\rho_2(T))\). We prove that \(F(\rho_2(T)) < F(T)\), using Lemma 7 again. Let \(x := v_1, x' := w, y := v\), and consider the notations associated to the rotation \(\rho_2 = (v_1v, vw) = (yx, yx')\). (Thus, in particular, \(z\) and \(z'\) are the neighbors of respectively \(v_1\) and \(w\) that are included in the unique \(v_1w\)-path.)
We have \( X' = X' = 1 \) since \( w \) is a leaf. Hence, by Lemma 7 to prove \( F(\rho_2(T)) < F(T) \) it suffices to show that \( X \subseteq \overline{X} \subseteq X \).

If \( v_1w \in E(T) \), then \( T_z \) is empty and \( \overline{Z} = \overline{Z}' = 1 \). Moreover, \( \ell \geq 1 \) because \( T_1 \) is not a path, implying \( \overline{X} \subseteq X \) and \( F(\rho_2(T)) < F(T) \).

Now, assume that \( v_1w \notin E(T) \). Since \( v_1 \) is a center of the tree of stars \( T_1^+ \), its distance to the leaf \( w \) is odd, and hence at least 3. This implies \( z \neq z' \). Also, the component of \( T_1^+-v_1 \) that includes \( w, z, \) and \( z' \) is also a tree of stars; hence, \( T_z \) is either a tree of stars, or almost a tree of stars, or an odd path.

In the first two cases, \( z \) and \( z' \) are respectively a leaf and a center of \( T_z \). Furthermore, \( z' \) is the exposed center of \( T_z \) in the second case. It follows \( Z > \phi \cdot \overline{Z} \) and \( Z < \phi \cdot \overline{Z}' \) from respectively Lemmas 4 and 5, implying \( \overline{Z} < \overline{Z}' \). This in turn implies \( X \subseteq \overline{X} \subseteq X \overline{Z} \overline{Z}' \) since trivially \( \overline{X} \subseteq X \).

In the third case, we have that \( \overline{Z} = \overline{Z}' \), because \( T_z - z \) is isomorphic to \( T_z - z' \). Since \( T_1 \) itself is not a path, we also have \( \ell = \deg_T(v_1) - 2 > 0 \), implying \( \overline{X} < X \). It follows again \( X \subseteq \overline{X} \subseteq X \overline{Z} \overline{Z}' \).

\[ \square \]

**Lemma 13.** Suppose that (2) holds for every \( i \in \{2, \ldots, k\} \). Then \( T \) admits a good rotation.

**Proof.** The proof involves two different rotations. First, let \( x := v, x' := w, y := v' \), and consider the rotation \( \rho_1 = (v^v, v'w) = (yx, yx') \) and the corresponding notations. (Thus, \( z = v_1 \) and \( z' \) is the unique neighbor \( w' \) of \( w \).) We have \( X' = X' = 1 \) and \( \overline{X} \subseteq X \).

If \( \overline{Z} \subseteq \overline{Z}' \), then \( F(\rho_1(T)) < F(T) \) by Lemma 7. Also, \( \alpha(T) = \alpha(\rho_1(T)) \) by Lemma 10, implying that \( \rho_1 \) is a good rotation. Thus, we may assume that \( \overline{Z} > \overline{Z}' \), that is,

\[ F(T_1 - \{w, v_1\}) > F(T_1 - \{w, w'\}). \]

(Observe that this implies \( w' \neq v_1 \).)

Now, let \( \rho_3 \) be the rotation \( \rho_3 := (vv_1, vw') \) (see Figure 6 for an illustration). We will show that \( \rho_3 \) is a good rotation. Similarly as before, let \( x := v_1, x' := w', y := v \), and consider the notations associated to \( \rho_3 = (vv_1, vw') = (yx, yx') \). We have

\[ F(T_1 - \{w, v_1\}) = X(X'Z + \overline{X} \overline{Z} \overline{Z}') \]

and

\[ F(T_1 - \{w, w'\}) = X'(XZ + \overline{X} \overline{Z} \overline{Z}). \]

Thus, (2) implies that \( X \overline{X} \overline{Z} \overline{Z}' > \overline{X} X' \overline{Z} \overline{Z} \), and it follows from Lemma 7 that \( F(\rho_3(T)) < F(T) \). Therefore, it remains to show that \( \alpha(T) = \alpha(\rho_3(T)) \).

Let \( S \) be a maximum stable set of \( T \). We may assume that \( w' \notin S \). (Indeed, if not, consider the set \( (S - \{w'\}) \cup \{w\} \) instead.) The set \( S \) is also a stable set of \( \rho_3(T) \), showing \( \alpha(T) \leq \alpha(\rho_3(T)) \).

Now, let \( S \) be a maximum stable set of \( \rho_3(T) \). If \( v \notin S \) or \( v_1 \notin S \), then \( S \) is a stable set of \( T \), showing \( \alpha(T) \geq \alpha(\rho_3(T)) \), and hence \( \alpha(T) = \alpha(\rho_3(T)) \). Thus, suppose that \( v, v_1 \in S \). The set \( \tilde{S} := (S - \{v_1\}) \cap V(T^+_1) \) is a stable set of the tree of stars \( T_1^+ \). Moreover, \( |\tilde{S}| < |S_1^+| \) since \( z \notin \tilde{S} \), where \( S_1^+ := V(T_1^+) - C(T_1^+) \) is the unique maximum stable set of \( T_1^+ \) (see Lemma 2). This implies that the set

\[ (S - V(T_1^+)) \cup S_1^+ \]
Figure 6. The rotation $\rho_3 = (vv_1, vw')$ in the proof of Lemma 13 (note that possibly $z = z'$).

has cardinality at least that of $S$, and furthermore is a stable set of $T$. Again, it follows that $\alpha(\rho_3(T)) = \alpha(T)$. \qed

3.3. Proof of Theorem 3

Let $T$ be an extremal tree. Arguing by contradiction, assume that $T$ is neither a path nor a tree of stars. (Recall that even paths are trees of stars.)

Root the tree $T$ at an arbitrary leaf $r$. Let $T_v (v \in V(T))$ be the subtree of $T$ rooted at vertex $v$. To each leaf $u$ of $T$ we associate a corresponding witness, defined as the highest ancestor $v$ of $u$ such that $T_v$ is a tree of stars and $v$ is a leaf of $T_v$. Since $u$ itself satisfies these two conditions, the latter vertex is well-defined.

Let us look at a few properties of witnesses: First, clearly $r$ is not a witness (for otherwise $T$ would be a tree of stars). Also, if $v$ and $w$ are two distinct witnesses, then $v$ is neither an ancestor nor a descendant of $w$. (In other words, the set of witnesses forms an antichain in the partial order implied the rooted tree.)

We may assume that $r$ has been chosen so that it satisfies:

\[ \text{(3) If the neighbor of $r$ has degree 2 in $T$, then $T - r$ is not a tree of stars.} \]

Indeed, let $v$ be the neighbor of $r$, and suppose $\deg_T(v) = 2$ and that $T - r$ is a tree of stars. Since $T - r$ is not a path, there is a vertex $w$ with degree at least 3 in $T - r$, which is therefore a center of $T - r$. Since $v$ is a leaf of $T - r$, it follows that the distance in $T$ between $r$ and $w$ is even. Now, consider a component of $T - w$ that does not contain $r$, and select a leaf $r'$ of that component. The tree $T - r'$ cannot be a tree of stars, because $w$ is at even distance from the leaf $r$ in $T - r'$. Thus, we deduce that (3) holds if we root $T$ at $r'$ instead of $r$.\]
Let \( u \) be a witness of maximum depth in \( T \), and let \( v_1 \) be the parent of \( u \). It follows from (3) that \( v_1 \neq r \), hence \( v_1 \) has a parent \( v \). Let \( T_1^+ \) be the tree obtained from \( T_{v_1} \) by adding the vertex \( v \) and the edge \( vv_1 \). We show:

\[
(4) \quad T_1^+ \text{ is a tree of stars.}
\]

This is trivially true if \( \text{deg}(v_1) = 2 \), hence assume \( \text{deg}(v_1) \geq 3 \), and let \( u_1, \ldots, u_\ell \) denote the children of \( v_1 \) distinct from \( u \). Let \( j \in \{1, \ldots, \ell\} \), and consider a leaf \( z \) of \( T \) which is contained in \( T_{u_j} \). In \( T \), the witness \( w \) of \( z \) cannot be higher than \( u_j \) (since otherwise \( w \) would be an ancestor of the witness \( u \)). Also, \( w \) cannot be a descendant of \( u_j \), because it would contradict the fact that \( u \) has maximum depth among all witnesses. Thus \( w = u_j \), and hence \( T_{u_j} \) is a tree of stars and \( u_j \) is a leaf of \( T_{u_j} \). It follows that \( T_1^+ \) is also a tree of stars.

The vertex \( v \) has at least two children (counting \( v_1 \)), since otherwise \( T_v = T_1^+ \), contradicting the fact that \( u \) is a witness. Observe that this implies \( v \neq r \). Let \( v_2, \ldots, v_\ell \) be the children of \( v \) that are distinct from \( v_1 \), and let \( T_i^+ (i \in \{2, \ldots, k\}) \) be the tree obtained from \( T_{v_i} \) by adding the vertex \( v \) and the edge \( vv_i \).

Let \( i \in \{2, \ldots, k\} \), and consider any witness \( u' \) contained in \( V(T_{v_i}) \) (observe that there must be at least one). Since the depth of \( u' \) is at most that of \( u \), either \( u' = v_i \) or \( u' \) is a child of \( v_i \). In the first case, \( T_{v_i} \) is a tree of stars and \( v_i \) is a leaf of that tree, by definition of a witness. In the second one, since \( u \) and \( u' \) have the same depth, the proof of (1) directly shows that \( T_i^+ \) is a tree of stars (one just needs to replace \( u \) by \( u' \) and \( v_1 \) by \( v_i \)).

Let \( T' \) be the component of \( T - v \) containing the root \( r \). Also, let \( T_i := T_{v_i} \) for \( i \in \{1, \ldots, k\} \). Let us summarize the previous observations: \( T_1^+ \) is a tree of stars, and for every \( i \in \{2, \ldots, k\} \), either \( T_i \) is a tree of stars and \( v_i \) is a leaf of \( T_i \), or \( T_i^+ \) is a tree of stars. It follows that \( T \) satisfies the requirements of Lemma 8, and therefore contains a good rotation, contradicting the fact that \( T \) is extremal. \( \square \)

4. Extremal Trees of Stars are Balanced

We have seen that every extremal tree is a tree of stars or an odd path (cf. Theorem 3). In this section, we refine this result by showing that, if an extremal tree \( T \) is a tree of stars, then \( T \) must be balanced. (Recall that a tree of stars is balanced if the degrees of every two of its centers differ by at most 1.)

**Theorem 14.** Every extremal tree is either a balanced tree of stars or an odd path.

We again resort to edge rotations to prove Theorem 14. To this aim, we need to introduce a few additional lemmas.

For an integer \( k \geq 2 \), let \( f_k : \mathbb{R} \to \mathbb{R} \) be the function defined as

\[
f_k(x) := x^k - x^{k-1} + 2x - 1.
\]

**Lemma 15.** The function \( f_k(\cdot) \) is strictly increasing on the interval \([0, 1]\).

*Proof.* We prove that the first derivative \( f'_k(\cdot) \) is strictly positive on \([0, 1]\), by induction on \( k \). Let \( x \in [0, 1] \). We have \( f'_k(x) > 0 \) when \( k = 2 \), since \( f'_2(x) = 2x + 1 > 0 \). For the inductive
step, let $k \geq 3$, and rewrite $f'_k(x)$ as follows:
\[
f'_k(x) = kx^{k-1} - (k - 1)x^{k-2} + 2
= 2 + x \left( (k - 1)x^{k-2} + x^{k-2} - (k - 2)x^{k-3} - x^{k-3} \right)
= 2 + x (f'_{k-1}(x) - 2) + x^{k-1} - x^{k-2}.
\]
Since $f'_{k-1}(x) - 2 > -2$ by the induction hypothesis, we deduce
\[
f'_k(x) > 2 - 2x + x^{k-1} - x^{k-2} = (1 - x)(2 - x^{k-2}) \geq 0,
\]
as claimed. \qed

Since $f_k(1/2) = -1/2^k < 0 < 1 = f_k(1)$ and $f_k(\cdot)$ is continuous, it follows from Lemma 15 that $f_k(\cdot)$ has a unique root in the open interval $(1/2, 1)$, which we denote by $R_k$.

**Lemma 16.** $R_k < 2^{-k/(k+1)}$ for every $k \geq 2$.

**Proof.** Let
\[
g(k) := f_k \left( 2^{-k/(k+1)} \right) = 2^{-k^2/(k+1)} - 2^{-k(k-1)/(k+1)} + 2 \cdot 2^{-k/(k+1)} - 1.
\]
By Lemma 15 it is enough to show $g(k) > 0$ for every integer $k \geq 2$. A quick hand-on computation shows that this is true for $k = 2$ and 3. Hence, we may assume $k \geq 4$.

We first rewrite $g(k)$ as follows:
\[
g(k) = 2^{-k^2/(k+1)} - 2^{-k(k-1)/(k+1)} + 2 \cdot 2^{-k/(k+1)} - 1
= 2^{-k^2/(k+1)} - 2^{k/(k+1)-k^2/(k+1)} + 2^{1/(k+1)} - 1
= 2^{1/(k+1)} - 1 - \frac{2^{k/(k+1)} - 1}{2^{k^2/(k+1)}}.
\]
Using $k^2/(k + 1) > k - 1$ and $2^{k/(k+1)} < 2$, we then obtain
\[
g(k) > 2^{1/(k+1)} - 1 - \frac{2^{k/(k+1)} - 1}{2^{k-1}}
> 2^{1/(k+1)} - 1 - 2^{-(k-1)}.
\]
For every real $y > 0$, we have $e^y > 1 + y$, and hence $2^y > 1 + y \ln 2$, since $2^y = e^{y \ln 2}$. It follows
\[
g(k) > \frac{\ln 2}{k + 1} - \frac{1}{2^{k-1}} = \frac{1}{k + 1} \left( \ln 2 - \frac{k + 1}{2^{k-1}} \right).
\]
Since the function $\ln 2 - (k + 1)/(2^{k-1})$ is increasing in $k$ and positive for $k = 4$, we deduce that $g(k) > 0$, as claimed. \qed

**Lemma 17.** Let $k \geq 2$ be an integer, let $T$ be a tree of stars such that all its centers have degree at least $k$, and let $v$ be a leaf of $T$. Then $F(T - v) < R_k \cdot F(T)$.

Observe that $R_2 = 1/\phi$; hence, this lemma generalizes Lemma 4 in the case of trees of stars.
Proof of Lemma 17. The proof is by induction on \(|T|\). The claim is true when \(|T| = 1\), since then \(F(T-v)/F(T) = 1/2 < R_k\).

For the inductive step, assume \(|T| > 1\), and let \(w\) be the unique neighbor of \(v\). Each component of \(T - \{v, w\}\) is a tree of stars; let us denote these trees by \(T_1, \ldots, T_\ell\) (thus \(\ell \geq k - 1\)). Let also \(v_i\) be the leaf of \(T_i\) that is adjacent to \(w\) in \(T\).

Letting

\[
\gamma := \prod_{i=1}^{\ell} \frac{F(T_i - v_i)}{F(T_i)},
\]

we deduce that

\[
\frac{F(T-v)}{F(T)} = \frac{\prod_{i=1}^{\ell} F(T_i) + \prod_{i=1}^{\ell} F(T_i - v_i)}{2 \prod_{i=1}^{\ell} F(T_i) + \prod_{i=1}^{\ell} F(T_i - v_i)} = 1 + \gamma.
\]

The induction hypothesis gives

\[0 < \gamma < (R_k)^\ell \leq (R_k)^{k-1} \]

By definition of \(R_k\), we have \((R_k)^k - (R_k)^{k-1} + 2R_k - 1 = 0\). It then follows

\[
\frac{F(T-v)}{F(T)} = \frac{1 + \gamma}{2 + \gamma} < \frac{1 + (R_k)^{k-1}}{2 + (R_k)^{k-1}} = R_k,
\]

as desired. \(\square\)

Now, we may prove Theorem 14.

Proof of Theorem 14. Let \(T\) be an extremal tree. We know by Theorem 3 that \(T\) is a tree of stars or an odd path. Arguing by contradiction, we assume that \(T\) is a tree of stars which is not balanced. We will show that \(T\) admits a good rotation, which contradicts the fact that \(T\) is extremal.

The tree \(T\) has at least two centers, as otherwise \(T\) is trivially balanced. Let \(x\) and \(x'\) be distinct centers of \(T\) maximizing the difference \(\deg(x) - \deg(x')\). Let also \(k := \deg(x)\) and \(\ell := \deg(x')\) (hence, \(k \geq \ell + 2\)).

Choose a neighbor \(y\) of \(x\) that is not on the unique \(xx'\)-path in \(T\), and consider the rotation \(\rho := (yx, yx')\) (see Figure 7 for an illustration). Observe that \(\rho(T)\) is also a tree of stars, and that \(T\) and \(\rho(T)\) have the same set of centers. In particular, \(\alpha(\rho(T)) = \alpha(T)\) by Lemma 2. Hence, to prove that \(\rho\) is a good rotation, it remains to show that \(F(\rho(T)) < F(T)\). This will be done by combining Lemma 7 with Lemmas 16 and 17.

![Figure 7. The rotation \(\rho\) used in the proof of Theorem 14.](image)
(Thus, $z$ and $z'$ are the neighbors of respectively $x$ and $x'$ that lie on the $xx'$-path in $T$, and possibly $z = z'$.) It follows from Lemma 17 that

$$\frac{Z_z}{Z} < R_\ell$$

and

$$\frac{X}{X} < (R_\ell)^{k-2} \leq (R_\ell)^{\ell}.$$ 

Also, we have that

$$\frac{Z_{z'}}{Z} \geq \frac{1}{2}$$

and

$$\frac{X'}{X'} \geq \frac{1}{2^{\ell-1}},$$

since $F(T') \leq 2F(T' - v)$ holds for every tree $T'$ and vertex $v$ of $T'$ (cf. Lemma 1). Combining these inequalities with Lemma 16, we obtain

$$\frac{XX'Z_{z'}}{XX'Z_z} > \frac{X}{(R_\ell)^{\ell}} \cdot \frac{X'}{2^{\ell-1}} \cdot \frac{Z_z}{2R_\ell} = \frac{XX'Z_z}{2^{\ell}} \cdot \frac{(R_\ell)^{-(\ell+1)}}{2^{\ell}} > \frac{XX'Z_z}{2^{\ell}} \cdot \left(\frac{2^{1/2}}{\ell+1}\right)^{-(\ell+1)} = \frac{XX'Z_z}{2^{\ell}}.$$ 

Therefore, $F(\rho(T)) < F(T)$ by Lemma 7. This completes the proof. □

5. Further Results

In Section 4, it has been shown that every extremal tree $T$ that is not an odd path must be a balanced tree of stars. An open problem is to understand how these stars are linked together. A few results in this direction are given here, but the general problem is far from being solved.

Let $T$ be a tree of stars. The center-tree of $T$ is the tree $C_T$ having the centers of $T$ as vertex set, and where two centers are adjacent if they are at distance 2 in $T$ (see Figure 8 for an illustration).

![Figure 8](image_url)

**Figure 8.** A tree of stars (left) and its corresponding center-tree (right).

Let $T$ be a balanced tree of stars with $n$ vertices and stability number $\alpha$. Since $T$ is balanced, every center of $T$ has degree either $\left\lceil \frac{n-1}{n-\alpha} \right\rceil$ or $\left\lfloor \frac{n-1}{n-\alpha} \right\rfloor - 1$. A center is said to be
heavy in the first case, light in the second. It can be checked that the number of heavy centers is given by
\[ h(n, \alpha) := \begin{cases} 
(n - 1) \mod (n - \alpha) & \text{if } (n - 1) \mod (n - \alpha) \neq 0 \\
n - \alpha & \text{otherwise} 
\end{cases} \]

The number of light centers of \( T \) is thus
\[ \ell(n, \alpha) := n - \alpha - h(n, \alpha). \]

The next theorem gives a simple characterization of the extremal trees when \( \ell(n, \alpha) \leq 2 \).

Let us remark that, if \( \alpha = n/2 \), then the path \( P_n \) is the only tree with \( n \) vertices and stability number \( \alpha \) that is extremal, as follows from the result of Prodinger and Tichy [18] mentioned in the introduction. Hence, we assume \( \alpha > n/2 \) in what follows.

**Theorem 18.** Let \( T \) be a tree with \( n \) vertices and stability number \( \alpha > n/2 \), and assume \( \ell(n, \alpha) \leq 2 \). Then, \( T \) is extremal if and only if \( T \) satisfies the following three conditions:

- \( T \) is a balanced tree of stars;
- the center-tree of \( T \) is isomorphic to a path \( P \), and
- each light center of \( T \) (if any) is an endpoint of \( P \).

Our proof of Theorem 18 is based on the following lemma.

**Lemma 19.** Let \( T \) be a balanced tree of stars, and assume that a leaf \( w \) of the center-tree \( C_T \) is heavy. If \( T \) is extremal, then \( C_T \) is a path, and all internal vertices of \( C_T \) are heavy.

**Proof.** Arguing by contradiction, suppose that \( T \) is extremal and contains a center \( v \) such that either \( v \) is heavy with degree at least 3 in \( C_T \), or light with degree at least 2 in \( C_T \). We may assume that \( v \) has been chosen so that every inner vertex of the \( vw \)-path in \( C_T \) is heavy and has degree 2 in \( C_T \).

Let \( P \) be the unique \( vw \)-path in the tree \( T \). Let \( y \) be a neighbor of \( v \) in the center-tree \( C_T \) such that \( y \notin V(P) \). Let \( x \) be the vertex of \( T \) that is adjacent to both \( v \) and \( y \). Let \( x' \) be a leaf of \( T \) that is adjacent to \( w \). (Thus, \( x' \notin V(P) \).) Let \( \rho \) be the rotation \( \rho := (yx, yx') \) (see Figure 9 for an illustration). The tree \( \rho(T) \) is a tree of stars and has the same number of centers as \( T \), thus \( \alpha(\rho(T)) = \alpha(T) \) by Lemma 2. Hence, to reach a contradiction, it is enough to show that \( F(\rho(T)) < F(T) \).

![Figure 9. The rotation \( \rho \) used in the proof of Lemma 19.](image-url)

Consider Notation 6 with respect to the rotation \( \rho \). Thus \( z = v, z' = w, \) and \( X = \overline{X} = X' = \overline{X'} = 1 \). By Lemma 7, \( F(\rho(T)) < F(T) \) if and only if \( \overline{Z}_z < \overline{Z}_{z'} \).
Let $u$ and $u'$ be the neighbors of respectively $z$ and $z'$ in the path $P$. Let $\bar{T}$ be the component of $T - \{z, z'\}$ that contains both $u$ and $u'$. Since every center of $T$ included in $V(P) - \{z, z'\}$ is heavy and has degree 2 in $C_T$, the trees $\bar{T} - u$ and $\bar{T} - u'$ are isomorphic.

Let $v_1, \ldots, v_k$ be the neighbors of $z$ (= $u$) in $T$ that are distinct from $u$ and $x$. Let $T_i^z$ $(1 \leq i \leq k)$ be the component of $T - \{z\}$ that includes $v_i$. Let $\ell := \deg_T(z') - 2$. Thus, $\ell = k$ or $k + 1$, depending on whether $z$ is heavy or light. Since the $\ell$ neighbors of $z'$ that are distinct from $u'$ and $x'$ are all leaves of $T$, it follows that

$$Z_z = \left( \prod_{i=1}^{k} F(T_i^z) \right) \cdot \left( 2^\ell F(\bar{T}) + F(\bar{T} - u') \right),$$

$$Z_{z'} = 2^\ell \cdot \left( F(\bar{T}) \prod_{i=1}^{k} F(T_i^z) + F(\bar{T} - u) \prod_{i=1}^{k} F(T_i^z - v_i) \right).$$

Using that $F(\bar{T} - u) = F(\bar{T} - u')$, we obtain

$$Z_{z'} - Z_z = F(\bar{T} - u) \cdot \left( 2^\ell \prod_{i=1}^{k} F(T_i^z - v_i) - \prod_{i=1}^{k} F(T_i^z) \right).$$

We have $F(T_i^z) \leq 2F(T_i^z - v_i)$ for every $i \in \{1, \ldots, k\}$, with strict inequality if $|T_i^z| > 1$. If $z$ is light, then $\ell = k + 1$, and

$$2^\ell \prod_{i=1}^{k} F(T_i^z - v_i) - \prod_{i=1}^{k} F(T_i^z) \geq (2^\ell - 2^k) \prod_{i=1}^{k} F(T_i^z - v_i) > 0.$$

Similarly, if $z$ is heavy, then $\ell = k$. Moreover, $|T_j^z| > 1$ holds for some $j \in \{1, \ldots, k\}$, because $z$ has degree at least 3 in $C_T$. This implies $F(T_j^z) < 2F(T_j^z - v_j)$, and

$$2^\ell \prod_{i=1}^{k} F(T_i^z - v_i) - \prod_{i=1}^{k} F(T_i^z) > (2^\ell - 2^k) \prod_{i=1}^{k} F(T_i^z - v_i) = 0.$$

Thus, $Z_z < Z_{z'}$ holds in both cases, as claimed. This concludes the proof. \qed 

**Proof of Theorem 18** First, we observe that there is a unique tree (up to isomorphism) satisfying the three conditions given in the statement of the theorem. Hence, it is enough to show that each of these three conditions is necessary for $T$ to be extremal.

Thus, suppose that $T$ is extremal. Then $T$ is a balanced tree of stars by Theorem 14 (note that $T$ cannot be an odd path since $\alpha > n/2$). If the center-tree of $T$ is not a path, then it has at least three leaves, and one of them is heavy. But then Lemma 19 implies that $T$ is not extremal, a contradiction. Hence, the center-tree is isomorphic to a path $P$. Furthermore, all internal vertices of $P$ are heavy centers of $T$, by the same lemma. The theorem follows. \qed 

When $\ell(n, \alpha) \geq 3$ and $\alpha > n/2$, every extremal tree $T$ is a balanced tree of stars by Theorem 14. However, the center-tree of $T$ is no longer necessarily a path. This can be seen on Figure 10 which provides a list of the center-trees of all extremal trees for $n = 24$ and $\alpha > 12$ (these have been computed with the system GraPHedron 13).
Figure 10. The center-trees of all trees of stars on $n = 24$ vertices which are extremal. Light centers are drawn in white and heavy centers in black.

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