Probability density function evolution of power systems subject to stochastic variation of renewable energy

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Abstract. As renewable energies are increasingly integrated into power systems, there is increasing interest in stochastic analysis of power systems. Better techniques should be developed to account for the uncertainty caused by penetration of renewables and consequently analyse its impacts on stochastic stability of power systems. In this paper, the Stochastic Differential Equations (SDEs) are used to represent the evolutionary behaviour of the power systems. The stationary Probability Density Function (PDF) solution to SDEs modelling power systems excited by Gaussian white noise is analysed. Subjected to such random excitation, the Joint Probability Density Function (JPDF) solution to the phase angle and angular velocity is governed by the generalized Fokker-Planck-Kolmogorov (FPK) equation. To solve this equation, the numerical method is adopted. Special measure is taken such that the generalized FPK equation is satisfied in the average sense of integration with the assumed PDF. Both weak and strong intensities of the stochastic excitations are considered in a single machine infinite bus power system. The numerical analysis has the same result as the one given by the Monte Carlo simulation. Potential studies on stochastic behaviour of multi-machine power systems with random excitations are discussed at the end.

1. Introduction

There are many stochastic factors in the operation of power system, such as generator outages, short-circuits or generation and load level fluctuations. In recent years, the stochastic factors in power system have shown high complexity and volatility with the expansion of renewable energy resources and widespread adoption of plug-in electric vehicles. These stochastic factors can cause unbalanced and stochastic power fluctuation in the operation of power systems, which brings great threat to stability of power systems, thus researching the stability of power system has a great realistic meaning[1-2]. The traditional methods of Differential Algebraic Equations (DAEs) cannot fully reflect the impact of stochastic excitations on power system; hence SDEs are employed to describe the stochastic characteristics of power system to better describe the stochastic factors[3-4]. The theory of SDEs has been applied widely in many fields, but it is still in the process of exploration on power system analysis. J.Y. Zhang constructed stochastic model for single machine connected to an infinite bus (SMIB) system under Gaussian type random excitation with the application of SDEs, then the operating trajectories were obtained through numerical method, finally the mean stability and the mean square stability were proved theoretically[1]. A stochastic model of wind turbine based on SDEs was established in [5], a criterion for stability under random excitation was put forward through Monte Carlo simulation subsequently. Because of the theoretical complexity of SDEs and the high dimensionality and strong nonlinearity of power system, it is difficult to obtain the analytic solution.
Most of the existing methods to obtain the solution generally refer to the numerical approach or Monte Carlo method. And it is generally believed that the numerical solution is a feasible tool to study the trajectory of stochastic model of power systems[6]. Among the numeric methods, Euler-Maruyama method and Milstein’s method are commonly used. A systematic introduction to the numerical simulation of SDEs can be found in [7-8].

Another effective method to study SDEs model is focus on the evolution of the probability density function (PDF) [9]. It has been proved that the solution of Itô SDE is a continuous Markov and diffusion process[10-11]. According to the theory of stochastic dynamics, if a diffusion proceed the solution of SDEs, then the PDF satisfies a Fokker-Planck-Kolmogorov (FPK) equation[6][9]. And the FPK equation can be used to determine the PDF and the steady probability density of the stochastic process, then by observing the shape of stationary PDF as parameter changes, stochastic bifurcation behaviour of the power systems can be explored [12]. However, studies focusing on this method is relatively rare. Reference[13] established the FPK equation of reliability function of SMIB system operation reliability, and then analysed the first passage problem. A dynamic model for SMIB system with stochastic excitation was established in [12], then through simplified the FPK equation properly, the approximate analytic solution of the stationary probability density was obtained, but numerical solution of the FPK equation was not given. It is difficult to obtain analytic solution of the FPK equation except special case, so numerical methods are necessary to obtain the numerical solution. Z.L. Huang proposed an algorithm for solving the numerical solution of the high dimensional stationary FPK equation [14]. And a systematic introduction to the numerical solution of FPK equation can be found in [15].

The remainder of this paper is arranged as follows. In the next section, the mathematical formulation of new energy power systems based on SDEs are introduced. In the third section, the stochastic probability density function and its evolution are discussed. Case studies for SMIB system are done in the fourth section. And in the last section, the conclusions on the stability problems caused by stochastic excitations of renewable energies and potential studies are discussed.

2. Mathematical Formulation of Power System with Stochastic Excitations by Renewables

2.1. Itô Stochastic Differential Equation
The general Itô stochastic differential equation is as follows [10-11]:

\[
\begin{align*}
\frac{dx(t)}{dt} &= a(x(t), t)dt + G(x(t), t)dB(t) \\
x(t_0) &= x_0
\end{align*}
\]

(1)

where \(x(t)\) and \(a\) are \(n \times 1\) vectors and \(x(t)\) is a stochastic process, \(G\) is an \(n \times m\) matrix, \(B(t)\) is a \(m \times 1\) vector of Wiener processes with covariance matrix \(V\). \(dB(t)\) is the Gaussian noise excitation corresponding to the Wiener process, which is also indicated as \(W(t)\).

2.2. SMIB Power Systems with Stochastic Excitation

![Figure 1: Single Machine Infinite Bus (SMIB) system.](image)

As the renewables connected to the power system shown in Figure 1, the stochastic model of SMIB system can be constructed as follows under proper assumptions [16]:

\[
M\ddot{\delta} + D\dot{\delta} = P_m - P_e + \sigma W(t)
\]

(2)

where \(\delta\) is the relative generator angle to the infinite bus. \(\dot{\delta}\) and \(\ddot{\delta}\) are derivative of first order and second order of \(\delta\), and \(\dot{\delta}\) is the generator velocity of the system. \(P_e\) is the electrical power and \(P_e = P_{\text{max}} \sin \delta\), where \(P_{\text{max}}\) is a constant. The constants \(M\) and \(D\) are the machine inertia constant, damping
coefficient and the mechanical power, respectively. \( W(t) \) is additive white noise which indicates random excitation of renewable energies, such as power fluctuation of wind turbines and solar panels, and/or load swing of plug-in electric vehicles. \( \sigma \) is the magnitude of \( W(t) \), which indicates intensity of the random excitation.

Using \( \delta \) to represent \( \frac{\rho_e}{M} \), and then equation (2) can be equivalently written as follows:

\[
\begin{align*}
\frac{d\delta}{dt} &= \omega \delta \\
\frac{d\omega}{dt} &= \left( \frac{\bar{P}_m}{M} - \bar{P}_e - \bar{D} \omega \right) dt + \sigma \delta \frac{dW(t)}{\sqrt{t}}
\end{align*}
\]

where \( \bar{P}_e = \frac{P_e}{M}, \bar{D} = \frac{D}{M}, \bar{P}_m = \frac{P_m}{M}, \sigma = \frac{\sigma}{M} \).

The vector form of equation (3) is:

\[
x' = f(x)dt + P_L
\]

where \( x = (\delta, \omega)^T, P_L = [0, \sigma W(t)^T] \), and \( f(x) = [f_1(x), f_2(x)]^T = (\omega, \bar{P}_m - \bar{P}_e - \bar{D} \omega)^T \).

For the power system described by equation (3), the stability of the system under small random excitation is of great interest. From the dynamic viewpoint, the power system can always maintain an equilibrium position (i.e., the equilibrium point of the system) to operate smoothly without perturbations, and the stability of the power system under small excitations is the ability of the system to continue to be in the equilibrium position or to achieve a new equilibrium state after impacted by random excitations. And the stability of the power system can be reflected by the stationary PDF.

3. Probability Density Function (PDF) and its Evolution

3.1. Fokker-Planck-Kolmogorov (FPK) Equation

In general, conditional transition probability density \( p(x, t|x_0, t_0) \) indicates the probability density of the state at time \( t \) under the condition of initial state \( (x_0, t_0) \). As mentioned earlier, \( p(x, t|x_0, t_0) \) satisfies following FPK equation [17]:

\[
\frac{\partial p(x, t|x_0, t_0)}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left[ a_i(x, t)p(x, t|x_0, t_0) \right] + \sum_{i,j=1}^{n} \frac{1}{2} \frac{\partial}{\partial x_i} \left[ b_{ij}(x, t)p(x, t|x_0, t_0) \right]
\]

where \( b_{ij}(x, t) = [G V G]^T \).

Solving equation (5) requires initial condition, boundary condition and normalization condition. Here the initial condition can be given by \( p(x, t|x_0) = \prod_{i=1}^{n} \delta(x_i - x_{0i}) \) as \( t \to t_0 \), where \( x_0 \) is the initial state and \( \delta \) is the Dirac delta function. For power system, the boundaries can be assumed to natural boundary conditions [16], which indicated that \( p(x, t|x_0, t_0) = 0, x \in \Gamma \), where \( \Gamma \) is the boundary of security domain. A normalization condition is also need to be satisfied:

\[
\int_{\Omega} p(x, t) dx = 1
\]

For the most part of engineering problems, the stationary solution of FPK equation, which means the behavior of the system dynamics as the states approaches steady state, is of greatest interest. If the stationary solution exists as \( t \to \infty \), then \( p(x|x_0) \) is indicated the stationary PDF and \( p(x|x_0) = \lim_{t \to \infty} p(x, t|x_0, t_0) \). Under this steady-state condition, \( \frac{\partial p}{\partial t} = 0 \), then equation (5) is called stationary FPK equation.

3.2. Numerical Methods for FPK Equation

As mentioned earlier, the analytic solution of FPK equation only can be obtained under restrictive conditions, thus numerical methods are necessary to calculate approximate solutions. Mathematically, FPK equation (5) is a parabolic Partial Differential Equation (PDE) with variable coefficients; therefore computational methods for PDE are adopted, such as path integration method, finite
difference method, finite element method and Monte Carlo method, etc[15]. Herein, finite difference method was used, and second-order central difference schemes were chosen to get the discrete equation [14]. Then linear algebraic equations can be obtained from the discretized equation as follows:

\[ AP = 0 \quad (7) \]

where \( A \) is coefficient matrix and \( P \) is an \( \bar{M} \times 1 \) vector connected by stationary PDF on the nodes that need to be calculated and \( \bar{M} \) is the number of the nodes of discretization.

Finally, the numerical solution of equation (5) can be obtained by solving equations (7). To solve the linear equations, we adopted the numerical solution combined the successive-over-relaxation (SOR) method. Corresponding iterative scheme of SOR is as follows[18]:

\[ x_i^{(n+1)} = x_i^{(n)} + \alpha (\sum_{j=1}^{n-1} A_{ij} x_j^{(n+1)} - \sum_{j=1}^{n} A_{ij} x_j^{(n)})/A_{ii} \quad (8) \]

where \( \alpha \) is the relaxation factor that takes value on interval \((0, 2)\), generally the appropriate value can be given by \( \alpha = 2 [1 + e(A)]^{-1/2} \), where \( e(A) \) is the condition number of \( A \). And note that the results should be normalized by equation (6).

4. Case Studies for SMIB System

4.1. The Stationary FPK Equation of SMIB System

For stability analysis of power system, the operation of the steady-state after being disturbed is focused. According to equation (3) and (5), the stationary FPK equation that SMIB systems satisfied can be given as follows:

\[ \frac{\sigma^2}{2} \frac{\partial^2 p(x)}{\partial \omega^2} - \frac{\partial [f_1(x)p(x)]}{\partial \delta} - \frac{\partial [f_2(x)p(x)]}{\partial \omega} = 0 \quad (9) \]

And the exact stationary solution of equation (9) is [17]:

\[ p(x) = c \exp \left[ -\frac{2\bar{D}}{\sigma^2} \left( \frac{\omega^2}{2} - \bar{P}_{\text{max}} \cos \delta - \bar{P}_{\text{m}} \delta \right) \right] \quad (10) \]

where \( c \) is the normalized constant.

4.2. Numerical Simulation of Stationary FPK Equation for SMIB System

For the stationary FPK equation (9) of SMIB system with the random excitation of renewable energies, numerical simulation was carried out according to the numerical method above. The system parameters are \( \bar{D} = 0.41; \bar{P}_{\text{max}} = 1; \bar{P}_{\text{m}} = 0.5; x_0 = (0, 1.5)^T \) and the equilibrium point was calculated as \( x_e = (0.524, 0)^T \).
Figure 4. Comparison of theoretical and numerical results

Figure 2 shows the exact stationary solution with $\sigma = 0.1$, the numerical results shown in Figure 3 and Figure 4 gives a comparison of the exact stationary solution and numerical solution of marginal probability. The results illustrate the effectiveness of the numerical method. As is shown in the figures, the stationary PDF is clustered in a peak around the equilibrium point of the system. It is shown that the system of equation (3) can maintain the stability under the stochastic perturbations with $\sigma = 0.1$.

Figure 5. Marginal probability with various intensities of stochastic excitation

Figure 6. Trajectories with various intensities of stochastic excitation

Figure 7. Simulation of stationary PDF by Monte Carlo method

Figure 8. Comparison of numerical solution and Monte Carlo simulation of marginal probability

Figure 5 shows impacts of the evolution on Marginal probability $p(\delta)$ under different intensities of random excitation. It can be concluded that stationary PDF are concentrated near the equilibrium point and the concentration trend weakens as the intensity getting stronger. The results indicate that when the power system is subjected to excitations from renewable energies or other factors, the greater the intensity of disturbances is, the lower the stability of the system is. Euler-Maruyama method was used to obtain the operating trajectories of the system under different intensities of stochastic excitation,
which is shown in Figure 6. It can be seen from the figure that each trajectory has a vibration due to the existence of disturbances, and the amplitude of vibration increases when the intensity gets stronger, which is consistent with the result in Figure 5.

In order to verify the correctness of the conclusions, Monte Carlo method is used to simulate the system of equation (4) 1000 times. Figure 7 and 8 show the result of Monte Carlo method. Figure 7 is the result of stationary PDF, which is corresponding to the solution obtained by FPK equation. And from the comparison in Figure 8, the results of marginal probability are basically ingood agreement, which proves the correctness of the method in this paper.

5. Conclusions and Further Studies
In this paper, the FPK equation and steady FPK equation are obtained from the stochastic model of a SMIB system with stochastic excitation of renewable energy, and the steady probability density of the system under random excitation of different intensities is analysed by numerical simulation. It can be captured from the results that the value of power angle is concentrated in the concentration trend weakens when the random excitation intensity of the system increases. The follow-up work is to study the evolution of probability density with time variable and the stochastic bifurcation problem of the system as the parameter changes.

The result herein gives us a clue to study dynamic behaviour of power systems with renewables. The method proposed here can be used to analyse multi-machine system. Apply the method given in [19-20], we can first divide the generators into two groups, and then get the equivalent two-machinesystem. After further simplification, the multiple-machine system will be equivalent to a SMIB system, which can be used to obtain the FPK equation and calculate the numerical solution according to the preceding method.

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