Minimal model of quantum measurements and emergence of classical domain

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We propose a minimalistic model of quantum measurements within a finite system that undergoes a regular unitary evolution. Based on information theory, the measurement process requires a sacrifice of information entropy. This loss in our model is represented by a single cubit being isolated from the rest of the system. As a result of the partitioning, the classical domain emerges and the measurement occurs. The effect of the measurement on the quantum state of the system can be described by the generalized projection operator, which in general is less restrictive than the traditional wave function collapse.

As it approaches its centennial, Quantum Mechanics (QM) is still commonly perceived as a counterintuitive and mysterious field of Physics. Most of its postulates and the overall mathematical structure, though exotic at the time of their development, are relatively straightforward. However, there is one important element of QM that remains puzzling and controversial, and is responsible for many of its paradoxical aspects: the measurement problem [1][4]. According to the conventional Copenhagen formulation of the QM, at the moment of a measurement, the regular unitary evolution stops working and the wave function "collapses," i.e. the system ends up in one of the eigenstates of the corresponding operator. This prescription is practical, but conceptually problematic since the observer is assumed to live in a classical world rather than being described by QM itself. From the point of view of information theory, at the moment of quantum measurement some information has to be sacrificed [1][7][9]. In this paper we present a minimalistic model in which this information loss is represented by a single quantum bit (cubit) that becomes unavailable for any future interactions. We demonstrate that such informational isolation of a part of the system leads to an emergence of the classical domain that eventually facilitates the measurement process.

In QM, the state of the system is describe as a vector in Hilbert space, called wave function $|\Psi\rangle$, and its time evolution is given by a linear unitary operator: $|\Psi\rangle \rightarrow \hat{T} |\Psi\rangle$. This evolution can be calculated, e.g. with the help of Schrödinger Equation, Heisenberg’s Matrix Mechanics or Dirac-Feynman path integrals. There is a long history of research into the topic quantum measurement, starting with John von Neumann’s scheme proposed in the early days of QM [1]. He demonstrated how a measuring apparatus operating according to the laws of QM, can be inserted between the measured system and the observer. He also formalized the wave function collapse process in the form of it Projection Postulate. According to it, the wave function, at the time of measurement, changes as $|\Psi\rangle \rightarrow \hat{P}_k |\Psi\rangle / \sqrt{p_k}$ where $\hat{P}_k = |k\rangle \langle k|$ is the projection operator associated with the observed eigenstate $|k\rangle$, and $p_k = \langle \Psi | \hat{P}_k |\Psi\rangle$ is the probability of that particular outcome.

To model the measuring process, von Neumann considered a combination of two quantum subsystems: the System S, and the Apparatus A. During the first step, which is called premeasurement, a quantum entanglement between these two subsystems is achieved. Namely, if S is in a quantum state $\sum_k \alpha_k |k\rangle$, and the Apparatus is originally in state $|0\rangle_A$, the premeasurement is the following unitary transformation:

$$\left( \sum_k \alpha_k |k\rangle \right) |0\rangle_A \rightarrow |\Psi\rangle = \sum_k \alpha_k |k\rangle |A \rangle$$  \hspace{1cm} (1)

Here $|k\rangle$ and $|A\rangle$ are states of S and A respectively. Following the premeasurement, the wave function collapse is described as a non-unitary transformation of the density operator, $\hat{\rho} = |\Psi\rangle \langle \Psi|$:

$$\hat{\rho} \rightarrow \sum_k \hat{P}_k \hat{\rho} \hat{P}_k$$  \hspace{1cm} (2)

Here $\hat{P}_k = |k\rangle_A \langle k|$ are projection operators of the Apparatus subsystem. If both the premeasurement Eq. (1), and the non-unitary projection process Eq. (2), are performed in the same basis of the Apparatus states, $|k\rangle_A$, they would transform the density operator of the combined system into the diagonal form:

$$\hat{\rho} = \sum_k |\alpha_k|^2 |k\rangle_A \langle k|$$  \hspace{1cm} (3)

Here $|\alpha_k|^2$ are probabilities of different results, and since all the off-diagonal terms are zeros, there is no interference between those outcomes, i.e. the superposition principle of the classical probability theory is recovered. It is easy to see that this procedure does not resolve the quantum measurement problem but simply restates mathematically the Copenhagen wave collapse prescription. In particular, von Neumann argues that an interface between the quantum and classical worlds is ultimately unavoidable since any measurement has to be eventually perceived by an observer who lives in a classical reality.

There was an exotic attempt by Bohm to couple wave function to real particle dynamics [3], as well as an esoteric multi-world interpretation by Everett [4][5], in
which observer’s own brain becomes a part of the theory. An important breakthrough was the so-called decoherence program introduced in the works of Zurek, Gell-Mann, Hartle, and others [10–13]. It is based on the observation that any practical measurement device interacts with an environment. Therefore, the natural next step beyond von Neumann scheme is to add Environment $E$ to the System and Apparatus. Let us imagine that the environment is coupled to the apparatus in such a way that the states $|k\rangle_A$ of the latter are preserved in time, but the evolution of the environment would depend on $|k\rangle_A$. As a result, following the premeasurement, Eq. (4), the composite $S + A + E$ system will evolve after some time into a new state:

$$|\Psi\rangle|0\rangle_E = \left( \sum_k \alpha_k |k\rangle_A \right) |0\rangle_E + \sum_k \alpha_k |k\rangle_A |k\rangle_E$$

The density operator of $S + A$ system is obtained by taking a partial trace of the overall $S + A + E$ density operator with respect to environment variables, $\hat{\rho}_{sa} = \text{Tr}_E \hat{\rho}_{sa}$. At this stage, one would recover the von Neumann non-unitary measurement process, Eq. (2), if the eventual states of environment that correspond to different indexes $k$ are mutually orthogonal: $\langle k|k'\rangle_E = \delta_{kk'}$. This orthogonality condition can indeed be proven directly for certain explicit models of the environment. However, the perfect environment-induced measurement is only achieved in the thermodynamic limit. So, even the simplest quantum system could be described self-consistently only as a part of an infinitely large one. As a result, within the decoherence program, the fundamental problem of quantum measurement becomes dependent on our ability to understand statistical properties of a complex system. Partially because of this, its generality and limits of applicability are not fully established.

Another insight into the nature of a quantum measurement is given by the information theory [11–13]. Quantum information entropy, also introduced by von Neumann, $S = -\text{Tr}(\hat{\rho} \log_2 \hat{\rho})$ is zero for the system in a pure quantum state, and is conserved by unitary evolution. Therefore, if the initial quantum state of an isolated system is known, its entropy information is $S = 0$. If we now perform a new measurement on the same system and record its result with a classical bit, the information entropy associated with that bit is $\Delta S = -p \log_2 p - (1 - p) \log_2 (1 - p)$, were $p$ is probability of it being in state 1. After an observer reads the bit, this information is being extracted, and the measured system once again returns to a pure state with $S = 0$. We come to a seemingly paradoxical conclusion that the new information is extracted from nowhere. The non-unitary von Neumann process and the decoherence program both provide a partial resolution to this paradox. In both cases, the information entropy of the system is being increased during the measurement. This allows one to reduce $S$ back to zero once the result of the measurement is read, and to record the new information about the system. In other words, either processes is needed to erase some information.

**a) “Fixed basis” scheme**

$$\begin{align*}
S: |\psi\rangle & \quad |0\rangle \\
A: |0\rangle & \quad \text{rotation operator} \\
B: |0\rangle & \quad \text{sink}
\end{align*}$$

**b) “Rotated basis” scheme**

$$\begin{align*}
S: |\psi\rangle & \quad |0\rangle \\
A: \hat{R}(\theta, \phi) |\psi\rangle & \quad \text{rotation operator} \\
B: |0\rangle & \quad \text{sink}
\end{align*}$$

![FIG. 1. Schematic representation of minimalistic model of quantum measurement](image)

Here we propose a minimalistic model in which the measurement and a classical domain naturally emerge within a small quantum system undergoing regular unitary evolution. In our model, the information cost of measurement is represented by a single qubit which is being “lost”, i.e. becomes unavailable for any future measurements or interactions. The simplest version of our model contains only 3 qubits: $S$, $A$ and $B$, which correspond to “System”, “Apparatus”, and “Black Box”. Prior to the measurement, $S$ is in an unknown state $|\psi\rangle = \alpha|1\rangle + \beta|0\rangle$ while both $A$ and $B$ are in their respective ”zero” states, $|0\rangle_A$ and $|0\rangle_B$. The measurement protocol is as follows: first, we perform the premeasurement with the help of quantum $\text{cnot}$ gate acting on $S$ and $A$, and then execute the same operation on $A$ and $B$. $\text{cnot}$ (or controlled-$\text{NOT}$) is one of the most common gates in quantum logic [14]. When acting on the System
and Apparatus with our initial conditions, it would results in von Neumann-style entanglement between them: $\alpha|1\rangle_A|0\rangle_B + \beta|0\rangle_A|0\rangle_B$. Following our two-step process, all three qubits would become entangled:

$$|\psi\rangle|0\rangle_A|0\rangle_B \rightarrow \alpha|1\rangle_A|1\rangle_B + \beta|0\rangle_A|0\rangle_B$$  \hspace{1cm} (5)

This is a unitary process fully consistent with QM. The overall measurement process is schematically shown in Fig. [1]. The only unconventional step in our protocol is that we assume $B$-qubit after this process to be "lost", i.e. unavailable for any future interactions. Mathematically, this would imply that instead of considering the full $S + A + B$ system, we should calculate the density operator of the reduced $SA$ system by taking partial trace with respect to $B$-qubit:

$$\rho_{sa} = \text{Tr}_B\rho_{sab} = |\alpha|^2|1\rangle_A|1\rangle + |\beta|^2|0\rangle_A|0\rangle$$  \hspace{1cm} (6)

This result is formally equivalent to the wave function collapse as given by von Neumann’s non-unitary projection process, Eq. [2].

The classical domain within the quantum system emerges as a direct consequence of loosing information about $B$-qubit. There are several ways in which this process can be interpreted and/or implemented. First, one can employ an additional device that acts as the entropy sink. Its role is to supply a new $B$-qubit in a pure quantum state $|0\rangle$, in exchange for the current one. Upon this exchange, we recover our original $S + A + B$ system, but the Apparatus now becomes a classical bit that has recorded the result of the measurement, $r$, while System $S$ ends up in a pure quantum state $|r\rangle$ for each of the outcomes. Information entropy associated with the measurement, $\Delta S = -\sum_r p_r \log_2 p_r$, is equal to the one adsorbed by entropy sink, together with $B$-qubit. If we consider $S + A + B$ as a single quantum system, its entropy would remain 0. This means that the positive entropy $2\Delta S$ of the two separated qubits is equal and opposite to the entropy associated with the classical correlation and quantum entanglement between them (each contributing $-\Delta S$ to the overall quantum information entropy) [7][9][15][16]. This negative mutual entropy is being lost due to separation. Furthermore, if the design of the entropy sink is such that any information about $B$-qubit is getting completely erased due to the underlying ergodicity [17][18], the ultimate entropy cost of a single-bit measurement is given by the maximum possible value of $\Delta S$, i.e. 1. This result is consistent with Landauer’s Principle that sets the lower bound for the energy cost needed for irreversible single-bit operation at finite temperature $T$: $E_{\text{min}} = k_B T \ln 2$ [19][20].

An alternative to introduction of the entropy sink would be the "gentleman’s agreement", according to which $B$-qubit simply becomes unavailable for any future interactions. Note that the choice between $A$ and $B$ is arbitrary: each of them can be used to record the result, while the other could be the "lost" qubit. Thus, one can imagine two independent observers, Alice and Bob, to have an access to $A$- and $B$-qubits, respectively. This way, each of them would have a classical record of the same measurement, as long as they are not allowed to communicate between each other. If the agreement is ever broken, i.e. any information is being exchanged between them, the non-classical correlations, such as violation of Bell’s inequality, would become possible [21][22].

An important aspect of the quantum measurement problem is the selection of the preferred basis. Indeed, the non-unitary transformation Eq. [2] is not invariant with respect to the choice of the basis states. This is one of the key properties of QM: for instance, according to Heisenberg’s uncertainty principle, one can choose to measure either particle’s position or its momentum, but not both. In order to perform a conventional measurement of the System with the help of the Apparatus, the basis in which they are entangled during the premeasurement, Eq. [1], should be the same as the basis in which the wave function collapse occurs. Under the decoherence program, it is suggested that basis of so-called "pointer" states is chosen by means of environment-induced superselection or einselection. It implies that the form of coupling between the Apparatus and the environment pre-determines the observable which is being measured.

Within our minimalistic model, it is straightforward to implement a scheme in which the basis of the Apparatus states during the two stages of the measurement process are not the same, and in fact completely arbitrary. Specifically, following the premeasurement, $\text{cnot}(S,A)$, and prior to the interaction with the $B$-qubit, we can apply a unitary rotation operator to the Apparatus qubit:

$$\hat{R}_A(\theta, \phi) = \begin{bmatrix} \cos \theta & e^{i\phi} \sin \theta \\ -e^{-i\phi} \sin \theta & \cos \theta \end{bmatrix}_A$$  \hspace{1cm} (7)

After that, $A$- and $B$-qubits are entangled by means of $\text{cnot}(A,B)$ quantum gate. As a result, our original state is transformed by three consecutive unitary operations, as shown in Fig. [1b]:

$$|\Psi\rangle_{\theta,\phi} = \text{cnot}(A,B) \times \hat{R}_A(\theta, \phi) \times \text{cnot}(S,A)|\psi\rangle|0\rangle_A|0\rangle_B$$  \hspace{1cm} (8)

As before, the key element of the measurement scheme is that qubit $B$ is "lost". We can calculate the resulting density operator of the $S + A$ system, by taking partial trace over $B$ subsystem:

$$\rho_{sa} = \text{Tr}_B|\Psi\rangle_{\theta,\phi}\langle\Psi|_{\theta,\phi} = \sum_{r=0,1} |r\rangle_A\langle S_r| \langle S_r|\langle r|_A$$  \hspace{1cm} (9)

Here $|1\rangle_A$ and $|0\rangle_A$ form the basis of the Apparatus states upon rotation, Eq. [7]. Just like in the previous case, the Apparatus becomes completely classical: there is no quantum interference between $|1\rangle_A$ and $|0\rangle_A$. This corresponds to the two alternative results of measurement,
The corresponding System states are:

\[ |S_r\rangle = \hat{P}_r(\theta, \phi)|\psi\rangle \]  \hspace{1cm} (10)

Here \( \hat{P}_r(\theta, \phi) \) can be interpreted as generalized projection operators for \( r = 1, 0 \):

\[
\hat{P}_r(\theta, \phi) = \sum_{k=1,0} R^*_r(\theta, \phi) \hat{P}_k = \begin{cases}
\hat{P}_1 \cos \theta + \hat{P}_0 e^{-i\phi} \sin \theta , & r = 1 \\
-\hat{P}_1 e^{i\phi} \sin \theta + \hat{P}_0 \cos \theta , & r = 0
\end{cases}
\]  \hspace{1cm} (11)

Here \( \hat{R}^r(\theta, \phi) \) is complex-conjugate to rotation matrix \( \hat{R}(\theta, \phi) \) given by Eq. (7). The two state vectors, \( |S_1\rangle \) and \( |S_0\rangle \), are in a general case not orthogonal, and not yet normalized. The probabilities of of the corresponding outcomes can be found in a standard QM manner:

\[ p_r = \langle S_r|S_r\rangle \]  \hspace{1cm} (12)

Importantly, von Neumann’s information entropy associated with the density operator \( \hat{\rho}_{sa} \) in Eq. (9), coincides with the Shannon’s entropy that can be calculated based on these probabilities:

\[ H = -\text{Tr}(\hat{\rho}_{sa} \log_2 \hat{\rho}_{sa}) = -\sum_{k=0,1} p_r \log_2 p_r \]  \hspace{1cm} (13)

This means that the density operator represents a so-called "ignorance interpretable" mixture of pure states. In other words, once the result of the measurement is known, the System ends up in pure quantum state, \( |S_r\rangle/\sqrt{p_r} \). However, in a general case, the specific states that correspond to each outcome are not pre-determined by the measurement. Indeed, \( \hat{P}_r(\theta, \phi) \) in Eq. (11) does not have the conventional form of a projection operator \( \hat{P}_k = |k\rangle \langle k| \), so the final states will depend on the state of the System prior to the measurement, \( |\phi\rangle \). In particular, one of the most important property of the projection operator, \( \hat{P}_k \hat{P}_k = \delta_{kk} \hat{P}_k \), is violated for \( \hat{P}_r(\theta, \phi) \). This implies that when two identical measurements are repeated one after another, on the same System but with two different Apparatuses, they are not guaranteed to give the same result. This is a spectacular violation of the Projection Postulate, although the latter is still valid for the combined \( S + A \) system.

Despite its unconventional properties, the generalized projection operator does describe the transformation of the system in a realistic, experimentally realizable, measurement procedure. An important difference from the regular wave function collapse is that the System, unlike the Apparatus, is not becoming classical immediately after the measurement, but ends up in a new coherent quantum state. Furthermore, one can imagine an experiment in which, following the premeasurement process, the Apparatus is stored at decoherence-free conditions for as much time as plausible. The measurement itself can be done later, in any basis of one’s choice. One can even completely "erase" the fact of the premeasurement. In order to do so, the rotation operator must be given by Hadamard quantum gate, \( \hat{H} = \hat{R}(\pi/4, 0) \). If now the result of the measurement is 1, it would mean that any effect of the premeasurement on the System has been successfully reverted. This is because \( \hat{P}_1(\pi/4, 0) = I/\sqrt{2} \). The probability of that outcome is 1/2. Its alternative, \( r = 0 \), also corresponds to a unitary transformation, Pauli-Z gate, applied to the System: \( \hat{P}_0(\pi/4, 0) = (\hat{P}_0 - \hat{P}_1)/\sqrt{2} = \hat{Z}/\sqrt{2} \).

In summary, we proposed a minimalistic, experimentally realizable scheme, in which the classical domain and the measurement process are emergent features resulting from the separation of a quantum system onto informationally isolated parts. This process is represented by the disposal of a single qubit, the process that can be facilitated by an external entropy sink, or simply be a matter of agreement that the cubit becomes unavailable for any future interactions. As a result, the Apparatus becomes a classical bit that records the result of the measurement. For each of the outcomes, the System ends up in a pure quantum state. However, in a general case, its transformation during measurement procedure is not a regular wave function collapse. Instead, it is given by the generalized projection operator \( \hat{P}_r \) which does not have to satisfy condition \( \hat{P}_r \hat{P}_r = \delta_{rr'} \hat{P}_r \), indicating a violation of the Projection Postulate in its strong form. In other words, the the System upon this type of measurement does not have to assume the eigenstate of certain operator, and the two consecutive measurements do not have to give the same result. Our simple model also challenges yet another prevailing idea about the nature of QM.
have to side with the famous "erroneous" quote by Einstein that God "does not play dice with the Universe". Indeed, QM by itself preserves information entropy of the system. The non-deterministic nature of a quantum measurement is the consequence of the information sacrifice needed for the measurement to be done.

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[1] J. von Neumann, Math. Foundation of Quantum Mechanics (Princeton University Press, Princeton, NJ, USA, 1955).
[2] P. A. M. Dirac, The Principles of Quantum Mechanics (Clarendon Press, Oxford, UK, 1930).
[3] D. Bohm, Physical Review 85, 166 (1952)
[4] H. Everett, Reviews of Modern Physics 29, 454 (1957).
[5] J. A. Wheeler, Reviews of Modern Physics 29, 463 (1957).
[6] E. P. Wigner, American Journal of Physics 31, 6 (1963).
[7] N. J. Cerf and C. Adami, Physical Review Letters 79, 5194 (1997).
[8] N. J. Cerf and C. Adami, Physica D-Nonlinear Phenomena 120, 62 (1998).
[9] K. Modi, A. Brodutch, H. Cable, T. Paterek, and V. Vedral, Reviews of Modern Physics 84, 1655 (2012).
[10] W. H. Zurek, Reviews of Modern Physics 75, 715 (2003).
[11] W. H. Zurek, Physics Today 44, 36 (1991).
[12] M. Gell-Mann and J. B. Hartle, Physical Review D 47, 3345 (1993).
[13] M. Schlosshauer, Reviews of Modern Physics 76, 1267 (2004).
[14] C. H. Bennett and D. P. DiVincenzo, Nature 404, 247 (2000).
[15] L. del Rio, J. Aberg, R. Renner, O. Dahlsten, and V. Vedral, Nature 474, 61 (2011).
[16] J. M. R. Parrondo, J. M. Horowitz, and T. Sagawa, Nature Physics 11, 131 (2015).
[17] R. Steinigeweg, A. Khodja, H. Niemeyer, C. Gogolin, and J. Gemmer, Phys. Rev. Lett. 112, 130403 (2014).
[18] S. Goldstein, J. L. Lebowitz, R. Tumulka, and N. Zanghì, Phys. Rev. Lett. 96, 050403 (2006).
[19] R. Landauer, IBM Journal of Research and Development 5, 183 (1961).
[20] C. H. Bennett, Studies in History and Philosophy of Modern Physics 34B, 501 (2003).
[21] J. S. Bell, Physics Physique Fizika 1, 195 (1964).
[22] J. S. Bell, Reviews of Modern Physics 38, 447 (1966).