Irregular fibers of complex polynomials in two variables

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Introduction

Let \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) be a polynomial. The bifurcation set \( \mathcal{B} \) for \( f \) is the minimal set of points of \( \mathbb{C} \) such that \( f : \mathbb{C}^n \setminus f^{-1}(\mathcal{B}) \rightarrow \mathbb{C} \setminus \mathcal{B} \) is a locally trivial fibration. For \( c \in \mathbb{C} \), we denote the fiber \( f^{-1}(c) \) by \( F_c \). The fiber \( F_c \) is irregular if \( c \) is in \( \mathcal{B} \). If \( s \notin \mathcal{B} \), then \( F_s \) is a generic fiber and is denoted by \( F_{\text{gen}} \). The tube \( T_c \) for the value \( c \) is a neighborhood \( f^{-1}(D^2(\varepsilon)) \) of the fiber \( F_c \), where \( D^2(\varepsilon) \) stands for a 2-disk in \( \mathbb{C} \), centered at \( c \), of radius \( \varepsilon \ll 1 \). We assume that affine critical singularities are isolated. The value \( c \) is regular at infinity if there exists a compact set \( K \) of \( \mathbb{C}^2 \) such that the restriction of \( f \) to \( T_c \setminus K \rightarrow D^2(\varepsilon) \) is a locally trivial fibration. Set \( n = 2 \). Let \( j_c : H_1(F_c) \rightarrow H_1(T_c) \) be the morphism induced by the inclusion of \( F_c \) in \( T_c \). One of the consequences of our study of \( j_c \) is the following:

**Theorem.** \( j_c \) is an isomorphism if and only if \( c \) is a regular value at infinity.

E. Artal-Bartolo, Pi. Cassou-Noguès and A. Dimca have proved this result in [ACD] for polynomials with a connected fiber \( F_c \). But for example the fiber \( F_0 \) of Broughton’s polynomial \( f(x, y) = x(xy + 1) \) is not connected. We also give necessary and sufficient conditions for \( j_c \) to be injective and surjective: let \( G_c \) the dual graph of \( F_c = f^{-1}(c) \) and \( \bar{G}_c \) the dual graph of a compactification of the fiber \( F_c \) obtained by a resolution at infinity of \( f \). We find that \( \text{rk Ker } j_c = n(F_c) - 1 + \text{rk } H_1(G_c) - \text{rk } H_1(\bar{G}_c) \) where \( n(F_c) \) is the number of connected components of \( F_c \). So \( j_c \) is injective if and only if \( F_c \) is connected and \( H_1(G_c) \) is isomorphic to \( H_1(\bar{G}_c) \).

We apply these results to the study of neighborhoods of irregular fibers. Set \( n \geq 2 \). Let \( F^\circ_c \) be the smooth part of \( F_c \): \( F^\circ_c \) is obtained by intersecting \( F_c \) with a large \( 2n \)-ball and cutting out a small neighborhood of the (isolated) singularities. Then \( F^\circ_c \) can be embedded in \( F_{\text{gen}} \). We study the following commutative diagram that links the three elements \( F^\circ_c \), \( F_{\text{gen}} \), and \( T_c \):

\[
\begin{array}{ccc}
H_q(F^\circ_c) & \xrightarrow{j_c} & H_q(T_c) \\
\ell_c \downarrow & & \downarrow k_c \\
H_q(F_{\text{gen}}) & & \\
\end{array}
\]

where \( \ell_c \) is the morphism induced in integral homology by the embedding; \( j_c \) and \( k_c \) are induced by inclusions. The morphism \( k_c \) is well-known and \( V_q(c) = \text{Ker } k_c \) are vanishing
cycles for the value $c$. Let $h_c$ be the monodromy induced on $H^q(F_{\text{gen}})$ by a small circle around the value $c$. Then we prove that the image of $\ell_c$ are invariant cycles by $h_c$:

$$\text{Ker}(h_c - \text{id}) = \ell_c(H^q(F^c_c)).$$

This formula for the case $n = 2$ has been obtained by F. Michel and C. Weber in [MW]. Finally we give a description of vanishing cycles with respect to eigenvalues of $h_c$ for homology with complex coefficients. For $\lambda \neq 1$ and $p$ a large integer the characteristic space $E(\lambda) = \text{Ker}(h_c - \lambda \text{id})^p$ is composed of vanishing cycles for the value $c$. For $\lambda = 1$ the situation is different. If $K_q(c) = V_q(c) \cap \text{Ker}(h_c - \text{id})$ are invariant and vanishing cycles we have

$$K_q(c) = \ell_c(\text{Ker} j^0_c).$$

And for $n = 2$ we get the formula

$$\text{rk} K_1(c) = \text{r}(F_c) - 1 + \text{rk} H_1(\bar{G}_c).$$

1 Irregular fibers and tubes

1.1 Bifurcation set

We can describe the bifurcation set $B$ as follows: let $\text{Sing} = \{z \in \mathbb{C}^n \mid \text{grad}_f(z) = 0\}$ be the set of affine critical points and let $B_{\text{aff}} = f(\text{Sing})$ be the set of affine critical values. The set $B_{\text{aff}}$ is a subset of $B$. The value $c \in \mathbb{C}$ is regular at infinity if there exists a disk $D$ centered at $c$ and a compact set $K$ of $\mathbb{C}^n$ with a locally trivial fibration $f : f^{-1}(D) \setminus K \to D$. The non-regular values at infinity are the critical values at infinity and are collected in $B_{\infty}$. The finite set $B$ of critical values is now:

$$B = B_{\text{aff}} \cup B_{\infty}.$$

In this article we always assume that affine singularities are isolated, that is to say that $\text{Sing}$ is an isolated set in $\mathbb{C}^n$. For $n = 2$ this hypothesis implies that the generic fiber is a connected set.

1.2 Statement of the result

In this paragraph $n = 2$. The inclusion of $F_c$ in $T_c$ induces a morphism $j_c : H_1(F_c) \to H_1(T_c)$.

**Theorem 1.** $j_c : H_1(F_c) \to H_1(T_c)$ is an isomorphism if and only if $c \notin B_{\infty}$.

When $F_c$ is a connected fiber this result has been obtained in [ACD]. We generalize this study and we give simple criteria to determine whether $j_c$ is injective or surjective. We firstly recall notations and results from [ACD].

Let denote $F_{\text{aff}} = F_c \cap B^1_R \ (R \gg 1)$ and $F_{\infty} = F_c \setminus F_{\text{aff}}$, thus $F_{\text{aff}} \cap F_{\infty} = K_c = f^{-1}(c) \cap S^3_R$ is the link at infinity for the value $c$. Similarly $T_{\text{aff}} = T_c \cap B^1_R$ and $T_{\infty} = T_c \setminus T_{\text{aff}}$. We denote $j_{\infty} : H_1(F_{\infty}) \to H_1(T_{\infty})$ the morphism induced by inclusion. The morphism

$\text{rk} K_1(c) = \text{r}(F_c) - 1 + \text{rk} H_1(\bar{G}_c)$. 
To compactify the situation, for 1.3 Resolution of singularities

\[ \phi \]

The set of components \( D \) points on the line at infinity \( L \)

We can blow-up more points to obtain the \( \phi \)

fibers of \( \phi \)

bamboos (possibly empty) (a

Mayer-Vietoris exact sequences for the decompositions \( F_c = F_{\text{aff}} \cup F_\infty \) and \( T_c = T_{\text{aff}} \cup T_\infty \) give the commutative diagram (D):

\[
\begin{array}{c}
0 \rightarrow H_1(F_\infty \cap F_{\text{aff}}) \xrightarrow{g} H_1(F_\infty) \oplus H_1(F_{\text{aff}}) \xrightarrow{h} H_1(F_c) \rightarrow 0 \\
0 \rightarrow H_1(T_\infty \cap T_{\text{aff}}) \xrightarrow{g'} H_1(T_\infty) \oplus H_1(T_{\text{aff}}) \xrightarrow{h'} H_1(T_c) \rightarrow H_0(T_\infty \cap T_{\text{aff}}).
\end{array}
\]

The 0 at the upper-right corner is provided by the injectivity of \( H_0(F_\infty \cap F_{\text{aff}}) \rightarrow H_0(F_\infty) \) (\( F_c \) need not to be a connected set) hence \( H_0(F_\infty \cap F_{\text{aff}}) \rightarrow H_0(F_\infty) \oplus H_0(F_{\text{aff}}) \) is injective.

1.3 Resolution of singularities

To compactify the situation, for \( n = 2 \), we need resolution of singularities at infinity [\( LW \)]:

\[
\begin{array}{ccc}
\mathbb{C}^2 & \rightarrow & \mathbb{C}P^2 \\
\uparrow f & & \downarrow \pi \\
\Sigma_w & \xrightarrow{j} & \mathbb{C}P^1 \\
\downarrow \phi_w & & \\
\mathbb{C} & \rightarrow & \mathbb{C}P^1
\end{array}
\]

\( j \) is the map coming from the homogenization of \( f \); \( \pi \) is the minimal blow-up of some points on the line at infinity \( L_\infty \) of \( \mathbb{C}P^2 \) in order to obtain a well-defined morphism \( \phi_w : \Sigma_w \rightarrow \mathbb{C}P^1 \): this the weak resolution. We denote \( \phi_w^{-1}(\infty) \) by \( D_\infty \), and let \( D_{\text{dic}} \) be the set of components \( D \) of \( \pi_w^{-1}(L_\infty) \) that verify \( \phi_w(D) = \mathbb{C}P^1 \). Such a \( D \) is a dicritical component. The degree of a dicritical component \( D \) is the degree of the branched covering \( \phi_w : D \rightarrow \mathbb{C}P^1 \). For the weak resolution the divisor \( \phi_w^{-1}(c) \cap \pi^{-1}(L_\infty), c \in \mathbb{C} \), is a union of bamboos (possibly empty) (a bamboo is a divisor whose dual graph is a linear tree). The set \( \mathcal{B}_\infty \) is the set of values of \( \phi_w \) on non-empty bamboos with the set of critical values of the restriction of \( \phi_w \) to the dicritical components.

We can blow-up more points to obtain the total resolution, \( \phi_t : \Sigma_t \rightarrow \mathbb{C}P^1 \), such that all fibers of \( \phi_t \) are normal crossing divisors that intersect the dicritical components transversally; moreover we blow-up affine singularities. Then \( D_\infty = \phi_t^{-1}(\infty) \) is the same as above and for \( c \in \mathcal{B} \) we denote \( D_c \) the divisor \( \phi_t^{-1}(c) \).

The dual graph \( G_c \) of \( D_c \) is obtained as follows: one vertex for each irreducible component of \( D_c \) and one edge between two vertices for one intersection of the corresponding components. A similar construction is done for \( D_\infty \), we know that \( G_\infty \) is a tree [\( LW \)]. The multiplicity of a component is the multiplicity of \( \phi_t \) on this component.

1.4 Study of \( j_\infty \)

See [\( ACD \)]. Let \( \phi \) be the weak resolution map for \( f \). Let denote by \( \text{Dic}_c \) the set of points \( P \) in the dicritical components, such that \( \phi(P) = c \). To each \( P \in \text{Dic}_c \) is associated one, and only one, connected component \( T_P \) of \( T_\infty \); \( T_P \) is the place at infinity for \( P \). We have \( T_\infty = \bigsqcup_{P \in \text{Dic}_c} T_P \) and we set \( F_P = T_P \cap F_\infty = T_P \cap F_c \) and \( K_P = \partial F_P \), finally \( n(F_P) \)
denotes the number of connected components of $F_P$. Let $\hat{F}_P$ be the strict transform of $c$ by $\phi$, intersected with $T_P$. The study of $j_\infty$ follows from the study of $j_P : H_1(F_P) \to H_1(T_P)$. Let $m_P$ be the intersection multiplicity of $F_P$ with the divisor $\pi^{-1}_w(L_\infty)$ at $P$.

**Case of $P \in \hat{F}_P$.** The group $H_1(T_P)$ is isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, $M_P$ being the boundary of a small disk with transversal intersection with the critical component. Moreover if $F_P = \coprod_{i=1}^{n(F_P)} F_P^i$ then $j_P([F_P^i]) = j_P([K_P^i]) = m_P^i[M_P]$.

**Case of $P$ being in a bamboo.** The group $H_1(T_P)$ is also isomorphic to $\mathbb{Z}$ and is generated by $[M_P]$, $M_P$ being the boundary of a small disk, with transversal intersection with the last component of the bamboo. Then $j_P[F_P^i] = j_P[K_P^i] = m_P^i, \ell_i[M_P]$. The integer $\ell_i$ only depends on the position where $F_P^i$ intersects the bamboo, moreover $\ell_i \geq 1$ and $\ell_i = 1$ if and only if $F_P^i$ intersects the bamboo at the last component. For a computation of $\ell_i$, refer to [ACD].

As a consequence $j_P$ is injective if and only if $n(F_P) = 1$ and $j_\infty$ is injective if and only if $n(F_P) = 1$ for all $P$ in $\text{Dic}_c$. In fact the rank of the kernel of $j_\infty$ is the sum of the ranks of the kernels of $j_P$ then

$$\text{rk} \ker j_\infty = \sum_{P \in \text{Dic}_c} (n(F_P) - 1).$$

Finally $j_\infty$ is surjective if and only if for all $P \in \text{Dic}_c$, $j_P$ is surjective.

### 1.5 Acyclicity

The value $c$ is *acyclic* if the morphism $\psi : H_0(T_\infty \cap T_{\text{aff}}) \to H_0(T_\infty) \oplus H_0(T_{\text{aff}})$ given by the Mayer-Vietorius exact sequence is injective.

Let give some interpretations of the acyclicity condition.

1. The injectivity of $\psi$ can be view as follows: two branches at infinity that intersect the same place at infinity have to be in different connected components of $F_c$.

2. Let $G_c$ be the *dual graph* of $F_c$ (one vertex for an irreducible component of $F_c$, two vertices are joined by an edge if the corresponding irreducible components have non-empty intersection, if a component has auto-intersection it provides a loop) and let $G_{c,P}$ be the graph obtained from $G_c$ by adding edges to vertices that correspond to the same place at infinity $T_P$. In other words $c$ is acyclic if and only if there is no new cycles in $G_{c,P}$, that is to say $H_1(G_c) \cong H_1(G_{c,P})$ for all $P$ in $\text{Dic}_c$.

3. Another interpretation is the following: $c$ is acyclic if and only if the morphism $h'$ of the diagram $(D)$ is surjective. This can be proved by the exact sequence:

$$H_1(T_\infty) \oplus H_1(T_{\text{aff}}) \xrightarrow{h'} H_1(T_c) \xrightarrow{\varphi} H_0(T_\infty \cap T_{\text{aff}}) \xrightarrow{\psi} H_0(T_\infty) \oplus H_0(T_{\text{aff}}) \to H_0(T_c).$$
4. Let consider the above Mayer-Vietoris exact sequence in reduced homology, the morphism \( \tilde{\psi} : \tilde{H}_0(T_\infty \cap T_{\text{aff}}) \rightarrow \tilde{H}_0(T_\infty) \oplus \tilde{H}_0(T_{\text{aff}}) \) is surjective because \( \tilde{H}_0(T_c) = \{0\} \). Moreover \( \tilde{\psi} \) is injective if and only if \( \psi \) is injective. As \( \tilde{\psi} \) is surjective, \( \psi \) is injective if and only if \( \text{rk} H_0(T_\infty \cap T_{\text{aff}}) = \text{rk} \tilde{H}_0(T_\infty) + \text{rk} \tilde{H}_0(T_{\text{aff}}) \), that is to say \( c \) is acyclic if and only if

\[
\sum_{P \in \text{Dic}_c} n(F_P) - 1 = \# \text{Dic}_c - 1 + n(F_c) - 1. \tag{*}
\]

This implies the lemma:

**Lemma 2.** \( j_\infty \) is injective \( \iff \) \( F_c \) is a connected set and \( c \) is acyclic.

**Proof.** If \( j_\infty \) is injective then \( n(F_P) = 1 \) for all \( P \) in \( \text{Dic}_c \), then \( H_0(T_\infty \cap T_{\text{aff}}) \cong H_0(T_\infty) \) and \( \psi \) is injective, hence \( c \) is acyclic and from equality \([\square]\), we have \( n(F_c) = 1 \) i.e. \( F_c \) is a connected set. Conversely, if \( c \) is acyclic and \( n(F_c) = 1 \) then equality \([\square]\) gives \( n(F_P) = 1 \) for all \( P \) in \( \text{Dic}_c \), thus \( j_\infty \) is injective. \( \square \)

Let us define a stronger notion of acyclicity. Let \( \tilde{G}_c \) be the dual graph of \( \phi^{-1}(c) \). The graph \( \tilde{G}_c \) can be obtained from \( G_c \) by adding edges between vertices that belong to the same place at infinity for all \( P \) in \( \text{Dic}_c \). The value \( c \) is strongly acyclic if \( H_1(\tilde{G}_c) \cong H_1(G_c) \).

Strong acyclicity implies acyclicity, but the converse can be false. However if \( F_c \) is a connected set (that is to say \( G_c \) is a connected graph) then both conditions are equivalent. This is implicitly expressed in the next lemma, which is just a result involving graphs.

**Lemma 3.** \( \text{rk} H_1(\tilde{G}_c) - \text{rk} H_1(G_c) = \sum_{P \in \text{Dic}_c} (n(F_P) - 1) - (n(F_c) - 1) \).

### 1.6 Surjectivity

**Proposition 4.** \( j_c \) surjective \( \iff \) \( j_\infty \) surjective and \( c \) acyclic.

**Proof.** Let us suppose that \( j_c \) is surjective then a version of the five lemma applied to diagram \((D)\) proves that \( j_\infty \) is surjective. As \( j_c \) and \( j_\infty \) are surjective, diagram \((D)\) implies that \( h' : H_1(T_\infty) \oplus H_1(T_{\text{aff}}) \rightarrow H_1(T_c) \) is surjective, that means that \( c \) is acyclic. Conversely if \( j_\infty \) is surjective and \( c \) is acyclic then \( h' \) is surjective and diagram \((D)\) implies that \( j_c \) is surjective. \( \square \)

### 1.7 Injectivity

**Proposition 5.** \( j_c \) is injective \( \iff \) \( F_c \) is a connected set and \( c \) is acyclic.

It follows from lemma \([\square]\) and from the next lemma.

**Lemma 6.** \( j_c \) injective \( \iff \) \( j_\infty \) injective.

Moreover the rank of the kernel is:

\[
\text{rk ker } j_c = \text{rk ker } j_\infty = \sum_{P \in \text{Dic}_c} (n(F_P) - 1) = n(F_c) - 1 + \text{rk} H_1(\tilde{G}_c) - \text{rk} H_1(G_c).
\]
The first part of this lemma can be proved by a version of the five lemma. However we shall only prove the equality of the ranks of \(\ker j_c\) and \(\ker j_\infty\). It will imply the lemma because we already know that \(\text{rk} \ker j_\infty = \sum_{P \in \text{Dic}_c} (n(F_P) - 1)\) and from lemma 3 we then have \(\text{rk} \ker j_\infty = n(F_c) - 1 + \text{rk} H_1(G_c) - \text{rk} H_1(G_c)\).

The study of the morphism \(j_c : H_1(F_c) \rightarrow H_1(T_c)\) is equivalent to the study of the morphism \(H_1(\text{aff}) \rightarrow H_1(T_c)\) induced by inclusion that, by abuse, will also be denoted by \(j_c\). To see this, it suffices to remark that \(F_c\) is obtained from \(F_{\text{aff}} = F_c \cap B_1^R\) by gluing \(F_c \cap S_3^R \times [0, +\infty[\) to its boundary \(F_c \cap S_2^R\). Then the morphism \(H_1(F_{\text{aff}}) \rightarrow H_1(F_c)\) induced by inclusion is an isomorphism; finally \(j_{\text{aff}} : H_1(F_{\text{aff}}) \rightarrow H_1(T_{\text{aff}})\) is also an isomorphism. The long exact sequence for the pair \((T_c, T_{\text{aff}})\) is:

\[
H_2(T_c) \rightarrow H_2(T_c, T_{\text{aff}}) \rightarrow H_1(T_{\text{aff}}) \xrightarrow{j_c} H_1(T_c)
\]

but \(H_2(T_c) = 0\) (see [ACL] for example) then the rank of \(\ker j_c\) is the rank of \(H_2(T_c, T_{\text{aff}})\). On the other hand, the study of \(j_\infty : H_1(F_\infty) \rightarrow H_1(T_\infty)\) is the same as the study of \(H_1(\partial T_\infty) \rightarrow H_1(T_\infty)\) induced by inclusion (and denoted by \(j_\infty\)) because the morphisms \(H_1(\partial F_\infty) \rightarrow H_1(F_\infty)\) and \(H_1(\partial F_\infty) \rightarrow H_1(\partial T_\infty)\) induced by inclusions are isomorphisms. The long exact sequence for \((T_\infty, \partial T_\infty)\) is:

\[
H_2(T_\infty) \rightarrow H_2(T_\infty, \partial T_\infty) \rightarrow H_1(\partial T_\infty) \xrightarrow{j_\infty} H_1(T_\infty).
\]

As \(H_2(T_\infty) = 0\) (see [ACL]), then the rank of \(\ker j_\infty\) is the same as \(H_2(T_\infty, \partial T_\infty)\). Finally the groups \(H_2(T_\infty, \partial T_\infty)\) and \(H_2(T_c, T_{\text{aff}})\) are isomorphic by excision, and then the ranks of \(\ker j_c\) and of \(\ker j_\infty\) are equal. That completes the proof.

1.8 Proof of the theorem

If \(c \notin B_\infty\), then the isomorphism \(j_{\text{aff}} : H_1(F_{\text{aff}}) \rightarrow H_1(T_{\text{aff}})\) implies that \(j_c\) is an isomorphism. Let suppose that \(c\) is a critical value at infinity and that \(j_c\) is injective. We have to prove that \(j_c\) is not surjective. As \(j_c\) is injective then by lemma 3, \(j_\infty\) is injective. By proposition 3 it suffices to prove that \(j_\infty\) is not surjective. Let \(P\) be a point of \(\text{Dic}_c\) that provides irregularity at infinity for the value \(c\), then \(n(F_P) = 1\) because \(j_\infty\) is injective. Let us prove that the morphism \(j_P\) is not surjective. For the case of \(P \in F_P\), the intersection multiplicity \(m_P\) is greater than 1, then \(j_P\) is not surjective. For the second case, in which \(P\) belongs to a bamboo, then \(m_P, l_i > 1\) except for the situation where only one strict transform intersects the bamboo at the last component. This is exactly the situation excluded by the lemma “bamboo extremity fiber” of [MW]. Hence \(j_\infty\) is not surjective and \(j_c\) is not an isomorphism.

1.9 Examples

We apply the results to two classical examples.

**Broughton polynomial.** Let \(f(x, y) = x(xy + 1)\), then \(B_{\text{aff}} = \emptyset, B = B_\infty = \{0\}\). Then for \(c \neq 0\), \(j_c\) is an isomorphism. The value 0 is acyclic since \(H_1(G_0) \cong H_1(G_0)\). The fiber \(F_0\) is not connected hence \(j_0\) is not injective. As the new component of \(G_0\) is of multiplicity 1 the corresponding morphism \(j_\infty\) is surjective, hence \(j_0\) is surjective.
Briançon polynomial. Let $f(x, y) = yp^3 + p^2s + a_1ps + a_0s$ with $s = xy + 1$, $p = x(xy + 1) + 1$, $a_1 = -\frac{5}{9}$, $a_0 = -\frac{1}{3}$. The bifurcation set is $B = B_\infty = \{0, c = -\frac{16}{9}\}$, moreover all fibers are smooth and irreducible. The value 0 is not acyclic then $j_0$ is neither injective nor surjective (but $j_\infty$ is surjective).

2 Situation around an irregular fiber

For $f : \mathbb{C}^n \rightarrow \mathbb{C}$ we study the neighborhood of an irregular fiber.

2.1 Smooth part of $F_c$

Let fix a value $c \in \mathbb{C}$ and let $B_{2n}^R$ be a large closed ball ($R \gg 1$). Let $B_{2n}^1, \ldots, B_{2n}^p$ be small open balls around the singular points (which are supposed to be isolated) of $F_c : F_c \cap \text{Sing}$. We denote $B_{2n}^1 \cup \ldots \cup B_{2n}^p$ by $B_j$. Then the smooth part of $F_c$ is

$$F_c^\circ = F_c \cap B_{2n}^R \setminus B_j.$$  

It is possible to embed $F_c^\circ$ in the generic fiber $F_{\text{gen}}$ (see [MW] and [NN]). We now explain the construction of this embedding by W. Neumann and P. Norbury. As $F_c$ has transversal intersection with the balls of $B_j$ and with $B_{2n}^R$, then there exists a small disk $D_c^2(c)$ such that for all $s$ in this disk, $F_s$ has transversal intersection with these balls. According to Ehresmann fibration theorem, $f$ induces a locally trivial fibration

$$f_1 : f^{-1}(D_c^2(c)) \cap B_{2n}^R \setminus B_j \rightarrow D_c^2(c).$$

In fact, as $D_c^2(c)$ is null homotopic, this fibration is trivial. Hence $F_c^\circ \times D_c^2(c)$ is diffeomorphic to $f^{-1}(D_c^2(c)) \cap B_{2n}^R \setminus B_j$. That provides an embedding of $F_c^\circ$ in $F_s$ for all $s$ in $D_c^2(c)$; and for such a $s$ with $s \neq c$, $F_s$ is a generic fiber. The morphism induced in homology
by this embedding is denoted by \( \ell_c \). Let \( j_c^o \) be the morphism induced by the inclusion of \( F_c^o \) in \( T_c = f^{-1}(D_\epsilon^2(c)) \). Similarly \( k_c \) denotes the morphism induced by the inclusion of the generic fiber \( F_{gen} = F_s \) (for \( s \in D_\epsilon^2(c), s \neq c \)) in \( T_c \). As all morphisms are induced by natural maps we have the lemma:

**Lemma 7.** The following diagram commutes:

\[
\begin{array}{ccc}
H_q(F_c^o) & \xrightarrow{j_c^o} & H_q(T_c) \\
\downarrow{\ell_c} & & \downarrow{k_c} \\
H_q(F_{gen}) & & \\
\end{array}
\]

### 2.2 Invariant cycles by \( h_c \)

Invariant cycles by the monodromy \( h_c \) can be recovered by the following property.

**Proposition 8.**

\[ \text{Ker} \left( h_c - \text{id} \right) = \ell_c(H_q(F_c^o)) \]

For \( n = 2 \), there is a similar formula in [MW], even for non-isolated singularities.

**Proof.** The proof uses a commutative diagram due to W. Neumann and P. Norbury [NN]:

\[
\begin{array}{ccc}
H_q(F_{gen}, F_c^o) & \xrightarrow{\sim} & V_q(c) \\
\downarrow{\varphi} & & \downarrow{i} \\
H_q(F_{gen}) & \xrightarrow{id - h_c} & H_q(F_{gen}) \\
\end{array}
\]

The morphism \( i \) is the inclusion and \( \psi \) is an isomorphism, so \( \text{Ker}(h_c - \text{id}) \) equals \( \text{Ker} \varphi \).

The long exact sequence for the pair \((F_{gen}, F_c^o)\) is:

\[
\cdots \rightarrow H_q(F_c^o) \xrightarrow{\ell_c} H_q(F_{gen}) \xrightarrow{\varphi} H_q(F_{gen}, F_c^o) \rightarrow \cdots
\]

So \( \text{Im} \ell_c = \text{Ker} \varphi = \text{Ker}(h_c - \text{id}) \).

We are able to apply this result to the calculus of the rank of \( \text{Ker}(h_c - \text{id}) \) for \( n = 2 \). Let denote the number of irreducible components in \( F_c \) by \( r(F_c) \), and let \( \text{Sing}_c \) be \( \text{Sing} \cap F_c \): the affine singularities on \( F_c \). Then \( H_2(F_{gen}, F_c^o) \) has rank the cardinal of \( \text{Sing}_c \) which is also the rank of \( \text{Ker} \ell_c \). Moreover \( \text{rk} H_1(F_c^o) = r(F_c) - \chi(F_c) + \# \text{Sing}_c \).

\[
\text{rk Ker} \left( h_c - \text{id} \right) = \text{rk Im} \ell_c \\
= \text{rk} H_1(F_c^o) - \text{rk Ker} \ell_c \\
= r(F_c) - \chi(F_c) + \# \text{Sing}_c - \# \text{Sing}_c \\
= r(F_c) - \chi(F_c).
\]

**Remark.** We obtain the following fact (see [MW]): if the fiber \( F_c \) \((c \in \mathcal{B})\) is irreducible then \( h_c \neq \text{id} \). The proof is as follows: if \( r(F_c) = 1 \) and \( h_c = \text{id} \) then from one hand \( \text{rk Ker}(h_c - \text{id}) = \text{rk} H_1(F_{gen}) = 1 - \chi(F_{gen}) \) and from the other hand \( \text{rk Ker}(h_c - \text{id}) = 1 - \chi(F_c) \); thus \( \chi(F_c) = \chi(F_{gen}) \) which is absurd for \( c \in \mathcal{B} \) by Suzuki formula.
2.3 Vanishing cycles

Now and until the end of this paper homology is homology with complex coefficients.

Vanishing cycles for eigenvalues \( \lambda \neq 1 \). Let \( E_\lambda \) be the space \( E_\lambda = \text{Ker}(h_c - \lambda \text{id})^p \) for a large integer \( p \).

**Lemma 9.** If \( \lambda \neq 1 \) then \( E_\lambda \subset V_q(c) \).

**Proof.** If \( \sigma \in H_q(F_{\text{gen}}) \) then \( h_c(\sigma) - \sigma \in V_q(c) \). This is just the fact that the cycle \( h_c(\sigma) - \sigma \) corresponds to the boundary of a “tube” defined by the action of the geometrical monodromy. We remark this fact can be generalized for \( j \geq 1 \) to \( h_c^j(\sigma) - \sigma \in V_q(c) \).

Let \( p \) be an integer that defines \( E_\lambda \), then for \( \sigma \in E_\lambda \):

\[
0 = (h_c - \lambda \text{id})^p(\sigma) = \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} h_c^j(\sigma)
\]

\[
= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} (h_c^j(\sigma) - \sigma) + \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} \sigma
\]

\[
= \sum_{j=0}^{p} \binom{p}{j} (-\lambda)^{p-j} (h_c^j(\sigma) - \sigma) + (1 - \lambda)^p \sigma.
\]

Each \( h_c^j(\sigma) - \sigma \) is in \( V_q(c) \), and a sum of such elements is also in \( V_q(c) \), then \( (1 - \lambda)^p \sigma \in V_q(c) \).

As \( \lambda \neq 1 \), then \( \sigma \in V_q(c) \).

Vanishing cycles for the eigenvalue \( \lambda = 1 \) We study what happens for cycles associated to the eigenvalue 1. Let recall that vanishing cycles \( V_q(c) = \text{Ker} k_c \) for the value \( c \), are cycles that “disappear” when the generic fiber tends to the fiber \( F_c \). Hence cycles that will not vanish are cycles that already exist in \( F_c \). From the former paragraph these cycles are associated to the eigenvalue 1.

Let \( (\tau_1, \ldots, \tau_p) \) be a family of \( H_q(F_{\text{gen}}) \) such that the matrix of \( h_c \) in this family is:

\[
\begin{pmatrix}
1 & 1 & (0) \\
1 & 1 \\
& 1 & \ddots \\
(0) & \ddots & 1 \\
& 1
\end{pmatrix}.
\]

Then, the cycles \( \tau_1, \ldots, \tau_{p-1} \) are vanishing cycles. It is a simple consequence of the fact that \( h_c(\sigma) - \sigma \in V_q(c) \), because for \( i = 1, \ldots, p - 1 \), we have \( h(\tau_{i+1}) - \tau_{i+1} = \tau_i \), and then \( \tau_i \) is a vanishing cycle. It remains the study of the cycle \( \tau_p \) and the particular case of Jordan blocks \((1)\) of size \( 1 \times 1 \). We will start with the second part.
Vanishing and invariant cycles. Let $K_q(c)$ be invariant and vanishing cycles for the value $c$: $K_q(c) = \text{Ker}(h_c - \text{id}) \cap V_q(c)$. Let us remark that the space $K_q(c) \oplus \bigoplus_{c' \neq c} V_q(c')$ is not equal to $\text{Ker}(h_c - \text{id})$. But equality holds in cohomology.

Lemma 10. $K_q(c) = \ell_c(\text{Ker} j_c^\circ)$.

This lemma just follows from the description of invariant cycles (proposition 8) and from the diagram of lemma [9]. For $n = 2$ we can calculate the dimension of $K_1(c)$.

Proposition 11. For $n = 2$, $\text{rk} K_1(c) = r(F_c) - 1 + \text{rk} H_1(\bar{G}_c)$.

Proof. The proof will be clear after the following remarks:

1. $K_1(c) = \ell_c(\text{Ker} j_c^\circ)$, by lemma [1].

2. $j_c^\circ = j_c \circ i_c$ with $i_c : H_1(F_c^0) \rightarrow H_1(F_c)$ the morphism induced by inclusion. It is consequence of the commutative diagram:

$$
\begin{array}{ccc}
H_1(F_c) & \xrightarrow{i_c} & H_1(F_c^0) \\
\downarrow{j_c} & & \downarrow{j_c^\circ} \\
H_1(T_c) & & H_1(T_c)
\end{array}
$$

3. $\text{rk} \text{Ker} j_c^\circ = \text{rk} \text{Ker} i_c + \text{rk} \text{Ker} j_c \cap \text{Im} i_c$, which is general formula for the kernel of the composition of morphisms.

4. $\text{Ker} j_c \cap \text{Im} i_c = \text{Ker} j_c$, because cycles of $H_1(F_c)$ that do not belong to $\text{Im} i_c$ are cycles corresponding to $H_1(G_c)$, so they already exist in $F_c$ and are not vanishing cycles.

5. $\text{rk} \text{Ker} i_c = \sum_{z \in \text{Sing}_c} r(F_{c,z})$, where $F_{c,z}$ denotes the germ of the curve $F_c$ at $z$.

6. $\text{rk} \text{Ker} j_c = \text{rk} \text{Ker} j_{\infty} = \sum_{P \in \text{Dic}_c} (n(F_P) - 1) = n(F_c) + \text{rk} H_1(\bar{G}_c) - \text{rk}(G_c)$, it has been proved in lemma [3].

7. $r(F_c) + \text{rk} H_1(G_c) = n(F_c) + \sum_{z \in \text{Sing}_c} (r(F_{c,z}) - 1)$. This a general formula for the graph $G_c$, the number of vertices of $G_c$ is $r(F_c)$, the number of connected components is $n(F_c)$, the number of loops is $\text{rk} H_1(G_c)$ and the number of edges for a vertex that correspond to an irreducible component $F_{\text{irr}}$ of $F_c$ is: $\sum_{z \in F_{\text{irr}}} (r(F_{\text{irr},z}) - 1)$.

8. $\text{rk} K_1(c) = \text{rk} \text{Ker} j_c^\circ - \# \text{Sing}_c$ because $\text{Ker} i_c$ is a subspace of $\text{Ker} \ell_c$ so $\text{rk} K_1(c) = \text{rk} \text{Ker} j_c^\circ - \text{rk} \ell_c$ and the dimension of $\text{Ker} \ell_c$ is $\# \text{Sing}_c$ (see paragraph 2.2).
We complete the proof:

\begin{align*}
\text{rk } K_1(c) &= \text{rk } \ell_c(\text{Ker } j_c^2) \\
&= \text{rk } \text{Ker } j_c^2 - \text{rk } \text{Ker } \ell_c \\
&= \text{rk } \text{Ker } j_c \circ i_c - \# \text{Sing}_c \\
&= \text{rk } \text{Ker } i_c + \text{rk } \text{Ker } j_c \cap \text{Im } i_c - \# \text{Sing}_c \\
&= \text{rk } \text{Ker } i_c - \# \text{Sing}_c - \text{rk } \text{Ker } j_c \\
&= \sum_{z \in \text{Sing}_c} (r(F_c,z) - 1) + n(F_c) + \text{rk } H_1(\bar{G}_c) - \text{rk } (G_c) \\
&= r(F_c) - 1 + \text{rk } H_1(\bar{G}_c). \\
\end{align*}

\[\square\]

**Filtration.** Let \(\phi\) be the map provided by the total resolution of \(f\). The divisor \(\phi^{-1}(c)\) is denoted by \(\sum_i m_i D_i\) where \(m_i\) stands for the multiplicity of \(D_i\). We associate to \(D_i\) a part of the generic fiber denoted by \(F_i\) (see [MW]). The filtration of the homology of the generic fiber is the sequence of inclusions:

\[W_{-1} \subset W_0 \subset W_1 \subset W_2 = H_1(F_{\text{gen}}).\]

with

- \(W_{-1}\): the boundary cycles, that is to say, if \(\bar{F}_{\text{gen}}\) is the compactification of \(F_{\text{gen}}\) and \(i_* : H_1(F_{\text{gen}}) \to H_1(\bar{F}_{\text{gen}})\) is induced by inclusion then \(W_{-1} = \text{Ker } i_*\).
- \(W_0\): these are gluing cycles: the homology group on the components of \(F_i \cap F_j\) \((i \neq j\).
- \(W_1\): the direct sum of the \(H_1(F_i)\).
- \(W_2 = H_1(F_{\text{gen}})\).

The subspaces \(W_0\) and \(W_1\) depend on the value \(c\).

**Jordan blocks for \(n = 2\).** For polynomials in two variables, the size of Jordan block for the monodromy \(h_c\) is less or equal to 2. Let denote by \(\sigma\) and \(\tau\) cycles of \(H_1(F_{\text{gen}})\) such that \(h(\sigma) = \sigma\) and \(h(\tau) = \sigma + \tau\). The matrix of \(h_c\) for the family \((\sigma, \tau)\) is \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\). We already know that the cycle \(\sigma\) vanishes.

A large cycle is a cycle of \(W_2 = H_1(F_{\text{gen}})\) that has a non-trivial class in \(W_2/W_1\). According to [MW] \(\tau\) is large cycle; moreover large cycles associated to the eigenvalue 1 are the embedding of \(H_1(G_c)\) in \(H_1(F_{\text{gen}})\). So large cycles are not vanishing cycles. The number of classes of large cycles in \(W_2/W_1\) is \(\text{rk } H_1(\bar{G}_c)\), this is also the number of Jordan 2-blocks for the eigenvalue 1.
Vanishing cycles. We describe vanishing cycles. For all the spaces $W_{-1}$, $W_0/W_{-1}$, $W_1/W_0$ and $W_2/W_1$ the cycles associated to eigenvalues different from 1 are vanishing cycles.

**Proposition 12.** Vanishing cycles for the eigenvalue 1 are dispatch as follows:

- for $W_{-1}$: $r(F_c) - 1$ cycles,
- for $W_0$: $\text{rk} H_1(\bar{G}_c)$ other cycles,
- $W_1$, $W_2$: no other cycle.

Let explain this distribution. We have already remark that large cycles associated to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ are not vanishing cycles, so vanishing cycles in $W_2$ are in $W_1$. Moreover there is $\text{rk} H_1(G_c)$ Jordan 2-blocks for the eigenvalue 1 that provide $\text{rk} H_1(\bar{G}_c)$ vanishing cycles (like $\sigma$) in $W_0$. The other vanishing cycles for the eigenvalue 1 are invariant cycles by $h_c$, in other words they belong to $K_1(c)$. We have $W_1 \cap K_1(c) = W_0 \cap K_1(c)$ because invariant cycles for $W_1$ that are not in $W_0$ correspond to the genus of the smooth part $F_c^\circ$ of $F_c$ (this is due to the equality $\text{Ker}(h_c - \text{id}) = \ell_c(H_1(F_c^\circ)))$. As they already appear in $F_c$, theses cycles are not vanishing cycles for the value $c$. Finally, if we have two distinct cycles $\sigma$ and $\sigma'$ in $W_0 \cap K_1(c)$, with the same class in $W_0/W_{-1}$, then $\sigma' = \sigma + \pi$, $\pi \in W_{-1}$; this implies that $\pi = \sigma' - \sigma$ is a vanishing cycle of $K_1(c)$. We can choose the $r(F_c) - 1$ remaining cycles of $K_1(c)$ in $W_{-1}$.

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