On the construction of unitary quantum
group differential calculus*

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Abstract
We develop a construction of the unitary type anti-involution for the quantized
differential calculus over $GL_q(n)$ in the case $|q| = 1$. To this end, we consider
a joint associative algebra of quantized functions, differential forms and Lie
derivatives over $GL_q(n)/SL_q(n)$, which is bicovariant with respect to
$GL_q(n)/SL_q(n)$ coactions. We define a specific non-central spectral extension
of this algebra by the spectral variables of three matrices of the algebra gen-

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group differential calculus over $GL_q(n)$.

1. Introduction

Soon after the invention of quantum group theory [6], the construction of the differential
calculi over quantum spaces and quantum groups became a hot topic in the noncommutative
geometry, accumulating much investigation activity. The general frameworks for these
investigations were given by the bicovariance postulates [32] and the $R$-matrix ideology [9].
Substantial progress was soon achieved in constructing the algebras of differential operators
(Lie derivatives) and differential forms over the general linear quantum groups (see, e.g.
[18, 22, 23, 26, 27, 29–31, 33]; thorough reviews of these investigations can be found in [13]

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and [19]). At the same time, serious difficulties were met in all attempts to complement the calculus with more sophisticated structures. This concerns the studies of the quantum group’s real forms, where no unitary quantum groups were found for the most interesting regime of the quantization parameter $|q| = 1$. This happened to a construction of the exterior algebra for the differential forms over quantum orthogonal and symplectic groups, where the no-go theorem was proved (see [3]). This was also the case with the construction of the de Rahm complex over special linear quantum groups in the frame of Woronowiczc’s approach. So it transpires that in the quantum group calculus, introducing any additional structure meets serious troubles and demands case-by-case investigations.

In the present paper, we address the problem of the construction of the unitary anti-involution for the differential calculus over linear quantum groups. One crucial hint for its solution was given already in [2] where the anti-involution map was looked for and found not in the quantum group, but in the larger Heisenberg double (HD) algebra. Another important ingredient of our construction is the spectral extension of the calculus algebra which was elaborated in [17] on the basis of the Cayley–Hamilton theorem for the quantum matrix algebras (see [11, 14, 15]).

The paper is organized as follows. In the next section we describe the differential calculi algebras over general and special linear quantum groups. We introduce algebras of quantized functions, differential forms and Lie derivatives over $GL_q(n)$ and consider their $SL_q(n)$ reduction and their bicovariant structure. We mainly follow the papers [16, 26], though we consider several different sets of generators and discuss the $SL_q(n)$ reduction conditions in detail.

In section 3, we briefly recall the Cayley–Hamilton theorem for the reflection equation (RE) algebras of the $GL_q(n)$ type. These are the algebras of the left- and right-invariant Lie derivatives. Extending the ideas of [17], we construct a special non-central extension of the differential calculus algebra with spectral variables—the eigenvalues of three matrices generating the RE subalgebras in the differential calculus (two of them are the matrices of the left- and right-invariant Lie derivatives). This is one of the two main results of this paper, which we present in theorem 3.2. Then, in the spectrally extended algebra, we introduce a three-parametric family of inner automorphisms. For certain integer values of their parameters these automorphisms reproduce the discrete time evolution of the q-deformed top [2].

Section 4 describes the construction of the Gauss decomposition for the RE algebras. Starting from this section, we restrict our consideration to the algebras associated with the Drinfeld–Jimbo R-matrices (see (2.1) below). All the previous constructions were carried out for a more general family of R-matrices of the $SL(n)$ type. Explanation of this notion and a collection of the R-matrix formulas are given in the appendix.

Section 5 is devoted to the construction of the unitary anti-involution in the (spectrally extended) differential calculus over $GL_q(n)$. This is the second major result of this work and we present it in theorem 5.5. Restriction of the anti-involution to the differential calculus over $SL_q(n)$ is also discussed.

We finally note that, although the exterior derivative is not considered in this paper, its construction suggested in [7] looks appropriate for the calculi we discuss. We leave a detailed consideration of the exterior derivative, its compatibility with the unitary structure and its possible BRST realization for a future publication.
In this section we describe associative unital algebras of differential calculi over \( GL_q(n) \) and \( SL_q(n) \)—quantizations of the classical calculi over \( GL(n)/SL(n) \). The quantum calculi algebras are generated by the components of four \( n \times n \) matrices:

- \( \| T^i_j \|_{i=1}^n \) — coordinate functions,
- \( \| \Omega^i_j \|_{i=1}^n \) — right-invariant 1-forms,
- \( \| L^i_j \|_{i,j=1}^n \) — right-invariant Lie derivatives,
- \( \| K^i_j \|_{i,j=1}^n \) — left-invariant Lie derivatives.

In the \( GL_q(n) \) case, one imposes on the generators a set of quadratic relations which fix classical values\(^1\) for the dimensions of the spaces of homogeneous polynomials in generators. These relations, in general, allow alphabetic ordering of the generators and therefore, we call them permutation relations. Transition to the \( SL_q(n) \) calculus is then achieved by imposing one more polynomial relation for each matrix of generators. We call these relations reduction conditions.

Both the permutation relations and the reduction conditions are given with the use of the \( R \)-matrix—an \( n^2 \times n^2 \) matrix \( R \in \text{Aut}(\mathbb{C}^n \otimes \mathbb{C}^n) \) satisfying Artin’s braid relation. We specify \( R \) to be of the \( SL(n) \) type which means, in particular, that its minimal polynomial is quadratic. All the necessary notions and the basics of the \( R \)-matrix techniques are recalled in the appendix. For a more detailed presentation of the \( R \)-matrix formalism, the reader is referred to paper \([17]\) and references therein. In what follows we use notation adopted there.

Our main motivating example is the calculus associated with the standard Drinfeld–Jimbo \( R \)-matrix

\[
R = \sum_{i,j=1}^n q^{k_i} (E_{ij} \otimes E_{ji}) + (q - q^{-1}) \sum_{i < j} E_{ii} \otimes E_{jj},
\]

(2.1)

Here \( q \in \mathbb{C} \setminus \{0\} \), and \((E_{ij})_{kl} = \delta_{ik} \delta_{jl}, \ i, j = 1, ..., n, \) are matrix units. As we will show, the quantum differential calculus associated with the matrix (2.1) admits unitary type specialization. This is the major result of our work. However, we stress that consistent \( SL_q(n) \) and \( GL_q(n) \) differential calculi can be associated with any \( SL(n) \) type \( R \)-matrix. Among those are multiparametric generalizations of the Drinfeld–Jimbo \( R \)-matrix \([25]\) (see also example 2.10 in \([17]\)) and the Cremmer–Gervais \( R \)-matrices \([5, 12]\). Therefore, we present a part of the construction in full generality and stick to considering the particular Drinfeld–Jimbo’s case starting from section 4.

We now proceed to writing explicit formulas.

### 2.1. Quantization of functions

Throughout this and the next sections we assume that \( R \in \text{Aut}(\mathbb{C}^n \otimes \mathbb{C}^n) \) is the \( SL(n) \) type \( R \)-matrix. This notion, together with the compressed matrix index notation and the notions of \( R \)-trace, \( \text{Tr}_R \), and of \( q \)-antisymmetrizers, \( A^{(n)} \), which appear in the formulas below, are explained in the appendix.

\(^1\) E.g. \( \binom{x^2 + x - 1}{x} \) and, respectively, \( \binom{x^2}{x} \) are dimensions of the spaces of \( k \)th order homogeneous polynomials of coordinate functions and, respectively, right-invariant 1-forms.
Permutation relations for the quantized coordinate functions over $GL_q(n)$ in the compressed matrix index notation read [6, 9]

$$ R T_1 T_2 = T_1 T_2 R. $$

(2.2)

Strictly speaking, one also assumes the invertibility of the quantum determinant of the matrix $T$

$$ \det_R T := \text{Tr}_{(1, \ldots, n)}(A^{(n)} T_1 T_2 \ldots T_n). $$

(2.3)

The $SL(n)$ type property of the $R$-matrix $R$ guarantees the centrality of $\det_R T$ and the $SL_q(n)$ reduction condition reads

$$ \det_R T = 1. $$

(2.4)

Further on we denote an associative unital algebra generated by elements $T_{ij}$ subject to relations (2.2), (2.4) as $\mathcal{F}[R]$ and call it the algebra of functions over $SL_q(n)$. It is also briefly called the RTT algebra.

In a standard way, $\mathcal{F}[R]$ is endowed with the Hopf algebra structure [6, 9] from which we recall formulas for the coproduct and for the antipode:

$$ \Delta(T_{ij}) = \sum_{k=1}^{n} T_{ik} \otimes T_{kj}, $$

(2.5)

$$ (T^{-1})_{ij} = q^{n(n-1)/2} \text{Tr}_{n \times n} (T_2 \ldots T_n A^{(n)}) (\det_R T)^{-1}, $$

(2.6)

where $n_q := (q^n - q^{-n})/(q - q^{-1})$ is the $q$-number. Using the symbol $T^{-1}$ for the antipode instead of the standard notation $S(T)$ is justified by equalities

$$ \sum_{k=1}^{n} T_{ik} (T^{-1})_{kj} = \sum_{k=1}^{n} (T^{-1})_{ik} T_{kj} = \delta_{ij} 1. $$

(2.7)

### 2.2. Quantization of forms

Permutation relations for the quantized external algebra of the right-invariant differential forms over $GL_q(n)$ were first suggested in [26, 33]

$$ R \Omega^i R \Omega^j = - \Omega^i R \Omega^j R^{-1}. $$

(2.8)

This algebra implies the following permutation relations for the R-trace of $\Omega^g$

$$ (\text{Tr}_R \Omega^g) \Omega^g + \Omega^g (\text{Tr}_R \Omega^g) = -(q - q^{-1}) (\Omega^g)^2 $$

thus making a naive $SL_q(n)$ reduction $\text{Tr}_R \Omega^g = 0$ impossible. Instead, one observes that the R-traceless matrix

$$ \Omega := \Omega^g - \frac{q^n}{n_q} (\text{Tr}_R \Omega^g) I $$

(2.9)

generates a subalgebra in (2.8) which does not contain element $\text{Tr}_R \Omega^g$:

$$ R \Omega_1 R \Omega_1 + \Omega_1 R \Omega_1 R^{-1} = \kappa_q (\Omega_1^2 + R \Omega_1^2 R). $$

(2.10)
Here we denote\(^2\)
\[
\kappa_q \equiv \frac{q^n(q - q^{-1})}{n_q + q^n(q - q^{-1})},
\]
assuming additionally \(n_q + q^n(q - q^{-1}) \neq 0.\) (2.11)

Permutation relations (2.2), (2.10) complemented with
\[
R \Omega_1 R^{-1} T_1 = T_2 \Omega_2
\]
define the algebra which is consistent with the \(SL_q(n)\) type reduction conditions for functions (2.4) and for 1-forms
\[
\text{Tr}_g \Omega = 0.
\]
(2.13)
We call it the external algebra of differential forms over \(SL_q(n)\) [16]. Analogously, relations (2.2), (2.8) complemented with (2.12), where \(\Omega\) is to be substituted by \(\Omega^g\), define the external algebra of differential forms over \(GL_q(n)\). As we will see, all relations containing matrices of 1-forms \(\Omega^\sigma/\Omega\) in the differential calculi over \(GL_q(n)/SL_q(n)\) are identical, only with the exception of their own permutations (2.8) and (2.10). Later on we will always write down relations for \(\Omega\) understanding that their analogues for \(\Omega^g\) have an identical form.

Equivalently, the external algebra can be written in terms of the left-invariant 1-forms
\[
\Theta^g_j = (T^{-1}(\Omega^g T))_j, \quad \Theta^g_j = (T^{-1}(\Omega T))_j,
\]
(2.14)
Deriving the permutation relations and the reduction condition for \(\Theta\) is a good exercise in the \(R\)-matrix techniques. They read:
\[
R^{-1} \Theta^g_2 R \Theta^g_1 = - \Theta^g_2 R \Theta^g_1 R,
\]
(2.15)
\[
R^{-1} \Theta_2 R \Theta_2 + \Theta_2 R \Theta_2 R = \kappa_q (\Theta_2^2 + R \Theta_2^2 R),
\]
(2.16)
\[
\text{Tr}_{g \rho} \Theta = 0,
\]
(2.17)
\[
\Theta_1 T_2 = T_2 R^{-1} \Theta_2 R, \quad \text{(for both, } \Theta \text{ and } \Theta^g).\]
(2.18)
Here \(R_{\rho} \equiv PRP\), and \(P \in \text{Aut}(\mathbb{C}^n \otimes \mathbb{C}^n)\) is the permutation operator: \(P(u \otimes v) = v \otimes u\).

Bi-invariant objects in the external algebra are given by the \(R\)-traces of powers of \(\Omega\). Their subalgebra is not affected by the quantization [8] and look like
\[
\text{Tr}_g \Omega^{2i} = \text{Tr}_{g \rho} \Theta^{2i} = 0 \quad \forall i \geq 1,
\]
(2.19)
\[
\omega_i \equiv \text{Tr}_g \Omega^{2i+1} = \text{Tr}_{g \rho} \Theta^{2i+1} \quad \forall i = 1, \ldots, n - 1,
\]
(2.20)
\[
\omega_i \omega_j = - \omega_j \omega_i.
\]
(2.21)

2.3. Quantization of Lie derivatives

The algebra of quantized right-invariant Lie derivatives (see equation (2.22) below) is widely known under the name of the RE algebra. It was introduced in the context of a factorized particle scattering on a half-line [4, 20] and has since found a number of applications in the theory of integrable systems in noncommutative geometry and in quantum groups.

\(^2\) Notice that in addition to the common for quantum groups condition that the quantization parameter \(q\) is not a root of unity (see equation (A.5) in the appendix) quantization of the differential calculus over the group assumes further restrictions on the possible values of \(q\) (see [8]). The condition appearing in formula (2.11) is one of these restrictions.
The common algebra of quantized functions and invariant Lie derivatives given by equation (2.2) and by equations (2.22), (2.23) below also has its own name—the HD algebra [1, 27, 28]. The action of Lie derivatives on the functions is identical to the action of the corresponding vector fields and so the HD algebra can be considered as the algebra of differential operators over a quantum group.

The whole set of relations for the right-invariant Lie derivatives was written for the first time in [26, 33]:

\[ R L_1 R L_1 = L_1 R L_1 R, \]  
(2.22)  
\[ R L_1 R T_1 = \gamma^2 T_1 L_2, \]  
(2.23)  
\[ R L_1 R \Omega_1 = \Omega_1 R L_1 R. \]  
(2.24)

For the SL_q(n) reduction we introduce the quantum determinant of L as

\[ \det_R L := \text{Tr}_R R^{a_1 a_2 \ldots a_n} (L^{a_n}) L_1 \ldots L_n, \]  
(2.25)

where concise notation

\[ L_1 := L_1, \quad L_i := R_{i-1} L_{i-1} R_{i-1}^{-1} \quad \forall i > 1. \]  
(2.26)

for the matrix L i-th copy is used. Further on, we assume invertibility and demand the centrality of \( \det_R L \)—the latter condition fixes the value of the parameter \( \gamma \) in (2.23)

\[ \gamma = q^{1/n}. \]  
(2.27)

The SL_q(n) reduction condition then reads:

\[ \det_R L = q^{-1} 1. \]  
(2.28)

A particular convenience of the normalization factor \( q^{-1} \) chosen here will become obvious later on (see section 3.3, the definition of the automorphism \( \phi_L \)).

Remark 2.1. In the algebra of Lie derivatives over GL_q(n) the element \( \det_L L \) cannot be central. Usually this is achieved by the choice \( \gamma = 1 \). Alternatively, one can keep \( \gamma = q^{2/n} \) extending the algebra with one more invertible generator \( \ell \), satisfying permutation relations

\[ \ell L = L \ell, \quad \ell T = q^{2/n} T \ell. \leqno{2.29} \]

Later on, we will not consider separately the algebra of the Lie derivatives over GL_q(n) having this possibility in mind.

Let us now turn to the discussion of the left-invariant Lie derivatives. Unlike vector fields, the left- and right-invariant Lie derivatives are independent and have to be introduced in the calculus separately. Notice, however, that keeping the mirror (left-right) symmetry of the calculus, one can uniquely reproduce permutation relations for the left-invariant objects (K, \( \Theta \), etc.) reading relations for right-invariant generators literally leftwards (note that the mirror image of \( R = R_{12} \) is \( R_{op} = R_{31} \)). In this way for the left-invariant Lie derivatives K we obtain

\[ R K_2 R K_3 = K_2 R K_3 R, \]  
(2.29)  
\[ T_2 R K_2 R = \gamma^2 K_1 T_2, \]  
(2.30)

Note that the difference in the definitions of the quantum determinants for the matrices \( T \) and \( L \) comes from the difference in their permutation relations.
These relations are to be complemented by the natural commutativity conditions of the left/right-invariant Lie derivatives with all the right/left-invariant objects:
\begin{align}
K_1 X_2 = X_2 K_1 & \quad \forall X = \Omega, L, \quad (2.32) \\
L_1 Y_2 = Y_2 L_1 & \quad \forall Y = \Theta, K. \quad (2.33)
\end{align}

Here the only new relation is the commutativity condition for \( L \) and \( K \). All the other commutativity conditions follow from the permutation relations imposed earlier.

The \( SL_q(n) \) reduction for the left-invariant Lie derivatives \( K \) reads
\[
\det_R K := \text{Tr}_{R^{1 \ldots n}} (A^{(n)} K_2 \ldots K_1) = q^{-1} 1, \quad (2.34)
\]
where \( i \)-th copy of the matrix \( K \) is defined as
\[
K_{ij} := K_n, \quad K_i := R_i K_{i+1} R_i^{-1} \quad \forall \ 1 \leq i \leq n. \quad (2.35)
\]

Here again, the formulae for \( \det_q K \) are obtained as mirror copies of those for \( \det_q L \).

### 2.4. Yet another RE algebra

Ideologically, permutation relations for the left- and right-invariant Lie derivatives given in the previous section describe an effect of, respectively, right and left shifts on the underlying quantum group ‘manifold’. It would be instructive to study distinctions in their actions on \((i.e. \) permutations with \( ) \) the other objects of the calculus. To this end, one may compare two objects obeying the same invariance properties: \( L \) and \( T K T^{-1} \). It turns out that they have identical permutation relations with functions, but not with forms. Let us analyze quantitatively this phenomenon looking at the permutation relations of their ratio
\[
F_{ij} := (L T K T^{-1})_{ij}. \quad (2.36)
\]

**Proposition 2.2.** The components of matrix \( \|F_{ij}\|_{i,j=1}^n \) generate the RE subalgebra in the calculi algebra and satisfy the following permutation relation with the other generators
\begin{align}
R F_1 R F_1 &= F_1 R F_1 R, \quad (2.37) \\
R F_1 R^{-1} T_1 &= T_1 F_2, \quad (2.38) \\
R F_1 R \Omega_1 &= \Omega_1 R F_1 R, \quad (2.39) \\
R L_1 R F_1 &= F_1 R L_1 R, \quad (2.40) \\
F_1 K_2 &= K_2 F_1. \quad (2.41)
\end{align}

The \( SL_q(n) \) reduction (2.27), (2.28), (2.34) implies the following condition on \( F \)
\[
\det_q F := \text{Tr}_{R^{1 \ldots n}} (A^{(n)} F_1 \ldots F_n) = q^{-n^2} 1. \quad (2.42)
\]

**Proof.** Checking permutation relations (2.37)–(2.41) is straightforward and we skip it. We shall consider in detail a more sophisticated calculation of the quantum determinant of \( F \).

First of all, denoting \( U = T K T^{-1} \) we separate \( L \) and \( U \) factors in \( \det_q F \):
\[
\det_q F = \text{Tr}_{R^{1 \ldots n}} (A^{(n)} (L_1 \ldots L_n)(U_1 \ldots U_n)).
\]
Here, since matrix $U$ satisfies a version of the RE with the inverse matrix $R$ in it $R^{-1}U_{ij}R^{-1}U_{ij} = U_{ij}R^{-1}UR^{-1}$, the definition of its overlined copy differs from that for $L$: $U_{ij} = R_{ij}^{-1}U_{ij}R_{ij}^{-1}$ (see equation (2.26)). Using condition $\text{rk}A^{(n)} = 1$ (see the appendix) we simplify the expression

$$\det_R F = \det_R L \cdot \text{Tr}_{(1, \ldots, n)}(A^{(n)} U_l \ldots U_l).$$

Next, using permutation relations for $T$ and $K$, we separate different matrix factors in the product of $U$-copies:

$$U_1 \ldots U_n = \gamma^{-2n(n-1)} \left( \prod_{i=1}^{n} J_i \right)^2 (T_1 \ldots T_n)(K_2 \ldots K_1)(T_n^{-1} \ldots T_1^{-1}),$$

where $\gamma = q^{1/n}$ (see equation (2.27)). Here $J_i$ are the $R$-matrix realizations of a commutative set of Jucys–Murphy elements in the braid group (see, e.g. [24]):

$$J_1 = I, \quad J_{i+1} = R_i J_i R_i, \quad \forall i \geq 1. \quad (2.43)$$

Evaluating $\prod_{i=1}^{n} J_i$ at $A^{(n)}$ as $q^{-n(n-1)}$ and exploiting again the rank $= 1$ property of $A^{(n)}$ we obtain

$$\det_R F = (\gamma q)^{-2n(n-1)} \det_R L \cdot \det_R T \cdot \text{Tr}_{(1, \ldots, n)}(A^{(n)} K_2 \ldots K_1) \cdot (\det_R T)^{-1}.$$

Finally, using equation (A.12) we substitute $\text{Tr}$ in this formula by $\text{Tr}_{K_n}$ and get

$$\det_R F = \gamma^{-2n(n-1)} q^{-n(n-2)} \det_R L \cdot \det_R T \cdot \det_R K \cdot (\det_R T)^{-1} = q^{-n^2} q^{2n} \det_R L \cdot \det_R K = q^{-n^2}.$$

Here in the last line we use relation

$$T \cdot \det_R K = q^{-2} \gamma^{2n} \det_R K \cdot T$$

to cancel $\det_R T$ and then, apply the $SL_q(n)$ reduction conditions.

2.5. Bicovariance

Up to now we have described certain unital associative algebra generated by the components of four matrices $T, \Omega, L$ and $K$. The possibility to endow it with a structure of the bicovariant bimodule over the Hopf algebra $\mathcal{F}[R]$ makes this algebra the differential calculus over the quantum group. This is a remarkable fact. In this section, we complement the construction of the differential calculus describing the $\mathcal{F}[R]$ comodule structures over it.

The left and right $\mathcal{F}[R]$ coactions—$\delta_l$ and $\delta_r$—on the algebra generators are defined as follows:

- on $T_{ij}$ they just reproduce the coproduct (2.5)

$$\delta_{l/r}(T_{ij}) = \Delta(T_{ij}); \quad (2.44)$$

- on the matrices of right-invariant generators, such as $\Omega$, $L$, or $F$, the coactions are given by

\[\text{For the definition of the bicovariant bimodule see, e.g. [19].}\]
where \( X = \Omega, L, F, \ldots \);

- on the matrices of left-invariant generators, such as \( \Theta \), or \( K \), they are defined as

\[
\delta_r(Y_{ij}) = 1 \otimes Y_{ij}, \quad \delta_r(Y_{ij}) = \sum_{k,p=1}^n (1 \otimes (T^{-1})_{kj})(Y_{kp} \otimes 1)(1 \otimes T_{pj}),
\]

where \( Y = \Theta, K, \ldots \).

The use of the terminology 'left/right-invariant' now becomes evident.

Notice that the co-transformation properties of the generators are preserved under matrix multiplication (e.g. \( \sum_k L_{kj} L_{ij} \) and \( \sum_k \Theta_{kj} K_{ij} \) are right- and left-invariant, respectively), whereas conjugation with \( T \) interchanges left and right co-transformations (e.g. \( \sum_{k,p} T_{kp} K_{ij} T_{pj} \) and \( \sum_{k,p} T_{kp}^{-1} L_{kp} T_{pj} \) transform as right- and left-invariant objects, respectively). The operation of taking the R-trace extracts bi-invariant objects, so that for \( \omega_i \) from equation (2.20) one has

\[
\delta_r(\omega_i) = 1 \otimes \omega_i, \quad \delta_r(\omega_i) = \omega_i \otimes 1.
\]

### 2.6. Summary

We collect considerations of this section into:

**Definition 2.3.** To any \( SL(n) \)-type \( R \)-matrix \( R \) there is a corresponding associative unital algebra \( D\mathbb{C}_R\mathbb{L}_G \) of the differential calculus over \( GL_q(n) \). This algebra is generated by the components of four matrices \( T, \Omega^g, L \) and \( K \), subject to the permutation relations

\[
(2.2), (2.8), (2.12), (2.22) - (2.24), (2.29), (2.30), (2.32),
\]

Substituting in this definition generators \( \Omega^g \mapsto \Omega \) and relations (2.8) \mapsto (2.10) and adding the \( SL_q(n) \) reduction conditions

\[
(2.4), (2.13), (2.11), (2.27), (2.28), (2.34).
\]

one obtains definition of the algebra \( D\mathbb{C}_R\mathbb{L}_G \) of the differential calculus over \( SL_q(n) \). The bicovariant \( \mathcal{F}[R] \)-bimodule structure on both algebras is given by equations (2.44)–(2.46).

**Remark 2.4.** As a generating set for \( D\mathbb{C}_R\mathbb{L}_G \) one can also choose quadruples of matrices \( \{ T, \Theta^g/\Theta, L, K \} \) and \( \{ T, \Omega^g/\Omega, L, F \} \). The permutation relations and the reduction conditions for these sets were presented earlier in this section.

**Remark 2.5.** The HD algebra over \( SL_q(n) \) investigated in [17] is a quotient algebra of \( D\mathbb{C}_R\mathbb{L}_G \) over relations

\[
F_{ij} = \delta_{ij} 1, \quad \Omega_{ij} = 0.
\]

The first relation imposes dependence of the left- and right-invariant vector fields in the HD.
3. Spectral extension and automorphisms

In this section we introduce three families of automorphisms on algebras $\mathfrak{D}(\mathfrak{gl}_q|R)$. Two of these automorphisms are generated by the actions of Lie derivatives $L, K$ and, as explained in [2], they reproduce a q-deformed version of an evolution of the Euler’s isotropic top. The third automorphism is related to matrix functions invariant. In section 5 we use these automorphisms for the construction of the unitary anti-involution over $\mathfrak{D}(\mathfrak{sl}_q|R)$, and so we have to define them as the algebra inner automorphisms. To this end, we define a spectral extension of the algebra—its extension by the eigenvalues of matrices $L, K$ and $F$. For the HD algebra the spectral extension was constructed in [17]. Here we present the generalization of that construction to the algebras $\mathfrak{D}(\mathfrak{gl}_q|R)$.

3.1. Characteristic identities and spectral variables

In this subsection, we collect structure results about the RE algebras of the $\text{SL}(n)$ type [11, 14, 15]. These results are necessary for the subsequent constructions.

Consider the RE algebra (2.22), (2.28) generated by the matrix of right-invariant Lie derivatives $L$. A set of elements $a_i, i = 0, \ldots, n$,

$$ a_0 := 1, \quad a_i := \text{Tr}_{K^{i-\cdots}}(A^0 L_1 \ldots L_i), \quad i \geq 1. \quad (3.1) $$

belongs to the center of the RE algebra; the last of them—$a_n$—is just the quantum determinant of $L$. These elements are the coefficients of the following matrix identity

$$ \sum_{i=0}^{n} (-q)^i a_i L^{n-i} = 0, \quad (3.2) $$

which is nothing but the RE algebra analogue of the Cayley–Hamilton theorem. We will introduce a special central extension of the RE algebra with the aim of bringing the characteristic identity (3.2) to a factorized form.

Consider an Abelian $\mathbb{C}$-algebra of polynomials in $n$ invertible indeterminates $\{\mu_{\alpha}^{\pm 1}\}_{\alpha=1}^{n}$ and in their differences $\{(\mu_{\alpha} - \mu_{\beta})^{\pm 1}: 1 \leq \beta < \alpha \leq n\}$, satisfying condition

$$ \prod_{\alpha=1}^{n} \mu_{\alpha} = q^{-1}. $$

We parameterize elements $a_i$ of the RE algebra by the elementary symmetric polynomials in $\mu_{\alpha}$;

$$ a_i = e_i(\mu_1, \ldots, \mu_n) := \sum_{1 \leq \alpha_1 < \cdots < \alpha_i \leq n} \mu_{\alpha_1} \mu_{\alpha_2} \cdots \mu_{\alpha_i}, \quad \forall i = 0, 1, \ldots, n, \quad (3.3) $$

assuming commutativity of indeterminates $\mu_{\alpha}$ with the elements of the RE algebra

$$ L \mu_{\alpha} = \mu_{\alpha} L. \quad (3.4) $$

The resulting central extension of the RE algebra is called its spectral extension, and the elements $\mu_{\alpha}$ are called eigenvalues of the ‘quantum’ matrix $L$. The characteristic identity in the completed RE algebra assumes a factorized form

$$ \prod_{\alpha=1}^{n} (L - q \mu_{\alpha} L) = 0. \quad (3.5) $$
It can be used for the construction of a set of mutually orthogonal matrix idempotents

\[ P_\alpha = \prod_{\beta=1}^{n} \frac{(L - q\mu_\beta I)}{q(\mu_\alpha - \mu_\beta)} : \quad p_\alpha p_\beta = \delta_{\alpha\beta} p_\alpha, \quad \sum_{\alpha=1}^{n} p_\alpha = I. \] (3.6)

By construction, evaluating \( L \) on the idempotents one obtains the eigenvalues

\[ L P_\alpha = p_\alpha L = q\mu_\alpha P_\alpha. \] (3.7)

Now we apply a similar spectral extension procedure for the RE algebras generated by matrices \( K \) and \( F \): see, respectively, equations (2.29), (2.34) and (2.37), (2.42). We parameterize coefficients \( b_i \) and \( c_i \) of their characteristic identities by elementary symmetric functions in \( n \) indeterminates \( \nu_\alpha \) and \( \rho_\alpha \), \( \alpha = 1, \ldots, n \):

\[ b_0 = 1, \quad b_1 = \text{Tr}_K \prod_{i=0}^{n} (K^{i} K_{1} \ldots K_{i}) = e_{i}(\nu_{1}, \ldots, \nu_{n}), \] (3.8)

\[ c_0 = 1, \quad c_1 = \text{Tr}_K \prod_{i=0}^{n} (K^{i} F_{1} \ldots F_{i}) = e_{i}(\rho_{1}, \ldots, \rho_{n}), \] (3.9)

where

\[ \prod_{\alpha=1}^{n} \nu_\alpha = q^{-1}, \quad \prod_{\alpha=1}^{n} \rho_\alpha = q^{-n}. \]

Assuming centrality of the eigenvalues

\[ \nu_\alpha K = K \nu_\alpha, \quad \rho_\alpha F = F \rho_\alpha, \] (3.10)

we factorize the characteristic identities

\[ \sum_{i=0}^{n} (-q)^{i} b_i K^{n-i} = \prod_{\alpha=1}^{n} (K - q\nu_\alpha I) = 0, \] (3.11)

\[ \sum_{i=0}^{n} (-q)^{i} c_i F^{n-i} = \prod_{\alpha=1}^{n} (F - q\rho_\alpha I) = 0. \] (3.12)

Imposing additional invertibility conditions on the eigenvalues and on their differences, we define associated sets of matrix idempotents

\[ Q_\alpha = \prod_{\beta=1}^{n} \frac{(K - q\mu_\beta I)}{q(\nu_\alpha - \nu_\beta)} : \quad q^{\alpha} q^{\beta} = \delta_{\alpha\beta} q^{\alpha}, \quad \sum_{\alpha=1}^{n} q^{\alpha} = I, \] (3.13)

\[ S_\alpha = \prod_{\beta=1}^{n} \frac{(F - q\rho_\beta I)}{q(\rho_\alpha - \rho_\beta)} : \quad s^{\alpha} s^{\beta} = \delta_{\alpha\beta} s^{\alpha}, \quad \sum_{\alpha=1}^{n} s^{\alpha} = I, \] (3.14)

so that

\[ K Q_\alpha = q^{\alpha} K = q^{\nu_\alpha} Q_\alpha, \] (3.15)

\[ F S_\alpha = q^{\alpha} F = q^{\rho_\alpha} S_\alpha. \] (3.16)
Finally, we can consistently set that the newly introduced spectral variables transform trivially under both left and right $\mathcal{F}[R]$ coactions

$$\delta_i \xi = \xi \otimes 1, \quad \delta_\xi = 1 \otimes \xi, \quad \forall \xi \in \{\mu_0, \nu_0, \rho_0\}. \quad (3.17)$$

### 3.2. Spectral extension

Our next step is to construct an extension of algebras $\mathfrak{gl}/\mathfrak{sl}[R]$ with the spectral vaiables. The result is definitely not a trivial central extension. Our goal is to define permutation relations for $m, n$ and $r$ in such a way that their elementary symmetric functions would commute exactly as the elements $a_i, b_i$ and $c_i$ do. In [17], a consistent definition for permutations of $\mu_0$ with $T$ was derived. Here we follow the same scheme. First, we calculate permutation relations of $a_i, b_i$ and $c_i$ with $T$ and $\Omega$.

**Proposition 3.1.** In algebras $\mathfrak{gl}/\mathfrak{sl}[R]$ elements $a_i$ (3.1) satisfy permutation relations

$$\gamma^{2i} T a_i = a_i T - (q^2 - 1) \sum_{j=1}^{i} (-q)^{-j} a_{i-j} (LT), \quad (3.18)$$

$$[\Omega, a_i] = (q^2 - 1) \sum_{j=1}^{i} (-q)^{-j} [\Omega, a_{i-j}]. \quad (3.19)$$

Here notation $[\cdot, \cdot]$ stands for the commutator. Relations for elements $b_i$ (3.8) are mirror images of those for $a_i$ with the substitution $a_i \mapsto b_i, L \mapsto K, \Omega \mapsto \Theta$. Namely,

$$\gamma^{2i} b_i T = T b_i - (q^2 - 1) \sum_{j=1}^{i} (-q)^{-j} (TK) b_{i-j}, \quad (3.20)$$

$$[\Theta, b_i] = (q^2 - 1) \sum_{j=1}^{i} (-q)^{-j} [\Theta, b_{i-j}]. \quad (3.21)$$

Permutation relations of $c_i$ (3.9) with $\Omega$ are identical to (3.19) with the substitution $a_i \mapsto c_i, L \mapsto F$, while with $T$ elements $c_i$ commute.

**Proof.** Equation (3.18) is proved in [17] in proposition 3.18. The proof of equation (3.19) is based on equality

$$q^{((i-1))} \text{Tr}_{\mathfrak{gl}/\mathfrak{sl}[R]}^{\mathfrak{sl}^{(i+1)}} (L_i J_i) \ldots (L_1 J_1) A^{(i)} = a_i I_1 - (q^2 - 1) \sum_{j=1}^{i} (-q)^{-j} a_{i-j} L_1^j, \quad (3.22)$$

where $X^1 = I \otimes X \in \text{End}(V^{(i+1)}) \forall X \in \text{End}(V^{(i)})$, and the Jucys–Murphy elements $J_i$ were defined in (2.43).

Equality (3.22), in turn, follows from the Cayley–Hamilton–Newton identity (see theorem 3.11 in [17]) and it is contained implicitly in the proof of proposition 3.18 in [17].

Now, as a consequence of permutation relations (2.24), one has

$$(L_j J_j) \Omega_1 = \Omega_1 (L_j J_j), \quad \forall j \geq 2,$$

and, hence the lhs of (3.22) commutes with $\Omega_1$. So does the rhs, which immediately leads to the equality (3.19).
Permutation relations for \( b_i \) (3.20), (3.21) follow from (3.18), (3.19) and the left-right symmetry of the calculus.

The identical form of the commutators of \( a_i \) and \( c_i \) with \( \Omega \) is a consequence of the identity of the permutation relations for \( L \) and \( F \) with \( \Omega \). Checking relation \([c_i, T] = 0\) is straightforward.

**Theorem-Definition 3.2.** Consider an Abelian \( \mathbb{C} \)-algebra of polynomials in \( 3n \) invertible indeterminates and in their differences

\[
\{ \mu_{\alpha}^{\pm 1}, \nu_{\alpha}^{\pm 1}, (\mu_{\alpha} - \mu_{\beta})^{\pm 1}, (\nu_{\alpha} - \nu_{\beta})^{\pm 1}, (\rho_{\alpha} - \rho_{\beta})^{\pm 1} : 1 \leq \beta < \alpha \leq n \},
\]

subject to relations

\[
\prod_{\alpha=1}^{n} \mu_{\alpha} = \prod_{\alpha=1}^{n} \nu_{\alpha} = q^{-1}, \quad \prod_{\alpha=1}^{n} \rho_{\alpha} = q^{-n^2}.
\] (3.23)

Spectral extensions \( \mathfrak{E}_{g/\mathfrak{gl}[R]} \) of the \( GL_q(n)/SL_q(n) \) differential calculi \( \mathfrak{E}_{g/\mathfrak{gl}1[R]} \) by this algebra are given by parameterization formulas (3.3), (3.8), (3.9) and by permutation relations

\[
\mu_{\alpha} X = X \mu_{\alpha}, \quad \forall X = L, K, \Theta, G : T^{-1}FT,
\] (3.24)

\[
\nu_{\alpha} Y = Y \nu_{\alpha}, \quad \forall Y = L, K, \Omega, F,
\] (3.25)

\[
\rho_{\alpha} Z = Z \rho_{\alpha}, \quad \forall Z = T, L, K, F, G,
\] (3.26)

\[
\gamma^2 (P^\beta T) \mu_{\alpha} = q^{2h_{\beta\alpha}} \mu_{\alpha} (P^\beta T),
\] (3.27)

\[
\gamma^2 \nu_{\alpha} (TQ^\beta) = q^{2h_{\beta\alpha}} (TQ^\beta) \nu_{\alpha}, \quad \text{(recall that } \gamma = q^{1/n}),
\] (3.28)

\[
q^{2h_{\beta\alpha}} (P^\beta X P^\sigma) \mu_{\alpha} = q^{2h_{\beta\alpha}} \mu_{\alpha} (P^\beta X P^\sigma), \quad \forall X = \Omega, F,
\] (3.29)

\[
q^{2h_{\beta\alpha}} (Q^\beta Y Q^\sigma) \nu_{\alpha} = q^{2h_{\beta\alpha}} \nu_{\alpha} (Q^\beta Y Q^\sigma), \quad \forall Y = \Theta, G,
\] (3.30)

\[
q^{2h_{\beta\alpha}} (S^\beta \Omega S^\sigma) \rho_{\alpha} = q^{2h_{\beta\alpha}} \rho_{\alpha} (S^\beta \Omega S^\sigma), \quad \forall \alpha, \beta, \sigma = 1, \ldots, n.
\] (3.31)

\([\rho_{\alpha}, \text{Tr}_\alpha X] = 0, \text{where } X \text{ is any matrix monomial in } \Omega \text{ and } F.\]

Here expressions for matrix idempotents \( P^\alpha, Q^\alpha, S^\alpha \) are given in (3.6), (3.13), (3.14).

Formulæ (3.27)–(3.31) can be equivalently written as

\[
\gamma^2 T \mu_{\alpha} = \mu_{\alpha} T + (q - q^{-1}) (LP^\alpha T),
\] (3.33)

\[
\gamma^2 \nu_{\alpha} T = T \nu_{\alpha} + (q - q^{-1}) (TKQ^\alpha),
\] (3.34)

\[
[X, \mu_{\alpha}] = (q - q^{-1}) [LP^\alpha, X] \quad \forall X = \Omega, F,
\] (3.35)

\[
[Y, \nu_{\alpha}] = (q - q^{-1}) [KQ^\alpha, Y] \quad \forall Y = \Theta, G,
\] (3.36)

\[
[\Omega, \rho_{\alpha}] = (q - q^{-1}) [FS^\alpha, \Omega] \quad \forall \alpha = 1, \ldots, n.
\] (3.37)

The extended algebras \( \mathfrak{E}_{g/\mathfrak{gl}[R]} \) are endowed with the structure of the \( \mathcal{F}[R] \) bicovariant bimodule by equations (2.44)–(2.46), (3.17).
Proof. As concerns formulae (3.27)–(3.31), one has to check that they are consistent with the parameterization formulas and the relations obtained in proposition 3.1. Relations (3.18) were checked for consistency in [17], theorem 3.27. Here we shall prove the consistency of the spectral extension with relations (3.19). The rest of the relations follow by similar considerations.

Let us calculate the permutation of the elementary symmetric function in spectral values $e_i(\mu)$ with the matrix of 1-forms $\Omega$ with the use of equation (3.29). In the calculations below we use the following notations

\[ e_i(\mu) := e_i(\mu)\big|_{\mu_a = 0}, \quad e_i(\mu^{\alpha\beta}) := e_i(\mu)\big|_{\mu_a = \mu_b = 0}. \]

\[
e_i(\mu) \Omega = \sum_{\alpha, \beta = 1}^n e_i(\mu^{\alpha\beta}) P^{\alpha\beta} \Omega P^{\beta} = \sum_{\alpha} P^{\alpha\beta} \Omega P^{\beta} e_i(\mu) + \sum_{\alpha \neq \beta} P^{\alpha\beta} \Omega P^{\beta} (e_i(\mu^{\beta\alpha}) - e_i(\mu^{\alpha\beta})) (q^{-2} \mu^{\alpha}_b + q^{2} \mu^{\beta}_b) + e_i(\mu^{\alpha\beta}) (\mu_a \mu_b)
\]

\[
= \Omega \ e_i(\mu) + (q^2 - 1) \sum_{\alpha = \beta} \left(P^{\alpha\beta} \Omega P^{\beta} e_i(\mu^{\beta\alpha}) - e_i(\mu^{\alpha\beta}) \mu_a \mu_b \right)
\]

\[
= \Omega \ e_i(\mu) + (q^2 - 1) \sum_{\alpha, \beta = 1}^n [\Omega, P^{\alpha\beta} e_i(\mu^{\beta\alpha}) \mu_a]
\]

\[
= \Omega \ e_i(\mu) + (q^2 - 1) \sum_{i = 1}^n [\Omega, e_i(\mu)(-L/q)^i].
\]

Here in the last line one uses formula $e_i(\mu^{\alpha\beta}) = \sum_{j = 0}^\infty e_i(\mu)(-\mu_a)^j$ and takes into account the fact that $\mu_a$ in the presence of $P^{\alpha\beta}$ can be substituted by $L/q$.

Comparing the first and the last lines of the calculation we see the consistency of the spectral extension with equation (3.19).

Finally, it is easy to see that equation (3.32) agrees with the algebraic relations in $\mathfrak{D}[R]$ observing that, by equations (2.37), (2.39), all components of matrix $F$ commute with elements $\text{Tr}_R X$ from (3.32).

Strictly speaking, relations (3.27)–(3.31) for spectral variables are not the permutations, since they are non-quadratic. Quite remarkably, they are consistent with the commutators (3.24)–(3.26). For instance, the permutation relations of $\mu_a$ with $T$ (3.33) and with $\Omega$ (3.35) lead to the trivial commutator for $\mu_a$ and $\Theta = T^{-1} \Omega T$.

Remark 3.3. Spectral variables $\mu_a, \nu_b$ satisfy a stronger version of equality (3.32); they commute with all bi-invariant elements of the calculus. In particular,

\[
[\xi, \text{Tr}_R X] = [\xi, \text{Tr}_R Y] = 0 \quad \forall \xi \in \{\mu_a, \nu_b\},
\]

where $X/Y$ could be any matrix monomial in $[\Omega, L, F]/[\Theta, K, G]$. This relation follows from the commutativity (3.24), (3.25) and the fact that the $R$-trace of any monomial in right-invariant matrices $L, \Omega, F$ can be reexpressed in terms of $R_{op}$-traces of left-invariant matrices $K, \Theta, G$, and vice-versa.

Remark 3.4. Equation (3.32) in the definitions of $\mathfrak{D}[\mathfrak{gl}(d)/[R]$ can be equally substituted by condition
where $X$ is any matrix monomial in $\Omega$ and $F$. Indeed:

$$ q^\alpha_{\beta} \text{Tr}_R (S^\alpha X S^\beta) = \text{Tr}_R (FS^\alpha X S^\beta) = \text{Tr}_R \left( R^{-1} S^\alpha X S^\beta R F \right) = \text{Tr}_R (S^\alpha X S^\beta) R \rho^\beta_{\beta}, $$

wherefrom (3.39) follows, if one commutes $\rho^\beta_{\beta}$ to the left and uses the invertibility of $(\rho^\beta_{\beta} = \rho^\beta_{\beta})$. The opposite implication follows from the presentation $\text{Tr}_R X = \sum_{\beta} \text{Tr}_R (S^\beta X S^\beta)$ and the permutation relations (3.31).

### 3.3. Automorphisms

In [2] an important discrete sequence of the HD algebra automorphisms was introduced. This sequence, generated by the right-invariant Lie derivatives, was interpreted there as a discrete time evolution of the q-deformed Euler top, and so, a problem of the construction of its evolution operator was posed. The problem was further addressed in [17], where it was shown that a solution can be found after the spectral extension of the initial algebra. Moreover, in an extended algebra, one has a continuous one-parametric family of the automorphisms—a continuous time evolution.

In the differential calculi algebras $\mathcal{D}C_{gl/dl}[R]$ one can define three independent series of such type automorphisms.

**Proposition 3.5.** Mappings $\phi_L$, $\phi_K$ and $\phi_F$, defined on generators as

$$ \phi_L : T \mapsto LT, \quad \Omega \mapsto L\Omega L^{-1}, \quad F \mapsto LFL^{-1}, \quad X \mapsto X \quad \forall X = L, K, \Theta; \quad (3.40) $$

$$ \phi_K : T \mapsto TK, \quad \Theta \mapsto K^{-1}\Theta K, \quad Y \mapsto Y \quad \forall Y = L, K, \Omega, F; \quad (3.41) $$

$$ \phi_F : T \mapsto T, \quad \Omega \mapsto F\Omega F^{-1}, \quad Z \mapsto Z \quad \forall Z = L, K, F. \quad (3.42) $$

generate the algebra automorphisms of the differential calculi $\mathcal{D}C_{gl/dl}[R]$. These automorphisms are mutually commutative.

**Proof.** Checking the compliance of the maps with the permutation relations in $\mathcal{D}C_{gl/dl}[R]$ and their mutual commutativity is straightforward. In proposition 4.1 [17] mappings $\phi_L$ and $\phi_K$ have been proven to comply with the $SL_q(n)$ reduction conditions on the Lie derivatives. It lasts to test the transformations of the reduction condition for the differential forms. We consider a calculation for $\phi_F(\text{Tr}_R \Omega)$:

$$ \phi_F(\text{Tr}_R \Omega) = \text{Tr}_R (F\Omega F^{-1}) = \text{Tr}_R \left( R^{-1} R F \right) = \text{Tr}_R \left( R^{-1} \Omega F^{-1} \right) = \text{Tr}_R \left( \Omega F^{-1} F \right) = \text{Tr}_R \Omega = 0, $$

where the underlined expression in the first line is moved to the right with the use of the permutation relations for $F$. 

In the spectrally completed algebras $\mathcal{D}C_{gl/dl}[R]$ these mappings can be generalized to a three-parametric family of inner algebra automorphisms. Strictly speaking, to this end one has to further extend the calculus, passing from the spectral generators $\{\mu_s, \nu_s, \rho_s\}$ to a new set of variables $\{x_s, y_s, z_s\}$. Normalizations (2.28), (2.34) of the matrices $L$ and $K$ were chosen in such a way that the transformation rules for $T$ here do not contain nontrivial coefficients. 

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5 Normalizations (2.28), (2.34) of the matrices $L$ and $K$ were chosen in such a way that the transformation rules for $T$ here do not contain nontrivial coefficients.
\[ \mu_n = q^{-1/n} \exp(2\pi i \nu_n), \quad \nu_n = q^{1/n} \exp(2\pi i \rho_n), \quad \rho_n = q^{-n} \exp(2\pi i \sigma_n), \] (3.43)

and considering formal power series in \( x_\alpha, y_\alpha, z_\alpha. \) In terms of these new variables the \( SL_q(n) \) reduction conditions for the spectral generators (3.23) read

\[ \sum_{\alpha = 1}^n x_\alpha = \sum_{\alpha = 1}^n y_\alpha = \sum_{\alpha = 1}^n z_\alpha = 0, \] (3.44)

and the permutation relations (3.27)–(3.30) take an additive form. For instance, permutations of \( x_\alpha \) with the matrices \( T \) and \( \Omega \) read

\[ (P^T) x_\alpha = (x_\alpha + 2\tau \delta_{\alpha \beta} - n^{-1})(P^T), \] (3.45)

\[ (P^2\Omega P^r) x_\alpha = (x_\alpha + 2\tau \delta_{\alpha \beta} - 2\tau \delta_{\alpha \beta})(P^2\Omega P^r), \] (3.46)

where we denote

\[ \tau = \frac{1}{2\pi i} \log q. \] (3.47)

The main result of this section is as follows.

**Theorem 3.6.** Consider three-parametric family \( \phi_{(t_1, t_2, t_3)} \) of \( \hat{D}C_{gl}([R]) \) inner automorphisms

\[ \phi_{(t_1, t_2, t_3)} : u \mapsto \varphi_{(t_1, t_2, t_3)} u (\varphi_{(t_1, t_2, t_3)})^{-1}, \quad \forall u \in \hat{D}C([R]), \]

\[ \varphi_{(t_1, t_2, t_3)} := \exp \left\{ -\frac{1}{2\tau} \sum_{\alpha = 1}^n \left( t_1 x_\alpha^2 - t_2 y_\alpha^2 + t_3 z_\alpha^2 \right) \right\}. \] (3.48)

Automorphisms \( \phi_L, \phi_K \) and \( \phi_F \) are elements of this family:

\[ \phi_L = \phi_{(1,0,0)}, \quad \phi_K = \phi_{(0,1,0)}, \quad \phi_F = \phi_{(0,0,1)}. \] (3.49)

**Proof.** For the proof one uses decompositions of matrix units (3.6), (3.13), (3.14) and permutation formulas like ones in (3.45), (3.46). An idea of the proof was suggested in [17], section 4.

In view of remark 3.3 and equation (3.32) one has:

**Proposition 3.7.** Bi-invariant elements of the calculi are invariant under two-parametric family of automorphisms \( \phi_{(t_1, t_2, 0)} \) \( \hat{D}C \) traces of matrix monomials in matrices \( \Omega \) and \( F \) are invariant under the whole family of automorphisms \( \phi_{(t_1, t_2, t_3)} \).

4. Gauss decomposition

We need one more structure to construct the unitary calculus, namely, the Gauss decomposition for the Lie derivatives. To our knowledge, such a decomposition is only known for the RE algebras associated with the Drinfeld–Jimbo \( R \)-matrix. So, from now on we consider the calculus associated with the \( R \)-matrix (2.1).

Following [9] we introduce two pairs of the RTT algebras generated by the upper/lower triangular matrices \( L^{+/\rightarrow} = \|l^{(\pm)}_{ij}\|_{i,j=1}^{n} \) and \( K^{+/\rightarrow} = \|k^{(\pm)}_{ij}\|_{i,j=1}^{n} \) subject to the permutation relations.
\[ RL_2^{(\pm)} L_1^{(\pm)} = L_2^{(\pm)} L_1^{(\pm)} R, \quad RK_2^{(\pm)} K_1^{(\pm)} = K_2^{(\pm)} K_1^{(\pm)} R, \]

and to the \( SL_q(n) \) reduction conditions\(^6\)

\[ \det_R L^{(\pm)} = \prod_{i=1}^{n} \ell_{ii}^{(\pm)} = 1, \quad \det_R K^{(\pm)} = \prod_{i=1}^{n} k_{ii}^{(\pm)} = 1, \]

and

\[ \ell_{ii}^{(-)} \ell_{ii}^{(+)}, k_{ii}^{(-)} k_{ii}^{(+)}, \quad \forall \ i = 1, 2, \ldots, n. \]

As is well known, the RE algebra can be realized in terms of these upper/lower triangular RTT algebras (see, e.g. [19], pp 345–7). So we do for the Lie derivatives \( L \) and \( K \):

\[ L = q^{n-1/n} (L^{(-)})^{-1} L^{(+)}; \quad K = q^{n-1/n} K^{(+)}, (K^{(-)})^{-1}. \]

Note that normalization factor \( q^{n-1/n} \) in these formulas is necessary for the compatibility of the \( SL_q(n) \) reductions (2.28), (2.34) and (4.3). Indeed, one can calculate

\[ \det_R L = q^{-1} (\det_R L^{(-)})^{-1} \det_R L^{(+)}, \]

and the same for matrix \( K \).

An extension of the Gauss decomposition to the spectral variables is obviously central. Less trivial is the extension for the algebras \( \widetilde{\mathcal{C}}_{g/l/d}[R] \). It was elaborated in [2, 27, 28]. Below we present a list of permutation relations for the matrices \( L^{(\pm)} \) and \( K^{(\pm)} \) derived in these papers

\[ L_1^{(\pm)} R^{\pm} T_1 = \gamma^{\pm} T_2 \ P \ L_2^{(\pm)}, \]

\[ T_2 R^{\pm} K_2^{(\pm)} = \gamma^{\pm} K_1^{(\pm)} \ P \ T_1, \]

Recall that \( \gamma = q^{1/n} \) and \( P \in \text{Aut}(V^\otimes 2) \) is the permutation matrix.

\[ [L_1^{(\pm)}, K_2^{(\mp)}] = [L_1^{(\pm)}, K_2^{(\mp)}] = 0, \]

\[ [\xi, L^{(\pm)}] = [\xi, K^{(\pm)}] = 0, \quad \forall \xi \in \{\mu, \nu, \rho\}, \]

\[ L_1^{(\pm)} R^{\pm} X_1 = X_2 \ L_1^{(\pm)} R^{\pm}, \quad K_1^{(\pm)} X_2 = X_2 \ K_1^{(\pm)} \quad \forall X = L, \Omega, F, \]

\[ Y_2 R^{\pm} X_2^{(\pm)} = X_2 Y_2 \ L_1^{(\pm)}, \quad L_1^{(\pm)} Y_2 = Y_2 L_1^{(\pm)} \quad \forall Y = K, \Theta. \]

One can check that these relations are (i) consistent with the previously defined permutation relations for \( L \) and \( K \), and (ii) respect reduction conditions (4.3), (4.4).

### 5. Unitary anti-involution

Now we are ready to construct a unitary anti-involution on \( \widetilde{\mathcal{C}}_{g/l/d}[R] \).

We fix the value of the quantization parameter \( q \) on a unit circle: \( q = e^{2\pi i \tau}, \tau \in \mathbb{R} \). In this case the Hermite conjugate of the Drinfeld–Jimbo \( R \)-matrix is

\[ R^\dagger = PR^{-1}P. \]

\(^6\) Here the quantum determinant for matrices \( L^{(\pm)}, K^{(\pm)} \) is defined by formula (2.3), which is universal for the RTT algebras.
As a starting point we take the Hermite conjugation of the triangular components of the Lie derivatives adopted in [2]:

\[
(L^{(\pm)})^\dagger = (L^{(\mp)})^{-1}, \quad (K^{(\pm)})^\dagger = (K^{(\mp)})^{-1},
\]

whereby ‘\dagger’ we understand the composition of the anti-linear algebra anti-involution and the matrix transposition. It is easy to check that this setting is compatible with permutation relations and reduction conditions (4.1)–(4.4) for \(L^{(\pm)}\) and \(K^{(\pm)}\).

**Remark 5.1.** The RTT algebras generated by matrices \(L^{(\pm)}\) and \(K^{(\pm)}\) can be endowed with the standard Hopf structure (2.5)–(2.7). However, the \(\dagger\) structure (5.2) does not make them exactly the Hopf \(^*\)-algebras. Instead, the compatibility condition for the coproduct and the Hermite conjugation reads

\[
(\Delta X)^{\dagger} = \sigma \circ (\Delta X^\dagger), \quad \forall X = L^{(\pm)}, K^{(\pm)},
\]

where \(\sigma\) is the transposition map (see, e.g. [21], p.101).

**5.1. Conjugation of spectral variables \(\mu_\alpha, \nu_\alpha\)**

In this subsection we calculate an effect of the Hermitean conjugation on the RE algebras of Lie derivatives and on their spectra.

For the matrices of generators their Hermite conjugates look like

\[
L^\dagger = L^{(\pm)}L^{-1}(L^{(\pm)})^{-1}, \quad K^\dagger = (K^{(\pm)})^{-1}K^{-1}K^{(\pm)}.
\]

Consider a set of bi-invariant elements, called *power sums*

\[
p_k := \text{Tr}_R L^k, \quad p_{-k} := \text{Tr}_R (L^{-k}) = q^{2n} \text{Tr}_R L^{-k} \quad (\text{see (A.11)}).
\]

They are related to the coefficients of characteristic polynomial (3.2) by a \(q\)-version of Newton relations [11]

\[
p_k - qa_k p_{-k-1} - \cdots + (-q)^{k-1}a_{k-1}p_1 + (-1)^k q_k a_k = 0, \quad \forall k \geq 1.
\]

These formulas are helpful for the following.

**Proposition 5.2.** On elements \(p_k, a_k, b_k\) conjugation gives

\[
p_k^\dagger = p_{-k}, \quad a_k^\dagger = a_{-k}/a_n, \quad b_k^\dagger = b_{-k}/b_n,
\]

**Proof.** Formulas for the power sums are obtained by a direct calculation:

\[
p_k^\dagger = q^{2n} \text{Tr}_R (L^{(\pm)})^k = q^{2n} \text{Tr}_R a_n \text{Tr}_R L^{(\pm)} L^{(\pm)} RL^{(\pm)}^{-1} (L^{(\pm)})^{-1} = q^{2n} \text{Tr}_R L^{-k} = p_{-k}.
\]

Here we used permutation relations and formulas (A.10) from the appendix. For clarity we underlined expressions which are transformed in the next step.

To get conjugation formulas for \(a_k\) we use the characteristic identity (3.2). Multiplying it by \(L^{-k}\) and taking \(\text{Tr}_R^{-1} = q^{2n} \text{Tr}_R\) we obtain
\[ q^{2n}(p_k - q^{a_k}_1 p_{k-1} + \cdots + (-q)^{a_{n-k} - 1} a_{n-k} - 1 p_1) + (-q)^{a_{n-k}} a_{n-k} \text{Tr}_{K^\perp} I \\
\] 
\[
+ (-q)^{a_{n-k+1}} a_{n-k+1} p_1 + \cdots + (-q)^{a_k} a_k p_k = 0.
\]

Simplifying the first term in brackets with the help of (5.5) and using (A.10) for the second term we find after collecting similar terms

\[
p_k = \frac{1}{q^{a_k}} a_k p_{k+1} + \cdots + \left( -\frac{1}{q} \right)^{k-1} \frac{a_{n-k}+1}{a_k} p_{n-k+1} + \left( -1 \right)^k k q^{a_k} a_k = 0.
\]

Comparing this formula with the result of the conjugation of (5.5) we conclude \[ a_k = a_{n-k}/a_n. \]

Formulas for \( b_k \) are obtained in the same way.

The proposition together with the parameterization formulae (3.3), (3.8) suggests:

**Corollary 5.3.** In the spectrally extended RE algebras conjugation rules (5.3) can be consistently complemented by

\[
\mu_\alpha^{(1)} = \mu_\alpha^{-1}, \quad \nu_\alpha^{(1)} = \nu_\alpha^{-1}.
\]

**Hermite conjugation of the corresponding matrix idempotents reads**

\[
(P^{\alpha})^\dagger = L^{(\pm)} P^{\alpha} (L^{(\pm)})^{-1}, \quad (Q^{\alpha})^\dagger = (K^{(\pm)})^{-1} Q^{\alpha} K^{(\pm)}.
\]

### 5.2. Conjugation ansatz for T, F and \( \rho_\alpha \)

The formula for the Hermite conjugation of \( T \) in the HD algebra was suggested in [2]. Generalizing it for the differential calculi algebras \( \widehat{D\mathcal{C}}_{sl}(R) \) we write down the following ansatz

\[
T^\dagger = q^{n-1/n}(K(-)^{-1}T^{\dagger})(L(-))^{-1}.
\]

Here we use shorthand notation \( \widetilde{X} \) for the image of \( X \) under some automorphism from the family (3.48). Its explicit form is to be specified later on. Our choice of numeric factor \( q^{n-1/n} \) will also be argued below.

The suggested \( T^\dagger \) has to satisfy the Hermite conjugates of permutation relations (2.2), (4.6), (4.7)

\[
R T_1^\dagger T_2^\dagger = T_2^\dagger T_1^\dagger R,
\]

\[
T_2^\dagger R^{(\pm)} L_2^{(\pm)} = \gamma^{(\pm)} L_1^{(\pm)} P T_1^\dagger,
\]

\[
K_1^{(\pm)} R^{(\pm)} T_1^\dagger = \gamma^{(\pm)} T_1^\dagger P K_2^{(\pm)}.
\]

It is a standard exercise in \( R \)-matrix calculations to verify these equalities. We only mention that while proving (5.10) one finds a remarkable relation

\[
T_1^\dagger T_2^\dagger = T_2^\dagger T_1^\dagger.
\]

It is also straightforward to test the compatibility of the ansatz with permutation relations (3.27), (3.28) of the spectrally extended algebras \( \widehat{D\mathcal{C}}_{sl}(R) \).

Less easy is checking the consistency of the ansatz with \( SL_q(n) \) reduction condition (2.4). It is suitable to consider the Hermite conjugate of its inverse
To calculate it we separate factors \( K^{(-)} \), \( T \) and \( L^{(-)} \) in the expression

\[
(T_n^{(-)})^{-1} \cdots (T_1^{(-)})^{-1} = \eta^{-n} (L_n^{(-)} T_n K_n^{(-)}) \cdots (L_1^{(-)} T_1 K_1^{(-)})
\]

where by \( \eta \) we denote the numeric factor in the ansatz: \( \eta = q^{n^{-1/2}} \). Substituting this expression in (5.14), evaluating Jucys–Murphy elements on \( A_n^{(n)} \) and using the rank \( = 1 \) property of \( A_n^{(n)} \) we obtain

\[
(\det_R T^{(-)})^\dagger = \eta^{-1} q^{-1} q^{n^{-1/2}} \det_R L^{(-)} \det_R T \det_R K^{(-)} = 1,
\]

where the last equality is satisfied due to \( SL_q(n) \) reduction conditions and due to our choice of normalization \( \eta \) in the ansatz.

Now we discuss the Hermite conjugation of matrix \( F \), postponing the investigation of the involutivity of the ansatz (5.9) to subsection 5.4.

Given formulae (5.3) for \( L^{\dagger} \) and \( K^{\dagger} \) and the ansatz for \( T^{\dagger} \) one can calculate \( F^{\dagger} \)

\[
F^{\dagger} = L^{(-)} F^{-1} (L^{(-)})^{-1},
\]

This expression is quite similar to those for \( L^{\dagger} \), \( K^{\dagger} \) and hence, considerations of section 5.1 can be repeated with little modifications for the eigenvalues of \( F \). We collect the results in:

**Proposition 5.4.** Formula (5.15) for the Hermite conjugation of matrix \( F \) determines the conjugation rules for the coefficients of its characteristic polynomial

\[
c_k^{\dagger} = c_{n-k}/c_n.
\]

These rules, in turn, agree with the unitary conjugation prescriptions for \( F \)’s eigenvalues

\[
\rho_k^{\dagger} = \rho_{n-k}^{-1},
\]

which result in the following Hermite conjugation for their corresponding matrix idempotents

\[
(S^{(\alpha)})^{\dagger} = L^{(-)} S^{\alpha} (L^{(-)})^{-1}.
\]

**Proof.** The only point which needs to be commented on here is the invariance of the r.h.s. of (5.16) under the automorphism from the ansatz. This fact follows by proposition 3.7.

5.3. Conjugation ansatz for differential forms

In this subsection we introduce an ansatz for Hermite conjugation in the algebra of differential forms. From now on things start to be different in cases \( GL_q(n) \) and \( SL_q(n) \). We show briefly the consistency of the ansatz with the algebra structure of \( \mathfrak{D} C[R] \) and consider the possibility of the \( SL_q(n) \) reduction of the conjugation.

For the Hermite conjugate of the matrix \( \Omega^{\delta} \) we write down the following ansatz

\[
(\Omega^{\delta})^{\dagger} = -L^{(-)} F^{-1} \Omega^{\delta} (L^{(-)})^{-1}.
\]

Here by \( \sim \) we denote an action of the same automorphism as in (5.9).

It is not hard to verify that the matrix elements of \( (\Omega^{\delta})^{\dagger} \) indeed satisfy the conjugated permutation relations (4.10), (2.12), (2.8):
Here, as an intermediate step in proving equation (5.21), one obtains a remarkable commutativity relation
\[
\Omega_1\Omega_2 = \Omega_2\Omega_1.
\]
Permutation relations for spectral variables \(\mu_a, \nu_a, \rho_a\) (3.29)–(3.32) are also compatible with the ansatz.

Consider now the action of the conjugation on the subalgebra generated by the \(R\)-traceless forms (2.9). In view of (A.14), under conjugation they go into the \(R_{op}\)-traceless matrices
\[
\Omega^f = (\Omega^g)^f - \frac{q^n}{n_q} \text{Tr}_{R_q}(\Omega^g) I,
\]
which satisfy permutation relations
\[
R^{-1}(\Omega_1^f) R^{-1}(\Omega_2^f) = \Omega_1^f R^{-1}(\Omega_2^f) + \kappa_{1/q} (\Omega_1^g)^2 + R^{-1}(\Omega_2^g)^2 R^{-1}.
\]
and hence, generate closed subalgebra in the external algebra (2.8), as well as \(\Omega\) did. However, the subalgebra generated by \(\Omega^f\) goes beyond the \(SL_q(n)\) differential calculus described earlier. Indeed, with the use of \(R\)-techniques one can express \(R_{op}\)-traces of the conjugated 1-forms in terms of the \(R\)-traces of non-conjugated ones
\[
\text{Tr}_{R_q}(\Omega^g) = -\text{Tr}(F^{-1}\Omega^g).
\]
Namely an appearance of the matrix factor \(F^{-1}\) in this formula shows clearly that the conjugation map \(\dagger\) does not preserve the \(SL_q(n)\) differential calculus. So we are left in a situation where two different mutually conjugate \(SL_q(n)\) calculi subalgebras lie inside the \(GL_q(n)\) calculus algebra.

### 5.4. Involutivity

In this subsection we fix uniquely the automorphism \(\varphi\) in the ansatz (5.9), (5.19) demanding involutivity of the conjugation \(\dagger\). We then summarize considerations of the present section in a theorem.

Here we expand notation
\[
\tilde{X} = \varphi X \varphi^{-1},
\]
where \(\varphi\) is one of automorphism’s generating elements (3.48). We assume \(\varphi^f = \varphi^{-1}\) keeping in mind the unitarity of the spectral variables. We now calculate \(T^{\dagger}\) and \(\Omega^{\dagger}\):
\[
(T^f)^{\dagger} = q^{1/n-n} (L^{++})^{-1} (\varphi^{-1})^f (T^{-1})^f \varphi^f (K^{++})^{-1}
\]
\[
= q^{1/n-n} (L^{++})^{-1} \varphi (q^{1/n-n} L^{-}) \varphi T \varphi^{-1} (K^{-})^{-1} (K^{++})^{-1}
\]
\[
= L^{-1} \varphi^2 T \varphi^{-2} K^{-1};
\]
\[
(\Omega^g)^{\dagger} = (L^{++}) (\varphi^{-1})^f (\Omega^g)^{f} (F^{-1})^f \varphi^f L^{++}
\]
\[
= (L^{++})^{-1} \varphi (L^{-}) \varphi F^{-1} \Omega \varphi^{-1} (L^{-})^{-1} (L^{-}) \varphi F \varphi^{-1} (L^{-})^{-1} \varphi^{-1} L^{++}
\]
\[
= L^{-1} F^{-1} \varphi^2 \Omega \varphi^{-2} F L.
\]
So we conclude that conditions $T^{\dagger\dagger} = T$ and $(\Omega^\dagger)^\dagger = \Omega^\dagger$ are satisfied with the choice

$$\varphi^2 = \varphi_{(1,1,1)}$$

that is

$$\varphi = \exp\left(-\frac{i\pi}{4\tau} \sum_{\alpha=1}^n (x^2_{\alpha} - y^2_{\alpha} + z^2_{\alpha})\right)$$

(5.28)

Now we are ready to formulate the final:

**Theorem 5.5.** For the Drinfeld–Jimbo R-matrix (2.1) consider spectrally extended algebra $\widehat{\mathfrak{D}}C_{sl}[R]$ of the differential calculus over $GL_q(n)$ taking parameter $q$ on a unit circle: $q = e^{2\pi i\tau}$, $\tau \in \mathbb{R}$.

The anti-linear algebra anti-homomorphism given on the generators by formulas

(5.2), (5.7), (5.9), (5.15), (5.17), (5.19), (5.27), (5.28)

defines unitary type anti-involution on $\widehat{\mathfrak{D}}C_{sl}[R]$. This unitary structure respects the bicovariance property of the calculus in a sense that the algebra $\widehat{\mathfrak{D}}C_{sl}[R]$ can be endowed with the structure of the bicovariant bimodule over Hopf algebra $F^\dagger[R]$ generated by the matrix components of $T^\dagger$. The left and right $F^\dagger[R]$ coactions $\delta^\dagger_{l/r}$ are defined on the generators as

$$\delta^\dagger_{l/r}(T^\dagger_{ij}) = \sum_{k=1}^n T^\dagger_{ik} \otimes T^\dagger_{kj},$$

(5.29)

$$\delta^\dagger_{l}(X^\dagger_{ij}) = \sum_{k,p=1}^n (1 \otimes (T^{-1})^\dagger_{kp})(X^\dagger_{kp} \otimes 1)(1 \otimes T^\dagger_{pj}), \quad \delta^\dagger_{l}(X^\dagger_{ij}) = X^\dagger_{ij} \otimes 1,$$

(5.30)

$$\delta^\dagger_{r}(Y^\dagger_{ij}) = \sum_{k,p=1}^n (T^\dagger_{ik} \otimes 1)(1 \otimes Y^\dagger_{kp}((T^{-1})^\dagger_{kp} \otimes 1), \quad \delta^\dagger_{r}(Y^\dagger_{ij}) = 1 \otimes Y^\dagger_{ij},$$

(5.31)

where $X^\dagger = (\Omega^\dagger, L^\dagger, F^\dagger, \ldots)$, $Y^\dagger = (\Theta^\dagger, K^\dagger, \ldots)$. Naturally, one can consider $F^\dagger[R]$ coactions $\delta_{l/r}$ as Hermite conjugates of the $F[R]$ coactions $\delta_{l/r}$, respectively.

Restriction of conjugation $\dagger$ to the subalgebra $\widehat{\mathfrak{D}}C_{sl}[R]$ results in the involutive anti-homomorphism of the two $SL_q(n)$ type subalgebras, generated by the $R/R_{op}$-traceless matrices of 1-forms $\Omega$ (2.9) and $\Omega^\dagger$ (5.24), respectively.

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**Appendix. R-matrices**

Throughout the paper we consider various matrices acting on tensor powers of some finite dimensional vector space $V$. For these matrices we use the, by now standard, compressed matrix notation. Namely, with any matrix $X \in \text{End}(V^\otimes k)$ we associate a series of matrices $X_i \in \text{End}(V^\otimes i)$, $n \geq k$,

$$X_i = I^\otimes(n-i) \otimes X \otimes I^\otimes(n-k+i-1), \quad i = 1, 2, \ldots, n-k+1,$$

(A.1)
where \( I \in \text{Aut}(V) \) is the identity. For \( X \in \text{End}(V^{\otimes 2}) \) in certain occasions we also use notation \( X_{ij} \) for the matrices acting nontrivially in spaces with labels \( i \) and \( j \), \( i < j \). In these notation \( X_{ii} \equiv X_{i,i+1} \).

An operator \( R \in \text{Aut}(V^{\otimes 2}) \) satisfying braid relation

\[
R_1 R_2 R_1 = R_2 R_1 R_2, \quad (A.2)
\]
is called an \( R \)-matrix. The permutation \( P(u \otimes v) = v \otimes u \), is an (example of an) \( R \)-matrix. If \( R \) is an \( R \)-matrix then so are the operators \( R^{-1} \) and \( R_{op} \equiv PRP \).

An \( R \)-matrix \( R \) is called skew invertible if there exists an operator \( \Psi_R \in \text{End}(V^{\otimes 2}) \) such that

\[
\text{Tr}(R_2 R_2 \Psi_{R_{23}}) = \text{Tr}(R_2 \Psi_{R_{12}} R_{23}) = P_{13}. \quad (A.3)
\]

Here \( \text{Tr} \) denotes trace operation in \( i \)th space. With a skew invertible \( R \)-matrix \( R \) one associates matrix \( D_R \in \text{End}(V) \):

\[
D_{R1} := \text{Tr}(R_2 \Psi_{R12}),
\]

by which one defines a notion of \( R \)-trace, \( \text{Tr}_R \). Namely, for any \( X \in \text{End}(V) \)

\[
\text{Tr}_R(X) := \text{Tr}(D_R X).
\]
The operation \( \text{Tr}_R \) is often called a quantum trace or, shortly, a \( q \)-trace. We use the name \( R \)-trace to emphasize the dependence of this operation on a choice of the \( R \)-matrix. Properties of the \( R \)-trace are listed in [17], section 2.2.

An \( R \)-matrix whose minimal polynomial is quadratic is called Hecke type. By an appropriate rescaling one can turn its minimal polynomial to a form

\[
(R - qa)(R + q^{-1}I) = 0, \quad (A.4)
\]

known under the name Hecke condition. Skew invertible Hecke type \( R \)-matrices are used for quantizing differential geometric constructions over linear (super)groups.

To specify \( GL/SL(n) \) cases we impose conditions on the \( R \)-matrix. First, we demand that the \( R \)-matrix eigenvalue \( q \in \mathbb{C} \setminus \{0\} \) does not coincide with certain roots of unity:

\[
i_q := (q^i - q^{-i})/(q - q^{-1}) \neq 0 \quad \forall i = 2, 3, \ldots, n. \quad (A.5)
\]

In this case by the Hecke type \( R \)-matrix one can construct a series of idempotents \( A^{(i)} \in \text{End}(V^{\otimes 2}), i = 1, \ldots, n \), called \( q \)-antisymmetrizers. Their inductive definition reads

\[
A^{(1)} = I, \quad \quad A^{(i)} = \frac{(i - 1)q}{i_q} A^{(i-1)} \left( \frac{q^{i-1}}{(i - 1)q} I - R_{i-1} \right) A^{(i-1)}, \quad (A.6)
\]

and their properties are listed in [17], section 2.4.

A skew invertible Hecke type \( R \)-matrice whose eigenvalues satisfies (A.5) is called \( GL(n) \) type if conditions

\[
\left( \frac{q^n}{n_q} I - R_n \right) A^{(n)} = 0, \quad (A.7)
\]

and

\[
\text{rk} A^{(n)} = 1 \quad (A.8)
\]
are fulfilled. The \( R \)-matrix is called \( SL(n) \) type if the additional condition
\[
\text{Tr}_{(2,\ldots,n+1)}(P_1 P_2 \ldots P_n A^{(n)}) \propto I_1
\] (A.9)
is satisfied. The latter condition guarantees the centrality of the element \( \det R \) \( T \) in the
differential calculus algebra \( \mathcal{O}C[R] \) [10] (see also [17]) and thus, makes the \( SL_q(n) \) reduction
possible.

If the \( R \)-matrix is \( GL(n)/SL(n) \) type, then so are the \( R \)-matrices \( R^{-1} \) and \( R_{op} \).

We complete the appendix with the list of formulas which are valid for the \( R \)-traces of the
\( GL(n) \) type \( R \)-matrices.
\[
\begin{align*}
\text{Tr}_{(1,\ldots,n)}(A^{(n)}) \cdots = q^{n^2} \text{Tr}_{R^{(n)} \cdots \cdots} (A^{(n)}) \cdots = q^{n^2} \text{Tr}_{R^{(n)} \cdots \cdots} (A^{(n)}) \cdots
\end{align*}
\] (A.12)

For the Drinfeld–Jimbo \( R \)-matrix (2.1) explicit expressions for the matrices of \( R \)-traces are
\[
\begin{align*}
D_R = \text{diag} \{ q^{1-2n}, q^{3-2n}, \ldots, q^{-1} \}, \\
D_{R_{op}} = \text{diag} \{ q^{-1}, q^{-3}, \ldots, q^{-2n+1} \},
\end{align*}
\] (A.13)
and in case \( |q| = 1 \) one has
\[
(\text{Tr}_R)^\dagger = q^{2n} \text{Tr}_{R_{op}}, \\
(A^{(n)})^\dagger = P^{(n)} A^{(n)} P^{(n)},
\] (A.14)
where \( P^{(n)} := P_1(P_2 P_1) \cdots \cdots (P_{n-1} \cdots P_n) \) is the operator inversing enumeration of vector
spaces in \( V^{\otimes n} \).

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