Research Article

Positive Solutions of a Nonlinear Three-Point Integral Boundary Value Problem

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We study the existence of positive solutions to the three-point integral boundary value problem

\[ u'' + a(t)f(u) = 0, \quad t \in (0, 1), \quad u(0) = 0, \quad \alpha \int_0^\eta u(s) \, ds = u(1), \]

where \( 0 < \eta < 1 \) and \( 0 < \alpha < 2/\eta^2 \). We show the existence of at least one positive solution if \( f \) is either superlinear or sublinear by applying the fixed point theorem in cones.

1. Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev \cite{1}. Then Gupta \cite{2} studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to \cite{3–19} and the references therein. However, all these papers are concerned with problems with three-point boundary condition restrictions on the slope of the solutions and the solutions themselves, for example,

\begin{align*}
  u(0) &= 0, \quad au(\eta) = u(1), \\
  u(0) &= \beta u(\eta), \quad au(\eta) = u(1), \\
  u'(0) &= 0, \quad au(\eta) = u(1), \\
  u(0) - \beta u'(0) &= 0, \quad au(\eta) = u(1), \\
  \alpha u(0) - \beta u'(0) &= 0, \quad u'(\eta) + u'(1) = 0,
\end{align*}

and so forth.
In this paper, we consider the existence of positive solutions to the equation

\[ u'' + a(t)f(u) = 0, \quad t \in (0,1), \]  

with the three-point integral boundary condition

\[ u(0) = 0, \quad \alpha \int_0^\eta u(s)ds = u(1), \]  

where \( 0 < \eta < 1 \). We note that the new three-point boundary conditions are related to the area under the curve of solutions \( u(t) \) from \( t = 0 \) to \( t = \eta \).

The aim of this paper is to give some results for existence of positive solutions to (1.2)-(1.3), assuming that \( 0 <\alpha < 2/\eta^2 \) and \( f \) is either superlinear or sublinear. Set

\[ f_0 = \lim_{u \to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}. \]  

Then \( f_0 = 0 \) and \( f_\infty = \infty \) correspond to the superlinear case, and \( f_0 = \infty \) and \( f_\infty = 0 \) correspond to the sublinear case. By the positive solution of (1.2)-(1.3) we mean that a function \( u(t) \) is positive on \( 0 < t < 1 \) and satisfies the problem (1.2)-(1.3).

Throughout this paper, we suppose the following conditions hold:

\( \text{(H1)} \) \( f \in C([0,\infty), [0,\infty)) \);

\( \text{(H2)} \) \( a \in C([0,1], [0,\infty)) \) and there exists \( t_0 \in [\eta,1] \) such that \( a(t_0) > 0 \).

The proof of the main theorem is based upon an application of the following Krasnoselskii’s fixed point theorem in a cone.

**Theorem 1.1** (see [20]). Let \( E \) be a Banach space, and let \( K \subset E \) be a cone. Assume \( \Omega_1, \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1, \quad \overline{\Omega}_1 \subset \Omega_2 \), and let

\[ A : K \cap \left( \overline{\Omega}_1 \setminus \Omega_2 \right) \to K \]  

be a completely continuous operator such that

(i) \( \|Au\| \leqslant \|u\|, \quad u \in K \cap \partial \Omega_1 \), and \( \|Au\| \geqslant \|u\|, \quad u \in K \cap \partial \Omega_2 \); or

(ii) \( \|Au\| \geqslant \|u\|, \quad u \in K \cap \partial \Omega_1 \), and \( \|Au\| \geqslant \|u\|, \quad u \in K \cap \partial \Omega_2 \).

Then \( A \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \).
2. Preliminaries

We now state and prove several lemmas before stating our main results.

Lemma 2.1. Let \( \alpha \eta^2 \neq 2 \). Then for \( y \in C[0, 1] \), the problem

\[
\begin{align*}
    u'' + y(t) &= 0, \quad t \in (0, 1), \quad (2.1) \\
    u(0) &= 0, \quad \alpha \int_0^\eta u(s)ds = u(1), \quad (2.2)
\end{align*}
\]

has a unique solution

\[
u(t) = \frac{2t}{2 - \alpha \eta^2} \int_0^1 (1 - s)y(s)ds - \frac{at}{2 - \alpha \eta^2} \int_0^\eta (\eta - s)^2 y(s)ds - \int_0^t (t - s)y(s)ds. \quad (2.3)\]

Proof. From (2.1), we have

\[
u''(t) = -y(t). \quad (2.4)\]

For \( t \in [0, 1] \), integration from 0 to \( t \), gives

\[
u'(t) = u'(0) - \int_0^t y(s)ds. \quad (2.5)\]

For \( t \in [0, 1] \), integration from 0 to \( t \) yields that

\[
u(t) = u'(0)t - \int_0^t \left( \int_0^x y(s)ds \right)dx, \quad (2.6)\]

that is,

\[
u(t) = u'(0)t - \int_0^t (t - s)y(s)ds. \quad (2.7)\]

So,

\[
u(1) = u'(0) - \int_0^1 (1 - s)y(s)ds. \quad (2.8)\]
Integrating (2.7) from 0 to \( \eta \), where \( \eta \in (0,1) \), we have

\[
\int_0^\eta u(s)ds = u'(0)\frac{\eta^2}{2} - \int_0^\eta \left( \int_0^x (x-s)y(s)ds \right)dx
\]

\[
= u'(0)\frac{\eta^2}{2} - \frac{1}{2} \int_0^\eta (\eta - s)^2 y(s)ds.
\]  \hspace{1cm} (2.9)

From (2.2), we obtain that

\[
u'(0) - \int_0^1 (1-s)y(s)ds = u'(0)\frac{\alpha \eta^2}{2} - \frac{\alpha}{2} \int_0^\eta (\eta - s)^2 y(s)ds. \hspace{1cm} (2.10)
\]

Thus,

\[
u'(0) = \frac{2}{2 - \alpha \eta^2} \int_0^1 (1-s)y(s)ds - \frac{\alpha}{2(2 - \alpha \eta^2)} \int_0^\eta (\eta - s)^2 y(s)ds.
\]  \hspace{1cm} (2.11)

Therefore, (2.1)-(2.2) has a unique solution

\[
u(t) = \frac{2t}{2 - \alpha \eta^2} \int_0^1 (1-s)y(s)ds - \frac{\alpha t}{2(2 - \alpha \eta^2)} \int_0^\eta (\eta - s)^2 y(s)ds - \int_0^t (t-s)y(s)ds.
\]  \hspace{1cm} (2.12)

\[\Box\]

**Lemma 2.2.** Let \( 0 < \alpha < 2/\eta^2 \). If \( y \in C(0,1) \) and \( y(t) \geq 0 \) on \( (0,1) \), then the unique solution \( u \) of (2.1)-(2.2) satisfies \( u \geq 0 \) for \( t \in [0,1] \).

**Proof.** If \( u(1) \geq 0 \), then, by the concavity of \( u \) and the fact that \( u(0) = 0 \), we have \( u(t) \geq 0 \) for \( t \in [0,1] \).

Moreover, we know that the graph of \( u(t) \) is concave down on \( (0,1) \), we get

\[
\int_0^\eta u(s)ds \geq \frac{1}{2} \eta u(\eta), \hspace{1cm} (2.13)
\]

where \( (1/2)\eta u(\eta) \) is the area of triangle under the curve \( u(t) \) from \( t = 0 \) to \( t = \eta \) for \( \eta \in (0,1) \).

Assume that \( u(1) < 0 \). From (2.2), we have

\[
\int_0^\eta u(s)ds < 0.
\]  \hspace{1cm} (2.14)

By concavity of \( u \) and \( \int_0^\eta u(s)ds < 0 \), it implies that \( u(\eta) < 0 \).
Lemma 2.3. Let $\alpha \eta^2 > 2$. If $y \in C(0,1)$ and $y(t) \geq 0$ for $t \in (0,1)$, then (2.1)-(2.2) has no positive solution.

Proof. Assume (2.1)-(2.2) has a positive solution $u$.

If $u(1) > 0$, then $\int_0^1 u(s)ds > 0$, it implies that $u(\eta) > 0$ and

$$\frac{u(1)}{1} = \alpha \int_0^\eta u(s)ds \geq \frac{\alpha \eta}{2} u(\eta) = \frac{\alpha \eta^2}{2} \frac{u(\eta)}{\eta} > \frac{u(\eta)}{\eta},$$

(2.15)

which contradicts the concavity of $u$.

Hence,

$$u(1) = \alpha \int_0^\eta u(s)ds \geq \frac{\alpha \eta}{2} u(\eta) > \frac{u(\eta)}{\eta},$$

(2.16)

which contradicts the concavity of $u$.

If $u(1) = 0$, then $\int_0^\eta u(s)ds = 0$, this is $u(t) \equiv 0$ for all $t \in [0,\eta]$. If there exists $\tau \in (\eta,1)$ such that $\eta > u(\eta) < u(\tau)$, which contradicts the concavity of $u$. Therefore, no positive solutions exist.

In the rest of the paper, we assume that $0 < \alpha \eta^2 < 2$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

Lemma 2.4. Let $0 < \alpha < 2/\eta^2$. If $y \in C[0,1]$ and $y \geq 0$, then the unique solution $u$ of the problem (2.1)-(2.2) satisfies

$$\inf_{t \in [\eta,1]} u(t) \geq \gamma \|u\|,$$

(2.17)

where

$$\gamma := \min \left\{ \eta, \frac{\alpha \eta^2}{2}, \frac{\alpha \eta (1-\eta)}{2-\alpha \eta^2} \right\}.$$

(2.18)

Proof. Set $u(\tau) = \|u\|$. We divide the proof into three cases.

Case 1. If $\eta < \tau < 1$ and $\inf_{t \in [\eta,1]} u(t) = u(\eta)$, then the concavity of $u$ implies that

$$\frac{u(\eta)}{\eta} \geq \frac{u(\tau)}{\tau} \geq u(\tau).$$

(2.19)

Thus,

$$\inf_{t \in [\eta,1]} u(t) \geq \eta \|u\|.$$
Case 2. If $\eta \leq \tau \leq 1$, and $\inf_{t \in [\eta, 1]} u(t) = u(1)$, then (2.2), (2.13), and the concavity of $u$ implies

$$u(1) = \alpha \int_0^\eta u(s) ds \geq \frac{\alpha \eta^2}{2} \frac{u(\eta)}{\eta} \geq \frac{\alpha \eta^2}{2} \frac{u(\tau)}{\tau} \geq \frac{\alpha \eta^2}{2} u(\tau).$$

Therefore,

$$\inf_{t \in [\eta, 1]} u(t) \geq \frac{\alpha \eta^2}{2} \|u\|. \tag{2.22}$$

Case 3. If $\tau \leq \eta < 1$, then $\inf_{t \in [\eta, 1]} u(t) = u(1)$. Using the concavity of $u$ and (2.2), (2.13), we have

$$u(\sigma) \leq u(1) + \frac{u(1) - u(\eta)}{1 - \eta} (0 - 1)$$

$$\leq u(1) \left[ 1 - \frac{2}{1 - \eta} \frac{\alpha \eta}{1 - \eta} \right]$$

$$= u(1) \frac{2 - \alpha \eta^2}{\alpha \eta (1 - \eta)}.$$ 

This implies that

$$\inf_{t \in [\eta, 1]} u(t) \geq \frac{\alpha \eta (1 - \eta)}{2 - \alpha \eta^2} \|u\|. \tag{2.24}$$

This completes the proof.

3. Main Results

Now we are in the position to establish the main result.

**Theorem 3.1.** Assume (H1) and (H2) hold. Then the problem (1.2)-(1.3) has at least one positive solution in the case

(i) $f_0 = 0$ and $f_\infty = \infty$ (superlinear), or

(ii) $f_0 = \infty$ and $f_\infty = 0$ (sublinear).
Proof. It is known that $0 < \alpha < 2/\eta^2$. From Lemma 2.1, $u$ is a solution to the boundary value problem (1.2)-(1.3) if and only if $u$ is a fixed point of operator $A$, where $A$ is defined by

$$Au(t) = \frac{2t}{2-\alpha t^2} \int_0^1 (1-s)a(s)f(u(s))ds - \frac{at}{2-\alpha t^2} \int_0^\eta (\eta - s)^2a(s)f(u(s))ds$$

$$- \int_0^t (t-s)a(s)f(u(s))ds.$$ \hspace{1cm} (3.1)

Denote that

$$K = \left\{ u \mid u \in C[0,1], \ u \geq 0, \ \inf_{\eta \leq t \leq 1} u(t) \geq \gamma \|u\| \right\},$$ \hspace{1cm} (3.2)

where $\gamma$ is defined in (2.18).

It is obvious that $K$ is a cone in $C[0,1]$. Moreover, by Lemmas 2.2 and 2.4, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

**Superlinear Case ($f_0 = 0$ and $f_\infty = \infty$).**

Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leq eu$, for $0 < u \leq H_1$, where $e > 0$ satisfies

$$\frac{2e}{2-\alpha t^2} \int_0^1 (1-s)a(s)ds \leq 1.$$ \hspace{1cm} (3.3)

Thus, if we let

$$\Omega_1 = \{ u \in C[0,1] \mid \|u\| < H_1 \},$$ \hspace{1cm} (3.4)

then, for $u \in K \cap \partial \Omega_1$, we get

$$Au(t) \leq \frac{2t}{2-\alpha t^2} \int_0^1 (1-s)a(s)f(u(s))ds$$

$$\leq \frac{2e}{2-\alpha t^2} \int_0^1 (1-s)a(s)u(s)ds$$

$$\leq \frac{2e}{2-\alpha t^2} \int_0^1 (1-s)a(s)ds\|u\|$$

$$\leq \|u\|.$$ \hspace{1cm} (3.5)

Thus $\|Au\| \leq \|u\|$, $u \in K \cap \partial \Omega_1$. 

Further, since \( f_\infty = \infty \), there exists \( \tilde{H}_2 > 0 \) such that \( f(u) \geq \rho u \), for \( u \geq \tilde{H}_2 \), where \( \rho > 0 \) is chosen so that

\[
\rho \frac{2\eta}{2 - \alpha \eta^2} \int_\eta^1 (1 - s)a(s)ds \geq 1. \tag{3.6}
\]

Let \( H_2 = \max\{2H_1, \tilde{H}_2/\gamma\} \) and \( \Omega_2 = \{u \in C[0,1] \mid \|u\| < H_2\} \). Then \( u \in K \cap \partial\Omega_2 \) implies that

\[
\inf_{\eta \leq t \leq 1} u(t) \geq \gamma \|u\| = \gamma H_2 \geq \tilde{H}_2,
\]

and so

\[
Au(\eta) = \frac{2\eta}{2 - \alpha \eta^2} \int_0^1 (1 - s)a(s)f(u(s))ds - \frac{\alpha \eta}{2 - \alpha \eta^2} \int_0^\eta (\eta - s)^2a(s)f(u(s))ds
\]

\[
- \int_0^\eta (\eta - s)a(s)f(u(s))ds
\]

\[
= \frac{2\eta}{2 - \alpha \eta^2} \int_0^1 (1 - s)a(s)f(u(s))ds - \frac{\alpha \eta}{2 - \alpha \eta^2} \int_0^\eta (\eta^2 - 2\eta s + s^2)a(s)f(u(s))ds
\]

\[
- \frac{1}{2 - \alpha \eta^2} \int_0^\eta (2 - \alpha \eta^2)(\eta - s)a(s)f(u(s))ds
\]

\[
= \frac{2\eta}{2 - \alpha \eta^2} \int_0^1 (1 - s)a(s)f(u(s))ds + \frac{\alpha \eta^2}{2 - \alpha \eta^2} \int_0^\eta sa(s)f(u(s))ds
\]

\[
- \frac{\alpha \eta}{2 - \alpha \eta^2} \int_0^\eta s^2a(s)f(u(s))ds - \frac{2\eta}{2 - \alpha \eta^2} \int_0^\eta a(s)f(u(s))ds
\]

\[
\geq \frac{2\eta}{2 - \alpha \eta^2} \int_\eta^1 (1 - s)a(s)f(u(s))ds
\]

\[
\geq \frac{2\eta \rho}{2 - \alpha \eta^2} \int_\eta^1 (1 - s)a(s)u(s)ds \geq \frac{2\eta \rho \gamma}{2 - \alpha \eta^2} \int_\eta^1 (1 - s)a(s)ds \|u\| \geq \|u\|.
\]

Hence, \( \|Au\| \geq \|u\|, u \in K \cap \partial\Omega_2 \). By the first past of Theorem 1.1, \( A \) has a fixed point in \( K \cap (\bar{\Omega}_2 \setminus \Omega_1) \) such that \( H_1 \leq \|u\| \leq H_2 \).
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Sublinear Case ($f_0 = \infty$ and $f_\infty = 0$).

Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geq Mu$ for $0 < u \leq H_3$, where $M > 0$ satisfies

$$\frac{2\eta \gamma M}{2 - \alpha \eta^2} \int_{\eta}^{1} (1 - s)a(s)ds \geq 1. \tag{3.9}$$

Let

$$\Omega_3 = \{ u \in C[0,1] \mid \| u \| < H_3 \}, \tag{3.10}$$

then for $u \in K \cap \partial \Omega_3$, we get

$$Au(\eta) = \frac{2\eta}{2 - \alpha \eta^2} \int_{0}^{\eta} (1 - s)a(s)f(u(s))ds - \frac{\alpha \eta}{2 - \alpha \eta^2} \int_{0}^{\eta} (\eta - s)^2 a(s)f(u(s))ds$$

$$- \int_{0}^{\eta} (\eta - s)a(s)f(u(s))ds$$

$$\geq \frac{2\eta}{2 - \alpha \eta^2} \int_{\eta}^{1} (1 - s)a(s)f(u(s))ds$$

$$\geq \frac{2\eta \gamma M}{2 - \alpha \eta^2} \int_{\eta}^{1} (1 - s)a(s)ds \| u \| \geq \| u \|. \tag{3.11}$$

Thus, $\| Au \| \geq \| u \|$, $u \in K \cap \partial \Omega_3$. Now, since $f_\infty = 0$, there exists $\tilde{H}_4 > 0$ so that $f(u) \leq \lambda u$ for $u \geq \tilde{H}_4$, where $\lambda > 0$ satisfies

$$\frac{2\lambda}{2 - \alpha \eta^2} \int_{0}^{1} (1 - s)a(s)ds \leq 1. \tag{3.12}$$

Choose $H_4 = \max\{2H_3, \tilde{H}_4 / \gamma\}$. Let

$$\Omega_4 = \{ u \in C[0,1] \mid \| u \| < H_4 \}, \tag{3.13}$$

then $u \in K \cap \partial \Omega_4$ implies that

$$\inf_{\eta \leq t \leq 1} u(t) \geq \gamma \| u \| = \gamma H_4 \geq \tilde{H}_4. \tag{3.14}$$
Therefore,

\[
Au(t) = \frac{2t}{2 - a\eta^2} \int_0^1 (1 - s)a(s) f(u(s))ds - \frac{at}{2 - a\eta^2} \int_0^\eta (\eta - s)^2 a(s)f(u(s))ds \\
- \int_0^t (t-s)a(s)f(u(s))ds \\
\leq \frac{2t}{2 - a\eta^2} \int_0^1 (1 - s)a(s)f(u(s))ds \\
\leq \frac{2\lambda\|u\|}{2 - a\eta^2} \int_0^1 (1 - s)a(s)ds \leq \|u\|.
\]

Thus \(\|Au\| \leq \|u\|, u \in K \cap \partial \Omega_1\). By the second part of Theorem 1.1, \(A\) has a fixed point \(u\) in \(K \cap (\overline{\Omega_1} \setminus \Omega_3)\), such that \(H_3 \leq \|u\| \leq H_4\). This completes the sublinear part of the theorem. Therefore, the problem (1.2)-(1.3) has at least one positive solution. \(\square\)

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