The singlet contribution to the Bjorken sum rule for polarized deep inelastic scattering

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Abstract

It is shown that the existing four-loop result for the Bjorken polarized sum rule for deep inelastic electron-nucleon scattering obtained within perturbative Quantum Chromodynamics should be supplemented by the calculation of the diagrams of the so called singlet type. We also give an explanation of the interesting coincidence of two different classes of diagrams, one of the non-singlet and one of the singlet type, contributing the $\alpha_s^4$-approximation to the total cross-section of electron-positron annihilation into hadrons.
Since the discovery of the asymptotic freedom [1] there was the enormous progress in perturbative calculations in Quantum Chromodynamics (QCD). In particular calculations of the Bjorken sum rule for polarized deep inelastic electron-nucleon scattering [2] have now some history. The leading \(O(\alpha_s)\) correction in the strong coupling constant \(\alpha_s\) was calculated in [3]. The next-to-leading \(O(\alpha_s^2)\) approximation was obtained in [4] and the \(O(\alpha_s^3)\) correction was found in [5]. Quite recently the \(O(\alpha_s^4)\) approximation was published [6].

In the present letter we demonstrate that the calculation [6] should be supplemented by the calculation of the diagrams of the so called singlet type. We determine this singlet contribution up to an overall constant using the Crewther relation [7].

We also give the explanation of the interesting coincidence of contributions of two different classes of diagrams, one of the non-singlet and one of the singlet type, contributing the \(\alpha_s^4\)-approximation to the total cross-section of electron-positron annihilation into hadrons.

The Bjorken polarized sum rule for polarized deep inelastic scattering has the following form

\[
\int_0^1 \left( g_1^{ep}(x, Q^2) - g_1^{en}(x, Q^2) \right) dx = \frac{g_A}{6} C_{Bjp}(a_s(Q^2)) + \text{nonperturbative terms},
\]

where \(g_1^{ep}\) and \(g_1^{en}\) are the structure functions of polarized electron-proton and electron-nucleon deep inelastic scattering, \(g_A \approx 1.22\) is the axial constant of the neutron \(\beta\)-decay, \(Q^2\) is the Euclidean momentum transfered squared, \(a_s = \alpha_s/\pi\) is the strong couplingant.

The coefficient function \(C_{Bjp}(a_s) = 1 + O(a_s)\) enters the following short-distance operator product expansion (OPE)

\[
i \int d^4xe^{iqx} T [J_\mu(x)J_\nu(0)] = (q_\mu q_\nu - g_{\mu\nu}q^2)\Pi^{EM}(Q^2) +
\]

\[
\epsilon_{\mu
u\lambda\rho} q_\rho q^2 \left[ C_{Bjp}^a(a_s) A_\lambda^a(0) + C_{EJ}(a_s) A_\lambda(0) \right] + \text{higher twists},
\]

where the summation over repeated indexes is assumed, \(J_\mu\) is the electromagnetic quark current, \(\Pi^{EM}(Q^2)\) is the polarization function, \(A_\lambda^a = \bar{\psi}\gamma_\lambda\gamma_5t^a\psi\) is the non-singlet (NS) axial quark current, \(t^a\) being the (diagonal) generator of the flavor \(SU(n_f)\)-group, \(n_f\) being the number of quark flavors. \(A_\lambda = \bar{\psi}\gamma_\lambda\gamma_5\psi\) is the singlet (SI) axial quark current.

To calculate the coefficient function \(C_{Bjp}^a(a_s)\) at the multiloop level one uses the method of projections [8] which gives

\[
i \int d^4xe^{iqx} < 0|T [\bar{\psi}(p)\gamma_\sigma\gamma_5t^a\psi(-p)J_\mu(x)J_\nu(0)] |0 > \bigg|_{p=0}^{\text{amputated}} =
\]
\[\text{const } \epsilon_{\mu\nu\rho\sigma} \frac{q^\mu}{q^2} C_{Bjp}^a(a_s) Z_A,\]

where \(\psi(p)\) is the Fourier transform of the quark field carrying the momentum \(p\). Quark legs are amputated. \(Z_A\) is the renormalization constant of the non-singlet axial current. \text{const} is the normalization constant. The technique how to deal with the \(\gamma_5\)-matrix in multiloop calculations within dimensional regularization and minimal subtraction scheme is given in \[9\].

The coefficient function \(C_{Bjp}^a(a_s)\) receives contributions from two types of diagrams. The first type, the non-singlet one (with both electromagnetic vertexes attached to the fermion line of external quark legs) produces the flavor factor \(\text{Tr}(Q_f^2 t^a)\), where \(Q_f\) is the (diagonal) quark charge matrix \(Q_f = \text{diag}(2/3, -1/3, -1/3, \ldots)\). The second, the singlet type (when one electromagnetic vertex is attached to the fermion line of external quark legs and another to the internal quark loop) gives the flavor factor \(\text{Tr}(Q_f)\text{Tr}(Q_f^2 t^a)\).

The ratio of these flavor factors does not depend on the index \(a\)

\[
\frac{\text{Tr}(Q_f)\text{Tr}(Q_f^2 t^a)}{\text{Tr}(Q_f^2 t^a)} = 3 \sum_{i=1}^{n_f} q_i
\]

where \(q_i\) are electromagnetic quark charges. That is why one can factorize from \(C_{Bjp}^a(a_s)\) the \(a\)-independent coefficient function \(C_{Bjp}(a_s)\) which enters the sum rule \(\text{II}\)

\[
C_{Bjp}^a(a_s) = \text{Tr}(Q_f^2 t^a) C_{NS}^a(a_s) + \text{Tr}(Q_f)\text{Tr}(Q_f t^a) C_{SI}^a(a_s) = \left( C_{NS}^a(a_s) + 3(3 \sum_{i=1}^{n_f} q_i) C_{SI}^a \right) \text{Tr}(Q_f^2 t^a) = C_{Bjp}(a_s) \text{Tr}(Q_f^2 t^a).
\]

It is the contribution of the singlet type \(C_{SI}^a\) which is missed in the calculation \[6\] of the \(\alpha_s^4\)-correction to the Bjorken polarized sum rule. It is interesting to note that individual diagrams of the singlet type give non-zero contributions to the sum rule already in the \(a_s^3\) order but their total sum nullifies \[5\] in this order. It can be explained by using the generalized Crewther relation \[7\]. The relation states that

\[
C_{Bjp}(a_s) D_{NS}^a(a_s) = d_R \left( 1 + \frac{\beta(a_s)}{a_s} K(a_s) \right),
\]

\[
K(a_s) = a_s K_1 + a_s^2 K_2 + a_s^3 K_3 + \ldots,
\]

where \(K_i\) are calculable in QCD coefficients, \(d_R\) is the dimension of the quark representation \((d_R = 3\) in QCD), \(\beta(a_s)\) is the renormalization group \(\beta\)-function

\[
\beta(a_s) = \mu^2 \frac{\partial a_s}{\partial \mu^2} = \sum_{i \geq 0} \beta_i a_s^{i+2}
\]
with the well known first coefficient \( \beta_0 = -\frac{11}{2}C_A + \frac{1}{3}T_F n_f \), \( C_A \) being the quadratic Casimir operator of the adjoint representation of the group and \( T_F \) being the trace normalization of the fundamental representation.

The Adler function \( D^{NS}(a_s) \) is defined as

\[
D^{EM}(a_s) = -12\pi^2 Q^2 \frac{d}{dQ^2} \Pi^{EM}(Q^2),
\]

\[
D^{EM}(a_s) = \left( \sum_i q_i^2 \right) D^{NS}(a_s) + \left( \sum_i q_i \right)^2 D^{SI}(a_s).
\]

The singlet diagrams contributing to \( C_{Bjp}(a_s) \) at the \( a_s^3 \) and the \( a_s^4 \) levels are proportional to the color factor \( d^{abc}d^{abc} \), where \( d^{abc} \) are the symmetric structure constants of the \( SU(N_c) \) color group (for QCD with the \( SU(3) \) group one gets \( d^{abc}d^{abc} = 40/3 \)). At the \( a_s^3 \) level the sum of the singlet diagrams should nullify since the color factor \( d^{abc}d^{abc} \) is the complete color factor for these diagrams and the coefficient \( \beta_0 \) can not be factorized which is in the contradiction with the Crewther relation (6). At the \( a_s^4 \) level there are enough loops (four) to generate the color structure \( \beta_0 d^{abc}d^{abc} \) in accordance with the relation (6). Thus one can get the non-zero singlet contribution to the Bjorken polarized sum rule in the order \( a_s^4 \)

\[
C_{Bjp}(a_s) = C^{NS}(a_s) + X a_s^4 \beta_0 \sum_{i=1}^{n_f} q_i d^{abc}d^{abc} + O(a_s^5),
\]

where the non-singlet contribution was calculated up to and including the order \( a_s^4 \) in (6). The numerical constant \( X \) is still to be calculated to get the complete \( O(a_s^5) \) correction.

In principle it is possible that after calculating the singlet contribution to \( C_{Bjp}(a_s) \) one can see at the \( a_s^4 \) level the validity of the interesting relation which connects different physical quantities

\[
\left[ C^{NS}(a_s) + n_f C^{SI}(a_s) \right] D^{NS}(a_s) = C_{GLS}(a_s) \left[ D^{NS}(a_s) + n_f D^{SI}(a_s) \right],
\]

here \( C_{GLS}(a_s) \) is the coefficient function of the Gross-Llewellyn Smith sum rule for deep inelastic neutrino-nucleon scattering (11). \( D^{NS}(a_s) + n_f D^{SI}(a_s) \equiv D(a_s)/n_f \), where \( D(a_s) \) is the Adler function corresponding to the correlator of the flavor singlet quark currents.

This relation is valid at the \( a_s^3 \) level. To show that it can be valid in all orders let us consider OPE for the following 3-point function

\[
T^{ab}_{\mu\nu\lambda}(p,q) = i \int <0|T[V_\mu(x)A^a_\lambda(y) V^b_\nu(0)]|0 > e^{ipx+iqy}dxdy,
\]
where $V_\mu = \overline{\psi} \gamma_\mu \psi$ is the vector singlet quark current, $V_\nu^b = \overline{\psi} \gamma_\nu t^b \psi$ is the vector non-singlet quark current, $A_\lambda^a$ is the axial vector current defined in eq.(2).

We can apply first the following OPE

$$i \int T [A_\lambda^a(y) V_\nu^b(0)] e^{i q y} dy = \delta^{ab} \epsilon_{\lambda \nu \alpha \beta} \frac{q^\beta}{Q^2} C_{GLS}(a_s) A_\alpha(0) + ...$$ (12)

and substitute it into eq.(11) to get

$$T^{ab}_{\mu \lambda}(p, q) = \delta^{ab} \epsilon_{\lambda \nu \alpha \beta} \frac{q^\beta}{Q^2} C_{GLS}(a_s) \int <0 | T [V_\mu(x) V_\alpha(0)] |0 > e^{i p x} dx + ...$$ (13)

For more formal derivation of the OPE for 3-point functions see [8].

On the other hand we can apply first the following OPE

$$i \int T [V_\mu(x) V_\nu^b(0)] e^{i p x} dx = \epsilon_{\mu \nu \alpha \beta} \frac{p^\beta}{P^2} \left[ C^{NS}(a_s) + n_f C^{SI}(a_s) \right] A_\alpha^b(0) + ...$$ (14)

and again substitute it into eq.(11) to obtain

$$T^{ab}_{\mu \lambda}(p, q) = \epsilon_{\mu \nu \alpha \beta} \frac{p^\beta}{P^2} \left[ C^{NS}(a_s) + n_f C^{SI}(a_s) \right] \times$$

$$\int <0 | T [A_\lambda^a(y) A_\alpha^b(0)] |0 > e^{i q y} dq + ...$$

Comparing eq.(13) and eq.(15) one can see a connection close to that of the relation (10). But presently we do not have a proof of this relation.

If eq.(10) is valid then one can determine the constant $X$ in eq.(9) without explicit calculations of the singlet contribution to $C_{Bjp}(a_s)$ using results of ref. [10]:

$$X = -\frac{179}{384} + \frac{25}{48} \zeta_3 - \frac{5}{24} \zeta_5.$$

We would like also, as a byproduct, to give an explanation of the interesting coincidence at the $a_s^4$ level [12] of contributions to the Adler function $D(a_s)$ of two different sets of (5-loop propagator) diagrams, one set of the non-singlet type and another set of the singlet type.

In diagrams of the non-singlet set both external electromagnetic vertexes are attached to the same quark circle and this quark circle is connected to another quark circle by four gluon lines in all possible ways. The contribution of this non-singlet set of diagrams to the Adler function $D(a_s)$ is (ref. [12], eq.(3.14))

$$a_s^4 \frac{3}{4} n_f d_F^{abcd} d_F^{abcd} \left( -\frac{13}{12} - \frac{4}{3} \zeta_3 + \frac{10}{3} \zeta_5 \right).$$ (16)

The exact definition of the color structure $d_F^{abcd} d_F^{abcd}$ is given in [13]. For QCD $d_F^{abcd} d_F^{abcd} = 5/12$. 

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In diagrams of the singlet set each external electromagnetic vertex is attached to its own quark circle and these quark circles are connected by three gluon lines (plus gluon propagator insertion in one of the circles by all possible ways). The contribution of this singlet class to the Adler function $D(a_s)$ is (ref. [12], eq.(3.16))

$$a_s^4 \frac{3}{4} n_f d^{abc} d^{abc} C_F \left( -\frac{13}{48} - \frac{1}{3} \zeta_3 + \frac{5}{6} \zeta_5 \right),$$

where $C_F$ is the quadratic Casimir operator of the fundamental representation of the gauge group.

After transition to the QED case (the gauge group $U(1)$) the contributions of the non-singlet and singlet sets of diagrams to $D(a)$ coincide.

To explain the coincidence let us use the following trick. We connect external vertexes for each (5-loop propagator) diagram from these sets with an extra photon propagator. The crucial observation is that as the result both the non-singlet and singlet sets of 5-loop propagator diagrams will produce one and the same set of 6-loop vacuum diagrams.

But in dimensional regularization massless vacuum diagrams are zero. We will introduce a non-zero photon mass $m$ to deal with non-zero diagrams. Thus we get

$$\int d^D q \frac{g_{\mu\nu}}{q^2 - m^2} S_{\mu\nu}^{NS}(q, m) \equiv \int d^D q \frac{g_{\mu\nu}}{q^2 - m^2} S_{\mu\nu}^{SI}(q, m),$$

(18)

here $S_{\mu\nu}^{NS}(q, m)$ and $S_{\mu\nu}^{SI}(q, m)$ are the contributions of the non-singlet and singlet sets of 5-loop propagator diagrams to the correlator $(q_{\mu}q_{\nu} - g_{\mu\nu}q^2)\Pi(-q^2)$ of the singlet fermion currents in QED. The propagator $\frac{g_{\mu\nu}}{q^2 - m^2}$ corresponds to the extra photon propagator introduced to generate 6-loop vacuum diagrams. For simplicity we choose the Feynman gauge. $D = 4 - 2\epsilon$ is the space-time dimension within dimensional regularization.

For both sets of diagrams $S_{\mu\nu}^{NS}$ and $S_{\mu\nu}^{SI}$ all ultraviolet subdivergences cancel due to gauge invariance and only simple $\frac{1}{\epsilon}$ poles remain. (These are the poles which generate contributions to the function $D(a)$.) Because of the equality (18) these poles coincide.

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