NON-LOCAL EQUIVARIANT STAR PRODUCT ON THE MINIMAL NILPOTENT ORBIT

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Dedicated to Dmitri Fuchs on his 60th birthday

Abstract. We construct a unique $G$-equivariant graded star product on the algebra $S(g)/I$ of polynomial functions on the minimal nilpotent coadjoint orbit $O_{\text{min}}$ of $G$ where $G$ is a complex simple Lie group and $g \neq \text{sl}(2, \mathbb{C})$. This strengthens the result of Arnal, Benamor and Cahen.

Our main result is to compute, for $G$ classical, the star product of a momentum function $\mu_x$ with any function $f$. We find $\mu_x \star f = \mu_x f + \frac{1}{2} \{\mu_x, f\} t + \Lambda^x(f) t^2$. For $g$ different from $\text{sp}(2n, \mathbb{C})$, $\Lambda^x$ is not a differential operator. Instead $\Lambda^x$ is the left quotient of an explicit order 4 algebraic differential operator $D^x$ by an order 2 invertible diagonalizable operator. Precisely, $\Lambda^x = -\frac{1}{4} \frac{1}{E'(E'+1)} D^x$ where $E'$ is a positive shift of the Euler vector field. Thus $\mu_x \star f$ is not local in $f$.

Using $\star$ we construct a positive definite hermitian inner product on $S(g)/I$. The Hilbert space completion of $S(g)/I$ is then a unitary representation of $G$. This quantizes $O_{\text{min}}$ in the sense of geometric quantization and the orbit method.

1. Introduction

The fundamental problem in equivariant quantization is the $G$-equivariant quantization of the coadjoint orbits of $G$, where $G$ is a simply-connected Lie group. In deformation quantization, there is a nice set of axioms for the star product $\star$ and then $G$-equivariance of $\star$ is a relation involving the momentum functions $\mu_x$, $x \in g$, where $g = \text{Lie}(G)$. In fact, this amounts to $G$-equivariance of the corresponding quantization map (see §2).

It was already recognized by Fronsdal ([9]) that the locality axiom for star products must be modified in order to accommodate equivariance. The locality axiom means, in either the smooth or algebraic setting, that the operators which define the star product are bidifferential.

One could simply exclude any constructions that are not local. But this would cast aside equivariant constructions (such as [4], §9, page 124), and, as we show, ([4]) which are unique and very natural; moreover these retain a strong flavor of locality. Figuring out what this “flavor” is and how to axiomatize it is a very interesting problem. It seems to involve “pseudo-differential” operators.

In this paper, we investigate the unique $G$-equivariant graded star product on the algebra $R$ associated to the minimal (non-zero) nilpotent coadjoint orbit $O_{\text{min}}$ in $g^*$, where $g$ is a simple complex Lie algebra different from $\text{sl}(2, \mathbb{C})$. Here $R = S(g)/I$ is the algebra of polynomial functions on $O_{\text{min}}$. The star product was constructed for $g$ different from $\text{sl}(n, \mathbb{C})$ by Arnal, Benamor, and Cahen in [4]. We strengthen their result.

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in our Proposition 3.1, after some preliminary work in §2. We find an analog of the Joseph ideal for \( \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), n \geq 3 \). (There is a 1-dimensional family of candidates, but only one of them produces a star product with parity.) We prove uniqueness whenever \( \mathfrak{g} \neq \mathfrak{sl}(2, \mathbb{C}) \).

To start off, we show (Proposition 1.1) that the star product of a momentum function \( \mu_x \) with any function \( f \) is the three term sum

\[
\mu_x \ast f = \mu_x f + \frac{1}{2} \{ \mu_x, f \} t + \Lambda^x(f) t^2
\]

(1.1)

where \( \Lambda^x \) are graded operators on \( \mathcal{R} \) of degree \(-1\). We compute \( \Lambda^x \) for \( \mathfrak{g} \) classical. For \( \mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}) \), we find (§3) some familiar order 2 differential operators (which appear in the Fock space model of the oscillator representation).

Our main result (Theorem 6.3) is a formula for \( \Lambda^x \) when \( \mathfrak{g} \) is classical but different from \( \mathfrak{sp}(2n, \mathbb{C}) \), i.e., when \( \mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C}) \) or \( \mathfrak{g} = \mathfrak{so}(n, \mathbb{C}) \). We find that \( \Lambda^x(x \neq 0) \) is not a differential operator but instead is the left quotient of an order 4 algebraic differential operator \( D^x \) by an order 2 invertible diagonalizable operator. Precisely, \( \Lambda^x = -\frac{1}{4} E'(E'+1) D^x \) where \( E' \) is a positive shift of the Euler vector field. So \( \mu_x \ast f \) is not local as an operator on \( f \). Thus \( \ast \) is not local.

The differential operators \( D^x \) were constructed by us earlier (for this purpose) in §2. It would be very interesting now to find formulas for the operators \( C_p(f, g) \) that define \( f \ast g \). For \( \mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C}) \), some progress toward this is made in §3 using results of Lecomte and Ovsienko ([13]). Also we think that the method of Levasseur and Stafford ([13]), which gave a new elegant construction of our \( D^x \) for \( \mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C}) \), might be extended to give the \( \Lambda^x \) and the \( C_p(\cdot, \cdot) \). These approaches are based on the fact that \( \mathcal{R} \) identifies with the algebra of regular functions on \( T^\ast(\mathbb{C}P^n) \).

The star product defines a representation \( \pi \) of \( \mathfrak{g} \oplus \mathfrak{g} \) on \( \mathcal{R} \). We write this out in Corollary 1.3 using the \( \Lambda^x \). In §4 we show that \( \ast \) gives rise to a positive definite hermitian inner product on \( \mathcal{R} \) compatible with \( \pi \) and the grading on \( \mathcal{R} \). In this way, \( \mathcal{R} \) becomes the Harish-Chandra module of a unitary representation of \( G \) on the Hilbert space completion \( \mathcal{H} = \hat{\mathcal{R}} = \bigoplus_{d=0}^\infty \mathcal{R}^d \). This quantizes \( \mathcal{O}_{\min} \), regarded as a real symplectic manifold, in the sense of geometric quantization. We compute the reproducing kernel of \( \mathcal{H} \) and deduce that \( \mathcal{H} \) is a Hilbert space of holomorphic functions on \( \mathcal{O}_{\min} \).

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2. EQUIVARIANT GRADED STAR PRODUCTS ON \( S(\mathfrak{g})/I \)

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra. The symmetric algebra \( S = S(\mathfrak{g}) \) is the algebra of polynomial functions on \( \mathfrak{g}^\ast \). Then \( S = \bigoplus_{d=0}^\infty S^d \) is a graded Poisson algebra in the natural way, where \( \{ S^d, S^p \} \subseteq S^{d+p-1} \). Let \( I = \bigoplus_{d=0}^\infty I^d \) be a graded Poisson ideal in \( S \). We are most interested in the case when \( I \) is the ideal \( \mathcal{I}(\mathcal{O}) \) of functions vanishing on a nilpotent coadjoint orbit \( \mathcal{O} \) in \( \mathfrak{g}^\ast \). The term “nilpotent” means that the corresponding adjoint orbit consists of nilpotent elements; this happens if and only if \( \mathcal{O} \) is stable under dilations.
Let \( R = S/I \) and \( R^d = S^d/I^d \). Then \( R = \oplus_{d=0}^{\infty} R^d \) is again a graded Poisson algebra. If \( I = I(\mathcal{O}) \), then \( R \) is the algebra of polynomial functions on the closure \( Cl(\mathcal{O}) \). In the sense of algebraic geometry, \( Cl(\mathcal{O}) \) is a closed complex algebraic subvariety of \( g^* \) and \( R \) is its algebra \( R(Cl(\mathcal{O})) \) of regular functions. The elements \( x \in g \) define momentum functions \( \mu_x \) in \( R^1 \) and \( \{\mu_x, \mu_y\} = \mu_{[x,y]} \). The natural graded linear \( G \)-action on \( R \) corresponds to the \( g \)-representation given by the operators \( \{\mu_x, \cdot\} \).

A graded star product on \( R \) is an associative \( \mathbb{C}[t] \)-linear product \( \ast \) on \( R[t] \) with the following properties. For \( f, g \in R \) we can write \( f \ast g = \sum_{p=0}^{\infty} C_p(f,g)t^p \) and then

\[
\begin{align*}
(i) \quad C_0(f,g) &= fg \\
(ii) \quad C_1(f,g) - C_1(g,f) &= \{f, g\} \\
(iii) \quad C_p(f,g) &= (-1)^p C_p(g,f) \\
(iv) \quad C_p(f,g) &\in R^{k+p} \text{ if } f \in R^k \text{ and } g \in R^l
\end{align*}
\]

Notice that (ii) and (iii) imply \( C_1(f,g) = \frac{1}{2} \{f,g\} \). Axiom (iii) is called the parity axiom.

Given \( \ast \), we define a new noncommutative product on \( R \) by \( f \circ g = f \ast g|_{t=1} \). Because of (iv), we can completely recover \( \ast \) from \( \circ \). It is easy to see that (iii) amounts to the relation \( (f \circ g)^{\alpha} = g^{\alpha} \circ f^{\alpha} \) where \( f \mapsto f^{\alpha} \) is the Poisson algebra anti-involution of \( R \) defined by \( f^{\alpha} = (-1)^d f \) if \( f \in R^d \).

The star bracket is given by \( [f,g]_\ast = f \ast g - g \ast f \). We say \( \ast \) is \( g \)-covariant if \( [\mu_x, \mu_y]_\ast = t\mu_{[x,y]} \), and \( \ast \) is \( G \)-equivariant (or strongly \( g \)-invariant) if we have the much stronger relation \( [\mu_x, f]_\ast = t\{\mu_x, f\} \). We say that a \( G \)-equivariant graded star product on \( R \) is an \( G \)-equivariant deformation quantization of \( R \).

Suppose \( \ast \) is a graded \( g \)-covariant star product on \( R \). Let \( U = U(g) \) be the universal enveloping algebra of \( g \) equipped with its canonical filtration \( \{U_d\}_{d=0}^{\infty} \); \( grU \) identifies naturally with \( S \). Then we have a noncommutative algebra homomorphism \( \Psi : U \to R \) defined by \( \Psi(x_1 \cdots x_d) = \mu_{x_1} \cdots \circ \mu_{x_d} \). Then \( \Psi \) is surjective in a filtered way, i.e., \( \Psi(U_p) = \oplus_{d=0}^{p} R^d \). The kernel of \( \Psi \) is a 2-sided ideal \( J \) such that \( grJ = I \), and so \( gr(U/J) \) identifies naturally with \( S/I \).

Thus we get a vector space isomorphism \( q : R \to U/J \) defined by

\[
q(\mu_{x_1} \cdots \circ \mu_{x_d} + J) = x_1 \cdots x_d + J \quad (2.1)
\]

Then \( q \) is a quantization map, i.e., \( q \) induces the identity maps \( R^d \to S^d/I^d \). We can recover \( \circ \) from \( q \) by the formula \( f \circ g = q^{-1}((qf)(qg)) \). Then \( \ast \) is given by \( f \ast g = q^{-1}((qf)(qg)) \) where \( q(f) = q(f)t^d \) if \( f \in R^d \).

Let \( \tau \) be the algebra anti-involution of \( U \) defined by \( x^{\tau} = -x \); this is the so-called principal anti-automorphism. The parity axiom (iii) implies that \( J \) is stable under \( \tau \), so that \( \tau \) descends to \( U/J \), and also \( q(f^{\alpha}) = q(f)^{\tau} \).

Clearly \( \ast \) is \( G \)-equivariant if and only if \( q \) is \( g \)-linear, i.e., \( q(\{\mu_x, f\}) = xq(f) - q(f)x \). This amounts to \( q \) being \( G \)-equivariant. In summary, this discussion gives

**Proposition 2.1.** Suppose \( \ast \) is a graded \( G \)-equivariant star product on \( R = S/I \). Then we obtain in a canonical way a 2-sided \( \tau \)-stable ideal \( J \) in \( U \) and a \( G \)-equivariant quantization map \( q : R \to U/J \) given by (2.1).
3. Construction of $\ast$ when $\mathcal{O} = \mathcal{O_{\min}}$

From now on we assume that $\mathfrak{g}$ is simple. Let $\mathcal{O}_{\min}$ be the minimal non-zero nilpotent coadjoint orbit in $\mathfrak{g}^\ast$. So $\mathcal{O}_{\min}$ corresponds to the adjoint orbit of highest root vectors, or equivalently, of highest weight vectors. We put $\mathcal{R} = \mathcal{S}/I$ where $I$ is the ideal of $\mathcal{O}_{\min}$.

Proposition 3.1. Assume $\mathfrak{g}$ is different from $\mathfrak{sl}(2, \mathbb{C})$. Then $\mathcal{R}$ admits a unique $G$-equivariant graded star product $\ast$.

This strengthens the result in [1] where they show that, if $\mathfrak{g}$ is different from $\mathfrak{sl}(n+1, \mathbb{C})$ for $n \geq 1$, then $\mathcal{R}$ admits a $\mathfrak{g}$-equivariant graded star product which is unique up to equivalence of star products. We need to exclude $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ because $\mathfrak{sl}(2, \mathbb{C})$ admits infinitely many such star products (in natural bijection with $\mathbb{C}$).

Proof. The discussion in [2] reverses easily to give a converse to Proposition [2.1]. Precisely, if $J$ is a 2-sided $\tau$-stable ideal of $\mathcal{U}$ such that $\text{gr} \ J = I$ and $q : \mathcal{R} \to \mathcal{U}/J$ is a $\mathfrak{g}$-equivariant quantization map such that $q(f^n) = q(f)^\tau$, then the formula $f \ast g = q^{-1}((q(f))(q(g)))$ defines a $\mathfrak{g}$-equivariant graded star product on $\mathcal{R}$. Thus it suffices to prove the following two statements.

(i) There exists a unique 2-sided ideal $J$ of $\mathcal{U}$ such that $\text{gr} \ J = I$ and $J^\tau = J$.

(ii) For such $J$, there exists a unique $G$-equivariant quantization map $q : \mathcal{R} \to \mathcal{U}/J$.

Notice that in (ii), $q(f^n) = q(f)^\tau$ follows automatically by uniqueness.

To prove (ii) we need only elementary facts about $\mathcal{R}$ (see e.g., [3]). The natural $G$-representation on $\mathcal{R}$ is multiplicity free. One can get a very quick abstract proof of (ii) just from this, but we will be more concrete.

$\mathcal{R}^d$ is irreducible and carries the $d$th Cartan power $\mathfrak{g}^\otimes d$ of the adjoint representation. Since the representation $\mathfrak{g}^\otimes d$ occurs just once in $\mathcal{S}^d$, $I$ has a unique graded $G$-stable complement $F = \oplus_{p=0}^{\infty} F^p$ in $\mathcal{S}$; then $F$ identifies with $\mathcal{R}$. We define a vector space isomorphism $\mathcal{F}^s : \mathcal{U} \to \mathcal{U}/J$ where $s : \mathcal{S} \to \mathcal{U}$ is the usual symmetrization map; here we only assume that $\text{gr} \ J = I$. Let $q$ be the corresponding map from $\mathcal{R}$ to $\mathcal{U}/J$. Then clearly $q$ is a $G$-equivariant quantization map. If $\mathfrak{h}$ is another such map, then the composition $L = q_{\mathfrak{h}}^{-1}$ satisfies: $f \in \mathcal{R}^d$ implies $L(f) = f + g$ where $g \in \mathcal{R}^{d-1}$. But also $L$ is $G$-linear and so the $G$-decomposition of $\mathcal{R}$ forces $L(\mathcal{R}^d) = \mathcal{R}^d$. Thus $L(f) = f$.

The proof of (i) breaks into two cases. If $\mathfrak{g}$ is different from $\mathfrak{sl}(n+1, \mathbb{C})$, then as in [1] we take $J$ to be the Joseph ideal constructed in [11, §5]. We may characterize $J$ as the unique 2-sided ideal in $\mathcal{U}$ whose associated graded is $I$. This is not the most familiar characterization, but it is immediate from the fact ([11, Prop. 10.2]) that $J$ is the only completely prime 2-sided ideal such that $\sqrt{\text{gr} \ J} = I$, and the equality ([10]) $\text{gr} \ J = I$. Then uniqueness of $J$ implies that $J = J^\tau$.

Now suppose that $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $n \geq 1$. Let $\mathcal{D}^\lambda(\mathbb{CP}^n)$ be the algebra of global sections of the sheaf of twisted differential operators acting on local sections of the $\lambda$th power of the canonical bundle on complex projective space; this makes sense for any complex number $\lambda$. We have a natural algebra homomorphism $\Phi^\lambda : \mathcal{U} \to \mathcal{D}^\lambda(\mathbb{CP}^n)$. It is easy to write nice formulas for the twisted vector fields $\Phi^\lambda_x$, $x \in \mathfrak{g}$, in local coordinates on the big cell $\mathbb{C}^n$; see e.g., [14].

Let $J^\lambda$ be the kernel of $\Phi^\lambda$. Then $\Phi^\lambda$ is surjective and $\text{gr} \ J^\lambda = I$ (see [3]). All 2-sided ideals $J$ in $\mathcal{U}$ with $\text{gr} \ J = I$ arise in this way. The principal anti-involution $\tau$ carries $J^\lambda$ to
Corollary 3.2. Suppose \( g = \mathfrak{sl}(2, \mathbb{C}) \). Then \( J^\lambda = \text{Ker} \Phi^\lambda \) (\( \lambda \in \mathbb{C} \)), corresponds to a \( G \)-equivariant graded star product \( \star_\lambda \) on \( \mathcal{R} \). All such star products arise in this way, and \( \star_\lambda \) is \( \tau \)-stable and \( J^\lambda = J^\mu \) iff \( \mu = 1 - \lambda \).

Proof. \( J^\lambda \) is generated by a maximal ideal in the center of \( \mathcal{U} \), it follows directly that \( J^\lambda \) is \( \tau \)-stable and \( J^\lambda = J^\mu \) iff \( \mu = 1 - \lambda \). \( \square \)

Proposition 3.3. In Proposition 3.1, the noncommutative algebra \( \mathcal{U}/J \) obtained by specializing \( \star \) at \( t = 1 \) is a simple ring.

Proof. The Joseph ideal is maximal by \([14]\), Th. 7.4], and this means \( \mathcal{U}/J \) is simple. If \( g = \mathfrak{sl}(n+1, \mathbb{C}) \), \( n \geq 2 \), then \( \mathcal{U}/J \) is isomorphic to \( \mathcal{D}_G(\mathbb{P}^n) \), which is simple by \([15]\). \( \square \)

4. The Operators \( \Lambda^x \)

Proposition 4.1. The star product of a momentum function \( \mu_x \), \( x \in g \), with an arbitrary function \( f \in \mathcal{R} \) is the three-term sum

\[
\mu_x \star f = \mu_x f + \frac{1}{2}\{\mu_x, f\}t + \Lambda^x(f)t^2
\]

where \( \Lambda^x \) are operators on \( \mathcal{R} \). The \( \Lambda^x \) commute, are graded of degree \( -1 \), and transform in the adjoint representation of \( G \).

Proof. We have \( \mu_x \star f = \mu_x f + \frac{1}{2}\{\mu_x, f\}t + \sum_{p = 2}^\infty M^x_p(f)t^p \) where \( M^x_p \) is graded of degree \( -p \). Then \( x \otimes f \mapsto M^x_p(f) \) defines a \( G \)-linear map \( \mathcal{M}_p : g \otimes \mathcal{R} \to \mathcal{R}^{d+1-p} \). We know \( \mathcal{R}_d \simeq g^\otimes d \) – see the proof of Proposition 3.1(ii). An easy fact about representations (from highest weight theory) is that if \( g^\otimes k \) appears \( g \otimes g^\otimes d \) then \( k \) lies in \( \{d+1, d, d-1\} \). So \( M^0_p = 0 \) if \( p \geq 3 \). Thus we get (4.1) where \( \Lambda^x = M^x_2 \).

We have \( (\mu_x \star f) \star \mu_y = \mu_x \star (f \star \mu_y) \). Computing the coefficients of \( t^4 \), we find \( \Lambda^x \Lambda^y(f) = \Lambda^y \Lambda^x(f) \). Computing the coefficients of \( t^5 \), we get the relation \([\eta^x, \Lambda^y] = \Lambda^{[x,y]}\) where \( \eta^x = \{\mu_x, \cdot\} \); so the \( \Lambda^x \) transform in the adjoint representation of \( g \). \( \square \)

Corollary 4.2. (i) The operators \( \Lambda^x \), \( x \in g \), completely determine \( \star \).

(ii) The \( \Lambda^x \) generate a graded commutative subalgebra \( \mathcal{A} \) of \( \text{End} \mathcal{R} \) isomorphic to \( \mathcal{R} \).

Proof. (i) Once we know (4.1), it is easy to compute \( \mu_{x_1} \cdots \mu_{x_k} \star f \) by induction on \( k \). (ii) This is easy, in fact \( \Lambda^{x_1} \cdots \Lambda^{x_k}(f) \) is the coefficient of \( t^{2d} \) in \( \mu_{x_1} \cdots \mu_{x_k} \star f \). Notice that \( \mathcal{A} = \bigoplus_{d=0}^\infty \mathcal{A}^{-d} \) is graded in negative degrees, so that \( \mathcal{A}^{-d} \) corresponds to \( \mathcal{R}^d \). \( \square \)
We have a representation $\pi$ of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{R}$ defined by $\pi^{x,y}(f) = \mu_x \circ f - f \circ \mu_y$.

**Corollary 4.3.** The representation $\pi$ is irreducible and we have

$$\pi^{x,y}(f) = \mu_x \circ f - f \circ \mu_y + \frac{1}{2} \{\mu_x \circ f, f\} + \Lambda^{x-y}(f) \quad (4.2)$$

**Proof.** $\pi$ is equivalent to the natural representation $\Pi$ of $\mathfrak{g} \oplus \mathfrak{g}$ on $\mathcal{U}/J$; indeed $\mathfrak{q}$ is an intertwining map. Proposition 3.3 implies that $\Pi$ is simple (and vice versa). □

**Remark 4.4.** Once we know the $\Lambda^x$, we can construct $J$ directly as the kernel of the algebra homomorphism $\mathcal{U} \to \text{End} \mathcal{R}$ defined by $x \mapsto \pi^{x,0} = \mu_x + \frac{1}{2} \{\mu_x , \cdot \} + \Lambda^x$. This is a noncommutative deformation of the fact that $I$ is the kernel of the algebra homomorphism $S \to \text{End} \mathcal{R}$ defined by $x \mapsto \mu_x$.

The rest of this paper is devoted to computing the operators $\Lambda^x$ when $\mathfrak{g}$ is classical.

5. THE CASE $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$

Suppose $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$. Let $\mathcal{A}$ be the Poisson algebra $\mathbb{C}[z_1, w_1, \ldots, z_n, w_n]$ where $\{z_i, z_j\} = \{w_i, w_j\} = 0$ and $\{z_i, w_j\} = \delta_{ij}$. We have a Poisson algebra grading $\mathcal{A} = \bigoplus_k \mathcal{A}^k$ where $\mathcal{A}^k$ is the space of homogeneous polynomials of total degree $k$. Then $\mathcal{A}^2$ is a Lie subalgebra, and this is a model for $\mathfrak{g}$ (i.e., $\mathcal{A}^2$ is isomorphic to $\mathfrak{g}$). Moreover, $\mathcal{A}^{\text{even}} = \bigoplus_{k=0}^{\infty} \mathcal{A}^{2k}$ is a model for $\mathcal{R}$. The Moyal star product on $\mathcal{A}$ restricts to $\mathcal{A}^{\text{even}}$; in this way we get a Moyal star product on $\mathcal{R}$.

We find a strengthened version of [1, Prop. 6].

**Proposition 5.1.** Let $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$. The Moyal star product is a $G$-equivariant graded star product on $\mathcal{R}$. If $n \geq 2$, then it corresponds to the Joseph ideal; if $n = 1$, then it corresponds to the ideal $J^1$.

The $\Lambda^x$ are order 2 algebraic differential operators and

$$\Lambda^{z_i z_j} = \frac{1}{4} \frac{\partial^2}{\partial w_i \partial w_j}, \quad \Lambda^{w_i w_j} = \frac{1}{4} \frac{\partial^2}{\partial z_i \partial z_j}, \quad \Lambda^{z_i w_j} = -\frac{1}{4} \frac{\partial^2}{\partial w_i \partial z_j} \quad (5.1)$$

6. COMPUTATION OF $\Lambda^x$

We assume from now on that $\mathfrak{g}$ is a classical complex simple Lie algebra different from $\mathfrak{sp}(2n, \mathbb{C})$, $n \geq 1$. This falls into two cases: (I) $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$ where $n \geq 2$, or (II) $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ where $n \geq 6$. It turns out that we can deal with both cases simultaneously by simply by introducing a parameter $\varepsilon$ and setting $\varepsilon = 0$ in (I) or $\varepsilon = 1$ in (II). We set $p_\varepsilon = p + \varepsilon$ and $p_{-\varepsilon} = p - \varepsilon$.

We put $G = SL_n(\mathbb{C})$ in (I) or $G = \text{Spin}_n(\mathbb{C})$ in (II). Notice that there is one coincidence between (I) and (II), namely $\mathfrak{g} = \mathfrak{sl}(4, \mathbb{C}) = \mathfrak{so}(6, \mathbb{C})$.

We define $m$ by $\text{dim} \mathcal{O}_{\text{min}} = 2m + 2$; so $m = n - 2$ in (I) or $m = n - 4$ in (II). Let $X, h, Y$ be a triple in $\mathfrak{g}$ such that $X \in \mathcal{O}_{\text{min}}$ and $[X, Y] = h$, $[h, X] = 2X$, $[h, Y] = -2Y$. Then $h$ is semisimple and $Y \in \mathcal{O}_{\text{min}}$. In this same setting we proved

**Theorem 6.1** ([3]). Let $\mathcal{D}_{4, -1}(\mathcal{O}_{\text{min}})$ denote the space of algebraic differential operators $D$ on $\mathcal{O}_{\text{min}}$ such that $D$ has order at most 4 and $D$ is graded of degree $-1$, i.e., $D(\mathcal{R}^p) \subseteq \mathcal{R}^{p-1}$.
Then $\mathcal{D}_{d-1}(O_{\text{min}})$ contains a unique copy of the adjoint representation of $G$. In other words, there is a non-zero $G$-equivariant complex linear map $g \mapsto \mathcal{D}_{d-1}(O_{\text{min}})$, $x \mapsto D^x$, and this map is unique up to scaling. For $x \neq 0$, $D^x$ has order exactly 4.

We can normalize the map $x \mapsto D^x$ so that, for $p \geq 0$, 

$$D^x(\mu_X^p) = \gamma_p \mu_X^{p-1}$$

(6.1)

where $\gamma_p = p(p + \frac{m-1}{2})p_\varepsilon(p_\varepsilon + \frac{m}{2})$. For $p \geq 1$, $D^x(\mu_X^p) \neq 0$.

Finally, the map $x \mapsto D^x$ extends naturally to an the operators $D^x$ generate a graded commutative subalgebra of $\mathcal{D}(O_{\text{min}})$ which is isomorphic to $R$. Thus for $f \in R$ we have the operator $D^f$, where $D^f D^g = D^f g$ and $D^\mu_x = D^x$.

Proof. This is a summary of the following results in [2]: Theorems 3.2.1 and 3.2.3, Corollary 3.2.4, Propositions 4.2.3 and 4.3.3, and Corollary 3.2.5. □

Remarks 6.2. (i) If $g = sl(4, \mathbb{C}) = so(6, \mathbb{C})$, then we can equally well choose $\varepsilon = 0$ or $\varepsilon = 1$ in computing $\gamma_p$. We of course end up with the same final answer.

(ii) $D^x$ defines an algebraic differential operator on $Cl(O_{\text{min}})$.

Let $E$ be the Euler vector field on $O_{\text{min}}$ so that $E$ operates on $R$ and $R^d$ is its $d$-eigenspace. We put $E' = E + \frac{m+1}{2}$. Notice that $E'$ is diagonalizable on $R$ with positive spectrum, and so $E' + k$ is invertible for any $k \geq 0$.

Theorem 6.3. For $x \in g$ we have $\Lambda^x = -\frac{1}{4E'(E' + 1)}D^x$.

Proof. This occupies §7. □

We found this formula for $\Lambda^x$ because we expected this shape $\Lambda^x = P^{-1} D^x$ where $P$ is a quantization of $4\lambda^2$ and $\lambda$ is the symbol of $E$; see [2, §1].

Remark 6.4. We can fit the case $g = sp(2n, \mathbb{C})$ discussed in §5 into this framework formally by putting $D^x = -4E'(E' + 1)\Lambda^x$ where the $\Lambda^x$ were given in (5.1) and again $E' = E + \frac{m+1}{2}$ for $m = \frac{1}{2} \dim O_{\text{min}} - 1 = n - 1$. Then the formula $D^x(\mu_X^p) = \gamma_p \mu_X^{p-1}$ in Theorem 6.3 still holds if we compute $\gamma_p$ for $\varepsilon = -\frac{1}{2}$. Here we may choose $X = -\frac{1}{2} w^2_1$, $Y = \frac{1}{2} z^2_1$, $h = z_1 w_1$.

7. Proof of Theorem 5.3

Lemma 7.1. We have $\Lambda^x = \phi D^x$ where $\phi$ is a linear operator on $R$ given by scalars $\phi_d$, $d \geq 0$, so that $\phi(f) = \phi_{d-1} f$ if $f \in R^d$. The scalars $\phi_d$ are unique.

Proof. Let $p \geq 1$. We have two $G$-linear maps $g \otimes R^p \rightarrow R^{p-1}$ defined by $\alpha_p(x \otimes f) = \Lambda^x(f)$ and $\beta_p(x \otimes f) = D^x(f)$. These must be proportional because Hom$_G(g \otimes \mathbb{R}^\otimes p, \mathbb{R}^\otimes (p-1))$ is 1-dimensional. We know that $\beta_p$ is non-zero by Theorem 5.1. So there is a unique scalar $\phi_{p-1}$ such that $\alpha_p = \phi_{p-1} \beta_p$. □

At this point, there is no guarantee that $\phi_p$ will be a nice function of $p$, in the sense that $\phi$ is a reasonable function of $E$. But Theorem 5.3 asserts $\phi = -\frac{1}{2} E'(E' + 1)$.

To prove this, we will write down a series of recursion relations for the $\phi_p$. To derive the recursions, we start with the bracket relation $[\pi^x, -x, \pi^y, -y] = \pi^z$ where $z = [x, y]$. 
By (12) we have $\pi^x, y = 2\mu_x + 2\Lambda^x$ and $\pi^z, z = \eta^z$ where $\eta^z = \{\mu_z, \cdot\}$. Since the operators $\Lambda^x$, like the $\mu_x$, commute among themselves, we get

$$[\mu_x, \Lambda^y] + [\Lambda^x, \mu_y] = \frac{1}{4} \eta^{[x,y]} \tag{7.1}$$

We choose $x = X$ and $y = Y$ so that $[x, y] = h$. Writing $\Lambda^x = \phi D^x$ and applying the operator identity (7.1) to a test function $f \in \mathcal{R}^p$, $p \geq 1$, we find

$$\phi_{p-1}(\mu_X D^Y(f) - \phi_{p}D^Y(\mu_X f) + \phi_{p}D^X(\mu_Y f) - \phi_{p-1}(\mu_Y D^X(f) = \frac{1}{4} \eta^h(f). \tag{7.2}$$

The recursions will arise by evaluating this for $f = \mu_s^X \mu_t^Y$, with $s + t = p$.

Before we can write down the recursions, we need some auxiliary computations, provided by the next result. (Unfortunately, (7.1) is not sufficient to determine all $\phi_p$.)

**Lemma 7.2.** For $s, t \geq 0$ we have

$$D^Y(\mu_s^X \mu_t^Y) = \alpha_{s,t} \mu_s^{X-1} \mu_t^Y + \beta_{s,t} \mu_s^{X-2} \mu_t^Y \mu_h^2 \tag{7.3}$$

$$D^X(\mu_s^X \mu_t^Y) = \alpha_{s,t} \mu_s^{X-1} \mu_t^Y + \beta_{s,t} \mu_s^{X-2} \mu_t^Y \mu_h^2 \tag{7.4}$$

where $\alpha_{s,t} = \gamma_s + \frac{1}{4} s t (2s + t + m)$ and $\beta_{s,t} = -\frac{1}{4} (s - 1) st (2s + t + m)$.

**Proof.** We have to go back into our explicit construction of $D^Y$ in [4, §4]. We worked over the Zariski open dense set $\mathcal{O}_{\min}^* = (\mu_y) \neq 0$ in $\mathcal{O}_{\min}$. We constructed $D^Y$ as the quotient $D^Y = \frac{1}{\mu_y} S$ where $S$ is a certain differential operator on $\mathcal{O}_{\min}^*$. More precisely,

$$s = \frac{1}{4} (T - q(\eta^Y)^2)$$

and $T = 2(E + \frac{r}{2} + r)(E + \frac{m}{2} - r)$ and $T$ is an explicit noncommutative polynomial in some vector fields on $\mathcal{O}_{\min}^*$ which annihilates $\mu_Y$. Also $\eta^Y$ annihilates $\mu_Y$.

It follows that for any $g \in R(\mathcal{O}_{\min}^*)$ we have $T(g \mu_s^X) = T(g) \mu_s^X$ and $\eta^Y(g \mu_s^X) = \eta^Y(g) \mu_s^X$.

Now we can compute $D^Y(\mu_s^X \mu_t^Y)$. We have $D^Y = A - B$ where $A = \frac{1}{4} \mu_y T$ and $B = \frac{1}{4} \mu_y q(\eta^Y)^2$. Then we find $D^Y(g \mu_s^X) = D^Y(g) \mu_s^X + B(g) \mu_s^X - B(g \mu_s^X)$. Let $g = \mu_s^X$. Then $D^Y(\mu_s^X) = \gamma_s \mu_s^{X-1}$ by (6.1). Also, since $\eta^Y(\mu_X) = \mu_{[X,X]} = -\mu_h$ and $\eta^Y$ is a vector field we find (as in [2, (67)])

$$(\eta^Y)^2(\mu_s^X) = (-2s \mu_X \mu_Y + s(s - 1) \mu_h^2) \mu_s^{X-2}$$

Using this we find

$$B(\mu_s^X \mu_t^Y) = \frac{1}{4} s q_{s+t} (-2 \mu_X + (s - 1) \mu_Y \mu_h^2) \mu_s^{X-2} \mu_t$$

where $q_p = (p + \frac{m}{2} + r)(p + \frac{m}{2} - r)$. Now we obtain

$$D^Y(\mu_s^X \mu_t^Y) = \gamma_s \mu_s^{X-1} \mu_t^Y - \frac{1}{4} s (q_{s+t} - q_s) \mu_s^{X-2} \mu_t^Y \mu_h^2$$

$$\quad = \alpha_{s,t} \mu_s^{X-1} \mu_t^Y + \beta_{s,t} \mu_s^{X-2} \mu_t^Y \mu_h^2$$

where $\alpha_{s,t} = \gamma_s + \frac{1}{4} s (q_{s+t} - q_s)$ and $\beta_{s,t} = -\frac{1}{4} (s - 1) s (q_{s+t} - q_s)$. This proves (7.3).

We can prove (7.4) by applying a certain automorphism. Let $\chi : SL(2, \mathbb{C}) \rightarrow G$ be the Lie group homomorphism corresponding to the Lie algebra inclusion $s \rightarrow g$ where $s$ is the span of $X, h$, and $Y$. The adjoint action of $\chi (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})$ defines a Lie algebra automorphism $\vartheta$ of $\mathfrak{g}$. Then $\vartheta(X) = -Y$, $\vartheta(Y) = -X$ and $\vartheta(h) = -h$. Clearly $\vartheta$ preserves $\mathcal{O}_{\min}$ and hence induces algebra automorphisms of $R$ and of $D(\mathcal{C}(\mathcal{O}_{\min}))$ which we again call $\vartheta$.

Then $\vartheta(\mu_x) = \mu_{\vartheta(x)}$ and $\vartheta(D^x) = D^{\vartheta(x)}$. Now applying $\vartheta$ to (7.4) we get (7.4). \qed
Remark 7.3. For $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, these calculations become much easier if we use the formulas for $D^x$ found in [13]. But there are no such formulas known when $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$.

Now we can obtain the recursions by plugging $f = \mu^s X^s \mu^t Y^t$, where $p = s + t$, into (7.2). We evaluate using (7.3), (7.4) and the fact $\eta^s(f) = 2(s - t)f$. The result only involves two functions, namely $f$ and $g = \mu^s X^s \mu^t Y^t \mu^2$. We find, for $s, t \geq 0$,

$$\phi_{p-1}[(\alpha_{s,t} - \alpha_{t,s})f + (\beta_{s,t} - \beta_{t,s})g] - \phi_p[(\alpha_{s+1,t} - \alpha_{t+1,s})f + (\beta_{s+1,t} - \beta_{t+1,s})g] = \frac{1}{2}(s - t)f$$

Equating coefficients of $f$ and $g$ we obtain the two recursions

$$\phi_{p-1}(\alpha_{s,t} - \alpha_{t,s}) - \phi_p(\alpha_{s+1,t} - \alpha_{t+1,s}) = \frac{1}{2}(s - t), \tag{7.5}$$

$$\phi_{p-1}(\beta_{s,t} - \beta_{t,s}) - \phi_p(\beta_{s+1,t} - \beta_{t+1,s}) = 0 \tag{7.6}$$

Both recursions are valid for $s, t \geq 1$, since $f$ and $g$ are linearly independent functions on $O_{\text{min}}$. Moreover (7.5) is valid for all $s, t \geq 0$, since $\beta_{i,j} = 0$ if $i = 0, i = 1$ or $j = 0$.

First we consider (7.6). Our formula for $\beta_{s,t}$ in Lemma 7.2 yields

$$\beta_{s,t} - \beta_{t,s} = -\frac{1}{4}st(s - t)(2s + 2t + m - 1)$$

$$\beta_{s+1,t} - \beta_{t+1,s} = -\frac{1}{4}st(s - t)(2s + 2t + m + 3) \tag{7.7}$$

For $p \geq 3$ we can write $p = s + t$ with $s, t \geq 1$ and $s \neq t$. Then (7.6) and (7.7) give

$$\phi_p = \frac{2p + m - 1}{2p + m + 3} \phi_{p-1}, \quad p \geq 3 \tag{7.8}$$

This is a very simple recursion with solution

$$\phi_p = \phi_2 \frac{(m + 5)(m + 7)}{(2p + m + 1)(2p + m + 3)}, \quad p \geq 2 \tag{7.9}$$

Our aim is to prove $\phi = -\frac{1}{4}E^{(E'+1)}$, which amounts to $\phi_p = -\frac{1}{4d_p(d_p+1)}$, $p \geq 0$, where $d_p = p + \frac{m+1}{2}$. So we are pleased that (7.9) gives

$$\phi_p = \frac{\omega}{4d_p(d_p + 1)}, \quad p \geq 2 \tag{7.10}$$

where $\omega$ is the constant $(m + 5)(m + 7)\phi_2$.

To determine $\phi_p$ at $p = 0, 1, 2$, we implement (7.3) for $t = 0$ and $p = s$. Since $\alpha_{p,0} = \gamma_p$ and $\alpha_{0,p} = 0$ we get

$$\phi_{p-1} \gamma_p - \phi_p(\gamma_{p+1} - \alpha_{1,p}) = \frac{1}{2}p, \quad p \geq 1 \tag{7.11}$$

To use this, we observe $\gamma_p = pd_p \nu_p$, $p \geq 0$, where $\nu_p = p (p - \frac{m}{2})$. After a little work, we find $\gamma_{p+1} - \alpha_{1,p} = p(d_p + 1)(\nu_p + 2d_p)$. Now (7.11) gives

$$\phi_p = \frac{\nu_p(d_p - 1)\phi_{p-1} - \frac{1}{2}}{(\nu_p + 2d_p)(d_p + 1)}, \quad p \geq 1 \tag{7.12}$$

If we put $\lambda_p = 4d_p(d_p + 1)\phi_p$ ($p \geq 0$) this simplifies nicely to give

$$\nu_p(\lambda_p - \lambda_{p-1}) = -2d_p(\lambda_p + 1), \quad p \geq 1 \tag{7.13}$$
Plugging in $\lambda_p = \omega$ for $p \geq 2$, we get $\omega = -1$, and then $\lambda_1 = \lambda_0 = -1$. Thus, for all $p \geq 0$, $\lambda_p = -1$ and so $\phi_p = -\frac{1}{4p(d_p+1)}$.

**Remark 7.4.** In this proof, we only used the fact that there exists some $J$ such that $\mathfrak{g} \mathfrak{r} J = I$ and $J = J^r$. But now (see Remark 4.4) we can recover $J$ as the kernel of the algebra homomorphism $U \to \text{End } \mathcal{R}$ defined by $x \mapsto \mu_x + \frac{1}{2} \{ \mu_x, \cdot \} - \frac{1}{4E'(E'+1)} D^x$. This gives a different proof that $J$ is unique.

8. Consequences of Theorem 6.3

We may rescale the complex Killing form $\langle \cdot, \cdot \rangle_\mathfrak{g}$ so that $\langle X, Y \rangle_\mathfrak{g} = \frac{1}{2}$.

**Corollary 8.1.** (i) We have $\Lambda^y(\mu_X^p) = \zeta_p \mu_X^{p-1}$ where $\zeta_p = -\frac{\gamma_p}{(2p+m+1)(2p+m+3)}$

(ii) The map $\Lambda^x : \mathcal{R}^p \to \mathcal{R}^{p-1}$ is non-zero if $p \geq 1$ and $x \neq 0$.

(iii) $\Lambda^x(y) = c \langle x, y \rangle_\mathfrak{g}$ where $c$ is a non-zero scalar; in fact $c = 2\zeta_1$.

**Proof.** (i) is immediate from (6.2). This gives (ii) if $x = Y$ (since $\gamma_p \neq 0$ if $p \geq 1$). Since the $\Lambda^x$ transform in the adjoint representation, we get (iii) for all $x$. Finally (iii) follows because the map $\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, $x \otimes y \mapsto \Lambda^x(\mu_y)$, is $G$-invariant and so must be a multiple of our normalized Killing form (see §3). Then $c$ is non-zero by (ii); choosing $x = Y$ and $y = X$ we find $c = 2\zeta_1$.

**Corollary 8.2.** For $x \neq 0$, $\Lambda^x$ fails to be a differential operator on $\mathcal{O}_{\text{min}}$. In fact, neither factor $E'$ nor $E' + 1$ left divides $D^x$.

**Proof.** Suppose one of $E'$ or $E' + 1$ left divides $D^x$ so that the quotient is a differential operator $A^x$ on $\mathcal{O}_{\text{min}}$. Since $D^x$ has order 4 (Theorem 6.1), $A^x$ has order 3. But then the $A^x$ span a copy of the adjoint representation in $\mathcal{D}_{4,1}(\mathcal{O}_{\text{min}})$ which is different from the copy spanned by the $D^x$. This contradicts uniqueness in Theorem 6.1.

Notice that the corollary implies that $\Lambda^x$ fails to be a differential operator on $\mathcal{R}$ (since otherwise $\Lambda^x$ would be a differential operator on $\text{Cl}(\mathcal{O}_{\text{min}})$).

**Remark 8.3.** Theorem 6.3 suggests that $C_2(\mu_x, \cdot) = \Lambda^x$ is “pseudo-differential” in some sense. This is different in character from the often cited example of “pseudo-differential” star product found [1, §9, page 124] for coadjoint orbits of the Euclidean group $E(2)$.

There Fronsdal obtains a star product where the operator $f \mapsto \mu_x \ast f$ is an infinite series of differential operators $C_k(\mu_x, \cdot) t^k$ with increasing order.

There is a unique (up to scaling) casimirs in $\mathcal{U}_2$, namely $Q = \sum_{i=1}^N x_i^2$, where $\{x_i\}_{i=1}^N$ is a basis of $\mathfrak{g}$ such that $\{ x_i, x_j \}_\mathfrak{g} = \delta_{ij}$. We next compute how $Q$ acts on $\mathcal{U}/J$ with respect to the left multiplication action of $\mathcal{U}$.

**Corollary 8.4.** $Q$ acts on $\mathcal{U}/J$ by the scalar $s = -\frac{(1 + \varepsilon)(m + 2 - 2\varepsilon)}{4(m + 3)} \dim \mathfrak{g}$.

**Proof.** The function $\sum_{i=1}^N \mu_x^2$ is $G$-invariant and so vanishes on $\mathcal{O}_{\text{min}}$. Now (6.1) and Corollary 6.4 give $\sum_{i=1}^N \mu_x \ast \mu_x = \sum_{i=1}^N (\mu_x^2 + c(x_i, x_i)_\mathfrak{g}) = 2\zeta_1 N$. This means (see §3) that $Q - 2\zeta_1 N$ lies in $J$, and so $Q$ acts by $2\zeta_1 N$. 


Remark 8.5. We conjecture that for the 5 exceptional simple Lie algebras, $\Lambda^x$ again has the form $-\frac{1}{4E(E+1)}D^x$ where $D^x$ are some (as yet unknown) order 4 algebraic differential operators on $O_{\text{min}}$.

9. Hermitian inner product on $\mathcal{R}$

We assume that $\mathfrak{g}$ is a complex simple Lie algebra different from $\mathfrak{sl}(2, \mathbb{C})$. Let $U$ be a maximal compact subgroup of $G$. Let $\sigma$ be the corresponding Cartan involution of $\mathfrak{g}$; so $\sigma$ is $\mathbb{C}$-antilinear. Then $\mathfrak{g}^\sharp = \{(x, \sigma(x)) \mid x \in \mathfrak{g}\}$ is a real form of $\mathfrak{g} \oplus \mathfrak{g}$. We have a $U$-invariant $\mathbb{C}$-antilinear algebra involution $f \mapsto \overline{f}$ on $\mathcal{R}$ defined by $\overline{f}(z) = \overline{f}(\sigma(z))$; see [2, §2.3].

Theorem 9.1. The formula

$$ (f|g) = \text{constant term in } f \circ \overline{g} \quad (9.1) $$

defines a $U$-invariant positive definite hermitian inner product on $\mathcal{R}$, with $(1|1) = 1$. In addition $(\cdot|\cdot)$ is $\mathfrak{g}^\sharp$-invariant, i.e., the operators $\pi^x, \sigma(x)$, $(x \in \mathfrak{g})$, are skew-adjoint.

Proof. The pairing $(\cdot|\cdot)$ is clearly sesquilinear and $U$-invariant with $(1|1) = 1$. It follows by $U$-invariance that $\mathcal{R}^j$ is orthogonal to $\mathcal{R}^k$ if $j \neq k$. Now to show $(\cdot|\cdot)$ is hermitian positive definite, it suffices to show that each number $\|\mu^p_Y\|^2 = (\mu^p_Y|\mu^p_Y)$, $p \geq 1$, is positive. We will use Remark 6.4 so that we can treat all cases simultaneously. We may assume now, by rechoosing $(X, h, Y)$ if needed, that $\sigma(Y) = -X$; see [2, §2.3]. Then by Corollary 4.2, the operator $\Lambda^j$ on $\mathcal{R}$, where $\Lambda^j$ is skew-adjoint. Then plainly

$$ (f|g) = \text{constant term in } \Lambda^j(\overline{g}) \quad (9.3) $$

This formula easily implies that the adjoint of (ordinary) left multiplication by $\mu^x$ is $\Lambda^{\sigma(x)}$. Hence the operators $\mu_x - \Lambda^{\sigma(x)}$ are skew-adjoint. But also the operators $\{\mu_{x+\sigma(x)}, \cdot\}$ are skew-adjoint since they correspond to the action of $U$. Thus, using (4.2), we see the operators $\pi^{x,\sigma(x)}$ are all skew-adjoint.

Notice that, since $\pi$ is irreducible by Corollary 4.3, $(\cdot|\cdot)$ is the unique $\mathfrak{g}^\sharp$-invariant hermitian pairing on $\mathcal{R}$ such that $(1|1) = 1$.

Corollary 9.2. The operators $\pi^{x,\sigma(x)}$ on $\mathcal{R}$ exponentiate to give a unitary representation of $G$ on the Hilbert space direct sum $\mathcal{H} = \bigoplus_{d=0}^{\infty} \mathcal{R}^d$. Then $\mathcal{R}$ is the Harish-Chandra module of this unitary representation.

Proof. This follows by a theorem of Harish-Chandra since $\mathcal{R}$ is an admissible $(\mathfrak{g} \oplus \mathfrak{g}, G)$-module where $\mathfrak{g} \oplus \mathfrak{g}$ acts by $\pi$ and $G$ acts corresponding to the operators $\{\mu^x, \cdot\}$. This quantizes $O_{\text{min}}$ in the sense of geometric quantization and the orbit method. We note that the shift from $E$ to $E'$ can be explained by half-forms in the same way as in [1, Prop. 5].
Corollary 9.3. The unitary representation of $G$ on $\mathcal{H}$ admits a reproducing kernel $\mathcal{K}$. Explicitly, $\mathcal{K}$ is the function $\mathcal{K}(x, y)$ on $O_{\text{min}} \times O_{\text{min}}$ given by the hypergeometric function
\[ \mathcal{K} = {}_1F_2 \left( \frac{m+3}{2}, 1+\varepsilon, 1-\varepsilon + \frac{m}{2}, 2T \right) \tag{9.4} \]
where $T(x, y) = -\langle x, \sigma(y) \rangle$. So $\mathcal{K}(x, y)$ is holomorphic in $x$ and anti-holomorphic in $y$. Consequently, $\mathcal{H}$ is a Hilbert space of holomorphic functions on $O_{\text{min}}$.

Proof. Going back to (9.2), we find
\[ \|\mu_p^\nu\|^2 = \frac{p!(1+\varepsilon)_p(1-\varepsilon + \frac{m}{2})_p}{4^p (\frac{m+3}{2})_p} \tag{9.5} \]
where we are using the classical notation $(a)_p = a(a+1)\cdots(a+p-1)$. By definition, $\mathcal{K} = \sum_{i=0}^{\infty} f_i \otimes \overline{f_i}$ where $f_0, f_1, \ldots$ is an orthonormal basis of $\mathcal{R}$ with respect to $(\cdot|\cdot)$. On the other hand, $T = \sum_{i=0}^{N} s_i \otimes \overline{s_i}$ where $s_0, \ldots, s_N$ is an orthonormal basis of $\mathcal{R}^1$ with respect to the hermitian inner product $\langle \mu_x|\mu_y \rangle = -\langle x, \sigma(y) \rangle$. This is positive definite since $\langle \mu_Y|\mu_Y \rangle = \langle Y, X \rangle = \frac{1}{2}$. It follows, as in [4, §8], that

\[ \mathcal{K} = \sum_{p=0}^{\infty} \frac{1}{\|\mu_p^\nu\|^2} \left( \frac{T}{2} \right)^p \]

So (9.5) gives (9.4).

Remark 9.4. In the case $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $\mathcal{H}$ is just the classical Fock space of even holomorphic functions $f(z_1, w_1, \ldots, z_n, w_n)$ with reproducing kernel $\mathcal{K} = \cosh(2\psi)$ where $\psi = \sum_{i=1}^{n} (|z_i|^2 + |w_i|^2)$. Indeed, $T = \frac{1}{2} \psi^2$ and the hypergeometric series collapses to
\[ {}_1F_2 \left( \frac{m+3}{2}, 1; \frac{m+3}{2}, 2T \right) = \cosh(\sqrt{8T}) = \cosh(2\psi) \tag{9.6} \]

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