Quantum phases and dynamics of geometric phase in a quantum spin chain under linear quench

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We study the quantum phases of anisotropic XY spin chain in presence and absence of adiabatic quench. A connection between geometric phase and criticality is established from the dynamical behaviour of the geometric phase for a quench induced quantum phase transition in a quantum spin chain. We predict XX criticality associated with a sequence of non-contractible geometric phases.

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Introduction: In recent times, the quantum phase transition (QPT) has become one of the prime research topics in condensed matter physics both from the theoretical and experimental perspective. QPT is associated with the fundamental changes that occur in the macroscopic nature of the matter at zero temperature due to the variation of an external parameter. Quantum phase transitions are characterized by the drastic change in the ground state properties of the system driven by the quantum fluctuations.

In this letter, we study the quantum phases and quantum phase transition of exactly solvable XY spin chain model. Although the exactly solvable XY model has been well studied [1] and is known to present a very rich structure but still the effect of linear quenching on the quantum phases has not been studied explicitly. The study of quantum phase transition and the nature of criticality through the analysis of the dynamics of geometric phase for linear quenching process is also rare for this model. The XY model exhibits different regions of criticality, like XX criticality, XY criticality and Ising criticality depending on the values of the parameters (anisotropic exchange interaction and magnetic field) of the system. In this letter, we study the nature of criticality explicitly through the dynamics of geometric phase when the system under consideration is under quench induced QPT. We also address the issue on the nature of the geometric phase in the context of XX criticality; it is shown that the XX region of criticality is characterized by the existence of non-contractible geometric phase.

Here we mention very briefly the essence of geometric phase in condensed matter: Geometric phases have been associated with a variety of condensed matter phenomena [2-7] since its inception [8]. Besides, various theoretical investigations, geometric phases have been experimentally tested in various cases, e.g. with photons [9-11], with neutrons [12-13] and with atoms [14]. The generation of a geometric phase (GP) is a witness of a singular point in the energy spectrum that arises in all non-trivial geometric evolutions. In this respect, the connection of geometric phase with quantum phase transition (QPT) has been explored very recently [15-17]. The geometric phase can be used as a tool to probe QPT in many body systems. Since response times typically diverge in the vicinity of the critical point, sweeping through the phase transition with a finite velocity leads to a breakdown of adiabatic condition and generate interesting dynamical (non-equilibrium) effects.

In the case of thermal phase transitions, the Kibble-Zurek (KZ) mechanism [18, 19] explains the formation of defects via rapid cooling. This idea of defect formation in second order phase transition has been extended to zero temperature quantum phase transition (QPT) [20, 21] by studying the spin models under linear quench. We will use this concept in our study.

Quantum Phase Analysis and Effect of Linear Quench: Let us start with the model Hamiltonian

\[
H = \sum_{i=-M}^{M} \left( \frac{1}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1}{2} \sigma_i^y \sigma_{i+1}^y + B(t) \sigma_i^z \right)
\]

(1)

where \(i\) is the site index, \(x, y\), and \(z\) denote components of spin. \(\alpha\) is the anisotropic coupling strength and \(B\) is the linear quench induced magnetic field in the \(z\) direction.

We recast the spinless fermions operators in terms of field operators by the relation

\[
\psi(x) = [e^{ik_F x} \psi_R(x) + e^{-ik_F x} \psi_L(x)]
\]

where \(\psi_R(x)\) and \(\psi_L(x)\) describe the second-quantized fields of right- and left-moving fermions respectively. One can express the fermionic fields in terms of bosonic fields by the relation

\[
\psi_r(x) = \sqrt{\frac{1}{2\pi}} e^{-i (r\phi(x) - \theta(x))},
\]

where \(\phi_r = \theta - \phi\) and \(\phi_L = \theta + \phi\). We finally get the

\[\phi = \frac{\sqrt{1+\alpha}}{\sqrt{2}} e^{-i \alpha x} \psi(x).\]
bosonized Hamiltonian as

\[ H = H_0 + \frac{\alpha}{2\pi^2 a^2} \int dx \cos(2\sqrt{\frac{\pi}{K}} \theta(x)) + B(t) \int dx \partial_x \phi(x). \tag{2} \]

In this derivation, we have used the following expressions for spin operators in terms of the bosonic fields:

\[ S_n^x = [ c_2 \cos(2\sqrt{\pi K} \phi) + (-1)^n c_3 \cos(\sqrt{\pi/K} \theta) , S_n^y = -[ c_2 \cos(2\sqrt{\pi K} \phi) + (-1)^n c_3 \sin(\sqrt{\pi/K} \theta) , S_n^z = \sqrt{\frac{\pi}{K}} \partial_x \phi + (-1)^n c_1 \cos(2\sqrt{\pi K} \phi) ] . K \] is the Luttinger liquid (LL) parameter, \( c_2 \) and \( c_3 \) are the constants.

Now we analyze the quantum phases of the system in the absence of linear quench: The second term of the Hamiltonian, which is the sine-Gordon coupling term is relevant when \( K > 1/2 \). The elementary excitation is gapped (one can estimate the mass gap by this relation \( M = \Lambda(\frac{\pi}{K})^{1/1-K} \), \( \Lambda \) is a cutoff parameter) and the system is in the staggered order phase. On the other hand, for \( K < 1/2 \), the system is in the Luttinger liquid phase. Next we interpret the quantum phases of the system by analyzing the two renormalization group equations. The renormalization group (RG) equation for the Hamiltonian (Eq. 2) is

\[ \frac{dK}{dl} = (2 - 1/K)\alpha, \quad \frac{d\theta}{dl} = \alpha^2/4. \tag{3} \]

We interpret from the first RG equation that sine-Gordon coupling with strength \( \alpha \) will be relevant and the system will flow from the weak coupling phase to the strong coupling phase, when \( K > 1/2 \) otherwise this coupling term is irrelevant and the system is in the LL phase of the system. The second RG equation reveals that as \( \alpha \) will increase, the LL parameter will increase, i.e., the flow of the second RG equation also support the flow of the first RG equation for its relevant phase, i.e., the system is in the staggered order phase. Results derived from Abelian bosonization method is consistent with the RG study because these RG equations have only trivial fixed point.

Now we analyze the effect of linear quenching (last term of Eq. 1), it modifies the quantum phases of the system. In the static limit for \( K < 1/2 \), the system is in the LL phase. The system drives to the ferromagnetic phase due to the presence of linear quench induced magnetic field in the \( z \) direction.

In the static limit, sine-Gordon coupling term is relevant \( (K > 1/2) \), the system has a excitation gap (stagger phase). The system drives to the ferromagnetic phase when the quench induced magnetic field is larger than the excitation gap of the system. The system is in the LL phase when the quench induce magnetic field is of the order of magnitude of the gap \( (B(t) \sim M = \Lambda(\frac{\pi}{K})^{1/1-K}) \). At time \( t = 0 \), system is in the staggered order magnetic phase.

**Geometric Phase and Criticality:** In this model, the geometric phase of the ground state is evaluated by applying a rotation of \( \phi \) around the \( Z \)-axis in a closed circuit to each spin \( \int d\phi \) [5] [22]. A new set of Hamiltonians \( H_\phi \) is constructed from the Hamiltonian (1) as

\[ H_\phi = U(\phi) H U^+(\phi) \tag{4} \]

where \( U(\phi) = \prod_{j=-M}^{M} \exp(i\sigma_j z/2) \) and \( \sigma_j^z \) is the \( z \) component of the standard Pauli matrix at site \( j \). The family of Hamiltonians generated by varying \( \phi \) has the same energy spectrum as the initial Hamiltonian and \( H(\phi) \) is \( \pi \)-periodic in \( \phi \). With the help of standard Jordan-Wigner transformations, which makes the spins to one dimensional spinless fermions via the relation \( a_j = (\prod_{i<j} \sigma_i^z) \sigma_j^\dagger \) and then using the Fourier transforms of the fermionic operator, \( d_k = \sqrt{\frac{\pi}{N}} \sum_{j} a_j \exp(-\frac{2\pi ikj}{N}) \) the Hamiltonian \( H_\phi \) can be diagonalized by transforming the fermionic operators in momentum space and then using Bogoliubov transformation. The ground state \( |g > \) of the system is expressed as

\[ |g > = \prod_{k>0} \frac{\theta_k}{\sqrt{1/k^2 + \sin^2 \theta_k}} |0 >_k |0 >_{-k} -ie^{2i\phi} \sin \frac{\theta_k}{2} |1 >_k |1 >_{-k} \tag{5} \]

where \( |0 >_k \) and \( |1 >_k \) are the vacuum and single fermionic excitation of the \( k \)-th momentum mode respectively. The angle \( \theta_k \) is given by

\[ \cos \theta_k = \frac{\cos k - B}{\Lambda_k} \tag{6} \]

and \( \Lambda_k = \sqrt{(\cos k - B)^2 + \alpha^2 \sin^2 k} \) is the energy gap above the ground state. The ground state is a direct product of \( N \) spins, each lying in the two-dimensional Hilbert space spanned by \( |0 >_k, |0 >_{-k} \) and \( |1 >_k, |1 >_{-k} \). For each value of \( k \), the state in each of the two dimensional Hilbert space can be represented as a Bloch vector with coordinates \((2\phi, \theta_k)\). The overall phase is given by the sum of the individual phases. One can also write the above Hamiltonian as single particle excitations \( H(\alpha, B(t), \phi) = \sum_{k=-M}^{M} \Lambda_k b_k^\dagger b_k \), where \( b_k = \cos(\theta_k/2) d_k -ie^{2i\phi}\sin(\theta_k/2) d_k^\dagger \). The pseudomomenta \( k \) take half integer values: \( k = \pm \frac{1}{2}, \pm \frac{3}{2}, ..., \pm \frac{N-1}{2} \). The direct calculation shows that the geometric phase for the \( k \)th mode, which represents the area in the parameter space enclosed by the loop determined by \((2\phi, \theta_k)\) is given by

\[ \Gamma_k = \pi(1 - \cos \theta_k) \tag{7} \]

The geometric phase of the state \(|g > \) is given by \( \Gamma_\phi = \sum_k \Gamma_k \). For an adiabatic evolution, if the initial state is an eigenstate, the evolved state remains in the eigenstate. So we may now derive the instantaneous geometric
phases of this system due to a gradually decreasing magnetic field. Let us explore the situation when the system (1) is driven adiabatically (slow transition) by a time dependent magnetic field \( B(t) \) such that

\[
B(t < 0) = -\frac{t}{\tau_q}
\]

(8)

\( B(t) \), driving the transition, is assumed to be linear with an adjustable time parameter \( \tau_q \) (\( 1/\tau_q \) is the quenching rate). Let the system be initially at time \( t(<0) \ll \tau_q \) such that \( B(t) \gg 1 \). The instantaneous ground state at any instant \( t \) is given by

\[
|\psi_0(t)\rangle = \prod_k (\cos \frac{\theta_k(t)}{2} |0_k \rangle + e^{i2\phi} \sin \frac{\theta_k(t)}{2} |1_k \rangle)
\]

(9)

We now use eqns. (7) and (6) to derive the geometric phase of the \( k^{th} \) mode which yields

\[
\Gamma_k(t) = \pi \left( 1 - \frac{\cos k + \frac{t}{\tau_q}}{\sqrt{\cos k + \frac{t^2}{\tau_q^2} + \alpha^2 \sin^2 k}} \right)
\]

(10)

The geometric phase for an isotropic system with \( \alpha = 0 \) and quantum Ising model with \( \alpha = 1 \) may now be easily obtained.

For \( \alpha = 0 \), \( \Gamma_k(t) = 2\pi \Theta(|t| - \tau_q) \) and

for \( \alpha = 1 \), \( \Gamma_k(t) = \pi \left( 1 - \frac{\cos k + \frac{t}{\tau_q}}{\sqrt{1 + \frac{t^2}{\tau_q^2} + 1 \cos k}} \right)
\]

(11)

For a system of size \( N \) the total geometric phase for the initial state is, \( \Gamma_{\text{initial}} = \sum_k \Gamma_k \). The magnetic field is gradually decreased by adjusting \( \tau_q \) and the critical point is attained at \( t = -\tau_q \) with \( B = 1 \). Then the geometric phase for the \( k^{th} \) mode is

\[
\Gamma_k(t = -\tau_q) = \pi \left( 1 - \frac{\cos k + 1}{\sqrt{\cos k + 1 + \alpha^2 \sin^2 k}} \right)
\]

(12)

At the critical point the total geometric phase is \( \Gamma_{\text{critical}} = \sum_k \Gamma_k(t = -\tau_q) \). Finally, at \( t = 0 \), when the magnetic field is gradually turned off, the situation is a bit different. The configuration of the final state will depend on the number of kinks generated in the system due to phase transition at or near \( t = -\tau_q \) and as such it will depend on the quench time \( \tau_q \) [10].

The number of kinks is the number of quasi-particles excited at \( B = 0 \) and is given by \( N = \sum_k p_k \) where \( p_k \), the excitation probability (for the slow transition) is given by the Landau Zener formula [23] \( p_k \approx \exp (-2\pi \tau_q k^2) \). As different pairs of quasi-particles \((k,-k)\) evolve independently, for large values of \( \tau_q \), it is likely that only one pair of quasi-particles with momenta \((k_0,-k_0)\) will be excited where \( k_0(=\pi/n) \) corresponds to the minimum value of the energy \( \Delta_k \). Thus the condition for adiabatic transition in a finite chain is given by, \( \tau_q >> \frac{\pi^2}{4\tau_q^2} \). Hence, well in the adiabatic regime, the final state at \( t = 0 \) is given by

\[
|\psi_{\text{final}}\rangle = |1_k \rangle \sum_{k,k\neq \pm k_0} \left( \cos \frac{\theta_k}{2} |0_k \rangle - e^{i2\phi} \sin \frac{\theta_k}{2} |1_k \rangle \right)
\]

This state is similar to direct product of only \( N-1 \) spins oriented along \((2\phi,\theta)\) where the state of the spin corresponding to momentum \( k_0 \) does not contribute to the geometric phase. The total geometric phase of this state is given by \( \Gamma_{\text{final}}(t = 0) = \sum_{k,k\neq \pm k_0} \pi (1 - \cos \theta_k) \). Now we study the geometric phase associated with the quench induced quantum phase transition. We study the variation of geometric phase \( (\Gamma_k) \) and its derivative with respect to the quench induced magnetic field \( (B) \) i.e. \( \frac{d\Gamma_k}{d\tau} \) with time. We find the non-analytic behavior of the derivative at \( t = \tau_q \). The analytical expression for the derivative is

\[
\frac{d\Gamma_k}{dB} = \frac{\pi \alpha^2 \sin^2 k}{((\cos k + t/\tau_q)^2 + \alpha^2 \sin^2 k)^{3/2}}
\]

(13)

In Fig. 1 we study the Berry phase \( \Gamma_k \) for different quench time \( \tau_q \), for the XX and XY spin chain system. For the XX (inset) model, we find the step function like behaviour at \( t = \tau_q \) where the sharp transition occurs. It can be seen that the system shows quantum criticality at that point. The sharp transition disappears for the finite anisotropy (XY) model. We observe from our study that the variation of \( \Gamma_k \) become flat for slower quenching rate. The upper panel of Fig. 2 shows the variation of \( \Gamma_k \) with \( \alpha \) and \( t \) to get the whole picture of variation of \( \Gamma_k \). It reveals from our study that sharp transition occurs for \( \alpha = 0 \) only. The lower panel of the Fig. 2, shows the total variation of \( \frac{d\Gamma_k}{dB} \). The non-analytical behaviour for \( \alpha = 0 \) at \( t = -\tau_q \) helps us to predict XX criticality under the linear quenching process. The analysis with different values of \( \tau_q \) shows that the appearance of XX criticality
is independent of $\tau_q$, i.e., independent of fast and slow quenching rate.

Finally, we discuss an important aspect of the XX criticality. It is shown that the XX region of criticality is characterized by the existence of non-contractible geometric phase. The phase is in the gapless excitations in the XX region of criticality (the gapless excitation is defined as $b_k^\dagger$ operator). Here $\cos(\frac{4\pi k\alpha}{N}) = B(t)(\Lambda_{k_0})$, $k_0 \in \{1, \ldots, M\}$. For finite $M$, we can write $\lim_{\alpha \to 0} \cos(\theta_k) = \pm 1$ and ground state $|g(\alpha, B(t), \phi) > = \otimes_{k < k_0} |0 >_k \otimes_{k > k_0} |1 >_k |1 >_{-k}$. The ground state of the system acquires the geometric phase due to the variation of $\phi$ for fixed $\alpha$ and $B(t)$.

$|g >$ is the tensor product of $M$ qubit states. We observe from the Fig. 1 and also from the analytical expression for $\Gamma_k$ that $\lim_{\alpha \to 0} \Gamma_k = 0, 2\pi$. In the thermodynamic limit there is a solution, $\cos\theta_{k_0} = 0$, which leads to the result $\Gamma_k = \pi$ for every $\alpha > 0$ and hence $\lim_{\alpha \to 0} \Gamma_{k_0} = \pi$. Therefore by using the relation of total geometric phase, we can write $\lim_{\alpha \to 0} \lim_{M \to \infty} \frac{\Gamma_k}{M} \neq 0$ which shows the sequence $\Gamma_k(\alpha, n)\to N$ is non-contractible in the thermodynamic limit. Hence, the non-contractible nature of the geometric phase associated with the XX criticality is proved.

To conclude, we have studied the various quantum phases of the XY spin model and also the effect of linear quench on these quantum phases. The dynamics of the associated geometric phases are studied for slow and rapid quenching and an intimate connection between geometric phase and quantum criticality is established. We have predicted XX criticality and also non-contractible geometric phase at criticality.

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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Color online, the left plan of this figure is for the geometric phase and right one is for the derivative of the geometric phase to study the quantum criticality. We present the variation of these quantities with anisotropy parameter and time.}
\end{figure}

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