Stability Analysis and Numerical Simulation of a Diffusive Prey-Predator Holling Type II Model

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Abstract. The prey predator interaction is fundamental and important process in a population dynamic. In this paper, a diffusive prey predator holling type II model considering limitation of the prey growth is presented. The model consists two distinct populations. Model is a nonlinear system of partial differential equations which is a initial boundary value problem. The behaviour solution of the model was analyzed by analyzing stability of the critical point. In solving this system, we use Homogeneous Neumann boundary conditions. Numerical solution was determined by using finite difference method. The results show that the diffusive model illustrates a spreading population over a limited area.

1. Introduction
The interaction of prey and predator is an important and fundamental process in a population dynamics. The mathematical model of prey predator is one of the mathematical models that is often studied. The prey predator model was introduced by the American biophysicist Alfred Lotka and the Italian mathematician Vito Volterra in the 1920s. The model states that the prey population grow at a certain rate and decrease because of predators. On the other hand, the predator population decrease at a certain rate and increase by prey consuming [6,7]. The prey population grow exponentially if there are no predators in a system. It is not possible because there is limitation of food and carrying capacity. Therefore, a function limiting the growth of the prey population is considered [5].

Furthermore, in the 1975s Holling and Tanner added different functional response to the prey predator model. The functional response is used to describe the density of the prey population. The functional response depends on many factors i.e the number of prey and predator, the carrying capacity, the saturation rate of predator, and competition between predators [2,3]. In ecological theory, the interaction between prey and predator is not only depend on time, but also depend on spatial. Diffusion theory is used to explain this phenomena. The diffusive prey predator model was studied [1,4,9,10]. The diffusive prey predator model is not easy by analytical solving. Numerical methods can be used to obtain approximate solutions of the model. One of the numerical methods used is finite difference method [8]. In this paper, the diffusive prey predator Holling type II model by limiting of prey population growth is studied. The behaviour solution of the model was analyzed by analyzing stability of the critical point. Furthermore, the analytical solution is confirmed by a numerical solution. The numerical solution is simulated by finite difference method. The results obtained can provide an overview of the spatial effect on the behavior of the prey predator system.
2. Mathematical Model

Prey predator Holling type II model lead to the following system of equations

\[
\begin{align*}
\frac{du}{dt} &= au \left( 1 - \frac{u}{K} \right) - \frac{buw}{1 + mu}, \\
\frac{dw}{dt} &= -dy + \frac{f_{uw}}{1 + mu},
\end{align*}
\]

where \( u = u(t) \) is prey population at time \( t \), \( w = w(t) \) is predator population at time \( t \), \( a \) is rate of prey growth, \( d \) is rate of predator death, \( b \) and \( f \) are rate of prey predator interaction, \( K \) is the capacity of the ecosystem, and \( m \) is saturation rate of predator.

The model (1) reformulated by adding the diffusive term. The model becomes

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_1 \frac{\partial^2 u}{\partial x^2} + au \left( 1 - \frac{u}{K} \right) - \frac{buw}{1 + mu}, \\
\frac{\partial w}{\partial t} &= D_2 \frac{\partial^2 w}{\partial x^2} - dw + \frac{f_{uw}}{1 + mu},
\end{align*}
\]

where \( D_1 \) and \( D_2 \) are diffusive constant, \( u = u(t) \) is prey population at position \( x \) and time \( t \), and \( w = w(t) \) is predator population at position \( x \) and time \( t \). The system of equation (2) is bounded with homogeneous Neumann boundary conditions

\[
\begin{align*}
u_x(0, t) &= u_x(L, t) = 0, \\
\nu_x(0, t) &= w_x(L, t) = 0,
\end{align*}
\]

and initial condition

\[
\begin{align*}
u(x, 0) &= 1.8e^{-(x-a)^2}, \\
w(x, 0) &= 0.3e^{-(x-0.7)^2}.
\end{align*}
\]

The homogeneous Neumann boundary condition is used because it is assumed that the system is closed and no migration across bound.

3. Stability Analysis of Diffusive Prey Predator Holling Type II Model

Dynamic behavior of the diffusive model (2) is analyzed by traveling wave solution.

Suppose \( u(x, t) = u(s) \) and \( w(x, t) = w(s) \), where \( s = x + ct \), then by the chain rule is obtained

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial s} \frac{\partial s}{\partial t} = cu', \\
\frac{\partial w}{\partial t} &= \frac{\partial w}{\partial s} \frac{\partial s}{\partial t} = cw',
\end{align*}
\]

\[
\begin{align*}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial s} \right) = u'', \\
\frac{\partial^2 w}{\partial x^2} &= \frac{\partial}{\partial x} \left( \frac{\partial w}{\partial s} \right) = w''.
\end{align*}
\]

Let \( u' = v \) and \( w' = z \) then \( u'' = v' \) and \( w'' = z' \), the model becomes

\[
\begin{align*}
v' &= cv - au \left( 1 - \frac{u}{K} \right) + \frac{buw}{1 + mu}, \\
w' &= cz, \\
z' &= cz + dw - \frac{f_{uw}}{1 + mu}.
\end{align*}
\]

The critical point of system (3) can be determined by solving \( u' = v' = w' = z' = 0 \). There are three critical points of system (3), \( E_1 = (0, 0, 0, 0) \), \( E_2 = (K, 0, 0, 0) \), and \( E_3 = \left( \frac{d}{f - dm}, 0, \frac{af(Kf - Kdm - d)}{bk(f - dm)}, 0 \right) \). Jacobian matrix of system (3) is
Jacobian matrix at critical point $E_1 = (0,0,0,0)$ is

$$J_{E_1} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-a & c & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & d & c
\end{bmatrix}. $$

Eigen values of Jacobian matrix $J_{E_1}$ are

$$\lambda_1 = \frac{c + \sqrt{c^2 - 4a}}{2},$$

$$\lambda_2 = \frac{c - \sqrt{c^2 - 4a}}{2},$$

$$\lambda_3 = \frac{c + \sqrt{c^2 + 4d}}{2},$$

$$\lambda_4 = \frac{c - \sqrt{c^2 + 4d}}{2}.$$ 

Jacobian matrix at critical point $E_2 = (K,0,0,0)$ is

$$J_{E_2} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
a & c & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & d & c
\end{bmatrix}. $$

Eigen values of Jacobian matrix $J_{E_2}$ are

$$\lambda_1 = \frac{c + \sqrt{c^2 - 4a}}{2},$$

$$\lambda_2 = \frac{c - \sqrt{c^2 + 4a}}{2},$$

$$\lambda_3 = \frac{c(Km + 1) + \sqrt{(K^2m^2 + 2Km + 1)(c^2 + 4d) - 4K(Kfm - f)} }{2(Km + 1)},$$

$$\lambda_4 = \frac{c(Km + 1) - \sqrt{(K^2m^2 + 2Km + 1)(c^2 + 4d) - 4K(Kfm - f)} }{2(Km + 1)}.$$ 

Jacobian matrix at critical point $E_3 = \left(\frac{d}{f - dm}, 0, \frac{a(Kfm - Km + d)}{bK(f - dm)^2}, 0\right)$ is

$$J_{E_3} = \begin{bmatrix}
0 & 1 & 0 & \frac{bf}{Kf (dm - f)} \\
-a\frac{(Kfm - Km + dm + f)}{Kf(dm - f)} & c & 0 & \frac{db}{f} \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & c
\end{bmatrix}. $$

Eigen values of Jacobian matrix $J_{E_3}$ are

$$\lambda_1 = \frac{1}{2A}\left(A + \sqrt{A^2 - 4B}\right),$$

$$\lambda_2 = \frac{1}{2A}\left(-A - \sqrt{A^2 - 4B}\right),$$

$$\lambda_3 = \frac{1}{2A}\left(-A + \sqrt{A^2 - 4B}\right),$$

$$\lambda_4 = \frac{1}{2A}\left(A - \sqrt{A^2 - 4B}\right).$$
\[ \lambda_2 = \frac{1}{2A} \left( cA + \sqrt{-A \left( B + 2\sqrt{ad(C + D + E)} \right)} \right), \]
\[ \lambda_3 = \frac{1}{2A} \left( cA + \sqrt{-A \left( B + 2\sqrt{ad(C + D + E)} \right)} \right), \]
\[ \lambda_1 = \frac{1}{2A} \left( cA - \sqrt{-A \left( B + 2\sqrt{ad(C + D + E)} \right)} \right). \]

where
\[ A = fK(dm - f), \]
\[ B = 2Kad^2m^2 - Kc^2dfm - 2Kadfm + Kc^2f^2 + 2amd^2 + 2adf, \]
\[ C = K^2ad^3m^4 - 2K^2ad^2fm^3 + 4K^2d^3fm^3 + K^2adf^2m^2, \]
\[ D = -12K^2f^2m^2 + 2Kad^3m^3 + 12K^2df^3m + 4Kdf^3m^2 - 4Kf^4, \]
\[ E = -2Kad^2m - 8Kad^2f^2m + ad^3m^2 + 4Kdf^3 + 2adf^2m + adf^2. \]

4. Finite Difference Method

Forward difference approximations for \( u_t \) and \( w_t \) are as follows:
\[ u_t = \frac{u_{i,j+1} - u_{i,j}}{\Delta t}, \]
\[ w_t = \frac{w_{i,j+1} - w_{i,j}}{\Delta t}. \]

Central different approximations for \( u_{xx} \) and \( w_{xx} \) are as follows:
\[ u_{xx} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}, \]
\[ w_{xx} = \frac{w_{i+1,j} - 2w_{i,j} + w_{i-1,j}}{(\Delta x)^2}. \]

Substitute the above approximation to the system (2), it is obtained
\[ u_{i,j+1} = \epsilon \left( u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \right) + u_{i,j} \left( 1 + a\Delta t - a\Delta t \frac{u_{i,j}}{K} - \frac{b\Delta tw_{i,j}}{1 + mu_{i,j}} \right), \]
\[ w_{i,j+1} = \epsilon \left( w_{i+1,j} - 2w_{i,j} + w_{i-1,j} \right) + w_{i,j} \left( 1 - d\Delta t - \frac{f\Delta tu_{i,j}}{1 + mu_{i,j}} \right), \]
where \( \epsilon = \frac{\Delta t}{(\Delta x)^2} \). Initial conditions for system (4) are
\[ u_{i,0} = u(i\Delta x, 0) = 1.8e^{-(i\Delta x-0.2)^2}, \]
\[ w_{i,0} = w(i\Delta x, 0) = 0.3e^{-(i\Delta x-0.7)^2}, \]
with boundary conditions for each \( j = 1, 2, \ldots \) are
\[ u_{1,j} = u_{n,j}, u_{i+1,j} = u_{i-1,j}, \]
\[ w_{1,j} = w_{n,j}, w_{i+1,j} = w_{i-1,j}. \]

5. Numerical Simulations

Let \( a = 1.1, b = 0.6, d = 0.7, f = 1.2, K = 1, D_1 = 1, D_2 = 1, m = 0.1 \) and \( c = 0.25 \) be the parameters will be used. By using these parameters, it can be concluded that the critical point \( E_1 \) is unstable, the critical point \( E_2 \) is unstable, and the critical point \( E_3 \) is unstable. Let the initial values \( v = 0 \) and \( z = 0 \), a phase portrait for the system (3) is shown as in Figure 1.
By using the given parameters and let $\Delta x = \frac{2}{10}$ and $\Delta t = \frac{1}{100}$, it is obtained values of $u_{i,j+1}$ and $w_{i,j+1}$ for $x \in [0,20]$ and $t \in [0,100]$. Figure 2 and 3 show solution of the system (2), Figure 4 and 5 show profile of prey and predator in spatial coordinate.

![Phase portrait of the system (3) at $v = 0$ and $z = 0$.](image)

**Figure 1.** Phase portrait of the system (3) at $v = 0$ and $z = 0$.

![Solution of the system (2) for prey.](image)

**Figure 2.** Solution of the system (2) for prey.
Figure 3. Solution of the system (2) for predator.

Figure 4. Profile of the system (2) for prey and predator at $t = 0$.

Figure 5. Profile of the system (2) for prey and predator at $t = 10$. 
6. Conclusion
In this paper, the stability of the diffusive prey predator holling type II model was discussed by limiting growth in the prey population. System (2) has three critical points, $E_1 = (0,0,0,0)$, $E_2 = (K,0,0,0)$, and $E_3 = \left( \frac{d}{f-dm}, 0, \frac{a(f-Kdm-d)}{bk(f-dm)^2}, 0 \right)$. Stability of the prey predator system depends on parameter of eigenvalues and diffusive term. The numerical solution for the system (2) is obtained using the finite difference method. Graph of the solution and profile are shown in the form of a three-dimensional figure.

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