THE QUASI-INV ARIANCE PROPERTY FOR THE
GAMMA KERNEL DETERMINANTAL MEASURE

GRIGORI OLSHANSKI

Abstract. The Gamma kernel is a projection kernel of the form
\( (A(x)B(y) - B(x)A(y))/(x - y) \), where \( A \) and \( B \) are certain func-
tions on the one-dimensional lattice expressed through Euler’s \( \Gamma \-
function. The Gamma kernel depends on two continuous param-
eters; its principal minors serve as the correlation functions of a
determinantal probability measure \( P \) defined on the space of in-
finite point configurations on the lattice. As was shown earlier
(Borodin and Olshanski, Advances in Math. 194 (2005), 141-202;
arXiv:math-ph/0305043), \( P \) describes the asymptotics of certain
ensembles of random partitions in a limit regime.

Theorem: The determinantal measure \( P \) is quasi-invariant with
respect to finitary permutations of the nodes of the lattice.
This result is motivated by an application to a model of infinite
particle stochastic dynamics.

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INTRODUCTION

0.1. Preliminaries: a general problem. Recall a few well-known notions from measure theory. Let \( \mathfrak{A} \) be a Borel space (that is, a set with a distinguished sigma-algebra of subsets). Two Borel measures \( P_1, P_2 \) on \( \mathfrak{A} \) are said to be equivalent if \( P_1 \) has a density with respect to \( P_2 \) and vice versa. They are said to disjoint or mutually singular if there exist disjoint Borel subsets \( B_1 \) and \( B_2 \) such that \( P_1 \) is supported by \( B_1 \) and \( P_2 \) is supported by \( B_2 \) (that is, \( P_1(\mathfrak{A} \setminus B_1) = P_2(\mathfrak{A} \setminus B_2) = 0 \)). Assume \( G \) is a group acting on \( \mathfrak{A} \) by Borel transformations; then a Borel measure \( P \) is said to be \( G \)-quasi-invariant if \( P \) is equivalent to its transform by any element \( g \in G \).

In practice, especially for measures living on “large” spaces, verifying the property of equivalence, disjointness or quasi-invariance, and explicit computation of densities (Radon–Nikodým derivatives) for equivalent measures can be a nontrivial task. There exist nice general results for particular classes of measures: infinite product measures (Kakutani’s theorem [Ka]), Gaussian measures on infinite-dimensional spaces (Feldman–Hajek’s theorem and related results, see [Kuo, Ch. II]), Poisson measures (see [Bro]).

Assume that \( \mathfrak{X} \) is a locally compact space, take as the “large” space \( \mathfrak{A} \) the space \( \text{Conf}(\mathfrak{X}) \) of locally finite point configurations on \( \mathfrak{X} \), and assume that the measures under consideration are probability measures on \( \mathfrak{A} = \text{Conf}(\mathfrak{X}) \); they are also called point processes on \( \mathfrak{X} \) (for fundamentals of point processes, see, e.g., [Le]). Poisson measures are just the simplest yet important example of point processes. The next by complexity example is the class of determinantal measures (processes). Determinantal measures are specified by their correlation kernels which
are functions $K(x, y)$ on $\mathfrak{X} \times \mathfrak{X}$. Note an analogy with covariation kernels of Gaussian measures which are also functions in two variables. Note also that, informally, Poisson measures can be viewed as a degenerate case of determinantal measures corresponding to kernels $K(x, y)$ concentrated on the diagonal $x = y$.

Many concrete examples of determinantal measures are furnished by random matrix theory and other sources, see, e.g., the surveys [So] and [Bor]. The interest to determinantal measures especially increased in the last years. However, to the best of my knowledge, the following problem was never discussed in the literature:

**Problem 1.** Assume we are given two determinantal measures, $P_1$ and $P_2$ on a common space $\text{Conf}(\mathfrak{X})$. How to test their equivalence (or, on the contrary, disjointness)? Is it possible to decide this by inspection of the respective correlation kernels $K_1(x, y)$ and $K_2(x, y)$?

One could imagine that equivalence $P_1 \sim P_2$ holds if the kernels are close to each other in an appropriate sense. However, there is a subtlety here, see Subsection 1.6 below.

Let $G$ be a group of homeomorphisms $g : \mathfrak{X} \to \mathfrak{X}$. Then $G$ also acts, in a natural way, on the space $\text{Conf}(\mathfrak{X})$ and hence on the space of probability measures on $\text{Conf}(\mathfrak{X})$. Observe that the latter action preserves the determinantal property: If $P$ is a determinantal measure on $\text{Conf}(\mathfrak{X})$ with correlation kernel $K(x, y)$, then the transformed measure $g(P)$ is determinantal, too, and $K(g^{-1}x, g^{-1}y)$ serves as its correlation kernel. Thus, the question of $G$-quasi-invariance of $P$ becomes a special instance of Problem 1.

**Problem 2.** Let $P$ and $G$ be as above. How to test whether $P$ is $G$-quasi-invariant? Is it possible to decide this by comparing the correlation kernels $K(x, y)$ and $K(g^{-1}x, g^{-1}y)$ for $g \in G$?

I think it would be interesting to develop general methods for solving Problem 2 and more general Problem 1. They seem to be nontrivial even in the case when $\mathfrak{X}$ is a countably infinite set with discrete topology.

**0.2. The Gamma kernel measure.** In the present paper we are dealing with a concrete model of determinantal measures, introduced in [BO2]. The space $\mathfrak{X}$ is assumed to be discrete and countable; it is convenient to identify it with the lattice $\mathbb{Z}' := \mathbb{Z} + \frac{1}{2}$ of half-integers. Then the space $\text{Conf}(\mathfrak{X}) = \text{Conf}(\mathbb{Z}')$ is simply the space of all subsets of $\mathbb{Z}'$. We consider a two-parameter family of kernels on $\mathbb{Z}' \times \mathbb{Z}'$. Following [BO2], we denote them as $K_{z, z'}(x, y)$; here $z$ and $z'$ are some
continuous parameters, and \( x, y \) are the arguments, which range over \( \mathbb{Z}' \). Each kernel is real-valued and symmetric. Moreover, it is a projection kernel meaning that it corresponds to a projection operator in the Hilbert space \( \ell^2(\mathbb{Z}') \). Like many examples of kernels from random matrix theory, our kernels can be written in the so-called integrable form \([\text{IIKS}], [\text{De}]\)

\[
\frac{A(x)B(y) - B(x)A(y)}{x - y},
\]

resembling Christoffel–Darboux kernels associated to orthogonal polynomials. In our situation \( A \) and \( B \) are certain functions on the lattice \( \mathbb{Z}' \), which are expressed through Euler’s \( \Gamma \)-function. For this reason we call \( K_{z,z'}(x,y) \) the \textit{Gamma kernel}. In \([\text{BO2}]\) we conjectured that the Gamma kernel might be a universal microscopic limit of the Christoffel–Darboux kernels for generic discrete orthogonal polynomials, in an appropriate asymptotic regime.

The Gamma kernel serves as the correlation kernel for a determinantal measure on \( \text{Conf}(\mathbb{Z}') \), called the \textit{Gamma kernel measure} and denoted as \( P_{z,z'} \). According to the general definition of determinantal measures (see \([\text{So}], [\text{Bor}]\)), the measure \( P_{z,z'} \) is characterized by its correlation functions

\[
\rho_n(x_1,\ldots,x_n) := P_{z,z'}\{X \in \text{Conf}(\mathbb{Z}') \mid X \ni x_1,\ldots,x_n\}
\]

which in turn are equal to principal \( n \times n \) minors of the kernel:

\[
\rho(x_1,\ldots,x_n) := \det[K_{z,z'}(x_i,x_j)]_{i,j=1}^n.
\]

Here \( n = 1, 2, \ldots \) and \( x_1, \ldots, x_n \) is an arbitrary \( n \)-tuple \( x_1, \ldots, x_n \) of pairwise distinct points from \( \mathbb{Z}' \).

As shown in \([\text{BO2}]\), the Gamma kernel measure arises from several models of representation–theoretic origin, through certain limit transitions.

A more detailed information about \( K_{z,z'}(x,y) \) and \( P_{z,z'} \) is given in Section 1 below, see also \([\text{BO2}], [\text{Ol2}]\).

0.3. The main result. We take as \( G \) the group \( \mathcal{S} \) of permutations of the set \( \mathbb{Z}' \) fixing all but finitely many points. Such permutations are said to be \textit{finitary}. Clearly, \( \mathcal{S} \) is a countable group. It is generated by the elementary transpositions \( \sigma_n \) of the lattice \( \mathbb{Z}' \). Here \( n \in \mathbb{Z} \) and \( \sigma_n \) transposes the points \( n - \frac{1}{2} \) and \( n + \frac{1}{2} \) of \( \mathbb{Z}' \). Each permutation \( \sigma \in \mathcal{S} \) induces, in a natural fashion, a transformation of the space \( \text{Conf}(\mathbb{Z}') \), which in turn results in a transformation \( P \mapsto \sigma(P) \) of probability measures on \( \text{Conf}(\mathbb{Z}') \).
The main result of the paper says that the Gamma kernel measure is quasi-invariant with respect to the action of the group $\mathcal{S}$:

**Main Theorem.** For any $\sigma \in \mathcal{S}$, the measures $P_{z,z'}$ and $\sigma(P_{z,z'})$ are equivalent. Moreover, the Radon–Nikodým derivative $\sigma(P_{z,z'})/P_{z,z'}$ can be explicitly computed.

This result gives a solution to the first question of Problem 2 in a concrete situation. As will be shown in another paper, the quasi-invariance property established in the theorem makes it possible to construct an equilibrium Markov process on $\text{Conf}(\mathbb{Z}')$ with determinantal dynamical correlation functions and equilibrium distribution $P_{z,z'}$. This application is one of the motivations of the present work.

It seems plausible that $P_{z,z'}$ is not quasi-invariant with respect to the transformations of $\text{Conf}(\mathbb{Z}')$ generated by the translations of the lattice. Note that the translation $x \mapsto x + k$ with $k \in \mathbb{Z}$ amounts to the shift $(z, z') \mapsto (z - k, z' - k)$ of the parameters (see Theorem 1.4). One can ask, more generally, whether any two Gamma kernel measures with distinct parameters are disjoint.

### 0.4. Scheme of proof of Main Theorem

The proof relies on the fact that for fixed $z, z'$, the measure $P_{z,z'}$ can be approximated by simpler measures which are $\mathcal{S}$–quasiinvariant and whose Radon–Nikodým derivatives (with respect to the action of the group $\mathcal{S}$) are readily computable.

The approximating measures depend on an additional parameter $\xi \in (0, 1)$ and are denoted as $P_{z,z',\xi}$. These are purely atomic probability measures supported by a single $\mathcal{S}$–orbit. They come from certain probability distributions on Young diagrams, and are called the $z$–measures (Kerov–Olshanski–Vershik [KOV], Borodin–Olshanski [BO1]). As $\xi$ goes to 1, the measures $P_{z,z',\xi}$ weakly converge to $P_{z,z'}$: this is simply the initial definition of $P_{z,z'}$ given in [BO2].

What we actually need to prove is that the convergence of the measures holds not only in the weak topology (that is, on bounded continuous test functions) but also in a much stronger sense: Namely,

$$\langle F, P_{z,z',\xi} \rangle \underset{\xi \to 1}{\longrightarrow} \langle F, P_{z,z'} \rangle$$

for certain test functions $F$ which, like the Radon–Nikodým derivatives, may be unbounded and not everywhere defined. Here and in the

1 A connection between quasi-invariance and existence of Markov dynamics, sometimes in hidden form, is present in various situations. See, e.g., [AKR], [SY].

2 The pair $(z, z')$ should be viewed as an unordered pair of parameters, because the transposition $z \leftrightarrow z'$ does not affect the measure, see Theorem 1.4.
sequel the angular brackets denote the pairing between functions and measures.

To explain this point more precisely we need some preparation.
First of all, it is convenient to transform all the measures in question by means of an involutive homeomorphism of the compact space $\text{Conf}(Z')$. This homeomorphism, denoted as “inv”, assigns to a configuration $X \in \text{Conf}(Z')$ its symmetric difference with the set $Z'_- = \{ \ldots, -\frac{3}{2}, -\frac{1}{2} \}$.

An equivalent description is the following. Regard $X$ as a configuration of charged particles occupying some of the sites of the lattice $Z'$, while the holes (that is, the unoccupied sites of $Z'$) are interpreted as anti–particles with opposite charge. Now, the new configuration $\text{inv}(X)$ is formed by the particles sitting to the right of 0 and the anti–particles to the left of 0. We call “inv” the particle/hole involution on $Z'_-$. For instance, if $X = Z'_-$ then $\text{inv}(X) = \emptyset$, the empty configuration. The configuration $X = Z'_-$ plays a distinguished role because the $\mathfrak{S}$–orbit of this configuration is the support of the pre–limit measures $P_{z,z',\xi}$. The map “inv” transforms this distinguished orbit into the set of all finite balanced configurations, that is, finite configurations with equally many points to the right and to the left of 0.

Note that the transform by “inv” leaves intact the action of all the elementary transpositions $\sigma_n$ with $n \neq 0$, only the action of $\sigma_0$ is perturbed.

Note also that if $P_\Sigma$ is a determinantal measure on $\text{Conf}(Z')$ then so is its push–forward $P := \text{inv}(P_\Sigma)$, and there is a simple relation between the correlation kernels of $P_\Sigma$ and $P$ ([BOU, Appendix]). If the kernel of $P_\Sigma$ is symmetric then that of $P$ has a different kind of symmetry: it is symmetric with respect to an indefinite inner product (see [BO1, Proposition 2.3 and Remark 2.4]), which reflects the presence of two kinds of particles.

Instead of the measures $P_{z,z',\xi}$ and $P_{z,z'}$ we will deal with their transforms by “inv”, denoted as $P_{z,z',\xi} := \text{inv}(P_{z,z',\xi})$ and $P_{z,z'} := \text{inv}(P_{z,z'})$. Clearly, the transform does not affect the formulation of the theorem, only the initial action of the group $\mathfrak{S}$ on $\text{Conf}(Z')$ has to be conjugated by the involution: an element $\sigma \in \mathfrak{S}$ now acts as the transformation

$$\tilde{\sigma} := \text{inv} \circ \sigma \circ \text{inv}. \quad (0.1)$$

An advantage of the transformed measures as compared to the initial ones is that the pre–limit measures $P_{z,z',\xi}$ live on finite configurations. In a weaker form, this property is inherited by the limit measures.
Namely, let us say that a configuration $X \in \text{Conf}(\mathbb{Z}')$ is sparse if
\[ \sum_{x \in X} |x|^{-1} < \infty. \]

Denote the set of all sparse configurations as $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$. There is a natural embedding $\text{Conf}_{\text{sparse}}(\mathbb{Z}') \hookrightarrow \ell^1(\mathbb{Z}')$ assigning to a sparse configuration $X$ its characteristic function multiplied by the function $|x|^{-1}$, and we equip $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$ with the “$\ell^1$–topology”, that is, the one induced by the norm of the Banach space $\ell^1(\mathbb{Z}')$. The $\ell^1$–topology is finer than the topology induced from the ambient space $\text{Conf}(\mathbb{Z}')$.

Now we are in a position to describe the scheme of proof.

**Claim 1.** The limit measures $P_{z,z'}$ are concentrated on the set of sparse configurations.

The claim makes sense because the set $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$ is a Borel subset in $\text{Conf}(\mathbb{Z}')$.

Given a function $f$ on the lattice $\mathbb{Z}'$ such that $|f(x)| = O(|x|^{-1})$, we define a function $\Phi_f(X)$ on the set of sparse configurations by the formula
\[ \Phi_f(X) = \prod_{x \in X} (1 + f(x)) \quad (0.2) \]
(the product is convergent). Such functions $\Phi_f$ will be called *multiplicative functionals* on configurations. Any multiplicative functional $\Phi_f$ is continuous in the $\ell^1$–topology.

Given a permutation $\sigma \in \mathfrak{S}$ and a measure $P$ on $\text{Conf}(\mathbb{Z}')$, we denote by $\tilde{\sigma}(P)$ the push–forward of $P$ under the transformation $\tilde{\sigma}$, see (0.1).

Let $z, z'$ be fixed and $\xi$ range over $(0, 1)$. For any $\sigma \in \mathfrak{S}$, let $\mu_{z,z',\xi}(\sigma, X)$ be the Radon–Nikodym derivative of the measure $\tilde{\sigma}(P_{z,z',\xi})$ with respect to the measure $P_{z,z',\xi}$. That is,
\[ \mu_{z,z',\xi}(\sigma, X) = \frac{\tilde{\sigma}(P_{z,z',\xi})(X)}{P_{z,z',\xi}(X)} = \frac{P_{z,z',\xi}(\tilde{\sigma}^{-1}(X))}{P_{z,z',\xi}(X)}; \]
here $X$ belongs to the countable set of finite balanced configurations.

**Claim 2.** Fix an arbitrary $\sigma \in \mathfrak{S}$.

(i) The function $\mu_{z,z',\xi}(\sigma, X)$ has a unique extension to a continuous function on $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$.

(ii) As $\xi \to 1$, the extended functions obtained in this way converge pointwise to a continuous function $\mu_{z,z'}(\sigma, X)$ on $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$.

(iii) The limit function $\mu_{z,z'}(\sigma, X)$ can be written as a finite linear combination of multiplicative functionals of the form (0.2).
Here continuity is assumed with respect to the $\ell^1$–topology. Actually, a somewhat stronger claim holds, see Proposition 3.1 and the subsequent discussion.

Claim 2 suggests that the limit function $\mu_{z,z'}(\sigma, X)$ might serve as the Radon–Nikodym derivative for the limit measure, that is,

$$\tilde{\sigma}(P_{z,z'}) = \mu_{z,z'}(\sigma, \cdot) P_{z,z'}.$$ 

This relation is indeed true. We reduce it to the following claim.

**Claim 3.** Let $f$ be an arbitrary function on $\mathbb{Z}'$ such that $|f(x)| = O(|x|^{-1})$. Then the multiplicative functional $\Phi_f$ given by (0.2) is absolutely integrable with respect to both the pre–limit and limit measures, and we have

$$\lim_{\xi \to 1} \langle \Phi_f, P_{z,z',\xi} \rangle = \langle \Phi_f, P_{z,z'} \rangle.$$ 

This claim is stronger than the assertion about the weak convergence of measures $P_{z,z',\xi} \to P_{z,z'}$ which was known previously. Indeed, weak convergence of measures on $\text{Conf}(\mathbb{Z}')$ means convergence on continuous test functions, while multiplicative functionals are, generally speaking, unbounded functions on $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$ and thus cannot be extended to continuous functions on the compact space $\text{Conf}(\mathbb{Z}')$.

To prove Claim 3 we use the well–known fact that the expectation of a multiplicative functional with respect to a determinantal measure can be expressed as a Fredholm determinant involving the correlation kernel. This makes it possible to reformulate the claim in terms of the correlation operators $K_{z,z',\xi}$ and $K_{z,z'}$ (these are operators in the Hilbert space $\ell^2(\mathbb{Z}')$ whose matrices are the correlation kernels of the measures $P_{z,z',\xi}$ and $P_{z,z'}$, respectively).

The reformulation is given in Claim 4 below. Represent the Hilbert space $H := \ell^2(\mathbb{Z}')$ as the direct sum of two subspaces $H_{\pm} = \ell^2(\mathbb{Z}'_{\pm})$ according to the splitting $\mathbb{Z}' = \mathbb{Z}'_+ \sqcup \mathbb{Z}'_-$ (positive and negative half–integers). Then any bounded operator in $H$ can be written as a $2 \times 2$ matrix with operator entries (or “blocks”). Let $L_{1/2}(H)$ denote the set (actually, algebra) of bounded operators in $H$ whose two diagonal blocks are trace class operators and two off–diagonal blocks are Hilbert–Schmidt operators. If $K \in L_{1/2}(H)$ then the Fredholm determinant $\det(1 + K)$ makes sense ([BOO, Appendix]). We equip $L_{1/2}(H)$ with the combined topology determined by the trace class norm on the diagonal blocks and the Hilbert–Schmidt norm on the off–diagonal ones.

**Claim 4.** Let $A$ stand for the operator of pointwise multiplication by the function $x \mapsto |x|^{-1/2}$ in the space $H$. The operator $AK_{z,z'}A$ lies
in $L_{1/2}(H)$, and, as $\xi$ goes to $1$, the operators $AK_{z,z',\xi}A$ approach the operator $AK_{z,z'}A$ in the combined topology of $L_{1/2}(H)$.

Note that the operator $A^2$ is not in $L_{1/2}(H)$, because the series $\sum |x|^{-1}$ taken over all $x \in Z'$ is divergent. This is the source of difficulties. For instance, the assertion that $AK_{z,z'}A$ belongs to $L_{1/2}(H)$ is not a formal consequence of the boundedness of $K_{z,z'}$.

Claim 4 is the key technical result of the paper. The proof relies on explicit expressions for the correlation kernels in terms of contour integrals and requires a considerable computational work.

I do not know whether the operators $AK_{z,z',\xi}A$ approach $AK_{z,z'}A$ simply in the trace class norm. The point is that the diagonal blocks are Hermitian nonnegative operators while the operators themselves are not. For nonnegative operators, one can use the fact that the convergence in the trace class norm is equivalent to the weak convergence together with the convergence of traces. For non–Hermitian operators, dealing with the trace class norm is difficult, while the Hilbert–Schmidt norm turns out to be much easier to handle. Fortunately, for the off–diagonal blocks, the convergence in the Hilbert–Schmidt norm already suffices.

0.5. **Organization of the paper.** Section 1 contains the basic notation and definitions related to the measures under consideration and their correlation kernels. Section 2 starts with basic facts related to multiplicative functionals and their connection to Fredholm determinants; then the proof of Claim 1 follows; it is readily derived from the explicit expression for the first correlation function of the measure $P_{z,z'}$. Section 3 is devoted to the proof of Claim 2. In Section 4, we formulate the main result (Theorem 4.1). Then we reduce to it to Theorem 4.2 and next to Theorem 4.3; they correspond to Claims 3 and 4, respectively. The main technical work is done in Sections 5 and 6, where we prove Theorem 4.3 (or Claim 4) separately for diagonal and off–diagonal blocks.

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1. **Z–measures and related objects**

1.1. **Partitions and lattice point configurations.** A partition is an infinite sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of nonnegative integers $\lambda_i$ such that $\lambda_i \geq \lambda_{i+1}$ and only finitely many $\lambda_i$’s are nonzero. We set $|\lambda| = \sum \lambda_i$. 
Let $X$ denote the set of all partitions; it is a countable set. Following Ma, we identify partitions and Young diagrams.

Let $Z'$ denote the set of all half-integers; that is, $Z' = \mathbb{Z} + \frac{1}{2}$. By $Z'_+$ and $Z'_-$ we denote the subsets of positive and negative half-integers, so that $Z'$ is the disjoint union of $Z'_+$ and $Z'_-$.

Subsets of $Z'$ are viewed as configurations of particles occupying the nodes of the lattice $Z'$. The unoccupied nodes are called holes. Let $\text{Conf}(Z')$ denote the space of all particle configurations on $Z'$. The space $\text{Conf}(Z')$ can be identified with the infinite product space $\{0, 1\}^{Z'}$ and we equip it with the product topology. In this topology, $\text{Conf}(Z')$ is a totally disconnected compact space.

Recall (see Subsection 0.4) that the particle/hole involution on $Z'_-$ is the involutive map $\text{Conf}(Z') \to \text{Conf}(Z')$ keeping intact particles and holes on $Z'_+ \subset Z'$ and changing particles by holes and vice versa on $Z'_- \subset Z'$. We denote the particle/hole involution by the symbol “inv”. In a more formal description, “inv” assigns to a configuration its symmetric difference with $Z'_-$. In particular, $\text{inv}(Z'_-) = \emptyset$.

To a partition $\lambda \in Y$ we assign the semi-infinite point configuration $X(\lambda) = \{\lambda_i - i + \frac{1}{2}\}_{i=1,2,\ldots} \in \text{Conf}(Z')$. Note that among $\lambda_i$’s some terms may repeat while the numbers $\lambda_i - i + \frac{1}{2}$ are all pairwise distinct. Clearly, the correspondence $\lambda \mapsto X(\lambda)$ is one-to-one. The configuration $X(\lambda)$ is sometimes called the Maya diagram of $\lambda$, see Miwa–Jimbo–Date [MJD].

For instance, the Maya diagram of the zero partition $\lambda = (0, 0, \ldots)$ is $Z'_-$. Any Maya diagram can be obtained from this one by finitely many elementary moves consisting in shifting one particle to the neighboring position on the right provided that it is unoccupied.

A finite configuration $X \subset Z'$ is called balanced if $|X \cap Z'_+| = |X \cap Z'_-|$. An important fact is that “inv” establishes a bijective correspondence between the Maya diagrams $X(\lambda)$ and the balanced configurations. We set

$$X(\lambda) = \text{inv}(X(\lambda)), \quad X_\pm(\lambda) = X(\lambda) \cap Z'_\pm.$$

An alternative interpretation of the balanced configuration $X(\lambda)$ is as follows: $X(\lambda) = X_+(\lambda) \cup X_-(\lambda)$ with

$$X_+(\lambda) = \{p_1 < \cdots < p_d\}, \quad X_-(\lambda) = \{-q_d < \cdots < -q_1\}$$

(recall that $|X_+(\lambda)| = |X_-(\lambda)|$), where the positive half-integers $p_i$ and $q_i$ are the modified Frobenius coordinates (Vershik–Kerov [VK]) of the Young diagram $\lambda$. They differ from the conventional Frobenius coordinates [Ma] by the additional summand $\frac{1}{2}$. A slight divergence
with the conventional notation is that we arrange the coordinates in the ascending order.

A direct explanation: \( d \) is the number of diagonal boxes in \( \lambda \) and
\[
(\lambda_1 - 1 + \frac{1}{2}, \lambda_2 - 2 + \frac{1}{2}, \ldots, \lambda_d - d + \frac{1}{2}) = (p_d, p_{d-1}, \ldots, p_1)
\]
\[
(\lambda'_1 - 1 + \frac{1}{2}, \lambda'_2 - 2 + \frac{1}{2}, \ldots, \lambda'_d - d + \frac{1}{2}) = (q_d, q_{d-1}, \ldots, q_1),
\]
where \( \lambda' \) is the transposed diagram.

Thus, we have defined two embeddings of the countable set \( Y \) into the space \( \text{Conf}(Z') \), namely, \( \lambda \mapsto X(\lambda) \) and \( \lambda \mapsto X(\lambda) \). These two embeddings are related to each other by the particle/hole involution on \( Z' \).

Note that each of the two embeddings maps \( Y \) onto a dense subset in \( \text{Conf}(Z') \).

1.2. \( Z \)-measures on partitions. Here we introduce a family \( \{M_{z,z',\xi}\} \) of probability measures on \( Y \), called the \( z \)-measures. The subscripts \( z \), \( z' \), and \( \xi \) are continuous parameters. Their range is as follows: parameter \( \xi \) belongs to the open unit interval \((0, 1)\), and parameters \( z \) and \( z' \) should be such that \( (z + k)(z' + k) > 0 \) for any integer \( k \). Detailed examination of this condition shows that either \( z \in \mathbb{C} \setminus \mathbb{R} \) and \( z' = \bar{z} \) (the principal series of values), or both \( z \) and \( z' \) are real numbers contained in an open interval \((N, N + 1)\) with \( N \in \mathbb{Z} \) (the complementary series of values).

We shall need the generalized Pochhammer symbol \((x)_\lambda\):
\[
(x)_\lambda = \prod_{i=1}^{\ell(\lambda)} (x - i + 1)_{\lambda_i}, \quad x \in \mathbb{C}, \quad \lambda \in Y,
\]
where \( \ell(\lambda) \) is the number of nonzero coordinates \( \lambda_i \) and
\[
(x)_k = x(x + 1) \ldots (x + k - 1) = \frac{\Gamma(x + k)}{\Gamma(x)}
\]
is the conventional Pochhammer symbol. Note that
\[
(x)_\lambda = \prod_{(i,j) \in \lambda} (x + j - i),
\]
where the product is taken over the boxes \((i,j)\) of the Young diagram \( \lambda \), and \( i \) and \( j \) stand for the row and column numbers of a box.

In this notation, the weight of \( \lambda \in Y \) assigned by the \( z \)-measure \( M_{z,z',\xi} \) is written as
\[
M_{z,z',\xi}(\lambda) = (1 - \xi)^{zz'} \xi^{\lambda|\lambda|} |\lambda(\lambda')_\lambda \left( \frac{\dim \lambda}{|\lambda|!} \right) |^2,
\]
(1.1)
where \( \dim \lambda \) is the dimension of the irreducible representation of the symmetric group of degree \( |\lambda| \) indexed by \( \lambda \).

Note that \( z \) and \( z' \) enter the formula symmetrically, so that their interchange does not affect the \( z \)-measure.

For the origin of formula (1.1) and the proof that \( M_{z,z',\xi} \) is indeed a probability measure, see Borodin–Olshanski [BO1], [BO2], [BO3] and references therein. Note that all the weights are strictly positive: this follows from the conditions imposed on parameters \( z \) and \( z' \). The \( z \)-measures form a deformation of the poissonized Plancherel measure and are a special case of Schur measures (see Okounkov [Ok]).

1.3. Limit measures. Throughout the paper the parameters \( z \) and \( z' \) are assumed to be fixed. If the third parameter \( \xi \) approaches 0, then the \( z \)-measures \( M_{z,z',\xi} \) converge to the Dirac measure at the zero partition: this is caused by the factor \( \xi^{|\lambda|} \).

A much more interesting picture arises as \( \xi \) approaches 1. Then the factor \((1 - \xi)^{zz'}\) forces each of the weights \( M_{z,z',\xi}(\lambda) \) to tend to 0 (note that \( zz' > 0 \)). This means that the \( z \)-measures on the discrete space \( \mathbb{Y} \) escape to infinity. However, the situation changes when we embed \( \mathbb{Y} \) into \( \text{Conf}(\mathbb{Z}') \). Recall that we have two embeddings, one producing semi-infinite configurations \( X(\lambda) \) and the other producing finite balanced configurations \( X(\lambda) \). Denote by \( P_{z,z',\xi} \) and \( P_{z,z',\xi} \) the push–forwards of the \( z \)-measure \( M_{z,z',\xi} \) under these two embeddings. Then the following result holds, see [BO2]:

**Theorem 1.1.** In the space of probability measures on the compact space \( \text{Conf}(\mathbb{Z}') \), there exist weak limits

\[
P_{z,z'} = \lim_{\xi \to 1} P_{z,z',\xi}, \quad P_{z,z'} = \lim_{\xi \to 1} P_{z,z',\xi}.
\]

Of course, \( P_{z,z',\xi} \) and \( P_{z,z',\xi} \) are transformed to each other under the particle/hole involution on \( \mathbb{Z}' \), and the same holds for the limit measures.

1.4. Projection correlation kernels. All the measures appearing in Theorem 1.1 are determinantal measures. Here we explain the structure of their correlation kernels (for a detailed exposition, see [BO2], [BO3], [BO4], and [Ol2]).

The key object is a second order difference operator \( D_{z,z',\xi} \) on the lattice \( \mathbb{Z}' \). This operator acts on a test function \( f(x), x \in \mathbb{Z}' \), according
to
\[ D_{z,z',\xi} f(x) = \sqrt{\xi(z + x + \frac{1}{2})(z' + x + \frac{1}{2})} f(x + 1) \]
\[ + \sqrt{\xi(z + x - \frac{1}{2})(z' + x - \frac{1}{2})} f(x - 1) \]
\[ - [x + \xi(z + z' + x)] f(x). \]
Since \( x \pm \frac{1}{2} \) is an integer for \( x \in \mathbb{Z}' \), the expressions under the square root are strictly positive, due to the conditions imposed on the parameters \( z \) and \( z' \).

As shown in [BO4], \( D_{z,z',\xi} \) determines an unbounded selfadjoint operator in the Hilbert space \( H = \ell^2(\mathbb{Z}') \). This operator has simple, purely discrete spectrum filling the subset \((1 - \xi)\mathbb{Z}' \subset \mathbb{R}\).

In the sequel we will freely pass from bounded operators in \( H \) to their kernels and vice versa using the natural orthonormal basis \( \{e_x\} \) in \( H \) indexed by points \( x \in \mathbb{Z}' \): If \( A \) is an operator in \( H \) then its kernel (or simply matrix) is defined as \( A(x,y) = (A e_y, e_x) \).

Let \( K_{z,z',\xi} \) denote the projection in \( H \) onto the positive part of the spectrum of \( D_{z,z',\xi} \), and let \( K_{z,z',\xi}(x,y) \) denote the corresponding kernel. (Here and below all projection operators are assumed to be orthogonal projections.)

**Theorem 1.2.** \( K_{z,z',\xi}(x,y) \) is the correlation kernel of the measure \( P_{z,z',\xi} \).

The operator corresponding to a correlation kernel of a determinantal measure will be called its correlation operator. Thus, the projection \( K_{z,z',\xi} \) is the correlation operator of \( P_{z,z',\xi} \).

Let \( D_{z,z'} \) denote the difference operator on \( \mathbb{Z}' \) which is obtained by setting \( \xi = 1 \) in the above formula defining \( D_{z,z',\xi} \). One can show that \( D_{z,z'} \) still determines a selfadjoint operator in \( \ell^2(\mathbb{Z}') \). Its spectrum is simple, purely continuous, filling the whole real line. Let \( K_{z,z'} \) denote the projection onto the positive part of the spectrum.

**Theorem 1.3.** (i) As \( \xi \) goes to 1, the projection operators \( K_{z,z',\xi} \) weakly converge to a projection operator \( K_{z,z'} \).

(ii) \( K_{z,z'} \) serves as the correlation operator of the limit measure \( P_{z,z'} \), that is, the kernel \( K_{z,z'}(x,y) \) is the correlation kernel of \( P_{z,z'} \).

Note that the weak convergence of operators in \( \ell^2(\mathbb{Z}') \) whose norms are uniformly bounded is the same as the pointwise convergence of the corresponding kernels. Note also that on the set of projections, the weak operator topology coincides with the strong operator topology.

The above definition of the operators \( K_{z,z',\xi} \) and \( K_{z,z'} \) through the difference operators \( D_{z,z',\xi} \) and \( D_{z,z'} \) is nice and useful but one often
needs explicit expressions for the correlation kernels. Various such expressions are available:

- Presentation in the integrable form \([BO1, BO2]\):
  \[
  \frac{A(x)B(y) - B(x)A(y)}{x - y}.
  \]

- Series expansion (or integral representation) involving eigenfunctions of the difference operators \([BO3, BO4]\).

- Double contour integral representation \([BO3, BO4]\).

In Sections 5 and 6 we will work with contour integrals. Theorems 1.4 and 1.5 below describe the integrable form for the limit kernel \(K_{z,z'}(x,y)\). This presentation will be used in Section 2.

**Theorem 1.4.** Assume \(z \neq z'\). For \(x, y \in \mathbb{Z}'\) and outside the diagonal \(x = y\),

\[
K_{z,z'}(x,y) = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \cdot \frac{P(x)Q(y) - Q(x)P(y)}{x - y},
\]

where

\[
P(x) = \frac{\Gamma(z + x + \frac{1}{2})}{\sqrt{\Gamma(z + x + \frac{1}{2})\Gamma(z' + x + \frac{1}{2})}},
\]

\[
Q(x) = \frac{\Gamma(z' + x + \frac{1}{2})}{\sqrt{\Gamma(z + x + \frac{1}{2})\Gamma(z' + x + \frac{1}{2})}}
\]

and \(\Gamma(\cdot)\) is Euler’s \(\Gamma\)-function.

On the diagonal \(x = y\),

\[
K_{z,z'}(x,x) = \frac{\sin(\pi z) \sin(\pi z')}{\pi \sin(\pi (z - z'))} \left(\psi(z + x + \frac{1}{2}) - \psi(z' + x + \frac{1}{2})\right),
\]

where \(\psi(x) = \Gamma'(x)/\Gamma(x)\) is the logarithmic derivative of the \(\Gamma\)-function.

See \([BO2]\) for a proof. In that paper, we called the kernel \(K_{z,z'}(x,y)\) the Gamma kernel.

In the case \(z = z'\) (then necessarily \(z \in \mathbb{R} \setminus \mathbb{Z}\)) an explicit expression can be obtained by taking the limit \(z' \to z\) (see \([BO2]\)), and the result is expressed through the \(\psi\) function (outside the diagonal) or its derivative \(\psi'\) (on the diagonal):

**Theorem 1.5.** Assume \(z = z' \in \mathbb{R} \setminus \mathbb{Z}\). For \(x, y \in \mathbb{Z}'\) and outside the diagonal \(x = y\),

\[
K_{z,z'}(x,y) = \left(\frac{\sin(\pi z)}{\pi}\right)^2 \frac{\psi(z + x + \frac{1}{2}) - \psi(z + y + \frac{1}{2})}{x - y}.
\]
On the diagonal \( x = y \),

\[
K_{z,z'}(x,x) = \left( \frac{\sin(\pi z)}{\pi} \right)^2 \psi'(z + x + \frac{1}{2}).
\]

Here is a simple corollary of the above formulas, which we will need later on:

**Corollary 1.6.** Let \( \rho^{(z,z')}_1(x) \) denote the density function of \( P_{z,z'} \). We have

\[
\rho^{(z,z')}_1(x) \sim \frac{C(z,z')}{|x|}, \quad |x| \to \infty,
\]

where

\[
C(z,z') = \begin{cases} 
\frac{\sin(\pi z) \sin(\pi z')(z-z')}{\pi \sin(\pi (z-z'))}, & z \neq z' \\
\left( \frac{\sin(\pi z)}{\pi} \right)^2, & z = z' \in \mathbb{R} \setminus \mathbb{Z}.
\end{cases}
\]

**Proof.** Recall that \( P_{z,z'} \) is related to \( P_{z,z'} \) by the particle/hole involution transformation on \( \mathbb{Z}_+ \). It follows that the density functions of the both measures coincide on \( \mathbb{Z}_+ \). By the very definition of determinantal measures, the density function of \( P_{z,z'} \) is given by the values of the correlation kernel on the diagonal \( x = y \). The formulas of Theorem 1.4 and Theorem 1.5 express \( K_{z,z'}(x,x) \) through the psi–function and its derivative. The asymptotic expansion of \( \psi(y) \) as \( y \to +\infty \) is given by formula 1.18(7) in Erdelyi [Er], which implies

\[
\psi(y) = \log y - (2y)^{-1} + O(y^{-2}), \quad \psi'(y) = y^{-1} + O(y^{-2}) \quad (y \to +\infty).
\]

Using this we readily get

\[
\rho^{(z,z')}_1(x) = \frac{C(z,z')}{x} + O(x^{-2}), \quad x \to +\infty,
\]

To handle the case \( x \to -\infty \) one can use the relation (see (1.2))

\[
K_{z,z'}(x,x) = 1 - K_{z,z'}(x,x), \quad x \in \mathbb{Z}_-,
\]

and then employ the identity ([Er, 1.7.1])

\[
\psi(y + \frac{1}{2}) - \psi(-y + \frac{1}{2}) = \pi \tan(\pi y).
\]

A simpler way is to use the symmetry property of \( P_{z,z'} \) discussed in Subsection 1.7 below. It immediately gives

\[
\rho^{(z,z')}_1(-x) = \rho^{(-z,-z')}_1(x), \quad x \in \mathbb{Z}_+.
\]

Since \( C(-z,-z') = C(z,z') \), we get the desired formula. \( \square \)
Remark 1.7. As is seen from Theorems 1.4 and 1.5, the limit kernel $K_{z,z'}(x,y)$ is real-valued. The same is true for the pre-limit kernels $K_{z,z',\xi}(x,y)$: this can be seen from their integrable form presentation or from the series expansion. The fact that the kernels are real-valued will be employed in Section 6.

1.5. $J$–Symmetric kernels and block decomposition. For technical reasons, it will be more convenient for us to deal, instead of $K_{z,z'}(x,y)$ and $K_{z,z',\xi}(x,y)$, with the correlation kernels for the measures $P_{z,z',\xi}$ and $P_{z,z'}$. The latter kernels will be denoted as $K_{z,z',\xi}(x,y)$ and $K_{z,z'}(x,y)$, respectively. The link between two kinds of kernels, the “$K$ kernels” and the “$\bar{K}$ kernels”, is given by the following relation (see [BOO, Appendix] for a proof):

$$
\varepsilon(x) K(x,y) \varepsilon(y) = \begin{cases} 
K(x,y), & x \in \mathbb{Z}'_+ \\
\delta_{xy} - K(x,y), & x \in \mathbb{Z}'_-
\end{cases},
$$

(1.2)

where

$$
\varepsilon(x) = \begin{cases} 
1, & x \in \mathbb{Z}'_+ \\
(-1)^{|x| - \frac{1}{2}}, & x \in \mathbb{Z}'_-
\end{cases}.
$$

Note that the factor $\varepsilon(x) = \pm 1$ does not affect the correlation functions (see Subsection 1.6). This factor becomes important in the limit regime considered in [BO1] and [BO3, §8], but for the purpose of the present paper, it is inessential and could be omitted; I wrote it only to keep the notation consistent with that of the previous papers [BO1], [BO2], [BO3].

Decompose the Hilbert space $H = \ell^2(\mathbb{Z}')$ into the direct sum $H = H_+ \oplus H_-$, where $H_\pm = \ell^2(\mathbb{Z}'_\pm)$. Then every operator $A$ in $H$ can be written in a block form,

$$
A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix},
$$

where $A_{++}$ acts from $H_+$ to $H_+$, $A_{+-}$ acts from $H_-$ to $H_+$, etc.

In terms of the block form, (1.2) can be rewritten as follows (below $A_\varepsilon$ denotes the operator of multiplication by $\varepsilon(x)$):

$$
K_{++} = K_{++}, \quad K_{+-} = K_{+-} A_\varepsilon, \\
K_{-+} = -A_\varepsilon K_{-+}, \quad K_{--} = 1 - A_\varepsilon K_{--} A_\varepsilon.
$$

It follows that if $K$ is an Hermitian operator in $H$ then $K$ is also Hermitian, but with respect to an indefinite inner product in $H$:

$$
[f,g] := (Jf,g), \quad f,g \in H, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$
Such operators are called $J$–Hermitian or $J$–symmetric operators. Thus, the operators $K_{z,z',\xi}$ and $K_{z,z'}$ are $J$–symmetric.

**Proposition 1.8.** The pre–limit operators $K_{z,z',\xi}$ belong to the trace class.

This claim is not obvious from the definition of the operators nor from the explicit expressions for the kernels, but can be easily derived from the results of [BO1] (it is immediately seen that the “$L$–operator” related to $K := K_{z,z',\xi}$ through the formula $K = L(1 + L)^{-1}$ is of trace class). The trace class property of $K_{z,z',\xi}$ is related to the fact that the measure $P_{z,z',\xi}$ lives on finite configurations (note that the trace of a correlation operator equals the expected total number of particles).

As for the limit measure $P_{z,z'}$, it lives on infinite configurations, and the limit operator $K_{z,z'}$ is not of trace class.

### 1.6. Gauge transformation of correlation kernels.

An arbitrary transformation of correlation kernels of the form

$$\mathcal{K}(x, y) \mapsto \phi(x)\mathcal{K}(x, y)\phi(y)^{-1}$$

with a nonvanishing function $\phi(x)$ does not affect the minors giving the values of the correlation functions. We call this a gauge transformation.

Thus, the correlation kernel is not a canonical object attached to a determinantal measure. This circumstance must be taken into account in attempting to solve Problem 1.

### 1.7. Symmetry.

Recall that by $\lambda \mapsto \lambda'$ we denote transposition of Young diagrams. Return to formula (1.1) for the $z$–measure weights and observe that $\dim \lambda' = \dim \lambda$ and $(z)_{\lambda} = (-1)^{|\lambda|}(\bar{z})_{\lambda'}$. This implies the important symmetry relation

$$M_{z,z',\xi}(\lambda') = M_{-z,-z',\xi}(\lambda), \quad \lambda \in \mathbb{Y}. \quad (1.3)$$

Next, observe that under transposition $\lambda \mapsto \lambda'$, the modified Frobenius coordinates interchange: $p_i \leftrightarrow q_i$. Together with the above symmetry relation this implies that the transformation of the measure $P_{z,z',\xi}$ induced by the reflection symmetry $x \mapsto -x$ of the lattice $\mathbb{Z}'$ amounts to the index transformation $(z, z') \mapsto (-z, -z')$. The same holds for the limit measures $P_{z,z'}$.

It is worth noting that the behavior of the measures $P_{z,z',\xi}$ and their limits under the reflection symmetry of $\mathbb{Z}'$ is more complex: besides the change of sign of $z$ and $z'$ one has to apply the particle/hole involution on the whole lattice.

The symmetry (1.3) is reflected in the following symmetry property for the kernels $K_{z,z',\xi}$:
Proposition 1.9. We have

\[ K_{z,z',\xi}(x,y) = (-1)^{\text{sgn}(x) \text{sgn}(y)} K_{-z,-z',\xi}(-x,-y). \]

This follows from [BO3, Theorem 7.2], see the comments to this theorem. Passing to the limit as \( \xi \to 1 \), we get the same property for the limit kernel \( K_{z,z'}(x,y) \).

2. Multiplicative functionals and Fredholm determinants

2.1. Generalities. Let \( \mathfrak{X} \) be a countable set. Below we will return to \( \mathfrak{X} = \mathbb{Z} \) but at this moment we do not need any structure on \( \mathfrak{X} \).

As in Subsection 1.1, we mean by a configuration in \( \mathfrak{X} \) an arbitrary subset \( X \subseteq \mathfrak{X} \) and we denote by \( \text{Conf}(\mathfrak{X}) \) the space of all configurations. Again, we equip \( \text{Conf}(\mathfrak{X}) \) with the topology determined by the identification \( \text{Conf}(\mathfrak{X}) \) with the infinite product space \( \{0,1\}^{\mathfrak{X}} \); then \( \text{Conf}(\mathfrak{X}) \) becomes a metrizable compact topological space. We also endow \( \text{Conf}(\mathfrak{X}) \) with the corresponding Borel structure.

Given a configuration \( X \in \text{Conf}(\mathfrak{X}) \), let \( 1_X \) denote its indicator function: for \( x \in \mathfrak{X} \), the value \( 1_X(x) \) equals 1 or 0 depending on whether \( x \) belongs or not to \( X \). Viewing \( 1_X \) as a collection of 0's and 1's indexed by points \( x \in \mathfrak{X} \) we just get the identification of \( \text{Conf}(\mathfrak{X}) \) with \( \{0,1\}^{\mathfrak{X}} \).

A function \( F(X) \) on \( \text{Conf}(\mathfrak{X}) \) is said to be a cylinder function if it depends only on the intersection \( X \cap Y \) with some finite subset \( Y \subset \mathfrak{X} \). Cylinder functions are continuous and form a dense subalgebra in \( C(\text{Conf}(\mathfrak{X})) \), the Banach algebra of continuous functions on the compact space \( \text{Conf}(\mathfrak{X}) \). We will need cylinder functions in Section 4.

2.2. Multiplicative functionals \( \Phi_f \). To a function \( f(x) \) on \( \mathfrak{X} \), we would like to assign a multiplicative functional on \( \text{Conf}(\mathfrak{X}) \) by means of the formula

\[ \Phi_f(X) = \prod_{x \in X} (1 + f(x)) = \prod_{x \in \mathfrak{X}} (1 + 1_X(x)f(x)), \quad X \in \text{Conf}(\mathfrak{X}). \]

Let us say that \( \Phi_f(X) \) is defined at \( X \) if the above product is absolutely convergent, which is equivalent to saying that the sum

\[ \sum_{x \in X} |f(x)| = \sum_{x \in \mathfrak{X}} 1_X(x)|f(x)| \quad (2.1) \]

is finite. Thus, the domain of definition for \( \Phi_f \) is the set of all configurations \( X \) for which (2.1) is finite.

Obviously, this set coincides with the whole space \( \text{Conf}(\mathfrak{X}) \) if and only if \( f \) belongs to \( \ell^1(\mathfrak{X}) \). In particular, this happens if \( f \) vanishes outside a finite subset \( Y \subset \mathfrak{X} \), and then \( \Phi_f \) is simply a cylinder function. However, we will need to deal with multiplicative functionals which
are defined on a proper subset of \( \text{Conf}(\mathcal{X}) \) only. Observe that for any function \( f \), the domain of definition of \( \Phi_f \) is a Borel subset in \( \text{Conf}(\mathcal{X}) \) (more precisely, a subset of type \( F_\sigma \)), and \( \Phi_f \) is a Borel function on this subset, because \( \Phi_f \) is a pointwise limit of cylinder functions.

Given a probability measure \( P \) on \( \text{Conf}(\mathcal{X}) \), it is important for us to see if the domain of definition of \( \Phi_f \) is of full \( P \)–measure. Here is a simple sufficient condition for this, expressed in terms of the density function \( \rho_1(x) \). Recall its meaning: \( \rho_1(x) \) is the probability that the random (with respect to \( P \)) configuration \( X \) contains \( x \).

**Proposition 2.1.** Let \( P \) be a probability measure on \( \text{Conf}(\mathcal{X}) \), \( \rho_1(x) \) be its density function, and \( f(x) \) be a function on \( \mathcal{X} \). If

\[
\sum_{x \in \mathcal{X}} \rho_1(x)|f(x)| < \infty
\]

then the multiplicative functional \( \Phi_f(X) \) is defined \( P \)–almost everywhere on \( \text{Conf}(\mathcal{X}) \).

**Proof.** Regard the quantity (2.1) as a function in \( \mathcal{X} \), with values in \([0, +\infty]\). This function is almost everywhere finite if its expectation is finite. Now, take the expectation of the right–hand side of (2.1). Since the expectation of \( 1_{\mathcal{X}}(x) \) is \( \rho_1(x) \), the result is \( \sum_{x \in \mathcal{X}} \rho_1(x)|f(x)| \), which is finite by the assumption. \( \square \)

Fix a function \( r(x) > 0 \) on \( \mathcal{X} \). Let us say that a configuration \( X \) is \( r \)–sparse (more precisely, sparse with respect to weight \( r^{-1} \)) if the series

\[
\sum_{x \in \mathcal{X}} r^{-1}(x) = \sum_{x \in \mathcal{X}} 1_{\mathcal{X}}(x)r^{-1}(x)
\]

converges. Let \( \text{Conf}_r(\mathcal{X}) \) denote the subset of all \( r \)–sparse configurations. If the series \( \sum_{x \in \mathcal{X}} r^{-1}(x) \) converges then obviously \( \text{Conf}_r(\mathcal{X}) = \text{Conf}(\mathcal{X}) \); otherwise \( \text{Conf}_r(\mathcal{X}) \) is a proper subset of \( \text{Conf}(\mathcal{X}) \). Note that it is a Borel subset (more precisely, a subset of type \( F_\sigma \)).

**Proposition 2.2.** Let \( P \) be a probability measure on \( \text{Conf}(\mathcal{X}) \) and \( \rho_1(x) \) be its density function. If

\[
\sum_{x \in \mathcal{X}} \rho_1(x)r^{-1}(x) < \infty
\]

then \( P \) is concentrated on the Borel subset \( \text{Conf}_r(\mathcal{X}) \).

**Proof.** The argument is the same as in the proof of Proposition 2.1. Consider the function

\[
\varphi(X) = \sum_{x \in \mathcal{X}} 1_{\mathcal{X}}(x)r^{-1}(x), \quad X \in \text{Conf}(\mathcal{X}),
\]

where \( \varphi(X) \) is the counting density of \( r \)–sparse configurations in \( X \).
which is allowed to take the value $+\infty$. We have to prove that $\varphi(X)$ is finite almost surely with respect to the measure $P$. This is obvious, because the expectation of $\varphi$ equals
\[
\sum_{x \in X} \rho_1(x)r^{-1}(x),
\]
which is finite by the assumption. □

Consider the correspondence $X \mapsto m_X$ that assigns to a configuration $X$ the function $m_X(x) = 1_X(x)r^{-1}(x)$ on $\mathcal{X}$. This correspondence determines an embedding of the set $\text{Conf}_r(\mathcal{X})$ into the Banach space $\ell^1(\mathcal{X})$. Using this embedding we equip $\text{Conf}_r(\mathcal{X})$ with the topology induced by the norm topology of $\ell^1(\mathcal{X})$. Let us call this topology the $\ell^1$–topology; of course, the definition depends on the choice of the function $r(x)$. If the series $\sum_{x \in X} r^{-1}(x)$ diverges then the $\ell^1$–topology on $\text{Conf}_r(\mathcal{X})$ is stronger than that induced by the canonical topology of the space $\text{Conf}(\mathcal{X})$.

**Proposition 2.3.** Let $f$ be a function on $\mathcal{X}$ such that the function $|f(x)|r(x)$ is bounded. Then the multiplicative functional $\Phi_f$ is well defined on $\text{Conf}_r(\mathcal{X})$. Moreover, $\Phi_f$ is continuous in the $\ell^1$–topology of $\text{Conf}_r(\mathcal{X})$ defined above.

**Proof.** The first claim is trivial. Indeed, set $g(x) = f(x)r(x)$. This function is in $\ell^\infty(\mathcal{X})$ while the function $m_X(x)$ is in $\ell^1(\mathcal{X})$. Since the quantity (2.1) can be represented as the pairing between $|g|$ and $m_X$, we conclude that (2.1) is finite.

To prove the second claim we observe that
\[
\Phi_f(X) = \prod_{x \in X} (1 + m_X(x)g(x)).
\]
Now the claim readily follows from the fact that the pairing between $g \in \ell^\infty(\mathcal{X})$ and $m_X \in \ell^1(\mathcal{X})$ is continuous in the second argument. □

**Example 2.4.** Let $\mathcal{X} = \mathbb{Z}'$ and $P = P_{z,z'}$. We know from Corollary 1.6 that in this concrete case the density function $\rho_1(x)$ on $\mathbb{Z}'$ decays as $|x|^{-1}$ as $x \to \pm \infty$. Then Proposition 2.2 says that $P_{z,z'}$ is concentrated on $\text{Conf}_r(\mathbb{Z}')$ provided that the function $r(x) > 0$ on $\mathbb{Z}'$ is such that the series $\sum_{x \in \mathbb{Z}'} r^{-1}(x)|x|^{-1}$ converges. For instance, one may take $r(x) = |x|^\delta$ with any $\delta > 0$. (Later on we will choose $r(x) = |x|$.)

Proposition 2.3 says that a multiplicative functional $\Phi_f$ is well defined on $\text{Conf}_r(\mathbb{Z}')$ if $f(x) = O(r^{-1}(x))$ as $x \to \pm \infty$. In particular, $\Phi_f(X)$ is well defined for $P_{z,z'}$–almost all configurations $X$ provided that $f$ satisfies the above condition for a positive function $r$ such that
\[ \sum_{x \in \mathbb{Z}} r^{-1}(x)|x|^{-1} < \infty. \] This condition on \( f \) essentially coincides with the condition of Proposition 2.1 (only that proposition avoids the intermediation of \( r \)), which is not surprising because the both propositions exploit the same idea.

2.3. **Condition of integrability for \( \Phi_f \).** Let again \( \mathcal{X} \) be a countable set and \( P \) be a probability Borel measure on \( \text{Conf}(\mathbb{X}) \). Here we give a condition for \( \Phi_f \) to be not only defined \( P \)-almost everywhere but also to have finite expectation. The condition involves the correlation functions of all orders. It is convenient to combine them into a single function \( \rho(X') \) defined on arbitrary finite subsets \( X' \subset \mathcal{X} \): By definition, \( \rho(X') \) equals the probability of the event that the random configuration \( X \) contains \( X' \).

Let \( X' \subset X \) mean that \( X' \) is a finite subset of \( X \). By \( E_P(. \cdot) \) we denote expectation with respect to \( P \).

**Proposition 2.5.** Let \( f(x) \) be a function on \( \mathcal{X} \) such that

\[ \sum_{X' \subset \mathcal{X}} \rho(X') \prod_{x \in X'} |f(x)| < \infty. \]

Then the multiplicative functional \( \Phi_f \) is defined almost everywhere with respect to \( P \), is absolutely integrable, and its expectation equals

\[ E_P(\Phi_f) = \sum_{X' \subset \mathcal{X}} \rho(X') \prod_{x \in X'} f(x). \] (2.2)

**Proof.** The above condition on \( f \) is stronger than the condition of Proposition 2.1 so that the first claim follows from Proposition 2.1. Checking the second and third claims uses the same argument as in that proposition.

Observe that

\[ \prod_{x \in X} (1 + |f(x)|) = \sum_{X' \subset X} \prod_{x \in X'} |f(x)| \]

in the sense that the both sides are simultaneously either finite or infinite, and if they are finite then they are equal.

Denote by \( \eta_{X'}(X) \) the function on \( \text{Conf}(\mathcal{X}) \) equal to 1 or 0 depending on whether \( X \) contains \( X' \) or not. The above equality can be rewritten as

\[ \prod_{x \in X} (1 + |f(x)|) = \sum_{X' \subset \mathcal{X}} \eta_{X'}(X) \prod_{x \in X'} |f(x)|. \]

Take the expectation of the both sides. Since \( E_P(\eta_{X'}) = \rho(X') \), we get

\[ E_P(\Phi_f) = \sum_{X' \subset \mathcal{X}} \rho(X') \prod_{x \in X'} |f(x)| < \infty. \]
Thus, we have checked the second and third claims for the function $|f|$. Since $|\Phi_f(X)| \leq \Phi_f(X)$, it follows that $\Phi_f$ is absolutely integrable. Now we can repeat the above argument with $f$ instead of $|f|$. The above computation with $|f|$ provides the necessary justification for manipulations with infinite sums.

2.4. Fredholm determinants. Let $\mathfrak{X}$ and $P$ be as in Subsection 2.3 and assume additionally that $P$ is determinantal with a correlation kernel $K(x, y)$ corresponding to a bounded operator $K$ in the Hilbert space $H := \ell^2(\mathfrak{X})$ (so $K$ is the correlation operator of $P$).

For a bounded function $f(x)$ on $\mathfrak{X}$, we denote by $Af$ the operator in $H$ given by multiplication by $f$.

Lemma 2.6. If $f$ is finitely supported then

$$\mathbb{E}_P(\Phi_f) = \det(1 + AfK).$$

Note that the determinant is well defined because, due to the assumption on $f$, the operator $AfK$ has finite-dimensional range.

Proof. This directly follows from (2.2). Indeed, according to the definition of determinantal measures, $\rho(X') = \det([K(x, y)]_{x, y \in X'})$. This implies

$$\rho(X') \prod_{x \in X'} f(x) = \det([f(x)K(x, y)]_{x, y \in X'}).$$

Then we employ a well-known identity from linear algebra: If $B = [B(x, y)]$ is a matrix then $\det(1 + B)$ equals the sum of the principal minors of $B$. The identity holds for matrices of finite size but we can apply it to $B = AfK$ because the matrix $[f(x)K(x, y)]$ has only finitely many nonzero rows. This gives us the equality

$$\sum_{X' \in \mathfrak{X}} \det([f(x)K(x, y)]_{x, y \in X'}) = \det(1 + AfK),$$

which concludes the proof. □

The hypothesis of the lemma is, of course, too restrictive: the above argument can be easily extended to the case when the operator $AfK$ is of trace class. A slightly more general fact is established below in Proposition 2.7, which is specially adapted to the application we need.

First, state a few general results from [BOO, Appendix].

Let $H = H_+ \oplus H_-$ be a $\mathbb{Z}_2$-graded Hilbert space. Any operator $A$ in $H$ can be written in block form,

$$A = \begin{bmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{bmatrix}$$
where \( A_{++} \) acts from \( H_+ \) to \( H_+ \), \( A_{+-} \) acts from \( H_- \) to \( H_+ \), etc. Let \( \mathcal{L}_{1|2}(H) \) be the set of bounded operators \( A \) whose diagonal blocks \( A_{++} \) and \( A_{--} \) are trace class operators while the off–diagonal blocks \( A_{+-} \) and \( A_{-+} \) are Hilbert–Schmidt operators. The set \( \mathcal{L}_{1|2}(H) \) is an algebra. We equip it with the corresponding combined topology: the topology of the trace class norm \( \| \cdot \|_1 \) for the diagonal blocks and the topology of the Hilbert–Schmidt norm \( \| \cdot \|_2 \) for the off–diagonal blocks.

There exists a unique continuous function on \( \mathcal{L}_{1|2}(H) \),

\[
A \mapsto \det(1 + A),
\]

coinciding with the conventional determinant when \( A \) is a finite rank operator. This function can be defined as

\[
\det(1 + A) = \det((1 + A)e^{-A})e^{\text{tr}(A_{++}) + \text{tr}(A_{--})},
\]

where the determinant in the right–hand side is the conventional one: the point is that \( A \mapsto (1 + A)e^{-A} - 1 \) is a continuous map from \( \mathcal{L}_{1|2}(H) \) to the set of trace class operators.

If \( \{E_N\}_{N=1,2,...} \) is an ascending chain of projection operators in \( H \) strongly convergent to 1 then

\[
\det(1 + A) = \lim_{N \to \infty} (1 + E_N AE_N). \tag{2.3}
\]

From now on we assume \( X = \mathbb{Z}' \) and we set

\[
H = \ell^2(\mathbb{Z}'), \quad H_+ = \ell^2(\mathbb{Z}_+'), \quad H_- = \ell^2(\mathbb{Z}_-').
\]

As before, \( \{e_x\}_{x \in \mathbb{Z}'} \) denotes the natural basis in \( H \).

**Proposition 2.7.** Let \( P \) be a determinantal probability measure on \( \text{Conf}(\mathbb{Z}') \), \( K(x,y) \) be its correlation kernel and \( K \) denote the corresponding correlation operator in \( H = \ell^2(\mathbb{Z}') \).

Further, assume that \( f(x) \) is a function on \( \mathbb{Z}' \) which can be written in the form \( f(x) = g(x)h^2(x) \), where \( g(x) \) is bounded and \( h \) is nonnegative and such that \( A_hKA_h \in \mathcal{L}_{1|2}(H) \).

Then the functional \( \Phi_f \) is defined almost everywhere with respect to \( P \), is absolutely integrable with respect to \( P \), and

\[
\mathbb{E}_P(\Phi_f) = \det(1 + A_gA_hKA_h).
\]

**Proof.** For \( N = 1, 2, \ldots \), let \( E_N \) be the projection in \( H \) onto the finite–dimensional subspace of functions concentrated on \( [-N, N] \cap \mathbb{Z}' \). As \( N \to \infty \), the projections \( E_N \) converge to 1.
Assume first that \( g \equiv 1 \). Then \( f(x) \) is nonnegative and the same argument as in Lemma 2.6 shows that

\[
\det(1 + E_N A_h K A_h E_N) = \det(1 + A_f E_N K E_N)
\]

\[
= \sum_{X' \subset [-N,N] \cap \mathbb{Z}'} \det ([K(x,y)]_{x,y \in X'}) \prod_{x \in X'} f(x)
\]

\[
= \sum_{X' \subset [-N,N] \cap \mathbb{Z}'} \rho(X') \prod_{x \in X'} f(x),
\]

where we used the fact that \( E_N \) and \( A_h \) commute. As \( N \to \infty \), the resulting quantity converges to the infinite sum

\[
\sum_{X' \in \mathbb{Z}'} \rho(X') \prod_{x \in X'} f(x).
\]

On the other hand, by virtue of (2.3),

\[
\det(1 + E_N A_g A_h K A_h E_N) \to \det(1 + A_h K A_h).
\]

Consequently,

\[
\det(1 + A_h K A_h) = \sum_{X' \in \mathbb{Z}'} \rho(X') \prod_{x \in X'} f(x) = \mathbb{E}_P(\Phi_f),
\]

where the last equality follows from Proposition 2.5.

For an arbitrary bounded \( g \) the computation is the same, and the above argument with \( f \geq 0 \) is used to check the absolute convergence of the arising infinite sum and to guarantee applicability of Proposition 2.5.

\[\square\]

3. **Radon–Nikodym derivatives**

Denote by \( \mathcal{G} \) the group of finitary permutations of the set \( \mathbb{Z}' \). This is a countable group generated by the elementary transpositions

\[\ldots, \sigma_{-1}, \sigma_0, \sigma_1, \ldots,\]

where \( \sigma_n \) transposes \( n - \frac{1}{2} \) and \( n + \frac{1}{2} \). The action of the group \( \mathcal{G} \) on \( \mathbb{Z}' \) induces its action on \( \text{Conf}(\mathbb{Z}') \). For \( \sigma \in \mathcal{G} \), we denote the corresponding transformation of \( \text{Conf}(\mathbb{Z}') \) by the same symbol \( \sigma \).

Let us represent configurations \( X \in \text{Conf}(\mathbb{Z}') \) as two–sided infinite sequences of black and white circles separated by vertical bars, like this:

\[
\ldots \bullet \mid \bullet \mid \circ \mid \bullet \mid \circ \mid \circ \ldots
\]

Here black and white circles represent particles and holes, respectively, and the subscripts under the bars are used to mark the positions of integers interlacing with half–integers. In this picture, the action of
σ_n affects only the two-circle fragment around the nth vertical bar and amounts to replacing “◦ | ◦” by “• | ◦” and vice versa (the fragments “◦ | ◦” and “• | ◦” remain intact).

This action preserves the set of Maya diagrams X(λ), so that we get an action of S on Y, which can be directly described as follows: application of the elementary transposition σ_n to a Young diagram λ amounts to adding or removing a box (i, j) with j − i = n, if this operation is possible. The transformation “• | ◦ → ◦ | •” corresponds to adding a box, and the inverse transformation corresponds to removing a box. In particular, σ_0 adds/removes boxes on the main diagonal of λ.

Our aim is to study the transformation of the measures P_{z,z',ξ} and their limits P_{z,z'} under the action of S. However, for technical reasons, it is more convenient to deal with the measures P_{z,z',ξ} and P_{z,z'} which are related to the former measures by the particle/hole involution on Z'_-. To this end we introduce the modified action of S on Conf(Z'): it differs from the natural one by conjugation with the particle/hole involution. Given σ ∈ S, we denote the modified action of σ on Conf(Z') by the symbol ˜σ. The relation between σ and ˜σ is

\[ ˜σ(X) = \text{inv}(σ(\text{inv}(X))), \quad σ ∈ S, \quad X ∈ \text{Conf}(Z'). \]

The modified transformations ˜σ : Conf(Z') → Conf(Z') induce transformations of measures denoted as P ↦ ˜σ(P).

Let us emphasize that the modified action is defined only on Conf(Z'), not on Z' itself.

In the case of elementary transpositions σ = σ_n, the modified action differs from the natural one for n = 0 only. Namely, the modified action of σ_0 amounts to switching “◦ | ◦ ↔ • | ◦”, while the fragments “◦ | ◦” and “• | ◦” remain intact.

Recall that a finite configuration X ∈ Z' has the form inv(X(λ)) with λ ∈ Y if and only if X is balanced in the sense that |X ∩ Z'_+| = |X ∩ Z'_−|. Since the initial action of S preserves the set of the semi-infinite configurations of the form X(λ), the modified action preserves the set of the finite balanced configurations. Obviously, if σ ∈ S, X = X(λ), and X = X(λ) = inv(X), then we have

\[ \frac{σ(P_{z,z',ξ})(X)}{P_{z,z',ξ}(X)} = \frac{˜σ(P_{z,z',ξ})(X)}{P_{z,z',ξ}(X)}. \]
We introduce a special notation for this Radon–Nikodým derivative:

$$\mu_{z,z',\xi}(\sigma, X) := \frac{\tilde{\sigma}(P_{z,z',\xi})(X)}{P_{z,z',\xi}(X)} = \frac{P_{z,z',\xi}(\tilde{\sigma}^{-1}(X))}{P_{z,z',\xi}(X)}, \quad \sigma \in \mathcal{G}, \quad (3.1)$$

where $X$ is a finite balanced configuration.

In the remaining part of the section we prove the following result.

**Proposition 3.1.** Fix an arbitrary couple $(z, z')$ of parameters belonging to the principal or complementary series. Let $\xi$ range over $(0, 1)$ and $X$ range over the set of finite balanced configurations on $\mathbb{Z}'$.

For any fixed $\sigma \in \mathcal{G}$, the Radon–Nikodým derivative $\mu_{z,z',\xi}(\sigma, X)$ can be written as a finite linear combination of multiplicative functionals of the form $\Phi_f$ multiplied by factors $\xi^k$ with $k \in \mathbb{Z}$, where each function $f(x)$ decays at infinity at least as $|x|^{-1}$:

$$\mu_{z,z',\xi}(\sigma, X) = \sum_{i=1}^{m} a_i \xi^{k_i} \Phi_{f_i}(X), \quad (3.2)$$

$$a_i \in \mathbb{R}, \quad k_i \in \mathbb{Z}, \quad f_i(x) = O(|x|^{-1}).$$

Let us emphasize that the right-hand side depends on $\xi$ through the factors $\xi^k$ only. Proposition 3.1 provides a refinement of Claim 2 of the Introduction, as explained after the end of the proof of the proposition.

**Proof.** Step 1. Given a subset $X' \subseteq \mathbb{Z}' \cap [-N, N]$, where $N \in \{1, 2, \ldots\}$, and a function $f^{\text{out}}$ on $\mathbb{Z}' \setminus [-N, N]$, we set

$$\eta_{N,X'}(X) = \begin{cases} 
1, & X \cap [-N, N] = X' \\
0, & \text{otherwise}
\end{cases}$$

$$\Phi_{N, f^{\text{out}}}(X) = \prod_{x \in X \setminus [-N,N]} (1 + f^{\text{out}}(x)).$$

We will prove that for any $\sigma \in \mathcal{G}$ and all $N$ large enough there exists a representation of the form

$$\mu_{z,z',\xi}(\sigma, X) = \sum_{i=1}^{m} a_i \xi^{k_i} \eta_{N,X'_i}(X) \Phi_{N, f^{\text{out}}_i}(X), \quad (3.3)$$

$$a_i \in \mathbb{R}, \quad k_i \in \mathbb{Z}, \quad X'_i \subseteq \mathbb{Z}' \cap [-N, N], \quad f^{\text{out}}_i(x) = O(|x|^{-1}).$$

Observe that (3.3) is equivalent to (3.2), because each function of the form $\eta_{N,X'}(X)$, being a cylinder functional depending on $X \cap [-N, N]$ only, can be written as a linear combination of functionals of the form

$$X \mapsto \prod_{x \in X \cap [-N,N]} (1 + f^{\text{in}}(x))$$
with appropriate functions $f^{in}$ on $\mathbb{Z}' \cap [-N, N]$.

**Step 2.** Next, we want to reduce the problem to the particular case when $\sigma$ is an elementary transposition. Since the elementary transpositions generate the whole group $\mathfrak{S}$, to perform the desired reduction, it suffices to prove that if the presentation (3.3) exists for two elements $\sigma, \tau \in \mathfrak{S}$ then it also exists for the product $\sigma \tau$.

It follows from the definition (3.1) that

$$
\mu_{z,z',\xi}(\sigma \tau, X) = \mu_{z,z',\xi}(\sigma, X) \cdot \mu_{z,z',\xi}(\tau, \bar{\sigma}^{-1}(X)).
$$

We may assume that $N$ is so large that the permutation $\sigma : \mathbb{Z}' \to \mathbb{Z}'$ does not move points outside $[-N, N]$. Then it is clear that if the function $X \mapsto \mu_{z,z',\xi}(\tau, X)$ admits a presentation of the form (3.3) then the same holds for the function $X \mapsto \mu_{z,z',\xi}(\tau, \bar{\sigma}^{-1}(X))$ as well. Thus, it remains to check that the set of functions admitting a representation of the form (3.3) (with $N$ fixed) is closed under multiplication. This is obvious, because the product of two functionals of the form $\Phi_{N,f^{out}}$ is a functional of the same kind:

$$
\Phi_{N,f^{out}} \Phi_{N,g^{out}} = \Phi_{N,h^{out}}
$$

with

$$
h^{out}(x) := f^{out}(x)g^{out}(x) + f^{out}(x) + g^{out}(x),
$$

and, moreover, if $f^{out}(x) = O(|x|^{-1})$ and $g^{out}(x) = O(|x|^{-1})$ then $h^{out}(x) = O(|x|^{-1})$.

**Step 3.** Thus, we have to analyze the ratio

$$
\mu_{z,z',\xi}(\sigma_n, X) = \frac{P_{z,z',\xi}(\bar{\sigma}_n^{-1}(X))}{P_{z,z',\xi}(X)} = \frac{P_{z,z',\xi}(\bar{\sigma}_n(X))}{P_{z,z',\xi}(X)},
$$

where the second equality holds because $\bar{\sigma}_n^{-1} = \bar{\sigma}_n$.

We aim to prove that for any $N > |n|$, there exists a single term representation

$$
\frac{P_{z,z',\xi}(\bar{\sigma}_n(X))}{P_{z,z',\xi}(X)} = a\xi^k \Phi_{N,f^{out}}(X),
$$

$k = 0, \pm 1$, $f^{out}(x) = O(|x|^{-1})$, (3.5)

where, in contrast to (3.3), $a$, $k$, and $f^{out}$ may depend on the intersection $X' := X \cap [-N, N]$. Observe that once (3.5) is established, we can combine various variants of (3.5) (which depend on $X'$) into a single representation (3.3) by making use of the factors $\eta_{N,X'}(X)$. Thus, it suffices to prove (3.5).
Step 4. Here we exhibit a convenient explicit expression for $P_{z,z',\xi}(X)$, where $X$ is an arbitrary balanced configuration. Employing the notation introduced in Subsection 1.1 we write

\[ X = \{-q_d, \ldots, -q_1, p_1, \ldots, p_d\}. \]

By definition, $P_{z,z',\xi}(X) = M_{z,z',\xi}(\lambda)$, where $\lambda$ is such that $X = X(\lambda)$. We have to rewrite the expression (1.1) for $M_{z,z',\xi}(\lambda)$ given in Subsection 1.2 in terms of the $p_i$'s and $q_i$'s. For the terms $(z)_\lambda$ and $(z')_\lambda$ this is easy, and for $\dim \lambda / |\lambda|!$ we employ the formula

\[
\dim \lambda = \prod_{1 \leq i < j \leq d} (p_j - p_i)(q_j - q_i) \cdot \prod_{i,j=1,\ldots,d} (p_i + q_j) / |\lambda|!
\]

given, e.g., in [Ol1, (2.7)]. The result is

\[
P_{z,z',\xi}(X) = (1 - \xi)^{zz'}\xi^{\sum_{i=1}^d (p_i + q_i)} (zz')^d 
\times \prod_{i=1}^d \frac{z + 1}{p_i - \frac{1}{2}}(z' + 1) p_i - \frac{1}{2} (-z + 1) q_i - \frac{1}{2} (-z' + 1) q_i - \frac{1}{2} 
\times \prod_{1 \leq i < j \leq d} (p_j - p_i)^2(q_j - q_i)^2 
\times \prod_{i,j=1,\ldots,d} (p_i + q_j)^2.
\]

(3.6)

At first glance formula (3.6) might appear cumbersome but actually it is well suited for our purpose, because it already has multiplicative form and after substitution into (3.4) many factors are cancelled out.

Below we examine separately the three cases: $n = 1, 2, \ldots, n = -1, -2, \ldots, n = 0$.

Step 5. Consider the case $n = 1, 2, \ldots$. Then the transformed configuration $\tilde{\sigma}_n(X)$ is the same as the configuration $\sigma_n(X)$, which in turn either coincides with the initial configuration $X$ or differs from it by shifting a single coordinate $p_i$ by $\pm 1$. The shift $p_i \to p_i + 1$ arises if there exists $i$ such that $p_i = n - \frac{1}{2}$ and either $i = d$ or $p_i + 1 \neq p_{i+1}$, which means that $X$ contains the fragment $\bullet|\circ$. The shift $p_i \to p_i - 1$ arises if there exists $i$ such that $p_i = n + \frac{1}{2}$ and either $i = 1$ or $p_i - 1 \neq p_{i-1}$, which means that $X$ contains the fragment $\circ|\bullet$. In all other cases $\sigma_n(X) = X$. Clearly, what of these possible variants takes place is uniquely determined by the intersection $X' := X \cap [-N,N]$ (recall that, by assumption, $N > |n|$).

If $\sigma_n(X) = X$ then the ratio (3.4) simply equals 1.
If \( \sigma_n \) transforms \( p_i \) to \( p_i \pm 1 \) then, as directly follows from (3.6), the ratio (3.4) equals
\[
\xi^{\pm 1} \left( \frac{(z + p_i \pm \frac{1}{2})(z' + p_i \pm \frac{1}{2})}{(p_i \pm \frac{1}{2})^2} \right)^{\pm 1} \prod_{j: j \neq i} \left( \frac{p_j - p_i}{p_j - p_i} \right)^2 \prod_j \left( \frac{q_j + p_i}{q_j + p_i} \right)^2.
\]
This has the desired form (3.5) with \( k = \pm 1 \) and
\[
1 + f^{\text{out}}(x) = \begin{cases} 
\left( 1 \mp \frac{1}{x - p_i} \right)^2, & x > N \\
\left( 1 \pm \frac{1}{|x| + p_i} \right)^{-2}, & x < -N.
\end{cases}
\]

Step 6. In the case \( n = -1, -2, \ldots \) one can repeat the argument of step 5. Alternatively, one can use the symmetry \( p_i \leftrightarrow q_i \) (Subsection 1.7).

Step 7. Finally, consider the case \( n = 0 \). Then either \( \tilde{\sigma}_0(X) = X \) or \( \tilde{\sigma}_0(X) \) differs from \( X \) by adding or removing the couple of coordinates \( p_1 = \frac{1}{2}, q_1 = \frac{1}{2} \). Therefore, ratio (3.4) either equals 1 or has the form
\[
(zz')^{\pm 1} \xi^{\pm 1} \left( \prod_j \frac{(p_j - \frac{1}{2})(q_j - \frac{1}{2})}{(p_j + \frac{1}{2})(q_j + \frac{1}{2})} \right)^{\pm 2}.
\]
This has the desired form (3.5) with \( k \) equal to 0 or \( \pm 1 \) and \( f^{\text{out}} \) equal to 0 or
\[
1 + f^{\text{out}}(x) = \left( 1 - \frac{1}{2|x|} \right)^{\pm 2} \left( 1 + \frac{1}{2|x|} \right)^{\mp 2}.
\]

\[\square\]

In the discussion below we use the notions introduced in Subsection 2.2.

**Definition 3.2.** Take the function \( r(x) = |x| \) on \( \mathbb{Z}' \). The corresponding subset \( \text{Conf}_r(\mathbb{Z}') \subset \text{Conf}(\mathbb{Z}') \) of \( r \)-sparse configurations will be denoted as \( \text{Conf}_{\text{sparse}}(\mathbb{Z}') \). We equip \( \text{Conf}_{\text{sparse}}(\mathbb{Z}') \) with the \( \ell^1 \)-topology.

By virtue of Proposition 2.3 any function of the form (3.2) is well defined and continuous on \( \text{Conf}_{\text{sparse}}(\mathbb{Z}') \). Thus, for any \( \sigma \in \mathcal{G} \), the function \( \mu_{z,z',\xi}(\sigma, X) \), initially defined on finite balanced configurations \( X \), admits a continuous extension to the larger set \( \text{Conf}_{\text{sparse}}(\mathbb{Z}') \). Although the presentation (3.2) is not unique, the result of the continuous extension provided by formula (3.2) does not depend on the specific presentation. Indeed, this follows from the fact that the finite balanced
configurations form a dense subset in $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$ (which is readily checked).

**Definition 3.3.** For any $\sigma \in \mathcal{S}$, let $\mu_{z,z'}(\sigma, X)$ stand for the function on $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$ obtained by specializing $\xi = 1$ in the right-hand side of formula (3.2).

Obviously, $\mu_{z,z'}(\sigma, X)$ coincides with the pointwise limit, as $\xi \to 1$, of the continuous extensions of the functions $\mu_{z,z'}(\sigma, X)$. This shows that $\mu_{z,z'}(\sigma, X)$ does not depend on a specific presentation (3.2). Moreover, Proposition 2.3 ensures that the limit function is continuous in the $\ell^1$-topology.

Thus, we have shown that Proposition 3.1 implies Claim 2 (Subsection 0.4) in a refined form.

4. **Main result: Formulation and beginning of proof**

The following theorem is the main result of the paper.

**Theorem 4.1.** (i) The measures $P_{z,z'}$ are quasiinvariant with respect to the modified action of the group $\mathcal{G}$ on probability measures on the space $\text{Conf}(\mathbb{Z}')$, as defined in Section 3.

(ii) For any permutation $\sigma \in \mathcal{S}$, the Radon–Nikodým derivative $\widetilde{\sigma}(P_{z,z'})/P_{z,z'}$ coincides with the limit expression $\mu_{z,z'}(\sigma, X)$ introduced in Definition 3.3 within a $P_{z,z'}$–null set.

Recall that each of the measures $P_{z,z'}$ is concentrated on the Borel subset $\text{Conf}_{\text{sparse}}(\mathbb{Z}')$ (see Example 2.4) and each of the functions $\mu_{z,z'}(\sigma, X)$ is well defined on the same subset and is a Borel function.

In this section, we will reduce Theorem 4.1 to Theorem 4.3 through an intermediate claim, Theorem 4.2, which is of independent interest. The proof of Theorem 4.3 occupies Sections 5 and 6.

As before, we use the angular brackets to denote the pairing between functions and measures.

**Theorem 4.2.** Let $f(x)$ be an arbitrary function on $\mathbb{Z}'$ such that $f(x) = O(|x|^{-1})$ as $x \to \pm \infty$. Then the multiplicative functional $\Phi_f$ is absolutely integrable with respect to the measures $P_{z,z',\xi}$ and $P_{z,z'}$ and

$$\lim_{\xi \to 1} \langle \Phi_f, P_{z,z',\xi} \rangle = \langle \Phi_f, P_{z,z'} \rangle.$$ 

*Derivation of Theorem 4.1 from Theorem 4.2.* Recall that the function $\mu_{z,z'}(\sigma, X)$ is a finite linear combination of the multiplicative functionals $\Phi_f$ with $f(x) = O(|x|^{-1})$, see Definition 3.3. Theorem 4.2 says that such functionals are absolutely integrable with respect to $P_{z,z'}$, which
implies that so is $\mu_{z,z'}(\sigma, X)$. Thus, $\mu_{z,z'}(\sigma, \cdot)P_{z,z'}$ is a finite Borel measure.

The claim of Theorem 4.1 is equivalent to the following one: For any $\sigma \in \mathcal{G}$,

$$\tilde{\sigma}(P_{z,z'}) = \mu_{z,z'}(\sigma, \cdot)P_{z,z'}.$$  

Recall that the cylinder functions on $\text{Conf}(\mathbb{Z}')$ are dense in $C(\text{Conf}(\mathbb{Z}'))$, see Subsection 2.1. Therefore, it suffices to prove that for any cylinder function $F$

$$\langle F, \tilde{\sigma}(P_{z,z'}) \rangle = \langle \mu_{z,z'}(\sigma, \cdot)F, P_{z,z'} \rangle.$$ 

(4.1)

Set

$$F^{\sigma}(X) = F(\tilde{\sigma}(X))$$

and observe that $F^{\sigma}$ is a cylinder function, too. We may rewrite the desired equality in the form

$$\langle F^{\sigma}, P_{z,z'} \rangle = \langle \mu_{z,z'}(\sigma, \cdot)F, P_{z,z'} \rangle.$$ 

(4.2)

By the very definition of $\mu_{z,z',\xi}$, we have

$$\tilde{\sigma}(P_{z,z',\xi}) = \mu_{z,z',\xi}(\sigma, \cdot)P_{z,z',\xi},$$

so that

$$\langle F^{\sigma}, P_{z,z',\xi} \rangle = \langle \mu_{z,z',\xi}(\sigma, \cdot)F, P_{z,z',\xi} \rangle.$$ 

(4.2)

A natural idea is to derive (4.1) from (4.2) by passing to the limit as $\xi$ goes to 1.

We know that the measures $P_{z,z',\xi}$ weakly converge to the measure $P_{z,z'}$ (Theorem 1.1 above). Therefore, the left–hand side of (4.2) converges to the left–hand side of (4.1).

Consequently, to establish (4.1) it remains to prove that the similar limit relation holds for the right–hand sides, namely

$$\lim_{\xi \to 1} \langle \mu_{z,z',\xi}(\sigma, \cdot)F, P_{z,z',\xi} \rangle = \langle \mu_{z,z'}(\sigma, \cdot)F, P_{z,z'} \rangle.$$ 

(4.3)

According to the definition of the function $\mu_{z,z'}(\sigma, X)$ (see Theorem 3.1 and Definition 3.3), the limit relation (4.3) can be reduced to the following one:

$$\lim_{\xi \to 1} \langle \xi^{k}F, P_{z,z',\xi} \rangle = \langle F, P_{z,z'} \rangle.$$ 

(4.4)

where $k \in \mathbb{Z}$, $F$ is a cylinder function, and $f(x) = O(|x|^{-1})$.

Obviously, the factor $\xi^{k}$, which tends to 1, is inessential and can be neglected, so that (4.4) can be simplified:

$$\lim_{\xi \to 1} \langle F, P_{z,z',\xi} \rangle = \langle F, P_{z,z'} \rangle.$$ 

(4.5)

Next, fix a finite subset $Y \subset \mathcal{X}$, so large that $F(X)$ depends on the intersection $X \cap Y$ only. It is readily verified that $F$ can be written
as a finite linear combination of multiplicative functionals $\Phi_{g_i}$, where each $g_i$ vanishes outside $Y$ (this claim actually concerns functions on the finite set $\{0,1\}^Y$). Observe that

$$\Phi_f \Phi_{g_i} = \Phi_{f_i}, \quad f_i := f + g_i + fg_i$$

we have already used such an equality in the proof of Proposition 3.1 step 2). It follows that the product $\Phi_f F$ can be written as a finite linear combination of multiplicative functionals $\Phi_{f_i}$, where each $f_i$ coincides with $f$ outside $Y$ and hence obeys the same decay condition, $f_i(x) = O(|x|^{-1})$. Thus, we have reduced (4.5) to the claim of Theorem 4.2.

The essence of difficulty in proving Theorem 4.2 is that, for generic $f$ decaying as $|x|^{-1}$, the multiplicative functional $\Phi_f$ is unbounded and so cannot be extended to a continuous function on the whole space $\text{Conf}(\mathbb{Z}')$. Thus, for our purpose, the fact of the weak convergence $P_{z,z',\xi} \to P_{z,z'}$, that is, convergence on continuous test functions, is insufficient: we have to enlarge the set of admissible test functions to include the functions like $\Phi_f$.\[3\]

The idea is to relate the required stronger convergence of the measures to an appropriate convergence of their correlation operators.

Set $h(x) = |x|^{-1/2}$, where $x \in \mathbb{Z}'$, and recall that $A_h$ denotes the operator of multiplication by $h$ in the Hilbert space $H = \ell^2(\mathbb{Z}')$. Below we use the notions introduced in Subsection 2.4

**Theorem 4.3.** (i) The operator $A_h K_{z,z'} A_h$ lies in $L_{12}(H)$.

(ii) As $\xi$ goes to 1, the operators $A_h K_{z,z',\xi} A_h$ converge to the operator $A_h K_{z,z'} A_h$ in the topology of the space $L_{12}(H)$.

**Comments.** Note that the pre-limit operators $A_h K_{z,z',\xi} A_h$ also lie in $L_{12}(H)$, because the operators $K_{z,z',\xi}$ are of trace class, see Proposition 1.8. According to the definition of $L_{12}(H)$ (see Subsection 2.4), the claim of the theorem means that the diagonal blocks converge in the trace class norm while the off–diagonal blocks converge in the Hilbert–Schmidt norm. I do not know whether, in the case of the off–diagonal blocks, the Hilbert–Schmidt norm can be replaced by the trace class norm.

**Derivation of Theorem 4.2 from Theorem 4.3.** The assumption on the function $f$ allows one to write it as $f = gh^2$, where $|g(x)|$ is bounded

---

3The situation is formally similar to that of weak convergence and moment convergence of probability measures on $\mathbb{R}$: In general, the former does not imply the latter.
THE QUASI-INVARIANCE PROPERTY

(recall that \( h(x) = |x|^{-1/2} \)). By virtue of Proposition 2.7 and claim (i) of Theorem 4.3, \( \Phi_f \) is absolutely integrable with respect to the limit measure \( P_{z,z'} \), and the same also holds for the pre-limit measures \( P_{z,z',\xi} \) (see the comments above). Moreover,

\[
\langle \Phi_f, P_{z,z'} \rangle = \det(1 + A_g A_h K_{z,z',\xi} A_h), \quad \langle \Phi_f, P_{z,z'} \rangle = \det(1 + A_g A_h K_{z,z'} A_h).
\]

Finally, claim (ii) of Theorem 4.2 implies that the operator \( A_g A_h K_{z,z',\xi} A_h \) converges to the operator \( A_g A_h K_{z,z'} A_h \) in the topology of \( L_{1/2}(H) \). Therefore, the corresponding determinants converge, too. Here we use the fact that the function \( A \mapsto \det(1 + A) \) is continuous on \( L_{1/2}(H) \), see Subsection 2.4.

Thus, we have reduced Theorem 4.2, and hence Theorem 4.1, to Theorem 4.3. The latter theorem is proved separately for the diagonal and off-diagonal blocks in Sections 5 and 6, respectively.

5. CONVERGENCE OF DIAGONAL BLOCKS IN THE TOPOLOGY OF THE TRACE CLASS NORM

Here we prove the claims of Theorem 4.3 for the diagonal blocks \( (\cdot)_{++} \) and \( (\cdot)_{--} \) of the operators in question. Due to the symmetry relation of Proposition 1.9, the latter block is obtained from the former one by a simple change of the basic parameters, \((z, z') \rightarrow (z, -z')\). Thus, it suffices to focus on the limit behavior of the block \( (\cdot)_{++} \). That is, we have to prove that the operator \( (A_h K_{z,z',\xi} A_h)_{++} \) in the Hilbert space \( \ell^2(\mathbb{Z}^+) \) is of trace class and

\[
\lim_{\xi^{-1}} \| (A_h K_{z,z',\xi} A_h)_{++} - (A_h K_{z,z'} A_h)_{++} \|_1 = 0,
\]

where \( \| \cdot \|_1 \) is the trace class norm.

Since the proof is long, let us describe its scheme. First of all, observe that, as long as we are dealing with the ++ block, there is no difference between \( K_{z,z',\xi} \) and \( K_{z,z',\xi} \) (and the same for the limit operators). Indeed, this follows from the relation between the both kind of operators, see Subsection 1.5.

In Proposition 5.1 we rederive the weak convergence \( K_{z,z',\xi} \rightarrow K_{z,z'} \), which implies the weak convergence \( (A_h K_{z,z',\xi} A_h)_{++} \rightarrow (A_h K_{z,z'} A_h)_{++} \).

Recall that \( K_{z,z',\xi} \) is a projection operator (see Subsection 1.4). This implies that its ++ block is a nonnegative operator, so that \( (A_h K_{z,z',\xi} A_h)_{++} \) is also a nonnegative operator. Therefore, the limit operator \( (A_h K_{z,z'} A_h)_{++} \) is nonnegative, too.

Consequently, to prove that the operator \( (A_h K_{z,z'} A_h)_{++} \) is of trace class it suffices to prove that its trace is finite. This is done in Proposition 5.2.
The similar fact for the pre-limit operator \((A_h K_{z,z',\xi} A_h)_{++}\) is a trivial consequence of Proposition 1.8.

In Proposition 5.4 we establish the convergence of traces,

\[
\lim_{\xi \to 1} \text{tr} \left( (A_h K_{z,z',\xi} A_h)_{++} \right) = \text{tr} \left( (A_h K_{z,z'} A_h)_{++} \right).
\]

This concludes the proof, because, for nonnegative operators, weak convergence together with convergence of traces is equivalent to convergence in the trace class norm (see, e.g., [BOO, Proposition A.9]).

Let us proceed to the detailed proof.

Our starting point is the double contour integral representation (5.1) (see below) for the kernel \(K_{z,z',\xi}(x,y)\) on the lattice \(\mathbb{Z}'\). Formula (5.1) is a particular case of a more general formula obtained in [BO3, §9, p. 148].

\[
K_{z,z',\xi}(x;y) = \frac{\Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2})}{\left( \frac{\Gamma(-z - x + \frac{1}{2}) \Gamma(-z' - x + \frac{1}{2}) \Gamma(-z - y + \frac{1}{2}) \Gamma(-z' - y + \frac{1}{2})}{(2\pi i)^2} \right)^{\frac{1}{2}}}
\times \frac{1 - \xi}{\omega_1 \omega_2 - 1} \int_{\omega_1} \int_{\omega_2} \left( 1 - \sqrt{\xi \omega_1} \right)^{z' + x - \frac{1}{2}} \left( 1 - \frac{\sqrt{\xi}}{\omega_1} \right)^{-z - x - \frac{1}{2}} \times \left( 1 - \sqrt{\xi \omega_2} \right)^{z + y - \frac{1}{2}} \left( 1 - \frac{\sqrt{\xi}}{\omega_2} \right)^{-z' - y - \frac{1}{2}} \frac{\omega_1^{-x - \frac{1}{2}} \omega_2^{-y - \frac{1}{2}}}{\omega_1 \omega_2 - 1} d\omega_1 d\omega_2. \tag{5.1}
\]

Let us explain the notation. Here \(\{\omega_1\}\) and \(\{\omega_2\}\) are arbitrary simple, positively oriented loops in \(\mathbb{C}\) with the following properties:

- Each of the contours surrounds the finite interval \([0, \sqrt{\xi}]\) and leaves outside the semi-infinite interval \([1/\sqrt{\xi}, +\infty)\) \(\subset \mathbb{R}\).
- On the direct product of the contours, \(\omega_1 \omega_2 \neq 1\), so that the denominator \(\omega_1 \omega_2 - 1\) in (5.1) does not vanish.

The simplest contours satisfying these conditions are the circles centered at 0, with radii slightly greater than 1. However, to pass to the limit as \(\xi \to 1\), we will deform these contours to a more sophisticated form, as explained below.

To make the integrand meaningful, we have to specify the branches of the power functions entering (5.1), and this is done in the following way. For the terms \((1 - \sqrt{\xi \omega})^\alpha\) (where \(\omega\) stands for \(\omega_1\) or \(\omega_2\) and \(\alpha\)

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\(^4\)Another integral representation, given in [BO4, Thm.3.3] and [BO3, Thm. 6.3], seems to be less suitable for our purpose.
equals $z' + x - \frac{1}{2}$ or $z + y - \frac{1}{2}$), we use the fact that $\{\omega\}$ is contained in the simply connected region $C \setminus [1/\sqrt{\xi}, +\infty)$ and specify the branch by setting $\arg(1 - \sqrt{\xi}\omega) = 0$ for real negative values of $\omega$. Likewise, the terms $\omega \mapsto (1 - \sqrt{\xi}/\omega)^\alpha$ are well defined in the simply connected region $(\mathbb{C} \cup \{\infty\}) \setminus [0, \sqrt{\xi}]$, with the convention that $\arg(1 - \sqrt{\xi}/\omega) = 0$ for real $\omega$ greater than $\sqrt{\xi}$.

Finally, note that the $\Gamma$–factors in the numerator are not singular because their arguments are not integers. Indeed, parameters $z$ and $z'$ are forbidden to take integral values while $x - \frac{1}{2}$ and $y - \frac{1}{2}$ are integers. As for the $\Gamma$–factors in the denominator, their product is strictly positive, again by virtue of the basic conditions on the parameters. Thus, we may and do assume that the square root extracted from this product is positive, too.

To perform the limit transition as $\xi \to 1$ we will need a special contour $C(R, r, \xi)$ in the complex $\omega$–plane. This contour depends on the parameters $R > 0$, $r > 0$, and $\xi$, and looks as follows (see Fig. 1):

![Figure 1. The contour $C(R, r, \xi)$: $A = Re^{i\theta}$, $B = 1/\sqrt{\xi}$, $C = \sqrt{\xi}$.](image)

Here we assume that the parameter $r > 0$ is small enough while the parameter $R > 0$ is big enough. The integration path starts at the point $A = Re^{i\theta}$ with small $\theta > 0$ such that $\Im \omega = R \sin \theta = r$, first goes along the circle $|\omega| = R$ in the positive direction till the point $Re^{-i\theta}$, then goes in parallel to the real line until the point $1/\sqrt{\xi} - ir$, further goes to the point $1/\sqrt{\xi} + ir$ along the left semicircle $|\omega - 1/\sqrt{\xi}| = r$ (so that 0 and $\sqrt{\xi}$ are left on the left), and finally returns to the initial point $Re^{i\theta}$ in parallel to the real line. In 

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5In [BO3] the definition of the contour was slightly different: there we assumed that before and after going around 0, the path goes exactly on the positive real axis. In the context of the present section such a definition works equally well but it is not suitable for the integral representation appearing in Section 6. This is why we have to slightly modify the definition of [BO3].
whole contour provided that $r$ is so small that $1/\sqrt{\xi} - r > 1$. This condition also implies that $\sqrt{\xi}$ lies inside the contour, as required.

Next, consider the following contour in the complex $u$–plane (see Fig. 2). Here $\rho > 0$ is the parameter, the integration path starts at infinity, goes towards 0 in the right half–plane, along the line $\Im u = -\rho$, then turns around 0 in the negative direction along the semicircle $|u| = \rho$, and finally returns to infinity in the right half–plane, along the line $\Im u = \rho$. Let us denote this contour as $[+\infty - i\rho, 0-, +\infty + i\rho]$.

Figure 2. The contour $[+\infty - i\rho, 0-, +\infty + i\rho]$.

Proposition 5.1. Assume $x, y \in \mathbb{Z}'$ are fixed. There exists the limit

$$
\lim_{\xi \to 1} K_{z, z', \xi}(x, y) = K_{z, z'}(x, y)
$$

with

$$
K_{z, z'}(x, y) = \frac{\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}}
\times \frac{1}{(2\pi i)^2} \oint_{\{u_1\}} \oint_{\{u_2\}} \frac{(-u_1)^{z' + x - \frac{1}{2}}(1 + u_1)^{-z + x - \frac{1}{2}}(-u_2)^{z + y - \frac{1}{2}}(1 + u_2)^{-z - y - \frac{1}{2}}}{u_1 + u_2 + 1} du_1 du_2,
$$

(5.2)

where both contours $\{u_1, 2\}$ are of the form $[+\infty - i\rho, 0-, +\infty + i\rho]$ as defined above, with $\rho < \frac{1}{2}$.

Comments. To give a sense to the function $(-u)^\alpha$ with $\alpha \in \mathbb{C}$ we cut the complex $u$–plane along $[0, +\infty)$ and agree that the argument of $-u$ equals 0 when $u$ intersects the negative real axis $(0, -\infty)$. This is equivalent to say that the argument of $-u$ equals $+\pi$ just below the cut and $-\pi$ just above the cut. Thus, one could remove the minus sign from $(-u_1)^{z' + x - \frac{1}{2}}$ and $(-u_2)^{z + y - \frac{1}{2}}$ and put instead in front of the integral the extra factor $e^{i\pi(z' + x + y - 1)}$, with the understanding that the argument of $u$ is equal to 0 just below the cut and to $-2\pi$ just above it.

We also assume that the branch of $(1 + u)^C$ (where $u$ is $u_1$ or $u_2$ and $C$ equals $-z - x - \frac{1}{2}$ or $z + y - \frac{1}{2}$) is defined with the understanding that the argument of $1 + u$ is in $(-\pi/2, \pi/2)$.
Proof. The existence of the limit kernel was first established in [BO]. In that paper we worked with the integrable form of the kernels $K_{z,z'}(x,y)$ and $K_{z,z'}(x,y)$, as written down above in Subsection 1.4. In the present form, the claim of the proposition is a particular case of a more general result obtained in [BO3, §9]. I will reproduce, with minor variations and with more details, the argument of [BO3] because all its steps will be employed in the sequel.

Step 1. Let us check that the double integral in (5.2) is absolutely convergent:

$$\int \int \frac{(-u_1)^{z'+x-\frac{1}{2}}(1+u_1)^{-z-x-\frac{1}{2}}(-u_2)^{z+y-\frac{1}{2}}(1+u_2)^{-z'-y-\frac{1}{2}}}{u_1 + u_2 + 1} du_1 du_2 < +\infty$$

(5.3)

First of all, the restriction $\rho < \frac{1}{2}$ guarantees that the denominator $u_1 + u_2 + 1$ remains separated from 0 as $u_1$ and $u_2$ range over the contours.

Set $\mu = \Re(z' - z)$. For the principal series $\mu = 0$, and for the complementary series $-1 < \mu < 1$.

Note that the modulus of the integrand is bounded from above, so that we have to check the convergence only in the case when at least one of the variables goes to infinity.

Fix a constant $C > 0$ large enough. In the region where $|u_1| \leq C$ and $|u_2| \leq C$, as was already pointed out, there is no problem of convergence.

Assume $|u_2| \leq C$ while $|u_1| \geq C$. Then we may exclude $u_2$. That is, we replace the quantity $(-u_2)^{z+y-\frac{1}{2}}(1+u_2)^{-z'-y-\frac{1}{2}}$ by an appropriate constant and also use the bound

$$\frac{1}{|u_1 + u_2 + 1|} \leq \text{const} \frac{1}{|u_1|}.$$ 

This allows one to discard integration over $u_2$. Further, for large $|u_1|$, on our contour, the arguments of $|u_1|$ and $|1 + u_1|$ are small, which makes it possible to replace both $u_1$ and $1 + u_1$ by the real variable $u = \Re u_1$. Then we are lead to the one–dimensional integral

$$\int_C^{+\infty} u^{\mu-2} du$$

whose convergence is obvious because $\mu < 1$. Interchanging $u_1 \leftrightarrow u_2$ gives the same effect, due to the symmetry $z \leftrightarrow z'$.

---

Incidentally I will also correct minor inaccuracies in [BO3 §9].
Assume now that both variables are large, \(|u_1| \geq C\) and \(|u_2| \geq C\). Then the same argument as above leads to the real integral

\[
\int\int_{u_1 \geq C, u_2 \geq C} \frac{u_1^{\mu-1} u_2^{\mu-1}}{u_1 + u_2} du_1 du_2.
\]

To handle it we use the bound

\[
\frac{1}{u_1 + u_2} \leq \frac{1}{u_1^{\nu} u_2^{1-\nu}},
\]

which holds for any \(\nu \in (0, 1)\). Let us choose \(\nu = \frac{1}{2} + \frac{1}{2} \mu\); then \(1 - \nu = \frac{1}{2} - \frac{1}{2} \mu\). Since \(\mu \in (-1, 1)\), the requirement \(\nu \in (0, 1)\) is satisfied. This bound reduces our double integral to the product of two simple integrals of the form

\[
\int_{u \geq C} u^{\pm \frac{1}{2} \mu - \frac{3}{2}} du,
\]

which are convergent.

**Step 2.** Let us turn to the kernel (5.1). Consider the contour \(\{\omega\} = C(R, r, \xi)\) where \(r = r(\xi) = (1 - \xi) \rho\) with \(\rho < \frac{1}{2}\), as above, and \(R > 0\) is large enough and fixed. It is readily verified that \(1/\sqrt{\xi} - r(\xi) > 1\). As mentioned above, this inequality guarantees that \(|\omega| > 1\) on the whole contour, so that the both contours in (5.1) can be deformed to the form \(C(R, (1 - \xi) \rho, \xi)\) without changing the value of the double integral.

Let us split each contour on two parts, the big arc on the circle \(|\omega| = R\), which we will denote as \(C^-(R, (1 - \xi) \rho, \xi)\), and the rest (inside the circle), denoted as \(C^+(R, (1 - \xi) \rho, \xi)\).

The \(\Gamma\)-factors in (5.1) and in (5.2) are the same, so that we may ignore them. On the contrary, the prefactor \(1 - \xi\) in (5.1) will play the key role.

Our plan for the remainder of the proof is as follows: First, we show that when we restrict the double integral in (5.1) (together with the prefactor \(1 - \xi\)) on the product of two copies of \(C^+(R, (1 - \xi) \rho, \xi)\) and pass to the limit as \(\xi \to 1\), we get the integral in (5.2). Next, we check that the contribution from the rest of the double integral is asymptotically negligible.

**Step 3.** Here we assume that both \(\omega_1\) and \(\omega_2\) range over the contour \(C^+(R, (1 - \xi) \rho, \xi)\). Make the change of variables \(\omega_1 \to u_1, \omega_2 \to u_2\) according to the relation

\[
\omega = \omega_\xi(u) = \frac{1}{\sqrt{\xi}} + (1 - \xi) u.
\]

After this transformation, the contour \(C^+(R, (1 - \xi) \rho, \xi)\) turns into a truncation of the contour \([+\infty - i \rho, 0-, +\infty + i \rho]\) (we have to impose
the constraint $\Re u \leq (1 - \xi)^{-1} \tilde{R}$, where $\tilde{R} = R - (1/\sqrt{\xi}) \approx R - 1$. As $\xi$ goes to 1, the threshold of the truncation shifts to the right, and in the limit we get the whole contour $[+\infty - i\rho, 0-, +\infty + i\rho]$. Substituting

$$1 - \sqrt{\xi \omega} = -(1 - \xi) \sqrt{\xi} \cdot u$$

$$\omega - \sqrt{\xi} = (1 - \xi) \left( u + \frac{1}{\sqrt{\xi}} \right)$$

$$\omega_1 \omega_2 - 1 = \frac{1 - \xi}{\xi} \left( \sqrt{\xi}(u_1 + u_2) + 1 + \xi(1 - \xi)u_1u_2 \right)$$

$$d\omega_1 d\omega_2 = (1 - \xi)^2 du_1 du_2$$

into the integrand of (5.1), we see that all the terms consisting of various powers of $1 - \xi$, including the prefactor $1 - \xi$, cancel out. The integrand that we get can be written as the product of the integrand of (5.2) with the following expression depending on $\xi$:

$$\xi^{\frac{1}{2}(z+z'+x+y-1)} \left( \frac{1 + u_1}{\sqrt{\xi} + u_1} \right)^{z+x+\frac{1}{2}} \left( \frac{1 + u_2}{\sqrt{\xi} + u_2} \right)^{z'+y+\frac{1}{2}} \frac{u_1 + u_2 + 1}{\sqrt{\xi}(u_1 + u_2) + 1 + \xi(1 - \xi)u_1u_2} (\omega_\xi(u_1))^z (\omega_\xi(u_2))^{z'}$$

(5.5)

As $\xi$ goes to 1, this expression converges to 1 pointwise (note that $\omega_\xi(u) \to 1$). On the other hand, as $u_1$ and $u_2$ range over the contour $[+\infty - i\rho, 0-, +\infty + i\rho]$, the modulus of this expression remains bounded, uniformly on $\xi$. Indeed, this is obvious for the first four factors. As for the last term, $|((\omega_\xi(u_1))^z (\omega_\xi(u_2))^{z'})$, it is bounded because $|\omega_\xi(u)| \leq R$ and the argument of $\omega_\xi(u)$ is close to 0 for $\xi$ close to 1 (recall that $\omega_\xi(u)$ is close to $[1, +\infty]$).

Consequently, by virtue of step 1, our integral converges absolutely and uniformly on $\xi$, so that we may pass to the limit under the sign of the integral, which results in the desired integral (5.2).

Step 4. On this last step, we will check that the contribution from the remaining parts of the contours, together with the prefactor $1 - \xi$, is asymptotically negligible. To do this, let us evaluate the modulus of the integrand in (5.1).

The crucial observation is that the factor $\omega_1 \omega_2 - 1$ in the denominator remains separated from 0. Indeed, when at least one of the variables ranges over the big arc $C^{-}(R, (1 - \xi)\rho, \xi)$ (which is just our case), we have $|\omega_1 \omega_2| \geq R > 1$. Consequently, we may ignore this factor,
and then our double integral factorizes into the product of two one-dimensional integrals: one is
\[ \oint_{\{\omega_1\}} \left(1 - \sqrt{\xi} \omega_1\right)^{z'+x-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z-x-\frac{1}{2}} \omega_1^{-x-\frac{1}{2}} d\omega_1 \]
and the other has the same form, only \(x\) is replaced by \(y\) and \(z\) is interchanged with \(z'\):
\[ \oint_{\{\omega_2\}} \left(1 - \sqrt{\xi} \omega_2\right)^{z+y-\frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{-z'-y-\frac{1}{2}} \omega_2^{-y-\frac{1}{2}} d\omega_2 \].

If one of the variables \(\omega_1, \omega_2\) ranges over the big arc \(C^-(R, (1-\xi)\rho, \xi)\), then the integrand of the corresponding integral is bounded uniformly on \(\xi\), so that this integral remains bounded. Thus, the case when both \(\omega_1\) and \(\omega_2\) range over the big arc is trivial: the prefactor \(1 - \xi\) forces the whole expression to go to 0.

Examine now the case when one of the variables (say, \(\omega_1\)) ranges over \(C^+(R, (1-\xi)\rho, \xi)\), while the other variable (hence, \(\omega_2\)) ranges over the big arc. Then we are left with the first integral together with the prefactor \(1 - \xi\).

Let us make the same change of a variable as above: \(\omega_1 = \omega \xi(u)\). Then the integral reduces to
\[ (1 - \xi)^\mu (\sqrt{\xi})^{z'+x-\frac{1}{2}} \oint_{\{u\}} (-u)^{z'+x-\frac{1}{2}} \left(u + \frac{1}{\sqrt{\xi}}\right)^{-z-x-\frac{1}{2}} (\omega \xi(u))^z du \].

The quantity \(|(\omega \xi(u))^z|\) is bounded uniformly on \(\xi\) and hence may be ignored. The factor \((\sqrt{\xi})^{z'+x-\frac{1}{2}}\) is inessential, too. Further, using the same argument as on step 1, we reduce our expression to the real integral
\[ (1 - \xi)^\mu \int_{C^1/(1-\xi)}^u \mu^{-1} du \]
where the upper limit arises due to the fact that \(\Re(\omega \xi(u)) \leq R\). Recall that \(\mu = \Re(z' - z) \in (-1, 1)\).

If \(\mu \in (-1, 0)\) then the integral is uniformly convergent, so that the whole expression grows as \((1 - \xi)^\mu\).

If \(\mu = 0\) then the integral grows as \(\log((1 - \xi)^{-1})\), and so is the growth of the whole expression.

If \(\mu \in (0, 1)\) then the integral grows as \((1 - \xi)^{-\mu}\) and the whole expression remains bounded.
In all these cases the small prefactor \((1 - \xi)\) in (5.1) dominates and makes the result asymptotically negligible. □

**Proposition 5.2.** The matrix

\[
\left[(m + \frac{1}{2})^{-1/2}(K_{z,z'})_{++}(m + \frac{1}{2}, n + \frac{1}{2})(n + \frac{1}{2})^{-1/2}\right]_{m,n=0,1,2,...}
\]

is of trace class.

*Proof.\* We have to prove that

\[
\sum_{m=0}^{\infty} (m + \frac{1}{2})^{-1}(K_{z,z'})_{++}(m + \frac{1}{2}, m + \frac{1}{2}) < +\infty.
\]

Since the \(++\) blocks of \(K_{z,z'}\) and \(K_{z,z'}\) are the same, we may use formula (5.2). On the diagonal \(x = y = m + \frac{1}{2}\), the ratio in (5.2) formed by the \(\Gamma\)-factors equals 1, so that we may omit them. Therefore,

\[
(K_{z,z'})_{++}(m + \frac{1}{2}, m + \frac{1}{2}) = \frac{1}{(2\pi i)^2} \oint_{\{u_1\}} \oint_{\{u_2\}} \frac{(-u_1)^z z^m (1 + u_1)^{-z - m - 1}(-u_2)^z z^m (1 + u_2)^{-z - m - 1} \, du_1 du_2}{u_1 + u_2 + 1}
\]

where the contours are the same as in (5.2). We have

\[
(K_{z,z'})_{++}(m + \frac{1}{2}, m + \frac{1}{2}) \leq \frac{1}{4\pi^2} \oint_{\{u_1\}} \oint_{\{u_2\}} \left| \frac{(-u_1)^z z^m (1 + u_1)^{-z - m - 1}(-u_2)^z z^m (1 + u_2)^{-z - m - 1} \, du_1 du_2}{u_1 + u_2 + 1} \right|
\]

\[
= \frac{1}{4\pi^2} \oint_{\{u_1\}} \oint_{\{u_2\}} \left| \frac{u_1 u_2}{(1 + u_1)(1 + u_2)} \right|^m \left| \frac{(-u_1)^z z^m (1 + u_1)^{-z - m - 1}(-u_2)^z z^m (1 + u_2)^{-z - m - 1} \, du_1 du_2}{u_1 + u_2 + 1} \right|.
\]

Introduce the factor \((m + \frac{1}{2})^{-1}\) inside the integral and sum over \(m = 0, 1, \ldots\). Observe that

\[
\sum_{m=0}^{\infty} (m + \frac{1}{2})^{-1} \left| \frac{u_1 u_2}{(1 + u_1)(1 + u_2)} \right|^m \leq 2 + 2 \sum_{m=1}^{\infty} m^{-1} \left| \frac{u_1 u_2}{(1 + u_1)(1 + u_2)} \right|^m
\]

\[
= 2 + 2 \log \left( \frac{1}{1 - \left| \frac{u_1 u_2}{(1 + u_1)(1 + u_2)} \right|} \right) =: F(u_1, u_2).
\]
Thus, we have to prove that
\[
\oint_{\{u_1\} \{u_2\}} F(u_1, u_2) \left| \frac{(-u_1')^z(1 + u_1)^{-z}(-u_2)(1 + u_2)^{-z}du_1du_2}{u_1 + u_2 + 1} \right| < +\infty.
\]

Without the factor \(F(u_1, u_2)\), this integral coincides with the integral (5.3) with \(x = y = \frac{1}{2}\), whose convergence has already been verified (see step 1 in the proof of Proposition 5.1). Let us show that the extra factor \(F(u_1, u_2)\) does not add very much. Indeed, it grows only as both \(u_1\) and \(u_2\) go to infinity, so that we may assume that both \(u_1\) and \(u_2\) are far from the origin. If \(u\) is a point on one of the contours, far from the origin, then writing \(u = |u| e^{i\theta}\) we have \(|u|\) large and \(\theta\) small. Then
\[
\left| \frac{u}{1 + u} \right|^2 = \frac{|u|^2}{|u|^2 + 2|u|\cos \theta + 1} = 1 - 2|u|^{-1}\cos \theta + O(|u|^{-2})
\]
and consequently
\[
\left| \frac{u}{1 + u} \right| = 1 - |u|^{-1}\cos \theta + O(|u|^{-2})
\]
with \(\cos \theta \to 1\) as \(u \to \infty\). This allows one to estimate the growth of the logarithm in \(F(u_1, u_2)\). Omitting unessential details we get that it behaves roughly as
\[
\log \left( \frac{1}{|u_1|^{-1} + |u_2|^{-1}} \right) \leq \text{const} \cdot |u_1|^{\delta} |u_2|^{\delta},
\]
where \(\delta > 0\) can be chosen arbitrarily small.

Using this bound and examining the argument of step 1 in Proposition 5.1 we see that the same arguments work equally well with the extra factor \(F(u_1, u_2)\).

In the sequel we use the notation
\[
\epsilon = 1 - \xi.
\]

**Lemma 5.3.** Fix an arbitrary \(c \in (0, \frac{1}{2})\). If \(R\) is large enough and \(\rho < \frac{1}{2}\) then for all sufficiently small \(\epsilon\) the following estimate holds
\[
\left| \frac{1 - \sqrt{\xi} \omega}{\omega - \sqrt{\xi}} \right| < 1 - c\epsilon, \quad \omega \in C(R, \epsilon, \rho, \xi).
\]

**Proof.** Consider the transform \(\omega \mapsto \omega' = \frac{1 - \sqrt{\xi} \omega}{\omega - \sqrt{\xi}}\). Its inverse has the form \(\omega = \frac{1 + \sqrt{\xi} \omega'}{\omega' + \sqrt{\xi}}\) and sends the interior part of the circle \(|\omega'| = 1 - c\epsilon\) to the exterior part \(S_{\text{ext}}\) of a circle \(S\). Thus, we have to check that \(C(R, \epsilon, \rho, \xi) \subset S_{\text{ext}}\).
The circle $S$ is symmetric relative the real axis and intersects it at the points $\omega_\mp$ corresponding to $\omega'_\mp = \mp(1 - c\varepsilon)$. Let us prove that both $\omega_-$ and $\omega_+$ are on the left of $\frac{1}{\sqrt{\xi}} - \varepsilon\rho$, the leftmost point of the contour $C^+(R, \varepsilon\rho, \xi)$.

We have

$$\omega_- = \frac{1 - \sqrt{\xi}(1 - c\varepsilon)}{-(1 - c\varepsilon) + \sqrt{\xi}} = \frac{1 - (1 - \frac{1}{2}c\varepsilon + O(\varepsilon^2))(1 - c\varepsilon)}{-(1 - c\varepsilon) + (1 - \frac{1}{2}c\varepsilon + O(\varepsilon^2))} = \frac{c + \frac{1}{2}}{c - \frac{1}{2}} + O(\varepsilon)$$

and

$$\omega_+ = \frac{1 + \sqrt{\xi}(1 - c\varepsilon)}{(1 - c\varepsilon) + \sqrt{\xi}} = \frac{1 + (1 - \sqrt{\xi})c\varepsilon}{(1 - c\varepsilon) + \sqrt{\xi}} = 1 + O(\varepsilon^2).$$

Since $c < \frac{1}{2}$, $\omega_-$ lies on the left of 0. As for $\omega_+$, it lies on the left of

$$\frac{1}{\sqrt{\xi}} - \varepsilon\rho \approx 1 + (\frac{1}{2} - \rho)\varepsilon,$$

because $\rho < \frac{1}{2}$ by the assumption.

This shows that the interior part of the contour, $C^+(R, \varepsilon\rho, \xi)$ lies in $S^{\text{ext}}$. The same also holds for the exterior part, $C^-(R, \varepsilon\rho, \xi)$, because $R$ can be made arbitrarily large. Indeed, as seen from the above expressions for the points $\omega_-$ and $\omega_+$, they do not run to infinity as $\xi \to 1$, so that if $R$ is chosen large enough, the circle of radius $R$ will enclose the circle $S$ for any $\xi$. □

**Proposition 5.4.**

$$\lim_{\xi \to 1} \left( \sum_{m \geq 0} \left( m + \frac{1}{2} \right)^{-1} (K_{z,z',\xi})_{++}(m + \frac{1}{2}, m + \frac{1}{2}) \right) = \sum_{m \geq 0} \left( m + \frac{1}{2} \right)^{-1} (K_{z,z'})_{++}(m + \frac{1}{2}, m + \frac{1}{2}).$$

**Proof.** We will argue as in the proof of Proposition 5.1, steps 2–4. The role of step 1 in that proof will be played by Proposition 5.2.

First of all, observe that on the diagonal $x = y$, the expression formed by the $\Gamma$–factors in front of the integrals in (5.1) and (5.2) equals 1. Consequently, we may ignore these factors.

Let $F_{m,\xi}(\omega_1, \omega_2)$ denote the integrand in the integral (5.1) corresponding to $x = y = m + \frac{1}{2}$. Likewise, let $F_{m}(u_1, u_2)$ denote the integrand in the integral (5.2) with $x = y = m + \frac{1}{2}$. We have to prove
that
\[
\lim_{\xi \to 1} \left( (1 - \xi) \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right)^{-1} \oint \oint F_{m;\xi}(\omega_1, \omega_2)d\omega_1 d\omega_2 \right)
\]
\[
= \sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right)^{-1} \oint \oint F_m(u_1, u_2) du_1 du_2. \tag{5.6}
\]

We already dispose of all the necessary information to conclude that (5.6) holds provided that we truncate the both contours in the left-hand side to \( C^+(R, \varepsilon \rho, \xi) \). Indeed, the ratio
\[
\frac{(1 - \xi)F_{m;\xi}(\omega_1, \omega_2)}{F_m(u_1, u_2)}
\]
is the particular case of (5.5) corresponding to \( x = y = m + \frac{1}{2} \). The argument of step 3 in Proposition 5.1 shows that this expression goes to 1 pointwise. Moreover, its modulus is bounded uniformly on both \( \xi \) and \( m \): To see this observe that
\[
\left| \frac{1 + u}{\sqrt{\xi} + u} \right| \leq 1, \quad u = u_1, u_2,
\]
and recall that \( |\omega_\xi(u)| \leq R_1, \ u = u_1, u_2. \)

Together with the result of Proposition 5.2 this implies the claim.

Now we have to check that the contribution of the remaining parts of the contours to the left-hand side of (5.6) is asymptotically negligible. This can be done by slightly modifying the argument of step 4 in Proposition 5.1.

Indeed, the integrand \( F_{m;\xi}(\omega_1, \omega_2) \) can be written in the form
\[
F_{m;\xi}(\omega_1, \omega_2) = F_{0;\xi}(\omega_1, \omega_2) \left( \frac{1 - \sqrt{\xi} \omega_1}{\omega_1 - \sqrt{\xi}} \right)^m \left( \frac{1 - \sqrt{\xi} \omega_2}{\omega_2 - \sqrt{\xi}} \right)^m.
\]
Compare our task with the one we had on step 4 in Proposition 5.1 (where we set \( x = y = \frac{1}{2} \)). The only difference is that now we have in the integral the extra factor equal to
\[
\sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right)^{-1} \left| \left( \frac{1 - \sqrt{\xi} \omega_1}{\omega_1 - \sqrt{\xi}} \right) \left( \frac{1 - \sqrt{\xi} \omega_2}{\omega_2 - \sqrt{\xi}} \right) \right|^m. \tag{5.7}
\]
By Lemma 5.3 the quantity (5.7) is majorated by
\[
\sum_{m=0}^{\infty} \left( m + \frac{1}{2} \right)^{-1} (1 - c \xi)^{2m},
\]
which grows as $\log(\varepsilon^{-1})$. Such an extra factor does not affect the estimates on step 4 of Proposition 5.1.

6. CONVERGENCE OF OFF–DIAGONAL BLOCKS IN HILBERT–SCHMIDT NORM

Here we prove the claims of Theorem 4.3 for the off–diagonal blocks. As in Section 5, we apply the symmetry relations of Proposition 1.9 to reduce the case of the $-+$ block to that of the $+-$ block, and due to the relations of Subsection 1.5 we may freely switch from $K_{z,z',\xi}$ to $K_{z',z,\xi}$.

However, to compute the Hilbert–Schmidt norm of the $-+$ block we cannot use anymore the basic contour integral representation (5.1) as we did in Section 5, because this would lead to divergent series in the integrand. The reason of this is that the variable $y$, which previously ranged over $\mathbb{Z}'_+$, will now range over $\mathbb{Z}'_-$. It turns out that we still may employ essentially the same estimates and arguments as in Section 5 but beforehand we have to change the integral representation (5.1).

Return to the initial version of the contours $\{\omega_1\}$ and $\{\omega_2\}$ in (5.1), when they were circles slightly greater than the unit circle, and make the change of a variable $\omega_2 \to \omega_2^{-1}$. Then the former condition $\omega_1 \omega_2 \neq 1$ will turn into the requirement that the second contour must lie inside the first contour. Further, we deform the contours so that they take the form $C(R_1, \varepsilon \rho_1, \xi)$ and $C(R_2, \varepsilon \rho_2, \xi)$, respectively, where $R_1 > R_2$ and $\rho_1 < \rho_2$: these inequalities just guarantee that the contour $C(R_2, \varepsilon \rho_2, \xi)$ lies inside $C(R_1, \varepsilon \rho_1, \xi)$.

After the transform $\omega_2 \to \omega_2^{-1}$ the formula (5.1) turns into

$$K_{z,z',\xi}(x; y) = \frac{\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}} \times \frac{1 - \xi}{(2\pi i)^2} \oint_{\{\omega_1\}} \oint_{\{\omega_2\}} \left(1 - \sqrt{\xi \omega_1}\right)^{z' + x - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_1}\right)^{-z - x - \frac{1}{2}} \times \left(1 - \sqrt{\xi \omega_2}\right)^{-z' - y - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{\omega_2}\right)^{z + y - \frac{1}{2}} \frac{\omega_1^{-x - \frac{1}{2}} \omega_2^{-y - \frac{1}{2}}}{\omega_1 - \omega_2} d\omega_1 d\omega_2. \quad (6.1)$$

According to Proposition 5.1, for any fixed $x, y \in \mathbb{Z}'$, there exists a limit

$$\lim_{\xi \to 1} K_{z,z',\xi}(x, y) = K_{z,z'}(x, y),$$
for which we dispose of the double contour integral representation (5.2). However, now we will need a different integral representation, which is consistent with (6.1):

**Proposition 6.1.** The following formula holds

\[
K_{z,z'}(x,y) = \frac{\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})}{(\Gamma(-z - x + \frac{1}{2})\Gamma(-z' - x + \frac{1}{2})\Gamma(-z - y + \frac{1}{2})\Gamma(-z' - y + \frac{1}{2}))^{\frac{1}{2}}}
\]

\[
\times \frac{1}{(2\pi i)^2} \oint_{\{u_1\}} \oint_{\{u_2\}} \frac{(u_1)^{z' + x - \frac{1}{2}(1 + u_1)}(1 + u_1)^{-z + x - \frac{1}{2}(-u_2)}(1 + u_2)^{z' - y - \frac{1}{2}}}{(u_1 - u_2)^{z + y - \frac{1}{2}}} du_1 du_2.
\]

(6.2)

where the contours have the form

\[\{u_1\} = [+\infty - i\rho_1, 0 -, +\infty + i\rho_1], \quad \{u_2\} = [+\infty - i\rho_2, 0 -, +\infty + i\rho_2]\]

and \(\rho_1 < \rho_2\).

**Proof.** The \(\Gamma\)-factors in (6.1) and (6.2) are the same, so that it suffices to prove that the integral in (6.1) together with the prefactor \((1 - \xi)\) converges to the integral in (6.2). We will follow the scheme of the proof of Proposition 5.1.

**Step 1.** Let us check that that the integral (6.2) is absolutely convergent. Arguing as on step 1 of Proposition 5.1 we reduce this claim to finiteness of the real integral

\[
\int \int \frac{u_1^{\mu-1} u_2^{\mu-1}}{|u_1 - u_2|^\rho} du_1 du_2,
\]

(6.3)

where \(\mu = \Re(z' - z) \in (-1, 1)\), \(\rho = \rho_2 - \rho_1 > 0\), and \(C\) is a positive constant.

By symmetry we may assume \(u_1 \geq u_2\). Making the change of variables \(u_1 = u + a, \ u_2 = u\), where \(u \geq C, a \geq \rho\), we get the integral

\[
\int_{a \geq \rho} a^{-\rho} \int_{u \geq C} (u + a)^{\mu-1} u^{-\mu-1} du
\]

\[
= \int_{a \geq \rho} a^{-2} \int_{u \geq C/a} (u + 1)^{\mu-1} u^{-\mu-1} du.
\]

Consider the interior integral in the second line: For large \(u\), the integrand decays as \(u^{-2}\), which is integrable near infinity, while for \(u\) near 0, the integrand behaves as \(u^{-\mu-1}\). It follows that if \(\mu < 0\) then the integral over \(u\) converges uniformly on \(a\); if \(\mu = 0\) then it grows as \(\log a\)
for large $a$; and if $\mu > 0$ then it grows as $a^\mu$. Consequently, the double integral can be estimated by one of the three convergent integrals
\[
\int_{a \geq \rho} a^{-2} da, \quad \int_{a \geq \rho} a^{-2} \log a \, da, \quad \int_{a \geq \rho} a^{\mu-2} \, da \quad (0 < \mu < 1).
\]

**Step 2.** Relying on the result of step 1, we can verify the required convergence of the integrals provided that we restrict the integration in (6.1) to interior parts of the contours. This is done exactly as on step 3 of Proposition 5.1. A minor simplification is that under the change of a variable (5.4), the quantity $\omega_1 - \omega_2$ is transformed simpler than $\omega_1 \omega_2 - 1$.

**Step 3.** Finally, we have to prove that the contributions from the remaining parts of the contours is asymptotically negligible due to the prefactor $1 - \xi$. Here we argue exactly as on step 4 of Proposition 5.1, with the only exception: In the case when $\omega_1$ ranges over the interior part of the contour, $C^+(R_1, \varepsilon \rho_1, \xi)$, while $\omega_2$ ranges over the exterior part, $C^-(R_2, \varepsilon \rho_2, \xi)$, we cannot automatically discard the denominator $\omega_1 - \omega_2$. The reason is that in this special case, the two parts of the contours come close at the distance of order $\varepsilon$, which happens near the point $R_2$.

This difficulty can be resolved in the following way. Dissect $C^+(R_1, \varepsilon \rho_1, \xi)$ into two parts by a vertical line $\Re \omega_1 = \text{const}$ so that the points on the left be separated from $R_2$ while the points on the right be separated from $1/\sqrt{\xi}$. Now we have two cases:

When $\omega_1$ ranges over the left part, we may discard the denominator $\omega_1 - \omega_2$ and argue as on step 4 of Proposition 5.1.

When $\omega_1$ ranges over the right part, the argument is different: Observe that the modulus of the whole integrand in (6.1), except the denominator $\omega_1 - \omega_2$, is bounded uniformly on $\xi$, while the quantity $|\omega_1 - \omega_2|^{-1}$ is integrable, so that the prefactor $1 - \xi$ makes the contribution negligible. Checking the integrability of $|\omega_1 - \omega_2|^{-1}$ is easy: Indeed, here it is even unessential that $C^-(R_2, \varepsilon \rho_2, \xi)$ does not contain a small arc around the point $R_2$. What is important is that the singularity $|\omega_1 - \omega_2|^{-1}$ arising near the point $R_2$ is of the same kind as the singularity $(a^2 + b^2)^{-1/2}$ in the real $(a, b)$-plane near the origin. \[\Box\]

Although formulas (6.1) and (6.2) are valid for any $x, y \in \mathbb{Z}'$, we will deal exclusively with $x \in \mathbb{Z}'_+$ and $y \in \mathbb{Z}'_-$. Below we set
\[
x = m + \frac{1}{2}, \quad y = -n - \frac{1}{2}, \quad m, n \in \mathbb{Z}_+.
\]
and use the notation
\[(K_{z,z',\xi})_{+}(m, n) = (-1)^{n}K_{z,z',\xi}(m + \frac{1}{2}, -n - \frac{1}{2})\]
\[(K_{z,z'})_{+}(m, n) = (-1)^{n}K_{z,z'}(m + \frac{1}{2}, -n - \frac{1}{2}).\]

Here the factor \((-1)^{n}\) comes from the factor \(\varepsilon(y)\) in (1.2).

The next result is similar to Proposition 5.2:

**Proposition 6.2.** The matrix
\[
\left[(m + \frac{1}{2})^{-1/2}(K_{z,z'})_{+}(m, n)(n + \frac{1}{2})^{-1/2}\right]_{m,n \in \mathbb{Z}'_+}
\]
belongs to the Hilbert–Schmidt class.

**Proof.** Since all the matrix entries are real (Remark 1.7), the claim means that the series
\[
\sum_{m,n=0}^{\infty} (m + \frac{1}{2})^{-1}(n + \frac{1}{2})^{-1}((K_{z,z'})_{+}(m, n))^{2}
\]
is finite.

To get the squared matrix entry we multiply out two copies of the double integral representation (6.2), where in the second copy we swap \(z\) and \(z'\). This operation does not change the kernel, as it follows from a series expansion for the kernel (see [BO4, (3.3)]) and the fact that the functions entering this expansion depend symmetrically on \(z\) and \(z'\). On the other hand, after this operation the \(\Gamma\)–factors in front of the integral will cancel out (this trick is borrowed from [BO3] and [BO4], see, e.g., the proof of Proposition 2.3 in [BO4]). The resulting expression for the sum (6.4) can be written in the form
\[
\sum_{m,n \in \mathbb{Z}'_+} (m + \frac{1}{2})^{-1}(n + \frac{1}{2})^{-1}
\times \frac{1}{16\pi^4} \int \int \int \int F_{mn}(u_1, u_2)F'_{mn}(u'_1, u'_2)du_1du_2du'_1du'_2,
\]
where
\[
\{u_1\} = \{u'_1\} = [+\infty - i\rho_1, 0-, +\infty + i\rho_1],
\{u_2\} = \{u'_2\} = [+\infty - i\rho_2, 0-, +\infty + i\rho_2]
\]
and
\[
F_{mn}(u_1, u_2) = \frac{(-u_1)^{z+m}(1 + u_1)^{-z-m-1}(-u_2)^{-z+n}(1 + u_2)^{z-n-1}}{u_1 - u_2},
\]
\[
F'_{mn}(u'_1, u'_2) = \frac{(-u'_1)^{z'+m}(1 + u'_1)^{-z'-m-1}(-u'_2)^{-z'+n}(1 + u'_2)^{z'-n-1}}{u'_1 - u'_2}.
\]
(in the second line $z$ and $z'$ are interchanged).

To show that the sum (6.5) is finite we replace the integrand by its modulus and then interchange summation and integration. Then we get the integral

$$
\frac{1}{16\pi^4} \oint \oint \oint \oint \sum_{m,n \in \mathbb{Z}_+} (m + \frac{1}{2})^{-1} (n + \frac{1}{2})^{-1} \times |F_{mn}(u_1, u_2)F'_{mn}(u'_1, u'_2) du_1 du_2 du'_1 du'_2|. \tag{6.6}
$$

It suffices to check that it is finite.

We have

$$
\sum_{m,n=0}^{\infty} (m + \frac{1}{2})^{-1} (n + \frac{1}{2})^{-1} |F_{mn}(u_1, u_2)F'_{mn}(u'_1, u'_2)| = |F_{00}(u_1, u_2)F'_{00}(u'_1, u'_2)|
$$

$$
\times \sum_{m,n=0}^{\infty} (m + \frac{1}{2})^{-1} (n + \frac{1}{2})^{-1} \left| \frac{u_1 u'_1}{u_1 + u'_1} \right|^m \left| \frac{u_2 u'_2}{u_2 + u'_2} \right|^n.
$$

The same argument as in the proof of Proposition 5.2 shows that for the latter sum there exists the upper bound of the form

$$
\sum_{m,n=0}^{\infty} (m + \frac{1}{2})^{-1} (n + \frac{1}{2})^{-1} \left| \frac{u_1 u'_1}{u_1 + u'_1} \right|^m \left| \frac{u_2 u'_2}{u_2 + u'_2} \right|^n \leq \text{const} |u_1|^\delta |u'_1|^\delta |u_2|^\delta |u'_2|^\delta,
$$

where $\delta > 0$ can be chosen as small as is needed.

Substituting this estimate into the 4-fold integral (6.6) leads to its splitting into the product of two double integrals, one of which is

$$
\oint \oint |u_1|^{\delta} |u_2|^{\delta} \frac{(-u_1)^z(1+u_1)^{-z-1}(-u_2)^{-z}(1+u_2)^z}{u_1 - u_2} du_1 du_2. \tag{6.7}
$$

and the other has the similar form, with $z$ and $z'$ interchanged. Therefore, it suffices to prove the finiteness of the integral (6.7).

Arguing as on step 1 of Proposition 5.1 we reduce this integral to the real integral

$$
\int \int_{u_1, u_2 \geq C, |u_1 - u_2| \geq \rho} \frac{u_1^{\mu + \delta - 1} u_2^{-\mu + \delta - 1}}{|u_1 - u_2|} du_1 du_2. \tag{6.8}
$$

This integral only slightly differs from the integral (6.3) examined on step 1 of Proposition 6.1 and the same argument as in Proposition 6.1 shows that (6.8) is finite provided that $\delta$ is chosen small enough. \qed
Proposition 6.3. As $\xi$ goes to 1, the matrices

$$\left[ (m + \frac{1}{2})^{-1/2}(K_{z,z',\xi})_{+,+} (m + \frac{1}{2}, -n + \frac{1}{2})^{-1/2} \right]_{m,n \in \mathbb{Z}'}$$

converge to the matrix

$$\left[ (m + \frac{1}{2})^{-1/2}(K_{z,z'})_{+,+} (m + \frac{1}{2}, -n + \frac{1}{2})^{-1/2} \right]_{m,n \in \mathbb{Z}'}$$

in the topology of the Hilbert–Schmidt norm.

Proof. We know that the convergence takes place in the weak operator topology, that is, for the matrix entries (this follows from Proposition 5.1). Consequently, it suffices to prove the convergence of the squared Hilbert–Schmidt norms:

$$\sum_{m,n=0}^{\infty} (m + \frac{1}{2})^{-1}(n + \frac{1}{2})^{-1} ((K_{z,z',\xi})_{+,+}(m,n))^2$$

$$\to \sum_{m,n=0}^{\infty} (m + \frac{1}{2})^{-1}(n + \frac{1}{2})^{-1} ((K_{z,z'})_{+,+}(m,n))^2$$

(recall that our matrices are real).

We will follow the arguments of Propositions 5.4, 6.1, and 5.1.

Write the left-hand side in the form similar to (6.5). Namely, let $F_{mn,\xi}(\omega_1, \omega_2)$ denote the integrand in (6.1), where we substitute $x = m + \frac{1}{2}$ and $y = -n - \frac{1}{2}$:

$$F_{mn,\xi}(\omega_1, \omega_2) = \left(1 - \sqrt{\xi \omega_1}\right)^{z'+m} \left(1 - \sqrt{\frac{\xi}{\omega_1}}\right)^{-z'-m-1} \times \left(1 - \sqrt{\xi \omega_2}\right)^{-z'+n} \left(1 - \sqrt{\frac{\xi}{\omega_2}}\right)^{z'-n-1} \frac{\omega_1^{-m-1} \omega_2^{-n-1}}{\omega_1 - \omega_2},$$

and let $F'_{mn,\xi}(\omega'_1, \omega'_2)$ be the similar quantity with $z$ and $z'$ interchanged:

$$F'_{mn,\xi}(\omega'_1, \omega'_2) = \left(1 - \sqrt{\xi \omega_1}\right)^{z+m} \left(1 - \sqrt{\frac{\xi}{\omega_1}}\right)^{-z'-m-1} \times \left(1 - \sqrt{\xi \omega_2}\right)^{-z+n} \left(1 - \sqrt{\frac{\xi}{\omega_2}}\right)^{z'-n-1} \frac{(\omega'_1)^{-m-1}(\omega'_2)^{-n-1}}{\omega'_1 - \omega'_2}.$$
As above, swapping $z$ and $z'$ kills the $\Gamma$–factors. In this notation, the series in question takes the form

$$\frac{1}{16\pi^4} \sum_{m,n \in \mathbb{Z}^+} (m + \frac{1}{2})^{-1} (n + \frac{1}{2})^{-1} \times \oint_{\{\omega_1\} \{\omega_2\} \{\omega_1'\} \{\omega_2'\}} (1 - \xi)^2 F_{mn;\xi}(\omega_1, \omega_2) F'_{mn;\xi}(\omega_1', \omega_2') d\omega_1 d\omega_2 d\omega_1' d\omega_2', \tag{6.9}$$

where

$$\{\omega_1\} = \{\omega_1'\} = C(R_1, \varepsilon \rho_1, \xi), \quad \{\omega_2\} = \{\omega_2'\} = C(R_2, \varepsilon \rho_2, \xi),$$

and $R_1 > R_2$, $r_1 < r_2$, $\varepsilon = 1 - \xi$, as before.

First, we truncate all the contours in (6.9) keeping only their interior parts $C^+(R_i, \varepsilon \rho_i, \xi)$, and then make the change of variables according to the rule (5.4), as we did before. The prefactor $(1 - \xi)^2$ disappears, we compare the resulting integrand with that in (6.5), and check that their ratio has uniformly bounded modulus and converges pointwise to 1. This is done exactly as in step 3 of the proof of Proposition 5.4. By virtue of Proposition 6.3, we get the desired convergence provided that the both contours in (6.9) are truncated.

Next, we check that the contribution from the remaining parts of the contours in (6.9) is asymptotically negligible due to the prefactor $(1 - \xi)^2$. Again, the proof goes as in the situation of Proposition 5.4. We replace the integrand by its modulus, interchange summation and integration, and evaluate the double sum over $m$ and $n$ using Lemma 5.3. This produces a factor growing like $(\log(\varepsilon^{-1}))^2$, which we add to our small prefactor $(1 - \xi)^2$. Due to this bound, the 4–fold integral reduces to the product of two double integrals,

$$\oint_{\{\omega_1\} \{\omega_2\}} |F_{00;\xi}(\omega_1, \omega_2) d\omega_1 d\omega_2| \quad \text{and} \quad \oint_{\{\omega_1'\} \{\omega_2'\}} |F'_{00;\xi}(\omega_1', \omega_2') d\omega_1' d\omega_2'|,$$

where each of the 4 variables ranges over $C^\pm(R_i, \varepsilon \rho_i, \xi)$, and for at least one of them the corresponding superscript has to be “−”.

These integrals were already examined in the proof of Proposition 6.1. For each of the integrals, there are two possible cases: Either (a) both variables range over interior parts of the contours or (b) one of the variables ranges over the interior part while the other variable ranges over the exterior part. We know that in case (a) the integral grows as $(1 - \xi)^{-1}$, while in case (b) the growth is suppressed by the small factor $(1 - \xi)$, even if one adds the extra growing factor $(\log(\varepsilon^{-1}))^2$. Since case
(b) occurs for at least one of the double integrals, we see that the small prefactor \((1 - \xi)^2\) suffices to make the whole contribution negligible. This completes the proof. □

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Institute for Information Transmission Problems, Bolshoy Karetny 19, Moscow 127994, Russia; Independent University of Moscow, Russia
E-mail address: olsh2007@gmail.com