KAUFFMAN-HARARY CONJECTURE HOLDS FOR MONTESINOS KNOTS

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ABSTRACT

The Kauffman-Harary conjecture states that for any reduced alternating diagram $K$ of a knot with a prime determinant $p$, every non-trivial Fox $p$-coloring of $K$ assigns different colors to its arcs. We generalize this conjecture by stating it in terms of homology of the double cover of $S^3$ branched along a link. In this way we extend the scope of the conjecture to all prime alternating links of arbitrary determinants. We first prove the Kauffman-Harary conjecture for pretzel knots and then we generalize our argument to show the generalized Kauffman-Harary conjecture for all Montesinos links. Finally, we speculate on the relation between the conjecture and Menasco’s work on incompressible surfaces in exteriors of alternating links.

Keywords: Kauffman-Harary conjecture, Fox coloring, alternating knot, double branched cover, incompressible surface.

1. Introduction

In this paper we consider the following conjecture by Kauffman and Harary [3].

Conjecture 1 (Kauffman-Harary Conjecture) Let $D$ be an alternating knot diagram with no nugatory crossings. If the determinant of $D$ is a prime number $p$ then every non-trivial Fox $p$-coloring of $D$ assigns different colors to different arcs.
of $\mathcal{D}$.

In the first section of the paper we prove the Kauffman-Harary conjecture for pretzel knots. In the second section we generalize the conjecture in terms of homology of the double branched covers of $S^3$ branched along links and illustrate it by examples. In the third section we prove Kauffman-Harary conjecture and its generalization for Montesinos links. In the last section we speculate about an approach to the generalized conjecture by relating it to Menasco’s results on incompressible surfaces in the exteriors of alternating links.

2. Pretzel Knots and Fox coloring

In this section we prove the Kauffman-Harary conjecture for pretzel knots. We deal with this special case in order to prepare a more general setting, in which we replace Fox coloring by homology of the double branched cover of $S^3$ branched along a link.

Definition 2

(i) We say that a link (or a tangle) diagram is $k$-colored if every arc is colored by one of the numbers $0, 1, \ldots, k-1$ (forming the cyclic group $\mathbb{Z}_k$) in such a way that at each crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing mod $k$; see Fig. 1.1.

(ii) The set of $k$-colorings of a diagram $\mathcal{D}$ forms an abelian group, denoted by $\text{Col}_k(\mathcal{D})$.

$$a \quad c = 2a-b \mod(k)$$
$$\quad b$$

Fig. 1.1

Proposition 3 (Fox) $\text{Col}_k(\mathcal{D}) = H_1(M^{(2)}_D, \mathbb{Z}_k) \oplus \mathbb{Z}_k$, where $M^{(2)}_D$ denotes the double cover of $S^3$ branched along $\mathcal{D}$.

The first class of knots for which the Kauffman-Harary conjecture has been proved is the family of rational (or 2-bridge) knots [5, 8]. L. Kauffman challenged us at AMS annual meeting at Baltimore in January 2003 to prove the conjecture for pretzel knots and he gave some ideas why it should hold [4]. Initially we were skeptical but after analyzing several examples (e.g. pretzel knot $P(11, 7, 5, 2)$ colored in Fig. 1.2) we became convinced that the conjecture holds for all alternating knots.
First we prove the theorem for pretzel knots, Fig. 1.3.

Theorem 4 The Kauffman-Harary conjecture holds for any alternating pretzel knot diagram, \( P(n_1, n_2, ..., n_k) \).

Proof. Without loss of generality we can assume that \( n_1, n_2, ..., n_k > 0 \); the case
3. The Generalized Kauffman-Harary Conjecture

It was noticed in [5] that Kauffman-Harary conjecture holds for any rational (2-bridge) knot without restrictions on the determinant of the knot. However, the formulation of the conjecture needs to be changed in this setting from “every nontrivial \(D\)-coloring...” to “there exists a \(D\)-coloring...”.

The coloring of 2-bridge tangle of type \(\frac{m}{n} (m > n)\) with the maxima colored by 0 and 1 is illustrated in Fig.2.1. Observe that the color of each arc strictly increases as one goes down along the diagram. As we close the tangle without introducing any new crossings (and not creating a nugatory crossing), we obtain a 2-bridge link of type \(\frac{m}{n}\) with determinant \(m\) (by Conway’s formula) where \(m > n\). From Fig.2.1
we see that the colors of the arcs do not exceed $m$ except for the last minimum $m + n \equiv n \mod m$, thus the conjecture holds for 2-bridge links. Note that in this case $H_1(M_{L_m}^{(2)}) = \mathbb{Z}_m$, so the first homology group is cyclic.

![Fig. 2.1. Rational $\frac{m}{n}$-tangle.](image)

$$\frac{m}{n} = a_k + \frac{1}{a_{k-1} + \ldots + \frac{1}{a_1}}$$

We will later use the consequence of this “propagation down” proof. Notice that if we color maxima by $x_0$ and $x_1$ in place of 0 and 1, then an arc colored before by $c$ is now colored by $x_0 + x_1c$. In Section 3 we consider rational tangles of type $\frac{m_i}{n_i}$, where $m_i \leq n_i$, in which case, if

$$\frac{m_i}{n_i} = a_{k,i} + \frac{1}{a_{k-1,i} + \ldots + \frac{1}{a_{1,i}}}$$

then $a_{k,i} = 0$. Our convention for the diagram of the rational $\frac{m}{n}$-tangle (after Conway [2]) is presented in Fig. 2.2.
The "internal" maximum is colored by $y$ and then $y_2 = m(y - y_1) + y_1$, $y_3 = (m + n)(y - y_1) + y_1$, and $y_4 = n(y - y_1) + y_1$.

The pretzel knot $P(15, 10, 6)$ has cyclic homology $H_1(M^{(2)}_{P(15,10,6)}) = \mathbb{Z}_{300}$. The coloring of this knot using different colors for each arc is illustrated in Fig. 2.3.

Fig. 2.3. 300-coloring of the pretzel knot, $P(15, 10, 6)$.

On the other hand the pretzel link $P(3, 3, 3)$ has the determinant $D = 27$ and does not allow a 27-coloring with every arc using a different color. Note that in this
case the group $H_1(M^{(2)}_{P(3,3,3)}) = \mathbb{Z}_9 \oplus \mathbb{Z}_3$ is not cyclic.

While the Kauffman-Harary conjecture requires the determinant of the knot to be a prime number, the above examples suggest that a weaker requirement could suffice. Namely, one could merely assume the homology group $H_1(M^{(2)}_L, \mathbb{Z})$ to be cyclic (equal to $\mathbb{Z}_D$, where $D$ is not necessary a prime number). The conjecture can be further extended by allowing the elements of the homology group to serve as colors. We checked the link $P(3,3,3)$ which has $H_1(M^{(2)}_{P(3,3,3)}) = \mathbb{Z}_9 \oplus \mathbb{Z}_3$ and the extended conjecture holds for this example. We give more details below.

If we decorate arcs of the diagram $L$ by commutative variables and we quotient the resulting free abelian group generated by these variables by relations of type $2a - b - c$ for every crossing, we get $Col(L) = H_1(M^{(2)}_L, \mathbb{Z}) \oplus \mathbb{Z}$. If we choose one arc to be decorated by 0 then we obtain the homology group $H_1(M^{(2)}_L, \mathbb{Z})$. The group $Col_n(L)$ of Fox $n$-colorings is the module which is $\mathbb{Z}_n$-dual to $Col(L)$ (or, equivalently it is the cohomology group with one additional $\mathbb{Z}_n$ factor, $H^1(M^{(2)}_L, \mathbb{Z}_n) \oplus \mathbb{Z}_n$). Motivated by this, we suggest that the proper conjecture, generalizing the Kauffman-Harary conjecture, should be the following.

**Conjecture 5 (The Generalized Kauffman-Harary (GKH) Conjecture)**

*If $L$ is an alternating diagram of a prime link without nugatory crossings then different arcs of $L$ represent different elements of $H_1(M^{(2)}_L, \mathbb{Z})$.***

**Remarks 6**

(1) For a knot $K$ with $H_1(M^{(2)}_K, \mathbb{Z}) = \mathbb{Z}_p$, the GKH conjecture is equivalent to the Kauffman-Harary conjecture.

(2) For a pretzel link $L = P(n_1, n_2, ..., n_k)$, the group $H_1(M^{(2)}_L, \mathbb{Z})$ is cyclic if and only if $gcd\{n_1 \cdots n_{i-1}n_{i+1} \cdots n_{j-1}n_{j+1} \cdots n_k | 1 \leq i < j \leq k\} = 1$; compare Proposition 7.

(3) Fig.2.4 illustrates the fact that the GKH conjecture holds for the pretzel link $P(3,3,3)$. $H_1(M^{(2)}_{P(3,3,3)}, \mathbb{Z}) = \mathbb{Z}_9 \oplus \mathbb{Z}_3$, and if we color maxima by 0, $x_1$, $x_2$ respectively, we obtain $x_1$ as a generator of $\mathbb{Z}_9$ part and $x_1 + x_2$ as a generator of $\mathbb{Z}_3$ part. Let us denote $x_1 + x_2$ by $u$. Then our 9 arcs use 9 different “colors”:

0, $x_1, 2x_1, 3x_1, 4x_1, 5x_1 + u, 6x_1 + 2u, 2x_1 + u, x_1 + 2u$.

(4) The conjecture obviously fails for non-prime knots because for connected sums of knots the connecting arcs always represent the same element of the homology group.

(5) The conclusion of the conjecture is equivalent to the statement that for any pair of arcs of the diagram there is a Fox coloring distinguishing them.
The first step to prove the GKH conjecture is to understand the homology group of the double branched cover and relate it to arc presentation (as in $Col(L)$ group). For the general pretzel link we have

**Proposition 7** For the pretzel link $L = P(n_1, n_2, ..., n_k)$ the first homology group of the double branched cover of $S^3$ branched along $L$ has the following canonical cyclic decomposition, $H_1(M^{(2)}_L) = \mathbb{Z}_{D_0/D_1} \oplus \mathbb{Z}_{D_1/D_2} \oplus ... \oplus \mathbb{Z}_{D_{k-2}}$, where $D_0 = \Sigma_{i=1}^{k} n_1 \cdot n_{i-1} \cdot n_{i+1} \cdot n_k$, $D_1 = \gcd\{n_1 \cdot n_{i-1} \cdot n_{i+1} \cdot n_j \cdot n_{j+1} \cdot n_k\}$, ..., $D_s = \gcd\{\text{products of } k-s-1 \text{ terms}\}$, ..., $D_{k-2} = \gcd\{n_1, n_2, ..., n_k\}$.

The proof in the more general setting of Montesinos links is given in Section 3 (Proposition 8).

4. Montesinos links

In this section we prove the GKH conjecture for alternating Montesinos links (including pretzel knots). We draw their diagrams in the manner similar to pretzel knots (rational $\frac{m}{n}$ tangle in place of a column which can be thought as $\frac{1}{n}$ rational tangle). Since we deal with alternating Montesinos links, we can assume that $0 < m_i \leq n_i$ and $\gcd(m_i, n_i) = 1$; see Fig. 3.1 and 3.2.
Kauffman-Harary conjecture holds for Montesinos Knots

Fig. 3.1; Alternating Montesinos link, $M\left(\frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k}\right)$.

Fig. 3.2; $M\left(\frac{3}{7}, \frac{2}{5}, \frac{1}{3}, \frac{1}{1}\right)$.

Generalizing Proposition 7, we first compute homology of the double branched cover along a Montesinos link. Conway’s formula gives the determinant as $D = \Sigma_{i=1}^{k} n_1 n_2 \ldots n_{i-1} m_i n_{i+1} \ldots n_k$.

**Proposition 8** The first homology group of the double cover of $S^3$ branched along the Montesinos link $L = M\left(\frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k}\right)$ has the following canonical decomposition into cyclic groups, $H_1(M_L^{(2)}, \mathbb{Z}) = \mathbb{Z}_{D_0/D_1} \oplus \mathbb{Z}_{D_1/D_2} \oplus \ldots \oplus \mathbb{Z}_{D_{k-2}}$, where $D_0 = D = \Sigma_{i=1}^{k} n_1 n_2 \ldots n_{i-1} m_i n_{i+1} \ldots n_k$, $D_s = gcd(W_{i,s})$, and $W_{i,s}$ is the set of products obtained from $n_1 n_2 \ldots n_{i-1} m_i n_{i+1} \ldots n_k$ by dropping $s$ letters from it. Finally, $D_{k-1} = 1$.

_Proof._ The group $H_1(M_L^{(2)}, \mathbb{Z})$ is an abelian group generated by elements $z_1, z_2, \ldots, z_k$ with relations

$n_1 z_1 = n_2 z_2 = \ldots = n_k z_k$, $m_1 z_1 + m_2 z_2 + \ldots + m_k z_k = 0$.

Here $z_i = y_i - x_{i-1}$, where $x_i$ are “connecting maxima” in the diagram, Fig.3.1 and $y_i$ are “internal” maxima of rational tangles (compare Fig.2.1, 2.2 and 3.2). We have $x_i - x_{i-1} = m_i (y_i - x_{i-1}) = m_i z_i$. The relations $n_1 z_1 = n_j z_j$ are obtained by comparing labels of the minima of the diagram. This presentation is described
by the matrix $A_k$, where rows represent relations of the group. That is, we have $H_1(M_L^{(2)}, \mathbb{Z}) = \mathbb{Z}^k/\text{Im}(A_k)$, where $A_k : \mathbb{Z}^k \to \mathbb{Z}^k$ is the linear map given by $v \mapsto (v)A_k$.

$$A_k = \begin{bmatrix}
n_1 & -n_2 & 0 & \cdots & 0 \\
n_1 & 0 & -n_3 & \cdots & 0 \\
n_1 & 0 & 0 & \cdots & -n_k \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
n_1 & 0 & \cdots & 0 & -n_k \\
m_1 & m_2 & \cdots & m_{k-1} & m_k
\end{bmatrix}.$$

The canonical decomposition of the group into cyclic groups can be obtained by finding elementary divisors of the matrix, that is, generators of ideals generated by minors of $A_k$ of codimension $s$. Elementary divisors can now be found by routine induction on $k$ to yield $D_s = \gcd(W_{i,s})$, where $W_{i,s}$ is the set of products obtained from $n_1 \ldots n_{i-1} m_i n_{i+1} \ldots n_k$ by dropping $s$ letters from it. □

The special form of the matrix representing $H_1(M_L^{(2)})$, where $L = M(\frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k})$, allows us to prove the GKH conjecture for such links.

**Theorem 9** The GKH conjecture holds for all alternating Montesinos links, $L = M(\frac{m_1}{n_1}, \ldots, \frac{m_k}{n_k})$.

**Proof.** We will show that no two different rational blocks (tangles) share the same element of the homology group. Different arcs inside each block represent different labels (see Fig.2.1 and the paragraph discussing it). It is enough to compare the first block with the $j$th block, where $1 < j < k$. Assume now that an arc of the first block represents the same homology as an arc of the $j$th block. Then, in $H_1(M_L^{(2)}, \mathbb{Z})$, we have for some $a$ and $b$, $0 \leq a \leq m_1 + n_1$ and $0 \leq b \leq n_j$:

$$a z_1 = b z_j + x_{j-1} \quad \text{or equivalently} \quad a z_1 = b z_j + m_{j-1} z_{j-1} + m_{j-2} z_{j-2} + \ldots + m_2 z_2 + m_1 z_1,$$

and further

$$(a - m_1) z_1 - m_2 z_2 - \ldots - m_{j-2} z_{j-2} - m_{j-1} z_{j-1} - b z_j = 0.$$

Adding this relation to the matrix of relations should keep the group unchanged. If the relation $n_1 z_1 - n_j z_j$ is deleted, the group cannot be smaller. Therefore if we replace the row $n_1 z_1 - n_j z_j$ by the row $(a - m_1) z_1 - m_2 z_2 - \ldots - m_{j-2} z_{j-2} - m_{j-1} z_{j-1} - b z_j$ we cannot decrease the (absolute value) of the determinant. On the other hand, now the matrix has the form:

$$\begin{bmatrix}
n_1 & -n_2 & 0 & 0 & \cdots & 0 & 0 \\
n_1 & 0 & -n_3 & 0 & \cdots & 0 & 0 \\
n_1 & 0 & 0 & -n_4 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a - m_1 & -m_2 & -m_3 & -m_4 & \cdots & -b & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
n_1 & 0 & 0 & 0 & \cdots & 0 & -n_k \\
m_1 & m_2 & m_3 & m_4 & \cdots & m_j & m_k
\end{bmatrix}.$$
One can check that the absolute value of its determinant is smaller than $D$ of Proposition 8, unless $a = m_1 + n_1, b = n_j$ and $j = 2$. To demonstrate this, one uses properties of the matrix $A_k$ and its blocks of codimension 1, and the fact that $0 < m_i < n_i$ (or $n_i = m_i = 1$).

5. Future directions

We expect that the method applied to prove the Generalized Kauffman-Harary Conjecture for Montesinos links can be extended to the case of 2-algebraic links (i.e. algebraic links in the sense of Conway) and also to closed 3-braids. However, the general case requires new ideas. Exploiting the connection of the GKH conjecture to incompressible surfaces in the way outlined below seems to be a promising idea.

Assume that the GHK conjecture fails for an irreducible alternating link $L$ in $S^3$. Then there are different arcs of its diagram labeled by $y_i$ and $y_j$ such that the element $y_i y_j^{-1}$ is homologically trivial in the double cover of $S^3$ branched along $L$. As the first step we analyze the possibility that this element is homologically trivial in the unbranched double cover $\tilde{M}$ of $S^3 - L$. In this case $y_i y_j^{-1}$ bounds a connected surface $\tilde{F}$ in $\tilde{M}$. Let $F$ be the projection of $\tilde{F}$ into $M$. In order to show that $y_i y_j^{-1}$ is homologically non-trivial it is sufficient to prove that $F$ contains a meridional curve. One may hope to show that by generalizing a theorem of Menasco stating that every closed, incompressible surface in the exterior of an irreducible alternating knot contains a meridional curve, [6, 7]. This approach will be discussed in the sequel to this paper [1].

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