Flat band states: disorder and nonlinearity

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We reveal the critical behaviour of Anderson localized modes near intersecting flat and dispersive bands in the quasi-one-dimensional diamond ladder with weak diagonal disorder W. The localization length $\xi$ of the flat band states scales with disorder as $\xi \sim W^{-\gamma}$, with $\gamma = 1.30 \pm 0.01$, in contrast to the dispersive bands with $\gamma = 2$. A small fraction of dispersive modes mixed with the flat band states is responsible for the unusual scaling. Anderson localization is therefore controlled by two different length scales. Nonlinearity couples these critical modes, resulting in qualitatively different wave spreading regimes, from enhanced expansion to resonant tunneling and self-trapping.

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Disorder in conventional media suppresses propagation of waves, resulting in the celebrated Anderson localization [1]. Interesting exceptions to this rule are systems containing dispersionless or flat bands, which can exhibit an “inverse” Anderson transition in three-dimensions (3D), where localized flat band states (FBS) delocalize with increasing disorder [2, 3]. It was argued that the localization of FBS should be qualitatively different when the flat band touches other dispersive bands [4]. This was demonstrated numerically in the weak disorder limit of a 2D lattice, with the FBS displaying critical, multifractal properties reminiscent of an Anderson transition. This is quite different from ordinary 1 and 2D lattices, which require long range coupling for critical behaviour to appear [2].

Flat bands can be realized in a variety of systems, including condensed matter physics [4, 5], ultracold atomic gases loaded in optical potentials [6, 7], and photonic lattices [10, 13]. Disorder inevitably appears in all these settings, so it is important to understand its effects. The above results highlight the unusual consequences of mixing macroscopically degenerate FBS via disorder. They also show that the mixing is sensitive to both the dimensionality of the system and the inclusion of a small number of modes which belong to dispersive bands. However, rigorous analytic results are scarce, and numerical studies in two or more dimensions are notoriously hard and imprecise due to finite size effects.

In this work, we study disorder induced wave localization in a quasi-1D system, the diamond ladder, which hosts intersecting flat and dispersive bands. A gap may be introduced without destroying the flat band, which allows us to unambiguously identify the drastic effects of mixing with the dispersive band states (DBS). An additional advantage in comparison to the higher dimensional lattices is that we can obtain rigorous numerical results, free of finite size effects, for the localization length and all other eigenstate properties. Furthermore, we investigate the interplay between disorder-induced mode mixing and nonlinearity.

Our main finding is that even weak mixing between the dispersive and flat bands, induced by weak disorder of strength W, completely changes FBS. The localization length $\xi$ at the flat band centre has the power law scaling, $\xi \sim W^{-\gamma}$ with exponent $\gamma = 1.30 \pm 0.01$. Corresponding modes are highly sparse. When the dispersive bands are separated by a gap $m > W$, the FBS no longer scale with disorder, $\gamma = 0$, with compact profiles resembling ordinary Anderson localization. In contrast, the DBS display conventional $\gamma = 2$ scaling [14]. The effect of nonlinearity on the FBS depends crucially on which types of modes they resonantly interact with, resulting in qualitatively different spreading regimes controllable by the interaction strength.

Model. We consider the quasi-1D diamond ladder shown in Fig. 1(a). Its band structure in Fig. 1(b) consists of two dispersive bands which touch a third, flat band at the edge of the Brillouin zone. The tight binding model describing the propagation of waves in this system is

$$i\hat{a}_n + (\epsilon_{a,n} + \beta|\hat{a}_n|^2)\hat{a}_n = -\nabla^2\hat{b}_{n+1},$$

$$i\hat{b}_n + (\epsilon_{b,n} + \beta|\hat{b}_n|^2)\hat{b}_n = -\nabla^2(\hat{a}_n + \hat{c}_n),$$

$$i\hat{c}_n + (\epsilon_{c,n} + \beta|\hat{c}_n|^2)\hat{c}_n = -\nabla^2\hat{b}_{n+1},$$

here $\nabla^2 f_n = f_n + f_{n-1}$ is the discrete Laplacian, $\beta$ is...
the nonlinearity coefficient, and \( \epsilon_n \) is the disorder potential. The dot denotes derivative with respect to time \( t \), which corresponds to the spatial propagation variable for optical waveguide arrays. The conserved norm \( \sum_n (|a_n|^2 + |b_n|^2 + |c_n|^2) \) is set to 1 without loss of generality.

In the linear, disorder-free limit \( \beta = \epsilon_{j,n} = 0 \), the mode profiles \( \psi_n = \{a_n, b_n, c_n\} \) are found from Eqs. (1)-(3) using \( \{a_n(t), b_n(t), c_n(t)\} = \psi_n e^{iEt} \). The linear spectrum \( E(k) = 0, \pm 2\sqrt{2} \cos(k/2) \) in Fig. (1b) is derived using plane wave expansion \( \psi_n = \psi_{ikn} \) with the wavenumber \( k \). We find \( \psi = \{1, 0, -1\}/\sqrt{2} \) for FBS with energy \( E = 0 \) which, in fact, can be generalized,

\[
\psi_n = \{1, 0, -1\} f_n, \quad (4)
\]

where \( f_n \) is an arbitrary function. In particular, \( f_n = \delta_{n,n_0} \) results in a single mode perfectly localized to a single unit cell \( n_0 \). The phase difference between the “a” and “c” sites causes destructive interference which effectively decouples sublattices and prevents diffraction. In contrast, DBS with \( E \neq 0 \) are infinitely extended, \( \psi_n = e^{ikn} [1, \pm 2\sqrt{2} \cos(k/2)]/2 \).

**Disorder.** We consider diagonal disorder with uncorrelated random variables \( \epsilon_{a,b,c} \) uniformly distributed in the range \([-\Delta, \Delta]\), with spectral width \( \Delta = 2\sqrt{2} + W/2 \).

It is convenient to solve Eqs. (1) and (3),

\[
a_n = \frac{b_n + b_{n+1}}{E - \epsilon_{a,n}}, \quad c_n = \frac{b_n + b_{n+1}}{E - \epsilon_{c,n}}, \quad (5)
\]

to obtain a single equation for a mode profile \( b_n \):

\[
\begin{align*}
\epsilon_n b_n &= C_n b_{n+1} + C_{n-1} b_{n-1} \\
C_n &= (\epsilon_n - E)^{-1} + (\epsilon_{c,n} - E)^{-1}, \quad (6)
\end{align*}
\]

\[
\begin{align*}
\epsilon_n &= \epsilon_{b,n} - E - C_n - C_{n-1}, \quad (8)
\end{align*}
\]

which resembles an ordinary periodic 1D lattice with both diagonal, \( \epsilon_n \), and coupling, \( C_n \), disorder \([12]\).

Written in this form, two distinct regime energies are apparent. When \( |E| < W/2 \), the couplings \( C_n \) can vanish or diverge, resulting in non-perturbative behaviour. This is because modes in this energy range are primarily composed of FBS and the non-perturbative behaviour is due to their macroscopic degeneracy when \( W = 0 \). In contrast, more regular behaviour occurs when \( |E| > W/2 \): here the dispersive bands provide the dominant contribution to the disordered modes. The couplings are all finite, so a perturbative treatment is possible, similar to an ordinary chain.

Numerically, we diagonalize Eqs. (1)-(3) for a disordered chain of finite size \( N \) with periodic boundary conditions \([17]\) and obtain the modes \( \psi_{\nu,n} = \{\alpha_{\nu,n}, b_{\nu,n}, c_{\nu,n}\} \), \( \nu = 1, 2, \ldots, 3N \). We characterize mode behaviour as a function of \( E \) using the following measures \([18]\). The participation ratio, \( P = 1/\sum_n (|\alpha_{\nu,n}|^4 + |b_{\nu,n}|^4 + |c_{\nu,n}|^4) \), measures the number of strongly excited sites. The second moment, \( m_2 = \sum_n (X_\nu - n)^2 |b_{\nu,n}|^2 + (X_\nu - n - 1/2)^2 (|\alpha_{\nu,n}|^2 + |c_{\nu,n}|^2) \), is sensitive to the distance between the tails of the eigenmode, here \( X_\nu = \sum_n (\nu b_{\nu,n^2} + (n + 1/2) (|\alpha_{\nu,n}|^2 + |c_{\nu,n}|^2) \) is the mode’s centre of mass. The compactness index, \( \zeta = P^2/m_2 \), reveals how uniformly the eigenstate excites the volume it occupies. We calculate the mean values of \( P \), \( m_2 \), and \( \zeta \) for each value of \( W \) by taking a sample of \( \sim 100,000 \) modes divided into 100 energy bins.

The localization length \( \xi \) is the asymptotic decay rate of the eigenmode tails, \( \psi_{\nu,n} \sim e^{-n/\xi} \), and is obtained via

\[
\xi^{-1}(E) = \lim_{N \to \infty} \frac{1}{N} \left\langle \sum_{n=1}^N \ln \frac{b_{n+1}}{b_n} \right\rangle, \quad (9)
\]

where \( \langle \cdot \rangle \) denotes averaging over different realizations of disorder.

Figure 2 shows the results for different disorder strengths. Clearly, the boundary \( |E| = W/2 \) separates modes with very different properties. For \( |E| > W/2 \) (high energy) we obtain compact, weakly localized modes with properties similar to those of an ordinary weakly disordered 1D chain.

When \( |E| < W/2 \) (low energy), the modes display remarkably different properties: \( \xi \), \( P \), and \( m_2 \) are orders of magnitude smaller, suggesting much stronger localization. In particular \( P \approx 7 \) does not change at all with \( W \). However, the compactness index \( \zeta \) is also very small, which indicates that the modes are sparse, consisting of well-separated peaks, completely different from conventional Anderson localized modes.

As the disorder strength is increased, the distinction between the low and high energy regimes becomes less
by shifting the onsite potentials, and opposite mass terms to the “a” and “c” sublattices demonstrate this, we create a gap by introducing equal properties become independent of the disorder strength. To sets the width of the disorder-broadened band introduce a gap between flat and dispersive bands. Then $m$ band component. As $W$ diverges.

As a result of this criticality the propagation of wavepackets at low energy is governed by heavy-tailed statistics. Most of the time a single-site flat band excitation Eq. (4) experiences an ordinary, diffusive expansion to a size on the order of a localization length. However, if highly sparse modes [Fig. 3(b)] are strongly excited, much larger expansion driven by tunnelling to distant sites occurs, including the oscillation of energy back and forth between well separated peaks illustrated in Figs. 3(c,d). In contrast to conventional 1D lattices, these sparse modes are statistically significant.

**Nonlinearity.** Nonlinear terms with $\beta \neq 0$ in Eqs. (1)-(3) become important at higher light intensities in optical waveguide arrays or with Feshbach-tuned two-body interactions in ultracold atomic gases in optical lattices. Nonlinearity couples modes of the linear chain [19, 20]. In ordinary lattices, this results in the destruction of Anderson localization, replaced by subdiffusive spreading [21], and probabilistic partial self-trapping [22]. Here, the coupled modes have qualitatively different properties, depending on their energy relative to the disorder strength.

The nonlinear Eqs. (1)-(3) can be represented in the basis of eigenmodes of the linear system $\psi_{\nu,n}$ [23],

$$i\dot{\phi}_\nu = E_\nu \phi_\nu + \beta \sum_{\nu_1,\nu_2,\nu_3} I_{\nu,\nu_1,\nu_2,\nu_3} \phi_{\nu_1}^* \phi_{\nu_2} \phi_{\nu_3},$$

where $\phi_\nu(t)$ is the complex amplitude of mode $\nu$ and $I$ is the overlap integral

$$I_{\nu,\nu_1,\nu_2,\nu_3} = \sum_n \sum_{\alpha=a,b,c} \alpha_{\nu,n}^* \alpha_{\nu_1,n} \alpha_{\nu_2,n} \alpha_{\nu_3,n},$$

with the summation over all unit cells and sublattices. $I$ determines the effective strength of coupling between different modes [15].

Nonlinearity introduces an additional energy scale that competes with the disorder, an energy shift $\delta E_\nu \approx s\beta I_{\nu,\nu,\nu,\nu} \approx s\beta/\sqrt{P}$, where $s = |\phi_\nu|^2$ is the occupation of a given mode. High energy modes have diverging $P$ in the limit of weak disorder, leading to small shifts. On the other hand, the low energy modes with $P \sim 7$ can experience significant energy shifts. Three distinct nonlinear regimes appear.

With weak nonlinearity, $s\beta < W$, the nonlinear frequency shift does not exceed the width of the low energy subspace. Strong resonant interaction and the associated enhanced spreading can only occur between overlapping low energy modes. Strong overlaps occur for pairs of

![FIG. 3. (color online) (a) Localization length $\xi(W)$ for flat ($E = 0$) and dispersive ($E = \sqrt{2}$) band modes, following the power laws $\xi \sim W^{-1.3}$ and $\xi \sim W^{-2}$ respectively, in comparison to $\xi(E = 0)$ when a gap of size $m = 1/2$ is introduced. (b) Mode profile of a sparse ($\zeta = 0.1$) flat band mode, $W = 0.5$, and the effective coupling $|C_{\nu}|$ defined by Eq. (6). (c) Intensity $L_n = |a_n|^2 + |b_n|^2 + |c_n|^2$ during propagation when sparse modes are strongly excited, $W = 1$. (d) Same as (c), but linear scale.](image)
In the intermediate regime, when the nonlinear energy shift exceeds the width of the low energy subspace, \( s \beta > W \), there is a strong energy transfer from low to high energy modes, resulting in enhanced spreading compared to the linear case. Each flat band mode can transfer energy into many dispersive band modes, resulting in Gaussian spreading statistics. Since \( s \) is decreasing, at some point it will tune out of strong interaction with the dispersive band, leaving a flat band component which remains strongly localized for potentially long times.

For very strong nonlinearity the energy shift \( \delta E \) can exceed the total band width, leading to self-trapping.

We verify these expectations by numerical simulations and present examples of propagation in the different regimes in Fig. 4 using an initial state that strongly excites a single peaked mode. Under weak nonlinearity there is no resonant interaction with the dispersive bands and the wavepacket expansion is similar to the linear case in Fig. 4(a). In the intermediate regime in Fig. 4(b), the expansion is driven by an initial transfer of energy to the dispersive bands and we see stronger spreading. With strong nonlinearity in Fig. 4(c) we observe the formation of a self-trapped state, which irregularly meanders between two quasi-stable positions. Energy is lost during this motion, and it eventually becomes trapped at a “b” sublattice site. We plot the intensity at the initially excited unit cell in all three cases.

Conclusions. The diamond ladder presents a test bed to explore the interplay between macroscopic degeneracy, disorder and nonlinearity. We showed how the mixing between macroscopically degenerate flat band modes and a small number of weakly localized modes of intersecting dispersive bands results in low energy modes with unusual properties. Consequently, the spreading of low energy wavepackets becomes sensitive to nonlinearity. Therefore, our results provide novel ideas for future studies in higher lattice dimensions and they highlight the importance of nonlinear interactions and many body quantum dynamics for weakly disordered flat band systems.

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