Towards a concept of property evaluation type

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Abstract. An appropriate characterization of property types is an important topic for measurement science. This paper proposes to derive them from evaluation types, and analyzes the consequences of this position for the VIM3.

1. Introduction

An important subset of concepts in the International Vocabulary of Metrology – Basic and General Concepts and Associated Terms (VIM3) [1] is built upon ‘property’, a primitive superordinate [2] of, in particular, ‘nominal property’, ‘quantity’, ‘ordinal quantity’, and the undefined and unnamed but widely used ‘quantity with unit’. As defined and used in the VIM3, the relations among such concepts can be depicted as in the following diagram:

This structure implies that ‘nominal property’ and ‘quantity’ are mutually exclusive – a nominal property is not a quantity and a quantity is not a nominal property – and, because of the negative connotation of ‘nominal property’ (“having no magnitude”, being ‘magnitude’ assumed as a primitive concept in the VIM3), exhaustive – each property is either a nominal property or a quantity. In set-theoretical terms, this implies that such two subordinate concepts define a partition of the superordinate ‘property’. The same conclusion can be plausibly drawn for ‘ordinal quantity’ and the undefined ‘quantity with unit’ – an ordinal quantity is not a quantity with unit and a quantity with unit is not an ordinal quantity; a quantity is either ordinal or with unit – with respect to ‘quantity’.

We argue that this structure by oppositions is not the best way to define the relevant subordinate concepts of ‘property’. The alternative concept structure that will be preliminarily presented in this paper is based on the so called “theory of scales of measurement” (the standard reference is a seminal paper by Stevens [3]), as presented according to the authors’ interpretation of it. This presentation will allow us:

- to highlight that the presented results apply both to quantities and nominal properties, and therefore to (generic) properties; accordingly, we will use the term “evaluation”, in the sense of value
assignment, to designate the superordinate concept of ‘measurement’ and ‘examination’ (the latter is
the term adopted by the VIM3 to designate the process of evaluating nominal properties – see def.
5.13);
- to remove the ambiguity whether the type of a property depends on the way the property is evaluated
(as the VIM3 assumes for ordinal quantities, that are “defined by a conventional measurement
procedure” – see def. 1.26), or it is instead a characteristic of the property “in itself” (as in the case of
nominal properties, that “have no magnitude” – see def. 1.30), thus paving the way to well-defined
concepts of property evaluation type (PET for short henceforth) and property type;
- to avoid the reference to the undefined and unclear concept of magnitude;
- to characterize the concepts of ordinal quantity and nominal property without the need of referencing
to undefined and unclear concepts, such as ‘conventional measurement procedure’ or by means of
negative connotations, such as ‘having no magnitude’.

The presentation emphasizes the priority of categorizing evaluations instead of properties. Indeed,
the features that characterize a property as object of a given evaluation process do not depend only on
the property “in itself”, but also critically on the evaluation process (for example, length, a
paradigmatic case of a quantity with unit, can be measured by means of a system able to acquire only
ordinal information: from the point of view of this specific evaluation, length behaves as an ordinal
quantity and as such its values should be dealt with in this particular case). Hence, a concept of
property type can be defined, as we will see in the following, but only as derived from the concept of
PET.

2. Backgrounder
An evaluation is basically (in this discussion uncertainty will be neglected, as the VIM3 does)
formalized as a mapping \( f: A \to S \), where:
- \( A \) is a set of objects (bodies, phenomena, …) on which the evaluation has to be performed;
- \( S \) is a set of symbols expressing the evaluation result; the VIM3 has a few terms for such symbols,
  “quantity value” and “measured quantity value”, but also the undefined “nominal property value” (on
  this matter, in a more specific analysis the distinction should be maintained between numbers and
  numerals);
- \( f \) is a function that maps objects to be evaluated to symbols, so that \( s = f(a) \) means that the symbol \( s \)
  is the result of the evaluation, as formalized by \( f \), of the object \( a \).

The mapping \( f \) admits a twofold interpretation, as it represents both the (general) property to be
evaluated and the process of its evaluation. Accordingly, in the following we will sometimes adopt the
overloaded notation “the property \( f \)” and “the evaluation \( f \)” for short. The application of any function \( f \)
induces a partition, i.e., a set of \( \approx \)-equivalence classes, on its domain \( A \), such that for any two objects
\( a_i \) and \( a_j \) of \( A \), \( f(a_i) = f(a_j) \) if and only if the two objects belong to the same class, i.e., \( a_i \approx a_j \), the
equivalence relation \( \approx \) thus meaning “having the same value according to \( f \)”. The set of the \( \approx \)-
equivalence equivalence classes is called the quotient set of \( A \) under \( f \) and is denoted as \( A/\approx \). Hence, any function
\( f: A \to S \) can be expressed as the composition of a function \( g: A \to A/\approx \) with a function \( h: A/\approx \to S \),
\( f(a) = h(g(a)) \), such that the value assigned to an object \( a \) by \( f \) can be obtained by mapping \( a \) to its \( \approx \)-
equivalence equivalence class \( [a] = g(a) \) and then mapping such class to the symbol \( h([a]) = h(g(a)) \) that is assigned,
by means of \( f \), to all objects belonging to the class itself.

Under the assumption that \( f \) formalizes an experimental process (as the VIM3 defines
‘measurement’ – see def. 2.1), the functions \( g \) and \( h \) have complementary features:
- the function \( g \) formalizes the experimental component of the evaluation, that consists in the
determination of the \( \approx \)-equivalence class which the given object belongs to, as typically obtained by
comparison among objects relatively to the given property; in general, \( g \) is not injective, i.e., different
objects can be mapped to the same \( \approx_f \)-equivalence class, since different objects can “have the same (specific) property”;
- the function \( h \) formalizes the \textit{representational} component of the evaluation, that consists in the \textit{assignment} of a symbol to the given \( \approx_f \)-equivalence class; \( h \) is always injective (and therefore it is bijective on the image \( f(A) \)), i.e., different \( \approx_f \)-equivalence classes are mapped to different symbols, since by construction each \( \approx_f \)-equivalence class has one (and only one) symbol associated.

The simple structure \( f = g^o h \) allows to expressively account for the well known fact that property values are not, in general, univocally assigned, in the sense that \( s = f(a) \) is assigned but a (scale) transformation (e.g., from meters to inches). Hence, each property evaluation is formalized by a family of functions \( f = g^o h_i \) such that:
- on the experimental side, a single function \( g \) formalizes the determination of the quotient set (the fact that two objects have the same length is independent of the way their length values are expressed, either in meters or in inches);
- on the representational side, each function \( h_i \) formalizes a different assignment of symbols to the classes of the quotient set (e.g., \( h_1 \) could stand for “length in meters”, \( h_2 \) for “length in inches”, and so on).

This characterization highlights that the evaluation of a property includes both an empirical and a symbolic component, and the empirical component generally does not completely determine the symbolic one, that maintains some arbitrariness. The mentioned (scale) transformations take into account this arbitrariness, and can be described as mappings \( \tau_{i,j} : S \to S \) such that \( \forall a \in A, f_j(a) = \tau_{i,j}(f_i(a)) \) and therefore, more specifically, \( h_i([a]) = \tau_{i,j}(h_j([a])) \), to be read, e.g., the length in inches (\( f_2 \)) of an object \( a \), \( f_2([a]) \), can be obtained by determining the class \([a]\) of the objects having the same length as \( a \), and then either directly mapping this class to a value in inches, \( h_2([a]) \), or mapping it to a value in meters, \( h_1([a]) \), and then transforming such value in inches, \( \tau_{1,2}(h_1([a])) \).

3. Basics on the concept of property evaluation type (PET)

Sometimes, together with \( \approx_f \), other relations are empirically meaningful because of the evaluation of a property, e.g., the relation \( <_f \) of “having a minor / greater (specific) property”. Let us suppose that such possible further relations on \( A \) are compatible with \( \approx_f \), so that, e.g., \( \forall a_1, a_2, a_3 \in A, a_1 \approx_f a_2 \) and \( a_2 \approx_f a_3 \), then \( a_1 \approx_f a_3 \). This implies that in its turn \( <_f \) is induced on the quotient set \( A/\approx_f \), in the sense that the \( \approx_f \)-equivalence classes can be considered as ordered according to \( <_f \).

A basic condition for the evaluation of a property \( f \) to be consistent is that for each empirical relation on \( A \) there is a symbolic relation on \( S \) such that \( f \) preserves all relation instances, e.g., if \( a_1 \approx_f a_2 \) then \( f \) is consistent only if \( f(a_1) < f(a_2) \) (the classical reference on this topic is [4]). This condition of preservation implies that any \( f_i \) is a homomorphism from \( A \) equipped with \( <_f \) into \( S \) equipped with <,
and therefore equivalently that any \( h_i \) is an isomorphism from \( A/\approx_f \) equipped with \(<_f\) into \( S \) equipped with \(<\). In its turn, this condition of morphism must be invariant under transformations \( \tau_{ij} \), i.e., if \( f_i \) is a homomorphism then also \( f_i = f_i \circ \tau_{ij} \) must be a homomorphism, and if \( h_i \) is an isomorphism then also \( h_i = h_i \circ \tau_{ij} \) must be an isomorphism.

On the other hand, not any transformation \( \tau_{ij} \) is invariant in this sense: for any relation to be preserved by \( f_i \), there is a set of “admissible” transformations such that the mappings \( h_i \) are isomorphisms. If \( a_1 <_f a_2 \) then the guarantee that \( f_i(a_1) < f_i(a_2) \) for all \( f_i \) is obtained by requiring that the transformations \( \tau_{ij} \) preserve the order \(<_f\). This implies that the \( \tau_{ij} \) must be monotonic.

Hence, the algebraic structure of the set of transformations that are permissible according to this condition of invariance determines a classification on the set of property evaluations, such that each class corresponds to a PET. Let us consider the two extreme cases:
- on the one hand, the mapping \( h_i \) is unconstrained, i.e., the transformation \( \tau_{ij} \) are all the injective mappings \( S \rightarrow S \). This is the case of evaluations performed by simply classifying the objects of \( A \), so that having the same (specific) property means only belonging to the same equivalence class. This PET is customarily called “nominal”;
- on the other hand, the mapping \( h_i \) is unique, i.e., the only transformation \( \tau_{ij} \) is the identity. This is the case of evaluations performed by counting predefined entities, so that having the same (specific) property means having the same number of such entities. This PET is customarily called “absolute”.

Between such cases other types are customarily acknowledged, in particular the ordinal, interval, and ratio type, that were already identified as such and characterized by Stevens [3]. The fundamental point here is that this concept of type applies to property evaluations, not directly to properties, so that the same property (in a sense to be further explored) can be evaluated by means of evaluations of different types. On this basis, a concept of property type can actually be derived, defined as the most specific (i.e., the algebraically richest) evaluation type allowed for the given property according to the currently available knowledge.

| PET   | Admissible transformations \( \tau \)                                                                 | Conditions of invariance                                                                 | Features                          |
|-------|-----------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------|-----------------------------------|
| nominal | injective transformations: if \( b_1 \neq b_2 \) then \( \tau(b_1) \neq \tau(b_2) \)                  | preservation of equivalence classes                                                     | classifications                   |
| ordinal | monotonic transformations: if \( b_1 < b_2 \) then \( \tau(b_1) < \tau(b_2) \)                     | preservation of order                                                                   | orderings                         |
| interval | linear transformations: \( \tau(b) = k_1 b + k_2, k_1 > 0 \)                                        | preservation of ratios of differences: \( (\tau(b_2) - \tau(b_1))/(b_2 - b_1) = k_1 \) | numerical mappings without a “natural zero” and a “natural unit”          |
| ratio   | similarity transformations: \( \tau(b) = kb, k > 0 \)                                               | preservation of ratios: \( \tau(b)/b = k \)                                             | numerical mappings with a “natural zero” but without a “natural unit”      |
| absolute | identity transformations: \( \tau(b) = b \)                                                     | preservation of values                                                                  | numerical mappings with a “natural zero” and a “natural unit”              |

4. On upgrading and downgrading property evaluation types

PETs are characterized by the structural information they preserve, and therefore by the algebraic structure under which the transformations \( \tau \) are invariant. Hence, the change from one type to an
algebraically stronger one can be interpreted as a *structural upgrade* of the evaluation, whereas the change from one type to an algebraically weaker one as a *structural downgrade* of the evaluation itself.

Downgrading from a stronger than nominal type is always possible, as it is obtained by neglecting some information that in principle could be preserved by a suitable evaluation. For example, a length evaluation can be downgraded from ratio to interval type by interpreting it as a distance evaluation, i.e., making the “zero” relative, but also to ordinal type, by assuming a ranking such as “long”, “average”, “short” for the set of values. On the other hand, upgrading is much more demanding, since it requires the introduction of appropriate structural information. In particular, the upgrade from interval to ratio type implies a “natural zero” to be set, and the upgrade from ratio to absolute type implies a “natural unit” to be set. In synthesis, downgrading is an operative choice, that can be always made for more-than-nominal evaluations; on the other hand, upgrading with respect to the currently known strongest type requires a scientific / technological advancement (as the one that allowed upgrading temperature measurement from ordinal to interval type), and therefore it is generally a notable historical event.

5. Property (evaluation) types and the VIM3: a comparison
The analysis can be now completed by proposing some comparisons between the concept of PET and the subordinate concepts of ‘property’ explicitly (‘nominal property’, ‘quantity’, ‘ordinal quantity’) or implicitly (‘quantity with unit’) introduced in the VIM3.
- The concept of PET focuses on property evaluation, and derives a concept of property type as a specific, “upper bound”, case, whereas the VIM3 directly (although implicitly) attributes a type to subordinates of ‘property’. The greater generality and correctness of the PET-based position do not seem in discussion here.
- PETs are unambiguously characterized in terms of structural invariance (see the table above), without recurring to unclear concepts such as ‘conventional measurement procedure’, or ‘having magnitude’, as instead the VIM3 does.
- The relations among PETs are of inclusion, not of opposition as instead in the VIM3. This better accounts for the (historical) phenomenon of the evaluation upgrade and the (operative) phenomenon of the evaluation downgrade. Furthermore, this clearly justifies why algebraically weak functions can be applied to strong evaluations (e.g. median to ratio evaluations) and instead algebraically strong functions are not to be applied, in general, to weak evaluations (e.g. mean to ordinal evaluations).
Hence, if expressed in terms of PETs the initial diagram can be reformulated as follows.

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nominal PET
    | ordinal PET
    | 
    | interval PET
    | 
    | ratio PET
    | 
    | absolute PET
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- Because of this relation of inclusion, the choice of restricting the usage of ‘measurement’ to specific PETs and property types (as the VIM3 does, when it assumes that nominal properties can be “examined” but cannot be measured) appears largely conventional.

On this basis, two alternative general strategies can be envisioned to revise the definitions for the subordinates of ‘property’ in the VIM3:
- an inclusive strategy, that applies to property types the structure of PETs, such that, e.g., an ordinal property is a nominal property (with the further constraint of order preservation) and an interval property is an ordinal property (with the further constraint of preservation of ratios of distances);
- an exclusive strategy, that maintains the relations as in the VIM3, such that, e.g., a nominal property is not an ordinal property nor vice versa; this can be obtained by defining nominal properties in terms of preservation of equivalence classes and nothing else, and so on.

According to the previous table on PETs, such definitions, in the inclusive version, could be, for example:

| type       | Definition expressed in terms of admissible transformations $\tau$                                      | Definition expressed in terms of conditions of invariance                                                                 | Definition expressed in terms of features                                                                 |
|------------|----------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------|
| nominal    | property whose values are determined but by an injective transformation                             | property that conveys information on equivalence classes of evaluated objects                                               | property that allows classifying objects                                                                  |
| ordinal    | property whose values are determined but by a monotonic transformation                             | property that conveys information on order of evaluated objects                                                              | property that allows ordering objects                                                                    |
| interval   | property whose values are determined but by a linear transformation                                | property that conveys information on ratios of differences between evaluated objects                                         | property that allows assigning numbers to objects but maintains both ‘0’ and ‘1’ relative                 |
| ratio      | property whose values are determined but by a similarity transformation                            | property that conveys information on ratios between evaluated objects                                                        | property that allows assigning numbers to objects together with a ‘0’ but maintains ‘1’ relative         |
| absolute   | property whose values are uniquely determined                                                    | property that conveys information on values of evaluated objects                                                              | property that allows assigning numbers to objects together with a ‘0’ and a ‘1’                         |

References
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