Introducing the Huber mechanism for differentially private low-rank matrix completion

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Abstract

Performing low-rank matrix completion with sensitive user data calls for privacy-preserving approaches. In this work, we propose a novel noise addition mechanism for preserving differential privacy where the noise distribution is inspired by Huber loss, a well-known loss function in robust statistics. The proposed Huber mechanism is evaluated against existing differential privacy mechanisms while solving the matrix completion problem using the Alternating Least Squares approach. We also propose using the Iteratively Re-Weighted Least Squares algorithm to complete low-rank matrices and study the performance of different noise mechanisms in both synthetic and real datasets. We prove that the proposed mechanism achieves $\epsilon$-differential privacy similar to the Laplace mechanism. Furthermore, empirical results indicate that the Huber mechanism outperforms Laplacian and Gaussian in some cases and is comparable, otherwise.

1 Introduction

Recovering a low-rank matrix from a set of limited observations is an important problem in machine learning and data analysis. It finds applications in multiple areas such as recommender systems Luo et al. [2014], image restoration He et al. [2015], and phase retrieval (For a full survey, see Nguyen et al. [2019]). Its use in recommender systems was popularized by Bennett et al. [2007] where the user fills in surveys regarding a small fraction of the items viewed and the system is expected to provide a recommendation by estimating the remaining entries of the matrix.

Many algorithms have been proposed in literature to solve this problem which typically involves matrix decomposition Lu et al. [2015], Liu et al. [2013]. In Jain et al. [2012], the target matrix $X$ is formulated as a bi-linear problem $X = UV^\top$; the authors propose alternating minimization over the matrices $U$ and $V$. The nuclear norm of the matrix is minimized in Cai et al. [2008] in order to recover the matrix with the lowest rank. Singular value decomposition is performed on the target matrix, and a threshold is applied iteratively over the singular values to converge to a proven unique solution. Most of these solution methods involve iterative procedures as a part of the optimization algorithm that converges to a desired solution.

Given that the user data, i.e., limited observations, available to the matrix completion algorithms are highly sensitive, there has been growing interest in providing privacy guarantees while handling such data. Differential privacy was introduced by Dwork et al. [2006] as a method of preserving sensitive data about an individual while providing statistical information regarding the dataset as a whole. The framework has been widely adapted to provide private versions of well-established algorithms in varied fundamental areas such as clustering, gradient descent, localization, deep learning, data mining

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and many more (Abadi et al. [2016], Xia et al. [2020], Song et al. [2013], Friedman and Schuster [2010]).

To that end, many differentially private low-rank matrix completion (LRMC) algorithms have been proposed (Liu et al. [2015], Jain et al. [2018]) where provable privacy (and sometimes, performance) guarantees are provided. The differentially private version of alternating least squares is proposed in Chien et al. [2021] where noise is introduced in the optimization procedure to provide privacy guarantees. The algorithms mentioned above use a combination of trimming techniques (clamping entries of high magnitude) and Gaussian noise addition to provide an \((\epsilon, \delta)\)-differential privacy guarantee which is a relaxed notion of privacy.

In this work, we explore the choice of using other noise addition mechanisms to achieve differentially private LRMC. We propose a novel addition mechanism, the Huber mechanism, that combines the advantages of the Gaussian and Laplace mechanisms and provide privacy guarantees for the proposed framework. We also implement the Alternating Least Squares algorithm with different noise addition mechanisms and compare their performance. In addition, we also propose a new optimization method inspired by the best way to post-process the noise added by the proposed mechanism so as to achieve accuracy without compromising on privacy. The contributions of our work are detailed as follows:

- We introduce a differential privacy mechanism, the Huber mechanism, which adds noise from the Huber distribution, which is a combination of Laplace and Gaussian distributions. This mechanism is proposed to achieve the advantages of both Laplacian and Gaussian mechanisms. We prove that the proposed mechanism achieves \(\epsilon\)-differential privacy.

- We propose two variations of differentially private low-rank matrix completion algorithms that use Alternating Least Squares (ALS) and Iteratively Re-weighted Least Squares (IRLS), where Huber noise is added to achieve privacy.

- We provide extensive simulation results on synthetic as well as real datasets comparing the performance of the Huber mechanism to other standard noise addition mechanisms for differentially private LRMC.

**Notation** Let \(\mathbb{N}_p\) be the set of first \(p\) natural numbers, \(\mathbb{N}_p = \{1, 2, \ldots, p\}\). \(\mathcal{N}(\mu, \sigma^2)\) denotes the normal distribution with mean \(\mu\) and variance \(\sigma^2\) and \(\mathcal{L}(\beta)\) denotes the Laplace distribution with scale parameter \(\beta\). Let \(\Phi(\cdot)\) denote the distribution function of the standard normal distribution.

Throughout the article, matrices are denoted by uppercase bold letters and vectors are denoted by lowercase bold letters. The \((i, j)\)-th entry, the \(i\)-th row (as a column vector) and the \(j\)-th column of the matrix \(\mathbf{B} \in \mathbb{R}^{p \times q}\) are respectively denoted as \(b_{ij}, \mathbf{b}_i\) and \(\mathbf{b}_j\), where \(i \in \mathbb{N}_p\) and \(j \in \mathbb{N}_q\). \(\mathbf{B}_{\mathcal{I}, \mathcal{J}}\) denotes the sub-matrix of \(\mathbf{B}\), formed with the entries \(b_{ij}, i \in \mathcal{I}\) and \(j \in \mathcal{J}\), where \(\mathcal{I}\) and \(\mathcal{J}\) are the index sets. Similarly, for a vector \(\mathbf{v} \in \mathbb{R}^p\), \(v_i\) denotes its \(i\)-th element, \(i \in \mathbb{N}_p\) and \(\mathbf{v}_\mathcal{I}\) denotes the sub-vector formed with the entries \(v_i, i \in \mathcal{I}\). Let \(\text{Diag}(\zeta_1, \zeta_2, \ldots, \zeta_p)\) denote the \(p \times p\) diagonal matrix with the diagonal entries \(\zeta_1, \zeta_2, \ldots, \zeta_p\). The \(\ell_p\)-norm of a vector \(\mathbf{b}\) is denoted as \(\|\mathbf{b}\|_p\) and the nuclear and Frobenius norms of the matrix \(\mathbf{B}\) are denoted by \(\|\mathbf{B}\|_*\) and \(\|\mathbf{B}\|_F\) respectively.

## 2 Differential privacy

### 2.1 Background

Differential privacy aims to preserve the privacy of an individual while allowing meaningful inferences from the entire dataset. A privacy mechanism \(\mathcal{M}\) is an algorithm that takes a data matrix as input and provides an output to a query. Ideally, the output provides accurate responses to the queries without compromising on individual data. Differential privacy is formally defined below and is reproduced from Dwork et al. [2014] for ease of reading.

**Definition 2.1.** A randomized mechanism \(\mathcal{M}\) is said to preserve \(\epsilon\)-differential privacy if for all datasets \(\mathcal{D}, \mathcal{D}' \in \mathcal{Z}\) that differ on a single element and for all possible sets \(S\),

\[
\Pr(\mathcal{M}(\mathcal{D}) \in S) \leq \exp(\epsilon) \Pr(\mathcal{M}(\mathcal{D}') \in S).
\]

A weaker notation of privacy is defined for cases when \(\epsilon\) differential privacy is obtained for a major probability excluding a small fraction \(\delta\).
Definition 2.2. A randomized mechanism $\mathcal{M}$ is said to preserve $(\epsilon, \delta)$-differential privacy if for all datasets $\mathcal{D}$ and $\mathcal{D}'$ that differ on a single element and for all possible sets $\mathcal{S}$,

$$\Pr(\mathcal{M}(\mathcal{D}) \in \mathcal{S}) \leq \exp(\epsilon) \Pr(\mathcal{M}(\mathcal{D}') \in \mathcal{S}) + \delta.$$ 

Sensitivity plays an important role in quantifying differential privacy. It measures the magnitude of change an output $f(\mathcal{D})$ incurs based on the change of a single data point in the worst-case scenario. The amount of noise required to preserve privacy depends on the sensitivity. It is often defined with respect to a specific norm. Here, we formally define sensitivity with a general norm.

Definition 2.3. The $\ell_p$-sensitivity of a function the function $f : \mathbb{Z} \to \mathbb{R}^K$ is given by

$$\Delta f_p = \max_{\mathcal{D}, \mathcal{D}' \in \mathcal{Z}} \| f(\mathcal{D}) - f(\mathcal{D}') \|_p,$$

where $\mathcal{D}, \mathcal{D}' \in \mathcal{Z}$ are neighbouring datasets differing only by a single data entry.

Differential privacy is typically achieved using the Gaussian or Laplacian mechanisms. As the names suggest, they involve the addition of Gaussian or Laplacian noise respectively. Let $t \in \mathbb{R}^K$ be the noise that is added to $f(\mathcal{D})$ to ensure privacy i.e,

$$\mathcal{M}(\mathcal{D}, f(\cdot)) = f(\mathcal{D}) + t.$$ 

In the Laplace mechanism, the coordinates of $t$ are i.i.d samples from $\mathcal{L}(\beta)$ and it is shown to provide $\epsilon$-DP with $\epsilon = \Delta f_1 / \beta$. Similarly in the Gaussian mechanism, the entries of $t$ are i.i.d samples from $\mathcal{N}(0, \sigma^2)$ and it offers $(\epsilon, \delta)$-DP with $\epsilon = \sqrt{2 \log(1.25/\delta) / \Delta f_2}$. The choice of the privacy mechanism depends on the application and its specific requirements. Note that there is always a trade-off between accuracy and privacy. Higher privacy is achieved by adding noise of a larger magnitude and this, in turn, will affect the accuracy of the outcome. Although the Laplace mechanism offers a higher degree of privacy, the Gaussian mechanism is often preferred for machine learning applications due to its various advantages. The primary advantage is that for applications in which $\ell_2$-sensitivity is much lower than $\ell_1$-sensitivity in higher dimensions, the Gaussian mechanism allows adding much less noise. We now propose a new mechanism termed the Huber mechanism which aims to combine the advantages of Gaussian and Laplace mechanisms and employ it to perform differentially private low-rank matrix completion in the further sections.

2.2 Introducing the Huber mechanism

In this work, we introduce a new mechanism for differential privacy that combines the advantages of the Laplacian and Gaussian mechanisms. The noise distribution is inspired by the Huber loss function proposed in Huber [1964] and is defined as follows

$$p(t; \alpha) = \kappa_\alpha \exp(-\rho_\alpha(t)),$$

where $\kappa_\alpha$ is the normalizing constant given by $\kappa_\alpha = \left(2^{\alpha^2} \exp\left(-\frac{\alpha^2}{2}\right) + \sqrt{2\pi} (2\Phi(\alpha) - 1)\right)^{-1}$ and

$$\rho_\alpha(t) = \begin{cases} \frac{t^2}{2} & , |t| \leq \alpha \\ \alpha(|t| - \frac{\alpha}{2}) & , |t| > \alpha \end{cases}$$

denotes the Huber loss function with the transition parameter $\alpha$. The derivative of the Huber loss function is known as the Huber influence function, which is given by

$$\psi_\alpha(t) = \text{sign}(t) \cdot \min(|t|, \alpha) = \begin{cases} -\alpha & , t < -\alpha \\ t & , -\alpha \leq t \leq \alpha \\ \alpha & , t > \alpha \end{cases}.$$

Huber distribution is symmetric with exponential tails and a Gaussian center. The parameter $\alpha$ offers flexibility to achieve the desired combination of Laplacian and Gaussian distributions. Given this unique property of the Huber distribution, we propose the Huber mechanism which adds Huber noise to assure privacy.
We derive the bounds for the two cases separately in Appendix A and verify that the upper bound is very close to the Laplace mechanism, especially for higher values of variance. Even with the relaxed definition of privacy, the Gaussian mechanism demands a huge privacy budget, making it a less preferable choice for applications with stricter budgets. The Huber mechanism provides a middle ground between the Laplacian and Gaussian mechanisms w.r.t accuracy and privacy.

We will now proceed to the application of the Huber mechanism to achieve differentially private LRMC.

### 3 Differentially private LRMC using Huber mechanism

Having introduced the Huber mechanism for differential privacy, we demonstrate its utility in the area of matrix completion. We aim to introduce privacy to the iterative optimization process, thereby resulting in a differentially private LRMC algorithm. The choice of privacy mechanism is critical,
especially in an algorithm involving iterative operations. Huber noise combines the advantage of Gaussian noise that works well with optimization procedures as most of them are tuned for a normal spread of inherent noise in the data, with the low privacy budget of Laplacian noise to give us a new desirable noise hybrid.

### 3.1 Alternating least squares

Let \( \mathbf{X} \in \mathbb{R}^{m \times n} \) be the sensitive data matrix that is partly filled. Let \( \Omega \subseteq \mathbb{N}^m \times \mathbb{N}^n \) be the set indices in which \( \mathbf{X} \) has complete entries. \( \mathcal{P}_{\Omega}(\cdot) \) denotes the mask function; if \((i,j) \in \Omega\) \( \mathcal{P}_{\Omega}(\mathbf{X})_{i,j} = x_{ij} \), otherwise \( \mathcal{P}_{\Omega}(\mathbf{X})_{i,j} = 0 \). As suggested by Candes and Plan [2010], to estimate the incomplete entries in \( \mathbf{X} \), we approximate it to a low-rank matrix, \( \mathbf{Z} \) and minimize over all such matrices.

\[
\min_{\mathbf{Z}} \mathcal{L}(\mathcal{P}_{\Omega}(\mathbf{X} - \mathbf{Z})) + \lambda \|\mathbf{Z}\|_\infty.
\]

Here, \( \mathcal{L}(\cdot) \) is the data fidelity loss function. This optimization ensures that the existing/complete entries of the matrix \( \mathbf{X} \) and the low-rank approximation \( \mathbf{Z} \) are similar.

Despite being convex, the nuclear norm makes the problem difficult to solve. To reduce the computational burden, \( \mathbf{Z} \) is bilinearly factorized as \( \mathbf{U} \mathbf{V}^\top \) where \( \mathbf{U} \in \mathbb{R}^{m \times r} \) and \( \mathbf{V} \in \mathbb{R}^{n \times r} \), \( r \ll \min(m,n) \) and the nuclear norm regularization is reduced to Frobenius-norm regularization in \( \mathbf{U} \) and \( \mathbf{V} \).

\[
\min_{\mathbf{U}, \mathbf{V}} \mathcal{L}(\mathcal{P}_{\Omega}(\mathbf{X} - \mathbf{U} \mathbf{V}^\top)) + \frac{\lambda}{2} \left( \|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2 \right) .
\]

Though this problem is not convex, it is biconvex and typically solved using Alternating minimization. When \( \mathcal{L}(\cdot) = \|\cdot\|_F^2 \), the alternating minimization corresponds to Alternating Least Squares (ALS) (Hastie et al. [2015]), which can be decomposed over each row \( \mathbf{u}_i \) of \( \mathbf{U} \) (and over each row \( \mathbf{v}_j \) of \( \mathbf{V} \)). ALS is found to be faster than the nuclear norm regularized problem. The minimization can be described as

\[
\min_{\mathbf{U}, \mathbf{V}} \|\mathcal{P}_{\Omega}(\mathbf{X} - \mathbf{U} \mathbf{V}^\top)\|_F^2 + \lambda \left( \|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2 \right) .
\]

For a given \( \mathbf{U}, \mathbf{v}_j \) can be obtained through the following optimization problem.

\[
\min_{\mathbf{v}_j} \frac{1}{2} \|\mathcal{P}_{\Omega_j}(\mathbf{x}_j) - \mathbf{U}_{\overline{\Omega}_j, \mathbf{N}_r} \mathbf{v}_j\|_2^2 + \frac{\lambda}{2} \|\mathbf{v}_j\|_2^2 ,
\]

which results in the estimate

\[
\mathbf{v}_j \leftarrow \left( (\mathbf{U}_{\overline{\Omega}_j, \mathbf{N}_r})^\top \mathbf{U}_{\overline{\Omega}_j, \mathbf{N}_r} + \lambda \mathbf{I} \right)^{-1} (\mathbf{U}_{\overline{\Omega}_j, \mathbf{N}_r})^\top \mathcal{P}_{\Omega_j}(\mathbf{x}_j) .
\]

Similarly, for a given \( \mathbf{V}, \mathbf{u}_i \) can be estimated through least squares. Thus, ALS updates the estimates \( \mathbf{U} \) and \( \mathbf{V} \) in an alternating fashion and is summarized in Algorithm 1.

Noise is also added to the ALS procedure for preserving privacy similar to Jain et al. [2018] and Chien et al. [2021], as it was observed during experimentation that adding noise to the output directly in each iteration did not allow the algorithm to converge leading to poor precision. In the simulations section, we experiment with ALS by adding all three noise types and going over their pros and cons.

### 3.2 IRLS for Huber mechanism

In the Huber mechanism, we add Huber noise which has a heavier tail compared to the Gaussian. The Huber loss has been used in robust statistics to make the estimator less sensitive to large deviations. We now use the Huber loss as the data fidelity loss i.e., \( \mathcal{L}(\cdot) \) to be chosen as \( \rho_\alpha(\cdot) \) as defined in (3). Thus,

\[
\min_{\mathbf{U}, \mathbf{V}} \rho_\alpha(\mathcal{P}_{\Omega}(\mathbf{X} - \mathbf{U} \mathbf{V}^\top)) + \lambda \left( \|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2 \right) . \tag{6}
\]

Similar to ALS, we solve (6) through alternating minimization which can be decomposed over each row of \( \mathbf{U} \) and \( \mathbf{V} \) to avoid complex matrix computations. Each of the alternating minimization steps
Algorithm 1: Noisy Alternating Least Squares (ALS).

**Input:** Incomplete data matrix $X$, regularization parameter $\lambda$, number of iterations $T_{ALS}$.

**Output:** The completed matrix $\hat{Z}$

1. Initialize $\hat{U}$ and $\hat{V}$ with random entries
2. for $T_{ALS}$ iterations do
   3. for $i \in \mathbb{N}_m$ do
      4. $\hat{u}_i \leftarrow (\hat{V}_{i}, \hat{N}_r) \top \hat{V}_{i}, \hat{N}_r + \lambda I_r)^{-1} (\hat{V}_{i}, \hat{N}_r) \top P_{\Omega_i}(x_i)$
   5. end
   6. for $j \in \mathbb{N}_n$ do
      7. Generate noise vector $t$
      8. $\hat{v}_j \leftarrow ((\hat{U}_{i}, \hat{N}_r) \top \hat{U}_{i}, \hat{N}_r + \lambda I_r)^{-1} ((\hat{U}_{i}, \hat{N}_r) \top P_{\Omega_i}(x_i) + t)$
   9. end
10. end
11. return $\hat{Z} = \hat{U} \hat{V} \top$

Involves a Huber loss minimization with a Frobenius norm regularization term,

$\hat{U} \leftarrow \arg\min_{U} \rho_{\alpha}(P_{\Omega}(X - U \hat{V} \top)) + \lambda \|U\|_F^2$  \hspace{1cm} (7a)

$\hat{V} \leftarrow \arg\min_{V} \rho_{\alpha}(P_{\Omega}(X - \hat{U} V \top)) + \lambda \|V\|_F^2$  \hspace{1cm} (7b)

In robust statistics, Iterative Re-weighted Least Squares (IRLS) (Holland and Welsch [1977]) is used whenever the optimization involves Huber loss. IRLS is an iterative estimator which is regaining popularity recently (Kümmerle et al. [2021]). The works Dollinger and Staudte [1991], Kalyani and Giridhar [2007] show that the IRLS procedure approaches the maximum likelihood estimator for Huber loss under certain conditions. We solve (7a) and (7b) through IRLS, which we term as Regularized Iterative Re-weighted Least Squares (R-IRLS)

To derive the step-wise iterations for R-IRLS, we equate the gradient of the loss function with respect to $u_i$ to 0, $-\nabla W_i(x_i - \hat{V}u_i) + \lambda u_i = 0$, where

$W_i = \text{Diag}(w_{i1}, w_{i2}, \ldots, w_{im})$,  \hspace{1cm} $w_{ij} = \frac{\psi_{\alpha}(x_{ij} - v_j \top u_i)}{x_{ij} - v_j \top u_i}$

and $\psi_{\alpha}()$ is the Huber influence function defined in (4). This is solved through the IRLS algorithm, proposed by Holland and Welsch [1977] where $u_i$ is computed iteratively with intermediate estimates $\hat{u}_i$ and weights $W_i$. Similarly, rows of $V$ can also be computed using R-IRLS when $U$ is fixed. Thus, (6) is solved using Alternating Minimization:

$u_i \leftarrow \text{R-IRLS}(P_{\Omega_i}(x_i), V_{i}, \lambda)$  \hspace{1cm} $\forall i \in \mathbb{N}_m$

$v_j \leftarrow \text{R-IRLS}(P_{\Omega_j}(x_j), U_{j}, \lambda)$  \hspace{1cm} $\forall j \in \mathbb{N}_n$ \hspace{1cm} (8)

Recall that the $i$-th row of $X$ is denoted by $x_i$, which corresponds to the data of the $i$-th user and $\hat{X}_i$ denotes the $i$-th column. As we are interested in preserving the privacy of the user, we choose to add noise to that particular R-IRLS iteration which deals with the rows of the data matrix instead of the columns. This ensures that privacy is preserved over the item embeddings $V$. Noise could be added to both sets of iterations; but since there is no significant privacy gained by adding noise to the columns of $X$, we instead opt for noise addition to just one of the variables. Thus, no noise is added while updating $\hat{u}_i$ and hence, we use least squares instead of R-IRLS to update $\hat{u}_i$. The iterative steps for the proposed method are given in Algorithm 2, which utilizes the function described in Algorithm 3.
Algorithm 2: IRLS+Huber

Input: Incomplete data matrix $X$, assumed rank $r$, Huber transition parameter $\alpha$, regularization parameter $\lambda$, number of iterations $N$.

Output: The completed matrix $\hat{Z}$

1. Initialize $\hat{U}$ and $\hat{V}$ with random entries
2. for $N$ iterations do
   3. for $i \in \mathbb{N}_m$ do
      4. $\hat{U}_i \leftarrow \left( (\hat{V}_{\Omega_i}, N_r) \left( \hat{V}_{\Omega_i}, N_r + \lambda I_r \right)^{-1} \right)^\top \hat{V}_{\Omega_i}(x_i)$
   5. end
   6. for $j \in \mathbb{N}_n$ do
      7. $\hat{V}_j \leftarrow \text{R-IRLS} \left( P_{\hat{\Omega}_j}(\tilde{x}_j), \hat{U}_{\hat{\Omega}_j, N_r, r, \alpha, \lambda} \right)$ ▶ Update estimate
   8. end
3. end
4. return $\hat{Z} = \hat{U}\hat{V}^\top$

Algorithm 3: R-IRLS $(y, A, r, \alpha, \lambda)$

Input: Targets $y$, data matrix $A$, assumed rank $r$, Huber transition parameter $\alpha$, regularization parameter $\lambda$, number of iterations $K$.

Output: The IRLS estimate $\hat{\theta}$

1. Initialize $\hat{\theta} \sim N(0, 1)$
2. for $K$ iterations do
   3. $W \leftarrow \text{Diag} \left( \psi_\alpha(y_i - a_i^\top \hat{\theta}), \ldots, \psi_\alpha(y_p - a_p^\top \hat{\theta}) \right)$ ▶ Update weights
   4. Generate noise vector $t$
   5. $\hat{\theta} \leftarrow \left( A^\top WA + \lambda I_q \right)^{-1} \left( A^\top Wy + t \right)$ ▶ Update estimate
5. end
6. return $\hat{\theta}$

The IRLS + Laplacian method swaps out the Huber noise for Laplacian noise drawn from the distribution $\mathcal{L}(\Delta f/\epsilon)$. We also experiment with noisy IRLS in the simulation section and examine its comparative performance with noisy ALS.

4 Simulation Results

We present the empirical results for the Huber mechanism of noise addition for the problem of low-rank matrix completion and compare it with the Gaussian and Laplace mechanisms optimized using the ALS and IRLS procedure. We consider 3 different datasets:

1. A synthetic dataset in which we generate matrices of a specific rank as $X = UV^\top$ with $U$ and $V$ generated randomly. A percentage of the entries are sampled at random and replaced with zeros to indicate the incomplete entries.
2. The MovieLens100k dataset (Harper and Konstan [2015]). The dataset consists of approximately 1000 users and 100000 entries. The percentage of observable entries is $\sim 5\%$.
3. The Sweet Recommender System dataset (Kidziński [2017]) which contains 2000 users and over 45000 ratings ($\sim 40\%$ of entries are observable).

The simulation parameters are tabulated in Table 2. The metric used for measuring the performance of the algorithms is Root Mean Squared Error (RMSE), which is determined as $\|X - UV^\top\|_F / \sqrt{mn}$ (as considered in Chien et al. [2021], Liu et al. [2013] and Liu et al. [2015]).
For synthetic data, we consider several cases of noise variance, fraction of observed entries and rank of $X$ to evaluate the performance of the various mechanisms using both the ALS and IRLS procedures to decide the best course of action to collect and compare results from real datasets. Please note that the variance of Huber noise approaches 1 asymptotically as $\alpha \rightarrow \infty$. But, it decays rapidly and is approximately $\frac{1}{\alpha}$ even at $\alpha = 3$, which is employed to generate Huber noise of unit variance.

**Synthetic datasets** We tabulate RMSE for synthetic data with various parameters (such as the variance of additive noise, percentage of observable entries and optimization procedure) in Table 3. The lowest RMSE in each case is provided in bold. From Table 3, there are two key observations. Firstly, Huber noise gives consistently better results across different variances and visible fractions. Recall from Table 1 (in page 4) that the Huber mechanism provides similar privacy to the Laplace mechanism for the same variance (for higher variance values). However, it seems to provide better accuracy for the same amount of noise being added as evidenced by Table 3. The Gaussian mechanism provides much lower privacy (greater value of $\epsilon$) but does not significantly reduce RMSE over the Huber and Laplace mechanisms. Secondly, IRLS gives significant improvement over ALS with regards to Huber noise, especially for data with lesser observable fraction. This could be because IRLS is able to handle large deviations brought about by the Huber noise unlike ALS, without affecting privacy.

It is also observed that an increase in the variance of added noise leads to an overall decrease in accuracy, as expected. However, the effect is heavily pronounced for data with a lesser fraction of observed entries. Moreover, this behaviour is more evident for Gaussian and Laplacian noise when compared to Huber noise which makes the Huber mechanism more preferable for adding noise of higher variance.

In the case of Huber noise, we observe that ALS gives a better performance than IRLS for lower variance and vice-versa for higher variances. This may occur due to the fact that the Huber and Gaussian distributions are quite similar at lower variances (larger $\alpha$). In line with theoretical expectations, ALS gives better performance consistently for Gaussian noise. However, in most cases, IRLS gives better accuracy than ALS for Laplacian noise making the former more suitable for the Laplace mechanism. We also analyze the variation in performance with rank of the matrix $X$; these results are provided in Appendix B. We notice that with an increase in rank, Gaussian mechanism provides the lowest RMSE in a few cases although Huber mechanism gives the best performance.
However, it is worth noting that the privacy guarantees for the Gaussian mechanism are much weaker than both Huber and Laplacian.

**Real datasets** Based on the results from Synthetic data, we compare the results of ALS + Gaussian noise, IRLS + Laplacian noise and IRLS + Huber noise for the MovieLens dataset and SweetRS dataset in Tables 4 and 5 respectively. No noise is added in the cases of baseline ALS and baseline IRLS and they are hence not private. Rank is set to be 32 for both the datasets. For MovieLens, 20 iterations of ALS are performed and for SweetRS $T_{ALS} = 100$. For SweetRS dataset, 40% of ratings are available. However, for experiments where a lower number of observable entries are required, the entries are sub-sampled to produce the desired fraction.

For synthetic datasets, we provide comparisons across noise addition mechanisms for the same optimization procedure. Here, as we are comparing across optimization procedures as well, We note that performing 20 iterations of IRLS requires much higher computation time when compared to ALS. Therefore, we also provide results for a variation of IRLS that perform only 2 IRLS iterations.

A key observation is that there is no significant increase in the RMSE when we limit to just two iterations of IRLS. Thus, reasonable accuracy is achieved with less complexity.

### Table 4: RMSE for MovieLens100k, Visible fraction = 5.1%

| Visible fraction | ALS  | IRLS | ALS + G | IRLS + L | IRLS + H | IRLS-2 + H |
|------------------|------|------|---------|----------|----------|------------|
| MSE              | 1.2463 | 1.2787 | 1.3883 | 1.3952 | 1.3755 | 1.3850 |

### Table 5: RMSE for SweetRS

| Visible fraction | ALS  | IRLS | ALS + G | IRLS + L | IRLS + H | IRLS-2 + H |
|------------------|------|------|---------|----------|----------|------------|
| MSE              | 2.1333 | 2.1344 | 2.2103 | 2.2096 | 2.2022 | 2.2137 |
| 5%               | 1.7103 | 1.7102 | 1.8830 | 1.8827 | 1.8757 | 1.8820 |
| 10%              | 1.6881 | 1.6981 | 1.7686 | 1.7756 | 1.7694 | 1.7791 |

Similar to the trend observed for synthetic data, we note that Huber mechanism results in the lowest RMSE for both MovieLens100k as well and SweetRS datasets when 5% of the entries are visible. As the percentage of observable entries increases, Gaussian mechanism takes over. As mentioned earlier, Gaussian mechanism provides weaker privacy guarantees.

### 5 Discussions

We can observe that Laplacian noise gives satisfactorily accurate results for the low privacy budget that it requires. But the only drawback is that the large tail of the Laplacian distribution makes the results imprecise if consistent performance is important, and in most cases it is. Gaussian also works well, especially for datasets with a small observable fraction, but its large privacy budget works against its favour. Huber noise gives the best trade-off overall, with the double benefits of both high accuracy and a low privacy budget.

Although Gaussian noise is preferred to achieve differentially private ALS, we observe from our simulation results that other noise mechanisms perform competitively while offering $\epsilon$-DP guarantee as compared to the $(\epsilon, \delta)$-DP guarantee provided by Gaussian. Therefore, we conclude that the choice of noise mechanism is not obvious and exploring other mechanisms is crucial in improving performance.

Though we have explored the efficacy of the Huber mechanism for differentially private matrix completion, we believe that the mechanism holds its own merits as a noise addition mechanism for differential privacy. While constructing private algorithms, typically there is a trade-off between accuracy and privacy. In this work, we have provided exact privacy guarantees of the Huber mechanism and empirically compared its accuracy with other noise mechanisms in the context of matrix completion. This is because the exact characterization of the accuracy of LRMC algorithms is difficult. It is worth exploring the merits of the Huber mechanism analytically and in the general context of differential privacy.
References

Xin Luo, Mengchu Zhou, Yunni Xia, and Qingsheng Zhu. An efficient non-negative matrix-factorization-based approach to collaborative filtering for recommender systems. *IEEE Transactions on Industrial Informatics*, 10(2):1273–1284, 2014.

Wei He, Hongyan Zhang, Liangpei Zhang, and Huanfeng Shen. Total-variation-regularized low-rank matrix factorization for hyperspectral image restoration. *IEEE transactions on geoscience and remote sensing*, 54(1):178–188, 2015.

Luong Trung Nguyen, Junhan Kim, and Byonghyo Shim. Low-rank matrix completion: A contemporary survey. *IEEE Access*, 7:94215–94237, 2019.

James Bennett, Charles Elkan, Bing Liu, Padhraic Smyth, and Domonkos Tikk. Kdd cup and workshop 2007. *Association for Computing Machinery*, 9(2):51–52, dec 2007. ISSN 1931-0145. doi: 10.1145/1345448.1345459. URL https://doi.org/10.1145/1345448.1345459.

Canyi Lu, Jinhui Tang, Shuicheng Yan, and Zhouchen Lin. Nonconvex nonsmooth low rank minimization via iteratively reweighted nuclear norm. *IEEE Transactions on Image Processing*, 25(2):829–839, 2015.

Yuanyuan Liu, L.C. Jiao, and Fanhua Shang. A fast tri-factorization method for low-rank matrix recovery and completion. *Pattern Recognition*, 46:163–173, 01 2013. doi: 10.1016/j.patcog.2012.07.003.

Prateek Jain, Praneeth Netrapalli, and Sujay Sanghavi. Low-rank matrix completion using alternating minimization. *Proceedings of the Annual ACM Symposium on Theory of Computing*, 12 2012. doi: 10.1145/2488608.2488693.

Jian-Feng Cai, Emmanuel J. Candés, and Zuowei Shen. A singular value thresholding algorithm for matrix completion, 2008. URL https://arxiv.org/abs/0810.3286.

Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. *Proceedings of the 2006 ACM international conference on Data engineering*, pages 1–12, 2006. ISBN 1-59593-180-3. doi: 10.1145/1102351.1102354.

Shuang Song, Kamalika Chaudhuri, and Anand D Sarwate. Stochastic gradient descent with differentially private updates. In *2013 IEEE Global Conference on Signal and Information Processing*, pages 245–248. IEEE, 2013.

Arik Friedman and Assaf Schuster. Data mining with differential privacy. In *Proceedings of the 16th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 493–502, 2010.

Ziqi Liu, Yu-Xiang Wang, and Alexander J. Smola. Fast differentially private matrix factorization, 2015. URL https://arxiv.org/abs/1505.01419.

Prateek Jain, Om Dipakbhai Thakkar, and Abhradeep Thakurta. Differentially private matrix completion revisited. In *International Conference on Machine Learning*, pages 2215–2224. PMLR, 2018.

Steve Chien, Prateek Jain, Walid Krichene, Steffen Rendle, Shuang Song, Abhradeep Thakurta, and Li Zhang. Private alternating least squares: Practical private matrix completion with tighter rates. In *International Conference on Machine Learning*, pages 1877–1887. PMLR, 2021.

Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Found. Trends Theor. Comput. Sci.*, 9(3-4):211–407, 2014.
Peter J Huber. Robust estimation of a location parameter. *The Annals of Mathematical Statistics*, pages 73–101, 1964.

Emmanuel J Candes and Yaniv Plan. Matrix completion with noise. *Proceedings of the IEEE*, 98 (6):925–936, 2010.

Trevor Hastie, Rahul Mazumder, Jason D Lee, and Reza Zadeh. Matrix completion and low-rank svd via fast alternating least squares. *The Journal of Machine Learning Research*, 16(1):3367–3402, 2015.

Paul W Holland and Roy E Welsch. Robust regression using iteratively reweighted least-squares. *Communications in Statistics-theory and Methods*, 6(9):813–827, 1977.

Christian Kümmerle, Claudio Mayrink Verdun, and Dominik Stöger. Iteratively reweighted least squares for basis pursuit with global linear convergence rate. *Advances in Neural Information Processing Systems*, 34, 2021.

Michael B Dollinger and Robert G Staudte. Influence functions of iteratively reweighted least squares estimators. *Journal of the American Statistical Association*, 86(415):709–716, 1991.

Sheetal Kalyani and Krishnamurthy Giridhar. Mse analysis of the iteratively reweighted least squares algorithm when applied to m estimators. In *IEEE GLOBECOM 2007-IEEE Global Telecommunications Conference*, pages 2873–2877. IEEE, 2007.

F. Maxwell Harper and Joseph A. Konstan. The movielens datasets: History and context. *ACM Trans. Interact. Intell. Syst.*, 5(4), dec 2015. ISSN 2160-6455. doi: 10.1145/2827872. URL https://doi.org/10.1145/2827872.

Łukasz Kidziński. Sweetrs: Dataset for a recommender systems of sweets, 2017. URL https://arxiv.org/abs/1709.03496.
A Appendix: Derivation of privacy for Huber mechanism

Case 1: $\Delta f \leq 2\alpha$  
We go over all the intervals of $x$ in order and find the value of the function $g(x)$.

i. For $x < -\Delta f - \alpha$, $|x| > \alpha$ and $|x + \Delta f| > \alpha$ and both $x$ and $x + \Delta f$ are negative.

$$g_1(x) = \alpha \left( -x - \Delta f - \frac{\alpha}{2} \right) - \alpha \left( -x - \frac{\alpha}{2} \right) = -\alpha \Delta f.$$  
Since $g_1(x)$ is constant, the maximum of $g_1(x)$ in this range, $g_{1,max} = -\alpha \Delta f$.

ii. For $-\Delta f - \alpha \leq x \leq -\alpha$, $|x| \geq \alpha$ whereas $|x + \Delta f| \leq \alpha$.

$$g_2(x) = \frac{(x + \Delta f)^2}{2} - \alpha \left( -x - \frac{\alpha}{2} \right) = \frac{(x + \alpha + \Delta f)^2}{2} - \alpha \Delta f.$$  
The function is then monotonically increasing. Thus, $g_{2,max}$, occurs at $x = -\alpha$, $g_{2,max} = g_2(x)|_{x = -\alpha} = \Delta f(\Delta f - 2\alpha)/2 \leq 0$, since $\Delta f \leq 2\alpha$ by the case definition.

iii. For $-\alpha < x \leq -\alpha - \Delta f$, $|x| \leq \alpha$ and $|x + \Delta f| \leq \alpha$. Hence,

$$g_3(x) = \frac{(x + \Delta f)^2}{2} - \frac{x^2}{2} = x \Delta f + \frac{\Delta f^2}{2}.$$  
As $g_3(x)$ is monotonically increasing, $g_{3,max} = g_3(x)|_{x = -\alpha - \Delta f} = \alpha \Delta f - \frac{\Delta f^2}{2} \leq \alpha \Delta f$.

iv. For $-\alpha - \Delta f < x \leq -\alpha$, $|x| \leq \alpha$ whereas $|x + \Delta f| > \alpha$. Now, $x + \Delta f > 0$. So,

$$g_4(x) = \alpha \left( x + \Delta f - \frac{\alpha}{2} \right) - \frac{x^2}{2} = \alpha \Delta f - \frac{1}{2}(x - \alpha)^2.$$  
The maximum value in this range occurs at $x = \alpha$. Hence, $g_{4,max} = \alpha \Delta f$.

v. For $x > \alpha$, both $|x| > \alpha$ and $|x + \Delta f| > \alpha$, both $x$ and $x + \Delta f$ are positive.

$$g_5(x) = \alpha \left( x + \Delta f - \frac{\alpha}{2} \right) - \alpha \left( x - \frac{\alpha}{2} \right) = \alpha \Delta f.$$  
Since the value of $g_5(x)$ comes out to be a constant, $g_{5,max} = \alpha \Delta f$.

Thus, the overall upper bound $g_{max}$ for Case 1 can be computed as

$$g_{max} = \max_{i=1,\ldots,5} g_{i,max} = \alpha \Delta f$$

Case 2: $\Delta f > 2\alpha$  
Similar to case 1, we compute the values for $g(x)$ for different intervals of $x$.

i. For $x < -\Delta f - \alpha$, $g_1(x)$ and $g_{1,max}$ are identical to those in Case 1.

ii. For $-\Delta f - \alpha \leq x \leq -\alpha - \Delta f$, $|x|$ remains greater than $\alpha$ whereas $|x + \Delta f| \leq \alpha$.

$$g_2(x) = \frac{(x + \Delta f)^2}{2} - \alpha \left( |x| - \frac{\alpha}{2} \right) = \frac{(x + \alpha + \Delta f)^2}{2} - \alpha \Delta f.$$  
The maximum value of $g_2(x)$ occurs at $x = -\alpha - \Delta f$. Hence, $g_{2,max} = g_2(x)|_{x = -\alpha - \Delta f} = \alpha(2\alpha - \Delta f) \leq 0$, since $\Delta f > 2\alpha$ by the case definition.

iii. For $-\alpha - \Delta f < x \leq -\alpha$, $|x| \geq \alpha$ and $|x + \Delta f| > \alpha$. Also, note that $x < 0$ but $x + \Delta f > 0$.

$$g_3(x) = \alpha \left( x + \Delta f - \frac{\alpha}{2} \right) - \alpha \left( -x - \frac{\alpha}{2} \right) = \alpha(2x + \Delta f).$$  
The maximum value occurs at $x = -\alpha$, $g_{3,max} = g_3(x)|_{x = -\alpha} = \alpha \Delta f - 2\alpha^2 \leq \alpha \Delta f$.

iv. For $-\alpha < x \leq \alpha$, $|x| \leq \alpha$. So,

$$g_4(x) = \alpha \left( |x + \Delta f| - \frac{\alpha}{2} \right) - \frac{x^2}{2} = \alpha \left( x + \Delta f - \frac{\alpha}{2} \right) - \frac{x^2}{2} = \alpha \Delta f - \frac{1}{2}(x - \alpha)^2.$$  
The maximum value occurs at $x = \alpha$, $g_{4,max} = \alpha \Delta f$.

v. For $x > \alpha$ too, $g_5(x)$ and $g_{5,max}$ are identical to those in Case 1.

The overall upper bound $g_{max}$ for Case 2 is

$$g_{max} = \max_{i=1,\ldots,5} g_{i,max} = \alpha \Delta f.$$
Appendix: Additional results for Synthetic data

We present the results for the synthetic data when the rank of $X$ is set to 10 and 20 in Tables 6 and 7 respectively. When compared to Table 3, these results show an increase in MSE owing to the increased complexity of a higher rank structure. However, the rest of the trends are similar to those observed in the rank 5 data.

Table 6: Results for Synthetic dataset of rank 10

| Variance | Observed fraction | Algorithm | Vanilla | Gaussian | Laplacian | Huber |
|----------|-------------------|-----------|---------|----------|----------|--------|
| 1        | 5%                | ALS       | 0.4058  | 0.6471   | 0.6452   | 0.6720 |
|          |                   | IRLS      | 0.4059  | 0.6376   | 0.6888   | 0.6479 |
|          | 10%               | ALS       | 0.2317  | 0.3015   | 0.3029   | 0.3016 |
|          |                   | IRLS      | 0.2317  | 0.3023   | 0.3037   | 0.3031 |
|          | 15%               | ALS       | 0.1602  | 0.2770   | 0.2774   | 0.2768 |
|          |                   | IRLS      | 0.1596  | 0.2773   | 0.2771   | 0.2767 |

Table 7: Results for Synthetic dataset of rank 20

| Variance | Observed fraction | Algorithm | Vanilla | Gaussian | Laplacian | Huber |
|----------|-------------------|-----------|---------|----------|----------|--------|
| 1        | 5%                | ALS       | 0.6902  | 1.5165   | 1.5330   | 1.5142 |
|          |                   | IRLS      | 0.6886  | 1.5493   | 1.5715   | 1.5191 |
|          | 10%               | ALS       | 0.4622  | 0.5087   | 0.5113   | 0.5083 |
|          |                   | IRLS      | 0.4600  | 0.5054   | 0.5189   | 0.5083 |
|          | 15%               | ALS       | 0.3961  | 0.4520   | 0.4529   | 0.4516 |
|          |                   | IRLS      | 0.3957  | 0.4513   | 0.4526   | 0.4494 |
| 2        | 5%                | ALS       | 0.6126  | 1.8677   | 1.8843   | 1.8770 |
|          |                   | IRLS      | 0.6136  | 1.9546   | 1.9966   | 1.9524 |
|          | 10%               | ALS       | 0.4648  | 0.5222   | 0.5248   | 0.5159 |
|          |                   | IRLS      | 0.4651  | 0.5220   | 0.5238   | 0.5154 |
|          | 15%               | ALS       | 0.3895  | 0.4470   | 0.4486   | 0.4448 |
|          |                   | IRLS      | 0.3960  | 0.4474   | 0.4488   | 0.4429 |