Irreducibility of periodic curves in cubic polynomial moduli space

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Abstract
In the moduli space of complex cubic polynomials with a marked critical point, given any $p \geq 1$, we prove that the loci formed by polynomials with the marked critical point periodic of period $p$ is an irreducible curve. Thus, answering a question posed by Milnor in the 1990s.

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1 | INTRODUCTION

In their celebrated articles [5, 6], Branner and Hubbard laid the foundations for the study of complex cubic polynomials as dynamical systems. These articles shed light on the structural importance of certain dynamically defined complex one-dimensional slices of cubic parameter space [10]. The systematic study of such slices was initiated by Milnor in a 1991 preprint, later published as [30].

Following Milnor, consider the moduli space $\text{poly}_c^m$ of affine conjugacy classes of cubic polynomials with a marked critical point. This moduli space is isomorphic to the quotient of $\mathbb{C}^2$ by a linear involution with a unique fixed point. For each $p \geq 1$, Milnor considered the algebraic subset of $\text{poly}_c^m$ formed by all polynomials such that the marked critical point is periodic of exact period
\( p \), denoted \( \mathcal{S}_p \). He established that \( \mathcal{S}_p \) is smooth for all \( p \geq 2 \). Therefore, connectedness and irreducibility of \( \mathcal{S}_p \) are equivalent properties. Computing explicit parameterizations, he proved that \( \mathcal{S}_1, \mathcal{S}_2, \) and \( \mathcal{S}_3 \) are connected and asked in general: Is \( \mathcal{S}_p \) connected? [30, Question 5.3].

Our aim here is to answer this question in the positive:

**Theorem 1.** For all \( p \geq 1 \), the affine algebraic set \( \mathcal{S}_p \) is irreducible.

Our result is similar to one known for quadratic dynatomic curves:

\[
\{(z, c) \in \mathbb{C}^2 \mid z \text{ has exact period } n \text{ for } z \mapsto z^2 + c\}.
\]

Dynatomic curves were shown to be smooth by Douady and Hubbard [11] and irreducible by Bousch [9]. Nowadays several proofs for the irreducibility of these curves are available. The proofs range from those relying more on algebraic methods like Bousch’s original proof (cf. Morton [32]) to those more strongly based on dynamical techniques by Schleicher–Lau [42] (cf. Buff–Tan [8]). Our methods rely on one-dimensional parameter space techniques and bear certain analogy with the work by Schleicher–Lau [42].

In further generality, one might consider the algebraic subsets of moduli space determined by critical orbit relations. Their importance is confirmed by recent results regarding the dynamical analogue of the André–Oort conjecture by Favre and Gauthier [16], and Ghioca and Ye [19]. Recently, Buff, Epstein, and Koch [3] studied the curves \( \mathcal{S}_{k,1} \subset \text{poly}^{cm}_3 \) formed by polynomials such that the marked critical point maps, in exactly \( k \) iterations, onto a fixed point. For any \( k \geq 1 \), they established that \( \mathcal{S}_{k,1} \) is irreducible. Also, they obtained an analogue result in the moduli space of quadratic rational maps. Their techniques are related to those employed by Bousch [9] for dynatomic curves.

There are still plenty of open questions about the global topology of \( \mathcal{S}_p \). As anticipated by Milnor [30, section 5D], once we know that \( \mathcal{S}_p \) is connected, it becomes meaningful to determine its genus. A formula for the Euler characteristic of \( \mathcal{S}_p \) was obtained in [7, Theorem 7.2]. However, a direct formula for the number of punctures of \( \mathcal{S}_p \) remains unknown. De Marco and Schiff [13] gave an algorithm to compute this number based on the work by De Marco and Pilgrim [12].

There is a rich interplay between the global topology of \( \mathcal{S}_p \) and how polynomials are organized within \( \mathcal{S}_p \) according to dynamics. As the curves \( \mathcal{S}_p \) are complex one-dimensional parameter spaces, they have been natural grounds to employ and further develop the wealth of ideas available for quadratic polynomials (cf. [15, 30, 39]). However, a relevant novelty here is that \( \mathcal{S}_p \) has nontrivial global topology, for \( p \geq 4 \).

The usual dichotomy between connected and disconnected Julia sets takes place on \( \mathcal{S}_p \), as well. The relative connectedness locus \( C(\mathcal{S}_p) \) is the set formed by all polynomials in \( \mathcal{S}_p \) with connected Julia set and its complement, the escape locus \( \mathcal{E}(\mathcal{S}_p) \), is a disjoint union of disks punctured at infinity (e.g., [7]). Hence, as an immediate consequence of our main result, we have the following:

**Corollary 2.** For all \( p \geq 1 \), the connectedness locus \( C(\mathcal{S}_p) \) is connected.

The dynamics of quadratic rational maps, in analogy with complex cubic polynomials, also strongly depends on the behavior of two critical points and quadratic rational moduli space is also a complex surface. The curves analogous to \( \mathcal{S}_p \) have been under intensive study during the last 25 years as well (e.g., see [2, 17, 21, 36–38, 40, 45]). Nevertheless, the irreducibility of these curves is still to be determined.
2 | OVERVIEW

The moduli space of complex cubic polynomials with a marked critical point, denoted \( \text{poly}_3^{cm} \), is formed by affine conjugacy classes of pairs \((f, a)\), where \( f : \mathbb{C} \to \mathbb{C} \) is a cubic polynomial and \( f'(a) = 0 \). Following Milnor, by passing to a double cover, one may avoid the nuances of working with conjugacy classes. More precisely, for \((a, v) \in \mathbb{C}^2\), let

\[
 f_{a,v}(z) = (z - a)^2(z + 2a) + v.
\]

The critical points of \( f_{a,v} \) are \( a \) and \( -a \). We say that \( a \) is the marked critical point and \( -a \) is the free critical point. Via conjugacy by \( z \mapsto -z \), the polynomials \( f_{a,v} \) and \( f_{-a,-v} \) represent the same element of \( \text{poly}_3^{cm} \). The quotient of \( \mathbb{C}^2 \) by the involution \( (a, v) \mapsto (-a, -v) \) is \( \text{poly}_3^{cm} \).

Denote by \( \mathcal{F}_p \subset \mathbb{C}^2 \) the affine algebraic curve formed by all \( f_{a,v} \) such that \( a \) has period exactly \( p \) under \( f_{a,v} \). It follows that \( \mathcal{F}_1 \) is a double cover of \( \mathcal{S}_1 \), ramified at the monomial map \( f_{0,0}(z) = z^3 \); and \( \mathcal{F}_p \to \mathcal{S}_p \) is a regular double cover for all \( p \geq 2 \). Thus, to establish our main result, it suffices to prove that \( \mathcal{F}_p \) is connected for all \( p \geq 2 \).

For the rest of the paper, let us fix \( p \geq 2 \) and consider the natural decomposition of \( \mathcal{F}_p \) into the connectedness locus \( \mathcal{C}(\mathcal{F}_p) \) and the escape locus \( \mathcal{E}(\mathcal{F}_p) \). The former consists of all \( f_{a,v} \in \mathcal{F}_p \) with connected Julia set. The latter is formed by maps with disconnected Julia set, or equivalently, maps such that the free critical point escapes to infinity.

It will also be convenient to drop the subscripts from \( f_{a,v} \in \mathcal{F}_p \). We will simply say that \( f \in \mathcal{F}_p \) has marked critical point \( a(f) \), free critical point \( -a(f) \) and, for \( j \geq 0 \), let \( a_j(f) = f^j(a(f)) \) be the corresponding periodic point in the orbit of \( a(f) \).

Given a map \( f \in \mathcal{F}_p \), let \( G_f \) be the Green function of the basin of infinity. For all \( f \in \mathcal{E}(\mathcal{F}_p) \),

\[
 \{ z : G_f(z) < G_f(-a(f)) \}
\]

is the union of two disjoint topological disks \( D_0 \) and \( D_1 \). We label these disks so that \( a(f) \in D_0 \). Although \( D_0 \) and \( D_1 \) depend on \( f \), it will cause no confusion to employ a notation that ignores this fact. The filled Julia set \( K(f) \) is contained in \( D_0 \cup D_1 \), in particular, the period \( p \) critical orbit points \( a_0(f), a_1(f), \ldots, a_{p-1}(f) \) lie either in \( D_0 \) or \( D_1 \).

The kneading word \( \kappa(f) \) of \( f \) is the binary sequence

\[
 \kappa(f) = t_1 \ldots t_{p-1}t_p,
\]

where \( t_j = 0 \) or 1 according to whether \( a_j(f) \in D_0 \) or \( D_1 \). Note that \( t_p = 0 \).

Connected components of the escape locus are called escape regions. They will play a central role in our work. From [30, section 5B] we may extract the following fundamental facts. Escape regions are in one-to-one correspondence with the ends of \( \mathcal{F}_p \). Maps within an escape region \( U' \) share the same kneading word, which we denote by \( \kappa(U') \). Different escape regions can have the same kneading word. However, there is exactly one region \( U \), called the distinguished escape region, such that \( \kappa(U') = 1^{p-1}0 \). We say that \( 1^{p-1}0 \) is the distinguished kneading word.

Every component of the algebraic set \( \mathcal{F}_p \) contains at least one escape region. Given an undistinguished escape region \( U' \) with kneading word \( \kappa \), our strategy consists on constructing a path in \( \mathcal{F}_p \) joining \( U' \) with another escape region \( U'' \) such that, in a certain sense, the kneading word \( \kappa' = \kappa(U'') \) is closer to the distinguished kneading word (cf. [1]). The kneading word \( \kappa' \) will be obtained from \( \kappa \) by a type \( A \) or \( B \) move described below.
To describe type $A$ and $B$ moves, for a kneading word $\kappa \neq 0^p$, let $\mu$ be such that $1^{\mu-1}$ is the largest string of 1’s that one encounters in $\kappa$; for $\kappa = 0^p$, let $\mu = 1$. If $f \in \mathcal{E}(S_p)$ has kneading word $\kappa$, then $\mu$ is the maximal time that takes an element of the periodic critical orbit in $D_0$ to return to $D_0$. We say that $\mu$ is the maximal return time of $\kappa$.

Let $\kappa = t_1 \ldots t_{p-1}0$ be an undistinguished word with maximal return time $\mu$.

(A) We say that $\kappa'$ is obtained from $\kappa$ by a type $A$ move if:

$$\kappa = 1^{\mu-1}0_{\mu+1} \cdots t_{p-1}0,$$

and

$$\kappa' = 0^{\mu-1}1_{\mu+1} \cdots t_{p-1}0,$$

for some $t'_{\mu+1}, \ldots, t'_{p-1} \in \{0, 1\}$.

(B) We say that $\kappa'$ is obtained from $\kappa$ by a type $B$ move if, for some $1 \leq k \leq p-1$:

$$\kappa = t_1 \cdots t_{k-1}0 \cdot 1^{\mu-1}t_{k+\mu} \cdots t_{p-1}0,$$

and

$$\kappa' = t_1 \cdots t_{k-1}1 \cdot 1^{\mu-1}t_{k+\mu} \cdots t_{p-1}0.$$

That is, the type $A$ move changes an initial string of $\kappa$ in a specific manner, but there is no restriction on how the tail changes. The type $B$ move just changes the $k$th symbol from a “0” to a “1” and leaves the rest of the symbols untouched. The former can only occur if the maximal return time is “attained” at the position $k = 0$, and the latter if it is “attained” at a position $k \neq 0$ in the kneading word $\kappa$.

It is not difficult to show that no matter how we successively apply type $A$ and $B$ moves, in at most $p^2 + p$ steps we arrive to the distinguished word (see Lemma 7.1). Type $A$ and $B$ moves will be associated to a special way of crossing certain types of hyperbolic components in $C(S_p)$, known as type $A$ and $B$ components.

Given an escape region $\mathcal{U}'$, our path within $S_p$ starts at a map with a ray connection. For a general background on external rays, we refer to Subsection 3.1. Denote by $V(a_k(f))$ the Fatou component of $f \in \mathcal{U}'$ containing $a_k(f)$. We say that a nonsmooth ray $R_\sigma^\mathcal{U}(\vartheta)$ where $\sigma = +$ or $-$ is a ray connection between $-a(f)$ and $a_k(f)$ if the following hold.

- $3\vartheta$ is periodic of period $p$ under multiplication by 3 in $\mathbb{R}/\mathbb{Z}$.
- $-a(f) \in R_\sigma^\mathcal{U}(\vartheta)$.
- $R_\sigma^\mathcal{U}(\vartheta)$ is a relatively (left or right) supporting ray of $V(a_k(f))$.

A detailed discussion about ray connections is given in the introduction to Section 4 where the notion of relatively supporting rays is introduced as a generalization of the usual notion of supporting rays. Then we prove that, for any undistinguished escape region $\mathcal{U}'$, there exists a map $f \in \mathcal{U}'$ that has a ray connection between $-a(f)$ and $a_k(f)$ for some $a_k(f) \in D_0$ such that $a_k(f)$ has maximal return time to $D_0$ (Theorem 4.3).

Once we have a map $f$ with an appropriate ray connection, we head toward the connectedness locus along the parameter ray $R_{\mathcal{U}'}(\vartheta)$ containing $f$. The landing behavior of rational parameter rays $R_{\mathcal{U}'}(\vartheta)$ is studied by Bonifant, Milnor, and Sutherland in a forthcoming paper.
These rays land at parabolic or postcritically finite (pcf) maps. For our purpose, we need to describe in greater detail the supporting properties of external rays, and to keep track of kneading words at the landing point of these parameter rays. To properly state our description, given a parabolic or pcf map $f_0$, we introduce the notions of a take-off argument $\theta$ of $f_0$ and its associated kneading word $\kappa(f, \theta)$, see Definitions 5.2 and 5.3. Rays landing at parabolic maps are the subject of Subsection 6.7, while rays landing at pcf maps are considered in Subsection 5.3. Special care is taken to describe the maps that are the landing point of parameter rays with ray connections. “Take-off” results are also relevant to our discussion. Namely, given any parabolic or pcf map $f_0$ with take-off argument $\theta$, we establish that there exists an escape region $U'$ such that $\kappa(U') = \kappa(f, \theta)$ and a parameter ray $R_{U'}(\theta)$ of $U'$ that lands at $f_0$. This follows directly from Bonifant and Milnor’s forthcoming work for $f_0$ pcf, and from Theorem 5.8 for $f_0$ parabolic.

According to Milnor [30], bounded hyperbolic components of $C(S_p)$ fall into four types: adjacent (A), bitransitive (B), capture (C), and disjoint (D). Our main interest will be on types A and B. A hyperbolic map $f \in C(S_p)$ is of type $A$ if the free critical point $-a(f)$ lies in the Fatou component $V(a(f))$. The map $f$ is of type $B$ if $-a(f) \in V(a_k(f))$ for some $0 < k < p$. For completeness, let us just mention that $f$ is of type $C$ if $-a(f)$ lies in a strictly preperiodic Fatou component, and of type $D$ if $-a(f)$ and $a(f)$ lies in disjoint cycles of Fatou components.

In Section 6, we start by discussing parameter internal rays and sectors of type $A$ and $B$ hyperbolic components (see Subsection 6.1 and Subsection 6.2). The key for the rest of the results in Section 6 is contained in Subsection 6.3, where we study the landing points of 0 and 1/2 parameter internal rays. For us, a “root” of a type $A$ or $B$ component will be the landing point of a 0-parameter internal ray, and a “co-root” will be the landing point of a 1/2-parameter internal ray of a type $A$ component. In Subsections 6.4 and 6.5, given a “root” or a “co-root” of a type $A$ and $B$ hyperbolic component, we compute the effect that crossing (trekking) to another “root” or “co-root” has in their take-off kneadings. In Subsections 6.6 and 6.7, we certify that the landing points of parameter rays with ray connections are in the boundary of a type $A$ or $B$ hyperbolic component. Loosely speaking, we are granted entry to such a hyperbolic component and say that these are “immigration” results. More precisely, we will show that in the presence of a periodic ray connection the landing point is a “root” of a type $A$ or $B$ component, see Subsection 6.6. In the presence of a preperiodic ray connection, the landing point will be a “co-root” of a type $A$ component, see Subsection 6.7.

In Section 7, we assemble the proof of our main result. Given an escape region $U'$ with kneading word $\kappa$, we consider the landing point $f_0$ of a parameter ray $R_{U'}(\theta)$ with a ray connection between $-a(f)$ and $a_k(f)$ where $a_k(f)$ has maximal return time to $D_0$. The immigration results guarantee that $f_0$ is in the boundary of a type $A$ or $B$ hyperbolic component $H$. The trekking theorems apply to find in $\partial H$ a parabolic or pcf map $f_1$ with a take-off argument $\theta$ so that the corresponding kneading $\kappa(f_1, \theta)$ is obtained from $\kappa(f_0, \theta) = \kappa$ by a type $A$ or $B$ move, according to whether $k = 0$ or $k \neq 0$. Finally, the take-off theorems yield an escape region $U''$ with kneading word $\kappa' = \kappa(f_1, \theta)$ and a parameter ray $R_{U''}(\theta)$ landing at $f_1$.

In Figure 1, we label parameters $\odot$ through $\otimes$ along a path crossing a type $B$ component. The path starts at $\odot$ that corresponds to a map in $R_{U'}(\theta)$. The parabolic maps $f_0$ and $f_1$ above correspond to $\odot$ and $\otimes$, respectively. The dynamical planes along this path are illustrated in Figure 2.

**Remark 2.1.** The figures contained in the paper are details of $S_3$ or Julia sets of this parameter space generated by FractalStream using the parameterization provided in [30]. The parameter and dynamical rays are handmade crude approximations. In parameter space illustrations, escape regions are colored white, type $A$, $B$, $C$ hyperbolic components are in gray, and type $D$ in black.
In dynamical plane illustrations, the basin of infinity is colored white, the basin of the marked critical periodic point $a(f)$ is in gray and, when existing, other Fatou components are in black.

### 3 RAYS AND SECTORS

For a general background on complex polynomial dynamics, we refer to [29]. In Subsection 3.1, we concentrate on external rays of disconnected Julia sets. External rays are often employed to subdivide the complex plane into regions for which it is convenient to introduce a notation explained...
in Subsection 3.2. Parameter external rays are the subject of Subsection 3.3. Internal rays are discussed in Subsection 3.4.

3.1 | External rays

We will intensively exploit external rays for maps with disconnected Julia sets. This brief section aims at recalling notations, definitions and summarizing results that apply to the context of disconnected Julia sets of maps in $S_p$ (see [18, 26, 27]).

Given $f \in S_p$, the Böttcher coordinate $\phi_f$, initially defined near $\infty$ (e.g., see [29, section 9]), is the unique conformal isomorphism asymptotic to the identity at infinity $(\phi_f(z)/z \to 1)$ such that $(\phi_f(z))^3 = \phi_f(f(z))$. There exists a unique continuous extension of $|\phi_f(z)|$ to the basin of infinity $V_f(\infty)$ respecting the functional relation $|\phi_f(z)|^3 = |\phi_f(f(z))|$. Denote by $V_f^*(\infty)$ the basin of infinity under the flow of the vector field $\nabla|\phi_f|$. The Böttcher coordinate extends along flow lines of $\nabla|\phi_f|$ to $V_f^*(\infty)$ (cf. [27]). The singularities $z$ of the vector field (i.e., $\nabla|\phi_f(z)| = 0$) coincide with the points $z$ that eventually map onto a critical point (in our case onto $-a(f)$).

If $f \in \mathcal{C}(S_p)$, then $V_f^*(\infty) = V_f(\infty)$, $\phi_f : V_f(\infty) \to \mathbb{C} \setminus \overline{D}$ is a conformal isomorphism, and $R_f(t) = \phi_f^{-1}(1, \infty[\exp(2\pi it))$ is called the external ray of $f$ at argument $t \in \mathbb{R}/\mathbb{Z}$.

When $f \in \mathcal{E}(S_p)$, for each $t \in \mathbb{R}/\mathbb{Z}$ there exists a unique maximal $\nabla|\phi_f|$-flow line, denoted $R^*_f(t)$, and a number $\rho(t) \geq 1$ such that

$$\phi_f : R^*_f(t) \to ]\rho(t), \infty[ \cdot \exp(2\pi it)$$

is a homeomorphism. If $\rho(t) > 1$, then the limit in the negative flow direction along $R^*_f(t)$ is a singularity $w$ of the flow (i.e., an iterated preimage of the critical point $-a(f)$). We say that $R^*_f(t)$ terminates at $w$. There is exactly one line $R_f^*(t)$ that contains the cocritical point $2a(f)$ (i.e., $f(2a(f)) = f(-a(f))$). Then $R_f^*(t)$ contains all points $z$ that eventually map onto a critical point (in our case onto $-a(f)$). These flow lines together with $-a(f)$ cut the complex plane into two regions. The one containing the periodic critical point $a(f)$ also contains the disk $D_0$ as well as $R_f^*(t)$ for all $t \in ]\theta - 1/3, \theta + 1/3[$. The critical point free region contains the disk $D_1$ as well as $R_f^*(t)$ for all $t \in ]\theta + 1/3, \theta - 1/3[$. Observe that $\rho(t) = 1$ if and only if $3^n t \neq \theta \pm 1/3$ for all $n \geq 0$ and, in this case, we say that $R_f(t) = \phi_f^{-1}(1, \infty[\exp(2\pi it))$ is the external ray of $f$ at argument $t \in \mathbb{R}/\mathbb{Z}$. Although the external ray $R_f(t)$ does not exist for arguments $t$ such that $\rho(t) > 1$, we may consider the left and right limit rays at $t$, denoted $R_f^{-}(t)$ and $R_f^{+}(t)$, respectively. They are defined as the limit of $R_f(s)$ when $s \to t$ from the left or from the right, as arcs parameterized by $|\phi_f|$ (see Goldberg and Milnor [18, Appendix A]). Left and right limit rays with arguments $t$ such that $\rho(t) > 1$ are usually called singular, bouncing, broken or nonsmooth. We adopt the convention that $R_f^\pm(t)$ simply denotes the smooth ray $R_f(t)$ when $\rho(t) = 1$. With this convention $f$ maps $R_f^\pm(t)$ onto $R_f^\pm(3t)$. It follows that every nonsmooth ray eventually maps onto $R_f^\pm(\theta \pm 1/3)$.

In this paper, multiplication by 3 acting in $\mathbb{R}/\mathbb{Z}$ will be denoted by $m_3 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Similarly, $m_d$ will denote multiplication by $d$ for $d \geq 2$.

In $S_p$, a map $f$ with disconnected Julia set $J(f)$ is hyperbolic. Thus, every ray lands (i.e., has a well-defined limit as it approaches the Julia set). Keep in mind that a repelling periodic point $z$ can be the landing point of infinitely many rays if $\{z\}$ is a connected component of $J(f)$; otherwise,
z is the landing point of finitely many periodic rays (some of which might be left or right limit rays) [26].

For any map $f \in S_p$, according to Roesch and Yin [41] we have that $\partial V(a(f))$ is a Jordan curve. It follows that $f^p : \partial V(a(f)) \to \partial V(a(f))$ is conjugate to multiplication by $m_d$ for $d = 2, 3$, or 4. Thus, $\partial V(a(f))$ contains at least one periodic point of period dividing $p$. The external rays landing at any one of these points have period $p$ (e.g., see Proposition 6.5).

### 3.2 Regions and sectors

Given $f \in S_p$, a connected and full subset $X$ of $K(f)$, and two rays $R_f(s), R_f(t)$ landing at $X$, we denote by $Z_f(X, s, t)$ the connected component of

$$
\mathbb{C} \setminus (R_f(s) \cup R_f(t) \cup X)
$$

containing the external rays with arguments in $]s, t[ \subset \mathbb{R}/\mathbb{Z}$. When $X$ is a singleton $\{z\}$ we simply write $Z_f(z, s, t)$. In this case, we say that $Z_f(z, s, t)$ is a sector at $z$ if no ray with argument in $]s, t[$ lands at $z$.

We freely employ this notation in the presence of nonsmooth rays, for example, if $R_f^-(s)$ and $R_f(t)$ land at $z$, then $Z_f(z, s, t)$ denotes the region containing all rays with arguments in $]s, t[$.

Consider a point $z$ in the boundary of a Fatou component $V$ of $f \in S_p$. Following Poirier [34], if $Z_f(z, s, t)$ is the sector based at $z$ containing $V$, we say that $s$ (resp., $t$) is the left (resp., right) supporting argument of $V$ at $z$.

### 3.3 Parameter external rays

For every $f \in \mathcal{E}(S_p)$, the cocritical point $2a(f)$ belongs to the domain $V_f^*(\infty)$ of the Böttcher coordinate $\phi_f$. Moreover, $f \to \phi_f(2a(f))$ depends holomorphically on $f \in \mathcal{E}(S_p)$. Furthermore, the map

$$
\Phi_{U^*} : \ U^* \to \mathbb{C} \setminus \overline{\mathbb{D}}
$$

$$
f \mapsto \phi_f(2a(f)),
$$

is a regular covering map (without branched points) of some degree called the multiplicity of $U^*$ [30, Lemma 5.9].

A parameter ray $R_{U^*}(\theta)$ with argument $\theta \in \mathbb{R}/\mathbb{Z}$ is defined as an arc in $U^*$ that maps bijectively onto $]1, \infty[\exp(2\pi i \theta)$ by $\Phi_{U^*}$. Be aware that there might be several parameter rays $R_{U^*}(\theta)$ associated to the same argument $\theta$. Note that $f$ lies in a ray $R_{U^*}(\theta)$ if and only if $R_f^-(\theta \pm 1/3)$ terminate at $-a(f)$.

### 3.4 Internal rays

We work under the assumption that $-a(f)$ does not belong to the periodic orbit of $a(f)$ unless otherwise stated. That is, the local degree of $f^p$ at $a_k(f)$ is 2, for all $k$ subscripts modulo $p$. There
exists a neighborhood $W$ of $a_k(f)$ and a unique univalent map $\varphi_{f,a_k} : W \to \mathbb{D}$ conjugating the action of $f^p$ with $z \mapsto z^2$. We say that $\varphi_{f,a_k} : W \to \mathbb{D}$ is the Böttcher coordinate at $a_k$. It follows that

$$\varphi_{f,a_{k+1}} \circ f(z) = \left( \varphi_{f,a_k}(z) \right)^{\deg_{a_k} f},$$

where $\deg_{a_k} f = 1$ if $k \neq 0$ and $\deg_{a_0} f = 2$. According to [29, Corollary 9.2], $|\varphi_{f,a_k}|$ has a unique continuous extension to $V_f(a_k(f))$ that satisfies the functional relation $|\varphi_{f,a_k}(f^p(z))| = |\varphi_{f,a_k}(z)|^2$. In analogy with what occurs in the basin of infinity, we let $V^s_f(a_k(f))$ be the basin of $a_k(f)$ under the flow of $-\nabla |\varphi_{f,a_k}|$. The Böttcher coordinate extends along flow lines to a conformal isomorphism

$$\varphi_{f,a_k} : V^s_f(a_k(f)) \to U^*_{f,a_k} \subset \mathbb{D},$$

where $U^*_{f,a_k}$ is a star-like domain around $w = 0$. The zeros of $\nabla |\varphi_{f,a_k}|$ are exactly the points in $V_f(a_k(f))$ that eventually map onto $-a(f)$.

If $-a(f) \notin V_f(a_j(f))$ for all $j$, or equivalently $f$ is not in a type $A$ or $B$ component, then the Böttcher coordinate extends to a conformal isomorphism $\varphi_{f,a_k} : V_f(a_k(f)) \to \mathbb{D}$. Moreover, $\partial V_f(a_k(f))$ is a Jordan curve, and $\varphi_{f,a_k}$ extends continuously to a conjugacy $\varphi_{f,a_k} : \overline{V_f(a_k(f))} \to \overline{\mathbb{D}}$ between $f^p$ and $z \mapsto z^2$. In particular, there is exactly one periodic point of period dividing $p$ in $\partial V_f(a_k(f))$. (This holds in further generality, e.g., see Proposition 6.5.)

Our main interest here is on maps $f$ that lie in a type $A$ or $B$ hyperbolic component. For such a map $f$, be aware that $V^s_f(a_k(f))$ is not dense in $V_f(a_k(f))$, in contrast with the discussion in Subsection 3.1. Given $t \in \mathbb{R}/\mathbb{Z}$, there exists a maximal $0 < \rho(t) \leq 1$ such that $[0,\rho(t)] \cdot \exp(2\pi it) \subset U^*_{f,a_k}$. Let

$$I^o_{f,a_k}(t) = \varphi_{f,a_k}^{-1}([0,\rho(t)] \cdot \exp(2\pi it)).$$

This arc starts at $a_k(f)$, and at the other end it approaches $\partial V_f(a_k(f))$ or an iterated preimage of $w$ of $-a(f)$. In the former case, $\rho(t) = 1$ and we say that $I^o_{f,a_k}(t)$ is an internal ray at $a_k$ with argument $t$. In the latter, $\rho(t) < 1$ and we say that $I^o_{f,a_k}(t)$ terminates at $w$. Right and left limit internal rays $I^\pm_{f,a_k}(t)$ are defined as in Subsection 3.1.

## 4 | RAY CONNECTIONS

The definition of ray connections relies on the notion of relatively supporting rays. This notion generalizes the definition of left and right supporting rays for Fatou components (see Subsection 3.2).

**Definition 4.1.** Given a bounded Fatou component $V$ of a map $f \in S_p$, consider an eventually periodic point $z \in \partial V$. There exist unique $t_1, t_2 \in \mathbb{Q}/\mathbb{Z}$ (possibly equal) in the same grand orbit under $m_3$ such that external rays (maybe bouncing) with arguments $t_1, t_2$ land at $z$, $V \subset Z(z, t_1, t_2)$ and, if a ray with argument $t \in [t_1, t_2]$ lands at $z$, then $t$ is not in the grand orbit of $t_1, t_2$. We say that $t_1$ (resp., $t_2$) is a relatively left (resp., right) supporting argument of $V$ at $z$. 


Figure 3 The lower left figure is a detail of $S_3$ with an external parameter ray and a 0-internal ray of a type B component, both landing at a parabolic map $\mathcal{P}$. When converging to $\mathcal{P}$, along each one of the two parameter rays, a fixed point with combinatorial rotation number $1/3$ and a period 3 periodic orbit coalesce. Relevant external rays are illustrated. For $\mathcal{P}$, the 0-internal rays are also drawn.

Remark 4.2. In Figure 3 $\mathcal{P}$, light gray rays and black rays are in different orbits. The rays meeting the lower edge of the figure are both relatively right supporting for the large gray Fatou component (containing $+a$). However, only the black ray is right supporting.

Recall that for any $f \in S_3$ and $0 \leq k < p$, there exists a period $p$ argument $t$ such that a ray with argument $t$ lands at $\partial V(a_k(f))$. Among the arguments in the orbit of $t$, we can always choose one that is relatively left (resp., right) supporting for $V(a_k(f))$. We are particularly interested in the situation when $f$ lies in an escape region and such a ray is one bouncing off the critical point $-a(f)$. When $k = 0$, we will also be interested in the situation when $t$ itself is not periodic but $3t$ has period $p$ and a ray with argument $t$ is relatively supporting for $V(a(f))$.

Consider an escape region $U$ and $f \in U$. Recall that an external ray $R^\sigma_3(\vartheta)$, where $\sigma \in \{+,-\}$, is a ray connection between $-a(f)$ and $a_k(f)$ if the following hold.

- $3\vartheta$ is periodic of period $p$ under $m_3$.
- $-a(f) \in R^\sigma_3(\vartheta)$. 
• $R^+_f(\vartheta)$ is a relatively (left or right) supporting ray of $V(a_k(f))$.

If a map $f$ in a parameter ray $R^U(\vartheta)$ has a ray connection $R^+_f(\vartheta)$ between $-a(f)$ and $a_k(f)$, then $\vartheta = \vartheta + 1/3$ or $\vartheta - 1/3$. Moreover, all the maps in $R^U(\vartheta)$ have such a ray connection. That is, ray connections are preserved along parameter rays.

A periodic ray connection $R^+_f(\vartheta)$ lands at the unique point $z_0$ of period dividing $p$ in $\partial V(a_k(f))$. A preperiodic ray connection lands at the unique nonperiodic point $w_0$ in $\partial V(a_0(f))$ such that $f(w_0)$ has period dividing $p$. That is, preperiodic ray connections only occur between $-a(f)$ and $a_0(f) = a(f)$.

Periodic (resp., preperiodic) ray connections between $-a(f)$ and $a_k(f)$ will be associated to parameter rays that land at parabolic (resp., pcf) maps. Type $A$ moves will be associated to ray connections such that $k = 0$, which may be periodic or preperiodic connections. Type $B$ moves will be associated to ray connections such that $k \neq 0$, which are always periodic connections.

The key to start our journey toward the distinguished escape region is the following result:

**Theorem 4.3.** Given an undistinguished escape region $U'$, there exists $f \in U'$ with a ray connection between $-a(f)$ and $a_k(f)$ such that the following statements hold.

1. $a_k(f) \in D_0$.
2. The return time of $a_k(f)$ to $D_0$ is maximal.

The rest of the section is devoted to the proof of this theorem.

Given an escape region $U'$, it will be convenient to consider a one-parameter family

$$\mathbb{R}/\mathbb{Z} \setminus \{0\} \rightarrow U'$$

$$\vartheta \mapsto f_\vartheta$$

such that:

1. $f_\vartheta = f_{a(\vartheta),v(\vartheta)}$, where $a(\vartheta)$ and $v(\vartheta)$ are real analytic functions;
2. $f_\vartheta$ lies in a parameter ray with argument $\vartheta$;
3. the escape rate of $-a(\vartheta)$ is independent of $\vartheta$.

The existence of such a family is guaranteed by Subsection 3.3. In the sequel, $f_\vartheta$ will always denote a family as above in an escape region $U'$, which will be clear from context. The external rays of $f_\vartheta$ will be simply denoted by $R_\vartheta(t)$, the Fatou component containing $a_k(f_\vartheta)$ simply by $V(a_k(\vartheta))$, and so on.

When the multiplicity of $U'$ is at least 2, the one-parameter family $f_\vartheta$ only covers a portion of an equipotential of $U'$. Namely, the curve of parameters $f \in U'$ with constant escape rate $G_f(-a(f))$ contains more than one map $f$ that belongs to a parameter ray with argument $\vartheta$, for all $\vartheta \in \mathbb{R}/\mathbb{Z}$ (as there are several parameter rays with the same argument $\vartheta$). In this case, the “endpoints” of the family $f_\vartheta$ are distinct. Working with a family $f_\vartheta$ that parameterizes only a portion of an equipotential will be sufficient, for our purpose.

It is not difficult to show that, for any $t \in \mathbb{R}/\mathbb{Z}$, as $\vartheta \nearrow \vartheta_0$, the rays $R_\vartheta(t)$ converge to $R^+_0(t)$, as arcs parameterized by $|\phi_\vartheta|$. A similar statement holds as $\vartheta \searrow \vartheta_0$.

Let us first show that ray connections are present in any escape region. However, it is essential (and harder) to obtain ray connections with the extra properties required in the theorem.
Lemma 4.4. If $U$ is an escape region, then there exists $f \in U$ with a ray connection between $-a(f)$ and $a_k(f)$ for some $k$.

Proof of Lemma 4.4. Pick any $0 < \theta < 1/3$ and let $t$ be the argument of a relatively supporting periodic ray $R_{\theta}^{\pm}(t)$ landing at the periodic point $w(\theta)$ in $\partial V(a(\theta))$ of period dividing $p$. If the ray is not smooth, then an iterate, say $R_{\theta}^{\pm}(3^k t)$, contains $-a(\theta)$. Hence, it is a ray connection between $-a(\theta)$ and $a_k(\theta)$. Thus, we may assume that $R_{\theta}(t)$ is smooth and relatively supporting for $V(a(\theta))$. Then $\theta - 1/3 < t < \theta + 1/3$. For some minimal $\theta' \in [\theta, t + 1/3]$, we have that rays at argument $t$ are not smooth. Such a $\theta'$ exists as for $\theta = t + 1/3$ the rays at arguments $t$ and $t + 2/3$ are not smooth. It follows that $R_{\theta'}^{+}(t)$ is relatively supporting for $V(a(\theta'))$ because it lands at the analytic continuation of $w(\theta)$. As before, we conclude that an iterate of $R_{\theta'}^{+}(t)$ is a ray connection between $-a(\theta')$ and $a_k(\theta')$ for some $k$. □

In the special case that $\kappa(U) = 0^p$, the existence of a ray connection as claimed in Theorem 4.3 follows from the above lemma. For a general escape component $U$, our search for ray connections with maximal return time starts by determining certain parameter intervals of $\theta$ such that the rays bouncing off $-a(\theta)$ land at points in $D_0$ with a prescribed return time to $D_0$.

Given $\theta \in \mathbb{R}/\mathbb{Z}$, we are interested on the dynamics of $m_3 : t \mapsto 3t$ according to the partition of $\mathbb{R}/\mathbb{Z}$ by the intervals:

$$\{I_0^+(\theta), I_1^+(\theta)\},$$

where $\sigma = +$ or $-$, and

$$I_0^+(\theta) = [\theta - 1/3, \theta + 1/3],$$
$$I_1^+(\theta) = [\theta + 1/3, \theta - 1/3],$$
$$I_0^-(\theta) = ]\theta - 1/3, \theta + 1/3[,$$
$$I_1^-(\theta) = ]\theta + 1/3, \theta - 1/3[.$$

These intervals are chosen so that $R_{\theta}^{+}(t)$ lands in $D_0$ or $D_1$ according to whether $t \in I_0^+(\theta)$ or $I_1^+(\theta)$. A similar statement holds for left limit rays. Now define

$$\text{itin}_{\theta}^+ : \mathbb{R}/\mathbb{Z} \to \{0, 1\}^{\mathbb{N}_0}$$
$$t \mapsto (i_n),$$

where $i_n$ is such that $m_3^n(t) \in I_{\theta}^+(\theta)$, for all $n \geq 0$. Note that $\text{itin}_{\theta}^+(t) = \text{itin}_{\theta}^-(t)$ for all $t$ that are not eventually mapped into $\{\theta - 1/3, \theta + 1/3\}$. That is, for all $t$ such that $R_{\theta}(t)$ exists and is smooth.

Lemma 4.5. For $\ell \geq 2$, consider the arguments:

$$\alpha_\ell = \frac{1}{3} - \frac{1}{3^\ell - 1},$$
$$\beta_\ell = \frac{1}{3} - \frac{1}{3(3^\ell - 1)}.$$
(1) If \( \theta \in ]\beta, 1/3[ \) or \( \theta \in ]-\beta, 1/3[ \), then for some \( i_n, j_n \):
\[
\text{itin}_\theta^-(\theta + 1/3) = 01^\ell i_{\ell+1}i_{\ell+2} \cdots ,
\]
\[
\text{itin}_\theta^+(\theta - 1/3) = 01^\ell j_{\ell+1}j_{\ell+2} \cdots .
\]

(2) If \( \theta \in ]\alpha, \beta[ \) or \( \theta \in ]-\alpha, \beta[ \), then for some \( i_n, j_n \):
\[
\text{itin}_\theta^-(\theta + 1/3) = 01^{\ell-1}0 i_{\ell+1}i_{\ell+2} \cdots ,
\]
\[
\text{itin}_\theta^+(\theta - 1/3) = 01^{\ell-1}0 j_{\ell+1}j_{\ell+2} \cdots .
\]

Let us denote the unique fixed point of \( f_\theta \) in \( D_1 \) by \( z(\theta) \). The arguments \( \alpha, \beta \) are closely related to the rays landing at \( z(\theta) \). In fact, \( \alpha - 1/3 = m_3(\beta + 1/3) \) has a period \( \ell \) orbit \( \mathcal{O} \) under \( m_3 \) contained in \( [\beta + 1/3, \alpha - 1/3] \) with combinatorial rotation number \(-1/\ell\). For all \( \theta \in ]\alpha, \beta[ \), it follows that \( \mathcal{O} \subset [\theta + 1/3, \theta - 1/3] \) and the external rays of \( f_\theta \) with arguments in \( \mathcal{O} \) land at \( z(\theta) \). Similar statements hold for \( \theta \in ]-\alpha, \beta[ \) with the difference that the orbit \( \mathcal{O} \) contained in \([-\alpha + 1/3, -\beta - 1/3] \) now has combinatorial rotation number \( 1/\ell \). The orbits \( \mathcal{O} \) are examples of “rotation sets” thoroughly discussed in [46].

Proof of Lemma 4.5. It is sufficient to prove assertion (1) for \( \theta \in ]\beta, 1/3[ \) and assertion (2) for \( \theta \in ]\alpha, \beta[ \) because \( m_3(\theta) \in I_1^\ell(\theta) \) if and only if \( m_3(-\theta) \in I_{-1}^\ell(-\theta) \).

If \( \theta \in ]\beta, 1/3[ \), then
\[
\frac{-1}{3(3^\ell - 1)} < \theta - \frac{1}{3} < 0,
\]
where we employ hereafter “\( < \)” to denote cyclic order in \( \mathbb{R}/\mathbb{Z} \). Therefore,
\[
\theta - \frac{2}{3} < 3^\ell \theta < 3^{\ell-1} \theta < \cdots < 3 \theta < \theta - \frac{1}{3} < 0,
\]
and (1) follows.

If \( \theta \in ]\alpha, \beta[ \), then
\[
\theta - \frac{1}{3} < 3^\ell \left( \theta - \frac{1}{3} \right) < \left( \theta - \frac{1}{3} \right) - \frac{1}{3} = \theta - \frac{2}{3}.
\]
Thus, \( m_3^\ell(\theta) \in I_0^\ell(\theta) \). For \( \ell = 2 \) it is easy to check that \( 3\theta \in I_1^\ell \) and (2) follows. For \( \ell \geq 3 \), we may apply (1) to obtain (2) because \( ]\alpha, \beta[ \subset ]\beta - 1, 1/3[ \).

Let us now locate in the dynamical plane of \( f_\theta \) the points with a given return time to \( D_0 \). Following Branner and Hubbard [5, 6], for \( \ell \geq 1 \), we say that
\[
L_{\theta}^{(\ell)} = \{ z \in \mathbb{C} : f_\theta^j(z) \in D_0 \cup D_1, 0 \leq j < \ell \}
\]
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is the level $\ell'$ set of $f_\vartheta$. Each connected component of a level $\ell'$ set is a Jordan domain called a level $\ell'$ disk. For $z \in L_0^{(\ell')}$, we denote by

$$D_0^{(\ell')} (z)$$

the unique level $\ell'$ disk containing $z$.

Given a word $w = (i_0, \ldots, i_{\ell'-1}) \in \{0, 1\}^{\ell'}$, let

$$L_0^{(\ell')} (w) = \{ z \in \mathbb{C} : f_\vartheta^j (z) \in D_{i_j}, 0 \leq j < \ell' \}.$$

We say that points in $L_0^{(\ell')} (w)$ have itineraries starting with $w$. Note that our numbering of the levels is so that words of length $\ell'$ determine subsets of the set of level $\ell'$. In particular,

$$L_0^{(1)} (0) = D_0, \quad L_0^{(1)} (1) = D_1.$$

Both level 1 disks have a common image $D$ under $f_\vartheta$. While $f_\vartheta : D_0 \to D$ has degree 2, the map $f_\vartheta : D_1 \to D$ has degree 1. Recall that $D_1$ contains a unique fixed point denoted $z(\vartheta)$. For all $\ell' \geq 1$,

$$L_0^{(\ell')} (1^\ell') = D_0^{(\ell')} (z(\vartheta)).$$

Moreover, each disk $D_0^{(\ell')} (z(\vartheta))$ contains exactly two disks of the next level. One of these disks is $D_0^{(\ell'+1)} (z(\vartheta))$ and the other one agrees with $L_0^{(\ell'+1)} (1^\ell')$.

Our main interest is to locate points in the orbit of $a(\vartheta)$ with maximal return time to $D_0$. These points have itineraries starting with $01^{\ell'-10}$ for some $\ell' \geq \mu_0$ where $\mu_0$ is the return time of $a(\vartheta)$ to $D_0$. Their location is closely related to the two preimages of the fixed point $z(\vartheta)$ in $D_0$.

Notation. Denote by $z'(\vartheta)$ and $z''(\vartheta)$ the two $f_\vartheta$-preimages in $D_0$ of the unique $f_\vartheta$-fixed point $z(\vartheta) \in D_1$.

For $\ell' \geq 2$,

$$L_0^{(\ell')} (01^{\ell'-1}) = D_0^{(\ell')} (z'(\vartheta)) \cup D_0^{(\ell')} (z''(\vartheta)), \quad (4.1)$$

where the union is disjoint if and only if $\ell' > \mu_0$; otherwise, the disks involved coincide with $D_0^{(\ell')} (a(\vartheta))$. The level $\mu_0 + 1$ disk $D_0^{(\mu_0+1)} (a(\vartheta))$ is the preimage in $D_0$ of $L_0^{(\mu_0)} (1^{\mu_0-1}0)$, therefore

$$L_0^{(\mu_0+1)} (01^{\mu_0-1}0) = D_0^{(\mu_0+1)} (a(\vartheta)). \quad (4.2)$$

If $\ell' > \mu_0$ and $z = z'(\vartheta)$ or $z''(\vartheta)$, then $D_0^{(\ell')} (z)$ contains exactly two disks of level $\ell' + 1$. One of these disks is $D_0^{(\ell'+1)} (z)$. We denote the other one by $C_0^{(\ell'+1)} (z)$ and call it the level $\ell' + 1$ companion disk of $z$. Thus, for $\ell' > \mu_0$,

$$L_0^{(\ell'+1)} (01^{\ell'-1}0) = C_0^{(\ell'+1)} (z'(\vartheta)) \cup C_0^{(\ell'+1)} (z''(\vartheta)). \quad (4.3)$$
Our aim now is to show that for \( \ell \geq \mu_0 \) we can adjust the parameter \( \theta \) in order to connect \(-a(\theta)\) with a level \( \ell \) disk around a given prefixed point \( z'(\theta) \) or \( z''(\theta) \). We label \( z'(\theta) \) and \( z''(\theta) \) such that:

\[
R_{1/3}^- (2/3) \text{ lands at } z'(1/3).
\]

For any \( \theta \in ]1/3, 2/3[ \), the external ray \( R_\theta(0) \) lands at the fixed point \( z(\theta) \). Also, the critical rays \( R_\theta^-(\theta + 1/3) \) and \( R_\theta^+(\theta - 1/3) \) land at points whose itinerary starts with \( 01^\ell \). That is, at a different level 2 disk than the points \( z'(\theta) \) and \( z''(\theta) \). However, for \( \theta < 1/3 \) and \( \theta > 2/3 \), the critical rays \( R_\theta^-(\theta + 1/3) \) and \( R_\theta^+(\theta - 1/3) \) land “close” to \( z'(\theta) \) and “close” to \( z''(\theta) \), respectively.

**Lemma 4.6.** Assume that \( \mu_0 > 1 \) where \( \mu_0 \) is the return time of \( a(\theta) \) to \( D_0 \). If \( \ell \geq \mu_0 \), then the following statements hold:

1. For all \( \theta \in ]\beta, 1/3[ \) the landing points of \( R_\theta^-(\theta + 1/3) \) and \( R_\theta^+(\theta - 1/3) \) belong to \( D(\ell + 1)(z'(\theta)) \).
2. For all \( \theta \in ]2/3, -\beta[, \) the landing points of \( R_\theta^-(\theta + 1/3) \) and \( R_\theta^+(\theta - 1/3) \) belong to \( D(\ell + 1)(z''(\theta)) \).

**Proof.** For all \( \theta \in ]\beta, 1/3[ \) and \( 1 \leq k \leq \ell \), a direct computation shows that \( m_3^k(\theta + 1/3) \notin \{\theta + 1/3, \theta - 1/3\} \). Hence, the rays \( R_\theta^-(\theta + 1/3) \) and \( R_\theta^+(\theta - 1/3) \) share a common sub-arc between \(-a(\theta)\) and \( L(\ell + 1) \theta \) because the sub-arc in each ray is free of iterated preimages of \(-a(\theta)\). Therefore, these rays land in the same level \( \ell + 1 \) disk, say \( D_\theta \). By Lemma 4.5(1), the landing point has itinerary starting with \( 01^\ell \). From (4.1), we have that \( D_\theta = D(\ell + 1)(z'(\theta)) \) or \( D(\ell + 1)(z''(\theta)) \). By continuity of the arcs and disks involved, we have that either \( D_\theta = D(\ell + 1)(z'(\theta)) \) for all \( \theta \in ]\beta, 1/3[ \) or \( D_\theta = D(\ell + 1)(z''(\theta)) \) for all \( \theta \in ]-\beta, 1/3[ \). Our choice of labeling for \( z'(\theta) \) and \( z''(\theta) \) implies that, for all \( t \in ]2/3 - 1/3\ell, 1/3[ \) such that the external ray \( R_{1/3}(t) \) is smooth, the ray \( R_{1/3}(t) \) lands in \( D(\ell + 1)(z'(1/3)) \). If \( \theta \in ]1/3 - 1/3\ell, 1/3[ \), then all the smooth rays \( R_\theta(t) \) with arguments in \( ]2/3 - 1/3\ell, 1/3[ \) land in the same level \( \ell + 1 \) disk \( D_\theta \) that depends continuously on \( \theta \). Therefore, \( D_\theta = D(\ell + 1)(z'(\theta)) \) and the first statement of the lemma holds.

For \( \theta \in ]1/3, 2/3[ \), the (smooth) fixed external ray \( R_\theta(0) \) lands at \( z(\theta) \) and the landing point of the smooth ray \( R_\theta(2/3) \) is \( z'(\theta) \) or \( z''(\theta) \). From our choice of labeling and continuity we conclude that \( R_\theta(2/3) \) lands at \( z'(\theta) \) for all \( \theta \in ]1/3, 2/3[ \). Therefore, \( R_{2/3}^+(2/3) \) lands at \( z'(2/3) \). Note also that \( R_{2/3}^+(0) \) lands at \( z(2/3) \). It follows that \( R_{2/3}^+(1/3) \) lands at \( z''(2/3) \) because this ray is the other preimage of \( R_{2/3}^+(0) \).

The second statement of the lemma now follows along similar lines. Indeed, for parameters \( \theta \in ]2/3, -\beta[, \) again \( R_\theta^+(\theta - 1/3) \) and \( R_\theta^-(\theta + 1/3) \) must land in a continuously varying level \( \ell + 1 \) disk \( D_\theta^\ell \). We may argue as above to conclude that \( D_\theta^\ell = D(\ell + 1)(z''(x)) \) because, from the previous paragraph, we know that for \( t \in ]1/3, 1/3 + 1/3\ell[ \) the rays \( R_{2/3}(t) \) land inside \( D(\ell + 1)(z''(x)) \).

**Corollary 4.7.** Suppose that \( \ell > \mu_0 > 1 \) where \( \mu_0 \) is the return time of \( a_0(\theta) \) to \( D_0 \). Let

\[
\begin{align*}
\theta_0 & \in ]\alpha, \beta[, \\
\theta_1 & \in -]\alpha, \beta[.
\end{align*}
\]
Then the following statements hold.

1. \( R^-_{\theta_0} (\theta_0 + 1/3) \) and \( R^+_{\theta_0} (\theta_0 - 1/3) \) land at points in the companion disk \( C_\theta^{(\varepsilon+1)} (z'(\theta_0)) \).
2. \( R^-_{\theta_1} (\theta_1 + 1/3) \) and \( R^+_{\theta_1} (\theta_1 - 1/3) \) land at points in the companion disk \( C_\theta^{(\varepsilon+1)} (z''(\theta_1)) \).

Proof. Let us just prove the first assertion as the proof of the second one is similar. Note that \( \alpha_{\theta_0 + 1/3}, \beta_{\theta_0 - 1/3} \subset \beta_{\theta_0 - 1/3}, 1/3 \). By the previous lemma, \( R^\pm_{\theta_0} (\theta_0 \pm 1/3) \) lands in \( D_\theta^{(\varepsilon)} (z'(\theta_0)) \). In view of Lemma 4.5, the itinerary \( \overline{\alpha_{\theta_0 + 1/3}, \beta_{\theta_0 - 1/3}} \) starts with \( 01 \). From (4.3), the corresponding rays land in the companion disk \( C_{\theta_0 + 1} (z'(\theta_0)) \).

Lemma 4.8. Let \( \mu \) be the maximal return time to \( D_0 \) and assume that \( 1 < \mu < p \). Then at least one of the following statements hold.

1. \( \Theta(a(\theta)) \cap D_\theta^{(\mu)} (z'(\theta)) \neq \emptyset \), for some \( \theta \), and there exists \( \theta_0 \in [\alpha_{\mu}, \beta_{\mu}] \) such that \( f_{\theta_0} \) has a ray connection between \( -a(\theta_0) \) and \( a_k(\theta_0) \), for some \( k \), such that \( a_k(\theta_0) \in D_\theta^{(\mu)} (z'(\theta_0)) \).
2. \( \Theta(a(\theta)) \cap D_\theta^{(\mu)} (z''(\theta)) \neq \emptyset \), for some \( \theta \), and there exists \( \theta_1 \in [\alpha_{\mu}, \beta_{\mu}] \) such that \( f_{\theta_1} \) has a ray connection between \( -a(\theta_1) \) and \( a_k(\theta_1) \), for some \( k \), such that \( a_k(\theta_1) \in D_\theta^{(\mu)} (z''(\theta_1)) \).

Proof. Points having itinerary starting with \( 01^{\mu-1} \) belong to \( D_\theta^{(\mu)} (z'(\theta)) \) or to \( D_\theta^{(\mu)} (z''(\theta)) \), in view of (4.1). Thus, if \( a_j(\theta) \in D_0 \) has return time \( \mu \) to \( D_0 \), then \( a_j(\theta) \in D_\theta^{(\mu)} (z'(\theta)) \) for all \( \theta \), or, \( a_j(\theta) \in D_\theta^{(\mu)} (z''(\theta)) \) for all \( \theta \). We assume the former and prove that the first assertion holds for some \( k \) not necessarily equal to \( j \). The other case follows along similar lines.

When \( \mu_0 = \mu \) we let \( j = 0 \), and observe that \( a_0(\theta) = a(\theta) \) belongs to \( D_\theta^{(\mu)} (z'(\theta)) = D_\theta^{(\mu)} (z''(\theta)) \).

Otherwise we choose any \( j \) such that \( a_j(\theta) \in D_\theta^{(\mu)} (z'(\theta)) \). Let \( \theta \in [\alpha_{\mu - 1/3}, \beta_{\mu + 1/3}] \) have period \( \mu \) under \( m_3 \). In particular, no period \( p \) ray is bouncing for \( \theta = \alpha_{\mu} \). Thus, we may consider a period \( p \) argument \( t \) such that the external ray \( R_{\alpha_{\mu}} (t) \) is smooth and relatively supporting for \( V(a_j(\alpha_{\mu})) \). Moreover, \( R_{\alpha_{\mu}} (t) \) is a relatively supporting ray of \( V(V(a_j(\alpha_{\mu})) \) for all \( \theta \) in a sufficiently small neighborhood of \( \alpha_{\mu} \).

Claim. If for all \( \theta \in [\alpha_{\mu}, \beta_{\mu}] \) the ray \( R_{\alpha_{\mu}} (t) \) exists (it is smooth), then there exists \( \theta_0 \in [\alpha_{\mu}, \beta_{\mu}] \) such that \( 3 \mu t = 3 \mu (\theta_0 - 1/3) \).

Proof of the Claim. Assume that the ray \( R_{\alpha_{\mu}} (t) \) is smooth for all \( \theta \in [\alpha_{\mu}, \beta_{\mu}] \). Consider the continuous functions \( \delta(\theta) \) and \( \delta(\theta) \geq 0 \) with domain \( [\alpha_{\mu}, \beta_{\mu}] \), where \( \delta(\theta) \) is defined as the arc length of \( [\theta - 1/3, m_{\mu}(\theta - 1/3)] \subset \mathbb{R}/\mathbb{Z} \) and \( \delta(\theta) \) is the arc length of the interval \( [\theta - 1/3, m_{\mu}(t)] \) \( \subset \mathbb{R}/\mathbb{Z} \). The arguments \( \alpha_{\mu} - 1/3 \) and \( \beta_{\mu} + 1/3 \) have period \( \mu \) and therefore \( \delta(\alpha_{\mu}) = 0 \) and \( \delta(\beta_{\mu}) = 2/3 \). Observe that \( R_{\alpha_{\mu}} (m_{\mu}(t)) \) lands in \( \partial V(a_{j+\mu}(\theta)) \). As \( a_{j+\mu}(\theta) \in D_0 \) we have that the external ray with argument \( m_{\mu}^\mu(t) \) lands in \( D_0 \). Therefore, \( \delta(\theta) \leq 2/3 \). Taking into account that \( t \) has period \( p \geq \mu \)
it follows that \(0 < \hat{\delta}(\theta) < 2/3\). By the Intermediate Value Theorem, there exists \(\theta_0 \in [\alpha_\mu, \beta_\mu]\) such that \(\delta(\theta_0) = \hat{\delta}(\theta_0)\). That is, \(3^t \mu t = 3^t (\theta_0 - 1/3)\) and the claim follows.

The proof continues by considering two situations. The first situation is when there exist \(\theta\in [\alpha_\mu, \beta_\mu]\) such that rays with argument \(t\) of \(f_\theta\) are not smooth. The second is when \(R_\theta(t)\) is smooth for all \(\theta \in [\alpha_\mu, \beta_\mu]\). In the latter situation, the claim allows us to work with a parameter \(\theta_0\) such that \(3^t \mu t = 3^t (\theta_0 - 1/3)\). Below, we show that the first situation yields a periodic ray connection while the second leads to a preperiodic one.

In the first situation, let \(\theta_0 \in [\alpha_\mu, \beta_\mu]\) be minimal such that the rays with argument \(t\) are not smooth. As \(\theta \in [\alpha_\mu, \theta_0]\) converges to \(\theta_0\), the rays \(R_\theta(t)\) converge to \(R_\theta^+(3^j t)\). Hence, \(R_\theta^+(3^j t)\) is a nonsmooth ray relatively supporting \(V(a_j(\theta_0))\). As every nonsmooth ray eventually maps onto one containing \(-a_\mu(\theta_0)\), there exists \(j^*\) such that \(R_\theta^+(3^j t)\) contains \(-a_\mu(\theta_0)\). It follows that \(3^j t = \theta_0 - 1/3\) or \(\theta_0 + 1/3\). The itinerary \(itin_{\theta_0}^+(3^j t)\) coincides with the itinerary of \(a_k(\theta_0)\) according to the disks \(D_0\) and \(D_1\). Thus, \(3^j t \neq \theta_0 + 1/3\), as \(itin_{\theta_0}^+(3^j t)\) starts with \(1^\theta_0\) and the maximal length for a string of 1’s in \(\chi(U^\theta)\) is \(\mu - 1\). Therefore, \(3^j t = \theta_0 - 1/3\) and \(R_\theta^+(\theta_0 - 1/3)\) is a periodic ray connection between \(-a_\mu(\theta_0)\) and \(a_k(\theta_0)\).

Lemma 4.6(2) yields that \(a_k(\theta_0)\) lies in \(D_\theta^{(\mu)}(z'_{\theta_0})\).

Now we consider the second situation in which \(R_\theta(t)\) is smooth for all \(\theta \in [\alpha_\mu, \beta_\mu]\) and \(\theta_0 \in [\alpha_\mu, \beta_\mu]\) is such that \(3^t \mu t = 3^t (\theta_0 - 1/3)\). To lighten notation, we omit the dependence on \(\theta_0\) of rays and disks. Recall that \(R^-(\theta_0 + 1/3)\) lands in \(D^{(\mu+1)}(a_j(\theta_0))\). Let \(\zeta\) be the intersection point of \(R^-(\theta_0 + 1/3)\) with \(\partial D^{(\mu+1)}(a_j(\theta_0))\). Note that \(\zeta\) also lies in \(R^+(\theta_0 - 1/3)\) because the portion of \(R^-(\theta_0 + 1/3)\) between the boundaries of levels 1 and \(\mu + 1\) is free of iterated preimages of \(-a_\mu(\theta_0)\). Also, \(f_{\theta_0}^\mu(\zeta)\in R(3^t \mu t)\) because \(3^t \mu t = 3^t (\theta_0 - 1/3)\). Moreover, the smooth ray \(R(t)\) intersects \(\partial D^{(\mu+1)}(a_j(\theta_0))\) at one point, say \(\xi\). If \(\mu > \mu_0\), then \(f_{\theta_0}^\mu : \partial D^{(\mu+1)}(a_j(\theta_0)) \to D_0\) is injective, and therefore \(\zeta = \xi \in R(t)\) that contradicts our assumption that \(R(t)\) is smooth. Thus, \(\mu = \mu_0\), \(a_j(\theta_0) = a(\theta_0)\), and \(f_{\theta_0}^\mu : \partial D^{(\mu+1)}(a_j(\theta_0)) \to \partial D_0\) is two-to-one. In this case, \(f_{\theta_0}^\mu : \partial D^{(\mu+1)}(a_j(\theta_0)) \to \partial D^{(\mu)}(a_j(\theta_0))\) is two-to-one. The gradient flow lines between \(\zeta\) and the Julia set as well as the flow line between \(\xi\) and the Julia set map under \(f_{\theta_0}^\mu\) onto the same sub-arc of \(R(3t)\). It follows that \(t = \theta_0\). Thus, \(R(\theta_0)\) lands at the unique point of period dividing \(p\) in \(\partial V(a(\theta_0))\) and the rays \(R^\pm(\theta_0 \mp 1/3)\) also land at a point in \(\partial V(a(\theta_0))\). Moreover, one of these rays is relatively supporting for \(V(a(\theta_0))\) because \(R(\theta_0)\) is relatively supporting. That is, \(R^+(\theta_0 - 1/3)\) or \(R^-(\theta_0 + 1/3)\) is a preperiodic ray connection between \(-a(\theta_0)\) and \(a(\theta_0)\).

Lemma 4.6(2) yields that \(a_k(\theta_0)\) lies in \(D_\theta^{(\mu)}(z'_{\theta_0})\).

For \(\theta = \theta_0\) or \(\theta_1\) as in the previous lemma, the return time of \(a_k(\theta)\) to \(D_0\) is \(\mu\) and hence maximal. Therefore, the lemma finishes the proof of Theorem 4.3 by establishing the existence of the corresponding ray connection.

5 | LANDING AND TAKE-OFF

The aim of this section is to discuss where rational parameter rays land and which rays land at a given map. According to Bonifant, Milnor, and Sutherland in a forthcoming paper, a parameter ray \(R_{L^\mu}(\theta)\) with \(\theta \in \mathbb{Q}/\mathbb{Z}\) lands at a parabolic or pcf map \(f_0\) whose basic features are summarized
in Theorem 2.8 of a forthcoming paper by Bonifant and Milnor, which in our notation reads as follows:

**Theorem 5.1** (Milnor, Bonifant and Sutherland). A parameter ray $R_{f_0}(\theta)$ with $\theta \in \mathbb{Q}/\mathbb{Z}$ lands at a map $f_0$. If $\theta' = \theta + 1/3$ or $\theta - 1/3$ is periodic, then $f_0$ is a parabolic map, the dynamical rays $R_{f_0}(\theta + 1/3), R_{f_0}(\theta - 1/3)$ land in $\partial V(-a(f_0))$, and $R_{f_0}(\theta')$ lands at a parabolic periodic point. If $\theta + 1/3$ and $\theta - 1/3$ are strictly preperiodic, then $f_0$ is a pcf map and $R_{f_0}(\theta + 1/3), R_{f_0}(\theta - 1/3)$ land at $-a(f_0)$.

For our purpose, we need to study in greater detail the supporting properties of the rays $R_{f_0}(\theta + 1/3), R_{f_0}(\theta - 1/3)$, the kneading words induced by these rays and to determine the existence of certain parameter rays landing at $f_0$. Our study is described in terms of the notions of take-off arguments and their associated kneading word defined below.

**Definition 5.2** (Take-off argument). Let $f_0$ be a map in $S_p$, with a parabolic periodic point $z_0$. Assume that $-a(f_0)$ is in the immediate basin of $z_0$ (i.e., $z_0 \in \partial V(-a(f_0))$). We say that $\theta$ is a take-off argument for $f_0$ if $R_{f_0}(\theta + 1/3)$ and $R_{f_0}(\theta - 1/3)$ are relatively supporting rays for $\partial V(-a(f_0))$, and one of these rays lands at $z_0$.

For a pcf map $f_0 \in \partial C(S_p)$, we say that $\theta$ is a take-off argument for $f_0$ if the external rays $R_{f_0}(\theta \pm 1/3)$ land at $-a(f_0)$. Equivalently, if $R_{f_0}(\theta)$ lands at the cocritical point $2a(f_0)$.

We stress that, by definition, if $\theta$ is a take-off argument of a parabolic map, then $\theta \pm 1/3$ are relatively supporting arguments for $\partial V(-a(f_0))$.

**Definition 5.3** (Take-off kneading). Given a parabolic (resp., pcf) map $f_0 \in S_p$ with take-off argument $\theta$, let $X = V(-a(f_0))$ (resp., $X = \{-a(f_0)\}$) and $\Gamma = R_{f_0}(\theta - 1/3) \cup X \cup R_{f_0}(\theta + 1/3)$. Then $C \setminus \Gamma$ consists of two connected components. Denote by $U_0$ the one containing $a(f_0)$ and by $U_1$ the other one. We say that

$$\kappa(f_0, \theta) = i_1i_2...i_{p-1}0$$

is the take-off kneading word of $f_0$ associated to $\theta$ if $a_j(f_0) \in U_{i_j}$ for $j = 1, ..., p - 1$.

Parameter rays landing at pcf maps are completely described by Theorem 5.2 of a forthcoming paper by Bonifant and Milnor:

**Theorem 5.4** (Bonifant and Milnor). If $\theta$ is a take-off argument of a pcf map $f_0 \in S_p$, then there exists a unique parameter ray $R_{U'}(\theta)$ of some escape region $U'$ with argument $\theta$ landing at $f_0$.

In Subsection 5.2, we analyze in greater detail the case in which $f_0$ is parabolic where we reprove part of Theorem 5.1. The techniques employed are a combination of the original techniques of the Orsay Notes [11] (cf. [25]) with elementary properties of the limit of rational dynamical rays. The aforementioned elementary properties are deduced in Subsection 5.1, following ideas from [22]. Ideas that are related to those employed by Petersen and Ryd [35] and by Bonifant, Milnor, and Sutherland in a forthcoming paper. In Subsection 5.3, we discuss how to deduce from the above theorems the statements that we need about parameter rays landing at pcf maps.
There is no claim of originality for the results or techniques in this section, which has a strong overlap with the ongoing work by Bonifant, Milnor, and Sutherland. Our particular emphasis is on controlling the combinatorics of maps that are the landing point of parameter rays with ray connections. This combinatorics is essential to determine how we cross the connectedness locus to exit at another escape region. More precisely, we heavily rely on the combinatorics to show that these landing points are in the boundary of a type A or B hyperbolic component, and to determine the effect that an appropriate exit from the hyperbolic component to the escape locus has on the kneading.

5.1 Limit of rational rays

Below we consider the behavior of $R_f^*(t)$ as $f \in R_U(\theta)$ approaches a map $f_0$ in the connectedness locus. In this context, $\limsup R_f^*(t)$ is defined as the set formed by all $z \in \mathbb{C}$ such that for every neighborhood $W$ of $z$ and any neighborhood $\mathcal{W}$ of $f_0$ we have that $R_f^*(t) \cap W \neq \emptyset$ for some $f \in R_U(\theta) \cap \mathcal{W}$. The set $\limsup I_{f,a_k}(t)$ is defined similarly.

Lemma 5.5. Let $R_U(\theta)$ be a parameter ray of an escape region $U$ such that $\theta \in \mathbb{Q}/\mathbb{Z}$. Then $R_U(\theta)$ lands at a parabolic or pcf map $f_0$. Moreover, the following statements hold.

1. If $t \in \mathbb{Q}/\mathbb{Z}$ is periodic under $m_3$, then $\{u_0\} = J(f_0) \cap \limsup R_f^*(t)$ where $u_0$ is the landing point of $R_{f_0}(t)$.
2. For all $k$, we have that $\{v_0\} = J(f_0) \cap \limsup I_{f,a_k}(0)$, where $v_0$ is the landing point of $I_{f_0,a_k}(0)$.
3. If $t \in \mathbb{Q}/\mathbb{Z}$ is strictly preperiodic and $w_0 \in J(f_0) \cap \limsup R_f^*(t)$, then $w_0$ is strictly preperiodic.
4. If $\theta' = \theta + 1/3$ or $\theta - 1/3$ is periodic, then $f_0$ is a parabolic map and $R_{f_0}(\theta')$ lands at a parabolic periodic point.

In (3), it would be interesting to know if $w_0$ is unique.†

Under the assumptions of the previous lemma, Bonifant, Milnor, and Sutherland in a forthcoming paper show that if $\theta + 1/3$ and $\theta - 1/3$ are strictly preperiodic, then the landing point of $R_U(\theta)$ is a pcf map. Moreover, when $\theta' = \theta + 1/3$ or $\theta - 1/3$ is periodic, they give a complete description of $\limsup R_f^*(\theta')$. Although we provide a self-contained proof here, it is not difficult to deduce the lemma directly from these facts.

Proof. To show that $R_U(\theta)$ lands, consider an accumulation point $g$ as we approach the connectedness locus along $R_U(\theta)$. Denote by $w$ the landing point of $R_{f_0}(\theta + 1/3)$. Then, for some $k \geq 0$, $g^k(w)$ is a parabolic periodic point or a critical point; for otherwise $w$ would be an eventually repelling periodic point (maybe periodic) with a critical point free orbit and for every $f$ close to $g$ the corresponding ray $R_f(\theta + 1/3)$ would be smooth, which is not the case for $f \in R_U(\theta)$ (e.g., see [18]). Thus, $g$ is a parabolic or pcf map. As the accumulation set of $R_U(\theta)$ in $C(S_p)$ is connected and there are only countably many parabolic and pcf maps in $S_p$, we have that $R_U(\theta)$ lands, say at $f_0$. It follows as well that if $\theta + 1/3$ or $\theta - 1/3$ is periodic, the landing point of the corresponding $f_0$-ray is a parabolic periodic point. That is, (4) holds.

Now, given $t \in \mathbb{Q}/\mathbb{Z}$, we study $X := J(f_0) \cap \limsup R_f^*(t)$. Provided that $t$ is periodic of period $q$, we start by showing that all the elements of $X$ are periodic of period dividing $q$. Afterward, taking into account that $f_0$ has only one free critical point we establish (1).

† This question was raised by an anonymous referee.
For \( f \in R_L(\theta) \), the Böttcher coordinate \( \phi_f : V_f^*(\infty) \to C \setminus \overline{D} \) has as image a star-like domain \( U_f^* \) around infinity. The domain \( U_f^* \) is obtained from \( C \setminus \overline{D} \) by removing the “needles”

\[ |1, |\phi_f(-a(f))|\ exp(2\pi i(\theta \pm 1/3)) \]

as well as all their iterated preimages under \( z \mapsto z^3 \). Recall that we let \( \rho(s) \) be such that the maximal radial line with argument \( s \) contained in \( U_f^* \) is \( \rho(s), \infty \exp(2\pi i s) \). Given \( R > 1 \), it is not difficult to show that there exists \( \delta > 0 \) such that for all \( f \in R_L(\theta) \) with \( |\phi_f(2a(f))| < R \), the sector

\[ S_t = \{ z \in C \setminus \overline{D} : |\arg(z - \rho(t) \exp(2\pi it)) - 2\pi t| < \delta \} \]

is contained in \( U_f^* \). Hence, for \( r > \rho(t) \), the hyperbolic distance in \( S_t \) from \( z = r \exp(2\pi it) \) to \( z^{3^q} \) is uniformly bounded above by a constant \( C \) independent of \( r > \rho(t) \). Denote by \( d_f \) the hyperbolic metric of \( V_f^*(\infty) \). The standard comparison between hyperbolic metrics yields that the hyperbolic distance in \( \phi_f^{-1}(S_t) \) bounds from above \( d_f \). Then for all \( f \in R_L(\theta) \) and all \( z \in R_f^*(t) \), we have \( d_f(z, f^q(z)) < C \) for some constant \( C \) independent of \( f \). Now if \( z_0 \in X \), then there exist \( f_n \to f \) and \( w_n \to w_0 \) such that \( w_n \in R_f^*(t) \). As \( \limsup J(f_n) \supset J(f_0) \), the Euclidean distance \( \varepsilon_n \) from \( w_n \) to \( J(f_n) \) converges to 0, which together with \( d_f(w_n, f_n^q(w_n)) < C \), implies that \( |w_n - f_n^q(w_n)| \to 0 \), using the comparison between Euclidean and hyperbolic metrics. Thus, \( f_0(q) = z_0 \). That is, every element of \( X \) is periodic of period dividing \( q \).

Now let \( u_0 \) be the landing point of \( R_f(\theta) \). When \( u_0 \) is a repelling periodic point, \( R_f(\theta) \) is a smooth ray for all \( f \) in a neighborhood of \( f_0 \), and \( R_f(\theta) \) converges uniformly to \( R_{f_0}(\theta) \), so (1) holds in this case (cf. [18]). When \( u_0 \) is a parabolic periodic point, consider the connected set \( Y := K(f_0) \cap \limsup R_f^*(t) \) that contains \( u_0 \) and \( X \). Let \( P \) be the union of the periodic Fatou components having \( u_0 \) in its boundary. Taking into account that the return map to any parabolic periodic Fatou component has degree 2, we conclude that \( u_0 \) is the unique periodic point in \( \partial P \) of period dividing \( q \) and \( X \subset Y \cap \partial P = \{ u_0 \} \). Hence, we have proven (1).

The second assertion of the lemma follows along similar lines but considering \( \varphi_{f,a_k} \) instead of \( \phi_f \).

For assertion (3), it is sufficient to consider the case in which \( t \in \mathbb{Q}/\mathbb{Z} \) is not periodic and \( 3t \) is periodic. Let \( w \in J(f_0) \cap \limsup R_f^*(t) \). We claim that \( w \) is strictly preperiodic. Proceeding by contraction, let us suppose that \( w \) is periodic. Let \( f_n \in R_L(\theta) \) and \( w_n \in R_{f_n}^*(t) \) be such that \( f_n \to f_0 \) and \( w_n \to w \). Let \( t' \) be the periodic preimage of \( 3t \) and \( w'_n \in R_{f_n}^*(t') \) be such that \( f_n(w'_n) = f_n(w_n) \). Then both \( w'_n \) and \( w_n \) converge to the unique periodic preimage \( w \) of \( f_0(w) \). Thus, \( f_0 \) is not locally injective around \( w \) for otherwise \( f_n \) would also be, but \( w'_n \neq w_n \). Hence, \( w \) is a periodic critical point of \( f_0 \) in \( J(f_0) \) which is impossible. Therefore, \( w \) is strictly preperiodic and the lemma follows.

\[ 5.2 \quad \text{Parabolic landing and take-off theorems} \]

**Theorem 5.6 (Periodic Landing Theorem).** Consider a parameter ray \( R_L(\theta) \) of an escape region \( U' \) such that \( \theta' = \theta + 1/3 \) or \( \theta - 1/3 \) is periodic. Then \( R_L(\theta) \) lands at a parabolic map with take-off
argument $\theta$ and $\chi(f_0, \theta) = \chi(U')$. Moreover, if $R_f^\ell(\theta')$ is a periodic ray connection between $-a(f')$ and $a_k(f)$ for $f \in R_U(\theta)$, then $\theta'$ is a relatively supporting argument for $V(a_k(f_0))$.

A similar result is proven by Bonifant and Milnor in Theorem 2.8 in a forthcoming paper. Under the assumptions of the theorem, they prove that $R_U(\theta)$ lands at a parabolic map $f_0$ such that the rays $R_{f_0}(\theta + 1/3)$ and $R_{f_0}(\theta - 1/3)$ land in $\partial V(-a(f_0))$. However, we also need to study the supporting properties of these rays. Indeed, the kneading word $\chi(f_0, \theta)$ is only defined under the condition that the arguments $\theta \pm 1/3$ are relatively supporting. Moreover, these supporting properties will be crucial, in Subsection 6.6, to identify the type A or B hyperbolic component containing $f_0$ in its boundary, when $f_0$ is the landing point of a ray with a periodic ray connection.

Our proof relies on the following:

**Lemma 5.7.** Consider $\theta \in \mathbb{Q}/\mathbb{Z}$ such that $\theta' = \theta + 1/3$ or $\theta - 1/3$ is periodic of exact period $q$. Let $R_U(\theta)$ be a parameter ray, of an escape region $U'$, landing at $f_0$. For $f \in R_U(\theta)$, denote by $z^\pm(f)$ the landing point of $R_f^\ell(\theta')$. Then all of the following statements hold.

1. $R_{f_0}(\theta')$ lands at a parabolic periodic point $z_0 \in \partial V_{f_0}(-a(f_0))$ of some period $\ell'$ that divides $q$.
2. $z^\pm(f) \to z_0$ as $f \in R_U(\theta)$ approaches $f_0$.
3. The periods of $z^+(f)$ and $z^-(f)$ are $\ell'$ and $q$, maybe not, respectively.
4. A ray $R_{f_0}(\ell')$ lands at $z_0$ if and only if a ray with argument $\ell'$ lands at the iterates of $z^+(f)$ or $z^-(f)$ under $f^{\ell'}$.
5. $f_0$ has take-off argument $\theta$.
6. $k(f_0, \theta) = k(U')$.

**Proof.** As a direct consequence of Lemma 5.5, the parameter ray $R_U(\theta)$ lands at a map $f_0$ such that the landing point $z_0$ of $R_{f_0}(\theta')$ is a parabolic periodic point, say of period $\ell'$. Moreover, as $-a(f_0)$ belongs to the connected set $K(f_0) \cap (\lim \sup R_f^\ell(\theta'))$ whose unique Julia set point is $z_0$, we have that $z_0 \in \partial V_{f_0}(-a(f_0))$. That is, (1) holds.

There is only one cycle of repelling petals at $z_0$ because $-a(f_0)$ is the unique free critical point of $f_0$. Hence, for $f$ converging to $f_0$, there exist one period $q/\ell'$ orbit of $f^{\ell'}$ and one fixed point of $f^{\ell'}$ that converge to $z_0$. Note that possibly $q = \ell'$, in this case two distinct fixed points of $f^q$ converge to $z_0$. See Figure 3, for a case in which $\ell' = 1$ and $q = 3$. No other periodic point can converge to $z_0$ (that is, given a period $N$, all other periodic points of period at most $N$ are bounded away from $z_0$, for $f$ close to $f_0$).

Let us prove that $z^\pm(f) \to z_0$. Denote by $d_f$ the hyperbolic metric in the basin of infinity of $f$, and by $\mathbb{H}_- \subset \mathbb{C}$ the lower half plane. It is convenient to consider the universal covering $h : \mathbb{H}_- \to \mathbb{C} \setminus \mathbb{D}$ given by $z \mapsto \exp(i z)$. Let $\tilde{U}_f := h^{-1}(U'_f)$. Given $R > 1$, there exists $\delta > 0$ such that for all $f \in R_U(\theta)$ with $|\phi_f(-a(f))| < R$, we have that $\tilde{U}_f$ contains sectors $S^+ = \{z \in \mathbb{H}_- : -\pi/2 < \arg(2\pi \theta' - z) < (\pi/2) + \delta\}$ and $S^- = \{z \in \mathbb{H}_- : (\pi/2) - \delta < \arg(z - 2\pi \theta') < -\pi/2\}$. Thus, for $\alpha \in [0, \delta]$ we may define

$$\gamma_f^\pm(\alpha) := \phi_f^{-1}(h(2\pi \theta' - |\alpha|, 0, 0 \cdot i \cdot \exp(\pm i \alpha))).$$

For each $\alpha$, the arc $\gamma_f^\pm(\alpha)$ is $f^q$ invariant. Moreover, $d_f(z, f^q(z))$ is uniformly bounded for all $z \in \gamma_f^\pm(\alpha)$ and all $f \in R_U(\theta)$ with $|\phi_f(-a(f))| < R$. In particular, $\gamma_f^\pm(\alpha)$ limit toward a peri-
odic point \( w^\pm(\alpha) \) of period dividing \( q \), as it approaches the Julia set. It follows that \( w^+(\alpha) \) (resp., \( w^-(\alpha) \)) is independent of \( \alpha \in ]0, \delta[ \) (e.g., see [29][Corollary 17.10]). Therefore, \( w^+(\alpha) = z^+(f) \) and \( w^-(\alpha) = z^-(f) \). Similarly, we may introduce arcs \( \gamma_{f_0}^\pm(\alpha) \) for \( f_0 \) and conclude that their limit point in the Julia set is \( z_0 \). We can argue as in the proof of Lemma 5.5, but considering \( \gamma_{f}^\pm(\alpha) \) instead of \( R_f^\pm(t) \), and prove that \( \{z_0\} = J(f_0) \cap (\limsup \gamma_{f}^\pm(\alpha)) \). As any limit point of \( z^\pm(f) \), as \( f \in R_V(\theta) \) approaches \( f_0 \), lies in this set, we deduce that \( z^+(f) \to z_0 \), which concludes the proof of (2).

Taking into account that only two periodic orbits under \( f^\ell \) can converge to \( z_0 \) and that their \( f^\ell \)-periods are 1 and \( q/\ell \), assertion (3) follows.

The \( f_0 \)-period of \( z_0 \) is \( \ell \), thus a ray \( R_{f_0}((3^n\theta') \) lands at \( z_0 \) if and only if \( \ell \) divides \( n \). That is, (4) holds for arguments in the orbit of \( \theta' \). Suppose that \( t \) is a period \( q \) argument not in the orbit of \( \theta' \). As \( f \in R_V(\theta) \) converges to \( f_0 \), from Lemma 5.5, \( R_{f_0}(t) \) lands at \( z_0 \) if and only if the landing point \( z(f) \) of \( R_f(t) \) converges to \( z_0 \). This is equivalent to \( z(f) \) being in the \( f^\ell \)-orbit of \( z^+(f) \) or \( z^-(f) \), as these are the only periodic points that converge to \( z_0 \). That is we have proven (4).

Let \( \theta'' = \theta + 1/3 \) or \( \theta - 1/3 \) be such that \( \theta'' \) is strictly preperiodic. We claim that \( R_{f_0}(\theta'') \) lands at \( \partial V(-a(f_0)) \). Denote by \( w_0 \) the unique nonperiodic preimage of \( f_0(z_0) \) in \( \partial V(-a(f_0)) \). Let \( P' \) be the union of the Fatou components that have \( w_0 \) in their boundaries. No other preperiodic boundary point of these Fatou components maps onto \( f_0(z_0) \). It follows that \( K(f_0) \cap \limsup R_{f_0}^\pm(\theta'') \) is contained in \( P' \). Therefore, \( J(f_0) \cap \limsup R_{f_0}^\pm(\theta'') = \{w_0\} \) and \( R_{f_0}(\theta'') \) lands at \( w_0 \in \partial V(-a(f_0)) \) that yields (5).

To prove (6), we must show that \( \theta' \) relatively supports \( V(-a(f_0)) \). From (4), we have a good description of the rays landing at \( z_0 \). By (3), either \( z^+(f) \) or \( z^-(f) \) has period \( q \). For simplicity, let us assume that \( z^+(f) \) has period \( q \). The period of \( \theta' \) is also \( q \). Therefore, the ray \( R_{f_0}^+(\theta') \) is the unique right limit ray with argument in the orbit of \( \theta' \) landing at \( z^+(f) \). Moreover, as \( z^-(f) \) and \( z^+(f) \) belong to distinct orbits, no left ray with argument of the form \( 3^m\theta' \) can land at \( z^+(f) \). In the dynamical plane of \( f \in R_V(\theta) \), consider the graph \( \Gamma \) formed by the flow lines \( R_f^\pm(\theta') \), \( R_f^\pm(\theta'') \) and \( -a(f) \). The point \( z^+(f) \) belongs to the connected component of \( C \setminus \Gamma \) containing rays with arguments in \( ]\theta', \theta''[ \) while \( z^-(f) \) lies in the other component. Rays (bouncing or not) with arguments different from \( \theta' \) and \( \theta'' \) are disjoint from \( \Gamma \). As \( z^-(f) \) is fixed under \( f^\ell \), the rays in the \( f^\ell \)-orbit of \( R_{f_0}^+(\theta') \) land in the same component as \( z^-(f) \), with the exception of \( R_{f_0}^+(\theta') \) lands at \( z^+(f) \). That is, \( f^\ell \)-orbit of \( z^+(f) \) with the sole exception of \( z^+(f) \) is contained in the component of \( C \setminus \Gamma \) corresponding to arguments in \( ]\theta''', \theta''[ \). It follows from (4) that every ray of \( f_0 \) with argument \( t \) in the orbit of \( \theta' \) landing at \( z_0 \) is such that \( t \in ]\theta''', \theta' \). Together with (5), this shows that \( \theta' \) is relatively left supporting. Similarly, if we assume that \( z^-(f) \) has period \( q \), then \( \theta' \) is relatively right supporting. That is, we have proven that \( \theta \) is a take-off argument for \( f_0 \).

To prove (7), let \( \kappa(f_0, \theta) = t_1 \ldots t_{p-1} 0 \) if \( t_j = 0 \) (resp., 1), consider \( t_j \in \theta - 1/3, \theta + 1/3[ \) (resp., \( \theta + 1/3, \theta - 1/3[ \) such that the ray \( R_{f_0}(t_j) \) lands at a repelling periodic point \( u_0 \) in \( \partial V(a_j(f_0)) \). As \( u_0 \) is at the same time the landing point of some periodic internal ray \( I_{f_0,a_j}(s_j) \), we have that the analytic continuation \( u(f) \) of \( u_0 \) lies in \( \partial V(a_j(f_0)) \), for \( f \) sufficiently close to \( f_0 \). Also, the external ray \( R_f(t_j) \) must be smooth and land at \( u(f) \) (e.g., [18]). We conclude that the kneading words \( \kappa(f_0, \theta) \) and \( \kappa(\theta') \) coincide. □

Proof of Theorem 5.6. Suppose that \( R_f^\pm(\theta') \) is a periodic ray connection between \(-a(f) \) and \( a_k(f) \), for all \( f \in R_V(\theta) \). To fix ideas we assume that \( \sigma = -1 \). In the notation of the previous lemma, \( z_0 \in \partial V(a_k(f_0)) \) because \( I_{f_0,a_k}(0) \) lands at \( z^-(f) \). From Lemma 5.5, we have that \( I_{f_0,a_k}(0) \) lands at
\[ z_0 = \lim z^{-}(f). \] Hence, the landing point \( z_0 \) of \( R_{f_0}(\theta') \) is in \( \partial V(a_k(f_0)) \). The theorem follows once we have proven that \( R_{f_0}(\theta') \) is a relatively supporting ray for \( V(a_k(f_0)) \).

Let us first consider the following general situation. Assume that \( \theta_1 \) and \( \theta_2 \) are arguments in the orbit of \( \theta' \) such that, for some \( f \in S_p \), there are rays with arguments \( \theta_1 \) and \( \theta_2 \) landing at a periodic point \( z \in \partial V(a_k(f)) \) of period \( q \). Also, assume that some ray with argument \( t \in ]\theta_1, \theta_2[ \) lands at a point \( w \in \partial V(a_k(f)) \) with \( w \neq z \). From Definition 4.1, it is not difficult to conclude that \( \theta_1 \) is relatively left supporting if and only if no argument of the form \( 3nq\theta_1 \) lies in \( ]t, \theta_2[ \).

Similarly, \( \theta_2 \) is relatively right supporting if and only if no argument of the form \( 3nq\theta_1 \) lies in \( ]\theta_1, t[ \).

Now, we prove that \( R_{f_0}(\theta') \) is a relatively supporting ray for \( V(a_k(f_0)) \). In view of (4) of the previous lemma, \( R_{f_0}(3n\theta') \) lands at \( z_0 \) if and only \( n \) divides \( \ell \). Consider a reference smooth periodic ray \( R_f(t) \) landing at some point \( v \in \partial V(a_k(f)) \) such that \( v \neq z_0 \). From Definition 4.1, it is not difficult to conclude that \( \theta_1 \) is relatively left supporting if and only if no argument of the form \( 3nq\theta_1 \) lies in \( ]t, \theta_2[ \).

Similarly, \( \theta_2 \) is relatively right supporting if and only if no argument of the form \( 3nq\theta_1 \) lies in \( ]\theta_1, t[ \).

Now, we prove that \( R_{f_0}(\theta') \) is a relatively supporting ray for \( V(a_k(f_0)) \). In view of (4) of the previous lemma, \( R_{f_0}(3n\theta') \) lands at \( z_0 \) if and only \( \ell \) divides \( n \). Consider a reference smooth periodic ray \( R_f(t) \) landing at some point \( u \in \partial V(a_k(f)) \) such that \( u \neq z_0 \). By the previous paragraph, if the period of \( z^{-}(f) \) is also \( \ell \), then \( ]t, \theta_2[ \) or \( ]\theta_1, t[ \) is disjoint from \( \{3n\ell\theta' : n \geq 1\} \), according to whether \( \theta' \) is a right or left relatively supporting argument of \( V(a_k(f)) \). If the period of \( z^{-}(f) \) is \( p \) and, therefore, \( z^+(f) \) has period \( \ell \), then \( ]t, \theta_2[ \) and \( \{3n\ell\theta' : n \geq 1\} \) are disjoint because \( z^+(f) \) lies in \( Z_f(V(a_k(f), \theta', t)) \) and all the rays \( R_f^+(3nq\theta') \) land at \( z^+(f) \). In all cases, \( R_{f_0}(\theta') \) relatively supports \( V(a_k(f_0)) \).

\[ \square \]

**Theorem 5.8** (Parabolic Take-off Theorem). Let \( f_0 \in S_p \) be a parabolic map with take-off argument \( \theta \). Then there exists an escape region \( \mathcal{U} \) with \( \kappa(\mathcal{U}) = \kappa(f_0, \theta) \) and a parameter ray \( R_{\mathcal{U}}(\theta) \) that lands at \( f_0 \).

Our proof of this theorem follows the original ideas introduced in the Orsay Notes [11] to characterize which external rays of the Mandelbrot set land at a given parabolic parameter. Our exposition employs perturbed Fatou coordinates in the spirit of Tan Lei’s article [25]. The existence of appropriate perturbed Fatou coordinates is guaranteed by [43, 44] (cf. [33]). We employ the Walz Theorem as stated and proved in [11].

**Proof.** Let \( f_0 \in S_p \) be a map such that \( z_0 \in \partial V(-a(f_0)) \) is a parabolic periodic point of period \( q_0 \) and multiplier a primitive \( q_1 \) root of unity. Let \( R_{f_0}^{q_1}(\theta') \) be a relatively supporting ray for \( V(-a(f_0)) \), landing at \( z_0 \), where \( \theta' = \theta + 1/3 \) or \( \theta - 1/3 \). It follows that \( \theta' \) has period \( q = q_0q_1 \).

From the fact that \( f_0 \) has one cycle of attracting petals, maybe after passing to a branched covering \( \lambda \mapsto f_{\lambda} \) from a disk \( \Lambda \subset \mathbb{C} \) centered at \( \lambda = 0 \) onto a neighborhood of \( f_0 \) in \( S_p \), we may assume that for all \( \lambda \in \Lambda \setminus \{0\} \) the map \( f_{\lambda} \) has a periodic point \( z(\lambda) \) of period \( q_0 \) and distinct periodic points \( \xi_1(\lambda), \ldots, \xi_{q_1}(\lambda) \) of period \( q_1 \), depending analytically on \( \lambda \), such that at \( \lambda = 0 \) all of them converge to \( z_0 \). For convenience, we change coordinates in the dynamical plane, via a translation \( z \mapsto z + (z(\lambda) - z_0) \), and abuse of notation by also calling \( f_{\lambda} \) the resulting family of monic critically marked cubic polynomials. With this normalization \( z_0 \) is a fixed point of \( f_{\lambda} \) for all \( \lambda \in \Lambda \). The free critical point of \( f_{\lambda} \) will be denoted by \( \omega_2 \). We continue denoting by \( \xi_1(\lambda), \ldots, \xi_{q_1}(\lambda) \) the period \( q \) points of \( f_{\lambda} \) close to \( z_0 \). It is useful to also consider the \( q \)th iterate of \( f_{\lambda} \) by introducing \( g_{q_1} = f_{\lambda}^{q_1} \) and setting \( d = 3^q \).

Let \( L_+ \) be the attracting direction at \( z_0 \) under iterations of \( g_{q_1} \) corresponding to the Fatou component \( V(\omega_0) \), and let \( L_- \) be the repelling direction corresponding to the external ray \( R_{f_0}^{q_1}(\theta') \). As \( R_{f_0}^{q_1}(\theta') \) is relatively supporting for \( V(\omega_0) \) and there is only one cycle of attracting/repelling
directions under $f_0^{q_0}$, the directions $L_+$ and $L_-$ are consecutive. That is, their angle at $z_0$ is $\pi/q_1$. Denote by $L$ the bisector of $L_+$ and $L_-$. That is, $L$ is a half-line emanating from $z_0$ making an angle of $\pi/2q_1$ with both directions $L_+$ and $L_-$. Given a small sector $\hat{L}$ based at $z_0$ around $L$, maybe after shrinking $\Lambda$ and relabeling $\zeta_j(\lambda)$, there exists a parameter space sector $\Lambda' \subset \Lambda$ based at $\lambda = 0$ such that $\zeta_1(\lambda) \in \hat{L}$ and $\pi/4 < |\arg((g_\lambda)'(z_0) - 1)| < 3\pi/4$, for all $\lambda \in \Lambda'$ (e.g., see [11, Exposé XI]).

According to Shishikura (see [44, Proposition 3.2.2] and [43, Appendix A.5]), given a neighborhood $U$ of $z_0$, after shrinking $\Lambda'$ if necessary, for all $\lambda \in \Lambda'$, we have that $U$ contains a Jordan domain $S_{+,\lambda}$ bounded by a closed arc $\ell_+$ and its image $g_\lambda(\ell_+)$ such that the endpoints of $\ell_+$ are $z_0$ and $\zeta_1(\lambda)$, and $S_{-,\lambda} \cap S_{+,\lambda} = \ell_+ \cap g_\lambda(\ell_+) = \{z_0, \zeta_1(\lambda)\}$. Moreover, for every $z \in S_{-,\lambda}$ there exists $n$ (depending on $\lambda$) such that $g_\lambda^n(z) \in S_{+,\lambda}$ and, for the smallest such $n$, we have that $g_\lambda^n(z)$ belongs to the Jordan domain $\Omega_\lambda$ enclosed by $\ell_- \cup g_\lambda(\ell_+)$ for all $0 \leq j < n$. Also, $\ell_{+,\lambda}$ converges to an arc $\ell_{+,0}$ as $\lambda \in \Lambda'$ converges to $\lambda = 0$. The arc $\ell_{-,0}$ bounds an attracting petal $\Omega_{0,-}$ in the direction $L_-$ and $\ell_{+,0}$ a repelling petal $\Omega_{0,+}$ in the direction $L_+$. We denote by $S_{+,0}$ the Jordan domain bounded by $\ell_{+,0}$ and $g_0(\ell_{+,0})$.

Let $m \geq 1$ be the smallest positive integer such that $g_0^m(\omega_0) \in S_{-,0}$. This integer exists, as $g_0^n(\omega_0)$ converges to $z_0$ tangentially to $L_-$. Shrinking $U$, we may assume that $g_0^m(\omega_0)$ lies in a small sector based at $z_0$ around $L_-$. Recalling the maximal $\nabla|\phi_f^\lambda|$ flow line asymptotic to the direction $\theta'$ at infinity by $R^\ast(\theta')$. This flow line is naturally parameterized via $|\phi_f^\lambda|$ by $|\rho_\lambda, \infty|$, for some $\rho_\lambda > 1$. Here it will be convenient to consider the continuous parameterization

$$\text{ray}_\lambda : [\log |\rho_\lambda|, \infty] \to R^\ast(\theta')$$

such that $r = \log |\phi_f^\lambda(\text{ray}_\lambda(r))|$ for all $r > \log |\rho_\lambda|$. Sometimes we abuse of notation and simply write $\text{ray}_\lambda$ instead of $R^\ast(\theta')$.

The ray $R_{f_0}(\theta')$ lands at $z_0$ tangentially to $L_+$. Thus, we may also assume that $R_{f_0}(\theta') \cap \Omega_{0,+}$ is contained in a small sector based at $z_0$ around $L_+$. Let $r_0 > 0$ be such that $\text{ray}_0(r_0) \in S_{+,0}$. Then, according to the Walz Theorem [11, Exposé XI], there exists $N_0$ with the property that for all $n \geq N_0$, we can find $\lambda_n \in \Lambda'$ such that

$$g_{\lambda_n}^n(\omega_{\lambda_n}) = \text{ray}_{\lambda_n}(r_0).$$

Moreover, $\lambda_n \to 0$ as $n \to \infty$.

Along the lines of the Orsay Notes (cf. [25]), two claims yield a proof for the take-off theorem:

**Claim 1.** $g_{\lambda_n}^n(\omega_{\lambda_n}) = \text{ray}_{\lambda_n}(r_0/d^n)$.  

**Claim 2.** $\omega_{\lambda_n} = \text{ray}_{\lambda_n}(r_0/d^{n+m})$.  

**Proof of Claim 1.** For $n$ sufficiently large, by analytic continuation, we may find an arc $\gamma : [r_0/d^n, r_0/d^{n-1}] \to \Omega_{\lambda_n}$ connecting $g_{\lambda_n}^n(\omega_{\lambda_n})$ with $g_{\lambda_n}^{n+1}(\omega_{\lambda_n})$ such that $g_{\lambda_n}^{n-1}(\gamma(r)) = \text{ray}_{\lambda_n}(rd^{n-1})$. By induction, for $j = 1, \ldots, n$, it follows that $g_{\lambda_n}^{n-j}(\gamma(r)) = \text{ray}_{\lambda_n}(rd^{(n-j)})$. In particular, when $j = n$ and $r = r_0$ we obtain Claim 1.
Proof of Claim 2. To prove the second claim, we strongly employ the fact that the parabolic basin contains a unique critical point. Indeed, observe that the attracting petal \( g_0(\Omega_{0,-}) \) is free of critical values of \( g_0^m \). Hence, arcs \( \beta \) connecting \( g_0^{m+1}(\omega_0) \) to \( g_0^{2m+1}(\omega_0) \) contained in \( g_0(\Omega_{0,-}) \) lift, under \( g_0^m \), to arcs \( \tilde{\beta} \) connecting some point \( w \) to \( g_0^{m+1}(\omega_0) \) where \( w \) is independent of the path \( \beta \) and the choice of the petal \( g_0(\Omega_{0,-}) \), modulo homotopy relative to the critical values of \( g_0^m \). The action of \( g_0 \) in \( V(\omega_0) \) is conformally conjugate to the map \( B : z \mapsto (3z^2 + 1)/(z^2 + 3) \) acting on \( \mathbb{D} \). As the critical point \( z = 0 \) of \( B \) has forward orbit contained in \([0,1]\), it is not difficult to conclude that \( \tilde{\beta} \) connects the critical value \( g_0(\omega_0) \) with \( g_0^{m+1}(\omega_0) \). By continuity, for \( n \) large enough, the portion \( \tilde{\beta}_n \) of \( \tilde{\beta} \), connecting \( g_0^{m+1}(\omega_0) \) to \( g_0^{2m+1}(\omega_0) \), lifts under \( g_0^m \) to an arc \( \tilde{\beta}_n \), contained in ray \( \lambda_n \), connecting \( g_0(\omega_{\lambda_n}) \) with \( g_0^{m+1}(\omega_{\lambda_n}) \). It follows that \( g_0(\omega_{\lambda_n}) \) lies in ray \( \lambda_n \). Finally, the portion of ray \( \lambda_n \) between \( g_0(\omega_{\lambda_n}) \) and \( g_0^2(\omega_{\lambda_n}) \) lifts, under \( g_0 \), to a path compactly contained in \( V(\omega_0) \) that necessarily connects \( \omega_{\lambda_n} \) to \( g_0(\omega_{\lambda_n}) \) because \( \omega_{\lambda_n} \) is the unique \( g_0 \) preimage of \( g_0(\omega_{\lambda_n}) \) in \( V(\omega_0) \), for \( n \) sufficiently large. It follows that \( \omega_{\lambda_n} \) is contained in ray \( \lambda_n \). That is, we have proven Claim 2.

From the second claim, we conclude that arbitrarily close to \( f_0 \) there exist \( f_{\lambda_n} \) that lies in a parameter ray \( R_{U_f}(\theta) \) for some escape region \( U_f \). As there are finitely many escape regions and each one has finitely many parameter rays with argument \( \theta \), after passing to a subsequence we may assume that both the escape region and the ray, say \( R_{U_f}(\theta) \), are independent of \( n \). Therefore, \( R_{U_f}(\theta) \) lands at \( f_0 \).

\section{Landing at pcf maps: Kneading and preperiodic ray connections}

We deduce from Theorem 5.1 the following:

**Theorem 5.9.** Consider a parameter ray \( R_{U_f}(\theta) \) of an escape region \( U \) such that \( \theta + 1/3 \) and \( \theta - 1/3 \) are strictly preperiodic arguments. Then \( R_{U_f}(\theta) \) lands at a pcf map \( f_0 \) with take-off argument \( \theta \) and \( \kappa(f_0,\theta) = \kappa(U) \). Moreover, if maps \( f \in R_{U_f}(\theta) \) have a preperiodic ray connection between \(-a(f)\) and \( a(f)\), then \(-a(f)\in \partial V(a(f_0))\) and \( \theta \) is a supporting argument for \( V(a(f_0)) \).

**Proof.** From Theorem 5.1, we know that \( R_{U_f}(\theta) \) lands at a pcf map \( f_0 \). In particular, all Julia cycles of \( f_0 \) are repelling and \( R_{U_f}(\theta) \) converges uniformly to \( R_{f_0}(\theta) \), as \( f \in R_{U_f}(\theta) \) converges to \( f_0 \). Therefore, \( R_{f_0}(\theta) \) lands at the cocritical point \( 2a(f_0) \) and \( R_{f_0}(\theta \pm 1/3) \) land at \(-a(f)\), so \( \theta \) is a take-off argument for \( f_0 \). Moreover, \( R^*_f(\theta \pm 1/3) \) converge to \( R_{f_0}(\theta \pm 1/3) \), thus \( \kappa(U) = \kappa(f_0,\theta) \). See Figure 5.

In the presence of a preperiodic ray connection between \(-a(f)\) and \( a(f)\), the periodic ray \( R_f(\theta) \) lands at \( \zeta(f) \in \partial V(a(f)) \) and it is relatively supporting for \( V(a(f)) \). As \( \zeta(f) \) converges to a repelling periodic point \( \zeta(f_0) \in \partial V(a(f_0)) \), we have that \( R_{f_0}(\theta) \) lands at \( \zeta(f_0) \) and it is relatively supporting for \( V(a(f_0)) \). The critical values of \( f_0 \) are not separated by rays landing at the orbit of \( \zeta(f_0) \) because \(-a(f_0)\in \partial V(a(f_0))\). Thus, we may apply [23, Theorem 3.2] to count the number of cycles of periodic rays landing at \( \zeta(f_0) \), after observing that, for this purpose, nonseparated critical values count as a single one. We conclude that the periodic orbit portrait of \( \zeta(f_0) \) consists of exactly one cycle of rays if the period \( \zeta(f_0) \) is a proper divisor of \( p \), and at most two cycles of rays if the period of \( \zeta(f_0) \) is \( p \). In both cases, \( R_{f_0}(\theta) \) is in fact a left or right supporting ray for \( V(a(f_0)) \).

\qed
6 | IMMIGRATION AND TREKKING

In Subsection 6.1, we introduce and discuss parameter internal rays of type A and B hyperbolic components. In Subsection 6.2, we discuss basic properties of maps in 0-parameter internal rays as well as in the sectors bounded by these rays. The landing of 0 and 1/2-parameter internal rays is the subject of Subsection 6.3. We will say that a “root” of a type A or B component is the landing point of a 0-parameter internal ray and a “co-root” is the landing point of a 1/2-parameter internal rays of a type A component. The effect that crossing from one root to another root of a type B hyperbolic component has on kneading words is the content of Subsection 6.4. The crossing of type A components is discussed in Subsection 6.5. In Subsection 6.6, we establish that in the presence of a periodic ray connection the corresponding parameter ray lands at a “root” of a type A or B component. In Subsection 6.7, we show that a parameter ray with a preperiodic ray connection lands at a “co-root” of a type A component.

6.1 | Parameter internal rays

Given a type A or B hyperbolic component \( \mathcal{H} \), let \( 0 \leq k < p \) be such that \(-a(f) \in V_f(a_k(f)) \) for all \( f \in \mathcal{H} \). There is exactly one map \( f_\star \in \mathcal{H} \) such that \(-a(f_\star) = a_k(f_\star) \) (see [31]). We say that the pcf map \( f_\star \) is the center of \( \mathcal{H} \). For each \( f \in \mathcal{H} \setminus \{f_\star\} \), there exists exactly one argument \( s \) such that \( I_{f,a_k}^*(s) \) terminates at \(-a(f) \). The Böttcher coordinate \( \varphi_{f,a_k} \) has a well-defined limit as \( z \in I_{f,a_k}^*(s) \) converges to \(-a(f) \) which, by abuse of notation, we write as \( \varphi_{f,a_k}(-a(f)) \). Thus, we may introduce:

\[
\Phi_{\mathcal{H}} : \mathcal{H} \to \mathbb{D}
\]

\[
f \mapsto \varphi_{f,a_k}(-a(f)), \ \text{if} \ f \neq f_\star,
\]

\[
f_\star \mapsto 0.
\]

**Proposition 6.1.** The map \( \Phi_{\mathcal{H}} : \mathcal{H} \to \mathbb{D} \) is a holomorphic branched covering with a unique branch point \( f_\star \). The degree of \( \Phi_{\mathcal{H}} \) is 2 if \( \mathcal{H} \) is of type A, and 3 if \( \mathcal{H} \) is of type B.

Given \( t \in \mathbb{R}/\mathbb{Z} \), we say that an arc \( I_{\mathcal{H}}(t) \) is a parameter internal ray with argument \( t \), if

\[
\Phi_{\mathcal{H}} : I_{\mathcal{H}}(t) \to [0,1[\exp(2\pi it)
\]

is a homeomorphism.

Auxiliary families of cubic and quartic polynomials with a fixed critical point \( \mu \) will be useful to apply the results contained in [31], in order to prove Proposition 6.1. Indeed, let

\[
A_\mu(z) = (z - \mu)^2(z + 2\mu) + \mu,
\]

and

\[
B_\mu(z) = (z^2 - \mu^2)^2 + \mu,
\]

where \( \mu \in \mathbb{C} \). Observe that \( S_1 \) corresponds exactly to the polynomials \( A_\mu \) with marked critical point \( \mu \) and free critical point \( \omega = -\mu \). The quartic polynomial \( B_\mu \) has critical points at \( \pm \mu \) and 0. Note that \( B_\mu(-\mu) = B_\mu(\mu) = \mu \), while \( \omega = 0 \) is the free critical point of the family.
The main hyperbolic component $\mathcal{H}_A$ of the $A_\mu$ family is the one containing the monomial $A_0(z) = z^2$. The main hyperbolic component $\mathcal{H}_B$ of the $B_\mu$ family is the one containing $B_0(z) = z^4$. We will rely on a result by Milnor [31] which implies that type $A$ and type $B$ hyperbolic components $\mathcal{H} \subset S_\mu$ possess a natural conformal isomorphism with the main hyperbolic components of the $A_\mu$ and $B_\mu$ families, respectively.

We let $T = A$ or $B$ and freely use the notation $T_\mu$ for the family $A_\mu$ or $B_\mu$. Similarly, $\mathcal{H}_T$ denotes the main hyperbolic component of the corresponding family. To describe Milnor’s conformal isomorphism, we choose a continuously varying fixed point $z_0(\mu)$ in the Julia set of $T_\mu \in \mathcal{H}_T$, and a continuously varying fixed point $z(\mu)$ of $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$ for $f \in \mathcal{H}$. Then, according to [31, Theorem 6.1], given $f \in \mathcal{H}$, there exists a unique map $T_\mu := \Psi_\mathcal{H}(f) \in \mathcal{H}_T$ such that $f^p : V(a_k(f)) \to V(a_k(f))$ is conjugate to $T_\mu : K(T_\mu) \to K(T_\mu)$ via a homeomorphism $h_f : V(a_k(f)) \to K(T_\mu)$, conformal in the interior, such that $h_f(z(\mu)) = z_0(\mu)$. Moreover, $\Psi_\mathcal{H} : \mathcal{H} \to \mathcal{H}_T$ is a conformal isomorphism.

Proofs of Proposition 6.1. It suffices to establish the corresponding assertions for $\mathcal{H}_A$ and $\mathcal{H}_B$. Although we may directly apply the content of Proposition 2.4(b) in [39] for $\mathcal{H}_A$, we briefly sketch here how to proceed in both cases: $A$ and $B$.

For $0 \neq \mu \in \mathcal{H}_T$, let $\varphi_\mu$ be the Böttcher coordinate that conjugates the action of $T_\mu$ near $z = \mu$ with $z \mapsto z^2$. Denote the basin of $\mu$, under $T_\mu$, by $V(\mu)$. As before, $|\varphi_\mu|$ extends continuously to $V(\mu)$ via the functional relation $|\varphi_\mu \circ T_\mu| = |\varphi_\mu|^2$, and $\varphi_\mu$ extends along the flow lines of $-\nabla \varphi_\mu$. For each $s \in \mathbb{R}/\mathbb{Z}$, denote by $I_\mu^s(0)$ the arc starting at $\mu$ such that $\varphi_\mu(I_\mu^s(0)) = [0, \rho(s)] \exp(2\pi i s)$ for some maximal $\rho(s) > 0$. There exists a unique $t \in \mathbb{R}/\mathbb{Z}$ such that $I_\mu^s(t)$ terminates at the free critical point $\omega$. Let $\Phi_T : \mathcal{H}_T \to \mathbb{D}$ be defined by $\Phi_T(\mu) = \rho(t) \exp(2\pi it)$ if $\mu \neq 0$, and $\Phi_T(0) = 0$. It is not difficult to show that $\Phi_T$ is a proper holomorphic map. The standard quasiconformal surgery argument shows that $\Phi_T$ is branch point free in $\mathcal{H}_T \setminus \{0\}$ (e.g., [4, section 4.2.2]). Parameter internal rays for $\mathcal{H}_T$ are now defined via $\Phi_T$ as above for $\mathcal{H}$. Moreover, if $\mathcal{H} \subset S_\mu$ is of type $T$, then $\Phi_\mathcal{H} = \Phi_T \circ \Psi_\mathcal{H}$.

We claim that the degree of $\Phi_A$ is $d_A := 2$ and of $\Phi_B$ is $d_B := 3$. In fact, it is sufficient to show that there are exactly $d_T$ parameter internal rays $I \subset \mathcal{H}_T$ with argument 0. Given such an internal ray $I$, any accumulation point $\mu_0$ of $I$ in $\partial \mathcal{H}_T$ is such that the internal ray $I_{\mu_0}(0)$ lands at a parabolic fixed point; for otherwise, $I_{\mu}(0)$ is smooth and lands at a repelling fixed point for all $\mu$ sufficiently close to $\mu_0$, which would contradict the definition of $I$. There are exactly $d_T$ parameters $\mu$ for which $T_\mu$ has a parabolic fixed point with multiplier 1. Namely the solutions of $\mu^2 = -4/9$ for $T = A$, and the solutions of $\mu^2 = -2/3$ for $T = B$. Thus, $I$ must limit one of these $d_T$ parameters as it approaches $\partial \mathcal{H}_T$.

We claim that distinct parameter internal rays with argument 0 must land at distinct parabolic parameters. Suppose the contrary. Then the complement of the closure of the 0 internal rays of $\mathcal{H}_T$ has a bounded connected component $W$. The region $W$ is contained in the connectedness locus of the family $T_\mu$, by the Maximum Principle. Therefore, $W \subset \mathcal{H}_T$ as $W$ is bifurcation free (e.g., see [28]). Moreover, $\partial W \setminus \mathcal{H}_T$ consists of at most $d_T$ parabolic maps. However, $\Phi_T$ has a nonconstant inverse branch from the simply connected domain $\mathbb{D} \setminus [0,1]$ into $W$. This is impossible, as at all points in $\partial \mathbb{D}$, the inverse branch would limit toward one of the $d_T$ parabolic maps.

The previous paragraph yields that the degree of $\Phi_T$ is at most $d_T$. Now, let $\zeta = \exp(2\pi i/d_T)$, and observe that $\zeta T_\mu(\zeta z) = T_{\zeta \mu}$. Hence, $\Phi_T(\zeta \mu) = \Phi_T(\mu)$ and we conclude that $\deg \Phi_T = d_T$. Proposition 6.1 now follows from the identity $\Phi_H = \Phi_T \circ \Psi_\mathcal{H}$. □
6.2  Sectors of type A and B components

We say that a sector $S$ of $\mathcal{H}$ is a connected component of the preimage under $\Phi_{\mathcal{H}}$ of $\{z \in \mathbb{D} \setminus \{0\} : s < \arg z < t\}$, for some $s, t \in \mathbb{R}/\mathbb{Z}$ not necessarily distinct.

We are particularly interested on sectors bounded by 0-rays. Consider a sector $S$ of $\mathcal{H}$ such that $\Phi_{\mathcal{H}} : S \to \mathbb{D} \setminus [0,1]$ is a bijection. If the 0-rays $I^\pm$ are such that the $s$-internal rays $I(s)$ contained in $S$ converge to $I^\pm$ as $s \to 0^\pm$, then we say that $S$ is the sector of $\mathcal{H}$ from $I^-$ to $I^+$. 

**Proposition 6.2.** Let $\mathcal{H}$ be a type A or B hyperbolic component. Consider a continuously varying fixed point $z(f)$ of $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$, for $f \in \mathcal{H}$. Then the following hold.

1. There exist unique 0-parameter internal rays $I^-$ and $I^+$ such that for all $f \in I^-$ (resp., $I^+$), the ray $I_{f,a_k}^-(0)$ (resp., $I_{f,a_k}^+(0)$) lands at $z(f)$.
2. The sector $S$ in $\mathcal{H}$ from $I^-$ to $I^+$ coincides with the set formed by all $f \in \mathcal{H}$ such that the 0-internal ray $I_{f,a_k}(0)$ is smooth and lands at $z(f)$.

**Definition 6.3.** In the above proposition, when $z(f)$ is the landing point of an external ray $R_f(\theta')$ for all $f \in \mathcal{H}$, the corresponding sector is denoted by $S(\mathcal{H}, \theta')$ and called the sector of $\mathcal{H}$ associated to $\theta'$.

The proof of the proposition will simultaneously yield additional information:

**Lemma 6.4.** Let $I$ be a parameter internal ray of $\mathcal{H}$ with argument 0. Denote by $z^\pm(f)$ the landing point of $I_{f,a_k}^\pm(0)$ for $f_+ \neq f \in I$. Then $z^-(f)$ is the successor of $z^+(f)$ among the fixed points of $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$ in the standard orientation of $\partial V(a_k(f))$.

Proof of Proposition 6.2 and Lemma 6.4. The parameter internal rays with argument 0 for $\mathcal{H}_A$ and $\mathcal{H}_B$ have an explicit description. Indeed, it is not difficult to check that, for $\mu \in ]0,2/3[ \cdot i \subset \mathcal{H}_A$, the arc $]-[|\mu|,|\mu|[(-i)$ is mapped by $\varphi_{\mu}$ into $]0,1[$. It follows that $I_A = ]0,2/3[ \cdot i$ is one parameter internal ray with argument 0 and $(-1) \cdot I_A = ]0,2/3[ \cdot i$ is the other one. Similarly, we let $\mu_0 = -(2/3)^{1/3} < 0$ and check that $[|\mu_0],0[ \subset \mathcal{H}_B$. Moreover, for $\mu_0 < \mu < 0$, we have that $[|\mu],0[ \subset \varphi_{\mu}$ into $]0,1[$. Hence, $I_B = ]|\mu_0],0[ \cdot i$ is one parameter internal ray with argument 0, and $\exp(\pm 2\pi i/3) \cdot I_B$ are the other ones.

We first prove the analogue of Proposition 6.2 for $\mathcal{H}_A$ and $\mathcal{H}_B$. Assume that $\mu \neq 0$ lies in a ray $I$ with argument 0. Denote by $z^\pm(\mu)$ the landing points of $I^\pm_\mu(0)$. It follows that $z^\pm(\mu)$ depend continuously on $\mu$. Denote by $z_j(\mu)$ the fixed point where external ray $R_{\mu}(j/d_T)$ lands, where $j = 0, ..., d_T - 1$. For $0 \neq \mu \in I_A$ (resp., $I_B$), exploiting the symmetry under reflection around the imaginary axis (resp., real axis) $z^-(\mu) = z_2(\mu)$ and $z^+(\mu) = z_0(\mu)$. (resp., $z^+(\mu) = z_1(\mu)$ and $z^-(\mu) = z_0(\mu)$). The multiplication by $\xi = \exp(2\pi i/d_T)$ conjugacy maps the landing point $z^\pm(\mu)$ of $I^\pm_\mu(0)$ for $0 \neq \mu \in I_T$ onto the landing point $z^\pm(\xi \cdot \mu)$ of $I^\pm_{\xi \cdot \mu}(0)$ for $\xi \cdot \mu \in \xi I_T$. The same conjugacy maps the fixed point $z_j(\mu)$ onto the fixed point $z_{j+1}(\mu)$, subscripts mod $d_T$. Hence, for any $z_j(\mu)$, there exists a unique parameter internal ray $I^\pm$ such that $z_j(\mu)$ is landing point of $I^\pm_\mu(0)$ for all $0 \neq \mu \in I^\pm$. Moreover, along any parameter internal ray with argument 0, the landing point of $I^\pm_\mu(0)$ is a successor of the landing point of $I^+\mu(0)$ among the fixed points in the Julia set $J(T_{\mu}) = \partial V(\mu)$. Now we may transfer these results to $\mathcal{H}$, via $\Psi_{\mathcal{H}}$, to conclude that Lemma 6.4 as well as (1) of the proposition hold.
For $0 < \mu \in \mathcal{H}_A$ it is not difficult to see that $I_\mu(0)$ is smooth and lands at $z_0(\mu)$. It follows that the sector $S$ of $\mathcal{H}_A$ from $(-1) \cdot I_A$ to $I_A$ (i.e., contained in the right half plane) is such that $I_\mu(0)$ lands at $z_0(\mu)$. Along $I_A$, we have that $I^{+}_\mu(0)$ lands at $z_0(\mu)$, and along $(-1) \cdot I_A$, we have that $I^{-}_\mu(0)$ lands at $z_0(\mu)$. Conjugacy via multiplication by $-1$ establishes a similar assertion for the sector in the left half plane. Each one of these sectors maps bijectively under $\Phi_A$ onto $\mathbb{D} \setminus \{0, 1\}$. Transferring these results to any hyperbolic component $\mathcal{H}$ of type $A$, part (2) of the proposition follows for type $A$ components. A similar analysis of maps $B_\mu$ with $\mu > 0$ in $\mathcal{H}_B$ proves assertion (2) of the proposition for type $B$ components.

6.3 Landing of 0 and 1/2-parameter internal rays

Here we describe where 0-parameter internal rays of type $A$ and $B$ hyperbolic components land. Landing points of 1/2-rays of type $A$ components are also considered. Before stating and proving the results, we summarize basic dynamical plane facts:

**Proposition 6.5.** Let $f \in C(S_p)$ be a hyperbolic map such that $-a(f) \in V(a_k(f))$, for some $0 \leq k < p$. Then the following statements hold.

1. $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$ is conjugate to $m_3 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ if $k = 0$ and to $m_4 : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ otherwise.

2. Suppose that $z_2$ is the successor of $z_1$ among the fixed points of $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$. Then the open arc from $z_2$ to $z_1$ in $\partial V(a_k(f))$ contains a unique point $w_1$ (resp., $w_2$) such that $f(w_1) = f(z_1)$ (resp., $f(w_2) = f(z_2)$). Moreover, the list $z_2, w_1, w_2, z_1$ respects cyclic order in $\partial V(a_k(f))$.

3. If $R_f(t)$ lands at a fixed point of $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$, then the period of $t$ is exactly $p$.

4. There exists at most one periodic point of $f$ in $\partial V(a_k(f))$ with more than one external ray landing at it.

5. There exists at most one periodic point of $f$ in $\partial V(a_k(f))$ of period strictly less than $p$.

**Proof.** As $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$ is an uniformly expanding degree 3 or 4 map of a Jordan curve, there exists a homeomorphism $h_k : \partial V(a_k(f)) \to \mathbb{R}/\mathbb{Z}$ conjugating $f^p$ with $m_3$ or $m_4$, respectively. Then, it is not difficult to construct $h_{k+1} : \partial V(a_{k+1}(f)) \to \mathbb{R}/\mathbb{Z}$ so that $d_k \cdot h_k(z) = h_{k+1}(f(z))$, where $d_k = 3$ if $k = 0$, and $d_k = 2$ if $k \neq 0$. Under $h_k$, the images of consecutive fixed points of $f^p$ are at distance $1/2$ if $k = 0$, and $1/3$ if $k \neq 0$. Assertion (2) now follows after analyzing the effect of multiplication by $d_k$ in intervals of these lengths.

Let $z$ be a fixed point of $f^p : \partial V(a_k(f)) \to \partial V(a_k(f))$. Then there exists a small arc $\gamma$ starting at $z$, contained in $\partial V(a_k(f))$, such that $f^p(\gamma) \supset \gamma$. As $f^p$ preserves the cyclic order of arcs emanating from $z$, every periodic ray landing at $z$ must be fixed under $f^p$, and we have proven (3).

Consider the set $X$ formed by the periodic points of $f$ that belong to $\partial V(a_j(f))$ for some $0 \leq j < p$ and have more than one ray landing at it. To prove (4), we assume that $X \neq \emptyset$. It is sufficient to show that $X$ has exactly one point in $\partial V(a_j(f))$ for some $0 \leq j < p$. In fact, let $Z_j(w, s, t)$ be a sector that has minimal angular length among all the sectors based at elements of $X$. Note that $Z_j(w, s, t)$ is disjoint from $X$. Every sector based at a periodic point contains a postcritical point (e.g., [23]). Hence, $Z_j(w, s, t)$ contains $V(a_{\ell}(f))$ for some $0 \leq \ell < p$ and $w$ is the unique element of $X$ in $\partial V(a_{\ell}(f))$. Statement (5) is a direct consequence of (4).
6.3.1 | Landing of 0-parameter rays

Consider a 0-ray $I_{\mathcal{H}}(0)$ of a type $A$ or $B$ hyperbolic component $\mathcal{H} \subset \mathcal{C}(S_p)$ centered at $f_*$. For $f \in I_{\mathcal{H}}(0)$ such that $f \neq f_*$, we will freely use the following notation (see Figure 4).

1. The landing points of $I_{f, a_k}(0)$ are denoted by $z^\pm(f)$. By Lemma 6.4, $z^-(f)$ is a successor of $z^+(f)$ among the fixed points of $f^p$ in $\partial V(a_k(f))$.
2. Denote by $w^\pm(f)$ the preimages of $f(z^\pm(f))$ in $\partial V(a_k(f))$ so that $z^-(f), w^+(f), w^-(f), z^+(f)$ are listed respecting cyclic order. These points exist and are well-defined due to Proposition 6.5.
3. For each one of the points $z = z^\pm(f)$ or $w^\pm(f)$, there exists a unique sector based at $z$ that contains $V(a_k(f))$. This sector is bounded at the left by a ray with argument $t(z)$ and at the right by one with argument $s(z)$. That is, we let $s(z), t(z) \in \mathbb{Q}/\mathbb{Z}$ be such that $V(a_k(f))$ is contained in the sector $Z_f(z, t(z), s(z))$ based at $z = z(f)$. Notice that $s(z)$ and $t(z)$ are not necessarily distinct. In fact, by Proposition 6.5(4), $s(z^-) = t(z^-)$ or $s(z^+) = t(z^+)$. Moreover, $s(z), t(z)$ are independent of $f \in I_{\mathcal{H}}(0) \setminus \{f_*\}$. The arguments $t(z)$ and $s(z)$ are left and right supporting for $V(a_k(f))$.

Proposition 6.6. Consider a type $A$ or $B$ hyperbolic component and an internal ray $I_{\mathcal{H}}(0)$ as above. Then $I_{\mathcal{H}}(0)$ lands at a parabolic map $f_0$ such that the landing point of $I_{f_0, a_k}(0)$ is a parabolic periodic point $z_0$. Moreover, all of the following statements hold.

1. The period of $z_0$ is the minimum $\ell'$ of the periods of $z^+(f)$ and $z^-(f)$ for any $f_* \neq f \in I_{\mathcal{H}}(0)$. The multiplier of $z_0$ is a primitive $p/\ell'$-root of unity.
(2) \( z^\pm(f) \to z_0 \) as \( f \in I_H(0) \) approaches \( f_0 \), and \( z_0 \in \partial V_{f_0}(-a(f_0)) \).

(3) A ray \( R_{f_0}(t) \) lands at \( z_0 \) if and only if \( R_f(t) \) lands at the \( f^- \)-orbit of \( z^+(f) \) or \( z^-(f) \) for all \( f \neq f_0 \in I_H(0) \).

(4) \( Z_{f_0}(z_0, t(z^+), s(z^-)) \) is the sector at \( z_0 \) that contains \( a_k(f_0) \).

(5) \( w^\pm(f) \to w_0 \) as \( f \in \mathbb{S}_{\mathbb{P}}(0) \) approaches \( f_0 \) where \( w_0 \) is the unique nonperiodic preimage of \( f_0(z_0) \) in \( \partial V_{f_0}(-a(f_0)) \).

(6) The external rays with arguments \( s(w^\pm) \) and \( t(w^\pm) \) of \( f_0 \) land at \( w_0 \).

(7) \( Z_{f_0}(z_0, t(z^-), s(z^+)) \) is a sector at \( z_0 \), and \( Z_{f_0}(w_0, t(w^-), s(w^+)) \) is a sector at \( w_0 \). Both of these sectors contain \( -a(f_0) \).

(8) If \( \theta \) is such that \( \{\theta + 1/3, \theta - 1/3\} \) is one of the pairs \( \{t(z^-), t(w^-)\}, \{s(z^-), s(w^-)\}, \{t(z^+), t(w^+)\}, \{s(z^+), s(w^+)\} \), then \( \theta \) is a take-off argument for \( f_0 \).

(9) For any pair of arguments \( s, t \in \{s(z^\pm), t(z^\pm), s(w^\pm), t(w^\pm)\} \), if \( a_j(f_\star) \in Z_{f_\star}(V(a_k(f_\star)), s, t) \), then \( a_j(f_0) \in Z_{f_0}(V(a_k(f_0)) \cup V(-a(f_0)), s, t) \).

As in the proof of Lemma 5.5, it follows that \( X \cap J(f_0) \) has a bounded Fatou component with two distinct periodic points of period dividing
In its boundary. As \( z_0 \in X \), we have that \( X \cap J(f_0) = \{ z_0 \} \). Moreover, \(-a(f_0) \in X\) so \( z_0 \in \partial V (-a(f_0)) \).

Now (1) and (2) will follow after proving that \( z^\pm(f) \to z_0 \). We argue in a similar fashion but instead of considering \( I_f^\pm(0) \), we consider arcs \( I_f^\pm \) landing at \( z^\pm(f) \) (cf. proof of Lemma 5.7). To introduce these arcs, let us consider \( h : \mathbb{H} \to \mathbb{D} \) given by \( z \mapsto \exp(iz) \), and let \( \tilde{U}_f = h^{-1}(U_f^+) \). For \( f \in I_H(0) \) such that \( -\log |\varphi_{f,a_k}(\alpha_f)| < 2\pi \), we have that \( \tilde{U}_f \) contains the sectors \( S^+ = \{ z \in \mathbb{H} : \pi/4 < \arg z < \pi/2 \} \) and \( S^- = \{ z \in \mathbb{H} : \pi/2 < \arg z < 3\pi/4 \} \). Given \( \alpha \in \pi/4, \pi/2 \) (resp., \( \pi/4, \pi/2 \)), the arc \( \gamma_f(\alpha) = \varphi_f^{-1}(h(0, \infty \cdot \exp(\text{i}\alpha))) \) is \( f \) invariant and \( d_f(z, f^p(z)) \) is uniformly bounded for all \( z \in \gamma_f(\alpha) \) and all \( f \in I_H(0) \) such that \( -\log |\varphi_{f,a_k}(-a(f))| < 2\pi \). It follows that \( \gamma_f(\alpha) \) limits to a periodic point \( w(\alpha) \) at one end and toward \( a_k(f) \) at the other. The periodic point \( w(\alpha) \) is independent of \( \alpha \in \pi/4, \pi/2 \) (resp., \( \pi/4, \pi/2 \)) (e.g., see [29, Corollary 17.10]).

Hence, \( w(\alpha) = z^-(f) \) (resp., \( z^+(f) \)) because \( \gamma_f(\alpha) \) converges to \( I_f^{-}(0) \) (resp., \( I_f^{+}(0) \)) as \( \alpha \to \pi/2^-(\text{resp., } \pi/2^+) \). The corresponding arc \( \gamma_f^+(\alpha) \), in the dynamical plane of \( f_0 \), is homotopic rel \( J(f_0) \) to the internal ray \( I_{f_0,a_k}(0) \). Hence, it limits toward \( z_0 \) at one end. A similar argument as the one used above for \( X \) yields that \( J(f_0) \cap (\limsup_R f(0)) = \{ z_0 \} \). Therefore, \( z^ \pm(f) \to z_0 \).

Let us now show that, if a periodic ray \( R(f) \) lands at \( z^\pm(f) \), then \( R_{f_0}(t) \) lands at \( z_0 \). Note that the connected set \( K(f_0) \cap (\limsup_R f(t)) \) contains \( z_0 \). We may again use the elementary hyperbolic distance estimate to show that \( J(f_0) \cap (\limsup_R f(t)) \) consists of points of period dividing \( p \).

Again, taking into account that periodic Fatou components of \( f_0 \) have degree 2 return maps and, therefore, only have one point of period dividing \( p \) on their boundaries, we conclude that \( J(f_0) \cap (\limsup_R f(t)) \) is just the singleton \( \{ z_0 \} \). Thus, \( R(f_0) \) lands at \( z_0 \).

From the previous paragraph, if \( R(f) \) lands at \( f^\ell m(z^\pm(f)) \), then \( R_{f_0}(t) \) lands at \( z_0 \). For (3), we claim that if a ray \( R_{f_0}(t) \) lands at \( z_0 \), then \( R(f) \) lands at \( f^\ell m(z^\pm(f)) \) for some \( m \). Observe that no ray landing at \( f^\ell m(z^\pm(f)) \) for \( j = 1, \ldots, \ell - 1 \) may land at \( z_0 \), as the period of \( z_0 \) is exactly \( \ell \).

Now, if a ray \( R(f) \) does not land in the orbit of \( z^\pm(f) \), then its landing point, say \( z(f) \), converges to a repelling periodic point \( z(f_0) \). Thus, \( R_{f_0}(t) \) lands at \( z(f_0) \neq z_0 \), which proves our claim and (3) follows.

We assert that the arguments of the rays of \( f_0 \) landing at \( z_0 \) are contained in \( \{ s(z^+), t(z^+) \} \cup \{ s(z^-), t(z^-) \} \). In fact, from Proposition 6.5(4), \( s(z^+) = t(z^+) \) or \( s(z^-) = t(z^-) \). For simplicity, assume that the latter holds. If \( \ell < p \), then the period of \( z^\pm(f) \) is \( \ell \), and the boundaries of \( V(a_k(f)), V(a_{k+\ell}(f)), \ldots, V(a_{k+p-\ell}(f)) \) meet at \( z^\pm(f) \). Hence, for \( 0 < m < p/\ell \) the sector \( Z_f(z^+(f), s(z^+(f)), t(z^+(f))) \) contains \( f^\ell m(z^-((f)) \). Thus, the rays landing at \( z_0 \) in the dynamical plane of \( f_0 \) are contained in \( \{ s(z^+), t(z^+) \} \cup \{ s(z^-), t(z^-) \} \).

From the previous paragraph, \( Z_{f_0}(z_0, t(z^+), s(z^-)) \) is a sector at \( z_0 \). Observe that in the dynamical plane of \( f \in I_H(0) \), the internal ray \( I_{f,a_k}(1/3) \) lands at a periodic point \( u(f) \) in the arc from \( z^+(f) \) to \( z^-(f) \) of \( \partial V(a_k(f)) \). The repelling periodic point \( u(f) \) is the landing point of an external ray \( R_f(t') \) with \( t' \in \{ t(z^+), s(z^-) \} \), and \( u(f) \) must converge to a repelling periodic point \( u(f_0) \) of \( f_0 \). Hence, \( I_{f_0,a_k}(1/3) \) and \( R_{f_0}(t') \) land at \( u(f_0) \). Therefore, \( a_k(f_0) \in Z_{f_0}(z_0, t(z^+), s(z^-)) \). That is, we have proven (4).

Consider \( f_* \neq f \in I_H(0) \). To study the convergence of \( w^\pm(f) \), let \( I_{f,a_k}^\pm \) be the portion of \( I_{f,a_k}^\pm \) connecting \( -a(f) \) with \( z^\pm(f) \). Then there exist arcs \( L^\pm_f \) connecting \( -a(f) \) with \( w^\pm(f) \) such that \( f : L^\pm_f \to R^\pm_f \) is a homeomorphism. As \( f \to f_0 \) along \( I_H(0) \), consider \( w^\pm \in J(f_0) \cap \limsup_L^\pm \). Then \( f(w^\pm) = f(z_0) \). We claim that \( w^+ \neq z_0 \) and \( w^- \neq z_0 \). Otherwise, for \( \sigma = + \) or \( - \), we could choose \( f_n \in I_H(0) \) converging to \( f_0 \), \( w_n \in L_f^\sigma_n \) converging to \( w^\sigma \) and \( z_n \in L_f^\sigma_n \) converging to \( \tilde{z}^\sigma \) such that \( f(z_n) = f(w_n) \). This would imply that \( f_0 \) is not locally injective at \( z_0 \), which is
impossible. We conclude that \( J(f_0) \cap \lim \sup L^\pm_f \) consists of strictly preperiodic points that map onto \( f(z_0) \). As Fatou components of \( f_0 \) contain at most one of these points in their boundary, and \(-a(f_0) \in \lim \sup L^+_f \cap \lim \sup L^-_f\), it follows that \( w^+ = w^- \) is the unique point \( w_0 \) in \( \partial V(-a(f_0)) \) with this property. Moreover, any limit point of \( w^\pm(f) \) lies in \( J(f_0) \cap \lim \sup L^\pm_f \). Hence, \( w^\pm(f) \to w_0 \), and we have proven (5).

Let \( \sigma = + \) or \(-\). For \( f \in I_H(0) \), consider \( t \) such that \( R_f(t) \) lands at \( w^\sigma(f) \). As \( f \in I_H(0) \) converges to \( f_0 \), we claim that \( J(f_0) \cap \lim \sup R_f(t) \) consists of \( f_0 \)-strictly preperiodic points \( w^\sigma \) such that \( f_0(w^\sigma) = f(z_0) \). In fact, we may apply a similar reasoning than in the previous paragraph to show that, if \( z_0 \in J(f_0) \cap \lim \sup R_f(t) \), then \( z_0 \) is critical, which is impossible. Thus, every point in \( J(f_0) \cap \lim \sup R_f(t) \) is a strictly preperiodic preimage of \( f_0(z_0) \). As there is at most one such preimage in the boundary of any bounded Fatou component of \( f_0 \), it follows that \( \{w_0\} = J(f_0) \cap \lim \sup R_f(t) \) that implies statement (6).

To establish (7), just observe that the rays of \( f_0 \) with arguments \( s(w^\pm) \) and \( t(w^\pm) \) all lie in the sector \( Z_{f_0}(z_0, t(z^-), s(z^+)) \) because \( w^\pm(f) \) lie in the arc from \( z^- \) to \( z^+ \) of \( \partial V(a_k(f)) \). Hence, \( w_0 \in Z_{f_0}(z_0, t(z^-), s(z^+)) \) and, therefore, \(-a(f_0) \in Z_{f_0}(z_0, t(z^-), s(z^+)) \). It follows that the sector at \( w_0 \) containing \(-a(f_0) \) must be bounded by rays with arguments that map onto \( 3t(z^-) \) and \( 3s(z^+) \). Thus, (7) holds.

As \( s(z^+) = t(z^+) \) or \( s(z^-) = t(z^-) \), the sectors \( Z_{f_0}(z_0, t(z^-), s(z^+)) \) and \( Z_{f_0}(z_0, t(z^+), s(z^-)) \) share a boundary ray. From (4) and (7), we conclude that the former sector contains \( a_k(f_0) \) and the latter \(-a(f_0) \). To fix ideas, let us assume that \( s(z^+) = t(z^+) \). Then, the unique ray with argument in \( ]t(z^-), s(z^-)[ \) landing at \( z_0 \) is the ray with argument \( s(z^+) \). The orbit of \( s(z^+) \) is distinct from the orbits of \( t(z^-) \) and \( s(z^-) \). Therefore, the rays with arguments \( s(z^+) \) and \( t(z^-) \) are all relatively supporting for \( V(-a(f_0)) \), and (8) follows.

Finally, to prove (9), just observe that if \( 0 < j < p \) and \( j \neq k \), then \( I_{f, a_j}(1/3) \) lands at a repelling periodic point that is the landing point of some external ray \( R_f(t') \). In the dynamical plane of the limit \( f_0 \), we also have that \( I_{f_0, a_j}(1/3) \) and \( R_{f_0}(t') \) land at a common repelling periodic point. From here, (9) follows.

6.3.2 | Landing of 1/2-parameter internal rays

Consider a ray \( I_H(1/2) \) of a type A hyperbolic component centered at \( f_* \). For any \( f \in I_H(1/2) \) distinct from \( f_* \), we will use the following notation.

1. Let \( z_0(f) \) be the landing point of \( I_{f, a}(0) \) and \( w^\pm(f) \) be the landing points of \( I_{f, a}^\pm(1/2) \).
2. For \( z = z_0(f) \) or \( w^\pm(f) \), let \( s(z), t(z) \) be such that \( Z_f(z, t(z), s(z)) \) is the sector at \( z \) containing \( a(f) \). Note that \( s(w^\pm) = s(z_0) \pm 1/3 \) and \( t(w^\pm) = t(z_0) \mp 1/3 \).

**Proposition 6.7.** Consider a type A hyperbolic component and an internal ray \( I_H(1/2) \) as above. Then \( I_H(1/2) \) lands at a pcf map \( f_0 \) such that \(-a(f_0) \) is the landing point of \( I_{f_0, a}(1/2) \). Moreover, all of the following statements hold.

1. As \( f \in I_H(1/2) \) converges to \( f_0 \), the periodic point \( z_0(f) \) converges to a repelling periodic point \( z_0(f_0) \) and \( w^\pm(f) \to -a(f_0) \).
2. The external rays \( s(w^\pm) \) and \( t(w^\pm) \) land at \(-a(f_0) \).
3. The sector at \(-a(f_0) \) containing \( a(f_0) \) is \( Z_{f_0}(-a(f_0), t(w^+), s(w^-)) \).
(4) If \( \theta = t(z_0) \) or \( s(z_0) \), then \( \theta \) is a take-off argument for \( f_0 \), and \( R_{f_0}(\theta) \) lands at \( z_0(f_0) \in \partial V(a(f_0)) \).

(5) For all \( f \neq f_0 \in I_H(1/2) \), if \( a_j(f) \in Z(V(a(f)), s, t) \) for some \( s, t \in \{s(z_0), t(z_0), s(w^\pm), t(w^\pm)\} \), then \( a_j(f_0) \in Z(V(a(f_0)), s, t) \).

Proof. Assume that \( f_n \in I_H(1/2) \) converges to \( g \in \partial H \). We claim that \( I_{g,a}(1/2) \) lands at \( -a(g) \). To prove this, consider \( L = (\limsup I_{f_n,a}(1/2)) \) and \( L_0 = (\limsup I_{f_n,a}(0)) \). Let \( z_0 \) be the landing point of \( I_{g,a}(0) \). As \( g(L) = L_0 \) and \( L_0 \cap J(g) = \{z_0\} \), if \( w \in L \cap J(g) \), then \( g(w) = g(z_0) \). Moreover, \( w \neq z_0 \); for otherwise, \( z_0 \) would be a critical point. Note that \( -a(g) \in L \).

Let us first show that \( -a(g) \) lies in the Julia set. We suppose that \( -a(g) \) lies in the Fatou set, and after some work arrive to a contradiction. Observe that \( V(-a(g)) \) has to be a periodic component of a parabolic basin. For otherwise, \( g \in \partial H \) would be hyperbolic because \( J(g) \) is critical point free. In particular, the continuum \( L \) that connects \( -a(g) \) and \( a(g) \) must contain the unique point \( w_0 \neq z_0 \) in \( \partial V(a(g)) \) such that \( f(w_0) = f(z_0) \). It follows that \( w_0 \in \partial V(-a(g)) \), for otherwise, there would exist a Fatou component distinct from \( V(\pm a(g)) \) containing two preimages of \( f(z_0) \) in its boundary. Let \( m \) be such that the Fatou components \( V(\pm a(g)) \) are fixed under \( g^m \). Then \( g^m(w_0) = w_0 \), which contradicts the fact that \( w_0 \) is strictly preperiodic. This contradiction shows that \( -a(g) \) lies in the Julia set.

It follows that \( g(-a(g)) = g(z_0) \). Therefore, \( -a(g) \) is the unique preimage of \( g(z_0) \) distinct from \( z_0 \). That is, \( L \cap J(g) \) is the singleton \( \{-a(g)\} \). As \( I_{g,a}(1/2) \subset L \), we conclude that the landing point of \( I_{g,a}(1/2) \) is \( -a(g) \).

Recall that \( g \) is an arbitrary accumulation point of \( I_H(1/2) \). As there are only finitely many \( g \in S_\beta \) such that \( g(-a(g)) \) is periodic of period dividing \( p \), we conclude that \( I_H(1/2) \) lands at a map \( f_0 \) such that \( -a(f_0) \) is the landing point of \( I_{f_0,a}(1/2) \). It follows that the landing point \( z_0(f_0) \) of \( I_{f_0,a}(0) \) is a repelling periodic point that is the limit of \( z_0(f) \). Moreover, the nonperiodic preimages \( w^\pm(f) \) of \( f(z_0(f)) \) converge to \( -a(f_0) \), the unique nonperiodic preimage of \( f_0(z_0(f_0)) \).

To establish (5), for \( f \in I_H(1/2) \), note that the landing point \( z(f) \) of the internal ray \( I_{f,a}(1/3) \) is a repelling periodic point that is also the landing point of some a periodic ray \( R_f(t') \). It follows that \( z(f) \) converges to a repelling periodic point of \( f_0 \) that is the common landing point of \( I_{f_0,a}(1/3) \) and \( R_{f_0}(t') \). Thus, if \( a_j(f) \) lies in the region \( Z_j(V(a(f)), s, t) \) as in the statement of the proposition, then \( t' \in ]s, t[ \) and \( R_{f_0}(t') \) lands at \( \partial V(a(f_0)) \). We conclude that (5) holds.

\[\square\]

6.4 Type B trekking theorem

Recall that \( S(H, \theta') \) denotes a sector of a type A or B hyperbolic component between 0-internal parameter rays as introduced in Definition 6.3.

Theorem 6.8. Assume that \( H \) is a type B component centered at \( f_* \) such that \( -a(f_*) = a_k(f_*) \). Suppose that \( R_{f_*}(\theta \pm 1/3) \) support \( \partial V(a_k(f)) \), and \( \theta' = \theta + 1/3 \) or \( \theta - 1/3 \) is a period \( p \) argument. Then:
(1) the 0-parameter internal rays that bound the sector $S(H, \theta')$ land at parabolic maps $f_0, f_1$ with take-off argument $\theta$, and
(2) the associated kneadings $\kappa(f_0, \theta)$ and $\kappa(f_1, \theta)$ differ at the $k$th symbol and coincide in the rest.

In Figures 1 and 2, the maps $f_0$ and $f_1$ of the previous theorem correspond to the parameters labeled by $\odot$ and $\otimes$.

**Proof.** For $\theta, \theta'$ as in the statement of the theorem, denote by $\theta''$ the argument distinct from $\theta, \theta'$ such that $3\theta'' = 3\theta$. That is, $\theta'$ and $\theta''$ support $V(a_k(f_\ast))$. Then $R_f(\theta')$ and $R_f(\theta'')$ land at points $z(f), w(f) \in \partial V(a_k(f))$, respectively, for all $f \in H$.

Consider the sector $S(H, \theta')$ that according to Proposition 6.2 is bounded by 0-parameter rays $I^\pm$. It will be convenient to rename the landing point of $I^\pm$ by $f^\pm$, respectively. Let $\sigma \in \{+, -\}$, also from Proposition 6.2, for all $f \in I^\sigma$ with $f \neq f_\ast$, the internal ray $I^{\sigma, a_k(0)}_f$ lands at $z(f)$. In the notation of Subsection 6.3.1, $z(f) = z^\sigma(f)$ and $w(f) = w^\sigma(f)$. Moreover, either $\theta' = s(z^\sigma)$ or $\theta' = t(z^\sigma)$ and $\theta'' = t(w^\sigma)$. In view of Proposition 6.6(8), $\theta$ is a take-off argument for the landing point $f_\sigma$ of $I^\sigma$.

From Proposition 6.6(9), if $j \neq k$, then $a_j(f_\ast) \in Z_{f_\ast}(V(a_k(f_\ast)), \theta', \theta'')$ if and only $a_j(f_\sigma) \in Z_{f_\sigma}(V(-a(f_\sigma)), \theta', \theta'')$. In particular, for $j \neq k$, the kneading symbol corresponding to $a_j(f_\sigma)$ is independent of whether $\sigma = +$ or $\sigma = -$.

To finish the proof, we have to show that the $k$th symbol of the kneading words of $\kappa(f_+, \theta)$ and $\kappa(f_-, \theta)$ are different. In fact, for $f$ along the ray $I^-$, we have that the landing point of $I^-(a_k(0))$ is $z^- = z(f)$, and the landing point of $I^-(a_k(0))$ is a point $z^+$, which belongs the arc in $\partial V(a_k(f))$ from $w^- = w(f)$ to $z^-$. Thus, $[t(z^+), s(z^-)] \subset \theta', \theta''$. Hence, $a_k(f_-) \in Z_{f_-}(z_0, t(z^+), s(z^-)) \subset Z_{f_-}(V(-a(f_-)), \theta', \theta'').$ Along the ray $I^+$, we have that the landing point of $I^+(a_k(0))$ is $z^+ = z(f)$, and the landing point of $I^+(a_k(0))$ is a point $z^-$ that belongs to the arc in $\partial V(a_k(f))$ from $w^+ = w(f)$ to $z^+$. As $[t(z^-), s(z^-)] \subset \theta', \theta''$, we have $a_k(f_+) \in Z_{f_+}(z_0, t(z^-), s(z^-)) \subset Z_{f_+}(V(-a(f_+)), \theta', \theta'')$. That is, we have proven that $a_k(f_+) \in Z_{f_+}(V(-a(f_+)), \theta', \theta'')$ and $a_k(f_-) \in Z_{f_-}(V(-a(f_-)), \theta', \theta'')$. Thus, the $k$th symbol of the kneading words of $\kappa(f_+, \theta)$ and $\kappa(f_-, \theta)$ are indeed different. □

### 6.5 Type A trekking theorem

**Theorem 6.9.** Assume that $H$ is a type A component centered at $f_\ast$, and $\theta'$ is a period $p$ supporting argument for $V(a(f_\ast))$. The 0-parameter rays of $H$ land at parabolic maps $f_1, f_2$, and the 1/2-parameter ray contained in $S(H, \theta')$ lands at a pcf map $f_0$ such that the following holds, modulo relabeling of $f_1$ and $f_2$.

1. For $j = 0, 1, 2$, the map $f_j$ has take-off argument $\theta' + j/3$.
2. $\kappa(f_j, \theta' + j/3) = t_0 ... t_{p-1} 0$,

where

$$t_n = \begin{cases} 1, & \text{if } 3^n \theta' \in j/3 + [3/3, 2/3], \\ 0, & \text{otherwise}. \end{cases}$$
After proving the previous theorem, in Corollary 6.10, we deduce that we can choose to cross from one root or co-root to another one in order to switch the symbols 0, 1 of an initial segment of the kneading words involved. See Figure 5.

Proof. Consider the sector $S(H, \theta')$ that, according to Proposition 6.2, is bounded by 0-parameter rays $I^\pm$. Let $f_1$ be the landing point of $I^+$ and $f_2$ the landing point of $I^-$. Consider $f \in I^+ \setminus \{f^*\}$, and denote by $z^\pm(f)$ the landing point of $I^\pm_{f,a}(0)$. According to Proposition 6.2, $R_f(\theta')$ lands at $z^+(f)$. By Subsection 6.3.1, the open arc in $\partial V(a(f))$ from $z^-(f)$ to $z^+(f)$ contains a unique point $w^+(f)$ with image $f(z^+(f))$. Necessarily, $R_f(\theta' + 2/3)$ lands at
As $\theta'$ is supporting, in the notation of Subsection 6.3.1(3), we have that $\theta' = s(z^+(f))$ or $t(z^+(f))$ and $\theta' + 2/3 = s(w^+(f))$ or $t(w^+(f))$, respectively. From Proposition 6.6(8), it follows that $\theta' + 1/3$ is a take-off argument for $f_1$, which proves assertion (1) for $j = 1$. Moreover, points $a_n(f_1)$ in the region $Z_{f_1}(V(-a(f_1)), \theta' + 2/3, \theta')$ correspond to symbols “1” in the kneading word $\kappa(f_1, x' + 1/3)$. Equivalently, as $R_{f_1}(3^m \theta')$ lands in $\partial V(a_n(f_1))$, symbols “1” correspond to rays $R_{f_1}(3^m \theta')$ contained in this region, that is, to $3^m \theta' \in 1/3 + \theta' + 1/3, \theta' + 2/3$. Thus, we have proven the theorem for $j = 1$.

A similar analysis for $f \in I^- \setminus f_*$, with the aid of Propositions 6.2 and 6.6, proves that the landing point $f_2$ of $I^-$ has the claimed properties.

Now consider the landing point $f_0$ of the $1/2$ internal ray contained in the sector $S(H, \theta')$. By Proposition 6.7(4), we have that $\theta'$ is a take-off argument for $f_0$ with the corresponding ray landing at $\partial V(a(f_0))$. Thus, $\kappa(f_0, \theta')$ as described in the statement of the theorem.

Corollary 6.10. Assume that $H$ is a type $A$ component centered at $f_*$, and $\theta'$ is a period $p$ supporting argument for $V(a(f_*))$. Let $g_0$ be the landing point of a $0$ or $1/2$ ray in $S(H, \theta')$ with take-off argument $\theta \in m^{-1}(3 \theta')$ and kneading word $\kappa(g_0, \theta) = 1^{\mu_0-11} \mu_{\mu_0+1} \cdots 0$. Then there exists a landing point $g_1$ of a $0$ or $1/2$ ray in $S(H, \theta')$ with take-off argument $\theta' \in m^{-1}(3 \theta')$ and kneading word $\kappa(g_1, \theta') = 0^{\mu_0-11} \mu_{\mu_0+1} \cdots 0$.

Proof. Given $g_0$, we have that $3^{2\mu_0} \in \theta - 1/3, \theta$ or $\theta, \theta + 1/3$. In the first case, choose $g_1$ with take-off argument $\theta' = \theta + 1/3$ and in the latter with take-off argument $\theta' = \theta - 1/3$. From the previous theorem, it follows that $\kappa(g_1, \theta')$ as desired.

6.6 | Parabolic immigration

Theorem 6.11. Let $f_0$ be a parabolic map with take-off argument $\theta$. Assume that the periodic argument $\theta' = \theta + 1/3$ or $\theta - 1/3$ is a period $p$ relatively supporting argument for $V(a_k(f_0))$. Then there exists a type $A$ or $B$ hyperbolic component $H$ centered at a map $f_*$ such that the following hold.

1. $a_k(f_*) = -a(f_*)$.
2. The rays $R_{f_*}(\theta \pm 1/3)$ support $V(-a(f_*))$.
3. $f_0$ is the landing point of a $0$-internal ray $I$ of $H$ contained in $\partial S(H, \theta')$.

By Theorem 5.6, the hypothesis of the previous theorem are fulfilled by maps $f_0$ that are the landing points of parameter rays $R_{f}(\theta)$ with a ray connection between $-a(f)$ and $a_k(f)$, for $f \in R_{f}(-\theta)$.

To prove the theorem above, we first identify parabolic maps in $S_p$ with the aid of critical portraits. Given a map $g \in S_p$ with a parabolic cycle, we say that the ordered pair of pairs of arguments in $Q/\mathbb{Z}$: $(\beta', \beta''), (\theta', \theta'')$ is a left (resp., right) supporting critical marking for $g$ if the following hold.

1. $\beta' \neq \beta''$ and $3 \beta' = 3 \beta'', \theta' \neq \theta''$ and $3 \theta' = 3 \theta''$.
2. $R_{g}(\beta')$ lands at the parabolic cycle. $R_{g}(\beta')$ and $R_{g}(\beta'')$ left (resp., right) support $V(-a(g))$.
3. The internal ray $I_{g,a}(0)$ and $R_{g}(\theta')$ land at the same point. $R_{g}(\theta')$ and $R_{g}(\theta'')$ left (resp., right) support $V(a(g))$. 

Lemma 6.12. Let \( g_0, g_1 \in S_p \) be maps with a parabolic cycle such that \( g_0 \) and \( g_1 \) have the same left (resp., right) supporting critical marking. Then \( g_0 = g_1 \).

The proof of the lemma presented here relies on Haïssinsky’s deformation of geometrically finite polynomials [20] and in Poirier’s uniqueness result for critically marked pcf polynomials [34]. However, using the pullback argument, one could give a longer, but more direct, proof that bypasses the use of Haïssinsky’s deformation and show, in the spirit of Poirier’s work, that \( g_0 \) and \( g_1 \) are conjugate by a quasiconformal map \( h \) that is conformal in the Fatou set to conclude that \( g_0 = g_1 \).

Proof. For simplicity of the exposition, we assume that the arguments involved are right supporting.

According to Haïssinsky [20], for \( i = 0, 1 \), there exist a continuous family \( G_{i, \eta} \in S_p \) parameterized by \( \eta \in [0, 1] \) such that \( G_{i, \eta} \) is hyperbolic for all \( \eta \in [0, 1] \), \( G_{i, 1} = g_i \), and the Caratheodory loop of \( G_{i, \eta} \) is independent of \( \eta \). For \( \eta < 1 \), the maps \( G_{i, \eta} \) have one attracting cycle that as \( \eta \to 1 \), converges to the parabolic cycle of \( g_i \). The multiplier of this attracting cycle converges to 1.

In particular, \( \beta', \beta'' \) are right supporting arguments for \( V(-a(G_{i, \eta})) \), and \( \delta', \delta'' \) are right supporting arguments for \( V(a(G_{i, \eta})) \).

Note that \( G_{i, \eta} \) belongs to a hyperbolic component \( \mathcal{H}_i \) of disjoint type, for \( i = 0, 1 \) and \( \eta < 1 \). According to Milnor, \( \mathcal{H}_i \) contains a unique pcf polynomial \( g_{i,*} \) [31]. As the Caratheodory loops of \( g_{i,*} \) and \( g_i \) agree, an admissible right supporting critical portrait for \( g_{i,*} \) is given by the pairs \( \{\delta', \delta''\} \) and \( \{\beta', \beta''\} \). According to Poirier, it follows that \( g_{0,*} = g_{1,*} \). Therefore, \( \mathcal{H}_0 = \mathcal{H}_1 \). As the hyperbolic component \( \mathcal{H}_0 = \mathcal{H}_1 \) is parameterized by the multiplier of the attracting cycle (not equal to the orbit of \( +a \)), it follows that \( g_0 = g_1 \). \( \square \)

Proof of Theorem 6.11. Suppose that \( f_0 \) is a parabolic map as in the statement of the theorem. That is, \( R_{f_0}(\theta') \) is a period \( p \) ray landing at a parabolic periodic point \( z_0 \) relatively supporting \( V(a_k(f_0)) \) and \( V(-a(f_0)) \), as \( \theta \) is a take-off argument (Definition 5.2). Here \( \theta' = \theta + 1/3 \) or \( \theta - 1/3 \), and \( R_{f_0}(\theta \pm 1/3) \) lands in \( \partial V(-a(f_0)) \).

It will be convenient to consider the period \( p \) arguments \( \alpha', \beta', \gamma' \in \mathbb{Q}/\mathbb{Z} \) such that the corresponding rays land at \( z_0 \), and the sectors at \( z_0 \) containing \( a_k(f_0) \) and \( -a(f_0) \) are \( Z_{f_0}(z_0, \alpha', \beta') \) and \( Z_{f_0}(z_0, \beta', \gamma') \), maybe not, respectively. Obviously, \( \theta' = \alpha' \) or \( \beta' \) or \( \gamma' \). If \( -a(f_0) \in Z_{f_0}(z_0, \alpha', \beta') \), then \( \beta' \) is right supporting for \( -a(f_0) \). Otherwise, \( -a(f_0) \in Z_{f_0}(z_0, \beta', \gamma') \), and \( \beta' \) is left supporting for \( -a(f_0) \).

Denote by \( w_0 \neq z_0 \) the point in \( \partial V(-a(f_0)) \) such that \( f_0(w_0) = f_0(z_0) \). Denote by \( \alpha'', \beta'', \gamma'' \) the arguments of the rays landing at \( w_0 \) such that \( 3\alpha'' = 3\alpha', 3\beta'' = 3\beta', 3\gamma'' = 3\gamma' \). For \( \theta, \theta' \) as in the statement of the theorem, it follows that \( \theta' = \alpha', \beta' \) or \( \gamma' \), and \( \{\theta + 1/3, \theta - 1/3\} \) is \( \{\alpha', \alpha''\}, \{\beta', \beta''\}, \) or \( \{\gamma', \gamma''\} \), respectively.

Observe that if \( \beta' \) is left supporting for \( -a(f_0) \), then

\[
\gamma', \alpha', \beta', \gamma'', \alpha'', \beta''
\]

are in cyclic order. Otherwise, if \( \beta' \) is right supporting for \( -a(f_0) \), then

\[
\beta', \gamma', \alpha', \beta'', \gamma'', \alpha''
\]

are in cyclic order.
To find a hyperbolic component with \( f_0 \) in its boundary, loosely speaking, we “unpinch” \( \beta' \). To formalize this idea, we use the language of laminations. According to [24], a closed equivalence relation \( \lambda \) in \( \mathbb{R} / \mathbb{Z} \) such that all equivalence classes are finite is called a real lamination of degree 3 if the following holds.

- If \( A \) is an equivalence class, then \( m_3(A) \) is an equivalence class.
- If \( A \) is an equivalence class and \([s, t] \) is a connected component of \( \mathbb{R} / \mathbb{Z} \setminus m_3(A) \) (i.e., \( m_3 \) is consecutive preserving on classes).
- If \( A \) and \( B \) are distinct equivalence classes, then \( A \) is contained in a connected component of \( \mathbb{R} / \mathbb{Z} \setminus B \) (i.e., classes are unlinked).

Given a map \( f \in C(S_p) \) with locally connected Julia set, the equivalence relation \( \lambda(f) \) in \( \mathbb{R} / \mathbb{Z} \) that identifies \( s \) and \( t \) if and only if \( R_f(s) \) and \( R_f(t) \) land at the same point is a real lamination. We say that \( \lambda(f) \) is the real lamination of \( f \).

As \( f_0 \) is geometrically finite, its Julia set is locally connected. Therefore, \( \lambda(f_0) \) is a real lamination. Let \( \lambda \) be the equivalence relation in \( \mathbb{R} / \mathbb{Z} \) that identifies two distinct arguments \( s, t \) if and only if \( s, t \) are \( \lambda(f_0) \)-equivalent and both \( s \) and \( t \) are not in the grand orbit of \( \beta' \). It is not difficult to check that \( \lambda \) is a real lamination. For this real lamination \( \lambda \), we have the following:

**Lemma 6.13.** There exists a hyperbolic pcf polynomial \( f_* \in S_p \) such that \( \lambda = \lambda(f_*) \), \( a_k(f_*) = -a(f_*) \), and the rays with arguments \( \alpha', \beta', \gamma', \alpha'', \beta'', \gamma'' \) land in \( \partial V(a_k(f_*)) \). Moreover, \( \alpha' \) is left supporting, \( \gamma' \) is right supporting, and \( \beta' \) is both left and right supporting for \( V(a_k(f_*)) \).

**Proof.** As every \( \lambda(f_0) \)-class maps bijectively onto its image, the same occurs with \( \lambda \)-classes. In particular, the action of \( m_3 \) on \( (\mathbb{R} / \mathbb{Z}) / \lambda \) has no periodic topological circle with bijective return map. Thus, we may apply [24, Theorem 1, Lemma 6.34], to obtain a polynomial \( f \) such that \( \lambda(f) = \lambda \), every cycle of \( f \) is repelling or superattracting, and every Fatou critical point is eventually periodic. As the Julia set of \( f \) is critical point free, \( f \) is a pcf hyperbolic map with locally connected Julia set naturally homeomorphic to \( (\mathbb{R} / \mathbb{Z}) / \lambda \). In a locally connected Julia set, the landing point of \( R_f(s) \) and \( R_f(t) \) lie in the boundary of the same bounded Fatou component if and only there is no \( \lambda(f) \)-class \( A \) such that \( s, t \) are in different components of \( \mathbb{R} / \mathbb{Z} \setminus A \). Hence, if \( R_{f_0}(s) \) and \( R_{f_0}(t) \) land in the boundary of a bounded Fatou component of \( f_0 \), then the corresponding rays \( R_f(s) \) and \( R_f(t) \) land in the boundary of some bounded Fatou component of \( f \). It follows that \( \alpha', \beta', \gamma', \alpha'', \beta'', \gamma'' \) land in the boundary of a Fatou component \( V_f(\omega) \) that has to be periodic and contain a critical point \( \omega \).

We claim that the period of \( V_f(\omega) \) is \( p \). It will be convenient to introduce the Carathéodory loops \( \gamma : \mathbb{R} / \mathbb{Z} \to J(f) \) and \( \gamma_0 : \mathbb{R} / \mathbb{Z} \to J(f_0) \) that map an argument \( t \) to the landing point of the ray \( R_f(t) \) and \( R_{f_0}(t) \), respectively. Let \( C \subset \mathbb{R} / \mathbb{Z} \) be the set formed by all arguments \( t \) such that \( \gamma_0(t) \) belongs to \( \partial V_{f_0}(-a(f_0)) \) or to \( \partial V_{f_0}(a_k(f_0)) \). From the previous paragraph, \( \gamma(C) \subset \partial V_f(\omega) \). We conclude that \( V_f(\omega) \) has period at most \( p \). To rule out the possibility of a period strictly smaller than \( p \), let \( 0 \leq j < p \) be such that \( \partial V_{f_0}(a_j(f_0)) \) has minimal length. It is sufficient to show that the region between the rays with arguments \( 3^j \alpha' \) and \( 3^j \gamma' \) contains the component \( V_{f_0}(a_j(f_0)) \) but no other component in its cycle. By contradiction, suppose that \( Z_{f_0}(f_0^j(z_0), 3^j \alpha', 3^j \gamma') \) contains \( V_{f_0}(a_n(f_0)) \) for some \( n \neq j \). Choose \( n \) such that \( 3^n \gamma' \), \( 3^n \alpha' \) has minimal length and \( 3^n \gamma', 3^n \alpha' \) [C] \( 3^j \alpha', 3^j \gamma' \). As every sector contains a postcritical point, the region \( Z_{f_0}(f_0^n(z_0), 3^n \gamma', 3^n \alpha') \) must contain a postcritical point, say \( a_m(f_0) \) with \( m \neq n, j \). Therefore, either \( 3^n \gamma', 3^n \gamma' \) or \( 3^m \gamma', 3^m \alpha' \) is contained in \( 3^j \alpha', 3^j \gamma' \),
which contradicts the choice of $j$ and $n$, and it implies that the period of $V_f(\omega)$ is $p$ as claimed.

Recall that $\gamma(C) \subset \partial V_f(\omega)$. If $k = 0$, we conclude that $\omega$ is a double critical point and $f$ is conjugate to a hyperbolic pcf map $f_* \in S_p$ with $-a(f_*) = a_0(f_*)$. If $k \neq 0$, then $V_f(\omega)$ is the image under the $k$th iterate of a critical Fatou component $V_f(\omega')$, and $f$ is conjugate to a map $f_* \in S_p$ such that the labeling of the critical points $\omega$ and $\omega'$ correspond to $-a(f_*)$ and $a(f_*)$.

The fact that the arguments $\alpha', \beta'$ and $\gamma'$ have the supporting properties of the statement is a direct consequence of the definition of $\lambda$.

To finish the proof of Theorem 6.11, let us assume that $\beta'$ is left supporting for $-a(f_0)$. The case in which is right supporting follows along similar lines. Under this assumption, $\vartheta'' = 3p - k \alpha'$ is left supporting for $V(\omega)$, and $\alpha''$ and $\vartheta''$ are in the same component of $\mathbb{R}/\mathbb{Z} \setminus \{\beta', \beta''\}$. It follows that $\{\beta', \beta''\}, \{\vartheta', \vartheta''\}$ is an admissible critical marking for $f_0$.

Let $H$ be the hyperbolic component centered at the map $f_*$ provided by the lemma. Given $f \in H$, denote by $z_{\alpha}(f)$ (resp., $w_{\alpha}(f)$) the landing point of the ray with argument $\alpha$ (resp., $\beta'$, $\alpha''$, $\beta''$). Then, $z_{\alpha}(f)$ is a successor of $z_{\alpha}(f)$ among the $f^p$-fixed points in $\partial V(a_k(f))$. By Proposition 6.2, there exists a unique 0-parameter internal ray $I_f$ of $H$ such that for all $f \neq f_*$ along $I_f$ the dynamical ray $I^{-}_{f, a_k(0)}$ lands at $z_{\alpha}(f)$. The same proposition guarantees that $I^+_{f, a_k(0)}$ lands at $z_{\beta}(f)$. From Proposition 6.6, it follows that the landing point $g$ of this ray $I$ is a parabolic map such that the ray $R_{f_0}(\theta')$ lands at a parabolic point $z_0(g)$, it is right supporting for $V(-a(g))$, and the ray $R_{g}(\theta'')$ lands at $w_0(g) \in \partial V(-a(g))$. Moreover, $R_{g}(\alpha')$ lands at $z_0(g)$, and it is right supporting for $a_k(g)$. It follows that $\{\beta', \beta''\}, \{\vartheta', \vartheta''\}$ is also an admissible critical marking for $g$. Thus, $g = f_0$.

To finish the proof, we check that $I$ is one of the rays $I^\pm$ that bound the sector $S(H, \vartheta')$. Indeed, applying Proposition 6.2, when $\vartheta' = \alpha'$ or $\gamma'$, we conclude that $I = I^-$, otherwise, $\vartheta' = \beta'$, and it follows that $I = I^+$.

6.7 | Prerepelling immigration

**Theorem 6.14.** Let $f_0 \in \partial C(S_p)$ be a pcf map with take-off argument $\theta$. Suppose that $\theta$ is also a period $p$ argument that supports $V(a(f_0))$. Then there exists a type $A$ hyperbolic component $H$ centered at a map $f_*$ such that the following hold:

1. $R_{f_*}(\theta)$ supports $V(-a(f_*))$.
2. the 1/2-internal ray $I$ of $H$ contained in $\text{S}(H, \theta)$ lands at $f_0$.

Observe that, due to Theorem 5.9, a landing point $f_0$ of a parameter ray with a preperiodic ray connection between $-a(f)$ and $a(f)$ fulfills the hypothesis of the previous theorem.

**Proof.** Consider a map $f_0$ as in the statement of Theorem 6.14. For simplicity of the exposition, we assume that the take-off argument $\theta$ is left supporting for $V(a(f_0))$. Note that the rays $R_{f_0}(\theta \pm 1/3)$ land at $-a(f_0) \in \partial V(a(f_0))$. Also, $\theta - 1/3$ is left supporting for $V(a(f_0))$. The map $f_0$ is pcf so is completely determined by a left supporting critical marking [34]. That is, if a pcf map $g \in S_p$ is such that $\{\theta, \theta - 1/3\}$ are left supporting arguments for $V(a(g))$ and $\{\theta + 1/3, \theta - 1/3\}$ land at $-a(g)$, then $g = f_0$. 


Either from Poirier’s work, or from the combinatorics of the multibrot set \( M_3 \) which is the connectedness locus of the family \( z \mapsto z^3 + c \), it follows that there exists \( f_\ast (z) = z^3 + c_\ast \) such that the critical point \( \pm a(f_\ast) = 0 \) has period \( p \) and \( \vartheta \) is supporting for \( V(-a(f_\ast)) \). In fact, one can use the combinatorics of \( f_0 \) to show that if an argument \( t \) has the same itinerary than \( \vartheta \) according to the partition \( \left[ \vartheta, \vartheta + 1/3 \right], \left[ \vartheta + 1/3, \vartheta + 2/3 \right], \left[ \vartheta + 2/3, \vartheta \right] \) of \( \mathbb{R}/\mathbb{Z} \), then \( t = \vartheta \). That is, \( \left( \vartheta, \vartheta + 1/3, \vartheta + 2/3 \right) \) is a left supporting critical marking of a pcf map \( f_\ast \), according to Poirier. Alternatively, using the results in [14], it follows that the multibrot external ray with argument \( 3\vartheta \) lands at the root of a hyperbolic component (of the family \( z \mapsto z^3 + c \)) centered at a pcf map \( f_\ast \) with the required properties.

Let \( H \subset S_p \) be the hyperbolic component centered at \( f_\ast \) and \( I \) its 1/2-internal ray contained in \( S(H, \vartheta) \). Let \( g \) be the landing point of \( I \). In the notation of Subsection 6.3.2, we have \( \vartheta = s(z) \).

From Proposition 6.7, we conclude that \( \{ \vartheta, \vartheta - 1/3 \} \) are supporting arguments for \( +a(g) \) and \( \{ \vartheta + 1/3, \vartheta - 1/3 \} \) land at \( -a(g) \). Therefore, \( g = f_0 \).

### 7 PROOF OF THEOREM 1

Consider an escape component \( U \) with undistinguished kneading word \( \kappa \) (i.e., \( \kappa \neq 1^{p-1}0 \)). Let \( \mu \) denote the maximal return time to \( D_0 \). According to Theorem 4.3, there exists \( f \in U \) with a ray connection \( R_{f_j}(\vartheta') \) between \( -a(f) \) and \( a_k(f) \in D_0 \) such that the return time of \( a_k(f) \) to \( D_0 \) is maximal. Denote by \( R_{U}(\vartheta) \) the parameter ray containing \( f \). Observe that \( \vartheta' = \vartheta + 1/3 \) or \( \vartheta - 1/3 \).

According to Theorems 5.6 and 5.9, the ray \( R_{U}(\vartheta) \) lands at a parabolic or a pcf map \( f_0 \) such that \( \kappa(f_0, \vartheta) = \kappa \). The map \( f_0 \) is pcf only when \( k = 0 \) and \( \vartheta' \) is preperiodic.

Our “immigration” results yield that the map \( f_0 \) is in the boundary of a type \( A \) or \( B \) hyperbolic component \( H \) according to whether \( k = 0 \) or \( k \neq 0 \). Indeed, such an assertion is granted by Theorem 6.11, when \( \vartheta' \) is periodic and \( k \) is arbitrary, and by Theorem 6.14, when \( \vartheta' \) is preperiodic and \( k = 0 \).

If \( k \neq 0 \), we apply Theorem 6.8 to find in \( \partial H \) a parabolic map \( f_1 \) with take-off argument \( \vartheta \) such that \( \kappa' = \kappa(f_1, \vartheta) \) is obtained from \( \kappa(f_0, \vartheta) \) by a type \( B \) move. If \( k = 0 \), we apply Corollary 6.10 to find in \( \partial H \) a parabolic or pcf map \( f_1 \) with take-off argument \( \vartheta = \vartheta + 1/3 \) or \( \vartheta - 1/3 \) such that \( \kappa' = \kappa(f_1, \vartheta) \) is obtained from \( \kappa(f_0, \vartheta) \) by a type \( A \) move.

Applying Theorem 5.8 when \( f_1 \) is a parabolic map and, Theorems 5.1 and 5.9 when \( f_1 \) is a pcf map, we obtain a new escape region \( U' \) such that \( \kappa(U') = \kappa' \) and \( f_1 \in \partial U' \). That is, \( U' \) and \( U \) are in the same connected component of \( S_p \).

Recall that there is only one escape region with the distinguished kneading word \( 1^{p-1}0 \). The theorem then follows from the lemma below which shows that after successively applying at most \( p^2 + p \) times type \( A \) or \( B \) moves, we obtain the distinguished kneading word \( 1^{p-1}0 \).

**Lemma 7.1.** Consider any sequence \( \kappa_1, ..., \kappa_{\ell} \) of kneading words such that \( \kappa_{i+1} \) is obtained from \( \kappa_i \) by a type \( A \) or a type \( B \) move. Then \( \ell \leq p^2 + p \).

**Proof.** Let us introduce two numbers \( b(\kappa) \) and \( w(\kappa) \) associated to a word \( \kappa = t_1 ... t_{p-1}0 \). The weight \( w(\kappa) \) is simply defined as the number of symbols 1 in \( \kappa \). For \( \kappa \neq 0^p \), let \( n \geq 0 \), \( m \geq 1 \) be such that \( \kappa = 0^n1^m0 \), and define \( b(\kappa) = n + m + 1 \). Let \( b(0^p) = 0 \). It follows that \( 0 \leq b(\kappa) \leq p \) and \( 0 \leq w(\kappa) \leq p - 1 \). That is, \( (b(\kappa), w(\kappa)) \in \{0, ..., p\} \times \{0, ..., p - 1\} \). It is not difficult to check that if \( \kappa' \) is obtained via a type \( A \) or \( B \) move from \( \kappa \neq 1^{p-1}0 \) then, in the lexicographic order, \((b(\kappa'), w(\kappa'))\)
is greater than \((b(\chi), w(\chi))\). Moreover, \(1^{p-1}0\) is the unique maximal kneading word with respect to this order. The lemma follows.

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