Asymptotic behaviour of global vortex rings

Daomin Cao\textsuperscript{1,2}, Jie Wan\textsuperscript{3,*}, Guodong Wang\textsuperscript{4} and Weicheng Zhan\textsuperscript{5}

\begin{itemize}
\item\textsuperscript{1} Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing 100190, People’s Republic of China
\item\textsuperscript{2} University of Chinese Academy of Sciences, Beijing 100049, People’s Republic of China
\item\textsuperscript{3} School of Mathematics and Statistics, Beijing Institute of Technology, Beijing 100081, People’s Republic of China
\item\textsuperscript{4} Institute for Advanced Study in Mathematics, Harbin Institute of Technology, Harbin 150001, People’s Republic of China
\item\textsuperscript{5} School of Mathematical Sciences, Xiamen University, Xiamen 361005, People’s Republic of China
\end{itemize}

E-mail: dmcao@amt.ac.cn, wanjie@bit.edu.cn, wangguodong@hit.edu.cn and zhanweicheng@amss.ac.cn

Received 13 April 2021, revised 25 March 2022
Accepted for publication 30 May 2022
Published 17 June 2022

Abstract

In this paper, we are concerned with nonlinear desingularization of steady vortex rings in $\mathbb{R}^3$ with a general nonlinearity $f$. Using the improved vorticity method, we construct a family of steady vortex rings which constitute a desingularization of the classical circular vortex filament in the whole space. The requirements on $f$ are very general, and it may not satisfy the Ambrosetti–Rabinowitz condition. Some qualitative and asymptotic properties are also established.

Keywords: incompressible Euler equations, vortex ring, variational method, desingularization
Mathematics Subject Classification numbers: 76B47 (35Q31).

1. Introduction

The motion of an incompressible steady ideal fluid in $\mathbb{R}^3$ is governed by the following Euler equations

$$ (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla P, $$

(1.1)
\[ \nabla \cdot \mathbf{v} = 0, \]  
\hspace{1cm} (1.2)

where \( \mathbf{v} = (v_1, v_2, v_3) \) is the velocity field and \( P \) is the scalar pressure. Let \( \{ \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z \} \) be the usual cylindrical coordinate frame. Then if the velocity field \( \mathbf{v} \) is axisymmetric, i.e., \( \mathbf{v} \) does not depend on the \( \theta \) coordinate, it can be expressed in the following way

\[ \mathbf{v} = v^r(r, z)\mathbf{e}_r + v^\theta(r, z)\mathbf{e}_\theta + v^z(r, z)\mathbf{e}_z. \]

The component \( v^\theta \) in the \( \mathbf{e}_\theta \) direction is called the swirl velocity. Let \( \omega := \nabla \times \mathbf{v} \) be the corresponding vorticity field. We shall refer to an axisymmetric non-swirling flow (\( v^\theta \equiv 0 \)) as ‘vortex ring’ if there is a toroidal region inside of which \( \omega \neq 0 \), and outside of which \( \omega = 0 \). Note that the conservation of mass equation (1.2) furnishes a Stokes stream function \( \Psi \) such that

\[ \mathbf{v} = \frac{1}{r} \left( \frac{\partial \Psi}{\partial z} \mathbf{e}_r + \frac{\partial \Psi}{\partial r} \mathbf{e}_z \right). \]  
\hspace{1cm} (1.3)

In terms of the Stokes stream function \( \Psi \), the problem can be reduced to a free boundary problem on the half plane \( \Pi = \{ (r, z) | r > 0 \} \) of the form:

\[ \begin{cases} 
L \Psi = 0 & \text{in } \Pi \setminus A, \\
L \Psi = \lambda f(\Psi) & \text{in } A, \\
\Psi(0, z) = -\mu \leq 0, & \text{in } A, \\
\Psi = 0 & \text{on } \partial A, \\
\frac{1}{r} \frac{\partial \Psi}{\partial r} \to -\mathcal{W} \text{ and } \frac{1}{r} \frac{\partial \Psi}{\partial z} \to 0 & \text{as } r^2 + z^2 \to \infty,
\end{cases} \]  
\hspace{1cm} (1.4)

where

\[ L := -\frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2}{\partial z^2}. \]

Here the vorticity function \( f \) and the vortex-strength parameter \( \lambda > 0 \) are prescribed; \( A \) is the \( (a \ priori \ unknown) \) cross-section of the vortex ring; \( \mu \) is the flux constant measuring the flow rate between the \( z \)-axis and \( \partial A \); the constant \( \mathcal{W} > 0 \) is the propagation speed, and the condition (1.8) means that the limit of the velocity field \( \mathbf{v} \) at infinity is \( -\mathcal{W} \mathbf{e}_z \). \( \zeta := L \Psi \) is called the potential vorticity of the flow. For a detailed presentation of this model the reader is referred to [18]. We shall say that \( \Psi \in C^1(\Pi) \cap C^2(\Pi \setminus \partial A) \) is a classical solution of (\( \mathcal{D} \)) if it solves the first two equations in (\( \mathcal{D} \)) almost everywhere. We remark that once the Stokes stream function \( \Psi \) is obtained, one can easily construct the corresponding solutions of the original variables \( (\mathbf{v}, P) \). In fact, let \( F : \mathbb{R} \to \mathbb{R} \) satisfy \( F' = f \). Then once we find the Stokes stream function \( \Psi \), the velocity of the flow is given by (1.3), and the pressure is given by \( P = -F(\Psi) - \frac{1}{2} |\nabla \Psi|^2 \), and the corresponding vorticity field is given by \( \omega = \lambda r f(\Psi) \mathbf{e}_\theta \).

In this paper, we are interested in the vortex desingularization problem of the steady vortex rings in the whole space. We want to find a family of Euler flow, such that the associated vorticities concentrate near a circular filament as the vorticity strength goes to infinity. The study of this kind of solutions can be traced back to Helmholtz [20] and Kelvin [32]. In 1858, Helmholtz [20] found that a vortex ring whose cross-section diameter is sufficiently small...
moves with a constant speed in the symmetric axis direction. In 1894, Hill [21] first constructed an explicit particular translating flow of the Euler equation (called Hill’s spherical vortex) whose support is a ball. Kelvin and Hicks then found a formula to show that if the vortex ring with circulation $\kappa$ has radius $r_*$ and its cross-section $\varepsilon$ is small, then the vortex ring translates approximately with the velocity (see [23, 32])

$$\frac{\kappa}{4\pi r_*} \left( \log \frac{8r_*}{\varepsilon} - \frac{1}{4} \right).$$

(1.9)

More generally, in addition to the circular vortex filaments, the law of motion of the general evolved curve has also been extensively studied. Under the assumption that the vorticity is uniformly distributed around the vortex filament $\Gamma(t) = \gamma(t, s)$ (s being the arc length parameter) with the cross-section radius $\varepsilon$, the vortex filament formally satisfies the following equation

$$\partial_t \gamma = 2c|\ln \varepsilon| (\partial_s \gamma \times \partial_{ss} \gamma),$$

(1.10)

where $c$ corresponds to the circulation of the velocity field on the boundary of sections to the filament. Let $\tau = |\ln \varepsilon| t$. If we denote $b_{\Gamma(t)}$ the binormal unit vectors and $K$ its curvature, (1.10) can also be written as

$$\partial_\tau \gamma = 2c(\partial_s \gamma \times \partial_{ss} \gamma) = 2cKb_{\Gamma(t)}.$$

(1.11)

This equation is also called the binormal curvature flow, which was first discovered by Da Rios [11] and then studied by Levi-Civita [24]. The vortex desingularization problem of (1.10) is also known as the vortex filament conjecture. So far, the vortex filament conjecture is still open, and it is only solved in a few special cases. The circular vortex filament considered in this paper is one of the special cases. For more numerical and theoretical results on the vortex filaments conjecture, we refer to [12, 22, 28] and reference therein.

From a mathematical point of view, many articles are devoted to studying the existence of vortex rings, see, e.g., [2–4, 12, 19, 22, 27, 35]. Fraenkel first proved that one can construct Euler flows such that its vorticity is supported in an arbitrarily small toroidal region (see [16, 17]). Global existence of vortex rings was first established by Fraenkel and Berger [18]. However, the vortex-strength parameter $\lambda$ in [18] arises as a Lagrange multiplier and hence is left undetermined. To overcome this shortcoming, existence of steady vortex rings are also studied by using the mountain pass theorem proposed by Ambrosetti and Rabinowitz [1]; see e.g., [2, 27]. In addition, [25, 26, 30] are some good historical reviews of vortex rings.

The purpose of this paper is to investigate the asymptotic behaviour of steady vortex rings. More precisely, we shall construct a family of solution pairs $(\Psi^{\lambda}, A^{\lambda})$ of $(\mathcal{P})$ under some assumptions on $f$ as $\lambda \to +\infty$, which constitute a desingularization of the classical circular vortex filament in $\mathbb{R}^3$. The class of $f$ we consider here is very general, which includes all $s_0^p$ with $p > 0$. And the Stokes stream function $\Psi^{\lambda}$ will bifurcate from the Green’s function of $\mathcal{L}$ as $\lambda \to +\infty$. This kind of bifurcation phenomenon is called ‘nonlinear desingularization’ (refer to [6]). Note that for special nonlinearities $f$, many results on desingularization have been obtained. In [19], Friedman and Turkington considered desingularization of steady vortex rings when $f$ is a step function. In [31], Tadie studied the asymptotic behaviour by letting the flux constant $\mu$ diverge. In [35], Yang proved the existence and asymptotic behaviour of vortex rings with some given $f$. We mention that their assumptions on $f$ are stronger than ours. One key assumption is that they need $f$ to satisfy the Ambrosetti–Rabinowitz condition
(see (f2) in [35]). Our approach does not need this condition. Moreover, their limiting objects are degenerate vortex rings with vanishing circulation. Our result provides a desingularization of singular vortex filaments with nonvanishing vorticity (see section 2 below). In [34], de Valeriola and Van Schaftingen also studied desingularization of steady vortex rings with \( f(s) = s^p \) for \( p > 1 \). In such a case, our results are similar to theirs (see theorem 1 in [34]). Our work here may be regarded as an extension of [34]. Recently, Cao et al. [10] further investigated desingularization of vortex rings when \( f \) is a step function and generalized the results in [19] to some extent. Besides the results mentioned above, we do not know any desingularization results for steady vortex rings in the whole space. Similar problems confined in other domains can be found in [10, 13, 14, 31, 34].

The paper is organized as follows. In section 2, we state our main result. We will give proof of our main result in section 3.

2. Main results

In this paper we shall use the following notations: Lebesgue measure on \( \mathbb{R}^2 \) is denoted by \( m \), is to be understood as the measure defining any \( L^p \) space, \( W^{1,p} \) space and \( W^{2,p} \) space, except when stated otherwise; \( \nu \) denotes the measure on \( \Pi \) having density \( r \) with respect to \( m \); \( |·| \) denotes the \( \nu \) measure; \( B_\delta(y) \) denotes the open ball in \( \mathbb{R}^2 \) of radius \( \delta \) centred at \( y \); \( \chi_\Omega \) denotes the characteristic function of \( \Omega \subseteq \mathbb{R}^2 \); for any real function \( h \), \( h^+ = \max\{h,0\} \) denotes the positive part of \( h \), and \( \text{supp}(h) \) denotes the support set of \( h \).

Let \( f : \mathbb{R} \to \mathbb{R} \) be a function. We say \( f \) is admissible if it satisfies the following assumptions:

(f1) \( f(s) \equiv 0 \) for \( s \leq 0 \), \( f(s) > 0 \) for \( s > 0 \), and \( f \) is non-decreasing in \( \mathbb{R} \); and one of the following holds

(f2) \( f \) is bounded;

(f3) There exist some numbers \( \delta_0 \in (0,1) \) and \( \delta_1 \geq 0 \) such that

\[
F(s) := \int_0^s f(t)dt \leq \delta_0 f(s)s + \delta_1 f(s), \quad \forall \ s \geq 0.
\]

In addition, for all \( \tau > 0 \),

\[
\lim_{s \to +\infty} (f(s)e^{-\tau s}) = 0.
\]

It should be noted that assumptions (f1) and (f3) imply \( \lim_{s \to +\infty} f(s) = +\infty \) (see [27]). Indeed, by (f1) and (f3) we have \( F(s) \leq \delta_0 F(s)s + \delta_1 F(s) \) for almost every \( s > 0 \), which implies that

\[
\left(\ln F'(s) = \frac{F'(s)}{F(s)} \geq \frac{1}{\delta_0 s + \delta_1}\right).
\]

So for any \( 0 < s_0 < s_1 \), we get

\[
\ln F(s_1) - \ln F(s_0) \geq \int_{s_0}^{s_1} \frac{1}{\delta_0 s + \delta_1}ds = \frac{1}{\delta_0} \ln \frac{\delta_0 s_1 + \delta_1}{\delta_0 s_0 + \delta_1}.
\]
that is,

\[ F(s_1) \geq F(s_0) \left( \frac{\delta_0 s_1 + \delta_1}{\delta_0 s_0 + \delta_1} \right) \frac{1}{\delta_0}. \]

Using (f2) again we have

\[ f(s_1) \geq \frac{1}{\delta_0 s_1 + \delta_1} F(s_1) \geq F(s_0) \left( \frac{\delta_0 s_0 + \delta_1}{\delta_0 s_0 + \delta_1} \right) ^{-\frac{1}{\delta_0}} \left( \delta_0 s_0 + \delta_1 \right) ^{\frac{1}{\delta_0}}. \]

This implies that \( f(s) \) grows without bound as \( s \) goes to infinity. Many unbounded functions satisfy the assumptions (f1) and (f3), such as \( f(s) = s^p \) for \( p \in (0, +\infty) \).

We note that for steady vortex rings, the vorticity function \( f \) can be very general. Hill’s spherical vortex is an exact solution of \( (\mathcal{P}) \) with Heaviside function \( f \). The case when \( f \) is Heaviside function corresponds to a steady vortex ring whose potential vorticity is constant throughout the core. We refer the interested reader to [2, 16–19]. The asymptotic behaviour of vortex rings with \( f(s) = s^p, p > 1 \) was considered in [34]. The vorticity function considered in [27, 35] satisfies (f1) and (f3) with \( \delta_0 \in (0, \frac{1}{2}) \) and \( \delta_1 = 0 \). We would like to point out that all the nonlinearities in these literature satisfy our assumptions.

Our main result in this paper is as follows.

**Theorem 2.1.** Suppose that \( f \) is admissible. Then for every \( W > 0, \kappa > 0 \) and all sufficiently large \( \lambda > 0 \), there exists a classical solution \( \Psi^\lambda \in C^1(\Pi) \cap C^2(\Pi \setminus \partial A^\lambda) \) of \( (\mathcal{P}) \) with

\( a \) \( A^\lambda = \{ (r, z) \in \Pi | \Psi^\lambda(r, z) > 0 \} \), \( \mu^\lambda = \frac{\kappa^2}{4\pi W} \log \lambda + o(\log \lambda) \) and \( \mu^\lambda = \frac{\kappa}{4\pi} \log \lambda \).

\( b \) Let \( \omega^\lambda = \lambda r f(\Psi^\lambda) \) be the azimuthal vorticity, then \( \int_{\Pi} \omega^\lambda dm = \kappa \) and \( \text{supp}(\omega^\lambda) = A^\lambda \).

\( c \) There exist some \( r_0, R_0 > 0 \) not depending on \( \lambda \) such that

\[ r_0 \lambda^{-1/2} \leq \text{diam}(A^\lambda) \leq R_0 \lambda^{-1/2}. \]

Moreover, it holds

\[ \lim_{\lambda \to +\infty} \text{dist} \left( A^\lambda, \left( \frac{\kappa}{4\pi W}, 0 \right) \right) = 0. \]

\( d \) As \( r \to 0 \), one has

\[ \frac{1}{r} \frac{\partial \Psi^\lambda}{\partial z} \to 0 \quad \text{and} \quad \frac{1}{r} \frac{\partial \Psi^\lambda}{\partial r} \text{ approaches a finite limit.} \]

**Remark 2.2.** One can see that our result is consistent with this Kelvin-Hicks formula (1.9). Indeed, it follows from (c) in theorem 2.1 that the diameter of cross-section \( A^\lambda \) is of order \( \lambda^{-\frac{1}{2}} \). Taking \( r_* = \kappa/4\pi W \) into (1.9), the translating speed of the vortex ring should be of the order of

\[ \frac{\kappa}{4\pi r_*} \log \lambda \leq -W \log \lambda, \]

which exactly coincides with (a) in theorem 2.1.
Remark 2.3. With these results in hand, one may expect to further study the asymptotic shape of the vortex core, see, e.g., [19, 33]. For the regularity of the free boundary, we refer to [9] for more discussion. Moreover, using the method in this paper, it is possible to get desingularization of vortex rings in other kinds of axisymmetric domains, such as infinite cylinders and exterior domains of a ball, see [10, 34] for instance.

Let us give some comments on the main result. Our strategy for the proof of theorem 2.1 is as follows. We first consider the case of $f$ being a bounded non-decreasing function. To this end, we transform the problem into a variational problem for the potential vorticity $\zeta := \omega^\theta / r$, the solutions of which define the desired steady vortex rings. This method is called the vorticity method, which is actually a dual variational principle (see, e.g., [5, 10]). Another method to study this problem is called the stream-function method, namely, finding a solution of $P$ directly (see, e.g., [2, 18, 27, 34, 35]). In the study of the existence and asymptotic behaviour of the vortex rings, our method is motivated by the work of Turkington [33] (see also [19]). The key idea is to estimate the order of energy as optimally as possible. We will show that in order to maximize the energy, these solutions must be concentrated. As for the case of unbounded nonlinearity $f$, based on the former analysis, we can use the truncation technique and accurate estimates of the stream function $\psi^{c, p}$ to get the result.

3. Proof of theorem 2.1

As in [3, 10], we define the inverse of $L$ as follows.

Definition 3.1. The Hilbert space $\mathcal{H}(\Pi)$ is the completion of $C^\infty_0(\Pi)$ with the scalar products

$$\langle u, v \rangle_{\mathcal{H}} = \int_{\Pi} \frac{1}{r^2} \nabla u \cdot \nabla v \, d\nu.$$  

We define inverse $K$ for $L$ in the weak solutions sense,

$$\langle Ku, v \rangle_{\mathcal{H}} = \int_{\Pi} uv \, d\nu \quad \text{for all } v \in \mathcal{H}(\Pi), \quad \text{when } u \in L^{10/7}(\Pi, r^3 \, dr \, dz).$$  

(3.1)

By the Riesz representation theorem, one can prove that for any $u \in L^{10/7}(\Pi, r^3 \, dr \, dz)$, $Ku \in \mathcal{H}(\Pi)$ is well-defined and $K$ is a bounded linear operator from $L^{10/7}(\Pi, r^3 \, dr \, dz)$ to $\mathcal{H}(\Pi)$, see lemmas 2.1 and 2.3 in [3] for instance.

3.1. Bounded case

In this subsection, we deal with the case when $f$ satisfies $(f_1)$ and $(f_2)$. Since $F$ is a convex function, we can define the conjugate function to $F$ as follows (see [29]):

$$G(s) = \sup_{s' \in \mathbb{R}} \left[ s s' - F(s') \right].$$  

It is easy to see that $G$ is a convex function. We denote by $\partial G(s)$ the subgradient of $G$ at $s$.

3.1.1. Variational problem. Let $\kappa > 0$, $W > 0$ be fixed and $0 < \varepsilon < 1$ be a parameter. Let $d = \kappa / (4\pi W) + 1$, $d' = \kappa / (8\pi W)$ and $D = \{(r, z) \in \Pi | d' \leq r \leq d, -1 \leq z \leq 1 \}$. Define

$$A := \left\{ \zeta \in L^\infty(\Pi) | \int_{\Pi} \zeta \, d\nu \leq \kappa, \supp(\zeta) \subseteq D \right\}.$$  

3685
Consider the maximization of the following functional over $\mathcal{A}$

$$E_{\varepsilon}(\zeta) = \frac{1}{2} \int_{D} \zeta K \zeta \, d\nu - \frac{W}{2} \log \frac{1}{\varepsilon} \int_{D} r^2 \zeta \, d\nu - \frac{1}{\varepsilon^2} \int_{D} G(\varepsilon^2 \zeta) \, d\nu. \quad (3.2)$$

For any $\zeta \in \mathcal{A}$, by the classical estimates for elliptic equations, we have $K \zeta \in W^{2,p}_{\text{loc}}(\Pi)$ $\cap C^{1,\alpha}_{\text{loc}}(\Pi)$ for any $1 < p < +\infty$, $0 < \alpha < 1$ (see [8, 15]). For given $\zeta$ we will use $\zeta^*$ to denote its Steiner symmetrization with respect to the line $z = 0$ in $\Pi$ (see appendix I of [18]).

**Lemma 3.2.** There exists $\zeta^* = (\zeta^*)^* \in \mathcal{A}$ such that

$$E_{\varepsilon}(\zeta) = \max_{\zeta \in \mathcal{A}} E_{\varepsilon}(\zeta^*) < +\infty.$$

Moreover,

$$0 \leq \zeta^* \leq \frac{\sup_{\mathbb{R}} f}{\varepsilon^2}, \quad \text{a.e. in } \Pi.$$

**Proof.** By the choice of $\mathcal{A}$, it is not hard to get that $E_{\varepsilon}$ is bounded from above. Notice that $G(s) = +\infty$ if $s > \sup_{\mathbb{R}} f$. So for any $\zeta \in \mathcal{A}$ which is not bounded almost everywhere by $\sup_{\mathbb{R}} f/\varepsilon^2$, we have $E_{\varepsilon}(\zeta) = -\infty$. Therefore we may take a maximizing sequence $\{\zeta_j\} \subset \mathcal{A}$ such that as $j \to +\infty$

$$E_{\varepsilon}(\zeta_j) \to \sup\left\{ E_{\varepsilon}(\zeta^*) : \zeta^* \in \mathcal{A} \right\}.$$

and $\zeta_j \to \zeta$ in $L^1(\Pi, \nu)$ weak topology, $L^\infty(\Pi, \nu)$ star weak topology and $0 \leq \zeta_j \leq \frac{\sup_{\mathbb{R}} f}{\varepsilon^2}$. It is easily checked that $\zeta \in \mathcal{A}$ and $0 \leq \zeta^* \leq \frac{\sup_{\mathbb{R}} f}{\varepsilon^2}$. Using the standard arguments (see, e.g., lemma 2 in [8]), we may assume that $\zeta_j = (\zeta_j)^*$, and hence $\zeta = \zeta^*$. By lemma 2.12 of [3], we know that the functional $E(\zeta) := \int_{D} \zeta K \zeta \, d\nu$ is sequentially continuous relative to $L'(D, \nu)$ weak topology for any $\frac{d}{d+1} < s < 2$. Since $\zeta_j \to \zeta$ in $L'(D, \nu)$ weak topology for any $\frac{d}{d+1} < s < 2$, we have

$$\lim_{j \to +\infty} \int_{D} \zeta_j K \zeta_j \, d\nu = \int_{D} \zeta K \zeta \, d\nu.$$

On the other hand, we have the lower semicontinuity of the rest of terms, namely,

$$\liminf_{j \to +\infty} \int_{D} r^2 \zeta_j \, d\nu \geq \int_{D} r^2 \zeta \, d\nu,$$

$$\liminf_{j \to +\infty} \int_{D} G(\varepsilon^2 \zeta_j) \, d\nu \geq \int_{D} G(\varepsilon^2 \zeta) \, d\nu,$$

where we have used the convexity of $G$. Consequently, we may conclude that $E_{\varepsilon}(\zeta) = \lim_{j \to +\infty} E_{\varepsilon}(\zeta_j) = \sup E_{\varepsilon}$, with $\zeta \in \mathcal{A}$. The proof is thus complete. \qed
We now turn to establish the Euler–Lagrange equations corresponding to the maximization problem. Hereafter we assume that
\[
\sup_{\mathcal{E}^2} f(D) > \kappa. \quad (3.3)
\]

**Lemma 3.3.** There holds
\[
\zeta = \frac{1}{\varepsilon} f \left( \mathcal{K} \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon} \right) \chi_D, \text{ a.e. in } \Pi, \quad (3.4)
\]
for some Lagrange multiplier $\mu \geq 0$. Moreover, we have $\int_D \zeta^2 \, d\nu = \kappa$ provided that $\zeta \neq 0$ and every number $\mu$ satisfying (3.4) is positive.

**Proof.** Let us introduce the function $\tau : \mathbb{R} \to \mathbb{R}$ defined by
\[
\tau(s) = \frac{1}{\varepsilon^2} \int_D \left( \mathcal{K} \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon} - s \right) \, d\nu, \quad s \in \mathbb{R}.
\]
It is clear that $\tau$ is non-increasing due to the monotonicity assumption of $f$. In view of (3.3), we have $\lim_{s \to -\infty} \tau(s) > \kappa$ and $\lim_{s \to +\infty} \tau(s) = 0$. Notice that $\zeta$ is Steiner symmetric, we check that $K \zeta$ is strictly symmetric decreasing with respect to $\zeta$. So every level set of $K \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon}$ has measure zero. Since the set of discontinuities of $f$ is at most countable, we check that $\tau$ is a continuous function. Therefore there exists a $\mu \in \mathbb{R}$ such that $\tau(\mu) = \int_D \zeta^2 \, d\nu$.

If $\mu < 0$, then
\[
\bar{\zeta} = \frac{1}{\varepsilon} f \left( \mathcal{K} \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon} \right) \chi_D \in \mathcal{A}.
\]
Let $\zeta = (\zeta + \bar{\zeta})/2 \in \mathcal{A}$. Using the convexity of $G$, we derive from $E_\varepsilon(\zeta) \geq E_\varepsilon(\zeta)$:
\[
\frac{1}{\varepsilon^2} \int_D G(\varepsilon^2 \bar{\zeta}) \, d\nu - \frac{1}{\varepsilon^2} \int_D G(\varepsilon^2 \zeta) \, d\nu 
\geq \int_D \left( \mathcal{K} \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon} \right) (\bar{\zeta} - \zeta) \, d\nu + \frac{1}{4} \int_D (\bar{\zeta} - \zeta) K(\bar{\zeta} - \zeta) \, d\nu. \quad (3.5)
\]
Note that $\partial G$ and $f$ are inverse graphs (see [29]). By the definition of $\bar{\zeta}$, we have
\[
K \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon} \in \partial G(\varepsilon^2 \bar{\zeta}), \quad \text{in } D.
\]
It follows from the convexity of $G$ that
\[
\frac{1}{\varepsilon^2} \int_D G(\varepsilon^2 \bar{\zeta}) \, d\nu - \frac{1}{\varepsilon^2} \int_D G(\varepsilon^2 \zeta) \, d\nu 
\geq \int_D \left( \mathcal{K} \zeta - \frac{W^2}{2} \log \frac{1}{\varepsilon} \right) (\bar{\zeta} - \zeta) \, d\nu. \quad (3.6)
\]
Combining (3.5) and (3.6), we get
\[ \int_D (\xi^\varepsilon - \zeta) K(\xi^\varepsilon - \zeta) d\nu \leq 0, \]
whence \( \xi^\varepsilon = \zeta \). So (3.4) holds with \( \mu^\varepsilon = 0 \).

If \( \mu^\varepsilon \geq 0 \), then we set
\[ \tilde{\xi}^\varepsilon = \frac{1}{\varepsilon^2} f \left( K \xi^\varepsilon - \frac{W r^2}{2} \log \frac{1}{\varepsilon} - \mu^\varepsilon \right) \chi_D \in \mathcal{A}. \]
Using a similar argument with \( \bar{\xi}^\varepsilon \) replaced by \( \tilde{\xi}^\varepsilon \), we now conclude that \( \xi^\varepsilon = \tilde{\xi}^\varepsilon \). So we get (3.4).

It remains to show that \( \int_D \xi^\varepsilon d\nu = \kappa \) if \( \xi^\varepsilon \neq 0 \) and every number \( \mu^\varepsilon \) satisfying (3.4) is positive. We argue by contradiction. Suppose \( \int_D \xi^\varepsilon d\nu < \kappa \), then we must have \( \tau(0) > \kappa \). Otherwise, as argued above, we see that (3.13) holds with \( \mu^\varepsilon = 0 \). So we can find a number \( \hat{\mu}^\varepsilon > 0 \) such that
\[ \hat{\xi}^\varepsilon = \frac{1}{\varepsilon^2} f \left( K \hat{\xi}^\varepsilon - \frac{W r^2}{2} \log \frac{1}{\varepsilon} - \hat{\mu}^\varepsilon \right) \chi_D \in \mathcal{A} \]
and \( \int_D \hat{\xi}^\varepsilon d\nu > \int_D \xi^\varepsilon d\nu \).

Using a similar argument, we can now obtain
\[ \int_D (\hat{\xi}^\varepsilon - \xi^\varepsilon) K(\hat{\xi}^\varepsilon - \xi^\varepsilon) d\nu < 0, \]
which is impossible. The proof of lemma 3.3 is thus complete.

3.1.2. Limiting behaviour of \( \xi^\varepsilon \). Let \( K(r, z, r', z') \) be the Green’s function of \( L \) in \( \Pi \), with respect to zero Dirichlet data and measure \( rdrdz \). Then
\[ K(r, z, r', z') = \frac{r r'}{4\pi} \int_{-\pi}^{\pi} \cos \theta' d\theta' \int_{r^2 + (z - z')^2}^{+\infty} \frac{1}{\sqrt{r^2 + (z - z')^2 - 2rr' \cos \theta'}}. \]
One can easily show that (see [3])
\[ K\zeta(r, z) = \int_D K(r, z, r', z') \zeta(r', z') r' dr' dz', \quad \forall \zeta \in \mathcal{A}. \quad (3.7) \]
Let us introduce
\[ \sigma = [(r - r')^2 + (z - z')^2]^{1/4}/(4rr')^{1/4}. \quad (3.8) \]
We have the following estimates about \( K \), see [10].

**Lemma 3.4. ([10], lemmas 3.1 and 3.2).** There holds
\[ 0 < K(r, z, r', z') \leq \frac{(rr')^{1/4}}{4\pi} \sinh^{-1}\left( \frac{1}{\sigma} \right), \quad \forall \sigma > 0. \quad (3.9) \]
Moreover, there exists a continuous function $\bar{h} \in L^\infty(\Omega \times \Omega)$ such that
\[
K(r, z, r', z') = \frac{\sqrt{rr'}}{2\pi} \log \frac{1}{\sigma} + \frac{\sqrt{rr'}}{2\pi} \log(1 + \sqrt{\sigma^2 + 1}) + \bar{h}(r, z, r', z') \sqrt{rr'}, \quad \text{in } \Omega \times \Omega.
\] (3.10)

To prove the main result, the first step is to give a lower bound of energy $E_\epsilon(\zeta^\epsilon)$. We adhere to the convention of denoting by $C, C_1, C_2, \ldots$ positive constants independent of $\epsilon$, whose values may change from line to line.

**Lemma 3.5.** For any $a \in (d', d)$, there exists $C > 0$ such that for all $\epsilon$ sufficiently small, there holds
\[
E_\epsilon(\zeta^\epsilon) \geq \left(\frac{\alpha \kappa^2}{4\pi} - \frac{\kappa Wa^2}{2}\right) \log \frac{1}{\epsilon} - C.
\]

**Proof.** Choose a test function $\tilde{\zeta}^\epsilon \in A$ defined by $\tilde{\zeta}^\epsilon = \frac{\eta_1}{4\pi} \chi_{D \setminus \Omega_{\sqrt{\epsilon}}(0, 0)}$. It is easy to see that $G$ is bounded on $[0, f(1)/2]$. Since $\zeta^\epsilon$ is a maximizer, we have $E_\epsilon(\zeta^\epsilon) \geq E_\epsilon(\tilde{\zeta}^\epsilon)$. By lemma 3.4, we obtain
\[
E_\epsilon(\tilde{\zeta}^\epsilon) = \frac{1}{2} \int_D \int_D \tilde{\zeta}^\epsilon(r, z) K(r, z, r', z') \tilde{\zeta}^\epsilon(r', z') r' \, dr' \, dz' \, d
\]
\[
- \frac{W}{2} \log \frac{1}{\epsilon} \int_D r \, \tilde{\zeta}^\epsilon \, d\nu - \frac{1}{\epsilon} \int_D G(\epsilon \tilde{\zeta}^\epsilon) \, d\nu \geq \frac{a + O(\epsilon)}{4\pi} \int_D \int_D \log \frac{1}{(r - r')^2 + (z - z')^2} \, d\tilde{\zeta}^\epsilon(r, z) \tilde{\zeta}^\epsilon(r', z') r' \, dr' \, dz' \, dr \, dz
\]
\[
- \frac{\kappa W [a + O(\epsilon)]^2}{2} \log \frac{1}{\epsilon} - C_1 \geq \left(\frac{\alpha \kappa^2}{4\pi} - \frac{\kappa Wa^2}{2}\right) \log \frac{1}{\epsilon} - C_2.
\]

We therefore complete the proof. \(\square\)

**Remark 3.6.** It is not hard to check that $\frac{\alpha \kappa^2}{4\pi} - \frac{\kappa Wa^2}{2}$, the coefficient of $\log \frac{1}{\epsilon}$, is a positive function of $a \in (0, \frac{\kappa}{4\pi W})$ and takes the unique maximum at $a = \frac{\kappa}{4\pi W}$. Indeed, in the following lemmas we will prove that support sets of maximizers $\zeta^\epsilon$ tend to $(\frac{\kappa}{4\pi W}, 0)$ as $\epsilon \to 0$.

We now turn to estimate the Lagrange multiplier $\mu^\epsilon$. Set
\[
\psi^\epsilon = K \zeta^\epsilon - \frac{W a^2}{2} \log \frac{1}{\epsilon} - \mu^\epsilon.
\] (3.11)

**Lemma 3.7.** We have
\[
\mu^\epsilon \geq 2\kappa^{-1} E_\epsilon(\zeta^\epsilon) + \kappa^{-1} \frac{W}{2} \log \frac{1}{\epsilon} \int_{\Pi} r^2 \zeta^\epsilon \, d\nu - C.
\]
Proof. We have

\[
2E_\varepsilon(\zeta^\varepsilon) = \int_\Pi \zeta^\varepsilon K \zeta^\varepsilon \, d\nu - W \log \left(\frac{1}{\varepsilon} \int_\Pi r^2 \zeta^\varepsilon \, d\nu\right) - \frac{2}{\varepsilon^2} \int_\Pi G(\varepsilon^2 \zeta^\varepsilon) \, d\nu
\]

\[
= \int_\Pi \zeta^\varepsilon \psi^\varepsilon \, d\nu - \frac{W}{2} \log \left(\frac{1}{\varepsilon} \int_\Pi r^2 \zeta^\varepsilon \, d\nu\right) - \frac{2}{\varepsilon^2} \int_\Pi G(\varepsilon^2 \zeta^\varepsilon) \, d\nu + \kappa \mu^\varepsilon. \tag{3.12}
\]

Let \(\psi^\varepsilon_+ \in H(\Pi)\) be a test function. By the definition of \(K\) (see (3.1)), one has

\[
\int_\Pi \frac{1}{r^2} |\nabla \psi^\varepsilon_+|^2 \, d\nu = \int_\Pi \zeta^\varepsilon \psi^\varepsilon \, d\nu. \tag{3.13}
\]

Set \(D_\varepsilon^1 = \{(r, z) \in D | \psi^\varepsilon(r, z) \geq 1\}\). Then we have \(|D_\varepsilon^1| \leq C \varepsilon^2\) since \(\zeta^\varepsilon \geq f(1)/\varepsilon^2\) almost everywhere on \(D_\varepsilon^1\). By Hölder’s inequality and Sobolev embedding

\[
\int_D \zeta^\varepsilon \psi^\varepsilon \, d\nu \leq \int_\Pi \zeta^\varepsilon (\psi^\varepsilon - 1)_+ \, d\nu + \kappa
\]

\[
\leq C \varepsilon^{-2} |D_\varepsilon^1|^1/2 \left(\int_D (\psi^\varepsilon - 1)_+^2 \, d\nu\right)^{1/2} + \kappa \tag{3.14}
\]

\[
\leq C \varepsilon^{-2} |D_\varepsilon^1|^1/2 \left(\int_D (\nabla (\psi^\varepsilon - 1)_+) \, d\nu + \int_D (\psi^\varepsilon - 1)_+ \, d\nu\right) + \kappa
\]

\[
\leq C \left(\int_\Pi \frac{1}{r^2} |\nabla \psi^\varepsilon_+|^2 \, d\nu\right)^{1/2} + C \varepsilon \int_D \zeta^\varepsilon \psi^\varepsilon \, d\nu + C.
\]

Combining (3.13) and (3.14), we conclude that \(\int_\Pi \zeta^\varepsilon \psi^\varepsilon \, d\nu\) is uniformly bounded with respect to \(\varepsilon\). On the other hand, by convexity we have

\[
G(\varepsilon^2 \zeta^\varepsilon) + F(\psi^\varepsilon) = \varepsilon^2 \zeta^\varepsilon \psi^\varepsilon, \quad \text{a.e on } D.
\]

Hence

\[
\frac{1}{\varepsilon^3} \int_\Pi G(\varepsilon^2 \zeta^\varepsilon) \, d\nu = \int_D \zeta^\varepsilon \psi^\varepsilon \, d\nu - \frac{1}{\varepsilon^3} \int_D F(\psi^\varepsilon) \, d\nu \leq 2 \int_D \zeta^\varepsilon \psi^\varepsilon \, d\nu \leq C.
\]

Here we have used the profile of \(\zeta^\varepsilon\) in lemma 3.3 and \(F(s) \leq sf(s)\). Now the desired result clearly follows from (3.12). \(\square\)

Combining lemma 3.7 with lemma 3.3, one can immediately prove that the circulation of the maximizer \(\zeta^\varepsilon\) is \(\kappa\). Now we analyse the limiting behaviour of the maximizer \(\zeta^\varepsilon\). To this end, we repeat the arguments in [10] to obtain the following asymptotics of \(\zeta^\varepsilon\). Set \(\Lambda = \sup_{\varepsilon} f\).
Corollary 3.8. If $\varepsilon$ is small enough, then $\int \Pi \zeta^e d\nu = \kappa$.

Proof. By lemma 3.5, one has

$$
\mu^e \geq 2\kappa^{-1} E_\varepsilon(\zeta^e) + \frac{\kappa^{-1} W}{2} \log \frac{1}{\varepsilon} \int \Pi r^2 \zeta^e d\nu - C,
$$

for any $a \in (d', d)$, so we have $\mu^e > 0$ as $\varepsilon$ sufficiently small. Since $\zeta^e \neq 0$, combining this with lemma 3.3, we get $\int \Pi \zeta^e d\nu = \kappa$. The proof is thus completed.

First we introduce the function $\Gamma_1$ as follows

$$
\Gamma_1(t) = \frac{\kappa t}{2\pi} - W t^2, \quad t \in [d', d].
$$

Set $r_* = \frac{\kappa}{4\pi W}$. It is not hard to check that $\Gamma_1(r_*) = \max_{t \in [d', d]} \Gamma_1(t)$. Set

$$
A_\varepsilon = \inf \{ r | (r, 0) \in \text{supp}(\zeta^e) \},
$$

$$
B_\varepsilon = \sup \{ r | (r, 0) \in \text{supp}(\zeta^e) \}.
$$

Then it is not hard to see that $A_\varepsilon$ and $B_\varepsilon$ describe the lower bound and upper bound of the distance between the origin and $\text{supp}(\zeta^e)$, respectively. Next, we show that in order to maximize the energy $E_\varepsilon$, the support sets of the maximizers $\zeta^e$ constructed above must be concentrated as $\varepsilon$ tends to zero. We reach our goal by several steps as follows.

Lemma 3.9. $\lim_{\varepsilon \to 0^+} A_\varepsilon = r_*.$

Proof. Let $\sigma$ be defined by (3.8) and $\gamma \in (0, 1)$. By lemma 3.3, for any $(r_\varepsilon, z_\varepsilon) \in \text{supp}(\zeta^e)$, we have

$$
K \zeta^e(r_\varepsilon, z_\varepsilon) - \frac{W(r_\varepsilon)^2}{2} \log \frac{1}{\varepsilon} \geq \mu^e.
$$

By the definition of $K$ (see (3.7)), one has

$$
K \zeta^e(r_\varepsilon, z_\varepsilon) = \int_D K(r_\varepsilon, z_\varepsilon, r', z') \zeta^e(r', z') r' dr' dz'
$$

$$
= \left( \int_{D \cap \{ \sigma > \varepsilon^2 \}} + \int_{D \cap \{ \sigma \leq \varepsilon^2 \}} \right) K(r_\varepsilon, z_\varepsilon, r', z') \zeta^e(r', z') r' dr' dz'
$$

$$
:= A_1 + A_2.
$$
On the one hand, by (3.9) in Lemma 3.4 we obtain

\[ A_1 = \int_{D \cap \{ \sigma < \epsilon \}} K(r, z, r', z') \zeta'(r', z') dr' dz' \]

\[ \leq \frac{(r_2)^{1/2}}{2\pi} \sinh^{-1}\left(\frac{1}{\epsilon}\right) \int_{D \cap \{ \sigma < \epsilon \}} \zeta'(r', z') r'^{3/2} dr' dz' \tag{3.19} \]

\[ \leq \frac{d}{2\pi} \sinh^{-1}\left(\frac{1}{\epsilon}\right) \int_{D \cap \{ \sigma < \epsilon \}} \zeta'(r', z') dr' dz' \leq \frac{kd}{2\pi} \sinh^{-1}\left(\frac{1}{\epsilon}\right). \]

On the other hand, by (3.8) we have \( D \cap \{ \sigma \geq \epsilon \} \subseteq B_{2d \epsilon} \cap ((r, z) \setminus (r_2, z_2)). \) Indeed, for any \((r', z') \in D \cap \{ \sigma \geq \epsilon \},\) we have \( \sigma = [(r_2 - r')^2 + (z_2 - z')^2]^{1/2} < \epsilon', \) which means that \([(r_2 - r')^2 + (z_2 - z')^2]^{1/2} \leq (4r_2 r' \epsilon')^{1/2} \leq 2d \epsilon'. \) Hence using (3.10) in Lemma 3.4, we get

\[ A_2 = \int_{D \cap \{ \sigma \leq \epsilon \}} K(r, z, r', z') \zeta'(r', z') dr' dz' \leq \frac{(r_2)^2 + C \epsilon'}{2\pi} \int_{D \cap \{ \sigma \leq \epsilon \}} \zeta'(r', z') dr' dz' \]

\[ \leq \frac{(r_2)^2 + C \epsilon'}{2\pi} \log \frac{1}{\epsilon} \int_{D \cap \{ \sigma \leq \epsilon \}} \zeta'(r', z') dr' dz' + C. \tag{3.20} \]

Notice that \( \log \frac{1}{\epsilon} \) is a radially symmetric monotone decreasing function, and \( \zeta'(r', z') \leq \frac{\Delta}{\epsilon}. \)

We choose a radially symmetric step function \( \zeta'(r', z') = \frac{\Delta}{\epsilon} \chi_{B_{r_2} \cap ((r, z))} \) such that \( \int_{D \cap \{ \sigma \leq \epsilon \}} \zeta'(r', z') dr' dz' = \int_{D \cap \{ \sigma \leq \epsilon \}} \zeta'(r', z') dr' dz'. \) Then by the rearrangement inequality and direct calculations we have

\[ A_2 \leq \frac{(r_2)^2 + C \epsilon'}{2\pi} \log \frac{1}{\epsilon} \int_{D \cap \{ \sigma \leq \epsilon \}} \zeta'(r', z') dr' dz' + C. \]

Since \( \text{supp}(\zeta') \subseteq D, \) we have \( r \geq d'. \) Using \( D \cap \{ \sigma \leq \epsilon \} \subseteq B_{2d \epsilon} \cap ((r, z)) \) again we obtain

\[ A_2 \leq \frac{(r_2)^2}{2\pi} \log \frac{1}{\epsilon} \int_{D \cap \{ \sigma \leq \epsilon \}} \zeta'(r', z') dr' dz' + C. \tag{3.21} \]

Therefore by (3.15), (3.19) and (3.21), we conclude that for any \( a \in (d', d) \) there holds

\[ \frac{r_2}{2\pi} \log \frac{1}{\epsilon} \int_{B_{2d \epsilon} \cap ((r, z))} \zeta' d\nu + \frac{kd}{4\pi} \sinh^{-1}\left(\frac{1}{\epsilon}\right) - \frac{W(r_2)^2}{2} \log \frac{1}{\epsilon} \geq \left(\frac{kd}{2\pi} - Wa^2\right) \log \frac{1}{\epsilon} + \frac{W}{2\epsilon} \log \frac{1}{\epsilon} \int_D \zeta' r'^2 d\nu - C. \]
Divide both sides of the above inequality by \( \log \frac{1}{\varepsilon} \), we obtain
\[
\Gamma_1(r_\varepsilon) \geq \frac{r_\varepsilon}{2\pi} \int_{B_{2r_\varepsilon}(0,0)} \zeta^c \, d\nu - W(r_\varepsilon)^2 \geq \Gamma_1(a) + \frac{W}{2\kappa} \times \left\{ \int_D \zeta^c r^2 \, d\nu - \kappa (r_\varepsilon)^2 \right\} - \frac{\kappa d}{4\pi} \sinh^{-1} \left( \frac{1}{\varepsilon} \right) / \log \frac{1}{\varepsilon} - C / \log \frac{1}{\varepsilon}. \tag{3.22}
\]

Note that
\[
\int_D \zeta^c r^2 \, d\nu \geq \kappa (r_\varepsilon)^2.
\]
Taking \((r_\varepsilon, z_\varepsilon) = (A_\varepsilon, 0)\) and letting \(\varepsilon\) tend to \(0^+\), we deduce from (3.22) that
\[
\liminf_{\varepsilon \to 0^+} \Gamma_1(A_\varepsilon) \geq \Gamma_1(a) - \gamma \kappa d / (2\pi).
\tag{3.23}
\]
Hence we get the desired result by letting \(a \to r_\star\) and \(\gamma \to 0\).

Next, we estimate the impulse of the flow.

**Lemma 3.10.** As \(\varepsilon \to 0^+\), one has
\[
\int_D \zeta^c r^2 \, d\nu \to \kappa r_\star^2. \tag{3.24}
\]
As a consequence, for any \(\eta > 0\), there holds
\[
\lim_{\varepsilon \to 0^+} \int_{D \cap \{ r \geq r_\star + \eta \}} \zeta^c \, d\nu = 0. \tag{3.25}
\]

**Proof.** From (3.22), we know that for any \(\gamma \in (0,1)\),
\[
0 \leq \liminf_{\varepsilon \to 0^+} \left[ \int_D \zeta^c r^2 \, d\nu - \kappa (r_\varepsilon)^2 \right] \leq \limsup_{\varepsilon \to 0^+} \left[ \int_D \zeta^c r^2 \, d\nu - \kappa (A_\varepsilon)^2 \right] \leq \frac{\kappa^2 d \gamma}{2\pi W}.
\]
Combining this with lemma 3.9 we get
\[
\lim_{\varepsilon \to 0^+} \int_D \zeta^c r^2 \, d\nu = \lim_{\varepsilon \to 0^+} \kappa (A_\varepsilon)^2 = \kappa r_\star^2.
\]
By (3.24) and a simple calculation, we immediately get (3.25).

The above lemma shows that the ‘major part’ of \(\zeta^c\) will concentrate near the line \(\{(r_\star, z) | z \in \mathbb{R}\}\) and on the set \(D \cap \{ r \geq r_\star + \eta \}\) for any \(\eta > 0\), the circulation of \(\zeta^c\) tends to 0 as \(\varepsilon \to 0\). Using this result, we can then get estimates of \(B_c\).
Lemma 3.11. \( \lim_{\varepsilon \to 0^+} B_{\varepsilon} = r_* \).

**Proof.** Clearly, \( \liminf_{\varepsilon \to 0^+} A_\varepsilon \geq \lim_{\varepsilon \to 0^+} B_\varepsilon = r_* \). Taking \((r_\varepsilon, z_\varepsilon) = (B_\varepsilon, 0)\) in (3.22), noting that \(0 < r_* < d\), we obtain from (3.22)
\[
\frac{d}{2\pi} \liminf_{\varepsilon \to 0^+} \int_{B_{2d\gamma}(B_\varepsilon, 0)} \zeta^c \, d\nu \geq \frac{W}{2} \liminf_{\varepsilon \to 0^+} (B_\varepsilon)^2 + \frac{W r_*^2}{2} - \frac{\kappa d \gamma}{4\pi}.
\]
\[
\geq \frac{\kappa r_*}{2\pi} - \frac{\kappa d \gamma}{4\pi}.
\]
Hence
\[
\liminf_{\varepsilon \to 0^+} \int_{B_{2d\gamma}(B_\varepsilon, 0)} \zeta^c \, d\nu \geq \frac{\kappa r_*}{2\pi} - \frac{\gamma \kappa}{2}.
\]
The desired result clearly follows from lemma 3.10 if we take \(\gamma\) so small that \(r_* / d - \gamma / 2 > 0\).

Having calculated that \( \lim_{\varepsilon \to 0^+} A_\varepsilon = \lim_{\varepsilon \to 0^+} B_\varepsilon = r_* \), we now estimate the upper bound of diameter of the support set of \(\zeta^c\), which is the order of \(\varepsilon^\gamma\) for any \(\gamma \in (0, 1)\).

Lemma 3.12. *For any number \( \gamma \in (0, 1) \), there holds*
\[
\text{diam} \left( \text{supp}(\zeta^c) \right) \leq 4d \varepsilon^\gamma
\]
*provided \( \varepsilon \) is small enough. Consequently,*
\[
\lim_{\varepsilon \to 0^+} \text{dist} \left( \text{supp}(\zeta^c), (r_*, 0) \right) = 0.
\]

**Proof.** Let us use the same notation as in the proof of lemma 3.9. Since \( \int_D \zeta^c \, d\nu = \kappa \), it suffices to prove that (see [10, 33])
\[
\int_{B_{2d\gamma}(r_\varepsilon, z_\varepsilon)} \zeta^c \, d\nu > \kappa / 2, \quad \forall (r_\varepsilon, z_\varepsilon) \in \text{supp}(\zeta^c).
\] (3.26)
From lemma 3.11 we know that \( r_\varepsilon \to r_* \) as \( \varepsilon \to 0^+ \). Taking this into (3.22) and by a direct calculation, we get
\[
\liminf_{\varepsilon \to 0^+} \int_{B_{2d\gamma}(r_\varepsilon, z_\varepsilon)} \zeta^c \, d\nu \geq \kappa - \frac{\kappa d \gamma}{r_*}. 
\] (3.27)
which implies (3.26) for all small \( \gamma \) such that \(1 - \frac{d \gamma}{r_*} > 1/2\). Then \( \text{diam} \left( \text{supp}(\zeta^c) \right) \leq 4d \varepsilon^\gamma \) for small \( \gamma \), which implies
\[
\text{diam} \left( \text{supp}(\zeta^c) \right) \leq C / \log \frac{1}{\varepsilon}
\]
provided \( \varepsilon \) is small enough. Thus we can improve (3.19) as follows

\[
A_1 = \int_{\mathcal{D} \cap \{ \sigma > \varepsilon \gamma \}} K(r, z, r', z') \zeta(r', z') \, dr' \, dz' \\
\leq \left( \frac{r \varepsilon}{4 \pi} \right)^2 \sinh^{-1} \left( \frac{1}{\varepsilon \gamma} \right) \int_{\mathcal{D} \cap \{ \sigma > \varepsilon \gamma \}} \zeta(r', z') r' \, dr' \, dz' \\
\leq \frac{K r}{4 \pi} \sinh^{-1} \left( \frac{1}{\varepsilon \gamma} \right) + C.
\]

Repeating the proof of (3.22), we can sharpen (3.27) as follows

\[
\liminf_{\varepsilon \to 0^+} \int_{B_{2 \varepsilon^{\gamma}((r, z))}} \zeta \, d\nu \geq \kappa - \frac{\gamma \kappa}{2}, \tag{3.28}
\]

which implies that for any \( \gamma \in (0, 1) \) and \( \varepsilon \) sufficiently small,

\[
\int_{B_{2 \varepsilon^{\gamma}((r, z))}} \zeta \, d\nu > \frac{\kappa}{2}.
\]

So we get \( \text{diam}(\text{supp}(\zeta)) \leq 4 \varepsilon^{\gamma} \).

Since \( \zeta = (\zeta^c)^c \), and \( \lim_{\varepsilon \to 0^+} A_2 = \lim_{\varepsilon \to 0^+} B_2 = r \Delta \), we have

\[
\lim_{\varepsilon \to 0^+} \text{dist}(\text{supp}(\zeta^c), (r, 0)) = 0.
\]

The proof is complete. \( \square \)

Actually, we have the following result.

Corollary 3.13. There holds

\[
\lim_{\varepsilon \to 0^+} \frac{\log \text{diam}(\text{supp}(\zeta^c))}{\log \varepsilon} = 1. \tag{3.29}
\]

Proof. Since \( \int_{\mathcal{D}} \zeta \, d\nu = \kappa \) and \( \zeta^c \leq \Lambda / \varepsilon^2 \), we get \( \kappa \leq C \varepsilon^{-2} (\text{diam}(\text{supp}(\zeta^c))^2 \), which implies that there exists a constant \( r_0 > 0 \) independent of \( \varepsilon, \Lambda \), such that

\[
\text{diam}(\text{supp}(\zeta^c)) \geq r_0 \varepsilon.
\]

Thus we have

\[
\limsup_{\varepsilon \to 0^+} \frac{\log \text{diam}(\text{supp}(\zeta^c))}{\log \varepsilon} \leq 1. \tag{3.30}
\]
On the other hand, by lemma 3.12, we obtain
\[
\liminf_{\varepsilon \to 0^+} \frac{\log \text{diam}(\text{supp}(\zeta^\varepsilon))}{\log \varepsilon} \geq \gamma, \quad \forall \gamma \in (0, 1).
\] (3.31)
Combining (3.30) and (3.31), we finish the proof. \qed

We now further study the profile of \( \zeta^\varepsilon \).

**Lemma 3.14.** For all sufficiently small \( \varepsilon \), there holds
\[
\zeta^\varepsilon = \frac{1}{\varepsilon^2} f(\psi^\varepsilon), \quad \text{a.e. on } \Pi.
\] (3.32)

**Proof.** In view of lemma 3.12, we have
\[
\text{dist}(\text{supp}(\zeta^\varepsilon), \partial D) > 0
\]
provided \( \varepsilon \) is sufficiently small. Hence
\[
\mathcal{L}\psi^\varepsilon = 0 \quad \text{in } \Pi \setminus \text{supp}(\zeta^\varepsilon),
\]
\[
\psi^\varepsilon \leq 0 \quad \text{on } \partial \Pi \cup \partial (\text{supp}(\zeta^\varepsilon)),
\]
\[
\psi^\varepsilon \leq 0 \quad \text{at } \infty.
\]
By the maximum principle, we conclude that \( \psi^\varepsilon \leq 0 \) in \( \Pi \setminus D \). The proof is completed. \qed

Furthermore, we can sharpen lemma 3.12 as follows. We can prove that the diameter of \( \text{supp}(\zeta^\varepsilon) \) is of the order of \( \varepsilon \). For the proof, we refer to [10]. First, combining lemmas 3.5, 3.7 and 3.12, we get the following asymptotic expansions.

**Lemma 3.15.** As \( \varepsilon \to 0^+ \), we have
\[
E_\varepsilon(\zeta^\varepsilon) = \left( \frac{\kappa^2 r_*}{4\pi} - \frac{\kappa W r_*^2}{2} \right) \log \frac{1}{\varepsilon} + O(1),
\] (3.33)
\[
\mu_\varepsilon = \left( \frac{\kappa r_*}{2\pi} - \frac{W r_*^2}{2} \right) \log \frac{1}{\varepsilon} + o \left( \log \frac{1}{\varepsilon} \right).
\] (3.34)

**Proof.** We first prove (3.33). According to lemma 3.12, there holds
\[
\text{supp}(\zeta^\varepsilon) \subseteq B_{4\varepsilon^{-\frac{1}{2}}}((A_\varepsilon, 0))
\]
for all sufficiently small \( \varepsilon \). Hence, by lemmas 3.3 and 3.4, we have
\[
\int_D \zeta^\varepsilon K \zeta^\varepsilon \, d\nu \leq \frac{(B_\varepsilon)^3}{2\pi} \int_{D \times D} \log \left[ (r - r')^2 + (z - z')^2 \right]^{-\frac{1}{2}}
\]
\[
\times \zeta^\varepsilon(r, z) \zeta^\varepsilon(r', z') \, dr \, dz \, dr' \, dz' + O(1).
\]
By the Riesz’s rearrangement inequality and simple calculation, we have
\begin{align*}
\int_{D \times D} \log \left[ (r - r')^2 + (z - z')^2 \right]^{-\frac{1}{4}} \zeta(\rho, \zeta) \zeta(\rho', \zeta') \, dr \, dr' \, dz' \\
\leq \log \left\{ \int_D \zeta \, dr \, dz \right\}^2 + O(1).
\end{align*}
Note that
\begin{align*}
\int_D \zeta \, dr \, dz \leq \frac{\kappa}{A_\varepsilon},
\end{align*}
thus by lemma 3.12, we have
\begin{align*}
\int_D \zeta \, K \zeta \, d\nu \leq \frac{\kappa \pi \varepsilon}{A_\varepsilon} \log \frac{1}{\varepsilon} + O(1) \leq \frac{\kappa^2 A_\varepsilon}{2\pi} \log \frac{1}{\varepsilon} + O(1),
\end{align*}
from which we conclude that
\begin{align*}
E_\varepsilon(\zeta) \leq \left( \frac{\kappa^2 A_\varepsilon}{4\pi} \right) \log \frac{1}{\varepsilon} + O(1)
\leq \left( \frac{\kappa^2 r_0}{4\pi} \right) \log \frac{1}{\varepsilon} + O(1).
\end{align*}
Combining this with lemma 3.5, we get (3.33). (3.34) clearly follows from (3.33) and (3.12). The proof is thus completed.

**Lemma 3.16.** There exist \( r_0, R_0 > 0 \) not depending on \( \varepsilon \) such that
\begin{align*}
\left( r_0 \varepsilon , R_0 \varepsilon \right) \leq \text{diam}(\text{supp}(\zeta)) \leq \left( r_0 \varepsilon, R_0 \varepsilon \right),
\end{align*}
provided \( \varepsilon \) is small enough.

**Proof.** In the proof of corollary 3.13, we have showed that \( \text{diam}(\text{supp}(\zeta)) \geq r_0 \varepsilon \) for some \( r_0 > 0 \). So it suffices to prove \( \text{diam}(\text{supp}(\zeta)) \leq R_0 \varepsilon \) for some \( R_0 > 0 \). Let us use the same notation as in the proof of lemma 3.12. Recalling that for any \((r, z) \in \text{diam}(\text{supp}(\zeta))\), we have
\begin{align*}
K \zeta^2 (r, z) - \frac{W(r_0)^2}{2} \log \frac{1}{\varepsilon} \geq \mu \varepsilon.
\end{align*}
(3.35)
According to lemma 3.12, there holds
\begin{align*}
\text{supp}(\zeta) \subseteq B_{\frac{1}{4d_k}} \left( (A_\varepsilon, 0) \right)
\end{align*}
for \( \varepsilon \) sufficiently small.
Let $R > 1$ be a positive number to be determined. On the one hand, we have

$$K\zeta(r, z) = \int_D K(r, z, r', z') \zeta(r', z') \, dr' \, dz'$$

$$= \left( \int_{D \cap \{ \sigma > R \epsilon \}} + \int_{D \cap \{ \sigma \leq R \epsilon \}} \right) K(r, z, r', z') \zeta(r', z') \, dr' \, dz'$$

$$:= B_1 + B_2.$$

By (3.10) in lemmas 3.4 and 3.12, we get

$$B_1 = \int_{D \cap \{ \sigma > R \epsilon \}} K(r, z, r', z') \zeta(r', z') \, dr' \, dz'$$

$$\leq \frac{(A \epsilon)^2 + O \left( \epsilon^2 \right)}{2\pi} \int_{D \cap \{ \sigma > R \epsilon \}} \log \frac{1}{\sigma} \zeta(r', z') \, dr' \, dz' + C$$

$$\leq \frac{(A \epsilon)^2 + O \left( \epsilon^2 \right)}{2\pi} \log \frac{1}{R \epsilon} \int_{D \cap \{ \sigma > R \epsilon \}} \zeta(r', z') \, dr' \, dz' + C$$

$$\leq \frac{A \epsilon}{2\pi} \log \frac{1}{R \epsilon} \int_{D \cap \{ \sigma > R \epsilon \}} \zeta^2 \, d\nu + C,$$

and

$$B_2 = \int_{D \cap \{ \sigma \leq R \epsilon \}} K(r, z, r', z') \zeta(r', z') \, dr' \, dz'$$

$$\leq \frac{(A \epsilon)^2 + O \left( \epsilon^2 \right)}{2\pi} \log \frac{1}{\epsilon} \int_{D \cap \{ \sigma \leq R \epsilon \}} \zeta(r', z') \, dr' \, dz' + C$$

$$\leq \frac{A \epsilon}{2\pi} \log \frac{1}{\epsilon} \int_{D \cap \{ \sigma \leq R \epsilon \}} \zeta^2 \, d\nu + C.$$

Taking (3.36) and (3.37) into (3.35), we get

$$\frac{A \epsilon}{2\pi} \log \frac{1}{R \epsilon} \int_{D \cap \{ \sigma > R \epsilon \}} \zeta^2 \, d\nu$$

$$+ \frac{A \epsilon}{2\pi} \log \frac{1}{\epsilon} \int_{D \cap \{ \sigma \leq R \epsilon \}} \zeta^2 \, d\nu - \frac{W(A \epsilon)^3}{2} \log \frac{1}{\epsilon} + C \geq \mu' .$$

(3.38)
On the other hand, one has
\[
\mu^e \geq \left( \frac{\kappa A_e}{2\pi} - \frac{W(A_e)^2}{2} \right) \log \frac{1}{\varepsilon} = C. \tag{3.39}
\]
Combining (3.38) and (3.39), we obtain
\[
\frac{\kappa A_e}{2\pi} \log \frac{1}{\varepsilon} \leq \frac{A_e}{2\pi} \int_{D^e(\sigma > R_e)} \zeta^e \, d\nu + \frac{A_e}{2\pi} \log \frac{1}{\varepsilon} \int_{D^e(\sigma \leq R_e)} \zeta^e \, d\nu + C.
\]
Hence we have
\[
\int_{D^e(\sigma \leq R_e)} \zeta^e \, d\nu \geq \kappa \left( 1 - \frac{C}{\log R} \right).
\]
Choosing \( R > 1 \) such that \( C(\log R)^{-1} < 1/2 \), we obtain
\[
\int_{D^e(\sigma > R_e)} \eta^e \, d\nu + \int_{D^e(\sigma \leq R_e)} \zeta^e \, d\nu > \frac{\kappa}{2}.
\]
Taking \( R_0 = 4dR \), we finish the proof of lemma 3.16. \( \square \)

### 3.1.3. Asymptotic shape.

Define the centre of \( \zeta^e \) to be
\[
x^e = \frac{\int x \zeta^e(x) \, d\mu(x)}{\int \zeta^e(x) \, d\mu(x)}.
\]
Let \( \eta^e(x) = \varepsilon^2 \zeta^e(x^e + \varepsilon x) \). We denote by \( (\eta^e)^\triangle \) the symmetric radially non-increasing Lebesgue-rearrangement of \( \eta^e \) centred on 0. The following result determines the asymptotic nature of \( \zeta^e \) in terms of its scaled version \( \eta^e \).

**Lemma 3.17.** As \( \varepsilon \to 0 \), every accumulation point of the family \( \{ \eta^e : \varepsilon > 0 \} \) in the weak topology of \( L^2 \) must be a radially non-increasing function. Namely, if there exists a subsequence \( \varepsilon_n \to 0 \) and a function \( \eta^\ast \) such that \( \liminf_{n \to +\infty} \varepsilon_n = 0 \) and \( \eta^e_n \to \eta^\ast \) in the \( L^2 \) weak topology, then \( \eta^\ast \) is radially non-increasing.

**Proof.** Let us assume that \( \eta^e \to \eta \) and \( (\eta^e)^\triangle \to h_0 \) weakly in \( L^2 \) for some functions \( \eta \) and \( h_0 \) as \( \varepsilon \to 0^+ \). By the Riesz’ rearrangement inequality, we have
\[
\int_{B_R(0) 	imes B_R(0)} \log \frac{1}{|x - x'|} |\eta^e(x)\eta^e(x')| \, dx \, dx' \leq \int_{B_{R_0}(0) 	imes B_{R_0}(0)} \log \frac{1}{|x - x'|} (\eta^e)^\triangle \times (\eta(x)^\triangle(\eta(x')) \, dx \, dx'.
\]
Hence
\[ \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \eta(x) \eta(x') \, dx \, dx' \leq \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \times h_0(x) h_0(x') \, dx \, dx'. \] (3.40)

Let \( \tilde{\zeta}_\varepsilon \) be defined as
\[ \tilde{\zeta}_\varepsilon(x) = \begin{cases} \varepsilon^2(\eta_\varepsilon^2)(\varepsilon^{-1}(x - x_\varepsilon)) & \text{if } x \in B_{\varepsilon}(x_\varepsilon), \\ 0 & \text{if } x \in D \setminus B_{\varepsilon}(x_\varepsilon). \end{cases} \]

Direct calculation yields that as \( \varepsilon \to 0^+ \),
\[ E_\varepsilon(\zeta_\varepsilon) = \frac{(A_\varepsilon)^3}{4\pi} \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \eta(x) \eta(x') \, dx \, dx' \]
\[ + \frac{(A_\varepsilon)^3}{4\pi} \left\{ \int_D \zeta_\varepsilon \, dm \right\}^2 - \frac{W(A_\varepsilon)^3}{2} \log \frac{1}{\varepsilon} \int_D \zeta_\varepsilon \, dm + \mathcal{R}_\varepsilon(\zeta_\varepsilon) + o(1), \]
and
\[ E_\varepsilon(\tilde{\zeta}_\varepsilon) = \frac{(A_\varepsilon)^3}{4\pi} \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \eta(x) \eta(x') \, dx \, dx' \]
\[ + \frac{(A_\varepsilon)^3}{4\pi} \left\{ \int_D \tilde{\zeta}_\varepsilon \, dm \right\}^2 - \frac{W(A_\varepsilon)^3}{2} \log \frac{1}{\varepsilon} \int_D \tilde{\zeta}_\varepsilon \, dm + \mathcal{R}_\varepsilon(\tilde{\zeta}_\varepsilon) + o(1), \]
where
\[ \lim_{\varepsilon \to 0^+} \mathcal{R}_\varepsilon(\zeta_\varepsilon) = \lim_{\varepsilon \to 0^+} \mathcal{R}_\varepsilon(\tilde{\zeta}_\varepsilon) < \infty. \]

Recalling that \( E_\varepsilon(\tilde{\zeta}_\varepsilon) \leq E_\varepsilon(\zeta_\varepsilon) \), we conclude that
\[ \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \eta(x) \eta(x') \, dx \, dx' \geq \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \times h_0(x) h_0(x') \, dx \, dx', \]
which together with (3.40) yields to
\[ \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \eta(x) \eta(x') \, dx \, dx' = \int \int_{B_{\varepsilon}(0) \times B_{\varepsilon}(0)} \log \frac{1}{|x - x'|} \times h_0(x) h_0(x') \, dx \, dx'. \]
By lemma 3.2 of [7], we know that there exists a translation $T$ in $\mathbb{R}^2$ such that $T\eta = h_0$. Note that

$$\int_{B_{R_0}(0)} x\eta(x)dm = \int_{B_{R_0}(0)} xh_0(x)dm = 0.$$ 

Thus $\eta = h_0$ and the proof is complete. \hfill \Box

Remark 3.18. With the above results in hand, one can further show that the core $\text{supp}(\zeta^\varepsilon)$ will be approximately a disk by using the standard scaling techniques; see [19, 33].

Now we are ready to prove theorem 2.1 when $f$ is bounded.

Proof of theorem 2.1. when $f$ is bounded. Let $\lambda = 1/\varepsilon^2$ and $\Psi^\lambda = \psi^\varepsilon$. The regularity of $\Psi^\lambda$ follows from classical elliptic regularity theory, see, e.g., [2, 3]. Now the desired results follow from the above lemmas and [3, 8]. Thus we complete the proof. \hfill \Box

3.2. Unbounded case

In this subsection, we deal with the remaining case. Suppose that $f$ is a function satisfying (f1) and (f3). The key idea is to truncate the function $f$ so that we can use the results already proved above. Let $\rho > 1$ and

$$f_\rho(s) = f(s) \quad \text{if } s \leq \rho; \quad f_\rho(s) = f(\rho) \quad \text{if } s > \rho.$$ 

Let

$$F_\rho(s) = \int_0^s f_\rho(s')ds'$$

and $G_\rho$ be the conjugate function to $F_\rho$. From the above discussion, we already know that there exists a family of solutions $(\zeta^\varepsilon, \rho, \psi^\varepsilon, \rho)$ satisfying

$$\zeta^\varepsilon = \frac{1}{\varepsilon^2} f_\rho(\psi^\varepsilon), \quad \text{a.e. on } \Pi.$$ (3.41)

All we have left to do is to seek a suitable parameter $\rho$ such that $\psi^\varepsilon, \rho \leq \rho$ almost everywhere on $\Pi$, and then (3.41) will be equivalent to

$$\zeta^\varepsilon = \frac{1}{\varepsilon^2} f(\psi^\varepsilon), \quad \text{a.e. on } \Pi.$$ 

In the following we will carefully demonstrate this in detail. In the sequel we shall write $C^*$ for a positive constant independent of $\varepsilon$ and $\rho$ whose value may change from line to line.

Lemma 3.19. Let $\rho > 1$ be fixed. Then for all sufficiently small $\varepsilon$, we have

$$K \zeta^\varepsilon(r, z) \leq \frac{K\rho}{2\pi} \log \frac{1}{\varepsilon} + C^* \log f(\rho) + C^*, \quad \forall (r, z) \in \text{supp}(\zeta^\varepsilon).$$
Proof. In view of lemma 3.4, one has
\[ 0 < K(r, z, r', z') \leq \frac{\sqrt{rr'}}{2\pi} \log \frac{1}{|r - r'|^2 + (z - z')^2} + C^* \quad \text{in} \ D \times D. \]

For any \( x = (r, z) \in \text{supp}(\zeta_{\varepsilon, \rho}) \) and taking \( \gamma = 1/2 \) in lemma 3.12, we have
\[ K\zeta_{\varepsilon, \rho}(x) \leq \frac{r^2 + C^* \varepsilon^2}{2\pi} \int_D \log \frac{1}{|x - y|} \zeta_{\varepsilon, \rho}(y) \, dm(y) + C^*. \]

By the rearrangement inequality, we get
\[ \int_D \log \frac{1}{|x - y|} \zeta_{\varepsilon, \rho}(y) \, dm(y) \leq \left( \log \frac{1}{\varepsilon} + \log f(\rho) \frac{1}{2} \right) \int_D \zeta_{\varepsilon, \rho} \, dm(y). \]

So we obtain
\[ K\zeta_{\varepsilon, \rho}(x) \leq \frac{r^2 + C^* \varepsilon^2}{2\pi} \int_D \log \frac{1}{|x - y|} \zeta_{\varepsilon, \rho}(y) \, dm(y) + C^* \]
\[ \leq \frac{kr}{2\pi} \log \frac{1}{\varepsilon} + C^* \log f(\rho) + C^*, \]
which completes the proof. \( \square \)

Denote
\[ E_{\varepsilon, \rho}(\zeta) = \frac{1}{2} \int_D \zeta K \zeta \, d\nu - \frac{W}{2} \log \frac{1}{\varepsilon} \int_D r^2 \zeta \, d\nu - \frac{1}{\varepsilon^2} \int_D G_{\rho}(\varepsilon^2 \zeta) \, d\nu. \]

By selecting a suitable competitor, we can easily get a lower bound of the energy.

Lemma 3.20. We have
\[ E_{\varepsilon, \rho}(\zeta_{\varepsilon, \rho}) \geq \left( \frac{\kappa^2 r_\rho}{4\pi} - \frac{\kappa W r_\rho^2}{2} \right) \log \frac{1}{\varepsilon} - C^*. \]

Next we turn to estimate the Lagrange multiplier \( \mu_{\varepsilon, \rho} \). Combining lemmas 3.19, 3.20 and 3.21, we get a priori upper bound of \( \psi_{\varepsilon, \rho} \) with respect to \( \rho \).

Lemma 3.21. Let \( \rho > 1 \) be fixed. Then for all sufficiently small \( \varepsilon \), we have
\[ \mu_{\varepsilon, \rho} \geq 2\kappa^{-1} E_{\varepsilon, \rho}(\zeta_{\varepsilon, \rho}) + \frac{W}{2\kappa} \log \frac{1}{\varepsilon} \int_D r^2 \zeta_{\varepsilon, \rho} \, d\nu - |2\delta_0 - 1| \kappa^{-1} \rho - C^*. \]

Proof. Recalling assumption (f3), there exists two positive numbers \( \delta_0 \in (0, 1) \) and \( \delta_1 \geq 0 \) such that
\[ F(s) \leq \delta_0 f(s)s + \delta_1 f(s), \quad \forall \ s \geq 0. \]
By definition, we have
\[ F_\rho(s) = F(s) \quad \text{if } s \leq \rho; \quad F_\rho(s) = F(\rho) + f(\rho)(s - \rho) \quad \text{if } s > \rho. \]

By convexity, it holds
\[ G_\rho(\varepsilon^2 \zeta^{\varepsilon, \rho}) + F_\rho(\psi^{\varepsilon, \rho}) = \varepsilon^2 \zeta^{\varepsilon, \rho} \psi^{\varepsilon, \rho}, \quad \text{a.e. on } D. \]

So we have
\[ 2E_{\varepsilon, \rho}(\zeta^{\varepsilon, \rho}) \leq \int_D \zeta^{\varepsilon, \rho} K \zeta^{\varepsilon, \rho} \, d\nu - W \log \frac{1}{\varepsilon} \int_D r^2 \zeta^{\varepsilon, \rho} \, d\nu - \frac{2}{\varepsilon^2} \int_D G_\rho(\varepsilon^2 \zeta^{\varepsilon, \rho}) \, d\nu \\
\leq \int_D \zeta^{\varepsilon, \rho} (\psi^{\varepsilon, \rho} - \rho)_+ \, d\nu - \frac{W}{2} \log \frac{1}{\varepsilon} \int_D r^2 \zeta^{\varepsilon, \rho} \, d\nu \\
+ |2\delta_0 - 1|\rho + 2\delta_1 \kappa + \kappa \mu^{\varepsilon, \rho}. \]

Let \((\psi^{\varepsilon, \rho} - \rho)_+ \in \mathcal{H}(\Pi)\) be a test function. Using a similar argument as that in the proof of lemma 3.7, we check that \(\int_D \zeta^{\varepsilon, \rho}(\psi^{\varepsilon, \rho} - \rho)_+ \, d\nu\) is uniformly bounded with respect to \(\varepsilon\) and \(\rho\).

The proof is thus complete. \(\square\)

**Lemma 3.22.** Let \(\rho > 1\) be fixed. Then for all sufficiently small \(\varepsilon\), we have
\[ \psi^{\varepsilon, \rho}(r, z) \leq C^* \log f(\rho) + |2\delta_0 - 1|\rho + C^*. \]

**Proof.** Set
\[ A_{\varepsilon, \rho} := \inf \{ r | (r, 0) \in \text{supp}(\zeta^{\varepsilon, \rho}) \}. \]

If \(\varepsilon\) is small enough, then \(\text{supp}(\zeta^{\varepsilon, \rho}) \subset B_{4d_{\varepsilon}^2}((A_{\varepsilon}, 0)).\) By lemmas 3.19, 3.20 and 3.21, for any \((r, z) \in \text{supp}(\zeta^{\varepsilon, \rho})\), it is not hard to get that
\[ \psi^{\varepsilon, \rho}(r, z) = K \zeta^{\varepsilon, \rho}(r, z) - \frac{W}{2} \log \frac{1}{\varepsilon} - \mu^{\varepsilon, \rho} \leq C^* \log f(\rho) + |2\delta_0 - 1|\rho + C^*. \]

By virtue of assumption \((f_3)\), there exists \(\rho_0 > 1\) sufficiently large such that
\[ C^* \log f(\rho_0) + |2\delta_0 - 1|\rho_0 + C^* \leq \rho_0. \]

Therefore if \(\varepsilon\) is sufficiently small, then \(\psi^{\varepsilon, \rho_0} \leq \rho_0\) almost everywhere on \(D\). In view of (3.41), we see that
\[ \zeta^{\varepsilon, \rho_0} = \frac{1}{\varepsilon^2} f(\psi^{\varepsilon, \rho_0}), \quad \text{a.e. on } \Pi, \]
provided \(\varepsilon\) is sufficiently small. In other words, we have already finished the proof of theorem 2.1.
Acknowledgments

The authors are grateful to the anonymous referee for his/her helpful comments and suggestion which helped to improve the presentation of the paper. D Cao was supported by NNSF of China Grant 11831009 and Chinese Academy of Sciences (No. QYZDJ-SSW-SYS021). J Wan was supported by NNSF of China Grant 12101045 and Beijing Institute of Technology Research Fund Program for Young Scholars (No. 3170011182016). G Wang was supported by NNSF of China (12001135, 12071098) and China Postdoctoral Science Foundation (2019M661261, 2021T140163).

References

[1] Ambrosetti A and Rabinowitz P H 1973 Dual variational methods in critical point theory and applications J. Funct. Anal. 14 349–81
[2] Ambrosetti A and Struwe M 1989 Existence of steady vortex rings in an ideal fluid Arch. Ration. Mech. Anal. 108 97–109
[3] Badia T V and Burton G R 2001 Vortex rings in $\mathbb{R}^3$ and rearrangements Proc. R. Soc. A 457 1115–35
[4] Benedetto D, Caglioti E and Marchioro C 2000 On the motion of a vortex ring with a sharply concentrated vorticity Math. Methods Appl. Sci. 23 147–68
[5] Benning T B 1976 The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics Applications of Methods of Functional Analysis to Problems of Mechanics (Lecture Notes in Math. vol 503) (Berlin: Springer) pp 8–29
[6] Berger M S and Fraenkel L E 1980 Nonlinear desingularization in certain free-boundary problems Commun. Math. Phys. 77 149–72
[7] Burchard A and Guo Y 2004 Compactness via symmetrization J. Funct. Anal. 214 40–73
[8] Burton G R 1987 Vortex rings in a cylinder and rearrangements J. Differ. Equ. 70 333–48
[9] Caffarelli L A and Friedman A 1980 The shape of axisymmetric rotating fluid J. Funct. Anal. 35 109–42
[10] Cao D, Wan J and Zhan W 2021 Desingularization of vortex rings in three dimensional Euler flows J. Differ. Equ. 270 1258–97
[11] Da Rios L S 1916 Vortici ad elica Nuovo Cimento 11 419–32
[12] Dávila J, del Pino M, Musso M and Wei J 2020 Travelling helices and the vortex filament conjecture in the incompressible Euler equations (arXiv:2007.00606)
[13] Dekeyser J 2019 Asymptotic of steady vortex pair in the lake equation SIAM J. Math. Anal. 51 1209–37
[14] Dekeyser J and Van Schaftingen J 2020 Vortex motion for the lake equations Commun. Math. Phys. 375 1459–501
[15] Douglas R J 1994 Rearrangements of functions on unbounded domains Proc. R. Soc. Edinburgh A 124 621–44
[16] Fraenkel L E 1970 On steady vortex rings of small cross-section in an ideal fluid Proc. R. Soc. A 316 29–62
[17] Fraenkel L E 1972 Examples of steady vortex rings of small cross-section in an ideal fluid J. Fluid Mech. 51 119–35
[18] Fraenkel L E and Berger M S 1974 A global theory of steady vortex rings in an ideal fluid Acta Math. 132 13–51
[19] Friedman A and Turkington B 1981 Vortex rings: existence and asymptotic estimates Trans. Am. Math. Soc. 268 1–37
[20] Helmholtz H 1858 On integrals of the hydrodynamics equations which express vortex motion Crelle’s J. 55 25–55
[21] Hill M J M 1894 On a spherical vortex Phil. Trans. R. Soc. A 185 213–45
[22] Jerrard R L and Seis C 2017 On the vortex filament conjecture for Euler flows Arch. Ration. Mech. Anal. 224 135–72
[23] Lamb H 1932 Hydrodynamics Cambridge Mathematical Library 6th edn (Cambridge: Cambridge University Press)
[24] Levi-Civita T 1932 Attrazione newtoniana dei tubi sottili e vortici filiformi *Annali della Scuola Normale Superiore di Pisa—Classe di Scienze* Ser. 2 vol 1 pp 229–50
[25] Lim T T and Nickels T B 1995 Vortex rings *Fluid Vortices* ed S I Green (Dordrecht: Kluwer) pp 95–153
[26] Meleshko V V, Gourjii A A and Krasnopolskaya T S 2012 Vortex rings: history and state of the art *J. Math. Sci.* 187 772–808
[27] Ni W-M 1980 On the existence of global vortex rings *J. Anal. Math.* 37 208–47
[28] Ricca R L 1991 Rediscovery of Da Rios equations *Nature* 352 561–2
[29] Rockafellar T 1972 *Convex Analysis* (Princeton, NJ: Princeton University Press)
[30] Shariff K and Leonard A 1992 Vortex rings *Annu. Rev. Fluid Mech.* 24 235–79
[31] Tadie 1994 On the bifurcation of steady vortex rings from a Green function *Math. Proc. Camb. Phil. Soc.* 116 555–68
[32] Thomson W 1910 *Mathematical and Physical Papers, IV* (Cambridge: Cambridge University Press)
[33] Turkington B 1983 On steady vortex flow in two dimensions, I *Commun. PDE* 8 999–1030
  Turkington B 1983 On steady vortex flow in two dimensions, II *Commun. PDE* 8 1031–71
[34] de Valeriola S and Van Schaftingen J 2013 Desingularization of vortex rings and shallow water vortices by a semilinear elliptic problem *Arch. Ration. Mech. Anal.* 210 409–50
[35] Yang J F 1995 Global vortex rings and asymptotic behaviour *Nonlinear Anal.* 25 531–46