Potts spin glasses with three, four and five states near \( T = T_c \): expanding around the replica symmetric solution

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Abstract

Expansion for the 1RSB free energy functionals of the mean-field (MF) Potts spin glass models up to the fourth order in \( \delta q_{\alpha\beta} \) around the replica symmetric (RS) solution is investigated. We consider models with three, four and five states and use a special quadrupole-like representation for them. The corresponding series for the 1RSB order parameters giving explicit temperature dependence are obtained in the vicinity of the point \( T = T_c \), where the RS solution becomes unstable. The crossover from continuous to jumpwise behavior with increasing number of states is derived analytically. A comparison is made between the free energy expansion for the Potts spin glass and that for other models.

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The spin glass corresponding to the random variant of the mean-field (MF) Potts model \([1]\) is one of the first models where the transition to the nonergodic state was discovered to be quite different from the standard \([2, 3]\) Sherrington–Kirkpatrick (SK) model \([4]\). Nevertheless, it remains the focus of investigations, and a number of questions are still not answered (see, e.g., the recent papers \([5, 6]\) that also deal with the MF Potts glasses with fewer number of states).

It is well known that in the case of the SK model, the so-called full replica symmetry breaking (FRSB) \([7]\) takes place at the instability point \( T_c \) of the replica symmetric (RS) solution. All other models with two-particle interaction and with reflection symmetry behave in the same way \([8]\). The Potts glass models (with \( p > 2 \)) are usually considered to be classical examples of models without reflection symmetry: FRSB does not work at \( T_c \) \([2, 3, 9, 10]\) and there is a region where the 1RSB solution is stable \([11]\) (see, however, \([5]\)). However, the Potts symmetry leads to the cancellation of some terms in the free energy, so that the RS solution is zero at high temperature. This is why the Potts models differ from other models without
reflection symmetry that have only one interaction term, in particular, from the SK-like models with the Hamiltonian
\[ H = -\frac{1}{2} \sum_{i \neq j} J_{ij} U_i U_j, \quad (1) \]
with quenched interactions \( J_{ij} \) distributed with Gaussian probability, but without reflection symmetry:
\[ \text{Tr}[U^{(2k+1)}] \neq 0 \quad (2) \]
for some integer \( k \). In this case, the disorder smears out the phase transition: there is no zero RS solution at high temperature (see, e.g., [12, 13]). At the same time, the Potts symmetry leads to the fact that the order parameter is one and the same for all \( p \) states, which makes it related to two-particle generalized SK-type models (1) and allows us to show that the dependence of the free energy on the order parameters in the vicinity of the bifurcation point \( T_c \) is universal in both cases (see below).

In [14], the 1RSB equations for the glass order parameters were solved numerically for a large set of integer values of \( p \). The authors confirmed the previously obtained result [9] that \( p = 4 \) is the largest integer value of states giving the continuous 1RSB transition in \( p \)-state MF Potts glass. At \( p = 5 \) the transition is discontinuous. Although the representation of the Potts model used by the authors is valid for all (not only integer) values of \( p \), their approach, being based on the numerical solution of equations, could not give the actual value of \( p \) that lies between \( p = 4 \) and \( p = 5 \) at which the crossover takes place. This would require a very high precision in the calculation of multiple integrals (see table 1 of [14]).

It therefore seems worthwhile to follow the crossover from continuous to jumpwise behavior of the order parameter in an analytic way, using bifurcation theory. This paper is devoted to this problem. To some extent we proceed along the same lines as the authors of [15] where the SK model was considered. Although our approach, based on a special quadrupole-like representation for Potts models, is valid only for integer values of \( p \), the actual analytical forms of the 1RSB glass order parameters near the bifurcation points of the RS solution (see figure 1) are such that it is plausible to assume that the point of crossover is just the point \( p = 4 \) itself, because the tangent to the order parameter is vertical at this point.
We use a special representation of Potts models that originates from the analogy between three-state Potts spin glass and isotropic quadrupole glass (see below). In our representation the number of integrations is fewer than in the usual approach, so it is more convenient for actual calculations. Moreover, this representation demonstrates explicitly the appearance of cubic terms in the free energy expansion. These terms play a crucial role in the order-parameter behavior.

The Potts glass model is defined by the Hamiltonian

\[ \hat{H} = -\frac{p}{2} \sum_{i \neq j} J_{ij} \delta_{\sigma_i,\sigma_j}, \]

where \( p \) is the number of states and the variables \( \sigma_i, \sigma_j \) can take values \( 0, 1, \ldots, p - 1 \). Here \( J_{ij} \) are random interactions distributed with Gaussian probability

\[ P(J_{ij}) = \frac{\sqrt{N}}{\sqrt{2\pi J}} \exp \left[ -\frac{(J_{ij})^2}{2J^2} \right], \]

where the factor \( N \) ensures a sensible thermodynamic limit.

The three-state Potts spin glass was described earlier in our papers (see, e.g., [11, 16, 17]) using a representation in terms of the operators of quadrupole momenta:

\[ Q = \left( -2, 0, 0 \right), \quad V = \sqrt{3} \left( J_x^2 - J_y^2 \right), \quad J = 1, \quad J = 1, 0, -1. \]

The operators \( Q \) and \( V \) commute, and in the representation where both of them are diagonal, have the following form

\[ Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & -\sqrt{3} \end{pmatrix}. \]

Throughout this paper we shall use the abbreviated notations \( Q_k = (-2, 1, 1) \) and \( V_k = (0, \sqrt{3}, -\sqrt{3}) \).

The equation

\[ [Q_k Q_l + V_k V_l] + 2 = 6 \delta_{k,l} \]

provides the equivalence of the Potts model with \( p = 3 \) to the model with the Hamiltonian

\[ H = -\frac{1}{2} \sum_{i \neq j} \left[ J_{ij} Q_i Q_j + G_{ij} V_i V_j \right], \]

(with \( J_{ij} \equiv G_{ij} \)). The quenched interactions are distributed with Gaussian probability.

Let us note that this representation is very useful for actual calculations and allowed us to obtain, in particular, the number of metastable states at zero temperature [17] and to determine the low-temperature boundary of the stability of the 1RSB solution [11].

Using the replica method, we obtain the average free energy of the system in the form

\[ \frac{F}{NkT} = \lim_{n \to 0} \frac{1}{n} \left[ -2t^2 n + \frac{1}{2} \sum_\alpha \langle x^\alpha \rangle^2 + \frac{t^2}{2} \sum_{\alpha \neq \beta} (q^{\alpha\beta})^2 \right. \]

\[ \left. - \ln \text{Tr} \exp \left[ \frac{t^2}{2} \sum_{\alpha \neq \beta} q^{\alpha\beta} (Q^\alpha Q^\beta + V^\alpha V^\beta) + t \sum_\alpha x^\alpha Q^\alpha \right] \right], \]

where \( q^{\alpha\beta} \) is the glass order parameter, \( x^\alpha \) is the ferroquadrupolar order parameter \( x^\alpha \sim \langle \langle Q^\alpha \rangle \rangle \) and \( t = J/kT \). Here the indices \( \alpha \) and \( \beta \) label replicas. The Potts symmetry permits describing the glass state with one glass order parameter for both kinds of operators.
The saddle point equations for the free energy (7) have the RS solution that is zero at high temperature. The RS solution becomes unstable at the point $T_c$ where it bifurcates, giving rise to other solutions. Let us note that in the general case of the Hamiltonian (6) with $J_{ij} \neq G_{ij}$, one of the glass order parameters, namely $q_1 \sim \langle Q^\alpha Q^\beta \rangle$, cannot be zero even at high temperatures. This fact can be easily seen by analyzing the high-temperature expansions for the RS equations for the order parameters [18]. For the free energy (7) we can obtain two kinds of solutions. One kind is a Potts-symmetrical solution with $q \neq 0$, $q_1 = q_2$, $q_2 \sim \langle V^\alpha V^\beta \rangle$ and the other with nonzero $x$ and $q_1 - q_2$. We consider only the solutions with Potts symmetry in the case $p = 3$ as well as for $p = 4, 5$.

Using the standard procedure we perform the first stage of the replica symmetry breaking (1RSB) according to Parisi ($n$ replicas are divided into $n/m$ groups with $m$ replicas in each group) and obtain the free energy in the form

$$F_{1RSB} = -NkT \left[ t^2 \left( 2 + (1 - m) \frac{v^2}{2} - 2v^2 \right) + \frac{1}{m} \ln \int d\varepsilon G \int d\varepsilon [\Psi]^m \right], \quad (8)$$

where we set $q^{\alpha \beta} = r_1 (0)$ if $\alpha$ and $\beta$ are from different groups and $q^{\alpha \beta} = r_1 + v (\equiv v)$ if $\alpha$ and $\beta$ belong to the same group,

$$\Psi = \exp \left( -2\theta_1 \right) + \exp \left( \theta_1 \right) \exp \left( \theta_2 \right) + \exp \left( -\theta_2 \right), \quad (9)$$

$$\theta_1 = zt \sqrt{v}, \quad \theta_2 = st \sqrt{v}, \quad (10)$$

$$\int \frac{dz}{\sqrt{2\pi}} (\cdots) \exp \left( -\frac{z^2}{2} \right) \equiv \int d\varepsilon G (\cdots). \quad (11)$$

Expanding the free energy (8) near $T_c$ up to $v^4$ and solving the equations that define the extremum conditions for the free energy, one can obtain [16]

$$v = 8t - 84t^2, \quad m = \frac{1}{2} = \frac{9}{2} \tau, \quad \Delta F = 16t^3 \tau. \quad (12)$$

where $\Delta F = F_{1RSB} - F_{RS}$ and $\tau = t - t_c$.

Let us consider now the case $p = 4$. In this case one can use the operators $Q' \sim 3J, 2 - J (J + 1)$, $V' \sim J$, for $J = 3/2$ and a third operator $P'$ defined as to be orthogonal to $Q'$ and $V'$. Namely $2Q'_z = (1, -1, -1, 1)$, $2\sqrt{3}V'_z = (-3, 1, -1, 3)$, $2\sqrt{3}P'_z = (1, 3, -3, -1)$.

It is easy to see that the following equations hold:

$$Q'^2 = \frac{1}{2}, \quad V'^2 = \frac{1}{4} + \frac{3}{2} Q', \quad P'^2 = \frac{1}{4} - \frac{3}{2} Q',$$

$$V'P' = -\frac{3}{\sqrt{3}} Q', \quad Q'V' = -\frac{3}{\sqrt{3}} P' + \frac{3}{2} Q', \quad Q'P' = \frac{3}{2} P' - \frac{3}{\sqrt{3}} Q',$$

so that

$$Q'_z Q'_l + V'_z V'_l + P'_z P'_l + \frac{1}{2} = \delta_{k,l}. \quad (13)$$

There is no reflection symmetry and we have

$$\text{Tr} P'^2 Q' = -\frac{2}{5}, \quad \text{Tr} V'^2 Q' = \frac{2}{3}, \quad \text{Tr} P'V' Q' = -\frac{3}{10}, \quad (14)$$

which leads to the appearance of cubic terms of the form
It is worth noting that our representation in terms of three matrices is not unique. For example, one can use the set $A_k = (1, -1, -1, 1)$, $B_k = (-1, 1, -1, 1)$, $C_k = (1, 1, -1, -1)$. We use, however, the quadrupole-like operators in order to retain the physical meaning of generalized anisotropic models and to preserve the analogy between even $p$ and odd $p$ cases.

\[ \cdots + b_3 (\delta q^{\alpha \beta})^3 + \cdots + b_2 \delta q^{\alpha \beta} \delta q^{\beta \gamma} \delta q^{\gamma \delta} \delta q^{\delta \alpha} \cdots \]

in the free energy expansion.\(^3\)

Now, using the replica method one can write the average free energy for the Hamiltonian

\[ H = -\frac{1}{2} \sum_{i \neq j} J_{ij} [Q_i' Q_j' + V_i' V_j' + P_i' P_j'], \quad (15) \]

in a form analogous to equation (7), find the RS solution and perform the 1RSB. The extremum conditions for the 1RSB free energy have the form of three equations that look different only at the first glance. In fact they are the equations from [14], integrated over different variables and so have a common solution $v = \langle \langle Q_i Q_j' \rangle \rangle = \langle \langle V_i' V_j' \rangle \rangle = \langle \langle P_i P_j' \rangle \rangle$. Expanding the 1RSB free energy in the neighborhood of $t_c$, up to $v^4$ we obtain (in standard variables)

\[ \Delta F / N k t = -\frac{1}{8} (-1 + m) t^2 v^2 + 6 + 6 t^2 + 4 (m - 1) t^4 v + (3 m^2 - 3 m - 7) t^4 v^2. \quad (16) \]

Here $t = t_c + \tau$, $m = m_0 + \delta m$ and $\tau$, $v$ and $\delta m$ are small. The extremum conditions relative to $v$ and $m$ give the system of equations

\[ -\frac{1}{2} (-1 + m) t^2 v [-3 + 3 \tau^2 + (m - 1) t^4 v + (3 m^2 - 3 m - 7) t^4 v^2] = 0, \]

\[ \frac{1}{8} t^2 v^2 [16 - 6 t^2 - 8 (m - 1) t^4 v + (4 + 12 m - 9 m^2) t^6 v^2] = 0. \quad (17) \]

To determine the relative orders of the small values $\tau$, $v$ and $\delta m$, one has to use the so-called bifurcation equation (see, e.g. [19]) for the system (17). One can also check directly that equations (17) become incompatible if $v$ and $\delta m$ are supposed to be series in integer powers of $\tau$. If $\delta m = m_1 \sqrt{\tau}$ and $v = v_1 \sqrt{\tau}$ (as follows from the bifurcation equation), then we obtain $t_c = 1$, $m_0 = 1$, and for $m_1$ and $v_1$ the following system of equations can be obtained from (17):

\[ 6 + 3 m_1 v_1 - 7 v_1^2 = 0, \]

\[ 12 + 8 m_1 v_1 - 7 v_1^2 = 0. \]

The system has two solutions that differ only by the sign. Taking into account that

\[ \Delta F / N k t = \frac{1}{2} m_1^2 v_1^4 \tau^{5/2}, \]

and that on the physical branch $m < 1$, we choose

\[ v = \sqrt{12 \tau / 35}, \quad m = 1 - \sqrt{21 \tau / 5}. \]

So in the case of the Potts spin glass with four states $m = 1$ at the point of bifurcation $t_c$, that is, $t_{1RSB} = t_c$.

Let us consider now the Potts glass with $p = 5$ in an analogous way. We shall use the following diagonal operators: $P_k^{(1)} = (-4, 1, 1, 1, 1)$, $P_k^{(2)} = (0, -3, 1, -1, 3)$, $P_k^{(3)} = \sqrt{5} (0, 1, -1, -1, 1)$, $P_k^{(4)} = (0, 1, 3, -3, -1)$. The equivalence to the Potts model follows from the relation

\[ P_k^{(1)} P_i^{(1)} + P_k^{(2)} P_i^{(2)} + P_k^{(3)} P_i^{(3)} + P_k^{(4)} P_i^{(4)} + 4 = 20 \delta_{k,l}. \]

The nonzero cubic terms are

\[ p_{i}^{(1)} = -12, \quad p_{i}^{(2)} = 4, \quad p_{i}^{(3)} = 4, \quad p_{i}^{(4)} = 4, \]

\[ p_{i}^{(4)} = 4, \quad p_{i}^{(2)} = 16/\sqrt{5}, \quad p_{i}^{(4)} = -16/\sqrt{5}, \quad p_{i}^{(1)} = -12/\sqrt{5}. \]

\(^3\) It is worth noting that our representation in terms of three matrices is not unique. For example, one can use the set $A_k = (1, -1, -1, 1, 1)$, $B_k = (-1, 1, -1, 1, 1)$, $C_k = (1, 1, -1, -1, 1)$. We use, however, the quadrupole-like operators in order to retain the physical meaning of generalized anisotropic models and to preserve the analogy between even $p$ and odd $p$ cases.
Now the 1RSB free energy expansion near $t_c$ has the following form:

$$\Delta F / NkT = -\frac{1}{12}(-1 + m)^2 t^2 v^2 \left[ -12 + 12 t^2 + 4(2m - 1)t^4v + (6m^2 + 6m - 35)t^6v^2 \right]. \quad (18)$$

The extremum conditions are

$$-\frac{1}{12}(-1 + m)^2 v[ -6 + 6 t^2 + 3(2m - 1)t^4v + (6m^2 + 6m - 35)t^6v^2 ] = 0,$$

$$\frac{1}{12} t^2 v^2 [12 - 12 t^2 - 4(2m - 1)t^4v + (41 - 18m^2)t^6v^2 ] = 0. \quad (19)$$

It is easy to obtain that $v$ and $\delta m$ can now be presented as a series in integer degrees of $\tau$:

$$v = v_1 \tau + v_2 \tau^2, \quad m = m_0 + m_1 \tau.$$ Here $v_1$, $v_2$, $m_0$ and $m_1$ satisfy the following equations:

$$6 - 3v_1 + 4m_0 v_1 = 0,$$

$$4 - v_1 + 2m_0 v_1 = 0$$

and

$$6v_2 - 12m_1 - 92 = 0,$$

$$6v_2 - 16m_1 - 43 = 0,$$

so that finally we obtain

$$t_c = 1, \quad m = \frac{3}{2} + \frac{40}{7} \tau, \quad v = -2\tau + \frac{39}{7} \tau^2$$

and the transition from RS to 1RSB cannot take place at the bifurcation point.

Here we would like to make a remark about the five-state model without claiming to be rigorous. Based on the fact that the numerically obtained [14] value of $t_{1\text{RSB}}$ for $m = 1$ is very close to our $t_c$, one can hope to obtain the jump to the 1RSB solution already from equation (18) and to estimate qualitatively the characteristics in question. In fact, if we expand (18) in the vicinity of $m = 1$ and let its first and second derivatives relative to $v$ be equal to zero, we obtain $t_{1\text{RSB}}^2 = 184/187$ and the jump $\Delta v = 561/8464$ (compare with the table in [14]).

The qualitative behavior of the solutions for $v$ near $t_c$ for $p = 3, 4, 5$ is presented in figure 1. Figure 1 demonstrates that the actual analytical forms of the 1RSB glass order parameters near the bifurcation points of the RS solution are such that it is plausible to assume that the point of crossover is just the point $p = 4$ itself because the tangent to the order parameter is vertical at this point.

Now let us consider the 1RSB free energy expansion near the bifurcation point $t_c$ from a more general point of view. It is interesting that the form of the series for $\Delta F$ in small deviations $â'q_{ab}^{\alpha \beta}$ from $q_{RS}$ up to the third order is one and the same for different models and coincides with the one obtained [13] for the random generalized Hamiltonian (1) where $U$ is an arbitrary diagonal operator. In the most general case (including (1) and (3)), $\Delta F$ depends on the following (the only nonzero) sums:

$$\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta} \langle \delta q_{ab}^{\alpha \beta} \rangle^2 = -[r - (m - 1)v]^2 - m(1 - m)v^2, \quad (20)$$

$$\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta, \delta} \langle \delta q_{ab}^{\alpha \beta} \delta q_{cd}^{\gamma \delta} \rangle = [r - (m - 1)v]^2, \quad (21)$$

$$\lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta} \langle \delta q_{ab}^{\alpha \beta} \delta q_{cd}^{\gamma \delta} \rangle^2 = -[r - (m - 1)v]^3 + 3m(m - 1)[r - (m - 1)v]v^2$$

$$+ m(m - 1)(2m - 1)v^3, \quad (22)$$
\[\lim_{n \to 0} \frac{1}{n} \sum_{a,\beta,\gamma} \delta q^{a\beta} \delta q^{\beta\gamma} \delta q^{\gamma\alpha} = 2[r - (m - 1)v]^3 - 3m(m - 1)(r - (m - 1)v)v^2 - m^2(m - 1)v^3, \quad (23)\]

\[\lim_{n \to 0} \frac{1}{n} \sum_{a,\beta,\gamma} \delta q^{a\beta} \delta q^{\beta\gamma} = [r - (m - 1)v]^3 - [r - (m - 1)v]v^2. \quad (24)\]

\[\lim_{n \to 0} \frac{1}{n} \sum_{a,\beta,\gamma,\delta} \delta q^{a\beta} \delta q^{\beta\gamma} \delta q^{\gamma\alpha} = \lim_{n \to 0} \frac{1}{n} \sum_{a,\beta,\gamma} \delta q^{a\beta} \delta q^{\beta\gamma} \delta q^{\gamma\delta} = -[r - (m - 1)v]^3, \quad (25)\]

with \( r = r_1 - q_{RS} \). The prime on the sum means that only the superscripts belonging to the same \( \delta q \) are necessarily different in \( \sum \).

So the deviation \( \Delta F \) of the free energy \( F_{RSB} \) from its RS part in the most general case is

\[
\frac{\Delta F}{NkT} = \frac{t^2}{4} [1 - t^2 W] [-[r - (m - 1)v]^2 - v^2(m - 1)]
\]

\[
- \frac{t^4}{2} L [r - (m - 1)v]^2 - t^6 [C[r - (m - 1)v]^3 + D[r - (m - 1)v]v^2m(m - 1) - B_3v^3m^2(m - 1) + B_4v^3m(m - 1)(2m - 1)] + \cdots, \quad (26)\]

where \( t = t_c + \tau \), and the parameters \( W, L, C, D, B_3, B_4 \) are some combinations of operators averaged over the RS solution. For example, the coefficient \( L \) enters \( \Delta F \) with the sum \((21)\):

\[
\frac{1}{n} \sum_{a,\beta,\gamma} \delta q^{a\beta} \delta q^{\beta\gamma}.
\]

Let us note that in the case of zero RS solutions for the order parameters, the expansion does not contain the terms where some indices occur only once. In the case of reflection symmetry, there is no term where some indices occur an odd number of times. If there is no reflection symmetry, using the extremum conditions for the free energy for the Hamiltonian (3) and taking into account the fact that \( L |_{m=1} \neq 0 \), the bifurcation condition gives \([13]\) \( r - (m - 1)v = 0 + o(\Delta t)^2 \), i.e. the condition that there is no linear term for the glass order parameters. In fact, there is no other linear term because

\[
[1 - t^2 W] |_{m=1} = \lambda_{RSB} |_{m=1} = 0 \quad (27)
\]

at the bifurcation point.

In the case of the Potts spin glass model, the situation is quite different, although the reflection symmetry is absent. Now \( L = 0 \), because it is the zero RS solution that bifurcates so that single indices mean that \( L \) is composed of zero values of averaged \( \text{Tr} \mathcal{Q} \text{Tr} V \ldots \). Analogously the coefficients in front of \((24)\) and \((25)\) are also zero. So we have \( C = 2B_3 - B_4 \) and \( D = 3(-B_3 + B_4) \). Here \( B_3 \) is the coefficient in front of \((23)\), and \( B_4 \) that in front of \((22)\). Since \( L |_{m=0} = 0 \), the condition \( r - (m - 1)v = 0 \) is not fulfilled (if \( p > 2 \), then \( r_1 = r = 0 \) \([14]\)).

In the general case, we obtain from the extremum conditions for the free energy \((26)\) the following equations for the order parameters:

\[
2m(m - 1)v Z \Delta t = t^6 m(m - 1)v[3[-B_4 + m(-B_3 + 2B_4)]v + 2D[r - (m - 1)v]], \quad (28)
\]

\[
(2m - 1)v^2 Z \Delta t = t^6 v^2[(2m - 1)[-B_4 + m(-B_3 + 2B_4)]v
\]

\[
+ m(m - 1)(-B_3 + 2B_4)v + D(2m - 1)[r - (m - 1)v]], \quad (29)
\]

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where
\[ Z = \frac{d}{dt} \left[ \frac{t^2}{4} \left( 1 - t^2 W \right) \right] |_{t_c}. \] (30)

Hence we obtain from (28) and (29),
\[ m(m - 1)[mB_3 - B_4] = 0. \] (31)

In fact, in the case of the Potts spin glass with \( p = 3 \) for the Hamiltonian (6) at \( t = t_c \), we have
\[ W = \frac{1}{2} [(\text{Tr} Q^2 / 3)^2 + (\text{Tr} V^2 / 3)^2] = 4, \]
\[ B_3 = \frac{1}{2} [(\text{Tr} Q^2 / 3)^3 + (\text{Tr} V^2 / 3)^3] = \frac{2}{3}, \]
\[ B_4 = \frac{1}{4} [(\text{Tr} Q^3 / 3)^2 + 3(\text{Tr} QV^2 / 3)^2] = \frac{4}{3}, \]
\[ D = -4. \] (32)

In this case, in the neighborhood of the bifurcation point, one can get from equations (28) and (29) that \( v \sim \tau \), and from equation (31) \( m = \frac{1}{2} \).

Analogously, in the case of four states (for the Hamiltonian (15)), we obtain \( D = 0, B_3 = B_4 = \frac{1}{128} \) so that \( m = 1 \). Equation (28) is an identity now. The rhs of (29) becomes zero and one has to take into account higher order terms in the expansion of \( F_{1RSB} \). This leads to \( v \sim \sqrt{\tau} \). For \( p = 5 \) we have \( D = 64, B_3 = \frac{128}{3}, B_4 = 64 \), so that \( m = \frac{3}{2} \) and \( v \sim -\tau \).

It is worth noting that from the free-energy expansion (26), it follows directly that the stability of the RS state is determined by the sign of \( \lambda_{RSrepl} \), just as in the case of the SK model [20]. To prove this fact, we consider small deviations from the RS solution \( q_{\alpha\beta} = q_{RS} + \delta q_{\alpha\beta} \) at arbitrary temperature and perform 1RSB. Using the notations introduced before, that is, \( \delta q_{\alpha\beta} = r \) if \( \alpha \) and \( \beta \) are from different groups and \( \delta q_{\alpha\beta} = r + v \) if \( \alpha \) and \( \beta \) belong to the same group, we can write \( F_{1RSB} - F_{RS} \) up to the second order in \( \delta q \) in the form (26) with arbitrary \( t \). Since \( [1 - t^2 W] = \lambda_{RSrepl} \), it is easy to see that for \( m < 1 \) and \( L \geq 0 \) the RS state is stable \((\Delta F < 0)\) for the values of temperature such that \( \lambda_{RSrepl} > 0 \). A similar result is valid for all subsequent stages of RSB because of the Parisi rule
\[ \lim_{n \to 0} \frac{1}{n} \sum_{\alpha, \beta} (\delta q_{\alpha\beta})^2 < 0. \]

To conclude, we introduced a representation for the Potts glass model based on the operators of the quadrupole momenta. With the help of this representation the expansions for the free energy functionals of the Potts spin glass models with three, four and five states up to the fourth order in \( \delta q_{\alpha\beta} \) around the replica symmetric solution were obtained. The temperature dependence of the 1RSB order parameters in the vicinity of the point \( T = T_c \), where the RS solution becomes unstable was derived analytically. The crossover from continuous to jumpwise behavior with growing number of states was traced. The analytical forms of the 1RSB glass order parameters near the bifurcation points of the RS solution lead to the conclusion that the crossover should occur at \( p = 4 \), since the tangent to the order parameter is vertical at this point. The comparison was made between the free energy expansions for the Potts spin glass and that for other models and the similarity of such expansions was demonstrated.

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