HILBERT SCHEME OF SKEW LINES ON CUBIC THREEFOLDS AND LOCUS OF PRIMITIVE VANISHING CYCLES

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Abstract. On a smooth cubic threefold $Y$, the set of pairs of skew lines determines an irreducible component of the Hilbert scheme of $Y$. We prove that such component is smooth and is isomorphic to the blowup of symmetric product of the Fano surface of lines along the diagonal.

The local system of integral vanishing cohomology $\mathcal{H}^2_{\text{van}}$ over the universal locus of hyperplane sections $U$ of $Y$ determines a covering space $T \rightarrow U$. It has a distinguished component $T'$ containing a vanishing cycle associated to a nodal degeneration. $T'$ admits a canonical normal completion $\bar{T}'$ by Stein. We show that $\bar{T}'$ is achieved by contracting finitely many curves on $\text{Bl}_0\Theta$ corresponding to Eckardt points on $Y$, where $\text{Bl}_0\Theta$ is the blowup of the theta divisor of the intermediate Jacobian of $Y$ at the isolated singularity.

We provide interpretations of boundary points of the completion space $\bar{T}'$ via minimal resolution of hyperplane sections with ADE singularities and the related theory on root systems. We also explore the relation to some Bridgland stable moduli spaces on $Y$.

1. INTRODUCTION

1.1. Hilbert scheme of skew lines. The Hilbert scheme $\text{Hilb}^{p(n)}(\mathbb{P}^m)$ parameterizes closed subschemes of $\mathbb{P}^m$ with fixed Hilbert polynomial $p(n)$. Grothendieck [Gro61] showed that it is a projective scheme and Hartshorne [Har66] showed that it is connected. According to the Morphy’s law, the Hilbert scheme can be in general highly pathological. For example, Mumford [Mum62] showed that there is an irreducible component of the smooth irreducible curves in $\mathbb{P}^3$ of degree 14 and genus 24 that is generically nonreduced. However, in some cases, the Hilbert schemes are known to contain smooth irreducible components. Piene and Schlessinger [PS85] showed that the set of smooth twisted cubics in $\mathbb{P}^3$ forms an open dense subspace of a 12 dimensional irreducible component $H$ of $\text{Hilb}^{3n+1}(\mathbb{P}^3)$. Moreover, $H$ is smooth and intersects the other component transversely.

Chen, Coskun and Nollet [CCN11] showed that the set of pairs of skew lines on $\mathbb{P}^m$ for $m \geq 3$ forms an open dense subspace of an irreducible component $H_m$ of the Hilbert scheme $\text{Hilb}^{2n+2}(\mathbb{P}^m)$. The other component $H'_m$ parameterizes a smooth conic and a disjoint point at its general point. Both $H_m$ and $H'_m$ are smooth and intersect transversely. Moreover, $H_3$ is smooth and isomorphic to the blow-up $\text{Bl}_\Delta\text{Sym}^2\text{Gr}(2,4)$ along the diagonal of the second symmetric product of the Grassmannian $\text{Gr}(2,4)$.
For a closed subscheme $Y$ of $\mathbb{P}^m$, the Hilbert scheme $\text{Hilb}^{p(n)}(Y)$ is a closed subscheme of $\text{Hilb}^{p(n)}(\mathbb{P}^m)$. When $Y$ is a smooth hypersurface in $\mathbb{P}^4$, there is a smooth surface $F$ of general type parameterizing lines on $Y$. There is an open dense subspace $\mathcal{M}$ of $F \times F$ parameterizing pairs of skew lines and such that its linear span intersects $Y$ transversely. There is a 2-to-1 map

$$\tau : \mathcal{M} \to \text{Hilb}(Y), \ (L_1, L_2) \mapsto \mathcal{O}_{L_1 \cup L_2},$$

whose image is a Zariski dense open subspace of an irreducible component $H(Y)$ of the Hilbert scheme of $Y$. $H(Y)$ is called the Hilbert scheme of skew lines on $Y$. Working over $\mathbb{C}$, we have the following result.

**Theorem 1.1.** $H(Y)$ is smooth and isomorphic to the blow-up $\text{Bl}_{\Delta_F} \text{Sym}^2 F$.

By taking the double cover branching along the exceptional divisor, we obtain $\widehat{H(Y)} \cong \text{Bl}_{\Delta_F} (F \times F)$, where $\widehat{H(Y)}$ can be informally regarded as the "Hilbert scheme" of ordered skew lines on $Y$. There is a "Hilbert-Chow morphism"

$$\widehat{H(Y)} \to F \times F,$$

identified with blowup along the diagonal $\Delta_F$.

$H(Y)$ parameterizes four types of schemes. A pair of skew lines $L_1, L_2$ is of type (I). As $L_1, L_2$ come to intersect at a point, the flat limit obtains an embedded point at the intersection, and this is a type (III) scheme. As $L_1, L_2$ come to coincide to a single line $L$, the flat limit has a double structure supported on $L$, and the scheme can be of type (II) or type (IV) depending on the normal bundle $N_{L|Y}$ being isomorphic to $\mathcal{O}_L \oplus \mathcal{O}_L$ or $\mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$.

By Clemens and Griffiths [CG72], the Abel-Jacobi map

$$\psi : F \times F \to JY, \ (L_1, L_2) \mapsto \int_{L_2}^{L_1},$$

by sending a pair of orderd lines to the linear functional that integrates the (2,1)-forms against a 3-chain whose boundary is $L_1 - L_2$. The image of $\psi$ is the theta divisor $\Theta$ of the intermediate Jacobian $JY$ of $Y$. According to Beauville [Bea82], $\Theta$ has 0 as its only singularity, and $\psi$ extends to a mophism on the blowup

$$\text{Bl}_{\Delta_F}(F \times F) \to \text{Bl}_0 \Theta,$$

whose restriction on the exceptional divisor is just the natural map of the incidence variety

$$\{p \in L\} \to Y.$$
To prove Theorem 1.1, one of the key points is to use the incidence variety to parameterize the type (II) and type (IV) schemes.

Just as a pair of skew lines spans a $\mathbb{P}^3$, each of the scheme $Z \in H(Y)$ determines a unique hyperplane in $\mathbb{P}^4$. Thus there is a morphism

$$\pi : H(Y) \rightarrow (\mathbb{P}^4)^*,$$

whose fiber at the hyperplane $H$ is identified with Hilbert schemes $H(S)$ of skew lines of the cubic surface $S$ where $S = Y \cap H$. $H(S)$ consists of 216 reduced points when $S$ is smooth, and is non-reduced when $S$ is singular. We will show that

**Theorem 1.2.** When $S$ at worst ADE singularities, $|H(S)_{\text{red}}| < \infty$; When $S$ has an elliptic singularity, $H(S)_{\text{red}} \cong \text{Sym}^2 E$, where $E$ is a smooth plane curve whose cone is isomorphic to $S$.

Equivalently, $\pi$ is generically finite, and the positive dimensional fibers of $\pi$ are correspondent to Eckardt points of $Y$. This allows us to characterize the following completion space.

1.2. **Locus of primitive vanishing cycles.** Let $\mathcal{O}^{\text{sm}}$ denote the open subspace of $\mathcal{O} := (\mathbb{P}^4)^*$ parameterizing smooth hyperplane sections. Denote $T'$ the set of pairs $(t, [L_1] - [L_2])$ such that $t \in \mathcal{O}^{\text{sm}}$ and that $L_1$ and $L_2$ are skew lines in the hyperplane section $Y \cap H_t$. $T'$ is called the locus of primitive vanishing cycles of $Y$. Equivalently, let $T$ denote the underlying analytic space of the local system $\mathcal{H}_{\text{van}}^2$ on $\mathcal{O}^{\text{sm}}$, whose stalk at $t$ is isomorphic to the vanishing cohomology $H^2(S_t, \mathbb{Z})_{\text{van}}$. $T$ has a distinguished component $T'$ containing a vanishing cycle of a nodal generation.

The projection to the first coordinate

$$T' \rightarrow \mathcal{O}^{\text{sm}}$$

is naturally a 72-to-1 covering space and is connected. The fiber is identified with the root system $R(\mathfrak{e}_6)$ of the Lie algebra $\mathfrak{e}_6$. By a theorem of Stein (Lemma 2.10), there is a unique normal analytic space $\tilde{T}'$ containing $T'$ as open dense subspace, and $\tilde{T}'/\mathcal{O}$ extends $T'/\mathcal{O}^{\text{sm}}$ as branched analytic covering spaces. Note that $\tilde{T}'$ is algebraic by GAGA’s theorem.

One can similarly compactify $T$ due to finite monodromy, and the compactified space $\bar{T}$ is exactly the compactification space defined by Schnell in Theorem 4.2 from [Sch14]. To be more precise, for the analytic space $T$ associated to a variation of Hodge structure $\mathcal{H}$ of weight $2n$ over a smooth base $U$, and for any smooth completion $\bar{U} \subseteq \bar{U}$, Schnell defined $\bar{T}$ as closure of $T$ inside certain analytic space defined by a $\mathcal{D}$-module as the minimal extension of bundle $\mathcal{F}^{n+1}\mathcal{H}$. In our case, all the primitive vanishing cycles are algebraic and the monodromy is finite. It turns out that such $\mathcal{D}$-module is trivial, and that the completion is
the same as the completion defined via Stein’s theorem. The space \( T \) will be more interesting when \( T \to \mathcal{O}^\text{sm} \) has infinite monodromy (for example, when \( Y \) is a quartic threefold, where the Hodge bundle is non-trivial), but it lacks an interpretation of the boundary points. To provide a concrete description to Schnell’s completion was our original motivation for this research.

Note that there is a 6-to-1 covering map

\[
e : \mathcal{M} \to T', \quad (L_1, L_2) \mapsto [L_1] - [L_2].
\]

\( \mathcal{M} \) naturally closes up in a branched double cover \( \overline{H(Y)} \cong \text{Bl}_{\Delta_F}(F \times F) \) of \( H(Y) \). It turns out that \( e \) extends to \( \mathcal{O}^\text{sm} \). As a consequence of Theorem 1.2 and the geometry of Abel-Jacobi maps, we obtained the following:

**Theorem 1.3.** There is a dominant morphism \( \text{Bl}_0 \Theta \to \overline{T}' \) contracting finitely many curves that are 1-1 correspondence with the Eckardt points on \( Y \). In particular, when \( Y \) is general and has no Eckardt point, \( \text{Bl}_0 \Theta \cong \overline{T}' \).

The boundary points on \( \overline{T}' \) has the following topological interpretation. Let \( B \) be a small ball around \( t_0 \). Pick a base point \( t' \in B^\text{sm} = B \cap \mathcal{O}^\text{sm} \). The fundamental group \( \pi_1(B^\text{sm}, t') \) acts on the root system \( R(\mathbb{E}_6) \) over \( t' \) via monodromy action. Then each point \( p \in \overline{T}' \) corresponds to one of the \( \pi_1(B^\text{sm}, t') \)-orbits, and it lies in the ramification locus of a connected component of the covering space \( T'|_{B^\text{sm}} \to B^\text{sm} \).

Let \( S \) be a cubic surface with at worst ADE singularities. Let

\[
\sigma : \tilde{S} \to S
\]

be its minimal resolution, then \( \tilde{S} \) is a weak del Pezzo surface of degree 3 and one can still define the root system \( R(\tilde{S}) \) as

\[
(2) \quad \alpha^2 = -2, \quad \alpha \cdot K_{\tilde{S}} = 0.
\]

It is known that \( R(\tilde{S}) \) is isomorphic to \( R(\mathbb{E}_6) \). Let \( W \) denote the Weyl group of \( \mathbb{E}_6 \), then \( W \) acts transitively on the 72 roots. Let \( W_e \) be the subgroup of \( W \) generated by the reflections corresponding to the effective (-2)-curves on \( \tilde{S} \) over the singular points of \( S \). \( W_e \) acts on the root system \( R(\mathbb{E}_6) \).

**Theorem 1.4.** The fiber of \( T' \to \mathcal{O} \) over \( t \) corresponding to the cubic surface \( S = H_i \cap Y \) is identified with the orbits on \( R(\mathbb{E}_6)/W_e \).

We need to use the Milnor fiber theory of isolated hypersurface singularities.

In [LLSvS17], the authors also used the orbits \( R(\mathbb{E}_6)/W(R_e) \) to study the reduced Hilbert schemes of generalized twisted cubics on \( S \).
We strengthen the result in [Sch11] stating that tube mapping on the component $T'$ is surjective in this special case, which was predicted by Herb Clemens that the topology of the locus of primitive vanishing cycles $T'$ is "complicated enough" to describe the middle dimensional primitive cohomology of $Y$.

**Proposition 1.5.** Let $Y$ be a general cubic threefold, and $\alpha \in H^2_{\text{van}}(Y_t, \mathbb{Z})$ a primitive vanishing cycle on a smooth hyperplane section. Then every element of $H^3(Y, \mathbb{Z})$ can be represented by a "tube" on $\alpha$, namely the trace of monodromy of the Poincare dual of $\alpha$ via a loop $l \subseteq \mathcal{O}^{\text{sm}}$ based at $t$ such that $l_* \alpha = \alpha$.

1.3. **Relation to moduli theory.** According to [APR19] and [BBF+20], the blowup of the theta divisor is isomorphic to a Bridgeland stable moduli space $\text{Bl}_0 \Theta \cong \mathcal{M}_\sigma(w)$ whose general point parameterizes $i_* \mathcal{O}_S(L - M)$ for $S$ a smooth hyperplane section of $Y$ and $L, M$ a pair of skew lines and $i : S \to Y$ is the inclusion. We have a natural map $\mathcal{M} \to \mathcal{M}_\sigma(w)$ by sending $(L, M) \mapsto i_* \mathcal{O}_S(L - M)$. Since this map uniquely extends to $\overline{\mathcal{M}}$, we have

**Proposition 1.6.** The Abel-Jacobi map $\widetilde{H(Y)} \to JY$ factors through the Bridgeland moduli space

$$
\begin{array}{ccc}
\widetilde{H(Y)} & \xrightarrow{\Psi} & \mathcal{M}_\sigma(w) \\
\downarrow \psi & & \downarrow \\
JY & & 
\end{array}
$$

where the vertical map is taking the second Chern class, followed by the standard Abel-Jacobi map. $\Psi$ is identified with $\overline{\mathcal{M}}$.

1.4. **Outline.** In section 2, we review the relation between cubic surfaces and the root system of $E_6$ and some basic facts about cubic threefolds. We also introduce the notion of locus $T'$ of primitive vanishing cycles on hyperplane sections of the cubic threefold $Y$ and its completion. In section 3 we review the work by [CCN11] on Hilbert schemes of skew lines on projective space, and give a proof of Theorem 1.1. In section 4 we study the Hilbert scheme of skew lines contained in a hyperplane section, and give a proof of Theorem 1.2. In section 5 we study compactification $\overline{T'}$ of the locus of primitive vanishing cycles and prove the Theorem 1.3. In section 6 we provide an interpretation of the boundary points of $\overline{T'}$ via the minimal resolution of cubic surfaces that have at worst ADE singularities. We will also prove Theorem 1.4. In section 7 we relate the "Hilbert scheme" $\widetilde{H(Y)}$ of ordered skew lines to the Bridgeland stable moduli space studied in [APR19]. We will prove Proposition 1.6. In section 8 we relate the construction of completion $\overline{T'}$ to Schnell’s completion and prove Proposition 1.5.
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2. Preliminaries

2.1. Roots on cubic surfaces. Let $S$ be a smooth cubic surface, then its Picard group is isomorphic to cohomology group $\text{Pic}(S) \cong H^2(S, \mathbb{Z}) \cong \mathbb{I}_{1,6}$, with basis $e_0, e_1, \ldots, e_6$ and the intersection pairing

$$e_i \cdot e_j = \begin{cases} 1, & \text{if } i = j = 0, \\ -1, & \text{if } i = j \neq 0, \\ 0, & \text{if } i \neq j. \end{cases}$$

The hyperplane class is $h = 3e_0 - e_1 - \cdots - e_6$, which is represented by a cubic curve passing through the 6 blow-up points in a planar representation. So the 27 lines are identified with the set of classes $\beta$ such that $\beta^2 = -1$ and $\beta \cdot h = 1$. The automorphism group $\text{Aut}(\mathbb{I}_{1,6})$ is isomorphic to the Weyl group $W(\mathbb{E}_6)$ of the $\mathbb{E}_6$ root system. The set

$$(3) \quad R_S = \{\alpha \in \text{Pic}(S)|\alpha^2 = -2, \alpha \cdot h = 0\},$$

together with the intersection pairing is isomorphic to the root system of $\mathbb{E}_6$, which contains 72 roots.

There is a one-to-one correspondence between the set of 72 roots and the six mutually disjoint lines. The set of six mutually disjoint lines uniquely determines a planar representation by blowing down the 6 exceptional lines, so there are 72 planar representations.

A root $\alpha$ and its negative counterpart $-\alpha$ correspond to a double-six $$\{L_1, \ldots, L_6; M_1, \ldots, M_6\}.$$ Moreover, by re-indexing the lines, we can assume $L_i$ is disjoint to $M_i$, for $i = 1, \ldots, 6$ and the root $\alpha$ can be written as the differences

$$\alpha = [L_i] - [M_i], \ i = 1, \ldots, 6.$$ For $i, j$ such that $1 \leq i \neq j \leq 6$, there is a unique line $L_{ij}$ on $S$ such that

$$[L_i] + [M_j] + [L_{ij}] = [L_j] + [M_i] + [L_{ij}] = h$$
is the hyperplane class. Each triple is called a tritangent trio, since the three lines mutually intersect and determine a hyperplane which is tangent to the cubic surface $S$ at three intersection points. The Weyl group action is transitive on the set of 72 roots, and is therefore on the six mutually disjoint lines. One refers to [Dol12], section 8.2.3, 9.1.1 and 9.1.2 for most of these facts.

Now we will discuss the relationship between roots and twisted cubics. Each of the planar representation corresponds to a rational equivalent class of twisted cubics. They are obtained by pullback of a general line on $\mathbb{P}^2$. So there are exactly 72 rational equivalent classes of twisted cubics on $S$. Conversely, if $C$ is a twisted cubic on $S$, then $C$ corresponds to a unique planar representation of $S$ such that the projection of $C$ to $\mathbb{P}^2$ is a line.

For each pair of skew lines $(L, M)$ on $S$, the difference $[L] - [M]$ is a root, which corresponds to a unique rational equivalent class of twisted cubic by adding the hyperplane class $h$. In sum, on a smooth cubic surface $S$, there is a one-to-one correspondence between

72 roots ↔ 72 six mutually disjoint lines ↔ 72 rational equivalent classes of twisted cubics

2.2. Basic facts about cubic threefolds. Let $Y$ be a smooth cubic threefold. The set of lines on $Y$ is parameterized by a smooth surface $F$ of general type.

The Albanese variety $\text{Alb}(F) = H^{1,0}(F)^*/H_1(F, \mathbb{Z})$ of $F$ is isomorphic to the intermediate Jacobian

$$JY = F^2H^3(X, \mathbb{C})^*/H_3(X, \mathbb{Z})$$

of $Y$ ([Bea82], Proposition 9), which is a principally polarized abelian variety of dimension 5, according to [CG72]. The theta divisor $\Theta$ of $JY$ has a unique singularity, which is a triple point and the projective tangent cone is isomorphic to the cubic threefold $Y$ itself [Bea82].

On a general cubic threefold $Y$, there are at most 6 lines passing through each given point $p \in Y$. However, a special cubic threefold may have points such that there are one-parameter families of lines through these points.

Definition 2.1. Let $Y$ be a smooth cubic threefold. An Eckardt point $p \in Y$ is a point where there are infinitely many lines on $Y$ passing through $p$.

Lemma 2.2. Let $Y$ be a smooth cubic threefold, then there are at most finitely many Eckardt points. $p \in Y$ is an Eckardt point if and only if the tangent hyperplane section $T_pY \cap Y$ is a cone over a smooth cubic plane curve. Moreover, a general cubic threefold $Y$ has no Eckardt points.

Proof. The first two statements can be derived from Lemma 8.1 in [CG72]. For the last statement, denote by $C$ the locus in the universal family $\mathbb{P}^{19} = \mathbb{P}(\text{Sym}^3 \mathbb{C}^4)$ of cubic surfaces that parameterizes cone over plane cubic curves. Then $\dim C = 12$. Let $W$ be the space of all cubic surfaces in $\mathbb{P}^4$. Since every cubic surface sits in exactly one hyperplane section,
then there is a natural projection

\[ p : W \to (\mathbb{P}^4)^*, \]

whose fiber is isomorphic to \( \mathbb{P}^19 \). Set \( \mathbb{P}^{34} = \mathbb{P}(\text{Sym}^3 \mathbb{C}^5) \) to be the space of all cubic hypersurfaces in \( \mathbb{P}^4 \). Then there is a map

\[ f : \mathbb{P}^{34} \times (\mathbb{P}^4)^* \to W \]

\[(Y, H) \mapsto Y \cap H,
\]

by sending a cubic threefold to a hyperplane section. Then \( f \) preserves the projection to \( (\mathbb{P}^4)^* \). Moreover, \( f \) is a fiber bundle over \( W \) and the fiber consists of cubic threefolds containing a fixed cubic surface, which has constant dimension 15. Let \( \mathcal{C} \subseteq W \) be the locus of cone over plane cubic curves, then \( \text{codim}_W \mathcal{C} = 7 \). Therefore the preimage \( f^{-1}(\mathcal{C}) \) has codimension 7 as well. It follows that its image in \( \mathbb{P}^{34} \) under the projection to the first coordinate has codimension at least 3, which completes the proof. \( \square \)

Next, we will discuss the lines on the hyperplane sections of a smooth cubic threefold \( Y \). According to Lemma 4.2, any hyperplane section \( S \) of \( Y \) will be normal. Moreover, either \( S \) has at worst ADE singularities, or \( S \) is a cone over a smooth plane cubic curve. Therefore, for a general hyperplane section \( S \), there are 27 lines on it. When the hyperplane section acquires an ordinary node, the double-six comes together and there are 6 lines through the singularity that have multiplicity two and 15 lines disjoint from the singularity. As cubic surface becomes slightly more singular, the number of lines goes down. When the hyperplane section acquires an elliptic singularity, the number of lines jumps to infinity. We would like to understand this phenomenon in pairs.

Let \( \mathcal{O} = (\mathbb{P}^4)^* \) and \( \mathcal{O}^{\text{sm}} \) be the open subspace parameterizing smooth hyperplane sections. Let’s consider the set of triples

\[ \mathcal{M} = \{(L_1, L_2, t) \in F \times F \times \mathcal{O}^{\text{sm}} \mid L_1, L_2 \subseteq Y_t, L_1 \cap L_2 = \emptyset\} \]

consisting of disjoint lines \( L_1 \) and \( L_2 \) on the smooth hyperplane section \( Y_t \). Then the natural projection \( \mathcal{M} \to \mathcal{O}^{\text{sm}} \) is a covering space with induced analytic structure.

**Proposition 2.3.** \( \mathcal{M} \to \mathcal{O}^{\text{sm}} \) is a connected covering space of degree 432.

**Proof.** There are 27 lines on a smooth cubic surface. For each line \( L \), there are exactly 16 lines disjoint from \( L \) on the cubic surface. Therefore, the degree of the covering map is \( 27 \times 16 = 432 \).

Let \( Y^{\text{sm}} \to \mathcal{O}^{\text{sm}} \) be the smooth family of cubic surfaces as smooth hyperplane sections of \( Y \), then the monodromy group permuting the 27 lines over the base is isomorphic to the Weyl group \( W(\mathbb{E}_6) \) [Che20], which is the full automorphism group \( \text{Aut}(I_{1,6}) \). Therefore, the monodromy action is transitive on the set of 6 mutually disjoint lines and the stabilizer
group is isomorphic to permutation group $S_6$ (Proposition 9.2.3 in [Dol12]). Moreover the stabilizer group action on the 6 lines is faithful, since for the 6 lines $\{e_1, \ldots, e_6\}$, the reflection given by the root $e_i - e_j$ permutes $e_i$ and $e_j$ and fixes other four lines. Therefore, the Weyl group $W(E_6)$ is transitive on the ordered 6-tuple $(L_1, \ldots, L_6)$ of mutually disjoint lines. In particular, the monodromy action is transitive on the disjoint pairs $(L_1, L_2)$, so the covering space $\mathcal{M}/\mathbb{O}^{sm}$ is connected. □

We are interested in the possible completion of $\mathcal{M}$. First, since any pair of disjoint lines uniquely determines the hyperplane that containing them by taking their span, there is an inclusion $\mathcal{M} \hookrightarrow F \times F$ and the image is a dense open subspace. Therefore $F \times F$ is a candidate for compactification of $\mathcal{M}$. Moreover, $F \times F$ parameterizes different configurations of pairs of lines in $Y$ and can be regarded as a Chow variety. It has a natural stratification $\Delta \subseteq D \subseteq F \times F$, where $\Delta$ is the diagonal parameterizing two lines that coincide, and general element in $D$ corresponds to two lines that intersect at a single point.

On the other hand, each disjoint pair $(L_1, L_2)$ corresponds to its ideal sheaf $I_{L_1 \cup L_2}$, and thus to the coherent sheaf $\mathcal{O}_{L_1 \cup L_2} = \mathcal{O}_Y/I_{L_1 \cup L_2}$ on $Y$, which has constant Hilbert polynomial $2n + 2$. Therefore we can look at flat limits of these ideals and compactify the space in the Hilbert scheme. Since the ideal sheaf does not distinguish the order of the lines, a double cover of the Hilbert scheme will compactify $\mathcal{M}$.

Lastly, we can take the closure of $\mathcal{M}$ inside the product $F \times F \times \mathbb{O}^{sm}$. We will discuss the relationship between the three compactifications, especially the first two.

2.3. Locus of primitive vanishing cycles. Let $X$ be a smooth projective variety of dimension $n$ embedded in a projective space $\mathbb{P}^N$. Let $\mathbb{O}^{sm}$ be the parameter space of smooth hyperplane sections, then there is a local system $\mathcal{H}^{n-1}_{\text{van}}$ whose stalk at $t$ is isomorphic to the vanishing cohomology

$$H^{n-1}_{\text{van}}(X_t, \mathbb{Z}) = \ker (H^{n-1}(X, \mathbb{Z}) \to H^{n-1}(X_t, \mathbb{Z}))$$

on the hyperplane section $X_t = X \cap H_t$. When $X$ is a smooth hypersurface in $\mathbb{P}^N$, $H^{n-1}_{\text{van}}(X_t, \mathbb{Z})$ coincides with the primitive cohomology $H^{n-1}_{\text{prim}}(X_t, \mathbb{Z})$.

The complement $\mathbb{O} \setminus \mathbb{O}^{sm}$ is isomorphic to the dual variety $X^\vee$ of $X$. According to a classical result by Lefschetz, a smooth point of $X^\vee$ corresponds to a hyperplane section that has only one ordinary node. Choose a holomorphic disk $\Delta \subset \mathbb{O}$ such that $\Delta^* \subset \mathbb{O}^{sm}$ and $\Delta$ intersects transversely to the boundary divisor at a smooth point. Then there is a one-parameter family $\{X_t\}_{t \in \Delta}$ of hyperplane sections with $X_0$ having a single node and $X_t$ smooth for $t \neq 0$. Let $\mathcal{X}_{\Delta}$ denote the total space, and $B_p \subseteq \mathcal{X}_{\Delta}$ a small neighborhood of the node $p \in X_0$. When $|t|$ is small enough, the Milnor fiber $X_t \cap B_p$ is diffeomorphic to the disk bundle of the tangent bundle $TS^{n-1}$ of a topological $(n - 1)$-sphere. Moreover, the
zero section $S^{n-1}$ specializes to the node $p$ as $t$ moves to 0. The fundamental class of the zero section defines a cohomology class $\delta_t \in H^{n-1}(X_t, \mathbb{Z})$, which is called a vanishing cycle associated to the degeneration $\{X_t\}_{t \in \Delta}$.

All vanishing cycles obtained above is conjugate to each other via a monodromy action ([Voit03], Proposition 3.23) due to irreducibility of $X^\vee$. We want to consider the entire orbit of vanishing cycle $\delta_t$ under the $\pi_1(\mathcal{O}^{sm}, t)$, so we introduce the following notion:

**Definition 2.4.** For any $t' \in \mathcal{O}^{sm}$, the class $\delta_{t'} \in H^{n-1}_{\text{van}}(X_{t'}, \mathbb{Z})$ is called a primitive vanishing cycle if it is conjugate to the vanishing cycle $\delta_t$ via monodromy action. Namely, there is a path $l \subseteq \mathcal{O}^{sm}$ joining $t'$ to $t$ which transports $\delta_{t'}$ to $\delta_t$ in the local system $\mathcal{H}^{n-1}_{\text{van}}$.

**Lemma 2.5.** The set of all primitive vanishing cycles generates the vanishing cohomology $H^{n-1}(X_t, \mathbb{Z})$. If $X_t$ has even dimension (resp. odd dimension), a primitive vanishing cycle on $X_t$ has self-intersection $\pm 2$ (resp. 0).

**Proof.** Fix $t \in \mathcal{O}^{sm}$. According to Lemma 2.26 in [Voit03], the vanishing cohomology $H^{n-1}(X_t, \mathbb{Z})$ is generated by a subset of primitive vanishing cycles which come from a Lefschetz pencil. Explicitly, take a general line $L$ through $t$, the associated family of hyperplane sections $\{X_t\}_{t \in L}$ has $k = \deg(X^\vee)$ nodal members, denoted as $t_1, ..., t_k$, so there are vanishing cycles $\delta_{t_i}$ on $X_{t_i}$ where $t_i'$ is close to $t_i$. By deleting a general point on $L$, there is a unique path $l_i$ joining $t_i'$ to $t_i$, and Lemma 2.26 says that the classes $(l_i)_*(\delta_{t_i})$, $i = 1, ..., k$ generates $H^{n-1}(X_t, \mathbb{Z})$, since they form a subset of primitive vanishing cycles, the first statement is proved. The second statement is due to Remark 3.21 in [Voit03].

Let $T$ denote the étale space of the local system $\mathcal{H}^{n-1}_{\text{van}}$, then $T \to \mathcal{O}^{sm}$ is an analytic covering space. The set of all primitive vanishing cycles on $X_t$ as $t$ varying over $\mathcal{O}^{sm}$ forms a connected component $T'$ of $T$.

**Definition 2.6.** We call the covering space $T' \to \mathcal{O}^{sm}$ the locus of primitive vanishing cycles on the hyperplane sections of $X$, with respect to the given projective embedding $X \subseteq \mathbb{P}^N$.

Now, we restrict to the case of cubic threefold. Let $Y$ be a smooth cubic threefold with standard embedding in $\mathbb{P}^4$. Then $\mathcal{O} = (\mathbb{P}^4)^*$ and $t \in \mathcal{O}^{sm} \subseteq \mathcal{O}$ correspond to a smooth cubic surface $Y_t$. Since $H^2(Y_t, \mathbb{Z}) \cong \text{Pic}(Y_t)$, all the primitive vanishing cycles are algebraic.

**Proposition 2.7.** Every primitive vanishing cycle $\alpha$ on $Y_t$ can be written as the difference $[L_1] - [L_2]$ for a pair of skew lines $L_1, L_2$ in $Y_t$ in exactly 6 different ways.

**Proof.** Consider a family $\{Y_t\}_{t \in \Delta}$ of hyperplane sections with $\Delta \subseteq \mathcal{O}$ and intersecting $Y^\vee$ transversely at a smooth point $0 \in \Delta$. There is a vanishing cycle $\delta$ on nearby $Y_t$.

By taking a double cover $\tilde{\Delta} \to \Delta$ branched at 0 and taking a small resolution, we get a smooth family of weak del Pezzo surfaces $\{\tilde{Y}_{\tilde{s}}\}_{\tilde{s} \in \tilde{\Delta}}$ with $s^2 = t$. The family can be represented...
by moving the six general blow-up points $p_1, ..., p_6$ on $\mathbb{P}^2$ to lie on a smooth conic $Q$, whose strict transform is a $(-2)$ curve, and its deformation is the vanishing cycle on the nearby $\tilde{Y}_s = Y_{s^2}$. Therefore, if we write $e_0$ as class of pullback of a general line on $\mathbb{P}^2$, and $e_i$ the class of exceptional curve $E_i$ of blow-up at $p_i$, for $i = 1, ..., 6$, the vanishing cycle

$$\alpha = 2e_0 - e_1 - ... - e_6.$$

Since $2e_0 - e_1 - ... - e_6 = [Q_1] - [E_i]$, where $[Q_i] = 2e_0 - e_1 - ... - e_6 + e_i$ is the class of strict transform of the conic $Q_i$ through the five points except for $p_i$, and $E_i$ is the $i$-th exceptional line representing $e_i$, we can write the vanishing cycle as the differences of classes of two skew lines in six ways. By comparing the classes of other lines, these are the only six ways.

Now let $\alpha'$ be a primitive vanishing cycle on $Y_t$ for a point $t' \in O^{\text{sm}}$, then the $\alpha'$ is conjugate to $\alpha$ via a path $l \subseteq O^{\text{sm}}$ joining $t$ to $t'$. Along the path $l$, the pullback family of cubic surfaces is smooth. The transport of $\alpha$ along the path $l$ can be written as difference of classes of skew lines in exactly six ways, so this property is satisfied by $\alpha'$.

**Corollary 2.8.** There is a degree-6 covering map $e: M \to T'$ by $(L_1, L_2) \mapsto [L_1] - [L_2]$, which fits into the commutative diagram Moreover, $\deg(\pi') = 72$ and the fiber of $\pi'$ is isomorphic to the root system on $\mathbb{E}_6$.

**Proof.** Since $\deg(\pi) = 432$, and $\deg(e) = 6$, we have $\deg(\pi') = 72$, as difference of classes of skew lines $[L_1] - [L_2]$ has self-intersection $-2$ and is orthogonal to the hyperplane class, the fiber of $\pi'$ is exactly the set of 72 roots $R_{Y_t}$ defined in (3).

**Remark 2.9.** In the root system of $\mathbb{E}_6$, the reflection along a root $\alpha$

$$s_\alpha(\beta) = \beta + (\alpha, \beta)\alpha,$$

corresponds to the Picard-Lefschetz formula in a nodal family $\{Y_t\}_{t \in \Delta}$.

Now we want to compactify the space $T'$. We have the following result due to Stein [Ste56] and Grauert-Remmert [GR58], and the result can be also found in [DG94], p.197.

**Lemma 2.10.** Let $U$ be a complex manifold, and $f: X \to U$ a finite analytic cover. Assume $\bar{U}$ is a normal completion of $U$, then there is a normal analytic space $\bar{X}$ containing $X$ as dense open subspace, together with finite analytic branched covering map $\bar{X} \to \bar{U}$, which agrees with $f$ on $X$. Moreover, when $\bar{U}$ is projective, $\bar{X}$ is also projective.
**Corollary 2.11.** There exists a normal algebraic variety $\overline{T}'$ together with a finite map $\overline{T}' \to \mathcal{O}$ which extends $T' \to \mathcal{O}^{\text{sm}}$.

A point $p \in T'$ over $t_0 \in \mathcal{O}$ can be interpreted as the following. Let $B$ be a small ball around $t_0$. Pick a base point $t' \in B^{\text{sm}} = B \cap \mathcal{O}^{\text{sm}}$. The fundamental group $\pi_1(B^{\text{sm}}, t')$ acts on the root system $R(\mathbb{E}_6)$ over $t'$ via monodromy action. Then $p$ corresponds to one of the orbit, and it lies in the ramification locus of a connected component of the covering space $T'|_{B^{\text{sm}}} \to B^{\text{sm}}$.

In Theorem 5.1, we will characterize the space $\overline{T}'$ geometrically. In Theorem 6.4 we will relate the boundary points on $\overline{T}'$ to the Lie theory of the root system of the minimal resolution of the singular cubic surfaces, in the case where they have at worst ADE singularities.

To study $T'$ and its compactification, we will use the covering map $e$, the completion of $\mathcal{M}$, together with the Abel-Jacobi map.

### 2.4. Abel-Jacobi map.

Let $Y$ be a smooth cubic threefold, and $F$ its Fano surface of lines, then there is an Abel-Jacobi map

$$\psi : F \times F \to JY, \ (p, q) \mapsto \int_{L_p}^{L_q} ,$$

where the integral is taken over a 3-chain $\Gamma$ whose boundary is $\partial \Gamma = L_p - L_q$.

It is shown in [CG72] that $\psi$ is generically 6:1 onto the theta divisor $\Theta$ of $JY$. Moreover, when the two lines are the same, the image is 0, so $\psi$ contracts the diagonal $\Delta_F$ to the point 0, which is the triple point singularity on $\Theta$.

Beauville showed that $\psi$ induces morphism on the blowup

$$\tilde{\psi} : \text{Bl}_{\Delta_F}(F \times F) \to \text{Bl}_0 \Theta.$$  

We will provide a modular interpretation of this morphism in the later sections.

To understand the restriction of $\tilde{\psi}$ on the exceptional divisor $E$, we look at the commutative diagram

$$\begin{array}{ccccccccc}
\mathbb{P}T_f & \xrightarrow{\cong} & \mathbb{P}N_{\Delta_F}(F \times F) & \hookrightarrow & \text{Bl}_{\Delta_F}(F \times F) & \xrightarrow{\tilde{\psi}} & \text{Bl}_0 \Theta & \hookrightarrow & \text{Bl}_0 JY \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F & \xrightarrow{\cong} & \Delta_F & \hookrightarrow & F \times F & \xrightarrow{\psi} & \Theta & \hookrightarrow & JY
\end{array}$$

$E$ is isomorphic to the associated projective bundle of the normal bundle $N_{\Delta_F}(F \times F)$. Since $N_{\Delta_F}(F \times F) \cong T_F$, we have an identification

$$E \cong \mathbb{P}N_{\Delta_F}(F \times F) \cong \mathbb{P}T_F.$$
On the other hand, denote $K$ the exceptional divisors on $\text{Bl}_0 \Theta$, then $K \cong Y$ due to Beauville \cite{Bea82} based on our previous discussion.

There is geometric interpretation of the map \cite{CG72} on the exceptional divisor $E$ described in \cite{CG72} Proposition 12.31. Denote $\Gamma \subseteq F \times Y$ the incidence variety of pairs $(t, y)$ of a line $L_t$ and a point $y \in L_t$, then

**Lemma 2.12.** There is a canonical isomorphism

\begin{equation}
\alpha : \mathbb{P}T_F \cong \Gamma,
\end{equation}

as $\mathbb{P}^1$-bundle over $F$.

One can find the proofs in \cite{CG72}, p.342-p.345, and \cite{Tju70}, and an algebraic proof in \cite{AT77}. In other words, the morphism $\alpha$ identifies the projective tangent space at $L \in F$ to the line $L$ in a canonical way.

We will describe this map explicitly. Over the locus of lines of the first type, $\alpha$ turns out to be equivalent to the following: for a line $L_t$ of first type on $Y$, the normal bundle is $\mathcal{O}_{L_t} \oplus \mathcal{O}_{L_t}$. Each section $s \in \mathbb{P}H^0(L, N_{L_t}|_Y)$ determines a line in $\mathbb{P}^4$ disjoint from $L_t$, and the span of the two lines is a hyperplane $H_s$ which is tangent to $Y$ at a unique point $p_s$ on $L_t$, so the map $\alpha$ is defined as

$$\alpha : (t, s) \mapsto p_s.$$ 

When the line $L = L_{t_0}$ becomes a line of the second type, the section $s_0$ becomes a section on $\mathcal{O}_L(1)$, so $s_0$ has a unique zero $\text{zero}(s_0) \in L$, and $\alpha$ sends $(t_0, s_0)$ to $\text{zero}(s_0)$, the conjugate point of $\text{zero}(s_0)$ under the dual map. This will be proved in Proposition 3.14

Based on the isomorphism \cite{CG72}, the restriction of $\tilde{\psi}$ to the exceptional divisor

$$\tilde{\psi}|_E : \mathbb{P}T_F \to Y$$

is isomorphic to

$$\Gamma \to Y, \ (t, y) \mapsto y,$$

by projection to the second coordinate.

**Corollary 2.13.** The map $\tilde{\psi}|_E$ is generically finite, and the only positive dimensional fibers are over Eckardt points on $Y$ and the fibers are isomorphic to elliptic curves. In particular, when $Y$ is general, $\tilde{\psi}|_E$ is finite.

*Proof.* Since $\tilde{\psi}|_E$ is isomorphic to the incidence projection $\Gamma \to Y$, the fiber is the set of lines through a given point. The fiber over $p$ has one-parameter family of lines if and only if $p$ is an Eckardt point by definition, and is an elliptic curve by Lemma 2.2. \hfill \Box
3. Hilbert Scheme of Skew Lines on Cubic Threefolds

3.1. Hilbert Scheme of Skew Lines on Projective Spaces. Consider a pair \((L_1, L_2)\) of skew lines on \(\mathbb{P}^3\). As a closed subvariety, \(Z = L_1 \cup L_2\) has Hilbert polynomial \(2n + 2\). It determines an irreducible component of the Hilbert scheme \(\text{Hilb}^{2n+2}(\mathbb{P}^3)\). According to [CCN11], the Hilbert scheme \(\text{Hilb}^{2n+2}(\mathbb{P}^3)\) has two irreducible components \(H_3\) and \(H'_3\). A general point in \(H_3\) parameterizes a pair of skew lines. A general point in \(H'_3\) parameterizes a smooth conic union an isolated point.

**Theorem 3.1.** ([CCN11], Theorem 1.1) Both \(H_3\) and \(H'_3\) are smooth and intersect transversely along the union of the locus of type (III) and type (IV) schemes. Moreover, \(H_3\) is isomorphic to \(\text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, 4)\), the blowup of the symmetric product of \(\text{Gr}(2, 4)\) along the diagonal. Moreover, the component \(H_3\) parameterizes four types of schemes:

- **Type (I):** A pair of skew lines;
- **Type (II):** A purely double structure supported on a line;
- **Type (III):** A pair of incident lines with an embedded point determined by the square of the ideal of the intersection point;
- **Type (IV):** A double structure contained in a plane and supported on a line, together with an embedded point determined by the square of the ideal of a point on the line.

![Type I](image1)

![Type II](image2)

![Type III](image3)

![Type IV](image4)

**Figure 1.** Schemes of the Four Types

One can write down the ideals of the schemes of the four types in the projective coordinate \(x_0, x_1, x_2, x_3\). The ideal of a type (I) scheme can be expressed as \((x_0, x_1) \cap (x_2, x_3) = (x_0x_2, x_0x_3, x_1x_2, x_1x_3)\), which is the ideal of two disjoint lines \(x_0 = x_1 = 0\) and \(x_2 = x_3 = 0\); The type (III) ideal can be written as \((x_0^2, x_0x_1, x_0x_2, x_1x_2) = (x_0, x_1x_2) \cap (x_0, x_1, x_2)^2\), which determines a pair of incident lines \(x_0 = x_1 = 0\) and \(x_0 = x_2 = 0\) with a spatially embedded point at the intersection point. The type (II) and (IV) ideals are all supported on a single
line \( x_0 = x_1 = 0 \). The corresponding ideal for type (II) scheme is \( (x_0^2, x_0x_1, x_1^2, x_0x_2 + x_1x_3) \), while the ideal of the type (IV) scheme is \( (x_0^2, x_0x_1, x_1^2, x_0x_2) = (x_0, x_1^2) \cap (x_0, x_1, x_2)^2 \). Equivalently, a type (II) scheme corresponds to the first-order neighborhood of a line in a smooth quadric surface, while a type (IV) scheme corresponds to the first-order neighborhood of a line in \( \mathbb{P}^2 \) union a spatially embedded point on the line.

The flat degenerations of the four types of schemes can be described geometrically as follows.

(I)\( \Rightarrow \) (II): As two disjoint lines come to coincide linearly;

(I)\( \Rightarrow \) (III): As two disjoint lines come to intersect at one point, an embedded point occurs at the intersection;

(III)\( \Rightarrow \) (IV): As two incident lines come to coincide;

(II)\( \Rightarrow \) (IV): As the smooth quadric surface degenerates to two planes intersecting transversely along a line \( M \). The support line \( L \) is contained in one of the plane and normal to the other. The embedded point occurs at the intersection \( L \cap M \).

One can interpret Theorem 3.1 via the Hilbert-Chow morphism

\[
\rho_3: H_3 \to \text{Sym}^2 Gr(2, 4),
\]

which sends type (I) and (III) schemes to their support, and sends type (II) and (IV) schemes to their support with multiplicity two. Denote \( D \) the subvariety of \( \text{Sym}^2 Gr(2, 4) \) parameterizing pairs of incident lines on \( \mathbb{P}^3 \). We have the stratification \( \Delta \subseteq D \subseteq \text{Sym}^2 Gr(2, 4) \). Then Theorem 3.1 states that \( \rho_3 \) is the blow-up along \( \Delta \). Over \( \Delta \) is a \( \mathbb{P}^3 \) bundle consisting of type (II) and type (IV) schemes (with type (IV) scheme forming a smooth quadric surface). The set of type (I) and type (III) scheme over \( D \setminus \Delta \) are type (III) schemes and over \( \text{Sym}^2 Gr(2, 4) \setminus D \) are type (I) schemes.

There is a similar result in higher dimensional projective spaces, one refers to [CCN11], Corollary 2.8 for precise statement. What we need here is that the irreducible component \( H_m \) of \( \text{Hilb}^{2n+2}(P_m) \) with a general point parameterizing a pair of skew lines is smooth, where \( m \geq 4 \). \( H_m \) still parameterizes the schemes of the four types defined in Theorem 3.1. Moreover, every scheme parameterized by \( H_m \) determines a unique linear subspace \( \mathbb{P}^3 \) of \( \mathbb{P}^m \) containing the scheme (Also see Lemma 3.5.3, [Lee00]). Therefore, there are two natural morphisms from \( H_m \).

\[
\begin{array}{ccc}
H_m & \xrightarrow{\pi} & Gr(4, m + 1) \\
\downarrow{\rho_m} & & \downarrow \\
\text{Sym}^2 Gr(2, m + 1) & & 
\end{array}
\]
\( \rho_m \) is again the Hilbert-Chow morphism, and the horizontal map \( \pi \) is to associate each scheme to the unique \( \mathbb{P}^3 \) containing the scheme. It turns out that \( \pi \) is a \( H_3 \) bundle.

We characterize the Hilbert-Chow morphism as successive blow-ups in the following proposition, which will be useful in proving the main theorem.

**Proposition 3.2.** The Hilbert-Chow morphism \( H_m \rightarrow \text{Sym}^2 \text{Gr}(2, m + 1) \) factors through

\[
H_m \xrightarrow{\sigma_2} \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, m + 1) \xrightarrow{\sigma_1} \text{Sym}^2 \text{Gr}(2, m + 1)
\]

where \( \sigma_1 \) blows up the diagonal, and \( \sigma_2 \) blows up the strict transform of the locus \( D \subseteq \text{Sym}^2 \text{Gr}(2, m + 1) \) parameterizing incident pairs of lines.

**Proof.** Let \( I_\Delta \) be the ideal sheaf of the diagonal \( \Delta \subseteq \text{Sym}^2 \text{Gr}(2, m + 1) \), then the pullback \( p^* I_\Delta \) is an ideal sheaf of a divisor, which is invertible since \( H_m \) is smooth ([CCN11], Corollary 2.8). So by the universal property of the blowup ([Har77], Proposition II.7.14), the Hilbert-Chow morphism \( p : H_m \rightarrow \text{Sym}^2 \text{Gr}(2, m + 1) \) factors through \( \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, m + 1) \) as in (9), where \( \sigma_1 \) blows up the diagonal, and \( \sigma_2 \) is birational.

Let \( D \subseteq \text{Sym}^2 \text{Gr}(2, m + 1) \) denote the locus of pair of incident lines and \( \tilde{D} \) the strict transform, which has codimension \( m - 2 \). For a type (III) supported on a pair of incident lines \( L_1 \cup L_2 \), the embedded point determines and is uniquely determined by a \( \mathbb{P}^3 \) containing \( L_1 \cup L_2 \), and there is a \( \mathbb{P}^{m-3} \)-family of such hyperplanes. Therefore the general fiber over \( \tilde{D} \) (and therefore over \( D \)) is isomorphic to \( \mathbb{P}^{m-3} \). Now \( \sigma_2^{-1}(\tilde{D}) \) is a divisor. By the same argument, \( \sigma_2 \) factors through \( W = \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, m + 1) \). We have a commutative diagram.

\[
\begin{array}{ccc}
H_m & \xrightarrow{\pi} & \text{Gr}(4, m + 1) \\
\downarrow{\sigma_3} & & \downarrow{\pi'} \\
W & \xrightarrow{\phi} & \text{Sym}^2 \text{Gr}(2, m + 1)
\end{array}
\]

\( \pi' \) is the morphism induced by the rational map \( \phi : (L_1, L_2) \mapsto \text{span}(L_1, L_2) \) and \( \pi = \pi' \circ \sigma_3 \). The fiber of \( \phi \) is a dense subset of \( \text{Sym}^2 \text{Gr}(2, 4) \), whose closure in \( W \) is isomorphic to \( \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, 4) \), so it is the fiber of \( \pi' \). On the other hand, we know that the fiber of \( \pi \) is also \( H_3 \cong \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, 4) \), so \( \sigma_3 \) is a bijective birational map. Therefore by Zariski’s main theorem, it is an isomorphism.

\[ \square \]

**3.2. Main theorem.** In this section, \( Y \) is a smooth cubic threefold. Recall that we defined a space \( \mathcal{M} \) consisting of set of triples \( (L_1, L_2, t) \) such that \( Y_t \) is a smooth hyperplane section
of $Y$ and $L_1, L_2$ are disjoint lines on $Y_t$. There is a 2-to-1 map
\[ \mathcal{M} \to \text{Hilb}^{2n+2}(Y), \quad (L_1, L_2, t) \mapsto \mathcal{O}_{L_1 \cup L_2} \]
and the closure of the image $H(Y)$ is an irreducible component of the Hilbert scheme $\text{Hilb}^{2n+2}(Y)$. In this section, we will characterize $H(Y)$.

Since the cubic threefold $Y$ is a closed subvariety in $\mathbb{P}^4$, $H(Y)$ is naturally a closed scheme of $H_4$. Recall that $F$ is the Fano surface of lines on cubic threefold, then via the natural inclusion $F \hookrightarrow Gr(2, 5)$, there is a following diagram.

\[
\begin{array}{ccc}
H(Y) & \xrightarrow{\rho} & H_4 \\
\downarrow & & \downarrow \\
\text{Sym}^2(F) & \xrightarrow{\rho_4} & \text{Sym}^2(Gr(2, 5))
\end{array}
\]

(10)

Since a general element in $\text{Sym}^2(F)$ and $\text{Sym}^2(Gr(2, 5))$ is a pair of skew lines, and only the reduced scheme structure has Hilbert polynomial $2n + 2$, therefore both $\rho$ and $\rho_4$ are birational morphisms. Our goal is to show the vertical map on the first column is identified with the blowup along the diagonal.

**Theorem 3.3.** $H(Y)$ is smooth and isomorphic to $\text{Bl}_{\Delta_F} \text{Sym}^2 F$.

A direct corollary is the characterization of the completion of $\mathcal{M}$.

**Corollary 3.4.** Denote $\overline{H(Y)}$ be the double cover of $H(Y)$ along the locus of type (II) and type (IV) schemes that are supported on a line, then $\overline{H(Y)} \cong \text{Bl}_{\Delta_F} (F \times F)$. So the completion of $\mathcal{M}$ with respect to $\tau$ is $\text{Bl}_{\Delta_F} (F \times F)$.

**Proof.** The $\mathbb{Z}_2$ quotient $g : F \times F \to \text{Sym}^2 F$ fixes the diagonal, so the action extends to the blowup of the diagonal and there is a commutative diagram.

\[
\begin{array}{ccc}
\text{Bl}_{\Delta_F} (F \times F) & \xrightarrow{\bar{g}} & \text{Bl}_{\Delta_F} \text{Sym}^2 F \\
\downarrow & & \downarrow \\
F \times F & \xrightarrow{g} & \text{Sym}^2 F
\end{array}
\]

3.3. **Some geometric preparations.** Let $Y$ be a smooth cubic threefold. We will prove the main theorem in this section. Namely, we will show $H(Y)$ is smooth and isomorphic to $\text{Bl}_{\Delta_F} \text{Sym}^2 F$. Our strategy is the following. We will show that each pair of incident lines on $Y$ supports a unique type (III) scheme. Given a double structure supported on
a single line, we will show that there is a \( \mathbb{P}^1 \)-family of double structures of the same type supported on that line. The \( \mathbb{P}^1 \)-bundle over the Fano surface will match with the exceptional divisor of the blowup \( \text{Bl}_{\Delta_p} \text{Sym}^2 F \). This can be globalized and leads to a bijective morphism \( \text{Bl}_{\Delta_p} \text{Sym}^2 F \to H(Y) \). Finally we show it is an isomorphism using smoothness of \( H_4 \).

One should note that, a line \( L \) on the cubic threefold \( Y \) supports a type (II) scheme if and only if it is a line of first type, and it supports a type (IV) scheme if and only if it is a line of second type. This is not the case in the projective space where every line supports both type (II) and type (IV) schemes.

**Definition 3.5.** Let \( W \hookrightarrow X \) be a closed immersion of schemes. Let \( I \subseteq \mathcal{O}_X \) be the ideal sheaf defining \( W \). The first-order infinitesimal neighborhood of \( W \) in \( X \) is defined to be the closed subscheme \( W' \subseteq X \) defined by \( I^2 \).

When both \( X \) and \( W \) are smooth, the scheme \( W' \) keeps track of the information normal bundle \( N_{W/X} \). As an example, the first-order infinitesimal neighborhood of a point \( p \) in a smooth variety \( X \) is the vector space \( T_p X \oplus \mathbb{C} \). The notion will be useful to characterize the schemes of the four types defined in Theorem 3.1 in a scheme theoretical way.

**Proposition 3.6.** Let \( m \geq 3 \) and a scheme \( Z \in H_m \) can be expressed in the following way depending on its type.

- **Type (I):** \( Z = Z_{\text{red}} \), a pair of skew lines;
- **Type (II):** \( Z = Z_Q \), the first-order infinitesimal neighborhood of the line \( L = Z_{\text{red}} \) in a smooth quadric surface \( Q \);
- **Type (III):** \( Z = Z_{\text{red}} \cup Z_{p,H} \), where \( Z_{\text{red}} \) is the union of a pair of lines incident at \( p \), and \( Z_{p,H} \) is the first-order infinitesimal neighborhood of \( p \) in linear subspace \( H \cong \mathbb{P}^3 \) of \( \mathbb{P}^m \);
- **Type (IV):** \( Z = Z_P \cup Z_{p,H} \), where \( Z_P \) is the first-order infinitesimal neighborhood of the line \( L = Z_{\text{red}} \) in a plane \( P \). \( p \in L \), and \( Z_{p,H} \) is the same as above.

Note that \( Z_P \) has Hilbert polynomial \( 2n + 1 \) and is called a non-reduced conic (see [KPS18] Lemma 2.1.1).

**Proposition 3.7.** Let \( X \) and \( Y \) be smooth projective subvarieties of \( \mathbb{P}^m \), and \( W \) is a smooth subvariety of both \( X \) and \( Y \). Denote \( Z \) the first-order infinitesimal neighborhood of \( W \) in \( X \), then \( Z \) is a closed subscheme of \( Y \) if and only if \( X \) is tangent to \( Y \) along \( W \). In other words, there is an inclusion \( T_X|_W \hookrightarrow T_Y|_W \) of tangent bundles restricted to \( W \).

**Proof.** The condition \( Z \subseteq Y \) is equivalent to surjectivity on the sheaves \( \mathcal{O}_Y \to \mathcal{O}_X/I^2_{W|X} \), but this map factors through

\[
\mathcal{O}_Y \to \mathcal{O}_Y/I^2_{W|Y} \to \mathcal{O}_X/I^2_{W|X}.
\]
By restricting to an affine chart, we have decomposition \( \mathcal{O}_X/I_{W|X}^2 \cong \mathcal{O}_W \oplus I_{W|X}/I_{W|X}^2 \). The surjectivity on \( \mathcal{O}_W \) follows from the assumption \( W \subseteq Y \), and the surjectivity
\[
I_{W|Y}/I_{W|Y}^2 \twoheadrightarrow I_{W|X}/I_{W|X}^2
\]
dualizes to the statement that there is an inclusion \( N_{W|X} \hookrightarrow N_{W|Y} \), which is equivalent to \( T_X|_W \hookrightarrow T_Y|_W \).

Now we study the elements in \( H(Y) \). First of all, for \( L_1 \) and \( L_2 \) disjoint lines on \( Y \), the scheme structure it supports is reduced. When \( L_1 \) and \( L_2 \) intersect at one point, we claim that

**Lemma 3.8.** There is a unique type (III) scheme \( Z \in H(Y) \) supported on \( L_1 \cup L_2 \), where the embedded point supported on \( p = L_1 \cap L_2 \) is the square of the maximal ideal in the tangent hyperplane \( T_pY \).

**Proof.** A type (III) subscheme \( Z \) of \( \mathbb{P}^4 \) is a union
\[
Z = Z_{\text{red}} \cup Z_{p,H},
\]
where \( Z_{\text{red}} = L_1 \cup L_2 \) is the reduced scheme and \( Z_{p,H} \) is the first-order infinitesimal neighborhood of \( p \) in a hyperplane \( H \in (\mathbb{P}^4)^* \). So if \( Z_{\text{red}} = L_1 \cup L_2 \) being contained in \( Y \) is given, then by Proposition 3.7, the condition \( Z_{p,H} \) being a subscheme of \( Y \) is equivalent to \( H = T_pY \) being the tangent hyperplane at \( p \).

**Corollary 3.9.** The Hilbert-Chow morphism
\[
\rho : H_4(Y) \rightarrow \text{Sym}^2 F
\]
is isomorphic over \( \text{Sym}^2 F \triangle F \).

**Proof.** \( \rho \) sends the set of type (I) and (III) schemes to \( \text{Sym}^2 F \triangle F \), which is bijective by Lemma 3.8, so by Zariski’s main theorem, it is an isomorphism.

Note that a similar result is obtained in Lemma 12.16, [CG72] using analytic method.

Next, we start to discuss the schemes parameterized by \( H(Y) \) and supported on a single line, that is the type (II) and type (IV) schemes. It turns out that they corresponds to the type of lines on cubic threefold by "dividing by 2". Recall the following definition from [CG72]

**Definition 3.10.** A line \( L \subseteq Y \) is called to be of first type if the normal bundle \( N_{L|Y} \cong \mathcal{O}_L \oplus \mathcal{O}_L \); \( L \) is called to be of second type if the normal bundle \( N_{L|Y} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \).

All lines in \( Y \) are in the two classes. The lines of first type are generic in \( F \), while the set of lines of second types forms a divisor of \( F \).
Proposition 3.11. Let $L$ be a line on $Y$, then

(1) $L$ is of the first type if and only if there is a smooth quadric surface in $\mathbb{P}^4$ tangent to $Y$ along $L$. Moreover, all such quadric surfaces are parameterized by $L$.

(2) $L$ is of the second type if and only if there is a unique plane $\mathbb{P}^2$ tangent to $Y$ along $L$.

Proof. When $L$ is of the first type, $N_{L|Y} \cong \mathcal{O}_L \oplus \mathcal{O}_L$ is a trivial rank two bundle. If we identify the normal bundle $N_{L|\mathbb{P}^4}$ with an Zariski open subspace $U$ of $Gr(2,5)$, then $N_{L|Y}$ is identified with a closed subspace of $U$. Therefore, a nonzero section $v$ is identified with a line $L_v$ disjoint from $L$ and they span a hyperplane $H_v$. We will show that there is a unique smooth quadric surface $Q_v$ contained in $H_v$ and is tangent to $Y$ along $L$.

Our idea is the following. As the section $v$ is scaled by a factor $\lambda \in \mathbb{C}$, then the set of $L_{\lambda v}$ sweeps out a surface, whose closure should be the quadric surface $Q_v$ that we are looking for. By construction $Q_v$ is tangent to $Y$ along $L$. (See [CG72] Lemma 6.18 for an equivalent description.)

Explicitly, suppose $L = \{x_2 = x_3 = x_4 = 0\}$ and use the local equation of $Y$ around $L$ given by

$$x_2x_0^2 + x_3x_0x_1 + x_4x_1^2 + \text{higher order terms in } x_2, x_3, x_4$$

as in (6.9) in [CG72], one can determine the local equation of the Fano surface $F$ at $L$ in the Grassmannian $Gr(2,5)$, as in (6.14) in [CG72]. By linearizing the equation in (6.14) in [CG72], we find that $L_v$ is the line

$$\lambda[1,0,0,a,-b] + \mu[0,1,-a,b,0], \ [\lambda,\mu] \in \mathbb{P}^1,$$

for some $a,b \in \mathbb{C}$ not both zero.

This allows us to determine the equation of the hyperplane $H_v$:

$$a^2x_4 + b^2x_2 + abx_3 = 0.$$  

Note that $H_v$ is determined by $v$ up to scaling. In other words, $H_v$ is determined by $[v] \in \mathbb{P}H^0(L,N_{L|Y})$.

Note that by (11), the line $L_{sv}$ satisfies the extra two equations

$$\begin{cases} sax_1 + x_2 = 0; \\ sbx_0 + x_4 = 0, \end{cases}$$

in addition to (12), where $s \in \mathbb{C}^*$ is a scalar. By cancelling out the factor $s$ and using (12), we find that the quadric equation is

$$\begin{cases} bx_1x_2 + ax_1x_3 + ax_0x_2, \quad \text{if } b \neq 0; \\ ax_0x_4 + bx_0x_3 + bx_1x_4 = 0, \quad \text{if } a \neq 0, \end{cases}$$

20
which uniquely determines a smooth quadric surface $Q_v$ tangent to $Y$ along $L$ and contained in $H_v$.

Conversely, if $Q$ is a smooth quadric surface tangent to $Y$ along $L$, then the ruling of $Q$ containing $L$ is a line in $Gr(2, 5)$, tangent to $F$ at $L$, and thus it corresponds to an element in $\mathbb{P}H^0(L, N_{L|Y})$.

Finally, note that by the equation (12), the hyperplane $H_v$ is the tangent hyperplane $T_{[b,a]}Y$ of $Y$ at point $[b,a] \in L$. So there is a one-to-one correspondence

$$\mathbb{P}H^0(L, N_{L|Y}) \leftrightarrow L \leftrightarrow \text{smooth quadric surfaces tangent to } Y \text{ along } L.$$  

For part (2), when $L$ of the second type, the image of dual map $L \to (\mathbb{P}^4)^*, \ x \mapsto T_xY$ along $L$ is a line. So the $\mathbb{P}^2 = \cap_{x \in LT_x Y}$ is a plane tangent to $Y$ along $L$. This uniquely characterizes lines of the second type. (See [CG72] 6.6, 6.7.) □

**Lemma 3.12.** Let $Z \subseteq H(Y)$ be a scheme supported on a line $L$. If $L$ is of the first type, then $Z$ is of type (II). If $L$ is of the second type, then $Z$ is of type (IV).

**Proof.** Let $Z$ be a type (II) subscheme of $Y$ supported on a line $L$, we will show that $L$ cannot be of the second type. Otherwise, let $Q$ be the corresponding smooth quadric surface associated to $Z$, by Proposition 3.7, the normal bundle $N_{L|Q} \cong \mathcal{O}_L$ admits an inclusion to $N_{L|Y} \cong \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1)$ as bundles. However, the sheaf map $\mathcal{O}_L \to \mathcal{O}_L(-1)$ is always trivial, and any inclusion of sheaves $\mathcal{O}_L \to \mathcal{O}_L(1)$ has torsion cokernel, so there is no inclusion $N_{L|Q} \hookrightarrow N_{L|Y}$ of normal bundles, which is a contradiction.

Let $Z$ be a type (IV) subscheme of $Y$ supported on a line $L$, we will show that $L$ cannot be of the first type. Otherwise, as $Z$ has a closed subscheme $Z_P$, which is a first-order neighborhood of $L$ in a plane $P$, $Y$ contains $Z_P$ as a closed subscheme, then by Proposition 3.7, there is a bundle inclusion

$$\mathcal{O}(1) \cong N_{L|P} \hookrightarrow N_{L|Y} \cong \mathcal{O}_L \oplus \mathcal{O}_L,$$

which is a contradiction. □

The proof above is inspired by Remark 2.1.2 and 2.1.7 in [KPS18], which proved that only lines of the second type can support the non-reduced conic structure.

Recall that the dual map of $Y$ along $L$ is a map

$$\mathcal{D} : L \to \emptyset, \ p \mapsto T_p Y.$$  

When $L$ is of the first type, $\mathcal{D}$ is one-to-one onto a conic. When $L$ is of the second type, $\mathcal{D}$ is two-to-one onto a line. (c.f. [CG72] Def. 6.6)

**Definition 3.13.** We call $q \in L$ a conjugate point of $p \in L$ if they have the same image under $\mathcal{D}$, and we denote $q = \bar{p}$.  

21
Proposition 3.14. The map \( \alpha : \mathbb{P}T_F \to \Gamma \) defined in (7) is described by the following:

\[
\alpha : (t, v) \mapsto \begin{cases} 
\text{the unique tangent point of } H_v \text{ with } Y \text{ on } L, \text{ when } L \text{ is of the first type;} \\
\text{zero}(v), \text{ when } L \text{ is of the second type.}
\end{cases}
\]

Proof. When \( L \) is of the first type, this is implicit on page 342-345 in [CG72]. Let’s make it explicit. By the construction of \( \alpha \) for \( (s, v) \in \mathbb{P}T_F \), with \( L = L_s \) and \( v \in H^0(L, N_{L|Y}) \), we can choose another line \( L_{s'} \) which intersects \( L_s \) such that the curve \( D_{s'} \) of lines incident to \( L_{s'} \) coincides with \( v \) in \( H^0(L, N_{L|Y}) \) (note \( s \) is a smooth point on \( D_{s'} \)). Use the same notation in [CG72], \( L_{\gamma_{i}(s)} \) is the third line in the triangle on \( Y \) determined by \( L \) and \( L_s \), then by (12.21), \( \alpha \) sends \( (s, v) \) to the point \( L_s \cap L_{\gamma_{i}(s)} \). This point, according to Lemma 12.20, is the point whose tangent hyperplane coincides with the limiting hyperplane \( \lim_{t \in D_{s'}, t \to 0} \text{Span}(L, L_t) \), which coincides with the hyperplane \( H_v \) constructed in (12).

However, \( \alpha \) is not explicit when \( L_t \) is of the second type in [CG72]. This is due to the dual map along a line of the second type is 2-to-1, i.e., for a general \( p \in L_t \), the tangent hyperplane \( T_P Y \) is also tangent to \( Y \) at a different point \( \bar{p} \in L_t \), so when we take the limit of a tangent point

As the locus of lines of the first type is dense in \( F \), the description on \( \alpha \) on the fibers of lines of the first type as above uniquely determines how \( \alpha \) acts on the pair \( (t, v) \), where \( L_t \) is a line of the second type. We can choose a one-parameter family of lines \( \{L_t\}_{t \in \Delta} \) with \( \Delta \) a holomorphic disk and \( t \neq 0 \) for lines of the first type and \( t = 0 \) a line of the second type.

There is a rank two vector bundle \( V \) on \( I_\Delta \cong L \times \Delta \), whose stalk over each \( t \in \Delta \) is \( \mathcal{O}_L \oplus \mathcal{O}_L \) when \( t \neq 0 \) and \( \mathcal{O}_L(1) \oplus \mathcal{O}_L(-1) \) when \( t = 0 \). Since the second projection \( (\pi_2)_* V \) to \( \Delta \) is a trivial bundle of rank two, we can choose two linearly sections \( v_1 \) and \( v_2 \) over \( \Delta \). When \( t \neq 0 \), \( v_1 \) and \( v_2 \) correspond to two lines \( L_{1,t}, L_{2,t} \) in a Zariski open subset of \( Gr(2, 5) \) that are disjoint from \( L \) and are infinitesimally close to \( Y \) when scaled by a small number (see proof of Proposition 3.11). When \( t = 0 \), the two lines \( L_{1,0} \) and \( L_{2,0} \) intersect \( L \) at two distinct points, corresponding to linearly independent sections on \( \mathcal{O}_L(1) \).

We want to keep track of the map \( \alpha \) when \( t \to 0 \). Similar to (6.9) and (6.10), we can choose a one-parameter family of automorphisms \( \sigma_t \) on \( \mathbb{P}^4 \), such that \( \sigma L_t = L \) and \( \sigma_t(Y) \) has local equations

\[
x_2x_0^2 + tx_3x_0x_1 + x_4x_1^2 + \text{higher order terms in } x_2, x_3, x_4
\]

around \( L \). Now, given a section \( v \in (\pi_2)_* V \), similar to the proof of Proposition 3.11, we find that the line \( L_{v, t} \) is given by

\[
\lambda[1, 0, 0, a, -tb] + \mu[0, 1, -ta, b, 0], \quad [\lambda, \mu] \in \mathbb{P}^1.
\]
Therefore, the hyperplane spanned by $L$ and $L_{v,t}$ is given by $a^2x_4 + b^2x_2 + t$ $abx_3 = 0$, which has the unique tangent point $[b,a]$ when $t \neq 0$. So $\alpha(0,v)$ should be $[b,a]$ as well by continuity.

As $t \to 0$, the limit hyperplane is $a^2x_4 + b^2x_2 = 0$, which is tangent to $Y$ at two points $[b,a]$ and $[b,-a]$. On the other hand, the limit line $L_0$ is given by $x_2 = x_4 = x_3 - ax_0 - bx_1 = 0$, which intersects $L$ at point $[b,-a]$, which is the conjugate point of $\alpha(0,v) = [b,a]$. This finishes the proof. □

3.4. Proof of the main theorem.

Proposition 3.15. Let $L$ be a line on the cubic threefold $Y$. Using the notation in ([10]), there is an isomorphism

$$\beta: \rho^{-1}(L, L) \cong L$$

which can be described in the following way. Let $Z \in \rho^{-1}(L, L)$ be a scheme.

(1) When $L$ is of the first type, the scheme $Z$ supported on $L$ has type (II) and is contained in a unique hyperplane $H_Z$, which is tangent to $Y$ at a unique point $p_Z$ along $L$, we have $\beta(Z) = p_Z$;

(2) When $L$ is of the second type, the scheme $Z$ supported on $L$ has type (IV) and has an embedded point supported at $p$, we have $\beta(Z) = \bar{p}$.

Moreover, the map is continuous with respect to $L \in F$. In other words, there is an identification

$$\beta: \rho^{-1}(\Delta_F) \cong \Gamma \cong \mathbb{P}T_F.$$

Proof. If $L$ is of first type, then by Lemma 3.12 it can only support type (II) schemes. Recall that a type (II) subscheme $Z_Q$ of $\mathbb{P}^4$ supported on $L$ is a first-order infinitesimal neighborhood of the line $L$ in a smooth quadric surface $Q \subseteq \mathbb{P}^4$. Therefore, if $Q$ is tangent to the cubic threefold $Y$ along $Y$, the first-order infinitesimal neighborhood of $L$ in $Q$ is contained in the first-order infinitesimal neighborhood of $L$ in $Y$, so the corresponding type (II) scheme $Z_Q$ is contained in $Y$. Conversely, every type (II) subscheme of $Y$ arises in this way. According to Proposition 3.11 (1), there is a $\mathbb{P}^1$-family of such smooth quadric surfaces, so there is a $\mathbb{P}^1$-family of type (II) subschemes of $Y$ supported on $L$, parameterized by

$$L \leftarrow \mathbb{P}H^0(L, N_{L,Y}) \rightarrow \rho^{-1}(L),$$

$$\left\{p \in L \mid H_p = T_p Y \right\} \leftrightarrow s \mapsto Z_Q_s.$$

Since the dual map along $L$ is one-to-one, each type (II) scheme over $L$ is contained in a unique hyperplane.

If $L$ is of the second type, then by Lemma 3.12 it can only support type (IV) schemes. Recall that a type (IV) subscheme $Z$ of $\mathbb{P}^4$ supported on $L$ can be written as

$$Z = Z_P \cup Z_{x,H},$$
where $Z_P$ is a first-order infinitesimal neighborhood of $L$ in a plane $P \subseteq \mathbb{P}^4$ and $Z_{x,H}$ is a first-order neighborhood of a point $x \in L$ in a hyperplane which contains $P$. We are looking for the condition such that $Z$ is a subscheme of $Y$. By Proposition 3.11 (2), there is a unique plane $P_L$ tangent to $Y$ along $L$, so $Z_{P_L}$ is a subscheme of $Y$. By Proposition 3.7 to require that $Z_{x,H}$ is a subscheme of $Y$ is the same as requiring that $H = T_x Y$ is the tangent hyperplane at $x$. This in return uniquely determines the scheme $Z_{x,H}$ and therefore the type (IV) scheme. Finally, note that $H^0(N_{L|Y}) = H^0(O_L(1))$, whose global sections are in bijection to their zeros on $L$ and therefore bijective to the conjugate points on the zeros, so there is a correspondence on set of type (IV) schemes supported on $L$:

$$\mathbb{P} H^0(L, N_{L|Y}) \to L \to \rho^{-1}(L),$$

$s \mapsto \text{Zero}(s) = x \mapsto Z_P \cup Z_{x,T_x Y}$.

Finally, to prove that the identification constructed above is consistent with degenerating a line of the first type to a line of the second type, we need to show that as the type (II) schemes $Z_t$ degenerate to a type (IV) scheme $Z_0$ on $Y$, the limit of the tangent points $p_t$ is the conjugate point of the support of the embedded point of $Z_0$.

We use the coordinates (15) and continue the computation similarly as in the proof of Proposition 3.11. We get the equation of quadric surfaces

$$\begin{cases}
  h_{a,b}(t) = a^2 x_4 + b^2 x_2 + tab x_3 = 0; \\
  q_{a,b}(t) = bx_1 x_2 + t ax_1 x_3 + ax_0 x_2 = 0,
\end{cases}$$

when $b \neq 0$ (and similarly for $a \neq 0$). The flat limit of the corresponding type (II) schemes is determined by the flat family of ideals

$$(h_{a,b}(t), (x_2, x_3, x_4)^2, q_{a,b}(t)) \Rightarrow (a^2 x_4 + b^2 x_2, (x_2, x_3, x_4)^2, x_2 (bx_1 + ax_0)),$$

where the quadric surface $h_{a,b}(0) = q_{a,b}(0)$ is reducible and is the union of the two planes $x_2 = a^2 x_4 + b^2 x_2 = 0$ and $bx_1 + ax_0 = a^2 x_4 + b^2 x_2 = 0$. Therefore the embedded point is supported at the intersection of $L$ with $x_2 = bx_1 + ax_0 = a^2 x_4 + b^2 x_2 = 0$, which is exactly $[b, -a] \in L$. It is the conjugate point of the limit tangent point. □

Proposition 3.15 above tells us that we can globalize the flat family of schemes of the four types over $\text{Bl}_{\Delta_F} \text{Sym}^2 F$.

**Corollary 3.16.** There is a bijective morphism $\delta : \text{Bl}_{\Delta_F} \text{Sym}^2 F \to H(Y)$.

**Proof.** Recall we have shown in Corollary 3.9 that the Hilbert-Chow morphism $\rho$ is an isomorphism off the diagonal.
Now we lift $\text{Bl}_{\Delta_F}\text{Sym}^2F$ to its double cover $\text{Bl}_{\Delta_F}(F \times F)$ and the exceptional divisor is identified with

$$E \cong \mathbb{P}N_{\Delta_F}(F \times F) \cong \mathbb{P}T_F.$$

For a one-parameter family of pair of lines $\{L_t\}_{t \in \Delta}$ where $\Delta$ is a holomorphic disk and with $L_t \neq L_0$ for $t \neq 0$, the flat limit of $\rho^{-1}(L_0, L_t)$ is a scheme with a double structure supported on $L$, which corresponds to a section $v \in \mathbb{P}H^0(L, N_{L|Y})$. According to the identification $PT_F \cong \Gamma$ as in (7) together with Proposition 3.15, the correspondence is continuous with respect to moving the one-parameter family and moving the line $L \in F$.

Therefore, $\text{Bl}_{\Delta_F}(F \times F)$ parameterizes flat families of skew lines. By the universal property of Hilbert schemes, there is a morphism $\text{Bl}_{\Delta_F}(F \times F) \rightarrow H(Y)$ which is two-to-one off the exceptional divisor $E$, and one-to-one on $E$, so it descends to 2:1 quotient and induces a bijective morphism $\text{Bl}_{\Delta_F}\text{Sym}^2F \rightarrow H(Y)$ as claimed. □

**Remark 3.17.** There is an alternative way to prove the Corollary 3.16 without Proposition 3.14. It is shorter but is relied on Abel-Jacobi map and Beauville’s characterization of singularity of the theta divisor. We sketch the proof here.

First we have the dominant birational map

$$\delta : \text{Bl}_{\Delta_F}\text{Sym}^2F \dashrightarrow H(Y),$$

by assigning $(L_1, L_2) \mapsto \mathcal{O}_{L_1 \cup L_2}$. We want to show it extends to a morphism.

One considers the rational map

$$\Phi : F \times F \dashrightarrow \mathcal{O}, \quad (L_1, L_2) \mapsto \text{Span}(L_1, L_2).$$

It factors through the Abel-Jacobi map (4)

$$F \times F \rightarrow \Theta \dashrightarrow \mathcal{O},$$

where the second map is the Gauss map, which associates each smooth point of the theta divisor $\Theta$ to the projective tangent hyperplane at that point.

According to the diagram (6), $\Phi$ extends to a morphism on $\text{Bl}_{\Delta_F}(F \times F)$ and factors through

$$\tilde{\Phi} : \text{Bl}_{\Delta_F}(F \times F) \rightarrow \text{Bl}_0\Theta \rightarrow \mathcal{O}.$$

$\tilde{\Phi}$ descends to $\mathbb{Z}_2$-quotient (just as $\Phi$ does), so it defines a morphism $\tilde{\Phi}' : \text{Bl}_{\Delta_F}\text{Sym}^2F \rightarrow \mathcal{O}$ and provides a continuous way of assigning hyperplanes.

Denote $\pi : H(Y) \rightarrow \mathcal{O}$ the natural projection. Then we have $\pi \circ \delta = \tilde{\Phi}'$. Use the properties of the dual map (14), there are at most two schemes of type (II) or (IV) which is supported on a line and is contained a hyperplane, indicating that the morphism $\overline{\Gamma(\delta)} \rightarrow \text{Bl}_{\Delta_F}\text{Sym}^2F$ from the graph closure $\overline{\Gamma(\delta)}$ of $\delta$ is finite, and thus is an isomorphism by Zariski’s theorem. It follows that $\delta$ is a morphism. Finally, injectivity can be checked on each fiber of $E \rightarrow \Delta_F$. 25
Now we are ready to prove our main theorem.

**Proof.** (Proof of Theorem 3.3) It suffices to show \( \delta : \text{Bl}_\Delta \text{Sym}^2 F \to H(Y) \) is an isomorphism. It suffices to show that the bijective morphism, composed with inclusion \( i : H(Y) \hookrightarrow H_4 \) is an immersion, namely, it is of maximal rank at each point.

By expressing \( H_4 \to \text{Sym}^2 \text{Gr}(2, 5) \) as successive blowups as in Proposition 3.2, we have a commutative diagram.

\[
\begin{array}{ccc}
\text{Bl}_\Delta \text{Sym}^2 F & \xrightarrow{\phi} & \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, 5) \\
\downarrow & & \downarrow \\
\text{Sym}^2 F & \xrightarrow{\sigma_1} & \text{Sym}^2 \text{Gr}(2, 5)
\end{array}
\]

(17)

\( \phi \) is the unique map which extends the rational map \( \text{Sym}^2 F \dashrightarrow \text{Bl}_\Delta \text{Sym}^2 \text{Gr}(2, 5) \) induced by the universal property of the blowup (Corollary II.7.15, [Har77]). Also note that the two sides of \( \phi \) are the Hilbert schemes of two points \( F[2] \) on \( F \), and \( \text{Gr}(2, 5)[2] \) on \( \text{Gr}(2, 5) \), respectively, so \( \phi \) is identified with the inclusion

\[ F[2] \hookrightarrow \text{Gr}(2, 5)[2]. \]

It follows that \( \sigma_2 \circ i \circ \delta = \phi \) is an immersion, so \( i \circ \delta \) has to be an immersion. \( \square \)

**Remark 3.18.** One can consider a different compactification of \( M \), namely by taking closure of \( M \) in the triple \( F \times F \times \mathcal{O} \). Since \( M \) can be regarded as the graph of the rational map \( \Phi : F \times F \dashrightarrow \mathcal{O} \), this compactification is identified with the graph closure \( \Gamma(\Phi) \). By the previous remark, \( \Phi \) extends to \( \tilde{\Phi} \) and \( \Gamma(\Phi) \) is the projection of graph of \( \tilde{\Phi} \) under the projection

\[ \text{Bl}_\Delta(F \times F) \to F \times F \times \mathcal{O}. \]

It is 1-to-1 on the exceptional fibers of lines of the first type, and 2-to-1 over on the exceptional fibers of lines of the second type. Therefore by Zariski’s main theorem, we have

**Proposition 3.19.** There is a normalization map \( \text{Bl}_\Delta(F \times F) \to \Gamma(\Phi) \).

We mistaken the above map as an isomorphism in the previous version of this paper. In fact, \( \Gamma(\Phi) \) is singular over the locus of lines of the second type.

4. **Hilbert scheme of skew lines on cubic surfaces**

In the last section, we showed the Hilbert scheme of skew lines \( H(Y) \) on a smooth cubic threefold \( Y \) is smooth and has two natural morphisms.
where the morphism $\pi$ is the composite of the horizontal map of $\text{Sym}^2F$ and inclusion $H(Y) \hookrightarrow H_4$.

In this section, we would like to understand the fiber of $\pi$. This is the same as understanding the schemes parameterized by $H_4$ that are contained in hyperplane sections of the cubic threefold $Y$.

**Definition 4.1.** Let $S \subseteq \mathbb{P}^3$ be a cubic surface. Define the Hilbert scheme of skew lines on $S$ to be

$$H(S) := \text{Hilb}^{2n+2}(S) \cap H_3.$$  

When $S$ is smooth, $H(S)$ is reduced and consists of 216 pairs of skew lines. However, when $S$ is singular, $H(S)$ is not reduced. We want to study the cardinality of the reduced scheme $H(S)_{\text{red}}$.

4.1. **Lines on cubic surfaces.** For smooth cubic surfaces, there are exactly 27 lines. For cubic surfaces with "mild" singularities, the number of lines is less than 27 and such number depends on the type of singularities as well as how lines pass through these singularities. However, for cubic surfaces that are "too singular", they contain infinitely many lines. The following result will make this more precise.

**Lemma 4.2.** Let $S$ be a cubic surface obtained from a hyperplane section of a smooth cubic threefold $Y$, then $S$ is normal and belongs to one of the following two cases:

(i) $S$ has at worse rational double points (RDPs);

(ii) $S$ has an elliptic singularity.

In case (i), the cubic surface $S$ contains at most 27 lines. In case (ii), $S$ is isomorphic to cone over a smooth plane cubic curve, therefore contains a one-parameter family of lines.

**Proof.** First we show that $S$ has to be normal. According to Theorem 9.2.1 in [Dol12], a non-normal cubic surface is either cone over singular cubic curve or projective equivalent to $t_0^2t_2 + t_1^2t_3 = 0$, or $t_2t_0t_1 + t_3t_0^2 + t_1^3 = 0$. In either case, $S$ has to be singular along a line $L$.

Now assume $S$ is the hyperplane section $t_4 = 0$, then $Y$ has defining equation

$$F_i(t_0, ..., t_3) + t_4Q(t_0, ..., t_4) = 0,$$

with $F_i(t_0, ..., t_3)$ the defining equation of $S$, and $Q(t_0, ..., t_4)$ a homogeneous quadric. Then by taking the partial derivatives and restricting to the line, one finds that $Y$ is singular at the
intersection between the line \( L \) and the quadric surface \( Q(t_0, \ldots, t_3, 0) = 0 \), which contradicts that \( Y \) is smooth. Therefore \( S \) is normal and has only isolated singularities.

By the classification theorem of cubic surfaces [BW79], \( S \) either has at worst RDPs (at worst \( \mathbb{E}_6 \) singularity) or is a cone over a smooth plane cubic curve. One refers to [Dol12], section 9.2.2 for number of lines on all cubic surfaces with at worst RDPs. □

Then by Lemma 4.2 and Lemma 2.2, we have

**Corollary 4.3.** Let \( Y \) be a smooth cubic threefold, and \( S = Y \cap H \) be a hyperplane section. Then \( S \) has only finitely many lines except when \( H \) is a tangent hyperplane \( T_pY \) of \( Y \) at an Eckardt point \( p \in Y \). In particular, when \( Y \) is general, all hyperplane sections \( S \) of \( Y \) has only finitely many lines.

### 4.2. Lines of first and second type.

From now on, we assume \( S \) is a normal cubic surface. Note that \( S \) can be embedded to a smooth cubic threefold as a hyperplane section. We want to characterize \( H(S)_{\text{red}} \). This requires analyzing how the schemes of the four types are supported on the pair of "skew" lines in different configuration in \( S \).

As long as \( S \) contains two skew lines \( L_1, L_2 \), it defines a type (I) subscheme of \( S \). For type (III) schemes, we have a direct observation as follows.

**Proposition 4.4.** Let \( L_1, L_2 \) be two lines on \( S \) that are incident at one point \( p \), then there is a type (III) subscheme of \( S \) supported on \( L_1 \cup L_2 \) if and only if \( S \) is singular at \( p \)

**Proof.** By Lemma 3.8, such a scheme is contained in the tangent hyperplane \( T_pY \), therefore \( Z \) lies in the unique hyperplane section \( Y \cap T_pY \), which is singular at the incident point. □

For type (II) and type (IV) schemes contained in \( S \), they affect the local geometry of the support line \( L \) inside \( S \). We need to introduce the following concepts.

**Definition 4.5.** Let \( L \) be a line in the cubic surface \( S \). Call \( L \) to be of the first type if there is a smooth quadric surface tangent to \( S \) along \( L \). Call \( L \) to be of the second type if there is a plane \( \mathbb{P}^2 \) tangent to \( S \) along \( L \).

The next proposition will explain the reason to introduce our definition.

**Proposition 4.6.** Let \( L \subseteq S \) be a line, then

1. \( L \) passes through at least one singularity of \( S \) if and only if \( L \) is of either the first type or the second type.
2. \( L \) is of the first type (resp. second type) if and only if the torsion-free part of the conormal sheaf \( N^*_L S = I_L/I^2_L \) is \( O_L \) (resp. \( O_L(-1) \)).
3. \( L \) is of the first type (resp. second type) in \( S \) if and only if \( S = H \cap Y \) is a hyperplane section of a smooth cubic threefold \( Y \) with \( H \) tangent to \( Y \) at some point on \( L \), and \( L \) is of the first type (resp. second type) in \( Y \).
Proof. Assume the line is defined by $x_2 = x_3 = 0$, then $S$ has equation

$$F = x_2 Q_0(x_0, x_1) + x_3 Q_1(x_0, x_1) + \text{higher order terms in } x_2, x_3,$$

where $Q_0$ and $Q_1$ are homogeneous quadrics in $x_0, x_1$. The dual map along $L$ is

$$\mathcal{D}|_L = [0, 0, Q_0(x_0, x_1), Q_1(x_0, x_1)].$$

$Q_0$ and $Q_1$ cannot both be zero because otherwise $S$ will be singular along the line, violating the normality assumption. So the image of the dual map is either a point or isomorphic to $\mathbb{P}^1$ in $(\mathbb{P}^3)^*$. In the first case, the point in dual space corresponds to a hyperplane $\mathbb{P}^2$ which is tangent to $S$ along $L$, so $L$ is of the second type. Moreover, $Q_0$ is parallel to $Q_1$, so they have nonempty common zero locus, where $S$ will be singular. In the second case, if $Q_0$ and $Q_1$ have no common zeros, then $S$ will be smooth along $L$. Otherwise, $Q_0$ and $Q_1$ has common reducible factor, then by canceling the factor and changing coordinate, the dual map becomes $\mathcal{D}|_L = [0, 0, x_0, x_1]$, and the quadric surface $x_0x_2 + x_1x_3 = 0$ is tangent to $S$ along $L$, which is a line of the first type.

For part (2), note that the conormal sheaf $I_L/J_L^2$ on $L$ has rank two at the singularities $L \cap S^{\text{sing}}$ and is locally free of rank one over the smooth locus. By definition, if $L$ is of the first type (resp. second type), and let $J_L$ be the ideal sheaf of $L$ in the smooth quadric surface (resp. $\mathbb{P}^2$), with $J_L/J_L^2$ the corresponding conormal bundle which is isomorphic to $\mathcal{O}_L$ (resp. $\mathcal{O}_L(-1)$), there is a short exact sequence

$$0 \rightarrow T \rightarrow N_{L|S}^* \rightarrow J_L/J_L^2 \rightarrow 0$$

with $T$ the torsion sheaf of rank one supported on $L \cap S^{\text{sing}}$. The sequence splits and the torsion-free part of $N_{L|S}^*$ is isomorphic to $J_L/J_L^2$.

For part (3), to build up the relationship with cubic threefold, assume $Y$ is a smooth cubic threefold and $L \subseteq Y$ is a line of the first type (resp. second type). Let $H$ be a hyperplane tangent to $Y$ at any point on $L$, then the hyperplane section $S = Y \cap H$ is singular at the tangent point. Moreover, there is a smooth quadric surface $Q$ (resp. $\mathbb{P}^2$) tangent to $Y$ along $L$ (Proposition 3.11) and is contained in the hyperplane $H$. So $L$ is of the first type (resp. second type) in the cubic surface $S$.

Conversely, we can embed $S$ into a smooth cubic threefold $Y$ as a hyperplane section $S = Y \cap H$, as long as $S$ is normal and has only isolated singularities. The line $L$ is regarded as a subvariety of both $S$ and $Y$. If $L$ is of the first type (resp. second type) in $S$, then there is a smooth quadric surface (resp. $\mathbb{P}^2$) tangent to $S$ along $L$, which will automatically imply that $L$ is of the first (resp. second type) in $Y$, by comparing the dual maps. Finally, the hyperplane $H$ will be tangent to $Y$ at some point on $L$, since otherwise $S$ will be smooth along $L$. \(\square\)
As a direct consequence, lines of the first type (resp. second type) on $S$ correspond to type (II) (resp. (IV)) schemes, just as cubic threefolds case in Proposition 3.12. Now we provide some explicit examples on how the schemes of the four types are contained in a normal cubic surface. For a smooth cubic surface $S$ has 216 pair of skew lines corresponding to 216 type (I) schemes. Below are some other typical examples.

**Example 4.7.** ($A_1$ singularity) Let $S$ be a cubic surface with a single singularity which is $A_1$. Then it has 21 lines, with 15 lines away from the nodes and 6 lines $L_1, ..., L_6$ passing through the node. There are 120 disjoint pair of lines, corresponding to type I scheme; each $L_i$ supports a unique type II scheme structure, and there are 6 such lines. The union $L_i \cup L_j$ with $i \neq j$ supports a unique type III structure with the embedded point supported at the intersection point, and there are 15 such pairs. $|H(S)_{red}| < \infty$.

**Example 4.8.** ($3A_2$ singularities) Let $S$ be defined by $xyz = w^3$. Then it has only 3 lines $x = w = 0, y = w = 0, z = w = 0$ with each line passing through two of the three $A_2$ singularities $[1, 0, 0, 0], [0, 1, 0, 0], [0, 0, 1, 0]$. In this case, each of the line supports type IV schemes in two different ways. The embedded point can be supported on either of the two singularities that the line passes. $S$ also contains type III schemes supported on every pair of the three lines with the embedded point supported at the intersection point. $|H(S)_{red}| < \infty$.

**Example 4.9.** (elliptic singularity) Let $S$ be a cone over a smooth cubic curve $E$. Each of the line only supports type IV scheme in a unique way: The embedded point is supported at the cone point. Moreover, each pair of the distinct lines intersects at the cone point, so their union supports a type III scheme. $H(S)_{red} \cong \text{Sym}^2 E$.

![Figure 2. Examples of Singular Cubic Surfaces](image)

Now we answer the question proposed at the beginning of this section.

**Theorem 4.10.** Denote $p_1, ..., p_k$ the Eckardt points on $Y$, then the only positive dimensional fibers of

$$\pi : H(Y) \to \emptyset$$

are at Eckardt hyperplane sections $H_{p_i}$. The reduced structure of such fiber is isomorphic to $\text{Sym}^2 E_i$, where $E_i$ is an elliptic curve. In particular, if $Y$ is general and has no Eckardt point, then $H(Y) \to \emptyset$ is finite.
Proof. If a hyperplane section $S$ is obtained by intersecting $Y$ with the tangent hyperplane $T_pY$ at some Eckardt point, then $S$ is the cone of an elliptic curve $E$. Then $H(S)_{red} \cong \text{Sym}^2 E$ as we have seen.

If $S = Y \cap H$, where $H$ is not a tangent hyperplane at Eckardt point, then by Corollary 4.3, it only has finitely many lines, and thus it has finitely many pairs of lines as well. Now by Proposition 4.4 and Proposition 4.6, there are finitely many schemes of the four types contained in $S$. In other words, $H(S)$ is zero dimensional.

Note that if a hyperplane is tangent to an Eckardt point, it cannot be tangent to any other point, so there is a one-to-one correspondence between the positive dimensional fibers of $H(Y) \to \mathbb{O}$ and the Eckardt points.

If a line $L$ on $S$ is away from the singularities of $S$, then $L$ is rigid in $S$. However, the occurrence of singularity along the line changes the local geometry of $L$ around $S$, according to Prop 4.6 (1) and (2). One may ask whether the singularities on $S$ that a line $L$ passes determines the type of the line.

Proposition 4.11. (1) If a line $L \subseteq S$ passes through only one singularity of $S$ and the singularity has type $A_1$, then $L$ is of the first type.

(2) If $L$ passes through more than one singularity, then $L$ is of the second type.

Proof. For (1), assume the line is given by $x_2 = x_3 = 0$ and the cubic surface has equation

$$x_0Q(x_1, x_2, x_3) + C(x_1, x_2, x_3) = 0,$$

with the cubic $C = x_1^3(ax_2 + bx_3) + \cdots$ and quadric $Q = x_1(cx_2 + dx_3) + \cdots$ intersecting transversely at 6 points, guaranteeing that $[1,0,0,0]$ is an $A_1$ singularity. The dual map along $L$ is $D|_L = [0,0,cx_0 + ax_1,dx_0 + bx_1]$. The transversality condition implies that the two linear forms are linearly independent, so it corresponds to dual map of a smooth quadric surface along $L$.

For (2), we regard $S$ as an hyperplane section of a smooth cubic threefold $Y$, then use the fact that the dual map on $Y$ along $L$ is 1-to-1 (resp. 2-to-1) when $L$ is of the first type (resp. second type).

Remark 4.12. Based on the examples that we studied and the previous proposition, the line of the first type tends to pass less singular points, and the line of the second type tends to pass through more singular points. If $L$ passes through only one singularity, one may wonder whether the type of the singularity that $L$ passes through determines the type of the line. However, this is not the case. There is a normal cubic surface defined by the equation $F = x_0x_1x_2 + x_2x_3^2 + x_3x_1^2$ (with an $A_4$ at $[1,0,0,0]$ and an $A_1$ singularity at $[0,0,1,0]$). Both of the lines $x_2 = x_3 = 0$, $x_1 = x_2 = 0$ only pass through the $A_4$ singularity. However,
by computing the dual map along the lines, we conclude that one of the lines is of first type and the other is of the second type.

**Remark 4.13.** For a smooth cubic threefold $Y$, its relative Hilbert scheme of lines $\mathcal{F}$ is flat over $\mathcal{O}\backslash\{H_1, \ldots, H_k\}$ where $\{H_1, \ldots, H_k\}$ is the set of the Eckardt hyperplanes, and has length 27 on each fiber (also see the introduction of [1zi01] and Example 1.1 (b) of [1zi02]). Classically, the lines for cubic surfaces with RDPs were studied by Cayley [Cay], and the number "27" is interpreted as the number of lines counted with multiplicities, and the multiplicity of a line depends on type of singularities it passes through.

Similarly, if we consider the pair of skew lines, again, the relative Hilbert scheme of skew lines $H(S/\mathcal{O})$ is flat over $\mathcal{O}\backslash\{H_1, \ldots, H_k\}$. Similarly, the length of the Hilbert schemes on $S$ is the constant number 216, which is the number of pairs of skew lines on a smooth cubic surface.

5. Monodromy Completion of Locus of Primitive vanishing cycles

Recall that $T'$ parameterizes the roots on the smooth hyperplane sections of $Y$, and $T' \to \mathcal{O}^{sm}$ is a 72-sheeted covering space. We have shown in Corollary 2.11 that $\tilde{T}'$ is the unique normal analytic space containing $T'$ as open dense subspace and such that $\tilde{T}' \to \mathcal{O}$ is finite and extends $T' \to \mathcal{O}^{sm}$. Moreover, the boundary points of $\tilde{T}'$ corresponds to monodromy orbits of the local fundamental groups.

In this section, we are going to characterize $\tilde{T}'$ via Abel-Jacobi map.

**Theorem 5.1.** There is a birational morphism $\text{Bl}_0\Theta \to \tilde{T}'$ contracting finitely many elliptic curves, which are one-to-one corresponding to the Eckardt points on the cubic threefold $Y$. In particular, when $Y$ is general, $\tilde{T}'$ is isomorphic to $\text{Bl}_0\Theta$.

We have the following commutative diagram from (6):

$$
\begin{array}{ccc}
\text{Bl}_{\Delta'}(F \times F) & \xrightarrow{\tilde{\psi}} & \text{Bl}_0\Theta \\
\downarrow{\tilde{\pi}} & & \downarrow{\lambda} \\
\mathcal{O} & & \\
\end{array}
$$

(19)

As a consequence of Theorem 4.10 shown in the previous section, $\tilde{\pi}$ contracts the surface $E_i \times E_i$ to the points $t_i$ corresponding to the Eckardt hyperplanes and is finite elsewhere. We want to use this to understand the fiber of $\lambda$. We first claim that this map contracts finitely many surfaces into curves indexed by Eckardt points, and in particular is finite when $Y$ is general.

Recall that for an Eckardt point, there is a curve $E$ family of lines passing through the point. So each Eckardt point corresponds to a curve $E \subseteq F$. 

Proposition 5.2. If $E \subseteq F$ is the curve of the lines corresponding to an Eckardt point on $Y$, then $C = \tilde{\psi}(E \times E)$ is an elliptic curve intersecting the exceptional divisor transversely.

Proof. Note that the composite $E \times E \hookrightarrow F \times F \xrightarrow{\tilde{\psi}} JY$ factors through

$$E \times E \rightarrow \text{Pic}^0(E), \ (p, q) \mapsto \mathcal{O}_E(p - q).$$

Since $\text{Pic}^0(E) \cong E$, this map is surjective and all fibers are isomorphic to $E$. Now since $E \times E \rightarrow JY$ is a morphism between abelian varieties, $\psi(E \times E)$ is either an elliptic curve or a point. However, due to the fact that the Albanese map $\text{alb}: F \rightarrow \text{Alb}(F)$ is an embedding, $\psi$ sends $E \times \{q\}$ isomorphically to its image, so $C$ is an elliptic curve. Finally, $\tilde{\psi}$ sends the strict transform of $E \times E$ to the strict transform of $C_i$ under the blowup map, which intersects the exceptional divisor transversely. □

Corollary 5.3. The only positive dimensional fibers of $\lambda$ are $C_i$.

Now we are ready to prove the Theorem 5.1.

Proof. The Abel-Jacobi map $\psi: F \times F \rightarrow JY$ is generically 6-to-1, and the image is the theta divisor $\Theta$. If we restrict the Abel-Jacobi map to $\mathcal{M}$, then $\psi|_{\mathcal{M}}$ factors through $T'$, due to the fact that the six differences of two disjoint lines are in the same rational equivalent classes, or by a topological fact ([Zhao15], Def. 2.1.2). Further, we note that these maps preserves projection to $\mathcal{O}_{\text{sm}}$, so we have the following diagram.

$$
\begin{array}{ccc}
\mathcal{M}/\mathcal{O}_{\text{sm}} & \xrightarrow{e} & T'/\mathcal{O}_{\text{sm}} \\
\downarrow{\psi|_{\mathcal{M}}} & & \downarrow{j} \\
\Theta^\circ/\mathcal{O}_{\text{sm}} & & \\
\end{array}
$$

Here $\Theta^\circ$ is the image of $\mathcal{M}$, which is dense open in $\Theta$. The evaluation map $e$ sends $(L_1, L_2) \mapsto [L_1] - [L_2]$ to the difference of the classes. $e$ is also 6-to-1 since around the smooth points on the boundary $\mathcal{O}\setminus\mathcal{O}_{\text{sm}}$, when a family of smooth cubic surfaces degenerate to a nodal cubic surface, there are exactly 6 way of expressing the primitive vanishing cycle as class of difference of two lines. Therefore $j: T'/\mathcal{O}_{\text{sm}} \cong \Theta^\circ/\mathcal{O}_{\text{sm}}$ is an isomorphism. Since $T' \hookrightarrow \Theta$ as open subspace, and the image is disjoint from $C_i$, we have natural inclusion $i: T' \hookrightarrow \text{Bl}_0\Theta$.

Now we look at the Stein factorization

$$\text{Bl}_0\Theta \xrightarrow{\lambda_1} W \xrightarrow{\lambda_2} \mathcal{O},$$

where $W$ is the relative spectrum of $\lambda_1_*\mathcal{O}_{\text{Bl}_0\Theta}$ so that $\lambda_1$ contracts the curves $C_i$ and $\lambda_2$ is finite. Moreover, $W$ is normal since $\text{Bl}_0\Theta$ is normal (p.213 in [GR84]).
Now, the composition $T' \hookrightarrow \text{Bl}_0 \Theta \rightarrow W$ is an inclusion and preserves the projection to $O$. So as branched covering maps, $W/O$ extends $T'/O^{\text{sm}}$. Due to the normality of $W$ and the uniqueness of extension analytic branched covering, Lemma 2.10, $W/O \cong T'/O$. This proves the theorem.

We note here that the exceptional divisor of $\text{Bl}_0 \Theta$ is isomorphic to $Y$. The restriction the projection $\lambda: \text{Bl}_0 \Theta \rightarrow O$ to $Y$ is isomorphic to the dual map

$$Y \rightarrow Y^\vee,$$

which associate each point $y$ to its tangent hyperplane $T_p Y$. This map is finite and birational, so is the normalization by Zariski’s main theorem.

Finally, we answer a question about extension of Abel-Jacobi map.

**Corollary 5.4.** There is a morphism $\bar{T}' \rightarrow JY$ which extends $T' \rightarrow JY$ if and only if the cubic threefold $Y$ has no Eckardt point.

**Proof.** When the cubic threefold $Y$ has no Eckardt point, $\bar{T}' \cong \text{Bl}_0 \Theta$ and the composite

(20)$$\text{Bl}_0 \Theta \rightarrow \Theta \hookrightarrow JY$$

is the extension of $T' \rightarrow JY$.

Conversely, when $Y$ has Eckardt points, we know that there is a birational morphism $\text{Bl}_0 \Theta \rightarrow \bar{T}'$ which blows down elliptic curves $E_i$ to points $u_i$, for $i = 1, \ldots, k$ which are in one-to-one correspondence to the Eckardt points on $Y$. However, the morphism $\text{Bl}_0 \Theta \rightarrow \bar{T}'$ sends $E_i$ isomorphically onto its image, so the rational map $\bar{T}' \rightarrow JY$ cannot be extended to $u_i$. □

6. **Boundary Points of $\bar{T}'$ and Minimal Resolutions**

For a smooth cubic threefold $Y$, we have shown that there is a unique completion $\bar{T}'$ of the locus of primitive vanishing cycle $T'$ by putting monodromy orbits of local fundamental groups to the boundary points on $\bar{T}'$ over $Y^\vee = O \setminus O^{\text{sm}}$. Our goal in this section is to build the relationship between the monodromy orbits and the minimal resolutions of the singular cubic surfaces corresponding to points in $Y^\vee$ and the related Lie theory of the root systems.

For each $t_0 \in O$, pick a small neighborhood $B$ around $t_0$. Fix another base point $t' \in B^{\text{sm}} := O^{\text{sm}} \cap B$, then the monodromy action permutes the 72 roots on the fiber $(\pi')^{-1}(t')$. Since it preserves the intersection pairing and polarization, it preserves the structure of root system. Thus, the monodromy representation is a homomorphism

(21)$$\rho: \pi_1(B^{\text{sm}}, t') \rightarrow W(E_6).$$

**Definition 6.1.** We call $\text{Im}(\rho)$ the monodromy group of $S_0$. 34
According to the definition of completion of finite analytic cover by [GR58] and [Ste56], we have the following monodromy interpretation of $\bar{T}'$.

**Lemma 6.2.** The fiber $(\pi')^{-1}(t_0)$ corresponds to the orbits of monodromy action induced by $\rho$.

Let $S_0$ denote the cubic surface as the hyperplane section of $Y$ corresponding to $t_0 \in \mathbb{O}$. We would like to relate the monodromy orbits to the root system on its minimal resolution $\tilde{S}_0 \rightarrow S_0$.

**6.1. Minimal resolution.** Let $S$ be a cubic surface with at worst ADE singularities. Let

$$\sigma : \tilde{S} \rightarrow S$$

be its minimal resolution, then $\tilde{S}$ is a weak del Pezzo surface of degree 3 and one can still define the root system $R(\tilde{S})$ as

$$\alpha^2 = -2, \; \alpha \cdot K_{\tilde{S}} = 0.$$

It is known that $R(\tilde{S})$ is isomorphic to $R(E_6)$. This can be seen as following. One can regard $\tilde{S}$ as blowing up six bubble points on $\mathbb{P}^2$ in almost general position. The six points can be deformed smoothly as they move to general position. As the equation (22) is topological invariant, the root system on $\tilde{S}$ is defined.

Note that each irreducible component $C$ of the exceptional divisor of $\sigma$ is a $(-2)$ curve and is orthogonal to the class $K_{\tilde{S}}$ since $\sigma$ is crepant, so $C$ defines a root and is effective as divisor class. We call such root an effective root. The set of all effective roots generates a sub-root system $R_e$ of $R(\tilde{S})$. Since each of the singularity $x_i$ of $S$ corresponds to a bunch of $(-2)$-curves on $\tilde{S}$ and they generate a sub-root system $R_i$, $R_e$ is isomorphic to the product $\prod_{i \in I} R_i$, where $I$ is the index set of singularities of $S$. Each $R_i$ corresponds to a connected sub-diagram of Dynkin diagram of $E_6$. One refers to [Dol12], section 8.1, 8.2, 8.3 and 9.1 for the detailed discussion.

Moreover, the reflections with respect to all of the effective roots define a subgroup $W(R_e)$ of the Weyl group $W(E_6)$, and $W(R_e)$ is isomorphic to the product $\prod_{i \in I} W_i$, where $W_i$ is the Weyl group generated by the reflections corresponding to the exceptional curves over the singularity $x_i$. One can consider the action of $W(R_e)$ on $R(E_6)$. The orbits that are contained in $R_e$ are naturally in bijection with the set of singularities on $S$. One can also define the maximal/minimal root of an orbit. In particular, the maximal root of the orbit corresponding to $x_i$ equals to the cohomology class of the fundamental cycle $Z_i$ at $x_i$. We refer to [LLSvS17], section 2.1 for the detailed discussion.

**Definition 6.3.** We define the $R(S)$ to be the set of the orbit in $R(E_6)$ under the action by $W(R_e)$. We informally call $R(S)$ to be the root system on $S$. 35
Note that $R(S)$ is just a set, without any intersection pairing. This set is used in Theorem 2.1 of [LLSvS17] to find the reduced Hilbert scheme of generalized twisted cubics on $S$.

Our main result is the following.

**Theorem 6.4.** There is a natural bijection between $(\bar{\pi'})^{-1}(t_0)$ and $R(S)$.

**Example 6.5.** Take a one-parameter family $\{S_t\}_{t \in \Delta}$ of cubic surface, where $S_0$ has an $A_1$ singularity, and $S_t$ is smooth when $t \neq 0$. Assume that total space is smooth. Then there is a vanishing cycle $\delta$ associated to the family. Fix a base point $t' \in \Delta^*$. The monodromy representation $\pi_1(\Delta^*, t') \rightarrow R(\mathbb{E}_6)$ is generated by the Picard-Lefschetz transformation

$$T_\delta : \alpha \mapsto \alpha + (\alpha, \delta)\delta.$$ 

As $\delta^2 = -2$, $T_\delta$ has order two, $T_\delta$ coincides with the reflection of root system along $\delta$. Therefore, the orbits of monodromy action are naturally identified with the orbits of Weyl group $W(A_1) = \mathbb{Z}_2$ action.

We can compute the cardinality of orbits, using the geometry of nodal cubic surface. The cubic surface $S_{t_0}$ can be represented by blowup of 6 general points on $\mathbb{P}^2$, and as it degenerate to $S_0$, the 6 points come to lie on a conic. Use the same notation as Proposition 2.7, we can choose the vanishing cycle $\delta = 2h - e_1 - \cdots - e_6$. The 72 roots can be expressed as classes

1. $\pm \delta$, 2 roots;
2. $\pm (h - e_i - e_j - e_k)$, $i, j, k$ distinct, 40 roots;
3. $e_i - e_j$, $i \neq j$, 30 roots.

The roots in classes (1) and (2) have nonzero intersections with $\delta$, while the classes in (3) are orthogonal to $\delta$, so by Picard-Lefschetz formula, the number of monodromy orbits is $1 + 20 + 30 = 51$.

From the topological point of view, each of the $W(R_e)$-orbit $R_t$ in $R_e$ corresponding to a singularity $x_i$ of $S$ is a root system corresponding to the type of $x_i$. It has root basis $\Delta_i$ of size $\mu_i$, given by a set of vanishing cycles of the Milnor fiber of $x_0$. The vanishing cycles can be represented by topological 2-spheres that are supported in an arbitrarily small neighborhood of $x_i$ in the total space of the family.

In [LLSvS17] Theorem 2.1, the authors showed that $R(S) = R(\mathbb{E}_6)/W(R_e)$ are in bijection with the connected components of the reduced Hilbert schemes of generalized twisted cubics on $S_0$. The orbits that contain an effective root corresponds to generalized twisted cubics that are not Cohen-Macaulay (whose reduced schemes are planar), while those orbits that don’t contain any effective roots corresponds to the generalized twisted cubics that are arithmetic Cohen-Macaulay (whose reduced schemes are not planar). In section 3 of [LLSvS17], it showed that there is a bijective between the $W(R_e)$-orbit on $R(S) \setminus R_e$ and the linear determinantal representations of cubic surfaces. The cardinality of such orbits is listed
in p.102, Table 1. On the other hand, we know that the cardinality of the orbits on \( R \) is exactly the number of the singularities. So we obtain the cardinality of the root system \( R(S) \) by adding up the two numbers.

**Corollary 6.6.** The cardinality \( |R(S)| \), which coincides with the cardinality of the fiber of \( T' \to \emptyset \) corresponding to the cubic surface \( S \), is listed in the table below.

| \( R \) | Type | \# | \( R \) | Type | \# | \( R \) | Type | \# |
|---|---|---|---|---|---|---|---|---|
| \( \emptyset \) | I | 72 | 4A_1 | XVI | 17 | \( A_1 + 2A_2 \) | XVII | 9 |
| \( A_1 \) | II | 51 | \( 2A_1 + A_2 \) | XIII | 15 | \( A_1 + A_4 \) | XIV | 6 |
| 2A_1 | IV | 36 | \( A_1 + A_3 \) | X | 12 | \( A_5 \) | XI | 5 |
| \( A_2 \) | III | 31 | \( 2A_2 \) | IX | 14 | \( D_5 \) | XV | 3 |
| \( 3A_1 \) | VIII | 25 | \( A_4 \) | VII | 9 | \( A_1 + A_5 \) | XIX | 3 |
| \( A_1 + A_2 \) | VI | 22 | \( D_4 \) | XII | 7 | \( 3A_2 \) | XXI | 5 |
| \( A_3 \) | V | 17 | \( 2A_1 + A_3 \) | XVIII | 8 | \( E_6 \) | XX | 1 |

Table 1: Numbers of roots on cubic surfaces of given singularity type.

6.2. **Proof of the main theorem.**

6.2.1. *A local argument.* Let \( p: Y \to B \) be the family of cubic surfaces over the ball \( B \) arising from hyperplane sections on \( Y \). Let \( x_0 \) be an isolated singularity of \( S_{t_0} \), where \( S_{t_0} \) is the hyperplane section of \( Y \) at \( H_{t_0} \). Take a small ball \( D_0 \) in the total space \( Y \) around \( x_0 \). Then by restricting to \( D_{0}^{sm} = D_0 \setminus p^{-1}(Y^\vee) \), the morphism

\[
p^{sm}: D_{0}^{sm} \to B^{sm}
\]

is a smooth fiber bundle. Let \( F \) be a fiber, then there is a monodromy representation

\[
\rho_0: \pi_1(B^{sm}, t) \to \text{Aut}H^2(F, \mathbb{Z}).
\]

We call \( \text{Im}(\rho_0) \) the monodromy group of \( x_0 \).

**Proposition 6.7.** Assume \( x_0 \) has type ADE. Then the monodromy group \( \text{Im}(\rho_0) \) of \( x_0 \) is isomorphic to the Weyl group \( W \) corresponding to singularity type of \( x_0 \).

To prove this proposition, we need to use the Milnor fiber theory. One refers to [Dur79] for a more detailed survey.

6.2.2. *Monodromy group on Milnor fiber.* Let \( f(x_1, ..., x_n) = 0 \) be a hypersurface in \( \mathbb{C}^n \) with an isolated singularity at 0, then the Milnor fiber \( F \) of \( f \) is the \( \{ f = w \} \cap B^n \) for a ball \( B^n \) around origin of small radius and \( w \in \mathbb{C} \) with a small magnitude. \( F \) has homotopy type of a bouquet of \( \mu \) spheres of dimension \( n - 1 \), where \( \mu \) is the Milnor number of the singularity, which coincides with the dimension of \( \mathbb{C} \)-vector space \( \mathbb{C}[x_1, ..., x_n]/(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \).
A deformation of $f$ is an analytic function

$$g(x_1, \ldots, x_n, w) : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$$

such that $g(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n)$, and $\tilde{f}(x_1, \ldots, x_n) = g(x_1, \ldots, x_n, 1)$ is called a perturbation of $f$. There is a perturbation $\tilde{f}$ of $f$ such that $\tilde{f}$ is a Morse function in the sense that all critical values of $\tilde{f}$ are distinct and all critical points are nondegenerate. There are exactly $\mu$ critical points $t_1, \ldots, t_\mu$ of $\tilde{f}$ and they are contained in a $\delta$-neighborhood $D_\delta$ of 0 in $\mathbb{C}$. The Milnor fibers of $f$ and $\tilde{f}$ are diffeomorphic.

We can choose a base point $t' \in D_\delta - \{t_1, \ldots, t_\mu\}$ and paths $p_i, 1 \leq i \leq \mu$ connecting $t'$ to $t_i$ such that its interior is contained in $D_\delta - \{t_1, \ldots, t_\mu\}$. We define a loop $l_i$ based at $t'$ where $l_i$ goes around $t_i$ anticlockwise along a small circle centered at $t_i$ and is connected by $p_i$. The loops $l_1, \ldots, l_\mu$ generate the fundamental group $\pi_1(D_\delta - \{t_1, \ldots, t_\mu\}, t')$. The loop $l_i$ induces monodromy action on the cohomology of fiber $H^{n-1}(F, \mathbb{Z})$ given by the Picard-Lefschetz formula

$$T_i : \alpha \mapsto \alpha + (\alpha, \delta_i)\delta_i,$$

where $\delta_i$ is the vanishing cycle associated to the critical value $t_i$. The set of all vanishing cycles $\{\delta_i\}_{i=1}^\mu$ generates $H^{n-1}(F, \mathbb{Z})$. When $n$ is odd, $(\delta_i, \delta_i) = \pm 2$, while when $n$ is even, $(\delta_i, \delta_i) = 0$.

**Definition 6.8.** We define monodromy group of the Milnor fiber of $f$ to be the subgroup of $\text{Aut} H^{n-1}(F, \mathbb{Z})$ generated by $T_1, \ldots, T_\mu$.

The monodromy group is independent of the choice of perturbation function and the loops $l_1, \ldots, l_\mu$. Moreover, in the case where $n = 3$ and $f(x_1, x_2, x_3) = 0$ has ADE singularity at origin, the following result is well known (one can refer to p.99 in [AGZV88], and [GZ95]).

**Lemma 6.9.** Vanishing cycles $\delta_1, \ldots, \delta_\mu$ can be naturally chosen to form a basis of root system of the corresponding ADE type in $H^2(F, \mathbb{Z})$. The monodromy group of $f$ is the Weyl group corresponding to the type of the singularity.

These vanishing cycles are obtained by a sequence of conjugation operations of paths $\{p_i\}_{i=1}^\mu$. Such operations are called Gabrielov operations.

Now let $S_0$ be the cubic surface arising from a hyperplane section of $Y$ with an affine chart defined by $f(x_1, x_2, x_3) = 0$ with an isolated singularity at $(0, 0, 0)$ of ADE type. The next result will show that deforming $f$ in the family of hyperplane sections is the "same" as considering Milnor fiber theory of $f$.

**Lemma 6.10.** Choose a linear 2-dimensional hyperplane sections family parameterized by $(\lambda, w) \in \mathbb{C}^2$ with $(0, 0)$ corresponds to $f(x_1, x_2, x_3) = 0$ with an ADE singularity, then there is an $\varepsilon > 0$ such that for all $\lambda, w$ with $|\lambda|, |w| < \varepsilon$, an affine chart of the total family is
analytically isomorphic to

\[(25)\quad f_\lambda(x_1, x_2, x_3) + w = 0,\]

where \(f_\lambda(x_1, x_2, x_3)\) is the affine equation of the hyperplane section at \((\lambda, 0)\).

**Proof.** \(f(x_1, x_2, x_3) = 0\) is an affine cubic surface with an isolated singularity at \((0, 0, 0)\) of ADE type. It fits into the cubic threefold \(Y\) defined by equation

\[F(x_1, x_2, x_3, w) = f(x_1, x_2, x_3) + wQ(x_1, x_2, x_3) + w^2L(x_1, x_2, x_3) + w^3\sigma,\]

where \(Q\) is quadratic, \(L\) is linear and \(\sigma\) is a constant. By varying \(w \in \mathbb{C}\), we have a pencil

\[(26)\quad \mathcal{Y} \to \mathbb{C}\]

of hyperplane sections \(f_w(x_1, x_2, x_3) = F(x_1, x_2, x_3, w)\) of \(Y\) through \(f(x_1, x_2, x_3) = 0\). Since the cubic threefold is smooth, \(Q(0, 0, 0) \neq 0\). Therefore, the equation of cubic threefold is

\[F(x_1, x_2, x_3) = f(x_1, x_2, x_3) + wG(x_1, x_2, x_3, w) = 0,\]

where \(G(x_1, x_2, x_3, w)\) is quadratic and is non-vanishing in a small neighborhood \(D\) of 0. Therefore, by restricting to \(D\) and setting \(g = f/G\), we get a family

\[g(x_1, x_2, x_3, w) + w = 0,\]

which is analytically equivalent to the family \((26)\) restricted to \(D\).

Now we choose a perturbation of \(f\) in the hyperplane section family transversal to the \(w\) direction. In other words, we choose a linear function

\[l = ax_1 + bx_2 + cx_3\]

with \(a, b, c \in \mathbb{C}\) being general, then

\[f_\lambda(x_1, x_2, x_3) = F(x_1, x_2, x_3, \lambda l), \quad \lambda \in \mathbb{C}\]

is a pencil of hyperplane sections through \(f\). We consider the two dimensional family spanned by \(l\) and \(w\). Then for \((l, w) \in \mathbb{C}^2\), the hyperplane section at \(\lambda l + w\) is defined by

\[(27)\quad f_{\lambda, w} = F(x_1, x_2, x_3, \lambda l + w) = f_\lambda(x_1, x_2, x_3) + wG(x_1, x_2, x_3, \lambda l) + \lambda lH(x, y, z, w),\]

where \(H(x, y, z, w) = G(x, y, z, w + \lambda l) - G(x, y, z, \lambda l) = wL(x_1, x_2, x_3) + (2w\lambda l + w^2)\sigma\) is divisible by \(w\).

Therefore, denote \(G' = G + \lambda lH/w\), we can express the two dimensional family \((27)\) as

\[\frac{f_{\lambda, w}}{G'} + w = 0,\]
in a small neighborhood $D^2$ of origin. It is analytically equivalent to the family
\[ f_\lambda(x_1, x_2, x_3) + w = 0. \]

Proof of Prop 6.7

Let $\Sigma_0$ be the discriminant locus $x_0$, namely the locus \{t $\in B | p^{-1}(t) \cap D_0$ is singular\}. $\Sigma_0 \subseteq \gamma$ $\cap B$ is an irreducible component (when $S_0$ has only one isolated singularity, they are the same).

Since the complement of the inclusion $B^{sm} \subseteq B \setminus \Sigma_0$ has real codimension at least two, there is a surjection
\[ \pi_1(B^{sm}, t') \twoheadrightarrow \pi_1(B \setminus \Sigma_0, t'), \]
where $t'$ is a fixed base point. Therefore, one reduces to the case where $S_0$ has only one singularity and $\Sigma_0 = \gamma \cap B$.

We choose a general line $L$ through $t'$ such that $L$ intersect $\Sigma_0$ transversely at smooth points, then $U = B^{sm} \cap L$ is an analytic open space. Moreover, by a local version of Zariski’s theorem on fundamental groups, there is a surjection
\[ \pi_1(U, t') \twoheadrightarrow \pi_1(B^{sm}, t'). \]
Therefore it suffices to show that the monodromy representations generated by the loops in the 1-dimensional open space $U$ is the entire Weyl group.

On the other hand, by Lemma 6.10 the hyperplane sections parameterized by $U$ is analytically equivalent to the family
\[ f'(x_1, x_2, x_3) + w = 0, \]
where $f'$ is the defining equation of hyperplane section at $t'$ and is a perturbation of $f$. Therefore, by Lemma 6.9 the monodromy group induced by $\pi_1(U, t')$ is the Weyl group corresponding to the type of $x_0$. □

6.2.3. Globalization.

Proposition 6.11. Let $S_0$ be a hyperplane section of $Y$ and $x_0$ be a singular point on $S_0$ of type ADE. Let $F$ denote the Milnor fiber of $x_0$, and $S_t$ a general nearby fiber, then there is an inclusion $F \hookrightarrow S_t$. The induced map on homology
\[ H_2(F, \mathbb{Z}) \rightarrow H_2(S_t, \mathbb{Z})_{van} \]
is injective.

Proof. This is due to Brieskorn’s theory [Bri70] and its globalization [Art74] (also see [KM], Theorem 4.43). Use the same notations as we introduced in the beginning of this section, there
exists a finite cover $B' \to B$, such that the total family admits simultaneous resolution in the category of algebraic spaces. In other words, there is a commutative diagram as following.

$$
\begin{array}{ccc}
\mathcal{Y}' & \xrightarrow{f} & \mathcal{Y} \\
\downarrow{g} & & \downarrow{g} \\
B' & \longrightarrow & B
\end{array}
$$

$\mathcal{Y}'$ is a complex analytic manifold, $f$ is bimeromorphic, and $g$ is a proper holomorphic submersion. (The resolution is in general not algebraic since the local gluing data is only analytic.)

$\mathcal{Y}' \to B'$ is diffeomorphic to the product $S_t \times B'$ by Ehresmann’s theorem, so the Milnor fiber $F = S_t \cap D_0$ is diffeomorphic to an open set $U$ of the central fiber $g^{-1}(0)$. The argument reduces to show that the homology group induced by the inclusion $U \hookrightarrow g^{-1}(0)$ is injective.

$g^{-1}(0)$ is isomorphic to the minimal resolution $\tilde{S}_0$ of $S_0$. Denote $V$ the exceptional curve in $\tilde{S}_0$ over $x_0$. Then $V$ is a bunch of $\mu(-2)$-curves, and corresponds to a connected sub-diagram of the Dynkin diagram of $E_6$. Since the image of $U$ in $S_0$ is a neighborhood of $x_0$, $U$ is a regular neighborhood of $V$. So the induced map $H_2(U, \mathbb{Z}) \to H_2(\tilde{S}_0, \mathbb{Z})$ is injective. ☐

**Remark 6.12.** This is false for elliptic singularity, since the Milnor number of such a singularity is 8, while the vanishing homology on $S_t$ has rank 6.

**Corollary 6.13.** Via the inclusion (29), $H_2(F, \mathbb{Z})$ is an irreducible sub-representation of $H_2(S_t, \mathbb{Z})_{\text{van}}$ as $\pi_1(B_{\text{sm}}, t')$-representations. It induces isomorphism between the monodromy group $\text{Im}(\rho_0)$ of $x_0$ defined in (24) and the monodromy group $\text{Im}(\rho)$ defined in (21).

The proof of the Theorem 6.4 reduces to the following argument.

**Proposition 6.14.** Assume that the cubic surface $S_0 = S_{t_0}$ has isolated singularities $x_1, ..., x_k$ of type ADE. Denote $W_i$ the Weyl group corresponding to the type of the singularity $x_i$, with $i = 1, ..., k$. Then the monodromy group of $S_0$ is isomorphic to

$$W_1 \times \cdots \times W_k.$$

Moreover, each of the factor $W_i$ is generated by the reflections corresponding to the exceptional curves over $x_i$.

**Proof.** Let $D_i$ denote a small ball in $\mathcal{Y}$ around $x_i$ such that $D_i \cap D_j = \emptyset$ for $i \neq j$. Let $\Sigma_i = \{t \in B | S_t \cap D_i \text{ is singular} \}$ be the discriminant of $x_i$. Then $\Sigma_i$ is an irreducible divisor of $B$ and $Y^\vee \cap B = \cup_i \Sigma_i(x_i)$. None of the $\Sigma_i$ contains $\Sigma_j$ for $i \neq j$, since otherwise the locus will extend to a proper curve, contradicting to the fact that the dual variety $Y^\vee$ is smooth in codimension one, and that the smooth locus parameterizes the hyperplane section with one ordinary nodal singularity.

41
Fix a general point $t' \in B_{sm}$. We take a general pencil $\mathbb{L}$ in $\mathbb{O}$ through $t'$ intersecting $Y^\vee \cap B$ transversely along the smooth locus. So $\mathbb{L}$ intersects each $\Sigma_i$ transversely at points $t'_i$, for $j = 1, \ldots, \mu_i$, where $\mu_i$ is the Milnor number of $x_i$. None of the $t'_i$ coincides with $t''_j$ unless $i = i'$ and $j = j'$. There is a vanishing cycle $\delta'_i \in H^2(S_{\tau}, \mathbb{Z})$ associate to $t'_i$. The monodromy action $T_{i'}^j$ induced by the simple loop around $t'_i$ on the 72 roots is given by the Picard-Lefschetz formula (23) associated to $\delta'_i$. Moreover, via the surjectivity

$$\pi_1(\mathbb{L} \cap B, t') \to \pi_1(B_{sm}, t'),$$

the monodromy group defined in (21) is generated by $T_{i'}^j$, $i = 1, \ldots, k$, $j = 1, \ldots, \mu_i$. By Proposition 6.7 and Corollary 6.13, the subgroup generated by $\{T_{i'}^j\}_{j=1}^{\mu_i}$ is the Weyl group $W_i$, which is also the subgroup generated by the reflections of corresponding to the exceptional curves over $x_i$.

Finally, since $\delta'_i$ can be represented by a topological 2-sphere contained in the neighborhood $D_i$ around $x_i$, so the intersection number

$$(\delta'_i, \delta''_{i'}) = 0, \ i \neq i'.$$

Therefore, the monodromy operators $T_{i'}^j$ and $T_{i''}^j$ commute for $i \neq i'$ by Picard-Lefschetz formula. Therefore, the subgroup corresponding to the monodromy group of $x_i$ commutes with the subgroup corresponding to the monodromy group of $x_j$. It follows that the monodromy group of $S_0$ is the product $W_1 \times \cdots \times W_k$. \hfill \Box

7. A Modular Interpretation

In this section, we discuss some relations between our results and the Bridgeland stable moduli spaces studied in [APR19] and [BBF+20].

In [APR19], the author studied the moduli space $M_{\sigma}(w)$ of Bridgeland stable objects in the Kuznetsov component with Chern character $w = H - \frac{1}{2}H^2 + \frac{1}{3}H^3$ for a smooth cubic threefold $Y$. Let $S$ denote a hyperplane section of $Y$. The moduli space $M_{\sigma}(w)$ parameterizes the following two objects:

1. $O_S(D)$, a reflexive sheaf or rank 1 associated to certain Weil divisor $D$ on $S$ (when $S$ is general, $D = L_1 - L_2$ for a pair of skew lines $L_1, L_2$ on $S$);
2. $I_{p|S}$, the ideal sheaf of a point in $S = Y \cap H$, where $H$ is the tangent hyperplane section at $p$.

In both cases, the stable object is contained in a unique hyperplane section, so there is a natural projection

$$M_{\sigma}(w) \to \mathbb{O}.$$ 

Proposition 7.1. The projection (30) is generically finite, and its only positive dimensional fibers are elliptic curves that correspond to the Eckardt points on $Y$.
Proof. It is shown in section 3.3 [APR19] that the $\mathcal{M}_\sigma(w)$ is isomorphic to the moduli space $\mathcal{M}_G(\kappa)$ of Gieseker-stable sheaves with Chern character $\kappa = (3, -H, \frac{1}{2}H^2, -\frac{1}{6}H^3)$ studied in [BBF+20]. According to Lemma 7.5 of [BBF+20], there is an isomorphism $\mathcal{M}_G(\kappa) \cong \text{Bl}_0 \Theta$. Via the isomorphisms, the projection $\lambda$ coincides with the Gauss map defined in [19]

$$\lambda : \text{Bl}_0 \Theta \to \mathcal{O},$$

because it agrees on the general points, where a pair of skew lines is sent to their spanning hyperplane section. Now the argument follows from Theorem 5.1. 

We reinterpret the map that we introduced in (5).

$$\tilde{\psi} : \text{Bl}_{\Delta_F}(F \times F) \to \text{Bl}_0 \Theta.$$

In corollary 3.4, we introduced a double cover $\widetilde{H(Y)}$ of the Hilbert scheme $H(Y)$. Informally, we can regard $\widetilde{H(Y)}$ as the "Hilbert scheme" that parameterizes universal flat families of ordered skew lines. There is an isomorphism

$$\widetilde{H(Y)} \cong \text{Bl}_{\Delta_F}(F \times F).$$

Thus there is an Abel-Jacobi map

$$\widetilde{AJ} : \widetilde{H(Y)} \to JY$$

by composing the blowup map and $F \times F \to JY$ as in (6).

For the stable moduli space, there is also an Abel-Jacobi map

$$AJ : \mathcal{M}_\sigma(w) \to JY$$

by sending $O_{S}D$ to $\sum_{D_0}^D$, for some fixed divisor $D_0$.

Regarding $\mathcal{M}_\sigma(w)$ as the blowup of the theta divisor, then according to Proposition 2.1 in [APR19], the exceptional divisor $K \cong Y$ parameterizes ideal sheaves $I_{p,S}$ of singular points on hyperplane sections $S$ of $Y$, and the complement of $K$ parameterizes coherent sheaves $O_{S}(D)$ with $D$ being certain Weil divisor on the hyperplane section $S$.

**Proposition 7.2.** The Abel-Jacobi map $\widetilde{AJ}$ factors through the moduli space $\mathcal{M}_\sigma(w)$ up to by adding a constant on the torus $JY$. In other words, there is a following commutative diagram (up to by adding a constant).

$$\begin{array}{ccc}
\widetilde{H(Y)} & \xrightarrow{\psi} & \mathcal{M}_\sigma(w) \\
& \searrow_{\lambda} & \downarrow_{AJ} \\
& & JY
\end{array}$$
By restricting $\Psi$ to the exceptional divisors on both sides, we have

$$\Psi|_{\mathbb{P}T_F} : \mathbb{P}T_F \to Y$$

by sending a scheme $Z$ of type (II) or (IV) to the ideal sheaf $I_p$, where $p$ is the unique point determined by $Z$ defined in Proposition 3.15. It is isomorphic to the projection from the incidence variety to $Y$ by the identification $[1]$.

**Proof.** We define a rational map $\Psi : (L_1, L_2) \mapsto \mathcal{O}_S(L_1 - L_2)$. Their Abel Jacobi images differ by a constant due to the presence of $D_0$. So we have a commutative diagram. Since $\Psi$

$$
\begin{array}{ccc}
\widehat{H(Y)} & \overset{\Psi}{\longrightarrow} & \mathcal{M}_\sigma(w) \\
\text{Bl}_{\Delta_F}(F \times F) & \overset{\psi}{\longrightarrow} & \text{Bl}_0\Theta
\end{array}
$$

agrees with $\tilde{\psi}$ at general points, $\Psi$ uniquely extends to a morphism. \qed

Although there is a complete classification of cubic surfaces $S$, and the Weil divisors on $S$ are pretty much controlled by the divisors on the minimal resolution $\tilde{S}$ of $S$, it is not explicitly known the expression of the divisor $D$ on singular hyperplane sections of $Y$ such that $\mathcal{O}_S(D) \in \mathcal{M}_\sigma(w)$.

As we know the off-diagonal part of $\widehat{H(Y)}$ parameterizes for type (I) and (III) schemes with an order, we conjecture that their supports determine the divisor $D$.

**Conjecture 7.3.** If $\mathcal{O}_S(D)$ is a stable object parameterized by $\mathcal{M}_\sigma(w)$, then $D$ has the form $L_1 - L_2$, where $L_1, L_2$ are two lines on $S$ (can be singular) and lie in one of the two cases:

(i) $L_1, L_2$ disjoint (Lines can pass through singularities);

(ii) $L_1$ and $L_2$ intersect at one point $p$, which is a singularity of $p$.

On the other hand, [BBF+20] uses the twisted cubics on $Y$ to characterize the moduli space $\mathcal{M}_G(\kappa)$. More precisely, a general point of $\mathcal{M}_G(\kappa)$ parameterizes a coherent sheaf $\mathcal{E}$ that fits into the exact sequence

$$0 \to \mathcal{E} \to \mathcal{O}_S^{\oplus 3} \to \mathcal{O}_S(C) \to 0,$$

where $C$ is a twisted cubic on hyperplane section $S$ (which is a Weil divisor when $S$ is singular).

Let $\mathcal{C}$ denote the open locus of smooth twisted cubics in the Hilbert scheme of $Y$. There is a commutative diagram.

The vertical arrows are Abel-Jacobi maps. For example, by fixing a planar cubic $C_0$, $\phi$ is defined by $C \to \int_{C_0}^C$, which factors through the moduli space $\mathcal{M}_G(\kappa)$. 44
Question 7.4. Is there a comparison map of the diagrams (31) and (32)? In other words, applying the isomorphism $M_\sigma(w) \cong M_G(\kappa)$ between the two moduli spaces, is there a rational map $\overline{H(Y)} \rightarrow C^\circ$ commuting with Abel-Jacobi maps?

8. Relation to Schnell’s results

8.1. $\mathcal{D}$-modules. For a polarized variation of Hodge structure $(\mathcal{H}, Q)$ of even weight over a quasi-projective variety $B_0$, as a Zariski open subset of a smooth projective variety $B$, Schnell [Sch14] constructed a completion of $T_Z$, the étale space of the local system $\mathcal{H}_Z$.

More explicitly, assume that $\mathcal{H}$ has weight $2n$. The data $(\mathcal{H}, Q)$ consists of a $\mathbb{Z}$-local system over $B_0$, a flat connection $\nabla$ on $\mathcal{H}_C = \mathcal{H} \times_{\mathbb{Z}} \mathcal{O}_{B_0}$, Hodge bundles $F^p\mathcal{H}_C$ and a nondegenerate pairing $Q : H^p \times H^p \rightarrow \mathbb{Q}$ satisfying the Hodge-Riemann conditions.

Consider $F^n\mathcal{H}$ the associated Hodge bundle, i.e., the subbundle whose fiber at $p \in B_0$ is $F^n\mathcal{H}_p$, the $n$-th Hodge filtration of the complex vector space $\mathcal{H}_p$. Then it is shown in Lemma 3.1 from [Sch14] that for each connected component $T_\lambda / B_0$ of $T_Z / B_0$, the natural mapping

$$T_\lambda \rightarrow T(F^n\mathcal{H})$$

$$\alpha \mapsto Q(\alpha, \cdot)$$

is finite, where $T(F^n\mathcal{H})$ is the underlying analytic space of the Hodge bundle.

Moreover, according to Saito’s Mixed Hodge Modules theory, there is a Hodge module $M$ underlying a filtered $\mathcal{D}_{B_0}$-module $(\mathcal{M}, F, \mathcal{M})$ supported on $B$, as the minimal extension of $(\mathcal{H}, \nabla)$.

Schnell considered the space $T(F_{n-1}\mathcal{M})$ as analytic spectrum of the $(n-1)$-th filtration of $\mathcal{M}$ and showed that the analytic closure of the image of the composite of

$$T_\lambda \rightarrow T(F^n\mathcal{H}) \rightarrow T(F_{n-1}\mathcal{M})$$

is still analytic, therefore it extends to a finite analytic covering by Grauert’s theorem and so there is a normal analytic space $\overline{T_\lambda}$ extending $T_\lambda$. Schnell defines $\overline{T_Z}$ as the union $\bigcup_{\lambda} \overline{T_\lambda}$.

In the case where the variation of Hodge structure comes from vanishing cohomology of hyperplane sections on a smooth cubic threefold, all $H^2$ on the general hyperplane section is
concentrated in \((1,1)\) part, so \(F^2\mathcal{H}\) is trivial and so is \(\mathcal{M}\). Therefore the analytic spectrum is nothing but the base space \(\mathcal{O}\). So the completion of the locus of primitive vanishing cycle \(T'\) is exactly the \(\tilde{T}'\) space we are discussing. In short, as a consequence of Theorem 5.1 we have:

**Corollary 8.1.** When the variation of Hodge structure comes from vanishing cohomology of hyperplane sections on a smooth cubic threefold, the completion space \(\tilde{T}_Z\) defined in [Sch14] p.10 contains an irreducible component which is biholomorphic to \(\tilde{T}'\) space, and is obtained by contracting finitely many curves in \(\text{Bl}_0\Theta\).

### 8.2. Tube Mapping

In [Sch11], Schnell studied the relationship between the primitive homology \(H_n(Y, Z)_{\text{prim}}\) of a smooth projective variety \(Y \subseteq \mathbb{P}^N\) of dimension \(n\) and the vanishing homology \(H_{n-1}(S, Z)_{\text{van}}\) of a smooth hyperplane section \(S = Y \cap H\). Let \(U \subseteq (\mathbb{P}^N)^*\) be the open set of smooth hyperplanes, and \(l \subseteq U\) be a loop based at \(t\), and \(\alpha \in H_{n-1}(S, Z)_{\text{van}}\), if \(l_\ast \alpha = \alpha\), then the trace of \(\alpha\) along the loop \(l\) is a topological \(n\)-chain on \(Y\) with boundary \(\alpha - l_\ast \alpha = 0\), so it is a \(n\)-cycle which is well-defined in the primitive homology. Since the \(n\)-cycle is a "tube" on \(\alpha\) over the loop \(l\), such map is called tube mapping. Schnell proved that if \(H_{\text{van}}^{n-1}(S, Z) \neq 0\), then the tube map

\[
\{(\lbrack l \rbrack, \alpha) \in \pi_1(U, t) \times H_{n-1}(S, Z)_{\text{van}} \mid l_\ast \alpha = \alpha \} \to H_n(Y, Z)_{\text{prim}}
\]

has cofinite image. Equivalently, the set of tubes on vanishing cohomology classes generates the middle dimensional primitive cohomology on \(Y\) over \(\mathbb{Q}\).

Herb Clemens conjectured that the theorem is still true by restricting the tube map to tubes on a single primitive vanishing cycle \(\alpha_0\) (a cycle deformable to a class representing a topological sphere when the hyperplane section is close to acquire a node), namely, Clemens conjectured that

**Conjecture 8.2.** under the same hypothesis, the image of

\[
(33) \quad \{(\lbrack l \rbrack, \alpha_0) \in \pi_1(U, t) \mid l_\ast \alpha_0 = \alpha_0 \} \to H_n(Y, Z)_{\text{prim}}
\]

is cofinite.

We will prove this conjecture when \(Y\) is a general cubic 3-fold.

**Theorem 8.3.** Let \(Y\) be a smooth cubic threefold, then the conjecture is true.

**Proof.** First note that \(H_3(Y, Z) = H_3(Y, Z)_{\text{prim}}\) due to \(H_1(Y) = 0\). Second, a primitive vanishing cycle \(\alpha_0 \in H_2(Y, Z)\) is represented by the difference of two lines, so \((t, \alpha_0)\) is a point on \(T'\). Also, recall that \(T'\) is a finite-sheet covering space of \(U = \mathbb{O}^\text{sm}\). A loop \(l \subseteq U\) such that \(l_\ast \alpha_0 = \alpha_0\) based at \(t\) corresponds to a loop \(\tilde{l} \subseteq T'\) based at \((t, \alpha_0)\), so by abusing
the notation \( \ast \) as the base point, the map \( (33) \) is the same as

\[
\pi_1(T', \ast) \to H_3(Y, \mathbb{Z}).
\]

The following result is proved in [Zhao15, p.26] in a more general setting. For reader’s convenience, we provide a self-contained proof here.

**Proposition 8.4.** The map \( (34) \) is induced on fundamental group by the topological Abel-Jacobi map \( \phi : T' \to JY \), followed by the isomorphism \( \pi_1(JY, \ast) \cong H_3(Y, \mathbb{Z}) \).

**Proof.** Let \( \tilde{l} \subseteq T' \) be a loop based at \( (t, \alpha_0) \), then its Abel-Jacobi image is determined by a family of 3-chains \( \Gamma_t \) indexed by \( t \in [0, 1] \) modulo 3-cycles on \( Y \), so we can choose \( \Gamma_t \) to be the union \( \Gamma_0 \cup \Gamma_t \) where \( \Gamma_t = \bigcup_{s \in [0, t]} \alpha_s \) as trace of primitive vanishing cycles along the path \( [0, 1] \). It follows that \( \Gamma_1 \) is a 3-chain such that \( \partial \Gamma_1 = \partial \Gamma_0 - \alpha_0 \), so the induced map on \( \pi_1 \) sends \( \tilde{l} \) to the image of the 3-cycle \( \Gamma_1 - \Gamma_0 = \bigcup_{t \in [0, 1]} \alpha_t \) in \( H_3(Y, \mathbb{Z}) \). \( \square \)

Finally, the proof of the theorem follows from the following argument.

**Proposition 8.5.** The map \( (34) \) induced by \( \phi \) is surjective.

**Proof.** First of all, \( \phi : T' \to JY \) factors through the inclusion \( T' \subseteq \text{Bl}_0 \Theta \). Moreover, \( T' \subseteq \text{Bl}_0 \Theta \) is a complement of a divisor in a smooth complex manifold, as a smooth loop based can be deformed to be disjoint from a real codimension-two set, there is a surjection \( \pi_1(T', \ast) \to \pi_1(\text{Bl}_0 \Theta, \ast) \). Therefore, it suffices to show that \( \pi_1(\text{Bl}_0 \Theta, \ast) \to \pi_1(JY) \) is surjective.

Next, choose \( p \in F \) such that its corresponding line \( L_p \) is of second type on \( Y \) and let \( D_p \) be the divisor of lines that are incident to \( L_p \). By Lemma 10.7 of [CG72, p \in D_p], and it follows that \( \{p\} \times F \setminus D_p \) is disjoint from the diagonal. In particular, let \( \sigma : \text{Bl}_{\Delta_p}(F \times F) \to F \times F \) be the blowup map, the restriction of \( \sigma^{-1} \) to the domain of \( \psi_p \) is an isomorphism. We define the restricted Abel-Jacobi map

\[
(35) \quad \psi_p : \{p\} \times F \setminus D_p \to JY.
\]

\( \psi_p \) lifts to the blowup, so the image of \( \pi_1(\text{Bl}_0 \Theta, \ast) \to \pi_1(JY) \) contains \( (\psi_p)_*(\pi_1(\{p\} \times F \setminus D_p, \ast)) \) as a subgroup. Thus it suffices to show that \( \psi_p \) induces surjectivity on fundamental groups.

To show this, note that \( \psi_p \) factors through the inclusion \( \{p\} \times F \setminus D_p \subseteq \{p\} \times F \) which induces surjective map on fundamental group for the same reason as in the first paragraph of the proof. Moreover the map \( \{p\} \times F \cong F \to JY \) factors through the Albanese map together with the isomorphism \( \text{Alb}(F) \xrightarrow{\cong} JY \) [CG72]. It follows that \( \psi \) induces an isomorphism between fundamental groups, therefore so does \( \psi_p \). \( \square \)
\[
F \xrightarrow{\psi} JY \\
\downarrow_{\text{alb}} \cong \\
\text{Alb}(F)
\]

(36)

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