Vacuum Polarization Effects from Fermion Zero-Point Energy

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Abstract

The influence of an external electromagnetic field on the vacuum structure of a quantized Dirac field is investigated by considering the quantum corrections to classical Maxwell’s Lagrangian density induced by fluctuations of the non-perturbative vacuum. Effective Lagrangian densities for Maxwell’s theory in (3+1) and (2+1) dimensions are derived from the vacuum zero-point energy of the fermion field in the context of a consistent Pauli-Villars-Rayski subtraction scheme, recovering Euler-Kockel-Heisenberg and Maxwell-Chern-Simons effective theories. Effective Scalar Quantum Electrodynamics as well as low temperature effects in both spinor and scalar theories are also discussed.

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1 Introduction

In the early 30’s, one of the more striking results from Dirac’s positron theory, in addition to the possibility of converting electromagnetic energy into matter through electron-positron pair production, in sufficiently strong electromagnetic fields, was the scattering of quantized electromagnetic radiation by other photons or an external electromagnetic field, the latter process known as Delbrück scattering. These nonlinear effects, which were not predicted by classical electromagnetic theory, were studied by Euler and Kockel and Heisenberg and Euler\cite{1}, who derived an effective Lagrangian density for such an extended version of Maxwell’s theory, induced by an external static and homogeneous electromagnetic field. Their work was refined by Weisskopf\cite{2}, the first to introduce the idea of charge renormalization, in the scope of the so-called physics of subtractions.

Although Weisskopf approach was not manifestly Lorentz invariant, he came up with an effective Lagrangian which is equivalent to the Lorentz and gauge invariant Lagrangian density derived by Schwinger\cite{3} some years latter. However, along his calculations, Weisskopf dealt with divergent sums and momentum integrals, without any regularization prescription to give them a precise meaning from the mathematica standpoint. Only in the late 40’s, short before the previously mentioned Schwinger’s approach, the physics of subtractions would be practiced with proper rigour in the works by Rayski and Pauli and Villars\cite{4}, who first introduced a Lorentz and gauge invariant regularization prescription to tackle the singular Green’s functions of quantum electrodynamics (QED).

Along the last two decades Effective theories have deserved special attention in different contexts, ranging from quantum electrodynamics in external fields\cite{5,6}, the linear sigma model\cite{7} and the Nambu-Jona-Lasinio\cite{8} in hadron physics to models for low-energy quantum chromodynamics (QCD)\cite{9} and gauge field theories in (2+1) space-time dimensions, in particular, quantum electrodynamics (QED$_3$)\cite{10} and the non-renormalizable gauged Thirring model\cite{11}, either at zero or at finite temperature\cite{12}.

The aim of this paper is to rederive Euler-Kockel-Heisenberg effective Lagrangian for QED and QED$_3$ following the approach of Berestetski, Lifshitz and Pitayeviski\cite{13}, which is inspired by Weisskopf’s essential ideas, employing Pauli-Villars-Rayski regularization prescription in order to keep under control divergent quantities which calls for a consistent subtraction proce-
dure. The next section presents the formalism and the discussion of the necessity of introducing auxiliary regulator masses into the subtracting scheme of divergences in the fermion zero-point energy for QED in the presence of a constant and uniform magnetic field. In section 3, the (2+1) dimensional case is studied, paying attention to the conditions that must be imposed on the regulator masses in order to eliminate the divergences of the theory and discuss the problem of parity violation in the effective Lagrangian density. The same analysis is also applied to scalar quantum electrodynamics (s-QED), in section 4. The last section is devoted to some concluding remarks.

2 QED Effective Lagrangian in (3+1) Dimensions

In order to determine the effective Lagrangian density for the electromagnetic field from the zero-point energy of matter, which in this case is represented by the electron-positron field, Dirac Hamiltonian must be written in Fock space as

\[ \mathcal{H} = \sum_{p, \sigma} \epsilon_{p\sigma}(a_{p\sigma}^\dagger a_{p\sigma} + b_{p\sigma}^\dagger b_{p\sigma}) + \varepsilon_0, \]  

where

\[ \varepsilon_0 = \langle \mathcal{H} \rangle = -\sum_{p, \sigma} \epsilon_{p\sigma} \]  

\[ = \sum_{p, \sigma} \int d^3x \psi_{p\sigma}(-)^* i \frac{\partial}{\partial t} \psi_{p\sigma} \]  

is the vacuum zero-point energy. Here \( \psi_{p\sigma}(-) \) are the negative frequency solutions of Dirac equation in the presence of the electromagnetic field, normalized in a unit volume, so that \( \varepsilon_0 \) is in fact a scalar density. The expectation value of the potential energy of an electron in these negative energy states is

\[ U_0 = \sum_{p, \sigma} \int d^3x \psi_{p\sigma}(-)^* e \phi \psi_{p\sigma} \]

\[ = E \sum_{p, \sigma} \int d^3x \psi_{p\sigma}(-)^* \frac{\partial \mathcal{H}}{\partial E} \psi_{p\sigma} = E \frac{\partial \varepsilon_0}{\partial E}, \]

where \( \phi = -E \cdot r \) is the scalar potential of a uniform electric field.
The above quantity must be subtracted from $\varepsilon_0$, since the physical vacuum is expected to have zero charge. Therefore, the total shift in the vacuum energy density induced by an external uniform electromagnetic field is

$$\delta W = \left( \varepsilon_0 - E \frac{\partial \varepsilon_0}{\partial E} \right) - \left( \varepsilon_0 - E \frac{\partial \varepsilon_0}{\partial E} \right)_{E=H=0}. \quad (4)$$

This represents an additional contribution to the total electromagnetic energy density $W$ due to vacuum polarization effects. This energy is related to the effective Lagrangian density $L_{\text{eff}}$ of the electromagnetic field by the transformation

$$W = E \frac{\partial L_{\text{eff}}}{\partial E} - L_{\text{eff}}, \quad (5)$$

so that its variation can be compared with expression (4). As a result,

$$\delta L = L_{\text{eff}} - L_0 = -\left[ \varepsilon_0 - (\varepsilon_0)_{E=H=0} \right], \quad (6)$$

where $L_0$ is the Lagrangian density for the applied external electromagnetic field.

The negative energy levels of an electron with charge $-|e|$ in a constant and uniform magnetic field $H_z = -H$ are

$$-\varepsilon_{p\sigma}^{(-)} = -\sqrt{m^2 + (2n+1 - \sigma)|e|H + p_z^2}, \quad (7)$$

where $n = 0, 1, 2, \ldots$ and $\sigma = \pm 1$. The sum over the $z$-components of the electron momenta in (2) will be evaluated in the continuum by considering the corresponding density of momentum states

$$\frac{|e|H}{2\pi} \frac{dp_z}{2\pi}.$$

Thus, the negative vacuum zero-point energy can be written as

$$-\varepsilon_0 = \frac{|e|H}{(2\pi)^2} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \sum_{\sigma=\pm 1} \varepsilon_{n,\sigma}(p_z)$$

$$= \frac{|e|H}{(2\pi)^2} \int_{-\infty}^{\infty} dp_z \left\{ \sqrt{m^2 + p_z^2} + 2 \sum_{n=1}^{\infty} \frac{1}{\sqrt{m^2 + 2|e|H n + p_z^2}} \right\}. \quad (8)$$

The above momentum integrals are quadratically divergent at the ultraviolet. In order to extract a finite result for $\delta L$, a consistent subtraction procedure is
necessary, starting from a mathematical meaningful expression for the zero-point energy, i.e., $\varepsilon_0$ must be regularized. This can be achieved in the sense of Pauli-Villars-Rayski regularization scheme, by introducing auxiliary masses of fictitious regulator fields, satisfying suitable conditions that eliminate the divergences of the original theory, which is ultimately recovered by letting the regulator masses go to infinite at the end of calculations.

The first step is to analyze the conditions fulfilled by the auxiliary masses according to the degree of singularity of the generalized function

$$D(m^2) = \int_{-\infty}^\infty dp \sqrt{m^2 + p^2}. \quad (9)$$

This may be interpreted in the sense of its analytical continuation

$$D_\delta(m^2) = \lim_{\delta \to 0} \int_{-\infty}^\infty dp (m^2 + p^2)^{\frac{1}{2} - \delta} \quad (10)$$

which can be differentiated with respect to the parameter $m^2$ before taking the limit $\delta \to 0$. The asymptotic expansion of integral (9) for $p^2 \gg m^2$,

$$D(m^2) \sim \int dp \left(1 + \frac{m^2}{2p^2}\right) + \text{(regular terms)},$$

shows that conditions

$$\sum_{i=0}^{N} c_i = 0, \sum_{i=0}^{N} c_i m_i^2 = 0 \quad (11)$$

must be imposed on the regulated expression

$$D^R(m^2) = \sum_{i=0}^{N} c_i D_i^R(m_i^2). \quad (12)$$

in order to eliminate the quadratic and logarithmic divergences, respectively. In the above expressions, $N$ is the total number of regulators, $c_0 = 1$ and $m_0 = m$ is the electron mass.

Instead of considering the regulated expression $\varepsilon_0^R$ for the zero-point energy (8), it's easier to cope with the function

$$\Phi^R(H) \equiv -\frac{\partial^2 \varepsilon_0^R}{\partial (m^2)^2} = \sum_i c_i \Phi_i(H), \quad (13)$$
where
\[
\Phi_i(H) = -\frac{|e|H}{2(2\pi)^2} \int_0^\infty dp_z \left\{ \left( m_i^2 + p_z^2 \right)^{-3/2-\delta} \right. \\
+ 2 \sum_{n=1}^{\infty} \left( m_i^2 + 2|e|Hn + p_z^2 \right)^{-3/2-\delta} \left. \right\}
\]
(14)
and perform the finite momentum integrals for \( \delta \to 0 \). As a result,
\[
\Phi_i(H) = -\frac{|e|H}{2(2\pi)^2} \left\{ \frac{1}{m_i^2} + 2 \sum_{n=1}^{\infty} \frac{1}{m_i^2 + 2|e|Hn} \right. \}
\]
(15)
The remaining sum still carries a divergence and can be evaluated after making use of the gamma function integral representation
\[
\frac{1}{A^{1+\delta}} = \frac{1}{\Gamma(1+\delta)} \int_{0^+}^{\infty} d\eta \eta^\delta e^{-A\eta},
\]
valid for \( \delta > -1 \). It follows that
\[
\Phi_i(H) = -\frac{|e|H}{8\pi^2} \frac{1}{\Gamma(1+\delta)} \int_{0^+}^{\infty} d\eta \eta^\delta e^{-m_i^2\eta} \left[ 2 \sum_{n=0}^{\infty} e^{-2|e|Hn\eta} - 1 \right]
\]
\[
= -\frac{|e|H}{8\pi^2} \frac{1}{\Gamma(1+\delta)} \int_{0^+}^{\infty} d\eta \eta^\delta e^{-m_i^2\eta} \coth(|e|H\eta). \quad (17)
\]
Two successive integrations with respect to \( m_i^2 \) give, for \( \delta \to 0 \),
\[
-\varepsilon_{0i} = -\frac{|e|H}{8\pi^2} \int_{0^+}^{\infty} \frac{d\eta}{\eta^2} e^{-m_i^2\eta} \coth(|e|H\eta) + C^{(0)}(H) + C^{(2)}(H)m_i^2, \quad (18)
\]
where \( C^{(0)} \) and \( C^{(2)} \) are eventually infinite constants which do not depend on \( m_i^2 \) and, therefore, can be absorbed into the coefficients \( c_i \)'s and eliminated in virtue of conditions (11).
Hence, from equation (6), the regularized correction to the electromagnetic Lagrangian density reads
\[
\delta \mathcal{L}^R = -\frac{1}{8\pi^2} \sum_i c_i \int_{0^+}^{\infty} d\eta \frac{e^{-m_i^2\eta}}{\eta^3} \left[ |e|H\eta \coth(|e|H\eta) - 1 \right]. \quad (19)
\]

For $H \ll m_i^2$, individual contributions to the above sum have the form $m_i^4 F \left( \frac{H^2}{m_i^4} \right)$, where $F$ is an adimensional function, whose power series expansion does not contain a divergent $H$ independent term. The remaining divergent terms, proportional to $H^2$, are eliminated from the sum by the first condition in (11). For this purpose, it’s sufficient to introduce two auxiliary masses $M_1$ and $M_2$ and $c_1 = c_2 = -1/2$, so that conditions (11) are always satisfied, even for large regulator masses. This can be achieved by choosing $M_1$ real and $M_2$ imaginary. The regulator contributions

$$c_1 \delta L_1 + c_2 \delta L = \frac{e^2 H^2}{3(8\pi^2)} \int_{0^+}^{\infty} d\eta \frac{e^{-\eta}}{\eta}$$

(20)

corresponds to the well-known charge renormalization counterterm.

Finally, letting $|M_1|, |M_2| \to \infty$ gives the expected finite result

$$\delta L = \frac{m^4}{8\pi^2} \int_{0^+}^{\infty} d\eta \frac{e^{-\eta}}{\eta^3} \left\{ -\eta b \coth(\eta b) + 1 - \frac{1}{3} b^2 \eta^2 \right\} ,$$

(21)

where $b = |e| H / m^2$. It can be shown$^{[13]}$ by dimensional analysis and from the invariants $a^2 - b^2$ and $a \cdot b$ of the electromagnetic theory that, in the presence of constant and uniform magnetic and electric fields, parallel to the $z$-direction,

$$\delta L = \frac{m^4}{8\pi^2} \int_{0^+}^{\infty} d\eta \frac{e^{-\eta}}{\eta^3} \left\{ -b \eta \coth(\eta b) a \eta \cotg(\eta a) + 1 - \frac{1}{3} (b^2 - a^2) \eta^2 \right\} ,$$

(22)

where $b = |e| H / m^2$ and $a = |e| E / m^2$, a result derived by Schwinger$^{[3]}$ using the Fock proper-time method$^{[14]}$.

Note that both integrals in expression (8) have the same degree of singularity and were regularized as a whole divergent object. However, if they were treated independently, the additional constraint

$$\sum_{i=0}^{N} c_i m_i^2 \log(m_i^2) = 0$$

(23)

would have to be imposed on each regularized integral in order to keep them finite, removing terms linear in $H$, which are not symmetric under reflections.
$H \rightarrow -H$. In fact, two successive integrations of

$$
\Phi^{(0)}_R \equiv \sum_i c_i \frac{1}{(m_i^2)^{1+\delta}} \\
= \frac{1}{\Gamma(1+\delta)} \sum_i c_i \int_{0+}^{\infty} d\eta \eta^\delta e^{-m_i^2 \eta}
$$

with respect to $m^2$ give

$$
\varepsilon^{(0)}_R = \frac{1}{\Gamma(1+\delta)} \sum_i c_i \int_{0+}^{\infty} d\eta \eta^\delta e^{-m_i^2 \eta}
= \frac{\Gamma(\delta-1)}{\Gamma(\delta+1)} \sum_i c_i (m_i^2)^{1-\delta} = \sum_i c_i m_i^2 \log(m_i^2) + \mathcal{O}(\delta),
$$

where conditions (11) have been used. In this way, the final result for $\delta L$ would be the same only if finite counterterms were added in order to preserve parity invariance. A quite analogous situation is found in scalar QED, and will be examined later.

### 3 QED Effective Lagrangian in (2+1) Dimensions

In (2+1) dimensions the electron energy levels in the presence of a constant and uniform magnetic field, derived from the 3-potential $A_\mu = (0, 0, xH)$, are given by

$$
|\epsilon_n\sigma| = \sqrt{m^2 + (2n + 1 - \sigma)|e|H}.
$$

(24)

This result is obtained by solving Dirac equation in the minimal representation of Clifford algebra, choosing $\gamma_0 = \sigma_z$, $\gamma_1 = i\sigma_x$ and $\gamma_2 = i\sigma_y$. In QED3, the coupling constant $e$ and the components $A_\mu$ have the same dimensionality $[e]=[A]=[m]^{1/2}$, so that $[H]=[m]^{3/2}$. This is of fundamental importance for the analysis of the functional dependence of $\delta L$ on $H$ and the fermion mass.

The vacuum zero-point energy follows from (2) and (24),

$$
-\varepsilon_0 = \frac{|e|H}{2\pi} \left[ |m| + 2 \sum_{n=1}^{\infty} \sqrt{m^2 + 2|e|Hn} \right] = \varepsilon_0^{(0)} + \varepsilon_0^{(1)},
$$

(25)
where $|e|H/(2\pi)$ amounts for the density of states in the (2+1)-dimensional phase space. In contrast with the four-dimensional case, the odd-parity term $\varepsilon_0^{(0)}$ is finite and does not need to be regularized. The divergent contribution must be replaced by its regularized counterpart

$$\varepsilon_0^{(1)\, R} = \sum_i c_i \varepsilon_0^{(1)}_i,$$

(26)

where each term, when differentiated with respect to $m_i^2$, yields a function

$$\Phi_i^{(1)}(H) = \frac{|e|H}{(2\pi)} \sum_{n=1}^{\infty} (m_i^2 + 2|e|Hn)^{-1/2-\delta}$$

$$= \frac{|e|H}{2\pi \Gamma(1/2)} \int_0^\infty d\eta \eta^{-1/2+\delta} e^{-m_i^2 \eta} \sum_{n=1}^{\infty} e^{-2|e|Hn\eta}$$

$$= \frac{|e|H}{4\pi \sqrt{\pi}} \int_0^\infty d\eta \eta^{-1/2+\delta} e^{-(m_i^2 + |e|H)\eta} \sinh(|e|H\eta),$$

(27)

which admits the integral representation (16). After integration of (27) with respect to $m_i^2$, the regularized zero-point energy reads

$$-\varepsilon_0^R = \frac{|e|H|m|}{2\pi} - \frac{|e|H}{4\pi \sqrt{\pi}} \sum_i c_i \int_0^\infty d\eta \eta^{-3/2} e^{-(m_i^2 + |e|H)\eta} \frac{e^{-|e|H|H\eta|}}{\sinh(|e|H\eta)},$$

(28)

where the integration constant has been suppressed by the first condition in (11), assumed to be valid also in the present case, so that any contribution to the regularized expression $\delta \mathcal{L}_R^R$ has the form $m_i^2 F(|e|H/m_i^2)$, where $F$ is an adimensional function whose $H$ series expansion does not allow even powers of the field masses to appear in the corrected electromagnetic Lagrangian density. On the other hand, this functional dependence of $F$ on the magnetic field $H$ indicates that the effective Lagrangian density for the (2+1)-dimensional electromagnetic field might not be invariant under parity transformations. This occurs in QED$_3$, as a consequence of a non-invariant fermion mass term in Dirac Lagrangian density, which is responsible for the dynamically generated odd-parity term in the renormalized electromagnetic sector$^{[15]}$.

Thus, from (6),

$$\delta \mathcal{L}^R = \frac{|e|H|m|}{2\pi} - \frac{1}{4\pi \sqrt{\pi}} \sum_i c_i \int_0^\infty d\eta \frac{\eta^{5/2} e^{-m_i^2 \eta}}{\sinh(|e|H\eta)} \left[ \frac{|e|H \eta e^{-|e|H\eta}}{\sinh(|e|H\eta)} - 1 \right].$$

(29)
For a weak applied magnetic field, the terms proportional to \( H \) in the power series expansion of the function in square brackets give rise to contributions

\[
\frac{|e|H|m_i|}{4\pi\sqrt{\pi}} \int_{0+}^{\infty} \frac{d\eta}{\eta^{3/2}} e^{-\eta},
\]

linear in the field masses, which can be eliminated by the subsidiary condition

\[
\sum_{i=0}^{N} c_i|m_i| = 0.
\]

This can be accomplished by introducing again two regulator masses \( M_1 \) and \( M_2 \), exactly as in the four-dimensional case, with \( c_1 = c_2 = -1/2 \). The remaining odd-parity contributions are finite.

Hence, for \( |M_1|, |M_2| \rightarrow \infty \),

\[
\delta \mathcal{L} = \frac{|e|H|m|}{2\pi} - \frac{1}{4\pi\sqrt{\pi}} \int_{0+}^{\infty} \frac{d\eta}{\eta^{3/2}} e^{-m^2\eta} \left[ \frac{|e|H\eta(e^{-|e|H\eta} + |e|H\eta)}{\sinh(|e|H\eta)} - 1 \right],
\]

where a counterterm had been added in virtue of (30). The term proportional to \( H^2 \) in the series expansion corresponds to a charge renormalization finite counterterm

\[
\delta \mathcal{L}_0 = -\frac{e^2}{12\pi|m|} H^2.
\]

4 Scalar QED

In s-QED, the energy levels of a charged boson in a constant and uniform magnetic field does not exhibit the spin degeneracy for \( n \neq 0 \). So, in four space-time dimensions, the negative vacuum zero-point energy turns out to be

\[
-\varepsilon_0 = \frac{|e|H}{(2\pi)^2} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \varepsilon_n(p_z)
= \frac{|e|H}{(2\pi)^2} \int_{-\infty}^{\infty} dp_z \sum_{n=0}^{\infty} \sqrt{m^2 + 2|e|Hn + p_z^2}.
\]

The regularized function to be evaluated is

\[
\Phi^R(H) = -\frac{\partial^2 \varepsilon_0^R}{\partial (m^2)^2} = \sum_i c_i \Phi_i(H),
\]
where

\[ \Phi_i(H) = -\frac{|e|H}{2(2\pi)^2} \int_0^\infty dp_\perp \sum_{n=0}^\infty \left( m_i^2 + 2|e|Hn + p_\perp^2 \right)^{-3/2 - \delta} \]

As in section 2, the above momentum integrals were performed under conditions (11) for the regularized zero-point energy \( \varepsilon_0^R \).

The functions \( \Phi_i(H) \) can be integrated with the help of (16). In this case, the resulting correction for the electromagnetic Lagrangian density is given by

\[ \delta L = -\frac{1}{8\pi^2} \int_{0+}^\infty \frac{d\eta}{\eta^2} e^{-m^2 \eta} \left\{ \frac{|e|H\eta}{\sinh(|e|H\eta)} \left[ e^{-|e|H\eta} + |e|H\eta + \ldots \right] - 1 \right\} \]

where both conditions (11) and (23) were applied to the regularized correction and finite counterterms had also to be added, so that parity invariance is retained. As expected, one ends up with (21), after charge renormalization.

5 Finite Temperature QED

Finite temperature effects may also be included in the present discussion of effective field theories. For this purpose it is convenient to consider, for example, the expectation value of the spinor QED Hamiltonian in a thermal vacuum, as the one defined in Thermo Field Dynamics[18]. In this case, the correction to Maxwell’s Lagrangian density reads

\[ \delta \mathcal{L} = -[\varepsilon_0^\beta - (\varepsilon_0^\beta)_{H=0}] , \]

where

\[ \varepsilon_0^\beta = \frac{|e|H}{(2\pi)^2} \sum_{n\sigma} \int_{-\infty}^\infty dp_\perp \epsilon_{n\sigma}(p) \left\{ \frac{1}{1 + e^{\beta\epsilon_{n\sigma}}} - \frac{1}{1 + e^{-\beta\epsilon_{n\sigma}}} \right\} . \]
In the low temperature limit, $\beta \to \infty$, the integrand in the above momentum integral may be expanded as

$$\varepsilon^\beta_0 = \varepsilon_0 - \frac{2|e|H}{(2\pi)^2} \sum_{n\sigma} \int_{-\infty}^{\infty} dp \varepsilon_{n\sigma}(p)e^{-2\beta_{n\sigma}}.$$  \hspace{1cm} (38)

The last expression can be evaluated in the case of a weak magnetic field $H$, by expanding the energy $\varepsilon_{n}(p)$ for $n \neq 0$,

$$\varepsilon_{n}(p) \sim \sqrt{p^2 + m^2} + \frac{|e|H}{\sqrt{p^2 + m^2}} n,$$

so that (38) becomes

$$\varepsilon^\beta_0 = \varepsilon_0 + \frac{2|e|H}{(2\pi)^2} \frac{\partial}{\partial \beta} \int_0^{\infty} dp \left\{ e^{-2\beta\sqrt{p^2 + m^2}} + 2 e^{-2\beta\sqrt{p^2 + m^2}} \right\}. \hspace{1cm} (39)$$

Performing the sum in the last term of the above equation gives, in the limit $\beta \to \infty$,

$$\varepsilon^\beta_0 = \varepsilon_0 + \frac{2|e|H}{(2\pi)^2} \frac{\partial}{\partial \beta} \int_0^{\infty} dp \left\{ e^{-2\beta\sqrt{p^2 + m^2}} + 2 e^{-2\beta\sqrt{p^2 + m^2}} \right\}. \hspace{1cm} (40)$$

Now, for a weak magnetic field, i.e., taking

$$e^{-\frac{2\beta|e|H}{\sqrt{p^2 + m^2}}} \approx 1 - \frac{2\beta|e|H}{\sqrt{p^2 + m^2}},$$

yields

$$\varepsilon^\beta_0 = \varepsilon_0 - \frac{3|e|H}{(2\pi)^2} \frac{\partial^2}{\partial \beta^2} K_0(2\beta m) - \frac{8\beta|e|^2H^2}{(2\pi)^2} \frac{\partial}{\partial \beta} K_0(2\beta m), \hspace{1cm} (41)$$

where

$$K_0 = \int_0^{\infty} dp \frac{e^{-2\beta\sqrt{p^2 + m^2}}}{\sqrt{p^2 + m^2}} \hspace{1cm} (42)$$

is the zero order modified Bessel function. Finally, substituting (41) in (36),

$$\delta L = \delta L_{\text{ren}} + \frac{12|e|Hm^2}{(2\pi)^2} \frac{\partial^2}{\partial (2\beta m)^2} K_0(2\beta m) + \frac{16\beta m|e|^2H^2}{(2\pi)^2} \frac{\partial}{\partial (2\beta m)} K_0(2\beta m). \hspace{1cm} (43)$$

The same procedure applies to the scalar case, giving

$$\delta L = \delta L_{\text{ren}} + \frac{16|e|Hm^2}{(2\pi)^2} \frac{\partial^2}{\partial (2\beta m)^2} K_0(2\beta m) + \frac{16\beta m|e|^2H^2}{(2\pi)^2} \frac{\partial}{\partial (2\beta m)} K_0(2\beta m). \hspace{1cm} (44)$$
6 Concluding Remarks

In the previous sections, radiative corrections to the electromagnetic Lagrangian densities in (3+1) and (2+1) dimensions were derived considering vacuum polarization effects induced by a constant and uniform electromagnetic background, which modify the zero-point energy of the quantized field of a relativistic charged particle. Divergent quantities were rendered finite from the very beginning and acquired an unambiguous mathematical meaning along each step of calculations. For this purpose, a consistent subtraction scheme had to be developed, introducing auxiliary regulator masses in the sense of Pauli-Villars-Rayski regularization.

As in QED, divergent physical quantities, such as the vacuum zero-point energy, must be regularized according to its singular degree. In four dimensions, the regulator masses satisfy relations (11), the same conditions that eliminate the quadratic and logarithmic divergences in the second-order vacuum polarization amplitude of QED, which has also to be regularized as a whole object, in order to preserve gauge invariance\cite{4}. The finite fourth-order Feynman amplitude for Delbrück scattering corresponds, in the low-energy limit, to the effective theory of Euler-Kockel-Heisenberg\cite{13}, rederived in section 2.

In the three-dimensional case, the requirements fulfilled by the auxiliary masses, namely

\[ \sum_i c_i = 0, \sum_i c_i |m_i| = 0, \]

agree with the corresponding constraints on the Pauli-Villars-Rayski regulators in the second-order vacuum polarization amplitude of QED$_3$, to account for the linear and logarithmic divergences\cite{16}. In section 3, only the divergent sector of the vacuum zero-point energy had to be regularized. The finite part lead to an odd-parity contribution to the effective Lagrangian density,

\[ \mathcal{L}_{\text{odd}} = \frac{|e| m H}{2\pi}, \]

which corresponds to a topological Chern-Simons term, radiatively induced in QED$_3$\cite{15}. If both the finite and divergent sectors had been treated in the same foot, condition (30) would have canceled the above mentioned contribution $\mathcal{L}_{\text{odd}}$. As a result, the correction to the electromagnetic Lagrangian density
would be written as

\[
\delta \mathcal{L} = -\frac{1}{4\pi\sqrt{\pi}} \int_{0}^{\infty} \frac{d\eta}{\eta^{5/2}} e^{-m^2\eta} \left[ |e| H\eta \coth(|e| H\eta) - 1 \right].
\] (45)

This is the same expression derived by Redlich\cite{17}, applying Schwinger's formalism\cite{3}, without a proper analysis of regularization constraints.

Weisskopf's zero-point energy method was also applied to s-QED. It was shown that, in virtue of the absence of spin degeneracy in the energy of charged bosons, the additional condition (23), together with the requirement of parity invariance, had to be imposed on the regularized electromagnetic correction, eliminating odd-parity contributions to the resulting effective Lagrangian density, which coincides with expression (21), derived from the fermion vacuum energy.

In the last section, it was discussed the natural extension of Weisskopf's formalism to finite temperature quantum electrodynamics, by considering the vacuum energy of Thermo Field Dynamics. At low temperatures, the Euler-Kockel-Heisenberg effective Lagrangian density factorizes from the corresponding momentum distributions, giving rise to ultraviolet finite temperature corrections. In both spinor and scalar cases, there appear odd-parity contributions which, together with higher order terms, exponentially vanish as the temperature approaches zero.

Finally, it's noteworthy to point out that Weisskopf's approach to effective Lagrangians might also shed light into recent discussions on radiative corrections in quantum field theories, such as the controversy on dynamical CPT violation and Lorentz symmetry breaking in the electroweak sector of a renormalizable extension of the Standard Model\cite{19}. This will be the subject of a forthcoming paper.

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