Pareto Driven Surrogate (ParDen-Sur) Assisted Optimisation of Multi-period Portfolio Backtest Simulations

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Abstract

Portfolio management is a multi-period multi-objective optimisation problem subject to a wide range of constraints. However, in practice, portfolio management is treated as a single-period problem partly due to the computationally burdensome hyper-parameter search procedure needed to construct a multi-period Pareto frontier. This study presents the Pareto Driven Surrogate (ParDen-Sur) modelling framework to efficiently perform the required hyper-parameter search. ParDen-Sur extends previous surrogate frameworks by including a reservoir sampling-based look-ahead mechanism for offspring generation in Evolutionary Algorithms (EAs) alongside the traditional acceptance sampling scheme. We evaluate this framework against, and in conjunction with, several seminal Multi-Objective (MO) EAs on two datasets for both the single- and multi-period use cases. Our results show that ParDen-Sur can speed up the exploration for optimal hyper-parameters by almost $2\times$ with a statistically significant improvement of the Pareto frontiers, across multiple EAs, for both datasets and use cases.
1 Introduction

Markowitz [1] formulated the portfolio selection problem as a Multi-Objective optimisation Problem (MOP), where an investor aims to satisfy the conflicting objectives of maximising return whilst minimising risk. Since this seminal work, the portfolio optimisation problem has been extended to include a range of additional constraints and objectives to reflect real-world considerations [2]. The competing objectives are typically combined linearly into a single function using trade-off parameters that reflect the relative importance of each objective (i.e. they “trade-off” different objectives). By varying the risk-aversion parameter and solving the optimisation problem, one can uncover the set of non-dominated solutions known as the Pareto frontier or, in the finance jargon, the efficient frontier\(^1\) [3, 4]. However, not all choices of trade-off parameters yield non-dominated solutions in the risk-return space, resulting in unnecessary computation. Thus, the key issue is determining the efficient frontier with as few evaluations as possible.

Multi-Objective (MO) Evolutionary Algorithms (EAs) have been extensively applied in portfolio optimisation [5]. However, portfolio optimisation is treated as a deterministic problem in these studies. In reality, the risk and return terms of the objective function are estimated from data and are hence stochastic. In the finance literature, the objective value of the optimised portfolio is considered an “in-sample”, and hence a highly inflated estimate of true performance [6]. Consequently, it is common practice that portfolio performance is estimated on separate test data using a process known as “backtesting”. This process is computationally burdensome since it involves repeatedly solving an optimisation problem at each time step in the backtest period for each choice of trade-off parameters [7].

In practice, portfolio management involves making a sequence of portfolio decisions over time and is thus more accurately represented as a multi-period optimisation problem [8]. Taking a multi-period perspective of the portfolio optimisation problem confers several advantages over the single-period counterpart. For example, Multi-period Portfolio Optimisation (MPO) allows for management of trading costs [9], time-varying forecasts, intertemporal hedging, and other intertemporal constraints [10]. However, realising these advantages requires that some challenges are overcome. Firstly, MPO requires multi-period forecasts adding difficulty to the already challenging task of single-period forecasting. Secondly, MPO is computationally burdensome,

\(^1\)We use the terms Pareto frontier and efficient frontier interchangeably throughout this work
especially when incorporating real-world considerations. Consequently, portfolio management in practice typically consists of repeatedly performing myopic single-period optimisations.

Traditionally portfolio optimisation is extended to a multi-period setting using stochastic optimal control as a framework and solved using Dynamic Programming (DP). However, formulations relying on DP are typically unable to adequately capture real-world market dynamics and investment considerations primarily due to the “curse of dimensionality”. Instead, approximate policies can be determined by considering simpler (although sub-optimal) Stochastic Programming (SP) formulations that do not depend on Bellman Optimality. For example, the single-period mean-variance problem can be extended to the multi-period case by including additional risk and return objectives and additional constraints for each period [2]. However, this extension negatively impacts the number of decision variables, and hence the computational complexity can grow exponentially with the investment horizon.

This work addresses the computational difficulties associated with back-testing when determining the efficient frontier during a realistic MPO problem. Following Boyd et al. [2], this study adopts the stochastic perspective of the portfolio optimisation problem and treats trade-off parameters as hyper-parameters to be optimised using backtest performance estimates. Nystrup et al. [11] shows that using the MO-CMA-ES EA for determining the efficient frontier improves upon the grid search procedure used by Boyd et al. [2]. Most recently, van Zyl et al. [7] compared several seminal MO EAs and proposed a procedure that includes a surrogate model, which further outperforms existing procedures.

This work extends that of van Zyl et al. [7] by investigating MPO. Most existing work in MPO tackles the problem of determining an approximately optimal policy, treating MPO as a single objective problem. Consideration of the efficient frontier is limited to the case when the problem is specified as a MO problem and an analytical solution is available (for example, [12, 13]). In contrast, this work considers the problem of trying to efficiently determine the Pareto frontier for a more realistic portfolio optimisation problem containing several constraints. The main aim of which is to improve not only the efficiency of the algorithms but also the overall quality and performance of the solutions. To this end, the contributions of this study are:

- an empirical performance comparison of several seminal MOP EA for both Single-period Portfolio Optimisation (SPO) and MPO,
- the demonstration of the superior efficacy of surrogate assisted optimisation to alleviate the computational burden of constructing the Pareto frontier,
- and the presentation of a new algorithm, Pareto Driven Surrogate (ParDen-Sur), which improves upon state-of-the-art in surrogate assisted portfolio optimisation by introducing a reservoir sampling based look-ahead mechanism.
2 Background

2.1 Mean-Variance Portfolio Optimisation

The original mean-variance optimisation problem is specified by a bi-objective criterion, where risk is measured by the variance of the portfolio’s return \([1]\). These objectives are combined linearly into a single optimisation problem:

\[
\text{maximise} \quad w'\mu - \frac{\gamma}{2}w'\Sigma w \quad (1)
\]

subject to \(w'1 = 1\) \( (2) \)

where \(w\) is the vector of portfolio weights, \(\mu\) is the vector of expected returns, \(\Sigma\) is the covariance matrix of returns, and \(\gamma\) is known as the risk-aversion parameter, which is used to trade-off the conflicting objectives. The resulting problem is a quadratic program that has an analytical solution.

A particular choice of the risk-aversion parameter yields a particular optimal solution. The efficient frontier can be determined by varying the trade-off parameters and solving the optimisation problem. It is often of interest to determine the portfolio from the efficient frontier that maximises the Sharpe Ratio, defined as the average return (in excess of the return on a risk-free asset) of the portfolio over the standard deviation of return. In this case, the optimisation procedure comprises two steps: first, determine the efficient frontier, and second, select the Sharpe-ratio maximising portfolio.

2.2 Approaches to Multi-period Portfolio Optimisation (MPO)

Starting with Merton \([14, 15]\), and others \([16–18]\), MPO has been studied as a Markov Decision Problem (MDP) where optimal policies are found using DP. Here, the portfolio optimisation problem is typically restricted to idealised settings to enable the construction and solution of the Bellman optimality equation. In particular, it is typical for the state-space to be limited to very few variables to avoid the curse of dimensionality and for the state transition function to be treated as known. In addition, the objective function is often chosen from a class of utility functions with convenient analytic properties, but which may render the problem as a single-objective optimisation problem where the risk-return trade-off is not made explicit. Lastly, problems are restricted to one or two constraints of theoretical interest. Thus, studies investigating MPO through DP are largely theoretical in nature, focusing on qualitative understanding of normative investor behaviour.

In contrast, SP is a highly flexible modelling framework for multi-period optimisation, which can incorporate a diverse array of real-world considerations, including alternative risk measures, risks and uncertainty hedging, hard and soft constraints, and intermediate goals (including stochastic and irregular outflows) \([2]\). Furthermore, SPs model uncertainty using scenarios,
defined as sequences of states, and hence are not restricted to MDP formulations. However, a SP becomes computationally intractable as its specification becomes extensive, and instead, an approximate program is used to find a policy. Common approximations include discretisation of the observed variables, aggregation of stages, reduction of the problem horizon, and replacing random quantities with their expectations [19]. When the data’s expectations are unknown, they are replaced with forecasts that are sequentially updated at each timestep, more broadly referred to as a rolling horizon procedure or Model Predictive Control (MPC) and widely adopted in practice.

2.3 A Convex Optimisation Framework for Portfolio Optimisation

Using the SP approach, Boyd et al. [2] presents a convex optimisation framework for MPO, which can be used for MPC and includes the single-period problem as a special case. Notably, trading and holding costs are treated as soft constraints selected to achieve some desired trading behaviour and are typically used as regularisation methods. The portfolio optimisation problem can be equivalently formulated by including the soft constraints as additional criteria in the objective function, with associated trade-off parameters:

\[
\begin{align*}
\text{maximise} & \quad \sum_{\tau=t+1}^{t+H} w_\tau' \mu_\tau - \frac{\gamma}{2} w_\tau' \Sigma_\tau w_\tau - \gamma_{\text{trade}} \phi_{\text{trade}}(w_\tau - w_{\tau-1}) \\
& \quad - \gamma_{\text{hold}} \phi_{\text{hold}}(w_\tau) \\
\text{subject to} & \quad w_\tau' 1 = 1, w_\tau - w_{\tau-1} \in Z_\tau, w_\tau \in W_\tau
\end{align*}
\]

where \( H \) is a natural number representing the investment horizon, \( \mu_\tau \) and \( \Sigma_\tau \) are the return and risk forecasts for each period \( \tau = t, \ldots, t+H \), \( W_\tau \) and \( Z_\tau \) are convex sets representing the holding and trading constraints respectively, and \( \gamma \), \( \gamma_{\text{trade}} \), and \( \gamma_{\text{hold}} \) are the trade-off parameters for risk, trading costs, and holding costs respectively. Note that setting \( H = 1 \) specifies the single-period problem.

The extended formulation of the mean-variance problem in Equation 3 is analytically intractable. Computational methods determine the efficient frontier by repeatedly solving the optimisation problem for different trade-off parameters, but, unlike Markowitz’s original formulation, not all selected trade-off parameters are associated with Pareto optimal solutions, resulting in unnecessary computation. This procedure of repeatedly optimising for different trade-off parameters becomes increasingly costly as the number of decision variables grows, thus limiting the investment horizon.
2.4 Evolutionary Algorithms (EAs)

There exists numerous evolutionary MOP algorithms in the literature [20, 21]. Amongst these, certain algorithms have been shown to be successful in applications across several different domains [22–26]. The use of EAs for finding solutions to portfolio optimisation problems has been covered extensively in the portfolio optimisation literature [27–29] with focus almost exclusively given to the single-period problem. The presented study aims to apply various representative EAs and demonstrate the utility of a surrogate-assisted evolutionary optimisation approach to solving portfolio optimisation problems described in this paper. To this end, this study implements the evolutionary MOP methods described below.

2.4.1 MO-CMA-ES

The Covariance Matrix Adaptation Evolution Strategy (CMA-ES) is an elitist evolutionary strategy for numerical optimisation. Multi-objective CMAES (MO-CMA-ES), maintains a population of individuals that adapt their search strategy as in the elitist CMA-ES. The elites are subjected to MO selection pressure. The selection pressure originates from non-dominated sorting using the crowding-distance first and then hypervolume as the second criterion [24].

2.4.2 NSGA-II

The NSGA-II algorithm follows the form of a Genetic Algorithm (GA) with modified crossover and survival. The surviving individuals are chosen Pareto front-wise first. There arises a situation where a front needs to be split since not all individuals can survive. When splitting the front, solutions are selected based on a Manhattan crowding distance in the objective space. However, the method seeks to hold onto the extreme points from each generation, and as a result, they are assigned a crowding distance of infinity. Furthermore, the method uses a binary tournament mating selection to increase selection pressure. Each individual is first compared by rank and then crowding distance [30].

2.4.3 R-NSGA-II

The R-NSGA-II algorithm follows NSGA-II with a modified survival selection mechanism. Unlike NSGA-II, solutions are selected based on rank in splitting the front. This rank is calculated based on the Euclidean distance to a set of supplied reference points. The solution closest to a reference point is assigned a rank of one. The algorithm then selects the points with the highest rank to each reference point as offspring. After each reference point has selected the solution with the best rank for survival, all solutions within the epsilon distance of surviving solutions are “cleared”, meaning they can not be selected for survival until all fronts have been processed. The cleared points are considered if more solutions still need to be selected. The free parameter epsilon has the effect that a smaller value results in a tighter set of solutions [30].
2.4.4 NSGA-III

The NSGA-III algorithm is based on reference directions instead of reference points. For the survival mechanism, after non-dominated sorting, the algorithm has a modified approach to dealing with the front splitting. The algorithm fills up the underrepresented reference direction first. If the reference direction does not have any solution assigned, then the solution with the smallest perpendicular distance in the normalised objective space is assigned as the survivor. In the case that additional solutions for a reference direction are required, these are assigned randomly. Consequently, when this algorithm converges, each reference direction attempts to find a suitable non-dominated solution [30].

2.4.5 R-NSGA-III

The R-NSGA-III algorithm follows the general NSGA-III procedure with a modified survival selection operator. First, non-dominated sorting is done as in NSGA-III. Solutions are associated with aspiration points based on perpendicular distance, and then solutions from the underrepresented reference direction are chosen. For this reason, when this algorithm converges, each reference line seeks to find a good representative, non-dominated solution [30].

2.4.6 U-NSGA-III

It has previously been shown that tournament selection performs better than random selection. Unlike NSGA-III, which selects parents randomly for mating, the U-NSGA-III algorithm uses tournament pressure as an improvement [30].

2.5 Backtesting

Studies applying EAs have largely treated the portfolio optimisation problem as a deterministic problem where the objective function is known. In reality, the objective function is unknown and is estimated from historical data. This reality has several important consequences for evaluating a portfolio, frontier, and solution method performance.

Firstly, the objective function value of the optimised portfolio is considered an “in-sample” and hence a highly inflated estimate of true performance. Consequently, it is common practice that portfolios are evaluated “out-of-sample”, that is, using return data after the period of optimisation. When evaluating an entire policy, the portfolio at the beginning of each time period is evaluated using the returns of that period, commonly referred to as “backtesting”. For policies constructed using a rolling window approach, a backtest necessarily involves an optimisation at each time period for each choice of trade-off parameters.

Secondly, there is no guarantee that in-sample Pareto optimal solutions are optimal out-of-sample, motivating the use of backtest estimates to determine the Pareto frontier. In particular, motivated by standard practice in machine
learning, Boyd et al. [2] considered the set of trade-off parameters as the hyper-parameters and the portfolio weights as the parameters in a hybrid approach. They construct the Pareto frontier through a grid-search over the hyper-parameters, which entails carrying out a backtest for every hyper-parameter in the set considered.

Thirdly, since backtest estimates are sample-specific, statistical comparison between solution methods needs to account for an additional source of randomness arising from sampling variation of the backtest data. This is in contrast to the typical use of statistical methods to compare algorithm performance, which only accounts for the variation due to the stochastic elements of the algorithm. However, proposing a statistical test that includes this additional source of variation is outside the scope of this study.

### 2.6 Surrogate-Assisted Optimisation

Surrogate models are used in optimisation problems where the objective function is unknown and costly to evaluate. In the surrogate-assisted approach, objective values (outputs) are first evaluated for a sample of different decision variable choices (inputs). Then, a supervised learning algorithm (the surrogate) is trained on the input-output pairs to estimate the objective function. The surrogate model can then be used to assist in numerically optimising the objective function. The surrogate model can also be sequentially improved by including input-output pairs for candidate solutions at each iteration as the optimisation algorithm proceeds [31]. This study’s unknown objective function is the out-of-sample performance over the hyper-parameter space of both SPO and MPO.
2.7 Motivation and Problem Statement

Real-world portfolio management can be described as a multi-period, MOP subject to a potentially wide range of constraints. Here, the goal is to find a multi-period Pareto frontier comprised of policies with an optimal risk-return trade-off over the entire investment horizon.

However, due to analytical and computational intractability, approximate single-period frontiers are solved at each time step. The multi-period problem is reduced to a single-period problem for two reasons: multi-period forecasting is difficult, and the computational burden of multi-period optimisation is proportional to the number of periods considered.

In this study, the results deal solely with the latter problem, addressing the computational difficulty of finding a multi-period efficient frontier. Most previous studies applying EAs do not recognise, firstly, that the Pareto frontier needs to be estimated on test data, and secondly, that most real-world considerations can be incorporated in a convex specification of the portfolio optimisation problem.

Recognising this, Boyd et al. [2] and Nystrup et al. [11] adopt a hybrid approach, where the construction of the Pareto frontier is split into a hyper-parameter selection problem and a convex portfolio optimisation problem. However, a poor strategy for selecting candidate hyper-parameters can yield many dominated solutions, resulting in unnecessary computation.

In response, this study adopts the hybrid methodology of Boyd et al. [2] and proposes including a surrogate model to conduct the hyper-parameter search more effectively when constructing the Pareto frontier.
3 Pareto Driven Surrogate (ParDen-Sur)

Surrogate methods have been extensively applied in single objective optimisation [32–39], whereas the applications of surrogate assisted methods within a MO setting are substantially less common. In particular, the literature for data-driven evolutionary MO surrogate assisted optimisation is especially sparse [40–43]. To this end, the presented ParDen-Sur framework is sufficiently general that it can be applied together with most evolutionary MO algorithms. ParDen-Sur also moves beyond generative models as surrogates, allowing for the integration of discriminative models [7].

ParDen-Sur relies on the use of supervised learning models to act as a surrogate drop-in replacement \( \hat{f}(\cdot) \) to evaluate a computationally resource intensive real-world experiment or simulation \( f(\cdot) \). The core idea of ParDen-Sur is to limit the evaluation of candidates using the simulation \( f(\cdot) \) to only those that have a high probability of advancing the Pareto front (the non-dominated solution set \( \mathcal{P} \)). The likelihood that a candidate would advance the Pareto front is determined by: i) whether the surrogate model predicts that the candidate advances the Pareto front and ii) the extent to which the surrogate is correct. In particular, ParDen-Sur attempts to estimate the probability that the surrogate is correct, which is used as a threshold within an acceptance sampling scheme, and to set the size of a reservoir sampling look-ahead scheme when selecting candidates.

In order to determine the extent to which the surrogate is correct, a mechanism is required for assessing the probability \( P[(x_i, y_i) \in \mathcal{P}|\hat{f}] \) that a pretender \((x_i, y_i)\) will be in the current non-dominated set \( \mathcal{P} \) given the surrogate’s prediction \( \hat{y}_i = \hat{f}(\cdot) \). ParDen-Sur employs \( k \)-fold cross-validation \( \text{CV}_k(\cdot) \) together with the non-dominating rank [44] of each candidate to estimate this probability as a non-dominated score (NDScore).

Calculating an NDScore requires fitting surrogates \( \hat{f}_{X_T,Y_T}(\cdot) \) to the training splits \((X_T,Y_T)\) with the MO \( Y_T \) as the target. For each point in a validation split \((X_V,Y_V)\), its non-dominating rank \( r \) in the split is calculated. The surrogates estimate the objective values on the validation split, \( \hat{Y}_V = \hat{f}_{X_T,Y_T}(X_V) \), and their non-dominating ranks are calculated, producing predicted ranks \( \hat{r} \). The performance of the predicted non-dominated ranks is evaluated using any error metric \( \mathcal{E}(\hat{r}, r) \), that is constrained to sum to one. Finally, the non-dominated score can be calculated by aggregating the errors calculated for the different splits: ParDen-Sur allows the use of any metric \( \mathcal{E}(\hat{r}, r) \) constrained to sum to one, to compare \( r \) with \( \hat{r} \):

\[
\text{NDScore}(\cdot) = \text{aggregation over } (X_T,Y_T), (X_V,Y_V) \in \text{CV}_k(X) \mathcal{E}(\hat{r}, r)
\]

with

\[
\hat{r} = [Y_V]^\dagger \quad \text{and} \quad r = [\hat{f}_{X_T,Y_T}(X_V)]^\dagger
\]
where $\lceil \cdot \rceil^\uparrow$ assigns the non-dominating rank to each vector. Since ParDen-Sur uses the $k$-fold cross-validation function $CV_k(\cdot)$ to estimate this value, the mean error across the $k$ splits is a suitable aggregation.

The presented method, ParDen-Sur, recognises several deficiencies in the originally developed ParDen algorithm \cite{7} and contains several extensions to overcome these. First, for the metric $\mathcal{E}(\hat{r}, r)$, ParDen-Sur uses the Kendall-$\tau$ rank instead of accuracy. Second, ParDen-Sur introduces a look-ahead mechanism that uses the surrogate model to iterate the EA forward several generations.

ParDen-Sur is especially cognisant of the limitations of adding additional free parameters to a surrogate assisted algorithmic framework. To limit the number of free parameters and ensure diversity of the population, ParDen-Sur uses reservoir sampling during the look-ahead procedure. Introducing a look-ahead procedure allows one to consider forgoing the acceptance sampling\footnote{Forgoing acceptance sampling without look-ahead is equivalent to the original EA algorithm}. Algorithm 1 presents the full details of the proposed method with Figure 3 giving a high-level overview of the details presented in the algorithm.

4 Methodology

In this study, the experimentation is split into two stages, referred to as the motivating and primary examples, respectively, for both the SPO and MPO cases. The motivating example is a scaled-down version of the problem presented in the primary example and is first explored to determine the most appropriate application of ParDen-Sur within the primary example. In the motivating example, several well-known MOP EAs are initially applied to the aforementioned problem (see Section 2.7) to determine which performs best and to provide a relevant baseline. After which, the study then applies ParDen-Sur with the two best performing EAs in the primary example.

All experiments are repeated ten times and average results are compared to the baseline EAs. The outperformance of the ParDen-Sur algorithm is assessed for statistical significance using one-sided Mann-Whitney U tests and adjusted for multiple testing using the Simes-Hochberg procedure. Hyper-parameters for the EAs are manually adapted from van Zyl et al. \cite{7}.

4.1 Method Metrics

This study considers three metrics as performance indicators for the solution set generated by each algorithm. These include the two modified distance calculations, Generational Distance Plus (GD+) and Inverted Generational Distance Plus (IGD+) \cite{45}, as well as the Hypervolume (HV) \cite{46}. For GD+ and IGD+, an optimal target set needs to be specified, which is specified in the results below as the Pareto frontier from a random grid search with 6510 hyper-parameters. For HV only a reference point needs to be provided which is chosen to be $(40, 0)$ in the risk-return space.
Algorithm 1 ParDen-Sur Algorithm

**Input:** EA (M), Terminating Criterion (TC), Look Ahead Terminating Criterion (TC_L), Simulation (f_e), Surrogate (H), Loss (L), Population Size (m), Use Acceptance Sampling (acceptance), Use Look Ahead (look-ahead)

**Output:** Approximate Optimal Frontier (P)

**Data:** Daily Trading Data (D)

```latex
\begin{algorithm}[H]
\SetAlgoLined
\KwResult{Approximate Optimal Frontier (P)}
\textbf{Function} \texttt{look-ahead}(\hat{f},G,\ldots):
\begin{algorithmic}[1]
\State \texttt{X}_P = \{\}  \hspace{1cm} /* \texttt{X}_P \text{ is an empty Reservoir} */
\State \texttt{r} = m \times \text{NDScore}(\hat{f},G)  \hspace{1cm} /* \texttt{r} \text{ is the Reservoir size as \% of non-dominated score} */
\State \texttt{M}_L = \text{copy}(M)  \hspace{1cm} /* \texttt{M}_L \text{ is a throw away copy of EA} */
\State \texttt{P}_C = \text{copy}(G)  \hspace{1cm} /* \texttt{P}_C \text{ is a throw away copy of Ground-truth} */
\While{\texttt{firstiteration} \text{ or } \neg \text{TC}_L(M_L)}
\State \texttt{X}_L \sim \text{M}_L.\text{infill}(m)  \hspace{1cm} /* \texttt{X}_L \text{ is a generate of } m \text{ look-ahead Candidates} */
\State \texttt{C} = \{(\texttt{X}_L,\hat{f}(\texttt{X}_L))\}  \hspace{1cm} /* \texttt{C} \text{ is an estimate of Candidates' with Surrogate} */
\State \texttt{M}_L.\text{advance}(\texttt{C})  \hspace{1cm} /* \texttt{M}_L \text{ advances EA with Candidates} */
\State \texttt{(X}_C,\hat{Y}_C) = \text{P}_C = \left[\{\texttt{C} \cup \texttt{P}_C\}\right] \cap C  \hspace{1cm} /* \texttt{X}_C,\hat{Y}_C \text{ are non-dominated Candidates} */
\State \texttt{X}_P = \text{reservoir}(\texttt{X}_P, \texttt{X}_C, \texttt{r})  \hspace{1cm} /* \texttt{X}_P \text{ is Reservoir sampling for Pretenders} */
\EndWhile
\State \textbf{return} \texttt{X}_P
\end{algorithmic}

\textbf{Main} \texttt{main}():
\begin{algorithmic}[1]
\State \texttt{G} = \{\}  \hspace{1cm} /* \texttt{G} \text{ is an Init Ground-truth} */
\State \texttt{X}_P \sim \text{M}.\text{infill}(m)  \hspace{1cm} /* \texttt{X}_P \text{ is a generate of } m \text{ warm start pretenders} */
\While{\texttt{firstiteration} \text{ or } \neg \text{TC}(M)}
\State \texttt{Y}_P = f_e(\texttt{X}_P, D)  \hspace{1cm} /* \texttt{Y}_P \text{ is a get pretenders' actual fitness} */
\State \texttt{G} = \{\texttt{G} \cup \{(\texttt{X}_P,\texttt{Y}_P)\}\}  \hspace{1cm} /* \texttt{G} \text{ is an add pretenders to ground-truth} */
\State \texttt{\hat{f}} = \text{argmin}_{h \in H} \mathbb{E}[L(h, G)]  \hspace{1cm} /* \texttt{\hat{f}} \text{ is a train surrogate on Ground-truth} */
\State \texttt{M}.\text{advance}(\{(\texttt{X}_P,\texttt{Y}_P)\})  \hspace{1cm} /* \texttt{M} \text{ advances the EA} */
\If{\texttt{look-ahead} }
\State \texttt{X}_P = \text{look-ahead}(\hat{f},G)  \hspace{1cm} /* \texttt{X}_P \text{ is look-ahead for Pretenders} */
\Else
\State \texttt{X}_L \sim \text{M}_L.\text{infill}(m)  \hspace{1cm} /* \texttt{X}_L \text{ is a generate of } m \text{ Candidates} */
\State \texttt{C} = \{(\texttt{X}_L,\hat{f}(\texttt{X}_L))\}  \hspace{1cm} /* \texttt{C} \text{ is an estimate of Candidates' with Surrogate} */
\State \texttt{(X}_P,\hat{Y}_P) = \text{P}_C = \left[\{\texttt{C} \cup \texttt{G}\}\right] \cap C  \hspace{1cm} /* \texttt{X}_C,\hat{Y}_C \text{ are non-dominated Candidates} */
\State s = m - |\texttt{X}_P|  \hspace{1cm} /* \texttt{s} \text{ is Get remaining spaces s in Pretenders} */
\If{\texttt{acceptance} }
\State \texttt{X}_F = \text{acceptance}(\sim \text{M}.\text{infill}(s), \text{NDScore}(\hat{f},G))  \hspace{1cm} /* \texttt{X}_F \text{ is acceptance sampling with non-dominated score threshold} */
\Else
\State \texttt{X}_P \sim \text{M}.\text{infill}(s)  \hspace{1cm} /* \texttt{X}_P \text{ is Generate s fill Candidates} */
\EndIf
\EndIf
\EndWhile
\State \textbf{return} \texttt{P} = [\texttt{G}]  \hspace{1cm} /* \texttt{P} \text{ is Non-dominated solutions} */
\end{algorithmic}
\end{algorithm}
```
It is often noted in the literature that, when looking into the solution quality of evolutionary approaches, no single metric can give a complete picture of which approach performs best in all circumstances. For this reason, evaluation across several different metrics is conducted. These metrics are defined by [47] as follows:

- **Success Rate (SR):** Success is defined as a required quality criterion that needs to be achieved, i.e. a solution within 99% or 95% of the known optimal value. In this case, the results consider the HV of the random 6510 grid search as the optimal solution. The success rate is defined as the percentage of algorithm executions in which the success criterion is met.
• **Average Evaluations to Success Rate (AESR):** Average number of evaluations of the EA required to reach a solution at a given SR, i.e. the AESR at a SR of 99% would be the average number of evaluations it took to reach the stated SR. The particular runs of the EA which do not meet the SR criterion are ignored.

• **Average Generations to Success Rate (AGSR):** Average number of generations of the EA required to reach a solution at a given SR, i.e. the AGSR at a SR of 99% would be the average number of generations it took to reach there. The particular runs of the EA which do not meet the SR criterion are ignored.

### 4.2 Hyper-parameters

The experiments use Latin Hyper-cube Sampling for random selection for all the EAs, real Uniform Crossover (UX) for crossover, and real Polynomial Mutation (PM) for mutation. For all comparisons, the results use a population size of 60 and an offspring size of 30. The EAs techniques all had a total evaluation budget of 510. The study uses Riesz s-Energy to generate reference directions or reference points. Table 1 shows the selected hyper-parameter values that are unique to the individual EAs. In the study the hyper-parameters are selected once and these values are re-used for both the SPO and MPO problems [30].

| Dataset | Method      | Crossover Rate | Mutation Rate | Selection          | Parameters |
|---------|-------------|----------------|---------------|--------------------|------------|
| DJIA    | MO-CMA-ES   | -              | -             | -                  | $\sigma = 0.1$ |
|         | NSGA-II     | 0.9            | 0.3           | Rank & Crowd       | -          |
|         | NSGA-III    | 0.9            | 0.3           | Tournament(2)      | -          |
|         | R-NSGA-II   | 0.9            | 0.3           | -                  | $\epsilon = 0.1$ |
|         | R-NSGA-III  | 0.9            | 0.3           | Tournament(2)      | $\mu = 0.1$ |
|         | U-NSGA-III  | 0.9            | 0.3           | Tournament(2)      | -          |
| S&P     | MO-CMA-ES   | -              | -             | -                  | $\sigma = 0.1$ |
|         | NSGA-II     | 0.9            | 0.2           | Rank & Crowd       | -          |
|         | NSGA-III    | 0.9            | 0.2           | Tournament(2)      | -          |
|         | R-NSGA-II   | 0.9            | 0.2           | -                  | $\epsilon = 0.1$ |
|         | R-NSGA-III  | 0.9            | 0.2           | Tournament(2)      | $\mu = 0.1$ |
|         | U-NSGA-III  | 0.9            | 0.2           | Tournament(2)      | -          |

For ParDen-Sur, the free parameter is the termination criteria for the look-ahead reservoir sampling. Multi-objective space tolerance termination was selected with a value of 0.0001 [30] set according to the scale of the solutions which are values in the range 0 to 60 with up to three decimal places being significant.
4.3 Research Instruments and Data

The motivating and primary examples for both the SPO and MPO cases are prepared using two different datasets. The dataset used in the primary examples is prepared exactly as in Boyd et al. [2], whereas a scaled-down version of this dataset is used in the motivating example.

In summary, in the primary example, consideration is limited to assets from the S&P 500 Index that has a full set of daily price data over the five years from January 2012 to December 2016. Daily volatilities used for determining transaction costs are calculated using the absolute difference between log open and close prices, while the bid-ask spread is fixed at 0.05%. Holding costs are fixed at 0.01%. These quantities are used to calculate the backtest performance of a policy.

Furthermore, risk, return, and trading cost forecasts are required at each period in the backtest period to construct the portfolio. Investigating different forecasting techniques is out of the scope of this study. Instead, return forecasts are imitated by adding zero-mean Gaussian noise to realised returns, whilst volatility and volume forecasts are specified using ten-day moving averages.

The datasets for the motivating example are prepared similarly, except that consideration is limited to assets from the Dow Jones Industrial Index over the eighteen months from July 2015 to December 2016. For further details, consult the supplementary materials.

All experiments use Ubuntu 20.04.1 LTS and are run on Intel(R) Xeon(R) CPU E5-2683 and E5-2695 v4 @ 2.10GHz. The study makes use of pymoo v0.5, pysamoo v0.1, and cvxportfolio [2]. The source code and data for reproduction of the results are available on GitHub, here.

5 Results

The following section discusses the results first for the SPO and then for the MPO.

5.1 Single-period Portfolio Optimisation (SPO)

5.1.1 Motivating Example

One of the objectives of the motivating example is to ascertain which two of the various seminal MO EA optimisation techniques are most appropriate for applying the ParDen-Sur algorithm within the SPO problem. Considering Table 2 we note that the results for NSGA-II and R-NSGA-II are mixed, with neither method outperforming the other across all indicators. It is worth noting that none of the implemented methods outperforms the 6510 random points.

Having identified the top-performing MO EA optimisation, the second objective of the motivating example is to determine which of the two selected methods is bested suited for the application of ParDen-Sur. The results in Table 2 highlight the superiority of ParDen-Sur applied to NSGA-II over R-NSGA-II. Further, the statistical tests indicate that ParDen-Sur results are
indeed significant and support an argument for ParDen-Sur’s superiority over the baseline EA as well as the superiority of using look-ahead with reservoir sampling over the original ParDen-Sur algorithm. Further, the results show that no matter which underpinning EA is used, the ParDen-Sur results are always statistically significantly superior.

We consider the quality indicators next, where the objective is to ascertain the degree to which ParDen-Sur is assisting the underpinning EA. Table 3 reinforces the selection of NSGA-II over R-NSGA-II given the superior SR and similar total evaluations as measured by AESR and AGSR. Further, the results show that no matter which underpinning EA is used, the superiority of ParDen-Sur results are always statistically significant.

### Table 2: SPO (Motivating Example): Performance indicators for the best observed value.

| Method          | GD+ | IGD+ | HV   |
|-----------------|-----|------|------|
| Random 6510     | -   | -    | 925.8 |
| Grid Search 510 [11] | .4488 | .4339 | 883.1 |
| NSGA-II         | .2298 | .2327 | 920.5 |
| R-NSGA-II       | .1272 | .4023 | 920.1 |
| NSGA-III        | .3646 | .4283 | 889.6 |
| R-NSGA-III      | .1987 | .3949 | 904.8 |
| U-NSGA-III      | .3699 | .3048 | 898.9 |
| MO-CMA-ES       | .3793 | .5156 | 882.4 |
| \(P\)NSGA-II\(^-\) | .1442 | .1359 | 933.8 |
| \(P\)NSGA-II\(^+\) | **.0963** | **.1443** | **938.3** |
| \(S\)NSGA-II\(^+\) | .1329 | **.1127** | 935.9 |
| \(P\)R-NSGA-II\(^-\) | .1856 | .1512 | 929.5 |
| \(P\)R-NSGA-II\(^+\) | .1343 | .2385 | 922.8 |
| \(S\)R-NSGA-II\(^+\) | **.1061** | .1631 | 931.9 |

- \(P\) ParDen-Sur with acceptance sampling
- \(S\) ParDen-Sur without acceptance sampling
- Algorithm applied with no-look
- Algorithm applied with look-ahead

#### 5.1.2 Primary Example

The main objective of the SPO primary example is to replicate the existing research and provide context for the current ParDen-Sur results. To this end Table 4 replicates the previous studies by [11] and [7]. The results reinforce the superior performance of ParDen-Sur over both NSGA-II and MO-CMA-ES as underpinning EAs. Further, the results highlight the superiority of look-ahead with reservoir sampling as found in ParDen-Sur over using just acceptance sampling. One interesting result requiring further investigation is the superior performance of R-NSGA-II when considering GD+.
Table 3: SPO (Motivating Example): Quality Indicators. Bold: best, grey: top two, red: warning indicator.

| Method                        | SR@95 | SR@90 | AESR@95 | AESR@90 | AGSR@95 |
|-------------------------------|-------|-------|---------|---------|---------|
| Grid Search 510               | 0.0   | 0.0   | -       | -       | -       |
| NSGA-II                       | 100.0 | 100.0 | 330.0   | 138.0   | 10.0    |
| R-NSGA-II                     | 90.0  | 100.0 | 283.4   | 144.0   | 8.5     |
| NSGA-III                      | 90.0  | 100.0 | 353.4   | 135.0   | 10.8    |
| R-NSGA-III                    | 60.0  | 100.0 | 293.4   | 131.0   | 8.7     |
| U-NSGA-III                    | 80.0  | 100.0 | 345.0   | 123.0   | 10.5    |
| MO-CMA-ES                     | 30.0  | 60.0  | 410.0   | 203.4   | 12.7    |

\[\mathcal{P}:\text{NSGA-II}^+\] 100.0 100.0 262.6 84.7 11.7
\[\mathcal{P}:\text{NSGA-II}^-\] 80.0 100.0 172.1 80.1 10.3
\[\mathcal{S}:\text{NSGA-II}^+\] 100.0 100.0 177.0 78.0 \textbf{4.9}
\[\mathcal{P}:\text{R-NSGA-II}^-\] 100.0 100.0 267.7 80.9 11.2
\[\mathcal{P}:\text{R-NSGA-II}^+\] 40.0 100.0 \textbf{163.0} 114.5 10.0
\[\mathcal{S}:\text{R-NSGA-II}^+\] 60.0 100.0 300.0 156.0 9.0

\(\mathcal{P}\): ParDen-Sur with acceptance sampling
\(\mathcal{S}\): ParDen-Sur without acceptance sampling
- Algorithm applied with no-look
+ Algorithm applied with look-ahead

Looking toward the quality indicators, the results in Table 5 reinforce the superior performance of ParDen-Sur with look-ahead over ParDen-Sur without for both NSGA-II and MO-CMA-ES. Having established the superiority of ParDen-Sur with look-ahead in the SPO problem space, the study now turns to the more challenging MPO problem.

5.2 Multi-period Portfolio Optimisation (MPO)

5.2.1 Motivating Example

The main objective of the motivating example was to establish which MO EA algorithms to use in conjunction with ParDen-Sur. Table 6, like in the SPO motivating example, indicates that NSGA-II and R-NSGA-II are superior choices for underpinning EAs. Both of these techniques significantly outperform MO-CMA-ES previously proposed by Nystrup et al. [11].

When considering the second objective of the motivating example as to which of the top two underpinning EAs are best suited to further exploration with ParDen-Sur, Table 6 shows a slight benefit in favour of NSGA-II. However, when we consider the quality indicators in Table 7, the results show that any gains by R-NSGA-II are also linked to substandard Success Rates at the 99% threshold. Interestingly, the quality indicators indicate that ParDen-Sur has lower AESR@99 and AGSR@99 than the underpinning EAs. However, the jump in SR@99 to 100% warrants further exploration and points to the superiority of ParDen-Sur not just for reducing the rate of convergence but also for improving the probability of convergence. This argument is further supported when considering the superiority of ParDen-Sur at the 95% threshold.
Table 4: SPO (Primary Example): Performance indicators for the best observed value.

| Method                      | GD+  | IGD+ | HV  |
|-----------------------------|------|------|-----|
| Random 6510                 | -    | -    | 1313.0 |
| Grid Search 510             | 0.59 | 0.573 | 1259.1 |
| NSGA-II [7]                 | 0.1137 | 0.1585 | 1308.4 |
| R-NSGA-II [7]               | **0.0259** | 0.2443 | 1309.7 |
| NSGA-III [7]                | 0.1062 | 0.2320 | 1305.0 |
| R-NSGA-III [7]              | 0.1215 | 0.2279 | 1304.3 |
| U-NSGA-III [7]              | 0.1314 | 0.2567 | 1302.7 |
| MO-CMA-ES [11]              | 0.1002 | 0.0965 | 1317.4 |
| \(\mathcal{P} : \text{MO-CMA-ES}^- [7]\) | 0.0912 | 0.0951 | 1320.0 |
| \(\text{S} : \text{MO-CMA-ES}^+\) | 0.0602 | **0.0757** | **1323.5** |
| \(\mathcal{P} : \text{NSGA-II}^- [7]\) | **0.0764** | 0.0901 | 1319.4 |
| \(\text{S} : \text{NSGA-II}^+\) | 0.0558 | **0.0604** | **1323.9** |

\(\mathcal{P}\) ParDen-Sur with acceptance sampling  
\(\text{S}\) ParDen-Sur without acceptance sampling  
\(\neg\) Algorithm applied with no-look  
\(\neg\) Algorithm applied with look-ahead

Table 5: SPO (Primary Example): Quality Indicators. Bold: best, grey: top two, red: warning indicator.

| Method                      | SR@99 | SR@95 | AESR@99 | AESR@95 | AGSR@99 |
|-----------------------------|-------|-------|---------|---------|---------|
| Grid Search 510 [11]        | 0.0   | 100.0 | -       | 510.0   | -       |
| NSGA-II [7]                 | 100.0 | 100.0 | 327.0   | 63.0    | 9.9     |
| R-NSGA-II [7]               | 100.0 | 100.0 | 297.0   | 63.0    | 8.9     |
| NSGA-III [7]                | 100.0 | 100.0 | 405.0   | 81.0    | 12.5    |
| R-NSGA-III [7]              | 60.0  | 100.0 | 415.3   | 75.2    | 12.8    |
| U-NSGA-III [7]              | 70.0  | 100.0 | 338.6   | 96.0    | 10.3    |
| MO-CMA-ES [11]              | 100.0 | 100.0 | 333.0   | 99.0    | 10.1    |
| \(\mathcal{P} : \text{MO-CMA-ES}^- [7]\) | 100.0 | 100.0 | 247.7   | 75.9    | 10.7    |
| \(\text{S} : \text{MO-CMA-ES}^+\) | 100.0 | 100.0 | **216.2** | 72.1    | **6.8** |
| \(\mathcal{P} : \text{NSGA-II}^- [7]\) | 100.0 | 100.0 | 254.7   | 66.4    | 10.1    |
| \(\text{S} : \text{NSGA-II}^+\) | 100.0 | 100.0 | **177.0** | 60.0    | **4.9**  |

\(\mathcal{P}\) ParDen-Sur with acceptance sampling  
\(\text{S}\) ParDen-Sur without acceptance sampling  
\(\neg\) Algorithm applied with no-look  
\(\neg\) Algorithm applied with look-ahead

Having established the superiority of ParDen-Sur coupled with NSGA-II on the MPO motivating example, the study now focuses on the MPO primary results.
Table 6: MPO (Motivating Example): Performance indicators for the best observed value.

| Method       | GD+  | IGD+ | HV    |
|--------------|------|------|-------|
| Random 6510  | -    | -    | 1632.9|
| Grid Search 510 | .5497 | .8091 | 1530.1|
| NSGA-II      | .1261| .1938| 1630.5|
| R-NSGA-II    | .1206| .2456| 1615.8|
| NSGA-III     | .1412| .2178| 1619.9|
| R-NSGA-III   | .1775| .3260| 1607.7|
| U-NSGA-III   | .2213| .2836| 1600.3|
| MO-CMA-ES    | .2360| .6529| 1555.4|
| P:NSGA-II−   | .0905| .1266| 1643.6|
| P:NSGA-II+   | .0910| .1048| 1640.7|
| S:NSGA-II+   | .1007| .1306| 1639.9|
| P:R-NSGA-II− | .1083| .1395| 1635.9|
| P:R-NSGA-II+ | .0765| .1159| 1640.3|
| S:R-NSGA-II+ | .0845| .1291| 1635.4|

P: ParDen-Sur with acceptance sampling
S: ParDen-Sur without acceptance sampling
−: Algorithm applied with no-look
+: Algorithm applied with look-ahead

Table 7: MPO (Motivating Example): Quality Indicators. Bold: best, grey: top two, red: warning indicator.

| Method       | SR@99 | SR@95 | AESR@99 | AESR@95 | AGSR@99 |
|--------------|-------|-------|---------|---------|---------|
| Grid Search 510 |     |     |         |         |         |
| NSGA-II      | 80.0  | 100.0| 315.0   | 138.0   | 9.5     |
| R-NSGA-II    | 60.0  | 100.0| 390.0   | 147.0   | 12.0    |
| NSGA-III     | 10.0  | 90.0 | 450.0   | 230.0   | 14.0    |
| R-NSGA-III   | 10.0  | 100.0| 452.0   | 191.0   | 14.0    |
| U-NSGA-III   | 10.0  | 100.0| 510.0   | 243.0   | 16.0    |
| MO-CMA-ES    | 0.0   | 40.0 |         |         |         |
| P:NSGA-II−   | 100.0 | 100.0| 321.0   | 117.0   | 9.7     |
| P:NSGA-II+   | 100.0 | 100.0| 327.0   | 105.0   | 9.9     |
| S:NSGA-II+   | 100.0 | 100.0| 327.0   | 120.0   | 9.9     |
| P:R-NSGA-II− | 100.0 | 100.0| 333.0   | 108.0   | 10.1    |
| P:R-NSGA-II+ | 90.0  | 100.0| 293.3   | 114.0   | 8.8     |
| S:R-NSGA-II+ | 90.0  | 100.0| 273.3   | 93.0    | 8.1     |

P: ParDen-Sur with acceptance sampling
S: ParDen-Sur without acceptance sampling
−: Algorithm applied with no-look
+: Algorithm applied with look-ahead
5.2.2 Primary Example

The main aim of this study is to evaluate the use of ParDen-Sur in the context of MPO on the primary example. Table 9 shows the out-performance of all variants of ParDen-Sur. Most interesting is the superior performance obtained using look-ahead. The p-values of the Mann-Whitney U tests (with multiple testing adjustments) in Table 8 add further weight by demonstrating the out-performance is statistically significant.

Table 8: MPO (Primary Example): p-values from testing differences in performance indicators between ParDen-Sur and baseline methods.

|             | HV        | GD+       | IGD+     |
|-------------|-----------|-----------|----------|
| NSGA-II     | 2.0 \times 10^{-4} | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} |
| R-NSGA-II   | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} |
| NSGA-III    | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} |
| R-NSGA-III  | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} |
| MO-CMA-ES   | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} | 9.1 \times 10^{-4} |

When considering the quality indicators in Table 10 the results show marked improvements in both AESR@99 and AGSR@99, with ParDen-Sur having an almost 2\times speedup over the underpinning EA in both these metrics. We also note that ParDen-Sur without acceptance sampling outperforms when considering the AESR@99. This result is interesting as it suggests the benefit of one less free parameter and a possible algorithm simplification. It is worth contrasting this with the results in Table 3 and 7 which have both supporting and conflicting support of this observation.

6 Discussion

The presented results demonstrate that the challenges of optimal hyper-parameter search related to multi-period optimisation are tractable within the context of surrogate assisted MO optimisation. Further, the results support the use of the ParDen-Sur algorithm with NSGA-II to significantly improve the rate at which one can arrive at a solution and the quality of the solutions found.

Previously, it has been shown that surrogate-assisted portfolio optimisation can speed up single-period hyper-parameter searches [7]. This study confirms those results and extends them to the multi-period optimisation use case. The outcome suggests that all versions of ParDen-Sur, the proposed method, lead to a speed-up in acquiring optimal frontiers in the portfolio optimisation task, as seen in Figure 4.

Previous work [7] showed that the ParDen framework was able to speed up both MO-CMA-ES and NSGA-II, and the results in this study show that this extends to R-NSGA-II. Further, the results are demonstrated on more than one dataset and for multiple periods. However, without testing this statement across the full battery of EAs, it is not possible to ascertain if the extensions to
Table 9: MPO (Primary Example): Performance indicators for the best observed value.

| Method                        | GD+  | IGD+  | HV   |
|-------------------------------|------|-------|------|
| Random 6510                   | -    | -     | 1699.9 |
| Grid Search [2]               | .8852| 1.3881| 1487.7 |
| Grid Search 510               | .4634| .4534 | 1644.7 |
| NSGA-II                       | .0998| .2110 | 1700.8 |
| R-NSGA-II                     | .0531| .2168 | 1694.9 |
| NSGA-III                      | .0852| .2361 | 1692.5 |
| R-NSGA-III                    | .1054| .2329 | 1690.1 |
| U-NSGA-III                    | .1189| .2378 | 1689.7 |
| MO-CMA-ES [11]                | .1751| .2248 | 1695.9 |
| $\mathcal{P}$:NSGA-II$^-$    | .0602| .0838 | 1711.2 |
| $\mathcal{P}$:NSGA-II$^+$    | .0428| .0731 | 1715.6 |
| $\mathcal{S}$:NSGA-II$^+$    | .0490| .0706 | 1715.4 |

$\mathcal{P}$: ParDen-Sur with acceptance sampling
$\mathcal{S}$: ParDen-Sur without acceptance sampling
$^-$: Algorithm applied with no-look
$^+$: Algorithm applied with look-ahead

ParDen as contained in ParDen-Sur are a general purpose surrogate assisted optimisation framework.

In the previous study, ParDen used only acceptance sampling to improve speed-up [7]. Here, the results demonstrate that the inclusion of look-ahead with acceptance sampling is a superior approach, as confirmed by Figure 5. Further, when AESR is not concerned, forgoing acceptance sampling in favour of looking ahead provides the best speed-up.

It is worth noting that the combination of acceptance sampling and look-ahead has a net positive impact on AESR. The case for minimal AESR would occur when one is, for instance, being charged for resources. Here using the smallest amount of evaluations of the simulation is desirable over the smallest number of generations as measured by AGSR.

7 Conclusion

This study presents the ParDen-Sur surrogate assisted optimisation algorithm, an extension to the original ParDen algorithm by van Zyl et al. [7], to address the computational difficulty of finding the efficient frontier for multiperiod portfolio optimisation. The extension takes the form of a reservoir sampling-enabled look-ahead mechanism for generating new offspring. Both the look-ahead and the acceptance sampling rely on the NDScore. Here the NDScore is extended to include Kendall-$\tau$. The extension and the original acceptance sampling mechanism are tested on the problems of SPO and MPO. The three variants of ParDen-Sur are evaluated across several EAs. Both performance and quality metrics are taken into account. The results demonstrate
a statistically significant improvement in both performance and quality of solutions when ParDen-Sur is applied to the aforementioned problems with the underpinning EAs considered. However, reservoir sampling-enabled look-ahead has the most impact on the results. The study has demonstrated the benefit of ParDen-Sur for the portfolio optimisation hyper-parameter search problem. Future work will look to explore additional markets as well as additional problem domains in which ParDen-Sur might be of value.

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**Declarations**

**7.1 Competing Interests**
The authors have no competing interests to declare.

**7.2 Availability of data and materials**
The datasets generated during and/or analysed during the current study are available on GitHub, [here](#).

**7.3 Code availability**
The source code used during the current study is available on GitHub, [here](#).
Fig. 4: ParDen-Sur Variants vs NSGA-II Hypervolume. All ParDen-Sur variants with Look-ahead on average can obtain an optimal solution in fewer generations. Indicated by their positions in the upper left-hand corners. Trial lines are in grey.
Fig. 5: Best ParDen-Sur Variants vs NSGA-II Hypervolume. The max-fill with
Look-ahead can obtain an optimal solution in the fewest generations. Indicated
by its dominant position in the upper left-hand corner.

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