ZETA FUNCTIONS OF THE 3-DIMENSIONAL ALMOST-BIEBERBACH GROUPS

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Abstract. The subgroup zeta function and the normal zeta function of a finitely generated virtually nilpotent group can be expressed as finite sums of Dirichlet series admitting Euler product factorization. We compute these series except for a finite number of local factors when the group is virtually nilpotent of Hirsch length 3. We deduce that they can be meromorphically continued to the whole complex plane and that they satisfy local functional equations. The complete computation (with no exception of local factors) is presented for those groups that are also torsion-free, that is, for the 3-dimensional almost-Bieberbach groups.

1. Introduction

The subgroup zeta function and the normal zeta function of a finitely generated group G are, respectively, the Dirichlet series

$$\zeta_G(s) = \sum_{n=1}^{\infty} \frac{a_n^\ast(G)}{n^s} \quad \text{and} \quad \zeta_G^\ast(s) = \sum_{n=1}^{\infty} \frac{a_n^\ast(G)}{n^s},$$

where $a_n^\ast(G)$ denotes the number of subgroups and $a_n^\ast(G)$ the number of normal subgroups of index n. They were introduced by Grunewald, Segal, and Smith in [8] as a means to study groups of polynomial subgroup growth (PSG). Indeed, these are precisely the groups for which these series are not only formal gadgets but also define analytic functions on some right-half plane of the complex plane. A natural problem is to understand how structural information of a group of PSG is encoded into arithmetical and analytic properties of its zeta functions and vice-versa.

Lubotzky, Mann and Segal characterized the finitely generated residually finite groups of PSG: these are precisely the virtually solvable groups of finite rank [11]. On the other hand, since they were introduced, zeta functions of groups have been extensively studied only in when the group is a finitely generated torsion-free nilpotent group: a $\Sigma$-group. The theory for these groups is rich and continues growing. We summarize some general results and refer the reader to the survey [24] and the references therein for more information and variations of the topic. Let $N$ be a $\Sigma$-group, let $* \in \{\leq, <\}$ and let $\alpha_N^*$ denote the abscissa of convergence of $\zeta_N^*(s)$. Then the following holds.

(N1) $\alpha_N^* \leq h(N)$ (= the Hirsch length of N) [8] Proposition 1.

(N2) $\alpha_N^* \in \mathbb{Q}$ and there is $\delta > 0$ such that $\zeta_N^*(s)$ admits a meromorphic continuation to the region \( \{ s \in \mathbb{C} : \text{Re}(s) > \alpha_N^* - \delta \} \) [5] Theorem 1.1.

(N3) $\zeta_N^*(s)$ has an Euler product factorization

$$\zeta_N^*(s) = \prod_{p \text{ prime}} \zeta_{N,p}^*(s)$$

where $\zeta_{N,p}^*(s) = \sum_{k=0}^{\infty} a_{p^k}^*(N) p^{-ks}$. In addition, each local factor $\zeta_{N,p}^*(s)$ is a rational function in $p^{-s}$ with coefficients in $\mathbb{Q}$ [8] Proposition 1.3 and Theorem 1.

(N4) For almost all prime $p$, $\zeta_{N,p}^*(s)$ satisfies the functional equation

$$\zeta_{N,p}^*(s)\mid_{p \rightarrow p^{-1}} = (-1)^h p^{h} \zeta_{N,p}^*(s),$$

where $h = h(N)$ [23] Corollary 1.1. If in addition $N$ is a $\Sigma_2$-group (a $\Sigma$-group of nilpotency class 2), then for almost all prime $p$, $\zeta_{N,p}^*(s)$ satisfies the functional equation

$$\zeta_{N,p}^*(s)\mid_{p \rightarrow p^{-1}} = (-1)^h p^{h} \zeta_{N,p}^*(s),$$

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where $d$ is the rank of $N/Z(N)$ \cite[Theorem C]{23}.

**Remark 1.1.** Properties (N1)-(N4) are, in fact, corollaries of the analogous properties established for the subring zeta function and the ideal zeta function of nilpotent Lie rings additively isomorphic to some $\mathbb{Z}^h$. The translation is done via the Mal’cev correspondence. The question of whether the ideal zeta function of a nilpotent Lie ring of nilpotency class $>2$ satisfies local functional equations led to the introduction and investigation of the submodule zeta function; see, e.g., \cite{17, 25, 10}. On the other hand, it has long been known that, for instance, the ideal zeta function of the filiform nilpotent Lie ring $\text{Fil}_4$ fails to satisfy local functional equations; see \cite[Theorem 2.39]{7}.

Regarding the computation of zeta functions, the following are classical examples.

1. For the free abelian group $\mathbb{Z}^h$, it holds that $\zeta^c_{\mathbb{Z}^h}(s) = \zeta(s)\zeta(s-1)\cdots\zeta(s-h+1)$, where $\zeta(s)$ is the Riemann zeta function. In \cite[Chap. 15]{12}, there are five different proofs of this elementary fact.

2. Let $H(\mathbb{Z})$ denote the integral points of the Heisenberg unipotent group scheme $H$. Then

$$\zeta_{H(\mathbb{Z})}(s) = \frac{\zeta(s)\zeta(s-1)\zeta(2s-2)\zeta(2s-3)}{\zeta(3s-3)}$$

\cite[Section 8]{8}. A general theory and calculation of $\zeta_{H(\mathbb{Z})}(s)$, for $K$ a number field and $\mathcal{O}_K$ its ring of integers, are presented in \cite{19} and \cite{20}.

Finding explicit formulae for the zeta functions of a $\mathcal{S}$-group is in general a difficult task. A substantial list of examples is recorded in \cite{7}, and new ones have emerged more recently; see, e.g., \cite{1, 19, 20, 26}. In \cite{16} and \cite{18}, Rossmann developed a method for computing certain zeta functions associated with $\mathcal{S}$-groups and other algebraic structures. This algorithm was implemented in \cite{15}.

This work studies and presents formulae for the zeta functions of the virtually nilpotent groups of Hirsch length 3 (see the next paragraph for a detailed description of the content). This is not the first attempt in dealing with zeta functions of groups that are not nilpotent. In \cite{1}, du Sautoy investigated zeta functions of compact $p$-adic analytic groups (= virtually uniform pro-$p$-groups) and proved that they are rational functions in $p^{-s}$. Then, in \cite{16}, the authors studied zeta functions of virtually abelian groups, proving for example, that they can be meromorphically continued to the whole complex plane (this property is not shared by zeta functions of $\mathcal{S}$-groups in general, as discussed in \cite[Chap. 7]{7}). The paper \cite{22} contains general properties of zeta functions of virtually nilpotent groups that we now recall. Let $G$ be a finitely generated virtually nilpotent group, and let $N \triangleleft G$ be a finite index normal subgroup that is a $\mathcal{S}$-group. It is easy to check that

$$\zeta^c_G(s) = \sum_{N \leq H \leq G} [G : H]^{-s}\zeta^H_G(s)$$

and

$$\zeta^G(s) = \sum_{N \leq H \leq G} [G : H]^{-s}\zeta^H_G(s),$$

where

$$\zeta^H_G(s) := \sum_{A \leq G : AN = H} [H : A]^{-s}$$

and

$$\zeta^G(s) := \sum_{A \leq G : AN = H} [H : A]^{-s}.$$

The series $\zeta^H_G(s)$ and $\zeta^G(s)$ will be referred to as partial zeta functions of $\zeta^G(s)$ and $\zeta^G(s)$ (with respect to $N$). Each of them is an Euler product of cone integrals \cite[Theorem 1]{22}, and hence it satisfies properties (N2) and (N3) above by the general theory of cone integrals developed in \cite{5}. Property (N1) also holds with the bound $h(N) = h(G)$ replaced by $h(G) + 1$ \cite[Proposition 2.12]{22}. The partial zeta functions were computed explicitly, except for a finite number of local factors, when the group $N$ is abelian; see \cite[Section 2]{6}. Computing the exceptional factors is, in general, more involved. The complete computation was done for the plane crystallographic groups. This is the main result in \cite{13}, and it is also summarized in \cite[Section 4]{6}. To the author’s knowledge, these are the only existing examples of computations of zeta functions of infinite groups that are not nilpotent.

The article is organized as follows. In Section 2 we obtain formulae for the local factors of $\zeta^G(s)$ and $\zeta^N(s)$ when $p \nmid [G : N]$ for every virtually nilpotent group of Hirsch length 3 (see Tables 1 and 2). This suffices to conclude that the subgroup and normal zeta functions of each of these groups extend to meromorphic functions on the whole plane (Corollary 2.8 and Corollary 2.19). We also deduce local functional equations, similar to but different from the ones presented in (N4) (Corollary 2.7 and Corollary 2.18). The arguments in this section are of a group-theoretical nature, akin to \cite[Section 2]{6}. In section 3...
we review a method for expressing local zeta functions of groups in terms of p-adic integrals. This method is used later to compute local factors at primes p | [G : N] (those not considered in Section 2). Section 4 presents the formulae for the zeta functions of the torsion-free virtually nilpotent groups of Hirsch length 3. A group like this is either a 3-dimensional Bieberbach group or a 3-dimensional almost-Bieberbach group (the fundamental group of a 3-dimensional infra-nilmanifold). Section 5 contains proofs of the formulae for the 3-dimensional Bieberbach groups. It also contains formulae for the zeta functions of a large family of Bieberbach groups with holonomy $C_2$, to illustrate how involved the method for computing local factors at bad primes is, even in the case of virtually abelian groups. Finally, Section 6 contains proofs of the formulae for the 3-dimensional almost Bieberbach groups.

Notation and conventions

$|S|$ denotes the cardinality of a set $S$. For a group $G$, $Z(G)$ denotes its center; $A \subseteq G$ means that $A$ is a subgroup; $A \triangleleft G$ means that $A$ is a normal subgroup; $[G : A]$ denotes the index of $A$ in $G$; $N_G(A)$ denotes the normalizer of $A$ in $G$. In sums involving $[G : A]$, only subgroups of finite index are considered. For a subset $S \subseteq G$, $(S)$ denotes the subgroup generated by $S$. When $G$ is profinite, $(S)$ denotes the closed subgroup generated by $S$. For $x, y \in G$, we denote $\hat{y} = xyx^{-1}$ and $[x, y] = xyx^{-1}y^{-1} = (\hat{y} \cdot y^{-1})$. For $S, T \subseteq G$ we denote $[S, T] = \{(s, t) | s \in S, t \in T\}$. We use without mentioning the fact that if $G$ is a finitely generated profinite group, then every finite index subgroup is open, and if $H$ and $K$ are closed subgroups, then $[H : K]$ is also closed; see [13].

The variable $p$ will be reserved for prime numbers. We denote by $v_p : \mathbb{Q}_p \rightarrow \mathbb{Z} \cup \{\infty\}$ the usual $p$-adic valuation, and by $| \cdot |_p = p^{-v_p(\cdot)}$ the $p$-adic norm. $\zeta_p(s)$ denotes $\frac{1}{1-p^s}$, the local factor of the Riemann zeta function $\zeta(s)$ at $p$.

2. Local factors at good primes and meromorphic continuation

Let $G$ be a finitely generated virtually nilpotent group, and let $N \triangleleft G$ be a finite index normal subgroup that is a $\mathbb{F}_2$-group. For a prime $p$, let $G_p$ be the completion of $G$ with respect to the family $\{B \triangleleft G_p : B \subseteq N_p, [N_p : B] \text{ a power of } p\}$. For $X \subseteq G$, we denote by $X_p$ the closure of $X$ in $G$. For a subquotient $X/Y$ of $G$ we denote $(X/Y)_p = X_p/Y_p$. Note that $N_p$ is the pro-$p$ completion of $N$, and that the inclusion $G \hookrightarrow G_p$ induces an isomorphism $G/N \cong G_p/N_p$.

As noted in the introduction, to study $\zeta_G^*(s)$, it is enough to focus on the partial zeta functions $\zeta_{G_p}^{H_p}(s) = \zeta_{G_p}^{H_p}(s)$ for $N \leq H \leq G$. There is no loss of generality if we only consider $\zeta_G^{H_p}(s)$. Similarly, to study $\zeta_G^{H_p}(s)$, it is enough to consider $\zeta_{G_p}^{H_p}(s)$ for $N \leq H \leq G$. The advantage is that we have

$$\zeta_G^{H_p}(s) = \prod_{p \text{ prime}} \zeta_{G_p}^{H_p}(s)$$

and

$$\zeta_G^{H_p}(s) = \prod_{p \text{ prime}} \zeta_{G_p}^{H_p}(s),$$

where $\zeta_{G_p}^{H_p}(s)$ and $\zeta_{G_p}^{H_p}(s)$ are computed with respect to $N_p$, [22, Proposition 2.2].

Along the section, unless otherwise specified, $N$ denotes a $\mathbb{F}_2$-group. We also fix the following notation:

$$Z = Z(N), \quad T = T/Z, \quad P = G/N, \quad \Lambda = \mathbb{Z}[P] \quad \text{and} \quad \Lambda_p = \mathbb{Z}_p[P].$$

We will first obtain general expressions for $\zeta_{G_p}^{H_p}(s)$ and $\zeta_{G_p}^{H_p}(s)$ when $p \nmid |P|$. Then we shall specialize to the case of Hirsch length 3, giving explicit formulae in terms of well-known series. The case $N \cong \mathbb{Z}^3$ (and more generally $N \cong \mathbb{Z}^h$ for any $h$) was settled in [6].

The action by conjugation of $G$ on $N ((g, n) \mapsto gng^{-1})$ induces structures of left $\Lambda$-modules on the abelian groups $T$ and $Z$ (and hence $T_p$ and $Z_p$ become $\Lambda_p$-modules). For this reason, we use additive notation when working with them. The commutator operation $[x, y] = xyx^{-1}y^{-1}$ induces a bilinear map $[\cdot, \cdot] : T \times T \rightarrow Z$ (and hence also a $\mathbb{Z}_p$-bilinear map $[\cdot, \cdot] : T_p \times T_p \rightarrow Z_p$) that is compatible with the action of $P$, that is, $[\cdot x, \cdot y] = \alpha \cdot [x, y]$ for all $\alpha \in P$ and $x, y \in T$. If $U_1$ and $U_2$ are $\mathbb{Z}_p$-submodules of $T_p$, then $U_1 \cap U_2$ denotes the $\mathbb{Z}_p$-submodule (=subgroup) of $Z_p$ generated by $[x_1, x_2]$ with $x_i \in U_i$.

For a ring $R$ and a left $R$-module $M$, we use the notation $X \leq R M$ to indicate that $X$ is a left $R$-submodule of $M$. For $r \in R$, $r_M : M \rightarrow M$ denotes the left multiplication by $r$. If $R = \Lambda$ or $\Lambda_p$, we denote by $\text{Der}(P, M)$ the set of derivations from $P$ to $M$. For a subset $S \subseteq P$, we denote $C_M(S) = \{x \in M : \alpha \cdot x = x, \forall \alpha \in S\}$. We will use the fact that if $p \nmid |P|$ and $M$ is a finitely generated $\Lambda_p$-module that is torsion-free as a $\mathbb{Z}_p$-module, then $M$ is a projective $\Lambda_p$-module, and all finite index $\Lambda_p$-submodules of $M$ are isomorphic. This follows from the fact that when $p \nmid |P|$, $\Lambda_p$ is a maximal order of $\mathbb{Q}_p[P]$. 


2.1. Counting subgroups. We begin with a general formula for the local factors of $\zeta_{G_p}(s)$ at “good” primes.

**Proposition 2.1.** If $p \nmid |P|$, then

$$
\zeta_{G_p}(s) = \sum_{U \leq T_p, [U,U] \subseteq A \subseteq N_p, V} |T_p : U|^{-s}[T_p : C_{T_p}(P) + U][Z_p : V]^{-s}|Z_p : C_{Z_p}(P) + V| |\text{Hom}_{A_p}(T_p, Z_p/V)|.
$$

**Proof.** For a subgroup $U \leq T_p$, we denote by $\hat{U}$ its pre-image in $N_p$. We set

$$
A := \{ A \leq G_p : AN_p = G_p, [G_p : A] < \infty \},
$$

$$
U := \{(U \leq A, T_p, V \leq A, Z_p) : [T_p : U] < \infty, [Z_p : V] < \infty, [U, U] \subseteq V \},
$$

$$
V := \{(U, V, C) : (U, V) \in U, C \in A, C \cap N_p = \hat{U} \},
$$

and define maps $\Phi : A \to V$, $\Psi : V \to U$ by

$$
\Phi(A) = (((A \cap N_p)/Z_p)/Z_p, A \cap Z_p, Z_p A) \text{ and } \Psi(U, V, C) = (U, V).
$$

It is straightforward to check that $\Phi$ is well-defined. Note that for $A \in (\Phi \Psi)^{-1}(U, V)$, it holds that $[G_p : A] = [N_p : A \cap N_p] = [N_p : \hat{U}]/[Z_p : V] = [T_p : U]/[Z_p : V]$; therefore,

$$
\zeta_{G_p}(s) = \sum_{A \in A} [G_p : A]^{-s} = \sum_{(U, V) \in U} [T_p : U]^{-s}[Z_p : V]^{-s} |(\Psi \Phi)^{-1}(U, V)|.
$$

We now fix $(U, V) \in U$ and show that $|\Psi^{-1}(U, V)| = |\text{Der}(P, T_p/U)|$. Note first that $\Psi^{-1}(U, V)$ is in a bijection with the set of complements of $N_p/\hat{U}$ in $G_p/\hat{U}$. Since $N_p/\hat{U}$ is a normal Hall subgroup of $G_p/\hat{U}$, there is at least one complement by the Schur-Zassenhaus Theorem, and therefore the number of complements is $|\text{Der}(P, N_p/\hat{U})| = |\text{Der}(P, T_p/U)|$ (cf. [21] Proposition 1, Chap. 3).

We now fix $(U, V, C) \in V$ and show that $|\Phi^{-1}((U, V, C))| = |\text{Hom}_{A_p}(U, Z_p/V)||\text{Der}(P, Z_p/V)|$. The subgroups $\hat{U}$ and $V$ are normal closed subgroups of $G_p$, and the condition $[U, U] \subseteq V$ implies that the quotient $U/V$ is abelian. Thus, $\hat{U}/V$ is a finitely generated $\mathbb{Z}_p$-module and $Z_p/V$ is clearly the torsion submodule. We use $C$ to give $U/V$ a structure of $\mathbb{Z}_p[P]$-module. Firstly, the action by conjugation of $C$ on $\hat{U}$ induces a structure of $C/\hat{U}$-module on $U/V$. Secondly, the inclusion $C \to G_p$ induces an isomorphism $C/\hat{U} \cong G_p/N_p = P$. Therefore, $\hat{U}/V$ becomes a $Z_p[P]$-module. Note that $Z_p/V$ has two structures of $Z_p[P]$-module: one as a subobject of $\hat{U}/V$ and one as a quotient of $Z_p$. It is easy to check that these two structures coincide. Similarly, the structure of $Z_p[P]$-module on $U/Z_p = U$ as a quotient of $\hat{U}/V$ and the one as a subobject of $T_p$ are the same.

Given $A \leq \Phi^{-1}(U, V, C)$, observe that $(A \cap N_p)/V$ is a complement of $Z_p/V$ in $\hat{U}/V$. We claim that this complement is $P$-invariant. Indeed, $A \cap N_p$ is normal in $A$ and in $(A \cap N_p)/Z_p$, so it is normal in $AZ_p = C$. Therefore, $(A \cap N_p)/V$ is $C/\hat{U}$-invariant, and hence $P$-invariant. Now, according to Lemma [22] below, the number of $P$-invariant complements of $Z_p/V$ in $U/V$ is $|\text{Hom}_{\mathbb{Z}_p}(U, Z_p/V)|$. Fix one such complement, say $B/V$. Note that $B$ is normal in $C$ (this follows from the condition of $B/V$ being $P$-invariant) and the set of those $A \in \Phi^{-1}(U, V, C)$ such that $A \cap N_p = B$ is in a bijection with the set of complements of $\hat{U}/B$ in $C/B$. Since $\hat{U}/B$ is a normal Hall subgroup of $C/B$, there is at least one such complement by the Schur-Zassenhaus Theorem, and then the number of complements is $|\text{Der}(C/\hat{U}, \hat{U}/B)| = |\text{Der}(P, \hat{U}/B)| = |\text{Der}(P, Z_p/V)|$. We conclude that $|\Phi^{-1}(U, V, C)| = |\text{Hom}_{\mathbb{Z}_p}(U, Z_p/V)| |\text{Der}(P, Z_p/V)|$.

To end the proof, we only need to show that $|\text{Der}(P, T_p/U)| = [T_p : U + C_{T_p}(P)]$, $|\text{Der}(P, Z_p/V)| = [Z_p : V + C_{Z_p}(P)]$ and $|\text{Hom}_{\mathbb{Z}_p}(U, Z_p/V)| = |\text{Hom}_{\mathbb{Z}_p}(T, Z_p/V)|$. Here we use again that $p \nmid |P|$. The first two equalities follow from [3] Lemma 2.4, and for the last one we use the fact that $U$ and $T$ are isomorphic as $\mathbb{Z}_p$-modules. □

The following lemma was used in the proof above.

**Lemma 2.2.** Let $R$ be a ring, let $M$ be a left $R$-module, and fix $X \leq R$. If there is a complement of $X$ in $M$ (i.e., there is $Y \leq R$ such that $X + Y = M$ and $X \cap Y = 0$), then the set of complements of $X$ in $M$ is in a bijection with $\text{Hom}_R(M/X, X)$. If $R = \mathbb{Z}_p[P]$ with $p \nmid |P|$, and if $M/X$ is finitely generated and torsion-free as a $\mathbb{Z}_p$-module, then the set of $(P$-invariant$)$ complements of $X$ in $M/X$ is in a bijection with $\text{Hom}_{\mathbb{Z}_p[P]}(M/X, X)$. 

Proof. Assume that there is a complement of $X$, say $Y$. Given any other complement $K \leq_R M$, we define $\varphi_K : Y \to X$ as follows. If $y \in Y$, then we can write $y = k + x$ uniquely with $k \in K$ and $x \in X$. We set $\varphi_K(y) = x$. It is clear that $\varphi_K \in \Hom_R(Y, X)$. Conversely, given $\varphi \in \Hom_R(Y, X)$, we define $K_\varphi = \{ y - \varphi(y) : y \in Y \}$. It is easy to check that $K_\varphi$ is a complement of $X$, and that $\varphi_K = \varphi$ and $K_{\varphi_K} = K$. Therefore, the set of complements of $X$ is in bijective with $\Hom_R(Y, X)$, which in turn is in a bijection with $\Hom_R(M/X, X)$ since $M/X$ and $Y$ are isomorphic. This proves the first part of the lemma.

We now show the second part, so we assume now that $R = \mathbb{Z}_p[P]$ with $p \nmid |P|$ and that $M/X$ is finitely generated and torsion-free as a $\mathbb{Z}_p$-module. Then $M/X$ is a free $\mathbb{Z}_p$-module of finite rank, and hence it is a projective $\mathbb{Z}_p[P]$-module (here we use the fact that $p \nmid |P|$). This implies that there is at least one complement of $X$; therefore, the set of complements is in a bijection with $\Hom_{\mathbb{Z}_p[P]}(M/X, X)$, as shown in the first part of the lemma. This completes the proof. \qed

The following lemma will be used to calculate $|\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)|$ in some particular cases.

Lemma 2.3. Let $M$ be a $\Lambda_p$-module and assume that $p \nmid |P|$. If there is $\alpha \in P$ such that $C_M(\alpha) = M$ and $C_{Z_p}(\alpha) = 0$, then $\Hom_{\Lambda_p}(M, \mathbb{Z}_p/V) = \{0\}$ for all $V \leq_{\Lambda_p} \mathbb{Z}_p$. Similarly, if $C_M(\alpha) = 0$ and $C_{Z_p}(\alpha) = \mathbb{Z}_p$, then also $\Hom_{\Lambda_p}(M, \mathbb{Z}_p/V) = \{0\}$.

Proof. Since $p \nmid |P|$ and since $\mathbb{Z}_p$ is finitely generated and torsion-free as a $\mathbb{Z}_p$-module, $\mathbb{Z}_p$ is a projective $\Lambda_p$-module. Thus, the lemma in both cases follows from the equality $\Hom_{\Lambda_p}(M, \mathbb{Z}_p) = 0$, whose verification is straightforward. \qed

We now specialize to the case when $N$ has Hirsch length 3. Recall that given a finite subgroup $F \subset \GL_2(\mathbb{Z})$, either $F$ is included in $\SL_2(\mathbb{Z})$, in which case $F$ is isomorphic to one of the cyclic groups $C_1, C_2, C_3, C_4, C_6$, or else there is $\beta \in F$ of order 2 with determinant $-1$, in which case $F$ is isomorphic to one of the dihedral groups $D_1(\cong C_2), D_2(\cong C_2 \times C_2), D_3, D_4, D_6$.

Lemma 2.4. Assume that $N$ is a $\Sigma_2$-group of Hirsch length 3 (so that $T \cong \mathbb{Z}^2$ and $Z \cong \mathbb{Z}$). Set $F := \text{Im}(P \to \GL(T))$ and $\eta := \text{rank}_F(C_T(P))$. Then the following holds.

1. $C_Z(P) = Z$ if and only if $F \subset \SL(T)$.

2. $\eta = \left\{ \begin{array}{ll} 2 & \text{if } F \text{ is trivial}, \\ 1 & \text{if } F \not\subset \SL(T) \text{ and } F \cong D_1, \\ 0 & \text{otherwise}. \end{array} \right.$

3. If $p \nmid |P|$, then for any $V \leq_{\Lambda_p} \mathbb{Z}_p$ of finite index, we have $|\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)| = [\mathbb{Z}_p : V]^\eta$.

Proof. Fix an ordered basis $\{x, y\}$ for $T$. Note that $[x, y] \in Z$ is non-zero. Given $\alpha \in P$, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the matrix of $\alpha_T$ with respect to $\{x, y\}$. We have $\alpha \cdot [x, y] = [ax + cy, bx + dy] = (ad - bc)[x, y]$. It follows that $C_Z(P) = Z$ if and only if $F \subset \SL(T)$. It also follows that $Z$ is naturally an $F$-module.

We now show (2) and (3). Fix a prime $p \nmid |P|$ and a subgroup $V \leq_{\Lambda_p} \mathbb{Z}_p$ of finite index. Note that (2) and (3) hold clearly for $F = C_1$. Assume next that $F \cong D_1$ and $F \not\subset \SL(T)$. Then the generator of $F$ has eigenvalues $1$ and $-1$, whence the $\Lambda$-submodules $T_+ := C_T(P)$ and $T_- := \{ x \in T : \beta \cdot x = -x \}$ have additive rank 1. It follows that $\eta = 1$. Now, since $p \neq 2$, the decomposition $x = \frac{x + \beta x}{2} + \frac{x - \beta x}{2}$ holds in $T_p$ and yields a decomposition $T_p = (T_+) \oplus (T_-)$. Since $C_Z(F) = 0$ by (1), Lemma 2.3 implies that $|\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)| = 1$. Thus, $|\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)| = |\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)| = [\mathbb{Z}_p : V]^\eta$.

Assume now that $C_1 \neq F = \langle \alpha \rangle \subset \SL(T)$. It holds that $C_T(\alpha) = 0$, and hence $\eta = 0$, since otherwise $1$ would be an eigenvalue of $\alpha$. However, since $\alpha \in \SL(T)$ the latter implies that $\alpha$ is the identity, which is not the case. Since $C_Z(\alpha) = Z$ by (1), Lemma 2.3 implies that $|\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)| = 1 = |\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)|^\eta$.

Assume finally that $F \cong D_d$ with $d \in \{2, 3, 4\}$. The intersection $F \cap \SL(T)$ is not the trivial group. Indeed, the product of any two elements of $F$ is in $\SL(T)$ and $F$ has more than two elements. It follows from the previous case that $\eta = 0$ and $|\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)| = 1 = |\Hom_{\Lambda_p}(T_p, \mathbb{Z}_p/V)|^\eta$. We have covered all the cases, so the proof is complete. \qed

Lemma 2.5. Assume that $N$ is a $\Sigma_2$-group of Hirsch length 3. If $U \leq T_p$ has finite index, then $\left\| Z : [U, U] \right\| = [Z : [N, N]]p^{-1}[T_p : U]$.

Proof. Let $\{x, y\}$ be a basis for the $\mathbb{Z}_p$-module $T_p$, and let $U \leq T_p$ be a $\mathbb{Z}_p$-submodule of finite index, say generated by $ax + by$ and $cx + dy$. Note that $[T_p, T_p] = \mathbb{Z}_p[x, y]$ and that $[U, U] = \mathbb{Z}_p[ax + by, cx + dy] = \mathbb{Z}_p(ad -$
Thus, \([Z_p : [U, U]] = [Z_p : [T_p, T_p]]|ad - bc|^{-1}_p\). On the other hand, clearly \([T_p : U] = |ad - bc|^{-1}_p\).

Thus, \([Z_p : [U, U]] = [Z_p : [T_p, T_p]][T_p : U]\). Finally, it is clear that \([Z_p : [T_p, T_p]] = [[Z : [T, T]]|^{-1}_p = [[Z, [N, N]]]|^{-1}_p\).

\[\textbf{Theorem 2.6. Assume that } N \text{ is a } \Sigma_2\text{-group of Hirsch length } 3. \text{ Let } F := \text{Im}(P \to GL(T)) \text{ and } E = T \rtimes K, \text{ which is a plane crystallographic group. If } p \nmid |P|, \text{ then}
\]
\[
\zeta_{G_p}^{(2, \leq)}(s) = \zeta_{p}(s - \eta) \left(\zeta_{E_p}^{(2, \leq)}(s) - p^{-s+\eta+\epsilon}[Z : [N, N]]|^{s-\eta-\epsilon}\zeta_{E_p}^{(2, \leq)}(2s - \eta - \epsilon)\right),
\]

where \(\eta := \text{rank}_2(C_T(F))\), \(\epsilon \in 0 \text{ or } 1\) and \(s\) is in the set \(SL(T)\) or not, and \(\zeta_{E_p}^{(2, \leq)}(s)\) is computed with respect to \(T_p \leq E_p\).

\[\textbf{Proof. Fix } p \nmid |P|. \text{ Any finite index subgroup } V \leq Z_p \text{ is } \mathcal{P}\text{-invariant, and } [Z_p : C_Z(p) + V] \leq 1 \text{ or }
\]
\([Z_p : V] \text{ according to whether } C_Z(p) \text{ is } Z \text{ or } 0 \text{ which in turn, by Lemma 2.4(1), is translated into whether }
\]
\(F \subseteq SL(T) \text{ or not. Thus, } [Z_p : C_Z(p) + V] = [Z : [V, V]]\text{, where } \epsilon \text{ is as in the proposition. Next, by Lemma 2.4(3), } |\text{Hom}_N(T_p, Z_p/V)| = [Z_p : V]|^\eta. \text{ Therefore, by Proposition 2.1}
\]
\[
\zeta_{G_p}^{(2, \leq)}(s) = \sum_{U \leq \Lambda_p, T_p} \left[\sum_{V \leq \Lambda_p, T_p} [T_p : U]^{-\eta}s[T_p : C^*_T(P) + U][Z_p : V]^{-s+\epsilon+\eta}\right]
\]
\[
= \sum_{U \leq \Lambda_p, T_p} \left[\sum_{V \leq \Lambda_p, T_p} [T_p : U]^{-\eta}s[T_p : C^*_T(P) + U]\left(\sum_{V \leq \Lambda_p, T_p} [Z_p : V]^{-s+\epsilon+\eta} - \sum_{V \leq p(U, U)} [Z_p : V]^{-s+\epsilon+\eta}\right)\right]
\]
\[
= \sum_{U \leq \Lambda_p, T_p} \left[\zeta_{p}(s - \eta) - [Z_p : p(U, U)]^{-s+\epsilon+\eta}\zeta_{p}(s - \eta - \epsilon)\right]
\]

where, in the last equality, we used Lemma 2.4. To complete the proof we have to show that \(\sum_{U \leq \Lambda_p, T_p} [T_p : U]^{-s}[T_p : C^*_T(F) + U] = \zeta_{E_p}^{(2, \leq)}(s)\). However, this follows from [6] Proposition 2.3 and Lemma 2.4.

\[\textbf{Corollary 2.7. If } p \nmid |P|, \text{ then } \zeta_{G_p}^{(2, \leq)}(s) \text{ is given as in Table 1, where } k = [Z : [N, N]] \text{ and for } d \in \{3, 4, 6\}, \]
\[
\chi_d : N \to \mathbb{C} \text{ is the extended residue class character, } \chi_d(n) = \begin{cases} 1 & \text{if } n \equiv 1 \mod d \\ -1 & \text{if } n \equiv -1 \mod d \\ 0 & \text{otherwise,} \end{cases}
\]

\((1 - \chi_d(p)p^{-s})^{-1} \text{ is the local factor at } p \text{ of the Dirichlet } L\text{-function of } \chi_d, L(s, \chi_d) = \sum_{n=1}^{\infty} \chi_d(n)n^{-s}. \)

Therefore, if \(p \nmid |P|[[Z : [N, N]]\), then \(\zeta_{G_p}^{(2, \leq)}(s)\) satisfies the functional equation

\[\begin{aligned}
\zeta_{G_p}^{(2, \leq)}(s)_{p^{-p-1}} &= (-1)^3p^{-3s+3}\zeta_{G_p}^{(2, \leq)}(s) & \text{if } F \cong C_1, C_2, D_2; \\
\zeta_{G_p}^{(2, \leq)}(s)_{p^{-p-1}} &= (-1)^3p^{-3s+2}\chi_d(p)\zeta_{G_p}^{(2, \leq)}(s) & \text{if } F \cong C_d, d \in \{3, 4, 6\}; \\
\zeta_{G_p}^{(2, \leq)}(s)_{p^{-p-1}} &= p^{-3s+3}\zeta_{G_p}^{(2, \leq)}(s) & \text{if } F \cong D_d, d \in \{3, 4, 6\}.
\end{aligned}\]

\[\textbf{Proof. The formula for } \zeta_{G_p}^{(2, \leq)}(s) \text{ in each case follows from Theorem 2.6 and uses the formula for } \zeta_{E_p}^{(2, \leq)}(s) \text{ obtained in [13] Chap. 5} \text{ as a step in the computation of the subgroup zeta functions of the plane crystallographic group } E \text{ (see also [6] Section 4.1)). The computation of } \eta \text{ was done in Lemma 2.4(2). If in addition } p \nmid |P|[[Z : [N, N]], then } [Z : [N, N]]_{|p} = 1 \text{ and the simplification of the formula is straightforward. The functional equation follows by inspection of formula.} \]

\[\textbf{Corollary 2.8. } \zeta_{G_p}^{(2, \leq)}(s) \text{ has abscissa of convergence in the set } \{\frac{3}{2}, 2, 3\} \text{ and admits a meromorphic continuation to the whole plane. The same holds for } \zeta_{G_p}^{(3)}(s). \]
Table 1. Local factors of $\zeta_G^{G_p,\leq}(s)$ at $p \nmid |P|$ ($F := \text{Im}(F \to GL(T))$ and $k := |Z : [N, N]|$).

| $F$ | $\zeta_G^{G_p,\leq}(s)$ |
|-----|-------------------------|
| $C_1$ | $\zeta_p(s-2)\left(\zeta_p(s-1) - p^{-s+2}|k_p^{-1}|^{s-2}\zeta_p(2s-2)\zeta_p(2s-3)\right)$ p \nmid |P| |
| $C_2$ | $\zeta_p(s)\left(\zeta_p(s-1)\zeta_p(2s-2) - p^{-s+1}|k_p^{-1}|^{s-1}\zeta_p(2s-3)\right)$ p \nmid |P| |
| $C_d$ | $\zeta_p(s)\left(L_p(s-1,\chi_d)\zeta_p(s-1) - p^{-s+1}|k_p^{-1}|^{s-1}\zeta_p(2s-2)\zeta_p(2s-3)\right)$ p \nmid |P| |
| $D_1$ | $\zeta_p(s-2)\left(\zeta_p(s-1) - p^{-s+2}|k_p^{-1}|^{s-2}\zeta_p(2s-2)\zeta_p(2s-3)\right)$ p \nmid |P| |
| $D_2$ | $\zeta_p(s-1)\left(\zeta_p(s-1) - p^{-s+1}|k_p^{-1}|^{s-1}\zeta_p(2s-2)\right)$ p \nmid |P| |
| $D_d$ | $\zeta_p(s-1)\left(\zeta_p(2s-2) - p^{-s+1}|k_p^{-1}|^{s-1}\zeta_p(4s-4)\right)$ p \nmid |P| |

Proof. According to Table 1, there is a Dirichlet series $Z(s) = \prod_p Z_p(s)$ such that: (a) it has abscissa of convergence in the set $\{\frac{3}{2}, 2, 3\}$; (b) it admits a meromorphic continuation to the whole plane; (c) for any finite set of primes, say $S$, $Z(s)$ and $\prod_{p \notin S} Z_p(s)$ have the same abscissa of convergence; and (d) $Z(s)$ coincides with $\zeta_G^{G_p,\leq}(s)$ except for a finite number of local factors. Since an exceptional local factor $\zeta_G^{G_p,\leq}(s)$ is a rational function of $p^{-s}$ with rational coefficients $[H]$, it follows from (b) and (d) that $\zeta_G^{G_p,\leq}(s)$ also has a meromorphic continuation to the whole plane. Moreover, $\zeta_G^{G_p,\leq}(s)$ and $Z(s)$ have the same abscissa of convergence. Indeed, by (c) and (d), it is enough to show that the abscissa of convergence of each local factor of $\zeta_G^{G_p,\leq}(s)$ is strictly less than the abscissa of convergence of $\zeta_G^{G_p,\leq}(s)$. Now, this follows from the fact that $\zeta_G^{G_p,\leq}(s)$ is an Euler product of cone integrals $[P]$ Theorem 1. This important property about cone integrals was established in $[5]$ Section 4], and it was a key point in obtaining analytic properties of global zeta functions.

To prove the last assertion of the corollary, we apply the first part to each of the partial zeta functions of $\zeta_G^{G_p,\leq}(s)$. □

Remark 2.9. If $F = C_1$ in Table 1 we recover $[5]$ Proposition 8.1]. See also $[H]$ for a generalization.

2.2. Counting normal subgroups. We begin with some preliminaries from elementary group theory. Let $G$ be for the moment any group, and let $N \triangleleft G$ be a normal subgroup. We define inductively a series of normal subgroups $\gamma_i(G, N) \supseteq \gamma_{i+1}(G, N)$ by setting

\[ \gamma_i(G, N) := N \text{ and } \gamma_{i+1}(G, N) := [\gamma_{i+1}(G, N), G], \text{ for } i \geq 2. \]

Lemma 2.10. Assume that $G$ is finite, and that $N$ is a normal Hall subgroup. If $\gamma_{c+1}(G, N) = 1$ for some $c$, then $N$ has a unique complement, say $C$. In addition, $G = N \times C$ and $\gamma_{c+1}(G) \cap N = \{1\}$. 

Proof. By the Schur-Zassenhaus theorem, there is at least one complement of $N$ in $G$, and they are all conjugated. We prove that there is only one by induction on $c$.

Assume that $c = 1$, and let $C$ be a complement of $N$ in $G$. Since $[N, C] \subseteq [N, G] = \gamma_2(G, N) = 1$, it follows that $C$ is normal in $NC = G$, and therefore, $C$ is the unique complement of $N$.

Assume now that $c > 1$. By the inductive hypothesis, there is a unique complement, say $C' \gamma \gamma_0(G, N)$, of $N/\gamma_0(G, N)$ in $G/\gamma_0(G, N)$. Now, given a complement $C$ of $N$ in $G$, clearly $(C\gamma_0(G, N))/\gamma_0(G, N)$ is a complement of $N/\gamma_0(G, N)$ in $G/\gamma_0(G, N)$, thus $C\gamma_0(G, N) = C'$. This implies that $C$ is a complement of $\gamma_0(G, N)$ in $C'$. However, $[\gamma_0(G, N), C'] \subseteq \gamma_0+1(G, N) = 1$, so by the case $c = 1$, there is only one possibility for $C$. This completes the induction.

The final part of the lemma is clear. □

Lemma 2.11. Assume that $N$ is nilpotent, say of class $c$, and let $A \lhd G$ such that $AN = G$. Then $\gamma_{c+1}(G) \subseteq A$; in particular, $\gamma_{c+1}(G, N) \subseteq A \cap N$.

Proof. Note that $B := A \cap N$ is also normal in $G$, and there is an identification $G/B = N/B \times A/B$. It follows that $\gamma_{c+1}(G/B) = \gamma_{c+1}(N/B) \times \gamma_{c+1}(A/B) = 1 \times \gamma_{c+1}(A/B) \subseteq A/B$. Since $\gamma_{c+1}(G/B) = (\gamma_{c+1}(G)B)/B$, we deduce that $\gamma_{c+1}(G) \subseteq A$. In particular, $\gamma_{c+1}(G, N) \subseteq \gamma_{c+1}(G) \cap N \subseteq A \cap N$. □

We return to the setting introduced at the beginning of the section except that we do not assume that the nilpotency class is 2 yet. We fix an intermediate normal subgroup $N \leq H \lhd G$ and consider the series $\zeta_{G, N}(s)$.

Note that $\gamma_i(H, N)$ is normal in $G$ for all $i$, and that $(\gamma_i(H, N))_p = \gamma_i(H_p, N_p)$ and $(\gamma_i(H))_p = \gamma_i(H_p)$ (cf. [14, Theorem 1.4]).

Proposition 2.12. Let $c$ be the nilpotency class of $N$. Set $G' := G/\gamma_{c+1}(H, N)$ and $N' := N/\gamma_{c+1}(H, N)$.

If $p \nmid |H/N|$, then $\zeta_{G, N}(s) = \zeta_{G, N}(s)^{N_p, c}(s)$. In particular, if $N_p \cong \mathbb{Z}_p/k\mathbb{Z}_p$ for some $k \in \mathbb{Z}$, then $\zeta_{G, N}(s) = \frac{1-p^{-e}|k|^s}{1-p^{-s}}$.

Proof. Let $H' := H/\gamma_{c+1}(H, N)$. By Lemma 2.11, we have $\zeta_{G, N}(s) = \zeta_{G', N}(s)$, where the series on the right is computed with respect to $N_p \lhd G'$ (this holds for all $p$). Assume now that $p \nmid |H/N|$. Given $A' \leq H'$ of finite index and normal in $G'$ such that $A'N_p = H'_p$, the intersection $B' := N'_p \cap A'$ is normal in $G'_p$ and $[N'_p : B'] = [H'_p : A']$. Conversely, given $B' \leq N'_p$ of finite index and normal in $G'_p$, we have $\gamma_{c+1}(H'_p, B', N'_p, B') = \gamma_{c+1}(H'_p, N'_p, B') = 1$ since $\gamma_{c+1}(H'_p, N'_p) = 1$. Therefore, we can apply Lemma 2.10 to $N'_p / B' \leq H'_p / B'$. It follows that there is a unique $A' \leq H'_p$ such that $A'N_p = H'_p$ and $A' \cap N'_p = B'$.

By the uniqueness, $A'/B'$ is normal in $G'_p / B'$, and hence $A'$ is normal in $G'_p$. We deduce that $\zeta_{G, N}(s) = \zeta_{G, N}(s)^{N_p, c}(s)$. This proves the first part of the lemma.

If in addition $N'' \cong \mathbb{Z}_p / k\mathbb{Z}_p$, then every finite index subgroup of $N''$ is characteristic; therefore, $\zeta_{G, N}(s) = \zeta_{\mathbb{Z}_p / k\mathbb{Z}_p}(s)$, which is clearly equal to $\frac{1-p^{-e}|k|^s}{1-p^{-s}}$. □

Corollary 2.13. $\zeta_{G}(s)$ and $\zeta_{G, N}(s)$ have the same abscissa of convergence.

Proof. We have to show that for each intermediate normal subgroup $N \leq H \lhd G$, the abscissa of convergence of $\zeta_{G, N}(s)$ is bounded by that of $\zeta_{G}(s)$. We set $G' := G/\sqrt{\gamma_{c+1}(H, N)}$ and $N' := N/\sqrt{\gamma_{c+1}(H, N)}$, where $\sqrt{\gamma_{c+1}(H, N)} := \{ x \in N : x^n \in \gamma_{c+1}(H, N) \text{ for some } n \in \mathbb{N} \}$. By Proposition 2.12, $\zeta_{G'}(s)$ and $\zeta_{G', N'}(s)$ have the same but a finite number of local factors. By [22, Theorem 1], both series are Euler products of cone integrals. As explained in the proof of Corollary 2.3, this suffices to ensure that $\zeta_{G'}(s)$ and $\zeta_{G', N'}(s)$ have the same abscissa of convergence. Finally, it is clear that the abscissa of convergence of $\zeta_{G', N'}(s)$ is bounded by that of $\zeta_{G}(s)$. □

Remark 2.14. We claim that if $p \nmid |H/N|$, then $\gamma_{c+1}(H_p, N_p) = N_p \cap \gamma_{c+1}(H_p)$; in particular, if $H_p / N_p$ is nilpotent of class $\leq c$, then $\gamma_{c+1}(H_p, N_p) = \gamma_{c+1}(H_p)$. Indeed, let $B \lhd G_p$ be a finite index normal subgroup such that $\gamma_{c+1}(H_p, N_p) \subseteq B \subseteq N_p$. Since $N_p / B$ is a normal Hall subgroup of $H_p / B$, and since $\gamma_{c+1}(H_p / B, N_p / B) = (\gamma_{c+1}(H_p, N_p)B) / B = 1$, we obtain by Lemma 2.10 that $\gamma_{c+1}(H_p / B) \cap N_p / B = 1$. It
follows that \((\gamma_{c+1}(H_p)B) \cap N_p \subseteq B\), and hence \(\gamma_{c+1}(H_p) \cap N_p \subseteq B\). Since \(N_p/\gamma_{c+1}(H_p, N_p)\) is residually finite, the obvious inclusion \(\gamma_{c+1}(H_p, N_p) \subseteq \gamma_{c+1}(H_p) \cap N_p\) must be an equality.

We now return to the case when \(N\) is a \(\mathbb{T}_2\)-group. Recall the notation introduced at the beginning of the section. Given \(V \leq Z_p\) of finite index, we denote by \(X(V) \leq T_p\) the largest subgroup such that \([T_p, X(V)] \subseteq V\). It has finite index in \(T_p\), and if \(V\) is in addition \(P\)-invariant, then \(X(V)\) is also \(P\)-invariant. Proposition 2.12 allows us to focus only on the first partial zeta function \(\zeta^{G^{'}, \alpha}(s) = \sum_{B \leq G, B \leq N}[N : B]^{-s}\), the other ones being of the form \(\zeta_{N^{'}, \alpha}(s)\) except for a finite number of local factors, where \(G^{'}\) is a quotient of \(G\).

**Proposition 2.15.** Fix a prime \(p\). Assume that all finite index \(\Lambda_p\)-submodules of \(T_p\) are isomorphic. Assume also that for every \(V \leq \Lambda_p, Z_p\) and \(V \leq \Lambda_p, T_p\) of finite index such that \(U \subseteq X(V)\), there exists \(B \trianglelefteq G_p\) of finite index such that \(B \leq N_p, (BZ_p)/Z_p = U\) and \(B \cap Z_p = V\). Then it holds that

\[
\zeta_{N^{'}, \alpha}(s) = \zeta_{E_p, \alpha}(s) \left( \sum_{V \leq \Lambda_p, Z_p} [T_p : X(V)]^{-s}[Z_p : V]^{-s} \right) \text{Hom}_{\Lambda_p}(T_p, Z_p/V)) \right).
\]

The above assumptions hold if \(p \nmid |P|\).

**Proof.** The proof is similar to that of [8 Lemma 6.1], so we omit it. We just point out that given \(U \subseteq \Lambda_p, T_p\) and \(V \leq \Lambda_p, Z_p\) of finite index with \(U \subseteq X(V)\), the number of normal subgroups \(B \trianglelefteq G_p\) such that \((BZ_p)/Z_p = U\) and \(B \cap Z_p = V\) is equal to \(\text{Hom}_{\Lambda_p}(U, Z_p/V)\). This uses the second assumption and Lemma 2.2. In addition, by the first assumption, \(\text{Hom}_{\Lambda_p}(U_p/V) = \text{Hom}_{\Lambda_p}(T_p/V)\).

The first assumption is satisfied when \(p \nmid |P|\) since in this case \(\Lambda_p\) is a maximal order of \(\mathbb{Q}_p[P]\), and the second assumption is satisfied by Lemma 2.2.

We now specialize to the case of Hirsch length 3.

**Lemma 2.16.** Assume that \(N\) is a \(\mathbb{T}_2\)-group of Hirsch length 3. Given \(V \leq Z_p\) of finite index, the following holds:

1. If \([T_p, T_p] \subseteq V\), then \([T_p : X(V)] = 1\).
2. If \(V \subseteq [T_p, T_p]\), then \([T_p : X(V)] = [Z : [N, N]]_p/Z_p/V]^2\).

**Proof.** The first assertion is trivial since in this case \(X(V) = T_p\). Assume now that \(V \subseteq [T_p, T_p]\). Let \((x, y)\) be a basis for \(T_p\). Given a \(\mathbb{Z}_p\)-submodule \(X \leq T_p\) of finite index, say generated by \(ax + by\) and \(cx + dy\), we have \([X, T_p] = \{a[x, y], b[x, y], c[x, y], d[x, y]\}\). Thus, \([X, T_p] \subseteq V\) if and only if \(a[x, y], b[x, y], c[x, y], d[x, y] \in V\). Since \([x, y]\) generates \([T_p, T_p]\), the previous condition holds if and only if \(a, b, c, d \in \frac{[Z : [N, N]]_p}{[Z_p : [T_p, T_p]]}\), which in turn is equivalent to saying that \(X \leq \frac{[Z_p : V]}{[Z_p : [T_p, T_p]]}T_p\). Thus, \(X(V) = \frac{[Z_p : V]}{[Z_p : [T_p, T_p]]}T_p\). The index of this subgroup is \([Z_p : V]^2[Z_p : [T_p, T_p]]^2 = [Z : [N, N]]_p/Z_p/V^2\).

**Theorem 2.17.** Assume that \(N\) is a \(\mathbb{T}_2\)-group of Hirsch length 3. Set \(F := \text{Im}(P \rightarrow GL(T))\) and \(E := T \times F\), which is a plane crystallographic group. If \(p \nmid |P|\), then

\[
\zeta_{N^{'}, \alpha}(s) = \zeta_{E_p, \alpha}(s) \left( \frac{1 - [Z : [N, N]]_p^{s-\eta}}{1 - p^{-s+\eta}} + [Z : [N, N]]_p^{s-\eta} \zeta_p(3s - \eta) \right),
\]

where \(\eta := \text{rank}_Z(C_T(P))\). In particular, if \(p \nmid |P||Z : [N, N]|\), then

\[
\zeta_{N^{'}, \alpha}(s) = \zeta_{E_p, \alpha}(s) \zeta_p(3s - \eta).
\]
Proof. By Lemma 2.3, \(|\text{Hom}_{\Lambda}(T_p, Z_p/V)| = [Z_p : V]|^\eta\) for any \(V \leq \Lambda_p\) \(Z_p\), where \(\eta = \text{rank}_2(C_T(P))\). It follows from Proposition 2.15 and Lemma 2.16 that

\[
\zeta_{G_p}^{N_p,<}(s) = \zeta_{E_p}^{T_p,<}(s) \left( \sum_{[T_p, T_p] \supset V \leq Z_p} [Z_p : V]^{-s+\eta} + \sum_{V \leq [T_p, T_p]} ||[Z : [N, N]]||_{p}^{-2s}[Z_p : V]^{-3s+\eta} \right)
\]

\[
= \zeta_{E_p}^{T_p,<}(s) \left( \sum_{k=0}^{\nu_p([T_p, T_p])-1} p^{(-s+\eta)k} + \sum_{V \leq [T_p, T_p]} ||[Z : [N, N]]||_{p}^{-2s}[Z_p : T_p]^{-3s+\eta}[[T_p, T_p] : V]^{-3s+\eta} \right)
\]

\[
= \zeta_{E_p}^{T_p,<}(s) \left( \frac{1 - ||[Z : [N, N]]||_{p}^{-\eta} + ||[Z : [N, N]]||_{p}^{-\eta} \gamma_p(3s - \eta)}{1 - p^{-s+\eta}} \right).
\]

If in addition \(p \mid [Z : [N, N]]\), then \([Z : [N, N]]_{p} = 1\), and the proof follows. \(\square\)

Corollary 2.18. If \(p \mid |P|\), then \(\zeta_{G_p}^{N_p,<}(s)\) is given as in Table 2, where \(k := [Z : [N, N]]\). Therefore, if \(p \mid |P|[Z : [N, N]],\) then \(\zeta_{G_p}^{N_p,<}(s)\) satisfies the following functional equation:

\[
\zeta_{G_p}^{N_p,<}(s)|_{p^{-2}} = \left( -1 \right)^{3} p^{-5s+3} \zeta_{G_p}^{N_p,<}(s)
\]

if \(F = C_1\),

\[
\zeta_{G_p}^{N_p,<}(s)|_{p^{-2}} = \left( -1 \right)^{3} p^{-5s+1} \zeta_{G_p}^{N_p,<}(s)
\]

if \(F \equiv C_2 = D_1\),

\[
\zeta_{G_p}^{N_p,<}(s)|_{p^{-2}} = \left( -1 \right)^{3} \chi_d(p) p^{-5s} \zeta_{G_p}^{N_p,<}(s)
\]

if \(F \equiv C_d, d \in \{3, 4, 6\}\),

\[
\zeta_{G_p}^{N_p,<}(s)|_{p^{-2}} = \left( -1 \right)^{3} p^{-5s} \zeta_{G_p}^{N_p,<}(s)
\]

if \(F \equiv D_2\),

\[
\zeta_{G_p}^{N_p,<}(s)|_{p^{-2}} = p^{-5s} \zeta_{G_p}^{N_p,<}(s)
\]

if \(F \equiv D_d, d \in \{3, 4, 6\}\).

Proof. This follows from Theorem 2.15 and uses the formula for \(\zeta_{E_p}^{T_p,<}(s)\) obtained in [13] Chap. 6) as a step in the computation of the normal zeta function of the plane crystallographic group \(E\) (see also [6] Section 4.2). The calculation of \(\eta\) was done in Lemma 2.3. The local functional equation follows by inspection of the formula. \(\square\)

Corollary 2.19. \(\zeta_{G}^{N,<}(s)\) has abscissa of convergence in the set \(\{1, 2\}\) and admits a meromorphic continuation to the whole plane. The same holds for \(\zeta_{G}^{H,<}(s)\).

Proof. The assertion about \(\zeta_{G}^{N,<}(s)\) follows by inspection of Table 2 and the same argument used in the proof of Corollary 2.18. Next, by Corollary 2.13, \(\zeta_{G}^{H,<}(s)\) has the same abscissa of convergence as \(\zeta_{G}^{N,<}(s)\). To complete the proof, it is enough to show that for each intermediate normal subgroup \(N \leq H \triangleleft G\) different from \(N\), the series \(\zeta_{G}^{H,<}(s)\) admits a meromorphic continuation to the whole plane. We use the notation from the proof of Corollary 2.13 with \(c = 2\). Since the local factor of \(\zeta_{G}^{H,<}(s)\) at a prime \(p\) is a rational function in \(p^{-s}\) by [3], it is enough to show that \(\zeta_{G}^{H,<}(s)\) admits a meromorphic continuation to the whole plane. If \(N' = N\), then we are in the same situation as in the case of \(\zeta_{G}^{N}(s)\), so we are done. If \(N' \neq N\), then either \(N' \cong \mathbb{Z}^2\), in which case the result follows from the formulae of the normal zeta functions of the plane crystallographic groups given in [13] Chap. 6), or else \(N' \cong \mathbb{Z}\) or \(N' = 0\), in which case the result is trivial. \(\square\)

2.3. Remark on the local functional equations. One natural problem is to decide whether local functional equations hold for \(\zeta_{G}^{H,<}(s)\) and \(\zeta_{G}^{N,<}(s)\) when the Hirsch length of \(N\) is higher than 3. When \(N\) is abelian, this follows from the explicit formulae obtained in [6] Section 2. In contrast to the situation of zeta functions of nilpotent groups, here the local functional equations are not uniform in \(p\). They depend on how \(p\) ramifies in certain number fields that arise in the decomposition of \(\mathbb{Q}[G/N]\) into simple algebras. Propositions 2.11 and 2.14 enable us to linearize the problem when the nilpotency class is 2 (indeed, the linearization for \(\zeta_{G}^{N,<}(s)\) holds more generally by the Mal’cev correspondence). We will not go into details. Briefly, the problem can be stated as follows. Let \(L\) be a 2-step nilpotent Lie ring additively isomorphic to \(\mathbb{Z}^n\), and let \(Z\) denote its center. Let \(P\) be a finite group acting on \(L\) by Lie ring automorphisms. We consider
Let $N$ be a torsion-free finitely generated nilpotent pro-$p$ group and let $G_p$ be a profinite group that includes $N_p$ as an open normal subgroup. Fix also an intermediate normal subgroup $N_p \leqslant H_p \lhd G_p$. We review the Dirichlet series

$$\zeta_p(N_p, s) = \sum [L : B]^{-s} |\text{Der}(P, L/B)| \quad \text{and} \quad \zeta_p^{\infty}(N_p, s) = \sum [L : B]^{-s},$$

where the first sum runs over the $P$-invariant finite index subrings, and the second one only over the $P$-invariant finite index ideals. Corollaries 2.7 and 2.18 imply that these series satisfy local functional equations when $L$ has additive rank 3. In analogy with the results of [23], we might ask:

**Question 1.** Do $\zeta_p^{\infty}(N_p, s)$ and $\zeta_p(N_p, s)$ satisfy local functional equations when $L$ is a 2-step nilpotent Lie ring of rank $> 3$? Does $\zeta_p^{\infty}(N_p, s)$ satisfy local functional equations for arbitrary rings (not necessarily nilpotent) that are additively isomorphic to some $\mathbb{Z}^h$?

We return to the case when $L$ is a 2-step nilpotent Lie ring additively isomorphic to $\mathbb{Z}^3$. It is not difficult to check, say by inspection of Table 1 and the results of [23 Section 2], that if we forget the structure of Lie ring on $L$ (obtaining $\mathbb{Z}^3$) and consider just $\zeta_p^{\infty}(N_p, s)$, then $\zeta_p^{\infty}(N_p, s)$ and $\zeta_p^{\infty}(N_p, s)$ satisfy the same local functional equations. By this, and again in analogy with (N4), we can also ask:

**Question 2.** Let $L$ be a ring additively isomorphic to $\mathbb{Z}^h$. Let $P$ be a finite group acting on $L$. If $\zeta_p^{\infty}(N_p, s)$ satisfies local functional equations, are these equations the same as those satisfied by $\zeta_p^{\infty}(N_p, s)$?

In a forthcoming paper, we show that Question 1 and Question 2 have positive answer for various nilpotent Lie rings of rank 4.

### 3. Local zeta functions as $p$-adic integrals

Let $N_p$ be a torsion-free finitely generated nilpotent pro-$p$ group and let $G_p$ be a profinite group that includes $N_p$ as an open normal subgroup. Fix also an intermediate normal subgroup $N_p \leqslant H_p \lhd G_p$. We review
a method, developed in [8] Section 2] for Σ-groups and extended to virtually nilpotent groups in [22], to express $\zeta_{H_{p_{<}}}^{\ast}(s)$ and $\zeta_{G_{p_{<}}}^{\ast}(s)$ as $p$-adic integrals. This method is used in the next sections to calculate local factors at "bad" primes in some cases.

3.1. Expressing $\zeta_{N_{p}}^{\ast}(s)$ and $\zeta_{G_{p}}^{\ast}(s)$ as $p$-adic integrals. An additional reference for this part is [12 Chap. 15], where the notation is more adapted to ours. Fix a Mal’cev basis $x = (x_1, \ldots, x_h)$ for $N_p$. By definition, the series of subgroups

$$N_p = \langle x_1, \ldots, x_h \rangle \supset \langle x_2, \ldots, x_h \rangle \supset \cdots \supset \langle x_h \rangle$$

is central, and each $x \in N_p$ determines a unique vector $a = (a_1, \ldots, a_h) \in \mathbb{Z}_{p}^h$ such that $x = x^{a_1} \cdots x^{a_h}$.

Let $T_h(Z_p)$ denote the set of $h \times h$-upper-triangular matrices with entries in $\mathbb{Z}_p$, and let $T_h^{+}(Z_p) \subset T_h(Z_p)$ denote the subset of those matrices with non-zero determinant. For $t \in T_h(Z_p)$, we set

$$B_t := \langle x^{t_1}, \ldots, x^{t_h} \rangle \leq N_p,$$

where $t_i$ denotes the $i$-th row of $t$. The subgroup $B_t$ is open if and only if $t \in T_h^{+}(Z_p)$.

Given an open subgroup $B \leq N_p$, we say that $t \in T_h^{+}(Z_p)$ represents a good basis for $B$ (with respect to $x$) if $B = B_t$ and $B = \{ (x^{t_1})^{\lambda_1} \cdots (x^{t_h})^{\lambda_h} : \lambda_1, \ldots, \lambda_h \in \mathbb{Z}_p \}$. In this case, $(x^{t_1}, \ldots, x^{t_h})$ is a Mal’cev basis for $B$ (also called a good basis for $B$). We set

$$\mathcal{M}(B) := \{ t \in T_h(Z_p) : t \text{ represents a good basis for } B \}.$$

Note that given $t = (t_{ij}) \in \mathcal{M}(B)$, the value $|t_{ii}|_p$ depends only on $B$ (and on the fixed Mal’cev basis $x$). Indeed, we have $|t_{ii}|_p^{-1} = |(x_{i_1}, \ldots, x_{i_h}) : B \cap \langle x_{i_1}, \ldots, x_{i_h} \rangle |$. We also see from this that

$$(3.1) \quad |N_p : B|^{-1} = \prod_{i=1}^{h} |t_{ii}|_p.$$

We collect results from [8] Section 2] in the next lemma. The topological group $T_h(Z_p) \cong \mathbb{Z}_p^{(h+1)/2}$ has a normalized Haar measure, which we denote by $\mu$.

**Lemma 3.1.** $\mathcal{M}(B)$ is an open subset of $T_h(Z_p)$, and for any $t = (t_{ij}) \in \mathcal{M}(B)$

$$(3.2) \quad \mu(\mathcal{M}(B)) = (1 - p^{-1})^h \prod_{i=1}^{h} |t_{ii}|_p.$$

Therefore, for a complex variable $s$,

$$(3.3) \quad |N : B|^{-s} = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}(B)} \prod_{i=1}^{h} |t_{ii}|_p^{s-1} d\mu.$$

**Proof.** The first part of the lemma is proved in [8 Lemma 2.5]; see also [12 Lemma 15.1.1]. We recall here the proof of [33]. We start from the right-hand side and use (3.1) and (3.2):

$$\frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}(B)} \prod_{i=1}^{h} |t_{ii}|_p^{s-1} d\mu = \frac{1}{(1 - p^{-1})^h} \left( \prod_{i=1}^{h} |t_{ii}|_p^{s-1} \right) \mu(\mathcal{M}(B)) = \prod_{i=1}^{h} |t_{ii}|_p^{s-1} \prod_{i=1}^{h} |t_{ii}|_p = |N_p : B|^{-s}. \tag{3.3}$$

We now set $\mathcal{M}_N := \cup_{B \leq N} \mathcal{M}(B)$ and $\mathcal{M}_N^+ := \cup_{B^{+} \leq N} \mathcal{M}(B)$, where the unions run only over open subgroups. These are open subsets of $T_h^{+}(Z_p)$, by Lemma 3.1 They both coincide with $T_h^{+}(Z_p)$ if $N$ is abelian. In the general case, the following descriptions were given in [8] Lemmas 2.3 and 2.4:

$$(3.4) \quad \mathcal{M}_{N_p} = \{ t \in T_h^{+}(Z_p) : [x^{t_i}, x^{t_j}] \in \langle x^{t_{j+1}}, \ldots, x^{t_h} \rangle, \ 1 \leq i < j \leq h \},$$

$$\mathcal{M}_{N_p}^+ = \{ t \in T_h^{+}(Z_p) : [x_i, x^{t_j}] \in \langle x^{t_{j+1}}, \ldots, x^{t_h} \rangle, \ 1 \leq i, j \leq h \}.$$ As an immediate consequence of formula (3.3), we obtain the following corollary.

**Corollary 3.2** (Proposition 2.7, [8]). For $s \in \{ \leq, < \}$,

$$\zeta_{N_p}^{\ast}(s) = \frac{1}{(1 - p^{-1})^h} \int_{\mathcal{M}_{N_p}^+} \prod_{i=1}^{h} |t_{ii}|_p^{s-1} d\mu.$$
3.2. Expressing $\zeta_{H_p}^{N_p}(s)$ and $\zeta_{G_p}^{N_p}(s)$ as $p$-adic integrals. We begin with the following particular case, which is an immediate consequence of (3.3).

**Corollary 3.3.** Let $\alpha_1, \ldots, \alpha_r \in G_p$ such that their classes modulo $N_p$ generate $G_p/N_p$. Then

$$
\zeta_{G_p}^{N_p}(s) = \frac{1}{1 - p^{-1}} \int_{t \in M_{N_p}^h} \prod_{i=1}^h |t_{ii}|^{s-1} \, d\mu.
$$

It remains to consider the case $H_p \neq N_p$. We set $F := H_p/N_p$ and denote by $e$ its identity element. The following is fixed in the rest of this section.

- A presentation $\langle f_1, \ldots, f_r | R_\lambda(f_1, \ldots, f_r) = e, \lambda \in J \rangle$ for the group $F$ ($f_j \neq e, \forall j$).
- Group words $w_f(X_1, \ldots, X_r), f \in F$, such that $f = w_f(f_1, \ldots, f_r)$. In the case $f = f_j$ ($j = 1, \ldots, r$), we just set $w_f = X_j$, and in the case $f = e$, we set $w_e = e$, the empty word.
- A transversal $\{ \beta_f : f \in F \}$ to the cosets of $N_p$ in $H_p$ such that $\beta_f N_p = f$ for all $f \in F$ and $\beta_e = 1$ (the identity of $N_p$). We denote $\beta := \beta_e$ (this is why we denoted the identity of $F$ by $e$ and not by 1).
- Elements $\alpha_1, \ldots, \alpha_r \in G_p$ whose classes modulo $N_p$ generate $G_p/N_p$. Assume that the first $t$ of them ($t \leq s$), $\alpha_1, \ldots, \alpha_t$, generate $G_p$ modulo $H_p$.

Given $A \leq H_p$ of finite index such that $AN_p = H_p$, the intersection $A \cap N_p \leq N_p$ is open and $A$ can be written as

$$
A = (A \cap N_p) \bigcup \left( \bigcup_{f \in F \setminus \{e\}} \beta_f n_f (A \cap N_p) \right)
$$

for some $n_f \in N_p$. Now, given an open subgroup $B \leq N_p$ and $n_f \in N_p$ for $f \in F \setminus \{e\}$, Lemma 3.4 below establishes necessary and sufficient conditions on the elements $n_f$ for the set

$$
A(B, (n_f)_{f \in F \setminus \{e\}}) := B \bigcup \left( \bigcup_{f \in F \setminus \{e\}} \beta_f n_f B \right)
$$

to be a subgroup of $G_p$. In this case, necessarily $A(B, (n_f)_{f \in F \setminus \{e\}})N_p = H_p$.

**Lemma 3.4.** Fix an open subgroup $B \leq N_p$ and $n_f \in N_p$ for each $f \in F \setminus \{e\}$. Let $n_f := n_{f_j}$. Then the set $A := A(B, (n_f)_{f \in F \setminus \{e\}})$ is a subgroup of $G_p$ if and only if the following three conditions are satisfied.

1. $\beta^{n_f} B \leq B$ for $1 \leq j \leq r$.
2. $B_{\alpha_i} \beta^{n_f} B$ for all $\lambda \in J$.
3. $\beta^{n_f} B \leq w_f(\beta^{n_{f_1}}, \ldots, \beta^{n_{f_r}}) B$ for all $f \in F \setminus \{e, f_1, \ldots, f_r\}$.

Moreover, $A$ is a normal subgroup of $G_p$ if and only if (2), (3), and the following four conditions are satisfied.

4. $B$ is normal in $N_p$.
5. $\alpha^k B \leq B$ for $1 \leq k \leq s$.
6. $[\beta^{n_{j_1}} B \leq B$ for $1 \leq j \leq r$ and $1 \leq i \leq h$.
7. $[\alpha_k, \beta^{n_{j_1}}] \leq w^{(\bar{\alpha}_k, f_j)(\beta^{n_{j_1}}, \ldots, \beta^{n_{j_r}})} B$ for $1 \leq k \leq t$ and $1 \leq j \leq r$, where $\bar{\alpha}_k$ denotes the class of $\alpha_k$ at $G_p/N_p$.

**Proof.** Assume first that $A$ is a subgroup of $G_p$. Then clearly $B = A \cap N_p$, so $B$ is normal in $A$; in particular, (1) holds. Note that $R_\lambda(\beta^{n_{j_1}}, \ldots, \beta^{n_{j_r}}) N_p = R_\lambda(f_1, \ldots, f_r) = e \in F$, and hence $R_\lambda(\beta^{n_{j_1}}, \ldots, \beta^{n_{j_r}}) \in A \cap N_p = B$; thus, (2) holds. Similarly, $\beta^{n_f} B$ and $w_f(\beta^{n_{f_1}}, \ldots, \beta^{n_{f_r}})$ have the same image at $F$, and hence (3) also holds.

Conversely, assume that the conditions (1), (2), and (3) are satisfied for the collection $(n_f)_{f \in F \setminus \{e\}}$. Condition (1) implies that $\beta^{n_f} B \in N_{H_p}(B)$, the normalizer of $B$ in $H_p$, and then (3) implies that $\beta^{n_f} B \in N_{H_p}(B)$ for all $f \in F \setminus \{e\}$. It follows that $A \subset N_{H_p}(B)$. Let $A'/B$ denote the subgroup of $N_{H_p}(B)/B$ generated by $A/B$. By (3), $A'/B$ is generated by the classes of $\beta^{n_{j_1}}, \ldots, \beta^{n_{j_r}}$, and by (2), these generators satisfy the relations defining the presentation of $F$. It follows that $|A'/B| \leq |F|$. On the other hand, $|A/B|$ has exactly $|F|$ elements. Thus, $A/B = A'/B$, whence $A = A'$. We conclude that $A$ is a subgroup of $G_p$. This completes the proof of the first part of the lemma.

Assume now that $A$ is a normal subgroup of $G_p$. According to the first part of the lemma, (2) and (3) are satisfied. Note that (4) and (5) also hold since $B = A \cap N_p$ is also normal in $G_p$. Now, the normality of
A implies that $[\beta, n_j, x_i] \in A$, and the normality of $N_p$ that $[\beta_j n_j, x_i] \in N_p$; thus $[\beta_j n_j, x_i] \in B = A \cap N_p$, whence (6) is also satisfied. Finally, $(w_{\alpha, f_j})^{-1}(\beta_1 n_1, \ldots, \beta_j n_j)\alpha_k, \beta_j n_j$ is in $A$ since $A$ is normal, and it is in $N_p$ since its image at $G_p/N_p$ is the identity. This shows that (7) also holds.

Conversely, assume that (2) to (7) are satisfied. Note that (4) and (5) imply (1); hence, according to the first part of the lemma, $A$ is a subgroup. They also imply that $B$ is normal in $G_p$. We saw in the second paragraph of the proof that $A/B$ is generated by the classes of the elements $\beta_j n_j$; thus, by (6), $N_p/B$ is included in $N_{G_p/B}(A/B)$. It follows that $A/B$ is normal in $H_p/B$. Finally, by (7), $A/B$ is normal in $G_p/B$, and hence $A$ is normal in $G_p$. This completes the proof of the lemma. □

We set more notation for the next proposition. We denote by $M_{k \times h}(Z_p)$ the set of $k \times h$-matrices with entries in $Z_p$. Given $v = (v_{ij}) \in M_{k \times h}(Z_p)$, its $i$-th row vector $(v_{1, i}, \ldots, v_{h, i}) \in Z_p^h$ is denoted by $v_i$. The normalized Haar measure of $T_h(Z_p) \times M_{k \times h}(Z_p) \cong Z_p^{kh+h(h+1)/2}$ is denoted by $\mu$. Elements of $T_h(Z_p) \times M_{k \times h}(Z_p)$ are denoted $(t, v)$.

**Proposition 3.5.** It holds:

$$\zeta_{H_p}^{H_p, \preceq}(s) = \frac{1}{(1-p^{-1})^h} \int_{T_{H_p, \preceq}^H} \prod_{i=1}^h |t_{ij}|^{s-i} \, d\mu \quad \text{and} \quad \zeta_{G_p}^{H_p, \preceq}(s) = \frac{1}{(1-p^{-1})^h} \int_{T_{G_p, \preceq}^H} \prod_{i=1}^h |t_{ij}|^{s-i} \, d\mu,$$

where $T_{H_p, \preceq}^H \subset T_h(Z_p) \times M_r(Z_p)$ is the set of pairs $(t, v)$ such that $t \in M_{N_p}^c$ and

$$\beta_j[x^{v_1}, x^{v_2}] \in [\beta_j x^{v_1}, x^{v_2}], \quad 1 \leq i \leq h, \quad 1 \leq j \leq r, \quad \lambda \in J,$$

and $T_{G_p, \preceq}^H \subset T_h(Z_p) \times M_r(Z_p)$ is the set of pairs $(t, v)$ such that $t \in M_{N_p}^c$ and

$$\alpha_k(x^{v_1}, \ldots, x^{v_r}), \quad \beta_j[x^{v_1}, x^{v_2}]^t \in [\beta_j x^{v_1}, x^{v_2}], \quad \beta_j[x^{v_1}, x^{v_2}]^t \in [\beta_j x^{v_1}, x^{v_2}], \quad 1 \leq k \leq s, \quad 1 \leq i \leq h, \quad \lambda \in J, \quad 1 \leq j \leq r, \quad 1 \leq i \leq t.$$

**Proof.** Throughout the proof, we shall refer repeatedly to conditions (1)-(7) of Lemma 3.3. We will also use the same notation. The first part of this lemma establishes a bijection between the family $F_{H_p, \preceq}^H$ of open subgroups $A \leq G_p$ such that $AN_p = H_p$ and the family of sequences $(n_f B)_{f \in F}$, where $B$ is an open subgroup of $N_p$ and the $n_f$’s are elements of $N$ with $n_\varepsilon = 1$ such that (1), (2), and (3) are fulfilled. As (3) simply expresses $n_f B$ in terms of $n_1 B, \ldots, n_r B$ (recall that $n_j$ denotes $n_{f_j}$), it follows that $F_{H_p, \preceq}^H$ is in a bijection with the set, say $\mathcal{F}$, of sequences $(B, n_1 B, \ldots, n_r B)$ for which conditions (1) and (2) are satisfied. Thus,

$$\zeta_{H_p}^{H_p, \preceq}(s) := \sum_{A \in F_{H_p, \preceq}^H} [H_p : A]^{-s} = \sum_{(B, n_1 B, \ldots, n_r B) \in \mathcal{F}} [N_p : B]^{-s}.$$

We now fix $(B, n_1 B, \ldots, n_r B) \in \mathcal{F}$ and set

$$S(B, n_1 B, \ldots, n_r B) := \{v \in M_{r \times h}(Z_p) : (B, x^{v_1} B, \ldots, x^{v_r} B) = (B, n_1 B, \ldots, n_r B)\}.$$

This is an open subset of $M_{r \times h}(Z_p)$ of Haar measure $[N_p : B]^{-r}$. In fact, the mapping $Z_p^h \to N_p$ given by $a \mapsto xa$ is a homeomorphism that preserves the (normalized) Haar measure (cf. [22, Lemma 2.4]), and the measure of each coset $n_f B$ is $[N_p : B]^{-1}$. Combining this with Lemma 3.3 and formula (3.1), we deduce that $M(B) \times S(B, n_1 B, \ldots, n_r B)$ is an open subset of $T_h(Z_p) \times M_{r \times h}(Z_p)$, and that for any $t = (t_{ij}) \in M(B)$

$$\mu(M(B) \times S(B, n_1 B, \ldots, n_r B)) = (1-p^{-1})^h \prod_{i=1}^h |t_{ij}|^r = (1-p^{-1})^h \prod_{i=1}^h |t_{ij}|^{r+i}.$$

It follows (again, using formula (3.1)) that

$$[N_p : B]^{-s} = \prod_{i=1}^h |t_{ij}|^r = (1-p^{-1})^{-h} \prod_{i=1}^h |t_{ij}|^{s-r-i} \mu(M(B) \times S(B, n_1 B, \ldots, n_r B))$$

$$= \frac{1}{(1-p^{-1})^h} \int_{M(B) \times S(B, n_1 B, \ldots, n_r B)} \prod_{i=1}^h |t_{ij}|^{s-r-i} \, d\mu.$$
This and (3.3) imply that

\[\zeta_{N_p,\kappa}^{H_p}(s) = \frac{1}{(1-p^{-1})^h} \int_{T^{H_p,\kappa}} \prod_{i=1}^{h} |t_{ii}|_{p}^{s-r-i} d\mu,\]

where \(T^{H_p,\kappa} = \bigcup_{(B,n_1B,\ldots,n_mB) \in \mathcal{S}} \mathcal{M}(B) \times S(B,n_1B,\ldots,n_mB).\) Note that this is the set of pairs \((t,v) \in T_b(\mathbb{Z}_p) \times M_{x \times h}(\mathbb{Z}_p)\) such that (a) \(t \in \mathcal{M}_{N_p}^{\kappa}\), and (b) the elements \(n_j := x^v_j\), for \(j = 1, \ldots, r\), satisfy conditions (1) and (2) with \(B = (x^v_1, \ldots, x^v_r)\). These are precisely the conditions listed in (3.5) since \(\beta_j[x^v_j, x^k] = [\beta_j x^v_j, x^k]\). This proves the integral expression for \(\zeta_{N_p,\kappa}^{H_p}(s)\).

The expression for \(\zeta_{G_p,\kappa}^{H_p}(s)\) is obtained similarly. This time we have to consider sequences \((n_jB)_{j \in F}\) such that conditions (2) to (7) are satisfied. Working as in the previous case, we arrive at the expression

\[\zeta_{G_p,\kappa}^{H_p}(s) = \frac{1}{(1-p^{-1})^h} \int_{T^{G_p,\kappa}} \prod_{i=1}^{h} |t_{ii}|_{p}^{s-r-i} d\mu,\]

where \(T^{G_p,\kappa} = \bigcup_{(B,n_1B,\ldots,n_mB) \in \mathcal{S}} \mathcal{M}(B) \times S(B,n_1B,\ldots,n_mB)\). and the union runs over those \((B,n_1B,\ldots,n_mB)\) with \(B\) open and normal in \(N_p\) and the \(n_j\)'s satisfying (2), (5), (6), (7). Therefore, \(T^{G_p,\kappa}\) consists of the pairs \((t,v)\) such that (a) \(t \in \mathcal{M}_{N_p}^{\kappa}\); (b) \(x^v_i \in (x^v_1, \ldots, x^v_r)\) for \(i = 1, \ldots, h\) and \(k = 1, \ldots, s\); and (c) the elements \(n_j := x^v_j\) (\(j = 1, \ldots, r\)) satisfy conditions (2), (6) and (7). Clearly (b) and (c) are equivalent to (3.5) since \(\beta_j[x^v_j, x_i]\). This completes the proof of the proposition. \(\square\)

### 3.3. A method to simplify cone integrals

We end this section with an elementary observation that we will use frequently to simplify the calculation of cone integrals.

Given \(m, n \in \mathbb{N}\), we consider subsets of \(\mathbb{Z}_p^m \times \mathbb{Z}_p^n\) obtained as follows.

(i) Fix a measurable subset \(D_0 \subset \mathbb{Z}_p^m\) and rational functions \(g_i(T) \in \mathbb{Q}_p(T) = \mathbb{Q}_p(T_1, \ldots, T_m), i = 1, \ldots, n\), such that \(g_i(t) \in \mathbb{Q}_p(T)\) whenever \(t \in D_0\).

(ii) Given an integer \(i\) such that \(1 \leq i \leq n\), assume that we have defined a measurable subset \(D_{i-1} \subset \mathbb{Z}_p^{m-1}\). Let \(k_i(T, V_1, \ldots, V_{i-1}), \lambda_i(T, V_1, \ldots, V_{i-1}) \in \mathbb{Q}_p(T, V_1, \ldots, V_{i-1})\) be rational functions such that \(k_i(t, v_1, \ldots, v_{i-1}) \in \mathbb{Z}_p\) and \(\lambda_i(t, v_1, \ldots, v_{i-1}) \in \mathbb{Z}_p\) whenever \((t, v_1, \ldots, v_{i-1}) \in D_{i-1}\). We define \(D_i\) in one of the following two ways:

\(\begin{align*}
\text{I:} & \quad D_1 = \left\{ (t, v_1, \ldots, v_i) \in \mathbb{Z}_p^m \times \mathbb{Z}_p^n : g_i(t) \mid k_i(t, v_1, \ldots, v_{i-1}) + \lambda_i(t, v_1, \ldots, v_{i-1}) \right\} \\
\text{II:} & \quad D_i = \left\{ (t, v_1, \ldots, v_i) \in \mathbb{Z}_p^m \times \mathbb{Z}_p^n : |g_i(t)|_p = |k_i(t, v_1, \ldots, v_{i-1}) + \lambda_i(t, v_1, \ldots, v_{i-1})|_p \right\}.
\end{align*}\)

(iii) We finally set \(D = D_n \subset \mathbb{Z}_p^m \times \mathbb{Z}_p^n\).

We say that \((D_0, g_1(T), \ldots, g_n(T))\) is the initial datum of definition of \(D\) and call \((v_1, \ldots, v_n)\) the sequence of pivots. A pivot \(v_i\) is said to be of type I or type II according to the way we choose to define \(D_i\).

**Proposition 3.6.** Let \(f_0, g_0 \in \mathbb{Q}_p(T) = \mathbb{Q}_p(T_1, \ldots, T_m)\) be non-zero polynomials, and consider the integral

\[\int_{D \subset \mathbb{Z}_p^m \times \mathbb{Z}_p^n} |f_0(t)|_p^n |g_0(t)|_p d\mu(t), \quad s \in \mathbb{C},\]

where \(D \subset \mathbb{Z}_p^m \times \mathbb{Z}_p^n\) is a measurable subset defined from a datum \((D_0, g_1(T), \ldots, g_n(T))\) and pivots \((v_1, \ldots, v_n)\). If the number of pivots of type II is \(r\), then

\[\int_{D} |f_0(t)|_p^n |g_0(t)|_p d\mu(t) = (1 - p^{-1})^r \int_{D_0} |f_0(t)|_p^n |g_0(t)|_p d\mu(t).\]

**Proof.** The integral (3.8) can be performed as follows. We start integrating with respect to the variable \(v_n\). Note that \(g_n(t) |k_n(t, v_1, \ldots, v_{n-1}) + \lambda_n(t, v_1, \ldots, v_{n-1})|_p\) if and only if \(v_n \in g_n(t) \mathbb{Z}_p - \frac{k_n(t, v_1, \ldots, v_{n-1})}{\lambda_n(t, v_1, \ldots, v_{n-1})}\), and the Haar measure of this set is \(|g_n(t)|_p\). Similarly, \(|g_i(t)|_p = |k_i(t, v_1, \ldots, v_{i-1}) + \lambda_i(t, v_1, \ldots, v_{i-1})|_p\) if and only if \(v_n \in g_n(t) \mathbb{Z}_p - \frac{k_i(t, v_1, \ldots, v_{i-1})}{\lambda_i(t, v_1, \ldots, v_{i-1})}\) and the Haar measure of the this set is \((1 - p^{-1})|g_n(t)|_p\). To sum up, after integrating with respect to \(v_n\), the integrand is multiplied by \(|g_n(t)|_p\) or by \((1 - p^{-1})|g_n(t)|_p\) according
to whether \( v_n \) is a pivot of type I or II, and the new domain of integration is clearly \( D_{n-1} \). We next integrate with respect to \( v_{n-1} \) and so on. After \( n \) steps, we arrive at the desired form of the integral. □

4. The Formulae

We now present the complete formulae for the subgroup and normal zeta functions of the 3-dimensional almost-Bieberbach groups, whose definition we recall below. Proofs of these formulae will be left for the next two and last sections. For details of the next discussion, we refer to [2].

Let \( \mathcal{N} \) be a connected and simply connected nilpotent Lie group, and let \( \Aut(\mathcal{N}) \) be the topological group of Lie group automorphisms of \( \mathcal{N} \). The semi-direct product \( \mathcal{N} \rtimes \Aut(\mathcal{N}) \) acts on \( \mathcal{N} \) in a canonical way by

\[
(n, \alpha)x = na(x), \quad \forall x, n \in \mathcal{N}, \quad \forall \alpha \in \Aut(\mathcal{N}).
\]

All maximal compact subgroups of \( \Aut(\mathcal{N}) \) are conjugate. We fix one of them, say \( \mathcal{C} \subset \Aut(\mathcal{N}) \). A torsion-free uniform discrete subgroup \( G \) of \( \mathcal{N} \times \mathcal{C} \) is called an almost-Bieberbach group (abbreviated as AB-group).

Note that the quotient space \( G/\mathcal{N} \) is a compact manifold of the same dimension as \( \mathcal{N} \) whose fundamental group is identified with \( G \). The dimension of \( G \) is defined as the dimension of \( \mathcal{N} \). When \( \mathcal{N} = \mathbb{R}^n \) and \( \mathcal{C} = O(\mathbb{R}^n) \), the orthogonal group, we recover the definition of Bieberbach group. The quotient space \( G/\mathbb{R}^n \), with the metric induced from the Euclidean space \( \mathbb{R}^n \), is a compact flat manifold.

The AB-groups were characterized algebraically as the finitely generated torsion-free virtually nilpotent groups (cf. [2] Theorem 3.1.3)). If \( G \) is an AB-group, then its Fitting subgroup \( \text{Fitt}(G) \) (the maximal normal nilpotent subgroup of \( G \)) is indeed maximal nilpotent. It follows that any intermediate subgroup \( \text{Fitt}(G) \leq H \leq G \) is also an AB-group with \( \text{Fitt}(H) = N \). An AB-group \( G \) is a Bieberbach group if and only if \( \text{Fitt}(G) \) is abelian.

4.1. The 3-dimensional Bieberbach groups and their zeta functions. There are only ten 3-dimensional Bieberbach groups up to isomorphism (cf. [27] Chap. 3). They are listed below with their corresponding zeta functions expressed in terms of the partial zeta functions with respect to the Fitting subgroup (which is always \( \langle x_1, x_2, x_3 \rangle \)). The first six are the fundamental groups of the orientable compact flat manifolds, and the last four are the fundamental groups of the non-orientable ones. We have arranged the presentation so that \( \zeta^G_\mathcal{C}(s) \) appears as the first term in the formula of \( \zeta^G_\mathcal{C}(s) \) and \( [G : N]^{-s-1} \zeta^G_\mathcal{C}(s) \) as the last one. The same holds for \( \zeta^G_\mathcal{C}(s) \).

\[
\mathcal{G}_1 = \langle x_1, x_2, x_3 : [x_1, x_2] = [x_1, x_3] = [x_2, x_3] = 1 \rangle \text{ has zeta functions}
\]

\[
\zeta^\mathcal{G}_1(s) = \zeta(s) = \zeta(s) \zeta(s-1) \zeta(s-2).
\]

\[
\mathcal{G}_2 = \langle \alpha, x_1, x_3 : x_1 , x_3 \rangle = 1 \forall i, j, \quad \alpha^2 = x_3, \quad \alpha x_1 = x_1^{-1}, \quad \alpha x_2 = x_2^{-1} \rangle \text{ has zeta functions}
\]

\[
\zeta^\mathcal{G}_2(s) = \zeta(s) \zeta(s-1) \zeta(s-2) = \zeta(s) \zeta(s-1)^2 \zeta(s-2),
\]

\[
\zeta^\mathcal{G}_2(s) = (1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s})(1 - 2^{-s}) \zeta(s) + 2^{-s} \zeta(s)^2 \zeta(s-1)(1 + 3 \cdot 2^{-s}).
\]

\[
\mathcal{G}_3 = \langle \alpha, x_1, x_3 : x_1 x_2 = x_1, \alpha x_2 = x_3, \alpha x_3 = x_2^{-1} x_3^{-1} \rangle \text{ has zeta functions}
\]

\[
\zeta^\mathcal{G}_3(s) = (1 - 3^{-s}) \zeta(s)(s-1)L(s-1, \chi_3) + 3^{-s} \zeta(s)(s-1) \zeta(s-2),
\]

\[
\zeta^\mathcal{G}_3(s) = (1 + 3 \cdot 3^{-s})(1 - 3^{-s}) \zeta(s) + 3^{-s} \zeta(s)^2 L(s, \chi_3)(1 + 2 \cdot 3^{-s}).
\]

\[
\mathcal{G}_4 = \langle \alpha, x_1, x_3 : x_1 x_2 = x_1, \alpha x_2 = x_3, \alpha x_3 = x_2^{-1} \rangle \text{ has zeta functions}
\]

\[
\zeta^\mathcal{G}_4(s) = \zeta(s) \zeta(s-1)L(s-1, \chi_3)(1 - 2^{-s}) + 2^{-s} \zeta(s)(s-1) \zeta(s-2)(1 - 2^{-s}) + 4^{-s} \zeta(s)(s-1) \zeta(s-2),
\]

\[
\zeta^\mathcal{G}_4(s) = (1 + 2 \cdot 2^{-s})(1 - 2^{-s}) \zeta(s) + 2^{-s}(1 + 2 \cdot 2^{-s} + 2 \cdot 4^{-s})(1 - 2^{-s}) \zeta(s) + 4^{-s} \zeta(s)^2 L(s, \chi_4)(1 + 2^{-s}).
\]

\[
\mathcal{G}_5 = \langle \alpha, x_1, x_2 : x_1 x_2 = x_1, \alpha x_2 = x_3, \alpha x_3 = x_2^{-1} x_3 \rangle \text{ has zeta functions}
\]

\[
\zeta^\mathcal{G}_5(s) = \zeta(s) \zeta(s-1)L(s-1, \chi_3)(1 - 2^{-s})(1 - 3^{-s}) + 2^{-s} \zeta(s)(s-1) \zeta(s-2)(1 - 2^{-s}) + 6^{-s} \zeta(s)(s-1) \zeta(s-2),
\]

\[
\zeta^\mathcal{G}_5(s) = (1 - 2^{-s})(1 - 3^{-s}) \zeta(s) + 2^{-s}(1 + 3^{-s})(1 - 3^{-s}) \zeta(s) + 6^{-s} \zeta(s)^2 L(s, \chi_3).
\]
\( B_6 = \left\langle \alpha, \beta, x_1, x_2, x_3 : \begin{array} {l} [x_i, x_j] = 1 \forall i, j, \alpha^2 = x_1, \alpha x_2 = x_2^{-1}, \alpha x_3 = x_3^{-1}, \\ \beta^2 = x_2, \beta x_1 = x_1^{-1}, \beta x_3 = x_3^{-1}, (\alpha \beta)^2 = x_3 \end{array} \right\rangle \) has zeta functions
\[
\zeta_{B_6}^G(s) = \zeta(s-1)^3(1-2^{-s})(1-2^{-s}) + 3 \cdot 2^{-s} \zeta(s)(s-2)(1-2^{-s}) + 4^{-s} \zeta(s)(s-1)(s-2), \\
\zeta_{B_6}^G(s) = 1 + 3 \cdot 2^{-s}(1-2^{-s})(1-2^{-s}) \zeta(s) + 4^{-s} \zeta(s)^3(1 + 4 \cdot 2^{-s} + 4^{-s}).
\]

\( \mathbb{B}_1 = \langle \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \forall i, j, \varepsilon^2 = x_1, \varepsilon x_2 = x_2, \varepsilon x_3 = x_3^{-1} \rangle \) has zeta functions
\[
\zeta_{\mathbb{B}_1}^G(s) = \zeta(s-1)^2(1-2^{-s})(1-2^{-s}) + 2^{-s} \zeta(s)(s-1)(s-2), \\
\zeta_{\mathbb{B}_1}^G(s) = (1-2^{-s})(1-2^{-s}) \zeta(s)(s-1) + 2^{-s}(1 + 3 \cdot 2^{-s}) \zeta(s)^2 \zeta(s-1).
\]

\( \mathbb{B}_2 = \langle \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \forall i, j, \varepsilon^2 = x_1, \varepsilon x_2 = x_2, \varepsilon x_3 = x_1 x_2 x_3^{-1} \rangle \) has zeta functions
\[
\zeta_{\mathbb{B}_2}^G(s) = \zeta(s)(s-1)^2(1-2^{-s})(1-2^{-s}) + 8 \cdot 4^{-s} \zeta(s)(s-1)(s-2), \\
\zeta_{\mathbb{B}_2}^G(s) = \zeta(s)(s-1)(1-2^{-s}) + 2^{-s} \zeta(s)^2 \zeta(s)(s-1)(1-2^{-s} + 4 \cdot 4^{-s}).
\]

\( \mathbb{B}_3 = \langle \alpha, \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \forall i, j, \alpha^2 = x_1, \alpha x_2 = x_2, \alpha x_3 = x_3^{-1}, \varepsilon^2 = x_2, \varepsilon x_1 = x_1, \varepsilon x_3 = x_3^{-1}, [\varepsilon, \alpha] = x_2 \rangle \) has zeta functions
\[
\zeta_{\mathbb{B}_3}^G(s) = \zeta(s)(s-1)^2(1-2^{-s})(1-4 \cdot 4^{-s}) + 2 \cdot 2^{-s} \zeta(s)(s-1)(2^{-s})(1 + 2 \cdot 2^{-s}) \\
+ 2^{-s} \zeta(s)(s-2)(s-3)(1-2^{-s}) + 4^{-s} \zeta(s)(s-1)(s-2), \\
\zeta_{\mathbb{B}_3}^G(s) = \zeta(s)(1-2^{-s})(1 + 3 \cdot 2^{-s}) + 2^{-s} \zeta(s)^2(1-2^{-s})(1 + 2^{-s} + 2 \cdot 4^{-s}) + 2^{-s} \zeta(s)(s-1)(2^{-s})(1 + 2 \cdot 2^{-s}) \\
+ 2^{-s} \zeta(s)^2(1-2^{-s})(1 + 5 \cdot 2^{-s} + 2 \cdot 4^{-s}) + 4^{-s} \zeta(s)^3(1 + 4 \cdot 2^{-s} + 4^{-s}).
\]

\( \mathbb{B}_4 = \langle \alpha, \varepsilon, x_1, x_2, x_3 : [x_i, x_j] = 1 \forall i, j, \alpha^2 = x_1, \alpha x_2 = x_2, \alpha x_3 = x_3^{-1}, \varepsilon^2 = x_2, \varepsilon x_1 = x_1, \varepsilon x_3 = x_3^{-1}, [\varepsilon, \alpha] = x_2 x_3 \rangle \) has zeta functions
\[
\zeta_{\mathbb{B}_4}^G(s) = \zeta(s)(s-1)^2(1-2^{-s})(1-2^{-s}) + 2^{-s} \zeta(s)(s-1)(s-2)(1-2^{-s}) \\
+ 2 \cdot 2^{-s} \zeta(s)(s-1)^2(1-2^{-s})(1 + 2^{-s} + 4^{-s}) + 4^{-s} \zeta(s)(s-1)(s-2), \\
\zeta_{\mathbb{B}_4}^G(s) = 4^{-s} \zeta(s)^3(1 + 4 \cdot 2^{-s} + 4^{-s}) + 2^{-s} \zeta(s)(s-1)(1 + 2 \cdot 2^{-s} + 4^{-s}) + 2^{-s} \zeta(s)^2(1-2^{-s})(1 + 2 \cdot 2^{-s}) + \zeta(s)(1-2^{-s}).
\]

4.2. The 3-dimensional almost-Bieberbach groups and their zeta functions. The 3-dimensional AB-groups with non-abelian Fitting subgroup were classified in [3]; see also [2, Chap. 7]. They are arranged into seven families according to the isomorphism type of \( G/Z(N) \) (\( G \) is the AB-group and \( N \) the Fitting subgroup), which is one of the plane crystallographic groups \( p_1, p_2, pg, p_3, p_4, p_6, p_{2gg} \). They are listed below with their zeta functions expressed in terms of the partial zeta functions with respect to the Fitting subgroup, which is always \( (x_1, x_2, x_3) \). This time the normal zeta function has been arranged so that \( [G : N]^{-s} \zeta_{G/N}^G(s) \) appears as the first term in the formula and \( \zeta_{G}^{G-N}(s) \) as the last one.

\[ N_k = \langle x_1, x_2, x_3 : [x_1, x_2] = x_1^k, [x_1, x_3] = [x_2, x_3] = 1 \rangle, \quad k \in \mathbb{N}, \text{ has zeta functions} \]
\[
\zeta_{N_k}^G(s) = \prod_{p \mid k} \zeta_p(s) \zeta_p(s-1) \zeta_p(2s-2) \zeta_p(2s-3) \times \prod_{p \mid k} \left( \zeta_p(s-2) \left( \zeta_p(s) \zeta_p(s-1) - p^{-s+2} |k_p|^{-2} \zeta_p(2s-2) \zeta_p(2s-3) \right) \right), \\
\zeta_{N_k}^G(s) = \prod_{p \mid k} \zeta_p(s) \zeta_p(s-1) \zeta_p(3s-2) \times \prod_{p \mid k} \zeta_p(s) \zeta_p(s-1) \left( \frac{1 - |k_p|^{-2}}{1 - p^{-s+2} + |k_p|^{-2}} \zeta_p(3s-\eta) \right).
\]
\[ G_{p^{2k}} = \left\{ \alpha, x_1, x_2, x_3 : \begin{array}{l}
x_2, x_1 = x_3^2, \\
x_1, x_3 = x_2, x_3 = x_1, \\
\end{array} \right\}, k \in \mathbb{N}, \text{ has zeta functions} \]

\[ \zeta_{G_{p^{2k}}}^2(s) = 2^{-s} \prod_{p \mid k} \zeta_p(s-1) \zeta_p(3s-3) \tau_{p_0}s(1-2^{-s}) \tau_{p_0} \zeta_p(2s-2) \zeta_p(2s-3) \]

\[ \zeta_{G_{p^{2k}}}^2(s) = 2^{-s} \prod_{p \mid k} \zeta_p(s-1) \zeta_p(3s-3) \tau_{p_0}s(1-2^{-s}) \tau_{p_0} \zeta_p(2s-2) \zeta_p(2s-3) \]

\[ G_{p^{3k}, \epsilon} = \left\{ \gamma, x_1, x_2, x_3 : \begin{array}{l}
x_2, x_1, x_2 = x_3^3, x_1, x_3 = x_2, x_3 = x_1, x_2 = x_1, x_2 = x_1 x_2 x_3^{-1} \end{array} \right\}, \epsilon \in \{1, -1\} \]

\[ k(\epsilon) \equiv 0 \mod 3, \text{ has zeta functions} \]

\[ \zeta_{G_{p^{3k}, \epsilon}}^2(s) = 3^{-s} \prod_{p \mid k} \zeta_p(s-1) \zeta_p(3s-3) L_p(s, \chi_0) L_p(2s-1, \chi_0) \mid L_p(3s-3, \chi_0) \]

\[ \zeta_{G_{p^{4k}, \epsilon}}^2(s) = 4^{-s} \prod_{p \mid k} \zeta_p(s-1) \zeta_p(3s) L_p(s, \chi_0) L_p(2s-1, \chi_0) \mid L_p(3s, \chi_0) \]

\[ \zeta_{G_{p^{4k}, \epsilon}}^2(s) = 4^{-s} \prod_{p \mid k} \zeta_p(s-1) \zeta_p(3s) L_p(s, \chi_0) L_p(2s-1, \chi_0) \mid L_p(3s, \chi_0) \]

\[ + 2^{-s} \prod_{p \mid k} \left( \frac{1}{1-p^{-s}} \right) \]

\[ \times \left( 1 + 2 \cdot 2^s + 2 \cdot 4^{-s} \right) \]
\[ G_{p62k,\epsilon} = \left\{ \gamma, x_1, x_2, x_3 : [x_2, x_1] = x_3^{2k}, [x_1, x_3] = [x_2, x_3] = 1, \right\}, k \in \mathbb{N} \text{ and } \epsilon \in \{1,-1\} \text{ verifying } k(\epsilon + k) \equiv 0 \mod 3, \text{ has zeta functions} \]
\[ \zeta_{G_{p62k,\epsilon}}(s) = \prod_{p \nmid k, \epsilon} \frac{\zeta_p(s - 1)\zeta_p(2s - 1)L_p(s - 1, \chi_3)L_p(2s - 1, \chi_3)}{L_p(3s - 2, \chi_3)} \times \prod_{p|k, \epsilon} \zeta_p(s) \left( L_p(s - 1, \chi_3)\zeta_p(s - 1) - p^{-s}|k_p|^sL_p(2s - 1, \chi_3)\zeta_p(2s - 1) \right) \times \zeta(2s - 2) \times 2^{-s} \zeta_{G_{p2k,\epsilon}}(s) + 6^{-s} \zeta_{G_{p2k,\epsilon}}(s) \]

5. Computing the zeta functions of the 3-dimensional Bieberbach groups

We prove the formulae presented in [11]. Throughout this section, we denote by \( G \) the Bieberbach group under consideration and by \( N \) the Fitting subgroup. We keep the notation introduced at the beginning of Section 2 except that here \( N \) is not a \( T_2 \)-group but rather an abelian group (and it is indeed an \( \mathbb{Z}[P] \)-module, where \( P = G/N \)). Instead, we denote \( Z = C_N(P), \quad E = G/Z, \quad T = N/Z. \)

The formula for the zeta functions of \( \vartheta_2 \) are already known (see Introduction), so we focus here on the other groups. We mentioned that an intermediate subgroup \( N \leq H \leq G \) is again a Bieberbach group with Fitting subgroup \( N \). Therefore, to prove the formula for \( \zeta_{G/H}(s) \), it is enough to prove the one for \( \zeta_{G/H}(s) \). The isomorphism classes of the intermediate subgroups \( N \leq H \leq G \) (if there is any) will be identified when computing the partial zeta functions \( \zeta_{G/H}(s) \). The following will be useful. If \( p|G/N \), then

\[ \zeta_{G_{p2k,\epsilon}}(s) \]
\[ = \prod_{p|k, \epsilon} \zeta_p(s - 1)^2 \zeta_p(2s - 1)^2 \times \prod_{p \nmid k, \epsilon} \zeta_p(s - 1) \left( (\zeta_p(s - 1))^2 - p^{-s+1}|k_p|^{s-1} \zeta_p(2s - 2) \right) \]
\[ + 2^{-s} \zeta_{G_{p2k,\epsilon}}(s) + 4^{-s} \zeta_{G_{p2k,\epsilon}}(s), \]

\[ \zeta_{G_{p2k,\epsilon}}(s) \]
\[ = 4^{-s} \prod_{p \nmid k, \epsilon} \zeta_p(s) \zeta_p(3s) \times \prod_{p|k, \epsilon} \zeta_p(s) \left( \frac{1 - |k_p|^s}{1 - p^{-s}} + |k_p|^s \zeta_p(3s) \right) \times \zeta(2s - 2) \left( \zeta(2s - 2) + 4^{-s} \right) \left( 1 + 5 \cdot 2^{-s} + 2 \cdot 4^{-s} + 8^{-s} - 16^{-s} - 2 \cdot 32^{-s} \right) \]
\[ + 2^{-s} \prod_{p \nmid k, \epsilon} \frac{1 - p^{-s}|k_p|^s}{1 - p^{-s}} \times (1 + 2 \cdot 2^{-s}) + 2 \cdot 2^{-s} \zeta(2s - 2) \left( 1 - 2^{-s} \right) \zeta(s) + 1. \]

This follow immediately from the analysis in [11] Section 2. When the rank of \( Z \) is 1, then \( E \) is a plane crystallographic group, and we will be able to apply the results of [13] (summarized in [6] Section 4).

When computing local factors at primes \( p|G/N \), we will sometimes use the method of \( p \)-adic integration with respect to the Malčev basis \( \{x_1, x_2, x_3\} \); specifically, Corollary 5.3 and Proposition 5.5. In this case, \( M_p^G = M_p^G = T_1^G(\mathbb{Z}_p) \). The following lemma, whose verification is straightforward, will be used to translate the conditions defining the domains of integration into cone conditions.
Lemma 5.1. An element $x^v \in N_p$ is in $(x^{t_1}, x^{t_2}, x^{t_3})$ if and only if the following holds:

1. $t_{11}t_{1} + t_{12} + t_{13}$
2. $t_{22} - \frac{t_{11}}{t_{11}}t_{12} + t_{23}$
3. $t_{33} - \frac{t_{11}}{t_{11}}t_{13} + t_{33}$

5.1. Computing zeta functions of a family of groups including $\mathcal{G}_2$ and $\mathcal{F}_1$. Given positive integers $a, b$, we shall compute the zeta functions of

$$G = \left\{ \alpha, x_1, \ldots, x_n, x_{a+1}, \ldots, x_{a+b} : \begin{align*}
\forall i, j, \alpha^2 &= x_{a+b}, \\
\alpha x_i &= x_{i+1} \quad (1 \leq i \leq n), \quad \alpha x_i &= x_i \quad (1 \leq i \leq n, b+1)
\end{align*} \right\}.$$ 

This, of course, includes the cases $G = \mathcal{G}_2$ and $G = \mathcal{F}_1$. Here $N = \langle x_1, \ldots, x_{a+b} \rangle$, and $\mathcal{G}_2$ is generated by the class of $\alpha$, and $Z = C_N(P) = \langle x_{a+1}, \ldots, x_{a+b} \rangle$. Note that if $U \leq Z \leq V \leq N$, then $U$ and $V$ are $P$-submodules. We first prove a series of lemmas. When working inside $T$ or $Z$, we shall use additive notation.

Lemma 5.2. For $V \leq Z$ of finite index, $\text{Hom}_T(T, Z/V) = \text{Hom}(C_2^a, Z/V)$, and this set has $[Z : 2Z + V]$ elements.

Proof. A group homomorphism $\varphi : T \rightarrow Z/V$ is of $P$-modules if and only if $\varphi(-x) = \varphi(x)$ for all $x \in T$, that is, $\varphi(2x) = 0$ for all $x \in T$, or $\varphi(2T) = 0$. This implies the first equality. Next, note that $\text{Hom}(C_2, Z/V) = \text{Hom}(\mu_2, \langle \frac{1}{2} \rangle Z/V)$, whose size is $([\frac{1}{2}] Z/V \cap Z : V)^a$. Thus, the second equality follows from the following calculation:

$$[\frac{1}{2} Z/V \cap Z : V] = [V \cap 2Z : 2V] = \frac{|V : 2V|}{|V : V \cap 2Z|} = \frac{|Z : 2Z|}{|V : V \cap 2Z|} = [Z : V + 2Z].$$

Lemma 5.3. Given $A \leq G$ of finite index such that $AN = G$, the subgroup $B := A \cap N$ is normal of finite index in $G$ and $x_{a+b} \in (B \cap Z) + 2Z$. If in addition $A$ is normal, then also $x^a_1, \ldots, x^a_b \in B$.

Conversely, given $B \leq N$ of finite index and normal in $G$ such that $x_{a+b} \in (B \cap Z) + 2Z$, the set $\{ A \leq G : AN = G, A \cap N = B \}$ has $[N : BZ][Z : 2Z + (B \cap Z)]$ elements. If in addition $x^a_1, \ldots, x^a_b \in B$, then every $A$ in the latter set is normal in $G$.

Proof. A subgroup $A \leq G$ such that $AN = G$ must contain $a$ for some $a \in N$. Then $(an)^2 \in A$. It is easy to check that $(an)^2 = x_{a+b} + 2z$ for some $z \in Z$. Thus $x_{a+b} + z \in B \cap Z$, where $B = A \cap N$, or $x_{a+b} \in (B \cap Z) + 2Z$. Note that $B$ is clearly normal, and if $A$ is also normal, then $[N : B] = (x^a_1, \ldots, x^a_b)$ is included in $B$ by Lemma 2.11. This proves the first part of the lemma.

We now fix $B \leq N$ of finite index and normal in $G$, and assume that $x_{a+b} + z \in B \cap Z$ for some $z \in Z$, which will be fixed. Let $\{w_1, \ldots, w_s\} \subset \langle x_1, \ldots, x_a \rangle$ and $\{z_1, \ldots, z_t\} \subset Z$ be transversals to the cosets of $BZ$ in $N$ and to the cosets of $B \cap Z$ in $Z$, respectively. Then $\{w_i, z_j : 1 \leq i \leq s, 1 \leq j \leq t\}$ is a transversal to the cosets of $B$ in $N$. A subgroup $A \leq G$ such that $AN = G$ and $A \cap N = B$ is of the form $A = B \cup \alpha w_i z_j B$ for uniquely determined $w_i$ and $z_j$. Now $A = B \cup \alpha w_i z_j B$ is a subgroup if and only if $(\alpha w_i z_j)^2 \in B$, that is $x_{a+b} + z_j \in B \cap Z$ in additive notation. In turn, this is equivalent to $2z_j - 2z \in B \cap Z$, that is, $z_j - z \in \langle \frac{1}{2} (B \cap Z) \cap Z \rangle$. Therefore, only $\{ \frac{1}{2} (B \cap Z) \cap Z : B \cap Z \}$ of the $z_j$‘s are allowed. Since there is no restriction on $w_i$’s, the number of possibilities for $A$ is $[N : BZ][\langle \frac{1}{2} (B \cap Z) \cap Z : B \cap Z \rangle]$, which is equal to $[N : BZ][Z : 2Z + (B \cap Z)]$ by Lemma 2.4. If in addition $[N : G] = (x^a_1, \ldots, x^a_b)$ is included in $B$, then $G/B$ is abelian and hence $A$ is normal. This completes the proof of the lemma.

Lemma 5.4. Fix $U \leq T$ and $V \leq Z$ of finite index. Then $[\{ B \leq G : (BZ)/Z = U, B \cap Z = V \}] = [Z : 2Z]^d$.

If in addition $2T \subseteq U$, then $[\{ B \leq G : (BZ)/Z = U, B \cap Z = V, x^a_1, \ldots, x^a_b \in B \}] = [Z : V + 2Z]^d$, where $d = \dim_{Z^2}(U/2T)$.

Proof. Let $U$ be the pre-image of $U$ in $N$. We are interested in the number of $P$-invariant complements of $Z/V$ in $\tilde{U}/V$. One of them is clearly $(U' + V)/V$, where $U'$ is the projection of $\tilde{U}$ onto $(x_1, \ldots, x_a)$, and hence by Lemma 2.4 the number of them is $|\text{Hom}_T(U/Z, Z/V)| = |\text{Hom}_T(U, Z/V)| = |\text{Hom}_T(P, Z/V)|$. This, in turn, is equal to $|Z : V + 2Z|^d$ by Lemma 5.2. This proves the first part of the lemma.

To show the second assertion, we use additive notation in $U/V$. There is no loss of generality if we assume that $U/2T$ is generated by the classes of $x_1, \ldots, x_a$. Let $\tilde{x}_i$ denote the class of $x_i$ modulo $V$, $i = 1, \ldots, a$. We are interested in the number of $P$-invariant complements of $Z/V$ in $\tilde{U}/V$ that contain $2\tilde{x}_1, \ldots, 2\tilde{x}_a$. One of them is $B_0/V$, where $B_0 = \langle \tilde{x}_1, \ldots, \tilde{x}_d, 2\tilde{x}_{d+1}, \ldots, 2\tilde{x}_a \rangle$. To count how many there are, we follow the proof of Lemma 2.2. A $P$-invariant complement is of the form $B_\varphi/V = \{ x - \varphi(x) : x \in B_0/V \} = \langle \tilde{x}_1 - \varphi(\tilde{x}_1), \ldots, \tilde{x}_d - \varphi(\tilde{x}_d), 2\tilde{x}_{d+1} - \ldots, 2\tilde{x}_a \rangle$. 

References:
\(\varphi(2\bar{x}_{d+1}, \ldots, 2\bar{x}_a - \varphi(2\bar{x}_a))\) for a uniquely determined \(\varphi \in \text{Hom}_{F}(B_0/V, Z/V) = \text{Hom}(B_0/(2B_0 + V), Z/V)\). The last equality uses Lemma 5.2, which is possible since clearly \(B_0/V \cong T\) as \(P\)-modules. Given \(\varphi\) as above, we have 2\(\bar{x}_i = 2\bar{x}_i - 2\varphi(\bar{x}_i) \in B_0/V\) for \(i = 1, \ldots, d\). If it is also required that 2\(\bar{x}_{d+1}, \ldots, 2\bar{x}_a \in B_0/V\), we need \(\varphi(2\bar{x}_i) = 0\) for all \(i = d + 1, \ldots, a\). Thus, the set of \(P\)-invariant complements of \(Z/V\) in \(U/V\) that contain 2\(\bar{x}_1, \ldots, 2\bar{x}_a\) is in a bijection with \(\text{Hom}(B_0/(2B_0 + 2\bar{x}_1, \ldots, 2\bar{x}_a + V), Z/V)\), that is, with \(\text{Hom}(\mathbb{F}_{2^{(d+1)\alpha}}(a_1, \ldots, a_d), Z/V)\). By Lemma 5.2 this set has \([Z : V + 2Z]^d\) elements. This completes the proof of the lemma. 

We now show a combinatorial lemma. Let \(M\) be a finite dimensional vector space over \(\mathbb{F}_2\). For a flag of subspaces \(\beta : 0 = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_l = M\), the length \(l\) is denoted by \(l(\beta)\). The set of all flags is denoted by \(\mathcal{F}(M)\).

**Lemma 5.5.** Let \(M\) be a non-zero finite dimensional vector space over \(\mathbb{F}_2\). Then it holds that

\[
\sum_{W \subseteq M} \sum_{\beta \in \mathcal{F}(M/W)} (-1)^{l(\beta)} = 0.
\]

Let \(k \geq 0\) and let \(M\) be a \(k\)-dimensional vector space over \(\mathbb{F}_2\). Then it holds that

\[
\sum_{\beta \in \mathcal{F}(M)} (-1)^{l(\beta)} = (-1)^k 2^\left(\begin{atoman} k \end{atoman}\right).
\]

**Proof.** If \(M\) has positive dimension, then any flag of \(M\) of length \(l\) produces a flag of length \(l - 1\) in \(M/W\) for some \(0 \neq W \subseteq M\), and conversely, any flag of \(M/W\) of length \(l - 1\) extends to a flag of \(M\) of length \(l\). This implies that

\[
\sum_{\beta \in \mathcal{F}(M)} (-1)^{l(\beta)} = \sum_{W \neq M} \sum_{\beta \in \mathcal{F}(M/W)} (-1)^{l(\beta)},
\]

which yields (5.3).

We now prove [5.4] by induction on \(k\). The case \(k = 0\) is obvious, so we assume that \(k \geq 0\). Let \(A_k\) be the left-hand side and \(B_k\) the right-hand side. Note that (5.3) implies that \(\sum_{i=0}^k A_i \left(\begin{atoman} i \end{atoman}\right) = 0\). Here \(\left(\begin{atoman} i \end{atoman}\right)\) denotes the \(2\)-binomial coefficient, which expresses the number of \(i\)-dimensional subspaces in a \(k\)-dimensional vector space over \(\mathbb{F}_2\). On the other hand, if we replace \(t\) by \(-1\) in the \(2\)-binomial theorem

\[
(1 + t)(1 + 2t) \cdots (1 + 2^k-t) = \sum_{i=0}^k 2^i \left(\begin{atoman} k \end{atoman}\right) t^i,
\]

we obtain that

\[
0 = \sum_{i=0}^k (-1)^i 2^i \left(\begin{atoman} i \end{atoman}\right)_2 = \sum_{i=0}^k B_k \left(\begin{atoman} i \end{atoman}\right)_2.
\]

Since \(A_i = B_i\) for \(i < k\) by the inductive hypothesis, and since \(\left(\begin{atoman} k \end{atoman}\right)_2 = 1\), it follows that \(A_k = B_k\). This completes the induction. 

**Lemma 5.6.** Let \(Z_1\) be a free abelian group of rank \(b\) and let \(Z_0 \subseteq Z_1\) be a subgroup such that \(2Z_1 \subseteq Z_0\). Then

\[
\sum_{Z_0 \subseteq Z_1, V + Z_0 \subseteq Z_1} [Z_1 : V]^{-s} = \left(\prod_{i=1}^{b} \zeta(s-i+1) \right) \left(\prod_{i=0}^{d-1} \left(1 - 2^{-i} - s\right)\right),
\]

where \(d = \dim_{\mathbb{F}_2} Z_1/Z_0\).

**Proof.** By (5.3), the left-hand side of (5.6) becomes

\[
\sum_{Z_0 \subseteq Z_1} [Z_1 : V]^{-s} \left(\sum_{Z_0 + V \subseteq Z_1} \sum_{\beta \in \mathcal{F}(Z_1/W)} (-1)^{l(\beta)}\right) = \sum_{Z_0 \subseteq W \subseteq Z_1} \sum_{V \subseteq W} [Z : W]^{-s} [W : V]^{-s} \left(\sum_{\beta \in \mathcal{F}(Z_1/W)} (-1)^{l(\beta)}\right).
\]

By (5.4), the latter becomes

\[
\prod_{i=1}^{b} \zeta(s-i+1) \left(\sum_{Z_0 \subseteq W \subseteq Z_1} [Z : W]^{-s} (-1)^{\dim_{\mathbb{F}_2} Z_1/W} 2^{\dim_{\mathbb{F}_2} Z_1/W} \right) = \prod_{i=1}^{b} \zeta(s-i+1) \left(\prod_{j=0}^{d-1} \left(1 - 2^{-j} - s\right)\right) \left(\prod_{j=0}^{d} \left(1 - 2^{-j} - s\right)\right).
\]

Finally, by using (5.5) with \(k = d\) and \(t = -2^{-s}\), the last expression becomes the right-hand side of (5.6). 

**Proposition 5.7.** The partial zeta functions of \(G\) with respect to \(N\) are as follows:

\[
\zeta_G^{c}^{c}(s) = \prod_{i=1}^{a} \zeta(s-i) \times \prod_{j=1}^{b} \zeta(s-j+1) \times \left(\prod_{k=1}^{b} 2^{(b-k)(a+1-s)} \left(\begin{atoman} b \end{atoman} - 1 \end{atoman}\right) \left(\begin{atoman} k \end{atoman} \right)_2 \left(1 - 2^{-s}\right) \cdots \left(1 - 2^{k-1-s}\right)\right),
\]

\[
\zeta_G^{c}^{N}(s) = \prod_{j=1}^{b} \zeta(s-j+1) \times \left(\prod_{i=0}^{a} 2^{(a-i-1)(1-s)} \left(\begin{atoman} a \end{atoman} - 1 \end{atoman}\right)_2 \left(\prod_{k=1}^{b} 2^{(b-k)(i+1-s)} \left(\begin{atoman} b \end{atoman} - 1 \end{atoman}\right) \left(\begin{atoman} k \end{atoman} - 1 \right)_2 \left(1 - 2^{-s}\right) \cdots \left(1 - 2^{k-1-s}\right)\right)\right).
\]
Proof. The first equality below uses Lemma 5.3, the second one Lemma 5.4, and the last one Lemma 5.6.

\[ \zeta_{G, \zeta}^{G, \zeta}(s) = \sum_{B \subseteq G : B \subseteq N, x_{a+b} \in (B \cap Z) + 2Z} [N : B]^{-s} [N : BZ] [Z : (B \cap Z) + 2Z] \]

\[ = \sum_{U \subseteq T, V \subseteq Z, x_{a+b} \in V + 2Z} [T : U]^{-s} [Z : V]^{-s} [Z : V + 2Z]^{-s} [T : U] [Z : V + 2Z] \]

\[ = \zeta_T(s - 1) \left( \sum_{2Z} \sum_{Z_1} [Z : Z_1]^{s+1-\varepsilon} \sum_{V \subseteq Z_1} [Z : V]^{-s} \right) \]

\[ = \prod_{i=1}^a \zeta(s - i) \left( \sum_{2Z} \sum_{Z_1} \sum_{V \subseteq Z_1} 2^{(k-k)(a+1-s)} [Z : V]^{-s} \right) \]

\[ = \prod_{i=1}^b \zeta(s - i) \times \prod_{j=1}^b \zeta(s - j + 1) \times \left( \sum_{k=1}^b \sum_{Z_1} 2^{(b-k)(a+1-s)} \left( \frac{b-1}{k-1} \right) (1 - 2^{-s}) \cdots (1 - 2^{b-k-s}) \right). \]

The proofs of the expressions for \( \zeta_{G, \zeta}^{G, \zeta}(s) \) and \( \zeta_{G, \zeta}^{G, \zeta}(s) \) are similar. In the first case we do not have to add the condition \( x_{a+b} \in V + 2Z \) or the factor \( [T : U][Z : V + 2Z] \) in the second line. In the second case, we have to add the condition \( 2T \leq U \) and replace \( [Z : V + 2Z] \) by \( [Z : V + 2Z]^{\dim_T(U/T)} \) in the second line. \( \square \)

5.2. Computing the zeta functions of \( \mathcal{G}_3, \mathcal{G}_4, \) and \( \mathcal{G}_5 \). Let \( G \) be any of the groups \( \mathcal{G}_3, \mathcal{G}_4, \) or \( \mathcal{G}_5 \). We set \( d = |G/N| \). Note that \( Z = \langle x_1 \rangle \) and that \( E \) is the plane crystallographic group \( pd. \)

5.2.1. Local factors of \( \zeta_{G, \zeta}^{G, \zeta}(s) \). If \( p \parallel d \), then by 5.1 we have \( \zeta_{G, \zeta}^{G, \zeta}(s) = \zeta_{E, \zeta}^{E, \zeta}(s) \zeta_{G, \zeta}^{G, \zeta}(s) \), and according to 13, 5.10, 5.13, 5.16, the latter is equal to \( \zeta_p(s - 1)L_p(s - 1, \chi_2)\zeta_p(s) \) if \( d \in \{3, 4\} \), and to \( \zeta_p(s - 1)L_p(s - 1, \chi_3)\zeta_p(s) \) if \( d = 6 \).

Assume now that \( p \) does not divide \( d \). We claim that if \( A \leq G \) is open and satisfies \( AN = G_p \), then \( Z_p \leq A \). Indeed, \( A \) contains \( \alpha \mathfrak{v} \) for some \( \mathfrak{v} \in \mathbb{Z}_p^G \). Then \( (\alpha \mathfrak{v})^2 = x_1^{dv_1+1} \in A \), and hence \( x_1 \in A \) since \( dv_1 + 1 \in \mathbb{Z}_p^G \). This proves the claim. It follows now that \( \zeta_{G, \zeta}^{G, \zeta}(s) = \zeta_{E, \zeta}^{E, \zeta}(s) \). According to 13, 5.10, 5.13, 5.16, this series is equal to \( \zeta_p(s - 1)L_p(s - 1, \chi_p) \), where \( q = d \) if \( d \in \{3, 4\} \) and \( q = 3 \) if \( d = 6 \).

5.2.2. Local factors of \( \zeta_{G, \zeta}^{G, \zeta}(s) \). If \( p \parallel d \), then by 5.1 we have \( \zeta_{G, \zeta}^{G, \zeta}(s) = \zeta_p(s)\zeta_{E, \zeta}^{E, \zeta}(s) \), and according to 13, 6.10, 6.13, 6.16 the latter is equal to \( \zeta_p(s)L_p(s, \chi_2)\zeta_p(s) \) if \( d \in \{3, 4\} \) and to \( \zeta_p(s)L_p(s, \chi_3)\zeta_p(s) \) if \( d = 6 \).

Assume now that \( p \parallel d \). By Corollary 3.3 we have

\[ \zeta_{G, \zeta}^{G, \zeta}(s) = \frac{1}{1 - p^{-1}} \int_\mathcal{T} |t_{11}|^{s-1} |t_{22}|^{s-2} |t_{33}|^{s-3} d\mu, \]

where \( \mathcal{T} \subseteq T^+_{Z_p} \) is defined by the following equivalent conditions:

\[ a(x^{t_1})x^{t_1}, a(x^{t_2}), a(x^{t_3}) \in (x^{t_1}, x^{t_2}, x^{t_3}) \]

\[ \Leftrightarrow x_2 t_{12} - t_{13}, x_2 t_{12} - (r+1)t_{13}, x_2 t_{13} - r_{23}, x_3 t_{23} - x_2 t_{13} \in (x^{t_1}, x^{t_2}, x^{t_3}), \]

where \( r = 1 \) if \( d = 3 \), \( r = 0 \) if \( d = 4 \), and \( r = -1 \) if \( d = 6 \). Therefore, by Lemma 5.1, the conditions defining \( \mathcal{T} \) are

\[ (1) \ t_{22} - t_{12} - t_{13}, \quad (2) \ t_{33} - t_{23} + t_{12} - (r+1)t_{13}, \quad (3) \ t_{22} t_{23}, \]

\[ (4) \ t_{33} t_{22} t_{23} + t_{22} - r_{23}, \quad (5) \ t_{22} - t_{33}, \quad (6) \ t_{33} t_{22} t_{23}. \]

Note that (3) implies (6), and (4) can be written as \( \frac{t_{23}^2 (t_{22} + t_{22})^2 - r_{23}^2 + 1}{t_{22}^2 + 1} \).

Assume first that \( d \in \{3, 4\} \). One easily checks that the equation \( T^2 - rT + 1 \equiv 0 \mod p \), in \( Z_p \), has \( -1 \) as a unique solution modulo \( p \), and there are no solutions in \( Z_p \) for \( T^2 - rT + 1 \equiv 0 \mod p^2 \). Therefore, (4) splits the domain of integration \( \mathcal{T} \) as a disjoint union \( \mathcal{T}^+ \cup \mathcal{T}^- \) according to the cases (4a) \( |t_{22}|_p = |t_{33}|_p \) and (4b) \( |pt_{22}|_p = |t_{33}|_p \).

If we assume (4a), then (1) and (3) reduce (2) to \( t_{33} |t_{22} + t_{22}| + 1 \), and then (1) reduces (2) again to \( t_{33} |t_{22} + t_{22}| + 1 \). Note that \( r + 2 = p \). The conditions defining \( \mathcal{T}^+ \) are therefore, \( t_{22} - t_{12} - t_{13}, \ t_{22} |t_{22}|, t_{22} t_{22} t_{23} \) and \( |t_{22}|_p = |t_{33}|_p \), and the integral over \( \mathcal{T}^+ \) becomes

\[ \frac{1}{1 - p^{-1}} \int_{\mathcal{T}^+} |t_{11}|^{s-1} |t_{22}|^{s-2} d\mu = \zeta_p(s) \frac{1}{1 - p^{-1}} \int_{t_{22} |t_{22}|} |t_{22}|^{s-2} d\mu = \zeta_p(s) \left( 1 + p^{1-2s} \zeta_p(2s) \right), \]

where in the first equality we integrated with respect to \( t_{11} \) and applied Proposition 5.6 with the pivots \( t_{12}, t_{23}, t_{33} \).
If we now assume (4b), then (3) and (5) are redundant, and since \( r + 2 = p \), we see that (2) can be written as \( t_{33}(\frac{22}{23} t_{12} + \frac{12}{23} t_{13} + \frac{1}{23} p t_{12}) \). Using (1) and the assumption that \( |t_{33}| = |p + |p + 1| \), we can reduce (2) to \( t_{33} t_{12} t_{13} \), which is equivalent to \( t_{22} t_{12} \). Thus, \( T^b \) is defined by the conditions \( t_{22} - t_{12} - t_{13}, t_{22} | t_{12}, pt_{22} | t_{23} + t_{22} \) and \( pt_{22} | t_{33} \), and the integral over \( T^b \) becomes

\[
\frac{1}{(1 - p^{-1})^3} \int_{t_{22}} |t_{11}|^{s-1} |t_{22}|^{2-s} p^{1-s} \, d\mu = \frac{1}{(1 - p^{-1})^2} \int |t_{11}|^{s-1} |t_{22}|^{2-s} p^{1-s} \, d\mu = p^{1-s} \zeta_p(s) \zeta_p(2s),
\]

where, in the first equality, we used Proposition 3.5 with the pivots \( t_{12}, t_{13}, t_{23}, t_{33} \).

We conclude that

\[ \zeta_{G_p}^{N_{G_p} \cdot \phi}(s) = \zeta_p(s) \left( 1 + p^{-2s} \zeta_p(2s) + p^{-s} \zeta_p(2s) \right) = \zeta_p(s)(1 + (p - 1)p^{-s}) \quad \text{for } d \in \{3, 4\} \text{ and } p|d. \]

Assume now that \( d = 6 \) (and hence \( r = -1 \)). If \( p = 3 \), then the analysis is the same as above except that the condition \( t_{22} t_{33} \) in the description of \( T^a \) must be replaced by \( t_{22} t_{13} \) since this time \( r = 2 + 1 = 3 \), and the condition (4b) is now \( |t_{33}| = |t_{33}| \text{ and } \frac{22}{23} t_{33} - 1 \). Since \( \frac{22}{23} \) is a unit, we can use the previous condition to reduce (1) to \( t_{22} t_{12} \). Thus, the conditions defining \( T^a \) are \( t_{22} t_{12}, 32 (|t_{22}| + 1) t_{12} + t_{22} t_{21}, 32 t_{33} t_{23} t_{22} t_{23} \), and the integral over \( T^a \) becomes

\[
\frac{1}{(1 - 3^{-1})^2} \int_{t_{22}} |t_{11}|^{s-1} |t_{22}|^{2-s} 3^{-1} \, d\mu = \frac{1}{(1 - 3^{-1})^2} \int |t_{11}|^{s-1} |t_{22}|^{2-s} 3^{-1} \, d\mu = 3^{-1} \zeta_3(s) \zeta_3(2s),
\]

where in the first equality we used Proposition 3.6 with the pivots \( t_{12}, t_{13}, t_{23}, t_{33} \).

We conclude that

\[ \zeta_{G_3}^{N_{G_3} \cdot \phi}(s) = \zeta_3(s) \zeta_3(2s) + 3^{-s} \zeta_3(s) \zeta_3(2s) = \zeta_3(s)^2. \]

Assume finally that \( p = 2 \). Since there are no solutions in \( \mathbb{Z}_2 \) for \( T^2 + T + 1 \equiv 0 \mod 2 \), (4) is equivalent to \( |t_{33}| = |t_{22}| t_{12} \) (and \( 2 \) is reduced to \( t_{22} \)). Thus, the conditions defining \( T \) are \( t_{22} - t_{12} - t_{13}, t_{22} t_{12}, t_{22} t_{23} \), and \( t_{22} t_{23} = |t_{33}| \), and the integral becomes

\[
\zeta_{G_2}^{N_{G_2} \cdot \phi}(s) = \frac{1}{(1 - 2^{-1})^2} \int |t_{11}|^{s-1} |t_{22}|^{2-s} \, d\mu = \frac{1}{(1 - 2^{-1})^2} \int |t_{11}|^{s-1} |t_{22}|^{2-s} \, d\mu = \zeta_2(s) \zeta_2(2s),
\]

where in the first equality we used Proposition 3.6 with the pivots \( t_{13}, t_{12}, t_{23}, t_{33} \).

5.2.3. Local factors of \( \zeta_{G}^{N \cdot \phi}(s) \). If \( p | d \), then \( |N_p, G_p \rangle = \langle x_2 x_3, x_2 x_3 x_4 x_5 \rangle = \langle x_2 x_3, x_4 x_5 \rangle = \langle x_2, x_3 \rangle \), where the last equality holds since \( r + 2 \in \mathbb{Z}_2 \). As \( N_p, G_p \rangle \cong \mathbb{Z}_p \), we have \( \zeta_{G_p}^{N \cdot \phi}(s) = \zeta_p(s) \) by Proposition 2.12.

Assume now that \( p | d \). Arguing as in the computation of \( \zeta_{G_p}^{N \cdot \phi}(s) \) with \( p \nmid d \), we find that \( \zeta_{G_p}^{N \cdot \phi}(s) = \zeta_{E_p}^{N \cdot \phi}(s) \), and according to the results of 13.6.10, 13.6.13, 13.6.16, this is equal to \( 1 + p^{-2} \) if \( d = 3 \) or 4, and to 1 if \( d = 6 \).

5.2.4. Local factors of \( \zeta_{G}^{H \cdot \phi}(s) \) for \( d = 4 \) and \( H = \langle x_2, x_1, x_2, x_3 \rangle \). Note that \( H = \mathcal{F}_d \). If \( p \not| d \), then \( |N_p, H_p \rangle = \langle x_2 x_3, x_2 x_3 x_4 x_5 \rangle = \langle x_2, x_3 \rangle \), and hence \( N_p, H_p \rangle \cong \mathbb{Z}_p \). Therefore, by Proposition 2.12, we have \( \zeta_{H_p}^{N \cdot \phi}(s) = \zeta_p(s) \).

Assume that \( p = 2 \). An open subgroup \( A \leq G_2 \) such that \( A \mathcal{N}_2 = H_2 \) must include \( Z_2 \). Indeed, \( A \) contains \( x^2 x^4 \) for some \( v \in \mathbb{Z}_2 \), and hence \( \langle x^2 \rangle = \langle x^2 \rangle = \langle x \rangle \), and \( x_1 \in A \). Since \( 2T_1 + 1 \in \mathbb{Z}_2 \) it follows that \( \zeta_{G_2}^{N \cdot \phi}(s) = \zeta_{E_2}^{N \cdot \phi}(s) \), and this is equal to \( 1 + 2^{-2} \cdot 4^{-s} \cdot 4^{-s} \), according to the results of 13.6.10.

5.2.5. Local factors of \( \zeta_{G}^{H \cdot \phi}(s) \) for \( d = 6 \) and \( H = \langle x_2, x_1, x_2, x_3 \rangle \). Note that \( H = \mathcal{F}_d \). Arguing as in the previous paragraph we find that \( \zeta_{G_p}^{N \cdot \phi}(s) = \zeta_p(s) \) if \( p \not| d \), and that \( \zeta_{G_2}^{N \cdot \phi}(s) = \zeta_{E_2}^{N \cdot \phi}(s) \), which is equal to \( 1 + 4^{-s} \), according to the results of 13.6.16.

5.2.6. Local factors of \( \zeta_{G}^{K \cdot \phi}(s) \) for \( d = 6 \) and \( K = \langle x_2, x_1, x_2, x_3 \rangle \). Note that \( K = \mathcal{F}_d \). Arguing as in the case \( d = 3 \) and \( K = G \), we obtain that \( \zeta_{G_p}^{N \cdot \phi}(s) = \zeta_p(s) \) if \( p \not| d \), and that \( \zeta_{G_2}^{N \cdot \phi}(s) = \zeta_{E_2}^{N \cdot \phi}(s) \), which is equal to \( 1 + 3^{-s} \), according to the results of 13.6.16.

5.3. Computing the zeta functions of \( G = \mathcal{F}_d \). Note that \( Z = 0 \) and that \( N = \langle x_1 \rangle \oplus \langle x_2 \rangle \oplus \langle x_3 \rangle \) is a decomposition of \( N \) into a direct sum of irreducible \( \mathbb{Z}[P] \)-modules that are pairwise non-isomorphic.
5.3.1. Local factors of $\zeta_G^{G,\infty}_{\mathbb{Z}}(s)$. If $p \neq 2$, then by using the above decomposition of $N$ and the results of Section 2, we obtain that $\zeta_{G,p}^{G,\infty}(s) = \zeta_p(s-1)\zeta_p(s-1)$.

We now compute $\zeta_{G_2}^{G,\infty}(s)$. An open subgroup $A \subseteq G_2$ such that $AN_2 = G_2$ must contain $\alpha x\beta\gamma$, $\beta x\alpha y$, and $\alpha \beta x\gamma$ for some $u, v, w \in \mathbb{Z}_2^3$, whence $A$ contains the squares of these elements, which are $x_1^{2u+1}$, $x_2^{2v+1}$, and $x_3^{2w+1}$. Since $2u+1, 2v+1, 2w+1 \in \mathbb{Z}_2^3$, it follows that $A$ includes $x_1, x_2, x_3$ and hence also $N_2$. This implies that $A = G_2$. We conclude that $\zeta_{G_2}^{G,\infty}(s) = 1$.

5.3.2. Local factors of $\zeta_G^{N,\infty}(s)$: If $p \neq 2$, arguing as in the calculation of $\zeta_{G_p}^{G,\infty}(s)$, we find that $\zeta_{G_p}^{N,\infty}(s) = \zeta_p(s)^3$.

Assume now that $p = 2$. By Corollary 5.1, we have

$$\zeta_{G_2}^{N,\infty}(s) = \frac{1}{(1-2^{-1})^3} \int_{T} |t_{11}|^2 |t_{22}|^2 |t_{33}|^2 \cdot \frac{d\mu}{(1-2^{-1})^2} \int_{T} |t_{11}|^2 |t_{22}|^2 |t_{33}|^2 \cdot \frac{d\mu}{(1-2^{-1})^2},$$

where $T \subseteq T_1^3(\mathbb{Z}_2)$ is defined by the following equivalent conditions:

\begin{align*}
\alpha(x_1)x^{-4}, \beta(x_2)x^{-3}, &\beta(x_3)x^{-2}, \gamma(x_1)x^{-1},
\end{align*}

\begin{align*}
&\Rightarrow x_2 \cdot x_3 \in \mathbb{Z}_2^3, \quad x_1, x_2, x_3, x_2^2, x_3^2, x_4 \in \mathbb{Z}_2^3.
\end{align*}

According to Lemma 5.1, these conditions are equivalent to

1. $t_{22}[2t_{12}, \quad 2] t_{33} = 2 t_{22} t_{23}$,
2. $t_{33}[2t_{13}, \quad 3] t_{33} = 2 t_{23}$,
3. $t_{33}[2t_{13}, \quad 4] t_{33} = 2 t_{23}$.

We express $T$ as a disjoint union $T_a \cup T_b$ according to the cases (4a) $t_{33}[t_{23}, \quad 2] t_{33} = 2 t_{23}$. In the first case, (2) is redundant and hence $T_a$ is defined by the conditions $t_{22}[2t_{12}, \quad 3] t_{33}[t_{23}, \quad 3]$ $t_{33}[t_{23}, \quad 3] t_{33}[t_{23}, \quad 3]$. Thus,

$$\frac{1}{(1-2^{-1})^3} \int_{T_a} |t_{11}|^2 |t_{22}|^2 |t_{33}|^2 \cdot \frac{d\mu}{(1-2^{-1})^2},$$

Assume now (4b). Note that (2) becomes equivalent to $t_{22}[t_{12}, \quad 2] t_{23}$, whence (1) is redundant. Note also that (3) is equivalent to $t_{23}[t_{13}, \quad 3]$. Therefore, the conditions defining $T_b$ are $t_{22}[t_{12}, \quad 2] t_{33}[t_{23}, \quad 3]$ and $t_{33}[t_{23}, \quad 3] = 2 t_{23}$. And the integral over $T_b$ becomes

$$\frac{1}{(1-2^{-1})^3} \int_{T_b} |t_{11}|^2 |t_{22}|^2 |t_{33}|^2 \cdot \frac{d\mu}{(1-2^{-1})^2},$$

where in the second equality we used Proposition 3.6 with the pivots $t_{12}, t_{13}, t_{33}$. We conclude that

$$\zeta_{G_2}^{N,\infty}(s) = \zeta_2(s) \left(1 + 2^{1-s} \zeta_2(s)^2 + 2^{1-s} \zeta_2(s)^2\right) = (1 + 4 \cdot 2^{-s} + 4^{-s}) \zeta_2(s)^3.$$

5.3.3. Local factors of $\zeta_G^{G,\infty}(s)$. If $p \neq 2$, then $[N_p, G_p] = (x_1, x_2, x_3) = (x_1, x_2, x_3)$, whence $\zeta_{G_p}^{G,\infty}(s) = 1$ by Proposition 2.12. If $p = 2$, then $\zeta_{G_2}^{G,\infty}(s) = 1$ since $\zeta_{G_2}^{G,\infty}(s) = 1$, as shown in 5.3.1.

5.3.4. Local factors of $\zeta_G^{H,\infty}(s)$ for $H = \langle x, x_1, x_2, x_3 \rangle$. Note that $H \cong \mathbb{F}_2$. Arguing as in 5.2.4, we find that $\zeta_{G_p}^{H,\infty}(s) = \zeta_p(s)$, and that $\zeta_{G_2}^{H,\infty}(s) = \zeta_{G_2}^{G,\infty}(s)$. One easily checks that $G/(x_1) \cong \mathbb{F}_2^2$; therefore, the latter series is equal to 1 + 2 + $2^{-s}$, according to the results of [13] 6.7.

5.3.5. Calculation of $\zeta_G^{\epsilon,\infty}(s)$ and $\zeta_G^{\epsilon,\infty}(s)$ for $K = \langle \beta, x_1, x_2, x_3 \rangle$ and $L = \langle \alpha \beta, x_1, x_2, x_3 \rangle$ is easy to check that $\zeta_G^{\epsilon,\infty}(s) = \zeta_G^{H,\infty}(s)$. This series was computed in the previous paragraph.

5.4. Computing the zeta functions of $G = \mathbb{F}_2$. We call $y_1 = x_3, y_2 = x_1$ and $y_3 = x_1 x_2$, so that now the relations are $\epsilon^2 = y_2, \epsilon y_3 = y_3$, and $\epsilon y_1 = y_1^2$. We will use this presentation. Note that $Z = \langle y_2, y_3 \rangle$, and hence $T \cong \mathbb{Z}$.
5.4.1. Local factors of $\zeta_G^{G,C}(s)$. If $p \neq 2$, then by (5.1) we have $\zeta_G^{G,C}(s) = \zeta_p(s)\zeta_p(s-1) = \zeta_p(s)\zeta_p(s-1)^2$.

Assume now that $p = 2$. By Proposition 3.5 with $H = G$, we have

$$\zeta_G^{G,C}(s) = \frac{1}{(1-2^{-s})^2} \int_{\mathcal{T}} |t_{11}|^{2s-2} |t_{22}|^{s-2} |t_{33}|^{2s-4} \, dt,$$

where $\mathcal{T} \subset T^+(\mathbb{Z}_2) \times \mathbb{Z}_2^3$ is the set of pairs $(t, v)$ satisfying the following equivalent conditions:

$$\epsilon(y^{t_1} y^{t_2}), \epsilon(y^{t_2}), \epsilon(y^{t_3}), (y^v)^2 \in \langle y^{t_1}, y^{t_2}, y^{t_3} \rangle$$

$$\Leftrightarrow y_2^{t_1 t_2}, y_3^{t_1 + 2t_3}, y_2^{2t_2 + 1}, y_3^{2t_3 + v} \in \langle y^{t_1}, y^{t_2}, y^{t_3} \rangle.$$

According to Lemma 5.1, these conditions are equivalent to

(1) $t_{22}|2t_{12}$,
(2) $t_{33} \neq -2t_{12}$
(3) $t_{22}|2v + 1$
(4) $t_{33} \neq -2t_{22}$

Note that (3) is equivalent to $t_{22} \in \mathbb{Z}_2^\times$, whence (1) is redundant. We now express $\mathcal{T}$ as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases: a) $t_{33} \in \mathbb{Z}_2^\times$ and b) $t_{33} \in \mathbb{Z}_2$.

Note that $\mathcal{T}^a$ is defined by $t_{22}, t_{33} \in \mathbb{Z}_2^\times$, and hence

$$\int_{\mathcal{T}^a} \left| t_{11} \right|^{2s-2} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-4} \, dt = \zeta_2(s-1).$$

In case b), (2) can be written as $2t_2 | t_{22} + t_1$ and it implies that $t_{11} \in \mathbb{Z}_2^\times$. By using Proposition 3.6 with the pivots $(t_{13}, v_1)$ the integral over $\mathcal{T}^b$ becomes

$$\int_{\mathcal{T}^b} \left| t_{11} \right|^{2s-2} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-4} \, dt = \int_{11 \cdot t_{33} \in \mathbb{Z}_2^\times} \left| t_{11} \right|^{2s-2} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-4} \, dt = 2^{3-2s} \zeta_2(s-1)^2.$$

We conclude that $\zeta_G^{G,C}(s) = \zeta_2(s-1)^2(1 + 2^{1-2s} \zeta_2(s-1)) = \zeta_2(s-1)^2(1 - 2^{1-s} + 8 \cdot 4^{-s})$.

5.4.2. Local factors of $\zeta_G^{N,C}(s)$. If $p \neq 2$, then by (5.1) we have $\zeta_G^{N,C}(s) = \zeta^{G,C}_p(s)\zeta_p(s) = \zeta_p(s)^2\zeta_2(s-1)$.

Assume now that $p = 2$. By Corollary 5.3, we have

$$\zeta_G^{N,C}(s) = \frac{1}{(1-2^{-s})^2} \int_{\mathcal{T}} \left| t_{11} \right|^{2s-1} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt,$$

where $\mathcal{T} = \{ t \in T^+(\mathbb{Z}_2) : \epsilon(y^{t_1} y^{t_2}), \epsilon(y^{t_2}), \epsilon(y^{t_3}) \in \langle y^{t_1}, y^{t_2}, y^{t_3} \rangle \} = \{ t \in T^+(\mathbb{Z}_2) : y_2^{t_1 t_2}, y_3^{t_1 + 2t_3}, y_2^{2t_2 + 1}, y_3^{2t_3 + v} \in \langle y^{t_1}, y^{t_2}, y^{t_3} \rangle \}$.

By Lemma 5.1 the conditions defining $\mathcal{T}$ are equivalent to

(1) $t_{22}|2t_{12}$,
(2) $t_{33} \neq -2t_{12}$

We can express $\mathcal{T}$ as a disjoint union $\mathcal{T}^a \cup \mathcal{T}^b$ according to the cases: (1a) $|t_{22}| = 2|t_{12}|$ and (1b) $t_{22}|t_{12}$.

In the first case the coefficient of $t_{33}$ in (2) is a unit, so the integral over $\mathcal{T}^a$ becomes

$$\frac{2^{2-s}}{(1-2^{-s})^2} \int_{\mathcal{T}^a} \left| t_{11} \right|^{2s-1} \left| t_{12} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt = \frac{2^{2-s}}{(1-2^{-s})^2} \int_{11 \cdot t_{33} \in \mathbb{Z}_2^\times} \left| t_{11} \right|^{2s-1} \left| t_{12} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt = 2^{3-2s} \zeta_2(s)^2 \zeta_2(s-1)^2.$$

where in the first equality we used Proposition 3.6 with the pivots $t_{22}, t_{12}$.

Assume now (1b). We express $\mathcal{T}^b$ as a disjoint union $\mathcal{T}^b = \mathcal{T}^{b,1} \cup \mathcal{T}^{b,2}$ according to the sub-cases (i) $t_{33} \in \mathbb{Z}_2^\times$ and (ii) $t_{33} \in \mathbb{Z}_2^\times$. Note that $\mathcal{T}^{b,1}$ is just defined by $t_{22}|t_{12}$ and $t_{33} \in \mathbb{Z}_2^\times$, so the integral over $\mathcal{T}^{b,1}$ becomes

$$\frac{1}{(1-2^{-s})^2} \int_{\mathcal{T}^{b,1}} \left| t_{11} \right|^{2s-1} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt = \frac{1}{(1-2^{-s})^2} \int_{11 \cdot t_{22} \in \mathbb{Z}_2^\times} \left| t_{11} \right|^{2s-1} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt = 2^{3-2s} \zeta_2(s)^2 \zeta_2(s-1)^2.$$

In sub-case (ii), condition (2) implies that $t_{11} \in \mathbb{Z}_2^\times$, whence $\mathcal{T}^{b,2}$ is defined by the conditions $2|t_{11}, 2|t_{33}, t_{22}|t_{12}$ and $t_{33} \neq -2t_{22} + t_1$ and $t_{12}$. By using Proposition 3.6 with the pivot $t_{13}$, the integral over $\mathcal{T}^{b,2}$ becomes

$$\frac{1}{(1-2^{-s})^2} \int_{\mathcal{T}^{b,2}} \left| t_{11} \right|^{2s-1} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt = \frac{1}{(1-2^{-s})^2} \int_{11 \cdot t_{13} \in \mathbb{Z}_2^\times} \left| t_{11} \right|^{2s-1} \left| t_{22} \right|^{s-2} \left| t_{33} \right|^{2s-3} \, dt = 2^{3-2s} \zeta_2(s)^2 \zeta_2(s-1).$$

We conclude that $\zeta_G^{N,C}(s) = \zeta_2(s)^2 \zeta_2(s-1)^2 + 2^{3-2s} \zeta_2(s-1)^2 = \zeta_2(s)^2 \zeta_2(s-1)(1 - 2^{1-s} + 8 \cdot 4^{-s})$.

5.4.3. Calculation of $\zeta_G^{G,N}(s)$. Note that $|N, G| = y_2^2 y_3$, and $G/[N, G] = \mathbb{Z}_2$. By Proposition 2.12 we have $\zeta_G^{G,N}(s) = \zeta_G^{G,[N, G]}(s)$ (with respect to $N/[N, G]$). Applying Lemma 5.6 with $Z_1 = G/[N, G]$ and $Z_0 = N/[N, G]$, we obtain that $\zeta_G^{G,N}(s) = \zeta(s)\zeta(s-1)(1 - 2^{-s})$.

5.5. Computing the zeta functions of $G = \mathcal{B}_3$. Note that $Z = (x_1)$. One can easily check that $E = G/Z$ is the plane crystallographic group $p2mg$. 


5.5.1. Local factors of $\zeta_{G_p}^{G_0}(s)$. If $p \neq 2$, then by (5.1) we have $\zeta_{G_p}^{G_0}(s) = \zeta_p(s)\zeta_{E_2}^{E_0}(s)$, which is equal to $\zeta_p(s)\zeta_p(s-1)^2$ according to the results of [13, 5.6].

We now compute $\zeta_{G_2}^{G_0}(s)$. An open subgroup $A < G_2$ such that $AN_2 = G_2$ must contain $a\xi^v$ for some $v \in Z_2^3$, and hence it contains $(a\xi x)^2 = x_1^{2v_1+1}$. Since $2v_1 + 1 \in Z_2$, we deduce that $Z_2 \subset A$. Therefore, $\zeta_{G_2}^{G_0}(s) = \zeta_{E_2}^{E_0}(s)$, which is equal to $(1 + 2^{1-s})\zeta_2(s-1)$ according to [13, 5.6].

5.5.2. Local factors of $\zeta_{G_p}^{N_{G_0}}(s)$. If $p \neq 2$, then by (5.1) we have $\zeta_{G_p}^{N_{G_0}}(s) = \zeta_p(s)\zeta_{E_2}^{T_2}(s)$, which is equal to $\zeta_p(s)^3$ according to [13, 6.6].

Assume now that $p = 2$. By Corollary 5.3

$$\zeta_{G_2}^{N_{G_0}}(s) = \frac{1}{(1 - 2^{-s})^2} \int_{T} |t_{11}|^{s-1} |t_{22}|^{s-2} |t_{33}|^{s-3} d\mu,$$

where $T \subset T_3(Z_2)$ is defined by the following equivalent conditions:

$$\begin{align*}
\alpha(x^{t_1})x^{-t_1}, \alpha(x^{t_2}), \alpha(x^{t_3}), \epsilon(x^{t_1})x^{-t_1}, \epsilon(x^{t_2}), \epsilon(x^{t_3}) & \in \langle x_1^{t_1}, x_2^{t_2}, x_3^{t_3} \rangle \\
\Leftrightarrow x_2^{2t_1}x_3^{2t_1}, x_2^{t_2}, x_2^{t_3}, x_3^{2t_2}, x_3^{2t_3}, x_3^{t_3} & \in \langle x_1^{t_1}, x_2^{t_2}, x_3^{t_3} \rangle \\
\Leftrightarrow x_2^{t_1}, x_3^{2t_1}, x_3^{2t_2}, x_3^{2t_3} & \in \langle x_1^{t_1}, x_2^{t_2}, x_3^{t_3} \rangle.
\end{align*}$$

This is the same integral that showed up in (3.3) when $p = 2$, so the result is $(1 + 4 \cdot 2^{-s} + 4^{-s})\zeta_2(s)^3$.

5.5.3. Local factors of $\zeta_{G_p}^{H_{G_0}}(s)$. If $p \neq 2$, then $[N_p, G_p] = \langle x_2^2, x_3^2 \rangle = \langle x_2, x_3 \rangle$, whence $N_p/[N_p, G_p] \cong \mathbb{Z}_p$. Thus, by Proposition 2.12 $\zeta_{G_p}^{H_{G_0}}(s) = \zeta(s)$.

Assume now that $p = 2$. By the same argument used in the calculation of $\zeta_{G_2}^{G_0}(s)$ in 5.5.1, we have $\zeta_{G_2}^{G_0}(s) = \zeta_{E_2}^{E_0}(s)$, which is equal to $1 + 4 \cdot 2^{-s}$ according to [13, 6.6].

5.5.4. Local factors of $\zeta_{G_p}^{H_{G_0}}(s)$ for $H = \langle \alpha, x_1, x_2, x_3 \rangle$. Note that $H = \mathbb{Z}_2$. Arguing as in 5.2.6.1 we find that $\zeta_{G_p}^{H_{G_0}}(s)$ if $p \neq 2$, and that $\zeta_{G_p}^{H_{G_0}}(s) = \zeta_{E_2}^{E_0}(s)$. The latter series is equal to $1 + 2^{1-s}$ according to [13, 6.6].

5.5.5. Local factors of $\zeta_{G_p}^{K_{G_0}}(s)$ for $K = \langle \epsilon, x_1, x_2, x_3 \rangle$. Note that $K = \mathbb{Z}_2$. If $p \neq 2$, then by (5.1) we have $\zeta_{G_p}^{K_{G_0}}(s) = \zeta_{E_2}^{E_0}(s)$, which is equal to $\zeta(s)$ according to [13, 6.6].

Assume now that $p = 2$. We claim that if $A < G_2$ is open and satisfies $AN_2 = K_2$, then it must include $\mathbb{Z}_2$. Indeed, $A$ includes $K_2, K_3 = \langle x_3 \rangle$ by Lemma 2.11. It also contains $\epsilon \mathbb{Z}_2$ for some $\epsilon \in Z_2^3$, whence it contains $[\alpha, \epsilon \mathbb{Z}_2] = x_2^{2v_1t_1}x_3^{v_2t_2}$.

5.5.6. Local factors of $\zeta_{G_p}^{H_{G_0}}(s)$ for $H = \langle \epsilon, x_1, x_2, x_3 \rangle$. Note that $K = \mathbb{Z}_2$. If $p \neq 2$, then by (5.1) we have $\zeta_{G_p}^{E_0}(s) = \zeta(s)$, which is equal to $\zeta(s)$ according to [13, 6.6].

Assume now that $p = 2$. By Corollary 5.3 (with $H = L$), we have

$$\zeta_{G_2}^{E_0}(s) = \frac{1}{(1 - 2^{-s})^2} \int_{T} |t_{11}|^{s-2} |t_{22}|^{s-2} |t_{33}|^{s-4} d\mu,$$

where $T \subset T_3(Z_2) \times Z_2^3$ is the set of pairs $(t, v)$ satisfying the following equivalent conditions:

$$\begin{align*}
\alpha(x^{t_1})x^{-t_1}, \alpha(x^{t_2}), \alpha(x^{t_3}), \epsilon(x^{t_1})x^{-t_1}, \epsilon(x^{t_2}), \epsilon(x^{t_3}) & \in \langle x_1^{t_1}, x_2^{t_2}, x_3^{t_3} \rangle \\
\Leftrightarrow x_2^{2t_1}x_3^{2t_1}, x_2^{t_2}, x_2^{t_3}, x_3^{2t_2}, x_3^{2t_3}, x_3^{t_3} & \in \langle x_1^{t_1}, x_2^{t_2}, x_3^{t_3} \rangle \\
\Leftrightarrow x_2^{t_1}, x_3^{2t_1}, x_3^{2t_2}, x_3^{2t_3} & \in \langle x_1^{t_1}, x_2^{t_2}, x_3^{t_3} \rangle.
\end{align*}$$

By Lemma 5.1, these conditions are equivalent to

1. $t_{33}|2t_{13}$, 2. $t_{33}|2t_{23}$, 3. $t_{11}|2v_1 + 1$, 4. $t_{22} = 2v_1 + 1/t_{11}$, 5. $t_{33} = 2v_1 + 1/t_{11}$, 6. $t_{22} = 2v_1 + 1/t_{11}$, 7. $t_{22} = 2v_1 + 1/t_{11}$, 8. $t_{22} = 2v_1 + 1/t_{11}$, 9. $t_{33} = 2v_1 + 1/t_{11}$. 10. $|t_{11}|^{s-2} |t_{22}|^{s-2} |t_{33}|^{s-4} d\mu.$
Note that (3) and (8) are equivalent to $t_{11}, t_{22} \in \mathbb{Z}_2^*$ and hence (4) and (6) are redundant. Notice also that (2) implies (7). To sum up, $T$ is defined by (1), (2), (5), (9) and $t_{11}, t_{22} \in \mathbb{Z}_2^*$. We express $T$ as a disjoint union $T^a \cup T^b$ according to the cases (2a) $t_{33}|t_{23}$ and (2b) $|t_{33}| = |2t_{23}|$.

If we assume (2a), then (9) can be replaced by $t_{33}|2t_{23}$ and then (5) can be replaced by $t_{33}|t_{13}$, which makes (1) redundant. Thus, the conditions defining $T^a$ are $t_{11}, t_{22} \in \mathbb{Z}_2^*$, $t_{33}|t_{13}$, $t_{33}|t_{23}$ and $t_{33}|2t_{23}$, and the integral over $T^a$ becomes

$$\int_{T^a} |t_{33}|^{-4} \, d\mu = \frac{1}{1 - 2^{-1}} \int_{t_{33}|t_{23}} |t_{33}|^{-4} \, d\mu = 1 + 2^{1-s} \zeta_2(s).$$

If we assume (2b), then (9) implies $2|t_{23}$. Note also that (1) and (9) can be replaced by $(1') t_{23}|t_{13}$ and $(9') t_{33}| \frac{t_{23}}{2 - t_{23}} - v_3$ and the latter implies that $t_{33}|2t_{23}$. This enables us to replace (5) by $(5') 2 \frac{t_{11} + t_{12}}{t_{23}} t_{23} - 2^{1-s} t_{23} + \frac{t_{23}}{2}$. The conditions defining $T^b$ are, therefore, $t_{11}, t_{22} \in \mathbb{Z}_2^*$, $2|t_{23}$, $|t_{33}| = |2t_{23}|$, (1'), $(5')$, and (9'), and the integral over $T^b$ becomes

$$\int_{T^b} |2t_{23}|^{-4} \, d\mu = \frac{1}{1 - 2^{-1}} \int_{2|t_{23}} |2t_{23}|^{-4} \, d\mu = 1 + 2^{1-s} \zeta_2(s),$$

where in the first equality we first applied Proposition [5.1] with the pivots $(t_{11}, t_{22}, t_{13}, t_{12}, t_{33}, v_3)$.

Therefore, $\zeta_2^{2-s}(s) = 1 + 2^{1-s} \zeta_2(s) + 2^{1-s} \zeta_2^2(s) = \zeta_2(s)(1 + 2^{-s} + 2^{1-2s})$.

### 5.6. Computing the zeta functions of $G = \mathbb{B}_4$.

Note that $Z = \langle x_1 \rangle$. One can easily check that $E = G/Z$ is the plane crystallographic group $\mathbb{P}2ggg$.

#### 5.6.1. Local factors of $\zeta_G^{K,c}(s)$.

If $p \neq 2$, then by [5.2.1] we have $\zeta_G^{p,c}(s) = \zeta_p(s)\zeta^{p,c}(s)$, and according to the results of [13] 5.7 the latter is equal to $\zeta_p(s)\zeta_p(s - 1)^2$.

Assume that $p = 2$. An open subgroup $A \leq G_2$ such that $AN_2 = G_2$ contains $\alpha x^v$ for some $v \in \mathbb{Z}_2^*$. Since $2v_1 + 1 \in \mathbb{Z}_2^*$, we conclude that $Z_2 \leq A$. It follows that $\zeta_G^{2,c}(s) = \zeta_2^{2,c}(s)$, which is 1, according to [13] 5.7.

#### 5.6.2. Calculus of $\zeta_G^{K,c}(s)$.

That $N$, as $\mathbb{Z}[P]$-module, is isomorphic to the analogue module in the case $G = \mathbb{B}_3$. Therefore, $\zeta_G^{K,c}(s) = (\zeta(s))^3(1 - 2^{-s} + 2^{-s})$ according to the result of [5.5.2].

#### 5.6.3. Local factors of $\zeta_G^{K,c}(s)$.

If $p \neq 2$, then $[N_p, G_p] = \langle x_2^2, x_3^2 \rangle = \langle x_2^2, x_3^2 \rangle$, whence $N_p/[N_p, G_p] \cong \mathbb{Z}_p$. Thus, by Proposition 5.1.11, $\zeta_G^{p,c}(s) = \zeta_p(s)\zeta^{p,c}(s)$.

Assume now that $p = 2$. If $A \leq G_2$ open and satisfies $AN_2 = G_2$, then $A$ contains $\alpha x^v$ and $e x^w$ for some $v, w \in \mathbb{Z}_2^*$, whence it contains both $(\alpha x^v)^2 = x_2^{1+v} + 1$ and $(\beta x^w)^2 = x_2^{1+v} x_2^{2x^w+1}$. Since $2v_1 + 1, 2w_1 + 1 \in \mathbb{Z}_2^*$, we conclude that $x_1, x_2 \in A$. In addition, $[G_2, G_2] \subseteq A$ by Lemma 2.11 in particular $x_2 x_3 \in A$. It follows that $x_1, x_2, x_3 \in A$ and hence $A = G_2$. Thus, $\zeta_G^{2,c}(s) = 1$.

#### 5.6.4. The local factors of $\zeta_G^{H,c}(s)$ for $H = \langle x_1, x_2, x_3 \rangle$.

Note that $H \cong \mathbb{B}_2$. Arguing as in 5.2.4, we find that $\zeta_G^{H,c}(s) = \zeta_p(s)\zeta^{p,c}(s) = \zeta_2^{2,c}(s)$, which is equal to $\zeta_2(s)\zeta_2(s - 1)^2$ according to the results of [13] 6.7.

#### 5.6.5. Local factors of $\zeta_G^{K,c}(s)$ for $K = \langle x, x_1, x_2, x_3 \rangle$.

Note that $K \cong \mathbb{B}_2$. If $p \neq 2$, then by [5.1] we have $\zeta_G^{p,c}(s) = \zeta_p(s)\zeta^{p,c}(s)$, which is equal to $\zeta_p(s)\zeta_p(s - 1)^2$ according to the results of [13] 6.7.

Assume now that $p = 2$. By Proposition 6.5 with $H = K$, we have

$$\zeta_G^{K,c}(s) = \frac{1}{1 - 2^{-1}} \int_{T^+} |t_{11}|^{-2} |t_{22}|^{-2} |t_{33}|^{-4} \, d\mu,$$

where $T^+ \subset \mathbb{T}_3^+(\mathbb{Z}_2) \times \mathbb{T}_3^+$ is the set of pairs $(t, v)$ satisfying the following equivalent conditions:

- $\alpha(x^1) x^t_1, \alpha(x^2) x^t_2, \alpha(x^3), (x^4)^t_1 x^t_1, (x^4)^t_2 x^t_2, \alpha(x^5) \in (x_1^2, x_2^2, x_3^2)$
- $x_2^{2x_1^3} x_3^t_1, x_2^{t_1}, x_3^{t_2}, x_3^t_3, x_2^{t_2}, x_3^{t_3}, x_1, x_2, x_3, \alpha(x^v) \in (x_1^2, x_2^2, x_3^2)$

By Lemma 5.1, these conditions are equivalent to

1. $t_{22}|2t_{12}$
2. $t_{33}|\frac{t_{12}}{t_{22}} t_{23}$
3. $t_{11}|2v_1$
4. $t_{22} - \frac{t_{21}}{t_{11}} t_{12}$
5. $t_{33}|\frac{t_{21}}{t_{11}} t_{12} - t_{33} - 1$
6. $t_{33}|2$
7. $t_{22} - 1 - 2v_2$
8. $t_{33}|\frac{1}{t_{22}} t_{23} - 1$

Note that (7) is equivalent to $t_{22} \in \mathbb{Z}_2^*$, whence (1) and (4) are redundant and (6) implies (2). We express $T$ as a disjoint union $T^a \cup T^b$ according to the cases (3a) $t_{11}|v_1$ and (3b) $t_{11}|2|v_1|$. 


In the case (3a), (6) reduces (5) to the condition $t_{31}| - 1$, that is $t_{31} \in \mathbb{Z}_2^*$; hence, $T^s$ is defined by $t_{11}v_1$, $t_{22}, t_{33} \in \mathbb{Z}_2^*$. The integral over $T^s$ becomes
\[
\frac{1}{1-2^{-s}} \int_{T^s} |t_{11}|^{-s-2} dt = \frac{1}{1-2^{-s}} \int |t_{11}|^{-s-1} dt = \zeta_2(s).
\]
We now assume (3b). Note that $T^b$ is defined by $|t_{11}| = |2v_1|, t_{22} \in \mathbb{Z}_2^*$, (5), (6), and (8), and the integral over $T^b$ becomes
\[
\frac{1}{1-2^{-s}} \int_{T^b} |2v_1|^{-s-2} |t_{33}|^{-s-4} dt = \frac{1}{1-2^{-s}} \int |t_{33}|^{-s-2} dt = (1 + 2^{-s})^2 \zeta_2(s),
\]
where in the second equality we applied Proposition 5.6 with the pivots $(t_{11}, t_{22}, t_{13}, t_{23})$.

We conclude that $G_{2}\zeta_2^{N\cdot <\cdot}(s) = \zeta_2(s) + (1 + 2^{-s-1})^2 \zeta_2(s) = (1 + 2^{-s} + 2\cdot 2^{-s}) \zeta_2(s).

5.6.6. Local factors of $\zeta_G^{L, <\cdot}(s)$ for $L = \langle \alpha, x_1, x_2, x_3 \rangle$. Note that $L \cong \mathbb{Z}_2$. If $\pi \neq 2$, then by [31], we have $\zeta_G^{L, <\cdot}(s) = \zeta_2(s)\zeta_2^{L, <\cdot}(s)$, which is equal to $\zeta_2(s)\zeta_2^{L, <\cdot}(s)$ according to the results of [13] 6.7.

Assume now that $p = 2$. We claim that any $A < G_2$ that is open and satisfies $AN_2 = L_2$ must include $(x_1x_2, x_2x_3, x_2^2)$. Indeed, $A$ includes $G_2[2] = (x_2^2)$ by Lemma 2.11. It also contains $\alpha x_2^e$ for some $e \in \mathbb{Z}_2$, whence it contains $(\alpha x_2^e)^2 = x_2^{e+1} x_3^{-1} x_3^{e+1}$ and $[\alpha, \alpha x_2^e] = x_2^{e+1} x_3^{-1} x_3^{e+1}$. We deduce that $x_2^{e+1} x_3^{-1} x_3^{e+1} \in A$, and using that $x_2^2 \in A$ we obtain that $(x_1x_2) x_2^{e+1} \in A$. It follows that $x_1x_2 \in A$ since $2v_1 + 1$ is a unit. With a similar argument, we find that $x_2 x_3 \in A$.

This proves the claim. Since $[\alpha, \alpha x_2^e] = (x_2x_3, x_2^2)$, the quotient group $G' := G_2/(x_1x_2, x_2x_3, x_2^3)$ is abelian, and therefore, $G_{2}^{G', <\cdot}(s) = \zeta_{L'}^{G', <\cdot}(s)$, where $L'$ is the image of $L_2$ at $G'$, and the partial zeta function on the right is computed with respect to $N'$, the image of $N_2$ at $G'$. Note that $H' \cong C_2 \times C_2$ and $N'$ has index 2. Thus, $G_{2}^{G', <\cdot}(s) = 1 + 2^{-s} - 2^{-s} - 2^{-s}.$

6. Computing the zeta functions of the 3-dimensional AB-groups

In this final section, we prove the formulae presented in [12]. We always denote by $G$ the AB-group under consideration and by $N$ the Fitting subgroup (which is generated by $x_1, x_2, x_3$). We keep the notation introduced at the beginning of Section 2. In particular,
\[
Z = Z(N), \quad T = N/Z, \quad E = G/Z, \quad P = G/N.
\]
The formulae for the zeta functions of $G = N = \langle x_1, x_2, x_3 : [x_2, x_3] = x_3^3, [x_1, x_3] = [x_2, x_3] = 1 \rangle$, where $k > 0$, are already computed; see Table 1 and 2 with $F = C_1$. We focus here on the cases $G \neq N$. If $N \leq H \leq G$ is an intermediate subgroup, then $H$ is also a 3-dimensional AB-group with Fitting subgroup $N$. Therefore, in order to compute $G_{2}^{N, <\cdot}(s)$, it will be enough to calculate $G_{2}^{N, <\cdot}(s)$ since the other partial zeta functions will have been computed in previous calculations. The isomorphism classes of the intermediate subgroups $N \cong G_i$ for any $i$ (if there is any) will be identified when computing the partial zeta functions $G_{2}^{N, <\cdot}(s)$.

When dealing with local factors at primes $p \mid |G : N|$, we will sometimes use the method of $p$-adic integration described in Section 3 specifically Corollary 3.3 and Proposition 3.5. The following lemma collects some information about the $T$-group $N$.

**Lemma 6.1.** Let $N = \langle x_1, x_2, x_3 : [x_2, x_3] = x_3^3, [x_1, x_3] = [x_2, x_3] = 1 \rangle$, with $k = |Z : [N, N]| > 0$.

1. For $a, b \in Z_p$ and $r \in Z_p$, it holds that
\[
(x^{a}, x^{b}) = x^{a+b+2a_3+3a_4+4a_5} = x^{a_1 x_2 a_2 x_3 a_4 + (\frac{a_5}{3}) a_1 a_2}, \quad [x^{a}, x^{b}] = x^{(b)(a_2-a_1)}.
\]
2. For $t \in T_1^+(Z_p)$, the following holds.
   (a) $t \in M_1^N$, if and only if $t_{33}|kt_{11} t_{22}.$
   (b) $t \in M_1^N$, if and only if $t_{33}|kt_{11} t_{22}$ and $t_{33}|kt_{12}$.
3. Fix $t \in M_1^N$. Given $a = (a_1, a_2, a_3) \in Z_p$, it holds that $x^a \in \langle x^{a_1}, x^{a_2}, x^{a_3} \rangle$ if and only if
   (a) $t_{11}|a_1$.
   (b) $t_{22} = t_{11} + a_2$.
   (c) $t_{33} = \frac{t_{11} t_{22} + t_{22} t_{33}}{t_{11} t_{22}}$. In the case that $t \in N_1^N$, (c) can be replaced by $t_{33} = \frac{t_{11} t_{22} + t_{22} t_{33}}{t_{11} t_{22}} + \frac{a_4 t_{11} - a_4 t_{22}}{t_{11} t_{22}}$.

**Proof.** The verification of (1) is straightforward, and (2) is a special case of (3.4). As for (3), given that $\{x^{a_1}, x^{a_2}, x^{a_3}\}$ is a basis for $\langle x^{a_1}, x^{a_2}, x^{a_3} \rangle$, the relation $x^a \in \langle x^{a_1}, x^{a_2}, x^{a_3} \rangle$ holds if and only $x^a = (x^{a_1})^{\lambda_1} (x^{a_2})^{\lambda_2} (x^{a_3})^{\lambda_3}$ for some $\lambda_1, \lambda_2, \lambda_3 \in Z_p$. Using (1), we find that
\[
(x^{a_1})^{\lambda_1} (x^{a_2})^{\lambda_2} (x^{a_3})^{\lambda_3} = x_1^{\lambda_1 a_1} x_2^{\lambda_1 a_2} x_3^{\lambda_1 a_3} x_2 x_3^{\lambda_2 a_2} t_{11} t_{12} t_{33} t_{33} a_3.
\]
The first part of (3) is now clear. In the case that \( t \in \mathcal{M}^{\alpha}_{\mathbf{G}} \) (and hence \((x_1^t, x_2^t, x_3^t)\) is normal), (2) implies that \( t_{33}|kt_{11} \). Therefore, the term \( \frac{1}{2}\kappa_{\mathbf{G}}(\frac{k}{t_{11}(t_{11} - 1)}t_{11}t_{12} = \left( \frac{kt_{11}}{2} \right)kt_{11}t_{12} \) can be deleted from (c). This completes the proof. \( \square \)

Finally, Table 3 below collects the results of some \( p \)-adic integrals that will appear in our calculations. The verification of these formulae is straightforward. The notation is as follows: \( k \) denotes an integer; the letters \( s, t, \) and \( u \) are complex variables; the variables of integration are \( p \)-adic integers; and \( \theta_p \) denotes \( \frac{1}{1-p^{-\infty}} \).

\[
\begin{array}{|c|c|}
\hline
p\text{-adic integral} & \text{result} \\
\hline
A_{p, k}(s, t) := \theta_{kp}^G \int_{x_{kp}|y} \left| x_{kp}^{s-1} \right| \left| y_{kp}^{t-1} \right| \, d\mu & \zeta_p(t) \left( \frac{1-|k|^s}{1-p^{-s}} + \frac{|k|^s}{1-p^{-t}} \right) \\
B_{p, k}(s, t) := \theta_{kp}^G \int_{x_{kp}|y} \left| x_{kp}^{s-1} \right| \left| y_{kp}^{t-1} \right| \, d\mu & \zeta_p(t) \left( \frac{1-|k|^s}{1-p^{-s}} + \frac{|k|^s}{1-p^{-t}} \right) \\
C_{p, k}(s, t, u) := \theta_{kp}^G \int_{x_{kp}|y} \left| x_{kp}^{s-1} \right| \left| y_{kp}^{t-1} \right| \left| z_{kp}^{u-1} \right| \, d\mu & \zeta_p(s)\zeta_p(t) \left( \frac{1-|k|^s}{1-p^{-s}} + \frac{|k|^s}{1-p^{-t}} \right) \\
D_{p, k}(s, t, u) := \theta_{kp}^G \int_{x_{kp}|y} \left| x_{kp}^{s-1} \right| \left| y_{kp}^{t-1} \right| \left| z_{kp}^{u-1} \right| \, d\mu & \zeta_p(s)\zeta_p(t) \left( \frac{1-|k|^s}{1-p^{-s}} + \frac{|k|^s}{1-p^{-t}} \right) \\
E_{p, k}(s, t, u) := \theta_{kp}^G \int_{x_{kp}|y} \left| x_{kp}^{s-1} \right| \left| y_{kp}^{t-1} \right| \left| z_{kp}^{u-1} \right| \, d\mu & \zeta_p(s)\zeta_p(t) \left( \frac{1-|k|^s}{1-p^{-s}} + \frac{|k|^s}{1-p^{-t}} \right) \\
\hline
\end{array}
\]

Table 3. List of \( p \)-adic integrals

6.1. Computing the zeta functions of \( G = G_{p2,2} \), \( k \in \mathbb{N} \).

6.1.1. Local factors of \( \zeta_G^{\mathcal{G}, \mathcal{C}}(s) \). If \( p \neq 2 \), then \( \zeta_{p}^{G, \mathcal{C}}(s) \) was given in Table 1 (the case \( F = C_2 \) and \( F \subset SL(T) \)).

Assume now that \( p = 2 \). If \( A \leq G_2 \) is open and satisfies \( AN_2 = G_2 \), then \( A \) contains \( \alpha \mathbf{v} \) for some \( \mathbf{v} \in \mathbb{Z}_2 \), whence \((\alpha \mathbf{v})^2 = x_1^{2 \mathbf{v} + 1} \in A \). Since \( 2t_3 + 1 \in \mathbb{Z}_2 \), we obtain that \( Z_p \subset A \). This implies that \( \zeta_{G_2}^{G, \mathcal{C}}(s) = \zeta_{G_2}^{G, \mathcal{C}}(s) \), which in turn is equal to \( \zeta_2(s - 1)\zeta_2(s - 2) \), according to the results of [13, 5.2].

6.1.2. Local factors of \( \zeta_G^{\mathcal{G}, \mathcal{C}}(s) \). If \( p \neq 2 \), then \( \zeta_{p}^{G, \mathcal{C}}(s) \) was given in Table 2 (the case \( F = C_2 \) and \( F \subset SL(T) \)).

Assume now that \( p = 2 \). We show that the assumptions of Proposition 2.15 are satisfied. Firstly, since the class of \( \alpha \) at \( T = G_2/\mathbb{N}_2 \) acts on \( T_2 \) as \(-i\mathbb{D}_2 \), any finite index subgroup of \( T_2 \) is a \( \mathbb{Z}_2[\mathcal{P}] \)-submodule, and they are all isomorphic. Fix \( U \subseteq T_2 \) and \( V \subseteq \mathbb{Z}_2 \) of finite index such that \([U, T_2] \subseteq V \). Let \( \tilde{U} \) be the pre-image of \( U \) in \( N_2 \). Note that \( \tilde{U} \) has a good basis of the form \( \{x_1^{t_1}, x_2^{t_2}, x_3^{t_3}\} \), and that \( \tilde{V} = \langle \mathbf{v} \rangle \) for some non-zero \( t_3 \). It is easy to check that the condition \([U, T_2] \subseteq V \) is translated into \( t_{33}|2kt_{11}, t_{32}|2kt_{12}, t_{32}|2kt_{22} \). Therefore, by Lemma 6.1.2, \( \{x_1^{t_1}, x_2^{t_2}, x_3^{t_3}\} \) represents a good basis for a normal subgroup \( B \leq N_2 \). Clearly, \( (BZ_2)/Z = U \) and \( B \cap Z_2 = V \). We now show that \( B \) is normal in \( G_2 \). Indeed, \( \alpha(x_1^{t_1}, x_2^{t_2}) = x_1^{-t_1}x_2^{-t_2} = (x_1^{t_1}x_2^{t_2})^{-1}x_3^{-2k_{2,1}t_3} \in B \) and \( \alpha(x_2^{t_2}) = x_2^{-t_2} \in B \).

We are now in position to apply Proposition 2.15. For the computation of \( |T_2 : X(V)| \) we use Lemma 2.10 which tells us that \( |T_2 : X(V)| = 1 \) if \( |Z_2 : V| < 2^{2v(2k))} \) and \( |T_2 : X(V)| = |2k_2^2|Z_2 : V^2 \) if \( V \subseteq \langle x_1^t \rangle \). For the computation of \( |\text{Hom}_{\mathbb{Z}_2}[\mathcal{P}](T_2, Z_2/V)| \) we use Lemma 5.2 which tells us that \( |\text{Hom}_{\mathbb{Z}_2}[\mathcal{P}](T_2, Z_2/V)| = 1 \) if \( V = Z_2 \) and \( V \) is \( 2 \)-otherwise. Thus, by Proposition 2.15 and the results of [13, 6.2],

\[
\zeta_G^{\mathcal{G}, \mathcal{C}}(s) = \zeta_{G_2}^{\mathcal{C}}(s) \left( 1 + \sum_{l=1}^{v(2k))} 2^{-4s}2^{2l} + \sum_{V \subseteq 2k_2} |2k_2|^{-2s}Z_2 : V^{-2s}Z_2 : V^{-s}2^l \right)
\]

6.1.3. Local factors of \( \zeta_G^{\mathcal{G}, \mathcal{C}}(s) \). If \( p \neq 2 \), then \( \gamma_3(G_2, N_2) = \langle x_1, x_2, x_3 \rangle \), whence \( N_2 / \gamma_3(G_2, N_2) \cong \mathbb{Z}_2 / \mathbb{Z}_p \). Thus, by Proposition 2.12, \( \zeta_{G_2}^{\mathcal{G}, \mathcal{C}}(s) = \frac{1-\frac{|k|^s}{1-p^{-s}}}{1-\frac{|k|^s}{1-p^{-s}}} \).

Assume now that \( p = 2 \). As shown in the calculation of \( \zeta_{G_2}^{\mathcal{G}, \mathcal{C}}(s) \), any \( A \leq G_2 \) that is open and satisfies \( AN_2 = G_2 \) must include \( Z_2 \). Thus, \( \zeta_{G_2}^{\mathcal{G}, \mathcal{C}}(s) = \zeta_{G_2}^{\mathcal{C}}(s) \), which is equal to \( 1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s} \) according to the results of [13, 6.2].

6.2. Computing the zeta functions of \( G = G_{pg,2k} \).
6.2.1. Local factors of $\zeta_{G}^{\psi,s}(s)$. If $p \neq 2$, then $\zeta_{G}^{\psi,s}(s)$ was given in Table 1 (the case $F = D_1$ and $F \not\subset SL(T)$).

Assume now that $p = 2$. By Proposition 6.3 and Lemma 6.1(2), we have

$$\zeta_{G}^{\psi,s}(s) = \frac{1}{2(1 - 2^{-1})} \int_{T} |t_{11}|^{s-2} |t_{22}|^{s-2} |t_{33}|^{s-3} d\mu,$$

where $T \subset T_{2}^{+}(\mathbb{Z}) \times \mathbb{Z}_{2}^{*}$ is the set of pairs $(t,v)$ satisfying $t_{33}|2kt_{11}t_{22}$ and the following equivalent conditions:

$$x_{11}^{t_{11}}, \beta(x_{11}), [\beta(x_{11}), x_{11}], \beta(x_{11}) \in \langle x_{11}, x_{12}, x_{13} \rangle,$$

$$x_{22}^{t_{22}}x_{33}^{t_{33}+1}t_{33}^{+1}t_{22}+1 \in \langle x_{11}, x_{12}, x_{13} \rangle.$$ 

By Lemma 6.1(3), the conditions defining $T$ are equivalent to

$$(1) \ t_{33}2kt_{11}t_{22}, \quad (2) \ t_{22}2t_{12}, \quad (3) \ t_{33}2t_{12}(-t_{33} + kvt_{22}) - kt_{11}(1 + 2t_{2}) - 2kt_{11}t_{22},$$

$$(4) \ t_{33}2(-t_{23} + kt_{22}v_{1}), \quad (5) \ t_{22}2v_{2} + 1, \quad (6) \ t_{33}2(kt_{22}v_{1})(-t_{23} + kvt_{22}).$$

Note that (5) is equivalent to $t_{22} \in \mathbb{Z}_{2}^{*}$, so (2) is redundant. Next, (6) implies (4), and (6) and (1) reduce (3) to the simpler condition $t_{33}|kt_{11}$. This makes (1) redundant. To sum up, the conditions defining $T$ are: $t_{22} \in \mathbb{Z}_{2}^{*}$, $t_{33}|kt_{11}$, $t_{33} - t_{23} + kvt_{22}$. Thus,

$$\zeta_{G}^{\psi,s}(s) = \frac{1}{2(1 - 2^{-1})} \int_{T} |t_{11}|^{s-2} |t_{33}^{s-3} d\mu = \frac{1}{2(1 - 2^{-1})} \int_{t_{33}|kt_{11}} |t_{11}|^{s-2} |t_{33}|^{s-3} d\mu = A_{2,k}(s - 2, s - 1),$$

where we used Proposition 6.6 with the pivots $t_{22}, t_{23}$.

6.2.2. Local factors of $\zeta_{G}^{\psi,s}(s)$. If $p \neq 2$, then $\zeta_{G}^{\psi,s}(s)$ was given in Table 2 (the case $F = D_1$ and $F \not\subset SL(T)$).

Assume now that $p = 2$. By Corollary 6.3 and Lemma 6.1(2),

$$\zeta_{G}^{\psi,s}(s) = \frac{1}{2(1 - 2^{-1})} \int_{T} |t_{11}|^{s-1} |t_{22}|^{s-2} |t_{33}|^{s-3} d\mu,$$

where $T \subset T_{2}^{+}(\mathbb{Z})$ is defined by $t_{33}|2kt_{11}, t_{33}|2kt_{12}, t_{33}|2kt_{22}$ and the following equivalent conditions:

$$x_{11}^{t_{11}}, \beta(x_{11}), [\beta(x_{11}), x_{11}], \beta(x_{11}) \in \langle x_{11}, x_{12}, x_{13} \rangle,$$

$$x_{22}^{t_{22}}x_{33}^{t_{33}+1}t_{33}^{+1}t_{22}+1 \in \langle x_{11}, x_{12}, x_{13} \rangle.$$ 

Therefore, by Lemma 6.1(3), the conditions defining $T$ are equivalent to

$$(1) \ t_{33}|2kt_{11}, \quad (2) \ t_{33}|2kt_{12}, \quad (3) \ t_{33}|2kt_{22}, \quad (4) \ t_{22}|2t_{12}, \quad (5) \ t_{33}|2t_{12}t_{23} + kt_{11}, \quad (6) \ t_{33}|2t_{23}.$$ 

We express $T$ as a disjoint union $T_{a,a} \cup T_{a,b} \cup T_{b}$ according to the following three cases: (1a, 4a) $t_{33}|kt_{11}$ and $t_{22}|t_{12}$; (1a, 4b) $t_{33}|kt_{11}$ and $t_{22}|2t_{12}$; and (1b) $t_{33}|2kt_{11}$.

In the first case, conditions (2) and (5) are redundant, and hence the conditions defining $T_{a,a}$ are $t_{33}|kt_{11}, t_{33}|2kt_{22}, t_{22}|t_{12}$ and $t_{33}|2kt_{12}$, and the integral over $T_{a,a}$ becomes

$$\frac{1}{2(1 - 2^{-1})} \int_{t_{33}|kt_{11}} |t_{11}|^{s-1} |t_{22}|^{s-2} |t_{33}|^{s-3} d\mu = \frac{1}{2(1 - 2^{-1})} \int_{t_{33}|kt_{11}} |t_{11}|^{s-1} |t_{22}|^{s-2} |t_{33}|^{s-3} d\mu = E_{2,k}(s, s, s - 2).$$

In the case (1a, 4b), condition (5) can be replaced by $t_{33}|t_{23}$, and hence the conditions defining $T_{a,b}$ are $t_{33}|kt_{11}$, $t_{33}|2kt_{12}, t_{33}|2t_{12}, t_{33}|2t_{23}$, and the integral over $T_{a,b}$ becomes

$$\frac{1}{2(1 - 2^{-1})} \int_{t_{33}|kt_{11}} |t_{11}|^{s-1} |2t_{12}|^{s-2} |t_{33}|^{s-3} d\mu = \frac{2^{-2}}{2(1 - 2^{-1})} \int_{t_{33}|kt_{11}} |t_{11}|^{s-1} |t_{12}|^{s-2} |t_{33}|^{s-3} d\mu = 2^{-2} D_{2,k}(s, s, s - 1),$$

where in the second equality we used Proposition 6.6 with the pivots $t_{23}, t_{22}$.

We now assume (1b). Since in this case $t_{33}|kt_{11}$, conditions (5) and (6) imply that $t_{22}|2t_{12}$. This, (1b) and (5) imply that $t_{33}|kt_{11}$. It is now easy to see that $T_{b}$ is defined by the conditions $|t_{33}|2kt_{11}, |t_{22}|2t_{12}, t_{33}|2kt_{12}, t_{33}|2kt_{22}$, or equivalently, $t_{33}|2kt_{11}, t_{22}|2t_{12}, t_{11}|t_{12}, t_{22}|2t_{12}$. Thus, the integral over $T_{b}$ becomes

$$\frac{1}{2(1 - 2^{-1})} \int_{t_{33}|kt_{11}} |t_{11}|^{s-1} |2t_{12}|^{s-2} |2kt_{11}|^{s-3} d\mu = \frac{2^{1-2s}k_{2}^{s-1}}{2(1 - 2^{-1})}\int_{t_{11}|t_{12}} |t_{11}|^{s-1} |t_{12}|^{s-2} |t_{11}|^{s-3} d\mu = 2^{1-2s} |k_{2}^{s-1} A_{2,1}(2s - 1, s),$$

where in the first equality we applied Proposition 6.6 with the pivots $(t_{23}, t_{12}, t_{33})$.

It follows from Table 3 that

$$\zeta_{G}^{\psi,s}(s) = \zeta_{2}(s)^{2}(\zeta_{2}(s - 1)(1 + 3 \cdot 2^{-s})(1 - |k|^{s-1} + \zeta_{2}(3s - 1)(1 + 2^{-s} + 3 \cdot 2^{-2s} - 2^{-3s} - 2^{-4s})|k|^{s-1}).$$
6.2.3. Local factors of \( \zeta_{G}^{G'}(s) \). If \( p \neq 2 \), then \( \gamma_3(G_p, N_p) = \langle x_1^2, x_3^2 \rangle = \langle x_1, x_3 \rangle \), whence \( N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p \). Thus, by Proposition 4.10 \( \zeta_{G}^{G'}(s) = \zeta_{G'}(s) \).

For the computation of \( \zeta_{G}^{G'}(s) \), we consider the cases \( k \) even or odd separately.

Assume that \( k \) is even. We claim that if \( A \leq G_2 \) is open and satisfies \( AN_2 = G_2 \), then \( A \) includes \( \langle x_1, x_2, x_3^2 \rangle \). In fact, any such \( A \) contains \( \langle x_v \rangle \) for some \( v \in \mathbb{Z}_2 \), whence \( A \cap \langle x_v \rangle = \langle x_v \rangle \). For \( x \in \mathbb{Z}_2 \), we deduce that \( x_1^2 x_3^{-2} x_9^{3k} x_9^{-1} \) and \( (x_9^2)^2 = x_9^{2v_9} x_9^{(2v_9 + 4) v_9} \). Since \( x_4 \) is even, and \( x_5 \) is the fact that \( k \) is even, we also deduce that \( x_2 v_9 + 1 \in A \), and hence \( x_3 \in A \) since \( 2v_9 + 1 \in \mathbb{Z}_2^* \). This proves the claim. It follows that \( \zeta_{G}^{G'}(s) = \zeta_{G'}(s) \), where \( G' \) is the quotient of \( G \) by \( \langle x_1^2, x_2, x_3^2 \rangle \), and \( \zeta_{G'}^{G'}(s) \) is computed with respect to \( N' \), the image of \( N_2 \) at \( G' \). Note that \( G' : = (\mathbb{Z}/22^3) \), the classes of \( \beta, x_1, x_2, x_3 \) being identified with \( (1, 0, 0), (0, 1, 0) \) and \( (0, 0, 1) \) respectively. The series \( \zeta_{G}^{G'}(s) \) enumerates the subgroups of \( G' \) that are not included in \( N' \). An easy calculation gives \( \zeta_{G'}^{G'}(s) = 1 + 6 \cdot 2^{-s} + 4 \cdot 4^{-s} \).

Assume now that \( k \) is odd. We claim that if \( A \leq G_2 \) is open and satisfies \( AN_2 = G_2 \), then \( A \) includes \( \langle x_1, x_2, x_3 \rangle \). It follows that \( \zeta_{G}^{G'}(s) = \zeta_{G'}^{G'}(s) \), where \( G' \) is the quotient of \( G \) by \( \langle x_1^2, x_2, x_3 \rangle \), and \( \zeta_{G'}^{G'}(s) \) is computed with respect to \( N' \), the image of \( N_2 \) at \( G' \). Note that \( G' \cong \mathbb{Z}/4 \times \mathbb{Z}/4 \), where the classes of \( \beta, x_1, x_2, x_3 \) are identified with \( (1, 0) \) and \( (0, 1) \), respectively. Thus, the series \( \zeta_{G'}^{G'}(s) \) enumerates the subgroups of \( \mathbb{Z}/4 \times \mathbb{Z}/4 \). A simple inspection shows that the only such subgroups are \( G' : = (\langle 1, 0 \rangle, (0, 2)), ((1, 1), (0, 2)), ((1, 0), (1, 1)), ((1, 2)), (1, 3)) \); thus, \( \zeta_{G'}^{G'}(s) = 1 + 2 \cdot 2^{-s} + 4 \cdot 4^{-s} \).

6.3. Computing the zeta functions of \( G_{p_3}, k, \ldots, G_{p_8}, k, \ldots, G_{p_{2k}}, k, \ldots \). The general form of these groups is

\[ G_{p, k, k, \ldots} = \langle \gamma, x_1, x_2, x_3 : [x_2, x_1] = x_3^3, [x_2, x_3] = x_2, \gamma x_1 = x_2, \gamma x_2 = x_1 x_2^{-1} x_3^2 \rangle \]

where \( k \in \mathbb{N}, (d, e, r) \in \{(3, 1, 1), (4, 2, 0), (6, 2, -1)\} \) and \( \epsilon = \begin{cases} 1 \text{ or } -1 & \text{if } k \equiv 0 \mod \frac{d}{3} \\ -1 & \text{if } k \equiv 0 \mod \frac{d}{3} \\ 1 & \text{otherwise} \end{cases} \). We let \( G \) be one of these groups.

6.3.1. Local factors of \( \zeta_{G}^{G'}(s) \). If \( p \parallel d \), then \( \zeta_{G_p}^{G_p}(s) \) was given in Table 1 (the case \( F = C_d \); note that \( \chi_6(p) = \chi_3(p) \) if \( p \parallel 6 \)).

Assume now that \( p \parallel d \). We claim that if \( A \leq G \) is open and satisfies \( AN_2 = G_p \), then \( Z_p \subset A_p \). Indeed, \( A \) contains \( \gamma x_v \) for some \( v \in \mathbb{Z}_p \), whence \( (\gamma x_v)^{d_2} = x_3^{d_2 v_2 + (r^2 - r + 1) v_2} \in A \). The claim will follow from the assertion that \( d_2 v_2 + (r^2 - r + 1) \in \mathbb{Z}_p \). If \( d = 4 \), then \( p = 2 \) and \( d = 2 \), and the assertion is clear. The same argument works for \( d = 6 \) and \( p = 2 \). Assume that \( d = 6 \) and \( p = 3 \). Since \( r = 1 \), we have \( r^2 - r + 1 = 3 \); hence the assertion is true in this case too. We finally assume that \( d = 3 \) and \( p = 3 \). In this case \( r = 1 \), whence \( r^2 - r + 1 = 1 \). Therefore, the assertion is that \( e - k \in \mathbb{Z}_p \). Since \( r_1 + r_2 + r_3 = 3 \), the assertion is clear. Otherwise, necessarily \( r_1 + r_2 + r_3 = 3 \mod 3 \), so the assertion in this case is equivalent to \( e - k \in \mathbb{Z}_p \), and this follows from the definition of \( e \) in terms of \( k \). This completes the proof of the claim. It follows that \( \zeta_{G}^{G_p}(s) = \zeta_{G_p}^{G_p}(s) \). According to [13] 5.10, 5.13, 5.16, the latter series is equal to \( \zeta_{G}(s - 1) \zeta_{L_p}(s - 1, \chi_d) \) if \( d \in \{3, 4\} \) and \( \zeta_{G}(s - 1) \zeta_{L_p}(s - 1, \chi_d) \) if \( d = 6 \).

6.3.2. Local factors of \( \zeta_{N}^{G}(s) \). If \( p \parallel d \), then \( \zeta_{N_p}^{G_p}(s) \) was given in Table 2 (the case \( F = C_d \)).

Assume now that \( p \parallel d \). By Corollary 3.3 and Lemma 6.1.2, we have

\[ \zeta_{N_p}^{G_p}(s) = \frac{1}{(1 - p^{-s})^4} \int_{T_1^+ (\mathbb{Z}_p)} t_{11}^{s-1} t_{12}^{s-2} t_{32}^{s-3} d\mu, \]

where \( T_1^+ (\mathbb{Z}_p) \) is defined by the conditions \( t_{33} \delta t_{11}, t_{33} \delta t_{12}, t_{33} \delta t_{22} \) and the following equivalent conditions:

\[ \langle x_1^2, x_2^2, x_3^2 \rangle \in \langle x_1, x_2, x_3 \rangle \]

\[ x_1 t_{11} x_2 t_{12} x_3 t_{32} \delta t_{11} \delta t_{12} \delta t_{22} \]
(4) \( t_{33}P_1 t_{13} + Q_1 t_{23} + (\ell_2^{12}) \delta k r, \)
(7) \( t_{33}P_2 t_{13} + Q_2 t_{23} + (\ell_2^{32}) \delta k r, \)
where \( P_1, P_2, Q_1 \) and \( Q_2 \) are rational functions that do not involve \( t_{13} \) or \( t_{23} \) and take values in \( \mathbb{Z}_p \) when restricted to \( T \). Note also that
\[
(6.1) \quad \det \begin{pmatrix} P_1 & Q_1 \\ P_2 & Q_2 \end{pmatrix} = -\frac{t_{12} + t_{11}}{t_{11}} \left( \frac{t_{12}}{t_{11}} - r - 1 \right) + \frac{t_{22}^{12}}{t_{11}} \frac{t_{11}}{t_{11}} - r t_{12} = -\frac{t_{12}^2}{t_{11}} - \frac{t_{12}}{t_{11}} + \frac{t_{12}}{t_{11}} + 1 + \frac{t_{12}}{t_{11}}^2 + 1 + \frac{t_{12}}{t_{11}} = 2 + r.
\]

We first assume that \((d, p) \in \{(3, 3), (4, 2), (6, 3)\}\). The terms \((\ell_2^{12}) \delta k r\) and \((\ell_2^{32}) \delta k r\) in conditions (4) and (7) can be deleted. In fact, they are already zero when \( r = 0 \), and if \( d \in \{3, 6\} \) and \( p = 3 \), then (1) implies that these two terms are divisible by \( t_{33} \). Now, one easily checks that the equation \( T^2 - r T + 1 \equiv 0 \mod p \) has \( 1 - r - r^2 \) as unique solution modulo \( p \) in \( \mathbb{Z}_p \), and that the equation \( T^2 - r T + 1 \equiv 0 \mod p^2 \) has no solutions in \( \mathbb{Z}_p \). Therefore, by (3), we can express \( T \) as a disjoint union \( T^a \cup T^b \) according to the cases (3a) \( |t_{11}|_p = |t_{22}|_p \), and (3b) \( |pt_{11}|_p = |t_{22}|_p \) and \( |pt_{12}|_1 = (1 - r - r^2) t_{11} \).

We describe \( T^b \). By (3a), the coefficient \( P_2 \) in (7) is a unit; therefore, by (6.1), we can replace (4) by \( t_{33}(1 + r) t_{23} \).

We conclude that \( T^b \) is defined by the conditions \( t_{33} | \delta k t_{22}, |t_{11}|_p = |t_{22}|_p, t_{22}|t_{12}, t_{33}(1 + r) t_{23} \) and (7), and the integral over \( T^b \) becomes
\[
\frac{1}{(1 - p^{-1})^2} \int_{\tau^b} |t_{22}|_{\delta k t_{22}}^{2-3} |t_{33}|_{\delta k t_{22}}^{3-3} d\mu = \frac{1}{(1 - p^{-1})^2} \int_{t_{33}(1 + r) t_{23}} |t_{22}|_{\delta k t_{22}}^{2-1} |t_{33}|_{\delta k t_{22}}^{3-1} d\mu = \zeta_p(2s) + \frac{1}{(1 - p^{-1})^2} \int_{|t_{33}|_{\delta k t_{22}}^{2-1}} |t_{22}|_{\delta k t_{22}}^{2-1} |t_{33}|_{\delta k t_{22}}^{3-1} d\mu,
\]
where in the first equality we used Proposition 3.6 with the pivots \((t_{12}, t_{11}, t_{13})\), and in the third one we used this proposition with the pivot \( t_{23} \).

We now describe \( T^a \). Condition (3b) implies that \( |t_{11}|_p = |t_{22}|_p \). This makes condition (2) redundant. Next, note that \( Q_1 = -\frac{t_{12}^2}{t_{11}} + \frac{t_{11}^2}{2t_{11}} \) is a unit of \( \mathbb{Z}_p \) since we are assuming that \( p | t_{11}^{22} \) and since \( T^2 - r T + 1 \equiv 0 \mod p^2 \) has no solutions in \( \mathbb{Z}_p \). Therefore, by (6.1), condition (7) can be replaced by the condition \( t_{33}(1 + r) t_{13} \).

We conclude that the conditions defining \( T^a \) are \( t_{33} | \delta k t_{22}, |t_{11}|_p = |t_{22}|_p, pt_{12} - (1 - r - r^2) t_{11}, t_{33}(1 + r) t_{13} \) and (4). The integral over \( T^a \) becomes
\[
\frac{1}{(1 - p^{-1})^2} \int_{\tau^a} |t_{11}|_{\delta k t_{11}}^{2-1} |t_{11}|_{\delta k t_{11}}^{3-2} d\mu = \frac{1}{(1 - p^{-1})^2} \int_{t_{33}(1 + r) t_{13}} |t_{22}|_{\delta k t_{22}}^{2-1} |t_{33}|_{\delta k t_{22}}^{3-2} d\mu = \zeta_p(s) + 2 + r |p|^{-1} (1 + p^{-s}) B_p, \delta k(s, 2s).
\]

We now assume that \((d, p) = (6, 2)\). The equation \( T^2 - r T + 1 \equiv 0 \mod 2 \) has no solutions in \( \mathbb{Z}_2 \), and hence (3) and (5) can be replaced by \( |t_{11}|_2 = |t_{22}|_2 \). It follows that the coefficient \( P_2 \) in condition (7) is a unit. Since the value of the determinant \((6.1)\) is \( 2 + r = -1 \), condition (4) can be replaced by one of the form \( t_{33}|t_{23} + R_1 \), where \( R_1 \) is a rational function that does not involve \( t_{13} \) or \( t_{23} \) and takes values in \( \mathbb{Z}_2 \) when restricted to \( T \). The conditions defining \( T \) are therefore \( t_{33}|2k t_{23}, t_{11}|t_{12}, |t_{11}|_2 = |t_{22}|_2, t_{33}|t_{23} + R_1 \), and (7). The integral over \( T \) becomes
\[
\zeta_p^{N_2} = \frac{1}{(1 - 2^{-2})^2} \int_{\tau^a} |t_{11}|_{\delta k t_{11}}^{2-3} |t_{33}|_{\delta k t_{11}}^{3-3} d\mu = \frac{1}{(1 - 2^{-2})^2} \int_{t_{33}|2k t_{11}} |t_{11}|_{\delta k t_{11}}^{2-1} |t_{33}|_{\delta k t_{11}}^{3-1} d\mu = A_{2,2k}(s, 2s),
\]
where in the first equality we used Proposition 4.6 with the pivots \((t_{12}, t_{22}, t_{23}, t_{13})\).

6.3. Local factors of \( \zeta_p^{G_p, \gamma} \). If \( p \neq d \), then \( \gamma_3(G_p, N_p) = \langle x_1^2, x_2, x_3^2, x_3^3 \rangle = \langle x_1, x_2, x_3^2 \rangle \), where the last equality holds since \( r + 2 \in \mathbb{Z}_p^* \). Thus, \( N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p/k\mathbb{Z}_p \), and by Proposition 2.12 we have \( \zeta_p^{G_p, \gamma}(s) = \frac{1}{1 - p^{-1}} \).

Assume now that \( p \neq d \). Arguing as in the calculation of \( \zeta_p^{G_p, \gamma}(s) \), we obtain that \( \zeta_p^{G_p, \gamma}(s) = \zeta_p^{E_p, \gamma}(s) \). According to the results of [13, 6.10, 6.13, 6.16], this series is equal to \( 1 + p \cdot p^{-s} \) if \( d = 3 \) or 4, and is 1 if \( d = 6 \).
6.3.4. Local factors of \( \zeta_G^{H,\zeta}(s) \) for \( d = 4 \) and \( H = (\gamma^2, x_1, x_2, x_3) \). Note that \( H = G_{p22k} \). If \( p \neq 2 \), then
\[
\gamma_3(H_p, N_p) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) = (x_1, x_2, x_3, 1),
\]
and hence \( N_p/\gamma_3(H_p, N_p) \cong \mathbb{Z}_p/k\mathbb{Z}_p \). By Proposition 6.1.6, we obtain
\[
\zeta_{\gamma_3(H_p, N_p)}^{H,\zeta}(s) = \frac{1 - p^{-s}|k|^{3s}}{1 - p^{-s}}.
\]
Assume now that \( p = 2 \). Arguing as in the calculation of \( \zeta_{G_{p22k}}^{H,\zeta}(s) \) in 6.1.3, we obtain that \( \zeta_{G_{2}^{22},\zeta}(s) = \zeta_{E_2}^{H,\zeta/2}(s) \), which is equal to \( 1 + 2 \cdot 2^{-s} + 2 \cdot 4^{-s} \) according to the results of [13, 6.10].

6.3.5. Local factors of \( \zeta_G^{H,\zeta}(s) \) for \( d = 6 \) and \( H := (\gamma^2, x_1, x_2, x_3) \). Note that \( H = G_{p22k} \). Arguing as in the previous paragraph, we obtain that
\[
\zeta_{\gamma_3(H_p, N_p)}^{H,\zeta}(s) = \frac{1 - p^{-s}|k|^{3s}}{1 - p^{-s}}
\]
if \( p \neq 2 \), and that \( \zeta_{G_{2}^{22},\zeta}(s) = \zeta_{E_2}^{H,\zeta/2}(s) \). The latter is equal to \( 1 + 4^{-s} \) according to the results of [13, 6.16].

6.3.6. Local factors of \( \zeta_G^{K,\zeta}(s) \) for \( d = 6 \) and \( K := (\gamma^2, x_1, x_2, x_3) \). Note that \( K = G_{p32k, n} \), where \( \eta = \epsilon \) if \( 3 \mid k \), and \( \eta = -\epsilon \) if \( 3 \nmid k \). If \( p \neq 3 \), then arguing as in 6.3.3, with \( G = G_{p32k, n} \), we find that
\[
\zeta_{\gamma_3(K_p, N_p)}^{K,\zeta}(s) = \frac{1 - p^{-s}|k|^{3s}}{1 - p^{-s}}
\]
and that \( \zeta_{\gamma_3(K_p, N_p)}^{K,\zeta}(s) = \zeta_{E_3}^{K,\zeta/2}(s) \), which is equal to \( 1 + 3^{-s} \) according to the results of [13, 6.16].

6.4. Computing the zeta functions of \( G = G_{p22k, 4k} \).

6.4.1. Local factors of \( \zeta_G^{K,\zeta}(s) \). If \( p \neq 2 \), then \( \zeta_G^{K,\zeta}(s) \) was given in Table 1 (the case \( F = D_2 \)).

Assume now that \( p = 2 \). Note that \((\alpha, x_1, x_2, x_3) = G_{p22k} \). Arguing as in 6.1.3 for \( G = G_{p22k, 4k} \), we find that
\[
\zeta_{\gamma_3(K_p, N_p)}^{K,\zeta}(s) = \zeta_{E_3}^{K,\zeta/2}(s)
\]
and the latter is equal to 1 according to the results of [13, 5.7].

6.4.2. Local factors of \( \zeta_G^{N,\zeta}(s) \). If \( p \neq 2 \), then \( \zeta_G^{N,\zeta}(s) \) was computed in Table 2 (the case \( F = D_2 \)).

Assume now that \( p = 2 \). By Corollary 6.3 and Lemma 6.1.2, we have
\[
\zeta_{\gamma_3(K_p, N_p)}^{N,\zeta}(s) = \frac{1}{(1 - 2^{-s})^3} \int_T |t_{11}|^{s-1}|t_{22}|^{s-2}|t_{33}|^{s-3} d\mu,
\]
where \( T \subset T_3^3(\mathbb{Z}_2) \) is defined by \( t_{33}, 4kt_{11}, t_{33}, 4kt_{12}, t_{33}, 4kt_{22} \) and the following equivalent conditions:
\[
\alpha(x^i) x^i, x^j, x^k \in (x^i, x^j, x^k), \ i = 1, 2, 3
\]
\[
\Leftrightarrow x_3^{2} t_{33} + 2k(t_{11} - t_{12}) - 2k t_{12} + 4kt_{22} - 2t_{22}, x_3^{2} t_{33} - 2t_{11} + 2k t_{12} + 4kt_{22} - 2t_{22} \in (x^i, x^j, x^k).
\]
Therefore, by Lemma 6.6, the conditions defining \( T \) are equivalent to
\[
(1) \ t_{33}, 4kt_{11}, (2) \ t_{33}, 4kt_{12}, (3) \ t_{33}, 2t_{11} + 2k(t_{11} - t_{12}), (4) \ t_{33}, 2t_{22}, (5) \ t_{22}, 2t_{12}, (6) \ t_{33}, 2t_{22}, 2t_{12} - 4t_{12} + 4kt_{11}, 4kt_{11}.
\]
Note that (2) follows from (5) and (7). Next, we express \( T \) as a disjoint union \( T^a \cup T^b \) according to the following cases:
\( a \) \( t_{33} \) \( \in \mathbb{Z}_2 \) \( \) and \( b \) \( 2t_{33} \).

In case \( a \), all the conditions but (5) are redundant, so
\[
\frac{1}{(1 - 2^{-s})^3} \int_T |t_{11}|^{s-1}|t_{22}|^{s-2}|t_{33}|^{s-3} d\mu = \frac{2^{2-s}}{1 - 2^{-s}} \int_{t_{22}} |t_{22}|^{s-2} d\mu = \zeta(s)(1 + 2^{1-s} \zeta_2(s)).
\]

In case \( b \), we split \( T^b \) as a union \( T^{b,a} \cup T^{b,b} \) according to the following cases: (5a) \( t_{22}, 2t_{11} \) and (5b) \( t_{22} = 2t_{12} \). Assume (5a). We sum the right-hand sides of (3) and (6) and obtain \( t_{33}, 2t_{22} + 2k t_{22} \). This and (4) imply that \( t_{33}, 2t_{22} \). Thus, (1) can be replaced by \( t_{33}, 2t_{22} \), and then (3) and (4) imply (6) clearly. To sum up, \( T^{b,a} \) is defined by the conditions 2 and 3 and \( t_{33}, 2t_{12} + 2k t_{12} \), and \( t_{33}, 2t_{22} \). Thus, (1), (2), (3), (4), (5), and (6) imply (4). Summarizing, the conditions defining \( T^{b,b} \) are \( 2t_{33}, 2t_{12} + 2k t_{12}, 2t_{12} \). Therefore,
\[
\frac{1}{(1 - 2^{-s})^3} \int_{T^{b,b}} |t_{11}|^{s-1}|t_{22}|^{s-2}|t_{33}|^{s-3} d\mu = \frac{2^{2-s}}{1 - 2^{-s}} \int_{t_{22}} |t_{22}|^{s-2} d\mu = 2^{2-s} \zeta_2(s),
\]
where in the first equality we used Proposition 5.6 with the pivots \( (t_{12}, t_{13}, t_{23}) \) and performed the change of variables \( t_{33} = 2u \). Assume now (5b). Observe that (2) and (7) are equivalent. Note also that (6) implies that \( t_{33}, 2t_{22} + 2k t_{22} \), and then that (1), (2), (3) imply that \( t_{33}, 4kt_{11} \). It follows that (1), (2), (3), (6) imply (4). Summarizing, the conditions defining \( T^{b,b} \) are \( 2t_{33}, 1, (2), 2, t_{33}, 2t_{12} + 2k t_{12} - t_{22} \). Therefore, the integral over \( T^{b,b} \) becomes
\[
\frac{1}{(1 - 2^{-s})^3} \int_{T^{b,b}} |t_{11}|^{s-1}|t_{22}|^{s-2}|t_{33}|^{s-3} d\mu = \frac{2^{1-2s}}{1 - 2^{-s}} \int_{t_{22}} |t_{22}|^{s-2} d\mu = 2^{1-2s} \zeta_2(s),
\]
In the first equality we used Proposition 5.6 with the pivots \( (t_{22}, t_{13}, t_{23}) \) and applied the change of variables \( t_{33} = 2u \).
We conclude that $\zeta_{G_2}^{N_2}(s) = \zeta_2(s)(1 + 2^{1-s} \zeta_2(s)) + 2^{1-2s} \zeta_2(s)^2 + (2^{2-s} + 2^{1-3s}) \zeta_{G_2}(s, s, s).

6.4.3. Local factors of $\zeta_{G_2}^{N_2}(s)$. If $p \neq 2$, then one can easily check that $\gamma_3(G_p, N_p) = \langle x_1, x_2, x_3 \rangle$, and hence $\zeta_{G_p}^{N_p}(s) = 1$. If $p = 2$, then $\zeta_{G_p}^{N_p}(s) = 1$ since already $\zeta_{G_2}^{N_2}(s) = 1$.

6.4.4. Local factors of $\zeta_{G}^{H}(s)$ for $H = \langle \alpha, x_1, x_2, x_3 \rangle$. Note that $H = G_{p^2.4k}$. Arguing as in [6.1.3] with $G = G_{p^2.4k}$, we find that $\zeta_{G_p}^{H_2}(s) = \frac{1}{1-p^{-s}} \zeta_{G_p}^{H_2}(1 + 2^{1-s})$ for $p \neq 2$, and that $\zeta_{G_2}^{H}(s) = \zeta_{G_2}^{H_2}(s)$. The latter is equal to $1 + 2 \cdot 2^{-s}$ according to the results of [13, 6.7].

6.4.5. Local factors of $\zeta_{K}^{G}(s)$ for $K = \langle \beta, x_1, x_2, x_3 \rangle$. Note that $K = G_{p^4.4k}$. If $p \neq 2$, then $\gamma_3(H_p, N_p) = \langle x_1^4, x_2^2 \rangle = \langle x_1^4, x_2 \rangle$, and hence $N_p/\gamma_3(G_p, N_p) \cong \mathbb{Z}_p$. Thus, by Proposition 2.12, we have $\zeta_{G_p}^{H}(s) = \zeta_{p}(s)$.

Assume now that $p = 2$. Arguing as in [6.2.3] with $G = G_{p^4.4k}$ we find that $\zeta_{G_2}^{K}(s) = \zeta_{G_2}^{K'}(s)$, where $G'$ is the quotient of $G_2$ by $\langle x_1, x_2^2, x_3^2 \rangle$, $K'$ is the image of $K$, and the new partial zeta function is computed with respect to $N'$, the image of $N$. Note that $K' \cong (\mathbb{Z}/2\mathbb{Z})^3$, the classes of $\beta$, $x_2$ and $x_3$ being identified with $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively. The action of the class of $\alpha$ on $K'$ is given by $(a, b, c) \mapsto (a, a+b, a+c)$.

It is easy to check that $\zeta_{G_2}^{K'}(s) = 1 + 2 \cdot 2^{-s}$.

6.4.6. Local factors of $\zeta_{L}(s)$ for $L = \langle \beta\alpha, x_1, x_2, x_3 \rangle$. One can easily check that the assignment $\alpha \mapsto \alpha^{-1}, \beta \mapsto \beta \alpha, x_1 \mapsto x_2, x_2 \mapsto x_1, x_3 \mapsto x_3$ extends to an automorphism of $G$ mapping $K$ onto $L$. Therefore, $\zeta_{G}^{K}(s) = \zeta_{G_2}^{K'}(s)$.

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