Magnetized deformation of neutron stars

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Abstract. The high magnetic field in neutron stars leads to interesting consequences. One of them is the deformation of neutron stars. The correction equations due to deformation effect that complemented the spherical Tolman-Oppenheimer-Volkoff equations can be calculated from Einstein field equations by including the magnetic contribution on the energy-momentum tensor of neutron star matter. Here, the multipole expansion is carried out on energy-momentum tensors containing magnetic fields as well as the corresponding metric around Schwarzschild one. The corresponding multipole expansions are performed only up to second order. We have shown that the effect of deformation appears more significantly in neutron stars with small masses. If we decrease the mass starting from maximum mass into the smaller one, the shape of neutron stars’ changes from spherical into oblate.

1. Introduction

The observations indicate that normal pulsars usually have a magnetic field at surface with intensity around $B_s \sim 10^8 - 10^{15}$ G [1]. There are some particular pulsars detected by Soft Gamma Repeaters (SGRs) [2], Anomalous X-Ray Pulsars (AXPs) and the most recent report is from Repeating Fast Radio Burst [3, 4]. They have magnetic field value at the surface $B_s \sim 10^{15}$ G. This magnetic field is significantly higher than those of normal pulsars called Magnetar. The value of magnetic field in the neutron stars core is not observed yet. However, it can be predicted theoretically around $B_c \sim 10^{18} - 10^{20}$ G [5].

The origin of high magnetic field inside neutron stars is still not clear until now. Magnetohydrodynamic (MHD) is the most accepted theory for describing the origin of magnetic fields on neutron star surface. The theory was developed by Duncan and Thompson based on an additional magnetic field is generated from a combination of rapid rotation and convective processes in the plasma during the proto-neutron star phase of the corresponding neutron stars [6].

The study of magnetic fields in neutron stars has been done by many researchers. In general, the presence of a magnetic field can be theoretically studied from microscopic and macroscopic aspects of the stars. Microscopically, the presence of a large magnetic field that significantly affects energy levels of charged particles is caused by Landau quantization. This affects the equation of state (EoS) for matter inside neutron stars. The effect of magnetic fields on the EoS model to describe hyperon stars [7, 8], quark stars [9] and hybrid stars [10] has been investigated. Macroscopically, the present of magnetic field can affect the structure of the star geometry. By solving the Einstein’s field equations using the corresponding metric, it shows that neutron star can be deformed due to magnetic field. Solutions for the later problem have been proposed by Konno et al. using an analytical method for
magnetized relativistic stars [11]. In otherwise, Mallick and Schramm using semi-numerical methods for hyperon stars [12].

This study is based on the method of Mallick and Schramm for energy-momentum tensor perturbation [12]. We obtain a rather different result on the corresponding solution of Einstein field equations. It was occurred because in our calculation, the pressure in energy-momentum tensor is expanded up to the quadrupole term while the energy density is expanded only up to the monopole term.

2. Formalism

The basic foundation of this study is based on the method Mallick and Schramm [12], by expanding the energy-momentum tensor in a form of multipolar expansion and using the Schwarzschild metric modification tensor of Hartle where the expansion is up to second order [13], we also using magnetic field profile ansatz proposed by Chakrabarty [7].

2.1. Multipole expansion in Tensor Energy-Momentum

We assumed the energy-momentum tensor in ideal fluid form. The additional magnetic field will be aligned along the z-axis, it is written in the energy-momentum tensor form as

$$ T_{\mu\nu} = \begin{pmatrix} -\varepsilon & 0 \\ P_\perp & P_\parallel \\ 0 & P_\parallel \end{pmatrix} $$

with $\varepsilon$ is the total energy density, $P_\perp$ and $P_\parallel$ is a perpendicular and parallel component of total pressure in respect to the direction of the magnetic field. The explicit expression of each term in equation (1) is $\varepsilon = \varepsilon_m + B^2/8\pi$, $P_\perp = P_m - MB + B^2/8\pi$ and $P_\parallel = P_m - B^2/8\pi$ [12], with $\varepsilon_m$ and $P_m$ are the energy density and pressure of the matter, $MB$ is the magnetization and $B^2/8\pi$ is a magnetic stress. Mallick and Schramm argue that the magnetic effects in matter EoS and magnetization term not significant [12]. Therefore, their contributions can be negligible. In this way each tensor element reduced into $\varepsilon = \varepsilon_m + B^2/8\pi$, $P_\perp = P_m + B^2/8\pi$ and $P_\parallel = P_m - B^2/8\pi$ [12]. For the pressures parts, they can written based on the direction of the magnetic field angle as equation (2) [12].

$$ P = P_m + \frac{B^2}{8\pi}(1 - 2\cos^2 \theta) $$

Equation (2) can be also written as equation (3-4) [12].

$$ P = P_m + \frac{B^2}{8\pi}\left[\frac{4}{3} - \frac{4}{3}P_2(\cos \theta)\right] $$

$$ P = P_m + [p_0 + p_2P_2(\cos \theta)] $$

where $P_2(\cos \theta)$ is the second order Legendre polynomial, $p_0 = B^2/24\pi$ is as monopole contributions and $p_2 = -4B^2/24\pi$ is as quadrupole contributions. So, the new energy-momentum tensor is written as

$$ T_{\mu\nu} = \begin{pmatrix} -\varepsilon & 0 \\ P & P \\ 0 & P \end{pmatrix} $$

where

$$ \varepsilon = \varepsilon_m + 3p_0 $$

$$ P = P_m + p_0 + p_2P_2(\cos \theta) $$
We can solve Einstein field equations using modified energy-momentum tensor that previously explained a multipolar expansion of metric tensor is also required. Similar to Ref. [12], we use the Hartle metric tensor expansion [13]. From these approximations we can calculate the deformation of magnetized neutron stars.

2.2. Hartle Approximation for Tensor Metric

Previously, we assume neutron stars as spherically symmetric. The interior solution of a static spherically symmetric object can be written in the Schwarzschild coordinate form \( t, r, \theta, \phi \).

\[
d s^2 = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)
\]

where the metric function for \( \nu \) and \( \lambda \) is a function for \( r \) only, with solution \( e^\lambda(r) = e^{-\nu(r)} = [1 - 2Gm(r)/r]^{-1} \), where \( G \) is the gravitational constant and \( m(r) \) is the mass within the radius \( r \) spherical objects. The author of reference [14] formulates the metric configuration for the slow rotational equilibrium of stationary and symmetric systems axially written until the second order (by excluding rotation term) as follows.

\[
d s^2 = -e^\nu(1 + 2h) dt^2 + e^\lambda \left[ 1 + \frac{e^4}{r} m \right] dr^2 + r^2(1 + 2k)(d\theta^2 + \sin^2 \theta d\phi^2)
\]

for \( h, m \) and \( k \) are functions of \( r \) and \( \theta \) defined as follows [15].

\[
h(r, \theta) = h_0(r) + h_2(r) P_2(\cos \theta) + \ldots
\]

(10)

\[
m(r, \theta) = m_0(r) + m_2(r) P_2(\cos \theta) + \ldots
\]

(11)

\[
k(r, \theta) = k_0(r) + k_2(r) P_2(\cos \theta) + \ldots
\]

(12)

where \( h_0, m_0, h_2, m_2, k_0 \) and \( k_2 \) are correction function up to second order, we set the value of \( k_0(r) = 0 \) according to coordinate transformation [14, 15]. To synchronize equation (9) with equation (6) and (7), the new metric tensor can be written up to quadrupole term as follows.

\[
d s^2 = -e^\nu(1 + 2[h_0(r) + h_2(r) P_2(\cos \theta)]) dt^2 + e^\lambda \left[ 1 + \frac{e^4}{r} [m_0(r) + m_2(r) P_2(\cos \theta)] \right] dr^2 + r^2[1 + 2k_2 P_2(\cos \theta)](d\theta^2 + \sin^2 \theta d\phi^2)
\]

(13)

Using these definitions of metric and energy-momentum tensor, we can obtain the solution for Einstein field equations.

2.3. Solution of Einstein Equations

Using equation (5) and equation (13), we can obtain from the corresponding Einstein equations the following first order differential equations.

\[
P_m' = -\frac{6}{r^2} (e_m + P_m)(m + 4\pi r^3 P_m) \left( 1 - \frac{2Gm}{r} \right)^{-1}
\]

(14)

\[
m' = 4\pi r^2 e_m
\]

(15)

Equation (14) and (15) are known as TOV equations with the correction for monopole term:

\[
m'_0 = 12\pi r^2 p_0
\]

(16)

\[
p'_0 = 4\pi G r e^4 p_0 + \frac{1}{r} G v^4 e^4 m_0 + \frac{1}{r^2} G e^4 m_0
\]

(17)

while for quadrupole term:

\[
h'_2 + k'_2 = h_2 \left( \frac{1}{r} - \frac{\nu'}{2} \right) + \frac{e^4}{r} \left( \frac{1}{r} + \frac{\nu'}{2} \right)
\]

(18)
\[ h_2 + \frac{e^4}{r} G m_2 = 0 \]  
\[ h'_2 + k'_2 \left( 1 + \frac{r}{2} \nu' \right) = 4 \pi G e^4 p_2 + \frac{1}{r^2} G e^4 m_2 + \frac{1}{r} G \nu^{m_2} + \frac{3}{r^2} e^4 h_2 + \frac{2}{r} e^4 k_2 \]  

(19)  
(20)

From the conservation of total energy-momentum, we obtained:
\[ \nu' = - \frac{2}{\varepsilon_m + p_m} P'_m \]  

for monopole term is
\[ p'_0 = -2 \nu' p_0 - (\varepsilon_m + p_m) h'_0 \]  

while for quadrupole term is
\[ p_2 = - (\varepsilon_m + p_m) h_2 \]  
\[ p'_2 = - \frac{1}{2} \nu' p_2 - (\varepsilon_m + p_m) h'_2 \]  

(22)  
(23)  
(24)

We can solve these differential equations simultaneously by using Runge-Kutta method. The corresponding boundary conditions are \( P_m(0) = P_c, P_m(R) = 0, m(0) = 0, m(R) = M \) and for the multipole factor according to reference [12,14], we use \( h_0(0) = 0, m_0(0) = 0, h_2(0) = 0 \) and \( k_2(0) = 0 \).

2.4. Magnetic Field Profile

The magnetic field profile of the neutron star is assumed depending on baryon density as follows [7]
\[ B(\rho) = B_s + B_0 \left\{ 1 - \exp \left[ -\beta \left( \frac{\rho}{\rho_0} \right)^\gamma \right] \right\} \]  

(25)

The ansatz is satisfying the general physical condition where the magnetic field in a star is not uniform. The model is constructed with condition the magnetic field in the center of the star that must be larger than the surface. \( \rho \) is a baryon density and \( \rho_o \) saturation density with value \( \rho_o = 0.15 \text{ fm}^{-3} \). Parameter \( \beta \) and \( \gamma \) control how fast the center of the magnetic field (\( B_s \)) fall to the asymptotic value at the surface (\( B_s \)). Here we set at \( \beta = 0.01 \) and \( \gamma = 2 \). According to the observations data, a magnetic field at the surface is \( \sim 10^{18} - 10^{19} \text{ G} \). The value of magnetic field in the center is estimated at \( 10^{18} - 10^{19} \text{ G} \) assuming the dynamo effect inside the star [6], for asymptotic value \( B_s = 4 \times 10^{18} \text{ G} \), where the density in the center is not greater than eight times the saturation density of symmetric nuclear matter, thus making the magnetic field value at the center is less than \( B_c = 1.75 \times 10^{18} \text{ G} \) [12], which is the value of \( B_c \) obtained from equation (25) when we set \( \rho \) at the \( R = 0 \).

3. Results

From the expansion parameters, we obtained the total mass and radius [12,15]:
\[ M_{\text{total}} = M_0 + \delta M, \]  
\[ R = R_0 + \delta R, \]  

(26)  
(27)

where \( M_0 \) is mass which exclude the magnetic field, \( \delta M = m_o \) as an additional correction mass due to the magnetic field, \( R_0 \) is radius without magnetic field and \( \delta R \) is an additional radius due to magnetic field effect. The later explicitly is defined as
\[ \delta R = \xi_0(r) + (\xi_2(r) + r k_2) P_2(\cos \theta), \]  

(28)

Therefore, for the specified angle \( R_e \equiv \xi_0(R) - (\xi_2(R) + R k_2)/2 \) and \( R_p \equiv \xi_0(R) + (\xi_2(R) + R k_2) \), \( R_e \) and \( R_p \) are equatorial and polar radius respectively, \( R \) is the radius of the spherical stars, \( \xi_0 \) and \( \xi_2 \) are defined as
\[ \xi_0(r) \equiv -\frac{r[r-2\hat{G}m(r)]}{G[4\pi r^3P_m+m(r)]}h_0, \quad (29) \]
\[ \xi_2(r) \equiv -\frac{r[r-2\hat{G}m(r)]}{G[4\pi r^3P_m+m(r)]}h_2. \quad (30) \]

Figure 1. (a) Relation between \( m_0 \) and energy density with different set value of \( B_0 \) (b) relation between \( m_0 \) and Magnetic field in the center of star for different set value \( P_c \).

The polar and equatorial radius of the star has the contribution of three parameters, which is \( \xi_0, \xi_2 \) and \( k_2 \). The contribution \( \xi_1 \) derived from the strong magnetic field at the surface of the magnetars, and \( k_2 \) is contribution of the integrated magnetic pressure inside the star [12]. The degree of deformation on a star is given by the deformation parameter called eccentricity or ellipticity defined as

\[ e = \sqrt{1 - \left(\frac{R_p}{R_e}\right)^2} \quad (31) \]

In this study we used EoS model from reference [16] where the hyperons are taken into account. In figure 1(a) we present a plot of energy density at the center of the star as a function of the mass correction term. For each fixed \( B_0 \) value, the mass correction increases with increasing energy density at the center of the star. The effect becomes significant if \( B_0 > 4 \times 10^{18} \text{ G} \). This indicates, a mass increasing if the stars are deformed, and according to equation (26), the value of maximum mass will increase along the increasing energy density in the center of the star. However, if we see the mass correction is lesser two orders magnitude than \( M_0 \), then the maximum mass is not significantly increased. Figure 1(b) describes the relation between \( m_0 \) and Magnetic field in the center of star for three different set value of fix pressure in center of the stars \( (P_c) \). As we see, \( m_0 \) increases with increasing magnetic field in the center of star, following the density energy in the center of the star trend. The Value of \( m_0 \) increase for \( P_c = 50 \text{ MeV/fm}^3 \) than the others rapidly, indicating significant deformation that occurs at a low value of \( P_c \).

Figure 2(a) describes the relation between the radius and the density energy in the center of the star with variant of magnetic field intensity \( B_0 \). The solid line represents a spherical symmetric radius for a neutron star without magnetic field. With the presence of a magnetic field term the polar and equatorial radius will be different where the difference can be observed also from equation (28). Along with the increase of density energy in the star center, the value of \( R_e \) always higher than the spherical radius, and the value \( R_p \) always below the value of the spherical radius so that the implication is an equatorial radius higher than the polar radius which geometrically makes the star tend to be oblate due to the existence of magnetic field. When we look at the changes in each set of values \( B_0 \), there is no significant change in \( R_e \) for every value of \( B_0 \), but \( R_p \) increases in respect to increasing energy density in the center of the star for \( B_0 = 4 \times 10^{18} \text{ G} \). Figure 2(b) shows that an increasing value of ellipticity leads to decreasing the total mass. If we refer to equation (31) where the value \( e \) is near to 0 means
there is no deformation and if it near to 1, deformation is maximum to oblate shape, then so it occurs at small total mass values significantly. The effect becomes more pronounced for \( B_0 = 4 \times 10^{18} \) G.

**Figure 2.** (a) relation between radius and energy density (b) relation between total mass and ellipticity with different set value of \( B_0 \).

**Figure 3.** Relation between total mass and radius for different set value of \( B_0 \).

**Figure 4.** Illustration of 3-dimensional deformation of neutron stars in each particular mass at \( B_0 = 4 \times 10^{18} \) G.
In figure 3 solid lines represent radius without taken into account magnetic field, and we also showed $R_e$ and $R_p$ with different set value of $B_0$. At the maximum mass of stars, it can be seen $R_e$ and $R_p$ overlap each other, indicating almost no deformation of the mass. Along with the maximum mass reduction of stars, there are differences among the three lines which is represent different value of magnetic field, the value of $R_e$ decreased to a low radius rapidly, otherwise $R_e$ increased to the high value. We also showed the 3-dimensional illustration of star profiles on the figure 4, the horizontal line represents $R_e$ and the vertical line represents $R_p$.

In figure 4 we make 3D illustration of neutron stars in each particular mass at $B_0 = 4 \times 10^{18}$ G. Figure 4(a) we show the deformation profile for small mass i.e., $M = 0.36 M_{\odot}$, where $R_e = 14.42$ km and $R_p = 9.15$ km. This result shows a very significant deformation. The star has an oblate shaped. For a higher mass of (b) $R_e = 13.12$ km, $R_p = 11.5$ km and (c) $R_e = 12.88$ km, $R_p = 11.95$ km, the equatorial star radius decreases, but the polar radius increases so the star still oblate, but the effect becomes less dramatic. Similarly, it happens for figure 4(c). Finally, at the maximum mass as shown figure 4(d) $R_e = 10.36$ km, $R_p = 10.36$ km, shows that the star becomes spherical.

4. Summary
We have studied numerically the deformation of neutron stars due to magnetic field using proposed method by reference [15]. The correction factors in mass and radius due to magnetic fields indicate the deformation of neutron stars occurs significantly for small mass. Furthermore, the effect of magnetic field is more pronounced for $B_0 > 4 \times 10^{18}$ G. Physically, for the small mass of the star causing the balanced between the gravity and magnetic field is significantly different for different directions. Therefore, the star becomes oblate-shaped due to the gravity pull of magnetic field on the z-axis more significantly than for other axes.

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