Destruction of Majorana modes in a topological nanowire by disorder in the superconducting substrate

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(Dated: February 11, 2013)

We study the proximity effect between a disordered superconductor substrate and a topological nanowire coupled via tunnel coupling. We illustrate how a finite pair-breaking term in the superconductor affects the local density of states in the nanowire. On one hand we show that the tunnel amplitude between the superconductor and the nanowire induces a topological transition in which the Majorana fermionic states can be destroyed when the coupling is very strong. On the other hand we note a non-trivial effect of the finite pair-breaking which can take the semiconducting nanowire between a topological and a non-topological phase in certain ranges of parameters. Our results have direct consequences for a nanowire coupled to an inhomogenous superconductor.

PACS numbers: 73.20.-r, 73.63.Nm, 74.45.+c, 74.50.+r,

Introduction

Majorana fermionic states have received a lot of interest because of their exotic properties [1, 2] such as non-Abelian statistics that open the perspective of using them for quantum computation [3–4]. A system which is expected to exhibit such states consists of a semiconducting wire with strong spin-orbit coupling such as InAs or InSb [5, 6], in the presence of an applied Zeeman field and in the proximity of an s-wave superconductor (SC) [6, 7]. The properties of this system have been extensively studied for an homogeneous SC [8–11].

A few recent experiments have reported the observation of zero-bias peaks [12–21] which are in good agreement but non completely consistent with the existence of Majorana fermions. Such experimental observations have been shown to also have simpler explanations because a zero bias peak could also arise without any Majorana fermions [22–26]. Most theoretical studies until now have focused on clean superconductors which generate a proximity effect in the semiconducting nanowire. The effect of an inhomogenous SC on the Majorana states has received less attention [27–30].

In this letter we consider a more general approach allowing one to characterize the physics of a semiconducting substrate and a topological nanowire coupled via tunnel coupling. We illustrate how a finite pair-breaking term in the superconductor affects the local density of states in the nanowire. On one hand we show that the tunneling rate between the superconductor and the nanowire induces a topological transition in which the Majorana fermionic states can be destroyed when the coupling is very strong. On the other hand we note a non-trivial effect of the finite pair-breaking which can take the semiconducting nanowire between a topological and a non-topological phase in certain ranges of parameters. Our results have direct consequences for a nanowire coupled to an inhomogenous superconductor.

Model

Our starting point is a semiconducting wire with strong spin-orbit coupling, in presence of a Zeeman field, and in proximity with an s-wave superconductor (see Fig. 1). The total Hamiltonian for such a system is of the form

\[ H = H_{NW} + H_S + H_T. \] (1)

In terms of the extended Nambu spinors the annihilation operator of the nanowire electrons is given by

\[ \hat{c}^\dagger = (c^\dagger_\uparrow, c^\dagger_d, c_\downarrow, -c_\uparrow), \] and the semiconducting nanowire Hamiltonian becomes

\[ H_{NW} = \int \hat{c}^\dagger \left[ \left( \frac{p^2}{2m} - \mu \right) \tau_z + \alpha \sigma \varphi \tau_z + V_x \sigma \right] \hat{c} \, dx, \] (2)

where \( \sigma \) and \( \tau \) are the Pauli matrices respectively in spin and particle-hole spaces, \( \mu \) is the chemical potential, \( \alpha \) is
the strength of the Rashba spin-orbit coupling and $V_z$ is the applied Zeeman magnetic field. The Hamiltonian of the bulk superconductor can be written as

$$H_S = \sum_{k,\sigma} \xi_k \Psi^\dagger_{k,\sigma} \Psi_{k,\sigma} + \Delta \Psi^\dagger_{k,\uparrow} \Psi^\dagger_{k,\downarrow} + h.c.$$  \hspace{1cm} (3)

with $\xi_k = \frac{k^2}{2m} - \mu$ and $\Psi_{k,\sigma}$ is the annihilation operator for an electron in the superconductor, having spin $\sigma$ and momentum $k$. The hopping term between the SC and the nanowire takes the following form

$$H_T = \sum_{i,\sigma} \hat{t}_i c^\dagger_{i,\sigma} \sum_k \Psi^\dagger_{k,\sigma} + h.c.$$  \hspace{1cm} (4)

where the operator $c^\dagger_{i,\sigma}$ creates an electron with spin $\sigma$ on site $i$. Since the total Hamiltonian is quadratic in the SC degrees of freedom, we can integrate out these modes, such that the effect of the SC substrate is taken into account by dressing the bare Green's function of the nanowire by a superconducting self-energy $\tilde{G}_T^{-1}(\omega)$

$$\tilde{G}_T^{-1}(\omega) = \tilde{G}_0^{-1}(\omega) - \Sigma_{i,R}^S(\omega).$$  \hspace{1cm} (5)

The local self energy depends on the tunneling amplitude $\hat{t}_i$ as well as on the retarded Green's function for the superconducting bulk electrons and can be written as

$$\Sigma_{i,R}^S(\omega) = |i\tau_i|^2 \tau_z \tilde{g}_R(\omega) \tau_z$$  \hspace{1cm} (6)

where $1$ is the unity matrix in spin/particle-hole space, and $\Gamma_{i,S} = \pi v(0) |\hat{t}_i|^2$ is the tunneling rate. We take this to be uniform along the nanowire for most of the rest of the paper, $\Gamma_{i,S} = \Gamma_S, \hat{t}_i = \hat{t}$. The retarded bare Green’s function of the semiconducting nanowire electrons is given by

$$\tilde{G}_R^{-1}(\omega) = (\omega + i\delta) 1 - H_{NW}$$  \hspace{1cm} (7)

where $\delta$ is an infinitesimal quasiparticle inverse lifetime which is introduced to avoid divergences in the numerical evaluations. For the purpose of our analysis, the Hamiltonian of the nanowire is best described using a lattice tight-binding model, such that Eq. (2) becomes

$$H_{NW} = \sum_i c^\dagger_i \left[(\mu - t)\tau_z + V_z \sigma_z\right] c_i
- \frac{1}{2} c^\dagger_i \left[t\tau_z + i\alpha \sigma_y \tau_z + h.c.\right] c_{i+1},$$  \hspace{1cm} (8)

where we defined the creation operator $c^\dagger_i$ of an electron in the nanowire on site $i$ in the Nambu basis as $c^\dagger_i = (c^\dagger_i, c^\dagger_{i+1}, c_{i+1}, -c_i)$. Here and throughout the remainder of the text, we work with physical dimensions corresponding to $\hbar = 1$, and in units in which the hopping term between sites is $t = 1$.

This general formalism allows us to model also a disordered as well as an inhomogeneous superconductor. Such properties are encoded in the superconducting self-energy which is related to the SC Green’s function. Thus we first use our model to study the effect of a finite pair-breaking which is taken into account via a pair-breaking parameter $\gamma$, phenomenologically introduced as an energy imaginary part $\delta 11$, $\omega \rightarrow \omega + i\gamma$ in Eq. (8).

In our calculations we focus on obtaining the local density of states (LDOS) (evaluated on a given site $i$) in the wire, which is given by the imaginary part of the dressed Green’s function

$$n_i(\omega) = -\frac{1}{\pi} \sum_{\beta=\uparrow,\downarrow} \text{Im}[\tilde{G}^{ii,\beta\beta}_R(\omega)].$$  \hspace{1cm} (9)

Throughout this letter we focus on the limit where the system is in the topological phase by choosing $V_z = 0.4, \Delta = 0.3, \mu = 0, and \alpha = 0.2$. A finite width ($\delta = 0.002 \hbar v_F/a$) is introduced in the numerical evaluations yielding a finite width of the peaks in the LDOS. We consider also that we are in the regime $l << \xi$ where $\xi$ is the superconducting coherence length and $l$ the length of the nanowire.

*Induced superconducting gap and the formation of Majorana modes* We focus first on the qualitative features of the proximity effect induced by the tunnel coupling with the SC substrate, and on the differences with the model in which the pairing term is put “by hand” directly in the nanowire $[6, 7, 14, 15]$. As described in Fig. 2 a), in the latter case a superconducting gap $\Delta$ is induced in the nanowire (in absence of Zeeman field and Rashba spin-orbit coupling). When we turn on the Zeeman field the effective gap becomes $\Delta - V_z$, and for $V_z$ equal to the gap closes. For values of $V_z$ larger than $\Delta$ and in the presence of spin-orbit coupling, the gap reopens and is topological in nature, and two Majorana states form at zero energy. The Rashba spin-orbit coupling localizes the two Majorana states more and more at the two ends of the nanowire.

When the proximity effect is modeled via a hopping term, in the absence of a Zeeman field and of Rashba spin-orbit coupling, a gap-like feature also appears at an energy $\Delta$, which can be described as a SC pseudo-gap (see Fig. 2 b) ). However, inside this gap we note the formation of Andreev bound states (ABS), the number of these states being determined by the ratio between the inverse nanowire length $1/l$ and the size of the gap (i.e. the energy difference between two ABSs is proportional to $1/l$ $[13]$).

In the absence of spin-orbit coupling and magnetic field, these ABSs have a uniform weight along the wire.
FIG. 2. LDOS of two models: DOS dependence on energy when the superconductivity is put "by hand" (a) and when the superconductivity is induced via a tunnel coupling (b). (see Fig. [a] a) ) and are a reminiscence of the quantized modes of the wire in the normal state [35]. Turning on the spin-orbit coupling localizes these states towards the ends of the wire (see Fig. [b] b), even in the non-topological state. Including a Zeeman field breaks the spin degeneracy and splits each of these states. The two ABSs come closer and closer with increasing the magnetic field until when, for a certain value of $V_z$, and in the presence of spin-orbit coupling, they merge (note this merging is also controlled by the chemical potential) (see Fig. [c] c)) and form two Majorana states. Increasing the spin-orbit coupling localizes these states more and more towards the ends of the wire (see Fig. [d] d). Thus we see that the proximity effect induced via the tunnel coupling between the SC and the nanowire also allows for the formation of localized Majorana modes, which can be seen evolving from the Andreev bound states in the wire.

Topological transitions In order to understand how these Majorana states are formed in a more quantitative manner, we analyze the effective Hamiltonian induced in the wire via the coupling with the superconducting substrate. Thus, we note that if we focus on the low-energy sector where we expect the Majorana modes to form, the self-energy induced in the wire is equivalent with an effective SC gap which can be evaluated by setting the energy to zero in the second term in Eq. (6), yielding $\Delta_{\text{eff}} = \Gamma_S \Delta / \sqrt{\Delta^2 + \gamma^2}$. It is this effective gap which governs the transition to a non-topological phase when the Zeeman field becomes smaller than a critical value, $V_z^2 < \Delta_{\text{eff}}^2 + \mu^2$.

Alternatively seen, for large values of $\Delta_{\text{eff}}$ a transition to a non-topological phase occurs. In terms of the tunnel coupling between the SC and the wire the corresponding condition to exit the topological phase is given by $\Gamma_S > \tilde{\Gamma}_S$, where $\tilde{\Gamma}_S = \sqrt{(V_z^2 - \mu^2)(1 + \frac{\gamma^2}{\Delta^2})}$. This can be checked numerically, and in Fig. [a] a we have plotted the LDOS at one end of the nanowire as a function of the energy and $\Gamma_S$. Indeed, we note that for values of $\Gamma_S$ larger than $\tilde{\Gamma}_S$ the Majorana peaks split in two, marking the exit from the topological phase. We have checked that the energy difference $\delta E$ between these ABSs depends linearly on $\Gamma_S$.

We also note that for very small tunnel coupling parameters the Majorana states are also split; this behavior can be explained by the fact that in this regime the superconductivity has not sufficiently penetrated the nanowire to have a real superconducting gap. By increasing the nanowire length the splitting of the Majorana states occurs for smaller and smaller values of $\Gamma_S$, confirming that this is a finite size effect; for large nanowire lengths we obtain a robust zero bias peak even for very small transmission rates.

While we do not detail it here, this formalism allows one to consider also a non-uniform tunneling rate which corresponds to disorder at the interface between the wire and the superconducting substrate. Such disorder yields a position dependent effective superconducting gap in the

FIG. 3. (Color online) ABS dependence on energy and position a) non-topological state, no Zeeman field and no spin-orbit coupling; b) non-topological state with a finite spin-orbit coupling; c) topological state, not too large spin-orbit coupling; d) topological state, strong spin-orbit coupling. The pair-breaking parameter is considered negligible, $\gamma = 0$. 

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wire, and the Majorana modes are destroyed for large disorder strengths.

![Graph](image)

**FIG. 4.** (Color online) LDOS of a Majorana peak (on the first site of the wire) as a function of energy and the tunneling rate $\Gamma_S$ (a) and pair-breaking strength (b). The dashed line denotes the transition between a topological and non-topological state which occurs at $\Gamma_S \approx 0.4$ (with our parameters). In b) we take a $\Gamma_S = 0.3$ in the main plot (no topological transition with varying $\gamma$). In the inset we take $\Delta = 0.15$, $V_z = 1$, and $\Gamma_S = 1.01$ (a Majorana peak is expected to form when $\gamma > \gamma \approx 0.021$).

We study also the effect of pair-breaking in the SC substrate. In terms of the pair-breaking parameter $\gamma$ the condition to have a topological phase is given by $\gamma > \tilde{\gamma} = \Delta \sqrt{\frac{\Gamma_S^2 - V_z^2}{\Gamma_S^2 - V_z^2}} - 1$. Thus, the position of the transition depends essentially on the difference $\Gamma_S^2 - V_z^2$. For large magnetic fields $V_z > \Gamma_S$ the topological condition is satisfied for all values of $\gamma$, and no transition to a non-topological state is possible. In Fig. 4b we have plotted the LDOS corresponding to a Majorana peak (i.e. the LDOS on the first site of the wire) as a function of energy and pair-breaking strength for this configuration ($V_z > \Gamma_S$). Note that the Majorana peaks widen and get dissolved in the bulk when the pair-breaking strength becomes of the same order of magnitude as the SC gap. This is not very surprising as for such strong pair breaking the BCS DOS is smoothed out and becomes similar to that of a normal metal; in this regime we expect our system to behave as a topological wire in the non-SC ("normal") regime (no Majorana modes).

When $\Gamma_S > V_z$ the Majorana fermions are destroyed at $\gamma = 0$, however, for certain parameter values chosen such that $\tilde{\gamma}$ is small with respect to the topological gap ($\Gamma_S$ very close to $\Gamma_S$, large $V_z$ with respect to $\Delta$) there exists a transition to a topological phase with increasing $\gamma$. Finding the range of parameters in which this can be realized requires fine tuning, and in the inset in Fig. 4b we have illustrated such a situation ($\Delta = 0.15$, $V_z = 1$, $\Gamma_S = 1.01$), for which a phase transition exists and the Majorana peak reforms for $\gamma > \gamma \approx 0.021$

**Inhomogenous junction** Last, but not least, we focus on an inhomogenous superconductor consisting of regions with different tunnel coupling between the wire and the SC, as well as with different pair-breaking strengths. In particular we consider a superconductor containing three regions, a central region in the topological phase ($\Gamma_S < \Gamma_S$, $\gamma = 0$), and two exteriors ones for which the pair-breaking strength as well as the tunnel coupling can be tuned such that the regions are in either the topological or the non-topological phase. When all regions are in topological phase, we restore a well-known case where we only have two Majorana states localized at the ends of the nanowire. When only the central region is in the topological phase, two Majorana fermions form at the extremities of the central part. An interesting situation is that of two exteriors regions with very strong pair-breaking. In this kind of configuration, the exteriors regions are quasi-normal, and we recover the behavior described in Ref. [14] for SN junctions, i.e., two Majorana fermionic states extended over the entire "normal" parts, as well as over a broad energy range.

**Conclusions** To summarize, we have used a general approach to describe the proximity effect induced by a bulk superconductor in a semiconducting nanowire via a tunnel coupling. We have applied our method to study the effect of the disorder in a superconductor (modeled as a pair-breaking term) on the physics of the formation of the Majorana fermions in the wire. We have shown that the nanowire can reach a non-topological phase for strong values of the tunneling rate between the SC and the wire. The pair-breaking strength can also take the system trough a topological phase transition in a finely tuned range of parameters. Finally we have shown that Majorana states can form in an inhomogenous superconductor. Our approach can be also applied to other kinds of disorder such as static or magnetic disorder.

D. C. and C. B. acknowledge discussions with T. Jonckheere and L. P. Kouwenhoven. The work of C. B. and D. C. is supported by the ERC Starting Independent Researcher Grant NANO-GRAPHENE 256965.
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