A solution to the zero-hamiltonian problem in 2-D gravity

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The zero-hamiltonian problem, present in reparametrization invariant systems, is solved for the 2-D induced gravity model. Working with methods developed by Henneaux et al. we find systematically the reduced phase-space physics, generated by an effective hamiltonian obtained after complete gauge fixing.

PACS. 11.10.Kk, 04.50.+h

I. INTRODUCTION

In theories of gravitation the canonical hamiltonian is a linear combination of constraints, meaning that after gauge fixing it turns into a strongly zero quantity. This is in brief the “zero-hamiltonian problem” (ZH problem), present, for a more general context, in diffeomorphism (Diff) invariant models.

Two-dimensional (2-D) gravity models has been under intensive investigation during the last two decades [1]. The old problem of quantum gravity, black holes physics and string dynamics were tested in these formulations. In particular, the so called induced gravity model was proposed in the eighties by Polyakov [2]; Emerging as a field theory of gravitation after integration of the matter fields. In spite of its peculiar features and successful quantum formulation, the ZH problem is also present in Polyakov’s model.

The ZH problem was analysed by Henneaux, Teitelboim and Vergara [3] in a broad context. The idea was to construct an extension on the original action that is invariant under gauge transformations not vanishing at the end-points; the boundary conditions were therefore modified through the gauge generators. The extension mentioned above is related to the physical (effective) hamiltonian of the theory that it going to rule the dynamics of the physical degrees of freedom. An alternative approach was proposed by Fulop, Gitman and Tyutin [4]; The main point here is that one works in the reduced phase-space. Once determined the simplectic structure, after complete gauge fixing, a time-dependent canonical transformation is performed, obtaining the dynamics generator for the physical variables.

In this work we apply these methods to the 2-D induced gravity model obtaining the dynamics and the effective hamiltonian in a sistematic way, working in reduced phase-space.

The manuscript is structured as follows. In Section II we make a panoramic description of the methods mentioned above to analyse the ZH problem. In section III we apply these techniques in the relativistic particle model, working in the proper-time gauge. In section IV we discuss our main interest, the 2-D induced gravity case. Finally, in Section V we display our conclusions.

II. SOLVING THE ZERO-HAMILTONIAN PROBLEM

In this section we make a short review of the techniques developed in [3] (and partially in [4]). These procedures solve the ZH problem, present in (Diff) invariant theories.

Given a gauge system with Diff invariance we can write its action as (we use finite-degrees-of-freedom’s notation for simplicity)

\[ S = \int L \mathrm{d}\tau = \int_{\tau_1}^{\tau_2} (P_i \dot{Q}^i - H_0 - \lambda_a G^a) \mathrm{d}\tau , \]

where the \( P_i \)'s \((i = 1, \ldots, N)\) represent the canonical momenta conjugated to the coordinates \( Q^i \), \( H_0 \) is the canonical Hamiltonian and the \( \lambda_a \) are Lagrange multipliers. As a consequence of the Diff invariance of the action, the system has a set of \( a \) \((a = 1, \ldots, R)\) first-class constraints. They satisfy, by definition, the Poisson algebra [5]

\[ \{ G_a, G_b \} = C_{ab}^c G_c \approx 0 \ , \]

that in the case of gravity theories turns into the well-known Diff algebra [6]. There are not second-class constraints.

The presence of the ZH problem is a consequence of the properties mentioned above. In fact, for Diff invariant systems the total hamiltonian \( (H_T) \) is a linear combination of constraints

\[ H_T = H_0 + \lambda^a G_a \approx 0 \ , \]

meaning that it is a strongly zero quantity after the (complete) gauge fixing procedure. This fact leaves no generator of dynamics in the reduced phase-space [6].

To solve the ZH problem, Henneaux Teitelboim and Vergara [3] (and in a different context Gitman and Tyutin [4]) proposed to perform an extension on \( S \) that takes into account end-point contributions. The action for the paths obeying these open boundary conditions (
the gauge parameters $\epsilon^a$ do not vanish at the end points) is
\[ S_E = \int_{\tau_1}^{\tau_2} (pq - H_0 - \lambda^a G_a) d\tau - \left[ P_i \frac{\partial G}{\partial P_i} - G \right]^{\tau_2}_{\tau_1}, \quad (4) \]
with $G \equiv \epsilon^a G_a$, where the $\epsilon$ are the gauge transformation parameters in the extended hamiltonian formalism \[9\]. The new boundary conditions are given by
\[ \bar{Q}(\tau_i) \equiv [Q - (Q, G)](\tau_i) \quad i = 1, 2, \quad (5) \]
with analogous expression for the new momenta $\bar{P}$.

The correspondent generating function $(M)$ is related to the gauge (Diff) generator. We have
\[ M = P_i \frac{\partial G}{\partial P_i} - G, \quad (6) \]
\[ S' = S + \int_{\tau_1}^{\tau_2} \frac{dM}{d\tau} d\tau, \quad (7) \]
and the action is invariant under gauge transformations with the open boundary conditions \[8\].

It is also possible to obtain a non-zero effective hamiltonian function ($H$) in reduced phase-space. After gauge fixing a new canonical transformation is performed, whose generator ($F$) is determined by the form of the gauge fixing constraints \[9\]. We can write
\[ \bar{H} = \left[ H_T + \frac{\partial F}{\partial \tau} \right]_{\text{fixed}}. \quad (8) \]

We obtain the last equality after (complete) gauge fixing, meaning that $\bar{H}$ is the hamiltonian function for the new variables, in reduced phase-space. The equations of motion will be
\[ \bar{Q} = \{Q, H\}_D \quad \bar{P} = \{\bar{P}, H\}_D, \quad (9) \]
where $D$ denotes the Dirac bracket operation. The constraints surface in the new variables is given by
\[ G^a(\bar{Q}, \bar{P}) \equiv G^a \left[ P(Q, \bar{P}), Q(Q, \bar{P}) \right] \quad (10) \]
with analogous expressions for the gauge fixing constraints (that will turn the weak equalities into strong ones).

III. THE RELATIVISTIC PARTICLE EXAMPLE

In this section we take the instructive relativistic particle example to apply the methods described in the precedent section. Essentially this analysis was done in \[9\] and partially in \[10\]. Here we reproduce some calculations for pedagogical reasons and also find correspondent results when the proper-time gauge fixing is considered.

As usual, the action is proportional to the world-line length
\[ S = -m \int ds, \quad (11) \]
from which follows the parametrized ($\tau$) lagrangian
\[ L = -m(-U^\nu U\nu)^\frac{1}{2}, \quad (12) \]
where the $U^\nu$s are the four-velocities ($U^\nu \equiv \frac{dx^\nu}{d\tau}$). The metric convention is diag $[-1, 1, 1, 1]$. As is well known, the action \[12\] is Diff invariant ($\tau \to \hat{\tau}$).

The total hamiltonian $H_T$ is easily constructed, being proportional to constraints as expected. The canonical hamiltonian $H_c$ is strongly zero
\[ H_T = H_c + \theta_1 \xi_1 = \theta_1 \xi_1. \quad (13) \]
The model has only one constraint, which is evidently first class ($\theta_1 = p^\mu \gamma_\mu + m^2$). $\xi_1$ is an arbitrary local multiplier.

The Hamilton-Jacobi equations of motion read
\[ \frac{dx^\mu}{d\tau} \approx \{x^\mu, H_T\} = 2 \xi_1 p^\mu \quad (14a) \]
\[ \frac{dp_\mu}{d\tau} \approx \{p_\mu, H_T\} = 0. \quad (14b) \]

The gauge transformations generated by the first-class constraint $\theta_1$ are analogous to the equations of motion above since the model is “pure gauge”
\[ \delta_c x^\mu = \{x^\mu, G\} = \{x^\mu, \epsilon \theta_1\} = 2 \epsilon p^\mu \quad (15a) \]
\[ \delta_c p_\mu = \{p_\mu, G\} = 0. \quad (15b) \]

The generating function $(M)$ is in this case, following the definition \[10\]
\[ \{p_\mu \frac{\partial G}{\partial P_\mu} - G\}^{\tau_2}_{\tau_1} = \epsilon (p^2 - m^2)|^{\tau_2}_{\tau_1}. \quad (16) \]
So, the improved action reads
\[ S = S + \frac{\delta \phi^0}{2 \phi^0} (p^2 - m^2)|^{\tau_2}_{\tau_1}, \quad (17) \]
with the new boundary conditions given by
\[ X^\mu(\tau_i) = [x^\mu - \frac{\delta \phi^0}{\phi^0} p^\mu](\tau_i) \quad i = 1, 2. \quad (18) \]
The next step is the reduced phase-space analysis. The proper-time gauge fixing condition is given by
\[ \theta_2 = x^0 \frac{p^0}{m} \tau. \quad (19) \]
Following the standard procedure, the Dirac brackets for the physical degrees of freedom (the “spatial” sector \( [x_i, p^j] \)) are
\[
\{x_i, x_j\}_D = 0 = \{p^i, p^j\}_D \quad \{x_i, p^j\}_D = \delta_{ij} .
\] (20)
The gauge fixing condition gives the form of the canonical transformation needed
\[
P_\mu = p_\mu \quad (21a)
\]
\[
X_i = x_i \quad (21b)
\]
\[
X_0 = x_0 - \frac{p_0}{m} \quad (21c)
\]
As a consequence the constraints in the new variables become
\[
\bar{\theta}_1 = \theta_1 \quad \bar{\theta}_2 = 0 .
\] (22)
The generator of the canonical transformation is a function of old momenta, new positions and time (Type \( F_3 \), see [2])
\[
F_3 = -X^\mu P_\mu + \left( \frac{P^0)^2}{2m} \right) .
\] (23)
So the hamiltonian in reduced phase-space is
\[
\bar{H} = H + \frac{\partial F_3}{\partial \tau} = \left( \frac{P^0)^2}{2m} \right) ,
\] (24)
giving the right Hamilton-Jacobi equations for the proper-time description
\[
\frac{dx^i}{d\tau} = \{x^i, \bar{H}\}_D = \frac{p^i}{m} \quad (25a)
\]
\[
\frac{dp^i}{d\tau} = \{p^i, \bar{H}\}_D = 0 .
\] (25b)

IV. THE INDUCED 2-D GRAVITY MODEL

In this part we apply the techniques described in section II to a field theory; of our interest is the 2-D induced gravity model proposed by Polyakov in [3].

We first want to find the generator of the local gauge transformations (\( \text{Diff} \)). The form variations in the fields are
\[
\delta \varphi = \epsilon^\mu \partial_\mu \varphi \quad (26a)
\]
\[
\delta g^{\mu \nu} = \partial_\sigma g^{\mu \nu} \epsilon^\sigma - g^{\mu \sigma} \partial_\sigma \epsilon^{\nu} + g^{\nu \sigma} \partial_\sigma \epsilon^{\mu} .
\] (26b)
It is possible to write the corresponding action as a local functional, introducing the auxiliary scalar field \( \varphi(x) \) \[4\]
\[
S = \int d^2x \sqrt{-g} \left( -\varphi \Box \varphi - \alpha R \varphi + \alpha^2 \beta \right) ,
\] (27)
where \( R \) is the 2-D scalar curvature. \( \alpha \) and \( \beta \) are scalars related to a central charge (gravity coupled to matter, before integration) and to a cosmological constant, respectively (for details see [2]).

The generator of the local invariances \[29\] must be a linear combination of the first class constraints that arise from \[27\] (there are not second class constraints). We have
\[
\omega_1 = \pi_{00} = 0 \quad (28a)
\]
\[
\omega_2 = \pi_{01} = 0 .
\] (28b)
where \( \pi^{\mu \nu} \) are the momenta conjugated to the metric components \( g_{\mu \nu} \). The time-consistency condition [5] applied to the primary constraints \[28\] gives two secondary constraints
\[
\phi_1 = \frac{1}{2} \left( \varphi'^2 - \frac{4}{\alpha^2} (g_{11} \pi^{11})^2 - \frac{4}{\alpha} (g_{11} \pi^{11}) \pi_{\varphi} \right)
\]
\[
- \alpha g_{11} + 2 \alpha \varphi'' + \alpha^2 \beta g_{11} \right) \quad (29a)
\]
\[
\phi_2 = \varphi \varphi' - 2 g_{11} \pi^{11'} - \pi^{11} g_{11'} ,
\] (29b)
and no more constraint generations appear. In accord with the discussion of section II, the \( \text{Diff} \) invariance implies into a hamiltonian functional that is proportional to \( \phi_1 \) and \( \phi_2 \)
\[
H_c = -\sqrt{-g} \frac{g_{01}}{g_{11}} \phi_1 + \frac{g_{01}}{g_{11}} \phi_2 .
\] (30)

Following the Anderson-Bergmann algorithm [6], the generators of the gauge transformations \[24\] must obey (taking for example the variations of the \( \varphi \) scalar field)
\[
\delta \varphi(x) = \{ \varphi(x), G \}
\]
\[
= \int dy \{ \varphi(x), \left( e^0 \tilde{G}_1^0 + e^0 \tilde{G}_0^0 + e^1 \tilde{G}_1^0 + e^0 \tilde{G}_0^1 \right) \}
\]
\[
= e^0 \partial_0 \varphi(x) + e^1 \partial_1 \varphi(x)
\]
\[
= e^0 \left( \frac{2\sqrt{-g}}{\alpha \pi^{11}} + \frac{g_{01}}{g_{11}} \varphi' \right) (x) + e^1 \partial_1 \varphi(x) ,
\] (31)
where we have used the definition of momentum \( \pi^{11} \)
\[
\pi^{11} = \frac{\alpha}{2\sqrt{-g}} \left[ \dot{\varphi} - \frac{g_{01}}{g_{11}} \varphi' \right] ,
\] (32)
to put the time derivatives in hamiltonian form. The \( \tilde{G} \) variables are linear combinations of contraints \[29\] and \[25\]. To determine their form we see that \( \tilde{G}_1^0 \) and \( \tilde{G}_0^1 \) cannot be proportional to \( \phi_1 \) nor to \( \phi_2 \) since the \( \varphi \) transformation has no \( \dot{\varphi} \) term. So, at maximum
\[
\tilde{G}_1^0 = A \omega_1 + B \omega_2 \quad (33a)
\]
\[
\tilde{G}_0^1 = C \omega_1 + D \omega_2 \quad (33b)
\]
\[
\tilde{G}_1^1 = E \Phi_1 + F \Phi_2 + L \omega_1 + M \omega_2 \quad (33c)
\]
\[
\tilde{G}_0^3 = H \Phi_1 + I \Phi_2 + N \omega_1 + P \omega_2 .
\] (33d)
Comparing with (31) we find that

\[ E = 0 \ , \ F = 1 \ . \]  

(34)

In a similar way we obtain the \( \epsilon_0 \) contribution and those from the metric components \( g_{\mu \nu} \) (using (26) and (29)). The final result is

\[
\begin{align*}
A &= 2g_{01} & N &= \partial_0 g_{00} & B &= g_{11} \\
C &= 2g_{00} & R &= 0 & D &= g_{11} \\
L &= \partial_1 g_{00} & T &= 0 & M &= \partial_0 g_{01} \\
E &= 0 & F &= 1 & P &= \partial_0 g_{01} \\
H &= \sqrt{-g} & I &= \frac{\partial_0}{g_{11}} & S &= g_{01}
\end{align*}
\]

with the gauge generator \( G \) at hand, following the ideas of section II, we go for the calculation of the generation function \( \hat{M} \)

\[
\hat{M} = \int mdy = \int dy [\frac{dG}{d\Pi^\alpha} F^\alpha - G] .
\]

(36)

After some algebra we obtain straightforwardly

\[
m = \sqrt{-g} \left[ \frac{\epsilon^2}{2} + \frac{\alpha}{\alpha^2} (g_{11} \pi^{11})^2 - \frac{g_{11}}{2g_{11}} \phi' \right. \\
+ \alpha \phi'' + \frac{2}{\alpha} g_{11} \pi^{11} \frac{\phi'}{g_{11}} \right] \\
+ (\epsilon^1 + \frac{g_{01}}{g_{11}}) \left[ -2g_{11}(\pi^{11})' - 2(g_{11})'' \pi^{11} \right] .
\]

(37a)

The next step is to construct the effective hamiltonian density after complete gauge fixing. We find first the reduced phase-space structure, using the Dirac bracket procedure. We choose as gauge fixing constraints

\[
\Gamma_1 = \pi^{11} - f(t) \quad \Gamma_6 = \partial_1 \phi - 1 .
\]

(38)

where \( f(t) \) is an arbitrary function of time. To obtain a more convenient form of the Dirac matrix we use the following linear combinations

\[
\begin{align*}
\Lambda_1 &= \phi_1 + \Gamma_5 \\
\Lambda_2 &= \phi_1 - \Gamma_5 ,
\end{align*}
\]

(39a, 39b)

whose Poisson brackets are

\[
\{ \Lambda_1(x), \Lambda_2(y) \} = -2\alpha \partial_x \delta(x-y) .
\]

(40)

The Dirac brackets for the physical degrees of freedom can be obtained in a two-steps procedure. First we fix the \( [\pi^{01}, \pi^{00}] \) sector (see expressions (28)) using the light-cone gauge fixing condition \( \bar{g} \), this is straightforward. In a second step we take the remaining set of contraints, namely \( \phi_3, \phi_2, \Lambda_1 \) and \( \Lambda_2 \). Finally we obtain, after a long calculation,

\[
\{ g_{11}(x), \pi^{11}(y) \}_D = \delta(x-y) ,
\]

(41)

the others are zero. It is important to notice that although the results obtained are the “expected” expressions (11), the gravitational field component \( g_{11} \) and its momentum are not independent quantities, they are linked by the constraints relations (29).

To find the effective hamiltonian in reduced phase-space we perform, as was explained in section II, a canonical transformation. In the gravity sector we have

\[
\Pi^{11} = \pi^{11} - f(t) \quad G_{11} = g_{11} .
\]

(42a, 42b)

The new lagrangian density reads

\[
L' = L + \partial_\mu F^\mu ,
\]

(43)

where \( F^\mu \) is the generator of the canonical transformation. The correct equations of motion are obtained when

\[
F^0 = \Pi_{11} g_{11} \quad F^1 = \alpha \left( 1 - \frac{1 + g_{11}}{2g_{11}} \right) \partial_1 g_{11} \Pi_{11} .
\]

(44)

Finally, the effective hamiltonian density is easily computed

\[
H_{\text{eff}} = g_{11} + \alpha \left( 1 - \frac{1 + g_{11}}{2g_{11}} \right) \Pi_{11} \partial_1 g_{11} ,
\]

(45)

and this density rules the dynamics of the gravitational field in the reduced phase-space. In fact, this result is in accord with the time derivative of \( g_{11} \) (obtained from the definition of momenta and the gauge fixing conditions (38))

\[
\dot{g}_{11}(x) = \left[ \frac{g_{11}}{\alpha^2} + 2 \left( 1 + \frac{g_{11} + 1}{2g_{11}} \right) \partial_1 g_{11} \right] (x) .
\]

(46)

V. CONCLUSIONS

The methods developed by Henneaux, Teitelboim and Vergara, and independently by Gitman and Tiutyn, offer a solution to the zero hamiltonian problem (ZH problem) in the 2-D induced gravity case. They permit to obtain an effective hamiltonian, which rules the evolution of physical degrees of freedom after complete gauge fixing. The key point is that open boundary conditions are necessary; the new hamiltonian arises naturally after a time-dependent canonical transformation is performed in reduced phase-space.

Acknowledgements

The authors would like to Thank Capes (Brazil) and the Physics Department of UFES (Vitória-Brazil) for financial support.
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