HÖRMANDER’S MULTIPLIER THEOREM FOR THE DUNKL TRANSFORM

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Abstract. For a normalized root system $R$ in $\mathbb{R}^N$ and a multiplicity function $k \geq 0$, let $N = N + \sum_{\alpha \in R} k(\alpha)$. Denote by $dw(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} dx$ the associated measure in $\mathbb{R}^N$. Let $F$ stands for the Dunkl transform. Given a bounded function $m$ on $\mathbb{R}^N$, we prove that if there is $s > N$ such that $m$ satisfies the classical H"ormander condition with the smoothness $s$, then the multiplier operator $T_m f = F^{-1}(mFf)$ is of weak type $(1,1)$, strong type $(p, p)$ for $1 < p < \infty$, and bounded on a relevant Hardy space $H^1$. To this end we study the Dunkl translations and the Dunkl convolution operators and prove that if $F$ is sufficiently regular, for example its certain Schwartz class seminorm is finite, then the Dunkl convolution operator with the function $F$ is bounded on $L^p(dw)$ for $1 \leq p \leq \infty$. We also consider boundedness of maximal operators associated with the Dunkl convolutions with Schwartz class functions.

1. Introduction and statements of the results

On the Euclidean space $\mathbb{R}^N$ we consider a normalized root system $R$ and a multiplicity function $k \geq 0$. Let

$$dw(x) = \prod_{\alpha \in R} |\langle x, \alpha \rangle|^{k(\alpha)} dx$$

be the associated measure in $\mathbb{R}^N$, where, here and subsequently, $dx$ stands for the Lebesgue measure in $\mathbb{R}^N$. Denote by $N = N + \sum_{\alpha \in R} k(\alpha)$ the homogeneous dimension of the system and by $G$ the Weyl group generated by the reflections $\sigma_{\alpha}, \alpha \in R$. Let $E(x, y)$ be the associated Dunkl kernel. The kernel $E(x, y)$ has a unique extension to a holomorphic function in $\mathbb{C}^N \times \mathbb{C}^N$. The Dunkl transform

$$Ff(\xi) = c_k^{-1} \int_{\mathbb{R}^N} E(-i\xi, x)f(x)dw(x),$$

where

$$c_k = \int_{\mathbb{R}^N} e^{-\|x\|^2} dw(x) > 0,$$

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originally defined for \( f \in L^1(dw) \), is an isometry on \( L^2(dw) \) and preserves the Schwartz class of functions \( S(\mathbb{R}^N) \) (see [8]). Its inverse \( \mathcal{F}^{-1} \) has the form
\[
\mathcal{F}^{-1} g(x) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) g(\xi) \, dw(\xi).
\]
The Dunkl transform \( \mathcal{F} \) is an analogue of the classical Fourier transform
\[
\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-i\langle \xi, x \rangle} f(x) \, dx.
\]
Let
\[
\|m\|_{W^s} = \|\hat{m}(x)(1 + \|x\|)^s\|_{L^2(dx)}
\]
for \( s \geq 0 \) be the classical Sobolev norm.
The metric measure space \((\mathbb{R}^N, \|x - y\|, dw)\) is doubling (see (2.4)). Let \( H^1_{\text{atom}} \) denote the atomic Hardy space in the sense of Coifman–Weiss [6] on the space of homogeneous type \((\mathbb{R}^N, \|x - y\|, dw)\) (see Section 7 for details).

We are in a position to state our main result.

**Theorem 1.2.** Let \( \psi \) be a smooth radial function such that \( \text{supp} \psi \subseteq \{ \xi : \frac{1}{4} \leq \|\xi\| \leq 4 \} \) and \( \psi(\xi) \equiv 1 \) for \( \{ \xi : \frac{1}{2} \leq \|\xi\| \leq 2 \} \). If \( m \) is a function on \( \mathbb{R}^N \) which satisfies the Hörmander condition
\[
(M = \sup_{t > 0} \|\psi(\cdot)m(t\cdot)\|_{W^s} < \infty)
\]
for some \( s > N \), then the multiplier operator
\[
T_m f = \mathcal{F}^{-1}(m \mathcal{F} f),
\]
originally defined on \( L^2(dw) \cap L^1(dw) \), is of
(A) weak type \((1, 1)\),
(B) strong type \((p, p)\) for \( 1 < p < \infty \),
(C) bounded on the Hardy space \( H^1_{\text{atom}} \).

Let us remark that we need the regularity of order \( s > N \) in the Hörmander’s condition (1.3). One might expect that a regularity \( s > N/2 \) would suffice (see the classical Hörmander’s multiplier theorem for the Fourier transform [17]). The price of \( N/2 \) we pay in the proof of the theorem is due to the fact that the so called Dunkl translation
\[
\tau_x f(y) = c_k^{-1} \int_{\mathbb{R}^N} E(i\xi, x) E(i\xi, y) \mathcal{F} f(\xi) \, dw(\xi)
\]
is bounded on \( L^2(dw) \) and it is an open problem if it is bounded on \( L^p(dw) \) for \( p \neq 2 \). If the translations \( \tau_x \) for a system of roots are uniformly bounded operators on \( L^1(dw) \), then the the regularity of order \( s > N/2 \) suffices. We elaborate this situation in Section 8. This happens e.g. in the case of the product system of roots or radial multipliers (see [4] and [7]). So in order to overcome the lack of knowledge about the Dunkl translation on the \( L^p(dw) \)-spaces for a general system of roots we use the only information we have, that is, the boundedness of \( \tau_x \) on \( L^2(dw) \) together with very important observation about supports of translations of \( L^2(dw) \)-functions, which is stated in the following our next main result.
Theorem 1.5. Let \( f \in L^2(dw) \), supp \( f \subseteq B(0, r) \), and \( \mathbf{x} \in \mathbb{R}^N \). Then

\[
\text{supp } \tau_{\mathbf{x}} f(-\cdot) \subseteq \mathcal{O}(B(x, r)),
\]

where \( \mathcal{O}(B(x, r)) = \bigcup_{\sigma \in G} B(\sigma(x), r) \) is the orbit of the Euclidean closed ball \( B(x, r) = \{ y \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{y}\| \leq r \} \).

The conclusion of Theorem 1.5 is known for \( f \) being \( L^2(dw) \)-radial functions supported by \( B(0, r) \) (see (2.9)). Our aim is to extend it for functions which are not necessary radial.

Let us also note that the Theorem 1.5 gives much precise information about the support of translations than that which follows from [1, Theorem 5.1]. Actually their result implies that \( \text{supp } \tau_{\mathbf{x}} f(-\cdot) \subseteq \{ y \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{y}\| - r \leq \|\mathbf{y}\| \leq \|\mathbf{x}\| + r \} \) for \( f \in L^2(dw) \), supp \( f \subseteq B(0, r) \).

2. Preliminaries

Dunkl theory is a generalization of Euclidean Fourier analysis. It started with the seminal article [11] and developed extensively afterwards (see e.g., [9], [10], [12], [13], [16], [18], [20], [19], [24], [23]). In this section we present basic facts concerning theory of the Dunkl operators. For details we refer the reader to [11], [21], and [22].

We consider the Euclidean space \( \mathbb{R}^N \) with the scalar product \( \langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{N} x_j y_j \), \( \mathbf{x} = (x_1, \ldots, x_N) \), \( \mathbf{y} = (y_1, \ldots, y_N) \), and the norm \( \|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle \). For a nonzero vector \( \alpha \in \mathbb{R}^N \) the reflection \( \sigma_\alpha \) with respect to the hyperplane \( \alpha \perp \) orthogonal to \( \alpha \) is given by

\[
\sigma_\alpha \mathbf{x} = \mathbf{x} - 2\frac{\langle \mathbf{x}, \alpha \rangle}{\|\alpha\|^2} \alpha.
\]

A finite set \( R \subseteq \mathbb{R}^N \setminus \{0\} \) is called a root system if \( \sigma_\alpha(R) = R \) for every \( \alpha \in R \). We shall consider normalized reduced root systems, that is, \( \|\alpha\|^2 = 2 \) for every \( \alpha \in R \). The finite group \( G \) generated by the reflections \( \sigma_\alpha \) is called the Weyl group (reflection group) of the root system. A multiplicity function is a \( G \)-invariant function \( k : R \to \mathbb{C} \) which will be fixed and \( \geq 0 \) throughout this paper.

The number \( N \) is called the homogeneous dimension of the system, since

\[
w(B(t\mathbf{x}, tr)) = t^N w(B(\mathbf{x}, r)) \quad \text{for } \mathbf{x} \in \mathbb{R}^N, t, r > 0,
\]

where here and subsequently \( B(\mathbf{x}, r) = \{ y \in \mathbb{R}^N : \|\mathbf{x} - \mathbf{y}\| \leq r \} \) denotes the (closed) Euclidean ball centered at \( \mathbf{x} \) with radius \( r > 0 \). Observe that

\[
w(B(\mathbf{x}, r)) \sim r^N \prod_{\alpha \in R} (|\langle \mathbf{x}, \alpha \rangle| + r)^{k(\alpha)}
\]

so \( dw(\mathbf{x}) \) it is doubling, that is, there is a constant \( C > 0 \) such that

\[
w(B(\mathbf{x}, 2r)) \leq C w(B(\mathbf{x}, r)) \quad \text{for } \mathbf{x} \in \mathbb{R}^N, r > 0.
\]

Moreover, by (2.3),

\[
C^{-1} \left( \frac{R}{r} \right)^N \leq \frac{w(B(\mathbf{x}, R))}{w(B(\mathbf{x}, r))} \leq C \left( \frac{R}{r} \right)^N \quad \text{for } 0 < r < R.
\]
The Dunkl operator $T_\xi$ is the following $k$-deformation of the directional derivative $\partial_\xi$ by a difference operator:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}} \frac{k(\alpha)}{2} \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$  

The Dunkl operators $T_\xi$, which were introduced in [11], commute and are skew-symmetric with respect to the $G$-invariant measure $dw$.

Let $e_j$, $j = 1, 2, ..., N$, denote the canonical orthonormal basis in $\mathbb{R}^N$ and let $T_j = T_{e_j}$.

For fixed $y \in \mathbb{R}^N$ the Dunkl kernel $E(x, y)$ is the unique solution of the system

$$T_\xi f = \langle \xi, y \rangle f, \quad f(0) = 1.$$  

In particular

$$T_{j\cdot x} E(x, y) = y_j E(x, y),$$  

where here and subsequently $T_{j\cdot x}$ denotes the action of $T_j$ with respect to the variable $x$. The function $E(x, y)$, which generalizes the exponential function $e^{(x\cdot y)}$, has a unique extension to a holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$. We have (see, e.g. [21], [22])

- $E(\lambda x, y) = E(x, \lambda y) = E(\lambda y, x) = E(\lambda \sigma(x), \sigma(y))$ for all $x, y \in \mathbb{C}^N$, $\sigma \in G$, and $\lambda \in \mathbb{C}$;
- $E(x, y) > 0$ for all $x, y \in \mathbb{R}^N$;
- $|E(-i \lambda, y)| \leq 1$ for all $x, y \in \mathbb{R}^N$;
- $E(0, y) = 1$ for all $y \in \mathbb{C}^N$.

Let us collect basic properties of the Dunkl transform $\mathcal{F}$ and the Dunkl translation $\tau_x$ defined in (1.1) and (1.4)

- $\mathcal{F}(T_\xi f) = i \langle \xi, \cdot \rangle \mathcal{F} f$, and $T_\xi (\mathcal{F} f) = -i \mathcal{F} (\langle \xi, \cdot \rangle f)$;
- the Dunkl transform of a radial function is again a radial function;
- $\mathcal{F}(f_\lambda)(\xi) = \mathcal{F}(\lambda \xi)$, where $f_\lambda(x) = \lambda^{-N} f(\lambda^{-1} x)$, $\lambda > 0$;
- each translation $\tau_x$ is a continuous linear map of $\mathcal{S}(\mathbb{R}^N)$ into itself, which extends to a contraction on $L^2(dw)$;
- (Identity) $\tau_0 = I$;
- (Symmetry) $\tau_x f(y) = \tau_y f(x)$ for all $x, y \in \mathbb{R}^N, f \in \mathcal{S}(\mathbb{R}^N)$;
- (Scaling) $\tau_x (f_\lambda) = (\tau_{\lambda^{-1} x} f)_\lambda$ for all $\lambda > 0$, $x \in \mathbb{R}^N$, $f \in \mathcal{S}(\mathbb{R}^N)$;
- (Commutativity) $T_\xi (\tau_x f) = \tau_x (T_\xi f)$;
- (Skew-symmetry)

$$\int_{\mathbb{R}^N} \tau_x f(y) g(y) dw(y) = \int_{\mathbb{R}^N} f(y) \tau_{-x} g(y) dw(y)$$

for all $x \in \mathbb{R}^N, f, g \in \mathcal{S}(\mathbb{R}^N)$.

The latter formula allows us to define the Dunkl translations $\tau_x f$ in the distributional sense for $f \in L^p(dw)$ with $1 \leq p \leq \infty$. Further,

$$\int_{\mathbb{R}^N} \tau_x f(y) dw(y) = \int_{\mathbb{R}^N} f(y) dw(y)$$

for all $x \in \mathbb{R}^N$, $f \in \mathcal{S}(\mathbb{R}^N)$.  

The following specific formula was obtained by Rösler [20]: for the Dunkl translations of (reasonable) radial functions \( f(x) = \tilde{f}(\|x\|) \):

\[
(2.7) \quad \tau_x f(-y) = \int_{\mathbb{R}^N} (\tilde{f} \circ A)(x, y, \eta) \, d\mu_x(\eta) \text{ for all } x, y \in \mathbb{R}^N.
\]

Here

\[
A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2(y, \eta)} = \sqrt{\|x\|^2 - \|\eta\|^2 + \|y - \eta\|^2}
\]

and \( \mu_x \) is a probability measure, which is supported in \( \text{conv} \, O(x) \).

It is not hard to see that \( A(x, y, \eta) \geq d(x, y) \) for \( x, y \in \mathbb{R}^N \) and \( \eta \in \text{conv} \, O(x) \), where here and subsequently,

\[
(2.8) \quad d(x, y) = \inf_{\sigma \in G} \|\sigma(x) - y\|
\]

denotes the distance of the orbits \( O(x) \) and \( O(y) \). Hence, (2.7) implies that if \( f \in L^2(dw) \) is a radial function supported by \( B(0, r) \), then

\[
(2.9) \quad \text{supp } \tau_x f(-\cdot) \subseteq O(B(x, r)).
\]

We prove first (2.9) for radial \( C^\infty_c \)-functions and then use a density argument and continuity of the Dunkl translation on \( L^2(dw) \).

The Dunkl convolution of two reasonable functions (for instance Schwartz functions) is defined by

\[
(f \ast g)(x) = c_k \mathcal{F}^{-1}[(\mathcal{F}f)(\mathcal{F}g)](x) = \int_{\mathbb{R}^N} (\mathcal{F}f)(\xi) (\mathcal{F}g)(\xi) E(x, i\xi) \, dw(\xi) \quad \text{for all } x \in \mathbb{R}^N,
\]

or, equivalently, by

\[
(f \ast g)(x) = \int_{\mathbb{R}^N} f(y) \tau_x g(-y) \, dw(y) = \int f(y) g(x, y) \, dw(y),
\]

where here and subsequently, we use the notation

\[
(2.10) \quad g(x, y) = \tau_x g(-y) = \tau_{-y} g(x)
\]

for a reasonable function \( g(x) \) on \( \mathbb{R}^N \). The last equality in (2.10) follows by symmetry of the Dunkl translation.

Let us collect some well-known formulæ for the Dunkl translation and convolution.

\[
(2.11) \quad \|\tau_y f\|_{L^2(dw)} \leq \|f\|_{L^2(dw)} \quad \text{for } f \in L^2(dw),
\]

\[
(2.12) \quad \|f \ast g\|_{L^2(dw)} \leq \|f\|_{L^1(dw)} \|g\|_{L^2(dw)} \quad \text{for } f \in L^1(dw), \ g \in L^2(dw).
\]

If \( g \) is a radial and continuous compactly supported, then \( \|\tau_y g\|_{L^1(dw)} = \|g\|_{L^1(dw)} \) (see (2.7)), in particular the translation \( \tau_y g \) can be uniquely extended to radial \( L^1(dw) \)-functions. Hence one can give a sense for the convolution \( f \ast g \), where \( f \in L^p(dw) \) and \( g \in L^1(dw) \) is radial.

The Dunkl Laplacian associated with \( G \) and \( k \) is the differential-difference operator

\[
\Delta = \sum_{j=1}^N T_j^2,
\]

which acts on \( C^2(\mathbb{R}^N) \) functions by

\[
\Delta f(x) = \Delta_{\text{eucl}} f(x) + \sum_{\alpha \in \mathbb{R}} k(\alpha) \delta_\alpha f(x),
\]
\[ \delta_\alpha f(x) = \frac{\partial f(x)}{\langle \alpha, x \rangle} - \frac{\|\alpha\|^2 f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}. \]

Clearly, \( \mathcal{F}(\Delta f)(\xi) = -\|\xi\|^2 \mathcal{F} f(\xi). \) The operator \( \Delta \) is essentially self-adjoint on \( L^2(dw) \) (see for instance [2, Theorem 3.1]) and generates the semigroup \( e^{t\Delta} \) of linear self-adjoint contractions on \( L^2(dw). \) The semigroup has the form

\[ e^{t\Delta} f(x) = \mathcal{F}^{-1}(e^{-t\|\xi\|^2} \mathcal{F} f(\xi))(x) = \int_{\mathbb{R}^N} h_t(x, y) f(y) \, dw(y), \]

where the heat kernel \( h_t(x, y) = \tau_x h_t(-y), h_t(x) = \mathcal{F}^{-1}(e^{-t\|\xi\|^2})(x) = c_k^{-1}(2t)^{-N/2} e^{-\|x\|^2/(4t)} \) is \( C^\infty \) function of all variables \( x, y \in \mathbb{R}^N, t > 0 \) and satisfies

\[ 0 < h_t(x, y) = h_t(y, x), \]

\[ \int_{\mathbb{R}^N} h_t(x, y) \, dw(y) = 1. \]

3. Properties of Dunkl translations - proof of Theorem 1.5

3.1. Properties of translations of the Dunkl heat kernel. We start this section by the list of further properties of the heat kernel. Set

\[ V(x, y, t) = \max(w(B(x, t)), w(B(y, t))). \]

The following estimates were proved in [5, Theorem 4.3].

**Theorem 3.1.** There are constants \( C, c > 0 \) such that for all \( x, y \in \mathbb{R}^N \) and \( t > 0 \) we have

\[ |h_t(x, y)| \leq C V(x, y, \sqrt{t})^{-1} e^{-c d(x, y)^2/t}, \]

\[ |h_t(x, y) - h_t(x, y')| \leq C \left( \frac{\|y - y'\|}{\sqrt{t}} \right) V(x, y, \sqrt{t})^{-1} e^{-c d(x, y)^2/t}. \]

Clearly, \( \tau_x \tau_y h_t(-z) \) is \( C^\infty \)-function of \( x, y, z \in \mathbb{R}^N \) and \( t > 0. \) The following formula is a direct consequence of (3.2).

\[ \|h_t(x, y)\|_{L^2(dw(y))} \leq \frac{C}{w(B(x, \sqrt{t}))^{1/2}}. \]

We have

\[ |\tau_x \tau_y h_t(-z)| \leq C w(B(x, \sqrt{t}))^{-1/2} w(B(y, \sqrt{t}))^{-1/2}, \]

\[ \int_{B(0,1/t)} |E(i\xi, x)|^2 \, dw(\xi) \leq \frac{C}{w(B(x, t))}. \]
Proof of (3.5). By Cauchy–Schwarz inequality, Plancharel theorem for the Dunkl transform $\mathcal{F}$, and (3.4) we have
\[
|τ_x τ_y h_t(−z)| = c_k^{-1} \left| \int E(iξ, x) E(iξ, y) E(−iξ, z) e^{-t||ξ||^2} \, dw(ξ) \right| \\
\leq c_k^{-1} \left( \int |E(iξ, x)|^2 e^{-t||ξ||^2} \, dw(ξ) \right)^{1/2} \left( \int |E(iξ, y)|^2 e^{-t||ξ||^2} \, dw(ξ) \right)^{1/2} \\
= \|h_t(x, z)\|_{L^2(dw(z))} \|h_t(y, z)\|_{L^2(dw(z))} \leq C \, w(B(x, \sqrt{t}))^{-1/2} w(B(y, \sqrt{t}))^{-1/2}. \]

Proof of (3.6). By (3.4) and Plancherel theorem for the Dunkl transform $\mathcal{F}$ we get
\[
\frac{C}{w(B(x, t))^{1/2}} \geq \|h_t^2(x, \cdot)\|_{L^2(dw)} = \|E(\cdot, x) e^{-t||\cdot||^2}\|_{L^2(dw)} \geq e^{-1}\|E(\cdot, x)\|_{L^2(B(0, t^{-1}), dw)}. \]

Proposition 3.7. If $m$ is a bounded function supported by $B(0, 1/t)$, then $(\mathcal{F}^{-1}m)(x, y)$ is a $C^\infty$-function of $x, y \in \mathbb{R}^N$ which satisfies
\[
|(|\mathcal{F}^{-1}m)(x, y)| \leq C\|m\|_{L^\infty} w(B(x, t))^{-1/2} w(B(y, t))^{-1/2}. \]

Proof. By the Cauchy–Schwarz inequality and (3.6),
\[
|(\mathcal{F}^{-1}m)(x, y)| = c_k^{-1} \left| \int_{B(0, 1/t)} m(ξ) E(−iξ, x) E(iξ, y) \, dw(ξ) \right| \\
\leq c_k^{-1} \|m\|_{L^\infty} \|E(\cdot, x)\|_{L^2(B(0, 1/t), dw)} \|E(\cdot, y)\|_{L^2(B(0, 1/t), dw)} \\
\leq C\|m\|_{L^\infty} w(B(x, t))^{-1/2} w(B(y, t))^{-1/2}. \]

3.2. Support of translations of compactly supported functions. Suppose that $f, g \in C^1(\mathbb{R}^N)$ and $g$ is radial. The following Leibniz rule can be confirmed by a direct calculation:
\[
T_j(fg) = f(T_j g) + g(T_j f) \text{ for all } 1 \leq j \leq N. \tag{3.9} \]
Let us denote the set of all polynomials of degree $d \geq 0$ by $\mathbb{P}_d$.

Proposition 3.10 ([22, Lemma 2.6]). Let $p \in \mathbb{P}_d$ and $1 \leq j \leq N$. Then $T_j p \in \mathbb{P}_{d−1}$.

Let $L$ be a positive integer. Let us denote
\[
g_L(x) = \max\{0, (1 − ||x||^2)^L\}. \]
The function $g_L$ is radial, belongs to $C^{L−1}(\mathbb{R}^N)$, and supp $g_L \subseteq B(0, 1)$.

For $\alpha = (α_1, α_2, \ldots, α_N) \in \mathbb{N}^N = (\mathbb{N} \cup \{0\})^N$ we define
\[
T^0_j = I, \quad T^α := T_1^{α_1} o T_2^{α_2} o \ldots o T_N^{α_N}. \]
Clearly, supp $T^α g_L \subseteq B(0, 1)$ for $||α|| < L$, where $||α|| = \sum_{j=1}^N α_j$. 
Lemma 3.11. Let \( L \in \mathbb{N} \) and \( p \) be a polynomial of degree \( d \). Then \( pg_L \) can be written in the form

\[
p(x)g_L(x) = \sum_{\ell=0}^{d} \sum_{|\alpha| \leq \ell} c_{\ell,\alpha} T^\alpha(g_{L+\ell})(x)
\]

for some \( c_{\ell,\alpha} \in \mathbb{C} \).

Proof. The proof is by induction on \( d \). The claim for \( d = 0 \) is obvious. Let us assume that for any polynomial \( p(x) \) of degree at most \( d \) and any positive integer \( L \in \mathbb{N} \) the function \( p(x)g_L(x) \) can be written in the form \( (3.12) \). We will prove the claim for any polynomial \( q(x) \) of degree \( d + 1 \) and any \( L \in \mathbb{N} \). By linearity, it is enough to prove the claim for \( q(x) = x_j p(x) \) with \( 1 \leq j \leq N \) and \( p \in \mathbb{P}_d \). Since \( g_{L+1} \in C^1(\mathbb{R}^N) \) is radial and \( p \in C^1(\mathbb{R}^N) \), by \( (3.9) \) we have

\[
T_j(pg_{L+1})(x) = p(x)T_j g_{L+1}(x) + g_{L+1}(x) T_j p(x)
\]

\[
= 2(L + 1)x_j p(x)g_L(x) + g_{L+1}(x) T_j p(x).
\]

By Proposition 3.10 we have \( T_j p \in \mathbb{P}_{d-1} \), so, by the induction hypothesis,

\[
g_{L+1}(x) T_j p(x) = \sum_{\ell=0}^{d-1} \sum_{|\alpha| \leq \ell} c_{\ell,\alpha} T^\alpha(g_{L+1+\ell})(x) = \sum_{\ell=0}^{d+1} \sum_{|\alpha| \leq \ell} c'_{\ell,\alpha} T^\alpha(g_{L+\ell})(x).
\]

Therefore, by \( (3.13) \), it is enough to check that \( T_j(pg_{L+1}) \) can be written in the form \( (3.12) \). Since \( p \in \mathbb{P}_d \), by the induction hypothesis

\[
p(x)g_{L+1}(x) = \sum_{\ell=0}^{d} \sum_{|\alpha| \leq \ell} d_{\ell,\alpha} T^\alpha(g_{L+1+\ell})(x),
\]

therefore

\[
T_j(pg_{L+1})(x) = \sum_{\ell=0}^{d} \sum_{|\alpha| \leq \ell} d_{\ell,\alpha} T_j \circ T^\alpha(g_{L+1+\ell})(x)
\]

\[
= \sum_{\ell=0}^{d} \sum_{|\alpha| \leq \ell} d_{\ell,\alpha} T^\alpha+j(g_{L+1+\ell})(x) = \sum_{\ell=0}^{d+1} \sum_{|\alpha| \leq \ell} d'_{\ell,\alpha} T^\alpha(g_{L+\ell})(x).
\]

\[
\Box
\]

Lemma 3.14. The set \( \bigcup_{d \in \mathbb{N}_0} \bigcup_{p \in \mathbb{P}_d} \{ p(\cdot) g_1(\cdot) \} \) is dense in \( L^2(B(0,1), dw) \).

Proof. Take any \( f \in L^2(B(0,1), dw) \) and fix \( \varepsilon > 0 \). There is \( \delta > 0 \) such that

\[
\|f - f \chi_{B(0,1-\delta)}\|_{L^2(dw)} < \varepsilon.
\]

Let \( \Phi \in C^\infty(B(0,1)) \) be a radial function such that \( \int \Phi(x) \, dw(x) = 1 \). Let us denote \( \Phi_t(x) = t^{-N} \Phi(t^{-1} x) \). Since \( \Phi_t \) is an approximate of the identity on \( L^2(dw) \), there is \( t > 0 \) such that

\[
\|\Phi_t * (f \chi_{B(0,1-\delta)}) - f \chi_{B(0,1-\delta)}\|_{L^2(dw)} < \varepsilon
\]
and supp $\Phi_t \ast (f \chi_{B(0,1-\delta)}) \subseteq B(0,1 - \delta/2)$. Since $f \chi_{B(0,1-\delta)} \in L^2(dw)$, the function $\Phi_t \ast (f \chi_{B(0,1-\delta)})$ is continuous. Moreover, there is $\eta > 0$ such that $g_1(x) > \eta$ for $x \in B(0,1 - \delta/2)$. This implies that $(\Phi_t \ast (f \chi_{B(0,1-\delta)}))g_1^{-1}$ is a continuous function on $B(0,1 - \delta/2)$, which extends to a continuous $h$ function on $B(0,1)$ (by putting 0 on $B(0,1) \setminus B(0,1 - \delta/2)$). Therefore, by the Stone–Weierstrass theorem, there is a polynomial $p$ such that

$$\|h - p\|_{L^\infty(B(0,1))} < \varepsilon. \tag{3.15}$$

Finally, by (3.15),

$$\|\Phi_t \ast (f \chi_{B(0,1-\delta)}) - pg_1\|_{L^2(dw)} = \|hg_1 - pg_1\|_{L^2(dw)} \leq w(B(0,1))^{1/2}\|hg_1 - pg_1\|_{L^\infty(B(0,1))} \leq w(B(0,1))^{1/2}\|g_1\|_{L^\infty(B(0,1))}\|h - p\|_{L^\infty(B(0,1))} \leq w(B(0,1))^{1/2}\varepsilon. \tag{3.16}$$

Proof of Theorem 1.5. It suffices to consider $r = 1$. Let $f \in L^2(B(0,1), dw)$. Fix $\varepsilon > 0$. By Lemma 3.14 there is a polynomial $p$ such that

$$\|f - pg_1\|_{L^2(dw)} < \varepsilon. \tag{3.17}$$

Since the Dunkl translation is bounded on $L^2(dw)$ and its norm is 1, we have

By Lemma 3.11 and the fact that the Dunkl translations commute with the Dunkl operators, the function $\tau_x(pg_1)$ can be written in the form

$$\tau_x(pg_1)(-y) = \tau_x \left( \sum_{\ell=0}^d \sum_{\|\alpha\| \leq \ell} c_{\ell,\alpha} T^\alpha(g_{1+\ell})(-y) \right) = \sum_{\ell=0}^d \sum_{\|\alpha\| \leq \ell} c_{\ell,\alpha} T^\alpha \tau_x(g_{1+\ell})(-y),$$

where $d$ is the degree of $p$. Since the functions $g_{1+\ell}$ are radial and supported by $B(0,1)$, by (2.9) we have that supp $\tau_x g_{\ell+1}(-\cdot) \subseteq O(B(x,1))$. This implies that supp $(T^\alpha \tau_x g_{\ell+1})(-\cdot) \subseteq O(B(x,1))$ and, finally,

$$\text{supp } \tau_x(pg_1)(-\cdot) \subseteq O(B(x,1)). \tag{3.18}$$

Since $\varepsilon > 0$ is taken arbitrarily, (3.16) and (3.18) imply the claim. \qed

4. Consequences of Theorem 1.5

**Corollary 4.1.** Suppose that $x, y \in \mathbb{R}^N$ satisfy $\|x\| + \|y\| < 1$. If $f$ is a continuous compactly supported function such that $f(z) = 0$ for all $z \in B(0,1)$, then $\tau_x f(y) = 0$.\n
**Proof.** The corollary follows from [1, Theorem 5.1]. We present here an alternative proof.
Take \( \varepsilon > 0 \) such that \( \|x\| + \|y\| + \varepsilon < 1 \). Let \( g \in L^2(dw) \), \( \text{supp} \ g \subseteq B(0, \|y\| + \varepsilon) \). We have

\[
\int \tau_x f(z) g(z) \, dw(z) = \int f(z) \tau_{-x} g(z) \, dw(z).
\]

By Theorem 1.5,

\[
\text{supp} \ \tau_{-x} g \subseteq \mathcal{O}(B(x, \|y\| + \varepsilon)) \subseteq B(0, \|x\| + \|y\| + \varepsilon) \subseteq B(0,1).
\]

By our assumption \( f(z) = 0 \) for all \( z \in B(0,1) \supseteq \text{supp} \ \tau_{-x} g \), so the second integral in (4.2) is zero. Thus \( \tau_x f \equiv 0 \) on \( B(0, \|y\| + \varepsilon) \). In particular, \( \tau_x f(y) = 0 \).

\[\square\]

**Lemma 4.3.** There is a constant \( C > 0 \) such that for any \( r > 0 \), \( x \in \mathbb{R}^N \), and any radial function \( \phi \in C_c(B(0,r)) \) we have

\[
\|\tau_x \phi(- \cdot)\|_{L^2(dw)} \leq C \frac{r^N \|\phi\|_{L^\infty}}{w(B(x,r))^{1/2}}.
\]

**Proof.** By (2.9) \( \text{supp} \ \tau_x \phi(- \cdot) \subseteq \mathcal{O}(B(x,r)) \), so

\[
\|\tau_x \phi\|_{L^2(dw)} \leq |G|^{1/2} w(B(x,r))^{1/2} \|\tau_x \phi\|_{L^\infty}.
\]

Furthermore, by [5, Corollary 3.10], there is a constant \( C > 0 \) such that

\[
\|\tau_x \phi\|_{L^\infty} \leq C \frac{r^N \|\phi\|_{L^\infty}}{w(B(x,r))},
\]

so the lemma follows. \[\square\]

**Proposition 4.4.** There is a constant \( C > 0 \) such that for any \( r_1, r_2 > 0 \), any \( f \in L^1(dw) \) such that \( \text{supp} \ f \subseteq B(0, r_2) \), any radial function \( \phi \in C_c(B(0,r_1)) \), and for all \( y \in \mathbb{R}^N \) we have

\[
\|\tau_y (f \ast \phi)\|_{L^1(dw)} \leq C (r_1(r_1 + r_2))^N \frac{\|\phi\|_{L^\infty}}{\|f\|_{L^1(dw)}}.
\]

**Proof.** By Theorem 1.5, \( \text{supp} \ \tau_y (f \ast \phi)(- \cdot) \subseteq \mathcal{O}(B(y, r_1 + r_2)) \). Therefore

\[
\|\tau_y (f \ast \phi)\|_{L^1(dw)} \leq |G|^{1/2} w(B(y, r_1 + r_2))^{1/2} \|\tau_y (f \ast \phi)\|_{L^2(dw)}.
\]

Since \( \tau_y (f \ast \phi) = f \ast (\tau_y \phi) \), we have

\[
\|\tau_y (f \ast \phi)\|_{L^2(dw)} = \left( \int \left| (f \ast \tau_y \phi)(x) \right|^2 \, dw(x) \right)^{1/2}
\]

\[
= \left( \int \left| \int \tau_x (\tau_y \phi)(-z) f(z) \, dw(z) \right|^2 \, dw(x) \right)^{1/2}
\]

\[
= \left( \int \left| \int \tau_{-z} (\tau_y \phi)(x) f(z) \, dw(z) \right|^2 \, dw(x) \right)^{1/2}.
\]
By Minkowski integral inequality
\begin{equation}
\left( \int \left| \int \tau_{-z}(\tau_{y}\phi)(x)f(z)\,dw(z) \right|^2 \,dw(x) \right)^{1/2} \leq \int \left| f(z) \right| \|\tau_{-z}\tau_{y}\phi\|_{L^2(dw)} \,dw(z).
\end{equation}
Since \( g \mapsto \tau_{-z}g \) is a contraction on \( L^2(dw) \) for all \( z \in \mathbb{R}^N \), by Lemma 4.3 we have
\begin{equation}
\|\tau_{-z}\tau_{y}\phi\|_{L^2(dw)} \leq \|\tau_{y}\phi\|_{L^2(dw)} \leq C \frac{r_1^N}{w(B(y, r_1))^{1/2}} \phi_{L^\infty}.
\end{equation}
Therefore, by (4.6) and (4.7),
\begin{equation}
\|\tau_{y}(f \ast \phi)\|_{L^2(dw)} \leq C \frac{r_1^N}{w(B(y, r_1))^{1/2}} \|f\|_{L^1(dw)}.
\end{equation}
Finally, by (4.5),
\begin{equation}
\|\tau_{y}(f \ast \phi)\|_{L^1(dw)} \leq C |G|^{1/2} w(B(y, r_1 + r_2))^{1/2} \frac{r_1^N}{w(B(y, r_1))^{1/2}} \|f\|_{L^1(dw)}
\leq C' (r_1(r_1 + r_2))^N |\phi|_{L^\infty} \|f\|_{L^1(dw)}.
\end{equation}

Let \( \Psi_0 \in C^\infty((-1, 1)) \) and \( \Psi \in C^\infty(\frac{1}{4}, 4) \) be such that
\begin{equation}
1 = \Psi_0(\|x\|) + \sum_{n=1}^{\infty} \Psi(2^{-n}\|x\|) = \sum_{n=0}^{\infty} \Psi_n(\|x\|) \text{ for all } x \in \mathbb{R}^N.
\end{equation}

**Proposition 4.9.** Fix \( \delta \geq 0 \). Assume that \( \phi \) is a continuous radial function such that
\begin{equation}
\sum_{n=0}^{\infty} 2^{n(N+\delta)} \|\phi(\cdot)\Psi_n(\|\cdot\|)\|_{L^\infty} = A < \infty
\end{equation}
and \( f \) is a measurable function on \( \mathbb{R}^N \) such that
\begin{equation}
\sum_{j=0}^{\infty} 2^{j(N/2+\delta)} \|f(\cdot)\Psi_n(\|\cdot\|)\|_{L^1(dw)} = B < \infty.
\end{equation}
Then \( f \ast \phi \in L^2(dw) \cap L^1(dw) \) and there is a constant \( C > 0 \) such that for every \( y \in \mathbb{R}^N \) we have
\begin{equation}
\int |\tau_{y}(f \ast \phi)(-x)|(1 + d(x, y))^\delta \,dw(x) \leq CAB.
\end{equation}

**Proof.** In the proof we will use the formula
\[
\tau_{x}(f \ast \phi)(-y) = \tau_{-y}(f \ast \phi)(x) = (f \ast \phi)(x, y).
\]
Let \( f_j(x) = f(x)\Psi_j(\|x\|), \phi_n(x) = \phi(x)\Psi_n(\|x\|). \) Observe that \( f = \sum_{j=0}^{\infty} f_j \) and the series converges in \( L^1(dw) \). Moreover, \( \|\phi_n\|_{L^2(dw)} \leq C 2^{nN/2} \|\phi_n\|_{L^\infty}, \) hence \( \phi = \sum_{n=0}^{\infty} \phi_n \).
and the convergence is in $L^2(dw)$. So, by (2.12), $f * \phi \in L^2(dw)$. Further, the double series

$$f * \phi = \sum_{j,n \in \mathbb{N}_0} f_j * \phi_n$$

is absolutely convergent in $L^2(dw)$, because

$$\sum_{j,n \in \mathbb{N}_0} \|f_j * \phi_n\|_{L^2(dw)} \leq \sum_{j,n \in \mathbb{N}_0} \|f_j\|_{L^1(dw)} \|\phi_n\|_{L^2(dw)} \leq C \sum_{j,n \in \mathbb{N}_0} \|f_j\|_{L^1(dw)} 2^{nN/2} \|\phi_n\|_{L^\infty} \leq CAB.$$

Using (2.11) and (2.12) we have

$$\tau_y (f * \phi) = \sum_{j,n \in \mathbb{N}_0} \tau_y (f_j * \phi_n) = \sum_{j,n \in \mathbb{N}_0} f_j * (\tau_y \phi_n)$$

and the convergence (absolute) is in $L^2(dw)$. Clearly, $\text{supp } f_j * \phi_n \subseteq B(0, 2^j + 2^n)$. Theorem 1.5 implies $\tau_y (f_j * \phi_n)(x) = 0$ for $d(x, y) > 2^j + 2^n$. Using Proposition 4.4 we obtain

$$\sum_{j,n \in \mathbb{N}_0} \int |(f_j * \phi_n)(x, y)|(1 + d(x, y))^{\delta} \, dw(x)$$

$$\leq C \sum_{j,n \in \mathbb{N}_0} 2^{Nn/2} (2^j + 2^n)^{\delta + N/2} \|f_j\|_{L^1(dw)} \|\phi_n\|_{L^\infty} \leq CAB.$$

Thus, the double series (4.13) converges in the $L^1((1 + d(x, y))^{\delta} \, dw(x))$-norm as well. The proof of the proposition is complete. \qed

**Corollary 4.14.** Assume that there is $\delta > 0$ such that $f(x)(1 + \|x\|)^{N/2 + \delta} \in L^1(dw)$ and $\phi(x)(1 + \|x\|)^{N+\delta} \in L^{\infty}(dw)$. Then for every $0 < \delta' < \delta$ there is a constant $C = C_{\delta, \delta'}$ such that

$$\|\tau_y (f * \phi)(x)(1 + d(x, y))^{\delta'}\|_{L^1(dw(x))}$$

$$\leq C \|f(\cdot)(1 + \|\cdot\|)^{N/2 + \delta}\|_{L^1(dw)} \|\phi(\cdot)(1 + \|\cdot\|)^{N+\delta}\|_{L^{\infty}}.$$

In particular,

$$\int_{\partial(B(y, r))^c} |\tau_y (f * \phi)(x)| \, dw(x)$$

$$\leq C r^{-\delta'} \|f(\cdot)(1 + \|\cdot\|)^{N/2 + \delta}\|_{L^1(dw)} \|\phi(\cdot)(1 + \|\cdot\|)^{N+\delta}\|_{L^{\infty}}.$$

**Corollary 4.17.** Let $s > N$ be a positive integer and $\varepsilon > 0$. Assume that a function $F \in C^{2s}(\mathbb{R}^N) \cap L^1(dw)$ satisfies:

$$B_1 = \|(I - \Delta)^s F(x)(1 + \|x\|)^{N/2 + \varepsilon}\|_{L^1(dw(x))} < \infty.$$

Then there is a constant $C_{s, \varepsilon} > 0$ such that

$$\sup_{y \in \mathbb{R}^N} \|\tau_y F\|_{L^1(dw)} \leq C_{s, \varepsilon} B_1.$$

In particular, for every $1 \leq p < \infty$ we have $\|g * F\|_{L^p(dw)} \leq C_{s, \varepsilon} B_1 \|g\|_{L^p(dw)}$. 
Proof. Set \( f = (I - \Delta)^s F, \) \( g(x) = c_k^{-1} \mathcal{F}^{-1}\{(1 + \|\xi\|^2)^{-s}\}(x). \) Then \( g \) is a radial function satisfying \(|g(x)| \leq C_L(1 + \|x\|)^{-L} \) for every \( L > 0. \) Clearly, \( F = f * g. \) Thus the corollary is a direct consequence of Proposition 4.9.

\[\square\]

**Corollary 4.18.** For every \( F \in \mathcal{S}(\mathbb{R}^N) \) there is a constant \( C > 0 \) such that
\[
\sup_{y \in \mathbb{R}^N} \|\tau_y F\|_{L^1(dw)} \leq C.
\]

**Theorem 4.19.** Let \( \Phi \in \mathcal{S}(\mathbb{R}^N). \) Then the maximal function
\[
\mathcal{M}_\Phi f(x) = \sup_{t > 0} |\Phi_t * f(x)| = \sup_{t > 0} \left| \int \Phi_t(x, y) f(y) \, dw(y) \right|
\]
where \( \Phi_t(x) = t^{-N}\Phi(t^{-1}x), \) is of weak type \((1, 1)\) and bounded on \( L^p(dw) \) for \( 1 < p \leq \infty. \)

**Proof.** It is enough to prove that there is a constant \( C = C_\Phi > 0 \) such that
\[
\mathcal{M}_\Phi f(x) \leq C \sum_{\sigma \in G} \mathcal{M}_{HL} f(\sigma(x)),
\]
where \( \mathcal{M}_{HL} \) is Hardy–Littlewood maximal function on the space of homogeneous type \((\mathbb{R}^N, \|x - y\|, dw).\) To this end it suffices to prove that there are constants \( C, \delta > 0 \) such that for all \( x, y \in \mathbb{R}^N \) we have
\[
|\Phi_t(x, y)| \leq C w(B(x, t)^{-1}\left(1 + \frac{d(x, y)}{t}\right)^{-N-\delta}.
\]

Let \( g(x) = c_k \mathcal{F}^{-1}\{(1 + \|\cdot\|^2)^{-N}\}. \) The function \( g \) is a radial and satisfies \(|g(x)| \leq C_1(1 + \|x\|)^{-2N-\delta}, \) so by [5, Corollary 3.10], we have
\[
|g_t(x, z)| \leq C_2 w(B(x, t)^{-1}\left(1 + \frac{d(x, z)}{t}\right)^{-s} \text{ for } s \in \{0, N + \delta\}.
\]

Set \( \Phi^{(1)} = \mathcal{F}^{-1}(\mathcal{F}(\Phi)(1 + \|\cdot\|^2)^{2N}). \) Then \( \Phi^{(1)} \in \mathcal{S}(\mathbb{R}^N), \) \( \Phi_t = \Phi^{(1)}_t * g_t * g_t, \) and
\[
|\Phi_t(x, y)| \leq \int |g_t(x, z)(\Phi^{(1)}_t * g_t)(z, y)| \, dw(z) \leq \int_{d(x, y) \leq 2d(x, z)} + \int_{d(x, y) \leq 2d(y, z)} = I_1 + I_2.
\]

Now, (4.21) with \( s = N + \delta \) and Corollary 4.14 with \( \delta' = 0 \) lead to
\[
I_1 \leq C_3 w(B(x, t)^{-1}\left(1 + \frac{d(x, z)}{t}\right)^{-N-\delta} |(\Phi^{(1)}_t * g_t)(z, y)| \, dw(z)
\]
\[
\leq C_3 w(B(x, t)^{-1}\left(1 + \frac{d(x, y)}{t}\right)^{-N-\delta} \int |(\Phi^{(1)}_t * g_t)(z, y)| \, dw(z)
\]
\[
\leq C_4 w(B(x, t)^{-1}\left(1 + \frac{d(x, y)}{t}\right)^{-N-\delta}.
\]

Further,
\[
I_2 \leq C_5 \left(1 + \frac{d(x, y)}{t}\right)^{-N-\delta} \int |g_t(x, z)(\Phi^{(1)}_t * g_t)(z, y)| \left(1 + \frac{d(y, z)}{t}\right)^{N+\delta} \, dw(z),
\]
so by (4.21) with \( s = 0 \) and Corollary 4.14 with \( \delta' = N + \delta, \) we obtain (4.20).
5. Multipliers

Let \( m \) be a function defined on \( \mathbb{R}^N \). In this section we assume that there exists \( s > N \) such that the multiplier \( m \) satisfies (1.3).

Fix \( \phi \) a radial \( C^\infty \) function on \( \mathbb{R}^N \) supported in the annulus \( \{ \xi \in \mathbb{R}^N : 1/2 \leq \|\xi\| \leq 2 \} \) such that \( 1 = \sum_{\ell \in \mathbb{Z}} \phi(2^{\ell} \xi) \). We define \( m_\ell(\xi) \), \( m_{\ell,1}(\xi) \), and \( m_{\ell,2}(\xi) \) as follows:

\[
(5.1) \quad m_\ell(\xi) = m(2^\ell \xi)\phi(\xi) = m_{\ell,1}(\xi) e^{-\|\xi\|^2} = m_{\ell,2}(\xi) e^{-\|\xi\|^2} e^{-\|\xi\|^2}.
\]

By assumption (1.3) there is \( C > 0 \) such that

\[
\sup_{\ell \in \mathbb{Z}} \|m_\ell\|_{W^2_2} \leq CM.
\]

Proposition 5.3 of [4] asserts that for any real numbers \( \alpha > \beta > 0 \) there is a constant \( C = C_{\alpha,\beta} \) such that

\[
(5.2) \quad \|Fm_\ell(x)(1 + \|x\|)^\beta\|_{L^2(dw(x))} \leq C\|m_\ell\|_{W^2_2} = C\|\hat{m}_\ell(x)(1 + \|x\|)^\alpha\|_{L^2(dx)},
\]

where \( \hat{m}_\ell \) denotes the classical Fourier transform of \( m_\ell \).

Set

\[
(5.3) \quad \tilde{K}_\ell(x,y) = \tau_{-y}(F^{-1}m_\ell)(x),
\]

\[
(5.4) \quad K_\ell(x,y) = \tau_{-y}F^{-1}(m(\cdot)\phi(2^{-\ell} \cdot))(x).
\]

By homogeneity,

\[
(5.5) \quad K_\ell(x,y) = 2^{N\ell} \tilde{K}_\ell(2^\ell x,2^\ell y).
\]

Obviously, \( \tilde{K}_\ell(x,y) \) and \( K_\ell(x,y) \) are \( C^\infty \) functions of \( x, y \), since \( m_\ell \) is, by assumption (1.3), a bounded compactly supported function.

Let \( T_m \) and \( T_\ell \) denote the Dunkl multiplier operators associated with \( m \) and \( m_\ell(2^{-\ell} \cdot) = \phi(2^{-\ell} \cdot)m(\cdot) \) respectively. Obviously, for \( f \in L^2(dw) \) one has

\[
(5.6) \quad T_\ell f(x) = \int K_\ell(x,y)f(y)\,dw(y).
\]

Clearly, \( \|m_{\ell,1}\|_{W^2_2} \leq CM \). Using (5.2) and the Cauchy–Schwarz inequality together with (2.2), we deduce that for every \( \delta' \geq 0 \) such that \( N + \delta' < s \) we have

\[
\|F^{-1}m_{\ell,1}(x)(1 + \|x\|)^{N/2+\delta'}\|_{L^1(dw(x))} \leq C_{\delta',\alpha} \|m_{\ell,1}\|_{W^2_2} \leq CM.
\]

Recall that

\[
(5.7) \quad F^{-1}m_\ell(x) = (F^{-1}m_{\ell,1}) \ast h_1(x),
\]

where \( h_1(x) = 2^{-N/2}c_\ell^{-1} \exp(-\|x\|^2/4) \) (see (2.13)). Therefore, by Corollary 4.14 together with (5.2) we have

\[
(5.8) \quad \int |\tilde{K}_\ell(x,y)|(1 + d(x,y))^{\delta'}\,dw(x) = \int |\tau_{-y}(F^{-1}m_\ell)(x)|(1 + d(x,y))^{\delta'}\,dw(x) \leq C_{\delta',s} \|m_{\ell,1}\|_{W^2_2} \leq CM.
\]
By the same arguments with $\delta' = 0$ we obtain

\begin{equation}
\int |\tau_{-y}(F^{-1}m_{\ell,1})(x)| \, dw(x) \leq C_{s,\delta'}\|m_{\ell,2}\|_{W^2_s} \leq CM. \tag{5.9}
\end{equation}

From (5.5) and (5.8) we conclude

\begin{equation}
\int |K_\ell(x, y)|(1 + d(x, y))^{\delta'} \, dw(x) \leq CM2^{-\delta\ell}. \tag{5.10}
\end{equation}

On the other hand, using (5.7) together with (5.9) we get

\begin{align}
\int |\tilde{K}_\ell(x, y) - \tilde{K}_\ell(x, y')| \, dw(x) \\
&= \int \left| \int \tau_{-z}(F^{-1}m_{\ell,1})(x) \left(h_1(z, y) - h_1(z, y')\right) \, dw(z) \right| \, dw(x) \\
&\leq C\|m_{\ell,2}\|_{W^2_s} \int \left| h_1(x, y) - h_1(z, y') \right| \, dw(z) \\
&\leq C'M\|y - y'\|,
\end{align}

where in the last inequality we have used (3.3). From (5.5) and (5.11) we easily deduce

\begin{equation}
\int |K_\ell(x, y) - K_\ell(x, y')| \, dw(x) \leq CM2^\ell\|y - y'\|. \tag{5.12}
\end{equation}

For a cube $Q \subset \mathbb{R}^N$ let $c_Q$ be its center and $\text{diam}(Q)$ be the length of its diameter. Let $Q^*$ denote the cube with center $c_Q$ such that $\text{diam}(Q^*) = 2\text{diam}(Q)$. The following proposition is a direct consequence of (5.10) and (5.12).

**Proposition 5.13.** There are constants $C, \delta' > 0$ such that for every cube $Q \subset \mathbb{R}^N$ and $y, y' \in Q$ we have

\begin{equation}
\int_{\mathbb{R}^N \setminus O(Q^*)} |K_\ell(x, y) - K_\ell(x, y')| \, dw(x) \leq CM \min \left((2^\ell\text{diam}(Q))^\delta', 2^\ell\text{diam}(Q)\right). \tag{6.1}
\end{equation}

6. Proof of Theorem 1.2 (A) and (B)

**Proof.** Having Proposition 5.13 already established, the proof of weak type $(1, 1)$ of the multiplier operator $T_m$ follows the standard pattern. Clearly, there is a constant $C_1 > 1$, which depends on the doubling constant and $N$, such that $w(Q) \leq C_1 w(Q')$, where $Q'$ is any sub-cube of $Q$, $\ell(Q') = \ell(Q)/2$, where $\ell(Q)$ denote the side length of $Q$.

Let $f \in L^1(dw) \cap L^2(dw)$. Fix $\lambda > 0$. Denote by $Q_\lambda$ the collection of all maximal (disjoint) dyadic cubes $Q_j$ in $\mathbb{R}^N$ satisfying

$$
\lambda < \frac{1}{w(Q_j)} \int_{Q_j} |f(x)| \, dw(x).
$$

Then

$$
\frac{1}{w(Q_j)} \int_{Q_j} |f(x)| \, dw(x) \leq C_1 \lambda.
$$
Set $\Omega = \bigcup_{Q_j \in \mathcal{Q}} Q_j$. Then $w(\Omega) \leq \lambda^{-1}\|f\|_{L^1(dw)}$. Form the corresponding Calderón–Zygmund decomposition of $f$, namely, $f = g + b$, where

$$
g(x) = f\chi_{\Omega^*}(x) + \sum_j w(Q_j)^{-1} \left( \int_{Q_j} f(y) \, dw(y) \right) \chi_{Q_j}(x),
$$

$$
b(x) = \sum_j b_j(x), \quad \text{where } b_j(x) = \left( f(x) - w(Q_j)^{-1} \int_{Q_j} f(y) \, dw(y) \right) \chi_{Q_j}(x).
$$

Clearly, $g, b \in L^1(dw) \cap L^2(dw)$, $|g(x)| \leq C_1\lambda$, $\|g\|_{L^2(dw)}^2 \leq C\lambda\|f\|_{L^1}$, $\sum_j \|b_j\|_{L^1(dw)} \leq C\|f\|_{L^1(dw)}$. Further,

$$
w(\{x \in \mathbb{R}^N : |T_m f(x)| > \lambda\}) \leq w(\{x \in \mathbb{R}^N : |T_m g(x)| > \lambda/2\}) + w(\{x \in \mathbb{R}^N : |T_m b(x)| > \lambda/2\}).
$$

Since $T_m$ is bounded on $L^2(dw)$, we obtain

$$
w(\{x \in \mathbb{R}^N : |T_m g(x)| > \lambda/2\}) \leq \frac{4}{\lambda^2} m L^\infty \|g\|_{L^2(dw)}^2 \leq \frac{4C_1}{\lambda^2} m L^\infty \|f\|_{L^1(dw)}.
$$

Let $Q_j^*$ be the cube with the same center $c_{Q_j}$ as $Q_j$ and two times larger side-length. Define $\Omega^* = \mathcal{O} \left( \bigcup_j Q_j^* \right)$. There is a constant $C_2 > 1$, which depends on the Weyl group, doubling constant, and $N$ such that

$$
w(\Omega^*) \leq C_2 w(\Omega) \leq C_2 \lambda^{-1}\|f\|_{L^1(dw)}.
$$

Thus it suffices to estimate $T_m b(x)$ on $\mathbb{R}^N \setminus \Omega^*$. Since $\sum_j b_j$ converges to $b$ in $L^2(dw)$ and $T_m b = \sum_{\ell \in \mathbb{Z}} T_\ell b$ in the $L^2(dw)$-norm, we have

$$
|T_m b(x)| \leq \sum_j \sum_{\ell \in \mathbb{Z}} |T_\ell b_j(x)|.
$$

By (5.6) and the fact that $\text{supp } b_j \subseteq Q_j$ and $\int b_j(y) \, dw(y) = 0$, we have

$$
\int_{\mathbb{R}^N \setminus \Omega^*} |T_\ell b_j(x)| \, dw(x) = \int_{\mathbb{R}^N \setminus \Omega^*} \left( \int_{Q_j} K_\ell(x,y) b_j(y) \, dw(y) \right) \, dw(x)
$$

$$
\leq \int_{\mathbb{R}^N \setminus \mathcal{O}(Q_j^*)} \left( \int_{Q_j} \left( K_\ell(x,y) - K_\ell(x,c_{Q_j}) \right) b_j(y) \, dw(y) \right) \, dw(x)
$$

$$
\leq CM \min \left( (2^\delta \text{diam}(Q_j))^{-\delta}, 2^\delta \text{diam}(Q_j) \right) \|b_j\|_{L^1(dw)},
$$

where in the last inequality we have used Proposition 5.13. Summing the inequalities (6.1) over $j$ and $\ell$ we end up with

$$
\int_{\mathbb{R}^N \setminus \Omega^*} |T_m b(x)| \, dw(x) \leq CM \sum_j \|b_j\|_{L^1} \leq CM \|f\|_{L^1(dw)},
$$

which, by the Chebyshev inequality, completes the proof of weak type $(1,1)$ of the operator $T_m$.  


The strong type \((p,p)\) of \(T_m\) follows from the Marcinkiewicz interpolation theorem and a duality argument.

7. Proof of Theorem 1.2 (C)

Hardy spaces \(H^1_\Delta\) in the Dunkl setting were studied in [5], [14], and for product systems of roots in [4]. They are extensions of the classical Hardy spaces on \(\mathbb{R}^N\) introduced and developed in [27], [15] (see also [26]).

We start this section by presenting three equivalent characterizations of the Hardy space \(H^1_\Delta\) associated with the Dunkl theory. Then we shall prove Theorem 1.2 (C).

**Definition 7.1.** A function \(a(x)\) is an atom ((1, \(\infty\))-atom) if there is a Euclidean ball \(B\) such that
(A) \(\text{supp } a \subseteq B\);
(B) \(\|a\|_{L^\infty} \leq w(B)^{-1}\);
(C) \(\int a(x) \, dw(x) = 0\).

**Definition 7.2.** A function \(f\) belongs to \(H^1_{\text{atom}}\) if there are \(\lambda_j \in \mathbb{C}\) and \((1, \infty)\)-atoms \(a_j\) such that \(f = \sum_{j=1}^{\infty} \lambda_j a_j\) and \(\sum_{j=1}^{\infty} |\lambda_j| < \infty\). Then
\[
\|f\|_{H^1_{\text{atom}}} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| \right\},
\]
where the infimum is taken over all representations of \(f\) as above.

**Definition 7.3.** We say that a function \(f\) belongs to the real Hardy space \(H^1_\Delta\) if the nontangential maximal function
\[
\mathcal{M}f(x) = \sup_{\|x-y\|<t} |\exp(t^2\Delta)f(x)|
\]
belongs to \(L^1(dw)\). The space \(H^1_\Delta\) is a Banach space with the norm
\[
\|f\|_{H^1_{\max,H}} = \|\mathcal{M}f\|_{L^1(dw)}.
\]

The following theorem was proved in [14, Theorem 1.6].

**Theorem 7.4.** The spaces \(H^1_\Delta\) and \(H^1_{\text{atom}}\) coincide and the corresponding norms are equivalent, that is, there is a constant \(C > 0\) such that
\[
C^{-1}\|f\|_{H^1_{\text{atom}}} \leq \|f\|_{H^1_{\max,H}} \leq C\|f\|_{H^1_{\text{atom}}}.
\]

**Definition 7.6.** The Riesz transforms are defined in the Dunkl setting by
\[
R_jf = T_j(-\Delta)^{-1/2}f \text{ for } 1 \leq j \leq N.
\]

The Riesz transforms are bounded operators on \(L^p(dw)\), for every \(1 < p < \infty\) (see [3], [25]). In the limit case \(p = 1\), they turn out to be bounded operators from \(H^1_\Delta\) into \(H^1_\Delta \subset L^1(dw)\). This leads to consider the space
\[
H^1_{\text{Riesz}} = \{f \in L^1(dw) \mid \|R_jf\|_{L^1(w)} < \infty \forall 1 \leq j \leq N\}.
\]

The following theorem was proved in [5, Theorem 2.11].
Theorem 7.7. The spaces $H^1_\Delta$ and $H^1_{\text{Riesz}}$ coincide and the corresponding norms $\|f\|_{H^1_{\text{max,H}}}$ and
$$\|f\|_{H^1_{\text{Riesz}}} := \|f\|_{L^1(dw)} + \sum_{j=1}^N \|R_j f\|_{L^1(dw)}.$$ are equivalent.

Proof of Theorem 1.2 (C). Let us check first that there is $C > 0$ such that $\|T_m a\|_{L^1(dw)} \leq C$ for any atom $a(\cdot)$. Without losing of generality we can assume that $a(\cdot)$ is associated with a cube $Q$. We have
$$\|T_m a\|_{L^1(dw)} \leq \|T_m a\|_{L^1(O(Q^*),dw)} + \|T_m a\|_{L^1((O(Q^*))^c,dw)},$$ where $Q^*$ is the cube with the same center as $Q$ and two times larger side-length. By the Cauchy–Schwarz inequality and property (B) of atom $a(\cdot)$ we have
$$\|T_m a\|_{L^1(O(Q^*),dw)} \leq w(O(Q^*))^{1/2} \|T_m a\|_{L^2((O(Q^*))^c,dw)} \leq C_1.$$ Thanks to properties (A) and (C) of $a(\cdot)$ we can use Proposition 5.13 and repeat argument presented in (6.1) with $a(\cdot)$ instead of $b_j$. This leads us to $\|T_m a\|_{L^1((O(Q^*))^c,dw)} \leq C_2$.

We turn now to complete the proof of part (C) of Theorem 1.2. Since $T_m$ maps continuously $L^1(dw)$ into $\mathcal{S}'(\mathbb{R}^N)$, it suffices (by Theorems 7.4 and 7.7) to check that there is a constant $C > 0$ such that $\|R_j T_m a\|_{L^1(dw)} \leq C M$ for any atom $a(\cdot)$ of $H^1_{\text{atom}}$ and $j = 1, 2, \ldots, N$. For this purpose note that the operator $R_j T_m$ is associated with the multiplier $n(\xi) = -\frac{i}{|\xi|} m(\xi)$ and there is a constant $C_3 > 0$ such that
$$\sup_{t>0} \|\psi(\cdot)n(t\cdot)\|_{W^2_2} \leq C_3 M.$$ Therefore, we can repeat the argument presented above to the operator associated with $n$ in place of $T_m$. \qed

8. Case of $L^1(dw)$ bounded translations

Theorem 8.1. Assume that for a root system $R$ and a multiplicity function $k \geq 0$ the translations $\tau_y$ are uniformly bounded operators on $L^1(dw)$, that is, there is a constant $C > 0$ such that for any $f \in L^1(dw)$ we have
$$\sup_{y \in \mathbb{R}^N} \|\tau_y f\|_{L^1(dw)} \leq C \|f\|_{L^1(dw)}.$$ If $m$ is a bounded function on $\mathbb{R}^N$ such that (1.3) is satisfied for some $s > N/2$, then the multiplier operator $T_m$ is of weak-type $(1,1)$, bounded on $L^p(dw)$ for $1 < p < \infty$, and bounded on the Hardy space $H^1_\Delta$.

Remark 8.3. The same analysis applies to any normalized system of roots $R$, $k \geq 0$, and radial multipliers, because the Dunkl transform of any radial function is radial and $\|\tau_y f\|_{L^1(dw)} \leq \|f\|_{L^1(dw)}$ for $f$ being radial.

Remark 8.4. The inequality (8.2) holds in the rank-one case (see, e.g. [22, Section 2.8] and hence in the product case.
Proof of Theorem 8.1. Since $L^2(B(0, r), dw)$ is dense in $L^1(B(0, r), dw)$, by Theorem 1.5 we have

$$\text{supp } \tau_x f(-\cdot) \subset \mathcal{O}(B(x, r)) \text{ for } f \in L^1(dw), \text{ supp } f \subseteq B(0, r).$$

From (8.5) we easily deduce that for any $\delta \geq 0$ there is $C > 0$ such that

$$\|f(x, y)(1 + d(x, y))^{\delta}\|_{L^1(dw(x))} \leq C\|f(x)(1 + \|x\|)^{\delta}\|_{L^1(dw(x))}.$$ 

Indeed, let $f_j = \Psi_j f$, where $\Psi_j$ are defined in (4.8). Then using (2.10), (8.2), and (8.5) we have

$$\|f(x, y)(1 + d(x, y))^{\delta}\|_{L^1(dw(x))} \leq \sum_{j=0}^{\infty} \|f_j(x, y)(1 + d(x, y))^{\delta}\|_{L^1(dw(x))}$$

$$\leq C\sum_{j=0}^{\infty} 2^{j\delta}\|f_j(x, y)\|_{L^1(dw(x))}$$

$$\leq C\sum_{j=0}^{\infty} 2^{j\delta}\|f_j(x)\|_{L^1(dw(x))}$$

$$\leq C\|f(x)(1 + \|x\|)^{\delta}\|_{L^1(dw(x))}.$$

Let $m_\ell, m_{\ell,1}, \tilde{K}_\ell(x, y), K_\ell(x, y)$ be defined by (5.1), (5.4), and (5.3) respectively. Take any $0 \leq \delta < s - \mathbb{N}/2$. Then

$$\|\mathcal{F}m_\ell(x)(1 + \|x\|^2)^{\mathbb{N}/2 + \delta}\|_{L^2(dw(x))} + \|\mathcal{F}m_{\ell,1}(x)(1 + \|x\|^2)^{\mathbb{N}/2 + \delta}\|_{L^2(dw(x))} \leq C_\delta M,$$

which, by the Cauchy-Schwartz inequality and (2.2), imply

$$\|\mathcal{F}m_\ell(x)(1 + \|x\|)^{\delta}\|_{L^1(dw(x))} + \|\mathcal{F}m_{\ell,1}(x)(1 + \|x\|)^{\delta}\|_{L^1(dw(x))} \leq C_\delta M.$$ 

By (8.7) and (5.5), the kernels $K_\ell(x, y)$ satisfy (5.10). Further, the Hölder regularity (5.12) hold for $K_\ell(x, y)$ as well (see (5.11) for the proof). Hence we easily deduce that conclusion of Proposition 5.13 is valid. Finally the weak-type (1, 1) estimate and the boundedness on $L^p(dw)$ of the multiplier operator $\mathcal{T}_m$ are obtained by the standard Calderón-Zygmund analysis presented in Section 6. The proof of boundedness of $\mathcal{T}_m$ on the Hardy space $H^1_{\text{atom}}$ is the same as in Section 7. \hfill $\Box$

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