Quantum Circuits Computing Unitary Transformations

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Abstract. Let \( \mathcal{H}_1, \mathcal{H}_2 \) be Hilbert spaces of the same finite dimension \( \geq 2 \), and \( C \) an arbitrary quantum circuit with (principal) input state in \( \mathcal{H}_1 \) and (principal) output state in \( \mathcal{H}_2 \). \( C \) may use ancillas and produce garbage which is traced out. \( C \) may employ classical channels and measurement gates. If \( C \) computes, for each computation path \( \mu \) through the circuit, a unitary transformation \( U_\mu : \mathcal{H}_1 \to \mathcal{H}_2 \) then, for each \( \mu \), the probability that a computation takes path \( \mu \) is independent of the input.

1. Introduction

A realistic quantum computer architecture only supports gates that are in its gate library. To implement a quantum algorithm, various quantum gates must be translated into library gates. Measurement gates and classical channels can be quite helpful in this process. Figure 1 gives an instructive example of a circuit with measurement gates and classical channels computing Controlled-NOT.

\[
|c\rangle \quad \underbrace{p := \text{PM}}_{H} \quad |0\rangle \quad \text{H} \quad \underbrace{q := \text{PM}}_{H} \quad |t\rangle \quad \text{H} \quad \underbrace{r := \text{SM}}_{H} \quad X \quad \text{Z} \quad q = p \oplus r = 1 \quad p \oplus r = 1
\]

Figure 1: A circuit for computing Controlled-NOT (a slight modification of a figure from [3]). PM is the qubit-parity measurement, and SM is the measurement in the standard basis. Implicit classical channels connect each measurement with the equation where it is used. The ancilla and the garbage to be discarded are shown on the middle line.

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Here we prove that, if a circuit computes a unitary transformation $U$ (or even a unitary transformation $U_\mu$ for each computation path $\mu$), then the probability that a computation takes a given path $\mu$ through the circuit is independent of the input.

Following Nielsen and Chuang [4, §2.2.3], we define a general measurement $M$ over a Hilbert space $\mathcal{H}$ as an indexed family $\langle L_j : j \in J \rangle$ of (linear) operators with $\sum_{j \in J} L_j^\dagger L_j$ being the identity operator $\text{Id}$. It is known [3] and easy to check that if $M$ does not disturb the state $\rho$ where it is performed, then no information about $\rho$ can be learned. In other words, if every $L_j$ is proportional to $\text{Id}$, so that if $M$ computes $\text{Id}$, then for each $j$ the probability of output $j$ is independent of $\rho$.

But, in quantum computing, the situation is more complex. Let’s consider a quantum circuit, possibly with classical channels. An input state of interest $\rho$, the *principal input*, is augmented with an ancilla $\alpha$. You run your circuit on the augmented state $\rho \otimes \alpha$ and then, crucially, you trace out garbage to obtain the principal output state. If the circuit computes a unitary transformation, is it still the case that the path probabilities do not depend on the principal input state? Vadym Kliuchnikov of Microsoft’s Quantum Architecture and Computing group surmised the positive answer. We prove that the answer is indeed positive.

Instead of working with quantum circuits directly, we introduce, in §2, a more powerful but simpler computation model. A *measurement tree* — in short, mtree — over a Hilbert space $\mathcal{H}$ is a finite tree where every non-leaf node has a measurement over $\mathcal{H}$ associated with it. Semantically, every quantum circuit, possibly with classical channels and measurement gates, is an mtree. That mtree may be exponentially large relative to the circuit size, but it is useful for analysis and verification of the circuit.

In §3 we formalize the notion that an mtree $T$ computes a unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$ where Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are of finite dimension $\geq 2$. In addition to the principal input in $\mathcal{H}_1$, $T$ may use an ancilla. In addition to the principal output in $\mathcal{H}_2$, $T$ may produce some garbage which is then traced out. There are different formalizations but, fortunately, we were able to prove them equivalent.

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1 We are preparing a pedagogical paper [2] where we use mtrees to explain the semantics of quantum circuits.
In §4, we prove our main result: If an mtree computes unitary operators along all possible computation paths, then the probability that the computation takes a given path depends only on the path but not on the input. Our proof also shows that, in this situation, the state of the garbage before tracing out is unentangled with the principal output, is independent of the input, and is pure. An important auxiliary theorem is relegated to the appendix.

Finally, in §5, we investigate to what extent our results generalize to computing (possibly nonunitary) linear transformations.

2. Measurement trees

Let $\mathcal{H}$ be a Hilbert space of finite dimension $\geq 2$. Following [4, §2.2.3], we define a measurement over $\mathcal{H}$ as an indexed family $M = \{L_j : j \in J\}$ of (linear) operators where $\sum_{j \in J} L_j^\dagger L_j$ is the identity operator $\text{Id}_\mathcal{H}$ on $\mathcal{H}$; elements of $J$ are outcomes of $M$. Notice that a measurement with only a single outcome is (a family consisting of) a unitary operator.

We take density operators over $\mathcal{H}$ to be nonzero, positive semidefinite, Hermitian operators on $\mathcal{H}$. The density operators form a convex set, denoted $\text{DO}(\mathcal{H})$. Its interior in the space of Hermitian operators is nonempty.

A density operator $\rho$ is normalized if its trace is 1.

Convention 2.1. We use possibly non-normalized density operators to represent states. A density operator $\rho$ represents the same state as its normalized version $\rho/\text{Tr}(\rho)$. If a measurement $M$ on the state represented by $\rho$ produces outcome $j$, then we will usually use density operator $L_j \rho L_j^\dagger$ (rather than any scalar multiple of it) to represent the post-measurement state. ◁

Accordingly, the transformation $\rho \mapsto L_j \rho L_j^\dagger$, resulting from $M$, is linear. Notice also that $\text{Tr}(L_j \rho L_j^\dagger)/\text{Tr}(\rho)$ is the probability of outcome $j$.

Definition 2.2 (Measurement trees). An mtree over $\mathcal{H}$ is a finite directed rooted tree where every non-leaf node $x$ has a measurement $M_x$ associated to it, and the edges emanating from $x$ are labeled in one-to-one fashion with the outcomes of $M_x$. ◁

Represent each node $x$ of an mtree $\mathcal{T}$ by the route $(j_1, \ldots, j_k)$ of (labels of) edges from the root to $x$, and view $\mathcal{T}$ as the following algorithm whose inputs are states of $\mathcal{H}$ given by density operators.

If the root is the only node, do nothing. Otherwise, given an input state (represented by) $\rho$, perform the root’s measurement on state $\rho_0 = \rho$ producing an outcome $j_1$ and post-measurement state $\rho_1$. If node $(j_1)$ is not
a leaf, perform its measurement on state $\rho_1$ producing an outcome $j_2$ and post-measurement state $\rho_2$. If node $(j_1, j_2)$ is not a leaf, perform its measurement on state $\rho_2$ producing an outcome $j_3$ and post-measurement state $\rho_3$, and so on until a leaf $(j_1, \ldots, j_n)$ is reached. Thus, the measurement performed at any step of the algorithm depends on the outcomes of the earlier measurements but not on the quantum states that they produced. This aspect of the definition mirrors the fact that, unlike classical states, quantum states cannot be inspected without measuring them and thereby possibly altering them.

The sequence $\rho_0, (j_1, \rho_1), (j_2, \rho_2), \ldots, (j_n, \rho_n)$ represents the resulting computation of $T$ on $\rho$, with $\rho_n$ representing the final state. We say that the computation follows the path $\mu = (j_1, \ldots, j_n)$ and that $\mu$ is attainable on the input $\rho$. A path is attainable if it is attainable on some input.

**Lemma 2.3.** Every attainable path $\mu$ is attainable on every input $\rho$ in the interior of $\text{DO}(\mathcal{H})$. And if functions $F_1, F_2$ from $\text{DO}(\mathcal{H})$ to a finite-dimensional vector space with standard topology are continuous and coincide in the interior of $\text{DO}(\mathcal{H})$, then they coincide on the whole $\text{DO}(\mathcal{H})$.

**Proof.** If $\mu$ is attainable on $\sigma$ with probability $p$, and if density operator $\rho \neq \sigma$ is in the interior of $\text{DO}(\mathcal{H})$, draw a straight line from $\sigma$ through $\rho$ to some $\tau$ in $\text{DO}(\mathcal{H})$. Then $\rho$ is a convex combination $a\sigma + b\tau$. Therefore $\mu$ is attainable on $\rho$ with probability $\geq ap$. And $F_1, F_2$ coincide on $\text{DO}(\mathcal{H})$ by continuity, because the interior is dense in $\text{DO}(\mathcal{H})$. □

For every $\mu \in \text{Paths}(T)$, the cumulative operator $C_\mu$ is the composition

$$C_\mu = A_n \circ A_{n-1} \circ \cdots \circ A_2 \circ A_1$$

where each $A_k$ is the operator that the measurement at node $(j_1, \ldots, j_{k-1})$ associates with outcome $j_k$. In the special case of the one-node tree, the length-zero path has $C_\mu = \text{Id}$.

Let $p(\mu|\rho)$ be the probability, according to quantum mechanics, that a computation of $T$ on input $\rho$ follows path $\mu$. The following claims can be verified by induction on the number of non-leaf nodes in $T$.

$$\sum_\mu C_\mu^\dagger C_\mu = \text{Id}, \quad (1)$$

$$p(\mu|\rho) = \frac{\text{Tr}(C_\mu \rho C_\mu^\dagger)}{\text{Tr}(\rho)} \quad \text{and} \quad (p(\mu|\rho) = 0) \iff (C_\mu \rho C_\mu^\dagger = 0). \quad (2)$$

**Definition 2.4.** The aggregate measurement of $T$ is the measurement $\langle C_\mu : \mu \in \text{Paths}(T) \rangle$. ◼

**Corollary 2.5** (Semantics). The aggregate measurement $M$ of $T$ is equivalent to $T$ in the following sense. For all $\rho \in \text{DO}(\mathcal{H})$ and $\mu \in \text{Paths}(T)$,
the probability of outcome $\mu$ of $M$ on state $\rho$ is the probability $p(\mu|\rho)$ that $T$ follows $\mu$ on $\rho$, and the post-measurement state of $M$ is the final state $C_\mu \rho C_\mu^\dagger$ of $T$.

3. Computing unitary operators

We shall define what it means for an mtree $T$ to compute a unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$, where Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ are of finite dimension $\geq 2$. $T$ may use an ancilla in addition to the principal input in $\mathcal{H}_1$ and may produce some garbage in addition to the principal output in $\mathcal{H}_2$. Accordingly, the initial state of $T$ is in a Hilbert space $\mathcal{H}_1 \otimes \mathcal{A}$, where $\mathcal{A}$ hosts the ancilla. While principal input $\rho$ varies, the initial state of the ancilla is a fixed normalized pure state $\alpha = |a\rangle\langle a|$ where $|a\rangle$ is a unit vector in $\mathcal{A}$. Accordingly, the initial state of the algorithm is $\rho \otimes \alpha$ which we abbreviate to $\hat{\rho}$. The final state of $T$ will similarly be in $\mathcal{H}_2 \otimes \mathcal{B}$, where $\mathcal{B}$ contains the garbage. The intention is to “forget” the garbage by tracing out $\mathcal{B}$.

Definition 3.1. $T$ statistically computes $U$ on principal input $\rho$ if the states $\text{Tr}_B \sum_{\mu \in \text{Paths}(T)} C_\mu \hat{\rho} C_\mu^\dagger$ and $U \rho U^\dagger$ coincide. $T$ statistically computes $U$ if it statistically computes $U$ on every $\rho \in \text{DO}(\mathcal{H}_1)$.

The reader may expect the weighted average, rather than the sum, in the preceding definition. Fortunately Convention 2.1 (with the subsequent paragraph) incorporates the weight factors (probabilities) into the density operators after the measurements.

Corollary 3.2. If $T$ statistically computes $U$ on $\rho$, then $\text{Tr}_B \sum_{\mu} C_\mu \hat{\rho} C_\mu^\dagger = U \rho U^\dagger$. And if $T$ statistically computes $U$ on all pure inputs then it statistically computes $U$ on all inputs.

Proof. The second claim follows from the first. To prove the first claim, it suffices to check that the two sides have the same trace. Using equations 1 and 2 in §2 we have

$$\text{Tr} \left( \text{Tr}_B \sum_{\mu} C_\mu \hat{\rho} C_\mu^\dagger \right) = \text{Tr} \sum_{\mu} C_\mu \hat{\rho} C_\mu^\dagger = \sum_{\mu} \text{Tr}(C_\mu \hat{\rho} C_\mu^\dagger)$$

$$= \sum_{\mu} (p(\mu|\hat{\rho}) \text{Tr}(\hat{\rho})) = \left( \sum_{\mu} p(\mu|\hat{\rho}) \right) \text{Tr}(\hat{\rho}) = \text{Tr}(U \rho U^\dagger). \quad \Box$$

Definition 3.3. $T$ deterministically computes $U$ principal on input $\rho$ if the states $\text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger)$ and $U \rho U^\dagger$ coincide for all $\mu \in \text{Paths}(T)$ attainable on $\hat{\rho}$. $T$ deterministically computes $U$ if it deterministically computes $U$ on all $\rho \in \text{DO}(\mathcal{H}_1)$.

$\Box$
Proposition 3.4. The following claims are equivalent for any unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$, any mtree $T$, and any pure state $\rho$ in $\mathcal{H}_1$.

1. $T$ deterministically computes $U$ on $\rho$.
2. $\text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger) = p(\mu|\hat{\rho}) \cdot U\rho U^\dagger$ for every $\mu \in \text{Paths}(T)$.
3. $T$ statistically computes $U$ on $\rho$.

Proof. (1) $\implies$ (2). It suffices to prove that the two sides of the equation in question have the same trace. By equation (2) in §2,

$$\text{Tr} \left( \text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger) \right) = \text{Tr} \left( C_\mu \hat{\rho} C_\mu^\dagger \right) = p(\mu|\hat{\rho}) \text{Tr}(\hat{\rho}) = \text{Tr} \left( p(\mu|\hat{\rho}) U\rho U^\dagger \right).$$

(2) $\implies$ (3). Suppose (2). Then

$$\text{Tr}_B \sum_\mu C_\mu \hat{\rho} C_\mu^\dagger = \sum_\mu \text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger) = \sum_\mu p(\mu|\hat{\rho}) \cdot U\rho U^\dagger = U\rho U^\dagger.$$ 

(3) $\implies$ (1). Assume (3). Without loss of generality $\text{Tr}(\rho) = 1$. Then the pure state $U\rho U^\dagger$ is the state

$$\text{Tr}_B \sum_\mu C_\mu \hat{\rho} C_\mu^\dagger = \sum_{p(\mu|\hat{\rho}) > 0} \text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger)$$

$$= \sum_{p(\mu|\hat{\rho}) > 0} p(\mu|\hat{\rho}) \frac{\text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger)}{p(\mu|\hat{\rho})}$$

On the right end, the fraction has trace 1, and so does the sum. But a pure state $U\rho U^\dagger$ cannot be expressed as a nontrivial convex combination of other states [1, §2.3]. Hence each state $\text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger)/p(\mu|\hat{\rho})$ is the state $U\rho U^\dagger$. It follows that $T$ deterministically computes $U$ on $\rho$.  

4. Independence

Theorem 4.1 (Independence). Suppose that an mtree $T$ deterministically (or statistically) computes a unitary transformation $U : \mathcal{H}_1 \to \mathcal{H}_2$ on the pure inputs. Then for every $\mu \in \text{Paths}(T)$, there is a pure density operator $\beta_\mu$ in $\mathcal{B}$ such that for all $\rho \in \text{DO}(\mathcal{H}_1)$,

1. $C_\mu \hat{\rho} C_\mu^\dagger = U\rho U^\dagger \otimes \beta_\mu$,
2. the probability $p(\mu|\hat{\rho})$ that a computation of $T$ follows path $\mu$ is $\text{Tr}(\beta_\mu)$ and thus is independent of $\rho$, and
3. if $\mu$ is attainable on some input, then it is attainable on all inputs.
Proof. Claim (3) obviously follows from claim (2). Claim (2) follows from claim (1):
\[ p(\mu|\hat{\rho}) = \frac{\text{Tr}(C_\mu \hat{\rho} C_\mu^\dagger)}{\text{Tr}(\hat{\rho})} = \frac{\text{Tr}(U\rho U^\dagger \otimes \beta_\mu)}{\text{Tr}(\hat{\rho})} = \frac{\text{Tr}(\rho) \times \text{Tr}(\beta_\mu)}{\text{Tr}(\hat{\rho})} = \text{Tr}(\beta_\mu). \]
It remains to prove claim (1).

By the linearity of the equation in (1), we may assume without loss of generality that \( \rho \) is pure. Then \( \text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger) = U\rho U^\dagger \) is pure. Since \( T \) deterministically computes \( U \) on \( \rho \), \( \text{Tr}(C_\mu \hat{\rho} C_\mu^\dagger) = \text{Tr}(\rho) \times \text{Tr}(\xi_\mu(\rho)) \). Notice that a mixed state \( \sigma \) is pure if and only if \( \text{Tr}(\sigma^2) = \text{Tr}(\sigma)^2 \). Since \( \rho \) is pure, the states \( \hat{\rho} \) and \( C_\mu \hat{\rho} C_\mu^\dagger \) are pure. We have
\[ (\text{Tr}(\rho))^2 \times (\text{Tr}(\xi_\mu(\rho)))^2 = \left(\text{Tr}(C_\mu \hat{\rho} C_\mu^\dagger)\right)^2 = \text{Tr} \left( (C_\mu \hat{\rho} C_\mu^\dagger)^2 \right) = \text{Tr} (\rho^2) \times \text{Tr} ((\xi_\mu(\rho))^2). \]
Cancelling \( \text{Tr}(\rho)^2 = \text{Tr}(\rho^2) \), we get \( \text{Tr} (\xi_\mu(\rho))^2 = (\text{Tr}(\xi_\mu(\rho)))^2 \), and so \( \xi_\mu(\rho) \) is pure as well.

Let \( \rho = |\psi\rangle\langle\psi| \). Then there is a \( B \) vector \( |f_\mu \psi\rangle \) such that \( \xi_\mu(\rho) = |f_\mu \psi\rangle\langle f_\mu \psi| \). Thus, \( C_\mu \hat{\rho} C_\mu^\dagger = \rho \otimes |f_\mu \psi\rangle\langle f_\mu \psi| \). Hence \( C_\mu(|\psi\rangle \otimes |a\rangle) = |\psi\rangle \otimes |f_\mu \psi\rangle \). Since \( |\psi\rangle \otimes |f_\mu \psi\rangle = C_\mu(|\psi\rangle \otimes |a\rangle) \) is linear in \( |\psi\rangle \), we can apply the constancy criterion in the Appendix with \( V_1 = \mathcal{H}_1, V_2 = \mathcal{H}_2, V_3 = \mathcal{B}, L = U \), and \( f = f_\mu \). We learn that \( f_\mu \) is constant, with value \( |b_\mu\rangle \) on the nonzero vectors of \( \mathcal{H}_1 \). So we have \( C_\mu(|\psi\rangle \otimes |a\rangle) = |\psi\rangle \otimes |b_\mu\rangle \) for every vector \( |\psi\rangle \) in \( \mathcal{H}_1 \). In terms of density operators, \( C_\mu \hat{\rho} C_\mu^\dagger = \rho \otimes |b_\mu\rangle\langle b_\mu| \) for every pure \( \rho \in \text{DO}(\mathcal{H}_1) \) and, by linearity, for all \( \rho \in \text{DO}(\mathcal{H}_1) \). The desired \( \beta_\mu = |b_\mu\rangle\langle b_\mu| \). \)

**Corollary 4.2.** If \( T \) computes \( U \) on pure inputs, either deterministically or statistically, then it computes \( U \) on all inputs, both deterministically and statistically.

**Corollary 4.3** (No information without state alteration). If an mtree computes an identity transformation, then the probability that a computation takes any given path does not depend on the input.

2Ballentine works with normalized density operators, but the theorem remains true, because any positive scalar factor can be shifted to \( \xi_\mu(\rho) \).
Definition 4.4. Let $\mathcal{U}$ be a family $\langle U_\mu : \mu \in \text{Paths}(\mathcal{T}) \rangle$ of unitary transformations $U_\mu : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ indexed by the paths of an mtree $\mathcal{T}$. Then $\mathcal{T}$ deterministically computes $\mathcal{U}$ on input $\rho$ if the states $\text{Tr}_B(C_\mu \hat{\rho} C_\mu^\dagger)$ and $U_\mu \rho U_\mu^\dagger$ coincide for all $\mu \in \text{Paths}(\mathcal{T})$ attainable on $\hat{\rho}$.

Theorem 4.5. Suppose that an mtree $\mathcal{T}$ deterministically computes a family $\langle U_\mu : \mu \in \text{Paths}(\mathcal{T}) \rangle$ of unitary transformations $U_\mu : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ on all pure inputs. Then for every $\mu \in \text{Paths}(\mathcal{T})$, there is a pure density operator $\beta_\mu$ in $\mathcal{B}$ such that for all $\rho \in \text{DO}(\mathcal{H}_1)$,

1. $C_\mu \hat{\rho} C_\mu^\dagger = U_\mu \rho U_\mu^\dagger \otimes \beta_\mu$,
2. the probability $p(\mu | \hat{\rho})$ that a computation of $\mathcal{T}$ follows path $\mu$ is $\text{Tr}(\beta_\mu)$ and thus is independent of $\rho$, and
3. if $\mu$ is attainable on some input, then it is attainable on all inputs.

Proof. Extend the mtree $\mathcal{T}$ as follows. For every path $\mu \in \text{Paths}(\mathcal{T})$, associate the leaf node $\ell_\mu$ of $\mu$ with the unitary $U_\mu^{-1} \otimes \text{Id}_\mathcal{B}$, viewed as a single-outcome measurement, and attach to $\ell_\mu$ one outgoing edge labeled with that single outcome. Let $\mu'$ be the one-edge extension of path $\mu$.

The extended mtree $\mathcal{T}'$ computes the identity operator $\text{Id}$ on the pure inputs, and the probability that a computation of $\mathcal{T}'$ follows path $\mu'$ on pure input $\hat{\rho}$ equals the probability that a computation of $\mathcal{T}$ follows path $\mu$ on $\hat{\rho}$. As a result, claims (2) and (3) follow from the corresponding claims of Theorem 4.1. Notice that the cumulative operator for the path $\mu'$ of $\mathcal{T}'$ is $U_\mu^\dagger C_\mu$ where $C_\mu$ is the cumulative operator for the path $\mu$ of $\mathcal{T}$. So the claim (1) of Theorem 4.1 with $U = \text{Id}$ says that

$$U_\mu^\dagger C_\mu \hat{\rho} C_\mu^\dagger U_\mu = \rho \otimes \beta_\mu$$

for all $\rho \in \text{DO}(\mathcal{H}_1)$. This implies claim (1). \qed

5. Computing linear operators

We shall now comment on potential generalizations of Theorem 4.1, first replacing unitary operators $U$ by arbitrary linear operators $L$, and second allowing linear operations $\mathcal{L}$ of density operators that are not of the form $\rho \mapsto X \rho X^\dagger$ for any linear operation $X$ on state vectors. In both situations, it is easy to define what it means for an mtree to compute — statistically or deterministically — such an operator: just replace $U_\mu \rho U_\mu^\dagger$ with $L_\mu \rho L_\mu^\dagger$ in the first case and with $\mathcal{L}(\rho)$ in the second. $\hat{\rho}$ and $p(\mu | \hat{\rho})$ have the same meaning as above.

Theorem 5.1. Let $L : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a linear transformation, and suppose that an mtree $\mathcal{T}$ computes $L$, deterministically or statistically, on pure inputs. Then $L$ is unitary up to a scalar factor.
Proof. Notice that the implication (3) \( \Rightarrow \) (1) of Proposition 3.4 does not utilize the unitarity of \( U \). Thus we may assume that \( T \) computes \( L \) deterministically on pure inputs.

Now walk through the proof of Theorem 4.1. Pure State Factor Theorem still can be applied because \( L\rho L^\dagger \) is pure when \( \rho \) is pure. In the subsequent computation, you just need to replace \( \operatorname{Tr}(\rho) \) and \( \operatorname{Tr}(\rho^2) \) with \( \operatorname{Tr}(L\rho L^\dagger) \) and \( \operatorname{Tr}((L\rho L^\dagger)^2) \). When you apply Theorem A.1, use the current \( L \) instead of \( L = U \). In the end, you get that
\[
C_\mu \hat{\rho} C_\mu^\dagger = L\rho L^\dagger \otimes |b_\mu\rangle \langle b_\mu| \quad \text{for all } \rho \in \DO(H_1).
\]

It follows that
\[
\operatorname{Tr}_B \sum_\mu (C_\mu \hat{\rho} C_\mu^\dagger) = L\rho L^\dagger \otimes \sum_\mu \operatorname{Tr}|b_\mu\rangle \langle b_\mu| = tL\rho L^\dagger,
\]
where \( t = \sum_{\mu} \langle b_\mu | b_\mu \rangle > 0 \). We show that \( U = cL \) is unitary for any \( c \) with \( |c|^2 = t \). For any state vector \( |\psi\rangle \) in \( H_1 \), consider \( \rho = |\psi\rangle \langle \psi| \). We have
\[
\langle U|U \rangle = \operatorname{Tr}(U|U\rangle \langle \psi|U^\dagger) = \operatorname{Tr}(tL\rho L^\dagger) = \operatorname{Tr}(\operatorname{Tr}_B \sum_\mu (C_\mu \hat{\rho} C_\mu^\dagger)) = \sum_\mu \operatorname{Tr}(C_\mu^\dagger C_\mu \hat{\rho}) = \operatorname{Tr}(\hat{\rho}) = \operatorname{Tr}(\rho) = \langle \psi | \psi \rangle.
\]
So \( U \) is an isometry. Since we work with finite dimensional Hilbert spaces, it follows that \( U \) is unitary. \( \square \)

We now turn to the second case, where a measurement tree computes (statistically or deterministically) a linear transformation \( L \) on density operators that is not necessarily of the form \( X\rho X^\dagger \) for any operator \( X \) on pure state vectors. The following observation and examples will show that this generalization is too broad; none of our previous work carries over to it.

Observe that every mtree statistically computes a linear operator
\[
L(\rho) = \operatorname{Tr}_B \left( \sum_\mu C_\mu \hat{\rho} C_\mu^\dagger \right).
\]

What about deterministic computations? First, the previous observation does not apply to them.

Example 5.2. Consider an mtree that is just measuring a given qubit in the computational basis \( \{|0\rangle, |1\rangle\} \). The mtree has two paths. Along one path, the input state \( \rho \) is projected to \( |0\rangle\langle 0| \), and, along the other path, \( \rho \) is projected to \( |1\rangle\langle 1| \). No single operator \( L \) is computed on both paths. (The mtree statistically computes the sum \( (|0\rangle\rho|0\rangle) |0\rangle\langle 0| + (|1\rangle\rho|1\rangle) |1\rangle\langle 1| \).) \( \triangleleft \)

Even when an mtree does deterministically compute \( L \), the independence property need not hold.
Example 5.3. Consider an mtree whose input consists of a single qubit of principal input $\rho$ and a single-qubit ancilla initially in state $|0\rangle$. The mtree measures $\rho$ in the computational basis and then traces out the (formerly) principal qubit and keeps only the (former) ancilla still in state $|0\rangle$.

Again, we have two paths, one for each of the measurement outcomes. On path 0, the post-measurement state is $((0|\rho|0)\rangle|00\rangle\langle00|$, where 00 means 0 in principal input and 0 in ancilla. Similarly, on path 1, the resulting state is $((1|\rho|1)\rangle|10\rangle\langle10|$. Then forgetting the principal input, we get $((0|\rho|0)\rangle|0\rangle\langle0|$, $Lx$ on path 0 and $((1|\rho|1)\rangle|0\rangle\langle0|$, $Ly$ on path 1. Thus, this mtree deterministically and statistically computes the linear operator $L(\rho) = \text{Tr}(\rho)|0\rangle\langle0|$. Yet it clearly does not have the independence property.

\textbf{Appendix A. A constancy criterion}

Let $V_1, V_2, V_3$ be vector spaces, $L : V_1 \to V_2$ a linear transformation of rank $\geq 2$, and $f$ a function $V_1 \to V_3$. Define $L = L \otimes f : V_1 \to V_2 \otimes V_3$ by $L(x) = L(x) \otimes f(x)$.

\textbf{Theorem A.1 (Constancy).} Suppose $L \otimes f$ is linear. Then $fx = fy$ for all $x, y$ with nonzero $Lx, Ly$.

\textbf{Proof.} It suffices to prove $fx = fy$ when $Lx, Ly$ are linearly independent. Indeed, there are vectors $v_1, v_2$ such that $Lv_1, Lv_2$ are independent because $\text{rank}(L) \geq 2$. If $Lx, Ly$ are linearly dependent but nonzero, then some $Lv_i$ is independent from both $Lx$ and $Ly$. Then $fx = fvi = fy$.

So suppose that $Lx, Ly$ are independent, and choose bases

- $v_1, v_2, \ldots$ for $V_2$, where $v_1 = Lx$ and $v_2 = Ly$, and
- $w_1, w_2, \ldots$ for $V_3$, where $w_1 = fx$ and $fy = aw_1 + bw_2$ for some scalars $a, b$.

Then $f(x + y) = \sum_j c_jw_j$ for some scalars $c_j$. We have

\[ (Lx + Ly) \otimes f(x + y) = L(x + y) \otimes f(x + y) = L(x + y) = Lx + Ly = (Lx \otimes fx) + (Ly \otimes fy), \]

and therefore

\[ (v_1 + v_2) \otimes \sum_j c_jw_j = v_1 \otimes w_1 + v_2 \otimes (aw_1 + bw_2). \]

Vectors $v_i \otimes w_j$ form a basis of $V \otimes W$. Comparing the coefficients of basis vectors $v_1 \otimes w_1, v_1 \otimes w_2, v_2 \otimes w_1, v_2 \otimes w_2$, we have $c_1 = 1, c_2 = 0, c_1 = a, c_2 = b$. Thus $a = c_1 = 1, b = c_2 = 0$, and $fy = 1w_1 + 0w_2 = w_1 = fx$. $\square$
References

[1] Ballentine, L. E. “Quantum mechanics: A modern development,” World Scientific 1998
[2] Blass A. and Gurevich Y., “Quantum circuits: Syntax and semantics” (tentative title), in preparation
[3] Busch P., “No information without disturbance: Quantum limitations of measurements,” In W.C. Myrvold and J. Christian (eds.), “Quantum reality, relativistic causality, and closing the epistemic circle,” Springer 2009
[4] Nielsen M. A. and Chuang I. L., “Quantum computation and quantum information,” Cambridge University Press, 10th Anniversary Edition, 2010
[5] Oded Zilberberg, Bernd Braunecker, and Daniel Loss, “Controlled-NOT gate for multiparticle qubits and topological quantum computation based on parity measurements,” Physical Review A 77 012327 2008

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