Solutions to the Wheeler DeWitt Constraint of Canonical Gravity Coupled to Scalar Matter Fields

Hans-Jürgen Matschull
II. Institut für theoretische Physik
Universität Hamburg
Luruper Chaussee 149
W-2000 Hamburg 50
Germany
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It is shown that the Wheeler DeWitt constraint of canonical gravity coupled to Klein Gordon scalar fields and expressed in terms of Ashtekar’s variables admits formal solutions which are parametrized by loops in the three dimensional hypersurface and which are extensions of the well known Wilson loop solutions found by Jacobson, Rovelli and Smolin.

Notation and Constraints

The action of four dimensional gravity coupled to $N$ real scalar matter fields is given by

$$S[E^A_{\mu}, \varphi^m] = \int d^4x \frac{1}{2} E \left( R - G^{\mu \nu} \partial_\mu \varphi^m \partial_\nu \varphi^m - m^2 \varphi^m \varphi^m \right),$$

where $\mu, \nu, \ldots$ denote four dimensional curved indices, $A, B, \ldots$ are flat indices raised or lowered by the Lorentzian metric with signature $(-, +, +, +)$. $E$ is the determinant and $G_{\mu \nu}$ the metric generated by the vierbein $E^A_{\mu}$. $R$ is the metric curvature scalar obtained from $G_{\mu \nu}$. Following the procedure in [1] and [2], but with the vierbein components as the basic variables instead.
of the metric, spacetime is split into space and time by decomposing the vierbein into a dreibein $e^\mu_a$, the lapse function $n$, and the shift vector $n^\alpha$:

$$E_\mu^A = \begin{pmatrix} n & 0 \\ n^\beta e_\beta a & e_{aa} \end{pmatrix}. \quad (2)$$

Thereby the four dimensional flat index $A$ splits into the three dimensional index $a$, labeling the three spatial unit vectors, and the index 0, indicating the timelike unit vector. As the three metric is positive definite, $a, b, \ldots$ will always appear as lower indices. Likewise, the curved index $\mu$ is split into the tangent space index $\alpha$ of the spatial three manifold and the index $t$, denoting the (parameter) time coordinate.

To treat this theory canonically ‘a la Dirac’[4], one has to express the action as a time integral of the Lagrangian and identify the canonical variables, which are most conveniently taken to be $\tilde{e}^\alpha_a := ee^\alpha_a$, $n$, $n^\alpha$, and $\phi^m$, where $n$ and $n^\alpha$ are Lagrange multipliers, whereas for the other fields we obtain the momenta

$$\pi^m = \frac{\delta L}{\delta \dot{\phi}^m} = en^{-1}(\phi^m - n^\alpha \partial_\alpha \phi^m),$$

$$p_{aa} = \frac{\delta L}{\delta \dot{\tilde{e}}_a^\alpha} = -\epsilon^\beta_a k_{\alpha\beta} \quad (3)$$

with the extrinsic curvature

$$k_{\alpha\beta} = \frac{1}{2} n^{-1}(\dot{g}_{\alpha\beta} - \nabla_\alpha n_\beta - \nabla_\beta n_\alpha). \quad (4)$$

The basic Poisson brackets are

$$\{\tilde{e}^\alpha_a, p_{\beta b}\} = \delta^\alpha_\beta \delta_{ab}, \quad \{\phi^m, \pi^n\} = \delta^{mn}, \quad (5)$$

which, of course, have to be supplemented by spatial delta functions. The derivatives of the Lagrangian with respect to $n$ and $n^\alpha$ as well as the symmetry of $k_{\alpha\beta}$ yield the Wheeler DeWitt[5, 6], diffeomorphism, and SO(3) constraints, respectively. Expressed in terms of Ashtekar’s variables[7], which are defined by

$$A_{aa} := ip_{aa} + \frac{1}{2} \epsilon_{abc} e_\beta^a \nabla_\alpha e_\beta b, \quad (6)$$

1For a detailed review of the space time split in terms of the vierbein and with similar conventions see also [3].
instead of the momentum variables $p_{aa}$, the constraints read

$$
H = \frac{1}{2} \varepsilon_{abc} \tilde{e}^a \tilde{e}^b \tilde{F}_{\alpha \beta c} - \frac{1}{2} \pi^m \pi^m
$$

$$
- \frac{1}{2} \tilde{e} m^2 \varphi^m \varphi^m - \frac{1}{2} \varepsilon^a_{\alpha \alpha} \frac{\partial_{\alpha} \varphi^m}{\delta} \frac{\partial_{\beta} \varphi^m}{\delta} \varphi^m,
$$

$$
H_\alpha = -i \tilde{e}^\beta F_{\alpha \beta a} - \partial_{\alpha} \varphi^m \pi^m,
$$

$$
C_a = D_\alpha \tilde{e}^\alpha_a,
$$

(7)

where $\tilde{e} = e^2$ is the determinant of $\tilde{e}^a_{\alpha a}$ and $A_{aa}$ is interpreted as a SO(3) connection defining a covariant derivative $D_\alpha V_a := \partial_{\alpha} V_a + \varepsilon_{abc} A_{ab} V_c$ and with curvature $F_{\alpha \beta a} = \partial_{\alpha} A_{\beta a} - \partial_{\beta} A_{\alpha a} + \varepsilon_{abc} A_{ab} A_{\beta c}$.

The non vanishing Poisson brackets are now given by

$$
\{\tilde{e}^a_{\alpha}, A_{\beta b}\} = i \delta^a_{\beta} \delta_{ab}, \quad \{\varphi^m, \pi^n\} = \delta^{mn}.
$$

(8)

Quantization and Observables

The quantum operators for the canonical variables are obtained by requiring the commutators to be $-i$ times the Poisson brackets. The wave functional $\Psi$ will be a function of $A_{aa}$ and $\varphi^m$, and the other fields are represented by

$$
\hat{\tilde{e}}^a_{\alpha} = \frac{\delta}{\delta A_{aa}}, \quad \hat{\pi}^m = i \frac{\delta}{\delta \varphi^m}.
$$

(9)

With this representation chosen and with the ordering such that all differential operators appear to the right, the SO(3) and diffeomorphism constraints in fact generate SO(3) rotations and diffeomorphisms on the wave functional, and thus they require it to be invariant under these transformations.

The Wheeler DeWitt constraint is represented by the operator

$$
\hat{\tilde{H}} = \frac{1}{2} (\varepsilon_{abc} F_{\alpha \beta c} - \delta_{ab} \partial_{\alpha} \varphi^m \partial_{\beta} \varphi^m) \frac{\delta}{\delta A_{aa}} \frac{\delta}{\delta A_{\beta b}} + \frac{1}{12} \varepsilon_{abc} \varepsilon_{\alpha \beta \gamma} m^2 \varphi^m \varphi^m \frac{\delta}{\delta A_{aa}} \frac{\delta}{\delta A_{\beta b}} \frac{\delta}{\delta A_{\gamma c}} + \frac{1}{2} \frac{\delta}{\delta \varphi^m} \frac{\delta}{\delta \varphi^m},
$$

(10)
which, of course, needs a regularization. This problem is discussed in detail in [8, 9], where one concludes that a the loop representation is needed to provide a rigorous regularization such that the solutions are annihilated by the constraints in the no-regularization limit. So let us first obtain the results formally.

In contrast to pure gravity, for the matter coupled model there are well defined observables in the sense of Dirac[4], i.e. operators commuting weakly with all the constraints. They are the conserved ‘angular momenta’ of the matter fields, i.e. the Noether charges of the global SO($N$) invariance of the Lagrangian:

$$J^{mn} := \int d^4x \varphi^{[m} \pi^{n]}.$$  \hspace{1cm} (11)

The corresponding quantum operator is

$$\hat{J}(M) := i \int d^4x \varphi^m M^{mn} \frac{\delta}{\delta \varphi^n},$$  \hspace{1cm} (12)

for any antisymmetric matrix $M$.

Solutions

The well known solutions of the Hamiltonian constraint for pure gravity parametrized by loops $\gamma$ are constructed out of the Wilson loop integral[8, 9]

$$T_\gamma(a, b) := \mathcal{P} \exp \left\{ \frac{i}{2} \int_a^b ds \gamma^\alpha A_{\alpha a} \sigma_a \right\},$$  \hspace{1cm} (13)

where path ordering is defined such that factors nearer to the upper border $b$ appear to the right, and $\sigma_a$ are the Pauli matrices obeying $\sigma_a \sigma_b = \delta_{ab} 1 + i \varepsilon_{abc} \sigma_c$. The prime denotes the derivative with respect to the curve parameter $s$. To obtain the action of the Hamiltonian constraint on $T_\gamma(a, b)$, one has to use

$$\frac{\delta}{\delta A_{\alpha a}(x)} \frac{\delta}{\delta A_{\beta b}(x)} T_\gamma(a, b) =$$

$$-\frac{1}{2} \int_a^b ds \int_s^b dt \, \delta(x, \gamma(s)) \delta(x, \gamma(t)) \gamma^{\alpha}(s) \gamma^{\beta}(t)$$

$$T_\gamma(a, s) \sigma_a T_\gamma(s, t) \sigma_b T_\gamma(s, b).$$  \hspace{1cm} (14)

As there is a contribution to the integral only for $s = t$, if the loop has no self intersections, the expression (14) becomes symmetric in $\alpha, \beta$ and we
may assume that $T_\gamma(s,t) = 1$. Defining the trace of the holonomy of the closed loop $\gamma$ by $T_\gamma := \text{Tr} T_\gamma(a,b)$, where $\gamma(a) = \gamma(b)$, we have

$$
\hat{H} T_\gamma = \frac{3}{8} T_\gamma \oint ds \oint dt \delta(x, \gamma(s)) \delta(x, \gamma(t)) \partial_s \varphi^m \partial_t \varphi^m.
$$

This expression has to be canceled against an equal expression that results from the momentum part of the Wheeler DeWitt constraint acting on a functional that depends on the matter fields. Define

$$
L^I_\gamma := \sqrt{\frac{3}{4}} \oint \gamma ds \varphi^m \partial_s \varphi^n I^{mn},
$$

where $I$ is a real antisymmetric matrix. Then

$$
\hat{H} \exp(iL^I_\gamma) = \frac{3}{8} \exp(iL^I_\gamma) \oint ds \oint dt \delta(x, \gamma(s)) \delta(x, \gamma(t)) \partial_s \varphi^m I^{mn} I^{np} \partial_t \varphi^p.
$$

Obviously, the wave functional

$$
\Psi^I_\gamma := T_\gamma \exp(iL^I_\gamma)
$$

is a solution to the Wheeler DeWitt constraint (10), if $I$ is a complex unit, i.e. $I^2 = -1$, as the various parts of (10) act either only on $T_\gamma$ or on $L^I_\gamma$.

Further solutions can be obtained by acting on the functional with the observables (12):

$$
\hat{J}(M) \Psi^I_\gamma = T_\gamma \exp(iL^I_\gamma) L^I_\gamma [I,M]
$$

is a new solution to (10).

We get nonvanishing state functionals with any antisymmetric matrix that does not commute with $I$. Those matrices are just the generators of SO($N$) which are not generators of the $U(1) \times SU(N/2)$ subgroup defined by the complex unit $I$. This is a feature of the matter coupled theory which is not present in pure gravity. See also [3] for similar observables in other kinds of matter coupled theories, e.g. extended supergravity and sigma models.

On the other hand, the main problems of the loop solutions remain: first of all, the metric determinant annihilate all the states, and, as a consequence, the states do not depend on the mass of the Klein Gordon field. Unfortunately, also the observables are mass independent and thus do not produce mass dependent states.
One might interpret this on a semiclassical level by saying that the functionals represent states of an empty universe, i.e. with no matter present, because only then a classical solution can be mass independent. But this is in contradiction with the fact, that there is an observable representing a classical constant of motion of the matter fields, which does not annihilate the state functional. Thus, in a slightly heuristical sense, there is matter in the universe. The reason for this apparent contradiction is that in Ashtekar’s formulation of gravity, even on the classical level, there are solutions to the constraints \( \tilde{c} \neq 0 \), but which are independent of \( m \), namely for \( \tilde{c} = \det g_{\alpha\beta} = 0 \), i.e. if the spatial metric is degenerate.

Another property of these solutions is that they are not diffeomorphism invariant, but this is exactly the same problem as for pure gravity and may be dealt with in the same way, i.e. by defining a loop representation and considering wave functionals defined on equivalence classes of loops\[9\]. Rovelli and Smolin showed that this procedure provides a naturally regularized Hamiltonian and it should be possible also for the matter coupled theory to make the formal results obtained here exact in a regularized framework.

A final question is whether there are similar solutions for other types of matter coupled theories. Apart from the somewhat awkward factor of \( \sqrt{3}/4 \) in the definition of the ‘matter loop’ in \( \tilde{c} \), the integrand consists essentially of two parts, which may be generalized to other models. The matrix \( I \) is a complex unit on the field space, and the one form \( \varphi^m \partial_\alpha \varphi^n \) integrated along the curve is the ‘lower \( \alpha \)’ component of the conserved current associated with the \( SO(N) \) symmetry of the matter fields. Both quantities have natural generalizations e.g. for sigma model type matter fields: \( I \) may be replaced by an (almost) complex structure and the global symmetry is provided by a killing vector field on the target space. A straightforward generalization of these solutions to such a sigma model, however, fails, because the killing vector field and the complex structure depend on \( \varphi^m \) in a nontrivial way and thus extra contributions to \( \tilde{c} \) appear which are not canceled by the corresponding ‘gravitational term’ \( \tilde{c} \).

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