The Optimal Packing of Eight Points in the Real Projective Plane

Dustin G. Mixon and Hans Parshall
Department of Mathematics, The Ohio State University, Columbus, OH, USA

ABSTRACT
How can we arrange \( n \) lines through the origin in three-dimensional Euclidean space in a way that maximizes the minimum interior angle between pairs of lines? Conway, Hardin, and Sloane (1996) produced line packings for \( n \leq 55 \) that they conjectured to be within numerical precision of optimal in this sense, but until now only the cases \( n \leq 7 \) have been solved. In this paper, we resolve the case \( n = 8 \). Drawing inspiration from recent work on the Tammes problem, we enumerate contact graph candidates for an optimal configuration and eliminate those that violate various combinatorial and geometric necessary conditions.

The contact graph of the putatively optimal numerical packing of Conway, Hardin, and Sloane is the only graph that survives, and we recover from this graph an exact expression for the minimum distance of eight optimally packed points in the real projective plane.

1. Introduction
Consider the fundamental line packing problem of packing \( n \) points in \( \mathbb{R}P^{d-1} \) or \( \mathbb{C}P^{d-1} \) so that the minimum distance is maximized. We refer to such \( n \)-point sets as projective \( n \)-packings, and we take the distance between two points in projective space to be the interior angle of the corresponding lines in affine space. The line packing problem has received considerable attention over the last century. The case of \( \mathbb{C}P^1 \) is equivalent to packing points in \( S^2 \), namely, the Tammes problem [Tammes 30]. The general line packing problem was originally formulated in [Fejes Tóth 65] and subsequently studied in [Welch 74, Delsarte et al. 75, Levenshtein 82]. Several putatively optimal packings in \( \mathbb{R}P^{d-1} \) were later provided in [Conway et al. 96], each conjectured to be within numerical precision of an optimal packing. Most of these conjectures remain open, even despite a surge of progress over the last decade (see [Fickus et al. 18] and references therein) that has been motivated in part by emerging applications in compressed sensing [Bandeira et al. 13], digital fingerprinting [Mixon et al. 13], quantum state tomography [Renes et al. 04], and multiple description coding [Strohmer and Heath 03].

In the present paper, we consider the special case of projective \( n \)-packings in \( \mathbb{R}P^2 \), that is, sets of \( n \) lines through the origin in \( \mathbb{R}^3 \). It is convenient to represent a projective \( n \)-packing by a set of \( n \) unit vectors \( \Phi \subseteq S^2 \) spanning the corresponding lines, where two such sets represent the same packing if and only if one can be obtained from the other by negating some of the vectors. We denote the resulting equivalence class by \( [\Phi] \). With this representation, we may define the coherence of the projective \( n \)-packing \( \Phi \) as

\[
\mu(\Phi) := \max_{x, y \in \Phi \atop x \neq y} |(x, y)|,
\]

which satisfies \( \mu(\Phi') = \mu(\Phi) \) for every \( \Phi' \in [\Phi] \). Setting

\[
\mu_n := \inf \{ \mu(\Phi) : \Phi \subseteq S^2, |\Phi| = n \},
\]

we say that a projective \( n \)-packing \( \Phi \) is optimal if \( \mu(\Phi) = \mu_n \). The existence of optimal projective \( n \)-packings is guaranteed for every \( n \) by a compactness argument, but they have only been classified for \( n \leq 7 \) [Fejes Tóth 65, Conway et al. 96, Cohn and Woo 12]; see Figure 1 for an illustration. In this paper, we compute \( \mu_8 \) and identify an optimal projective 8-packing, which is unique up to isometry.

Theorem 1. The projective 8-packing with minimum coherence \( \mu_8 \) is unique up to isometry. Furthermore, \( \mu_8 \) is the largest root of

\[
1 + 5x - 8x^2 - 80x^3 - 78x^4 + 146x^5 - 80x^6 - 584x^7 + 677x^8 + 1537x^9,
\]

which is given numerically by \( \mu_8 \approx 0.6475889787 \).
This improves upon the prior state of the art, which is summarized by the following lemma. The lower bound follows from the Levenshtein bound [Levenshtein 82], while the upper bound follows from an explicit numerical packing given in [Conway et al. 96], which was recently given exact coordinates in [Mixon and Parshall 19].

Lemma 2. The coherence \( \mu_8 \) of an optimal projective 8-packing satisfies

\[
0.6 \leq \mu_8 \leq 0.647588979.
\]

A set \( X \subseteq S^2 \) is said to be antipodal if \( X = -X \), and we say \( X \) is an antipodal 2n-packing if \( X \) is antipodal with \( |X| = 2n \). We may identify any projective \( n \)-packing \( \Phi \) with an antipodal 2n-packing, namely

\[
X(\Phi) := \{ x \in S^2 : x \in \Phi \text{ or } -x \in \Phi \}.
\]

For \( x, y \in S^2 \), we define their geodesic distance to be

\[
d(x, y) := \arccos(\langle x, y \rangle),
\]

and we define the minimum distance of any finite \( X \subseteq S^2 \) to be

\[
\psi(X) := \min_{x, y \in X, x \neq y} d(x, y).
\]

Of course, minimizing \( \mu(\Phi) \) over projective \( n \)-packings \( \Phi \) is equivalent to maximizing \( \psi(X) \) over antipodal 2n-packings \( X \). Both formulations play a role in our proof of the main result. The minimum distance \( \psi(X) \) is the natural quantity for us to work with in Sections 2–4 where we consider antipodal 2n-packings \( X \).

However, we state Theorem 1 in terms of the coherence due to its prevalence in the literature, and it is most natural to use \( \mu_8 \) in Section 5 when we compute the Gram matrix for an optimal projective 8-packing.

The problem of maximizing \( \psi(X) \) over antipodal 2n-packings \( X \) bears resemblance to the Tammes problem, originating in [Tammes 30], of maximizing \( \psi(X) \) over arbitrary packings of \( n \) points \( X \subseteq S^2 \). The earliest solutions to the Tammes problem, for \( n = 3, 4, 6, 12 \), are given by regular triangulations of the sphere that achieve equality in a general inequality on the optimal minimum distance [Fejes Tóth 49]. For sufficiently large values of \( n \), this inequality was sharpened in [Robinson 61], leading to the solution for \( n = 24 \). Further progress has been made by studying the contact graph of \( X \subseteq S^2 \), denoted \( G(X) \), which is drawn on the sphere with vertices \( X \) and with a geodesic edge between \( x, y \in X \) exactly when \( d(x, y) = \psi(X) \). Classifying contact graphs led to solutions to the Tammes problem for \( n = 5, 7, 8, 9 \) [Schütte and van der Waerden 51] and \( n = 10, 11 \) [Danzer 86]. More recently, computer enumeration and optimization of over a billion contact graph candidates led to the solutions for \( n = 13, 14 \) [Musin and Tarasov 12, 15b]. The authors write in related work [Musin and Tarasov 15a] that “the direct approach to the solution of the Tammes problem based on computer enumeration of irreducible contact graphs ... has actually been exhausted.” All other cases of the Tammes problem remain open. For instance, the \( n = 20 \) case is open, though it has long been known that the vertices of the dodecahedron are suboptimal [van der Waerden 52].

Figure 1. Optimal antipodal packings of 2 points in \( S^2 \) for \( n \in \{3, \ldots, 8\} \), along with the geodesic edges that appear in the corresponding contact graphs. For the packings with \( n \leq 6 \), unique optimality was established in [Fejes Tóth 65]. Optimality in the case \( n = 7 \) was proven in [Conway et al. 96], while the uniqueness of this optimal packing was established in [Cohn and Woo 12]. Our main contribution is a computer-assisted proof of a conjecture in [Conway et al. 96] that the above packing for \( n = 8 \) is optimal (and uniquely so).
To prove our main result, we adapt the approach of contact graph enumeration and elimination. To make the enumeration feasible, we leverage the anti-podal constraint to significantly reduce the number of graphs that we need to generate and consider. To this end, Section 2 identifies a set $C_1$ of 547 graphs with the property that every maximal connected set of optimal antipodal $2^n$-packings has a representative packing with contact graph in $C_1$:

In Section 3, we consider all planar embeddings of members of $C_1$ modulo homeomorphism and reflection. These equivalence classes can be represented as combinatorial embeddings, which we identify using the SPQR-tree data structure developed in [Di Battista and Tamassia 89]. After removing embeddings that violate various necessary conditions on the contact graph’s facial structure, we arrive at a set $E_1$ of 217 combinatorial embeddings of contact graph candidates. Section 4 then imposes geometric constraints in order to eliminate all but four of these candidates, three of which are subgraphs of the fourth. We then demonstrate in Section 5 that the three subgraphs violate the Karush–Kuhn–Tucker optimality conditions, thereby isolating a single contact graph candidate $G$. For the sake of reproducibility, we offer a SageMath worksheet and a Mathematica notebook (Supplementary Material) for our computations. By [Mixon and Parshall 19, Theorem II.2], there is a unique antipodal $2^n$-packing (up to isometry) with coherence satisfying Lemma 2 and whose contact graph is isomorphic to $G$. The singleton set containing this packing is a maximal connected set of optimal packings, thereby implying uniqueness. We conclude in Section 6 with a discussion of possible directions for future work.

2. Enumerating contact graph candidates

In this section, we discuss various combinatorial features of the contact graph of an antipodal $2^n$-packing, and then we apply these features to enumerate contact graph candidates for the case where $n = 8$. First, for any $X \subseteq S^2$, notice that no two edges of $G(X)$ may cross, since otherwise, a simple application of the triangle inequality demonstrates that two points in $X$ are closer than $\psi(X)$ apart. This implies the following.

Lemma 3. For any $X \subseteq S^2$, $G(X)$ is a planar graph.

We define the projective contact graph $G(\Phi)$ of a projective $n$-packing $\Phi$ to have vertex set $\Phi$ and $x \leftrightarrow y$ precisely when $|\langle x, y \rangle| = \mu(\Phi)$. Provided $n > 3$ so that $\mu(\Phi) > 0$, the contact graph $G(X(\Phi))$ of the corresponding antipodal packing provides a canonical double cover of $G(\Phi)$, where the covering map amounts to identifying antipodal pairs. This covering map naturally provides an embedding of $G(\Phi)$ into $\mathbb{RP}^2$, and we call a graph with an embedding into $\mathbb{RP}^2$ projective planar.
Lemma 4. For any projective packing \( \Phi, G(\Phi) \) is a projective planar graph.

A fundamental tool of spherical geometry is the spherical law of cosines.

Lemma 5. If \( a, b, c \) are geodesic side lengths for a spherical triangle with interior angle \( h \) opposite \( a \), then
\[
\cos a = \cos b \cos c + \sin b \sin c \cos(h).
\]

For finite \( X \subseteq S^2 \), we say \( x \in X \) can be shifted if every open neighborhood \( U \subseteq S^2 \) of \( x \) contains a point \( y \in U \) such that \( d(x', y) > \psi(X) \) for all \( x' \in X \setminus \{x\} \). If the isolated vertices of \( G(X) \) are the only ones that can be shifted, then we call \( G(X) \) irreducible. Irreducible contact graphs were introduced in [Schütte and van der Waerden 51] in order to study the Tammes problem. We record here some of their observations and include short proofs (in English) for the reader's convenience.

Lemma 6. Suppose \( X \subseteq S^2 \) with \( |X| > 6 \) and \( G(X) \) irreducible.

i. If \( x \in X \) is not isolated in \( G(X) \), then \( 3 \leq \deg(x) \leq 5 \).

ii. Each face of \( G(X) \) is a convex spherical polygon.

iii. Each face of \( G(X) \) is incident to at most \( \lfloor 2\pi/\psi(X) \rfloor \) edges.

Figure 3. The four combinatorial embeddings of contact graph candidates remaining after Section 4. Antipodal vertices share the same vertex label. \( H_1 \) is obtained from \( G \) by deleting two antipodal edges, \( H_2 \) is obtained from \( H_1 \) by deleting two antipodal edges, and \( H_3 \) is obtained from \( G \) by isolating the antipodal vertices with label 2.

Lemma 7. For every convex spherical \( B \subseteq S^2 \) and convex spherical polygon \( A \subseteq B \), the perimeter of \( A \) is at most the perimeter of \( B \).

Proof. There exist finitely many hemispheres \( \{H_i\} \) such that \( A = \cap_i H_i \). Put \( S_0 := B \) and \( S_{k+1} := S_k \cap H_{k+1} \) for each \( k \geq 0 \). Then by the triangle inequality, the perimeter of \( S_{k+1} \) is at most that of \( S_k \) for every \( k \), implying the result.

Proof of Lemma 6. If \( \deg(x) = 1 \), then \( x \) can be shifted. Similarly, if \( \deg(x) = 2 \) and \( \psi(X) < \pi/2 \), which holds for \( |X| > 6 \), then \( x \) can be shifted. Hence, \( \deg(x) \geq 3 \). Suppose now, for a contradiction, that \( \deg(x) > 5 \). Then at least six faces meet at \( x \), and the average of their interior angles meeting at \( x \) is at most \( \pi/3 \). Then there exist two vertices \( y, z \in X \) adjacent to
\( x \) where the edges \( x \leftrightarrow y \) and \( x \leftrightarrow z \) of length \( \psi(X) \) meet at \( x \) at an angle of at most \( \pi/3 \). Letting \( a \) denote the geodesic distance from \( y \) to \( z \), Lemma 5 implies
\[
\cos(a) \geq \cos^2(\psi(X)) + \frac{1}{2} \sin^2(\psi(X)).
\]

It is easily verified that for any \( \theta \in (0, 2\pi) \), \( \cos^2(\theta) + (1/2) \sin^2(\theta) > \cos(\theta). \) This shows \( \cos(a) > \cos(\psi(X)). \) As both \( a \) and \( \psi(X) \) lie in \( (0, \pi) \), we see \( a < \psi(X). \) We have contradicted the definition of the minimum distance \( \psi(X) \), and so we conclude \( \deg(x) \leq 5. \) This establishes (i). For (ii), if any face of \( G(X) \) is not convex, then the non-convexity is witnessed by one of its vertices having interior angle greater than \( \pi. \) This vertex can then be shifted into the face, contradicting irreducibility. Finally, (iii) follows from the fact that each face has perimeter at most \( 2\pi \), which in turn follows from Lemma 7 by taking \( A \) to be the face in question, which is allowed by (ii), and \( B \) to be any hemisphere that contains that face. □

Suppose \( X \subseteq S^2 \) contains a pair of antipodal points \( x, -x \in X. \) As long as \( |X| > 6 \) so that \( \psi(X) < \pi/2, x \) can be shifted if and only if \( -x \) can be shifted as well. It follows that by shifting antipodal vertices simultaneously, every antipodal \( X \subseteq S^2 \) with \( |X| > 6 \) and \( G(X) \) reducible can be transformed to an antipodal \( X' \subseteq S^2 \) with \( \psi(X') \geq \psi(X) \) such that either \( G(X') \) is irreducible or \( \psi(X') > \psi(X). \) Hence, when classifying the contact graphs of optimal antipodal 2\( n \)-packings \( X, \) it suffices to consider irreducible candidates for \( G(X) \) and possible locations for isolated vertices. In what follows, we make a few additional observations specific to antipodal sets, including a strengthening of Lemma 6(iii).

**Lemma 8.** Suppose \( X \subseteq S^2 \) is antipodal with \( |X| > 6 \) and \( G(X) \) irreducible.

i. The shortest path between an antipodal pair of vertices in \( G(X) \) has at least \( \lceil \pi/\psi(X) \rceil \) edges.

ii. Any two vertices incident to a common face of \( G(X) \) are not antipodal.

iii. Each face of \( G(X) \) is incident to at most
\[
\left\lfloor \frac{2\pi}{\psi(X)} \sin\left(\frac{\pi - \psi(X)}{2}\right) \right\rfloor
\]
edges.

**Proof.** For (i), note that we need at least \( \pi/\psi(X) \) edges of length \( \psi(X) \) to form a path between \( x, -x \in X. \) For (ii), suppose that \( x \) and \( -x \) were incident to a common face of \( G(X). \) This face must then consist of two disjoint paths of at least \( \lceil \pi/\psi(X) \rceil \) edges. Together with Lemma 6(iii), we see that each path must form a geodesic of length \( \pi \) and the face must form a spherical lune. Since \( \psi(X) < \pi/2, \) we could then shift a point distinct from \( x, -x \) into the face, demonstrating the reducibility of \( G(X). \) Hence, (ii) follows. For (iii), observe that if a pair of points \( x, y \in X \) is not an antipodal pair, then \( d(x, y) \leq \pi - \psi(X); \) otherwise, the distance from \( x \) to \( -y \) would be less than \( \psi(X). \) Then every face of \( G(X) \) lies in a spherical cap of geodesic diameter at most \( \pi - \psi(X), \) which has perimeter at most \( 2\pi \sin\left(\frac{\pi - \psi(X)}{2}\right). \) Lemma 7 then gives an upper bound on the perimeter of each convex face of \( G(X), \) from which (iii) follows. □

A short argument involving Lemma 7 forbids the isolated vertices of a contact graph from residing within a triangular face. Additional restrictions for isolated vertices were observed in [Böröczky and Szabó 03, Lemma 2, Lemma 9].

**Lemma 9.** Suppose \( X \subseteq S^2 \) and \( |X| \geq 13. \)

i. Every face of \( G(X) \) containing an isolated vertex is incident to at most 6 edges.

ii. No hexagonal face of \( G(X) \) contains two isolated vertices.

### 2.1. Generating contact graph candidates

We now construct a finite set of graphs which contains, up to isomorphism, the contact graph of an optimal antipodal 16-packing. For the remainder of this section, let \( \Phi \) denote an optimal projective 8-packing with corresponding optimal antipodal 16-packing \( X = X(\Phi), \) where the discussion preceding Lemma 6 allows us to insist that \( G(X) \) be irreducible without loss of generality. By Lemma 6(i), each non-isolated vertex in \( X \) has degree 3, 4, or 5 in \( G(X). \) As \( G(X) \) provides a double cover for \( G(\{\Phi\}), \) each non-isolated vertex in \( G(\{\Phi\}) \) has degree 3, 4, or 5 as well.

By deleting isolated vertices from a graph, we obtain its essential part. Observe that the essential part of \( G(\{\Phi\}) \) is necessarily connected, since otherwise, \( G(X) \) exhibits a face that is not simply connected, and therefore not a convex spherical polygon, violating Lemma 6(ii). Moreover, the essential part of \( G(\{\Phi\}) \) must have at least six vertices. Otherwise, \( G(\{\Phi\}) \) would have three isolated vertices, which would imply that the essential part of \( G(X) \) consists of at most five antipodal pairs. By Lemma 9(ii), it would also contain a face incident to six edges, which is impossible since Lemma 8(ii) forbids antipodal pairs from lying on the same face.
We use nauty [McKay and Piperno 14] within SageMath [The Sage Developers 18] to generate connected graphs with order 6, 7, or 8, with minimum degree at least 3, and with maximum degree at most 5. This produces a set $\Gamma_0$ of 1007 graphs, one of which must be isomorphic to the essential part of $G(\emptyset)$. To obtain candidates for $G(X)$, we generate all planar double covers for the graphs in $\Gamma_0$. Given a graph on vertices $V$ with edges $E$, every double cover of the graph can be represented on the vertex set $V \times \{0, 1\}$ in the following way. For each edge $\{v, w\} \in E$, either include in the double cover the edges $\{(v, 0), (w, 0)\}$, $\{(v, 1), (w, 1)\}$ or the edges $\{(v, 0), (w, 1)\}$ or the edges $\{(v, 1), (w, 0)\}$. In this way, a graph with $m$ edges has $2^m$ double covers, many of which are often isomorphic.

We need only retain double covers that are connected, planar, and consistent with the constraints of this section. It will be helpful to set $\psi_8 := \arccos(\mu_8)$; observe that $\psi(X) = \psi_8$ since $X$ is optimal by assumption. Lemma 2 provides the bounds

$$0.86638 \leq \psi_8 \leq 0.9273.$$  

(2-1)

The upper bound together with Lemma 8(i) ensures that every path between antipodal vertices $\{v, 0\}$ and $\{v, 1\}$ must travel along at least four edges; we eliminate graphs that violate this condition. We also eliminate any graphs that contain a wheel subgraph, or a cycle whose vertices all share a common neighbor. Indeed, since the edge lengths of $G(X)$ are all equal, it could only contain a wheel graph on $n+1$ vertices if the interior angle of each triangular face was $2\pi/n$. This is only possible on $S^2$ for $n = 3, 4, 5$, and so there must be an equilateral spherical triangle of side length $\psi(X)$ and interior angle at least $2\pi/5$. Lemma 5 then implies

$$\cos(\psi(X)) \leq \cos^2(\psi(X)) + \sin^2(\psi(X)) \cos(2\pi/5),$$

leading to an edge length of $\psi(X) \geq 1.107$ and violating (2-1). This justifies eliminating those graphs with a wheel subgraph. Retaining only one isomorphic copy of each double cover that meets all of our requirements so far, we obtain a set $\Gamma_1$ of 547 planar graphs that contains an isomorphic copy of the essential part of $G(X)$.

Since we are considering reasonably small graphs, nauty generates $\Gamma_0$ very quickly. However, extracting $\Gamma_1$ from $\Gamma_0$ is rather slow due to repeated graph isomorphism queries, taking several hours on a 3.4 GHz Intel Core i5. Lemma 4 indicates that we could first trim $\Gamma_0$ by insisting that each member be projective planar. As the anonymous referee has helpfully brought to our attention, it is likely much more efficient to generate our contact graph candidates with surftri [Sulanke 17]; we discuss this further in Section 6.

3. Enumerating combinatorial graph embeddings

At this point, we have a sizable collection of contact graph candidates, and later, we will eliminate most of these by geometric considerations. Before we can do this, we must identify the different planar embeddings of our contact graph candidates up to homeomorphism. Such equivalence classes can be encoded combinatorially by associating with each vertex a counterclockwise cyclic ordering of its neighbors. Given a planar embedding of a graph modulo homeomorphism, we call this encoding the corresponding combinatorial embedding of that graph. In this section, we describe a practical method to enumerate all combinatorial embeddings of a planar graph, which will allow us to recover all combinatorial embeddings for the graphs of $\Gamma_1$.

We first summarize some standard language from graph theory. A cut vertex for $G$ is a vertex whose deletion disconnects $G$; we say $G$ is separable if it has a cut vertex or biconnected otherwise. The blocks of a graph $G$ are its maximal biconnected subgraphs. A separation pair for $G$ is a pair of vertices whose simultaneous deletion disconnects $G$, and we say $G$ is triconnected if it has no separation pair.

It is well known [Whitney 33] that the triconnected planar graphs are exactly those with a unique combinatorial embedding, up to reversing the cyclic ordering of each vertex simultaneously. As observed in [Mac Lane 37], it is “natural to try to reduce a graph $G$ which is not triply connected to triply connected constituents.” For our purposes, since the combinatorial embeddings of the triconnected constituents are unique modulo reflection, such a decomposition would enable one to enumerate all combinatorial embeddings of the original graph. Similar decompositions are crucial to efficient graph algorithms such as linear-time planarity tests [Hopcroft and Tarjan 74].

We will encode the triconnected constituents of graphs with the SPQR-tree data structure. This was introduced in [Di Battista and Tamassia 89] and provides an efficient representation of the set of all combinatorial embeddings of a biconnected planar graph [Di Battista and Tamassia 96]. The main idea is to decompose a biconnected graph $G$ into edge-disjoint subgraphs $\{G_i\}$; the combinatorial embeddings of $G$ can be obtained from those of $\{G_i\}$ by considering their possible relative arrangements. If any of $\{G_i\}$ is triconnected, we can report their two combinatorial
embeddings. Otherwise, we can iterate the procedure for each biconnected \( G_i \) and for each block of each separable \( G_i \). An example of such a decomposition is depicted in Figure 2. While the formal definition of SPQR-trees is quite technical, we found the presentation in [Gutwenger 10] to be particularly readable. Following this resource, we provide in our SageMath worksheet (Supplementary Material) an implementation of SPQR-trees that allows us to enumerate all of the combinatorial embeddings of a given biconnected planar graph. We note that the most recent version of SageMath now has some built-in support for constructing SPQR-trees, but it does not yet have methods for enumerating combinatorial embeddings.

### 3.1. Generating combinatorial embeddings

In Section 2.1, we generated a set \( \Gamma_1 \) consisting of 547 planar graphs, one of which is guaranteed to be the essential part of \( G(X) \) for some optimal antipodal 16-packing \( X \). We now generate combinatorial embeddings of these graphs, and one of these necessarily corresponds to the planar embedding of \( G(X) \) modulo homeomorphism and reflection. Of the 547 graphs in \( \Gamma_1 \), 263 are triconnected, each producing a unique combinatorial embedding modulo reflection, and for these it suffices to produce the embedding provided by the linear time planarity algorithm [Boyer and Myrvold 04] implemented in SageMath. The remaining 284 graphs are biconnected, and so we can produce reflection representatives of their combinatorial embeddings by constructing their SPQR-trees. All together, we generate a set \( \mathcal{E}_0 \) of 2540 combinatorial embeddings, but only a fraction of these exhibit the necessary facial structure identified in the previous section. In particular, by Lemma 8(iii), all faces must be incident to at most 6 edges, and by Lemma 9, there must be a hexagonal face for every isolated vertex. The members of \( \mathcal{E}_0 \) meeting these requirements form a set \( \mathcal{E}_1 \) consisting of only 217 combinatorial embeddings. Five of these have order 14, while the remaining 212 have order 16. One of these must represent the combinatorial embedding of \( G(X) \).

### 4. Geometric constraints

At this point, we have identified a set \( \mathcal{E}_1 \) consisting of only 217 combinatorial embeddings of contact graph candidates, and at least one of these corresponds to an optimal antipodal 16-packing. In this section, we leverage geometric constraints to eliminate almost all of these embeddings from consideration. In principle, determining \( \psi \) amounts to maximizing \( \psi \) such that an embedding from \( \mathcal{E}_1 \) can be drawn on \( S^2 \) with equilateral edge length \( \psi \). To help describe the feasibility region of this optimization problem, we take the interior angles \( \{\theta_i\} \) of the faces of a given antipodal embedding to be the decision variables. These angles must satisfy certain relationships imposed by spherical geometry, which results in linear and nonlinear constraints. We will relax each of these constraints to produce a linear program that must be feasible if the embedding corresponds to an optimal antipodal 16-packing. Note that this same approach was taken in [Musin and Tarasov 12, 15b] to solve instances of the Tammes problem.

Let \( \alpha_n(d) \) denote the interior angle of a spherical regular \( n \)-gon of side length \( d \). We first record some straightforward consequences of Lemma 5 which have already appeared in recent work on the Tamnes problem [Musin and Tarasov 12, 15b].

**Lemma 10.** An equilateral spherical triangle with side length \( \psi \) has interior angle

\[
\alpha_3(\psi) = \arccos\left( \frac{\cos(\psi)}{1 + \cos(\psi)} \right).
\]

The opposite interior angles within a spherical rhombus are equal. If a rhombus has side length \( \psi \), then its adjacent interior angles \( 0, \theta \) satisfy

\[
\tan(\theta/2) \tan(\theta'/2) = \sec(\psi).
\]

In general, for an equilateral convex spherical polygon with \( n \) sides of length \( \psi \), we can express each of its interior angles \( \theta_1, \ldots, \theta_n \) as a function of \( \psi \) and any \( n-3 \) of its interior angles.

We have the following additional restrictions on the interior angles of a contact graph.

**Lemma 11.** Let \( X \subseteq S^2 \) and set \( \psi = \psi(X) \).

i. Every interior angle \( \theta \) of \( G(X) \) satisfies \( \alpha_3(\psi) \leq \theta \leq \pi \).

ii. The sum of the interior angles that meet at a vertex in \( G(X) \) equals \( 2\pi \).

iii. The sum of the interior angles of a face of \( G(X) \) incident to \( n \) edges is at most \( n\alpha_n(\psi) \).

iv. Suppose \( \alpha, \beta, \gamma, \delta, \epsilon \) are the interior angles of a pentagonal face of \( G(X) \) ordered cyclically. Then \( \alpha \geq \epsilon \) only if \( \delta \geq \beta \).

v. Suppose \( \alpha, \beta, \gamma, \delta, \epsilon, \zeta \) are the interior angles of a hexagonal face of \( G(X) \) ordered cyclically. Then \( \alpha \geq \delta \) only if both \( \gamma \geq \zeta \) and \( \epsilon \geq \beta \).

**Proof.** For (i), note that every interior angle \( \theta \) must be at least \( \alpha(\psi) \) since vertices of \( G(X) \) must be of geodesic distance \( \psi \) apart, while the convexity of the faces...
guarantees $\theta \leq \pi$. Next, (ii) is obvious. For (iii), recall that each face of $G(X)$ is a convex spherical $n$-gon by Lemma 6(ii), and so its interior angles $\theta_1, \ldots, \theta_n$ determine its area $\sum_{j=1}^{n} \theta_j - (n-2)\pi$. The convex spherical $n$-gon of perimeter $n\psi$ with maximum area is the regular convex spherical $n$-gon with area $n\pi_2(\psi) - (n-2)\pi$; see [Fejes Tóth 64, Section 30]. Comparing these two areas yields (iii). The proofs of (iv) and (v) can be found in [Danzer 86, Proposition 5] and [Böröczky and Szabó 03, Proposition 12], respectively. \hfill \Box

Antipodal packings satisfy additional constraints that do not appear in the relevant Tammes literature.

**Lemma 12.** Let $X \subseteq S^2$ be antipodal and set $\psi = \psi(X)$. Suppose $G(X)$ contains a path from $x \in X$ to $-x$ along four edges $e_1, e_2, e_3, e_4$. Denote the angle between $e_1$ and $e_2$ by $\alpha$, the angle between $e_2$ and $e_3$ by $\beta$, and the angle between $e_3$ and $e_4$ by $\gamma$. Then

$$\cos(\alpha) + \cos(\beta) = -2\cot^2(\psi) \quad \text{and} \quad \cos(\gamma) = -2\cot^2(\psi).$$

**Proof.** Suppose the path along $e_1, e_2, e_3, e_4$ passes through the vertices $x, x_1, x_2, x_3, -x$. Connecting $x$ to $x_3$ by a geodesic segment results in a spherical triangle with side lengths $\psi, \psi, d(x, x_3)$, where $d(x, x_3)$ has opposite angle $\alpha$. Similarly, connecting $x_3$ to $-x$ by a geodesic segment results in a spherical triangle with side lengths $\psi, \psi, d(x_3, -x)$, where $d(x_3, -x)$ has opposite angle $\gamma$. Moreover, we have $d(x, x_3) + d(x_3, -x) = \pi$ since $x$ and $-x$ are antipodal. Applying Lemma 5,

$$\cos(\alpha) + \cos(\beta) = \frac{\cos(d(x, x_3)) - \cos^2(\psi)}{\sin^2(\psi)} + \frac{\cos(d(x_3, -x)) - \cos^2(\psi)}{\sin^2(\psi)} = -2\cot^2(\psi).$$

Similarly, connecting $x_1$ to $x_3$ results in a spherical triangle with side lengths $\psi, \psi, d(x_1, x_3)$, where $d(x_1, x_3)$ has opposite angle $\beta$. Moreover, we know that $d(x_1, x_3) \geq \pi - 2\psi$ since we are traveling between antipodal points $x$ and $-x$. Applying Lemma 5,

$$\cos(\beta) = \frac{\cos(d(x_1, x_3)) - \cos^2(\psi)}{\sin^2(\psi)} \leq \frac{-\cos(2\psi) - \cos^2(\psi)}{\sin^2(\psi)} = 1 - 2\cot^2(\psi).$$

\hfill \Box

### 4.1. Eliminating combinatorial embeddings

From Section 3.1, we have a set $\mathcal{E}_1$ of 217 combinatorial embeddings. Fix a combinatorial embedding $E$ in $\mathcal{E}_1$ and denote its (variable) interior angles by $\theta_1, \ldots, \theta_r$. For each $1 \leq i \leq r$, we set

$$m_i := \inf \theta_i \quad \text{and} \quad M_i := \sup \theta_i,$$

where the infimum and supremum are taken over all drawings of $E$ on $S^2$ with equilateral geodesic edge length $\psi$. We suppose that such a drawing exists, but if we reach a conclusion of the form $m_i > M_i$ for any $1 \leq i \leq r$, then we arrive at a contradiction and may eliminate $E$ from consideration. For the remainder of this section, we describe our method for eliminating most of $\mathcal{E}_1$ through linear programming. For speed, we first perform the following computations numerically. At the cost of several more hours on a 3.4 GHz Intel Core i5, we verified all of our linear programming queries with exact arithmetic as implemented in GLPK. Our code is available in our SageMath worksheet (Supplementary Material).

Recall that Lemma 2 implies $0.86638 \leq \psi_8 \leq 0.9273$, and so Lemma 10 gives $1.1668 \leq \theta_3(\psi_8) \leq 1.1864$. Moreover, by Lemma 11(i), each interior angle $\theta_i$ satisfies $1.1668 \leq \theta_i \leq \pi$. This provides us with initial lower bounds on each $m_i$ and initial upper bounds on each $M_i$. Next, we relax by treating the edge length $\psi$ as another decision variable. Denote

$$m_0 := \inf \psi \quad \text{and} \quad M_0 := \sup \psi,$$

where the infimum and supremum are taken over all drawings of the embedding with edge length $\psi$ satisfying $0.86638 \leq \psi \leq 0.9273$ and with each interior angle $\theta_i$ satisfying $\theta_i \in [m_i, M_i]$. Of course, if the combinatorial embedding corresponds to an optimal contact graph, then $M_0 = \psi_8$. We can use our bounds so far and simple interval arithmetic to relax each of the nonlinear constraints from the previous section to linear constraints and use linear programming to determine improved bounds on each of $m_i$ and $M_i$ for $0 \leq i \leq r$. For instance, suppose we currently have bounds of the form $m_0 \geq a$ and $M_0 \leq b$. Then Lemma 10 and Lemma 11(i) together imply

$$m_i \geq \alpha_3(\psi) \geq \arccos \left( \frac{\cos(a)}{1 + \cos(b)} \right)$$

for every $1 \leq i \leq r$. In a similar way, the remainder of Lemma 10 and also Lemma 12 can be used to improve estimates of various $m_i$ and $M_i$ from existing estimates of other $m_i$ and $M_i$.

In addition, every part of Lemma 11 converts existing bounds on $m_0$ and $M_0$ into inequality constraints on each $\theta_i$, allowing one to improve existing estimates on each $m_i$ and $M_i$ for $1 \leq i \leq r$ by linear programming. As one might expect, this process benefits from iteration. If our estimates of $m_i$ and $M_i$ for any $0 \leq \theta_0$.
$i \leq r$ ever imply $m_i > M_i$, then we may eliminate the corresponding embedding from consideration. For each member of $E_1$, we follow this general procedure for several iterations, and as a result, only 17 combinatorial embeddings remain.

For the remaining 17 embeddings, we improve our bounds on each $m_i$ and $M_i$ for $1 \leq i \leq r$ as follows. Suppose that we know $m_0 \geq a$ and $M_0 \leq b$. Then we can partition $[a, b]$ into several subintervals $[a_i, b_i] = [a_1, b_1] \cup \cdots \cup [a_k, b_k]$ and repeat the same procedure as in the previous paragraph under the assumption $\psi \in [a_j, b_j]$. Each linear program involves relaxations of several nonlinear constraints on each interior angle $\theta_i$ involving $\psi$, so it comes as no surprise that we have stronger bounds on each $\theta_i$ when we restrict $\psi$ to a small subinterval. For several embeddings, each subinterval $[a_i, b_i]$ results in an infeasible program, allowing us to eliminate that embedding. In other cases, we instead obtain improved bounds on several $m_i$ and $M_i$. After this stage, only seven combinatorial embeddings remain.

We remove a few more embeddings from consideration by simultaneously bisecting feasibility regions for $\psi$ and each interior angle for a given face. For a face incident to $n$ edges, this results in $2^{n-1}$ regions that can be checked for feasibility individually. Two of our embeddings lead to infeasible programs in all $2^7$ regions corresponding to one of their hexagonal faces. For others, this again leads to improved upper bounds on $\psi$, and iterating our procedure finally shows that one last embedding leads to an infeasible program.

At this point, we have reduced our search to four combinatorial embeddings, which we denote by $G$, $H_1$, $H_2$, and $H_3$; see Figure 3 for an illustration. One may verify that $G$ corresponds to the numerical packing obtained by Conway, Hardin, and Sloane [Conway et al. 96], which we made exact in [Mixon and Parshall 19]. Each $H_i$ can be obtained from $G$ by deleting appropriate edges. In the following section, we show that none of $H_1$, $H_2$, or $H_3$ is optimal by passing to a dual optimization program.

5. Optimality conditions and uniqueness

To complete the proof of the main result, we first apply optimization theory to analyze each of the remaining four contact graph candidates (eliminating all but one), and then we leverage ideas from real algebraic geometry to demonstrate uniqueness. For the first step, we follow [Musin and Tarasov 15b] by testing whether each candidate is consistent with the necessary Karush–Kuhn–Tucker optimality conditions [Boyd and Vandenberghe 04, §5.5.3]. In words, these conditions imply that optimal packings are infinitesimally rigid and therefore must have an associated equilibrium stress matrix, which we denote by $(w_{ij})$.

**Lemma 13.** Let $X = \{x_1, \ldots, x_{2n}\}$ be an optimal antipodal 2n-packing. Set

$$J(i) := \{j : x_i \rightarrow x_j \text{ in } G(X)\},$$

and for each $j \in J(i)$, let $v_{ij}$ denote the unit vector at $x_i$ tangent to the geodesic from $x_i$ to $x_j$. There exist coefficients $\omega_{ij} = \omega_{ji} \in \mathbb{R}$ for $1 \leq i, j \leq 2n$ with $i \neq j$ such that

i. $\omega_{ij} \geq 0$ for all $i, j$ with $i \neq j$,
ii. $\omega_{ij} = 0$ whenever $x_i$ and $x_j$ are not adjacent in $G(X)$,
iii. $\sum_{i,j : i \neq j} \omega_{ij} = 1$, and
iv. for each $i$, $\sum_{j \in J(i)} \omega_{ij} v_{ij} = 0$.

**Proof.** For any antipodal 2n-packing $Z = \{z_1, \ldots, z_{2n}\}$, we may take $z_{j+n} = -z_j$ for $1 \leq j \leq n$ without loss of generality by reordering if necessary. As such, the antipodal 2n-packing problem in $S^2$ is equivalent to the following program, whose decision variables are $z_1, \ldots, z_{2n} \in \mathbb{R}^3$ and $\delta \in \mathbb{R}$:

maximize $\delta$

subject to $\delta - |z_i - z_j|^2 \leq 0$ for all $i, j$ with $i \neq j$,

$|z_i|^2 - 1 = 0$ for all $i$, and

$z_{i+n} + z_i = 0$ for all $1 \leq i \leq n$.

For the optimal packing $X$, we denote the square of its minimum Euclidean distance by $\Delta := 2 - 2 \cos(\psi(X))$. Then $\Delta$ is the maximum value of the above program with maximizer $(x_1, \ldots, x_{2n}, \Delta)$. For what follows, it is convenient to put $z := (z_1, \ldots, z_{2n})$ and define

$$f(z, \delta) = \delta - g_0(z, \delta) = \delta - |z_i - z_j|^2 \text{ for } i, j \text{ with } i \neq j,$$

$$h_1(z, \delta) = |z_i|^2 - 1 \text{ for } 1 \leq i \leq 2n,$$

$$h_i'(z, \delta) = |z_{i+n} + z_i|^2 \text{ for } 1 \leq i \leq n.$$

The Karush–Kuhn–Tucker conditions guarantee coefficients $\omega_{ij}, \lambda_i, \lambda_i'$ with

$$\nabla f(x, \Delta) = \sum_{i=1}^{2n} \sum_{j=1}^{2n} \omega_{ij} \nabla g_{ij}(x, \Delta)$$

$$+ \sum_{i=1}^{2n} \lambda_i \nabla h_i(x, \Delta) + \sum_{i=1}^{n} \lambda_i' \nabla h_i'(x, \Delta).$$

(5–1)

By symmetry, we may replace both $\omega_{ij}$ and $\omega_{ji}$ with their average, thereby ensuring $\omega_{ij} = \omega_{ji}$. Dual feasibility ensures $\omega_{ij} \geq 0$ for all $i, j$ with $i \neq j$, implying
(i). Complementary slackness ensures \( \omega_{ij}g_j(x, \Delta) = 0 \). As such, when \( j \notin J(i) \) and \( j \neq i \), then \( g_j(x, \Delta) \neq 0 \), and so \( \omega_{ij} = 0 \). This gives (ii). Notice that (iii) follows by considering the \( \delta \)-component of the gradient vector in (5–1), while its \( z_r \)-component yields

\[
0 = \sum_{j \in J(i)} 2\omega_{ij}(x_j - x_i) + 2\delta x_i + 2\xi(x_{i+n} + x_i)
= \sum_{j \in J(i)} 2\omega_{ij}(x_j - x_i) + 2\delta x_i.
\]

(5–2)

We now fix \( i \) and, for each \( j \in J(i) \), we let \( p_j \) denote the projection of \( x_j - x_i \) onto the 2-dimensional subspace parallel to the tangent plane of \( S^2 \) at the point \( x_i \). Projecting (5–2) onto the same subspace demonstrates

\[
\sum_{j \in J(i)} \omega_{ij}p_j = 0.
\]

(5–3)

We claim that \( |p_j| \) is identical for all \( j \in J(i) \). Indeed, for every \( j \in J(i) \), each vector \( x_j - x_i \) has the same norm, and therefore the same component in the \( x_i \) direction since \( x_i, x_j \in S^2 \). Scaling (5–3) by the common factor of \( |p_j| \) proves (iv).

\[\square\]

5.1. Eliminating subgraphs

Lemma 13 reports that every optimal antipodal packing \( X \) must have a corresponding stress matrix \( (\omega_{ij}) \). Note that we can verify whether a given \( (\omega_{ij}) \) is a stress matrix once we have the graph \( G(X) \) and the tension directions \( (v_{ij}) \), both of which are determined by \( X \). In this subsection, we argue that for each combinatorial embedding \( H_\ell \) illustrated in Figure 3, then for every choice of tension directions \( (v_{ij}) \) that are consistent with the bounds derived in the previous section, there is no stress matrix \( (\omega_{ij}) \) that is consistent with both \( H_\ell \) and \( (v_{ij}) \).

Given \( H_\ell \), suppose for the sake of contradiction that there exists an optimal antipodal 16-packing \( X \) such that the essential part of \( G(X) \) belongs to the homeomorphism equivalence class of either \( G \) or its reflection. Furthermore, no other irreducible graph is the contact graph of an optimal antipodal 16-packing. Since \( G \) is irreducible with no isolated vertices, we may conclude that the contact graph of every optimal antipodal 16-packing is isomorphic to \( G \).

5.2. The optimal configuration

By the process of elimination, every maximal connected set of optimal antipodal 16-packings in \( S^2 \) has a representative packing \( X \) whose contact graph belongs to the homeomorphism equivalence class of either \( G \) or its reflection. Furthermore, no other irreducible graph is the contact graph of an optimal antipodal 16-packing. Since \( G \) is irreducible with no isolated vertices, we may conclude that the contact graph of every optimal antipodal 16-packing is isomorphic to \( G \).

By selecting one vector from each antipodal pair in \( X \), we obtain a representation of an optimal projective 8-packing by unit vectors \( \Phi = \{\varphi_1, \ldots, \varphi_8\} \subseteq S^2 \). If we abuse notation by treating \( \Phi \) as a matrix with columns \( \varphi_1, \ldots, \varphi_8 \), then its Gram matrix is given by \( \Phi^T\Phi \). Notice that \( (\Phi^T\Phi)_{ij} = \mu_{ij} \) precisely when \( \varphi_i \) and \( \varphi_j \) are adjacent in \( G(X) \), while \( (\Phi^T\Phi)_{ij} = -\mu_{ij} \) precisely when \( \varphi_i \) and \(-\varphi_j \) are adjacent in \( G(X) \). By inspecting \( G \) in Figure 3, we see that we may select \( \Phi \) so that \( \Phi^T\Phi \) takes the form of the matrix \( M \) appearing in Theorem 14 below. The proof of Theorem 1 is then complete after applying Theorem 14. For completeness, we see, we can apply the relationships in Lemma 13 to further improve these bounds.

Notice that bounds on the interior angles \( \{\theta_k\} \) imply bounds on the angles \( \theta_{ij} = \sum_{k \in K_{ij}} \theta_k \), which in turn imply bounds on the coordinates of \( u_{ij} \) that take the form \( \cos(\theta_{ij}) = [c_{ij}, C_{ij}] \) and \( \sin(\theta_{ij}) = [S_{ij}, S_{ij}] \). If \( (\omega_{ij}) \) satisfies Lemma 13(iv), then it must be the case that

\[
\sum_{j \in J(i)} \omega_{ij}c_{ij} \leq 0 \leq \sum_{j \in J(i)} \omega_{ij}C_{ij},
\]

and

\[
\sum_{j \in J(i)} \omega_{ij}S_{ij} \leq 0 \leq \sum_{j \in J(i)} \omega_{ij}S_{ij}.
\]

(5–4)

By linear programming, we can determine whether the set \( \Omega \) consisting of all \( (\omega_{ij}) \) that satisfy both Lemma 13(i)–(iii) and (5–4) is empty. Overall, we can sharpen the bounds from Section 4.1 by imposing the additional constraint that \( \Omega \) is nonempty. With this approach, we repeatedly partition the admissible intervals for each variable from Section 4.1, improving the bounds on each \( m_k \) and \( M_k \) until we demonstrate \( m_k > M_k \) for some \( k \). This procedure delivers the desired contradiction for each of \( H_1, H_2, \) and \( H_3 \). Recalling that the existence of an optimal antipodal 16-packing is guaranteed to exist by a compactness argument, this leaves us considering only the optimal contact graph \( G \).
describe here the computer-assisted proof of Theorem 14 that appeared first in [Mixon and Parshall 19].

**Theorem 14.** There is a unique real matrix of the form

\[
M = \begin{pmatrix}
1 & a_1 & \mu & a_2 & a_3 & -\mu & a_4 & -\mu \\
\mu & -\mu & 1 & a_5 & a_6 & -\mu & a_7 & a_8 \\
\mu & -\mu & 1 & a_9 & a_{10} & -\mu & a_{11} & a_{12} \\
a_2 & a_5 & 1 & a_{13} & a_{14} & a_6 & a_7 & a_8 \\
a_3 & -\mu & a_8 & 1 & a_{11} & a_{12} & \mu & -\mu \\
-\mu & a_6 & -\mu & a_{12} & \mu & 1 & a_{13} & \mu \\
a_4 & -\mu & a_9 & \mu & \mu & \mu & a_{14} & 1 \\
-\mu & a_7 & a_{10} & \mu & -\mu & a_{13} & a_{14} & 1 \\
\end{pmatrix}
\]  

(5–6)

with the following properties:

i. \( \text{rank}(M) = 3 \),

ii. \( M \) is positive semidefinite,

iii. \( 0.6 \leq \mu \leq 0.64759 \), and

iv. \( |a_j| < \mu \) for all \( 1 \leq j \leq 14 \).

Moreover \( \mu \) is the largest root of

\[
1 + 5x - 8x^2 - 80x^3 - 78x^4 + 146x^5 - 80x^6 - 584x^7 + 677x^8 + 1537x^9
\]

and is given numerically by \( \mu \approx 0.6475889787 \).

**Proof.** Observe that if (i) holds, then each of the 4 \( \times 4 \) minors of \( M \) must vanish, and Sylvester’s criterion states that (ii) is equivalent to each principal minor of \( M \) being nonnegative. It follows that admissible values of \( \mu \) and \( a_1, \ldots, a_{14} \) must satisfy a system of polynomial equalities and inequalities. In principle, the cylindrical algebraic decomposition (CAD) algorithm [Collins 75] constructs the corresponding semialgebraic solution set in \( \mathbb{R}^{15} \). Unfortunately, the runtime of CAD can be doubly exponential in the number of variables, and so it is not surprising that a naive application of CAD does not yield the result in a reasonable amount of time. We instead iteratively run smaller instances of CAD arising from subsystems corresponding to the vanishing of several 4 \( \times 4 \) minors and conditions (iii) and (iv). After several steps, this proves that there is a unique matrix \( M \) of the form (5–5) satisfying \( \text{rank}(M) \leq 3 \) and conditions (iii) and (iv). Moreover, this procedure computes the entries of \( M \) exactly, including the reported value of \( \mu \). These entries can be found and reproduced within our Mathematica notebook (Supplementary Material). For a detailed account of these computations, see [Mixon and Parshall 19].

As we have already observed, there is a Gram matrix of an optimal projective 8-packing of the form (5–5) which has rank 3 and satisfies (iii) and (iv). As \( M \) is unique, we see that (i) and (ii) must also hold. We may also verify (i) and (ii) directly in order to keep Theorem 14 self-contained. With the exact entries of \( M \) in hand, it is relatively quick to check symbolically in Mathematica that \( \text{rank}(M) = 3 \), so (i) holds. Asking Mathematica directly if \( M \) is positive semidefinite does not terminate quickly. Instead, we check symbolically that each of the 3 \( \times 3 \) principal minors of \( M \) is positive, and we claim this is enough to establish (ii). Indeed, since \( \text{rank}(M) = 3 \), we know each of the 4 \( \times 4 \) minors of \( M \) vanish. Moreover, each of the diagonal entries are 1, and the bounds (iii) and (iv) are enough to ensure that the 2 \( \times 2 \) principal minors of \( M \) are positive as well. Then (ii) follows from Sylvester’s criterion.

\[ \Box \]

**6. Future work**

In this paper, we identified an optimal projective 8-packing in \( \mathbb{RP}^2 \), and we established that it is unique up to isometry. The unique optimality of this packing was predicted in [Conway et al. 96], where numerical solutions for putatively optimal projective \( n \)-packings in \( \mathbb{RP}^2 \) were presented for \( n \leq 55 \). For the \( n = 9 \) case, they conjecture that there are two distinct isometry classes of optimal projective packings. It would be natural to extend our work to study these larger packings, and such extensions would benefit from a more computationally efficient procedure. The first bottleneck appears in Section 2, where, given a projective contact graph candidate in \( \Gamma_0 \) on \( m \) edges, we naively produced all of its \( 2^m \) double covers, and then we retained a single isomorphic copy of each double cover that happened to be planar and consistent with other combinatorial constraints. As indicated by Lemma 4, we only needed to consider members of \( \Gamma_0 \) that are projective planar. The anonymous referee has helpfully brought to our attention the software package surftri [Sulanke 17], which generates projective planar embeddings. We expect that our procedure for generating contact graph candidates can be sped up with surftri, and we will address this in future work on larger packings. The second bottleneck appears when we use exact arithmetic for our linear programming computations in Sections 4.1 and 5.1, and in total these take us roughly 24 hours on a 3.4 GHz Intel Core i5. We suspect it will be more efficient for larger packings to work numerically and argue that the computational error does not negatively impact our results. Finally, it would be interesting to investigate comparable approaches for efficiently packing points in \( \mathbb{F}^{d-1} \) for \( F \in \{ \mathbb{R}, \mathbb{C} \} \) with \( d \geq 3 \) and, more generally, packing higher-dimensional subspaces in \( \mathbb{F}^d \).
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