Quantum Modifications to Gravity Waves in de Sitter Spacetime

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Abstract. We study the effect of the vacuum fluctuations of a massless conformally invariant field on tensor perturbations of spatially flat de Sitter spacetime. We find that de Sitter spacetime is stable within the regime of the semiclassical theory. Next we investigate the modification of linearized plane gravity waves by the effects of the quantum stress tensor. We find a correction term whose amplitude depends on the initial time \( \eta_0 \). If this initial time is the beginning of inflation, then as long as the energy scale of inflation and the proper frequency of the mode at \( \eta_0 \) are well below the Planck scale, the fractional correction is small. Nonetheless, modes which are transplanckian at the onset of inflation may have a significant correction. The increase in amplitude can be potentially observed through a modification of the power spectrum of tensor perturbations in inflationary cosmology. This enhancement of the power spectrum depends upon the initial time, and is greater for shorter wavelengths.

1. Introduction
Most of the inflationary models assume that the universe has experienced a period of exponential expansion and is approximately described by de Sitter spacetime. Quantum fields in de Sitter spacetime account for the primordial spectrum of scalar and tensor perturbations. In addition, quantum effects may modify the duration of inflation and possibly introduce instabilities [1, 2, 3].

We examine some of the quantum field effects in the semiclassical theory, where gravity couples with the renormalized expectation value of a matter field stress tensor \( \langle T_{\mu\nu} \rangle \).
The semiclassical theory has been extensively applied to scalar perturbations of de Sitter spacetime [4], but the topic of this paper focuses on tensor perturbations. We will treat a model in which the matter field is a conformal field, and address two physical questions: the stability of de Sitter spacetime under tensor perturbations, and the effects of one-loop quantum matter field corrections on gravity waves in de Sitter spacetime. We adopt the sign conventions of Ref. [5], and use units in which $h = c = 1$.

2. Weakly perturbed de Sitter spacetime

We are concerned with a portion of global de Sitter spacetime which can be represented as a spatially flat Robertson-Walker universe with the metric

$$ds^2 = a^2(\eta)(-d\eta^2 + dx^2 + dy^2 + dz^2),$$

where $a(\eta) = -1/(H\eta)$ and $\eta < 0$ is the conformal time coordinate. We would like to consider tensor perturbations of this geometry, which describe gravitational waves on the de Sitter background. Let the perturbed metric be $g_{\mu\nu} = \gamma_{\mu\nu} + h_{\mu\nu}$, where $\gamma_{\mu\nu}$ is the background metric of Eq. (1), and $h_{\mu\nu}$ is the perturbation. We employ the transverse trace-free gauge defined by $h_{\mu\nu} u^\nu = 0$, $h_{\mu} = 0$ and $h_{\mu\nu} u^\nu = 0$. Here $u^\mu = \delta_0^\mu$ is the four velocity of a comoving observer, covariant derivatives are taken respect to the fixed de Sitter background, and indices are raised and lowered by the background metric. To first order in the perturbation $h_{\mu\nu}$ [6, 7], we find that the local terms in $\langle T_{\mu\nu} \rangle$ amount to shifts in the cosmological constant and Newton’s constant, so we are left with only the effects of the $P_{\mu\nu}$ and nonlocal $Q_{\mu\nu}$ terms on the tensor perturbations, such that in the transverse trace-free gauge the semiclassical Einstein equation reduces to

$$\Box_s h_{ij} = -16\pi\ell_p^2 (P_{ij} + Q_{ij}),$$

with $\ell_p$ being the Planck length.

3. Spatially homogeneous solutions

We study the stability of the tensor perturbations of de Sitter spacetime in the presence of the quantum stress tensor of the conformal field. It is sufficient to examine spatially homogeneous solutions of Eq. (2), as these will be the most rapidly growing modes if there is an instability. Let

$$h_{ij} = a^{-2} h_{ij} = h_i^l = e_i^l (-\eta)^{-b},$$

where $e_i^l$ is a constant polarization tensor and $b$ is a constant. A solution for which $b > 0$ will grow as a power of conformal time as $\eta \to 0^-$. Substituting Eq. (3) into Eq. (2) yields

$$b(3 + b) \simeq -\xi \left\{ 6[1 + \ln(H\lambda/2)] b + [5 + \pi^2 + 11\ln(H\lambda/2)] b^2 + O(b^3) \right\},$$

in terms of small parameter $\xi = 64\pi^2 \ell_p^2 H^2\alpha$. Within the domain of validity of the semiclassical theory, the only possibility for an unstable solution is one with a small positive value of $b$. We find that there are no solutions with $b > 0$ so long as $\xi \ll 1$. Hence we conclude that de Sitter spacetime is stable in the semiclassical theory against tensor perturbations.

4. Effects on gravity waves

Now we examine the effect of the quantum stress tensor on gravity waves in de Sitter spacetime. The plane wave solutions of the Lifshitz equation, $\Box_s h_i^l = 0$, are of the form

$$h_{\mu}^\nu = c_0 e_{\mu}^\nu (1 + i k\eta) e^{i(kx - k\eta)},$$
where $c_0$ is a constant and $e^\mu_\nu$ is the polarization tensor. In the presence of the quantum stress tensor, the modified gravity wave may be expressed as $h^\mu_\nu + h^\mu_\nu$, and in the limit that $|\eta_0| \gg |\eta|$, 

$$h^\mu_\nu(x) \sim 64\pi^2 i e^\mu_\nu c_0 \alpha H^2 k^2 |\eta_0| (1 + i k \eta) e^{i(kx-k\eta)}, \tag{6}$$

which has the same functional form as $h^\mu_\nu$, but is out of phase by $\pi/2$. The modified wave is no longer constant when the mode has a proper wavelength larger than the horizon size, $k|\eta| < 1$. This is in contrast to the unperturbed mode, Eq. (5), whose magnitude is approximately constant when it is outside the horizon. The most striking feature of Eq. (6) is that the correction term due to the quantum stress tensor is proportional to $|\eta_0|$, and hence is larger the earlier the coupling between the quantum stress tensor and the metric perturbation is switched on. We can write the ratio of the magnitude of the correction to that of the original wave as

$$\Gamma = \left| \frac{h^\mu_\nu}{h^\mu_\nu} \right| = 64\pi^2 \alpha H^2 k^2 |\eta_0| = 64\pi^2 \alpha H k_P \ell_p^2. \tag{7}$$

Here $k_P = k/a(\eta_0) = kH|\eta_0|$ is the physical wavenumber of the mode as measured by a comoving observer at time $\eta = \eta_0$. If we require that the curvature of the de Sitter spacetime be well below the Planck scale ($H\ell_p \ll 1$), and that the mode in question be always below the Planck scale ($k\ell_p \ll 1$) while it interacts with the quantum stress tensor, then these two conditions together imply that $|h^\mu_\nu/h^\mu_\nu| < 1$, and hence the quantum correction to the gravity wave is smaller than the original wave. However, if inflation lasts for a sufficiently long time, then modes which are cosmological interest today appear to have been above the Planck scale at the onset of inflation. This is the cosmological version of the transplanckian problem.

The inflationary model predicts a Gaussian and nearly scale invariant spectrum of tensor perturbations [8, 9], which might be found in polarization measurements of the CMB. The tensor perturbations from inflation are less model dependent than the density perturbations. At this time, they leave an imprint on the CMB in the form of a power spectrum of tensor perturbations given by

$$\delta_\nu^2 \approx \frac{8}{\pi} \ell_p^2 H^2. \tag{8}$$

This is an approximately flat spectrum. The effect of the conformal stress tensor is to modify this power spectrum by a factor $|1 - i\Gamma|^2 = 1 + \Gamma^2$.

Let us choose the scale factor to be unity at the end of inflation, $E_R$ be the reheating energy, and $S$ be the factor by which the universe expands from the initial conformal time $\eta = \eta_0$ to the end of inflation. We have

$$\Gamma^2 = 1.34 \times 10^{-78} \left( \frac{10^{25} \text{ cm}}{\ell_0} \right)^2 \left( \frac{E_R}{10^{15} \text{ GeV}} \right)^6 S^2, \tag{9}$$

if we use the value of $\alpha = 1/(320\pi^3)$ corresponding to the electromagnetic field. Recall that $\ell_0 \simeq 10^{25}$ cm corresponds to angular scales of the order of $1^\circ$ of the present horizon size.

If one has only the minimal inflation needed to solve the horizon and flatness problems, so $S \simeq 10^{23}$, then the effects of the one-loop correction on the tensor perturbation spectrum is negligible. However, larger values of $S$ have the potential to produce significant corrections. For example, $E_R \simeq 10^{15}$GeV and $S \simeq 10^{30}$ would lead to an effect of order unity at $1^\circ$ scales. One should expect the one-loop approximation to begin to break down, but this can serve as an order of magnitude estimate. In contrast to the nearly flat spectrum, Eq. (8), due to free graviton fluctuations, the one-loop effect is highly tilted toward the blue end of the spectrum. The effect is potentially observable. Thus this possibility does require one to take seriously the contribution of modes which were transplanckian at the beginning of inflation.
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