1-loop correction to the SU(3) symmetric chiral soliton

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Abstract

Masses of the SU(3) chiral soliton in tree approximation turn out at much too high energies typically around 2 GeV. It is shown that 1-loop corrections reduce this value drastically with results in the region of the empirical nucleon mass.

1 Introduction

In this short letter we discuss the effect of 1-loop corrections to the mass of the SU(3) symmetric chiral soliton. It is well-known that the soliton’s tree mass turns out too high already in SU(2) and the SU(3) extension adds further a large kaonic rotational energy such that the situation is worsened with values which typically lie around or even above 2 GeV. In SU(2) it was shown [1, 2] that pionic 1-loop corrections are capable to reduce the too high tree mass to a reasonable number (Table 1).

The main concern of this paper is to answer the question whether the same may happen also in SU(3) where we have to start from a much larger soliton mass, and if that actually occurs how can it be in accordance with $N_C$ counting? In the course of this investigation we will recognize as often in soliton models the decisive role of the Wess-Zumino-Witten (WZW) term.

A detailed discussion of the SU(2) case is found in [2] and the SU(3) extension requires only minor changes in the formulation. The renormalisation procedure is identical to that used in chiral perturbation theory and relies on chiral counting together with dimensional regularisation which preserves chiral symmetry. Starting point is always an exact numerical solution of the classical equations of motion around which fluctuations for the 1-loop calculation are considered and the exactness of the solution guarantees that the terms linear in the fluctuations vanish. These requirements limit the applicability of the procedure considerably: in SU(3) this is the rotating hedgehog

$$U = AU_0A^\dagger, \quad U_0 = \begin{pmatrix} e^{i\hat{T}_i\hat{F}(r)} & 1 \\ 1 & 1 \end{pmatrix}, \quad A \in SU(3)$$

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in the symmetric case $m_K = m_\pi$. Here $F(r)$ denotes the chiral angle and $A$ is a SU(3) rotation matrix depending on Euler angles $\alpha_a$, $a = 1, \ldots, 8$. Already for weak symmetry breaking, $m_K \gtrsim m_\pi$, the ansatz (1) does not remain exact. Allowing the profile to become Euler angle dependent, $F(r, \alpha_a)$, improves the situation but does not solve the problem: kaon and eta solitonic components are induced already in lowest order ($m_K^2 - m_\pi^2$). To consider fluctuations around such a SU(3) deformed object is certainly beyond present possibilities and we are therefore not in the position to calculate 1-loop in the rotator approach. On the other hand, for strong symmetry breaking $m_K \gg m_\pi$ the hedgehog rotating in SU(2) only, $A \in$ SU(2), becomes an exact solution. However there the assumption $m_K \gg m_\pi$ is in conflict with chiral counting and the adopted regularisation scheme. Whether the standard chiral lagrangian may nevertheless be used in connection with the bound-state approach will be subject of a separate investigation. At present our procedure applies only to SU(2) ($m_K \to \infty$) and to SU(3) symmetry ($m_K = m_\pi$) which is treated in the following. The actual nucleon mass should lie in between these two limiting cases.

2 Formulation

The standard chiral SU(3) lagrangian expressed in terms of the matrix $U$ which contains the dynamical fields and the mass matrix $M$

$$
\mathcal{L} = \frac{f^2}{4} \text{tr} \left[ \partial_\mu U \partial^\mu U^\dagger + M(U + U^\dagger) \right]
$$

$$
+ \left( L_1 + L_2 + \frac{1}{2} L_3 \right) \text{tr} \partial_\mu U \partial^\mu U^\dagger + \frac{1}{2} L_3 \text{tr} \left[ [U^\dagger \partial_\mu U, U^\dagger \partial_\nu U] \right]^2
$$

$$
+ \left( L_3 + 3L_2 \right) \left[ \text{tr} \partial_\mu U \partial^\mu U^\dagger \partial_\nu U \partial^\nu U^\dagger - \frac{1}{2} \left( \text{tr} \partial_\mu U \partial^\mu U^\dagger \right)^2 \right]
$$

$$
+ L_4 \text{tr} M(U + U^\dagger) \text{tr} \partial_\mu U \partial^\mu U^\dagger + L_5 \text{tr} (UM + MU^\dagger) \partial_\mu U \partial^\mu U^\dagger
$$

$$
+ L_6 \left( \text{tr} M(U + U^\dagger) \right)^2 + L_7 \left( \text{tr} M(U - U^\dagger) \right)^2
$$

$$
+ L_8 \text{tr} (MUMU + MU^\dagger MU^\dagger)
$$

$$
\equiv \frac{f^2}{4} \text{tr} \left[ \partial_\mu U \partial^\mu U^\dagger + M(U + U^\dagger) \right] + \sum_{i=1}^{8} L_i \mathcal{L}_i^{(4)}
$$

comprises the familiar non-linear sigma ($N(\sigma)$) model of (chiral order) ChO2 and eight terms of ChO4 which are relevant in the soliton sector without external fields. At scale $\mu = m_\rho = 770$ MeV which should provide the lagrangian in leading order $N_C$ the renormalized low energy constants (LECs) are chosen

$$
L_1 + L_2 + \frac{1}{2} L_3 = 0, \quad L_3 + 3L_2 = 0, \quad L_2 = \frac{1}{16\pi^2}
$$
\[ L_4 = L_6 = 0, \quad L_5 = 2L_8 = -6L_7 = \frac{f_K^2 - f_\pi^2}{8(m_K^2 - m_\pi^2)} = 2.3 \cdot 10^{-3} \quad (3) \]

to be in accordance with the standard values \[6, 7\] (within error bars) with one exception: \( L_2 \) has to be fixed by an effective Skyrme parameter \( e = 4.25 \) (the standard value would correspond to \( e \approx 7 \)) in order to simulate the missing higher ChOs generated by vector mesons. A detailed justification of this choice is found in ref. \[2\]. The SU(2) reduction of (2) yields exactly the lagrangian employed in that reference (and the LECs which additionally appear in SU(3) take their standard values). As a consequence the soliton as well as the pionic 1-loop results do not have to be recalculated, but may just be taken from there. It should be mentioned that although the LECs are chosen such that many of the ChO4 terms in \(2\) vanish at scale \( \mu = m_\phi \) all these terms are switched on and do contribute when the scale is changed.

In the SU(3) symmetric case under consideration the mass matrix \( M = m^2 \cdot 1 \) is diagonal and leads to identical kaon and pion masses and decay constants

\[
\begin{align*}
L_4^2 &= f_\pi^2 = f^2 + 8(3L_4 + L_5)m^2 & f &= 91.1 \text{ MeV} \\
L_5^2 m_K^2 &= f_\pi^2 m_\pi^2 = f^2 m^2 + 16(3L_6 + L_8)m^4 & m &= 138 \text{ MeV}.
\end{align*}
\quad (4)
\]

Because the symmetry breakers are absent, the nucleon mass in tree approximation

\[
E_{\text{tree}} = M_0 + \frac{J(J+1)}{2\Theta_\pi} + \frac{1}{2\Theta_K} \left[ C_2 - J(J+1) - \frac{N_C^2}{12} \right] \\
= M_0 + \frac{3}{8\Theta_\pi} + \frac{N_C}{4\Theta_K}, \quad N_C \text{ odd}
\quad (5)
\]
comprises the soliton mass \( M_0 \) of order \( N_C \), the pionic rotational energy of order \( N_C^{-1} \) (\( \Theta_\pi \) pionic moment of inertia) and the kaonic rotational energy of order \( N_C^0 \) (\( \Theta_K \) kaonic moment of inertia). The non-trivial \( N_C \) assignment to the kaonic rotational energy is caused by the WZW term which selects the lowest lying multiplet depending on the number of colors. For odd \( N_C \) the "nucleon" with spin and isospin \( 1/2 \) and hypercharge \( N_C/3 \) sits in the multiplet with the labels

\[(p, q) = (1, \frac{N_C - 1}{2}), \quad C_2 = \frac{N_C^2}{12} + \frac{N_C}{2} + \frac{3}{4}. \quad (6)\]

With that eigenvalue \( C_2 \) of the Casimir operator eq. (5) is immediately verified.

For the 1-loop calculation fluctuations \( \eta_a \) are introduced through the ansatz

\[ U = A \sqrt{U_0} e^{i\lambda_a \eta_a / f} \sqrt{U_0} A^\dagger, \quad a = 1, \ldots, 8 \quad (7) \]
and the corresponding equations of motion (e.o.m.) which according to their time dependence \( \sim e^{-i \omega t} \) may be written as

\[ h_{ab}^2 \eta_b = \omega^2 n_{ab} \eta_b \quad (8) \]
\( h^2_{ab} \) is a differential operator and \( n^2_{ab} \) the metric) have to be solved for the phase-shifts. Because the e.o.m. (8) decouple for the different meson species into partial waves characterized by phonon spin \( L \) and parity the pionic, kaonic and eta phase-shifts may be summed up separately over the various channels (\( L_c \))

\[
\delta^x(p) = \sum_{L_c} (2L+1)\delta^x_{L_c}(p) \quad x = \pi, K, \eta. 
\]  

(9)

The ultra-violet divergencies contained in the Casimir energy are related to the high momentum behaviour of these phaseshifts

\[
\delta^x(p) \xrightarrow{p \to \infty} a^x_0 p^3 + a^x_1 p + \frac{a^x_2}{p} + \cdots
\]  

(10)

with expansion coefficients \( a^x_0, a^x_1, a^x_2 \) known analytically for the \( N\ell\sigma \) model (the explicitly denoted terms give rise to at least logarithmically divergent expressions). These coefficients obey the important ChO4 relation

\[
\sum_x \left[ 3\pi m_x^4 a^x_0 - 4\pi m_x^2 a^x_1 + 8\pi a^x_2 \right] = \sum_{i=1}^8 \Gamma_i \int d^3r \mathcal{L}_i^{(4)},
\]  

(11)

where the \( \Gamma_i \)'s are simple numerical factors given in [3] and which is used below for regularisation of the Casimir energy. For the full model (2) the coefficients have to be determined numerically and the challenge is to calculate the phase-shifts with great precision up to \( p_{\text{max}} \simeq 25m_\pi \) where \( L_{\text{max}} \simeq 100 \) partial waves are needed (for details see [2]). With these informations at hand the divergencies in the 1-loop contribution may be isolated using dimensional regularisation

\[
E_{\text{cas}} = \frac{1}{2\pi} \sum_x \left\{ -\int_0^\infty \frac{pd\rho}{\sqrt{p^2 + m_x^2}} [\delta^x(p) - a^x_0 p^3 - a^x_1 p - \frac{a^x_2}{p}] - m_x \delta^x(0) \right\}
\]

\[
+ \frac{3m_x^4 a^x_0}{16} \left( \frac{1}{6} + \ell n \frac{m^2_x}{\mu^2} \right) - \frac{m_x^2 a^x_1}{4} \ell n \frac{m^2_x}{\mu^2} + \frac{a^x_2}{2} \left( 1 + \ell n \frac{m^2_x}{\mu^2} \right)
\]

\[
+ \Lambda(\mu) \sum_x \left[ 3\pi m_x^4 a^x_0 - 4\pi m_x^2 a^x_1 + 8\pi a^x_2 \right]
\]

\[
\equiv \sum_x E_{\text{cas}}^x(\mu) + \Lambda(\mu) \left[ \sum_{i=1}^8 \Gamma_{i} \int d^3r \mathcal{L}_i^{(4)} + \text{higher ChOs} \right]
\]  

(12)

which involves a scale \( \mu \) to render the arguments in the logarithms dimensionless. The divergencies as \( d \to 4 \) reside in

\[
\Lambda(\mu) = \frac{\mu^{d-4}}{16\pi^2} \left[ \frac{1}{d-4} - \frac{1}{2}(\Gamma'(1) + \ell n(4\pi) + 1) \right]
\]  

(13)
and may finally be absorbed into a redefinition of the LECs

\begin{align}
L_i^r(\mu) &= L_i - \Gamma_i \Lambda(\mu), \\
L_i^r(\mu) &= L_i^r(m_\rho) - \frac{\Gamma_i}{32\pi^2} \ln\left(\frac{\mu^2}{m_\rho^2}\right)
\end{align}

which become scale-dependent. The renormalisation scheme is identical to that used in chiral perturbation theory.

From (12) it is also noticed that the regularisation scheme must fail for \( m_K \gg m_\pi \): in the limit \( m_K \to \infty \) we would obtain an infinite contribution from the second row containing the chiral logarithms (the term \( m_K \delta^K(0)/2\pi \) would cancel the bound state contribution \( \frac{1}{2} \sum_{\omega} \omega \) which has to be added in that case because the infinitesimal kaonic rotations appear at finite energies).

For \( m_K = m_\pi \) with the finite contributions in (12) the nucleon mass in tree + 1-loop is finally determined

\begin{align}
E_{\text{tree+1-loop}} = M_0(\mu) + \frac{3}{8\Theta_\pi(\mu)} + \frac{N_C}{4\Theta_K(\mu)} + \sum_x E^x_{\text{cas}}(\mu).
\end{align}

All quantities involved become scale-dependent in a non-trivial way, this will be investigated. Because the contributions of the various mesons enter additively we may consider them separately.

### 2.1 Pions

The pionic contribution is the same as in SU(2), this was mentioned already. In ref. [2] we obtained a 1-loop contribution \( E^\pi_{\text{cas}}(m_\rho) = -680 \text{ MeV} \) at scale \( \mu = m_\rho \). Further it was found that the scale-dependences of \( M_0(\mu) \) and \( E^\pi_{\text{cas}}(\mu) \) cancel almost exactly over a wide region of scales, compare Fig. 3.2 in that reference. This finding was interpreted as strong evidence for the reliability of the renormalisation procedure and also for the reasonable choice of the effective Skyrme parameter \( e \). The pionic rotational energy in (5) is very small by itself and its scale-dependence is relatively weak such that it does not destroy this property.

### 2.2 Eta

The coupling of the \( \eta \) to the soliton proceeds through the mass terms only and consequently is extremely weak. The resulting 1-loop contribution \( E^\eta_{\text{cas}}(m_\rho) = +0.5 \text{ MeV} \) is tiny.

### 2.3 Kaons

Because in the kaonic sector we have four infinitesimal kaonic rotations the phase-shift starts at \( \delta^K(0) = 4\pi \) according to Levinson’s theorem. This fact, although
the zero-modes do not contribute by themselves (because they are located at zero energy), leads to a large negative Casimir energy which at scale $\mu = m_\rho$ amounts to $E^K_{\text{cas}}(m_\rho) = -425 \text{ MeV}$. This contribution compensates nicely for the kaonic rotational energy $N_C/4\Theta_K(m_\rho) = +390 \text{ MeV}$ which is of the same order $N^0_C$ (Table 1). However, the scale-dependence of the rotational energy is enhanced further if the 1-loop contribution is included as is noticed from Fig.1. This indicates that there is an important term missing.

### 2.4 Kaons with WZW term

Inclusion of the WZW term has two effects (i) it adds a contribution to the kaonic moment of inertia and (ii) it modifies the e.o.m. for the fluctuations (8) introducing a term linear in $\omega$.

(i) Kaonic moment of inertia

The WZW term provides the driving term for an induced soliton component proportional to the kaonic angular velocity which leads to a contribution to the moment of inertia $I$. Because we have already taken into account the collective rotation and in order to avoid double counting we have to impose the constraint

$$\int d^3r \, z_a^e \eta^2_{ab} \eta_b = 0 , \quad z_a^e = \frac{2f_K}{\sqrt{2\Theta_K}} \sin\left(\frac{F}{2}\right) f_{aei} \hat{r}_i$$

(16)
that the induced component $\eta$ be orthogonal to the infinitesimal rotation (for $m_K = m_\pi$ the corresponding equation without constraint would not even have a unique solution because the zero-mode as solution of the homogeneous equation can always be added with arbitrary strength [9]). The constraint reduces the induced component considerably. For $m_K = m_\pi$ and scale $\mu = m_\rho$ we obtain $\Theta_K(m_\rho) = (1.92 + 0.21) \text{GeV}^{-1} = 2.13 \text{GeV}^{-1}$ a 10% contribution from the induced component which however decreases rapidly with increasing kaon mass justifying that this contribution is normally neglected. In our case it helps to soften the scale-dependence of the kaonic rotational energy as is noticed by comparing Fig.1 with Fig.2.

(ii) Phase-shifts

The WZW term introduces a term linear in $\omega$ into the e.o.m. and as a consequence not all infinitesimal rotations remain at zero energy. For physical kaon mass, two of them appear as bound states and are interpreted as kaonic excitations of the nucleon (bound-state approach [3]). For $m_K = m_\pi$ these states appear as resonances in the continuum and consequently the kaonic phase-shift starts only at $\delta^K(0) = 2\pi$. Again, because these states are already considered in our approach in the kaonic collective rotation they have to be projected from the space of allowed fluctuations by implementing the constraint (16). Then the phase-shift starts again at $\delta^K(0) = 4\pi$ as it should.

Figure 2: Same as Fig.1 but with the WZW term considered. The tree + 1-loop contribution is almost scale-independent over a wide region of scales.
With inclusion of the WZW term the Casimir energy $E_{\text{cas}}(m_\rho) = -600$ MeV at scale $\mu = m_\rho$ over-compensates the kaonic rotational energy $N_C/4\Theta_K(m_\rho) = +350$ MeV (Table 1). In contrast to the case without WZW the scale-dependences of these quantities are opposite such that the kaonic energy in tree + 1-loop becomes almost scale-independent over a wide region (Fig.2) quite similar to the pionic contributions (for very small scales the soliton becomes unstable due to a too strong symmetric ChO4 term). This confirms that the WZW term plays an important role also in this context.

3 Results

Table 1 comprises the results for the nucleon mass in the SU(3) symmetric chiral soliton model with and without the WZW term taken into account. It should be kept in mind that there are no additional adjustable parameters in the game, the effective Skyrme parameter was taken as in the SU(2) calculation. It is noticed that in both cases the tree mass of $\simeq 2$ GeV is appreciably reduced into the region of the physical nucleon mass. This is compatible with $N_C$ counting because for the kaonic contributions tree and 1-loop are of the same order $N_C^0$ and because a strong cancellation between the two occurs. As was discussed in the previous section, scale-independence requires the inclusion of the WZW term. Thus, the results for the nucleon mass are in case of SU(3) symmetry 770 MeV and for SU(2) 1020 MeV such that the empirical nucleon mass actually lies in between these two limiting values. Once the lagrangian is fixed the calculation presented is exact to order $N_C^0$.

|                | SU(2) | SU(2) without WZW | SU(2) with WZW |
|----------------|-------|-------------------|---------------|
| soliton mass   | $N_C^0$ | 1630              | 1630          |
| pionic rotation| $N_C^{-1}$ | 70               | 70            |
| kaonic rotation| $N_C^0$ | 390               | 350           |
| total tree     | 1700  | 2090              | 2050          |
| $\pi$ 1-loop   | $N_C^0$ | −680              | −680          |
| $K$ 1-loop     | $N_C^0$ | −425              | −600          |
| $\eta$ 1-loop  | $N_C^0$ | +0.5              | +0.5          |
| total tree + 1-loop | 1020 | 985               | 770           |
there are no other contributions to this order. Unfortunately we were not in the position to calculate 1-loop corrections for finite \( m_K \neq m_\pi \) for the reasons discussed in the introduction. Possibly one can estimate the Casimir energy for \( m_K \gtrsim m_\pi \) in the rotator approach where one finds Euler angle dependent scattering equations \[\text{[10]}\], but it is difficult to control the necessary approximations (neglection of the terms linear in the fluctuations which arise from the rotating hedgehog being not an exact solution). On the other hand for \( m_K \gg m_\pi \) the 1-loop contributions could possibly be calculated using the bound-state approach \[\text{[5]}\] which also remains to be investigated.

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