The ambiguity function and the displacement operator basis in quantum mechanics

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Abstract
We present a method for calculating expectation values of operators in terms of a corresponding c-function formalism which is not the Wigner–Weyl position-momentum phase-space, but another space. Here, the quantity representing the quantum system is the expectation value of the displacement operator, parametrized by the position and momentum displacements, and expectation values are evaluated as classical integrals over these parameters. The displacement operator is found to offer a complete orthogonal basis for operators, and some of its other properties are investigated. Connection to the Wigner distribution and Weyl procedure are discussed and examples are given.

Keywords: Wigner Weyl, phase space quantum mechanics, radar Ambiguity function, displacement operator completeness orthogonality

1. Dedication
Wolfgang Schleich is a master of quantum mechanics in phase-space, and of physics in general. He is always full of insight and excels in finding simplicity. We believe we present an interesting way to do quantum mechanics in phase-space, a field in which Prof Schleich has made enormous contributions, and tried investigating its source to the bottom, in the spirit of Prof Schleich. We are elated to dedicate these results to him.

2. Introduction
In the early 1930s, Wigner pioneered the phase-space formulation of quantum mechanics, introducing the Wigner distribution \([1]\). He did so in an attempt to make quantum mechanics more classical (statistical mechanics) looking, and indeed, to calculate some quantum properties of gasses. Since then, other phase-space formulations of quantum mechanics were created. Besides to construct a quantum theory of statistical mechanics, the drive to create phase-space distributions was due to several aspects: one was a fundamental aspect—trying to create new formulations of quantum mechanics and studying the uncertainty principle; another, not too distant motivation was the study of the classical-quantum interface; still other reasons are for mathematical, as well as conceptual simplicity. Indeed, Wolfgang Schleich is notorious for utilizing the Wigner distribution to simplify physical problems and to give insight into their inner-workings \([2]\).

Independently of these developments, there was a push to determine which operators should be used in quantum mechanics to describe systems. That is, starting with a system which we might describe classically by some quantity \(A_C(q, p)\), what is the quantum analog? Due to the quantum commutation relations between the position and momentum operators, it is far from obvious how to obtain the ‘quantum version’ of \(A_C(q, p)\). Several answers were given, including one by Weyl \([3]\). In the late 1940s, Moyal realized the
connection [4] between Weyl’s procedure and Wigner’s distribution. Namely, that
\[ \text{Tr}(\rho A) = \iint dq dp \hat{P}(q, p) \hat{A}(q, p), \]
if \( \hat{P} \) is Wigner’s distribution coming from \( \rho \) and \( A \) is the operator coming from \( \hat{A} \) via Weyl’s procedure. In the mid
1960’s, Cohen found the connection between all of the distributions and the operators, and gave a way to generate arbitrary distributions [5].

We present the theory behind one such distribution, the ambiguity distribution, which is seldom used in quantum mechanics, and present many of its properties and the properties of its accompanying classical operator, including the transformations (both directions) between operator and the corresponding c-number function. We also show that it has multiple advantages over the widely-used distributions.

Royer [6] has shown that the Wigner distribution [1, 7] is the expectation value of a seed operator
\[ W(q, p) = 2 \exp[2i(p - q) \cdot (q - q)/\hbar ], \]
also called the ‘displaced parity operator,’ and Englert [8] has found the above elegant expression for \( W \) in terms of operator-ordered exponentials. We [9] have reviewed these expressions for \( W \) and found some additional ones, and showed how operator-ordering emerges naturally from the seed operator. It is in general interesting to see how such distributions come about [9, 10].

Now we find that the displacement operator, which we denote by \( \Theta \), is another such seed operator; not for Wigner functions, rather for their characteristic function, also known as the ambiguity function or as the Shirey function [12]. We thus call \( \Theta \) the ambiguity seed-operator. This operator has some attractive properties, such as operator orthogonality [equation (6)] and operator completeness [equation (7)], and could be used for calculating expectation values [equation (5)], for evolving quantum states in time [equation (9)], and for studying and representing quantum mechanics in phase-space. As we show in section 5, it happens that \( \Theta \) is intimately-related to the Wigner seed operator \( W \). This formalism could be used for mathematical manipulations of operators, and the distribution was used in [13] for solving several problems in laser spectroscopy, in [14] for studying the decoherence of a harmonic oscillator in a heat bath, and in [15, 16], for studying fundamental issues in relativistic quantum field theory. Already in [17] it was suggested that the ambiguity function could be used for quantum mechanics, and since submission, we have been made aware of a recent article on this distribution [18].

Compared to the Wigner distribution, this formalism has some advantages. In our formalism, calculation of the expectation value of (polynomial) operators does not involve any integration—just derivatives and multiplication as in equation (36). This is a clear advantage over the Wigner distribution. Also, in some cases, e.g. in [14], the distribution we present is easier to evolve in time. The Wigner c-number corresponding to a Hermitian operator is real, while in our formalism it need not be; this may or may not be advantageous—for example in some cases, obtaining the ambiguity function from an operator might be easier than obtaining the Wigner function from the same operator. Like the Wigner distribution, the ambiguity function transforms nicely under canonical transformations. In contrast with the Wigner distribution, the ambiguity distribution does not satisfy the marginals. The properties of the distribution come from its seed operator.

3. c-numbers for use in quantum mechanics

We study the properties of the displacement operator and show that it gives rise to a quantum phase-space formalism. We present a method for obtaining the c-number distribution (ambiguity function) \( A(\eta, \xi) \) for any arbitrary operator \( A \), and show how it can be used for calculating quantum mechanical expectation values. Like the Wigner function [1, 2, 7, 19], this c-number distribution is also obtained from the expectation value of some seed operator, \( \Theta(\eta, \xi) \), the ambiguity seed operator. Starting with an arbitrary operator \( A \), we define the c-number
\[ A(\eta, \xi) = \text{Tr}[A \Theta(\eta, \xi)], \]
from which the operator \( A \) could be recovered by
\[ A = \iint d\eta d\xi \Theta(\eta, \xi) \Theta(-\eta, -\xi), \]
where the operator \( \Theta(\eta, \xi) \) is
\[ \Theta(\eta, \xi) = \frac{e^{i(\eta \eta + \xi \xi)/\hbar}}{\sqrt{2\pi \hbar}}, \]
which is known as the displacement operator (divided by \( \sqrt{2\pi \hbar} \)). We note that \( \eta \) and \( \xi \) have dimensions of momentum and position, respectively. The c-number \( A(\eta, \xi) \) in equation (2) is also known as the ambiguity function in the field of radar [20, 21], and in section 5, we show that it is the double Fourier transform, or characteristic function, of the Wigner function \( \hat{A}(q, p) \) corresponding to the operator \( A \).

We may use our definition in equation (2) for the c-number functions in order to compute quantum mechanical expectation values, or more generally, traces of operator products. We find that the trace of the product of two arbitrary operators, \( A \) and \( B \), could be computed by
\[ \text{Tr}[AB] = \iint d\eta d\xi \ A(\eta, \xi) B(-\eta, -\xi). \]
The expectation value of \( A \) is obtained when \( B \) is the density matrix \( \rho \).

As we show in appendices A and B, the formalism is a consequence of the completeness and orthogonality of \( \Theta \); that is, (1) the operator orthogonality
\[ \text{Tr}[\Theta(\eta, \xi) \Theta(-\eta', -\xi')] = \delta(\eta - \eta') \delta(\xi - \xi'), \]
5 All integrations range from \(-\infty\) to \(\infty\) unless otherwise specified.
and (2) the operator completeness
\[ \int d\eta d\xi \langle k' | \Omega(\eta, \xi) | x' \rangle \langle x | \Omega(-\eta', -\xi') | k \rangle = \delta(x - x') \delta(k - k'), \]

where \( | x \rangle \) and \( | k \rangle \) are position and momentum eigenstates, respectively. Both equations (6) (orthogonality) and (7) (completeness) follow directly from equation (4), and are used in appendices A and B to derive equations (2), (3), and (5).

### 4. Time evolution

We may use this formulation to evolve the quantum state \( P(\eta, \xi) = \text{Tr}(\rho \Theta) \) in time using the Schrödinger (von Neumann) equation, or to evolve an arbitrary operator \( A(\eta, \xi) = \text{Tr}(\rho \Theta) \) in time using the Heisenberg equation. In particular, we give a prescription purely in terms of ambiguity quantities.

\[
\frac{\partial}{\partial t} P(\eta, \xi, t) = \frac{2}{\hbar} \int d\eta' d\xi' \sin \frac{\eta - \eta'}{2\hbar} \times H(\frac{\eta}{2} + \eta', \frac{\xi}{2} + \xi') P(\eta, \xi, t),
\]

for the quantum state. To evolve an arbitrary operator, \( A(\eta, \xi) \), in time, take \( t \to -t \) and replace \( P \) by \( A \) in equation (9). For example, (using equations (9) and (33)) time evolution under the constant force Hamiltonian is

\[
\frac{\partial}{\partial t} P(\eta, \xi, t) = \left\{ \frac{\eta}{m} \frac{\partial}{\partial \xi} + i \frac{F}{\hbar} \right\} P(\eta, \xi, t),
\]

which could also be obtained from

\[
i \hbar \frac{\partial}{\partial t} \text{Tr}(\rho \Theta) = \text{Tr}(\{\text{H\rho} - \rho \text{H}\Theta\}).
\]

The solution to equation (10) is

\[
P(\eta, \xi, t) = e^{i m F t^2 \frac{\hbar^2}{2m} e^{-i \eta t} e^{i \frac{m \hbar}{\eta} \chi^2}} \text{P} \left( \frac{\eta}{m} t, \eta, 0 \right)
\]

\[
= e^{i \eta t \chi^2 / 2m \hbar} P \left( \frac{\eta}{m} t, \eta, 0 \right).
\]

That is, the ambiguity function of the quantum state is evolved to some time \( t \) by simple substitution at time \( t = 0 \) and multiplication by a phase.

In [14], the time evolution enters through the \( \Theta \) operator. Particularly, \( \Theta(\eta, \xi, t) \) is found by using the time-evolution of the position and momentum operators. Using the formula for the derivative of the exponential of an operator

\[
\frac{d}{dt} e^{B(t)} = \int_{0}^{1} d\lambda e^{\lambda B(t)} \frac{dB(t)}{dt} e^{(1-\lambda)B(t)},
\]

it was found that

\[
\frac{d}{dt} P(\eta, \xi) = \frac{i}{\hbar} \text{Tr} \left\{ P(\eta, \xi) \right\} \frac{\hbar}{\sqrt{2\pi}} \left[ \eta \frac{dA(t)}{dt} + \xi \frac{dP(t)}{dt} \right] e^{(1-\lambda)(|\eta|^{2} + |\xi|^{2})/\hbar}.
\]

\[
(15)
\]

### 5. Connection to Wigner functions and to the Weyl procedure

The Weyl procedure is a procedure introduced by Weyl for obtaining quantum operators \( A \) from c-numbers \( A_{W} \) (which Moyal showed that \( A_{W} \) is the Wigner function \( A \) corresponding to \( A \) [4]). Weyl was interested in determining what quantum operators one should use given classical analogs. He proposed [3, 19]

\[
A = \int d\eta d\xi \Theta(-\eta, -\xi) \int d\eta dp \tilde{A}(q, p) e^{(iq + p\eta)/\hbar}.
\]

\[
(16)
\]

Therefore, comparing equations (3) and (16), we find that the c-number \( A \) which we defined in equation (2) is the double Fourier transform of the Wigner function \( \tilde{A} \) of \( A \).

An interesting connection also exists between the seed operators for the ambiguity function, \( \Theta(\eta, \xi) \), and the seed operators for the Wigner function, \( W(q, p) \). The Wigner function \( \tilde{A} \) is obtained from \( A \) via [8, 9]

\[
\tilde{A}(q, p) = \text{Tr}(\rho \tilde{W}(q, p)),
\]

\[
(17)
\]

and \( \Theta(\eta, \xi) \) is connected to \( W(q, p) \) by Fourier transform (see appendix C)

\[
W(q, p) = \frac{1}{\sqrt{2\pi \hbar}} \int d\eta d\xi e^{-i(q\eta + p\xi)/\hbar} \Theta(\eta, \xi).
\]

\[
(18)
\]

So we see it is no accident that also \( A(\eta, \xi) \) and \( \tilde{A}(q, p) \) are related by Fourier transform.

Another interesting connection could be found when calculating \( \tilde{A} \) and \( \tilde{A} \) from \( A \). In the position representation, the ambiguity function is

\[
A(\eta, \xi) = \int dq e^{-i(q\eta + p\xi)/\hbar} | \eta - \xi/2 | A(q + \xi/2),
\]

\[
(19)
\]

while the Wigner–Weyl function is

\[
\tilde{A}(q, p) = \int d\xi e^{-ip\xi/\hbar} | \eta - \xi/2 | A(q + \xi/2).
\]

\[
(20)
\]

### 6. Examples

**Ex: Position states.** Because \( A(\eta, \xi) \) is complex, it describes the position state \( A = | x \rangle \langle x | \) as a function with dependence on both \( \eta \) and \( \xi \),

\[
A(\eta, \xi) = \delta(\xi) e^{i\eta y},
\]

\[
(21)
\]

6 There is also interest in the inverse procedure—obtaining the classical function \( \tilde{A}(q, p) \) from the operator \( A \) [22, 23].
in contrast to the Wigner function of $\mathbf{A}$,
\[ \hat{A}(q, p) = \delta(q - x), \]  
(22)
which has no momentum dependence. The superposition of two positions $|\psi\rangle = \alpha|x_1\rangle + \beta|x_2\rangle$ is
\[
A(\eta, \xi) = |\alpha|^2 e^{i\eta q_2} \delta(x_2 - x_1 - \xi) + |\beta|^2 e^{i\eta q_2} \delta(x_2 - x_1 - \xi) + \beta\alpha^* e^{i\eta q_2} \delta(x_2 - x_1 - \xi) + |\beta|^2 e^{i\eta q_2} \delta(x_2 - x_1 - \xi).
\]
(23)
and to the Wigner function
\[
\rho(\eta, \xi) = \frac{e^{i\eta q_2 + i\eta p_2}}{\sqrt{2\pi\hbar}} \exp \left\{ -\frac{\eta^2}{4\hbar^2/\Delta} - \frac{\xi^2}{4\Delta} \right\},
\]
(26)
corresponding to the ambiguity function
\[
P(\eta, \xi) = \frac{e^{i\eta q_2 + i\eta p_2}}{\sqrt{2\pi\hbar}} \exp \left\{ -\frac{\eta^2}{4\hbar^2/\Delta} - \frac{\xi^2}{4\Delta} \right\},
\]
(27)
and to the Wigner function
\[
\hat{P}(q, p) = \frac{1}{\hbar} \exp \left\{ -\frac{(q - x)^2}{\Delta} - \frac{(p - k)^2}{\hbar^2/\Delta} \right\},
\]
(28)
where the semicolon ($\cdot$) in the exponent of equation (26) is Schwinger-ordered-exponential notation, as in equation (1).

**Ex: Gaussian state.** The density matrix of the Gaussian state wavefunction peaked about the position $q = x$, and having average momentum $k$,
\[
\psi(q) = \left( \frac{\Delta}{\pi} \right)^{1/4} \exp \left\{ -\frac{(q - x)^2}{2\Delta} - i q k / h \right\},
\]
(25)
is
\[
\rho = \sqrt{2} \exp \left\{ \frac{(q - x)^2}{-2\Delta} \right\} e^{i q x_0 - k_1 / h} \exp \left\{ \frac{(p - k)^2}{-2\hbar^2 / \Delta} \right\},
\]
(29)
and the Wigner function
\[
\hat{P}(q, p) = \frac{1}{\hbar} \exp \left\{ -\frac{(q - x)^2}{\Delta} - \frac{(p - k)^2}{\hbar^2 / \Delta} \right\},
\]
(30)
which are like Bopp operators [24], which could be used for obtaining Wigner functions from operators in a simple way [9]. Using equations (29), the ambiguity function $H(\eta, \xi)$ is
\[
\frac{H(\eta, \xi)}{\sqrt{2\pi\hbar}} = \mathbf{H} \left( \frac{\hbar}{\iota \eta} + \frac{\xi}{2}, \frac{\hbar}{\iota \xi} - \frac{\eta}{2} \right) \delta(\eta) \delta(\xi)
\]
(31)
and the expectation value of the Hamiltonian is calculated via equation (5)
\[
\langle \mathbf{H} \rangle = \int d\eta d\xi \mathbf{P}(\eta, \xi) H(-\eta, -\xi)
\]
(32)
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\]
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\]
(34)
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\[
\langle \mathbf{H} \rangle = \int d\eta d\xi \mathbf{P}(\eta, \xi) H(-\eta, -\xi)
\]
(35)
and the expectation value of the Hamiltonian is calculated via equation (5)
\[
\langle \mathbf{H} \rangle = \int d\eta d\xi \mathbf{P}(\eta, \xi) H(-\eta, -\xi)
\]
(36)
where in the last equality in equation (36), the expression is evaluated at $\eta = \xi = 0$, and $P(\eta, \xi)$ is the ambiguity function which is obtained from the state’s density matrix via equation (2). Interestingly, it is the value of the operated-on quantum state $P(\eta, \xi)$ at zero displacement ($\eta = \xi = 0$) that gives the expectation value of the operator. Amazingly, no integrals are required; this expression involves only derivatives and multiplication. This is generally the case for operators which are polynomial in $q$ and $p$.

**7. Properties of $\Theta(\eta, \xi)$**

We now discuss some properties of the ambiguity seed operator.

The ambiguity seed operator has the property that its form is unchanged under a canonical transformation
\[
q = \alpha Q + \beta P, \quad p = \gamma Q + \delta P,
\]
(37)
that is to say, the new parameters become
\[
\eta \longrightarrow \alpha \eta + \gamma \xi, \quad \xi \longrightarrow \beta \eta + \delta \xi.
\]
(38)
Specifically,
\[
\Theta_{\eta, p}(\eta, \xi) = \Theta_{Q, p}(\alpha \eta + \gamma \xi, \beta \eta + \delta \xi),
\]
(39)
where $\Theta_{Q, P}(\eta, \xi)$ is the operator $\exp[i(\eta Q + \xi P)/\hbar]/\sqrt{2\pi\hbar}$. If the Jacobian $J$ of the transformation in equation (38) is unity, that is, $J = \alpha \delta - \beta \gamma = 1$, then the commutators $[q, p] = J [Q, P]$ are equal (which makes the transformation canonical), and we have the symmetry that
\[
A(\eta, \xi) = \mathbf{H} \left( \frac{\hbar}{\iota \eta} + \frac{\xi}{2}, \frac{\hbar}{\iota \xi} - \frac{\eta}{2} \right) \delta(\eta) \delta(\xi)
\]
(40)
and that
\[
\Theta_{Q, p}(\eta, \xi) = \Theta_{Q, p}(\alpha \eta + \gamma \xi, \beta \eta + \delta \xi)
\]
(41)
and
\[
\Theta_{Q, p}(\eta, \xi) = \Theta_{Q, p}(\alpha \eta + \gamma \xi, \beta \eta + \delta \xi)
\]
(42)
where we have used equations (3) and (39), and $\eta$ and $\xi$ instead of $-\eta$ and $-\xi$. Using equation (6), we find that
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\begin{equation}
\text{Tr}[A(Q, P)\Theta_{q,p}(-\eta, -\xi)] = A_{q,p}(\alpha\eta + \gamma\xi, \beta\eta + \delta\xi),
\end{equation}

which means that if we calculate \(A_{q,p}(\eta, \xi)\) corresponding to some operator \(A(q, p)\), then we can obtain the c-number \(A_{Q,P}(\eta, \xi)\) corresponding to \(A(Q, P)\) via the simple substitution, equation (38)

\begin{equation}
A_{Q,P}(\eta, \xi) = A_{q,p}(\alpha\eta + \gamma\xi, \beta\eta + \delta\xi).
\end{equation}

The trace of \(\Theta(\eta, \xi)\) is

\begin{equation}
\text{Tr}[\Theta(\eta, \xi)] = \sqrt{2\pi\hbar} \delta(\eta)\delta(\xi),
\end{equation}

and its integral is

\begin{equation}
\iint \frac{d\eta d\xi}{\sqrt{2\pi\hbar}} \Theta(\eta, \xi) = 2e^{-2a_{q,p}/\hbar},
\end{equation}

which is the parity operator [8], where the semicolon (\(\;\)) in the exponent has the same meaning as it does in equation (26) [equation (1)]. This is not shocking, because it offers another connection to the Wigner distribution which is the expectation value of the displaced parity operator [6].

Integrating the ambiguity seed over one variable, we get

\begin{align}
\int \frac{d\eta}{\sqrt{2\pi\hbar}} \Theta(\eta, \xi) &= \left| q = -\frac{\xi}{2}\right| \left| q = \frac{\xi}{2}\right| \\
\int \frac{d\xi}{\sqrt{2\pi\hbar}} \Theta(\eta, \xi) &= \left| p = \frac{\eta}{2}\right| \left| p = -\frac{\eta}{2}\right|. 
\end{align}

which means that

\begin{align}
\int \frac{d\eta}{\sqrt{2\pi\hbar}} A(\eta, \xi) &= \left| q = \frac{\xi}{2}\right| A \left| q = \frac{\xi}{2}\right| \\
\int \frac{d\xi}{\sqrt{2\pi\hbar}} A(\eta, \xi) &= \left| p = \frac{\eta}{2}\right| A \left| p = \frac{\eta}{2}\right|. 
\end{align}

**Composition rule.** Since the ambiguity seed operators are phase-space displacement operators, we would expect that they could be combined into a different displacement. This is true, however, up to a phase

\begin{equation}
\Theta(\eta_1, \xi_1)\Theta(\eta_2, \xi_2) = \frac{e^{i(\eta_1\xi_2 - \eta_2\xi_1)/2\hbar}}{\sqrt{2\pi\hbar}} \Theta(\eta_1 + \eta_2, \xi_1 + \xi_2),
\end{equation}

which comes from the Campbell–Baker–Hausdorff relation. However, when actually displacing an operator \(f(q, p)\), the overall phase factor in equation (49) cancels

\begin{equation}
\Theta(f(q, p))\Theta(\eta, \xi)/2\pi\hbar = \Theta(\eta_1, \xi_1)\Theta(\eta_2, \xi_2)f(q, p)\Theta(\eta_1, \xi_1)\Theta(\eta_2, \xi_2) = f(q + \xi, p + \eta)/(2\pi\hbar)^2,
\end{equation}

where here \(\eta = \eta_1 + \eta_2\) and \(\xi = \xi_1 + \xi_2\). Using equation (49), we see that the orthogonality property, equation (6), follows from equation (45).

\footnote{The ‘\(\frown\)’ is boldface, indicating that operator-ordering is important.}

\section{8. Conclusions}

We have studied the properties of the displacement operator and have shown that it forms a complete and orthogonal basis of operators. We have also shown how it could yield c-function distributions corresponding to quantum states and to quantum operators, and how those could be used for studying quantum mechanics in phase space, e.g. for calculation of expectation values or time evolution. Surprisingly, in contrast with the usual phase-space formalisms (Wigner, Kirkwood–Rihaczek, etc.), expectation values of (polynomial) operators do not involve any integration—only derivatives. We have shown that the ambiguity distribution possesses attractive features, such as symmetries under canonical quantum transformations. All properties of the c-functions come from the displacement operator \(\sqrt{2\pi\hbar}\Theta\).

There are prospects for generalizing this treatment using ideas from [5, 25].

\section{Acknowledgments}

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\section{Appendix A. The relationship between equations (2) and (3)}

To see that equations (2) and (3) are correct, we insert one into the other to check for consistency. First, To verify equation (3), we consider the position–momentum matrix elements of the arbitrary operator \(A\)

\begin{equation}
\langle x|A|k\rangle = \iint d\eta d\xi \iint dx'dk' \langle x'|A|k'\rangle \times \langle k'|\Theta(\eta, \xi)|x'\rangle \langle x|\Theta(-\eta', -\xi')|k\rangle,
\end{equation}

where we have inserted equation (2). Using equation (7), we find that equation (A1) is self-consistent

\begin{equation}
\langle x|A|k\rangle = \iint dx'dk' \langle x'|A|k'\rangle \delta(x - x')\delta(k - k').
\end{equation}

The final check of the consistency of equations (2) and (3) involves inserting equation (3) into equation (2)

\begin{equation}
A(\eta, \xi) = \text{Tr}\left\{\Theta(\eta, \xi) \iint d\eta' d\xi' A(\eta', \xi') \Theta(-\eta', -\xi')\right\}
\end{equation}

(A3)
\[ = \int \frac{d\eta}{2\pi} \int d\xi \, A(\eta', \xi') \text{Tr}[\Theta(\eta, \xi) \Theta(-\eta', -\xi')], \]  
\tag{A4}

and using equation (6), we find that equation (A4) is self-consistent

\[ A(\eta, \xi) = \int \frac{d\eta'}{2\pi} \int d\xi' \, A(\eta', \xi') \delta(\eta - \eta') \delta(\xi - \xi'). \]  
\tag{A5}

Equations (6) and (7) follow directly from the definition of the ambiguity seed operator \( \Theta \), equation (4). This is easiest to show by using the position–momentum and momentum–position matrix elements of \( \Theta \),

\[ \langle x | \Theta(\eta, \xi) | k \rangle = e^{i(kx + \eta/2)/\hbar} \frac{1}{2\pi\hbar} e^{i(\eta k + \xi)/\hbar} \]
\[ \langle k | \Theta(\eta, \xi) | x \rangle = \frac{e^{-i(kx + \eta/2)/\hbar}}{2\pi\hbar} e^{i(\eta k + \xi)/\hbar}, \]  
\tag{A6}

which come from the Campbell–Baker–Hausdorff relation [26]

\[ \sqrt{2\pi\hbar} \Theta(\eta, \xi) = e^{i\eta k/\hbar} e^{i\eta k/\hbar/2\hbar} = e^{i\eta k/\hbar} e^{i\eta k/\hbar} e^{-i\eta k/2\hbar}. \]  
\tag{A7}

we can obtain equations (6) and (7) in a straight-forward manner.

**Appendix B. The expectation value method**

Equation (5) is obtained from the same principles. Taking equation (2) for \( A \) and \( B \), we take the trace and use equation (6) to find that

\[ \text{Tr}[AB] = \int \int \frac{d\eta}{2\pi} \int d\xi \, A(\eta', \xi') B(\eta', \xi') \times \text{Tr}[\Theta(-\eta, -\xi) \Theta(-\eta', -\xi')] \]
\[ = \int \frac{d\eta}{2\pi} d\xi \, A(\eta, \xi) B(-\eta, -\xi), \]  
\tag{B1}

which is equation (5), where we have used equation (6).

**Appendix C. Relationship between Ambiguity and Wigner seeds**

In equation (18), we stated that the ambiguity function seed operator, equation (4) and the Wigner seed operator,

\[ W = 2e^{2i(p \cdot p' - q \cdot q')/\hbar}, \]  
\tag{C1}

(where the semicolon denotes Schwinger exponent ordering, as we mention in a footnote on the first page—see equation (1)) are related via Fourier transform. Here we provide the proof. Starting with equation (18) (reproduced here for clarity),

\[ \frac{1}{\sqrt{2\pi\hbar}} \int \frac{d\eta}{2\pi} e^{-i(2q \cdot q')/\hbar} \Theta(\eta, \xi) \]
\[ = \int \frac{d\eta}{2\pi\hbar} e^{i(p \cdot p' - \eta/2)/\hbar} e^{i(q \cdot q')/\hbar}, \]  
\tag{C2}

where we used equation (A7). Integrating over \( \xi \), we have

\[ 2 \int \frac{d\eta}{2\pi\hbar} \delta(2(p \cdot p' - q \cdot q')/\hbar) e^{i(q \cdot q')/\hbar} = 2e^{2i(p \cdot p' - q \cdot q')/\hbar} = W, \]  
\tag{C3}

which is the Wigner seed operator.

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