NOTES ON THE QUASI-GALOIS CLOSED SCHEMES

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Abstract. Let \( f : X \rightarrow Y \) be a surjective morphism of integral schemes. Then \( X \) is said to be quasi-galois closed over \( Y \) by \( f \) if \( X \) has a unique conjugate over \( Y \) in an algebraically closed field. Such a notion has been applied to the computation of étale fundamental groups. In this paper we will use affine coverings with values in a fixed field to discuss quasi-galois closed and then give a sufficient and essential condition for quasi-galois closed. Here, we will avoid using affine structures on a scheme since their definition looks copious and fussy.

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Introduction

The quasi-galois closed schemes are introduced and then discussed by affine structures in \cite{2}. That is, let \( X/Y \) be integral schemes with a surjective morphism \( f \). Then \( X \) is said to be quasi-galois closed (or \( qc \) for short) over \( Y \) by \( f \) if \( X \) has a unique conjugate over \( Y \) in an algebraically closed field. The \( qc \) schemes behave like quasi-galois extensions of fields and have many nice properties, which can be regarded as a generalization of pseudo-galois schemes in the sense of Suslin-Voevodsky (see \cite{9, 10}).

The notion of \( qc \) schemes introduced there is mainly for us to understand the étale fundamental group of a scheme. In deed, by \( qc \) schemes we have the computation of étale fundamental groups of schemes (see \cite{4, 6}) and get a splitting homotopy exact sequence of these profinite groups in the sense of Grothendieck (see \cite{4}). By \( qc \) covers of a scheme, we will also obtain a profinite group, a \( qc \) fundamental group of the scheme that contains the étale fundamental group as a normal subgroup (see \cite{5, 6}).

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However, the definition for affine structures on a scheme looks rather copious and fussy. In this paper we will avoid using affine structures to get the key property for \(qc\) schemes. Instead, we will use affine coverings with values in an algebraically closed field to discuss \(qc\) schemes and then obtain a sufficient and essential condition for a \(qc\) scheme.

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1. Preliminaries

1.1. Conventions. As usual, for an integral scheme \(X\), we let \(k(X)\) denote the function field of \(X\).

For an integral domain \(D\), let \(Fr(D)\) denote the field of fractions of \(D\). In particular, \(Fr(D)\) will be assumed to be contained in \(\Omega\) if \(D\) is contained in a field \(\Omega\).

1.2. Affine covering with values. Let \(X\) be a scheme. An affine covering of \(X\) is a family \(\mathcal{C}_X = \{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}\), where for each \(\alpha \in \Delta\), \(\phi_\alpha\) is an isomorphism from an open set \(U_\alpha\) of \(X\) onto the spectrum \(Spec A_\alpha\) of a commutative ring \(A_\alpha\).

Each \((U_\alpha, \phi_\alpha; A_\alpha)\) is called a local chart. For the sake of brevity, a local chart \((U_\alpha, \phi_\alpha; A_\alpha)\) will be denoted by \(U_\alpha\) or \((U_\alpha, \phi_\alpha)\).

An affine covering \(\mathcal{C}_X\) of \((X, \mathcal{O}_X)\) is said to be reduced if \(U_\alpha \neq U_\beta\) holds for any \(\alpha \neq \beta\) in \(\Delta\).

Let \(\mathcal{Comm}\) be the category of commutative rings with identity. For a given field \(\Omega\), let \(\mathcal{Comm}(\Omega)\) be the category consisting of the subrings of \(\Omega\) and their isomorphisms.

**Definition 1.1.** Let \(\mathcal{Comm}_0\) be a subcategory of \(\mathcal{Comm}\). An affine covering \(\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}\) of \(X\) is said to be with values in \(\mathcal{Comm}_0\) if for each \(\alpha \in \Delta\) there are \(\mathcal{O}_X(U_\alpha) = A_\alpha\) and \(U_\alpha = Spec(A_\alpha)\), where \(A_\alpha\) is a ring contained in \(\mathcal{Comm}_0\).

In particular, let \(\Omega\) be a field. An affine covering \(\mathcal{C}_X\) of \(X\) with values in \(\mathcal{Comm}(\Omega)\) is said to be with values in the field \(\Omega\).

**Remark 1.2.** The affine open subschemes of a given scheme are usually flexible and unspecified. Here, the field \(\Omega\) above enables the affine open subschemes of \(X\) to be measurable while the other open subschemes of \(X\) are still unmeasurable.
2. Definition for qc Schemes

Assume that $\mathcal{O}_X$ and $\mathcal{O}'_X$ are two structure sheaves on the underlying space of an integral scheme $X$. The two integral schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$ are said to be essentially equal provided that for any open set $U$ in $X$, we have

\[ U \text{ is affine open in } (X, \mathcal{O}_X) \iff \text{ so is } U \text{ in } (X, \mathcal{O}'_X) \]

and in such a case, $D_1 = D_2$ holds or there is $Fr(D_1) = Fr(D_2)$ such that for any nonzero $x \in Fr(D_1)$, either

\[ x \in D_1 \bigcap D_2 \]

or

\[ x \in D_1 \setminus D_2 \iff x^{-1} \in D_2 \setminus D_1 \]

holds, where $D_1 = \mathcal{O}_X(U)$ and $D_2 = \mathcal{O}'_X(U)$.

Two schemes $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are said to be essentially equal if the underlying spaces of $X$ and $Z$ are equal and the schemes $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}_Z)$ are essentially equal.

Let $X$ and $Y$ be integral schemes and let $f : X \rightarrow Y$ be a surjective morphism of finite type. Denote by $Aut(X/Y)$ the automorphism group of $X$ over $Y$.

A integral scheme $Z$ is said to be a conjugate of $X$ over $Y$ if there is an isomorphism $\sigma : X \rightarrow Z$ over $Y$.

Definition 2.1. $X$ is said to be quasi-galois closed (or qc for short) over $Y$ by $f$ if there is an algebraically closed field $\Omega$ and a reduced affine covering $C_X$ of $X$ with values in $\Omega$ such that for any conjugate $Z$ of $X$ over $Y$ the two conditions are satisfied:

- $(X, \mathcal{O}_X)$ and $(Z, \mathcal{O}_Z)$ are essentially equal if $Z$ has a reduced affine covering with values in $\Omega$.
- $C_Z \subseteq C_X$ holds if $C_Z$ is a reduced affine covering of $Z$ with values in $\Omega$.

There are many qc schemes. For example, let $t$ be a variable over $\mathbb{Q}$. Then $Spec(\mathbb{Z}[2^{1/2}, t^{1/3}])$ is qc over $Spec(\mathbb{Z})$, where we take $\Omega = \mathbb{Q}(2^{1/2}, t^{1/3})$.

Remark 2.2. Now let us give some notes on the algebraically closed field $\Omega$ in Definition 2.1.

(i) By $\Omega$, the rings of affine open sets in $X$ are taken to be as subrings of the same ring $\Omega$ so that they can be compared with each other.

(ii) By $\Omega$, we can restrict ourselves only to consider the function fields which have the same variables over a given field.
Remark 2.3. The affine covering $C_X$ of $X$ in Definition 2.1 is maximal by set inclusion, which is indeed the natural affine structure of $X$ with values in $\Omega$. Conversely, it can be seen that a qc integral scheme has a unique natural affine structure with values in $\Omega$ (see [2]).

Remark 2.4. It is seen that qc schemes behave always like quasi-galois extensions of fields (see [2]). In general, there are an infinite number of qc schemes over a given integral scheme. See [3, 4, 5] for the criterion and existence of qc schemes.

3. A Condition Equivalent to qc Schemes

Let $K$ be an extension of a field $k$. Note that here $K$ is not necessarily algebraic over $k$.

Definition 3.1. $K$ is said to be quasi-galois over $k$ if each irreducible polynomial $f(X) \in F[X]$ that has a root in $K$ factors completely in $K[X]$ into linear factors for any subfield $F$ with $k \subseteq F \subseteq K$.

Definition 3.2. $D_1$ is said to be a conjugation of $D_2$ over $D$ if there is an $F$–isomorphism $\tau : Fr(D_1) \rightarrow Fr(D_2)$ such that $\tau(D_1) = D_2$, where $F \triangleq k(\Delta)$, $k \triangleq Fr(D)$, $\Delta$ is a transcendental basis of the field $Fr(D_1)$ over $k$, and $F$ is contained in $Fr(D_1) \cap Fr(D_2)$.

Now let $X$ and $Y$ be two integral schemes and let $f : X \rightarrow Y$ be a surjective morphism of finite type. Fixed an algebraic closure $\Omega$ of the function field $k(Y)$.

Definition 3.3. A reduced affine covering $C_X$ of $X$ with values in $\Omega$ is said to be quasi-galois closed over $Y$ by $f$ if the condition below is satisfied:

There exists a local chart $(U'_\alpha, \phi'_\alpha; A'_\alpha) \in C_X$ such that $U'_\alpha \subseteq \varphi^{-1}(V_\alpha)$ holds for any $(U_\alpha, \phi_\alpha; A_\alpha) \in C_X$, for any affine open set $V_\alpha$ in $Y$ with $U_\alpha \subseteq f^{-1}(V_\alpha)$, and for any conjugate $A'_\alpha$ of $A_\alpha$ over $B_\alpha$, where $B_\alpha$ is the canonical image of $\mathcal{O}_Y(V_\alpha)$ in the function field $k(X)$ via $f$.

Definition 3.4. An affine covering $\{(U_\alpha, \phi_\alpha; A_\alpha)\}_{\alpha \in \Delta}$ of $X$ is said to be an affine patching of $X$ if the map $\phi_\alpha$ is the identity map on $U_\alpha = \text{Spec}A_\alpha$ for each $\alpha \in \Delta$.

Evidently, an affine patching is reduced.

Theorem 3.5. Let $X$ be qc over $Y$ by $f$ and let the function field $k(Y)$ be contained in $\Omega$. Then there is a unique maximal affine covering $C_X$ of $X$ with values in $\Omega$ such that $C_X$ is quasi-galois closed over $Y$ by $f$. 
Proof. Trivial. □

It follows that we have the following corollaries.

**Corollary 3.6.** Let $C_X$ and $\Omega$ be assumed as in *Definition 2.1*. Then there are the following statements.

- The field $\Omega$ is an algebraic closure of the function field $k(X)$.
- The affine covering $C_X$ is a unique maximal affine covering with values in $\Omega$ that is quasi-galois closed over $Y$ by $f$.

**Corollary 3.7.** Any qc integral scheme has a unique maximal affine structure with values in an algebraic closure of its function field (see [1, 2]).

Finally, we give a sufficient and essential condition for qc schemes.

**Condition 3.8.** Let $(W, O_W)$ be a scheme. For any open set $U$ in $W$, we identify the stalk of $O_W$ at a point $x \in U$ with the stalk of the restriction of $O_W$ to $U$ at the point $x$.

**Theorem 3.9.** Let $X, Y$ be integral schemes and let $f : X \to Y$ be a surjective morphism of finite type. Suppose that the function field $k(Y)$ is contained in $\Omega$. Then under Condition 3.6, the following statements are equivalent:

- The scheme $X$ is qc over $Y$ by $f$.
- There is a unique maximal affine patching $C_X$ of $X$ with values in $\Omega$ such that $C_X$ is quasi-galois closed over $Y$ by $f$.

*Proof.* It is immediate from Lemma 2.13 (in [4]) and Theorem 3.5 above. □

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