CONTRACTION PROPERTY OF DIFFERENTIAL OPERATOR ON FOCK SPACE

DAVID KALAJ

ABSTRACT. In the recent paper, [8] Nicola and Tilli proved the Faber-Krahn inequality, which for \( p = 2 \), states the following. If \( f \in F^2_\alpha \) is an entire function from the corresponding Fock space, then

\[
\frac{1}{\pi} \int_{\Omega} |f(z)|^2 e^{-\alpha |z|^2} dx dy \leq (1 - e^{-|\Omega|}) \| f \|^2_{2,\alpha}.
\]

Here \( \Omega \) is a domain in the complex plane and \( |\Omega| \) is its Lebesgue measure. This inequality is sharp and equality can be attained. We prove the following sharp inequality

\[
\int_{\Omega} \frac{|f^{(n)}(z)|^2 e^{-\alpha |z|^2}}{\pi^n n! L_n(\pi |z|^2)} dx dy \leq (1 - e^{-(n+1)|\Omega|}) \| f \|^2_{2,\alpha},
\]

where \( L_n \) is Laguerre polynomial, and \( n \in \{0, 1, 2, 3, 4\} \). For \( n = 0 \) it coincides with the result of Nicola and Tilli.

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1. INTRODUCTION

Let \( \mathbb{C} \) be the complex plane. Denote by \( dA(z) (= dx dy) \) the Lebesgue measure on the complex plane. Throughout the paper, we consider the Gaussian-probability measure

\[
d\mu_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha |z|^2} dA(z),
\]

where \( \alpha \) is a positive parameter. Also let \( dA_\alpha(z) = \frac{\alpha}{\pi} dA(z) \).

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For $1 \leq p < \infty$, let $L^p(\mathbb{C}, d\mu_\alpha)$ denote the space of all Lebesgue measurable functions $f$ on $\mathbb{C}$ such that
\[
\|f\|_{p,\alpha}^p = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\frac{p\alpha|z|^2}{2}} dA(z) < \infty.
\]

The Segal–Bargmann space also known as the Fock space, denoted by $\mathcal{F}^2_{\alpha}$, consists of all entire functions $f$ in $L^2(\mathbb{C}, d\mu_\alpha)$. For any $p \geq 1$, $\mathcal{F}_p^\alpha$ is a closed subspace of $L^p(\mathbb{C}, d\mu_\alpha)$. Therefore $\mathcal{F}_p^\alpha$ is a Banach space.

We refer to the book [9] for a good setting of the Fock space on the plane and the papers [2], [3] and [5] for a higher-dimensional setting.

Since $L^2(\mathbb{C}, d\mu_\alpha)$ is a Hilbert space with the inner product
\[
(f, g)_\alpha = \int_{\mathbb{C}} f(z)\overline{g(z)} d\mu_\alpha(z),
\]
the Fock space $\mathcal{F}^2_{\alpha}$ as its closed subspace determines a natural orthogonal projection $P_\alpha : L^2(\mathbb{C}, d\mu_\alpha) \to \mathcal{F}^2_{\alpha}$.

It can be shown (see [9]) that $P_\alpha$ is an integral operator induced by the reproducing kernel
\[
K_\alpha(z, w) = e^{\alpha z\overline{w}}.
\]

More precisely,
\[
P_\alpha f(z) = \int_{\mathbb{C}} K_\alpha(z, w) f(w) d\mu_\alpha(w), f \in L^2(\mathbb{C}, d\mu_\alpha),
\]
and in particular
\[
f(z) = \int_{\mathbb{C}} K_\alpha(z, w) f(w) d\mu_\alpha(w), f \in \mathcal{F}^2_{\alpha}.
\]

Laguerre polynomial is defined by
\[
L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k.
\]
For $n = 0$, $L_0(x) = 1$.

Let $F = _1F_1$ be the Kummer function defined by
\[
F(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k k!} z^k,
\]
where $(c)_k = c(c + 1) \cdots (c + k - 1)$ is denoted the shifter factoriel. The connection between Kummer function and Laguerre polynomial is given by
\[
F(1 + n, 1, r) = e^r L_n(-r).
\]

For $\alpha > 0$ and $n \geq 0$ an integer, we define the Hilbert space of entire functions $\mathcal{F}^2_{n,\alpha} = L^2(\mathbb{C}, d\mu_{n,\alpha})$, where
\[
d\mu_{n,\alpha}(z) = \frac{1}{n!\alpha^n L_n(-\alpha|z|^2)} d\mu_\alpha(z).
\]
In this paper we consider the differential operator \( D_n[f](z) = f^{(n)}(z) \), and show that it maps \( F^2_{\alpha} \) into \( F^2_{n,\alpha} \). Moreover, we show that it is a contraction satisfying the local contraction property.

We start with a result of Haslinger, author, and Vujadinovic in [6], which can be stated as the contraction property of differential operator from \( F^2_{\alpha} \) into \( F^\infty_{n,\alpha} \), where the last space is defined in a standard fashion. Note that the space \( F^\infty_{n,\alpha} \) is defined to be the space of all entire functions \( f \) such that

\[
\|f\|_{\infty,n,\alpha} := \alpha^{-n/2}n^{-1/2} \sup \{ |f(z)|e^{-\alpha|z|^2}L^{-1/2}_n(-\alpha|z|^2) : z \in \mathbb{C} \} < \infty.
\]

**Proposition 1.1.** Let \( f \in F^2_{\alpha} \). Then we have the sharp point-wise inequality

\[
|f^{(n)}(z)| \leq e^{\alpha|z|^2/2} \sqrt{\alpha^n n! L_n(-\alpha|z|^2)} \|f - T_n(f)\|_{2,\alpha},
\]

where \( n \in \mathbb{N}_0 \) and \( T_n(f)(w) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} w^k \), and \( L_n \) is Laguerre polynomial. The extremal functions are of the form

\[
f(w) = \sum_{k=0}^{n-1} a_k w^k + \alpha^n e^{\alpha \bar{w}w} w^n.
\]

In particular, for \( n = 1 \) we get the following best estimates

\[
|f'(z)| \leq \sqrt{\alpha (1 + \alpha|z|^2)} e^{\alpha|z|^2/2} \|f\|_{2,\alpha}, \quad z \in \mathbb{C}.
\]

In other words we have the sharp inequality

\[
\|f^{(n)}\|_{\infty,n,\alpha} \leq \|f\|_{2,\alpha}.
\]

Proposition 1.1 for \( n = 0 \) coincides with a corresponding result [9, Theorem 2.7] or [4, Theorem 3.4.2], and that result was crucial for proving the following important result in [8], by Nicola and Tilli.

**Theorem 1.2.** For \( f \in F^2_{\alpha} \), and a domain \( \Omega \) with a finite measure, we have

\[
\int_{\Omega} |f(z)|^2 e^{-\pi|z|^2} dxdy \leq (1 - e^{-|\Omega|}) \|f\|_{2,\alpha}^2.
\]

This result is sharp and equality is attained for certain balls and certain specific functions related to extremal functions in Proposition 1.1.

Theorem 1.2 is further extended in [8] for \( p \geq 2 \), and proved local Lieb’s uncertainty inequality for the STFT (short-time Fourier transforms see [8, Theorem 5.2]), by using Lieb’s uncertainty inequality for the STFT. Theorem 1.2 proves a conjecture of Abreu and Speckbacher [11, Conjecture 1] (for \( p = 2 \)). We extend Theorem 1.2 for higher derivatives. We believe that our results can be also applied to STFT, especially because of the Laguerre connection [2, Chapter 1]. By using the method developed in [8], Kulikov in [7] proved some conjectures for Bergman and Hardy space of holomorphic mappings. We also use the method developed in [8].

In this paper, we consider the following conjecture...
Conjecture 1.3. For \( f \in F_\alpha^2 \), and a domain \( \Omega \) with a finite measure, we have

\[
\int_{\Omega} |f^{(n)}(z)|^2 d\mu_{n,\alpha}(z) \leq (1 - e^{-(n+1)|\Omega|}) \|f\|^2_{2,\alpha},
\]

where \( n \geq 0 \).

We reduce Conjecture 1.3 to the following conjecture

Conjecture 1.4. For \( f \in F_\alpha^2 \), we have

\[
\|f^{(n)}\|^2_{2,n,\alpha} := \int_{C} |f^{(n)}(z)|^2 d\mu_{n,\alpha}(z) \leq \|f\|^2_{2,\alpha}.
\]

Further Conjecture 1.4 is reduced to the following

Conjecture 1.5. Let \( k \geq n \) and

\[
a_k = \frac{\Gamma[1+k]}{n!\Gamma[1+k-n]} \int_{0}^{\infty} \frac{e^{-r}r^{k-n}}{L_n(-r)} dr,
\]

where \( L_n \) is Laguerre polynomial, and \( n \geq 0 \).

Then \( a_k < 1 \).

Then we prove Conjecture 1.3 for \( n \leq 4 \). In other words, we prove

Theorem 1.6. For \( f \in F_\alpha^2 \), and a domain \( \Omega \subset \mathbb{C} \) with a finite Lebesgue measure \( |\Omega| \), we have

\[
\frac{\alpha}{\pi} \int_{\Omega} \frac{|f^{(n)}(z)|^2 e^{-\alpha|z|^2}}{\alpha^n n! L_n(-\alpha|z|^2)} dxdy \leq (1 - e^{-(n+1)|\Omega|}) \|f\|^2_{2,\alpha},
\]

where \( L_n \) is Laguerre polynomial, and \( 0 \leq n \leq 4 \). The inequality is sharp but is never attained. In particular for \( n = 1 \), we have

\[
\int_{\Omega} \frac{|f'(z)|^2 e^{-\alpha|z|^2}}{\alpha(1 + \alpha|z|^2)} dA_{\alpha}(z) \leq (1 - e^{-2|\Omega|}) \|f\|^2_{2,\alpha}.
\]

In other words the differential operator \( D_n : F_\alpha^2 \to F_{n,\alpha}^2 \), is a contraction and satisfies local contraction property on sub-domains of complex plane.

Remark 1.7. Note that for \( n = 0 \) the previous theorem is exactly the main result of Nicola and Tilli in [8]. Note also that \( f \in F_\alpha^2 \not\iff f' \in F_\alpha^2 \) (see a counterexample in [6]), so Laguerre polynomials are inevitable.

Corollary 1.8. Under conditions of Theorem 1.6 we have

\[
\int_{\Omega} \frac{|f'(z)|^2 e^{-\alpha|z|^2}}{\alpha(1 + \alpha|z|^2)} dA_{\alpha}(z) \leq (1 - e^{-2|\Omega|}) \|f\|^2_{2,\alpha}.
\]

where \( C \) is the space of all constants \( d_{\alpha} \) is the induced metric in the Fock space \( F_\alpha^2 \).
2. PROOF OF MAIN RESULTS

We need the following corollary of Proposition 1.1.

**Corollary 2.1.** If \( f \in F^{2}_{\alpha} \), then
\[
\lim_{|z| \to \infty} \frac{|f^{(n)}(z)|}{\alpha^n n! L_n(-\alpha|z|^2)} e^{-\alpha|z|^2/2} = 0.
\]
and in particular, for \( n = 1 \), we have
\[
\lim_{|z| \to \infty} \frac{|f'(z)|}{\alpha(1 + \alpha|z|^2)} e^{-\alpha|z|^2/2} = 0.
\]

**Proof:** Let us prove for \( n = 1 \) and note that the same proof works for every \( n \).
Since the space of polynomials is dense in the Fock space, for given \( \epsilon > 0 \), there is
a polynomial \( P \) so that
\[
\|f - P\|_{2,\alpha} \leq \epsilon.
\]
Then we have
\[
\|f\|_{2,\alpha} \leq \|f - P\|_{2,\alpha} + \|P\|_{2,\alpha}.
\]
From (1.2), we get
\[
|f'(z) - P'(z)| \leq \sqrt{\alpha (1 + \alpha|z|^2)} e^{\alpha|z|^2/2} \|f - P\|_{2,\alpha} \leq \sqrt{\alpha (1 + \alpha|z|^2)} e^{\alpha|z|^2/2} \epsilon.
\]
On the other hand
\[
\lim_{|z| \to \infty} \frac{|P'(z)|}{\alpha(1 + \alpha|z|^2)} e^{-\alpha|z|^2/2} = 0.
\]
So
\[
\limsup_{|z| \to \infty} \frac{|f'(z)|}{\alpha(1 + \alpha|z|^2)} e^{-\alpha|z|^2/2} \leq \epsilon.
\]
Thus
\[
\limsup_{|z| \to \infty} \frac{|f'(z)|}{\alpha(1 + \alpha|z|^2)} e^{-\alpha|z|^2/2} = 0
\]
as it is stated. \( \square \)

**Lemma 2.2.** Let
\[
u(z) = \frac{|f^{(n)}(z)|^2 e^{-\alpha|z|^2}}{\alpha^n n! L_n(-\alpha|z|^2)}
\]
and \( A_t = \{ z : \nu(z) > t \} \). If \( |\Omega| = s = \mu(t) \), then
\[
\int_{\Omega} \nu(z) dx dy \leq \int_{A_t} \nu(z) dx dy.
\]

**Proof of Lemma 2.2** As in [8], let \( \Omega_1 = \Omega \cap A_t \). Then \( \Omega = \Omega_1 \cup (\Omega \setminus \Omega_1) \) and
\( A_t = \Omega_1 \cap (A_t \setminus \Omega_1) \). We also have
\[
|\Omega \setminus \Omega_1| = |\Omega| - |\Omega_1| = |A_t| - |\Omega_1| = |A_t \setminus \Omega_1|.
\]
If \( z \in (\Omega \setminus \Omega_1) \), then \( \nu(z) \leq t \) and if \( w \in A_t \setminus \Omega_1 \) then \( \nu(w) > t \).
Thus
\[
\int_{\Omega} u(z) dxdy = \int_{\Omega_1} u(z) dxdy + \int_{\Omega \setminus \Omega_1} u(z) dxdy
\]
\[
\leq \int_{\Omega_1} u(z) dxdy + s \int_{\Omega \setminus \Omega_1} dxdy
\]
\[
= \int_{\Omega_1} u(z) dxdy + s \int_{A \setminus \Omega_1} dxdy
\]
\[
\leq \int_{\Omega_1} u(z) dxdy + \int_{A \setminus \Omega_1} u(z) dxdy
\]
\[
= \int_{A} u(z) dxdy.
\]

\[\square\]

2.1. Proof Theorem 1.6 for \( n = 1 \). We need the following general global estimate.

**Lemma 2.3.** Assume that \( \alpha > 0 \) and that \( f \in F^2_{\alpha} \). Then
\[
\frac{1}{\pi} \int_{\mathbb{C}} \frac{|f'(z)|^2 e^{-\alpha|z|^2}}{1 + \alpha|z|^2} dxdy \leq \|f\|_{2,\alpha}^2.
\]

**Proof.** Since \( f \) is an entire holomorphic functions, it has a Taylor representation:
\[
f(w) = \sum_{k=0}^{\infty} a_k w^k.
\]

Then
\[
f'(w) = \sum_{k=1}^{\infty} k a_k w^{k-1}.
\]

Thus
\[
\|f\|_{2,\alpha}^2 = \sum_{k=0}^{\infty} |a_k|^2 \frac{k!}{\alpha^k}.
\]

Moreover
\[
\frac{\alpha}{\pi} \int_{\mathbb{C}} \frac{|f'(z)|^2 e^{-\alpha|z|^2}}{\alpha(1 + \alpha|z|^2)} dxdy = 2\alpha \sum_{k=1}^{\infty} k^2 |a_k|^2 \int_{0}^{\infty} \frac{r^{2(k-1)+1} e^{-\alpha r^2}}{\alpha(1 + \alpha r^2)} dr
\]
\[
= \sum_{k=1}^{\infty} k^2 |a_k|^2 e E_k \frac{\Gamma(k)}{\alpha^k}
\]
\[
= \sum_{k=0}^{\infty} g_k |a_k|^2 \frac{k!}{\alpha^k},
\]

where
\[
E_k = E_k(1) = \int_{1}^{\infty} \frac{e^{-t}}{t^k} dt,
\]
and \( g_k = keE_k \). We need the following

**Claim.** For every \( k \geq 1 \) we have \( g_k < 1 \). Moreover

(2.1) \[ \lim_{k \to \infty} g(k) = 1. \]

By using partial integration we get

\[ g_{k+1} = \frac{k+1}{k^2} (k - g_k). \]

Further \( \frac{1}{2} \leq g_1 = 0.596347 < 1 \). Now we prove by induction that

\[ \frac{k}{k+1} < g_k < 1 \]

for \( k \geq 2 \). In order to do so let

\[ h(y) := \frac{k+1}{k^2} (k - y). \]

Then it is clear \( h\left(\frac{k}{k+1}\right) = 1 \) and

\[ h(1) = \frac{k^2 - 1}{k^2} \geq \frac{k+1}{k+2}. \]

Thus the claim follows.

Since \( ekE_k < 1 \), we obtain that

\[ \frac{1}{\pi} \int_{\mathbb{C}} \frac{|f'(z)|^2 e^{-\alpha |z|^2}}{1 + \alpha |z|^2} \, dx \, dy \leq \|f\|^2_{2,\alpha}, \]

as claimed.

\[ \square \]

**Proof of Theorem 1.6 for \( n = 1 \).** Let

\[ u(z) = \frac{|f'(z)|^2 e^{-\alpha |z|^2}}{\alpha (1 + \alpha |z|^2)} \]

and define \( \mu(t) = |A_t| \). Then by Corollary 2.1 \( A_t = \{ z : u(z) > t \} \) is compactly supported in \( \mathbb{C} \) and has a smooth boundary \( \partial A_t = \{ z : u(z) = t \} \) for almost every \( t \in (0, \max u) \). Also \( \lim_{|z| \to \infty} u(z) = 0 \) uniformly provided that \( f \in L^2(\mathbb{C}, d\mu_\alpha) \), where

\[ d\mu_\alpha = \frac{\alpha}{\pi} e^{-\alpha |z|^2} dx \, dy. \]

Let, \( s = \mu(t) \) and define

\[ I(s) = \frac{\alpha}{\pi} \int_{A_t} u(z) \, dx \, dy. \]

Then as in [8] Lemma 3.4] we can prove that \( I \) is smooth and

\[ I'(s) = \mu^{-1}(s). \]

In view of Lemma 2.2 we need to prove that

\[ I(s) \leq (1 - e^{-2|A_t|}) \|f\|_{2,\alpha}. \]
Assume as we may that $\|f\|_{2,\alpha} = 1$. Further we have $|A_t| = s$. Therefore we need to show that

$$I(s) \leq (1 - e^{-2s}).$$

Then by using the Cauchy inequality to the length of $\partial A_t$ we obtain

$$|\partial A_t|^2 \leq \int_{\partial A_t} |\nabla u|^{-1} ds \cdot \int_{\partial A_t} |\nabla u| ds.$$

Further by [8, Lemma 3.2] we have

$$\int_{\partial A_t} |\nabla u|^{-1} ds = -\mu'(t).$$

Let $\nu$ be the out-pointing unit normal at the smooth curve $\partial A_t$. Note that this curve is smooth for almost every $t > 0$. This can be shown by using a similar approach as in [8]. Now

$$|\nabla u| = -\langle \nabla u, \nu \rangle = -t \langle \nabla \log u, \nu \rangle,$$

for $z \in \partial A_t = \{ z : u(z) = t \}$. Thus

$$\int_{\partial A_t} |\nabla u| ds = -t \int_{\partial A_t} \langle \nabla \log u, \nu \rangle ds.$$

By Green theorem we have

$$\int_{\partial A_t} \langle \nabla \log u, \nu \rangle ds = \int_{A_t} \Delta \log u dA(z)$$

Since

$$\Delta \log u = -4\alpha \left( 1 + \frac{1}{(1 + \alpha|z|^2)^2} \right),$$

because $f'(z)$ is nonvanishing holomorphic function on $A_t$, we obtain

$$\int_{\partial A_t} \langle \nabla \log u, \nu \rangle ds = 4t\alpha \int_{A_t} \left( 1 + \frac{1}{(1 + \alpha|z|^2)^2} \right) dA(z)$$

$$= 4t\pi \int_{A_t} \left( 1 + \frac{1}{(1 + \alpha|z|^2)^2} \right) dA(z)$$

$$\leq 8t\pi |A_t|.$$
is increasing, because 

$$Q'(s) = e^{2s}(2q(s) + q'(s)) \geq 0.$$ 

Therefore 

$$G(\sigma) = I(-1/2 \log \sigma)$$ 

is a convex function. Namely 

$$G'(\sigma) = -\frac{\mu^{-1}(-1/2 \log \sigma)}{2\sigma} = -e^{2s}q(s) = -Q(s),$$ 

where \( s = e^{-2\sigma} \). So 

$$G''(\sigma) = 2Q'(s)e^{-2\sigma} > 0.$$ 

Moreover \( G \) satisfies the conditions \( G(0) \leq 1 \) and \( G(1) = 0 \). The first condition is satisfied because of Lemma 2.3. Namely 

$$G(0) = I(+\infty) = \int_{A_0} u(z) dA_\alpha \leq 1.$$ 

On the other hand 

$$G(1) = I(0) = \int_{A_\infty} u(z) dA_\alpha = 0.$$ 

By using the convexity of \( G \) we have \( G(\sigma) \leq (1 - \sigma) \) and this implies the inequality 

(2.3) 

$$I(s) \leq (1 - e^{-2s}).$$ 

To prove the sharpness, let \( f(z) = z^m \). Then in view of (2.1) we have 

$$\lim_{R \to \infty, m \to \infty} J(R, m) = 1,$$ 

where 

$$J(R, m) = \frac{\int_{|z| < R} |f'(z)|^2 e^{-\alpha |z|^2} \, dx \, dy}{\int_{|z| < R} (1 - e^{-2\pi R^2}) \|f\|_{2, \alpha}^2}.$$

\[ \square \]

2.2. The proof of Theorem 1.6 for \( n \geq 2 \). The proof is just an imitation of the case \( n = 1 \), so we omit some details. We begin by the following lemma

Lemma 2.4. Let \( L(x, y) = \log F[1 + n, 1, \alpha(x^2 + y^2)] \), where \( F \) is Kummer function. Then 

$$\Delta L(x, y) \leq 4\alpha(1 + n).$$ 

Proof. First of all, we have \( \Delta L(x, y) = g(t) \) where \( t = (x^2 + y^2) \) and 

$$g(t) = \frac{g(t)}{2(1 + n)\alpha} F[1 + n, 1, \alpha t]^2 = -2(1 + n)\alpha t F[2 + n, 2, \alpha t]^2$$

$$+ F[1 + n, 1, \alpha t] (2F[2 + n, 2, \alpha t] + (2 + n)\alpha t F[3 + n, 3, \alpha t]).$$

We need to prove that \( g(t) \leq g(0) = 4\alpha(1 + n) \). It is equivalent with the following 

$$k(t) = g(t)F[1 + n, 1, \alpha t]^2 - 4\alpha(1 + n)F[1 + n, 1, \alpha t]^2 \leq 0.$$
Then after some straightforward computations we get
\[ k(t) = -4\alpha^2 ntF[1 + n, 2, \alpha t](F[1 + n, 1, \alpha t] + nF[1 + n, 2, \alpha t]) \]
which is certainly a non-positive function. This finishes the proof of the lemma.

Now define
\[ J(s) = \frac{\alpha}{\pi} \int_{A_t} u_n(z) dxdy, \]
where
\[ u_n = \frac{|f^{(n)}(z)|^2 e^{-\alpha|z|^2}}{\alpha^n n! L_n(-\alpha|z|^2)}. \]
Now instead of (2.2), we use
\[ -t \Delta u \leq 4\alpha(1 + n), \]
and instead of (2.3) we have
\[ J_n(s) \leq (1 - e^{-ms}), \]
provided that Conjecture 1.5 is correct. This finishes the proof of Theorem 1.6 for the case \( n \geq 2 \), up to Lemma 2.5 below. The sharpness part can be proved in a similar way as in the case \( n = 1 \), but using, in this case, the equation (2.4) below.

**Lemma 2.5.** For \( f \in \mathcal{F}_\alpha^2 \), we have the following sharp inequality
\[ I_n(f) := \frac{\alpha}{\pi} \int_{C} \frac{|f^{(n)}(z)|^2 e^{-\alpha|z|^2}}{\alpha^n n! L_n(-\alpha|z|^2)} dxdy \leq \| f \|^2_{2,\alpha}, \]
provided that Conjecture 1.5 is true.

**Proof.** Since
\[ f^{(n)}(w) = \sum_{k=n}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-n+1)} a_k w^{k-n}, \]
and because the polynomials \( w^n \) and \( w^m \) are orthogonal the Fock space for \( n \neq m \), it is clear that it is enough to prove the inequality for such functions. So we need to show that
\[ I_n(w^k) \leq \| w^k \|_{2,\alpha}^2. \]
Let \( k \geq n \) and
\[ a_k = \frac{I_n(w^k)}{\| w^k \|_{2,\alpha}^2}. \]
Then after straightforward computation we get
\[ I_n(w^k) = \frac{a^{-k}\Gamma(1 + k)}{n!\Gamma^2(1 + k - n)} \int_0^\infty e^{-s} \frac{s^{k-n}}{L_n(-s)} ds \]
and
\[ \| w^k \|_{2,\alpha}^2 = \alpha^{-k}k!. \]
Thus
\[ a_k = \frac{\Gamma[1 + k]}{n!\Gamma[1 + k - n]^2} \int_0^\infty \frac{e^{-r}r^{k-n}}{L_n(-r)} dr. \]
We need to show that
(1) \( \lim_{k \to \infty} a_k = 1. \)
We will prove the first assertion, while the second is proved for few small integers. This is the content of the following lemma.

**Lemma 2.6.** Let \( p \geq n \) and

\[
A(p) = \frac{\Gamma[1 + p]}{n!\Gamma[1 + p - n]^2} \int_{0}^{\infty} \frac{e^{-r} r^{p-n} \ln(1 + r)}{L_{n}(-r)} dr,
\]

where \( L_{n}(-r) \) is the Laguerre polynomial. Then

\[
\lim_{p \to \infty} A(p) = 1,
\]

and if \( n \leq 4 \), then \( A(p) < 1 \).

**Proof.** First of all we have

\[
L_{n}(-r) = \sum_{m=0}^{n} \frac{\Gamma(1 + n)}{\Gamma(1 + m)^2 \Gamma(1 + n - m)} r^m.
\]

Fix \( k \) and let \( p = k + n \). It follows from the definition of \( A(h + k), h \in \{0, \ldots, n\} \) that

\[
\sum_{h=0}^{n} \frac{A(h + k) \Gamma(1 + h + k - n)^2}{\Gamma(1 + h)^2 \Gamma(1 + h + k) \Gamma(1 + n - h)} = \frac{\Gamma(1 + k - n)}{(n!)^2}.
\]

In order to prove \( \lim_{k \to \infty} A(k + n) = 1 \), observe that, in view of (2.5), for every \( n \) we have

\[
A(k + n) = C(k, n) - \sum_{h=0}^{n-1} C_h(k, n) A(k + h),
\]

where

\[
C(k, n) = \frac{(k - n)!(k + n)!}{(k!)^2}
\]

and

\[
C_h(k, n) = \frac{((h + k - n)!(n!)^2(k + n)!}{(h!)^2(k!)^2(h + k)!(-h + n)!}.
\]

Then we easily conclude that

\[
A(k + n) < C(k, n) \leq \frac{(2n)!}{(n!)^2}.
\]

So \( A(k + n) \) is bounded for fixed \( n \). The same hold for \( A(k + h) \) for \( h \leq n \). Namely for \( k \geq 2n \),

\[
A(k + h) \leq \frac{(2h)!}{(h!)^2} \leq \frac{(2n)!}{(n!)^2}.
\]

Further we have

\[
\lim_{k \to \infty} C(k, n) = 1
\]

and

\[
\lim_{k \to \infty} C_h(k, n) = 0.
\]
This and (2.6) implies that

\[
\lim_{k \to \infty} A(k + n) = 1.
\]

Let us now prove \( A(k) < 1 \) for first few integers \( n \). For \( n = 2 \) we have

\[
A(2 + k) = \frac{(2 + k)(-k + k^3 - 2(1 + k)A(k) - 4(-1 + k)^2A(1 + k))}{(-1 + k)^2k^2}.
\]

Let

\[
B(x, y) = \frac{(2 + k)(-k + k^3 - 2(1 + k)x - 4(-1 + k)^2y)}{(-1 + k)^2k^2}.
\]

Further \( A(2) = 0.427393, A(3) = 0.662691, A(4) = 0.784876, A(5) = 0.852841, A(6) = 0.893707, A(7) = 0.919951 \). So we get \( A(j) < 1 \) for \( j \leq 7 \) and \( a(j) > j/(j + 1) \) for \( 4 \leq j \leq 7 \). Now we use mathematical induction to prove that \( \frac{m}{m+1} \leq A(m) < 1 \) for \( m > 7 \). Let \( m = k > 7 \) and assume that \( \frac{m}{m+1} \leq A(m) < 1 \) and \( \frac{m+1}{m+2} \leq A(m+1) < 1 \). Then \( A(m+2) \geq B(1, 1) \), and

\[
(1 + m)^2m^2(2 + m)(3 + m)(-18 + m(9 + (-8 + m)m)) \geq 0,
\]

and the last inequality is true for every \( m \geq 8 \). Moreover

\[
A(m+2) \leq B(m/(m+1), (m+1)/(m+2))
\]

\[
= \frac{(2 + m)
(3m + m^3 - 4(1+m)^2(1+m))}{(-1 + m)^2m^2}
\]

\[
< 1,
\]

because the last inequality is equivalent with

\[
(1 + m)^2m^2(2 + m) > 0.
\]

For \( n = 3 \), we have

\[
A(3 + k) = \frac{(1 + k)(2 + k)(3 + k)}{(-2 + k)(-1 + k)k} - \frac{(1 + k)(2 + k)(3 + k)}{(-2 + k)(-1 + k)k}
\]

\[
\times \left(\frac{6A(k)}{(-2 + k)(-1 + k)k} + \frac{18(-2 + k)A(1 + k)}{(-1 + k)k(1 + k)} + \frac{9(-2 + k)(-1 + k)A(2 + k)}{k(1 + k)(2 + k)}\right).
\]

By proceeding similarly as above, we can prove that \( \frac{k}{k+1} \leq A(k) < 1 \) for \( k \geq 14 \), and that \( A(k) < 1 \) for \( k \in \{1, \ldots, 13\} \).
For $n = 4$, we have

\[(2.7)\]

\[
A(4 + k) = X - \left( \frac{24XA(k)}{(−3 + k)(−2 + k)(−1 + k)k} \right)
\]

\[
+ \frac{96X(−3 + k)A(1 + k)}{(−2 + k)(−1 + k)k(1 + k)} + \frac{72X(−3 + k)(−2 + k)A(2 + k)}{(−1 + k)k(1 + k)(2 + k)}
\]

\[
+ \frac{16X(−3 + k)(−2 + k)(−1 + k)A(3 + k)}{k(1 + k)(2 + k)(3 + k)}
\]

where

\[
X = \frac{(1 + k)(2 + k)(3 + k)(4 + k)}{(−3 + k)(−2 + k)(−1 + k)k}.
\]

Then, by using Mathematica software we obtain $A(j) < j/(j + 1)$, $j = 1, \ldots, 16$ but $A(j) > j/(j + 1)$ for $j = 17, \ldots, 27$. By induction, and previous formula we obtain that $j/(j + 1) < A(j) < 1$, for $j > 27$. Namely, LHS of (2.7), for $A(j) ≡ 1$ is equal to $(4+k)^{(-13680+k(11316+k(−5464+k(1921+k(-520+k(106+(16+k)k)))))))$, and the last expression is bigger than $(k + 4)/(k + 5)$ if and only if $k \geq 28$. Further the LHS of (2.7), for $A(j) = j/(j + 1)$ is equal to $-52832+k(14544+k(-2016+k(180+k(453+k(-216+k(58+(−12+k)k)))))))$ and the last expression is smaller than 1 for every $k$.

\[\square\]

**Remark 2.7.** Similar to the proof of the Lemma 2.5, one can proceed to prove the statement for some constants greater than 4, but this still does not solve the general problem. We believe that the recurrent formula (2.6) is suitable for proving the general case, and we also believe that (2.6) implies $j/(j + 1) < A(j) < 1$ for $j > n^2$. This and some good estimations of $A(j)$ for $j \leq n^2$ would conclude the proof of the general case, but we fail to do it.

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**References**

[1] L. D. Abreu and M. Speckbacher. **Donoho-Logan large sieve principles for modulation and polyanalytic Fock spaces.** Bulletin des Sciences Mathematiques, 10.1016/j.bulsci.2021.103032, 2021, [arXiv:1808.02258](http://arxiv.org/abs/1808.02258).

[2] G. B. Folland. **Harmonic analysis in phase space.** Ann. of Math. Stud., 122, Princeton University Press, Princeton, 1989.

[3] B. C. Hall, **Holomorphic methods in analysis and mathematical physics.** Contemp. Math., 260, 1-59 (1999).

[4] K. Gröchenig, **Foundations of time-frequency analysis.** Applied and Numerical Harmonic Analysis. Boston, MA: Birkhäuser. xv, 359 p. (2001).

[5] S. Janson, J. Peetre, R. Rochberg, **Hankel forms and the Fock space.** Rev. Mat. Iberoam., 3 n. 1, 61-138 (1987).

[6] F. Haslinger, D. Kalaj, David, Dj. Vujadinovic, **Sharp pointwise estimates for Fock spaces.** Comput. Methods Funct. Theory 21, No. 2, 343-359 (2021).
[7] A. Kulikov, *Functionals with extrema at reproducing kernels*, [arXiv:2203.12349](https://arxiv.org/abs/2203.12349) to appear in GAFA.

[8] F. Nicola, P. Tilli, *The Faber-Krahn inequality for the short-time Fourier transform*. Invent. math. (2022). https://doi.org/10.1007/s00222-022-01119-8.

[9] K. Zhu, *Analysis on Fock Spaces*, Graduate Texts in Mathematics, 263, Springer, New York 2012.

University of Montenegro, Faculty of Natural Sciences and Mathematics, Cetinjski put b.b. 81000 Podgorica, Montenegro

Email address: davidk@ucg.ac.me