Non-Commutative GUTs, Standard Model and $C, P, T$

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Abstract

Noncommutative Yang-Mills theories are sensitive to the choice of the representation that enters in the gauge kinetic term. We constrain this ambiguity by considering grand unified theories. We find that at first order in the noncommutativity parameter $\theta$, $SU(5)$ is not truly a unified theory, while $SO(10)$ has a unique noncommutative generalization. In view of these results we discuss the noncommutative SM theory that is compatible with $SO(10)$ GUT and find that there are no modifications to the SM gauge kinetic term at lowest order in $\theta$.

We study in detail the reality, hermiticity and $C, P, T$ properties of the Seiberg-Witten map and of the resulting effective actions expanded in ordinary fields. We find that in models of GUTs (or compatible with GUTs) right-handed fermions and left-handed ones appear with opposite Seiberg-Witten map.
1 Introduction

Noncommutative (NC) space-time and in general noncommutative geometry seem a natural arena where to study a quantum theory of general relativity; however one does not need to invoke quantum gravity to motivate noncommutative space-time models. In M-theory and in open string theory, in the presence of a nonvanishing NS $B$ field, the effective physics on D-branes can be described by a noncommutative gauge theory [1]. Here the source of noncommutativity is the two-form $B$. The easiest and most studied example is the case of constant $B$, this induces the following noncommutativity $[x^\mu \star x^\nu] \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = i\theta^{\mu\nu}$ with $\theta^{\mu\nu}$ constant (and depending on $B$). $\theta^{\mu\nu}$ fixes directions in space-time. With respect to fixed $\theta^{\mu\nu}$ we see that the Lorentz group is broken in a spontaneous way; in a bigger theory where the $B$ field is dynamical and not frozen to a constant value we have Lorentz covariance. Also, at low energies ($E^2\theta^{\mu\nu} \ll 1$) Lorentz symmetry should be recovered. The product $\star$ is the Moyal star product. On functions $f, g$ we have $f \star g = f e^{i\sum_\mu \theta_{\mu\nu} \partial_{\mu} \partial_{\nu}} g$.

Recently there has been a lot of interest in the study of realistic particle models based on the $[x^\mu \star x^\nu] = i\theta^{\mu\nu}$ noncommutative space-time [2–4]. The general idea being that a noncommutative space-time structure is not necessarily a Planck length phenomenon. Bounds on the noncommutative scale from collider physics can be as low as a few TeV’s [5] and it is therefore interesting to compare the theoretical predictions of these models [5,6] with near future experiments. Bounds from low energy physics, in particular from clock comparison experiments are in general much higher [7]. These bounds must be interpreted with care, however, for several reasons: 1) Due to the nature of the experiments, the bound concerns the spatial and temporal average of the noncommutativity tensor $\theta^{\mu\nu}$. Non-constant (slowly varying) components of $\theta$ may not be directly affected by the bound. 2) The bounds are based on loop calculations in noncommutative field theory and in particular in NC QCD. These theories are so far not well understood as full fledged quantum theories. There may, e.g., be additional terms in the quantum action (that are consistent with the symmetries) and whose coefficients (rather than the overall noncommutativity scale) are bounded by the experiments. Among the different NC (star product) generalization of the Standard Model (SM) ([2, 3]), the one in [3] is the only one based directly on the SM gauge group $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$. It also has the same particle content as the ordinary SM. The major new aspect of [3], with respect to the ordinary SM, is the appearence of new $\theta$-dependent interactions that are dictated by requiring that both noncommutative and commutative gauge transformations are a symmetry of the action. The construction of this model is based on Seiberg-Witten map (SW map) between commutative and noncommutative gauge transformations and fields. SW map was initially introduced for $U(N)$ gauge fields in [9], in the context of open string theory (and the zero slope limit $\alpha' \to 0$ [9]). It has then been studied in the case.

\footnote{See also [8] for a way to avoid the bounds.}
of arbitrary gauge groups [10, 11, 13–16] and space-time dependent parameters \( \theta^{\mu\nu} \) [17].

The SW map and the \( \star \) product allow us to expand the noncommutative action order by order in \( \theta \) and to express it in terms of ordinary commutative fields, so that one can then study the property of this \( \theta \)-expanded commutative action. It turns out that given a commutative YM theory, SW map and commutative/noncommutative gauge invariance are in general not enough in order to single out a unique noncommutative generalization of the original YM theory. One can follow different criteria in order to select a specific noncommutative generalization. We here focus on a classical analysis, in particular imposing the constraint that the noncommutative generalization of the Standard Model should be compatible with noncommutative GUT theories. Another issue would be to single out a noncommutative SM or GUT that is well behaved at the quantum level.

We refer to the problems relative to renormalization and chiral gauge anomalies. For example \( \theta \)-expanded massless QED is not renormalizable [19], however the photon self energy is renormalizable to all orders [18], and, adding just one extra fermion – gauge boson interaction, one obtains one-loop renormalizability of the full theory [20]. Chiral anomalies in the context of \( \theta \)-expanded actions (as far as we know) have not yet been studied (see however [23]). The analysis in the case of \( U(N) \) gauge theories (no SW map, no \( \theta \)-expansion) shows that chiral theories are not anomaly free [21, 22].

In this paper we first present a general study of the ambiguities that are present when constructing NCYM theories. We then see that at first order in \( \theta \) there is no ambiguity in \( SO(10) \) NCYM theory. In particular no triple gauge bosons coupling of the kind \( \theta FFF \) (indices arbitrarily contracted) is present. We further study the noncommutative SM compatible with \( SO(10) \): it is constructed using just left handed fermions and antifermions, so that adding a left handed antineutrino \( (\bar{\nu}_L = -i\sigma_2 \nu_R^*) \) one obtains the \( SO(10) \) chiral fermion multiplet. If (as it is natural) one considers just the adjoint representation \( \rho_{adj} \) and the fermion representation \( \rho_f \) in this SM noncommutative gauge kinetic term \( \sum \rho \text{Tr}(\rho(\hat{F})\rho(\hat{F})) \), then here too no \( \theta FFF \) term is present. This is not the case for the NCSM in [3] because there the non-chiral vector \( \Psi' = (u_L^i, d_L^i, u_R^i, d_R^i, \nu_L, e_L^-, e_R^-) \) is considered.

We next study the reality, hermiticity, charge conjugation, parity and time reversal properties of the SW map and of \( \theta \)-expanded NCYM theories. This constraints the possible freedom in the choice of a “good” SW map. There are in principle two choices for the result of the combined charge parity and time reversal transformation on the NC algebra (the star product): the CPT operator maps the star product to the opposite star product (with \( -\theta \) in place of \( \theta \)); alternatively one can define a \( \text{cpt} \) transformation that leaves the \( \star \) product invariant. In [24] the \( C, P, T \) properties of NCQED were studied assuming the usual \( C, P \) and \( T \) transformations also for noncommutative fields. Here we show that the usual \( C, P, T \) transformation on commutative spinors and nonabelian gauge potentials imply, via SW map, the same \( C, P, T \) transformations for the noncommutative spinors and gauge potentials. We also see that \( CPT \) is always a symmetry of
noncommutative actions. The $CPT$ operator is compatible with the SW map. The $cpt$ operator on the other hand maps the SW map to the opposite SW map. It is also not difficult to construct NCYM actions that are even under $\theta \rightarrow -\theta$ and thus invariant under this $cpt$ transformation.

The reality property of the SW map is used to analyze the difference between the SM in [3] and the GUT inspired SM proposed here. It is a basic one, and can be studied also in a QED model. While in [3], and in general in the literature, left and right handed components of a noncommutative spinor field are built with the same SW map, we here use and advocate a different choice: if noncommutative left handed fermions are built with the $+\theta$ SW map then their right handed companions should be built with the $-\theta$ SW map; this implies that both noncommutative $\psi_L$ and $\psi^{C_L} \equiv -i\sigma_2 \psi^*_R$ are built with the $+\theta$ SW map. In other words, with this choice, noncommutativity does not distinguish between a left handed fermion and a left handed antifermion, but does distinguish between fermions with different chirality. This appears to be the only choice compatible with GUT theories.

The paper is organized as follows. In Section 2 we construct and discuss the ambiguities in noncommutative $SO(10)$, $SU(5)$ and $U(1)_Y \otimes SU(2)_L \otimes SU(3)_C$ YM theories with fermion matter. We then study the Higgs sector of the noncommutative SM and the Higgs sector of $SO(10)$. In sections 3 and 4 we study the reality, hermiticity and $C,P,T$ properties of NCYM actions. In Subsection 4.1 the difference between actions built respectively with the $+\theta$ and the $-\theta$ choice for right handed fermions is described. In Section 5 we see that if $\theta$ properly transforms under $C,P,T$, then NCYM actions have the same $CP$ and $T$ symmetries as their corresponding commutative ones. The $cpt$ transformation is then considered. In the Appendix a general expression of the SW map at first order in $\theta$ is given; tensor products of noncommutative gauge transformations are also considered.

## 2 Building NCYM theories

Consider an ordinary “commutative” YM action with gauge group $G$, where $G$ is a compact simple Lie group, and one fermion multiplet $\Psi$

$$S = \int d^4 x \frac{1}{2g^2} \text{Tr}(F_{\mu \nu} F^{\mu \nu}) + \overline{\Psi} \gamma \partial \Psi$$

This action is gauge invariant under

$$\delta \Psi = i \rho_\Psi (\Lambda) \Psi$$

3
where $\rho_\Psi$ is the representation of $G$ determined by the multiplet $\Psi$. Following [11] the noncommutative generalization of (1) is given by

$$\hat{S} = \int d^4x \frac{-1}{2g^2} Tr(\hat{F}_{\mu\nu} \ast \hat{F}^{\mu\nu}) + \overline{\Psi} \ast \hat{i} \hat{D} \Psi$$  \hspace{1cm} (3)$$

where the noncommutative field strength $\hat{F}$ is defined by

$$\hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i[\hat{A}_\mu \ast \hat{A}_\nu],$$  \hspace{1cm} (4)$$

and where both the noncommutative fields $\hat{A}$ and $\hat{F}$ are hermitian: $\hat{F}^{\dagger}_{\mu\nu} = \hat{F}_{\mu\nu}$.

The covariant derivative is given by

$$\hat{D}_\mu \hat{\Psi} = \partial_\mu \hat{\Psi} - i\rho_\Psi(\hat{A}_\mu) \ast \hat{\Psi}.$$  \hspace{1cm} (5)$$

The action (3) is invariant under the noncommutative gauge transformations

$$\delta \hat{\Psi} = i\rho_\Psi(\hat{A}) \ast \hat{\Psi} \hspace{1cm} (6)$$
$$\delta \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i[\hat{\Lambda} \ast \hat{A}_\mu] \Rightarrow \delta \hat{F}_{\mu\nu} = i[\hat{\Lambda} \ast \hat{F}_{\mu\nu}] \hspace{1cm} (7)$$

The fields $\hat{A}$, $\hat{\Psi}$ and $\hat{\Lambda}$ are functions of the commutative fields $A, \Psi, \Lambda$ and the noncommutativity parameter $\theta$ via the SW map [9]. At first order in $\theta$ we have (see Section 3 for the freedom in the choice of SW map; see the appendix for the most general SW map at first order in $\theta$)

$$\hat{A}_\xi[A, \theta] = A_\xi + \frac{1}{4} \theta^{\mu\nu} \{ A_\nu, \partial_\mu A_\xi \} + \frac{1}{4} \theta^{\mu\nu} \{ F_{\mu\xi}, A_\nu \} + O(\theta^2) \hspace{1cm} (8)$$
$$\hat{\Lambda}[\Lambda, A, \theta] = \Lambda + \frac{1}{4} \theta^{\mu\nu} \{ \partial_\mu \Lambda, A_\nu \} + O(\theta^2) \hspace{1cm} (9)$$
$$\hat{\Psi}[\Psi, A, \theta] = \Psi + \frac{1}{2} \theta^{\mu\nu} \rho_\Psi(A_\nu) \partial_\mu \Psi + i \frac{1}{8} \theta^{\mu\nu} [\rho_\Psi(A_\mu), \rho_\Psi(A_\nu)] \Psi + O(\theta^2) \hspace{1cm} (10)$$

In terms of the commutative fields the action (3) is also invariant under the ordinary gauge transformation $\delta A_\mu = \partial_\mu \Lambda + i[\Lambda, A_\mu]$, $\delta \Psi = i\rho_\Psi(\Lambda)\Psi$.

In (3) the information on the gauge group $G$ is through the dependence of the noncommutative fields on the commutative ones. The commutative gauge potential $A$ and gauge parameter $\Lambda$ are valued in the $G$ Lie algebra, $A = A^a T^a, \Lambda = \Lambda^a T^a$. It follows that $\hat{A}$ and $\hat{\Lambda}$ are valued in the universal enveloping algebra of the $G$ Lie algebra. Due to the SW map, the degrees of freedom of $\hat{A}$ are the same as that of $A$. Similarly to $\hat{A}$, also $\hat{F}$ is valued in the universal enveloping algebra of $G$, and we write

$$\hat{F}_{\mu\nu} = \sum_{s=1}^{\infty} \sum_{a_1,\ldots,a_s} F^{(a_1\ldots a_s)}_{\mu\nu}(\theta, \partial, A^{(a)}) T^{a_1} T^{a_2} \ldots T^{a_s}$$  \hspace{1cm} (11)$$
where $F^{(a_1,\ldots,a_s)}_{\mu\nu}(\theta,\partial, A^{(a)})$ is a function homogeneous and of order $s$ in the gauge potentials $A^{(a)}$. By dimensional analysis it is at least of order $s$ in $\theta$. From (11) it is clear that expression (3) is ambiguous because in $Tr(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu})$ we have not specified the representation $\rho(T^a)^2$. We can render explicit the ambiguity in (3) by writing

$$\frac{1}{g^2} Tr(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}) = \sum_{\rho} c_{\rho} Tr(\rho(\hat{F}_{\mu\nu}) \star \rho(\hat{F}^{\mu\nu}))$$  \hspace{1cm} (12)$$

where the sum is extended over all unitary irreducible and inequivalent representations $\rho$ of $G$. The real coefficients $c_{\rho}$ parametrize the ambiguity in (12). They are constrained by the condition

$$\frac{1}{g^2} = \sum_{\rho} c_{\rho} Tr(\rho(T^a) \rho(T^a))$$

that is obtained by requiring that in the commutative limit, $\theta \to 0$, (12) becomes the usual commutative gauge kinetic term $\frac{1}{2g^2} \sum_a F^a_{\mu\nu} F^{a\mu\nu}$. Notice that in Euclidean space the action (3) should be negative definite. Now for each irrep. we have

$$\int d^4x \; Tr(\rho(\hat{F}_{\mu\nu}) \rho(\hat{F}^{\mu\nu})) = \int d^4x \; Tr(\rho(\hat{F}_{\mu\nu}) \rho(\hat{F}^{\mu\nu})) = \int d^4x \; Tr(\rho(\hat{F}_{\mu\nu}) \rho(\hat{F}^{\mu\nu})) \geq 0$$  \hspace{1cm} (13)$$

because $\rho(\hat{F}_{\mu\nu}) \rho(\hat{F}^{\mu\nu})$ is an hermitian positive operator. In particular (3) is negative definite if the coefficients $c_{\rho}$ are positive.

The ambiguity (12) in the action (3) can also be studied by expanding (12) in terms of the commutative fields $\Psi, A, F$. By using (8),(10) one obtains ([3, 11]),

$$\hat{S}_{gauge} = -\frac{1}{2g^2} \int d^4x \; Tr(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu})$$

$$= -\frac{1}{4g^2} \sum_{a=1}^{\dim G} F^a_{\mu\nu} F^{a\mu\nu} + \frac{\theta^{\mu\nu}}{2g^2} \int d^4x \; Tr(F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}) - \frac{\theta^{\mu\nu}}{g^2} \int d^4x \; Tr(F_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}) + O(\theta^2)$$

The cubic terms in this formula can be further simplified by observing that

$$\theta^{\mu\nu} F^a_{\mu\rho} F^b_{\nu\sigma} F^{c\rho\sigma} Tr(T^a[T^b, T^c]) = 0$$  \hspace{1cm} (14)$$

(use $Tr(T^a[T^b, T^c]) = Tr(T^c[T^a, T^b])$ and that $\theta^{\mu\nu} F^a_{\mu\rho} F^b_{\nu\sigma} F^{c\rho\sigma}$ is symmetric in $a \leftrightarrow b$). We thus arrive at the expression

$$\hat{S}_{gauge} + O(\theta^2) =$$

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[2] We denote with $T$ an abstract Lie algebra generator, with $\rho(T)$ a generic representation normalized such that $Tr(\rho(T^a) \rho(T^a)) = \frac{1}{2} \delta^{ab}$, and with $t$ a generator in the fundamental representation.
\[
= -\frac{1}{4g^2} \int d^4x \sum_{a=1}^{\dim G} F^{a\mu}_\nu F^{a\mu}_\nu + \frac{\theta^{\mu\nu}}{8g^2} \int d^4x \text{Tr}(F_{\mu\nu} \{ F_{\rho\sigma}, F^{\rho\sigma} \}) - \frac{\theta^{\mu\nu}}{2g^2} \int d^4x \text{Tr}(F_{\mu\rho} \{ F_{\nu\sigma}, F^{\rho\sigma} \})
\]

\[
= -\frac{1}{4g^2} \int d^4x \sum_{a=1}^{\dim G} F^{a\mu}_\nu F^{a\mu}_\nu + (\sum_\rho c_\rho D^{abc}_\rho) \frac{\theta^{\mu\nu}}{4} \int d^4x \frac{1}{4} F^{a\mu}_\nu F^{b\rho}_\sigma F^{c\rho}_\sigma - F^{a\mu}_\nu F^{b\rho}_\sigma F^{c\rho}_\sigma
\]

where
\[
\frac{1}{2} D^{abc}_\rho \equiv \text{Tr}(\rho(T^a) \{ \rho(T^b), \rho(T^c) \}) = A(\rho) \text{Tr}(t^a \{ t^b, t^c \}) \equiv \frac{1}{2} A(\rho) d^{abc}. \tag{16}
\]

Here \( t^a \) denotes the fundamental representation, and we are using that the completely symmetric \( D^{abc}_\rho \) tensor in the representation \( \rho \) is proportional to the \( d^{abc} \) one defined by the fundamental representation. In particular for all simple Lie groups, except \( SU(N) \) with \( N \geq 3 \), we have \( D^{abc}_\rho = 0 \) for any representation \( \rho \). Thus from (15) we see that at first order in \( \theta \) the ambiguity (12) is present just for \( SU(N) \) Lie groups, and it is equivalent to the choice of the real number \( \sum_\rho c_\rho A(\rho) \).

Among the possible representations that one can choose in (12) there are two natural ones. The fermion representation and the adjoint representation.

The adjoint representation is particularly appealing if we just have a pure gauge action, then, since only the structure constants appear in the commutative gauge kinetic term \( \sum_a F^{a\mu}_\nu F^{a\mu}_\nu \), a possible choice is indeed to consider only the adjoint representation. This is a minimal choice in the sense that in this case only structure constants enter (11) and (12). In Subsection 3.1 we show that in this case the gauge action is even in \( \theta \).

If we also have matter fields then from (5) we see that we must consider the particle representation \( \rho_\Psi \) given by the multiplet \( \Psi \) (and inherited by \( \hat{\Psi} \)). In (12) one could then make the minimal choice of selecting just the \( \rho_\Psi \) representation.

Along the lines of the above NCYM theories framework we now examine the SO(10), the \( SU(5) \) and the Standard Model noncommutative gauge theories.

### 2.1 Noncommutative SO(10)

We first consider only one fermion generation because this fits in one multiplet: the 16-dimensional spinor representation of SO(10) usually denoted \( 16^+ \). We write the left handed multiplet as
\[
\Psi_L^+ = (u^i, d^i, -u^C_i, d^C_i, \nu, e^-, e^+, -\nu^C)_{L} \tag{17}
\]

where \( i \) is the \( SU(3) \) color index and \( \nu^C_{L} = -i\sigma_2 \nu^*_R \) is the charge conjugate of the neutrino particle \( \nu_R \) (not present in the Standard Model). The gauge and fermion
sector of noncommutative $SO(10)$ is then simply obtained by replacing $\hat{\Psi}$ with $\hat{\Psi}_L^+$ in (3).

Next we consider all three fermion families. In this case we do not have a single multiplet, the kinetic term for the fermions reads

$$\sum_{B=1}^{3} \overline{\Psi}^{(B)}_L \star i\hat{D} \Psi^{(B)}_L$$

(18)

and in principle in the gauge kinetic term we could have different weights $c_{\rho_{\Psi}(i)}$ for each of the three 16-dimensional representations $\rho_{\Psi(1)}, \rho_{\Psi(2)}, \rho_{\Psi(3)}$. But these representations are three identical copies and therefore only the combination $c_{\rho_{\Psi(1)}} + c_{\rho_{\Psi(2)}} + c_{\rho_{\Psi(3)}}$ enters.

In conclusion we expect the noncommutative $SO(10)$ gauge kinetic term to contain at most a combination of the adjoint representation and of the 16$^+$ particle representation; indeed it is difficult to conceive a mechanism that generates other representations than these two. This holds especially if one considers the noncommutative $SO(10)$ action as a fundamental one in the sense that no other fermion has been integrated out in order to obtain (3).

Finally let us repeat that in (15), whatever representation $\rho$ one considers, no linear term in $\theta$, i.e. no cubic term in $F$ can appear. This is so because $SO(10)$ is anomaly free: $D^\rho_{abc} = 0$ for all $\rho$. In other words, at first order in $\theta$, noncommutative $SO(10)$ gauge theory is unique.

2.2 Noncommutative $SU(5)$

The fermionic sector of $SU(5)$ is made by two multiplets for each family. The $\psi^C L$ multiplet transforms in the 5 of $SU(5)$, while the $\chi_L$ multiplet transforms according to the 10 of $SU(5)$.

In this case we expect that the adjoint, the 5 and the 10 representations enter in (12). In principle one can consider the coefficients $c_5 \neq c_{10}$, i.e. while the $(\psi^C L, \chi_L)$ fermion rep. is $5 \oplus 10$, in (12) the weights $c_\rho$ of the 5 and the 10 can possibly be not the same. Only if $c_5 \neq c_{10}$ then $\sum_\rho c_\rho D^{abc}_\rho \neq 0$ in (15). Proof: The adjoint rep. $C^a$, defined by $[t^a, t^b] = C^{abc} t^c$, is antisymmetric, $(C^a)^t = -C^a$, and therefore we have

$$\text{Tr}(C^a \{ C^b, C^c \}) = \text{Tr}(C^a \{ C^b, C^c \})^t = -\text{Tr}(C^a \{ C^b, C^c \}) = 0.$$  

(19)

The representation $\overline{5} \oplus 10$ is anomaly free because $D^{abc}_5 = -D^{abc}_{10}$. If we consider $c_5 \neq c_{10}$ then $\sum_\rho c_\rho D^{abc}_\rho = (c_5 - c_{10}) D^{abc}_5 \neq 0$.

We see that, already at first order in the noncommutativity parameter $\theta$, noncommutative $SU(5)$ gauge theory is not uniquely determined by the gauge coupling constant $g$, but also by the value of $\sum_\rho c_\rho D^{abc}_\rho$. It is tempting to set $c_5 = c_{10}$ so that $\sum_\rho c_\rho D^{abc}_\rho = 0$ and exactly the fermion representation $\overline{5} \oplus 10$ (and eventually the adjoint one) enter
We also have

\[ \text{SU(5)} \] and similarly for \( \text{Tr}(\rho) \). Because \( \text{Tr}(\rho) \) in this case the group is not a simple group. We denote by \( \mathcal{T}^A \) the generators of \( U(1) \otimes SU(2) \otimes SU(3) \). They are \( \{\mathcal{T}^A\} = \{Y, T_L^a, T_S^l\} \) with \( a = 2, 3, 4 \) and \( l = 5, \ldots, 12 \). Any irrep. of \( U(1) \otimes SU(2) \otimes SU(3) \), is a product of an irrep. of \( U(1) \), of \( SU(2) \) and of \( SU(3) \). We write

\[ \{\rho(\mathcal{T}^A)\} = \{\rho_1(Y) \otimes 1_{\rho_2} \otimes 1_{\rho_3}, 1 \otimes \rho_2(T_L^a) \otimes 1_{\rho_3}, 1 \otimes 1_{\rho_2} \otimes \rho_3(T_S^l)\} \]

We also have

\[ \text{Tr}(\rho(\mathcal{T}^1)\rho(\mathcal{T}^a)) = \text{Tr}(\rho(\mathcal{T}^1)\rho(\mathcal{T}^i)) = \text{Tr}(\rho(\mathcal{T}^a)\rho(\mathcal{T}^i)) = 0 \] (20)

because \( \text{Tr}((\rho_1(Y) \otimes 1_{\rho_2} \otimes 1_{\rho_3})(1 \otimes \rho_2(T_L^a) \otimes 1_{\rho_3})) = \text{Tr}(\rho_1(Y))\text{Tr}(\rho_2(T_L^a))\text{Tr}(1_{\rho_3}) = 0 \) and similarly for \( \text{Tr}(\rho(\mathcal{T}^1)\rho(\mathcal{T}^i)) \) and \( \text{Tr}(\rho(\mathcal{T}^a)\rho(\mathcal{T}^i)) \). The gauge kinetic term is

\[ \hat{S}_{\text{gauge}} = -\frac{1}{2} \int d^4x \sum_{\rho} c_\rho \text{Tr}(\rho(\hat{F}_{\mu\nu}) \ast \rho(\hat{F}^{\mu\nu})) \]

\[ = -\frac{1}{2} \int d^4x \sum_{\rho_1,\rho_2,\rho_3} c_{\rho_1 \otimes \rho_2 \otimes \rho_3} \text{Tr}((\rho_1 \otimes \rho_2 \otimes \rho_3)(\hat{F}_{\mu\nu}) \ast (\rho_1 \otimes \rho_2 \otimes \rho_3)(\hat{F}^{\mu\nu})) \]

where the sum is over all inequivalent irrep.‘s of \( U(1) \otimes SU(2) \otimes SU(3) \). In the commutative limit only terms quadratic in \( \mathcal{T} \) enter the traces, and using (20) we obtain

\[ \hat{S}_{\text{gauge}} \stackrel{\theta \to 0}{\longrightarrow} S_{\text{gauge}}^{\text{cl}} = -\frac{1}{2} \int d^4x \left[ \sum_{\rho_1,\rho_2,\rho_3} c_{\rho_1 \otimes \rho_2 \otimes \rho_3} d(\rho_2) d(\rho_3) \rho_1(Y) \rho_1(Y) \right] F_{\mu\nu}^1 F^{1\mu\nu} \]

\[ + \frac{1}{2} \sum_{\rho_1,\rho_2,\rho_3} c_{\rho_1 \otimes \rho_2 \otimes \rho_3} \rho_2(\rho_3) \left[ \sum_a F_{\mu\nu}^a F^{a\mu\nu} \right] \]

\[ + \frac{1}{2} \sum_{\rho_1,\rho_2,\rho_3} c_{\rho_1 \otimes \rho_2 \otimes \rho_3} \rho_2(\rho_3) \left[ \sum_l F_{\mu\nu}^l F^{l\mu\nu} \right] \]

\[ \equiv -\int d^4x \left[ \frac{1}{4g^2} F_{\mu\nu}^1 F^{1\mu\nu} \right] + \frac{1}{4g^2} \sum_a F_{\mu\nu}^a F^{a\mu\nu} + \frac{1}{4g_S^2} \sum_m F_{\mu\nu}^m F^{m\mu\nu} \] (22)

where \( d(\rho) \) is the dimension of the irrep. \( \rho \). From the last line we read off the three structure constants \( g^2, g^2, g_S^2 \) in terms of the coefficients \( c_{\rho_1 \otimes \rho_2 \otimes \rho_3} \) and of the irrep. \( \rho_1, \rho_2, \rho_3 \).^4

^3In [25] only \( c_\rho \neq 0 \) and therefore triple gauge boson couplings like \( \theta FFF \) are present.

^4With these general formulas one can recover the results of [3, Appendix C].
At first order in \( \theta \) the only nonvanishing symmetric traces are \( \text{Tr}(\rho(\mathcal{T}^l)\rho(\mathcal{T}^a)\rho(\mathcal{T}^u)) \), \( \text{Tr}(\rho(\mathcal{T}^l)\rho(\mathcal{T}^l)\rho(\mathcal{T}^l)) \), \( \text{Tr}(\rho(\mathcal{T}^l)\{\rho(\mathcal{T}^u),\rho(\mathcal{T}^l)\}) \). We now recall (21) and (15) and obtain

\[
\hat{S}_{\text{gauge}} = S_{\text{gauge}}^{cl} + \int d^4x \, \nu_1 (\theta \cdot F^1 F^1 + \theta \cdot \tilde{F}^1 F^1) \\
+ \nu_2 \sum_a (\theta \cdot F^a F^a + 2 \theta \cdot \tilde{F}^a F^a + \theta \cdot \tilde{F}^1 F^1 + 2 \theta \cdot \tilde{F}^a F^a) \\
+ \nu_3 \sum_l (\theta \cdot F^l F^l + 2 \theta \cdot F^l F^l + \theta \cdot \tilde{F}^l F^l + 2 \theta \cdot \tilde{F}^l F^l) \\
+ \frac{1}{4} \sum_{\rho_1, \rho_2, \rho_3} c_{\rho_1 \otimes \rho_2 \otimes \rho_3} d(\rho_2) D_\rho_3^{l l'} \theta^{\mu
u} \left[ \frac{1}{4} F_{\mu\rho} F_{\nu\rho} - F_{\mu\rho} F_{\nu\rho} \right] \quad (23)
\]

where \( \theta \cdot F \equiv \theta_{\mu\nu} F^{\mu\nu} \), \( \theta \cdot \tilde{F} \equiv \theta_{\mu\nu} \tilde{F}^{\mu\nu} \) and \( \tilde{F}^{\mu\nu} \equiv \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \). In (23) we have used

\[
\theta_{\nu\mu} F^{1 \mu \rho} F^{a \sigma \nu} = \frac{1}{4} [\theta \cdot F^a F^a + \theta \cdot \tilde{F}^a F^a + \theta \cdot \tilde{F}^1 F^1] \\
\theta_{\nu\mu} F^{a \mu \rho} F^{1 \sigma \nu} = \frac{1}{2} \theta \cdot F^a F^a + \frac{1}{4} \theta \cdot \tilde{F}^1 F^a
\]

and similar formulas with \( a \to l \) and \( a \to 1 \). The coefficients \( \nu_1, \nu_2, \nu_3 \) depend on \( c_{\rho_1 \otimes \rho_2 \otimes \rho_3} \) and the irrep. \( \rho_1, \rho_2, \rho_3 \).

In the particle representations, only the trivial, the fundamental and the conjugate of the fundamental appear for colour \( SU(3) \). (This accounts for invariance under charge conjugation: if we replace \( u \) and \( d \) with \( u^c \) and \( d^c \) the \( SU(3) \) lagrangian is unaffected). In the noncommutative case it is natural to preserve this symmetry between representations and conjugate representations. In other words any irrep. \( \rho_3 \) of \( SU(3) \) should appear together with its conjugate irrep. \( \rho_3^* \) and with \( c_{\rho_1 \otimes \rho_2 \otimes \rho_3^*} = c_{\rho_1 \otimes \rho_2 \otimes \rho_3} \). The last line in (23) then vanishes. The proof easily follows from

\[
\frac{1}{2} D_\rho_3^{l l'} = \text{Tr}(\rho_3^* (\mathcal{T}^l) \{\rho_3^* (\mathcal{T}^u),\rho_3^* (\mathcal{T}^l)\}) = -\text{Tr}(\rho_3 (\mathcal{T}^l) \{\rho_3 (\mathcal{T}^u),\rho_3 (\mathcal{T}^l)\}) = -\frac{1}{2} D_\rho_3^{l l'}
\]

where we used \( \rho^* \mathcal{T} = -\rho \mathcal{T} \) since \( \mathcal{T} \) is hermitian.

About the fermion kinetic term, the fermion vector \( \Psi^{(B)}_L \), where \( B = 1, 2, 3 \) is the family index, is given by (cf. \( \Psi^+_L \) above) \( \Psi_L = (u^i, d^i, -u_{i}^c, d_{i}^c, \nu, e^-, e^+)_L \). The covariant derivative is as in (5), with \( \Psi \rightarrow \Psi_L \) and with \( A_\mu = A_\mu^A \mathcal{T}^A \). The fermion kinetic term is then as in (18) (with \( \Psi^+_L \rightarrow \Psi_L \)). This Standard Model is built using only left handed fermions and antifermions. We call it GUT inspired because its noncommutative structure can be embedded in \( SO(10) \) GUT. Indeed \( \Psi_L \) and \( \Psi^+_L \) differ just by the extra neutrino \( \nu_{L}^c = -i \sigma_2 \nu_{R}^* \); moreover under an infinitesimal gauge transformation
all fermions in $\Psi_L$ transform with $\hat{\Lambda}$ on the left. This GUT inspired Standard Model differs from the one considered in \cite{3}; indeed here we started from the chiral vector $\Psi_L$, while there the vector $\Psi' = (u_L^i, d_L^i, u_R^i d_R^i, \nu_L, e_L^-, e_R^-)$ is considered. In the commutative case $\int \overline{\Psi'} \not{\partial} \Psi' = \int \overline{\Psi_L} \not{\partial} \Psi_L$ but in the noncommutative case, as we discuss in Subsection 4.1 (see also (27) and the last lines of the next subsection), this is no more true: $\int \overline{\Psi'} \not{\partial} \Psi' \neq \int \overline{\Psi_L} \not{\partial} \Psi_L$, if we change $\theta$ into $-\theta$ in the right handed sector of $\int \overline{\Psi'} \not{\partial} \Psi'$, then the two expressions coincide.

Finally, if in the gauge kinetic term (23) we consider only the adjoint rep. and the fermion rep. we have

$$\hat{S}_{\text{gauge}} = S_{\text{gauge}}^{cd} + O(\theta^2).$$

(24)

This is so because the fermion rep. is anomaly free: $D_{\rho_{\text{fermion}}}^{AA'A''} = 0 (A = 1, a, l)$, because for $U(1)$ the adjoint rep. is trivial, and for the last line in (23), because the adjoint rep. of $SU(3)$ has also $D_{\rho_{\text{adj}}}^{ll''l'''} = 0$ (a proof is as in (19)).

2.4 Higgs Sector in the (GUT inspired) Noncommutative Standard Model

In the commutative SM, the Yukawa terms can be written as

$$W^{BB'} \phi^\dagger L_L^{(B)} e_R^{(B')} + G_u^{BB'} \phi^\dagger Q_L^{(B)} u_R^{(B')} + G_d^{BB'} \phi^\dagger Q_L^{(B)} d_R^{(B')} + \text{herm. conj.}$$

(25)

where $L_L = (\nu_L^i, e_L^i)$, $Q_L = (u_L^i, d_L^i)$, $\phi = (\varphi^+_0, \phi^0)$ and $\phi = i\tau_2 \phi^*$ with $\tau_2$ the Pauli matrix in $SU(2)$ gauge group space. The matrices $W^{BB'}, G_u^{BB'}, G_d^{BB'}$ are the Yukawa couplings ($B, B' = 1, 2, 3$), and the sum over group and spinor indices is understood, so that $\phi^\dagger L_L e_R = e_R^\dagger \phi^L L_L$, $\phi^\dagger Q_L u_R = u_R^\dagger \phi^L Q_L$, etc..

A noncommutative version of (25) is not straightforward; for example, the noncommutative fields

$$\hat{\phi}^\dagger, \hat{L}_L, \hat{e}_R^\dagger = i\sigma_2 e_L^C = i\sigma_2 e_L^C,$$

under an infinitesimal $U(1) \otimes SU(2) \otimes SU(3)$ gauge transformation $\Lambda$, transform as

$$\delta \hat{\phi}^\dagger = -i \hat{\phi}^\dagger \star \rho_\phi (\hat{\Lambda}) \ , \ \delta \hat{L}_L = i \rho_{L_L}(\hat{\Lambda}) \star \hat{L}_L ,$$

(26)

$$\delta \hat{e}_R^\dagger = \delta i \sigma_2 e_L^C = i \rho_{i\sigma_2 e_L^C} (\hat{\Lambda}) \star i \sigma_2 e_L^C = i \rho_{i\sigma_2 e_L^C} (\hat{\Lambda}) \star \hat{e}_R^\dagger,$$

(27)

and therefore the term $\int d^4x \hat{\phi}^\dagger \star \hat{L}_L \star \hat{e}_R^\dagger = \int d^4x e_R^\dagger \star \hat{\phi}^\dagger \star \hat{L}_L$ cannot be gauge invariant because, for example, $\rho_{e_R^\dagger} (\hat{\Lambda})$ does not commute with $\hat{L}_L$. 

10
A solution is to consider hybrid Seiberg-Witten maps \( \tilde{H} \) [12] on \( L_L \) and \( Q_L \). The noncommutative Yukawa terms are then

\[
W^{BB'} \tilde{\phi}^\dagger \star \tilde{L}_L^{H(B)} \star \tilde{e}_R^{(B')} + G_u^{BB'} \tilde{\phi}^\dagger \star \tilde{Q}_L^{H(B)} \star \tilde{u}_R^{(B')} + G_d^{BB'} \tilde{\phi}^\dagger \star \tilde{Q}_L^{H(B)} \star \tilde{d}_R^{(B')} + \text{herm. conj.}
\]

(28)

with \( \tilde{\phi} \equiv \tilde{\phi}[\phi, A, \theta] \) and \( \tilde{\phi} \equiv \tilde{\phi}[\tilde{\phi}, A, \theta] \). Under an infinitesimal \( U(1) \otimes SU(2) \otimes SU(3) \) gauge transformation \( \Lambda \), \( \tilde{L}_L^H \), \( \tilde{Q}_L^H \), and \( \tilde{Q}_L^H \) transform as

\[
\begin{align*}
\delta \tilde{L}_L^H &= i\rho_\phi(\hat{\Lambda}) \star \tilde{L}_L^H - iL_L^H \star \rho_{e_R}(\hat{\Lambda}) \\
\delta \tilde{Q}_L^H &= i\rho_\phi(\hat{\Lambda}) \star \tilde{Q}_L^H - iQ_L^H \star \rho_{u_R}(\hat{\Lambda}) \\
\delta \tilde{Q}_L^H &= i\rho_\phi(\hat{\Lambda}) \star \tilde{Q}_L^H - iQ_L^H \star \rho_{d_R}(\hat{\Lambda})
\end{align*}
\]

(29)-(31)

We see that in the hybrid SW map \( \hat{\Lambda} \) appears both on the left and on the right of the fermions. We also see that the representation of \( \Lambda \) is inherited from the Higgs and fermions that respectively sandwich \( \tilde{L}_L^H \), \( \tilde{Q}_L^H \) and \( \tilde{Q}_L^H \). The Yukawa terms (28) are thus invariant under noncommutative gauge transformations. Of course in the \( \theta \to 0 \) limit (29)-(31) become \( \delta L_L = \rho_{L_L}(\Lambda) L_L \), \( \delta Q_L = \rho_{Q_L}(\Lambda) Q_L \). At first order in \( \theta \) we have [3,12]

\[
\tilde{\Psi}^H = \Psi + \frac{1}{2} \theta^{\mu\nu} A_{\nu} \left( \partial_\mu \Psi - \frac{i}{2} (A_\mu \Psi - \Psi A_\mu) \right) + \frac{1}{2} \theta^{\mu\nu} \left( \partial_\mu \Psi - \frac{i}{2} (A_\mu \Psi - \Psi A_\mu) \right) A'_\nu + O(\theta^2)
\]

(32)

where \( A \) carries the representation of the fields on the left of \( \tilde{\Psi}^H \), while \( A' \) carries the representation of the fields on the right of \( \tilde{\Psi}^H \). The choice of Yukawa terms (28) differs from those studied in [3]. There the hybrid SW map is considered on \( \phi \), in particular there \( \tilde{\phi}^H \) is not invariant under \( SU(3) \) gauge transformations; here, as in the commutative case, \( \delta \tilde{\phi} = 0 \) under \( SU(3) \) transformations.\(^5\) Another main difference (cf. also Subsection 4.1) is that in [3] \( \delta \tilde{e}_R = -i\tilde{e}^* \star \rho_{e_R}(\hat{\Lambda}) \), i.e. contrary to (27), \( \hat{\Lambda} \) appears on the right and not on the left of the right handed electron.

Finally the Higgs kinetic and potential terms are given by

\[
(\bar{D}_\mu \tilde{\phi})^\dagger \star \bar{D}^\mu \tilde{\phi} + \mu^2 \tilde{\phi}^\dagger \star \tilde{\phi} - \lambda \tilde{\phi}^\dagger \star \tilde{\phi} \star \tilde{\phi}^\dagger \star \tilde{\phi}.
\]

(33)

\[\text{2.5 Higgs Sector in Noncommutative SO(10)}\]

\(^5\)This implies, that in [3] gluons couple directly to the Higgs field, which is not the case here.
Up to now we have examined three different noncommutative gauge theories, SO(10), SU(5) and SM. One can also consider the spontaneous symmetry breaking $SO(10) \rightarrow G \rightarrow U(1) \otimes SU(2) \otimes SU(3) \rightarrow U(1) \otimes SU(3)$. There are many patterns for SSB depending on the choices of the Higgses and of the intermediate symmetry group $G$ (e.g. $SU(4) \otimes SU(2)_L \otimes SU(2)_R$, $SU(5)$ etc., see for example [26]). In general one can construct a noncommutative version of a given Higgs potential using the SW map and the hybrid SM map. The noncommutative $SO(10)$ invariant Yukawa terms are built using similar techniques. In the commutative case we have the Yukawa term [27]

$$i\Phi^*_10 \Psi^+_L \sigma_2 \Psi^+_L - i\Psi^+_L \sigma_2 \Psi^+_L \Phi_{10}$$

(34)

where here transposition is just in the spin indices, $\Phi^*_10 \Psi^+_L \sigma_2 \Psi^+_L = \Phi^*_10 \Psi^+_L \sigma_2 \Psi^+_L$; moreover we have suppressed the family indices, so that the term $\Psi^+_L \sigma_2 \Psi^+_L \Phi_{10}$ stands for $W^{BB'} \Psi^+_L \Phi_{10}$ of similar terms are obtained with the Higgs multiplets $\Phi_{126}$ and $\Phi_{120}$. Here 10, 126 and 120 are the irrep. contained in $16 \times 16 = 10 \oplus 126 \oplus 120$, so that the Yukawa term (34) is an $SO(10)$ singlet. A noncommutative generalization of (34) is obtained requiring that the noncommutative version of $\Psi^+_L \sigma_2 \Psi^+_L$ transforms as $16 \times 16$. This is achieved with the term

$$\hat{\Psi}^+_L \star \hat{\Psi}^+_L \sigma_2 \alpha \beta = \hat{\Psi}^+_L \star \hat{\Psi}^+_L \sigma_2 \alpha \beta$$

(35)

where 1 is the $16 \times 16$ unit matrix, $1 \otimes \hat{\Psi}^+_L$ is the standard SW map on $\hat{\Psi}^+_L$ and $\hat{\Psi}^+_L \otimes 1$ is the hybrid SW map (cf. (32)); by definition, under an infinitesimal gauge transformation (cf. (6)) $\delta \hat{\Psi}^+_L = i\rho_{\Psi}^+_L (\hat{\Lambda}) \star \hat{\Psi}^+_L$ and

$$\delta (\hat{\Psi}^+_L \otimes 1) = i (\rho_{\Psi}^+_L (\hat{\Lambda}) \otimes 1 + 1 \otimes \rho_{\Psi}^+_L (\hat{\Lambda})) \star (\hat{\Psi}^+_L \otimes 1) - i (\hat{\Psi}^+_L \otimes 1) \star (1 \otimes \rho_{\Psi}^+_L (\hat{\Lambda})) .$$

We see that in the commutative limit $\hat{\Psi}^+_L \otimes 1$ transforms as $\Psi^+_L$. The noncommutative Yukawa term then reads

$$i \hat{\Phi}^*_10 \star \hat{\Psi}^+_L \sigma_2 \alpha \beta - i \hat{\Psi}^+_L \star \hat{\Phi}^*_10 \sigma_2 \alpha \beta$$

(36)

where $\delta \hat{\Phi}_{10} = i\rho_{10} (\hat{\Lambda}) \star \hat{\Phi}_{10}$. Similarly for $\hat{\Phi}_{126}$ and $\hat{\Phi}_{120}$.

3 Hermiticity and reality of SW map

In the previous section we used that if $A$ is hermitian then $\hat{A}$ is hermitian too. At first order in $\theta$ this is indeed the case, see (8). In this section we show, to all orders in $\theta$, that
\( \hat{A} \) and \( \hat{\Lambda} \) can be chosen hermitian if \( A \) and \( \Lambda \) are hermitian. More generally we show that SW map commutes with hermitian conjugation as well as with complex conjugation.

Given a (not necessarily unitary) rep. \( \rho_\Psi \) of \( G \) defined by the multiplet \( \Psi \), we can always consider the multiplet \( \Upsilon \) that transforms according to the inverse hermitian representation \( \rho_\Upsilon \) given by \( \rho_\Upsilon(g) = (\rho_\Psi(g))^{-1} \), for all \( g \in G \). Similarly we consider the multiplet \( \Psi^* \) that transforms according to the conjugate rep. \( \rho_{\Psi^*}(g) = \rho_\Psi(g)^\dagger \). \(^6\) Since \( g = e^{iA} = e^{iA^aT^a} \), with \( A^a \) real, and \( A = A^aT^a \) with \( A^a \) real, at the Lie algebra level we have

\[
\rho_\Upsilon(\Lambda) = (\rho_\Psi(\Lambda))^\dagger, \quad \rho_\Upsilon(A) = (\rho_\Psi(A))^\dagger, \quad \rho_{\Psi^*}(\Lambda) = -\rho_\Psi(\Lambda), \quad \rho_{\Psi^*}(A) = -\rho_\Psi(A). \tag{37}
\]

Commutativity of SW map with hermitian conjugation and with complex conjugation means

\[
\rho_{\Psi^*}(\hat{A}) = \rho_\Psi(\hat{A})^\dagger, \quad \rho_{\Psi^*}(\hat{\Lambda}) = \rho_\Psi(\hat{\Lambda})^\dagger \quad \text{i.e.} \quad (\rho_\Psi(\hat{A}))^\dagger = \rho_{\Psi^*}(\hat{A})^\dagger, \quad (\rho_\Psi(\hat{\Lambda}))^\dagger = \rho_{\Psi^*}(\hat{\Lambda})^\dagger, \tag{38}
\]

that for short we rewrite \( \hat{A}^\dagger = \hat{A}^\dagger \), \( \hat{\Lambda}^\dagger = \hat{\Lambda}^\dagger \), and

\[
\hat{\Psi}^* = \hat{\Psi}^* \quad \rho_{\Psi^*}(\hat{A}) = -\rho_\Psi(\hat{A}) \quad \rho_{\Psi^*}(\hat{\Lambda}) = -\rho_\Psi(\hat{\Lambda}). \tag{39}
\]

In (38) and (39) we used the following notation (cf. Section 2)

\[
\rho_\Psi(\hat{A}) \equiv \rho_\Psi(\hat{A}) \equiv \hat{A}[\rho_\Psi(A), \theta] \quad \rho_\Psi(\hat{\Lambda}) \equiv \rho_\Psi(\hat{\Lambda}) \equiv \hat{\Lambda}[\rho_\Psi(A), \rho_\Psi(\Lambda), \theta] \tag{40}
\]

\[
\rho_{\Psi^*}(\hat{A}) \equiv \rho_{\Psi^*}(\hat{A}) \equiv \hat{A}[\rho_{\Psi^*}(A), -\theta] \quad \rho_{\Psi^*}(\hat{\Lambda}) \equiv \rho_{\Psi^*}(\hat{\Lambda}) \equiv \hat{\Lambda}[\rho_{\Psi^*}(A), \rho_{\Psi^*}(\Lambda), -\theta] \tag{41}
\]

\[
\hat{\Psi} \equiv \text{SW}[\Psi, \rho_\Psi(A), \theta] \quad \hat{\Upsilon} \equiv \text{SW}[\Upsilon, \rho_\Upsilon(A), \theta] \quad \hat{\Psi}^* \equiv \text{SW}[\Psi^*, \rho_{\Psi^*}(A), -\theta] \tag{42}
\]

Notice that in (42) \( \theta \) appears with the opposite sign w.r.t. \( \theta \) in (40), similarly in (43). Consistency requires that with the representation \( \rho_{\Psi^*} \), we must consider the opposite star product \( *^\theta \) i.e. the star product built with \( -\theta \) instead of \( \theta \). The \( -\theta \) in (42) is also consistent with the charge conjugation operator defined in (65).

For the proof of (38) and (39), in the case of constant \( \theta \), we can use SW differential equation [9]. As discussed in [28], SW differential equation is not unique. Hermiticity and reality indeed constrain the freedom in the choice of SW map. Hermiticity and reality are physical requirements, since we want the noncommutative action (obtained via SW map from the commutative one) to be real if the commutative one is real. However there are two extra ambiguities. One is related to gauge transformations, SW map is

\(^6\)In order to avoid possible confusions with \( \Psi^\dagger \gamma^\alpha \), on multiplets we denote complex conjugation with \( * \) instead of \( ^\dagger \). Also, in this section, the multiplet \( \Psi \) is not necessarily a fermion multiplet.
defined to map orbits of the commutative gauge group to orbits of the noncommutative one, therefore there is no unique way to associate to a given commutative gauge potential a given noncommutative gauge potential. The other ambiguity is related to field redefinitions of the noncommutative gauge potential.

We expect that in both cases these ambiguities are not physical because the noncommutative $S$ matrix is gauge invariant and is expected to be independent from field redefinitions.

Here we choose a specific SW differential equation, it reads [9]:

\[ \delta_\theta \hat{A}_\mu = \delta \theta^{\rho\sigma} \frac{\partial}{\partial \theta^{\rho\sigma}} \hat{A}_\mu = -\frac{1}{4} \delta \theta^{\rho\sigma} \{ \hat{A}_\rho \hat{A}_\mu + \hat{F}_{\sigma\mu} \} \]  

(44)

\[ \delta_\theta \hat{\Lambda} = \delta \theta^{\rho\sigma} \frac{\partial}{\partial \theta^{\rho\sigma}} \hat{\Lambda} = -\frac{1}{4} \delta \theta^{\rho\sigma} \{ \partial_\rho \hat{\Lambda} \hat{A}_\sigma \} \]  

(45)

in these expressions $\hat{A}$ and $\hat{\Lambda}$ are valued in the universal enveloping algebra of $G$. As in [9], (44) and (45) are obtained by requiring that gauge equivalence classes of the $\hat{A}$ gauge theory, with noncommutativity $\ast$ given by $\theta' \equiv \theta + \delta \theta$, correspond to gauge equivalence classes of the $\hat{A}$ gauge theory, with noncommutativity $\ast$ given by $\theta$. In formulas, writing $\hat{A} = \hat{A}[A, \theta] = \hat{A}[\hat{A}, \delta \theta]$ the condition reads

\[ \delta_\Lambda \hat{A}' \equiv \hat{\Lambda}' + i[\Lambda' \ast \hat{A}] = \delta_\Lambda \hat{A}[\hat{A}, \delta \theta] , \]  

(46)

where $\delta_\Lambda \hat{A}[\hat{A}, \delta \theta] \equiv \hat{A}[\hat{A} + \delta_\Lambda \hat{A}, \delta \theta] - \hat{A}[\hat{A}, \delta \theta]$. SW map for the multiplet $\Psi$ can similarly be obtained by requiring $\delta_\Lambda \hat{\Psi} \equiv i\Lambda' \ast \hat{\Psi} = \delta_\Lambda \hat{\Psi}[\hat{A}, \hat{\Psi}]$; we have

\[ \delta_\theta \hat{\Psi} = \delta \theta^{\rho\sigma} \frac{\partial}{\partial \theta^{\rho\sigma}} \hat{\Psi} = -\frac{1}{2} \delta \theta^{\mu\nu} \rho_\Psi(A_\mu) \ast \partial_\nu \hat{\Psi} + i\frac{1}{8} \delta \theta^{\mu\nu} [\rho_\Psi(A_\mu) \ast \rho_\Psi(A_\nu)] \ast \hat{\Psi} . \]  

(47)

In order to show (38) we notice that for generic space-time dependent matrices $M$ and $N$, under complex conjugation, transposition and hermitian conjugation we have (recall $f \ast g = f e^{i\theta^{\mu\nu} \partial_\mu \partial_\nu} g$)

\[ (M \ast N) = \overline{M} \ast N \]  

(48)

We now apply $\dagger$ to (44) and (45) in the $\rho_\Psi$ representation and obtain

\[ \delta_\theta \hat{\rho}_\Psi(A_\mu) = -\frac{1}{4} \delta \theta^{\rho\sigma} \{ \overline{\rho_\Psi(A_\rho)} \dagger \ast \partial_\sigma \overline{\rho_\Psi(A_\mu)} \dagger + \overline{\rho_\Psi(F_{\sigma\mu})} \dagger \} \]  

(49)

\[ \delta_\theta \hat{\rho}_\Psi(\Lambda) = -\frac{1}{4} \delta \theta^{\rho\sigma} \{ \overline{\partial_\rho \rho_\Psi(\Lambda)} \dagger \ast \overline{\rho_\Psi(A_\sigma)} \dagger \} \]  

(50)

If at order $\mathcal{O}(\delta \theta^n)$ we have $(\rho_\Psi(A))\dagger = \overline{\rho_\Psi(A)} \dagger$, $(\rho_\Psi(\Lambda))\dagger = \overline{\rho_\Psi(\Lambda)} \dagger$, then (49), (50) show that this is also true for $\rho_\Psi(A) = \rho_\Psi(A) + \delta_\theta \rho_\Psi(A)$ and $\rho_\Psi(\Lambda) = \rho_\Psi(\Lambda) + \delta_\theta \rho_\Psi(\Lambda)$,
i.e. (38) holds also at order $O(\delta \theta^{n+1})$. Now, since for $\theta = 0$ (38) trivially holds, we conclude that (38) holds for finite $\theta$.

In particular if $\rho_\Psi$ is a unitary rep.: $\rho_\Psi(\Lambda)^\dagger = \rho_\Psi(\Lambda)$, then hermiticity of $\rho_\Psi(A)$ implies hermiticity of $\overline{\rho_\Psi(A)}$ and of $\overline{\rho_\Psi(\Lambda)}$.

In order to show reality of SW map, see (39), we complex conjugate (44), (45) and (47) in the $\rho_\Psi$ representation and obtain

\[
\delta_\theta \rho_\Psi(A_\mu) = -\frac{1}{4}(\delta \theta)_{\alpha\sigma} \{ \rho_\Psi(A_\rho) \, \rho_\Psi^\dagger \partial_\sigma \rho_\Psi(A_\mu) + \rho_\Psi(F_{\sigma\mu}) \} \\
\delta_\theta \rho_\Psi(\Lambda) = -\frac{1}{4}(\delta \theta)_{\rho\sigma} \{ \partial_\rho \rho_\Psi(\Lambda) \, \rho_\Psi^\dagger \rho_\Psi(\Lambda_\sigma) \} \\
\delta_\theta \Psi^* = -\frac{1}{2}(\delta \theta)^{\mu\nu} \rho_\Psi(A_\mu) \, \rho_\Psi^\dagger \partial_\nu \Psi^* + \frac{i}{8}(\delta \theta)^{\mu\nu} \{ \rho_\Psi(A_\mu) \, \rho_\Psi^\dagger \rho_\Psi(A_\nu) \} \rho_\Psi^\dagger \Psi^* 
\]

Comparison of (51), (52) and (53) with SW differential equation for the $\rho_\Psi^*$ representation

\[
\delta_\theta \overline{\rho_\Psi^*}(A_\mu) = -\frac{1}{4}(\delta \theta)^{\rho\sigma} \{ \rho_\Psi^*(A_\rho) \, \rho_\Psi^\dagger \partial_\sigma \rho_\Psi^*(A_\mu) + \rho_\Psi^*(F_{\sigma\mu}) \} \\
\delta_\theta \overline{\rho_\Psi^*}(\Lambda) = -\frac{1}{4}(\delta \theta)^{\rho\sigma} \{ \partial_\rho \rho_\Psi^*(\Lambda) \, \rho_\Psi^\dagger \rho_\Psi^*(A_\sigma) \} \\
\delta_\theta \overline{\Psi}^* = -\frac{1}{2}(\delta \theta)^{\mu\nu} \rho_\Psi^*(A_\mu) \, \rho_\Psi^\dagger \partial_\nu \overline{\Psi}^* + \frac{i}{8}(\delta \theta)^{\mu\nu} \{ \rho_\Psi^*(A_\mu) \, \rho_\Psi^\dagger \rho_\Psi^*(A_\nu) \} \rho_\Psi^\dagger \overline{\Psi}^* 
\]

shows that if (39) holds at order $O(\delta \theta^n)$ then it is also true for $\overline{\Psi} = \Psi + \delta_\theta \Psi$, for $\rho_\Psi(A)^\dagger = \rho_\Psi(A) + \delta_\theta \rho_\Psi(A)$ and for $\rho_\Psi(\Lambda)^\dagger = \rho_\Psi(\Lambda) + \delta_\theta \rho_\Psi(\Lambda)$, i.e. (39) holds also at order $O(\delta \theta^{n+1})$. Now, since for $\theta = 0$ (39) holds, we conclude that (39) holds for finite $\theta$.

We end this section observing that if $\rho_\Psi$ is a unitary representation, then the SW differential equation for $\Psi^*$ reads

\[
\delta_\theta \overline{\Psi}^* = \frac{1}{2}(\delta \theta)^{\mu\nu} \rho_\Psi(A_\mu) \, \rho_\Psi^\dagger \partial_\nu \overline{\Psi}^* + \frac{i}{8}(\delta \theta)^{\mu\nu} \{ \rho_\Psi(A_\mu) \, \rho_\Psi^\dagger \rho_\Psi(A_\nu) \} \rho_\Psi^\dagger \overline{\Psi}^* .
\]

Since the components of $\overline{\Psi}^\dagger$ and $\overline{\Psi}^*$ are the same, comparison of (57) with the hermitian conjugate of (47) shows again that SW the map commutes with complex conjugation.

### 3.1 Gauge kinetic terms $\int Tr(\hat{F} \hat{F})$ that are invariant under $\theta \to -\theta$

It is not difficult to derive from (39) the property

\[
\overline{\rho_\Psi(F)} = -\rho_\Psi^*(F) 
\]
where (as in (40) and (42)) we recall that on the l.h.s., the SW map with \( +\theta \) is used, while the SW map with \(-\theta\) enters the r.h.s.. Reality of the gauge kinetic term then implies
\[
\int d^4x \text{Tr}(\hat{\rho}_\Psi (F) \rho_\Psi (F)) = \int d^4x \text{Tr}(\hat{\rho}_\Psi (F^\dagger) \rho_\Psi (F^\dagger)) = \int d^4x \text{Tr}(\rho_\Psi^* (F) \rho_\Psi (F))
\]
where in the last expression the SW map with \(-\theta\) is used. We thus see that the gauge kinetic term \( \int Tr(\hat{F} \hat{F}) \) can be associated with \(+\theta\) or \(-\theta\) depending on the representation used. In particular for representations that are real (i.e. \( \rho = \rho^* \) up to a similarity transformation) equality (59) implies that the gauge kinetic term is even in \( \theta \). An important example of real representations is given by the adjoint representation.

4 \( C, P \) and \( T \)

We consider the noncommutativity parameter \( \theta \) as a two-tensor that transforms covariantly under Lorentz rotations and more generally under \( C, P \) and \( T \). These transformation properties of \( \theta \) are compatible with the relation \([9, 29]\) between \( \theta \), the closed string metric \( g \) and the NS \( B \)-field (that transforms as a field strength) \( \theta = \frac{1}{g_{+B}} A \) where \( (\ )_A \) denotes the antisymmetric part of the matrix. Under time inversion and parity we explicitly have \( \Lambda^T \rightarrow \Lambda^T = \Lambda \), \( \Lambda^P \rightarrow \Lambda^P = \Lambda \) and\(^7\)
\[
\begin{align*}
\theta^{\mu\nu} \xrightarrow{T} \theta^{T\mu\nu} &= \begin{cases} 
\theta^{0j} 
\end{cases} 
A_\mu \xrightarrow{T} A_\mu^T &= \begin{cases} 
A_0 
\end{cases} 
\end{align*}
\]
\[
\begin{align*}
\Psi_L \xrightarrow{T} \Psi_L^T &= -i\sigma_1\sigma_3\Psi_L 

\partial_\mu \xrightarrow{T} \partial^T_\mu &= \begin{cases} 
-\partial_0 
\end{cases} 
\end{align*}
\]
\[
\begin{align*}
\theta^{\mu\nu} \xrightarrow{P} \theta^{P\mu\nu} &= \begin{cases} 
-\theta^{0j} 
\end{cases} 
A_\mu \xrightarrow{P} A_\mu^P &= \begin{cases} 
A_0 
\end{cases} 
\end{align*}
\]
\[
\begin{align*}
\Psi_L \xrightarrow{P} \Psi_L^P &= \Psi_R 

\partial_\mu \xrightarrow{P} \partial^P_\mu &= \begin{cases} 
\partial_0 
\end{cases} 
\end{align*}
\]
Under charge conjugation we have\(^8\)
\[
\theta^{\mu\nu} \xrightarrow{C} \theta^{C\mu\nu} = -\theta^{\mu\nu}
\]
\(^7\)We use two component spinor notation and the Weil representation \( \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \), \( \gamma^1 = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \), \( \gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \). We also write \( \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \sigma^\mu & 0 \end{pmatrix} \).
\(^8\)Charge conjugation does not act on the Lorentz structure, therefore it maps left (right) handed fermions to left (right) handed fermions.
\[ A_\mu = A^{a}_\mu T^a \rightarrow A^C_\mu = \overline{A}_\mu \]  
\[ \Lambda = \Lambda^a T^a \rightarrow \Lambda^C = -\Lambda \]  
(65)

\[ \Psi_L \rightarrow \Psi_L^C = -i\sigma_2 \Psi_R^* = \Psi^C_L \] ,  
\[ \Psi_R \rightarrow \Psi_R^C = i\sigma_2 \Psi_L^* = \Psi^C_R \] .  
(66)

where \( A_\mu = A^{a}_\mu T^a \) and \( \Lambda = \Lambda^a T^a \) are shorthand notations for the complex conjugate representation. More precisely

\[ \rho\Psi_L(T^a) \rightarrow (\rho\Psi_L(T^a))^C = -\rho\Psi_L^*(T^a) = \rho\Psi^*_L(T^a) \] ,  
(67)

where we used (37). Similarly \( \rho\Psi_R(T^a) \rightarrow -\rho\Psi_R^*(T^a) = \rho\Psi^*_R(T^a) \).

Let us consider the SW map

\[ \hat{\Psi}_L = \text{SW}[\Psi_L, \rho\Psi_L(A), \theta, \partial, i] \]

where we have written explicitly also the dependence on the partial derivatives; the imaginary unit \( i \) in the last slot marks that the coefficients in the SW map are in general complex coefficients. Since \( T \) is antilinear and multiplicative we have

\[ \hat{\Psi}_L^T = \text{SW}[\Psi_L^T, \rho\Psi_L^T(A^T), \theta^T, \partial^T, -i] \]  
(68)

where now \(-i\) means that we are considering the complex conjugates of the coefficients in the SW map. Similarly

\[ \hat{\Lambda}^T = \hat{\Lambda}[A^T, \theta^T, \partial^T, -i] \] ,  
(69)

more precisely we should write \( \rho\Psi_L(A) \rightarrow \hat{\rho}\Psi_L(A) \rightarrow -\rho\Psi_L^*(A) = \rho\Psi_R^*(A) \) and similarly for \( \hat{\Lambda} \).

We now show that the \( T \) operation on hatted variables has the same expression as on unhatted variables:

\[ \hat{\Psi}_L^T = -i\sigma_1\sigma_3\hat{\Psi}_L \] ,  
\[ \hat{A}_\mu^T = \left\{ \begin{array}{ll} -\hat{A}_0 & , \hat{\Lambda}^T = \hat{\Lambda} \end{array} \right. \]  
(70)

We first notice that (39) implies (use (37) and replace \( \Psi^* \) with \( \Psi_L \))

\[ \text{SW}[\Psi_L, \rho\Psi_L(A), \theta, \partial, -i] = \text{SW}[\Psi_L, -\rho\Psi_L(A), -\theta, \partial, i] \]  
(71)

\[ \hat{\Lambda}[\rho\Psi_L(A), \theta, \partial, -i] = -\hat{A}[-\rho\Psi_L(A), -\theta, \partial, i] \]  
(72)

and similarly for \( \hat{\Lambda} \). We then have

\[ \hat{\Psi}_L^T = \text{SW}[-i\sigma_1\sigma_3\Psi_L, -\rho\Psi_L(A_0), \rho\Psi_L(A_i), -\theta^0, \theta^j, -\partial_0, \partial_i, i] = -i\sigma_1\sigma_3\hat{\Psi}_L \]  
(73)
where in the last passage we factored $-i\sigma_1\sigma_3$ and noticed that the $-$ signs appear together with the index 0 and that since the SW map preserves the space-time index structure, the $-$ signs appear always in pairs. One proceeds similarly with $\hat{A}^T$ and $\hat{\Lambda}^T$.

In order to discuss parity and charge conjugation on noncommutative spinors we first consider noncommutative QED with just a 4-component Dirac spinor $\psi$, and decompose it into its Weil spinors $\psi_L$ and $\psi_R$. Their charge conjugate spinors are $\psi_L^C = \psi_L^\ast = -i\sigma_2\psi_R^\ast$ and $\psi_R^C = \psi_R^\ast = i\sigma_2\psi_L^\ast$. Once we define

$$\hat{\psi}_L = \text{SW}[\psi_L, \rho_{\psi_L}(A), \theta, \partial, i],$$

we then have the choice

$$\hat{\psi}_R = \text{SW}[\psi_R, \rho_{\psi_R}(A), \pm\theta, \partial, i]. \quad (74)$$

In the literature the choice $+\theta$ is usually considered so that for the 4-component Dirac spinor $\psi$ we can write $\hat{\psi} = \text{SW}[\psi, A, \theta, \partial, i]$, $\partial \hat{\psi} = i\Lambda \ast \hat{\psi}$. With this choice, the gauge potential $A$ and the noncommutativity parameter $\theta$ appear with the same sign in $\hat{\psi}_L$ and $\hat{\psi}_R$. We here advocate the opposite choice ($-\theta$) in (74). Indeed we have that

$$\hat{\psi}_R = \text{SW}[\psi_R, \rho_{\psi_R}(A), -\theta, \partial, i] \iff \hat{\psi}_L^C = \text{SW}[\psi_L^C, \rho_{\psi_L^C}(A), +\theta, \partial, i] \quad (75)$$

so that with the $-\theta$ choice in (74), both left handed fermions $\hat{\psi}_L$, $\hat{\psi}_L^C$ are associated with $\theta$ while the right handed ones $\hat{\psi}_R$, $\hat{\psi}_R^C$ are associated with $-\theta$. In GUT theories we have multiplets of definite chirality (see e.g. (17)) and therefore this is the natural choice to consider in this setting. Property (75) is easily proven (recall (39) with $-\theta$ instead of $\theta$)

$$\hat{\psi}_L^C = \text{SW}[-i\sigma_2\psi_R^\ast, \rho_{\psi_R^\ast}(A), \theta, \partial, i] = -i\sigma_2 \text{SW}[\psi_R^\ast, \rho_{\psi_R^\ast}(A), \theta, \partial, i]$$

$$= -i\sigma_2 \text{SW}[\psi_R, \rho_{\psi_R}(A), -\theta, \partial, i] = -i\sigma_2 \hat{\psi}_R^\ast. \quad (76)$$

In the following, in order to describe both $\pm\theta$ choices, we write

$$\hat{\Psi}_R = \text{SW}[\Psi_R, \rho_{\Psi_R}(A), \sigma_{\Psi_R}(\theta), \partial, i] \quad (77)$$

where $\sigma_{\Psi_R}(\theta) = \pm\theta$ depending on (74). The $T, P, C$ transformed spinors then read

$$\hat{\Psi}_R^T = \text{SW}[\Psi_R^T, \rho_{\Psi_R^T}(A^T), \sigma_{\Psi_R^T}(\theta^T), \partial^T, -i] \quad (78)$$

and

$$\hat{\Psi}_L^P = \text{SW}[\Psi_L^P, \rho_{\Psi_L^P}(A^P), \theta^P, \partial^P, i], \quad \hat{\Psi}_R^P = \text{SW}[\Psi_R^P, \rho_{\Psi_R^P}(A^P), \sigma_{\Psi_R^P}(\theta^P), \partial^P, i] \quad (79)$$

$$\hat{\Psi}_L^C = \text{SW}[\Psi_L^C, (\rho_{\Psi_L^C}(A))^C, \theta^C, \partial, i], \quad \hat{\Psi}_R^C = \text{SW}[\Psi_R^C, (\rho_{\Psi_R^C}(A))^C, \sigma_{\Psi_R^C}(\theta^C), \partial, i] \quad (80)$$
Consistently with (78)-(80) we also have \( \hat{\rho}_\Psi R (A)^T = \hat{A} [\rho_\Psi R (A^T), \sigma_\Psi R (\theta^T), \partial^T, -i] \) and
\[
\hat{\rho}_\Psi R (A)^P = \hat{A} [\rho_\Psi R (A^P), \theta^P, \partial^P, i], \quad \hat{\rho}_\Psi L (A)^C = \hat{A} [\rho_\Psi L (A)^C, \theta^C, \partial, i]
\]
(81)
and similarly for \( \hat{\Lambda} \).

If we replace \( \theta \) and \( \Psi_L \) with \( \sigma_\Psi R (\theta) \) and \( \Psi_R \) in (73) we immediately have that the \( T \) transformation on right handed hatted variables has the same expressions as on unhatted variables
\[
\hat{\psi}_R^T = -i \sigma_1 \sigma_3 \hat{\psi}_R, \quad \hat{\rho}_\Psi R (A_\mu)^T = \begin{cases} \rho_\Psi R (A_0) & , \quad \rho_\Psi R (\Lambda)^T = \rho_\Psi R (\Lambda) \end{cases}
\]
(82)
and similarly for \( \hat{\Lambda} \).

We now show that for the \(+\theta\) choice of equation (74) parity and charge conjugation on hatted variables have the same expressions as on unhatted variables:
\[
\hat{\psi}_L^P = \hat{\psi}_R, \quad \hat{\psi}_R^P = \hat{\psi}_L, \quad \hat{\Lambda}_\mu^P = \hat{\Lambda}^P = \hat{\Lambda}, \quad \hat{\rho}_\Psi R (A_0)^P = \hat{\rho}_\Psi R (\Lambda)^P = \hat{\rho}_\Psi R (\Lambda)
\]
(83)

We proceed similarly in the case of \( C \). For example reality of the SW map (39) leads to
\[
\hat{\psi}_L^C = \text{SW} [\Psi_L, -\rho_\Psi L (A), \theta^C, \partial, i] = -i \sigma_2 \text{SW} [\Psi_R, \rho_\Psi R (A), -\theta, \partial, i] = -i \sigma_2 \hat{\psi}_R^* \]
(89)
where we again used $\rho \Psi_L = \rho \Psi_R$, a necessary condition for charge conjugation symmetry. As in (89) we also have $\Psi_R^C = i\sigma_2 \Psi_R^*$. The proof of (87) easily follows from (39).

In the $-\theta$ case of (74) the equalities (84)-(87) do not hold because $\theta$ appears with the wrong sign. We can cure this by defining $\theta^P$ and $\theta^C$ with an extra $-$ sign (i.e. $\theta^P_{ij} = -\theta_{ij}$, $\theta^P_{0i} = -\theta_{0i}$ and $\theta^C = \theta$) then (84)-(87) hold.

Finally notice that independently from the $\pm \theta$ choice (and from the $CP$ symmetry of commutative actions) we can always consider the $CP$ transformed SW map, and we have
\[
\begin{align*}
\Psi_L^{CP} &= i\sigma_2 \Psi_L^* \\
\Psi_R^{CP} &= -i\sigma_2 \Psi_R^* \\
A_\mu^{CP} &= \begin{cases} 
-A_0 \\
A_i 
\end{cases}, \\
\Lambda^{CP} &= -\Lambda.
\end{align*}
\] (90)

We have concentrated on the case of constant theta in this section but the results should still be valid in the general $\theta(x)$ case. To show this one needs to consider the methods of [17] in place of the SW differential equation (which is limited to the Moyal-Weyl star product).

4.1 Noncommutative QED$_+$ and QED$_-$

QED$_\pm$ are the two different QED theories obtained with the two different $\pm \theta$ choices (74). It is easy to compare the two constructions. We have (up to gauge kinetic terms)
\[
S_{QED_+} = \int \bar{\psi} i\slashed{D} \psi = \int \bar{\psi}_L \star i\slashed{D} \psi_L + \bar{\psi}_R \star i\slashed{D} \psi_R.
\] (91)

On the other hand, the GUT inspired QED$_-$ is obtained considering the left handed spinor $\chi_L = (\psi_L \psi_C)$ so that $\bar{\psi}_C^{\LH} = \text{SW}[\psi_L^C, \rho \psi^C(A), \theta, \partial, \slashed{D}]$. We have
\[
S_{QED_-} = \int \bar{\psi}_L \star i\slashed{D} \psi_L + \bar{\psi}_C^{\LH} \star i\slashed{D} \psi_C^{\LH}.
\] (92)

Now, from (76) and $\sigma$ matrices algebra we have $\int \bar{\psi}_C^{\LH} \star i\slashed{D} \psi_C^{\LH} = \int \bar{\psi}_R^{\text{op}} \star^{\text{op}} i\slashed{D} \psi_R^{\text{op}}$, where we have emphasized that we are using the $-\theta$ convention in the SW map by writing $\sim^{\text{op}}$ instead of $\sim$. We conclude that in order to obtain QED$_-$ from QED$_+$ we just need to change $\theta$ into $-\theta$ in the right handed fermion sector of QED$_+$. 

5 $C, P, T$ properties of NCYM actions.

In this section we derive the transformations properties of NCYM actions. We assume that $\theta$ transforms as in (60), (62) and (65). Then in the $+\theta$ choice (74) we have that
NCYM actions are invariant under $C, P$ and $T$ iff in the commutative limit they are invariant. On the other hand, in the $-\theta$ choice NCYM actions are invariant under $CP$ and $T$ iff in the commutative limit they are invariant. For the fermion kinetic term these statements are a straightforward consequence of $\int \hat{\Psi}_L^\dagger \star \partial \hat{\Psi}_L = \int \hat{\Psi}_L^\dagger \bar{\partial} \hat{\Psi}_L$ (and similarly for $\hat{\Psi}_R$). Since $\hat{F}$ transforms like $F$ under $CP$ and $T$, and in the $+\theta$ case also under $C$ and $P$ separately,\(^9\) the $C, P, T$ properties of the gauge kinetic term $\int Tr(\hat{F} \star \hat{F}) = \int Tr(\hat{F} \bar{\partial} \hat{F})$ easily follow. Inspection of the fermion gauge bosons interaction term leads also to the same conclusion. For sake of clarity we treat separately the $+\theta$ case and the $-\theta$ choice (cf. (74)).

**+$\theta$ case** For a 4-component Dirac spinor the interaction term is $\bar{\psi} \gamma^\mu \hat{A}_\mu \star \psi$. Invariance under $P$ and $T$ transformations is straightforward since $P$ and $T$ do not change the $\star$ product. Invariance under charge conjugation follows from

\[
\left( \bar{\psi} \gamma^\mu \hat{A}_\mu \star \psi \right)^C = \bar{\psi}^C \star^C \gamma^\mu \hat{A}_\mu \star \psi^C = i(\gamma^0 \gamma^2 \gamma^2) \star^C i(\gamma^0 \gamma^2) \star \left( \bar{\psi} \gamma^\mu \hat{A}_\mu \star \psi \right)^t
\]

where we used hermiticity of $\hat{A}$, the standard gamma matrix algebra and, as usual, that spinors anticommute.

**-$\theta$ case** In two components notation we have the interaction terms

\[
\hat{\Psi}_L^\dagger \star \hat{A} \star \hat{\Psi}_L + \hat{\Psi}_R^\dagger \star^p \hat{A} \star^p \hat{\Psi}_R
\]  

(93)

Notice that $\hat{\Psi}_R$ (consistently with (74)) commands the opposite star product $\star^p$. Invariance under time reversal is straightforward since $T$ leaves invariant the $\star$ and $\star^p$ products. Under $CP$ we have

\[
\left( \hat{\Psi}_L^\dagger \star \hat{A} \star \hat{\Psi}_L \right)^{CP} = \hat{\Psi}_L^\dagger^{CP} \star^{CP} \sigma \hat{A}^{CP} \star^{CP} \hat{\Psi}_L^{CP} = \left( i\sigma_2 \hat{\Psi}_L^{\star} \right)^t \star^p \sigma \left( -\hat{A} \right)^P \star^p i\sigma_2 \hat{\Psi}_L^{\star}
\]

\[
= \hat{\Psi}_L^t i\sigma_2 \star^p \sigma \hat{A}^t \star^p i\sigma_2 \hat{\Psi}_L^* = -\hat{\Psi}_L^t \star^p \sigma \hat{A}^t \star^p \hat{\Psi}_L^*
\]

(94)

Similarly for $\hat{\Psi}_R$.

We have studied the $C, P$ and $T$ symmetry properties of NCYM actions where the $\theta$ transformations under $C, P$ and $T$ are given in (62),(60) and (65). Viceversa, if we

\(^9\) $\hat{F}$ transforms like $F$ under $P$ and $T$ because $P$ and $T$ leave invariant the $\star$-product (indeed $(i\theta^{\mu
u} \partial_\mu \partial_\nu)^T = -i (\theta^{\mu
u} \partial_\mu \partial_\nu)^T = i\theta^{\mu\nu} \partial_\mu \partial_\nu$). Under charge conjugation we also have $\hat{F}^C = -\hat{F}$, indeed $i[\hat{A}^C \gamma^C \hat{A}^C] = -i[\hat{A} \star \hat{A}]$ because the action of $C$ on $\star$ equals complex conjugation.
keep $\theta$ fixed under $C, P$ and $T$ transformations, we in general have that NCYM theories break $C, P$ and $T$ symmetries. Notice however that in the $-\theta$ case, if we keep $\theta$ fixed under $C$, then the SW map is well behaved under $C$, and $C$ is a symmetry of a NCYM action if it is a symmetry of the corresponding commutative one and the gauge kinetic term $\int Tr(\hat{F}\hat{F})$ is even in $\theta$. For example one can check that when $\rho_{\Psi_L} = \rho_{\Psi_R}$, the sum in (93) is invariant under $C$.

Finally, from (62), (60) (65) (and (61), (63), (66)) it follows that under the combined $CPT$ transformation, $\theta$ does not change, and therefore $CPT$ (with fixed $\theta$) is always a symmetry of NCYM actions. In models where $\theta$ changes sign under $CPT$, e.g., for nonconstant $\theta^{\mu\nu} = C^\mu_\rho x^\rho$ with fixed background $C^\mu_\rho$, we do expect spontaneous breaking of $CPT$.

cpt breaking

Here we do not consider the $CPT$ operator, but the $cpt$ one. $CPT$ and $cpt$ differ only in their action on $\theta$, we have $\theta^{cpt} = -\theta$ (while $\theta^{CPT} = \theta$). In particular, in the commutative case $\theta = 0$, we have $CPT = cpt$. The transformation $\theta^{cpt} = -\theta$ can be justified by a quantum mechanics analogy. In QM the antiunitary $cpt$ operator acts on the $x^i$ and $p^i$ operators via conjugation so that $cpt(x\circ p) cpt^{-1} = cpt x\circ cpt^{-1} \circ cpt p\circ cpt^{-1} = -x\circ p$, (and the $[x^i, p^j] = i\hbar \delta^i_j$ relations are invariant under $cpt$). It is then natural to define $cpt(x^\mu \circ x^\nu) cpt^{-1} = cpt x^\mu cpt^{-1} \circ cpt x^\nu cpt^{-1}$, that using the $\star$-product representation reads $cpt(x^\mu \star x^\nu) cpt^{-1} = cpt x^\mu cpt^{-1} \star cpt x^\nu cpt^{-1}$. From here we see that $cpt$ does not act on the $\star$-product. Since $\star \sim e^{\frac{i}{\hbar} \theta^{\mu\nu}\delta_\mu \delta_\nu}$ we must have $(i\theta^{\mu\nu})^{cpt} = cpt (i\theta^{\mu\nu})^{cpt^{-1}} = i\theta^{\mu\nu}$ and therefore $cpt (\theta^{\mu\nu})^{cpt^{-1}} = -\theta^{\mu\nu}$. These considerations may be generalized to an $x$ dependent $\theta$. If $\theta^{\mu\nu} = C^{\mu\rho, \nu}_\rho$ this means $(C^{\mu\rho, \nu}_\rho)^{cpt} = C^{\mu\rho, \nu}_\rho$; or we may consider $\theta^{\mu\nu} = b^\mu x^\nu - b^\nu x^\mu$ with $(b^\mu)^{cpt} = b^\mu$.

The NCYM actions we have studied are invariant under $CPT$ and therefore under $cpt$ they are invariant iff they are even in $\theta^{\mu\nu}$. Usually this is not the case, for example $\bar{\Psi} \star \hat{A} \star \bar{\Psi}$ has a nonvanishing term linear in $\theta$. We conclude that under $cpt$ the NCYM actions we have studied are not invariant, and $cpt$ is explicitly broken. One could consider NCYM actions even in $\theta$, for example the most general $SU(2)$ pure gauge kinetic term is even in $\theta$ because $SU(2)$ has only real representations (see Subsection 3.1). In this case the action is invariant under $cpt$ but $cpt$ is spontaneously broken because $\theta$ itself is not invariant under $cpt$. Viceversa if $\theta^{\mu\nu} = C^\mu_\rho x^\rho$ we have $(C^{\mu\nu}_\rho)^{cpt} = C^{\mu\nu}_\rho$ and with respect to the fixed background $C^{\mu\nu}_\rho$, $cpt$ is not broken.

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22
**Appendix: Seiberg-Witten maps and tensor products**

**Gauge parameter:** The most general solution (up to redefinition of the ordinary field $A$, and $\Lambda$) to the consistency relation (CR) [11] 

$$[\hat{\Lambda}_\alpha[A] \dagger \hat{\Lambda}_\beta[A]] + i\delta_\alpha \hat{\Lambda}_\beta[A] - i\delta_\beta \hat{\Lambda}_\alpha[A] = \hat{\Lambda}_{[\alpha,\beta]}[A],$$

(95)

to first order in $\theta$ (which may be non-constant) is

$$\hat{\Lambda}_\Lambda[A] = \Lambda + \frac{1}{2} \theta^{ij} \{ A_j, \partial, \Lambda \}_c + O(\theta^2),$$

(96)

where (cf. (9)) $\hat{\Lambda}_\alpha[A] = \hat{\Lambda}[\alpha, A, \theta]$, and for any two matrices $P$ and $Q$

$$\{ P, Q \}_c \equiv cP \cdot Q + (1 - c)Q \cdot P = \frac{1}{2} \{ P, Q \} + (c - \frac{1}{2})[P, Q].$$

(97)

The requirement of hermiticity singles out the preferred choice $c = 1/2$ plus possibly a purely imaginary function of space-time. The function $c$ also appears in the following paragraphs.

**Covariantizing map:** The most general differential operator (up to ordinary field redefinitions) that is a local function of the ordinary gauge potential $A_i$ and turns a function $f$ into a covariant function $D[A](f)$, with

$$\delta_\Lambda D[A](f) = i[\hat{\Lambda}_\Lambda[A] \dagger D[A](f)],$$

(98)

to second order in $\theta$ (which may be non-constant) is\(^{10}\)

$$D[A] = \text{id} + \theta^{ij} A_j \partial_i + \frac{1}{2} \theta^{ij} \{ A_j, \partial, (\theta^{kl} A_l) \}_c \partial_k + \frac{1}{2} \theta^{ij} \theta^{kl} A_j A_l \partial_i \partial_k + \frac{1}{2} \theta^{ij} \theta^{kl} \{ F_{il}, A_j \}_c \partial_k + O(\theta^3).$$

(99)

**NC gauge potential:** Given the covariantizing map $D[A]$ the noncommutative gauge potential $\tilde{A}^i$ can be read off from the equation

$$D[A](x^i) = x^i + \tilde{A}^i.$$  

(100)

This gives the following expression for $\tilde{A}^i$ valid to first order in $\theta$ (which may be non-constant)

$$\tilde{A}^i = \theta^{il} A_l + \frac{1}{2} \theta^{kli} \{ A_j, \partial_k (\theta^{il} A_l) \}_c + \frac{1}{2} \theta^{kl} \theta^{il} \{ F_{il}, A_j \}_c + O(\theta^2).$$

(101)

For constant non-degenerate $\theta^{il}$ it is more convenient to work with $\hat{A}_l$, where $\tilde{A}^i = \theta^{il} \hat{A}_l$.

\(^{10}\)Note: $\frac{1}{2} \theta^{ij} \theta^{kl} \{ A_j, A_l \}_c \partial_i \partial_k \equiv \frac{1}{2} \theta^{ij} \theta^{kl} A_j A_l \partial_i \partial_k$. 

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**NC Matter field:** The most general SW map (up to ordinary field redefinitions) for (scalar) matter field is

\[
\hat{\Psi} = \psi + \frac{1}{2} \theta^{ij} A_j \partial_i \psi + \frac{1}{4} \theta^{ij}(\partial_i A_j) \psi - \frac{i}{4} \theta^{ij}\{A_i, A_j\}c\psi + O(\theta^2) .
\] (102)

Note: We have used the classical field redefinition freedom \(\psi \mapsto \psi + \mu \theta^{ij} F_{ij} \psi\) with \(\mu = c/4 - 1/8\) to obtain the above formula (where all products of matrices appear within brackets \(\{,\}_c\)).

**Tensor products \(G \times G'\)**

Consider the Seiberg-Witten map \(\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A']\) for the gauge parameter corresponding to a product \(G \times G'\) of gauge groups. By linearity in the ordinary gauge parameters \(\Lambda\) and \(\Lambda'\) we have

\[
\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A'] = \hat{\Lambda}_A[A, A'] + \hat{\Lambda}_{A'}[A, A'] .
\] (103)

The combined gauge parameter \(\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A']\) should satisfy the CR (95). This implies that in general \(\hat{\Lambda}_A[A, A']\) and \(\hat{\Lambda}_{A'}[A, A']\) satisfy CR’s individually and that there are also new mixed CR’s:

\[
[\hat{\Lambda}_A, \hat{\Lambda}_{A'}] + i\delta_A \hat{\Lambda}_{A'} - i\delta_{A'} \hat{\Lambda}_A = 0
\] (104)

and ditto with \(\hat{\Lambda} \leftrightarrow \hat{\Lambda}'\). Note that there is no inhomogeneous term on the RHS because \([\Lambda, \Lambda'] = 0\). The most general hermitian solution to these equations to order \(\theta\) is

\[
\hat{\Lambda}_{(\Lambda, \Lambda')}[A, A'] = \Lambda + \Lambda' + \frac{1}{2} \theta^{ij} \left(\{A_j, \partial_i \Lambda\}_c + \{A'_j, \partial_i \Lambda'\}_d\right) + (1 - \gamma) \theta^{ij} A'_j \partial_i \Lambda + \frac{\gamma}{2} \theta^{ij} A_j \partial_i \Lambda' + O(\theta^2)
\] (105)

where \(\gamma\) is a real function on space-time and \(c-1/2, d-1/2\) are pure imaginary functions. Comparing (105) with (96) we see that we had to use the freedom of field redefinitions \(A \rightarrow A + (4 - 2\gamma) A', A' \rightarrow 2\gamma A' + A\), to find the gauge parameter for \(G \times G'\). An important special case is given by \(\gamma = 1\) and \(d = c\): The corresponding \textbf{symmetric solution} can be obtained by applying the formula (96) to \(\Lambda + \Lambda'\) and \(A + A'\). Other interesting special cases are the \textbf{asymmetric solutions} for \(\gamma = 2\) and \(\gamma = 0\): Here one of the two terms in (103) is given by the ordinary SW map (96).

For the product of (scalar) fields \(\Psi \Psi'\), where \(\Psi\) transforms under \(G\) and \(\Psi'\) transforms under \(G'\) we can also construct a SW map \(\hat{\Psi}[\Psi, \Psi', A, A']\).\(^{11}\) The most general solution

\(^{11}\)Note that the naive choice \(\hat{\Psi} \star \hat{\Psi}'\) does not work, because the gauge parameter \(\hat{\Lambda}'\) does not \(\star\)-commute with \(\hat{\Psi}\) in the second term of \(\delta(\hat{\Psi} \star \hat{\Psi}') = i\hat{\Lambda} \star \hat{\Psi} \star \hat{\Psi}' + i\hat{\Psi} \star \hat{\Lambda}' \star \hat{\Psi}'\).
to order $\theta$ is
\[
\hat{\Psi} [\Psi, \Psi', A, A'] = \Psi \Psi' + \theta^{\mu\nu} \rho_{\mu\nu} \Psi \Psi' - \frac{i}{2} (1 + \gamma) \theta^{\mu\nu} \partial_\nu \Psi \partial_\mu \Psi' + \frac{1}{2} \theta^{\mu\nu} (A'_\nu + \gamma A_\nu) \Psi \partial_\mu \Psi' \\
+ \frac{1}{2} \theta^{\mu\nu} (A_\nu + (2 - \gamma) A'_\nu) \partial_\mu \Psi \Psi' + \frac{1}{2} \theta^{\mu\nu} ((1 - d) \partial_\mu A'_\nu + (1 - c) \partial_\mu A_\nu) \Psi \Psi' + O(\theta^2) \quad (106)
\]
where $\gamma, c, d$ are as in equation (105) and $\rho_{\mu\nu}$ is any function (or differential operator) that may depend on the gauge potentials $A, A'$ and that satisfies $\delta_\Lambda (\rho_{\mu\nu} \Psi \Psi') = i \Lambda \rho_{\mu\nu} \Psi \Psi'$ and $\delta_{\Lambda'} (\rho_{\mu\nu} \Psi \Psi') = i \Lambda' \rho_{\mu\nu} \Psi \Psi'$. A possibility is $\rho_{\mu\nu} = \rho_{\mu\nu} (F_{\mu\nu}, F'_{\mu\nu}, x)$. A similar somewhat lengthy expression exists also for the noncommutative gauge potential.

An alternative strategy for the construction of the SW map for products of fields can be based on the hybrid SW map (32): The SW map for the product of fields $\Psi$ and $\Psi'$ can be written as (cf. (35))
\[
\hat{\Psi} [\Psi, \Psi', A, A'] = \hat{\Psi}^H [\Psi, A + A', A'] \ast \hat{\Psi}' [\Psi', A']. \quad (107)
\]
The gauge transformation $\delta \Psi = i \Lambda \Psi, \delta \Psi' = i \Lambda' \Psi', \delta A_\mu = \partial_\mu \Lambda + i [\Lambda, A_\mu], \delta A'_\mu = \partial_\mu \Lambda' + i [\Lambda', A'_\mu]$ induces the desired transformation
\[
\delta \hat{\Psi} [\Psi, \Psi', A, A'] = i \hat{\Lambda}_{(\Lambda + \Lambda')} [A + A'] \ast \hat{\Psi} [\Psi, \Psi', A, A'], \quad (108)
\]
where we have used that $\Psi$ and $\Lambda'$ commute. We see that the version of the hybrid SW map under consideration corresponds exactly to the symmetric solution for the gauge parameter.

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