Loop Variables and Gauge Invariant Interactions in String Theory

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Abstract

We describe a method of writing down the exact interacting gauge invariant equations for all the modes of the bosonic open string. It is a generalization of the loop variable approach that was used earlier for the free, and lowest order interacting cases. The generalization involves, as before, the introduction of a parameter to label the different strings involved in an interaction. The interacting string has thus becomes a “band” of finite width. As in the free case, the fields appear to be massless in one higher dimension. Although a proof of the consistency and gauge invariance to all orders (and thus of equivalence with string theory) is not yet available, plausibility arguments are given. We also give some simple illustrations of the procedure.

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1 Introduction

The loop variable approach introduced in \[\text{I}\] (hereafter I) is an attempt to write down gauge invariant equations of motion for both massive and massless modes. This method being rooted in the sigma model approach \[\text{II, III, IV, V, VI}\], the computations are expected to be simpler and the gauge transformation laws more transparent. This hope was borne out at the free level and also to a certain extent in the interacting case \[\text{II}\] (hereafter II). The gauge transformations at the free level can be summarized by the equation

\[k(t) \rightarrow k(t)\lambda(t)\]  

(1.1)

Here \(k(t)\) is the generalized momentum Fourier-conjugate to \(X\) and \(\lambda\) is the gauge parameter. This clearly has the form of a rescaling and one can speculate on the space-time interpretation of the string symmetries as has been done for instance in I.

In II the interacting case was discussed. It was shown that the leading interactions could be obtained by the simple trick of introducing an additional parameter ‘\(\sigma\)’ as \(k(t) \rightarrow k(t, \sigma)\), parametrizing different interacting strings. Thus, for instance, \(k_{i}^{\mu}(\sigma_{1})k_{j}^{\nu}(\sigma_{2})\) could stand for two massless photons when \(\sigma_{1} \neq \sigma_{2}\), but when \(\sigma_{1} = \sigma_{2}\) it would represent a massive “spin 2” excitation of one string. The gauge transformations admit a corresponding generalization

\[k(t, \sigma) \rightarrow k(t, \sigma) \int d\sigma_{1} \lambda(t, \sigma_{1})\]  

(1.2)

It was shown, however that this prescription introduces only the leading interaction terms.

In this talk we show that there is a natural generalization of this construction to include the full set of interactions that one expects based on the operator product expansion (OPE) of vertex operators. We verify the gauge invariance (under 1.2) in one non-trivial equation. We give arguments why we expect this to hold for all the equations. We do not have a complete proof of this yet.

This paper is organized as follows. In section II we give a short review of II and elaborate on the role of the parameter \(\sigma\). In section III we describe the generalization referred to above. In section IV we give some examples. Section V contains some concluding remarks.
2 Review

In I the following expression was the starting point to obtain the equations of motion at the free level:

\[ A = e^{\frac{k_0}{2} \Sigma + k_0 \partial_x \Sigma + \sum_{n,m} k_n k_m (\frac{\partial^2}{\partial x_n \partial x_m} - \frac{\partial^2}{\partial x_{n+m}}) \Sigma + i k_n Y_n} \tag{2.3} \]

The prescription was to vary w.r.t \( \Sigma \) and evaluate at \( \Sigma = 0 \) to get the equations of motion. Here, \( 2\Sigma \equiv < Y(z) Y(z) > \) and \( Y = \sum_n \alpha_n \frac{\partial^n X}{(n-1)!} \equiv \sum_n \alpha_n \tilde{Y}_n \). \( \alpha_n \) are the modes of the einbein \( \alpha(t) \) used in defining the loop variable

\[ e^{i \int_0^t \alpha(t) k(t) \partial_x X(z+t) dt} \equiv e^{i \sum_n \alpha_n \tilde{Y}_n} \]

One can also show easily that \( Y_n = \frac{\partial Y}{\partial x_n} \). \( \Sigma \) is thus a generalization of the Liouville mode, and what we have is a generalization of the Weyl invariance condition on vertex operators.

There is an alternative way to obtain the \( \Sigma \) dependence \[11\]. This is to perform a general conformal transformation on a vertex operator by acting on it with \( e^{\sum_n \lambda_n L_n + \sum_n \lambda_{-n} L_{-n}} \) using the relation \[13\]:

\[ e^{\sum_n \lambda_n L_n + \sum_n \lambda_{-n} L_{-n}} \equiv e^{K_m \lambda_{-n-m} + \sum_n \tilde{Y}_m \lambda_{n+m} + imK_n \lambda_{-n-m} e^{iK_m \tilde{Y}_m} \tag{2.5} \]

The anomalous term is \( K_n \lambda_{-n-m} \) and the classical term is \( mK_n \lambda_{n+m} \). We will ignore the classical piece: this can be rewritten as a \( (mass)^2 \) term which will be reproduced by performing a dimensional reduction and other pieces involving derivatives of \( \Sigma \) (defined below) that correspond to field redefinitions \[1\]. We can apply (2.3) to the loop variable (2.4) by setting \( K_m = \sum_n k_{m-n} \alpha_n \). Defining

\[ \Sigma = \sum_{p,q} \alpha_p \alpha_q \lambda_{-p-q} \tag{2.6} \]

we recover (??). It is the approach described above that generalizes more easily to the interacting case.

The equations thus obtained are invariant under

\[ k_n \rightarrow \sum_m k_{n-m} \lambda_m \tag{2.7} \]

\(^1\)This relation is only true to lowest order in \( \lambda \). The exact expression is given in \[13\]
which is just the mode expansion of (1.1).

That this is an invariance of the equations of motion derived from (??) follows essentially from the fact that the transformation (2.7), applied to (??)
changes it by a total derivative.

$$\delta A = \sum_n \lambda_n \frac{\partial}{\partial x^n}[A] \quad (2.8)$$

Thus the equations obtained from (??) cannot be affected.

However there are some caveats. In proving (2.8) one needs to use equations such as

$$\left(\frac{\partial}{\partial x^1} - \frac{\partial}{\partial x^2}\right) \Sigma = \left(\frac{\partial^3}{\partial x_1^3} - \frac{\partial^2}{\partial x_1 \partial x_2}\right) \Sigma = 2 \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_3}\right) \Sigma \quad (2.9)$$

which follow from the basic definitions \([1]\). This results in certain constraints having to be satisfied in order for the equations to be invariant. An example is the tracelessness constraint

$$\lambda_1 k_1 k_1 = 0 \quad (2.10)$$

described in I. For further details we refer the reader to \([1]\) and \([3]\).

In II this approach was generalized to include some interactions. The basic idea was to introduce a new parameter $\sigma : 0 \leq \sigma \leq 1$ to label different strings and to replace each $k_n$ in the free equation by $\int_0^1 d\sigma k_n(\sigma)$. The next step was to assume that

$$<k_1^\mu(\sigma_1)k_1^\nu(\sigma_2)> = S^{\mu\nu}\delta(\sigma_1 - \sigma_2) + A^\mu A^\nu \quad (2.11)$$

where $<...>$ denotes $\int Dk(\sigma)...\psi[k(\sigma)]$, $\psi$ being the “string field” defined in I.\(^2\) This corresponds to saying that when $\sigma_1 = \sigma_2$, both the $k_1$’s belong to the same string and otherwise to different strings where they represent two photons at an interaction point.\(^3\) The gauge transformation is replaced by (1.2). This is easily seen to give interacting interacting equations. However the fact is that this is only a leading term in the infinite set of interaction vertices.

\(^2\)No special property of $\psi$ is assumed other than this.

\(^3\)It is conceivable that (2.11) may have to be generalized by replacing the $\delta$-function on the RHS by something else, when we go to the fully interacting case. However in this talk we will not do so - in the example considered (albeit a very simple one) (2.11) seems to be sufficient.
As a prelude to generalizing this construction, let us explain more precisely the nature of the replacement \( k_n \rightarrow \int_0^1 d\sigma k_n(\sigma) \). Let us split the interval \((0, 1)\) into \(N\) bits of width \(a = \frac{1}{N}\). We will assume that when \(\sigma\) satisfies \(\frac{n}{N} \leq \sigma \leq \frac{n+1}{N}\) it represents the \((n+1)\)th string. Let us also define a function

\[
D(\sigma_1, \sigma_2) = \begin{cases} 
1 & \text{if } \sigma_1, \sigma_2 \text{ belong to the same interval} \\
0 & \text{if } \sigma_1, \sigma_2 \text{ belong to different intervals.} 
\end{cases}
\] (2.12)

Thus \(\int_0^1 d\sigma_1 \int_0^1 d\sigma_2 D(\sigma_1, \sigma_2) = a = \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 D(\sigma_1, \sigma_2)\).

Then we set

\[
<k_\mu(\sigma_1)k_\nu(\sigma_2)> = \frac{D(\sigma_1, \sigma_2)}{a} S^\mu\nu + A^\mu A^\nu
\] (2.13)

In the limit \(N \rightarrow \infty, a \rightarrow 0\), \(\frac{D(\sigma_1, \sigma_2)}{a} \approx \delta(\sigma_1 - \sigma_2)\) and we recover (2.11).

In effect (2.12) has been modified to

\[
e^{\int_0^1 d\sigma_1 \int_0^1 d\sigma_2 [k_0(\sigma_1)k_0(\sigma_2)\Sigma + k_n(\sigma_1)k_0(\sigma_2)\partial_{\sigma_n}\Sigma + \Sigma + \sum_{n,m}(\partial_{\sigma_n}\partial_{\sigma_m} - \frac{a}{\sigma_{n+m}})\Sigma] + \int_0^1 d\sigma k_n(\sigma)Y_n}
\] (2.14)

The final step (which is also necessary in the free case), is to dimensionally reduce to obtain the massive equations. For details we refer the reader to I.

The modification (2.11), that replaces \(S^\mu\nu\) by \(S^\mu\nu + A^\mu A^\nu\) can be understood in terms of the OPE. Consider a correlation function involving two vector vertex operators and any other set of operators, that we represent as

\[
\mathcal{A} = <V_1V_2...V_N : k_1^\mu \partial_z X^\mu e^{ik_0 X} : q_1^\nu \partial_w X^\nu e^{iq_0 X} >
\] (2.15)

The OPE of \(k_1^\mu \partial_z X^\mu(z)e^{ik_0 X}\) and \(q_1^\nu \partial_w X^\nu(w)e^{iq_0 X}\) is given by

\[
: k_1^\mu \partial_z X^\mu(z)e^{ik_0 X} :: q_1^\nu \partial_w X^\nu(w)e^{iq_0 X} :=
\]

\[
k_1^\mu q_1^\nu \partial_z X^\mu \partial_w X^\nu e^{i(k_0 X(z)+q_0 X(w))} + \text{ terms involving contractions.}
\] (2.16)

We can Taylor expand

\[
X(w) = X(z) + (w - z)\partial_z X + O(w - z) + ...
\] (2.17)

This gives for the leading term in (2.15)

\[
\mathcal{A} = <V_1V_2...V_N : k_1^\mu q_1^\nu \partial_z X^\mu \partial_z X^\nu e^{i(k_0 X(z)+q_0 X(w))} >
\] (2.18)
Compare this with the correlation involving $S^{\mu\nu}$:

$$A' = <V_1 V_2 \ldots V_N : k_1^\mu k_1^\nu \partial_2 X^\mu \partial_2 X^\nu e^{ik_0 X} :>$$  \hspace{1cm} (2.19)$$

We see that $A$ and $A'$ give identical terms except that $S^{\mu\nu}$ is replaced by $A^\mu A^\nu$. It is in this sense that the substitution given in II, gives the leading term in the OPE. The crucial point is that, while in (2.14) we have introduced the parameter $\sigma$ in the $k_n$'s we have not done so for the $Y_n$'s. This is equivalent to approximating $X(w)$ by $X(z)$ in (2.17). Clearly, the generalization required to get all the terms is to introduce the parameter $\sigma$ in $Y$ also. We turn to this in the next section.

3 Introducing $\sigma$-dependence in the loop variable

We will introduce the parameter $\sigma$ in all the variables keeping in mind the basic motivation that $\sigma$ labels different vertex operators. Thus all the variables that are required to define a vertex operator become $\sigma$ dependent. Thus

$$X^\mu(z) \to X^\mu(z(\sigma))$$  \hspace{1cm} (3.20)

$$x_n \to x_n(\sigma)$$  \hspace{1cm} (3.21)

in addition to

$$k_n^\mu \to k_n^\mu(\sigma)$$  \hspace{1cm} (3.22)

(3.20) and (3.21) imply that

$$\frac{\partial}{\partial x_n} Y \to \frac{\partial}{\partial x_n(\sigma)} Y(z(\sigma), x_n(\sigma))$$  \hspace{1cm} (3.23)

Note that $X$ need not be an explicit function of $\sigma$ since at a given location $z$, on the world sheet there can only be one $X(z)$. As an example of the above consider the case when we have regions $(0,1/2)$ and $(1/2,1)$. When $0 \leq \sigma \leq 1/2$ one has $z(\sigma) \equiv z$ and for $1/2 \leq \sigma \leq 1$ one has $z(\sigma) \equiv w$. Similarly $x_n(\sigma)$ could be called $x_n, y_n$ in the two regions and $k_n(\sigma)$ could be called $k_n, p_n$ in the two regions. Thus in this example the vertex operator $k_n(\sigma) Y_n(z(\sigma), x_n(\sigma)) e^{ik_0(\sigma)} Y(\sigma)$ stands for $k_n \frac{\partial Y}{\partial x_n}(z,x) e^{ik_0 Y(z,x_n)}$ and $p_n \frac{\partial Y}{\partial y_n}(w,y) e^{ip_0 Y(w,y_n)}$ in the two regions.
We will further assume that the integration measure of the free theory $[dx_1dx_2....]$ is replaced by $[Dx_1(\sigma)Dx_2(\sigma)....]$ so that we can continue to integrate by parts. However we have to clarify what we mean by a derivative w.r.t $x_n(\sigma)$: In (B.23) we have $\frac{\partial Y(z(\sigma),x_i(\sigma))}{\partial x_n(\sigma)}$: One has to specify the meaning of $\frac{\partial x_n(\sigma)}{\partial x_n(\sigma')}$.

Clearly what we want is: If $\sigma, \sigma'$ belong to the same interval, then $\frac{\partial x_n(\sigma)}{\partial x_n(\sigma')} = 1$ and zero otherwise. Thus using (2.12)

$$\frac{\partial x_n(\sigma)}{\partial x_n(\sigma')} = D(\sigma, \sigma')$$  \hspace{1cm} (3.24)

or more generally

$$\frac{\partial x_n(\sigma)}{\partial x_m(\sigma')} = \delta_{mn}D(\sigma, \sigma')$$  \hspace{1cm} (3.25)

Note that this is not the same as the conventional functional derivative. However we can define

$$\frac{\delta x_n(\sigma)}{\delta x_m(\sigma')} \equiv \frac{D(\sigma, \sigma')}{a}$$  \hspace{1cm} (3.26)

which, in the limit $a \to 0$ becomes the usual functional derivative. Thus

$$\int d\sigma' \frac{\delta Y(\sigma)}{\delta x_n(\sigma')} = \frac{\partial Y(\sigma)}{\partial x_n(\sigma)}$$  \hspace{1cm} (3.27)

We can now write down the generalization of (??)

$$\exp\{\int \int d\sigma_1d\sigma_2\left(k_0(\sigma_1).k_0(\sigma_2)\right)\left[\tilde{\Sigma}(\sigma_1, \sigma_2) + \tilde{G}(\sigma_1, \sigma_2)\right]
$$

$$+ \sum_{n>0} \int d\sigma_3 k_n(\sigma_1).k_0(\sigma_2) \frac{\delta}{\delta x_n(\sigma_1)}\left[\tilde{\Sigma}(\sigma_3, \sigma_2) + \tilde{G}(\sigma_3, \sigma_2)\right]
$$

$$+ \int \int d\sigma_3d\sigma_4 \sum_{n,m>0} k_n(\sigma_1).k_m(\sigma_2)
$$

$$\frac{1}{2} \left[\frac{\delta^2}{\delta x_n(\sigma_1)\delta x_m(\sigma_2)} - \delta(\sigma_1 - \sigma_2)\frac{\delta}{\delta x_{n+m}(\sigma_1)}\right]\left[\tilde{\Sigma}(\sigma_3, \sigma_4) + \tilde{G}(\sigma_3, \sigma_4)\right]\right\}
$$

$$\exp\{i \int d\sigma k_n(\sigma)Y_n(\sigma)\}$$  \hspace{1cm} (3.28)
In (3.28) \( G(\sigma_1, \sigma_2) = \tilde{G}(z(\sigma_1), z(\sigma_2)) = \langle Y(z(\sigma_1))Y(z(\sigma_2)) \rangle \) is the Green function which starts out as \( \ln(z_1 - z_2) \). More precisely, if we define: [Using the notation \( z_i = z(\sigma_i) \)]

\[
D_{z_1} = D_{z(\sigma_1)} \equiv 1 + \alpha_1(\sigma_1) \frac{\partial}{\partial z(\sigma_1)} + \alpha_2 \frac{\partial^2}{\partial z^2(\sigma_1)} + ... \tag{3.29}
\]

so that

\[
Y(z(\sigma)) = D_{z(\sigma)} X(z(\sigma)) \tag{3.30}
\]

then,

\[
\tilde{G}(z_1, z_2) = D_{z_1} D_{z_2} G(z_1, z_2) \tag{3.31}
\]

\[
\tilde{\Sigma}(\sigma_1, \sigma_2) = D_{z_1} D_{z_2} \Sigma(\sigma_1, \sigma_2) \tag{3.32}
\]

where

\[
\Sigma(\sigma_1, \sigma_2) = \frac{\lambda(z(\sigma_1)) - \lambda(z(\sigma_2))}{z(\sigma_1) - z(\sigma_2)} \tag{3.33}
\]

is the generalization of the usual \( \Sigma(\sigma) \) which is equal to \( \frac{d\lambda}{dz} \). The \( \tilde{\Sigma} \) dependence in (3.28) is obtained by the following step:

\[
e^{\frac{1}{2} \int du \lambda(u) [\partial_x X(z+u)]^2} e^{ik_n \frac{\partial}{\partial x_n} D_{z_1} X} e^{ip_m \frac{\partial}{\partial x_m} D_{z_2} X} \tag{3.34}
\]

defines the action of the Virasoro generators on the two sets of vertex operators.

\[
= e^{ik_n p_m \partial_{x_n} \partial_{y_m} D_{z_1} D_{z_2} \frac{\lambda(u)}{z_1 - z_2}} \frac{1}{z_1 - u - 1} \frac{1}{z_2 - u} \tag{3.35}
\]

\[
= e^{ik_n p_m \partial_{x_n} \partial_{y_m} \tilde{\Sigma}} \tag{3.36}
\]

This expression is only valid to lowest order in \( \lambda \) which is all we need here.\footnote{The exact expression is given in \cite{13}}

The expression

\[
\int \int d\sigma_1 d\sigma_2 \frac{1}{2} \frac{\delta^2}{\delta x_n(\sigma_1) \delta x_m(\sigma_2)} - \delta(\sigma_1 - \sigma_2) \frac{\delta}{\delta x_{n+m}(\sigma_1)} [\tilde{\Sigma}(\sigma_3, \sigma_4) + \tilde{G}(\sigma_3, \sigma_4)] \tag{3.37}
\]

can easily be seen to be equal to

\[
\frac{\partial^2}{\partial x_n(\sigma_3) \partial x_m(\sigma_3)} \tilde{\Sigma}(\sigma_3, \sigma_4) \tag{3.38}
\]
In the limit $\sigma_3 = \sigma_4 = \sigma$ this is just equal to $1/2[ \frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2}] \Sigma(\sigma, \sigma)$ and reduces to the free field case described by (2.14) (provided the limit is taken after differentiation).

One can show \cite{14} that the gauge transformation (1.2) changes (3.28) by a total derivative

$$\delta A = \int d\sigma \lambda(\sigma) \frac{\delta}{\delta x_n(\sigma)} A$$

By the argument given in Section II this should mean that the equations are gauge invariant. However there is one fact that has to be kept in mind. Consider the following expression: $[\Sigma(\sigma, \sigma)$ is the generalization of the Liouville field that was used in Section II$]$

$$2(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial}{\partial x_1}) \Sigma A + (\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_1}) \Sigma \frac{\partial A}{\partial x_1}$$

Using (2.9) we get

$$= \frac{\partial}{\partial x_1} [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_1}) \Sigma A]$$

which is a total derivative. However if we vary (3.40) w.r.t. $\Sigma$, one gets

$$2\delta \Sigma(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial}{\partial x_1}) A + \delta \Sigma(\frac{\partial^2}{\partial x_1^2} + \frac{\partial}{\partial x_1}) \frac{\partial A}{\partial x_1}$$

which is not zero. On the other hand if we rewrite (3.40) as (using (2.9))

$$\frac{\partial}{\partial x_1} [(\frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_1}) \Sigma A]$$

and vary w.r.t $\Sigma$ we get

$$\delta \Sigma(-\frac{\partial^3}{\partial x_1^3} + \frac{\partial^2}{\partial x_1 \partial x_2}) A + \delta \Sigma(\frac{\partial^3}{\partial x_1^2} + \frac{\partial^2}{\partial x_1 \partial x_2}) A$$

which is zero.

Thus one has to be careful about varying w.r.t $\Sigma$ indiscriminately. There are certain constraints (such as (2.9)) that $\Sigma$ obeys and hence it cannot be varied without taking these into account. In I and II there was a tracelessness condition on the gauge parameters that saved the day, but this will not always be the case in the interacting cases considered here. One possible solution to the problem (in the example considered above) is to ensure that the form (3.43) is used, rather than (3.40). This solution was not available in the
free case because (3.43) will generically, lead to higher derivative equations of motion, whereas for consistency of field propagation it is always desirable to have equations that are quadratic in derivatives - which (3.40) implies. However in interaction terms one can allow higher derivative terms. So our solution to this problem will be to choose the form that ensures gauge invariance. To be more precise, constraints such as (2.9), imply that \( \Sigma \) cannot be varied arbitrarily. Since the constraints depend on \( x_1, x_2, x_3 \) let us assume that \( f(x_1, x_2, x_3) \) is a function that satisfies (2.9). Then \( \Sigma(x_1, \ldots x_n) \) is of the form \( f(x_1, x_2, x_3)\Sigma'(x_4, \ldots x_n, \ldots) \). Then, if the constraint (2.9) is written symbolically as \( Df = 0 \), a term of the form \( Df \delta \Sigma' A = 0 \) (3.45) can be added to (3.28) since it is identically zero. On the other hand, on integration by parts, it adds a piece \( D^\dagger A \) to the equation of motion obtained by varying \( \Sigma \). Thus, in any equation, for each such constraint, there is an ambiguity parametrized by one parameter \( x \) of the form \( xD^\dagger A \). We have to fix these parameters by requiring gauge invariance. In the examples that we have checked, this is a consistent prescription. We do not have an argument that guarantees the consistency in all cases.

The entity \( \tilde{\Sigma}(\sigma_1, \sigma_2) \) is a "bilocal" field as can be seen from (3.32) and (3.33). In the limit \( \sigma_1 \to \sigma_2 \) \( \Sigma(\sigma_1, \sigma_2) \to \frac{d\lambda}{dz} \), which can be identified with the usual Liouville mode. Our strategy will be to Taylor expand \( \Sigma(\sigma_1, \sigma_2) \) in powers of \( z(\sigma_1) - z(\sigma_2) \), so that we get \( \tilde{\Sigma}(\sigma_1, \sigma_1) \) and derivatives. In order to preserve the covariance of the equations of motion we will have to use a covariant Taylor expansion of the form (derivation is given in [14]):

\[
\tilde{\Sigma}(\sigma_1, \sigma_2) = \tilde{\Sigma}(\sigma_1, \sigma_1) + [z(\sigma_2) - z(\sigma_1)]\frac{\partial \Sigma}{\partial x_1} \bigg|_{\sigma_2 = \sigma_1} + O(z(\sigma_2) - z(\sigma_1))^2 \quad (3.46)
\]

A priori, \( Y \) is a function of \( \sigma \), implicitly through \( z(\sigma) \) and also through \( x_n(\sigma) \). However we will set \( x_n(\sigma) \) to be independent of \( \sigma \). This is because gauge transformations correspond to translations of \( x_n(\sigma) \). The gauge parameter should depend on \( z(\sigma) \) but not directly on \( \sigma \). Thus we do not lose anything by setting the \( x_n \)'s to be equal. This does not affect earlier arguments that show that the gauge variation (3.28) is a total derivative in \( x_n(\sigma) \) (see (3.33)).

Finally once the equations obtained from (3.28) have been written down in terms of \( \tilde{G}(\sigma_1, \sigma_2) \) one has to actually perform the Koba-Nielsen integration over \( z(\sigma_i) \). At this point our strategy will be to set all the \( x_n \)'s to zero,
where upon the integrals reduce to the usual S-matrix type Koba-Nielsen integration. The gauge invariance does not depend on the form of $\tilde{G}$ and hence will continue to hold. In actually performing the Koba-Nielsen integrals one will encounter divergences. These have to be regulated in the usual way [18, 19]. Thus we have a prescription for writing down gauge invariant interacting equations for all the modes of the string. In the next section we will illustrate the method with some simple examples.

4 Examples

4.1 Tachyon

We will start by considering the tachyon. The tachyon, being the ground state is a little different from the other ones, in that it is difficult to keep track of how many vertex operator insertions there are in a term. Thus the term $e^{\int_0^1 d\sigma k(\sigma)Y(\sigma)}$ can stand for any number of exponentials $e^{ik.X}$. To circumvent this, let us introduce an identity operator $\int d\sigma I(\sigma)$, which is just equal to 1 but formally serves the purpose of counting tachyon insertions. Thus $\int d\sigma_1 \int d\sigma_2 I(\sigma_1)I(\sigma_2)$ will signal the presence of two tachyons. (In all calculations below integration over all $\sigma$’s is understood whether or not they are stated explicitly. Furthermore integration over $z(\sigma)$ (Koba Nielsen integration) is also understood.) And we can let

$$<I(\sigma)k_0^\mu(\sigma_1)>=\partial_\mu \phi \frac{D(\sigma,\sigma_1)}{a}$$

$$<I(\sigma_1)I(\sigma_2)>=\phi^2$$

In evaluating the action of $e^{\sum_n \lambda_n k_n}$ in (2.7) we had neglected the “classical” piece which was really the naive classical dimension - the oscillator number. This was incorporated as a $D + 1^{th}$ component of momentum [1]. In I it was called $q_0$ and we set $q_0^2 = m^2$. We will continue to do that in the interacting case. However we have to interpret terms of the form $q_0.p_0$ that arise from $k_0(\sigma_1).k_0(\sigma_2)$ when $\sigma_1 \neq \sigma_2$. It turns out that this can be given a natural interpretation as the sum of the oscillator numbers of the oscillators that have been contracted. It thus gives $(z_1 - z_2)^{k_0(\sigma_1)k_0(\sigma_2)+q_0.p_0}$. Similarly in the coefficient of $\Sigma(\sigma_1,\sigma_2)$ we have a term $q_0.p_0$. Here again it will stand for
the number of contractions. But we add one because $dz_1$, which has a dimension one, also contributes to the Liouville mode dependence. Thus $\Sigma(\sigma_1, \sigma_2)$ measures the dimension not only of the derivatives $\partial_z$ in the correlation but also the dimension of $dz_1$. Having incorporated this prescription we can treat all particles as massless, just as in the free case. Let us proceed with the tachyon.

\[
\int d\sigma_1 I(\sigma_1) e^{i \int d\sigma_3 d\sigma_4 k_0(\sigma_3) k_0(\sigma_4) [\tilde{\Sigma}(\sigma_3, \sigma_4) + \tilde{G}(\sigma_3, \sigma_4)]} e^{i \int d\sigma k_0(\sigma) Y(\sigma)} +
\]

\[
\int \int d\sigma_1 d\sigma_2 \frac{1}{2!} I(\sigma_1) I(\sigma_2) e^{i \int d\sigma_3 d\sigma_4 k_0(\sigma_3) k_0(\sigma_4) [\tilde{\Sigma}(\sigma_3, \sigma_4) + \tilde{G}(\sigma_3, \sigma_4)]} e^{i \int d\sigma k_0(\sigma) Y(\sigma)}
\]

are the first two terms. The first term gives, using

\[
<k_0(\sigma_1) k_0(\sigma_3) I(\sigma_1) >= \frac{D(\sigma_4, \sigma_1) D(\sigma_3, \sigma_1)}{a} k_0^2 \phi(k_0)
\]

and $\tilde{G}(\sigma, \sigma) = 0$ due to normal ordering.

\[
k_0(\sigma_1) k_0(\sigma_1) \phi(k_0) = (k_0^2 + q_0^2) \phi(k_0) = (k_0^2 - 2) \phi(k_0)
\]

where $q_0^2$ has been set to $-2$. The interacting piece gives:

\[
\frac{1}{2!} \phi \int \int \left[ \frac{D(\sigma_3, \sigma_1)}{a} \frac{D(\sigma_4, \sigma_1)}{a} + \frac{D(\sigma_3, \sigma_2)}{a} \frac{D(\sigma_4, \sigma_2)}{a} + \frac{2 D(\sigma_3, \sigma_1) D(\sigma_4, \sigma_2)}{a} \right] 1/2 k_0(\sigma_3) k_0(\sigma_4) \tilde{\Sigma}(\sigma_3, \sigma_4) e^{i \int k Y}
\]

\[
= 1/4 \phi(\sigma_1) \phi(\sigma_2) (k_0 + p_0)^2 \tilde{\Sigma}(\sigma_1, \sigma_1) \int e^{k.p \tilde{G}(\sigma_1, \sigma_2) d\sigma_2} e^{i(k+p)Y}
\]

We have used (4.11) in the exponent also. Further we have Taylor expanded $Y(\sigma_2), \Sigma(\sigma_2, \sigma_2)$ and $\Sigma(\sigma_1, \sigma_2)$ around $\sigma_1$, and have kept the lowest term.

Varying wrt $\Sigma$ gives

\[
1/4 (k + p)^2 \phi(k) \phi(p) \int dz_2(z_1 - z_2)^k e^{i(k+p)Y}
\]
Set \( q.p = 1 \) (for \( \int dz^2 \)) and \( q^2 = p^2 = -2 \) in \( (k + p)^2 \) which multiplies \( \Sigma \). Also in \( (z_1 - z_2)^{k+p+q.p} \) we set \( q.p = 0 \) (since there are no contractions) as per the above prescription. The net result is

\[
\frac{1}{4} \frac{(k + p)^2}{k.p + 1} \phi(k)\phi(p)e^{i(k+p)Y} \tag{4.9}
\]

In the on-shell limit, \( (k + p)^2 = 2k.p + 2 \) and we get the quadratic term in the Tachyon equation \([8]\). Of course, in this problem, since no gauge invariance is involved, all this seems longwinded. However it will be necessary with the other modes where gauge invariance is involved.

### 4.2 Vector

For the vector, we keep terms of total dimension \( \leq 2 \). This gives:

\[
\exp\left(\frac{k_0(\sigma_1)k_0(\sigma_2)}{2}[\bar{\Sigma}(\sigma_1, \sigma_2) + \bar{G}(\sigma_1, \sigma_2)] + k_1(\sigma_1), k_0(\sigma_2)\frac{\partial}{\partial x_1(\sigma_1)}[\bar{\Sigma}(\sigma_1, \sigma_2) + \bar{G}(\sigma_1, \sigma_2)] + \frac{k_1(\sigma_1)k_1(\sigma_2)}{2} \frac{\partial^2}{\partial x_1(\sigma_1)\partial x_1(\sigma_2)}[\bar{\Sigma}(\sigma_1, \sigma_2) + \bar{G}(\sigma_1, \sigma_2)] + k_2(\sigma_1), k_0(\sigma_2)\frac{\partial}{\partial x_2(\sigma_1)}[\bar{\Sigma}(\sigma_1, \sigma_2) + \bar{G}(\sigma_1, \sigma_2)]\right) e^{i k_0(\sigma)Y(\sigma)} \tag{4.10}
\]

We emphasize that the dimensional reduction and identification of \( q_0^2 \) with the masses of the particles is necessary for making contact with string theory. None of that is necessary for verifying the gauge invariance of the equations obtained from \((4.10)\). This is in fact a ‘mysterious’ feature of this construction.

In this talk we will only consider the simplest interacting example with two vectors contributing to the vector equation of motion. Now, for Abelian fields there are no quadratic corrections to the equations at this order in derivatives. Thus the answer we are looking for is in fact zero. In this sense the calculation is trivial. However the vanishing of the quadratic corrections happens only after Bose symmetrizing on the vector fields, whereas, gauge invariance of the equation can be verified even before this is done. In this sense, we have a non trivial test of gauge invariance.
At the free level we get:

\[
\int \int \frac{1}{2} k_0(\sigma_1).k_0(\sigma_2) \Sigma(\sigma_1, \sigma_2) i \int k_1(\sigma) Y_1 + k_1(\sigma_1).k_0(\sigma_2) \frac{\partial}{\partial x_1(\sigma_1)} [\Sigma(\sigma_1, \sigma_2)] e^i \int k_0 Y
\]

(4.11)

Varying w.r.t $\Sigma$, after Taylor expanding using,

\[
\frac{\partial}{\partial x_1(\sigma_1)} [\Sigma(\sigma_1, \sigma_2)] = \frac{1}{2} \frac{\partial}{\partial x_1(\sigma_1)} [\Sigma(\sigma_1, \sigma_1)] + O(z(\sigma_1) - z(\sigma_2)) + ...
\]

we get for (1.11)

\[
\int \int d\sigma_1 d\sigma_2 \frac{1}{2} k_0(\sigma_1).k_0(\sigma_2) i \int k_1(\sigma) Y_1 - \frac{1}{2} k_1(\sigma_1).k_0(\sigma_2) \int d\sigma ik_0(\sigma) \frac{\partial Y(\sigma)}{\partial x_1(\sigma_1)}
\]

(4.12)

Since $x_1(\sigma_1)$ is independent of $\sigma_1$, $\frac{\partial Y(\sigma)}{\partial x_1(\sigma_1)} \equiv \frac{\partial Y(\sigma)}{\partial x_1} = Y_1(\sigma)$. Thus we get Maxwell’s equations:

\[
i(k^0_0 k^\mu_1 - k_1 k^\mu_0) = 0
\]

(4.13)

We now turn to the interacting case. Terms involving two $k_1$’s or one $k_2$ are:

\begin{itemize}
  \item[i)] $1/2 \int \int k_1(\sigma_1).k_0(\sigma_2) \frac{\partial}{\partial x_1(\sigma_1)} (\Sigma + \tilde{G}) e^{1/2} \int k_0(\sigma_3).k_0(\sigma_4)(\tilde{G} + \Sigma)e^i \int k_0 Y$
  \item[ii)] $1/2 \int \int k_1(\sigma_1).k_1(\sigma_2) \frac{\partial^2}{\partial x_1(\sigma_1)\partial x_2(\sigma_2)} (\Sigma + \tilde{G}) e^{1/2} \int k_0(\sigma_3).k_0(\sigma_4)(\tilde{G} + \Sigma)e^i \int k_0 Y$
  \item[iii)] $\int \int k_2(\sigma_1).k_0(\sigma_2) \frac{\partial}{\partial x_2(\sigma_1)} (\Sigma + \tilde{G}) e^{1/2} \int k_0(\sigma_3).k_0(\sigma_4)(\tilde{G} + \Sigma)e^i \int k_0 Y$
  \item[iv)] $\int \int k_1(\sigma_1).k_0(\sigma_2) \frac{\partial}{\partial x_1(\sigma_1)} (\Sigma + \tilde{G}) e^{1/2} \int k_0(\sigma_3).k_0(\sigma_4)(\tilde{G} + \Sigma)e^i \int k_0 Y \int i k_1 Y_1$
  \item[v)] $e^{1/2} \int k_0(\sigma_3).k_0(\sigma_4)(\tilde{G} + \Sigma)e^i \int k_0 Y \int i k_2 Y_2$
\end{itemize}

At the interacting level $k^\mu_2$ also transforms into $A_\mu$ [2], therefore we need to include terms involving $k^\mu_2$ for gauge invariance. Consider the terms contributing to $Y^\mu_1$. Taylor expanding $\Sigma$ we find:

\begin{itemize}
  \item[i)] $[(\frac{1}{2} k_1(\sigma_1).k_0(\sigma_2) + \frac{1}{2} k_1(\sigma_2).k_0(\sigma_1)) \frac{1}{2} \frac{\partial}{\partial x_1(\sigma_1)} \Sigma(\sigma_1, \sigma_1)]$
\end{itemize}
\[
[k_1(\sigma_3), k_0(\sigma_4)] \frac{\partial \tilde{G}^{(\sigma_3, \sigma_4)}}{\partial x_1(\sigma_3)} e^i \int k_0 Y
\]  

(4.14)

Varying w.r.t \( \Sigma \)

i) \[-1/4[k_1(\sigma_1), k_0(\sigma_2) + k_1(\sigma_2), k_0(\sigma_1)]k_1(\sigma_3), k_0(\sigma_4) \frac{\partial \tilde{G}^{(\sigma_3, \sigma_4)}}{\partial x_1(\sigma_3)} e^i \int k_0 Y i \int k_0 Y_1 \]

(4.15)

Using

\[
\frac{\partial^2 \Sigma(\sigma_1, \sigma_2)}{\partial x_1(\sigma_1) \partial x_1(\sigma_2)} = \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial}{\partial x_2} \right) \tilde{\Sigma}(\sigma_1, \sigma_2) + O(z(\sigma_2) - z(\sigma_2))
\]



ii) \[1/2k_1(\sigma_1), k_1(\sigma_2) \frac{\partial^2}{\partial x_1(\sigma_1) \partial x_1(\sigma_2)}(\tilde{\Sigma} + \tilde{G}) e^{1/2} \int \int k_0(\sigma_3), k_0(\sigma_4)(\tilde{G} + \tilde{\Sigma}) e^i \int k_0 Y \]

(4.16)

Varying w.r.t. \( \Sigma \) gives:

\[ii) \ 1/4k_1(\sigma_1), k_1(\sigma_2), k_0(\sigma_3), k_0(\sigma_4) \frac{\partial \tilde{G}}{\partial x_1(\sigma_1)} i \int k_0 Y_1 \]

(4.17)

(iii) and (v) do not contribute to \( Y_1 \). (iv) gives

\[iv) \ k_1(\sigma_1), k_0(\sigma_2) \frac{\partial}{\partial x_1(\sigma_1)} \tilde{G}^{1/2} k_0(\sigma_3), k_0(\sigma_4) \tilde{\Sigma}(\sigma_3, \sigma_4) \int i k_1 Y_1 + 1/2[k_1(\sigma_1), k_0(\sigma_2) + k_1(\sigma_2), k_0(\sigma_1)]1/2 \frac{\partial}{\partial x_1(\sigma_1)} \tilde{\Sigma} e^{1/2} \int \int k_0(\sigma_3), k_0(\sigma_4) \tilde{G} e^i \int k_0 Y \int i k_1 Y_1 \]

(4.18)

Varying w.r.t. \( \Sigma \) gives

\[iv) \ 1/2k_0(\sigma_3), k_0(\sigma_4) k_1(\sigma_1), k_0(\sigma_2) \frac{\partial \tilde{G}^{(\sigma_1, \sigma_2)}}{\partial x_1} \int i k_1 Y_1 - 1/4[k_1(\sigma_1), k_0(\sigma_2) + k_1(\sigma_2), k_0(\sigma_1)]1/2 k_0(\sigma_3), k_0(\sigma_4) \frac{\partial \tilde{G}}{\partial x_1(\sigma_1)} i \int k_1 Y_1 \]

(4.19)

The final result, (4.15) + (4.17) + (4.19) is manifestly gauge invariant under

\[k_1(\sigma_1) \rightarrow k_0(\sigma_1) \int d\sigma \lambda_1(\sigma) \]
Note that the expressions are multiplied by $e^{k_0(\sigma_i)\cdot k_0(\sigma_j)}\tilde{G}(\sigma_i,\sigma_j)$ which is just $(z(\sigma_i) - z(\sigma_j))k_0(\sigma_i)\cdot k_0(\sigma_j)$ once we set $x_n = 0$.

To get the final answer in terms of space-time fields, and to get their gauge transformation laws one has to use (2.13). We will not work out the details here. In the present example because of the antisymmetry of $z(\sigma_1) - z(\sigma_2)$ under $\sigma_1 \leftrightarrow \sigma_2$, it is clear that the final answer cannot be symmetric under exchange of the two photons and therefore the final answer is zero. This is just a reflection of the fact that $F^{\mu\nu}F_{\nu\rho}F^{\rho\mu}$ is identically zero for Abelian fields and there is no quadratic correction to Maxwell’s equations. Of course if we consider higher derivatives or if there are group indices on the gauge fields this will not be true.

There is one remark that needs to be made. At first sight it is a little puzzling that the gauge transformation law is so simple in an interacting theory, and furthermore that it is the same in the fully interacting version described in this talk as it was in the earlier case (see section II of this talk [2]). The resolution of this puzzle is that the gauge transformation law expressed in terms of spacetime fields is more complicated in the interacting case. For e.g. the transformation of $\partial_\mu S^{\mu\nu}$ as given by the rules described above, cannot be obtained by differentiating that of $S^{\mu\nu}$ given in II! Thus, the exact transformation law of $S^{\mu\nu}$ is written as an expansion in powers of momentum and has to be worked out by considering the transformations of $k_1k_1, k_1k_1k_0, k_1k_1k_0k_0,\ldots$. In other words, the gauge transformation law of $S^{\mu\nu}$, or any other field, as given by the law (1.1) is only exact for constant fields, or equivalently at any one space-time point. One has to apply the prescription (1.1) and (2.11) to the derivatives separately. One can combine these to get a Taylor expansion that can be worked out to the desired degree of accuracy. These details and other examples of interacting equations will be presented elsewhere.

5 Conclusions

We have described a general construction that gives gauge invariant equations of motion, the gauge transformation prescription (in terms of loop variables) being the same as in II. The main advantages are that the prescription for writing down the equations and gauge transformation laws are fairly straightforward. The gauge transformations written in terms of loop variables seem
to have some geometric meaning - they look like local scale transformations. The interactions look as if they have the effect of converting a string to a membrane. The fields also appear massless in one higher dimension. These are intriguing features. There is no requirement of being on-shell, in this method. It should therefore be possible to show that it is equivalent to (open) string field theory. [16, 15, 17]

There are several questions that need to be answered before one can claim that these are the interacting (tree level) string equations. One is the resolution of the ambiguity described in Section III. It has to be checked whether the dimensional reduction works in all cases. Formally since the calculations are identical on-shell with the S-matrix calculation, one should not have any problems. Whether problems arise off-shell, needs to be checked. As shown in [12], even U(1) gauge invariance of the massless vector is violated when a finite cutoff is introduced in order to go off-shell, and one needs to introduce massive modes to restore gauge invariance. In the loop variable formalism all the modes are present from the start and there should not be a problem. Gauge invariance seems to be exact, on or off-shell. However the exact value of the Koba-Nielsen integral will depend on the cutoff prescription. This issue needs to be clarified. Also, one needs to prove that in all cases, and to all orders, the transition to space-time fields from the loop variable representation can be made unambiguously. Finally, assuming the above issues are resolved satisfactorily, one has to see whether this formalism provides any insight into the various other issues that have become pressing in string theory, such as duality.

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References
[1] B. Sathiapalan, Nucl. Phys. B326 (1989) 376.
[2] B. Sathiapalan, hep-th/9512224; Mod. Phys. Lett. A11(1996)571.
[3] B. Sathiapalan, Nucl. Phys. B415 (1994)332.
[4] C. Lovelace, Phys. Lett. 135B (1984) 75.
[5] C. Callan, D. Friedan, E. Martinec and M. Perry, Nucl. Phys. B262 (1985) 593.

[6] A. Sen, Phys. Rev. D 32 (1985) 2102.

[7] E. Fradkin and A. Tseytlin, Phys. Lett. B 163 (1985) 123.

[8] S. Das and B. Sathiapalan, Phys. Rev. Lett. 56 (1985) 2664.

[9] A. Tseytlin, Nucl. Phys. B276 (1986) 391.

[10] J. Hughes, J. Liu and J. Polchinski, Nucl. Phys. B316 (1989) 15.

[11] B. Sathiapalan, hep-th/9511153; Mod. Phys. Lett. A11 (1996) 317.

[12] B. Sathiapalan, hep-th/9409023; Int. J. Mod. Phys. A10 (1995) 4501.

[13] B. Sathiapalan, hep-th/9207051; Nucl. Phys. B405 (1993) 367.

[14] B. Sathiapalan, in preparation.

[15] T. Banks and M. Peskin, Nucl. Phys. B264 (1986) 513.

[16] W. Siegel and B. Zwiebach, Nucl. Phys. B263 (1986) 105.

[17] E. Witten, Nucl. Phys. B268 (1986) 253.

[18] B. Sathiapalan, Nucl. Phys. B294 (1987) 733.

[19] B. Sathiapalan, hep-th/9509097; Int. J. Mod. Phys. A11 (1996) 2887.