EXOTIC ITERATED DEHN TWISTS

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Abstract. The cotangent bundle of any exotic sphere admits compactly supported symplectic automorphisms which generalize certain powers of Dehn twists. We introduce these automorphisms, and prove that they have infinite order up to compactly supported symplectic isotopy.

Introduction

This note shows the following:

Theorem 1. Let $T^*L$ be the cotangent bundle of an exotic $n$-sphere $L$, $n \neq 4$, with its canonical symplectic structure. Let $\text{Symp}^c(T^*L)$ be the group of compactly supported symplectic automorphisms. Then $\pi_0(\text{Symp}^c(T^*L))$ contains an element of infinite order.

By an exotic $n$-sphere, we mean a homotopy sphere not diffeomorphic to $S^n$. There are no such spheres for $n = 1, 2, 3, 5, 6$, so Theorem 1 concerns $n \geq 7$ (if there are exotic 4-spheres, our technique won’t apply to them, since we start by choosing a Morse function with exactly two critical points; any homotopy 4-sphere with that property is known to be standard). For comparison, recall that on the cotangent bundle of the standard sphere, the Dehn twist

$$[\tau_{S^n}] \in \pi_0(\text{Symp}^c(T^*S^n))$$

has infinite order in any dimension $n > 0$ (this is classical for odd $n$; the first proof for $n = 2$ was given in [14], and that extends to higher even $n$ as well, even though it was not stated there; see also [15, 16, 11, 2]). The symplectic automorphisms $\phi_L \in \text{Symp}^c(T^*L)$ which we construct, using Lefschetz fibration techniques, are analogues of certain even negative powers $\tau_{S^n}^{-2k}$ of [11]. More precisely, for even $n$ the analogy is with $k = 1$, while for odd $n$ one takes the smallest $k$ such that the connected sum of $2k$ copies

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of $L$ is diffeomorphic to $S^n$. This explains the terminology \textit{exotic iterated Dehn twists}.

One can get some insight into the situation from the following heuristic argument. The action of the standard Dehn twist $\tau_{S^n}$ on a cotangent fibre $F \subset T^*S^n$ can be described as a Lagrangian surgery (or Lagrangian connected sum, see [13] and [14, Appendix]):

$$\tau_{S^n}(F) \simeq S^n \# F,$$

where on the right hand side $S^n$ is the zero-section, and $\simeq$ is compactly supported Lagrangian isotopy. Next, one can arrange that $S^n \# F$ intersects $S^n$ transversally in a single point, and then

$$(3) \quad \tau_{2S^n}(F) \simeq S^n \# S^n \# F.$$ 

If $n$ is odd, the two copies of the zero-section $S^n$ contribute to (3) with the same orientation, while for even $n$ the orientations are opposite. There can be no direct analogue of (2) for exotic spheres $L$, since $L \# F$ is not even diffeomorphic to $F$ (here and later on, we mean diffeomorphisms which are compatible with the obvious identifications at infinity). However, in the analogue of (3) for even $n$ one would have $L \# \bar{L} \# F$, which is always diffeomorphic to $F$. Indeed, our $\phi_L^{-1}$ is analogous to $\tau_{2S^n}$ in even dimensions. For odd $n$, the generalization of (3) would be $L \# L \# F$, which is still not necessarily diffeomorphic to $F$. However, by iterating further one will eventually get back to $F$, because the Kervaire-Milnor group of homotopy $n$-spheres is finite [10].

Our proof that $[\phi_L] \in \pi_0(\text{Symp}^c(T^*L))$ has infinite order uses a Floer cohomology computation, adapting an idea from [15, Lemma 5.7], which means that gradings play a central role. It turns out that for half the dimensions there is a purely topological reason, again in parallel with the situation for standard Dehn twists:

\textbf{Proposition 2.} If $n$ is odd, $\phi_L$ is not homotopic to the identity by a compactly supported homotopy, and the same holds for its iterates. On the other hand, if $n$ is even, then $[\phi_L] \in \pi_0(\text{Diff}^c(T^*L))$ has finite order.

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Homotopy spheres

A homotopy $n$-sphere is a closed smooth manifold $L$ homotopy equivalent to $S^n$. Assume from now on that $n \geq 6$. Let $\Theta_n$ be the set of diffeomorphism classes of oriented homotopy $n$-spheres. This is a finite abelian group under connected sum, with the inverse map being orientation-reversal $L \mapsto \bar{L}$.

Given $f \in \text{Diff}^+(S^{n-1})$, one can form the homotopy $n$-sphere

$$L_f = B_0 \cup_f B_1.$$  

Here, $B_0$ and $B_1$ are two copies of the closed unit ball in $\mathbb{R}^n$, and one identifies $x \in \partial B_0$ with $f(x) \in \partial B_1$. Equip $L_f$ with the orientation coming from $B_0$. This yields a group homomorphism

$$\pi_0(\text{Diff}^+(S^{n-1})) \longrightarrow \Theta_n.$$  

The $h$-cobordism theorem shows that (5) is onto, and Cerf’s theorem \[4\] that it is injective.

**Lemma 3.** Take $f \in \text{Diff}^+(S^{n-1})$, and an orientation-reversing $a \in O(n)$. Then $afa^{-1}$ is isotopic to $f^{-1}$.

**Proof.** Applying $a$ to both hemispheres in (4) yields a diffeomorphism $\bar{L}_f \rightarrow L_{afa^{-1}}$. On the other hand $\bar{L}_f \cong L_{f^{-1}}$, by exchanging the hemispheres. Because of the injectivity of (5), this implies the desired result. \[\square\]

**Weinstein handles**

We begin by recalling some basic facts about handle decompositions in symplectic topology \[5\]. Let $M^{2n}$ be a symplectic manifold which is Liouville. By this we mean that $M$ carries a Liouville (symplectically expanding) vector field $Z$, whose flow exists for all times, and an exhausting function $h$ such that $Z.h > 0$ outside a compact subset. In the special case where $h$ is Morse and $Z$ is gradient-like on all of $M$, we say that $M$ is Weinstein. The Weinstein structure gives rise to a description of $M$ by iterated attachment of Weinstein handles \[18\].

We will be interested in a very special class of Weinstein manifolds, namely those for which $h$ has only two critical points $x_0, x_1$, of index 0 and $n$, respectively. In that case, the resulting description of $M$ has the following particularly simple form: start with the standard closed symplectic ball $B^{2n}$; attach a Weinstein handle along a Legendrian sphere $K \subset S^{2n-1} = \partial B^{2n}$; and then complete the result by adding an infinite cone along the boundary,
which recovers $M$ up to symplectic isomorphism. More precisely, what one needs for the handle attachment process is a Legendrian sphere $K$ together with a parametrization, which means a class

$$[f] \in \pi_0(\text{Diff}(K, S^{n-1})/O(n)).$$

The core of the handle is a standard ball $B^n$, whose boundary will be attached to $K$ via (6).

One can describe the handle data more explicitly, in terms of the flow of $Z$. Namely, let $M(x_0, x_1)$ be the space of (unparametrized) flow lines of $Z$ with asymptotics $x_0, x_1$. For technical simplicity, suppose that there are Darboux coordinates around $x_0$ in which $Z = \frac{1}{2}p \partial_p + \frac{1}{2}q \partial_q$, and similarly Darboux coordinates around $x_1$ in which $Z = -\frac{1}{2}p \partial_p + \frac{3}{2}q \partial_q$. Then, by intersecting flow lines with small spheres in those coordinates, one obtains two maps

$$\epsilon_0, \epsilon_1 : M(x_0, x_1) \rightarrow S^{2n-1}.$$

Each is an embedding, whose image is a Legendrian sphere. However, while the image of $\epsilon_1$ is always the same, namely $S^{n-1} = \{q = 0\} \subset S^{2n-1}$, that of $\epsilon_0$ is the sphere $K$ used in the handle attachment process. The parametrization (6) is given by

$$f = \epsilon_1 \epsilon_0^{-1}.$$

**Lemma 4.** Let $L$ be a homotopy $n$-sphere, $n \geq 6$. Then $T^*L$ is an instance of the previously described handle attachment process, where $K = S^{n-1} \subset S^{2n-1}$ is the standard Legendrian sphere, and the parametrization (6) is the preimage of $L$ under (5). \[ \square \]

The proof is elementary, and we will only give an outline. Take a Morse function on $L$ which has only two critical points, a minimum $x_0$ and a maximum $x_1$. Starting from that, one constructs a Morse function on $T^*L$, and a Liouville vector field which is gradient-like for that function. This is done so that all flow lines on $M(x_0, x_1)$ lie inside the zero-section $L$, and will be gradient flow lines of the original Morse function. Then, the construction of the parametrization from (7) recovers the description of $L$ as in (3).

**LEFSCHETZ FIBRATIONS**

Let $\pi : E \rightarrow D$ be an exact symplectic Lefschetz fibration over the closed disc $D$. We pick a base point $* \in \partial D$, and denote the fibre over that point by $F$. The fibre is a Liouville domain. The total space $E$ itself is a compact manifold with corners, but one can round off those corners to produce a
Liouville domain. Denote the Liouville manifold obtained by attaching an infinite cone to the boundary of that domain by $M$.

To be precise, the definition of exact symplectic Lefschetz fibration used here is a slightly modified version of that in [17, Section 15]. The modifications, which are introduced for technical simplicity, are the following ones:

• (Topological restrictions) We assume that $\dim(E) = 2n \geq 6$ and $H^1(F, \partial F) = 0$, $H^1(F) = 0$.

• (Local structure near the critical points) We also assume that our Lefschetz fibrations are standard near each critical point; this means that

$$\pi(w) = w_1^2 + \cdots + w_n^2 + \zeta,$$

in local complex coordinates $w = (w_1, \ldots, w_n)$ in which the symplectic form $\omega_E$ is the standard constant Kähler form on $\mathbb{C}^n$.

• (Triviality near the horizontal boundary) Recall that the horizontal boundary $\partial_h E \subset E$ is the union of the boundaries of all the fibres. The local topological structure near that boundary can be described as follows: there is a diffeomorphism

$$D \times N_F \cong N_E$$

where $N_F$ is a neighbourhood of $\partial F \subset F$, and $N_E$ a neighbourhood of $\partial_h E \subset E$. We additionally require that $\nu$ can be chosen such that

$$\nu^* \omega_E = \omega_D + \omega_F|N_F,$$

where $\omega_D$ is some positive multiple of the standard volume form on $D \subset \mathbb{C}$.

A vanishing path is an embedded path in $D$ starting from one of the critical values, and ending at *. To each such path belongs a Lefschetz thimble, which is an embedded Lagrangian disc $\Delta \subset E$ fibered over that path, and its vanishing cycle, which is the Lagrangian sphere $V = \partial \Delta \subset F$. Because of the way in which it arises as a boundary, the vanishing cycle comes with a natural parametrization in the sense of [13]. Now suppose that we are given a basis of vanishing paths $(\gamma_0, \ldots, \gamma_{r-1})$ (this is often called a distinguished basis in the Picard-Lefschetz theory literature, see for instance [17, Section 16d]), and the associated vanishing cycles $(V_0, \ldots, V_{r-1})$ with their parametrizations
([f_0], \ldots, [f_r-1]). From this, one can reconstruct the symplectic Lefschetz fibration up to a suitable notion of deformation \cite{17} Lemma 16.9, hence the Liouville manifold $M$ up to symplectic isomorphism. In fact, as explained in \cite{3} Section 8, one can translate that process into the language of Weinstein handle attachment. As a special case of that translation, one obtains the following:

**Lemma 5.** Take a Lefschetz fibration whose fibre $F$ is the disc cotangent bundle $D^*S^{n-1}$, and which has two vanishing cycles $(V_0, V_1)$, which are both copies of the zero-section $S^{n-1}$ but with arbitrary parametrizations $([f_0], [f_1])$. Then, the associated Liouville manifold $M$ can also be obtained by taking the standard ball $B^{2n}$ and attaching a handle along the standard Legendrian sphere $S^{n-1} \subset S^{2n-1}$, with parametrization $f_1^{-1}f_0$.

As before, we omit the details, and only give an outline of the argument. As a preliminary observation, note that passing from $(f_0, f_1)$ to $(gf_0, gf_1)$ for any $g \in \text{Diff}(S^{n-1})$ does not change $M$; it merely changes the way in which one thinks of the fibre $F$ as being identified with $D^*S^{n-1}$. Hence, we may assume without loss of generality that $(f_0, f_1) = (f, \text{id})$.

Applying the strategy from \cite{3} directly yields a slightly more complicated Weinstein handle decomposition, which is as follows. Start with $D^2 \times D^*S^{n-1}$, and round off its corners to obtain a Liouville domain $U$. Then, one obtains $M$ by attaching handles along the Legendrian spheres $K_0 = \{z_0\} \times S^{n-1}$, $K_1 = \{z_1\} \times S^{n-1}$ in $\partial U$ (and then adding an infinite cone to the boundary). Here, $z_0 \neq z_1$ are points on $S^1 = \partial D^2$; both $S^{n-1}$ are the zero-section; and one additionally equips $K_0, K_1$ with the parametrizations $(f_0, f_1) = (f, \text{id})$. A simple handle cancellation argument (see \cite{5} Proposition 12.22) for the general notion) shows that attaching a handle along $K_1$ results in a Liouville domain which is deformation equivalent to $D^{2n}$. The deformation happens in such a way that $K_0$ becomes the standard Legendrian sphere $S^{n-1} \subset \partial D^{2n}$, still carrying the parametrization $f$.

**Lemma 6.** In the situation of Lemma 5, $M$ is symplectically isomorphic to the cotangent bundle $T^*L$ of the homotopy $n$-sphere associated to $f_1^{-1}f_0 \in \text{Diff}(S^{n-1})$, in the sense of \cite{3}.

This is immediate by combining Lemmas 4 and 5. Alternatively, one can view it as an instance of a more general relation between real and complex Morse theory, explored in \cite{8, 7}. However, while it is highly plausible that the total spaces of the complexified Morse functions constructed in those papers are cotangent bundles, that fact has not actually been proved \cite{7}.
Remark 1.3. This is the reason why we have adopted an approach which is less direct, but uses only standard tools.

**Lifting the Global Monodromy**

Let’s return to a general exact Lefschetz fibration $\pi : E \to D$, with fibre $F = E_*$. Denote by $\text{Symp}(F, \partial F)$ the group of symplectic automorphisms which are the identity near the boundary. The monodromy maps of loops in $(D \setminus \{\text{critical values of } \pi\}, \ast)$ are elements of this group. In particular, by going once around $\partial D$ one gets an automorphism $\mu$, called the global monodromy of the Lefschetz fibration.

**Assumption 7.** There is some integer $k$ such that for each vanishing cycle $V$, $\mu^k(V) \simeq V$. Here, $\simeq$ means isotopy of Lagrangian spheres in $F$ compatible with parametrizations as in \((6)\).

It is worth while exploring the meaning of this a little. Generally speaking, if $L_0, L_1$ are two Lagrangian spheres (with parametrizations), and $\xi$ a symplectic automorphism such that $\xi(L_0) \simeq L_0, \xi(L_1) \simeq L_1$, then it follows that

\[
\xi(\tau_{L_0}(L_1)) \simeq \tau_{\xi(L_0)}(\xi(L_1)) \simeq \tau_{L_0}(L_1),
\]

and the same holds for $\tau_{L_0}^{-1}(L_1)$. By applying this to the situation above, and using the Hurwitz moves that relate different vanishing cycles, one sees that it is enough to assume that $\mu^k(V_i) \simeq V_i$ for the vanishing cycles in some fixed basis ($V_0, \ldots, V_{r-1}$). Note also that by the Picard-Lefschetz formula,

\[
\mu \simeq \tau_{V_0} \cdots \tau_{V_{r-1}},
\]

where $\simeq$ is isotopy inside $\text{Symp}(F, \partial F)$. Hence, Assumption 7 can be equivalently reformulated as

\[
(\tau_{V_0} \cdots \tau_{V_{r-1}})^k(V_i) \simeq V_i \quad \text{for } i = 0, \ldots, r - 1.
\]

Let $\psi$ be the left-handed (negative) Dehn twist along a loop parallel to $\partial D$. This is an area-preserving diffeomorphism of $D$ which is the identity near $\partial D$, and also leaves a slightly smaller disc pointwise fixed; that smaller disc should contain all critical values of $\pi$. To make the discussion more concrete, it may be convenient to write down an explicit model, say

\[
\psi(z) = e^{i\alpha(|z|)}z, \quad \alpha(r) = 0 \quad \text{for } r \leq 1 - 2\varepsilon, \quad \alpha(r) = 2\pi \quad \text{for } r \geq 1 - \varepsilon,
\]
for some choice of small constant $\epsilon$ and function $\alpha$ satisfying the constraints in (15).

**Lemma 8.** Under Assumption 7, there is a $\phi \in \text{Symp}(E, \partial E)$ with the following property. If $\Delta$ is the Lefschetz thimble associated to any vanishing path $\gamma$, then $\phi(\Delta)$ is isotopic to the Lefschetz thimble $\Delta'$ for the path $\gamma' = \psi^k(\gamma)$ (the isotopy being one of Lagrangian submanifolds in $E$ with boundary in $F$).

**Proof.** Using parallel transport, we can lift $\psi^k$ to a diffeomorphism of the subset $\{ |\pi(x)| \geq 1 - 3\epsilon \} \subset E$ which is fibrewise symplectic, and equal to the identity on each fibre $E_z$, $|z| \geq 1 - \epsilon$. Then, its restrictions to the fibres $E_z$, $|z| \leq 1 - 2\epsilon$, are isotopic to $\mu^{-k}$. Using Assumption 7, one can now extend it to a diffeomorphism of the whole of $E$ which is still a lift of $\psi^k$, is fibrewise symplectic, and is given by some real orthogonal matrix in each local coordinate system (9). Denote this diffeomorphism by $\xi$. By construction, this is the identity near the vertical boundary $\partial v E = \pi^{-1}(\partial D)$. If we use the trivialization given by (10), the behaviour near $\partial h E$ is the trivial lift

\[(16) \quad \xi(z, x) = (\psi^k(z), x).\]

In particular, $\xi^*\omega_E - \omega_E$ vanishes near $\partial E$, hence can be written as the exterior derivative of a one-form which vanishes near $\partial E$ (this is obvious under our assumptions, because $H^2(E, \partial E; \mathbb{R}) \cong H_{2n-2}(E; \mathbb{R})$ always vanishes).

If one replaces the given symplectic form $\omega_E$ with $\tilde{\omega}_E = \omega_E + C\pi^*\omega_D$, where $C$ a sufficiently large constant, then the linear deformation from $\tilde{\omega}_E$ to $\xi^*\tilde{\omega}_E$ is through symplectic forms. Moreover, since $\psi^k$ preserves $\omega_D$, we have $\xi^*\tilde{\omega}_E - \tilde{\omega}_E = \xi^*\omega_E - \omega_E$. Using a Moser argument based on that, one can find an isotopy supported in the interior of $E$, which deforms $\xi$ to a diffeomorphism $\tilde{\xi}$ that is symplectic with respect to $\tilde{\omega}_E$.

The next step is to make $\tilde{\xi}$ equal to the identity near the boundary. One can find a function $H_D \in C^\infty(D, \mathbb{R})$ which vanishes near $\partial D$ and whose Hamiltonian flow, with respect to the symplectic form $(1 + C)\omega_D$, has time-1 map equal to $\psi^{-k}$ (here, we are using the fact that the base is a disc, since otherwise the boundary-parallel Dehn twist would not be isotopic to the identity in the mapping class group rel boundary). Take a function $H_E \in C^\infty(E, \mathbb{R})$ which vanishes near $\partial_v E$ and such that $H_E(z, x) = H_D(z)$ near $\partial_h E$, again using (10). If we deform $\tilde{\xi}$ by composing it with the Hamiltonian flow of $H_E$ for the symplectic form $\tilde{\omega}_E$, the outcome at time 1 will become compactly supported.
The final step of the construction, again using a Moser argument as in [17, Lemma 7.1], is to deform the identity map to a conformally symplectic embedding of \((E, \tilde{\omega}_E)\) into \((E, \omega_E)\). This yields a deformation of the \(\tilde{\omega}_E\)-symplectic automorphism to one which is symplectic for the original \(\omega_E\), which is our desired \(\phi\).

Concerning Lefschetz thimbles, let’s first look at \(\xi(\Delta)\). By definition, this is fibered over \(\gamma'\) and Lagrangian with respect to the symplectic form \((\xi^{-1})^*\tilde{\omega}_E\). Hence, it is automatically the Lefschetz thimble associated to that path and symplectic form. Following through the Moser isotopy then shows that \(\tilde{\xi}(\Delta)\) is isotopic to \(\Delta'\) through submanifolds of \(E\), with boundary in \(F\), which are Lagrangian with respect to \(\tilde{\omega}_E\). The last step, going back to the original \(\omega_E\), uses the same idea. \(\square\)

We want to consider briefly the topological aspects of the maps \(\phi\) constructed in the previous Lemma. Let \((\Delta_0, \ldots, \Delta_{r-1})\) be a basis of Lefschetz thimbles. Choose an orientation of each \(\Delta_i\), and equip \(V_i = \partial \Delta_i\) with the induced orientations. The classes \([\Delta_i] \in H_n(E, F)\) form a basis of that group, so our choice yields an identification

\[
H_n(E, F) \cong \mathbb{Z}^r.
\]

Set \(s_i = +1\) if the isotopy \(\mu^k(V_i) \simeq V_i\) from Assumption 7 preserves orientations, and \(s_i = -1\) otherwise. Let \(S\) be the diagonal matrix with entries \(S_{ii} = s_i\). Take the lower-triangular matrix \(A \in GL(r, \mathbb{Z})\) with entries

\[
A_{ij} = \begin{cases} 
(-1)^{(n+1)n/2} V_i \cdot V_j & i > j, \\
1 & i = j, \\
0 & i < j. 
\end{cases}
\]

Here, \(V_i \cdot V_j\) is the intersection number in \(F\), and the dimension-dependent sign \((-1)^{(n+1)n/2}\) has been added to simplify computations with the Picard-Lefschetz formula.

**Lemma 9.** The action of \(\phi\) on (17) is given by \((-1)^{nk} S((A^t)^{-1} A)^k\).

**Proof.** It is convenient to view \(\phi\) as the composition of two other maps \(\phi_0, \phi_1^k\) (both are diffeomorphisms of \(E\) preserving \(F\), and the precise statement is that \(\phi\) is isotopic to \(\phi_0 \phi_1^k\), within the group of such diffeomorphisms). First, using Assumption 7, one can define a diffeomorphism \(\phi_0\) of \(E\) which takes every fibre of \(\pi\) to itself, and whose restriction to the base fibre \(F\) equals \(\mu^{-k}\). In local coordinates 81 near the \(i\)-th critical point, this diffeomorphism will
be given by an orthogonal matrix with determinant $s_i$. This implies that $(\phi_0)_*[\Delta_i] = s_i[\Delta_i]$, hence that $(\phi_0)_* = S$ in terms of (17).

The negative Dehn twist $\psi$ is the time-one map of a flow on $D$ which rotates the boundary positively (and leaves a large part of the interior fixed). Lifting this flow to $E$, and then taking its time-one map, yields a diffeomorphism $\phi_1$ whose restriction to $F$ is $\mu$. As a form of the classical monodromy formula ([1, vol. 2, Theorem 2.6] or [12, Hauptsatz]), we have $(\phi_1)_* = (-1)^n(A^n)^{-1}A$. The fact that $\phi$ is isotopic to $\phi_0\phi_1^k$ is easy to see, since it is essentially a reinterpretation of our construction in Lemma 8.

**Gradings**

We will now add one more assumption to the setup above, namely that $c_1(E) = 0$. This allows one to introduce graded Lagrangian submanifolds in the sense of [15]. Since our Lefschetz fibrations always have $H^1(E) = 0$ (as a consequence of the assumptions $2n \geq 6$ and $H^1(F) = 0$), the theory of such submanifolds is essentially unique. Note that a grading of a Lefschetz thimble $\Delta \subset E$ induces a grading of the vanishing cycle $V = \partial \Delta \subset F$, see [17, Section 16f] for details.

**Lemma 10.** Let $\gamma$ be any vanishing path, and $\gamma' = \psi^k(\gamma)$, with $\psi$ as before [15], and $k$ any integer. Write $\Delta$, $\Delta'$ for their Lefschetz thimbles. Choose gradings for $\Delta$ and $\Delta'$ which agree near the unique critical point of $\pi$ lying on those submanifolds. Then, the resulting gradings of the vanishing cycles $V = \partial \Delta$, $V' = \partial \Delta'$ satisfy

\[ \mu^k(V) \simeq V'[2k], \]

where $\mu$ is given its canonical grading (which vanishes near $\partial F$), and $[2k]$ is the downwards shift by $2k$. \[ \square \]

This comes down to the following two observations. First, the monodromy maps for our Lefschetz fibration can be lifted canonically to graded symplectic automorphisms. Second, the rotation number of the path $\gamma'$ in the plane differs from that of $\gamma$ by $k$, and this accounts for the additional shift in (19). We omit the details.

**Assumption 11.** There are integers $k$ and $\sigma$ such that for each vanishing cycle $V$, $\mu^k(V) \simeq V[\sigma]$. Here, $\simeq$ means isotopy of graded Lagrangian spheres in $F$ compatible with parametrizations as in (19).
This is obviously a version of Assumption \ref{assumption:gradings} including gradings. In parallel with \ref{eq:gradings}, it is equivalent to saying that for some distinguished basis \((V_0, \ldots, V_{r-1})\), we have
\begin{equation}
(\tau_{V_0} \cdots \tau_{V_{r-1}})^k(V_i) \simeq V_i[\sigma] \quad \text{for } i = 0, \ldots, r - 1.
\end{equation}

Here, each Dehn twist \(\tau_{V_i}\) carries its distinguished grading (which is zero away from a tubular neighbourhood of the twisting sphere). The corresponding version of Lemma \ref{lemma:gradings} is this:

**Lemma 12.** Suppose that Assumption \ref{assumption:gradings} holds. Equip the symplectic automorphism \(\phi\) from Lemma \ref{lemma:gradings} with the unique grading which is zero near \(\partial E\). Equip the Lefschetz thimbles \(\Delta, \Delta'\) with gradings as in Lemma \ref{lemma:gradings}. Then the isotopy \(\phi(\Delta) \simeq \Delta'\) from Lemma \ref{lemma:gradings} lifts to an isotopy of graded Lagrangian submanifolds,
\begin{equation}
\phi(\Delta) \simeq \Delta'[2k - \sigma].
\end{equation}

**Proof.** Let’s temporarily forget about the gradings, and return to the situation from Lemma \ref{lemma:gradings}. Consider the vanishing cycles \(V = \partial \Delta, V' = \partial \Delta'\). By definition of \(\gamma', V'\) is isotopic to \(\mu^k(V)\). On the other hand, \(\mu^k(V)\) is isotopic to \(V\) by assumption. We therefore have an isotopy
\begin{equation}
V \simeq V'
\end{equation}
of (parametrized) Lagrangian spheres in \(F\), and this is canonical up to homotopy rel endpoints. Inspection of the proof of Lemma \ref{lemma:gradings} shows that the isotopy
\begin{equation}
\phi(\Delta) \simeq \Delta'
\end{equation}
is constructed in such a way that its restriction to \(F\) reproduces \ref{eq:isotopy}. We now add gradings to the argument. If we use the same gradings of \(\Delta\) and \(\Delta'\) as before, then \(V'[2k]\) is isotopic to \(\mu^k(V)\), by Lemma \ref{lemma:gradings}. On the other hand \(\mu^k(V)\) is isotopic to \(V[\sigma]\), by Assumption \ref{assumption:gradings}. Hence \ref{eq:isotopy} lifts to a graded Lagrangian isotopy \(V \simeq V'[2k - \sigma]\). Hence, the same applies to \ref{eq:isotopy} when we equip it with gradings. \(\square\)

We remain in the situation of this Lemma, and derive the desired consequence.

**Lemma 13.** Suppose that \(E\) contains a closed symplectic manifold \(\Lambda\) with \(H^1(\Lambda) = 0\); that there is a Lefschetz thimble \(\Delta\) such that the Floer cohomology \(HF^*(L, \Delta)\) is nonzero; and that \(\sigma \neq 2k\) in Assumption \ref{assumption:gradings}. Then \([\phi] \in \pi_0(\text{Symp}(E, \partial E))\) has infinite order.
Proof. If we pick gradings, $HF^*(\Lambda, \Delta)$ is a well-defined $\mathbb{Z}$-graded vector space over $\mathbb{Z}/2$, which is finite-dimensional, and invariant under Lagrangian isotopies. Let’s argue by contradiction, first assuming that $\phi$ is isotopic to the identity in $\text{Symp}(E, \partial E)$. This isotopy can be lifted to the graded symplectic automorphism group, with the consequence that

$$HF^*(\Lambda, \phi(\Delta)) \cong HF^*(\Lambda, \Delta).$$

But Lemma 12 implies that

$$HF^*(\Lambda, \phi(\Delta)) \cong HF^{*+2k-\sigma}(\Lambda, \Delta),$$

which is a contradiction. The same argument can be applied to the iterates of $\phi$. \hfill \Box

Conclusion

We are now ready to construct our symplectic automorphisms. Take the Lefschetz fibration $\pi : E \to D$ from Lemma 5, where the fibre is $F = D^*S^{n-1}$, and the two vanishing cycles $V_0 = V_1 = S^{n-1}$ are parametrized by $f_0 = \text{id}$ and $f_1 = f \in \text{Diff}^+(S^{n-1})$. The associated Dehn twists in $\text{Symp}(F, \partial F)$ are

$$\tau_{V_0} = \tau_{S^n},$$

$$\tau_{V_1} = (T^*f)^{-1}\tau_{S^n}(T^*f),$$

where $\tau_{S^n}$ is the standard Dehn twist, and $T^*f$ is the symplectic automorphism of $T^*S^{n-1}$ induced by $f$. In particular, since the restriction of $\tau_{S^n}$ to the zero-section is the antipodal map $a$, we have

$$\tau_{V_0}\tau_{V_1}|_{S^{n-1}} = a f^{-1} a f.$$ 

Suppose first that $n$ is even. Then $a$ is orientation-preserving, hence can be deformed to the identity in $\text{Diff}(S^{n-1})$. By (28), the same holds for $\tau_{V_0}\tau_{V_1}|_{S^{n-1}}$. This allows one to apply Lemma 8 with $k = 1$, yielding $\phi \in \text{Symp}(E, \partial E)$. After rounding off corners and adding a cone, as in Lemma 6 and extending $\phi$ by the identity to the cone, we get a compactly supported symplectic automorphism of $T^*L$, denoted by $\phi_L$.

In the odd-dimensional case $a$ is orientation-reversing, hence $af^{-1}a$ is isotopic to $f$ by Lemma 5 which in view of (28) means that $\tau_{V_0}\tau_{V_1}|_{S^{n-1}}$ is isotopic to $f^2$. Take the smallest $k$ such that $f^{2k}$ is isotopic to the identity, or equivalently such that the connected sum of $2k$ copies of the associated homotopy sphere $L$ is diffeomorphic to $S^n$. Then, the rest of the construction goes through as before, leading again to a $\phi_L \in \text{Symp}^c(T^*L)$. 

Proof of Proposition 2. To any continuous map $\phi : E \to E$ which is the identity near the boundary, one can associate its variation operator $\text{Var}(\phi)$, which is a homomorphism from $H_*(E, \partial E)$ to $H_*(E)$. It fits into a commutative diagram

\[
\begin{array}{ccc}
H_*(E, F) & \xrightarrow{\phi_* - \text{id}} & H_*(E, F) \\
\downarrow & & \uparrow \\
H_*(E, \partial E) & \xrightarrow{\text{Var}(\phi)} & H_*(E)
\end{array}
\]

In our case, the only possibly nontrivial degree is $* = n$. Moreover, in that degree the $\downarrow$ map is onto, and the $\uparrow$ map injective. Hence, one can compute the variation using Lemma 9.

Suppose first that $n$ is odd. Choose orientations of the Lefschetz thimbles so that the induced orientations of $V_0 = V_1 = S^{n-1}$ coincide. Then the matrices appearing in Lemma 9 are

\[
A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad S = I, \quad -(A^t)^{-1}A = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} = I + 2 \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix},
\]

where $I$ is the identity matrix. This implies that

\[
\phi_* = I + 2k \begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \end{pmatrix},
\]

hence $\text{Var}(\phi)$ is nontrivial. The same argument applies to any iterate of $\phi$. Since the variation operator is constant under homotopies supported in the interior of $E$, it follows that no iterate of $\phi$ admits such a homotopy. The same applies to $\phi_L$, inside the group of maps $T^*L \to T^*L$ which are the identity at infinity.

Now take $n$ even. The counterpart of (30) is

\[
A = S = I.
\]

By the same computation as before, $\phi_*$ is the identity on $H_n(E, F)$, and $\text{Var}(\phi)$ must vanish. Note that $T^*L$ is always diffeomorphic to the cotangent bundle of the standard sphere $S^3$. It then follows from [13, Theorem 3] (together with the table on p. 189 of that paper) that some iterate of $\phi_L$ is isotopic to the identity in $\text{Diff}^c(T^*L)$.

Proof of Theorem 1. By [15, Lemma 5.7], we have $\tau_{V_i}(V_i) = V_i[2-n]$. Hence, Assumption [11] applies with $\sigma = 2k(2-n) \neq 2k$. 

\[\square\]
In our construction, $T^*L$ was obtained by rounding off the corners of $E$ and attaching a cone to the boundary. By using the Liouville flow for negative times, one can therefore compress any compact subset of $T^*L$ into $E$. In particular, we can apply this to the zero-section, the result being a Lagrangian homotopy sphere $\Lambda \subset E$, such that $[\Lambda]$ is a generator for $H_n(E)$. Because we know its homology class, it follows easily that the intersection number $\Lambda \cdot \Delta_i = \pm 1$ is nonzero, hence so is the Floer cohomology $HF^*(\Lambda, \Delta_i)$. Applying Lemma 13 shows that $[\phi] \in \pi_0(Symp(E, \partial E))$ has infinite order. But by using the same compression trick as before, this implies the same property for $[\phi_L] \in \pi_0(Symp^c(T^*L))$.

\[\square\]

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