Symmetries and dynamics in constrained systems

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Abstract

We review in detail the Hamiltonian dynamics for constrained systems. Emphasis is put on the total Hamiltonian system rather than on the extended Hamiltonian system. We provide a systematic analysis of (global and local) symmetries in total Hamiltonian systems. In particular, in analogue to total Hamiltonians, we introduce the notion of total Noether charges. Grassmannian degrees of freedom are also addressed in details.

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1 Introduction

Symmetries have always been a determinant guide for the understanding of Nature, because they seem to enable the simultaneous concretization of the two ideals underlying the scientific quest: simplicity and beauty.\(^1\) The symmetry principles have been essential for the development of modern physics, \textit{e.g.} in the birth of both relativity theories or in the building of the standard model. The importance of symmetries has been recognized since the very beginning of scientific inquiry but mankind waited until the twentieth century for a new paradigm to emerge: the gauge symmetry principle.\(^2\) The aphorism “symmetry dictates interaction” can be considered as the cornerstone of modern theoretical physics. Both in classical general relativity and in quantum field theory, the gauge symmetries are the deep geometrical foundations of fundamental interactions. Indeed, gauge symmetries determine the terms which may appear in the action. Nevertheless, some qualifications need to be made because, unfortunately, the symmetries rarely fix uniquely the interactions although this dream underlies most unification models. Even though, at first sight, gauge transformations could have been naively dismissed as auxiliary -if not irrelevant- tools since they are in some sense “unphysical,” they actually proved to be almost unavoidable! For instance, from a field theoretical point of view the light-cone formulation is perfectly consistent by itself, but it is extremely convenient to introduce spurious unphysical (in other words, “gauge”) degrees of freedom in order to write down Lagrangians for massless particles which are manifestly local and covariant under Lorentz transformations. Another example is general relativity where the decisive role played by the requirement of covariance under the diffeomorphisms does not need to be stressed, even though a superficial glance at this issue would dismiss this requirement as irrelevant since any theory can be formulated independently of the coordinate system by introducing an affine connection.\(^3\)

Like for every deep and fundamental concept in physics, gauge symmetries exhibit many faces and can be approached in different ways. The investigations of Dirac on the Hamiltonian formulation of gravity opened a new door for entering into the world of gauge theories. As explained in his seminal works [3], the presence of gauge symmetries in the Lagrangian framework implies, from a Hamiltonian point of view, the existence of “constraints” on the phase space variables. Conversely, the study of constraints in the Hamiltonian framework may serve as a path leading towards some understanding of gauge symmetries. The present lecture notes are intended to be

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\(^1\)One may recommend the collection of inspiring lectures given by Chandrasekhar or the celebrated book of Weyl on this topic [1].

\(^2\)The many developments of this crucial chapter in the history of physics are very well summarized in the book [2].

\(^3\)A student, scared by some of the conceptual subtleties arising from the gauge symmetry principle, could find some recomfort in the following surprising anecdote: during his quest for a reconciliation between gravity and relativity, Einstein himself initially argued that the equations of motion for the metric must \textit{not} be diffeomorphism covariant! The “physical” bases of this wrong initial requirement were related to the subtle issues mentioned above.
a self-contained introduction to the Hamiltonian formulation of systems with constraints. Since the seminal investigations of Dirac, the development of this topic has been so dramatic that we would not pretend to be complete. At most, we do hope that these notes might be useful to newcomers searching for a pedestrian and concrete approach on the interplay between Lagrangian vs Hamiltonian systems from the point of view of gauge symmetries vs constraints. The main particularity of the present notes is that the (rigid and gauge) symmetries and their associated Noether charges are discussed with many details. For instance, the various possible definitions according to the choice of formalisms (Lagrangian, total or extended Hamiltonian) are introduced and compared with each other. Their explicit relationship is provided, due to its importance for applications. Another original feature is that the fermionic case is included in the presentation from the very beginning. This case is so relevant in physics that we found better to discuss the general case immediately in order to allow a uniform treatment of all physical cases, rather than devote later a specific section to this ‘particular’ case. We also insert all the details and proofs of the properties of supermatrices that are used in this text. More generally, pretty much all results presented here are given with their proofs, in order to be entirely self-contained. Nevertheless, we have not aimed at complete mathematical rigour (in the sense that all symbols written in the text are supposed to exist under suitable regularity conditions and that the formal manipulations they are subject to, are allowed). The major emphasis is on the classical level, though we provide some flavour of the quantization process at the end of these notes, for both first and second class constraints.

In contradistinction to most of the fundamental textbooks on the subject, such as [4–7], we focus on the total Hamiltonian instead of the extended Hamiltonian.\(^4\) On the one hand, the main advantage of this choice is that the dynamics determined by the former is always equivalent to the Lagrangian dynamics. On the other hand, its drawback is precisely that the primary constraints play a privileged role, while such a distinction is not relevant from a purely Hamiltonian perspective. Of course, the evolution of the physical quantities (i.e. the observables) through the dynamics of either the Lagrangian, the total Hamiltonian or the extended Hamiltonian always agrees. Therefore, the preference between total and extended Hamiltonian is somehow ‘philosophical’, in the sense that it reflects the opinion whether, respectively, the Lagrangian formulation is more fundamental than the Hamiltonian one, or the contrary. Mathematically, one may argue that none of these opinions is more valid than the others because some Lagrangian systems do not allow an Hamiltonian formulation, and conversely. (Both types of examples are reviewed in this text.) Physically, the quantization process seems to rely heavily on the Hamiltonian formulation, even if Feynmann’s path integral could plead in favour of the Lagrangian as well. Still, a rigorous

\(^4\)Of course, these textbooks do include very detailed discussions on the total Hamiltonian formalism, we only mean that it is not their chief emphasis and guideline. Of course, the formalism of constraints is discussed in many other textbooks, e.g. [8, 9].
The plan of these notes is as follows: The Lagrangian dynamics is reviewed in Section 2 under complete generality, scrutinizing on various aspects of the symmetries of the action principle which are not always addressed in details (higher derivatives, finite versus infinitesimal transformation, invertibility, etc). The canonical Hamiltonian formalism for a dynamical system with constraints is reviewed in Section 3 and the conditions of the equivalence between the Lagrangian and the Hamiltonian formalism are mentioned. In the section 4, the total and extended Hamiltonian dynamics are introduced together with the distinct types of constraints: primary or secondary, first or second class. Although these distinctions become somewhat irrelevant at some deeper level from the Hamiltonian point of view, they are very important when addressing the quantization process or the concrete relation between the Lagrangian and the Hamiltonian formulations. The symmetries of dynamical systems and their associated conserved charges are discussed thoroughly in Section 5 from several perspectives. The equivalences between the many approaches are not shown through general theorems but through direct computations in order to provide for the reader a concrete grasp of the formalism. The Dirac quantization, suitable for second class constraints and reviewed in Section 6, has a rather straight interpretation: eliminate the spurious degrees of freedom by making use of the Dirac bracket. The main drawback of this method is that, in general, one is unable to compute the Dirac bracket explicitly. While the BRST quantization, suitable for first class constraints and presented in Section 7, is more subtle conceptually (because it is formulated as a cohomological problem) and technically (because many fields such as ghosts, etc, have to be added) it possesses at least one great virtue: if the Hamiltonian constraints (or the Lagrangian gauge symmetries) have been entirely determined, then this method can always be settled concretely in order to write down the gauge-fixed path integral, even if the BRST cohomology group cannot be computed explicitly. At the end come some appendices: some proofs of various properties are placed in the appendix A in order to lighten the core of the text. A rigorous treatment of the fermionic variables is provided in Appendix B through a review of the Grassmann algebras, while all the necessary definitions and basic properties of supermatrices are provided in Appendix C. The proofs of some propositions on the canonical forms of supermatrices are presented in details in the appendix D. Finally, the appendix E is devoted to a very simple and illustrative example of the general discussion contained in the body of these notes. We advise the reader to progressively go through this example, while (s)he goes through the general material in the core of the text.

These notes are an expanded version of some lectures given by JHP at Sogang University
during the years 2007 and 2008.
2  Lagrangian dynamics: symmetry and Grassmann variables

2.1  Euler-Lagrange equations

We first consider a generic Lagrangian depending on \( N \) variables, \( q^A, 1 \leq A \leq N \), their time derivatives, and time is allowed to appear explicitly,

\[
\mathcal{L}(q_n, t),
\]

where

\[
q^A_n = \left( \frac{d}{dt} \right)^n q^A, \quad n = 0, 1, 2, 3, \ldots.
\]

The \( q^A_n \)'s form the coordinates of the so called “jet space”. Note that some of the variables can be fermionic.\(^6\) In our conventions, unless explicitly mentioned, all derivatives act from the left to the right,

\[
\frac{\partial F}{\partial q^A} = \overleftarrow{\partial} F = (-1)^{\#_A(\#_F + \#_A)} \overrightarrow{\partial} F,
\]

where \( \#_A \) is the \( \mathbb{Z}_2 \)-grading of the \( 2N \)-dimensional tangent space with coordinates \( (q^A, \dot{q}^B) \),

\[
\#_A = \begin{cases} 0 & \text{for bosonic } A \\ 1 & \text{for fermionic } A \end{cases}.
\]

From the variation of the action

\[
S[ q^A ] = \int \mathcal{L}(q_n, t) \, dt,
\]

and up to the boundary terms, one obtains

\[
\int dt \, \delta \mathcal{L}(q_n, t) = \int dt \, \delta q^A \left[ \sum_{n=0}^{\infty} \left( \frac{d}{dt} \right)^n \frac{\partial}{\partial q^A_n} \right] \mathcal{L}(q_m, t),
\]

so that the corresponding Euler-Lagrange equations

\[
\frac{\delta \mathcal{L}(q_m, t)}{\delta q^A} \equiv 0,
\]

\(^5\)For some issues in the continuous limit \( N \to \infty \), see e.g. [9].

\(^6\)One of the only prerequisite of these lectures is that the reader is supposed to be familiar with graded algebras and related super objects. A good self-contained introduction to supersymmetry is [10].
are given by acting on the Lagrangian with a linear differential operator called the Euler-Lagrange operator,
\[ \delta \frac{\partial}{\partial q^A} := \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \frac{\partial}{\partial q^A_n} . \tag{2.8} \]
In the jet space, the submanifold defined by Eq.(2.7) is called the “stationary (hyper)surface”. We have introduced the symbol \( \equiv \) which will stand, from now on, for ‘equal on the stationary surface’ or, equivalently, ‘equal modulo the Lagrangian equations of motion (2.7)’. Remark that the Euler-Lagrange operator (2.8) is not a derivation, i.e. it does not obey to the Leibniz rule.

It is useful to understand that in the jet space the total derivative \( \frac{d}{dt} \) is defined as
\[ \frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{n=0}^{\infty} q^n_{n+1} \frac{\partial}{\partial q^n_n} , \tag{2.9} \]
so that
\[ \frac{\partial}{\partial q^A_n} \frac{d}{dt} = \frac{d}{dt} \frac{\partial}{\partial q^A_n} , \quad \frac{\partial}{\partial q^n_n} \frac{d}{dt} = \frac{d}{dt} \frac{\partial}{\partial q^n_n} + \frac{\partial}{\partial q^n_{n-1}} , \quad n \geq 1 . \tag{2.10} \]
This implies that the Euler-Lagrange equations of a total derivative term are identically vanishing,
\[ \delta \frac{\partial}{\partial q^A} \left( \frac{dK}{dt} \right) = 0 . \tag{2.11} \]
The converse is also true, therefore
\[ \frac{\delta F}{\delta q^A}(q_n, t) = 0 \iff F(q_n, t) = \frac{dK(q_n, t)}{dt} . \tag{2.12} \]
We present a proof of these statements in Appendix A. Note that a generalization of Eq.(2.12) holds for field theories \( (\mathcal{N} = \infty) \) too, in which case it is referred to as “algebraic Poincaré lemma”.

### 2.2 Symmetries in the Lagrangian formalism

In general, a symmetry of the action involves a certain change of variables,
\[ q^A \longrightarrow \tilde{q}^A(q_n, t) , \tag{2.13} \]
\[ ^7 \text{For the fermionic degrees of freedom, there arises a subtle point which we discuss in Appendix B.} \]
\[ ^8 \text{For more details on this lemma, on jet space, etc, see e.g. the section 4 of [11] and references therein.} \]
which may explicitly depend on the $q_n^B$'s and the time $t$. It corresponds to a symmetry of the action if the Lagrangian is invariant under the transformation up to total derivative terms,

$$ L(\tilde{q}_n, t) = L(q_n, t) + \frac{d}{dt}K(q_n, t), $$

(2.14)

where

$$ \tilde{q}_n^A = \left(\frac{d}{dt}\right)^n q_n^A = \left(\frac{\partial}{\partial t} + \sum_{m=0}^{\infty} q_n^{B_{m+1}} \frac{\partial}{\partial q_n^B}\right)^n q_n^A(q_l, t). $$

(2.15)

Surely this imposes nontrivial conditions on the form of $\tilde{q}(q_n, t)$ in terms of the Lagrangian, $L(q_n, t)$.

One useful identity for an arbitrary function $F$ is\(^9\), for $m \geq 1$,

$$ \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left( \frac{\partial q_m^B}{\partial q_n^A} F \right) = \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left[ \frac{\partial q_0^B}{\partial q_n^A} \left( -\frac{d}{dt} \right)^m F \right], $$

(2.16)

which gives the following algebraic identity for any change of variables,

$$ \frac{\delta L(\tilde{q}_l, t)}{\delta q^A} = \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left( \sum_{m=0}^{\infty} \frac{\partial q_m^B}{\partial q_n^A} \frac{\partial L(\tilde{q}_l, t)}{\partial \tilde{q}_m^B} \right) = \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left( \frac{\partial q_0^B}{\partial q_n^A} \frac{\delta L(\tilde{q}_l, t)}{\delta \tilde{q}_B} \right). $$

(2.17)

Infinitesimally, $q_0^A \rightarrow q_0^A + \delta q_0^A(q_n, t)$, Eq.(2.17) implies

$$ \left[ \frac{\delta}{\delta q^A} , \delta \right] L(q_l, t) = \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left( \frac{\partial (\delta q_0^B)}{\partial q_n^A} \frac{\delta L(q_l, t)}{\partial q^B} \right), $$

(2.18)

since

$$ \left( \frac{\delta L(\tilde{q}_l, t)}{\delta q^A} - \frac{\delta L(q_l, t)}{\delta q^A} \right) - \left( \frac{\delta L(\tilde{q}_l, t)}{\delta \tilde{q}^A} - \frac{\delta L(q_l, t)}{\delta q^A} \right) $$

$$ = \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left( \frac{\partial q_0^B}{\partial q_n^A} \frac{\delta L(\tilde{q}_l, t)}{\delta q^B} \right) - \frac{\delta L(\tilde{q}_l, t)}{\delta q^A}. $$

(2.19)

Eq.(2.17) will be used later in order to show that the Euler-Lagrange equations are preserved by symmetries of the action, a fact which is natural to expect but is non trivial to prove.

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\(^9\)See Eq.(A.10) for a proof.
Now assuming the symmetry (2.13), acting with the Euler-Lagrange operator on both sides of (2.14), using (2.11) and (2.17), we get
\[
\frac{\delta L(q_l, t)}{\delta q^A} = \sum_{n=0}^{\infty} \left( -\frac{1}{n!} \right)^n \left( \frac{\partial \tilde{q}^B_0 \delta L(\tilde{q}_l, t)}{\partial q^n_A} \right).
\]

Before we discuss the generic cases, we first consider the simple case where \( \tilde{q}^B_0(q, t) \) depends on \( q \) and \( t \) only, being independent of \( q \n (n \geq 1) \) and invertible i.e. \( \det(\partial \tilde{q}/\partial q) \neq 0 \). We have
\[
\frac{\delta L(\tilde{q}_l, t)}{\delta \tilde{q}^A} = \frac{\partial \tilde{q}^B(\tilde{q}, t)}{\partial q^A} \frac{\delta L(q_l, t)}{\delta q^B},
\]
using (2.20) with \( q \) and \( \tilde{q} \) exchanged. Hence, if \( q(t) \) is a solution of the equations of motion, then so is \( \tilde{q}(q, t) \), where \( \tilde{q}(q, t) \) depends on \( q \) and \( t \) only, i.e. not on the derivatives \( q^n (n \geq 1) \).

Now, for the generic cases where \( \tilde{q}(q_n, t) \) depends on the \( q^n \)'s \( (n \geq 0) \): if there exists an inverse map \( q(\tilde{q}_n, t) \) - most likely depending on the infinite set of variables\(^{10} q_n (n = 0, 1, 2, \cdots) \) - then the (inverse relation of) Eq.(2.20) indeed shows that if \( q(t) \) is a solution of the equations of motion, then so is \( \tilde{q}(q_n, t) \).

### 2.2.1 Invertible transformations

The existence of an inverse map is always guaranteed when there exists a corresponding infinitesimal transformation,
\[
q^A \rightarrow q^A + \delta q^A, \quad \delta q^A = f^A(q_n, t).
\]

Consequently, the infinitesimal transformation of the coordinates of the jet space are given by
\[
q^n_A \rightarrow q^n_A + \delta q^n_A,
\]
where
\[
\delta q^n_A = \left( \frac{1}{n!} \right)^n f^n_A(q_m, t) = \left( \frac{\partial}{\partial t} + \sum_{l=0}^{\infty} q^B_{l+1} \frac{\partial}{\partial q^B_l} \right)^n f^A(q_m, t) =: f^n_A(q_m, t).
\]

More explicitly, we define an exponential map with a real parameter \( s \),
\[
\tilde{q}^A(s, q_n, t) = \exp \left( s \sum_{l=0}^{\infty} f^B_l(q_m, t) \frac{\partial}{\partial q^B_l} \right) q^A, \quad \tilde{q}^A(0, q_n, t) = q^A.
\]

\(^{10}\)For example, the usual translational symmetry reads
\[
\tilde{q}^A(t) = q^A(t + a) = \sum_{n=0}^\infty \frac{a^n}{n!} q^n_A(t) \quad \iff \quad q^A(t) = \tilde{q}^A(t - a) = \sum_{n=0}^\infty \frac{(-a)^n}{n!} q^n_A(t).
\]
From (2.10) we first note the commutativity property,
\[
\frac{d}{dt} \left( \sum_{l=0}^{\infty} f_l^B \frac{\partial}{\partial q_l^B} \right) = \sum_{l=0}^{\infty} \left( f_{l+1}^B \frac{\partial}{\partial q_l^B} + f_l^B \frac{d}{dt} \frac{\partial}{\partial q_l^B} \right) = \left( \sum_{l=0}^{\infty} f_l^B \frac{\partial}{\partial q_l^B} \right) \frac{d}{dt},
\]
and hence,
\[
\tilde{q}_n^A(s, q_m, t) = \left( \frac{d}{dt} \right)^n \tilde{q}^A(s, q_m, t) = \exp \left( s \sum_{l=0}^{\infty} f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} \right) q_n^A.
\]

The main claim is then,
\[
\frac{d\tilde{q}_n^A}{ds} = \sum_{l=0}^{\infty} f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} \tilde{q}_n^B = f_n^A(\tilde{q}_m, t).
\]

From (2.26), the first equality in (2.27) is obvious. The derivation of the other relation is carried in Eq.(A.12) of the appendix. The equation (2.27) implies that the following differential operator is ‘s’-independent,
\[
\sum_{l=0}^{\infty} f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} = f_n^A(\tilde{q}_m, t).
\]

Using the above identities, it is straightforward to obtain the explicit inverse map, \( \tilde{q}^A \rightarrow q^A \),
\[
q^A(s, \tilde{q}_n, t) = \exp \left( -s f_l^B(\tilde{q}_m, t) \frac{\partial}{\partial q_l^B} \right) \tilde{q}_n^A.
\]

### 2.2.2 Local symmetries

In the case of a “local symmetry”, namely if the transformation involves some arbitrary time dependent functions \( \alpha_i(t) \) as
\[
q^A \rightarrow \tilde{q}^A(q_n, t, \alpha(t)),
\]
it is possible to have ‘different’ solutions by varying the arbitrary functions \( \alpha_i(t) \), even though one starts from the same initial data \( \tilde{q}_n(t_0) \). However, one may consider that the Lagrangian alone dictates the whole dynamics of the given system (and nothing else) and, furthermore, that the dynamics is deterministic (i.e. there is a unique solution to the Cauchy problem). As long
as one takes this viewpoint for granted, then one must regard different trajectories as the *same physical state*.

Namely any local symmetry must be a ‘gauge’ (i.e. unphysical) symmetry. *In mathematical terms, a physical state is given by an equivalence class for which the local symmetry defines the equivalence relation.*\(^{11}\) Obviously, the presence of the gauge symmetries complicates the correct counting\(^{12}\) of the number of physical degrees of freedom (especially if the gauge symmetries are not independent, *etc*). An “observable” quantity is a function on the space of physical states, hence it must be gauge invariant. We will turn back to these issues in the Hamiltonian context.

### 2.3 Second order field equations

Henceforth, we focus on the standard Lagrangian, \(L(q, \dot{q}, t)\), which depends on \(q^A, \dot{q}^A, t\) only, and derive some algebraic identities for the later use.

When the infinitesimal symmetry transformation, \(q^A \to q^A + \delta q^A(q, \dot{q}, t)\), also depends on \(q^A, \dot{q}^A, t\) only, we have\(^{13}\)

\[
\delta \dot{q}^A = \frac{d}{dt} \delta q^A(q, \dot{q}, t) = \ddot{q}^B \frac{\partial (\delta q^A)}{\partial \dot{q}^B} + \dot{q}^B \frac{\partial (\delta q^A)}{\partial q^B} + \frac{\partial (\delta q^A)}{\partial t}.
\]  

(2.31)

Thus, the function \(\delta K(q, \dot{q}, t)\) in

\[
\delta L = \frac{d}{dt} \delta K
\]

(2.32)

---

\(^{11}\)A complete and detailed treatment of the gauge invariance of an action can be found in the chapter 3 of [7].

\(^{12}\)The Hamiltonian framework enables a precise and ‘algorithmic’ computation of the number of degrees of freedom, which leads to a precise and completely general criterion for counting the physical degrees of freedom directly from the form of gauge transformations in the Lagrangian formalism itself [12]. This rigorous treatment clarifies the origin of some maxims from the physicist folklore (such as “gauge shoots twice”) and thereby provides a supplementary argument in favor of the fruitful interplay between Hamiltonian and Lagrangian formalisms.

\(^{13}\)Actually, in the case of a regular second-order Lagrangian, this can be assumed without loss of generality, as explained in the exercise 3.8 of the book [7].
must depend on \(q^A, \dot{q}^A, t\) only too, as
\[
\delta L = \delta q^A \frac{\partial L(q, \dot{q}, t)}{\partial q^A} + \delta q^A \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A}
\]
\[
= \delta q^A \frac{\partial L(q, \dot{q}, t)}{\partial q^A} + \left( \dot{q}^B \frac{\partial (\delta q^A)}{\partial q^B} + \dot{q}^B \frac{\partial (\delta q^A)}{\partial \dot{q}^A} + \frac{\partial (\delta q^A)}{\partial t} \right) \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A}
\]
\[
= \frac{d}{dt} \delta K = \dot{q}^A \frac{\partial (\delta K)}{\partial q^A} + \dot{q}^A \frac{\partial (\delta K)}{\partial \dot{q}^A} + \frac{\partial (\delta K)}{\partial t}.
\]
This implies
\[
\frac{\partial (\delta q^B)}{\partial q^A} \frac{\partial L(q, \dot{q}, t)}{\partial q^A} = \frac{\partial (\delta K)}{\partial \dot{q}^A}, \tag{2.34}
\]
and
\[
\delta q^B \frac{\partial L}{\partial q^B} + \left( \dot{q}^C \frac{\partial (\delta q^B)}{\partial q^C} + \frac{\partial (\delta q^B)}{\partial \dot{q}^B} \right) \frac{\partial L}{\partial \dot{q}^B} = \dot{q}^B \frac{\partial (\delta K)}{\partial q^B} + \frac{\partial (\delta K)}{\partial \dot{q}^B}. \tag{2.35}
\]
By taking the partial derivative of the latter equation with respect to \(\dot{q}^A\), making use firstly of Eq. (2.31) and secondly of Eq. (2.34), we get
\[
\frac{\partial (\delta q^B)}{\partial q^A} \frac{\partial L}{\partial q^B} + \delta q^B \frac{\partial^2 L}{\partial q^B \partial q^A} + \frac{\partial (\delta q^B)}{\partial \dot{q}^B} \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} + \left( \frac{\partial (\delta q^B)}{\partial q^A} + \dot{q}^C \frac{\partial^2 (\delta q^B)}{\partial q^C \partial q^A} + \frac{\partial^2 (\delta q^B)}{\partial \dot{q}^A \partial t} \right) \frac{\partial L}{\partial \dot{q}^B}
\]
\[
= \frac{\partial (\delta K)}{\partial q^A} + \dot{q}^B \frac{\partial^2 (\delta K)}{\partial q^A \partial \dot{q}^A} + \frac{\partial (\delta K)}{\partial \dot{q}^B} \frac{\partial^2 L}{\partial \dot{q}^B \partial q^A}
\]
\[
= \frac{\partial (\delta K)}{\partial q^A} + \left( \dot{q}^B \frac{\partial}{\partial q^A} + \frac{\partial}{\partial \dot{q}^B} \right) \left( \frac{\partial (\delta q^C)}{\partial q^C} \frac{\partial L}{\partial q^A} + \dot{q}^C \frac{\partial (\delta q^B)}{\partial q^C} \frac{\partial^2 L}{\partial q^B \partial q^A} \right). \tag{2.36}
\]
From Eq. (2.34) we find that the coefficient of \(\dot{q}^C\) must vanish, which implies an integrability condition:
\[
\frac{\partial (\delta q^B)}{\partial q^C} \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} - (-1)^{\#A\#C} \frac{\partial (\delta q^B)}{\partial \dot{q}^A} \frac{\partial^2 L}{\partial q^B \partial q^C}
\]
\[
= (-1)^{\#A\#C} \frac{\partial}{\partial q^A} \left( \frac{\partial (\delta q^B)}{\partial q^C \partial q^B} \frac{\partial L}{\partial q^A} \right) - \frac{\partial}{\partial q^C} \left( \frac{\partial (\delta q^B)}{\partial q^A \partial \dot{q}^B} \frac{\partial L}{\partial q^A} \right) = 0. \tag{2.37}
\]
The transformation of the ‘momenta’ \( \partial L / \partial \dot{q}^A \) is given by

\[
\delta \left( \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A} \right) = \delta q^B \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} + \delta \dot{q}^B \frac{\partial^2 L}{\partial \dot{q}^B \partial \dot{q}^A}
\]

\[
= \frac{\partial (\delta K)}{\partial q^A} - \frac{\partial (\delta q^B)}{\partial q^A} \frac{\partial L}{\partial \dot{q}^B} + \delta \dot{q}^B \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^B} \right) - \frac{\partial L}{\partial \dot{q}^B} \right],
\]

where Eqs. (2.36) and (2.37) have been used in the derivation at the second line. Finally, for the transformation of the Hamiltonian we get

\[
\delta \left( \dot{q}^A \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A} - L(q, \dot{q}, t) \right) = \dot{q}^A \frac{\partial (\delta q^B)}{\partial \dot{q}^A} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^B} \right) - \frac{\partial L}{\partial \dot{q}^B} \right] + \frac{\partial (\delta q^A)}{\partial t} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial (\delta K)}{\partial t}.
\]

These relations will be used later in Section 5.1 where we analyze the symmetries in the Hamiltonian formalism.

The corresponding Noether charge is defined by

\[
Q = \delta q^A \frac{\partial L}{\partial \dot{q}^A} - \delta K.
\]

Up to the Euler-Lagrange equation,

\[
\frac{\partial L}{\partial q^A} \equiv \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right),
\]

the Noether charge is conserved,

\[
\frac{dQ}{dt} = \delta q^A \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) - \frac{\partial L}{\partial \dot{q}^A} \right] \equiv 0,
\]

due to Eq.(2.32). Further discussions on symmetries in the Lagrangian dynamics are carried out in Section 5.1.
3 From Lagrangian to Hamiltonian and vice versa

3.1 Canonical momenta

Given a standard Lagrangian $L(q, \dot{q}, t)$, depending on $N$ bosonic or fermionic variables $q^A$ (with $1 \leq A \leq N$) and their first time derivatives (and, possibly, on time as well), the equations of motion read

$$\frac{\partial L}{\partial q^A} \equiv \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^A} \right) = \ddot{q}^B \frac{\partial^2 L}{\partial \dot{q}^B \partial \dot{q}^A} + \dot{q}^B \frac{\partial^2 L}{\partial q^B \partial \dot{q}^A} + \frac{\partial^2 L}{\partial t \partial \dot{q}^A}. \quad (3.1)$$

If the $N \times N$ supermatrix, $\frac{\partial^2 L}{\partial \dot{q}^B \partial \dot{q}^A}$, is nondegenerate, all the $\ddot{q}^A$’s are uniquely determined by $q$ and $\dot{q}$. Namely all the variables are completely determined by the initial data, and also all the ‘velocities’ $\dot{q}^A$ may be expressed in terms of $q$ and the canonical momenta $p_A := \frac{\partial L}{\partial \dot{q}^A}$.

Henceforth, we focus on the degenerate case,

$$\text{sdet} \left( \frac{\partial^2 L}{\partial \dot{q}^B \partial \dot{q}^A} \right) = 0. \quad (3.2)$$

In a large class of examples, it may still possible that all the variables are uniquely determined from the initial data through the equations of motion, e.g. for a Lagrangian $L(q, \dot{q}, t) = \omega_{AB} \dot{q}^A q^B - V(q, t)$, linear in the ‘$\dot{q}^A$’, and where the constant graded symmetric matrix, $\omega_{[AB]} = \omega_{AB}$, is non-degenerate. The (anti)symmetrization has weight one, i.e.

$$\omega_{[AB]} := \frac{1}{2} \left( \omega_{AB} - (-)^{#A#B} \omega_{BA} \right). \quad (3.3)$$

We will not attempt to analyze and classify all cases here in the Lagrangian formalism, but we will do so in the Hamiltonian formalism later.

3.2 Primary constraints

Now, let us start from the expressions for the $N$ momenta in terms of $q, \dot{q}, t$,

$$p_A = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A} = f_A(q, \dot{q}, t) \quad A = 1, 2, \ldots, N, \quad (3.4)$$

and try to invert the map in order to express the velocities $\dot{q}^A$ in terms of $q, t$ and the momenta $p$.

We first consider a bosonic system having bosonic variables only. If one of the $N$ momenta (3.4), say $p_a$, depends nontrivially on a certain velocity, say $\dot{q}^\hat{a}$, then this velocity can be expressed in terms of $p_a$ and the remaining velocities, collectively denoted by $\dot{q}^\hat{m}$, as well as $q^A$ and $t$. Then, substituting the expression

$$\dot{q}^\hat{a} = h^\hat{a}(q^A, p_a, \dot{q}^\hat{m}, t), \quad (3.5)$$
into the other momenta than \( p_a \), collectively denoted by \( p_m \), we get
\[
p_m = g_m(q^A, p_a, \dot{q}^\hat{m}, t),
\]
(3.6)
This procedure can be repeated until the expressions for the momenta \( p_m \) do not depend on any of the velocities. Performing the procedure \textit{step by step} a finite number of times, we get finally
\[
\dot{q}^\hat{a} = h^\hat{a}(q^A, p_a, \dot{q}^\hat{m}, t), \quad p_m = f_m(q^A, p_a, t),
\]
where the sets of momenta and velocities split into two disjoint groups each,
\[
p_A = (p_a, p_m) : \{a\} \cup \{m\} = \{1, 2, \ldots, N\}, \quad \{a\} \cap \{m\} = \emptyset, \quad \{\hat{a}\} \cup \{\hat{m}\} = \emptyset,
\]
(3.7)
and in particular our procedure defines a one-to-one correspondence of \( \{a\} \leftrightarrow \{\hat{a}\} \).\(^{14}\) In words, on one side the velocities have hatted indices, on the other side the momenta have unhatted indices. For the velocities, the \( \hat{m} \)'s correspond to the velocities which remain independent while the \( \hat{a} \)'s correspond to the velocities which are determined in terms of the former velocities and momenta. For the momenta, the situation is opposite: the \( a \)'s correspond to the ones which are independent while the \( m \)'s correspond to the momenta which are expressed in terms of the latter momenta.

Notice that \( p_m(q, \dot{q}) = f_m(q^A, p_a(q, \dot{q}), t) \) are identities on the tangent space of coordinates \( (q^A, \dot{q}^B) \) and that there exists an invertible map between the momenta \( p_a \) and the velocities \( \dot{q}^{\hat{a}} \) (keeping \( q^A, \dot{q}^\hat{m} \) and \( t \) fixed),
\[
\dot{q}^{\hat{a}} = h^{\hat{a}}(q^A, p_a, \dot{q}^\hat{m}, t) \iff p_a = f_a(q^B, \dot{q}^A, t).
\]
(3.9)
The latter means that
\[
\det \left( \frac{\partial p_a(q, \dot{q}, t)}{\partial \dot{q}^{\hat{a}}} \right) \neq 0,
\]
(3.10)
thus the rank of \( \partial p_A/\partial \dot{q}^B \) is equal\(^{15}\) to the dimension of the set \( \{a\} \). Also, one may say that there is a one-to-one map\(^{16}\)
\[
\begin{pmatrix}
q^A \\
\dot{q}^A \\
q^B \\
\dot{q}^B \\
\dot{q}^\hat{m}
\end{pmatrix}
\longleftrightarrow
\begin{pmatrix}
p_a \\
\dot{q}^B
\end{pmatrix}.
\]
(3.11)
\(^{14}\)They are distinguished because \( p_a \) is not necessarily the conjugate momentum of \( \dot{q}^{\hat{a}} \).
\(^{15}\)This fact is an alternative starting point for getting the constraints and the decomposition of the indices in disjoint set. We preferred to provide a concrete explanation of the result (3.7) instead of a slightly more abstract one in terms of the rank of the (super)Jacobian \textit{via} the implicit function theorem.
\(^{16}\)See (3.34) for a more general result.
Now we return to the generic systems having both bosons and fermions. In contrast to the bosonic system, the above procedure which expresses the velocities in terms of the momenta may not work even if the momenta depend on velocities nontrivially, mainly due to the non-existence of an inverse for any fermionic variable. We consider an example,

\[ L = i \dot{\theta} \dot{x}, \]  

(3.12)

which gives \( p_\theta = i \dot{\theta} \dot{x} \) and \( p_x = i \dot{\theta} \theta \). When \( \theta \) is fermionic while \( x \) is bosonic (which is the case with the usual notations) none of the expressions can be inverted. Furthermore, the corresponding Hamiltonian reads \( H = i \dot{\theta} \dot{x} = L \) which again cannot be reexpressed by the momenta and coordinates only.\(^{17}\) If \( \theta \) were bosonic, then \( H = -ip_\theta p_\theta \theta^{-1} \). In the present paper we do not consider this case. We always assume that when the the momenta depend on velocities nontrivially, one can always obtain the inverse function until we achieve the expression (3.7). Namely we will restrict\(^{18}\) our analysis to systems of bosons and fermions, where the \( 2N \) coordinates \((p_A, q^B)\) of the phase space \( (1 \leq A, B \leq N) \), are subject to \( \mathcal{M} \) functionally independent constraints,

\[ \phi_m(p, q, t) = 0, \quad 1 \leq m \leq \mathcal{M}. \]  

(3.13)

The \( \mathcal{M} \) primary constraints are independent in the sense that the following \( \mathcal{M} \) vectors are linearly independent,

\[ \bar{\partial}_p \phi_m \bigg|_V = \left( \frac{\partial \phi_m}{\partial p_1}, \frac{\partial \phi_m}{\partial p_2}, \ldots, \frac{\partial \phi_m}{\partial p_N} \right) \bigg|_V, \quad 1 \leq m \leq \mathcal{M}, \]

(3.14)

where \( |_V \) means that the left-hand-side is evaluated on the hypersurface defined by the system (3.13), after taking the partial derivatives. The implicit function theorem actually ensures that it is possible to solve Eq.(3.13) for \( \mathcal{M} \) of the momenta, as in Eq.(3.7). Also, the \( \mathcal{M} \) primary constraints naturally define a \( (2N - \mathcal{M}) \)-dimensional hypersurface \( V \) in the phase space, called the “primary constraint (hyper)surface”.

\[ V = \{ (p, q) \mid \phi_m(p, q, t) = 0, \quad 1 \leq m \leq \mathcal{M} \}. \]

(3.15)

\(^{17}\)One possible way to circumvent the obstacle is to employ explicitly the Grassmann algebra basis and work strictly with the real number coefficients, namely \([q^A], [p_A]\), as discussed in Appendix B. One can then apply the above procedure in the bosonic system without any problem until one gets a similar expression to (3.7). However, the corresponding Hamiltonian dynamics for all the coefficients, especially the Poisson bracket, will have to be decomposed into a complicated expression, and this will not be done here. We will always assume that the expressions of the constraints do not need any explicit use of the Grassmann algebra basis.

\(^{18}\)If the constraints are not independent from each other, they are said to be “reducible.” This more general case is treated in the section 1.3.4 of [7].
We furthermore assume that at fixed time $t$ all the constraints can in principle be solved to express any point on $V$ by $2N - M$ independent variables $x^i$ ($1 \leq i \leq 2N - M$),

$$V = \{ (p, q) = f(x, t) \} ,$$

which provide a local coordinate chart on $V$ (the time dependence is due to the fact that the primary constraints may depend explicitly on time). On the other hand, $M$ constraints of all $\phi_m$’s can be taken as coordinates for the linearly independent directions to $V$ in the full phase space. The entire $2N$-dimensional phase space has then two sets of coordinate charts,

$$\begin{align*}
(p_A, q^B), & \quad 1 \leq A, B \leq N \\
\iff (x^i, \phi_m), & \quad 1 \leq i \leq 2N - M, \quad 1 \leq m \leq M .
\end{align*}$$

Note that there may exist some freedom in choosing different sets of the independent momenta $\{p_a\}$ from the $\mathcal{M}$ primary constraints.\(^{19}\)

For an arbitrary function $F(p, q, t)$ on the $2N$-dimensional phase space, with the coordinate system $(x, \phi)$ in (3.17), we define $\tilde{F}(x, \phi, t) := F(p, q, t)$ and a set of $\mathcal{M}$ functions, $F^m(p, q, t)$, by

$$F(p, q, t) = \tilde{F}(x, \phi, t) = \tilde{F}(x, 0, t) + \phi_m F^m(p, q, t) = F(p, q, t)|_V + \phi_m F^m(p, q, t) ,$$

where $F(p, q, t)|_V = \tilde{F}(x, 0, t)$. In other words, $|_V$ means that we substitute $(p, q)$ by its expression in terms of $(x, t)$ on the stationary surface $V$.

Finally note already that when we study the Hamiltonian dynamics, there can appear more constraints, namely the “secondary constraints”. In this case, all the constraints will define a smaller hypersurface, $\mathcal{V} \subset V$, in the phase space.

### 3.3 Prior to the Hamiltonian formulation: change of variables

In this subsection, instead of $(q^A, \dot{q}^B)$ we regard $(q^\hat{a}, \dot{q}^{\hat{a}}, p_a, \dot{\phi}^{\hat{m}})$ in Eqs.(3.7) as the independent variables, and discuss briefly the time evolution of them. The Lagrangian equations of motion are

\(^{19}\)For more details on the regularity conditions and the properties they imply, some of which are used here, the reader is referred to the subsection 1.1.2 of [7]. Notice also that more general regularity conditions (e.g. where the momenta do not play a distinguished role, or where the constraints are not assumed to be independent) can be defined, as is done in the first chapter of [7].
equivalent to
\[
\frac{d p_a}{dt} = \left( \frac{\partial L(q, \dot{q}, t)}{\partial q^a} \right) \dot{q}^a = h^a(q^A, p_a, \dot{q}^m, t), \tag{3.19}
\]
\[
\frac{df_m(q^A, p_a, t)}{dt} = \left( \frac{\partial L(q, \dot{q}, t)}{\partial q^m} \right) \dot{q}^a = h^a(q^A, p_a, \dot{q}^m, t), \tag{3.20}
\]
provided
\[
\frac{d \dot{q}^a}{dt} = h^a(q, p_a, \dot{q}^m, t), \quad \frac{d \dot{q}^m}{dt} = \dot{q}^m. \tag{3.21}
\]
Essentially these equations lead to a set of $M$ algebraic relations on $(q^A, p_a, \dot{q}^m)$ by substituting (3.19) and (3.21) into (3.20):\[\tag{3.22}\]
\[
h^a \frac{\partial f_m}{\partial \dot{q}^a} + \dot{q}^m \frac{\partial f_m}{\partial \dot{q}^m} + \left( \frac{\partial L}{\partial \dot{q}^a} \right)_{\dot{q}^a = h^a} \frac{\partial f_m}{\partial p_a} + \frac{\partial f_m}{\partial t} = \left( \frac{\partial L}{\partial p^m} \right)_{\dot{q}^m = h^a},
\]
where $m$ runs from 1 to $M$. Now these $M$ algebraic relations can be thought as constraints for the $M$ variables $\dot{q}^m$. Such constraints fix some of the $\dot{q}^m$’s but may leave others as completely free parameters. Once all the $\dot{q}^m$’s are determined as functions of other variables or as free parameters, the time evolution of the remaining (here, taken to be independent) variables $(p_a, q^a, \dot{q}^m)$ follows from (3.19) and (3.21). However, the constraints (3.22) are in general nonlinear in $\dot{q}^m$ and so they are difficult to solve. Below, we move to the Hamiltonian formalism where the independent variables are $(q^A, p_a, \dot{q}^m)$ rather than $(q^A, p_a, q^m)$. One advantage is that the corresponding constraints will be linear in $\dot{q}^m$ so that we can do a more explicit analysis (see Subsection 4.3).

### 3.4 From Lagrangian to Hamiltonian

Suppose that a given Lagrangian leads to $M$ primary constraints, say (3.7):
\[
\phi_m(q^A, p_a, t) := p_m - f_m(q^A, p_a, t). \tag{3.23}
\]
\[\text{In terms of the Hamiltonian } H(q^A, p_a, t) \text{ and the Poisson bracket } [\cdot, \cdot]_{P.B.} \text{ defined later (in Eq.(3.24) and (4.1) respectively), the equation (3.22) can be reexpressed in a compact form:}
\[
\left[ H(q, p_a, t) + \dot{q}^m(p_n - f_n), p_m - f_m \right]_{P.B.} + \frac{\partial f_m}{\partial t} = 0,
\]
where $f_m = f_m(q, p_a, t)$ and the explicit velocities $\dot{q}^n$ are taken to be constant with respect to the phase space derivatives of the Poisson bracket. Anticipating a bit, one may realize that, in such a way, there are $M$ linear equations (3.29) for the $M$ variables $\dot{q}^m$ rather than the $M$ algebraic nonlinear equations (3.22) for $\dot{q}^m$.\[\tag{3.29} \]
Replacing \( \dot{q}^\alpha \) by \( h^\alpha(q, p_a, \dot{q}^m, t) \) in (3.23) we take again \((q^A, p_a, \dot{q}^m, t)\) as the independent variables. We write the “canonical Hamiltonian” as

\[
H(q^A, p_a, t) := \dot{q}^A p_A - L(q, \dot{q}, t)
\]

\[
= h^\alpha(q^A, p_a, \dot{q}^m, t) p_\alpha + \dot{q}^m p_\dot{m} - L(q^A, h^\alpha(q^A, p_a, \dot{q}^m, t), \dot{q}^\dot{m}) \tag{3.24}
\]

where we made use of Eq.(3.7). Since there may exist some freedom in choosing different sets of the independent \( N-M \) momenta, the Hamiltonian is not uniquely specified, in general, from a given Lagrangian, but depends on this choice. However, on \( V \), these Hamiltonians are all equal.

From the fact that

\[
\frac{\partial H}{\partial \dot{q}^m} = \frac{\partial h^\alpha}{\partial \dot{q}^m} p_\alpha + \frac{\partial h^\dot{m}}{\partial \dot{q}^\dot{m}} p_\dot{m} - \frac{\partial L}{\partial \dot{q}^\alpha} \frac{\partial \phi^m}{\partial p_\alpha} = 0, \tag{3.25}
\]

one can see that the canonical Hamiltonian is indeed a function of \( q^A, p_a \) and \( t \) only, i.e. it is independent of \( \dot{q}^m \). Further direct calculations can lead to

\[
\frac{\partial H(q, p_b, t)}{\partial p_a} = (-1)^{\#_a} \dot{q}^a - (-1)^{\#_a} \#_m q^m \frac{\partial \phi^m}{\partial p_a},
\]

\[
\frac{\partial H(q, p_b, t)}{\partial p_n} = (-1)^{\#_n} \dot{q}^n - (-1)^{\#_n} \#_m q^m \frac{\partial \phi^m}{\partial p_n} = 0, \tag{3.26}
\]

\[
\frac{\partial H(q, p_b, t)}{\partial q^A} = - \left( \frac{\partial L(q, \dot{q}, t)}{\partial q^A} \right) _{\dot{q}^\alpha = h^\alpha} - (-1)^{\#_A} \#_m q^m \frac{\partial \phi^m}{\partial p_A},
\]

where all the velocities are to be understood as functions of \((q^A, p_a, \dot{q}^m, t)\) by the substitution \( \dot{q}^\alpha = h^\alpha(q, p_a, \dot{q}^m, t) \). The first two equations are easily obtained by making use of the Legendre transform philosophy, i.e. the canonical Hamiltonian does not really depend on the velocities. Concretely, it is enough to perform the partial differentiation only of the momenta in the term \( h^\alpha p_A \) in order to compute the right-hand-side of the first lines from (3.26). In a unified manner, any velocity can be expressed as a function of \((q, p_a, \dot{q}^m, t)\)

\[
\dot{q}^A(q, p_a, \dot{q}^m, t) := (-1)^{\#_A} \frac{\partial H(q, p_b, t)}{\partial p_A} + (-1)^{\#_A(1+\#_m)} \dot{q}^m \frac{\partial \phi^m}{\partial p_A}. \tag{3.27}
\]
Now this formula suggests that we can take not only \((q, p, \dot{q}^m, t)\) but, alternatively, \((q, p, \dot{q}^m, t)\) as independent variables. As follows from Subsection 3.3, the Hamiltonian dynamics is consistent with the Euler-Lagrangian equations only if
\[
\frac{dp_a}{dt} = -\frac{\partial H(q, p_b, t)}{\partial q^a} + (-1)^{\#_a\#_n} \dot{q}^n \frac{\partial f_n}{\partial q^a}, \\
\frac{df_m}{dt} = -\frac{\partial H(q, p_b, t)}{\partial q^m} + (-1)^{\#_m\#_n} \dot{q}^n \frac{\partial f_n}{\partial q^m},
\]
where we made use of the definition (3.24) of the canonical Hamiltonian and of the constraint (3.23). The equation (3.29) leads to \(M\) linear constraints on the \(M\) variables \(\dot{q}^m\). Once we know the complete solution of \(\dot{q}^m\), as we will do in Sec.4.3, the other equations (3.27) and (3.28) determine the time evolution of the remaining variables \(p_a, q^a\).

For an equivalent but more compact description of the Hamiltonian dynamics for the variables \((q^A, p_a, \dot{q}^m, t)\), we introduce the “total Hamiltonian” defined by
\[
H_T(q^A, p_B, u^m, t) := H(q^A, p_A, t) + \phi_m(q^A, p_B, t) u^m,
\]
where \(H_T\) is indeed a function on the ‘total’ phase space with coordinates \((q^B, p_A)\) but demand that
\[
\frac{\partial H_T}{\partial p_A} \bigg|_V = (-1)^{\#_A} \dot{q}^A, \\
\frac{\partial H_T}{\partial q^A} \bigg|_V = -\frac{\partial L(q, \dot{q})}{\partial q^A}.
\]
Combining (3.23) and (3.31) we identify
\[
u^m = (-1)^{\#_m} \dot{q}^m,
\]
hence \(\phi_m \nu^m = \dot{q}^m \phi_m\). We also have
\[
H_T|_V = H|_V, \\
\frac{\partial H_T}{\partial \nu^m} \bigg|_V = 0.
\]
In summary we note that there exist two one-to-one maps:
\[
\begin{pmatrix} q^A \\ p_b \\ \dot{q}^m \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} q^A \\ \dot{q}^B \end{pmatrix} \quad \leftrightarrow \quad \begin{pmatrix} q^A \\ p_b \\ \nu^m \end{pmatrix}.
\]
The Lagrangian dynamics is equivalent to the Hamiltonian one with the primary constraints. In the former system, the dynamical variables are by definition \( \{q^A, \dot{q}^B\} \), while in the latter the set of independent variables can be chosen to be \( \{q^A, p_b, u^m\} \) for convenience.

As we will shortly show in Subsection 4.3, in the Hamiltonian dynamics there is a systematic way of identifying the variables \( \{u^m\} \). Once the most general solution \( u^m(q, p, t) \) preserving the primary constraints is obtained, as in (4.45), the Hamiltonian dynamics determines the time evolution of the other variables \( \{q^A, p_b\} \), with the restriction on \( V \). After the generalization to the ‘total Hamiltonian system’, it can govern the dynamics of the \( 2N \)-dimensional whole phase space, with variables \( \{q^A, p_B\} \), free from any restriction.

### 3.5 From Hamiltonian to Lagrangian

We start\(^{21}\) with a given Hamiltonian \( H(p, q) \) on the \( 2N \)-dimensional phase space of coordinates \( \{p_A, q^B\} \), and \( M \) independent arbitrary but fixed primary constraints, \( \phi_m(p, q, t) = 0 \), which can be solved as \( p_m = f_m(q^A, p_b, t) \) to express \( M \) of the momenta in terms of the positions and the \( N-M \) other momenta. There can be some freedom in choosing different sets of the independent momenta, \( \{p_a\} \). The relevant phase space reduces to a \((2N-M)\)-dimensional hyperspace \( V \), as in (3.15), which will be further restricted to its sub-manifold \( V \subset V \) if there occurs ‘secondary constraints’ (see Subsection 4.3).

Introducing \( M \) new variables \( u^m \), we define the total Hamiltonian,

\[
H_T(p, q, u, t) = H(p, q, t) + \sum_{m=1}^{M} \phi_m(p, q, t)u^m.
\]  

(3.35)

The action principle is derived from the action

\[
S[q^A, p_B, u^m] = \int dt \left( p_A \dot{q}^A - H_T \right).
\]  

(3.36)

For instance, this leads to

\[
\dot{q}^A(q, p_i, u^l, t) \equiv (-1)^{\#A} \left. \frac{\partial H_T}{\partial p_A} \right|_V = (-1)^{\#A} \left. \left( \frac{\partial H}{\partial p_A} + \frac{\partial \phi_m}{\partial p_A} u^m \right) \right|_V,
\]  

(3.37)

where \( |_V \) means that we substitute \( p_m \) by \( p_m = f_m(q^A, p_b, t) \) after taking the partial derivatives.

\( ^{21} \)This subsection is provided for completeness, and may be skipped at the first reading.
We assume that there exists an inverse map, \((q^A, \dot{q}^B) \rightarrow (q^A, p_b, u^m)\), to write
\[
p_a(q, \dot{q}, t), \quad p_m = f_m\left(q, p_b(q, \dot{q}, t), t\right), \quad u^m(q, \dot{q}, t).
\] (3.38)

Provided with these functions, \(p_A(q, \dot{q}, t)\), we define
\[
L(q, \dot{q}, t) := \left(\dot{q}^A p_A - H(p, q, t)\right)|_V = \left(\dot{q}^A p_A - H_T(p, q, u, t)\right)|_V,
\] (3.39)
to recover the Lagrangian dynamics,
\[
\frac{\partial L(q, \dot{q}, t)}{\partial q^A} = \left(p_A + (-1)^\#_A\#_B \dot{q}^B \frac{\partial p_B}{\partial q^A} - \frac{\partial p_B}{\partial q^A} \frac{\partial H_T}{\partial p_B}\right)|_V = p_A(q, \dot{q}, t), \tag{3.40}
\]
\[
\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A} = \left((-1)^\#_A\#_B \dot{q}^B \frac{\partial p_B}{\partial \dot{q}^A} - \frac{\partial p_B}{\partial \dot{q}^A} \frac{\partial H_T}{\partial p_B} - \frac{\partial H_T}{\partial \dot{q}^A}\right)|_V = -\frac{\partial H_T(p, q, u, t)}{\partial q^A}|_V, \tag{3.41}
\]
along with the \(M\) constraints, \(p_m = f_m(q^A, p_b, t)\).

This analysis shows the equivalence between the Hamiltonian and the Lagrangian formalism, up to the technical assumption on the existence of the inverse map (3.38).\(^{22}\)

\(^{22}\)Notice that the equivalence between the Lagrangian and total Hamiltonian system can be shown in a large number of ways, see e.g. the exercises 1.2 and 1.4 of the book [7] for some alternatives. The latter exercise is based on the general result about the elimination of “auxiliary fields” (in the present case, the ‘auxiliary’ fields are \(p_b\) and \(u^m\)) via their own equations of motion.
4 Total Hamiltonian dynamics

4.1 Poisson bracket

On the $2N$-dimensional phase space, with coordinates $(q^A, p_B)$, equipped with the $\mathbb{Z}_2$-grading, we define the Poisson Bracket as

$$[F, G]_{P.B.} = (-1)^{\#_A\#_F} \frac{\partial F}{\partial q^A} \frac{\partial G}{\partial p_A} - (-1)^{\#_A(\#_F+1)} \frac{\partial F}{\partial p_A} \frac{\partial G}{\partial q^A}. \quad (4.1)$$

The Poisson bracket can be rewritten in a more compact form,

$$[\ , \ }_{P.B.} = (-1)^{\#_A} \overleftarrow{\frac{\partial}{\partial q^A}} \overrightarrow{\frac{\partial}{\partial p_A}} - \overrightarrow{\frac{\partial}{\partial p_A}} \overleftarrow{\frac{\partial}{\partial q^A}}, \quad (4.2)$$

where the arrows indicate the direction the derivatives act. It satisfies the graded skew-symmetry property,

$$[F, G]_{P.B.} = -(-1)^{\#_F\#_G} [G, F]_{P.B.}, \quad (4.3)$$

the Leibniz rule,

$$[F, GK]_{P.B.} = [F, G]_{P.B.} K + (-1)^{\#_G\#_K} G [F, K]_{P.B.}, \quad (4.4)$$

and the Jacobi identity,

$$[[F, G]_{P.B.}, K]_{P.B.} = [F, [G, K]_{P.B.}]_{P.B.} - (-1)^{\#_F\#_G} [G, [F, K]_{P.B.}]_{P.B.}, \quad (4.5)$$

or equivalently,

$$(-1)^{\#_F\#_H} [F, [G, H]_{P.B.}]_{P.B.} + (-1)^{\#_G\#_F} [G, [H, F]_{P.B.}]_{P.B.} + (-1)^{\#_H\#_G} [H, [F, G]_{P.B.}]_{P.B.} = 0. \quad (4.6)$$

Let $\dagger$ denote the Hermitian conjugation such that $(a b)\dagger = b\dagger a\dagger$, i.e. $\dagger$ is an involution on the algebra of functions on the phase space. Reality condition on the phase space reads,

$$q^A\dagger = q^A, \quad p_A\dagger = (-1)^{\#_A} p_A, \quad (4.7)$$

because the symplectic form $\dot{q}^A p_A$ must be real and

$$(\dot{q}^A p_A)\dagger = p_A\dagger (\dot{q}^A)\dagger = (-1)^{\#_A} (\dot{q}^A)\dagger p_A\dagger. \quad (4.8)$$
Hence we have
\[
\left( \frac{\partial F}{\partial q^A} \right)^\dagger = (-1)^{\#_A(\#_F+1)} \frac{\partial (F^\dagger)}{\partial q^A},
\]
\[
\left( \frac{\partial F}{\partial p_A} \right)^\dagger = (-1)^{\#_F \#_A} \frac{\partial (F^\dagger)}{\partial p_A},
\]
(4.9)
and hence,
\[
\left[ F, G \right]_{P.B.}^\dagger = (-1)^{\#_F \#_G} [F^\dagger, G^\dagger]_{P.B.} = -[G^\dagger, F^\dagger]_{P.B.}.
\]
(4.10)

4.2 Time derivatives - preliminary

With a given total Hamiltonian,
\[
H_T(p, q, u(p, q, t), t) = H(p, q, t) + \sum_{m=1}^{M} \phi_m(p, q, t) u^m(p, q, t),
\]
(4.11)
and the generalization motivated at the end of Subsection 3.4, we let the following formulae govern the dynamics of the full phase space,
\[
\dot{q}^A = (-1)^{\#_A} \frac{\partial H_T}{\partial p_A}, \quad \dot{p}_A = -\frac{\partial H_T}{\partial q^A},
\]
(4.12)
where the variables \( u^m(p, q, t) \) are to be understood as functions of \( p, q, t \), of which the explicit forms are not yet specified. Note that with the restriction on the hypersurface \( V \), i.e. putting \( \phi_m = 0 \) after taking the derivatives, the above equations reduce to (3.37,3.40,3.41) which already indicates some equivalence between the total Hamiltonian dynamics and the Lagrangian dynamics.\(^{23}\)

The time derivative of an arbitrary quantity \( F(p, q, t) \) takes a simple form\(^{24}\) in terms of the Poisson bracket (4.1),
\[
\frac{dF(p, q, t)}{dt} = [F, H_T]_{P.B.} + \frac{\partial F}{\partial t}.
\]
(4.14)

\(^{23}\)It is worthwhile to note that the above equations (4.12) indeed govern the full dynamics of all the coefficients \([p_A]_J\) and \([q^A]_J\) of the Grassmann algebra (see Appendix B).

\(^{24}\)One may generalize Eq.(4.14) by adding an arbitrary quantity proportional to the constraints to the right hand side,
\[
\frac{dF}{dt} = [F, H_T]_{P.B.} + \frac{\partial F}{\partial t} + \phi_m C^m(F).
\]
(4.13)
See (4.48) and also the Dirac bracket (6.10) for further discussion. The Dirac bracket gives an alternative dynamics which coincides with the dynamics of the Poisson bracket only on \( V \), but it is more suitable for quantization.
The crucial viewpoint we adopt here is the following: Though the above equations supplemented by the primary constraints are equivalent to the Lagrangian formalism (as shown in Subsection 3.4) we consider them to be more fundamental. Namely, without restriction on the hypersurface, we let them govern the entire phase space. Then, we try to make sure that the dynamics can be consistently truncated to the hypersurface. Namely we will look for the necessary and sufficient conditions to maintain the constraints throughout the time evolution. By imposing so, it may well be the case that we determine some of the unknown variables $u^m$ completely, at least the values on the hypersurface, and obtain further consistency conditions or ‘secondary constraints’. In the latter case, both the primary constraints and the secondary constraints should be imposed to define the hypersurface, say $\mathcal{V}$. In this way, we indeed make our Hamiltonian dynamics be consistent with the Lagrangian dynamics if one considers the space of physical states to live on the constraint surface $\mathcal{V}$.

### 4.3 Preserving the constraints - primary and secondary constraints

On the hypersurface $\mathcal{V}$, we have
\[
\left( \frac{d}{dt} F(p, q, t) \right) \simeq [F, H]_{P.B.} + [F, \phi_m]_{P.B.} u^m + \left( \frac{\partial F}{\partial t} \right),
\]
where we introduced the notation $\simeq$ for the “weak equality” defined via the equivalence relation
\[
f \simeq g \iff f|_\mathcal{V} = g|_\mathcal{V} \iff f = g + \phi_m z^m.
\]

We remind the reader that $|_\mathcal{V}$ means the restriction on the primary constraint surface $\mathcal{V}$, or the expression of $(p, q)$ in terms of the variables $x^i$ as in (3.16), after taking the partial derivatives. In other words, the symbol $\simeq$ stands for “equal on the primary constraint surface”. This symbol makes the distinction with the “strong equality”, which is the usual equality throughout all phase space, and proves to be convenient in order to avoid repetitions of the restriction on $\mathcal{V}$ for every term in a lengthy expression. On the left hand side, we get from (3.18) that
\[
\left( \frac{d}{dt} F(p, q, t) \right) |_{\mathcal{V}} = \left( \frac{d}{dt} \left( F(p, q, t)|_{\mathcal{V}} + \phi_m F^m \right) \right) |_{\mathcal{V}} = \frac{d}{dt} (F(p, q, t)|_{\mathcal{V}}) + \dot{\phi}_m F^m .
\]

For the consistency with the Lagrangian formalism, the time derivative of the primary constraints must vanish on $\mathcal{V}$,
\[
\frac{d\phi_m}{dt} = \left( [\phi_m, H]_{P.B.} + \frac{\partial \phi_m}{\partial t} + [\phi_m, \phi_n]_{P.B.} u^n \right) \simeq 0 , \quad (m, n = 1, 2, \cdots, \mathcal{M}) ,
\]
which precisely corresponds to (3.29). We focus on the following supermatrix defined on \((q^A, p_b, t)\),

\[
\Omega_{mn} := [\phi_m, \phi_n]_{P.B.}|_V = -(-1)^{\#_m\#_n} [\phi_n, \phi_m]_{P.B.}|_V = \begin{pmatrix}
A_- & \Psi \\
-\Psi^T & iA_+
\end{pmatrix}_{mn},
\] (4.19)

where \(A_\pm\) are symmetric or anti-symmetric (\(A^T_\pm = \pm A_\pm\)) bosonic matrices while \(\Psi\) is a fermionic matrix. Now Eq.(4.18) can be taken as a set of linear equations with the \(M\) unknown variables \(u^m (1 \leq m \leq M)\),

\[
[\phi_n, H]_{P.B.} + \frac{\partial \phi_n}{\partial t} \simeq -\Omega_{nm} u^m,
\] (4.20)

where, surely, the lefthand-side and \(\Omega\) are given by fixed functions on \(V\) of either \((x, t)\) or, equivalently, \((p_b, q^A, t)\)\(^{25}\).

Let us analyze the process more concretely. Without loss of generality, assuming that all the primary constraints are real,

\[
(\phi_m)^\dagger = \phi_m, \quad (u^m)^\dagger = (-1)^{\#_m u^m}.
\] (4.21)

the supermatrix is anti-Hermitian, \(\Omega_{ml} = - (\Omega_{lm})^\dagger\), so that all the matrices are real, \(A_\pm = A_\pm^*, \Psi^* = \Psi\). The forthcoming analysis is rather technical and complicate since one includes fermions. The main results are summarized in Subsection 4.4 to which the reader may jump directly in a first reading.

Under the real linear transformation,\(^{26}\)

\[
(L^*)_k^m = (-1)^{\#_m(\#_k+\#_m)} (L_k^m)^* = L_k^m,
\]

\[
\phi_m \longrightarrow L_m^k \phi_k = \phi_k (L^T)^k_m,
\] (4.22)

\[
u^m \longrightarrow (L^T)^{-1} m^k u^k = (-1)^{\#_m+\#_k} u^k (L^{-1})^k_m,
\]

the contraction \(\phi_m u^m\) is invariant and the reality condition (4.21) is preserved, and the supermatrix transforms as

\[
[\phi, \phi]_{P.B.}|_V \longrightarrow L [\phi, \phi]_{P.B.}|_V L^T,
\] (4.23)

\(^{25}\)For a given set of the generating elements of the Grassmann algebra \(\Lambda_N\) (as in Appendix B) the above formula (4.20) can be, in principle, completely analyzed. Furthermore, for the consistency of the Lagrangian mechanics, there must be a solution thereof. Indeed, we always implicitly assumed that we disregard any inconsistent Lagrangian like \(L = q\) which would lead to \(\delta L/\delta q = 1 = 0\). In the same spirit, we can also expect that, if necessary, there may occur some more extra constraints on \(V\), that is ‘secondary constraints’.

\(^{26}\)Note that, in general, \((L^T)_k^m = (-1)^{\#_m(\#_n+\#_k)} L_k^m, (L^*)_k^m = (-1)^{\#_m(\#_n+\#_k)} (L_k^m)^*,\) see Eq.(C.2).
where we have set $L|_V = L$ for simplicity. As shown in (D.31), one can always transform any anti-Hermitian supermatrix into the following ‘canonical’ form by a real linear transformation,\(^{27}\)

$$
L\Omega L^T = \begin{pmatrix}
    b_- & 0 & 0 & 0 \\
    0 & s_- & 0 & \psi \\
    0 & 0 & i b_+ & 0 \\
    0 & -\psi^T & 0 & i s_+
\end{pmatrix},
$$

(4.24)

where all the matrices are real, $b_\pm = b_\pm^*$, $s_\pm = s_\pm^*$, $\psi = \psi^*$; $b_\pm$ are nondegenerate bosonic matrices so that $b_\pm^{-1}$ exist; $s_\pm$ are bosonic products of fermions (i.e. even number of products of fermions); and $b_\pm = \pm b_\pm^T$, $s_\pm = \pm s_\pm^T$. It may be the case that $s_\pm$ and/or $\psi$ vanish.

With the decomposition of the index $m$ into $c$, $\dot{c}$ for bosonic variables and $\alpha$, $\dot{\alpha}$ for fermionic ones, which should be obvious from the inspection of Eqs.(4.25)-(4.28), the consistency condition (4.18) now splits into

$$
[\phi_c, H]_{P.B.} + \frac{\partial \phi_c}{\partial t} \simeq -(b_-)_{\alpha d} u^d,
$$

(4.25)

$$
[\phi_\alpha, H]_{P.B.} + \frac{\partial \phi_\alpha}{\partial t} \simeq -i(b_+)_{\alpha\beta} u^\beta,
$$

(4.26)

and

$$
[\phi_{\dot{c}}, H]_{P.B.} + \frac{\partial \phi_{\dot{c}}}{\partial t} \simeq -(s_-)_{\dot{\alpha} \dot{d}} u^{\dot{d}} - \psi_{\dot{c}\dot{\beta}} u^{\dot{\beta}},
$$

(4.27)

$$
[\phi_{\dot{\alpha}}, H]_{P.B.} + \frac{\partial \phi_{\dot{\alpha}}}{\partial t} \simeq \psi_{\dot{c}\dot{\alpha}} u^{\dot{c}} - i(s_+)_{\dot{\alpha}\dot{\beta}} u^{\dot{\beta}}.
$$

(4.28)

The meaning of the first two equations, (4.25) and (4.26), is clear. Since $b_\pm$ are nondegenerate, they fix the unknown variables, $u^c$, $u^\alpha$, completely as functions of $(q^A, p_b, t)$ or $(x, t)$ on the hypersurface $V$.

The analysis of the last two equations, (4.27) and (4.28), is somewhat tricky. Before the full analysis, we first focus on the purely bosonic systems, which was the case studied by Dirac [4].

\(^{27}\)Contrary to the usual complex number valued Hermitian matrix, a Hermitian supermatrix may not be completely diagonalizable. However, if the Hermitian supermatrix is nondegenerate, it is diagonalizable. See our Lemma 4 in (D.31).
• Bosonic systems.

In the bosonic systems, the equations (4.26) and (4.28) simply do not appear, and the essential relations are

\[
\begin{align*}
\left[ \phi_c, H \right]_{P.B.} + \frac{\partial \phi_c}{\partial t} & \simeq - \left( (b_-)_{cd} 0 \right) \\
\left[ \phi_c, H \right]_{P.B.} + \frac{\partial \phi_c}{\partial t} & \simeq - \left( 0 0 \right) \\
& \simeq - \begin{pmatrix}
(b_-)_{cd} & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
u^d \\
u^d
\end{pmatrix}.
\end{align*}
\] (4.29)

Since \( b_- \) is nondegenerate, the variables \( u^c \) are completely determined on \( V \) in terms of the variables \( (p_a, q^B, t) \), while the other variables \( u^\hat{c} \) remain as locally free variables at this stage. The vanishing of the second row in the left-hand-side can give some new, namely ‘secondary’, constraints. In this case, the primary and secondary constraints define together a smaller hypersurface, say \( V' \subset V \), and some of the \( p_a \)'s can be expressed in terms of others. Then the already determined variables \( u^c \) should be further restricted on \( V' \), making the first row hold on \( V' \) too. We note that the number of secondary constraints, say \( n \), are not greater that the number of the yet free variables, \( m \equiv \dim \{ u^\hat{c} \} \), \( n \leq m \).

The next step is to consider the time derivatives of the secondary constraints, analogously to (4.18). Regarding them as linear equations in the variables \( u^\hat{c} \) and taking some linear transformations to the canonical form, (D.21),

\[
\begin{pmatrix}
x |_{V'} \\
x' |_{V'}
\end{pmatrix} = \begin{pmatrix} 1_{k \times k} & 0_{k \times (m-k)} \\
0_{(n-k) \times k} & 0_{(n-k) \times (m-k)}
\end{pmatrix}
\begin{pmatrix}
u^c \\
u^\hat{c}
\end{pmatrix},
\] (4.30)

one can determine the variables \( u^\hat{c} \) completely on \( V' \), and there may appear new, namely “tertiary”, constraints \( x' |_{V'} \neq 0 \). Again the number of the tertiary constraints, are not greater that the number of the surviving locally free variables \( u^\hat{c} \), since \( (n-k) \leq (m-k) \).

The procedure may go on, but it should terminate at certain point, since the total number of constraints should not exceed the dimension of the whole phase space for any consistent dynamics. By a slight abuse of terminology, one refers to all these new constraints as ‘secondary’.

Eventually, we end up with a set of constraints,

\[
\left\{ \phi_h = 0 , \ 1 \leq h \leq \mathcal{M} + \mathcal{M}' \right\},
\] (4.31)

of which the ranges \( 1 \leq h \leq \mathcal{M} \) and \( \mathcal{M} + 1 \leq h \leq \mathcal{M} + \mathcal{M}' \) respectively correspond to the primary and secondary constraints. They define the hypersurface, \( \mathcal{V} \),

\[
\mathcal{V} := \left\{ (p, q) \mid \phi_h = 0 , \ 1 \leq h \leq \mathcal{M} + \mathcal{M}' \right\}.
\] (4.32)
All the constraints $\phi_h$ ($1 \leq h \leq M + M'$) are static on $V$, in the sense that the restriction on $V$ is preserved by the time evolution, $\left.\dot{\phi}_h\right|_V = 0$. In other words, they satisfy

$$-\left(\{\phi_h, H\}_{\text{P.B.}} + \frac{\partial \phi_h}{\partial t}\right)\Big|_V = \sum_{m=1}^{M} \left(\{\phi_h, \phi_m\}_{\text{P.B.}}, u^m\right)\Big|_V$$

$$\iff \left(\begin{array}{c} \times \\ 0 \end{array}\right) = \left(\begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array}\right) \left(\begin{array}{c} u^{m'} \\ u^{m''} \end{array}\right) : \text{in the canonical form on } V. \quad (4.33)$$

Some of the $u^m$ variables, i.e. the $u^{m'}$'s, are completely determined\footnote{Strictly speaking, what we have determined are the variables on the hypersurfaces, $\{u^{m'}\}_V$. For the generic dynamics in the full phase space, we may either employ them literally as they are, or use the continuously extended functions which have nontrivial dependence on the orthogonal directions to the hypersurfaces. In any case the dynamics on $V$ is the same.} on $V$ as functions of $(p, q, t)$, while the others, the $u^{m''}$'s, if any, remain as locally free (that is, arbitrarily time dependent variables). The latter correspond to the zero eigenvectors of the $(M + M') \times M$ matrix, $[\phi_h, \phi_m]_{\text{P.B.}}|_V$.

**Generic systems.**

Now we return to the equations, (4.27) and (4.28), which are relevant to the generic systems of bosons and fermions,

$$\left(\begin{array}{c} [\phi_{\dot{a}}, H]_{\text{P.B.}} + \frac{\partial \phi_{\dot{a}}}{\partial t} \\ [\phi_{\dot{\alpha}}, H]_{\text{P.B.}} + \frac{\partial \phi_{\dot{\alpha}}}{\partial t} \end{array}\right) \simeq \left(\begin{array}{cc} (s_\gamma)_{\dot{a}\dot{b}} & \psi_{\dot{\alpha}\dot{\beta}} \\ -\psi_{b\dot{a}} & i(s_+)^{\dot{\alpha}\dot{\beta}} \end{array}\right) \left(\begin{array}{c} u^b \\ u^{\dot{\beta}} \end{array}\right). \quad (4.34)$$

In general, the complete analysis of the above formulae is always possible, if we introduce explicitly the Grassmann algebra basis of (B.2). By expanding all the quantities in terms of the basis accompanied with the real or complex number coefficients, i.e. $[q^A]_J, [p_A]_J$, one can convert them into the linear equations in $[u^a]_J, [u^\alpha]_J$, over $\mathbb{R}$ or $\mathbb{C}$. The linear equations can be completely analyzed, essentially in the same way as in the previous bosonic case. After all the finitely repeated procedures, the results will be parallel: some of the coefficients, $[u^a]_J, [u^\alpha]_J$, are completely determined in terms of $(q^A)_J, [p_A]_J, t)$, while others remain as locally free parameters, implying that the time evolution in the phase space is not
deterministic. There may appear secondary constraints in terms of the coefficients, \([q^A]_J, [p_A]_J\).

However, in practice we favor the Lagrangian systems which do not require any explicit use of the basis for the Grassmann algebra. In such ‘good’ systems of both bosons and fermions, all the expressions can be written collectively in terms of \((p, q, u, t)\), rather than \(([p]_J, [q]_J, [u]_J, t)\), as if in the bosonic system. We summarize the results in the following separate subsection.

### 4.4 Hamiltonian dynamics after analyzing the constraints - summary

In all the bosonic systems as well as all the ‘good’ systems for bosons and fermions, in the sense that the explicit use of the basis of the Grassmann algebra is not required, we have the following generic situation:

- The Hamiltonian, \(H(p, q, t)\), as well as the primary constraints, \(\phi_m(p, q, t)\), are given as functions on the \(2N\)-dimensional full phase space with coordinates \((p_A, q^B)\) with \(1 \leq A, B \leq N\).
  
  In particular, the primary constraints define a \((2N - M)\)-dimensional hypersurface
  \[
  V = \{(p, q) \mid \phi_m(p, q, t) = 0, \ 1 \leq m \leq M\}.
  \]  
  \(\text{(4.35)}\)

- The total Hamiltonian is a sum of the canonical Hamiltonian and linear combinations of the primary constraints,
  \[
  H_T(p, q, u(p, q, t), t) = H(p, q, t) + \sum_{m=1}^{M} \phi_m(p, q, t) u^m(p, q, t).
  \]  
  \(\text{(4.36)}\)

- The dynamics of the whole \(2N\)-dimensional phase space is subject to
  \[
  \dot{q}^A = (-1)^{\#A} \frac{\partial H_T}{\partial p_A}, \quad \dot{p}_A = -\frac{\partial H_T}{\partial q^A},
  \]  
  \(\text{(4.37)}\)
  so that the time derivative of an arbitrary function, say \(F(p, q, t)\), on the \(2N\)-dimensional phase space reads
  \[
  \frac{dF(p, q, t)}{dt} = [F, H_T]_{PB} + \frac{\partial F}{\partial t}.
  \]  
  \(\text{(4.38)}\)

where the variables \(u^m(p, q, t)\) are not yet specified, but by looking for the necessary and sufficient conditions to maintain the hypersurface \(V\), in order to be consistent with the dynamics, we may determine some of them completely. In general we encounter the following situation:
• The whole primary constraint surface $\mathcal{V}$ may not be consistent with the dynamics, in the sense that it may not be preserved by the time evolution. It may well be the case that only a subset of $\mathcal{V}$, say $\mathcal{V} \subseteq \mathcal{V}$, is preserved. The constraint surface $\mathcal{V}$ is specified by the primary as well as the secondary constraints,

$$\mathcal{V} = \{(p, q) \mid \phi_h(p, q, t) \approx 0, \ 1 \leq h \leq \mathcal{M} + \mathcal{M}'\}, \quad (4.39)$$

where $\{\phi_h\}$ denote the complete set of constraints indexed by $h$, such that $\mathcal{M} + 1 \leq h \leq \mathcal{M} + \mathcal{M}'$ correspond to the secondary constraints. We introduced the notation $\approx$ for the other “weak equality”, that is defined as ‘equal on the constraint surface $\mathcal{V}$’,

$$f \approx g \iff f|_\mathcal{V} = g|_\mathcal{V} \iff f = g + \phi_h z^h. \quad (4.40)$$

We emphasize the important distinction between the two weak equalities, because (4.16) implies (4.40) but the converse is not always true since $\mathcal{V} \subsetneq \mathcal{V}$. All the constraints, in principle, can be solved to express any point on $\mathcal{V}$ by $2\mathcal{N} - \mathcal{M} - \mathcal{M}'$ independent variables, say $y^i$ (with $1 \leq i \leq 2\mathcal{N} - \mathcal{M} - \mathcal{M}'$),

$$\mathcal{V} = \{(p, q) = (f(y, t), g(y, t))\}, \quad (4.41)$$

which provide a local coordinate chart on $\mathcal{V}$. On the other hand, $\mathcal{M} + \mathcal{M}'$ of all $\phi_h$’s can be taken as complementary coordinates for the orthogonal directions to $\mathcal{V}$ in the full phase space. The entire $2\mathcal{N}$-dimensional phase space has then two sets of coordinate charts,

$$(p_A, q_B), \quad 1 \leq A, B \leq \mathcal{N}$$

$$\iff (y^i, \phi_h), \quad 1 \leq i \leq 2\mathcal{N} - \mathcal{M} - \mathcal{M}', \ 1 \leq h \leq \mathcal{M} + \mathcal{M}' \quad (4.42)$$

• There exists at least one set of solutions for $\{u^m, 1 \leq m \leq \mathcal{M}\}$ and a $(\mathcal{M} + \mathcal{M}') \times (\mathcal{M} + \mathcal{M}')$ supermatrix $T^{h'h}$ satisfying for all the constraints,

$$\dot{\phi}_h = [\phi_h, H + \phi_m u^m]_{P.B.} + \frac{\partial \phi_h}{\partial t} = \phi_h T^{h'h}, \quad 1 \leq h \leq \mathcal{M} + \mathcal{M}', \quad (4.43)$$

or equivalently

$$[\phi_h, H]_{P.B.} + \frac{\partial \phi_h}{\partial t} \approx -[\phi_h, \phi_m]_{P.B.} u^m. \quad (4.44)$$

where $\approx$ indicates that the equality strictly holds after the restriction on $\mathcal{V}$ or, equivalently, putting $(p, q) = (f(y, t), g(y, t))$, after taking the derivatives.
The most general solution of Eq.(4.44) reads
\[ u^m(p,q,t) = U^m(p,q,t) + V^m_i(p,q,t) v^i(t), \] (4.45)
where \( v^i(t) \) (with \( 1 \leq i \leq I \leq M \)) are arbitrary time dependent functions, \( U^m(p,q,t) \) is one particular solution, and \( V^m_i(p,q,t) \) span a basis of the kernel of the \((M + M') \times M\) supermatrix \( \{\phi_h, \phi_m\}_{P,B,\mathcal{V}} \),
\[ \sum_{m=1}^{M} \{\phi_h, \phi_m\}_{P,B,\mathcal{V}} V^m_i \approx 0, \quad \text{for all} \quad 1 \leq h \leq M + M'. \] (4.46)

- When \( \phi_m = p_m - f_m(q, p_a, t) \), substituting the most general solution (4.45) for \( u^m \) into the expression of the velocity
\[ \dot{q}^A = \dot{q}^A(q^B, p_a, (-1)^{# m} u^m(p,q,t), t) \]
given by (3.27), the total Hamiltonian reads simply:
\[ H_T(p_A,q^B,t) = \dot{q}^A p_A - L(q^B, \dot{q}^A, t). \] (4.47)

- As in (4.13), one can generalize the total Hamiltonian dynamics by adding terms proportional to the constraints. In particular, still preserving the Poisson bracket structure of the time evolution, (4.38), - which is essential for the quantization - one can modify the total Hamiltonian alone by adding freely terms quadratic (or higher) in the constraints, both primary and secondary,
\[ H_T = H + \phi_m u^m \quad \Rightarrow \quad H_T = H + \phi_m u^m + \frac{1}{2} \phi_h \phi_{h'} w^{h'h'}. \] (4.48)
where \( 1 \leq h, h' \leq M + M' \), while \( 1 \leq m \leq M \), and \( w^{h'h'} = (-1)^{# h # h'} w^{h'h} \) are newly introduced local parameters, being arbitrary time dependent functions. The modification does not affect our previous analysis at all, and the resulting dynamics remains the same on the hypersurface \( \mathcal{V} \).

- Other characteristic features of the total Hamiltonian dynamics are discussed in the following subsections.
4.5 First-class and second-class

We define a dynamical quantity, say $F(p, q, t)$ a function on phase space, to be “first-class”, if it has zero Poisson brackets with all the constraints, both primary and secondary, on $V$,

$$\left[ F, \phi_h \right]_{P.B.} \approx 0, \quad 1 \leq h \leq M + M'. \quad (4.49)$$

Otherwise it is said to be “second-class”. Physically, this distinction is extremely important, particularly for the constraints, as will be explained later on in this subsection.

4.5.1 Main properties of first-class quantities

We enunciate some useful properties of first-class quantities:

- If the function $F$ on phase space is first-class, then the Poisson bracket $\left[ F, \phi_h \right]_{P.B.}$ must be a linear combination of the constraints,

$$\left[ F, \phi_h \right]_{P.B.} = f^h_{h'} \phi_{h'} . \quad (4.50)$$

- If $F$ is first-class, then for any arbitrary dynamical variable, say $G(p, q, t)$,

$$\left[ F, G \right]_{P.B.} = \left[ F, \tilde{G}(y, \phi, t) \right]_{P.B.} . \quad (4.51)$$

**Proof:** For an arbitrary analytic function, $G(p, q, t)$, with the new coordinates for the $2N$-dimensional phase space, $(y^i, \phi_h)$ from (4.42), we define $\tilde{G}(y, \phi, t) := G(p, q, t)$. Then from the property of being first-class and the Leibniz rule (4.4) of the Poisson bracket, one can derive the result,

$$\left[ F, \tilde{G}(y, \phi, t) \right]_{P.B.} = \left[ F, \tilde{G}(y, 0, t) \right]_{P.B.} . \quad (4.52)$$

- The Poisson bracket of two first-class quantities is also first-class.
Proof: This can be shown from the Jacobi identity (4.6). For two first-class quantities, say $F_1$ and $F_2$,

$$\left[ \left[ F_1, F_2 \right]_{P.B., \phi_h} \right]_{P.B.} \approx 0.$$  

$$\left[ \left[ F_1, F_2 \right]_{P.B., \phi_h} \right]_{P.B.} \approx (-1)^{\#F_1 \#F_2} \left[ F_2, \left[ F_1, \phi_h \right]_{P.B.} \right]_{P.B.} \approx -\partial \phi_h \partial t \bigg|_V.$$  

(4.53)

4.5.2 First-class constraints as gauge symmetry generators

From Eq.(4.46), the complete set of primary first-class constraints is given by

$$\phi_i^{1st} := \phi_m V^m_i, \quad \left[ \phi_h, \phi_i^{1st} \right]_{P.B.} \approx 0.$$  

(4.54)

In virtue of (4.48) and (4.54) the total Hamiltonian reads now

$$H_T = H' + \phi_i^{1st} v^i(t) + \frac{1}{2} \phi_h \phi_{hh'} w^{hh'},$$  

(4.55)

where $H'$ is defined as

$$H' := H + \phi_m U^m,$$  

(4.56)

and satisfies, for any $1 \leq h \leq \mathcal{M} + \mathcal{M}'$,

$$\left[ \phi_h, H' \right]_{P.B.} \approx \left[ \phi_h, H_T \right]_{P.B.} \approx -\frac{\partial \phi_h}{\partial t} \bigg|_V.$$  

(4.57)

Thus, when the constraints do not have any explicit time dependence, both $H'$ and $H_T$ are first-class, and up to quadratic terms in the constraints the total Hamiltonian is a sum of the first-class Hamiltonian $H'$ plus a linear combination of the primary, first-class constraints with arbitrary functions of time as coefficients. Notice that such a decomposition is not unique since $U^m$ can be any solution of the inhomogeneous equation (4.44).
The time derivative of an arbitrary function \( F(p, q, t) \), c.f. (4.38), on the \( 2N \)-dimensional phase space can be rewritten as

\[
\frac{dF(p, q, t)}{dt} = \{ F, H \}_\text{P.B.} + \{ F, \phi_{\text{1st}}^i \}_\text{P.B.} v^i(t) + \frac{\partial F}{\partial t}.
\] (4.58)

As a result of the appearance of the arbitrary time dependent functions \( v^i(t) \), the dynamical variables at future times are not completely determined by the initial dynamical variables.

We should recall the following viewpoint by Dirac [4]: “all those values for the \( p \)’s and \( q \)’s at a certain time which can evolve from one initial state must correspond to the same physical state at that time”. A natural definition of the space of physical states is thus as the set of initial variables that should be given at some given moment of time, say \( t_0 \), in order to determine completely the time evolution via the equations of motion. The presence of arbitrary functions of time, \( v^i(t) \), in the total Hamiltonian \( H_T \) signals that the phase space contains some unphysical degrees of freedom. Indeed two different choices of arbitrary functions, say \( v^i_1 \) and \( v^i_2 \), would lead to distinct total Hamiltonians and thus different time change of a dynamical variable, say \( F \). After some time interval \( \delta t \), the evolutions of \( F \) would differ by

\[
\delta F = \{ F, \phi^i_{\text{1st}} \} (v^i_2 - v^i_1) \delta t.
\] (4.59)

The key philosophy we stick to is the standard one that different choices of the local gauge parameters correspond to different total Hamiltonian systems, which nevertheless should be taken equivalent, describing the same physics. Following the viewpoint advocated in subsection 2.2.2 for Lagrangian systems, this means that Eq.(4.59) defines an ambiguity in the time evolution that should be physically irrelevant. In other words, the transformation (4.59) is a gauge symmetry. In modern terminology, this implies that:

(i) a physical state is represented by an equivalence class, where one mods out by the gauge symmetries, therefore

(ii) the space of physical states must be understood as the quotient of the constraint surface by the gauge orbits and

(iii) an observable is a gauge invariant function on the constraint surface.

The space of physical states is also a symplectic manifold\(^{29} \) and is sometimes called “reduced phase space.” As a short dictionary for physicists on mathematical jargon, we may say that the

\(^{29}\) The section 1.4.2 of [7] is devoted to the subtle counting of physical degrees of freedom in the Hamiltonian context, which is equal to half the dimension of the symplectic manifold.
former quotient space is usually called “symplectic reduction” by mathematicians while they would introduce gauge transformations via a “Lie group action” and might refer to Noether’s theorem as the “moment map” (see e.g. [13]).

A remarkable property is that the Poisson bracket \( \{ H', \phi_i^{\text{1st}} \} \) between the first class Hamiltonian \( H' \) and any primary first class constraint \( \phi_i^{\text{1st}} \) is also a gauge symmetry generator. This can be shown by comparing the time evolution successively determined by (i) the total Hamiltonian \( H_T \) during an interval \( \delta t_1 \) and after by the first-class Hamiltonian \( H' \) during an interval \( \delta t_2 \), or (ii) the same operations, but in the reverse order. The net difference must be a gauge transformation since \( H_T \) and \( H' \) define the same evolution of physical states. By using the Jacobi identity, one may check explicitly that this gauge transformation is indeed generated by \( \{ H', \phi_i^{\text{1st}} \} \).

As one can see in (4.59), the primary first-class constraints generate gauge symmetries. A natural question is whether the converse is true: are all gauge symmetries generated by primary first-class constraints? In full generality, the answer is no. This can be understood from the fact that the Poisson bracket \( \{ H', \phi_i^{\text{1st}} \} \) is also a gauge symmetry generator. From Eq.(4.50) we know that this Poisson bracket is a linear combination of first-class constraints, but it is not guaranteed that only primary constraints appear. Therefore, some secondary constraints may also generate gauge transformations. Then another question arises: do all the secondary first-class constraints generate gauge symmetries? Dirac conjectured that the answer would be yes. But, again, in full generality the answer is no, although in most physical applications the answer is yes.\(^{30}\) This is the reason why first-class quantities have such a distinct status.

### 4.6 Extended Hamiltonian dynamics

One can generalize the total Hamiltonian system further to the so-called “extended Hamiltonian” system. We define the extended Hamiltonian in a similar way to the total one except that the former includes all first-class constraints (the primary as well as the secondary, tertiary, etc).

\[
H_E(p, q, t, v^i, v'^i, w) := H_T + \phi_i^{\text{1st}} v^i(t) + \phi_i^{\text{1st}} v'^i(t) + \phi_i^{\text{1st}} w(t) + \frac{1}{2} \phi_{hh'} w^{hh'},
\]

where \( \phi_i^{\text{1st}} \) (with \( 1 \leq i' \leq \mathcal{I}' \leq \mathcal{M}' \)) denotes the secondary first-class constraints.

The extended Hamiltonian is usually preferred because, from the Hamiltonian point of view, the distinction between primary and secondary constraints is actually irrelevant. The distinction becomes important only if one wants to make contact with the Lagrangian formulation, as in

\(^{30}\)A counterexample of the Dirac conjecture is given in subsection 1.2.2 of the book [7]. A proof of the Dirac conjecture under some hypotheses is provided in its subsection 3.3.2.
Subsection 3.5. So, one may let $H_E$ governs the whole dynamics rather than $H_T$.

\[
\frac{dF}{dt} = [F, H_E]_{PB} + \frac{\partial F}{\partial t}.
\] (4.61)

Compared to the total Hamiltonian dynamics, the constraint surface $\mathcal{V}$, is still preserved but a generic other object, say $F$ (rather than the constraint $\phi_h$), undergoes a different time evolution from the total Hamiltonian dynamics, even on $\mathcal{V}$. Indeed, $[F, \phi_{i^\prime}^{\text{1st}}]_{PB}|_{\mathcal{V}} \neq 0$ in general. Thus, unlike the total Hamiltonian dynamics, the extended Hamiltonian dynamics is in general different from the Lagrangian dynamics. Still, if the Poisson bracket of a quantity $F$ with any secondary first-class constraint is zero, then its evolution on $\mathcal{V}$ are the same.

With an arbitrary local parameter $\varepsilon(t)$, if there exists a gauge symmetry generator $Q_{\text{1st}} = Q_{\text{1st}} \varepsilon(t)$ which takes any solution $(q^A, p_B)(t)$ of the extended Hamiltonian dynamics to another by

\[
\delta q^A = [q^A, Q_{\text{1st}}]_{PB} \varepsilon(t), \quad \delta p_B = [p_B, Q_{\text{1st}}]_{PB} \varepsilon(t),
\] (4.62)

then at any time, say $t_0$, one can start to transform the solution to another without changing the initial data $(q^A, p_B)(t_0)$ by simply setting $\varepsilon(t_0) = 0$. Thus, again the future dynamical variables are not uniquely determined by the initial data. From the point of view of the gauge symmetries, the main difference between the total and extended Hamiltonian dynamics is that the latter assumes the Dirac conjecture applies. In this case, a dynamical quantity $F(q^A, p_B, t)$ on the phase space defines an observable if and only if its Poisson bracket with any first-class constraint vanishes weakly

\[
[F, \phi_{i^\prime''}^{\text{1st}}] \approx 0, \quad (1 \leq i'' \leq I + I')
\] (4.63)

Notice that an observable has been defined as a function on the constraint surface, so one should identify two functions that coincide on $\mathcal{V}$, i.e. the observable corresponding to the first-class quantity $F$ is the equivalence class for the weak equality. The conclusion is that the physical quantities (that is, the observables) undergo the same evolution under the total and extended Hamiltonian dynamics.\(^{31}\)

### 4.7 Time independence of the Poisson bracket

For a given set $\{v^i(t)\}$ of the local functions, the dynamical variables $(p, q)$ follow a unique and invertible trajectory in the phase space such that there exists a one to one map between $(p, q)$ and the initial data,

\[
p_A(t, p_0, q_0), \quad p_A(t_0, p_0, q_0) = p_{0A}; \quad q^B(t, p_0, q_0), \quad q^B(t_0, p_0, q_0) = q_0^B.
\] (4.64)

\(^{31}\)For more comments on the relationship between the total and extended Hamiltonian formalisms the reader may look, e.g. at [14].
A crucial fact follows, proven in Eq. (A.15): The Poisson bracket is independent of time,

\[
\{ , \}_\text{P.B.} = \{ , \}_\text{P.B.}|_{t=0},
\]

or

\[
(1)^A \frac{\partial}{\partial q^A} \frac{\partial}{\partial p_A} \text{P.B.} - \frac{\partial}{\partial p_A} \frac{\partial}{\partial q^A} \text{P.B.} = (1)^A \frac{\partial}{\partial q^0_A} \frac{\partial}{\partial p_{0A}} \text{P.B.} - \frac{\partial}{\partial p_{0A}} \frac{\partial}{\partial q^0_A}. \tag{4.66}
\]

Namely the time evolution generated by the total Hamiltonian is a symplectic transformation.

### 4.8 Other remarks on the total Hamiltonian formalism

- A useful identity.
  For an arbitrary function \( F(p, q, t) \), we have the following identity,\(^{32}\)

\[
\frac{d}{dt} \left. \left( F(p, q, t) \right) \right|_{\mathcal{V}} = \left. \left( \frac{d}{dt} F(p, q, t) \right) \right|_{\mathcal{V}}, \tag{4.67}
\]

proven in Eq. (A.13). In words, the following two actions, taking the time derivative and restricting on \( \mathcal{V} \), commute with each other. Intuitively, this is obvious since we imposed that \( \mathcal{V} \) be preserved under the time evolution.

- The time derivatives of the primary first-class constraints \( \phi_1^{1st} \) are of the general form, using Eqs. (4.43) and (4.55),

\[
\frac{d\phi_1^{1st}}{dt} := \phi_h T_i^h = \left( \left[ \phi_1^{1st}, H' \right] \right)_\text{P.B.} + \frac{\partial \phi_1^{1st}}{\partial t} + v_1^{1st},
\]

\[
v_1^{1st} := \left[ \phi_1^{1st}, \phi_j^{1st} \right]_\text{P.B.} v^i, \quad v^i v_1^{1st} = 0, \tag{4.68}
\]

\[
\left[ v_1^{1st}, \phi_h \right]_\text{P.B.} \approx 0 \quad \text{for all } h = 1, 2, \cdots, M + M'.
\]

Namely the time derivative of any primary, first-class constraint decomposes into two parts, one being independent of the local gauge parameters \( v^i(t) \), and the other one being first-class and orthogonal to the local gauge parameter.

\(^{32}\)The above relation (4.67) should be compared with

\[
\left. \frac{\partial}{\partial t} \left( F \right) \right|_{\mathcal{V}} = \left. \left( \frac{\partial F}{\partial t} \right) \right|_{\mathcal{V}} + \frac{\partial p_A}{\partial t} \left. \left( \frac{\partial F}{\partial p_A} \right) \right|_{\mathcal{V}} + \frac{\partial q^A}{\partial t} \left. \left( \frac{\partial F}{\partial q^A} \right) \right|_{\mathcal{V}}.
\]
In the static case, where there is no explicit time dependence in $\phi_h(p, q)$ and $H'(p, q)$ defined in (4.56), the time derivative of any primary as well as secondary first-class constraint, $\phi_{1st}$, is first-class too, since $\dot{\phi}_{1st} = [\phi_{1st}, H_T]_{P.B.}$ so that

$$\left\{ \dot{\phi}_{1st}, \phi_h \right\}_{P.B.} = [\phi_{1st}, [H_T, \phi_h]]_{P.B.} - [H_T, [\phi_{1st}, \phi_h]]_{P.B.} \approx 0.$$  \hspace{1cm} (4.69)

Hence, in the static case, for all the first-class constraints, both the primary ones, $\phi_{1st}^i$, and secondary ones, $\phi_{1st}'$, we can write

$$\dot{\phi}_{1st}^i = [\phi_{1st}^i, H_T]_{P.B.} = \phi_{1st}^j T_j^i + \phi_{1st}' T_{i'}^i,$$

$$\dot{\phi}_{1st}' = [\phi_{1st}', H_T]_{P.B.} = \phi_{1st}' T_{i'}^i + \phi_{1st} T_j^j T_{i'}.$$  \hspace{1cm} (4.70)

Combining the above two results in the static case, $[\phi_{1st}^i, H']_{P.B.}$ and $\nu_{1st}^i$ are separately first-class constraints.

---

33 Actually, this is an example of the fact that the Poisson bracket of two first-class quantities is also first-class (4.53).
5 Symmetry in the total Hamiltonian system

5.1 Symmetry from the Lagrangian system - revisited

In the Lagrangian formalism, the notion of symmetry corresponds to a change of variables \( q^A \rightarrow q'^A \) which leaves the Lagrangian invariant up to the total derivative term as in (2.14),

\[
L(q', \dot{q'}, t) = L(q, \dot{q}, t) + \frac{dK}{dt}.
\]

(5.1)

As a consequence, the symmetry takes one solution of the Euler-Lagrange equations to a new one as in (2.21),

\[
\frac{\partial L(q, \dot{q}, t)}{\partial q^A} \equiv \frac{d}{dt} \left( \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A} \right) \Rightarrow \frac{\partial L(q', \dot{q}', t)}{\partial q'^A} \equiv \frac{d}{dt} \left( \frac{\partial L(q', \dot{q}', t)}{\partial \dot{q}'^A} \right).
\]

(5.2)

As discussed in Section 2.2, if the infinitesimal symmetry transformation \( q^A \rightarrow q^A + \delta q^A(q, \dot{q}, t) \) depends on \( q^A, \dot{q}^A, t \) only, then the quantity \( \delta K \) in \( \delta L = \frac{d}{dt} \delta K \) must also depend only on \( q^A, \dot{q}^A, t \), and hence so is the corresponding Noether charge:

\[
Q(q, \dot{q}, t) = \delta q^A(q, \dot{q}, t)p_A(q, \dot{q}, t) - \delta K(q, \dot{q}, t).
\]

(5.3)

5.1.1 Change of variables

Henceforth in the present subsection, i.e. until Eq.(5.15), we take \((q, p_a, \dot{q}^\hat{m}, t)\) as the independent variables for any quantity which carries a hat symbol. For instance, we set

\[
\hat{p}_A := \left( \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^A} \right) \dot{q}^\hat{a} = \hat{h}^\hat{a}(q^B, p_a, \dot{q}^\hat{m}, t), \quad \hat{p}_a := p_a, \quad \hat{p}_m := f_m(q^B, p_a, t),
\]

\[
\delta \hat{q}^A(q^B, p_a, \dot{q}^\hat{m}, t) := \delta q^A \left( q^B, h^\hat{a}(q, p_a, \dot{q}^\hat{m}, t), \dot{q}^\hat{m}, t \right),
\]

\[
\delta \hat{K}(q^B, p_a, \dot{q}^\hat{m}, t) := \delta K \left( q^B, h^\hat{a}(q, p_a, \dot{q}^\hat{a}, t), \dot{q}^\hat{m}, t \right),
\]

(5.4)

where we substituted \( \dot{q}^\hat{a} \) by \( h^\hat{a}(q^B, p_a, \dot{q}^\hat{m}, t) \) according to the relation in Eq.(3.7). Notice that

\[
\frac{\partial \hat{p}_B}{\partial \dot{q}^\hat{m}} = 0,
\]

(5.5)

\(^{34}\)This subsection is parallel to Sec.3.4 where the Hamiltonian corresponds to the time translational symmetry generator.

\(^{35}\)In order to avoid confusion with some subsequent notations, we slightly changed the convention by using ‘prime’ instead of ‘tilde’ to denote the new variables.
as follows from (3.7). Furthermore, in agreement with Eq.(2.38) we define another function depending on \((q, p, \dot{q}^m, t)\),

\[
\frac{\delta p_A(q, p, \dot{q}^m, t)}{\delta q^A} := \left[ \frac{\partial L}{\partial \dot{q}^A} - \frac{\partial (\delta q^B)}{\partial q^A} \left( \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^B} \right) - \frac{\partial L}{\partial q^B} \right) \right] \dot{q}^a = h^a(q, p, \dot{q}^m, t) \tag{5.6}
\]

which has been defined in such a way that the r.h.s. is independent of the accelerations, as it should in the Hamiltonian formalism to come. Similarly, notice that the evaluation of (2.34) at \(\dot{q}^a = h^a(q, p, \dot{q}^m, t)\), gives

\[
\left( \frac{\partial (\delta K)}{\partial \dot{q}^A} \right) \dot{q}^a = h^a \tag{5.7}
\]

From the next equation (5.10) until the equation (5.15), the partial derivatives acting on any quantity with a hat symbol, say

\[
\hat{A}(q^B, p, \dot{q}^m, t) := \left( A(q, \dot{q}, t) \right)_{\dot{q}^a = h^a(q, p, \dot{q}^m, t)}, \tag{5.8}
\]

is taken regarding \((q^A, p_B, \dot{q}^m, t)\) as independent variables, while for unhatted quantities the independent variables are \((q^A, \dot{q}^B, t)\). Concretely, this means that one should make use of the chain rule, so that

\[
\frac{\partial \hat{A}}{\partial q^A} = \left( \frac{\partial A}{\partial q^A} \right)_{\dot{q}^a = h^a} + \frac{\partial h^a}{\partial q^A} \left( \frac{\partial A}{\partial \dot{q}^b} \right)_{\dot{q}^a = h^a}, \tag{5.9}
\]

On-shell, the variation (5.6) is equal to the variation of the momenta. Moreover, it satisfies

\[
\frac{\partial \delta p_A(q, p_a, \dot{q}^m, t)}{\partial \dot{q}^m} = \frac{\partial \delta q^B(q, p_a, \dot{q}^m, t)}{\partial \dot{q}^m} \left( \frac{\partial^2 L(q, \dot{q}, t)}{\partial q^B \partial q^A} \right)_{\dot{q}^a = h^a(q, p_a, \dot{q}^m, t)} \tag{5.10}
\]

\[
= (-1)^{\#_A \#_n} \frac{\partial \delta q^n(q, p_a, t, \dot{q}^m)}{\partial \dot{q}^n} \frac{\partial f_n(q, p_a, t)}{\partial q^A}.
\]
The proof of these two equalities is provided in Appendix A. Similarly to the corresponding steps, we also have the following identities:

\[
(-1)^{\#_A \#_B} \frac{\partial \hat{Q}^B}{\partial \hat{q}^m} \partial h^\dot{a} \hat{q}^a \frac{\partial^2 L}{\partial p_A \partial h^\dot{a} \partial \hat{q}^B} = 0 ,
\]

\[
(-1)^{\#_A \#_B} \frac{\partial \hat{Q}^B}{\partial \hat{q}^m} \partial h^\dot{a} \hat{q}^a \frac{\partial^2 L}{\partial q^A \partial h^\dot{a} \partial \hat{q}^B} = (-1)^{(\#_A \#_B) \#_C} \frac{\partial h^\dot{a} \partial \delta q^B \partial \hat{p}_B}{\partial q^A \partial h^\dot{a} \partial p_C} .
\]

The above substitution induces a Noether charge depending on \((q, p_a, t)\):

\[
\hat{Q}(q^A, p_a, t) = \hat{q}^A \hat{p}_A - \hat{\delta}K .
\]

From (5.7) one can easily verify that the Noether charge is indeed a function of \(q^A, p_a\) and \(t\) only (i.e. it is independent of \(\hat{q}^m\)),

\[
\frac{\partial \hat{Q}}{\partial \hat{q}^m} = \left( \frac{\partial h^\dot{a} \partial (\delta q^A)}{\partial \hat{q}^a} + \frac{\partial (\delta q^A)}{\partial \hat{q}^m} \right) \hat{p}_A - \left( \frac{\partial h^\dot{a} \partial (\delta K)}{\partial \hat{q}^m} + \frac{\partial (\delta K)}{\partial \hat{q}^m} \right) = 0 .
\]

Furthermore, with (5.7) we get

\[
\frac{\partial \hat{Q}(q, p_a, t)}{\partial p_a} = \frac{\partial h^\dot{a} \partial (\delta q^A)}{\partial \hat{q}^a} \hat{p}_A + (-1)^{\#_a} \hat{q}^a + (-1)^{\#_a \#_m} \hat{q}^m \frac{\partial f_m}{\partial p_a} \frac{\partial h^\dot{a} \partial (\delta K)}{\partial \hat{q}^m} = 0 .
\]

and with (5.6),

\[
\frac{\partial \hat{Q}(q, p_a, t)}{\partial q^A} = \left( \frac{\partial (\delta q^B)}{\partial q^A} + \frac{\partial h^\dot{a} \partial (\delta q^B)}{\partial h^\dot{a}} \right) \hat{p}_A + (-1)^{\#_A \#_m} \hat{q}^m \frac{\partial f_m}{\partial q^A} - \frac{\partial (\delta K)}{\partial q^A} - \frac{\partial h^\dot{a} \partial (\delta K)}{\partial h^\dot{a}} = -\hat{q}^m \frac{\partial f_m}{\partial q^A} .
\]

In order for the Noether charge \(\hat{Q}(q, p_a, t)\) to generate the symmetry transformations \((\hat{\delta}q, \hat{\delta}p)\) via the Poisson bracket in the corresponding Hamiltonian system, the derivatives of \(f_m\) in the r.h.s. of (5.14) and (5.15) should be absent. Instead, in a spirit similar to the total Hamiltonian, we will define a “total Noether charge” which will be a function of \((p_A, q^B, \hat{q}^m, t)\) rather than\(^{36}\) \((p_A, q^B, \hat{q}^m, t)\).

---

\(^{36}\)Note that the equation (3.27) implies that \(\frac{\partial \hat{q}^m}{\partial p_A \partial \hat{q}^m} = (-1)^{\#_A \#_m} \frac{\partial \hat{q}^m}{\partial p_A \partial \hat{q}^m} = (-1)^{\#_A \#_m} \frac{\partial \hat{q}^m}{\partial p_A \partial \hat{q}^m} .\)
5.1.2 Total Noether charge

Let us denote by $\tilde{v}^A$ the explicit expression of $\dot{q}^A$ in terms of the variables $q^B, p_a, \dot{q}^m$ and $t$, as in (3.27). Substituting the velocities $\dot{q}$ by their explicit form $\tilde{v}(q, p_a, \dot{q}^m, t)$ henceforth, we take $(q^B, p_a, \dot{q}^m, t)$ as the independent variables for any quantity which carries a tilde symbol. In other words, we set

$$\tilde{p}_A := \left( \frac{\partial L(q, \dot{q})}{\partial q^A} \right) = \tilde{p}_A(q, p_a, q^m, t), \quad \tilde{p}_a := p_a, \quad \tilde{p}_m := f_m(q, p_a, t),$$

\[ \tilde{\delta}K(q, p_a, \dot{q}^m, t) := \delta K(q, \tilde{v}, t) = \tilde{\delta}K(q, p_a, \tilde{v}^m, t), \]

\[ \tilde{\delta}q^A(q, p_a, \dot{q}^m, t) := \delta q^A(q, \tilde{v}, t) = \tilde{\delta}q^A(q, p_a, \tilde{v}^m, t), \]

\[ \tilde{\delta}p_A(q, p_a, \dot{q}^m, t) := \tilde{\delta}p_A(q, p_a, \tilde{v}^m, t). \]

The total Noether charge is then defined as

$$\tilde{Q}_T(p, q, \dot{q}^m, t) := \tilde{\delta}q^A(q, p_a, \dot{q}^m, t) p_A - \tilde{\delta}K(q, p_a, \dot{q}^m, t)$$

\[ = \tilde{Q}(q, p_a, t) + \tilde{\delta}q^n(q, p_a, \dot{q}^m, t) \left( p_n - f_n(q, p_a, t) \right). \] (5.17)

Contrary to the quantity $\tilde{Q}(q, p_a, t)$ in (5.12), in the above expression of $\tilde{Q}_T(p, q, t, \dot{q}^m)$ the constrained momenta $p_m$ have not been substituted by the primary constraints $p_m = f_m(q, p_a, t)$, since it is the untilded momenta which multiplies $\delta q$. It follows from Eqs.(5.13)-(5.15) that\(^{37}\)

$$\frac{\partial \tilde{Q}_T(p, q, t, \dot{q}^m)}{\partial \dot{q}^m} = \frac{\partial \tilde{\delta}q^n}{\partial \dot{q}^m} \left( p_n - f_n(q, p_a, t) \right),$$

$$\frac{\partial \tilde{Q}_T(p, q, t, \dot{q}^m)}{\partial p_A} = (-1)^{#A} \tilde{\delta}q^A + \frac{\partial (\tilde{\delta}q^n)}{\partial p_A} \left( p_n - f_n(q, p_a, t) \right),$$

$$\frac{\partial \tilde{Q}_T(p, q, t, \dot{q}^m)}{\partial q^A} = - \tilde{\delta}p_A + \frac{\partial (\tilde{\delta}q^n)}{\partial q^A} \left( p_n - f_n(q, p_a, t) \right).$$ (5.18)

\(^{37}\)Similar equations to (5.18) can be straightforwardly obtained either for the case where we take $(q, p_a, t, \dot{q}^m)$ as independent variables or for the case where all the $\dot{q}^m = (-1)^{#n} u^m$ are completely determined in terms of $(q, p_a, t)$ and free parameters $\nu^i(t)$ after solving all the constraints (4.45).
In particular, in terms of the Poisson bracket we have

\[
\{\tilde{Q}_T, q^A\}_{P.B.} = -\tilde{\delta}q^A - \{q^A, \tilde{q}^m\}_{P.B.}\left(p_m - f_m(q, p_a, t)\right),
\]

\[
\{\tilde{Q}_T, p_A\}_{P.B.} = -\tilde{\delta}p_A - \{p_A, \tilde{q}^m\}_{P.B.}\left(p_m - f_m(q, p_a, t)\right).
\]

(5.19)

Note that the expressions are consistent with the integrability relations e.g. \(\partial_{p_A} \partial_{q^m} \tilde{Q}_T = (-1)^{\#_A \#_m} \partial_{q^m} \partial_{p_A} \tilde{Q}_T\), thanks to (5.10) and (5.11). Obviously on the primary constraint surface \(V\), where \(p_m = f_m(q, p_a, t)\), the above relations get simplified: the first relation in (5.18) means that \(\tilde{Q}_T\) becomes independent of \(\dot{q}^m\) on \(V\), and the other equations lead to

\[
\{\tilde{Q}_T, q^A\}_{P.B.} \simeq -\tilde{\delta}q^A, \quad \{\tilde{Q}_T, p_A\}_{P.B.} \simeq -\tilde{\delta}p_A.
\]

(5.20)

In the Hamiltonian formalism to come, the relations (5.20) will be interpreted as the property that the total Noether charge \(\tilde{Q}_T\) generates the infinitesimal symmetry transformations on \(V\) and on-shell. \(^{38}\)

By making use, first of Eq.(5.20) and then of Eq.(5.6), one can show that the primary constraint surface \(V\) is preserved under the infinitesimal symmetry transformations, at least on-shell. Indeed, \(^{39}\)

\[
\{\tilde{Q}_T, p_m - f_m(q, p_a, t)\}_{P.B.} \simeq -\tilde{\delta}p_m + \tilde{\delta}q^A \frac{\partial f_m}{\partial q^m} + \tilde{\delta}p_a \frac{\partial f_m}{\partial p_a}
\]

\[
\simeq -\left\{ \delta \left( \frac{\partial L}{\partial \dot{q}^m} \right) - \frac{\partial (\delta q^B)}{\partial q^m} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^m} \right) - \frac{\partial L}{\partial \dot{q}^m} \right] \right\} + \tilde{\delta}q^A \frac{\partial f_m}{\partial q^m} + \left\{ \delta \left( \frac{\partial L}{\partial \dot{q}^a} \right) - \frac{\partial (\delta q^B)}{\partial q^a} \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^a} \right) - \frac{\partial L}{\partial \dot{q}^a} \right] \right\} \frac{\partial f_m}{\partial p_a}.
\]

(5.21)

Furthermore, we notice that the identity

\[
\delta f_m(q^A, p_a, t) = \tilde{\delta}q^A \frac{\partial f_m}{\partial q^A} + \delta \left( \frac{\partial L}{\partial \dot{q}^a} \right) \frac{\partial f_m}{\partial p_a}.
\]

(5.22)

\(^{38}\)The interplay between symmetries and conserved charges in the Lagrangian vs total Hamiltonian formalisms is briefly discussed in various exercises of the book [7] as particular cases of very general results on the elimination of auxiliary fields (mentioned in Footnote 22). More precisely, see e.g. Exercises 3.17, 3.25, 3.28 and 18.15 of [7]. An analogous derivation of such results for the extended Hamiltonian formalism should follow the general procedures introduced in [14]. As mentioned in the introduction, in the present text we prefer a more direct and pedestrian approach.

\(^{39}\)In each line of (5.21) and (5.24), the velocity should be replaced by \(\tilde{v}(q, p_a, \dot{q}^m, t)\) similarly to (3.27) and the expression is independent of the acceleration via cancellation due to (2.37), as it must be.
combined with the expression of momenta (3.4) when $A = m$,

$$\frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}^m} = f_m(q, \dot{q}, t),$$

leads to

$$\left[ \tilde{Q}_m, p - f_m(q, p_a) \right]_{P.B.} \approx \left( \frac{\partial (\delta q^B)}{\partial \dot{q}^m} - (-1)^{#_a + #_m} \sum_a \frac{\partial f_m}{\partial p_a} \frac{\partial (\delta q^B)}{\partial q^a} \right) \left( \frac{\partial L}{\partial \dot{q}^B} - \frac{\partial L}{\partial q^B} \right) = 0.$$  

(5.23)

(5.24)

### 5.1.3 Phase space variables

As discussed in Sec. 4.4, after solving the constraints, all the functions $\dot{q}^m = (-1)^m u^m(p, q, t)$ are completely determined in terms of time and the phase space variables, together with the local free parameters $v^i(t)$, as in Eq. (4.45). Therefore, all the velocities have been removed so that the only independent variables are phase space variables. In other words, in this sense we are again working in the genuine Hamiltonian formalism. Substituting the general solution given in (4.45) into (5.16) reduces the infinitesimal transformations and the total Noether charge to be functions of $(p, q, t)$:

$$\Delta q^A(p, q, t) := \tilde{\delta} q^A(q, p_a, \dot{q}^m(p, q, t), t),$$

$$\Delta p_A(p, q, t) := \tilde{\delta} p_A(q, p_a, \dot{q}^m(p, q, t), t),$$

$$Q_T(p, q, t) := \tilde{Q}_T(q, p_a, \dot{q}^m(p, q, t), t) = \Delta q^A p_A - \tilde{\delta} K(q, p_a, \dot{q}^m(p, q, t), t),$$

which satisfy

$$[Q_T, q^A]_{P.B.} = -\Delta q^A - [q^A, \Delta q^m]_{P.B.} \left( p - f_m(q, p_a, t) \right),$$

$$[Q_T, p_A]_{P.B.} = -\Delta p_A - [p_A, \Delta q^m]_{P.B.} \left( p - f_m(q, p_a, t) \right).$$

(5.25)

(5.26)

The total Hamiltonian is equal to $H_T := \tilde{v}^A p_A - L(q, \tilde{v}, t)$ according to (4.47), where the expression of $u^m$ given by (4.45) is substituted. Now, we get on the primary constraint surface $V$ that

$$\left[ Q_T, H_T \right]_{P.B.} \simeq [Q_T, \tilde{v}^A]_{P.B.} p_A - \tilde{v}^A \Delta p_A + \Delta q^A \frac{\partial L(q, \tilde{v}, t)}{\partial \dot{q}^A} - [Q_T, \tilde{v}^A]_{P.B.} \frac{\partial L(q, \tilde{v}, t)}{\partial \tilde{v}^A},$$

(5.27)
due to (5.26). But the expressions of the momenta (3.4) show that the sum
\[
\left[ Q_T, \tilde{v}^A \right]_{P.B.} \left( p_A - \frac{\partial L(q, \tilde{\nu}, t)}{\partial \tilde{v}^A} \right) = \left[ Q_T, \tilde{v}^m \right]_{P.B.} \left( p_m - \frac{\partial L(q, \tilde{\nu}, t)}{\partial \tilde{v}^m} \right) \simeq 0 ,
\] (5.28)
vanishes on the primary constraint surface \( V \). Therefore,
\[
\left[ Q_T, H_T \right]_{P.B.} \simeq -\tilde{v}^A \Delta p_A + \Delta q^A \frac{\partial L(q, \tilde{\nu}, t)}{\partial q^A}
\] (5.29)
where we made use of Eq. (5.6) to get the third line and of Eq. (2.35) to obtain the fourth line.

In terms of the very definition of the total Noether charge (5.25), we have thus shown that the Noether charge is conserved on the primary constraint surface,
\[
\left[ Q_T, H_T \right]_{P.B.} + \frac{\partial Q_T}{\partial t} \simeq 0 ,
\] (5.30)
It is worth noting that this result is off-shell and parallel to the off-shell invariance of the action under the symmetry transformation. Of course, on-shell \( dQ_T/\partial t \equiv \left[ Q_T, H_T \right]_{P.B.} + \frac{\partial Q_T}{\partial t} \simeq 0 \) to be compared with the on-shell conservation (2.42) of the Noether charge. Another way of expressing (5.30) is to say that \( \left[ Q_T, H_T \right]_{P.B.} + \frac{\partial Q_T}{\partial t} \) is a linear combination of the constraints. In other words, the total Noether charge generates a transformation which preserves the Hamiltonian on the primary constraint surface.

Furthermore, from the last formula in (5.18), not only \( \tilde{Q}_T \) but also \( Q_T \) preserves the primary constraints on-shell as in (5.24). More precisely,
\[
\left[ Q_T, \phi_m(q, p_a, t) \right]_{P.B.} \simeq 0 , \quad \text{on-shell.}
\] (5.31)
Therefore, the time evolution of this condition also vanishes on the primary constraint surface and on-shell, i.e.

\[
\frac{d}{dt} \{ Q_T, \phi_m(q, p_a, t) \}_P.B. = \left\{ \{ Q_T, \phi_m \}_P.B., H_T \right\}_P.B. + \frac{\partial}{\partial t} \{ Q_T, \phi_m \}_P.B. \simeq 0, \quad \text{on-shell.} \tag{5.32}
\]

For the secondary constraints which essentially stem from \( \{ \phi_m, H_T \}_P.B. + \frac{\partial \phi_m}{\partial t} \), we notice

\[
\left\{ Q_T, \left[ \phi_m, H_T \right]_P.B. + \frac{\partial \phi_m}{\partial t} \right\}_P.B. = \left\{ \left[ Q_T, \phi_m \right]_P.B., H_T \right\}_P.B. + \left[ \phi_m, \left[ Q_T, H_T \right]_P.B. + \frac{\partial Q_T}{\partial t} \right\}_P.B.,
\]

and hence, from (5.30) and (5.32) we deduce that not only the primary constraints but also the secondary constraints are preserved on-shell by \( Q_T \) if \( \{ Q_T, H_T \}_P.B. + \frac{\partial Q_T}{\partial t} \) corresponds to a first class constraint. In this case, \( Q_T \) is first class on-shell. As we see in the next subsection, the condition further implies that \( Q_T \) preserves the solution space too.

### 5.2 Symmetry in the total Hamiltonian system

In this subsection, motivated by the results in the previous subsection where we studied the general properties of the total Noether charge which originates from a symmetry in a Lagrangian system, we discuss the symmetry in the total Hamiltonian system directly without referring to any Lagrangian system. In order to make the analysis concise, we introduce a single letter, \( x^M \), \( 1 \leq M \leq 2N \), to denote both \( p \) and \( q \),

\[
x^A = q^A, \quad x^{N+A} = p_A. \tag{5.34}
\]

We define a \( 2N \times 2N \) constant non-degenerate graded skew-symmetric matrix by

\[
J^{MN} = \left\{ x^M, x^N \right\}_P.B. = -(-1)^{#_M #_N} J^{NM} = \begin{pmatrix}
0 & (-1)^{#_A} \delta^A_B \\
\delta_A^B & 0
\end{pmatrix}, \tag{5.35}
\]

which gives

\[
\left\{ F, G \right\}_P.B. = \frac{\partial F}{\partial x^M} J^{MN} \frac{\partial G}{\partial x^N}. \tag{5.36}
\]

In particular, \( \left\{ x^M, G \right\}_P.B. = J^{MN} \partial_N G \), where \( \partial_N = \frac{\partial}{\partial x^N} \).
5.2.1 Definition of symmetry transformations

Now we define a symmetry of the total Hamiltonian system as a coordinate transformation on the jet space that

1. depends on the phase space only, \( i.e. \) it should not depend on \( \dot{x}^M, \ddot{x}^M, etc \),
   \[ x^M \longrightarrow x'^M(x, t) ; \quad (5.37) \]
2. preserves the symplectic structure
   \[ J^{MN} = \bigl( \partial x'^M / \partial x^K \bigr) J^{KL} \bigl( \partial x'^N / \partial x^L \bigr) = \{ x'^M, x'^N \}_{P.B.} ; \quad (5.38) \]
3. takes any physical solution to another which means the preservation of both the on-shell relations and the constraints. Namely, if \( x(t) \) is a solution of the time evolution governed by a total Hamiltonian \( H_T(x, t; v, w) \), given by (4.55),
   \[ \dot{x}^M = \{ x^M, H_T(x, t; v, w) \}_{P.B.} = J^{MN} \partial_N H_T(x, t; v, w) , \quad (5.39) \]
   then so must be \( x'^M(x(t), t) \) for the same total Hamiltonian, up to the change of the local parameters \( u^i(t) \) and \( w^{hh'}(t) \),
   \[ \dot{x}'^M = \dot{x}^N \partial_N x'^M + \partial_t x'^M = J^{MN} \partial_N H_T(x', t; u', w') . \quad (5.40) \]
   Furthermore, such a symmetry must preserve the constraint surface \( \mathcal{V} \) on-shell,
   \[ \phi_h(x', t) \approx c^h(x, t) \phi_g(x, t) \quad \text{on-shell.} \quad (5.41) \]

Infinitesimally, the second requirement (5.38) reads
   \[ J^{-1}_{ML} \partial_N (\delta x^L) = (-1)^{#M\#N} J^{-1}_{NL} \partial_M (\delta x^L) . \quad (5.42) \]
In other words, the ‘super’ one-form \( \delta x_M := J^{-1}_{ML} \delta x^L \) is closed, hence exact. Therefore, there exists a “generating function” \( Q_H(x) \) on the phase space such that, \( c.f. \) (5.26),
   \[ \delta x^M = \{ x^M, Q_H \}_{P.B.} . \quad (5.43) \]
Conversely, any such transformation leaves the symplectic structure invariant.
In comparison to the above definition of symmetry transformations, which refers to a specific given total Hamiltonian, a “canonical transformation,” \(x \rightarrow x'(x, t)\) is defined as a coordinate transformation on phase space such that for every total Hamiltonian \(H_T(x, t)\) there must exist another (not necessarily equal) total Hamiltonian \(H'_T(x', t)\) obeying

\[
\dot{x'}^M = \dot{x}^N \partial_N x'^M + \partial_t x'^M = J^{MN} \partial'_N H'_T(x', t).
\]  

(5.44)

As the time evolution is generated by the total Hamiltonian, \(\dot{x}^N = J^{NK} \partial_K H_T(x, t)\), the condition (5.44) is equivalent to

\[
\left[x'^M, x'^N\right]_{P.B.} \partial'_N H_T(x, t) + \partial_t x'^M = J^{MN} \partial'_N H'_T(x', t),
\]

(5.45)

since

\[
\left[x'^M, x'^N\right]_{P.B.} = \frac{\partial x'^M}{\partial x^K} J^{KL} \frac{\partial x'^N}{\partial x^L}.
\]  

(5.46)

The analysis of Eq.(5.45) leads essentially to an integrability condition on the left hand side for arbitrary \(H_T(x, t)\), in order to be consistent with \(\partial_M \partial_N H_T = (-1)^{#M#N} \partial'_M \partial'_N H'_T\). Namely with the notation \(x'_M := J^{-1}_{MN} x'^N\), the integrability condition reads

\[
(\partial'_M \left[x'_N, x'^K\right]_{P.B.}) \partial'_K H_T + (-1)^{#M#N} \left[x'_N, x'^K\right]_{P.B.} \partial'_M \partial'_K H_T + \partial'_M x^K \partial_K \partial_t x'_N
\]

\[
= (-1)^{#M#N} \left(x'_M, x'^K\right)_{P.B.} \partial'_K H_T + \left[x'_M, x'^K\right]_{P.B.} \partial'_M \partial'_K H_T + (-1)^{#M#N} \partial'_N x^K \partial_K \partial_t x'_M.
\]  

(5.47)

This must hold for arbitrary \(H_T(x, t)\) and hence we have three independent relations:

\[
(\partial'_M \left[x'_N, x'^K\right]_{P.B.}) \partial'_K H_T = (-1)^{#M#N} \partial'_N \left[x'_M, x'^K\right]_{P.B.} \partial'_K H_T,
\]  

(5.48)

\[
\left(x'_M, x'^K\right)_{P.B.} \partial'_K \partial'_M H_T = (-1)^{#M#N} \left[x'_M, x'^K\right]_{P.B.} \partial'_K \partial'_N H_T,
\]  

(5.49)

\[
\partial'_M x^K \partial_K \partial_t x'_N = (-1)^{#M#N} \partial'_N x^K \partial_K \partial_t x'_M.
\]  

(5.50)

Firstly, the second relation (5.49) with the quadratic choice \(H_T = x'^P x'^Q\) shows that \(\left[x'_M, x'^K\right]_{P.B.}\) is proportional to \(\delta'_N^K\) or

\[
\left[x'^M, x'^N\right]_{P.B.} = f(x, t) J^{MN}.
\]  

(5.51)

Secondly, Eq.(5.48) with the linear choice \(H_T = x'^P\) further reveals that \(f(x, t)\) is independent of \(x\), i.e.

\[
\partial_M f(x, t) = 0.
\]  

50
Finally, the last relation (5.50) shows that there exists a bosonic function $\Phi(x', t)$ satisfying
\[ \partial_t x'_M = \partial'_{M'} \Phi(x', t). \]
Using this and from (4.3), (5.51) we note that the explicit time derivative of $f(t)$ vanishes as
\[ \partial_t f(t) J^{-1}_{MN} = \{\partial'_M \Phi, x'_N\}_{\text{P.B.}} + \{[x'_M, \partial'_{N'} \Phi]\}_{\text{P.B.}} = 0. \]

Thus, from Eq.(5.51) one notices that canonical transformations leave the symplectic structure invariant up to a constant $\{[x'_M, x'^N]\}_{\text{P.B.}} = \text{constant} \times J^{MN}$. Shortly, up to rescalings, canonical transformations are symplectic transformations.

Finally we note that, if we require the preservation of the $\mathbb{Z}_2$-gradation and of the reality properties $(\delta x^M)^\dagger = \delta (x^M)^\dagger$, we may set $Q_H$ to be bosonic and Hermitian $Q_H = Q^\dagger_H$, so that
\[ (\delta x^M)^\dagger = \{x^M, Q_H\}_{\text{P.B.}} = \delta (x^M)^\dagger. \]

### 5.2.2 Criteria for symmetry generators

In order to clarify the criteria for the generating function $Q_H$ in (5.43) to meet the remaining conditions (5.40) and (5.41) as to be a symmetry generator, we investigate the infinitesimal version of them which are given by
\[ [\delta x^M, H_T]_{\text{P.B.}} + \partial_i (\delta x^M) = J^{MN} \left( \delta x^L \partial_L H_T + (\partial_N \phi^1_{\text{1st}}) \delta v^i + (\partial_N \phi_h) \phi_h' \delta w^{hh'} \right) \]
\[ = \delta x^L \partial_L [x^M, H_T]_{\text{P.B.}} + \left[ x^M, \phi^1_{\text{1st}} \delta v^i + \frac{1}{2} \phi_h \phi_h' \delta w^{hh'} \right]_{\text{P.B.}}, \]
and for $1 \leq h \leq M + M'$,
\[ \delta x^M \partial_M \phi_h = \{x^M, Q_H\}_{\text{P.B.}} \partial_M \phi_h = \{Q_H, \phi_h\}_{\text{P.B.}} \approx 0 \quad \text{on-shell.} \]

\[ \text{[From the Leibniz rule of the Poisson bracket (4.4), it is worth to note an identity for an arbitrary function } F(x, t), \]
\[ [Q_H, F(x, t)]_{\text{P.B.}} = \{Q_H, x^M\}_{\text{P.B.}} \partial_M F(x, t). \]

51
The latter simply implies that \( Q_H \) must be first-class on-shell. The former condition (5.54) must hold for arbitrary solutions of the total Hamiltonian. In particular, it should hold at the initial time, say \( t_0 \), at which the initial data \( x_0 \) can be taken arbitrarily. Thus the condition (5.54) should be interpreted off-shell, and the general solution \( \delta x^M(x, t) \) of the partial differential equation (5.54) may lead to all the symmetries in a given total Hamiltonian system. Rather, we translate the condition (5.54) in terms of the symmetry generator \( Q_H \), as done in (5.55),

\[
\left[ [x^M, Q_H], H_T \right]_{P.B.} = - \left[ Q_H, [x^M, H_T] \right]_{P.B.} + \left[ [x^M, \phi_i^{1\text{st}} \delta v^i + \frac{1}{2} \phi_{h} \phi_{h'} \delta w^{hh'} - \partial_t Q_H \right]_{P.B.}.
\]

(5.56)

Due to Jacobi identity this is equivalent to

\[
\left[ [x^M, Q_H, H_T]_{P.B.} + \partial_t Q_H - \phi_i^{1\text{st}} \delta v^i - \frac{1}{2} \phi_{h} \phi_{h'} \delta w^{hh'} \right]_{P.B.} = 0.
\]

(5.57)

This condition should hold for every \( x^M, 1 \leq M \leq 2N \). Therefore,

\[
\left[ Q_H, H_T \right]_{P.B.} + \partial_t Q_H - \phi_i^{1\text{st}} \delta v^i - \frac{1}{2} \phi_{h} \phi_{h'} \delta w^{hh'} = f(t),
\]

(5.58)

where \( f(t) \) is an arbitrary time dependent, but phase-space independent function. This function can be removed by a redefinition of the generator,

\[
Q_H \rightarrow Q_H + \int_{t_0}^{t} dt' f(t'),
\]

(5.59)

as the shift has no effect on the symmetry transformation, \( \delta x^M = [x^M, Q_H]_{P.B.} \).

Thus, the necessary and sufficient condition for an on-shell first class quantity \( Q_H(p, q, t) \) to be a symmetry generator of a given total Hamiltonian \( H_T(p, q, t; v, w) \) reads

\[
\left[ Q_H, H_T \right]_{P.B.} + \frac{\partial Q_H}{\partial t} = \phi_i^{1\text{st}} \delta v^i + \frac{1}{2} \phi_{h} \phi_{h'} \delta w^{hh'},
\]

(5.60)

which is consistent with (5.30). The usual Hamiltonian version of the Noether theorem in unconstrained systems states that any conserved charge \( (dQ_H/dt = 0) \) is a symmetry generator. The formula (5.60) is the corresponding generalization to constrained systems.

### 5.3 Solutions

- **Every quantity, which is first-class and conserved on-shell, is a symmetry generator.**

  Indeed, the fact that \( Q_H \) remains constant under the time evolution reads

\[
\frac{dQ_H}{dt} \equiv \left[ Q_H, H_T \right]_{P.B.} + \frac{\partial Q_H}{\partial t} = 0,
\]

(5.61)
which is stronger than (5.60).

- **The total Noether charge** $Q_T$ (5.25) **originating from a Lagrangian system is a symmetry generator** if $[Q_T, H_T]_{P.B.} + \frac{\partial Q_T}{\partial t}$ corresponds to a first class constraint. In this case, the full expressions of the right hand sides of (5.25) should correspond to $\delta x^M$ and it follows that $Q_T$ is first class on-shell, as we saw in Sec.5.1.

- **Static examples:**

In the static case, there is no explicit time dependence in the constraints $\phi_h(p, q)$ and the quantity $H'(p, q)$ defined in (4.56). The time derivative of any first-class constraint $\phi_{1st}^i$ is then first-class too, as shown in (4.69).

  - **In the static case, the total Hamiltonian itself corresponds to a symmetry generator**

    \[
    \frac{dH_T}{dt} \equiv \frac{\partial H_T}{\partial t} = 0 .
    \]  

    (5.62)

  - **If there is no secondary first-class constraint in the given system (so that all the first class constraints are linear in $\phi_{1st}^i$), then**

    \[
    \dot{\phi}_{1st}^i = [\phi_{1st}^i, H_T]_{P.B.} = \phi_{1st}^j T^j_i .
    \]  

    Hence $\phi_{1st}^i \varepsilon^i(t)$ corresponds to a gauge symmetry generator with arbitrary time dependent functions $\varepsilon^i(t)$, i.e.

    \[
    Q_H = \phi_{1st}^i \varepsilon^i(t) ,
    \]  

    (5.64)

    satisfying (5.60).

  - **Alternatively if the time derivative of $\phi_{1st}^i$ is quadratic in the constraints $\phi_h$ (and hence first-class)**

    \[
    \dot{\phi}_{1st}^i = [\phi_{1st}^i, H_T]_{P.B.} = \frac{1}{2} \phi_h \phi_{hh'} T^{hh'}_i ,
    \]  

    then again $\phi_{1st}^i \varepsilon^i(t)$ corresponds to a gauge symmetry generator,

    \[
    Q_H = \phi_{1st}^i \varepsilon^i(t) .
    \]  

    (5.66)
Combining the above two cases we have more general solutions. Namely, if the time derivative of \( \dot{\phi}_{1}^{\text{st}} \) is a sum of terms linear in \( \dot{\phi}_{2}^{\text{st}} \) and quadratic in \( \phi_{h} \),

\[
\dot{\phi}_{1}^{\text{st}} = \left[ \dot{\phi}_{1}^{\text{st}}, H_{T} \right]_{\text{P.B.}} = \dot{\phi}_{2}^{\text{st}} T_{a}^{b} + \frac{1}{2} \phi_{h} \phi_{h} T_{hh}^{a} ,
\]

(5.67)

then \( \dot{\phi}_{1}^{\text{st}} \varepsilon^{a}(t) \) corresponds to a gauge symmetry generator,

\[
Q_{H} = \dot{\phi}_{1}^{\text{st}} \varepsilon^{a}(t) .
\]

(5.68)

For example, we consider the Lagrangian \( L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} e^{y} x^{2} \), whose equations of motion leave \( y \) arbitrary (so \( y \) is pure gauge) but fix \( x \) in time \( x = x_{0} \). This Lagrangian produces the Hamiltonian \( H = \frac{1}{2} e^{-y} p_{x}^{2} \), one primary first-class constraint \( \phi_{1}^{\text{st}} = p_{y} \), one secondary first-class constraint \( p_{x} \), and the total Hamiltonian \( H_{T} = \frac{1}{2} e^{-y} p_{x}^{2} + p_{y} v(t) \). This is a counterexample to Dirac’s conjecture (see e.g. the subsection 1.2.2 of [7]) because the secondary first-class constraint \( p_{x} \) does not generate any gauge symmetry as \( x \) is fixed by the equations of motion. However, the time derivative of the primary first-class constraint satisfies Eq.(5.65),

\[
\dot{p}_{y} = \left[ p_{y}, H_{T} \right]_{\text{P.B.}} = \frac{1}{2} e^{-y} (p_{x})^{2} ,
\]

(5.69)

and generates arbitrary shifts of the pure gauge variable \( y \).

A linear combination of the primary and secondary first-class constraints, \( \dot{\phi}_{1}^{\text{st}} \) and \( \dot{\phi}_{2}^{\text{st}} \), can be a gauge symmetry generator,

\[
Q_{H} = \dot{\phi}_{1}^{\text{st}} \varepsilon^{a}(t) + \dot{\phi}_{2}^{\text{st}} \varepsilon^{s}(t) ,
\]

(5.70)

if the local functions, \( \varepsilon^{a}(t) \) and \( \varepsilon^{s}(t) \) satisfy, with (4.70),

\[
\frac{d\varepsilon^{s}(t)}{dt} + T_{s}^{s} \varepsilon^{s}(t) + T_{a}^{s} \varepsilon^{a}(t) = 0 .
\]

(5.71)

For example we consider the Maxwell theory of which the Lagrangian and the Hamiltonian read\(^{41}\)

\[
\mathcal{L} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} ,
\]

\[
H = H' = \int d^{D-1} x \left[ \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \Pi^{i} \Pi^{i} - A_{0} \partial_{i} \Pi^{i} \right] ,
\]

(5.72)

\[
H_{T} = \int d^{D-1} x \left[ \frac{1}{4} F_{ij} F_{ij} + \frac{1}{2} \Pi^{i} \Pi^{i} - A_{0} \partial_{i} \Pi^{i} + v(x) \Pi^{0} \right] .
\]

\(^{41}\)This example is also handled in the section 19.1.1 of the reference [7].
The gauge symmetry is one example of (5.70) and (5.71) as

$$Q_H = \int dx^{D-1} \left[ \varepsilon(x) \partial_i \Pi^i - \partial_0 \varepsilon(x) \Pi^0 \right],$$  \hspace{1cm} (5.73)

where $\Pi^\mu = F_{\mu 0}$ are the gauge invariant canonical momenta for $A_\mu$, and $\Pi^0$ is the primary first-class constraint, while $\partial_i \Pi^i$ is the secondary first-class constraint. There appears no other constraint.

- In the extended Hamiltonian formalism of Sec.4.6, every first-class constraint corresponds to a gauge symmetry, if the system is static. Namely $\phi^{1st}_i \varepsilon^i(t) + \phi^{1st}_\mu \varepsilon^\mu(t)$ is a gauge symmetry generator with arbitrary time dependent functions, $\varepsilon^i(t)$ and $\varepsilon^\mu(t)$,

$$Q_E = \phi^{1st}_i \varepsilon^i(t) + \phi^{1st}_\mu \varepsilon^\mu(t).$$  \hspace{1cm} (5.74)

As one can see, the generators of local (i.e. gauge) symmetries are linear combinations of the constraints, therefore they vanish weakly in contradistinction with the generators of global symmetries. In this sense, only global symmetries lead to non-trivial conserved charges.

### 5.4 Dynamics with the arbitrariness - gauge symmetry

We remind the reader that the key philosophy we stick to is that different choices of the local gauge parameters correspond to different total Hamiltonian systems, which nevertheless should be taken equivalent, describing the same physics (see Subsection 4.5.2).

A somewhat less drastic - though equivalent - perspective is to consider only one total Hamiltonian throughout the time evolution, with a single set of local gauge parameters. The local functions should be continuous all the time but infinitely differentiable, i.e. $C^\infty$, only piecewise in time. This discontinuity in the derivatives corresponds to changes of local gauge parameters at different times. As long as the local functions $v^\mu(t)$ are continuous, one can change them arbitrarily at any moment. The continuity guarantees the continuity of the first order time derivative of the dynamical variables $(\dot{p}, \dot{q})$. However, in the Hamiltonian dynamics, there is no reason to require the continuities for the higher order time derivatives.

Explicitly, expressing the dynamical variable, $F$, at a future time, $t + dt$, as a power expansion
of \( dt \) around the present time, \( t \), we have
\[
F(t + dt) = F(p(t'), q(t'), t')_{t'=t+dt}
\]
\[
= F + dt\dot{F} + \frac{1}{2} dt^2 \ddot{F} + \mathcal{O}(dt^3)
\]
\[
= F + dt \left( \frac{\partial F}{\partial t} + \left[F, H_T\right]_{P.B.} \right)
\]
\[
+ \frac{1}{2} dt^2 \left( \frac{\partial^2 F}{\partial t^2} + 2 \left[\frac{\partial F}{\partial t}, H_T\right]_{P.B.} + \left[F, \frac{\partial H_T}{\partial t}\right]_{P.B.} + \left[\left[F, H_T\right], H_T\right]_{P.B.} \right)
\]
\[
= \hat{F} + dt \left[\hat{F}, \hat{H}_T\right]_{P.B.} + \frac{1}{2} dt^2 \left[\hat{F}, \hat{H}_T\right]_{P.B.} + \mathcal{O}(dt^3),
\]
where we have assumed that \( \hat{F} \) exists or \( v^i(t) \) is differentiable, and we have set
\[
\hat{F} = F + dt \frac{\partial F}{\partial t} + \frac{1}{2} dt^2 \frac{\partial^2 F}{\partial t^2},
\]
\[
\hat{H}_T = H_T + \frac{1}{2} dt \frac{\partial H_T}{\partial t} = H' + \phi^{1st}_i v^i + \frac{1}{2} dt \left( \frac{\partial H'}{\partial t} + \phi^{1st}_i \frac{\partial v^i}{\partial t} + \phi^{1st}_i \frac{d v^i}{dt} \right).
\]

The coefficients \( v^i(t) \) are completely arbitrary and at our disposal. We recall that different choices of the local parameters mean different total Hamiltonian systems, which nevertheless should be regarded equivalent, i.e. describing the same physics. For two different choices of the coefficients, say \( v \) and \( v + \Delta v \), the dynamical variable at the future time differs by
\[
\Delta F(t + dt) = dt \left[\hat{F}, \phi^{1st}_i\right]_{P.B.} \Delta v^i
\]
\[
+ \frac{1}{2} dt^2 \left[\hat{F}, \frac{\partial \phi^{1st}_i}{\partial t}\right]_{P.B.} \Delta v^i + \frac{1}{2} dt^2 \left[\hat{F}, \phi^{1st}_i\right]_{P.B.} \frac{d \Delta v^i}{dt}
\]
\[
+ \frac{1}{2} dt^2 \left[\left[\hat{F}, \phi^{1st}_i\right], H'\right]_{P.B.} \Delta v^i + \frac{1}{2} dt^2 \left[\left[\hat{F}, H'\right], \phi^{1st}_i\right]_{P.B.} \Delta v^i
\]
\[
+ \frac{1}{2} dt^2 \left[\left[\hat{F}, \phi^{1st}_i\right], \phi^{1st}_j\right]_{P.B.} \left(v^i \Delta v^j + v^j \Delta v^i + \Delta v^j \Delta v^i\right)
\]
\[
+ \mathcal{O}(dt^3, \Delta v).
\]
Thus, the leading order in the difference appears at the first order in $dt$ or the velocity, when $\Delta v(t) \neq 0$. Namely, different velocities for the same initial configuration can still correspond to the same physical state. This may correspond to the choice of the Noether charge, $Q_T = \phi_1^{1st} \varepsilon^a(t)$ such that $\varepsilon^i(t) = 0$ and $\dot{\varepsilon}^i(t) = \Delta v^i$ at time $t$.

On the other hand, if at time $t$ one has $\Delta v(t) = 0$, then the time derivative $(\dot{p}, \dot{q})$ of all the dynamical variables are the same in the two different total Hamiltonian systems, and the first nontrivial difference appears at the order of $dt^2$ or the ‘acceleration’,

$$\Delta F(t + dt) \sim \frac{1}{2} dt^2 \left[ \hat{F}, \phi_1^{1st} \right]_{P.B.} \frac{d\Delta v^i}{dt}.$$  \hspace{1cm} (5.78)

Again, this may correspond to the choice of the Noether charge, $Q_T = \phi_1^{1st} \varepsilon^i(t)$ such that $\varepsilon^i(t) = 0$, $\dot{\varepsilon}^i(t) = 0$ and $\ddot{\varepsilon}^i(t) = \frac{d\Delta v^i}{dt}$ at time $t$.

\footnote{For $Q_H = \phi_1^{1st} \varepsilon^i(t)$ to be actually a symmetry generator, meaning it preserves the solution space, $\{p(t), q(t)\}$, some extra conditions should be satisfied as (5.67) or (5.70).}

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6 Dirac quantization for second class constraints

6.1 Dirac bracket

On the $2N$-dimensional phase space with variables $q^A$ and $p^B$ ($1 \leq A, B \leq N$), we consider a set of functions $\rho_s(p,q)$ (where $1 \leq s \leq \dim\{\rho_s\} \leq 2N$) such that the following supermatrix is non-degenerate,
\[
\Omega_{st} := [\rho_s, \rho_t]_{P.B.} = -(-1)^{s+t} \Omega_{ts} ,
\] (6.1)
or its inverse exists,
\[
\Omega_{st} (\Omega^{-1})^{tu} = \delta^u_s \iff (\Omega^{-1})^{st} \Omega_{tu} = \delta^s_t .
\] (6.2)

We note
\[
(\Omega^{-1})^{st} = -(-1)^{s+t} \Omega^{-1}_{st} ,
\] (6.3)
and for an arbitrary quantity, $F$,
\[
[F, (\Omega^{-1})^{st}]_{P.B.} = -(-1)^{s+t} \Omega^{-1}_{st} [F, \Omega_{uv}]_{P.B.} (\Omega^{-1})^{tu} .
\] (6.4)

We define the “Dirac bracket” associated with the functions $\rho_s$ as,
\[
[F, G]_{\text{Dirac}} := [F, G]_{P.B.} - [F, \rho_s]_{P.B.} \Omega_{st} [\rho_t, G]_{P.B.} .
\] (6.5)

Some crucial identities follow. We first note that for an arbitrary object, $F$,
\[
[\rho_s, F]_{\text{Dirac}} = 0 , \quad [F, \rho_s]_{\text{Dirac}} = 0 .
\] (6.6)

This property is the raison d’être of the Dirac bracket. It means that one may impose $\rho_s = 0$ either before or after computing the Dirac bracket, whichever one prefers. Just like the Poisson bracket, the Dirac bracket satisfies the symmetric property,
\[
[F, G]_{\text{Dirac}} = -(-1)^{s+t} [G, F]_{\text{Dirac}} ,
\] (6.7)
and the Leibniz rule,
\[
[F, GK]_{\text{Dirac}} = [F, G]_{\text{Dirac}} K + (-1)^{s+t} G [F, K]_{\text{Dirac}} ,
\] (6.8)
\[
[FG, K]_{\text{Dirac}} = F [G, K]_{\text{Dirac}} + (-1)^{s+t} [F, K]_{\text{Dirac}} G .
\]

Moreover, from the Jacobi identity for the Poisson bracket, (4.6) and (6.3, 6.4), one can verify the Jacobi identity for the Dirac bracket after some tedious calculations,
\[
[[F, G]_{\text{Dirac}}, H]_{\text{Dirac}} = [F, [G, H]_{\text{Dirac}}]_{\text{Dirac}} - (-1)^{s+t} G [F, [H]_{\text{Dirac}}]_{\text{Dirac}} .
\] (6.9)

In mathematical terms, one says that the Dirac bracket obeys to the axioms of a graded Poisson bracket.
6.2 Total Hamiltonian dynamics with Dirac bracket

One can prove by contradiction that the Poisson bracket between all the second class constraints is non-degenerate. For constrained systems, the Dirac bracket is defined for all the second-class constraints by identifying $\rho_s$ with $\phi_{2nd}^s$. It is convenient to let the Dirac bracket governs the dynamics rather than the Poisson bracket,

$$\frac{dF(p,q,t)}{dt} = [F, H_T]_{\text{Dirac}} + \frac{\partial F}{\partial t},$$

(6.10)

because in such case one may impose the second-class constraints before computing the evolution of the system. It follows that one may omit the second-class primary constraints while adding to the Hamiltonian or to the Noether charge in the definition of the total Hamiltonian (3.30) or the total Noether charge (5.17). We note that

$$[F, H_T]_{P.B.} - [F, H_T]_{\text{Dirac}} = \sum_{\text{bosons}} \left( \frac{\partial F}{\partial q^b} \frac{\partial G}{\partial p_b} - \frac{\partial F}{\partial p_b} \frac{\partial G}{\partial q^b} \right) + \sum_{\text{fermions}} \frac{1}{2} (-1)^{#F} L^{-1/2} \alpha \beta \frac{\partial F}{\partial \psi^\alpha} \frac{\partial G}{\partial \psi^\beta},$$

(6.13)

where the factor 1/2 comes from the factor two in the last equation of (6.12). Notice that it is important in Eq.(6.13) to treat the partial derivatives $(q^b, p_b, \psi^\alpha, \phi_{2nd}^\beta)$ as the independent variables rather than $(q^b, p_b, \psi^\alpha, p_\alpha)$, and hence

$$[\phi_{2nd}^\alpha, G]_{\text{Dirac}} = 0, \quad [\psi^\alpha, \psi^\beta]_{\text{Dirac}} = -\frac{1}{2} L^{-1/2} \delta^\alpha_\beta, \quad [p_\alpha, \psi^\beta]_{\text{Dirac}} = -\frac{1}{2} \delta^\beta_\alpha, \quad \text{etc.}$$

(6.14)

• First order kinetic terms

In most of the cases, the momenta $p_\alpha$ for the fermions are linear in the spinor field $\psi^\alpha$, resulting in the primary second-class constraints,

$$\phi_{2nd}^\alpha := p_\alpha - L_{\alpha\beta} \psi^\beta, \quad L_{\alpha\beta} = L_{\beta\alpha}, \quad [\phi_{2nd}^\alpha, \phi_{2nd}^\beta]_{P.B.} = 2 L_{\alpha\beta}.$$

(6.12)

Using Eq.(6.5), the Dirac bracket of an unconstrained bosonic system coupled with fermions read

$$[F, G]_{\text{Dirac}} = \sum_{\text{bosons}} \left( \frac{\partial F}{\partial q^b} \frac{\partial G}{\partial p_b} - \frac{\partial F}{\partial p_b} \frac{\partial G}{\partial q^b} \right) + \sum_{\text{fermions}} \frac{1}{2} (-1)^{#F} L^{-1/2} \alpha \beta \frac{\partial F}{\partial \psi^\alpha} \frac{\partial G}{\partial \psi^\beta},$$

(6.13)

Hence, as long as the second-class constraints have no explicit time dependence, the Poisson brackets $[\phi_{2nd}^\alpha, H_T]_{P.B.}$ in the right-hand-side vanish on the constraint hypersurface $V$. In such case, the dynamics with the Dirac bracket and the other with the Poisson bracket, are identical on $V$. Namely both reduce to the same Lagrangian dynamics.
The last expression should be compared with \[ \{ p_\alpha, \psi^\beta \}_{PB} = -\delta_\alpha^\beta. \] This ‘halfness’ of the Dirac bracket is not related to the fermionic character of \( \psi \), instead it is typical for the system with first order Lagrangians (such as the variational principle of the Hamiltonian formulation itself). If \( \psi \) is complex, then the ‘halfness’ of the quantization is ‘doubled’ and one recovers the standard naive rule of the canonical quantization for a Dirac spinor.

- **Quantization**

The quantization can be straightforwardly performed by replacing the Dirac bracket by the super-commutator with a factor \(-i\),

\[
\{ , \}_\text{Dirac} \quad \Rightarrow \quad -i \{ , \},
\]

which gives the standard convention, \( \{ q^A, p_B \} = +i \delta^A_B \). The point is that, from

\[
\{ \phi^{2\text{nd}}, F \}_\text{Dirac} = 0 \quad \forall F,
\]

the second-class constraints are central, even after the quantization. Therefore, one can simultaneously impose the second-class constraints on the physical states. This would not be possible if one had naively performed the correspondence rule in terms of the Poisson bracket itself. The second-class constraints should be represented by identically vanishing operators on the physical Hilbert space. In practice, this may be realized by solving explicitly the constraints in terms of some set \( \{ y^w \} \) of independent variables

\[
\phi^{2\text{nd}}(x^M, t) = 0 \quad \iff \quad x^M = f^M(y^v)
\]

and try to represent the algebra \( \{ y^v, y^w \}_\text{Dirac} = g^{vw}(y) \) on the Hilbert space of functions of the \( y \)'s only.

Although this way of quantizing second-class constrained systems looks pretty straightforward and conceptually clear (one imposes all the constraints), in most practical cases, second-class constraints are most tedious because in general either we are not able to invert the matrix \( \Omega_{st} \) and the Dirac bracket is not known explicitly, so that nothing can be done at all, or we are not able to find a faithful representation of the Dirac bracket algebra.\(^{43}\) Somehow surprisingly, first-class constraints are preferable because there is an algorithmic - though involved and subtle - way to quantize the theory in terms of the Poisson bracket (which is easy to represent).

\(^{43}\) More comments on the quantization of second-class constraints can be found in the section 13.1 of [7]. Notice that some systems admit only the Dirac method of quantization and not the so-called “reduced phase space” method [18].
7 BRST quantization for first class constraints

The BRST procedure is motivated through the Faddeev-Popov construction. Here we review the essential features of them in a self-contained manner.

7.1 Integration over a Lie group - Haar measure

For a Lie group $G$ of dimension $N_G$, we parameterize its elements $g$ by the coordinates $\theta^a$ ($1 \leq a \leq N_G$) of the corresponding Lie algebra of a basis $\{T_a\}$,

$$g(\theta) = e^{i\theta^a T_a}, \quad [T_a, T_b] = iC_{ab}^c T_c. \quad (7.1)$$

We also define a set of $N_G$ functions $\zeta^a (\theta_1, \theta_2)$ from the multiplication,

$$g(\theta_1) g(\theta_2) = g (\zeta(\theta_1, \theta_2)). \quad (7.2)$$

From the Baker-Campbell-Hausdorff formula,

$$\ln (e^x e^y) = x + y + \frac{1}{2} [x, y] + \text{higher order commutators}, \quad (7.3)$$

we obtain explicitly,

$$\zeta^a (\theta_1, \theta_2) = \theta_1^a + \theta_2^a - \frac{1}{2} \theta_1^b \theta_2^c C_{bc}^a + \text{higher order terms}. \quad (7.4)$$

The left invariant measure for the integration over the Lie group $G$ is denoted by

$$D_{LG} := D\theta W_L(\theta) = \prod_{a=1}^{N_G} d\theta^a W_L(\theta). \quad (7.5)$$

By definition, it must satisfy the property of left invariance, i.e. for an arbitrary fixed element $g_0 \in G$ and any function on the group $F(g)$,

$$\int D_{LG} F(g) = \int D_{LG} F(g_0 g). \quad (7.6)$$

Hence, the following identity must hold for any $\theta_0$ and $\theta$,

$$W_L(\theta) = \det \left( \frac{\partial \zeta^a (\theta_0, \theta)}{\partial \theta^b} \right) W_L (\zeta (\theta_0, \theta)). \quad (7.7)$$
Some simple choices like $\theta = 0$ or $\theta_0 = -\theta$ give explicitly

$$W_L(\theta) = W_L(0) \det \left( \frac{\partial \zeta^a(\theta, \theta_0)}{\partial \theta^b} \right) \bigg|_{\theta_0 = -\theta} = W_L(0) \det^{-1} \left( \frac{\partial \zeta^a(\theta, \vartheta)}{\partial \theta^b} \right) \bigg|_{\vartheta = 0}. \quad (7.8)$$

Similarly one can define the right invariant measure $D_R g = D \theta W_R(\theta)$ to obtain

$$W_R(\theta) = W_R(0) \det \left( \frac{\partial \zeta^a(\theta, \theta_0)}{\partial \theta^b} \right) \bigg|_{\theta_0 = -\theta} = W_R(0) \det^{-1} \left( \frac{\partial \zeta^a(\vartheta, \theta)}{\partial \theta^b} \right) \bigg|_{\vartheta = 0}. \quad (7.9)$$

Now we are going to show that both measures can be set equal. The chain rule for $\theta' := \zeta(\theta, v)$ gives

$$\left( \frac{\partial \zeta^a(\zeta(\theta, \vartheta), \theta_0)}{\partial \vartheta^c} \right) \bigg|_{\vartheta = 0} = \left( \frac{\partial \zeta^a(\theta', \theta_0)}{\partial \theta^b} \right) \bigg|_{\theta' = \theta} \left( \frac{\partial \zeta^b(\theta, \vartheta)}{\partial \vartheta^c} \right) \bigg|_{\vartheta = 0}, \quad (7.10)$$

because $\theta' = \theta$ when $v = 0$. The associativity property explicitly reads

$$\zeta(\theta_1, \theta_2, \theta_3) := \zeta(\zeta(\theta_1, \theta_2), \theta_3) = \zeta(\theta_1, \zeta(\theta_2, \theta_3)). \quad (7.11)$$

Therefore, evaluating Eq. (7.10) at $\theta_0 = -\theta$, we obtain

$$\det \left( \frac{\partial \zeta^a(\theta, \vartheta, -\theta)}{\partial \vartheta^c} \right) \bigg|_{\vartheta = 0} = \det \left( \frac{\partial \zeta^a(\theta, \theta_0)}{\partial \theta^b} \right) \bigg|_{\theta_0 = -\theta} \det \left( \frac{\partial \zeta^b(\theta, \vartheta)}{\partial \vartheta^c} \right) \bigg|_{\vartheta = 0}. \quad (7.12)$$

From

$$g(\zeta(\theta, \vartheta, -\theta)) = g(\theta)g(\vartheta)g(\theta)^{-1} = e^{i\theta^a T_{\theta a}}, \quad T_{\theta a} := g(\theta)T_a g(\theta)^{-1} =: (M_{\theta})^b_a T_b, \quad (7.13)$$

it follows that

$$\zeta^a(\theta, \vartheta, -\theta) = \vartheta^b (M_{\theta})_b^a, \quad (7.14)$$

Consequently,

$$\text{tr}(T_a T_b) = (M_{\theta})^c_a (M_{\theta})^d_b \text{tr}(T_c T_d). \quad (7.15)$$

Thus, as long as $\text{tr}(T_a T_b)$ is invertible as a $N_G \times N_G$ matrix, (e.g. when $T_a$ are in the adjoint representation of a semisimple\footnote{If it is compact, then one can further take $\text{tr}(T_a T_b) \propto \delta_{ab}$.} Lie group) we have $(\det M_{\theta})^2 = 1$. Furthermore, from the
continuity at $\theta = 0$ we disregard the possibility of $\det M_\theta$ being minus one. Thus, we obtain from (7.14)
\[
\det \left( \frac{\partial \zeta^a(\theta, \vartheta, -\theta)}{\partial \vartheta_c} \right) = \det M_\theta = +1.
\] (7.16)
Inserting this relation on the left-hand-side of (7.12) and setting $W_L(0) = W_R(0)$, one has shown that the left and right invariant measures may be chosen identical for the Lie groups where $\det \left( \tr(T_aT_b) \right) \neq 0$:
\[
W_H(\theta) := W_L(\theta) = W_R(\theta).
\] (7.17)
The corresponding measure is known as “Haar measure” [19] and satisfies
\[
\int \mathcal{D}g F(g) = \int \mathcal{D}g F(g_0g) = \int \mathcal{D}g F(gg_0) = \int \mathcal{D}\theta W_H(\theta)F(g(\theta)).
\] (7.18)

In the case of the local gauge symmetry in field theories with the gauge group $G$, the parameters $\theta^a(x)$ are in fact arbitrary local functions. We assume that there exists a countable complete set in the commutative algebra of local functions,
\[
\{ f_n(x) \}, \quad f_n(x)f_m(x) := d_{nm}^l f_l(x), \quad \{ f_n(x)|f_m(x) \} = \delta_{nm},
\] (7.19)
where $d_{nm}^l$ are the structure constants of the algebra with the pointwise product. Then we can write $\theta^a(x) = \theta^{an} f_n(x)$ so that
\[
\theta^a(x)T_a = \theta^{an}T_{an}(x), \quad T_{an}(x) := T_a f_n(x), \quad [T_{an}(x), T_{bm}(x)] = iC^{c}_{ab}d_{anm}^{l}T_{cl}(x).
\] (7.20)
As the structure constants for the set $\{T_{an}(x)\}$ are independent of the coordinates $x$, the above relations induce a novel group $\hat{G} = \{ \hat{g}(\hat{\theta}^a) \}$, which is defined by the representation $g(\theta(x))$, with the parameters $\hat{\theta}^a$. Namely, though the representation is given for some fixed coordinates system $x$, there exists an abstract group independent of the coordinate choice. We have
\[
\hat{g}(\hat{\theta}_1)\hat{g}(\hat{\theta}_2) = \hat{g} \left( \hat{\zeta}(\hat{\theta}_1, \hat{\theta}_2) \right), \quad \hat{\zeta}^{an}(\hat{\theta}_1, \hat{\theta}_2) := \{ f_n(x)|\zeta^a(\theta_1(x), \theta_2(x)) \}.
\] (7.21)
The group $\hat{G}$ is infinite-dimensional since it is a local (i.e. position dependent) group.
Now we are ready to straightforwardly apply the left/right invariant measure to this local group:

\[ \int \mathcal{D} \hat{g} F(\hat{g}) = \int \mathcal{D} \hat{\theta} \hat{W}_H(\hat{\theta}) F(\hat{g}(\hat{\theta})) = \int \mathcal{D} \hat{g} F(\hat{g}_0 \hat{g}) = \int \mathcal{D} \hat{g} F(\hat{g} \hat{g}_0) , \]

\[ \hat{W}_H(\hat{\theta}) = \det -1 \left( \frac{\partial \hat{\zeta}^{am}(\hat{\theta}, \hat{\vartheta})}{\partial \hat{\vartheta}^b} \right) \bigg|_{\hat{\theta} = 0} = \det -1 \left( \frac{\partial \hat{\zeta}^{am}(\hat{\theta}, \hat{\vartheta})}{\partial \hat{\vartheta}^b} \right) \bigg|_{\hat{\theta} = 0} , \quad \mathcal{D} \hat{\theta} = \prod_{a,n} d \hat{\theta}^a . \]

The situation in gauge field theories is that \( F(\hat{g}) \) is given by a spacetime integral of a functional \( F(g, \partial_\mu g) \) which depends on the local group element \( g \) and its spacetime derivatives \( \partial_\mu g \),

\[ F(\hat{g}) = \int \! d^D x \, F(g, \partial_\mu g) . \]  

(7.23)

In the remaining of the paper, for short notation we drop the hat symbol and simply denote e.g.

\[ \int \mathcal{D} g F[g] = \int \mathcal{D} \hat{g} F(\hat{g}) . \]

(7.24)

**7.2 Faddeev-Popov method**

We consider a dynamical system where a finite-dimensional Lie group \( \mathcal{G} \), acts on the dynamical variables which we denote collectively by \( \Phi \). For each element, \( g \in \mathcal{G} \), we define a map (i.e. here a gauge transformation),

\[ g : \Phi \longrightarrow \Phi_g . \]  

(7.25)

For later purpose, we write the successive gauge transformation in the following order, \( \Phi \rightarrow \Phi_{g_1} \rightarrow \Phi_{g_2 g_1} \). In other words, one has a left action of \( \mathcal{G} \) on the space of dynamical variables.

We introduce a Lie algebra valued functional \( h[\Phi] = h^a[\Phi] T_a \) of \( \Phi \) (which may depend on its derivatives as well). We assume that \( h[\Phi] \) is non-degenerate under the gauge transformations as\(^{45}\)

\[ \det \left( \frac{\partial h^a[\Phi_g(\theta)]}{\partial \theta^b} \right) \bigg|_{\theta = 0} \neq 0 , \]

(7.26)

in which case it is called the “gauge-fixing functional”.

\(^{45}\)It is not necessary to require the non-degeneracy for \( \theta \neq 0 \).
For an arbitrary function $B(h)$ of $h^a$, the Faddeev-Popov functional $B_{F.P.}[\Phi]$ of $\Phi$ reads

$$B_{F.P.}[\Phi] := B(h[\Phi]) \det \left( \frac{\partial h^a[\Phi_g(\theta)]}{\partial \theta^b} \right) \bigg|_{\theta=0}.$$  

(7.27)

By making use of the definition (7.22) of the Haar measure, one finds the following crucial identity,

$$\int Dg B_{F.P.}[\Phi_g] = \int Dg B(h[\Phi_g]) \det \left( \frac{\partial h^a[\Phi_g(\theta)]}{\partial \theta^b} \right) \bigg|_{\theta=0}$$

$$= \int D\theta \det \left( \frac{\partial \zeta^b(\theta, \theta^0)}{\partial \theta^c} \right) \bigg|_{\theta^0=-\theta} B(h[\Phi_g(\theta)]) \det \left( \frac{\partial h^a[\Phi_g(\zeta(\theta, \theta))]}{\partial \theta^b} \right) \bigg|_{\theta=0}$$

$$= \int D\theta \det \left( \frac{\partial h^a[\Phi_g(\theta)]}{\partial \theta^b} \right) B(h[\Phi_g(\theta)])$$

$$= \int Dh B(h),$$  

(7.28)

where

$$Dh := N_G \prod_{a=1}^{N_G} dh^a.$$  

(7.29)

Thus, the integral of $B_{F.P.}[\Phi_g]$ over $G$ is just a $c$-number, being independent of $\Phi$ or the choice of the gauge-fixing functionals $h^a[\Phi]$, provided that the integral domain for $h$ is $\Phi$ independent.

Now we consider a gauge invariant functional,

$$\mathcal{F}(\Phi) = \mathcal{F}(\Phi_g),$$  

(7.30)

and further assume that the path integral measure is gauge invariant,

$$\mathcal{D} \Phi = \mathcal{D} \Phi_g, \quad \det \left( \frac{\mathcal{D} \Phi_g}{\mathcal{D} \Phi} \right) = 1,$$  

(7.31)

which is always satisfied for any dynamical system where the fields are in the adjoint representation or there are equal number of fundamental and anti-fundamental fields, transformed by $g$ and
The Faddeev-Popov path integral method prescribes to multiply the gauge invariant functional integrand by the Faddeev-Popov functional,

\[ \int D\Phi \mathcal{F}(\Phi) \rightarrow \int D\Phi \mathcal{F}(\Phi) \mathcal{B}_{F-P} [\Phi] = \int D\Phi \mathcal{F}(\Phi) B(h[\Phi]) \det \left( \frac{\partial h^a[\Phi_g(\theta)]}{\partial \theta^b} \right) \bigg|_{\theta=0}, \]

to satisfy

\[ \int Dg \int D\Phi \mathcal{F}(\Phi) \mathcal{B}_{F-P} [\Phi] = \int Dg \int D\Phi \mathcal{F}(\Phi) \mathcal{B}_{F-P} [\Phi_g] = \left( \int Dh B(h) \right) \int D\Phi \mathcal{F}(\Phi), \]

where we made use of (7.28), (7.30) and (7.31). Equivalently we have

\[ \int D\Phi \mathcal{F}(\Phi) \mathcal{B}_{F-P} [\Phi] = \left( \int Dh B(h) \right) \frac{\int D\Phi \mathcal{F}(\Phi)}{\int Dg 1} \propto \int D\Phi \mathcal{F}(\Phi). \]

In gauge field theories, the volume integral of the gauge group is often divergent, and the Faddeev-Popov method [16] provides a regularization scheme for that by adding the Faddeev-Popov factor to the original gauge invariant action. In the path integral, the gauge invariant functional is given by the exponential of the gauge symmetric action,

\[ \mathcal{F}[\Phi] = e^{+iS[\Phi]}, \quad S[\Phi] = S[\Phi_g]. \]

Furthermore, in this context, the arbitrary function \( B(h) \) is reformulated as a Fourier transformation,

\[ B(h[\Phi]) = \int \mathcal{D}k e^{+i(-V(k) + k_a h^a[\Phi])}, \]

where the fields \( k_a \) are said to be “auxiliary” and \( V(k) \) is proportional to the logarithm of the Fourier transform \( \tilde{B}(k) \) of the function \( B(h) \) (anyway, the coice of \( V \) is as arbitrary as the one of \( B \)). Moreover, the Faddeev-Popov determinant can be written by introducing a pair of fermionic scalar fields, called “ghosts”, \( \bar{\omega}_a \) and \( \omega^b \) (\( 1 \leq a, b \leq N_G \)),

\[ \det \left( \frac{\partial h^a[\Phi_g(\theta)]}{\partial \theta^b} \right) \bigg|_{\theta=0} = \int \mathcal{D}\bar{\omega} \mathcal{D}\omega e^{+i\bar{\omega}_a \Delta^a_{\Phi}[\Phi] \omega^b}, \]
where we set
\[ \Delta_{ab}^{\Phi} := \frac{\partial h^a[\Phi_{g(\theta)}]}{\partial \theta^b} \bigg|_{\theta=0}. \] (7.38)

Note that \( \bar{\omega}_a \) and \( \omega^b \) are not necessarily complex conjugate to each other. We further assign the ghost number, +1 for \( \omega^a \), −1 for \( \bar{\omega}_b \), and 0 for the other fields \( \Phi \) and \( k^a \).

Combining them all, the Faddeev-Popov action reads
\[ S_{F.P.}[\Phi, \omega, \bar{\omega}, k] = S[\Phi] - V(k) + k^a h^a[\Phi] + \bar{\omega}_a \Delta_{ab}^{\Phi} \omega^b. \] (7.39)

For example, in the Yang-Mills theory (on a curved spacetime), if we take the Lorentz gauge,
\[ h[A] := \nabla_{\mu} A^\mu, \] (7.40)
then we can set
\[ S_{F.P.}[\Phi, \omega, \bar{\omega}, k] = S_{YM}[\Phi] + \int dx^D \sqrt{g} \text{tr} \left( -V(k) + k^a h^a[A] - \nabla^\mu \bar{\omega} D_\mu \omega \right). \] (7.41)

Another example is the gauged Hermitian one-matrix model. Diagonal gauge choice leads to a Faddeev-Popov determinant which is nothing but the Vandermonde determinant [17].

### 7.3 BRST symmetry

Remarkably, even after choosing a gauge\(^{47}\), the path integral still does have a symmetry related to the gauge invariance. Indeed, the Faddeev-Popov action (7.39) possesses a fermionic nilpotent rigid symmetry known as “BRST symmetry”, after its discoverers, Becchi-Rouet-Stora [20] and, independently, Tyutin [21].

To discuss the BRST symmetry it is useful to note that the infinitesimal transformation associated with the Lie algebra element \( \delta g = i \partial^\mu T_a \) is given by
\[ \delta \Phi = \frac{d}{ds} \Phi_{g(s\theta)} \bigg|_{s=0}, \] (7.42)

\(^{46}\)All these relations are valid for theories in the Minkowskian spacetime. For the Euclidean theories we only need to replace the factor, \(+i\) by \(-1\), and consider the Laplace transformation rather than the Fourier transformations.

\(^{47}\)For example, the choice \( V(k) = 0 \) leads to a delta function to fix the gauge as \( h^a[\Phi] = 0 \).
and, from (7.3), the Faddeev-Popov matrix $\Delta^a_b[\Phi]$ transforms as,

$$\delta \Delta^a_b[\Phi] = \frac{d}{ds} \Delta^a_b[\Phi_{g(s\theta)}] \bigg|_{s=0}$$

$$= \frac{d}{ds} \left( \frac{\partial h^a[\Phi_{g(\zeta,s\theta)}]}{\partial \theta^b} \right) \bigg|_{\zeta=0, s=0}$$

$$= \frac{d}{ds} \left( \frac{\partial \zeta^c(\theta,s\vartheta)}{\partial \theta^b} \right) \bigg|_{\theta=0, s=0}$$

$$= -\frac{1}{2} \Delta^a_c[\Phi] C^{bc}_d \vartheta^d, \quad (7.43)$$

while $h^a[\Phi]$ transforms as

$$\delta h^a[\Phi] = \Delta^a_b[\Phi] \vartheta^b. \quad (7.44)$$

With a fermionic rigid (i.e. independent of $x$) scalar parameter $\varepsilon$, the BRST transformation reads as a ‘gauge transformation’ generated by $\vartheta^a = \varepsilon \omega^a$ or, equivalently by

$$\delta g = i \varepsilon \omega^a T_a, \quad (7.45)$$

so that

$$\delta \Phi = \frac{d}{ds} \Phi_{g(s\omega)} \bigg|_{s=0}, \quad \delta k_a = 0,$$

$$\delta \omega^a = -\frac{1}{2} \varepsilon C^a_{bc} \omega^b \omega^c, \quad \delta \bar{\omega}_a = -\varepsilon k_a,$$  

where we also defined the transformation of the ghosts $\omega$ and $\bar{\omega}$.  

In Yang-Mills theories the dynamical variables $\{ \Phi \} = \{ A_\mu, \phi, \bar{\psi}, \psi \}$ consist of a vector field $A_\mu$, matter fields $\psi, \bar{\psi}$ in the fundamental, anti-fundamental representations and matter fields $\phi$ in the adjoint representation,\(^{48}\) such that the standard gauge transformations are

$$A_{g\mu} = g A_\mu g^{-1} - i \partial_\mu gg^{-1}, \quad \phi_g = g \phi g^{-1}, \quad \psi_g = g \psi, \quad \bar{\psi}_g = \bar{\psi} g^{-1}, \quad \text{etc.}$$  

Explicit expressions for the infinitesimal BRST transformations read [22]

$$\delta A_\mu = \varepsilon D_\mu \omega, \quad \delta \phi = i[\varepsilon \omega, \phi], \quad \delta \psi = i \varepsilon \omega \psi, \quad \delta \bar{\psi} = -i \bar{\psi} \varepsilon \omega. \quad (7.48)$$

\(^{48}\)In our analysis, the matter fields $\phi, \psi$ and $\bar{\psi}$ can be either bosonic or fermionic.
where we set
\[ \omega := \omega^a T_a. \] (7.49)

We introduce the BRST charge \( Q_{\text{BRST}} \) which is the fermionic generator of the BRST transformations satisfying \( \delta \Phi = \varepsilon [Q_{\text{BRST}}, \Phi] \). More explicitly,

\[
\begin{align*}
[Q_{\text{BRST}}, A_\mu] &= +D_\mu \omega, & [Q_{\text{BRST}}, \phi] &= +i[\omega, \phi], \\
[Q_{\text{BRST}}, \psi] &= +i\omega \psi, & [Q_{\text{BRST}}, \bar{\psi}] &= \mp i\bar{\psi}\omega, \\
\{Q_{\text{BRST}}, \omega^a\} &= -\frac{1}{2}C^a_{bc}\omega^b\omega^c, & \{Q_{\text{BRST}}, \bar{\omega}_a\} &= -k_a, \\
[Q_{\text{BRST}}, k_a] &= 0,
\end{align*}
\] (7.50)

where the sign, \( \mp \) depends on whether \( \bar{\psi} \) is bosonic or fermionic.

In particular, the transformation was designed to satisfy \( \{Q_{\text{BRST}}, \Delta^b_a[\bar{\phi}]\omega^b\} = 0 \) in order for the Faddeev-Popov action (7.39) to be BRST closed, and such that
\[ \{Q_{\text{BRST}}, \omega\} = +i\omega^2. \] (7.51)

The BRST charge \( Q_{\text{BRST}} \) increases the ghost number by +1.

Because \( \varepsilon^2 = 0 \), the finite transformations are then given by
\[ e^{\varepsilon \text{ad} Q_{\text{BRST}}} = 1 + \varepsilon \text{ad} Q_{\text{BRST}}, \] (7.52)

and, with \( g := e^{i\varepsilon \omega} = 1 + i\varepsilon \omega \), each field transforms to
\[
\begin{align*}
\Phi_g, & \quad \omega_g = g\omega = \omega + i\varepsilon \omega^2, & \bar{\omega}_g = \bar{\omega}_a - \varepsilon k_a, & \quad k_g = k_a.
\end{align*}
\] (7.53)

Further under the successive transformations, \( e^{\varepsilon_1 \text{ad} Q_{\text{BRST}}} e^{\varepsilon_2 \text{ad} Q_{\text{BRST}}} \), each field transforms as
\[
\begin{align*}
\Phi & \rightarrow \Phi_{g_2} & \rightarrow \Phi_{g_1 g_2} = \Phi_{g_3}, \\
\omega & \rightarrow \omega_{g_2} = \omega + i\varepsilon_2 \omega^2 & \rightarrow \omega_{g_1} + i\varepsilon_2 (\omega_{g_1})^2 = \omega + i(\varepsilon_1 + \varepsilon_2)\omega^2 = \omega_{g_3}, \\
\bar{\omega}_a & \rightarrow \bar{\omega}_{g_2a} = \bar{\omega}_a - \varepsilon_2 k_a & \rightarrow \bar{\omega}_{g_1} - \varepsilon_2 k_{g_1a} = \bar{\omega}_a - (\varepsilon_1 + \varepsilon_2)k_a = \bar{\omega}_{g_3a}, \\
k_a & \rightarrow k_{g_2a} = k_a & \rightarrow k_{g_1a} = k_a = k_{g_3a}.
\end{align*}
\] (7.54)
where we set
\[ g'_2 := 1 + i\varepsilon_2 \omega g_1 = 1 + i
\varepsilon_2 \omega + \varepsilon_1 \varepsilon_2 \omega^2, \quad g_3 := e^{i(\varepsilon_1 + \varepsilon_2)\omega} = 1 + i(\varepsilon_1 + \varepsilon_2)\omega = g_1 \circ g'_2. \]

Thus, we note
\[ e^{\varepsilon_1 \text{ad}_{\text{BRST}}} e^{\varepsilon_2 \text{ad}_{\text{BRST}}} = e^{(\varepsilon_1 + \varepsilon_2)\text{ad}_{\text{BRST}}}, \]
and hence, the BRST charge is nilpotent,
\[ (\text{ad}_{\text{BRST}})^2 = 0, \quad \text{or} \quad [\text{BRST}, [\text{BRST}, \text{Field}]] = 0. \]

The nilpotent property can be directly checked from (7.50), using the Jacobi identity.

From the gauge invariance of the original action it follows that \( S[\Phi] \) is \( Q_{\text{BRST}} \)-closed,
\[ \left[ Q_{\text{BRST}}, S[\Phi] \right] = 0. \]

Hence, the Faddeev-Popov action reads as a sum of \( Q_{\text{BRST}} \)-closed and \( Q_{\text{BRST}} \)-exact terms,
\[ S_{\text{F.P.}}[\Phi, \omega, \bar{\omega}, k] = S[\Phi] + \left\{ Q_{\text{BRST}}, \bar{\omega} A^a(k) - \bar{\omega} h^a[\Phi] \right\}, \]
where, without loss of generality we have shifted the arbitrary function of the auxiliary field, \( V(k) \), by a constant in order to satisfy \( V(0) = 0 \) and to write it as
\[ V(k) := k_a V^a(k) = -\left\{ Q_{\text{BRST}}, \bar{\omega} A^a(k) \right\}. \]

In particular, for the Yang-Mills theory with the Lorentz gauge (7.40), one can rewrite the whole action (7.41) as
\[ S_{\text{YM}}[\Phi] + \int dx^D \sqrt{g} \text{tr} \left( - V(k) + k \nabla_\mu A^\mu - \nabla^\mu \bar{\omega} D_\mu \right) \]
\[ = S_{\text{YM}}[\Phi] + \int dx^D \left\{ Q_{\text{BRST}}, \sqrt{g} \text{tr} (\bar{\omega} V(k) - \bar{\omega} \nabla_\mu A^\mu) \right\}. \]
7.4 Hodge charge and a number operator

The Hodge charge is defined to be fermionic and acts only on the auxiliary fields \( k_a \) as

\[
[Q_{\text{Hodge}} , k_a] = -\bar{\omega}_a , \quad [Q_{\text{Hodge}} , \text{others}] = 0 .
\]  

(7.62)

Hence, it is nilpotent,

\[
(ad Q_{\text{Hodge}})^2 = 0 ,
\]

(7.63)

and satisfies

\[
\{ Q_{\text{Hodge}} , Q_{\text{BRST}} \} = N_{\bar{\omega}, k} ,
\]

(7.64)

where \( N_{\bar{\omega}, k} \) is the number operator counting the total number of \( \bar{\omega}_a \) and \( k_b \) fields,

\[
[N_{\bar{\omega}, k} , \bar{\omega}_a] = \bar{\omega}_a , \quad [N_{\bar{\omega}, k} , k_a] = k_a , \quad [N_{\bar{\omega}, k} , \text{others}] = 0 .
\]

(7.65)

Both of \( Q_{\text{BRST}} \) and \( Q_{\text{Hodge}} \) do not change the total number of \( \bar{\omega}_a \) and \( k_b \) fields since they do not annihilate the quantity such as \( \{ Q_{\text{Hodge}} , \bar{\omega}_a \} = 0 . \) One can show straightforwardly that

\[
[Q_{\text{BRST}} , N_{\bar{\omega}, k}] = 0 , \quad [Q_{\text{Hodge}} , N_{\bar{\omega}, k}] = 0 .
\]

(7.66)

Consider a \( Q_{\text{BRST}} \)-closed quantity \( \Upsilon , \)

\[
[Q_{\text{BRST}} , \Upsilon] = 0 .
\]

(7.67)

The latter condition is extension of the gauge invariance condition for functionals depending on \( \Phi \) only. We can decompose it as a sum of \( N_{\bar{\omega}, k} \) eigenstates,

\[
\Upsilon = \sum_{N=0}^{\infty} \Upsilon_N , \quad [N_{\bar{\omega}, k} , \Upsilon_N] = N \Upsilon_N .
\]

(7.68)

From \( [Q_{\text{BRST}} , N_{\bar{\omega}, k}] = 0 \) one can deduce that

\[
[Q_{\text{BRST}} , \Upsilon_N] = 0 .
\]

(7.69)

Therefore, any \( Q_{\text{BRST}} \)-closed quantity can be written as

\[
\left[ Q_{\text{BRST}} , \Upsilon \right] = 0 \iff \Upsilon [\Phi, \omega, \bar{\omega}, k] = \Upsilon_0 [\omega, \Phi] + \left[ Q_{\text{BRST}} , \bar{\Upsilon} [\Phi, \omega, \bar{\omega}, k] \right] ,
\]

(7.70)
where
\[
\left[ Q_{\text{BRST}} , \Upsilon_0[\omega, \Phi] \right] = 0 , \quad \left[ N_{\bar{\omega},k} , \Upsilon_0[\omega, \Phi] \right] = 0 , \quad \bar{\Upsilon} = \sum_{N=1}^{\infty} \frac{1}{N} \left[ Q_{\text{Hodge}} , \Upsilon_N \right] .
\]  
(7.71)

In other words, the cohomology group of the BRST charge is trivial at non-vanishing grading \( N_{\bar{\omega},k} \). This reasoning is a particular example of a standard procedure for computing cohomology groups. In mathematical terms, the Hodge differential \( Q_{\text{Hodge}} \) is called a “contracting homotopy” for the number operator \( N_{\bar{\omega},k} \) with respect to the BRST differential \( Q_{\text{BRST}} \). More materials on general cohomology groups can be found in the introduction for physicists to graded differential algebras in the chapter 8 of [7].

The importance of the BRST cohomology sits in the general theorem that the BRST cohomology group at ghost number zero is isomorphic\(^{49}\) to the algebra of observables of the theory. In other words, for the quantum theory it is isomorphic to the physical spectrum. Apart from this, as was roughly shown in the above example, the importance of the BRST formalism in gauge theories is that it allows to write a gauge-fixed path integral and to make sure that the final results are independent of the choice of gauge (because the corresponding terms are BRST trivial), see e.g. [23]. Nevertheless, the BRST cohomology is of high interest already at the classical level, as explained in the report [11] where some examples of applications are given. For further discussion on the ‘Hamiltonian’ BRST formalism presented here, we refer to the chapters 9 till 12 and the chapter 14 of [7]. The ‘Lagrangian’ BRST formalism of Batalin and Vilkovisky [24] has the advantage of being entirely general (it includes the case of gauge algebras that close only on-shell) and covariant (since it is Lagrangian). The Batalin-Vilkovisky (also called “antifield”) formalism is explained in the specific reviews [25] and in the chapters 17 & 18 of [7].

\(^{49}\)The reader may consult the section 11.1 of [7] for a proof and for more comments.
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Appendix

A Some proofs

Here we present the proofs of some facts discussed in the main body of the text.

- Eq.(2.12).

Proposition. An arbitrary function on the jet space $F(q_n, t)$ is a total derivative if and only if its Euler-Lagrange equations vanish identically,

$$\frac{\delta F}{\delta q^A}(q_n, t) = 0 \iff F(q_n, t) = \frac{dK(q_n, t)}{dt}.$$  \hfill (A.1)

Proof: The proof of the necessity ‘$\Leftarrow$’ is straightforward from (2.10). In order to show the sufficiency, ‘$\Rightarrow$’, we filter the set of all functions on the jet space by sets $\mathcal{F}_N$,

$$\mathcal{F}_N := \{F(q^A_n, t) : 0 \leq n \leq N\}, \quad N = 0, 1, 2, 3, \cdots.$$ \hfill (A.2)

We prove the sufficiency by mathematical induction on $N$:

- When $N = 0$, the left hand side of the claim (A.1) implies that the function depends, at most, only on the explicit time, $t$, being independent of $q^A_0$, i.e. $F(t)$.
  We can simply set $K(t) = \int_0^t \frac{d}{dt'} F(t')$.

- Now we assume that the converse is true for any $0 \leq N < M$, and consider the case, $N = M$. It is useful to note that if $F \in \mathcal{F}_N$, then \(\left(\frac{d}{dt}\right)^n F \in \mathcal{F}_{N+n}\) and its only dependence on $q_{N+n}$ appears as

$$\left(\frac{d}{dt}\right)^n F = q^A_{N+n} \frac{\partial F}{\partial q^A_N} + \mathcal{O}_{N+n-1}, \quad \mathcal{O}_{N+n-1} \in \mathcal{F}_{N+n-1}.$$  \hfill (A.3)

Hence for $F \in \mathcal{F}_M$, from

$$0 = \frac{\delta F(q_n, t)}{\delta q^A} = \sum_{n=0}^M \left(\frac{d}{dt}\right)^n \frac{\partial F}{\partial q^A_n} = (-1)^M q^B_{2M} \frac{\partial^2 F}{\partial q^B_M q^A_M} + \mathcal{O}_{2M-1},$$  \hfill (A.4)

we first note that $F \in \mathcal{F}_M$ is at most linear in $q_M$, i.e.

$$F(q_n, t) = q^A_M F_A(q_n, t) + \mathcal{O}_{M-1}, \quad F_A \in \mathcal{F}_{M-1}. \quad (A.5)$$
Consequently,

\[
0 = \frac{\delta F(q_n, t)}{\delta q^A} = \left( -\frac{d}{dt} \right)^M F_A + (-1)^{#A#B} \left( -\frac{d}{dt} \right)^{M-1} \left( q^B_M \frac{\partial F_B}{\partial q^A_{M-1}} \right) + \mathcal{O}_{2M-2}
\]

\[
= (-1)^M q^B_{2M-1} \frac{\partial F_A}{\partial q^B_{M-1}} + (-1)^{M-1}(-1)^{#A#B} q^B_{2M-1} \frac{\partial F_B}{\partial q^A_{M-1}} + \mathcal{O}_{2M-2}.
\]

(A.6)

Thus,

\[
\frac{\partial F_A}{\partial q^B_{M-1}} - (-1)^{#A#B} \frac{\partial F_B}{\partial q^A_{M-1}} = 0,
\]

(A.7)

so, by the usual Poincaré lemma in the space of one-forms \( F_A(q^B_{M-1}) \), there exists a function \( K(q_n, t) \), such that

\[
F_A = \frac{\partial K(q_n, t)}{\partial q^A_{M-1}}, \quad K(q_n, t) \in \mathcal{F}_{M-1}.
\]

(A.8)

Finally, if we define \( F' := F - \frac{dK}{dt} \), then

\[
\frac{\delta F'}{\delta q^A} = 0, \quad F' \in \mathcal{F}_{M-1}.
\]

(A.9)

Thus, from the induction hypothesis, \( F' \) is a total derivative, and hence so is \( F \) itself.

This completes our proof. \( \square \)

- Eq.(2.16).

Using Eq.(2.11), direct manipulation shows the following chain of identities, for an ar-
arbitrary function $F(q_n, t)$ on the jet space and for $m \geq 1$,

$$\sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left( \frac{\partial q_m^B}{\partial q_n^A} F \right) = \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left[ \left\{ \frac{d}{dt} \left( \frac{\partial q_{m-1}^B}{\partial q_n^A} \right) + \frac{\partial q_{m-1}^B}{\partial q_n^A} \right\} F \right]$$

$$= \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left[ \frac{\partial q_{m-1}^B}{\partial q_n^A} \left( -\frac{d}{dt} \right)^m F \right]$$

$$= \sum_{n=0}^{\infty} \left( -\frac{d}{dt} \right)^n \left[ \frac{\partial q_{m-1}^B}{\partial q_n^A} \left( -\frac{d}{dt} \right)^m F \right] \; .$$

(A.10)

- Eq.(2.27).

We show the relation (2.27) by induction on the power of $s$. We assume that the following relation is true up to the power $N - 1$ in $s$,

$$f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} \tilde{q}_n^A = f_n^A(\tilde{q}_m, t) + \mathcal{O}(s^N) \; ,$$

(A.11)

which is clearly true for $N = 1$, as $\tilde{q}_m^A = q_m^A$ if $s = 0$. Now differentiating the left hand side with respect to $s$, we get, up to the power $N$ in $s$,

$$\frac{d}{ds} \left( f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} \tilde{q}_n^A \right) = f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} \left( f_p^B(q_k, t) \frac{\partial}{\partial q_p^B} \tilde{q}_n^A \right)$$

$$= f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} f_n^A(\tilde{q}_m, t) + \mathcal{O}(s^N)$$

$$= f_l^B(q_m, t) \frac{\partial}{\partial q_l^B} \tilde{q}_k^C \frac{\partial}{\partial q_k^C} f_n^A(\tilde{q}_m, t) + \mathcal{O}(s^N) \; .$$

(A.12)
Thus, the relation holds up to power $N$, and this completes our proof.

• Eq.(4.67).

**Proposition.** For an arbitrary quantity, $F(p, q, t)$, the two actions, namely taking the time derivative and taking the restriction on $\mathcal{V}$, commute each order.

**Proof:** Using the coordinates, $(x, \phi)$ for the whole phase space, (4.42), we first define $\tilde{F}(x, \phi, t) = F(p, q, t)$. Then

\[
\left( \frac{d}{dt} F(p, q, t) \right) \bigg|_{\mathcal{V}} = \left( \frac{d}{dt} \tilde{F}(x, \phi, t) \right) \bigg|_{\mathcal{V}} = \left( \dot{x}^i \frac{\partial \tilde{F}}{\partial x^i} + \dot{\phi}_n \frac{\partial \tilde{F}}{\partial \phi_n} + \frac{\partial \tilde{F}}{\partial t} \right) \bigg|_{\mathcal{V}}
\]

\[
= \left( \dot{x}^i \frac{\partial \tilde{F}(x, 0, t)}{\partial x^i} + \frac{\partial \tilde{F}(x, 0, t)}{\partial t} \right) \bigg|_{\mathcal{V}} = \frac{d}{dt} \tilde{F}(x, 0, t)
\]

\[
= \frac{d}{dt} \left( F(p, q, t) \big|_{\mathcal{V}} \right).
\]

\[\square\]

• Eq.(4.66).

**Proposition.** The Poisson bracket is independent of time, 

\[
\{ \ , \ \} \_{P.B.} = \{ \ , \ \} \_{P.B.} \big|_{t_0}, \quad \text{(A.14)}
\]

or

\[
(-1)^{\#_A} \frac{\partial}{\partial q^A} \frac{\partial}{\partial p_A} - \frac{\partial}{\partial p_A} \frac{\partial}{\partial q^A} = \frac{\partial}{\partial q_0^A} \frac{\partial}{\partial p_0 A} - \frac{\partial}{\partial p_0 A} \frac{\partial}{\partial q_0^A}, \quad \text{(A.15)}
\]

Roughly speaking, the Poisson bracket is independent of time, \textit{i.e.} preserved, because time evolution is a symplectic transformation generated by the Hamiltonian.
**Proof:** We show the proposition for a set \( \{ \psi^i(t) \} \) of local functions which are piecewise infinitely differentiable. Then the time independence holds globally, since \((p, q)\) are globally continuous. A direct manipulation gives

\[
\left[ , \right]_{P.B.} \big|_{t_0} = (-1)^{\#_A} \partial_A \frac{\partial}{\partial q_0^A} \left. \frac{\partial}{\partial p_0 A} \frac{\partial}{\partial q_0^B} \right|_{t_0} - \left. \frac{\partial}{\partial q_0^A} \frac{\partial}{\partial p_0 A} \frac{\partial}{\partial q_0^B} \right|_{t_0} \\
= \left. \frac{\partial}{\partial q_0^A} \left[ q^A, q^B \right]_{P.B.} \big|_{t_0} \frac{\partial}{\partial q_0^B} + \left. \frac{\partial}{\partial q_0^A} \left[ q^A, p_B \right]_{P.B.} \big|_{t_0} \frac{\partial}{\partial p_B} \right|_{t_0} \\
+ \left. \frac{\partial}{\partial p_A} \left[ p_A, q^B \right]_{P.B.} \big|_{t_0} \frac{\partial}{\partial q_0^B} + \left. \frac{\partial}{\partial p_A} \left[ p_A, p_B \right]_{P.B.} \big|_{t_0} \frac{\partial}{\partial p_B} \right|_{t_0}.
\]

(A.16)

Obviously, the equality we want to show holds for the zeroth order in \( t - t_0 \). Now we suppose that it holds up to the order \((t - t_0)^k\), so that up to the power \( k \),

\[
\left[ q^A, q^B \right]_{P.B.} \big|_{t_0} \simeq 0, \quad \left[ q^A, p_B \right]_{P.B.} \big|_{t_0} \simeq (-1)^{\#_A} \delta^A_B, \\
\left[ p_A, q^B \right]_{P.B.} \big|_{t_0} \simeq -\delta^B_A, \quad \left[ p_A, p_B \right]_{P.B.} \big|_{t_0} \simeq 0.
\]

(A.17)

Also for two generic functions, \( F(p, q) \) and \( G(p, q) \), which do not have explicit time dependence, we get up to the power, \( k \),

\[
\frac{d}{dt} \left( \left[ F, G \right]_{P.B.} \big|_{t_0} \right) = \left[ \frac{dF}{dt}, G \right]_{P.B.} \big|_{t_0} + \left[ F, \frac{dG}{dt} \right]_{P.B.} \big|_{t_0} \\
\simeq \left[ \frac{dF}{dt}, G \right]_{P.B.} + \left[ F, \frac{dG}{dt} \right]_{P.B.} \\
= \left[ \left[ F, H_T \right]_{P.B.}, G \right]_{P.B.} + \left[ F, \left[ G, H_T \right]_{P.B.} \right]_{P.B.} \\
= -\left[ H_T, \left[ F, G \right]_{P.B.} \right]_{P.B.}.
\]

(A.18)

This shows that the relations, (A.17), actually hold up to the power, \( k + 1 \), completing the proof. \( \Box \)

- Eq.(5.10)

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The first equality in (5.10) follows from the algebraic identity (2.34) on the tangent space variables \((q, \dot{q}, t)\). Firstly, one considers (5.7) for \(A = \hat{m}\),

\[
\frac{\partial (\delta q^B)}{\partial \hat{q}^m} \frac{\partial L}{\partial q^B} = \frac{\partial (\delta K)}{\partial \hat{q}^m},
\]  

(A.19)

Secondly, one takes the partial derivative of each side of (A.19) with respect to \(q^A\),

\[
\frac{\partial^2 (\delta q^B)}{\partial q^A \partial \hat{q}^m} \frac{\partial L}{\partial q^B} + (-1)^{\#A(\#B+\#m)} \frac{\partial (\delta q^B)}{\partial \hat{q}^m} \frac{\partial^2 L}{\partial q^A \partial \hat{q}^B} = \frac{\partial^2 (\delta K)}{\partial q^A \partial \hat{q}^m}.
\]  

(A.20)

Thirdly, the partial derivative of each side of (5.6) reads explicitly as

\[
\frac{\partial \hat{p}_A}{\partial \hat{q}^m} = \frac{\partial^2 (\delta K)}{\partial \hat{q}^m \partial \hat{q}^A} - \frac{\partial^2 (\delta q^B)}{\partial \hat{q}^m \partial \hat{q}^B} \hat{p}_B,
\]  

(A.21)

where we made use of (5.5). Fourthly, making use of (A.20) in (A.21) leads to (5.10).

The second equality in (5.10) holds because of the integrability condition (2.37). Proceeding step by step, one may start by using the chain rule in order to show the identity

\[
(-1)^{\#A \#B} \frac{\partial (\delta q^B)}{\partial \hat{q}^m} \frac{\partial \hat{h}^a}{\partial q^A} \frac{\partial^2 L}{\partial q^B \partial \hat{q}^B} = (-1)^{\#A + \#a \#m} \frac{\partial \hat{h}^a}{\partial \hat{q}^m} \left( \frac{\partial (\delta q^B)}{\partial \hat{q}^m} + \frac{\partial \hat{h}^b}{\partial \hat{q}^m} \frac{\partial (\delta q^B)}{\partial \hat{q}^B} \right) \frac{\partial^2 L}{\partial \hat{q}^B \partial \hat{q}^B}.
\]  

(A.22)

Then, the relation (2.37) is used to exchange some indices in Eq.(A.22) as follows

\[
(-1)^{\#A + \#a \#m} \frac{\partial (\delta q^B)}{\partial \hat{q}^m} \left( \frac{\partial \hat{h}^b}{\partial \hat{q}^m} \frac{\partial (\delta q^B)}{\partial \hat{q}^B} \right) \frac{\partial^2 L}{\partial \hat{q}^B \partial \hat{q}^B} \hat{p}_B = (-1)^{\#A + \#a \#m} \frac{\partial \hat{h}^b}{\partial \hat{q}^m} \left( \frac{\partial^2 L}{\partial \hat{q}^m \partial \hat{q}^B} + \frac{\partial \hat{h}^b}{\partial \hat{q}^m} \frac{\partial^2 L}{\partial \hat{q}^B \partial \hat{q}^B} \right).
\]  

(A.23)

Finally, one observes that the sum of terms in the parenthesis of Eq.(A.23) vanishes since

\[
\frac{\partial^2 L}{\partial \hat{q}^m \partial \hat{q}^B} + \frac{\partial \hat{h}^b}{\partial \hat{q}^m} \frac{\partial^2 L}{\partial \hat{q}^B \partial \hat{q}^B} = \frac{\partial \hat{p}_B(q, p_a, t)}{\partial \hat{q}^m} = 0,
\]  

(A.24)

due to (5.5). The set of Eqs.(A.22)-(A.24) implies that

\[
\left( \frac{\partial^2 L(q, \dot{q}, t)}{\partial q^A \partial \dot{q}^B} \right) \dot{\hat{q}}^a = \hat{h}^a(q, p_a, \hat{q}^m, t)
\]  

(A.25)

which ends the proof of Eq.(5.10) due to (3.7).
B Grassmann algebra

In principle, in order to be able to discuss rigorously a generic dynamical system (i.e. which contains both bosons and fermions), one needs to introduce the “Grassmann algebra” $\Lambda_{\hat{N}}$ which is generated by the $\hat{N}$ anti-commuting Grassmann variables [10],

$$\zeta^\alpha \zeta^\beta + \zeta^\beta \zeta^\alpha = 0, \quad \alpha, \beta = 1, 2, \cdots, \hat{N}. \quad (B.1)$$

They generate the following basis for the Grassmann algebra

$$\begin{array}{c}
1 \\
\zeta^\alpha \\
\zeta^\alpha \zeta^\beta, \quad \alpha < \beta \\
\vdots \\
\zeta^1 \zeta^2 \cdots \zeta^{\hat{N}} \\
\end{array} \quad \text{“body”}$$

$$\begin{array}{c}
\zeta^\alpha \zeta^\beta, \quad \alpha < \beta \\
\vdots \\
\zeta^1 \zeta^2 \cdots \zeta^{\hat{N}} \\
\end{array} \quad \text{“soul”}$$

(B.2)

which has the dimension, $\dim(\Lambda_{\hat{N}}) = 2^{\hat{N}}$ while $\hat{N}$ can be infinity.

Any quantity in the Grassmann algebra, $\Lambda_{\hat{N}}$, can be expressed as an expansion in terms of the above basis over the field $\mathbb{R}$ of real numbers field (or the field $\mathbb{C}$ of complex numbers),

$$x = x_0 + \sum_{n=1}^{\hat{N}} \frac{1}{n!} x_{\alpha_1 \alpha_2 \cdots \alpha_n} \zeta^{\alpha_1} \zeta^{\alpha_2} \cdots \zeta^{\alpha_n}, \quad (B.3)$$

where $x_0$ and $x_{\alpha_1 \alpha_2 \cdots \alpha_n}$ are real (or complex numbers) carrying totally anti-symmetric indices. Naturally, the bosons allow the expansion of even $n$’s only, while fermions allow only odd $n$’s.

*It is crucial to note that if and only if $x_0 \neq 0$, the inverse, $x^{-1}$, exists.*

It is convenient to rename the elements in the basis with a given ordering as

$$\{ Z^J, \ 1 \leq J \leq 2^{\hat{N}} \} = \{ 1, \zeta^\alpha, \cdots, \zeta^1 \zeta^2 \cdots \zeta^{\hat{N}} \}, \quad (B.4)$$
and to write
\[ x = \sum_{J=1}^{2^\hat{N}} x_J Z^J. \] (B.5)

We also introduce the following notation to pick up the real or complex number coefficient,
\[ [x]_J = x_J. \] (B.6)

In terms of the Grassmann algebra, the Lagrangian is a bosonic variable, but not necessarily a pure ‘body’. Furthermore the actual dynamical variables are the real numbers, \([q^A]_J\), leading to a much bigger phase space.

In deriving the equations of motion for a Lagrangian, what we actually encounter is the expression,
\[ \int dt \delta L(q_n, t) = \int dt \delta [q^A]_J Z^J \frac{\delta}{\delta q^A} L(q_m, t). \] (B.7)

This implies that for the bosons, by considering especially the variations of the pure body, the equations of motion can be indeed collectively expressed as usual (2.7), but the equations of motion for the fermions should be refined to hold in a weaker form,
\[ \left[ \frac{\delta L}{\delta q^A}(q_m, t) \right]_J \equiv 0, \quad J \neq 2^\hat{N}, \] (B.8)

when \( \hat{N} \) is odd. Namely for the fermions, the usual equation of motion is true except the highest order in ‘soul’. However this subtle issue can be neglected either by imposing the missing equation for \( J = 2^\hat{N} \) by hand, or by letting \( \hat{N} \to \infty \).

The purpose of the present subsection was to provide a rigorous way to analyze the dynamical systems containing both bosons and fermions. Nevertheless, in practice we will favor the Lagrangian systems which do not require any explicit use of the basis for the Grassmann algebra, especially when they are transformed into Hamiltonian form for constrained systems.
C Basics on supermatrices

A generic \((n_1 + n_2) \times (m_1 + m_2)\) supermatrix, \(M\), over a Grassmann algebra, \(\Lambda_{\hat{N}}\), (see Section B), is of the form,

\[
M = \begin{pmatrix}
A_{n_1 \times m_1} & \Psi_{n_1 \times m_2} \\
\Theta_{n_2 \times m_1} & B_{n_2 \times m_2}
\end{pmatrix},
\]

(C.1)

where \(A\), \(B\) are bosonic and \(\Psi\), \(\Theta\) are fermionic.

The complex conjugation, transpose, and the Hermitian conjugation read respectively [10],

\[
M^* = \begin{pmatrix}
A & \Psi \\
\Theta & B
\end{pmatrix}^* = \begin{pmatrix}
A^* & -\Psi^* \\
\Theta^* & B^*
\end{pmatrix} \quad \text{or} \quad (M^*)_{ab} = (-1)^{\#b(\#a + \#b)}(M_{ab})^*,
\]

\[
M^T = \begin{pmatrix}
A & \Psi \\
\Theta & B
\end{pmatrix}^T = \begin{pmatrix}
A^T & \Theta^T \\
-\Psi^T & B^T
\end{pmatrix} \quad \text{or} \quad (M^T)_{ab} = (-1)^{\#a(\#a + \#b)}M_{ba},
\]

\[
M^\dagger = (M^*)^T = \begin{pmatrix}
A & \Psi \\
\Theta & B
\end{pmatrix}^\dagger = \begin{pmatrix}
A^\dagger & \Theta^\dagger \\
\Psi^\dagger & B^\dagger
\end{pmatrix} \quad \text{or} \quad (M^\dagger)_{ab} = (M_{ba})^*.
\]

(C.2)

Note that

\[
(M^*)^* = M, \quad (M^\dagger)^\dagger = M, \quad (M^T)^\dagger = M^*, \quad (M_1M_2)^* = M_1^*M_2^*, \quad (M_1M_2)^T = M_2^TM_1^T, \quad (M_1M_2)^\dagger = M_2^\daggerM_1^\dagger.
\]

(C.3)

However,

\[
(M^T)^T \neq M, \quad (M^*)^T \neq (M^T)^*, \quad (M^\dagger)^T \neq M^*, \quad \text{etc.}
\]

(C.4)

In particular, a real supermatrix is of the generic form,

\[
M = M^* = \begin{pmatrix}
A & i\Psi \\
\Theta & B
\end{pmatrix}, \quad M^\dagger = M^T,
\]

(C.5)

where every variable is real, \(A = A^*, B = B^*, \Psi = \Psi^*, \Theta = \Theta^*\).
For the \((n_1 + n_2) \times (n_1 + n_2)\) square supermatrix, \(M\),

\[
M = \begin{pmatrix}
A_{n_1 \times n_1} & \Psi_{n_1 \times n_2} \\
\Theta_{n_2 \times n_1} & B_{n_2 \times n_2}
\end{pmatrix},
\]

the inverse can be expressed as

\[
M^{-1} = \begin{pmatrix}
(A - \Psi B^{-1} \Theta)^{-1} & -A^{-1} \Psi (B - \Theta A^{-1} \Psi)^{-1} \\
-B^{-1} \Theta (A - \Psi B^{-1} \Theta)^{-1} & (B - \Theta A^{-1} \Psi)^{-1}
\end{pmatrix},
\]

where we may write

\[
(A - \Psi B^{-1} \Theta)^{-1} = A^{-1} + \sum_{p=1}^{\infty} (A^{-1} \Psi B^{-1} \Theta)^p A^{-1}.
\]

Note that due to the fermionic property of \(\Psi, \Theta\), the power series terminates at \(p \leq n_1 n_2 + 1\).

The supertrace and the superdeterminant of \(M\) are defined as \([10]\)

\[
\text{str } M = \text{tr } A - \text{tr } B, \tag{C.9}
\]

\[
\text{sdet } M = \det (A - \Psi B^{-1} \Theta) / \det B = \det A / \det (B - \Theta A^{-1} \Psi). \tag{C.10}
\]

From Eq.(C.7),

\[
\text{sdet } M \neq 0 \iff \det A \det B \neq 0
\]

is the necessary and sufficient condition for the existence of \(M^{-1}\).

The supertrace and the superdeterminant have the properties,

\[
\text{str } (M_1 M_2) = \text{str } (M_2 M_1), \quad \text{sdet } (M_1 M_2) = \text{sdet } M_1 \text{ sdet } M_2. \tag{C.11}
\]

For a generic \(n \times n\) bosonic matrix, \(A\), in a similar fashion to above, decomposing it into the ‘body’ and ‘soul’ (see Section B),

\[
A = A_{\text{body}} + A_{\text{soul}}, \tag{C.12}
\]

\(^{50}\)The last equality comes from \(\det(1 - A^{-1} \Psi B^{-1} \Theta) = \det^{-1}(1 - B^{-1} \Theta A^{-1} \Psi)\), which can be shown using \(\det(1 - X) = \exp \left(- \sum_{p=1}^{\infty} \frac{1}{p} \text{tr } X^p \right)\), and observing \(\text{tr } (A^{-1} \Psi B^{-1} \Theta)^p = -\text{tr } (B^{-1} \Theta A^{-1} \Psi)^p\).
we have

\[ A^{-1} = A_{\text{body}}^{-1} + \sum_{p=1}^{n^2+1} (-A_{\text{body}}^{-1} A_{\text{soul}})^p A_{\text{body}}^{-1}. \]  

(C.13)

Thus, \( A^{-1} \) exists if and only if \( A_{\text{body}}^{-1} \) exists.
D Lemmas on the canonical transformations of supermatrices

In this appendix, we do not explicitly state which entries in the supermatrices which are Grassmann even or odd. The way the supermatrices have been written is supposed to be self-explanatory.

Fact 1.
For any $n \times m$ bosonic matrix $M$ over a Grassmann algebra $\Lambda_N$ (see Section B),

\[
M = \begin{pmatrix} v_1 & v_2 & \cdots & v_m \end{pmatrix}, \quad v_j = (v_{1j} \ v_{2j} \ \cdots \ v_{nj})^T,
\]

there exists an $m \times m$ nondegenerate matrix $Q$ satisfying

\[
MQ = \begin{pmatrix} w_1 & w_2 & \cdots & w_k & s_1 & s_2 & \cdots & s_{m-k} \end{pmatrix},
\]

where all vector (i.e. columns) $w_i$ are orthogonal to each other and each body of them is nonzero, while the $s_j$’s are pure souls or zero.

Proof: To show this, one needs to separate the vectors $v_i$ into two groups: pure soul ones and other ones with nontrivial bodies. Then one only needs to orthogonalize\(^{51}\) the latter.

Fact 2.
For any $n \times m$ bosonic matrix $M$ over a Grassmann algebra $\Lambda_N$, there exists two nondegenerate matrices, $P$ and $P'$, which transform $M$ into the canonical form,

\[
P'MP = \begin{pmatrix} \Lambda_{k \times k} & 0_{k \times (m-k)} \\ 0_{(n-k) \times k} & s_{(n-k) \times (m-k)} \end{pmatrix},
\]

where $\Lambda_{k \times k}$ is a $k \times k$ nondegenerate diagonal matrix,

\[
\Lambda_{k \times k} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_k), \quad (\lambda_i)_{\text{body}} \neq 0, \quad 1 \leq i \leq k,
\]

of rank $k$ so that $k \leq m$ and $k \leq n$, and the $(n-k) \times (m-k)$ matrix $s_{(n-k) \times (m-k)}$ is a pure soul.

\(^{51}\)Note that, since the body is nonzero, the inverse of the scalar product $(w_i^\dagger w_j)^{-1}$ exists, and the orthogonalization can be done by a generalization of the Gram-Schmidt procedure for the graded case.
Proof: From Fact 1, we construct $P'$ out of the orthogonal vectors $\{w_1^+, w_2^+, \cdots, w_k^+\}$ and their complementary vectors to get

$$P' MP = \begin{pmatrix} \Lambda & s' \\ 0 & s \end{pmatrix}. \quad (D.5)$$

Now we only need to take one more step for the final result,

$$\begin{pmatrix} \Lambda & s' \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & -\Lambda^{-1}s' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda & 0 \\ 0 & s \end{pmatrix}. \quad (D.6)$$

Fact 3. - Corollary
For any nondegenerate $n \times n$ bosonic matrix $M$ over a Grassmann algebra $\Lambda_N$, meaning

$$\det M_{\text{body}} \neq 0, \quad (D.7)$$

there exist two nondegenerate matrices $P$ and $P'$ which transforms $M$ into the identity,

$$P' MP = 1. \quad (D.8)$$

Proof: The proof is straightforward from Fact 2 and its Proof, since the matrix is non-degenerate and $n = m = k. \square$

Fact 4.
For any nondegenerate $(n_1 + n_2) \times (n_1 + n_2)$ square supermatrix $M$ over a Grassmann algebra, $\Lambda_N$,

$$M = \begin{pmatrix} A & \psi \\ \chi & B \end{pmatrix}, \quad \det A \det B \neq 0, \quad (D.9)$$

there exists two nondegenerate supermatrices $P$ and $P'$ satisfying

$$P' MP = 1. \quad (D.10)$$
Proof: After the transformation,
\[
\begin{pmatrix}
1 & 0 \\
-\chi A^{-1} & 1
\end{pmatrix}
\begin{pmatrix}
A & \psi \\
\chi & B
\end{pmatrix}
\begin{pmatrix}
1 & -A^{-1}\psi \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
A & 0 \\
0 & B - \chi A^{-1}\psi
\end{pmatrix},
\] (D.11)
we only need to apply Fact 3. \(\square\)

Fact 5.
For any nondegenerate, bosonic, real, symmetric or anti-symmetric matrix \(A_{\pm}\) over a Grassmann algebra \(\Lambda_{\hat{N}}\),
\[
A_{\pm} = A_0 + A_{\text{soul}},
A_{\pm}^* = A_{\pm},
\] (D.12)
\((\det A_{\pm})_{\text{body}} = (\det A_0)_{\text{body}} \neq 0\), \(A_{\pm}^T = \pm A_{\pm}\),
there exists a nondegenerate real matrix \(P\), the transformation induced by which, removes the pure soul \(A_{\text{soul}}\) completely,
\[
PA_{\pm}P^T = A_0.
\] (D.13)

Remark: The point of Fact 5 is the removal or addition of any soul to the original matrix, \(A_{\pm}\), via the insertion between two appropriately chosen real matrices (D.13).

Proof: We present explicitly the real transformation,
\[
P = P^* = 1 + \sum_{n=1}^{\infty} a_n (A_{\text{soul}}A_0^{-1})^n,
\] (D.14)
where the coefficients are given by a recurrence relation [15], with \(a_0 = 0, a_1 = -\frac{1}{2}\),
\[
a_{n+1} = -a_n - \frac{1}{2} \sum_{j=1}^{n} a_j (a_{n+1-j} + a_{n-j}), \quad \text{for } n \geq 1.
\] (D.15)

Due to the Grassmannian property, the sum in (D.14) terminates at a finite order. \(\square\)

Fact 6.
For any bosonic, real, symmetric or anti-symmetric matrix, \(A_{\pm}\), over a Grassmann algebra, \(\Lambda_{\hat{N}}\),
\[
A_{\pm} = A_{\pm}^*, \quad A_{\pm}^T = \pm A_{\pm},
\] (D.16)
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there exists a nondegenerate real matrix, $P$, which transforms $A$ into the canonical form,

$$PA_{\pm}P^T = \begin{pmatrix} b_{\pm} & 0 \\ 0 & s_{\pm} \end{pmatrix}, \quad P = P^*, \quad (D.17)$$

where the bosonic matrix, $b_{\pm} = \pm b^T_{\pm}$, is nondegenerate, $(\det b_{\pm})_{\text{body}} \neq 0$, while $s = \pm s^T$ is a pure soul or zero.

**Proof:** First, using a real orthonormal matrix $O$, one can transform the ‘body’ of $A$ into the canonical form $\begin{pmatrix} b & 0 & 0 & 0 \\ 0 & 0 & s_{\pm} & 0 \end{pmatrix}$, which gives

$$OA_{\pm}O^T = \begin{pmatrix} b + s_1 & s_2 \\ \pm s_2^T & s_3 \end{pmatrix}, \quad (D.18)$$

where $b = \pm b^T$ is nondegenerate and real, while $s_1 = \pm s_1^T, s_2, s_3 = \pm s_3^T$ are all real and pure souls. We further transform it as

$$\begin{pmatrix} 1 & 0 \\ \mp s_2^T(b + s_1)^{-1} & 1 \end{pmatrix} \begin{pmatrix} b + s_1 & s_2 \\ \pm s_2^T & s_3 \end{pmatrix} \begin{pmatrix} 1 & -(b + s_1)^{-1}s_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b + s_1 & 0 \\ 0 & s_3 \mp s_2^T(b + s_1)^{-1}s_2 \end{pmatrix}. \quad (D.19)$$

To complete the proof, we only need to apply Fact 5 to $b + s_1$. □

**Fact 7.**

For a generic $(n_1 + n_2) \times (m_1 + m_2)$ supermatrix $M$ over a Grassmann algebra $\Lambda_N$,

$$M = \begin{pmatrix} A_{n_1 \times m_1} & \Psi_{n_1 \times m_2} \\ \Theta_{n_2 \times m_1} & B_{n_2 \times m_2} \end{pmatrix}, \quad (D.20)$$

there exists two nondegenerate supermatrices $P$ and $P'$ which transforms it into the canonical form,

$$P'MP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & s_1 & 0 & \psi \\ 0 & 0 & 1 & 0 \\ 0 & \chi & 0 & s_2 \end{pmatrix}, \quad (D.21)$$

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where $s_1, s_2$ are bosonic pure soul, and $\psi, \chi$ are fermionic. The partition of the canonical form reads,

$$[k_1 + (n_1 - k_1) + k_2 + (n_2 - k_2)] 	imes [k_1 + (m_1 - k_1) + k_2 + (m_2 - k_2)], \tag{D.22}$$

where $k_1, k_2$ are respectively the ranks of the bosonic matrices $A, B$.

**Proof:** From Fact 2, we can first transform the $A_{n_1 \times m_1}$ into the canonical form, in order to put $M$ into the form

$$
\begin{pmatrix}
1 & 0 & \psi_1 \\
0 & s_1 & \psi_2 \\
\chi_1 & \chi_2 & B
\end{pmatrix}, \tag{D.23}
$$

and to further have

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-\chi_1 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 0 & \psi_1 \\
0 & s_1 & \psi_2 \\
\chi_1 & \chi_2 & B
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & -\psi_1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s_1 & \psi_2 \\
0 & \chi_2 & B - \chi_1 \psi_1
\end{pmatrix}. \tag{D.24}
$$

Now we apply Fact 3 to $(B - \chi_1 \psi_1)$ to get

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s_1 & \psi_3 & \psi_4 \\
0 & \chi_3 & 1 & 0 \\
0 & \chi_4 & 0 & s_2
\end{pmatrix}. \tag{D.25}
$$

Finally, one completes the proof by the equality,

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & -\psi_3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s_1 & \psi_3 & \psi_4 \\
0 & \chi_3 & 1 & 0 \\
0 & \chi_4 & 0 & s_2
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -\chi_3 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & s_1 - \psi_3 \chi_3 & 0 & \psi_4 \\
0 & 0 & 1 & 0 \\
0 & \chi_4 & 0 & s_2
\end{pmatrix}. \tag{D.26}
$$
Remark: Note also that Fact 3 follows as a corollary too.

Fact 8.
Consider a \((n_- + n_+) \times (n_- + n_+)\) anti-Hermitian supermatrix, over a Grassmann algebra \(\Lambda_{\hat{N}}\), which has the symmetry property, \(\Omega_{ab} = -(1)^{\#_a \#_b} \Omega_{ba}\), or equivalently
\[
\Omega = -\Omega^\dagger, \quad \Omega^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \Omega.
\] (D.27)

It is of the general form,
\[
\Omega = \begin{pmatrix} A_- & \Psi \\ -\Psi^T & i A_+ \end{pmatrix},
\] (D.28)
where every variable is real, \(A_\pm = A^\ast_\pm\), \(\Psi = \Psi^*\), and \(A_\pm = \pm A^T_\pm\).

We note that the anti-Hermiticity and symmetry properties (D.27) are preserved under the transformations by a real supermatrix, (C.5),
\[
\Omega \implies L\Omega L^T, \quad L = L^*,
\] (D.29)
since
\[
L^\dagger = (L^*)^T, \quad (L^T)^\dagger = L^*, \quad (L^T)^T = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} L \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\] (D.30)

The claim is that there exists a nondegenerate real supermatrix, \(L = L^*\), which transforms \(\Omega\) into the following canonical form,
\[
L\Omega L^T = \begin{pmatrix} b_- & 0 & 0 & 0 \\ 0 & s_- & 0 & \psi \\ 0 & 0 & ib_+ & 0 \\ 0 & -\psi^T & 0 & is_+ \end{pmatrix},
\] (D.31)
where all the variables are real, \(b_\pm = b^{\ast}_\pm\), \(s_\pm = s^{\ast}_\pm\), \(\psi = \psi^*\); \(b_\pm\) are nondegenerate bosonic matrices, \((\det b_\pm)_{\text{body}} \neq 0\); \(s_\pm\) are pure souls; and \(b_\pm = \pm b^{\ast T}_\pm\), \(s_\pm = \pm s^{T}_\pm\).

The partition reads
\[
\left[ k_- + (n_- - k_-) + k_+ + (n_+ - k_+) \right]^2,
\] (D.32)
where \(k_\mp\) are respectively the ranks of the bosonic matrices, \(A_\mp\), so that \((b_\pm)_{k_\pm \times k_\mp}\).
Proof: From Fact 6, we transform $A_-$ into the canonical form, in such a way that

$$RΩR^T = \begin{pmatrix} b_- & 0 & ψ_1 \\ 0 & s'_- & ψ_2 \\ −ψ_1^T & −ψ_2^T & iA_+ \end{pmatrix}, \quad (D.33)$$

where $R = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}$. We take it further to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ψ_1^{-1}b_- & 0 & 1 \end{pmatrix} \begin{pmatrix} b_- & 0 & ψ_1 \\ 0 & s'_- & ψ_2 \\ −ψ_1^T & −ψ_2^T & iA_+ \end{pmatrix} \begin{pmatrix} 1 & 0 & −b^{-1}_ψ_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (D.34)$$

$$= \begin{pmatrix} b_- & 0 & 0 \\ 0 & s'_- & ψ_2 \\ 0 & −ψ_2^T & iA_+ + ψ_1^{-1}b_ψ_1 \end{pmatrix}.$$ 

Now apply Fact 6 to $(iA_+ + ψ_1^{-1}b_ψ_1)$ to get

$$\begin{pmatrix} b_- & 0 & 0 & 0 \\ 0 & s'_- & ψ_2 & χ \\ 0 & −χ^T & ib_ψ & 0 \\ 0 & −ψ_2^T & 0 & is_ψ \end{pmatrix}. \quad (D.35)$$

Finally, to complete the proof, we only need to take the following transformation,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & iχb_ψ^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} b_- & 0 & 0 & 0 \\ 0 & s'_- & ψ_2 & χ \\ 0 & −χ^T & ib_ψ & 0 \\ 0 & −ψ_2^T & 0 & is_ψ \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & −ib_ψ^{-1}χ^T & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (D.36)$$

$$= \begin{pmatrix} b_- & 0 & 0 & 0 \\ 0 & s'_- − iχb_ψ^{-1}χ^T & 0 & ψ \\ 0 & 0 & ib_ψ & 0 \\ 0 & −ψ_2^T & 0 & is_ψ \end{pmatrix}.$$ 

$\square$
E  A paradigmatic example

As an illustration of the general case, discussed in the core of the text, a most simple case is presented here: a Lagrangian

\[ L(q^A, \dot{q}^B) = T(\dot{q}^B) - V(q^A), \]  

(E.1)

which

1. is a function on a bosonic tangent space of finite dimension \( N \),
2. is of homogeneity degree equal to two (i.e. the system is free),
3. leads to a positive definite energy, and
4. does not lead to tertiary constraints.

Hopefully, this example combines three virtues: (i) its simplicity should allow to displace the focus from the technical onto the conceptual, (ii) it includes both cases of first and second class constraints, and (iii) it provides the starting point of usual perturbative expansion, so it is not merely academical.

The Lagrangian is assumed to be quadratic, therefore the kinetic energy \( T(\dot{q}) = \frac{1}{2} T_{AB} \dot{q}^A \dot{q}^B \) and the potential energy \( V(q) = \frac{1}{2} V_{AB} q^A q^B \) are both quadratic forms. The Lagrangian (E.1) leads to a conserved energy equal to

\[ E(q, \dot{q}) = T(\dot{q}) + V(q), \]  

(E.2)

which is positive definite, \( E(q, \dot{q}) \geq 0 \), if and only if the kinetic and potential energy are separately positive definite: \( T(\dot{q}) \geq 0, V(q) \geq 0 \). Without loss of generality, one may assume that the variables \( q^A \) have been ‘rotated’ so that the symmetric matrix \( T_{AB} = \partial^2 L/\partial q^A \partial q^B \) is diagonal:

\[ T(\dot{q}^a, \dot{q}^m) = \frac{1}{2} \sum_a M_a (\dot{q}^a)^2, \]  

(E.3)

where the index \( a \) corresponds to the \( N - M \) strictly positive eigenvalues \( M_a > 0 \), while the index \( m \) corresponds to the remaining \( M \) zero eigenvalues. The corresponding momentas are respectively given by \( p_a = M_a \dot{q}^a \) (no sum on the index \( a \)) and \( p_m = 0 \).\(^{52}\) Therefore, one gets \( M \) primary constraints \( \phi_m(q, p, t) = p_m \) and the primary constraint surface \( V \) is the hyperplane \( p_m = \)

\(^{52}\)In this example, notice that the distinction between hatted and unh indices is not necessary.
of codimension $\mathcal{M}$ embedded in the $2\mathcal{N}$-dimensional phase space. The canonical Hamiltonian (3.24) reads

$$H(q^A, p_b, t)\big|_V = \sum_a \frac{(p_a)^2}{2m_a} - \frac{1}{2} \sum_{A,B} V_{AB} q^A q^B.$$  \hspace{1cm} (E.4)

The total Hamiltonian (3.30) is thus given by

$$H_T(q^A, p_B, u^m, t) = H + u^m p_m.$$  \hspace{1cm} (E.5)

The time evolution of $q^m$ through the Poisson bracket with the total Hamiltonian leads to the equality

$$\dot{q}^m = \{q^m, H_T\}_{PB} = u^m.$$  \hspace{1cm} \text{(4.18)}

Now the point is that the preservation (4.43) of the secondary constraints under the time evolution would lead to new, i.e. tertiary constraints if some entries $V_{ma}$ were non-vanishing. Therefore, in the particular example we are considering, one assumes that the potential does not include mixed terms:

$$V(q^a, q^m) = \frac{1}{2} \sum_{a,b} V_{ab} q^a q^b + \frac{1}{2} \sum_{m,n} V_{mn} q^m q^n.$$  \hspace{1cm} (E.7)

For that reason, one may perform a rotation in the plane of the variables $q^a/\sqrt{M_a}$ in order to make the symmetric matrix $V_{ab}$ diagonal without modifying the kinetic energy. Without loss of generality, the symmetric matrix $V_{mn}$ may also be assumed to be diagonal. Therefore,

$$V(q^a, q^\alpha, q^\bar{\alpha}, q^{\mu}) = \frac{1}{2} \sum_\pi (\omega_\pi)^2 (q^\pi)^2 + \frac{1}{2} \sum_\bar{\alpha} (q^{\bar{\alpha}})^2.$$  \hspace{1cm} (E.8)

where the ‘barred’ indices correspond to the strictly positive eigenvalues while the ‘Greek’ indices correspond to the vanishing eigenvalues of the matrices $V_{ab}$ and $V_{mn}$. The secondary constraints (E.6) become simply

$$\phi'_m(q, p, t) = V_{mn} q^m + \frac{1}{2} \sum_{n\neq m} V_{mn} q^n + \frac{1}{2} \sum_a V_{ma} q^a.$$  \hspace{1cm} (E.6)

The preservation (4.44) of the secondary constraints under the time evolution only leads to the fact that the Lagrange multipliers $u^\mu = 0$, while the $u^\mu$‘s can be arbitrary functions of time, which signals the presence of some gauge freedom. The constraint surface $\mathcal{V}$ is thus the hyperplane defined by the system $p_\mu = p^\mu = q^\pi = 0$. It is straightforward to check that the constraints $p_\mu$ are (primary) first class constraints and that the set $\{p^\mu, q^\pi\}$ contains all the second class constraints. Notice that, in
this example, the total and extended Hamiltonians are identical since there are no secondary first class constraints.

Under the sole hypotheses stated above, the Lagrangian and the Hamiltonian have been decomposed into a sum of four pieces

\[ L = L_{\text{free}} + L_{\text{harmonic}} + L^{1\text{st}} + L^{2\text{nd}}, \]
\[ H = H_{\text{free}} + H_{\text{harmonic}} + H^{1\text{st}} + H^{2\text{nd}}, \]

(E.9)

where each piece corresponds to one of the following four distinct physical cases:

- **Free particles:**
  \[ L_{\text{free}}(q^\alpha) = \frac{1}{2} \sum_\alpha M_\alpha (q^\alpha)^2, \]
  \[ H_{\text{free}}(p_\alpha) = \sum_\alpha \frac{(p_\alpha)^2}{2 M_\alpha}. \]

- **Harmonic oscillators:**
  \[ L_{\text{harmonic}}(q^\pi, \dot{q}^\pi) = \frac{1}{2} \sum_\pi \left[ M_\pi (q^\pi)^2 - (\omega_\pi)^2 (\dot{q}^\pi)^2 \right], \]
  \[ H_{\text{harmonic}}(q^\pi, p_\pi) = \frac{1}{2} \sum_\alpha \left[ \frac{(p_\pi)^2}{M_\pi} + (\omega_\pi)^2 (q^\pi)^2 \right]. \]

- **First class variables:**
  \[ L^{1\text{st}} = 0, \]
  \[ H_{T}^{1\text{st}}(p_\mu, u^\nu) = \sum_\mu u_\mu p_\mu = H_{E}^{1\text{st}}(p_\mu, u^\nu). \]

- **Second class variables:**
  \[ L^{2\text{nd}}(q^m) = -\frac{1}{2} \sum_m (q^m)^2, \]
  \[ H_{T}^{2\text{nd}}(q^m, p_\pi, u^\pi) = \frac{1}{2} \sum_\pi (p_\pi)^2 = H_{E}^{2\text{nd}}(q^m, p_\pi). \]

---

The present “paradigmatic” example has been inspired from the two examples given in the section 1.6.2 of [7] which correspond to \( L^{1\text{st}} \) and \( L^{2\text{nd}} \). The straightforward quantization procedure for these two cases is respectively carried on in the sections 13.1.1 and 13.1.2.
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