ON THE REGULARITY OF $D$-MODULES GENERATED BY RELATIVE CHARACTERS

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Abstract. Following the ideas of Ginzburg, for a subgroup $K$ of a connected reductive $\mathbb{R}$-group $G$ we introduce the notion of $K$-admissible $D$-modules on a homogeneous $G$-variety $Z$. We show that $K$-admissible $D$-modules are regular holonomic when $K$ and $Z$ are absolutely spherical. This framework includes: (i) the relative characters attached to two spherical subgroups $H_1$ and $H_2$, provided that the twisting character $\chi_i$ factors through the maximal reductive quotient of $H_i$, for $i = 1, 2$; (ii) localization on $Z$ of Harish-Chandra modules; (iii) the generalized matrix coefficients when $K(\mathbb{R})$ is maximal compact. This complements the holonomicity proven by Aizenbud–Gourevitch–Minchenko. The use of regularity is illustrated by a crude estimate on the growth of $K$-admissible distributions based on tools from subanalytic geometry.

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1. Introduction

Let $G$ be a connected reductive group over $\mathbb{R}$ and denote its opposite group by $G^\text{op}$. Differential equations with regular singularities have played an important role in representation theory of the Lie group $G(\mathbb{R})$. One significant example is Harish-Chandra’s study of invariant eigendistributions on $G(\mathbb{R})$, which includes the character

$$f \mapsto \Theta_\pi(f) := \text{tr} \pi(f), \quad f \in C^\infty_c(G(\mathbb{R}))$$

of an SAF representation $\pi$ of $G(\mathbb{R})$ as a typical case. Our terminology of SAF representation follows [6], meaning smooth admissible Fréchet of moderate growth. Another example is the study of asymptotics of the matrix coefficients $h_{\hat{v}, \cdot}v_{\cdot} \in C^\infty_c(G(\mathbb{R}))$ for some $v \in V, v \in V$. Generalizing the characters or matrix coefficients to the relative setting, one can also consider similar distributions on $Z(\mathbb{R})$, where $Z$ is an $\mathbb{R}$-variety with $Z(\mathbb{R}) \neq \emptyset$, homogeneous under right $G$-action, satisfying finiteness condition under some subgroup $K \subset G$ and the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. Of course, $Z$ and $K$ must be subject to some geometric conditions. It turns out that sphericity is a reasonable requirement. In this article, we say $Z$ is spherical if $Z_{\mathbb{C}} := Z \times_{\mathbb{R}} \mathbb{C}$ has an open dense orbit under any Borel subgroup of $G_{\mathbb{C}} := G \times_{\mathbb{R}} \mathbb{C}$, and we say $K \subset G$ is spherical if the homogeneous variety $K \backslash G$ is; this is also known as being absolutely spherical. We single out two motivating families of such distributions.

(1) The notion of matrix coefficients of $\pi$ generalizes to the relative case: given

$$\eta \in N_\pi := \text{Hom}_{G(\mathbb{R})}(\pi, C^\infty_c(Z(\mathbb{R})))$$

the space $\eta(V_{\pi})$ consists of $Z(\mathfrak{g})$-finite $C^\infty$-functions on $Z(\mathbb{R})$. Let $K = G^\theta$ where $\theta$ is a Cartan involution of $G$, so that $K(\mathbb{R})$ is maximal compact in $G(\mathbb{R})$. If we consider only $K(\mathbb{R})$-finite vectors in $\pi$, the generalized matrix coefficients are $\ell$-finite as well. These coefficients are the subject matter of relative harmonic analysis over $\mathbb{R}$; see [28] and the references therein. Unsurprisingly, differential equations with regular singularities entered there.

Note that the space $N_\pi$ differs from that in [31, §4.1] where one considered $C^\infty$ half-densities instead.

(2) Let $H_i \subset G$ be spherical subgroups and $\chi_i : H_i(\mathbb{R}) \to \mathbb{C}^\times$ be smooth characters ($i = 1, 2$). The relative characters are certain $Z(\mathfrak{g})$-finite distributions on $G(\mathbb{R})$ which are left $(H_1, \chi_1)$-equivariant and right $(H_2, \chi_2)$-equivariant. Specifically, Let $\phi_1$ (resp. $\phi_2$) be a continuous $(H_1(\mathbb{R}), \chi_1)$-equivariant (resp. $(H_2(\mathbb{R}), \chi_2^{-1})$-equivariant) linear functional of $V_{\pi}$ (resp. $V_{\pi}$). The corresponding relative character is
\[ \Theta_{\phi_1,\phi_2} : f \mapsto \langle \phi_1, \pi(f)\phi_2 \rangle, \quad f \in C_c^\infty(G(\mathbb{R})). \]

They appear in the local Archimedean components in the spectral side of relative trace formula. Endowing \( \mathbb{Z} := G \) with the right \( G^{\text{op}} \times G \)-action \( x \mapsto (a,b) \mapsto axb \) and taking \( K := H_1^{\text{op}} \times H_2 \subset G^{\text{op}} \times G \), we may regard relative characters as \( Z(\mathbb{R})_-(\chi_1, \chi_2^{-1}) \)-equivariant distributions on \( Z(\mathbb{R}) \), thereby fitting into the previous framework.

The modern theory of algebraic differential systems is phrased in terms of \( \mathcal{D} \)-modules. Any distribution on \( Z(\mathbb{R}) \) generates a \( \mathcal{D}_Z \)-module, and taking complexification yields a \( \mathcal{D}_Z \mathbb{C} \)-module. The \( \text{regular holonomic} \) \( \mathcal{D}_Z \)-modules generalize the systems with regular singularities, and they are related to perverse sheaves on complex analytic manifolds via the Riemann–Hilbert correspondence. For example, Harish-Chandra’s differential system for eigendistributions are studied in [22] from this perspective. In the recent work [3], the relative characters are shown to be holonomic. The matrix coefficients in the group case are also related to the wonderful compactifications in [5] using the language of \( \mathcal{D} \)-modules.

**Main results**

Let \( Z \) be a spherical homogeneous \( G \)-variety, \( Z(\mathbb{R}) \neq \emptyset \) and \( K \subset G \) be a spherical subgroup as alluded to above. We call a \( \mathcal{D}_Z \)-module regular holonomic if its complexification is. The aim of this article is to show that a large class of \( \mathcal{D}_Z \)-modules with suitable equivariant or monodromic structures are regular holonomic. This includes the \( \mathcal{D}_Z \)-modules generated by

1. the relative characters \( \Theta_{\phi_1,\phi_2} \) with \( Z = G \) and \( K = H_1^{\text{op}} \times H_2 \subset G^{\text{op}} \times G \),
assuming that the differential of \( \chi_i \) factors through the maximal reductive quotient of \( \mathfrak{h}_i \), for \( i = 1, 2 \);
2. the \( K(\mathbb{R}) \)-finite generalized matrix coefficients (with \( K = G^\theta \)) on \( Z(\mathbb{R}) \).

This strengthens the holonomicity of relative characters proven in [3]. Specifically, in Theorem 5.6 and Corollary 5.7, we will prove the regularity for \( K \)-admissible \( \mathcal{D}_Z \)-modules, as explicated below.

Let \( K \subset G \) be a subgroup. We say a character \( \chi : \mathfrak{k} \to \mathbb{C} \) between Lie algebras is *reductive* if \( \chi \) factors through the maximal reductive quotient of \( \mathfrak{k} \); we say a smooth character \( K(\mathbb{R}) \to \mathbb{C}^\times \) is reductive if its differential is reductive. A \( \mathcal{D}_Z \)-module \( \mathcal{M} \) is said to be \( K \)-admissible (Definition 5.3) if it is generated by a \( D_Z \)-module \( M \), where \( D_Z := \Gamma(Z, \mathcal{D}_Z) \), such that

- \( M \) is finitely generated over \( D_Z \),
- each element of \( M \) is \( Z(\mathfrak{g}) \)-finite, i.e., \( M \) is locally \( Z(\mathfrak{g}) \)-finite,
- \( \mathcal{M} \) carries a \((K, \chi)\)-monodromic structure (see Definition 2.2) for some reductive character \( \chi : \mathfrak{k} \to \mathbb{C} \).

The definition is global in the sense that it only depends on \( M \). The aforementioned monodromic structure can be regarded as a twisted variant of \( K \)-equivariance; see [4], [15]. If the \((K, \chi)\)-monodromic structure is weakened to local \( \mathfrak{k} \)-finiteness, the resulting notion is called \( \mathfrak{k} \)-admissibility (Definition 3.1).

The notion of \( K \)-admissibility is inspired by Ginzburg’s works [18, 19] which consider the case \( Z = G/K \) for a symmetric subgroup \( K \subset G \). One may imagine
that the present work is a direct generalization of loc. cit. to two spherical subgroups $H, K$ that are not necessarily equal nor symmetric. The regularity is obtained by the same strategy: we pass to a doubled basic affine space of $G$ via the horocycle transform (also known as Harish-Chandra transform), then apply the results à la Beilinson–Bernstein, for which we refer to [15, §2.5]. Nonetheless, there are also some differences.

- The holonomicity for $\mathfrak{k}$-admissible $\mathcal{D}_Z$-modules is proved in [18] by a parity argument for symmetric subgroups. We prove this in Corollary 3.9 for all spherical subgroups by applying the same criterion from loc. cit. twice, with the help of Springer resolutions. This is inspired by [3].
- We do not study the local characterization of admissible modules as done in [18, Thm. 1.4.2(ii)⇒(i)], so the analogues of [18, §3.4] are not needed.
- We work with $(K, \chi)$-monodromic $\mathcal{D}_Z$-modules (an extra structure on $\mathcal{D}$-modules), whereas [18] considered locally $\mathfrak{k}$-finite ones (a property of $\mathcal{D}$-modules). The permanence of $(K, \chi)$-monodromicity under various operations is easier to assure.
- The reductivity of $\chi$ is necessary in the proof; see Remark 5.2 and the discussion below on $\Theta_{\phi_1, \phi_2}$.

The result on regularity is directly applicable to relative characters whenever $\chi_i : H_i(\mathbb{R}) \to \mathbb{C}^\times$ is reductive for $i = 1, 2$. As for generalized matrix coefficients, we will actually prove that for any Harish-Chandra module $V$, its localization on $Z$

$$\text{Loc}_Z(V) := \mathcal{D}_Z \otimes_{U(g)} V$$

is generated by the $K$-admissible $D_Z$-module $D_Z \otimes_{U(g)} V$; here $\chi$ is the trivial character. Taking $V = V^K(\mathbb{R})_{\text{fini}}$ for some SAF representation $\pi$, the $K(\mathbb{R})$-finite generalized matrix coefficients of $\pi$ appear in subquotients of $\text{Loc}_Z(V)$, therefore generate regular holonomic submodules. For further discussions on the localization functor, we refer to [5].

Note that in the case of relative characters $\Theta_{\phi_1, \phi_2}$, the reductivity assumption on $\chi_1, \chi_2$ excludes the Whittaker-induced case, for example when $G$ is quasi-split, $H_1 = H_2 = U$ is maximal unipotent and $\chi_1 = \chi_2^{-1}$ is a non-degenerate character of $U(\mathbb{R})$; we refer to [20] for a description of the resulting Whittaker category of $D_G$-modules in terms of nil-Hecke algebras.

Without the reductivity of $\chi_1, \chi_2$, regularity fails according to the final paragraph of Example 9.4, and we can only conclude from $\mathfrak{h}_1^{\text{op}} \times \mathfrak{h}_2$-admissibility that $\Theta_{\phi_1, \phi_2}$ generates a holonomic $D_G$-module, which is already proven in [3].

Applications

We give only the simplest consequences of regularity to illustrate its usage. For results which can be deduced by holonomicity alone, we refer to [3].

- Functions, distributions or hyperfunctions (in Sato’s sense) on $Z(\mathbb{R})$ generating a regular holonomic $\mathcal{D}_Z$-module have a quite rigid structure; we refer to [14, III.1] [32, IX] for further discussions. Let us begin with the simplest properties.

(1) Suppose that a hyperfunction $u$ generates a regular holonomic $\mathcal{D}_Z$-module, for example when $D_Z \cdot u$ is a subquotient of a $K$-admissible module. First, by
holonomicity, there exists a Zariski-open dense $U \subset Z$ on which the $\mathcal{D}_Z$-module is an integrable connection. Then $u|_{U(\mathbb{R})}$ is analytic. In some cases it is easy to write $U$ down. This is indeed the case for Labesse’s twisted space (Example 3.10), which is the main subject of twisted harmonic analysis.

(2) Secondly, in this case it is well-known that $u$ is automatically a distribution; on the other hand, if $u$ is $C^\infty$ then it is automatically analytic (Theorem 9.2).

(3) Variant: Suppose that $K = G^\theta$ and the hyperfunction $u$ generates a subquotient of a $\mathfrak{t}$-admissible $D_Z$-module. Elliptic regularity theorem implies that $u$ is always analytic, even when $Z$ is non-spherical (Proposition 9.7).

The remaining applications concern growth estimates. It is well-known that $u$ is of at most polynomial growth on the smooth locus $U$, but we have to recast this into a convenient form. Definition–Proposition 7.4 provides a notion of moderate growth of $u$ at infinity. Loosely speaking, this means that $p^a u|_{U(\mathbb{R})} = O(1)$ for any reasonable function $p : U(\mathbb{R}) \to \mathbb{R}_{>0}$ that decays to zero at infinity, where $a > 0$ depends on $p$ and $u$; the “infinity” here is defined using any smooth compactification $U \hookrightarrow X$. After reducing to the case where $X \setminus U$ has normal crossings, the moderate growth at infinity for $u$ (Theorem 8.4) follows readily from the standard estimates from Deligne [14]. A flexible framework for such arguments is provided by subanalytic geometry, in particular by Lojasiewicz’s inequality recalled in Theorem 6.4.

The weakness of these growth estimates is the implicit exponent $a$. Take the character $\Theta_\pi$ of an SAF representation $\pi$ for example. Our general result asserts that $|D^G|^{a} \Theta_\pi$ is locally bounded, where $D^G$ is the Weyl discriminant on $G$; on the other hand, Harish-Chandra obtained this for $a = 1/2$.

When applied to generalized matrix coefficients, our “soft” method furnishes an estimate that is akin to [28, Thm. 7.2], but without any information on the exponent; see Theorem 10.5 and Corollary 10.6. Since those results are also easy consequences of the moderate growth of SAF representations, we omit their proofs.

Incidentally, we also prove in Proposition 10.2 that $\text{Hom}_\mathbb{C}(V/\mathfrak{h}V, \mathbb{C})$ is finite-dimensional whenever $H \subset G$ is a spherical subgroup and $V$ is a Harish-Chandra module. It implies that $N_\pi$ is finite-dimensional. Although these results have been proven in stronger forms, see [2], our regularity-based proof can express $\text{Hom}_\mathbb{C}(V/\mathfrak{h}V, \mathbb{C})$ in terms of the stalks of the solution complex of $\text{Loc}_Z(V)$.

Structure of this article

The first part of this article aims at regularity. In §2 we collect and review the required notions of monodromic $\mathcal{D}$-modules from [4, 15], together with several instances for later use. In §3, the notion of $\mathfrak{t}$-admissible modules is defined, and we show their holonomicity when $\mathfrak{t}$ is a spherical subalgebra, by invoking Ginzburg’s criterion. §4 is a review of the horocycle correspondence, which is used in §5 to prove the regularity of $K$-admissible modules.

The second part concerns applications. §6 and §7 introduce some vocabularies from subanalytic geometry, in order to state the notion of moderate growth at infinity. This is then related to solutions of regular holonomic systems in §8, following Deligne’s work. §9 presents some immediate applications of regularity to harmonic analysis, including the basic examples and an estimate on admissible
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Conventions

- Real manifolds in this article are equi-dimensional, but need not be connected. Unless otherwise specified, $C^\infty$ functions on a real manifold and continuous functions on a topological space are $\mathbb{C}$-valued.

The dual of a vector space $V$ is denoted by $V^\vee$. The underlying vector space of a representation $\pi$ is denoted as $V_\pi$.

When it is necessary to distinguish the derived functors from their non-derived versions, or to indicate their cohomologies, we use the prefix $L$ (resp. $R$) to denote the left (resp. right) derived ones, such as $R\mathcal{H}om$.

- Let $A$ be a ring, or more generally a ring object in a topos. We denote by $A$-$\text{Mod}$ the category of left $A$-modules. For any $A$-module $M$, write $\text{Sym}(M)$ and $\wedge M$ for its symmetric and exterior algebras, respectively. When $A$ is an algebra over a field $k$, we say $M$ is locally finite under $A$ if every $m \in M$ is contained in an $A$-submodule which is finite-dimensional over $k$.

- Let $k$ be a field. By a $k$-variety we mean an integral, separated scheme of finite type over $\text{Spec}(k)$. If $k'$ is an extension of $k$, we write $Z_{k'} := Z \otimes_k k'$ for any $k$-variety $Z$. The set of $k$-points of $Z$ is denoted by $Z(k)$. The sheaf of regular functions is denoted by $\mathcal{O}_Z$, and $\mathcal{O}_{Z,x}$ is the local ring at $x$.

The cotangent bundle of a smooth $k$-variety $Z$ is denoted by $T^*Z$. For a subvariety $W \subset Z$ we denote by $T^*_WZ$ its conormal bundle.

If $Z$ is a $\mathbb{C}$-variety, $Z^{an}$ will denote its analytification; likewise for $\mathcal{O}_Z$-modules.

- Group objects in the category of $k$-varieties are called $k$-groups. Subgroups of $k$-groups are understood as closed $k$-subgroups. The opposite group (resp. derived subgroup, identity connected component, unipotent radical) of $G$ is denoted by $G^{op}$ (resp. $G_{\text{der}}$, $G^\circ$, $R_u(G)$); the same convention on $G^\circ$ pertains to Lie groups as well.

For any affine $k$-group $H$, define the additive groups $X^*(H) := \text{Hom}(H, \mathbb{G}_m)$ and $X_*(H) := \text{Hom}(\mathbb{G}_m, H)$. For $\tau \in \text{Aut}(G)$, denote by $G^\tau$ the fixed locus of $\tau$ in $G$.

- Unless otherwise specified, $k$-groups act on $k$-varieties on the right, written as $(x, g) \mapsto xg$; accordingly, groups and Lie algebras act on the left of function spaces. The stabilizer of a point $x$ under $G$ is denoted by $\text{Stab}_G(x)$. When an affine $k$-group $G$ acts on a normal $k$-variety $Z$, we say $Z$ is a $G$-variety; when $G$ acts transitively, $Z$ is said to be a homogeneous $G$-variety. If $H \subset G$ is a subgroup and $X$ is an $H$-variety, we define the quotient

$$X^H := X \times G/(xh, h^{-1}g) \sim (x, g), \quad h \in H$$
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which exists as a G-variety under mild conditions; see [37, Thm. 2.2]. Denote the image of \((x, g)\) in \(X \times G\) as \([x, g]\).

- The Lie algebra of a \(k\)-group is denoted as \(\mathfrak{g} := \text{Lie} G\), and its dual by \(\mathfrak{g}^*\). The center of the universal enveloping algebra \(U(\mathfrak{g})\) is denoted by \(Z(\mathfrak{g})\). By a character of \(\mathfrak{g}\), we mean a homomorphism of Lie algebras \(\mathfrak{g} \to k\) (i.e., homomorphisms of \(k\)-algebras \(U(\mathfrak{g}) \to k\)), which is automatically zero on \([\mathfrak{g}, \mathfrak{g}]\). A character of \(\mathfrak{g}\) is called reductive if it factors through the maximal reductive quotient. The adjoint action of \(G\) on \(\mathfrak{g}, \mathfrak{g}\) or \(G\) itself is denoted as \(\text{Ad}\).

\(\text{For a field } k\text{ of characteristic zero and a smooth } k\text{-variety, } \mathcal{D}_Z\text{ denotes the sheaf (actually étale-local) of algebraic differential operators on } Z, \text{ and } D_Z := \Gamma(Z, \mathcal{D}_Z); \text{ the stalk at } x \text{ is denoted by } D_{Z,x}. \text{ The same rule applies to modules: } \mathcal{D}_Z\text{-modules will be denoted by symbols like } \mathcal{M}, \text{ and } D_Z\text{-modules by } M, \text{ and so forth. We will only consider left } D_Z\text{-modules.}

\text{Integrable connections will be understood in the algebraic sense. The analytiﬁcation of a } \mathcal{D}_Z\text{-module } \mathcal{M} \text{ is written as } \mathcal{M}^{an}. \text{ For ane } Z, \text{ we will switch freely between } \mathcal{D}_Z\text{-modules and } D_Z\text{-modules by using the functor } (Z, \cdot).\n
2. Equivariant and monodromic \(\mathcal{D}\)-modules

Let \(k\) be a field of characteristic zero with algebraic closure \(\overline{k}\). For a smooth \(k\)-variety \(Z\), the formation of \(\mathcal{D}_Z\) and \(D_Z := \Gamma(Z, \mathcal{D}_Z)\) is compatible with change of base field \(k\), and we will mostly be concerned with the case when \(k = \overline{k}\).

\textbf{Example 2.1.} Let \(k = \mathbb{R}\) and let \(Z\) be a smooth \(\mathbb{R}\)-variety. In this case \(Z(\mathbb{R})\) is Zariski-dense in \(Z\) if it is nonempty; see [33, 1.A]. Therefore \(Z\) is \(\mathbb{R}\)-dense in the sense of [27]. Any \(C^\infty\)-function \(u : Z(\mathbb{R}) \to \mathbb{C}\) generates a \(\mathcal{D}_Z\)-module \(\mathcal{D}_Z \cdot u\), which in turn gives a \(\mathcal{D}_{Z,\mathbb{C}}\)-module by base change. The same holds for distributions, or more generally for hyperfunctions on \(Z(\mathbb{R})\).

\text{Let } G \text{ be an ane } k\text{-group and } Z \text{ be a } G\text{-variety. Consider the action morphism } a : Z \times G \to Z, \text{ the projection } pr_1 : Z \times G \to Z, \text{ the morphisms}

\[ p_0, p_1, p_2 : Z \times G \times G \to Z \times G, \quad p_0(x, g, h) = (xg, h), \quad p_1(x, g, h) = (x, gh), \quad p_2(x, g, h) = (x, g), \]

and \(i : Z \to Z \times G\) given by \(i(x) = (x, 1)\). We have \(ap_1 = ap_0, \text{ pr}_1p_1 = \text{ pr}_1p_2, \text{ pr}_1p_0 = ap_2\) and \(ai = \text{ pr}_1i = \text{id}_Z\). The following notions are well-known, see, e.g., [4, 1.8.5] or [15, 2.5].

\textbf{Definition 2.2.} Let \(Z\) be a smooth \(G\)-variety. The \(G\)-action on \(Z\) induces a homomorphism \(U(\mathfrak{g}) \to D_Z\) of \(k\)-algebras. Consider a \(\mathcal{D}_Z\)-module \(\mathcal{M}\).

\(1\) We say that \(\mathcal{M}\) is \(G\)-equivariant, if it is endowed with an isomorphism of \(\mathcal{D}_Z\times_G\)-modules

\[ \varphi : a^*\mathcal{M} \overset{\sim}{\to} \text{pr}_1^*\mathcal{M} = \mathcal{M} \boxtimes \mathcal{O}_G \]
subject to the cocycle condition that

\[\begin{array}{c}
p_1^*a^*\mathcal{M} \\ \Downarrow \varphi \\
p_0^*a^*\mathcal{M}
\end{array} \xrightarrow{\sim} \begin{array}{c}
p_1^*p_1^*\mathcal{M} \\ \Downarrow \\
p_0^*p_1^*\mathcal{M}
\end{array} \]

are commutative diagrams.

(2) Let $\chi : \mathfrak{g} \to \mathbb{k}$ be a character of Lie algebras; let $\mathcal{O}_{G,\chi}$ be the trivial line bundle $\mathcal{O}_G$ equipped with the integrable connection $\nabla_{\theta u} = \theta u - \chi(\theta)u$ for all $\theta \in \mathfrak{g}$, viewed as a right invariant vector field. Then $\mathcal{O}_{G,\chi}$ is a $\mathcal{D}_G$-module: $\theta$ maps $f \in \mathcal{O}_{G,\chi}$ to $\theta f$ (the usual derivative in $\mathcal{O}_G$) plus $\chi(\theta)f$. We say that $\mathcal{M}$ is $(G,\chi)$-monodromic if it is endowed with an isomorphism of $\mathcal{D}_{Z \times G}$-modules

$\varphi : a^*\mathcal{M} \xrightarrow{\sim} \mathcal{M} \otimes \mathcal{O}_{G,\chi}$

subject to cocycle condition. For trivial $\chi$ we recover the notion of $G$-equivariance.

(3) We say that $\mathcal{M}$ is weakly $G$-equivariant, if the $\varphi$ above is only an isomorphism of $\mathcal{D}_Z \otimes \mathcal{O}_{G}$-modules.

Note that if $\chi$ lifts to a character $\tilde{\chi} : G \to \mathbb{G}_m$, we have $\mathcal{O}_{G,\chi'} \xrightarrow{\sim} \mathcal{O}_{G,\chi + \chi'}$ by $f \mapsto \tilde{\chi}f$ for any $\chi'$.

The $G$-equivariant (resp. weakly $G$-equivariant, $(G,\chi)$-monodromic) $\mathcal{D}_Z$-modules form an abelian category for any given $\chi$: the morphisms are required to respect $\varphi$.

If $Z = \{\text{pt}\}$, a $G$-equivariant (resp. weakly $G$-equivariant) $\mathcal{D}_Z$-module is nothing but a locally finite algebraic representation of $\pi_0(G)$ (resp. $G$) over $\mathbb{k}$.

In concrete terms, if $\mathcal{M}$ is viewed just as a quasi-coherent sheaf on $Z$, then $\varphi : a^*\mathcal{M} \xrightarrow{\sim} \mathcal{p}_1^*\mathcal{M}$ with cocycle conditions encodes a $G$-equivariance structure on $\mathcal{M}$; see [35, 38.12]. As a $\mathcal{D}_Z$-module, weak equivariance means that the $G$-action on $\mathcal{M}$ is compatible with the $G$-action on $\mathcal{D}_Z$, namely the transport of structure $D \xrightarrow{g} g^{-1}Dg$; here $g \in G$ acts on $\mathcal{D}_Z$ by the regular representation $f(x) \mapsto f(xg)$, so $g^{-1}Dg$ makes sense in $\mathcal{D}_Z$.

Equivariance as a $\mathcal{D}_Z$-module means that the $U(\mathfrak{g})$-action on $\mathcal{M}$ given by $U(\mathfrak{g}) \to \mathcal{D}_Z$ coincides with that given by the $G$-action on $\mathcal{M}$, i.e., $\varphi$ is also $\mathcal{D}_G$-linear.

**Lemma 2.3.** Suppose $\mathcal{M}$ is $(G,\chi)$-monodromic for some $\chi$. Then for each open subset $U \subset Z$ and each $s \in \Gamma(U,\mathcal{M})$, the $\mathbb{k}$-vector space $U(\mathfrak{g}) \cdot s$ is finite-dimensional.

**Proof.** Using the isomorphism $\varphi$ above, the required $U(\mathfrak{g})$-finiteness property is transferred to the case of local sections of $\mathcal{O}_{G,\chi}$ under $\nabla$, which is evident. \(\square\)

We present several examples for later use.
Example 2.4 (Function spaces). Take \( k = \mathbb{R} \) and let \( Z \) be a smooth \( K \)-variety where \( K \) is an affine \( \mathbb{R} \)-group. Let \( V \) be a \( \mathbb{C} \)-vector space of \( C^\infty \) functions on \( Z(\mathbb{R}) \). Suppose that

- \( V \) is stable under the regular representation \( u(x) \mapsto u(xk) \) of \( K(\mathbb{R}) \) on \( C^\infty(Z(\mathbb{R})) \);
- the \( K(\mathbb{R}) \)-representation \( V \) extends to a locally finite, algebraic representation of \( K(\mathbb{C}) \) on \( V \).

Then the \( \mathcal{D}_{Z\mathbb{R}} \)-module generated by \( V \) is \( K \)-equivariant. To see this, we use the "concrete" interpretation of equivariance. First, the \( K(\mathbb{R}) \)-action on \( V \) is clearly compatible with its action on \( \mathcal{D}_{Z\mathbb{R}} \) by transport of structure. The \( k \)-actions from \( U(k) \to \mathcal{D}_{Z\mathbb{R}} \) and that from the action on \( V \) also coincide, for similar reason. These compatibilities extend algebraically to \( K(\mathbb{C}) \) since the \( K(\mathbb{R}) \)-representation on \( V \) extends. The formula for \( \varphi \) reads:

\[
\varphi(u(x)) = \sum_{i=1}^{m} u_i(x) f_i(k) \implies P \cdot a^*(u) \mapsto P \cdot \sum_{i=1}^{m} u_i(x) \otimes f_i(k) \quad (2.1)
\]

where \( P \in \mathcal{D}_{Z\times K} \), \( u, u_i \in V \) and \( f_i \in \mathcal{O}_K \).

The same holds for distributions and hyperfunctions on \( Z(\mathbb{R}) \) as well.

Remark 2.5. Here is a typical application of Example 2.4: \( G \) is a connected reductive \( \mathbb{R} \)-group, \( Z \) is a smooth \( G \)-variety, \( K = G^\theta \) for some Cartan involution \( \theta \) of \( G \), and \( V \subset C^\infty(Z(\mathbb{R})) \) is an admissible \((g, K(\mathbb{R}))\)-module with respect to the regular representation \( f(x) \mapsto f(xg) \) of \( G(\mathbb{R}) \) on \( C^\infty(Z(\mathbb{R})) \). In this case, extendibility of \( V \) to a locally finite algebraic \( K(\mathbb{C}) \)-representation follows from Weyl’s unitarian trick, as locally finite representations of \( K(\mathbb{R}) \) are algebraic. Again, the same holds for distributions and hyperfunctions.

Example 2.6 (Relative invariants). Let \( Z \) be a smooth \( K \)-variety as in Example 2.4. Let \( u \) be a \( C^\infty \)-function (or distribution, hyperfunction) on \( Z(\mathbb{R}) \) and let \( \chi: \mathfrak{k} \to \mathbb{C} \) be a character, satisfying

\[
\forall \theta \in \mathfrak{k}, \quad \theta \cdot u = \chi(\theta) u.
\]

Then \( u \) generates a \((K, \chi)\)-monodromic \( \mathcal{D}_{Z\mathbb{R}} \)-module. Specifically, one takes the isomorphism \( \varphi \) to be

\[
P \cdot a^*(u) \mapsto P \cdot (u \otimes 1), \quad P \in \mathcal{D}_{Z\times G}.
\]

Example 2.4 is not applicable to this scenario when \( \chi \) does not come from a character \( K \to \mathbb{G}_m \), for example when \( K \) is unipotent and \( \chi \) is nontrivial. Even when \( \chi \) lifts, the formula above differs from (2.1) by the character of \( K \).

Example 2.7 (Localizations). Let \( Z \) be a \( G \)-variety over a field \( k \) of characteristic zero, and \( K \) be a subgroup of \( G \), so that the notion of \((g, K)\)-module is defined. The localization functor (non-derived) is

\[
\text{Loc}_Z : U(g)\text{-Mod} \to \mathcal{D}_{Z\mathbb{C}}\text{-Mod}, \quad W \mapsto \mathcal{D}_{Z\mathbb{C}} \otimes_{U(g)} W.
\]
When $V$ is a $(g, K)$-module, $\text{Loc}_Z(V)$ acquires a weakly $K$-equivariant structure by letting $k \in K$ act via

$$k \cdot (P \otimes v) = kPk^{-1} \otimes kv, \quad P \in \mathcal{D}_Z, \ v \in V.$$  

This is readily seen to be well-defined. It is actually equivariant: the $K$-action induces an $\mathfrak{k}$-action on $\text{Loc}_Z(V)$, which is

$$P \otimes v \mapsto (\theta P - P\theta) \otimes v + P \otimes (\theta v) = (\theta P) \otimes v$$

for all $\theta \in \mathfrak{k}$ and $P \otimes v \in \text{Loc}_Z(V)$.

Another perspective on monodromic modules from [4, 2.5] will be needed. Assume henceforth $k = \overline{k}$. Let $T$ be a $k$-torus and $\pi : \tilde{X} \to X$ be a $T$-torsor; $X$ is smooth. Put $\mathcal{D} := (\pi_* \mathcal{D}_{\tilde{X}})^T$.

(1) For any ideal $a$ of $\text{Sym}(t)$ and any $\mathcal{D}$-module $\mathcal{M}$, write $\mathcal{M}[a] \subset \mathcal{M}$ for the subsheaf annihilated by $a$, which is seen to be a $\mathcal{D}$-submodule. Every $\xi \in t^*$ corresponds to a maximal ideal $m_\xi \subset \text{Sym}(t)$, and we write

$$\mathcal{M}_\xi := \mathcal{M}[m_\xi], \quad \mathcal{M}_\xi^\circ := \bigcup_{n \geq 1} \mathcal{M}[m_\xi^n].$$

Define $\mathcal{M}_\text{fin} := \bigcup_a \mathcal{M}[a]$ where $a$ ranges over the ideals of finite codimension. Then $\mathcal{M}_\text{fin} = \bigoplus_{\xi} \mathcal{M}_\xi$.

(2) Since $\pi$ is affine, the study of $\mathcal{D}_{\tilde{X}}$-modules is the same as that of $\pi_* \mathcal{D}_{\tilde{X}}$-modules. Let $t^*_Z \subset t$ be the lattice of characters from $X^*(T)$. For any ideal $a \subset \text{Sym}(t)$ and a $\pi_* \mathcal{D}_{\tilde{X}}$-module $\mathcal{N}$, we define the submodule

$$\mathcal{N}[\overline{a}] := \sum_{\xi \in t^*_Z} \mathcal{N}[\xi^* a],$$

where

$$\xi^* \in \text{Aut}_k(\text{Sym}(\mathfrak{h})) : \mathfrak{h} \ni \chi \mapsto \xi + \xi(\chi).$$

The same recipe above yields, for each $\overline{\xi} \in t^*/t^*_Z$ one defines,

$$\mathcal{N}_{\overline{\xi}} \subset \mathcal{N}_{\overline{\xi}} \subset \mathcal{N}_\text{fin} := \bigcup_{a: \text{ideal codim} \ll \infty} \mathcal{N}[\overline{a}].$$

Fix $\xi \in t^*$ and let $\overline{\xi}$ be its class modulo $t^*_Z$. We are interested in the modules $\mathcal{M}$ (resp. $\mathcal{N}$) satisfying

$$\mathcal{M} = \mathcal{M}_\text{fin}, \quad \mathcal{M} = \mathcal{M}_{\overline{\xi}}, \quad \text{or} \quad \mathcal{M} = \mathcal{M}_\xi \quad (\text{resp.} \ \mathcal{N} = \mathcal{N}_\text{fin}, \text{etc.})$$
By [4, 2.5.3 and 2.5.4], this gives rise to a diagram of abelian (sub)categories

\[ \mathcal{D}_X - \text{Mod}_\xi \subset \mathcal{D}_X - \text{Mod}_\xi \subset \mathcal{D}_X - \text{Mod}_{\text{fin}} = \prod_{\eta} \mathcal{D}_X - \text{Mod}_\eta \]

\[ \pi^{-1} \downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \]

\[ (\pi_* \mathcal{D}_X) - \text{Mod}_\xi \subset (\pi_* \mathcal{D}_X) - \text{Mod}_\xi \subset (\pi_* \mathcal{D}_X) - \text{Mod}_{\text{fin}} = \prod_{\eta} (\pi_* \mathcal{D}_X) - \text{Mod}_\eta \]  \tag{2.2}

\[ (\mathcal{D} - \text{Mod}_\xi) \otimes \mathcal{D} \downarrow \rho_\xi \quad \Downarrow \quad \Downarrow \quad \Downarrow \]

\[ \mathcal{D} - \text{Mod}_\xi \subset \mathcal{D} - \text{Mod}_\xi \subset \mathcal{D} - \text{Mod}_{\text{fin}} = \prod_{\eta} \mathcal{D} - \text{Mod}_\eta \]

in which:

- the categories in the last two rows have just been defined;
- the pair \((\pi^{-1}, \pi_*)\) realizes an equivalence between \(\mathcal{D}_X - \text{Mod}\) and \((\pi_* \mathcal{D}_X) - \text{Mod}\), and this defines the categories in the first row;
- the “induction” functor \((\pi_* \mathcal{D}_X) \otimes -\) also turns out to give equivalences

\[ (\pi_* \mathcal{D}_X) - \text{Mod}_\xi \rightarrow \mathcal{D} - \text{Mod}_\xi \text{ and } (\pi_* \mathcal{D}_X) - \text{Mod}_\xi \rightarrow \mathcal{D} - \text{Mod}_\xi, \]

with quasi-inverses

\[ \rho_\xi : \mathcal{N} \mapsto \bigcup_{n \geq 1} \mathcal{N}[m_\xi^n], \quad \rho_\xi : \mathcal{N} \mapsto \mathcal{N}[m_\xi] \]

respectively.

Let us link the first and the third rows in (2.2). According to [4, 1.8.9], the inclusions \(\mathcal{O}_X \hookrightarrow \mathcal{D}_X\) and \(\mathcal{O}_X \hookrightarrow \mathcal{D}\) induce

\[ \mathcal{O}_X \otimes \pi^{-1} \mathcal{M} \cong \mathcal{D}_X \otimes \pi^{-1} \mathcal{M} \cong \pi^{-1} \left( \pi_* \mathcal{D}_X \otimes \mathcal{M} \right), \quad \mathcal{M} \in \mathcal{D} - \text{Mod}. \]  \tag{2.3}

Furthermore, \(\mathcal{D} - \text{Mod}_\xi\) is equivalent to \(\mathcal{D}_\xi - \text{Mod}\), where \(\mathcal{D}_\xi := \mathcal{D}/m_\xi \mathcal{D}\) is the sheaf on \(X\) of locally trivial twisted differential operators (TDO’s) associated with \(\xi \in t^*\); see [4, 2.1]. In this article, we prefer to connect \(\mathcal{D} - \text{Mod}_\xi\) to \((T, \xi)\)-monodromic \(\mathcal{D}_X\)-modules as follows.

**Proposition 2.8.** Fix \(\xi \in t^*\). Every object \(\mathcal{N}\) of \(\mathcal{D}_X - \text{Mod}_\xi\) carries a canonical \((T, \xi)\)-monodromic structure. This realizes an equivalence of abelian categories

\[ \{(T, \xi)\text{-monodromic modules}\} \cong \mathcal{D} - \text{Mod}_\xi \cong \mathcal{D}_\xi - \text{Mod}. \]

**Proof.** Our routine arguments are built on the mutually quasi-inverse functors

\[ \pi^{-1} \left( \pi_* \mathcal{D}_X \otimes - \right) : \mathcal{D} - \text{Mod} \leftrightarrow \left\{ \text{weakly } T\text{-equivariant } \mathcal{D}_X\text{-modules} \right\} : \pi_*(\mathcal{D}_X) \rightarrow \mathcal{D} - \text{Mod} \]
This is the content of [4, 1.8.10], where the weakly $T$-equivariant modules are called weak $(\mathcal{D}_X, T)$-modules (see 1.8.5 of loc. cit.).

Define the action and projection morphisms $a, \text{pr}_1 : \tilde{X} \times T \to \tilde{X}$. Let $\mathcal{N}$ be realized as $\mathcal{O}_{\tilde{X}} \pi^{-1} \mathcal{M}$ via (2.3), where $\mathcal{M}$ is a $\mathcal{D}$-module. As $\pi a = \pi \text{pr}_1$, the isomorphism $\varphi$ in Definition 2.2 can be explicitly given using

$$a^*\mathcal{N} \simeq \mathcal{O}_{\tilde{X}} \pi^{-1} \otimes \mathcal{O}_{\tilde{X}} \pi^{-1} \mathcal{M},$$

$$\text{pr}_1^*\mathcal{N} \simeq \mathcal{O}_{\tilde{X}} \pi^{-1} \otimes \mathcal{O}_{\tilde{X}} \pi^{-1} \mathcal{M}$$

$$= \pi^{-1} \mathcal{M} \otimes \mathcal{O}_T.$$

We have to show that weak equivariance structure is $(T, \xi)$-monodromic when $\mathcal{M} \in \tilde{\mathcal{D}}\text{-Mod}_\xi$ through the isomorphisms above. We have $\mathcal{D}_T = \mathcal{O}_T \cdot \text{Sym}(t)$. Given $\theta \in t$, it acts on $a^*\mathcal{N}$ by Leibniz rule (see [23, §1.3]); the result is the sum of

1. the effect of $\theta$ on $\mathcal{O}_{\tilde{X}} \pi^{-1} \mathcal{M}$ through the second slot,
2. the effect on $a^{-1} \pi^{-1} \mathcal{M}$: note that $\theta$ induce an operator in $\mathcal{D}_X$ which actually comes from $\pi^{-1} \tilde{\mathcal{D}}$, so the $\theta$-action on $a^{-1} \pi^{-1} \mathcal{M}$ equals the scalar $\xi(\theta)$.

The same applies to the $\theta$-action on $\text{pr}_1^*\mathcal{N}$, except that the effect on $\text{pr}_1^* \pi^{-1} \mathcal{M}$ is trivial. To make $\varphi : a^*\mathcal{N} \to \mathcal{N} \otimes \mathcal{O}_{T, \xi}$ commute with $\mathcal{D}_T$-action, one replaces the $\mathcal{O}_T$ in $\text{pr}_1^*\mathcal{N}$ by $\mathcal{O}_{T, \xi}$.

Conversely, consider a $(T, \xi)$-monodromic $\mathcal{N}$. Being weakly $T$-equivariant, it is canonically isomorphic to $\pi^{-1} \left( \pi_* \mathcal{D}_X \otimes \mathcal{M} \right)$ where $\mathcal{M} := \pi_*(\mathcal{N})^T$ is a $\tilde{\mathcal{D}}$-module. Let $\theta := \gamma'(1) \in t$ where $\gamma \in X_*(T)$. The $\theta$-action on $\mathcal{M} \subset \pi_*\mathcal{N}$ is determined from $\mathcal{N}$: it is the sum of

1. the derivative at $t = 1$ of the $\gamma(t)$-action, which is 0 since $\mathcal{M} = \pi_*(\mathcal{N})^T$, and
2. the scalar multiplication by $\xi(\theta)$, since $\mathcal{N}$ is monodromic.

By varying $\theta$ (or $\gamma$), we see $\mathcal{M} = \mathcal{M}_\xi$, hence $\mathcal{M} \in \tilde{\mathcal{D}}\text{-Mod}_\xi$ as required. Finally, the equivalence with $\mathcal{D}_\xi\text{-Mod}$ has already been remarked.

Now consider a general field $\mathbb{k}$ of characteristic zero and an affine $\mathbb{k}$-group $G$.

**Definition 2.9.** For a smooth $G$-scheme $Z$ and a character $\chi : \mathfrak{g} \to \mathbb{k}$, we denote the bounded equivariant derived category of $(G, \chi)$-monodromic $\mathcal{D}_Z$-modules as $\mathcal{D}_G^{b, \chi}(Z)$: this is a triangulated category with a $t$-structure satisfying the following properties.

- The heart of $\mathcal{D}_G^{b, \chi}(Z)$ is equivalent to the abelian category of $(G, \chi)$-monodromic $\mathcal{D}_Z$-modules.
- For any subgroup $L \subset G$ with $\eta := |t|$, we have the forgetful functor $\text{obl} : \mathcal{D}_G^{b, \chi}(Z) \to \mathcal{D}_{L, \eta}^{b, \chi}(Z)$.
- The functors $\text{obl}$ are $t$-exact, and induce the usual forgetful functors on cohomologies (i.e., forgetting the monodromic structure).
The usual operations on complexes of \( D \)-modules (such as \( f^! \), \( f_* \), etc.) relative to \( G \)-equivariant morphisms lift to the monodromic setting, with the caveat that \((G, \chi)\) and \((G, -\chi)\) are exchanged under duality (cf. Definition 2.2); we will not make direct use of the duality functor. All these operations commute with forgetful functors. The usual adjunction relations also hold in this generality.

When \( \chi \) is trivial, we obtain the \( G \)-equivariant derived category \( \mathcal{D}^b_{\mathcal{G}}(Z) \) and the forgetful functor to \( \mathcal{D}^b(Z) \). When \( G = \{1\} \), we recover \( \mathcal{D}^b(Z) \).

A few remarks are in order. The classical accounts on \( D \)-modules often impose quasi-projectivity on the varieties. This constraint can be safely removed in view of recent theories, such as that of crystals [16]; see also [17, Chap. 4]. When \( \chi \) is trivial, \( \mathcal{D}^b_{\mathcal{G}}(Z) \) is originally defined in [7, §4], and the six operations in this framework are in [7, Thm. 3.4.1]. This theory can also be understood in terms of \( D \)-modules (more accurately: crystals) on quotient stacks \([Z/G]\), within the formalism of stable \( \infty \)-categories. Passing to the homotopy category yields the required equivariant derived categories.

For example, \( \text{oblv} : \mathcal{D}^b_{\mathcal{G}}(Z) \to \mathcal{D}^b(Z) \) is “locally the same” as pull-back of \( D \)-modules along various \( G \)-torsors \( P \to S \) such that \( P \) maps equivariantly to \( Z \); this operation is clearly \( t \)-exact and induces the usual pull-back on cohomologies since \( G \) is smooth. The case with nontrivial \( G \)-monodromy \( \chi \) is explained in [16, §6.5] by employing the formalism of TDO’s, and this includes our setting of \( (G, \chi) \)-monodromic modules by Proposition 2.8, by considering the \( G/G_{\text{der}} \)-torsor \([Z/G_{\text{der}}] \to [Z/G]\).

We will only make mild use of equivariant derived categories as a blackbox in §5, and leave these issues aside.

3. \( \mathfrak{t} \)-admissible \( D \)-modules: holonomicity

Fix an algebraically closed field \( k \) of characteristic zero and a connected reductive \( k \)-group \( G \). In what follows, \( Z \) will be a homogeneous \( G \)-variety over \( k \).

**Definition 3.1 (V. Ginzburg [18, Def. 1.2]).** Let \( K \) be a subgroup of \( G \). A \( D_Z \)-module \( M \) is called \( \mathfrak{t} \)-admissible if

- \( M \) is finitely generated over \( D_Z \);
- for every \( m \in M \), the dimensions of \( U(\mathfrak{t}) \cdot m \) and \( \mathcal{Z}(\mathfrak{g}) \cdot m \) are both finite — in other words, \( M \) is locally finite under \( U(\mathfrak{t}) \) and \( \mathcal{Z}(\mathfrak{g}) \).

Denote the \( \mathcal{D}_Z \)-module generated by \( M \) as \( \mathcal{M} \). Quotients and finitely generated submodules of a \( \mathfrak{t} \)-admissible \( D_Z \)-module are still admissible.

**Remark 3.2.** The definition of \( \mathfrak{t} \)-admissibility is of a global nature. In loc. cit., \( Z \) is assumed to be affine so that the global sections functor \( \Gamma : \mathcal{D}_Z\text{-Mod} \to D_Z\text{-Mod} \) is an equivalence. Our \( \mathcal{D}_Z \)-modules \( \mathcal{M} \) are globally generated by construction, and the properties of \( \mathcal{M} \) such as holonomicity, etc. will be accessed through \( M \).

**Remark 3.3.** A \( D_Z \)-module \( M \) is \( \mathfrak{t} \)-admissible if and only if \( M \) is generated by a \( k \)-subspace \( M_0 \) such that \( M_0 \) is finite-dimensional and closed under the actions of both \( U(\mathfrak{t}) \) and \( \mathcal{Z}(\mathfrak{g}) \).
Consider the cotangent bundle $T^* \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}$. Every $v \in \mathfrak{g}$ induces a vector field $\xi_v$ on $\mathbb{Z}$. One can evaluate $\xi_v$ at any point of $T^* \mathbb{Z}$, giving rise to the moment map

$$\mu = \mu_{\mathfrak{g}} : T^* \mathbb{Z} \to \mathfrak{g}^*.$$ 

Recall that $T^* \mathbb{Z}$ carries a natural symplectic structure and a $G$-action. For a smooth $k$-variety $X$ carrying a symplectic structure and a closed subvariety $Y \subset X$, we say $Y$ is co-isotropic (resp. isotropic, Lagrangian) if $T_y Y$ is a co-isotropic (resp. isotropic, Lagrangian) subspace of $T_y X$, at every smooth point $y$ of $Y$. For example, the conormal bundle $T^*_W \mathbb{Z}$ is Lagrangian in $T^* \mathbb{Z}$ for any locally closed $W \subset \mathbb{Z}$. The characteristic variety $\text{Ch}(\mathcal{M})$ of any coherent $D_{\mathbb{Z}}$-module $\mathcal{M}$ is a conic, co-isotropic closed subvariety of $T^* \mathbb{Z}$ (see [23, Thm. 2.3.1]). When $\dim \text{Ch}(\mathcal{M}) = \dim \mathbb{Z}$, we say $\mathcal{M}$ is holonomic. By abuse of notation, $\mathcal{M}$ is also said to be holonomic.

Let $\mathcal{N} \subset \mathfrak{g}^*$ be the nilpotent cone. It is the zero locus of all $f \in k[\mathfrak{g}^*]^G$ without constant terms.

**Proposition 3.4** (V. Ginzburg [18, Lem. 2.1.2]). Let $K \subset G$ be a subgroup and let $\mathcal{M}$ be a $\mathfrak{t}$-admissible $D_{\mathbb{Z}}$-module, then

$$\text{Ch}(\mathcal{M}) \subset \mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp).$$

**Proof.** Let us sketch the arguments in *loc. cit.* briefly. Take $M_0$ as in Remark 3.3 so that $\mathcal{M} = \mathcal{D}_{\mathbb{Z}} \cdot M_0$. Define the good filtration $F^i \mathcal{M} := \mathcal{D}_{\mathbb{Z}} \cdot M_0$ (see [23, Def. 2.1.2]) where $\mathcal{D}_{\mathbb{Z}}^i$ is the filtration of $\mathcal{D}_{\mathbb{Z}}$ by degrees. Denote by $Z^+(\mathfrak{g}) \subset Z(\mathfrak{g})$ the augmentation ideal, with the filtration induced from $U(\mathfrak{g})$. One checks that each $F^i \mathcal{M}$ is stable under $U(\mathfrak{t})$ and $Z^+(\mathfrak{g})$. It follows that $\text{gr}_{F^i \mathcal{M}}$ is annihilated by $\text{gr} Z^+(\mathfrak{g})$ and $\mathfrak{t}$ (more precisely, under their images in $\text{gr} \mathcal{D}_{\mathbb{Z}} \simeq \pi_* \mathcal{O}_{T^* \mathbb{Z}}$). By considering their zero loci, one can infer that $\text{Supp}(\text{gr}_{F^i \mathcal{M}}) \subset \mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp)$. $\square$

By choosing $x_0 \in Z(k)$ and putting

$$H := \text{Stab}_{G}(x_0), \quad H \backslash G \xrightarrow{\sim} Hg \xrightarrow{h \mapsto x_0h} Z,$$

we can identify

$$T^*_{x_0} \mathbb{Z} \simeq \mathfrak{h}^\perp, \quad T^* \mathbb{Z} \simeq \mathfrak{h}^\perp \times G \quad (G\text{-equivariant}).$$

The group $G$ acts on $\mathfrak{g}^*$ through the coadjoint action. Writing elements of $\mathfrak{h}^\perp \times G$ as equivalence classes $[\omega, g]$, where $\omega \in \mathfrak{h}^\perp$, $g \in G$ and impose the relation $[\omega, hg] = [h^{-1} \omega h, g]$ for all $h \in H$, the moment map becomes

$$\mu : \mathfrak{h}^\perp \times G \to \mathfrak{g}^*$$

$$[\omega, g] \mapsto g^{-1} \omega g.$$

We have the following criterion due to Ginzburg. First, recall that $\mathcal{N}$ is the union of all nilpotent coadjoint orbits in $\mathfrak{g}^*$, which are finite in number. Each coadjoint orbit $\mathcal{O}$ is endowed with the Kirillov–Kostant–Souriau symplectic structure; cf. [12, Prop. 1.1.5]. By stipulation, $\mathcal{O}$ is Lagrangian in any smooth symplectic variety.
Proposition 3.5 (V. Ginzburg). Let $H, K \subset G$ be subgroups and let $Z := H \backslash G$. Define $\mu : T^*Z \to \mathfrak{g}^*$ as before. For every nilpotent coadjoint orbit $O \subset \mathcal{N}$, the following are equivalent:

- $\mu^{-1}(O \cap \mathfrak{t}^\perp)$ is isotropic (resp. co-isotropic, Lagrangian) in $T^*Z$;
- $O \cap \mathfrak{h}^\perp$ and $O \cap \mathfrak{t}^\perp$ are both isotropic (resp. co-isotropic, Lagrangian) in $O$.

These intersections are set-theoretic, i.e., they are reduced schemes.

Proof. This is [18, Prop. 1.5.1], whose proof is in §3.1 of loc. cit. □

Corollary 3.6. Suppose that $O \cap \mathfrak{h}^\perp$ and $O \cap \mathfrak{t}^\perp$ are both isotropic for every nilpotent coadjoint orbit $O \subset \mathcal{N}$. Then

(i) $\mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp)$ is Lagrangian in $T^*Z$;
(ii) all $\mathfrak{t}$-admissible $D_2$-modules are holonomic.

Proof. By the finiteness of nilpotent coadjoint orbits, $\mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp) = \bigcup O \mu^{-1}(O \cap \mathfrak{t}^\perp)$ is isotropic in $T^*Z$; in particular its dimension cannot exceed $\dim Z$. Let $M$ be any $\mathfrak{t}$-admissible $D_2$-module. From $\text{Ch}(M) \subset \mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp)$ we see $\text{Ch}(M)$ is isotropic by [12, Prop. 1.3.30]. Hence $\text{Ch}(M)$ is Lagrangian and $M$ is holonomic. This proves (ii).

Now

$$\dim Z = \dim T^*_ZZ \leq \dim \mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp) \leq \dim Z$$

implies that $\dim \mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp) = \dim Z$, so $\mu^{-1}(\mathcal{N} \cap \mathfrak{t}^\perp)$ is Lagrangian, proving (i). □

We are now ready to prove holonomicity of admissible $D$-modules in the spherical case.

Definition 3.7. A $G$-variety $Z$ is said to be spherical if it has an open $B$-orbit, for some (equivalently, any) Borel subgroup $B$ of $G$. A subgroup $H \subset G$ is said to be spherical if $H \backslash G$ is spherical; this property depends only on $\mathfrak{h}$.

For non-algebraically closed fields $\mathbb{k}$, we say a $G$-variety $Z$ over $\mathbb{k}$ is spherical if $Z_{\mathbb{k}}$ is. Such $G$-varieties are often called absolutely spherical, for example in [28].

Let $\mathcal{B}$ denote the flag variety, i.e., the $G$-variety of Borel subgroups of $G$. Note that $H \subset G$ is spherical if and only if $H$ has only finitely many $H$-orbits.

Theorem 3.8. If $K$ is a spherical subgroup of $G$, then $O \cap \mathfrak{t}^\perp$ is isotropic in $O$ for every nilpotent coadjoint orbit $O$, where $O$ is endowed with the Kirillov–Kostant–Souriau symplectic structure.

Proof. We may assume $K$ connected. Consider the moment map for the $G$-variety $\mathcal{B}$, denoted as $r$. By fixing a Borel subgroup $B_0$, we have

$$B_0 \backslash G \xrightarrow{\sim} \mathcal{B}$$
$$B_0 g \mapsto g^{-1}B_0 g,$$
$$r : T^*\mathcal{B} \simeq B_0^\perp \times G \to \mathcal{N} \subset \mathfrak{g}^*$$
$$[x, g] \mapsto g^{-1}xg.$$
In other words, \( r \) is the Springer resolution for \( N \). Now fix a nilpotent coadjoint orbit \( O \). By applying Proposition 3.5 to the given subgroup \( K, H := B_0 \) and \( Z = B \), we obtain

\[
r^{-1}(O \cap \mathfrak{t}^\perp) \text{ is isotropic in } T^*B \iff \text{ so are } O \cap b_0^\perp, O \cap \mathfrak{t}^\perp \text{ in } O.
\]

Note that \( O \cap b_0^\perp \) is known to be Lagrangian [12, Thm. 3.3.7].

It remains to show \( r^{-1}(O \cap \mathfrak{t}^\perp) \) is isotropic. Set \( L := r^{-1}(k^\perp) \); it consists precisely of the cotangent vectors of \( B \) that are orthogonal to the vector fields induced by \( k \). Let \( F_1, \ldots, F_k \) be the \( K \)-orbits in \( B \), so that

\[
L = \bigsqcup_{i=1}^k T_{F_i}^*B.
\]

The \( T_{F_i}^*B \) are defined as in [23, p.65] and are Lagrangian in \( T^*B \), for \( i = 1, \ldots, k \). Hence \( L \) is Lagrangian as well. It follows from [12, Prop. 1.3.30] that \( r^{-1}(O \cap \mathfrak{t}^\perp) \) is isotropic in \( T^*B \), since \( r^{-1}(O \cap \mathfrak{t}^\perp) \subseteq L \).

We remark that the usage of \( L \) in the foregoing arguments is inspired by the proof of [3, Lem. 2.2]. In the following cases, \( O \cap \mathfrak{h}^\perp \) is even known to be Lagrangian:

1. \( \mathfrak{h} \) is spherical and solvable, see [12, Thm. 1.5.7];
2. \( \mathfrak{h} = \mathfrak{g}^\theta \) for some involution \( \theta \) of \( G \), see the proof of [18, Prop. 3.1.1].

**Corollary 3.9.** Let \( Z \) be a spherical homogeneous \( G \)-variety, and let \( K \) be a spherical subgroup of \( G \). Then every \( \mathfrak{k} \)-admissible \( D_Z \)-module \( M \) is holonomic. In particular, there is a \( K \)-invariant Zariski open dense subset \( U \subset Z \) on which \( M \) is an integrable connection.

If \( G, Z, K \) are defined over a subfield \( \mathbb{k}_0 \subset \mathbb{k} \) and \( Z(\mathbb{k}_0) \neq \emptyset \), one can choose \( U \) to be defined over \( \mathbb{k}_0 \) as well.

**Proof.** Choose \( x_0 \in Z(\mathbb{k}) \). Apply Theorem 3.8 to the spherical subgroups \( K \) and \( H := \text{Stab}_G(x_0) \), then conclude holonomicity by using Corollary 3.6. It is well-known that there is an open dense \( U \subset Z \) over which \( \mu^{-1}(N \cap \mathfrak{t}^\perp) \) (hence \( \text{Ch}(M) \)) reduces to the zero section, e.g., [23, Prop. 3.1.6]. By the equivariance of \( \mu \), one can replace \( U \) by \( U \cdot K \) to assume \( K \)-invariance. The last assertion follows immediately.

In concrete applications, it is often important to determine the \( U \) in Corollary 3.9. In the symmetric case \( K = G^\theta \) and \( Z = K \setminus G \), where \( \theta \) is an involution of \( G \), we refer to [18, Prop. 3.5.1] for such a description. Below is another explicit and easier instance.

**Example 3.10** (Twisted spaces). Let us illustrate the description of \( U \) by the case of twisted spaces of Labesse; a detailed discussion can be found in [30, I.3]. Take \( \mathbb{k} \) to be a field of characteristic zero; a *twisted space* \( \tilde{G} \) under a connected reductive \( \mathbb{k} \)-group \( G \) is the following data:
• $\tilde{G}$ is a $G^{\text{op}} \times G$-homogeneous variety, $\tilde{G}(k) \neq \emptyset$, with action written as $\gamma \cdot (a, b) = a \gamma b$;
• $\tilde{G}$ is simultaneously a $G^{\text{op}}$-torsor and a $G$-torsor, and there exists $\text{Ad} : \tilde{G} \to \text{Aut}(G)$ such that
$$\gamma g = \text{Ad}(\gamma)(g) \gamma, \quad \gamma \in \tilde{G}, \ g \in G.$$  

It follows that $\text{Ad}(a \gamma b) = \text{Ad}(a) \text{Ad}(\gamma) \text{Ad}(b)$. Steinberg’s theorem [30, Thm. I.3.7.1] implies that $\text{Ad}(\gamma)$ stabilizes a Borel subgroup over $k$ for every $\gamma \in \tilde{G}(k)$, therefore $\tilde{G}$ is a spherical $G^{\text{op}} \times G$-variety by Bruhat decomposition. When there exists $\gamma_0$ with $\text{Ad}(\gamma_0) = \text{id}$, we are reduced to the well-studied “group case” $\tilde{G} \simeq G$.

Let $\ell$ be the absolute rank of $\tilde{G}$ defined in [30, p.60]. For each $\gamma \in \tilde{G}$, define $D^{\tilde{G}}(\gamma)$ to be the coefficient of $X^\ell$ in $\det(X - \text{Ad}(\gamma) + 1|g)$ where $X$ is an indeterminate. Then $D^{\tilde{G}}$ is a regular function on $\tilde{G}$. Define $\tilde{G}_{\text{reg}} := \{D^{\tilde{G}} \neq 0\} \subset \tilde{G}$; the elements thereof are called regular elements. This generalizes the notion of regular semisimple elements in $G$. A basic fact is that $\tilde{G}_{\text{reg}}$ is open dense in $\tilde{G}$.

Choose $g_0 \in \tilde{G}(k)$ and put $\tau := \text{Ad}(g_0)^{-1} \in \text{Aut}(G)$. Then
$$H := \text{Stab}_{G^{\text{op}} \times G}(g_0) = \{(a, b) : g_0 \tau(a) b = g_0\} = \{(g, \tau(g)^{-1}) : g \in G\}.$$  

Thus $\mathfrak{h} = \text{im}(\text{id}, -\tau) \subset \mathfrak{g}^{\text{op}} \times \mathfrak{g}$ and $\mathfrak{h}^{\perp} = \text{im}(\text{id}, \tau) \in \mathfrak{g}^{\text{op}*} \times \mathfrak{g}^*$, where we write $\tau$ for the induced automorphisms on $\mathfrak{g}$ and $\mathfrak{g}^*$. Summing up:
$$\mathfrak{g}^* \times G \xrightarrow{\sim} \mathfrak{h}^{\perp} \times (G^{\text{op}} \times G) \xrightarrow{\sim} \text{T}^* \tilde{G} \longrightarrow \tilde{G}$$  

Identify $\text{T}^* \tilde{G}$ with $\mathfrak{g}^* \times G$. Then
$$\mu(\lambda, g) = (\lambda, g^{-1}(\tau \lambda) g) = (\lambda, \text{Ad}(g^{-1})\tau \lambda).$$

Next, take the spherical subgroup $K := \{(g^{-1}, g) : g \in G\}$ of $G^{\text{op}} \times G$. We have $\mathfrak{t}^{\perp} = \{(\mu, \mu) : \mu \in \mathfrak{g}^*\}$. By the definition of $\tau$,
$$\mu(\lambda, g) \in \mathfrak{t}^{\perp} \iff \text{Ad}(g) \lambda = \text{Ad}(g_0)^{-1}(\lambda) \iff \text{Ad}(g_0) \lambda = \lambda.$$  

Fix an invariant bilinear form $\mathfrak{g} \times \mathfrak{g} \to k$ to identify $\mathfrak{g} \simeq \mathfrak{g}^*$. Assume $\gamma := \gamma_0 g$ is regular, then:
• $\mu(\lambda, g) \in \mathfrak{t}^{\perp}$ is equivalent to $\lambda \in \mathfrak{g}^{\text{Ad}(\gamma)}$, whilst $G^{\text{Ad}(\gamma)^*}$ is a torus by [30, Lemme I.3.11.2];
• $\mu(\lambda, g) \in \mathcal{N}$ is equivalent to $\lambda$ being nilpotent.

The conjunction of the two properties above is thus $\lambda = 0$. This shows that $\mu^{-1}(\mathfrak{t}^{\perp} \cap \mathcal{N})$ reduces to zero section over $\tilde{G}_{\text{reg}}$. Hence we may choose $U := \tilde{G}_{\text{reg}}$. 
4. Review of horocycle correspondence

The definitions below follow [18, §8]. Consider an algebraically closed field \( \mathbb{k} \) of characteristic zero and a connected reductive \( \mathbb{k} \)-group \( G \). Fix a Borel subgroup \( B \subset G \) with \( U := R_u(B) \), and let \( T := B/U \). Define

\[
\begin{align*}
Y & := U \setminus G, \quad \text{right } G\text{-action, left } T\text{-action;} \\
Y^{\text{op}} & := G/U, \quad \text{left } G\text{-action, right } T\text{-action;} \\
B & := T \setminus Y = B \setminus G, \quad \text{right } G\text{-action;} \\
\mathcal{Y} & := Y^{\text{op}} T \setimes Y, \quad \text{right } G^{\text{op}} \times G\text{-action.}
\end{align*}
\]

Here, the horocycle space \( \mathcal{Y} \) is formed by taking the quotient of the right \( T\)-action on \( Y^{\text{op}} \times Y \) via \( (y, y')t = (yt, t^{-1}y') \). Observe that \( \mathcal{Y} \) carries the free \( T\)-action \( \mathcal{Y} \) with quotient \( B \times B \).

Consider the morphisms

\[
\begin{array}{ccc}
G & \xleftarrow{p} & B \times G \\
g & \mapsto & (Ty, g) \mapsto [y^{-1}, yg].
\end{array}
\]

Let \( G^{\text{op}} \times G \) act on the right of \( B \times G \) (resp. of \( G \)) by \( (\beta, g) \cdot (a, b) = (\beta a^{-1}, agb) \) for all \( a, b, g \in G \) and \( \beta \in B \) (resp. by bilateral translation).

**Lemma 4.1.** The morphisms \( p, q \) are both \( G^{\text{op}} \times G\)-equivariant. Moreover, \( q \) is smooth affine and surjective, and for all \( g_1, g_2 \in G \) we have

\[
q^{-1}([g_1U, U g_2]) = \{ Bg_1^{-1} \} \times g_1U g_2.
\]

In particular, \( q^{-1}([g_1U, g_2]) \) is naturally a left \( g_1U g_1^{-1} \)-torsor.

**Proof.** The following diagram is Cartesian

\[
\begin{array}{ccc}
Y \times G & \xrightarrow{\beta} & B \times G \\
\downarrow{\alpha} & & \downarrow{q} \\
Y \times Y & \xrightarrow{\gamma} & \mathcal{Y}
\end{array}
\]

where \( \alpha(y, g) = (y, yg) \), \( \beta(y, g) = (Ty, g) \) and \( \gamma(y_1, y_2) = [y_1^{-1}, y_2] \). Therefore, by descent along \( T\)-torsors, it suffices to show \( \alpha \) is smooth affine surjective. Smoothness and surjectivity are straightforward. To show \( \alpha \) is affine, we use another Cartesian diagram:

\[
\begin{array}{ccc}
\{ (g, h_1, h_2) \in G^3 : h_1 g h_2^{-1} \in U \} & \xrightarrow{\alpha'} & Y \times G \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
G \times G & \xrightarrow{} & Y \times Y
\end{array}
\]

The upper-left corner is closed in \( G^3 \), hence \( \alpha' \) is affine. This property descends to \( \alpha \) along \( G \times G \rightarrow Y \times Y \).

The description of \( q^{-1}([g_1U, U g_2]) \) follows from \( \alpha^{-1}(U g_1^{-1}, U g_2) = \{ U g_1^{-1} \} \times g_1U g_2 \). \( \Box \)

The equivariance of \( p, q \) justifies the following
**Definition 4.2.** Let $L \subset G^{\text{op}} \times G$ be any subgroup and $\chi : I \to k$ be a character. In the setting above, we set

$$HC_{L,\chi} := q_*p^!,$$  

$$CH_{L,\chi} := pq^*$$

in the equivariant derived categories of $(L, \chi)$-monodromic $\mathcal{D}$-modules (Definition 2.9). They give rise to a pair of adjoint functors

$$\begin{align*}
CH_{L,\chi} : \mathcal{D}_L^b(Y) &\rightleftarrows \mathcal{D}_L^b(h(G)) : HC_{L,\chi}.
\end{align*}$$

Here $\mathcal{D}_L^b$ denotes the full triangulated subcategory of $\mathcal{D}_L^b(h(G))$ formed by complexes with holonomic cohomologies.

When $L = \{1\}$, we denote them simply as $CH, HC$.

For $L = \{1\}$ we obtain the non-equivariant version $\mathcal{D}_L^b(h(Y)) \rightleftarrows \mathcal{D}_L^b(h(G))$, whilst for trivial we obtain the pair $\mathcal{D}_L^b(h(Y)) \rightleftarrows \mathcal{D}_L^b(h(G))$ for complexes of equivariant $\mathcal{D}$-modules. Finally, these functors are also compatible with forgetful functors $\mathcal{D}_L^b(h(G)) \rightleftarrows \mathcal{D}_L^b(Y)$ when $L_0 \subset L$ is a subgroup and $\chi'_0 = \chi|_{L_0}$. Cf. the discussions after Definition 2.9.

**Theorem 4.3 (See [18, Thm. 8.5.1]).** The identity functor of $\mathcal{D}_L^b(h(G))$ is a direct summand of $CH \circ HC$; this splits the adjunction co-unit $CH \circ HC \to \text{id}$.

In fact, it is shown in *loc. cit.* that in the setting of constructible sheaves, $CH \circ HC$ is given by convolution with the Springer sheaf $Spr \in \text{Perv}(G)$, that is, $Rr_*k_{\overline{N}}[\dim \overline{N}]$ where $r : \overline{N} \to N \subset G$ is the Springer resolution. Moreover $Spr$ is known to contain the skyscraper sheaf centered at 1 as a direct summand, see [12, 8.9.17]. These are transcribed to the $\mathcal{D}$-module setting in [18, §8.7].

5. $K$-admissible $D$-modules: regularity

Retain the conventions from §4; in particular $k$ is algebraically closed. Denote the $T$-torsor $Y \to B \times B$ as $\pi$.

For the notion of regular holonomic $\mathcal{D}$-modules or complexes, we refer to standard references such as [10, VII] or [23, Chap. 6].

**Lemma 5.1.** Let $L \subset G^{\text{op}} \times G$ be a spherical subgroup, and $\chi$ be a reductive character of $I$. Suppose that $N$ is a simple $(L, \chi)$-monodromic $\mathcal{D}_Y$-module, $N$ is holonomic, and that there exists a covering $B \times B = W_1 \cup \cdots \cup W_r$ by affine open subsets such that the sections of $N|_{\pi^{-1}W_i}$ are all $t$-finite, for $i = 1, \ldots, r$. Then $N$ is a regular holonomic $\mathcal{D}_Y$-module.

**Proof.** Apply the discussions around (2.2) to the $T$-torsor $\pi$. In view of the $t$-finiteness assumption, one can decompose $N$ into components indexed by $\xi \in t^*/t^*_L$ according to (2.2). Since the actions of $L$ and $T$ commute, each component is still $(L, \chi)$-monodromic, and the simplicity implies that $N$ is a simple object in $\mathcal{D}_X{\text{-Mod}}_\xi$ for some $\xi$; in fact, it must belong to $\mathcal{D}_X{\text{-Mod}}_\xi$.

Pick a representative $\xi \in t^*$ of $\overline{\xi}$. By Proposition 2.8, $N$ acquires a canonical $(T, \xi)$-monodromic structure. Since $L$ and $T$ commute, $N$ is actually $(L \times T, (\chi, \xi))$-monodromic. Noting that $L \times T$ acts on $Y$ with finitely many orbits, one applies [15, Lem. 2.5.1] to conclude the regularity of $N$. This is legitimate by the Remark below. $\square$
Remark 5.2. The result [15, Lem. 2.5.1] cited above asserts that if $Q$ is a connected reductive group, $\psi : q \to k$ is a character, and $X$ is a smooth $Q$-variety with finitely many $Q$-orbits, then every $(Q, \psi)$-monodromic $\mathcal{D}_Q$-module $\mathcal{N}$ is regular holonomic. The case when $\psi$ is trivial and $Q$ is an arbitrary affine group is well-known; see [23, Thm. 11.6.1]. What we need is the case $Q := L \times T$ acting on $X := \mathcal{Y}$ and $\psi := (\chi, \xi)$. This can be extracted from the proof of [15, Lem. 2.5.1] as follows. Choose a smooth $G$-equivariant compactification $j : \mathcal{Y} \to \overline{\mathcal{Y}}$. The goal is to show that $j_*\mathcal{N}$ is regular holonomic at every point. In loc. cit., this is reduced to the assertion that $\mathcal{O}_{Q, \psi}$ is regular holonomic as a $\mathcal{D}_{\overline{Q}}$-module, which is then established for connected reductive $Q$.

To treat our case, first replace $L$ by $L^\circ$ to ensure $Q$ is connected. Take a Levi decomposition $Q = Q' \rtimes R_u(Q)$. Since $\psi$ is a reductive character, it decomposes into $\psi' \rtimes 0$. The regularity reduces to (a) the standard case of $\mathcal{O}_{R_u(Q)}$, and (b) the case of $\mathcal{O}_{Q', \psi'}$ which is addressed in loc. cit.

Note that reductivity is necessary in these arguments. If we take $Q = \mathbb{G}_a$ and $\psi$ nontrivial, then $\mathcal{O}_{Q, \psi} \simeq \mathcal{D}_Q/\mathcal{D}_Q (d/dt - a)$ for some $a \neq 0$. It is holonomic, yet irregular at $\infty$.

Definition 5.3. Let $Z$ be a homogeneous $G$-variety. Let $K$ be a subgroup of $G$. A $D_Z$-module $M$ is called $K$-admissible if

- $M$ is finitely generated over $D_Z$;
- $M$ is locally finite under $\mathcal{Z}(g)$;
- the $\mathcal{D}_Z$-module $\mathcal{M}$ generated by $M$ is equipped with a $(K, \chi)$-monodromic structure for some reductive character $\chi$ of $\mathfrak{t}$.

Quotients and finitely generated submodules of a $K$-admissible $D_Z$-module are still admissible, provided that they are $K$-stable. By Lemma 2.3, every $K$-admissible $D_Z$-module is $\mathfrak{t}$-admissible in the sense of Definition 3.1. For any $D_Z$-module $M$, we write $\mathcal{M} := \mathcal{D}_Z \cdot M$.

Remark 5.4. One can view $G$ as a $G^{\text{op}} \times G$-variety by $x \cdot (a, b) = axb$. The definition above can therefore be applied to $D_G$-modules under the action of a subgroup $L \subset G^{\text{op}} \times G$. Note that the local finiteness under $\mathcal{Z}(g \times g)$ and $\mathcal{Z}(g)$ are equivalent.

Example 5.5. The examples mentioned in §2 are directly related to $K$-admissibility. In what follows, we view $Z$ as a smooth variety over $\mathbb{C}$ which is definable over $\mathbb{R}$.

(i) Function spaces In Example 2.4, suppose furthermore that $V$ is finitely generated over $D_Z$ and locally finite under $\mathcal{Z}(g)$, then $V$ generates a $K$-admissible $D_Z$-module with trivial $\chi$. Indeed, $\mathcal{D}_Z \cdot V$ is equipped with a $K$-equivariant structure.

(ii) Relative invariants with reductive character $\chi$ In Example 2.6, suppose that $u$ is $\mathcal{Z}(g)$-finite, then $D_Z \cdot u$ is $K$-admissible; in this case, $\mathcal{D}_Z \cdot u$ is equipped with a $(K, \chi)$-monodromic structure.

(iii) Localizations In Example 2.7, suppose that the $(g, K)$-module $V$ is a Harish-Chandra module; see [6, §4]. In this case $V$ is finitely generated over $U(g)$ and locally finite under $\mathcal{Z}(g)$. Hence $D_Z \otimes_{U(g)} V$ is $K$-admissible with trivial $\chi$. The $\mathcal{D}_Z$-module it generates is $\text{Loc}_Z(V)$ which is $K$-equivariant.
Theorem 5.6. Let $L \subset G^{\text{op}} \times G$ be a spherical subgroup. Then every $L$-admissible $D_G$-module $M$ is regular holonomic.

Proof. By Corollary 3.9 applied to $Z := G$ and $L$, we see $M$ is holonomic. Theorem 4.3 implies that $\mathcal{M}$ (as a $D_G$-module) is a direct summand of $\text{CH} \circ \text{HC}(\mathcal{M})$. Since the functors $p_!$ and $q^*$ in derived categories preserve regular holonomic complexes (see [23, Thm. 6.1.5]), the regularity of $\mathcal{M}$ will follow from that of the cohomologies of $\text{HC}(\mathcal{M})$, which we prove below.

Since
$$\text{HC}(\mathcal{M}) := \text{HC}(\text{obl}(\mathcal{M})) \simeq \text{obl}(\text{HC}_{L, \chi}(\mathcal{M})),$$
the cohomologies $\text{HC}^i(\mathcal{M})$ of $\text{HC}(\mathcal{M})$ are endowed with $(L, \chi)$-monodromic structures (cf. Definition 2.9 and the subsequent discussions), for any given $i \in \mathbb{Z}$. The $\mathcal{D}_Y$-module $\text{HC}^i(\mathcal{M})$ is holonomic, thus of finite length in the $(L, \chi)$-monodromic category. It suffices to show that each simple $(L, \chi)$-monodromic subquotient $N$ of $\text{HC}^i(\mathcal{M})$ is regular holonomic.

In view of Lemma 5.1, it remains to check the $t$-local finiteness of $N|_{\pi^{-1}W}$, where $W \subset B \times B$ ranges over some finite affine open covering; note that $\pi^{-1}W$ is still affine. This will follow from the same property for $\text{HC}^i(\mathcal{M})$. As $p^!\mathcal{M} \simeq \mathcal{O}_B \boxtimes \mathcal{M}[\dim B]$ by [10, VII.9.14 Corollary], it remains to show the local $t$-finiteness of sections of $R^j q_* (\mathcal{O}_B \boxtimes \mathcal{M})$ over $\pi^{-1}W$, for all $j \geq 0$ and suitably chosen $W$.

The required argument for the last step is given in [19, p.156–158]. Let us conclude by a very brief sketch. Using the fact that $q$ is affine smooth and the description of its fibers (Lemma 4.1), one computes $R^j q_* (\mathcal{O}_B \boxtimes \mathcal{M})$ by an explicit relative de Rham resolution. The local $t$-finiteness is thus related to the known local $Z(\mathfrak{g})$-finiteness of $M$ by the following observation. The $T$-action $[y_1, y_2] \mapsto [y_1, ty_2]$ on $\mathcal{Y}$ lifts to $B \times G$ by
$$t : (Ty, g) \mapsto (Ty, y^{-1}tyg), \quad t \in T$$
by which one computes the $t$-action on $R^j q_* (\mathcal{O}_B \boxtimes \mathcal{M})$. A standard fact says
$$Z(\mathfrak{g}) \subset U(t) \oplus U(\mathfrak{g})u \tag{5.1}$$
and the resulting projection $Z(\mathfrak{g}) \to U(t)$ so obtained is the Harish-Chandra map without shifting by the half-sum of positive roots.

Analogously, one may let $u$ act via $u : (Ty, G) \mapsto (Ty, y^{-1}uyg)$ by choosing local sections for $Y \to B$. However $y^{-1}uy$ is the “vertical direction” over $q(Ty, g) = [y^{-1}, yg]$ relative to $q$ by Lemma 4.1. In view of (5.1), this will eventually enable us to employ the local $Z(\mathfrak{g})$-finiteness of $M$. □

Corollary 5.7 (Cf. [18, Cor. 8.9.1]). Let $Z$ be a spherical homogeneous $G$-variety, and $K \subset G$ be a spherical subgroup. Then every $K$-admissible $D_Z$-module $M$ generates a regular holonomic $\mathcal{D}_Z$-module.

Proof. We may assume $Z = H \backslash G$ where $H$ is a spherical subgroup of $G$. The quotient map $f : G \to Z$ is an $H$-torsor, hence smooth. By Corollary 3.9, $\mathcal{M}$ is holonomic, and so is $\mathcal{N} := f^* \mathcal{M}$. Note that $\mathcal{N}$ is concentrated at degree 0 by the
flatness of \( f \), and it is generated by \( N \), the finitely generated \( D_G \)-module formed by \( f^* \)-images of the elements of \( M \).

Let \( L := H^{\text{op}} \times K \subset G^{\text{op}} \times G \). We contend that \( N \) is \( L \)-admissible. Indeed, the local \( \mathcal{Z}(g) \)-finiteness is inherited from \( M \); so is the \( H^{\text{op}} \times K \)-monodromic structure on \( \mathcal{N} \) since it is pulled back from \( H \backslash G \).

Note that \( L \) is a spherical subgroup of \( G^{\text{op}} \times G \). Therefore \( \mathcal{N} \) is regular holonomic by Theorem 5.6. This implies the regularity of \( M \) by [10, VII. 12.9].

6. Subanalytic sets and maps

We will use the notion of subanalytic subsets and subanalytic functions on real analytic manifolds; the relevant theory can be found in [9] or [25, §8.2].

**Definition 6.1.** Let \( M \) be a real analytic manifold. A subset \( X \subset M \) is said to be semianalytic if each \( x \in M \) has an open neighborhood \( U \) such that \( X \setminus U = \bigcup_{i=1}^p \bigcap_{j=1}^q X_{ij} \), where each \( X_{ij} \) is described by \( f_{ij} = 0 \) or \( f_{ij} > 0 \) for some family of analytic functions \( f_{ij} : U \to \mathbb{R} \).

We say \( X \subset M \) is subanalytic if any \( x \in X \) has an open neighborhood \( U \) in \( M \) such that \( X \setminus U = \text{pr}_1(A) \) for some relatively compact semianalytic subset \( A \subset M \times N \), where \( N \) is a real analytic manifold and \( \text{pr}_1 : M \times N \to M \) is the projection.

Below is a summary of the basic properties we need. See the paragraph after [9, Def. 3.1],

- Locally closed analytic submanifolds are semianalytic, hence subanalytic.
- Finite unions and finite intersections of subanalytic sets are subanalytic.
- Connected components of a subanalytic set are locally finite and subanalytic.
- The closure of a subanalytic subset is subanalytic.
- Complements of subanalytic sets are subanalytic; this is [9, Thm. 3.10].

**Definition 6.2.** Let \( X \subset M \) be a subset, \( N \) be a real analytic manifold. We say a function \( f : X \to N \) is subanalytic if its graph \( \Gamma_f \subset M \times N \) is subanalytic.

- Morphisms between analytic manifolds are subanalytic.
- The image of a relatively compact subanalytic set under a subanalytic mapping remains subanalytic; see the remark after [9, Def. 3.2].
- Composites of subanalytic maps are subanalytic.
- Let \( X \subset \mathbb{R}^n \) be a subanalytic subset, then the Euclidean distance \( d(x, X) \) is subanalytic on \( \mathbb{R}^n \); this is [9, Rems. 3.11 (1)].

We are ready to state our main technical tool, Lojasiewicz’s inequality.

**Definition 6.3.** Let \( M \) be a set and \( f, g : M \to \mathbb{R}_{\geq 0} \). We write \( f \preceq g \) if there exist constants \( a, C \in \mathbb{R}_{\geq 0} \) such that \( f \leq C g^a \). If both \( f \preceq g \) and \( g \preceq f \) hold, we write \( f \sim g \) and say they are power-equivalent.

**Theorem 6.4** (S. Lojasiewicz; see [9, Thm. 6.4]). Let \( M \) be a real analytic manifold, \( E \subset M \) a subanalytic subset and let \( f, g : E \to \mathbb{R} \) be subanalytic functions with compact graphs in \( E \times \mathbb{R} \). If \( f^{-1}(0) \subset g^{-1}(0) \), then \( |g| \preceq |f| \).
As a particular case, the assumptions hold when $E$ is compact subanalytic and $f, g : E \to \mathbb{R}_{\geq 0}$ are continuous subanalytic functions. In that case, $f^{-1}(0) = g^{-1}(0)$ if and only if $f$ and $g$ are power-equivalent.

We record some easy observations for later use.

**Lemma 6.5.** Let $\pi : X \to Y$ be a locally trivial fibration between real analytic manifolds.

(i) Let $E \subset Y$ be a subset. If $\pi^{-1}(E)$ is subanalytic in $X$, then $E$ is subanalytic in $Y$.

(ii) Let $f_Y : Y \to \mathbb{R}$ be a function such that $f_X := f_Y \circ \pi$ is subanalytic on $X$, then $f_Y$ is subanalytic on $Y$.

**Proof.** Consider (i). By the local nature of Definition 6.1, upon retracting $Y$ we may assume $X = Y \times F$ and $\pi$ is the first projection, where $F$ is some real analytic manifold. For every $y \in Y$, pick $f \in F$. Since $E \times F$ is subanalytic in $Y \times F$, there exist

- an open neighborhood $U_Y \times U_F$ of $(y, f)$ in $Y \times F$,
- a real analytic manifold $N$,
- a relative compact semianalytic subset $A \subset (Y \times F) \times N$,

such that $(E \times F) \cap (U_Y \times U_F) = \text{pr}_{12}(A)$, where $\text{pr}_{12} : (Y \times F) \times N \to Y \times F$ is the projection. It follows that $E \cap U_Y = \pi(\text{pr}_{12}(A))$.

Taking the “$N$” in Definition 6.1 to be the $N \times F$ above, the preceding discussion shows that $E$ is subanalytic, by varying $y$.

As for (ii), we have to show the graph $\Gamma_{f_Y} \subset Y \times \mathbb{R}$ is subanalytic. Observe that $\Gamma_{f_X} = (\pi \times \text{id}_\mathbb{R})^{-1}(\Gamma_{f_Y})$. We conclude by applying (i) to $\pi \times \text{id}_\mathbb{R} : X \times \mathbb{R} \to Y \times \mathbb{R}$. □

### 7. Growth conditions

**Definition 7.1.** Consider a set $M$, its subset $V$ and a function $p : M \to \mathbb{R}_{\geq 0}$. We say a function $f : V \to \mathbb{C}$ has $p$-bounded growth relative to $M$, if there exists $a \in \mathbb{R}_{>0}$ such that $p^a |f|$ is bounded on $V$. This notion depends only on the power-equivalence class of $p$ (Definition 6.3).

**Lemma 7.2.** Let $M$ be a topological space, $V$ an open subset and $p : M \to \mathbb{R}_{\geq 0}$ be continuous. Let $\pi : M' \to M$ be a continuous map. If $f : V \to \mathbb{C}$ has $p$-bounded growth, then $f \pi : \pi^{-1}(V) \to \mathbb{C}$ has $\pi^*p$-bounded growth; the converse holds if $f$ is continuous and $\pi : \pi^{-1}(V) \to V$ has dense image.

**Proof.** Immediate. □

The utility of this notion is explained by the following result.

**Lemma 7.3.** Suppose that $M$ is a compact real analytic manifold, and $U \subset M$ is an open subanalytic subset. Then there exists a subanalytic continuous function $p : M \to \mathbb{R}_{\geq 0}$ such that $U = \{ x \in M : p(x) > 0 \}$. 

Proof. Without loss of generality we may assume $M$ connected. Recall that $M \setminus U \subseteq M$ is closed and subanalytic. There exists a closed immersion $i : M \to \mathbb{R}^N$ of real analytic spaces, by [1, Thm. 1]. Now take

$$p(x) := d(i(x), i(M \setminus U)), \quad x \in M$$

where $d$ is the Euclidean distance function on $\mathbb{R}^N$. Since $i(M \setminus U) \subseteq \mathbb{R}^n$ is subanalytic, $d(\cdot, i(M \setminus U))$ is subanalytic continuous on $\mathbb{R}^N$, hence so is $p$. \qed

Now we turn to the case of real algebraic varieties.

**Definition–Proposition 7.4.** Let $X$ be a smooth $\mathbb{R}$-variety.

- There exists an open immersion $j : X \to \overline{X}$ with Zariski-dense image, such that $\overline{X}$ is smooth and $\overline{X}(\mathbb{R})$ is compact.
- For each $j$ above, there exists a continuous subanalytic function $p : \overline{X}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ such that

$$j(X)(\mathbb{R}) = \{ x \in \overline{X}(\mathbb{R}) : p(x) > 0 \}.$$

In this case, $p$ is said to be adapted to $j$.

Let $V$ be a connected component of $X(\mathbb{R})$. We say a continuous function $f : V \to \mathbb{C}$ has moderate growth at infinity if $f$ has $p$-bounded growth relative to $\overline{X}(\mathbb{R})$. This notion is independent of $j : X \to \overline{X}$ and $p$.

**Proof.** The existence of $j$ is ensured by Nagata’s theorem followed by Hironaka’s resolution of singularities. The existence of $p$ follows from Lemma 7.3. Consider the category $\mathcal{C}$ of open immersions $j : X \to \overline{X}$ as above, the morphisms from $j_1 : X \to \overline{X}_1$ to $j_2 : X \to \overline{X}_2$ being morphisms $\pi : \overline{X}_1 \to \overline{X}_2$ such that $\pi j_1 = j_2$.

We claim that $\mathcal{C}$ is co-filtrant.

Indeed, given $j_i : X \to \overline{X}_i$ for $i = 1, 2$, take $\overline{X}'$ to be the schematic closure of the diagonal image of $X$ in $\overline{X}'_1 \times \overline{X}'_2$. Then we obtain an open dense immersion $j' = j_1 \times j_2 : X \to \overline{X}'$; thus $\overline{X}'(\mathbb{R})$ is compact, but $\overline{X}'$ is not necessarily smooth. To remedy this, take a resolution of singularities $\rho : \overline{X} \to \overline{X}'$ which is proper and restricts to $\rho^{-1}(j'(X)) \cong j'(X)$. Then $j : X \to \overline{X}$ dominates both $j_1$ and $j_2$ in $\mathcal{C}$.

Let $f : V \to \mathbb{C}$ be a continuous function. Extending $f$ by zero, we may assume $V = X(\mathbb{R})$. Consider a morphism $\pi$ in $\mathcal{C}$ from $j' : X \to \overline{X}'$ to $j : X \to \overline{X}$. Let $p$ (resp. $p'$) be adapted to $j$ (resp. $j'$). We claim that $f$ has $p$-bounded growth relative to $\overline{X}(\mathbb{R})$ if and only if it has $p'$-bounded growth relative to $\overline{X}'(\mathbb{R})$. In view of the previous step, this will entail that the notion of moderate growth at infinity is independent of all choices.

We first show that $\pi^{-1}(j(X)) = j'(X)$. This is because $\pi : \pi^{-1}(j(X)) \to j(X)$ has a section $\sigma : j(X) \to \overline{X} \cong j'(X) \subset \pi^{-1}(j(X))$, thus $\sigma$ is a closed immersion by [35, 28.3.1] since $\pi^{-1}(j(X))$ is separated. As $j'(X)$ is also open in the irreducible subset $\pi^{-1}(j(X))$, we see $\pi^{-1}(j(X)) = j'(X)$. Now $p', p\pi : \overline{X}'(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ both have $j'(X)(\mathbb{R})$ as their non-zero locus. Theorem 6.4 implies that $p'$ and $p\pi$ are power-equivalent. Hence $p'$-bounded growth and $p\pi$-bounded growth relative to $\overline{X}(\mathbb{R})$ are equivalent.
Moreover, $p\pi$-bounded growth relative to $\bar{X}'(\mathbb{R})$ is equivalent to $p$-bounded growth relative to $\bar{X}(\mathbb{R})$ for continuous functions on $X(\mathbb{R})$ (Lemma 7.2). This establishes our claim. □

For a systematic treatise of tempered functions on manifolds definable in polynomially bounded $\sigma$-minimal structures, see [34]. The author is grateful to one of the referees for this suggestion.

Below is an intrinsic characterization of the functions $p|_{X(\mathbb{R})}$, or rather their inverses.

**Proposition 7.5.** Let $j : X \hookrightarrow \bar{X}$ be as in Definition–Proposition 7.4. Suppose that $w : X(\mathbb{R}) \to \mathbb{R}_{>0}$ satisfies

(i) $w$ is continuous and subanalytic;

(ii) for each constant $B > 0$, the subset \{ $x \in X(\mathbb{R}) : w(x) \leq B$ \} is compact.

Then $w^{-1}$ extends continuously to a unique subanalytic function $p : \bar{X}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ adapted to $j$. Conversely, for any $p$ adapted to $j$, the function $w(x) := p(x)^{-1}$ on $X(\mathbb{R})$ satisfies (i) and (ii).

**Proof.** Put $\partial X := \bar{X} \setminus X$. We have to extend $w^{-1} : X(\mathbb{R}) \to \mathbb{R}_{>0}$ to a continuous subanalytic function on $\bar{X}(\mathbb{R})$ adapted to $j$. By (ii), we see $w(x) \to +\infty$ when $x$ tends to $\partial X(\mathbb{R})$. Let $\Gamma_{w^{-1}} \subset X(\mathbb{R}) \times \mathbb{R}_{>0}$ be the graph of $w^{-1}$, which is subanalytic by (i). Its closure $\bar{\Gamma}_{w^{-1}}$ in $\bar{X}(\mathbb{R}) \times \mathbb{R}$ is still subanalytic; moreover $\bar{\Gamma}_{w^{-1}} \cap (\partial X(\mathbb{R}) \times \mathbb{R}) = \partial X(\mathbb{R}) \times \{0\}$. Hence $w^{-1}$ extends to a continuous subanalytic function $p : \bar{X}(\mathbb{R}) \to \mathbb{R}$ by zero. The converse is easy. □

**Remark 7.6.** The real algebraic structures of $X$ and $\bar{X}$ play no roles in the proof above. Furthermore, we do not need to assume $\bar{X}(\mathbb{R})$ is smooth: it suffices to embed it into a real analytic manifold in order to talk about subanalyticity.

When $X$ is a homogeneous $G$-variety for a connected reductive $\mathbb{R}$-group $G$, we shall choose $w$ with additional properties. To begin with, define the norm $\| \cdot \| : G(\mathbb{R}) \to \mathbb{R}_{>0}$ as in [6, 2.1.2]. More precisely, choose an algebraic embedding $\iota : G \to \text{GL}(\mathbb{N})$ and set

$$\|g\| := \text{tr} (\iota(g) \cdot ^t \iota(g)) + \text{tr} (\iota(g^{-1}) \cdot ^t \iota(g)^{-1}), \quad g \in G(\mathbb{R}).$$

(7.1)

On the other hand, we also have the function on the connected component $G(\mathbb{R})^\circ$ defined by $\|g\|_{\text{max}} := \exp(d(g, 1))$ where $d$ comes from a left-invariant Riemannian metric on $G(\mathbb{R})^\circ$. According to [6, Lem. 2.1], $\| \cdot \|_{\text{max}}$ and $\| \cdot \|$ are power-equivalent as functions on $G(\mathbb{R})^\circ$.

The following is a variant of the weight functions discussed in [29, 5.3], which is suitable for harmonic analysis on $G$-varieties.

**Lemma 7.7.** Suppose that $X$ is a homogeneous $G$-variety. There exists a function $w : X(\mathbb{R}) \to \mathbb{R}_{\geq 1}$ such that the properties (i) and (ii) in Proposition 7.5 are satisfied, and that there exists $C > 0$ and $N \in \mathbb{Z}_{\geq 1}$ such that $w(xg) \leq C\|g\|^N w(x)$ for all $g \in G(\mathbb{R})^\circ$ and $x \in X(\mathbb{R})$. 
Proof. Choose points \( x_1, \ldots, x_n \in X(\mathbb{R}) \) so that \( X(\mathbb{R}) \) decomposes into connected components

\[
X(\mathbb{R}) = \bigsqcup_{i=1}^{n} x_i G(\mathbb{R})^\circ \simeq \bigsqcup_{i=1}^{n} H_i \setminus G(\mathbb{R})^\circ, \quad H_i := \text{Stab}_{G(\mathbb{R})^\circ}(x_i).
\]

Set \( X_i := H_i \setminus G(\mathbb{R})^\circ \). It suffices to fix \( 1 \leq i \leq n \) and define a function \( w : X_i \to \mathbb{R}_{>0} \) with the required properties.

Consider \( \hat{w}(x_i g) := \exp(d(H_i, g)) \). It is continuous and subanalytic in \( g \in G(\mathbb{R})^\circ \) since \( d(H_i, g) \) is. Indeed, \( d(\cdot, \cdot) \) is subanalytic on \( G(\mathbb{R})^\circ \times G(\mathbb{R})^\circ \) by a general result [36, Thm. 3.5.2]; as for \( d(H_i, \cdot) \), repeat the arguments in [9, Rems. 3.11].

The function \( \hat{w} \) factors through \( w : X_i \to \mathbb{R}_{>0} \). Lemma 6.5 (ii) implies \( w \) is subanalytic, and \( w \) is clearly continuous. Recall that \( d(\cdot, \cdot) \) is left invariant. For any \( B \), the closed subset \( \{ x \in X_i : w(x) \leq B \} \) is compact since it is contained in the image of the compact \( \{ g \in G(\mathbb{R})^\circ : \|g\|_{\max} \leq 2B \} \) under \( g \mapsto x_i g \).

Suppose \( g, t \in G(\mathbb{R})^\circ \). From \( d(H_i, gt) \leq d(H_i, g) + d(g, gt) \) we obtain \( \hat{w}(x_i gt) \leq \|t\|_{\max} \hat{w}(x_i g) \). The required estimate on \( w(x g) \) follows from the power-equivalence between \( \| \cdot \| \) and \( \| \cdot \|_{\max} \). \( \square \)

Observe that if \( w \) satisfies the requirements of Lemma 7.7, so does \( w^\alpha \) for any \( \alpha > 0 \).

8. Growth of regular holonomic solutions

We apply the formalism of §7 to the solutions of regular holonomic systems. As the first step, we relate the notion of \( p \)-bounded growth to the following growth condition taken from [14], [26] that appears frequently in microlocal analysis.

**Definition 8.1.** Let \( M \) be a real analytic manifold and \( V \subset M \) be an open subset. We say a continuous function \( f \) on \( V \) has polynomial growth at \( x \in M \) if for any sufficiently small compact neighborhood \( K \ni x \) in \( M \), there exists \( N \in \mathbb{Z}_{\geq 1} \) such that

\[
\sup_{y \in K \cap V} d(y, K \setminus V)^N |f(y)| < +\infty; \tag{8.1}
\]

here \( d \) is the Euclidean distance relative to an analytic coordinate chart on \( K \), and the \( \sup := 0 \) when \( K \cap V = \emptyset \) or \( K \subset V \). We say \( f \) has polynomial growth relative to \( M \) if it is so at every \( x \).

It follows from Lojasiewicz’s inequality (Theorem 6.4, and also [9, Rem. 6.5]) that the foregoing definition is independent of local coordinate charts. Besides, only the behavior of \( f \) around the boundary \( \partial V \) matters. Its relation to \( p \)-bounded growth is explicated as follows.

**Proposition 8.2.** Let \( V \) be a subanalytic open subset of a compact real analytic manifold \( M \), and \( p : M \to \mathbb{R}_{>0} \) be a continuous subanalytic function. Suppose that \( V \subset \{ x \in M : p(x) > 0 \} \). If a continuous function \( f : V \to \mathbb{C} \) has polynomial growth relative to \( M \), then \( f \) has \( p \)-bounded growth.
Proof. Cover ∂V by finitely many compact neighborhoods $K_1, \ldots, K_m$ as above in Definition 8.1; for each $i$ we have chosen an analytic coordinate chart $K_i \hookrightarrow \mathbb{R}^n$ where $n = \dim M$, with the corresponding distance function $d_i$ and the exponent $N_i$ in (8.1); we may also assume $d_i(y, K_i \setminus V) \leq 1$ and $p(y) \leq 1$ for all $y \in K_i \cap V$. Since $p(y) = 0 \implies y \not\in V \implies d_i(y, K_i \setminus V) = 0$ for all $y \in K_i$, Theorem 6.4 (see also [9, Rem. 6.5]) then implies $p(y)^{r_i} \geq c_i d_i(y, K_i)^{N_i}$ for some constants $c_i, r_i > 0$, for all $1 \leq i \leq m$ and $y \in K_i \cap V$. Taking $r := \max\{r_1, \ldots, r_m\}$, we see $p^r|f|$ is bounded on $V$. □

Corollary 8.3. Let $X$ be a smooth $\mathbb{R}$-variety and $V$ be a connected component of $X(\mathbb{R})$. Every continuous function $f : V \to \mathbb{C}$ of polynomial growth automatically has moderate growth at infinity.

Suppose $U$ is a smooth $\mathbb{R}$-variety, and let $\mathcal{M}$ be a $\mathcal{D}_U$-module generated by some global section $\mu$. Let $V$ be an open subset of $U(\mathbb{R})$ and $u : V \to \mathbb{C}$ be an analytic function. Therefore $u$ extends holomorphically to an open subset $\mathcal{V} \subset U^{an}$ containing $V$. We say $u$ is an analytic solution to $\mathcal{M}$, if $\mu \mapsto u$ induces a homomorphism of $\mathcal{D}_\mathcal{V}$-modules for some $\mathcal{V}$ as above; here we also employ the language of analytic $\mathcal{D}$-modules on complex manifolds.

Theorem 8.4. Let $U$ be a smooth $\mathbb{R}$-variety and $\mathcal{M}$ be a regular holonomic $\mathcal{D}_U$-module generated by some $\mu \in \Gamma(U, \mathcal{M})$. Let $V$ be a connected component of $U(\mathbb{R})$. Then every analytic solution $u : V \to \mathbb{C}$ to $\mathcal{M}$ has moderate growth at infinity in the sense of Definition–Proposition 7.4.

Proof. Since $\mathcal{M}$ is holonomic, there exists an open $U_0 \subset U$ such that $\mathcal{M}$ is an integrable connection on $U_0$. Our aim is to show that $u$ is of $p$-bounded growth relative to $X(\mathbb{R})$, for any data $(X, U, U_0, V, \mathcal{M}, u, p)$ where

- $X$ is a smooth proper $\mathbb{R}$-variety, together with an open dense immersion $U \hookrightarrow X$;
- $U_0 \subset U$ is open dense;
- $V$ is a connected component of $U(\mathbb{R})$;
- $\mathcal{M}$ is a regular holonomic $\mathcal{D}_U$-module, generated by some global section $\mu$ and $\mathcal{M}|_{U_0}$ is an integrable connection;
- $u$ is an analytic solution to $\mathcal{M}$ on $V$;
- $p : X(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ is adapted to $U \hookrightarrow X$ in the sense of Definition–Proposition 7.4.

Here we require $X$ to be a proper $\mathbb{R}$-scheme, which is stronger than the compactness of $X(\mathbb{R})$.

Consider a proper surjective morphism $\pi : X' \to X$ between $\mathbb{R}$-varieties. Set $U' := \alpha^{-1}(U)$, $U'_0 := \alpha^{-1}(U_0)$, $u' := u \circ \pi$, $p' := p \circ \pi$.

Let $\mathcal{M}'$ be the $\mathcal{D}_{U'}$-module $L^0\pi^*\mathcal{M}$. It is still regular holonomic, generated by $\mu' := 1 \otimes \mu$, and is an integrable connection on $U'_0$; then $u'$ is an analytic solution to $\mathcal{M}'|_{U'}$. To estimate $u'$, we restrict it to a connected component $V'$ of $\alpha^{-1}(V)$. Lemma 7.2 and Definition–Proposition 7.4 entail that the case for $(X', U', U'_0, V', \mathcal{M}', u', p')$,
for various connected components $V'$, will imply the case for $(X, U, U_0, V, \mathcal{M}, u, p)$. Some preliminary reductions are in order.

(1) First, we reduce to the case where $X \setminus U_0$ and its closed subset $X \setminus U$ are both divisors. This is easily achieved by blowing up.

(2) Next, we take $\pi : X' \to X$ so that $\pi^{-1}(X \setminus U_0)$ is a divisor with normal crossings. This can be done by Hironaka's theorem, but de Jong's alteration [13, Thm. 4.1] suffices for our purpose as $X$ is proper. Then $\pi^{-1}(X \setminus U)$ is also a divisor, as any preimage of a divisor does.

Now study the behavior of $u$ around some $x \in \partial V$ in $X(\mathbb{R})$. Let $\mathcal{D}$ denote the unit open disc in $\mathbb{C}$. We may choose local coordinates $z_1, \ldots, z_n$ on an open neighborhood $O \ni x$ in $X(\mathbb{C})$, such that

$$(z_1, \ldots, z_n) : O \xrightarrow{\sim} \mathcal{D}^n, \quad x \mapsto (0, \ldots, 0),$$

$$O \cap (X \setminus U_0) = \{z_1 \cdots z_n = 0\}, \quad O \cap (X \setminus U) = \{z_1 \cdots z_a = 0\},$$

for some $0 \leq a \leq n$. Therefore $O \cap U = \{z_1 \cdots z_a \neq 0\}$ and $O \cap V$ is a union of connected components of $O \cap U$.

The section $u|_{O \cap V \cap U_0(\mathbb{R})}$ of the local system associated with $\mathcal{M}|_{U_0}$ extends to a multi-valued section on $O \cap (X \setminus U_0)(\mathbb{C})$, i.e., a section on the universal covering. It is a well-known virtue of regular holonomic systems (see, e.g., [14, III.1], [32, IX.2.2]) that the analytically continued $u$ can be expressed as a finite sum

$$u = \sum_{s, m} \Phi_{s, m}(z) z^s \log^m(z), \quad (8.2)$$

with

$$s = (s_1, \ldots, s_n) \in \mathbb{C}^n, \quad m = (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}^n,$$

$$z^s := \prod_{i=1}^n z_i^{s_i}, \quad \log^m(z) := \prod_{i=1}^n (\log z_i)^{m_i},$$

$$\Phi_{s, m} : \text{holomorphic functions on } \mathcal{D}^n,$$

where we take the usual branches of log and $z^s$. To ensure uniqueness, we may assume $s$ ranges over representatives of $\mathbb{C}^n/\mathbb{Z}^n$ (see the Remark after the cited result in [32]). The standard generators $g_1, \ldots, g_n$ of $\pi_1(O \cap (X \setminus U_0)(\mathbb{C}), x) \simeq \mathbb{Z}^n$ act as

$$z_i^{s_i} \xrightarrow{g_i} \exp\left(2\pi \sqrt{-1} s_i\right) z_i^{s_i}, \quad \log z_i \xrightarrow{g_i} \log z_i + 2\pi \sqrt{-1}, \quad i = 1, \ldots, n. \quad (8.3)$$

The $g_i$-action on $u$ for $a < i \leq n$ is realized by analytic continuation along the loop $z_i = \epsilon \exp\left(2\pi \sqrt{-1} \theta\right)$ where $0 < \epsilon \ll 1$ and $\theta \in [0, 2\pi]$; the other coordinates $z_j$ are nonzero constants. But $u$ is analytic on $V$, hence holomorphic in some open neighborhood of $(z_1, \ldots, 0, \ldots, z_n) \in O \cap V$ inside $X(\mathbb{C})$. The monodromic action $g_i$ is thus trivial on $u$ when $a < i \leq n$.

By comparison with (8.3), we conclude that (8.2) involves only terms with

$$s = (s_1, \ldots, s_a, 0, \ldots, 0), \quad m = (m_1, \ldots, m_a, 0, \ldots, 0).$$
Therefore \( u|_{V \cap O} \) has polynomial growth relative to \( X(\mathbb{R}) \) by (8.2); multi-valuedness is not an issue since \( V \cap O \) has contractible connected components. Apply Proposition 8.2 to deduce \( p \)-bounded growth. \( \square \)

**Remark 8.5.** The case \( U = U_0 \) of Theorem 8.4 (see the proof) is recorded in [14, Thm. II.4.1].

### 9. Applications to admissible distributions

Throughout this section, the connected reductive group \( G \), its subgroups and homogeneous spaces are all defined over \( \mathbb{R} \), but the \( \mathcal{D} \)-modules will live over \( \mathbb{C} \). It is thus convenient to adopt the classical viewpoint that the groups and varieties are over \( \mathbb{C} \), but also carry \( \mathbb{R} \)-structures. In particular, the \( \mathcal{D} \)-modules in question live on \( \mathbb{C} \)-varieties. We write \( \mathcal{D}_{Z^{\text{an}}} \) for the sheaf of analytic differential operators on \( Z^{\text{an}} \), the \( \mathbb{C} \)-analytic variety associated with \( Z \).

For any smooth variety \( Z \) defined over \( \mathbb{R} \), we view \( Z(\mathbb{R}) \) as a real analytic manifold. For a \( \mathcal{D}_Z \)-module \( M \), we denote the \( \mathcal{D}_Z \)-module it generates as \( M \) as usual. Hereafter, \( Z \) will be a homogeneous \( G \)-variety and \( K \subset G \) will be a subgroup.

**Definition 9.1.** Let \( u \) be a distribution, or more generally a hyperfunction on \( Z(\mathbb{R}) \) in Sato’s sense. We say \( u \) is \( K \)-admissible (resp. \( \mathfrak{k} \)-admissible) if there exist

- a \( K \)-admissible (resp. \( \mathfrak{k} \)-admissible) \( \mathcal{D}_Z \)-module \( M \),
- a subquotient \( N \) of \( M \) in \( \mathcal{D}_Z \)-Mod and an isomorphism \( \mathcal{D}_Z \cdot u \cong N \).

When \( Z \) is spherical and \( K \) is a spherical subgroup, Corollary 5.7 implies that every \( K \)-admissible hyperfunction generates a regular holonomic \( \mathcal{D}_Z \)-module, and Corollary 3.9 implies that every \( \mathfrak{k} \)-admissible hyperfunction generates a holonomic \( \mathcal{D}_Z \)-module.

We shall study the \( K \)-admissible hyperfunctions through the solution complexes of \( \mathcal{D}_Z \)-modules. Following [25, XI], let

\[
\mathcal{B}_Z(\mathbb{R}) \supset \mathcal{D}b_Z(\mathbb{R}) \supset \mathcal{C}^\infty_Z(\mathbb{R}) \supset \mathcal{A}_Z(\mathbb{R})
\]

denote the sheaves of hyperfunctions, distributions, \( C^\infty \)-functions, and analytic functions on \( Z(\mathbb{R}) \), respectively. Extending by zero, they are also viewed as sheaves on \( Z^{\text{an}} \); in fact they are \( \mathcal{D}_{Z^{\text{an}}} \)-modules. Note that

\[
\mathcal{A}_Z(\mathbb{R}) = \mathcal{O}_Z^{\text{an}}|Z(\mathbb{R})|, \quad \mathcal{B}_Z(\mathbb{R}) = R^\dim Z \mathcal{G}_Z(\mathbb{R}) (\mathcal{O}_Z^{\text{an}}) \otimes \mathcal{O}_Z(\mathbb{R}),
\]

where \( \mathcal{O}_Z(\mathbb{R}) \) is the orientation sheaf.

Hereafter, assume \( Z \) is a spherical homogeneous \( G \)-variety and \( K \subset G \) is a spherical subgroup. For every regular holonomic \( \mathcal{D}_Z \)-module \( M \), we obtain from [24, Cor. 8.3 and 8.5] the quasi-isomorphisms

\[
\text{RHom}_{\mathcal{D}_Z} (\mathcal{M}^{\text{an}}, \mathcal{B}_Z(\mathbb{R})) \cong \text{RHom}_{\mathcal{D}_Z} (\mathcal{M}^{\text{an}}, \mathcal{O}_Z(\mathbb{R})),
\]

\[
\text{RHom}_{\mathcal{D}_Z} (\mathcal{M}^{\text{an}}, \mathcal{A}_Z(\mathbb{R})) \cong \text{RHom}_{\mathcal{D}_Z} (\mathcal{M}^{\text{an}}, \mathcal{C}^\infty_Z(\mathbb{R})).
\]
These \( \text{R} \text{Hom}_{\mathcal{D}\mathcal{Z}^\text{an}}(\mathcal{M}^\text{an}, \cdot) \) are the solution complexes of \( \mathcal{M} \) valued in various function spaces.

**Theorem 9.2.** Assume \( Z \) is spherical homogeneous and \( K \subset G \) is a spherical subgroup. Every \( K \)-admissible hyperfunction on \( Z(\mathbb{R}) \) is a distribution, and every \( K \)-admissible \( C^\infty \)-function on \( Z(\mathbb{R}) \) is analytic.

**Proof.** Consider a hyperfunction \( u \) on \( Z(\mathbb{R}) \), a \( K \)-admissible \( D_Z \)-module \( \mathcal{M} \) and a subquotient \( \mathcal{N} \) of \( \mathcal{M} \) such that \( \mathcal{D}_Z \cdot u \simeq \mathcal{N} \). Since \( \mathcal{M} \) is regular holonomic by Corollary 5.7, so is \( \mathcal{N} \). On the other hand \( u \) can be identified as an element of \( \text{Hom}_{\mathcal{D}\mathcal{Z}^\text{an}}(\mathcal{N}^\text{an}, \mathcal{B}_Z(\mathbb{R})) \). The same holds for distributions, \( C^\infty \)-functions and analytic functions. It remains to take \( H^0 \) in the quasi-isomorphisms above. \( \square \)

In the next two examples, the group acting on homogeneous spaces is always \( G^\text{op} \times G \), and the action is written as \( \gamma(a, b) = a \gamma b \).

**Example 9.3 (Twisted characters).** Take \( \tilde{G} \) to be a twisted space under \( G \) (Example 3.10 with \( k = \mathbb{R} \)) and take \( K = \{(g^{-1}, g) : g \in G \} \). We also fix a smooth character \( \omega : G(\mathbb{R}) \to \mathbb{C}^\times \) and consider the distributions \( \Theta \) on \( \tilde{G}(\mathbb{R}) \) satisfying

\[
\Theta(gf) = \omega(g^{-1})\Theta(f), \quad gf(\gamma) = f(g^{-1}\gamma g)
\]

for all \( g \in G(\mathbb{R}) \). A typical source of such distributions on \( \tilde{G}(\mathbb{R}) \) is the \( \omega \)-twisted character. We follow [30, I.2.6] to define them. First, we define a smooth \( \omega \)-representation to be a pair \( (\pi, \tilde{\pi}) \) where \( \pi \) is an SAF representation of \( G(\mathbb{R}) \) (see [6, p.46]) with underlying Fréchet space \( V_\pi \), and \( \tilde{\pi} : \tilde{G}(\mathbb{R}) \to \text{Aut}_\mathbb{C}(V_\pi) \) is such that

- \( \tilde{\pi}(a \gamma b) = \pi(a)\tilde{\pi}(\gamma)\pi(b) \cdot \omega(b) \) for all \( a, b \in G(\mathbb{R}) \) and \( \gamma \in \tilde{G}(\mathbb{R}) \),
- for some \( \gamma \) (equivalently, for any \( \gamma \)) in \( \tilde{G}(\mathbb{R}) \), the endomorphism \( \tilde{\pi}(\gamma) \) of \( V_\pi \) is invertible and continuous.

For every \( f \in C_c^\infty(\tilde{G}(\mathbb{R})) \), set

\[
\tilde{\pi}(f) := \int_{\tilde{G}(\mathbb{R})} f(\gamma)\tilde{\pi}(\gamma) \, d\ell\gamma \in \text{End}_\mathbb{C}(V_\pi)
\]

by fixing a left \( G(\mathbb{R}) \)-invariant measure \( d\ell\gamma \) on \( \tilde{G}(\mathbb{R}) \). If we fix \( \gamma_0 \in \tilde{G}(\mathbb{R}) \) and set

\[
f_0(g) := f(g\gamma_0), \quad A := \tilde{\pi}(\gamma_0)
\]

so that \( f_0 \in C_c^\infty(\tilde{G}(\mathbb{R})) \), then

\[
\tilde{\pi}(f) = \pi(f_0) \circ A.
\]

Note that \( A \) is an intertwining operator from \( \omega \otimes \pi \) to \( \pi \circ \text{Ad}(\gamma_0) \). This will allow us to define the \( \omega \)-twisted character of \( \tilde{\pi} \) as the distribution

\[
\Theta_{\tilde{\pi}} : f \mapsto \text{tr}(\tilde{\pi}(f)) = \text{tr}(\pi(f_0) \circ A : V_\pi \to V_\pi).
\]

To be precise, one has to embed \( V_\pi \) into a Hilbert globalization of the associated Harish-Chandra module in order to talk about the trace; see [6, §5.1].
The distribution $\Theta_\pi$ is K-admissible. Indeed, it satisfies the equivariance (9.1) under $G(\mathbb{R}) \simeq K(\mathbb{R})$, and is clearly $Z(g^{\text{op}} \times g)$-finite. When $\gamma_0$ can be chosen with $\text{Ad}(\gamma_0) = \text{id}$, we revert to the Harish-Chandra characters.

**Example 9.4** (Relative characters). For $i = 1, 2$, let $H_i \subset G$ be a spherical subgroup and $\chi_i : h_i \to \mathbb{C}$ be a reductive character. Let $\tilde{\pi}$ be the contragredient of an SAF representation $\pi$ of $G(\mathbb{R})$. Consider continuous linear functionals that are equivariant under the Lie algebras $h_1$ and $h_2$:

$$\phi_1 \in \text{Hom}_{h_1}(V_\pi, \chi_1), \quad \phi_2 \in \text{Hom}_{h_2}(V_\pi, -\chi_2).$$

Noting that $\pi^{\vee\vee} = \pi$, the corresponding relative character is the distribution on $G(\mathbb{R})$ (cf. §1)

$$\Theta_{\phi_1, \phi_2} : f \mapsto \langle \phi_1, \pi(f) \phi_2 \rangle.$$  

These distributions are studied thoroughly in [3], and the holonomicity has been established there; in loc. cit., $\Theta_{\phi_1, \phi_2}(f)$ is extended to all Schwartz functions $f$ on $G(\mathbb{R})$.

The conditions on $\phi_1, \phi_2$ and $\chi_1, \chi_2$ imply that $\Theta_{\phi_1, \phi_2}$ is an $H_1^{\text{op}} \times H_2$-admissible distribution on $G(\mathbb{R})$; in fact $D_G \cdot \Theta_{\phi_1, \phi_2}$ is an $H_1^{\text{op}} \times H_2$-admissible $D_G$-module by Example 5.5 (ii).

If the reductivity assumption on $\chi_1, \chi_2$ is dropped, $D_G \cdot \Theta_{\phi_1, \phi_2}$ is only $h_1^{\text{op}} \times h_2$-admissible. Suppose for instance that $\chi_2$ is non-reductive, then $\Theta := \Theta_{\phi_1, \phi_2}$ is an irregular holonomic $D_G$-module. To see this, note that there exists an $H_1^{\text{op}} \times H_2$-invariant open dense subset $U \subset G$ on which $\Theta$ is analytic (see below). There exists a copy of $G_a$ in $R_u(H_2)$ on which $\chi_2$ is nontrivial. Were $D_G \cdot \Theta$ regular holonomic, so would be its pullback to any $G_a$-orbit in $U$. However, $\Theta$ restricts to an exponential function on $G_a$, whose $D$-module is irregular at $\infty$.

For the next result, we return to general $G$, $K$ and $Z$.

**Theorem 9.5.** Assume $Z$ is spherical homogeneous and $K \subset G$ is a spherical subgroup. Let $u$ be a $\mathfrak{t}$-admissible distribution on $Z(\mathbb{R})$. There is a $K$-invariant open dense subset $U \subset Z$ such that $u$ is analytic on $U(\mathbb{R})$. When $u$ is $K$-admissible, $P \cdot u|_{U(\mathbb{R})}$ has moderate growth at infinity in the sense of Definition–Proposition 7.4, for all $P \in D_U$.

**Proof.** Take a $\mathfrak{t}$-admissible module $\tilde{M}$ that contains $u$ in a subquotient. Corollary 3.9 implies the existence of $U$.

When $u$ is $K$-admissible, Theorem 5.7 implies $\tilde{M}$ is regular; so is its restriction to $U$, hence $M_P := \mathcal{D}_U \cdot Pu|_{U(\mathbb{R})}$ are also regular holonomic, for any $P \in D_U$. Apply Theorem 8.4 to $M_P$, $\mu := Pu|_{U(\mathbb{R})}$, its analytic solution $Pu|_{U(\mathbb{R})}$; and to each connected component $V$ of $U(\mathbb{R})$ to deduce the moderate growth at infinity.

**Example 9.6.** Consider the twisted character $\Theta_\tilde{\pi}$ (Example 9.3) for instance. As seen in Example 3.10 (with the notations therein), one can take $U := \tilde{G}_{\text{reg}}$ inside $\tilde{G}$. Let us re-define $\Theta_\pi$ to be zero on $(\tilde{G} \setminus U)(\mathbb{R})$. We claim that Theorem 9.5 implies
that $|D\tilde{G}|^a \Theta_\tilde{\pi}$ is locally bounded on $\tilde{G}(\mathbb{R})$ for some $a > 0$. To see this, start with any smooth compactification $\tilde{G} \hookrightarrow \mathcal{G}$ and any $p : \mathcal{G}(\mathbb{R}) \to \mathbb{R}_{\geq 0}$ adapted to $U \hookrightarrow \mathcal{G}$ as in Definition–Proposition 7.4. For each $g \in G(\mathbb{R})$, take a compact subanalytic neighborhood $E \ni g$ inside $\tilde{G}(\mathbb{R})$. Since $E \cap (\mathcal{G} \setminus U)(\mathbb{R}) = E \cap (\tilde{G} \setminus U)(\mathbb{R})$, we have $p(\gamma) = 0 \iff |D\tilde{G}(\gamma)| = 0$ for all $\gamma \in E$. Theorem 6.4 implies that $p$ and $|D\tilde{G}|$ are power-equivalent over $E$. Therefore $\Theta_\tilde{\pi}|_E$ is of $|D\tilde{G}|$-bounded growth. This is considerably weaker than Harish-Chandra’s result [21, Thm. 3] which attains $a = 1/2$. The same estimates work for $P \cdot \Theta_\tilde{\pi}$ for any $P \in D_U$.

Let $\theta$ be a Cartan involution of $G$. When $K = G^\theta$, so $K(\mathbb{R}) \subset G(\mathbb{R})$ is a maximal compact subgroup. The classical technique of elliptic regularity applies. We rephrase it in the language of $\mathcal{D}$-modules as follows. It does not require sphericity of $Z$.

**Proposition 9.7.** Let $M$ be a $\mathfrak{g}$-admissible $D_Z$-module, then $\mathcal{M}^{\text{an}}$ is elliptic in the sense of [25, Def. 11.5.5]. The same is true for all $\mathcal{D}_Z$-submodules $\mathcal{N}$ of $\mathcal{M}$. Consequently,

$$\text{RHom}_{\mathcal{D}_Z^{\text{an}}} \left( \mathcal{N}^{\text{an}}, \mathcal{A}(Z(\mathbb{R})) \right) \to \text{RHom}_{\mathcal{D}_Z^{\text{an}}} \left( \mathcal{N}^{\text{an}}, \mathcal{R}(Z(\mathbb{R})) \right)$$

is a quasi-isomorphism. Consequently, $\mathfrak{g}$-admissible hyperfunctions on $Z(\mathbb{R})$ are analytic.

**Proof.** To show the ellipticity of $\mathcal{M}$, we have to show that

$$T^*_Z(\mathbb{R}) Z(\mathbb{C}) \cap \text{Ch} \mathcal(M) = T^*_Z(\mathbb{R}) Z(\mathbb{R}).$$

Here $Z(\mathbb{C})$ is viewed as a real manifold, $T^* Z(\mathbb{C})$ denotes the real cotangent bundle of $Z(\mathbb{C})$, containing the conormal bundle $T^*_Z \mathbb{R}$ to $Z(\mathbb{R})$. We have $T^* (Z^{\text{an}}) \simeq T^* Z(\mathbb{C})$ as real analytic manifolds by forgetting complex structures. Hence the intersection above makes sense.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition into $\pm 1$-eigenspaces of $\theta$, where all vector spaces are over $\mathbb{R}$. There exists a non-degenerate bilinear form $\beta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which is negative definite (resp. positive definite) on $\mathfrak{k}$ (resp. on $\mathfrak{p}$). Let $\Omega \in Z(\mathfrak{g})$ and $\Omega_K \in Z(\mathfrak{k})$ be the Casimir elements corresponding to $\beta$ and $\beta|_K$. Let $\Delta := \Omega - 2\Omega_K$. It is homogeneous of degree 2 in $U(\mathfrak{g})$, and Proposition 3.4 (or its proof) gives $\text{Ch} \mathcal(M) \subset \mu^{-1}(\Delta = 0)$.

Take a basis $X_1, \ldots, X_a$ of $\mathfrak{k}$ and $X_{a+1}, \ldots, X_b$ of $\mathfrak{p}$ under which $\beta$ becomes

$$\text{diag}(-1, \ldots, -1, 1, \ldots, 1).$$

Then $\Delta = \sum_{i=1}^b X_i^2$ is negative definite on $\sqrt{-1} \cdot \mathfrak{g}^* \subset \mathfrak{g} \otimes \mathbb{C}$. Therefore the image of $T^*_Z(\mathbb{R}) Z(\mathbb{C})$ under $\mu$ lies in $(\sqrt{-1} \cdot \mathfrak{g}^*) \cap \{ \Delta = 0 \} = \{ 0 \}$. Upon recalling the definition of $\mu$, we deduce $T^*_Z(\mathbb{R}) Z(\mathbb{C}) \cap \text{Ch} \mathcal(M) = T^*_Z(\mathbb{R}) Z(\mathbb{R})$. 


For any subquotient \( \mathcal{N} \) of \( \mathcal{M} \), we have \( \text{Ch}(\mathcal{N}) \subset \text{Ch}(\mathcal{M}) \) thus \( \mathcal{N} \) is elliptic as well. The quasi-isomorphism for solution complexes for \( \mathcal{N} \) follows from [25, p. 468]. By taking \( \mathcal{N} \) to be the module generated by a \( \mathfrak{t} \)-admissible hyperfunction \( u \) on \( Z(\mathbb{R}) \), we infer that \( u \) is analytic. \( \square \)

10. The case of generalized matrix coefficients

The conventions from §9 remain in force. Moreover, we assume:

- \( G \) is a connected reductive \( \mathbb{R} \)-group,
- \( Z \) is a spherical homogeneous \( G \)-variety with \( Z(\mathbb{R}) \neq \emptyset \),
- \( K = G^\theta \) for some Cartan involution \( \theta \) of \( G \).

We may choose \( x_0 \in Z(\mathbb{R}) \) to identify \( Z' = H_n G \) where \( H \) is a spherical subgroup. As symmetric subgroups are spherical [37, Thm. 26.14], \( K \)-admissible distributions or hyperfunctions on \( Z(\mathbb{R}) \) generate regular holonomic \( D_Z \)-modules (Definition 9.1).

Next, we fix an SAF representation \( \pi \) of \( G(\mathbb{R}) \) (see [6]). Set

\[
\mathcal{N}_\pi := \text{Hom}_{G(\mathbb{R})}(\pi, C^\infty(Z(\mathbb{R})))
\]

where we take the continuous \( \text{Hom} \) of continuous \( G(\mathbb{R}) \)-representations, and \( C^\infty(Z(\mathbb{R})) \) is topologized as in [31, §4.1]. Let \( V^K_{\text{fini}} \) denote the Harish-Chandra module of \( K \)-finite vectors in \( V_\pi \).

It is well-known that \( \text{dim}_C \mathcal{N}_\pi \) is finite as \( Z \) is spherical, see [3, Thm. E] and the references therein, where stronger versions are obtained. We are going to show that the finiteness is an outright consequence of regularity, thereby giving a somewhat more geometric proof of this result. First, recall the localization functor \( \text{Loc}_Z \) from Example 2.7.

Lemma 10.1. Write \( \hat{\mathcal{O}}_{Z,x_0} \) for the formal completion of \( \mathcal{O}_{Z,x_0} \). For any \( U(\mathfrak{g}) \)-module \( V \), we have an isomorphism of \( C \)-vector spaces

\[
\text{Hom}_{\mathcal{O}_{an,x_0}} \left( \text{Loc}_Z(V)^{an}, \hat{\mathcal{O}}_{Z,x_0} \right) \cong \text{Hom}_C \left( V/\mathfrak{h}V, C \right)
\]

\[
\Phi \mapsto [v + \mathfrak{h}V \mapsto \Phi(1 \otimes v)(x_0)]
\]

where \( \Phi(1 \otimes v)(x_0) \) means the evaluation at \( x_0 \) of the formal function \( \Phi(1 \otimes v) \).

Proof. An element of the left-hand side is the same as a \( U(\mathfrak{g}) \)-homomorphism \( V \to \hat{\mathcal{O}}_{Z,x_0} \). Realize \( \hat{\mathcal{O}}_{G,1} \) as the \( C \)-algebra of linear functions \( U(\mathfrak{g}) \to C \). Observing that \( Z \simeq H \setminus G \), we may identify \( \hat{\mathcal{O}}_{Z,x_0} \) with the \( C \)-subalgebra \( \mathcal{F} \) of \( \hat{\mathcal{O}}_{G,1} \) consisting of linear functions which are zero on \( U(\mathfrak{h})U(\mathfrak{g}) \).

Note that the left \( U(\mathfrak{g}) \)-action on \( \hat{\mathcal{O}}_{Z,x_0} \) transcribes to \( (Xf)(Y + U(\mathfrak{h})U(\mathfrak{g})) = f(YX + U(\mathfrak{h})U(\mathfrak{g})) \), where \( X, Y \in U(\mathfrak{g}) \) and \( f \in \mathcal{F} \). Define:

\[
\text{Hom}_{U(\mathfrak{g})}(V, \mathcal{F}) \xrightarrow{\Phi} \text{Hom}_C \left( V/\mathfrak{h}V, C \right)
\]

\[
\Phi \mapsto \Phi(\cdot) (1 + U(\mathfrak{h})U(\mathfrak{g}))
\]

\[
[v \mapsto \Psi((\cdot)v) \in \mathcal{F}] \longmapsto \Psi.
\]
It is routine to check that both arrows are well-defined, $C$-linear and mutually inverse. The assertion follows. □

**Proposition 10.2.** Let $V$ be a Harish-Chandra module of $G$ and $K$. Then the space $\text{Hom}_C(V/\mathfrak{h}V, C)$ is finite-dimensional. In fact it is isomorphic to the space $\text{Hom}_{\mathscr{D}_{\mathbb{Z}^\text{an},x_0}}(\text{Loc}_Z(V)^\text{an}, \hat{\mathcal{O}}_{\mathbb{Z}^\text{an},x_0})$.

**Proof.** Let $\mathcal{M} := \text{Loc}_Z(V)$. Example 5.5 (iii) together with Corollary 5.7 imply that $\mathcal{M}$ is regular holonomic. There is a natural homomorphism $\nu_{x_0}$ from the analytic local ring $\mathcal{O}_{\mathbb{Z}^\text{an};x_0}$ to $\hat{\mathcal{O}}_{x_0}$. The comparison theorem [23, Prop. 7.3.1] implies that $\nu_{x_0}$ induces $\text{Hom}_{\mathcal{D}_{\mathbb{Z}^\text{an},x_0}}(\mathcal{M}^\text{an}, \hat{\mathcal{O}}_{x_0})$.

Note that the automatic continuity for $\mathfrak{h} \subset \mathfrak{g}$ discussed in [6, §11.2] is still unreachable by these results.

**Corollary 10.3.** For every SAF representation $\pi$ of $G(\mathbb{R})$ we have $\dim_C N_\pi < +\infty$, and $\mathfrak{h}V_{\pi}$ is of finite codimension in $V_\pi$.

**Proof.** Put $V := V_{\pi}^{K-\text{fini}}$. We have $\text{Hom}_C(V/\mathfrak{h}V, C) \subset \text{Hom}_{H(\mathbb{R})}(\pi, C)$ where the right hand side indicates the continuous Hom. By Frobenius reciprocity in this setting, we have $\text{Hom}_{H(\mathbb{R})}(\pi, C) \simeq \text{Hom}_{G(\mathbb{R})}(\pi, C^\infty(H(\mathbb{R})\backslash G(\mathbb{R})))$. Write $Z(\mathbb{R}) = \bigsqcup_{i=1}^r x_i G(\mathbb{R})$ with $H_i := \text{Stab}_G(x_i)$, then

$$C^\infty(Z(\mathbb{R})) = \bigoplus_{i=1}^r C^\infty(H_i(\mathbb{R})\backslash G(\mathbb{R})).$$

Lemma 10.1 applied to $H = H_1, \ldots, H_r$ gives $\dim_N \pi < +\infty$. □

Let us turn to the functions $\eta(v) \in C^\infty(Z(\mathbb{R}))$ for $\eta \in N_\pi$ and $v \in V_\pi$. They are called the generalized matrix coefficients of $\pi$.

**Proposition 10.4.** For every $v \in V_{\pi}^{K-\text{fini}}$ and $\eta \in N_\pi$, the function $\eta(v)$ on $Z(\mathbb{R})$ is $K$-admissible.

**Proof.** This can be seen in two ways. Either apply Example 5.5 (i) together with Remark 2.5 to see that $\eta(v)$ generates a $K$-admissible $D_Z$-module, or apply (iii) to see that $\mathscr{D}_Z \otimes V_{\pi}^{K-\text{fini}} \mid_{U(\mathfrak{g})}$ is a $K$-admissible $\mathscr{D}_Z$-module and note that each $\eta \in N_\pi$ induces a well-defined homomorphism

$$\mathscr{D}_Z \otimes V_{\pi}^{K-\text{fini}} \rightarrow C_Z^\infty, \quad P \otimes v \mapsto P\eta(v)$$

between $\mathscr{D}_Z$-modules. □
By combining Theorem 8.4 and Proposition 10.4, it is possible to deduce the estimate below for generalized matrix coefficients. Recall from [6, p.51] that a continuous semi-norm $q : V_\pi \to \mathbb{R}_{\geq 0}$ is called $G$-continuous if $G(\mathbb{R}) \times V_\pi \to V_\pi$ is continuous with respect to the $q$-topology on $V_\pi$.

**Theorem 10.5.** Let $\eta \in \mathbb{N}_\pi$. There exist

- a function $w : Z(\mathbb{R}) \to \mathbb{R}_{\geq 1}$ as in Lemma 7.7,
- a G-continuous semi-norm $q : V_\pi \to \mathbb{R}_{\geq 0}$,

such that $|\eta(v)(x)| \leq w(x)q(v)$ for all $v \in V_\pi$ and $x \in Z(\mathbb{R})$. They depend on $\pi$ and $\eta$.

Using the theory of toroidal embeddings [27, §7] together with Lojasiewicz’s inequality, this can be upgraded to an estimate in terms of the weak polar decomposition [27, §13]. Some further definitions are in order.

Fix a minimal parabolic $\mathbb{R}$-subgroup $P \subset G$, a Levi component $M_P$ of $P$, and let $A$ be the maximal split central torus in $M_P$. Using the local structure theorems for $Z$ (see [27, 4.6 Cor. and (4.15)]), one attaches to $Z$ an affine smooth subvariety $Z_{el} \subset Z$ (the elementary kernel), $Z_{el}(\mathbb{R}) \neq \emptyset$, on which $A$ acts with kernel $A_0$. Set $A_Z := A/A_0$ which acts freely on $Z_{el}$. We may take $x_0 \in Z_{el}(\mathbb{R})$.

In [27, (10.9)] is defined the set of simple restricted spherical roots $\Sigma_R(Z) \subset X^*(A)$. Following [27, (13.1), (13.5)] we define

$$A_Z(\mathbb{R})^- := \{a \in A_Z(\mathbb{R}) : \forall \sigma \in \Sigma_R(Z), |\sigma(a)| \leq 1\},$$

$$a_Z^* := X^*(A_Z) \otimes \mathbb{R}.$$

For all $\lambda = \sum_i \lambda_i \otimes t_i \in a_Z^*$ and $a \in A_Z(\mathbb{R})$, we write $|a|^\lambda := \prod_i |\lambda_i(a)|^{t_i}$, which is well defined.

Also needed is the affine group of $\mathbb{R}$-central automorphisms $\mathfrak{A} := \mathfrak{A}_R(Z)$ acting on the left of $Z$; see [27, (8.5)]. Its identity connected component $\mathfrak{A}^\circ$ is a split torus embedded in $A_Z$. Given an SAF representation $\pi$, note that $\mathfrak{A}(\mathbb{R})$ acts linearly and continuously on $\mathbb{N}_\pi$ by

$$(a\eta)(v)(x) = \eta(v)(a^{-1}x), \quad a \in \mathfrak{A}(\mathbb{R}), v \in V_\pi, x \in Z(\mathbb{R}).$$

The eigen-embeddings in $\mathbb{N}_\pi$ are defined to be the eigenvectors under $\mathfrak{A}^\circ(\mathbb{R})$.

**Corollary 10.6.** Let $\pi$ be an SAF representation, and $\eta \in \mathbb{N}_\pi$ be an eigen-embedding. For any closed subanalytic subset $\Omega \subset G(\mathbb{R})$, there exist $\lambda \in a_Z^*$ and a $G$-continuous semi-norm $q : V_\pi \to \mathbb{R}_{\geq 0}$, both depending on $(\pi, \eta)$, such that

$$|\eta(v)(x_0a\omega)| \leq |a|^\lambda q(v), \quad a \in A_Z(\mathbb{R})^-, \omega \in \Omega.$$

This is comparable to [28, Thm. 7.2]. However, the crude estimate above can be deduced more directly from the notion of SAF representations: they are of moderate growth. In loc. cit., one obtains the optimal exponent $\lambda$, and the approach thereof can be extended to all real spherical homogeneous spaces; see [8]. For this reason, the proofs of both Theorem 10.5 and Corollary 10.6 are omitted here.
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