Penetration depth of an electric field in a semi-infinite classical plasma

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February 12, 2019

Abstract

It is shown that the penetration of an oscillating electric field in a semi-infinite classical plasma obeys the standard exponential attenuation law $e^{-x/\lambda_e}$ (besides oscillations), where $x$ is the distance from the wall and $\lambda_e$ is the extinction length (penetration depth, attenuation length). The penetration depth is computed here explicitly; it is shown that it is of the order $\lambda_e \simeq \| \varepsilon \|/(1 - \varepsilon)^{1/3}v_{th}/\omega$, where $\varepsilon$ is the dielectric function, $\omega$ is the frequency of the field and $v_{th} = \sqrt{T/m}$ is the thermal velocity ($T$ being the temperature and $m$ the particle (electron) mass). The result is obtained by including explicitly the contribution of the surface term.

Key words: Landau damping, semi-infinite plasma; electric field; penetration depth

1 Introduction

It is well known that there exists a mechanism of energy transfer between collective modes and individual particles in collisionless classical plasmas, governed by the Landau damping. The origin of this mechanism is the causal
character of the response of the plasmas to external excitations. The Landau damping received much interest, due to its application to heating plasmas by radiofrequency electric fields.\[2\]-\[6\] Also, the Landau damping enjoyed controversies along the years, as a consequence of the counter-intuitive character of an energy loss in collisionless plasmas.\[7\]-\[19\] Apart from theoretical and experimental investigations, numerical-analysis\[24\]-\[5\] and mathematical studies are devoted to the phenomenon,\[20\]-\[23\] which show both the complexity of the concept and difficulties related to its understanding at the fundamental level.

In semi-infinite plasmas the Landau damping appears as attenuated spatial oscillations (vibrations). This phenomenon, with its characteristic penetration depth, has a particular relevance for surface effects. Specifically, the Landau damping in semi-infinite plasmas implies an attenuated electric field, with spatial and temporal oscillations, besides a uniform component, as a response to a uniform oscillating external electric field perpendicularly applied to the plasma surface. The calculation of the exact form of this response is complicate, due, on one hand, to the difficulties related to the Landau damping, and, on the other hand, as a consequence of the presence of the surface. The latter point is particularly interesting, because the response is discontinuous at the surface, and the usual Fourier or Laplace techniques may not include properly this discontinuity. In addition, the surface boundary conditions may bring further complications. These difficulties have been analyzed recently in a clear formulation in Ref. [26]. In various approximations (see, for instance, Refs. [24]-[27]), including the original calculation in Ref. [1], the asymptotically attenuated field is presented as being proportional to $x^{2/3}e^{-\frac{4}{3}(\omega x/v_{th})^{2/3}}$, where $x$ is the distance from the wall, $\omega$ is the frequency of the field and $v_{th} = \sqrt{T/m}$ is the thermal velocity, $T$ being the temperature and $m$ being the particle mass (electrons); sometimes, an exponential attenuation $\sim e^{-\omega_{0}x/v_{th}}$ is included, where $\omega_{0}$ is the plasma frequency. A non-linear $x$-dependence ($\sim x^{2/3}$) is related to model assumptions made upon the surface and an asymptotic treatment of the Landau damping for the Boltzmann kinetic equation (see, for instance, Ref. [26]). We show here that, when the surface condition (surface term) is included explicitly, the attenuated field obeys the standard exponential attenuation law $e^{-x/\lambda_{e}}$ (apart from factors oscillating in space), where $\lambda_{e}$ is an extinction length (penetration depth, attenuation length) which is computed here explicitly; up to immaterial numerical factors, it is of the order $\lambda_{e} \simeq \frac{\| \varepsilon \|\sqrt{1 - \varepsilon}}{(1 - \varepsilon)}^{1/3}v_{th}/\omega$,\[26\].
where $\varepsilon$ is the dielectric function.

## 2 Semi-infinite plasma

We consider a classical plasma at thermal equilibrium consisting of mobile charges $q$ with mass $m$ and concentration $n$ (electrons) moving in a rigid neutralizing background. We confine this plasma to a semi-infinite space (half-space) $x > 0$, bounded by a plane surface $x = 0$. The plasma is subject to a uniform oscillating external electric field $E_0 e^{-i\omega t}$, where $E_0$ is directed along the $x$-direction (capacitively coupled plasma). The plasma is governed by the Maxwell distribution. The mean thermal velocity is sufficiently small to consider plasma unmagnetized. Since the field is directed along the $x$-direction we may integrate over the transverse velocities and use for the Maxwell distribution $F = n(\beta m/2\pi)^{1/2} e^{-\frac{1}{2} \beta m v^2}$, where $v$ is the velocity along the $x$-direction and $\beta = 1/T$ is the reciprocal temperature. In the collisionless regime the change $f(x,v)e^{-i\omega t}$ in the Maxwell distribution is governed by the Boltzmann (Vlasov) equation

$$-i\omega f + v \frac{\partial f}{\partial x} + \frac{q}{m} (E_0 + E + E_1) \frac{\partial F}{\partial v} = 0$$  \hspace{1cm} (1)

where $E$ is a uniform internal electric field and $E_1$ is another internal electric field, which may vary in space; these fields are generated by internal charges and currents. The uniform reaction field $E$ occurs in an infinite space too, i.e. a space bounded by surfaces at infinity (it is a bulk reaction field), while the non-uniform field $E_1$ is due to the presence of the surface (it is a surface field). We seek the solution of equation (1) as $f(x,v) = f_0(v) + f_1(x,v)$, where

$$-i\omega f_0 + \frac{q}{m} (E_0 + E) \frac{\partial F}{\partial v} = 0$$ \hspace{1cm} (2)

and

$$-i\omega f_1 + v \frac{\partial f_1}{\partial x} + \frac{q}{m} E_1 \frac{\partial F}{\partial v} = 0$$ \hspace{1cm} (3)

The uniform part $f_0$ of the solution does not generate charge density in plasma; it generates a current density. Therefore, it should satisfy the equation

$$i\omega E = 4\pi q \int dv \cdot v f_0$$  \hspace{1cm} (4)
it is easy to see that this equation arises from the general equation $\partial E / \partial t + 4\pi j = 0$, where $j$ is the current density; this equation ensures the vanishing of the (internal) magnetic field, as expected. The non-uniform part $f_1$ of the solution generates a charge density in plasma; it satisfies the equation

$$\frac{\partial E_1}{\partial x} = 4\pi q \int dv f_1 .$$

The solution of equations (2) and (4) is

$$f_0 = -\frac{i q \omega E_0}{m(\omega^2 - \omega_0^2)} \frac{\partial F}{\partial v}$$

and

$$E = \frac{\omega_0^2}{\omega_2 - \omega_0^2} E_0 , \ E_t = E_0 + \frac{\omega_0^2}{\omega_2 - \omega_0^2} E_0 ,$$

where $\omega_0 = (4\pi n q^2 / m)^{1/2}$ is the plasma frequency; we recognize here the response of a boundless plasma to an electric field (restricted to $x > 0$), where $\varepsilon = 1 - \omega_0^2 / \omega^2$ is the dielectric function and $E_t$ is the total field in plasma ($P = \chi E_t$ is the polarization and $\chi = (\varepsilon - 1)/4\pi = -n q^2 / m \omega^2$ is the electric susceptibility).

In order to deal conveniently with the boundary condition at the surface we multiply equation (3) by the step function $\theta(x)$ ($\theta(x) = 1$ for $x > 0$, $\theta(x) = 0$ for $x < 0$) and restrict ourselves to the solution for $x > 0$; equation (3) becomes

$$-i \omega f_1 + v \frac{\partial f_1}{\partial x} + \frac{q}{m} E_1 \frac{\partial F}{\partial v} = v f_s \delta(x) ,$$

where $f_s = f_s(v) = f_1(x = 0, v)$; we can check directly this surface term by integrating equation (8) along a small distance perpendicular to the surface $x = 0$. Similarly, equation (5) becomes

$$\frac{\partial E_1}{\partial x} - E_{1s} \delta(x) = 4\pi q \int dv f_1 ,$$

where $E_{1s} = E_1(x = 0)$. The inclusion of the surface $\delta$-terms in equations (8) and (9) is the main point of this paper.

In equations (8) and (9) we use the Fourier transforms with respect to the coordinate $x$ (and restrict ourselves to $x > 0$); we get

$$f_1(k, v) = \frac{i}{\omega - vk + i\gamma} \left[ v f_s(v) - \frac{q}{m} \frac{\partial F}{\partial v} E_1(k) \right]$$

(10)
and

\[
E_1(k) = \frac{4\pi q \int dv \frac{v f_s(v)}{\omega -vk + i0^+} - iE_{s1}}{k + \frac{4\pi q^2}{m} \int dv \frac{\partial F/\partial v}{\omega -vk + i0^+}},
\]

(11)

where \( \gamma \to 0^+ \). It is worth noting that in the Fourier transforms we replace \( \omega \) by \( \omega + i\gamma \), \( \gamma \to 0^+ \), in order to ensure the causal behaviour (i.e., zero response for time \( t < 0 \), which requires a pole in the lower \( \omega \)-half-plane). This procedure gives a pole in the upper \( k \)-half-plane (this is the connection between the Landau damping and the spatial decay). At the same time, in the integrals with respect to \( v \) we may take the limit \( \gamma \to 0^+ \), which avoids the singularity \( \omega = vk \); the insertion of the parameter \( \gamma \) produces the Landau damping. We denote by \( A \) the denominator in equation (11); it can be estimated as

\[
A = k + \frac{4\pi q^2}{m} \int dv \frac{\partial F/\partial v}{\omega -vk + i0^+} = k + \frac{4\pi q^2}{m} P \int dv \frac{\partial F/\partial v}{\omega -vk} - i\frac{4\pi^2 g^2}{mk} \frac{\partial F}{\partial v} \bigg|_{v=\omega/k} \]

\[
\simeq k(1 - \omega_0^2/\omega^2) - i\frac{4\pi^2 g^2}{mk} \frac{\partial F}{\partial v} \bigg|_{v=\omega_0/k};
\]

(12)

we can see that the zeroes of \( A \) give the damped collective eigenmodes \( \omega = \pm \omega_0 - i\Gamma \) (plasma frequency), where \( \Gamma \) is given by the imaginary part in equation (12) \( (\Gamma \simeq -2\pi^2 q^2 \omega_0/mk^2)(\partial F/\partial v) \big|_{v=\omega_0/k}); \) this is the Landau damping.

3 Penetrating electric field

In order to estimate the field \( E_1(x) \) we need the zeroes of \( A \) with respect to \( k \) in equation (11). It is convenient to introduce the variable \( \xi = \sqrt{\beta m/2\omega/k} \). We can see easily that the zeroes of \( A \) are given by \( \xi^2 \big| \xi \mid e^{-\xi^2} = -i\alpha \), where \( \alpha = | \varepsilon | /2\sqrt{\pi(1 - \varepsilon)}; \) we consider the case \( \omega < \omega_0 \) (\( \varepsilon < 0 \); the rather unrealistic case \( \omega > \omega_0 \) can be treated similarly, by using the equation \( \xi^2 \big| \xi \mid e^{-\xi^2} = i\alpha \). For small values of \( \alpha \) we get two roots of the equation \( A = 0 \), given by \( k_{1,2} \simeq \pm \frac{1}{2\alpha^{1/3}} \sqrt{\beta m \omega(1 + i)} \); only \( k_1 \) (placed in the upper half-plane) contributes to the \( k \)-integration for \( x > 0 \). In estimating the integral in the numerator of equation (11) we may leave aside the contribution of the
principal value. For \( k \) near \( k_1 \) the field \( E_1(k) \) has the form

\[
E_1(k) \simeq \frac{B}{k-k_1+i\pi/2},
\]

(13)

\[
B = \frac{8\sqrt{2}\pi q\alpha^{2/3}v_{th}^2}{5\pi|\varepsilon|\omega_0} \frac{1}{(1+i)f_s(1+i)} + \frac{2i\varepsilon}{5\pi|\varepsilon|} E_{1s}.
\]

The reverse Fourier transformation leads to

\[
E_1(x) = E_{1s} e^{i(\omega-1)x/2a^{1/3}v_{th}}.
\]

(14)

with the relationship

\[
E_{1s} = -\frac{8\sqrt{2}\pi q\alpha^{2/3}v_{th}^2}{(2+5|\varepsilon|)\omega_0} (1+i)f_s(1+i)\alpha^{1/3}v_{th}(1-i)
\]

(15)

(or \( E_{1s} = iB \)). The final result is given by \( E_1(t, x) = Re [E_1(x)e^{-i\omega t}] \). We can see that an additional, non-uniform, electric field \( E_1(x) \) appears as a result of the presence of the surface. This field oscillates in space and is attenuated with an attenuation length (penetration depth, extinction length) \( \lambda_e \simeq (1/\pi)^{1/6}[(\varepsilon |/(1-\varepsilon)]^{1/3}v_{th}/\omega \). It is worth noting that the penetration depth and the wavelength of the spatial oscillations have the same order of magnitude.

### 4 Discussion and conclusions

Making use of \( E_1(k) \) given by equations (11) and (13) we can calculate the change \( f_1(x, v) \) in the distribution function (equation (10)); if we limit ourselves to slow spatial oscillations, we get

\[
f_1(x, v) \simeq -\frac{iq}{m\omega} \text{sgn}(v) \frac{\partial F}{\partial v} E_1(x)
\]

(compare with equations (6) and (11)). Within this approximation \( f_s(v) = -(iq/m\omega)\text{sgn}(v)(\partial F/\partial v)E_{1s} \) and the polarization charge and current densities are zero (as expected for slow oscillations).

The amplitude of the field \( E_1(x) \) depends on the parameter \( E_{1s} \), which accounts for the boundary condition at \( x = 0 \). It is related to \( f_s(v) = \frac{1}{2\pi} \int dk f_1(k, v) \) by equation (15), where \( f_1(k, v) \) is given by equation (10); it
is easy to see that the integration of the first term in equation (10) gives $f_s$, while, making use of equations (13), the integration with respect to $k$ of the term which includes $E_1(k)$ is zero.

Within the kinetic approach we may estimate the local change in temperature by $\delta T = 2T f / F$, where the overbar implies an integration over velocities (thermal average). We can see that only $f_1$ contributes to this integration. Making use of equation (16) we get $\delta T = 0$. However, if we keep the contribution of the fast oscillations, we get a surface change of temperature

$$\delta T \simeq \frac{2iT}{n\omega} \int dv \cdot v f_s(v) \cdot \delta(x) + \ldots.$$  \hspace{1cm} (17)

($i.e., Re(\delta T e^{-i\omega t})$). The $\delta$-type contribution in equation (17) corresponds to the surface sheath in plasma heating models,[6, 25]

Similar calculations of the penetration depth can be made for a plasma confined between two plane-parallel walls (or other geometries); the result depends on the boundary conditions incorporated in parameters like $f_s$.\[24\]

The boundary parameter $f_s$ is a model parameter; we may take $f_0 + f_s = 0$ ($f(x = 0, v) = 0$) as a natural assumption, an equation which provides the parameter $f_s$. For $f_s = -f_0$ the field $E_1$ at the surface (maximum value) is of the order $E_1 \simeq E / |\varepsilon|$, where $E$ is the internal uniform field given by equation (7). The surface change in temperature (equation (17)) can be written in this case as

$$\delta T = \frac{1}{2\pi} \left( \frac{E_0}{q/a^2} \right) T \cdot a \delta(x)$$ \hspace{1cm} (18)

(for $\omega \ll \omega_0$), where $a$ is the mean separation distance between the particles ($a = n^{-1/3}$); $q/a^2 \gg E_0$ is an electric field of the order of the microscopic (inter-particle) field.

In conclusion, it is shown in this paper that the penetration of an oscillating electric field in a semi-infinite classical plasma obeys the standard exponential penetration law $e^{-x/\lambda_e}$ (beside a uniform component), which may exhibit spatial oscillations, the extinction length $\lambda_e$ (penetration depth, attenuation length) being of the order $\lambda_e \simeq [|\varepsilon| / (1 - \varepsilon)]^{1/3} v_{th} / \omega$; ($\varepsilon$ is the dielectric function, $\omega$ is the frequency of the field and $v_{th} = \sqrt{T/m}$ is the thermal velocity). The surface term is included explicitly in these calculations.
Acknowledgments. The author is indebted to the members of the Laboratory of Theoretical Physics at Magurele-Bucharest for many fruitful discussions. This work has been supported by the Scientific Research Agency of the Romanian Government through Grants 04-ELI/2016 (Program 5/5.1/ELI-RO), PN 16 42 01 01/2016 and PN (ELI) 16 42 01 05/2016.

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