Special Kähler structures, cubic differentials and hyperbolic metrics

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Abstract

We obtain necessary conditions for the existence of special Kähler structures with isolated singularities on compact Riemann surfaces. We prove that these conditions are also sufficient in the case of the Riemann sphere and, moreover, we determine the whole moduli space of special Kähler structures with fixed singularities. The tool we develop for this aim is a correspondence between special Kähler structures and pairs consisting of a cubic differential and a hyperbolic metric.

1 Introduction

For reader’s convenience, let us recall the definition of the affine special Kähler structure, which is the main object of study of this preprint.

Definition 1 ([Fre99]). An (affine) special Kähler structure on a manifold $\Sigma$ is a quadruple $(g, I, \omega, \nabla)$, where $(\Sigma, g, I, \omega)$ is a Kähler manifold with Riemannian metric $g$, complex structure $I$, and symplectic form $\omega(\cdot, \cdot) = g(I\cdot, \cdot)$, and $\nabla$ is a flat symplectic torsion-free connection on the tangent bundle $T\Sigma$ such that

$$ (\nabla_X I)Y = (\nabla_Y I)X $$

holds for all vector fields $X$ and $Y$.

If $I$ is fixed, which is always assumed to be the case below, we say for simplicity that $(g, \nabla)$ is a special Kähler structure.

The notion of a special Kähler structure has its origin in physics [Gat84, dWVP84] and is the natural structure of the base of an algebraic integrable system [Fre99]. In particular, algebraic integrable systems appear naturally in gauge theory [Hit87, SW94, SW94b, Nek03, Gai12], where a special instance of an algebraic integrable system—the Seiberg–Witten curve—plays a central rôle in the (physical) Seiberg–Witten theory. Very recently, special Kähler structures on Riemann surfaces have been extensively studied from the perspective of $\mathcal{N} = 2$ superconformal field theory, see [ALM17] and references therein.

Examples of special Kähler structures can be found in [Fre99, Hay15, CH18]. An elementary introduction to special Kähler geometry on Riemann surfaces can be found in [CH17].

Note that a complete special Kähler metric is necessarily flat [Lu99, BC01]. Besides, singularities of fibers of an algebraic integrable system lead to singularities of the corresponding special Kähler structure. This motivates studies of singular special Kähler metrics as a natural structure on bases of algebraic integrable systems.
Associated to a special Kähler structure is the period map \( \tau \), which takes values in the Siegel upper half-space \([\text{Fre99}]\). If \( \Sigma \) is a Riemann surface, which is assumed to be the case below unless otherwise stated explicitly, \( \tau \) takes values in the upper half-plane \( \mathcal{H} := \{ z \in \mathbb{C} \mid \text{Im} \, z > 0 \} \), which is endowed with the standard hyperbolic metric \( g_{\mathcal{H}} = (\text{Im} \, z)^{-2} |dz|^2 \).

**Definition 3.** Let \( \Sigma \) be a Riemann surface. For a special Kähler structure \((g, \nabla)\) on \( \Sigma \) with the period map \( \tau \) we call \( \hat{g} := \tau^* g_{\mathcal{H}} \) the associated hyperbolic metric.

Notice that \( \tau \) depends on certain choices and, moreover, is defined locally only (or, equivalently, on the universal covering of \( \Sigma \)), however \( \hat{g} \) is well-defined. Also, \( \hat{g} \) may degenerate at isolated points, hence, strictly speaking, \( \hat{g} \) is a metric only outside of some discrete subset of \( \Sigma \). This is not a concern for us, since we are interested in singular special Kähler structures, which involves singular metrics anyway.

If \( \hat{g} \) is any hyperbolic metric on \( \Sigma \), we say that \( \hat{g} \) represents a divisor \(^1\sum_{j=1}^{n} (\alpha_j - 1) p_j \) with \( 0 \leq \alpha_j \neq 1 \) if the following holds: If \( \alpha_j = 0 \), then \( \hat{g} \) has a cusp singularity at \( p_j \); If \( \alpha_j > 0 \), \( \hat{g} \) has a cone singularity of order \( \alpha_j \).

Recall [Fre99] that for any special Kähler structure we can also construct the associated cubic form \( \Xi \), which is a holomorphic section of \( K_{\Sigma}^2 \), where \( K_{\Sigma} \) is the canonical bundle of \( \Sigma \). Throughout this manuscript we assume that \( \Xi \) is non-zero. This means that we exclude special Kähler structures \((g, \nabla)\), where \( g \) is flat and \( \nabla \) is the Levi–Civita connection of \( g \).

Thus, to any special Kähler structure on a Riemann surface we can associate a pair \((\hat{g}, \Xi)\) as above. Our main result, **Theorem 4** below, states, roughly speaking, that for any pair \((\hat{g}, \Xi)\) consisting of a hyperbolic metric possibly singular at isolated points and a meromorphic cubic form we can construct a special Kähler structure, whose associated hyperbolic metric and associated cubic form are \( \hat{g} \) and \( \Xi \) respectively.

**Theorem 4.** Let \( \Xi \) be a meromorphic cubic differential on a Riemann surface \( \Sigma \) (not necessarily compact) with the divisor \( \Xi = \sum_{p \in \Sigma} \text{ord}_p \Xi \cdot p \). Let also \( \hat{g} \) be a hyperbolic metric on \( \Sigma \) representing a divisor \( D \). Then there is a unique special Kähler structure \((g, \nabla)\) on \( \Sigma \) whose associated hyperbolic metric and associated cubic form are \( \hat{g} \) and \( \Xi \) respectively. Moreover, \((g, \nabla)\) is smooth on \( \Sigma_0 := \Sigma \setminus (\text{supp}(\Xi) \cup \text{supp} D) \) and the following also holds:

(i) A cusp singularity \( p \) of \( \hat{g} \) is a logarithmic singularity of \((g, \nabla)\) of order \( \frac{1}{2} (\text{ord}_p \Xi + 1) \);

(ii) A cone singularity \( p \) of \( \hat{g} \) of order \( \alpha \) is a cone singularity of \((g, \nabla)\) of order \( \frac{1}{2} (\text{ord}_p \Xi - \alpha) \);

(iii) \((g, \nabla)\) has a cone singularity of order \( \frac{1}{2} \text{ord}_p \Xi \) at a point \( p \in \text{supp}(\Xi) \setminus \text{supp} D \).

A somewhat more precise version of this result is **Theorem 25**, which is proved in Section 3.

We would like to note that the correspondence of **Theorem 4** is pretty much explicit. To demonstrate this, pick any local holomorphic coordinate \( z \) and write \( \hat{g} = e^{2u} |dz|^2 \) and \( \Xi = \Xi_0(z) \, dz^3 \). Then the special Kähler metric of **Theorem 4** is given by

\[ g = e^{-u} |dz|^2, \quad \text{where} \quad u = v - \log |\Xi_0| - 2 \log 2. \]

Using [CH18, (9),(11)], one can also obtain an explicit formula for a connection one-form of \( \nabla \) in terms of \( v \) and \( \Xi_0 \). We leave details to the reader.

Furthermore, pick integers \( k \geq 3, \ell \in [0, k], \) a \( k \)-tuple \( p = (p_1, \ldots, p_k) \) of pairwise distinct points in \( \Sigma \) as well as a \( k \)-tuple \( b = (\beta_1, \ldots, \beta_k) \) of real numbers.

\(^1\)This notion of a divisor differs from the one usually met in algebraic geometry. Both notions are well established in the literature and unfortunately we will have to use both of them.
**Definition 5.** We call
\[
\mathcal{M}^\ell_k (p, b) := \left\{ (g, \nabla) \text{ special Kähler structure on } \Sigma, \text{ ord}_{p_j} (g, \nabla) = \frac{1}{2} \beta_j \right\} / \mathbb{R}_{>0}
\]
the moduli space of special Kähler structures with fixed singularities (or, simply the moduli space of special Kähler structures for short), where \( \text{ord}_{p_j} (g, \nabla) \) is the order of \((g, \nabla)\) at \(p_j\) in the sense of [CH18, Def. 6], the first \(\ell\) points of \(p\) are of conic type, and the remaining points are all of logarithmic type (in particular, if \(\ell = 0\) all points are of logarithmic type, whereas for \(\ell = k\) all points are of conic type), the group \(\mathbb{R}_{>0}\) acts by the rescalings of the metric.

In addition, we call
\[
\mathcal{M}^0_k (p, b) := \left\{ [g] \mid \exists \nabla \text{ such that } [g, \nabla] \in \mathcal{M}^\ell_k (p, b) \right\}
\]
the moduli space of special Kähler metrics.

Notice that at this point both \(\mathcal{M}^\ell_k (p, b)\) and \(\mathcal{M}^0_k (p, b)\) are defined as sets only. We justify the name by introducing a topology on these sets in Section 5 below.

**Theorem 6.** For any compact Riemann surface \(\Sigma\), if \(\mathcal{M}^\ell_k (p, b)\) is non-empty, then the following inequalities hold:
\[
4(\gamma - 1) < \beta_1 + \cdots + \beta_k, \\
[\beta_1] + \cdots + [\beta_\ell] + \beta_{\ell+1} + \cdots + \beta_k \leq 6(\gamma - 1) + k - \ell,
\]
where \([\beta]\) is the greatest integer not exceeding \(\beta\).

For \(\Sigma = \mathbb{P}^1\) the space \(\mathcal{M}^\ell_k (p, b)\) is non-empty if and only if (7) holds. Moreover, under this condition, \(\mathcal{M}^\ell_k (p, b)\) is homeomorphic to an open dense subset of a sphere of dimension \(2N + 1\), where \(N = -6 + k - \ell - \sum_{j=1}^k [\beta_j]\). The space \(\mathcal{M}^0_k (p, b)\) is homeomorphic to a Zariski open subset of \(\mathbb{C}P^N\).

In the special case \(\Sigma = \mathbb{P}^1\) and \(\ell = k\), i.e., all singularities are of conic type, \(\mathcal{M}^k_k (p, b)\) is homeomorphic to a sphere of dimension \(2N + 1\), where \(N = -6 - \sum_{j=1}^k [\beta_j]\). The space \(\mathcal{M}^0_k (p, b)\) is homeomorphic to \(\mathbb{C}P^N\).

The proof of this theorem can be found in Section 5.

We also establish necessary and sufficient conditions for the existence of special Kähler structures on elliptic curves as well as describe the corresponding moduli spaces in Corollary 38.

While proving our main statements we obtain also other results, which may be of some interest. In particular, we construct an example of a special Kähler structure whose associated cubic form has essential singularities, see Example 26. To the best of our knowledge this is the first example of an associated cubic form with an essential singularity. We also describe all special Kähler structures compatible with a fixed metric, see Section 4. Furthermore, let \((g, \nabla)\) be a special Kähler structure on a compact Riemann surface with finitely many prescribed singularities. Then the maps which assigns to \((g, \nabla)\) the associated cubic form \(\Xi\) is injective, see Theorem 31 for a more precise statement. This is surprising, since there is no reason to believe that \(\Xi\), which a priori encodes the difference between the Levi–Civita and the flat symplectic connections only, should determine the whole special Kähler structure. Moreover, this is a truly global statement in the sense that the corresponding local statement is clearly false.

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2 Preliminaries

Let \( \Omega \subset \mathbb{C} \) be any domain, which is viewed as being equipped with a holomorphic coordinate \( z = x + yi \) and the flat Euclidean metric \(|dz|^2 = dx^2 + dy^2\). We assume that any element of \( H^1(\Omega; \mathbb{R}) \) can be represented by a co-closed 1-form.

Write a special Kähler metric \( g \) on \( \Omega \) in the form

\[
g = e^{-u}|dz|^2.
\]

Using the global trivialisation of \( T\Omega \) provided by the real coordinates \((x, y)\) the connection \( \nabla \) is described by its connection 1-form \( \omega^\nabla \in \Omega^1(\Omega; gl(2, \mathbb{R})) \). A computation shows that \( \omega^\nabla \) can be written in the form

\[
\omega^\nabla = \begin{pmatrix}
\omega_{11} - * \omega_{11} \\
* \omega_{22} \\
\omega_{22}
\end{pmatrix},
\]

(8)

where

\[
2\omega_{11} = e^u (dh + \psi) - du, \quad 2\omega_{22} = -e^u (dh + \psi) - du.
\]

(9)

Here \( * \) denotes the Hodge star operator with respect to the flat metric, \( h \) is a smooth function, and \( \psi \) is a 1-form. These data are subject to the equation

\[
\Delta h = 0, \quad (d + d^*)\psi = 0, \quad \Delta u = |dh + 2\psi|^2 e^{2u},
\]

(10)

where \( \Delta = \partial_{xx}^2 + \partial_{yy}^2 \). Moreover, given any triple \((h, u, \psi)\) satisfying (10) the metric \( g = e^{-u}|dz|^2 \) together with \( \omega^\nabla \), which is given by (8) and (9), constitutes a special Kähler structure on \( \Omega \) (with it’s complex structure inherited from \( \mathbb{C} \)).

If \( \Omega \) is the punctured disc \( B_1^* \), any closed and co-closed 1-form can be written as \( a\varphi \), where \( \varphi \) is a generator of \( H^1(B_1^*; \mathbb{R}) \). For example, we can fix

\[
\varphi = \frac{y \, dx - x \, dy}{x^2 + y^2} = -d \left( \arg(x + iy) \right).
\]

Hence, a special Kähler structure on the punctured disc can be described in terms of solutions of the following equations

\[
\Delta h = 0, \quad \Delta u = |dh + a\varphi|^2 e^{2u}, \quad \Delta u = |dh + 2\psi|^2 e^{2u},
\]

(11)

where \( h, u \in C^\infty(B_1^*) \) and \( a \in \mathbb{R} \).

If \((h, u, a)\) is a solution of (11), the associated holomorphic cubic form of the corresponding special Kähler structure is

\[
Ξ = \Xi_0 \, dz^3 = \frac{1}{2} \left( \frac{a}{2z} - \frac{\partial h}{\partial z} \right) \, dz^3.
\]
Remark 12. Tracing through the description of special Kähler structures in terms of solutions of (11) as given in [CH17], it is easy to see that the function $h$ is defined only up to a constant. In other words, if $c$ is any real constant, $(h, u, a)$ and $(h+c, u, a)$ determine equal special Kähler structures.

A straightforward computation shows that $|dh + a\varphi|^2 = 16|\Xi_0|^2 = 16|\Xi|^2$. Hence, the second equation of (11) can be written as
\[ \Delta u = 16 |\Xi|^2 e^{2u}. \] (13)

Furthermore, write $\Xi_0(z) = \hat{\Xi}_0(z) + A z^{-1}$, where $A \in \mathbb{C}$ is the residue of $\Xi_0$ at the origin and denote by $H$ a primitive of $\hat{\Xi}_0$. Notice that $H$ is well-defined up to a constant. Define
\[ h := -4 \text{Im} H - 4 \text{Im} A \log |z| \quad \text{and} \quad a := 4 \text{Re} A. \]

Using $\partial_z \text{Im} H = \frac{1}{2} \partial_z H$ we compute
\[ \frac{1}{2} \left( \frac{a}{2z} - \frac{\partial_h}{\partial z} i \right) = \Xi_0(z). \]

The upshot of this computation is that $\Xi_0$ determines and is determined by $h$ and $a$.

Slightly more generally, let $\Omega = \hat{\Omega} \setminus \{p_1, \ldots, p_k\}$, where $\hat{\Omega}$ is a simply connected domain in $\mathbb{C}$. Then any closed and co-closed 1-form $\psi$ can be written as $\sum_{j=1}^k a_k \varphi_k$, where $\varphi_k := -d \left( \text{arg} (z - p_k) \right)$. It is easy to see that the above discussion can be repeated verbatim in this case too leading to the following result.

**Proposition 14.** Let $\Omega = \hat{\Omega} \setminus \{p_1, \ldots, p_k\}$, where $\hat{\Omega}$ is a simply connected domain in $\mathbb{C}$. Any pair $(u, \Xi)$ satisfying (13) determines a special Kähler structure on $\Omega$ such that the corresponding associated cubic form is $\Xi$. Conversely, any special Kähler structure on $\Omega$ determines a solution of (13). \hfill $\Box$

For the sake of clarity, let us spell the correspondence in the above proposition. Thus, if $(u, \Xi = \Xi_0 dz^3)$ is a solution of (13), put $g = e^{-u} |dz|^2$. Also, write $\Xi_0(z) = \hat{\Xi}_0(z) + \sum_j A_j (z - p_j)^{-1}$, where $A_j$ is the residue of $\Xi_0$ at $p_j$. If $H$ is a primitive of $\hat{\Xi}_0$, put
\[ h := -4 \text{Im} H - 4 \sum_{j=1}^k (\text{Im} A_j) \log |z - p_j| \quad \text{and} \quad a_j := 4 \text{Re} A_j. \]

Then the corresponding special Kähler structure is given by (8) and (9) with $\psi = \sum_j a_j \varphi_j$.

### 3 Special Kähler structures and the period maps

Let $(\Sigma, g, l, \omega, \nabla)$ be a special Kähler structure, where $\dim_{\mathbb{C}} \Sigma = n$. Denote by $\mathcal{U}$ the corresponding affine structure. This means that $\mathcal{U}$ is a covering of $M$ by open sets; Moreover, each $U \in \mathcal{U}$ is equipped with a $2n$-tuple of holomorphic functions $(z_1, \ldots, z_n; w_1, \ldots, w_n)$, where $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n)$ are conjugate special holomorphic coordinates on $U$ [Fre99]. If $U \in \mathcal{U}$ is another open set equipped with $(\hat{z}, \hat{w})$, then we have a relation
\[
\begin{pmatrix} z \\ w \end{pmatrix} = P \begin{pmatrix} \hat{z} \\ \hat{w} \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix},
\]
where \( P \in \text{Sp}(2n; \mathbb{R}) \) and \( a, b \in \mathbb{C}^n \) are some constants.

Denote
\[
\tau_{jk} = \frac{\partial w_k}{\partial z_j}.
\]

Then the matrix \( \tau := (\tau_{jk}) \) is symmetric and \( \text{Im} \, \tau \) is positive definite. In fact, \( \omega = \frac{i}{2} \sum \text{Im} \tau_{jk} d z_j \wedge d \bar{z}_k \). In particular, we have a holomorphic map
\[
\tau: U \to \mathcal{H}_n := \{ Z \in M_n(\mathbb{C}) \mid Z^t = Z, \text{ Im } Z \text{ is positive definite } \}
\]
whose target space is the Siegel upper half-space.

Recall that the group \( \text{Sp}(2n, \mathbb{R}) \) acts on \( \mathcal{H}_n \) via
\[
P \cdot Z = (AZ + B)(CZ + D)^{-1}, \quad \text{where } P = \begin{pmatrix} A & B \\ C & D \end{pmatrix},
\]
and the unique \( \text{Sp}(2n, \mathbb{R}) \)-invariant metric is given by \( g_{\mathcal{H}_n} = \text{tr}((Y^{-1} dZ)(Y^{-1} d\bar{Z})) \), where \( Y = \text{Im } Z \).

If \( \tilde{\tau} \) is a map corresponding to the chart \( \tilde{U} \), then the corresponding period maps are related by
\[
\tau = (D \tilde{\tau} + C)(A \tilde{\tau} + B)^{-1} = \tilde{P} \cdot \tau, \quad \text{where } \tilde{P} = \begin{pmatrix} D & C \\ B & A \end{pmatrix} \in \text{Sp}(2n, \mathbb{R}).
\]

Hence, \( \tau^* g_{\mathcal{H}_n} \) does not depend on the choice of an affine patch. As already explained in the introduction, if \( n = 1 \), \( \tilde{g} := \tau^* g_{\mathcal{H}_1} \) is a constant negative curvature metric, which we call the associated hyperbolic metric.

While the pull-back metric is defined in any dimension, the case \( n = 1 \) has some special features. Indeed, in this case \( \Sigma \) is a Riemann surface, \( \mathcal{H} = \mathcal{H}_1 \) is the upper half-plane so that \( \tau \) is a local biholomorphism except perhaps at isolated points. Hence, \( \tilde{g} \) is non-degenerate on \( \Sigma \) outside of some discrete subset. Moreover, the subset where \( \tilde{g} \) degenerates is easy to describe, see Proposition 20.

More importantly, in the case \( n = 1 \) the metric \( g_{\mathcal{H}_1} \) coincides with the standard hyperbolic metric \( (\text{Im } z)^{-2} |dz|^2 \). Hence, the pull-back metric \( \tilde{g} \) is also hyperbolic where it is non-degenerate.

Remark 15. Recall, that a holomorphic map \( \tau: \Sigma \to \mathcal{H} \), which may be multi-valued, is called a developing map of a hyperbolic metric \( \tilde{g} \), if \( \tilde{g} = \tau^* g_{\mathcal{H}} \). Hence, the very definition yields that the period map of a special Kähler structure is a developing map of the associated hyperbolic metric.

Example 16. Consider the following local example: \( \Sigma \) is the punctured unit disc in \( \mathbb{C} \) equipped with the metric \( g = -\log |z| |dz|^2 \), which is special Kähler. Then \( z \) is a special holomorphic coordinate with the conjugate given by \( w = 2i(z \log z - z) \). Hence, the period map is \( \tau = 2i \log z \). Of course, \( \tau \) is multivalued, but all values of \( \tau \) are related by Möbius transformations and therefore \( \tau^* g_{\mathcal{H}} \) is well defined and equals \( (|z| \log |z|)^{-2} |dz|^2 \), which is the standard Poincaré metric on the punctured disc.

Example 17. Let \( \Sigma \) be the upper half-plane \( \mathcal{H} \) equipped with the following special Kähler structure \([\text{Fre99}, \text{Rem. 1.20}]\):
\[
g = y |dz|^2, \quad \omega = \frac{1}{y} \begin{pmatrix} dy & dx \\ 0 & 0 \end{pmatrix},
\]
where \( z = x + yi \) is a coordinate on \( \mathcal{H} \).

It is easy to check that \((-iz, -\frac{1}{2}z^2)\) is a pair of special holomorphic conjugate coordinates. Hence, \( \tau(z) = z \), which means that \( \tau^* g_\mathcal{H} = g_\mathcal{H} \).

It will be useful below to have a relation between \( \Xi \) and \( \tau \). Thus, if \( Z \) is a special holomorphic coordinate, we have

\[
\Xi = 1 + \frac{1}{4} \left( \frac{d\tau}{dz} \right)^2 dZ^3. 
\] (18)

Then, for an arbitrary holomorphic coordinate \( z \) we obtain

\[
\Xi = \Xi_0 dz^3 = \frac{1}{4} \frac{d\tau}{dZ} dZ^3 = \frac{1}{4} \frac{d\tau}{dz} \left( \frac{dZ}{dz} \right)^3 dz^3 = \frac{1}{4} \frac{d\tau}{dz} \left( \frac{dZ}{dz} \right)^2 dz^3, 
\]

which yields in turn

\[
\frac{d\tau}{dz} = 4 \Xi_0 \left( \frac{dZ}{dz} \right)^{-2}. 
\] (19)

A useful corollary of this computation is the following.

**Proposition 20.** Let \( p \) be a regular point of a special Kähler structure on a Riemann surface. Then the associated hyperbolic metric degenerates at \( p \) if and only if \( \Xi(p) = 0 \). \( \square \)

The next result is the key ingredient in the proof of our main result, Theorem 4.

**Lemma 21.** Let \( \Omega \) be as in Proposition 14.

(i) Let \( \{ g = e^{-u}|dz|^2, \nabla \} \) be a special Kähler structure on \( \Omega \). Then the associated hyperbolic metric, which is defined on \( \Omega \setminus \Xi^{-1}(0) \), is given by \( \tilde{g} = e^{2v}|dz|^2 \), where

\[
v = u + \log |\Xi_0| + 2 \log 2. 
\] (22)

(ii) Given any hyperbolic metric \( \tilde{g} = e^{2v}|dz|^2 \) and any holomorphic cubic form \( \Xi = \Xi_0(z) dz^3 \) on \( \Omega \), there is a unique special Kähler structure \( \{ g, \nabla \} \) on \( \Omega \setminus \Xi^{-1}(0) \) such that \( g = e^{-u}|dz|^2 \), where \( u \) is determined by (22), and \( \Xi \) is the associated cubic form.

**Remark 23.** We would like to point out that in the statement of Lemma 21, the domain \( \Omega \) is allowed to have no punctures, i.e., \( k = 0 \) is allowed.

**Proof of Lemma 21.** Notice that since \( \Xi_0 \) is holomorphic, \( \log |\Xi_0| \) is harmonic on \( \Omega \setminus \Xi^{-1}(0) \). By (13), for \( v := u + \log |\Xi_0| + 2 \log 2 \) we have

\[
\Delta v = \Delta u = 16 |\Xi_0|^2 e^{2v-2 \log |\Xi_0|-4 \log 2} = e^{2v}. 
\]

Hence, \( \tilde{g} = e^{2v}|dz|^2 \) is a metric of constant curvature \(-1\) on \( \Omega \setminus \Xi^{-1}(0) \).

Furthermore, we claim that \( \tau^* g_\mathcal{H} = \tilde{g} \). To see this, notice that if \( Z \) is a special holomorphic coordinate (in a neighbourhood of some point), we have

\[
g = e^{-u}|dz|^2 = (\text{Im} \tau) |dZ|^2 = (\text{Im} \tau) |\partial_z Z|^2 |dz|^2 = (\text{Im} \tau) \frac{4 |\Xi_0|}{|\partial_z \tau|} |dz|^2. 
\]

Here the last equality follows from (19). Hence,

\[
u = \log |\partial_z \tau| - \log (\text{Im} \tau) - \log |\Xi_0| - 2 \log 2 \quad \Leftrightarrow \quad v = \log |\partial_z \tau| - \log (\text{Im} \tau),
\]
which yields in turn
\[ \tilde{g} = e^{2u} |dz|^2 = \frac{|\partial_z \tau|^2}{(\text{Im} \, \tau)^2} |dz|^2 = \tau^* g_{H_1}. \]
This clearly proves (i). The last part, (ii), is obtained by reading the above computation backwards. \(\square\)

**Corollary 24.** Let \( g \) be a special Kähler metric on the punctured disc \( B_1^* \) such that the associated holomorphic cubic form \( \Xi \) has order \( n \in \mathbb{Z} \) at the origin. Let \( \tilde{g} \) be the associated hyperbolic metric. Then the following holds:

(i) \( g \) is conical of order \( \beta/2 \) if and only if \( \tilde{g} \) is conical of order \( n - \beta \in (-1, +\infty) \), i.e.,
\[ g = r^\beta (C + o(1)) |dz|^2 \iff \tilde{g} = r^{2(n-\beta)} (C' + o(1)) |dz|^2; \]

(ii) \( g \) has a logarithmic singularity if and only if \( \tilde{g} \) has a cusp, i.e.,
\[ g = -r^{n+1} \log r (C + o(1)) |dz|^2 \iff \tilde{g} = \frac{C'}{r \log r} |dz|^2. \]

**Theorem 25.** Let \( \Sigma \) be a Riemann surface (not necessarily compact) and \( \Sigma_0 \subset \Sigma \) be an open subset. For any holomorphic cubic form \( \Xi \) and any smooth hyperbolic metric \( \tilde{g} \) on \( \Sigma_0 \) there is a unique special Kähler structure \( (g, \nabla) \) on \( \Sigma_0 \setminus \Xi^{-1}(0) \) whose associated hyperbolic metric and associated cubic form are \( \tilde{g} \) and \( \Xi \) respectively.

If \( \Xi \) is meromorphic on \( \Sigma \) with the divisor \( (\Xi) = \sum_{p \in \Sigma} \text{ord}_p \Xi \cdot p \) and \( \tilde{g} \) represents a divisor \( D \), then for the special Kähler structure \( (g, \nabla) \) on \( \Sigma_0 := \Sigma \setminus (\text{supp}(\Xi) \cup \text{supp} D) \) as above the following holds:

(i) A cusp singularity \( p \) of \( \tilde{g} \) is a logarithmic singularity of \((g, \nabla)\) of order \( \frac{1}{2} (\text{ord}_p \Xi + 1) \);

(ii) A cone singularity \( p \) of \( \tilde{g} \) of order \( \alpha \) is a cone singularity of \((g, \nabla)\) of order \( \frac{1}{2} (\text{ord}_p \Xi - \alpha) \);

(iii) \((g, \nabla)\) has a cone singularity of order \( \frac{1}{2} \text{ord}_p \Xi \) at a point \( p \in \text{supp}(\Xi) \cup \text{supp} D \).

**Proof.** Pick a point \( p \in \Sigma \) and an open set \( U \) together with a holomorphic coordinate \( z \) centered at \( p \). If \( p \notin \text{supp}(\Xi) \cup \text{supp} D \), we may think of \( U \) as a disc \( \{|z| < 1\} \). Otherwise, \( U \) can be chosen to be the punctured disc.

By **Lemma 21**, \( \tilde{g} = e^{2u} |dz|^2 \) and \( \Xi = \Xi_0(z) \cdot dz^3 \) determine a unique special Kähler structure \((g, \nabla)\) on \( U \), where \( g = e^{-u} |dz|^2 \) with \( u = v - \log |\Xi_0| - 2 \log 2 \). Moreover, \( u \) satisfies (13). Since this construction of \((g, \nabla)\) involves a local coordinate, \((g, \nabla)\) in a priori depends on this choice. We prove, however, that it is in fact immaterial, i.e., different choices yield equal special Kähler structures.

To this end, choose another holomorphic coordinate \( \hat{z} \) on \( U \). If \( \hat{z} = f(z) \), where \( f \) is holomorphic, the local representations \( \hat{\Xi}_0(\hat{z}) \cdot d\hat{z}^3 \) and \( \Xi_0(z) \cdot dz^3 \) of \( \Xi \) are related by \( \hat{\Xi}_0(\hat{z}) = \Xi_0(z) (f'(z))^{-3} \). Also, for the flat metric \( g_1 = |d\hat{z}|^2 \) and the corresponding Laplacian \( \Delta_1 = \partial_{\hat{z}\hat{z}}^2 + \partial_{\hat{g}\hat{g}}^2 \) we have
\[ g_1 = |f'(z)|^2 |dz|^2 \quad \text{and} \quad \Delta_1 = |f'(z)|^{-2} \Delta. \]

Multiply (13) by \( |f'(z)|^{-2} \) to obtain
\[ \Delta_1 u = |f'(z)|^{-2} |\Xi_0(z)|^2 e^{2u} = |f'(z)|^4 |\hat{\Xi}_0(\hat{z})|^2 e^{2u}, \]
where the subscript “0” indicates the norm induced by $|dz|^2$. Furthermore, for $\hat{u} := u + 2 \log |f'(z)|$ we compute

$$\Delta_1 \hat{u} = \Delta_1 u = |\hat{\Xi}_0|^2 e^{2\hat{u}}.$$ 

Hence, for the unique special Kähler structure $(\hat{g}, \hat{\nabla})$ determined by $(\hat{u}, \Xi)$ in the coordinate $\hat{z}$ as in Proposition 14, we have

$$\hat{g} = e^{-\hat{u}}|d\hat{z}|^2 = e^{-u}|dz|^2 = g.$$ 

Since a special Kähler metric and the associated cubic form determine the flat symplectic connection uniquely, we conclude that $(g, \nabla)$ and $(\hat{g}, \hat{\nabla})$ coincide (more precisely, this means $(g, \nabla) = f^*(\hat{g}, \hat{\nabla})$). By the construction, $(\hat{g}, \hat{\nabla})$ is the special Kähler structure determined by $\Xi$ and the hyperbolic metric

$$\exp \left(2\hat{u} + 2 \log |\hat{\Xi}_0(\hat{z})|\right) |d\hat{z}|^2 = e^{2u}|dz|^2 = \tilde{g},$$

where the above equality follows from the definition of $\hat{u}$. Thus, the choice of the local coordinate used in Proposition 14 is immaterial as claimed. This proves the existence of a special Kähler structure for given $\Xi$ and $\tilde{g}$.

The uniqueness of the special Kähler structure corresponding to $(\hat{g}, \Xi)$ follows immediately from the corresponding local statement. The other properties claimed follow directly from Corollary 24.}

**Example 26** (A special Kähler structure whose associated cubic form has an essential singularity). Let $\Xi(z) := e^{1/6}dz^3$ be a cubic holomorphic form on $\mathbb{C}^*$. $\Xi$ may be thought of as a holomorphic cubic form on $\mathbb{P}^1$ with two singularities: one essential and the other one of degree $-6$. Pick a hyperbolic metric singular at 3 points, for example 0, $\infty$, and some $\lambda \in \mathbb{C}^*$. By Theorem 25 we obtain a special Kähler structure on $\mathbb{P}^1$ with three singularities such that $\Xi$ is the associated cubic form. To the best of our knowledge, this is the first example of a special Kähler structure whose associated cubic form has essential singularities.

### 4 Special Kähler metrics versus special Kähler structures

Theorem 25 allows to construct inequivalent special Kähler structures such that the corresponding Riemannian metrics are equal. Indeed, fix a pair $(\hat{g}, \Xi)$ as in Theorem 25 and let $g$ be the corresponding special Kähler metric. It is then clear from (22) that the pair $(\hat{g}, \lambda \Xi)$ leads to the metric $|\lambda| \cdot g$, where $\lambda \in \mathbb{C}^*$. Hence, specializing to $|\lambda| = 1$ we obtain a family of special Kähler structures parameterized by $S^1$ such that all corresponding Riemannian metrics are equal.

**Example 27.** Fix arbitrarily a hyperbolic metric $\tilde{g}$ on the punctured unit disc $B_1^*$. Choose a holomorphic cubic differential $\Xi$ on $B_1^*$ such that $\Xi$ is of order $-3$ at the origin. Observe that the leading coefficient $\xi_{-3}$ in the expansion $\Xi_0(z) = \xi_{-3} z^{-3} + \xi_{-2} z^{-2} + \ldots$ is independent of the choice of a local coordinate. Hence, the family $\{ \lambda \Xi \mid |\lambda| = 1 \}$ consists of holomorphic cubic differentials that are pairwise inequivalent even up to a change of coordinates. Hence, for the corresponding family of special Kähler structures $(g, \nabla_\lambda)$ the metric is independent of $\lambda$ and the corresponding structures are pairwise inequivalent.

**Proposition 28.** Let $\Sigma$ be a Riemann surface.
(i) Let \((g, \nabla)\) and \((\hat{g}, \hat{\nabla})\) be two special Kähler structures on \(\Sigma\), whose associated cubic forms are \(\Xi\) and \(\hat{\Xi}\) respectively. If \(g = \hat{g}\), then
\[
\hat{\Xi} = \lambda \cdot \Xi,
\]  
where \(\lambda \in \mathbb{C}\) is of absolute value 1;

(ii) If \((g, \nabla)\) is a special Kähler structure on \(\Sigma\) whose associated cubic form is \(\Xi\), then for each \(\lambda \in S^1\) there is a unique special Kähler structure \((g, \nabla_\lambda)\), whose associated cubic form is \(\lambda \Xi\).

**Proof.** Clearly, to prove (i) it is enough to check (29) in a neighborhood of a regular point \(p \in \Sigma\). Thus, let \(z\) be a local holomorphic coordinate in a neighborhood \(U\) of \(p\).

If \((g, \nabla)\) and \((\hat{g}, \hat{\nabla})\) are two special Kähler structures with the associated cubic forms \(\Xi = \Xi_0 dz^3\) and \(\hat{\Xi} = \hat{\Xi}_0 dz^3\) respectively such that \(g = \hat{g}\), then (13) implies \(|\Xi_0| = |\hat{\Xi}_0|\). Since both \(\Xi_0\) and \(\hat{\Xi}_0\) are holomorphic on \(U\), there is \(\lambda \in S^1 \subset \mathbb{C}\), such that \(\Xi_0 = \lambda \cdot \hat{\Xi}_0\). This proves (i).

Claim (ii) follows from Theorem 25 by setting \(\hat{g}\) to be the associated hyperbolic metric of \((g, \nabla)\).

\[\square\]

5 A necessary and sufficient condition for the existence of special Kähler structures on compact Riemann surfaces

Just like in the introduction, pick integers \(k \geq 3\), \(\ell \in [0, k]\), a \(k\)-tuple \(p = (p_1, \ldots, p_k)\) of pairwise distinct points in \(\Sigma\) as well as a \(k\)-tuple \(b = (\beta_1, \ldots, \beta_k)\) of real numbers. Denote
\[
D = D(p, b) := -\sum_{j=1}^{\ell} [\beta_j]p_j - \sum_{j=\ell+1}^{k} (\beta_j - 1)p_j, \quad L = L(p, b) := O(K_\Sigma^3 + D), \quad \text{and} \quad H(p, b) := \{ z \in \mathbb{H}(L) \mid \Xi \neq 0, \text{ord}_{p_j} \Xi = \beta_j - 1 \text{ for } \ell + 1 \leq j \leq k \},
\]
where \(K_\Sigma\) is the canonical bundle of \(\Sigma\). In other words, \(H(p, b)\) consists of all non-trivial meromorphic cubic differentials \(\Xi\) which are holomorphic on \(\Sigma \setminus \{p_1, \ldots, p_k\}\) and satisfy
\[
\text{ord}_{p_j} \Xi \geq [\beta_j] \text{ if } j \leq \ell \quad \text{and} \quad \text{ord}_{p_j} \Xi = \beta_j - 1 \text{ if } j > \ell.
\]  

**Theorem 31.** Let \(\Sigma\) be a compact Riemann surface of genus \(\gamma\). Then \(M_k^\ell(p, b) \neq \emptyset\) if and only if the following two conditions hold:

(i) \(4(\gamma - 1) < \beta_1 + \cdots + \beta_k\);

(ii) \(H(p, b) \neq \emptyset\).

Moreover, the map that assigns to a special Kähler structure \((g, \nabla)\) as above its associated cubic form \(\Xi \in H(p, b)\) is a bijection.

**Proof.** If \(M_k^\ell(p, b) \neq \emptyset\), then the associated cubic form \(\Xi\) of any special Kähler structure \((g, \nabla)\) such that \([g, \nabla] \in M_k^\ell(p, b)\) lies in \(H(p, b)\), hence (ii) holds.

Furthermore, since \(\text{ord}_{p_j} \Xi - \beta_j \geq -1\), the associated hyperbolic metric \(\hat{g}\) has either a cone singularity with positive angle or a cusp at each \(p_j\). Let \(p_{k+1}, \ldots, p_m\) be further points on \(\Sigma\) such that \(\hat{g}\) is singular. By Corollary 24, (i) each \(p_j\) with \(j \geq k + 1\) is conical singularity of \(\hat{g}\) of order \(\text{ord}_{p_j} \Xi > 0\). Hence, by the Gauß–Bonnet theorem applied to \(\hat{g}\) we have
\[
\sum_{j=1}^{k} (\text{ord}_{p_j} \Xi - \beta_j) + \sum_{j=k+1}^{m} \text{ord}_{p_j} \Xi + 2 - 2\gamma < 0.
\]
Hence, we obtain
\[ \sum_{j=1}^{k} \beta_j > \sum_{j=1}^{m} \ord_{p_j} \Xi + 2 - 2\gamma = 4(\gamma - 1). \]

It remains to show that (i) and (ii) yield a special Kähler structure. Indeed, notice that (i) and (ii) imply
\[ \sum_{j=1}^{k} \left( \ord_{p_j} \Xi - \beta_j \right) + \sum_{\Xi(q) = 0} \ord_q \Xi + 2 - 2\gamma < 0. \]

Hence, by [Hei62] there exists a hyperbolic metric \( \tilde{g} \) which has conical singularities at each zero of \( \Xi \) of order \( \ord_q \Xi \), and has either a cone singularity or a cusp at each \( p_j \) for all \( 1 \leq j \leq k \). The proof is finished by appealing to Theorem 25.

**Corollary 32.** Let \( (g, \nabla) \) be a special Kähler structure on \( \mathbb{P}^1 \) such that the associated cubic form \( \Xi \) is non-trivial and meromorphic. Then \( (g, \nabla) \) must be singular at least at 3 points.

**Proof.** By (7) for \( \ell = 0 \) we obtain \(-4 < \sum \beta_j \leq -6 + k \), i.e., there must be at least \( k \geq 3 \) singularities provided all of them are logarithmic. Hence, if \( k = 1 \) we must have one conic singularity of order \( \beta/2 \) such that \( \beta > -4 \) and \( \lfloor \beta \rfloor \leq -6 \), which is clearly impossible.

If \( k = 2 \) and \( \ell = 1 \), we have
\[ -4 < \beta_1 + \beta_2 \leq [\beta_1] + \beta_2 + 1 \leq -6 + 1 + 1, \]
which is a contradiction. The case \( k = \ell = 2 \) is excluded in a similar manner.

**Remark 33.** For any \( n \in \mathbb{Z} \) the metric \( g = r^n |dz|^2 \) is flat on \( \mathbb{C} \setminus \{0\} \), hence, can be thought of as a special Kähler structure on \( \mathbb{P}^1 \) singular at most at 2 points, namely 0 and \( \infty \). Notice that the corresponding cubic form is trivial, hence this example does not contradict Corollary 32.

Denote by \( \pi : \mathcal{M}_k^*(p, b) \to \mathcal{M}_k^0(p, b) \) the natural projection, which has been studied in Section 4. In particular, each fiber of \( \pi \) is isomorphic to the circle.

We have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}_k^0(p, b) & \xrightarrow{\Xi} & H(p, b)/\mathbb{R}_{>0} \\
\pi \downarrow & & \downarrow \\
\mathcal{M}_k^*(p, b) & \xrightarrow{\xi} & H(p, b)/\mathbb{C}^*,
\end{array}
\] (34)

where slightly abusing notations \( \Xi \) stays for the map, which assigns to a special Kähler structure its associated cubic form, and \( \xi \) is just the induced map. By Proposition 28 and Theorem 31 both \( \Xi \) and \( \xi \) are bijections. This can be used to define topologies on \( \mathcal{M}_k^0(p, b) \) and \( \mathcal{M}_k^*(p, b) \). Indeed, \( H(p, b) \) is naturally a subset of a vector space \( H^0(L) \), which can be equipped with a topology by introducing a scalar product (notice that the origin is not contained in \( H(p, b) \)).

**Proof of Theorem 6.** The fact that (7) holds provided \( \mathcal{M}_k^0(p, b) \neq \emptyset \) follows directly from Theorem 31.

Given a \( k \)-tuple of points on \( \Sigma = \mathbb{P}^1 \) as well as a \( k \)-tuple \( b = (\beta_1, \ldots, \beta_k) \) of real numbers such that (7) holds, it is easy to construct explicitly a meromorphic cubic differential \( \Xi \) on \( \mathbb{P}^1 \) such that (30) holds and \( \Xi \) is holomorphic on \( \mathbb{P}^1 \setminus \{p_1, \ldots, p_k\} \). Hence, Theorem 31 also implies that \( \mathcal{M}_k^0(p, b) \) is non-empty provided (7) holds.

The remaining claims follow directly from Diagram 34. \( \Box \)
Example 35. Let $\mathcal{M}^0_{24}$ denote the moduli space of all special Kähler metrics with $k$ singular points all of logarithmic type. $\mathcal{M}^0_{24}$ fibers over $\text{Sym}^{24}(\mathbb{P}^1) \setminus \{\text{Diagonal subset}\}$, where each fiber is homeomorphic to a Zariski open subset of $\mathbb{C}P^{18}$. Hence, $\mathcal{M}^0_{24}$ has complex dimension 42. If we also mod out by the natural action of $\text{PGL}(2, \mathbb{C})$, the resulting space is of complex dimension 39. This space is of interest for elliptic K3 surfaces [GW00].

Remark 36. It is possible to define topologies, or even smooth structures, on $\mathcal{M}^k_0(p, b)$ and $\mathcal{M}^k_0(p, b)$ directly along the lines of [MW17]. This would then require to prove that the map $\Xi$ is a homeomorphism, which seems to be excessive for our modest aims.

6 Existence of special Kahler metrics on Riemann surfaces with positive genera

Corollary 37. Let $\Sigma$ be a compact Riemann surface with genus $\gamma > 0$. Then $\mathcal{M}^k_0(p, b)$ is non-empty if and only if the following three conditions hold:

(i) $\beta_1 + \cdots + \beta_k > 4\gamma - 4$,

(ii) the line bundle $L = L(p, b)$ has a non-trivial holomorphic section, and

(iii) for all $\ell + 1 \leq j \leq k$ we have $\dim_{\mathbb{C}} H^0(L) > \dim_{\mathbb{C}} H^0(L - p_j)$.

Moreover, under the above conditions, $\mathcal{M}^k_0(p, b)$ is homeomorphic to an open dense subset of a sphere of dimension $2N + 1$, where $N := \dim_{\mathbb{C}} H^0(L) - 1$. The space $\mathcal{M}^k_0(p, b)$ is homeomorphic to a Zariski open subset of $\mathbb{C}P^N$.

Proof. Suppose that these three conditions hold. By the last two ones, there exists a meromorphic cubic differential $\Xi$ which is holomorphic outside $\{p_1, \cdots, p_k\}$ and satisfies $\text{ord}_{p_j} \Xi \geq [\beta_j]$, for $1 \leq j \leq \ell$ and $\text{ord}_{p_j} \Xi = \beta_j - 1$, for $\ell + 1 \leq j \leq k$. The conclusion follows from the first condition and Theorem 31.

Suppose that $\mathcal{M}^k_0(p, b)$ is non-empty. The first two conditions follows from Theorem 31. We show the third one by contradiction. Suppose that there exist $\ell + 1 \leq j \leq k$ such that

$$\dim_{\mathbb{C}} H^0(L) \leq \dim_{\mathbb{C}} H^0(L - p_j).$$

Then we find $H^0(L - p_j) = H^0(L)$. We pick a special Kähler structure $(g, \nabla)$ in $\mathcal{M}^k_0(p, b)$ with associated cubic differential $\Xi$. By Theorem 31, we know that $\Xi$ has order $\beta_j - 1$ at $p_j$. On the other hand, since $\Xi$ belongs to $H^0(L) = H^0(L - p_j)$, the order of $\Xi$ at $p_j$ should be greater than or equal $\beta_j$. This is a contradiction.

In the case $\Sigma$ is an elliptic curve (compact Riemann surface of genus one), the statement of the above corollary can be made more explicit.

Corollary 38. For an elliptic curve $E$ the space $\mathcal{M}^k_0(p, b)$ is non-empty if and only if the following three conditions hold:

(i) $\beta_1 + \cdots + \beta_k > 0$;

(ii) The line bundle $L = L(p, b)$ has a non-trivial holomorphic section;

(iii) If $\deg L = 1$, then for all $\ell + 1 \leq j \leq k$, the divisor $-D(p, b) + p_j$ is not equivalent to zero.
Moreover, under these conditions, $\mathcal{M}_k^s(p, b)$ is homeomorphic to an open dense subset of a sphere of dimension $2N + 1$, where $N := \dim_\mathbb{C} H^0(L) - 1$. The space $\mathcal{M}_k^s(p, b)$ is homeomorphic to a Zariski open subset of $\mathbb{C}P^N$.

Proof. While a proof of this corollary could be obtained from Corollary 37, we prefer a more direct approach.

Thus, suppose there exists a special Kähler structure $(g, \nabla)$ in the statement of this corollary. The first two conditions follow from Theorem 31 directly. Suppose $\deg L = 1$. Then $H^0(L)$ has dimension one by the Riemann-Roch theorem. If there exists $\ell + 1 \leq j \leq k$ such that the divisor $-D + p_j := -D(p, b) + p_j$ is equivalent to zero, then there exists an elliptic function $f$ such that $(f) = -D + p_j$ and $H^0(L)$ is generated by $f \, dz^3$, where $dz$ is a nowhere vanishing holomorphic one-form on $E$. Furthermore, the associated cubic differential $\Xi$ equals $\text{Const.} \, f \, dz^3$ and has order $\beta_j$ at $p_j$. This is a contradiction, which finishes the proof of the “if” part.

Assume that $(i)$–$(iii)$ hold. We divide the proof of the “if” part into the following two cases.

Case 1. Suppose $\deg L = 0$. Then $L$ is trivial, in particular $L$ has a non-trivial holomorphic section, and there exists an elliptic function $f$ on $E$ such that $(f) = -D$. By Theorem 31, there exists a special Kähler structure $(g, \nabla)$ whose associated cubic differential is $f \, dz^3$.

Case 2. Suppose $d := \deg L > 0$. By the Abel-Jacobi theorem, we can find a point $q \in E = \mathbb{C}/\Lambda$ and an elliptic function $f$ on $E$ such that $(f) = dq - D$. If $q \notin \{p_{\ell+1}, \ldots, p_k\}$, then by Theorem 31 there exists such a special Kähler structure $(g, \nabla)$ whose associated cubic differential is $f \, dz^3$, where $dz$ is a non-where vanishing holomorphic 1-form on $E$. Hence, it remains to consider the case $q \in \{p_{\ell+1}, \ldots, p_k\}$.

Subcase 2.1. Suppose $d \geq 2$. Since there exist $q_1, \ldots, q_d$ in $E \setminus \{p_{\ell+1}, \ldots, p_k\}$ such that $q_1 + \cdots + q_d \equiv dq \, (\text{mod } \Lambda)$, we can find an elliptic function $f$ on $E$ such that $(f) = q_1 + \cdots + q_d - D$. We are done.

Subcase 2.2. Suppose $d = 1$. Then there exists $\ell + 1 \leq j \leq k$ such that $-D + p_j \sim 0$. However, this possibility is excluded by $(iii)$. \hfill \Box

Conjecture 39. Let $\Sigma$ be a compact oriented two-manifold of genus $\gamma > 1$.

(i) There exists a complex structure $I$ on $\Sigma$ and a special Kähler structure $(g, \nabla)$ with exactly $k \geq 1$ logarithmic singularities of orders $\beta_1/2, \ldots, \beta_k/2$ on $\Sigma$ if and only if

$$4\gamma - 4 < \beta_1 + \cdots + \beta_k \leq 6\gamma - 6 + k.$$ 

(ii) There exists a complex structure $I$ on $\Sigma$ and a special Kähler structure $(g, \nabla)$ with $k$ singularities of orders $\beta_1/2, \ldots, \beta_k/2$ such that exactly the first $\ell \geq 1$ of them are conic if and only if

$$4\gamma - 4 < \beta_1 + \cdots + \beta_k, \quad [\beta_1] + \cdots + [\beta_\ell] + \beta_{\ell+1} + \cdots + \beta_k \leq 6\gamma - 6 + k - \ell.$$ 

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