Heterotic Hyper-Kähler flux backgrounds

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ABSTRACT: We study Heterotic supergravity on Hyper-Kähler manifolds in the presence of non-trivial warping and three form flux with Abelian bundles in the large charge limit. We find exact, regular solutions for multi-centered Gibbons-Hawking spaces and Atiyah-Hitchin manifolds. In the case of Atiyah-Hitchin, regularity requires that the circle at infinity is of the same order as the instanton number, which is taken to be large. Alternatively there may be a non-trivial density of smeared five branes at the bolt.

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1 Introduction

Heterotic flux backgrounds are interesting models of string backgrounds. There is no in principle impediment from using the R-NS string worldsheet formalism although compact models with minimal or no supersymmetry remain challenging to construct.

In this work we study local Heterotic flux backgrounds on Hyper-Kähler four manifolds: in particular the Gibbons-Hawking spaces [1] and the Atiyah-Hitchin manifold [2]. Key to our configurations is that the gauge fields are Abelian and we take the large charge limit such that $\text{Tr} F \wedge F$ dominates $\text{Tr} R \wedge R$ in the Bianchi identity. This large-charge limit has been previously studied on the Eguchi-Hanson space [3] and the conifold [4, 5] and this type of limit is familiar from the large-charge supergravity limit crucial to the development of holography [6] in type II and M-theory.

Our strategy is to first compute explicit solutions to the Hermitian-Yang-Mills equations on Hyper-Kähler spaces and then backreact them on the geometry. This backreaction affects only the conformal mode of the metric but generates a non-trivial three-form flux.

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According to Aspinwall [7] we are studying \textit{Goldilocks} theories with just the right amount of supersymmetry; perhaps then not surprisingly we solve the supergravity background exactly. For the Atiyah-Hitchin background it is nonetheless somewhat impressive that we can both solve Hermitian-Yang-Mills exactly and integrate analytically the resulting Poisson equation for the backreaction of this instanton.

The instanton we use on the Atiyah-Hitchin manifold is well known from a classic duality paper by Ashoke Sen [8]. Perhaps the central result of our work is that the Heterotic backreaction of this instanton can be made regular. This is somewhat non-trivial since the negative mass of Atiyah-Hitchin induces a negative warp factor thus violating the desired signature of space-time. We circumvent this in two ways: first by allowing the asymptotic circle to be large and secondly by including smeared five-brane sources.

We also study the presence of electric $H$-flux and fundamental strings. The electric flux modifies the BPS equations in a straightforward way and for each solution with magnetic $H$-flux, the electric flux can be added through a harmonic function on the Hyper-Kähler manifold. We analyze limits in which we recover AdS$_3$ geometries but these reduce to the known AdS$_3 \times S^3 \times HK_4$.

Upon completing this work we were made aware that our BPS equations have turned up in five dimensional supergravity. The local equations we study can essentially be found in [9, 10] but with very different global and regularity requirements. This is not surprising since we can dimensionally reduce our solutions on $\mathbb{R}^5$ to get solutions of ungauged five dimensional supergravity. In addition, these equations also turn up in type II supergravity for $T^2$ fibrations over Hyper-Kähler spaces and the type II analogue of the Gibbons-Hawking solutions we find have been analyzed in [11]. It is straightforward to convert our solutions on the Atiyah-Hitchin manifold to such type II backgrounds.

\section{Hyper-Kähler heterotic backgrounds}

The primary backgrounds we consider are of the form $\mathbb{R}^{1,5} \times HK_4$ where $HK_4$ is a warped Hyper-Kähler-four manifold. We consider a non-trivial three-form flux $H_{(3)}$, dilaton $\Phi$ and Heterotic gauge field $F$. The background metric ansatz is:

$$ds_{10}^2 = ds_{1,5}^2 + H ds_4^2$$

where $ds_4^2$ is an Hyper-Kähler metric on a four-manifold $HK_4$ and $H$ a conformal factor. The BPS equations are fairly standard:

\begin{align}
\phi &= H \\
H_{(3)} &= - *_4 dH \\
J_{ab} F &= 0, \quad a = 1, 2, 3
\end{align}

where the $*_4$ is the Hodge dual w.r.t. the Hyper-Kähler metric on $HK_4$ and $J_a$ are the three Kähler forms.

A major difficulty in finding explicit solutions of Heterotic supergravity with non-trivial three-form flux is to satisfy the Bianchi identity at the appropriate order in $\alpha'$. Following
earlier works by some of the authors [3–5], our strategy will be to work in a large (fivebrane) charge limit, ensuring that the contribution of the $\text{Tr} R \wedge R$ term is subdominant and can be consistently neglected. The Heterotic Bianchi identity simplifies to

$$dH(3) = \alpha' \text{Tr} F \wedge F$$

(2.3)

implying from the three-form ansatz (2.2b) that:

$$d *_4 dH = -\alpha' \text{Tr} F \wedge F.$$  

(2.4)

### 2.1 Principal torus bundles and type IIA/IIB solutions

We consider a more general class of backgrounds, that can be viewed as local models of the principal torus bundles over wrapped K3 surfaces introduced in [12] and discussed in many works including [13–16]. They generalize the solutions based on Eguchi-Hanson space presented in [17]. The general ansatz for such principal two-torus bundle $T^2 \rightarrow M_6 \rightarrow HK_4$ over a Hyper-Kähler four-manifold is of the form

$$ds^2_{10} = ds^2_{1,3} + H ds^2_4 + \frac{U_2}{T_2} (dx + Tdy + \pi^* \alpha)^2,$$

(2.5)

where $\alpha$ is a connection one-form on $HK_4$ such that $\vartheta = dx + Tdy + \pi^* \alpha$ is a globally defined one-form on $M_6$ with

$$\frac{1}{2\pi} d\vartheta = \pi^* \varpi, \quad \varpi = \varpi_1 + T \varpi_2, \quad \varpi_i \in H^2(HK_4, \mathbb{Z}),$$

(2.6)

and by supersymmetry

$$J^a \wedge \varpi = 0, \quad a = 1, 2, 3.$$

(2.7)

The expression for the three-form becomes then

$$H(3) = -*_4 dH - \frac{\alpha' U_2}{T_2} \text{Re} (*_4 d\vartheta \wedge \bar{\vartheta}).$$

(2.8)

By an appropriate choice of $\varpi \in H^2(HK_4, \mathbb{Z})$ one can find solutions with $dH(3) = 0$, which can also be obtained as supersymmetric solutions of type IIA or type IIB supergravity with NS-NS fluxes, as was discussed in [3] and [18].

### 3 Gibbons-Hawking: ALE and ALF

We can solve explicitly (2.2) and (2.4) for the multicentered Gibbons-Hawking ALE and ALF spaces, that we denote collectively by $M_{GH}$. The corresponding Hyper-Kähler metrics are given by:

$$ds^2 = V(x)^{-1} (d\tau + \omega)^2 + V(x) dx \cdot dx,$$

(3.1a)

$$dV = *_3 d\omega,$$

(3.1b)

$$V = \epsilon + 2m \sum_{i=1}^{k} \frac{1}{|x - x_i|},$$

(3.1c)

---

1 As usual $*_3$ is the Hodge dual on $\mathbb{R}^3$. 
where $\epsilon = 0$ gives the ALE (multi Eguchi-Hanson) series and $\epsilon = 1$ the ALF (multi Taub-NUT) series. The periodicity of $\tau$ is determined by expanding around a pole of $V(x)$ to be:

$$\tau \sim \tau + 8\pi m, \quad (3.2)$$

and the triplet of Kähler forms is given by

$$J_a = (d\tau + \omega) \wedge dz^a - V *_3 dz^a, \quad a = 1, 2, 3. \quad (3.3)$$

We will consider heterotic supergravity solutions for warped ALE or ALF spaces supported by Abelian gauge bundles. To explicitly write the gauge fields we denote

$$V_i = \frac{2m}{|x - x_i|}, \quad d\omega_i = *_3 dV_i; \quad (3.4)$$

Then a representative of the topologically non-trivial gauge fields is locally given by

$$A_i = \omega_i - \frac{V_i}{V}(d\tau + \omega). \quad (3.5)$$

We note that

$$\sum_{i=1}^{k} A_i = \epsilon \frac{d\tau + \omega}{V} - d\tau \quad (3.6)$$

is topologically trivial since $V^{-1}(d\tau + \omega)$ is globally defined. Thus there are $(k - 1)$ non-trivial gauge fields, in agreement with the $(k - 1)$ non-trivial two-cycles. The corresponding field strengths are

$$F_i = dA_i = V *_3 d \left[ \frac{V_i}{V} \right] - d \left[ \frac{V_i}{V} \right] \wedge (d\tau + \omega). \quad (3.7)$$

It is straightforward to see that $F_i$ are anti-self dual and solve Hermitian Yang-Mills\footnote{For a one form $\alpha$ on $\mathbb{R}^3$ we have $*_4 \alpha = -(d\tau + \omega) \wedge *_3 \alpha$. In particular this means that a function which is invariant under the U(1) generated by $\partial_\tau$ is harmonic on the Gibbons-Hawking space iff it is harmonic on $\mathbb{R}^3$.}

$$*_4 F_i = -F_i, \quad J_a \wedge F_j = 0. \quad (3.8)$$

For the Bianchi identity (2.4) we compute

$$F_i \wedge F_j = -2V *_3 d \left[ \frac{V_i}{V} \right] \wedge d \left[ \frac{V_j}{V} \right] \wedge (d\tau + \omega) \quad (3.9)$$

$$d*_4 \left[ \frac{V_i V_j}{V} \right] = -F_i \wedge F_j \quad (3.10)$$

so that if we take

$$F = \frac{1}{4m} \sum_{i=1}^{k} dA_i \cdot \mathbf{q}_i \cdot \mathcal{T}, \quad (3.11)$$
where \( \mathcal{T} \in U(1)^{16} \) is in the Cartan subalgebra of \( E_8 \times E_8 \) or \( SO(32) \) and \( \mathbf{q}_i \) the corresponding charge vectors, and solve (2.4), we get the general solution:\(^3\)

\[
H = \delta + h(\mathbf{x}) + \frac{\alpha'}{8m^2V} \sum_{i,j=1}^{k} V_i V_j \mathbf{q}_i \cdot \mathbf{q}_j.
\]

(3.12)

Here \( \delta = 0,1 \) is an integration constant (not to be confused with \( \epsilon \) a similar integration constant in the Gibbons-Hawking warp factor \( V \)) and \( h(\mathbf{x}) \) is any harmonic function on \( M_{GH} \). Taking \( h(\mathbf{x}) \) to be invariant under \( \partial_r \) we have

\[
h(\mathbf{x}) = \frac{1}{m} \sum_{\alpha} \frac{q_\alpha}{|\mathbf{x} - \mathbf{x}_\alpha|}
\]

(3.13)

corresponding to mobile neutral five-brane sources inserted at \( \mathbf{x}_\alpha \).

In appendix A we show how the two center solution is related to the Eguchi-Hanson solution that was discussed in particular in \cite{3}.

3.1 Five-brane and magnetic charges

At infinity we can compute the five-brane charge using (2.2b) and (3.12). We have

\[
\mathcal{H}_{(3)} = (d\tau + \omega) \wedge \ast_3 dH
\]

(3.14)

and so\(^4\)

\[
dH = -\frac{\alpha'}{4mk\tau^2} \left( \sum_{i,j=1}^{k} \mathbf{q}_i \cdot \mathbf{q}_j \right) d\tau - \frac{1}{m} \sum_{\alpha} \frac{q_\alpha}{r^2} + \ldots
\]

(3.15a)

\[
\mathcal{H}_{(3)} = \left[ -\frac{\alpha'}{4mk} \sum_{i,j=1}^{k} \mathbf{q}_i \cdot \mathbf{q}_j - \frac{1}{m} \sum_{\alpha} q_\alpha \right] (d\tau + \omega) \wedge \Omega_2 + \ldots
\]

(3.15b)

and the Maxwell five-brane charge is

\[
Q_M = \frac{1}{4\pi^2\alpha'} \int_{S^3/\mathbb{Z}_k} \mathcal{H}_{(3)} = -\frac{2}{k^2} \sum_{i,j=1}^{k} \mathbf{q}_i \cdot \mathbf{q}_j - \sum_{\alpha} \frac{8q_\alpha}{k}.
\]

(3.16)

One can also define a Page charge, which is quantized, as:

\[
Q_P = \frac{1}{4\pi^2\alpha'} \int_{S^3/\mathbb{Z}_k} (\mathcal{H}_{(3)} - A \wedge F)
\]

(3.17)

\[= -\sum_{\alpha} \frac{8q_\alpha}{k} \in \mathbb{Z}.\]

---

\(^3\)we have chosen to work with Hermitian gauge fields, normalized as \( \text{Tr} \mathcal{T}_\alpha \mathcal{T}_\beta = 2\delta_{\alpha\beta} \).

\(^4\)The volume form of a three-sphere is

\[
ds_{S^3}^2 = \frac{1}{4} \left[ \frac{1}{4m^2} (d\tau + \omega)^2 + d\Omega_2^2 \right]
\]

\[2\pi^2 = \int \text{vol}(S^3) = \frac{1}{8} \int \frac{1}{2m} (d\tau + \omega) \wedge \Omega_2.\]
**Magnetic charges.** We take a basis of two cycles to be $\Delta_i$ where the poles of the $\Delta_i$ are at $x_i$ and $x_{i+1}$. Then the matrix of magnetic charges associated with the Abelian gauge bundle are:

$$q_{i,j} = \frac{1}{2\pi} \int_{\Delta_j} F_i = \frac{q_i \cdot T}{8\pi m} \left( \int_{x_j}^{x_{j+1}} d\tau \right) \left. \frac{V_i}{V} \right|_{x_j} = q_i \cdot T \left[ \delta_{j+1,i} - \delta_{j,i} \right].$$ (3.18)

### 3.2 Partial blow-down limits and fivebranes

The function $V(x)$ has $k$-poles, now suppose that $k'$ of these poles are co-incident, which correspond to a partial blow-down limit of the ALE or ALF space. In the present situation some of the Abelian instantons (3.7) become point-like as the corresponding two-cycles shrink and we expect heterotic five-brane to appear. We now check that in the region around such a pole we obtain the near horizon of five-brane solution of Callan, Harvey and Strominger [19, 20] where the three-sphere is orbifolded by $\mathbb{Z}_{k'}$.

For simplicity we set $x_j = 0$ for $j = 1, \ldots, k'$, and in the neighborhood of this pole the functions $H$ and $V$ behaves like

$$H \stackrel{r \to 0^+}{\sim} \frac{1}{r^2} \frac{\alpha'}{2k'm} Q_5, \quad V \stackrel{r \to 0^+}{\sim} \frac{4nk'}{r^2}$$

hence the solution approaches

$$d_{10}^2 \stackrel{r \to 0^+}{\sim} d_{1,5}^2 + 2\alpha' Q_5 \left[ \frac{dr^2}{r^2} + \frac{r^2}{4} \left( \sigma_1^2 + \sigma_3^2 + \left( \frac{\sigma_3}{2k'm} \right)^2 \right) \right]$$

$$H_{(3)} \stackrel{r \to 0^+}{\sim} \frac{\alpha' Q_5}{2} \sigma_1 \wedge \sigma_2 \wedge \frac{\sigma_3}{2k'm}$$

where the five-brane charge is given by:

$$Q_5 = \sum_{i,j=1}^{k'} q_i \cdot q_j.$$ (3.21)

### 3.3 Double scaling limit

For the two-center Eguchi-Hanson solution ($k = 2$) there exists an interesting double scaling limit [3], defined as:

$$g_s \to 0, \quad \lambda := \frac{g_s \sqrt{\alpha'}}{a} \text{ fixed and finite},$$ (3.22)

where $a$ is the distance between the two centers. This limit decouples the asymptotically locally Euclidean region, and $\lambda$ becomes the effective coupling constant of the interacting string theory.

In the spherical coordinates reviewed in appendix A the metric of the solution becomes

$$d\bar{s}^2 = d_{1,5}^2 + \frac{\alpha' Q_5}{2} \left[ \frac{dr^2}{r^2(1 - \frac{a^4}{r^2})} + \frac{1}{4} \left( 1 - \frac{a^4}{r^2} \right) \sigma_3^2 + d\Omega_2^2 \right]$$ (3.23)
and the corresponding heterotic string theory admits an exactly solvable worldsheet CFT. This space has an asymptotic linear dilaton hence admits a holographic description as a little string theory [21] but, unlike the CHS background, is given by a smooth solution of heterotic supergravity.

A double scaling limit can be described in principle for arbitrary $k$. Let us define $x_i = ay_i$ where the coordinates $y_i$ are dimension-less and $a$ is a common scale factor. The double-scaling limit can be then described exactly as before by eq. (3.22); in practice the double scaling limit amounts to setting $\delta \to 0$ in (3.12).

It would be interesting to check if one could derive a worldsheet CFT for the double scaled solutions when $k > 2$, in particular whenever the centers are arranged following a simple pattern, for instance a homogeneous distribution on a circle.

4 Atiyah-Hitchin

The Atiyah-Hitchin space $M_{AH}$ is a four-dimensional smooth manifold with an explicit Hyper-Kähler metric which at long distances approximates Taub-NUT with a negative mass parameter. The original work is [2, 22] and an interesting simplification was given in [23]. Our notation will follow a more recent work [10] where $M_{AH}$ was used as a potential Euclidean Hyper-Kähler base manifold for five dimensional supergravity solutions. In [10] regularity required the absence of closed time-like curves and this effectively excluded physical solutions whereas for our computations the non-trivial regularity conditions are essentially just positivity of the warp factor and we will find regular solutions.

The metric is

$$ds^2_{AH} = \frac{1}{4} a_1^2 a_2 a_3^2 d\eta^2 + \frac{1}{4} a_1^2 \sigma_1^2 + \frac{1}{4} a_2 \sigma_2^2 + \frac{1}{4} a_3 \sigma_3^2$$  \hspace{1cm} (4.1)$$

with the SU(2) invariant one-forms satisfying $d\sigma_i = \frac{1}{2} \epsilon_{ijk} \sigma_j \wedge \sigma_k$ given by

$$\sigma_1 = \cos \psi d\theta + \sin \psi \sin \theta d\phi$$

$$\sigma_2 = \sin \psi d\theta - \cos \psi \sin \theta d\phi$$

$$\sigma_3 = d\psi + \cos \theta d\phi$$

and the $a_i$ are subject to the following system of ODE’s:

$$\frac{d a_1}{a_1} = \frac{1}{2} \left[ (a_2 - a_3)^2 - a_1^2 \right]$$  \hspace{1cm} (4.2)$$

and cyclic permutations (dot is the derivative with respect to $\eta$). One defines new functions quadratic in the $a_i$:

$$w_1 = a_2 a_3, \quad w_2 = a_1 a_3, \quad w_3 = a_1 a_2$$  \hspace{1cm} (4.3)$$

and then the system of ODE’s is then

$$(w_1 + w_2)' = -2 \frac{w_1 w_2}{u^2}$$  \hspace{1cm} (4.4)$$

$$(w_2 + w_3)' = -2 \frac{w_2 w_3}{u^2}$$  \hspace{1cm} (4.5)$$

$$(w_3 + w_1)' = -2 \frac{w_3 w_1}{u^2}$$  \hspace{1cm} (4.6)$$
(prime is derivative with respect to $\theta$) with solution

$$w_1 = -uu' - \frac{1}{2} u^2 \csc \theta$$

$$w_2 = -uu' + \frac{1}{2} u^2 \cot \theta$$

$$w_3 = -uu' + \frac{1}{2} u^2 \csc \theta,$$

where

$$u = \frac{1}{\pi} \sqrt{\sin \theta K \left( \sin^{\frac{3}{2}} \frac{\theta}{2} \right)}$$

and $\eta$ is given in terms of $\theta$ through

$$u^2 d\eta = d\theta, \quad \eta = - \int_0^\pi \frac{d\theta}{u^2}.$$  \hspace{1cm} (4.10)

For our gauge field anazts we need an anti-self dual two form on $M_{AH}$, this is then guaranteed to solve Hermitian Yang-Mills without the need to construct the explicit Hyper-Kähler structure.\footnote{One could in principle write down the Hyper-Kähler structure using the results of [24] or by computing the Killing spinors.}

In a classic paper on dualities [8], Sen gave an integral expression for exactly such an anti-self dual, harmonic two-form on $M_{AH}$ but the appearance of this two-form dates back to the works [25–27]. Interestingly, from the work [10] we have the closed-form expression of this two-form

$$\Omega = h \left( a_1^2 dr \wedge \sigma_1 - \sigma_2 \wedge \sigma_3 \right),$$

$$h = \frac{u^2}{w_1 \sin \frac{\theta}{2}}.$$ \hspace{1cm} (4.12)

In [10] they consider self-dual forms but with a small modification of the frames this is made anti-self dual. More precisely our choice of frames is

$$e_0 = \frac{a_1 a_2 a_3}{2} d\eta, \quad e_i = \frac{a_i}{2} \sigma_i,$$ \hspace{1cm} (4.13)

whereas in [10] an additional minus sign in $e_0$ was used. So we have locally

$$\Omega = -d(h_1 \sigma_1).$$ \hspace{1cm} (4.14)

In fact one can construct a triplet of anti-self dual forms $\Omega_-$ and a triplet of self-dual forms $\Omega_+$ in a similar manner:

$$\Omega_{i-} = -d(h_i \sigma_i), \quad \Omega_{i+} = d(h_i^{-1} \sigma_i)$$ \hspace{1cm} (4.15)

with

$$h_1 = \frac{u^2}{w_1 \sin \frac{\theta}{2}}, \quad h_2 = \frac{u^2}{w_2}, \quad h_3 = \frac{u^2}{w_3 \cos \frac{\theta}{2}},$$ \hspace{1cm} (4.16)

however only $\Omega_{1-} = \Omega$ is normalizable. Given that there is a single non-trivial two-cycle in $M_{AH}$ one might be pleased to know that this normalizable form is dual to this two-cycle but there was no guarantee that the dual two-form would have an SU(2) invariant representative.
4.1 Bianchi identity

We take our gauge field to be
\[ F = \Omega \mathbf{q} \cdot \mathbf{H} \] (4.18)
where \( \mathbf{H} \in U(1)^{16} \) is in the Cartan subalgebra of \( E_8 \times E_8 \) or \( SO(32) \) and \( \mathbf{q} \) the corresponding charge vector. The three-form flux is
\[ \mathbf{H}(3) = -\frac{H'}{4} \sigma^1 \wedge \sigma^2 \wedge \sigma^3 \] (4.19)
and the Bianchi identity is
\[ d \ast_4 dH = -2q^2 \Omega \wedge \Omega, \] (4.20)
where \( q^2 = \mathbf{q} \cdot \mathbf{q} \).

Quite remarkably, one can integrate this Poisson equation explicitly
\[ H = h_0 + h_1 \eta + \frac{2q^2}{w_1} \] (4.21)
where \( \{h_0, h_1\} \) are constant coefficients of the s-wave harmonic functions on \( M_{AH} \). The last term is manifestly negative definite for the whole region \( 0 \leq \theta \leq \pi \) but we will see that one can compensate for this by a choice of harmonic function and obtain a positive definite warp factor.

4.2 Regularity

The regularity of \( M_{AH} \) has been previously studied in detail, we repeat it here to help determine regularity of our warp factor.

In the region \( \theta \sim \pi \), we define a radial co-ordinate \( r = -\log \cos \frac{\theta}{2} \) and using
\[ K = r + \log(4) + \ldots \] (4.22)
\[ u = \frac{\sqrt{2}}{\pi} re^{-r/2} + \ldots \] (4.23)
\[ w_1 = -\frac{r}{\pi^2} \] (4.24)
we find that the metric is
\[ ds^2_{AH} = dr^2 + r^2 (\sigma_1^2 + \sigma_2^2) + \sigma_3^2 + \ldots \] (4.25)
and
\[ \frac{1}{w_1} = -\frac{\pi^2}{r} + \mathcal{O}(r^{-2}) \] (4.26)
\[ \eta = -\frac{\pi^2}{r} + \frac{\pi \log(4)}{r^2} + \mathcal{O}(r^{-3}) \] (4.27)
so that the asymptotic expansion of the warp factor is
\[ H = h_0 - \frac{h_1 + 2q^2}{r} + \ldots \] (4.28)
In the region $\theta \sim 0$, we define a new radial variable $\rho = \frac{\theta^2}{64}$ and the metric is
\[
d s^2_{AH} = d\rho^2 + 4\rho^2 \sigma_1^2 + \frac{1}{16}(\sigma_2^2 + \sigma_3^2) + \ldots \tag{4.29}
\]
with
\[
\frac{1}{w_1} = -4 + 32\rho^2 + \ldots \tag{4.30}
\]
\[
\eta = \log \rho^2 + \ldots \tag{4.31}
\]
so that the IR expansion of the warp factor is
\[
H = (h_0 - 8q^2) + 64q^2 \rho^2 + \ldots \tag{4.32}
\]
From these expansions we see that with
\[
h_0 > 8q^2, \quad h_1 = 0 \tag{4.33}
\]
we have a positive warp factor which is regular everywhere. We define a rescaled radial coordinate near $\theta \sim \pi$ to be \( \hat{\theta} = h_0^{1/2} r \) and \( \hat{\rho} = h_0^{1/2} \rho \) near $\theta = 0$ so that
\[
\begin{align*}
\theta \sim \pi & : \quad d s_{10} = d s_{1,5} + d \hat{\theta}^2 + \hat{\theta}^2 (\sigma_1^2 + \sigma_2^2) + h_0 \sigma_3^2 + \ldots \tag{4.34} \\
\theta \sim 0 & : \quad d s_{10} = d s_{1,5} + d \hat{\rho}^2 + 4\hat{\rho}^2 \sigma_1^2 + \frac{h_0}{16}(\sigma_2^2 + \sigma_3^2) + \ldots \tag{4.35}
\end{align*}
\]
and see that the cost of a positive warp factor is that both the circle at infinity and the two-sphere at the bolt are large.

It is also important that the $\text{Tr} R_- \wedge R_-$ term in the Bianchi identity remains small compared to $\text{Tr} F \wedge F$. From explicit computations we find that the only possible divergences in $\text{Tr} R_- \wedge R_-$ appear through the warp factor as$^6$ $\{H'/H, H''/H\}$ which by tuning $h_0$ can be made sufficiently small with respect to $\text{Tr} F \wedge F$. This confirms that our large charge approximation remains valid and these warped Atiyah-Hitchin solutions are good Heterotic backgrounds at leading order.

Alternatively we can obtain a positive warp factor through$^7$
\[
h_0 = 1, \quad h_1 < -2q^2. \tag{4.36}
\]
This corresponds to smearing neutral five-branes on the $S^2$ at $\theta = 0$. Note that due to this smearing, at the IR ($\theta = 0$) the harmonic function parameterized by $h_1$ scales like a source in $\mathbb{R}^2$. In the UV ($\theta = \pi$), due to the finite circle, the harmonic function scales like $\frac{1}{r}$ which is that of a source in $\mathbb{R}^3$ not $\mathbb{R}^4$. The solution is of course singular for the usual reason that smeared branes are singular but this is of a good type and is resolved in string theory.

$^6$An explicit computation using the Chern connection can be found in [14] and agrees with our conclusion here.

$^7$The value of $h_0$ could be chosen to be another non-zero number $O(q^0)$. 

\[ - 10 - \]
4.3 Five-brane charge

Computing the five-brane charge requires understanding some global features of $M_{AH}$. From (4.25) and (4.29) we see that there are two inequivalent, emergent U(1) symmetries in the UV and IR, which are broken in the bulk. From [23] we know that a regular manifold requires the periodicities to be

$$0 \leq \psi \leq 2\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi$$

(4.37)

as well as that the free $\mathbb{Z}_2$ symmetry

$$I_1 : \quad \theta \rightarrow \pi - \theta, \quad \phi \rightarrow \pi + \phi, \quad \psi \rightarrow -\psi$$

(4.38)

is enforced. The horizontal space in the UV is thus $\mathbb{RP}^3/I_1$.

Using (4.28) we have

$$\mathcal{H}_{(3)} = [(h_1 + 2q^2) + \ldots] \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3$$

(4.39)

compute the Maxwell five-brane charge to be

$$Q_M = \frac{1}{4\pi\alpha'} \int_{\mathbb{RP}^3/I_1} \mathcal{H}_{(3)} = h_1 + 2q^2.$$ (4.40)

This is not required to be quantized. The Page charge is defined as in (3.17) and we find

$$Q_P = h_1$$ (4.41)

which must be integral.

4.4 Gauge field charge

The gauge field charge is computed using (4.12) and (4.18) and the IR expansion

$$h = -2 + \ldots$$

(4.42)

Under the symmetry (4.38), the bolt remains a two sphere$^8$ whose volume is $4\pi$. We find

$$\frac{1}{2\pi} \int_{S^2} F = 2q \cdot \mathcal{H} \frac{1}{2\pi} \int_{S^2} \sigma_2 \wedge \sigma_3$$

$$= 4q \cdot \mathcal{H} \in \mathbb{Z}.$$ (4.43)

5 Fundamental string sources and AdS$_3$ solutions

Heterotic backgrounds with an $\mathbb{R}^{1,1}$ factor allow for the inclusion of F1-strings along $\mathbb{R}^{1,1}$ in addition to the magnetic five-branes. The electric source of three form flux induces a non-trivial warp factor and allows for AdS$_3$ solutions. To include these fundamental strings,

---

$^8$As explained in [23] there is an additional, optional $\mathbb{Z}_2$ symmetry usually denoted $I_3$ which would convert the bolt into an $\mathbb{RP}^3$. 

---
we first consider internal eight-manifolds $X_8$ and then specialize the internal manifold to be a product of Hyper-Kähler manifolds.

The metric and three form are

$$ds_{10}^2 = e^{2A} ds_{1,1}^2 + ds_8^2$$

(5.1)

$$H^{(3)} = \text{vol}_2 \wedge h^{(1)} + h^{(3)}$$

(5.2)

where $\text{vol}_2 = e^{2A} dx^0 \wedge dx^1$. Then we find that the BPS equations are a slight embellishment of those found in [28]:

$$\Psi_i d\Psi = d(\phi - A)$$

(5.3)

$$H^{(3)} = h^{(1)} \wedge \text{vol}_2 + h^{(3)}$$

(5.4)

$$h^{(1)} = -2dA$$

(5.5)

$$h^{(3)} = *_8 e^{2(\phi - A)} d\left(e^{2(A-\phi)} \Psi\right)$$

(5.6)

where $\Psi$ is the Spin(8) structure on $M_8$:\footnote{We note that with canonical holomorphic frames $E_i = e_{2i-1} + i e_{2i}$ such that $ds_8^2 = E_i \otimes \overline{E}_i$, the SU(4) structure is $J = \frac{1}{2} E_i \wedge \overline{E}_i$, $\Omega = E_1 \wedge E_2 \wedge E_3 \wedge E_4$ and

$$\Psi = \frac{1}{2} (J \wedge J + \Omega + \overline{\Omega})$$

(5.7)

We must supplement the BPS equations with the Bianchi identity (2.3) and then due to the non-trivial warp-factor $A$, one must also impose the three form flux equation of motion:

$$0 = d\left( e^{-2\phi} *_{10} H^{(3)} \right) \quad \Rightarrow \quad \begin{cases} 0 = d\left( e^{-2\phi} *_8 h^{(1)} \right) \\ 0 = d\left( e^{2(A-\phi)} *_8 h^{(3)} \right) \end{cases}$$

(5.9)

5.1 Product of Hyper-Kähler manifolds

Our solutions with string and five-brane charges have a natural splitting of the internal eight manifold into a product of Hyper-Kähler manifolds\footnote{One might consider an additional warp factor in front of $ds_{M_1}^2$ however from [29, 30] we know that this must be constant.}

$$ds_{10}^2 = e^{2A} ds_{M_1}^2 + ds_{M_2}^2$$

(5.10)

where $M_i$ are both HyperKähler four manifolds, whose triplet of Kähler forms we denote

$$\{J_i, \text{Re} \Omega_i, \text{Im} \Omega_i\}.$$ 

(5.11)

The functions $A, B$ depend only on the co-ordinates $y_i$ of $M_2$. The Spin(8) structure is given by

$$\Psi = \frac{1}{2} (J_1 \wedge J_1 + 2e^{2B} J_1 \wedge J_2 + e^{A} J_2 \wedge J_2 + e^{2B} (\Omega_1 \wedge \Omega_2 + \overline{\Omega}_1 \wedge \overline{\Omega}_2))$$

(5.12)
We find the BPS conditions, Bianchi identity and equations of motion give\footnote{Note that is $E = e^B \tilde{E}$ is an 8d frame $*_8 E \wedge J_1 \wedge J_1 = 2 *_8 E \wedge vol_{M_1} = 2 e^{3B} *_{M_2} E \tilde{E} = 2 e^{2B} *_{M_2} E$.}

\begin{align}
\phi &= A + B \\
h_{(3)} &= -*_{M_2} d e^{2B} \\
d *_{M_2} d e^{2B} &= -\frac{1}{2} \alpha' \text{Tr} F \wedge F \\
0 &= d *_{M_2} d e^{-2A} \\
0 &= J_{2L} F
\end{align}

so we see that the only additional pieces of data from the equations in section 2 is that $e^{2A}$ is harmonic on $M_2$ and the dilaton receives a shift proportional to $A$. For the Atiyah-Hitchin manifold we can smear F1-strings on the $S^2$ bolt in much the same way as we have described for smearing 5-branes on the bolt around (4.36), that is by

\begin{equation}
e^{2A} \sim \eta.
\end{equation}

We will now be somewhat more explicit for the Gibbons-Hawking spaces.

5.2 AdS$_3$ from Gibbons-Hawking

When $M_2$ is a Gibbons-Hawking space, the U(1) invariant harmonic functions are

\begin{equation}
e^{-2A} = 1 + \sum_r \frac{\hat{q}_r}{|x - x_r|}
\end{equation}

corresponding to strings placed along $\mathbb{R}^{1,1}$ and at fixed points of $\partial_r$ on $M_2$.

If in addition we choose to place these strings at poles of $V$ we recover AdS$_3$ geometries near such a pole. We put $k'$ poles of $V$ as well as the strings at $x_r = x_i = 0$ then in the vicinity of $x_i$ we have

\begin{align}
e^{2A} &= \frac{r}{q_0} \ldots e^{2B} = \frac{1}{r} \frac{\alpha'}{4m} Q_5 + \ldots, \\
V &= \frac{2mk'}{r} + \ldots
\end{align}

so that

\begin{align}
ds^2_{10} &= \frac{r}{q_0} ds_{1,1}^2 + ds_{M_2}^2 + 2\alpha' k'^2 Q_5 \left[ \frac{1}{4} \frac{dr^2}{r^2} + ds_{S^3/Z_k}^2 \right] \\
&= 2\alpha' k'^2 Q_5 \left[ ds_{AdS_3}^2 + ds_{S^3/Z_k}^2 \right] + ds_{M_1}^2 \\
\frac{e^{2\phi}}{4m\hat{q}_0} &= \frac{\alpha' Q_5}{4m\hat{q}_0}
\end{align}

where $r = \rho^2$. The F1-charge is given as usual by

\begin{equation}
Q_1 = \frac{4m\hat{q}_0 \text{vol}(M_1)}{\alpha' 3}.
\end{equation}

The gauge field vanishes in this limit and the background is sourced by three-form flux.
6 Conclusions

The key aspect of our solutions with Abelian gauge bundles is that we have taken a large charge limit and consistently suppressed the Tr $R \wedge R$ term in the Bianchi identity, which is subdominant at leading order in the expansion in $\frac{1}{q^2}$. We have shown how this large charge limit can lead to exact supersymmetric flux backgrounds, and it is particularly interesting the Atiyah-Hitchin manifold can provide a regular background. This configuration requires some ingenuity to counteract the negative mass and result in a background of the correct signature. This Atiyah-Hitchin based solution is distinctly different from those based on Gibbons-Hawking; while the latter can be viewed as marginal deformations of the orbifold of the CHS solutions the finite two-cycle in the Atiyah-Hitchin manifold cannot be blown down. As such we do not have a worldsheet theory from which we can imagine obtaining this as the background geometry.

In these backgrounds, the gauge fields are completely solved for by using the Hermitian Yang-Mills equations which then provide a source for the three-form flux. It is conceivable that non-Abelian bundles could be constructed such that Tr $F \wedge F$ dominates Tr $R \wedge R$ everywhere. Since the Kronheimer-Nakajima construction [33] gives a solution of all instantons on ALE Gibbons-Hawking spaces, one could possibly even construct such instantons, however most instantons will provide a source the Bianchi identity whose solution is a general function of four variables and thus unsolvable. A particularly neat class of instantons is based on the ’t Hooft ansatz [34]:

$$A_0 = \frac{1}{2} \vec{G} \cdot \vec{\sigma}, \quad \vec{A} = \frac{1}{2} \left[ \vec{\omega}(\vec{G} \cdot \vec{\sigma}) - V(\vec{G} \times \vec{\sigma}) \right]$$



with $f$ harmonic on $\mathbb{R}^3$. For finite action, the centers of $f$ are constrained to lie at the poles of $V$. These instantons can have large Tr $F \wedge F$ in the limit of large number of poles of $f$ but Tr $R \wedge R$ will not be suppressed.

There are numerous directions for progress on the worldsheet description of these backgrounds. The elliptic genus for type II on ALE spaces has been computed recently [35] based on general developments in this field [36] and we expect to be able to provide a similar solution for these Heterotic models or the type II flux backgrounds of section 2.1. It would also be interesting to provide an exactly solvable worldsheet model of the near-horizon region of the multi-center Gibbons-Hawking backgrounds, generalizing the gauged WZW model of the two-centered solution.

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12 Interesting five dimensional solutions with non-Abelian gauge fields have appeared recently [31] and the lift to the Heterotic string has been discussed [32]. However it is not clear to us how these solutions will solve the exact Bianchi identity.
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A Eguchi-Hanson

When \( k = 2 \) and \( \epsilon = 0 \), the explicit co-ordinate transformation is known \([37]\) from the Gibbons-Hawking space to the Eguchi-Hanson space \([38]\). In Cartesian co-ordinates the two center ALE Gibbons-Hawking space has

\[
\omega = \left[ \frac{z - a^2/8}{\sqrt{x^2 + y^2 + (z - a^2/8)^2}} + \frac{z + a^2/8}{\sqrt{x^2 + y^2 + (z + a^2/8)^2}} \right] d\left( \tan^{-1} \frac{y}{x} \right) \quad (A.1)
\]

\[
V = \frac{1}{\sqrt{x^2 + y^2 + (z - a^2/8)^2}} + \frac{1}{\sqrt{x^2 + y^2 + (z + a^2/8)^2}}. \quad (A.2)
\]

Following \([37]\) we have \((a \leq r)\):

\[
x = \frac{a^2}{8} \sqrt{\frac{r^4}{a^4} - 1} \sin \theta \cos \psi
\]

\[
y = \frac{a^2}{8} \sqrt{\frac{r^4}{a^4} - 1} \sin \theta \sin \psi
\]

\[
z = \frac{1}{8} r^2 \cos \theta
\]

so that

\[
V = \frac{16}{a^2} \frac{r^2}{a^2 - \cos^2 \theta},
\]

\[
\omega = \frac{2 \cos \theta (\frac{r^2}{a^2} - 1)}{r^2 - \cos \theta} d\psi. \quad (A.3)
\]

As an example, we write explicitly the solution for Heterotic five-branes on Eguchi-Hanson with additional F1-strings.\(^{13}\)

\[
ds^2_{M_2} = \frac{dr^2}{f^2} + \frac{r^2}{4} \left[ \sigma_1^2 + \sigma_2^2 + f^2 \sigma_3^2 \right] \quad (A.4)
\]

\[
f^2 = 1 - \frac{a^4}{r^4} \quad (A.5)
\]

\[
h_{(3)} = 2 f^2 r^3 (e^{2B})' \sigma_1 \wedge \sigma_2 \wedge \sigma_3
\]

\[
F = d \left( \frac{a^2}{r^2} \eta \right) \quad (A.6)
\]

\[
e^{2B} = 1 + \frac{8 \alpha' Q_5}{r^2} + \frac{Q_5}{8 a^2} \log \left[ \frac{r^2 / a^2 - 1}{r^2 / a^2 + 1} \right] \quad (A.7)
\]

\(^{13}\)One can take \(M_1\) to be \(T^4\) or \(K3\) with the Ricci-flat metric.
\[ e^{-2A} = 1 + \frac{Q_1}{a^2} \log \left[ \frac{r^2/a^2 - 1}{r^2/a^2 + 1} \right] \]  
\[ e^{2\Phi} = e^{2(A + B)} \]  
(A.8)  
(A.9)

In addition to the Heterotic five-branes which resolve the singularity, there are \( Q_1 \) mobile F1-strings and \( Q_5 \) NS5-branes smeared on the blown-up \( S^2 \). Due to the smearing of the strings, the near horizon limit has a log-singularity at \( r \sim a \) in the warp factor \( e^{2A} \) and thus there is no enhancement to AdS\(_3\). In the blow-down limit \( a \to 0 \) where the Eguchi-Hanson space becomes \( \mathbb{C}^2/\mathbb{Z}_2 \), the gauge field vanishes and we get the \( \mathbb{Z}_2 \) orbifold of the usual F1-NS5-solution, the near-horizon limit is AdS\(_3 \times S^3/\mathbb{Z}_2 \times M_1 \).

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