Aligned Magnetic Field on Williamson Fluid over a Stretching Sheet with Newtonian Heating

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Abstract. Boundary layer flow and heat transfer on Williamson fluid over a stretching sheet is numerically investigated. This paper has been conducted with aim to address how to extend the mathematical formulation for the aligned magnetic field on Williamson fluid over a stretching sheet with Newtonian Heating and how does the effects of several parameters with thermal boundary condition of Newtonian heating in the temperature and velocity profiles. The main objectives of this study are to extend the mathematical formulation and solve numerical algorithms for computations by using Runge-Kutta-Fehlberg (RKF45) technique and analyze the influence aligned angle, magnetic field parameter, Williamson fluid, Prandtl number and conjugate parameter in the term of distribution velocity and temperature are presented graphically. The finding revealed that increasing Williamson fluid parameter, resulted in negative value due to opposite direction between fluid flow and stretching sheet. Increasing on values of Prandtl number results to a decrease in temperature profile. Further, increasing the conjugate parameter resulted to the increased in boundary layer thickness and increasing the magnetic field parameter results to reduced the velocity profile.

Keywords: Aligned angle, magnetic field, Williamson fluid, Newtonian Heating

1. Introduction

Recent studies have shown substantial development of the non-Newtonian fluid. This development can be monitored by the complex nature of the fluid used in various industrial applications such as manufacturing of polymer sheets, paper production and the aerodynamic extrusion of plastic sheets [1]. Unlike Newtonian fluid, the relationship between stress levels and non-Newtonian fluid strain is non-linear due to the time or deformation dependence of the fluid viscosity. The most commonly mentioned non-Newtonian model include in literature are the micropolar fluid model, viscoelastic fluid model, Jeffrey fluid model, Casson fluid model, second grade fluid model and Williamson fluid model. Others researchers who take part in non-Newtonian fluid are, [1-5]. In this study Williamson fluid will be consider.
[6] the first lead to developed a model equation of pseudoplastic fluid and solve the flow to describe the fluid flow of pseudoplastic. Other researchers who considered the Williamson fluid are [4,7] and [8]. Hydromagnetic boundary layer flow of Williamson fluid in the presence of thermal radiation and Ohmic dissipation was examined by [9]. Next, radiation effect on MHD Williamson fluid flow over stretching cylinder through porous medium with heat source has been deliberate by [10,11] studied dual solutions on thermal radiation effects on Williamson fluid flow due to an expanding/contracting cylinder with nanomaterials.

The term magnetohydrodynamics (MHD) refers to magneto (magnetic field), hydro (water) and dynamics (movement) which clearly represent an electrically fluid’s motion under the magnetic field [12]. As continuance of the pioneer studies of magnetic field, [13] examined MHD orthogonal stagnation-point flow of a micropolar fluid with the magnetic field parallel to the velocity at infinity. Next, effect of an inclined magnetic field on peristaltic flow of Williamson fluid in an inclined channel with convective conditions was carry out by [14,15] investigate MHD mixed convective stagnation point flow and heat transfer of an incompressible nanofluid over an inclined stretching sheet with chemical reaction and radiation.

This study was conducted with the goal of examine the non-Newtonian Williamson fluid flow with associated aligned magnetic field and Newtonian heating boundary condition, based on the above literature. The findings mentioned here are new because this issue was not published before.

2. Mathematical Formulation
Consider a steady two-dimensional and incompressible Williamson fluid flow over a vertical stretching sheet with $x -$ axes being positioned vertically in upward direction along the sheet and $y -$ axes normal to it. The origin of the sheet is placed at the leading edge of the $x -$ axes. It is assumed the sheet is stretched with uniform velocity $u_a(x) = ax$ where $a$ is constant and the bottom of the sheet is heated by Newtonian heating. An aligned magnetic field with an aligned angle $\phi$ (in the range of $0^\circ$ to $90^\circ$) is applied to the flow as illustrate in Figure 1. The boundary layer equations [16,17] as follow:

\[
\frac{\partial T}{\partial y} = -h_s T(NH)
\]

\[
u = u_a = ax
\]

\[
\begin{align*}
\phi & \quad \text{Momentum boundary layer} \\
\theta & \quad \text{Thermal boundary layer}
\end{align*}
\]

**FIGURE 1:** Schematic diagram for aligned magnetic field on Williamson fluid over a stretching sheet with Newtonian heating
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 
\] (1)

\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + \sqrt{2} \nu \Gamma \frac{\partial^2 u}{\partial y^2} - \sigma u \beta^2 \sin^2 \phi
\] (2)

\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2}
\] (3)

corresponds to the following conditions

\[
u = u_w(x), \quad v = 0, \quad \frac{\partial T}{\partial y} = -h_T \text{ at } y = 0,
\] (4)
\[u = 0, \quad T \to T_\infty \text{ as } y \to \infty
\]

where \(u\) and \(v\) are the velocities components along the \(x\) and \(y\) directions, respectively. Further, \(T\) is the fluid temperature in the boundary layer, \(\Gamma\) is the time constant, \(\beta\) is the magnetic field strength, \(\nu\) is the kinematic viscosity, \(\alpha\) is the thermal diffusivity and \(h_T\) is the heat transfer coefficient.

The similarity variables are as follows [16]

\[
\eta = \left(\frac{a}{v}\right)^{\frac{1}{2}} y, \quad \psi = (av)^{\frac{1}{2}} xf(n), \quad \theta(\eta) = \frac{T - T_\infty}{T_\infty},
\] (5)

where \(\eta\) and \(\theta(\eta)\) is the non-dimensional variables, while \(\psi\) is the stream function. Then, \(u\) and \(v\) can be defined as \(u = \frac{\partial \psi}{\partial y}\) and \(v = -\frac{\partial \psi}{\partial x}\), which satisfies the equation (1).

Further, noticed that

\[
u = axf'(\eta), \quad v = -(av)^{\frac{1}{2}} f(\eta)
\] (6)

where prime refer to differentiation with respect to \(\eta\). By substitute equations (5) and (6) into equations (2) and (3), the following equations are obtain:

\[
f'' + ff' + \lambda f'' f'' - f'^2 - (M \sin^2 \phi)f' = 0
\] (7)
\[\frac{1}{Pr} \theta'' + f \theta' = 0
\] (8)

where \(\Pr = \frac{\nu}{\alpha}\) is the Prandtl number, \(\lambda = x \Gamma \sqrt{\frac{2e^3}{\nu}}\) is the non-Newtonian Williamson fluid parameter and \(M = \frac{\sigma B_0^2}{\alpha}\) is the Magnetic field parameter.

A derivative based on the use of equations (5) and (6) into boundary conditions (4) become
\[ f(0) = 0, \quad f'(0) = 1, \quad \theta'(0) = -\gamma(1 + \theta(0)) \]  

(9)

\[ f'(\eta) \to 0, \quad \theta(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \]

where \( \gamma = h \left( \frac{v}{a} \right)^{\frac{1}{2}} \) is the conjugate parameter for Newtonian heating. Noticed that \( \gamma = 0 \) is for the insulated plate and \( \gamma \to \infty \) is when the surface temperature remains constant. The physical quantities of interest are the skin friction coefficient \( C_f \) which are given by

\[ C_f = \frac{\tau_w}{\rho u_w^2}, \]

(10)

where \( \rho \) is the fluid density. The surface shear stress \( \tau_w \) are given by

\[ \tau_w = \mu \frac{\partial u}{\partial y} \left[ 1 + \Gamma \sqrt{\frac{1}{2} \frac{\partial u}{\partial y}} \right], \]

(11)

with \( \mu = \rho \nu \) being the dynamic viscosity. Using the similarity variables in (5) give

\[ C_f \text{Re}^{\frac{1}{2}} = f^*(0) + \frac{1}{2} \left( f^*(0) \right)^2 \]

(12)

where \( \text{Re} = \frac{u_\infty x}{\nu} \) is the local Reynolds number.

3. Results and Discussion

Runge-Kutta-Fehlberg (RKF45) technique with the aid of Maple tolos to solve the system of ordinary differential equations (7) and (8) with boundary conditions (9). For better understanding the flow and heat transfer properties, pertinent parameters, namely the Prandtl number \( Pr \), the non-Newtonian Williamson fluid parameter \( \lambda \), the magnetic parameter \( M \), aligned angle \( \phi \) and the conjugate parameter \( \gamma \) are considered. For computing purposes, the boundary layer thickness from 3.5 to 10 is considered to provide the accurate numerical result.

The numerical results obtained are validated by comparison values of \( \theta(0) \). The comparison with [16] and [18] can be seen in Table 1. Both solved analytically with using the Keller-box method which is programmed in MATLAB software. The numerical results have been found to be well agreed, so we are sure of the precision of the findings.

| Pr   | [16]  | [18]  | Present  |
|------|-------|-------|----------|
| 3    | 6.0258| 6.0513| 6.05159  |
| 5    | 1.7659| 1.7604| 1.76039  |
| 7    | 1.1351| 1.1168| 1.11681  |
| 10   | 0.7653| 0.7645| 0.76452  |
| 100  | 0.1612| 0.1478| 0.14782  |
Table 2 presents the values of heat transfer coefficient \(-\theta(0)\) and reduced skin friction coefficient \(C_f\,Re_i^{1/2}\) for a different values of the non-Newtonian Williamson fluid parameter \(\lambda\). Noticed that the values of given are in negative value due to opposite direction between fluid flow and stretching sheet. From table 2, It is found that the increase of \(\lambda\) reduced the values of \(C_f\,Re_i^{1/2}\).

**Table 2.** Values of \(Nu, Re_i^{1/2}\) and \(C_f\,Re_i^{1/2}\) for the various values of \(\lambda\) when \(Pr = 7, \gamma = 1, \phi = \frac{\pi}{6}\) and \(M = 1.5\)

| \(\lambda\) | \(-\theta(0)\)   | \(C_f\,Re_i^{1/2}\) |
|-------------|-------------------|----------------------|
| 0.1         | 2.17873           | -1.14957             |
| 0.15        | 2.18523           | -1.13578             |
| 0.2         | 2.19243           | -1.12127             |
| 0.25        | 2.20049           | -1.10589             |
| 0.3         | 2.20965           | -1.08947             |

Figure 2 captures the effects of Prandtl number on temperature profiles for various values of Pr. Increasing on values of Pr results to a reduction in the temperature profile due to a large value of Pr signifies highly viscous fluid with low thermal conductivity. Physically the Prandtl number is termed as the correlation between momentum and thermal diffusivities, larger values of Pr have low conductivity while small Pr values possess thermal conductivity. Figure 3 demonstrate the temperature profile for several values of conjugate parameter \(\gamma\) respectively. The increasing in \(\gamma\) lead to an increase in the thickness of boundary layer. Besides, from the boundary conditions (9), the heat transfer coefficients also increase as well as the wall temperature.

**FIGURE 2:** Temperature profiles \(\theta(\eta)\) for various \(Pr\) when \(\phi = \frac{\pi}{6}, \lambda = 0.1, \gamma = 0.5\) and \(M = 1\)

**FIGURE 3:** Temperature profiles \(\theta(\eta)\) for values of \(\gamma\) when \(\phi = \frac{\pi}{6}, Pr = 7, \lambda = 0.1\) and \(M = 1\)
Figure 4 presents the velocity profile for different values of magnetic field parameter $M$. It is observed that the increase of $M$ results in reduced velocity profile. Physically, the increase of $M$ has reduced a thickness of the boundary layer which implies increasing manner of the magnitude of the velocity gradient, thus enhanced the reduced skin friction coefficient. Figure 5 presents the velocity profile for various values aligned angle parameter $\phi$. The important findings is the effects of aligned and can be varies from 0° to 90° and if $\phi=0$ is consider absent of magnetic field effect. It is found that the increasing aligned angle makes velocity gradient decrease. Physically, the increase of $\phi$ enhanced the intensity of magnetic field thus provide the similar effects as in figure 4.

![FIGURE 4: Velocity profile $f'(\eta)$ for various values $M$ when $\phi=\frac{\pi}{6}$, $Pr=7$, $\gamma=0.5$ and $\lambda=0.1$](image1)

![FIGURE 5: velocity profile $f'(\eta)$ for several values of $\phi$ when $Pr=7$, $\gamma=0.5$, $\lambda=0.1$ and $M=1.5$](image2)

Figure 6 displays the variation of the Nusselt number $tNu_x Re_x^{-1/2}$ with $\lambda$ for several values of $\phi$ when $Pr=7$, $\gamma=0.5$ and $M=1.5$. The result observed from figure $Nu_x Re_x^{-1/2}$ increase with the increases values of $\phi$, while as the $Nu_x Re_x^{-1/2}$ increase with increase in $\lambda$ for fixed values of $\phi$. Similar result obtain in Figure 7, $Nu_x Re_x^{-1/2}$ increase with the increase values of $M$, while as the $Nu_x Re_x^{-1/2}$ increase with increase in $\lambda$ for fixed values of $M$. 
FIGURE 6: Variation of Nusselt number $Nu_x Re_x^{-1/2}$ with $\lambda$ for several values of $\phi$ when $Pr = 7, \gamma = 0.5$ and $M = 1.5$

FIGURE 7: Variation of Nusselt number $Nu_x Re_x^{-1/2}$ with $\lambda$ for several values of $M$ when $Pr = 7, \gamma = 0.5$ and $\phi = \pi/6$

4. Conclusion
The present study examined the aligned magnetic field on Williamson fluid over a stretching sheet with Newtonian heating. The values of skin friction coefficient as well as the temperature and velocity profiles affected by Prandtl number $Pr$, magnetic parameter $M$, non-Newtonian Williamson fluid parameter $\lambda$, aligned angle parameter $\phi$ numerically studied. The important findings as follows:

- Increasing Williamson fluid parameter, resulted to $C_f Re_x^{1/2}$ in negative value due to opposite direction between fluid flow and stretching sheet. Noted that, Williamson fluid parameter give small influence to temperatura and velocity distribution.
- Increasing on values of $Pr$ results to a decrease in temperature profile. Further in considering of conjugate parameter, increasing in $\gamma$ resulted to the increased in boundary layer thickness.
- Increasing of $M$ results to reduced the velocity profile meanwhile for aligned angle parameter, increasing $\phi$ makes velocity gradient decrease.

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