Embeddings into Free Heyting Algebras and
Translations into Intuitionistic Propositional
Logic

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Abstract

We find a translation with particularly nice properties from intuitionistic propositional logic in countably many variables to intuitionistic propositional logic in two variables. In addition, the existence of a possibly-not-as-nice translation from any countable logic into intuitionistic propositional logic in two variables is shown. The nonexistence of a translation from classical logic into intuitionistic propositional logic which preserves $\land$ and $\lor$ but not necessarily $\top$ is proven. These results about translations follow from additional results about embeddings into free Heyting algebras.

1 Introduction

Intuitionistic logic has been explored for many years as a language for computer science, with a guiding principle being the Brouwer-Heyting-Kolmogorov interpretation, under which intuitionistic proofs of implication are functions and existence proofs require witnesses. Higher-order intuitionistic systems which can express a great deal of mathematics, such as Girard’s System $F$
and Martin-Löf’s type theory (good references are [8] and [3]), have been
developed and implemented by prominent computer scientists such as Con-
stable, Huet and Coquand (see [2] and [1]). With all this development and
with the existence of well-established topological, Kripke, and categorical se-
manics for intuitionistic systems, it may come as a surprise that many fun-
damental structural properties of intuitionistic propositional calculus have
not been developed. By way of contrast, corresponding issues for classical
logics have been settled for at least 75 years.

Heyting algebras are an equationally defined class of algebras with op-
erations $\lor$, $\land$, and $\rightarrow$ and constants $\bot$ and $\top$ (representing “or,” “and,” “im-
plies,” “false,” and “true” respectively) that stand in the same relation to
intuitionistic propositional logic that Boolean algebras do to classical pro-
sitional logic. What follows is a very brief introduction to free Heyting a-
lgebras and a summary of the results that will be presented in this paper.

For each $n \in \mathbb{N}$, let $V_n = \{x_1, \ldots, x_n\}$ and let $F_n$ be the set of propo-
sitional sentences in variables $V_n$. Let $\simeq^n_i$ and $\simeq^n_c$ be the intuitionistic and
classical logical equivalence relations respectively.

The classical Lindenbaum algebra $B_n$ is then defined as $F_n / \simeq^n_c$ and the
intuitionistic Lindenbaum algebra $H_n$ is defined as $F_n / \simeq^n_i$. The operations
$\land$ and $\lor$ and the constants $\top$ and $\bot$ are naturally defined on $B_n$ and the
operations $\land$, $\lor$, and $\rightarrow$ and the constants $\top$ and $\bot$ are naturally defined on
$H_n$. $B_n$ is then isomorphic to the free Boolean algebra on $n$ generators and
$H_n$ is the free Heyting algebra on $n$ generators. As usual, the order $\leq$ may
be defined from $\land$ (or from $\lor$). Like all Heyting algebras, each $H_n$ is also a
distributive lattice.

The analogous statements are true for $V_\omega = \{x_1, x_2, \ldots\}$, $F_\omega$, $B_\omega$, and
$H_\omega$.

The structure of each $B_n$ and of $B_\omega$ is well understood. However, among
the free Heyting algebras, only $H_1$ is completely understood. It is known
from [10] that if we let $\phi_i = \neg x_i$, $\psi_i = x_i$, $\phi_{i+1} = \phi_i \rightarrow \psi_i$, and $\psi_{i+1} = \phi_i \lor \psi_i$,
then each propositional formula in the single variable $x_1$ is intuitionistically
equivalent to exactly one formula in $\{\bot\} \cup \{\phi_i \mid i \in \omega\} \cup \{\psi_i \mid i \in \omega\} \cup \{\top\}$. Further, we can easily write down conditions characterizing the order on
those formulas, so that the structure of $H_1$ is completely characterized.

Although the structure of $H_n$ for $n \geq 2$ and of $H_\omega$ is not fully understood,
there are a number of facts known. Although not a complete list, the reader
is referred to [4], [7], [5], and [6]. A very useful construction is contained
in [4] which will we avail ourselves of in this paper and which is described in
Section 2 below. The results of this paper are as follows: There is a lattice-embedding from $H_\omega$ into $H_2$. This obviously implies that there is a lattice-embedding from $H_m$ into $H_n$ for any $m \geq 1$, $n \geq 2$, but in these cases, more is true: For any $n \geq 2$ and $m \geq 1$, there is a $\phi$ and a $\psi \in H_n$ such that $[\phi, \psi] := \{ \rho \in H_n \mid \phi \leq \rho \leq \psi \}$ and $H_m$ are isomorphic as lattices. In addition, the isomorphism from $[\phi, \psi]$ to $H_m$ can be extended to all of $H_n$, so that there is a surjective lattice-homomorphism from $H_n$ to $H_m$.

Furthermore, we will show that any countable partial order can be order-embedded into $H_2$, and that the countable atomless boolean algebra $B_\omega$ cannot be lattice-embedded into $H_\omega$.

Some of these results also have significance in terms of translations into intuitionistic propositional logic, a notion which we now define.

Let $\vdash$ be the intuitionistic consequence relation. Define a consequence-preserving translation from $n$-variable intuitionistic logic into $m$-variable intuitionistic logic to be a function $f : F_n \rightarrow F_m$ such that for all $\emptyset \neq \Gamma \subseteq F_n$, $\phi \in F_n$, $\Gamma \vdash \phi$ iff $f(\Gamma) \vdash f(\phi)$.

Define a tautology-respecting translation from $n$-variable intuitionistic logic into $m$-variable intuitionistic logic to be a function $f : F_n \rightarrow F_m$ such that for all $\phi \in F_n$, $\vdash \phi$ iff $\vdash f(\phi)$.

The term “respecting” is used to emphasize that the property of being consequence-respecting and the property of being tautology-respecting are stronger than the property of being consequence-preserving and tautology-preserving respectively.

We define a $\land, \lor$-preserving translation from $n$-variable propositional logic to $m$-variable propositional logic to be a function $f : F_n \rightarrow F_m$ such that for all $\phi, \psi \in F_n$, $f(\phi \land \psi) = f(\phi) \land f(\psi)$ and $f(\phi \lor \psi) = f(\phi) \lor f(\psi)$.

We make the obvious modifications to the definitions for translations from classical logic to intuitionistic logic and for $\omega$-variable logics.

Thus, Gödel’s double-negation translation (see [9] or [3]) is a tautology-respecting but not consequence-respecting or $\land, \lor$-preserving translation from $\omega$-variable classical logic to $\omega$-variable intuitionistic logic and Gentzen’s translation (again, see [9] or [3]) is a tautology-respecting and consequence-respecting but not $\land, \lor$-preserving translation from $\omega$-variable classical logic to $\omega$-variable intuitionistic logic. Both of these translations may be restricted to be from $n$-variable classical logic to $n$-variable intuitionistic logic for any $n$.

Some of the results of this paper may then be restated as follows: There
is a consequence- and tautology-respecting, \((\land, \lor)\) -preserving translation of \(\omega\)-variable intuitionistic logic into 2-variable intuitionistic logic. We also get consequence- and tautology-respecting translations (which aren’t \((\land, \lor)\)-preserving) of \(\omega\)-variable classical logic into 2-variable intuitionistic logic by composing with Gentzen’s translation. This translation may be read off explicitly from the proof contained in this paper together with the construction of [3].

The disjunction property of intuitionistic logic implies that there can be no tautology-respecting translation of classical logic into intuitionistic logic. In addition, we will show that there is no merely consequence-respecting, \((\land, \lor)\)-preserving translation of \(\omega\)-variable classical logic into \(\omega\)-variable intuitionistic logic (and thus not into \(n\)-variable intuitionistic logic for any \(n\)).

The result that any countable partial order can be embedded in \(H_2\) implies that any logic may be translated in a consequence- and tautology-respecting but not necessarily \((\land, \lor)\)-preserving way into 2-variable intuitionistic logic, as long as the logic is countable.

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2 Notation, Terminology, and Bellissima’s Construction

As above, let \(V_n = \{x_1, x_2, \ldots, x_n\}\).

We will use Bellissima’s construction ([4]) of, for each \(n\), a Kripke model \(K_n\) over \(V_n\) satisfying Propositions 1 and 2 below. The construction and relevant facts about it will be stated here.

Given a Kripke model \(K\) over \(V_n\), let \(\text{Nodes}(K)\) be the set of nodes of \(K\) and let \(\leq(K)\) be the (non-strict) partial order on \(\text{Nodes}(K)\) given by \(K\). If no confusion will result, we may use \(K\) in place of \(\text{Nodes}(K)\). Given \(\alpha \in K\), let \(w(\alpha) = \{x_i \in V_n \mid \alpha \vdash x_i\}\).

We will define a Kripke model \(K_n\) in stages, so that \(K_n = \bigcup_i K_n^i\) where the \(K_n^i\) are defined as follows:

\[\text{Nodes}(K_n^0) = \mathcal{P}(V_n)\text{ and }\leq(K_n^0) = \{(\alpha, \alpha) \mid \alpha \in \text{Nodes}(K_n^0)\}.\]

For clarity, when we want to emphasize that we are thinking of \(U \subseteq V_n\) as a node, we may write \(\text{node}(U)\) or \(\text{node}_{K_n}(U)\). If we want to also note that it is in \(K_n^0\),
we may write node\(^0(U)\) or node\(^0_{K_n}(U)\). For \(U \subseteq V_n\), we let \(w(\text{node}(U)) = U\).

Given \(K_n^0, \ldots, K_n^i\), let \(T_{i+1}\) be the set of subsets \(T\) of \(\bigcup_{j=0}^i K_n^j\) such that \(T \cap (K_n^i - K_n^{i-1}) \neq \emptyset\) and such that the elements of \(T\) are pairwise incomparable with respect to \(\leq(K_n^i)\). Then we define Nodes\((K_n^{i+1}) - \text{Nodes}(K_n^i)\) to be

\[
\{\langle T, U \rangle \mid T \in T_{i+1}, U \subseteq V_n, U \subseteq \bigcap_{\alpha \in T} w(\alpha) \text{ and if } T = \{\beta\}, \text{ then } U \not\subseteq w(\beta)\}
\]

For clarity, when we want to emphasize that we are thinking of \(\langle T, U \rangle\) as a node, we may write node\((\langle T, U \rangle)\) or node\(_{K_n}(\langle T, U \rangle)\). If we want to also note that it is in \(K_n^{i+1}\), we may write node\(^{i+1}(\langle T, U \rangle)\) or node\(^{i+1}_{K_n}(\langle T, U \rangle)\).

We declare that \(w(\langle T, U \rangle) = U\) and we let \(\leq(K_n^{i+1})\) be the reflexive transitive closure of \(\leq(K_n^i) \cup \{\langle(T, U), \beta \rangle \mid \langle T, U \rangle \in K_n^{i+1} - K_n^i, \beta \in T\}\).

Let \(k(\phi)\) denote the set of nodes in \(K_n\) which force \(\phi\), for \(\phi\) a propositional formula in \(n\) variables.

**Proposition 1** ([4]). For \(\phi\) and \(\psi\) propositional formulas in \(x_1, \ldots, x_n\), \(\phi \vdash \psi\) iff \(k(\phi) \subseteq k(\psi)\).

**Proposition 2** ([4]). For each node \(\alpha \in K_n\), there is a \(\phi_\alpha\) such that \(k(\phi_\alpha) = \{\beta \in K_n \mid \beta \geq \alpha\}\) and there is a \(\phi'_\alpha\) such that \(k(\phi'_\alpha) = \{\beta \in K_n \mid \beta \not\leq \alpha\}\).

We now fix some terminology.

If \(\alpha < \beta\) are nodes in some Kripke model, then \(\beta\) is called a successor of \(\alpha\) and \(\alpha\) a predecessor of \(\beta\). If there is no \(\gamma\) with \(\alpha < \gamma < \beta\), then \(\beta\) is called an immediate successor of \(\alpha\). The assertions “\(\alpha\) is above \(\beta\)” and “\(\beta\) is below \(\alpha\)” both mean \(\alpha \geq \beta\).

For any node \(\alpha\) in any Kripke model, \(s(\alpha)\) is the set of \(\alpha\)'s immediate successors.

If \(\alpha \in K_n\), then \(\phi_\alpha\) and \(\phi'_\alpha\) are as in Proposition [2].

For each \(m\), \(\text{Lev}_n^m = K_n^m - K_n^{m-1}\). This may also be called \(\text{Lev}_m\) if \(n\) is clear from context and may be denoted in English as “level \(m\).”

If \(\alpha \in K_n\), then \(\text{Lev}(\alpha)\) is the unique \(i\) such that \(\alpha \in \text{Lev}_i^m\). Note that if \(\alpha \leq \beta\), \(\text{Lev}(\alpha) \geq \text{Lev}(\beta)\).

If \(T\) is a set of nodes in \(K_n\), let \(r(T) = \{\alpha \in T \mid \neg(\exists \beta \in T) (\beta < \alpha)\}\). Thus, for example, for any \(T\) with \(|T| \geq 2\), \(\langle r(T), \emptyset \rangle \in K_n\). The following facts will be used below and follow without much difficulty directly from the construction.
Fact 3. For $n \geq 2$ and $m \geq 0$, $|\text{Lev}_{m+1}^n| > |\text{Lev}_m^n|$. In particular, there are arbitrarily large levels of $K_n$.

The following fact is a more general version of the preceding fact.

Fact 4. Let $S \subseteq K_n$, $|S| \geq 3$ and let each element of $S$ be at the same level. Let $S'$ be the downward closure of $S$. Then $|S' \cap \text{Lev}_{m+1}^n| > |S' \cap \text{Lev}_m^n|$ for any $m$ greater than or equal to the common level of the elements of $S$.

3 A Lattice Embedding from $H_m$ to $H_n$ for $m \geq 1, n \geq 2$

Theorem 5. Let $n \geq 2$, $m \geq 1$. Then there are $\phi, \psi \in H_n$ such that $H_m$ is isomorphic to $[\phi, \psi]$. In addition, the isomorphism from $[\phi, \psi]$ to $H_m$ can be extended to a surjective lattice-homomorphism from $H_n$ to $H_m$.

Proof. The main work is contained in the following proposition.

Proposition 6. Let $m \geq 2$ and $n$ be such that there is a level $\text{Lev}_i^n$ of $K_n$ and a set $A \subseteq \text{Lev}_i^n$ such that $|A| = m$ and each $\alpha \in A$ has some immediate successor not above any other $\alpha' \in A$. Then there is a $\phi, \psi \in H_n$ such that $H_m$ is lattice-isomorphic to $[\phi, \psi]$ and in addition, the isomorphism from $[\phi, \psi]$ to $H_m$ can be extended to a surjective lattice-homomorphism from $H_n$ to $H_m$.

Proof. Fix $A$ and $i$ from the hypothesis.

Let $A = \{\alpha_1, \ldots, \alpha_m\}$. Let $\phi$ be

$$\bigvee_i \phi_{\alpha_i}.$$ 

For each $A' \subseteq A$, let $\gamma_{A'}$ be the node $(r(T), \emptyset)$ where $T = A' \cup \bigcup_{\alpha \in A'} s(\alpha)$. This is valid as the elements of $r(T)$ are pairwise incomparable and $|r(T)| \geq 2$ since $m \geq 2$.

Note that $\gamma_{A'}$ is at level $i + 1$ if $A'$ is nonempty and at level $i$ if $A'$ is empty. Since each $\alpha_i$ has a successor not above any other $\alpha_j$, if $A' \neq A''$, $\gamma_{A'} \neq \gamma_{A''}$. 

6
Let $S = \{ \rho \in K_{n+1}^i \mid (\forall A' \subseteq A) (\rho \not\geq \gamma_{A'}) \}$ and let $\psi$ be $\psi_0 \land \psi_1$ where $\psi_0$ is
\[
\neg\left( \bigvee_{A' \subseteq A} \phi_{\gamma_{A'}} \right)
\]
and $\psi_1$ is
\[
\bigwedge_{\rho \in S} \phi'_\rho
\]
Define a function $g$ with domain $K_m$ as follows:
1. $g(\text{node}_K(U)) = \gamma_{A'}$ where $A' = \{ \alpha_k \mid x_k \in V_m - U \}$.
2. $g(\text{node}_{K_m}^i(\langle T, U \rangle)) = \langle r(T'), \emptyset \rangle$, where $T' = \{ g(\delta) \mid \delta \in T \} \cup \{ \alpha_k \mid x_k \in V_m - U \}$.

We will show that the range of $g$ is contained in $K_n$. By induction, what we must show is that $\langle r(T'), \emptyset \rangle$ is in $K_n$, which will hold as long as $|r(T')| \geq 2$.

**Lemma 7.** The function $g$ is into $K_n$ and preserves order and nonorder. For all $\beta \in K_m$ and $x_k \in V_m$, $\beta \models x_k$ iff $g(\beta) \not\leq \alpha_k$.

**Proof.** We will prove by induction on $i$ that $g$ restricted to $K_{m}^i$ satisfies the conditions in the statement of the lemma.

For $i = 0$, observe that $\{ g(\text{node}^0(U)) \mid U \subseteq V_m \}$ is pairwise incomparable and that if $U \neq U'$, $g(\text{node}^0(U)) \neq g(\text{node}^0(U'))$ as they have different immediate successors. It is also the case that for all $\text{node}^0(U) \in K_m^0$ and $x_k \in V_m$, $\text{node}^0(U) \models x_k$ iff $x_k \in U$ if $\gamma_{A'} \not\leq \alpha_k$, where $A' = \{ \alpha_k \mid x_k \in V_m - U \}$.

Finally, since each $\gamma_{A'}$ is in $K_n$, the range of $g$ restricted to $K_{m}^0$ is contained in $K_n$.

Now suppose $g$ restricted to $K_{m}^i$ satisfies the hypotheses in the statement of the lemma.

We first show that the range of $g$ restricted to $K_{m}^{i+1}$ is contained in $K_n$. Let $\langle T, U \rangle \in \text{Lev}_{m}^{i+1}$. If $|T| \geq 2$, then $|r(T)| \geq 2$ and we are done. If $|T| = \{ \beta \}$, then $U \subseteq w(\beta)$ and $T'$ must contain both $g(\beta)$ and $\alpha_k$, where $x_k \in w(\beta) - U$. Since $\beta \models x_k$, $g(\beta) \not\leq \alpha_k$. Since it is fairly easy to see that each $\alpha_k$ is not less than any element of the range of $g$, we must have $|r(T')| \geq 2$.

It is immediate then that $g$ restricted to $K_{m}^{i+1}$ is preserves order and the immediate successor relation. Each element of $\text{Lev}_{m}^{i+1}$ is of the form node$^{i+1}(T, U)$. Observe that if $U \neq U'$ and $\langle T, U \rangle, \langle T, U' \rangle \in K_m$, then $g(\text{node}^{i+1}(\langle T, U \rangle)) \neq g(\text{node}^{i+1}(\langle T, U' \rangle))$ as they have different immediate
successors. Similarly, if \( r(T) \neq r(T') \) then \( g(\text{node}^{i+1}((r(T), U))) \neq g(\text{node}^{i+1}((r(T'), U'))) \) as they have different immediate successors. We can now conclude that \( g \) preserves nonorder by using the inductive hypothesis and the fact that \( g \) preserves the immediate successor relation. \( \square \)

**Lemma 8.** The sets \( \text{ran}(g) \) and \( k(\phi) \) are disjoint and \( \text{ran}(g) \cup k(\phi) = k(\psi) \).

**Proof.** It is immediate that \( \text{ran}(g) \) and \( k(\phi) \) are disjoint.

We will first show that \( \text{ran}(g) \cup k(\phi) \subseteq k(\psi) \). It is clear that \( k(\phi) \subseteq k(\psi) \). Since every node in \( \text{ran}(g) \) is at level \( i \leq i + 1 \), every node in \( \text{ran}(g) \) forces \( \psi_1 \). Since \( \psi_0 \) is doubly negated and every successor of a node in \( \text{ran}(g) \) is in \( \text{ran}(g) \) or \( k(\phi) \), by induction every element of \( \text{ran}(g) \) forces \( \psi_0 \) and \( \text{ran}(g) \subseteq k(\psi) \).

We will now show that \( k(\psi) \subseteq \text{ran}(g) \cup k(\phi) \). By construction, \( k(\psi) \cap K^n_i = k(\phi) \cup \{\gamma_A\} \) and \( k(\psi) \cap \text{Lev}^n_{i+1} = \text{ran}(g) \cap \text{Lev}^n_{i+1} = \text{ran}(g|\text{Lev}^0_{m}) - \gamma_A \). We will show that \( k(\psi) \cap \text{Lev}^n_j \subseteq \text{ran}(g|\text{Lev}^m_{j-(i+1)}) \) for all \( j \geq i + 1 \) by induction on \( j \). We just observed that this holds for \( j = i + 1 \).

Suppose it holds for \( j \). A node of \( k(\psi) \cap \text{Lev}^n_{j+1} \) must be of the form \( \text{node}^{i+1}((\langle T, \emptyset \rangle)) \) for \( T \subseteq k(\psi) \cap K^n_i \). Since \( T \) must contain an element of \( k(\psi) \cap \text{Lev}^n_j \) and every such node is below every element of \( k(\psi) \cap \text{Lev}^n_{i-1} \), \( T \) must be a subset of \( k(\psi) \cap (K^n_i - K^n_{i-1}) \). Let \( S = g^{-1}(T \cap (K^n_i - K^n_{i-1})) \) and \( U = \{x_k \mid \alpha_k \in T \cap \text{Lev}_i\} \). Then \( g^{-1}(\text{node}^{j+1}(\langle T, \emptyset \rangle)) \) is \( \text{node}^{j-1}_{K^n_m}(\langle S, \bigcap_{\mu \in S} w(\mu) - U \rangle) \) 

It follows from Lemmas 7 and 8 that \( g \) is an order-isomorphism from \( K_m \) to \( k(\phi) - k(\psi) \).

Define \( f : F_m \to F_n \) by:

1. \( f(\bot) = \phi \)
2. \( f(x_i) = (\phi'_i \lor \phi) \land \psi \)
3. \( f(\rho_0 \land \rho_1) = f(\rho_0) \land f(\rho_1) \)
4. \( f(\rho_0 \lor \rho_1) = f(\rho_0) \lor f(\rho_1) \)
5. \( f(\rho_0 \rightarrow \rho_1) = (f(\rho_0) \rightarrow f(\rho_1)) \land \psi \).

**Lemma 9.** For any \( \rho \in F_m \), \( \phi \vdash f(\rho) \vdash \psi \). If \( \delta = g(\gamma) \) then \( \gamma \vdash \rho \) iff \( \delta \vdash f(\rho) \).

**Proof.** The proof that \( \phi \vdash f(\rho) \vdash \psi \) is an easy proof by induction on \( \rho \).

We now prove the second part of the lemma by induction on \( \rho \).

For \( \rho = \bot \), the result is immediate. The observation that \( \gamma \vdash x_i \) iff \( \delta \not\vdash \alpha_i \) furnishes the case where \( \rho \) is \( x_i \). The inductive steps follow from the
existence of the order-isomorphism \( g \) from \( K_m \) to \( k(\psi) - k(\phi) \) and the fact that \( \phi \models f(\rho) \) for all \( \rho \).

Note that it follows from Lemma 9 that \( f \) is injective and hence an embedding.

We now define a function from \( F_n \) to \( F_m \) that is an inverse to \( f \) when restricted to \([\phi, \psi]\). Define \( h \) from \( F_n \) to \( F_m \) as follows:

1. \( h(\bot) = h(x_i) = \bot \).
2. \( h(\rho_0 \land \rho_1) = h(\rho_0) \land h(\rho_1) \).
3. \( h(\rho_0 \lor \rho_1) = h(\rho_0) \lor h(\rho_1) \).
4. If there is some \( \delta \in k(\phi) \cap K_n^{i-1} \) such that \( \delta \not\models \rho_0 \rightarrow \rho_1 \), then \( h(\rho_0 \rightarrow \rho_1) = \bot \). Otherwise,

\[
h(\rho_0 \rightarrow \rho_1) = (h(\rho_0) \rightarrow h(\rho_1)) \land \bigwedge \{ x_i \mid \alpha_i \not\models \rho \}
\]

**Lemma 10.** Let \( \delta = g(\gamma) \). For all \( \rho \in F_n \), \( \delta \models \rho \) iff \( \gamma \models h(\rho) \).

**Proof.** We will prove this by induction on the level of \( \gamma \) and the structure of \( \rho \).

If \( \rho \) is \( \bot \) or \( x_i \), then \( \delta \not\models \rho \) and \( \gamma \not\models h(\rho) \).

The inductive step for \( \rho = \rho_0 \lor \rho_1 \) and \( \rho = \rho_0 \land \rho_1 \) is straightforward.

Let \( \rho \) be \( \rho_0 \rightarrow \rho_1 \). Suppose \( \delta \models \rho \). Then, since for every \( \mu \in k(\phi) \cap K_n^{i-1} \), \( \delta < \mu \), \( h(\rho) = (h(\rho_0) \rightarrow h(\rho_1)) \land \bigwedge \{ x_i \mid \alpha_i \not\models \rho \} \). Since \( \delta \models \rho \), if \( \alpha_i \not\models \rho \), \( \delta \not\models \alpha_i \). It follows that \( \gamma \models x_i \). Thus \( \gamma \) forces the right conjunct of \( h(\rho) \).

Suppose \( \delta \models \rho_0 \) and \( \delta \models \rho_1 \). Then we are done by the inductive hypothesis on the structure of \( \rho \). Otherwise, suppose \( \delta \not\models \rho_0 \). Then we are done by the inductive hypothesis on the structure of \( \rho \) and the level of \( \gamma \).

Now suppose \( \delta \not\models \rho \). Then there is some \( \mu \geq \delta \) such that \( \mu \models \rho_0 \) and \( \mu \not\models \rho_1 \). If \( \mu \) is in the range of \( g \) then we are done by induction. If \( \mu \in K_n^{i-1} \), then \( h(\rho) = \bot \) and we are done. Otherwise \( \mu \in K_n^i \) and is some \( \alpha_j \). Since \( \delta < \alpha_j \), \( \gamma \not\models x_j \) and \( \gamma \not\models h(\rho) \).

It follows from Lemma 10 and Lemma 9 that if \( \phi \models \rho \models \psi \), then \( f(h(\rho)) = \rho \).

If \( n \geq 2, m \geq 2 \) by Fact 3 we can find a level in \( K_n \) satisfying the hypotheses of the Proposition. For example, we may pick a level in \( K_n \) of cardinality greater than \( 2m \), call \( 2m \) of its elements \( \beta_1, \ldots, \beta_{2m} \), and let \( A = \{ \{ \beta_1, \beta_2 \}, \emptyset \}, \ldots, \{ \beta_{2m-1}, \beta_{2m} \}, \emptyset \} \).
If $m = 1$, then we may let $\phi$ be $\bot$ and $\psi$ be $x_2 \land \ldots \land x_n$. The embedding $f$ from $H_1$ to $[\phi, \psi] \subseteq H_n$ sends $\rho$ to $\rho \land x_2 \land \ldots \land x_n$. We may define a surjective lattice homomorphism $h$ from $H_n$ to $H_1$ that is an inverse to $f$ as follows:

$h(x_1) = x_1$
$h(x_i) = \top$ for $1 < i \leq n$
$h(\phi \land \psi) = h(\phi) \land h(\psi)$
$h(\phi \lor \psi) = h(\phi) \lor h(\psi)$
$h(\phi \rightarrow \psi) = h(\phi) \rightarrow h(\psi)$

Corollary 11. There is a consequence-respecting, $(\land, \lor)$-preserving translation but not tautology-respecting from $m$-variable intuitionistic logic to $n$-variable intuitionistic logic for $n \geq 2$.

Proof. Immediate. □

Corollary 12. There is a consequence- and tautology-respecting translation from $m$-variable intuitionistic logic to $n$-variable intuitionistic logic for $n \geq 2$.

Proof. Let $f : F_m \rightarrow F_n$ be a consequence-respecting translation from $m$-variable intuitionistic logic to $n$-variable intuitionistic logic. Define $f'$ by

$f'(\phi) = f(\top) \rightarrow f(\phi)$

Then $f'$ is consequence- and tautology-respecting. To see that it is consequence-respecting: If $\Gamma \vdash \phi$ then $f(\Gamma) \vdash f(\phi)$, so $f(\top) \rightarrow f(\Gamma)$, $f(\top) \vdash f(\phi)$ and $f(\top) \rightarrow f(\Gamma) \vdash f(\phi)$, where $f(\top) \rightarrow f(\Gamma)$ is an abbreviation of $\{ f(\top) \rightarrow \psi \mid \psi \in f(\Gamma) \}$.

Conversely, if $f(\top) \rightarrow f(\Gamma)$, $f(\top) \vdash f(\phi)$, then $f(\Gamma) \vdash f(\phi)$ since $f(\Gamma) \vdash f(\top) \rightarrow f(\Gamma)$ and $f(\Gamma) \vdash f(\top)$ (this last fact is due to the fact that $f$ is consequence-preserving). □

Corollary 13. There is a consequence- and tautology-respecting, $(\land, \lor)$-preserving translation from $m$-variable intuitionistic logic to $n$-variable intuitionistic logic for $n \geq 2$.

Proof. Let $f : F_m \rightarrow F_n$ be a consequence-respecting and $(\land, \lor)$-preserving translation from $m$-variable intuitionistic logic to $n$-variable intuitionistic logic. Define $f'$ by

$$f'(\phi) = \begin{cases} f(\phi) & \text{if } \not\vdash^m_I \phi \\ \top & \text{if } \vdash^m_T \phi \end{cases}$$

This is clearly still consequence-respecting and $\land$-preserving. The disjunction property of intuitionistic logic implies that it is also $\lor$-preserving. □
Note that the translations given in Corollaries 11 and 12 can be done in linear time, while the one given in Corollary 13 cannot, as it requires deciding whether the given formula is a tautology.

By [4], $H_n$ for $n \geq 2$ has an infinite descending chain, while $H_1$ does not, so there is no embedding of $H_n$ into $H_1$ for $n \geq 2$.

4 A Lattice-Embedding from $H_\omega$ to $H_n$ for $n \geq 2$

Theorem 14. There is a lattice-embedding from $H_\omega$ into $H_2$

Proof. Pick $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in K_2$, all at the same level, say $i$. This may be done by Fact 3. Let $S = \{ \rho \in K_i | \forall i \in \{1, 2, 3, 4\} \rho \not\geq \alpha_i \}$. Let $\phi$ be

$$(\neg\neg(\alpha_1 \lor \alpha_2 \lor \alpha_3 \lor \alpha_4)) \land \bigwedge_{\beta \in S} \phi'_\beta.$$

Let $T = \{ \rho \in K_2 | \forall i \in \{1, 2, 3, 4, 5\} \rho \not\geq \alpha_i \}$. Let $\psi$ be

$$(\neg\neg(\alpha_1 \lor \alpha_2 \lor \alpha_3 \lor \alpha_4 \lor \alpha_5)) \land \bigwedge_{\beta \in T} \phi'_\beta.$$

Define a sequence $\{ \beta_i^j | i \in \omega, j \in \{1, 2, 3, 4\} \}$ as follows: let $\beta_0^i = \alpha_j$. For $i \geq 0$, let $\{ \beta_{i+1}^j | j = 1, 2, 3, 4 \}$ be a collection of four distinct nodes of the same level, with $\text{Lev}(\beta_{i+1}^j) > \text{Lev}(\beta_j^i)$ and such that they all force $\neg\neg(\beta_i^2 \lor \beta_i^3 \lor \beta_i^4)$. For example, we may take $\beta_1^1 = \langle \{\beta_i^2, \beta_i^3\}, \emptyset \rangle$, $\beta_2^2 = \langle \{\beta_i^2, \beta_i^1\}, \emptyset \rangle$, $\beta_3^3 = \langle \{\beta_i^3, \beta_i^4\}, \emptyset \rangle$, and $\beta_4^4 = \langle \{\beta_i^2, \beta_i^3, \beta_i^4\}, \emptyset \rangle$.

As in [2] (where a very similar construction is done), the nodes of $\{ \beta_i^1 | i \in \omega \}$ are pairwise incomparable, and they all force $\phi$.

Define a Kripke model $K$ over the language $V_\omega = \{ x_i | i \in \omega \}$ as follows: The set of nodes of $K$ is the set $k(\psi) - k(\phi)$ and a node $\alpha$ forces $x_i$ iff $\alpha \not\leq \beta_i^1$.

For all $\phi \in F_\omega$, let $k(\phi) = \{ \alpha \in K | \alpha \models \phi \}$.

Lemma 15. For all $\phi, \psi \in F_\omega$, $k(\phi) \subseteq k(\psi)$ iff $\phi \vdash \psi$.

Proof. Since $K$ is a Kripke model, if $\phi \vdash \psi$, $k(\phi) \subseteq k(\psi)$.

Suppose $\phi \not\vdash \psi$. Then there is a rooted finite Kripke model $K'$ over $V_\omega$ such that $K' \models \phi$ and $K' \not\models \psi$. Since variables not occurring in $\phi$ or $\psi$
are irrelevant, we may assume that each node of \( K' \) forces cofinitely many propositional variables.

Define a map \( a: K' \rightarrow K \) inductively on \( K' \) as follows: If \( \gamma \in K' \) is a node such that \( a(\gamma') \) has defined for all immediate successors of \( \gamma \), then let \( a(\gamma) \) be a node whose set of successors in \( K_2 \) is the upward-closure of the set

\[
\{ \beta_i \mid \gamma \not\models x_i \} \cup \{ a(\gamma') \mid \gamma' \geq \gamma \} \cup \{ \alpha_5 \}.
\]

For each \( i \), \( \gamma \models x_i \) iff \( a(\gamma) \models x_i \). Since \( a \) is also order-preserving and its range is upward-closed in \( K \), we have that if \( \gamma \) is the root of \( K' \), \( a(\gamma) \models \phi \) and \( a(\gamma) \not\models \psi \). \( \square \)

Now, as before, define \( f: F_\omega \rightarrow F_2 \) by:

1. \( f(\bot) = \phi \)
2. \( f(x_i) = (\phi'_i \lor \phi) \land \psi \)
3. \( f(\rho_0 \land \rho_1) = f(\rho_0) \land f(\rho_1) \)
4. \( f(\rho_0 \lor \rho_1) = f(\rho_0) \lor f(\rho_1) \)
5. \( f(\rho_0 \rightarrow \rho_1) = (f(\rho_0) \rightarrow f(\rho_1)) \land \psi \)

By precisely the same argument as before, this is an embedding. \( \square \)

Note that, by [4], in any interval \([\phi, \psi] \subseteq H_n\), there are atomic elements. As there are no atomic elements in \( H_\omega \), \( H_\omega \) cannot be embedded in \( H_n \) as an interval.

5 Impossibility of Lattice-Embedding \( B_\omega \) into \( H_\omega \)

Let \( B_\omega \) be the countable atomless Boolean algebra. We will think of it as the Lindenbaum algebra of classical propositional logic on countably infinitely many variables.

Proposition 16. There is no lattice embedding from \( B_\omega \) into \( H_n \) for any \( n \) or into \( H_\omega \).

Proof. By the previous theorem, it suffices to prove the proposition for \( H_2 \). Suppose there is a lattice embedding of \( B_\omega \) into \( H_2 \). Call it \( f \).

Let \( f(\top) \) have \( n \) subformulas. Consider the \( 2^n \) formulas \( \phi_1 = x_1 \land \cdots \land x_n \), \( \phi_2 = x_1 \land \cdots \land \lnot x_n \), ..., \( \phi_{2^n} = \lnot x_1 \land \cdots \land \lnot x_n \). Since \( f \) preserves \( \land \) and \( \lor \) we must have that \( \{ k(f(\phi_i)) \mid 1 \leq i \leq 2^n \} \) is a partition of \( k(f(\top)) - k(f(\bot)) \) and that \( k(f(\phi_i)) \cap (k(f(\top)) - k(f(\bot))) \) is non-empty for each \( i \).
Lemma 17. Let \( \alpha \) be a node in a Kripke model with exactly two immediate successors, \( \alpha_1 \) and \( \alpha_2 \). Let \( \phi \) be a formula. Suppose that for each subformula \( \phi' \) of \( \phi \), \( \alpha_1 \models \phi' \) iff \( \alpha_2 \models \phi' \) and that for all propositional variables \( v \) appearing in \( \phi \), if \( \alpha_1 \) and \( \alpha_2 \) force \( v \), then \( \alpha \models v \). Then for each subformula \( \phi' \) of \( \phi \), \( \alpha \models \phi' \) iff \( \alpha_1 \models \phi \) iff \( \alpha_2 \models \phi \). In particular, \( \alpha \models \phi \) iff \( \alpha_1 \models \phi \) iff \( \alpha_2 \models \phi \).

Proof. By induction on the structure of \( \phi \). The conclusion is immediate if \( \phi \) is atomic, and the \( \land \) and \( \lor \) cases are straightforward.

Suppose \( \phi \) is \( \phi_1 \rightarrow \phi_2 \). By induction, we can conclude that \( \alpha_1 \models \phi' \) iff \( \alpha_2 \models \phi' \) if \( \phi' \) is a subformula of \( \phi_1 \) or \( \phi_2 \). We just have to verify that \( \alpha \models \phi \) iff \( \alpha_1 \models \phi \).

Suppose \( \alpha \models \phi \). Then, as \( \alpha_1 \geq \alpha \), \( \alpha_1 \models \phi \).

Now suppose \( \alpha \not\models \phi \). Thus, there must be some \( \alpha' \geq \alpha \) such that \( \alpha' \models \phi_1 \) and \( \alpha' \not\models \phi_2 \). If \( \alpha' = \alpha \) then we are done by induction. Otherwise, we must have \( \alpha' \geq \alpha_1 \) or \( \alpha' \geq \alpha_2 \), and thus \( \alpha_1 \not\models \phi \).

For each \( i \), let \( \beta_i \in k(f(\phi_i)) \cap (k(f(\top)) - k(f(\bot))) \). By the pigeonhole principle, there must be some \( i \) and \( j \), \( i \neq j \), such that \( \beta_i \models \phi' \) iff \( \beta_j \models \phi' \) for all subformulas \( \phi' \) of \( f(\top) \). Let \( \beta \) be \( \langle \{\beta_i, \beta_j\}, w(\beta_i) \cap w(\beta_j) \rangle \). We can easily verify that \( \beta \in K_2 \). By the lemma, \( \beta \in f(\top) \). Thus, \( \beta \) is in \( k(f(\phi_m)) \cap (k(f(\top)) - k(f(\bot))) \) for some \( m \). Without loss of generality, say \( m \neq i \). Then \( \beta_i \models f(\phi_m) \) and \( \beta_i \models f(\phi_i) \) but \( \beta_i \not\models f(\bot) \), a contradiction.

\[ \square \]

6 Order-Embeddings

Proposition 18. Any countable partial ordering can be order-embedded into \( H_2 \) (and, therefore, into \( H_n \) for any \( n \geq 2 \)).

Proof. We first make the following definition:

Definition 19 (\( \psi(\alpha_1, \ldots, \alpha_m) \), Permissive formulas). Let \( \{\alpha_1, \ldots, \alpha_m\} \) be a set of nodes of \( K_2 \) all with the same level. Let \( S(\alpha_1, \ldots, \alpha_m) = \{\delta \in K_2 \mid \text{Lev}(\delta) \leq \text{Lev}(\alpha_1) \text{ and } \forall i \delta \not\geq \alpha_i\} \).

We define \( \psi(\alpha_1, \ldots, \alpha_m) \) to be

\[
\left( \neg \bigvee_{i=1}^{m} \phi_{\alpha_i} \right) \land \left( \bigwedge_{\delta \in S(\alpha_1, \ldots, \alpha_m)} \phi'_\delta \right)
\]
If $T = \{\alpha_1, \ldots, \alpha_m\}$ then $\psi(T)$ will denote $\psi(\alpha_1, \ldots, \alpha_m)$. If some $\alpha_i$ is at a different level than some $\alpha_j$, $\psi(\alpha_1, \ldots, \alpha_m)$ is not defined.

A formula of the form $\psi(\alpha_1, \ldots, \alpha_m)$ where $m \geq 3$ will be called permis-

sive. The set $\{\alpha_1, \ldots, \alpha_m\}$ is called the set of generators of $\psi(\alpha_1, \ldots, \alpha_m)$ and $\text{Lev}(\alpha_1)$ is called the level of $\psi(\alpha_1, \ldots, \alpha_m)$.

**Lemma 20.** Given any permissive formula $\psi$, there exist permissive formulas $\psi_n$ for $n \in \{0, 1\}$ such that for each $n \in \{0, 1\}$, $k(\psi_n) \subseteq k(\psi)$ and $k(\psi_0) \cap k(\psi_1)$ is finite.

**Proof.** Let $i$ be greater than the level of $\psi$ with $|\text{Lev}_i \cap k(\psi)| \geq 6$. We can find $i$ by Fact 4. Let $\text{Lev}_i \cap k(\psi) = \{\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3, \ldots\}$, let $\psi_0 = \psi(\alpha_1, \alpha_2, \alpha_3)$ and $\psi_1 = \psi(\beta_1, \beta_2, \beta_3)$. □

**Definition 21 ($\psi_\sigma$).** We define $\psi_\sigma$, for $\sigma \in \{0, 1\}^{<\omega}$ as follows: Let $\psi_\varepsilon = T$. Given $\phi_\sigma$, define $\phi_\sigma_n$ for $n \in \{0, 1\}$ so that $\phi_\sigma_n$ is permissive, $k(\phi_\sigma_n) \subseteq k(\phi_\sigma)$, and $k(\phi_{\sigma_0}) \cap k(\phi_{\sigma_1})$ is finite as in the above lemma.

Note also that $\psi_\sigma \vdash \psi_\sigma'$ iff $\sigma$ is an initial segment of $\sigma'$ as a binary string. Note also that $\psi_i \vdash \bigvee_i \psi_\sigma_i$ iff there is an $i$ such that $\sigma = \sigma_i$.

**Definition 22 (Complete Sets).** A set $S \subseteq H_2$ such that each element of $S$ is a disjunction of the form $\bigvee_{i=1}^n \psi_\sigma_i$ is called complete if it satisfies the following property: Let $S_1, S_2$ be such that $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \emptyset$, $S_1$ is upward closed, and $S_2$ is downward closed. Then there is some $\sigma(S_1, S_2)$ such that $|\sigma(S_1, S_2)| > \max\{|\sigma| \mid \psi_\sigma \text{ a disjunct of a formula in } S\}$, $\psi_\sigma(s_1, s_2)$ implies every element of $S_1$, and $\psi_\sigma(s_1, s_2)$ implies no element of $S_2$.

Note that the condition that

$$|\sigma(S_1, S_2)| > \max\{|\sigma| \mid \psi_\sigma \text{ a disjunct of a formula in } S\}$$

means that no $\psi_\sigma \in S$ can imply $\psi_\sigma(s_1, s_2)$.

Note also that if $\sigma_1$ is such that $|\sigma_1| > \max\{|\sigma| \mid \psi_\sigma \text{ a disjunct of a formula in } S\}$, $\psi_\sigma_1$ implies every element of $S_1$, and $\psi_\sigma_1$ implies no element of $S_2$ then so does $\sigma_1 \sigma_2$ for any $\sigma_2$ (where the juxtaposition indicates concatenation). Without loss of generality, then, we may assume that each $\sigma(S_1, S_2)$ has the same length.

The proposition will follow from the following lemma.
Lemma 23. Suppose $P$ is a finite partial order, $S \subseteq H_2$ is a complete set, and $h$ is an isomorphism from $P$ to $(S, \leq)$. For any partial order $P'$ such that $|P'| = |P| + 1$, there is a $\phi$ of the form $\bigvee_i \psi_{\sigma_i}$ such that $S \cup \{\phi\}$ is complete, $P' \simeq (S \cup \{\phi\}, \leq)$ via an isomorphism extending $h$.

Proof. If $P = \emptyset$, let $\phi$ be $\psi_0$. This is complete as we may let $\sigma(\{\phi\}, \emptyset) = 00$ and $\sigma(\emptyset, \{\phi\}) = 10$.

Suppose $S = T_1 \cup T_2 \cup T_3$ where $T_1$ is downward closed and $T_2$ is upward closed, and we would like to find $\phi$ so that $\phi$ is above all the elements of $T_1$, below all the elements of $T_2$ and incomparable with the elements of $T_3$.

Let $S$ be the collection of all partitions $(S_1, S_2)$ of $S$ such that $S_1$ is upward closed and $S_2$ is downward closed.

Let $\phi$ be

$$
\bigvee_{\chi \in T_1} \chi \lor \bigvee \{\psi_{\sigma(S_1, S_2)0} \mid (S_1, S_2) \in S \text{ and } T_2 \subseteq S_1\}
$$

Clearly, $\phi$ is above every $\chi \in T_1$. We also have that $\phi$ is below every $\rho \in T_2$, since every $\chi \in T_1$ must be below every $\rho \in T_2$, and by definition every $\psi_{\sigma(S_1, S_2)0}$ with $T_2 \subseteq S_1$ is below every $\rho \in T_2$.

$\phi$ is not above any element in $T_2 \cup T_3$: As noted above, $\psi_{\sigma} \vdash \bigvee \psi_{\sigma_i}$ implies $\sigma = \sigma_i$ for some $i$. But the disjuncts of $\phi$ are either elements of $T_1$ (which cannot be implied by elements of $T_2$ or $T_3$) or of the form $\psi_{\sigma}$ where the length of $\sigma$ is greater than the length of any $\sigma'$ for $\psi_{\sigma'}$ some disjunct of a formula in $T_2 \cup T_3$.

$\phi$ is not below any element in $T_1 \cup T_3$: Let $\mu \in T_1 \cup T_3$. Let $S_2 = \{\mu' \mid \mu' \leq \mu\}$ and $S_1 = S - S_2$. Then $\psi_{\sigma'(S_1, S_2)0}$ is a disjunct of $\phi$ which does not imply $\mu$.

To see that $S \cup \{\phi\}$ is complete: Let $(S_1, S_2) \in S$ with $\phi \in S_1$. Since $S_1$ is upward closed, we must have $T_2 \cup S_1$. Thus $\psi_{\sigma(S_1 - \{\phi\}, S_2)0}$ is a disjunct of $\phi$. We may therefore take $\sigma(S_1, S_2)$ to be $\sigma(S_1 - \{\phi\}, S_2)0$ concatenated with enough zeroes to make its length greater than $\max\{\sigma \mid \psi_{\sigma} \text{ a disjunct of a formula in } S \cup \{\phi\}\}$.

Let $(S_1, S_2) \in S$ with $\phi \in S_2$. Thus $T_1 \subseteq S_2$. If there is any member of $T_2$ in $S_2$, then we may take $\sigma(S_1, S_2)$ to be any sufficiently long extension of $\sigma(S_1, S_2 - \{\phi\})$, since $\psi_{\sigma(S_1, S_2 - \{\phi\})}$ cannot imply $\phi$ since $\phi$ implies each element of $S_2$. 

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Thus we may assume that $T_2 \subseteq S_1$. Therefore, $\psi_{\sigma(S_1,S_2-\{\phi\})_0}$ is a disjunct of $\phi$. By construction, there are no disjuncts of $\phi$ above it. We may take $\sigma(S_1,S_2)$ to be any sufficiently long extension of $\sigma(S_1,S_2-\{\phi\})_1$.

References

[1] The Coq proof assistant, http://coq.inria.fr.

[2] The PRL project, http://www.nuprl.org.

[3] Michael Beeson, Foundations of constructive mathematics: Metamathematical studies, Springer, Berlin/Heidelberg/New York, 1985

[4] Fabio Bellissima, Finitely generated free Heyting algebras, Journal of Symbolic Logic 51 (1986), 152–165

[5] Carsten Butz, Finitely presented Heyting algebras, http://www.itu.dk/~butz/research/heyting.ps.gz, 1998

[6] Luck Darnière and Markus Junker, On finitely generated Heyting algebras, http://home.mathematik.uni-freiburg.de/junker/preprints/heyting-221005.pdf, 2005

[7] Silvio Ghilardi and Marek Zawadowski, A sheaf representation and duality for finitely presented Heyting algebras, Journal of Symbolic Logic 60 (1995), 911–939

[8] Jean-Yves Girard, Yves Lafont, and Paul Taylor, Proofs and types, Cambridge University Press, Cambridge, 1989

[9] Anil Nerode, George Odifreddi, and Richard Platek, Constructive logics and lambda calculi, in preparation

[10] Iwao Nishimura, On formulas of one variable in intuitionistic propositional calculus, Journal of Symbolic Logic 25 (1960) 327–331

[11] Alasdair Urquhart, Free Heyting Algebras, Algebra Universalis 3 (1973) 94–97

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