LOWER BOUNDS AND INFINITY CRITERION FOR
BRAUER $p$-DIMENSIONS OF FINITELY-GENERATED
FIELD EXTENSIONS

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Abstract. Let $E$ be a field, $p$ a prime number and $F/E$ a finitely-generated extension of transcendency degree $t$. This paper shows that if the absolute Galois group $G_E$ is of nonzero cohomological $p$-dimension $\text{cd}_p(G_E)$, then the field $F$ has Brauer $p$-dimension $\text{Br}_{p}(F) \geq t$ except, possibly, in case $p = 2$, the Sylow pro-2-subgroups of $G_E$ are of order 2, and $F$ is a nonreal field. It announces that $\text{Br}_{p}(F)$ is infinite whenever $t \geq 1$ and the absolute Brauer $p$-dimension $\text{abr}_{p}(E)$ is infinite; moreover, for each pair $(m, n)$ of integers with $1 \leq m \leq n$, there exists a central division $F$-algebra of exponent $p^m$ and Schur index $p^n$.

1. Introduction and index-exponent relations over finitely-generated field extensions

Let $E$ be a field, $\text{Br}(E)$ its Brauer group, $s(E)$ the class of finite-dimensional associative central simple $E$-algebras, and $d(E)$ the subclass of division algebras $D \in s(E)$. It is known that $\text{Br}(E)$ is an abelian torsion group (cf. [16], Sect. 14.4), so it decomposes into the direct sum of its $p$-components $\text{Br}(E)_p$, where $p$ runs across the set $\mathbb{P}$ of prime numbers. Denote by $[A]$ the equivalence class in $\text{Br}(E)$ of any $A \in s(E)$. The degree $\text{deg}(A)$, the Schur index $\text{ind}(A)$, and the exponent $\text{exp}(A)$ (the order of $[A]$ in $\text{Br}(E)$) are important invariants of $A$. Note that $\text{deg}(A) = n \text{ind}(A)$, and $\text{ind}(A)$ and $\text{exp}(A)$ are related as follows (cf. [16], Sects. 13.4, 14.4 and 15.2):

\[(1.1) \text{exp}(A) \text{ divides } \text{ind}(A) \text{ and is divisible by every } p \in \mathbb{P} \text{ dividing } \text{ind}(A). \text{ For each } B \in s(E) \text{ with } \text{ind}(B) \text{ relatively prime to } \text{ind}(A), \text{ind}(A \otimes_E B) = \text{ind}(A)\text{ind}(B); \text{ in particular, the tensor product } A \otimes_E B \text{ lies in } d(E), \text{ provided that } A \in d(E) \text{ and } B \in d(E).\]

As shown by Brauer, (1.1) fully describe the generally valid restrictions between Schur indices and exponents:

\[(1.2) \text{Given a pair } (m, n) \text{ of positive integers, such that } n \mid m \text{ and } n \text{ is divisible by any } p \in \mathbb{P} \text{ dividing } m, \text{ there is a field } F \text{ and } D \in d(F) \text{ with } \text{ind}(D) = m \text{ and } \text{exp}(D) = n \text{ (Brauer, see [16], Sect. 19.6). One can take as } F \text{ any rational (i.e. purely transcendental) extension of infinite transcendency degree over an arbitrary field } F_0.\]

A field $E$ is said to be of Brauer $p$-dimension $\text{Br}_{p}(E) = n$, where $n \in \mathbb{Z}$, if $n$ is the least integer for which $\text{ind}(D) \leq \text{exp}(D)^n$ whenever $D \in d(E)$.

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and $[D] \in \text{Br}(E)_p$. We say that $\text{Brd}_p(E) = \infty$, if there exists a sequence $D_\nu \in d(E), \nu \in \mathbb{N}$, such that $[D_\nu] \in \text{Br}(E)_p$ and $\text{ind}(D_\nu) > \exp(D_\nu)^\nu$, for each index $\nu$. By an absolute Brauer $p$-dimension (abbr, $\text{abrd}_p(E)$) of $E$, we mean the supremum $\sup\{\text{Brd}_p(R) : R \in \text{Fe}(E)\}$. Here and in the sequel, $\text{Fe}(E)$ denotes the set of finite extensions of $E$ in a separable closure $E_{\text{sep}}$. In what follows, we denote by $\text{trd}(F/E)$ the transcendency degree and $I(F/E)$ stands for the set of intermediate fields of any extension $F/E$.

Clearly, $\text{Brd}_p(E) \leq \text{abrd}_p(E)$, for every field $E$ and $p \in \mathbb{P}$. It is known that $\text{Brd}_p(E) = \text{abrd}_p(E) = 1$, for every $p \in \mathbb{P}$, in the following cases:

(1.3) (i) $E$ is a global or local field (by class field theory, see, e.g., [2], Chs. VI and Ch. VII, by Serre and Tate, respectively);
(ii) $E$ is the function field of an algebraic surface (defined) over an algebraically closed field $E_0$ [3], [12];
(iii) $E$ is the function field of an algebraic curve over a pseudo algebraically closed field $E_0$ with $\text{cd}_p(\text{G}_{E_0}) > 0$ [4].

By a Brauer dimension and an absolute Brauer dimension of $E$, we mean the suprema $\text{Brd}(E) = \sup\{\text{Brd}_p(E) : p \in \mathbb{P}\}$ and $\text{abrd}(E) = \sup\{\text{abrd}_p(E) : p \in \mathbb{P}\}$, respectively. It would be of interest to know whether the function fields of algebraic varieties over a global, local or algebraically closed field are of finite absolute Brauer dimensions. Note also that fields of finite absolute Brauer $p$-dimensions, for all $p \in \mathbb{P}$, are singled out for their place in research areas like Galois cohomology (cf. [9], Sect. 3, [3], Remark 3.6, and [4], the end of Section 3 and Corollary 5.7) and the structure theory of their locally finite-dimensional central division algebras (see [3], Proposition 1.1 and the paragraph at the bottom of page 2). These observations draw one’s attention to the following open problem:

(1.4) Find whether the class of fields $E$ of finite absolute Brauer $p$-dimensions, for a fixed $p \in \mathbb{P}$ different from $\text{char}(E)$, is closed under the formation of finitely-generated extensions.

The following result of [3] is used there for proving that the class of fields $E$ with $\text{Brd}(E) < \infty$ is not closed under taking finitely-generated extensions:

**Theorem 1.1.** Let $E$ be a field, $p \in \mathbb{P}$ and $F/E$ a finitely-generated extension, such that $\text{trd}(F/E) = t \geq 1$. Then:

(i) $\text{Brd}_p(F) \geq \text{abrd}_p(E) + t - 1$, if $\text{abrd}_p(E) < \infty$ and $F/E$ is rational;
(ii) When $\text{abrd}_p(E) = \infty$, there are $\{D_{n,m} \in d(F) : n \in \mathbb{N}, m = 1, \ldots, n\}$ with $\exp(D_{n,m}) = p^m$ and $\text{ind}(D_{n,m}) = p^n$, for each admissible pair $(n,m)$;
(iii) $\text{Brd}_p(F) = \infty$, provided $p = \text{char}(E)$ and the degree $[E : E^p]$ is infinite, where $E^p = \{e^p : e \in E\}$, if $\text{char}(E) = p$ and $[E : E^p] = p^\nu < \infty$, then $\nu + t - 1 \leq \text{Brd}_p(F) < \nu + t$.

Theorem 1.1 is supplemented in [3], Sect. 3, as follows:

(1.5) Given a finitely-generated field extension $F/E$ with $\text{trd}(F/E) = t \geq 1$ and $\text{abrd}_p(E) < \infty$ when $p$ runs across some nonempty subset $P \subseteq \mathbb{P}$, there exists a finite subset $P(F/E)$ of $P$, such that $\text{Brd}_p(F) \geq \text{abrd}_p(E) + t - 1$, for each $p \in P \setminus P(F/E)$.
It is worth noting that there exist field extensions $F/E$ satisfying the conditions of (1.5), for $P = \mathbb{P}$, such that $P(F/E)$ is necessarily nonempty.

**Example.** Let $E$ be a real closed field, $F$ the function field of the Brauer-Severi variety corresponding to the symbol $E$-algebra $A = A_{-1}(-1, -1; E)$, and $F' = F \otimes E E(\sqrt{-1})$. By the Artin-Schreier theory (cf. [11], Ch. XI, Theorem 2), then $E(\sqrt{-1}) = E_{\text{sep}}$, whence $\text{abrd}_p(E) = 0$, for all $p \in \mathbb{P} \setminus \{2\}$. Since $-1$ does not lie in the norm group $N(E(\sqrt{-1})/E)$, it also follows that $A \in d(E)$. Note further that $\text{trd}(F/E) = 1$, $[A \otimes E F] = 0$ in $\text{Br}(F)$, and $F'/E(\sqrt{-1})$ is a rational extension (see [18], Theorem 13.8 and Corollaries 13.9 and 13.16). In view of Tsen’s theorem (cf. [16], Sect. 19.4), the noted property of $F'$ ensures that it is a $C_1$-field, so it follows from [19], Ch. II, Proposition 6, that $\text{cd}(G_{F'}) \leq 1$. As $A \otimes E F \cong A_1(-1, -1; F)$ over $F$, the equality $[A \otimes E F] = 0$ implies $F$ is a nonreal field, so it follows from the Artin-Schreier theory that $G_F$ is a torsion-free group. Observing finally that $G_F$ embeds in $G_E$ as an open subgroup, one obtains from [19], Ch. I, 4.2, Corollary 3, that $\text{cd}(G_E) \leq 1$, which means that $\text{abrd}(F) = 0 < \text{abrd}_p(E) = 1$.

Statement (1.1), Theorem 1.1 and basic properties of finitely-generated field extensions (cf. [11], Ch. X) imply the following:

(1.6) If the answer to (1.4) is affirmative, for some $p \in \mathbb{P}$, $p \neq \text{char}(E)$, and each finitely-generated extension $F'/E$ with $\text{trd}(F/E) = t \geq 1$, then there exists $c_t(p) \in \mathbb{N}$, such that $\text{Brd}_p(\Phi) \leq c_t(p)$ whenever $\Phi/E$ is a finitely-generated extension and $\text{trd}(\Phi/E) < t$ (see also [3], Proposition 4.6).

Theorem 1.1 (i) shows that the solution to [11], Problem 4.5, concerning the possibility to find a good definition of a field dimension $\text{dim}(E)$, is negative except, possibly, in the case of $\text{abrd}(E) < \infty$. In addition, it implies that if $\text{abrd}(E) < \infty$ and [11], Problem 4.5, is solved affirmatively, for all finitely-generated extensions $F'/E$, then the fields $F$ satisfy much stronger conditions than the one conjectured by (1.6) (see [3], (1.5)). As to our next result (for a proof, see [3], Proposition 5.8), it indicates that the answer to (1.4) will be positive, for finitely-generated extensions $F'/E$ with $\text{trd}(F/E) \leq n$, for some $n \in \mathbb{N}$, if this is the case in zero characteristic (see also [3], Remark 5.9, for an application of de Jong’s theorem [8]):

(1.7) Let $E$ be a field of characteristic $q > 0$ and $F/E$ a finitely-generated extension. Then there exists a field $E'$ with char($E') = 0$ and a finitely-generated extension $F'/E'$ satisfying the following:

(i) $G_{E'} \cong G_E$ and $\text{trd}(F'/E') = \text{trd}(F/E)$;

(ii) $\text{Brd}_p(F') \geq \text{Brd}_p(F)$, $\text{abrd}_p(F') \geq \text{abrd}_p(F)$, $\text{Brd}_p(E') = \text{Brd}_p(E)$ and $\text{abrd}_p(E') = \text{abrd}_p(E)$, for each $p \in \mathbb{P}$ different from $q$.

The proof of Theorem 1.1 in [3] relies on the following two lemmas. When $\mu = 1$, the former one is a theorem due to Albert. Besides in [3], Sect. 3, a proof of the former lemma can be found in [15], Sect. 1.

**Lemma 1.2.** A field $E$ satisfies the inequality $\text{abrd}_p(E) \leq \mu$, for some $p \in \mathbb{P}$ and $\mu \in \mathbb{N}$, if and only if, for each $E' \in \text{Fe}(E)$, $\text{ind}(\Delta_{E'}) \leq p^\mu$ whenever $\Delta_{E'} \in d(E')$ and $\exp(\Delta) = p$. 

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Lemma 1.3. Let $K$ be a field, $F = K(X)$ a rational extension of $K$ with $\text{trd}(F/K) = 1$, $f(X) \in K[X]$ a separable and irreducible polynomial over $K$, $L$ an extension of $K$ in $K_{\text{sep}}$ obtained by adjunction of a root of $f$, $v$ a discrete valuation of $F$ acting trivially on $K$ with a uniform element $\bar{v}$, and $(F_v, \bar{v})$ a Henselization of $(F, v)$. Suppose that $\bar{D} \in d(L)$ is an algebra of exponent $p$. Then $L$ is $K$-isomorphic to the residue field of $(F_v, \bar{v})$, and there exist $D' \in d(F_v)$ and $D \in d(F)$, such that $\exp(D') = p$, $[D \otimes_F F_v] = [D']$, and $D'$ is an inertial lift of $\bar{D}$ over $F_v$.

2. The main result

The purpose of this paper is to prove the following assertion which applied to a field with $\text{abr}_{p}(E) = 0$, improves the inequality in Theorem 1.1 (i):

Theorem 2.1. Let $F$ be a finitely-generated extension of a field $E$ with $\text{cd}_{p}(G_E) \neq 0$. Then $\text{Brd}_{p}(F) \geq \text{trd}(F/E)$ except, possibly, when $p = 2$, the Sylow pro-2-subgroups of $G_E$ are of order 2, and $F$ is a nonreal field.

The following result is contained in [3], Propositions 4.6 and 5.10, and is obtained by the method of proving Theorem 2.1 (see also [4], (4.10) and Proposition 4.3):

Theorem 2.2. Assume that $E$ is a field of type pointed out in (1.3). Then $\text{Brd}_{p}(F) \geq 1 + \text{trd}(F/E)$, for every finitely-generated extension $F/E$.

Remark 2.3. (i): Theorem 2.1 ensures that $\text{Brd}_{p}(\Phi) \geq n$, $p \in \mathbb{P}$, if $\Phi$ is a finitely-generated extension of a quasifinite field $\Phi_0$, and $\text{trd}(\Phi/\Phi_0) = n$. Therefore, one obtains following the proof of [3], Proposition 5.10, that the conclusion of Theorem 2.2 remains valid, if $E$ is endowed with a Henselian discrete valuation whose residue field is quasifinite.

(ii): Given a finitely-generated field extension $F/E$ with $\text{trd}(F/E) = k$, Theorem 2.1 implies Nakayama’s inequalities $\text{Brd}_{p}(F) \geq k - 1$, $p \in \mathbb{P}$ (cf. [8], Sect. 2). When $\text{cd}_{p}(G_E) = 0$, for some $p$, and $E$ is perfect in case $p = \text{char}(E)$, we have $\text{Brd}_{p}(F) = k - 1$ if and only if this holds in case $E$ is algebraically closed. The claim that $\text{Brd}(F) = k - 1$ when $E$ is algebraically closed is the content of the so called Standard Conjecture, for function fields of algebraic varieties over an algebraically closed field (see [12], Sect. 1, [13], page 3, and for relations with (1.4), the end of [3], Sect. 4).

The proof of Theorem 2.1 is based on the same idea as the one of Theorem 1.1. It relies on the following lemmas proved in [3].

Lemma 2.4. Let $(K, v)$ be a nontrivially real-valued field, and $(K_v, \bar{v})$ a Henselization of $(K, v)$. Assume that $\Delta_v \in d(K_v)$ has exponent $p \in \mathbb{P}$. Then there exists $\Delta \in d(K)$, such that $\exp(\Delta) = p$ and $[\Delta \otimes_K K_v] = [\Delta_v]$.

Lemma 2.4 is essentially due to Saltman [17], and our next lemma is a special case of the Grunwald-Wang theorem (cf. [14], Theorems 1 and 2).
Lemma 2.5. Let $F$ be a field, $S = \{v_1, \ldots, v_s\}$ a finite set of non-equivalent nontrivial real-valued valuations of $F$, and for each index $j$, let $F_{v_j}$ be a Henselization of $K$ in $K_{\text{sep}}$ relative to $v_j$, and $L_j/F_{v_j}$ be a cyclic field extension of degree $p^e_j$, for some $p \in \mathbb{P}$ and $e_j \in \mathbb{N}$. Put $\mu = \max\{\mu_1, \ldots, \mu_s\}$ and suppose that $\sqrt{-1} \in F$ in case $\mu \geq 3$, $p = 2$ and $\text{char}(F) = 0$. Then there exists a degree $p^\mu$ cyclic field extension $L/F$, such that $L_{v_j}$ is $F_{v_j}$-isomorphic to $L_j$, where $v_j'$ is a valuation of $L$ extending $v_j$, for $j = 1, \ldots, s$.

In the rest of this Section, we recall some general results on Henselian valuations which are used (often implicitly, like Lemma 1.3) for proving Theorem 2.1. A Krull valuation $v$ of a field $K$ is called Henselian, if $v$ extends uniquely, up-to an equivalence, to a valuation $v_L$ on each algebraic extension $L$ of $K$. Assuming that $v$ is Henselian, denote by $v(L)$ the value group and by $\hat{L}$ the residue field of $(L, v_L)$. It is known that $\hat{L}/\hat{K}$ is an algebraic extension and $v(\hat{L})$ is a subgroup of $v(L)$. When $L/K$ is finite and $e(L/K)$ is the index of $v(L)$ in $v(L)$, by Ostrowski’s theorem \[. \] Theorem 17.2.1, $[L: K][L: \hat{K}]^{-1}e(L/K)$ is not divisible by any $p \in \mathbb{P}$, $p \neq \text{char}(\hat{K})$. In particular, if $\text{char}(\hat{K})$ does not divide $[L: \hat{K}]$, then $[L: \hat{K}] = [\hat{L}: \hat{K}]e(L/K)$. Ostrowski’s theorem implies that there are group isomorphisms $\psi(K)/\psi(\hat{K}) \cong v(L)/pv(L)$, $p \in \mathbb{P}$, and in case $\text{char}(\hat{K}) \nmid [L: \hat{K}]$, they are canonically induced by the natural embedding of $K$ into $L$.

As usual, a finite extension $R$ of $K$ is called inertial, if $[R: K] = [\hat{R}: \hat{K}]$ and $\hat{R}$ is separable over $\hat{K}$. It follows from the Henselity of $v$ that the composite $K_{ur}$ of inertial extensions of $K$ in $K_{\text{sep}}$ has the following properties:

(2.1) (i) $v(K_{ur}) = v(K)$ and finite extensions of $K$ in $K_{ur}$ are inertial;
(ii) Each finite extension of $\hat{K}$ in $\hat{K}_{\text{sep}}$ is $\hat{K}$-isomorphic to the residue field of an inertial extension of $K$; hence, $\hat{K}_{ur}$ is $\hat{K}$-isomorphic to $\hat{K}_{\text{sep}}$;
(iii) $K_{ur}/K$ is a Galois extension with $\mathcal{G}(K_{ur}/K) \cong \mathcal{G}_{\hat{K}}$.

Similarly, it is known that each $\Delta \in d(K)$ has a unique, up-to an equivalence, valuation $v_\Delta$ extending $v$ so that the value group $v(\Delta)$ of $(\Delta, v_\Delta)$ is abelian (see \[. \]). Note that $v(\Delta)$ includes $v(K)$ as an ordered subgroup of index $e(\Delta/K) \leq [\Delta: K]$, the residue division ring $\hat{\Delta}$ of $(\Delta, v_\Delta)$ is a $\hat{K}$-algebra, and $[\hat{\Delta}: \hat{K}] \leq [\Delta: K]$. Moreover, by Ostrowski-Dradl’s theorem (cf. \[. \], (1.2)), $e(\Delta/K)[\hat{\Delta}: \hat{K}] \mid [\Delta: K]$, and in case $\text{char}(\hat{K}) \nmid [\Delta: K]$, $[\Delta: K] = e(\Delta/K)[\hat{\Delta}: \hat{K}]$. An algebra $D \in d(K)$ is called inertial, if $[D: K] = [\hat{D}: \hat{K}]$ and $\hat{D} \in d(\hat{K})$. In what follows, we also need the following results (see \[. \], Remark 3.4 and Theorems 2.8 and 3.1):

(2.2) (i) Each $\hat{D} \in d(\hat{K})$ has a unique, up-to an $F$-isomorphism, inertial lift $D$ over $K$ (i.e. $D \in d(K)$, $D$ is inertial over $K$ and $\hat{D} = \hat{D}$).
(ii) The set $I\text{Br}(K)$ of Brauer equivalence classes of inertial $K$-algebras forms a subgroup of $\text{Br}(K)$ canonically isomorphic to $\text{Br}(\hat{K})$.
(iii) For each $\Theta \in d(K)$ inertial over $K$, and any $R \in I(K_{ur}/K)$, $[\Theta \otimes_K R] \in I\text{Br}(R)$ and $\text{ind}(\Theta \otimes_K R) = \text{ind}(\hat{\Theta} \otimes_{\hat{K}} \hat{R})$. 

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3. Proof of Theorem 2.1

Let $E$ be a field with $\text{cd}_p(G_E) > 0$, for some $p \in \mathbb{P}$, and let $F/E$ be a finitely-generated extension. Throughout this Section, $E_{\text{sep}}$ is identified with its $E$-isomorphic copy in $F_{\text{sep}}$, and for any field $Y$, $r_p(Y)$ denotes the rank of the Galois group $\mathcal{G}(Y(p)/Y)$ of the maximal $p$-extension $Y(p)$ of $Y$ (in $Y_{\text{sep}}$) as a pro-$p$-group. Assuming that $\text{trd}(F/E) = t$ and $G_p$ is a Sylow pro-$p$-subgroup of $G_E$, we deduce Theorem 2.1 by proving the following:

(3.1) There exists $D \in d(F)$, such that $\exp(D) = p$, $\text{ind}(D) = p^t$ and $D$ is presentable as a tensor product of cyclic $F$-algebras of degree $p$ except, possibly, in the case where $p = 2$, $G_2$ is of order 2 and $F$ is a nonreal field.

Let $E_p$ be the fixed field and $o(G_p)$ the order of $G_p$. Our assumptions show that $r_p(E_p) \geq 1$, which implies the existence of a field $M \in \text{Fe}(E)$ with $r_p(M) \geq 1$ (apply the method of proving 10, Sect. 13.2, Proposition b). Moreover, $M$ can be chosen to be nonreal unless $p = 2$ and $o(G_p) = 2$.

Assuming that $M$ is nonreal, one obtains from 20, Theorem 2, that there exists a $\mathbb{Z}_p$-extension $\Phi$ of $M$ in $E_{\text{sep}}$. Hence, by Galois theory and the fact that $\mathbb{Z}_p$ is continuously isomorphic to its open subgroups, $\Phi M'/M'$ is Galois with $\mathcal{G}(\Phi M'/M') \cong \mathbb{Z}_p$, for each $M' \in \text{Fe}(E)$. This makes it easy to obtain from basic properties of valuation prolongations on finite extensions that $M$ can be chosen as an $E$-isomorphic copy of the residue field of a height $t$ valuation $v$ of $F$, trivial on $E$ with $v(F) = \mathbb{Z}_t$. Here $\mathbb{Z}_t$ is viewed as an ordered abelian group with respect to the inversely-lexicographic ordering.

Let $(F_v, \hat{v})$ be a Henselization of $(F, v)$. Suppose first that $t = 1$ and take $\pi \in F$ so that $\langle v(\pi) \rangle = v(F)$. Then $v$ lies in an infinite system of nonequivalent discrete valuations of $F$ trivial on $E$ (cf. 2, Ch. II, Lemma 3.1). In view of Lemma 2.5, this implies the existence of degree $p$ cyclic extensions $F_n$, $n \in \mathbb{N}$, of $F$, such that $F_1/F$ is inertial relative to $v$, and $F_n \subset F_v$, $n \geq 2$. Let $\varphi_n$ be a generator of $\mathcal{G}(F_n/F)$, for each $n \in \mathbb{N}$. It follows from the choice of $F_1$ that the cyclic $F$-algebra $(F_1/F, \sigma_1, \pi)$ lies in $d(F)$ and $(F_1/F, \sigma_1, \pi) \otimes_F F_v \in d(F_v)$, which proves (3.1) in case $t = 1$.

Assume now that $t \geq 2$, and fix elements $\pi_1, \ldots, \pi_t \in K$ so that $v(F)$ be generated by the set $\{v(\pi_j) \colon j = 1, \ldots, t\}$, and $H = \langle v(\pi_i) \rangle$ be the minimal nontrivial isolated subgroup of $v(F)$. Then $v$ and $H$ induce canonically on $F$ a valuation $v_H$ with $v_H(F) = v(F)/H$; also, they give rise to a valuation $\hat{v}_H$ of the residue field $F_H$ of $(F, v_H)$ with $\hat{v}_H(F_H) = H$ and a residue field equal to $M$ (cf. 5, Sect. 5.2). In addition, it is easily verified that $F_H/E$ is a finitely-generated extension with $\text{trd}(F_H/E) = 1$. Hence, by the proof of the already considered special case of Theorem 2.1, there exist $D_H \in d(F_H)$ and $\Psi_H \in I(F_{H, \text{sep}}/F_H)$, such that $\text{ind}(D_H) = p$, $D_H \otimes_{F_H} \Psi_H \in d(\Psi_H)$, and $I(\Psi_H/F_H)$ contains infinitely many degree $p$ cyclic extensions of $F_H$. Observing now that $v_H$ is of height $t - 1$, and using repeatedly (2.2), Lemmas 2.4, 2.5 and 3.1 (i), one proves that there exists a cyclic $F$-algebra $D'_H \in d(F)$, such that $\text{ind}(D'_H) = p$, $D'_H \otimes_F F_{v_H} \in d(F_{v_H})$ and $D'_H \otimes_F F_{v_H}$ is an inertial lift of $D_H$ over a Henselization $F_{v_H}$ of $F$ relative to $v_H$. Similarly, it can be deduced from (2.2) that each degree $p$ cyclic extension of $F_H$ is realizable as the residue field of an inertial cyclic degree $p$ extension of $F$ relative to $v_H$. This implies the existence of an
inertial extension \( (F', v'_H)/(F, v_H) \), such that \([F': F] = [F'v_H: Fv_H] = p^{t-1}, D' \otimes_F F' \in d(F) \) and \( F' = F_2 \ldots F_t \), where \( F_i/F \) is a degree \( p \) cyclic extension of \( F \), for \( i = 2, \ldots, t \). In view of Morandi’s theorem (cf. [7], Proposition 1.4), it is now easy to construct an algebra \( \Delta \subseteq \hat{\mathbb{F}} \) is finitely-generated, \( \text{trd}(E) = \hat{\mathbb{F}} \otimes \mathbb{F} \), \( E \) is a discrete valuation of \( \hat{\mathbb{F}} \), \( v \) valuation \( v \) of \( \hat{\mathbb{F}} \), \( D_t \subseteq \hat{\mathbb{F}} \) that cosets \((3.1)\), under the hypothesis that \( o \) is trivial on \( (\mathbb{Z}/2\mathbb{Z}) = 2 \). By the Artin-Schreier theory, \( o(G_2) = 2 \) if and only if the fixed field \( E_2 \) is real closed. Our proof also relies on the following lemma.

**Lemma 3.1.** Let \( E \) be a formally real field and \( F \) a finitely-generated extension of \( E \) with \text{trd}(F/E) = 1. Then \( F \) is formally real if and only if it has a discrete valuation \( v \) trivial on \( E \), whose residue field \( \hat{\mathbb{F}} \) is formally real.

**Proof.** It is known and easy to prove (cf. [10], Lemma 1) that if \( F \) is a nonreal field and \( \omega \) is a discrete valuation of \( F \) trivial on \( E \), then the residue field of \( (F, \omega) \) is nonreal as well. Assume now that \( F \) is formally real, fix a real closure \( F' \) of \( F \) in \( F_{\text{sep}} \), and put \( E' = E_{\text{sep}} \cap F' \). Observe that \( E_{\text{sep}} F'/F' \) is a Galois extension with \( G(E_{\text{sep}} F'/F') \cong G_{E'} \). Since, by the Artin-Schreier theory, \( F_{\text{sep}} = F'(\sqrt{-1}) = E_{\text{sep}} F' \), this means that \( E_{\text{sep}} = E' (\sqrt{-1}) \), whence, \( E' \) is a real closure of \( E \) in \( E_{\text{sep}} \). Note also that \( E' F'/E' \) is finitely-generated, \( \text{trd}(E' F'/E') = 1 \) and \( E' F' \subseteq F' \), i.e. the extension \( E' F'/E' \) satisfies the conditions of Lemma 3.1. This enables one to deduce from [10], Theorem 6 and Proposition, that \( E' F' \) has a discrete valuation \( v' \) trivial on \( E' \) and with a residue field \( E' \). It is now easy to see that the valuation \( v \) of \( F \) induced by \( v' \) has the properties required by Lemma 3.1. Specifically, \( \hat{F} \) is \( E \)-isomorphic to a finite extension of \( E \) in \( E' \). \( \square \)

We are now in the remaining case of (3.1). Suppose first that \( t = 1 \), put \( F_0 = E(X) \), for some \( X \in F \) transcendental over \( E \), and denote by \( \Omega_0 \) the extension of \( F_0 \) in \( F_{\text{sep}} \) generated by the square roots of the totally positive elements of \( F_0 \) (i.e. those realizable over \( F_0 \) as finite sums of squares, see [11], Ch. XI, Proposition 2). Then \( F \Omega_0 \) is formally real, which implies \( A_{-1}(-1, -1; \Omega) \in d(\Omega) \), for each \( \Omega \in I(F \Omega_0/F_0) \), proving the assertion of (3.1). Note also that \( \Omega_0/F_0 \) is an infinite Galois extension with \( G(\Omega_0/F_0) \) of exponent 2. This follows from Kummer theory and the fact that cosets \( (X^2 + a^2)F_0^2, a \in E^+ \), generate an infinite subgroup of \( F_0^*/F_0^{*2} \).

Assume now that \( t \geq 2 \), define \( F_0 \) and \( \Omega_0 \) as above and denote by \( F_1 \) the algebraic closure of \( F_0 \) in \( F \). Applying Lemma 3.1 and proceeding by induction on \( t \), one concludes that \( F \) has valuation \( v \) trivial on \( F_1 \), such that \( v(F) = \mathbb{Z}^{t-1}, v \) is of height \( t - 1 \) and \( \hat{F} \) is a formally real finite extension of \( F_1 \). Fix a Henselization \( (F, v) \) and an \( F_1 \)-isomorphic copy \( F_1' \) of \( \hat{F} \) in \( F_{\text{sep}} \). It is easily verified that \( F_1' \Omega_0 \) is a formally real field and \( F_1' \Omega_0/F \) is a Galois extension. As \( v \) is of height \( t - 1 \), one proves, using repeatedly (2.1) and Lemma 2.5, that \( I(F \Omega_0/F) \) contains infinitely many quadratic and inertial
extensions of $F$ relative to $v$. Therefore, there exist fields $Y_n \in I(F\Omega_0/F)$, $n \in \mathbb{N}$, such that $[Y_n : F] = 2$, $[Y_1 \cdots Y_n : F] = 2^n$ and $Y_1 \cdots Y_n$ is inertial over $F$ relative to $v$, for each index $n$. Fix a generator $q_j$ of $G(Y_j/F)$, and take elements $\pi_j \in F$, $j = 2, \ldots, t$, so that $(v(\pi_2), \ldots, v(\pi_t)) = v(F)$. Put $\Delta_1 = A_{-1}(-1, -1; F)$ and consider the cyclic $F$-algebras $\Delta_j = (Y_j/F, q_j, \pi_j)$, $j = 2, \ldots, t$. Since $F^t\Omega_0$ is formally real, $A_{-1}(-1, -1; F) \otimes_F F^t\Omega_0 \in d(F^t\Omega_0)$, so it follows from Morandi’s theorem, the noted properties of the fields $Y_n$, $n \in \mathbb{N}$, and the choice of $\pi_2, \ldots, \pi_t$, that the $F$-algebra $\Delta = \Delta_1 \otimes_F \cdots \otimes_F \Delta_t$ lies in $d(F)$ and $\Delta \otimes_F F_v \in d(F_v)$. This yields $\exp(\Delta) = 2$ and $\ind(\Delta) = 2^t$, so (3.1) and Theorem 2.1 are proved.

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