Query Order

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Abstract

We study the effect of query order on computational power, and show that $P_{BH_j}^{1}[BH_k]^{-tt}(NP)$ if $j$ is even and $k$ is odd, and equals $R_{j+2k-1}^{p}$ otherwise. Thus, unless the polynomial hierarchy collapses, it holds that for each $1 \leq j \leq k$: $P_{BH_j}^{1}[BH_k] = P_{BH_k}^{1}[BH_j] \iff (j = k) \lor (j \text{ is even } \land k = j + 1)$.

We extend our analysis to apply to more general query classes.

1 Introduction

This paper studies the importance of query order. Everyone knows that it makes more sense to first look up in your on-line datebook the date of the yearly computer science conference and then phone your travel agent to get tickets, as opposed to first phoning your travel agent (without knowing the date) and then consulting your on-line datebook to find the date. In real life, order matters.

This paper seeks to determine—for the first time to the best of our knowledge—whether one’s everyday-life intuition that order matters carries over to complexity theory.

In particular, for classes $C_1$ and $C_2$ from the boolean hierarchy [10,11], we ask whether one question to $C_1$ followed by one question to $C_2$ is more powerful than one question to $C_2$ followed by one question to $C_1$. That is, we seek the relative powers of the classes $P_{C_1}^{1}[C_2]^{-tt}$ and $P_{C_2}^{1}[C_1]^{-tt}$.

As is standard [28], for any constant $m$ we say $A$ is $m$-truth-table reducible to $B$ ($A \leq_{m-tt} B$) if there is a polynomial-time computable function that, on each input $x$, computes both

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(a) $m$ strings $x_1, x_2, \cdots, x_m$ and (b) a predicate, $\alpha$, of $m$ boolean variables, such that $x_1, x_2, \cdots, x_m$ and $\alpha$ satisfy:

$$x \in A \iff \alpha(\chi_B(x_1), \chi_B(x_2), \cdots, \chi_B(x_m)),$$

where $\chi_B$ denotes the characteristic function of $B$. For any $a$ and $b$ for which $\leq_a^b$ is defined and for any class $C$, let $R_a^b(C) = \{L \mid (\exists C \in C)[L \leq_a^b C]\}$.

We prove via the mind change technique that, for $j, k \geq 1$:

$$p^{BH_j[1]:BH_k[1]} = \begin{cases} R_j^{p^j+2k-1-\text{tt}}(\text{NP}) & \text{if } j \text{ is even and } k \text{ is odd} \\ R_j^{p^j+2k-\text{tt}}(\text{NP}) & \text{otherwise.} \end{cases}$$

Informally, this says that the second query counts more towards the power of the class than the first query does. In particular, assuming that the polynomial hierarchy does not collapse, we have that if $1 \leq j \leq k$ then $p^{BH_j[1]:BH_k[1]}$ and $p^{BH_k[1]:BH_j[1]}$ differ unless $(j = k) \lor (j$ is even $\land k = j + 1)$.

The interesting case is the strength of $p^{BH_j[1]:BH_k[1]}$ when $j$ is even and $k$ is odd. In some sense, $j + 2k$ NP questions underpin this class. However, by arguing that a certain underlying graph must contain an odd cycle, we show that one can always make do with $j + 2k - 1$ queries. We generalize our results to apply broadly to classes with tree-like query structure.

The work of this paper (especially Theorem 3.6), which first appeared in [21], should be compared with the independent work of Agrawal, Beigel, and Thierauf, which first appeared in [1]. In particular, let $p^{BH_j[1]:BH_k[1]}$ denote the class of languages recognized by some polynomial-time machine making one query to a BH$_j$ oracle followed by one query to a BH$_k$ oracle and accepting if and only if the second query is answered “yes.” Agrawal, Beigel, and Thierauf prove (using different notation):

$$p^{BH_j[1]:BH_k[1]} = \begin{cases} \text{BH}^{j+2k-1} & \text{if } j \not\equiv k \pmod{2} \\ \text{BH}^{j+2k} & \text{otherwise.} \end{cases}$$

Note that this result is incomparable with the results of Theorem 3.6, as their result deals with a different and seemingly more restrictive acceptance mechanism. Some insight into the degree of restrictiveness of their acceptance mechanism, and its relationship to ours, is given by the following claim (Corollary 3.7), which follows immediately from Theorem 5.7 and Lemma 5.9 of [1] and Theorem 3.6 of the present paper:

$$R_j^{p^j-1-\text{tt}}(p^{BH_j[1]:BH_k[1]+}) = \begin{cases} p^{BH^j-1[1]:BH_k[1]} & \text{if } j \text{ is odd and } k \text{ is even} \\ p^{BH_j[1]:BH_k[1]+} & \text{otherwise.} \end{cases}$$

2 Preliminaries

For standard notions not defined here, we refer the reader to any computational complexity textbook, e.g., [6, 2, 22, 27]. Let $\chi_A$, $m$-truth-table reducibility (“$\leq_m^{\text{tt}}$”), $R_a^b(C)$, and $p^{BH_j[1]:BH_k[1]+}$ be as defined in Section 1.
The boolean hierarchy $\text{BH}_i$ is defined as follows, where $C_1 \oplus C_2 = \{ L \mid (\exists A \in C_1)(\exists B \in C_2) [L = A - B] \}$.

$\text{BH}_1 = \text{NP}$,

$\text{BH}_k = \text{NP} \ominus \text{BH}_{k-1}$, for $k > 1$,

$\text{coBH}_k = \{ L \mid L \in \text{BH}_k \}$, for $k > 0$, and

$\text{BH} = \bigcup_{i \geq 1} \text{BH}_i$.

The boolean hierarchy has been intensely investigated, and quite a bit has been learned about its structure (see, e.g., [10,11,9,25,24,30,14,3,5]). Recently, various results have also been developed regarding boolean hierarchies over classes other than NP [7,13,5,22].

For any language classes $C_1$ and $C_2$, define $\text{PC}^{C_1[1]}:C_2[1]$ to be the class of languages accepted by polynomial-time machines making one query to a $C_1$ oracle followed by one query to a $C_2$ oracle. For any language classes $C_1$, $C_2$, and $C_3$, define $\text{PC}^{C_1[1]}:C_2[1],C_3[1]$ to be the class of languages accepted by polynomial-time machines making one query to a $C_1$ oracle followed in the case of a “no” answer to this first query by one query to a $C_2$ oracle, and in the case of a “yes” answer to the first query by one query to a $C_3$ oracle.

3 The Importance of Query Order

We ask whether the order of queries matters. We will study this in the setting of the boolean hierarchy. In particular, does $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1]$ equal $\text{PC}^{\text{BH}_k[1]}:\text{BH}_j[1]$, or are they incomparable, or does one strictly contain the other?

For clarity of presentation, we will in this section handle classes only of the form $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1]$. We show that for no $j$, $k$, $j'$, and $k'$ are $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1]$ and $\text{PC}^{\text{BH}_{j'}[1]}:\text{BH}_{k'}[1]$ incomparable. In Section 4 we will handle the more general case of classes of the form $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1],\text{BH}_l[1]$, and even classes with a more complicated tree-like query structure.

Indeed, we show in this section that in almost all cases, $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1]$ is so powerful that it can do anything that can be done with $j + 2k$ truth-table queries to $\text{NP}$. Since, based on the answer to the first $\text{BH}_j$ query, there are two possible $\text{BH}_k$ queries that might follow, $j + 2k$ is exactly the number of queries asked in a brute force truth-table simulation of $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1]$. Thus, our result shows that (in almost all cases) the power of the class is not reduced by the nonlinear structure of the $j + 2k$ queries underlying $\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1]$—that is, the power is not reduced by the fact that in any given run only $j + k$ underlying $\text{NP}$ queries will be even implicitly asked (via the $\text{BH}_j$ query and the one asked $\text{BH}_k$ query). We say “in almost all cases” as if $j$ is even and $k$ is odd, we prove there is a power reduction of exactly one level.

All the results of the previous paragraph follow from a general characterization that we prove. For $j, k \geq 1$:

$$\text{PC}^{\text{BH}_j[1]}:\text{BH}_k[1] = \begin{cases} \text{R}^p_{j+2k-1-\text{tt}}(\text{NP}) & \text{if } j \text{ is even and } k \text{ is odd} \\ \text{R}^p_{j+2k-\text{tt}}(\text{NP}) & \text{otherwise.} \end{cases}$$
Our proof employs the mind change technique, which predates complexity theory. In particular we show that $P^{BH_j[1]:BH_k[1]}$ has at most $j + 2k$ ($j + 2k - 1$ if $j$ is even and $k$ is odd) mind changes, and that $BH_{j+2k}$ ($BH_{j+2k-1}$ if $j$ is even and $k$ is odd) is contained in $P^{BH_j[1]:BH_k[1]}$.

The mind change technique or equivalent manipulation was applied to complexity theory in each of the early papers on the boolean hierarchy, including the work of Cai et al. ([10], see also [32,12]), Köbler et al. [25], and Beigel [3]. These papers use mind changes for a number of purposes. Most crucially they use the maximum number of mind changes (what a mind change is will soon be made clear) of a class as an upper bound that can be used to prove that the class is contained in some other class. In the other direction, they also use the number of mind changes that certain classes—especially the classes of the boolean hierarchy due to their normal form as nested subtractions of telescoping sets [10]—possess to show that they can simulate other classes. Even for classes that have the same number of mind changes, relativized separations are obtained via showing that the mind changes are of different character (mind change sequences are of two types, depending on whether they start with acceptance or rejection). The technique has also proven useful in many other more recent papers, e.g., [14,13,5].

To make clear the basic nature of mind change arguments, in a simple form, we give an example. We informally argue that each set that is $k$-truth-table reducible to NP is in fact in $R_{p}^{p}1$-tt$(BH_k)$.

Lemma 3.1 For every $k \geq 1$, $R_{k$-tt$}^{p}(NP) = R_{p}^{p}1$-tt$(BH_k)$.

Proof This fact (stated slightly differently) is due to Köbler et al. [25], and the proof flavor presented here is most akin to the approach of Beigel [3]. Consider a $k$-truth-table reduction to an NP set, $F$. Let $L$ be the language accepted by the $k$-truth-table reduction to $F$. Consider some input $x$ and without loss of generality assume $k$ queries are generated. Let us suppose for the moment that the reduction rejects when all $k$ queries receive the answer “no.” Consider the $k$-dimensional hypercube such that one dimension is associated with each query (0 in that dimension means the query is answered no and 1 means it is answered yes). So the origin is associated with all queries being getting the answer no and the point $(1,1,\ldots,1)$ is associated with all queries getting the answer yes. Now, also label each vertex with either A (accept) or R (reject) based on what the truth-table would do given the answers represented by that vertex. So under our supposition, the origin has the label R. Finally, label each vertex with an integer as follows. Label the origin with 0. Inductively label each remaining vertex with the maximum integer induced by the vertices that immediately precede it (i.e., those that are the same as it except one yes answer has been changed to a no answer). A preceding vertex $v$ with integer label $i$ induces in a successor $v'$ the integer $i + 1$ if $v$ and $v'$ have different A/R labels, and $i$ if they have the same label. Note that vertices given even labels correspond to rejection and those given odd labels correspond to acceptance. Informally, a mind change is just changing one or more strings from no to yes in a way that moves us from a vertex labeled $i$ to one labeled $i + 1$. For $1 \leq i \leq k$, let $B_i$ be the NP set that accepts $x$ if (in the queries/labeling generated by
the action of the truth-table on input \( x \) for some vertex \( v \) labeled \( i \) all the queries \( v \) claims are yes are indeed in the NP set \( F \). Note that \( B_1 \supseteq B_2 \supseteq B_3 \supseteq \cdots \), as if a node labeled \( v \) is in \( B_j \), \( j \geq 2 \), then certainly its predecessor node with label \( j - 1 \) must be in \( B_{j-1} \), as that predecessor represents a subset of the strings \( v \) represents. But now note that \( L \) is exactly \( B_1 - (B_2 - (B_3 - (\cdots - (B_{k-1} - B_k) \cdots))) \). Why? Let the vertex \( w \) (say with integer label \( i_w \)) represent the true answers to the queries. Note that by construction, \( x \in B_q \) for all \( q \leq i_w \) but \( x \notin B_q \) for any \( q > i_w \). As the \( B_i \) were alternating in terms of representing acceptance and rejection, and given the format \( B_1 - (B_2 - (B_3 - (\cdots - (B_{k-1} - B_k) \cdots))) \), the set \( B_1 - (B_2 - (B_3 - (\cdots - (B_{k-1} - B_k) \cdots))) \), will do exactly what \( B_{i_w} \) represents, namely, the action on the correct answers. Thus, we have just given a proof that a \( k \)-truth-table reduction that rejects whenever all answers are no can be simulated by a set in BH\(_k\). Of course, one cannot validly assume that the reduction rejects whenever all answers are no. But it is not hard to see (analogously to the above) that the complement of a BH\(_k\) \( \leq \text{tt} \) BH\(_k\) reduction accepts when all answers are no can (analogously to the above) be handled via a set in RH\(_k\) and that (since what the truth-table reduction does when all answers are no is itself polynomial-time computable) via a set in RH\(_{1\text{-tt}}\) (BH\(_k\)) we can accept an arbitrary set in RH\(_{1\text{-tt}}\) (BH\(_k\)). Of course, it is clear by brute force simulation that RH\(_{1\text{-tt}}\) (BH\(_k\)) \( \subseteq \) RH\(_{k\text{-tt}}\) (NP), and so it holds that RH\(_{1\text{-tt}}\) (BH\(_k\)) = RH\(_{k\text{-tt}}\) (NP).  

What actually is being shown above is that RH\(_{1\text{-tt}}\) (BH\(_k\)) can handle \( k \) appropriately structured mind changes, starting either from reject or accept. In the following theorem, the crucial things we show are that (a) P\(^{\text{BH\(_k\)[1]}\text{-BH\(_k\)[1]}}\) can simulate, starting at either accept or reject, \( j + 2k \) (respectively, \( j + 2k - 1 \)) mind changes if \( j \) is odd or \( k \) is even (respectively, if \( j \) is even and \( k \) is odd), and (b) for \( j \) even and \( k \) odd, P\(^{\text{BH\(_k\)[1]}\text{-BH\(_k\)[1]}}\) can never have more than \( j + 2k - 1 \) mind changes. We achieve (b) by examining the possible mind change flow of a P\(^{\text{BH\(_k\)[1]}\text{-BH\(_k\)[1]}}\) machine, \( j \) even and \( k \) odd, and showing that either a mind change is flagrantly wasted, or a certain underlying graph has an odd length directed cycle (which thus is not two-colorable, and from this will lose one mind change).

Since our arguments in the proofs of this section use paths in hypercubes, we will find useful the concept of an ascending path in a hypercube. Let \( K = \{0, 1\}^d \) be the \( d \)-dimensional hypercube. Then every path \( p \) in \( K \) can be described as a linear combination of unit vectors \( u_1, \ldots, u_d \), where \( u_i \) is the \( i \)th unit vector. We call \( p \) an ascending path in \( K \) leading from \((0, 0, \cdots, 0) \) to \( v \) if and only if it can be identified with a sum

\[ u_{i_1} + u_{i_2} + \cdots + u_{i_n} \]

of distinct unit vectors \( u_{i_v} \) such that the vertices of this path \( p \) are

\[ v_0 = (0, \cdots, 0), v_1 = u_{i_1}, v_2 = u_{i_1} + u_{i_2}, \cdots, v = u_{i_1} + u_{i_2} + \cdots + u_{i_n}. \]

We will call this sum the description of \( p \). Note that the order of the \( u \)'s matters, as a permutation of the \( u \)'s results in another path. We call \( p \) an ascending path (without specifying starting point and endpoint) if \( p \) is an ascending path leading from \((0, 0, \cdots, 0) \) to \((1, 1, \cdots, 1) \).
Before turning to results, we will first study the structure of ascending paths in labeled hypercubes and give some necessary definitions. Building upon them, we will then prove Lemma 3.3, which states that \( P^{B[1]:B[1]} \) can handle exactly \( j + 2k \) (\( j + 2k - 1 \) if \( j \) is even and \( k \) is odd) mind changes.

Let \( M \) be a \( P^{B[1]:B[1]} \) machine with oracles \( A \in BH_j \) and \( B \in BH_k \) and let \( x \in \Sigma^* \). On input \( x \), \( M \) first makes a query \( q_1(x) \) to \( A \) and then if the answer to the first query was “no” asks query \( q_2(x) \) to \( B \) and if the answer to the first query was “yes” asks query \( q_3(x) \) to \( B \). Without loss of generality assume that on every input \( x \) exactly two queries are asked.

Every set \( C \in BH_l \) can be written as the nested difference of sets \( C_1, C_2, \ldots, C_l \in NP \)
\[
C = C_1 - (C_2 - (\cdots - (C_{l-1} - C_l) \cdots))
\]
and following Cai et al. [10] we even can assume that
\[
C_l \subseteq C_{l-1} \subseteq \cdots \subseteq C_2 \subseteq C_1.
\]

Hence a query “\( q \in C? \)” can certainly be solved via \( l \) queries “\( q \in C_1? \),” “\( q \in C_2? \),” \( \cdots \), “\( q \in C_l? \)”

In light of this comment, we let
\[
A = A_1 - (A_2 - (\cdots - (A_{j-1} - A_j) \cdots)) \quad \text{where} \quad A_i \in NP \quad \text{for} \quad i = 1, 2, \ldots, j
\]
and \( A_j \subseteq \cdots \subseteq A_1 \), and
\[
B = B_1 - (B_2 - (\cdots - (B_{k-1} - B_k) \cdots)) \quad \text{where} \quad B_i \in NP \quad \text{for} \quad i = 1, 2, \ldots, k
\]
and \( B_k \subseteq \cdots \subseteq B_1 \).

For the sake of definiteness let us assume that the queries
\[
q_1(x) \in A_1, \ldots, q_1(x) \in A_j, q_2(x) \in B_1, \ldots, q_2(x) \in B_k, q_3(x) \in B_1, \ldots, q_3(x) \in B_k
\]
correspond in this order to the \( j + 2k \) dimensions of the \( (j + 2k) \)-dimensional hypercube \( H = \{0, 1\}^{j+2k} \). More precisely, a vector \( (a_1, \ldots, a_{j+2k}) \in H \) is understood to consist of the answers to the above-mentioned queries, where 0 means “no” and 1 means “yes.”

Since a query “\( q \in C? \)” for some \( C \in BH_l \) and \( C = C_1 - (C_2 - (\cdots (C_{l-1} - C_l)) \cdots) \) can be solved by evaluating the answers to “\( q \in C_1? \),” “\( q \in C_2? \),” \( \cdots \), “\( q \in C_l? \)” every node \( v \in H \) gives us answers to “\( q_1(x) \in A? \)” (by evaluating the first \( j \) components of \( v \)), to “\( q_2(x) \in B? \)” (by evaluating the \( k \) components of \( v \) that immediately follow the first \( j \) components of \( v \)) and to “\( q_3(x) \in B? \)” (by evaluating the last \( k \) of \( v \)’s components). This gives us a labeling of all vertices of \( H \). We simply assign label \( A \) (Accept) to vertex \( v \in H \) if \( M^{A[1]:B[1]}(x) \) accepts if the answers to the two asked questions are as determined by \( v \). If \( M^{A[1]:B[1]}(x) \) rejects in this case we assign label \( R \) (Reject) to \( v \).

So let \( H_M(x) \) be the \( (j + 2k) \)-dimensional hypercube labeled according to \( M^{A[1]:B[1]}(x) \). The number of mind changes on an ascending path \( p \) of \( H_M(x) \) leading from \( (0, 0, \cdots, 0) \)
to a vertex $t$ is by definition the number of label changes when moving from $(0, 0, \ldots, 0)$ to $t$ along $p$. The number of mind changes of an internal node $v$ of $H_M(x)$ is the maximum number of mind changes on an ascending path leading from $(0, 0, \ldots, 0)$ to $v$. And finally, the number of mind changes of a $\text{P}^{\text{BH}_j[1]:\text{BH}_k[1]}$ machine $M$ is by definition the maximum number (we take the maximum over all $x \in \Sigma^*$) of mind changes of the vertex $(1, 1, \cdots, 1)$ in $H_M(x)$; in other words, this number is the maximum number of label changes on an ascending path in $H_M(x)$ for some $x \in \Sigma^*$.

We say we lose a mind change (between two adjacent vertices $v_i$ and $v_{i+1}$) along an ascending path if when moving from $v_i$ to $v_{i+1}$ the machine does not change its acceptance behavior.

One can easily verify the following fact:

**Fact 3.2** If $M$ is a $\text{P}^{\text{BH}_j[1]:\text{BH}_k[1]}$ machine such that on input $x$ the acceptance behavior is independent of the answer to one or more of the two possible second queries (that is, if for at least one of the second queries both a “yes” and a “no” answer yield the same acceptance or rejection behavior), then we lose at least one mind change on every path in $H_M(x)$.

So from now on, in light of Fact 3.2 let $M^{A[1]:B[1]}(x)$ be a $\text{P}^{\text{BH}_j[1]:\text{BH}_k[1]}$ machine that has, on input $x$, one of the following four acceptance schemes (the scheme may depend on the input).

1. $M$ accepts if and only if exactly one of the two sequential queries is answered “yes.”
2. $M$ accepts if and only if either both or neither of the two asked queries is answered “yes.”
3. $M$ accepts if and only if the second query is answered “yes.”
4. $M$ accepts if and only if the second query is answered “no.”

**Fact 3.3** If $p$ is an ascending path in $H_M(x)$ such that $p$ contains adjacent vertices $v$ and $v + u_d$ such that

$$d \leq j$$

and the $(d')$th component of $v$ is 0 for some $d' < d$,

then $p$ loses a mind change.

**Proof** Since $A \in \text{BH}_j$ and thus $A = A_1 - (A_2 - (\cdots - (A_{j-1} - A_j) \cdots))$ and there is a 0 in the $(d')$th component of $v$ and $v + u_d$, both vertices yield the same answer to “$q_1(x) \in A$?” The 1 in the $d$th component of $v + u_d$ has no effect at all on the answer to “$q_1(x) \in A$?” and so on the outcome of $M^{A[1]:B[1]}(x)$. Hence, both vertices have the same label and $p$ loses a mind change.

Similarly, one can prove that if $p$ is an ascending path and $p$ contains two adjacent vertices $v$ and $v + u_d$ such that $j < d' < d \leq j + k$ and the $(d')$th component of $v$ is 0 or $j + k < d' < d \leq j + 2k$ and the $(d')$th component of $v$ is 0 then $p$ also loses one mind change.
Furthermore, in light of Fact 3.3, let us focus only on paths $p$ that change their first $j$, second $k$, and last $k$ dimensions from the smallest to the highest dimension in each group. This allows us to simplify the description of paths as follows. Let $e_1$ be the following operator on $H$:

$$e_1((a_1, \ldots, a_{j+2k})) = \begin{cases} (a_1, \ldots, a_{i-1}, 1, \ldots, a_{j+2k}) & \text{if } i \leq j, a_i = 0 \land (\forall j : j < i)[a_j = 1] \\
(a_1, \ldots, a_{j+2k}) & \text{otherwise.} \end{cases}$$

The operators $e_2$ and $e_3$ act on the index groups $(j+1, \ldots, j+k)$ and $(j+k+1, \ldots, j+2k)$, respectively, in the same manner: the zero component with smallest index among the zero components is incremented by 1. The only reasonable paths to consider are those emerging from repeated applications of $e_1$, $e_2$ and $e_3$ to $(0, \ldots, 0)$. We will use $(e_{i_1}, e_{i_2}, \ldots, e_{i_{j+2k}})$ to denote the path with vertices $v_0 = (0, \ldots, 0)$, $v_1 = e_{i_1}(v_0), v_2 = e_{i_2}(v_1), \ldots$, $v_{j+2k} = e_{i_{j+2k}}(v_{j+2k-1}) = (1, 1, \ldots, 1)$.

The next fact gives sufficient conditions for an ascending path to lose a mind change, namely:

**Fact 3.4** On any ascending path $p$ a mind change loss occurs if:

**Case 1.1** there is an $e_2$ after an odd number of $e_1$’s in the description of $p$, or

**Case 1.2** there is an $e_3$ after an even number of $e_1$’s in the description of $p$, or

**Case 2** the description of $p$ contains a sequence of odd length at least 3 that starts and ends with $e_1$ and contains no other $e_1$’s.

**Proof** We will call the occurrence of Case 1.1 (Case 1.2) in $p$ an “$e_2$-loss” (“$e_3$-loss”) and the occurrence of Case 2 an “odd episode.” In general we call a subpath of $p$ of length at least 3 that starts and ends with $e_1$ and contains no other $e_1$ an episode.

Intuitively $p$ loses a mind change in the case of Case 1.1 (1.2), since in the actual computation $M(x)$ does not really ask query $q_2(x)$ ($q_3(x)$) and so a change in the answers to the $k$ underlying NP queries of $q_2(x)$ ($q_3(x)$) does not affect the outcome of the overall computation.

Intuitively in Case 2 the following argument holds. If the description of $p$ contains an odd episode, say starting with $e_{i_1} = e_1$ and ending with $e_{i_{i'}} = e_1$, then $v_{l-1}, v_l, \ldots, v_{i'}$ form an even-length subpath $p'$ of $p$. If the odd episode contains both $e_2$’s and $e_3$’s then note that Case 1 applies and we are done. In fact due to Case 1, we may hence forward assume the odd episode, between the starting and the ending $e_1$’s, has only $e_2$’s (respectively $e_3$’s), if we have an even (respectively odd) number of $e_1$’s up to and including the $e_1$ starting the odd episode. So in this case $v_{l-1}$ and $v_{i'}$ have the same label Accept/Reject. The acceptance behavior of $M^{[1]:B[1]}(x)$ due to $v_{l-1}$ and $v_{i'}$ is the same, because after two $e_1$’s the answer to “$q_1(x) \in A$?” is the same as it was before the two $e_1$’s, and the $e_2$’s ($e_3$’s) have not influenced the answer to $q_3(x)$ ($q_2(x)$). Thus we have a subpath of even length, namely $v_{l-1}, v_l, \ldots, v_{i'}$, whose starting point and endpoint have the same Accept/Reject label. To assign to each vertex of this path an Accept/Reject label in such a way that no
mind changes are lost is equivalent to the impossible task of 2-coloring an odd cycle. Hence we lose at least one mind change for every occurrence of an odd episode.

Before proving the main theorem of this section, we show the following lemma, Lemma 3.5, which tells how many mind changes \( P^{BH_j[1]:BH_k[1]} \) can handle. We say a complexity class \( P^{BH_j[1]:BH_k[1]} \) can handle exactly \( m \) mind changes if and only if (a) no \( P^{BH_j[1]:BH_k[1]} \) machine has more than \( m \) mind changes and (b) there is a specific \( P^{BH_j[1]:BH_k[1]} \) machine that has \( m \) mind changes. It is known (see, e.g., [10,25,3]) that \( R_{k-tt}(NP) \) can handle exactly \( k \) mind changes.

**Lemma 3.5** The class \( P^{BH_j[1]:BH_k[1]} \) can handle exactly \( m \) mind changes, where

\[
m = \begin{cases} 
  j + 2k - 1 & \text{if } j \text{ is even and } k \text{ is odd} \\
  j + 2k & \text{otherwise.}
\end{cases}
\]

**Proof** We first consider the case in which \( j \) is even and \( k \) is odd.

We want to argue that for every \( P^{BH_j[1]:BH_k[1]} \) machine \( M \) and every \( x \in \Sigma^* \), on every ascending path in the \( j + 2k \) dimensional, appropriately labeled hypercube \( H_M(x) \) there are at most \( j + 2k - 1 \) mind changes. Let \( x \in \Sigma^* \) and \( M \) be a \( P^{BH_j[1]:BH_k[1]} \) machine with the oracles \( A \) and \( B \). Due to Facts 3.3 and 3.2, it suffices to consider a \( P^{BH_j[1]:BH_k[1]} \) machine \( M \) with one of the four previously mentioned acceptance schemes on input \( x \) and to show that every path \( p \) having the introduced description loses at least one mind change. Let \( M(x) \) be such a machine and \( p \) be such a path. There are two possibilities.

**Case A** The description of \( p \) contains an \( e_2 \)-loss or an \( e_3 \)-loss.

According to Fact 3.4, \( p \) loses at least one mind change.

**Case B** The description of \( p \) contains neither an \( e_2 \)-loss nor an \( e_3 \)-loss.

Hence the description of \( p \) consists of blocks of consecutive \( e_2 \)'s and \( e_3 \)'s separated by blocks of \( e_1 \)'s. Since the description of \( p \) contains \( k \) \( e_3 \)'s and \( k \) is odd, there is a block of \( e_3 \)'s of odd size in \( p \). Since we have no \( e_3 \)-loss and \( j \) is even this block is surrounded by \( e_1 \)'s. Thus we have an odd episode in the description of \( p \) and, according to Fact 3.4, \( p \) loses a mind change.

So no \( P^{BH_j[1]:BH_k[1]} \) machine can realize more than \( j + 2k - 1 \) mind changes.

It remains to show that there is a deterministic \( P^{BH_j[1]:BH_k[1]} \) machine and an input \( x \in \Sigma^* \) such that in the associated hypercube \( H_M(x) \) there is a path having exactly \( j + 2k - 1 \) mind changes.

Let us consider the path \( p_0 \),

\[
p_0 = (e_2,e_2,\cdots,e_2,e_1,\cdots,e_1,e_3,e_3,\cdots,e_3,e_1).
\]

Consider the deterministic oracle machine \( W \) that asks two sequential queries and accepts an input \( x \) if and only if the second query of \( W(x) \) was answered “yes” (acceptance scheme (3)). We know as just shown that all ascending paths of \( H_W(x) \) have at most
$j + 2k - 1$ mind changes. Note that for every $x \in \Sigma^*$ the path $p_0$ loses only one mind change and thus $P^{\BH_j[1]:\BH_k[1]}$ can handle exactly $j + 2k - 1$ mind changes.

This completes the proof of the case “$j$ is even and $k$ is odd.” We now turn to the “$j$ is odd or $k$ is even” case of the lemma being proven.

Since our hypercube has (in all cases) $j + 2k$ dimensions, certainly $P^{\BH_j[1]:\BH_k[1]}$ can handle (in all cases) no more than $j + 2k$ mind changes.

If $j$ is odd, we consider the path

$$p_1 = (e_2, e_2, \cdots, e_2, e_1, e_1, \cdots, e_1, e_3, \cdots, e_3)$$

and— using the acceptance scheme numbering introduced just after Fact 3.3—we consider the machine having for every input $x$ acceptance scheme (3) or (1) for $k$ odd or even, respectively. If $j$ is even and $k$ is even, we consider path $p_0$ and we consider the machine having acceptance scheme (1) for every input.

In each of these cases the considered machine changes its mind along the associated path exactly $j + 2k$ times. Hence for $j$ odd or $k$ even the class $P^{\BH_j[1]:\BH_k[1]}$ can handle exactly $j + 2k$ mind changes.

Now we are ready to prove our main theorem of this section.

**Theorem 3.6** For $j, k \geq 1$,

$$P^{\BH_j[1]:\BH_k[1]} = \begin{cases} R^{p_{j+2k-1-tt}}_j(NP) & \text{if } j \text{ is even and } k \text{ is odd} \\ R^{p_{j+2k-tt}}_j(NP) & \text{otherwise.} \end{cases}$$

**Proof** In order to avoid unnecessary case distinctions we prove the fact for arbitrary $j$ and $k$ and simply denote the appropriate number of mind changes by $m$, namely (see Lemma 3.5) $j + 2k - 1$ if $j$ is even and $k$ is odd and $j + 2k$ otherwise. First we would like to show that $P^{\BH_j[1]:\BH_k[1]} \subseteq R^{p_m}_m(NP)$. We show this by explicitly giving the appropriate truth-table reduction.

Let $A \in P^{\BH_j[1]:\BH_k[1]}$ and let $m$ be the number of mind changes (according to Lemma 3.5) $j + 2k - 1$ if $j$ is even and $k$ is odd and $j + 2k$ otherwise. Let $M$ be a deterministic oracle machine, witnessing $A \in P^{\BH_j[1]:\BH_k[1]}$, via the sets $S_1 \in \BH_j$ and $S_2 \in \BH_k$. As noted by Beigel [3], the set $Q = \{\langle x, k \rangle \mid M(x) \text{ has at least } k \text{ mind changes} \}$ is an NP set. Note that if $M^{S_1[1]:S_2[1]}(x)$ on a particular input $x$ rejects (respectively accepts) if both queries have the answer “no” then $M^{S_1[1]:S_2[1]}(x)$ accepts if and only if the node (of the implicit hypercube) associated with the actual answers has an odd (respectively even) number of mind changes.

Define the variables $o, y_1, y_2, \cdots, y_m$ and the $m$-ary boolean function $\alpha$:

- $o = 0$ if $M^{S_1[1]:S_2[1]}(x)$ rejects if both queries are answered “no,”
- $o = 1$ if $M^{S_1[1]:S_2[1]}(x)$ accepts if both queries are answered “no,”

$$y_1 = \langle x, 1 \rangle,$$
$$y_2 = \langle x, 2 \rangle,$$
$$y_3 = \langle x, 3 \rangle,$$
$$\vdots$$
Lemma 3.1. Since the class $P$ the form $B$ $BH$ $∈$ the function $j$ $α$ $⊆$ $· · ·$ $m$ $A$ $∈$ $m$. We show $x$ $BH$ $x$, $BH$ $⊆$ $· · ·$ $B$ $m$ $BH$ $y$. Clearly we can compute the just defined variables for a given $x$ and also evaluate the function $α$ at $(χQ(y_1), χQ(y_2), \cdots, χQ(y_m))$ in polynomial time. And finally we have $x ∈ A$ $⇐ ⇒$ $α(χQ(y_1), χQ(y_2), \cdots, χQ(y_m)) = 1$. Thus $A ∈ R^p_{m-tt}(NP)$. It remains to show that $R^p_{m-tt}(NP) ⊆ P^{BH_1[1]:BH_2[1]}$. Recall $R^p_{k-tt}(NP) = R^p_{1-tt}(BH_k)$ from Lemma [3]. Since the class $P^{BH_1[1]:BH_2[1]}$ is closed under $≤^p_{1-tt}$ reductions it suffices to prove $BH_m ⊆ P^{BH_1[1]:BH_2[1]}$. 

So let $B ∈ BH_m$. Following Cai et al. [10] we may assume that the set $B$ is of the form $B = B_1 - (B_2 - (B_3 - (\cdots - (B_{m-1} - B_m) \cdots))))$ with $B_1, B_2, \cdots, B_m ∈ NP$ and $B_m ⊆ \cdots ⊆ B_2 ⊆ B_1$. We show $B ∈ P^{BH_1[1]:BH_2[1]}$ by using ideas of the second part of the proof of Lemma 4.1, namely by implementing the specific good path $p_0$, respectively $p_1$. $B$ is accepted by a $P^{BH_1[1]:BH_2[1]}$ machine $M$ as follows:

Case 1 $j$ is odd.

Define the two oracle sets $O_1$ and $O_2$:

$O_1 = B_{k+1} - (B_{k+2} - (\cdots - (B_{k+j-1} - B_{k+j}) \cdots))$, and

$O_2 = \{(y, 2) \mid y ∈ B_1 - (B_2 - (\cdots - (B_{k-1} - B_k) \cdots)) \}$

$∪\{(y, 3) \mid y ∈ B_{j+k+1} - (B_{j+k+2} - (\cdots - (B_{j+2k-1} - B_{j+2k}) \cdots))\}$. Note that $O_1 ∈ BH_j$ and $O_2 ∈ BH_k$. On input $x$ $M$ first queries “$x ∈ O_1$.” In case of a “no” answer $M(x)$ queries $⟨x, 2⟩ ∈ O_2$ and in case of a “yes” answer to the first query $M(x)$ asks $⟨x, 3⟩ ∈ O_2$.

Case 1.1 $k$ is odd.

$M(x)$ accepts if and only if the second query is answered “yes.”

Case 1.2 $k$ is even.

$M(x)$ accepts if and only if exactly one of the two queries is answered “yes.”

Case 2 $j$ is even.

Define the two oracle sets $O_1$ and $O_2$:

$O_1 = B_{k+1} - (B_{k+2} - (\cdots - (B_{k+j-1} - B_m) \cdots))$, and

$O_2 = \{(y, 2) \mid y ∈ B_1 - (B_2 - (\cdots - (B_{k-1} - B_k) \cdots)) \}$

$∪\{(y, 3) \mid y ∈ B_{j+k} - (B_{j+k+1} - (\cdots - (B_{m-2} - B_{m-1}) \cdots))\}$. Note that $O_1 ∈ BH_j$ and $O_2 ∈ BH_k$. On input $x$ $M$ first queries “$x ∈ O_1$.” In case of a “no” answer $M(x)$ queries $⟨x, 2⟩ ∈ O_2$ and in case of a “yes” answer to the first query $M(x)$ asks $⟨x, 3⟩ ∈ O_2$.

Case 2.1 $k$ is odd.

$M(x)$ accepts if and only if the second query is answered “yes.”

Case 2.2 $k$ is even.

$M(x)$ accepts if and only if exactly one of the two queries is answered “yes.”

$y_m = \langle x, m \rangle$,
Case 2.2 $k$ is even.

$M(x)$ accepts if and only if exactly one of the two queries is answered “yes.”

It is interesting to note which properties of NP are actually required in the above proof for the result to hold. The proof essentially rests on the fact that the key set $Q$ (describing that, for given $x$ and $m$, the $P^{BH_j}[1]:BH_k[1]$ machine $M$ on input $x$ has at least $m$ mind changes) is an NP set. So considering an arbitrary underlying class $C$, for proving $Q \in C$ it suffices to note that $Q$ is in the class $\exists^b \cdot R^p_{C-btt}(C)$ and to assume that $C$ be closed under $\exists^b$ and conjunctive bounded truth-table reductions. Indeed, the $\exists^b$ quantifier describes that there is a path in the boolean hypercube $H_M(x)$, and via the $\leq^p_{C-btt}$-reduction it can be checked that this path is an ascending path and all the answers the vertices on that path claim to be “yes” answers indeed correspond to query strings that belong to the class $C$. Similar observations have been stated in earlier papers [3, 5].

Theorem 3.6 allows us to derive a relationship between classes of the form $P^{BH_j[1]:BH_k[1]}$ and $P^{BH_j[1]:BH_k[1]} +$. Classes of the latter form were studied in [1].

Corollary 3.7 For every $j, k \geq 1$,

$$R^p_{1-tt}(P^{BH_j[1]:BH_k[1]} +) = \begin{cases} P^{BH_{j-1}[1]:BH_k[1]} & \text{if } j \text{ is odd and } k \text{ is even} \\ P^{BH_j[1]:BH_k[1]} & \text{otherwise.} \end{cases}$$

The proof is immediate by the results of Theorem 3.6 of this paper and Theorem 5.7 and Lemma 5.9 of [1].

From Theorem 3.6 we can immediately conclude that order matters for queries to the boolean hierarchy unless the boolean hierarchy itself collapses.

Corollary 3.8

1. If $(j = k) \lor (j \text{ is even and } k = j + 1)$, $1 \leq j \leq k$, then $P^{BH_j[1]:BH_k[1]} = P^{BH_k[1]:BH_j[1]}$.

2. Unless the boolean hierarchy (and thus the polynomial hierarchy) collapses: for every $1 \leq j \leq k$, $P^{BH_j[1]:BH_k[1]} \neq P^{BH_k[1]:BH_j[1]}$ unless $(j = k) \lor (j \text{ is even and } k = j + 1)$.

The corollary holds, in light of the theorem, simply because the boolean hierarchy and the truth-table hierarchy are interleaved [25] in such a way that the boolean hierarchy levels are sandwiched between levels of the bounded-truth-table hierarchy, and thus if two different

1Here, $\leq^p_{C-btt}$ denotes the conjunctive bounded truth-table reducibility, and for any class $\mathcal{K}$, $\exists^b \cdot \mathcal{K}$ is defined to be the class of languages $A$ for which there exists a set $B \in \mathcal{K}$ and a constant bound $m$ such that $x \in A$ if and only if there exists a string $y$ of length at most $m$ with $\langle x, y \rangle \in B$. 

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levels of the bounded-truth-table hierarchy are the same (say levels r and s, \( r < s \)), then some level (in particular, \( BH_{r+1} \)) of the boolean hierarchy is closed under complementation, and thus, by the downward separation property of the boolean hierarchy \([10]\), the boolean hierarchy would collapse. Furthermore, Kadin \([24]\) has shown that if the boolean hierarchy collapses then the polynomial hierarchy collapses, and Wagner, and Chang and Kadin, and Beigel, Chang, and Ogihara have improved the strength of this connection \([28,29,14,5]\).

The strongest known connection is: If \( BH_q = \text{coBH}_q \), then \( PH = \left( P^{\text{NP}}_{(q-1)-\text{tt}} \right)^{\text{NP}} \), where \( (P^{\text{NP}}_{m-\text{tt}})^{\text{NP}} \) denotes the class of languages accepted by \( P \) machines given \( m \)-truth-table access to an \( \text{NP}^{\text{NP}} \) oracle and also given unlimited access to an \( \text{NP} \) oracle (note that \( (P^{\text{NP}}_{1-\text{tt}})^{\text{NP}} \) is equal to \( P^{\text{NP}}_{\min(\alpha(j,k), \alpha(j',k'))-\text{tt}}^{\text{NP}} \), where \( \alpha(a, b) = a + 2b - 1 \) if \( a \) is even and \( b \) is odd and \( a + 2b \) otherwise.

In light of this discussion, we can make more clear exactly what collapse is spoken of in the second part of the above corollary. In particular, the collapse of the polynomial hierarchy is (at least) to \( (P^{\text{NP}}_{(k+2j)-\text{tt}})^{\text{NP}} \).

Of course, Theorem 3.6 applies far more generally. From it, for any \( j, k, j', k' \), one can either immediately conclude equality, or can immediately conclude that the classes are not equal unless the polynomial hierarchy collapses to \( (P^{\text{NP}}_{\min(a(j,k), a(j',k'))-\text{tt}})^{\text{NP}} \), where \( \alpha(a, b) = a + 2b - 1 \) if \( a \) is even and \( b \) is odd and \( a + 2b \) otherwise.

The point of Theorem 3.6 is that from the even/odd structure of \( P^{BH_j[1]:BH_k[1]} \) classes one can immediately tell their number of mind changes, and thus their strength, without having to do a separate, detailed, mind change analysis for each \( j \) and \( k \) pair. However, note that one can, via a time-consuming but mechanical procedure, analyze almost any class with a query tree structure (namely by looking at the full tree of possible queries and answers, and for each of the huge number of possible ways its leaves can each be labeled accept-reject compute the number of mind changes that labeling creates, and then look at the maximum over all these numbers). For example, one can quickly see that one query to \( DP \) followed by 4-tt access to \( \text{NP} \) yields exactly the languages in \( R^{10-\text{tt}}(\text{NP}) \).

4 General Case

In the previous section, we studied classes of the form \( P^{BH_j[1]:BH_k[1]} \). We completely characterized them in terms of reducibility hulls of \( \text{NP} \) and noted that in this setting the order of access to different oracles matters quite a bit. What can be said about, for example, the class \( P^{BH_j[1]:BH_k[1]:BH_l[1]} \)? Is it equal to \( P^{BH_j[1]:BH_k[1]:BH_l[1]} \)? (We’ll see that the answer is “no” in certain cases.) Even more generally, what can be said about the classes of languages that are accepted by deterministic oracle machines with tree-like query structures and with

\(^2\)Though one level is gained by the \( q-1 \) in the \( [5] \) connection between the boolean hierarchy and the polynomial hierarchy, one level is lost in the collapse of the boolean hierarchy that follows from a given collapse in the truth-table hierarchy. We speculate that it might be possible for the \( k + 2j \) claim to be strengthened by one level by applying the \( [5] \) technique directly to the truth-table hierarchy.
each query being made to a (potentially) different oracle from a (potentially) different level of the boolean hierarchy? Is it possible that with a more complicated query structure we might lose even more than the one mind change lost in the case of $P^{BH_j[1]:BH_k[1]}$ with $j$ even and $k$ odd? (From the results of the section, it will be clear that the answer to this question is “yes”; mind changes can, in certain specific circumstances, accumulate.)

First of all, we can immediately derive a characterization of the class $P^{BH_j[1]:BH_k[1]}$, from the results of the previous section, namely:

**Theorem 4.1** For $j, k, l \geq 1$,

$$P^{BH_j[1]:BH_k[1],BH_l[1]} = \begin{cases} R^p_{j+k+l-1-tt}(NP) & \text{if } j \text{ is even and } l \text{ is odd} \\ R^p_{j+k+l-tt}(NP) & \text{otherwise.} \end{cases}$$

**Proof** Note that in Lemma 3.5 we handle the special case of $k = l$. However, notice that the mind change loss for $j$ even and $k$ odd is due only to the fact that the query made after the first query is answered “yes” is made to an oracle from an odd level, namely $k$, of the boolean hierarchy. In particular the mind change loss is not tied to the query we ask in case the first query is answered “no.” Thus we have

**Claim** The class $P^{BH_j[1]:BH_k[1],BH_l[1]}$ can handle exactly $m$ mind changes where

$$m = \begin{cases} j + k + l - 1 & \text{if } j \text{ is even and } l \text{ is odd} \\ j + k + l & \text{otherwise.} \end{cases}$$

Similarly to the proof of Theorem 3.6 one can now show the equality we claim.

Note that for every $j, k, l \geq 1$, we obviously have

$$P^{BH_j[1]:BH_k[1],BH_l[1]} = P^{R^p_{i-tt}(BH_j)[1]:R^p_{i-tt}(BH_k)[1],R^p_{i-tt}(BH_l)[1]}$$

and thus the following corollary holds.

**Corollary 4.2** For $j, k, l \geq 1$,

$$P^{R^p_{i-tt}(BH_j)[1]:R^p_{i-tt}(BH_k)[1],R^p_{i-tt}(BH_l)[1]} = \begin{cases} R^p_{j+k+l-1-tt}(NP) & \text{if } j \text{ is even and } l \text{ is odd} \\ R^p_{j+k+l-tt}(NP) & \text{otherwise.} \end{cases}$$

The last corollary is the key tool to use in evaluating any class of languages that are accepted by deterministic oracle machines with tree-like query structures and with each query being made to a (potentially) different oracle from a (potentially) different level of the boolean hierarchy.

We formalize some notions to use in studying this. Let $T$ be a binary tree, not necessarily complete, such that each internal node $v_i$ (a) has exactly two children, and (b) is labeled by a natural number $n_i$ (whose purpose will be explained below). For such a tree $T$, define $f_T$ by $f_T(v_i) = n_i$. Henceforward, we will write $f$ for $f_T$ in contexts in which $T$ is clear. Let $root_T$ be the root of the tree (we will assign to this node the name $v_1$) and let $LT_T$
and $RT_T$ respectively be the left and right subtrees of the root. We will denote the class of sets that are accepted by a deterministic oracle machine with a $T$-like query structure by $P^{(T)}$. Here the structure of the tree $T$ gives the potential computation tree of every $P^{(T)}$ machine in the sense that inductively if a query at node $v$ is answered “no” (“yes”) we keep on moving through the tree in the left (right) subtree of $v$. And at each internal node $v_i$ of $T$ the natural number $n_i$ gives the level of the boolean hierarchy from which the oracle queried at that node is taken.

For example consider the tree $T$ (see Figure 1), in which $f(v_1) = 2$, $f(v_2) = 2$, $f(v_3) = 4$, $f(v_4) = 1$, and $f(v_5) = 3$. A $P^{(T)}$ machine works as follows. The first query is made to a DP oracle. If the answer to that first query is “no” a second query is made to the DP oracle associated with $v_2$, and if the answer to the first query is “yes” the second query is made to the BH$_4$ oracle associated with $v_3$. A third query is made only if the answer to the first query is “yes”; in this case, the oracle set of the third query is in NP if the answer to the second query is “no,” and is in BH$_3$ if the answer to the second query is “yes.” Note that for every input $x \in \Sigma^*$ every $P^{(T)}$ machine $M(x)$ assigns a label A (Accept) or R (Reject) to each leaf of $T$ with its own specific acceptance behavior (which, in particular, may depend on $x$).
If $T$ is the complete tree of depth 1 (i.e., a root plus two leaves), then by definition $m(T) = f(root_T)$, and otherwise define

$$m(T) = \begin{cases} 
  f(root_T) + m(LT_T) + m(RT_T) - 1 & \text{if } f(root_T) \equiv 0 \pmod{2} \text{ and } m(RT_T) \equiv 1 \pmod{2} \\
  f(root_T) + m(LT_T) + m(RT_T) & \text{otherwise}.
\end{cases}$$

For our example tree $T$ we have $m(T) = 10$. The main theorem of this section will prove $m(T)$ determines the number of bounded truth-table accesses to NP that completely characterize the class $P^T$. It follows from the main theorem that, for example, $P^T = R^P_{\text{10-tt}}(NP)$.

**Theorem 4.3** $P^T = R^P_{m(T)-\text{tt}}(NP)$.

**Proof** The proof consists of an obvious induction over the depth $d$ of the tree. Note that the correctness of the base case of the induction, $d = 2$, is given by Theorem 4.1. The proof of the inductive step follows immediately from the obvious fact that

$$P^T = \text{pBH}_{f(root_T), P^{LT_T}, P^{RT_T}},$$

combined with Lemma 3.1 ($R^P_{k\text{-tt}}(NP) = R^P_{\text{tt}}(BH_k)$) and Corollary 4.2.

Finally, we mention that a study of query order in the polynomial hierarchy (as opposed to the boolean hierarchy) has very recently been initiated by E. Hemaspaandra, L. Hemaspaandra, and H. Hempel ([19], see also [31,4]) and this study has led to a somewhat surprising downward translation result: For $k > 2$, $\Sigma^P_k = \Pi^P_k \iff P^{\Sigma^p_kannie} = P^{\Sigma^p_k[2]}$ ([16], see also the extensions obtained in [8,20]). Query order (see also the survey [17]) has also recently proven useful in studying the structure of complete sets [18] and in characterizing bottleneck-computation classes [23].

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