Dynamical speedup of a two-level system induced by coupling in the hierarchical environment

Kai Xu,1 Guo-Feng Zhang,1,∗ and Wu-Ming Liu2

1Key Laboratory of Micro-Nano Measurement-Manipulation and Physics (Ministry of Education), School of Physics and Nuclear Energy Engineering, Beihang University, Xueyuan Road No. 37, Beijing 100191, China
2Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing, China

(Dated: November 6, 2018)

We investigate the dynamics of a two-level system in the presence of an overall environment composed of two layers. The first layer is just one single-mode cavity which decays to memoryless reservoir while the second layer is the two coupled single-mode cavities which decay to memoryless or memory-keeping reservoirs. In the weak-coupling regime between the qubit and the first-layer environment, our attention is focused on the effects of the coupling in the hierarchical environment on the non-Markovian speedup dynamics behavior of the system. We show that, by controlling the coupling in the second-layer environment, the multiple dynamics crossovers from Markovian to non-Markovian and from no-speedup to speedup can be realized. This results hold independently on the nature of the second-layer environment. Differently, we find that how the coupling between the two layers affects the non-Markovian speedup dynamics behavior depends on the nature of the second-layer environment.

PACS numbers: 03.65.Yz, 03.67.Lx, 42.50.-p

I. INTRODUCTION

Recently, motivated by real physical systems, understanding the evolution of open quantum systems that are coupled to the environment has drawn more and more interest [1–5]. In general, such physical systems can interchange with its environment particles, energy or information and therefore lead to decoherence and dissipation of the system’s quantum properties [6]. In this situation, a monotonic one-way flow of information from the system to the environment tends to the appearance what is called Markovian dynamics. However, in many scenarios, due to the increasing capability to manipulate quantum systems, leading to the occurrence of a backflow of information from the environment to the system, what is called non-Markovian dynamics [7–27]. The non-Markovian effects not only suppress the decay of the coherence or the entanglement of quantum systems [28, 29] but also play a leading role in many real physical processes such as quantum state engineering, quantum control [30–32] and the quantum information processing [33–37]. For example, recent experiments [38] have shown that non-Markovianity can improve the probability of success of the Deutsch-Jozsa algorithm in diamonds.

The non-Markovianity has received great attention in the form of quantitative measurement [39,40], experimental demonstration [41–43] and the impact on the speedup evolution of quantum system [44–46]. For instance, a new characterization of non-Markovian quantum evolution based on the concept of non-Markovianity degree has been proposed [39]. The experimental realization of a non-Markovian process where system and environment are coupled through a simulated transverse Ising model has been reported [41]. The non-Markovian effect could induce speedup dynamics process in the strong system-environment coupling regime for the damped Jaynes-Cummings model [58]. And this novel has been realized by increasing the system-environment coupling strength and the number of atoms in a controllable environment [59].

In the previous studies, some researchers have considered the quantum system coupled to a single-layer environment. However, the system can be influenced by multilayer environments [60–64] in the realistic scenarios. For example, the electron spin in a quantum dot may be influenced strongly by the surrounding nuclei [60]. The surrounding nitrogen impurities constitute the principal bath for a nitrogen-vacancy center, while the carbon-13 nuclear spins also have some influences on it [61, 63]. Based on these, multilayered environments have been considered for the study of non-Markovian dynamics of the system. A qubit that is coupled to a hierarchical environment, which contains a single-mode cavity and a reservoir consisting of an infinite number of modes has been investigated [65]. They show the non-Markovian character of the system is influenced by the coupling strength between the qubit and cavity and the correlation time of the reservoir. Besides, the hierarchical environment model where the first layer is just a single lossy cavity while the second layer consists of a number of lossy cavities has been considered [66]. In this model, the increase the number of the lossy cavities and the coupling between the two layers can trigger the non-Markovian dynamics of the system. However, in the experiment, the
coupling relationship between different parts of a complex environment has a great influence on the dynamic behavior of the system. For example, a single spin interacting with an adjustable spin bath shows that both the internal interactions of the bath and the coupling between the central spin and the bath can be tuned in situ, allowing access to regimes with surprisingly different behavior \[60\]. So in the treatments of the composite environments, the influence of the coupling relationship between various parts of the overall environment on the dynamic behavior of the system has to be taken into account.

Based on these, we mainly investigate the dynamics of a qubit coupled with the overall environment composed of two layers. By using the quantum speed limit (QSL) time \[58, 67, 68\] to define the speedup evolitional process, the influence of the coupling in the second-layer environment and the coupling between the two layers on the quantum evolitional speed of the qubit are discussed in the weak-coupling regime between the qubit and the first-layer environment. In this paper, by considering the second-layer environment with different properties, we elaborate how the non-Markovian speedup dynamics of the system can be obtained by controlling the coupling in the the hierarchical environment. Besides, it also explains the process that the coupling in the hierarchical environment affects the non-Markovian speedup dynamics of the system when the second-layer environment has different properties.

II. THEORETICAL MODEL

We consider that the entire system consists of a qubit and the hierarchical environment where the cavity \(m_0\) and its corresponding memoryless reservoir \(R_0\) serve as the first-layer environment for the two-level system and the two coupled cavities \(m_1, m_2\) and reservoirs \(R_1, R_2\) involved act as the second-layer environment, as depicted in Fig. 1. More precisely, the two-level system is coupled with strength \(\kappa_0\) to the mode \(m_0\) which decays to a memoryless reservoir \(R_0\) with a lossy rate \(\Gamma_0\) and then the mode \(m_0\) is further coupled with strength \(\kappa\) to the modes \(m_1, m_2\) which decay to their respective reservoirs \(R_1, R_2\) with lossy rates \(\Gamma_{1,2} = \Gamma\). Furthermore, the coupling strength between cavities \(m_1\) and \(m_2\) is \(\Omega\). For the sake of simplicity, we assume that the frequency \(\omega_n\) of the mode \(m_n\) is equal to the qubit transition frequency \(\omega_0\), i.e., \(\omega_n = \omega_0\). The total Hamiltonian is given by \(H = H_0 + H_I\), reads

\[
H_0 = \frac{\omega_0}{2}\sigma_z + \sum_{n=0}^{2} \omega_n b_n^\dagger b_n + \sum_{n=0}^{2} \sum_{k} \omega_{n,k} c_{n,k}^\dagger c_{n,k},
\]

\[
H_I = \kappa_0 (\sigma_+ b_0 + \sigma_- b_0^\dagger) + \sum_{n=1}^{2} \kappa (b_n b_0^\dagger + b_0^\dagger b_n) + \Omega (b_1^\dagger b_2 + b_2^\dagger b_1) + \sum_{n=0}^{2} \sum_{k} g_{n,k} (b_n c_{n,k}^\dagger + b_n^\dagger c_{n,k})\tag{1}
\]

In Eq (1), \(\omega_0\) is the transition frequency of the qubit system, \(\sigma_z\) denote the raising and lowering operators of the qubit, \(b_n^\dagger (b_n)\) is the bosonic creation (annihilation) operators for the mode \(m_n\) with frequency \(\omega_n\), while \(\kappa_0\), \(\kappa\), \(\Omega\) represent the corresponding couplings. Furthermore, \(c_{n,k}^\dagger (c_{n,k})\) is the creation (annihilation) operator of field mode \(k\) with frequency \(\omega_{n,k}\) of reservoir \(R_n\) and \(g_{n,k}\) denotes the coupling of the mode \(m_n\) with the mode \(k\) of its own reservoir \(R_n\). In the interaction picture, Eq. (1) can be written as

\[
H_{int} = \kappa_0 (\sigma_+ b_0 + \sigma_- b_0^\dagger) + \sum_{n=1}^{2} \kappa (b_n b_0^\dagger + b_0^\dagger b_n) + \Omega (b_1^\dagger b_2 + b_2^\dagger b_1) + \sum_{n=0}^{2} \sum_{k} g_{n,k} (b_n c_{n,k}^\dagger + b_n^\dagger c_{n,k}) e^{i\Delta_{n,k}t} + b_n^\dagger c_{n,k} e^{-i\Delta_{n,k}t},\tag{2}
\]

where \(\Delta_{n,k} = \omega_{n,k} - \omega_0\).
The reservoirs $R_1$ and $R_2$ in the second layer environment may be memoryless or memory-keeping. Selecting different types of the reservoir correspond to different methods to obtain the dynamic evolution of the qubit. Then as we all known, if there is no second layer environment of our system, the dynamics of the qubit depends on the parameters $k_0$ and $\Gamma_0$ in such a way that $k_0 < \Gamma_0/4$ ($k_0 > \Gamma_0/4$), identified as the weak (strong) coupling regime. Below, in the weak qubit-m$_0$ coupling regime, we take the second-layer environment with different properties as examples to study the dynamical evolution of the system.

### III. MEMORYLESS NATURE OF THE SECOND-LAYER ENVIRONMENT

In the section, we consider the reservoirs $R_1$ and $R_2$ in the second layer environment are Markovian (memoryless). In other words, the correlation times of the reservoirs $R_1$ and $R_2$ are much smaller than the single-mode relaxation time. In this case, the density operator $\rho$ of the total system is

$$\frac{d\rho}{dt} = -i[H, \rho] - \frac{\Gamma_0}{2}(b_0^\dagger b_0\rho - 2b_0\rho b_0^\dagger + \rho b_0^\dagger b_0)$$

$$- \sum_{n=1}^3 \frac{\Gamma_n}{2}(b_n^\dagger b_n\rho - 2b_n\rho b_n^\dagger + \rho b_n^\dagger b_n),$$

(3)

where $\Gamma_0$ and $\Gamma_n$ denote the dissipation rate of the cavities $m_0$ and $m_n$, respectively. For simplicity, we assume that the qubit is initially in the excited state $|1\rangle_s$ and three modes are in the ground state $|0\rangle_s$, i.e., the total system initial state is $\rho(0) = [1000](0100)$ with $\psi(0) = |0\rangle$. Since there exists at most one excitation in the total system at any time, then at time $t$ the evolutional state of the total system can be written as $|\psi(t)\rangle = a(t)|1000\rangle + c_0(t)|0100\rangle + c_1(t)|0010\rangle + c_2(t)|0001\rangle$, where $a(t)$, $c_0(t)$, $c_1(t)$, $c_2(t)$ correspond to probability amplitudes of the excited state for the atom or the modes $m_0$, $m_1$, $m_2$ with $a(0) = 1$ and $c_0(0) = c_1(0) = c_2(0) = 0$. Besides, the probability amplitudes $a(t)$, $c_0(t)$, $c_1(t)$, $c_2(t)$ are governed by the Hamiltonians in Eq. (1), and determined by a set of differential equations as

$$i\dot{a}(t) = \kappa_0 c_0(t),$$

$$i\dot{c}_0(t) = -\frac{i}{2}\Gamma_0 c_0(t) + \kappa_0 a(t) + \kappa(c_1(t) + c_2(t)),$$

$$i\dot{c}_1(t) = -\frac{i}{2}\Gamma_1 c_1(t) + \kappa c_0(t) + \Omega c_2(t),$$

$$i\dot{c}_2(t) = -\frac{i}{2}\Gamma_2 c_2(t) + \kappa c_0(t) + \Omega c_1(t).$$

(4)

The solutions of the above equations can be obtained by Laplace transformation and Laplace inverse transformation combined with numerical simulation. Then the reduced density matrix of the qubit in the atomic basis $\{|1\rangle_s, |0\rangle_s\}$ can be expressed as

$$\rho^s(t) = \begin{pmatrix} \rho_{11}(0)|a(t)|^2 & \rho_{10}(0)a(t) \\ \rho_{01}(0)a(t)^* & \rho_{00}(0) + \rho_{11}(0)(1 - |a(t)|^2) \end{pmatrix}$$

(5)

where $\rho_{11}(0) = 1$, $\rho_{00}(0) = \rho_{01}(0) = \rho_{10}(0) = 0$.

A measure $\mathbf{N}(\Phi)$ of non-Markovianity based on the distinguishability between the evolutions of two different initial states of the system has been defined by Breuer et al. [39]. For a quantum process $\Phi(t)$, $\rho^s(t) = \Phi(t)\rho^s(0)$, with $\rho^s(0)$ and $\rho^s(t)$ denote the density operators at time $t = 0$ and at any time $t > 0$ of the quantum system, respectively, this suggests defining the measure $\mathbf{N}(\Phi)$ for the non-Markovianity of the quantum process $\Phi(t)$ through $\mathbf{N}(\Phi) = \max_{\rho_{1,2}(0)}\int_{\sigma > 0} dt\sigma[t, \rho_{1,2}^*(0)]$, with $\sigma[t, \rho_{1,2}^*(0)] = \frac{1}{2}\mathcal{D}(\rho_{1,2}^*(t), \rho_{1,2}^*(0))$ is the rate of change of the trace distance. $\mathcal{D}(\rho^1, \rho^2) = \frac{1}{2}\|\rho^1 - \rho^2\|$, where $\|M\| = Tr(\sqrt{M^* M})$ and $0 \leq \mathcal{D} \leq 1$. And $\sigma[t, \rho_{1,2}^*(0)] = 0$ corresponds to all dynamical semigroups and all time-dependent Markovian processes. To evaluate the non-Markovianity, we should find a specific pair of optimal

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{(Color online) (a), (b) The non-Markovianity $\mathbf{N}(\Phi)$ of the atomic system dynamics process from $\rho_0^s = |1\rangle_s\langle 1|$ to $\rho^s_*$ as a function of the coupling strength $\kappa$ between the two layers and the coupling strength $\Omega$ in the second-layer environment with memoryless effects for the weak qubit-m$_0$ coupling regime (i.e., $\kappa_0 = 0.2\Gamma_0$). The parameters are: $\Gamma = \Gamma_0$, $\tau = 4$.}
\end{figure}
initial states \( \rho_{1,2}(0) \) to maximize the time derivative of the trace distance. The states of these optimal pairs must be orthogonal and lie on the boundary of the space of physical states[18]. Through a numerical simulation, it is proven that the optimal state pair of the initial states can be chosen as \( \rho_{1}(0) = (|0\rangle_s + |1\rangle_s)/\sqrt{2} \) and \( \rho_{2}(0) = (|0\rangle_s - |1\rangle_s)/\sqrt{2} \) [19, 20]. Here, for our model, by selecting this optimal state pair, the rate of change of the trace distance can be derived in a simple form as \( \sigma[t, \rho_{1,2}(0)] = \partial_t |a(t)| \). Then the non-Markovianity of the quantum system dynamics process from \( \rho^s(0) \) to \( \rho^s(t) \) can be calculated by \( N(\Phi) = \int_0^t |\partial_t |a(t)|| dt \).

Without the other-layer environment, the system experiences the Markovian dynamics in the weak coupling regime \( (\Gamma_0 > 4\kappa_0) \). In the case of adding the second-layer environment with different properties, the atomic dynamics process from \( \rho_0^s \) to \( \rho_2^s \) would be considered in the weak qubit-m_0 coupling regime (here \( \tau \) is the actual evolution time). Firstly, in the case of the second-layer environment with the memoryless nature, the non-Markovianity of the atom dynamics as function of the controllable hierarchical environment parameters \( (\kappa, \Omega) \) has been plotted in Fig. 2. By fixing \( \Omega/\Gamma_0 \) in Fig. 2(a), a remarkable dynamical crossover from Markovian behavior to non-Markovian behavior can occur at a certain critical coupling strength \( \kappa_c \). When \( \kappa < \kappa_c \), the system exhibits Markovian dynamics behavior, and then the non-Markovianity increases monotonically with increasing \( \kappa/\Gamma_0 \). Differently, the variations of the non-Markovianity can be abundant with respect to the scaled coupling strength \( \Omega/\Gamma_0 \), as shown in Fig. 2(b). For relatively small values of \( \kappa \) (e.g., \( \kappa = 1.8\Gamma_0 \)), the non-Markovianity decreases monotonically with increasing \( \Omega \), and then the dynamics abides Markovian behavior. For larger values of \( \kappa \), the non-Markovianity of the atom dynamics can be nonmonotonic as increasing \( \Omega \). In this case, the non-Markovianity can experience successive decreasing and increasing behaviors with increasing \( \Omega \). For particular values of the coupling strength between the two layers (e.g., \( \kappa = 2\Gamma_0, 2.2\Gamma_0 \)), the non-Markovianity may vanish within a finite interval of \( \Omega \) and revive again, eventually vanishing at relatively large \( \Omega \). In other words, the successive transitions between non-Markovian and Markovian regimes for the system dynamics can be induced by controlling the coupling strength \( \Omega \) in the second-layer environment. Furthermore, by fixing \( \kappa/\Gamma_0 \) in Fig. 2(b), we can be surprised to find that, compared with the absence of coupling in the second-layer environment (i.e., \( \Omega = 0 \)), the introduction of coupling between the modes \( m_1 \) and \( m_2 \) cannot enhance the non-Markovianity of the system. Finally, it needs to be emphasized that, by considering the second-layer environment with the memoryless nature in the weak qubit-m_0 coupling regime, manipulations the coupling strength in the hierarchical environment can trigger the non-Markovian dynamics behavior of the system.

To comprehensively understand the impacts of the hierarchical environment parameters mentioned above in the weak qubit-m_0 coupling regime, Fig. 3 describes the \( \kappa-\Omega \) phase diagrams of the transitions between the non-Markovian and Markovian dynamics. It is clear that, by fixing \( \Omega \), the crossover between Markovian and non-Markovian dynamics can be occurred as increasing \( \kappa \).

**IV. MEMORY-KEEPING NATURE OF THE SECOND-LAYER ENVIRONMENT**

In the previous section, we have considered the qubit is coupled to a mode \( m_0 \) which decays to a memoryless reservoir and then the mode \( m_0 \) is interacting with two coupled modes, \( m_1 \) and \( m_2 \), which are dissipated respectively by two memoryless reservoirs \( R_1 \) and \( R_2 \). We have already known that in the weak qubit-m_0 coupling regime, three modes in the hierarchical environment assume full responsibility for arousing the environmental memory effect. However, it is necessary to study the overall memory effects of the two-level system due to the coupling changes if these three modes are only part of the overall environmental memory. Based on this, in the weak qubit-m_0 coupling regime, we consider the modes \( m_1 \) and \( m_2 \) in the second-layer environment are dissipated by structured reservoirs \( R_1, R_2 \) exhibiting memory effects.

In the following, we assume the qubit is initially in the excited state while the three modes and the corresponding to reservoirs are in their ground state. So the evolvemental initially state of the total system is \( |\varphi(0)\rangle = |1\rangle_s |000\rangle_{m_0m_1m_2} |00\rangle_{R_1} |0\rangle_{R_2} \) with \( \prod_k |0\rangle_{R_k} \). Then the evolution of the total system at time \( t \) can be given as
\[ |\varphi(t)\rangle = h(t)|1\rangle_s|000\rangle_{m_0m_1m_2} + |0\rangle_s c_0(t)|100\rangle + c_1(t)|010\rangle + \frac{2}{\hbar} \sum_{n=0}^{k} \sum_k c_{n,k} |1_k\rangle_{R_n} |\bar{n}\rangle_{R_{\pi,n}}, \]

where \(|1_k\rangle_{R_n} = |00\cdots\bar{1}_k\cdots00\rangle_{R_n}\) denotes that one excitation in the \(k\)th mode of the reservoir \(R_n\) and \(\bar{n}\) is complimentary to \(n\). And when \(t = 0\), \(h(0) = 1, c_0(0) = c_1(0) = c_2(0) = c_{n,k}(0) = 0\). Besides, the non-Hermitian Hamiltonian, which includes the additional terms \(-\frac{i\hbar\alpha}{2}\), is considered to approximate the dissipative effect of the lossy cavity \(m_0\). Put Eq. (6) into the Schrödinger equation, the Hamiltonian of the total system in the interaction picture is determined by the following equations

\[ \begin{align*}
\dot{h}(t) &= -i\hbar\alpha h(t), \\
\dot{c}_0(t) &= -\frac{1}{2} \Gamma_0 c_0(t) - i\hbar\alpha h(t) - i\hbar(c_1(t) + c_2(t)), \\
\dot{c}_1(t) &= -i\hbar\alpha c_0(t) - i\hbar\alpha c_2(t) - ig_{1,k} e^{i\Delta_1,ik} c_1(t), \\
\dot{c}_2(t) &= -i\hbar\alpha c_0(t) - i\hbar\alpha c_1(t) - ig_{2,k} e^{i\Delta_2,ik} c_2(t), \\
\dot{c}_{0,k}(t) &= 0,
\end{align*} \tag{7} \]

Integrate the last two equations above with the initial condition \(c_{n,k} = 0\) and bring the results into the third and fourth equations above, we get the following equation

\[ \begin{align*}
\dot{c}_1(t) &= -i\hbar\alpha c_0(t) - i\hbar\alpha c_2(t) \\
&\quad - \int_0^t \sum_k |g_{1,k}|^2 e^{-i\Delta_1,ik(t-t')} c_1(t') dt', \\
\dot{c}_2(t) &= -i\hbar\alpha c_0(t) - i\hbar\alpha c_1(t) \\
&\quad - \int_0^t \sum_k |g_{2,k}|^2 e^{-i\Delta_2,ik(t-t')} c_2(t') dt'.
\end{align*} \tag{8} \]

The \(\sum_k |g_{n,k}|^2 e^{-i\Delta_n,ik(t-t')}\) in the above equation is recognized as a correlation function \(f_k(t-t')\) of the reservoir \(R_n\) in the second-layer environment. The correlation function \(f_k(t-t') = \int d\omega J_n(\omega) e^{i(\omega_0-\omega)(t-t')}\) is related to the spectral density \(J_n(\omega)\) of the reservoir \(R_n\). Now suppose that the reservoir \(R_n\) \((n = 1, 2)\) has a Lorentzian spectral \(J_n(\omega) = \frac{1}{\pi} \frac{\Gamma_n/2}{(\omega-\omega_n)^2 + \Gamma_n^2}\), where the coupling strength between the mode \(m_0\) and the reservoir \(R_n\) is \(\Upsilon_n\) and the correlation time of the reservoir \(R_n\) is \(\lambda_n^{-1}\). Then the two-point correlation function of the reservoir \(R_n\) in the second-layer environment can be written as \(f_k(t) = \frac{1}{\hbar} \int_0^T \sum_n \Upsilon_n e^{-\lambda_n t}\). The solutions of the amplitude \(h(t)\) can be obtained by solving Eqs. (7) and (8). Then we can analysis the dynamical evolution of the system.

In the weak qubit-\(m_0\) coupling regime, Figs. 4(a) and 4(b) show how the non-Markovianity of the atomic dynamics is affected by the hierarchical environment parameters \(\Omega\) or \(\kappa\) when the second-layer environment has a memory-keeping effect. It is worth noting that, by fixing \(\Upsilon_1 = \Upsilon_2 = \Gamma_0, \lambda_1 = \lambda_2 = 0.1\Gamma_0, \tau = 4\); (b) \(\Upsilon_1 = \Upsilon_2 = \Gamma_0, \lambda_1 = \lambda_2 = \Gamma_0, \tau = 4\). (8). Then we can analysis the dynamical evolution of the system.
when $\kappa = 1.5\Gamma_0$ is relative small in the Fig. 4(b), the non-Markovianity monotonically decreases with increasing $\Omega$, and then the system always remains Markovian dynamics behavior. While $\kappa$ is larger (e.g., $\kappa = 1.7\Gamma_0$, $1.9\Gamma_0$, $2.1\Gamma_0$), the increase of $\Omega$ can induce continuous increase and decrease behaviors of the non-Markovianity. Finally, it is worth noting that, compared with the memoryless effect in the second-layer environment, the introduction of memory-keeping effect gives $\kappa$ a double impact (i.e., the enhancement of $\kappa$ can improve and weaken non-Markovianity).

V. QUANTUM SPEEDUP OF THE ATOMIC DYNAMICS

In this section, we will use the definition of the QSL time for an open quantum system, which can be helpful to analyze the maximal speed of evolution of an open system. The QSL time between an initial state $\rho^s(0) = |0\rangle\langle 0|$ and its target state $\rho^s(\tau)$ (the evolutive state of the system at the actual evolution time $\tau$) for open system is defined by \[ \tau_{QSL} = \frac{2}{k^2} \text{Tr} \left[ \ln \left( \frac{\rho^s(0), \rho^s(\tau)}{\rho^0(0), \rho^0(\tau)} \right) \right] \], where $\Omega(\rho^s(0), \rho^s(\tau)) = \text{arccos} \left( \sqrt{\langle 0 | \rho^s(\tau) | 0 \rangle} \right)$ denotes the Bures angle between the initial and target states of the system, and $\Delta^\infty_{s} = \frac{\tau^{-1}}{\int_0^\infty \| \dot{\rho}^s(t) \|^2 dt}$ with the operator norm $\| \dot{\rho}^s(t) \|^2$ equaling to the largest singular value of $\dot{\rho}^s(t)$. $\tau_{QSL}/\tau = 1$ means the quantum system evolution is already along the fastest path and possesses no potential capacity for further quantum speedup. While for the case $\tau_{QSL}/\tau < 1$, the speedup evolution of the quantum system may occur and the much shorter $\tau_{QSL}/\tau$, the greater the capacity for potential speedup will be.

In the light of Eq. 5, the relationship between non-Markovianity and the QSL times has been given by $\tau_{QSL} = \frac{2}{k^2} \text{Tr} \left[ \ln \left( \frac{\rho^s(0), \rho^s(\tau)}{\rho^0(0), \rho^0(\tau)} \right) \right]$. The above equation means that the larger the non-Markovianity would lead to the lower the QSL times (that is to say, the greater the capacity for potential speedup could be). In this model, by the controllable non-Markovianity discussed above, the speedup of the quantum system can also be achieved. Below, in the case of the second-layer environment with the different properties, we mainly focus on the influence of the coupling in the hierarchical environment on the dynamical speedup of quantum system in the weak qubit-$m_0$ coupling regime.

For the second-layer environment with memoryless nature in the weak qubit-$m_0$ coupling regime, the QSL time $\tau_{QSL}/\tau$ as a function of the coupling strength $\Omega$ and $\kappa$ have been plotted in Figs. 5 (a) and (b). By fixing $\Omega = \Gamma_0$ in the Fig. 5(a), the no-speedup evolution ($\tau_{QSL}/\tau = 1$) could be followed, and the speedup evolution ($\tau_{QSL}/\tau < 1$) would occur when the coupling strength $\kappa$ is larger than a certain critical coupling strength $\kappa_{c1}$. As for Fig. 5(b), by fixing $\kappa = 1.8\Gamma_0$, the speedup evolution of the system can be induced by decreasing $\Omega$. However, in the case $\kappa = 2.4\Gamma_0$, no matter how we adjust the parameter $\Omega$, the system always remains speedup evolution. Besides, when the coupling strength $\kappa$ takes the particular values (e.g., $\kappa = 2\Gamma_0$, $2.2\Gamma_0$), the quantum system can experience successive transforms from speedup evolution to no-speedup evolution as increasing $\Omega$. This means the speed of evolution for the system can be controlled to a speed-up or speed-down process by manipulating the coupling strength $\Omega$. So in the case of the second-layer environment with the memoryless nature in the weak qubit-$m_0$ coupling regime, the purpose of accelerating evolution can be achieved by controlling the coupling strength in the hierarchical environment.

For the second-layer environment with memory-keeping nature in the weak qubit-$m_0$ coupling regime, the variation of the QSL time with respect to $\kappa$ and $\Omega$ have been shown in Figs. 5 (c) and (d). When the value $\Omega$ is confirmed in Fig. 5(c), in the case $\kappa < \kappa_{c2}$ ($\kappa_{c2}$ means the critical value of $\kappa$), the QSL time is always equal to the actual evolution time, but the QSL time can be nonmonotonic with increasing $\kappa$ when $\kappa > \kappa_{c2}$. That is to say, by manipulating $\kappa$, the evolution of the system can be accelerated and its evolution speed can also be controlled. Besides, by confirming $\kappa = 1.5\Gamma_0$ in Fig. 5(d), the QSL time monotonically increases as increasing $\Omega$ and then remains at the actual evolution time. This means that the speedup evolution may appear as decreasing $\Omega$. However, in the case $\kappa = 1.7\Gamma_0$, $1.9\Gamma_0$, $2.1\Gamma_0$, the QSL time monotonically increases with increasing
VI. CONCLUSION

In conclusion, for the weak qubit-$m_0$ coupling regime, we have investigated the dynamics behavior of the qubit in a controllable hierarchical environment where the first-layer environment is a mode $m_0$ which decays to a memoryless reservoir and the second-layer environment is two coupled modes $m_1$ and $m_2$ which decay to memoryless or memory-keeping reservoirs. In the case of the memoryless nature of the second-layer environment, the three modes ($m_0, m_1, m_2$) can be regarded as the memory source of the overall environment. In the case of the memory-keeping nature of the second-layer environment, the three modes can only be a part of the total memory source of the overall environment. In the above two cases, by controlling the coupling strength $\kappa$ between the two layers and the coupling strength $\Omega$ in the second-layer environment, two dynamical crossovers of the quantum system, from Markovian to non-Markovian dynamics and from no-speedup evolution to speedup evolution, have been achieved in the weak qubit-$m_0$ coupling regime. And it is worth noting that, the coupling in the second-layer environment can stimulate the multiple transitions from Markovian to non-Markovian dynamics and from no-speedup evolution to speedup evolution. This results hold independently on the nature of the second-layer environment. Besides, we also can be surprised to find that, compared with the absence of the coupling between the modes $m_1$ and $m_2$, the introduction of coupling in the second-layer environment cannot play a beneficial role on the non-Markovian speedup dynamics behavior of the system. To further illustrate the reasons behind the above results, we try to give a discussion for this problem based on the competitive relationship between $\kappa$ and $\Omega$. To be concrete, when the coupling strength $\Omega$ plays the dominating role in the evolution of system, the non-Markovian speedup dynamics behavior of the system is inhibited and therefore eventually leads to Markovian no-speedup dynamics behavior. While $\kappa$ acts mainly, the non-Markovian dynamics behavior of the system can be activated. So the alternating effects of the coupling strength $\kappa$ and $\Omega$ on the dynamic behaviors of the system can lead to the increasing and decreasing behaviors of the non-Markovianity. Furthermore, it also explains how the coupling between the two layers affects the non-Markovian speedup dynamics behavior depends on the nature of the second-layer environment. For the memoryless nature of the second-layer environment, the non-Markovianity and the capacity for potential speedup of the system become greater as increasing $\kappa$. Differently, when the second-layer environment has a memory-keeping nature, the non-Markovianity and the capacity for potential speedup of the system can be improved and weakened as increasing the coupling strength $\kappa$ between the two layers.

VII. ACKNOWLEDGEMENTS

This work was supported by NSFC under grants Nos. 11574022, 11434015, 61227902, 61835013, 11611530676, KZ201610005011, the National Key R&D Program of China under grants Nos. 2016YFA0301500, SFRPCAS under grants No. XDB01020300, XDB21030300.

K. Xu and G. F. Zhang contributed equally to this work.

[1] F. Caruso, V. Giovannetti, C. Lupo, and S. Mancini, Rev. Mod. Phys. 86, 1203 (2014).
[2] R. Lo Franco, B. Bellomo, S. Maniscalco, and G. Compagno, Int. J. Mod. Phys. B 27, 1345053 (2013).
[3] A. Rivas, S. F. Huelga, and M. B. Plenio, Rep. Prog. Phys. 77, 094001 (2014).
[4] H. Lee, Y. C. Cheng, and G. R. Fleming, Science 316, 1462 (2007).
[5] L. S. Cederbaum, E. Gindensperger, and I. Burghardt, Phys. Rev. Lett. 94, 113003 (2005).
[6] H. P. Breuer and F. Petruccione, Theory of Open Quantum Systems (Oxford University Press, New York, 2002).
[7] I. de Vega and D. Alonso, Rev. Mod. Phys. 89, 015001 (2017).
[8] H. P. Breuer, E. M. Laine, J. Piilo, and B. Vacchini, Phys. Rev. A 88, 021002 (2016).
[9] S. Lorenzo, F. Plastina, and M. Paternostro, Phys. Rev. A 88, 020102(R) (2013).
[10] A. Rivas, S. F. Huelga, and M. B. Plenio, Phys. Rev. Lett. 105, 050403 (2010).
[11] F. Caruso, V. Giovannetti, C. Lupo, and S. Mancini, Rev. Mod. Phys. 86, 1203 (2014).
[12] E. M. Laine, J. Piilo, and H. P. Breuer, Europhys. Lett. 92, 60010 (2010).
[13] J. Dajka, and J. Luczka, Phys. Rev. A 82, 012341 (2010).
[14] A. Smirne, H. P. Breuer, J. Piilo, and B. Vacchini, Phys. Rev. A 82, 062114 (2010).
[15] S. Luo, S. Fu, and H. Song, Phys. Rev. A 86, 064410 (2012).
[16] X. M. Lu, X. Wang, and C. P. Sun, Phys. Rev. A 82, 042103 (2010).
[17] D. Chruscinski, and S. Maniscalco, Phys. Rev. Lett. 112, 120404 (2014); C. Addis, B. Bylicka, D. Chruscinski, and S. Maniscalco, Phys. Rev. A 90, 052103 (2014).
[18] S. Wilfmann, A. Karlsson, E. M. Laine, J. Piilo, and H. P. Breuer, Phys. Rev. A 86, 062108 (2012).
[19] J. G. Li, J. Zou, and B. Shao, Phys. Rev. A 81, 062124 (2010).
