Modular Nuclearity and Entanglement Measures

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Abstract

In the framework of Algebraic Quantum Field Theory, several operator algebraic notions of entanglement entropy can be associated to any couple of causally disjoint and distant spacetime regions \(S_A\) and \(S_B\). One of them, known as canonical entanglement entropy, is defined as the von Neumann entropy on some canonical intermediate type I factor. In this work we show the canonical entanglement entropy of the vacuum state to be finite on a wide family of conformal nets including the \(U(1)\)-current model and the \(SU(n)\)-loop group models. More in general, such a finiteness property is expected to rely on some nuclearity condition of the system. To support this conjecture, we show that the mutual information is finite in any local QFT verifying a modular \(p\)-nuclearity condition for some \(0 < p < 1\). A similar result is proved for another entanglement entropy introduced in this work. We conclude with some personal considerations on 1 + 1-dimensional integrable models with factorizing S-matrices and remarks for future works in this direction.

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1 Introduction

In classical information theory, a standard notion is that of Shannon entropy, a quantity that measures the amount of information carried by a given state of a system. Its quantum mechanical analogue was defined by and named after von Neumann. In this context, the von Neumann entropy is connected to the quantum phenomenon of entanglement. Entanglement occurs on a composite system when a state is not the product of two independent states on the subsystems. Entanglement has been investigated profoundly as a means of probing the foundations of quantum mechanics (as in the EPR paradox and Bell’s inequalities) as well as a resource for quantum information theory. On type I factors, the von Neumann entropy of a state \(\varphi\) with density matrix \(\rho\) is

\[
S(\varphi) = -\text{tr} \rho \log \rho.
\]

However, in the axiomatic approach to Quantum Field Theory (QFT) utilizing Haag-Kastler nets one can show under very general conditions that the algebras of observables are type III von Neumann algebras [15], where no density matrices exist and a different entropy-type functional is required. The role of entanglement in QFT is more recent and increasingly important [25]. It appears in relations with several primary research topics in theoretical physics as area theorems [17], c-theorems [9] and quantum null energy inequalities [27, 32].

In the study of entropy in local QFT, nuclearity conditions play an important role [17, 28, 31]. These conditions are predicated on the compactness criterion proposed by Haag and Swieca [10]. This criterion

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is an abstraction of the insight that a QFT model must have a bound on the number of its local degrees of freedom to show regular thermodynamical behaviour. Based on similar heuristic arguments, Buchholz and Wichmann strengthened this assumption and suggested the first nuclearity condition \([5]\), which nowadays is known as energy nuclearity condition. Explicitly, let \( \mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subseteq B(\mathcal{H}) \) be some local Haag-Kastler net describing a local QFT on the Minkowski space. Denote by \( \Omega \) the vacuum vector and by \( \omega \) the corresponding vacuum state. One says that the energy nuclearity condition holds if

\[
\Theta_{\beta, \mathcal{O}}: \mathcal{A}(\mathcal{O}) \mapsto \mathcal{H}, \quad \Theta_{\beta, \mathcal{O}}(a) = e^{-\beta H_0}\Omega,
\]

is nuclear for any bounded region \( \mathcal{O} \) and any inverse temperature \( \beta > 0 \), where \( H = P_0 \) denotes the Hamiltonian with respect to the time direction \( x_0 \). The nuclear norm of (1) can be interpreted as the partition function of the restricted system at some fixed inverse temperature, hence it is natural to expect such nuclear norms to measure the entropy of the state. At a later time, a related nuclearity condition has been found by use of modular theory \([6, 7]\). According to this second nuclearity condition, one considers an inclusion \( \mathcal{O} \subset \tilde{\mathcal{O}} \) of spacetime regions and requires the map

\[
\Xi: \mathcal{A}(\mathcal{O}) \mapsto \mathcal{H}, \quad \Xi(x) = \Delta^{1/4}\Omega,
\]

to be nuclear, with \( \Delta \) the modular operator of the bigger local algebra \( \mathcal{A}(\tilde{\mathcal{O}}) \) with respect to \( \Omega \). In this case one speaks of modular nuclearity condition. If the map (2) is \( p \)-nuclear then one will say that the modular \( p \)-nuclearity condition is satisfied. If modular \( p \)-nuclearity holds for some \( 0 < p \leq 1 \) then the modular nuclearity condition is satisfied, and if so then it is well known that the split property holds, namely there is an intermediate type I factor \( \mathcal{A}(\mathcal{O}) \subset \mathcal{F} \subset \mathcal{A}(\tilde{\mathcal{O}}) \) \([6, 7]\).

The split property is an expression of the statistical independence of two spacelike separated regions. It is considerably stronger than (Einstein) causality, which is included in the Haag-Kastler axioms. While causality expresses the independence of measurement apparatuses located in spacelike separated regions, the split property additionally takes the stage of preparation into account. It signifies that no choice of a state prepared in one region can prevent the preparation of any state in the other region. Among all intermediate type I factors, a canonical one can be chosen by using standard representation arguments \([13]\). When the split property holds, one natural entanglement measure to consider is the von Neumann entropy on the canonical intermediate type I-factor \([25]\), namely what we here call canonical entanglement entropy. As \( \omega \) is pure on \( B(\mathcal{H}) = \mathcal{F} \vee \mathcal{F}' \), the canonical entanglement entropy is given by

\[
E_C(\omega) = S_{\mathcal{F}}(\omega) = S_{\mathcal{F}'}(\omega).
\]

In this work we extend some results of \([25]\) and we show this entanglement entropy to be finite on a wide family of conformal nets including the \( U(1) \)-current. In order to do so, we first notice where the proof of \([25]\) can be extended and we then construct a vacuum preserving conditional expectation between canonical intermediate type I factors.

More in general, it is a common belief that this entanglement entropy can be bounded from above by assuming some nuclearity property of the system. A result of this type is strongly suggested by previous works \([25, 28, 31]\) and shows interesting applications in AdS/CFT contexts as pointed out in \([14]\), where it appeared as splitting entropy and it was used to conjecture a bound on the reflected entropy. This type of considerations can be extended to a wide family of entanglement measures defined on a generic bipartite system \( A \otimes B \) like the mutual information \([17]\)

\[
E_I(\omega) = S(\omega||\omega_A \otimes \omega_B),
\]

which quantifies how much knowledge on the second system one can gain from measuring the first one. In this work we prove a relation between the mutual information and the modular \( p \)-nuclearity condition.

The paper is organised as follows. In \section{section 2}, \section{section 3} and \section{section 4} we collect various notions about entropy, entanglement and modular nuclearity. In \section{section 5} we apply the theory of standard
representations to construct a cpu map between canonical intermediate type I factors, with a focus on twisted-local nets on the circle. We then use this construction to extend [25]. In section 6 we study the connection between modular \( p \)-nuclearity conditions and certain entanglement measures. In particular, we show that the modular \( p \)-nuclearity with \( 0 < p < 1 \) implies the finiteness of the mutual information and of one more entanglement measure inspired by [30]. We also add remarks concerning area laws [17]. Finally, in section 7 we apply these considerations to a family of 1 + 1-dimensional integrable models with factorizing S-matrices [20] and investigate the asymptotic behaviour of different entanglement measures as the distance between two causally disjoint wedges diverges. In section 8 we present a few remarks that might be useful for future research in this direction.

2 Quantum entropy basics

Let \( \mathcal{M} \) be a von Neumann algebra in standard form on some Hilbert space \( \mathcal{H} \) and let \( \varphi, \psi \) be two normal positive linear functionals on \( \mathcal{M} \) represented by the vectors \( \xi \) and \( \eta \). Denote by \( s(\varphi) = [\mathcal{M}^\prime \xi] \) and \( s(\psi) = [\mathcal{M}^\prime \eta] \) the central supports of \( \varphi \) and \( \psi \) respectively. We define the Tomita relative operator

\[
S_{\xi,\eta}(x\eta + \zeta) = s(\psi)x^*\xi, \quad x \in \mathcal{M}, \quad \zeta \in [\mathcal{M}\eta]^\perp.
\]

This densely defined conjugate-linear operator is closable. Its closure will be equally denoted and its polar decomposition is given by

\[
S_{\xi,\eta} = J_{\xi,\eta}\Delta_{\xi,\eta}^{1/2},
\]

where \( \text{supp} \Delta_{\xi,\eta} = s(\varphi)s'(\psi) \), \( J_{\xi,\eta}J_{\xi,\eta}^* = s(\varphi)s'(\psi), \quad J_{\xi,\eta}J_{\xi,\eta}^* = s'(\varphi)s(\psi). \)

In the case \( \xi = \eta \) we will write \( S_{\xi} = S_{\xi,\xi} \), and similarly \( J_{\xi} = J_{\xi,\xi} \) and \( \Delta_{\xi} = \Delta_{\xi,\xi} \). If \( \xi \) and \( \eta \) are both in the natural cone \( \mathcal{P}_2 \), then we also have the polar decomposition

\[
S_{\xi,\eta} = J\Delta_{\xi,\eta}^{1/2},
\]

with \( J \) the modular conjugation of the standard form. We recall that if \( \varphi \) is faithful then \( \xi \) is cyclic, and if so then \( \mathcal{P}_2 = \mathcal{P}_2^{\prime} \) where

\[
\mathcal{P}_2^{\prime} = \left\{ \Delta_{\xi}^{1/4}x^*x\xi, \ x \in \mathcal{M} \right\} = \left\{ xJx^*J\xi, \ x \in \mathcal{M} \right\}.
\]

Finally, by using primes to denote the modular operators of the commutant, we have the identities

\[
J_{\xi,\eta}\Delta_{\xi,\eta}^{1/2}J_{\xi,\eta} = \Delta_{\eta,\xi}^{-1/2}, \quad J_{\xi,\eta}^* = J_{\eta,\xi}, \quad (\Delta_{\eta,\xi})^z = \Delta_{\xi,\eta}^z.
\]

The relative entropy between \( \varphi \) and \( \psi \) is defined by

\[
S(\varphi\|\psi) = -(\xi|\log \Delta_{\eta,\xi}\xi)
\]

if \( s(\varphi) \leq s(\psi) \), otherwise \( S(\varphi\|\psi) = +\infty \) by definition. Equation (4) does not depend on the choice of the representing vectors. If \( \mathcal{M} \) is not in standard form, then equation (4) holds if the relative modular operator is replaced with a spatial derivative [30].

The scalar product (4) has to be intended by applying the spectral theorem to the relative modular operator \( \Delta_{\eta,\xi} \), namely we have

\[
S(\varphi\|\psi) = -\int_{0}^{1} \log \lambda \, d(E_{\eta,\xi}(\lambda)\xi) - \int_{1}^{\infty} \log \lambda \, d(E_{\eta,\xi}(\lambda)\xi),
\]

where the second integral is always finite by the estimate \( \log \lambda \leq \lambda \). In particular, \( S(\varphi\|\psi) \) is finite if and only if the first integral appearing in (5) is finite. By this remark it follows that [30]

\[
S(\varphi\|\psi) = \frac{d}{dt} \varphi( (D\psi: D\varphi)_{t} ) \bigg|_{t=0} = -i\frac{d}{dt} \varphi( (D\psi: D\psi)_{t} ) \bigg|_{t=0},
\]

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where \((D\varphi| D\psi)_t = (D\psi| D\varphi)^*_t\) is the Connes cocycle. Identity \([31]\) can be proved by using the dominated convergence theorem if \(S(\varphi\|\psi)\) is finite and by the Fatou’s lemma if \(S(\varphi\|\psi) = +\infty\) as shown in \([10]\). Let us now recall some properties of the relative entropy \([30]\).

(i) \(S(\varphi\|\psi) \geq \varphi(I)(\log \varphi(I) - \log \psi(I))\), and \(S(\lambda\varphi\|\mu\psi) = \lambda S(\varphi\|\psi) - \lambda \varphi(I) \log(\mu/\lambda)\) for any \(\lambda, \mu \geq 0\).

Moreover, \(S(\varphi\|\psi) \geq \|\varphi - \psi\|^2/2\), so that \(S(\varphi\|\psi) = 0\) if and only if \(\varphi = \psi\).

(ii) \(S(\varphi\|\psi)\) is lower semi-continuous in the \(\sigma(\mathcal{M}_*, \mathcal{M})\)-topology.

(iii) \(S(\varphi\|\psi)\) is convex in both its variables. By (i) this is equivalent to the subadditivity of \(S(\varphi\|\psi)\) in both its variables.

(iv) \(S(\varphi\|\psi)\) is superadditive in its first argument. Furthermore, \(S(\varphi\|\psi) \leq S(\varphi'|\psi')\) if \(\psi \geq \psi'\) and \(\varphi \geq \varphi'\) with \(\|\varphi\| = \|\varphi'\|\).

(v) If \(\alpha: \mathcal{M}_1 \to \mathcal{M}_2\) is a Schwarz mapping such that \(\varphi_2 \cdot \alpha \leq \varphi_1\) and \(\varphi_2 \cdot \alpha \leq \psi_1\), then \(S_{\mathcal{M}_1}(\varphi_1\|\psi_1) \leq S_{\mathcal{M}_2}(\varphi_2\|\psi_2)\). In particular, \(S(\varphi\|\psi)\) is monotone increasing with respect to inclusions of von Neumann algebras.

(vi) Let \((\mathcal{M}_i)\), be an increasing net of von Neumann subalgebras of \(\mathcal{M}\) with the property \(\bigcup_i \mathcal{M}_i = \mathcal{M}\). Then the increasing net \(S_{\mathcal{M}_i}(\varphi\|\psi)\) converges to \(S(\varphi\|\psi)\).

(vii) Let \(\varepsilon: \mathcal{M} \to \mathcal{N}\) be a faithful normal conditional expectation. If \(\varphi\) and \(\psi\) are normal states on \(\mathcal{M}\) and \(\mathcal{N}\) respectively, then \(S_{\mathcal{M}}(\varphi\|\psi \cdot \varepsilon) = S_{\mathcal{N}}(\varphi\|\psi) + S_{\mathcal{M}}(\varphi\|\varphi \cdot \varepsilon)\).

(viii) Let \(\varphi\) be a normal state on the spatial tensor product \(\mathcal{M}_1 \otimes \mathcal{M}_2\) with partials \(\varphi_i = \varphi|_{\mathcal{M}_i}\). Consider then normal states \(\psi_i\) on \(\mathcal{M}_i\). As a corollary of (vi), we have \(S(\varphi\|\psi_1 \otimes \psi_2) = S(\varphi_1\|\psi_1) + S(\varphi_2\|\psi_2) + S(\varphi\|\psi_1 \otimes \psi_2)\).

By using the universal representation, the relative entropy can be defined on a generic \(C^*\)-algebra. If we replace the strong closure with the norm closure in (i) and the \(\sigma(\mathcal{M}_*, \mathcal{M})\)-topology with the weak topology in (i), then properties from (i) to (v) still hold in the \(C^*\)-algebraic setting \([30]\). As shown in \([30]\), if \(\psi\) is a positive normal functional of \(\mathcal{M}\) and \(t \in \mathbb{R}\), then the sublevel

\[
\mathcal{K}(\psi, t) = \{\varphi \in \mathcal{M}_+^* : S(\varphi\|\psi) \leq t\}
\]

consists of normal functionals and it is a convex compact set with respect to the \(\sigma(\mathcal{M}_*, \mathcal{M})\)-topology. The following lemma is original.

**Lemma 1.** \(\mathcal{K}(\psi, t)\) is sequentially \(\sigma(\mathcal{M}_*, \mathcal{M})\)-compact and its set of extremal points is

\[
\mathcal{E}(\psi, t) = \{\varphi \in \mathcal{M}_+^* : S(\varphi\|\psi) = t\}.
\]

Moreover, after a restriction to \(\mathcal{M}_{s(\psi)}\), the union \(\mathcal{K}(\psi) = \bigcup_t \mathcal{K}(\psi, t)\) is norm dense in the set of normal positive functionals of the reduced algebra \(\mathcal{M}_{s(\psi)}\).

**Proof.** The first two claims follow from the Eberlein-Smulian theorem and Donald’s identity \([30], Proposition 5.23\). The last point holds if \(\psi\) is faithful, since in this case the set of positive normal functionals \(\varphi\) such that \(\varphi \leq \alpha \psi\) for some \(\alpha > 0\) is norm dense in \(\mathcal{M}_+^*\) \([31], Theorem 2.3.19\). The general case follows by noticing that \(S_{\mathcal{M}}(\varphi\|\psi) = S_{\mathcal{M}_{s(\psi)}}(\varphi\|\psi)\) if \(s(\varphi) \leq s(\psi)\), which is a necessary condition for \(S_{\mathcal{M}}(\varphi\|\psi)\) to be finite.

**Definition 2.** If \(\varphi\) is a state on a \(C^*\)-algebra \(A\), then the **von Neumann entropy** of \(\varphi\) is defined by

\[
S_A(\varphi) = \sup \left\{ \sum_i \lambda_i S(\varphi_i\|\varphi) : \sum_i \lambda_i \varphi_i = \varphi \right\},
\]

where the supremum is over all decompositions of \(\varphi\) into finite (or equivalently countable) convex combinations of other states. If \(A\) is clear, we will simply write \(S_A(\varphi) = S(\varphi)\).
Some properties of $S(\varphi)$ are immediate from those of the relative entropy: $S(\varphi)$ is nonnegative, vanishes if and only if $\varphi$ is a pure state and it is weakly lower semicontinuous. On type I factors, the von Neumann entropy of a normal state $\varphi$ with density matrix $\rho$ is given by $S(\varphi) = -\text{tr} \rho \log \rho$. We now list a few properties of the von Neumann entropy \[30\]. The notation $\eta(t) = -t \log t$ is standard in information theory.

(s0) (concavity) Given states $\varphi$ and $\omega$, then $\lambda S(\varphi) + (1 - \lambda)S(\omega) \leq S(\lambda \varphi + (1 - \lambda) \omega) \leq \lambda S(\varphi) + (1 - \lambda)S(\omega) + \eta(\lambda) + \eta(1 - \lambda)$ for each $0 < \lambda < 1$.

(s1) (strong subadditivity) On a three-fold-product $B(\mathcal{H}_1) \otimes B(\mathcal{H}_2) \otimes B(\mathcal{H}_3)$, a normal state $\omega_{123}$ with marginal states $\omega_{ij}$ satisfies $S(\omega_{123}) + S(\omega_2) \leq S(\omega_{12}) + S(\omega_{23})$.

(s2) $S(\psi) = \inf \{ -\sum \eta(\lambda_i) \}$, where the infimum is taken over all the possible decompositions into pure states.

(s3) (tensor product) On the projective tensor product $A \otimes B$, we have the identity $S(\varphi_1 \otimes \varphi_2) = S(\varphi_1) + S(\varphi_2)$.

**Definition 3.** Consider an inclusion of $C^*$-algebras $A \subseteq B$ and a state $\varphi$ on $B$. The *entropy of $\varphi$ with respect to $A$* is

$$H_\varphi^B(A) = \sup \left\{ \sum \lambda_i S_A(\varphi_i \| \varphi) : \varphi = \sum \lambda_i \varphi_i \right\},$$

where the supremum is over all finite (countable) convex decompositions $\varphi = \sum \lambda_i \varphi_i$ on $B$. If the bigger $C^*$-algebra $B$ is clear, we will briefly use the notation $H_\varphi^B(A) = H_\varphi(A)$.

The entropy of a subalgebra is actually a particular case of what is known as *conditional entropy*. We list a few of its properties \[11, 30\].

(c0) (monotonicity) $H_\varphi^B(A_1) \leq H_\varphi^B(A_2)$ if $A_1 \subseteq A_2 \subseteq B_1 \subseteq B_2$.

(c1) (semincontinuity) $\varphi \mapsto H_\varphi^B(A)$ is weakly lower semicontinuous.

(c2) (martingale property) $\lim H_\varphi^B(A_i) = H_\varphi^B(A)$ if $(A_i)_i$ is an increasing net of $C^*$-subalgebras of $B$ with union norm dense in $A$.

(c3) (concavity) $\lambda H_\varphi^B(A) + (1 - \lambda) H_\varphi^B(A) \leq H_\varphi^B(A) \leq \lambda H_\varphi^B(A) + (1 - \lambda) H_\varphi^B(A) + \eta(\lambda) + \eta(1 - \lambda)$ for $\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2$ on $B$ and $\lambda$ in $(0, 1)$.

In (c2), the union $\cup_i A_i$ can be strongly dense if all the $C^*$-algebras are replaced with von Neumann algebras and the state $\varphi$ is normal. We point out that the concavity of $H_\varphi(A)$ mentioned in (c3) certainly holds whenever $A$ is AF \[30, Theorem 5.29 and Proposition 10.6\], but the general case is a bit unclear to the authors \[11\]. What is clear instead, is the following original simple lemma which says whenever the inequality (c0) reduces to an equality in the case $A_1 = A_2$.

**Lemma 4.** Consider the $C^*$-algebra inclusions $A_1 \subseteq B_1 \subseteq B_2$ and $A_1 \subseteq A_2 \subseteq B_2$. Let $\varphi$ be a state on $B_2$. If there is a $\varphi$-preserving conditional expectation $\varepsilon : B_2 \to B_1$, then

$$H_\varphi^{B_1}(A_1) \leq H_\varphi^{B_2}(A_2).$$

**Proof.** We can follow Proposition 6.7 of \[30\]. Indeed, if $\psi$ is a state of $B_1$ then $\psi \cdot \varepsilon$ is a state of $B_2$. Therefore, if $\varphi = \sum \lambda_i \varphi_i$ on $B_1$ for some states $\varphi_i$ of $B_1$ then $\varphi = \sum \lambda_i \varphi_i \cdot \varepsilon$ is a decomposition of $\varphi$ into states of $B_2$. The rest follows from $S_A(\varphi \| \varphi) \leq S_A(\varphi_i \cdot \varepsilon \| \varphi_i)$.

In general, the existence of some $\varphi$-preserving conditional expectation between von Neumann algebras is not always true. A well known necessary and sufficient condition has been provided by Takesaki \[34, Theorem IX.4.2\]. However, if it exists then it satisfies the following structure theorem, which is a partial generalization of Proposition 3.1.4 and Proposition 3.1.5 of \[18\] to the non-tracial case.
Theorem 5. Let $\mathcal{M}$ be a von Neumann algebra in standard form on $\mathcal{H}$. Let $\mathcal{N}$ be a von Neumann subalgebra of $\mathcal{M}$, $\varphi$ a normal faithful state of $\mathcal{M}$ and $\xi$ the unique vector in the natural cone representing $\varphi$. If there exists a $\varphi$-preserving conditional expectation $\varepsilon: \mathcal{M} \to \mathcal{N}$, then there is a unique projection $e$ in $\mathcal{N}'$ such that

(i) $ex\xi = \varepsilon(x)\xi$ and $exe = \varepsilon(x)e$ for $x$ in $\mathcal{M}$.

Furthermore,

(ii) $\mathcal{N}e = e(\mathcal{M} \vee e)e$,

(iii) $\mathcal{N}' = \mathcal{M}' \vee e$,

(iv) there is a $*$-isomorphism $\phi: \mathcal{N}e \to \mathcal{N}$ such that $\varepsilon(x) = \phi(exe)$ for $x$ in $\mathcal{M}$.

Proof. (i) Recalling that $\xi$ is standard for $\mathcal{M}$, one can first show the map $x\xi \mapsto \varepsilon(x)\xi$ to be an orthogonal projection onto $[\mathcal{N}\xi]$ since $\varepsilon$ is a $\varphi$-preserving conditional expectation. It is then easy to notice that $e$ belongs to $\mathcal{N}'$, and the identity $exe = \varepsilon(x)e$ follows. The uniqueness follows by construction. (ii) This follows from $\mathcal{N}e = e(\mathcal{M} \vee e)e$, which is a consequence of (i) as well. (iv) We show the map $y \mapsto ye$ for $y$ in $\mathcal{N}$ to be injective and hence an isomorphism. But if $ye = 0$ then $ey^* = 0$, hence $\varphi(y^*y) = 0$ which implies $y = 0$ by faithfulness. □

Remark 6. If we denote by $s(\varphi)$ the central support of $\varphi$ and we assume $\mathcal{N}$ to be a von Neumann subalgebra of $s(\varphi)M(s(\varphi))$, then the previous theorem still holds in the nonfaithful case.

Corollary 7. Let $\mathcal{M}, \mathcal{N}, \varphi, \xi$ and $\varepsilon$ be as in the previous theorem. If $P^k_{\mathcal{M}}$ and $P^k_{\mathcal{N}}$ are the natural cones of $\mathcal{M}$ and $\mathcal{N}$ respectively, then

$$eP^k_{\mathcal{M}} = P^k_{\mathcal{N}} \subseteq P^k_{\mathcal{M}}.$$  

In particular, the elements of $P^k_{\mathcal{N}}$ correspond to the $\varepsilon$-invariant normal positive functionals of $\mathcal{M}$.

Proof. Let $S = J\Delta^{1/2}$ be the modular operator of $\mathcal{M}$ with respect to $\xi$. By the previous theorem we have $\varepsilon(x^*) = \varepsilon(x)^*$ and hence $eS = Se$, which implies that $e\Delta = \Delta e$ is the modular operator of $\mathcal{N}e$. Similarly, $eJ = Je$ is the modular conjugation of $\mathcal{N}e$. The thesis follows from the first identity of equation (3), where the last claim is a consequence of the uniqueness of the representative vector. □

3 Entanglement Measures

In this section we discuss entanglement in a general setting and we review some quantitative measures of entanglement and their properties [17].

Let $A, B$ be a couple of commuting von Neumann algebras on some Hilbert space $\mathcal{H}$. We shall say that the pair $(A, B)$ is split if there exists a von Neumann algebra isomorphism $\phi: A \vee B \to A \otimes B$ such that $\phi(ab) = a \otimes b$. We will refer to split pairs also as bipartite systems. In quantum field theory, bipartite systems are associated to causally disjoint regions. Clearly the spatial tensor product $A \otimes B$ has a natural structure of bipartite system since $A \cong A \otimes 1$ and $B \cong 1 \otimes B$. If $A \vee B$ is $\sigma$-finite, then the pair $(A, B)$ is split if and only if for any given normal states $\varphi_A$ on $A$ and $\varphi_B$ on $B$ there exists a normal state $\varphi$ on $A \vee B$ such that $\varphi(ab) = \varphi_A(a)\varphi_B(b)$ [23].

A state $\omega$ on $A \otimes B$ is said to be separable if there are positive normal functionals $\varphi_j$ on $A$ and $\psi_j$ on $B$ such that $\omega = \sum_j \varphi_j \otimes \psi_j$, where the sum is assumed to be norm convergent. Separable states are normal and satisfy the following original lemma inspired by [29].

Lemma 8. Given two von Neumann algebras $A$ and $B$, consider a state $\omega$ on the bipartite system $A \otimes B$. If $\omega$ is separable, then

$$S_A(\omega) = H^A_{\omega \otimes B}(A).$$

Proof. If $\omega = \sum_j \phi_j \otimes \psi_j$ and $\pi$ is the GNS representation of $A$ associated to the marginal state $\omega_A = \omega|_A$, then we have an equivalence of GNS representations $\pi \cong \oplus_j \pi_j$, where $\pi_j$ is the GNS representation of $A$ given by $\psi_j(1)\phi_j$. We can define a cpu map $\varepsilon_j: A \otimes B \to \pi_j(A)$ by $\varepsilon_j(a \otimes b) = \pi_j(a)\psi_j(b)/\psi_j(1)$, and this
lead us to define a conditional expectation \( \varepsilon : A \otimes B \to pAp \) by \( \varepsilon = \pi^{-1} \cdot \oplus_j \varepsilon_j \), where \( p \) is the support projection of \( \omega_A \). The claim follows from the identity \( S_A(\omega) = S_{pAp}(\omega) \) (see the proof of Proposition 6.8 of [30]) and Lemma 4.

A normal state which is not separable is called *entangled*. Therefore, an entanglement measure for a bipartite system should be a state functional that vanishes on separable states.

**Definition 9.** The *relative entanglement entropy* of a normal state \( \omega \) on a bipartite system \( A \otimes B \) is given by

\[
E_R(\omega) = \inf \{ S(\omega\|\sigma) : \sigma \text{ is a separable state} \}.
\]

The *mutual information* \( E_I(\omega) \) is given by

\[
E_I(\omega) = S(\omega\|\omega_A \otimes \omega_B).
\]

where \( \omega_A = \omega|_A \) and similarly for \( B \).

Clearly \( E_R(\omega) \leq E_I(\omega) \). As an example, let us consider a bipartite system given by \( A = B(\mathcal{H}) \) and \( B = B(\mathcal{H}') \), with \( \mathcal{H} \) and \( \mathcal{H}' \) finite dimensional Hilbert spaces. The mutual information is given by

\[
E_I(\omega) = S(\omega_A) + S(\omega_B) - S(\omega). \tag{11}
\]

We point out that, without any finiteness assumption, on hyperfinite type I factors we can only write \( E_I(\omega) + S(\omega) = S(\omega_A) + S(\omega_B) \). The mutual information is non-negative and independent of the order of \( A \) and \( B \) [17]. For separable states \( \omega = \sum_j \lambda_j \varphi_j \otimes \psi_j \), with \( \varphi_j \) and \( \psi_j \) normal states, we also have [17]

\[
E_I(\omega) \leq \sum_j \eta(\lambda_j), \tag{12}
\]

with \( \eta(t) = -t \ln t \) the information function. Moreover, we can use (11) to deduce the following “concavity” property of the mutual information.

**Lemma 10.** Let \( A \) and \( B \) be AF factors and \( \omega = \sum_j \lambda_j \omega_j \) be a convex decomposition of a normal state \( \omega \) on \( A \otimes B \), with \( \omega_j \) normal states. Then

\[
\sum_j \lambda_j E_I(\omega_j) - \sum_j \eta(\lambda_j) \leq E_I(\omega) \leq \sum_j \lambda_j E_I(\omega_j) + 2 \sum_j \eta(\lambda_j).
\]

**Proof.** This result is known for type I-factors, e.g., [33]. If \( A \) and \( B \) are generic hyperfinite factors, then there is a family of bipartite systems of finite-dimensional (and hence type I) factors that is weakly dense in \( A \otimes B \). On each of these finite-dimensional systems, the statement holds by the previous argument. We conclude the statement using the approximation property (vi) of the relative entropy.

It is an easy remark to notice that, always by assuming \( A \) and \( B \) to be finite dimensional type I factors, if \( \omega = \sum_j \lambda_j \omega_j \) is a convex decomposition of a state \( \omega \) in states \( \omega_j \), then

\[
\sum_j \lambda_j E_I(\omega_j) - \sum_j \eta(\lambda_j) \leq E_I(\omega) \leq \sum_j \lambda_j E_I(\omega_j) + 2 \sum_j \eta(\lambda_j).
\]

By monotonicity of the relative entropy, the same inequalities hold if \( \omega \) is normal and \( A \) and \( B \) are both hyperfinite type I factors. Furthermore, if \( \omega \) is pure then \( E_I(\omega) = 2S(\omega_A) = 2S(\omega_B) \) (Proposition 6.5. of [30]) while the relative entanglement entropy between \( A \) and \( B \) is [35]

\[
E_R(\omega) = S(\omega_A) = S(\omega_B).
\]

**Definition 11.** A cp map \( \mathcal{F} : A_1 \otimes B_1 \to A_2 \otimes B_2 \) between two bipartite systems will be called *local* if it is of the form

\[
\mathcal{F}(a \otimes b) = \mathcal{F}_A(a) \otimes \mathcal{F}_B(b),
\]

where \( \mathcal{F}_A \) and \( \mathcal{F}_B \) are normal cp maps. More generally, a *separable operation* is by definition a family of normal, local cp maps \( \mathcal{F}_j \) such that \( \sum_j \mathcal{F}_j(1) = 1 \). We think of such an operation as mapping a state \( \omega \) with probability \( p_j = \omega(\mathcal{F}_j(1)) \) to \( \mathcal{F}_j^* \omega / p_j \).
Separable operations map separable states to separable states. The relative entanglement entropy \( E_R(\omega) \) of a bipartite system \( A \otimes B \) has the following properties \([17]\).

1. **Symmetry** \( E_R(\omega) \) is independent of the order of the systems A and B.
2. **Non-negative** \( E_R(\omega) \in [0, \infty] \), with \( E_R(\omega) = 0 \) if \( \omega \) is separable and \( E_R(\omega) = \infty \) when \( \omega \) is not normal. Furthermore, if \( E_R(\omega) = 0 \) then \( \omega \) is norm limit of separable states.
3. **Continuity** Let \( (\mathfrak{A}_i)_i \) and \( (\mathfrak{B}_i)_i \) be two increasing nets of subalgebras of A and B respectively, with \( \mathfrak{A}_i \cong \mathfrak{B}_i \cong M_{n_i}(\mathbb{C}) \). Let \( \omega_i \) and \( \omega'_j \) be normal states on \( \mathfrak{A}_i \otimes \mathfrak{B}_i \) such that \( \lim_i \| \omega_i - \omega'_j \| = 0 \). Then
   \[
   \lim_{i \to \infty} \frac{E_R(\omega'_j) - E_R(\omega_i)}{\ln n_i} = 0.
   \]
4. **Convexity** \( E_R \) is convex.
5. **tensor products** Let \( A_i \otimes B_i \) with \( i = 1, 2 \) be two bipartite systems, and let \( \omega_i \) be states on \( A_i \otimes B_i \). Then
   \[
   E_R(\omega_1 \otimes \omega_2) \leq E_R(\omega_1) + E_R(\omega_2).
   \]

The mutual information \([11]\) clearly satisfies (e0) and (e5), and it is shown in \([17]\) that it also satisfies properties (e2) and (e4). Property (e1) does not follow in a straightforward way from the definitions, since we can state inequality \([12]\) at most. Property (e3) does not hold in general.

### 4 Modular nuclearity conditions

We will say that a split pair \((A, B)\) is **standard** if \( A, B \) and \( A \vee B \) are in standard form with respect to some vector \( \Omega \). We set \( J_A = J_{A, \Omega}, J_B = J_{B, \Omega} \), and similarly \( \Delta_A = \Delta_{A, \Omega}, \Delta_B = \Delta_{B, \Omega} \). As \( A \otimes B \) is in standard form with respect to \( \Omega \otimes \Omega \), the isomorphism \( \phi : A \otimes B \rightarrow A \otimes B \) has a standard implementation, namely is uniquely implemented by some unitary \( U \) which maps the natural cone of \( A \otimes B \) onto the natural cone of \( A \otimes B \) \([13]\). It can also be shown that \( J_A \otimes J_B = U^* J U^{-1} \), with \( J = J_{A \otimes B, \Omega} \). The **canonical intermediate type I factors** are \( F = U^{-1}(B(H) \otimes 1)U \) and \( F' = U^{-1}(1 \otimes B(H))U \). By construction, \( F \) is the unique \( J \)-invariant type I factor \( A \subset F \subset B' \), and similarly for \( F' \). If \( A \) and \( B \) are both factors then \( F = A \vee JAJ = B' \cap JB'J \), and therefore \( F' = B \vee JBJ = A' \cap JA'J \) \([13]\).

An inclusion \( N \subset M \) of von Neumann algebras is said to be **split** if the pair \((N, M')\) is split. We shall often pass from a split inclusion to a split pair and back. The trivial inclusion \( N = M \) is split if and only if \( N \) is a type I factor \([23]\). The inclusion \( N \subset M \) is said to be **standard** if there is a vector \( \Omega \) which is standard for \( N, M \) and the relative commutant \( N' \cap M \). If \( N \subset M \) is a standard split inclusion then each intermediate type I factor \( R \) is \( \sigma \)-finite and hence separable, therefore the Hilbert space \( H \) has to be separable as \( R \Omega \) is dense in \( H \). If \( N \vee M' \) has a cyclic and separating vector, then the pair \((N, M')\) is split if and only if there is an intermediate type I factor \( N \subset R \subset M \) \([23]\).

\(^1\)More precisely, we should say that \( E_R(\omega) = E_R(\omega \cdot \pi) \), with \( E_R(\omega \cdot \pi) \) the relative entanglement entropy on \( B \otimes A \) and \( \pi \) the natural permutation isomorphism.
Definition 12. Consider an inclusion \( N \subseteq M \) of von Neumann algebras on a Hilbert space \( \mathcal{H} \). Assume the existence of a standard vector \( \Omega \) for \( M \) and denote by \( \Delta \) the corresponding modular operator. We will say that the inclusion \( N \subseteq M \) satisfies the modular nuclearity condition if the map

\[
\Xi : N \to \mathcal{H}, \quad \Xi(x) = \Delta^{1/4}x\Omega,
\]

is nuclear.

A modular nuclear inclusion of factors is split, and a split inclusion of factors implies the compactness of the map (13) [6]. This motivates the interest in the split property in local quantum field theory contexts, where the split property amounts to some form of statistical independence between causally disjoint spacetime regions [17, 20].

The previous nuclearity condition can be easily generalized as follows. Consider a linear map \( \Theta : \mathcal{E} \to \mathcal{F} \) between Banach spaces. The map \( \Theta \) is said to be of type \( l_p \), \( p > 0 \), if there exists a sequence of linear mappings \( \Theta_i : \mathcal{E} \to \mathcal{F} \) of rank \( i \) such that

\[
\sum_i \| \Theta - \Theta_i \|^p < +\infty.
\]

The map \( \Theta \) will be said to be of type \( s \) if it is of type \( l_p \) for any \( p > 0 \). Each mapping \( \Theta \) of type \( l_p \) for some \( 0 < p \leq 1 \) is nuclear. Indeed, there are sequences of linear functionals \( e_i \in \mathcal{E}^* \) and of elements \( f_i \in \mathcal{F} \) such that

\[
\Theta(x) = \sum_i e_i(x)f_i, \quad x \in \mathcal{E},
\]

is an absolutely convergent series for each \( x \) in \( \mathcal{E} \), with

\[
\Theta(x) = \sum_i e_i(x)f_i, \quad \Theta(x) = \sum_i \| e_i \|^p\| f_i \|^p < +\infty.
\]

The induced quasi-norm, also called \( p \)-norm, is given by

\[
\| \Theta \|_p = \inf \left( \sum_i \| e_i \|^p\| f_i \|^p \right)^{1/p},
\]

where the infimum is taken over all possible representations of \( \Theta \) of the form (4). The above nuclearity condition can be then rephrased as modular \( p \)-nuclearity condition if the map (13) is of type \( l_p \).

Definition 13. Let \((A, B)\) be a standard split pair with standard vector \( \Omega \) representing a state \( \omega \). Denote by \( \Delta_A \) and \( \Delta_B \) the corresponding modular operators. We define

\[
\Xi_A(a) = \Delta_B^{1/4}a\Omega, \quad \Xi_B(b) = \Delta_A^{1/4}b\Omega.
\]

with \( a \) in \( A \) and \( b \) in \( B \). Given \( p > 0 \) we define the \( p \)-partition function as

\[
z_p = \min\{\| \Xi_A \|_p, \| \Xi_B \|_p\}.
\]

We will say that the pair \((A, B)\) satisfies the \( p \)-modular nuclearity condition if the \( p \)-partition function is finite. In the case \( p = 1 \) we will simply talk of partition function and of modular nuclearity condition.

The \( p \)-modular nuclearity condition implies the modular nuclearity condition if \( p \leq 1 \). In order to motivate our definition, we notice that if \((A, B)\) satisfies the modular nuclearity condition then it is split.

5 Results in chiral CFT

We begin with a few definitions. Let \( \mathcal{K} \) be the family of all the open, nonempty and non dense intervals of the circle. For \( I \) in \( \mathcal{K} \), \( I' \) denotes the interior of the complement. The Möbius group \( \text{Möb} \) acts on the circle by linear fractional transformations. A Möbius covariant net \((A, U, \Omega)\) consists of a family \( \{A(I)\}_{I \in \mathcal{K}} \).
of von Neumann algebras acting on a separable Hilbert space $H$, a strongly continuous unitary representation $U$ of Möb and a vector $\Omega$ in $H$, called the vacuum vector, satisfying the following properties \cite{12}:

(i) $\mathcal{A}(I_1) \subseteq \mathcal{A}(I_2)$ if $I_1 \subseteq I_2$ (isotony),
(ii) $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ for every $g$ in Möb and $I$ in $\mathcal{K}$ (Möbius covariance),
(iii) the representation $U$ has positive energy, namely the generator of rotations has non-negative spectrum (positivity of the energy),
(iv) $\Omega$ is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{K}} \mathcal{A}(I)$, and up to a scalar $\Omega$ is the unique Möbius-invariant vector of $H$ (vacuum).

A Möbius covariant net is said to be twisted-local if the following axiom is satisfied:

(v) there exists a unitary $Z$ commuting with the representation $U$ and such that $Z\Omega = \Omega$ and $ZA(I')Z^* \subseteq \mathcal{A}(I)'$ (twisted-locality).

In the following, twisted-local Möbius covariant nets will be briefly referred to as twisted-local nets. A few consequences of the axioms (i)-(iv) are \cite{12}:

(vi) $\Omega$ is cyclic and separating for each $\mathcal{A}(I)$ (Reeh Schlieder property),
(vii) $\mathcal{A}(I) \subseteq \bigvee_\alpha \mathcal{A}(I_\alpha)$ if $I \subseteq \bigcup_\alpha I_\alpha$ (additivity),

while if we also assume (v) then we have \cite{12}:

(viii) $\mathcal{A}(I') = Z\mathcal{A}(I)'Z^*$ for every $I$ in $\mathcal{K}$ (twisted-duality),
(ix) if $I_\alpha$ is the upper half of the circle and $\Delta$ is the modular operator associated to $\mathcal{A}(I_\alpha)$ and $\Omega$, then for every $t$ in $\mathbb{R}$ we have

$$\Delta^t = U(\delta_{-2\pi t}),$$

(16)

where $\delta$ is the one parameter group of dilations (Bisognano-Wichmann),

(x) each local algebra $\mathcal{A}(I)$ is a type III factor and $\bigvee_{I \in \mathcal{I}_R} \mathcal{A}(I) = B(H)$, with $\mathcal{I}_R$ the set of all the open, nonempty and non dense intervals of $S^1 \setminus \{-1\}$ (irreducibility).

A twisted-local net is said to be local if $Z$ is the identity. A particular class of local nets is certainly that of conformal nets \cite{32}. In a conformal net the following additional property is automatic \cite{26}:

(xi) if $\bar{I} \subset J$ then there is a type I factor $\mathcal{R}$ such that $\mathcal{A}(I) \subset \mathcal{R} \subset \mathcal{A}(J)$ (split property).

As noticed above, every intermediate type I factor $\mathcal{R}$ is separable, hence the split property ensures the separability of the Hilbert space on which the local algebras $\mathcal{A}(I)$ act.

Let $(\mathcal{A}, U, \Omega)$ be a twisted-local net on a Hilbert space $H$. We call a family $\mathcal{B} = \{B(I)\}_{I \in \mathcal{K}}$ of von Neumann subalgebras $B(I) \subseteq \mathcal{A}(I)$ a subnet of $\mathcal{A}$ if it satisfies isotony and Möbius covariance with respect to $U$. We will use the notation $B \subseteq A$ to denote the subnets $B$ of $A$. If $\mathcal{A}(I') \cap B(I) = \mathbb{C}$ for one (and hence for all) interval $I$ in $\mathcal{K}$, then the inclusion $B \subseteq A$ is said to be irreducible. If we denote by $e = [H_B]$ the orthogonal projection onto $H_B = \bigvee_{I \in \mathcal{K}} B(I)\Omega$, then it is easy to notice that $e$ is in the commutant of all the von Neumann algebras $B(I)$ and that it commutes with $U$. Then $B$ is itself a Möbius covariant net on $eH$ with unitary representation the restriction of $U$ on $eH$. The projection $e$ does not depend on the choice of the interval $I$ and is the Jones projection of the inclusion $B(I) \subseteq \mathcal{A}(I)$. Therefore, how noticed in Theorem \cite{5} $B(I)$ is naturally isomorphic to the reduced von Neumann algebra $eB(I)e$. If we have an inclusion of nets $B \subseteq A$, then for each interval $I$ there is a canonical faithful normal conditional expectation $e_I : \mathcal{A}(I) \to B(I)$ which preserves the vacuum state $\omega$ by Bisognano-Wichmann, Möbius covariance for the subnet $B$ and Takesaki’s theorem (\cite{31}, Theorem IX.4.2.).
Proposition 14. Let \( \mathcal{B} \subseteq \mathcal{A} \) be an inclusion of Möbius covariant nets. If \( \mathcal{A} \) is twisted-local, then all the conditional expectations \( \varepsilon_I : \mathcal{A}(I) \to \mathcal{B}(I) \) extend to a unique vacuum-preserving conditional expectation \( \varepsilon : \mathfrak{A} \to \mathfrak{B} \). Here the \( C^* \)-algebra \( \mathfrak{A} \) is the norm closure of the union of the local algebras \( \mathcal{A}(I) \), and similarly for \( \mathfrak{B} \). The projection \( e = [\mathfrak{B}\Omega] \) satisfies \( \varepsilon(x\Omega) = \varepsilon(x)\Omega \) and \( exe = \varepsilon(x)e \) for all \( x \) in \( \mathfrak{A} \).

Proof. Denote by \( \omega(\cdot) = (\Omega \cdot \Omega) \) the vacuum state and by \( e \) the orthogonal projection onto \( \mathcal{H}_B = \bigvee_{I \in \mathcal{K}} \mathcal{B}(I)\Omega \). As mentioned above, for each interval \( I \in \mathcal{K} \) we have that \( e(x\Omega) = \varepsilon_I(x)\Omega \) for \( x \) in \( \mathcal{A}(I) \). As the vacuum is locally faithful by Reeh-Schlieder, this implies that the conditional expectations are compatible, namely \( \varepsilon_I \) is an extension of \( \varepsilon_J \) whenever \( I \subseteq J \). Therefore, for \( x \) in the union \( \bigcup_{I \in \mathcal{K}} \mathcal{A}(I) \) we can define \( \varepsilon(x) \) by setting \( \varepsilon(x) = \varepsilon_I(x) \) whenever \( x \) is in \( \mathcal{A}(I) \). The map \( \varepsilon \) is bounded since every \( \varepsilon_I \) has unital norm, hence we can continuously extend \( \varepsilon \) to \( \mathfrak{A} \) (this procedure is sometimes known as the BLT theorem). Finally, \( \varepsilon \) is a conditional expectation since by continuity \( \varepsilon \) is a positive \( \mathfrak{B} \)-linear projection, and the identity \( \omega = \omega \cdot \varepsilon \) follows as well. To prove the last statement, one first shows that \( x\Omega \mapsto \varepsilon(x)\Omega \), with \( x \) in \( \mathfrak{A} \), is a well defined projection onto \( e\mathcal{H} \). The identity \( exe = \varepsilon(x)e \) follows.

Lemma 15. Let \( \mathcal{B} \subseteq \mathcal{A} \) be an inclusion of Möbius covariant nets with Jones projection \( e = [\mathfrak{B}\Omega] \). Assume \( (\mathcal{A}, U, \Omega) \) to be twisted-local with twist operator \( Z_A \). If \( eZ_A = Z_Ae \), then \( \mathcal{B} \) is twisted-local.

Proof. We show \( Z_B = eZ_Ae \) to be a twist operator for \( \mathcal{B} \). Clearly \( Z_B \) fixes the vacuum vector and commutes with \( U \). Then, by Theorem 6 and twisted locality we have the chain of inclusions \( Z_B\mathcal{B}(I')Z_B = Z_B\mathcal{A}(I')Z_B \subseteq e\mathcal{A}(I')e \subseteq e\mathcal{B}(I')e \). The thesis follows.

Definition 16. Let \( (\mathcal{A}, U, \Omega) \) be a twisted-local net satisfying the split property. Given a couple of distant open intervals \( I \) and \( J \) of the circle, denote by \( \mathcal{F} \) the canonical intermediate type I factor associated to the bipartite system \( \mathcal{A}(I) \vee \mathcal{A}(J) \). We define the canonical entanglement entropy of \( \omega \) with respect to \( (I, J) \) the von Neumann entropy

\[
    E_C(\omega) = S_F(\omega) = S_{F'}(\omega).
\]

Remark 17. In the notation of the previous definition, assume \( (\mathcal{A}, U, \Omega) \) to be local. Since \( \omega \) is a pure state on \( B(\mathcal{H}) = \mathcal{F} \vee \mathcal{F}' \), then by the monotonicity of the relative entropy and (11) we find that

\[
    E_I(\omega) \leq 2E_C(\omega),
\]

with \( E_I(\omega) \) the mutual information of the bipartite system \( \mathcal{A}(I) \vee \mathcal{A}(J) \).

Theorem 18. Let \( (\mathcal{A}, U, \Omega) \) be a twisted-local net on some Hilbert space \( \mathcal{H} \) with twist operator \( Z_A \). Consider a twisted-local subnet \( \mathcal{B} \) of \( \mathcal{A} \) satisfying the assumptions of Lemma 13. Assume also \( \mathcal{A} \) and \( \mathcal{B} \) to both satisfy the split property. Denote by \( \mathcal{F}_A \) and \( \mathcal{F}_B \) the canonical intermediate type I factors corresponding to the inclusions \( \mathcal{A}(I) \subseteq \mathcal{A}(\bar{I}) \) and \( \mathcal{B}(I) \subseteq \mathcal{B}(\bar{I}) \) respectively, with \( \bar{I} \subseteq I \). If \( \varepsilon \) and \( e \) are as in Proposition 14 then \( \varepsilon(\mathcal{F}_A)e = \mathcal{F}_B \). In particular,

\[
    S_{\mathcal{F}_B}(\omega) \leq S_{\mathcal{F}_A}(\omega). \tag{17}
\]

Proof. We denote by \( J_A \) the modular conjugation of \( \mathcal{A}(I) \vee \mathcal{A}(\bar{I}) \) and by \( J_B \) the modular conjugation of \( \mathcal{B}(I) \vee \mathcal{B}(\bar{I}) \). By Proposition 14 we have a conditional expectation \( \varepsilon : \mathcal{A}(I) \to \mathcal{B}(I) \), but by Lemma 15 we also have that \( e\mathcal{A}(I')e = Z_AB(I')Z_A' \). Therefore, by Theorem 6 we have that \( \varepsilon \) restricts to a conditional expectation \( \varepsilon : \mathcal{A}(I') \cap \mathcal{A}(\bar{I}) \to \mathcal{A}(I') \cap \mathcal{A}(\bar{I}) \), and thanks to the considerations described in the proof of Corollary 7 we can claim that \( eJ_A = J_Ae = eJ_B \). We can now state that the conditional expectation \( \varepsilon \) maps \( \mathcal{A}(I) \cap J_A\mathcal{A}(I)J_A \) onto \( \mathcal{B}(I) \cap J_B\mathcal{B}(I)J_B \); clearly \( \varepsilon(\mathcal{A}(\bar{I})) = \mathcal{B}(\bar{I}) \), but thanks to Theorem 8 and the previous remark we also have \( \varepsilon(J_A\mathcal{A}(I)J_A) = J_B\mathcal{B}(I)J_B \). It follows that \( \varepsilon(\mathcal{F}_A)e = \mathcal{F}_B \). Finally, inequality (17) is a consequence of Theorem 13 and Lemma 3.

We think Theorem 13 to be one of the main results of this work. We point out that, even though we focused on twisted-local nets on the circle, Proposition 14 and Theorem 13 easily extend to any inclusion on twisted-local Haag-Kastler nets, provided that the Bisognano-Wichmann property is satisfied.
Remark 19. Let \((\mathcal{A}, U, \Omega)\) be a twisted-local net satisfying the split property. Given a couple of disjoint and distant open intervals \(I\) and \(J\) of \(S^1 \setminus \{-1\}\), by stereographic projection we can identify them with open intervals \(\tilde{I}\) and \(\tilde{J}\) of the real line \([32]\). Define \(s = \text{dist}(\tilde{I}, \tilde{J})\) and denote by \(E_C(s)\) the corresponding canonical entanglement entropy. By monotonicity of the relative entropy, one can apply Theorem 17 of \([17]\) to provide a lower bound for \(E_C(s)\) in the limit \(s \to 0\). Similarly, always by assuming dilation covariance, in higher dimension we can claim that the canonical entanglement entropy satisfies lower bounds of area law type.

Theorem 20. Let \((\mathcal{A}, U, \Omega)\) be one of the following twisted-local conformal nets:

1. the free Fermi net,
2. some LSU(\(n\))-conformal net of level \(\ell \geq 1\),
3. the \(U(1)\)-current,
4. the Virasoro net with central charge given by
   \[ c = \frac{(n^2 - 1)}{\ell + n}, \]
   with \(\ell \geq 1\) and \(n \geq 2\) integers.

If \(\mathcal{F}\) is the canonical intermediate type I factor given by the inclusion \(\mathcal{A}(I) \subseteq \mathcal{A}(\tilde{I})\) with \(T \subseteq \tilde{I}\), then
\[ S_F(\omega) < +\infty. \]

Proof. The finiteness property \((19)\) has been proved on the free Fermi net in \([25]\), where explicit estimates can be found. However, by similarity of the first quantization Hilbert spaces, the same proof can be replicated on any LSU(\(n\))-conformal net of level \(\ell = 1\) \([32, 36]\). The explicit estimates provided in \([25]\) still apply in this setting. Therefore, since the embedding LSU(\(n\)) \(\subseteq\) LSU(\(n\ell\)) gives rise to all the LSU(\(n\))-conformal nets of level \(\ell \geq 1\) \([36]\), the estimates of \([25]\) apply to any LSU(\(n\))-conformal net. Since LSU(\(n\))-conformal nets are local, we can apply Theorem 18 to any conformal subnet of these models like the Virasoro net with central charge given by \((18)\) \([32]\). Theorem 18 cannot be applied to any conformal subnet of the free Fermi net, since in this case graded locality rather than locality holds \([25]\). However, it can be easily checked that such a requirement holds at least for the \(U(1)\)-current model \([25]\).

Theorem 20, which can be summarized as a generalization of \([25]\) by using Theorem 18, is the main result of this work. We point out that the proof exhibited in \([25]\) heavily depends on the structure of the free Fermi net. However, more in general the finiteness property \((19)\) is expected to rely on some nuclearity condition of the system such as the trace-class property \([17, 31]\). To support this conjecture, in the next section we provide a few results motivated by Remark 17.

6 Modular nuclearity and entanglement

In the previous section we proved the finiteness of some entanglement entropy to be finite on some twisted-local nets on the circle. Works on this topic suggest such an entanglement measure to be finite by assuming some modular nuclearity condition \([17, 25]\). Even though a general proof on a model independent ground is still lacking, in this section we provide a few results in this direction.

Lemma 21. Let \((A, B)\) be a standard split pair with standard vector \(\Omega\) inducing a state \(\omega\). We assume modular \(p\)-nuclearity to hold for some \(0 < p \leq 1\), namely the \(p\)-partition function \((15)\) is finite for some \(0 < p \leq 1\). Given \(\epsilon > 0\), there are sequences of normal linear functionals \(\phi_j\) on \(A\) and \(\psi_j\) on \(B\) such that
\[ \omega(ab) = \sum_j \phi_j(a)\psi_j(b), \quad a \in A, \ b \in B, \]
and \(\sum_j \|\phi_j\|^p\|\psi_j\|^p < z^p + \epsilon\).
Proof. We assume $z_p = \|\Xi_A\|_p$ and we follow Lemma 3 of [17]. Given $a$ in $A$ and $b$ in $B$, we note that
\[
\omega(ab) = (\Omega(ab\Omega) = ((\Delta^{1/4} + \Delta^{-1/4})^{-1}(1 + \Delta^{-1/2})b^*\Omega|\Delta^{1/4}a\Omega)
\]
\[
= ((\Delta^{1/4} + \Delta^{-1/4})^{-1}(b^* + Jb\Omega)\Omega|\Xi_A(a)),
\]
where $\Delta = \Delta_{B',\Omega}$ and $J = J_{B',\Omega}$. If $z_p$ is finite and $\epsilon > 0$, then there are sequences of positive normal functionals $\phi_j$ on $A$ and vectors $\xi_j$ in $\mathcal{H}$ such that
\[
\Xi_B(a) = \sum_j \phi(a)\xi_j, \quad a \in A,
\]
and $\sum_j \|\phi_j\|^p\|\xi_j\|^p < z_p^p + \epsilon$. Define now normal functionals $\psi_j$ on $B$ by
\[
\psi_j(b) = ((\Delta^{1/4} + \Delta^{-1/4})^{-1}(b^* + Jb\Omega)\Omega|\xi_j),
\]
and note that $\|\psi_j\| \leq \|\xi_j\|$ because of the estimate $\|(\Delta^{1/4} + \Delta^{-1/4})^{-1}\| \leq 1/2$ and the spectral calculus. Putting both paragraphs together we find the conclusion.

Before providing a corollary of the previous lemma, we describe a general procedure known as polarization of a functional. Let $\omega$ be a continuous functional. We will say that $\omega$ is self-adjoint if $\omega = \omega^*$, with $\omega^*(x) = \overline{\omega(x^*)}$ the conjugate of $\omega$. By use of $\omega^*$ one can write $\omega = \phi + i\psi$, with $\phi$ and $\psi$ self-adjoint. Then, after applying a Jordan decomposition on both $\phi$ and $\psi$, we can write $\omega = \sum_{k=0}^3(i)^a\omega_k$, with $\omega_k$ positive. The inequality $\|\omega_\pm\| \leq \|\omega\|$ can also be proved, and $\omega_\pm$ are all normal if $\omega$ is.

Corollary 22. With the notation of the previous lemma, for every $\epsilon > 0$ we can write $\omega = (1 + \lambda)\omega_+ - \lambda\omega_-$, where $\omega_\pm$ are separable states and $(1 + \lambda)^p \leq 4(z_p^p + \epsilon)$.

Proof. By polarization, we can decompose $\phi_j = \sum_{\alpha=0}^3(i)^\alpha \phi_j^\alpha$ and $\psi_j = \sum_{\alpha=0}^3(i)^\alpha \psi_j^\alpha$ in four positive normal functionals. One also has that $\|\phi_j^\alpha\| \leq \phi_j$ holds, and similarly for $\psi_j$. Since $\omega$ is positive, then after the identification $A \vee B \cong A \otimes B$ we find
\[
\omega = \sum_j \sum_{\alpha=0}^3 \phi_j^\alpha \otimes \psi_j^{4-\alpha} - \sum_j \sum_{\alpha=0}^3 \phi_j^\alpha \otimes \psi_j^{2-\alpha},
\]
namely $\omega$ is difference of two separable functionals. The thesis follows.

Lemma 23. With the hypotheses of Lemma 21 assume $\omega$ to have an expression like in (20) and assume $\mu_p = \sum_j \|\phi_j\|^p\|\psi_j\|^p$ to be finite for some $0 < p \leq 1$. Then there is a separable positive linear functional $\sigma$ such that $\sigma \geq \omega$ on $A \vee B$ and $\|\sigma\|^p = \mu_p \leq \mu_p$.

Proof. We follow [17] Lemma 4. By polar decomposition there are partial isometries $u_j$ in $A$ such that $\phi(u_j \cdot) \geq 0$ on $A$ and $\phi_j(u_j u_j^*) = \phi_j$. It follows in particular that $\phi_j(u_j) = \|\phi_j\|$ and
\[
\bar{\phi}_j(a) = \bar{\phi}_j(u_j u_j^* a^*) = \phi_j(u_j u_j^* a^*) = \phi_j(u_j a u_j)
\]
for all $a$ in $A$, where we used the fact that $\phi_j(u_j)$ is hermitian (here $\bar{\psi}(a) = \overline{\psi(a^*)}$). Similarly, there are partial isometries $v_j$ in $B$ such that $\psi_j(v_j \cdot) \geq 0$ and $\psi_j(v_j v_j^*) = \psi_j$. Note that the positive linear functional $\rho_j = \phi_j(u_j \cdot) \otimes \psi_j(v_j \cdot)$ is separable. Writing $w_j = u_j \otimes w_j$ we then define
\[
\sigma_j(\cdot) = \frac{1}{2} \rho_j(\cdot) + \frac{1}{2} \rho_j(w^* \cdot w),
\]
which is also separable, because $w$ is a simple tensor product. Furthermore,
\[
\|\sigma_j\| = \sigma_j(1) = \|\phi_j\| \|\psi_j\|,
\]

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Let \( \omega \) be a standard vector on a wide family of 1 + 1-dimensional integrable models with factorizing S-matrices \([2, 20, 22]\). All these models also satisfy the hyperfiniteness of the local algebras, hence Theorem 25 can be applied in these settings. We now follow \([31]\) and we show the finiteness of some “tailored” entanglement entropy under the assumption of modular p-nuclearity for some \(0 < p < 1\).

**Definition 27.** Let \((A, B)\) be a standard split pair of von Neumann algebras on a Hilbert space \(\mathcal{H}\) with standard vector \(\Omega\). If \(u: \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}\) is a unitary implementing the natural isomorphism \(A \vee B \cong A \otimes B\), then we will denote by \(R_u = u^{-1}(B(\mathcal{H}) \otimes \mathbb{1})u\) the corresponding type I factor. We define an *intermediate pair* any such pair \((u, R_u)\).

**Definition 28.** Let \((A, B)\) be a standard split pair as in the previous definition. Given a state \(\psi\) on \(B(\mathcal{H})\), we call the *intermediate entanglement entropy of \(\psi\) the functional*

\[
I(\psi) = \sup_{(u, R_u)} \inf_{\phi, \lambda} \frac{1}{\lambda} S(\phi),
\]

where supremum is over all intermediate pairs, the infimum is over all states \(\phi\) on \(B(\mathcal{H})\) and real numbers \(0 < \lambda \leq 1\) such that \(\phi \geq \lambda \psi\) on \(A \vee B\) and \(S(\phi)\) is the von Neumann entropy of \(\phi\) on the intermediate type I factor \(R_u\).
Theorem 29. Let $(A, B)$ be a standard split pair with standard vector $\Omega$ inducing a state $\omega$. Denote by $z_p$ the $p$-partition function [15]. If $z_p$ is finite for some $0 < p < 1$, then the intermediate entanglement entropy is finite. Explicitly,
\[
I(\omega) \leq z_p \ln z_p + c_p z_p^p. \tag{23}
\]

Proof. The proof consists of a computation that does not depend on the choice of the intermediate pair, which is therefore implicit in what follows. Through the natural isomorphism $A \vee B \cong A \otimes B$ we will identify $A$ with $A \otimes 1$ and $B$ with $1 \otimes B$. Lemma \cite{23} gives a separable dominating normal functional $\sigma \geq \omega$ with $\|\sigma\|^p \leq z_p^p + \epsilon$ for $\epsilon > 0$ arbitrarily small. We utilize the separability of $\hat{\sigma} = \sigma/\|\sigma\|$ over the bipartite system $A \otimes B$ and decompose it into positive, normal functionals, say $\hat{\sigma} = \sum_j \phi_j \otimes \psi_j$. Without loss of generality we can assume $\phi_j$ to be states on $A$. Now we notice that $\phi_j \otimes \psi_j$ is a normal positive functional on $A \otimes B$, hence it can be extended by taking a representative vectors. Since such extension has same norm, we can extend $\sigma$ to a separable positive functional on $B(\mathcal{H}) \otimes B(\mathcal{H})$ in such a way that still $\|\sigma\|^p \leq z_p^p + \epsilon$. We introduce some further notation by setting $\eta(t) = -t \ln t$ and $1/c_p = (1 - p)e$. Therefore, we have
\[
\|\sigma\|S(\hat{\sigma}) \leq \|\sigma\| \ln \|\sigma\| + \sum_j \eta(\|\psi_j\|) \leq \|\sigma\| \ln \|\sigma\| + c_p \sum_j \|\psi_j\|^p,
\]
and the claimed estimate follows from the arbitrariness of $\sigma$. \hfill \Box

7 Application to integrable models

Definition 30. A local quantum field theory $(\mathcal{A}, U, \Omega)$ on the Minkowski space is said to satisfy the split property (for double cones) if the inclusion $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ is split whenever $\mathcal{O}_1 \subset \mathcal{O}_2$ is an inclusion of double cones such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$.

The split property ensures some statistical independence property of the considered model. The split property does not hold for unbounded regions like wedges in more than two spacetime dimensions [4, 20]. In the literature there are several criteria which are known to imply the split property, where many of them are referred to as “nuclearity conditions”. In order to formulate this condition in a local Haag-Kastler net on a $d$-dimensional Minkowski space, one considers a region $\mathcal{O} \subseteq \mathbb{R}^d$ and a parameter $\beta > 0$ representing the inverse temperature. One defines
\[
\Theta_{\beta, \mathcal{O}}: \mathcal{A}(\mathcal{O}) \to \mathcal{H}, \quad \Theta_{\beta, \mathcal{O}}(A) = e^{-\beta H} A \Omega, \tag{24}
\]
where $H = P_0$ denotes the Hamiltonian with respect to the time direction $x_0$.

Definition 31. A local quantum field theory on the Minkowski space $\mathbb{R}^d$ is said to satisfy the energy nuclearity condition if the maps [24] are nuclear for any bounded region $\mathcal{O}$ and any inverse temperature $\beta > 0$. Moreover, there must exist constants $\beta_0, n > 0$ depending on $\mathcal{O}$ such that the nuclear norm of $\Theta_{\beta, \mathcal{O}}$ is bounded by
\[
\|\Theta_{\beta, \mathcal{O}}\|_1 \leq e^{(\beta_0/\beta)^n}, \quad \beta \to 0.
\]

Definition 32. A local quantum field theory on the Minkowski space is said to satisfy the modular nuclearity condition (for double cones) if the inclusion $\mathcal{A}(\mathcal{O}_1) \subseteq \mathcal{A}(\mathcal{O}_2)$ is modular nuclear whenever $\mathcal{O}_1 \subset \mathcal{O}_2$ is an inclusion of double cones such that $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$.

A modular nuclear inclusion of factors is split, and a split inclusion of factors implies the compactness of the map [15]. As mentioned in section 4, the previous nuclearity conditions can be changed into modular $p$-nuclearity conditions by requiring to the considered maps to be of type $ip^l$ for some $0 < p \leq 1$. Modular $p$-nuclearity has been proved in the theory of a scalar free field for any $p > 0$ [21] and holds on conformal nets satisfying the trace class property [8].
A class of integrable models on $\mathbb{R}^{1+1}$ that satisfies modular $p$-nuclearity for wedges was constructed in [20]. The only input needed are the mass $m$ and a 2-body scattering matrix. We review the structure and some properties of these models. The statements and the construction in greater detail can be found in [20].

**Definition 33.** A (2-body) scattering function is an analytic function $S_2: S(0, \pi) \to \mathbb{C}$ which is bounded and continuous on the closure of this strip and satisfies the equations

$$
\overline{S_2(\theta)} = S_2(\theta)^{-1} = S_2(-\theta) = S_2(\theta + i\pi), \quad \theta \in \mathbb{R}.
$$

The set of all the scattering functions will be denoted by $S$. For $S_2$ in $S$, we define

$$
\kappa(S_2) = \inf\{\text{Im } \zeta : \zeta \in S(0, \pi/2), \quad S_2(\zeta) = 0\}.
$$

The subfamily $S_0 \subset S$ consists of those scattering functions $S_2$ with $\kappa(S_2) > 0$ and for which

$$
||S_2||_\infty = \sup\{|S_2(\zeta)| : \zeta \in \overline{S(-\kappa, \pi + \kappa)}\} < +\infty, \quad \kappa \in (0, \kappa(S_2)) .
$$

The families of scattering functions $S$ and $S_0$ can then be divided into “bosonic” and “fermionic” classes according to

$$
S^\pm = \{ S_2 \in S : S_2(0) = \pm 1 \}, \quad S = S^+ \cup S^-,
$$

$$
S_0^\pm = \{ S_2 \in S_0 : S_2(0) = \pm 1 \}, \quad S = S_0^+ \cup S_0^-.
$$

In general, the $S$-matrix of a QFT model with interaction is difficult to handle and has been subject of research (see e.g. [19] and references therein). In particular, there is few knowledge about the higher $S$-matrix elements $S_{n,m}$ with $n, m > 2$. In $1 + 1$ dimensions, there exist $S$-matrices, called factorising $S$-matrices, that are completely determined by the two-particle $S$-matrix. The entries $S_{n,m}$ vanish for $n \neq m$ and $S_n = S_{n,n}$ is the product of two-particle $S$-matrices. We now paraphrase the main steps of constructing a net of von Neumann algebras on $\mathbb{R}^{1+1}$ from a (2-body) scattering function $S_2$ [20].

One can parameterize the mass-shell $\Omega_m$ with the rapidity $p(\theta) = m(\cosh \theta, \sinh \theta)$, with $\theta \in \mathbb{R}$, so that the one-particle space simplifies to $\mathcal{H}_1 = L^2(\mathbb{R}, d\theta)$. The next step is to define a “$S_2$-symmetric” Fock-space over $\mathcal{H}_1$. For that, we define a unitary representation $D_n$ of the symmetric group $S_n$ on $\mathcal{H}_1^\otimes n$ that takes the $S$-matrix $S_2$ into account. The complete $S_2$-symmetric projection is defined as $E_n = (1/n!) \sum_{\sigma \in S_n} D_n(\sigma)$. The “$S_2$-symmetric” $n$-particle space is $\mathcal{H}_n = E_n \mathcal{H}_1^\otimes n$ and the “$S_2$-symmetric” Fock space is the direct sum of the $n$-particle spaces: $\mathcal{H} = \mathbb{C} \Omega \bigoplus_{n \geq 1} \mathcal{H}_n$. For $\chi \in \mathcal{H}$, we define the annihilation $z(\chi)$ and creation operator $z^\dagger(\chi)$ with the $S_2$-symmetric projection analogous to the usual creation and annihilation operators. The creation and annihilation operators are closable and have a common core and are mutual adjoints of each other. For a Schwartz function $f \in \mathcal{S}(\mathbb{R}^{1+1})$, one-particle vectors $f^\pm$ are defined as the (anti) Fourier transformation of $f$ restricted to the mass shell. Furthermore, we define an involution $J$ by $(J\Psi)_n(\theta_1, \ldots, \theta_n) = \Psi_n(\theta_n, \ldots, \theta_1)$. With $f^*(x) = f(-x)$, the (closable and essentially self-adjoint for real $f$) field operators $\phi(f)$ and $\phi'(f)$ are defined as

$$
\phi(f) = z^\dagger(f^+) + z(f^-), \quad \phi'(f) = J\phi(f^*).J.
$$

If $f$ and $g$ belong to $\mathcal{S}(\mathbb{R}^{1+1})$, with $f$ supported in $W_R$ and $g$ supported in $W_L$, then $[\phi'(f), \phi(g)]\Psi = 0$ for any $\Psi$ belonging to a dense common core. Finally, this construction gives to a local net $W \mapsto \mathcal{A}(W)$ on wedges defined by

$$
\mathcal{A}(W_L + x) = \{ e^{i\phi(f)} : f \in \mathcal{S}(W_L + x, \mathbb{R}) \}'' ,
$$

$$
\mathcal{A}(W_R + x) = \{ e^{i\phi'(f)} : f \in \mathcal{S}(W_R + x, \mathbb{R}) \}'' .
$$

The algebra of observables localized in a double cone $\mathcal{O} = W_1 \cap W_2$ is defined as

$$
\mathcal{A}(W_1 \cap W_2) = \mathcal{A}(W_1) \cap \mathcal{A}(W_2) ,
$$

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and for arbitrary open regions $Q \subseteq \mathbb{R}^{1+1}$ we put $\mathcal{A}(Q)$ as the von Neumann algebra generated by all the local algebras $\mathcal{A}(O)$ with $O \subseteq Q$. In [20], the author shows that the above models with scattering function $S_2$ in $S^\text{0}_0$ satisfy the axioms of AQFT. Moreover, the inclusion $\mathcal{A}(W_R+s) \subset \mathcal{A}(W_R)$ is modular $p$-nuclear for large enough $s$, where $s = (0,s)$.

**Theorem 34.** [2, 20, 22] Let $(\mathcal{A}, U, \Omega)$ be an integrable quantum field theory on $\mathbb{R}^2$ with factorizing $S$-matrix $S_2 \in S$. Define

$$\Xi(s) : \mathcal{A}(W_R) \to \mathcal{H}, \quad \Xi(s)A = \Delta^{1/4}U(s)A\Omega, \quad s > 0,$$

where $U(s)$ is the unitary associated to the translation of $s = (0, s)$ and $\Delta$ is the modular operator of $(\mathcal{A}(W_R), \Omega)$. If $S_2 \in S^\text{0}_0$, then there exist some finite splitting distance $s_{\text{min}} < \infty$ such that $\Xi(s)$ is $p$-nuclear for all $p > 0$ and $s > s_{\text{min}}$.

**Lemma 35.** The map (25) satisfies $\|\Xi(s)\|_p \to 1$ as $s \to +\infty$.

**Proof.** This fact can be noticed by reading the thesis [20] and the subsequent works [2] [22].

**Remark 36.** By studying carefully [2], [20] and [22], it should also follow that the maps

$$\Sigma(s) : \mathcal{A}(W_{Rs})s\Omega \to \mathcal{H}, \quad \Sigma(s)A\Omega = \Delta^{1/4}U(s)A\Omega, \quad s > 0,$$

are $p$-nuclear for all $p > 0$ and $s > s_{\text{min}}$.

We now compare the asymptotic behaviour of two different entanglement measures in the setting of these integrable models. If we denote by $E_R(s)$ the vacuum relative entropy of entanglement corresponding to the wedge inclusion mentioned in Theorem 34, then by Lemma 21 and Lemma 23 it is easy to notice that [17]

$$E_R(s) \leq \ln \|\Xi(s)\|_1 \to 0, \quad s \to +\infty.$$  

However, if we denote by $E_I(s)$ the associated mutual information and we apply the estimate (21) for any $0 < p < 1$ then we can state at most that

$$\limsup_{s \to +\infty} E_I(s) \leq 1/e.$$  

Analogously, we can apply (23) to estimate the intermediate entanglement entropy in the limit $s \to +\infty$. With the same arguments and similar notation we find

$$\limsup_{s \to +\infty} I(s) \leq 1/e.$$  

8 Conclusions

We close this work with additional remarks that might be useful for future research in this area. In particular, we list a few conditions which are equivalent to the finiteness of the canonical entanglement entropy.

The techniques of section 5 rely on the presence of a separable state $\sigma$ on the bipartite system $A \otimes B$ that dominates $\omega$. If $F$ is an intermediate type I factor $A \subseteq F \subseteq B'$ arising from the natural isomorphism $A \vee B \cong A \otimes B$ as in Definition 27, then it is possible to construct a separable functional on $B(\mathcal{H}) \otimes B(\mathcal{H}) \cong F \vee F'$ that dominates $\omega$ on $F$ and on $F'$ by use of generalized conditional expectations [1].

More specifically, let $(A, B)$ be a standard split pair with finite $p$-partition function for some $0 < p < 1$. As discussed in [1] [30], one has two $\omega$-preserving cpu maps, say $\varepsilon$ and $\varepsilon'$, induced by the inclusions $A \subseteq F$ and $B \subseteq F'$ respectively. By the isomorphism $B(\mathcal{H}) \otimes B(\mathcal{H}) \cong F \vee F' = B(\mathcal{H})$ we can then define a map $\varepsilon \otimes \varepsilon'$ on $B(\mathcal{H})$ extending both $\varepsilon$ and $\varepsilon'$. If $\sigma$ is the dominating separable functional from Lemma 23, then $\sigma_0 = \sigma \cdot (\varepsilon \otimes \varepsilon')$ dominates $\omega_0 = \omega \cdot (\varepsilon \otimes \varepsilon')$. Notice that $\omega = \omega_0$ on $F$ and on $F'$, but in general not
on $B(H)$. The functional $\sigma_0 = \sum \phi_j \cdot \varepsilon \otimes \psi_j \cdot \varepsilon'$ is separable with $\sum_j \|\phi_j \cdot \varepsilon\|_p \|\psi_j \cdot \varepsilon'\|_p = \mu_p$ finite (cf. Lemma 24 for notation), and in the notation of Theorem 25 we have
\[ E_I(\omega_0) = S(\omega_0 \| \omega_B \otimes \omega_B') \leq c_p z_p + \eta(z_p) - \eta(z_p), \]
where the r.h.s. is finite by assumption. Unfortunately, this does not imply the finiteness of the canonical entanglement entropy since $\omega_0$ is not a pure state on $B(H)$. But we can make use of generalized conditional expectations to give an equivalent description of the canonical entanglement entropy. In particular, by use of equation (11) and Lemma 8 we can claim that
\[ 2E_C(\omega) = S_B(\omega \| \omega_F \otimes \omega_F') = 2S_B(\omega_0) = 2H_\omega(F), \]
with $H_\omega(F) = H^{B(\mathcal{H})}_\omega(F)$ the conditional entropy. The authors of [14] argued on grounds of physical arguments that
\[ E_C(\omega) \approx E^{F \vee B'}_I(\omega) = S(\omega \| \omega_F \otimes \omega_B'), \]
and indeed it is reasonable to expect that the results of this work can be properly strengthened. For example, Theorem 25 implies that $E^{F \vee B'}_I(\omega)$ is finite if the $\omega$-preserving generalized conditional expectation from $F \vee B'$ onto $A \vee B'$ is a separable operation in the terminology of Definition 11. Another strategy could be that of estimating the entanglement entropy of some energy cutoff of the vacuum state like in [31] and then to operate some limit procedure. A different approach is the one of [25], in which the authors use the language of standard subspaces. Unfortunately, even if completely rigorous, this last work heavily depends on the structure of the free Fermi nets, and a generalization of it seems quite challenging up to now. In the context of conformal nets, the authors expect the trace-class property to be a good assumption to start with [23, 31].

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