An interesting link between linear feasibility, linear programming, support vector margin and convex hull.

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April 10, 2019

Abstract

This short paper presents an algorithm based on simple projections and linear feasibility ($Ax = 0, x > 0$) queries which solves both linear programming ($\min_x c^T x$), and, support vector margin ($\min_w w^T w$).

In addition, this algorithm is strongly polynomial on special instance of both linear program and support vector margin, characterized by a similar convex hull property.

Thus, this algorithm could be interesting as a link between all these four notions: linear feasibility, linear programming, support vector margin and convex hull.

1 Introduction

Linear programming is the very studied task of solving $\min_x c^T x$ with $A$ a matrix, $b$ and $c$ some vectors. This problem can be solved in polynomial time since [14, 7, 12]. State of the art algorithm to solve linear program are today interior point algorithms (e.g. [17]). In addition, several special case of linear program can be solved more efficiently:

- combinatorial linear program [20]
- linear program with at most two variables per inequality [10]
- markov chain [19]
- system having binary solution [4]

Support vector margin [6] used to be a central tool of machine learning (before deep learning [15]), and, consist to solve $\min_w w^T w$. Complexity of support vector margin seems to be polynomial as a special problem of semi
definite programming complexity [8], although semi definite complexity without assumption under degeneracy is not that clear [9] (when solving is about producing an exact solution plus a corresponding certificate). And, in addition, approximate solutions for support vector margin exist like [11] (theoretically proven) or [13, 3] (efficient implementations).

Independently, [5] recently shows that linear feasibility i.e. \( Ax = 0, \ x > 0 \) can be solve in strongly polynomial time. This result is especially interesting, because [2] shows that major interior point family does not solve linear program in strong polynomial time. And, so, the question of solving linear program as a strong polynomial sequence of linear feasibility queries is relevant.

This short paper focus on this last point, and, presents an algorithm which solves both linear programming and support vector margin by solving a sequence of linear feasibility on maximal subset (section 2). Although, the complexity of this algorithm may be exponential, it is showed, in section 3, that special instance of both linear programming and support vector margin can be solved in strong polynomial time under a common assumption linked to convex hull properties.

Thus, this short paper provides at least an interesting links between linear feasibility, linear programming, support vector margin and geometric complexity of convex hull, which is a very studied subject (e.g. [1, 16, 18]).

Notations

\( \mathbb{N}, \mathbb{Q} \) are the sets of integer and rational numbers. \( \setminus \) is the ensemble subtraction. \( \forall n \in \mathbb{N}\setminus\{0\}, \ R(n) \) is the range of integer from 1 to n i.e. \( R(n) = \{ k \in \mathbb{N} ; 1 \leq k \leq n \} = \{1, ..., n\} \).

\( \forall I, J \in \mathbb{N}\setminus\{0\}, \mathbb{Q}^I \) is the set of \( I \) dimensional vector of \( \mathbb{Q} \), and, \( \mathbb{Q}_{I,J} \) is the set of matrix with \( I \) rows and \( J \) columns, with values in \( \mathbb{Q} \). \( \mathbb{Q}^I \) would be matched with \( \mathbb{Q}^1 \) i.e. vector are seen as a column vector.

\( \forall i \in \mathbb{N}\setminus\{0\}, \ A_i \) designs the \( i \) component: a rational for vector or a row for a matrix. Rows are seen as row vector i.e. if \( \forall A \in \mathbb{Q}_{I,J}, \ A_i \in \mathbb{Q}_{1,J} \). Also \( A^T \) is the transposition operation i.e. \( \forall I, J \in \mathbb{N}\setminus\{0\}, \forall A \in \mathbb{Q}_{I,J}, \ A^T \in \mathbb{Q}_{J,I} \) with \( \forall (i,j) \in R(I) \times R(J), \ A^T_{j,i} = A_{i,j} \). For all set \( S \subset \mathbb{N}, \ A_S, b_S \) is the sub matrix or sub vector obtained when keeping only rows or components \( s \in S \).

\( \mathbf{0} \) and \( \mathbf{1} \) are the 0 and 1 vector i.e. vector contains only 0 or only 1, and \( \mathbf{I} \) is the identity matrix.

\( \mathbb{U} \) is the set of normalized vector from \( \mathbb{Q}^I \) i.e. such that \( v^Tv = 1 \) i.e. \( \mathbb{U} = \{ v \in \mathbb{Q}^I, v^Tv = 1 \} \). \( \mathbb{U}_{I,J} \) is the set of matrix from \( \mathcal{M}_{I,J} \) whose rows (after transposition) are in \( \mathbb{U}_J \) (only the rows, not necessarily the columns).

Also, if \( A \in \mathcal{M}_{I,J} \), then, the null vector space of \( A \) (i.e. the kernel) is written \( \text{Ker}(A) = \{ v \in \mathbb{Q}^I / Av = \mathbf{0} \} \).

All notations are quite classical except \( R(n) \) for the range of integer from 1 to \( n \), \( \mathbb{Q}_{I,J} \) is the set of matrix with \( I \) rows and \( J \) columns (more often written \( \mathcal{M}_{I,J}(\mathbb{Q}) \) and \( \mathbb{U} \) to indicate normalization of vector and/or matrix.
2 Algorithm

2.1 Slide and jump algorithm

Let quickly consider the following lemma:
if it is possible to solve \( \exists x \mid Ax = 0, \ x > 0 \) when \( A \) has full rank, then it is possible to solve \( \exists x, Ax > 0 \). It is sufficient for that to consider the matrix \( A \) formed by \( A \) concatenate with \( -A \) concatenate with \(-I\) (which has full rank due to the identity bloc) ; proof is in section 4.

With the ability to call a strong polynomial time sub solver dedicated to \( Ax > 0 \), it is easy to get a small improvement from a not optimal admissible either for linear program or support vector margin. Indeed, if \( w \) verifying \( Aw \geq 1 \) has not minimal norm, then it is possible to find \( u \) such that \( \begin{pmatrix} A \\ -w^T \end{pmatrix} u > 0 \) (this will be proven more formally in section 2.4). Now, it could take infinite time to reach the optimal solution by computing such \( u \) and updating \( w = w + \varepsilon u \). These moves can be see as jump because its allow to exit a situation but there are not sufficient.

Inversely, the algorithm considers sliding moves: it considers the current most problematic constraints e.g. \( D \) such that \( ADw = 1 \), and it solves the easy optimization problem derived by freezing this set i.e. it solves \( \min_w w^T w \). This problem can be trivially solved because it is just a projection on a vectorial space (can be done for example with gram schmidt basis transform).

Yet, sliding moves will allow the algorithm to explore completely a particular combinatorial situation \( (D) \) and jumping moves will allow to exit this structure. So the termination will be guarantee because \( D \) can not be observed again as it has been completely explored (this will be proven more formally in section 2.4).

The main key point of this short paper is that the same slide and jump idea can solve different problems, at least linear program and support vector margin.

Unfortunately, the algorithm may be exponential because there is a lot of possible situations \( D \) (of course exploration can not lead to visit all of them, but, this point is not enough investigated in this short paper). Still, this algorithm is also interesting because it is strongly polynomial for both these two problems under the same kind of assumptions linked with convex hull properties.
2.2 Pseudo code for linear program

Let assume the input linear program is \( \min_{x \in \mathbb{Q}^N} c^T x \) with \( A \in \mathbb{U}_{M,N}, \) \( b \in \mathbb{Q}^M, \) \( c \in \mathbb{U}_N, \) \( Ac = \frac{3}{5} \mathbf{1}, \) \( x_{start} \) being a trivial admissible point e.g. \( (1 + \frac{a}{b} \max b_m)c, \) and, \( c^T x \) being bounded by 0.

Currently, this does not restrict generality because all linear program can be derived under this form (see section 4).

The pseudo code of the algorithm is:

1. compute \( d = \min_{m \in R(M)} A_m x - b_m \)
2. compute \( D = \{ m \in R(M) / A_m x - b_m = d \} \) the set of constraints at distance \( d \)
3. let \( u \) be the orthogonal projection of \(-c\) on \( \text{Ker}(A_D) \)
4. if \( cu < 0, A_D u = 0 \)
   (a) compute \( g = \min_{m \in R(M) / A_m u < 0} \frac{A_m x - b_m - d}{A_m u} \)
   (b) let \( x = x + gu \) and GO TO 1
5. call sub solver \( \exists v / (A_D - c^T) v > 0 \)
6. let \( z = x - \frac{5d}{3} c \)
7. if \( v \) exists, \( v = \min_{m \in R(M)} \frac{1}{A_m v} v \)
   (a) \( h = \min_{m \in R(M)/A_m v < 0} \frac{A_m z - b_m}{A_m v} \)
   (b) let \( x = z + \frac{h}{v} v \)
   (c) GO TO 1
8. return \( z \) and the certificate
2.3 Pseudo code for support vector margin

Let assume the input problem is \( \min_{w \in \mathbb{Q}^N, \ Aw \geq 1} w^T w \) with \( A \in \mathbb{U}_{M,N} \). Again this does not restrict generality because in fact the algorithm can solve any \( \min_{w \in \mathbb{Q}^N, \ Aw \geq b > 0} w^T w \) (so normalization of section 4 is also possible). Yet, considering \( \min_{w \in \mathbb{Q}^N, \ Aw \geq 1} w^T w \) with \( A \in \mathbb{U}_{M,N} \) is relevant from machine learning point of view (this assumption gives equivalent importance to all vectors) and allows to a simpler expression of the hypothesis leading to strong polynomial time.

Also, a single call to the sub folder allows to know if \( Aw \geq 1 \) admits a solution, so, algorithm can assume to start from an admissible \( w \).

The pseudo code of the algorithm is:

1. \( w = \frac{1}{\min_{m \in R(M)} A_m w} \)
2. compute \( D = \{ m \in R(M) / A_m w = 1 \} \)
3. let \( u \) be the orthogonal projection of \(-w\) on \( \text{Ker}(A_D) \)
4. if \(wu < 0, A_D u = 0 \)
   (a) if \( A(w + u) \geq 1 \)
      i. let \( w = w + u \) and GO TO 2
   (b) compute \( g = \min_{m \in R(M) / A_m u < 0} \frac{A_m w - 1}{A_m u} \)
   (c) let \( v = v + gu \) and GO TO 2
5. call sub solver \( \exists v / \left( \begin{array}{c} A_D \\ -w^T \end{array} \right) v > 0 \)
6. if \( v \) not exists, return the certificate and \( w \)
7. if \( Av \geq 0 \)
   (a) let \( w = w + \frac{wu}{w^T v} \) and GO TO 1
8. \( v = \frac{1}{\min_{m \in D} A_m v} \)
9. \( h = \min_{m / A_m w < 0} \frac{A_m w - 1}{A_m v} \)
10. let \( w = w + \frac{h}{4} v \)
11. GO TO 1
2.4 Termination on linear program

2.4.1 Basic properties

Steps 1, 2, 5, 6, 8 are well defined. Step 3 is well defined because \( D \) (step 2) can not be empty.

Now, by assumption \( cx \) is bounded i.e. \( \forall x, Ax \geq b \Rightarrow cx \geq 0 \). So, if there is \( y \) such that \( cy < 0 \) and \( \forall m, A_m y \geq 0 \), then, one could produce an unbounded admissible point \( x + \lambda y \) as \( A(x + \lambda y) \geq Ax \geq b \) and \( c(x + \lambda y) \xrightarrow[\lambda \to \infty]{} -\infty \). So, in steps 4 and 7, the minimum is not taken on an empty set.

The algorithm is well defined.

In step 4, let \( m \) such that \( A_m v < 0 \), seeing test leading to step 4 it implies that \( m \notin D \), and so that \( A_m x - b_m > d \) by definition of \( d, D \). So \( g > 0 \).

So, \( \frac{A_m x - b_m - d}{A_m v} \geq g > 0 \) And, so \( \frac{A_m x - b_m - d}{A_m v} A_m v \leq g A_m v \) (product by a negative). So, \( A_m x + g A_m v - b_m \geq A_m x + \frac{A_m x - b_m - d}{A_m v} A_m v - b_m = \frac{d}{m} \).

Also, if \( A_m v \geq 0 \), then \( A_m (x + g v) - b_m \geq A_m x - b_m \geq d > 0 \).

In step 7, let \( m \) such that \( A_m w < 0 \), seeing test leading to step 7 it implies that \( m \notin D \), which implies that \( A_m x - b_m > d \), and so that \( A_m z - b_m > 0 \) as \( z = x - \frac{2d}{3} c \) and by assumption \( A_m c = \frac{\rho}{3} \). So \( h > 0 \).

Again, \( A_m z + \frac{1}{4} h A_m w - b_m \geq A_m z + \frac{1}{4} \frac{A_m z - b_m}{A_z w} A_m w - b_m = \frac{3}{4} (A_m z - b_m) > 0 \)

Also, if \( A_m w = 0 \), \( A_m (x + hw) - b_m = A_m x - b_m > 0 \). And, if \( A_m z - b_m = 0 \), \( A_m w > 0 \) so \( A_m (x + hw) - b_m = h A_m w > 0 \).

So, if \( x \) is in the inner space \( x/\lambda > b \), then, it will again be in the inner space after steps 4 or 7. Then, \( A z = A(x - \frac{2d}{3} c) \geq b + d \frac{1}{3} - \frac{2d}{3} \frac{1}{3} \geq b \) because \( Ac = \frac{\rho}{3} \).

The algorithm works with inner points, and, if it terminates, it returns an admissible point.

Now, obviously, all moves strictly decreases \( cx \) (because by construction \( wc < 0 \) and \( cw < 0 \) and \( g, h \neq 0 \) - minimization excluding index from \( D \)). Let assume the algorithm returns a non optimal admissible point. It means that the step 5 found no solution. Yet, let consider \( \theta = x^* + \frac{cz}{w} c - z \) with \( x^* \) being a solution (i.e. optimal). As, \( x^* \) verifies \( Ax^* \geq b \), let define \( \rho = x^* + \frac{cz}{w} c \). So, \( \rho \) verifies \( A\rho \geq b \), and, \( c\rho = \frac{cz}{w} < cz \) (because \( cz > cx^* \) otherwise it would have been a solution). So, \( c(\rho - z) < 0 \) and \( A_D (\rho - z) = A_D \rho - b_D \). So, \( \theta = x^* + \frac{cz}{w} c - z \) was a possible solution for the sub solver. Of course, the solver may not return this particular solution, but still, it should have returned something.

If the algorithm terminates, it returns a solution.
2.4.2 Breaking the loop

In step 4, let consider the index \( k \notin D \) such that \( \frac{A_k x - b_k - d}{A_k v} \geq g \) is minimal. So, \( \frac{A_k x - b_k - d}{A_k v} A_k v = g A_k v \) (product by a negative). And, so, \( A_k x + \frac{A_k x - b_k - d}{A_k v} A_k v - b_k = A_k x + \frac{A_k x - b_k - d}{A_k v} A_k v - b_k = d \).

After a step 4.b some constraint outside \( D \) enters in \( D \).

But, \( D \) is bounded by \( \{1, \ldots, M\} \). So algorithm can not loop more than \( M \) times between two iterations reaching step 5. Now, imagine algorithm reaches two times the step 5 with twice the same value for \( D \) (in point \( x_1 \) and \( x_2 \)). Then, let consider the point \( z_1 \) and \( z_2 \) from step 6. Then, it holds both \( cz_2 < cz_1 \) (algorithm strictly decreases the score) and \( A_D z_2 = A_D z_1 = 0 \). So, step 3 should at least have returned \( z_2 - z_1 \), and, the algorithm should not have pass the step 4 when meeting \( x_1 \).

Yet, number of \( D \) is bounded by \( 2^M \). So the number of loops is less than \( M 2^M \).

The algorithm terminates.

More precisely, let \( D_1, ..., D_T \) be the set observed during the algorithm. For all \( t \), if \( D_t \) reaches step 5 it means there does not exist \( v \in \text{Ker}(A_D) \) such that \( vc < 0 \). Now, for all \( t, \tau \), if \( D_t \subset D_\tau \), it means the opposite: \( A_D(z_\tau - z_t) = 0 \) and \( cz_\tau < cz_t \) as \( cx \) is strictly decreasing.

This can even be refined by stressing that if \( D_t \) reaches step 5, then \( D_{t-1} \) is never again a subset of \( D_\tau \) (\( t < \tau \)). Indeed, if \( D_{t-1} \) is a subset of \( D_\tau \), then, let consider \( \theta = z_\tau - \sum_{i \in I_u \setminus I_{t-1}} \alpha_i A_i \). By definition of \( z_\tau \) and \( v_{t-1} \), \( A_D z_\tau, \theta = 0 \) but also \( A_D z_{t-1}, \theta = 0 \) by construction. Of course, \( \theta \) may not be admissible (\( A\theta \geq b \) may be false), but anyway, \( \theta - z \) should have been considered in step \( t \).

If \( D_t \) reaches step 5, then, \( D_{t-1} \notin D_\tau \) for all \( \tau > t \).

3 Complexity analysis

Termination is proven by forbidding value for \( D_t \). Yet, as there have a large number of possible value for \( D_t \) this is not very useful. In this section, I observe that in some situation, some constraints with definitely be rejected from the algorithm which is much more useful as there is only \( M \) constraints.

3.1 Dominated constraint

Let write \( \Gamma_R = \{ u \in \mathbb{Q}^R / u \geq 0, 1u = 1 \} \) the simplex of dimension \( R \). The size of \( \Gamma \) will be omitted.

Let \( A \in \mathcal{U}(\mathcal{M}_{M,N}(\mathbb{Q})) \) i.e. \( \forall m, A_m A_m = 1 \).

\( \forall u \in \mathbb{Q}^N \) such that \( A u \geq 1 \), let write \( I_u = \{ m / A_m u = 1 \} \), and, \( \Omega_u = \{ \sum_{i \in I_u} \alpha_i A_i / \alpha \in \Gamma \} \).
Now, let assume that both \( \exists v \in \mathbb{Q}^N \) such that \( Av \geq 1 \) and \( \exists k / A_k v > 1 \), with \( \frac{1}{A_k v} A_k \in \Omega_v \), and, \( \exists w \in \mathbb{Q}^N / \forall m \neq k 0 < A_k w < A_m w \).

As \( \frac{1}{A_k v} A_k \in \Omega_v \), then \( \exists \alpha \in \Gamma \) such that \( \frac{1}{A_k v} A_k = \sum_{i \in I_n} \alpha_i A_i \), and so, such that \( \frac{1}{A_k v} A_k w = \sum_{i \in I_n} \alpha_i A_i w \) but by assumption \( A_i w > A_k w \) so \( \frac{1}{A_k v} A_k w = \sum_{i \in I_n} \alpha_i A_i w > \sum_{i \in I_u} \alpha_i A_k w \) but \( \sum_{i \in I_u} \alpha_i A_k w = ( \sum_{i \in I_u} \alpha_i ) A_k w = A_k w \). So, \( A_k w < A_k w \), this is impossible.

For any, \( A \in U(\mathcal{M}_{M,N}(\mathbb{Q})) \), and any \( k \), if \( \exists v \in \mathbb{Q}^N \) such that \( Av \geq 1 \), \( A_k v > 1 \) and \( \frac{1}{A_k v} A_k \in \Omega_v \), then \( \exists w \in \mathbb{Q}^N / \forall m \neq k, 0 < A_k w < A_m w \).

This lemma can be interpreted: \( \Omega_v \) is the convex hull of \( A_i \) such that \( A_i v = 1 \) in the hyper plane \( H = \{ u / vu = 1 \} \). Now, all \( \frac{1}{A_k v} A_k \) are in \( H \) and more precisely in \( H \cap B \) with \( B = \{ u / uu \leq 1 \} \) and \( A_i \) such that \( A_i v = 1 \) are border point of \( H \cap B \).

Let stress that this is true for any \( u \), typically, the one that minimize \( uu \) (with \( A_k v > 1 \)) and/or the one that minimize \( \max A_m v \) and/or \( \max A_m v - \min A_m v \) and/or the one that maximize the area of \( \Omega_v \) regarding the area of \( H \cap B \).

In other words, \( \Omega_v \) is a convex hull of support vectors, all \( \frac{1}{A_k v} A_k \) are in the excircle of this convex hull, and, all \( \frac{1}{A_k v} A_k \) in the convex hull are dominated by support vectors.

The lemma can be interpreted as:

\[ A_k \text{ is non dominated only if } \frac{1}{A_k v} A_k \text{ is strictly between the convex hull } \Omega_v \text{ and its excircle.} \]

### 3.2 Application to the algorithm

Now, let consider \( \left( \begin{array}{c} A_D \\ -c^T \end{array} \right) \) \( w > 0 \), and, so let assume \( \exists d \in D_t \) such that \( \exists w \in \mathbb{Q}^N / \forall m \in D_t \{ k \} 0 < A_k w < A_m w, cw < 0 \).

If \( D_t \) reaches a step 5 and \( k \) enters in \( D_r (\tau > t) \), let consider \( x_t \) and \( x_r - d_r A_k \) and \( \theta = x_r - d_r A_k - z_r \). By construction \( A_k \theta = 0 \) while \( \forall m \in D_t \{ k \} A_m \theta > 0 \).

And, \( c \theta < 0 \) because \( x_r < z_r \) (due to step 7.b and the decreasing property of the algorithm) and \( -c A_k < 0 \).

So there is a contradiction.

For any set \( D \) observed in step 7, if the hypothesis of the lemma 3.1 are valid, then there is a constraint \( k \in D \) such that \( k \) can never enters in \( D \) in following steps 2.

If all sets reaching step 7 validate the hypothesis of the lemma 3.1, then the algorithm can not call the sub solver more than \( M \) times (plus \( M \) loop between each call of the sub solver). Thus, under this strong assumption, the algorithm is strongly polynomial itself, as loops are either simple projection or call to a strongly polynomial time algorithm.
Under the strong hypothesis that all vertex observed during the execution validate the hypothesis of the lemma 3.1, then the algorithm is strongly polynomial.

4 Lemmas

This section provide lemmas about the required assumption on linear program and the link between sub solver and Chubanov algorithm.

4.1 Linear feasibility and linear separability

[5] presents an algorithm to solve in strongly polynomial time the following problem: \( \exists x \in \mathbb{Q}^N / Ax = 0, x > 0 \) under the assumption that \( A \in M_{M,N}(\mathbb{Q}) \) has a rank of \( M \).

Let \( A \) a matrix without any assumption. Let consider the matrix \( A = (A \ -A \ -I) \) formed with \( A \) concat with \( -A \) concat with \( -I \) the opposite of identity matrix.

First, this matrix has full rank \( M \) due to the identity block.

Then, applying the Chubanov algorithm (or any linear feasibility solver) to this matrix \( A \) will lead (if a solution exists) to \( x_1, x_2, x_3 \) such that

\[
\begin{pmatrix}
  A & -A & -I
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} = 0
\]

and
\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix} > 0
\]

So, let \( x = x_1 - x_2 \), it holds that \( Ax = Ix_3 = x_3 > 0 \). And, if no solution exists, then Chubanov algorithm will provide a certificate.

Chubanov algorithm can be applied on a derived problem to solve linear separability \( \exists x \in \mathbb{Q}^N / Ax > 0 \) with \( A \in M_{M,N}(\mathbb{Q}) \) (under no assumption on \( A \), especially on the rank).

4.2 Normalizing linear program

If the linear program given as input is \( \min_{Ax \geq b} cx \) and verifies \( A \in U(M_{J,I}(\mathbb{Q})) \), \( b \in \mathbb{Q}^M \) with \( i \neq j \Rightarrow (A_i, b_i) \neq (A_j, b_j) \), \( c \in U(\mathbb{Q}^N) \) and \( Ac = \gamma 1 \) with \( \gamma > 0 \), and, \( cx \) being bounded by 0, then the offered algorithm of section 2 can be directly used.

Otherwise, the linear program has to be normalized with the following scheme:

1. If the linear program is as an optimisation problem (e.g. \( \max_{Ax \leq b, x \geq 0} cx \)), it should first be converted into an inequality system \( A'x \geq b' \). This could be done by combining primal and dual.
2. After that (or directly is input was an inequality system), an other normalisation is performed to reach required property (here with $\gamma = \frac{3}{5}$)

3. the important point is that from any linear program, pre processing can form an equivalent linear program meeting these requirements

4. In addition, if needed redundant rows are removed (trivial pre processing). Currently, algorithm can just be updated to directly handle identical rows. But this pre processing is that trivial that it is better to perform it: if $A_i = A_j$ after all normalisation, either $b_i = b_j$ and row $j$ can be removed either the row with smaller $b$ can be removed, because $A_i x \geq b_i > b_j \Rightarrow A_i x \geq b_j$...

### 4.2.1 Primal dual

The conversion of an optimisation linear program into a linear inequality system is quite classical. A brief recall is provided below.

Let assume original goal is to solve $\max \ A_{\text{raw}} x \leq b_{\text{raw}}, x \geq 0 \ c_{\text{raw}} x$. It is well known that the dual problem is $\min \ A_{\text{raw}}^T y \geq c_{\text{raw}}, y \geq 0$. Now, the primal dual is formed by combining all constraints: $A_{\text{raw}} x \leq b_{\text{raw}},$ and, $x \geq 0,$ and, $A_{\text{raw}}^T y \geq c_{\text{raw}},$ and $c_{\text{raw}} x = b_{\text{raw}} y$, and finally, $y \geq 0$.

So, the problem $\max \ A_{\text{raw}} x \leq b_{\text{raw}}, x \geq 0 \ c_{\text{raw}} x$ can be folded into $A_{\text{big}} x \geq b_{\text{big}}$ with $A_{\text{big}} = \begin{pmatrix} -A_{\text{raw}} & 0 \\ I & 0 \\ 0 & A_{\text{raw}}^T \\ 0 & I \\ c_{\text{raw}} & -b_{\text{raw}} \\ -c_{\text{raw}} & b_{\text{raw}} \end{pmatrix}$ and $b_{\text{big}} = \begin{pmatrix} -b_{\text{raw}} \\ 0 \\ c_{\text{raw}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

### 4.2.2 Normalized primal dual error minimization

This normalisation step takes a linear program $\Gamma x \geq \beta$ as input, and, produces an equivalent linear program $\min \ c x$ with $A \in \mathbb{U}(M_{J,1}(\mathbb{Q})), \ b \in \mathbb{Q}^M, \ c \in \mathbb{U}(\mathbb{Q}^N)$, and, $Ac = \frac{\beta}{2} 1$, and, $cx$ being bounded by 0.

It is sufficient to consider:

$$A = \begin{pmatrix} 4 \ & \ 4 \ & \ 4 \\ 5(\frac{1}{2} + \frac{1}{M} \ + 1) \ & \ 5(\frac{1}{2} + \frac{1}{M} \ + 1) \ & \ 5(\frac{1}{2} + \frac{1}{M} \ + 1) \\ \cdots \ & \ \cdots \ & \ \cdots \\ 4 \ & \ 4 \ & \ 4 \\ 5(\frac{1}{2} + \frac{1}{M} \ + 1) \ & \ 5(\frac{1}{2} + \frac{1}{M} \ + 1) \ & \ 5(\frac{1}{2} + \frac{1}{M} \ + 1) \\ 0 \ & \ 0 \ & \ 0 \\ 0 \ & \ 0 \ & \ 0 \\ 0 \ & \ 0 \ & \ 0 \\ \end{pmatrix}$$
and
\[
b = \begin{pmatrix}
\frac{4}{5(\Gamma_m + 1)} \beta_1 \\
\vdots \\
\frac{4}{5(\Gamma_m + 1)} \beta_M \\
0 \\
0 \\
0 \\
0
\end{pmatrix}, \quad c = \begin{pmatrix}
0 \\
\vdots \\
0 \\
0 \\
1
\end{pmatrix}
\]

First, the produced linear program is in the desired form: \( \min \ cx \) with \( A x \geq b \).

Trivially, \( A c = \frac{2}{3} 1 \) by construction, and, all rows of \( A \) are normalized either directly because \( (\frac{2}{3})^2 + (\frac{2}{3})^2 = 1 \), or, because of that, and the fact that,
\[
(\frac{1}{\Gamma_m + 1})^2 \Gamma_m \Gamma + (\frac{1}{\Gamma_m + 1})^2 (\Gamma_m \Gamma + 1)^2 (\frac{1}{\Gamma_m + 1})^2 \times (\Gamma_m \Gamma + 1) = (\frac{1}{\Gamma_m + 1})^2 \times (\Gamma_m \Gamma + 1)^2 \text{ which is } 1!
\]

Then, the 4 last constraint prevent \( x_{N+3} \) to be negative so \( cx \) is well bounded by 0. Indeed, if \( x_{N+2} + x_{N+3} \geq 0 \) and \( -x_{N+2} + x_{N+3} \geq 0 \), then \( x_{N+3} \geq 0 \), and, when \( x_{N+3} = 0 \) these constraints force \( x_{N+1} = x_{N+2} = 0 \) because the three constraints \( x_{N+2} + x_{N+3} \geq 0 \), \( -x_{N+2} + x_{N+3} \geq 0 \), and \( x_{N+3} = 0 \) can be reduced to \( x_{N+2} \geq 0 \) and \( -x_{N+2} \geq 0 \) which force \( x_{N+2} = 0 \).

Now, the goal is to minimize \( cx = x_{N+3} \). So, either the minimum is \( x_{N+3} = 0 \) or either there is no such solution. In the case \( A x \geq b, x_{N+3} = 0 \), it holds \( x_{N+1} = x_{N+2} = x_{N+3} = 0 \), and, \( x_1, ..., x_N = \chi \) with \( 4(\frac{1}{\Gamma_m + 1}) \Gamma_m \chi \geq 5(\frac{4}{\Gamma_m + 1}) \beta_m \). But this last inequality can be reduced to \( \Gamma_m \chi \geq \beta_m \). So, if the solution of the derived linear program is \( x \) with \( A x \geq b, x_{N+3} = 0 \), then \( x_1, ..., x_N = \chi \) is a solution of the original set of inequality.

And, inversely, if there is a solution \( \chi \), then, \( x = \chi, 0, 0, 0 \) is a solution of the optimisation problem (because \( x_{N+3} \) is bounded by 0).

So, this derived linear program is equivalent to the inequality set.

For any linear program, it is possible to create a derived form meeting the requirement of the offered algorithm.

All this normalization is entirely done in \( \mathbb{Q} \) i.e. no square root are needed.

**Discussion**

This short paper presents an algorithm based on the idea of greedy optimisation with frozen combinatorial structure, and, linear feasibility queries which allows to refine the combinatorial structure. To give a metaphor, greedy moves slides on linear constraints, and linear feasibility allows to jump on new ones.

The first interest of this *slide and jump* idea is it ability to solve both linear program and support vector margin (both algorithms are almost identical).
And, the second is that algorithms are strongly polynomial for both problems under a same convex hull assumption. Thus, this short paper presents a link between linear feasibility, linear program, support vector margin and geometric complexity of convex hull.

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