Enumerating randoms

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Abstract

We investigate enumerability properties for classes of random reals which permit recursive approximations from below. For four classical notions of randomness, Martin-Löf randomness, computable randomness, Schnorr randomness, and Kurtz randomness, as well as for bi-immunity, we detail whether the left-recursive enumerable members can be enumerated, and similarly for the complementary left-r.e. classes. We prove a general equivalence between arithmetic complexity and existence of numberings for classes of left-r.e. reals and give optimal arithmetic hardness results.

1 Effective randomness

Think of a real number between 0 and 1. Is it random? In order to give a meaningful answer to this question, one must first obtain an expression for the real number in mind. Any coherent language contains no more than countably many expressions, and therefore we must always settle for a language with uncountably many indescribable reals. On the other hand,

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there exists a natural and robust class of real numbers which admit recursive increasing approximations. We call such numbers left-r.e. reals. Brodhead and Kjos-Hanssen [3] observed that there exists an effective enumeration, or numbering, of the left-r.e. reals, and Chaitin [4] showed that some left-r.e. reals are Martin-Löf random. Random left-r.e. reals thus serve as a friction point between definability and pure randomness.

In the following exposition we examine which classes of left-r.e. randoms and non-randoms admit numberings (and are therefore describable). A related definability question also arises, namely how difficult is it to determine whether a real is random? As a means of classifying complexity, we place the index sets for left-r.e. randoms inside the arithmetic hierarchy. One can view this program as a continuation of work by Hitchcock, Lutz, and Terwijn [9] which places classes of (non-effective) randoms inside the Borel hierarchy. In contrast with the case of r.e. sets, we shall find a close connection between numberings and arithmetic complexity for classes of left-r.e. reals.

Notation. Some standard notation used in this article includes ∀∞ which denotes “for all but finitely many” and ∃∞ which means “there exist infinitely many.” X↾n is the length n prefix of X, and denotes concatenation. For finite sequence σ and τ, σ ≤ τ means σ is a prefix of τ, σ < τ indicates that σ is a proper prefix of τ, and |σ| is the length of σ. For non-negative integers x, |x| is the ceiling of log(x + 1). ⟨·, ·⟩ : ω × ω → ω is some recursive pairing function which we fix for rest of the paper. For sets A and B, A ⊕ B = {2n : n ∈ A} ∪ {2n + 1 : n ∈ B}. ′ is the jump operator, µ is the unbounded search operator, ↓ means converge, and A ≤T B means A Turing reduces to B. As usual, 0′ denotes the halting set. For further background on recursion theory and algorithmic randomness, see [23] and [1].

A real is an infinite sequence of 0’s and 1’s and corresponds to the binary expansion of a number in [0, 1]. A machine M is said to be prefix-free if for any distinct finite strings σ, τ ∈ dom M, σ is not a prefix of τ. The prefix-free complexity of a string σ with respect to a prefix-free machine M is given by K_M(σ) = min{|p| : M(p) = σ}. Furthermore, there exists a universal prefix machine U such that for any prefix-free machine M, K_U(σ) ≤ K_M(σ) + O(1) for all σ ∈ {0, 1}*. We fix K = K_U for the remainder of this exposition.

Definition 1.1. A real X is called Martin-Löf random [16] if

(∃c) (∀n) [K(X↾n) ≥ n − c].

(1.1)
Intuitively, every prefix of the string \( X \) in (1.1) is incompressible and therefore admits no simple description.

A martingale \( M : \{0,1\}^* \to \mathbb{R}^+ \) is a function satisfying the fairness condition: for all \( \sigma \in 2^{<\omega} \),

\[
M(\sigma) = \frac{M(\sigma0) + M(\sigma1)}{2}.
\]

The martingale \( M \) succeeds on a sequence \( X \) if \( \limsup M(X \upharpoonright n) = \infty \). If \( M \) succeeds on \( X \) and there exists an unbounded recursive function \( g \) satisfying

\[
(\exists \infty n) [g(n) \leq M(X \upharpoonright n)],
\]

we say that \( M \) Schnorr-succeeds on \( X \). The idea behind Definition 1.2 is that no gambling strategy can achieve arbitrary wealth by betting on a random sequence.

**Definition 1.2** \((20, 21)\). A real \( X \) is called computably random \(20\) if no recursive martingale succeeds on \( X \). A real \( X \) is Schnorr random if no recursive martingale Schnorr-succeeds on \( X \).

\( C \) is a \( \Pi^0_1 \)-class if there exists a recursive predicate \( R \) such that

\[
C = \{X : (\forall n) [R(X \upharpoonright n)]\}.
\]

**Definition 1.3.** A real \( X \) is Kurtz random if it does not belong to any null \( \Pi^0_1 \)-class. In other words, \( X \) belongs to every recursive open class of Lebesgue measure 1.

The classes of randoms mentioned above relate to each other as follows:

**Theorem 1.4** (see \([17\) or \([7\]). Martin-Löf randomness \(\Longrightarrow\) computable randomness \(\Longrightarrow\) Schnorr randomness \(\Longrightarrow\) Kurtz randomness.

## 2 Acceptable numberings

A numbering \( \varphi \) is a partial-recursive (p.r.) function \( (e,x) \mapsto \varphi_e(x) \). A numbering \( \varphi \) is precisely a programming language, and \( \varphi_e \) is the \( e \)th program in that language. A real number \( X \in [0,1] \) is called left-r.e. if it can be written in the form

\[
X = \sum_{x \in \text{dom} \varphi_e} 2^{-|x|}.
\]

for some numbering \( \varphi \). Chaitin showed \([4\] that there exists a left-r.e. Martin-Löf random real which he called \( \Omega \), hence each of the classes in Theorem 1.4 contains a left-r.e. member.
**Definition 2.1.** Let \( C \) be a class of left-r.e. reals. A *left-r.e. numbering* of \( C \) is a function with range \( C \) given by

\[
e \mapsto \sum_{\sigma \in \text{dom } \varphi_e} 2^{-|\sigma|}
\]

for some numbering \( \varphi \). A left-r.e. numbering of all left-r.e. reals is called *universal*. Similarly, an *r.e. numbering* of a class \( C \) is a mapping \( e \mapsto \text{dom } \varphi_e \) for some numbering \( \varphi \), and an r.e. numbering is *universal* if every r.e. set appears in its range.

Universal left-r.e. numberings do indeed exist \[3\]: if \( \varphi_e \) induces a universal r.e. numbering, then \( \varphi_e \) also induces a universal left-r.e. numbering. Garden variety numberings in recursion theory satisfy the so-called *s-m-n Theorem* \[23\] and are called *acceptable* numberings:

**Definition 2.2.** A (left-r.e.) numbering \( \varphi \) is called a (left-r.e.) *Gödel numbering* or acceptable (left-r.e.) numbering if for every (left-r.e) numbering \( \psi \) there exists a recursive function \( f \) such that \( \varphi_f(e) = \psi_e \) for all \( e \).

Intuitively, the function \( f \) in Definition 2.2 translates code from program \( \psi \) into program \( \varphi \). Thus acceptable numberings are maximal: any given numbering can be uniformly translated into any acceptable one. Furthermore, any two acceptable numberings are isomorphic in the sense of \[19\]. These two properties make the notion of an acceptable numbering rather robust.

We shall use capital letters to denote sets of reals, but we reserve the capital letter \( W \) for r.e. numberings. Greek letters \( \sigma \) and \( \tau \) will denote finite binary strings, \( \varphi \) and \( \psi \) will denote numberings, and \( \alpha, \beta, \gamma, \) and \( \zeta \) will be left-r.e. numberings.

**Definition 2.3.** A set \( A \subseteq \omega \) is a called a \( \Sigma_n \) set if it is \( \Sigma_n^0 \) in the usual sense of recursion theory. The complement of a \( \Sigma_n \) set is a \( \Pi_n \) set. We say that a set \( A \) many-to-one reduces to a set \( B \), or \( A \leq_m B \), if there exists a recursive function \( f \) such that for all \( x, x \in A \iff f(x) \in B \). A set \( A \) is called \( \Sigma_n \)-hard (resp. \( \Pi_n \)-hard) if for every \( \Sigma_n \) (resp. \( \Pi_n \)) set \( X \), \( X \leq_m A \). A set \( A \) is \( \Sigma_n \) (resp. \( \Pi_n \)) complete if \( A \) is a \( \Sigma_n \) (resp. \( \Pi_n \)) set and \( A \) is \( \Sigma_n \)-hard (resp. \( \Pi_n \)-hard).

We shall make use of the following classical theorem, and we will prove an analogue for left-r.e. index sets in Theorem \[3.5\]
Theorem 2.4 (Σ₃-Representation Theorem [23]). Let \( W₀, W₁, W₂, \ldots \) be an acceptable universal r.e. numbering, and let \( A \) be a Σ₃-set. Then there exists a recursive function \( f \) such that for all \( x \),

\[
x \in A \iff (\forall y) [W_{f(x,y)} = \omega];
\]

\[
x \notin A \iff (\forall y) [W_{f(x,y)} \text{ is finite}].
\]

We first show that there is no canonical way to number random sets via acceptable left-r.e. numberings. The class of left-r.e. random reals is a natural example of a class which has a left-r.e. numbering but no maximal (i.e. acceptable) numbering.

Definition 2.5. Let \( C \) be a class of reals. A real \( X \) is a shift-persistent element of \( C \) if \( \sigma \downarrow X \in C \) for every prefix \( \sigma \).

We call a real infinite if its binary expansion contains both infinitely many 1’s and infinitely many 0’s and finite otherwise. This definition for reals highlights an important distinction between sets and reals. For any prefix \( \sigma \), the real number \( \sigma 01111 \ldots \) equals \( \sigma 10000 \ldots \). Hence there is no difference between the set of “finite” reals and the set of “co-finite” reals. For the same reason, and unlike the case for sets, there is no difference between “infinite” and “co-infinite” reals. The existence of a left-r.e. numbering for the infinite r.e. sets (under the usual definition of infinite) is thus a triviality: any left-r.e. numbering of the r.e. sets is also an enumeration of the infinite r.e. sets.

Theorem 2.6. Assume that a family \( C \) has a shift-persistent element and does not contain all infinite left-r.e. reals \( R \) with \( 0 < R < 1 \). Then \( C \) does not have an acceptable left-r.e. numbering.

Proof. Let \( X \) be a shift-persistent member of \( C \), let \( R \) be the missed out infinite left-r.e. real with \( 0 < R \leq 1 \), and let \( \sigma₀, \sigma₁, \sigma₂, \ldots \) be a recursive approximation of \( R \) from the left such that all \( n \) satisfy \( \sigma_n111 \ldots < \sigma_{n+1}000 \ldots < R \) (as reals). Every infinite left-r.e. real has such an approximation. Suppose \( \alpha₀, \alpha₁, \alpha₂ \ldots \) is an acceptable left-r.e. numbering of \( C \).

Now there is a 0'-recursive function \( F \) such that \( F(n) \) is the first \( m \) such that the first \( m \) bits of \( R \) differ from the first \( m \) bits of every \( \alpha_k \) with \( k \leq c₀(n) \) where \( c₀ \) is the convergence module of \( \Omega \); note that \( c₀ \) dominates every recursive function. This function \( F \) has an approximation \( F_s \) and now one takes the set \( \beta_n = \sigma_s \downarrow X \) for the first stage \( s \) such that for all \( t \geq s \) it holds that \( F_t(n) = F_s(n) \) and the first \( F_s(n) \) bits of \( \sigma_t \) exist and are equal.
to those of $\sigma_s$. Note that this $\sigma_s$ can be found as the function values $F_t(n)$ converge to $F(n)$ and similarly the $\sigma_t$ converge to $R$.

Each set $\beta_n$ is in the list $\alpha_0, \alpha_1, \alpha_2, \ldots$ by definition of $X$. Furthermore, $\beta_n$ coincides with $R$ on its first $F(n)$ bits while every $\alpha_k$ with $k \leq c_{\Omega}(n)$ differs from $R$ on its first $F(n)$ bits. Hence $\beta_n \notin \{\alpha_0, \alpha_1, \ldots, \alpha_{c_{\Omega}(n)}\}$. It follows that there is no recursive function $f$ with $\beta_n = \alpha_{f(n)}$ for all $n$ as $c_{\Omega}$ would dominate $f$. Thus the numbering $\alpha_0, \alpha_1, \alpha_2, \ldots$ cannot be an acceptable numbering of the left-r.e. sets of its type.

\[ \square \]

**Corollary 2.7.** There is no acceptable left-r.e. numbering of either the left-r.e. randoms or the left-r.e. non-randoms (under any reasonable definition of random).

### 3 Arithmetic classification via numberings

Unlike r.e. numberings, the existence of left-r.e. numberings admits a neat characterization in terms of $\Sigma_3$ sets. As a corollary, we will get that the left-r.e. Martin-Löf random reals are enumerable but not co-enumerable. For convenience we introduce the following operator on finite strings.

**Definition 3.1.** For any finite binary string $\sigma$, $\sigma \downarrow$ denotes the string $\sigma$ with the maximum 1 changed to a 0 (if it exists). If $\sigma$ consists of all zeros, then $\sigma \downarrow = \sigma$.

A refinement of the following result appears in [17, Theorem 3.5.21] using an alternate proof.

**Lemma 3.2** (Nies [17]). Let $X$ be a real which infinitely often has a prefix of length $n$ followed by $(n + 2) \cdot 2^n$ 1’s. Then $X$ is not Schnorr random.

**Proof.** We exhibit a martingale which Schnorr-succeeds on $X$. The betting strategy is as follows. For simplicity, let us assume that we start with $3.00. For the initial bet, place $1 on the “1” outcome. Now suppose we have already seen a string $\sigma$ of length $n$. If the last digit of $\sigma$ is “0,” then bet $2^{-n}$ dollars on the “1” outcome. Otherwise, make the same bet that was made the last time.

We claim this martingale succeeds on $X$. The martingale loses at most $2^{-n}$ dollars from betting on the $(n + 1)^{st}$ digit of $X$. Thus the total money lost from playing over an infinite amount of time is at most $2.00$. On the other hand, we are bound to eventually reach a string of length $(n + 2) \cdot 2^n$ immediately following $X \upharpoonright n$. At this point, $2^{-n}$ dollars will be wagered
\((n + 2) \cdot 2^n\) times in a row, for a net gain of at least \(n\) dollars over the interval of zeros. By assumption on \(X\) we reach such points infinitely often, and therefore the winnings go to infinity.

Finally we exhibit a recursive function which infinitely often is a correct lower bound for the gambler’s capital. Define a recursive function which guesses at each position that we are at the end of an interval of \((n + 2) \cdot 2^n\) zeros. The function always outputs \(n\) where \(n\) is the length of the corresponding interval that would have preceded the long string of zeros. If no such integer \(n\) exists, then output 1. Infinitely often this guess will be correct and, as noted in the previous paragraph, we will indeed have at least \(n\) dollars at this point.

**Theorem 3.3.** Let \(A \subseteq \omega\) be a \(\Sigma_3\)-set, and let \(\alpha\) be an acceptable universal left-r.e. numbering. Then there exist a recursive functions \(g\) such that

\[
x \in A \implies \alpha_{g(x)} \text{ is Martin-Löf random;}
\]

\[
x \notin A \implies \alpha_{g(x)} \text{ is not Schnorr random.}
\]

**Proof.** Let \(W\) be an acceptable universal r.e. numbering. Without loss of generality, assume that for all \(e\) at most one element of \(e\) enters \(W_e\) at each stage of its enumeration \(\{W_{e,s}\}\) and furthermore at least one \(W_e\) increases at each stage. By the \(\Sigma_3\)-Representation Theorem 2.4, there exists a function \(f\) satisfying:

\[
x \in A \implies W_{f(x,n)} \text{ is infinite for some } n;
\]

\[
x \notin A \implies W_{f(x,n)} \text{ is finite for all } n.
\]

For each \(x\) and \(s\), let

\[
\sigma_{0,s}^x = \Omega_s \mathrel{\upharpoonright} |W_{f(x,0),s}|,
\]

let

\[
m(e, s) = \text{greatest stage } t + 1 < s \text{ such that } \max\{x : \Omega_{e,t+1}(x) = 1\} \neq \max\{x : \Omega_{e,t}(x) = 1\},
\]

and inductively define

\[
\sigma_{n+1,s}^x = 1^{(|\sigma_{n,s}^x|+2) \cdot 2^{|\sigma_{n,s}^x|}} \cdot (\Omega_s \mathrel{\upharpoonright} |W_{f(x,n+1),m[f(x,n+1),s]}|). \tag{3.1}
\]

Roughly speaking, (3.1) consists of a long string of 1’s followed by an approximation of \(\Omega\). By Lemma 3.2 there are enough 1’s that if all the \(\sigma_n^x\)’s
remain finite, then (3.2) is not Schnorr random. On the other hand, if some $\sigma_n^x$ does blow up to infinity, then (3.2) becomes the Martin-Löf random $\Omega$ with some finite prefix attached.

Define the recursive function $g$ by

$$\alpha_{g}(x) = \lim_{s} \sigma_{0,s}^x \overline{\sigma}_{1,s}^x \overline{\sigma}_{2,s}^x \ldots$$

(3.2)

We verify that the approximation in (3.2) is left-r.e. by analyzing the change between stages $s$ and $s+1$. By induction, the length of $\sigma_{n,t}^x$ is increasing in $t$ for every $n$. Let $e$ be the least index such that $\sigma_{e,s+1}^x$ is longer than $\sigma_{e,s}^x$. By minimality, the prefix of 1’s at the beginning of this string must remain unchanged but the approximation to $\Omega$ increases. In particular,

$$|W_{f(x,e),m[f(x,e),s]}| \neq |W_{f(x,e),m[f(x,e),s+1]}| \downarrow,$$

Due to the $\downarrow$ operator, the 0 at some existing position changes to a 1 in stage $s+1$. Hence $\sigma_{e}^x$ can expand in stage $s+1$ while permitting a left-r.e. approximation for (3.2). Finally, the limit in (3.2) exists because the sequence of reals is increasing and bounded from above.

Suppose that $W_{f(x,e),m[f(x,e),s]}$ is infinite for some $n$, and let $e$ be the least such index. By minimality, $\sigma_{j}^x = \lim_{s} \sigma_{j,s}^x$ is finite for all $j < e$. Hence for $e > 0$,

$$\alpha_{g}(x) = \sigma_{0}^x \overline{\sigma}_{1}^x \overline{\sigma}_{2}^x \ldots \overline{1(|\sigma_{e}^x|+2)2^{\sigma_{e}^x}2^{2\sigma_{e}^x}1}' \Omega,$$

which is Martin-Löf random. All $\sigma_{n}^x$ with $n > e$ get “kicked to infinity.” The case $e = 0$ is similar.

On the other hand, suppose that $W_{f(x,n)}$ is finite for all $n$. In this case $\sigma_{0}^x$ is finite, and

$$\sigma_{n+1}^x = 1(|\sigma_{n}^x|+2)2^{\sigma_{n}^x}2^{2\sigma_{n}^x}1 \Omega_{s_{n}} \downarrow |W_{f(x,n+1),m[f(x,n+1),s_{n}]}| \downarrow,$$

where $s_{n}$ is the final stage where $W_{f(x,n+1)}$ increases. Thus infinitely often $\alpha_{g}(x)$ has a prefix of length $|\sigma|$ followed by $(|\sigma|+2)2^{\sigma}$ 1’s. By Lemma 3.2, $\alpha_{g}(x)$ is not Schnorr random.

Corollary 3.4. In any acceptable universal left-r.e. numbering, the indices of the left-r.e. Martin-Löf random reals, computable random reals, and Schnorr random reals are $\Sigma_3$-hard.

A [left-r.e. or r.e.] numbering is called a [left-r.e. or r.e.] Friedberg numbering if every member in its range has a unique index. Friedberg initiated
the study of these numberings in 1958 when he showed that the r.e. sets can be enumerated without repetition [8]. More recently Kummer [13] gave a simplified proof of Friedberg’s result, and Brodhead and Kjos-Hanssen [3] adapted his idea to show that there exists a left-r.e. Friedberg numbering of the left-r.e. Martin-Löf random sets. We now show that left-r.e. Friedberg numberings can be used to characterize $\Sigma_3$-index sets.

**Theorem 3.5.** Let $C$ be a class of infinite left-r.e. reals which contains a shift-persistent element. Then for any universal left-r.e. numbering $\alpha$, the following are equivalent:

(i) $\{e : \alpha_e \in C\}$ is a $\Sigma_3$-set.

(ii) There exists a left-r.e. numbering of $C$.

(iii) There exists a left-r.e. Friedberg numbering of $C$.

**Proof.** Let $\alpha$ be any universal left-r.e. numbering, and let $C_{\alpha} = \{e : \alpha_e \in C\}$.

(i) $\iff$ (ii). Suppose that $\beta$ is a left-r.e. numbering for $C$. Then

$$C_{\alpha} = \{e : (\exists d) [\alpha_e = \beta_d]\} = \{e : (\exists d) (\forall n, s) (\exists t > s) [\alpha_{e, t} \upharpoonright n = \beta_{d, t} \upharpoonright n]\}.$$

Conversely, assume that $C_{\alpha} \in \Sigma_3$ and let $\gamma$ be an acceptable universal left-r.e. numbering. By Theorem 3.3 there exists a recursive function $g$ such that $e \in C_{\alpha} \iff \gamma_{g(e)}$ is Martin-Löf random.

For reals $X$, let

$$r_b(X) = \max\{n : (\forall m \leq n) [K(X \upharpoonright m) \geq m - b]\},$$

and without loss of generality, assume that $\alpha_{e,s}$ has finitely many 1’s at each stage $s$ of the recursive approximation. Let $C$ be a shift-persistent element of $C$, and let $C_0, C_1, C_2 \ldots$ be a left-r.e. approximation for $C$. Since we want to avoid dealing with $\alpha_e$’s which are equal to 0, let

$$f(e) = e^{th} \alpha\text{-index found to be nonzero},$$

and let $t(e)$ be the first stage at which $\alpha_{f(e)}$ appears to be nonzero. For notational convenience, let

$$q(e) = \min\{x : \alpha_{f(e), t(e)}(x) = 1\},$$
and let

\[
\xi_{(e,b),s} = \begin{cases}
0^{g(e)} & \text{if } \left| r_b(\gamma_{g[f(e)],s}) \right| \leq q(e); \\
\alpha_{f(e),s} \upharpoonright r_b(\gamma_{g[f(e)],s}) & \text{otherwise}
\end{cases}
\]

be the prefix of \(\alpha_{f(e),s}\) that has the length of \(\gamma_{g[f(e)]}\)'s prefix which looks random at stage \(s\). Let

\[
m(e,s) = \text{greatest stage } t + 1 < s \text{ such that }
\max \{ x : \alpha_{f(e),t+1}(x) = 1 \} \neq \max \{ x : \alpha_{f(e),t}(x) = 1 \}.
\]

Now define a further left-r.e. numbering \(\beta\) by

\[
\beta_{e,b,s+1} = \xi_{(e,b),m(e,s)} \upharpoonright \gamma_{g[f(e)],s+1}.
\]

(3.3)

The operator \(\upharpoonright\) in (3.3) is needed to ensure that \(\beta\) is a left-r.e. numbering: whenever \(r_b(\gamma_{g[f(e)],m(e,s+1)}) \neq r_b(\gamma_{g[f(e)],m(e,s)})\), this expansion is handled by replacing a “0” with “1” which clears the higher indices, making room for \(C_{s+1}\).

Finally, \(\beta_0, \beta_1, \ldots\) is a left-r.e. numbering for \(C\). Indeed,

\[
f(e) \in C_{\alpha} \implies (\exists b) \left[ \gamma_{g[f(e)]} \right. \text{ is Martin-Löf random with constant } b \]
\[
\implies \beta_{(e,b)} = \alpha_{f(e)}.
\]

Of course a \(\beta\)-index for the real \(0\) can be added if necessary.

In the case where \(\gamma_{g[f(e)]}\) is not Martin-Löf random with constant \(b\), \(C_s\) does not get “kicked to infinity” but then \(\beta_{(e,b)} \in C\) because \(C\) is a shift-persistent member of \(C\). \(\square\)

(ii) \iff (iii). Assume that \(C\) has a numbering \(\gamma\). Let \(C\) be a shift-persistent element of \(C\), and let

\[
B = \{ 1^n \upharpoonright C : n \in \omega \} \cup \{ X \in C : X < C \}
\]

be a subclass of \(C\). Note that

\[
A := \{ X : X \in C - B \} = \{ X \in C : (\exists n) \left[ 1^n \upharpoonright C < X < 1^{n+1} \upharpoonright C \right] \}
\]

has a left-r.e. numbering \(\alpha\) given by:

\[
\alpha_{(e,n,k),s} = \begin{cases}
1^n \upharpoonright C_s + 2^{-k} & \text{if } (\gamma_{e,s} \upharpoonright k)^0 \leq (1^n \upharpoonright C_s \upharpoonright k)^0; \\
\gamma_{e,s} & \text{if } (1^n \upharpoonright C_s \upharpoonright k)^0 < (\gamma_{e,s} \upharpoonright k)^0 < (1^{n+1} \upharpoonright C_s \upharpoonright k)^0; \\
1^{n+1} \upharpoonright C_s - 2^{-k} & \text{if } (1^{n+1} \upharpoonright C_s \upharpoonright k)^0 \leq (\gamma_{e,s} \upharpoonright k)^0.
\end{cases}
\]
where the triple \( \langle e, n, k \rangle \) ranges over values \( k \) which are greater than or equal to the index of the least 0 in \( 1^n \sim C \). Strictly speaking, every tail of \( C \) must be a shift-persistent element in order that each \( \alpha \)-index yields a member of \( C \). Since every member of \( C \) is infinite, however, we can overcome this shortcoming by modifying the tails for \( \alpha_{(e,n,k),s} \) to be \( C_s \) in the first and third cases. Let \( \beta \) be a left-r.e. numbering for \( B \).

Using \( \alpha \) and \( \beta \), we now exhibit a Friedberg numbering \( \zeta \) for \( A \cup B = C \). Let

\[
M = \{ e : (\forall j < e) [\alpha_j \neq \alpha_e] \}.
\]

Every member of \( A \) has a unique index in \( M \). Since \( M \) is a \( \Sigma_2 \)-set, there exists a 0'-recursive function \( m \) whose domain is \( M \). Let \( m_0, m_1, m_2, \ldots \) be a recursive approximation to \( m \). Using this approximation, we shall design \( \zeta \) in such a way that each \( \alpha \)-indexed real in \( M \) occurs at exactly one \( \zeta \)-index, and the remaining \( \zeta \)-indices will be home to the \( \beta \)-indexed reals.

We define a function \( f : \omega \mapsto (\omega \cup \{\infty\}) \times \{\alpha, *\} \) which maps \( \zeta \)-indices to either \( \alpha \)-indices or *'s. The \( \infty \) symbol is used for destroyed \( \beta \) indices which are (or never were) attached to \( \zeta \)-indices, and the \( \alpha \) and * symbols indicate whether the particular \( \zeta \)-index is following an \( \alpha \)-index or a \( \beta \)-index. If \( f(e) = (x, *) \) for some \( x \), \( f(e) \) "explodes" and we say that the \( \zeta \)-index \( e \) has been destroyed. \( f_s : \omega \mapsto ([\omega \cup \{\infty\}] \times \{\alpha, *\}) \cup \{\uparrow\} \) will be a recursive approximation to \( f \) based on the recursive approximation \( m_s \). \( \zeta \)-indices that are destroyed at some stage take on \( \beta \)-indices in the limit (rather than \( \alpha \)-indices). We shall also keep track of which \( \beta \)-indices have been taken on by \( \zeta \)-indices: \( G_s \) will be the set of \( \beta \)-indices which have been \( \zeta \)-used by stage \( s \). We will achieve \( \lim G_s = \omega \). Since \( C \) contains only infinite sets, every \( \alpha \)-indexed real is less than some \( \beta \)-indexed real, and therefore we can use \( \beta \) as a garbage can to collect for those approximations \( m_s(e) \) which turned out to be wrong. We shall also have an auxiliary recursive function \( r(s) \) which marks the boundary between the \( \zeta \)-indices which are following values in \( \omega \cup \{*\} \) and those whose value is \( \uparrow \) at stage \( s \).

The construction is as follows:

**Stage 0.**

Set \( G_0 = \emptyset \), \( r(0) = 0 \), \( f_0(e) = \uparrow \), and \( \zeta_{e,0} = 0 \) for all \( e \geq 0 \).

**Stage \( s + 1 \).** Let

\[
A = \{ x < s : m_{s+1}(x) \uparrow \text{ and } m_s(x) \downarrow \},
\]

\[
X = \{ x < s : m_{s+1}(x) \downarrow \text{ and } m_s(x) \uparrow \}.
\]
let \( \{a_1, a_2, \ldots, a_k\} \) be the indices below or equal to \( r(s) \) satisfying \( f_s(e_i) \in A \), and let \( \{x_1, x_2, \ldots, x_d\} \) be the indices below or equal to \( r(s) \) satisfying \( f_s(e_i) \in X \). We destroy all followers of \( \{x_1, \ldots, x_d\} \), and create new followers for \( \{a_1, \ldots, a_k\} \):

\[
f_{s+1}(n) = \begin{cases} 
\langle x_i, * \rangle & \text{if } f_s(n) = \langle x_i, \alpha \rangle \text{ for some } 1 \leq i \leq d; \\
\langle a_i, \alpha \rangle & \text{if } n = r(s) + i \text{ for some } 1 \leq i \leq k; \\
\langle s, \alpha \rangle & \text{if } n = r(s) + k + 1; \\
\langle \infty, * \rangle & \text{if } n = r(s) + k + 2; \\
f_s(n) & \text{otherwise.}
\end{cases}
\] (3.4)

The \( \zeta \)-index \( r(s) + k + 1 \) is used to introduce a new \( \alpha \)-index, and the \( \zeta \)-index \( r(s) + k + 2 \) is used to ensure that some new \( \beta \)-index is taken up at this stage. Set \( r(s+1) = r(s) + k + 2 \).

Next, assign new reals from \( B \) to the \( \zeta \)-indices that were destroyed in this stage.

- Let

\[ y_1 = (\mu n)[\beta_{n,s} > \zeta_{x_1,s} \land n \notin G_s] \]

and inductively for \( 0 \leq i \leq d \),

\[ y_{i+1} = (\mu n)[\beta_{n,s} > \max\{\zeta_{x_{i+1,s}, \beta_{y_i}}\} \land n \notin G_s]. \]

Choose the least \( \beta \)-index not yet assigned to a \( \zeta \)-index and call it \( z \):

\[ z = \min\{n : n \notin \{y_1, y_2, \ldots, y_d\} \text{ and } n \notin G_s\}. \] (3.5)

This choice of \( z \) ensures that every member of \( B \) will have some index in \( \zeta \).

- Set

\[ \zeta_{n,t} = \begin{cases} 
\beta_{y_i,t} & \text{if } f_{s+1}(n) = x_i \text{ for some } 1 \leq i \leq d; \\
\beta_{z,t} & \text{if } n = r(s + 1).
\end{cases} \]

for all \( t > s \).

- Set \( G_{s+1} = G_s \cup \{y_0, y_1, \ldots, y_k, z\} \).

For the remaining \( \zeta \)-indices which have not been destroyed in this stage or some previous stage, continue following \( \alpha \)-indices:

\[
\zeta_{e,s+1} = \begin{cases} 
0 & \text{if } f_{s+1}(e) = \uparrow; \\
\alpha_{f_{s+1}(e),s+1} & \text{if } f_{s+1}(e) \notin \{(n, *) : n \in \omega\} \cup \{\uparrow\}. 
\end{cases}
\] (3.6)

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By induction on stages, (3.4) and (3.6) ensure that for all $s$ and $e \leq s$, there exists a unique $n$ such that

$$
\zeta_{n, s + 1} = \alpha_{\text{first projection of } f_{s+1}(e)}.
$$

Since each sequence $\{\alpha_{\text{first projection of } f_{s+1}(e)}\}$ converges to a unique member in the range of $\alpha$ on the set of indices $e \in \text{dom } m$, it follows that there is a unique $\zeta$-index for each real in $A$. Indeed for $e \notin \text{dom } m$, the approximation for $m(e)$ may oscillate between convergence and divergence infinitely often, but we simply introduce a fresh $\zeta$-index for an unused member of $B$ each time this happens and therefore $\alpha_e$ will not occupy a $\zeta$-index in the limit. Furthermore (3.4) and (3.5) ensure that there is a unique $\zeta$-index for each real in $B$.

Finally, $\zeta_e \in A \cup B$ for all $e$. If the index $e$ is destroyed at some stage in the construction, then some $\beta$-index $n$ is assigned at that stage and $\zeta_e = \beta_n$. On the other hand if index $e$ is never destroyed, then $\zeta_e$ takes an $\alpha$-index, namely $\zeta_e = \alpha_{\text{first projection of } f(e)}$.

Hence (i) $\iff$ (ii) $\iff$ (iii).

**Corollary 3.6.** The following classes have left-r.e. numberings:

(i) the left-r.e. Martin-Löf random reals,

(ii) the left-r.e. Kurtz non-random reals, and

(iii) the infinite left-r.e. reals.

Corollary 3.6(iii) contrasts with the fact that there is no r.e. numbering of the infinite r.e. sets. This is not too surprising as the recursive sets are also enumerable if viewed as r.e. sets. It remains to show that the hypothesis “contains a shift-persistent element” is necessary in Theorem 3.5.

**Lemma 3.7.** There is no r.e. numbering of the infinite r.e. sets.

*Proof.* Suppose that $A_0, A_1, A_2, \ldots$ were an r.e. numbering of the infinite r.e. set. Search for an $a_0 \in A_0$, and let $b_0 = a_0 + 1$. Next, search for an $a_1 \in A_1$ which is greater than $b_0$, and let $b_1 = a_1 + 1$. Continuing the diagonalization, find $a_2 \in A_2$ which is greater than $b_1$ and let $b_2 = a_2 + 1$, and proceed similarly for $b_3, b_4, \ldots$. Now $\{b_0 < b_1 < b_2 < \ldots\}$ is an infinite r.e. set which disagrees from the $n$th r.e. set at $a_n$. $\square$

**Theorem 3.8.** There exists a $\Sigma_3$-class of infinite left-r.e. reals which contains no shift-persistent element and has no left-r.e. numbering.
Proof. Let $W_0, W_1, W_2, \ldots$ be a universal r.e. numbering with $W_0 = \emptyset$, and let

$$\mathcal{C} = \{1^n0^\omega \oplus \omega : W_n \text{ is infinite}\}.$$ 

For any universal left-r.e. numbering $\alpha$, the set $\{ e : \alpha_e \in \mathcal{C} \}$ is $\Sigma_3$ because

$$\alpha_e \in \mathcal{C} \iff (\exists n)(\forall d > n)(\exists x, s > d)[\alpha_e, s \mid d = 1^n0^{d-n} \oplus 1^d \& x \in W_{n,s}].$$

Furthermore, each member of $\mathcal{C}$ contains infinitely many 0’s and 1’s but $0^\omega \not\in \mathcal{C}$ for any $X$, hence $\mathcal{C}$ contains no shift-persistent element.

Suppose that $\mathcal{C}$ has a left-r.e. numbering $\beta$, and define a function $i$ on sets $X$:

$$i(X) = \text{greatest } n \text{ such that } 1^{2n-1}0 \text{ is a prefix of } X.$$ 

We construct an r.e. numbering $B_0, B_1, B_2, \ldots$ of the infinite r.e. sets as follows.

Stage 0.

Set $f_0(e) = 0$ for all $e$ and $r(0) = 0$. $f_0(e), f_1(e), f_2(e), \ldots$ will be an increasing recursive approximation of the index $f(e)$ such that $f(e) = i(\beta_e)$. $r$ will be an increasing, recursive function which protects computations of $B$ that are underway.

Stage $s + 1$.

Let $e_1, e_2, \ldots, e_n$ be the indices below $r(s)$ such that $i(\beta_{e,s+1}) \neq i(\beta_{e,s})$. For all $0 \leq k \leq n$ and $0 \leq j \leq r(s)$, set

$$B_{ek} = \omega; \quad (3.7)$$

$$f_{s+1}[r(s) + k] = i(\beta_{e_k,s+1}); \quad (3.8)$$

$$f_{s+1}[r(s) + n + 1] = s; \quad (3.9)$$

$$f_{s+1}(j) = f_s(j). \quad (3.10)$$

For all indices $e$ not destroyed in (3.7) in this stage or some previous stage, set

$$B_{e,s+1} = W_{f_{s+1}(e)}.$$ 

Finally,

$$r(s + 1) = r(s) + n + 1.$$ 

The construction of $B_e$ is indeed r.e. Each $B_e$ starts off as an empty set, follows some r.e. set $W_{f_1(e)}$ for a while, then (possibly) becomes gets destroyed and becomes $\omega$. In all stages $s$ of the construction, $B_{e,s} = W_{f_s(e)}$ or $B_e = \omega$. 

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In particular, there is a unique $B$-index for each set $W_{f+s+1(e),s+1}$ and the remaining indices below $r(s)$ have been destroyed. Thus, in the limit, there is a unique $B$-index for each $W_{f(e)} = W_i(\beta_e)$ and the rest of the $B_i$'s are $\omega$. Since $W_i(\beta_0), W_i(\beta_1), W_i(\beta_2), \ldots$ is an enumeration of the infinite r.e. sets, $B_0, B_1, B_2, \ldots$ is indeed an r.e. numbering of the infinite r.e. sets, contrary to Lemma 3.7.

Remark. If “$\omega$” is removed from the definition of $C$ in Theorem 3.8, then by the same argument we get a purely finite $\Sigma_3$ class of reals which contains no shift-persistent element and has no left-r.e. numbering. It is also clear that the union of this purely finite class with the following class of finite reals (which has a shift-persistent element) 

$$\{0^{n+1}1^\omega : n \in \omega\}$$

is also $\Sigma_3$ and has no left-r.e. numbering.

More along the lines of randomness, we note that the class of left-r.e. reals $X$ satisfying $X + \Omega \leq 1$ has a $\Pi_1$ index set (in any numbering), has no shift-persistent element, and has no left-r.e. numbering. Indeed if this class had a left-r.e. numbering, then $\Omega$ would be recursive.

Corollary 3.9. The left-r.e. Martin-Löf non-random reals, computable non-random reals, and Schnorr non-random reals have no left-r.e. numberings. Hence none of these classes has a $\Sigma_3$ index set in any universal left-r.e. numbering.

Proof. These classes are $\Pi_3$-hard in any acceptable numbering by Corollary 3.4. It follows from Theorem 3.5 that none of these classes are effectively enumerable and hence cannot be $\Sigma_3$ in any universal left-r.e. numbering.

4 Kurtz randomness and weaker notions

Downey, Griffiths, and Reid showed that every non-zero r.e. degree contains a left-r.e. Kurtz random real. Since the Schnorr random reals only occur in the high degrees, it follows that some left-r.e. Kurtz random real is not Schnorr random. We shall now see that left-r.e. Kurtz random reals are not enumerable and obtain a weaker separation by means of enumerations.

Lemma 4.1. For each recursive function $f$, there is a left-r.e. Kurtz random real $X$ such that for infinitely many $n$, $(X \upharpoonright n)^{\sim f(n)}$ is a prefix of $X$. 
Proof. At stage $s$ we have a preliminary finite binary sequence which is decomposed as $\sigma_0 \preceq \sigma_1 \preceq \cdots \preceq \sigma_s$, where $\sigma_i$ monitors the $i$th $\Pi^0_1$ class $C_i$ given by a recursive tree $T_i$. Initially $\sigma_0 = \sigma_{i-1} \preceq 0^{j(|\sigma_{i-1}|)}$. The action is that if at stage $s$ there is a string $\tau$ of length at most $s$ extending $\sigma_{i-1}$ and to the right of $\sigma_i$ such that $\tau$ is not in $T_i$, then let the new $\sigma_i$ be such a $\tau$. If the strings $\sigma_0, \ldots, \sigma_{i-1}$ have stabilized and $T_i$ truly has measure zero then such a string $\tau$ must eventually appear. For infinitely many $i, T_i = 2^{<\omega}$ of course, so then we will make no such move and so the resulting left-r.e. sequence does indeed have $(X | n)^- 0^{j(n)} \prec x$ for infinitely many $n$. \qed

**Theorem 4.2** (Nies [17], Exercise 1.4.23). Let $C$ be a class of r.e. sets closed under finite variants that contains the recursive sets but not all the r.e. sets. If the index set of $C$ is $\Sigma_3$ in some acceptable universal r.e. numbering, then it is $\Sigma_3$-complete in that numbering.

As the next argument shows, Theorem 4.2 also holds for left-r.e. numberings.

**Theorem 4.3.** In every acceptable universal left-r.e. numbering, the set of indices that are Kurtz random is $\Pi_3$-hard.

*Proof.* Let $\alpha$ be an acceptable universal left-r.e. numbering, and let $W$ be an acceptable universal r.e. one using the canonical map (2.1). Let $C$ denote the class of r.e. sets whose associated left-r.e. real is not Kurtz random. Since left-r.e. Kurtz randoms exist and non-Kurtz randoms are closed under finite variants, it follows from Theorem 4.2 that $C = \{ e : W_e \in C \}$ is $\Sigma_3$-complete, hence so is the index set of the left-r.e. Kurtz non-randoms, $\{ y : (\exists x \in C) [ x \mapsto \alpha_y] \}$. \qed

*Sketch of a direct proof.* Given an r.e. set $W$, we want to build a left-r.e. real $X$ such that all the columns of $W$ are finite if $x$ is Kurtz random. At stage $s$ we have a sequence $\sigma_0 \prec \sigma_1 \cdots \prec \sigma_s$. Building $\sigma_i$ is a corresponding strategy $S_i$.

Strategies $S_{2i}$ try to ensure that if column $i$ of $W$, $W[i]$, is infinite, then $x$ is equal to a finite string follow by infinitely many 1s, hence $x$ is recursive and not Kurtz random. It does this by checking whether another element has been added to $W[i]$, and if so adding another 1 to its string $\sigma_{2i}$. This may injure lower priority strategies by moving their strings to the right. If $i$ is minimal such that $W[i]$ is infinite, and all the higher priority strategies have stopped acting, then $S_{2i}$ will ensure its objective is met.

Strategies $S_{2i+1}$ try to ensure that $X$ is Kurtz random. They do this by first letting $\sigma_{2i+1}$ be $\sigma_{2i}$ followed by a 0. (This may be called “creating
room". Then at each subsequent stage it monitors the $i$th $\Pi^0_1$ class $T_i$ to see whether it can now see a way to leave $T_i$ by making an extension and/or moving right, while still respecting (extending) $\sigma_{2i}$. If $T_i$ truly has measure zero, and all higher priority strategies have stopped acting, then this will eventually be possible. We have to "create room" to ensure that there is a fixed-positive-measure remaining search space for $\sigma_{2i+1}$ that is untouched by the actions of lower priority strategies.

If some $W^{[i]}$ is infinite then $S_{2i}$ infinitely often injures its subordinates, and ensures its objective. If each $W^{[i]}$ is finite, then the construction is finite injury and $X$ is Kurtz random.

**Corollary 4.4.** There is no left-r.e. numbering of the left-r.e. Kurtz randoms. Hence the index set of the left-r.e. Kurtz randoms is $\Pi_3 - \Sigma_3$ in every universal left-r.e. numbering.

**Proof.** Otherwise the index set would be both $\Sigma_3$ (by Theorem 3.5) and $\Pi_3$-hard (by Theorem 4.3). Applying Theorem 3.5 again yields the desired conclusion.

Since the left-r.e. Martin-Löf randoms are enumerable (Corollary 3.6), we get:

**Corollary 4.5.** There exists a left-r.e. Kurtz random real which is not Martin-Löf random.

We turn to immunity notions which (except for hyperimmunity) are implied by randomness.

**Definition 4.6.** An infinite set is **immune** if it contains no infinite recursive subset. Even stronger, a set $A = \{a_0 < a_1 < \cdots\}$ is **hyperimmune** if there exists no recursive function $f$ such that $f(n) > a_n$ for all $n$. It is **bi-(hyper)immune** if both $A$ and the complement $\overline{A}$ are (hyper)immune.

From the point of view of recursion theory, bi-immune is the absolute weakest possible notion of randomness: any real which is not bi-immune contains an infinite recursive subset on which any recursive martingale could succeed.

While a left-r.e. real cannot be 1-generic [17], we can use an argument along the lines of Theorem 3.3 to establish the existence of bi-hyperimmune left-r.e. reals.
Theorem 4.7. Let $A \subseteq \omega$ be a $\Sigma_3$-set, and let $\alpha$ be an acceptable universal left-r.e. numbering. Then there exist a recursive function $g$ such that

\begin{align*}
    x \in A &\implies \alpha_g(x) \text{ is finite}; \\
    x \notin A &\implies \alpha_g(x) \text{ is bi-hyperimmune}.
\end{align*}

Proof. Let $\varphi$ be a universal numbering, and let $W$ be an acceptable universal r.e. numbering. Without loss of generality, assume $W_e$ enumerates at most one element at each stage of its recursive approximation. By the $\Sigma_3$-Representation Theorem 2.4, there exists a recursive function $f$ such that:

\begin{align*}
    x \in A &\implies W_{f(x,n)} \text{ is infinite for some } n; \\
    x \notin A &\implies W_{f(x,n)} \text{ is finite for all } n.
\end{align*}

The idea now is make $\alpha_g(x)$ an alternating series of intervals composed of 0’s or 1’s

$$\alpha_g(x) = 0^{n_0} \cdot 1^{n_1} \cdot 0^{n_2} \cdot 1^{n_3} \cdot \ldots$$

which are sufficiently long to ensure that $\alpha_g(x)$ is bi-hyperimmune in the case that $W_{f(x,n)}$ is finite for all $n$. If on the other hand $W_{f(x,n)}$ is infinite for some $n$, then a corresponding interval will blow up to infinity and $\alpha_g(x)$ will be finite, i.e. a diadic rational.

The fly in the ointment now is that we must ensure our construction is left-r.e. For this reason, it will be convenient to always blow up intervals consisting of 1’s rather than intervals of 0’s. In the construction below, we use markers at each stage $s$: $a_{0,s} \leq a_{1,s} \leq a_{2,s} \leq \cdots$. Between indices $a_{2n,2s}$ and $a_{2n+1,2s}$, $\alpha_g(x)$ will contain 1’s, and $\alpha_g(x)$ will contain 0’s between $a_{2n+1,2s}$ and $a_{2n+2,2s}$.

Stage 0 (initialize).

Set $\alpha_{g(x),0} = 0$ and $a_{0,0} = a_{1,0} = a_{2,0} = \cdots = 0$.

Stage $2s + 1$ (make room).

Find the least $e < s$ (if it exists) such that $|W_{e,s+1}| > |W_{e,s}|$ or

$$(\exists j < e) \left[ \varphi_{j,s+1}(a_{e,s}) \downarrow \quad \& \quad a_{e+1,s} \leq \varphi_{j,s+1}(a_{e,s}) \right],$$

and let $k_s$ be the greatest integer such that $2k_s < e$. Set

$$\alpha_{g(x),2s+1} = \left( \alpha_{g(x),2s} \uparrow a_{2k_s+1,s} \right) \downarrow 0^\omega.$$

If no such $e$ exists, do nothing and skip stage $2s + 2$. 

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Stage 2s + 2 (diagonalize and expand).

Set $a_{2k,s+1} = \max \{ \varphi_{j,s+1}(a_{2k,s}) : \varphi_{j,s+1} \downarrow \} \cup \{ |W_{2k,s,s+1}| \} \cup \{ a_{2k,s} + 1 \}$.

For all $n > 2k_s$, set

$$a_{n,s+1} = \max \{ \varphi_{j,s+1}(a_{n,s}) : \varphi_{j,s+1} \downarrow \} \cup \{ |W_{n,s+1}| \}.$$  

For all $j < 2k_s$, set $a_{j,s+1} = a_{j,s}$. For notational convenience, let

$$d_n = a_{n+1,s+1} - a_{n,s+1},$$

and set

$$\alpha_{g(x),2s+2} = (\alpha_{g(x),s+1} \uparrow a_{2k_s,s+1} - 1) \sim 1^{d_{2k_s}} \sim 0^{d_{2k_s+1}} \sim 1^{d_{2k_s+2}} \sim \ldots$$

In the construction above, $\{ \alpha_{g(x),s} \}$ is indeed a left-r.e. approximation. In stage $2s+1$ some 0 is changed to a 1, and only higher indices are modified in the subsequent stage $2s + 2$. Hence the limit $\alpha_{g(x)} = \lim_s \alpha_{g(x),s}$ exists.

Suppose that $W_f(x,n)$ is infinite for some $n$, let $e$ be the least such index for which this is true, and let $k$ be the greatest integer such that $2k < e$. By induction, $a_{j,s}$ converges to a finite value for all $j \leq 2k$. On the other hand, $\lim_s a_{2k+1,s} = \infty$, hence $\alpha_{g(x)} = \sigma \sim 1^\omega$ for some prefix $\sigma$ of length $a_{2k} = \lim_s a_{2k,s}$.

Now assume that $W_f(x,n)$ is finite for all $n$. In this case, for any recursive $\varphi_k$, we have

$$\varphi_k(a_{2k+1}) < a_{2k+2} \leq \left[ a_{2k+1}^{th} \text{ member of } \alpha_{g(x)} \right]$$

since $\alpha_{g(x)}$ contains only 0’s on the indices between $a_{2k+1}$ and $a_{2k+2}$. A similar argument holds for $1 - \alpha_{g(x)}$, the complement of $\alpha_{g(x)}$. Hence $\alpha_{g(x)}$ is bi-hyperimmune.

Corollary 4.8. In any acceptable universal left-r.e. numbering, the index sets for the following classes are $\Pi_3$-complete:

(i) the left-r.e. immune reals,

(ii) the left-r.e. hyperimmune reals,

(iii) the left-r.e. bi-immune reals,

(iv) the left-r.e. bi-hyperimmune reals.
Corollary 4.8 can be deduced from either Theorem 4.7 or, given the existence of left-r.e. bi-hyperimmune reals (Theorem 4.10), from Theorem 4.2. Now from Theorem 3.5, we have the following result.

**Corollary 4.9.** In any universal left-r.e. numbering, the following classes have \( \Sigma_3 - \Pi_3 \) index sets:

1. the left-r.e. non-immune reals,
2. the left-r.e. non-hyperimmune reals,
3. the left-r.e. non-bi-immune reals,
4. the left-r.e. non-bi-hyperimmune reals

Moreover, there exists a left-r.e. numbering for each of these classes.

It is known that every Kurtz random is bi-immune [14], but the reverse inclusion does not hold [2]. We now separate the left-r.e. versions of these notions.

**Theorem 4.10.** There exists a bi-hyperimmune left-r.e. real which is not Kurtz random.

**Proof.** A simple modification to the construction in Theorem 4.7, for the case where \( x \notin A \), yields the corollary. This case builds a sequence of the form

\[
0^n \sim 1^n \sim 0^n \sim 1^n \sim \ldots ,
\]

and as long as we additionally ensure that \( n_{2k} = n_{2k+1} \) and \( n_k \geq 3 \) for all \( k \), the resulting left-r.e. real \( X \) is still bi-hyperimmune.

A martingale \( M \) Kurtz-succeeds on a set \( A \) if \( M \) succeeds on \( A \) and there exists a recursive function \( f \) such that \( M(A \upharpoonright n) > f(n) \) for all \( n \). Wang [25], [6] showed that a sequence is Kurtz random if and only if some recursive martingale Kurtz succeeds on it. Due to the simple form of (4.1), a recursive gambler can always bet a dollar that the next bit is the same as the previous one seen and be guaranteed to gain, over the course of each interval, at least the number of bits in the interval minus 2. Since the length of each interval is at least 3, \( f(n) = n/3 \) is a lower bound on the winnings for this gambler. Hence \( X \) is not Kurtz random.

Chaitin’s \( \Omega \) is an example of a left-r.e. Kurtz random which, by Lemma 3.2, is not bi-hyperimmune.

### 5 Classes of higher complexity

We now investigate the complex randomness notions of Schnorr randomness and computable randomness. As we shall see, neither of these left-r.e. 

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A set $A$ is called high if $A' \geq_T \emptyset''$. A theorem of Nies, Stephan, and Terwijn [18] shows the existence of left-r.e. Schnorr randoms which are not Martin-Löf random:

**Theorem 5.1** (Nies, Stephan, Terwijn [18]). The following statements are equivalent for any set $A$:

1. $A$ is high.
2. There is a set $B \equiv_T A$ which is computably random but not Martin-Löf random.
3. There is a set $C \equiv_T A$ which is Schnorr random but not computably random.

In the case that $A$ is left-r.e. and high, the sets $B$ and $C$ can be chosen as left-r.e. sets as well.

Furthermore, Downey and Griffith [5, 7] proved that every left-r.e. Schnorr random real is high. Therefore

**Fact 5.2.** A left-r.e. $X$ is high $\iff$ $X$ Turing equivalent to a left-r.e. Schnorr random $\iff$ $X$ is Turing equivalent to a left-r.e. computable random.

In his PhD thesis [22], Schwarz characterized the complexity of the high r.e. degrees:

**Theorem 5.3** (Schwarz [22], [23]). In any acceptable universal r.e. numbering $W_0, W_1, W_2, \ldots$, \{e : $W_e$ is high\} is $\Sigma_5$-complete.

Using this Schwarz’s theorem, we obtain the following enumeration result.

**Theorem 5.4.** Let $C$ be a class of left-r.e. reals such that:

1. Every member of $C$ is high, and
2. every high set is Turing equivalent to some member of $C$.

Then for any universal left-r.e. numbering $\alpha$, \{e : $\alpha_e \in C$\} is not a $\Sigma_4$-set and hence is neither enumerable nor co-enumerable.

**Proof.** Let $C$ be a class satisfying the hypothesis of the theorem, let $W$ be an acceptable universal r.e. numbering, let $\Phi$ denote a Turing functional, and suppose that

$$\alpha_i \in C \iff (\exists n_1) (\forall n_2) (\exists n_3) (\forall n_4) [P(i, n_1, n_2, n_3, n_4)]$$. 

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for some recursive predicate $P$.

For convenience assume that whenever a computation $\Phi_{j}^{W_{e},t}$ is injured, it is undefined for at least one stage; then

$$W_{e} \text{ is high } \iff (\exists i, j) \left[ \alpha_{i} \in \mathcal{C} \quad \& \quad \alpha_{i} = \Phi_{j}^{W_{e}} \right]$$

$$\iff (\exists i, j, n_{1}) (\forall x, n_{2}) (\exists t) (\exists n_{3}) (\forall u > t) (\forall n_{4})$$

$$P(i, n_{1}, n_{2}, n_{3}, n_{4}) \quad \& \quad \alpha_{i,u}(x) = \Phi_{j,u}^{W_{e}}(x).$$

Thus $\{ e : W_{e} \text{ is high} \}$ is a $\Sigma_{4}$-set, contrary to Theorem 5.3. \hfill \Box

**Corollary 5.5.** Neither the Schnorr random reals nor the computably random reals are $\Sigma_{4}$ in any universal left-r.e. numbering. Hence neither class nor its complement has a left-r.e. numbering.

**Proof.** By Fact 5.2 the left-r.e. Schnorr random reals and the left-r.e. computably random reals satisfy the hypothesis of the Theorem 5.4. Apply Theorem 3.5. \hfill \Box

It remains to characterize the hardness of computable randomness and Schnorr randomness in an acceptable universal left-r.e. numbering. For the remainder of this paper, we fix an acceptable universal left-r.e. numbering $\alpha$ and an acceptable universal r.e. numbering $W$. The *principal function* of a set $A = \{ a_{0} < a_{1} < a_{2} < \ldots \}$ is given by $n \mapsto a_{n}$; we write $p_{A}(n) = a_{n}$. We will be particularly interested in the principal functions of co-r.e. sets, so we use the abbreviation $p_{e}$ for $p_{\neg e}$. We say that a function $f : \omega \to \omega$ is *dominating* if it dominates all recursive functions.

**Theorem 5.6.** There is a Turing reduction procedure $\Phi$ and a recursive function $g$ such that for all $e$,

(i) $\Phi^{p_{e}}$ is a left-r.e. real,

(ii) $\alpha_{g(e)} = \Phi^{p_{e}},$ and

(iii) $\Phi^{f}$ is computably random if $f$ is dominating.

**Proof.** This fact follows from the proof of Nies, Stephan, Terwijn [18, Theorem 4.2, (I) implies (II), r.e. case] which appears as Theorem 5.1 in this paper. \hfill \Box

A set $A$ is *low* if $A' \leq_{T} \emptyset'$, and a function is *low* if it is computable from a low set. A function $f$ is *diagonally non-recursive (DNR)* if for some numbering $\varphi$ and every $e$, the value $\varphi_{e}(e)$, if defined, differs from $f(e)$. 22
Lemma 5.7. A low left-r.e. real cannot compute a Schnorr random.

Proof. Suppose such a real $A$ computes a Schnorr random set $X$. Since $X$ is not high, $X$ must also be Martin-Löf random (by Theorem 5.1). Kučera showed that every Martin-Löf random set computes a DNR function [12], [11, Theorem 6], so $A$ computes a DNR function. Moreover $A$ has r.e. Turing degree because it is truth-table equivalent to the r.e. set $\{\sigma : \sigma \uparrow \leq A\}$. An r.e. set computes a DNR-function if and only if the set is Turing complete [1, 10, 11, Corollary 9], hence $A \equiv_T \emptyset$. This contradicts the fact that $A$ is low.

An r.e. set $A$ is maximal if for each r.e. set $W$ with $A \subseteq W$, either $\omega \setminus W$ or $W \setminus A$ is finite. Friedberg [8] proved that maximal sets exist.

Theorem 5.8. For every $A \in \Pi_4$, there exists a recursive function $f$ such that for all $e$,

$$e \in A \implies \alpha_{f(e)} \text{ is computably random; } \quad (5.1)$$
$$e \notin A \implies \alpha_{f(e)} \text{ is not Schnorr random. } \quad (5.2)$$

Proof. Let us fix a $\Pi_4$-complete set $A$; By [23, XII. Exercise 4.26], there is a recursive function $h$ such that

$$e \in A \iff W_{h(e)} \text{ is maximal } \iff W_{h(e)} \text{ is not low.}$$

Martin and Tennenbaum showed that the principal function of the complement of a maximal set dominates all recursive functions [23, XI. Proposition 1.2]. Using this result and the function $g$ given by Theorem 5.6,

$$W_{h(e)} \text{ is maximal } \implies p_{h(e)} \text{ is dominating}$$
$$\implies \alpha_{g[h(e)]} = \Phi^{p_{h(e)}} \text{ is computably random,}$$

and by Lemma 5.7 with Theorem 5.6(1),

$$W_{h(e)} \text{ is not maximal } \implies p_{h(e)} \text{ is low}$$
$$\implies \alpha_{g[h(e)]} = \Phi^{p_{h(e)}} \text{ is not Schnorr random.}$$

The function $f = g \circ h$ witnesses the conclusion of this theorem.

Note that if we replaced “computably random” with “Martin-Löf random” in (5.1), we would obtain a characterization of $\Sigma_3$ sets rather than $\Pi_4$ sets (care of Theorem 3.3). Since every computable random is Schnorr random (Theorem 1.4), we obtained an optimal hardness result:
Corollary 5.9. In any acceptable universal left-r.e. numbering, both the indices of the Schnorr randoms and the indices of the computably randoms are $\Pi_4$-complete.

We summarize our main results in Table 1. A theorem in a forthcoming paper [24] states that every $0'$-recursive 1-generic set has a co-r.e. indifferent set which is retraceable by a recursive function. It follows that for each the families of randoms listed in Table 1, there exists a universal left-r.e. numbering which makes the set of the indices for that class 1-generic. Therefore we cannot obtain any arithmetic hardness results for index sets in the general case of universal left-r.e. numberings.

| Left-r.e. family      | Complexity | Hardness* |
|-----------------------|------------|-----------|
| Martin-Löf randoms    | $\Sigma_3 - \Pi_3$ | 5.3       | $\Pi_3$-hard 5.1 |
| computable randoms    | $\Pi_4 - \Sigma_4$ | 5.5       | $\Pi_4$-hard 5.9 |
| Schnorr randoms       | $\Pi_4 - \Sigma_4$ | 5.5       | $\Pi_4$-hard 5.9 |
| Kurtz randoms         | $\Pi_3 - \Sigma_3$ | 4.4       | $\Pi_3$-hard 4.3 |
| bi-immune sets        | $\Pi_3 - \Sigma_3$ | 4.9       | $\Pi_3$-hard 4.8 |

Table 1: Complexities listed hold for any universal left-r.e. numbering. *Hardness results are for acceptable universal left-r.e. numberings.

In Table 1, each left-r.e. family is a proper subset of the one below it. Among the families in this table, only the Martin-Löf randoms have a left-r.e. numbering, and among the complementary families only the Kurtz non-randoms and non-bi-immune sets do (by Theorem 3.5).

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