Hamiltonian forms of ‘Carroll’ gravity

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Abstract

We develop a Hamiltonian description of the ‘Carroll’ (Levy Leblond-Sen Gupta) limit of gravity in the first order formalism. Two inequivalent formulations are obtained, containing the so called ‘magnetic’ and ‘electric’ Hamiltonian constraints of metric gravity. Through a constraint analysis, the number of physical degrees of freedom are shown to be two per spacetime point in these singular limits. Whereas the Hamiltonian constraint in the first limit depends only on the densitized triad fields and their space derivatives, the second limit provides a simple canonical formulation of the BKL behaviour near spacelike singularities in terms of these triad and their conjugate. Owing to their attractive features, both the limiting Hamiltonian theories emerge as interesting candidates for a quantization.

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I. INTRODUCTION

The Lorentz transformations exhibit an intriguing limit opposite to the Galilean one. The corresponding set of boosts relating different frames, discovered independently by Levy-Leblond \[1\] and Sen Gupta \[2\], are given by:

\[
t' = t - \frac{x}{w}, \quad x' = x, \quad y' = y, \quad z' = z,
\]

where \(w\) is a parameter with the dimension of velocity. These transformations are obtained by replacing the relative frame velocity by \(\frac{c^2}{w}\) in the original Lorentz transformations and then taking the limits of a small speed of light \((c << w)\) as well as small temporal separations between events \((c\Delta t << \Delta x)\). The parameter \(w\), however, does not admit a natural interpretation as a physical velocity. Rather, it is the rate of motion \(\frac{\Delta x}{\Delta t}\) in the unprimed frame of an event that is fixed in time in the primed frame. These boosts, along with the spatial rotations and spacetime translations, form the so-called ‘Carroll’ group. As shown by Levy Leblond \[1, 3\], this ten-parameter group could be obtained through a Inönü-Wigner contraction of the Poincare group.

Despite the nonstandard causal structure of the Levy Leblond-Sen Gupta spacetimes where the lightcone shrinks (almost) to a line, the Carrollian limit is found to be intimately connected to a remarkably wide range of physical contexts. For instance, a special case of this limit \[4, 5\] in the second order metric formulation of gravity could be related to the BKL behaviour \[6\] near a spacelike singularity where time derivatives of fields dominate over spatial ones. Carrollian physics is also relevant to the description of cosmological billiards \[7\], null surfaces \[8, 9\], tachyons \[10\], gravitational waves \[11\], non-AdS holography \[12\], inflationary cosmology \[13\] and so on.

The Levy Leblond-Sen Gupta spacetimes have been typically shown to arise in gravity theories where the metric is either exactly degenerate \[14\] or becomes so in a limiting sense \[4, 5\]. The latter case has also been interpreted as a \(c \rightarrow 0\) (or ‘strong coupling’) limit. However, since \(c\) has dimensions, such an interpretation demands caution since operationally it may lead to more than one inequivalent limits of field theories, depending on the units chosen to define the fundamental variables. In fact, examples of such inequivalent phases of gravity have appeared in the earlier literature within approaches based on the gauging of Carroll algebra \[15, 17\] or through expansions in powers of \(c^2\) within a Lagrangian framework \[18\].
Recently, it has been elucidated that relativistic field theories in general could admit two inequivalent Carroll limits. This feature is exactly analogous to the ‘electric’ and ‘magnetic’ limits in Galilean electromagnetism, the basic variables in the two cases being assigned different units through relative factors of $c$.

Here, we develop a Hamiltonian formulation of Carrollian gravity within a first order framework, where the canonical variables as well as the constraint structure differ nontrivially from the metric (ADM) formalism. After solving the second-class constraints, we obtain two inequivalent limits, where the associated Hamiltonian constraints could be related to the so-called ‘magnetic’ and ‘electric’ Hamiltonian constraints in the Carrollian phases of metric gravity.

The form of the limiting constraints are significantly different for the two limiting theories as compared to the original Hilbert-Palatini constraints. However, through a full constraint analysis, we explicitly demonstrate that the limiting theories exhibit the same number of physical degrees of freedom as in Hilbert-Palatini (or, Einstein-Hilbert) gravity.

The final form of the Hamiltonian constraints in either of the two Carroll limits are considerably simpler than the full constraint in pure gravity and are of interest from the perspective of a canonical quantization of gravity. Also, the latter case provides a remarkably simple Hamiltonian representation of the BKL conjecture. A successful quantization of the associated canonical constraints here could provide an answer as to whether quantum gravity effects could really resolve a singularity, at least in the cosmological case.

In the next section, we develop a Hamiltonian set up for the so called ‘magnetic’ limit following a brief outline of the Hamiltonian structure of Hilbert-Palatini theory. Based on an appropriate scaling limit, we obtain the Carrollian phase of the constraints without making any gauge choice. This is followed by a constraint analysis based on their Poisson algebra. The constraints in the time gauge are also found. In section-III, we introduce a different scaling prescription and find the resulting Hamiltonian theory. In time gauge, the associated Hamiltonian constraint is found to be the first order counterpart of the ‘electric’ Hamiltonian constraint in the metric formulation. In section-IV, we unravel the connection between the limits considered and the Levy Leblond-Sen Gupta spacetimes. The concluding section contains a few relevant remarks.
II. HILBERT-PALATINI GRAVITY: LIMIT A

The first order or \(SO(3,1)\) gauge formulation of Einstein gravity is based on the tetrad \(e^I_{\mu}\) and spin-connection fields \(\omega^I_{\mu}\). The action principle involves the following Lagrangian density:

\[
\mathcal{L}(\dot{e}, \dot{\omega}) = \frac{1}{2\kappa} \dot{e}^I_{\mu} \dot{e}^J_{\nu} \hat{R}^{IJ}_{\mu\nu}(\dot{\omega})
\]

where \(\kappa\) is the gravitational coupling, \(\dot{e} \equiv \det \dot{e}^I_{\mu}\), \(\dot{e}^I_{\mu}\) is the inverse tetrad field and \(\hat{R}^{IJ}_{\mu\nu}(\dot{\omega}) = \partial_{[\mu} \dot{\omega}^{IJ}_{\nu]} + \dot{\omega}^I_{[\mu} \dot{\omega}^{J]}_{\nu]}\) is the \(SO(3,1)\) field strength. To begin with, let us consider the following limit leading to a realization of the Carroll algebra, where the basic variables go to the new ones as:

\[
\begin{align*}
\dot{e}^0_{\mu} &= \epsilon \dot{e}^0_{\mu}, \quad \dot{e}^i_{\mu} = \dot{e}^0_{\mu}, \quad \dot{\omega}^{0i}_{\mu} = \epsilon \omega^{0i}_{\mu}, \quad \dot{\omega}^{ij}_{\mu} = \omega^{ij}_{\mu}; \\
\dot{\epsilon}^0_{\epsilon} &= \frac{1}{\epsilon} \dot{\epsilon}^0_{\epsilon}, \quad \dot{\epsilon}^i_{\epsilon} = \dot{\epsilon}^i_{\epsilon}.
\end{align*}
\]

The unhatted variables are assumed to remain finite in the limit \(\epsilon \to 0\), implying that the determinant \(\dot{e}\) approaches zero. In these redefined variables, the Hilbert-Palatini Lagrangian density becomes:

\[
\mathcal{L} = \frac{\epsilon}{2\kappa} \epsilon e^I_{\mu} e^J_{\nu} R^{IJ}_{\mu\nu}(\omega)
\]

where the new field strengths \(R^{0i}_{\mu\nu} = D[i\omega^0_{\nu}]^i := \partial_{[\mu} \omega^{0i}_{\nu]} + \omega^k_{[i\omega^{0k}_{\nu]}^i} \) and \(R^{ij}_{\mu\nu} = \partial_{[\mu} \omega^{ij}_{\nu]} + \omega^k_{[i\omega^{jk}_{\nu]}^i} \) represent the \(SO(3)\) counterparts of the original ones (\(D_\mu\) denotes the \(SO(3)\) covariant derivative). Evidently, the Lagrangian density is well-defined provided the coupling scales as \(\kappa = \epsilon \kappa\). In what follows next, we set up a Hamiltonian formulation of the above limit within the first-order framework.

We employ the standard Hamiltonian decomposition of the tetrad fields as \([22,23]\):

\[
\begin{align*}
\dot{e}^I_{t} &= \sqrt{\dot{\epsilon} N} \dot{M}^I + \dot{N}^a \dot{V}^I_a, \quad \dot{e}^I_a = \dot{V}^I_a; \\
\dot{M}_I \dot{V}^I_a &= 0, \quad \dot{M}_I \dot{M}^I = -1; \\
\dot{e}^I_t &= -\frac{\dot{\epsilon}}{\sqrt{\epsilon} N} \dot{M}_I, \quad \dot{e}^a_I = \dot{V}^a_I + \frac{\dot{N}^a_I}{\sqrt{\epsilon} N}; \\
\dot{\epsilon}^I_t \dot{V}^a_I := 0, \quad \dot{\epsilon}^a_I \dot{V}^b_I := \delta^a_I \dot{V}^b_I := \delta^a_I + \dot{M}_I \dot{M}^I.
\end{align*}
\]

The spatial metric and its determinant are defined as: \(\dot{q}_{ab} := \dot{V}_a \dot{V}_b\) and \(\dot{q} := \det \dot{q}_{ab}\), with \(\dot{\epsilon} := det(\dot{e}^I_{\mu}) = \dot{N} \dot{q}\). Using the identity,

\[
\dot{e}^I_{[a} \dot{e}^b_{;j]} = \dot{N} \dot{\epsilon} e^I_{[a} e^b_{;j]} e^I_{a} \eta^{KL} + \dot{N}^{[a} \dot{e}^b_{;j]} e^I_{[a} \dot{e}^I_{j]};
\]
the original Lagrangian density (2) becomes:

\[ \mathcal{L} = \frac{\hat{e}^t e^a_J \hat{R}_{ab} IJ}{\kappa} + \frac{1}{2} (\hat{N} \hat{e}^t_I e^a_J e^b_L K_I J^L \eta K L + \hat{N}^a e^b_J \hat{e}^a_L) \hat{R}_{ab} IJ \]  

(4)

Defining the momenta conjugate to \( \hat{\omega}_a^{IJ} \) as \( \hat{\Pi}^a_{IJ} := \frac{\hat{e}^t e^a_J}{\kappa} \), we may rewrite the above as:

\[ \mathcal{L} = \frac{1}{2} \hat{\Pi}^a_{IJ} \partial_t \hat{\omega}_a^{IJ} - \mathcal{H} \]  

(5)

where the Hamiltonian density \( \mathcal{H} \) is completely constrained owing to the absence of velocities corresponding for the fields \( \hat{N}^a, \hat{\omega}_a^{IJ} \):

\[ \mathcal{H} = \hat{N} \hat{H} + \hat{N}^a \hat{H}_a + \frac{1}{2} \hat{\omega}_a^{IJ} \hat{G}_{IJ} \]  

(6)

The constraints read:

\[ \hat{H} = \frac{\hat{\kappa}}{2} \hat{\Pi}^a_{IK} \hat{\Pi}^b_{JL} \eta^{KL} \hat{R}_{ab} IJ \approx 0, \]
\[ \hat{H}_a = \frac{1}{2} \hat{\Pi}^b_{IJ} \hat{R}_{ab} IJ \approx 0, \]
\[ \hat{G}_{IJ} = -\hat{D}_a \hat{\Pi}^a_{IJ} \approx 0 \]  

(7)

In addition, the momenta are associated with six primary constraints, reflecting that only twelve of its components are independent:

\[ \hat{C}^{ab} := \frac{1}{2} \epsilon^{IJKL} \hat{\Pi}^a_{IJ} \hat{\Pi}^b_{KL} \approx 0 \]  

(8)

We may implement the Carroll limit (3) through the following redefinitions of the canonical variables, where, as earlier, the unhatted variables remain finite in the limit \( \epsilon \to 0 \):

\[ \hat{\omega}^{0i}_\mu = \epsilon \omega^{0i}_\mu, \quad \hat{\omega}^{ij}_\mu = \omega^{ij}_\mu, \quad \hat{\Pi}^a_{0i} = \frac{1}{\epsilon} \pi^a_{0i}, \quad \hat{\Pi}^a_{ij} = \Pi^a_{ij}; \]
\[ \hat{N} = \epsilon N, \quad \hat{N}^a = N^a. \]  

(9)

Note that these scalings preserve the symplectic form, as is necessary. The transformation of the lapse follows from the scaling of the tetrad determinant. Let us now analyse the Hamiltonian structure of gravity theory resulting from the above Carroll limit.

The full Hamiltonian constraint in the new variables reads:

\[ \hat{H} = \frac{\epsilon \kappa}{2} \left[ -\frac{1}{\epsilon^2} \hat{\Pi}^a_{0i} \hat{\Pi}^b_{0j} R_{ab}^{ij} + \hat{\Pi}^a_{ik} \hat{\Pi}^b_{jk} R_{ab}^{ij} + 2 \hat{\Pi}^a_{0k} \hat{\Pi}^b_{ik} R_{ab}^{0i} \right] \]  

\[ - \frac{1}{2} \frac{1}{\epsilon} \hat{\Pi}^a_{0i} \hat{\Pi}^b_{0j} R_{ab}^{ij} + \hat{\Pi}^a_{ik} \hat{\Pi}^b_{jk} R_{ab}^{ij} + \frac{1}{\epsilon} \hat{\Pi}^a_{0k} \hat{\Pi}^b_{ik} R_{ab}^{0i} \right] \]  

(5)
where the new field strength decomposes into its $SO(3)$ counterpart $\bar{R}_{ab}^{ij}$ and the rest: $R_{ab}^{ij} = \bar{R}_{ab}^{ij} + \epsilon^2 \omega_{[a}^{0i} \omega_{b]}^{0j}$. Redefining the lapse as in (9), then using $\bar{N}\bar{H} = NH$ and finally implementing the limit $\epsilon \to 0$, we obtain from the above:

$$H = -\frac{\kappa}{2} \Pi_{0i}^b \Pi_{0j}^a \bar{R}_{ab}^{ij}$$

(10)

The right hand side is clearly well-defined since the variables are all finite in the limit. An immediate fact to be gleaned from the above is the vanishing Poisson bracket $[H, H] = 0$, a characteristic of the Carroll algebra.

The new vector constraint also retains only the $SO(3)$ covariant derivative and field-strength in the limit:

$$H_a = \Pi_{0i}^b D_{[a} \omega_{b]}^{0i} + \frac{1}{2} \Pi_{ij}^b \bar{R}_{ab}^{ij}$$

(11)

Proceeding similarly, we find the following limiting forms of the rotation and boost constraints:

$$G_{ij} = -\bar{D}_a \Pi_{ai}^a + \omega_{a[0| \Pi_{0j}^{a0}], G_{0i} = -\bar{D}_a \Pi_{0i}^a}$$

(12)

Evidently, the rotation constraint is the only one that exhibits no change when compared with the full Hilbert-Palatini constraints.

### A. Algebra of Carrollian constraints

Following a tedious but straightforward analysis, we summarize the algebra of the Carrollian constraints below:

$$\left[ \int MH, \int N^a H_a \right] = \int \left[ (M \partial_a N^a - N^a \partial_a M) H + MN^a \Pi_{0j}^b \bar{R}_{ab}^{ij} G^{0i}\right],$$

$$\left[ \int N^a H_a, \int M^b H_b \right] = \int \left[ (M^d \partial_d N^b - N^d \partial_d M^b) H_b - N^a M^b R_{ab}^{IJ} G_{IJ}\right],$$

$$\left[ \int MH, \int \Lambda_{IJ} G^{IJ} \right] = 0 = \left[ \int M^a H_a, \int \Lambda_{IJ} G^{IJ}\right],$$

$$\left[ \int \Lambda_{IJ} G^{IJ}, \int \Lambda_{KL} G^{KL} \right] = 4 \left[ \int \Lambda_{IJ} \Lambda_{KL} G^{IJ} \right],$$

$$\left[ \int M^a H_a, \int \lambda_{cd} C^{cd} \right] = \int \left[ (M^c \partial_c \lambda_{cd} + 2 \lambda_{ac} \partial_a M^c - \lambda_{ab} \partial_c M^c) C^{ab} - \frac{1}{2} \epsilon_{ijk} \lambda_{ab} N^a (\Pi_{0i}^b G^{jk} + \Pi_{0j}^b G^{0i}) \right],$$

$$\left[ \int MH, \int \lambda_{ab} C^{ab} \right] = 2 \int M \lambda_{ab} \epsilon^{ijk} \left[ \Pi_{0k}^c \Pi_{0j}^b \bar{D}_c \Pi_{0i}^a + \Pi_{0k}^b \Pi_{0i}^a G_{0j} \right].$$

(13)
Note that the last bracket prevents a closure of the algebra, implying a set of six secondary constraints:

\[ D^{ab} := \epsilon_{ijk} \Pi^c_{0k} \Pi^{(b}_{0j} \hat{D}_c \Pi^{a)}_{0i} \approx 0. \]  \hspace{1cm} (14)

As could be checked explicitly, there are no further (secondary) constraints. With this, the only second-class pair is given by \((C^{ab}, D^{cd})\):

\[ \left[ \int \lambda_{ab} C^{ab}, \int \mu_{cd} D^{cd} \right] = 4 \int \lambda_{[a}[\mu_{d]c} \Pi^a_{0i} \Pi^b_{0k} \Pi^c_{0k} \Pi^d_{0i}. \]

**B. Solution of second-class constraints**

The second-class constraints could be solved by eliminating the six redundant variables in \(\hat{\Pi}^a_{IJ}\) through the parametrization \(\Pi^a_{0i} = E^a_i, \, \Pi^a_{ij} = \chi_{[i} E^a_{j]}\), following a standard approach in first order gravity \[22\]. The momenta \(E^a_i\) essentially turn out to be the densitized triad, as could be seen from the definition of \(\Pi^a_{IJ}\). The resulting symplectic form now depends only on the twelve independent canonical pairs:

\[ \Omega = \frac{1}{2} \Pi^a_{IJ} \partial_t \omega_a^{IJ} = E^a_i \partial_t Q^i_a + \zeta_j \partial_t \chi^j \]  \hspace{1cm} (15)

where we have introduced the new fields as: \(Q^i_a = \omega_a^{0i} - \chi_j \omega_a^{ij}\), \(\zeta^j = E^a_i \omega_a^{ij}\). The last equality also implies that only three components \((\zeta^i)\) of \(\omega_a^{ij}\) are dynamical, allowing for the following parametrization:

\[ \omega_a^{ij} = \frac{1}{2} E^{[ij}_a \zeta^j] + \epsilon^{ijk} E^d_a N^{kl} \]  \hspace{1cm} (16)

Clearly, the velocities of the six fields \(N_{kl} = N_{lk}\) do not appear in the Lagrangian density. The vanishing of their conjugate momenta \((P^{kl} \approx 0)\) leads to a further set of secondary constraints:

\[ [H, P^{kl}] \approx 0 \]  \hspace{1cm} (17)

Solution to these constraints (along with the boost) is equivalent to the vanishing of (spatial) torsion, determining the fields \(\omega_a^{ij}\) in terms of the momenta \(E^a_i\).
Following the solution of the second-class constraints as above, the Carrollian family of first-class constraints in the new variables finally become:

\begin{align*}
H &= -\frac{\kappa}{2} E_i^a E_j^b \bar{R}^{ij}_{ab} \approx 0, \\
H_a &= E_i^b \left[ \bar{D}_{[a} Q_{b]}^i + \omega_{[i}^k \partial_{a]} \chi_k \right] \approx 0, \\
G_{ij} &= -\left[ \partial_a (\chi_i E^a_{j}) + \chi_{[j} \zeta_{i]} - Q_{a[i} E^a_{j]} \right] \approx 0, \\
G_{0k} &= -\partial_a E^a_k + \zeta_k \approx 0.
\end{align*}

\text{(18)}

C. Time gauge

We may simplify the theory even further by setting the time gauge $\chi_i = 0$, a condition which forms a second-class pair with the boost constraint given by eq. \text{(18)}. In this gauge, the solution of boost thus turns the associated conjugate momenta into a dependent variable as: $\zeta_i = \partial_a E^a_i$.

Let us emphasize that in order to obtain the Carrollian constraints directly in time gauge, the limit should be taken after implementing the gauge. However, in this particular case here, this procedure is equivalent to taking the limit before imposing the gauge.

In this gauge, the Carroll limit of the constraints could be read off from eq. \text{(17)} as:

\begin{align*}
H &= -\frac{\kappa}{2} E_i^a E_j^b \bar{R}^{ij}_{ab} \approx 0, \\
H_a &= E_i^b \bar{D}_{[a} Q_{b]}^i \approx 0, \\
G_{ij} &= Q_{a[i} E^a_{j]} \approx 0. 
\end{align*}

\text{(19)}

These form a system of first-class constraints, while obeying the Carroll algebra.

With these set of constraints, one may directly proceed towards a canonical quantization.

Let us add that the rotation constraint in time gauge could be solved as $Q^k_a = E^k_a M^{kl}$ where $M_{kl} = M_{lk}$ is an arbitrary field. This indeterminacy of the connection $\omega_a^{i0}$ resembles a similar feature found in the solutions of gravity theory where the tetrad fields are degenerate in an exact sense \text{\cite{14}} (as opposed to the limiting sense as here). There it had been demonstrated that the Lagrangian equations of motion leave both $\omega_a^{i0}$ and $\omega_a^{ij}$ under-determined up to two symmetric fields $M^{kl}$ and $N^{kl}$. On the other hand, this also shows that the degenerate limit of gravity is not really equivalent to a gravity theory based on exactly degenerate tetrad fields.
We may also note that this ambiguity up to $M_{kl}$ is equivalent to the one appearing within a Lagrangian description of the ‘magnetic’ Carroll limit of gravity [16], where a similar field shows up as a multiplier in the second order action.

D. Physical degrees of freedom

Based on the Poisson algebra presented earlier, it is straightforward to count the physical degrees of freedom in the limiting Hamiltonian theory. Before setting the time gauge, the canonical pairs $(\omega^{I}_{a}, \Pi^{a}_{IJ})$ are subject to 10 first class constraints $H, H_{a}, G^{0}_{ik}, G^{ij}$ and six second-class pairs $(C^{ab}, D^{cd})$. This leaves two degrees of freedom per spacetime point in the Carroll phase of gravity. This is same as in Hilbert-Palatini gravity, despite the fact that the first case represents a singular limit of the latter and the explicit canonical form and details of the constraints exhibit nontrivial differences between the two cases. A counting after fixing the time gauge leads to the same result.

III. LIMIT B

In this section we demonstrate that in time gauge, the original Hilbert-Palatini Hamiltonian admits another limit inequivalent to the earlier limit A. To this end, we consider the following rescalings on the canonical variables:

$$\hat{\omega}^{0}_{\mu} = \frac{1}{\epsilon} \omega^{0}_{\mu}, \quad \hat{\omega}^{ij}_{\mu} = \omega^{ij}_{\mu}, \quad \hat{\Pi}^{a}_{0i} = \epsilon \pi^{a}_{0i}, \quad \hat{\omega}^{ij}_{a} = \omega^{ij}_{a},$$

$$\hat{\mathcal{N}} = \frac{1}{\epsilon} N, \quad \hat{\mathcal{N}}^{a} = N^{a}. \quad (20)$$

In terms of the tetrad fields and their inverse, the limit above translates to:

$$\hat{e}^{0}_{\mu} = \frac{1}{\epsilon} e^{0}_{\mu}, \quad \hat{e}^{i}_{\mu} = e^{i}_{\mu}, \quad \hat{e}^{\mu}_{0} = \epsilon e^{\mu}_{0}, \quad \hat{e}^{\mu}_{i} = e^{\mu}_{i}. \quad (21)$$

Under these, the original Lagrangian density (2) remains form invariant and well-defined provided the coupling scales as: $\hat{\kappa} = \frac{\kappa}{\epsilon}$. The limit $\epsilon \to 0$ may be called a ‘strong coupling limit’ in the $\hat{\kappa}$ space.

Next, we set the time gauge after obtaining all the constraints. In order to avoid repetitions, we only display the new constraints directly in the unhatted (nonsingular) variables using the same procedure elucidated in the earlier section.
In time gauge the boost constraint is already solved. Along with the vanishing torsion condition, this implies that the spatial connection $\omega^{ij}_a(E)$ is to be treated as a functional of the momenta $E^a_i$. We obtain:

$$H = -\frac{\kappa}{2} E^a_i E^b_j \left[ \epsilon^2 \tilde{R}^{ij}_a(\omega(E)) + Q^i_a Q^j_b \right] \approx 0,$$

$$H_a = E^b_i \tilde{D}^i_{[a} Q^j_{b]} \approx 0,$$

$$G_{ij} = Q_{a[i} E^a_{j]} \approx 0 \quad (22)$$

Now we can impose the limit $\epsilon \to 0$, which changes the Hamiltonian constraint to the one below while preserving the rest:

$$H = -\frac{\kappa}{2} E^a_i E^b_j Q^i_a Q^j_b \approx 0 \quad (23)$$

With this, we obtain a Hamiltonian theory inequivalent to the one in limit A in the earlier section. This readily reflects the fundamental feature of Carroll algebra: $[H, H] = 0$.

The Hamiltonian constraint (23) could be identified with the one in second order metric gravity (corresponding to different canonical variables and constraint structure), where only the extrinsic curvature dependent terms survive and the spatial curvature term drops out, called a ‘strong-coupling’ or ‘zero-signature’ or ‘electric’ limit in the literature [4, 5, 15].

### IV. THE SCALING LIMITS AND LEVY LEBLOND-SEN GUPTA SPACETIMES

After this demonstration of the fact that the scaling of the basic variables as introduced in (20) does lead to a different Hamiltonian formulation of Carroll gravity, we now discuss why this should be so. Let us first consider how the phase B could be connected to the Levy Leblond- Sen Gupta limit $c \to 0$.

Note that this limit involving $c$ makes sense provided there exists some characteristic velocity parameter built out of the fundamental field variables (e.g. $\frac{|E|}{|B|}$ for electromagnetic fields). In first order gravity theory, there does exist such a scale given by the ratio of $|e^j_\mu|$ and either $|e^0_\mu|$ or $|e_{\mu0}|$, depending upon the units chosen.

Let us now consider the identity: $-1 = c^2 \hat{e}^0_\mu \hat{e}^{\mu0} = \left[ ce^0_\mu \right] \left[ -\frac{1}{c} e^\mu_0 \right]$ where the appropriate factors of $c$ have been restored in the internal metric $\eta_{IJ}$. A natural way to define a singular limit from the above is to keep both factors $ce^0_\mu := \hat{e}^0_\mu$ and $\frac{1}{c} c^\mu_0 = \hat{e}^\mu_0$ finite as $c \to 0$, which implies:

$$\hat{e}^0_\mu \to \infty, \quad \hat{e}^\mu_0 \to 0.$$  \hspace{1cm} (24)
This is precisely the scaling prescription introduced in (21).

The other alternative towards a singular limit, however, could be changing the units of the basic variables and keeping the factors $\frac{1}{c} e^{\mu}_0 := e^{\mu}_0$ and $c e^{\mu}_0 = e^{\mu}_0$ finite instead, as $c \to 0$. This evidently corresponds to the limit A (eq.(3)), leading to the other (‘magnetic’) Carrollian phase of Hamiltonian gravity.

V. CONCLUSIONS

In general, a singular limit of a field theory need not preserve the number of degrees of freedom of the original theory. Here, we settle this issue in the context of possible Carrollian limits of Hilbert-Palatini gravity theory. By setting up a Hamiltonian formulation of the possible limits and carrying out a detailed constraint analysis, we find that the number of physical degrees of freedom is two, which is the same as in the original theory. This answers an important as well as oft quoted question [15, 16] in the literature relating to ‘Carroll’ gravity.

While doing so, we introduce a scaling prescription (limit B) which naturally leads to a Hamiltonian constraint within first order gravity resembling the so-called ‘electric’ Carroll limit of metric gravity. The limiting canonical structure obtained in terms of the densitized triad and its conjugate is a remarkably simple representation of the celebrated BKL conjecture. Even though the corresponding Hamiltonian constraint, when quantized, exhibits an ordering ambiguity, it is purely algebraic (contains no derivatives of coordinate or momenta). This is quite an attractive feature, as compared to the full Hamiltonian constraint of Hilbert-Palatini gravity. A quantization of this Hamiltonian theory (limit B here) could lead to new insights regarding a possible connection between quantum gravity and a potential resolution of (spacelike, and in particular, cosmological) singularities [24, 25].

The magnetic Hamiltonian constraint in its canonical form here, on the other hand, is free from any ordering ambiguity. It depends only on the momenta ($E^{a}_{\mu}$) and their spatial derivatives. Naturally, this is an interesting candidate Hamiltonian to quantize as well.

To sum up, the Hamiltonian theories in phases A and B altogether provide an intriguing arena from a canonical quantization perspective, modulo issues such as operator ordering, regularization or interpretational aspects of quantum gravity.

As a final observation, our analysis also shows that the Carrollian phase of gravity is in
general inequivalent to a gravity theory based on an exactly degenerate tetrad (as opposed to a tetrad that degenerates only in a limit). To be precise, only one of the two undetermined symmetric fields which had been found to appear in the generic spacetime solutions with noninvertible tetrads [14] shows up in the Carrollian phase. It is plausible that there exist further limiting phases of Hilbert-Palatini gravity connected to the ones considered here, where the other symmetric field found in ref. [14] could enter the description. These intriguing questions deserve further studies.

ACKNOWLEDGMENTS

Support (in part) of the SERB, DST, Govt. of India, through the MATRICS project grant MTR/2021/000008 is gratefully acknowledged. I also thank Marc Henneaux for helpful clarifications on ref. [19] through a private communication.

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1 After the completion of this work, we noted the eprint in [26] that has appeared very recently, aiming to demonstrate the equivalence of the ‘magnetic’ Carroll limit of the first and second order gravity actions. While it is not concerned with a Hamiltonian analysis involving the first order constraints (without fixing any gauge) as here, there appears a discussion on the ‘magnetic’ limit in time gauge whose results have a partial overlap with ours here in one of the subsections. Our perspective and approach however is different.
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