Prediction of operating dynamics in floating-zone crystal growth using Gaussian mixture model

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ABSTRACT
We have applied a Gaussian mixture regression to the prediction of operation dynamics in floating zone crystal growth as an example of a materials process. From only five demonstration trajectories, we successfully predicted the operating dynamics using the Gaussian mixture model with better precision than obtained by using linear regression or neural networks. The current results indicate that the Gaussian mixture regression is suitable for predicting the operation dynamics of materials processes in which it is preferable to avoid large changes from stable operating conditions. Furthermore, precise prediction by the Gaussian mixture regression will lead to the optimization of operation trajectories and automatic control of materials processes.

1. Introduction
The advances and application of informatics has ushered in a new era in materials processing [1]. Materials processes have been upgraded, automated, and efficiently optimized by applying various kinds of informatics algorithms such as Bayesian optimization, and neural networks [2–9]. Application of Bayesian optimization to materials processes has successfully reduced the number of trials for optimization and suggested better operation conditions for high performance and yields [10–15]. Surrogate modelling of materials processes by neural networks can efficiently search for the optimal conditions, [2,16–18]. Although the parameters for most materials processes are independently given, which means that a set of input parameters gives a single output, some materials processes are manually controlled according to the information obtained during operation. For example, in crystal growth by the optical floating zone (FZ) method, an operator monitors the status of the melt in a furnace by a camera and changes the input parameters to maintain suitable conditions for single-crystal growth (details of crystal growth by the FZ method are described later). In this case, it is necessary to conduct a huge number of trials when conventional Bayesian optimization is applied, and massive numbers of simulation results are required to construct a surrogate model by neural networks [18].
In other research fields such as robotics, chemical engineering and aerospace engineering, trajectory optimization has been studied for a long time for the development of various kinds of methods [19–24]. Commonly, trajectory optimization begins with predicting the dynamics of a system from demonstration experiments. In the present study, we predicted the dynamics of crystal growth by the optical FZ method as an example using Gaussian mixture regression (GMR) with consideration of the characteristics of the trajectory of materials processing in manufacturing.

2. FZ method and the characteristic of its trajectory

The FZ method without using a crucible was developed to grow high-purity silicon single crystals without the molten zone coming in contact with any foreign materials [25,26]. Silicon wafers up to 200 mm diameter are manufactured by the FZ method using RF heating, for the production of semiconductors. Not only silicon crystals but also numerous other crystals such as semiconductors [27], metals [28], alloys [29], intermetallic compounds [30–32], oxides [33–36], borides [37,38], carbides [39] and silicides [40–42], especially those with a high melting temperature, have been grown by the optical FZ method [43]. Figure 1 shows a schematic illustration and photograph of a typical optical FZ apparatus. The FZ crystal growth process is usually monitored by a video camera installed in front of a hole in the focusing mirrors. Two polycrystalline rods are mounted so that their tips meet at the focal point of the ellipsoidal mirrors. Halogen or xenon lamps are placed at the other focal points of these mirrors. The optical FZ crystal growth begins with melting the tips of the polycrystalline rods and bringing them together to create a liquid melt called the floating zone between the lower (seed) rod and the upper (feed) rod. After the floating zone is created, the zone begins to move upwards by moving the seed and feed downward (or by moving the mirrors upward), the liquid melt cools, and the crystal finally grows on the seed rod. During crystal growth, the rods rotate either in the same direction or opposite directions at certain speeds. An operator controls the input parameters such as the power of the lamps and the movement speed of the feed so that the liquid melt is not separated or dripped off (the speed at which the seed moves, which is assumed to be crystal growth rate, is usually fixed during crystal growth). If the lamp power is too low, the tips of the seed and feed collapse and the liquid melt can easily separate. On the other hand, if the power is too high, the liquid melt has difficulty maintaining the shape of the floating zone by its surface tension, and it can easily drip off. In this situation, the operator attempts to grow a single crystal by forming a certain shape in which the crystal diameter is first reduced (called ‘necking’) and then gradually increased to the necessary size.

The reason why the operator controls the input parameters according to the state of the liquid melt is that the same operation trajectories never yield the exact same melt conditions leading to different results due to the slight deviations, such as the condition and configuration of the feed and seed rods, quartz tube, halogen lamp and other components in the FZ furnace. Although the response of the melt conditions to the input parameters is almost the same, the slight differences in melt conditions are accumulated and the results become gradually different in each operation. This is true of other materials processes that are manually controlled by human operation according to the monitored conditions.

In summary, the trajectories of the crystal growth process by the FZ method and other materials processes manually controlled according to monitored conditions have the following characteristics that should be reflected in the algorithm:

1. The number of demonstrations (operation trajectories for model construction) is limited.
2. Each trajectory is similar, and is sufficient to predict the dynamics within a limited parameter space.

![Figure 1](image-url)
Since materials processes such as FZ crystal growth experiments generally take a long time for preparation and it is difficult to generate ‘big data’, a sample-efficient algorithm is necessary. On the other hand, it is not necessary to predict the dynamics in all parameter space because for these materials processes, especially for mass production, it is preferable not to deviate greatly from the stable operation trajectory. Considering these characteristics of the trajectories for these materials processes, prediction of the dynamics using GMR is a good choice since GMR can efficiently construct nonlinear dynamics within a neighbourhood around the space covered by the demonstrations, which was successfully demonstrated in learning the non-linear dynamics of robot motion [23,44].

3. Emulation of FZ crystal growth trajectory for learning

Since the real dynamics of FZ crystal growth are unknown, prediction of the dynamics was validated by virtual experiments using an emulator of the FZ crystal growth experiment with a given set of dynamics. Note that whether the given dynamics are consistent with the real dynamics is not relevant to validation of the prediction. Thus, we first made an emulator imitating the FZ crystal growth experiment in which the lamp power (P) and movement speeds of the feed (u) and seed (v) can be controlled as input parameters, and the height of the liquid melt (h) and the diameter of the grown crystal (d) were determined according to the following dynamics:

\[
\begin{align*}
    h &= aP, \\
    \frac{1}{2}hd^2 &= vd_0 - ud,
\end{align*}
\]

where a is a constant and \(d_0\) is the diameter of the feed. Here we assumed P is independent of time and the volume of a feed, melt and a crystal are represented by the 2-dimensional area size. The left-hand side of Equation (2) represents the time differential of melt area, and the right-hand side represents the speed of increment of melt by changing the feed into the melt and the speed of decrement of melt by changing the melt into the crystal. The value of h, which is independent of time, is determined by P, and the value of d depending on time is determined by Equation (2). The system states of the FZ crystal growth with the fixed values of P and u were described by d and \(\dot{d}\).

Figure 2 shows a schematic illustration of the model for FZ crystal growth and its parameters, and a screenshot of the emulator program with a graphical user interface (GUI). The program can emulate the FZ crystal growth experiments and acquire the trajectories of input and output parameters. We manually prepared 10 trajectories for learning (training) (#1~5) and validation (testing) (#6~10) of the prediction of the dynamics with the actual FZ crystal growth experiment in mind. In the present study, we fixed both the movement speed of the feed and the lamp power as 1.0. Figure 3 shows the 5 trajectories for learning in which we aimed to create the following crystal shape:

1. At the initial stage (100 < t), the diameter of the crystal is kept the same as that of the feed rods (d = 1.0)
2. Then (100 < t < 300), the diameter is decreased to 0.1 corresponding to the “necking” process.
3. After the necking process (300 < t < 500), the diameter is increased to 1.0.
4. Finally, the crystal diameter is kept at 1.0 until the crystal growth experiment is over (500 < t < 1000).

Since the trajectories were manually prepared using the emulator program, they were different from each other and did not completely fulfill the above aim.

4. Prediction of dynamics by GMR

For prediction of the dynamics of the FZ crystal growth, we utilized GMR, which was reported to efficiently learn non-linear dynamics for robot motions [23–45]. Here we described the prediction of the dynamics by GMR assuming FZ crystal growth. The state of the liquid melt at time \((t+1)\), which is composed of the height and diameter of the liquid melt and described as \(x_{t+1} = (h_{t+1}, d_{t+1})\), is assumed to be determined by the state of the liquid melt \((x_t)\) and inputs composed of the lamp power and moving speeds of the feed and seed \((y_t = (P, u_t, v_t))\) at time t as follows:

\[
\begin{align*}
    x_{t+1} &= f(z_t), \\
    z_t &= (x_t, y_t).
\end{align*}
\]

The function f was represented by a locally linear function as follows:

\[
    x_{t+1} = F_z \cdot z_t + F_0.
\]

\(F_z\) and \(F_0\) are the coefficient matrix and vector, respectively. In linear regression (LR), these coefficients are determined independently of \(z_t\) so as to minimize the sum of the squared errors. In GMR, however, these coefficients are estimated by a Gaussian mixture model (GMM), which is a non-linear combination of a finite set of Gaussian kernels constructed from the set of demonstration trajectories. GMM defines a joint probability distribution \(P(x_t, x_{t+1})\) of the current state, input and next state over a set of demonstration trajectories as a mixture of a finite set of n Gaussians.
where $\Sigma_k$ and $\pi_k$ were optimized to fit the demonstration trajectories by an expectation maximization (EM) algorithm [46]. Taking the posterior mean estimate of $P(x_{t+1}|z_t)$, one can estimate the function $f$ as a nonlinear sum of linear dynamics as follows [45]:

$$x_{t+1} = f(x_t) = \sum_{k=1}^{n} \pi_k G(x_{t+1}|\mu_k, \Sigma_k)$$

with $\mu^k$ and $\Sigma^k$ being the mean value and covariance matrix of the $k$th Gaussian as follows:

$$\mu^k = \begin{bmatrix} \mu^k_1 \\ \mu^k_2 \\ \vdots \\ \mu^k_n \end{bmatrix}, \Sigma^k = \begin{bmatrix} \Sigma^k_{11} & \Sigma^k_{12} & \cdots & \Sigma^k_{1n} \\ \Sigma^k_{21} & \Sigma^k_{22} & \cdots & \Sigma^k_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma^k_{n1} & \Sigma^k_{n2} & \cdots & \Sigma^k_{nn} \end{bmatrix}$$

where $\pi_k$ is a prior. Gaussian mixtures were optimized to fit the demonstration trajectories by an expectation maximization (EM) algorithm. The initial values of $\mu^k$, $\Sigma^k$ and $\pi_k$ were set. In the present study, the mean value and covariance matrix for a demonstration trajectory, and a uniform prior distribution was used as the initial values $\mu_k$, $\Sigma_k$ and $\pi_k$. The values of $\mu^k$, $\Sigma^k$ and $\pi_k$ were optimized to fit the demonstration trajectories by an expectation maximization (EM) algorithm [46]. Taking the posterior mean estimate of $P(x_{t+1}|z_t)$, one can estimate the function $f$ as a nonlinear sum of linear dynamics as follows [45]:

$$x_{t+1} = f(x_t) = \sum_{k=1}^{n} \pi_k G(x_{t+1}|\mu_k, \Sigma_k)$$

where

$$\pi_k = \frac{P(z_t|x_{t+1}, \mu^k, \Sigma^k)}{\sum_{k=1}^{n} P(z_t|x_{t+1}, \mu^k, \Sigma^k)}$$

Equation (9) shows that the next state ($x_{t+1}$) is described by the current state ($x_t$), the inputs ($y_t$) and the optimized Gaussian mixtures by EM algorithm.

In the present study we set the number of Gaussian mixtures as $n=50$ for prediction of the dynamics of FZ crystal growth as a result of parameter search.
5. Results and discussion

Among the 10 trajectories prepared by the emulator, the GMM was trained from the 5 demonstration trajectories (Train-1~5) and validated by the other remaining 5 trajectories (Test-1~5). Figure 4 shows the Gaussian mixtures optimized to the 5 demonstration trajectories. The Gaussian mixtures are formed to cover the demonstration trajectories, which implies that the dynamics predicted by the GMM reproduce actual dynamics near the demonstration trajectories. Note that some Gaussians have very narrow width indicating the overfitting to the training trajectories. However, the overfitting is acceptable since we aim to make a model which can precisely predict the dynamics near the stable operation trajectories.

Figure 5 shows the trajectory and absolute error of the diameter of the grown crystal \((d)\) predicted by the GMM in comparison with the actual trajectories as well as the trajectories predicted by linear regression and a neural network (NN). The GMM...
reproduced the actual trajectories better than the neural network and linear regression. From the viewpoint of crystal growth, the trajectory predicted by using GMM is acceptable except for Test-4, in which the values of \(d\) are deviating to the negative values at around \(t = 400\). The negative values of \(d\) correspond to the melt is separated. Most of the trajectories predicted by NN tend to deviate from the true value when the value of \(d\) decreases at around \(t = 150\) or increases at around \(t = 400\), and the value of \(d\) tends to increase rapidly, which make the quality of single crystal worse. The trajectories predicted by linear regression seems to much larger deviation than GMM.

Figure 6 shows the mean absolute error (MAE) for the trajectories predicted by GMM. Compared to the other methods, the GMM can more accurately predict the trajectories. Figure 7 shows the relative errors in the value of \(d\) in the next time step predicted from the value of \(d\) and \(v\) in the current time step accommodating the training trajectories. Relative error was calculated as the difference between the value of \(d\) in the next time step calculated from Equation (2) and GMM. The prediction is sufficiently precise, and the relative errors are less than 0.01 near the training trajectories. The non-linear local dynamics of the operation trajectories for the FZ crystal growth were accurately predicted by the GMM near the training trajectories. Although it is difficult to strictly define whether the trajectory in operation is near the training trajectories or not, we can notice the accuracy of the prediction of the dynamics during the operation since we can compare the actual state and predicted state from the previous state and the GMM.

In the present study, we successfully demonstrated that the GMR is an efficient method for the prediction of operation dynamics for FZ crystal growth. From only five operation trajectories, the GMM can predict the actual trajectories of FZ crystal growth with relatively high accuracy. Although the current result was demonstrated by operation trajectories made by an emulator program, it is expected that the dynamics of the actual FZ crystal growth can be well-predicted by the GMM. The present demonstration is the first step of the automation of materials process including FZ crystal growth with optimized trajectory.

6. Conclusion
We have applied the GMR to the prediction of FZ crystal growth, as an example of materials processes in which it is preferable to avoid deviating greatly from the stable operation trajectory. The GMR can predict the dynamics of the FZ crystal growth process from only five operation trajectories better than neural networks and linear regression. Although there is still a gap between the demonstration in the simple emulation program and its application to the actual materials processes which is complex and probabilistic, the present results imply the feasibility of the prediction of the actual dynamics for the FZ crystal growth process, which will lead to the optimization of operation trajectories and automatic control of materials processes.

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Author contributions
SS conceptualized the basic idea and SH conceived the application to the materials process. RO constructed algorithm and programs for analysis under the guidance of SS, with the assistance of YT and in continuous discussion with all authors. The manuscript was written by SH in discussion with all authors.

References
[1] Sha W, Guo Y, Yuan Q, et al. Artificial intelligence to power the future of materials science and engineering. Adv Intell Syst. 2020;2(4):1900143. DOI:10.1002/aisy.201900143.
[2] Tsunooka Y, Kokubo N, Hatasa G, et al. High-speed prediction of computational fluid dynamics simulation in crystal growth. Cryst Eng Comm. 2018;20(41):6546–6550. DOI:10.1039/C8CE00977E.
[3] Dropka N, Holena M, Eckele B, et al. Fast forecasting of VGF crystal growth process by dynamic neural networks. J Crystal Growth. 2019;5219–14.
[4] Wang L, Sekimoto A, Takehara Y, et al. Optimal control of SiC crystal growth in the RF-TSSG system using reinforcement learning. Crystals (Basel). 2020;10(9):791. DOI:10.3390/cryst10090791.
[5] Takehara Y, Sekimoto A, Okano Y, et al. Bayesian optimization for a high- and uniform-crystal yield rate in the top-seeded solution growth process of silicon carbide under applied magnetic field and seed rotation. J Crystal Growth. 2020;532:125437.
[6] Wang C, Tan XP, Tor SB, et al. Machine learning in additive manufacturing: state-of-the-art and perspectives. Additive Manuf. 2020;36:101538.
[7] Yu W, Zhu C, Tsujimori Y, et al. Geometrical design of a crystal growth system guided by a machine learning algorithm. Cryst Eng Comm. 2021;23(14):2695–2702. DOI:10.1039/D1CE00106F.
[8] Kawata A, Murayama K, Sumitani S, et al. Design of automatic detection algorithm for dislocation contrasts in birefringence images of SiC wafers. Jpn J Appl Phys. 2021;60(SB):SBBD06. DOI:10.35848/1347-4065/abde29.
[9] Harada S, Tsujimori K, Matsushita Y. Automatic detection of basal plane dislocations in a 150-mm SiC epitaxial wafer by photoluminescence imaging and template-matching algorithm. J Electron Mater. 2021;51(1):243–248.
[10] Shimizu R, Kobayashi S, Watanabe Y, et al. Autonomous materials synthesis by machine learning and robotics. APL Mater. 2020;8(11):111110. DOI:10.1063/5.0020370.
[11] Osada K, Kutsukake K, Yamamoto J, et al. Adaptive Bayesian optimization for epitaxial growth of Si thin films under various constraints. Mater Today Commun. 2020;25:101538.
[12] Miyagawa S, Gotoh K, Kutsukake K, et al. Application of Bayesian optimization for improved passivation performance in TiOx/x/SiOy/c-Si heterostructure by hydrogen plasma treatment. Appl Phys Express. 2021;14(2):025503. DOI:10.35848/1882-0786/abd869.
[13] Miyagawa S, Gotoh K, Kutsukake K, et al. Application of Bayesian optimization for high-performance TiOx/sioy/c-Si passivating contact. Sol Energy Mater Sol Cells. 2021;230:11251.
[14] Ohkubo I, Hou Z, Lee JN, et al. Realization of closed-loop optimization of epitaxial titanium nitride thin-film growth via machine learning. Mater Today Phys. 2021;16:100296.
[15] Wakabayashi YK, Otsuka T, Krockenberger Y, et al. Machine-Learning-Assisted thin-film growth: Bayesian optimization in molecular beam epitaxy of SrRuO3 thin films. Apl Mater. 2019;7(10):101114. DOI:10.1063/1.5123019.
[16] Dropka N, Holena M. Optimization of magnetically driven directional solidification of silicon using artificial neural networks and Gaussian process models. J Crystal Growth. 2017;471:53–61.
[17] Dang Y, Liu L, Li Z. Optimization of the controlling recipe in quasi-single crystalline silicon growth using artificial neural network and genetic algorithm. J Crystal Growth. 2019;522:195–203.
[18] Dang Y, Zhu C, Ikumi M, et al. Adaptive process control for crystal growth using machine learning for high-speed prediction: Application to SiC solution growth. Cryst Eng Comm. 2021;23(9):1982–1990. DOI:10.1039/D0CE01824D.
[19] Betts JT. Survey of numerical methods for trajectory optimization. J Guidance Control Dyn. 2012;35(2):193–207.
[20] Eaton JW, Rawlings JB. Model-predictive control of chemical processes. Chem Eng Sci. 1992;47(4):705–720.
[21] Mayne DQ, Rawlings JB, Rao CV, et al. Constrained model predictive control: stability and optimality. Automatica. 2000;36(6):789–814. DOI:10.1016/S0005-1098(99)00214-9.
[22] Chettibi T, Lehtihet HE, Haddad M, et al. Minimum cost trajectory planning for industrial robots. Eur J Mech A/solids. 2004;23(4):703–715. DOI:10.1016/j.euromechsol.2004.02.006.
[23] Khansari-Zadeh SM, Billard A. Learning stable nonlinear dynamical systems with Gaussian mixture models. IEEE Trans Rob. 2011;27(5):943–957.
[24] Posa M, Kuindersma S, Tedrake R. Optimization and stabilization of trajectories for constrained dynamical systems. Proceedings - IEEE International Conference on Robotics and Automation; Stockholm, Sweden; 2016 June. p. 1366–1373.
[25] Theuerer HC Method of processing semiconductive materials. United States: United States Patent Office; 1952.
[26] Riemann H, Luedde A. Floating zone crystal growth. In: Nakajima K, Usami N, editors. Crystal growth of Si for solar cells. Berlin, Heidelberg: Springer; 2009. pp. 41–53.
[27] Campbell TA, Schweizer M, Dold P, et al. Float zone growth and characterization of Ge1-xSix (x<10 at%) single crystals. J Crystal Growth. 2001;226(2–3):231–239. DOI:10.1016/S0022-0248(01)01394-X.
[28] Calverley A, Lever RF. The floating-zone melting of refractory metals by electron bombardment. J Sci Instrum. 1957;34(4):142.

[29] Glebovsky VG, Semenov VN. Electron-Beam floating-zone melting of refractory metals and alloys: art and science. Int J Refract Metals Hard Mater. 1993;12(5):295–301.

[30] Inui H, Oh MH, Nakamura A, et al. Room-Temperature tensile deformation of polycrystalline twinned (PST) crystals of TiAl. Acta Metallurgica Et Materialia. 1992;40(11):3095–3104. DOI:10.1016/0956-7151(92)90472-Q.

[31] Hirano T, Mawari T. Unidirectional solidification of Ni3Al by a floating zone method. Acta Metallurgica Et Materialia. 1993;41(6):1783–1789.

[32] Tanaka K, Ohashi T, Kishida K, et al. Single-crystal elastic constants of Co3(Al,W) with the L12 structure. Appl Phys Lett. 2007;91(18):181907. DOI:10.1063/1.2805020.

[33] Kimura S, Shindo I. Single crystal growth of YIG by the floating zone method. J Crystal Growth. 1977;41(2):192–198.

[34] Balbashov AM, Egorov SK. Apparatus for growth of single crystals of oxide compounds by floating zone melting with radiation heating. J Crystal Growth. 1981;52:498–504.

[35] Koohpayeh SM, Fort D, Abell JS. The optical floating zone technique: A review of experimental procedures with special reference to oxides. Prog Cryst Growth Charact Mater. 2008;54(3–4):121–137.

[36] Harada S, Kosaka N, Yagi T, et al. Crossover from incoherent to coherent thermal conduction in bulk titanium oxide natural superlattices. Scr Mater. 2022;208:114326.

[37] Otani S, Honma S, Yajima Y, et al. Preparation of LaB6 single crystals from a boron-rich molten zone by the floating zone method. J Crystal Growth. 1993;126(2–3):466–470. DOI:10.1016/0022-0248(93)90052-X.

[38] Otani S, Ishizawa Y. Preparation of ZrB2 single crystals by the floating zone method. J Crystal Growth. 1996;165(3):319–322.

[39] Christensen AN. The crystal growth of the transition metal compounds TiC, TiN, and ZrN by a floating zone technique. J Crystal Growth. 1976;33(1):99–104.

[40] Nerlund Christensen A. Crystal growth and characterization of the transition metal silicides MoSi2 and WSi2. J Crystal Growth. 1993;129(1–2):266–268.

[41] Harada S, Hoshikawa H, Kuwabara K, et al. Crystal structure refinement of ReSi 1.75 with an ordered arrangement of silicon vacancies. Philos Mag. 2011;91(23):3108–3127. DOI:10.1080/14786435.2011.570278.

[42] Harada S, Tanaka K, Kishida K, et al. Direct observation of vacancies and local thermal vibration in thermoelectric rhenium silicide. Appl Phys Express. 2012;5(3):035203. DOI:10.1143/APEX.5.035203.

[43] Dabrowska HA, Dabkowski AB, et al. Crystal growth of oxides by optical floating zone technique. In: Dhanaraj G, Kullaiah B, Prasad Veditors. Springer handbook of crystal growth. Berlin, Heidelberg: Springer; 2010. pp. 367–391.

[44] Levine S, Abbeel P. Learning neural network policies with guided policy search under unknown dynamics. Proceedings - Twenty-eighth conference on Neural Information Process System, Montreal, Canada; 2014 Dec. p. 1071–1079.

[45] Khansari-Zadeh SM, Billard A, BM: An iterative algorithm to learn stable non-linear dynamical systems with Gaussian mixture models. Proceedings - IEEE International Conference on Robotics and Automation; Anchorage, Alaska, USA; 2010. p. 2381–2388.

[46] Dempster AP, Laird NM, Rubin DB. Maximum likelihood from incomplete data via the EM algorithm. J Royal Stat Soc Ser B (Methodological). 1977;39:1–22.