Optical soliton solutions of the conformable time fractional Radhakrishnan–Kundu–Lakshmanan Model

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Abstract
In the present study, we have obtained different kinds of wave solutions which possess distinctive physical characteristics of the nonlinear conformable time fractional Radhakrishnan–Kundu–Lakshmanan model by utilizing the generalized Jacobi elliptic function method. 3-D surfaces to some of the reported solutions are plotted and the dependence of the behaviour of the solutions on the fractional derivative has also been analyzed in the present study. In addition to providing physical explanation of Radhakrishnan-Kundu-Lakshmanan equation, the solutions presented here may also provide an insight into the study of wave propagation in various conformable fractional nonlinear models arising in nonlinear sciences.

Keywords Radhakrishnan–Kundu–Lakshmanan model · Conformable fractional derivative · Generalized Jacobi elliptic function method · Optical soliton solutions

Mathematics Subject Classification 35B30 · 35C08 · 26A33

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1 Introduction
Over the last few years, significant progress has been made in the field of nonlinear partial differential equations (NLPDEs). Many NLPDEs are used in the mathematical formulation of complex phenomena in several nonlinear physics fields like plasma physics, nonlinear fiber optics, electromagnetism, fluid dynamics and optics (Biswas et al. 2014; Debnath 2011). One interesting topic to investigate is the theory of optical solitons and their propagation through nonlinear optical fibers. The optical soliton represents a pulse which travels without any distortion due to dispersion or other factors.

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A soliton in optics is used to illustrate any optical field that does not change during propagation as a result of the delicate balance between nonlinear and linear phenomena in the medium. Throughout a nonlinear medium, electromagnetic pulses propagate in multiple dimensions. Different physical factors such as dispersion, material dispersion, diffraction and nonlinear response affect the pulse dynamics (Ali et al. 2018). Soliton propagation has been represented mathematically by a variety of models such as Fokas–Lenells equation (Ismael et al. 2020), Sasa–Satsuma equation (Seadawy et al. 2021), Chen–Lee–Liu equation (Younas and Younis 2020), nonlinear Schrödinger’s equation (Hosseini et al. 2018), Kundu–Mukherjee–Naskar equation (Rezazadeh et al. 2021), Kundu–Eckhaus equation (Rezazadeh et al. 2019), Schrödinger–Hirota equation (Yépez-Martínez and Gómez-Aguilar 2019) and Ginzburg–Landau model (Elbo-re 2020), (2 + 1)-dimensional Heisenberg ferromagnetic spin chain model (Hosseini et al. 2021), Calogero-Degasperis equation and Sharma-Tasso-Olver equation (Kaur and Wazwaz 2022).

Obtaining the exact solutions of the NLPDEs helps in understanding the relationship between a differential equation and its mathematical and physical applications. The exact traveling-wave solutions of NLPDEs have been constructed via various effective approaches which include exponential rational function method (Ghanbari et al. 2019), extended trial function method (Biswas et al. 2018), improved Bernoulli sub equation function method (Baskonus and Bulut 2016; Bulut et al. 2017), Riccati-Bernoulli sub-ODE method (Ozdemir et al. 2021).

This study aims to examine numerous exact solutions including bright-dark soliton solutions, Jacobi elliptic solutions of the conformable time-fractional perturbed Radhakrishnan–Kundu–Lakshmanan (RKL) equation (Bansal et al. 2018; Gonzalez-Gaxiola and Biswas 2019; Sulaiman et al. 2018) via GJEF method for the first time in the conformable time fractional RKL model. The dynamics of the light pulses were studied and represented by various NLPDEs including RKL equation. According to the Kerr law nonlinearity, conformable time-fractional perturbed RKL equation has the following dimensionless form Sulaiman and Bulut (2019), Biswas et al. (2018):

\[
iD_\alpha^\psi + aD_\psi_{xx} + b|\psi|^2\psi - i\Omega\psi_x - i\Lambda(|\psi|^2\psi)_x - i\sigma(|\psi|^2)_x\psi - iY\psi_{xxx} = 0, \quad 0 < \alpha \leq 1
\]

(1)

here \(\psi\) is the complex-valued wave function of \(x\); space and \(t\); time. The fractional temporal evolution of the nonlinear wave is represented by the term \(D_\alpha^\psi\). The coefficient \(a\) represents the group-velocity dispersion (GVD), \(b\) represents the coefficient of nonlinearity, \(\Omega\) symbolises the inter-modal dispersion (IMD) and \(\Lambda, \sigma, Y\) signify the coefficient of self-steepening for short pulses, the higher-order dispersion coefficient and the third order dispersion term respectively.

As soon as we substitute \(\alpha = 1\) in the time fractional perturbed RKL equation, we obtain the original Radhakrishnan–Kundu–Lakshmanan equation (Gonzalez-Gaxiola and Biswas 2019).

This paper is structured as follows: In sect. 2, we have given a brief overview of the conformable fractional derivative. In sect. 3, the GJEF method has been described. In sect. 4, exact solution of conformable time-fractional RKL model have been obtained. In sect. 5, we have briefly analysed the obtained exact solutions using their graphs and finally concluding remarks have been given in sect. 6.
2 Conformable fractional derivative

In this section, we present an overview on the conformable fractional derivative. In 2014, Khalil et al. (2014) gave a new definition of the fractional derivative called the conformable fractional derivative (CFD). In recent times, many authors have used this CFD in various ways. Abdeljawad worked the conception of fractional versions of chain rule, Gronwall’s inequality, exponential functions, Taylor power series expansions, integration by parts, Laplace transforms and linear differential systems (Abdeljawad 2015). Oqielat et al. (2020) compared conformable and Caputo derivatives in the solutions of nonlinear time-fractional version of Schrödinger equations. Some new properties and theorems of this definition of CFD were investigated and some new definitions were given by Atangana et al. (2015). The CFD of order \( \alpha \) is defined for a function \( g : (0, \infty) \to \mathbb{R} \) by Khalil et al. (2014):

\[
D^{\alpha} g(t) = \lim_{\varepsilon \to 0} \frac{g^{[\alpha]-1}(t + \varepsilon t^{[\alpha]-\alpha}) - g^{[\alpha]-1}(t)}{\varepsilon}, \quad m - 1 < \alpha \leq m, \, t > 0
\]

where \( m \in \mathbb{N} \) and \( |\alpha| \) is the smallest integer number greater than or equal \( \alpha \), provided that \( D^{\alpha} g(0) = \lim_{t \to 0^+} D^{\alpha} g(t) \), \( g(t) \) is \( m \)-differentiable and \( D^{\alpha} g(0) = \lim_{t \to 0^+} D^{\alpha} g(t) \) exists. As a special case, if \( 0 < \alpha \leq 1 \), then we have

\[
D^{\alpha} g(t) = \lim_{\varepsilon \to 0} \frac{g(t + \varepsilon t^{1-\alpha}) - g(t)}{\varepsilon}, \quad t > 0
\]

provided \( D^{\alpha} g(0) = \lim_{t \to 0^+} D^{\alpha} g(t) \), \( g(t) \) is differentiable and \( D^{\alpha} g(0) = \lim_{t \to 0^+} D^{\alpha} g(x) \) exists.

For functions \( g(t) \) and \( h(t) \) the conformable fractional derivative bears the following properties for \( 0 < \alpha \leq 1 \), Khalil et al. (2014):

1. \( D^{\alpha}(pg + qh) = pD^{\alpha}(g) + qD^{\alpha}(h), \; p, q \in \mathbb{R} \)
2. \( D^{\alpha}(t^{A}) = \Lambda t^{A-\alpha}, \; \Lambda \in \mathbb{R} \)
3. \( D^{\alpha}(gh) = gD^{\alpha}(h) + hD^{\alpha}(g) \)
4. \( D^{\alpha} \left( \frac{g}{h} \right) = \frac{hD^{\alpha}(g) - gD^{\alpha}(h)}{h^{2}} \)

A significant and useful rule is that if \( g(t) \) is a \( m \)-differentiable function at \( t > 0 \) and \( m - 1 < \alpha \leq m \), then, \( D^{\alpha} g(t) = [t^{[\alpha]-\alpha} g^{[\alpha]}](t) \). The corresponding conformable fractional integral has the following properties:

1. Let \( \alpha \in (m, m+1] \) then the \( \alpha \)-order conformable fractional integral is given by
   \[
   I^{\alpha} f(t) = I^{m+1} \left[ t^{m-\alpha} g(t) \right] = \frac{1}{m!} \int_{0}^{t} \frac{f(\tau)}{\tau^{m+\alpha}} g(\tau) d\tau, \quad \text{where } I^{m+1} \text{ is the } (m+1)\text{-order integral.}
   \]
2. \( I^{\alpha} [g(t)] = I^{[\alpha]-1} g(t) \) is the usual Riemann improper integral and \( \alpha \in (0, 1] \).
3. \( D^{\alpha} I^{\alpha} [g(t)] = g(t), \; \text{for } t \geq 0 \), where \( g(t) \) is any continuous function in the domain of \( I^{\alpha} \).
4. \( I^{\alpha} D^{\alpha} [g(t)] = g(t) - \sum_{k=0}^{n} \frac{g^{(k)}(0) t^{k}}{k!} \), for \( t \geq 0 \), where \( g(t) \) is an \((m+1)\)-times differentiable function, \( n < \alpha \leq n + 1 \).
5. Let \( g : [0, b) \to \mathbb{R} \) be differentiable and \( 0 < \alpha \leq 1 \). Then, for all \( t > 0 \) we have
   \[
   I^{\alpha} D^{\alpha} [g(t)] = g(t) - g(0).
   \]
The main advantage of using the CFD is that it complies with all the rules and concepts of an ordinary derivative, like product, quotient and chain rules, whereas the other fractional definitions do not. It also helps in the generalisations of some of the well known transforms like the Sumudu and Laplace transforms and can be utilized as a tool for solving the fractional differential equations. It opens the door to extend and create new definitions such as the non-conformable fractional derivative (Guzman et al. 2018), M-conformable fractional derivative (Sousa and De Oliveira 2018), class conformable fractional derivatives (Anderson and Ulness 2015), Fuzzy generalized conformable fractional derivative (Harir et al. 2020), modified conformable fractional derivative (El-Ajou 2020) and truncated $\Omega$ fractional derivative (Sabi’u et al. 2022).

3 Description of generalized Jacobi elliptic function method

In the following section, we have briefly described the steps of generalized Jacobi elliptic function method (GJEFM) (Hua-Mei 2005) for conformable fractional equation.

Step 1 Consider a general time conformable fractional evolution equation as follows:

$$F\left(\Psi, \frac{\partial^\alpha \Psi}{\partial t^\alpha}, \frac{\partial \Psi}{\partial x}, \frac{\partial^{2\alpha} \Psi}{\partial t^{2\alpha}}, \frac{\partial^2 \Psi}{\partial x^2}, \ldots\right) = 0$$

(2)

By using the wave transformation

$$\Psi(x, t) = \chi(\zeta)e^{i\Phi}, \zeta = \kappa(x - c \frac{t^\alpha}{\alpha}), \Phi = -vx + \sigma \frac{t^\alpha}{\alpha} + \theta$$

and using the chain rule (Abdeljawad 2015), (2) reduces into an ordinary differential equation (ODE).

$$f\left(\chi, \chi', \chi'', \ldots\right) = 0$$

(3)

where $'$ denotes first order derivative w.r.t. $\zeta$.

Step 2 Assume that the solution of (3) takes the following form:

$$\chi(\zeta) = a_0 + \sum_{i=1}^{s} a_i F^i(\zeta) + \sum_{i=1}^{s} b_j F^{-j}(\zeta)$$

(4)

Here, the functions $F(\zeta)$ are the solution of the following ODE:

$$\left(F'\right)^2(\zeta) = \xi F^4(\zeta) + \tau F^2(\zeta) + \eta$$

(5)

which are given in Table 1 for different values of constants $\xi$, $\tau$ and $\eta$.

Step 3 By using the balancing principle, $s$ in (4) can be determined. Then inserting (4) into (3), and equating the coefficients of polynomial in $F$ to zero, a system of algebraic equations is yielded. On solving that system, we can determine $a_i$, $b_j$.

Step 4 By utilizing all values of $\zeta$, $\tau$, $\eta$ and gathering all the results given in Table 1, the exact solutions of (2) can be obtained.

In the Table 1, we have
Table 1 Solutions of (5) for some values of $\xi$, $\tau$ and $\eta$ (Hua-Mei 2005)

| $\xi$ | $\tau$ | $\eta$ | $F$ |
|-------|--------|--------|-----|
| $m^2$ | $-(1 + m^2)$ | 1 | $sn$, $cd$ |
| $-m^2$ | $2m^2 - 1$ | $1 - m^2$ | $cn$ |
| $-1$ | $2 - m^2$ | $m^2 - 1$ | $dn$ |
| 1 | $-(1 + m^2)$ | $m^2$ | $ns$, $dc$ |
| $1 - m^2$ | $2m^2 - 1$ | $-m^2$ | $nc$ |
| $m^2 - 1$ | $2 - m^2$ | $-1$ | $nd$ |
| $1 - m^2$ | $2 - m^2$ | 1 | $sc$ |
| $-m^2(1 - m^2)$ | $2m^2 - 1$ | 1 | $sd$ |
| 1 | $2 - m^2$ | $1 - m^2$ | $cs$ |
| $2m^2 - 1$ | $-m^2(1 - m^2)$ | $ds$ |
| $-\frac{1}{4}$ | $\frac{m^2+1}{2}$ | $\frac{-(1-m^2)^2}{4}$ | $mcn \mp dn$ |
| $\frac{1}{4}$ | $\frac{1-2m^2}{2}$ | $\frac{1}{4}$ | $ns \mp cs$ |
| $1-m^2$ | $\frac{m^2+1}{2}$ | $\frac{1-m^2}{4}$ | $nc \mp sc$ |
| $\frac{1}{4}$ | $\frac{m^2-2}{2}$ | $\frac{m^2}{4}$ | $ns \mp ds$ |
| $\frac{m^2}{4}$ | $\frac{m^2-2}{2}$ | $\frac{m^2}{4}$ | $sn \mp tcn$, $\frac{dn}{\sqrt{1-m^2 sn \mp en}}$ |
| $\frac{1}{4}$ | $\frac{1-2m^2}{2}$ | $\frac{1}{4}$ | $mcn \mp udn$, $\frac{sn}{\sqrt{1-m^2 en}}$ |
| $\frac{m^2}{4}$ | $\frac{m^2-2}{2}$ | 1 | $\frac{sn}{1+dn}$ |
| $\frac{1}{4}$ | $\frac{1-m^2}{2}$ | $\frac{1-m^2}{4}$ | $\frac{en}{dn+cn}$ |
| $\frac{m^2-1}{4}$ | $\frac{m^2+1}{2}$ | $\frac{1}{4}$ | $\frac{sn}{1+cn}$ |
| $\frac{1}{4}$ | $\frac{m^2}{2}$ | $\frac{m^2}{4}$ | $\frac{sn}{en}$ |
| $\frac{m^4}{4}$ | $\frac{m^2-1}{2}$ | $\frac{1}{4}$ | $\frac{en}{dn}$ |

$sn = sn(\xi, m), cd = cd(\xi, m), cn = cn(\xi, m), dn = dn(\xi, m), ns = ns(\xi, m), cs = cs(\xi, m), ds = ds(\xi, m), sc = sc(\xi, m), sd = sd(\xi, m)$ are the Jacobi elliptic functions with the modulus $0 < m < 1$. These functions degenerate into hyperbolic functions, when $m \to 1$ as follows:

$$sn(\xi, 1) = \tanh(\xi), cn(\xi, 1) = \sech(\xi), dn(\xi, 1) = \sech(\xi),$$

$$ns(\xi, 1) = \coth(\xi), cs(\xi, 1) = \csch(\xi),$$

$$ds(\xi, 1) = \csch(\xi), sc(\xi, 1) = \sinh(\xi), sd(\xi, 1) = \sinh(\xi),$$

$$nc(\xi, 1) = \cosh(\xi), cd(\xi, 1) = 1$$

and into trigonometric functions, when $m \to 0$, as follows:

$$sn(\xi, 0) = \sin(\xi), cd(\xi, 0) = \cos(\xi),$$

$$ns(\xi, 0) = \csc(\xi), cs(\xi, 0) = \cot(\xi),$$

$$ds(\xi, 0) = \csc(\xi), sc(\xi, 0) = \tan(\xi),$$

$$nc(\xi, 0) = \sec(\xi), dn(\xi, 0) = 1$$
4 Exact solution of conformable time-fractional RKL equation by GJEFM

In this section, we use the GJEFM to the conformable time-fractional perturbed RKL equation. Consider the complex conformable time-fractional travelling wave transformation

\[ \Psi(x,t) = \chi(\zeta)e^{i\phi}, \quad \zeta = \kappa \left( x - c \frac{t^\alpha}{\alpha} \right), \quad \Phi = -\nu x + \sigma \frac{t^\alpha}{\alpha} + \theta. \]  

(6)

Substituting (6) into (1) and comparing the real and imaginary parts, we get

\[ \kappa^2(a + 3\nu Y)\chi'' + (b - \nu \Lambda)\chi^3 - \left( \sigma + av^2 + \Omega v + Y v^3 \right)\chi = 0 \]  

(7)

from the real part, and

\[ \kappa^2 Y \chi'' - \left( c + 2av + \Omega + 3v^2 Y \right) \chi' - (3\Lambda + 2\theta)\chi^2 \chi' = 0 \]  

(8)

from the imaginary part. Integrating (8) once, we have

\[ 3\kappa^2 Y \chi'' - 3\left( c + 2av + \Omega + 3v^2 Y \right) \chi - (3\Lambda + 2\theta)\chi^3 = 0 \]  

(9)

As the function \( \chi \) satisfies both (7) and (9), the following constraint condition is given:

\[ \frac{a + 3\nu Y}{3Y} = \frac{\sigma + av^2 + \Omega v + Y v^3}{3\left( c + 2av + \Omega + 3v^2 Y \right)} = -\frac{b - \nu \Lambda}{3\Lambda + 2\theta} \]  

(10)

Solving (10) for \( \nu \) and \( c \), we obtain

\[ \nu = -\frac{(3bY + 2a\theta + 3a\Lambda)}{6Y(\Lambda + \theta)}, \quad c = \frac{Y\left( \sigma + av^2 + \Omega v + Y v^3 \right)}{a + 3\nu Y} - \left( 2av + \Omega + 3Yv^2 \right). \]  

(11)

By using the homogeneous balance principle (Fan and Zhang 1998) and balancing \( \chi'' \) and \( \chi^3 \) in (7), yields \( s = 1 \). Therefore, we assume that the solution of (7) is of the form

\[ \chi(\zeta) = a_0 + a_1 F(\zeta) + \frac{b_1}{F(\zeta)}. \]  

(12)

Substituting (12) into (7) and comparing the coefficients of various powers of \( F(\zeta) \), the following system of equations is obtained
Optical soliton solutions of the conformable time fractional thermal wave equation: An extension of the Bäcklund transformation method

\(- b_2^3(v\Lambda - b) = 0,\)
\(- 3b_2b_3^2(v\Lambda - b) = 0,\)
\(18b_2 (\left( Y \eta \kappa^2 - 1/6 \Lambda a_0b_2 - 1/6 \Lambda b_2^2 \right) \nu + 1/3 \eta \kappa^2 + 1/6 a_0b_2b + 1/6 b_2^2b) = 0,\)
\((6\eta Y \kappa b_2 - 6 \Lambda a_0b_2 - 3 \Lambda a_1b_2^2 - \Lambda b_2^3) \nu + 3a_1b_2^2b + 6a_0b_2b + b_1(2\eta \kappa^2a + \nu b_2) = 0,\)
\(- b_2 Y v^3 - b_2av^2 + (\left( -3a_0b_2^2 + (12Y \tau \kappa^2 - 3a_0^2\Lambda - 6\Lambda a_1b_1 - \Omega) b_2 - 3\Lambda a_0b_2^2 \right) \nu + 3a_1b_2^2b + 6a_0b_2b + a_1b_2b - b_1(\tau \kappa^2 - 3ba_0^2 - 3ba_1b_1 + \sigma) = 0,\)
\(- a_0 Y v^3 - a_0av^2 + \left( 6\xi Y \kappa^2 - 6a_0b_2 - 3a_1^2\Lambda \right) a_2 + \left( 6\xi Y \kappa - 3\Lambda a_1^2 \right) b_2 - a_0\left( a_0^2\Lambda + 6a_1b_1 + \Omega \right) \nu + 2a_0b_2a_1 + a_1b_2a_2 - a_1(\tau \kappa^2 - 3ba_0^2 - 3ba_1b_1 + \sigma) = 0,\)
\(- Y v^3a_2 - av^2a_2 + \left( -3a_0b_2^2 + (12Y \tau \kappa^2 - 3a_0^2\Lambda - 6\Lambda a_1b_1 - \Omega) a_2 - 3\Lambda a_0a_1^2 \right) \nu + 3a_1b_2^2b + 4\tau \kappa^2a_2 + 3ba_0^2 + 6ba_1b_1 + \sigma) a_2 + 3ba_0a_1^2 = 0,\)
\((6\xi Y \kappa^2a_1 - 6\Lambda a_0a_1a_2 - \Lambda a_1^3 - 3\Lambda a_2^2b_1 \nu + 2\xi \kappa^2a_1 + 6ba_0a_1a_2 + ba_1^3 + 3ba_0^2b_1 = 0,\)
\(18a_2 \left( \left( 6\xi Y \kappa^2 - 1/6 \Lambda a_0a_2 - 1/6 \Lambda a_1^2 \right) \nu + 1/3 \xi \kappa^2 + 1/6 a_0b_2a_2 + 1/6 b_2a_2^2 \right) = 0,\)
\(- 3a_1a_2^2(v\Lambda - b) = 0,\)
\(- a_2^3(v\Lambda - b) = 0,\)

On solving the above obtained system with the aid of Maple, the following values are obtained

**SET I**

\[ a_0 = 0, \quad a_1 = i\sqrt{\frac{6\xi Y \nu + 2\xi a}{b - v\Lambda}} \kappa, \quad b_1 = 0, \quad \sigma = 3\tau Y \nu \kappa^2 + \tau \kappa^2 - Y \nu^3 - av^2 - \Omega \nu, \]

**SET II**

\[ a_0 = 0, \quad a_1 = 0, \quad b_1 = i\sqrt{\frac{6\eta Y \nu + 2\eta a}{b - v\Lambda}} \kappa, \quad \sigma = 3\tau Y \nu \kappa^2 + \tau \kappa^2 - Y \nu^3 - av^2 - \Omega \nu, \]
SET III

\[ a_0 = 0, \quad a_1 = i \sqrt{\frac{6 \xi Y \nu + 2 \xi a}{b - \nu A}} \kappa, \quad b_1 = i \sqrt{\frac{6 \eta Y \nu + 2 \eta a}{b - \nu A}} \kappa \] and

\[ \sigma = 3 \tau Y \nu \kappa^2 + \tau a \kappa^2 - Y \nu^3 - 3 \nu A \delta_1 \kappa^2 \delta_2 - a \nu^2 + 3 b \delta_1 \kappa^2 \delta_2 - \Omega \nu, \]

where \( \delta_1 = i \sqrt{\frac{6 \xi Y \nu + 2 \xi a}{b - \nu A}} \), and

\[ \delta_2 = i \sqrt{\frac{6 \eta Y \nu + 2 \eta a}{b - \nu A}}. \]  

(17)

Subsequently, using the corresponding values of \( \xi \), \( \tau \) and \( \eta \) from Table 1 and values of \( \nu \) and \( c \) from (11) and utilising them together with (11), we obtain the following solutions of (1)

Case (i) \( \xi = m^2 \), \( \tau = -(1 + m^2) \), \( \eta = 1 \), \( \zeta = \kappa \left( x - \frac{c^2}{a} \right) \)

Solution corresponding to the values obtained in set (I)

\( \sigma = -3 (1 + m^2) Y \nu \kappa^2 - (1 + m^2) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu \)

\[ \Psi_{1,1}(x,t) = \left( i \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - \nu A}} \kappa \text{sn} (\zeta) \right) e^{i \left( -\nu x + \frac{\sigma}{\nu} + \theta \right)}, \]  

(18)

and

\[ \Psi_{1,2}(x,t) = \left( i \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - \nu A}} \kappa \text{cd} (\zeta) \right) e^{i \left( -\nu x + \frac{\sigma}{\nu} + \theta \right)}, \]  

(19)

Solution corresponding to the values obtained in set (II)

\( \sigma = -3 (1 + m^2) Y \nu \kappa^2 - (1 + m^2) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu \)

\[ \Psi_{1,3}(x,t) = \left( i \sqrt{\frac{6 Y \nu + 2 a}{b - \nu A}} \kappa \frac{1}{\text{sn} (\zeta)} \right) e^{i \left( -\nu x + \frac{\sigma}{\nu} + \theta \right)}, \]  

(20)

and

\[ \Psi_{1,4}(x,t) = \left( i \sqrt{\frac{6 Y \nu + 2 a}{b - \nu A}} \kappa \frac{1}{\text{cd} (\zeta)} \right) e^{i \left( -\nu x + \frac{\sigma}{\nu} + \theta \right)}, \]  

(21)

Solution corresponding to the values obtained in set (III)

\( \sigma = 3 \tau Y \nu \kappa^2 + \tau a \kappa^2 - Y \nu^3 - 3 \nu A \delta_1 \kappa^2 \delta_2 - a \nu^2 + 3 b \delta_1 \kappa^2 \delta_2 - \Omega \nu \)

\[ \Psi_{1,5}(x,t) = \left( i \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - \nu A}} \kappa \text{sn} (\zeta) + i \sqrt{\frac{6 Y \nu + 2 a}{b - \nu A}} \kappa \left( \frac{1}{\text{sn} (\zeta)} \right) \right) e^{i \left( -\nu x + \frac{\sigma}{\nu} + \theta \right)}, \]  

(22)

and

\[ \Psi_{1,6}(x,t) = \left( i \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - \nu A}} \kappa \text{cd} (\zeta) + i \sqrt{\frac{6 Y \nu + 2 a}{b - \nu A}} \kappa \left( \frac{1}{\text{cd} (\zeta)} \right) \right) e^{i \left( -\nu x + \frac{\sigma}{\nu} + \theta \right)}, \]  

(23)
Case (ii) $\xi = -m^2$, $\tau = 2m^2 - 1$, $\eta = 1 - m^2$, $\zeta = \kappa \left( x - c\frac{\alpha}{a} \right)$

Solution corresponding to the values obtained in set (I)

$\sigma = 3 (2m^2 - 1)Y\nu k^2 + (2m^2 - 1)ak^2 - Y\nu^3 - av^2 - \Omega v$

$$\Psi_{2,1}(x,t) = \left( -\sqrt{\frac{6m^2 Y\nu + 2m^2 a}{b - vA}} \kappa \text{cn}(\zeta) \right) e^{i\left( -\nu x + m \frac{\alpha}{a} + \theta \right)}.$$

(24)

Solution corresponding to the values obtained in set (II)

$\sigma = 3 (2m^2 - 1)Y\nu k^2 + (2m^2 - 1)ak^2 - Y\nu^3 - av^2 - \Omega v$

$$\Psi_{2,2}(x,t) = \left( i\sqrt{\frac{6(1 - m^2) Y\nu + 2(1 - m^2)a}{b - vA}} \kappa \left( \frac{1}{\text{cn}(\zeta)} \right) \right) e^{i\left( -\nu x + m \frac{\alpha}{a} + \theta \right)}.$$

(25)

Solution corresponding to the values obtained in set (III)

$\sigma = 3 \tau Y\nu k^2 + \tau ak^2 - Y\nu^3 - 3\nu A \beta_1 k^2 \beta_2 - av^2 + 3b \delta_1 k^2 \delta_2 - \Omega v$

$$\Psi_{2,3}(x,t) = \left( -\sqrt{\frac{6m^2 Y\nu + 2m^2 a}{b - vA}} \kappa \text{dn}(\zeta) \right) e^{i\left( -\nu x + m \frac{\alpha}{a} + \theta \right)}.$$

(26)

Case (iii) $\xi = -1$, $\tau = (2 - m^2)$, $\eta = m^2 - 1$, $\zeta = \kappa \left( x - c\frac{\alpha}{a} \right)$

Solution corresponding to the values obtained in set (I)

$\sigma = 3 (2 - m^2)Y\nu k^2 + (2 - m^2)ak^2 - Y\nu^3 - av^2 - \Omega v$

$$\Psi_{3,1}(x,t) = \left( -\sqrt{\frac{6 Y\nu + 2a}{b - vA}} \kappa \text{dn}(\zeta) \right) e^{i\left( -\nu x + m \frac{\alpha}{a} + \theta \right)}.$$

(27)

Solution corresponding to the values obtained in set (II)

$\sigma = 3 (2 - m^2)Y\nu k^2 + (2 - m^2)ak^2 - Y\nu^3 - av^2 - \Omega v$

$$\Psi_{3,1}(x,t) = \left( i\sqrt{\frac{6(m^2 - 1) Y\nu + 2(m^2 - 1)a}{b - vA}} \kappa \left( \frac{1}{\text{dn}(\zeta)} \right) \right) e^{i\left( -\nu x + m \frac{\alpha}{a} + \theta \right)}.$$

(28)

Solution corresponding to the values obtained in set (III)

$\sigma = 3 \tau Y\nu k^2 + \tau ak^2 - Y\nu^3 - 3\nu A \beta_1 k^2 \beta_2 - av^2 + 3b \delta_1 k^2 \delta_2 - \Omega v$

$$\Psi_{3,2}(x,t) = \left( -\sqrt{\frac{6 Y\nu + 2a}{b - vA}} \kappa \text{dn}(\zeta) \right) e^{i\left( -\nu x + m \frac{\alpha}{a} + \theta \right)}.$$

(29)

Case (iv) $\xi = 1$, $\tau = -(1 + m^2)$, $\eta = m^2$, $\zeta = \kappa \left( x - c\frac{\alpha}{a} \right)$

Solution corresponding to the values obtained in set (I)

$\sigma = -3 (1 + m^2)Y\nu k^2 - (1 + m^2)ak^2 - Y\nu^3 - av^2 - \Omega v$
\[ \Psi_{4,1}(x, t) = \left( I \sqrt{\frac{6 Y \nu + 2 a}{b - v \Lambda}} \kappa \eta s(\zeta) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (30)

and

\[ \Psi_{4,2}(x, t) = \left( I \sqrt{\frac{6 Y \nu + 2 a}{b - v \Lambda}} \kappa d c(\zeta) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (31)

Solution corresponding to the values obtained in set (II)

\[ \sigma = -3 (1 + m^2) Y \nu k^2 - (1 + m^2) a k^2 - Y \nu^3 - av^2 - \Omega \nu \]

\[ \Psi_{4,3}(x, t) = \left( I \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - v \Lambda}} \kappa \left( \frac{1}{ns(\zeta)} \right) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (32)

and

\[ \Psi_{4,4}(x, t) = \left( I \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - v \Lambda}} \kappa \left( \frac{1}{dc(\zeta)} \right) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (33)

Solution corresponding to the values obtained in set (III)

\[ \sigma = 3 \tau Y \nu k^2 + \tau a k^2 - Y \nu^3 - 3 \nu \Lambda \delta_1 k^2 \theta_2 - av^2 + 3 b \theta_1 k^2 \theta_2 - \Omega \nu, \]

\[ \Psi_{4,5}(x, t) = \left( I \sqrt{\frac{6 Y \nu + 2 a}{b - v \Lambda}} \kappa \eta s(\zeta) + I \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - v \Lambda}} \kappa \left( \frac{1}{ns(\zeta)} \right) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (34)

and

\[ \Psi_{4,6}(x, t) = \left( I \sqrt{\frac{6 Y \nu + 2 a}{b - v \Lambda}} \kappa d c(\zeta) + I \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - v \Lambda}} \kappa \left( \frac{1}{dc(\zeta)} \right) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (35)

Case (v) \( \xi = 1 - m^2, \tau = 2 m^2 - 1, \eta = -m^2, \kappa = \kappa \left( x - c \frac{\tau}{a} \right) \)

Solution corresponding to the values obtained in set (I)

\[ \sigma = 3 (2 m^2 - 1) Y \nu k^2 + (2 m^2 - 1) a k^2 - Y \nu^3 - av^2 - \Omega \nu \]

\[ \Psi_{5,1}(x, t) = \left( I \sqrt{\frac{6 (1 - m^2) Y \nu + 2 (1 - m^2) a}{b - v \Lambda}} \kappa n c(\zeta) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (36)

Solution corresponding to the values obtained in set (II)

\[ \sigma = 3 (2 m^2 - 1) Y \nu k^2 + (2 m^2 - 1) a k^2 - Y \nu^3 - av^2 - \Omega \nu \]

\[ \Psi_{5,2}(x, t) = \left( I \sqrt{\frac{6 m^2 Y \nu + 2 m^2 a}{b - v \Lambda}} \kappa \left( \frac{1}{nc(\zeta)} \right) \right) e^{i(-\nu x + \omega \frac{\zeta}{\tau} + \theta)}, \] (37)

Solution corresponding to the values obtained in set (III)

\[ \sigma = 3 \tau Y \nu k^2 + \tau a k^2 - Y \nu^3 - 3 \nu \Lambda \delta_1 k^2 \theta_2 - av^2 + 3 b \theta_1 k^2 \theta_2 - \Omega \nu, \]
\[ \Psi_{5,3}(x,t) = \left( i \sqrt{\frac{6(1-m^2)}{b-vA}} Y \nu + 2(1-m^2) \frac{a}{b-vA} \kappa nc(\xi) - \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{b-vA}} \frac{1}{\kappa(\xi)} \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{38} \]

**Case (vi)** \( \xi = m^2 - 1, \tau = 2 - m^2, \eta = -1, \zeta = \kappa \left( x - \frac{c}{a} \right) \)

Solution corresponding to the values obtained in set (I)
\[ \sigma = 3 (2 - m^2) Y \nu k^2 + (2 - m^2) \alpha k^2 - Y \nu^3 - av^2 - \Omega v \]
\[ \Psi_{6,1}(x,t) = \left( i \sqrt{\frac{6(m^2 - 1)}{b-vA}} Y \nu + 2(m^2 - 1) \frac{a}{b-vA} \kappa nd(\xi) - \sqrt{\frac{6 Y \nu + 2 a}{b-vA}} \frac{1}{\kappa(\xi)} \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{39} \]

Solution corresponding to the values obtained in set (II)
\[ \sigma = 3 (2 - m^2) Y \nu k^2 + (2 - m^2) \alpha k^2 - Y \nu^3 - av^2 - \Omega v \]
\[ \Psi_{6,2}(x,t) = \left( - \sqrt{\frac{6 Y \nu + 2 a}{b-vA}} \kappa \left( \frac{1}{nd(\xi)} \right) \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{40} \]

Solution corresponding to the values obtained in set (III)
\[ \sigma = 3 \tau Y \nu k^2 + \tau \alpha k^2 - Y \nu^3 - 3 \nu \Lambda \delta \kappa^2 \theta_2 - av^2 + 3b \theta_1 \kappa^2 \theta_2 - \Omega v, \]
\[ \Psi_{6,3}(x,t) = \left( i \sqrt{\frac{6(m^2 - 1)}{b-vA}} Y \nu + 2(m^2 - 1) \frac{a}{b-vA} \kappa nd(\xi) - \sqrt{\frac{6 Y \nu + 2 a}{b-vA}} \kappa \left( \frac{1}{nd(\xi)} \right) \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{41} \]

**Case (vii)** \( \xi = 1 - m^2, \tau = 2 - m^2, \eta = 1, \zeta = \kappa \left( x - \frac{c}{a} \right) \)

Solution corresponding to the values obtained in set (I)
\[ \sigma = 3 (2 - m^2) Y \nu k^2 + (2 - m^2) \alpha k^2 - Y \nu^3 - av^2 - \Omega v \]
\[ \Psi_{7,1}(x,t) = \left( i \sqrt{\frac{6(1-m^2)}{b-vA}} Y \nu + 2(1-m^2) \frac{a}{b-vA} \kappa sc(\xi) \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{42} \]

Solution corresponding to the values obtained in set (II)
\[ \sigma = 3 (2 - m^2) Y \nu k^2 + (2 - m^2) \alpha k^2 - Y \nu^3 - av^2 - \Omega v \]
\[ \Psi_{7,2}(x,t) = \left( i \sqrt{\frac{6 Y \nu + 2 a}{b-vA}} \kappa \left( \frac{1}{sc(\xi)} \right) \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{43} \]

Solution corresponding to the values obtained in set (III)
\[ \sigma = 3 \tau Y \nu k^2 + \tau \alpha k^2 - Y \nu^3 - 3 \nu \Lambda \delta \kappa^2 \theta_2 - av^2 + 3b \theta_1 \kappa^2 \theta_2 - \Omega v, \]
\[ \Psi_{7,3}(x,t) = \left( i \sqrt{\frac{6(1-m^2)}{b-vA}} Y \nu + 2(1-m^2) \frac{a}{b-vA} \kappa sc(\xi) + i \sqrt{\frac{6 Y \nu + 2 a}{b-vA}} \kappa \left( \frac{1}{sc(\xi)} \right) \right) e^{i\left(-\nu x + \sigma \frac{c}{a} + \theta\right)}, \tag{44} \]
Case (viii) \(\xi = -m^2(1-m^2), \tau = 2m^2 - 1, \eta = 1, \zeta = \kappa \left( x - c \frac{e^a}{a} \right)\)

Solution corresponding to the values obtained in set (I)
\(\varpi = 3 (2m^2 - 1) Y \nu k^2 + (2m^2 - 1) \alpha k^2 - Y \nu^3 - av^2 - \Omega v\)

\[
\Psi_{9,1}(x, t) = \left( -\sqrt{6m^2(1-m^2)Y \nu + 2m^2(1-m^2)a \over b - \nu A} \kappa^2 \gamma \left( x - c \frac{e^a}{a} \right) \right) e^{i(-\nu x + \sigma^{e^a}/e + \theta)}, \quad (45)
\]

Solution corresponding to the values obtained in set (II)
\(\varpi = 3 (2m^2 - 1) Y \nu k^2 + (2m^2 - 1) \alpha k^2 - Y \nu^3 - av^2 - \Omega v\)

\[
\Psi_{9,2}(x, t) = \left( i \sqrt{6(1-m^2)Y \nu + 2(1-m^2)a \over b - \nu A} \kappa \left( x - c \frac{e^a}{a} \right) \right) e^{i(-\nu x + \sigma^{e^a}/e + \theta)}, \quad (46)
\]

Solution corresponding to the values obtained in set (III)
\(\varpi = 3 \tau Y \nu k^2 + \alpha \tau k^2 - Y \nu^3 - 3 \nu \Delta \sigma_1 \frac{e^a}{a} \theta_2 - av^2 + 3 b \theta_1 \kappa^2 \theta_2 - \Omega v,\)

\[
\Psi_{9,3}(x, t) = \left( i \sqrt{6(1-m^2)Y \nu + 2(1-m^2)a \over b - \nu A} \kappa \left( x - c \frac{e^a}{a} \right) \right) e^{i(-\nu x + \sigma^{e^a}/e + \theta)}, \quad (47)
\]

Case (ix) \(\xi = 1, \tau = 2 - m^2, \eta = 1 - m^2, \zeta = \kappa \left( x - c \frac{e^a}{a} \right)\)

Solution corresponding to the values obtained in set (I)
\(\varpi = 3 (2-m^2) Y \nu k^2 + (2-m^2) \alpha k^2 - Y \nu^3 - av^2 - \Omega v\)

\[
\Psi_{9,1}(x, t) = \left( i \sqrt{6(1-m^2)Y \nu + 2(1-m^2)a \over b - \nu A} \kappa \left( x - c \frac{e^a}{a} \right) \right) e^{i(-\nu x + \sigma^{e^a}/e + \theta)}, \quad (48)
\]

Solution corresponding to the values obtained in set (II)
\(\varpi = 3 (2-m^2) Y \nu k^2 + (2-m^2) \alpha k^2 - Y \nu^3 - av^2 - \Omega v\)

\[
\Psi_{9,2}(x, t) = \left( i \sqrt{6(1-m^2)Y \nu + 2(1-m^2)a \over b - \nu A} \kappa \left( x - c \frac{e^a}{a} \right) \right) e^{i(-\nu x + \sigma^{e^a}/e + \theta)}, \quad (49)
\]

Solution corresponding to the values obtained in set (III)
\(\varpi = 3 \tau Y \nu k^2 + \alpha \tau k^2 - Y \nu^3 - 3 \nu \Delta \sigma_1 \frac{e^a}{a} \theta_2 - av^2 + 3 b \theta_1 \kappa^2 \theta_2 - \Omega v,\)

\[
\Psi_{9,3}(x, t) = \left( i \sqrt{6(1-m^2)Y \nu + 2(1-m^2)a \over b - \nu A} \kappa \left( x - c \frac{e^a}{a} \right) \right) e^{i(-\nu x + \sigma^{e^a}/e + \theta)}, \quad (50)
\]
\[ \Psi_{10,1}(x, t) = \left( \sqrt[10]{\frac{6Y v + 2a}{b - vA}} \right) e^{-\frac{\kappa ds}{\text{I}} + \theta}, \]  

(51)

Solution corresponding to the values obtained in \textit{set (II)}
\[ \sigma = 3(2m^2 - 1)Y \nu k^2 + (2m^2 - 1)ak^2 - Y \nu^3 - av^2 - \Omega \nu \]
\[ \Psi_{10,2}(x, t) = \left( -\sqrt{\frac{6m^2(1 - m^2)}{b - vA}} \right) e^{-\frac{\kappa ds}{\text{II}} + \theta}, \]
(52)

Solution corresponding to the values obtained in \textit{set (III)}
\[ \sigma = 3\tau Y \nu k^2 + \tau ak^2 - Y \nu^3 - 3vA \delta \kappa^2 \theta_2 - av^2 + 3b \delta \kappa^2 \theta_2 - \Omega \nu, \]
\[ \Psi_{10,3}(x, t) = \left( -\sqrt{\frac{6m^2(1 - m^2)}{b - vA}} \right) e^{-\frac{\kappa ds}{\text{III}} + \theta}, \]
(53)

Case (xi) \( \xi = \frac{1}{4}, \tau = \frac{m^2 + 1}{2}, \eta = \frac{1 - m^2}{4}, \zeta = \kappa \left( x - \frac{c^2}{a} \right) \)

Solution corresponding to the values obtained in \textit{set (I)}
\[ \sigma = 3\left( \frac{m^2 + 1}{2} \right) Y \nu k^2 + \left( \frac{m^2 + 1}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]
\[ \Psi_{11,1}(x, t) = \left( -\sqrt{\frac{6Y v + 2a}{4(b - vA)}} \right) \kappa(mcn(\zeta) \mp dn(\zeta)) e^{-\frac{\kappa ds}{\text{I}} + \theta}, \]
(54)

Solution corresponding to the values obtained in \textit{set (II)}
\[ \sigma = 3\left( \frac{m^2 + 1}{2} \right) Y \nu k^2 + \left( \frac{m^2 + 1}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]
\[ \Psi_{11,2}(x, t) = \left( -\sqrt{\frac{6(1 - m^2)^2 Y \nu v + 2(1 - m^2)^2 a}{4(b - vA)}} \right) \kappa(mcn(\zeta) \mp dn(\zeta)) e^{-\frac{\kappa ds}{\text{II}} + \theta}, \]
(55)

Solution corresponding to the values obtained in \textit{set (III)}
\[ \sigma = 3\tau Y \nu k^2 + \tau ak^2 - Y \nu^3 - 3vA \delta \kappa^2 \theta_2 - av^2 + 3b \delta \kappa^2 \theta_2 - \Omega \nu, \]
\[ \Psi_{11,3}(x, t) = \left( -\sqrt{\frac{6Y v + 2a}{4(b - vA)}} \right) k(mcn(\zeta) \mp dn(\zeta)) \left( -\sqrt{\frac{6(1 - m^2)^2 Y \nu v + 2(1 - m^2)^2 a}{4(b - vA)}} \right) \kappa(mcn(\zeta) \mp dn(\zeta)) e^{-\frac{\kappa ds}{\text{III}} + \theta}, \]
(56)

Case (xii) \( \xi = \frac{1}{4}, \tau = \frac{1 - 2m^2}{2}, \eta = \frac{1}{4}, \zeta = \kappa \left( x - \frac{c^2}{a} \right) \)

Solution corresponding to the values obtained in \textit{set (I)}
\[ \sigma = 3\left( \frac{1 - 2m^2}{2} \right) Y \nu k^2 + \left( \frac{1 - 2m^2}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]
\[ \Psi_{12,1}(x,t) = \left( t \sqrt[4]{ \frac{6Y}{4(b-vA)} \kappa (nsc(\zeta) \mp cs(\zeta)) } \right) e^{i(-\nu_s + \omega_s^a + \theta)}, \] 

Solution corresponding to the values obtained in set (II)

\[ \sigma = 3 \left( \frac{1-2m}{2} \right) Y \nu k^2 + \left( \frac{1-2m}{2} \right) \nu k^2 - Y \nu^3 - \nu^2 - \Omega \nu \]

\[ \Psi_{12,2}(x,t) = \left( t \sqrt[4]{ \frac{6Y}{4(b-vA)} \kappa (nsc(\zeta) \mp cs(\zeta)) } \right) e^{i(-\nu_s + \omega_s^a + \theta)}, \] 

Solution corresponding to the values obtained in set (III)

\[ \sigma = 3 \tau Y \nu k^2 + \tau \nu k^2 - Y \nu^3 - \nu^2 - \nu^2 - \Omega \nu, \]

\[ \Psi_{12,3}(x,t) = \left( t \sqrt[4]{ \frac{6Y}{4(b-vA)} \kappa (nsc(\zeta) \mp cs(\zeta)) } + \sqrt[4]{ \frac{6Y}{4(b-vA)} \kappa (nsc(\zeta) \mp cs(\zeta)) } \right) e^{i(-\nu_s + \omega_s^a + \theta)}, \]

\[ \text{Case (xiii)} \quad \xi = \frac{1-m^2}{4}, \tau = \frac{m^2+1}{2}, \eta = \frac{1-m^2}{4}, \zeta = \kappa \left( x - c_s \frac{\eta}{a} \right) \]

Solution corresponding to the values obtained in set (I)

\[ \sigma = 3 \left( \frac{m^2+1}{2} \right) Y \nu k^2 + \left( \frac{m^2+1}{2} \right) \nu k^2 - Y \nu^3 - \nu^2 - \nu^2 - \Omega \nu \]

\[ \Psi_{13,1}(x,t) = \left( t \sqrt[4]{ \frac{6(1-m^2)Yv + 2(1-m^2)a}{4(b-vA)} \kappa (ncs(\xi) \mp cs(\zeta)) } \right) e^{i(-\nu_s + \omega_s^a + \theta)}, \]

Solution corresponding to the values obtained in set (II)

\[ \sigma = 3 \left( \frac{m^2+1}{2} \right) Y \nu k^2 + \left( \frac{m^2+1}{2} \right) \nu k^2 - Y \nu^3 - \nu^2 - \nu^2 - \Omega \nu \]

\[ \Psi_{13,2}(x,t) = \left( t \sqrt[4]{ \frac{6(1-m^2)Yv + 2(1-m^2)a}{4(b-vA)} \kappa (ncs(\xi) \mp cs(\zeta)) } + \sqrt[4]{ \frac{6(1-m^2)Yv + 2(1-m^2)a}{4(b-vA)} \kappa (ncs(\xi) \mp cs(\zeta)) } \right) e^{i(-\nu_s + \omega_s^a + \theta)}, \]

Solution corresponding to the values obtained in set (III)

\[ \sigma = 3 \tau Y \nu k^2 + \tau \nu k^2 - Y \nu^3 - \nu^2 - \nu^2 - \Omega \nu, \]

\[ \Psi_{13,3}(x,t) = \left( \sqrt[4]{ \frac{6(1-m^2)Yv + 2(1-m^2)a}{4(b-vA)} \kappa (ncs(\xi) \mp cs(\zeta)) } \right) + \left( \sqrt[4]{ \frac{6(1-m^2)Yv + 2(1-m^2)a}{4(b-vA)} \kappa (ncs(\xi) \mp cs(\zeta)) } \right) e^{i(-\nu_s + \omega_s^a + \theta)}, \]

\[ \text{Case (xiv)} \quad \xi = \frac{1}{4}, \tau = \frac{m^2-2}{2}, \eta = \frac{m^2}{4}, \zeta = \kappa \left( x - c_s \frac{\eta}{a} \right) \]

Solution corresponding to the values obtained in set (I)

\[ \sigma = 3 \left( \frac{m^2-2}{2} \right) Y \nu k^2 + \left( \frac{m^2-2}{2} \right) \nu k^2 - Y \nu^3 - \nu^2 - \nu^2 - \Omega \nu \]
\[
\Psi_{14,1}(x, t) = \left( i \sqrt{\frac{6 Y \nu + 2 a}{4(b - \nu \Lambda)}} \kappa(ns(\zeta) \mp ds(\zeta)) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (63)
\]

Solution corresponding to the values obtained in set (II)
\[
\sigma = 3 \left( \frac{m^2 - 2}{2} \right) Y \nu^2 + \left( \frac{m^2 - 2}{2} \right) a k^2 - Y \nu^3 - a^2 - \Omega \nu
\]
\[
\Psi_{14,2}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa \left( \frac{1}{(ns(\zeta) \mp ds(\zeta))} \right) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (64)
\]

Solution corresponding to the values obtained in set (III)
\[
\sigma = 3 \tau Y \nu k^2 + \tau a k^2 - Y \nu^3 - 3 \nu A \partial_1 k^2 \partial_2 - a^2 + 3b \partial_1 k^2 \partial_2 - \Omega \nu
\]
\[
\Psi_{14,3}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa(ns(\zeta) \mp ds(\zeta)) + i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa \left( \frac{1}{(ns(\zeta) \mp ds(\zeta))} \right) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (65)
\]

Case (xv) \( \xi = \frac{m^2}{4} \), \( \tau = \frac{m^2 - 2}{2} \), \( \eta = \frac{m^2}{4} \), \( \zeta = \kappa \left( x - \frac{a}{\kappa} \right) \)

Solution corresponding to the values obtained in set (I)
\[
\sigma = 3 \left( \frac{m^2 - 2}{2} \right) Y \nu k^2 + \left( \frac{m^2 - 2}{2} \right) a k^2 - Y \nu^3 - a^2 - \Omega \nu
\]
\[
\Psi_{15,1}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa(sn(\zeta) \mp icn(\zeta)) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (66)
\]

and
\[
\Psi_{15,2}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa \left( \frac{dn(\zeta)}{\sqrt{1 - m^2 sn(\zeta) \mp cn(\zeta)}} \right) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (67)
\]

Solution corresponding to the values obtained in set (II)
\[
\sigma = 3 \left( \frac{m^2 - 2}{2} \right) Y \nu k^2 + \left( \frac{m^2 - 2}{2} \right) a k^2 - Y \nu^3 - a^2 - \Omega \nu
\]
\[
\Psi_{15,3}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa \left( \frac{1}{sn(\zeta) \mp icn(\zeta)} \right) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (68)
\]

and
\[
\Psi_{15,4}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa \left( \frac{1}{sn(\zeta) \mp icn(\zeta)} \right) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (69)
\]

Solution corresponding to the values obtained in set (III)
\[
\sigma = 3 \tau Y \nu k^2 + \tau a k^2 - Y \nu^3 - 3 \nu A \partial_1 k^2 \partial_2 - a^2 + 3b \partial_1 k^2 \partial_2 - \Omega \nu
\]
\[
\Psi_{15,5}(x, t) = \left( i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa(sn(\zeta) \mp icn(\zeta)) + i \sqrt{\frac{6m^2 Y \nu + 2m^2 a}{4(b - \nu \Lambda)}} \kappa \left( \frac{1}{sn(\zeta) \mp icn(\zeta)} \right) \right) e^{i\left(-\nu x + \frac{m}{a} \frac{\zeta}{\kappa} + \theta\right)} , \quad (70)
\]
and

$$\Psi_{15,6}(x, t) = \left( i \sqrt{\frac{6m^2 Y v + 2m^2 a}{4(b - vA)} \kappa \left( \frac{d_n(\xi)}{\sqrt{1 - m^2 sn(\xi) \mp cn(\xi)}} \right) \right) e^{i(-v\lambda + m_{\text{out}}^2 + \theta)} \right),$$

(71)

Case (xvi) $\xi = \frac{1}{4}, \tau = \frac{3m_2}{2}, \eta = \frac{1}{4}, \xi = \kappa \left( x - c \frac{v^2}{a} \right)$

Solution corresponding to the values obtained in set (I)

$$\sigma = 3 \left( \frac{m_2}{2} \right) Y v^2 + \frac{3m_2}{2} a k^2 - Y v^3 - a v^2 - \omega v$$

$$\Psi_{16,1}(x, t) = \left( i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{mcn(\xi) \mp idn(\xi)}{1 \mp cn(\xi)} \right) \right) e^{i(-v\lambda + m_{\text{out}}^2 + \theta)},$$

(72)

and

$$\Psi_{16,2}(x, t) = \left( i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{sn(\xi)}{1 \mp cn(\xi)} \right) \right) e^{i(-v\lambda + m_{\text{out}}^2 + \theta)},$$

(73)

Solution corresponding to the values obtained in set (II)

$$\sigma = 3 \left( \frac{m_2}{2} \right) Y v^2 + \frac{3m_2}{2} a k^2 - Y v^3 - a v^2 - \omega v$$

$$\Psi_{16,3}(x, t) = \left( i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{1}{mcn(\xi) \mp idn(\xi))} \right) \right) e^{i(-v\lambda + m_{\text{out}}^2 + \theta)},$$

(74)

and

$$\Psi_{16,4}(x, t) = \left( i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{1}{mcn(\xi)} \mp idn(\xi))} \right) e^{i(-v\lambda + m_{\text{out}}^2 + \theta)},$$

(75)

Solution corresponding to the values obtained in set (III)

$$\sigma = 3 \tau Y v k^2 + \tau a k^2 - Y v^3 - 3vA \theta_1 k^2 \theta_2 - a v^2 + 3b \theta_1 k^2 \theta_2 - \omega v,$$

$$\Psi_{16,5}(x, t) = \left( i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{mcn(\xi) \mp idn(\xi))} {1 \mp cn(\xi)} \right) \right) e^{i(-v\lambda + m_{\text{out}}^2 + \theta)},$$

(76)

and

$$\Psi_{16,6}(x, t) = \left( i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{sn(\xi)}{1 \mp cn(\xi)} \right) \right),$$

$$+ i \sqrt{\frac{6 Y v + 2 a}{4(b - vA)} \kappa \left( \frac{1 \mp cn(\xi)}{sn(\xi)} \right)} e^{i(-v\lambda + m_{\text{out}}^2 + \theta)},$$

(77)

Case (xvii) $\xi = \frac{m^2}{4}, \tau = \frac{m_2}{2}, \eta = \frac{1}{4}, \xi = \kappa \left( x - c \frac{v^2}{a} \right)$

Solution corresponding to the values obtained in set (I)

$$\sigma = 3 \left( \frac{m_2}{2} \right) Y v k^2 + \left( \frac{m_2}{2} \right) a k^2 - Y v^3 - a v^2 - \omega v$$
\[ \Psi_{17,1}(x,t) = \left( \sqrt[1/4]{2m^3 Y \nu + 2m^2 a} \frac{a}{b - \nu A} \kappa \left( \frac{\text{sn}(\zeta)}{1 \mp \text{dn}(\zeta)} \right) \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)}, \] (78)

Solution corresponding to the values obtained in set (II)
\[ \sigma = 3 \left( \frac{m^2 - 2}{2} \right) Y \nu \kappa^2 + \left( \frac{m^2 - 2}{2} \right) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu 

\[ \Psi_{17,2}(x,t) = \left( \sqrt[1/4]{6Y \nu + 2a} \frac{a}{b - \nu A} \kappa \left( \frac{1 \mp \text{dn}(\zeta)}{\text{sn}(\zeta)} \right) \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)}, \] (79)

Solution corresponding to the values obtained in set (III)
\[ \sigma = 3 \tau Y \nu \kappa^2 + (\frac{m^2 + 1}{2}) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu, \]

\[ \Psi_{17,3}(x,t) = \left( \sqrt[1/4]{6m^3 Y \nu + 2m^2 a} \frac{a}{b - \nu A} \kappa \left( \frac{\text{sn}(\zeta)}{1 \mp \text{dn}(\zeta)} \right) + \sqrt[1/4]{6Y \nu + 2a} \frac{a}{b - \nu A} \kappa \left( \frac{1 \mp \text{dn}(\zeta)}{\text{sn}(\zeta)} \right) \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)}, \] (80)

Case (xviii) \( \xi = \frac{m^2 - 1}{4}, \tau = \frac{m^2 + 1}{2}, \eta = \frac{m^2 - 1}{4}, \zeta = \kappa \left( x - \frac{c^2}{\sigma} \right) \)

Solution corresponding to the values obtained in set (I)
\[ \sigma = 3 \left( \frac{m^2 + 1}{2} \right) Y \nu \kappa^2 + \left( \frac{m^2 + 1}{2} \right) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu 

\[ \Psi_{18,1}(x,t) = \left( \sqrt[1/4]{6m^3 Y \nu + 2m^2 a} \frac{a}{b - \nu A} \kappa \left( \frac{\text{dn}(\zeta)}{1 \mp \text{msn}(\zeta)} \right) \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)}, \] (81)

Solution corresponding to the values obtained in set (II)
\[ \sigma = 3 \left( \frac{m^2 + 1}{2} \right) Y \nu \kappa^2 + \left( \frac{m^2 + 1}{2} \right) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu 

\[ \Psi_{18,2}(x,t) = \left( \sqrt[1/4]{6m^3 Y \nu + 2m^2 a} \frac{a}{b - \nu A} \kappa \left( \frac{\text{sn}(\zeta)}{1 \mp \text{dn}(\zeta)} \right) \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)}, \] (82)

Solution corresponding to the values obtained in set (III)
\[ \sigma = 3 \tau Y \nu \kappa^2 + (\frac{m^2 + 1}{2}) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu, \]

\[ \Psi_{18,3}(x,t) = \left( \sqrt[1/4]{6m^3 Y \nu + 2m^2 a} \frac{a}{b - \nu A} \kappa \left( \frac{\text{dn}(\zeta)}{1 \mp \text{msn}(\zeta)} \right) \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)} + \sqrt[1/4]{6m^3 Y \nu + 2m^2 a} \frac{a}{b - \nu A} \kappa \left( \frac{\text{sn}(\zeta)}{1 \mp \text{dn}(\zeta)} \right) e^{(-\nu x + \sigma \frac{\zeta}{\sigma} + \theta)}, \] (83)

Case (xix) \( \xi = \frac{1 - m^2}{4}, \tau = \frac{m^2 + 1}{2}, \eta = \frac{1 - m^2}{4}, \zeta = \kappa \left( x - \frac{c^2}{\sigma} \right) \)

Solution corresponding to the values obtained in set (I)
\[ \sigma = 3 \left( \frac{m^2 + 1}{2} \right) Y \nu \kappa^2 + \left( \frac{m^2 + 1}{2} \right) a \kappa^2 - Y \nu^3 - a \nu^2 - \Omega \nu \]
\[ \Psi_{19,1}(x, t) = \left( i \sqrt{\frac{6(1 - m^2) Y \nu + 2(1 - m^2) a}{4(b - \nu A)}} \frac{cn(\zeta)}{\sqrt{1 + sn(\zeta)}} \right) e^{i(-\nu \tau + \sigma \frac{\zeta}{\alpha} + \theta)}, \quad (84) \]

Solution corresponding to the values obtained in set (II) 
\[ \sigma = 3 \left( \frac{m^2+1}{2} \right) Y \nu k^2 + \left( \frac{m^2+1}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]

\[ \Psi_{19,2}(x, t) = \left( i \sqrt{\frac{6(1 - m^2) Y \nu + 2(1 - m^2) a}{4(b - \nu A)}} \frac{cn(\zeta)}{\sqrt{1 + sn(\zeta)}} \right) e^{i(-\nu \tau + \sigma \frac{\zeta}{\alpha} + \theta)}, \quad (85) \]

Solution corresponding to the values obtained in set (III) 
\[ \sigma = 3 \tau Y \nu k^2 + \tau ak^2 - Y \nu^3 - 3v\Lambda \beta_1 k^2 \beta_2 - av^2 + 3b \beta_1 k^2 \beta_2 - \Omega \nu, \]

\[ \Psi_{19,3}(x, t) = \left( i \sqrt{\frac{6(1 - m^2) Y \nu + 2(1 - m^2) a}{4(b - \nu A)}} \frac{cn(\zeta)}{\sqrt{1 + sn(\zeta)}} \right) e^{i(-\nu \tau + \sigma \frac{\zeta}{\alpha} + \theta)}, \quad (86) \]

Case (xx) \( \xi = \frac{(1-m^2)^2}{4}, \tau = \frac{m^2+1}{2}, \eta = \frac{1}{4}, \zeta = \kappa \left( x - \frac{c \tau}{\alpha} \right) \)
Solution corresponding to the values obtained in set (I) 
\[ \sigma = 3 \left( \frac{m^2+1}{2} \right) Y \nu k^2 + \left( \frac{m^2+1}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]

\[ \Psi_{20,1}(x, t) = \left( i \sqrt{\frac{6((1 - m^2)^2) Y \nu + 2((1 - m^2)^2) a}{4(b - \nu A)}} \frac{sn(\zeta)}{dn(\zeta) \mp cn(\zeta)} \right) e^{i(-\nu \tau + \sigma \frac{\zeta}{\alpha} + \theta)}, \quad (87) \]

Solution corresponding to the values obtained in set (II) 
\[ \sigma = 3 \left( \frac{m^2+1}{2} \right) Y \nu k^2 + \left( \frac{m^2+1}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]

\[ \Psi_{20,2}(x, t) = \left( i \sqrt{\frac{6 Y \nu + 2a}{4(b - \nu A)}} \frac{dn(\zeta) \mp cn(\zeta)}{sn(\zeta)} \right) e^{i(-\nu \tau + \sigma \frac{\zeta}{\alpha} + \theta)}, \quad (88) \]

Solution corresponding to the values obtained in set (III) 
\[ \sigma = 3 \tau Y \nu k^2 + \tau ak^2 - Y \nu^3 - 3v\Lambda \beta_1 k^2 \beta_2 - av^2 + 3b \beta_1 k^2 \beta_2 - \Omega \nu, \]

\[ \Psi_{20,3}(x, t) = \left( i \sqrt{\frac{6((1 - m^2)^2) Y \nu + 2((1 - m^2)^2) a}{4(b - \nu A)}} \frac{sn(\zeta)}{dn(\zeta) \mp cn(\zeta)} \right) e^{i(-\nu \tau + \sigma \frac{\zeta}{\alpha} + \theta)}, \quad (89) \]

Case (xxi) \( \xi = \frac{m^2}{4}, \tau = \frac{m^2-1}{2}, \eta = \frac{1}{4}, \zeta = \kappa \left( x - \frac{c \tau}{\alpha} \right) \)
Solution corresponding to the values obtained in set (I) 
\[ \sigma = 3 \left( \frac{m^2-1}{2} \right) Y \nu k^2 + \left( \frac{m^2-1}{2} \right) ak^2 - Y \nu^3 - av^2 - \Omega \nu \]
\[ \Psi_{21,1}(x,t) = \left( i\sqrt{\frac{6m^4 Y v + 2m^4 a}{4(b - v\Lambda)}} \kappa \left( \frac{\text{cn}(\xi)}{\sqrt{1 - m^2 \mp \text{dn}(\xi)}} \right) \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)}, \]  
(90)

Solution corresponding to the values obtained in set (II)
\[ \sigma = 3 \left( \frac{n-1}{2} \right) Y v k^2 + \left( \frac{m-1}{2} \right) a k^2 - Y v^3 - av^2 - \Omega v \]
\[ \Psi_{21,2}(x,t) = \left( i\sqrt{\frac{6 Y v + 2 a}{4(b - v\Lambda)}} \kappa \left( \frac{\sqrt{1 - m^2 \mp \text{dn}(\xi)}}{\text{cn}(\xi)} \right) \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)}, \]  
(91)

Solution corresponding to the values obtained in set (III)
\[ \sigma = 3 \tau Y v k^2 + \tau a k^2 - Y v^3 - 3\nu\Lambda \delta_1 k^2 \theta_2 - av^2 + 3b\delta_1 k^2 \theta_2 - \Omega v, \]
\[ \Psi_{21,3}(x,t) = \left( i\sqrt{\frac{6m^4 Y v + 2m^4 a}{4(b - v\Lambda)}} \kappa \left( \frac{\text{cn}(\xi)}{\sqrt{1 - m^2 \mp \text{dn}(\xi)}} \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)} \right. \]
\[ \left. + i\sqrt{\frac{6 Y v + 2 a}{4(b - v\Lambda)}} \kappa \left( \frac{\sqrt{1 - m^2 \mp \text{dn}(\xi)}}{\text{cn}(\xi)} \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)} \right), \]  
(92)

**Note** By using the limiting value of \( m \) in the Table 1, we can obtain both solitonic and trigonometric solutions. Listed below are a few such solutions.

**Bright soliton solution**
\[ \sigma = 3 Y v k^2 + a k^2 - Y v^3 - av^2 - \Omega v \]
\[ \Psi(x,t) = \left( -\sqrt{\frac{6 Y v + 2 a}{b - v\Lambda}} \kappa \text{sech}(\xi) \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)}, \]  
(93)

**Dark soliton solution**
\[ \sigma = 3 \tau Y v k^2 + \tau a k^2 - Y v^3 - 3\nu\Lambda \delta_1 k^2 \theta_2 - av^2 + 3b\delta_1 k^2 \theta_2 - \Omega v, \]
where \( \delta_1 = i\sqrt{\frac{6 Y v + 2 a}{b - v\Lambda}} \), and \( \delta_2 = i\sqrt{\frac{6 Y v + 2 a}{b - v\Lambda}} \).
\[ \Psi(x,t) = \left( i\sqrt{\frac{6 Y v + 2 a}{b - v\Lambda}} \kappa \text{tanh}(\xi) + i\sqrt{\frac{6 Y v + 2 a}{b - v\Lambda}} \kappa \left( \frac{1}{\text{tanh}(\xi)} \right) \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)}, \]  
(94)

**Singular soliton solution**
\[ \sigma = 3 Y v k^2 + a k^2 - Y v^3 - av^2 - \Omega v \]
\[ \Psi(x,t) = \left( i\sqrt{\frac{6 Y v + 2 a}{b - v\Lambda}} \kappa \text{csch}(\xi) \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)}, \]  
(95)

**Combined singular soliton solution**
\[ \sigma = \frac{-3}{2} Y v k^2 + \frac{-1}{2} a k^2 - Y v^3 - av^2 - \Omega v \]
\[ \Psi(x,t) = \left( i\sqrt{\frac{6 Y v + 2 a}{4(b - v\Lambda)}} \kappa (\text{coth}(\xi) \mp \text{csch}(\xi)) \right) e^{i(-\nu x + \omega \frac{\text{sech}(\xi)}{\kappa} + \theta)}, \]  
(96)
5 Analysis of the exact solution

In this section we provide a summary of the analysis of exact solution obtained in the Sect. 4. The graphical representation of these solutions have been done by choosing the suitable values of various parameters to depict the spatio temporal extension of the derived wave solutions. 3D plots of the solution represented by (96) shown in the Fig. 1a, b show that the solution exhibits lump solitons for these parametric values. It is evident from Fig. 1c, d that the width of the wave decreases as the value of $\alpha$ increases and the wave decays towards 0 rapidly for higher values of $\alpha$ as compared to lower values of $\alpha$. This shows the dependence of the behaviour of the wave on the value of the fractional derivative i.e $\alpha$.

3D plots of $\Psi(x, t)$ given in (93) are presented in the Fig. 2a, b. Figure 2c, d shows the change in curvature and the rate of decay of the wave solution which increases as the value of $\alpha$ increases. 3D plots shown in Fig.3 is the representation of $\Psi(x, t)$ given in (18) and the graph shows that in this case the solution is periodic in nature. 3D plots of the solution $\Psi(x, t)$ in (57) are presented in the Fig. 4a and b show that two parallel breather type waves

![Fig. 1](image-url)
Fig. 2 Plots of (93) with the parametric values $Y = 0.75, \alpha = 0.65, \beta = 0.55, \gamma = 0.8, \Lambda = 0.45, \sigma = 0.3, \Omega = 0.5, \theta = 0, \kappa = 1$

Fig. 3 Plots of (18) with the parametric values $Y = 0.75, \alpha = 0.65, \beta = 0.55, m = 0.2, \gamma = 0.8, \Lambda = 0.45, \sigma = 0.3, \Omega = 0.5, \theta = 0, \kappa = 1$
Fig. 4 Plots of (57) with the parametric values $Y = 0.75, a = 0.65, b = 0.55, m = 0.2, \alpha = 0.65, \Lambda = 0.45, \sigma = 0.3, \Omega = 0.5, \theta = 0, \kappa = 1$

Fig. 5 Plots of (80) with the parametric values $Y = 0.75, a = 0.65, b = 0.55, \alpha = 0.8, m = 0.65, \Lambda = 0.45, \sigma = 0.3, \Omega = 0.5, \theta = 0, \kappa = 1$
travel parallel in case of real part of the solution and in case of the imaginary part exhibits a single breather type wave.

3D plots for the real and imaginary parts of the solution $\Psi(x, t)$ in (80) are shown in Fig.5a and b show that the solution exhibits periodic lump-type breather soliton for these parametric values. As the value of parameter $\alpha$ increases, an increase in amplitude of the wave function and associated phase shifts in observed in Fig. 5c, d and 3D plots of (87) are given in Fig.6.

A similar analysis of all the other solutions obtained in sect. 4 can also be done.

6 Conclusion

Wave solutions with rich physical structures of complex nonlinear Radhakrishnan-Kundu-Lakshmanan equation with conformable fractional time derivative have been obtained by utilizing to the GJEF method. The solutions obtained in this study can have potential applications in the explanation of physical interpretation of the studied nonlinear model in the field of nonlinear optics and fluid dynamics. By restricting the modulus of Jacobi elliptic functions, we obtain solitary wave solutions, bright-dark wave optical solitons, singular solitons, etc. The dependence of the behaviour of the solutions on the fractional derivative has also been analyzed in the present study. It becomes apparent from the computations in this study that the GJEF method provides us a powerful tool that can be used in the construction of optical soliton solutions of conformable fractional nonlinear models in mathematical science.

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Declarations

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