Edge States and Entanglement Entropy

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Abstract

It is known that gauge fields defined on manifolds with spatial boundaries support states localized at the boundaries. In this paper, we demonstrate how coarse-graining over these states can lead to an entanglement entropy. In particular, we show that the entanglement entropy of the ground state for the quantum Hall effect on a disk exhibits an approximate “area” law.

I. INTRODUCTION

Recently there has been renewed interest in the origin of the so-called area law for black hole entropy. One reason for this interest is that it can shed some light on the possible substructure for a quantum theory of gravity. In this regard one often invokes the hypothesis that there are excitations of the black hole horizon leading to surface states associated to the horizon. In a previous paper \textsuperscript{1}, using a very simple treatment of the constraints of canonical gravity, we have shown that these states arise in a manner similar to the edge states in quantum Hall effect (QHE) \textsuperscript{2,3}. We also speculated there that tracing over these edge states might lead to an entropy for black holes as happens with the calculation of entanglement entropy \textsuperscript{4,5}.

A complete calculation to verify the above speculation is a rather formidable task. However, as a step forward, we can try to find evidence in other models where edge states arise and lead to important physics. With this in mind, in this paper we study the effective field theory describing QHE \textsuperscript{7} on a disk and calculate the entanglement entropy \textsuperscript{6,7} that arises when we trace out the edge degrees of freedom. More specifically, we see that the ground state of the system turns out to have non-trivial correlations between the bulk and the edge degrees of freedom. For this reason, tracing over the edge degrees of freedom leads to an impure density matrix. The entanglement entropy that we calculate is the entropy associated
with this density matrix. The result we find is quite interesting - the entanglement entropy obeys an approximate area law [that is the entropy is approximately proportional to the perimeter of the disk] for weak coupling between the edge and the bulk. An area law of the similar type has also been discovered for the thermodynamic entropy (2+1) dimensional black holes recently [8,9].

The paper is arranged as follows. In Section 2, to keep the paper self-contained, we discuss how edge states arise for gauge theories defined on manifolds with spatial boundaries. In Section 3, we discuss our model, namely Maxwell-Chern-Simons (MCS) theory defined on a disk coupled to a chiral scalar field living on the edge of the disk. This is a phenomenological model that describes QHE (along with the edge currents present in a Hall system). In Section 4, we find the Hamiltonian for the system by eliminating the unphysical modes using the Gauss law. This Hamiltonian is quadratic and hence in principle, one can find ground state wavefunction $\Psi$. In Section 5, we calculate the ground state using perturbation theory and the entanglement entropy associated with this ground state. We find that, in the above weak-coupling limit, this entropy is approximately proportional to the perimeter of the disk ("area law"). This result can also be checked using the so-called replica trick [10], which we discuss in the appendix.

**II. EDGE OBSERVABLES**

In this Section, we will see how edge observables occur for gauge theories on manifolds with boundaries. We will take the MCS theory on a disk as a typical example [11] since we will be using this model also in the later Sections.

Here, we use the following conventions:

1. Greek and Latin indices take values 0,1,2 and 1,2 respectively.

2. The three-dimensional metric $\eta_{\mu\nu}$ is specified by its nonvanishing entries
   \[ \eta_{00} = -1, \quad \eta_{11} = \eta_{22} = +1 \] while the three-dimensional Levi-Civita symbol is $\epsilon^{\mu\nu\lambda}$ with $\epsilon^{012} = +1$.

3. The spatial metric is given by the Kronecker $\delta_{ij}$ symbol while the two-dimensional spacetime and spatial Levi-Civita symbols are $\epsilon^{\mu\nu}$ and $\epsilon^{ij} \equiv \epsilon^{0ij}$.

We will also assume that the disk has a circular boundary and radius $R$. The MCS Lagrangian is

\[
L = \int_D d^2 x \mathcal{L} = -\frac{t}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\sigma H}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \, .
\]

The Poisson brackets (PB's) for (2.1) are

\[\{F_{\mu\nu}, \phi\} = \epsilon^{\rho\sigma\lambda} \partial_\rho \phi \epsilon_{\sigma\lambda\nu} \]

\[\{F_{\mu\nu}, F_{\rho\sigma}\} = \epsilon^{\rho\sigma\lambda} \partial_\lambda F_{\mu\nu} \]

\[\{\phi, F_{\mu\nu}\} = \epsilon_{\rho\sigma\lambda} \partial_\rho \phi \partial_\sigma F_{\mu\lambda} \]

\[\{\phi, \phi\} = 0\]

\[\{F_{\mu\nu}, F_{\rho\sigma}\} = 0\]

\[\{F_{\mu\nu}, \phi\} = 0\]

\[\{\phi, \phi\} = 0\]

We thank R. Sorkin for emphasizing this point to us.
\{A_i(x), A_j(y)\} = \{\Pi_i(x), \Pi_j(y)\} = 0,
\{A_i(x), \Pi_j(y)\} = \delta_{ij}\delta^2(x - y). \tag{2.2}

Here and in what follows, all fields are to be evaluated at some fixed time while the PB’s are at equal times. Thus \(x^0 = y^0\) in (2.2). As usual these PB’s will be replaced by commutation relations (CR’s) in the quantized theory. Also \(A\) and \(\Pi\) give the magnetic field \(B\) and the components \(F_{0i} := E_i\) of the electric field by the formulae

\[ B = \epsilon_{ij}\partial_j A_j, \]
\[ E_i = \frac{1}{t}(\Pi_i + \frac{\sigma H}{2}\epsilon_{ij} A_j). \tag{2.3} \]

The Hamiltonian and Gauss law for (2.1) in quantum theory are

\[ H = \int_D d^2x \mathcal{H}, \]
\[ \mathcal{H} = \frac{1}{2t}[ (\Pi_i + \frac{\sigma H}{2}\epsilon_{ij} A_j)^2 + t^2(\epsilon_{ij}\partial_i A_j)^2], \]
\[ G(\chi)|\cdot\rangle = 0 \quad \text{for} \quad \chi|_{\partial D} = 0, \tag{2.4} \]

where

\[ G(\chi) = -\int_D d^2x \partial_i \chi^{(0)}[\Pi_i - \frac{\sigma H}{2}\epsilon_{ij} A_j] \tag{2.5} \]

and \(|\cdot\rangle\) is any physical state.

The reason for the boundary condition on \(\chi\) is that it is only with this condition that vanishing of \(G(\chi)|\cdot\rangle\) is implied by the usual Gauss law \(\partial_i(\Pi_i - \frac{\sigma H}{2}\epsilon_{ij} A_j) \approx 0\) (after a partial integration). Note that it is necessary to rewrite this classical Gauss law in terms of \(G(\chi)\) in quantum theory. This is because \(\Pi_i - \frac{\sigma H}{2}\epsilon_{ij} A_j\) are operator valued distributions in quantum theory so that their derivatives have to be interpreted by smearing them with the derivatives of suitable test functions \([12]\). But it is only when the test functions satisfy the above boundary conditions that the smeared Gauss law restricted to the classical context vanishes by the classical Gauss law \([12]\).

The edge observables \(Q(\Lambda)\) are obtained from \(G\) by changing the boundary conditions on \(\chi\). They are

\[ Q(\Lambda) = -\int_D d^2x \partial_i \Lambda[\Pi_i - \frac{\sigma H}{2}\epsilon_{ij} A_j], \quad \Lambda|_{\partial D} = \text{not necessarily} \ 0. \tag{2.6} \]

They are the generators of the affine Lie groups \(\tilde{LU}(1)\) and have the commutators

\[ [Q(\Lambda), Q(\Lambda')] = -i\sigma_H \int_D d^2x \epsilon_{ij} \partial_i \Lambda \partial_j \Lambda' = -i\sigma_H \int_{\partial D} \Lambda d\Lambda'. \tag{2.7} \]

The action of \(Q(\Lambda)\) on \(|\cdot\rangle\) depends only on the boundary value of \(\Lambda\) since the difference between \(Q(\Lambda)\) and \(Q(\Lambda')\) when \(\Lambda\) and \(\Lambda'\) coincide at the boundary is a constraint and hence annihilates the physical states. Furthermore, it commutes with observables localized within \(D\), its action on these observables being that of a \(G(\chi)\) \([12]\). Hence it can be regarded as localized at the edge.
III. MCS ACTION WITH THE CHIRAL SCALAR FIELD

The MCS action we have considered till now suffers however from an “anomaly”. Under the gauge transformations

\[ A_i \rightarrow A_i + \partial_i \Lambda, \]
\[ E_i \rightarrow E_i, \]  
(3.1)

for arbitrary \( \Lambda \), the MCS action \( S_{bulk} \) is not gauge invariant:

\[ S_{bulk} \rightarrow S_{bulk} - \frac{\sigma H}{2} \int_{\partial \mathcal{M}} d^2 x \epsilon^{\mu \nu} \partial_\mu \Lambda A_\nu \]  
(3.2)

(\( \mathcal{M} \) being \( D \times \mathbb{R} \)). If we require that physics be gauge invariant, then we must modify the MCS action \( S_{bulk} \) by a surface term as follows:

\[ S_{tot} = S_{bulk} + \frac{\sigma H}{2q} \int_{\partial \mathcal{M}} d^2 x \epsilon^{\mu \nu} \partial_\mu \phi A_\nu - \frac{l^2}{8\pi} \int_{\partial \mathcal{M}} d^2 x (D_\mu \phi)(D^\mu \phi), \]  
(3.3)

\[ D_\mu \phi = \partial_\mu \phi - q A_\mu. \]  
(3.4)

Here \( l^2 \) is a positive constant and \( q \) is the charge by which the field \( \phi \) living at the boundary is gauged. Also \( d^2 x = dt R d\theta \), \( R \) being the radius of \( \partial D \) and \( \theta \in [0, 2\pi] \) its angular variable. Under the gauge transformations (3.1), \( \phi \) transforms as

\[ \phi \rightarrow \phi + q \Lambda, \]  
(3.5)

so that \( \Phi = e^{i\phi} \) transforms like a complex scalar field with charge \( q \). [Since \( \Phi \) is the true charged excitation, we also have the identification \( \phi \approx \phi + 2\pi \).]

With this transformation law for \( \phi \), we see that

\[ S_{tot} \rightarrow S_{tot}. \]  
(3.6)

So far, the coefficient \( l^2 \) in (3.3) is arbitrary. However, if this action is to describe QHE \[ \text{(3.3)} \], then the currents due to the edge scalar field \( \phi \) are required to be “chiral”. In this case, it can be shown \[ \text{(3.3)} \] that the coefficient \( l^2 \) gets uniquely fixed according to \[ \text{(13)} \]

\[ l^2 = 2\pi \frac{\sigma H}{q^2} \]  
(3.7)

With \( l^2 \) fixed in this way, we can impose the “chirality” constraint that the edge fields are left-moving, namely that

\[ D_0 \phi - D_\theta \phi = 0. \]  
(3.8)

Thus the total action that correctly describes QHE is

\[ S_{tot} = S_{bulk} + \frac{\sigma H}{2q} \int_{\partial \mathcal{M}} d^2 x \epsilon^{\mu \nu} \partial_\mu \phi A_\nu - \frac{\sigma H}{4q^2} \int_{\partial \mathcal{M}} d^2 x (D_\mu \phi)(D^\mu \phi). \]  
(3.9)
IV. THE HAMILTONIAN AND QUANTIZATION

In this Section, we will find the Hamiltonian for the action (3.9) and then set up the formalism for the quantization of this interacting system.

From the action (3.9), we find the momentum conjugate to $A_i$ to be

$$\Pi_i = tE_i - \frac{\sigma_H}{2} \epsilon_{ij} A_j$$ (4.1)

(where $E_i := F_{0i}$) and the momentum conjugate to $\phi$ to be

$$\Pi_{\phi} = \frac{\sigma_H}{2q} D_\theta \phi + \frac{\sigma_H}{2q} A_\theta.$$ (4.2)

The Hamiltonian for this system is

$$H_{\text{tot}} = H_{\text{bulk}} + H_{\text{edge}}$$

$$H_{\text{bulk}} = \frac{1}{2t} \int_D d^2x [(\Pi_i + \frac{\sigma_H}{2} \epsilon_{ij} A_j)^2 + t^2 (\epsilon_{ij} \partial_i A_j)^2]$$

$$H_{\text{edge}} = \frac{1}{2} \int_{\partial D} dx \left[ \frac{2q^2}{\sigma_H} (\Pi_\phi - \frac{\sigma_H}{2q} A_\theta)^2 + \frac{\sigma_H}{2q^2} (\phi' - q A_\theta)^2 \right]$$ (4.3)

and the Gauss law constraint is

$$\mathcal{G}(\chi) := - \int_D d^2x \partial_i \chi (\Pi_i - \frac{\sigma_H}{2} \epsilon_{ij} A_j) - q \int_{\partial D} Rd\theta \chi (\Pi_\phi + \frac{\sigma_H}{2q} \phi') \approx 0,$$ (4.4)

$R$ being the radius of the disk and $\theta (\text{mod } 2\pi)$ the angle coordinate for $\partial D$. To quantize the system described by (4.3) and (4.4), we now need to choose mode expansions for the fields $A_i, \Pi_i, \phi$ and $\Pi_\phi$. As usual, the coefficients of these modes will play the role of creation and annihilation operators in the quantized theory.

In an earlier work [11,12], we have quantized the bulk theory alone (namely the MCS action without the chiral scalar field). There we showed that among the various permissible mode expansions parametrised by a real parameter $\lambda$, only the one characterized by $\lambda = 0$ allows the existence of edge observables. This is because each $\lambda$ specifies a domain $\mathcal{D}_\lambda$ for the space of one-forms ($A_i$ or $\Pi_i$) and the edge observables do not leave this domain invariant unless $\lambda = 0$. Since edge observables are crucial in our work here, we will use only the mode expansions specified by the choice $\lambda = 0$.

The corresponding mode expansions for $A, \Pi$ and $E$ are [11]

$$A = (a_{nm} \frac{\Psi_{nm}}{\sqrt{t}} + a^{(\ast)}_{nm} \frac{\Psi_{nm}}{\sqrt{t}} + \alpha_{n} \frac{h_{n}}{\sqrt{\sigma_{H}}}) + \text{ h.c.},$$ (4.5)

$$\Pi = (\pi_{nm} \sqrt{t} \Psi_{nm} + \pi^{(\ast)}_{nm} \sqrt{t} \ast \Psi_{nm} + p_{n} \sqrt{\sigma_{H}} h_{n}) + \text{ h.c.},$$ (4.6)

$$E = (e_{nm} \frac{\Psi_{nm}}{\sqrt{t}} + e^{(\ast)}_{nm} \frac{\Psi_{nm}}{\sqrt{t}}) + \sqrt{\frac{\sigma_{H}}{t}} (p_{n} + i \frac{\alpha_{n}}{2}) h_{n} + \text{ h.c.},$$ (4.7)

$$e_{nm} := \pi_{nm} - \frac{\sigma_{H}}{2t} a^{(\ast)}_{nm},$$

$$e^{(\ast)}_{nm} = \pi^{(\ast)}_{nm} + \frac{\sigma_{H}}{2t} a_{nm}.$$
summation over repeated indices \([ n \) over positive and \( m \) over non-negative integers\] being understood. Here, we have used (4.1) and also the notation of forms while \( \ast \) refers to the Hodge dual [14]. Further we have scaled the modes of [11] for later convenience and dropped the superscript (1) employed in [11] to emphasize one-forms. The modes used above are defined as follows (see [11] for details):

\[
\Psi_{nm} := N_{nm} * d(J_n(\omega_{nm}^r)e^{in\theta}),
\]

\[
h_n := \frac{1}{\sqrt{2\pi n R^n}} d(r^n e^{in\theta}).
\] (4.8)

The \( d \) refers to the exterior derivative, the \( J_n \)'s refer as usual to the cylindrical Bessel functions [15] while the \( \omega_{nm} \)'s are such that \( J_n(\omega_{nm}^R) = 0 \). Also the \( N_{nm} \)'s are normalization constants chosen such that

\[
\int_{\mathbb{D}} \bar{\Psi}_{nm} * \Psi_{nm} = -1, \text{ bar denoting complex conjugation.}
\]

In (4.5-4.7), it is understood that \( n \) is summed over all non-negative integers for the first two terms and all the positive integers for the third term while \( m \) is summed over all positive integers.

From the PB relations (2.2) and using also (2.3), we get the following non-vanishing CR’s:

\[
[a_{nm}, \pi_{n'm'}^\dagger] = [a_{nm}^\dagger, \pi_{n'm'}] = [a_{nm}, e_{n'm'}^\dagger] = [a_{nm}^\dagger, e_{n'm'}] = i\delta_{nn'}\delta_{mm'},
\]

\[
[a_{(s)nm}, \pi_{n'm'}^\dagger] = [a_{(s)nm}^\dagger, \pi_{n'm'}] = [a_{(s)nm}, e_{n'm'}^\dagger] = [a_{(s)nm}^\dagger, e_{n'm'}] = i\delta_{nn'}\delta_{mm'},
\]

\[
[e_{nm}, e_{n'm'}^\dagger] = [e_{nm}^\dagger, e_{n'm'}] = i\frac{\sigma_H}{t} \delta_{nm'}\delta_{mm'},
\]

\[
[\alpha_n, p_{n'}^\dagger] = [\alpha_n^\dagger, p_{n'}] = i\delta_{nn'}.
\] (4.9)

We also mode expand the edge fields \( \phi \) and \( \Pi_\phi \) as follows:

\[
\phi = (\phi_n \frac{q}{\sqrt{2\pi n_\sigma_H}} e^{in\theta}) + h.c.,
\] (4.10)

\[
\Pi_\phi = (\pi_n \frac{q}{2\pi n_\sigma_H} e^{in\theta}) + h.c..
\] (4.11)

We have suppressed the winding modes in writing (4.10,4.11) as they are not important for this paper. Also \( n \) here is summed only over non-negative integers. Though (4.10) looks singular for \( n = 0 \), it is all right for our purposes since what occurs in the Hamiltonian and the Gauss law is \( \phi' \) (and not \( \phi \)) and this itself admits a well-defined mode expansion even for \( n = 0 \). Once again, the only non-zero CR’s are

\[
[\phi_n, \pi_{n'}^\dagger] = i\delta_{nn'}.
\] (4.12)

The Gauss law (4.4) in terms of these modes takes the form

\[
(e_{nm}^{(s)} - \frac{\sigma_H}{t} a_{nm}) |\cdot\rangle = (e_{nm}^{(s)} - \frac{\sigma_H}{t} a_{nm}^\dagger) |\cdot\rangle = 0
\] (4.13)

and

\[
\{(p_n - \frac{i}{2} \alpha_n) + (\pi_n + \frac{i}{2} \phi_n) \} |\cdot\rangle = \{(p_n^\dagger + \frac{i}{2} \alpha_n^\dagger) + (\pi_n^\dagger - \frac{i}{2} \phi_n^\dagger) \} |\cdot\rangle = 0,
\] (4.14)
\(|·⟩\) being any physical state. Here (4.13) is the condition on the modes \(\Psi_{nm}\) while (4.14) arises as the constraint on the modes \(h_n, \bar{h}_n\).

We can use (4.13) to eliminate \(e^*_{nm}\) in favour of \(a_{nm}\). In that case the Hamiltonian (4.3) in terms of these modes becomes

\[
H_{\text{tot}} = H_{\text{bulk}} + H_{\text{edge}}
\]

\[
H_{\text{bulk}} = \frac{1}{2} \left\{ e_{nm}^\dagger e_{nm} + \left( \omega_{nm}^2 + \left( \frac{\sigma H}{t} \right)^2 \right) a_{nm}^\dagger a_{nm} \right\} + \frac{\sigma H}{2t} (c_n^\dagger c_n + n^\dagger n)
\]

\[
H_{\text{edge}} = \frac{n}{R} \left[ (\pi_n + \frac{i}{2} \phi_n - i \alpha_n + R \sqrt{\frac{2 \pi \sigma H}{nt}} \omega_{nm} N_{nm} J_n' (\omega_{nm} R) a_{nm})^\dagger
\right.
\]

\[
(p_n + \frac{i}{2} \phi_n - i \alpha_n + R \sqrt{\frac{2 \pi \sigma H}{nt}} \omega_{nm'} N_{nm'} J_n' (\omega_{nm'} R) a_{nm'}) + (\pi_n - \frac{i}{2} \phi_n)^\dagger (\pi_n - \frac{i}{2} \phi_n) \right] (4.15)
\]

where

\[
c_n \equiv (p_n + \frac{i}{2} \alpha_n)^\dagger, \quad (4.16)
\]

and

\[
J_n'(x) \equiv \frac{d}{dx} J_n(x). \quad (4.17)
\]

If we are in one chiral sector of the edge theory (the physically relevant sector when dealing with QHE), then we can also impose the condition

\[
(\pi_n - \frac{i}{2} \phi_n) |·⟩ = 0 \quad (4.18)
\]

on physical states |·⟩ (such a condition arising from the mode expansion of the chirality constraint (3.3)).

We can now also use the Gauss law (4.14) to eliminate \((\pi_n + \frac{i}{2} \phi_n)\) in terms of \((p_n - \frac{1}{2} \alpha_n)\). The edge Hamiltonian then reduces (in the above chiral sector) to

\[
H_{\text{edge}} = \frac{n}{R} \left[ (p_n - \frac{1}{2} \alpha_n - R \sqrt{\frac{2 \pi \sigma H}{nt}} \omega_{nm} N_{nm} J_n' (\omega_{nm} R) a_{nm})^\dagger \right.
\]

\[
(p_n - \frac{1}{2} \alpha_n - R \sqrt{\frac{2 \pi \sigma H}{nt}} \omega_{nm'} N_{nm'} J_n' (\omega_{nm'} R) a_{nm'}) \right]. (4.19)
\]

Therefore the task of quantization reduces to the diagonalization of the system described by the following Hamiltonian :

\[
H = \frac{1}{2} \left\{ e_{nm}^\dagger e_{nm} + \left( \omega_{nm}^2 + \left( \frac{\sigma H}{t} \right)^2 \right) a_{nm}^\dagger a_{nm} \right\} + \frac{\sigma H}{t} (c_n^\dagger c_n + n^\dagger n)
\]

\[
\frac{n}{R} \left( c_n^\dagger - R \sqrt{\frac{2 \pi \sigma H}{nt}} \omega_{nm} N_{nm} J_n' (\omega_{nm} R) a_{nm} \right) - \frac{n}{R} \left( c_n^\dagger - R \sqrt{\frac{2 \pi \sigma H}{nt}} \omega_{nm'} N_{nm'} J_n' (\omega_{nm'} R) a_{nm'}^\dagger \right). \quad (4.20)
\]

The Hamiltonian in (4.20) has also been normal ordered, the vacuum \(|0⟩\) being defined by \(e_{nm}|0⟩ = a_{nm}|0⟩ = c_n|0⟩ = 0\). Note that \(H\) preserves the constraints (4.13), (4.17) and (4.18).
V. THE GROUND STATE AND ITS ENTANGLEMENT ENTROPY

We are not interested here in finding the entire spectrum of the Hamiltonian (4.20). Our objective is in finding the ground state of the system. As is clear even from (4.20), this system has correlations between the edge degrees of freedom and bulk degrees of freedom $e_{nm}$, $a_{nm}$. The density matrix obtained by tracing out the edge states will therefore be impure and will have a non-zero entropy associated to it.

We will now find the ground state of this interacting system by using perturbation theory with $q$ being the perturbing parameter. Before starting the calculations, it is important to note that in physical situations in QHE, the coefficient $\sigma_H$ can be written as $k q^2$ where $k$ is a number of order unity. We will therefore replace $\sigma_H$ in (4.20) by $k q^2$ to make the order in $q$ explicit:

$$H = \frac{1}{2} \left\{ e_{nm}^2 e_{nm} + (\omega_{nm}^2 + (k q^2)^2) a_{nm}^\dagger a_{nm} \right\} + \frac{k q^2}{t} c_n^\dagger c_n +$$

$$\frac{n}{R} (c_n^\dagger - q R) \sqrt{\frac{2 \pi k}{n t}} \omega_{nm} N_{nm} J_n' (\omega_{nm} R) a_{nm} (c_n - q R \sqrt{\frac{2 \pi k}{n t}} \omega_{nm} N_{nm'} J_n' (\omega_{nm'} R) a_{nm'}). \quad (5.1)$$

Keeping terms only to order $q^2$ and defining $\hat{e}_{nm} = e_{nm}/\sqrt{\omega_{nm}}$ and $\hat{a}_{nm} = a_{nm} \sqrt{\omega_{nm}}$, the above Hamiltonian can be rewritten as

$$H = \frac{1}{2} \omega_{nm}[\hat{e}_{nm}^\dagger \hat{e}_{nm} + \hat{a}_{nm}^\dagger \hat{a}_{nm}] + \left( \frac{n}{R} + \frac{k q^2}{t} \right) c_n^\dagger c_n - q \sqrt{\frac{2 \pi \omega_{nm} J_n' (\omega_{nm} R)}{n t}} (c_n \hat{a}_{nm} + c_n^\dagger \hat{a}_{nm}^\dagger) +$$

$$q^2 \frac{2 \pi k R}{t} \sqrt{\omega_{nm} \omega_{nm'}} N_{nm} N_{nm'} J_n' (\omega_{nm} R) J_n' (\omega_{nm'} R) \hat{a}_{nm} \hat{a}_{nm'}^\dagger. \quad (5.2)$$

We now define annihilation operators $A_{nm}$ and $B_{nm}$ as follows:

$$A_{nm} := \frac{1}{\sqrt{2}} (\hat{e}_{nm} - i \hat{a}_{nm})$$

$$B_{nm} := \frac{1}{\sqrt{2}} (\hat{e}_{nm}^\dagger - i \hat{a}_{nm}^\dagger). \quad (5.3)$$

Their non-vanishing commutators are contained in

$$[A_{nm}, A_{nm'}^\dagger] = \delta_{nm} \delta_{nm'},$$

$$[B_{nm}, B_{nm'}^\dagger] = \delta_{nm} \delta_{nm'}. \quad (5.4)$$

In terms of these variables the Hamiltonian is

$$H = \frac{1}{2} \omega_{nm}[A_{nm}^\dagger A_{nm} + B_{nm}^\dagger B_{nm}] + \frac{n}{R} c_n^\dagger c_n -$$

$$\frac{q}{i} \sqrt{\frac{\pi k \omega_{nm}}{t}} N_{nm} J_n' (\omega_{nm} R) (c_n (B_{nm}^\dagger - A_{nm}) + c_n^\dagger (A_{nm}^\dagger - B_{nm})) +$$

$$q^2 \frac{k}{t} c_n^\dagger c_n + \frac{\pi k R}{t} \sqrt{\omega_{nm} \omega_{nm'}} N_{nm} N_{nm'} J_n' (\omega_{nm} R) J_n' (\omega_{nm'} R) (A_{nm}^\dagger A_{nm'} + B_{nm}^\dagger B_{nm}) +$$

$$- A_{nm}^\dagger B_{nm'}^\dagger - B_{nm}^\dagger A_{nm'} \right]. \quad (5.5)$$
The dependence on \( R \) follows:

\[ |\Psi\rangle_{\text{gnd}} = \prod_{n} |\Psi, n\rangle_{\text{gnd}}, \quad (5.6) \]

where,

\[ |\Psi, n\rangle_{\text{gnd}} = |0\rangle[1 - q^{2} \sum_{m} \frac{\pi nk\omega_{nm}N_{nm}^{2}(J_{n}'(\omega_{nm}R))^{2}}{t(\frac{n}{R} + \frac{\omega_{nm}}{2})^{2}}] + \]

\[ \sum_{m} |1\rangle_{A_{nm}}(1)_{\text{gnd}} (iq) \sqrt{\frac{\pi nk\omega_{nm}N_{nm}J_{n}'(\omega_{nm}R)}}{t} \frac{\omega_{nm}R}{(\frac{n}{R} + \frac{\omega_{nm}}{2})^{2}} - \]

\[ q^{2} \sum_{m,m'} |1\rangle_{A_{nm}}(1)_{\text{gnd}} \langle 2\rangle_{A_{nm}}(1)_{\text{gnd}} \frac{\pi nk}{t} \sqrt{\frac{\omega_{nm}\omega_{nm'}N_{nm}N_{nm'}J_{n}'(\omega_{nm}R)J_{n}'(\omega_{nm'}R)}{(\frac{n}{R} + \frac{\omega_{nm}}{2})(\frac{n}{R} + \frac{\omega_{nm} + \omega_{nm'}}{2})}. \quad (5.7) \]

Here \( |0\rangle \) refers to the unperturbed ground state:

\[ A_{nm}|0\rangle = B_{nm}|0\rangle = c_{n}|0\rangle = 0. \quad (5.8) \]

Also the state \( |1\rangle_{A_{nm}} \) refers to the first excited state created by the creation operator \( A_{nm}^{\dagger} \), a similar notation being used for the other states.

From this ground state, we can also now define the corresponding density matrix \( \hat{\rho} \) and then obtain the reduced density matrix by tracing over the bulk states. [Actually, we may want to trace over the edge states if the objective is to calculate the entropy due to a lack of knowledge of the edge states. But the answer for the entropy itself is the same whether we trace over one or the other.] To order \( q^{2} \), we have for this reduced density matrix,

\[ \hat{\rho}_{\text{red}} := Tr_{(\text{bulk})}(\hat{\rho}) = |0\rangle_{a_{n}a_{n}}\langle 0| - p_{n} + |1\rangle_{a_{n}a_{n}}\langle 1|p_{n} \quad (5.9) \]

where

\[ p_{n} = q^{2} \sum_{m} \left[ \frac{\pi nk}{t} \frac{\omega_{nm}N_{nm}^{2}J_{n}'^{2}(\omega_{nm}R)}{(\frac{n}{R} + \frac{\omega_{nm}}{2})^{2}} \right]. \quad (5.10) \]

The dependence on \( R \) factors out in the expression for \( p_{n} \) as can be seen by rewriting it as follows:

\[ p_{n} = q^{2} \left( \frac{\pi nk}{t} \right) \frac{R}{t} \sum_{m} \frac{(\omega_{nm}R)N_{nm}^{2}J_{n}'^{2}(\omega_{nm}R)}{(n + \frac{\omega_{nm}}{2})^{2}} \]

\[ = q^{2} \left( \frac{\pi nk}{t} \right) \frac{R}{t} \sum_{m} \frac{4\chi_{nm}N_{nm}^{2}J_{n}'^{2}(\chi_{nm})}{(2n + \chi_{nm})^{2}} \]

\[ \rightarrow q^{2} \frac{4\pi nkR}{t} \sum_{m} \frac{1}{(2n + \chi_{nm})^{2}} \chi_{nm}, \quad \text{for large } \chi_{nm} := \omega_{nm}R, \quad (5.11) \]
\( \chi_{nm} \) being zeroes of \( J_n \). This shows that \( p_n \) is proportional to \( R \). Note that \( p_n \) is finite. This is because for large \( m \),

\[
|\chi_{nm}| \sim \frac{\pi}{2} (2m + n - \frac{1}{2}), \quad N_{nm}J'_n(\chi_{nm}) \sim O\left(\frac{1}{\chi_{nm}}\right)
\]

so that

\[
\frac{\chi_{nm}N^2_{nm}J^2_n(\chi_{nm})}{(2n + \chi_{nm})^2} = O\left(\frac{1}{m^3}\right)
\]

Here the above behavior of \( \chi_{nm} \), and the behavior of \( J_n(\chi_{nm}) \) and \( J'_n(\chi_{nm}) \) for large \( |\chi_{nm}| \) are known from the literature \[16\] to be

\[
J_n(\chi_{nm}) \sim \sqrt{\frac{2}{\pi \chi_{nm}}} \cos(\chi_{nm} - \frac{n\pi}{2} - \frac{\pi}{4}) = O\left(\frac{1}{\sqrt{\chi_{nm}}}\right), \\
J'_n(\chi_{nm}) \sim -\sqrt{\frac{2}{\pi \chi_{nm}}} \sin(\chi_{nm} - \frac{n\pi}{2} - \frac{\pi}{4}) = O\left(\frac{1}{\sqrt{\chi_{nm}}}\right)
\]

while that of \( N_{nm} \) is obtained from the normalization condition

\[
\int_D N^2_{nm} \left[ \frac{\chi_{nm}^2}{R} J^2_n\left(\frac{\chi_{nm} r}{R}\right) + \frac{n^2}{r^2} \frac{J^2_n(\chi_{nm} r)}{R} \right] r dr d\theta = 1
\]

on \( \Psi_{nm} \). Hence,

\[
\frac{1}{N^2_{nm}} = 2\pi \int_0^{\chi_{nm}} \left[ \left(\frac{dJ_n(u)}{du}\right)^2 + \frac{n^2}{u^2} (\frac{n}{u} J_n(u))^2 \right] u du,
\]

where \( u = \frac{\chi_{nm} r}{R} \). It shows that \( N^2_{nm} \) depends only on \( \chi_{nm} \) and gives

\[
\frac{\partial}{\partial \chi_{nm}} \left(\frac{1}{N^2_{nm}}\right) = \text{constant for large } |\chi_{nm}|
\]

Hence

\[
N_{nm} = O\left(\frac{1}{\sqrt{\chi_{nm}}}\right) \quad \text{as } |\chi_{nm}| \to \infty
\]

Putting these estimates together we get (5.13)

We can now infer that \( p_n \) being finite, the entropy of the \( n \)th mode

\[
S_n := -p_n \log p_n - (1 - p_n) \log(1 - p_n)
\]

is also finite for generic \( p_n \).

[However, the total entropy defined as \( S := \sum_n S_n \) can diverge unless one imposes a cut-off (for example, on the maximum allowed \( n \)).]
VI. CONCLUDING REMARKS

In this concluding section, we draw attention to the approximate “area law” obeyed by the above entanglement entropy. From the expression (5.19) for the entropy $S_n$, we see that in the limit of $q$ being small, we can approximate it by just the first term. Namely,

$$S_n \approx p_n - p_n \log p_n \quad \text{for } q \text{ small}$$

Since we know from (5.11) that $p_n$ itself scales as $R$, we see that $S_n$ also scales as $R$ (apart from the logarithmic term). For a disk, this is therefore a statement that the entropy approximately scales like the perimeter of the disk (or the area of the boundary of the disk).

It is interesting that such an “area law” has emerged for the entanglement entropy of edge states. Before we draw general conclusions, however, we should check whether such a law is a mere coincidence for the 2+1 theory describing Hall effect or whether it holds more generally. One possibility is to find the entanglement entropy for a similar theory in 3+1 dimensions, but without a Chern-Simons term. If the “area law” does hold in such more general situations, it seems reasonable to conclude that the black hole entropy is at least partially the creation of the entanglement entropy arising out of a lack of knowledge of the edge states describing excitations of the black hole horizon.

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APPENDIX: THE REPLICA TRICK IN ACTION

In this section we show that the results given above also follows from another method, namely the replica trick \[10\]. Our Hamiltonian, given by (4.20), is quadratic in the various field modes and their conjugate momenta. The ground state wavefunction for this system will then be a generalized Gaussian. To simplify the problem and to highlight the important aspects, we will restrict ourselves to one tower of radial modes \(a_{nm}\) with any fixed \(m\), which we will call \(a_n\), and keep all the edge modes \(\alpha_n\). The measure defining the scalar product is \(\prod_n da_n da_n^\dagger d\alpha_n d\alpha_n^\dagger\). The calculation is performed using the Schrödinger representation where we have the substitutions

\[ e_n \rightarrow -i \frac{\partial}{\partial a_n^\dagger}, \quad e_n^\dagger \rightarrow -i \frac{\partial}{\partial a_n} \quad (A1) \]

\[ p_n \rightarrow -i \frac{\partial}{\partial \alpha_n^\dagger}, \quad p_n^\dagger \rightarrow -i \frac{\partial}{\partial \alpha_n}. \quad (A2) \]

The ground state wavefunctional is found by using the following Gaussian ansatz which is consistent with angular momentum conservation:

\[ \Psi(\{a_n\}, \{a_n^\dagger\}; \{\alpha_n\}, \{\alpha_n^\dagger\}) = \prod_n N_n e^{-(A_n a_n a_n^\dagger + B_n a_n^\dagger a_n + C_n a_n^\dagger \alpha_n + D_n \alpha_n^\dagger a_n)}. \quad (A3) \]

Here \(\prod_n N_n\) is the normalization factor for the wavefunction and \(A_n, B_n, C_n, D_n\) are real constants to be determined. It is also understood that \(\dagger\) denotes complex conjugation here.

The Schrödinger equation is

\[ H \Psi(\{a_n\}, \{\alpha_n\}) = E \Psi(\{a_n\}, \{\alpha_n\}) \quad (A4) \]

where our Hamiltonian is given by (4.20). The constants \(A_n, B_n, C_n\) and \(D_n\) in the wavefunctional \(\Psi\) can be determined from (A4) and requiring the wavefunction to be normalizable. \(A_n, C_n\) and \(D_n\) turn out to be

\[ A_n = \frac{1}{2}, \]

\[ C_n = -\sqrt{\frac{2\pi \sigma H}{nt}} \sqrt{\frac{2m \omega_n N_{nm} J'_n(\chi_{nm})}{(2E_n + \frac{\sigma H}{t} + \frac{n}{R})}}, \]

\[ D_n = 0, \quad (A5) \]

while \(B_n\) is determined by a complicated cubic equation. The solutions to this equation are rather cumbersome and are not illuminating. However, in the weak coupling (\(\frac{\sigma H}{t} \rightarrow 0\)) and large radius limit (\(R \rightarrow \infty\)), one gets the unperturbed value,

\[ B_n = \omega_n. \quad (A6) \]

The ground state energy is given by \(E = \sum_n E_n\) where

\[ E_n = \left(\frac{\sigma H}{t} + \frac{n}{R}\right)A_n + \frac{1}{2}B_n, \quad (A7) \]
Hereafter we will write our wavefunctional as

$$\Psi(\{a_n\}; \{a_n^\dagger\}; \{\alpha_n\}; \{a_n^\dagger\}) = \prod_n N_n e^{-(A_n a_n a_n^\dagger + B_n a_n a_n^\dagger + C_n a_n a_n^\dagger + A_n a_n a_n^\dagger) + B_n (a_n a_n^\dagger + a_n^\dagger a_n^\dagger) + C_n (a_n a_n a_n^\dagger + a_n a_n^\dagger a_n)}.$$  \hfill (A8)

as \(D_n = 0\).

Given the above form for the wavefunctional, one can readily form the ground state density matrix \(\rho\):

$$\rho(\{a_n\}; \{\alpha_n\}|\{a_n'\}; \{\alpha_n'\}) = \Psi(\{a_n\}; \{\alpha_n\})\Psi^*(\{a_n'\}; \{\alpha_n'\}) = \prod_n |N_n'|^2 e^{-(A_n (a_n a_n^\dagger + a_n' a_n^\dagger) + B_n (a_n a_n^\dagger + a_n^\dagger a_n')) + C_n (a_n a_n a_n^\dagger + a_n a_n^\dagger a_n')}}$$

$$\equiv \prod_n \rho^{(n)}(a_n, \alpha_n|a_n', \alpha_n').$$  \hfill (A9)

The reduced density matrix is found by tracing over the set of variables which is not being observed, say \(a_n\)’s. Therefore, the reduced density matrix in this case would be given by

$$\rho_{\text{red}}(\{\alpha_n\}; \{\alpha_n'\}) = \prod_n da_n d\bar{a}_n \rho^{(n)}(a_n; \alpha_n; a_n').$$  \hfill (A10)

A straightforward Gaussian integration over the complex variables \(a_n\) leads to

$$\rho_{\text{red}}(\{\alpha_n\}; \{\alpha_n'\}) = \prod_n |N_n'|^2 e^{-(A_n \alpha_n^\dagger \alpha_n + \alpha_n'^\dagger \alpha_n')} + \gamma_n \alpha_n^\dagger \alpha_n},$$

$$\equiv \rho^{(n)}_{\text{red}}(\alpha_n; \alpha_n')$$  \hfill (A11)

where \(N_n' = N_n \sqrt{2B_n}\) and \(\gamma_n = |c_n|^2\). The normalization factor \(N'\) can be computed easily by recalling that for any density matrix \(\rho\), we require \(\text{tr} \rho = 1\). This gives

$$|N_n'|^2 = \frac{1}{\pi}(2A_n - \gamma_n)$$  \hfill (A12)

The entanglement entropy \(S = -\text{tr} \rho_{\text{red}} \ln \rho_{\text{red}}\) can be computed from the reduced density matrix using the so-called replica trick \(\square\) according to which

$$S = -\frac{d}{dN} \left(\text{tr} (\rho_{\text{red}}^N)\right)_{N=1},$$  \hfill (A13)

\(\rho_{\text{red}}\) here being the operator with the kernel \(\square\).

Now, one can find for real and positive \(A\) the following equation

$$\rho^{(N)}_{\text{red}}(\{\alpha_n\}; \{\alpha_n'\}) \equiv \prod_n \int d\bar{z}_{1,n} d\bar{z}_{1,n} \rho^{(n)}_{\text{red}}(\alpha_n; \bar{z}_{1,n}) \rho^{(n)}_{\text{red}}(\bar{z}_{1,n}; \bar{z}_{2,n}) \cdots \rho^{(n)}_{\text{red}}(\bar{z}_{N-1,n}; \alpha_n')$$

$$= \prod_n |N_n'|^2 e^{-(A_n \alpha_n a_n^\dagger) + 2A_n \bar{z}_{1,n} \bar{z}_{1,n} + \gamma_n \bar{z}_{1,n} \alpha_n} \int d\bar{z}_{2,n} d\bar{z}_{2,n} e^{2A_n \alpha_n^\dagger \alpha_n + \alpha_n^\dagger \alpha_n} \int d\bar{z}_{3,n} d\bar{z}_{3,n} \times \cdots$$

$$\cdots \times \int d\bar{z}_{N-1,n} d\bar{z}_{N-1,n} e^{-(2A_n \bar{z}_{N-1,n} \bar{z}_{N-1,n} + \gamma_n \alpha_n^\dagger \alpha_n + \alpha_n^\dagger \alpha_n)} e^{-(A_n \alpha_n a_n^\dagger)}$$
where we have used (A12). This expression is invariant under the simultaneous rescaling using equations (A7,A5) (A13), it readily follows that the entanglement entropy vanishes as well.

where we have repeated used the identity

The bars here denote complex conjugation (just as the daggers).

Hence, we can see

\[
\int dz d\bar{z} e^{-Az + \beta z + c\bar{z}} = \frac{\pi}{A} e^{\frac{\pi^2}{4A}}.
\] (A15)

The bars here denote complex conjugation (just as the daggers).

Hence, we can see

\[
\text{tr} \left( \rho_{\text{red}}^N \right) = \prod_n \int d\alpha_n d\alpha_n^\dagger \rho_{\text{red}}^{(n)}(\alpha_n; \alpha_n).
\]

\[
= \prod_n |\mathcal{N}_n'|^{2N} \left( \frac{\pi}{2A_n} \right)^{N-1} \frac{\pi}{2A_n - \gamma_n(\frac{\gamma_n}{2A_n})^{N-1}}
\]

\[
= \prod_n \left( \frac{2A_n - \gamma_n}{2A_n} \right)^N \left( \frac{\gamma_n}{(2A_n)^N} \right)^N
\]

\[
= \prod_n \left( 1 - \frac{C_n^2}{4A_n B_n} \right)^N
\] (A16)

where we have used (A12). This expression is invariant under the simultaneous rescaling \(A_n, B_n, C_n \rightarrow \lambda A_n, \lambda B_n, \lambda C_n\) of the constants. Hence so will be the entropy. Using it in (A13), it readily follows that

\[
S = \sum_n \left[ \frac{1}{2A_n - \gamma_n} (2A_n \ln 2A_n - \gamma_n \ln \gamma_n) - \ln(2A_n - \gamma_n) \right].
\] (A17)

The constants \(C_n\) are determined by the coupling between the variables \(\{\alpha_n\}\) and \(\{a_n\}\). When the couplings vanish, \(C_n\) are zero and so are \(\gamma_n\) and hence, as one can see from (A17), the entanglement entropy vanishes as well.

In the weak coupling limit \((q \rightarrow 0)\), the following simplifications occur. One can write using equations (A7,A3)

\[
C_n = -\sqrt{\frac{2\pi \sigma_H}{nt}} \frac{2i n \omega_n N_{nm} J_n'(\chi_{nm})}{n t} \left( \omega_n + \frac{2n \pi}{R} + \frac{2n \pi}{\rho} \right),
\]

\[
\rightarrow -\sqrt{\frac{2\pi k q^2}{nt}} \frac{2i n \omega_n N_{nm} J_n'(\chi_{nm})}{n t} \left( \omega_n + \frac{2n \pi}{R} \right),
\] (A18)
where we have dropped the term proportional to $q^2$ in the denominator as it would be small compared to the other terms in the weak coupling limit.

Therefore, one finds

$$\gamma_n = \frac{|C_n|^2}{2B_n} = \frac{4\pi k q^2 R n N_m^2 J_n^2(\chi_{nm})\chi_n}{t (\chi_n + 2n)^2}, \quad (A19)$$

where we have used the relation $\chi_n = \omega_n R$. Note that $\gamma_n$ is nothing but $p_n$ for a single bulk mode. Due to the relation $A_n = \frac{1}{2}$, the entropy expression now reduces to

$$S = \sum_n \left[ \frac{1}{(1 - \gamma_n)} (-\gamma_n \ln \gamma_n) - \ln(1 - \gamma_n) \right]. \quad (A20)$$

For small $\gamma_n$ this expression reduces to that appearing in equation (6.1) and hence shows exactly the same scaling behavior.