Rotating black hole shadow in asymptotically safe gravity

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Abstract

We investigate the rotating black hole shadow and its observables in asymptotically safe gravity (ASG). The shape and size of shadow depends on various parameters such as black hole mass ($M$), spin parameter ($a$) and ASG parameters ($\zeta$) and ($\gamma$). We derive the complete null geodesics using Hamilton-Jacobi equation and Carter separable method. In our study, we find that the ASG parameters effectively change the black hole shadow as it appears smaller and more distorted in ASG as compared to their general relativity counterparts. We estimate the spin and ASG parameter from the study of shadow observables.

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I. INTRODUCTION

Einstein formulated the theory of General Relativity (GR) long back in 1915, which is still considered one of the most profound physical theory of all time. The physicists are struggling to develop a fully consistent theory of quantum gravity. That Einstein theory of GR is perturbatively non-renormalizable (except for pure gravity with no interaction with scalar fields at one-loop level) [1–3], therefore it is consider merely as an effective field theory. We can put a suitable cut-off at high energy limit, which as a result gives a correct description of gravity up to a certain energy scale and length scale, i.e. near singularity of black hole the classical description of GR cease to validate and demands a new physical theory.

The consistent theory of quantum gravity is phenomenologically important at least for two crucial reasons, namely for the unified theory of fundamental interaction (GUT), and for the origin of the Universe [4, 5]. However, in the process of formation of such theory using perturbative calculations the intangible hurdle comes in the form of non-renormalizability. Particularly this inevitable problem is the manifestation of the fact that the gravitational coupling constant is a dimensionful (dimension $[M]^{-2}$) quantity, rendering the infinite series of counter-terms to eliminate the divergences in the theory [6]. Indeed at each loop correction in perturbation theory, the ultraviolet (UV) divergences in quantum gravity get worse and worse, as a result, the divergence has unbounded growth. To tackle this problem of non-renormalizability and to construct a full UV complete quantum theory of gravity physicist have explored various possibilities, e.g., string theory [7, 8], loop quantum gravity [9, 10] and noncommutative geometry [11]. The UV completion of gravity would validate it to all energy scale (including arbitrary high energy scale) but it demands theory to be renormalizable. Even considering the GR as low energy limit of a more general theory called supergravity [12], the problem of non-renormalizability could not be resolved completely, although it alleviates the UV divergence of theory by reducing the number of diverging terms due to the underlying supersymmetry. The standard perturbative quantization of gravity was ended up in an impasse until a non-perturbative approach for renormalization of gravitational theory discovered called the asymptotic safety gravity [13, 14]. They suggested that there exist a fixed point in the UV limit in the renormalization group flow [15, 16]; the running of a gravitational coupling constant approach fixed point in UV limit such that
physical quantities become safe of unphysical divergences. The resulting theory would be asymptotically safe in the sense that at high energies unphysical divergences are likely to be absent [17, 18]. The asymptotically safe gravity (ASG) extend the outreach of effective field theory approach such that we can safely remove the UV cut-off from the theory and can have a complete description at all energy scale (including arbitrarily high energies) [14, 19–21]. Since the ASG has significant effects at short distances (or high energy scale) and black hole provides a natural testbed to study the gravity in its strongest regime, therefore it would also be interesting to study the black hole shadows in this context both from the phenomenological view to understand the theory of quantum gravity and from the observational perspective. Black hole solutions in the ASG theory have been extensively studied [22–25, 69? ] and discovered that the black hole properties change remarkably in the ASG theory.

Here, we wish to understand what impact the quantum corrections from ASG on the morphology of a supermassive black hole shadow and on the near spacetime geometry. Two major projects for black hole shadow observation, namely Event Horizon Telescope [26] and BlackHoleCam [27] are targeting supermassive black hole Sagittarius A* (Sgr A*) at our galactic center and likely to soon release first observational outcome. Therefore, a rational study of black hole shadow may reveal the near-horizon structure and may open a window to quantum gravity. In a pioneering study, Synge [28] and Luminet [29], for the first time ever determined the shadow of Schwarzschild black hole over a bright background, which for Kerr black hole was later determined by Bardeen [30]. Indeed, shadow is a gravitationally lensed image of the photon captured region accounting for those photons which inevitably end up in the black hole over a finite affine parameter, thus, its silhouette appears as a sharp boundary between dark and bright region. Having the assertion that a black hole shadow observation will be available in near future, a large comprehensive literature has been addressed the shadow study either within GR or modified gravities [31–51]. In addition, the effect of surrounding thin plasma medium on the rotating black hole shadow also has been studied [52–54], the Kerr black hole shadow with scalar hair by ray tracing method was reported in [55, 56] and it is also extended to the higher dimensional spacetime [57–60]. Moreover, characterization of shadow may provide a potential way to estimate the black hole parameters [38, 61–63]. Recently, it is found that shadow observations may offer a way to distinguish noncommutative geometry inspired black hole from the Kerr one [64]. It is also
believed that quantum modifications to black hole solutions may propagate to the horizon scale and may have observable implications for astrophysical phenomena that originate in the vicinity of horizons [65, 66]. Indeed, metric fluctuation have similar observable effects on black hole shadow [67]. Moreover, a detailed comprehensive analysis of black hole shadow can provide a better opportunity to study the physical processes at the very vicinity of horizon and also to determine the correct theory of gravity in its strongest regime.

The paper is organized as follow. In Sec. II, we discuss the rotating black hole in the ASG, the null geodesics in this spacetime are discussed in Sec. III. Further in Sec. IV, we obtain the black hole shadow. In Sec. V, we summarize our main results.

II. ROTATING BLACK HOLE IN ASG

In the ASG theory, the modification in action comes in the form of including higher order derivative terms. The rotating black hole in ASG [68], in the Boyer-Lindquist coordinate takes the form

$$ds^2 = -\left(1 - \frac{2MG(r)r}{\Sigma}\right)dt^2 - \frac{4MG(r)ar}{\Sigma}sin^2\theta dt d\phi + \frac{\Delta}{\Sigma} dr^2 + \Sigma d\theta^2 + \frac{\Delta_1 sin^2\theta}{\Sigma} d\phi^2$$ (1)

with

$$\Sigma = r^2 + a^2 cos^2\theta, \quad \Delta = r^2 + a^2 - 2MG(r)r, \quad \Delta_1 = (r^2 + a^2)^2 - a^2 \Delta sin^2\theta,$$ (2)

where $G(r) = \frac{G_0 r^3}{r^2 + \zeta (r + \gamma MG_0)}$ is running (or scale-dependent) coupling constant which at asymptotically large $r$ matches with the Newton’s gravitational constant $G_0$, i.e., $r \rightarrow \infty \Rightarrow G(r) \rightarrow G_0$, while at small length scale significantly differ from $G_0$. The parameters $\zeta$ and $\gamma$ are new variable of ASG. Eq. (1) represent the Kerr black hole solution in the ASG theory.

In the infra-red limit the solution (1) takes the form [69]

$$ds^2 = -\left(1 - \frac{2MG_0 r}{\Sigma}\left(1 - \frac{\zeta}{r^2}\right)\right)dt^2 - \frac{4MG_0 ar}{\Sigma}sin^2\theta dt d\phi + \frac{\Sigma}{\Delta_2} dr^2 + \Sigma d\theta^2 + \frac{\Delta_1}{\Delta_2} sin^2\theta d\phi^2,$$ (3)

where $\Delta_2 = r^2 - 2Mr + a^2 + \frac{2M\zeta}{r^2}$. Equation (1) represent the exterior spacetime of a stationary, rotating and axially symmetric black hole in the ASG theory, which reduces to
the Kerr black hole in the absence of asymptotic safety $\zeta \to 0$. The static and spherically symmetric black hole solution in infra-red limit can be retained from Eq. (3) as the limiting case of $a = 0$. The horizon of a black hole is a null hypersurface generated by null geodesics, which defined by the solution of equation $g^{rr} = 0$. The spacetime (3) admit two horizon solutions corresponds to Cauchy and event horizon, which are closely spaced as compared to GR solution. Indeed, horizon properties change significantly when one consider the ASG modifications, even in the infra-red limit.

III. NULL GEODESICS AROUND ROTATING BLACK HOLE IN ASG

In this section, we study the test particle trajectory in the rotating black hole spacetime (1) in ASG. We can safely assume that the black hole is in the presence of a luminous background (or surrounded by glowing accreting gas). The light coming from this source will get deflected due to a strong gravitational field near black hole before reaching to the distant observer. This deflection of light has explicit dependency over various parameters i.e., black hole mass $M$, spin parameter $a$, photon energy $\mathcal{E}$, angular momentum $\mathcal{L}$, and impact parameters. The photon trajectory can be classified into three categories namely, capture orbit, scattering orbit, and unstable circular and spherical orbit [33, 38]. The photons moving on unstable orbits will form the apparent shape of the event horizon, which further get gravitationally lensed to depict the black hole shadow. The apparent shadow of the black hole is always larger than the actual geometrical size of the event horizon. To solve geodesic equations of motion we choose Hamilton-Jacobi equation and a separable method which was originally given by Carter [70]. The Jacobean action $S = S(x^\mu, \tau)$ as a function of spatial co-ordinate $x^\mu$ and affine parameter along null geodesics $\tau$, and the Hamilton-Jacobi equation can be reads

$$\frac{\partial S}{\partial \tau} = -\frac{1}{2} g^{\alpha\beta} \frac{\partial S}{\partial x^\alpha} \frac{\partial S}{\partial x^\beta}. \quad (4)$$

The stationary and axially symmetric spacetime described in Eq. (1) admits the time-translational and rotational invariance, which guarantees the existence of two conserved quantities associated with the test particle along geodesics namely, energy $\mathcal{E}$ and angular momentum $\mathcal{L}$. Therefore, we can choose a separable solution in the following form [71]

$$S = \frac{1}{2} m_0^2 \tau - \mathcal{E} t + \mathcal{L} \phi + S_r(r) + S_\theta(\theta), \quad (5)$$
where $m_0$ is the mass of the test particle which is zero for the photon. By using the Hamilton-Jacobi equation and the variable separable solution we analytically derived the geodesic equations, which for photons can be written as

\[
\Sigma \frac{dt}{d\tau} = \frac{1}{\Delta} \left( \Delta_1 \mathcal{E} - 2MG(r)a r \mathcal{L} \right),
\]

(6)

\[
\Sigma \frac{dr}{d\tau} = \sqrt{\mathcal{R}(r)},
\]

(7)

\[
\Sigma \frac{d\theta}{d\tau} = \sqrt{\Theta(\theta)},
\]

(8)

\[
\Sigma \frac{d\phi}{d\tau} = \frac{1}{\Delta} \left( 2MG(r)ar \mathcal{E} + (\Sigma - 2MG(r)r) \mathcal{L} \csc^2 \theta \right),
\]

(9)

where $\mathcal{R}(r)$ and $\Theta(\theta)$ in Eq. (7) and (8) are the pure functions of radial and angular coordinate, respectively and have the following form

\[
\mathcal{R}(r) = \left[ (r^2 + a^2) \mathcal{E} - ar \mathcal{L} \right]^2 - (r^2 - 2MG(r)r + a^2) \left[ (a \mathcal{E} - \mathcal{L})^2 + \mathcal{K} \right],
\]

(10)

\[
\Theta(\theta) = \mathcal{K} - \left( \mathcal{L}^2 \csc^2 \theta - a^2 \mathcal{E}^2 \right) \cos^2 \theta,
\]

(11)

with a separable Carter constant $\mathcal{K}$. These equations completely define the photon geodesics in the spacetime of rotating black hole in ASG. Since the metric (1) is asymptotically flat at $(r \to \infty)$, therefore the null geodesics will be a straight line at spatial infinity. In order to study the general photon orbits we define two dimensionless quantities or impact parameters in terms of constants of motion as $\xi = \mathcal{L}/\mathcal{E}$ and $\eta = \mathcal{K}/\mathcal{E}^2$. From the radial equation of motion (7) we can rewrite the function $\mathcal{R}(r)$ for photon case and also in terms of impact parameters $\eta$ and $\xi$ in the following way

\[
\mathcal{R}(r) = \frac{1}{\mathcal{E}^2} \left[ ((r^2 + a^2) - a \xi)^2 - (r^2 - 2MG(r)r + a^2) \left( (a - \xi)^2 + \eta \right) \right].
\]

(12)

From Eq. (7), we can get the functional form of effective potential experienced by photons in the black hole spacetime. In the vicinity of horizon, the gravitational field is so mighty that even photons are enforced to move in the bound orbits, which can be characterized by the maximum of effective potential. It has been studied that photons moving with higher angular momentum than those moving on circular orbit feels scattering, while those with smaller angular momentum falls into the black hole. Therefore, the unstable orbits, which separate the scattering and capture orbit are crucial for the study of shadow silhouette. Thus photons which account for the shadow silhouette experience radial turning point in
their trajectory and also correspond for local maximum of effective potential:

\[ \mathcal{R}(r) = \frac{\partial \mathcal{R}(r)}{\partial r} = 0, \quad \text{and} \quad \frac{\partial^2 \mathcal{R}(r)}{\partial r^2} \leq 0. \]  

(13)

Every orbit around a black hole is characterized by constant \( \xi \) and \( \eta \), which for the unstable orbit took the value

\[ \eta = \frac{-r_0^4}{a^2((M - r_0)r_0^5 + r_0^2(-2r_0^2 + Mr_0(3 - 2\gamma) + 4M^2\gamma)\zeta - (r_0 + M\gamma)^2\zeta^2)^2}\left[4a^2M(r_0^3 + r_0\zeta + M\gamma\zeta)^2(-r_0^3 + r_0\zeta + 2M\gamma\zeta) + (r_0^5(-3M + r_0) + r_0^3(2r_0 + M(-1 + 2\gamma)))\zeta + (r_0 + M\gamma)^2\zeta^2\right], \]  

(14)

\[ \xi = \frac{1}{a((M - r_0)r_0^5 + r_0^2(-2r_0^2 + Mr_0(3 - 2\gamma) + 4M^2\gamma)\zeta - (r_0 + M\gamma)^2\zeta^2)^2}\left[r_0^5(r_0^2(-3M - r_0) + a^2(M + r_0)) + r_0^2(a^2(r_0(3M + 2r_0) + 2M(2M + r_0)\gamma) + r_0^3(2r_0 + M(-1 + 2\gamma)))\zeta + (a^2 + r_0^2)(r_0 + M\gamma)^2\zeta^2\right], \]  

(15)

where \( r_0 \) is the unstable orbit radius satisfying Eq. (13). The trajectory of photon orbits around the black hole in ASG can be defined by the above Eqs. (14) and (15). In the limit \( (\zeta \rightarrow 0) \) the expressions of impact parameters exactly reduce to the Kerr spacetime which takes the following form

\[ \xi = \frac{r_0^2(r_0 - 3M) + a^2(M + r_0)}{a(M - r_0)}, \]  

(16)

\[ \eta = \frac{r_0^3(4a^2M - r_0(r_0 - 3M)^2)}{a^2(M - r_0)^2}. \]  

(17)

The expressions \( \eta \) and \( \xi \) in Eqs. (14) and (15) give the critical values of impact parameters for photon undergoing unstable orbits around black hole. In order to define the shadow on the celestial sphere of a far distant observer we need to define celestial coordinates in observer frame which we will discuss in the next section.

**IV. BLACK HOLE SHADOW AND OBSERVABLES**

A far distant observer perceive black hole shadow as a dark region over a luminous background. The shadow of non-rotating black hole appears as a perfectly circular disc while the shapes of rotating black hole shadow get distorted due to the additional spin parameter. The black hole shadow can be better visualized by celestial coordinate \( \alpha \) and \( \beta \),
FIG. 1: Plot showing the black hole shadow in ASG theory for different values of $\zeta$, $\gamma$ and $a$.

which are the apparent angular distance of shadow silhouette from the direction of line of sight projected on the celestial sphere, and are defined as [38]

$$\alpha = \lim_{r_* \to \infty} \left( -r_*^2 \sin \theta_0 \frac{d\phi}{dr} \right), \quad \beta = \lim_{r_* \to \infty} r_* \frac{d\theta}{dr},$$

(18)

where $r_*$ is the distance between the observer and the black hole while $\theta_0$ is the inclination angle between the line of sight of the observer and the rotational axis of the black hole, for practical purpose we can consider $r_* \to \infty$. Using geodesic equations and celestial coordinate equations one can directly relate coordinate $\alpha$ and $\beta$ to the impact parameter as

$$\alpha = -\frac{\xi}{\sin \theta_0}, \quad \beta = \sqrt{\eta + a^2 \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0}.$$

(19)
The effect of black hole shadow is most significant at the equatorial plane \((\theta_0 = \pi/2)\) and decrease further as we move towards a polar end, therefore keeping the observer at the equatorial plane the above equations of celestial coordinate reduces to

\[
\alpha = -\xi, \quad \beta = \pm \sqrt{\eta},
\]

which satisfy the following relation

\[
\alpha^2 + \beta^2 = \frac{1}{a^2 (\zeta r_0^2 (4\gamma M^2 + (3-2\gamma)Mr_0 - 2r_0^2) + r_0^5 (M - r_0) - \zeta^2 (\gamma M + r_0)^2)) + ((r_0^5 (r_0 - 3M) + \zeta r_0^3 (2\gamma - 1)M + 2\zeta r_0^4) + \zeta^2 (\gamma M + r_0)^2)) + (r_0^5 (r_0^2 (-3M + r_0) + a^2 (M + r_o)) + r_0^2 (a^2 + r_0^2) (2\gamma M (2M + r_0) + r_0 (3M + 2r_0))) (\gamma M + r_0)^2 + r_0^3 (2\gamma - 1)M + 2r_0)) \zeta^2] .
\]

The contour of the above equation traces the apparent shape of the rotating black hole shadow in the ASG, which for the Kerr black hole can be revert back in the limit of \(\zeta \to 0\), where the above equation reduces to

\[
\alpha^2 + \beta^2 = \frac{a^2 (M + r_0)^2 - 6M^2 r_0^2 + 2r_0^4}{(M - r_0)^2}.
\]

The shadow of black hole in ASG has explicit dependency over the parameter \(a\) and \(\zeta\). We have shown the contour of rotating black hole shadow in Fig. 1 and 2 for different values of \(\zeta\), \(a\) and \(\gamma\). In Fig. 1 we plotted shadow for various values of \(\zeta\) by fixing \(a\) and \(\gamma\), while plots in Fig. 2 shows contour of black hole shadow for different values of \(a\) by keeping \(\zeta\) and \(\gamma\) fixed. To characterize the shape and size of the shadow, we adopt the observables: radius of shadow \(R_s\) and distortion parameter \(\delta_s\) [38]. Indeed, \(R_s\) is the radius of reference circle passing by three points (top, bottom, and extreme right) of the shadow boundary, whereas \(\delta_s\) characterizes its deviation from the perfect circle. The first observable \(R_s\) can be written in terms of the celestial coordinates of some specific points on the shadow as follow [38]

\[
R_s = \frac{\left(\alpha_t - \alpha_r\right)^2 + \beta_t^2}{2|\alpha_t - \alpha_r|} ,
\]

where \((\alpha_t, \beta_t)\) is the topmost position where the contour of black hole shadow and the reference circle cut the vertical axis and similarly \((\alpha_r, \beta_r)\) is the rightmost position where
FIG. 2: Plot showing the black hole shadow in ASG theory for different values of $\zeta$ and $a$.

FIG. 3: Plots showing the variation of observables $R_s$ and $\delta_s$ in ASG theory with $\zeta$ and $a$ for $\gamma = 0.1$. 
FIG. 4: Plot showing the contour of $R_s$ (solid red line) and $\delta_s$ (dashed black line) in the $(a, \zeta)$ plane where each curve labeled with corresponding values of $R_s$ and $\delta_s$.

FIG. 5: Plots showing the non-rotating black hole shadow and its observable $R_s$ in ASG theory for different values of $\zeta$. 
the reference circle and contour of shadow cut the horizontal axis. The second observable $\delta_s$ completely depends upon dent $D_s(= \alpha_l - \tilde{\alpha}_l)$ and shadow radius $R_s$ which can be expressed in the following form [38]

$$\delta_s = \frac{D_s}{R_s}. \quad (24)$$

Here, $\alpha_l$ and $\tilde{\alpha}_l$ are the coordinates where shadow and the reference circle cut the negative $\alpha$ axes. In our study we find that the spin $a$, and ASG parameters $\zeta$ and $\gamma$ sufficiently affect the shape and the size of the shadow. For instance, with increasing values of $a$ and $\zeta$ the black hole shadow gets distorted and a dent appears in the left side of the shadow. Rotating black hole shadow in ASG is considerable different from Kerr black hole, though the effects of ASG parameters are more prominent for higher values of black hole spin (c.f. Fig. 1).

In Fig. 3, we show the variation of shadow radius $R_s$ and the distortion parameter $\delta_s$ with ASG parameter $\zeta$ for different spin parameter. It is very evident that the radius of shadow decreases monotonically while the distortion parameter gets increases with increasing $\zeta$.

Since, the shadow observables for black hole in ASG are notably different from those in GR, therefore, it would be interesting to extract out black hole parameters from the analysis of shadow. In order to estimate these parameters, we make an analysis in Fig. 4, where the intersection of constant curves of $R_s$ and $\delta_s$ gives the exact value of black hole spin and ASG parameter $\zeta$. We have also plotted non-rotating black hole shadows in Fig. 5 for different values of $\zeta$. The first silhouette in Fig. 5 (full black line) corresponds to the Schwarzschild black hole in GR ($\zeta = 0$), and it can be clearly inferred from the figure that with increasing values of $\zeta$ the shadow size decreases. From the study of the non-rotating case, we conclude that the black hole shadow in ASG is still circular but appears smaller as compared to the shadow of the Schwarzschild black holes in standard GR.

V. CONCLUSION

The usual perturbative approach for quantization of gravity faces the problem of infinite numbers of diverging terms due to the dimensionality dependency of gravitational coupling constant. In the expedition of constructing a fully consistent theory of quantum gravity, Weinberg made an important headway by fixing the running gravitational constant at UV scale [13]. The short distance behavior of gravity is characterized by a non-trivial fixed point of running gravitational coupling constant in the renormalization group flow. As far as the
effects of asymptotic safety on black hole spacetime are concerned, the quantum effects are small for the massive black hole and also diminishes at very large distance. Nevertheless, even in the IR limit, the black hole in ASG theory are significantly differ from their GR counterparts. The generic absence of curvature singularity at center results in the zero temperature finite mass black hole remnant which formed at the final stage of black hole evaporation.

In this paper, we investigate these quantum corrections in the ASG propagate beyond the black hole horizon and left any imprints on the shadow. We study the black hole shadow in ASG as perceived by a far distant observer sitting on the equatorial plane. The ASG parameters affect the near horizon geometry, and so does the black hole shadow. In particular, the shadow of black hole in ASG is considerably different from their GR counterpart. Hence, similar to the noncommutativity and the metric perturbations the quantum corrections manifested in the ASG theory may also have observational effects in the shadow detection. Shadow radius decrease with increasing $\zeta$, whereas the intrinsic distortion in rotating black hole shadow gradually increase with increasing $\zeta$. It is found that the effect of asymptotic safety on rotating black hole shadow is more prominent for higher values of the spin parameter, whereas for slowly rotating black holes it would be difficult to distinguish shadow in ASG theory from the corresponding one in standard GR.

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