ON THE INTERIOR APPROXIMATE CONTROLLABILITY FOR FRACTIONAL WAVE EQUATIONS

VALENTIN KEYANTUO AND MAHAMADI WARMA

University of Puerto Rico
Rio Piedras Campus
Department of Mathematics
P.O. Box 70377, San Juan PR 00936-8377, USA

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Abstract. We study the interior approximate controllability of fractional wave equations with the fractional Caputo derivative associated with a non-negative self-adjoint operator satisfying the unique continuation property. Some well-posedness and fine regularity properties of solutions to fractional wave and fractional backward wave type equations are also obtained. As an example of applications of our results we obtain that if \(1 < \alpha < 2\) and \(\Omega \subset \mathbb{R}^N\) is a smooth connected open set with boundary \(\partial \Omega\), then the system

\[
\begin{align*}
D_\alpha^t u + A_B u &= f \quad \text{in } \Omega \times (0, T), \\
B u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) &= u_0, \quad \partial_t u(\cdot, 0) = u_1 \quad \text{in } \Omega,
\end{align*}
\]

where \(T > 0\), \(1 < \alpha < 2\), \(D_\alpha^t u\) denotes the Caputo fractional derivative of order \(\alpha\) of the function \(u\) (see (3.1) below), the operator \(A\) subject to the boundary condition \(B u = 0\) on \(\partial \Omega\) is a closed non-negative linear self-adjoint operator on \(L^2(\Omega)\) with compact resolvent. Then, \(A\) can be a symmetric and uniformly elliptic operator with bounded measurable coefficients with Dirichlet or Robin type boundary conditions. Another example for the operator is the fractional Laplace operator \((-\Delta)^s\) \((0 < s < 1)\) with the Dirichlet boundary condition \(u = 0\) on \(\mathbb{R}^N \setminus \Omega\). In (1.1), \(u = u(x, t)\) is the state to be controlled and \(f = f(x, t)\) is the control function which is localized

1. Introduction. Let \(\Omega \subset \mathbb{R}^N\) be a bounded domain with smooth boundary \(\partial \Omega\). Of concern in the present paper is the controllability of the fractional wave equation. More precisely, we consider the following initial boundary value problem

\[
\begin{align*}
D_\alpha^t u + A u &= f \quad \text{in } \Omega \times (0, T), \\
B u &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(\cdot, 0) &= u_0, \quad \partial_t u(\cdot, 0) = u_1 \quad \text{in } \Omega,
\end{align*}
\]

where \(T > 0\), \(1 < \alpha < 2\), \(D_\alpha^t u\) denotes the Caputo fractional derivative of order \(\alpha\) of the function \(u\) (see (3.1) below), the operator \(A\) subject to the boundary condition \(B u = 0\) on \(\partial \Omega\) is a closed non-negative linear self-adjoint operator on \(L^2(\Omega)\) with compact resolvent. Then, \(A\) can be a symmetric and uniformly elliptic operator with bounded measurable coefficients with Dirichlet or Robin type boundary conditions. Another example for the operator is the fractional Laplace operator \((-\Delta)^s\) \((0 < s < 1)\) with the Dirichlet boundary condition \(u = 0\) on \(\mathbb{R}^N \setminus \Omega\). In (1.1), \(u = u(x, t)\) is the state to be controlled and \(f = f(x, t)\) is the control function which is localized

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in an open set \( \omega \subset \Omega \). We shall generally write \( A_B \) for the realization in \( L^2(\Omega) \) of \( A \) with the boundary condition \( Bu = 0 \) on \( \partial \Omega \).

When \( \alpha = 2 \), the system (1.1) is the well-known wave type equation. The exact or/and approximate controllability of such a system is well-known and has been investigated by several authors when \( A \) is a uniformly elliptic operator with various boundary conditions (Dirichlet, Neumann and Robin). For instance we refer to the monograph [23] by E. Zuazua for a complete overview. The case where \( A = (-\Delta)^s \) \((0 < s < 2)\) is the fractional Laplace operator with the Dirichlet boundary condition has been recently investigated in [4] where the interior exact controllability has been obtained for the Shrödinger equation \((\alpha = 1)\) and the wave equation \((\alpha = 2)\) if \( 1 \leq s < 2 \). Moreover the author has proved that the associated Schrödinger equation \((\alpha = 1)\) is not exact controllable if \( 0 < s < 1/2 \).

We mention that there are very few references (except [6, 7, 13]) where controllability of fractional diffusion or fractional wave equations has been investigated. The main problem is that the nature of solutions of the systems for \( 0 < \alpha < 1 \) (resp. \( 1 < \alpha < 2 \)) differs from the classical case \( \alpha = 1 \) (resp. \( \alpha = 2 \)). Most recently, it has been shown in [13] that for any \( \alpha > 0 \), \((\alpha \notin N)\) such a system is not exact controllable in the classical sense, that is, for example if \( 0 < \alpha < 1 \), there is no control function \( f \) such that the solution \( u \) of the associated system can rest at some time \( T > 0 \). The interior approximate controllability of the case \( 0 < \alpha < 1 \) has been recently investigated in [7] where the authors have shown that for a symmetric non-negative uniformly elliptic operator with the Dirichlet boundary condition, the corresponding fractional diffusion system is approximately controllable for any \( T > 0 \), \( \omega \subset \Omega \) any open set and any \( f \in C^\infty_0(\omega \times (0, T)) \).

The boundary approximate controllability of the fractional diffusion equation for a symmetric non-negative uniformly elliptic operator with non-homogeneous Dirichlet boundary condition has also been studied in [6] with a positive result.

Motivated by these results, we propose in this paper to investigate the case of the fractional wave equation as stated in (1.1). We say that the system (1.1) is approximately controllable if for any \( u_0 \in V_1^\alpha = D(\frac{1}{2} A_1^\alpha) \), \( u_1 \in L^2(\Omega) \), \( T > 0 \), and \( \varepsilon > 0 \), there exists a control function \( f \) such that the corresponding unique solution \( u \) of the system (1.1) satisfies

\[
\|u(\cdot, T) - u_0\|_{V_1^\alpha} + \|\partial_t u(\cdot, T) - u_1\|_{L^2(\Omega)} \leq \varepsilon.
\]

The main result obtained in the present paper reads as follows: if \( 1 < \alpha < 2 \) and \( \omega \subset \Omega \) is any open set, \((u_0, u_1) \in V_1^\alpha \times L^2(\Omega)\), \( T > 0 \) and for any \( f \in C^\infty_0(\omega \times (0, T)) \), the system (1.1) is approximately controllable. Given that the system cannot be exactly controllable (by [13]), the approximate controllability is the best possible result that can be expected in the study of the controllability of the system (1.1). In order to obtain our result, we shall use some tools from the classical case \( \alpha = 2 \) (see e.g [23, Chapter 2] and the references therein) and also exploit some ideas recently used in the fractional diffusion equation in [6, 7]. More precisely, we shall use the explicit representation of the solutions which relies on the eigenfunction expansion.

The paper is organized as follows. In Section 2 we state the main results and give some examples of operators that apply to our situation. Section 3 contains some intermediate results that are needed in the proofs of our main results. More precisely, we prove the existence, uniqueness and regularity of strong solutions to the system (1.1) and its associated adjoint system. Finally in Section 4 we give the proofs of the results stated in Section 2.
2. Main results and examples. In this section we state the main results of the paper and give some examples. First, we introduce our assumptions on the operator \( A \). Let \( A_B \) be the realization in \( L^2(\Omega) \) of the operator \( A \) with the boundary condition \( Bu = 0 \) on \( \partial \Omega \).

**Assumption 1.** We assume that \( A_B \) is a non-negative self-adjoint operator with domain \( D(A_B) \) in \( L^2(\Omega) \) and that the embedding \( D(A_B) \hookrightarrow L^2(\Omega) \) is compact.

It follows from Assumption 1 that \( A_B \) is given by a bilinear symmetric, continuous, elliptic and closed form \( A_B \) with domain \( D(A_B) := V_{\frac{1}{2}} = D(A_{\frac{1}{2}}^1) \) and \( A_B(u, u) \geq 0 \) for every \( u \in V_{\frac{1}{2}} \). We denote by \( V_{\frac{1}{2}}^* \) the dual of \( V_{\frac{1}{2}} \). Then \( V_{\frac{1}{2}} \hookrightarrow L^2(\Omega) \hookrightarrow V_{\frac{1}{2}}^* \). Moreover, we have that \( A_B \) has a compact resolvent, hence, its eigenvalues form a non-decreasing sequence of real numbers \( 0 \leq \lambda_1 \leq \cdots \leq \lambda_n \cdots \) such that \( \lim_{n \to \infty} \lambda_n = \infty \). We assume that the first eigenvalue \( \lambda_1 > 0 \). We denote by \( \{ \varphi_n, n \in \mathbb{N} \} \) the orthonormal basis of normalized eigenfunctions associated with the eigenvalues \( \{ \lambda_n, n \in \mathbb{N} \} \). Then \( \varphi_n \in D(A_B) \) for every \( n \in \mathbb{N} \) and \( (\varphi_n)_{n \in \mathbb{N}} \) is total in \( V_{\frac{1}{2}} \) and in \( L^2(\Omega) \).

Throughout the remainder of the article, for a real number \( \gamma \geq 0 \), we let \( V_{\gamma} := D(A_B^\gamma) \) be equipped with the norm

\[
\| u \|_{V_{\gamma}} := \| A_B^\gamma u \|_{L^2(\Omega)},
\]

and we denote by \( V_{-\gamma} \) the dual of \( V_{\gamma} \). Note that \( (\varphi_n)_{n \in \mathbb{N}} \) is also total in \( V_{\gamma} \) if \( 0 \leq \gamma \leq 1 \). Moreover, \( (\cdot, \cdot) \) will designate the scalar product in \( L^2(\Omega) \).

### 2.1. Main results.

First, we introduce the notion of strong solutions to the system (1.1).

**Definition 2.1.** Let \( u_0, u_1 \) and \( f \) be given functions. A function \( u \) is said to be a strong solution of the system (1.1), if for a.e. \( t \in (0, T) \), for every \( T > 0 \), and for some \( 0 < \gamma < 1 \), the following properties hold.

- **Regularity:**
  \[
  \begin{cases}
    u \in C([0, T]; V_{\gamma}) \cap C^1([0, T]; L^2(\Omega)), \\
    D_t^\gamma u \in C((0, T]; L^2(\Omega)),
  \end{cases}
  \tag{2.1}
  \]
  and \( u(\cdot, 0) = u_0, \partial_t u(\cdot, 0) = u_1 \).

- **Variational identity:** for every \( \varphi \in V_{\gamma} \) and for a.e. \( t \in (0, T) \), we have
  \[
  (D_t^\gamma u(\cdot, t), \varphi) + A_B(u(\cdot, t), \varphi) - (f(\cdot, t), \varphi) = 0.
  \tag{2.2}
  \]

It follows from Definition 2.1 that for a strong solution \( u \) of (1.1), the values \( u(\cdot, T) \) and \( \partial_t u(\cdot, T) \) make sense and belong to \( V_{\gamma} \) and \( L^2(\Omega) \), respectively.

Using the fractional integration by parts formula (see (3.4) below) we have that the following backward system

\[
\begin{align*}
  D_t^\alpha v + Av &= 0 & \text{in } \Omega \times (0, T), \\
  Bu &= 0 & \text{on } \partial \Omega \times (0, T), \\
  I_t^{2-\alpha} v(\cdot, T) &= v_0, & D_t^{\alpha-1} v(\cdot, T) &= v_1 & \text{in } \Omega,
\end{align*}
\tag{2.3}
\]

can be viewed as the adjoint system associated with (1.1). In (2.3) for \( \beta > 0 \), \( I_t^\beta \) (resp. \( D_t^\beta \)) is the right Riemann-Liouville fractional integral (resp. the right
Let \( \text{Theorem 2.3.} \) interior approximate controllability of the system (1.1) defined in (3.2) below (resp. in (3.3) below) and the initial conditions have the interpretation:

\[
I_{t,T}^{2-\alpha} v(\cdot, T) = \lim_{t \to T} \frac{1}{1-(2-\alpha)} \int_t^T (\tau-t)^{1-\alpha} v(\cdot, \tau) \, d\tau = v_0,
\]

and

\[
D_{t,T}^{2-\alpha} v(\cdot, T) = - \lim_{t \to T} \frac{1}{1-(2-\alpha)} \frac{d}{dt} \int_t^T (\tau-t)^{1-\alpha} v(\cdot, \tau) \, d\tau = v_1.
\]

We adopt the following definition of strong solutions for the backward system (2.3).

**Definition 2.2.** Let \( v_0, v_1 \) be given functions and \( T > 0 \). A function \( v \) is said to be a strong solution of the system (2.3), if for a.e. \( t \in (0, T) \), and for some \( 0 < \gamma < 1 \), the following properties hold.

- **Regularity:**
  \[
  \begin{align*}
  I_{t,T}^{2-\alpha} v &\in C([0, T]; V_1), \\
  D_{t,T}^{2-\alpha} v &\in C([0, T]; L^2(\Omega)), \\
  D_{t,T}^\gamma v &\in C([0, T]; L^2(\Omega)),
  \end{align*}
  \]  \hfill (2.4)

  and \( I_{t,T}^{2-\alpha} v(\cdot, T) = v_0, D_{t,T}^{2-\alpha} v(\cdot, T) = v_1 \).

- **Variational identity:** for every \( \varphi \in V_\gamma \) and for a.e. \( t \in (0, T) \), we have
  \[
  (D_{t,T}^\gamma v(\cdot, t), \varphi) + A_B(v(\cdot, t), \varphi) = 0.
  \]  \hfill (2.5)

Next, we show that the adjoint system (2.3) satisfies the unique continuation principle.

**Proposition 2.1.** Let \( 1 < \alpha < 2, \gamma := \frac{1}{\alpha} \), \( v_0 \in V_\gamma \), \( v_1 \in L^2(\Omega) \) and let \( \omega \subset \Omega \) be an arbitrary open set. Assume that \( A_B \) has the unique continuation property in the sense that if \( \lambda \) is an eigenvalue of \( A_B \), and \( (A_B - \lambda)u = 0 \) in \( \Omega \) and \( u = 0 \) in \( \omega \) then \( u = 0 \) in \( \Omega \). Let \( v \) be the unique strong solution to the system (2.3). If \( v = 0 \) on \( \omega \times (0, T) \), then \( v = 0 \) on \( \Omega \times (0, T) \).

As a direct consequence of Proposition 2.1, we have the following result of the interior approximate controllability of the system (1.1).

**Theorem 2.3.** Let \( 1 < \alpha < 2, \gamma := \frac{1}{\alpha} \) and assume that the operator \( A_B \) has the unique continuation property. Then the system (1.1) is approximately controllable for any given \( T > 0 \), an arbitrary open set \( \omega \subset \Omega \) and \( f \in C_0^\infty(\omega \times (0, T)) \). That is,

\[
\left\{ (u(\cdot), \partial_t u(\cdot, T)) : f \in C_0^\infty(\omega \times (0, T)) \right\} V_\gamma \times L^2(\Omega) = V_\gamma \times L^2(\Omega),
\]

where \( u \) is the unique strong solution of the system (1.1).

2.2. **Some examples of operators.** We conclude this section by giving some examples of operators satisfying our assumptions.

**Example 1 (Elliptic operators).** We consider the operator \( A \) given formally by

\[
Au(x) = - \sum_{i,j=1}^N \frac{\partial}{\partial x_j} \left( a_{i,j}(x) \frac{\partial u}{\partial x_i}(x) \right) + b(x)u(x), \quad x \in \Omega,
\]

where the coefficients satisfy the following conditions:

\[
a_{ij}(x) = a_{ji}(x), \quad a_{ij} \in W^{1,\infty}(\Omega), \quad 1 \leq i, j \leq N, \quad b \in L^\infty(\Omega), \quad b(x) \geq 0, \quad x \in \Omega,
\]
and there exists a constant $\mu > 0$ such that
\[
\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^N.
\]

(a) The Dirichlet boundary condition. Let $A_D$ be the self-adjoint operator on $L^2(\Omega)$ associated with the closed bilinear symmetric form
\[
A_D(u, v) := \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Omega} b(x)uv \, dx, \quad u, v \in W_0^{1,2}(\Omega),
\]
that is,
\[
D(A_D) := \{ u \in W_0^{1,2}(\Omega), \exists w \in L^2(\Omega), A_D(u, v) = (w, v) \forall v \in W_0^{1,2}(\Omega) \}, \quad A_Du = w.
\]
Then $A_D$ is the realization of $A$ with the Dirichlet boundary condition $Bu = u|_{\partial \Omega} = 0$ on $\partial \Omega$ and it satisfies all the assumptions.

(b) The Robin boundary condition. Let $\beta \in L^\infty(\partial \Omega)$ satisfy $\beta(x) \geq \beta_0 > 0$ on $\partial \Omega$. Let $A_R$ be the self-adjoint operator on $L^2(\Omega)$ associated with the closed bilinear symmetric form $A_R$ given for $u, v \in W^{1,2}(\Omega)$ by:
\[
A_R(u, v) := \sum_{i,j=1}^{N} \int_{\Omega} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_{\Omega} b(x)uv \, dx + \int_{\partial \Omega} \beta(x)uv \, d\sigma.
\]
As for the Dirichlet boundary condition, we have that
\[
D(A_R) := \{ u \in W^{1,2}(\Omega), \exists w \in L^2(\Omega), A_R(u, v) = (w, v) \forall v \in W^{1,2}(\Omega) \}, \quad A_Ru = w,
\]
and $A_R$ is the realization of $A$ with the Robin boundary condition $Bu = \partial_{\nu,a}u + \beta u = 0$ on $\partial \Omega$ and it also satisfies all the assumptions, where here,
\[
\partial_{\nu,a}u(x) = \sum_{i,j=1}^{N} a_{ij}(x) \frac{\partial u(x)}{\partial x_i} \nu_j(x).
\]
Moreover, the operator $A_R$ also enjoys the unique continuation property. For more details we refer to [8] and the references therein.

**Example 2** (The fractional Laplacian). Let $0 < s < 1$ and define the fractional Laplacian $(-\Delta)^s u$ of a function $u$ by
\[
(-\Delta)^s u(x) := \lim_{\epsilon \to 0} C_{N,s} \int_{\{y \in \mathbb{R}^N : |x-y| > \epsilon \}} \frac{u(x) - u(y)}{|x-y|^{N+2s}} \, dy, \quad x \in \mathbb{R}^N,
\]
provided that the limit exists, where $C_{N,s}$ is a normalization constant (see e.g. [18, 19, 20, 21, 22]). Let $(-\Delta)_{\Omega}^s$ be the realization in $L^2(\Omega)$ of $(-\Delta)^s$ with the Dirichlet boundary condition, that is, $(-\Delta)_{\Omega}^s$ is the self-adjoint operator in $L^2(\Omega)$ associated with the closed symmetric form
\[
\mathcal{F}(u, v) = \frac{C_{N,s}}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y))}{|x-y|^{N+2s}} \, dxdy, \quad u, v \in W_0^{s,2}(\Omega),
\]
where here
\[
W_0^{s,2}(\Omega) := \{ u \in W^{s,2}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \}.
\]
That is,
\[
\begin{aligned}
D^{\alpha} := \{ u \in W^{\alpha,2}(\Omega), \exists w \in L^{2}(\Omega), \mathcal{F}(u, v) = (w, v) \forall v \in W^{\alpha,2}(\Omega) \}, \\
(-\Delta)^{\alpha} u = w.
\end{aligned}
\]

By [19] (see also [18, 20, 21]) the operator \((-\Delta)^{\alpha}\) has compact resolvent and its eigenvalues form a non-decreasing sequence of real numbers \(0 < \lambda_1 \leq \lambda_2 \leq \cdots \lambda_n \leq \cdots\) that converges to \(\infty\). It also follows from [5, Theorem 1.4] that \((-\Delta)^{\alpha}\) satisfies the unique continuation property.

3. Preliminary results. In this section we give some notations and intermediate results as they are needed in the proofs of our main results. In particular we introduce the Caputo and Riemann-Liouville fractional derivatives and prove the existence, uniqueness and regularity of solutions to the systems (1.1) and (2.3).

3.1. Fractional derivatives and the Mittag-Leffler functions. Throughout the remainder of this section, \(X\) denotes a Banach space. The Caputo fractional derivative of order \(\alpha\) with \(n-1 \leq \alpha < n\), \(n \in \mathbb{N}\), of a locally integrable function \(u : [0, T] \to X\) is defined by
\[
D^{\alpha}_t u(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds, \quad \forall t \in (0, T).
\]

Then, when \(\alpha = 1\) and \(u\) is differentiable on \((0, T)\) (resp. \(\alpha = 2\) and \(u\) is twice differentiable on \((0, T)\)), we get that \(D^1_t u = \frac{du}{dt}\) (resp. \(D^2_t u = \frac{d^2 u}{dt^2}\)). The right Riemann-Liouville fractional integral of order \(\alpha > 0\) of a locally integrable function \(u : (0, T] \to X\) is defined by
\[
I^{\alpha}_{t,T} u(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} u(s) ds, \quad \forall t \in (0, T).
\]

The right Riemann-Liouville fractional derivative of order \(\alpha\) with \(n-1 \leq \alpha < n\), \(n \in \mathbb{N}\), of a locally integrable function \(u : (0, T] \to X\) is defined for \(t \in (0, T)\) by
\[
D^{\alpha}_{t,T} u(t) = (-1)^n \frac{d^n}{dt^n} I^{\alpha-n}_{t,T} u(t)
\]
\[
= (-1)^n \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_t^T (t-s)^{n-\alpha-1} u(s) ds.
\]

From (3.3), we have that if \(u\) is differentiable (resp. twice differentiable), then \(D^1_{t,T} u = \frac{du}{dt}\) (resp. \(D^2_{t,T} u = \frac{d^2 u}{dt^2}\)). We have the following fractional integration by parts formula taken from [1] (see also [2]). Let \(1 < \alpha < 2\). Then,
\[
\int_0^T v(t) D^{\alpha}_{t,T} u(t) \ dt = \int_a^b u(t) D^{\alpha}_{t,T} v(t) \ dt
\]
\[
+ \left[ I^{\alpha-1}_{t,T} v(t) u'(t) + D^{\alpha-1}_{t,T} v(t) u(t) \right]^{t=T}_{t=0},
\]
provided that the left and right-hand side expressions make sense.

The Mittag-Leffler function with two parameters is defined as follows:
\[
E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C}, \quad z \in \mathbb{C}.
\]
It is well-known that $E_{\alpha,\beta}(z)$ is an entire function. The following estimate of the Mittag-Leffler function will be useful. Let $0 < \alpha < 2$, $\beta \in \mathbb{R}$ and $\mu$ be such that $\frac{\alpha \pi}{2} < \mu < \min\{\pi, \alpha \pi\}$. Then there exists a constant $C = C(\alpha, \beta, \mu) > 0$ such that

\[ |E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|}, \quad \mu \leq |\arg(z)| \leq \pi. \]  

(3.6)

In the literature, frequently the notation $E_{\alpha} = E_{\alpha,1}$ is used. The Laplace transform of the Mittag-Leffler function is given by:

\[ \int_0^\infty e^{-\lambda t^{\alpha k + \beta - 1}} E_{\alpha,\beta}^{(k)}(\pm \omega t^\alpha) dt = \frac{k! \lambda^{\alpha - \beta}}{(\lambda^n \mp \omega)^{k+1}}, \quad \Re(\lambda) > |\omega|^{1/\alpha}. \]  

(3.7)

Here, $k \in \mathbb{N} \cup \{0\}$ and $\omega \in \mathbb{R}$. Using (3.7), we obtain for $0 < \alpha < 2$:

\[ D_{t}^\alpha E_{\alpha,1}(zt^\alpha) = z E_{\alpha,1}(zt^\alpha), \quad t > 0, z \in \mathbb{C}, \]  

(3.8)

that is, for every $z \in \mathbb{C}$, the function $u(t) := E_{\alpha,1}(zt^\alpha)$ is a solution of the scalar valued problem

\[ D_{t}^\alpha u(t) = zu(t), \quad t > 0, 0 < \alpha < 2. \]

Moreover, we have that for $\lambda > 0$ and $m \in \mathbb{N}$,

\[ \frac{d^m}{dt^m} E_{\alpha,1}(-\lambda t^\alpha) = -\lambda t^{\alpha - m} E_{\alpha,\alpha - m + 1}(-\lambda t^\alpha), \quad t > 0. \]  

(3.9)

The proofs of (3.6), (3.7), (3.8) and (3.9) are contained in [16]. For more details on fractional derivatives and integrals and the Mittag-Leffler functions we refer to [1, 3, 9, 10, 14, 15, 16] and the references therein.

3.2. Well-posedness and fine regularity of solutions to fractional wave equations. Let $1 < \alpha < 2$ and consider the following fractional order homogeneous wave equation

\[ \begin{cases} 
D_{t}^\alpha u + A_B u = 0, & \text{in } \Omega \times (0, T), \\
u(\cdot, 0) = u_0, & \partial_t u(\cdot, 0) = u_1 \text{ in } \Omega, 
\end{cases} \]  

(3.10)

where $u_0, u_1 \in L^2(\Omega)$. This is exactly the system (1.1) with $f \equiv 0$.

We introduce the notion of classical solutions of the homogeneous system that can be viewed as the motivation of our definition of strong solutions to the system (1.1).

**Definition 3.1.** Let $T > 0$. A function $u \in C([0, T]; D(A_B)) \cap C^1([0, T]; L^2(\Omega))$ is said to be a classical solution of (3.10), if $D_{t}^\alpha u \in C((0, T]; L^2(\Omega))$ and (3.10) is satisfied.

We shall make frequent use of the following well known inequality (which expresses that Hilbert space has Rademacher type 2, or can be viewed simply as a consequence of the parallelogram law): If we denote by $\mathbb{B}^n$, $n \in \mathbb{N}$ the set of all $n$–tuples $e = (\epsilon_1, \cdots, \epsilon_n)$ with $\epsilon_k = \pm 1$, then

\[ \frac{1}{2^n} \sum_{e \in \mathbb{B}^n} \left\| \sum_{k=1}^n \epsilon_k \xi_k \right\|^2 \leq \sum_{k=1}^n \left\| \xi_k \right\|^2 \]  

(3.11)

We have the following result of existence and uniqueness of classical solutions.
Theorem 3.2. Let $1 < \alpha < 2$, $T > 0$ and $\delta := \frac{\alpha - 1}{\alpha}$. Then for every $u_0 \in D(A_B)$ and $u_1 \in V_\delta$, the system (3.10) has a unique classical solution $u$ given by

$$ u(\cdot, t) = \sum_{n=1}^{\infty} (u_0, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^{\infty} (u_1, \varphi_n) t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n. $$

(3.12)

Proof. Let $1 < \alpha < 2$, $T > 0$ and $\delta := \frac{\alpha - 1}{\alpha}$. It has been shown in [17, Theorem 2.3] that for every $u_0 \in D(A_B)$ and $u_1 \in V_{1/2} = D(A_B^{1/2})$, the function $u$ given by (3.12) is the unique classical solution of the system (3.10). We show that the result holds for $u_1 \in V_\delta \supset V_{1/2}$. Let $u_0 \in D(A_B)$ and $u_1 \in V_\delta$. We show that $u$ satisfies the regularity given in the theorem. Let $u_{0,n} := (u_0, \varphi_n)$ and $u_{1,n} := (u_1, \varphi_n)$. Then using (3.11), we have the following estimate:

$$ \|u(\cdot, t)\|_{D(A_B)}^2 = \|A_B u(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2 \left\| \sum_{n=1}^{\infty} \lambda_n u_{0,n} E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n \right\|^2 + 2 \left\| \sum_{n=1}^{\infty} u_{1,n} \lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n \right\|^2 \leq 2 \sum_{n=1}^{\infty} |\lambda_n u_{0,n}|^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + 2 \sum_{n=1}^{\infty} |\lambda_n^{\alpha-1} u_{1,n}|^2 |\lambda_n^{\frac{\alpha}{2}} t E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \leq C \left( \|u_0\|_{D(A_B)}^2 + \|u_1\|_{V_\delta}^2 \right), $$

(3.13)

where we have used (3.6) and the fact that

$$ |\lambda_n^{\frac{\alpha}{2}} t E_{\alpha,2}(-\lambda_n t^\alpha)| \leq C_1 \frac{\lambda_n^{\frac{\alpha}{2}} t}{1 + \lambda_n t^\alpha} \leq C_1 \frac{\alpha - 1}{\alpha} \lambda_n t. $$

It is easy to verify that the series in (3.12) converges in $D(A_B)$ uniformly in $t \in [0, T]$. Hence, $u \in C([0, T]; D(A_B))$. Next, we show that $\partial_t u \in C([0, T]; L^2(\Omega))$. Using (3.12) and (3.9) we have that for a.e. $t \in (0, T)$,

$$ \partial_t u(\cdot, t) = \sum_{n=1}^{\infty} u_{0,n} \lambda_n t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha) + \sum_{n=1}^{\infty} u_{1,n} E_{\alpha,1}(-\lambda_n t^\alpha). $$

(3.14)

From (3.14), using (3.6) and again (3.11), we obtain that

$$ \|\partial_t u(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2 \sum_{n=1}^{\infty} |\lambda_n u_{0,n}|^2 |t^{\alpha-1} E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + 2 \sum_{n=1}^{\infty} |u_{1,n}|^2 |E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \leq C t^{2(\alpha-1)} (\|u_0\|_{D(A_B)}^2 + \|u_1\|_{L^2(\Omega)}^2) \leq C (t^{2(\alpha-1)} \|u_0\|_{D(A_B)}^2 + \|u_1\|_{V_\delta}^2). $$

Since $\alpha \geq 1$ and the series in (3.14) converges in $L^2(\Omega)$ uniformly in $t \in [0, T]$, it follows that $\partial_t u \in C([0, T]; L^2(\Omega))$ and we have shown that $u \in C^1([0, T]; L^2(\Omega))$. 


Finally, it follows from (3.13) that
\[ \|D_t^\alpha u(\cdot,t)\|_{L^2(\Omega)}^2 = \|A_B u(\cdot,t)\|_{L^2(\Omega)}^2 \leq C(\|u_0\|_{H^\alpha(\Omega)}^2 + \|u_1\|_{V_\alpha}^2). \]
As above we have shown that $D_t^\alpha u \in C([0,T];L^2(\Omega))$. The uniqueness of the solution is easy to verify and the proof is finished.

In order to show the existence of strong solutions, we shall frequently use the following estimates that follow from the general relations (3.6), (3.7), (3.9) and some straightforward computations.

**Lemma 3.3.** Let $1 < \alpha < 2$ and $\alpha' > 0$. Then the following assertions hold.
(a) Let $0 \leq \beta < 1$ and $0 < \gamma < \alpha$. Then there exist two constants $C > 0$, $C_1 > 0$ such that for every $\lambda > 0$ and $t > 0$,
\[ |\lambda^\beta t^\gamma E_{\alpha,\alpha'}(-\lambda t^\alpha)| \leq C_1 \frac{\lambda^\beta t^\gamma}{1 + \lambda t^\alpha} \leq C \min\{t^{\gamma-\alpha\beta}, \lambda^{\beta-\frac{\alpha}{2}}\}. \]
(b) Let $0 \leq \gamma < 1$ and $\lambda > 0$. Then there exist two constants $C > 0$, $C_1 > 0$ such that for every $t > 0$,
\[ |\lambda^{1-\gamma} t^{\alpha-2} E_{\alpha,\alpha'}(\lambda t^\alpha)| \leq C_1 \frac{\lambda^{1-\gamma} t^{\alpha-2}}{1 + \lambda t^\alpha} \leq C t^{\alpha-2}. \]
(c) Let $T > 0$ and $\lambda > 0$. Then
\[ \int_0^T |t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)| \, dt = -\frac{1}{\lambda} \int_0^T \frac{d}{dt} E_{\alpha,1}(-\lambda t^\alpha) \, dt \]
\[ = \frac{1}{\lambda} (1 - E_{\alpha,\alpha}(-\lambda T^\alpha)). \]
(d) Let $\lambda > 0$ and $t > 0$. Then
\[ \int_0^t E_{\alpha,1}(-\lambda t^\alpha) \, dt = t E_{\alpha,2}(-\lambda t^\alpha). \]

Next, we show the existence and uniqueness of strong solutions.

**Theorem 3.4.** Let $1 < \alpha < 2$, $\gamma := \frac{1}{\alpha}$, $u_0 \in V_\gamma$, $u_1 \in L^2(\Omega)$ and let the function $f \in W^{1,\infty}((0,T);L^2(\Omega))$. Then the system (1.1) has a unique strong solution $u$ given by
\[ u(\cdot,t) = \sum_{n=1}^\infty (u_0, \varphi_n) E_{\alpha,1}(-\lambda_n t^\alpha) \varphi_n + \sum_{n=1}^\infty (u_1, \varphi_n) t E_{\alpha,2}(-\lambda_n t^\alpha) \varphi_n \]
\[ + \sum_{n=1}^\infty \left( \int_0^t (f(\cdot,\tau), \varphi_n)(t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right) \varphi_n, \]
where $E_{\alpha,1}$, $E_{\alpha,2}$ and $E_{\alpha,\alpha}$ are the Mittag-Leffler functions given in (3.5). Moreover, there are some constants $C_1, C_2, C_3 > 0$ such that for all $t \in (0,T)$,
\[ \|u(\cdot,t)\|_{V_\gamma}^2 \leq C_1 \left( \|u_0\|_{V_\gamma}^2 + \|u_1\|_{L^2(\Omega)}^2 + \|f\|_{L^{\infty}((0,T);L^2(\Omega))}^2 \right), \]
\[ \|\partial_t u(\cdot,t)\|_{L^2(\Omega)}^2 \leq C_2 \left( \|u_0\|_{V_\gamma}^2 + \|u_1\|_{L^2(\Omega)}^2 + t^{2(\alpha-1)} \|f(\cdot,0)\|_{L^2(\Omega)}^2 + \|f\|_{L^{\infty}((0,T);L^2(\Omega))}^2 \right), \]
\[ \|D_t^\alpha u(\cdot,t)\|_{L^2(\Omega)}^2 \leq C_3 \left( t^{-2\alpha} \|u_0\|_{V_\gamma}^2 + t^{2(1-\alpha)} \|u_1\|_{L^2(\Omega)}^2 + \|f\|_{L^{\infty}((0,T);L^2(\Omega))}^2 \right). \]
Proof. Let \(1 < \alpha < 2\), \(\gamma = \frac{1}{\alpha}\), \(u_0 \in V_\gamma\), \(u_1 \in L^2(\Omega)\) and \(f \in W^{1,\infty}((0,T);L^2(\Omega))\). It has been shown in [17] (see also [12]) that the function \(u\) given in (3.19) is the unique weak solution to the problem (3.10). Next, we show that \(u\) satisfies the properties given in (2.1). We adopt the notation: \(u_{0,n} := (u_0, \varphi_n)\), \(u_{1,n} := (u_1, \varphi_n)\) and \(f_{n}(t) = (f(\cdot, t), \varphi_n)\).

(a) First, we claim that \(u \in C([0,T]; V_\gamma)\). Using (3.19), (3.15), (3.17) and (3.11), we get the following estimates:

\[
\|u(\cdot,t)\|^2_{L_\gamma^2} = 4 \sum_{n=1}^{\infty} |u_{0,n}\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + 4 \sum_{n=1}^{\infty} |u_{1,n}\lambda_n E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \\
+ 4 \sum_{n=1}^{\infty} \int_0^t (f(\cdot, \tau), \varphi_n)\lambda_n (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha)\,d\tau \leq c_1 \sum_{n=1}^{\infty} |\lambda_n u_{0,n}|^2 + 4 \sum_{n=1}^{\infty} |u_{1,n}|^2 |\lambda_n E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \\
+ 4 \sum_{n=1}^{\infty} \sup_{t \in [0,T]} |f_n(t)|^2 \left( \int_0^t \lambda_n^\gamma \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n \tau^\alpha)\,d\tau \right)^2 \\
\leq c_1 \|u_0\|^2_{L_\gamma^2} + c_2 t^{2(1-\alpha\gamma)} \|u_1\|^2_{L_\gamma^2} \Omega \\
+ 4 \sum_{n=1}^{\infty} \sup_{t \in [0,T]} |f_n(t)|^2 \left( \lambda_n^{\gamma-1}(1-E_{\alpha,1}(-\lambda_n T^\alpha)) \right)^2 \\
\leq C \left( \|u_0\|^2_{L_\gamma^2} + \|u_1\|^2_{L_\gamma^2} + \lambda_1^{\gamma-1} \|f\|^2_{L^{\infty}((0,T);L^2(\Omega))} \right). \tag{3.21}
\]

Since \(1 - \alpha \gamma \geq 0\), \(\gamma - 1 \leq 0\) and the series in (3.19) converges in \(V_\gamma\) uniformly in \(t \in [0,T]\), it follows from (3.21) that \(u \in C([0,T]; V_\gamma)\) and the claim is proved.

(b) Next, we show that \(u \in C^1([0,T]; L^2(\Omega))\). We notice that a simple calculation gives that for a.e. \(t \in (0,T)\),

\[
\partial_t u(\cdot,t) = \sum_{n=1}^{\infty} u_{0,n}\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)\varphi_n + \sum_{n=1}^{\infty} u_{1,n}E_{\alpha,1}(-\lambda_n t^\alpha)\varphi_n \\
+ \sum_{n=1}^{\infty} f_n(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha) \\
+ \sum_{n=1}^{\infty} \int_0^t f_n(\tau) \lambda_n (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \,d\tau \varphi_n. \tag{3.22}
\]

Using (3.22), (3.6), (3.15), (3.17) and (3.11), we get the following estimates:

\[
\|\partial_t u(\cdot,t)\|^2_{L_\gamma^2} = 8 \sum_{n=1}^{\infty} |u_{0,n}\lambda_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\
+ 8 \sum_{n=1}^{\infty} |u_{1,n}E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \\
+ 8 \sum_{n=1}^{\infty} |f_n(0) t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \\
+ 8 \int_0^t f_n(\tau) \lambda_n (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \,d\tau \varphi_n. \tag{3.23}
\]
\[ + 8 \sum_{n=1}^{\infty} \left| \int_0^t f_n'(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right|^2 \]
\[ \leq 8 \sum_{n=1}^{\infty} |\lambda_n^2u_0n|^2|\lambda_n^{1-\gamma}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n t^\alpha)|^2 \]
\[ + c_1 \sum_{n=1}^{\infty} |u_{1,n}|^2 + c_2 t^{2(\alpha-1)} \sum_{n=1}^{\infty} |f_n(0)|^2 \]
\[ + 8 \sum_{n=1}^{\infty} \sup_{t \in [0,T]} |f_n'(t)|^2 \left( \int_0^t \lambda_n \tau^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n \tau^\alpha) \, d\tau \right)^2 \]
\[ \leq C \left( \|u_0\|^2_{L^2(\Omega)} + \|u_1\|^2_{L^2(\Omega)} + t^{2(\alpha-1)}\|f(\cdot,0)\|^2_{L^2(\Omega)} \right. \]
\[ \left. + \|f\|^2_{L^\infty((0,T);L^2(\Omega))} \right). \quad (3.23) \]

It follows from (3.21) and (3.23) and the fact that the series in (3.22) converges in \( L^2(\Omega) \) uniformly for \( t \in [0,T] \) that \( u \in C^1([0,T];L^2(\Omega)). \)

(c) Next, we prove that \( D\alpha_t u \in C((0,T];L^2(\Omega)). \) Since \( D\alpha_t u = -A_Bu + f \), it follows from (3.19) that

\[ D\alpha_t u(\cdot,t) = -\sum_{n=1}^{\infty} u_{0,n}\lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)\phi_n - \sum_{n=1}^{\infty} u_{1,n}\lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha)\phi_n \]
\[ - \sum_{n=1}^{\infty} \left( \int_0^t f_n(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right) \phi_n + f(t). \quad (3.24) \]

Using (3.6), (3.15), (3.16), (3.17) and (3.11), we get from (3.24) that

\[ \|D\alpha_t u(\cdot,t)\|^2_{L^2(\Omega)} \leq 8 \sum_{n=1}^{\infty} |u_0n\lambda_n^2\lambda_n^{1-\gamma}E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \]
\[ + 8 \sum_{n=1}^{\infty} |u_{1,n}\lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \]
\[ + 8 \sum_{n=1}^{\infty} \left| \int_0^t f_n(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha-1}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right|^2 \]
\[ + C\|f\|^2_{L^\infty((0,T);L^2(\Omega))} \]
\[ \leq 8 \sum_{n=1}^{\infty} |u_0n\lambda_n|^2|\lambda_n^{1-\gamma}E_{\alpha,1}(-\lambda_n t^\alpha)|^2 \]
\[ + 8 \sum_{n=1}^{\infty} |u_{1,n}|^2|\lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha)|^2 \]
\[ + 8 \sum_{n=1}^{\infty} \left| \int_0^t f_n(\tau)\lambda_n(t-\tau)^{\alpha-1}E_{\alpha,\alpha-1}(-\lambda_n(t-\tau)^\alpha) \, d\tau \right|^2 \]
\[ + C\|f\|^2_{L^\infty((0,T);L^2(\Omega))} \]
\[ \leq C_1 t^{-2\alpha}\|u_0\|^2_{V_{\gamma}} + C_2 t^{2(1-\alpha)}\|u_1\|^2_{L^2(\Omega)} \]
\[ + C_3\|f\|^2_{L^\infty((0,T);L^2(\Omega))}. \quad (3.25) \]
Since the series in (3.24) converges in $L^2(\Omega)$ uniformly on any compact subset of $(0, T]$, we have that $D_t^n u \in C((0, T]; L^2(\Omega))$.

(d) Using (3.19) and (3.22), we get that

$$u(\cdot, 0) = \sum_{n=1}^{\infty} u_{0,n} \varphi_n = u_0 \quad \text{and} \quad \partial_t u(\cdot, 0) = \sum_{n=1}^{\infty} u_{1,n} \varphi_n = u_1.$$  

We have shown that $u$ is the unique strong solution of the system (1.1). Now, given that $u$ satisfies the required regularity, we are allowed to multiply the first identity in (3.10) by a test function $\varphi \in V_\gamma$ and integrate over $\Omega$ to get the variational identity (2.2). Finally the estimate (3.20) follows from (3.21), (3.23) and (3.25). The proof of the theorem is finished. \hfill \Box

We mention that existence, regularity and representation of solutions of the corresponding system (1.1) for the case $0 < \alpha < 1$ has been completely studied in [11] and the references therein.

Remark 1. We mention that we have the following additional regularity for the strong solution $u$ of the system (1.1). Let $\gamma = \frac{1}{\alpha}$ and recall that $V_{-\gamma} = (V_\gamma)^*$.

(a) In fact, $D_t^n u \in C([0, T]; V_{-\gamma})$. Indeed, using (3.24), (3.6), (3.15), (3.16) and (3.17), we get from (3.24) that

$$\|D_t^n u(\cdot, t)\|_{V_{-\gamma}}^2 \leq \sum_{n=1}^{\infty} |u_{0,n} \lambda_n^{-\gamma} \lambda_n E_{\alpha,1}(-\lambda_n t^\alpha)|^2 + \sum_{n=1}^{\infty} |u_{1,n} \lambda_n^{-\gamma} \lambda_n t E_{\alpha,2}(-\lambda_n t^\alpha)|^2$$

$$+ \sum_{n=1}^{\infty} \left| \int_0^t f_n(\tau) \lambda_n^{-\gamma} \lambda_n (t - \tau)^{\alpha-1} E_{\alpha,\alpha-1}(-\lambda_n (t - \tau)^\alpha) \, d\tau \right|^2$$

$$+ C\|f\|_{L^\infty(0,T); L^2(\Omega)}^2$$

$$\leq C_1 \lambda_1^{1-2\gamma} \|u_0\|_{V_\gamma}^2 + C_2 t^{\alpha\gamma + 1 - \alpha} \|u_1\|_{L^2(\Omega)}^2$$

$$+ C_3 \|f\|_{L^\infty(0,T); L^2(\Omega)}^2.$$

(b) Since $D_t^n u = -A_B u + f \in C((0, T], L^2(\Omega))$, we have that $u \in C([0, T]; D(A_B))$.

(c) If $u_0 = u_1 = 0$, then $D_t^n u, A_B u$ and $f$ belong to $C([0, T], L^2(\Omega))$, and hence, $u \in C([0, T]; D(A_B))$.

Proceeding exactly as in the proof of Theorem 3.4 we get the following variant of strong solutions with weak initial data that we will call weak-strong solutions.
Theorem 3.5. Let $1 < \alpha < 2$, $\gamma := \frac{1}{\alpha}$, $u_0 \in L^2(\Omega)$, $u_1 \in V_{-\gamma}$ and $f \in W^{1,\infty}(0,T; V_{-\gamma})$. Then the system (1.1) has a unique weak-strong solution $u$ given by (3.19), that is,
\begin{align}
\begin{cases}
u \in C([0,T]; L^2(\Omega)) \cap C^1([0,T]; V_{-\gamma}),
\mathcal{D}_t^\alpha \nu \in C((0,T); V_{-\gamma}),
\end{cases}
\end{align}
and $u(\cdot,0) = u_0$, $\partial_t u(\cdot,0) = u_1$. Moreover, there exist constants $C_1, C_2, C_3 > 0$ such that for all $t \in (0,T)$,
\begin{align}
\begin{cases}
\|u(\cdot,t)\|_{L^2(\Omega)}^2 \leq C_1 \left(\|u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{V_{-\gamma}}^2 + \|f\|_{L^\infty((0,T); V_{-\gamma})}^2\right),
\|\partial_t u(\cdot,t)\|_{V_{-\gamma}}^2 \leq C_2 \left(\|u_0\|_{L^2(\Omega)}^2 + \|u_1\|_{V_{-\gamma}}^2 + t^{2(\alpha-1)}\|f(\cdot,0)\|_{V_{-\gamma}}^2 + \|f\|_{L^\infty((0,T); V_{-\gamma})}^2\right),
\|\mathcal{D}_t^\alpha u(\cdot,t)\|_{V_{-\gamma}}^2 \leq C_3 \left(t^{-2\alpha}\|u_0\|_{L^2(\Omega)}^2 + t^{2(1-\alpha)}\|u_1\|_{V_{-\gamma}}^2 + \|f\|_{L^\infty((0,T); V_{-\gamma})}^2\right).
\end{cases}
\end{align}

Next, we show the existence, uniqueness and regularity of solutions to the backward system (2.3).

Theorem 3.6. Let $1 < \alpha < 2$, $\gamma := \frac{1}{\alpha}$, $v_0 \in V_{\gamma}$ and $v_1 \in L^2(\Omega)$. Then the system (2.3) has a unique strong solution $v$ given by
\begin{align}
v(\cdot,t) = \sum_{n=1}^\infty (v_0, \varphi_n)(T-t)^{-\alpha+1}(-\lambda_n(T-t)')^{\alpha+1} \varphi_n,
\end{align}
and there exist two constants $C_1, C_2 > 0$ such that
\begin{align}
\begin{cases}
\|I_{t,T}^\alpha v(\cdot,t)\|_{V_{\gamma}}^2 + \|D_{t,T}^{\alpha-1} v(\cdot,t)\|_{L^2(\Omega)}^2 \leq C_1 \left(\|v_0\|_{V_{\gamma}}^2 + \|v_1\|_{L^2(\Omega)}^2\right),
\|D_{t,T}^\alpha v(\cdot,t)\|_{L^2(\Omega)}^2 \leq C_2 (T-t)^{-2} \left(\|v_0\|_{V_{\gamma}}^2 + \|v_1\|_{L^2(\Omega)}^2\right),
\end{cases}
\end{align}
Moreover, the unique strong solution $v$ can be analytically extended to the half-plane $\Sigma_T := \{z \in \mathbb{C} : \text{Re}(z) < T\}$.

Proof. Let $1 < \alpha < 2$, $\gamma := \frac{1}{\alpha}$, $v_0 \in V_{\gamma}$ and $v_1 \in L^2(\Omega)$. First we show uniqueness of solutions. Indeed, let $v$ be a solution of the system (2.3) with $v_0 = v_1 = 0$. Taking the inner product of (2.3) with $\varphi_n$ and setting $v_n(t) = (v(t), \varphi_n)$, we get that (given that $A_B$ is self-adjoint)
\begin{align}
D_{t,T}^\alpha v_n(t) = -\lambda_n v_n(t), \text{ for a.e. } t \in (0,T).
\end{align}
Since $I_{t,T}^\alpha v \in C([0,T]; V_{\gamma}) \subset C([0,T]; L^2(\Omega))$, we have that the function $I_{t,T}^{-\alpha} v_n = (I_{t,T}^{-\alpha} v, \varphi_n) \in C([0,T])$ and
\begin{align}
|I_{t,T}^{-\alpha} v_n(t)|^2 \leq \sum_{n=1}^\infty |I_{t,T}^{\alpha} v_n|^2 \leq \|I_{t,T}^{\alpha} v\|_{L^2(\Omega)}^2 \to 0 \text{ as } t \to T.
\end{align}
This implies that
\begin{align}
I_{t,T}^{-\alpha} v_n(T) = 0.
\end{align}
Similarly we get that
\begin{align}
D_{t,T}^{\alpha-1} v_n(T) = 0.
\end{align}
Since the fractional differential equation (3.30) with the initial conditions (3.31)-
(3.32) has zero as its unique solution (see e.g. [3]), it follows that \( v_n(t) = 0 \),
n = 1, 2, \ldots. Since \( \varphi_n \) is a complete orthonormal system in \( L^2(\Omega) \), we have that
\( v = 0 \) in \( \Omega \times (0, T) \) and the proof of the uniqueness is complete.

Next, we show the existence of the solution. Let once more \( u_{0,n} := (u_0, \varphi_n) \),
u_{1,n} := (u_1, \varphi_n), 1 \leq n \leq N \) where \( N \in \mathbb{N} \), and set
\[
v_N(x, t) := \sum_{n=1}^{N} v_{0,n}(T-t)^{\alpha-1} E_{\alpha,\alpha-1}(-\lambda_n(T-t)^{\alpha}) \varphi_n(x) + \sum_{n=1}^{N} v_{1,n}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \varphi_n(x).
\]

(a) Let \( \nu \) be given by (3.28). We claim that \( I_{t,T}^{2-\alpha} \nu \in C([0, T]; V_\gamma) \). Integrating
termwise, we have that
\[
I_{t,T}^{2-\alpha} v_N(t) = \sum_{n=1}^{N} v_{0,n} E_{\alpha,1}(-\lambda_n(T-t)^{\alpha}) \varphi_n + \sum_{n=1}^{N} v_{1,n} \int_{t}^{T} E_{\alpha,1}(-\lambda_n(T-\tau)^{\alpha}) \, d\tau \varphi_n, \tag{3.33}
\]
in \( V_\gamma \). Using (3.6) and the estimates in Lemma 3.3, we have that for every
\( t \in [0, T] \) and \( M, N \in \mathbb{N} \) with \( M > N \),
\[
\| I_{t,T}^{2-\alpha} v_N(t) - I_{t,T}^{2-\alpha} v_M(t) \|_{V_\gamma}^2 = 2 \sum_{n=N+1}^{M} \lambda_n^\gamma |v_{0,n} E_{\alpha,1}(-\lambda_n(T-t)^{\alpha})|^2
+ 2 \sum_{n=N+1}^{M} |v_{1,n} \lambda_n^\gamma \int_{t}^{T} E_{\alpha,1}(-\lambda_n(T-\tau)^{\alpha}) \, d\tau|^2
= 2 \sum_{n=N+1}^{M} |\lambda_n^\gamma v_{0,n} E_{\alpha,1}(-\lambda_n(T-t)^{\alpha})|^2
+ 2 \sum_{n=N+1}^{M} |v_{1,n} \lambda_n^\gamma (T-t) E_{\alpha,2}(-\lambda_n(T-t)^{\alpha})|^2
\leq C \left( \sum_{n=N+1}^{M} |\lambda_n^\gamma v_{0,n}|^2 + \sum_{n=N+1}^{M} |v_{1,n}|^2 \right) \rightarrow 0 \text{ as } N, M \rightarrow \infty.
\]
We have shown that the series
\[
\sum_{n=1}^{\infty} v_{0,n} E_{\alpha,1}(-\lambda_n(T-t)^{\alpha}) \varphi_n + \sum_{n=1}^{\infty} v_{1,n} \int_{t}^{T} E_{\alpha,1}(-\lambda_n(T-\tau)^{\alpha}) \, d\tau \varphi_n \rightarrow I_{t,T}^{2-\alpha} \nu(\cdot, t) \text{ in } V_\gamma,
\]
and that the convergence is uniform in \( t \in [0, T] \). Hence, \( I_{t,T}^{2-\alpha} \nu \in C([0, T]; V_\gamma) \).

Using (3.6) and Lemma 3.3 again, we get that
Finally, since \( E \) the two functions

\[(b) \text{ We show that } D_T^{a-1}v \in C([0, T]; L^2(\Omega)). \text{ Integrating termwise, we also get that}
\]

\[
D_T^{a-1}v_N(t) = \sum_{n=1}^{N} v_{\alpha,n}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \phi_n
\]

\[
+ \sum_{n=1}^{N} v_1, n E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \phi_n.
\]

Proceeding as in part (a), we get that

\[
\sum_{n=1}^{\infty} v_{\alpha,n}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \phi_n
\]

\[
+ \sum_{n=1}^{\infty} v_1, n E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \phi_n \rightarrow D_T^{a-1}v(t) \text{ in } L^2(\Omega),
\]

and the convergence is uniform in \( t \in [0, T] \). Using (3.6) and Lemma 3.3 we get the following estimates: there exist constants \( c_1, c_2 > 0 \) such that

\[
\|D_T^{a-1}v(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2 \sum_{n=1}^{\infty} |\lambda_n^\gamma v_{0,n}|^2 |\lambda_n^{1-\gamma}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})|^2
\]

\[
+ 2 \sum_{n=1}^{\infty} |v_1,n|^2 |E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})|^2
\]

\[
\leq c_1 \|v_0\|_{V_{\gamma}}^2 + c_2 \|v_1\|_{L^2(\Omega)},
\]

and the proof is complete.

\[(c) \text{ We prove that } D_T^{a}v \in C([0, T]; L^2(\Omega)). \text{ This follows as in the proof of part (a) or part (b) with the difference that here the convergence is only uniform on any compact subset of } [0, T]. \text{ Since } D_T^\alpha v(\cdot, t) = -A_B v(\cdot, t), \text{ we have that}
\]

\[
\|D_T^{a}v(\cdot, t)\|_{L^2(\Omega)}^2 \leq 2 \sum_{n=1}^{\infty} |\lambda_n^\gamma v_{0,n}|^2 |\lambda_n^{1-\gamma}(T-t)^{\alpha-2} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})|^2
\]

\[
+ 2 \sum_{n=1}^{\infty} |v_1,n|^2 |\lambda_n(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})|^2
\]

\[
\leq C(T-t)^{-2} \left( \|v_0\|^2_{V_{\gamma}} + \|v_1\|^2_{L^2(\Omega)} \right),
\]

\[(d) \text{ It follows from (3.33) and (3.35) that}
\]

\[
I_T^{\alpha}v(T) = \sum_{n=1}^{\infty} v_{0,n} = v_0 \text{ and } D_T^{a-1}v(T) = \sum_{n=1}^{\infty} v_{1,n} \phi_n = v_1.
\]

Finally, since \( E_{\alpha,\alpha}(-\lambda_n z) \) and \( E_{\alpha,\alpha}(-\lambda_n z) \) are entire functions, it follows that the two functions

\[(T-t)^{\alpha-2} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha}) \text{ and } (T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^{\alpha})
\]
can be analytically extended to the half-plane $\Sigma_T$. This implies that the functions
\[
\sum_{n=1}^{N} v_{0,n}(T-t)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(T-t)^\alpha) \varphi_n
\]
and
\[
\sum_{n=1}^{N} v_{1,n}(T-t)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha) \varphi_n
\]
are analytic in $\Sigma_T$. Let $\delta > 0$ be fixed but otherwise arbitrary. Let $z \in \mathbb{C}$ satisfy $\text{Re}(z) \leq T - \delta$. Then using Lemma 3.3, we get that
\[
\left\| \sum_{n=N+1}^{\infty} v_{0,n}(T-z)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(T-z)^\alpha) \varphi_n \right\|_{L^2(\Omega)}^2 \\
+ \left\| \sum_{n=N+1}^{\infty} v_{1,n}(T-z)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-z)^\alpha) \varphi_n \right\|_{L^2(\Omega)}^2
\]
\[
\leq C_1 \sum_{n=N+1}^{\infty} |v_{0,n}|^2 |T-z|^{2(\alpha-2)} \left( \frac{1}{1+\lambda_n|T-z|^\alpha} \right)^2 \\
+ C_2 \sum_{n=N+1}^{\infty} |v_{1,n}|^2 |T-z|^{2(\alpha-1)} \left( \frac{1}{1+\lambda_n|T-z|^\alpha} \right)^2
\]
\[
\leq C \delta^{2(\alpha-2)} \sum_{n=N+1}^{\infty} |v_{0,n}|^2 + C |T-z|^{2(\alpha-1)} \sum_{n=N+1}^{\infty} |v_{0,n}|^2 \rightarrow 0 \text{ as } N \rightarrow \infty.
\]
We have shown that
\[
v(\cdot, z) := \sum_{n=1}^{\infty} (v_{0,n}(T-z)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(T-z)^\alpha) \varphi_n \\
+ \sum_{n=1}^{\infty} (v_{1,n}(T-z)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(T-z)^\alpha) \varphi_n
\]
is uniformly convergent in any compact subset of $\Sigma_T$. Hence, $v$ is also analytic in $\Sigma_T$. The proof of the theorem is finished. \hfill \square

**Remark 2.** We notice that the solution $v$ of the backward system (2.3) satisfies the following additional regularity: there exists a constant $C > 0$ such that
\[
\|v(\cdot, t)\|_{L^2(\Omega)}^2 \leq C \left( (T-t)^{2(\alpha-2)} \|v_0\|_{V_\gamma}^2 + (T-t)^{2(\alpha-1)} \|v_1\|_{L^2(\Omega)}^2 \right). \tag{3.38}
\]
Using (3.38) we also get that $v \in C([0, T); L^2(\Omega)) \cap L^1((0, T); L^2(\Omega))$.

Here also, we have the following weak-strong solutions to the backward system whose proof follows the lines of the proof of Theorem 3.6.

**Theorem 3.7.** Let $1 < \alpha < 2$, $\gamma := \frac{1}{\alpha}$, $v_0 \in L^2(\Omega)$ and $v_1 \in V_{-\gamma}$. Then the system (2.3) has a unique weak-strong solution $v$ given by (3.28) in the sense that, $I_{t,T}^{\alpha-2} v \in C([0, T]; L^2(\Omega))$, $D_{t,T}^{\alpha-1} v \in C([0, T]; V_{-\gamma})$, $D_{t,T}^{\alpha} v \in C([0, T); V_{-\gamma})$. 

\[
\sum_{n=1}^{N} v_{0,n}(T-t)^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n(T-t)^\alpha) \varphi_n
\]
and
and $I_{r,T}^{2-\alpha}v(\cdot, t) = v_0$, $D_{r,T}^{\alpha-1}v(\cdot, t) = v_1$. Moreover, there exist two constants $C_1, C_2 > 0$ such that
\[
\begin{align*}
&
norm{I_{r,T}^{2-\alpha}v(\cdot, t)}^2_{L^2(\Omega)} + \norm{D_{r,T}^{\alpha-1}v(\cdot, t)}^2_{V_{\gamma}} \leq C_1 \left(\norm{v_0}^2_{L^2(\Omega)} + \norm{v_1}^2_{V_{\gamma}}\right), \quad t \in [0, T], \\
&\norm{D_{r,T}^\alpha v(\cdot, t)}^2_{V_{\gamma}} \leq C_2 (T-t)^{-2} \left(\norm{v_0}^2_{L^2(\Omega)} + \norm{v_1}^2_{V_{\gamma}}\right), \quad t \in [0, T].
\end{align*}
\]

(3.39)

In addition, the unique weak-strong solution $v$ can be analytically extended to the half-plane $\Sigma_T := \{z \in \mathbb{C} : \text{Re}(z) < T\}$.

4. Proof of the main results. In this section we give the proof of the main results stated in Section 2. Let $\gamma = \frac{1}{\alpha}$. We recall that the system (1.1) is approximately controllable if, for every $\varepsilon > 0$ and $(u_0, u_1), (w_0, w_1) \in V_\gamma \times L^2(\Omega)$, there exists a control function $f \in L^2(\omega \times (0, T))$ such that the corresponding solution $u$ of the system (1.1) with initial data $(u_0, u_1)$ satisfies
\[
\norm{u(\cdot, T), \partial_t u(\cdot, T) - (w_0, w_1)}_{V_\gamma \times L^2(\Omega)} \leq \varepsilon.
\]

(4.1)

It is sufficient to consider the case $(u_0, u_1) = (0, 0)$ and we assume this throughout the remainder of this section.

Now we are ready to give the proof of the first main result.

Proof of Proposition 2.1. Let $1 < \alpha < 2$, $\gamma := \frac{1}{\alpha}$, $v_0 \in V_\gamma$, $v_1 \in L^2(\Omega)$ and let $\omega \subset \Omega$ be an arbitrary open set. Let $v$ be the unique strong solution to the system (2.3) and assume that $v = 0$ in $\omega \times (0, T)$. Since $v = 0$ in $\omega \times (0, T)$ and $v : [0, T) \to L^2(\Omega)$ can be analytically extended to the half-plane $\Sigma_T$, it follows that
\[
v(x, t) = \sum_{n=1}^{\infty} v_{0,n}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n(T-t)^\alpha)\varphi_n(x) \\
+ \sum_{n=1}^{\infty} v_{1,n}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n(T-t)^\alpha)\varphi_n(x)
\]
\[
= 0, \quad x \in \omega, \quad t \in (-\infty, T).
\]

(4.2)

Let $\{\lambda_k\}_{k \in \mathbb{N}}$ be the set of all eigenvalues of the operator $A_B$. Let $\{\psi_{k_j}\}_{1 \leq j \leq m_k}$ be an orthonormal basis for Ker$(\lambda_k - A_B)$. Then (4.2) can be rewritten as
\[
v(x, t) = \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} v_{0,k_j}\psi_{k_j}(x) \right) (T-t)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_k(T-t)^\alpha) \\
+ \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} v_{1,k_j}\psi_{k_j}(x) \right) (T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-t)^\alpha)
\]
\[
= 0, \quad x \in \omega, \quad t \in (-\infty, T).
\]

(4.3)

Let $z \in \mathbb{C}$ with $\text{Re}(z) := \eta > 0$ and let $N \in \mathbb{N}$. Since the system $\{\psi_{k_j}\}$, for $1 \leq j \leq m_k$, $1 \leq k \leq N$ is orthonormal, we have that
\[
\left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} v_{0,k_j}\psi_{k_j}(x) \right) e^{z(T-t)}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_k(T-t)^\alpha) \right\|_{L^2(\Omega)}^2
\]
By the Lebesgue dominated convergence theorem, we have that
\[ \leq \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} |v_{0,k_j}|^2 \right) e^{2\eta(t-T)}(T-t)^{(\alpha-2)}E_{\alpha,\alpha-1}(-\lambda_k(T-t)^\alpha)^2 \]
\[ \leq C e^{2\eta(t-T)}(T-t)^{2(\alpha-2)}\|v_0\|^2_{V_\gamma}, \]
and
\[ \left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} v_{1,k_j} \psi_{k_j}(x) \right) e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-t)^\alpha) \right\|_{L^2(\Omega)}^2 \]
\[ \leq C e^{2\eta(t-T)}(T-t)^{2(\alpha-1)}\|v_1\|^2_{L^2(\Omega)}. \]

Letting
\[ w_N(\cdot, t) := \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} v_{0,k_j} \psi_{k_j}(x) \right) e^{z(t-T)}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_k(T-t)^\alpha) \]
\[ + \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} v_{1,k_j} \psi_{k_j}(x) \right) e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-t)^\alpha), \]
we have shown that
\[ \|w_N(\cdot, t)\|_{L^2(\Omega)} \]
\[ \leq \left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} v_{0,k_j} \psi_{k_j}(x) \right) e^{z(t-T)}(T-t)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_k(T-t)^\alpha) \right\|_{L^2(\Omega)} \]
\[ + \left\| \sum_{k=1}^{N} \left( \sum_{j=1}^{m_k} v_{1,k_j} \psi_{k_j}(x) \right) e^{z(t-T)}(T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-t)^\alpha) \right\|_{L^2(\Omega)} \]
\[ \leq C e^{\eta(t-T)} \left[ (T-t)^{\alpha-2}\|v_0\|_{V_\gamma} + (T-t)^{\alpha-1}\|v_1\|_{L^2(\Omega)} \right]. \quad (4.4) \]

Since 1 < \alpha < 2, we have that the right hand side of (4.4) is integrable over \( t \in (-\infty, T) \). More precisely, we have that
\[ \int_{-\infty}^{T} e^{\eta(t-T)} \left[ (T-t)^{\alpha-2}\|v_0\|_{V_\gamma} + (T-t)^{\alpha-1}\|v_1\|_{L^2(\Omega)} \right] \, dt \]
\[ = \int_{0}^{\infty} e^{-\tau} \frac{\tau^{\alpha-2}}{\eta^{\alpha-1}} \, d\tau\|v_0\|_{V_\gamma} + \int_{0}^{\infty} e^{-\tau} \frac{\tau^{\alpha-1}}{\eta^\alpha} \, d\tau\|v_1\|_{L^2(\Omega)} \]
\[ = \frac{\Gamma(\alpha-1)}{\eta^{\alpha-1}} \|v_0\|_{V_\gamma} + \frac{\Gamma(\alpha)}{\eta^{\alpha}} \|v_1\|_{L^2(\Omega)}. \]

By the Lebesgue dominated convergence theorem, we have that
\[ \int_{-\infty}^{T} e^{z(t-T)} \left[ \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} v_{0,k_j} \psi_{k_j}(x) \right) (T-t)^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_k(T-t)^\alpha) \right] \, dt \]
\[ + \sum_{k=1}^{\infty} \left( \sum_{j=1}^{m_k} v_{1,k_j} \psi_{k_j}(x) \right) (T-t)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_k(T-t)^\alpha) \, dt \]
\[
\begin{align*}
&= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} v_{0,k,j} \psi_{k,j}(x) \left( \int_{-\infty}^{T} e^{z(t-T)}(T-t)^{\alpha-2} E_{\alpha-1}(-\lambda_k(T-t)^\alpha)dt \right) \\
&+ \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} v_{1,k,j} \psi_{k,j}(x) \left( \int_{-\infty}^{T} e^{z(t-T)}(T-t)^{\alpha-1} E_{\alpha}(-\lambda_k(T-t)^\alpha)dt \right) \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{z v_{0,k,j}}{z^\alpha + \lambda_k} \psi_{k,j}(x) + \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \frac{v_{1,k,j}}{z^\alpha + \lambda_k} \psi_{k,j}(x), \ x \in \Omega, \ Re(z) > 0 \\
&= \sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \left( \frac{z v_{0,k,j}}{z^\alpha + \lambda_k} + \frac{v_{1,k,j}}{z^\alpha + \lambda_k} \right) \psi_{k,j}(x), \ x \in \Omega, \ Re(z) > 0. \tag{4.5}
\end{align*}
\]

To arrive at (4.5), we have used the fact that
\[
\int_{-\infty}^{T} e^{z(t-T)}(T-t)^{\alpha-2} E_{\alpha-1}(-\lambda_k(T-t)^\alpha)dt = \int_{0}^{\infty} e^{-z\tau\alpha-2} E_{\alpha-1}(-\lambda_k\tau^\alpha) \ d\tau = \frac{z}{z^\alpha + \lambda_k}
\]
and
\[
\int_{-\infty}^{T} e^{z(t-T)}(T-t)^{\alpha-1} E_{\alpha}(-\lambda_k(T-t)^\alpha)dt = \int_{0}^{\infty} e^{-z\tau\alpha-1} E_{\alpha}(-\lambda_k\tau^\alpha) \ d\tau = \frac{1}{z^\alpha + \lambda_k}.
\]

These relations follow from a simple change of variable and (3.7). It follows from (4.3) and (4.5) that
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \left( \frac{z v_{0,k,j}}{z^\alpha + \lambda_k} + \frac{v_{1,k,j}}{z^\alpha + \lambda_k} \right) \psi_{k,j}(x) = 0, \ x \in \omega, \ Re(z) > 0.
\]

Letting \( \eta := z^\alpha \), we have shown that
\[
\sum_{k=1}^{\infty} \sum_{j=1}^{m_k} \left( \frac{\eta \xi v_{0,k,j}}{\eta + \lambda_k} + \frac{\xi v_{1,k,j}}{\eta + \lambda_k} \right) \psi_{k,j}(x) = 0, \ x \in \omega, \ Re(\eta) > 0. \tag{4.6}
\]

Using the analytic continuation in \( \eta \), we have that the identity (4.6) holds for every \( \eta \in \mathbb{C} \setminus \{-\lambda_k\}_{k \in \mathbb{N}} \). Taking a suitable small circle about \(-\lambda_l\) and not including \{-\lambda_k\}_{k \neq l} and integrating (4.6) on that circle we get that
\[
w_l := \sum_{j=1}^{m_l} \left[ (-\lambda_l)^{\frac{1}{\alpha}} v_{0,l,j} + v_{1,l,j} \right] \psi_{l,j}(x) = 0, \ x \in \omega,
\]
where
\[
(-\lambda_l)^{\frac{1}{\alpha}} = e^{\frac{1}{\alpha} \log(-\lambda_l)} = e^{\frac{1}{\alpha} \log(\lambda_l) + i \pi} = \lambda_l^{\frac{1}{\alpha}} \left[ \cos \left( \frac{\pi}{\alpha} \right) + i \sin \left( \frac{\pi}{\alpha} \right) \right].
\]

Since \((A_B - \lambda_l)w_l = 0 \in \Omega\), \( w_l = 0 \) in \( \omega \), and by assumption \( A_B \) has the unique controllability property, it follows that \( w_l = 0 \) in \( \Omega \) for every \( l \). Since \( \{\psi_{l,j}\}_{1 \leq j \leq m_k} \) is linearly independent in \( L^2(\Omega) \), we get that \((-\lambda_l)^{1/\alpha} v_{0,j} + v_{1,j} = 0\) for \( 1 \leq j \leq m_k \),
\[ 0 = (-\lambda)^{1/\alpha} v_0 + v_1 = \lambda^{\frac{1}{2}} \left[ \cos \left( \frac{\pi}{\alpha} \right) + i \sin \left( \frac{\pi}{\alpha} \right) \right] v_0 + v_1 = \lambda^{\frac{1}{2}} \cos \left( \frac{\pi}{\alpha} \right) v_0 + v_1 + i \lambda^{\frac{1}{2}} \sin \left( \frac{\pi}{\alpha} \right) v_0 = 0. \] (4.7)

It follows from (4.7) that \( v_0 = 0 = v_1 \) and hence, \( v = 0 \) in \( \Omega \times (0, T) \). The proof is finished.

**Proof of Theorem 2.3.** Let \( u \) be the unique strong solution of (1.1) with \( u_0 = u_1 = 0 \) and \( v \) the unique strong solution of (2.3) with \( v_0 \in V_\gamma \) and \( v_1 \in L^2(\Omega) \). First, it follows from Remark 1 that both \( D_t^\alpha u \) and \( A_B u \) belong to \( L^\infty((0, T); L^2(\Omega)) \). Second, it follows from Remark 2 that \( v \in L^1((0, T); L^2(\Omega)) \). Therefore, integrating by parts, we get that (using (3.4) and the fact that \( (A_B u, v) = (u, A_B u) \)),

\[
0 = \int_0^T \int_\Omega (D_t^\alpha u + A_B u - f) v \, dx dt \\
= \int_0^T \int_\Omega v D_t^\alpha u \, dx dt + \int_0^T \int_\Omega v A_B u \, dx dt - \int_0^T \int_\Omega f v \, dx dt \\
= \int_0^T \int_\Omega u D_t^\alpha v \, dx dt + \int_\Omega \left[ I_{t,T}^{2-\alpha} v(x, T) \partial_t u(x, T) + D_t^{\alpha - 1} v(x, T) u(x, T) \right] dx \\
+ \int_0^T \int_\Omega u A_B v \, dx dt - \int_0^T \int_\Omega f v \, dx dt \\
= \int_0^T \int_\Omega (D_t^\alpha v + A_B v) u \, dx dt \\
+ \int_\Omega \left[ I_{t,T}^{2-\alpha} v(x, T) \partial_t u(x, T) + D_t^{\alpha - 1} v(x, T) u(x, T) \right] \, dx - \int_0^T \int_\Omega f v \, dx dt \\
= \int_\Omega \left[ I_{t,T}^{2-\alpha} v(x, T) \partial_t u(x, T) + D_t^{\alpha - 1} v(x, T) u(x, T) \right] \, dx - \int_0^T \int_\Omega f v \, dx dt. \quad (4.8)
\]

Thus, we have shown that

\[
\int_\Omega [\partial_t u(x, T)v_0 + u(x, T)v_1] \, dx = \int_0^T \int_\Omega f v \, dx dt. \quad (4.9)
\]

To prove that the set \( \{ (u(\cdot, T), \partial_t u(\cdot, T)) : f \in C_0^\infty(\omega \times (0, T)) \} \) is dense in \( V_\gamma \times L^2(\Omega) \), we have to show that if \((v_0, v_1) \in V_\gamma \times L^2(\Omega)\) is such that

\[
\int_\Omega [\partial_t u(x, T)v_0 + u(x, T)v_1] \, dx = 0 \quad (4.10)
\]

for any \( f \in C_0^\infty(\omega \times (0, T)) \), then \( v_0 = v_1 = 0 \). Indeed, it follows from (4.9) and (4.10) that

\[
\int_0^T \int_\omega f v \, dx dt = 0
\]

for any \( f \in C_0^\infty(\omega \times (0, T)) \). By the fundamental lemma of the calculus of variations, we have that

\[
v(x, t) = 0, \quad (x, t) \in \omega \times (0, T).
\]

It follows from Proposition 2.1 that

\[
v(x, t) = 0, \quad (x, t) \in \Omega \times (0, T).
\]
Since the solution of (2.3) is unique, it follows that $v_0 = v_1 = 0$ on $\Omega$. The proof of the theorem is finished.

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E-mail address: valentin.keyantuo@upr.edu
E-mail address: mahamadi.warma@upr.edu, mjwarma@gmail.com