STABILITY OF EXTREMAL KÄHLER MANIFOLDS

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Dedicated to Professor Shoshichi Kobayashi on his seventieth birthday

1. Introduction

In Donaldson’s study [10] of asymptotic stability for polarized algebraic manifolds 
\((M, L)\), critical metrics originally defined by Zhang [39] (see also [22]) are referred to 
as balanced metrics and play a central role when the polarized algebraic manifolds admit 
Kähler metrics of constant scalar curvature. Let \( T \cong (\mathbb{C}^*)^k \) be an algebraic torus in 
the identity component \( \text{Aut}^0(M) \) of the group of holomorphic automorphisms of \( M \). In 
this paper, we define the concept of critical metrics relative to \( T \), and as an application, 
choosing a suitable \( T \), we shall show that a result in [26] on the asymptotic approximation 
of critical metrics (see [10], [39]) can be generalized to the case where \((M, L)\) admits an 
extremal Kähler metric in the polarization class. Then in our forthcoming paper [27], we 
shall show that a slight modification of the concept of stability (see Theorem A below) 
allows us to obtain the asymptotic stability of extremal Kähler manifolds even when the 
obstruction as in [26] does not vanish. In particular, by an argument similar to [10], an 
extremal Kähler metric in a fixed integral Kähler class on a projective algebraic manifold 
\( M \) will be shown to be unique\(^1\) up to the action of the group \( \text{Aut}^0(M) \).

2. Statement of results

Throughout this paper, we fix once for all an ample holomorphic line bundle \( L \) on a 
connected projective algebraic manifold \( M \). Let \( H \) be the maximal connected linear algebraic 
subgroup of \( \text{Aut}^0(M) \), so that \( \text{Aut}^0(M)/H \) is an abelian variety. The corresponding 
Lie subalgebra of \( H^0(M, \mathcal{O}(T^{1,0}M)) \) will be denoted by \( \mathfrak{h} \). For the complete linear system 
\( |L^m|, m \gg 1 \), we consider the Kodaira embedding 
\[
\Phi_m = \Phi_{|L^m|} : M \hookrightarrow \mathbb{P}^*(V_m), \quad m \gg 1,
\]

where \( \mathbb{P}^*(V_m) \) denotes the set of all hyperplanes through the origin in \( V_m := H^0(M, \mathcal{O}(L^m)) \). Put \( N_m := \dim V_m - 1 \). Let \( n \) and \( d \) be respectively the dimension of \( M \) and the degree of 
the image \( M_m := \Phi_m(M) \) in the projective space \( \mathbb{P}^*(V_m) \). Put \( W_m = \{\text{Sym}^d(V_m)\}^\otimes n+1 \). 
Then to the image \( M_m \) of \( M \), we can associate a nonzero element \( \hat{M}_m \) in \( W_m^* \) such that the 
corresponding element \( [\hat{M}_m] \) in \( \mathbb{P}^*(W_m) \) is the Chow point associated to the irreducible

\(^1\)For this uniqueness, we choose \( \mathbb{Z}^d \) (cf. Section 2) as the algebraic torus \( T \).

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reduced algebraic cycle $M_m$ on $\mathbb{P}^s(V_m)$. Replacing $L$ by some positive integral multiple of $L$ if necessary, we fix an $H$-linearization of $L$, i.e., a lift to $L$ of the $H$-action on $M$ such that $H$ acts on $L$ as bundle isomorphisms covering the $H$-action on $M$. For an algebraic torus $T$ in $H$, this naturally induces a $T$-action on $V_m$ for each $m$. Now for each character $\chi \in \text{Hom}(T, \mathbb{C}^*)$, we set

$$V(\chi) := \{ s \in V_m : t \cdot s = \chi(t) s \text{ for all } t \in T \}.$$ 

Then we have mutually distinct characters $\chi_1, \chi_2, \ldots, \chi_{\nu_m} \in \text{Hom}(T, \mathbb{C}^*)$ such that the vector space $V_m = H^0(M, \mathcal{O}(L^m))$ is uniquely written as a direct sum

$$V_m = \bigoplus_{k=1}^{\nu_m} V(\chi_k).$$

Put $G_m := \Pi_{k=1}^{\nu_m} \text{SL}(V(\chi_k))$, and the associated Lie subalgebra of $\text{sl}(V_m)$ will be denoted by $\mathfrak{g}_m$. More precisely, $G_m$ and $\mathfrak{g}_m$ possibly depend on the choice of the algebraic torus $T$, and if necessary, we denote these by $G_m(T)$ and $\mathfrak{g}_m(T)$, respectively. The $T$-action on $V_m$ is, more precisely, a right action, while we regard the $G_m$-action on $V_m$ as a left action. Since $T$ is Abelian, this $T$-action on $V_m$ can be regarded also as a left action.

The group $G_m$ acts diagonally on $V_m$ in such a way that, for each $k$, the $k$-th factor $\text{SL}(V(\chi_k))$ of $G_m$ acts just on the $k$-th factor $V(\chi_k)$ of $V_m$. This induces a natural $G_m$-action on $W_m$ and also on $W_m^*$.

**Definition 2.2.** (a) The subvariety $M_m$ of $\mathbb{P}^s(V_m)$ is said to be *stable relative to $T$* or *semistable relative to $T$*, according as the orbit $G_m \cdot \hat{M}_m$ is closed in $W_m^*$ or the closure of $G_m \cdot \hat{M}_m$ in $W_m^*$ does not contain the origin of $W_m^*$.

(b) Let $\mathfrak{t}_c$ denote the Lie subalgebra of the maximal compact subgroup $T_c$ of $T$, and as a real Lie subalgebra of the complex Lie algebra $\mathfrak{t}$, we define $\mathfrak{t}_\mathbb{R} := \sqrt{-1} \mathfrak{t}_c$.

Take a Hermitian metric for $V_m$ such that $V(\chi_k) \perp V(\chi_\ell)$ if $k \neq \ell$. Put $N_m := \dim V_m - 1$ and $n_k := \dim V(\chi_k)$. We then set

$$l(k, i) := (i - 1) + \sum_{j=1}^{k-1} n_j, \quad i = 1, 2, \ldots, n_k; \quad k = 1, 2, \ldots, \nu_m,$$

where the right-hand side denotes $i - 1$ in the special case $k = 1$. Let $\| \| \|$ denote the Hermitian norm for $V_m$ induced by the Hermitian metric. Take a $\mathbb{C}$-basis $\{s_0, s_1, \ldots, s_{N_m} \}$ for $V_m$.

**Definition 2.3.** We say that $\{s_0, s_1, \ldots, s_{N_m} \}$ is an *admissible normal basis* for $V_m$ if there exist positive real constants $b_k, k = 1, 2, \ldots, \nu_m$, and a $\mathbb{C}$-basis $\{s_{k,i} ; i = 1, 2, \ldots, n_k \}$ for $V(\chi_k)$, with $\Sigma_{k=1}^{\nu_m} n_k b_k = N_m + 1$, such that

1. $s_{l(k,i)} = s_{k,i}, \quad i = 1, 2, \ldots, n_k; \quad k = 1, 2, \ldots, \nu_m$;
2. $s_l \perp s_{l'}$ if $l \neq l'$;
Then the real vector $b := (b_1, b_2, \ldots, b_{\nu_m})$ is called the *index* of the admissible normal basis \{s_0, s_1, \ldots, s_{N_m}\} for $V_m$.

We now specify a Hermitian metric on $V_m$. For the maximal compact subgroup $T_c$ of $T$ above, let $\mathcal{S}$ be the set (≠ ∅) of all $T_c$-invariant Kähler forms in the class $c_1(L)_{\mathbb{R}}$. Let $\omega \in \mathcal{S}$, and choose a Hermitian metric $h$ for $L$ such that $\omega = c_1(L; h)$. Define a Hermitian metric on $V_m$ by

\[(s, s')_{L^2} := \int_M (s, s')_{h_m} \omega^n, \quad s, s' \in V_m,\]

where $(s, s')_{h_m}$ denotes the function on $M$ obtained as the the pointwise inner product of $s, s'$ by the Hermitian metric $h_m$ on $L^m$. Now, let us consider the situation that $V_m$ has the Hermitian metric (2.4). Then

\[V(\chi_k) \perp V(\chi_\ell), \quad k \neq \ell,\]

and define a maximal compact subgroup $(G_m)_c$ of $G_m$ by $(G_m)_c := \prod_{k=1}^{\nu_m} SU(V(\chi_k))$. Again by this Hermitian metric $(\ , \ )_{L^2}$, let \{s_0, s_1, \ldots, s_{N_m}\} an admissible normal basis for $V_m$ of a given index $b$. Put

\[(2.5) \quad E_{\omega,b} := \sum_{i=0}^{N_m} |s_i|_{h_m}^2,\]

where $|s|_{h_m} := (s, s)_{h_m}$ for all $s \in V_m$. Then $E_{\omega,b}$ depends only on $\omega$ and $b$. Namely, once $\omega$ and $b$ are fixed, $E_{\omega,b}$ is independent of the choice of an admissible normal basis for $V(\chi_k)$ of index $b$. Fix a positive integer $m$ such that $L^m$ is very ample.

**Definition 2.6.** An element $\omega$ in $\mathcal{S}$ is called a *critical metric relative to $T$*, if there exists an admissible normal basis \{s_0, s_1, \ldots, s_{N_m}\} for $V_m$ such that the associated function $E_{\omega,b}$ on $M$ is constant for the index $b$ of the admissible normal basis. This generalizes a *critical metric* of Zhang [39] (see also [5]) who treated the case $T = \{1\}$. If $\omega$ is a critical metric relative to $T$, then by integrating the equality (2.5) over $M$, we see that the constant $E_{\omega,b}$ is $(N_m + 1)/c_1(L)^n[M]$.

For the centralizer $Z_H(T)$ of $T$ in $H$, let $Z_H(T)^0$ be its identity component. For $m$ as above, the following generalization of a result in [39] is crucial to our study of stability:

**Theorem A.** The subvariety $M_m$ of $\mathbb{P}(V_m)$ is stable relative to $T$ if and only if there exists a critical metric $\omega \in \mathcal{S}$ relative to $T$. Moreover, for a fixed index $b$, a critical metric $\omega$ in $\mathcal{S}$ relative to $T$ with constant $E_{\omega,b}$ is unique up to the action of $Z_H(T)^0$.

We now fix a maximal compact connected subgroup $K$ of $H$. The corresponding Lie subalgebra of $\mathfrak{h}$ is denoted by $\mathfrak{t}$. Let $\mathcal{S}_K$ denote the set of all Kähler forms $\omega$ in the class $c_1(L)_{\mathbb{R}}$ such that the identity component of the group of the isometries of $(M, \omega)$ coincides
with $K$. Then $S_K \neq \emptyset$, and an extremal Kähler metric, if any, in the class $c_1(L)_{\mathbb{R}}$ is always in $H$-orbits of elements of $S_K$. For each $\omega \in S_K$, we write

$$\omega = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha,\beta} g_{\alpha\beta} dz^\alpha \wedge d\bar{z}^\beta$$

in terms of a system $(z^1, \ldots, z^n)$ of holomorphic local coordinates on $M$. Let $K_\omega$ be the space of all real-valued smooth functions $u$ on $M$ such that $\int_M u \omega^n = 0$ and that

$$\text{grad}_\omega^u := \frac{1}{\sqrt{-1}} \sum_{\alpha,\beta} g_{\bar{\beta}\alpha} \frac{\partial u}{\partial z^\beta} \frac{\partial}{\partial z^\alpha}$$

is a holomorphic vector field on $M$. Then $K_\omega$ forms a real Lie subalgebra of $\mathfrak{h}$ by the Poisson bracket for $(M, \omega)$. We then have the Lie algebra isomorphism

$$K_\omega \cong \mathfrak{t}, \quad u \leftrightarrow \text{grad}_\omega^u.$$ 

For the space $C^\infty(M)_{\mathbb{R}}$ of real-valued smooth functions on $M$, we consider the inner product defined by $(u_1, u_2)_\omega := \int_M u_1 u_2 \omega^n$ for $u_1, u_2 \in C^\infty(M)_{\mathbb{R}}$. Let $\text{pr} : C^\infty(M)_{\mathbb{R}} \to K_\omega$ be the orthogonal projection. Let $\mathfrak{z}$ be the center of $\mathfrak{t}$. Then the vector field

$$\mathcal{V} := \text{grad}_\omega^u \text{pr}(\sigma_\omega) \in \mathfrak{z}$$

is called the extremal Kähler vector field of $(M, \omega)$, where $\sigma_\omega$ denotes the scalar curvature of $\omega$. Then $\mathcal{V}$ is independent of the choice of $\omega$ in $S$, and satisfies $\exp(2\pi \gamma \mathcal{V}) = 1$ for some positive integer $\gamma$ (cf. [13], [32]). Next, since we have an $H$-linearization of $L$, there exists a natural inclusion $H \subset \text{GL}(V_m)$. By passing to the Lie algebras, we obtain

$$\mathfrak{h} \subset \mathfrak{gl}(V_m).$$

Take a Hermitian metric $h$ for $L$ such that the corresponding first Chern form $c_1(L; h)$ is $\omega$. As in [23], (1.4.1), the infinitesimal $\mathfrak{h}$-action on $L$ induces an infinitesimal $\mathfrak{h}$-action on the complexification $\mathcal{H}^C_m$ of the space of all Hermitian metrics $\mathcal{H}_m$ on the line bundle $L^m$. The Futaki-Morita character $F : \mathfrak{h} \to \mathbb{C}$ is given by

$$F(\mathcal{V}) := \frac{\sqrt{-1}}{2\pi} \int_M h^{-1}(\mathcal{V} h) \omega^n,$$

which is independent of the choice of $h$ (see for instance [15]). For the identity component $Z$ of the center of $K$, we consider its complexification $Z^C$ in $H$. Then the corresponding Lie algebra is just the complexification $\mathfrak{z}^C$ of $\mathfrak{z}$ above. We now consider the set $\Delta$ of all algebraic tori in $Z^C$. Let $T \in \Delta$. Put

$$q := 1/m.$$

For $\omega = c_1(L; h) \in S_K$, we consider the Hermitian metric (2.4) for $V_m$. We then choose an admissible normal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for $V_m$ of index $(1, 1, \ldots, 1)$. By the asymptotic
expansion of Tian-Zelditch (cf. [33], [38]; see also [4]) for \( m \gg 1 \), there exist real-valued smooth functions \( a_k(\omega) \), \( k = 1, 2, \ldots \), on \( M \) such that

\[
(2.7) \quad \frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|_{h_0}^{2m} = 1 + a_1(\omega)q + a_2(\omega)q^2 + \cdots .
\]

Then \( a_1(\omega) = \sigma_\omega / 2 \) by a result of Lu [20]. Let \( \mathcal{Y} \in \mathfrak{t}_\mathbb{R} \), and put \( g := \exp^C \mathcal{Y} \in T \), where the element \( \exp(\mathcal{Y}/2) \) in \( T \) is written as \( \exp^C \mathcal{Y} \) by abuse of terminology. Recall that the \( T \)-action on \( V_m \) is a right action, though it can be viewed also as a left action. Put \( h_g := h \cdot g \) for simplicity. Using the notation in Definition 2.3, we write

\[
Z(q, \omega; \mathcal{Y}) := \frac{n!}{m^n} \sum_{j=0}^{N_m} |s_j|_{h_0}^{2m} = g^* \left\{ \frac{n!}{m^n} \sum_{k=1}^{\nu_m} |\chi_k(\exp^C \mathcal{Y})|^{-2} \sum_{i=1}^{n_k} |s_{k,i}|_{h_0}^{2m} \right\}, \quad \mathcal{Y} \in \mathfrak{t}_\mathbb{R}.
\]

For extremal Kähler manifolds, the following generalization of [20] allows us to approximate arbitrarily some critical metrics relative to \( T \):

**Theorem B.** Let \( \omega_0 = c_1(L; h_0) \) be an extremal Kähler metric in the class \( c_1(L)_{\mathbb{R}} \) with extremal Kähler vector field \( \mathcal{V} \). Then for some \( T \in \Delta \), there exist a sequence of vector fields \( \mathcal{Y}_k \in \mathfrak{t}_\mathbb{R} \), a formal power series \( C_q \) in \( q \) with real coefficients (cf. Section 6), and smooth real-valued functions \( \varphi_k \), \( k = 1, 2, \ldots \), on \( M \) such that

\[
(2.8) \quad Z(q, \omega(\ell); \mathcal{Y}(\ell)) = C_q + 0(q^{\ell+2}),
\]

where \( \mathcal{Y}(\ell) := (\sqrt{-1} \mathcal{V}/2) q^2 + \sum_{k=1}^\ell q^{k+2} \mathcal{Y}_k \), \( h(\ell) := h_0 \exp(-\sum_{k=1}^\ell q^k \varphi_k) \), and \( \omega(\ell) := c_1(L; h(\ell)) \).

The equality (2.8) above means that there exists a positive real constant \( A_\ell \) independent of \( q \) such that \( \|Z(q, \omega(\ell); \mathcal{Y}(\ell)) - C_q\|_{C^0(M)} \leq A_\ell q^{\ell+2} \) for all \( q \) with \( 0 \leq q \leq 1 \). By [38], for every nonnegative integer \( j \), a choice of a larger constant \( A = A_{j, \ell} > 0 \) keeps Theorem B still valid even if the \( C^0(M) \)-norm is replaced by the \( C^j(M) \)-norm.

3. A STABILITY CRITERION

In this section, some stability criterion will be given as a preliminary. In a forthcoming paper [27], we actually use a stronger version of Theorem 3.2 which guarantees the stability only by checking the closedness of orbits through a point for special one-parameter subgroups "perpendicular" to the isotropy subgroup. Now, for a connected reductive algebraic group \( G \), defined over \( \mathbb{C} \), we consider a representation of \( G \) on an \( N \)-dimensional complex vector space \( W \). We fix a maximal compact subgroup \( G_c \) of \( G \). Moreover, let \( \mathbb{C}^* \) be a one-dimensional algebraic torus with the maximal compact subgroup \( S^1 \).
Definition 3.1. (a) An algebraic group homomorphism \( \lambda : \mathbb{C}^* \to G \) is said to be a special one-parameter subgroup of \( G \), if the image \( \lambda(S^1) \) is contained in \( G_c \).

(b) A point \( w \neq 0 \) in \( W \) is said to be stable, if the orbit \( G \cdot w \) is closed in \( W \).

Later, we apply the following stability criterion to the case where \( W = W_m^* \) and \( G = G_m \). Let \( w \neq 0 \) be a point in \( W \).

Theorem 3.2. A point \( w \) as above is stable if and only if there exists a point \( w' \) in the orbit \( G \cdot w \) of \( w \) such that \( \lambda(\mathbb{C}^*) \cdot w' \) is closed in \( W \) for every special one-parameter subgroup \( \lambda : \mathbb{C}^* \to G \) of \( G \).

Proof. We prove this by induction on \( \dim(G \cdot w) \). If \( \dim(G \cdot w) = 0 \), the statement of the above theorem is obviously true. Hence, fixing a positive integer \( k \), assume that the statement is true for all \( 0 \neq w \in W \) such that \( \dim(G \cdot w) < k \). Now, let \( 0 \neq w \in W \) be such that \( \dim(G \cdot w) = k \), and the proof is reduced to showing the statement for such a point \( w \). Let \( \Sigma(G) \) be the set of all special one-parameter subgroups of \( G \). Fix a \( G_c \)-invariant Hermitian metric \( \| \| \) on \( W \). The proof is divided into three steps:

Step 1: First, we prove “only if” part of Theorem 3.2. Assume that \( w \) is stable. Since \( G \cdot w \) is closed in \( W \), the nonnegative function on \( G \cdot w \) defined by

\[
G \cdot w \ni g \cdot w \mapsto \| g \cdot w \| \in \mathbb{R}, \quad g \in G,
\]

has a critical point at some point \( w' \) in \( G \cdot w \). Let \( \lambda \in \Sigma(G) \), and it suffices to show the closedness of \( \lambda(\mathbb{C}^*) \cdot w' \) in \( W \). We may assume that \( \dim(\lambda(\mathbb{C}^*) \cdot w') > 0 \). Then by using the coordinate system associated to an orthonormal basis for \( W \), we can write \( w' \) as \( (w_0', \ldots, w_r', 0, \ldots, 0) \) in such a way that \( w'_{\alpha} \neq 0 \) for all \( 0 \leq \alpha \leq r \) and that

\[
\lambda(e^t) \cdot w' = (e^{t\gamma_0}w_0', \ldots, e^{t\gamma_r}w_r', 0, \ldots, 0), \quad t \in \mathbb{C},
\]

where \( \gamma_\alpha, \alpha = 0, 1, \ldots, r \), are integers independent of the choice of \( t \) in \( \mathbb{C} \). Since the closed orbit \( G \cdot w \) does not contain the origin of \( W \), the inclusion \( \lambda(\mathbb{C}^*) \cdot w' \subset G \cdot w \) shows that \( r \geq 1 \) and that the coincidence \( \gamma_0 = \gamma_1 = \cdots = \gamma_r \) cannot occur. In particular,

\[
f(t) := \log \| \lambda(e^t) \cdot w' \|^2 = \log \left( e^{2t\gamma_0}|w_0'|^2 + e^{2t\gamma_1}|w_1'|^2 + \cdots + e^{2t\gamma_r}|w_r'|^2 \right), \quad t \in \mathbb{R},
\]

satisfies \( f''(t) > 0 \) for all \( t \). Moreover, since the function in (3.3) has a critical point at \( w' \), we have \( f'(0) = 0 \). It now follows that \( \lim_{t \to +\infty} f(t) = +\infty \) and \( \lim_{t \to -\infty} f(t) = +\infty \). Hence \( \lambda(\mathbb{C}^*) \cdot w' \) is closed in \( W \), as required.

Step 2: To prove “if” part of Theorem 3.2, we may assume that \( w = w' \) without loss of generality. Hence, suppose that \( \lambda(\mathbb{C}^*) \cdot w \) is closed in \( W \) for every \( \lambda \in \Sigma(G) \). It then suffices to show that \( G \cdot w \) is closed in \( W \). For contradiction, assume that \( G \cdot w \) is not closed in \( W \). Since the closure of \( G \cdot w \) in \( W \) always contains a closed orbit \( O_1 \) in \( W \),
by \( \dim O_1 < \dim (G \cdot w) = k \), the induction hypothesis shows that there exists a point \( \hat{w} \in O_1 \) such that

\[
\lambda(\mathbb{C}^*) \cdot \hat{w} \text{ is closed in } W \text{ for every } \lambda \in \Sigma(G).
\]

Moreover, there exist elements \( g_i, i = 1, 2, \ldots, \) in \( G \) such that \( g_i \cdot w \) converges to \( \hat{w} \) in \( W \). Then for each \( i \), we can write \( g_i = \kappa'_i \cdot \exp(2\pi A_i) \cdot \kappa_i \) for some \( \kappa_i, \kappa'_i \in G_c \) and for some \( A_i \in \mathfrak{a} \), where \( 2\pi\sqrt{-1} \mathfrak{a} \) is the Lie algebra of some maximal compact torus in \( G_c \). Let \( 2\pi\sqrt{-1} \mathfrak{a}_Z \) be the kernel of the exponential map of the Lie algebra \( 2\pi\sqrt{-1} \mathfrak{a} \), and put \( \mathfrak{a}_Q := \mathfrak{a}_Z \otimes \mathbb{Q} \). Replacing \( \{ \kappa_i \} \) by its subsequence if necessary, we may assume that

\[
\kappa_i \to \kappa_{\infty} \text{ and } \{ \exp(2\pi A_i) \cdot \kappa_i \} \cdot w \to w_{\infty}, \quad \text{as } i \to \infty,
\]

for some \( \kappa_{\infty} \in G_c \) and \( w_{\infty} \in G_c \cdot \hat{w} \). Then by (3.4), the orbit \( \lambda(\mathbb{C}^*) \cdot w_{\infty} \) is also closed in \( W \) for every \( \lambda \in \Sigma(G) \). Let \( \mathfrak{a}_{\infty} \) denote the Lie subalgebra of \( \mathfrak{a} \) consisting of all elements in \( \mathfrak{a} \) whose associated vector fields on \( W \) vanish at \( \kappa_{\infty} \cdot w \). For a Euclidean metric on \( \mathfrak{a} \) induced from a suitable bilinear from on \( \mathfrak{a}_Q \) defined over \( \mathbb{Q} \), we write \( \mathfrak{a} \) as a direct sum \( \mathfrak{a}_{\infty}^+ \oplus \mathfrak{a}_{\infty}^- \), where \( \mathfrak{a}_{\infty}^+ \) is the orthogonal complement of \( \mathfrak{a}_{\infty} \) in \( \mathfrak{a} \). Let \( \tilde{A}_i \) be the image of \( A_i \) under the orthogonal projection

\[
\text{pr}_1 : \mathfrak{a} (= \mathfrak{a}_{\infty}^+ \oplus \mathfrak{a}_{\infty}^-) \to \mathfrak{a}_{\infty}^+, \quad A \mapsto \tilde{A} := \text{pr}_1(A).
\]

Note that \( \{ \exp(2\pi A_i) \cdot \kappa_{\infty} \} \cdot w = \{ \exp(2\pi \tilde{A}_i) \cdot \kappa_{\infty} \} \cdot w \). Hence,

\[
\limsup_{i \to \infty} \| \exp \{ 2\pi \text{Ad}(\kappa_{\infty}^{-1}) \tilde{A}_i \} \cdot w \| = \limsup_{i \to \infty} \| \{ \exp(2\pi A_i) \cdot \kappa_{\infty} \} \cdot w \|
\leq \lim_{i \to \infty} \| \{ \exp(2\pi A_i) \cdot \kappa_i \} \cdot w \| = \| w_{\infty} \| < +\infty.
\]

**Step 3:** Since \( \lambda(\mathbb{C}^*) \cdot w \) is closed in \( W \) for every \( \lambda \in \Sigma(G) \), by the boundedness in (3.6), \( \{ \tilde{A}_i \} \) is a bounded sequence in \( \mathfrak{a}_{\infty}^+ \) (see Remark 3.7 below). Hence, for some element \( A_{\infty} \) in \( \mathfrak{a}_{\infty}^- \), replacing \( \{ \tilde{A}_i \} \) by its subsequence if necessary, we may assume that \( \tilde{A}_i \to A_{\infty} \) as \( i \to \infty \). Then by (3.5),

\[
w_{\infty} = \lim_{i \to \infty} \{ \exp(2\pi \tilde{A}_i) \cdot \kappa_i \} \cdot w = \{ \exp(2\pi \tilde{A}_{\infty}) \cdot \kappa_{\infty} \} \cdot w.
\]

Since we have \( \exp(2\pi \tilde{A}_{\infty}) \in G \), the point \( w_{\infty} \) in \( O_1 \) belongs to the orbit \( G \cdot w \). This contradicts \( O_1 \cap (G \cdot w) = \emptyset \), as required. The proof of Lemma 3.2 is now complete.

**Remark 3.7.** The boundedness of the sequence \( \{ \tilde{A}_i \} \) in \( \mathfrak{a}_{\infty}^+ \) in Step 3 above can be seen as follows: For contradiction, we assume that the sequence \( \{ \tilde{A}_i \} \) is unbounded. Put \( v := \kappa_{\infty} \cdot w \) for simplicity. Then by (3.6), we first observe that

\[
\limsup_{i \to \infty} \| \exp(2\pi \tilde{A}_i) \cdot v \| < +\infty.
\]

Since \( 2\pi\sqrt{-1} \mathfrak{a}_{\infty} \) is the Lie algebra of the isotropy subgroup of the compact torus \( \exp(2\pi\sqrt{-1} \mathfrak{a}) \) at \( v \), both \( \mathfrak{a}_{\infty} \) and \( \mathfrak{a}_{\infty}^+ \) are defined over \( \mathbb{Q} \) in \( \mathfrak{a} \). By choosing a complex coordinate system
of $W$, we can write $v$ as $(v_0, \ldots, v_r, 0, \ldots, 0)$ for some integer $r$ with $0 \leq r \leq \dim W - 1$ such that $v_\alpha \neq 0$ for all $0 \leq \alpha \leq r$ and that

$$\exp (2\pi \bar{A}) \cdot v = (e^{2\pi \chi_0(A)} v_0, \ldots, e^{2\pi \chi_r(A)} v_r, 0, \ldots, 0), \quad \bar{A} \in a_\infty^\perp,$$

where $\chi_\alpha : a_\infty^\perp \to \mathbb{R}$, $\alpha = 0, 1, \ldots, r$, are additive characters defined over $\mathbb{Q}$. Put $n := \dim_{\mathbb{R}} a_\infty^\perp$, and let $(a_\infty^\perp)_\mathbb{Q}$ denote the set of all rational points in $a_\infty^\perp$. Let us now identify

$$a_\infty^\perp = \mathbb{R}^n \quad \text{and} \quad (a_\infty^\perp)_\mathbb{Q} = \mathbb{Q}^n,$$

as vector spaces. Since the orbit $\lambda(\mathbb{C}^*) \cdot w$ is closed in $W$ for all special one-parameter subgroups $\lambda : \mathbb{C}^* \to G$ of $G$, the same thing is true also for $\lambda(\mathbb{C}^*) \cdot v$. Hence,

$$\mathbb{Q}^n \setminus \{0\} \subset \bigcup_{\alpha, \beta = 0}^r U_{\alpha \beta},$$

where $U_{\alpha \beta} := \{ A \in a; \chi_\alpha(A) > 0 > \chi_\beta(A) \}$. Note that the boundaries of the open sets $U_{\alpha \beta}$, $1 \leq \alpha \leq r$, $1 \leq \beta \leq r$, in $\mathbb{R}^n$ sit in the union of $\mathbb{Q}$-hyperplanes

$$H_{\alpha} := \{ \chi_\alpha = 0 \}, \quad \alpha = 0, 1, \ldots, r,$$

in $\mathbb{R}^r$. Since an intersection of any finite number of hyperplanes $H_{\alpha}$, $\alpha = 0, 1, \ldots, r$, has dense rational points, (3.10) above easily implies

$$\mathbb{R}^n \setminus \{0\} = \bigcup_{\alpha, \beta = 0}^r U_{\alpha \beta}.$$

Replacing $\{ \bar{A}_i \}$ by its suitable subsequence if necessary, we may assume that there exists an element $A_\infty$ in $a_\infty^\perp (= \mathbb{R}^n)$ with $\| A_\infty \|_a = 1$ such that

$$\lim_{i \to \infty} \| \bar{A}_i \|_a = A_\infty,$$

where $\| \|_a$ denotes the Euclidean norm for $a$ as in Step 2 in the proof of Theorem 3.2. By (3.11), there exist $\alpha, \beta \in \{0, 1, \ldots, r\}$ such that $A_\infty \in U_{\alpha \beta}$, and in particular $\chi_\alpha(A_\infty) > 0$. On the other hand, $\limsup_{i \to \infty} \| \bar{A}_i \|_a = +\infty$ by our assumption. Thus,

$$\limsup_{i \to \infty} \chi_\alpha(\bar{A}_i) = \limsup_{i \to \infty} \{ \| \bar{A}_i \|_a \cdot \chi_\alpha(\bar{A}_i/\| \bar{A}_i \|_a) \} = (\limsup_{i \to \infty} \| \bar{A}_i \|_a) \chi_\alpha(A_\infty) = +\infty,$$

in contradiction to (3.8) and (3.9), as required.

4. The Chow norm

Take an algebraic torus $T \subset \text{Aut}^0(M)$, and let $\iota : \text{SL}(V_m) \to \text{PGL}(V_m)$ be the natural projection, where we regard $\text{Aut}^0(M)$ as a subgroup of $\text{PGL}(V_m)$ via the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$, $m \gg 1$. In this section, we fix a $\bar{T}_c$-invariant Hermitian metric $\rho$ on $V_m$, where $\bar{T}_c$ is the maximal compact subgroup of $\bar{T} := \iota^{-1}(T)$. Obviously, in terms of this metric, $V(\chi_k) \perp V(\chi_\ell)$ if $k \neq \ell$. Using Deligne’s pairings (cf. [8], 8.3), Zhang
(39, 1.5) defined a special type of norm on $W_m^*$, called the Chow norm, as a nonnegative real-valued function

\[(4.1) \quad W_m^* \ni w \mapsto ||w||_{CH(\rho)} \in \mathbb{R}_{\geq 0},\]

with very significant properties described below. First, this is a norm, so that it has the only zero at the origin satisfying the homogeneity condition

\[||c \cdot w||_{CH(\rho)} = |c| \cdot ||w||_{CH(\rho)} \quad \text{for all } c, w \in \mathbb{C} \times W_m^*.\]

For the group SL($V_m$), we consider the maximal compact subgroup SU($V_m; \rho$). For a special one-parameter subgroup

\[\lambda : \mathbb{C}^* \to SL(V_m)\]

of SL($V_m$), there exist integers $\gamma_j$, $j = 0, 1, \ldots, N_m$, and an orthonormal basis $\{s_0, s_1, \ldots, s_{N_m}\}$ for ($V_m, \rho$) such that, for all $j$,

\[(4.2) \quad \lambda_\zeta \cdot s_j = e^{\zeta \gamma_j} s_j, \quad z \in \mathbb{C},\]

where $\lambda_\zeta := \lambda(e^z)$. Recall that the subvariety $M_m$ in $\mathbb{P}^*(V_m)$ is the image of the Kodaira embedding $\Phi_m : M \hookrightarrow \mathbb{P}^*(V_m)$ defined by

\[(4.3) \quad \Phi_m(p) = (s_0(p) : s_1(p) : \cdots : s_{N_m}(p)), \quad p \in M,\]

where $\mathbb{P}^*(V_m)$ is identified with $\mathbb{P}^{N_m}(\mathbb{C}) = \{(z_0 : z_1 : \cdots : z_{N_m})\}$. Put $M_{m,t} := \lambda_\zeta(M_m)$ for each $t \in \mathbb{R}$. As in Section 2, $\hat{M}_{m,t} := \lambda_\zeta \cdot \hat{M}_m$ is the nonzero point of $W_m^*$ sitting over the Chow point of the irreducible reduced cycle $M_{m,t}$ on $\mathbb{P}^*(V_m)$. Then (cf. [39], 1.4, 3.4.1)

\[(4.4) \quad \frac{d}{dt} \left( \log \|\hat{M}_{m,t}\|_{CH(\rho)} \right) = (n + 1) \frac{\sum_{j=0}^{N_m} \gamma_j |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} (\Phi_m^* \lambda_\zeta^* \omega_{FS})^n,\]

where $\omega_{FS}$ is the Fubini-Study form $((\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\sum_{j=0}^{N_m} |z_j|^2))$ on $\mathbb{P}^*(V_m)$, and we regard $\lambda_t$ as a linear transformation of $\mathbb{P}^*(V_m)$ induced by (4.2). Note that the term $\Phi_m^* \lambda_\zeta^* \omega_{FS}$ above is just $((\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2))$. Put $\Gamma := 2\pi \sqrt{-1} \mathbb{Z}$. By setting

\[\mathbb{C}/\Gamma = \{ t + \sqrt{-1} \theta : t \in \mathbb{R}, \theta \in \mathbb{R}/(2\pi \mathbb{Z}) \},\]

we consider the complexified situation. Let $\eta : M \times \mathbb{C}/\Gamma \to \mathbb{P}^*(V_m)$ be the map sending each $(p, t + \sqrt{-1} \theta)$ in $M \times \mathbb{C}/\Gamma$ to $\lambda_{t+\sqrt{-1} \theta} \cdot \Phi_m(p)$ in $\mathbb{P}^*(V_m)$. For simplicity, we put

\[Q := \frac{\sum_{j=0}^{N_m} \gamma_j e^{2\gamma_j} |s_j|^2}{\sum_{j=0}^{N_m} e^{2\gamma_j} |s_j|^2} \left( = \frac{\sum_{j=0}^{N_m} \gamma_j |\lambda_t \cdot s_j|^2}{\sum_{j=0}^{N_m} |\lambda_t \cdot s_j|^2} \right).\]

We further put $z := t + \sqrt{-1} \theta$. For the time being, on the total complex manifold $M \times \mathbb{C}/\Gamma$, the $\partial$-operator and the $\bar{\partial}$-operator will be written simply as $\partial$ and $\bar{\partial}$ respectively, while on $M$, they will be denoted by $\partial_M$ and $\bar{\partial}_M$ respectively. Then

\[\eta^* \omega_{FS} = \Phi_m^* \lambda_\zeta^* \omega_{FS} + \frac{\sqrt{-1}}{2\pi} (\partial_M Q \wedge d\bar{z} + dz \wedge \bar{\partial}_M Q) + \frac{\sqrt{-1}}{4\pi} \frac{\partial Q}{\partial t} dz \wedge d\bar{z}.\]
For \( 0 \neq r \in \mathbb{R} \), we consider the 1-chain \( I_r := [0, r] \), where \( [0, r] \) means the 1-chain \(-[r, 0]\) if \( r < 0 \). Let \( \text{pr} : \mathbb{C}/\Gamma \to \mathbb{R} \) be the mapping sending each \( t + \sqrt{-1}\theta \) to \( t \). We now put \( B_r := \text{pr}^* I_r \). Then \( \int_{M \times B_r} \eta^* \omega_{FS}^{n+1} \) is nothing but
\[
(n + 1) \int_0^r dt \int_M \left( \frac{\partial Q}{\partial t} \Phi^*_m \lambda'^*_i \omega_{FS}^n + \frac{\sqrt{-1}}{\pi} \bar{\partial} M Q \wedge \partial M Q \wedge n \Phi^*_m \lambda'^*_i \omega_{FS}^{n-1} \right)
\]
\[
= \int_0^r \frac{d^2}{dt^2} \left( \log \|\hat{M}_{m,t}\|_{CH(\rho)} \right) dt = \frac{d}{dt} \left( \log \|\hat{M}_{m,t}\|_{CH(\rho)} \right) \bigg|_{t=0}^{t=r},
\]
and by assuming \( r \geq 0 \), we obtain the following convexity formula:

**Theorem 4.5.**
\[
\frac{d}{dt} \left( \log \|\hat{M}_{m,t}\|_{CH(\rho)} \right) \bigg|_{t=0}^{t=r} = \int_{M \times B_r} \eta^* \omega_{FS}^{n+1} \geq 0.
\]

**Remark 4.6.** Besides special one-parameter subgroups of \( SL(V_m) \), we also consider a little more general smooth path \( \lambda_t, t \in \mathbb{R} \), in \( GL(V_m) \) written explicitly by
\[
\lambda_t \cdot s_j = e^{r \gamma_j + \delta_j} s_j, \quad j = 0, 1, \ldots, N_m,
\]
where \( \gamma_j, \delta_j \in \mathbb{R} \) are not necessarily rational. In this case also, we easily see that the formula (4.4) and Theorem 4.5 are still valid.

5. PROOF OF THEOREM A

The statement of Theorem A is divided into “if” part, “only if” part, and the uniqueness part. We shall prove these three parts separately.

Proof of “if” part. Let \( \omega \in \mathcal{S} \) be a critical metric relative to \( T \). Then by Definition 2.6, in terms of the Hermitian metric defined in (2.4), there exists an admissible normal basis \( \{s_0, s_1, \ldots, s_{N_m}\} \) for \( V_m \) of index \( b \) such that the associated function \( E_{\omega,b} \) has a constant value \( C \) on \( M \). By operating \((\sqrt{-1}/2\pi) \bar{\partial} \log \) on the identity \( E_{\omega,b} = C \), we have
\[
\Phi^*_m \omega_{FS} = m \omega.
\]

Besides the Hermitian metric defined in (2.4), we shall now define another Hermitian metric on \( V_m \). By the identification \( V_m \cong \mathbb{C}^N_m \) via the basis \( \{s_0, s_1, \ldots, s_{N_m}\} \), the standard Hermitian metric on \( \mathbb{C}^N_m \) induces a Hermitian metric \( \rho \) on \( V_m \). As a maximal compact subgroup of \( G_m \), we choose \( (G_m)_c \) as in Section 2 by using the metric defined in (2.4). Then the Hermitian metric \( \rho \) is also preserved by the \( (G_m)_c \)-action on \( V_m \). Let
\[
\lambda : \mathbb{C}^* \to G_m
\]
be a special one-parameter subgroup of \( G_m \). By the notation \( l(k, i) \) as in Definition 2.3, we put \( s_{k,i} := s_{l(k,i)} \). If necessary, replacing \( \{s_0, s_1, \ldots, s_{N_m}\} \) by another admissible normal basis for \( V_m \) of the same index \( b \), we may assume without loss of generality that there exist integers \( \gamma_{k,i}, i = 1, 2, \ldots, n_k \), satisfying
\[
\lambda_t \cdot s_{k,i} = e^{t \gamma_{k,i}} s_{k,i}, \quad t \in \mathbb{C},
\]
where \( \lambda_t := \lambda(e^t) \) is as in (4.2), and the equality \( \sum_{i=1}^{n} \gamma_{k,i} = 0 \) is required to hold for every \( k \). Put \( \gamma_{k,i} = \gamma_{l(i,j)} \) for simplicity. Then by (4.4) and (5.1),

\[
\frac{d}{dt} \left( \log \| \hat{M}_{m,t} \|_{\text{CH}(\rho)} \right) |_{t=0} = (n+1) \int_{M} \left( \sum_{j=0}^{N} |s_j|^2 \right)^{\frac{n}{2}} \left( \Phi_m^* \omega_{FS} \right)^{n} = (n+1) m^n \int_{M} \left( \sum_{j=0}^{N} |s_j|^2 \right)^{\frac{n}{2}} \omega_i = (n+1) m^n \int_{M} \left( \sum_{j=0}^{N} |s_j|^2 \right)^{\frac{n}{2}} \omega_i = \left( n+1 \right) m^n \int_{M} \left( \sum_{j=0}^{N} |s_j|^2 \right)^{\frac{n}{2}} \omega_i = \left( n+1 \right) m^n \int_{M} \left( \sum_{j=0}^{N} |s_j|^2 \right)^{\frac{n}{2}} \omega_i = 0.
\]

Note also that, by Theorem 4.5, we have \( c := (d^2/dt^2)(\log \| \hat{M}_{m,t} \|_{\text{CH}(\rho)}) |_{t=0} \geq 0 \).

**Case 1:** If \( c \) is positive, then \( \lim_{t \to -\infty} \| \hat{M}_{m,t} \|_{\text{CH}(\rho)} = +\infty = \lim_{t \to +\infty} \| \hat{M}_{m,t} \|_{\text{CH}(\rho)} \), and in particular \( \lambda(\mathbb{C}^*) \cdot \hat{M}_m \) is closed.

**Case 2:** If \( c \) is zero, then by applying Theorem 4.5 infinitesimally, we see that \( \lambda(\mathbb{C}^*) \) preserves the subvariety \( M_m \) in \( \mathbb{P}^*(V_m) \), and moreover by \( (d/dt)(\log \| \hat{M}_{m,t} \|_{\text{CH}(\rho)}) |_{t=0} = 0 \), the isotropy representation of \( \lambda(\mathbb{C}^*) \) on the complex line \( \mathbb{C} \hat{M}_m \) is trivial. Hence, \( \lambda(\mathbb{C}^*) \cdot \hat{M}_m \) is a single point, and in particular closed.

Thus, these two cases together with Theorem 3.2 show that the subvariety \( M_m \) of \( \mathbb{P}^*(V_m) \) is stable relative to \( T \), as required.

**Remark 5.3.** About the one-parameter subgroup \( \{ \lambda_t; t \in \mathbb{R} \} \) of \( G_m \), we consider a more general situation that \( \gamma_{k,i} \) in (5.2) are just real numbers which are not necessarily rational. The above computation together with Remark 4.6 shows that, even in this case, \( (d/dt)|_{t=0}(\log \| \hat{M}_{m,t} \|_{\text{CH}(\rho)}) \) vanishes.

Proof of “only if” part. Assume that the subvariety \( M_m \) in \( \mathbb{P}^*(V_m) \) is stable relative to \( T \). Take a Hermitian metric \( \rho \) for \( V_m \) such that \( V(\chi_k) \perp V(\chi_\ell) \) for \( k \neq \ell \). For this \( \rho \), we consider the associated Chow norm. Since the orbit \( G_m \cdot \hat{M}_m \) is closed in \( W_m \), the Chow norm restricted to this orbit attains an absolute minimum. Hence, for some \( g_0 \in G_m \),

\[
0 \neq \| g_0 \cdot \hat{M}_m \|_{\text{CH}(\rho)} \leq \| g \cdot \hat{M}_m \|_{\text{CH}(\rho)}, \quad \text{for all } g \in G_m.
\]

By choosing an admissible normal basis \( \{ s_0, s_1, \ldots, s_{N_m} \} \) for \( (V_m; \rho) \) of index \( (1, 1, \ldots, 1) \), we identify \( V_m \) with \( \mathbb{C}^{N_m} = \{ (z_0, z_1, \ldots, z_{N_m}) \} \). Then \( SL(V_m) \) is identified with \( SL(N_m + 1; \mathbb{C}) \). Let \( g_m \) be the Lie subalgebra of \( \mathfrak{sl}(N_m + 1; \mathbb{C}) \) associated to the Lie subgroup \( G_m \) of \( SL(N_m + 1; \mathbb{C}) \). We can now write \( g_0 = \kappa' \cdot \exp \{ \text{Ad}(\kappa) D \} \) for some \( \kappa, \kappa' \in G_m \) and a real diagonal matrix \( D \) in \( g_m \). By \( \| \exp \{ \text{Ad}(\kappa) D \} \cdot \hat{M}_m \|_{\text{CH}(\rho)} = \| g_0 \cdot \hat{M}_m \|_{\text{CH}(\rho)} \), we have

\[
\| \exp \{ \text{Ad}(\kappa) D \} \cdot \hat{M}_m \|_{\text{CH}(\rho)} \leq \| \exp \{ t \text{Ad}(\kappa) A \} \cdot \exp \{ \text{Ad}(\kappa) D \} \cdot \hat{M}_m \|_{\text{CH}(\rho)}, \quad t \in \mathbb{R},
\]

for every real diagonal matrix \( A \) in \( g_m \). For \( j = 0, 1, \ldots, N_m \), we write the \( j \)-th diagonal element of \( A \) and \( D \) above as \( a_j \) and \( d_j \), respectively. Put \( c_j := \exp d_j \) and \( s_j' := \kappa^{-1} \cdot s_j \).
Then \( \{ s'_0, s'_1, \ldots, s'_{N_m} \} \) is again an admissible normal basis for \((V_m, \rho)\) of index \((1, 1, \ldots, 1)\). By the notation in Definition 2.3, we rewrite \( s'_j, a_j, c_j, z_j \) as \( s'_{k,i}, a_{k,i}, c_{k,i}, z_{k,i} \) by

\[
  s'_{k,i} := s'_l(k,i), \quad a_{k,i} := a_l(k,i), \quad c_{k,i} := c_l(k,i), \quad z_{k,i} := z_l(k,i),
\]

where \( k = 1, 2, \ldots, \nu_m \) and \( i = 1, 2, \ldots, n_k \). By (5.4), the derivative at \( t = 0 \) of the right-hand side of (5.4) vanishes. Hence by (4.4) together with Remark 4.6, fixing an arbitrary real diagonal matrix \( A \) in \( \mathfrak{g}_m \), we have

\[
  (5.5) \quad \int_M \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_{k,i} c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = 0
\]

where we set \( \Theta := (\sqrt{-1}/2\pi) \partial \bar{\partial} \log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |z_{k,i}|^2) \). Let \( k_0 \in \{1, 2, \ldots, \nu_m\} \) and let \( i_1, i_2 \in \{1, 2, \ldots, n_k\} \) with \( i_1 \neq i_2 \). Using Kronecker’s delta, we specify the real diagonal matrix \( A \) by setting

\[
  a_{k,i} = \delta_{kk_0} (\delta_{i_1i} - \delta_{i_2i}), \quad k = 1, 2, \ldots, \nu_m; \quad i = 1, 2, \ldots, n_k.
\]

Apply (5.5) to this \( A \), and let \((i_1, i_2)\) run through the set of all pairs of two distinct elements in \( \{1, 2, \ldots, n_k\}\). Then there exists a positive constant \( b_k > 0 \) independent of the choice of \( i \) in \( \{1, 2, \ldots, n_k\}\) such that

\[
(5.6) \quad \frac{N_m + 1}{m^n c_1(L)^n [M]} \int_M \frac{c_{k,i}^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2} \Phi_m^*(\Theta^n) = b_k, \quad k = 1, 2, \ldots, \nu_m.
\]

The following identity (5.7) allows us to define (cf. [39]) a Hermitian metric \( h_{FS} \) on \( L^m \) by

\[
(5.7) \quad |s|^2_{h_{FS}} := \frac{(N_m + 1)}{c_1(L)^n [M]} \frac{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |(s, s'_{k,i})_\rho|^2 |s'_{k,i}|^2}{\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2}, \quad s \in V_m.
\]

Then for this Hermitian metric, it is easily seen that

\[
(5.8) \quad \Sigma_{j=0}^{N_m} |c_j s'_{j}|^2 h_{FS} = \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} c_{k,i}^2 |s'_{k,i}|^2 h_{FS} = (N_m + 1)/c_1(L)^n [M].
\]

By operating \((\sqrt{-1}/2\pi) \partial \bar{\partial} \log \) on both sides of (5.8), we obtain \( \Phi_m^* \Theta = c_1(L^m; h_{FS}) \). We now set \( h := (h_{FS})^{1/m} \) and \( \omega := c_1(L; h) \). Then

\[
  \omega = (1/m) \Phi_m^* \Theta.
\]

Put \( s''_{k,i} := c_{k,i} s'_{k,i} \), and as in Definition 2.3, we write \( s''_{k,i} \) as \( s''_{l(k,i)} \). Then by (5.8), we have the equality \( \Sigma_{j=0}^{N_m} |s''_{j}|^2 h_m = (N_m + 1)/c_1(L)^n [M] \). Moreover, in terms of the Hermitian metric defined in (2.4), the equality (5.6) is interpreted as

\[
  \|s''_{k,i}\|_{L^2}^2 = b_k, \quad k = 1, 2, \ldots, \nu_m; \quad i = 1, 2, \ldots, n_k,
\]

while by this together with (5.8) above, we obtain \( \Sigma_{k=1}^{\nu_m} n_k b_k = N_m + 1 \), as required.
Proof of uniqueness. Let $\omega = c_1(L; h)$ and $\omega' = c_1(L; h')$ be critical metrics relative to $T$, and let $\{s_j \mid j = 0, 1, \ldots, N_m\}$ and $\{s'_j \mid j = 0, 1, \ldots, N_m\}$ be respectively the associated admissible normal bases for $V_m$ of index $b$. We use the notation in Definition 2.3. Then

$$E_{\omega, b} := \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}|^2_{h^m}$$

and

$$E_{\omega', b} := \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s'_{k,i}|^2_{h^m}$$

take the same constant value $C := (N_m + 1)/c_1(L)^n[M]$ on $M$. Note here that, by operating $(\sqrt{-1}/2\pi)\partial\bar{\partial}\log$ on both of these identities, we obtain

$$m\omega = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s_{k,i}|^2)$$

and

$$m\omega' = (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |s'_{k,i}|^2).$$

If necessary, we replace each $s_{k,i}$ by $\zeta_k s_{k,i}$ for a suitable complex number $\zeta_k$, independent of $i$, of absolute value 1. Then for each $k = 1, 2, \ldots, \nu_m$, we may assume that there exist a matrix $g^{(k)} = (g_{i,i}^{(k)}) \in \text{GL}(n_k; \mathbb{C})$ satisfying

$$s'_{k,i} = \sum_{i=1}^{n_k} s_{k,i} g_{i,i}^{(k)},$$

where $i$ and $\hat{i}$ always run through the integers in $\{1, 2, \ldots, n_k\}$. Then the matrix $g^{(k)}$ above is written as $\kappa^{(k)} \cdot (\exp A^{(k)}) \cdot (\kappa'^{(k)})^{-1}$ for some real diagonal matrix $A^{(k)}$ and

$$\kappa^{(k)} = (\kappa_{i,i}^{(k)})$$

and $\kappa'^{(k)} = (\kappa'_{i,i}^{(k)})$ in $\text{SU}(n_k)$. Let $a_{i}^{(k)}$ be the $i$-th diagonal element of $A^{(k)}$. For each $i$, we put $\tilde{s}_{k,i} := \sum_{i=1}^{n_k} s_{k,i} a_{i}^{(k)}$ and $\tilde{s}'_{k,i} := \sum_{i=1}^{n_k} s_{k,i} a_{i}^{(k)}$. If necessary, we replace the bases $\{s_{k,1}, s_{k,2}, \ldots, s_{k,n_k}\}$ and $\{s'_{k,1}, s'_{k,2}, \ldots, s'_{k,n_k}\}$ for $V(k)$ by the bases $\{\tilde{s}_{k,1}, \tilde{s}_{k,2}, \ldots, \tilde{s}_{k,n_k}\}$ and $\{\tilde{s}'_{k,1}, \tilde{s}'_{k,2}, \ldots, \tilde{s}'_{k,n_k}\}$, respectively. Then we may assume, from the beginning, that

$$s'_{k,i} = \{\exp(a_{i}^{(k)})\} s_{k,i}, \quad i = 1, 2, \ldots, n_k.$$

We now set $\tau_{k,i} := s_{k,i}/\sqrt{b_k}$, and the Hermitian metric for $V_m$ defined in (2.4) will be denoted by $\rho$. Then $\{\tau_{k,i} \mid k = 1, 2, \ldots, \nu_m; i = 1, 2, \ldots, n_k\}$ is an admissible normal basis of index $(1, 1, \ldots, 1)$ for $(V_m, \rho)$. Let $\{\lambda_t \mid t \in \mathbb{C}\}$ be the smooth one-parameter family of elements in $\text{GL}(V_m)$ defined by

$$\lambda_t \cdot \tau_{k,i} = \{\exp(t a_{i}^{(k)})\} \sqrt{b_k} \tau_{k,i}, \quad k = 1, 2, \ldots, \nu_m; \quad i = 1, 2, \ldots, n_k.$$

Put $\dot{M}_{m,t} := \lambda_t \cdot \dot{M}_m$, $0 \leq t \leq 1$. Then by Remark 4.6 applied to the formula (4.4), the derivative $\partial(t) := (d/dt)(\log \|\dot{M}_{m,t}\|_{\text{CH}(\rho)})/(n + 1)$ at $t \in [0, 1]$ is expressible as

$$\int_M \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_{i}^{(k)} \frac{|\lambda_t \cdot \tau_{k,i}|^2}{\lambda_t \cdot \tau_{k,i}} \left\{ (\sqrt{-1}/2\pi)\partial\bar{\partial}\log(\sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} |\lambda_t \cdot \tau_{k,i}|^2) \right\}^n$$

Hence at $t = 0$, we see that

$$\partial(0) = \int_M \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} \{a_{i}^{(k)}|s_{k,i}|^2_{h^m}/C\}(m\omega)^n = (m^n/C) \sum_{k=1}^{\nu_m} \sum_{i=1}^{n_k} a_{i}^{(k)}.$$
while at \( t = 1 \) also, we obtain
\[
\vartheta(1) = \int_M \sum_{k=1}^{\nu_{m}} \sum_{i=1}^{n_k} \{a_i^{(k)}|s_{k,i}^{2}|_{H^{m}}/C\}(m\omega')^{n} = (m^{n}/C) \sum_{k=1}^{\nu_{m}} \{b_k \sum_{i=1}^{n_k} a_i^{(k)}\}.
\]
Thus, \( \vartheta(0) \) coincides with \( \vartheta(1) \), while by Remark 4.6, we see from Theorem 4.5 that
\[
(d^2/dt^2)\{\log \|\hat{M}_{m,t}\|_{CH(\rho)}\} \geq 0 \text{ on } [0, 1]. \text{ Hence, for all } t \in [0, 1],
\]
\[
(d^2/dt^2)\{\log \|\hat{M}_{m,t}\|_{CH(\rho)}\} = 0, \quad \text{on } M.
\]
By Remark 4.6, the formula in Theorem 4.5 shows that \( \lambda_t, t \in [0, 1] \), belong to \( H \) up to a positive scalar multiple. Since \( \lambda_1 \) commutes with \( T \), the uniqueness follows, as required.

6. Proof of Theorem B

Throughout this section, we assume that the first Chern class \( c_1(L)^e \) admits an extremal Kähler metric \( \omega_0 = c_1(L; h_0) \). Then by a theorem of Calabi [3], the identity component \( K \) of the group of isometries of \((M, \omega_0)\) is a maximal compact connected subgroup of \( H \), and we obtain \( \omega_0 \in \mathcal{S}_K \) by the notation in the introduction.

Definition 6.1. For a \( K \)-invariant Kähler metric \( \omega \in \mathcal{S}_K \) on \( M \) in the class \( c_1(L)^e \), we choose a Hermitian metric \( h \) on \( L \) such that \( \omega = c_1(L; h) \). Then the power series in \( q \) given by the right-hand side of (2.8) will be denoted by \( \Psi(\omega, q) \). Given \( \omega \) and \( q \), the power series \( \Psi(\omega, q) \) is independent of the choice of \( h \).

Let \( \mathcal{D}_0 \) be the Lichnérówicz operator as defined in [3], (2.1), for the extremal Kähler manifold \((M, \omega_0)\). Then by \( \mathcal{V} \in \mathfrak{k} \), the operator \( \mathcal{D}_0 \) preserves the space \( \mathcal{F} \) of all real-valued smooth \( K \)-invariant functions \( \varphi \) such that \( \int_M \varphi \omega_0^n = 0 \). Hence, we regard \( \mathcal{D}_0 \) just as an operator \( \mathcal{D}_0 : \mathcal{F} \rightarrow \mathcal{F} \), and the kernel in \( \mathcal{F} \) of this restricted operator will be denoted simply by \( \text{Ker} \mathcal{D}_0 \). Then \( \text{Ker} \mathcal{D}_0 \) is a subspace of \( \mathcal{K}_{\omega_0} \), and we have an isomorphism
\[
(6.2) \quad e_0 : \text{Ker} \mathcal{D}_0 \cong \mathfrak{z}, \quad \varphi \leftrightarrow e_0(\varphi) := \text{grad}_{\omega_0}^C \varphi.
\]
By the inner product \( (\ , \)\)\(_{\omega_0} \) defined in the introduction, we write \( \mathcal{F} \) as an orthogonal direct sum \( \text{Ker} \mathcal{D}_0 \oplus \text{Ker} \mathcal{D}_0^\perp \). We then consider the orthogonal projection
\[
P : \mathcal{F} (= \text{Ker} \mathcal{D}_0 \oplus \text{Ker} \mathcal{D}_0^\perp) \rightarrow \text{Ker} \mathcal{D}_0.
\]
Now, starting from \( \omega(0) := \omega_0 \), we inductively define a Hermitian metric \( h(k) \), a Kähler metric \( \omega(k) := c_1(L; h(k)) \in \mathcal{S}_K \), and a vector field \( \mathcal{Y}(k) \in \sqrt{-1} \mathfrak{z}, k = 1, 2, \ldots, \) by
\[
(6.3) \quad \begin{cases}
h(k) := h(k - 1) \exp(-q^k \varphi_k), \\
\omega(k) = \omega(k - 1) + (\sqrt{-1}/2\pi) q^k \partial \bar{\partial} \varphi_k, \\
\mathcal{Y}(k) = \mathcal{Y}(k - 1) + \sqrt{-1} q^k \partial \varphi_k + e_0 \partial \varphi_k.
\end{cases}
\]
for appropriate \( \varphi_k \in \text{Ker } D_0^\perp \) and \( \zeta_k \in \text{Ker } D_0 \), where \( \omega(k) \) and \( \mathcal{Y}(k) \) are required to satisfy the condition (2.8) with \( \ell \) replaced by \( k \). We now set \( g(k) := \exp^C \mathcal{Y}(k) \). Then

\[
\{ h(k) \cdot g(k) \}^{-m} h(k)^m \{ Z(q, \omega(k); \mathcal{Y}(k)) - C_q \}
\]

\[
= \frac{n!}{m^n} \{ \sum_{j=0}^{N^m} |s_j| h(k)^m \} - C_q \{ g(k) \cdot h(k)^{-m} \} h(k)^m
\]

\[
= \Psi(\omega(k), q) - C_q h(k)^m \{ (\exp^C \mathcal{Y}(k)) \cdot h(k)^{-m} \},
\]

\[
= \Psi(\omega(k), q) - C_q \{ 1 + h(k) (\mathcal{Y}(k)/q) \cdot h(k)^{-1} + R(\mathcal{Y}(k); h(k)) \},
\]

where \( C_q = 1 + \sum_{k=0}^{\infty} \alpha_k q^{k+1} \) is a power series in \( q \) with real coefficients \( \alpha_k \) specified later, and the last term \( R(\mathcal{Y}(k); h(k)) := h(k)^m \sum_{j=2}^{\infty} \{ \mathcal{Y}(k)^j/j! \} \cdot h(k)^{-m} \) will be taken care of as a higher order term in \( q \). Consider the truncated term \( C_{q, \ell} = 1 + \sum_{k=0}^{\ell} \alpha_k q^{k+1} \). Put

\[
\Xi(\omega(k), \mathcal{Y}(k), C_{q, k}) := \Psi(\omega(k), q) - C_{q, k} \{ 1 - (\mathcal{Y}(k)/q) \cdot \log h(k) + R(\mathcal{Y}(k); h(k)) \}
\]

for each \( k \). Then, in terms of \( \omega(k), \mathcal{Y}(k) \) and \( C_{q, k} \), the condition (2.8) with \( \ell \) replaced by \( k \) is just the equivalence

\[(6.4) \quad \Xi(\omega(k), \mathcal{Y}(k), C_{q, k}) \equiv 0, \quad \text{modulo } q^{k+2}.
\]

We shall now define \( \omega(k), \mathcal{Y}(k) \) and \( C_{q, k} \) inductively in such a way that the condition (6.4) is satisfied. If \( k = 0 \), then we set \( \omega(0) = \omega_0, \mathcal{Y}(0) = \sqrt{-1} q^2 \mathcal{N}/2 \) and \( C_{q, 0} = 1 + \alpha_0 q \), where we put \( \alpha_0 := (2c_1(L)^n[M]^{-1} \{ \int_{M} \sigma_{\omega} \omega^n + 2\pi F(V) \} \) for \( \omega \in \mathcal{S}_K \). This \( \alpha_0 \) is obviously independent of the choice of \( \omega \) in \( \mathcal{S}_K \). Then, modulo \( q^2 \),

\[
\Psi(\omega(k), q) - C_{q, 0} \{ 1 - (\mathcal{Y}(0)/q) \cdot \log h(0) + R(\mathcal{Y}(0); h(0)) \}
\]

\[
\equiv \left( 1 + \frac{\sigma_{\omega_0}}{2} q \right)^2 - (1 + \alpha_0 q) \left\{ 1 - q h_0^{-1} \sqrt{-1} (\mathcal{Y}/2) \cdot h_0 \right\}
\]

\[
\equiv \left( 1 + \frac{\sigma_{\omega_0}}{2} - (1 + \alpha_0 q) \left\{ 1 + (\frac{\sigma_{\omega_0}}{2} \cdot q) - (1 + \alpha_0 q) \right\} \equiv 0,
\]

and we see that (6.4) is true for \( k = 0 \). Here, the equality \( h_0^{-1} \sqrt{-1} (\mathcal{Y}/2) \cdot h_0 = \alpha_0 - (\sigma_{\omega_0}/2) \) follows from a routine computation (see for instance [23]).

Hence, let \( \ell \geq 1 \) and assume (6.4) for \( k = \ell - 1 \). It then suffices to find \( \varphi_\ell, \zeta_\ell \) and \( \alpha_\ell \) satisfying (6.4) for \( k = \ell \). Put \( \mathcal{Y}_\ell := \sqrt{-1} c_0(\zeta_\ell) \). For each \( (\varphi_\ell, \zeta_\ell, \alpha_\ell) \in \text{Ker } D_0^\perp \times \text{Ker } D_0 \times \mathbb{R} \), we consider

\[
\Phi(q; \varphi_\ell, \zeta_\ell, \alpha_\ell) := \Psi \left( \omega(\ell - 1) + (\sqrt{-1}/2\pi)q^\ell \partial \varphi_\ell, q \right) -
\]

\[
(C_{q, \ell-1} + \alpha_\ell q^{\ell+1}) \left\{ 1 - (\mathcal{Y}(\ell - 1)/q + q^{\ell+1} \mathcal{Y}_\ell) \cdot \log \{ h(\ell - 1) \exp(\mathcal{Y}_\ell \cdot q) \}
\]

\[
+ R \left( \mathcal{Y}(\ell - 1)/q + q^{\ell+1} \mathcal{Y}_\ell; h(\ell - 1) \exp(-q^\ell \varphi_\ell) \right) \right\}.
\]
By the induction hypothesis, \( \Xi(\omega(\ell - 1), \mathcal{Y}(\ell - 1), C_{q,\ell-1}) \equiv 0 \) modulo \( q^{\ell+1} \). Since 
\[
\Phi(q; 0, 0, 0) = \Xi(\omega(\ell - 1), \mathcal{Y}(\ell - 1), C_{q,\ell-1}),
\]
we have
\[
\Phi(q; 0, 0, 0) \equiv u_\ell q^{\ell+1}, \quad \text{mod} \ q^{\ell+2},
\]
for some real-valued \( K \)-invariant smooth function \( u_\ell \) on \( M \). Let \( (\varphi_\ell, \zeta_\ell, \alpha_\ell) \in \text{Ker} \mathcal{D}_0^+ \times \text{Ker} \mathcal{D}_0 \times \mathbb{R} \). Since \( \varphi_k \) is \( K \)-invariant, by \( \mathcal{V} \in \mathfrak{k} \), we see that \( \sqrt{-1}\mathcal{V}\varphi_k \) is a real-valued function on \( M \). Note also that \( \mathcal{Y}(0) = (\sqrt{-1}\mathcal{V}/2)q^2 \). Then the variation formula for the scalar curvature (see for instance [3], (2.5)) shows that, modulo \( q^{\ell+2} \),
\[
\Phi(q; \varphi_\ell, \zeta_\ell, \alpha_\ell) \equiv \Phi(q; 0, 0, 0) + \frac{q^{\ell+1}}{2} (-\mathcal{D}_0 + \sqrt{-1}\mathcal{V})\varphi_\ell - \alpha_\ell q^{\ell+1} + q^{\ell+1}h_0^{-1}(\mathcal{Y}_\ell \cdot h_0) - \frac{\sqrt{-1}}{2}\mathcal{V}\varphi_\ell q^{\ell+1}
\]
\[
\equiv \left\{ u_\ell - \mathcal{D}_0(\varphi_\ell/2) - \alpha_\ell - \hat{F}_m(\mathcal{Y}_\ell) + e_0^{-1}(\sqrt{-1}\mathcal{V}_\ell) \right\} q^{\ell+1},
\]
where we put \( \hat{F}(\mathcal{V}) := \{c_1(L)^n[M]\}^{-1}2\pi F(\sqrt{-1}\mathcal{V}) \) for each \( \mathcal{V} \in \sqrt{-1}\mathfrak{g} \). By setting \( \mu_\ell := \{c_1(L)^n[M]\}^{-1}(\int_M u_\ell \omega_0^n) \), we write \( u_\ell \) as a sum
\[
u u_\ell = \mu_\ell + u'_\ell + u''_\ell,
\]
where \( u'_\ell := (1 - P)(u_\ell - \mu_\ell) \in \text{Ker} \mathcal{D}_0^+ \) and \( u''_\ell := P(u_\ell - \mu_\ell) \in \text{Ker} \mathcal{D}_0 \). Now, let \( \varphi_\ell \) be the unique element of \( \text{Ker} \mathcal{D}_0^+ \) such that \( \mathcal{D}_0(\varphi_\ell/2) = u'_\ell \). Moreover, we put
\[
\zeta_\ell := u''_\ell, \quad \alpha_\ell := \mu_\ell - \hat{F}(\mathcal{V}_\ell).
\]
Then by \( \mathcal{Y}_\ell = \sqrt{-1}e_0(\zeta_\ell) = \sqrt{-1}e_0(u''_\ell) \), we obtain
\[
\Phi(q; \varphi_\ell, \zeta_\ell, \alpha_\ell) \equiv \left\{ \mu_\ell + u'_\ell + u''_\ell - \mathcal{D}_0(\varphi_\ell/2) - \alpha_\ell - \hat{F}_m(\mathcal{Y}_\ell) + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell) \right\} q^{\ell+1}
\]
\[
\equiv \left\{ u'_\ell + e_0^{-1}(\sqrt{-1}\mathcal{Y}_\ell) \right\} q^{\ell+1} \equiv 0, \quad \text{mod} \ q^{\ell+2},
\]
as required. Write \( \sqrt{-1}\mathcal{V}/2 \) as \( \mathcal{Y}_\ell \) for simplicity. Now, for the real Lie subalgebra \( \mathfrak{b} \) of \( \mathfrak{g} \) generated by \( \mathcal{Y}_k \), \( k = 0, 1, 2, \ldots \), its complexification \( \mathfrak{b}^C \) in \( \mathfrak{g}^C \) generates a complex Lie subgroup \( B^C \) of \( Z^C \). Then it is easy to check that the algebraic subtorus \( T \) of \( Z^C \) obtained as the closure of \( B^C \) in \( Z^C \) has the required properties.

Remark 6.5. In Theorem C, assume that \( \omega_0 \) is a Kähler metric of constant scalar curvature, and moreover that the actions \( \rho_{\mu_{\nu}(\cdot)} \), \( \nu = 1, 2, \ldots \), coincide for all sufficiently large \( \nu \). Then by [20], the trivial group \( \{1\} \) can be chosen as the algebraic subtorus \( T \) above of \( Z^C \).

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