The Eshelby stress tensor, angular momentum tensor and scaling flux in micropolar elasticity

Markus Lazar a,∗, Helmut O.K. Kirchner b,c,†

a Emmy Noether Research Group, Department of Physics, Darmstadt University of Technology, Hochschulstr. 6, D-64289 Darmstadt, Germany
b Université Paris-Sud, UMR8 182, Orsay, F-91405
c CNRS, Orsay, F-91405

December 2, 2015

Abstract

The (static) energy momentum tensor, angular momentum tensor and scaling flux vector of micropolar elasticity are derived within the framework of Noether’s theorem on variational principles. Certain balance (or broken conservation) laws of broken translational, rotational and dilatational symmetries are found including inhomogeneities, elastic anisotropy, body forces, body couples and dislocations and disclinations present. The non-conserved \( J \)-, \( L \)- and \( M \)-integrals of micropolar elasticity are derived and discussed. We give explicit formulae for the configurational forces, moments and work terms.

Keywords: Micropolar elasticity; Dislocations; Disclinations; Peach-Koehler force; Mathisson-Papapetrou force.

∗Corresponding author. E-mail address: lazar@fkp.tu-darmstadt.de (M. Lazar).
†E-mail address: kirchnerhok@hotmail.com (H.O.K. Kirchner).
1 Introduction

Symmetries and conservation laws of micropolar elasticity are of important interest in mathematical physics, material science and engineering science. Jarić (1978, 1986) and Dai (1986) studied conservation laws in micropolar elastostatics. Vukobrat (1989) obtained some conservation laws for micropolar elastodynamics. The Noether theorem was applied by Pucci and Saccomandi (1990); Nikitin and Zubov (1998); Maugin (1998); Lubarda and Markenscoff (2003) to obtain conservation laws and the corresponding conserved currents for linear micropolar elasticity. For couple stress elasticity the Noether theorem was used by Lubarda and Markenscoff (2000). All these investigations were restricted to homogeneous, source-free and compatible micropolar elasticity. The derived conservation laws correspond to variational invariance of the strain energy with respect to translation and rotation symmetries. Therefore, the $J$- and $L$-integrals are conserved in such a version of micropolar elasticity. On the other hand, the scaling symmetry is not a variational symmetry in micropolar elasticity, thus, the $M$-integral is not conserved (see, e.g., Lubarda and Markenscoff (2003)).

Not so many results on conservation laws are known for micropolar elasticity with defects. Only the Eshelby stress tensor and the configurational forces caused by defects are known (Kluge, 1969a,b). The Eshelby stress tensor corresponds to translation symmetry and it may be identified with the (static) energy-momentum tensor (EMT). But nothing is known in the literature for nonhomogeneous micropolar elasticity with body forces, body couples, dislocations and disclinations present.

It is the purpose of the present paper to extend the results of these earlier results on micropolar elasticity to account for material nonhomogeneity, anisotropy, defects, body forces and body couples. We derive balance laws breaking the translation, rotation and scaling symmetries. The symmetry breaking terms are called configurational forces, configurational moments and configurational work. In turn, we find the expressions for the Eshelby stress tensor, angular momentum tensor and scaling flux in micropolar elasticity.

2 Basic equations of micropolar elasticity

In this section, we recall the basics of micropolar elasticity (Eringen, 1999). We consider the general case of anisotropic linear micropolar elasticity theory for non-homogeneous and incompatible media with defects. The strain energy for a micropolar material reads

\[ W = \int w \, dV \]  \hspace{1cm} (2.1)

with the energy density

\[ w = \frac{1}{2} A_{ijkl} \gamma_{ij} \gamma_{kl} + B_{ijkl} \gamma_{ij} \kappa_{kl} + \frac{1}{2} C_{ijkl} \kappa_{ij} \kappa_{kl}, \]  \hspace{1cm} (2.2)

where $\gamma_{ij}$ denotes the elastic micropolar distortion tensor and $\kappa_{ij}$ is the elastic wryness tensor. These elastic ‘strain’ tensors are given in terms of a displacement vector $u_i$ and
a microrotation $\phi_i$. Additionally, the total ‘strains’ may be decomposed into elastic and plastic parts according to

$$\begin{align*}
\gamma_{ij}^T &= \partial_j u_i + \epsilon_{ijk} \phi_k = \gamma_{ij} + \gamma_{ij}^P, \\
\kappa_{ij}^T &= \partial_j \phi_i = \kappa_{ij} + \kappa_{ij}^P.
\end{align*}$$

(2.3)

(2.4)

Here $\gamma_{ij}^P$ is the plastic distortion and $\kappa_{ij}^P$ is the plastic wryness. For simplicity, we have assumed a linear relationship but that is not at all necessary. The constitutive relations for full anisotropy read:

$$
\begin{align*}
t_{ij} &= \frac{\partial w}{\partial \gamma_{ij}} = A_{ijkl} \gamma_{kl} + B_{ijkl} \kappa_{kl}, \\
m_{ij} &= \frac{\partial w}{\partial \kappa_{ij}} = B_{klij} \gamma_{kl} + C_{ijkl} \kappa_{kl},
\end{align*}
$$

(2.5)

(2.6)

where $A_{ijkl}$, $B_{ijkl}$ and $C_{ijkl}$ are the elastic tensors of micropolar elasticity with the symmetries

$$A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij}.$$  

(2.7)

Dimensionally, $[C_{ijkl}] = \ell^3 [A_{ijkl}]$, where $\ell$ is a material length parameter. For the non-homogeneous medium under consideration, they depend on position, $A_{ijkl}(x)$, $B_{ijkl}(x)$ and $C_{ijkl}(x)$. $t_{ij}$ is the force stress tensor and $m_{ij}$ is the couple stress tensor. For an isotropic micropolar material the elastic tensors simplify to

$$
\begin{align*}
A_{ijkl} &= \lambda \delta_{ij} \delta_{kl} + \mu \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + \mu_c \left( \delta_{il} \delta_{jk} - \delta_{ij} \delta_{kl} \right), \\
C_{ijkl} &= \alpha \delta_{ij} \delta_{kl} + \beta \left( \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) + \gamma \left( \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl} \right), \\
B_{ijkl} &= 0,
\end{align*}
$$

(2.8)

where $\mu$ is the shear modulus, $\lambda$ denotes the Lamé constant, and $\mu_c$, $\alpha$, $\beta$ and $\gamma$ are additional material constants for micropolar elasticity.

The field equations in presence of an external force $f_i$ and an external couple $l_i$ are given by

$$
\begin{align*}
\partial_j t_{ij} + f_i &= 0, \\
\partial_j m_{ij} - \epsilon_{ijk} t_{jk} + l_i &= 0.
\end{align*}
$$

(2.9)

(2.10)

In linear micropolar elasticity, the incompatibility equations are

$$
\begin{align*}
\epsilon_{jkl} \left( \partial_k \gamma_{il} + \epsilon_{ikm} \kappa_{ml} \right) &= \alpha_{ij}, \\
\epsilon_{jkl} \partial_k \kappa_{il} &= \Theta_{ij},
\end{align*}
$$

(2.11)

(2.12)

and

$$
\begin{align*}
-\epsilon_{jkl} \left( \partial_k \gamma_{il}^P + \epsilon_{ikm} \kappa_{ml}^P \right) &= \alpha_{ij}, \\
-\epsilon_{jkl} \partial_k \kappa_{il}^P &= \Theta_{ij},
\end{align*}
$$

(2.13)

(2.14)

3
where \( \alpha_{ij} \) and \( \Theta_{ij} \) are the dislocation density and disclination density tensors, respectively. By differentiating we obtain the conservation laws for the dislocation density and disclination density tensors:

\[
\partial_j \alpha_{ij} - \epsilon_{ijk} \Theta_{jk} = 0, \quad (2.15)
\]
\[
\partial_j \Theta_{ij} = 0, \quad (2.16)
\]

which mean that the disclination density tensor is divergence free in the second index and the divergence of the dislocation density tensor is determined by the skew-symmetric part of the disclination density tensor. Equivalently, upon multiplication of (2.11) and (2.12) with the Levi-Civita tensor

\[
\partial_k \gamma_{ij} - \partial_j \gamma_{ik} + \epsilon_{ikl} \kappa_{lj} - \epsilon_{ijl} \kappa_{lk} = \epsilon_{kjl} \alpha_{il}, \quad (2.17)
\]
\[
\partial_k \kappa_{ij} - \partial_j \kappa_{ik} = \epsilon_{kjl} \Theta_{il}. \quad (2.18)
\]

### 3 The Eshelby stress tensor and configurational forces in micropolar elasticity

Let us take an arbitrary infinitesimal functional derivative \( \delta W \) of the elastic energy density. We follow the procedure given by Kirchner (1999) in order to construct the energy-momentum tensor and the corresponding configurational forces. From Eqs. (2.1), (2.2) and (2.7) we get

\[
\delta W = \frac{1}{2} \int \left\{ \left[ \delta A_{ijkl} \gamma_{ij} \gamma_{kl} + 2A_{ijkl} \gamma_{ij} \left[ \delta \gamma_{kl} \right] + 2\left[ \delta B_{ijkl} \gamma_{ij} \kappa_{kl} \right] + 2B_{ijkl} \left[ \delta \gamma_{ij} \right] \kappa_{kl} \right] + 2B_{ijkl} \gamma_{ij} \left[ \delta \kappa_{kl} \right] + \left[ \delta C_{ijkl} \kappa_{ij} \kappa_{kl} \right] + 2C_{ijkl} \kappa_{ij} \left[ \delta \kappa_{kl} \right] \right\} dV. \quad (3.1)
\]

With the constitutive relations (2.5) and (2.6) there remains

\[
\delta W = \int \left\{ t_{ij} \left[ \delta \gamma_{ij} \right] + m_{ij} \left[ \delta \kappa_{ij} \right] + \frac{1}{2} \left[ \delta A_{ijkl} \gamma_{ij} \gamma_{kl} \right] + \left[ \delta B_{ijkl} \gamma_{ij} \kappa_{kl} \right] + \frac{1}{2} \left[ \delta C_{ijkl} \kappa_{ij} \kappa_{kl} \right] \right\} dV. \quad (3.2)
\]

Having configurational forces in mind, we specify the functional derivative to be translational:

\[
\delta = (\delta x_k) \partial_k, \quad (3.3)
\]

where \( (\delta x_k) \) is an infinitesimal shift in the \( x_k \)-direction. On the left hand side of Eq. (3.1) we write

\[
\delta W = \int \delta w \, dV = \int \left[ \partial_k w \right] (\delta x_k) \, dV = \int \partial_i \left[ w \delta_{ik} \right] (\delta x_k) \, dV \quad (3.4)
\]
with the energy density (2.2). On the right hand side of Eq. (3.1) we have

\[
\delta W = \int \left\{ t_{ij} \partial_k \gamma_{ij} - \partial_j \gamma_{ik} + t_{ij} \partial_j \gamma_{ik} + m_{ij} \partial_k \kappa_{ij} - \partial_j \kappa_{ik} + m_{ij} \partial_j \kappa_{ik} \right\} \, dV,
\]

where the second, third, fifth and sixth terms have been subtracted and added. The purpose is to obtain the square brackets with the meaning of Eqs. (2.17) and (2.18). The third and sixth terms may be written

\[
t_{ij} \partial_j \gamma_{ik} = \partial_j \left[ t_{ij} \gamma_{ik} \right] + f_i \gamma_{ik},
\]

\[
m_{ij} \partial_j \kappa_{ik} = \partial_j \left[ m_{ij} \kappa_{ik} \right] - \partial_j \left[ m_{ij} \kappa_{ik} \right] - \epsilon_{ijkl} t_{lj} \kappa_{ik} + l_i \kappa_{ik}.
\]

Now we add and subtract the term \( \epsilon_{ijkl} t_{lj} \kappa_{ik} \) in order to get the structure of the dislocation tensor (2.17). In addition, we introduce and use the following tensor

\[
\bar{\gamma}_{ij} = \gamma_{ij} - \epsilon_{ijk} \phi_k = \partial_j u_i - \gamma_{ij}^P.
\]

The purpose is to obtain an Eshelby stress tensor which is divergenceless in the limit to isotropic, source-free and homogeneous micropolar elasticity. We obtain the expression

\[
\int \left\{ \epsilon_{kjl} t_{ij} \alpha_{kl} + \epsilon_{kjl} m_{ij} \Theta_{kl} - \epsilon_{kjl} t_{ij} \kappa_{ik}^P + f_i \gamma_{ik} + l_i \kappa_{ik} + \frac{1}{2} \gamma_{ij} \left[ \partial_k A_{ijmn} \right] \gamma_{mn} + \gamma_{ij} \left[ \partial_k B_{ijmn} \right] \kappa_{mn} + \frac{1}{2} \kappa_{ij} \left[ \partial_k C_{ijmn} \right] \kappa_{mn} \right\} \, dV = \int \partial_j \left[ w \delta_{jk} - t_{ij} \bar{\gamma}_{ik} - m_{ij} \kappa_{ik} \right] \, dV = J_k.
\]

The first integral contains the terms breaking the translational symmetry called configurational forces. The integrand of the second integral in Eq. (3.9) is the divergence of the (canonical) energy-momentum tensor (EMT)

\[
P_{kj} = w \delta_{jk} - t_{ij} \bar{\gamma}_{ik} - m_{ij} \kappa_{ik},
\]

which we call the Eshelby stress tensor of micropolar elasticity. It is a generalization of the Eshelby stress tensor (Eshelby, 1951, 1975) in elasticity towards micropolar elasticity. Eq. (3.10) is in agreement with the so-called Maxwell stress tensor of Cosserat theory given by Kluge (1969a,b). The divergence of the EMT (3.10) can be integrated out with Gauss, and we find the \( J \)-integral for micropolar elasticity

\[
J_k = \int P_{kj} n_j \, dS.
\]

The usual argument of forming a penny shaped volume across the surface between two media I and II gives

\[
J_k = \int \left[ P_{kj}^I - P_{kj}^{II} \right] n_j \, dS.
\]
Here $P^I_{kj}$ and $P^{II}_{kj}$ are the energy-momentum tensors on the sides I and II of the interface, respectively. The first integral in Eq. (3.9) defines a sum of configurational force densities:

$$F_k = \epsilon_{kji} t_{ij} \alpha_{il} + \epsilon_{kji} m_{ij} \Theta_{il} - \epsilon_{kji} \tau_{ji} \kappa^P_{li} + f^i \bar{\gamma}_{ik} + l_i \kappa_{ik} + f_{k}^{inh},$$  \hspace{1cm} (3.13)

where the inhomogeneity force density or Eshelby force density is due to the gradient of the elastic tensors (see also Eshelby (1951); Maugin (1993)):

$$f_{k}^{inh} = \frac{1}{2} \gamma_{ij} \partial_k A_{ijmn} \gamma_{mn} + \gamma_{ij} \partial_k B_{ijmn} \kappa_{mn} + \frac{1}{2} \kappa_{ij} \partial_k C_{ijmn} \kappa_{mn}. \hspace{1cm} (3.14)$$

The first term in Eq. (3.13) is the configurational force density on a dislocation density $\alpha_{il}$ in presence of a force stress $t_{ij}$,

$$\epsilon_{kji} t_{ij} \alpha_{il}. \hspace{1cm} (3.15)$$

This expression is the same as the Peach-Koehler force density in elasticity (Peach and Koehler, 1950). The second term is the configurational force density on a disclination density $\Theta_{il}$ in presence of a couple stress $m_{ij}$,

$$\epsilon_{kji} m_{ij} \Theta_{il}. \hspace{1cm} (3.16)$$

The expression (3.16) is the Mathisson-Papapetrou force density due to disclinations. The third terms has a similar form as the Peach-Koehler force. It is a configurational force density on a plastic wryness $\kappa^P_{li}$, caused by disclinations, in presence of the force stress $t_{ji}$,

$$-\epsilon_{kji} \tau_{ji} \kappa^P_{li}. \hspace{1cm} (3.17)$$

These three configurational forces caused by defects have been already discovered by Kluge (1969a,b). The fourth term is the configurational force density on a body force $f_i$ in presence of the distortion $\bar{\gamma}_{ik}$

$$f_i \bar{\gamma}_{ik}. \hspace{1cm} (3.18)$$

It is similar in form as the Cherepanov force density (see, e.g., Cherepanov (1981); Eischen and Herrmann (1987)). The fifth term is the configurational force density on a body couple $l_i$ in presence of an elastic wryness $\kappa_{ik}$

$$l_i \kappa_{ik}. \hspace{1cm} (3.19)$$

For homogeneous, source-free and compatible micropolar elasticity we recover the divergenceless Eshelby stress tensor as

$$P_{kj} = w \delta_{jk} - t_{ij} \partial_i u_k - m_{ij} \partial_i \phi_k, \hspace{1cm} \partial_j P_{kj} = 0. \hspace{1cm} (3.20)$$

This formula is in agreement with the Eshelby stress tensor given by Lubarda and Markenscoff (2003). The corresponding Eshelby stress tensor for finite theory of polar elasticity has been given by Maugin (1998) and Nikitin and Zubov (1998).
The angular momentum tensor (AMT) and configurational moments in micropolar elasticity

Now, having the AMT in mind, we specify the functional derivative to be rotational:

$$\delta = (\delta x_k)\epsilon_{kji}x_j \partial_i,$$

(4.1)

where $(\delta x_k)$ denotes the $x_k$-direction of the axis of rotation. Using the same manipulations as in section 3, we find:

$$\epsilon_{kji}x_j J_i = \int \epsilon_{kji} x_j F_i \, dV = \int \epsilon_{kji} [\partial_n (x_j P_{in}) - P_{ij}] \, dV$$

(4.2)

with Eq. (3.13). Now we rewrite the part

$$\epsilon_{kji} P_{ij} = -\epsilon_{kji} (t_{ij} \gamma_{li} + m_{ij} \kappa_{li}).$$

(4.3)

When we subtract and add the terms $\epsilon_{kji} t_{iil} \gamma_{jl}$ and $\epsilon_{kji} m_{ii} \kappa_{lj}$, and use the equilibrium equations (2.9) and (2.10), and the decompositions of the strain tensors (2.3) and (2.4), we obtain the result

$$\int \epsilon_{kji} \{ x_j F_i + f_j \phi_i + t_{il} \gamma_{jl} + m_{il} \kappa_{jl} + t_{il} \gamma_{lj} - t_{lj} \gamma_{li} + m_{il} \kappa_{lj} - m_{ij} \kappa_{li} + \epsilon_{ilk} t_{lk} \phi_j \} \, dV$$

$$= \int \epsilon_{kji} \partial_n (x_j P_{in} + u_j t_{in} + \phi_{i} m_{in}) \, dV = L_k.$$

(4.4)

The first integral contains terms breaking the rotational symmetry which we call configurational moment densities. The first term is a configurational vector moment produced by the configurational forces $F_j$. The other terms are intrinsic vector moments as a result of the manipulation during the calculation in order to obtain an angular momentum tensor. The integrand of the second integral in Eq. (4.4) is the divergence of the (canonical) AMT

$$M_{kn} = \epsilon_{kji} [x_j P_{in} + u_j t_{in} + \phi_j m_{in}].$$

(4.5)

This is the incompatible generalization of the compatible result given by Pucci and Saccomandi (1990); Lubarda and Markenscoff (2003). Eq. (4.5) consists of two parts: the first one is the orbital AMT given in terms of the EMT and the second one is the spin AMT given in terms of the force and couple stresses. Eq. (4.4) may be transformed into a surface integral

$$L_k = \int M_{kj} n_j \, dS.$$

(4.6)

The tensor $M_{ki}$ is related to a tensor of third rank as follows:

$$M_{kn} = \frac{1}{2} \epsilon_{kji} M_{jin}.$$

(4.7)
It may be decomposed according

\[ M_{jin} = L_{jin} + S_{jin}, \]  

(4.8)

with the orbital angular momentum tensor

\[ L_{jin} = x_j P_{in} - x_i P_{jk} \]  

(4.9)

and the spin angular momentum tensor

\[ S_{jin} = u_j t_{in} - u_i t_{jn} + \phi_j m_{in} - \phi_i m_{jn}. \]  

(4.10)

It is the canonical (or Noether) spin AMT. Furthermore, the spin part can be written as:

\[ S_{jin} = u_\alpha (\Sigma_{ji})^{\alpha\beta} t_{\beta n} + \phi_\alpha (\Sigma_{ji})^{\alpha\beta} m_{\beta n}, \]  

(4.11)

with the infinitesimal generator of the finite-dimensional irreducible representation of the rotation group for a vector field (spin-1 field):

\[ (\Sigma_{ji})^{\alpha\beta} = \delta_j^\alpha \delta_i^\beta - \delta_i^\alpha \delta_j^\beta. \]  

(4.12)

Since

\[ \partial_j M_{kj} \neq 0, \]  

(4.13)

the configurational moments break the rotational symmetry. Only for source-less \((f_i = 0, l_i = 0)\), isotropic, compatible and homogeneous micropolar elasticity, the AMT is divergenceless. The isotropy condition is:

\[ \epsilon_{kji} [t_{ij} \gamma_{jl} - t_{ij} \gamma_{li} + m_{ij} \kappa_{jl} - m_{ij} \kappa_{li} + \epsilon_{ikl} t_{ik} \phi_j] = 0. \]  

(4.14)

Eq. (4.14) has to be fulfilled by the isotropic constitutive relations (2.5), (2.6) and (2.8). It is the generalization of the isotropy condition of elasticity (see, e.g., Eshelby (1975)).

5 The scaling flux and configurational work in micropolar elasticity

Having the \(M\)-integral in mind, we specify the functional derivative to be dilatational:

\[ \delta = x_k \partial_k. \]  

(5.1)

Using Eq. (3.9), we find

\[ x_k J_k = \int x_k F_k \, dV = \int [\partial_j (x_k P_{kj}) - P_{kk}] \, dV \]  

(5.2)
with
\[ P_{kk} = \partial_j \left[ \frac{n-2}{2} u_{ij} + \frac{n}{2} \phi_{mi} \right] + \frac{n-2}{2} (f_i u_i - t_{ij} \gamma_{ij}^p) + \frac{n}{2} (l_i \phi_i - m_{ij} \kappa_{ij}^p) - m_{ij} \kappa_{ij} \] (5.3)

and \( \delta_{kk} = n \). Thus, \( n = 3 \) for three dimensions and \( n = 2 \) for two dimensions. The \( M \)-integral generalized for micropolar elasticity of an anisotropic, non-homogeneous medium with body forces and couples is
\[
\int \left\{ x_k F_k + \frac{n-2}{2} f_i u_i + \frac{n}{2} l_i \phi_i - \frac{n-2}{2} t_{ij} \gamma_{ij}^p - \frac{n}{2} m_{ij} \kappa_{ij}^p - m_{ij} \kappa_{ij} \right\} dV = \int \partial_j \left[ x_k P_{kj} - \frac{n-2}{2} u_{kj} t_{kj} - \frac{n}{2} \phi_k m_{kj} \right] dV = M. \] (5.4)

It can be seen that the (configurational work) terms appearing in the first integral in Eq. (5.4) break the dilatation (or scaling) invariance. The first term is built from the configurational forces by multiplication with \( x_k \). The other terms are called intrinsic scalar moments. The integrand of the second integral in Eq. (5.4) is the divergence of the scaling flux vector
\[ Y_j = \left[ x_k P_{kj} - \frac{n-2}{2} u_{kj} t_{kj} - \frac{n}{2} \phi_k m_{kj} \right]. \] (5.5)

Eq. (5.5) is the incompatible generalization of the compatible result given by Lubarda and Markenscoff (2003). The term \(-(n-2)/2\) is the scale (or canonical) dimension of the displacement vector \( u_k \) and \(-n/2\) is the scale (or canonical) dimension of the axial vector field \( \phi_k \). The first term in Eq. (5.5) is the ‘orbital’ piece and the other two terms are the ‘intrinsic’ parts of the scaling flux vector.

Eq. (5.5) can be transformed into a surface integral
\[ M = \int Y_j n_j dS. \] (5.6)

In general, the dilatation current is not divergenceless
\[ \partial_j Y_j \neq 0. \] (5.7)

Therefore, in this case the scaling symmetry is broken. Even in the compatible, homogeneous and source-free case the scaling symmetry is broken; namely
\[ M = -\int m_{ij} \kappa_{ij} dV. \] (5.8)

The reason is that field theories like gradient elasticity, micropolar elasticity and micromorphic elasticity are theories with internal length scales. Because the material tensors have different dimensions, such constants with the dimension of length appearing in the Lagrangian (strain energy density) violate the dilatational (scaling) invariance.
6 Conclusion

In this paper, we have derived broken conservation laws of micropolar elasticity by using the framework of the Noether theorem on invariant variational principles. Earlier results obtained by Kluge (1969a,b); Pucci and Saccomandi (1990); Lubarda and Markenscoff (2003) have been extended to account for material nonhomoogeneity, anisotropy, defects (dislocations, disclinations), body forces and body couples. We calculated the Eshelby stress tensor, angular momentum tensor and scaling flux vector, which are not divergenceless, for such an extended micropolar theory. Additionally, we have given the $J$-, $L$- and $M$-integrals for this extension. The terms breaking the translational, rotational and scaling invariance are called configurational forces, moments and work, respectively.

Acknowledgement

M.L. has been supported by an Emmy-Noether grant of the Deutsche Forschungsgemeinschaft (Grant No. La1974/1-2).

References

Cherepanov, G.P., 1981. Invariant $\Gamma$ integrals. Engineering Fracture Mechanics 14, 39–58.

Dai, T.-M., 1986. Some path independent integrals for micropolar media. Int. J. Solids Structures 22, 729–735.

Eischen, J.W., Herrmann, G., 1987. Energy release rates and related balance laws in linear elastic defect mechanics. Journal of Applied Mechanics 54, 388–392.

Eringen, A.C., 1999. Microcontinuum Field Theories I: Foundations and Solids. Springer, New York.

Eshelby, J.D., 1951. The force on an elastic singularity. Phil. Trans. Roy. Soc. London A 244, 87–112.

Eshelby, J.D., 1975. The elastic energy-momentum tensor. Journal of Elasticity 5, 321–335.

Jarić, J.P., 1978. Conservation laws of the $J$-integral type in micropolar elastostatics, Int. J. Engng. Sci. 16, 967–984.

Jarić, J.P., 1986. The energy release rate in quasi-static crack propagation and $J$-integral. Int. J. Solids Structures 22, 767–778.

Kirchner, H.O.K., 1999. The force on an elastic singularity in a nonhomogenous medium. J. Mech. Phys. Solids 47, 993–998.
Kluge, G., 1969a. Über den Zusammenhang der allgemeinen Versetzungstheorie mit dem Cosserat-Kontinuum. Wissenschaftliche Zeitschrift der Technischen Hochschule Otto von Guericke Magdeburg 13, 377–380.

Kluge, G., 1969b. Zur Dynamik der allgemeinen Versetzungstheorie bei Berücksichtigung von Momentenspannungen. Int. J. Engng. Sci. 7, 169–183.

Lubarda, V.A., Markenscoff, X., 2000. Conservation integrals in couple stress elasticity. Journal of the Mechanics and Physics of Solids 48, 553–564.

Lubarda, V.A., Markenscoff, X., 2003. On conservation integrals in micropolar elasticity. Philosophical Magazine 83, 1365–1377.

Maugin, G.A., 1993. Material Inhomogeneities in Elasticity. Chapman and Hall, London.

Maugin, G.A., 1998. On the structure of the theory of polar elasticity. Phil. Trans. R. Soc. Lond. A 356, 1367–1395.

Nikitin, E., Zubov, L.M., 1998. Conservation laws and conjugate solutions in the elasticity of simple materials and materials with couple stress. J. Elasticity 11, 1–22.

Peach, M.O., Koehler, J.S. 1950. Forces extended on dislocations and the stress field produced by them. Phys. Rev. 80, 436–439.

Pucci, E., Saccomandi, G, 1990. Symmetries and conservation laws in micropolar elasticity. Int. J. Engng. Sci. 28, 557–562.

Vukobrat, M., 1989. Conservation laws in micropolar elastodynamics and path-independent integrals. Int. J. Engng. Sci. 27, 1093–1106.