AFFINE EQUIVALENCE AND SADDLE CONNECTION
GRAPHS OF TRANSLATION SURFACES

HUIPING PAN

ABSTRACT. To every translation surface, we associate a saddle connection graph, which is a subgraph of the arc graph. We prove that every isomorphism between two saddle connection graphs is induced by an affine homeomorphism between the underlying translation surfaces. We also investigate the automorphism group of the saddle connection graph, and the corresponding quotient graph.

Keywords: translation surfaces, affine homeomorphisms, saddle connection graphs, isomorphisms

AMS MSC2010: 30F60, 30F30, 54H15

1. INTRODUCTION

1.1. Arc complex. For a compact oriented topological surface with marked points, the arc complex is a simplicial complex whose $k$-simplices correspond to the set of $k + 1$ isotopy classes of properly embedded non-trivial arcs which can be realized pairwise disjointly outside the marked points. The arc complex is an important and useful tool for the study of mapping class group ([12, 13]). Masur and Schleimer ([25]) proved that the arc complex is $\delta$-hyperbolic (see also [15]). Later, Hesel-Przytycki-Webb ([14]) proved that the arc complex is uniformly 7-hyperbolic.

The mapping class group acts naturally on the arc complex. İrmak-McCarthy ([20]) proved that every injective simplicial map from the arc complex is induced by a self-homeomorphism. Based on this, they described completely the automorphism group of the arc complex. (These results also hold for the arc complex of non-orientable surface, [18, 19].)

1.2. Translation surfaces. A translation surface is a pair $(X, \omega)$ where $X$ is a closed Riemann surface with genus at least one and $\omega$ is a holomorphic one form on $X$. The interest in translation surfaces comes from three parts. First, it comes from Teichmüller theory. The cotangent space of Teichmüller space can be identified with the space of holomorphic quadratic differentials, whose canonical double cover give rise to holomorphic one forms. Very recently, based on the $SL(2, \mathbb{R})$ dynamics of (half-) translation surfaces, Eskin,
McMullen, Mukamel and Wright (27,6,38) discovered and studied new remarkable totally geodesic primitive subvarieties of low dimensional moduli spaces. Based on the L-shaped pillowcases, Markovic (22) proved that the Carathéodory metric and the Teichmüller metric disagree on the Teichmüller space of closed surface with genus at least two. Second, it comes from the interval exchange transformations and polygonal billiards. It is known that both of them correspond to the directional flows on (half-) translation surfaces. Third, it comes from the $GL(2,\mathbb{R})$ dynamics of the Hodge bundle over the moduli space, which is an analogue of the unipotent flow on homogeneous spaces. Eskin-Mirzakhani (14) proved that every $SL(2,\mathbb{R})$ invariant ergodic measure in each connected component is affine. Together with Mohammadi, they proved (8) proved that every $GL(2,\mathbb{R})$ orbit closure in each connected component is an affine invariant submanifold. (For a general introduction to translation surfaces, we refer to [10, 26, 36, 37, 39].)

1.3. Saddle connection graph. In parallel with the arc graph, which is the 1-skeleton of the arc complex for a topological surface with marked points, we consider the saddle connection graph for a translation surface with marked points. Let $(X, \omega; \Sigma)$ be a translation surface with marked points $\Sigma$ which contains all the zeros of $\omega$. The holomorphic one form $\omega$ induces a singular flat metric on $X$. A saddle connection on $(X, \omega; \Sigma)$ is a $|\omega|$-geodesic segment connects two points in $\Sigma$ such that the interior is disjoint from $\Sigma$. The saddle connection graph of $(X, \omega; \Sigma)$, denoted by $S(X, \omega; \Sigma)$, is a graph such that the vertices are saddle connections and the edges are pairs of interiorly disjoint saddle connections. This graph has infinite diameter (see Proposition 2.7). Moreover, it follows from an observation due to Minsky-Taylor (28) that the saddle connection graph is isometrically embedded into the arc graph. In particular, it is connected and $\delta$-hyperbolic (see Section 2).

The main difference between the arc graph and the saddle connection graph is that the geodesic representative of a generic topological arc consists of several saddle connections instead of one saddle connection, which makes the saddle connection graph more complicated.

1.4. Statement of Results. The aim of this paper is to investigate isomorphisms between two saddle connection graphs, and the automorphism group of the saddle connection graph. A homeomorphism between two translation surfaces with marked points is called affine if it is affine outside the set of the marked points with respect to the coordinates defined by integrating the corresponding holomorphic one forms. Every affine homeomorphism induces an isomorphism between the corresponding saddle connection graphs. The main result of this paper shows that the converse is also true.

**Theorem 1.1.** Let $(X, \omega; \Sigma), (X', \omega'; \Sigma')$ be two translation surfaces with marked points such that $|\Sigma| \geq 1, |\Sigma'| \geq 1$. Then every isomorphism $F$ :
$S(X, \omega; \Sigma) \to S(X', \omega'; \Sigma')$ is induced by an affine homeomorphism $f : (X, \omega; \Sigma) \to (X', \omega'; \Sigma')$.

**Remark 1.** A direct consequence of Theorem 1.1 is that every saddle connection graph determines a Teichmüller disk. In other words, there is an one-to-one correspondence between the set of isomorphism classes of saddle connection graphs and the set of Teichmüller disks.

**Theorem 1.2.** Let $(X, \omega; \Sigma)$ be a translation surface with marked points such that $|\Sigma| \geq 1$.

1. If $(X, \omega; \Sigma)$ is not a torus with one marked point, then the automorphism group of $S(X, \omega; \Sigma)$ is isomorphic to the group of affine homeomorphisms of $(X, \omega; \Sigma)$.
2. If $(X, \omega; \Sigma)$ is a torus with one marked point, then the automorphism group of $S(X, \omega; \Sigma)$ is an index two subgroup of the group of affine homeomorphisms of $(X, \omega; \Sigma)$.

**Remark 2.** For a generic translation surface which does not belong to any hyperelliptic component, the group of affine homeomorphisms is trivial. As a consequence, the corresponding saddle connection graph has no nontrivial automorphisms, which is different from the arc graph.

**Theorem 1.3.** Let $(X, \omega; \Sigma)$ be a translation surface with marked points. Let $\mathcal{G}(X, \omega; \Sigma)$ be the quotient of $S(X, \omega; \Sigma)$ by its automorphism group.

1. $\mathcal{G}(X, \omega; \Sigma)$ has infinite edges if and only if $(X, \omega; \Sigma)$ is not a torus with one marked point.
2. If $(X, \omega; \Sigma)$ is a Veech surface, then $\mathcal{G}(X, \omega; \Sigma)$ has finite vertices.

**Remark 3.** We don’t know whether the converse to the second statement in Theorem 1.3 is true or not.

1.5. **Related results.** To each Teichmüller disk, Smillie-Weiss ([32]) introduced the spine graph, which is a tree in the hyperbolic plane. They proved that the spine graph has compact quotient by the Veech group if and only if the Veech group is a lattice. The dual of the spine graph is a graph whose vertices are the directions of saddle connections and whose edges are pairs of directions which are the directions of the shortest saddle connections of some translation surface in the Teichmüller disk. Nguyen studied the graph of degenerate cylinders for translation surfaces in genus two ([30]) and the graph of periodic directions for translation surfaces satisfying the Veech dichotomy ([31]). He proved that both of them are hyperbolic, and that every automorphism which comes from the mapping class group is induced by an affine self-homeomorphism. Moreover, based on the graph of periodic directions, Nguyen ([31]) gave an algorithm to “determine” a coarse fundamental domain and a generating set for the Veech group of a Veech surface.
1.6. Outline. In Section 2, we prove some basic properties of the saddle connection graph including the connectedness, hyperbolicity and infinite diameter. In Section 3-6, we prove Theorem 1.1. The proof consists of three steps:

Step 1. We start with an observation that if $\gamma_1, \gamma_2, \gamma_3$ bound a triangle on $(X, \omega; \Sigma)$, then any two of $\{F(\gamma_1), F(\gamma_2), F(\gamma_3)\}$, together with one more saddle connection, will bound a triangle on $(X', \omega'; \Sigma')$ whose interior is disjoint from the third one (Corollary 3.2). Based on this, we prove that the isomorphism $F$ preserves triangles (Theorem 4.1).

Step 2. Fix a triangulation of $(X, \omega; \Sigma)$. The correspondence between triangles obtained in step 1 induces an affine map between triangles. We show that these affine maps have orientation consistency (Proposition 5.4), which allows us to glue these affine maps between triangles to obtain a homeomorphism from $(X, \omega; \Sigma)$ to $(X', \omega'; \Sigma')$ (Theorem 5.1). It then follows from the connectedness of the triangulation graph that the isotopy class of the resulting homeomorphism is independent of the choices of triangulations (Proposition 5.9).

Step 3. We prove that the induced homeomorphism obtained in step 2 is isotopic to an affine homeomorphism (Proposition 6.1).

Step three is a standard argument (see [4, 30, 31]). The technical part of this paper is step one and step two. In Section 6, we also prove Theorem 1.2. In Section 7, we prove Theorem 1.3. In Section 8, we propose two questions.

Convention. In this paper, whenever we mention “intersection” between two subsets of a translation surface, we mean the intersection between the interior of these subsets.

Acknowledgements. We thank Duc-Manh Nguyen for pointing out a mistake in the proof of Theorem 1.3 in an earlier version and for informing us the references [3, 17]. We thank Kasra Rafi for informing us the reference [28]. We also thank Lixin Liu and Weixu Su for useful discussions.

2. Saddle connection graph

2.1. Translation surfaces.

Definition 1 (Translation surface). A translation surface is a pair $(X, \omega)$ where $X$ is a closed orientable Riemann surface with genus at least 1, and $\omega$ is a holomorphic one form one $X$. A translation surface with marked points is a triple $(X, \omega; \Sigma)$ such that $(X, \omega)$ is a translation surface, and $\Sigma$ is a finite subset of $X$ which contains the zero set of $\omega$. 
The holomorphic one form $\omega$ induces a singular flat metric $|\omega|$ on $X$, which is flat outside the zero set of $\omega$. The cone angle at each zero point of $\omega$ is a multiple of $2\pi$.

**Definition 2 (Saddle connection).** A **saddle connection** on $(X, \omega; \Sigma)$ is a $|\omega|$-geodesic segment connects two points in $\Sigma$ such that the interior is disjoint from $\Sigma$.

For a saddle connection $\alpha$, the integral $\int_\alpha \omega$ is called the **holonomy** of $\alpha$.

**Proposition 2.1** ([16] [24] [35]). Let $(X, \omega; \Sigma)$ be a translation surface with marked points.

- The set of holonomies of saddle connections on $(X, \omega; \Sigma)$ is a discrete subset of $\mathbb{R}^2$.
- The set of directions of saddle connections on $(X, \omega; \Sigma)$ is a dense subset of the unit circle.

**Definition 3 (Cylinder).** A **cylinder** on $(X, \omega; \Sigma)$ is an open subset disjoint from $\Sigma$, which is isometric to $(\mathbb{R}/c\mathbb{Z}) \times (0, h)$ with $c, h \in \mathbb{R}_{>0}$ and not properly contained in any other subset with the same property. A cylinder is called **simple** if each of its boundary components consists of one saddle connection. It is called **semisimple** if at least one of its boundary components consists of one saddle connection. A boundary component of a semisimple cylinder is called **simple** if it consists of only one saddle connection.

**2.2. Saddle connection graph.**

**Definition 4 (Saddle connection graph).** The **saddle connection graph** of $(X, \omega; \Sigma)$, denoted by $\mathcal{S}(X, \omega; \Sigma)$, is a graph such that the vertices $\mathcal{S}^0(X, \omega; \Sigma)$ are saddle connections and the edges $\mathcal{S}^1(X, \omega; \Sigma)$ are pairs of interiorly disjoint saddle connections.

For a translation surface with marked points $(X, \omega; \Sigma)$, let $\Lambda(X, \omega; \Sigma)$ be the set of shortest saddle connections on $(X, \omega)$.

**Lemma 2.2.** Every two saddle connections in $\Lambda(X, \omega; \Sigma)$ are disjoint.

**Proof.** Suppose to the contrary that there exist $\alpha, \beta \in \Lambda(X, \omega; \Sigma)$ such that they intersect each other in the interior. Then they would introduce a shorter saddle connection.

**Remark 4.** The set of shortest saddle connections on a translation surface has been studied by several people, see [2] [32].

**Proposition 2.3 (Connectedness).** For any translation surface with marked points $(X, \omega; \Sigma)$, the saddle connection graph $\mathcal{S}(X, \omega; \Sigma)$ is connected.

**Proof.** Let $\alpha, \beta$ be two arbitrary saddle connections on $(X, \omega; \Sigma)$. If they are parallel, they are disjoint. Hence, they are connected by an edge in $\mathcal{S}(X, \omega; \Sigma)$. 
If \( \alpha \) and \( \beta \) are not parallel, we may suppose that they are horizontal and vertical, respectively. Let \( a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \). There exists \( t_0 > 0 \) large enough, such that the shortest saddle connections on \( a_{-t_0}(X, \omega; \Sigma) \) and \( a_{t_0}(X, \omega; \Sigma) \) are horizontal and vertical, respectively. Let \( c : [0, 1] \to \text{SL}(2, \mathbb{R}) \cdot (X, \omega; \Sigma) \) be a path such that \( c(0) = a_{-t_0}(X, \omega; \Sigma) \) and \( c(1) = a_{t_0}(X, \omega; \Sigma) \). Let \( \Lambda(s) \) be the set of shortest saddle connections on \( c(s). \) There exist \( 0 = s_0 < s_1 < s_2 < \cdots < s_k = 1 \) such that \( \Lambda(s) = \Lambda(s') \) for \( s, s' \in (s_i, s_{i+1}) \), and that \( \Lambda(s) \cap \Lambda(s') \neq \emptyset \) for \( s, s' \in [s_i, s_{i+1}] \). Let \( \beta_i \) be a saddle connection in \( \Lambda(s_i) \cap \Lambda(s_{i+1}) \). For each \( 0 \leq i \leq k-1 \), \( \beta_i \) and \( \beta_{i+1} \) are disjoint. Hence they are connected by an edge in \( \mathcal{S}(X, \omega). \) Therefore, the sequence \( \alpha = \beta_0, \beta_1, \beta_2, \cdots, \beta_{k-1}, \beta_k = \beta \) corresponds to a path in \( \mathcal{S}(X, \omega; \Sigma) \) connecting \( \alpha \) and \( \beta \).

\[ \square \]

**Remark 5.** The proof above is inspired by the work of Smillie and Weiss (\cite{32}).

### 2.3. Arc graph

Let us forget about the translation structure for a while, and focus on the underlying topological surface with marked points \((X, \Sigma)\). An arc \( \underline{a} \) on \((X, \Sigma)\) is called *properly embedded* if \( \partial \underline{a} \subset \Sigma \) and the interior is disjoint from \( \Sigma \). The *arc graph* \( \mathcal{A}(X, \Sigma) \), denoted by \( \mathcal{A}(X, \Sigma) \), is a graph such that the vertices \( \mathcal{A}^0(X, \Sigma) \) are isotopy classes of properly embedded non-trivial arcs on \((X, \Sigma)\), the edges \( \mathcal{A}^1(X, \Sigma) \) are pairs of isotopy classes of properly embedded non-trivial arcs which can be realized interiorly disjoint.

It is clear that the saddle connection graph \( \mathcal{S}(X, \omega; \Sigma) \) is a subgraph of the arc graph \( \mathcal{A}(X, \Sigma) \). Let \( I : \mathcal{S}(X, \omega; \Sigma) \to \mathcal{A}(X, \omega) \) be the canonical embedding.

For any topological arc \( \underline{a} \in \mathcal{A}(X, \omega) \), the \(|\omega|-\text{geodesic} \) representative consists of several saddle connections which appear in the \(|\omega|-\text{geodesic} \) representative of \( \underline{a} \). The following lemma is from \cite{28}.

**Lemma 2.4** (\cite{28}). For any two disjoint arcs \( \alpha, \beta \) in \( \mathcal{A}^0(X, \Sigma) \), saddle connections from \( s(\alpha) \) and \( s(\beta) \) are either disjoint or equal.

**Remark 6.** The converse is not true. In fact, there exist \( \alpha \in \mathcal{A}^0(X, \Sigma), \beta \in \mathcal{S}^0(X, \omega; \Sigma) \subset \mathcal{A}^0(X, \Sigma), \) such that \( \beta \in s(\alpha) \) and \( \alpha, \beta \) intersect transversely as elements in \( \mathcal{A}^0(X, \Sigma) \).

### 2.4. Hyperbolicity

Let each edge in \( \mathcal{A}(X, \Sigma) \) and \( \mathcal{S}(X, \omega; \Sigma) \) be of length one. In this way, we get two geodesic metric graphs, \( (\mathcal{A}(X, \Sigma), d_A) \) and \( (\mathcal{S}(X, \omega; \Sigma), d_S) \), in the sense that every two vertices can be connected by a path whose length is exactly their distance.

**Definition 5.** A geodesic metric space \((X, d)\) is called \( \delta\)-hyperbolic if for every geodesic triangle \([xy] \cup [yz] \cup [zx]\), each geodesic in \{\([xy]\), \([yz]\), \([zx]\)\} is contained in the \( \delta\)-neighbourhood of the union of the other two.
Theorem 2.5 ([25] [15] [14]). The arc graph \((A(X, \Sigma), d_A)\) is \(\delta\)-hyperbolic.

Remark 7. In fact, Hesel-Przytycki-Webb ([14]) show that \((A(X, \Sigma), d_A)\) is 7-hyperbolic.

The following proposition is a direct consequence of Lemma 7.1.

Proposition 2.6. The inclusion map
\[
(S(X, \omega; \Sigma), d_S) \hookrightarrow (A(X, \omega; \Sigma), d_A)
\]
is an isometric embedding, such that for any two saddle connections \(\alpha, \beta\), there is a geodesic in \((A(X, \omega), d_A)\) connecting them which is contained in \((S(X, \omega; \Sigma), d_S)\). In particular, \((S(X, \omega; \Sigma), d_S)\) is connected and \(\delta\)-hyperbolic.

Proof. Let \(\alpha, \beta\) be two vertices in \(S(X, \omega; \Sigma)\). It is clear that \(d_S(\alpha, \beta) \leq d_A(\alpha, \beta)\). Let \(\alpha_0 = \alpha, \alpha_1, \ldots, \alpha_l = \beta\) be a geodesic in \((A(X, \omega), d_A)\). Let \(\gamma_i\) be a saddle connection in the \(|\omega|\)-geodesic representative of \(\alpha_i\). It follows from Lemma 7.1 that \(\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_l = \beta\) is a path in \((S(X, \omega; \Sigma), d_S)\), which implies that \(d_S(\alpha, \beta) \leq l = d_A(\alpha, \beta)\). Therefore, \(d_S(\alpha, \beta) = d_A(\alpha, \beta)\).

As a consequence, \(\gamma_0 = \alpha, \gamma_1, \ldots, \gamma_l = \beta\) is a geodesic in \((S(X, \omega; \Sigma), d_S)\).

Together with Theorem 2.5 this implies that \((S(X, \omega; \Sigma), d_S)\) is \(\delta\)-hyperbolic. \(\square\)

Remark 8. For a translation surface whose Veech group contains a pseudo-Anosov element, Minsky-Taylor ([28]) proved a stronger result which states that the inclusion map from a subset \(V\) of \(S(X, \omega)\) which consists of saddle connections in some “veering triangulation” of the corresponding mapping torus, to the arc and curve graph, is an isometric embedding, such that for any two saddle connections \(\alpha, \beta\) in \(V\), there is a geodesic in the arc and curve graph connecting them which is contained in \(V\). We thank Kasra Rafi for informing us this result.

2.5. Measured foliations. A measured foliation on a surface is a foliation equipped with a transversely invariant measure. A measured foliation is called uniquely ergodic if every two transverse measures supported on the underlying foliation differ by a positive scalar. The directional flow in each direction \(\theta\) of a translation surface \((X, \omega)\) determines a measured foliation \(\mu_\theta\). Let us denote by \(\mathcal{MF}(X, \omega)\) the set of measured foliation arising from the directional flows on \((X, \omega)\). It is known ([21] Theorem 1]) that for almost every direction \(\theta\), the measured foliation \(\mu_\theta \in \mathcal{MF}(X, \omega)\) is uniquely ergodic. Moreover, if \(\mu_\theta \in \mathcal{MF}(X, \omega)\) is uniquely ergodic, then there is no saddle connection on \((X, \omega)\) in the direction \(\theta\), and every leaf in this direction is dense. Since there are countably many saddle connections on \((X, \omega; \Sigma)\), for almost every direction \(\theta\), there is no saddle connection in \(\theta\) direction on \((X, \omega; \Sigma)\), and the measured foliation \(\mu_\theta \in \mathcal{MF}(X, \omega)\) is uniquely ergodic. Let \(\mathcal{MF}_\theta(X, \omega; \Sigma)\) be the set of all such directional measured foliations on \((X, \omega)\). (See [9] [33] for more details on measured foliations.)
2.6. **Infinite diameter.**

**Proposition 2.7.** Let \((X, \omega; \Sigma)\) be a translation surface with marked points. The saddle connection graph \(S(X, \omega; \Sigma)\) has infinite diameter.

*Proof.* We use an argument of F. Luo as explained in [23 §4.3]. Suppose to the contrary that the diameter of \(S(X, \omega; \Sigma)\) is finite.

After rotation, we may assume that the horizontal measured foliation of \((X, \omega; \Sigma)\) belongs to \(\text{MF}_u(X, \omega; \Sigma)\). Fix a non-horizontal saddle connection \(\alpha\). Let \(\{\beta_i\}_{i \geq 1}\) be a sequence of saddle connections, such that the limit of the directions is horizontal. This implies that for every non-horizontal segment \(I\), there exists \(T > 0\), such that \(\beta_i\) intersects \(I\) for all \(i > T\).

By assumption, the diameter of \(S(X, \omega; \Sigma)\) is finite, there is a subsequence, still denoted by \(\{\beta_i\}_{i \geq 1}\) for convenience, such that \(d_S(\alpha, \beta_i) = N\) for some \(N < \infty\). For each \(i\), consider a geodesic \(\beta_i^0 = \alpha, \beta_i^1, \ldots, \beta_i^{N-1}, \beta_i^N = \beta_i\) in the saddle connection graph from \(\alpha\) to \(\beta_i\). On the other hand, for every non-horizontal segment \(I\), \(\beta_i\) intersects \(I\) for all large enough \(i\). Therefore, the limit of the directions of \(\beta_i^{N-1}\) has to be horizontal. Otherwise, \(\beta_i = \beta_i^N\) would intersect \(\beta_i^{N-1}\) for large enough \(i\). Inductively, we obtain that the limit of the directions of \(\beta_i^1\) has to be horizontal. As a consequence, \(\beta_i^1\) would intersect the non-horizontal saddle connection \(\alpha = \beta_i^0\) for large enough \(i\), which contradicts that \(d_S(\alpha, \beta_i^1) = 1\).

\[\square\]

**Remark 9.** We thank Duc-Manh Nguyen for asking the question that whether the diameter \(S(X, \omega; \Sigma)\) is always infinite.

3. **Semisimple cylinder preserving**

Throughout this section, we assume that \((X, \omega; \Sigma)\) and \((X', \omega'; \Sigma')\) are translations surfaces with marked points which are not tori with one marked point, and that \(F : S(X, \omega; \Sigma) \to S(X', \omega'; \Sigma')\) is an isomorphism. For convenience, we denote \(F(\gamma)\) by \(\gamma'\) for any saddle connection \(\gamma\) on \((X, \omega; \Sigma)\).

3.1. **Pentagons and Quadrilaterals.**

![Figure 1](image)

**Figure 1.** Convex pentagon associated with the triangle \(\Delta\).
Lemma 3.1. Let \( \gamma_1, \gamma_2, \gamma_3 \) be three saddle connections bounding a triangle \( \Delta \) on \((X, \omega; \Sigma)\) which is not contained in a simple cylinder. Then there is a convex pentagon \( \mathcal{P} = (P_1P_2P_3P_4) \subset \mathbb{R}^2 \) and a locally isometric immersion \( I : \mathcal{P} \to (X, \omega; \Sigma) \) (see Figure 1), such that \( \gamma \) is embedded in the interior of \( \mathcal{P} \), \( I(P_1P_3) = \gamma_1 \), \( I(P_3P_4) = \gamma_2 \), \( I(P_4P_1) = \gamma_3 \), the angle bounded by \( \overline{P_3P_4} \) and \( \overline{P_3P_5} \) at \( P_3 \) is less than \( \pi \), i.e. \( \mathcal{P} \) is strictly convex at \( P_3 \).

Proof. Suppose that \( \gamma_1, \gamma_2, \gamma_3 \) are arranged in the counterclockwise order with respect to \( \Delta \). After rotation, we may assume that \( \gamma_3 \) is horizontal. Develop the triangle \( \Delta \) to \((X, \omega; \Sigma)\) in \( \mathcal{P} \). Let \( P \) be the inverse of this developing map. Let \( H \) be a bi-infinite horizontal strip which contains \( P_1, P_3, P_4 \) in the boundary. Consider the image \( H := I(\bar{H}) \) of \( \bar{H} \) in \((X, \omega; \Sigma)\).

Case 1: the interior of \( H \) contains at least one singularity. It is clear that \( H - \Delta \) has two components, denoted by \( \bar{H}_1 \) and \( \bar{H}_2 \), such that \( \bar{H}_1 \) is on the left side and \( \bar{H}_2 \) is on the right side. Let \( P_2 \in \bar{H}_1 \) (resp. \( P_5 \in \bar{H}_2 \)) be the singularity whose horizontal distance to \( P_1 \) is minimal among all the singularities in \( \bar{H}_1 \) (resp. \( \bar{H}_2 \)). If there are more than one such points, we chose the one with minimal vertical distance to \( P_1 \) among them.

The five points, \( P_1, P_2, P_3, P_4, P_5 \) determine a convex pentagon \( \mathcal{P} = (P_1P_2P_3P_4P_5) \) in \( H \). To prove the lemma in this case, it remains to show that the map \( I \) is embedded in the interior of the pentagon \( \mathcal{P} = (P_1P_2P_3P_4P_5) \). If the interior is not embedded, either the wedge bounded by \( \overline{P_1P_3} \) and \( \overline{P_4P_5} \) would cross \( \overline{P_4P_5} \cup \overline{P_4P_1} \), or the wedge bounded by \( \overline{P_3P_5} \) and \( \overline{P_3P_1} \) would cross \( \overline{P_3P_5} \cup \overline{P_3P_4} \), which contradicts the assumption that the horizontal distances from \( P_2, P_5 \) to \( P_1 \) are minimal.

Case 2: the interior of \( H \) contains no singularities. In this case, \( H \) is a cylinder (see Figure 1(b)). By assumption, \( H \) is not a simple cylinder. Denote by \( \partial_1 \) the boundary component of \( H \) which contains \( P_1 \), and by \( \partial_2 \) the other boundary component. Let \( P_2 \in \partial_1 \) be the first singularity which is on the left side of \( P_1 \). Let \( P_5 \in \partial_2 \) be the first singularity on the right side of \( P_4 \). The resulting pentagon \( \mathcal{P} = (P_1P_2P_3P_4) \) satisfies all the properties listed in the lemma.

Corollary 3.2. Let \( \gamma_1, \gamma_2, \gamma_3 \) be three saddle connections bounding a triangle on \((X, \omega; \Sigma)\) which is not contained in a simple cylinder. Then there is an immersed strictly convex quadrilateral \( \mathcal{Q}' = (Q_1'Q_2'Q_3'Q_4') \) on \((X, \omega'; \Sigma')\) which is embedded in the interior, such that

1. the interior of \( \mathcal{Q}' \) is disjoint from \( \gamma_3' \),
2. \( Q_1'Q_3' = \gamma_1' \), \( Q_1'Q_4' = \gamma_2' \).
In particular, \( \gamma'_i, \gamma'_{i+1} \), together with one more saddle connection, bound a triangle on \( (X', \omega'; \Sigma') \) whose interior is disjoint from \( \gamma'_{i-1} \), where \( i = 1, 2, 3 \) and \( \gamma_4 = \gamma_1, \gamma_0 = \gamma_3 \).

**Proof.** Consider the convex pentagon \( \mathcal{P} = (P_1 P_2 P_3 P_4 P_5) \) in Lemma 3.1. Let \( \beta_1, \beta_2 \) be the saddle connections in \( \mathcal{P} \) connect \( P_2 \) to \( P_4, P_5 \), respectively. Let us extend \( \{P_1 P_2, P_2 P_3, P_3 P_4 = \gamma_3, P_4 P_5, P_5 P_1, P_1 P_3 = \gamma_1, P_1 P_4 = \gamma_2\} \) to a triangulation \( \Gamma' \) of \( (X, \omega; \Sigma) \). It is clear that \( \Gamma' := F(\Gamma) \) is a triangulation of \( (X', \omega'; \Sigma') \). Since \( \beta_1 \) intersects \( \gamma_1 \in \Gamma \), \( \beta_1 \) intersects no other elements in \( \Gamma, \beta'_1 \) intersects \( \gamma'_1 \in \Gamma' \) and intersects no other elements in \( \Gamma' \). Therefore, there is a strictly convex quadrilateral \( \tilde{Q}' \subset (X', \omega'; \Sigma') \) containing \( \gamma'_1 \) as a diagonal. Now, let us consider \( \beta_2 \). It intersects \( \gamma_1, \gamma_2 \in \Gamma \), and intersects no other elements in \( \Gamma \). Therefore, \( \beta'_2 \) intersects \( \gamma'_1, \gamma'_2 \in \Gamma' \), and intersects no other elements in \( \Gamma' \). This implies that \( \gamma'_2 \) is one of the boundary saddle connections of \( \tilde{Q}' \). Let us label the vertices of \( \tilde{Q}' \) by \( Q'_1, Q'_2, Q'_3, Q'_4 \), such that \( Q'_1 Q'_3 = \gamma'_1, Q'_1 Q'_4 = \gamma'_2 \).

\( \square \)

### 3.2. Semisimple Cylinders.

Notice that every interior saddle connection of a cylinder connects its two boundary components. For a semisimple cylinder, all of its interior saddle connections share a common endpoint which is the singular point of a simple boundary component (see Definition 3 for related definitions).

**Definition 6** (Consecutive saddle connections). Let \( C \) be a semisimple cylinder, let \( A \) be the singular point of a simple boundary of \( C \). Two interior saddle connections of \( C \) are called consecutive with respect to \( A \) if they, together with a saddle connection from the other boundary component of \( C \), bound a triangle in \( C \).

**Remark 10.** For a simple cylinder, there are two candidate reference points for the definition of consecutive saddle connections. It is not hard to see that two saddle connections are consecutive with respect to one reference point if and only if they are consecutive with respect to the other one. Therefore, there is no ambiguity if we say two saddle connections of a semisimple cylinder are consecutive without mentioning the reference point.

**Proposition 3.3.** Let \( C \subset (X, \omega; \Sigma) \) be a semisimple cylinder. Then there is a corresponding semisimple cylinder \( C' \subset (X', \omega'; \Sigma') \) such that

(i) \( \beta \) is an interior saddle connection of \( C \) if and only if \( \beta' \) is an interior saddle connection of \( C' \), and

(ii) the non-simple boundary components of \( C \) and \( C' \) have the same number of saddle connections.

(iii) let \( \gamma^+ \) be the saddle connection of a simple boundary component of \( C \), then \( F(\gamma^+) \) is the saddle connection of a simple boundary component of \( C' \).

In particular, if \( C \) is a simple cylinder, so is \( C' \).
Proof. Part (i). After applying the $SL(2,\mathbb{R})$ action, we may assume that $C$ is a horizontal cylinder. Let $\{\gamma^+\}$ and $\{\gamma^-_1, \ldots, \gamma^-_n\}$ be the boundary components of $C$. Without loss of generality, assume that $\gamma^+$ is the top boundary component and that $\gamma^-_1, \ldots, \gamma^-_n$ are arranged from left to right cyclically.

Let $A$ be the left endpoint of $\gamma^+$. Notice that every interior saddle connection of $C$ connects the top boundary component and the bottom boundary component. Therefore, we can assume that all of these interior saddle connections start from $A$, and hence are downward. Now, let us label the interior saddle connections of $C$ by $\{\beta_j : j \in \mathbb{Z}\}$ such that they appear in the counterclockwise order with respect to $A$, when $j$ varies from $-\infty$ to $+\infty$. In particular, $\beta_i$ and $\beta_{i+1}$ are consecutive for any $i \in \mathbb{Z}$. It is clear that

$$(1) \quad i(\beta_i, \beta_j) = 0 \text{ if and only if } |i - j| \leq n,$$

and that

$$(2) \quad i(\beta_i, \beta_{i+n+1}) = 1, \forall i \in \mathbb{Z}.$$ 

In fact, we have,

$$(3) \quad i(\beta_i, \beta_j) = \left\lfloor \frac{|j - i| - 1}{n} \right\rfloor,$$

where $\lfloor \cdot \rfloor$ represents the Euclidean norm and $[x]$ represent the largest integer not exceeding $x \in \mathbb{R}$.

Now we split the remaining of the proof into three claims.

Claim 0. For any $n+1$ consecutive saddle connections $\beta'_i, \beta'_{i+1}, \ldots, \beta'_{i+n+1}$, the complement $(X', \omega'; \Sigma') \setminus (\beta'_{i+1} \cup \cdots \cup \beta'_{i+n+1})$ contains no triangle components.

**Proof of Claim 0.** If $n = 1$, i.e. $C$ is a simple cylinder, then there is no disjoint triples in $\{\beta_j, j \in \mathbb{Z}\}$. The claim follows. Assume that $n \geq 2$. Suppose to the contrary that there exist three saddle connections $\beta'_i, \beta'_j, \beta'_k$ with $1 \leq i < j < k \leq n+1$ such that they bound a triangle on $(X', \omega'; \Sigma')$. It follows from Equation (3) that

$$(4) \quad i(\beta'_{i+n+1}, \beta'_i) = 1, \quad i(\beta'_{i+n+1}, \beta'_j) = 0, \quad i(\beta'_{i+n+1}, \beta'_k) = 0,$$

$$(5) \quad i(\beta'_{j+n+1}, \beta'_j) = 1, \quad i(\beta'_{j+n+1}, \beta'_k) = 0, \quad i(\beta'_{j+n+1}, \beta'_i) = 0,$$

and that $i(\beta'_{i+n+1}, \beta'_{j+n+1}) = 0$. But Equations (4) and (5) together with the fact that $\beta'_i, \beta'_j, \beta'_k$ bound a triangle imply that $i(\beta'_{i+n+1}, \beta'_{j+n+1}) \neq 0$.

Claim 1. All the saddle connections $\{\beta'_i\}$ share a common vertex $A'$ such that they appear in the clockwise with respect to $A'$ when $i$ varies from $-\infty$ to $+\infty$, or they appear in the counterclockwise order with respect to $A'$ when $i$ varies from $-\infty$ to $+\infty$.

**Proof of Claim 1.** If $n = 1$, i.e. $C$ is a simple cylinder, the claim follows. Assume that $n \geq 2$. We first show that any three consecutive saddle connections $\beta'_i, \beta'_{i+1}$ and $\beta'_{i+2}$ share a common vertex $A'$ and that they appear
in the clockwise or counterclockwise order such that the angle bounded by $eta_i$, $eta_{i+2}$ at $A'$ is less than $\pi$. Consider the $n + 1$ consecutive saddle connections $\beta_{i-n-1}, \cdots, \beta_{i-1}$. Let us extend $\gamma^+, \gamma_1^-, \cdots, \gamma_n^-, \beta_{i-1-n}, \cdots, \beta_{i-1}$ to a triangulation $\Gamma$ of $(X, \omega; \Sigma)$. It is clear that $\Gamma' = F(\Gamma)$ is a triangulation of $(X', \omega'; \Sigma')$. Notice that $\beta_i$ intersects $\beta_{i-1-n}$ and intersects no other saddle connections in $\Gamma$. Therefore, $\beta'_i$ intersects $\beta_{i-1-n}'$ and intersects no other saddle connections in $\Gamma'$, which means that $\beta'_i, \beta'_{i-1-n}$ are the two diagonals of a convex quadrilateral $Q'$ which is the union of two triangles from $(X', \omega'; \Sigma') \setminus \Gamma'$. On the other hand, $\beta_{i+1}$ intersects both $\beta_{i-1-n}$ and
\( \beta_{i-n} \), and intersects no other saddle connections in \( \Gamma \). Therefore, \( \beta'_{i+1} \) intersects both \( \beta'_{i-1-n} \) and \( \beta'_{i-n} \), and intersects no other saddle connections in \( \Gamma' \), which implies that \( \beta'_{i-n} \) is one of the boundary saddle connections of \( Q' \).

There are two possibilities in the sense that \( \beta'_i \) and \( \beta'_{i-n} \) can appear in the clockwise or counterclockwise order with respect to the triangle determined by them (since \( C \) is not a simple cylinder, \( \beta_i \) and \( \beta_{i-n} \) determine a unique triangle in \( C \)). Correspondingly, \( \beta'_i \) and \( \beta'_{i-n} \) determine a unique triangle in \( C' \), see Figure 2(b). In the following, we assume that they appear in the counterclockwise order. The clockwise case is similar.

Let \( \delta'_1, \delta'_2, \delta'_3 \) be the remaining boundary saddle connections of \( Q' \) such that \( \beta'_{i-n}, \delta'_1, \delta'_2 \) and \( \delta'_3 \) appear in the counterclockwise order (see the left figure in Figure 2(b)). Let \( A' \) be the vertex of \( Q' \) bounded by \( \delta'_1 \) and \( \delta'_2 \). Then \( \beta'_i \) and \( \beta'_{i+1} \) share the common vertex \( A' \). Consider the triangle \( \Delta'_1 \) of \( (X', \omega'; \Sigma') \cap \Gamma' \) not contained in \( Q' \) which contains \( \beta'_{i-n} \) in the boundary. Let \( \delta'_1, \delta'_3 \) be the remaining saddle connections of \( \Delta'_1 \) such that \( \gamma'_{i-n}, \delta'_1, \delta'_3 \) appear in the counterclockwise order. Notice that \( \beta_{i+2} \) intersects all of \( \beta_{i-1-n}, \beta_{i-n} \) and \( \beta_{i-n+1} \), and intersects no other saddle connections in \( \Gamma' \cup \{ \beta_i, \beta_{i+1} \} \). Therefore, \( \beta'_{i-n+1} \) has to be one of \( \{ \delta'_1, \delta'_3, \delta'_4 \} \). On the other hand, \( \beta_{i-2n-1} \) intersects both \( \beta_{i-n} \) and \( \beta'_{i-n+1} \), but it is disjoint from other saddle connections in \( \Gamma' \). This implies that \( \beta'_{i-n+1} \) can not be \( \delta'_1 \). Therefore, it has to be one of \( \{ \delta'_1, \delta'_3 \} \). Since \( \beta'_{i+2} \) intersects all of \( \beta'_{i-1-n}, \beta'_{i-n} \) and \( \beta'_{i-n+1} \), and intersects no other saddle connections in \( \Gamma' \), it must contain \( A' \) as one of its endpoints. Finally, consider the triangulation \( \{ \beta'_i, \beta'_{i+1}, \beta'_{i+2} \} \cup \Gamma' \\setminus \{ \beta'_{i-1-n}, \beta'_{i-n}, \beta'_{i-n+1} \} \). \( \beta'_{i-n+2} \) intersects both \( \beta'_i \) and \( \beta'_{i+1} \), but it is disjoint from other saddle connections in this triangulation. In particular, it is disjoint from \( \beta'_{i+2} \). Therefore, \( \{ \beta'_i, \beta'_{i+1}, \beta'_{i+2} \} \) must appear in the counterclockwise order with respect to \( A' \), which implies that the angle bounded by \( \beta'_i, \beta'_{i+2} \) at \( A' \) is less than \( \pi \), and that \( \beta'_{i-n+1} \) is \( \delta'_3 \).

That \( \beta'_{i-n-1} \) is \( \delta'_3 \) implies that \( \beta'_{i+1} \) and \( \beta'_{i-n+1} \) appear in the counterclockwise order with respect to the triangle determined by them. Repeating the argument above, we obtain that \( \{ \beta'_{i+1}, \beta'_{i+2}, \beta'_{i+3} \} \) appear in the counterclockwise order with respect to \( A' \). Inductively, this shows that all the saddle connections \( \{ \beta'_i \} \) share a common vertex \( A' \) such that they appear in the counterclockwise order when \( j \) varies from \(-\infty \) to \( +\infty \), based on the assumption that for some \( i_0, \beta'_{i_0} \) and \( \beta'_{i_0-n} \) appear in the counterclockwise order with respect to the triangle determined by them. If for some \( i_0, \beta'_{i_0} \) and \( \beta'_{i_0-n} \) appear in the clockwise order with respect to the triangle determined by them, the argument above shows that all the saddle connections \( \{ \beta'_i \} \) share a common vertex \( A' \) such that they appear in the clockwise order with respect to \( A' \) when \( j \) varies from \(-\infty \) to \( +\infty \).

Claim 2. All the saddle connections \( \{ \beta'_i \} \) are interior saddle connections of a semisimple cylinder \( C' \) on \( (X', \omega'; \Sigma') \).
Proof of Claim 2. By Claim 1, all the saddle connections \( \{ \beta_j \} \) share a common vertex \( A' \) such that they appear in the clockwise or counterclockwise order with respect to \( A' \) when \( j \) varies from \(-\infty\) to \(+\infty\). Here we assume that they appear in the counterclockwise order. The clockwise case is similar.

Consider the \( n+1 \) consecutive saddle connections \( \beta_0, \ldots, \beta_n \). By the same argument as in the proof of claim 1, \( \beta'_0 \) and \( \beta'_n \) are the two diagonals of a convex quadrilateral \( Q' \) which is the union of two triangles from \((X', \omega'; \Sigma')\setminus\Gamma'\), and that \( \beta_0 \) is one of the boundary saddle connections of \( Q' \) (see Figure 2(c)). Moreover, \( \beta'_1 \) dose not intersect \( Q' \).

Let \( \delta'_1, \delta'_2, \delta'_3 \) be the remaining boundary saddle connections of \( Q' \) such that \( \beta'_0, \delta'_1, \delta'_2 \) and \( \delta'_3 \) appear in the counterclockwise order. It follows from Claim 1 and Equation [3] that \( \delta'_2 \) is exactly \( \beta'_{n-1} \), see Figure 2(c). Choose an orientation of \( \beta'_0 \). After applying the \( SL(2, \mathbb{R}) \) action, we may assume that \( \beta'_0 \) is vertically downward from \( A' \) and that \( \delta'_1 \) is horizontal. By assumption, \( \beta'_0, \beta'_1, \beta'_2, \beta'_3, \ldots, \beta'_n, \ldots \) appear in the counterclockwise order with respect to \( A' \). This implies that \( \beta'_1, \beta'_2, \beta'_3, \ldots, \beta'_n, \ldots \) are downward to the right from \( A' \). Let \( (x_i, y_i) \in \{(x, y) \in \mathbb{R}^2 : y < 0\} \) be the holonomy vector of \( \beta'_j \), \( j \geq 1 \), from \( A' \). To prove Claim 2, it suffices to prove that \( y_1 = y_2 = \cdots = y_n \).

Suppose that there exist \( j \neq j' \in [1, n] \), such that \( y_j \neq y_{j'} \), then there exists \( i \in (1, 2n) \) such that \( y_i < \min\{y_{i-1}, y_{i+1}\} \), which implies that the saddle connection connecting \((x_{i-1}, y_{i-1})\) and \((x_{i+1}, y_{i+1})\) intersects \( \beta'_i \) but does not intersect \( \beta'_{i-1} \) nor \( \beta'_{i+1} \). But this is impossible, because there is no saddle connection on \((X, \omega; \Sigma)\) which intersects \( \beta_i \) but does not intersect \( \beta_{i-1} \) nor \( \beta_{i+1} \).

**Part (ii).** Notice that the number of saddle connections of a non-simple boundary component of a semisimple cylinder \( C \) is the maximum number \( n \) such that \( C \) has \( n + 1 \) interior saddle connection which are pairwise disjoint.

**Part (iii).** The proof is split into two steps. In step 1, we show that \( F(\gamma^+) \) is a boundary saddle connection of \( C' \); in step 2, we show that \( F(\gamma^+) \) is one of the two boundary components of \( C'' \).

**Step 1.** Cut \( C \) along the interior saddle connection \( \beta_n \), we obtain a generalized parallelogram \( \mathcal{P} \) containing \( \beta_n \) as a pair of opposite edges (see Figure 2(d)). Let \( H \) be the infinite (closed) strip obtained by extending \( \mathcal{P} \) in the direction of \( \beta_n \). \( H \setminus \mathcal{P} \) has two components, denoted by \( H^+ \) and \( H^- \) respectively the top one and the bottom one. Let us move \( \gamma^+ \) in \( H^+ \) in the direction of \( \beta_n \). Denote by \( \gamma^+(t) \) the position of \( \gamma^+ \) at time \( t > 0 \). Since \((X, \omega; \Sigma)\) has finite area, \( \gamma^+(t) \) will meet zeros of \( \omega_1 \) during this process. Let \( \hat{A} \) the first such zero. Connect \( \hat{A} \) to the endpoints of \( \gamma^+ \) by two saddle connections \( \eta_1, \eta_2 \).

Let \( \Omega \) be the pentagon bounded by \( \eta_1, \beta_n, \gamma_1^-, \beta_1, \) and \( \eta_2 \). We claim that the interior of \( \Omega \) is embedded in \((X, \omega; \Sigma)\). Otherwise, both \( \eta_1 \) and \( \eta_2 \) will intersect \( \gamma_1^- \) or \( \beta_1 \). This contradicts the assumption that \( \hat{A} \) is the first zero.
met by $\gamma^+(t),\ t > 0$, because it would meet the common endpoints of $\gamma_1^-$ and $\beta_1$ before it meets $\hat{A}$.

Notice that there is a diagonal saddle connection $\xi$ in the interior of $Q$ intersecting $\gamma^+$ and $\beta_0$. Now, let us extend the set $\{\eta_2, \eta_1, \beta_n, \gamma_1^-, \beta_1, \beta_0, \gamma^+\}$ to a triangulation $\Gamma_1$ of $(X, \omega; \Sigma)$. It is clear that $\xi'$ intersects $F(\gamma^+)$ and $\beta_0'$, and intersects no other elements in $\Gamma_1 = F(\Gamma_1)$. On the other hand, $\beta_0'$ is an interior saddle connection of the semisimple cylinder $C'$ by the part (i) above. Therefore, $F(\gamma^+)$ is a boundary saddle connection of $C'$.

Step 2. Notice that for any $i \in \mathbb{Z}, \beta_i, \beta_{i+n}$ and $\gamma^+$ bound a triangle on $(X, \omega; \Sigma)$. It follows from Corollary 3.2 that for any $i \in \mathbb{Z}$, $F(\gamma^+)$ and $\beta_i'$, together with one more saddle connection, bound a triangle on $(X', \omega'; \Sigma')$. Therefore, $F(\gamma^+)$ is contained in a boundary component of $C'$ which contains at most two saddle connections.

Suppose that $F(\gamma^+)$ is contained in a boundary component of $C'$ which contains exactly two saddle connections (see Figure 2(d)). Let $\gamma'$ be the other saddle connection of this boundary component. Let $A_3'$ be the vertex as indicated in Figure 2(d). By part (i), there exist four consecutive interior saddle connections $\beta_1', \beta_2', \beta_3', \beta_4'$ such that each of $\{\beta_1', \beta_2', F(\gamma^+)\}, \{\beta_2', \beta_3', \gamma'\}, \{\beta_3', \beta_4', F(\gamma^+)\}$ bounds a triangle. On the other hand, it follows from part (ii) that the non-simple boundary component of $C$ consists of exactly two saddle connections. Therefore, $\beta_1, \beta_3$ and $\gamma^+$ bound a triangle on $(X, \omega; \Sigma)$. By Corollary 3.2, this implies, $\beta_3^+$ and $F(\gamma^+)$, together with one more saddle connection, bound a triangle whose interior is disjoint from $\beta_1'$, which is impossible, see Figure 2(d).

For the convenience of reference, we summarize Claim 1 and Claim 2 in the proof of Proposition 3.3 as following.

**Corollary 3.4.** The images of two consecutive interior saddle connections of a semisimple cylinder on $(X, \omega; \Sigma)$ under the isomorphism $F$ are two consecutive interior saddle connections of a corresponding semisimple cylinder on $(X', \omega'; \Sigma')$.

Combining Corollary 3.2 and Proposition 3.3, we get the following corollary, which is an improvement of Corollary 3.2.

**Corollary 3.5.** Let $\gamma_1, \gamma_2, \gamma_3$ be three saddle connections bounding a triangle on $(X, \omega; \Sigma)$ which is not contained in a simple cylinder. Then there is an immersed pentagon $Q' = (Q_1'Q_2'Q_3'Q_4'Q_5')$ on $(X', \omega'; \Sigma')$ which is embedded in the interior, such that

1. The interior of $Q'$ is disjoint from $\gamma_3'$, and
2. $Q_1'Q_2' = \gamma_1', Q_1'Q_4' = \gamma_2'$, and
3. The interior angles at $Q_1'$, $Q_3'$ are less than $\pi$, i.e. $Q$ is strictly convex at $Q_1'$ and $Q_3'$.

**Proof.** Consider the convex pentagon $P = (P_1P_2P_3P_4P_5)$ in Lemma 3.1. Let $\beta_1, \beta_2$ be the saddle connections in $P$ connect $P_2$ to $P_4, P_5$, respectively. Let
us extend \( \{P_1P_2, P_2P_3, P_3P_4 = \gamma_3, P_4P_5, P_5P_1, P_1P_3 = \gamma_1, P_1P_4 = \gamma_2\} \) to a triangulation \( \Gamma \) of \( (X, \omega; \Sigma) \).

Consider the strictly convex quadrilateral \( Q' = (Q'_1Q'_2Q'_3Q'_4) \) obtained in Corollary 3.2. We claim that there is a triangle \( \Delta' \) in \( (X', \omega'; \Sigma') \) which is not contained in \( \tilde{Q'} = (Q'_1Q'_2Q'_3Q'_4) \), and which contains \( \gamma'_2 \) in the boundary. Otherwise, the two triangles in \( (X', \omega'; \Sigma') \) which contain \( \gamma'_2 \) in the boundary are exactly the two triangles in \( \tilde{Q'} \), which implies that \( \tilde{Q'} \) corresponds to a simple cylinder on \( (X', \omega'; \Sigma') \) with \( \overline{Q'_2Q'_3} \) and \( \overline{Q'_1Q'_4} \) identified. By Proposition 3.3, \( \gamma_1 \) and \( \gamma_2 \) are contained in a simple cylinder on \( (X, \omega; \Sigma) \). Therefore, the triangle bounded by \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) is also contained in this simple cylinder, which contradict the assumption that the triangle bounded by \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) is not contained in a simple cylinder.

Denote by \( Q'_5 \) the other vertex of \( \Delta'_3 \). Consider the pentagon \( Q' = (Q'_1Q'_2Q'_3Q'_4Q'_5) \), where \( \overline{Q'_1Q'_3} = \gamma'_1, \overline{Q'_1Q'_4} = \gamma'_2 \). That \( \beta' \) intersects only \( \gamma'_1 \) and \( \gamma'_2 \) for saddle connections in \( \Gamma' \) implies that \( Q' \) contains \( \beta' \) as a diagonal connecting \( Q'_2 \) and \( Q'_5 \). Therefore, the interior angle of \( Q' \) at \( Q'_1 \) is less than \( \pi \). That the interior angle at \( Q'_3 \) is less than \( \pi \) follows from the fact that \( (Q'_1Q'_2Q'_3Q'_4) \) is a strictly convex quadrilateral.

4. Triangle preserving

Throughout this section, we assume that \( (X, \omega; \Sigma) \) and \( (X', \omega'; \Sigma') \) are translations surfaces with marked points, and that \( F : \mathcal{S}(X, \omega; \Sigma) \rightarrow \mathcal{S}(X', \omega'; \Sigma') \) is an isomorphism. For convenience, we denote \( F(\gamma) \) by \( \gamma' \) for any saddle connection \( \gamma \) on \( (X, \omega; \Sigma) \). The goal of this section is to prove the following triangle preserving theorem.

**Theorem 4.1.** Let \( \gamma_1, \gamma_2, \gamma_3 \) be three saddle connections bounding a triangle \( \Delta \) on \( (X, \omega; \Sigma) \). Then \( \gamma'_1, \gamma'_2, \gamma'_3 \) also bound a triangle \( \Delta' \) on \( (X', \omega'; \Sigma') \).

**Proof.** If \( (X, \omega; \Sigma) \) is a torus with one marked point, \( \{\gamma_1, \gamma_2, \gamma_3\} \) is a triangulation of \( (X, \omega; \Sigma) \). Therefore, \( \{\gamma'_1, \gamma'_2, \gamma'_3\} \) is a triangulation of \( (X', \omega'; \Sigma') \). In particular, they bound a triangle on \( (X', \omega'; \Sigma) \).

If \( \gamma_1, \gamma_2, \gamma_3 \) are contained in the interior of a simple cylinder on \( (X, \omega; \Sigma) \), then, by Proposition 3.3, \( \gamma'_1, \gamma'_2, \gamma'_3 \) are also contained in the interior of some simple cylinder on \( (X', \omega'; \Sigma') \). This implies that \( \gamma'_1, \gamma'_2, \gamma'_3 \) bound a triangle on \( (X', \omega'; \Sigma') \).

In the following, we assume that \( (X, \omega; \Sigma) \) is not a torus with one marked point and that the triangle bounded by \( \gamma_1, \gamma_2, \gamma_3 \) is not contained in the interior of a simple cylinder. It follows from Corollary 3.2, \( \gamma'_i, \gamma'_{i+1} \), together with one more saddle connection, bound a triangle \( \Delta'_{i-1} \) on \( (X', \omega'; \Sigma') \) whose interior is disjoint from \( \gamma'_{i-1} \), where \( i = 1, 2, 3 \) and \( \gamma_4 = \gamma_1, \gamma_0 = \gamma_3, \Delta_3 = \Delta'_0 \). After applying the \( \text{SL}(2, \mathbb{R}) \) action, we may assume that \( \Delta \) and \( \Delta'_0 \) are equilateral triangles. Since the the interior of the triangle \( \Delta'_i \) determined by \( \gamma'_2 \) and \( \gamma'_3 \) is disjoint from \( \gamma'_1 \), there are four possibilities for the position of
Figure 3. Four possibilities of $\gamma'_1, \gamma'_2, \gamma'_3$, where the red region in each case represents the triangle $\Delta'_1$, the dashed lines are saddle connections.

$\Delta'_1$ which are illustrated in Figure 3. Let us denote these types by I, II, III and IV, respectively.

Notice that in the case of type I, $\gamma'_1, \gamma'_2, \gamma'_3$ bound a triangle on $(X', \omega'; \Sigma')$ which is what we want. So we need to rule out the other three possibilities. The remaining of the proof will be split into three lemmas: Lemma 4.2, Lemma 4.3 and Lemma 4.4.

**Lemma 4.2.** Type II is equivalent to Type III or IV.

**Proof.** Consider the triangle $\Delta'_2$ determined by $\gamma'_1$ and $\gamma'_3$, whose interior is disjoint from $\gamma'_2$. There are five possibilities based on the position of $\gamma'_1$ and the order of $(\gamma'_3, \gamma'_1)$ with respect to $\Delta'_2$: type II-a, II-b, II-c, II-d, and II-e as in Figure 4, where $(\gamma'_3, \gamma'_1)$ appear in the counterclockwise order in type II-a and II-b, they appear in the clockwise order in type II-c, II-d, and II-e. For type II-b and II-c, the interior of the triangle $\Delta'_1$ determined by $\gamma'_2$ and $\gamma'_3$ intersects $\gamma'_1$, which contradicts Corollary 3.2. For type II-a and II-d, instead of observing the pair $(\gamma'_1, \gamma'_2)$, we can observe the pair $(\gamma'_2, \gamma'_3)$. In this viewpoint, type II-a and II-d are equivalent to type III and type IV, respectively. For type II-e, we translate $\gamma'_3$ to the other endpoint of $\gamma'_1$ than $A$ (see Figure 4 II-e)). Then we observe the pair $(\gamma'_2, \gamma'_1)$ instead of $(\gamma'_1, \gamma'_2)$. In this viewpoint, type II-e is equivalent to type IV.

**Lemma 4.3.** Type III is not realizable.

**Proof.** Consider the triangle $\Delta'_2$ determined by $\gamma'_1$ and $\gamma'_3$ whose interior is disjoint from $\gamma'_2$. There are five possibilities based on the position of $\gamma'_1$ and the the order of $(\gamma'_3, \gamma'_1)$ with respect to $\Delta'_2$: III-a, III-b, III-c, III-d, and III-e, see Figure 5, where $(\gamma'_3, \gamma'_1)$ appear in the counterclockwise order in type II-a, II-b and II-c, they appear in the clockwise order in type II-d, and II-e. We will show that none of them is realizable. In each case, let $\delta'_1, \delta'_2, \delta'_3$ be the third saddle connection of $\Delta'_1, \Delta'_2, \Delta'_3$ respectively.
Figure 4. The gray region represents the triangle $\Delta_2'$.

**Type III-a.** In this case, the interior of the triangle $\Delta_2'$ determined by $\gamma_1'$ and $\gamma_3'$ intersects $\gamma_2'$ (see Figure 5 (III-a)), which is impossible by Corollary 3.2.

**Type III-b.** Let $A_0'$, $A_1'$, $A_2'$, be the points as illustrated in Figure 5 (III-b). Let $\theta_1$ be the angle at $A_1'$ from $\delta_2'$ to $\delta_1'$ in the counterclockwise order. Let $\theta_2$ be the angle at $A_2'$ from $\delta_1'$ to $\delta_2'$ in the counterclockwise order. By Corollary
there is a pentagon $Q'$ (resp. $Q'$) on $(X', \omega'; \Sigma')$ containing $\gamma'_1, \gamma'_2$ (resp. $\gamma'_3, \gamma'_1$) as diagonals whose interior angle at $A'_1$ (resp. $A'_2$) is less than $\pi$, and whose interior does not intersect $\gamma'_3$ (resp. $\gamma'_2$). Therefore, $0 < \theta_1, \theta_2 < \pi$. Hence $0 < \theta_1 + \theta_2 < 2\pi$. On the other hand, since the two saddle connections labeled by $\delta'_2$ are parallel, $\theta_1 = (2\pi - \theta_2)$, i.e. $\theta_1 + \theta_2 = 2\pi$. As a consequence, type III-b is not realizable.

**Type III-c.** The proof in this case is similar to that of type III-b by looking at the angles at $A'_3, A'_4$. We omit the proof.
Type III-d. In this case, the interior of the triangle $\Delta'_1$ bounded by $\gamma'_2$, $\delta'_1$ and $\gamma'_3$ intersects $\gamma'_1$ (see Figure 5 (III-d)), which is impossible by Corollary 3.2.

Type III-e. Let $A'_6, A'_0, A'_7, A'_6, A'_9$ be the vertices indicated in Figure 6 (III-e). Again, it follows from Corollary 3.2 that there is a pentagon containing $\gamma'_2, \gamma'_3$ as diagonals, whose interior is disjoint from $\gamma'_1$ and whose interior angle at $A'_6$ is less than $\pi$. As a consequence, the angle at $A'_6$ from $\delta'_2$ to $\delta'_3$ in the counterclockwise order is less than $\pi$. Combined with the fact that $\Delta'_1, \Delta'_2, \Delta'_3$ are embedded triangles, this implies that there is a saddle connection $\eta'$ connecting $A'_5$ and $A'_7$, which intersects each of $\gamma'_2$ and $\gamma'_3$. Consider the parallelogram $\mathcal{P}'$ bounded by $\gamma'_1, \eta', \gamma'_1$ and $\delta'_1$. Since $(X', \omega'; \Sigma')$ is a translation surface, $\eta'$ is embedded in $(X', \omega'; \Sigma')$. Hence, $\eta'$ is the bottom boundary saddle connection of a semisimple cylinder $C'$ on $(X', \omega'; \Sigma')$. On the other hand, there are no zeros of $\omega'$ in the interior of $\mathcal{P}'$. As a consequence, the parallelogram $\mathcal{P}'$ corresponds exactly to the cylinder $C'$ and that $C'$ is a simple cylinder. In particular, $C'$ contains $\gamma'_1$ as an interior saddle connection.

Next, we show that on $(X', \omega'; \Sigma')$, each of $\gamma'_2$ and $\gamma'_3$ crosses $C'$ exactly once. Otherwise, if $\gamma'_2$ (resp. $\gamma'_3$) crosses $C'$ more than once on $(X', \omega'; \Sigma')$, then $\gamma'_2$ (resp. $\gamma'_3$) would intersect $\delta'_1$ in the interior, which contradicts the fact that the triangle $\Delta'_1$ bounded by $\gamma'_2, \gamma'_3$ and $\delta'_1$ is embedded on $(X', \omega'; \Sigma')$.

By Proposition 3.3 there is a corresponding simple cylinder $C$ on $(X, \omega; \Sigma)$ containing $\gamma_1$ as an interior saddle connection, and whose boundary saddle connections are $\eta := F^{-1}(\eta')$ and $\delta_1 := F^{-1}(\delta'_1)$. Since both $\gamma'_2$ and $\gamma'_3$ do not intersect $\delta'_1$, both $\gamma_2$ and $\gamma_3$ do not intersect $\delta_1$. Therefore each of $\gamma_2$ and $\gamma_3$ crosses $C$ exactly once. But this is impossible, because $\gamma_1, \gamma_2$ and $\gamma_3$ bound a triangle on $(X_1, \omega_2)$ and that $\gamma_1$ is an interior saddle connection of $C$. As a consequence, Type III-e is not realizable.

Lemma 4.4. Type IV is not realizable.

Proof. Consider again the triangle $\Delta'_2$ determined by $\gamma_1$ and $\gamma_3$ which is disjoint from $\gamma_2$. There are seven possibilities based on the position of $\gamma'_1$ and the order of $(\gamma'_3, \gamma'_1)$ with respect to $\Delta'_2$, IV-a, IV-b, · · · , IV-g, see Figure 6 where $(\gamma'_3, \gamma'_1)$ appear in the counterclockwise order in type IV-b, IV-f, and IV-g, they appear in the clockwise order in the remaining cases. We will show that none of them is realizable. In each case, let $\delta'_1, \delta'_2, \delta'_3$ be the third saddle connection of $\Delta'_1, \Delta'_2, \Delta'_3$ respectively.

Type IV-a. Cut the triangle $\Delta'_2$ in IV-a in Figure 6 along $\gamma'_3$, then glue it back along $\gamma'_1$. The resulting picture is IV-a' in Figure 6. Now, instead of observing the pair $(\gamma'_1, \gamma'_2)$, we can observe the pair $(\gamma'_3, \gamma'_2)$. In this viewpoint, it is equivalent to type IV-e.
Figure 6. Seven possibilities of type IV, where each red region represents a semisimple cylinder.
**Type IV-b.** This type is not realizable, because in this case, the interior of the triangle $\Delta'_2$ determined by $\gamma'_1$ and $\gamma'_3$ would intersect $\gamma'_2$, which is impossible by Corollary 3.2.

**Type IV-c.** This type is not realizable, because in this case, the interior of the triangle $\Delta'_3$ determined by $\gamma'_1$ and $\gamma'_2$ would intersect $\gamma'_3$, which is impossible by Corollary 3.2.

**Type IV-d.** Instead of observing $(\gamma'_1, \gamma'_2)$, we can observe the pair $(\gamma'_1, \gamma'_3)$. In this viewpoint, type IV-d is equivalent to type III-c. Therefore, this type is also not realizable.

**Type IV-e.** Connecting the equivalent points on $\gamma'_1$ near the endpoints of $\delta'_1$ by straight segments parallel to $\delta'_1$ (see Figure 14 (IV-e)), we obtain a (maximal) semisimple cylinder $C'$ on $(X', \omega'; \Sigma')$ such that:

- $\delta'_1$ is one of the two boundary components, and
- the interior of $C'$ intersects $\gamma'_1$, but disjoint from $\gamma'_2, \gamma'_3$.

To show that $\gamma'_2$ is disjoint from the interior of $C'$, we look at the possible intersection pattern. Notice that $\gamma'_2$ is disjoint from $\delta'_1$. If it intersects the interior of $C'$, it can not escape from $C'$ after intersection, which means that it would end at the vertex $A'_2$ as illustrated in the figure through the interior of $C'$. But this is impossible, because otherwise it would intersect the interior of the triangle $\Delta'_2$ determined by $\gamma'_1$ and $\gamma'_3$ before it reaches $A'_2$. The reason for the case of $\gamma'_3$ is similar.

It then follows from Proposition 3.3 that there is a corresponding semisimple cylinder $C$ on $(X, \omega; \Sigma)$ whose interior intersects $\gamma_1$, but disjoint from $\gamma_2, \gamma_3$. But this is impossible, because $\gamma_1, \gamma_2, \gamma_3$ bound a triangle $\Delta$ on $(X, \omega; \Sigma)$. Therefore, type IV-e is not realizable.

**Type IV-f.** This type is not realizable, because in this case, the interior of the triangle $\Delta'_1$ determined by $\gamma'_2$ and $\gamma'_3$ would intersect $\gamma'_1$, which is impossible by Corollary 3.2.

**Type IV-g.** Connecting the equivalent points on $\gamma'_1$ near the endpoints of $\gamma'_2$ by straight segments parallel to $\gamma'_2$ (see Figure 6 (IV-g)), we obtain a (maximal) semisimple cylinder $C''$ on $(X', \omega'; \Sigma')$ such that:

1. the interior of $C'$ intersects both $\gamma'_1$ and $\gamma'_3$, and
2. $\gamma'_2$ is one of the two boundary components.

By Proposition 3.3 there is a corresponding semisimple cylinder $C$ on $(X, \omega; \Sigma)$ such that: (1) the interior of $C$ intersects both $\gamma_1$ and $\gamma_3$, (2) $\gamma_2$ is one of the two boundary components.

Recall $\gamma_1, \gamma_2, \gamma_3$ bound a triangle on $(X, \omega; \Sigma)$, therefore, $\gamma_1$ is an interior saddle connection of $C$ if and only if $\gamma_3$ is an interior saddle connection of $C$. By Proposition 3.3 $\{\gamma_1, \gamma_3, \gamma'_1, \gamma'_3\}$ are interior saddle connection in $C$ or $C'$ respectively if and only if any one of them is an interior saddle connection in $C$ or $C'$ respectively.
Case 1: none of \{γ_1, γ_3, γ_1', γ_3'\} is an interior saddle connection (see Figure 7(a-b)). In this case, for any interior saddle connection β' on (X', ω'; Σ'), either it intersects both of γ_1' and γ_3', or it intersects none of them (see Figure 7(b)). But on (X, ω; Σ), there exists at least one interior saddle connection intersection only one of \{γ_1, γ_3\} (Figure 7(a)). As a consequence, this case is not realizable.

Case 2: all of \{γ_1, γ_3, γ_1', γ_3'\} are interior saddle connections in C or C' (see Figure 7(c-d)). By Corollary 3.2, γ_3, γ_1', with a saddle connection from the boundary component of C' other than γ_2', bound a triangle. Hence, they are consecutive in the sense of Definition 6. By Corollary 3.4, γ_3, γ_1 are also consecutive in C. On the other hand, they bound a triangle with γ_2 which is a simple boundary component of C. Therefore, C is a simple cylinder. It then follows from Proposition 3.3 that C' is also a simple cylinder. But C' can not be a simple cylinder, because otherwise γ_1', γ_2' and γ_3' would bound a triangle in C'.

\[\square\]

5. Homeomorphism

The goal of this section is to prove the following theorem.

**Theorem 5.1.** Every isomorphism \(F : \mathcal{S}(X, \omega; \Sigma) \to \mathcal{S}(X', \omega'; \Sigma')\) is induced by a homeomorphism \(f : (X, \omega; \Sigma) \to (X', \omega'; \Sigma')\) up to isotopy, such that

1. \(f(\Sigma) = \Sigma'\), and
2. for any saddle connection \(\gamma\) on \((X, \omega; \Sigma)\), \(f(\gamma)\) is isotopic to \(F(\gamma)\).
Moreover, if \((X, \omega; \Sigma)\) is not a torus with one marked point, then such a homeomorphism is unique up to isotopy. If \((X, \omega; \Sigma)\) is a torus with one marked point, then there are two such homeomorphisms up to isotopy.

5.1. Triangulation graphs.

**Definition 7.** Two triangulations \(\Gamma_1\) and \(\Gamma_2\) of \((X, \omega; \Sigma)\) differ by an elementary move if

- there exist \(\beta_1 \in \Gamma_1\) and \(\beta_2 \in \Gamma_2\) such that \(\Gamma_1 \setminus \beta_1 = \Gamma_2 \setminus \beta_2\), and
- there exist \(\gamma_1, \gamma_2, \gamma_3, \gamma_4 \in \Gamma_1 \setminus \beta_1 = \Gamma_2 \setminus \beta_2\) such that they bound a convex quadrilateral on \((X, \omega; \Sigma)\) which contains \(\beta_1, \beta_2\) as diagonals.

**Definition 8.** The triangulation graph of \((X, \omega; \Sigma)\), denoted by \(\mathcal{T}(X, \omega; \Sigma)\), is a graph whose vertices are triangulations of \((X, \omega; \Sigma)\), and whose edges are pairs of triangulations which differ by an elementary move.

**Proposition 5.2** ([3, 17, 29, 34]). For any translation surface with marked points \((X, \omega; \Sigma)\), the triangulation graph \(\mathcal{T}(X, \omega)\) is connected.

**Remark 11.** The above proposition holds for general flat surfaces (simplicial surfaces) (see [3, 17, 34]). Nguyen ([29]) provides an elementary proof for the case of (half-)translation surfaces.

5.2. Orientation consistency. By Theorem 4.1, we know that if the saddle connections \(\gamma_1, \gamma_2, \gamma_3\) bound a triangle \(\Delta\) on \((X, \omega; \Sigma)\), their images \(F(\gamma_1), F(\gamma_2), F(\gamma_3)\) also bound a triangle on \((X', \omega'; \Sigma')\), which is denoted by \(\Delta'\). This correspondence induces an affine homeomorphism between \(\Delta\) and \(\Delta'\), which is called the \(F\)-induced affine homeomorphism and denoted by \(f_\Delta\). Our goal is to “glue” these \(F\)-induced affine homeomorphisms between triangles according to some triangulation of \((X, \omega; \Sigma)\) to obtain a globally well defined homeomorphism from \((X, \omega; \Sigma)\) to \((X', \omega'; \Sigma')\). To do this, we need to clarify the orientation consistency among affine homeomorphisms between triangles.

**Definition 9.** Two triangles on \((X, \omega; \Sigma)\) with boundary saddle connections \(\{\gamma_1, \gamma_2, \gamma_3\}, \{\gamma_3, \gamma_4, \gamma_5\}\), respectively, are called coconvex if their interior are disjoint, and that the quadrilateral bounded by \(\gamma_1, \gamma_2, \gamma_4, \gamma_5\) is convex and is embedded in the interior on \((X, \omega; \Sigma)\).
Lemma 5.3. Let $\Delta_1$ be an embedded triangle on $(X, \omega; \Sigma)$ with boundary saddle connections $\{\gamma_1, \gamma_2, \gamma_3\}$. Let $A$ be the vertex opposite to $\gamma_3$. Let $\vec{v} \in \mathbb{R}^2$ be a vector such that the ray directed by $\vec{v}$ from $A$ intersects the interior of $\gamma_3$. Let $H$ be the half-infinite strip directed by $\vec{v}$ which contains $\gamma_3$ as one boundary end. Then there is an embedded triangle $\Delta_2$ on $(X, \omega; \Sigma)$ containing $\gamma_3$ in the boundary, whose interior is disjoint from $\Delta_1$, and which is contained in $H$. In particular, $\Delta_1$ and $\Delta_2$ are coconvex (see Figure 8).

Proof. The proof is similar to that of Lemma 3.1. □

Proposition 5.4. Let $F : S(X, \omega; \Sigma) \rightarrow S(X', \omega'; \Sigma')$ be an isomorphism. Then either

1. for every triangle on $(X, \omega; \Sigma)$, the induced affine homeomorphism between triangles is orientation preserving, or
2. for every triangle on $(X, \omega; \Sigma)$, the induced affine homeomorphism between triangles is orientation reversing.

Proof. Notice that by Proposition 5.2, in order to prove the proposition, it suffices to prove the orientation consistency for any pair of triangles on $(X, \omega; \Sigma)$ which share a common boundary saddle connection, and whose interior are disjoint.

Let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ be five saddle connections on $(X, \omega; \Sigma)$ such that $\gamma_1, \gamma_2, \gamma_3$ (resp. $\gamma_3, \gamma_4, \gamma_5$) bound a triangle $\Delta_1$ (resp. $\Delta_2$), such that $\Delta_1$ and $\Delta_2$ are on the different sides of $\gamma_3$. By Theorem 4.1, the corresponding images $F(\gamma_1), F(\gamma_2), F(\gamma_3)$ (resp. $F(\gamma_3), F(\gamma_4), F(\gamma_5)$) also bound a triangle $\Delta'_1$ (resp. $\Delta'_2$).

Without loss of generality, we may assume that the $F-$induced affine homeomorphism from $\Delta_2$ to $\Delta'_2$ is orientation preserving. In the following,
we shall prove that the $F$–induced affine homeomorphism from $\Delta_1$ to $\Delta'_1$ is also orientation preserving. The proof will be divided into two cases.

**Case 1:** $\Delta_1$ and $\Delta_2$ are coconvex, i.e. the quadrilateral $Q$ bounded by $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ is convex. Suppose that $\gamma_1, \gamma_2, \gamma_4, \gamma_5$ appear in the counterclockwise order. To prove the proposition for this case, it suffices to prove that $F(\gamma_1), F(\gamma_2), F(\gamma_4), F(\gamma_5)$ also appear in the counterclockwise order with respect to the quadrilateral $Q'$ bounded by $F(\gamma_1), F(\gamma_2), F(\gamma_4), F(\gamma_5)$. Notice that $\gamma_3$ is a diagonal of $Q$. Let us denote by $\gamma_6$ the other diagonal of $Q$. It is clear that $\gamma_2, \gamma_4, \gamma_6$ bound a triangle contained in $Q$. Therefore, $F(\gamma_2), F(\gamma_4), F(\gamma_6)$ also bound a triangle contained in $Q'$. This implies that $F(\gamma_1), F(\gamma_2), F(\gamma_4), F(\gamma_5)$ appear in the counterclockwise order with respect to the quadrilateral $Q'$.

**Case 2:** $\Delta_1$ and $\Delta_2$ are not coconvex, i.e. the quadrilateral $Q$ bounded by $\gamma_1, \gamma_2, \gamma_5, \gamma_4$ is not convex.

After applying the $\text{SL}(2, \mathbb{R})$ action, we may assume that $\Delta_1$ is equilateral. By symmetry, we may also assume that the angle bounded by $\gamma_2$ and $\gamma_4$ with respect to $Q$ is at least $\pi$ (but smaller than $2\pi$) (see Figure 9).

**Step 1.** Consider the infinite strip $H_0$ parallel to $\gamma_1$, which contains $\gamma_5$ as one boundary end, and which is disjoint from $\Delta_2$ (see Figure 9(a)). Now let us move $\gamma_5$ parallelly within $H_0$. Since the surface $(X, \omega; \Sigma)$ is of finite area, it would meet some singularity during this process, possibly on the boundary of $H_0$. Let $B_1$ be the first such singularity. Connect $B_1$ to the endpoints of $\gamma_5$ within $H_0$. It is clear that they are saddle connections, denoted by $\delta_1, \beta_1$ respectively. Let $\tilde{\Delta}^1$ be the triangle bounded by $\gamma_5, \delta_1, \beta_1$. It follows from Lemma 5.3 that $\tilde{\Delta}^1$ is disjoint from the interior of $\Delta_2$. Moreover, it is disjoint from the interior of $\Delta_1$. Otherwise, the wedge $W \subset \tilde{\Delta}^1$ bounded by $\delta_1$ and $\beta_1$ would cross $\Delta_1$. If wedge $W$ enters $\Delta_1$ through $\gamma_1$ (resp. $\gamma_2$), since the slope of $\delta_1$ (resp. $\beta_1$) is bigger than that of $\gamma_2$ (resp. $\gamma_1$), $W$ will cross $\gamma_3$ instead of $\gamma_2$ (resp. $\gamma_1$), which implies that $W$ would enter $\Delta_2$. This contradicts that $\tilde{\Delta}^1$ is disjoint from the interior of $\Delta_2$.

Notice that the union of $\Delta_2$ and $\tilde{\Delta}^1$ is a convex quadrilateral, denoted by $Q_1$. Let $\alpha_1$ be the diagonal of $Q_1$ other than $\gamma_5$, let $\Delta^1$ be the triangle bounded by $\gamma_3, \alpha_1, \beta_1$. By assumption, the $F$–induced affine homeomorphism on $\Delta_2$ is orientation preserving. By the case 1 above, the $F$–induced affine homeomorphism on $\tilde{\Delta}^1$ is also orientation preserving. Therefore, the $F$–induced homeomorphism on $Q_1$ is orientation preserving. This implies that the $F$–induced homeomorphism on $\Delta^1$ is also orientation preserving.

**Step 2.** If the cone angle bounded by $\gamma_2$ and $\alpha_1$ inside $\Delta_1 \cup \Delta^1$ is less than $\pi$, i.e. $\Delta^1$ and $\Delta_1$ are coconvex, we are done. Otherwise, consider the triangle $\Delta^1$ instead of $\Delta_2$, and repeat step 1. In this way, we construct a sequence of triangles $\Delta^i$ with boundary saddle connections $\{\gamma_3, \alpha_i, \beta_i\}$ such that

- $\Delta^i$ is contained in $H_0$, and the interior is disjoint from $\Delta_1$, 
- $\Delta^i$ and $\Delta^{i-1}$ are coconvex, we are done. Otherwise, consider the triangle $\Delta^{i+1}$ instead of $\Delta^i$, and repeat step 1.

This process generates an infinite sequence $\{\Delta^i\}$ which converges to a limit $\Delta^\infty$. If $\Delta^\infty$ is not convex, it would cross some singularity, possibly on the boundary of $H_0$. Let $B_2$ be the first such singularity. Connect $B_2$ to the endpoints of $\gamma_5$ within $H_0$. It is clear that they are saddle connections, denoted by $\delta_2, \beta_2$ respectively. Let $\tilde{\Delta}^\infty$ be the triangle bounded by $\gamma_5, \delta_2, \beta_2$. It follows from Lemma 5.3 that $\tilde{\Delta}^\infty$ is disjoint from the interior of $\Delta_2$. Moreover, it is disjoint from the interior of $\Delta_1$. Otherwise, the wedge $W \subset \tilde{\Delta}^\infty$ bounded by $\delta_2$ and $\beta_2$ would cross $\Delta_1$. If wedge $W$ enters $\Delta_1$ through $\gamma_1$ (resp. $\gamma_2$), since the slope of $\delta_2$ (resp. $\beta_2$) is bigger than that of $\gamma_2$ (resp. $\gamma_1$), $W$ will cross $\gamma_3$ instead of $\gamma_2$ (resp. $\gamma_1$), which implies that $W$ would enter $\Delta_2$. This contradicts that $\tilde{\Delta}^\infty$ is disjoint from the interior of $\Delta_2$.

Notice that the union of $\Delta_2$ and $\tilde{\Delta}^\infty$ is a convex quadrilateral, denoted by $Q_2$. Let $\alpha_2$ be the diagonal of $Q_2$ other than $\gamma_5$, let $\Delta^{\infty}$ be the triangle bounded by $\gamma_3, \alpha_2, \beta_2$. By assumption, the $F$–induced affine homeomorphism on $\Delta_2$ is orientation preserving. By the case 1 above, the $F$–induced affine homeomorphism on $\tilde{\Delta}^\infty$ is also orientation preserving. Therefore, the $F$–induced homeomorphism on $Q_2$ is orientation preserving. This implies that the $F$–induced homeomorphism on $\Delta^{\infty}$ is also orientation preserving.

**Step 3.** If the cone angle bounded by $\gamma_2$ and $\alpha_2$ inside $\Delta_1 \cup \Delta^{\infty}$ is less than $\pi$, i.e. $\Delta^{\infty}$ and $\Delta_1$ are coconvex, we are done. Otherwise, consider the triangle $\Delta^{\infty}$ instead of $\Delta_2$, and repeat step 1. In this way, we construct a sequence of triangles $\Delta^{\infty}$ with boundary saddle connections $\{\gamma_3, \alpha_2, \beta_2\}$ such that

- $\Delta^{\infty}$ is contained in $H_0$, and the interior is disjoint from $\Delta_1$,
• that the $F$–induced homeomorphism on $\Delta^i$ is also orientation preserving.

**Terminating.** We claim that the algorithm above will terminate after finite steps. In Figure [9](a), let us extend $\gamma_2$ from the left endpoint of $\gamma_3$, through $\Delta_2$ until it hits one of the infinite boundary rays of the strip $H_0$. Denote this distance by $L$. It is clear that, if $\Delta^i$ and $\Delta_1$ are not coconvex, then for all $1 \leq j \leq i$, the length of $\alpha_j$ is less than $L + |\gamma_4|$, where $|\gamma_4|$ represents the length of $\gamma_4$. On the other hand, there is an upper bound $N$ for the number of saddle connections on $(X, \omega; \Sigma)$ with lengths no longer than $L + |\gamma_4|$. As a consequence, there exists $1 \leq i_0 \leq N + 1$ such that $\Delta^{i_0}$ and $\Delta_1$ are coconvex. It then follows from the case 1 above that the $F$–induced homeomorphism on $\Delta_1$ is orientation preserving.

**Corollary 5.5.** Let $F : S(X, \omega; \Sigma) \rightarrow S(X', \omega'; \Sigma')$ be an isomorphism. Then any triangulation $\Gamma$ of $(X, \omega; \Sigma)$ induces a homeomorphism $f_\Gamma$ between $(X, \omega; \Sigma)$ and $(X', \omega'; \Sigma')$, such that

1. $f_\Gamma(\Sigma) = \Sigma'$, and
2. for any saddle connection $\gamma \in \Gamma$, $f_\Gamma(\gamma) = F(\gamma)$.

**Proof.** It follows from Theorem 4.1 and Proposition 5.4.

The homeomorphism $f_\Gamma$ obtained in Corollary 5.5 is called the *F*-induced homeomorphism with respect to $\Gamma$. In the following, we prove that the isotopy class of $f_\Gamma$ is independent of the choices of triangulations.

**Proposition 5.6.** Let $F : S(X, \omega; \Sigma) \rightarrow S(X', \omega'; \Sigma')$ be an isomorphism. For any two triangulations $\Gamma_1$ and $\Gamma_2$, the $F$-induced homeomorphisms $f_{\Gamma_1}$ and $f_{\Gamma_2}$ are isotopic.

**Proof.** By Proposition 5.2 it suffices to prove it for the case that $\Gamma_1$ and $\Gamma_2$ differ by an elementary move. Let $\Gamma_1 = \{\alpha_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \cdots, \gamma_k\}$ and $\Gamma_2 = \{\beta_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \cdots, \gamma_k\}$ such that $\gamma_2, \gamma_3, \gamma_4, \gamma_5$ bound a convex quadrilateral $Q$ on $(X, \omega; \Sigma)$ whose diagonals are $\alpha_1, \beta_1$. Correspondingly, $F(\gamma_2), F(\gamma_3), F(\gamma_4), F(\gamma_5)$ bound a convex quadrilateral $Q'$ on $(X', \omega'; \Sigma')$ whose diagonals are $F(\alpha_1), F(\beta_1)$. By construction, $f_{\Gamma_1}|_{X_1\setminus Q} = f_{\Gamma_2}|_{X_1\setminus Q}$. Notice that $\alpha_1$ and $\beta_1$ divide $Q$ into four triangles. Let $f_Q$ be the piecewisely affine map from $Q$ to $Q'$ whose restriction to each of these four triangles is affine. Let $f_{12} : (X, \omega; \Sigma) \rightarrow (X', \omega'; \Sigma')$ be a homeomorphism such that $f_{12}|_{X_1\setminus Q} = f_{\Gamma_1}|_{X_1\setminus Q} = f_{\Gamma_2}|_{X_1\setminus Q}$ and $f_{12}|_{Q} = f_Q$. It is clear that both $f_{\Gamma_1}$ and $f_{\Gamma_2}$ are isotopic to $f_{12}$. Therefore, $f_{\Gamma_1}$ and $f_{\Gamma_2}$ are isotopic.

**Proof of Theorem 5.1.** If $(X, \omega; \Sigma)$ is not a torus with one marked point, they every triple of saddle connections bound at most one triangle. Then the Theorem follows from Corollary 5.5 and Proposition 5.6.

If $(X, \omega; \Sigma)$ is a tours with one marked point. Then every triangulation $\Gamma$ consists of three saddle connections, which bound two triangles on
Therefore, the $F$-induced homeomorphisms with respect to $\Gamma$ has two choices, which results in two isotopy classes of homeomorphisms satisfying the condition in the theorem.

6. **Affine Homeomorphism**

Our goal in this section is to prove the following proposition, which is the last piece for proving the our main theorem. (See Section 2.5 for related definitions and notations.)

**Proposition 6.1.** Let $f : (X, \omega; \Sigma) \rightarrow (X', \omega'; \Sigma')$ be a homeomorphism which induces an isomorphism $f_* : \mathcal{S}(X, \omega; \Sigma) \rightarrow \mathcal{S}(X', \omega'; \Sigma')$. Then $f$ is isotopic to an affine homeomorphism.

**Proof of Proposition 6.1.** Let $\mu \in \mathcal{MF}_u(X, \omega; \Sigma)$ be a uniquely ergodic measured foliation. We claim that $f(\mu)$ is also a measured foliation in some direction on $(X', \omega'; \Sigma')$. After applying the SL(2, $\mathbb{R}$) action, we may assume that $\mu$ is horizontal. Not let $l : [0, \infty) \rightarrow (X, \omega; \Sigma)$ be a horizontal ray emanating from a singulary point. It is known that $l$ is dense in $(X, \omega; \Sigma)$. For each positive integer $n$, let $t_n > 0$ be the smallest $t$ such that there is a singular point, denoted by $A_n$, in the $1/n$ neighbourhood of $l(t)$ which is also vertically above it. Let $\delta_n$ be the vertical segment in this neighbourhood which connects $l(t_n)$ to $A_n$. Let $\beta_n$ be the oriented saddle connection from $l(0)$ to $A_n$ which is homotopic to $l_{[0,t_n]} * \delta_n$, where $*$ means concatenation. It is clear that the angle from the real positive axis to $\beta_n$ is decrease to zero. Therefore, the angle from the real positive axis to $f_*(\beta_n)$ is monotone on $(X', \omega'; \Sigma')$. Let $l' : [0, \infty) \rightarrow (X', \omega'; \Sigma')$ be the limit ray, $\theta'$ be the limit angle, and $\mu'$ be the measured foliation in this limit direction. By construction, $f(l)$ and $l'$ are homotopic. As a consequence, the intersection number between $f(\mu)$ and $\mu'$ is zero. This implies that $f(\mu) = \mu'$, since $\mu$ is uniquely ergodic. The claim follows.

Now, let $\nu \in \mathcal{MF}_u(X, \omega; \Sigma)$ be another uniquely ergodic measured foliation. A similar argument as above shows that $f(\nu) \in \mathcal{MF}_u(X', \omega'; \Sigma')$. After applying the action of SL(2, $\mathbb{R}$), we may assume that $\mu$ and $f(\mu)$ are horizontal, $\nu$ and $f(\nu)$ are vertical. It then follows from [11, Theorem 3.1] that $f$ is isotopic to an affine homeomorphism whose derivative is $
abla (e^{t/2} 0 0 e^{-t/2}) \in \text{SL}(2, \mathbb{R})$. \hfill $\Box$

**Remark 12.** The proof above follows essentially the argument of [4, Lemma 22]. Also, Nguyen proved similar results in [30, 31].

**Proof of Theorem 1.1.** It follows from Theorem 5.1 and Proposition 6.1 \hfill $\Box$

**Proof of Theorem 1.2.** It follows from Theorem 5.1 and Proposition 6.1 \hfill $\Box$

7. **Quotient Graph**

The goal of this section is to prove Theorem 1.3. We start with the following lemma.
Figure 10. The red region represents a simple cylinder determined by $\gamma^\pm_1$ and the saddle connection connecting the left endpoints of $\gamma^\pm_1$.

Lemma 7.1. If $(X,\omega;\Sigma)$ is not a torus with one marked point, then there are saddle connections $\alpha, \beta_1, \beta_2, \cdots$, such that $\beta_i$ is disjoint from $\alpha$ for all $i \geq 1$, and that $\lim_{i \to \infty} |\int_\alpha \omega \wedge \int_{\beta_i} \omega| = \infty$.

Proof. It follows from [24] that $(X,\omega;\Sigma)$ has infinitely many cylinders. First, we claim that there exists a cylinder $C_0$ such that the closure is a proper subset of $(X,\omega;\Sigma)$. Indeed, let $C \subset (X,\omega;\Sigma)$ be a cylinder. Without loss generality, we may assume that $C$ is horizontal. If $(X,\omega;\Sigma) \neq C$, the claim follows. If $(X,\omega;\Sigma) = C$, then for every saddle connection $\delta^+$ in the upper boundary component of $C$, there is a corresponding saddle connection $\delta^-$ in the lower boundary component, such that $\int_{\gamma^+} \omega = \int_{\gamma^-} \omega$ (see Figure 10). Let $(\gamma^+_1, \gamma^-_1)$ be such a pair, they determine a simple cylinder $C_1$ as illustrated in Figure 10 By assumption, $(X,\omega;\Sigma)$ is not a torus with one marked point, each boundary component of $C$ contains at least two saddle connections. In particular, $C_1$ is a proper subset of $C = (X,\omega;\Sigma)$.

Next, let $\alpha$ be a non-horizontal saddle connection on $(X,\omega;\Sigma) \setminus C_0$, let $\{\beta_i\}_{i \geq 1}$ be a sequence of interior saddle connections of $C_0$. It is clear that $\beta_i$ is disjoint from $\alpha$ for all $i \geq 1$, and that $\lim_{i \to \infty} |\int_\alpha \omega \wedge \int_{\beta_i} \omega| = \infty$.

Proof of Theorem 1.3. (1) Notice that every edge in the saddle connection graph is represented by a pair of disjoint saddle connections.

If $(X,\omega;\Sigma)$ is a torus with one marked point, the group $\text{Aff}^+(X,\omega;\Sigma)$ of orientation-preserving affine homeomorphisms of $(X,\omega;\Sigma)$ is isomorphic to $SL(2,\mathbb{Z})$, and the set of pairs of disjoint saddle connections has one $\text{Aff}^+(X,\omega;\Sigma)$-orbit. In particular, $\mathcal{G}(X,\omega;\Sigma)$ has one vertex and one edge.

If $(X,\omega;\Sigma)$ is not a torus with one marked point, let $\{\alpha_1, \beta_1\}$, $\{\alpha_2, \beta_2\}$ be two pairs of non-parallel, disjoint saddle connections on $(X,\omega;\Sigma)$. They represent two edges $e_1, e_2$ of $\mathcal{S}(X,\omega;\Sigma)$. Suppose that there is an automorphism $F$ of $\mathcal{S}(X,\omega;\Sigma)$ such that $F(e_1) = e_2$. By Theorem 1.1, there is an affine homeomorphism $f$ of $(X,\omega;\Sigma)$ such that $f(\{\alpha_1, \beta_1\}) = \{\alpha_2, \beta_2\}$. Therefore, $|\int_{\alpha_1} \omega \wedge \int_{\beta_1} \omega| = |\int_{\alpha_2} \omega \wedge \int_{\beta_2} \omega| \neq 0$. On the other hand, by Lemma 7.1, the set

$$\{ |\int_\alpha \omega \wedge \int_\beta \omega| : \alpha, \beta \text{ are disjoint saddle connections} \}$$
is an infinite set. As a consequence, $\mathcal{G}(X,\omega;\Sigma)$ has infinite edges.

(2) If $(X,\omega;\Sigma)$ is Veech surface, it follows from [35] (see also [32]) that the set of (embedded) triangles on $(X,\omega;\Sigma)$ has finite $\text{Aff}(X,\omega;\Sigma)$-orbits, where $\text{Aff}(X,\omega;\Sigma)$ is the group of affine homeomorphisms of $(X,\omega;\Sigma)$. This implies that the set of saddle connections has finite $\text{Aff}(X,\omega;\Sigma)$-orbits, since each saddle connection is contained in at least one triangle. In particular, $\mathcal{G}(X,\omega;\Sigma)$ has finite vertices.

8. Questions

8.1. Characterization of Veech surfaces. It follows from Theorem 1.3 that if $(X,\omega;\Sigma)$ is a Veech surface, the quotient graph $\mathcal{G}(X,\omega;\Sigma)$ has finite vertices.

**Question 1.** Is it true that $\mathcal{G}(X,\omega;\Sigma)$ has finite vertices if and only if $(X,\omega;\Sigma)$ is a Veech surface?

**Remark 13.** Let $\mathbb{H}^2\setminus T(X,\omega;\Sigma)$ be the spine tree defined by Smillie-Weiss (see [32] §4 for the definition). Suppose that $\mathcal{G}(X,\omega;\Sigma)$ has finite vertices. Then the set of directions of saddle connections has finite $\text{Aff}(X,\omega;\Sigma)$-orbits, which implies that the set of components of $\mathbb{H}^2\setminus T(X,\omega;\Sigma)$ has finite $\text{Aff}(X,\omega;\Sigma)$-orbits. To prove that $(X,\omega;\Sigma)$ is a Veech surface, it suffices to prove that every component of $\mathbb{H}^2\setminus T(X,\omega;\Sigma)$ has finite quotient area by $\text{Aff}(X,\omega;\Sigma)$. This is equivalent to show that every saddle connection has non-trivial stabilizers in $\text{Aff}(X,\omega;\Sigma)$.

8.2. Injective simplicial maps. Irmak-MacCarthy ([20]) proved that every injective simplicial map from an arc graph to itself is induced by some self-homeomorphism of the underlying surface. We may ask a similar question for the saddle connection graph.

**Question 2.** Let $(X,\omega;\Sigma)$ be a translation surface with marked points. Is it true that every injective simplicial map $F : \mathcal{S}(X,\omega;\Sigma) \to \mathcal{S}(X,\omega;\Sigma)$ is induced by some affine homeomorphism $f : (X,\omega;\Sigma) \to (X,\omega;\Sigma)$?

References

[1] J. Aramayona, Simplicial embeddings between pants graphs, Geom. Dedicata 144 (2010), 115-128.
[2] C. Boissy and S. Geninska, Systoles in translation surfaces, [arXiv:1707.05050](http://arxiv.org/abs/1707.05050).
[3] A. I. Bobenko and B. A. Springborn, A discrete Laplace-Beltrami operator for simplicial surfaces, Discrete Comput. Geom. 38:4 (2007), 740-756.
[4] M. Duchin, C. Leininger and K. Rafi, Length spectra and degeneration of flat metrics, Invent math. 182(2010), 231-277.
[5] V. Erlandsson and F. Fanoni, Simplicial embeddings between multicurve graphs. Michigan Math. J. 66 (2017), no. 3, 549-567.
[6] A. Eskin, C. T. McMullen, R. E. Mukamel and A. Wright, Billiards, quadrilaterals and moduli spaces, preprint, available on [http://www-personal.umich.edu/~alexmw/papers.html](http://www-personal.umich.edu/~alexmw/papers.html).
[7] A. Eskin and M. Mirzakhani, Invariant and stationary measures for the $SL(2, \mathbb{R})$ action on Moduli space, arXiv:1302.3320v6, to appear in Publications l’IHES.

[8] A. Eskin, M. Mirzakhani and A. Mohammadi, Isolation, equidistribution, and orbit closure for the $SL(2, \mathbb{R})$ action on moduli space, Ann. of Math. 182 (2015), 673-721.

[9] A. Fathi, F. Laudenbach and V. Poénaru, Thurston’s work on surfaces. Translated from the 1979 French original by Djun M. Kim and Dan Margalit. Mathematical Notes, 48. Princeton University Press, Princeton, NJ, 2012. ISBN: 978-0-691-14735-2.

[10] G. Forni and C. Matheus, Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards. J. Mod. Dyn. 8 (2014), 3-3, 271–436.

[11] F. P. Gardiner and H. Masur, Extremal length geometry of Teichmüller space. Complex Var. Theory Appl. 16(23), 209237 (1991).

[12] J. L. Harer, Stability of the homology of the mapping class groups of orientable surfaces. Ann. of Math. (2) 121 (1985), no. 2, 215–249.

[13] J. L. Harer, The virtual cohomological dimension of the mapping class group of an orientable surface. Invent. Math. 84 (1986), no. 1, 157–176.

[14] S. Hensel, P. Przytycki and R. C. H. Webb, 1-slim triangles and uniform hyperbolicity for arc graphs and curve graphs. J. Eur. Math. Soc. (JEMS) 17 (2015), no. 4, 755–762.

[15] A. Hilion and C. Horbez, The hyperbolicity of the sphere complex via surgery paths. J. Reine Angew. Math. 730 (2017), 135–161.

[16] P. Hubert, and T. A. Schmidt, An introduction to Veech surfaces. Handbook of dynamical systems. Vol. 1B, 501–526, Elsevier B. V., Amsterdam, 2006.

[17] C. Indermitte, Th. M. Liebling, M. Troyanov, and H. Clemenon, Voronoi diagrams on piecewise flat surfaces and an application to biological growth. Theor. Comput. Sci. 263, 263-274 (2001).

[18] E. Irmak, Injective Simplicial Maps of the Arc Complex on Nonorientable Surfaces, Algebr. Geom. Topol. 9 (2009) 2055-2077.

[19] E. Irmak, Injective Simplicial Maps of the Complexes of Curves of Nonorientable Surfaces, Topology Appl. 153 (2006), no. 8, 1309-1340.

[20] E. Irmak and J. D. McCarthy, Injective Simplicial Maps of the Arc Complex, Turkish J. Math. 34 (2010), no. 3, 339-354.

[21] S. Kerckhoff, H. Masur and J. Smillie, Ergodicity of Billiard Flows and Quadratic Differentials, Ann. of Math., Vol. 124, No. 2 (1986), pp. 293-311.

[22] V. Markovic, Carathéodory’s metrics on Teichmüller spaces and L-shaped pillowcases. Duke Math. J. 167 (2018), no. 3, 497–535.

[23] H. Masur and Y. Minsky, Geometry of the complex of curves I: Hyperbolicity, Invent. math. 138, 103-149 (1999).

[24] H. Masur, Closed trajectories for quadratic differentials with an application to billiards. Duke Math. J. 53 (1986), no. 2, 307–314.

[25] H. Masur and S. Schleimer, The geometry of the disk complex. J. Amer. Math. Soc. 26 (2013), no. 1, 1–62.

[26] H. Masur and S. Tabachnikov, Rational billiards and flat structures. Handbook of dynamical systems, Vol. 1A, 1015–1089, North-Holland, Amsterdam, 2002.

[27] C. T. McMullen, R. E. Mukamel and A. Wright, Cubic curves and totally geodesic subvarieties of moduli space. Ann. of Math. (2) 185 (2017), no. 3, 957–990.

[28] Y. Minsky and S. J. Taylor, Fibered faces, veering triangulations, and the arc complex. Geom. Funct. Anal. 27 (2017), no. 6, 1450-1496.

[29] D.-M. Nguyen, Triangulations and volume form on moduli space of flat surfaces, Geom. Funct. Anal. Vol. 20 (2010) 192-228.

[30] D.-M. Nguyen, Translation surfaces and the curve graph in genus two. Algebr. Geom. Topol. 17 (2017), no. 4, 2177-2237.
[31] D.-M. Nguyen, Veech dichotomy and tessellations of the hyperbolic plane, preprint, \texttt{arXiv:1808.09329}.

[32] J. Smillie and B. Weiss, Characterizations of lattice surfaces. Invent. Math. 180 (2010), no. 3, 535-557.

[33] K. Strebel, Quadratic differentials. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 5. Springer-Verlag, Berlin, 1984. ISBN: 3-540-13035-7.

[34] G. Tahar, flat triangulations and flip, \texttt{arXiv:1701.00310v2}.

[35] Y. B. Vorobets, Plane structures and billiards in rational polygons: the Veech alternative. (Russian) Uspekhi Mat. Nauk 51 (1996), no. 5(311), 3-42; translation in Russian Math. Surveys 51 (1996), no. 5, 779-817.

[36] A. Wright, Translation surfaces and their orbit closures: an introduction for a broad audience. EMS Surv. Math. Sci. 2 (2015), no. 1, 63–108.

[37] A. Wright, From rational billiards to dynamics on moduli spaces. Bull. Amer. Math. Soc. (N.S.) 53 (2016), no. 1, 41–56.

[38] A. Wright, Totally geodesic submanifolds of Teichmüller space, \texttt{arXiv:1702.03249} to appear in J. Diff. Geom.

[39] A. Zorich, Flat surfaces. Frontiers in number theory, physics, and geometry. I, 437–583, Springer, Berlin, 2006.

Huiping Pan, Department of Mathematics, Jinan University, 510632, Guangzhou, China

E-mail address: panhp@jnu.edu.cn