Generalized coarse graining applied to a $\phi^4$-theory: A model reduction-renormalization group synthesis

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Although it is already known that the renormalization group (RG) is not a black box routine, the purpose of this letter is to make it more algorithmic. It is easy to be misled into thinking that the RG already is algorithmic because its key ingredients are coarse graining and rescaling the system variables $[1,2]$. However, fully algorithmic implementations of the RG fail for large classes of problems because it is not possible to ignore the physics of a system and expect to obtain meaningful results. Capturing the essential physics requires isolating the appropriate models and the structure of perturbations and uncertainties. Consequently, this process is system-specific. Additionally, the scale on which the physics is observed must be specified. For instance, for bosonic theories the long-wavelength physics is investigated, while for fermionic system it is the physics near the Fermi surface. These considerations suggest that the primary obstacles to automation are the model identification and coarse graining processes. In this article we present a RG procedure that accounts for these system-specific obstacles and apply it to investigate the equilibrium properties of a $\phi^4$-theory.

The perspective adopted in this letter is that the obstacles mentioned above arise from the identification of physical observables. We use control theoretic techniques for open systems to systematically identify observables. Specifically, consider an open system of the form

$$\dot{x} = f(x) + u,$$  \hspace{1cm} (1)

where $x$ and $u$ are vectors in possibly infinite dimensional spaces. From a field theoretic or statistical mechanical point of view, without $u$, equation (1) represents the “classical” equations of motion of the system. $u$ represents generic driving as well as a possible noise to which the system is exposed. If $u$ is generated by a continuous stochastic process, this imposes a specific structure on the noise. Theories with this restriction on $u$ may be mapped to a field theory via Martin-Siggia-Rose (MSR)/closed-time-path (CTP) methods $[3,4,5]$. In these instances, coarse graining is typically dictated by conserved quantities and thermodynamic considerations. Consequently such systems are locally coarse grained. Suppose $u$ is an arbitrary input (e.g. $u \in L^2$) to the system. We consider the states or regions in phase space that are most accessible via driving to be responsible for describing the the essential characteristics of the system. This is analogous to the energy landscape picture in statistical mechanics where fluctuations govern which states contribute the most to the statistics of the system. We use this control theoretic notion of the importance of states to specify how to coarse grain and, consequently, to generate RG equations. The key step in generalizing the RG lies in ascertaining how to coarse grain.

While equation (1) addresses the effects of perturbations to the system, it does not allow for the possibility of multiscale or constrained observation. We can remedy this by considering more general open systems of the form

$$\dot{x} = f(x) + B u, \hspace{1cm} y = C x,$$  \hspace{1cm} (2)

where $y$ reflects that only some subspace or, more generally, subset of phase space is directly measurable. The operators $B$ and $C$ respectively specify the structure of how noise may enter the system and which states can be measured. The additional structure provides a practical way to model real systems and consider experimental constraints. The constraint imposed by only measuring $y$ strongly influences the relative importance of the internal states $x$ and hence coarse graining. For instance, if we measure a projected subspace of $x$ over a finite but short time horizon, we can expect that those states responsible for the transient dynamics will be the most important. In this case, conservation laws may play an insignificant role in determining how to coarse grain.

In $[6]$ we proposed to coarse grain linear systems based on retaining the states contributing most to the response of the system to disturbances. Following common practice, we coarse grain equation (2) based on its linearization about a particular solution with $u = 0$. In particular, to simplify analysis, we only consider linearizations about equilibrium solutions. The linearizations are
generically described by
\[
\dot{x} = Ax + \tilde{B} \dot{u}, \\
\dot{y} = Cx. 
\]

Associated with equation (3) are invariants known as Hankel singular values (HSV’s) [9, 11]. The HSV are non-negative real numbers, \( \sigma_{\text{max}} \geq \sigma_{\kappa} \geq \sigma_{\text{min}} \) that comprise the spectrum of the operator \( W \). For systems considered over an finite time horizon, \( t_f \), consider the positive operators \( X \) and \( Y \), called gramians, that are determined by the equations
\[
\frac{dX}{dt_f} = AX + XA^\dagger + \tilde{B} \tilde{B}^\dagger; \quad X(0) = 0, \\
\frac{dY}{dt_f} = A^\dagger Y + YA + C^\dagger C; \quad Y(0) = 0.
\]

\( W^2 \) may be factored as
\[
W^2 = XY. 
\]

HSV’s provide a precise measure of the error incurred by approximating the effect \( \dot{u} \) on \( \dot{y} \) with reduced order models. The HSV’s may be interpreted as supplying a measure of the importance of the internal states \( \dot{x} \). If \( W \) is invertible, it is always possible to find a coordinate system, called balanced coordinates, such that \( X = Y = \text{diag}(\sigma_{\text{max}}, \ldots, \sigma_{\text{min}}) \). When equation (3) is transformed to balanced coordinates, the best reductions are those that project out the states corresponding to small HSV. In other words, the ordering of the HSV, at least locally around an equilibrium configuration in phase space, specifies how to coarse grain a system. An in depth treatment of this material may be found in [8, 10]. It is also sometimes possible to “balance” the full nonlinear system [12].

The RG can easily be adapted for HSV-based coarse graining. Operator theoretic approaches to the RG [3, 4] demonstrate that coarse graining in the Wilsonian RG is equivalent to multiplying operators or states by projection operators [17]. The essence of this work is to use HSV’s to identify the projection operator. As before, suppose that \( \kappa \) is a vector index that orders the HSV’s \( \sigma_{\kappa} \) for equation (3) from largest to smallest. A generalized Wilsonian RG procedure is obtained by: 1) transforming the system to balanced coordinates (about an equilibrium solution),
\[
\dot{\phi}(\kappa, t) = \int R(\kappa, x) \phi(x, t) dx, 
\]
so that the partition function takes the form,
\[
Z \{ \{ \phi \} \} = \int D\phi \exp \left( -S \left( \mathbf{g}, \{ \phi \} \right) \right) = \int D\dot{\phi} \exp \left( -S \left( \mathbf{g}, \{ \dot{\phi} \} \right) \right),
\]

where \( \mathbf{g} \) is the original set of coupling constants/functions, \( \mathbf{\mathcal{J}} \) is the Jacobian from equation (7), and \( \mathbf{g} \) is the resulting transformed set of coupling constants; 2) integrating out \( \kappa \)-shells about \( \sigma_{\text{min}} \) analogously to wavevector shells; and 3) rescaling \( \kappa \) and \( \phi \) appropriately. An interesting but technically challenging variant of this procedure is to integrate out \( \sigma_{\kappa} \)-shells instead of \( \kappa \)-shells about \( \sigma_{\text{min}} \). The remarkable feature of this variant is that \( \sigma_{\kappa} \) does not respect spatial dimension. For instance, integrating out a single \( \sigma_{\kappa} \)-shell may entail integrating out an entire subspace in \( \mathbf{x} \)-space. This seems to provide a natural way to understand how three dimensional systems may have regions (e.g. affine subspaces) exhibiting one or two dimensional critical behavior. The technical challenge lies in rescaling \( \sigma_{\kappa} \). It is not clear that rescaling \( \sigma_{\kappa} \) will recover the full \( \kappa \)-space thereby generating a meaningful RG.

Before applying this procedure to the nonlinear wave equation, we apply it to some trivial examples to build intuition. We first consider the (driven) diffusion equation
\[
\partial_t \phi = D \nabla^2 \phi + \gamma u. 
\]

In this example, \( B = \gamma, \ C = 1 \), and we take \( t_f \rightarrow \infty \). By considering a stable system over an infinite time horizon, we only need to solve the Lyapunov equations,
\[
AX + XA^\dagger + \tilde{B} \tilde{B}^\dagger = 0, \\
A^\dagger Y + YA + C^\dagger C = 0,
\]

instead of equations (11). By taking the Fourier transform of equations (10)–(11), it is easy to derive that \( W \), from equation (3), in balanced coordinates is given by
\[
W_{\mathbf{k}}^{\mathbf{bal}} = \frac{|\gamma|}{2D|\mathbf{k}|^2}.
\]

FIG. 1: Schematic of shell in \( \kappa \) space that is integrated out in the (generalized) Wilsonian RG.
physics. The smallest error is incurred by projecting out large wavevectors. We will not complete the RG analysis here because the standard RG treatments based on local coarse graining are applicable. For a field theoretic treatment it is possible to obtain the coarse grained action from the MSR/CTP formalism \[13\], however alternative formulations may be found in \[14\].

As a second example, we consider the (driven) linear wave equation

\[ \partial_t^2 \phi = \nu^2 \nabla^2 \phi + \gamma u, \quad y = \phi. \]  

(13)

By the units of \( u \), it represents a true force acting on \( \phi \). This, in addition to the fact that only \( \phi \) is the “measurable” quantity, implies that we have isolated our attention on \( \phi \)-based observables. This is choice is a very statistical equilibrium and thermodynamic one. We have completely neglected \( \pi \), the field conjugate to \( \phi \), that represents the kinetic contributions to the system. When posed as a set of first order equations, equation (13) becomes

\[
\begin{bmatrix}
\partial_t \phi \\
\partial_t \pi \\
y
\end{bmatrix}
=
\begin{bmatrix}
0 & 1 & 0 \\
\nu^2 \nabla^2 & 0 & \gamma \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\phi \\
\pi \\
y
\end{bmatrix},
\]

(14)

By smoothing out the time-cutoff at \( t_f \) with a damped exponential in the integral representation of the solution of equations (13) and (14), the problem simplifies to solving Lyapunov equations. This smoothing process is also known as exponential discounting. With the given form of \( B \) and \( C \) in this problem, we find that the matrix of HSV’s, \( W \), is approximately given by

\[ W^{\text{bal}}(k) \approx \frac{|\gamma|}{4u^2|k|} \otimes I_{2 \times 2}, \]

(15)

where \( I_{2 \times 2} \) is the \( 2 \times 2 \) matrix identity, \( a \sim 1/t_f \), and \( \otimes \) is the dyadic (algebraic tensor) product. As with the diffusion equation, short-wavelength physics does not significantly contribute to the response, so locally coarse graining is appropriate. We will refrain from treating this example in more detail because the standard RG treatment of the wave equation and its \( \phi^4 \) nonlinear generalization may be readily found in field theory textbooks \[13\].

The main point of the previous examples is to convey that the generalized Wilsonian RG consistently reproduces the results of the standard RG for systems that we already intuitively know should be locally coarse grained. A great strength of the generalized RG is that it permits us to tackle less intuitive problems. We apply the generalized RG to determine the statistical equilibrium properties of a nonlinear wave equation with a particular nonequilibrium choice of observables. As will be seen, a surprising result is that this choice of observables forces us to nonlocally coarse grain. The nonlocality of the coarse graining has very interesting implications with regard to the resulting induced RG flow. In the remainder of the paper, we then consider some of these novel implications.

The (driven) equations of motion that we are considering are

\[
\begin{align*}
\partial_t \phi &= \pi + \alpha_1 u_1 \\
\partial_t \pi &= \nabla^2 \phi + \frac{\lambda}{\pi} \phi^3 + \alpha_2 u_2 \\
y &= \begin{bmatrix} \beta_1 \\ \beta_2 \pi \end{bmatrix},
\end{align*}
\]

(16)

where \( \phi \) and \( \pi \) are real-valued fields. The driving now includes generalized forces in addition to “true” forces. By expanding around equilibrium solutions of \( \nabla^2 \phi = 0 \) we find that for each real-space position \( x \),

\[
B = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}.
\]

(17)

This driving allows for more states in \( (\phi, \pi) \)-phase space to be accessible compared to the driving in equation (14). This, in combination with the form of \( y \), ensures that both \( \phi \) and \( \pi \)-dependent observables are being considered. By using exponential discounting, as mentioned earlier, we find that the diagonal operator of HSV’s is given by

\[ W_k \approx \frac{1}{4} \left[ (\alpha_1^2 |k|^{-1} + \alpha_2^2 |k|) \times (\beta_1^2 |k|^{-1} + \beta_2^2 |k|) \right]^{1/2} \otimes I_{2 \times 2}. \]

This \( W_k \) does not have the HSV’s ordered from largest to smallest, so it is not expressed in balanced coordinates. It is immediately apparent that the HSV’s are large for both\( u_1 \) and \( u_2 \) and small for both \( \alpha_1 \) and \( \alpha_2 \). In this case, equation (19) becomes

\[
W_k \approx \frac{1}{4} \left[ (\alpha_1^2 |k|^{-1} + \alpha_2^2 |k|) \times (\beta_1^2 |k|^{-1} + \beta_2^2 |k|) \right]^{1/2} \otimes I_{2 \times 2}.
\]

\[
\prod_{k} |k| \ll \prod_{k} |k|.
\]

where \( I_{2 \times 2} \) is the \( 2 \times 2 \) matrix identity, \( a \sim 1/t_f \), and \( \otimes \) is the dyadic (algebraic tensor) product. As with the diffusion equation, short-wavelength physics does not significantly contribute to the response, so locally coarse graining is appropriate. We will refrain from treating this example in more detail because the standard RG treatment of the wave equation and its \( \phi^4 \) nonlinear generalization may be readily found in field theory textbooks \[13\].

The main point of the previous examples is to convey that the generalized Wilsonian RG consistently reproduces the results of the standard RG for systems that we already intuitively know should be locally coarse grained. A great strength of the generalized RG is that it permits us to tackle less intuitive problems. We apply the generalized RG to determine the statistical equilibrium properties of a nonlinear wave equation with a particular nonequilibrium choice of observables. As will be seen, a surprising result is that this choice of observables forces us to nonlocally coarse grain. The nonlocality of the coarse graining has very interesting implications with regard to the resulting induced RG flow. In the remainder of the paper, we then consider some of these novel implications.

The action for this system, without driving, in Fourier
space is given by

\[ S(g, \{ \dot{\phi} \}) = \frac{1}{(2\pi)^D} \int \frac{d\text{k}}{2\pi^D} \delta \left( \sum_{j=1}^{n} k_j \right) \dot{\phi}(k_1) \dot{\phi}(k_2) \dot{\phi}(k_3) \dot{\phi}(k_4), \]

where \( D \) is the spatial dimension of the system since we are only considering the statistical equilibrium properties of the system. If we wished to do a full nonequilibrium treatment, we should coarse grain the MSR/CTP-action for the system. In order to coarse grain, we let \( \dot{\phi} = \dot{\phi}_c + \dot{\phi}_m \), when \( \dot{\phi}_c \) is only nonzero for \( |\text{k}| \leq \Lambda \), \( \dot{\phi}_c \) is only nonzero for \( \Lambda < |\text{k}| < \Lambda^{-1} \), and \( \dot{\phi}_m \) is only nonzero for \( |\text{k}| \geq \Lambda^{-1} \), where \( \Lambda < 1 \). With this decomposition, the path integral measure factors as \( \mathcal{D}\dot{\phi} = \mathcal{D}\dot{\phi}_c \mathcal{D}\dot{\phi}_m \mathcal{D}\dot{\phi}_c \). The RG equations are then induced by integrating out \( \dot{\phi}_m \) and then rescaling the wavevectors and fields. For this problem, the rescaling procedure requires that

\[ \dot{\phi}_c(k) = Z_c \phi_1(\Lambda^{-1}k), \]
\[ \dot{\phi}_c(k) = Z_c \phi_2(\Lambda k), \]

and \( p = \Lambda^{-1}k \) for \( |k| \leq \Lambda \) and \( p = \Lambda k \) for \( |k| \geq \Lambda^{-1} \). Naive power counting breaks down as a direct result of rescaling in the two disjoint wavevector regimes.

Although we start with a theory where \( g = (1, \lambda, 0, \ldots) \) we can expect that the RG transformations may generate new nonlinear terms and that the coupling constants may become coupling functions. In fact, \( g \) flows towards having an infinite number of nontrivial components. In particular, the coupling constant \( \lambda \) becomes a coupling function, \( \lambda(p_1, p_2, p_3, p_4) \), that may be decomposed into the coupling functions \( \{\lambda_{i,j} \}_{i=1} \). Here the notation indicates that \( \lambda_{i,j} \) has \( i \) wavevectors with \( |p_i| < 1 \) and \( j \) wavevectors with \( |p_j| > 1 \). If we let \( \Lambda = e^{-d_i} \), then to first loop order the RG equations for \( \lambda_{i,j} \) are

\[ \partial_t \lambda_{i,j} = \alpha_{i,j} \lambda_{i,j} + 4^{-1} \lambda_{i,j}^2 J(p_1, p_2, p_3, p_4) + O(\lambda^3), \]

where \( \alpha_{0,4} = D - 4, \alpha_{4,0} = 4 - D, \alpha_{2,2} = -D, \alpha_{3,3} = -2(D - 1) \), and \( \alpha_{1,3} = -2 \). Also, \( J(p_1, p_2, p_3, p_4) = K(p_1 + p_2) + K(p_1 + p_3) + \ldots + K(p_3 + p_4) \), where \( K(q) \) comes from the one-loop contribution to the 4-point vertex and is given by

\[ K(q) = \frac{2}{\Lambda^{D-1}} \int_{|q|<\Lambda^{-1}} \frac{dQ}{(2\pi)^D} \frac{Q^2}{Q^2 + |q|^2} \]
\[ = \frac{1}{2^{D-1} \Gamma^{D-1}} B \left( \frac{D-1}{2}, \frac{D-1}{2} \right) \]
\[ \times 2F_1 \left( D - 1, \frac{D-1}{2}; 1; \frac{4|q|^2}{(1+|q|^2)} \right), \]

where \( B(x, y) \) is the Euler beta function and \( 2F_1 \) is the hypergeometric function. Although we do not start with a "mass" term in the action (i.e. \( m^2 \phi^2 \)), such a term is generated by the RG flow. If we denote the mass terms for \( |q| < 1 \) and \( |q| > 1 \) respectively by \( m^2_c(q) \) and \( m^2_c(q) \), then the associated RG equations for these terms are given by

\[ \partial_t m^2_c = 2m^2_c + \frac{S_D \lambda}{2(2\pi)^D} + O(\lambda^2) \]
\[ \partial_t m^2_c = -2m^2_c + \frac{S_D \lambda}{2(2\pi)^D} + O(\lambda^2) \]

The first thing to notice in equation (22) is that the contribution from tree level, the linear term, indicates that the coupling functions involving a mixing of wavevectors (i.e. \( i, j \neq 0 \)) are irrelevant. This is the first piece of evidence that the theory decouples at small and large wavelength. The second piece of evidence for this is that \( K(q) \) in equation (22) diverges as \( |q| \to 1 \). Due to \( K(q) \)-dependence \( \lambda \) acquires through \( J(p_1, p_2, p_3, p_4) \), this divergence seems to act like a barrier in the effective coarse grained theory to prevent the large and small wavevector physics from mixing. As alluded to earlier, this decoupling of the small and large wavelength physics is a manifestation of independence of \( \phi \) and \( \pi \) as physical observables. Rather than being general, we consider this decoupling to be special because we are only considering a real scalar \( \phi^4 \)-theory. Without driving, this model lacks continuous symmetries to be broken. In a more general model, the existence of such symmetries and the associated set of gauge transformations would provide means of coupling \( \phi \) and \( \pi \).

It is not possible to ignore the wavevector dependence that \( \lambda \) acquires because they are relevant for \( |p_i| > 1 \). Specifically, higher derivative perturbations, \( p^n \phi^2, n > 2 \) and \( p^n \phi^m, n > 0, m > 4 \), in addition to higher order nonlinearities, \( \phi^4, n > 4 \), become relevant when \( |p_i| > 1 \). However, the couplings at small wavevector, \( |p_i| > 1 \), obey the standard RG equations obtained by local coarse graining. Physically this means that just as long as the system is not exposed to short-wavelength perturbations, the long-wavelength, "thermodynamic" physics will remain robustly observable. If the system is perturbed to its large-wavevector regime, then it will flow to a short-wavelength fixed point instead of the more familiar Gaussian and Wilson-Fisher fixed points. This reflects that the dynamically faster short-wavelength perturbations are able to excite the conjugate field \( \pi \), thereby driving the system away from its standard statistical equilibrium. Were the conjugate field not accessible to the "noise" \( \alpha_1 = 0 \), or not an observable, \( \beta_2 = 0 \), this phenomena would not have occurred.

In this letter we have presented a new RG procedure and have applied it to a \( \phi^4 \) toy model. This procedure provides, to the author's knowledge, the first systematic means to identify the RG projection operator. When both equilibrium and nonequilibrium observables are chosen, this RG procedure predicts that naive power counting breaks down and that terms that are ordinarily irrelevant become relevant at large wavevector. Table II summarizes these results for the \( \phi^4 \)-theory. The generalized Wilsonian RG developed here is applicable
to nonequilibrium and heterogeneous systems, finite or infinite dimensional systems, and systems with various perturbations and uncertainties. Although the RG is still formally an uncontrolled approximation, the coarse graining is chosen such that the effective, coarsened system is close to the original one. Despite the versatility of this method, it is often difficult to analytically determine the balancing transformations. However, since there are very efficient numerical algorithms for finding balanced coordinates, this generalized RG remains a numerically useful and practical algorithm.

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| coarse | \(|p| < 1\) | \(p^n \hat{\phi}^2, n \geq 4\) |
|---|---|---|
| graining | \(p^n \hat{\phi}^m, n > 0\), \(\hat{\phi}^n\) | \(p^n \hat{\phi}^m, n > 0\), \(\hat{\phi}^n\) |
|\(n > 2\), \(m \geq 4\) | \(n > 2\), \(m \geq 4\), \(n > 4\) |
| local | no | no | yes |
| nonlocal | no | no | yes |

TABLE I: Relevance of perturbations

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[1] N. Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, (Perseus, Reading, MA, 1992).  
[2] R. Shankar, *Rev. Mod. Phys.* **66**, 129 (1994).  
[3] J. Müller and J. Rau, *Phys. Lett. B* **386**, 274-278 (1996).  
[4] V. Bach, J. Fröhlich, and I.M. Sigal, *Adv. Math.* **137**, 205-298 (1998).  
[5] V. Bach, T. Chen, J. Frölich, and I.M. Sigal, J. Funct. Anal. **203**, 44-92 (2003).  
[6] P.C. Martin, E.D. Siggia, and H.A. Rose, *Phys. Rev. A* **8**, 423 (1973).  
[7] F. Cooper, A. Khare, and H. Rose, *Phys. Lett. B* **515**, 463 (2001).  
[8] D.E. Reynolds, cond-mat/0309116.  
[9] K. Glover, *Int. J. Control* **39**, 1115 (1984).  
[10] G.E. Dullerud and F. Paganini, *A Course in Robust Control Theory*, (Springer-Verlag, New York, 2000).  
[11] V.V. Peller, *Hankel Operators and their Applications*, (Springer-Verlag, New York, 2003).  
[12] J.M.A. Scherpen, *Syst. Contr. Lett.* **21**, 143-153 (1993).  
[13] J. Zanella and E. Calzetta, *Phys. Rev. E* **66**, 036134 (2002).  
[14] M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, (Addison-Wesley Publishing Company, New York, 1995).  
[15] J. Bricmont and A. Kupiainen, in *Constructive Physics. Results in Field Theory, Statistical Mechanics and Solid State Physics*, ed. V. Rivasseau, Springer Lecture Notes in Physics, **446**, 83-115 (1995).  
[16] Some subtleties arise for infinite dimension systems, but otherwise similar results hold.  
[17] As is indicated in [5], we need not limit our attention to projection operators.  
[18] \(\dot{\xi}\) is pathologically fast compared to \(\xi\)