Regularized Integrals on Riemann Surfaces and Modular Forms

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Abstract

We introduce a simple procedure to integrate differential forms with arbitrary holomorphic poles on Riemann surfaces. It gives rise to an intrinsic regularization of such singular integrals in terms of the underlying conformal geometry. Applied to products of Riemann surfaces, this regularization scheme establishes an analytic theory for integrals over configuration spaces, including Feynman graph integrals arising from two dimensional chiral quantum field theories. Specializing to elliptic curves, we show such regularized graph integrals are almost-holomorphic modular forms that geometrically provide modular completions of the corresponding ordered $A$-cycle integrals.

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1 Introduction

The present work aims to develop analytic tools for integrals on configuration spaces of Riemann surfaces arising from two dimensional chiral quantum field theories.

Regularized integrals on Riemann surfaces

Let \( \Sigma \) be a closed Riemann surface without boundary. Let \( \omega \) be a 2-form on \( \Sigma \) which is smooth away from a finite subset \( D \subset \Sigma \) but may admit holomorphic poles of arbitrary order\(^1\) along \( D \). In this paper, we introduce the notion of regularized integral (Definition 2.5)

\[
\hat{\int}_\Sigma \omega
\]

as a recipe to integrate the singular form \( \omega \) on \( \Sigma \). This is defined as follows.

We first decompose \( \omega \) into (Lemma 2.1)

\[
\omega = \alpha + \partial \beta
\]

where \( \alpha \) is a 2-form with at most logarithmic pole along \( D \), \( \beta \) is a \((0,1)\)-form with arbitrary order of poles along \( D \), and \( \partial \) is the holomorphic de Rham differential. Then we define

\[
\hat{\int}_\Sigma \omega := \int_\Sigma \alpha
\]

where the right hand side is absolutely integrable.

Here are a few remarks.

(1) The integral \( \int_\Sigma \alpha \) does not depend on the choice of the decomposition \( \omega = \alpha + \partial \beta \) (Proposition 2.7). Therefore it is reasonable to denote it by \( \int_\Sigma \omega \).

(2) The regularized integral is extended to the case when \( \Sigma \) has boundary. Then

\[
\int_\Sigma \omega := \int_\Sigma \alpha + \int_{\partial \Sigma} \beta
\]

which again does not depend on the choice of the decomposition (Theorem 2.4).

(3) \( \int_\Sigma \) is invariant under conformal transformations (Proposition 2.8).

(4) \( \int_\Sigma \) gives an intrinsic meaning of the Cauchy principal value (Theorem 2.9). Such regularized integral can be generalized to the \( n \)-th Cartesian product \( \Sigma^n \) (see discussions below), even though the meaning of Cauchy principal value is not clear there.

The above properties can be summarized as

the conformal geometry of \( \Sigma \) gives an intrinsic regularization of the integral \( \int_\Sigma \omega \).

The regularized integral enjoys many other nice properties. For example,

\(^1\)To be a little more precise, \( \omega \) is an element of \( \mathcal{A}^{1,1}(\Sigma \ast D) \), see Section 2.1 for details.
• A version of Stokes Theorem holds (Theorem 2.14)
\[ \int_{\Sigma} d\alpha = -2\pi i \text{Res}_{\Sigma}(\alpha) + \int_{\partial \Sigma} \alpha. \]

• Riemann-Hodge bilinear type relation holds (Proposition 2.18).

• A version of push-forward map exists which intertwines the holomorphic de Rham differential (Theorem 2.29).

Regularized integrals on configuration spaces

Let $\Sigma$ be a compact Riemann surface without boundary and $\Sigma^n$ be the $n$-th Cartesian product of $\Sigma$. Let $\Delta_{ij} := \{(z_1, \cdots, z_n) \in \Sigma^n | z_i = z_j\}$ and $\Delta$ be the collection of all such diagonal divisors called the big diagonal
\[ \Delta = \bigcup_{1 \leq i \neq j \leq n} \Delta_{ij}. \]

Let $\omega$ now be a $2n$-form on $\Sigma^n$ which is smooth away from $\Delta$ but may admit holomorphic poles of arbitrary order along $\Delta$. Such $\omega$ defines a smooth $2n$-form on $\Sigma^n - \Delta$, which is the configuration space of $n$ points on $\Sigma$. We can decompose $\omega$ (Lemma 2.36) $\omega = \alpha + \partial \beta$ where $\alpha$ has at most logarithmic pole along $\Delta$ (in the sense of Definition 2.30) and thus is absolutely integrable on $\Sigma^n$. This allows us to define the regularized integral
\[ \int_{\Sigma^n} \omega := \int_{\Sigma^n} \alpha \]
in the same fashion as above (Definition 2.38).

Unlike the $n = 1$ case, it is not clear how to identify the above $\int_{\Sigma^n} \omega$ as an intrinsic Cauchy principal value in general. Nevertheless it can be shown that such regularized integral is equal to the $n$-times iterated regularized integral on $\Sigma$ (Theorem 2.37)
\[ \int_{\Sigma^n} \omega = \int_{\Sigma} \cdots \int_{\Sigma} \omega. \]
This can be viewed as a canonical regularization of $\int_{\Sigma^n} \omega$ via the conformal structure of $\Sigma$.

Such $\omega$ arises in chiral quantum field theories on $\Sigma$, such as chiral bosons, chiral $\beta\gamma$-systems, chiral $bc$-systems and their deformations. The diagonal singularities of $\omega$ come from the local behavior of propagators. The integrations on $\Sigma^n$ correspond to Feynman graph integrals. Due to the existence of diagonal singularities, the naive Feynman graph integrals on $\Sigma^n$ are problematic and require regularizations. In the particular case when $\Sigma$ is an elliptic curve, integrals over product of disjoint $A$-cycles instead of over $\Sigma^n$ are often studied. Such $A$-cycle integrals are mathematically well-defined and are expected to capture essential aspects of the original chiral theories via the mechanism of contact terms [Dou95, Dij97]. Our construction of $\int_{\Sigma^n} \omega$ fills the gap by providing a geometric regularization of Feynman graph integrals in two dimensional chiral theories. Furthermore, we build a precise link between our regularized integrals and the $A$-cycle integrals in Theorem 1.3 below.

Remark 1.1. The situation here is very different from the finiteness phenomenon of Feynman graph integrals in topological field theories considered by Kontsevich [Kon94, Kon03], Axelrod-Singer [AS93] and Getzler-Jones [GJ94]. There we usually have an integral
\[ \int_{\Sigma^n} \Omega \]

\[ \text{The technique of finding logarithmic representatives for top-degree cohomology classes is classical, see e.g., } [\text{Ler}59]. \]
which is convergent by the reason that $\Omega$ extends to a smooth form on a real version of the Fulton-MacPherson compactification of the configuration space of $n$ points on $X$. In our case, such extension is impossible since the naive integral $\int_{\Sigma^n} \omega$ is not absolutely convergent in general. Instead, the regularized integral $\int_{\Sigma^n} \omega$ can be viewed as the holomorphic counterpart in dimension two of the above construction in topological field theories.

**Remark 1.2.** A more algebraic approach in dealing with a singular integral of the form $\int_{\sigma} \omega$, where $\sigma$ is a cycle intersecting with poles of an algebraic form $\omega$, is to suitably interpret the integral as the pairing between certain relative homology and relative cohomology. This involves studies of mixed Hodge structures on relative (co)homologies, as well as algebraic structures on graph complexes and graph (co)homologies, see for example [CK00, BEK06, Blo07, BK08, Blo08, Mar09, Blo15]. To a large extent, this approach avoids having to integrate singular forms, but can bring in additional complicated combinatorics and thus make exact computations difficult. This algebraic setting does not seem to apply directly to our case where $\omega$ is a smooth top-form with holomorphic singularities, and we hope to connect our analytic approach to this algebraic one in a future investigation.

**Quasi-modularity and geometric modular completion**

In this paper, we mainly apply the notion of regularized integrals to the case when

$$\Sigma = E = C / \Lambda, \quad \Lambda := \mathbb{Z} \oplus \mathbb{Z} \tau$$

is an elliptic curve. Here $\tau$ is a point on the upper-half plane $\mathbb{H}$. We shall show that regularized integrals lead to tremendous geometric constructions of modular objects.

As a prototypical example, let $\Phi(z; \tau)$ be a meromorphic elliptic function on $C \times H$ which is modular of weight $k \in \mathbb{Z}$ (Definition 3.2). Hence $\Phi(-; \tau)$ defines a meromorphic function on $E$. Assume $\Phi(-; \tau)$ does not have residue at any pole, so $\varphi = \Phi(z; \tau)dz$ defines a 2nd kind Abelian differential on $E$. Then the following holds (Proposition 2.22)

$$\int_{E} \frac{d^2z}{\text{im} \tau} \Phi = \int_{A} dz \Phi(z; \tau) - \frac{1}{2i\text{im} \tau} \cdot 2\pi i \langle \varphi, dz \rangle_{P}, \quad d^2z := \frac{i}{2} dz \wedge d\bar{z}.$$  

Here $A$ is a representative of the $A$-cycle class that does not intersect the poles of $\Phi(-; \tau)$, and $\langle \varphi, dz \rangle_{P}$ is the Poincaré residue pairing. In this expression,

$$\int_{E} \frac{d^2z}{\text{im} \tau} \Phi \quad \text{is modular of weight } k$$  

while

$$\int_{A} dz \Phi(z; \tau) \quad \text{is quasi-modular of weight } k.$$  

We see that the regularized integral gives the modular completion of the $A$-cycle integral.

Such phenomenon actually occurs in great generality for integrals on configuration spaces. Let us first recall the following notion of holomorphic limit (Definition 3.1)

$$\lim_{\tau \to \infty} : \mathcal{O}_{\mathbb{H}}[\frac{1}{\text{im} \tau}] \to \mathcal{O}_{\mathbb{H}}, \quad f(\tau, \tau) = \sum_{i=0}^{N} \frac{f_i(\tau)}{(\text{im} \tau)^i} \to f_0(\tau).$$

Here the $f_i(\tau)$'s are holomorphic in $\tau$. It sends modular quantities to quasi-modular ones. See Appendix [A] for the basics on modularity, quasi-modularity, and their relations.
The main application for us in this paper is summarized in the following theorem.

**Theorem 1.3** (Theorem 3.4, Theorem 3.9). Let $\Phi(z_1, \ldots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ (Definition 3.2) which is holomorphic away from diagonals (Definition 3.3). Let $A_1, \ldots, A_n$ be $n$ disjoint representatives of the $A$-cycle class on $E_\tau$ ordered as in Fig. 1. Then

1. The regularized integral
   
   $$
   \int_{E_\tau} \left( \prod_{i=1}^{n} \frac{d^2z_i}{\text{im } \tau} \right) \Phi(z_1, \ldots, z_n; \tau) \quad \text{lies in } \mathcal{O}_\mathbb{H}\left[ \frac{1}{\text{im } \tau} \right].
   $$

   Its holomorphic limit is given by the average of ordered $A$-cycle integrals (Definition 3.5)

   $$
   \lim_{\tau \to \infty} \int_{E_\tau} \left( \prod_{i=1}^{n} \frac{d^2z_i}{\text{im } \tau} \right) \Phi(z_1, \ldots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \ldots, z_n; \tau).
   $$

2. If $\Phi$ is modular of weight $m$ on $\mathbb{C}^n \times \mathbb{H}$, then
   
   $$
   \int_{E_\tau} \left( \prod_{i=1}^{n} \frac{d^2z_i}{\text{im } \tau} \right) \Phi \quad \text{is modular of weight } m \text{ on } \mathbb{H}
   $$

   and thus

   $$
   \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \ldots, z_n; \tau) \quad \text{is quasi-modular of weight } m \text{ on } \mathbb{H}.
   $$

3. If $\Phi$ is modular of weight $m$ on $\mathbb{C}^n \times \mathbb{H}$, then for any $\sigma \in S_n$ the ordered $A$-cycle integral

   $$
   \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \ldots, z_n; \tau) \quad \text{is quasi-modular of mixed weight on } \mathbb{H}
   $$

   with leading weight $m$. Moreover, there exists an explicit formula for each weight component.

The 1st and 2nd statements of Theorem 1.3 say that the average of ordered $A$-cycle integrals

$$
\frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \ldots, z_n; \tau)
$$

...
for a modular $\Phi$ is quasi-modular of the same weight as $\Phi$. Its modular completion is precisely given by

$$\int_{E^3} \left( \prod_{i=1}^{n} \frac{d^2 z_i}{\text{im} \, \tau} \right) \Phi.$$  

This generalizes the previous result on the modular completion of a single $A$-cycle integral.

The 3rd statement in Theorem 1.3 says that in general the ordered $A$-cycle integral

$$\int_{A_1} \cdots \int_{A_n} \Phi(z_1, \ldots, z_n; \tau)$$

is quasi-modular of mixed weight. While the quasi-modularity property has been discussed intensively in the literature [Di95, KZ95, RY09, Li12, BBBM17, GM20, OP18] (see also [BO00, EO01, RY10, Zag16, CMZ18, CMSZ20]), the mixed-weight phenomenon was only recently discovered in [GM20, OP18]. It is also proved in [OP18] by a different method that the average of ordered $A$-cycle integrals gives a quasi-modular form of pure weight.

Our result offers a geometric origin of the mixed-weight quasi-modularity. In fact, unlike the ordered $A$-cycle integrals, where specifying the $A$-cycles breaks the symmetry under the action of the modular group, our regularized integral is over the whole elliptic curve and therefore respects modularity. Theorem 1.3 offers a precise connection between regularized integrals and ordered $A$-cycle integrals. It not only shows the mixed-weight quasi-modularity of each ordered $A$-cycle integral, but also provides explicit combinatorial formulae for all the components of different weights. Such formulae arise (see Section 3.1) from the Poincaré-Birkhoff-Witt Theorem applied to the standard algebraic fact

\[ \text{tensor algebra} = \text{universal enveloping algebra of free Lie algebra}. \]

This fact tells one can express any ordered tensor into symmetric tensors of multi-commutators. Explicit formulae can be obtained from the result in [Sol68]. The multi-commutators of $A$-cycle integrations lead to multi-residues, while symmetric tensors are organized into quasi-modular objects of pure weight by Theorem 1.3. Each residue operation decreases the modular weight by one, leading to the mixed-weight phenomenon.

Here is an example (Example 3.11) with $n = 3$.

\[
\int_{A_1} dz_1 \int_{A_2} dz_2 \int_{A_3} dz_3 \Phi = \lim_{\tau \to \infty} \left\{ \int_{E_1} \frac{d^2 z_1}{\text{im} \, \tau} \int_{E_2} \frac{d^2 z_2}{\text{im} \, \tau} \int_{E_3} \frac{d^2 z_3}{\text{im} \, \tau} \Phi + \frac{1}{2} \int_{E_1} \frac{d^2 z_1}{\text{im} \, \tau} \int_{E_2} \frac{d^2 z_2}{\text{im} \, \tau} \int_{E_3} \frac{d^2 z_3}{\text{im} \, \tau} \Phi + \frac{1}{2} \int_{E_1} \frac{d^2 z_1}{\text{im} \, \tau} \int_{E_2} \frac{d^2 z_2}{\text{im} \, \tau} \int_{E_3} \frac{d^2 z_3}{\text{im} \, \tau} \Phi \right\}.
\]

Terms in different lines of the above formula have different modular weights. By the 1st statement in Theorem 1.3 each term is given by an average of ordered $A$-cycle integrals.

We also present an example (Example 3.12) with $n = 4$ and

\[
\Phi(z_1, z_2, z_3, z_4; \tau) = \wp(z_1 - z_2; \tau) \wp(z_2 - z_3; \tau) \wp(z_3 - z_4; \tau) \wp(z_4 - z_1; \tau).
\]

Here $\wp(z; \tau)$ is the Weierstrass $\wp$-function. All of the inequivalent ordered $A$-cycle integrals are explic-
itenly computed to be quasi-modular forms of mixed weight (here $'=\frac{1}{2\pi i}\partial$).

$$
\int_{A_1} dz_4 \int_{A_2} dz_3 \int_{A_3} dz_2 \int_{A_4} dz_1 \Phi = \left(\frac{\pi^8}{34} E_4^4 - \frac{2^5\pi^8}{3^2} E_2''\right) + \left(\frac{2^5\pi^8}{32\cdot5} E_4^4\right), \\
\int_{A_1} dz_3 \int_{A_2} dz_4 \int_{A_3} dz_2 \int_{A_4} dz_1 \Phi = \left(\frac{\pi^8}{34} E_4^4 - \frac{2^5\pi^8}{3^2} E_2''\right) + \left(-\frac{2^4\pi^8}{32\cdot5} E_4^4\right), \\
\int_{A_1} dz_4 \int_{A_2} dz_2 \int_{A_3} dz_3 \int_{A_4} dz_1 \Phi = \left(\frac{\pi^8}{34} E_4^4 - \frac{2^5\pi^8}{3^2} E_2''\right) + \left(-\frac{2^4\pi^8}{32\cdot5} E_4^4\right).
$$

It is illuminating to see directly here that averaging the ordered $A$-cycle integrals leads to cancellation of lower-weight terms and we find a quasi-modular form of pure weight

$$
\frac{1}{4!} \sum_{c' \in S_4} \int_{A_1} dz_{c'(1)} \int_{A_2} dz_{c'(2)} \int_{A_3} dz_{c'(3)} \int_{A_4} dz_{c'(4)} \Phi = \frac{\pi^8}{34} E_4^4 + \frac{2\pi^8}{34} \left(3E_2^2E_4 - 4E_2E_6 + E_4^2\right).
$$

\textbf{Remark 1.4.} Theorem 1.3 clarifies mathematically several aspects of chiral deformations of two dimensional conformal field theories in the sense of [Dij97]. The integral $\int_{E_4^4}$ can be viewed as a direct computation of correlation functions on the elliptic curve $E_4$ using Feynman rules, while the ordered $A$-cycle integrals can be viewed as computations from the operator formalism point of view. These two computations are not exactly the same in general but are related to each other by contact terms and the holomorphic limit $\tau \to \infty$ [Rud94, Dou95, Dij95, Dij97]. This explains why the operator formalism usually leads to quasi-modularity and how it is related to the modularity inherited from the geometry of the elliptic curve. In the theory of chiral deformations of free boson [Dij95, Dij97, Li16], the appearance of $\tau$-dependence is a two-dimensional example of the holomorphic anomaly in the context of Kodaira-Spencer gravity on Calabi-Yau manifolds [BCOV94, CL12].

\textbf{Regularized vs. $A$-cycle Feynman graph integrals: chiral boson example}

We consider Feynman graph integrals in chiral boson theory on $E_4$ (see Section 3.3). Let

$$
\bar{P}(z_1, z_2; \tau, \bar{\tau}) := \wp (z_1 - z_2; \tau) + \frac{\pi^2}{3} \hat{E}_2(\tau, \bar{\tau}), \quad \hat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi} \frac{1}{\Im \tau}.
$$

Here $\wp(z; \tau)$ is the Weierstrass $\wp$-function and $E_2$ is the 2nd Eisenstein series. Let

$$
P(z_1, z_2; \tau) := \wp (z_1 - z_2; \tau) + \frac{\pi^2}{3} E_2(\tau).
$$

Let $\Gamma$ be an oriented graph with no self-loops. Let $E(\Gamma)$ be its set of edges, and $V(\Gamma)$ be its set of vertices with cardinality $n = |V(\Gamma)|$. We label the vertices by fixing an identification

$$
V(\Gamma) \to \{1, 2, \cdots, n\}.
$$

Consider the following quantity associated to the labeled graph

$$
\Phi_\Gamma(z_1, \cdots, z_n; \tau, \bar{\tau}) := \prod_{e \in E(\Gamma)} \bar{P}(z_{l(e)}, z_{h(e)}; \tau, \bar{\tau}).
$$

Here $h(e)$ is the head of the edge $e$ and $l(e)$ is the tail. Denote

$$
\lim_{\tau \to \infty} \Phi_\Gamma(z_1, \cdots, z_n; \tau, \bar{\tau}) := \prod_{e \in E(\Gamma)} P(z_{l(e)}, z_{h(e)}; \tau).
$$
The regularized Feynman graph integral on $E$ for $\Gamma$ in our context is (Definition 3.19)

$$\hat{I}_\Gamma := \int_{E^n} \left( \prod_{i=1}^{n} \frac{d^2 z_i}{\text{im } \tau} \right) \Phi_\Gamma(z_1, \cdots, z_n; \tau, \bar{\tau}).$$

**Theorem 1.5** (Theorem 3.22). $\hat{I}_\Gamma$ is an almost-holomorphic modular form of weight $2|E(\Gamma)|$. Its holomorphic limit is given by

$$\lim_{\bar{\tau} \to \infty} \hat{I}_\Gamma = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \lim_{\bar{\tau} \to \infty} \Phi_\Gamma(z_1, \cdots, z_n; \tau, \bar{\tau})$$

which is a holomorphic quasi-modular form of the same weight. Here $A_1, \cdots, A_n$ are $n$ disjoint representatives of the $A$-cycle class on $E$. The corresponding ordered $A$-cycle integrals, especially those associated to trivalent graphs, have attracted a lot of attention in recent years [RY09, Li12, BBBM17, GM20, OP18] since their introduction [Dij95] to the studies of mirror symmetry for elliptic curves.

**Theorem 1.5** above connects regularized Feynman graph integrals to the corresponding ordered $A$-cycle integrals via the holomorphic limit $\lim_{\bar{\tau} \to \infty}$. In particular, it provides a very practical way to compute regularized Feynman graph integrals from ordered $A$-cycle integrals. This is demonstrated through several examples (Examples 3.23, Example 3.24, Example 3.25).

**Remark 1.6.** There is another approach [Li12] to study such graph integrals on $E$ using the heat kernel regularization, following the effective renormalization method developed in [C11]. We expect that the heat kernel regularization there and the regularization discussed in this paper produce the same regularized graph integrals.

**Organization of the paper**

In Section 2 we introduce the notion of regularized integrals on Riemann surfaces and on configuration spaces of Riemann surfaces, and establish their main properties.

In Section 3 we apply our theory to elliptic curves. We prove the modularity of regularized integrals on configuration spaces and relate it to the quasi-modularity of ordered $A$-cycle integrals. As a byproduct, we offer a geometric proof of the mixed-weight phenomenon of ordered $A$-cycle integrals and provide concrete formulae for each weight component. We then illustrate our results through examples of Feynman graph integrals.

In Appendix A we review the basics of modular forms and elliptic functions. Some combinatorial arguments in proving our main results and details in evaluating certain regularized/ordered $A$-cycle integrals are relegated to Appendix B and Appendix C, respectively.

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2 Regularized integrals on Riemann surfaces

In this section we introduce the notion of regularized integrals on Riemann surfaces (Definition 2.5), and on the configuration spaces of Riemann surfaces (Definition 2.38). We explore aspects of their geometric properties which will play important roles in Section 3 for our application.

The main goal is to give a meaning (see Definition 2.38 and Theorem 2.37) to the divergent integral

\[ \int_{\Sigma^n} \omega \]

where \( \omega \) is a differential form on the product \( \Sigma^n \) with arbitrary meromorphic poles along the diagonals, and thus defines a smooth form on the \( n \)-point configuration space of \( \Sigma \). These integrals arise naturally as correlation functions of non-local operators in a chiral CFT on \( \Sigma \) through the form

\[ \langle \int_{\Sigma} \mathcal{O}_1(z_1) \cdots \int_{\Sigma} \mathcal{O}_n(z_n) \rangle = \int_{\Sigma^n} \langle \mathcal{O}_1(z_1) \cdots \mathcal{O}_n(z_n) \rangle . \]

Here the \( \mathcal{O}_i \)'s are 2-form valued operators on \( \Sigma \). Then our notion of regularized integrals can be viewed as providing a mathematical tool to study such correlation functions of non-local operators.

2.1 Regularized integrals

Compact surface

Let \( \Sigma \) be a compact Riemann surface, possibly with boundary \( \partial \Sigma \). We will concentrate on regularized integrals on compact surfaces and briefly discuss modifications to non-compact surfaces at the end of this subsection.

Let \( \mathcal{O}_\Sigma \) be the sheaf of holomorphic functions and \( \Omega^\bullet_\Sigma \) be the sheaf of holomorphic forms on \( \Sigma \). We sometimes write \( \mathcal{O}, \Omega^\bullet \) for simplicity when \( \Sigma \) is clear from the context. Let

\[ \mathcal{A}^{p,q}(\Sigma) := \mathcal{A}^{0,q}(\Sigma, \Omega^p), \quad p, q = 0, 1 \]

be the space of smooth \((p, q)\)-forms on \( \Sigma \) and

\[ \mathcal{A}^k(\Sigma) := \bigoplus_{p+q=k} \mathcal{A}^{p,q}(\Sigma) \]

be the space of smooth \( k \)-forms.

Let \( D \subset \Sigma \) be a finite subset of points which does not meet \( \partial \Sigma \). Let

\[ \Omega^\bullet_\Sigma(*)D) := \bigcup_{n \geq 0} \Omega^\bullet_\Sigma(nD) \]
be the sheaf of meromorphic forms which are holomorphic on \( \Sigma - D \) but possibly with arbitrary order of poles along \( D \). Let \( \Omega^1_{\Sigma}(\log D) \) be the subsheaf of \( \Omega^1_{\Sigma}(\ast D) \) consisting of 1-forms that are logarithmic along \( D \). Since \( \Sigma \) is complex one-dimensional, we have

\[
\Omega^1_{\Sigma}(\log D) = \Omega^1_{\Sigma} \otimes \mathcal{O}_{\Sigma}(D).
\]

Let

\[
\Omega^0_{\Sigma}(\log D) := \Omega^0_{\Sigma} \oplus \Omega^1_{\Sigma}(\log D).
\]

We use

\[
A^{p,q}(\Sigma, \ast D) := A^{0,q}(\Sigma, \Omega^p(\ast D)), \quad A^{p,q}(\Sigma, \log D) := A^{0,q}(\Sigma, \Omega^p(\log D)), \quad p, q = 0, 1
\]

for the corresponding \((p, q)\)-forms with specified poles along \( D \) and

\[
A^k(\Sigma, \ast D) := \bigoplus_{p+q=k} A^{p,q}(\Sigma, \ast D), \quad A^k(\Sigma, \log D) = \bigoplus_{p+q=k} A^{p,q}(\Sigma, \log D).
\]

By definition, elements of \( A^k(\Sigma, \ast D) \) are \( k \)-forms \( \omega \) which are smooth on \( \Sigma - D \) and are of the form

\[
\omega = \frac{\alpha}{z^n} \quad \text{in a small neighborhood of } p \in D.
\]

Here \( z \) is a local holomorphic coordinate with \( z(p) = 0 \), \( n \) is a non-negative integer and \( \alpha \) is a smooth \( k \)-form around \( p \). The form \( \omega \) lies in \( A^k(\Sigma, \log D) \) if it is locally of the form

\[
\omega = \alpha + \beta \frac{dz}{z}
\]

where \( \alpha, \beta \) are smooth \((0, \bullet)\)-forms around \( p \).

The complex \( A^{\bullet, \bullet}(\Sigma, \ast D) \) is a bi-graded complex with two natural differentials

\[
\bar{\partial} : A^{\bullet, \bullet}(\Sigma, \ast D) \to A^{\bullet, \bullet+1}(\Sigma, \ast D), \quad \partial : A^{\bullet, \bullet}(\Sigma, \ast D) \to A^{\bullet+1, \bullet}(\Sigma, \ast D).
\]

Moreover, \( A^{\bullet, \bullet}(\Sigma, \log D) \subset A^{\bullet, \bullet}(\Sigma, \ast D) \) is a bi-graded subcomplex. The total differential

\[
d = \bar{\partial} + \partial
\]

is the de Rham differential.

The goal of this subsection is to explain that the following integral

\[
\int_{\Sigma} \omega, \quad \omega \in A^2(\Sigma, \ast D)
\]

has a canonical meaning although \( \omega \) may have poles along \( D \). This will be called regularized integral. To avoid possible confusion, the regularized integral will be denoted by

\[
\hat{\int}_{\Sigma} \omega.
\]

It will extend the usual integral for smooth forms, i.e., the following diagram is commutative

\[
\begin{array}{ccc}
A^2(\Sigma) & \xrightarrow{\int_{\Sigma}} & A^2(\Sigma, \ast D) \\
\downarrow f_\Sigma & & \downarrow f_\Sigma \\
\mathbb{C} & \xrightarrow{} & \mathbb{C}
\end{array}
\]

We start with two useful lemmas.
Lemma 2.1. Any $\omega \in A^{1,\bullet}(\Sigma, \star D)$ can be written as
\[ \omega = \alpha + \partial \beta, \quad \text{where} \quad \alpha \in A^{1,\bullet}(\Sigma, \log D), \quad \beta \in A^{0,\bullet}(\Sigma, \star D). \]

The supports of $\alpha$ and $\beta$ can be chosen to be contained in the support of $\omega$.

Proof. Let $D = \{p_1, \ldots, p_m\}$. Let $\{U_i\}$ be disjoint open subsets such that $p_i \in U_i$ for each $i$. Let $V_i \subset U_i$ be smaller open subsets with $\overline{V_i} \subset U_i$ such that
\[ U_0 := \Sigma - \bigcup_i U_i, \quad U_1, \ldots, U_m \]
define an open cover of $\Sigma$. Let
\[ 1 = \rho_0 + \rho_1 + \cdots + \rho_m, \quad \text{Supp}(\rho_k) \subset U_k, \quad k = 0, \ldots, m \]
be a partition of unity subordinate to this open cover. Let $\omega_j = \rho_j \omega$. We have
\[ \omega = \omega_0 + \omega_1 + \cdots + \omega_m. \]
Since $\omega_0$ is smooth, we only need to show that each $\omega_i$ can be written as
\[ \omega_i = \alpha_i + \partial \beta_i, \quad i = 1, \ldots, m. \]
Then we can choose
\[ \alpha = \omega_0 + \sum_{i=1}^m \alpha_i, \quad \beta = \sum_{i=1}^m \beta_i. \]

The problem is local and we focus on the small neighborhood $U_i$ with local holomorphic coordinate $z$ such that $z(p_i) = 0$. Assume
\[ \omega_i = \frac{dz}{z^n} \wedge g \]
where $g$ is smooth with compact support in $U_i$. If $n = 1$, then we can choose
\[ \alpha_i = \omega_i, \quad \beta_i = 0. \]
If $n > 1$, then
\[ \omega_i = -\frac{1}{n-1} \partial \left( \frac{g}{z^{n-1}} \right) + \frac{\partial g}{(n-1)z^{n-1}}. \]
We repeat this process to reduce the order of pole and eventually find $\alpha_i, \beta_i$ as required. \hfill \square

Lemma 2.2. Let $f$ be a smooth function around the origin $0 \in \mathbb{C}$. Let $n$ be a positive integer. Then
\[ \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \frac{f \, dz}{z^n} = 0, \quad \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \frac{f \, dz}{z^n} = \frac{2\pi i}{(n-1)!} \partial_z^{n-1} f(0). \]
Here the integration contour is counter-clockwise oriented.

Remark 2.3. When $f$ is holomorphic, this reduces to the usual residue formula.

Proof. Let
\[ h = \sum_{k,m \geq 0 \atop k+m \leq n} a_{km} z^k \bar{z}^m, \quad a_{km} = \frac{1}{k!m!} \partial_z^k \partial_{\bar{z}}^m f|_{z=0} \]
be the Taylor approximation of \( f \) at \( z = 0 \) up to order \( n \). From the remainder estimate
\[
f = h + o(|z|^n),
\]
we have
\[
limit_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{f dz}{z^n} = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{hdz}{z^n}, \quad \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{f dz}{z^n} = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{hdz}{z^n}.
\]
Therefore we only need to analyze the case when \( f = z^k \) is a single monomial. Then
\[
\lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{z^k dz}{z^n} = \lim_{\epsilon \to 0} e^{k+m+1-n} (-i)^2 \pi e^{(k-m-1)n-i\theta} d\theta = \lim_{\epsilon \to 0} e^{k+m+1-n} (-2\pi i) \delta_{k,m+1+n} = 0.
\]
Similarly
\[
\lim_{\epsilon \to 0} \int_{|z|=\epsilon} \frac{z^k dz}{z^n} = \lim_{\epsilon \to 0} e^{k+m+1-n} i \int_{0}^{2\pi} e^{(k-m+1-n)i\theta} d\theta = 2\pi i \lim_{\epsilon \to 0} e^{k+m+1-n} \delta_{k,m-1+n} = 2\pi i \delta_{m,0} \delta_{k,n-1}.
\]

**Theorem 2.4.** Let \( \omega \in \mathcal{A}^2(\Sigma, \star D) \). Then there exist \( \alpha \in \mathcal{A}^2(\Sigma, \log D), \beta \in \mathcal{A}^{0,1}(\Sigma, \star D) \) such that \( \omega = \alpha + \partial \beta \). The integral \( \int_{\Sigma} \alpha \) is absolutely convergent and the sum
\[
\int_{\Sigma} \alpha + \int_{\partial \Sigma} \beta
\]
does not depend on the choice of \( \alpha, \beta \).

**Proof.** Such \( \alpha, \beta \) exist by Lemma 2.1, \( \int_{\Sigma} \alpha \) is absolutely convergent since \( \alpha \) is logarithmic. \( \int_{\partial \Sigma} \beta \) is also well-defined since \( D \) does not meet \( \partial \Sigma \). Assume we have two expressions
\[
\omega = \alpha + \partial \beta = \alpha' + \partial \beta'.
\]
Let \( z_i \) be a local coordinate around \( p_i \in D \) such that \( z_i(p_i) = 0 \). Let \( B_{\epsilon} = \{|z| \leq \epsilon\} \) be a small \( \epsilon \)-ball centered at \( p_i \). Then
\[
\int_{\Sigma} (\alpha - \alpha') = \lim_{\epsilon \to 0} \int_{\Sigma \setminus \bigcup_i B_{\epsilon}} (\alpha - \alpha') = -\lim_{\epsilon \to 0} \int_{\Sigma \setminus \bigcup_i B_{\epsilon}} \partial (\beta - \beta')
\]
\[
= -\lim_{\epsilon \to 0} \int_{\Sigma \setminus \bigcup_i B_{\epsilon}} d(\beta - \beta') = -\int_{\partial \Sigma} (\beta - \beta') + \sum_i \lim_{\epsilon \to 0} \int_{\partial B_i} (\beta - \beta')
\]
\[
= -\int_{\partial \Sigma} (\beta - \beta').
\]
Here we have used Lemma 2.2 in the last step. This proves the theorem. \( \square \)

**Definition 2.5.** We define the regularized integral
\[
\int_{\Sigma} : \mathcal{A}^\bullet(\Sigma, \star D) \to \mathbb{C}
\]
by
\[
\int_{\Sigma} \omega := \begin{cases} 0 & \text{if } \omega \in \mathcal{A}^{\leq 1}(\Sigma, \star D), \\ \int_{\Sigma} \alpha + \int_{\partial \Sigma} \beta & \text{if } \omega = \alpha + \partial \beta \in \mathcal{A}^2(\Sigma, \star D). \end{cases}
\]
Here \( \alpha \in \mathcal{A}^2(\Sigma, \log D), \beta \in \mathcal{A}^{0,1}(\Sigma, \star D) \).
The regularized integral is well-defined by Theorem 2.4. It clearly extends the usual integration of smooth forms

\[ \int_{\Sigma} \omega = \int_{\Sigma} \omega, \quad \text{for} \quad \omega \in \mathcal{A}^2(\Sigma). \]

**Example 2.6.** Let \( E_\tau = \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau), \) \( \text{im} \tau > 0 \) be an elliptic curve and \( \wp(z; \tau) \) be the Weierstrass \( \wp \)-function. We now evaluate the regularized integral

\[ \int_{E_\tau} \wp(z; \tau) \frac{i}{2 \text{im} \tau} dz \wedge d\bar{z}. \]

It is a standard fact from the theory of elliptic functions (see Appendix A) that

\[ \hat{P}(z; \tau) := \wp(z; \tau) + \eta_1 + \frac{-\pi}{\text{im} \tau} = -\partial_z \hat{Z}, \quad \hat{Z} := \zeta(z; \tau) - z \eta_1 + \frac{-\pi}{\text{im} \tau}(z - z), \]

where \( \eta_1 = \frac{\pi^2}{4} E_2 \) is a semi-period, \( \zeta(z; \tau) \) the Weierstrass \( \zeta \)-function. Note that both \( \hat{P} \) and \( \hat{Z} \) are elliptic functions on \( \mathbb{C} \) and thus descend to functions on \( E_\tau \), while this is not so for \( \zeta \). It follows that

\[ \hat{P}(z; \tau) \frac{i}{2 \text{im} \tau} dz \wedge d\bar{z} = 0 - \partial \left( \hat{Z} \cdot \frac{i}{2 \text{im} \tau} d\bar{z} \right). \]

Hence by Definition 2.5 one has

\[ \int_{E_\tau} \hat{P}(z; \tau) \frac{i}{2 \text{im} \tau} dz \wedge d\bar{z} = 0, \]

and thus

\[ \int_{E_\tau} \wp(z; \tau) \frac{i}{2 \text{im} \tau} dz \wedge d\bar{z} = -\int_{E_\tau} (\eta_1 + \frac{-\pi}{\text{im} \tau}) \frac{i}{2 \text{im} \tau} dz \wedge d\bar{z} = \left( \eta_1 + \frac{-\pi}{\text{im} \tau} \right). \]

**Proposition 2.7.** Assume \( \partial \Sigma = \emptyset \). Then the regularized integral factors through the quotient

\[ \int_{\Sigma} : \frac{\mathcal{A}^2(\Sigma, \star D)}{\partial \mathcal{A}^{0,1}(\Sigma, \star D)} \to \mathbb{C}. \]

**Proof.** This follows by construction. \( \square \)

**Proposition 2.8.** Let \( f : (\Sigma_1, \partial \Sigma_1) \to (\Sigma_2, \partial \Sigma_2) \) be a diffeomorphism which is bi-holomorphic. Let \( D_1 \subset \Sigma_1 \) be a finite subset which does not meet \( \partial \Sigma_1 \) and \( D_2 = f(D_1) \). Let \( \omega \in \mathcal{A}^2(\Sigma_2, \star D_2) \). Then the pull-back \( f^* \omega \in \mathcal{A}^2(\Sigma_1, \star D_1) \) and

\[ \int_{\Sigma_1} f^* \omega = \int_{\Sigma_2} \omega. \]

**Proof.** It is clear that \( f^* \omega \in \mathcal{A}^2(\Sigma_1, \star D_1) \). Let \( \alpha \in \mathcal{A}^2(\Sigma_2, \text{log} \, D_2), \beta \in \mathcal{A}^{0,1}(\Sigma_2, \star D_2) \) such that \( \omega = \alpha + \partial \beta \). Since \( f \) is holomorphic,

\[ f^* \omega = f^* \alpha + f^* (\partial \beta) = f^* \alpha + \partial (f^* \beta). \]

Therefore

\[ \int_{\Sigma_1} f^* \omega = \int_{\Sigma_1} f^* \alpha + \int_{\partial \Sigma_1} f^* \beta = \int_{\Sigma_2} \alpha + \int_{\partial \Sigma_2} \beta = \int_{\Sigma_2} \omega. \]

\( \square \)
**Cauchy principal value**

The regularized integral can be viewed as a version of Cauchy principal value (see e.g., [Dem12]). In other words, the conformal structure of the Riemann surface leads to a regularization scheme that defines intrinsically the principal value of integrals of forms with holomorphic poles. This is demonstrated by the following theorem.

**Theorem 2.9.** Suppose $\omega \in A^2(\Sigma, \star D)$. Let $z_i$ be a local holomorphic coordinate around $p_i \in D$ such that $z_i(p_i) = 0$. Let $B^i_\epsilon = \{|z_i| \leq \epsilon\}$ be a small $\epsilon$-ball centered at $p_i$. Then

$$\int_{\Sigma} \omega = \lim_{\epsilon \to 0} \int_{\Sigma - \cup_i B^i_\epsilon} \omega .$$

In particular, the limit $\lim_{\epsilon \to 0} \int_{\Sigma - \cup_i B^i_\epsilon} \omega$ exists and does not depend on the choice of the local coordinate $z_i$'s.

**Proof.** Let $\omega = \alpha + \partial \beta$ where $\alpha \in A^2(\Sigma, \log D), \beta \in A^{0,1}(\Sigma, \star D)$. Then $\omega = \alpha + d\beta$ also holds and

$$\lim_{\epsilon \to 0} \int_{\Sigma - \cup_i B^i_\epsilon} \omega = \lim_{\epsilon \to 0} \int_{\Sigma - \cup_i B^i_\epsilon} (\alpha + d\beta) = \lim_{\epsilon \to 0} \int_{\Sigma - \cup_i B^i_\epsilon} \alpha + \int_{\partial \Sigma} \beta - \sum_i \lim_{\epsilon \to 0} \int_{\partial B^i_\epsilon} \beta = \int_{\Sigma} \alpha + \int_{\partial \Sigma} \beta .$$

Here $\lim_{\epsilon \to 0} \int_{\partial B^i_\epsilon} \beta = 0$ by Lemma 2.2.

**Non-compact surface**

Let $\Sigma$ be a non-compact Riemann surface without boundary.\(^3\) Let

$$A^k_c(\Sigma, \star D) \subset A^k(\Sigma, \star D)$$

denote the space of forms with compact support, i.e., forms that vanish outside a compact subset in $\Sigma$.

**Definition 2.10.** Given $\omega \in A^2_c(\Sigma, \star D)$, we define the regularized integral by

$$\int_{\Sigma} \omega := \int_{\Sigma} \alpha .$$

Here $\omega = \alpha + \partial \beta$ where $\alpha \in A^2_c(\Sigma, \log D), \beta \in A^{0,1}_c(\Sigma, \star D)$.

The existence of $\alpha, \beta$ is guaranteed by Lemma 2.1. The regularized integral for compactly supported forms is similar to that on compact surfaces without boundary. It factors through

$$\int_{\Sigma} : \frac{A^2_c(\Sigma, \star D)}{\partial A^{0,1}_c(\Sigma, \star D)} \to \mathbb{C} .$$

### 2.2 Residues and Stokes theorem

In this subsection, we establish a version of Stokes formula for regularized integrals.

Let us first describe an extension of residue in our context.

---

\(^3\)For non-compact Riemann surfaces with boundaries, the same formula in Definition 2.5 defines the regularized integral for compactly support forms.
Lemma/Definition 2.11. Let $\alpha \in A^1(\Sigma, \star D)$. Let $p \in D$ and $z$ be a local holomorphic coordinate around $p$ such that $z(p) = 0$. Then the following limit
\[ \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha \]
onexists and does not depend on the choice of the local coordinate $z$. Here the integration contour is counter-clockwise oriented. We will denote this limit by
\[ \oint_p \alpha := \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha. \]
Moreover, if $\alpha = \partial \beta$ for some $\beta \in A^0(\Sigma, \star D)$ or $\alpha \in A^{0,1}(\Sigma, \star D)$, then $\oint_p \alpha = 0$.

Proof. Let $z$ be a local holomorphic coordinate with $z(p) = 0$. By Lemma 2.2, $\lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha$ exists and will vanish if $\alpha$ is a $(0,1)$-form. If $\alpha = \partial \beta$ for some $\beta \in A^0(\Sigma, \star D)$, then
\[ \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \partial \beta = -\lim_{\epsilon \to 0} \int_{|z|=\epsilon} \bar{\partial} \beta = 0. \]

We next show such limit is independent of the choice of the local coordinate $z$. We can assume $\alpha$ is a $(1,0)$-form and work locally near $z = 0$. As in the proof of Lemma 2.2, it is enough to approximate $\alpha$ by a truncated Taylor approximation. For such a polynomial, one then has the expansion
\[ \alpha = \alpha_0 + \bar{z} \beta, \]
where $\alpha_0$ is a meromorphic $(1,0)$-form and $\beta$ is smooth with possibly holomorphic pole at $z = 0$. A similar calculation as in Lemma 2.2 shows that
\[ \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \bar{z} \beta = 0. \]
Hence
\[ \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha_0. \]
Let $w$ be another local holomorphic coordinate around $p$ with $w(p) = 0$. We have a similar decomposition as above
\[ \alpha = \bar{\alpha}_0 + \bar{w} \bar{\beta}. \]
Since the coordinate transformation $z \to w$ is holomorphic, by type reasons one has
\[ \alpha_0 = \bar{\alpha}_0 \]
and they are closed 1-forms. By Stokes theorem,
\[ \int_{|z|=\epsilon} \alpha_0 = \int_{|w|=\epsilon} \alpha_0, \]
and this integral is independent of the sufficiently small $\epsilon$. It follows that
\[ \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha = \lim_{\epsilon \to 0} \int_{|z|=\epsilon} \alpha_0 = \int_{|w|=\epsilon} \alpha_0 = \lim_{\epsilon \to 0} \int_{|w|=\epsilon} \bar{\alpha}_0 = \lim_{\epsilon \to 0} \int_{|w|=\epsilon} \alpha. \]

Remark 2.12. When $\alpha$ is a closed 1-form, the value $\int_{|z|=\epsilon} \alpha$ does not depend on $\epsilon$ by Stokes theorem. In general, $\int_{|z|=\epsilon} \alpha$ will depend on $\epsilon$, and our notation $\oint_p \alpha$ only refers to the limit value as $\epsilon \to 0$. 

\[ \square \]
**Definition 2.13.** Let $\alpha \in A^1(\Sigma, \star D)$ and $p \in D$. We define the local residue of $\alpha$ at $p$ by

$$\text{Res}_p \alpha := \frac{1}{2\pi i} \oint_p \alpha.$$ 

We define the global residue of $\alpha$ on $\Sigma$ by

$$\text{Res}_\Sigma \alpha := \sum_{p \in D} \text{Res}_p \alpha.$$ 

Lemma 2.2 says that $\text{Res}_p$ factors through

$$\text{Res}_p : \frac{A^1(\Sigma, \star D)}{A^0,1(\Sigma, \star D) + \partial A^0(\Sigma, \star D)} \to \mathbb{C}.$$ 

It coincides with the usual residue when $\alpha$ is a meromorphic $(1,0)$-form.

The next theorem describes a version of Stokes formula for a regularized integral. It generalizes the global residue theorem for meromorphic 1-forms on Riemann surfaces.

**Theorem 2.14.** Let $\Sigma$ be a compact Riemann surface possibly with boundary $\partial \Sigma$. Let $\alpha \in A^1(\Sigma, \star D)$. Then we have the following version of Stokes formula for the regularized integral

$$\oint_{\Sigma} d\alpha = -2\pi i \text{Res}_\Sigma (\alpha) + \oint_{\partial \Sigma} \alpha.$$ 

**Proof.** Let $D = \{p_1, \cdots, p_m\}$ and $z_i$ be a local coordinate around $p_i$ with $z_i(p_i) = 0$. Let $B^i_\epsilon = \{|z_i| \leq \epsilon\}$ be a small $\epsilon$-ball centered at $p_i$. By Theorem 2.9, we have

$$\oint_{\Sigma} d\alpha = \lim_{\epsilon \to 0} \oint_{\Sigma - \cup_i B^i_\epsilon} d\alpha = -\lim_{\epsilon \to 0} \sum_{i} \int_{|z_i| = \epsilon} \alpha + \oint_{\partial \Sigma} \alpha = -2\pi i \text{Res}_\Sigma (\alpha) + \oint_{\partial \Sigma} \alpha.$$ 

**Remark 2.15.** When $\Sigma$ is a compact surface without boundary and $\alpha$ is a meromorphic $(1,0)$-form so that $d\alpha = 0$, Theorem 2.14 reduces to the usual global residue formula

$$\text{Res}_\Sigma \alpha = 0.$$ 

**Theorem 2.16.** Let $\Sigma$ be a non-compact Riemann surface. Let $\alpha \in A^1_c(\Sigma, \star D)$. Then

$$\oint_{\Sigma} d\alpha = -2\pi i \text{Res}_\Sigma (\alpha).$$ 

**Proof.** Similar to the proof of Theorem 2.14.

We end this subsection with a simple proposition that will be useful for computations.

**Proposition 2.17.** Let $\alpha \in A^1(\Sigma, \star D)$, $p \in D$ and $f$ be a holomorphic function on $\Sigma$. Then

$$\oint_p (f \alpha) = f(p) \oint_p \alpha.$$ 

Here $f$ is the complex conjugate of $f$.

**Proof.** This follows from Lemma 2.2.
2.3 Riemann-Hodge bilinear formula

The Stokes formula allows one to express certain regularized integral via Riemann-Hodge bilinear type formula. We illustrate the basic idea in this subsection by computing

\[ \int_{\Sigma} \omega \]

where \( \Sigma \) is a compact genus-\( g \) Riemann surface without boundary, and \( \omega \) is of the form

\[ \omega = \varphi \wedge \alpha, \quad \varphi \in \Omega^1(\Sigma, \star D), \quad \alpha \in \mathcal{A}^1(\Sigma), \quad d\alpha = 0. \]

That is, \( \varphi \) is a meromorphic \((1, 0)\)-form with poles along \( D \), and \( \alpha \) is a smooth closed 1-form.

We follow the method presented in \cite{GH14} to compute the above integral. Fix once and for all a canonical basis of \( H_1(\Sigma, \mathbb{Z}) \). Fix a reference point \( p_0 \in \Sigma \). Let \( \delta_1, \cdots, \delta_{2g} \) be cycles representing the canonical basis that are issued from \( p_0 \): \( \delta_1, \cdots, \delta_g \) correspond to \( A \)-cycles, and \( \delta_{g+1}, \cdots, \delta_{2g} \) correspond to \( B \)-cycles. We choose these cycles such that they do not intersect \( D \). The complement of these cycles on \( \Sigma \) is a simply connected region \( \Delta \) on \( \Sigma \).

![Figure 2: Riemann surface with cut locus.](image)

Let

\[ \pi : \Delta \to \Sigma \]

denote the inclusion. The divisor \( D \) can be viewed as a finite subset of points lying in the interior of \( \Delta \) via the quotient \( \pi \). The pull-back under \( \pi \) defines a map

\[ \pi^* : \mathcal{A}^k(\Sigma, \star D) \to \mathcal{A}^k(\Delta, \star D). \]

It is straightforward to check (e.g., by using Theorem 2.9) that

\[ \int_{\Delta} \pi^* \omega = \int_{\Sigma} \omega, \quad \forall \omega \in \mathcal{A}^2(\Sigma, \star D). \]

**Proposition 2.18.** Let \( \varphi \) be a meromorphic \((1, 0)\)-form with poles along \( D \), and \( \alpha \) be a smooth closed 1-form. Let \( u \) be the function on \( \Delta \) defined by

\[ u(z) = \int_{p_0}^{z} \pi^* (\alpha), \quad z \in \Delta. \]

Here the integral \( \int_{p_0}^{z} \) is over an arbitrary curve inside \( \Delta \) that connects \( p_0 \) to \( z \). Then

\[ \int_{\Sigma} \varphi \wedge \alpha = \sum_{i=1}^{g} \left( \int_{\delta_i} \pi^* (\varphi) \right) \left( \int_{\delta_{g+i}} \pi^* (\alpha) \right) - \sum_{i=1}^{g} \left( \int_{\delta_i} \pi^* (\alpha) \right) \left( \int_{\delta_{g+i}} \pi^* (\varphi) \right) + 2\pi i \sum_{p \in D} \text{Res}_p (\pi^* (\varphi) u). \]
Applying Theorem 2.14, we find

\[ \phi \wedge \alpha \]

is zero since \( \phi \wedge \alpha = 0 \) by type reasons. The formula above reduces to the usual Riemann-Hodge bilinear relation.

**Proof.** Since \( \int_\Sigma \phi \wedge \alpha = \int_\Delta \pi^*(\phi \wedge \alpha) \), we shall work with \( \Delta \). Observe that

\[ \pi^*(\phi \wedge \alpha) = -d(u\pi^*(\phi)) . \]

Applying Theorem 2.14, we find

\[ \int_\Delta \pi^*(\phi \wedge \alpha) = -\int_{\partial \Delta} u\pi^*(\phi) + 2\pi i \sum_{p \in \Delta} \text{Res}_p (\pi^*(\phi)u) . \]

Evaluating \( \int_{\partial \Delta} u\pi^*(\phi) \) directly, one finds

\[ -\int_{\partial \Delta} u\pi^*(\phi) = \sum_{i=1}^g \left( \int_{\delta_i} \pi^*(\phi) \int_{\delta_{g+i}} \pi^*(\alpha) - \int_{\delta_i} \pi^*(\alpha) \int_{\delta_{g+i}} \pi^*(\phi) \right) . \]

This leads to the desired formula. \( \square \)

As a consistent check, the bilinear formula in Proposition 2.18 is independent of the choices of cycles representing the canonical basis, and the choice of canonical basis itself. In fact, although different choices can give rise to different line integrals individually (differ by the residues of \( \phi \)), the overall sum remains unchanged. The formula is also independent of the choice of the reference point \( p_0 \) by the global residue theorem.

In the case when \( \alpha \) is an anti-holomorphic 1-form, Proposition 2.18 has an intrinsic formulation as follows. Consider the 1st homology \( H_1(\Sigma - D) \) of the complement \( \Sigma - D \) and the 1st relative homology \( H_1(\Sigma, D) \). Lefschetz-Poincaré duality implies that we have a perfect pairing via intersection

\[ \cap : H_1(\Sigma - D) \times H_1(\Sigma, D) \to \mathbb{Z} . \]

**Proposition 2.20.** Let \( \phi \) be a meromorphic \((1,0)\)-form with poles along \( D \), and \( \psi \) be an anti-holomorphic 1-form. Let \( \{\gamma_i\} \) be a basis of \( H_1(\Sigma - D) \) and \( \{\gamma^j\} \) be the dual basis of \( H_1(\Sigma, D) \) such that \( \gamma_i \cap \gamma^j = \delta_i^j \). Then

\[ \int_\Sigma \phi \wedge \psi = \sum_i \int_{\gamma_i} \phi \int_{\gamma^j} \psi . \]

**Proof.** It is enough to show this for a particular choice of basis. Then linearity implies the results for all the other choices. We present one choice inside \( \Delta \) as follows. Let \( D = \{p_1, \ldots, p_m\} \) which lie in the interior of \( \Delta \). For each \( 1 \leq i \leq m - 1 \), let \( c_i \) be a small counter-clockwise oriented loop around \( p_i \), and let \( b_i \) be a path in \( \Delta \) that start from \( p_m \) and ends at \( p_i \). We require all \( c_i \)'s do not intersect, and \( b_i \) only intersects \( c_i \) at one point. Then

\[ \{\gamma_i\} = \{\delta_1, \ldots, \delta_g, \delta_{g+1}, \ldots, \delta_{2g}, c_1, \ldots, c_{m-1}\} \]

is a basis of \( H_1(\Sigma - D) \), and

\[ \{\gamma^j\} = \{\delta_{g+1}, \ldots, \delta_{2g}, -\delta_1, \ldots, -\delta_g, b_1, \ldots, b_{m-1}\} \]
is the dual basis of $H_1(\Sigma, D)$. Let 
\[ u(z) = \int_{p_0}^{z} \pi^*(\tilde{\psi}). \]

By Proposition 2.17, 
\[ \text{Res}_{p_0} (\pi^*(\varphi)u) = (\text{Res}_p \pi^*(\varphi)) \left( \int_{p_0}^{p} \pi^*(\tilde{\psi}) \right). \]

Since \( \sum_{i=1}^{m} \text{Res}_{p_i} \pi^*(\varphi) = 0 \), we can write 
\[ 2\pi i \sum_{p \in D} \text{Res}_p (\pi^*(\varphi)u) = 2\pi i \sum_{i=1}^{m} (\text{Res}_{p_i} \pi^*(\varphi)) \cdot \int_{p_0}^{p_i} \pi^*(\tilde{\psi}) = \sum_{i=1}^{m} \int_{c_i}^{b_i} \pi^*(\varphi) \int_{b_i}^{\pi^*(\tilde{\psi})}. \]

This leads to the desired formula (cf. the “Double Copy Formula” in [BD21 Corollary 3.19]) 
\[ \left\{ \sum_{i=1}^{s} \left( \int_{\delta_i}^{\pi^*(\varphi)} \int_{\delta_i}^{\tilde{\psi}} - \int_{\delta_i}^{\pi^*(\tilde{\psi})} \int_{\delta_i}^{\pi^*(\varphi)} \right) + \sum_{i=1}^{m-1} \int_{c_i}^{b_i} \pi^*(\varphi) \int_{b_i}^{\pi^*(\tilde{\psi})} \right\} = \sum_{i=1}^{s} \int_{\gamma_i}^{\varphi} \int_{\gamma_i}^{\tilde{\psi}}. \]

\[ \square \]

**Application: prototypical example on \( A \)-cycle integral and quasi-modularity**

As an application, we discuss the quasi-modularity of certain \( A \)-cycle integrals on elliptic curves. Systematic studies and generalizations will be presented in Section 3.

Let \( \tau \) be a point on the upper half-plane \( \mathbb{H} \). Let 
\[ E_\tau = \mathbb{C} / \Lambda_\tau, \quad \Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau \]
be the corresponding elliptic curve. We will use \( z \) as the linear holomorphic coordinate on the universal cover \( \mathbb{C} \). We consider the following action of \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{C} \times \mathbb{H} \) by 
\[ \gamma : \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C} \times \mathbb{H}, \]
\[ (z; \tau) \mapsto (\gamma z; \gamma \tau) := \left( \frac{z}{c\tau + d}, \frac{az + b}{c\tau + d} \right). \]

Let \( \Phi(z; \tau) \) be a meromorphic function on \( \mathbb{C} \times \mathbb{H} \) which is elliptic: 
\[ \Phi(z + \lambda; \tau) = \Phi(z; \tau), \quad \forall \lambda \in \Lambda_\tau, \]
and modular of weight \( k \in \mathbb{Z} \) 
\[ \Phi(\gamma z; \gamma \tau) = (c\tau + d)^k \Phi(z; \tau), \quad \forall \gamma \in \text{SL}_2(\mathbb{Z}). \]

Then \( \Phi(-; \tau) \) defines a meromorphic function on \( E_\tau \) for each \( \tau \in \mathbb{H} \).

**Proposition 2.21.** The regularized integral 
\[ f(\tau) = \int_{E_\tau} \frac{d^2z}{\text{im} \tau} \Phi(z; \tau), \quad d^2z := \frac{i}{2} dz \wedge d\bar{z} \]
is modular of weight \( k \) as a function of \( \tau \in \mathbb{H} \), i.e., 
\[ f(\gamma \tau) = (c\tau + d)^k f(\tau), \quad \forall \gamma \in \text{SL}_2(\mathbb{Z}). \]
Proof. Given a fixed \( \tau \), we choose a parallelogram \( \square_c \) in \( \mathbb{C} \) with vertices \( \{c, c + 1, c + 1 + \tau, c + \tau\} \) such that the poles \( D_\tau \) of \( \Phi(-; \tau) \) do not lie on the boundary of \( \square_c \).

Then \( \square_c \) is a fundamental domain for \( E_\tau \). We have
\[
\int_{E_\tau} \frac{d^2z}{\text{im} \tau} \Phi(z; \tau) = \int_{\square_c} \frac{d^2z}{\text{im} \tau} \Phi(z; \tau).
\]

Let \( \gamma \square_c \) be the image of \( \square_c \) under the \( \gamma \)-action. Then \( \gamma \square_c \) is a fundamental domain for \( E_{\gamma\tau} \) whose boundary does not intersect poles of \( \Phi(z; \gamma \tau) \). Hence
\[
f(\gamma \tau) = \int_{E_{\gamma\tau}} \frac{d^2z}{\text{im} (\gamma \tau)} \Phi(z; \gamma \tau) = \int_{\gamma \square_c} \frac{d^2z}{\text{im} (\gamma \tau)} \Phi(z; \gamma \tau) = \int_{\square_c} \frac{d^2(\gamma z)}{\text{im} (\gamma \tau)} \Phi(\gamma z; \gamma \tau)
\]
\[= (c\tau + d)^k \int_{E_\tau} \frac{d^2z}{\text{im} \tau} \Phi(z; \tau) = (c\tau + d)^k f(\tau).
\]

Here we have used Proposition 2.8 in the third equality.

As an example, we consider the case when \( \Phi(-; \tau) \) has no residue at any pole. Then
\[
\varphi = \Phi(z; \tau)dz
\]
defines a 2nd kind Abelian differential on \( E_\tau \). Let \( \{A, B\} \in H_1(E_\tau, \mathbb{Z}) \) be the basis with representatives as in the figure below. We assume such representatives do not intersect the poles \( D_\varphi \) of \( \varphi \). Otherwise we perturb the chosen representatives slightly to avoid poles.

Since \( \varphi \) has no residues, the \( A \)-cycle integral
\[
\int_A \varphi
\]
does not depend on the representative \( A \) of its cycle class. By Proposition 2.18,
\[
\int_{E_\tau} \frac{d^2z}{\text{im} \tau} \Phi = \int_{E_\tau} \varphi \wedge \frac{d \text{im} z}{\text{im} \tau} = \int_A \varphi \int_B \frac{d \text{im} z}{\text{im} \tau} - \int_B \varphi \int_A \frac{d \text{im} z}{\text{im} \tau} + 2\pi i \sum_{p \in D} \text{Res}_p \left( \frac{\varphi \text{im} z}{\text{im} \tau} \right).
\]
Since \( \text{Res}_p(\varphi) = 0 \), we have
\[
\text{Res}_p(\varphi z) = 0, \quad \sum_{p \in D} \text{Res}_p(\varphi z) = -\langle \varphi, dz \rangle_P,
\]
where $\langle \varphi, dz \rangle_p$ is the Poincaré residue pairing. It follows that

$$\int_{E_\tau} \frac{d^2 z}{\mathrm{im} \tau} \Phi = \int_A \varphi - \frac{1}{2i \mathrm{im} \tau} \cdot 2\pi i \langle \varphi, dz \rangle_p.$$  

The following proposition regarding the quasi-modularity (in the sense of [KZ95], see Definition A.1) of $\int_A \varphi$ is immediate.

**Proposition 2.22.** Assume $\Phi$ is modular of weight $k$ and $\varphi = \Phi dz$ is a 2nd kind Abelian differential on $E_\tau$. Then the $A$-cycle integral

$$\int_A \varphi$$

is quasi-modular of weight $k$ whose “modular completion” (see Appendix A) is given by the regularized integral

$$\int_{E_\tau} \frac{d^2 z}{\mathrm{im} \tau} \Phi = \int_A \varphi - \frac{1}{2i \mathrm{im} \tau} \cdot 2\pi i \langle \varphi, dz \rangle_p.$$  

**Proof.** By Proposition 2.21, the regularized integral $\int_{E_\tau} \frac{d^2 z}{\mathrm{im} \tau} \Phi$ is modular. One can prove similarly that $\langle \varphi, dz \rangle_p$, as a meromorphic function in $\tau$, is also modular. Quasi-modularity of $\int_A \varphi$ then follows from the modularity of $\int_{E_\tau} \frac{d^2 z}{\mathrm{im} \tau} \Phi$, the modularity of $\langle \varphi, dz \rangle_p$, and the relation between $\int_{E_\tau} \frac{d^2 z}{\mathrm{im} \tau} \Phi$ and $\int_A \varphi$ above. 

**Remark 2.23.** The integral $\int_A \varphi$ being quasi-modular instead of modular is due to the fact that specifying the $A$-cycle breaks the symmetry under the action of the modular group. Our geometric perspective offers a natural way to obtain its modular completion in terms of regularized integral on the whole elliptic curve. This phenomenon will be vastly generalized in Section 3 (Theorem 3.4).

**Example 2.24.** Consider $\Phi = \wp^m$ with $\varphi = \wp^m dz$, where $m \geq 1$ and $\wp$ is the Weierstrass $\wp$-function. The computations of integrals $\int_A \wp^m dz$ are standard, see e.g., [Siv09], by using the Weierstrass equation (A.5) and the relation (A.9).

For example, for $m = 1$ one has from (A.9)

$$\int_A \wp(z; \tau) dz = -\frac{\pi^2}{3} E_2(\tau).$$

By the Weierstrass equation (A.5) we have $2\wp'' = 12\wp^2 - \frac{4}{3} \pi^4 E_4$ and hence

$$\wp^2 dz = \frac{1}{6} d\wp' + \frac{1}{9} \pi^4 E_4 dz.$$

This gives

$$\int_A \wp^2(z; \tau) dz = \int_A \left( \frac{1}{6} d\wp' + \frac{1}{9} \pi^4 E_4 dz \right) = \frac{\pi^4}{9} E_4.$$

Similarly, for the $m = 3$ case we multiply the relation $2\wp'' = 12\wp^2 - \frac{4}{3} \pi^4 E_4$ by $\wp$ and obtain

$$12\wp^3 dz - \frac{4}{3} \pi^4 E_4 \wp dz = 2\wp \wp'' dz = 2d(\wp \wp') - 2(\wp')^2 dz = 2d(\wp \wp') - 2(4\wp^3 dz - \frac{4}{3} \pi^4 E_4 \wp dz - \frac{8}{27} \pi^6 E_6 dz).$$
It follows that
\[
\int_A \varphi^3 dz = \frac{1}{20} \int_A (4\pi^4 E_4 \varphi + \frac{16}{27} \pi^6 E_6) dz = -\frac{1}{15} \pi^6 E_2 E_4 + \frac{4}{5 \cdot 27} \pi^6 E_6.
\]

Using (A.6) and (A.7), one has
\[
\langle \varphi dz, dz \rangle_p = -1, \quad \langle \varphi^2 dz, dz \rangle_p = 0, \quad \langle \varphi^3 dz, dz \rangle_p = -\frac{3}{20} \cdot \frac{8}{15} = -\frac{4}{5} E_4.
\]

Therefore, we obtain (cf. Example 2.6)
\[
\begin{align*}
\int_A \varphi(z; \tau) dz - \frac{1}{2i \text{im} \tau} \cdot 2\pi i \langle \varphi(z; \tau) dz, dz \rangle_p &= -\frac{\pi^2}{3} \bar{E}_2(\tau) + \frac{\pi}{\text{im} \tau} = -\frac{\pi^2}{3} \bar{E}_2(\tau), \\
\int_A \varphi^2(z; \tau) dz - \frac{1}{2i \text{im} \tau} \cdot 2\pi i \langle \varphi^2(z; \tau) dz, dz \rangle_p &= \frac{\pi^4}{9} E_4, \\
\int_A \varphi^3(z; \tau) dz - \frac{1}{2i \text{im} \tau} \cdot 2\pi i \langle \varphi^3(z; \tau) dz, dz \rangle_p &= -\frac{1}{15} \pi^6 \bar{E}_2 E_4 + \frac{4}{5 \cdot 27} \pi^6 E_6.
\end{align*}
\]

**Remark 2.25.** One can also evaluate these $A$-cycle integrals by first lifting the function $\varphi(z)$ along the Picard uniformization
\[
C^* \to E_\tau = C^* / \mathbb{Z}, \quad q = e^{2\pi i \tau}.
\]

The resulting integrand is expressed in terms of the holomorphic coordinate $u$ on the cover $C^*$ that is related to the coordinate $z$ on the universal cover $C$ by $u = \exp(2\pi i z)$. Then one computes the degree zero term in the $u$-expansion on $C^*$. The formulae in (A.1) allow us to express the resulting $q$-series in terms of the Eisenstein series. See [RY09, BBBM17, GM20] for discussions along these lines. Using the “computing twice” trick, one can then produce many nontrivial identities between $q$-series and quasi-modular forms.

For example, the $m = 2$ case leads to the identity
\[
(2\pi i)^4 \cdot 2 \sum_{k \geq 1} \frac{k^2 q^k}{(1 - q^k)^2} = \frac{1}{9} \pi^4 (E_4 - E_2^3),
\]
which can be proved directly by using (A.1) and the Ramanujan identities (A.4).

The $m = 3$ case leads to
\[
(2\pi i)^6 \sum_{k, \ell \geq 1} k\ell(k + \ell) \frac{q^{k+\ell}}{(1 - q^k)(1 - q^\ell)(1 - q^{k+\ell})} = \frac{\pi^6}{26 \cdot 345} (5E_2^3 - 3E_2 E_4 - 2E_6),
\]
which seems to be less easy to prove directly.

### 2.4 Variation property

In this subsection, we establish several variation properties of regularized integrals.

We assume $\Sigma$ is a compact Riemann surface without boundary. Let $X$ be a complex manifold. Let $\{s_i : X \to \Sigma\}$ be a set of distinct holomorphic maps. Their graphs define a set of smooth divisors
\[
D_i = \{(z, x) \in \Sigma \times X \mid z = s_i(x)\} \subset \Sigma \times X.
\]

The intersection
\[
D_i \cap D_j
\]

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can be viewed as a divisor $D_{ij}$ on $X$ under the projection $\pi : \Sigma \times X \to X$. Equivalently, $D_{ij}$ is given by the fiber product

\[
\begin{array}{ccc}
D_{ij} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\Sigma & \delta & \Sigma \times \Sigma
\end{array}
\]

where $\delta : \Sigma \to \Sigma \times \Sigma$ is the diagonal map. Let us denote

\[ D = \bigcup_i D_i, \quad D^{(2)} = \bigcup_{i \neq j} D_{ij}. \]

Here $D$ can be viewed as a family of divisors on $\Sigma$ parametrized by $X$.

We can similarly define the spaces

\[ A^\bullet(\Sigma \times X, \ast D), \quad A^\bullet(X, \ast D^{(2)}) \]

consisting of forms with arbitrary order of poles along the specified divisors. Locally, in a small open subset $U \times V \subset \Sigma \times X$ such that each $D_i$ is defined by $z = s_i(x)$, we have

\[ A^\bullet(U \times V, \ast D) = A^\bullet(U \times V)[(z - s_i(x))^{-1}], \quad A^\bullet(V, \ast D^{(2)}) = A^\bullet(V)[(s_i(x) - s_j(x))^{-1}]. \]

Lemma 2.26. Let

\[ \omega = \frac{f(z, y)dz \wedge d\bar{z}}{(z - z_0(y))^n} \]

be a family of 2-forms on $z \in \mathbb{C}$ parametrized by $y \in \mathbb{R}$. Here $z_0(y)$ is smooth in $y$; $f(z, y)$ is smooth in $z, y$ and we have omitted the $\bar{z}$-dependence for convenience; $f(-, y)$ has compact support along $z \in \mathbb{C}$ for any fixed $y$. Then the regularized integral $\int_C \omega$ is smooth in $y$ and

\[
\partial_y \int_C \omega = \int_C \partial_y \omega + \partial_y z_0(y) \int_{z_0(y)} \frac{f(z, y)}{(z - z_0(y))^n} dz.
\]

In particular, if $t$ is a complex variable, $z_0(t)$ is holomorphic in $t$, and $\omega = \frac{f(z, t)dz \wedge d\bar{z}}{(z - z_0(t))^n}$, then

\[
\partial_t \int_C \omega = \int_C \partial_t \omega, \quad \partial_t \int_C \omega = \int_C \partial_t \omega + \partial_t z_0(t) \int_{z_0(t)} \frac{f(z, t)}{(z - z_0(t))^n} dz.
\]

Proof. A change of coordinate $z \to z + z_0(y)$ by shifting and Proposition 2.8 imply

\[
\int_C \omega = \int_C \frac{f(z + z_0(y), y)dz \wedge d\bar{z}}{z^n}.
\]

Let $\delta$ denote the holomorphic de Rham differential in $z$. We can write

\[
\frac{f(z + z_0(y), y)dz \wedge d\bar{z}}{z^n} = \frac{1}{(n - 1)!} \partial_z^{n-1} f(z + z_0(y), y) dz \wedge d\bar{z} + \partial \beta
\]

for some $(0, 1)$-form $\beta$. It follows that

\[
\int_C \frac{f(z + z_0(y), y)dz \wedge d\bar{z}}{z^n} = \frac{1}{(n - 1)!} \int_C \partial_z^{n-1} f(z + z_0(y), y) dz \wedge d\bar{z}.
\]

This immediately implies the smoothness of $\int_C \omega$ in $y$. 

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We can apply $\partial_y$ to equation $(\dagger)$ and use the fact that $\partial_y$ commutes with $\partial$ to find
\[
\partial_y \int_C \omega = \int_C \partial_y \left( \frac{f(z + z_0(y), y)dz \wedge d\bar{z}}{z^n} \right) \\
= \int_C \left( \partial_y f(z + z_0(y), y)dz \wedge d\bar{z} \right) + \int_C \frac{\partial_y f(z + z_0(y), y)dz \wedge d\bar{z}}{z^n} \\
+ \int_C \left( \partial_y f(z + z_0(y), y)dz \wedge d\bar{z} \right).
\]

The first two terms on the last expression give
\[
(\partial_y z_0(y)) \int_C \frac{\partial_y f(z + z_0(y), y)dz \wedge d\bar{z}}{(z - z_0(y))^n} + \int_C \frac{\partial_y f(z, y)dz \wedge d\bar{z}}{(z - z_0(y))^n} \\
= \int_C n(\partial_y z_0(y))f dz \wedge d\bar{z} + \int_C \frac{\partial_y f(z, y)dz \wedge d\bar{z}}{(z - z_0(y))^n} = \int_C \partial_y \omega.
\]

The last term is
\[
\int_C \left( \partial_y f(z + z_0(y), y)dz \wedge d\bar{z} \right).
\]

Here we have used Theorem 2.16 in the last step.

Given a form $\omega \in A^\bullet(\Sigma \times X, \star D)$, we can perform the regularized integral
\[
\int_\Sigma \omega
\]
along $\Sigma$ first to end up with a form on $X$. By Lemma 2.26 and a use of partition of unity, $\int_\Sigma \omega$ is smooth on $X - D^{(2)}$. We next analyze the singularity of $\int_\Sigma \omega$ around $D^{(2)}$.

**Proposition 2.27.** Let $\omega \in A^\bullet(\Sigma \times X, \star D)$. Then
\[
\int_\Sigma \omega \in A^{\bullet-2}(X, \star D^{(2)}).
\]
In other words, $\int_\Sigma \omega$ has holomorphic poles along $D^{(2)}$.

**Proof.** The problem is local. Let us assume we are in a neighborhood $V$ of $x$ with $s_1(x) = \cdots = s_k(x)$. Let
\[
g = \prod_{1 \leq i \neq j \leq k} (s_i(x) - s_j(x))^N
\]
where $N$ is a sufficient large integer to be determined later. Since $g$ does not depend on $\Sigma$,
\[
\int_\Sigma \omega = \frac{1}{g} \int_\Sigma g \omega.
\]
Locally $\omega$ has the form

$$\omega = \frac{dz}{\prod_{i=1}^{k} (z - s_i(x))^m_i} \varphi$$

where $\varphi$ is smooth. By a repeated use of the relation

$$\frac{1}{(z - s_i(x))(z - s_j(x))} = \frac{1}{(s_i(x) - s_j(x))}\left( \frac{1}{z - s_i(x)} - \frac{1}{z - s_j(x)} \right)$$

we can find a large enough $N$ such that

$$g\omega = \sum_{i} \frac{dz}{(z - s_i(x))^m_i} \varphi_i$$

with $\varphi_i$ smooth. By Lemma 2.26, $f_{\Sigma} g\omega$ is smooth. It follows that

$$\int_{\Sigma} \omega = \frac{1}{g} \int_{\Sigma} g\omega \in A^{*-2}(X, \ast D^{(2)}) .$$

It follows that the regularized integral defines a push-forward map

$$\int_{\Sigma} : A^{*}(\Sigma \times X, \ast D) \to A^{*-2}(X, \ast D^{(2)}) .$$

**Remark 2.28.** A version of regularized integral on $\mathbb{C}P^1 \times \mathbb{R}^2$ was defined in [BSV20] for the 4d Chern-Simons action in a similar fashion. It can be viewed as $\int_{\mathbb{R}^2} \int_{\mathbb{C}P^1}$ in our terminology.

**Theorem 2.29.** Assume $\Sigma$ is a compact Riemann surface without boundary. Then the push-forward map $f_{\Sigma} : A^{*}(\Sigma \times X, \ast D) \to A^{*-2}(X, \ast D^{(2)})$ intertwines the holomorphic de Rham differential

$$\int_{\Sigma} \partial_{\Sigma \times X} \omega = \partial_X f_{\Sigma} \int_{\Sigma} \omega .$$

Here $\partial_{\Sigma \times X}, \partial_X$ are the holomorphic de Rham differentials on $\Sigma \times X$ and $X$, respectively.

**Proof.** Let us write $\partial_{\Sigma \times X} = \partial_{\Sigma} + \partial_X$. Then

$$\int_{\Sigma} \partial_{\Sigma \times X} \omega = \int_{\Sigma} \partial_{\Sigma} \omega + \int_{\Sigma} \partial_X \omega .$$

Since $\Sigma$ has no boundary, $f_{\Sigma} \partial_{\Sigma} \omega = 0$. The pole locations $s_i(x)$ vary holomorphically. Using a partition of unity and Lemma 2.26, we find $f_{\Sigma} \partial_X \omega = \partial_X f_{\Sigma} \omega$. The theorem follows.

### 2.5 Integrals on configuration spaces

Let $\Sigma$ be a compact Riemann surface without boundary. Let $\Sigma^n$ be the $n$-th Cartesian product of $\Sigma$. Fix once and for all a global enumeration $1, 2, \cdots, n$ for the factors of $\Sigma^n$. Given an index subset $I = \{i_1, \cdots, i_k\} \subset \{1, \cdots, n\}$, let

$$\Delta_I := \{(z_1, \cdots, z_n) \in \Sigma^n | z_{i_1} = \cdots = z_{i_k}\}.$$
The collection of all such diagonal divisors, denoted by

\[ \Delta = \bigcup_{1 \leq i \neq j \leq n} \Delta_{ij}, \]

is called the big diagonal.

In this subsection, we will generalize the notion of regularized integrals on \( \Sigma \) to define integration of forms on \( \Sigma^n \) with arbitrary holomorphic poles along \( \Delta \).

Let

\[ \Omega_{\Sigma^n}^*(\cdot \Delta) \]

denote the sheaf of meromorphic forms which are holomorphic on \( \Sigma^n - \Delta \) but with arbitrary order of poles along \( \Delta \). Then \( \Omega_{\Sigma^n}^*(\cdot \Delta) \) is a sheaf of differential graded (dg) algebra where the dg structure is given by the holomorphic de Rham differential \( \partial \). Let

\[ \Omega_{\Sigma^n}^*(\log \Delta_{ij}) \subset \Omega_{\Sigma^n}^*(\cdot \Delta) \]

denote the subsheaf of logarithmic forms along the smooth divisor \( \Delta_{ij} \).

**Definition 2.30.** We define \( \Omega_{\Sigma^n}^*(\log \Delta) \) (or simply \( \Omega^*(\log \Delta) \)) to be the sheaf of subalgebra of \( \Omega_{\Sigma^n}^*(\cdot \Delta) \) generated by all \( \Omega_{\Sigma^n}^*(\log \Delta_{ij}) \)’s.

In a local neighborhood \( U \) of a point \( (z_1, \cdots, z_n) \) with \( z_i = z_j \) for \( i, j \in I \subset \{1, \cdots, n\} \), we have

\[ \Omega_U^*(\log \Delta) = \Omega_U^* \left[ \frac{dz_i - dz_j}{z_i - z_j} \right]_{i, j \in I, i \neq j}. \]

Here \( \Omega_U^* \) is the sheaf of holomorphic forms over \( U \).

**Remark 2.31.** The divisor \( \Delta \) is not normal crossing, so \( \Omega_{\Sigma^n}^*(\log \Delta) \) is not the usual log-forms associated to normal crossing divisors. We still use the notation \( \log \Delta \) to illustrate its explicit meaning as above and hope it will not cause confusion.

We also consider smooth forms. Let

\[ A^{p,q}(\Sigma^n, \cdot \Delta) := A^{0,q}(\Sigma^n, \Omega^p(\cdot \Delta)), \quad A^{p,q}(\Sigma^n, \log \Delta) := A^{0,q}(\Sigma^n, \Omega^p(\log \Delta)) \]

and

\[ A^k(\Sigma^n, \cdot \Delta) = \bigoplus_{p+q=k} A^{p,q}(\Sigma^n, \cdot \Delta), \quad A^k(\Sigma^n, \log \Delta) = \bigoplus_{p+q=k} A^{p,q}(\Sigma^n, \log \Delta). \]

Our first goal in this subsection is to define a regularized integral

\[ \int_{\Sigma^n} : A^{2n}(\Sigma^n, \cdot \Delta) \to C \]

in the same fashion as in Section 2.1. As we will prove later (Theorem 2.37), such defined integral will be equal to the iterated regularized integral over the factors of \( \Sigma^n \):

\[ \int_{\Sigma^n} = \int_{\Sigma_1} \cdots \int_{\Sigma_n} \]

where the right hand side is well-defined by Proposition 2.27. This would imply a Fubini type theorem for regularized integrals on \( \Sigma^n \) (see Corollary 2.39 below).
Local theory

Let us first work locally in \( U^n \) where \( U \) is a small open disk around the origin in \( \mathbb{C} \). Let \((z_1, \cdots, z_n)\) be the holomorphic coordinates on \( U^n \). Denote \( z_{ij} := z_i - z_j \).

Then
\[
\Omega^\bullet_{U^n}(\star \Delta) = \Omega^\bullet_{U^n}[z_{ij}^{-1}], \quad \Omega^\bullet_{U^n}(\log \Delta) = \Omega^\bullet_{U^n}\left[ \frac{dz_{ij}}{z_{ij}} \right].
\]

and
\[
\mathcal{A}^\bullet(U^n, \star \Delta) = \mathcal{A}^\bullet(U^n)[z_{ij}^{-1}], \quad \mathcal{A}^\bullet(U^n, \log \Delta) = \mathcal{A}^\bullet(U^n) \left[ \frac{dz_{ij}}{z_{ij}} \right].
\]

For simplicity, we also denote
\[
\theta_{ij} = \frac{dz_{ij}}{z_{ij}}.
\]

It is useful to observe that
\[
\frac{1}{z_{ij}z_{jk}} + \frac{1}{z_{jk}z_{ki}} + \frac{1}{z_{ki}z_{ij}} = 0, \quad \text{for} \ i, j, k \ \text{distinct}.
\]

This implies that the 1-forms \( \{\theta_{ij}\} \) satisfy the Arnold relation
\[
\theta_{ij}\theta_{jk} + \theta_{jk}\theta_{ki} + \theta_{ki}\theta_{ij} = 0, \quad \text{for} \ i, j, k \ \text{distinct}.
\]

**Lemma 2.32.** A basis of \( \mathbb{C}[\theta_{ij}] \) is
\[
\{ \theta_{i_1 j_1} \theta_{i_2 j_2} \cdots \theta_{i_k j_k} \mid i_1 < i_2 < \cdots < i_k \ \text{and} \ i_1 < j_1, i_2 < j_2, \cdots, i_k < j_k \}.
\]

**Lemma 2.32** is well-known [Arn69] and the list above gives a basis of the cohomology of configuration space of \( n \) points on \( \mathbb{C} \).

**Lemma 2.33.** Any element in \( \mathbb{C}[z_{ij}^{-1}] \) can be written as a linear combination of
\[
\left\{ \frac{1}{z_{i_1 j_1}^{m_1} z_{i_2 j_2}^{m_2} \cdots z_{i_k j_k}^{m_k}} \right| i_1 < i_2 < \cdots < i_k \ \text{and} \ i_1 < j_1, i_2 < j_2, \cdots, i_k < j_k, \ m_1, \cdots, m_k \geq 0 \right\}.
\]

**Proof.** We prove by induction on the number \( n \) of variables.

Any monomial \( \frac{1}{f} \) in \( \mathbb{C}[z_{ij}^{-1}] \) has the form
\[
\frac{1}{f} = \frac{1}{z_{j_1}^{m_1} \cdots z_{j_s}^{m_s} g}, \quad m_1, \cdots, m_s > 0
\]

where \( k < j_1 < \cdots < j_s \) and \( g \) contains only factors of \( z_{ab} \) with \( a, b > k \).

If \( s = 1 \), we apply the induction hypothesis to \( \frac{1}{s} \). Assume \( s > 1 \). We can use the relation
\[
\frac{1}{z_{j_1} z_{j_2}} = \frac{1}{z_{j_1} z_{j_1} z_{j_2}} - \frac{1}{z_{j_2} z_{j_1} z_{j_2}}
\]

to write \( \frac{1}{s} \) as a linear combination of terms with either \( m_1 \) or \( m_2 \) being decreased. Repeating this process, we will eventually arrive at the situation \( s = 1 \). Then the induction applies.

\( \square \)
Lemma 2.34. Let $\omega \in \mathcal{A}_C^{2n}(U^n, \log \Delta)$ be a top-form with compact support. Then the integral
$$\int_{U^n} \omega$$
is absolutely convergent.

**Proof.** By Lemma 2.32 we can assume $\omega$ has the form
$$\omega = \alpha \theta_{i_1j_1} \theta_{i_2j_2} \cdots \theta_{i_kj_k}$$
where $i_1 < i_2 < \cdots < i_k$, $i_1 < j_1, i_2 < j_2, \cdots, i_k < j_k$, and $\alpha$ is a smooth $(2n-k)$-form. We can consider a linear change of coordinate such that $z_{i_1j_1}, \cdots, z_{i_kj_k}$ are part of the new coordinates. Then $\omega$ has only logarithmic pole so the integral is absolutely convergent. \(\square\)

Lemma 2.35. Any $\omega \in \mathcal{A}_C^{2n}(U^n, \star \Delta)$ can be expressed as
$$\omega = \alpha + \partial \beta$$
where $\alpha \in \mathcal{A}_C^{2n}(U^n, \log \Delta)$ and $\beta \in \mathcal{A}_C^{n-1,n}(U^n, \star \Delta)$. The supports of $\alpha$ and $\beta$ can be chosen to be contained in the support of $\omega$.

**Proof.** By Lemma 2.33 we can assume $\omega$ has the form
$$\omega = \frac{dz_{i_1j_1}}{z_{i_1j_1}^{m_1}} \cdot \frac{dz_{i_2j_2}}{z_{i_2j_2}^{m_2}} \cdots \frac{dz_{i_kj_k}}{z_{i_kj_k}^{m_k}} \varphi, \quad \varphi \text{ is a smooth } (2n-k)\text{-form}$$
where $i_1 < i_2 < \cdots < i_k$ and $i_1 < j_1, i_2 < j_2, \cdots, i_k < j_k$, $m_1, \cdots, m_k > 0$. Notice that $z_{i_1j_1}, \cdots, z_{i_kj_k}$ can be extended to become part of a set of linear coordinates.

If $m_1 > 1$, then we can write
$$\omega = -\frac{1}{(m_1-1)} \partial \left( \frac{dz_{i_2j_2}}{z_{i_2j_2}^{m_2}} \cdot \frac{dz_{i_3j_3}}{z_{i_3j_3}^{m_3}} \cdots \frac{dz_{i_kj_k}}{z_{i_kj_k}^{m_k}} \varphi \right) - (-1)^k \frac{1}{z_{i_1j_1}^{m_1-1}} \frac{dz_{i_2j_2}}{z_{i_2j_2}^{m_2}} \cdots \frac{dz_{i_kj_k}}{z_{i_kj_k}^{m_k}} \partial \varphi$$
$$= -\frac{1}{(m_1-1)} \partial \left( \frac{dz_{i_2j_2}}{z_{i_2j_2}^{m_2}} \cdot \frac{dz_{i_3j_3}}{z_{i_3j_3}^{m_3}} \cdots \frac{dz_{i_kj_k}}{z_{i_kj_k}^{m_k}} \varphi \right) + \frac{dz_{i_1j_1}}{z_{i_1j_1}^{m_1-1}} \frac{dz_{i_2j_2}}{z_{i_2j_2}^{m_2}} \cdots \frac{dz_{i_kj_k}}{z_{i_kj_k}^{m_k}} \varphi$$
where $\varphi$ is another smooth $(2n-k)$-form. Repeating this process, we can reduce $\omega$ to a form with $m_1 = m_2 = \cdots = m_k = 1$ up a $\partial$-exact term. This proves the lemma. \(\square\)

**Global theory**

We first have the analogue of Lemma 2.1

Lemma 2.36. Any $\omega \in \mathcal{A}_C^{2n}(\Sigma^n, \star \Delta)$ can be expressed as
$$\omega = \alpha + \partial \beta$$
where $\alpha \in \mathcal{A}_C^{2n}(\Sigma^n, \log \Delta)$ and $\beta \in \mathcal{A}_C^{n-1,n}(\Sigma^n, \star \Delta)$.

**Proof.** This follows from Lemma 2.35 and a use of partition of unity. \(\square\)

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Given $\omega \in A^\bullet(\Sigma^n, \ast \Delta)$, we consider its push-forward along a factor of $\Sigma$ by performing a regularized integration. Proposition 2.27 implies that
\[
\int_\Sigma : A^\bullet(\Sigma^n, \ast \Delta) \rightarrow A^{\bullet-2}(\Sigma^{n-1}, \ast \Delta).
\]
By Theorem 2.29 we have
\[
\int_\Sigma \partial \alpha = \partial \int_\Sigma \alpha, \quad \text{for} \quad \alpha \in A^\bullet(\Sigma^n, \ast \Delta).
\]
Here $\partial$ is the holomorphic de Rham differential on the corresponding space.

**Theorem 2.37.** Let $\omega \in A^{2n}(\Sigma^n, \ast \Delta)$. Choose a decomposition (as guaranteed by Lemma 2.36)
\[
\omega = \alpha + \partial \beta, \quad \text{where} \quad \alpha \in A^{2n}(\Sigma^n, \log \Delta), \beta \in A^{n-1,n}(\Sigma^n, \ast \Delta).
\]
Then the integral $\int_{\Sigma^n} \alpha$ is absolutely convergent and is equal to the iterated regularized integral
\[
\int_{\Sigma^n} \alpha = \int_{\Sigma} \int_{\Sigma} \cdots \int_{\Sigma} \omega.
\]
In particular, the value $\int_{\Sigma^n} \alpha$ does not depend on the choice of $\alpha$ and $\beta$.

**Proof.** By Corollary 2.34, $\alpha$ is absolutely integrable on $\Sigma^n$. It is logarithmic along any factor of $\Sigma$. The push-forward along a factor of $\Sigma$ gives
\[
\int_\Sigma \omega = \int_\Sigma \alpha + \int_\Sigma \partial \beta = \int_\Sigma \alpha + \partial \int_\Sigma \beta \in A^{2n-2}(\Sigma^{n-1}, \ast \Delta).
\]
The form $\int_\Sigma \alpha$ again lies in $A^{2n-2}(\Sigma^{n-1}, \log \Delta)$. This can be proved by the same method as in the proof of Proposition 2.27 Iterating this process, we eventually arrive at
\[
\int_{\Sigma^n} \alpha = \int_{\Sigma} \int_{\Sigma} \cdots \int_{\Sigma} \alpha.
\]
It follows that
\[
\int_{\Sigma^n} \alpha = \int_{\Sigma} \int_{\Sigma} \cdots \int_{\Sigma} \omega.
\]
In particular, this value only depends on $\omega$, but not on the choice of the decomposition. \qed

**Definition 2.38.** We define the regularized integral
\[
\int_{\Sigma^n} : A^{2n}(\Sigma^n, \ast \Delta) \rightarrow \mathbb{C} \quad \text{by} \quad \int_{\Sigma^n} \omega := \int_{\Sigma^n} \alpha
\]

where $\omega = \alpha + \partial \beta$ for $\alpha \in A^{2n}(\Sigma^n, \log \Delta)$ and $\beta \in A^{n-1,n}(\Sigma^n, \ast \Delta)$.

Such regularized integral is well-defined by Theorem 2.37.

**Corollary 2.39.** Let $\sigma$ be any permutation of $\{1, 2, \cdots, n\}$. Then the iterated regularized integral
\[
\int_{\Sigma_{\sigma(1)}} \int_{\Sigma_{\sigma(2)}} \cdots \int_{\Sigma_{\sigma(n)}} \omega, \quad \omega \in A^{2n}(\Sigma^n, \ast \Delta)
\]
does not depend on the choice of $\sigma$.

**Proof.** By Theorem 2.37 all such iterated regularized integrals are equal to $\int_{\Sigma^n} \omega$. \qed
3 Application: regularized integrals and modular forms

In this section, we apply our theory to elliptic curves and construct a large class of modular objects, including those coming from Feynman graph integrals. We obtain a precise connection between quasi-modular forms arising from $A$-cycle integrals and regularized integrals on configuration spaces of elliptic curves (Theorem 3.4). This leads to a simple geometric proof of the mixed weight quasi-modularity of $A$-cycle integrals, as well as novel combinatorial formulae for all the components of different weights (Theorem 3.9).

We recall and fix some notations. Let $\tau$ be a point on the upper half-plane $\mathbf{H}$. Let

$$E_\tau = \mathbb{C}/\Lambda_\tau, \quad \Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau$$

be the corresponding elliptic curve. Recall that $z$ is linear holomorphic coordinate on the universal cover $\mathbb{C}$ in terms of which one has

$$\int_{E_\tau} \frac{d^2z}{\text{im} \tau} = 1, \quad d^2z := \frac{i}{2} dz \wedge d\bar{z}.$$

We will fix a basis $\{A, B\}$ for $H_1(E_\tau, \mathbb{Z})$. In the universal cover $\mathbb{C} \to E_\tau$, $A$ is represented by the segment $[\tau, \tau + 1]$ and $B$ is represented by the segment $[1, 1 + \tau]$. Such $A, B$ will be called the canonical representatives. The fundamental domains and $A, B$-cycles on both the universal cover and the Picard uniformization $\mathbb{C}^* \to E_\tau$ are displayed in Fig. 3 below.

The following notion of holomorphic limit plays an important role in this work.

**Definition 3.1.** Let $\mathcal{O}_{\mathbf{H}}[\frac{1}{\text{im} \tau}]$ denote functions $f(\tau, \bar{\tau})$ on the upper half-plane $\mathbf{H}$ of the form

$$f(\tau, \bar{\tau}) = \sum_{i=0}^{N} \frac{f_i(\tau)}{\text{im} \tau}^i, \quad N < \infty.$$

Here the $f_i(\tau)$’s are holomorphic in $\tau$. We define the holomorphic limit, denoted by

$$\lim_{\tau \to \infty} : \mathcal{O}_{\mathbf{H}}[\frac{1}{\text{im} \tau}] \to \mathcal{O}_{\mathbf{H}},$$

as

$$\lim_{\tau \to \infty} f(\tau, \bar{\tau}) := f_0(\tau).$$

We say $f$ is almost-holomorphic on $\mathbf{H}$ if $f \in \mathcal{O}_{\mathbf{H}}[\frac{1}{\text{im} \tau}]$. 

![Figure 3: Fundamental domains and canonical representatives.](image)
The notion of holomorphic limit can be generalized straightforwardly to the space \( \mathcal{M}_H[1/m_{\tau}] \) consisting of polynomials in \( 1/im_{\tau} \) with coefficients being meromorphic functions in \( \tau \). We still call it holomorphic limit by abuse of language.

In Appendix \( \text{A} \) we collect basics of modular forms and elliptic functions that will be frequently used in this section.

### 3.1 Regularized integrals v.s. \( \Lambda \)-cycle integrals

#### Regularized integrals and modularity

We consider the following action of \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{C}^n \times \mathcal{H} \) \((n \geq 0)\) by

\[
\gamma : \mathbb{C}^n \times \mathcal{H} \to \mathbb{C}^n \times \mathcal{H},
\]

\[
(z_1, \ldots , z_n; \tau) \mapsto (\gamma z_1, \ldots , \gamma z_n; \gamma \tau) := \left( \frac{z_1}{ct+d}, \ldots , \frac{z_n}{ct+d}; \frac{a\tau+b}{ct+d} \right).
\]

**Definition 3.2.** A function \( \Phi(z_1, \ldots , z_n; \tau) \) on \( \mathbb{C}^n \times \mathcal{H} \) is modular of weight \( k \in \mathbb{Z} \) if

\[
\Phi(\gamma z_1, \ldots , \gamma z_n; \gamma \tau) = (c\tau + d)^k \Phi(z_1, \ldots , z_n; \tau), \quad \forall \gamma \in \text{SL}_2(\mathbb{Z}).
\]

It is said to be elliptic if

\[
\Phi(z_1 + \lambda_1, \ldots , z_n + \lambda_n; \tau) = \Phi(z_1, \ldots , z_n; \tau), \quad \forall (\lambda_1, \ldots , \lambda_n) \in \Lambda^\tau.
\]

An elliptic function \( \Phi \) defines a function \( \Phi(-; \tau) \) on \( E^\tau_n \) for generic fixed \( \tau \). We will use the same symbol \( \Phi \) to denote such a function on \( \mathbb{C}^n \times \mathcal{H} \) and the induced function on \( E^\tau_n \) when the meaning is clear from the context.

**Definition 3.3.** A meromorphic function \( \Phi(z_1, \ldots , z_n; \tau) \) on \( \mathbb{C}^n \times \mathcal{H} \) is said to be holomorphic away from diagonals if the poles of \( \Phi \) are contained in the union of all the following divisors

\[
\{z_i - z_j - \lambda = 0\} \subset \mathbb{C}^n \times \mathcal{H}, \quad 1 \leq i \neq j \leq n, \quad \lambda \in \Lambda^\tau.
\]

Let \( \Phi(z_1, \ldots , z_n; \tau) \) be a meromorphic elliptic function on \( \mathbb{C}^n \times \mathcal{H} \) which is holomorphic away from diagonals. Then the meromorphic function \( \Phi(-; \tau) \) on \( E^\tau_n \) has possible poles only along the big diagonal

\[
\Delta = \bigcup_{1 \leq i \neq j \leq n} \Delta_{ij} \subset E^\tau_n.
\]

So \( \Phi(-; \tau) \) defines a holomorphic function on the configuration space of \( n \) points on \( E_{\tau} \).

We are interested in the following regularized integral

\[
\int_{E^\tau_n} \left( \prod_{i=1}^n \frac{d^2z_i}{\text{im}\ \tau} \right) \Phi(z_1, \ldots , z_n; \tau)
\]

which is defined by Definition 2.38. By Theorem 2.37, this integral can be expressed as an iterated regularized integral on \( E_{\tau} \)

\[
\int_{E^\tau_n} \left( \prod_{i=1}^n \frac{d^2z_i}{\text{im}\ \tau} \right) \Phi(z_1, \ldots , z_n; \tau) = \int_{E_{i_1}} \frac{d^2z_{i_1}}{\text{im}\ \tau} \cdots \int_{E_{i_n}} \frac{d^2z_{i_n}}{\text{im}\ \tau} \Phi(z_1, \ldots , z_n; \tau),
\]

where \( \{i_1, \ldots , i_n\} \) is an arbitrary permutation of \( \{1, \ldots , n\} \). By Corollary 2.39 its value does not depend on the choice of the ordering for integration, i.e., the choice of \( i_1, \ldots , i_n \).

The main theorem for our application in this section is the following.
Theorem 3.4. Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ which is holomorphic away from diagonals. Then

(1) The regularized integral
\[
\int_{E_\tau^k} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im} \tau} \right) \Phi(z_1, \cdots, z_n; \tau) \text{ lies in } \mathcal{O}_\mathbb{H}[\frac{1}{\text{im} \tau}].
\]

(2) Let $A_1, \cdots, A_n$ be $n$ disjoint representatives of the $A$-cycle class on $E_\tau$. Then
\[
\lim_{\bar{\tau} \to \infty} \int_{E_\tau^k} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im} \tau} \right) \Phi(z_1, \cdots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \cdots, z_n; \tau).
\]

(3) If $\Phi$ is modular of weight $m$ on $\mathbb{C}^n \times \mathbb{H}$, then
\[
\int_{E_\tau^k} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im} \tau} \right) \Phi \text{ is modular of weight } m \text{ on } \mathbb{H}.
\]

The proof of Theorem 3.4 will be deferred to Section 3.4 after we have developed several techniques in the next subsections.

A-cycle integrals and quasi-modularity

Definition 3.5. Let $f(z_1, \cdots, z_k)$ be a meromorphic function on $E_\tau^k$ which is holomorphic when all $z_i$’s are distinct. Let $\{i_1, \cdots, i_k\}$ be a permutation of $\{1, 2, \cdots, k\}$. We define the ordered A-cycle integral to be the following integral
\[
\int_A dz_{i_1} \cdots \int_A dz_{i_k} f := \int_{A_1} dz_{i_1} \cdots \int_{A_k} dz_{i_k} f.
\]

Here $A_1, A_2, \cdots, A_k$ are representatives of the $A$-cycle class on the fundamental domain of $E_\tau$ that are ordered and oriented by
\[
A_i = \text{interval from } \epsilon_i \tau \text{ to } \epsilon_i \tau + 1,
\]
where
\[
0 < \epsilon_1 < \epsilon_2 < \cdots < \epsilon_k < 1.
\]

Remark 3.6. Since $f$ is meromorphic, the integral $\int_{A_1} dz_{i_1} \cdots \int_{A_k} dz_{i_k} f$ does not depend on the precise values of $\epsilon_i$’s. So we write the ordered A-cycle integral as $\int_A dz_{i_1} \cdots \int_A dz_{i_k} f$ without specifying the locations. It is also invariant under the cyclic permutation
\[
\int_A dz_{i_1} \cdots \int_A dz_{i_k} f = \int_A dz_{i_2} \cdots \int_A dz_{i_k} \int_A dz_{i_1} f.
\]
Observe that switching the order of an iterated $A$-cycle integral results in a difference related to the residue. This is illustrated by deforming the integration contour as in Fig. 4.

\[
\int_A dz_2 \int_A dz_1 (\cdots) - \int_A dz_1 \int_A dz_2 (\cdots) = \int_A dz_1 \int_A dz_2 (\cdots). 
\]

\[\begin{array}{c}
\int dz_2 \\
\int dz_1 \\
\hline
\end{array} = \begin{array}{c}
\oint dz_2 \\
\bullet z_1 \\
\hline
\end{array}
\]

Figure 4: Commutator of ordered $A$-cycle integrations.

Therefore we find the following identity for two different integral operations

\[
\int_A dz_1 \oint_{z_1} dz_2 = \left[ \int_A dz_2, \int_A dz_1 \right]
\]

where the right hand side is the commutator of two ordered $A$-cycle integrals defined above.

The same consideration proves the following lemma that expresses commutators of ordered $A$-cycle integrals in terms of residue operations.

**Lemma 3.7.** Let $f(z_1, \cdots, z_k)$ be a meromorphic function on $E^k_\tau$ which is holomorphic when all $z_i$’s are distinct. Then

\[
\int_A dz_1 \oint_{z_1} \cdots \oint_{z_k} dz_k f(z_1, \cdots, z_k) = \left[ \int_A dz_2, \left[ \int_A dz_3, \cdots, \left[ \int_A dz_k, \int_A dz_1 \right] \right] \right] f(z_1, \cdots, z_k).
\]

Let $\Phi$ be a function as in Theorem 3.4. Then Theorem 3.4 can be equivalently described in terms of the ordered $A$-cycle integrals as

\[
\lim_{\tau \to \infty} \int_{E_\tau} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im } \tau} \right) \Phi = \frac{1}{n!} \sum_{\sigma \in S_n} \int_A dz_{\sigma(1)} \cdots \int_A dz_{\sigma(n)} \Phi.
\]

This actually leads to a similar description for any ordered $A$-cycle integral as follows.

Let $V = \mathbb{R}^n$ be a real vector space of dimension $n$. Let

\[
T(V) = \bigoplus_{k \geq 0} V^k
\]

denote the tensor algebra over $\mathbb{R}$. Let $\text{Lie}(V)$ denote the free Lie algebra over $\mathbb{R}$ generated by $V$. We equip $T(V)$ with the Lie algebra structure whose Lie bracket is the commutator

\[
[a, b] := a \otimes b - b \otimes a, \quad a, b \in T(V).
\]

Then $\text{Lie}(V)$ can be viewed as the Lie subalgebra of $T(V)$ generated by $V$ and the bracket operation $[\cdot, \cdot]$. It is a classical result that the tensor algebra $T(V)$

\[
T(V) = \mathcal{U}(\text{Lie}(V))
\]
is the universal enveloping algebra of \( \text{Lie}(V) \). By the Poincaré-Birkhoff-Witt Theorem, we can identify

\[
T(V) = \mathcal{S}(\text{Lie}(V))
\]
as vector spaces, where \( \mathcal{S} \) refers to symmetric tensors. This identity gives a natural way to connect any ordered \( A \)-cycle integrals to regularized integrals.

Explicitly, let \( x_1, \ldots, x_n \) denote a basis of \( V \). Then \( T(V) \) can be identified as the free \( \mathbb{R} \)-algebra generated by \( x_i \)'s

\[
T(V) = \mathbb{R} \langle x_1, \ldots, x_n \rangle.
\]

Let \( \{ Y_i^{(s)} \}_{i,s} \) be a basis of \( \text{Lie}(V) \) such that each \( Y_i^{(s)} \) is of the form

\[
[x_i, [x_{i_2}, \ldots, [x_{i_{s-1}}, x_{i_s}] \ldots]], \quad I = (i_1, i_2, \ldots, i_s).
\]

Using \( T(V) = \mathcal{S}(\text{Lie}(V)) \), we can write

\[
x_1 x_2 \cdots x_n = \sum_{k=1}^{n} R_n^{(k)}(Y).
\]

Here \( R_n^{(k)}(Y) \) is a degree-\( k \) polynomial (viewed as symmetric tensor) in \( Y_i^{(\leq n)} \)'s. Explicit formula can be obtained from the result in [Sol68] applied to \( \text{Lie}(V) \).

**Example 3.8.** Here is the example when \( n = 3 \) (see [Sol68]). We choose

\[
Y^{(1)} = \{ x_1, x_2, x_3 \},
\]

\[
Y^{(2)} = \{ [x_1, x_2], [x_1, x_3], [x_2, x_3] \},
\]

\[
Y^{(3)} = \{ [x_1, [x_2, x_3]], [x_2, [x_1, x_3]] \}.
\]

Then

\[
x_1 x_2 x_3 = \frac{1}{6} (x_1 x_2 x_3 + x_1 x_3 x_2 + x_2 x_1 x_3 + x_2 x_3 x_1 + x_3 x_1 x_2 + x_3 x_2 x_1)
\]

\[+ \frac{1}{4} (x_1 [x_2, x_3] + [x_2, x_3] x_1) + \frac{1}{4} (x_2 [x_1, x_3] + [x_1, x_3] x_2) + \frac{1}{4} (x_3 [x_1, x_2] + [x_1, x_2] x_3)
\]

\[+ \frac{1}{3} [x_1, [x_2, x_3]] - \frac{1}{6} [x_2, [x_1, x_3]].
\]

We have

\[
R_3^{(3)} = Y_1^{(1)} Y_2^{(1)} Y_3^{(1)},
\]

\[
R_3^{(2)} = \frac{1}{2} Y_1^{(1)} Y_2^{(2)} + \frac{1}{2} Y_2^{(1)} Y_1^{(2)} + \frac{1}{2} Y_3^{(1)} Y_1^{(2)},
\]

\[
R_3^{(1)} = \frac{1}{3} Y_1^{(3)} + \frac{1}{6} Y_2^{(3)} + \frac{1}{213} Y_3^{(3)}.
\]

For each

\[
Y_i^{(s)} = [x_i, [x_{i_2}, \ldots, [x_{i_{s-1}}, x_{i_s}] \ldots]],
\]

we associate an operation

\[
\oint_{Y_i^{(s)}} := \begin{cases} 
\oint_{z_{i_2}} dz_{i_2} \oint_{z_{i_3}} dz_{i_3} \cdots \oint_{z_{i_{s-1}}} dz_{i_{s-1}} & I = (i_1, i_2, \ldots, i_s), \quad s \geq 2, \\
\text{identity operator} & I = (i), \quad s = 1.
\end{cases}
\]
Then for each $R_n^{(k)}(Y)$ that appears in the above decomposition for $x_1 x_2 \cdots x_n$, we write

$$R_n^{(k)}(\oint_Y)$$

for the operation that replaces each $\gamma_i^{(s)}$ by $\oint_Y$ in $R_n^{(k)}$. Therefore, one can write

$$R_n^{(k)}(\oint_Y) \Phi = \sum_{l=(i_1, \cdots, i_k)} \phi_l^{(k)}(z_{i_1}, \cdots, z_{i_k}; \tau)$$

for some functions $\phi_l^{(k)}$.

**Theorem 3.9.** Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ which is holomorphic away from diagonals. Let

$$R_n^{(k)}(\oint_Y) \Phi = \sum_{l=(i_1, \cdots, i_k)} \phi_l^{(k)}(z_{i_1}, \cdots, z_{i_k}; \tau).$$

1. The ordered $A$-cycle integral is given by the holomorphic limit

$$\int_A dz_1 \cdots \int_A dz_n \Phi(z_1, \cdots, z_n; \tau) = \lim_{\tau \to \infty} \sum_{k=1}^n \sum_{l=(i_1, \cdots, i_k)} \oint_{E_k^i} \left( \prod_{j=1}^k \frac{d^2z_{i_j}}{\text{im} \tau} \right) \phi_l^{(k)}(z_{i_1}, \cdots, z_{i_k}; \tau).$$

2. If $\Phi$ is modular of weight $m$ on $\mathbb{C}^n \times \mathbb{H}$, then each

$$\oint_{E_k^i} \left( \prod_{j=1}^k \frac{d^2z_{i_j}}{\text{im} \tau} \right) \phi_l^{(k)}(z_{i_1}, \cdots, z_{i_k}; \tau)$$

is modular of weight $m + k - n$. In particular, the ordered $A$-cycle integral

$$\int_A dz_1 \cdots \int_A dz_n \Phi(z_1, \cdots, z_n; \tau)$$

is quasi-modular of mixed weight with each weight $\leq m$, and the leading weight-$m$ component is

$$\lim_{\tau \to \infty} \oint_{E_k^i} \left( \prod_{j=1}^k \frac{d^2z_{i_j}}{\text{im} \tau} \right) \Phi.$$ 

**Remark 3.10.** If $\Phi$ is modular of weight $m$ on $\mathbb{C}^n \times \mathbb{H}$, then for any $\sigma \in S_n$,

$$\int_A dz_{\sigma(1)} \cdots \int_A dz_{\sigma(n)} \Phi(z_1, \cdots, z_n; \tau)$$

is quasi-modular of mixed weight with each weight $\leq m$. This follows by applying Theorem 3.9 to $\Phi_{\sigma}(z_1, \cdots, z_n; \tau) := \Phi(z_{\sigma^{-1}(1)}, \cdots, z_{\sigma^{-1}(n)}; \tau)$. Theorem 3.4 says that averaging all such ordered $A$-cycle integrals leads to cancellation of all lower-weight components. Such cancellation phenomenon was also proved in [OP18] using a different method.

**Proof of Theorem 3.9.** The algebraic identity $x_1 x_2 \cdots x_n = \sum_{k=1}^n R_n^{(k)}(Y)$ together with Lemma 3.7 and Theorem 3.4 imply

$$\int_A dz_1 \cdots \int_A dz_n \Phi(z_1, \cdots, z_n; \tau) = \lim_{\tau \to \infty} \sum_{k=1}^n \sum_{l=(i_1, \cdots, i_k)} \oint_{E_k^i} \left( \prod_{j=1}^k \frac{d^2z_{i_j}}{\text{im} \tau} \right) \phi_l^{(k)}(z_{i_1}, \cdots, z_{i_k}; \tau).$$
Assume $\Phi$ is modular of weight $m$ on $\mathbb{C}^n \times H$. Each $\varphi^{(k)}_i$ is obtained from $\Phi$ by applying residue $(n-k)$ times. Since each residue map along a diagonal decreases the weight by 1, $\varphi^{(k)}_i$ is modular of weight $m + k - n$. By Theorem 3.4,

$$
\int_{E^+_r} \left( \prod_{j=1}^{k} \frac{d^2 z_i}{\text{im} \, \tau} \right) \varphi^{(k)}_i (z_i, \ldots, z_i; \tau)
$$

is modular of weight $m + k - n$.

\[ \square \]

**Example 3.11.** In the case $n = 3$ as in Example 3.8, Theorem 3.9 implies

$$
\int_{A} dz_1 \int_{A} dz_2 \int_{A} dz_3 \Phi = \lim_{\tau \to \infty} \left\{ \int_{E_r} \frac{d^2 z_1}{\text{im} \, \tau} \int_{E_r} \frac{d^2 z_2}{\text{im} \, \tau} \int_{E_r} \frac{d^2 z_3}{\text{im} \, \tau} \Phi + \frac{1}{2} \int_{E_r} \frac{d^2 z_1}{\text{im} \, \tau} \int_{E_r} \frac{d^2 z_2}{\text{im} \, \tau} \int_{E_r} \frac{d^2 z_3}{\text{im} \, \tau} \Phi \right\}.
$$

**Example 3.12.** Consider the case $n = 4$, with

$$
\Phi(z_1, z_2, z_3, z_4; \tau) = \varphi(z_1 - z_2; \tau) \varphi(z_2 - z_3; \tau) \varphi(z_3 - z_4; \tau) \varphi(z_4 - z_1; \tau).
$$

The function $\Phi$ is invariant under the dihedral group action on the 4 arguments $z_1, z_2, z_3, z_4$. Hence among the $4! = 24$ ordered $A$-cycles integrals it suffices to consider the following 3 integrals:

$$
\int_{A} dz_4 \int_{A} dz_3 \int_{A} dz_2 \int_{A} dz_1 \Phi, \quad \int_{A} dz_3 \int_{A} dz_4 \int_{A} dz_2 \int_{A} dz_1 \Phi, \quad \int_{A} dz_4 \int_{A} dz_3 \int_{A} dz_2 \int_{A} dz_1 \Phi.
$$

Following the method outlined in Remark 2.25, we obtain

$$
\int_{A} dz_4 \int_{A} dz_3 \int_{A} dz_2 \int_{A} dz_1 \Phi = (2\pi i)^8 \sum_{k \geq 1} \frac{k^4 q^k + q^{3k}}{(1 - q^k)^4} + \frac{\pi^2}{3} E_2^4,
$$

$$
\int_{A} dz_3 \int_{A} dz_4 \int_{A} dz_2 \int_{A} dz_1 \Phi = (2\pi i)^8 \sum_{k \geq 1} \frac{2q^{2k}}{(1 - q^k)^4} + \frac{\pi^2}{3} E_2^4,
$$

$$
\int_{A} dz_4 \int_{A} dz_3 \int_{A} dz_2 \int_{A} dz_1 \Phi = (2\pi i)^8 \sum_{k \geq 1} \frac{k^4 q^{2k}}{(1 - q^k)^4} + \frac{\pi^2}{3} E_2^4.
$$

On the other hand, using (A.1), (A.4), we obtain

$$
\sum_{k \geq 1} \frac{k^4 q^k + 4q^{2k} + q^{3k}}{(1 - q^k)^4} = -\frac{1}{24} E''_2 = \frac{1}{2^2 3^3} (3E_2^2 E_4 - 4E_2 E_6 + E_4^2),
$$

$$
\sum_{k \geq 1} \frac{k^4 q^k}{(1 - q^k)^2} = \frac{1}{240} E'_4 = \frac{1}{2^4 3^5} (4E_2 E_6 - E_8),
$$

where $' = \frac{1}{2\pi i} \frac{d}{d\tau}$. Combining the above two sets of relations, we see that the above 3 ordered $A$-cycle integrals are holomorphic quasi-modular forms of mixed weight

$$
\int_{A} dz_4 \int_{A} dz_3 \int_{A} dz_2 \int_{A} dz_1 \Phi = \frac{\pi^8}{3^4} E_2^4 + 2^8 \pi^8 \left( \frac{1}{3} \cdot \frac{1}{24} E''_2 + \frac{2}{3} \cdot \frac{1}{240} E'_4 \right),
$$

\[ \text{4This can be seen from the modular transformation on the variables } z_1, \ldots, z_n \text{ in Definition 3.2.} \]
\[
\int_A dz_3 \int_A dz_4 \int_A dz_2 \int_A dz_1 \Phi = \frac{\pi^8}{34} E_2^4 + 2^8 \pi^8 \left( \frac{1}{3} \cdot \frac{1}{24} E_2'' - \frac{1}{3} \cdot \frac{1}{240} E_4' \right),
\]

\[
\int_A dz_4 \int_A dz_2 \int_A dz_3 \int_A dz_1 \Phi = \frac{\pi^8}{34} E_2^4 + 2^8 \pi^8 \left( \frac{1}{3} \cdot \frac{1}{24} E_2'' - \frac{1}{3} \cdot \frac{1}{240} E_4' \right).
\]

It follows that

\[
\frac{1}{4!} \sum_{\sigma \in S_4} \int_A dz_{\sigma(1)} \cdots \int_A dz_{\sigma(4)} \Phi = \frac{\pi^8}{34} E_2^4 + 2^8 \pi^8 \frac{1}{3} \cdot \frac{1}{24} E_2''
\]

\[
= \frac{\pi^8}{34} E_2^4 + \frac{2^8 \pi^8}{34} (3E_2^2 E_4 - 4E_2 E_6 + E_4^2).
\]

It is straightforward to compute the iterated residues. For example, we have (here \(\tau = \partial_z\))

\[
\text{Res}_{z_2 = z_3} \text{Res}_{z_1 = z_2} \Phi = \varphi(z_3 - z_4) \varphi''(z_3 - z_4),
\]

\[
\text{Res}_{z_2 = z_4} \text{Res}_{z_1 = z_2} \Phi = -\varphi(z_3 - z_4) \varphi''(z_3 - z_4),
\]

\[
\text{Res}_{z_3 = z_4} \text{Res}_{z_1 = z_2} \Phi = \varphi'(z_2 - z_4) \varphi'(z_3 - z_4),
\]

\[
\text{Res}_{z_4 = z_3} \text{Res}_{z_1 = z_2} \Phi = -\varphi''(z_3 - z_2) \varphi(z_2 - z_3).
\]

The result

\[
\int_A \varphi^3 dz = -\frac{1}{15} \pi^6 E_2 E_4 + \frac{2^2}{3^3} \pi^6 E_6
\]

from Example 2.24 implies that

\[
\int_A \varphi \varphi'' dz = -\int_A \varphi' \varphi' dz = \frac{2^3}{3^2} \pi^6 (-E_4 E_2 + E_6).
\]

By Proposition 2.27, \(\text{Res} \text{Res} \text{Res} \text{Res} \Phi = 0\). By the pure-weight reason in Theorem 3.16, the iterated regularized integrals of \(\text{Res} \Phi\) and \(\text{Res} \text{Res} \text{Res} \Phi\) are almost-holomorphic modular forms of odd weight and hence vanish. Both of these two claims can be confirmed in the current example from direct computations. Applying Theorem 3.9 to the ordered \(A\)-cycle integral \(\int_A dz_4 \int_A dz_3 \int_A dz_2 \int_A dz_1 \Phi\), a tedious calculation shows that the holomorphic limits of the \(\text{Res} \text{Res} \Phi\) terms in Theorem 3.9 combine to

\[
\frac{2^3}{3^2} \pi^8 \pi' E_4'.
\]

These match the above results for \(\int_A dz_4 \int_A dz_3 \int_A dz_2 \int_A dz_1 \Phi\) and \(\frac{1}{4!} \sum_{\sigma \in S_4} \int_A dz_{\sigma(1)} \cdots \int_A dz_{\sigma(4)} \Phi\).

### 3.2 Modularity of regularized integrals

In this subsection we establish statement (1) and (3) of Theorem 3.4.

**Definition 3.13.** We say a function \(\Psi\) on \(\mathbb{C}^n \times H\) is almost-meromorphic if \(\Psi\) can be written as a finite sum

\[
\Psi(z_1, \ldots, z_n; \tau) = \sum_{k_1, \ldots, k_n, m \geq 0} \Psi_{k_1, \ldots, k_n, m}(z_1, \ldots, z_n; \tau) \left( \frac{\text{im } z_1}{\text{im } \tau} \right)^{k_1} \cdots \left( \frac{\text{im } z_n}{\text{im } \tau} \right)^{k_n} \left( \frac{1}{\text{im } \tau} \right)^m,
\]

where each \(\Psi_{k_1, \ldots, k_n, m}(z_1, \ldots, z; \tau)\) is a meromorphic function on \(\mathbb{C}^n \times H\).

**Definition 3.14.** Let \(\mathcal{R}^E_n\) denote the space of functions \(\Psi\) on \(\mathbb{C}^n \times H\) such that
• Ψ is elliptic and almost-meromorphic.
• Each component Ψ_{k_1,...,k_n,m} as in Definition 3.13 is holomorphic away from diagonals.

Let Ψ ∈ ℛ^{E_n}. Then Ψ(−; τ) defines a function on E^n with possible poles only along all the diagonals of E^n. We consider the following regularized integral

$$\int_{E^n} \frac{d^2z_n}{\text{im} \tau} Ψ$$

which is a well-defined function on C^{n-1} × H by Proposition 2.27.

**Proposition 3.15.** The regularized integral defines a map

$$\int_{E^n} \frac{d^2z_n}{\text{im} \tau} : ℛ^{E_n} \to ℛ^{E_{n-1}}.$$

If Ψ ∈ ℛ^{E_n} is modular of weight k, then \( \int_{E^n} \frac{d^2z_n}{\text{im} \tau} Ψ \) is also modular of weight k.

**Proof.** Let Ψ ∈ ℛ^{E_n}. It is clear that \( \int_{E^n} \frac{d^2z_n}{\text{im} \tau} Ψ \) is elliptic. By Proposition 2.27 it also has the required location of poles. We need to show that \( \int_{E^n} \frac{d^2z_n}{\text{im} \tau} Ψ \) is almost-meromorphic. Let

$$f(z_1,\cdots,z_{n-1}; τ) = \int_{E^n} \frac{d^2z_n}{\text{im} \tau} Ψ(z_1,\cdots,z_n; τ).$$

Given \( z_1,\cdots,z_{n-1} \), we choose a parallelogram □_c in C with vertices \{c, c + 1, c + 1 + τ, c + τ\} such that the poles \( D \) of \( Ψ(z_1,\cdots,z_{n-1},-; τ) \) as a function of \( z_n \) do not lie on the boundary of □_c. Let \( A_c \) denote the interval from \( c + τ \) to \( c + 1 + τ \), and \( B_c \) denote the interval from \( c + 1 \) to \( c + 1 + τ \), as illustrated in Fig. 5 below.

![Figure 5: Parallelogram □_c.](image)

Then we have

$$f(z_1,\cdots,z_{n-1}; τ) = \int_{□_c} \frac{d^2z_n}{\text{im} \tau} Ψ(z_1,\cdots,z_n; τ),$$

and its value does not depend on the choice of \( c \) by the translation invariance of the regularized integral as shown in Proposition 2.8. Let

$$Ψ = \sum_k Ψ_k \left( \frac{\text{im} z_n}{\text{im} \tau} \right)^k,$$
where $\Psi_k$ is meromorphic in $z_n$ and almost-meromorphic in $\{z_1, \ldots, z_{n-1}, \tau\}$. Using Theorem 2.14, we find

$$\int_{\Box_c} \frac{d^2 z_n}{\text{im} \tau} \Psi(z_1, \ldots, z_n; \tau) = \int_{\Box_c} \frac{dz_n \wedge d(\text{im} z_n)}{\text{im} \tau} \Psi(z_1, \ldots, z_n; \tau)$$

$$= - \int_{\Box_c} d \left( \sum_k \frac{\Psi_k(z_n; \tau)}{\text{im} \tau} \right)$$

$$= \int_{\mathcal{A}_c} dz_n \sum_k \Psi_k(-z_n; \tau) \left( \frac{\text{im} z_n}{\text{im} \tau} \right) + \int_{\mathcal{A}_c} dz_n \sum_k \Psi_k(-z_n - \tau; \tau) \left( \frac{\text{im} (z_n - \tau)}{\text{im} \tau} \right)$$

$$+ \sum_{p \in \mathcal{D}} \sum_k \Psi_k \left( \frac{\text{im} z_n}{\text{im} \tau} \right).$$

Here the $B$-cycle integration is cancelled out by the periodicity of $\Psi$ under $z_n \mapsto z_n + 1$.

Observe that at a pole $p \in \mathcal{D}$,

$$\int_{p} dz_n \Psi_k \left( \frac{\text{im} (z_n - p) + \text{im} p}{\text{im} \tau} \right) = \sum_{a+b=k+1} \binom{k+1}{a} \int_{p} dz_n \Psi_k \left( \frac{\text{im} (z_n - p)}{\text{im} \tau} \right) \left( \frac{\text{im} p}{\text{im} \tau} \right)^b$$

$$= \sum_{a+b=k+1} \binom{k+1}{a} \left( \frac{\text{im} p}{2\text{im} \tau} \right)^b \int_{p} dz_n \Psi_k \left( \frac{z_n - p}{2\text{im} \tau} \right)^a.$$

Here in the last step we have used Proposition 2.17. Since all the poles $p$ inside $\Box_c$ are of the form $p = z_i + \lambda$ for some $\lambda \in \Lambda_{\gamma}$, $\frac{\text{im} p}{\text{im} \tau}$ has the form $\frac{\text{im} z_i + \frac{\text{im} \lambda}{\text{im} \tau}}{\text{im} \tau}$.

Now let us assume $\Psi$ is modular of weight $k$ and $\gamma \in \text{SL}_2(\mathbb{Z})$. Let $\gamma \Box_c$ be the image of $\Box_c$ under the $\gamma$-action. Then $\gamma \Box_c$ is a fundamental domain for $\Psi(\gamma z_1, \ldots, \gamma z_{n-1}, z_n; \gamma \tau)$ regarded as a function of $z_n$. Therefore

$$f(\gamma z_1, \ldots, \gamma z_{n-1}; \gamma \tau) = \int_{\gamma \Box_c} \frac{d^2 z_n}{\text{im} (\gamma \tau)} \Psi(\gamma z_1, \ldots, \gamma z_{n-1}, z_n; \gamma \tau).$$

Using Proposition 2.8 and the modularity of $\Psi$, this is equal to

$$\int_{\Box_c} \frac{d^2 (\gamma z_n)}{\text{im} (\gamma \tau)} \Psi(\gamma z_1, \ldots, \gamma z_{n-1}, \gamma z_n; \gamma \tau) = \int_{\Box_c} \frac{d^2 z_n}{\text{im} \tau} \Psi(z_1, \ldots, z_n; \tau)$$

$$= (c\tau + d)^k \int_{\Box_c} \frac{d^2 z_n}{\text{im} \tau} \Psi(z_1, \ldots, z_n; \tau) = (c\tau + d)^k f(z_1, \ldots, z_{n-1}; \tau).$$

$\square$

**Theorem 3.16.** Let $\Phi(z_1, \ldots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$ which is holomorphic away from diagonals and modular of weight $k$. Then

$$\int_{E_\tau} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im} \tau} \right) \Phi$$
is modular of weight $k$ and almost-holomorphic on $\mathbf{H}$. Its holomorphic limit

$$
\lim_{\tau \to \infty} \oint_{E_\tau} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im } \tau} \right) \Phi
$$
is quasi-modular of weight $k$ and holomorphic on $\mathbf{H}$.

Proof. By Theorem 2.37, we have

$$
\oint_{E_\tau} \left( \prod_{i=1}^n \frac{d^2 z_i}{\text{im } \tau} \right) \Phi = \oint_{E_\tau} \frac{d^2 z_1}{\text{im } \tau} \cdots \oint_{E_\tau} \frac{d^2 z_n}{\text{im } \tau} \Phi.
$$
The modularity follows by applying Proposition 3.15 $n$ times. The last statement on the quasi-modularity follows from a general fact about holomorphic limit in the theory of modular forms [KZ95].

Remark 3.17. In the language of the theory of modular forms, the holomorphic limits of such iterated regularized integrals are called weakly holomorphic quasi-modular forms, see Definition A.3.

3.3 Regularized Feynman graph integrals

One of the main motivation of this paper is to develop analytic methods for 2d chiral quantum field theories. Perturbative correlation functions of such theories are given by sums of Feynman graph integrals, which are integrals on product of Riemann surfaces of differential forms with holomorphic poles on the big diagonal. When the theory is put on elliptic curves, Theorem 3.4 provides a powerful tool both for theoretical constructions and for practical computations. We illustrate this by the example of chiral boson. The same method applies to other theories such as chiral $bc$-systems and chiral $\beta\gamma$-systems.

Let $\wp(z; \tau)$ be the Weierstrass $\wp$-function

$$
\wp(z; \tau) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda_{\tau}, (0,0)} \left( \frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right).
$$

Let

$$
\tilde{P}(z_1, z_2; \tau, \bar{\tau}) := \wp(z_1 - z_2; \tau) + \frac{\pi^2}{3} \tilde{E}_2(\tau, \bar{\tau}), \quad \tilde{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi} \frac{1}{\text{im } \tau},
$$

where $E_2$ is the 2nd Eisenstein series. See Appendix A for more details. Here we have specified the $\bar{\tau}$-dependence in $\tilde{P}(z_1, z_2; \tau, \bar{\tau})$ which defines an elliptic function on $\mathbb{C}^2 \times \mathbf{H}$ and is modular of weight 2. As a meromorphic function on $E_\tau^2$, $\tilde{P}(-; \tau)$ has order 2 pole along the diagonal.

Remark 3.18. Let $\phi$ be the field of free chiral boson. Then $\tilde{P}$ is the two-point function on $E_\tau$ (see e.g., [Dou95, Dij97])

$$
\tilde{P}(z_1, z_2; \tau, \bar{\tau}) = \langle \partial \phi(z_1) \partial \phi(z_2) \rangle_{E_{\tau}}.
$$

Mathematically it is known as the Schiffer kernel [Lyu78, Lak01] and is essentially given by the 2nd derivative of the Green’s function associated to the flat metric on $E_\tau$. Its holomorphic limit gives the Bergman kernel associated to our canonical marking $\{A, B\}$ on $E_\tau$.

Let $\Gamma$ be an oriented graph with no self-loops. Let $E(\Gamma)$ be its set of edges, and $V(\Gamma)$ be its set of vertices with cardinality $n = |V(\Gamma)|$. We label the vertices by fixing an identification

$$
V(\Gamma) \to \{1, 2, \cdots, n\}.
$$

5The data of orientation on the graph is not strictly necessary for the Feynman graph integrals here, since $\tilde{P}$ is an even function. We however reserve this notion for potential generalizations such as chiral $bc$-systems and chiral $\beta\gamma$-systems.
The Feynman rule assigns to the graph $\Gamma$ a quantity

$$\Phi_{\Gamma}(z_1, \ldots, z_n; \tau, \bar{\tau}) := \prod_{e \in E(\Gamma)} \hat{P}(z_{t(e)}, z_{h(e)}; \tau, \bar{\tau}),$$

where $h(e)$ is the head of the edge $e$ and $t(e)$ the tail. It is clear that $\Phi_{\Gamma}$ can be written as

$$\Phi_{\Gamma}(z_1, \ldots, z_n; \tau, \bar{\tau}) = \sum_{m=0}^{\lfloor E(V) \rfloor} \Phi_{\Gamma,m}(z_1, \ldots, z_n; \tau) \frac{1}{(\text{im } \tau)^m},$$

where $\Phi_{\Gamma,m}(z_1, \ldots, z_n; \tau)$'s are meromorphic functions on $\mathbb{C}^n \times H$. Let us denote

$$\lim_{\tau \to \infty} \Phi_{\Gamma} := \Phi_{\Gamma,0}.$$

**Definition 3.19.** We define the regularized Feynman graph integral $\tilde{I}_{\Gamma}$ for $\Gamma$ to be

$$\tilde{I}_{\Gamma} := \oint_{E} \left( \prod_{i=1}^{n} \frac{d^2 z_i}{\text{im } \tau} \right) \Phi_{\Gamma}(z_1, \ldots, z_n; \tau, \bar{\tau}).$$

By Corollary 2.39, $\tilde{I}_{\Gamma}$ does not depend the choice of the labeling on $V(\Gamma)$.

The next lemma shows that these graph integrals satisfy a regularity condition at $\tau = i\infty$.

**Lemma 3.20 (Regularity).** Let $f(x_{ij}, y_{ij}) \in \mathbb{C}[x_{ij}, y_{ij}]_{1 \leq i < j \leq n}$ be a polynomial. Let

$$\Phi(z_1, \ldots, z_n; \tau) = f(\varphi(z_i - z_j; \tau), \frac{1}{2\pi i} \varphi'(z_i - z_j; \tau)).$$

Then for any $\sigma \in S_n$ and disjoint representatives $A_1, \ldots, A_n$ of $A$-cycles, the $A$-cycle integral

$$\int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi(z_1, \ldots, z_n; \tau)$$

is holomorphic on $H$ and extends to $\tau = i\infty$ by

$$\lim_{\tau \to i\infty} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi = f(x_{ij} = -\frac{\pi^2}{3}, y_{ij} = 0).$$

**Proof.** By assumption, $\Phi$ is a meromorphic elliptic function on $\mathbb{C}^n \times H$ which is holomorphic away from diagonals. The $A$-cycle integral $\int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \Phi$ is obviously holomorphic in $\tau$ on $H$. We only need to show that it is holomorphic at $\tau = i\infty$.

If $f$ is a constant, then

$$\int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} 1 = 1$$

as desired. By linearity, we can assume $f$ is a non-constant monomial

$$f = \prod_{i>j} x_{ij}^{m_{ij}} y_{ij}^{\epsilon_{ij}}.$$

Let

$$\tilde{\Phi} = f(P(z_i - z_j), \frac{1}{2\pi i} P'(z_i - z_j)), \text{ where } P(z) = \varphi(z) + \frac{\pi^2}{3} E_2.$$
Since \( \lim_{\tau \to i\infty} E_2 = 1 \), proving the desired claims about the asymptotic at \( \tau = i\infty \) is equivalent to showing that

\[
\lim_{\tau \to i\infty} \int_{A_1} dz_{c(1)} \cdots \int_{A_n} dz_{c(n)} \Phi = 0, \quad \forall \sigma \in S_n.
\]

We follow the approach outlined in Remark 2.25 to evaluate the above integral. We first lift the functions \( P(z), P'(z) \) along the Picard uniformization 

\[
u = \exp(2\pi i z), \quad q = e^{2\pi i \tau}.
\]

By (A.8) in Appendix A, we have the absolutely convergent series expansion in \( u \)

\[
P(u) = (2\pi i)^2 \sum_{k \geq 1} \frac{ku^k}{1 - q^k} + (2\pi i)^2 \sum_{k \geq 1} \frac{kq^ku^{-k}}{1 - q^k}, \quad \text{valid in the region } |q| < |u| < 1.
\]

This can be written as

\[
P(u) = \sum_{k \neq 0} c_k u^k, \quad c_k = (2\pi i)^2 \frac{k}{1 - q^k}, \quad |q| < |u| < 1.
\]

We only consider the case when \( \sigma = 1 \in S_n \), and the \( A \)-cycles are ordered and represented on the \( u \)-plane within the region \( |q| < |u| < 1 \) as in Fig. 6. The same argument applies to other cases in the way explained in Remark 3.10. Then

\[
\int_{A_1} \cdots \int_{A_n} dz_1 \cdots dz_n \Phi = \frac{1}{(2\pi i)^n} \int_{A_1} \frac{du_1}{u_1} \cdots \int_{A_n} \frac{du_n}{u_n} \prod_{i \neq j}(P(u_{ij}))^{m_{ij}}(u_{ij} \partial_u P(u_{ij}))^{\epsilon_{ij}}.
\]

Here \( u_{ij} = u_i/u_j \). Notice that \( |q| < |u_{ij}| < 1 \) if \( i > j, \quad u_i \in A_i \).

So we can use the above power series expression for \( P(u_{ij}) \). We are interested in the region when \( |q| \) is very small (\( \tau \to i\infty \)). We fix the radius of each \( A_i \), say between \( \frac{1}{2} \) and 1, and assume \( |q| < \frac{1}{4} \). Then the power series expansion of \( \prod_{i \neq j}(P(u_{ij}))^{m_{ij}}(u_{ij} \partial_u P(u_{ij}))^{\epsilon_{ij}} \) is uniformly absolutely convergent within the integration region and \( |q| \) small, so we can integrate term by term and compute the limit \( q \to 0 \).
We find the desired vanishing property

\[ \frac{1}{(2\pi i)^n} \int_{A_1} \frac{du_1}{u_1} \cdots \int_{A_n} \frac{du_n}{u_n} u_1^{k_1} u_2^{k_2} \cdots u_n^{k_n} = \delta_{k_1,0} \delta_{k_2,0} \cdots \delta_{k_n,0}, \]

the value of the $A$-cycle integral is given by the coefficient of the constant term

\[ u_1^0 u_2^0 \cdots u_n^0 \]

of the series expansion of $\prod_{i<j}(P(u_{ij}))^{m_{ij}}(u_{ij}\partial_u P(u_{ij}))^{\epsilon_{ij}}$.

Let us assume $\sum_i (m_{ni} + \epsilon_{ni}) > 0$. Otherwise the integral does not depend on $u_n$, so we can integrate out $u_n$ first and repeat this process to arrive at this situation. Consider the series expansion that involves $u_n$

\[ \prod_{i<n} (P(u_{ni}))^{m_{ni}}(u_{ij}\partial_u P(u_{ij}))^{\epsilon_{ni}}, \quad P(u_{ni}) = \sum_{k \neq 0} c_k u_{ni}^k = \sum_{k \neq 0} c_k \frac{u_n^k}{u_i^k}. \]

Each term that has $u_n^0$-order contains at least one factor of $c_k$ with $k < 0$. Since

\[ \lim_{q \to 0} c_k = \begin{cases} (2\pi i)^2 k & k > 0, \\ 0 & k < 0. \end{cases} \]

We find the desired vanishing property

\[ \lim_{q \to 0} \frac{1}{(2\pi i)^n} \int_{A_1} \frac{du_1}{u_1} \cdots \int_{A_n} \frac{du_n}{u_n} \prod_{i<j} (P(u_{ij}))^{m_{ij}}(u_{ij}\partial_u P(u_{ij}))^{\epsilon_{ij}} = 0. \]

\[ \square \]

**Remark 3.21.** The regularity result Lemma 3.20 actually holds for more general meromorphic elliptic functions. For example, one also has the holomorphicity at $\tau = i\infty$ for the $A$-cycle integral of a meromorphic elliptic function of the form

\[ \Phi(z_1, \ldots, z_n; \tau) = \prod_{1 \leq i < j \leq n} \frac{\theta(z_i - z_j + c_{ij}; \tau)}{\theta(z_i - z_j; \tau)}. \]

Here $\theta(z; \tau)$ is the unique theta function with odd characteristic of genus one that vanishes at $z = 0$, $c_{ij}$ are constants that could depend linearly in $\tau$. This can be proved in a similar way as Lemma 3.20 by using the Jacobi triple product formula for $\theta$ on $\mathbb{C}^*$. Regularized integrals of functions of this form include Feynman graph integrals that appear in chiral $bc$-systems and chiral $\beta\gamma$-systems.

**Theorem 3.22.** For each oriented graph $\Gamma$ with no self-loops, the regularized Feynman graph integral $\hat{I}_\Gamma$ is an almost-holomorphic modular form of weight $2|E(\Gamma)|$. Its holomorphic limit is given by

\[ \lim_{\tau \to \infty} \hat{I}_\Gamma = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \lim_{\tau \to \infty} \Phi_{\Gamma}(z_1, \ldots, z_n; \tau, \tau) \]

which is a holomorphic quasi-modular form of the same weight. Here $A_1, \ldots, A_n$ are $n$ disjoint representatives of the $A$-cycle class on $E_\tau$.

**Proof.** $\hat{I}_\Gamma$ is modular of weight $2|E(\Gamma)|$ since $\hat{P}$ is modular of weight 2. By Theorem 3.4

\[ \int_{E_\tau^1} \left( \prod_{i=1}^{n} \frac{d^2z_i}{\text{im} \tau} \right) \Phi_{\Gamma,m}(z_1, \ldots, z_n; \tau) \in \mathcal{O}_H \left[ \frac{1}{\text{im} \tau} \right] \]
for each $m$. Therefore $\hat{I}_\Gamma \in \mathcal{O}_H\left[\frac{1}{\im \tau}\right]$ and

$$
\lim_{\tau \to \infty} \hat{I}_\Gamma = \lim_{\tau \to \infty} \int_{E^m_\Gamma} \left( \prod_{i=1}^{n} \frac{d^2z_i}{\im \tau} \right) \Phi_{1,0}(z_1, \cdots, z_n; \tau) = \frac{1}{n!} \sum_{\sigma \in S_n} \int_{A_1} dz_{\sigma(1)} \cdots \int_{A_n} dz_{\sigma(n)} \lim_{\tau \to \infty} \Phi_{1}(z_1, \cdots, z_n; \tau, \tau).
$$

By Lemma 3.20, $\lim_{\tau \to \infty} \hat{I}_\Gamma$ is holomorphic on $H$ and holomorphically extended to $\tau = i\infty$. Therefore $\hat{I}_\Gamma$ is an almost-holomorphic modular form and $\lim_{\tau \to \infty} \hat{I}_\Gamma$ is a holomorphic quasi-modular form, both of weight $2|E(V)|$.

We discuss some examples to illustrate how to compute $\hat{I}_\Gamma$ using Theorem 3.22.

**Example 3.23.** Consider the following graph $\Gamma_\ell$ in Fig. 7 with 2 vertices and $\ell$ edges.

![Figure 7: The banana graph $\Gamma_\ell$: 2 vertices and $\ell$ edges.](image)

The regularized Feynman graph integral is

$$
\hat{I}_{\Gamma_\ell} = \int_{E^m_\Gamma} \left( \prod_{i=1}^{2} \frac{d^2z_i}{\im \tau} \right) \hat{P}^{\ell}(z_1, z_2; \tau, \ell) = \int_{E^m_\Gamma} \left( \prod_{i=1}^{2} \frac{d^2z_i}{\im \tau} \right) \left( \varphi(z_1 - z_2; \tau) + \frac{\pi^2}{3} E^2(\tau, \ell) \right)^\ell.
$$

Translation symmetry implies that

$$
\lim_{\tau \to \infty} \hat{I}_{\Gamma_\ell} = \int_{A_1} dz_1 \int_{A_2} dz_2 \left( \varphi(z_1 - z_2; \tau) + \frac{\pi^2}{3} E^2(\tau) \right)^\ell = \int_{A} dz \left( \varphi(z; \tau) + \frac{\pi^2}{3} E^2(\tau) \right)^\ell.
$$

Here $A$ is a representative of the $A$-cycle class away from the origin $O \in E_\tau$ which is the pole of the 2nd kind Abelian differential

$$
\varphi = \left( \varphi(z; \tau) + \frac{\pi^2}{3} E^2(\tau) \right)^\ell dz.
$$

By Proposition 2.22,

$$
\hat{I}_{\Gamma_\ell} = \int_{A} \varphi \frac{1}{2i \im \tau} \cdot 2\pi i \langle \varphi, dz \rangle_{\varphi}.
$$

Theorem A.5 implies that $\hat{I}_{\Gamma_\ell}$ is recovered from $\lim_{\tau \to \infty} \hat{I}_{\Gamma_\ell}$ by writing the latter in terms of a polynomial in $E_2, E_4, E_6$, then replacing $E_2$ by $\hat{E}_2$. 

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We list the results for the first few graphs up to $\ell = 3$ as follows. From Example 2.24 we have the following formulae in terms of holomorphic quasi-modular forms

\[
\int_A dz \left( \varphi(z; \tau) + \frac{\pi^2}{3} E_2(\tau) \right) = 0,
\]
\[
\int_A dz \left( \varphi(z; \tau) + \frac{\pi^2}{3} E_2(\tau) \right)^2 = \pi^4 \frac{-E_2^2 + E_4}{9},
\]
\[
\int_A dz \left( \varphi(z; \tau) + \frac{\pi^2}{3} E_2(\tau) \right)^3 = \pi^6 \frac{-10E_2^3 + 6E_2E_4 + 4E_6}{5 \cdot 27}.
\]

By Theorem 3.22 the regularized graph integrals are given by their modular completions

\[
\tilde{I}_{\Gamma_1} = 0, \quad \tilde{I}_{\Gamma_2} = \pi^4 \frac{-E_2^2 + E_4}{9}, \quad \tilde{I}_{\Gamma_3} = \pi^6 \frac{-10E_2^3 + 6E_2E_4 + 4E_6}{5 \cdot 27}.
\]

**Example 3.24.** Consider the following graph $\Gamma_\Delta$ with 3 vertices and 3 edges (Fig. 8).

\[\text{Figure 8: The triangle graph $\Gamma_\Delta$: 3 vertices and 3 edges.}\]

The regularized Feynman graph integral is

\[
\tilde{I}_{\Gamma_\Delta} = \int_{E_1^3} \left( \prod_{i=1}^{3} \frac{d^2z_i}{\text{im} \, \tau} \right) \Phi_{\Gamma_\Delta}, \quad \Phi_{\Gamma_\Delta} = \tilde{P}(z_1, z_2) \tilde{P}(z_2, z_3) \tilde{P}(z_3, z_1).
\]

By Theorem 3.22 and Theorem A.5 it suffices to compute the iterated $A$-cycle integrals. In the current case there is a permutation symmetry on the graph, hence iterated $A$-cycle integrals with different orderings give rise to the same result. Therefore we have

\[
\lim_{\tau \to \infty} \tilde{I}_{\Gamma_\Delta} = \int_A dz_1 \int_A dz_2 \int_A dz_1 P(z_1 - z_2) P(z_2 - z_3) P(z_3 - z_1), \quad P(z) = \varphi(z) + \frac{\pi^2}{3} E_2.
\]

Following the approach outlined in Remark 2.25 we obtain

\[
\int_A dz_1 P(z_1 - z_2) P(z_2 - z_3) P(z_3 - z_1) = (2\pi i)^4 P(z_2 - z_3) \sum_{k \geq 1} \frac{k^2 q^k}{(1 - q^k)^2} \left( \frac{u_3}{u_2} \right)^k,
\]
\[
\int_A dz_2 \int_A dz_1 P(z_1 - z_2) P(z_2 - z_3) P(z_3 - z_1) = (2\pi i)^6 \sum_{k \geq 1} \frac{k^3 (q^k + q^{2k})}{(1 - q^k)^3}.
\]

Using (A.1), (A.4), we obtain

\[
(2\pi i)^6 \sum_{k \geq 1} \frac{k^3 (q^k + q^{2k})}{(1 - q^k)^3} = -\frac{1}{24} (2\pi i)^6 (q \frac{d}{dq})^2 E_2 = \frac{1}{123} (2\pi i)^6 (-E_2^3 + 3E_2E_4 - 2E_6).
\]
Using Theorem 3.22 and Theorem A.5 we then have

$$\hat{I}_{\Gamma} = \frac{1}{12^3} (2\pi i)^6 (-\tilde{E}_2^3 + 3\tilde{E}_2 E_4 - 2E_6) .$$

As a comparison, a straightforward way of evaluating the iterated regularized 2d integrals is presented in Appendix C.

Example 3.25. Consider the graphs $\Gamma_1, \Gamma_2$ in Fig. 9 below. These are the only two trivalent graphs with 4 vertices and no self-loops.

![Figure 9: Two trivalent graphs $\Gamma_1, \Gamma_2$ with 4 vertices each.](image)

The ordered $A$-cycle integrals for these graphs are studied in e.g., [RY09, Section 6], [BBBM17, Example 3.5], [GM20, Section 5.4, Section 9.3]. In particular, the mixed-weight phenomenon for the ordered $A$-cycle integrals discovered in [GM20, OP18] are demonstrated on these examples.

As pointed out in the above-cited works, changing the labeling of the vertices is equivalent to changing the ordering for the iterated $A$-cycle integrals. This is also evident from our definition of ordered $A$-cycle integrals. Hence we can stick to a particularly chosen labeling for the graph as we are going to consider all possible orderings.

Consider first the Feynman graph integral associated to $\Gamma_1$. We fix the labeling of vertices to be the one indicated in Fig. 9. The function $\Phi_{\Gamma_1}(z_1, z_2, z_3, z_4; \tau)$ associated to this labeled graph is

$$\Phi_{\Gamma_1}(z_1, z_2, z_3, z_4; \tau) = \tilde{P}(z_1 - z_2) \tilde{P}(z_2 - z_3) \tilde{P}(z_3 - z_1) \tilde{P}(z_1 - z_4) \tilde{P}(z_2 - z_4) \tilde{P}(z_3 - z_4) .$$

It is invariant under the action of $S_4$ which permutes the vertices labeled by 1, 2, 3, 4. Since all these ordered $A$-cycles integrals are equal, according to Theorem 3.4 they have pure modular weight 12. Indeed, their holomorphic limits are given by (see e.g., [GM20, Section 9.3]) the following quasi-modular form of pure weight 12

$$C := \frac{(2\pi i)^{12}}{2^{11} \cdot 3^8} (-E_2^6 + 3E_4^4 E_4 - 3E_2^2 E_4^2 + E_4^3) .$$

Note that the results in [GM20] are expressed in terms of the basis $G_{2k} = -\frac{B_{2k}}{4k} E_{2k}, k \geq 1$, here we use the $E_{2k}$’s whose normalizations are more convenient in consideration of Lemma 3.20. The average of such quantities is thus $C$ itself

$$\frac{1}{4!} \sum_{\sigma \in S_4} \int_A dz_{\sigma(1)} \cdots \int_A dz_{\sigma(4)} \lim_{\tau \to \infty} \Phi_{\Gamma_1} = \frac{(2\pi i)^{12}}{2^{11} \cdot 3^8} (-E_2^6 + 3E_4^4 E_4 - 3E_2^2 E_4^2 + E_4^3) .$$
Applying Theorem [3.4], we obtain the following result for the regularized integral

\[ \mathcal{I}_1 = \int_{E^\perp} \left( \prod_{i=1}^{4} \frac{d^2z_i}{\text{im } \tau} \right) \Phi_{\mathcal{I}_1} = \frac{(2\pi i)^{12}}{2^{10} \cdot 3^5} ( -\bar{E}_2^6 + 3\bar{E}_2^4 E_4 - 3\bar{E}_2^3 E_4^2 + E_4^3 ) . \]

For the regularized Feynman graph integral associated to \( \Gamma_2 \), we fix the labeling of vertices to be the one indicated in Fig. [9] The associated function \( \Phi_{\Gamma_2}(z_1, z_2, z_3, z_4; \tau) \) is

\[ \Phi_{\Gamma_2}(z_1, z_2, z_3, z_4; \tau) = \hat{P}(z_1 - z_2) \hat{P}^2(z_2 - z_3) \hat{P}(z_3 - z_4) \hat{P}^2(z_4 - z_1) . \]

It is invariant under the automorphism group \( G \) of the labeled graph which is generated by horizontal and vertical flips. Among the \( 4! = 24 \) ordered \( A \)-cycles integrals it suffices to consider \( 4! / |G| = 6 \) of them. The same reasoning as in the \( \Gamma_1 \) case tells that the holomorphic limits of these ordered \( A \)-cycle integrals are linear combinations of quasi-modular forms of weight 12 and 10. The results for these integrals, which we quote from [GM20, Section 5.4, Section 9.3], are as follows. Let

\[
\begin{align*}
\mathbf{a} : & = \frac{(2\pi i)^{12}}{2^{10} \cdot 3^5} ( -3E_2^6 + 6E_2^4 E_4 + 4E_2^3 E_4^2 - 3E_2^2 E_4^3 - 12E_2 E_4 E_6 + 4E_4^3 + 4E_6^2 ) , \\
\mathbf{b} : & = \frac{(2\pi i)^{12}}{2^6 \cdot 3^3 \cdot 5 \cdot 7} ( -7E_2^3 E_4 - 3E_2^4 E_6 + 3E_2 E_4^3 + 7E_4 E_6 ) .
\end{align*}
\]

Then 2 out of the 6 inequivalent ordered \( A \)-cycle integrals are equal to

\[
\mathbf{A} := \int_A \text{dz}_4 \int_A \text{dz}_3 \int_A \text{dz}_2 \int_A \text{dz}_1 \lim_{\tau \to \infty} \Phi_{\mathcal{I}_1} = \mathbf{a} - 2\mathbf{b} .
\]

The other 4 are equal to

\[
\mathbf{B} := \int_A \text{dz}_2 \int_A \text{dz}_3 \int_A \text{dz}_4 \int_A \text{dz}_1 \lim_{\tau \to \infty} \Phi_{\mathcal{I}_1} = \mathbf{a} + \mathbf{b} .
\]

The average of the holomorphic limits of ordered \( A \)-cycle integrals is then

\[
\frac{1}{4!} \sum_{\sigma \in S_4} \int_A \text{dz}_{\sigma(1)} \cdots \int_A \text{dz}_{\sigma(4)} \lim_{\tau \to \infty} \Phi_{\Gamma_2} = \frac{1}{4!} \cdot |G| \cdot (\mathbf{A} \cdot 2 + \mathbf{B} \cdot 4) = \mathbf{a} .
\]

Applying Theorem [3.4], we obtain the following result for the regularized integral

\[ \mathcal{I}_2 = \int_{E^\perp} \left( \prod_{i=1}^{4} \frac{d^2z_i}{\text{im } \tau} \right) \Phi_{\mathcal{I}_2} = \frac{(2\pi i)^{12}}{2^{10} \cdot 3^5} ( -3\bar{E}_2^6 + 6\bar{E}_2^4 E_4 + 4\bar{E}_2^3 E_4^2 - 3\bar{E}_2^2 E_4^3 - 12\bar{E}_2 E_4 E_6 + 4E_4^3 + 4E_6^2 ) . \]

### 3.4 Proof of Theorem [3.4]

In this subsection, we complete the proof of Theorem [3.4] The statements (1) and (3) of Theorem [3.4] follow from Theorem [3.16] We next compute the limit

\[
\lim_{\tau \to \infty} \int_{E^\perp} \left( \prod_{i=1}^{n} \frac{d^2z_i}{\text{im } \tau} \right) \Phi(z_1, \cdots, z_n; \tau) .
\]

Let \( \Box_c (c \in \mathbb{C}) \) denote the parallelogram in \( \mathbb{C} \) with vertices \( \{ c, c + 1, c + 1 + \tau, c + \tau \} \). Let \( A_c^+, A_c^-, B_c^+, B_c^- \) denote the intervals as illustrated in Fig. [10]
Lemma 3.26. Let $\Psi$ be an almost-meromorphic elliptic function on $\mathbb{C} \times \mathbb{H}$. Let us write

$$\Psi = \sum_k \Psi_k \left( \frac{\text{im} z}{\text{im} \tau} \right)^k, \quad \Psi_k = \sum_{m=0}^{n_k} \frac{\Psi_{k,m}}{(\text{im} \tau)^m},$$

where $\Psi_{k,m}$ is meromorphic on $\mathbb{C} \times \mathbb{H}$.

Let $\square_c$ be a parallelogram whose boundary does not meet poles of $\Psi$. Then

$$\int_{E_c} \frac{d^2z}{\text{im} \tau} \Psi = \sum_k \frac{1}{k+1} \int_{A_c^+} dz \Psi_k(z) + \sum_k \frac{1}{k+1} \sum_{w \in D} \left( \frac{\text{im} w}{\text{im} \tau} \right)^{k+1} \int_w dz \Psi_k(z)$$

$$+ \frac{1}{k+1} \sum_{j=1}^{k+1} \frac{1}{(\text{im} \tau)^j} \binom{k+1}{j} \sum_{w \in D} \left( \frac{\text{im} w}{\text{im} \tau} \right)^{k+1-j} \int_w dz \Psi_k(z) \left( \frac{z-w}{2i} \right)^j.$$

Here $D$ consists of the poles of $\Psi$ inside $\square_c$.

Proof. Let

$$Y = \sum_k \frac{\Psi_k}{k+1} \left( \frac{\text{im} z}{\text{im} \tau} \right)^{k+1}, \quad \partial_z Y = \frac{i}{2\text{im} \tau} \Psi.$$

The relation $\Psi(z+1) = \Psi(z)$ implies $Y(z+1) = Y(z)$. The relation $\Psi(z+\tau) = \Psi(z)$ implies

$$\partial_z (Y(z+\tau) - Y(z)) = \frac{i}{2\text{im} \tau} (\Psi(z+\tau) - \Psi(z)) = 0.$$

This further implies

$$Y(z+\tau) - Y(z) = \sum_k \frac{\Psi_k(z+\tau)}{k+1}.$$
By Theorem 2.14 and Proposition 2.17,

\[
\int_{E_1} d^2z \Psi = \int_{\Box_c} d^2z \Psi = \int_{\Box_c} \Psi dz \wedge \frac{d\imath z}{\imath \tau} = -\int_{\Box_c} d(Ydz)
\]

\[
= \int_{A_1} dz (Y(z + \tau) - Y(z)) - \int_{B_1} dz (Y(z + 1) - Y(z)) + \sum_{w \in D} \oint_{\gamma w} Ydz
\]

\[
= \int_{A_1} dz \sum_{k} \Psi_k(z + \tau) + \sum_{k} \frac{1}{k+1} \sum_{w \in D} \oint_{\gamma w} \left( \Psi_k(z) \left( \frac{z - \bar{z}}{2i\imath \tau} \right)^{k+1} \right)
\]

\[
= \sum_{k} \frac{1}{k+1} \int_{A_1} dz \Psi_k(z) + \sum_{k} \frac{1}{k+1} \sum_{w \in D} \oint_{\gamma w} \left( \Psi_k(z) \left( \frac{z - \bar{w}}{2i\imath \tau} \right)^{k+1} \right)
\]

\[
= \sum_{k} \frac{1}{k+1} \int_{A_1} dz \Psi_k(z) + \sum_{k} \frac{1}{k+1} \sum_{w \in D} \left( \frac{\imath w}{\imath \tau} \right)^{k+1} \oint_{\gamma w} dz \Psi_k(z)
\]

\[
+ \sum_{k} \frac{1}{k+1} \sum_{j=1}^{k+1-1} \left( \frac{1}{\imath \tau} \right)^{j} \sum_{w \in D} \left( \frac{\imath w}{\imath \tau} \right)^{k+1-j} \oint_{\gamma w} dz \Psi_k(z) \left( \frac{z - \bar{w}}{2i\imath \tau} \right)^{j}.
\]

\[
\square
\]

Our strategy to prove Theorem 3.4 is to apply Lemma 3.26 \( n \) times to

\[
\int_{E_1^n} \left( \prod_{i=1}^{n} d^2z_i \right) \Phi(z_1, \cdots, z_n; \tau) = \int_{E_1^n} d^2z_1 \cdots \int_{E_1^n} d^2z_n \Phi(z_1, \cdots, z_n; \tau)
\]

as an iterated integral over parallelogram \( \Box_c \)'s

\[
\int_{\Box_{c_1}} d^2z_1 \cdots \int_{\Box_{c_n}} d^2z_n \Phi(z_1, \cdots, z_n; \tau).
\]

One immediate difficulty is that we need to ensure all poles lie in the interior of parallelograms in each step of integration. Therefore we have to choose the shift \( c_i \)'s suitably.

Let \( c_0 \in \Box_0 \) be a chosen point such that

\[
z \in \Box_{-c_0} \Rightarrow \frac{1}{2}z \in \text{interior of } \Box_{-c_0}.
\]

Such \( c_0 \) always exists and can be chosen to be independent of \( \tau \) under a small perturbation of \( \tau \). Let \( \epsilon_1, \cdots, \epsilon_{n-1} \in (0, 1) \) be small enough positive real numbers satisfying

\[
\epsilon_{i+1} + \cdots + \epsilon_{n-1} + \frac{1}{2}\epsilon_i, \quad \forall 1 \leq i \leq n - 2.
\]

Consider the following linear change of variables \( z_i \mapsto w_i \)

\[
\begin{pmatrix}
  z_1 \\
  z_2 \\
  \vdots \\
  z_n
\end{pmatrix} =
\begin{pmatrix}
  1 & 0 & 0 & \cdots & 0 \\
  \epsilon_1 & 1 & 0 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  \epsilon_1 & \epsilon_2 & \cdots & \epsilon_{n-1} & 1
\end{pmatrix}
\begin{pmatrix}
  w_1 \\
  w_2 \\
  \vdots \\
  w_n
\end{pmatrix}.
\]

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Let
\[
\begin{align*}
  c_1 &= -c_0 \\
  c_2 &= c_2(z_1) = -c_0 + \epsilon_1 w_1 \\
  c_3 &= c_3(z_1, z_2) = -c_0 + \epsilon_1 w_1 + \epsilon_2 w_2 \\
  &\vdots \\
  c_n &= c_n(z_1, \ldots, z_{i-1}) = -c_0 + \epsilon_1 w_1 + \epsilon_2 w_2 + \cdots + \epsilon_{n-1} w_{n-1}.
\end{align*}
\]
Here \(c_i\) is viewed as a function of \(z_1, \ldots, z_{i-1}\) under the inverse of the above transformation.

**Lemma 3.27.** For any \(2 \leq i \leq n\), if \(z_1, \ldots, z_{i-1} \in \square_{c_i'}\), then \(z_1, \ldots, z_{i-1}\) lie in the interior of \(\square_{c_i}\).

**Proof.** Observe that
\[
z_j = w_j + c_0 + c_j.
\]
The condition \(z_j \in \square_{c_j}\) is the same as
\[
w_j \in \square_{-c_0}.
\]
For any \(j < i\),
\[
z_j - (\epsilon_1 w_1 + \cdots + \epsilon_{i-1} w_{i-1}) = (1 - \epsilon_j) w_j - \epsilon_{j+1} w_{j+1} - \cdots - \epsilon_{i-1} w_{i-1}
\]
\[
= (1 - \epsilon_j) w_j + (2\epsilon_{j+1})(-\frac{1}{2} w_{j+1}) + \cdots + (2\epsilon_{i-1})(-\frac{1}{2} w_{i-1}).
\]
By our choice of \(c_0\), all points \(w_j, (-\frac{1}{2} w_{j+1}), \ldots, (-\frac{1}{2} w_{i-1})\) lie in \(\square_{-c_0}\). Since
\[
(1 - \epsilon_j) + (2\epsilon_{j+1}) + \cdots + (2\epsilon_{i-1}) < 1
\]
and \(\square_{-c_0}\) is a convex set containing the origin, the value \(z_j - (\epsilon_1 w_1 + \cdots + \epsilon_{i-1} w_{i-1})\) lies in the interior of \(\square_{-c_0}\). So \(z_j\) lies in the interior of \(\square_{c_i}\) for any \(j < i\). This proves the lemma.

Now we can write
\[
\int_{E_{\tilde{\tau}}} \left( \prod_{i=1}^{n} d^2 z_i \right) \Phi = \int_{\square_{c_1}} d^2 z_1 \cdots \int_{\square_{c_n}} d^2 z_n \Phi
\]
as an ordered regularized integral where \(c_i\) depends on \(z_1, \ldots, z_{i-1}\) as chosen above. It ensures that \(z_1, \ldots, z_{i-1}\) lies in the interior of the integration parallelogram of \(z_j\) for any \(j \geq i\). The value of this iterated integral does not depend on the choice of \(c_i'\)’s.

We now apply Lemma 3.26 \(n\) times to this integral and keep the leading term in the \(\frac{1}{\text{im } \tau}\)-expansion in order to compute the limit \(\tilde{\tau} \to \infty\). It is not hard to see that for an almost-meromorphic elliptic function \(\Psi\)
\[
\int_{E_{\tilde{\tau}}} d^2 z \Psi = \sum_{k} \frac{1}{k+1} \int_{A_{\tilde{\tau}}} dz \Psi_k(z) + \sum_{k} \frac{1}{k+1} \sum_{w \in D} \left( \frac{\text{im } w}{\text{im } \tau} \right)^{k+1} \int_{w} dz \Psi_k(z) + O\left( \frac{1}{\text{im } \tau} \right).
\]
and hence we only need to keep the first two terms on the right hand side at each step of integration in the limit \(\tilde{\tau} \to \infty\).

The answer will become a combinatorial expression in terms of \(A\)-cycle integrals and residues. Let us first introduce some notations in order to describe this combinatorial result.

**Definition 3.28.** A tree is a connected undirected graph with no simple circuits. A rooted tree is a tree in which one vertex has been designated as the root.
Example 3.29. Here is an example of rooted tree.

Let $T$ be a rooted tree. Let $V(T)$ = vertices of $T$, $rt(T) \in V(T)$ the root vertex.

The level $l(v)$ of a vertex $v \in V(T)$ is the length of the unique path from the root to $v$. The level of the root vertex is 0. A vertex $v'$ is called a child of $v$ if there is an edge from $v$ to $v'$ and $l(v') = l(v) + 1$. In this case, $v$ is called the parent of $v'$. A vertex $v'$ is called a descendant of $v$ if there is a path from $v$ to $v'$, with strictly increasing levels along the consecutive vertices lying on the path. In the above example, the root has two children and five descendants.

Definition 3.30. A rooted forest $F = \{T_1, \cdots, T_m\}$ is a disjoint union of rooted trees $T_1, \cdots, T_m$. Let $V(F)$ denote the disjoint union of vertices of rooted trees of $F$.

Example 3.31. Here is an example of a rooted forest consisting of two rooted trees

![Figure 11: A rooted forest with two trees.](image)

Definition 3.32. Let $F$ be a rooted forest with $n$ vertices. We define a normal marking of $F$ to be an one-to-one map

$$
\chi : V(F) \to \{1, 2, \cdots, n\}
$$

such that

$$
\chi(v) < \chi(v') \text{ if } v' \text{ is a child of } v.
$$

An isomorphism between $(F_1, \chi_1)$ and $(F_2, \chi_2)$ is a graph isomorphism

$$
g : F_1 \to F_2
$$

such that the following diagram commutes

$$
\begin{array}{ccc}
F_1 & \xrightarrow{g} & F_2 \\
\chi_1 & & \chi_2 \\
\{1, \cdots, n\} & & \{1, \cdots, n\}
\end{array}
$$
**Definition 3.33.** Let $\Gamma_n$ denote the isomorphism classes of pairs $(F, \chi)$ where

- $F$ is a rooted forest with $n$ vertices.
- $\chi$ is a normal marking of $F$.

Given $(F, \chi) \in \Gamma_n$, we order the rooted trees $T_1, \ldots, T_m$ of $F$ such that

$$\chi(rt(T_1)) < \chi(rt(T_2)) < \cdots < \chi(rt(T_m)).$$

**Example 3.34.** Fig. 12 below gives an example of an element in $\Gamma_{10}$. The left tree is $T_1$ and the right tree is $T_2$.

![Figure 12: A forest with normal marking.](image)

For each vertex $v_0$ in $F$, it defines a rooted tree $T_{v_0}$ consisting of $v_0$ and all its descendants. Then $T_{v_0}$ is rooted at $v_0$. We define a residue operation

$$\oint_{T_{v_0}}$$

as follows. Let $v_1, \ldots, v_k$ be all the children of $v_0$ ordered by

$$\chi(v_1) < \chi(v_2) < \cdots < \chi(v_k).$$

Then $\oint_{T_{v_0}}$ is recursively defined by

$$\oint_{T_{v_0}} := \left( \oint_{z_{\chi(v_1)}} dz_{\chi(v_1)} \oint_{T_{v_1}} \right) \cdots \left( \oint_{z_{\chi(v_k)}} dz_{\chi(v_k)} \oint_{T_{v_k}} \right).$$

For a normally marked forest $(F, \chi)$, we denote the following operation

$$\int_{(F, \chi)} := \left( \int_{A_{i_1}^+} dz_{\chi(v_1)} \oint_{T_{i_1}} \right) \cdots \left( \int_{A_{i_m}^+} dz_{\chi(v_k)} \oint_{T_{i_m}} \right),$$

where $i_k = \chi(rt(T_k))$.

Given $(F, \chi)$, we let $v \in V(F)$ with marking $\chi(v) = i$. We denote

$$p_\chi(i) = \begin{cases} 
\text{the marking of the parent of } v & \text{if } v \text{ is not a root,} \\
0 & \text{if } v \text{ is a root.}
\end{cases}$$

We assign the following rational number to $(F, \chi)$

$$p(F, \chi) := \int_0^1 dx_1 \int_0^{x_{p_\chi(2)}} dx_2 \cdots \int_0^{x_{p_\chi(i)}} dx_i \cdots \int_0^{x_{p_\chi(n)}} dx_n, \quad x_0 \equiv 1.$$
Example 3.35. For Example 3.34 in Fig. 12 one has
\[
\int_{T_1} = \left( \oint_{z_1} dz_2 \oint_{z_2} dz_5 \oint_{z_2} dz_{10} \right) \left( \oint_{z_1} dz_6 \oint_{z_6} dz_8 \right), \quad \int_{T_2} = \oint_{z_3} dz_4 \oint_{z_3} dz_7 \oint_{z_3} dz_9.
\]
\[
\int_{(F,\chi)} = \left( \int_{A_1^+} d z_1 \oint_{z_1} d z_2 \oint_{z_2} d z_5 \oint_{z_2} d z_{10} \right) \left( \int_{A_3^+} d z_3 \oint_{z_3} d z_4 \oint_{z_3} d z_7 \oint_{z_3} d z_9 \right).
\]
\[
p(F,\chi) = \int_0^1 dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \int_0^{x_3} dx_4 \int_0^{x_4} dx_5 \int_0^{x_5} dx_6 \int_0^{x_6} dx_7 \int_0^{x_7} dx_8 \int_0^{x_8} dx_9 \int_0^{x_9} dx_{10} = \frac{1}{144}.
\]

It is easy to see that \( p(F,\chi) \) does not depend on the choice of \( \chi \). Therefore we write
\[
p(F) = p(F,\chi).
\]

In particular, we can define \( p(T) \) for any rooted tree.\(^6\) If \( F = \{T_1, \ldots, T_m\} \), then
\[
p(F) = p(T_1) \cdots p(T_m).
\]

The following lemma gives a useful recursive formula for \( p(T) \).

Lemma 3.36. Let \( T \) be a rooted tree. Let \( v_1, \ldots, v_m \) be all the children of the root vertex. Then
\[
p(T) = \frac{1}{|V(T)|} p(T_{v_1}) \cdots p(T_{v_m}).
\]

Recall \( T_{v_i} \) is the rooted tree consisting of \( v_i \) and all its descendants in \( T \).

Proof. Let \( n_i = |V(T_{v_i})| \). From the integration formula, it is easy to see that
\[
p(T) = \int_0^1 dx \prod_{i=1}^m (x^{n_i} p(T_{v_i})) = \frac{1}{1 + \sum_{i=1}^m n_i} p(T_{v_1}) \cdots p(T_{v_m}) = \frac{1}{|V(T)|} p(T_{v_1}) \cdots p(T_{v_m}).
\]

The next lemma is the key combinatorial formula for computing the holomorphic limit.

Lemma 3.37. Let \( \Phi(z_1, \ldots, z_n; \tau) \) be a meromorphic elliptic function on \( \mathbb{C}^n \times \mathbb{H} \). Then
\[
\lim_{\tau \to \infty} \int_{\square, \tau} d^2 z_1 \cdots d^2 z_n \Phi = \sum_{(F,\chi) \in \Gamma_n} p(F) \int_{(F,\chi)} \Phi.
\]

Proof. This follows by applying Lemma 3.26 \( n \) times. In each step, we only need to keep the first two terms on the right hand side in
\[
\int_{E_1} \frac{d^2 z}{i m \tau} \Psi = \sum_k \frac{1}{k+1} \int_{A_1^+} dz \Psi_k(z) + \sum_k \frac{1}{k+1} \sum_{w \in D} \left( \frac{\text{im} w}{\text{im} \tau} \right)^{k+1} \int_{w} dz \Psi_k(z) + O\left( \frac{1}{\text{im} \tau} \right).
\]

The operation \( \int_{(F,\chi)} \Phi \) keeps track of the residues and \( A \)-cycle integrals. The combinatorial factor \( p(F) \) keeps track of those \( \frac{1}{\text{im} \tau} \)-factors that appear at each step of integration. \(\square\)

\(^6\) The quantity \( 1/p(T) \) also appears as the tree factorial in [Kre00]. We thank the anonymous referee for pointing this out.
We can further express this combinatorial formula in terms of the ordered $A$-cycle integrals as defined in Definition 3.5. Recall from Lemma 3.7 that

$$\int_A dz_1 \oint_{z_2} \int_A dz_2 \oint_{z_3} \int_A dz_3 \cdots \oint_{z_k} \int_A dz_k = \left[ \int_A dz_2 , \left[ \int_A dz_3 , \cdots , \left[ \int_A dz_k , \int_A dz_1 \right] \right] \right] .$$

Let $R_n = \mathbb{C} \langle x_1, \cdots, x_n \rangle$ be the tensor algebra in $n$ variables $x_1, \cdots, x_n$. The generators $x_i$'s do not commute with each other, and each element in $R_n$ is expressed in terms of a linear combination of words in $x_i$'s. For any $a, b \in R_n$, we denote $[a, b] := ab - ba$.

Let $(F, \chi) \in \Gamma_n, v_0 \in V(F)$ and $T_{v_0}$ be the tree rooted at $v_0$ as defined above. We define

$$x_{T_{v_0}} \in R_n$$

as follows. Let $v_1, \cdots, v_k$ be all the children of $v_0$ such that

$$\chi(v_1) < \chi(v_2) < \cdots < \chi(v_k) .$$

Then $x_{T_{v_0}}$ is recursively defined by

$$x_{T_{v_0}} := \left[ x_{T_{v_1}} , \cdots , \left[ x_{T_{v_k}} , x_{\chi(v_0)} \right] \right] .$$

Let $T_1, \cdots, T_m$ be rooted trees of $F$ ordered by the marking as above. We define

$$x_{(F, \chi)} := x_{T_1} \cdots x_{T_m} .$$

**Example 3.38.** For Example 3.34 in Fig. 12 one has

$$x_{T_1} = [[x_5, [x_{10}, x_2]], [[x_8, x_6], x_1]], \quad x_{T_2} = [x_4, [x_7, [x_9, x_3]]],$$

and

$$x_{(F, \chi)} = x_{T_1} x_{T_2} .$$

Given such $x_{(F, \chi)}$, we associate an ordered $A$-cycle integral

$$x_{(F, \chi)}(\oint_A)$$

by the substitution

$$x_i \mapsto \oint_A dz_i .$$

Lemma 3.39 below is basically a reformulation of Lemma 3.37 by the above discussion. The ordering of $A$-cycles comes from the ordering in the iterated regularized integrals adapted to our choice of $c_i$'s.

**Lemma 3.39.** Let $\Phi(z_1, \cdots, z_n; \tau)$ be a meromorphic elliptic function on $\mathbb{C}^n \times \mathbb{H}$. Then one has

$$\lim_{\tau \to \infty} \int_{\Box_{c_1}} d^2 z_1 \cdots \int_{\Box_{c_n}} d^2 z_n \Phi = \sum_{(F, \chi) \in \Gamma_n} p(F) x_{(F, \chi)}(\oint_A) \Phi .$$

Lemma 3.40 below together with Lemma 3.39 will prove statement (2) of Theorem 3.4.
Lemma 3.40. One has
\[
\sum_{(F, \chi) \in \Gamma_n} p(F)\chi(F, \chi) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.
\]

Example 3.41. Here is an illustration of Lemma 3.40 when \(n = 3\). There are 6 elements of \(\Gamma_3\)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \downarrow & \downarrow \\
2 & 3 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 2 & 1 \\
\downarrow & \downarrow & \downarrow \\
3 & 2 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
1 & 1 & 2 \\
\downarrow & \downarrow & \downarrow \\
3 & 1 & 2 \\
\end{array}
\]

The sum \(\sum_{(F, \chi) \in \Gamma_n} p(F)\chi(F, \chi)\) gives
\[
x_1 x_2 x_3 + \frac{1}{2} [x_2, x_1] x_3 + \frac{1}{2} [x_3, x_1] x_2 + \frac{1}{2} x_1 [x_3, x_2] + \frac{1}{3} [x_2, [x_3, x_1]] + \frac{1}{6} [[x_3, x_2], x_1]
\]
\[
= \frac{1}{6} (x_1 x_2 x_3 + x_1 x_3 x_2 + x_2 x_1 x_3 + x_2 x_3 x_1 + x_3 x_1 x_2 + x_3 x_2 x_1).
\]

We are only left to prove Lemma 3.40.

Proof of Lemma 3.40

We prove Lemma 3.40 by induction on \(n\). We first need a recursion formula.

Lemma 3.42. Assume Lemma 3.40 holds for \(n - 1\). Then
\[
\sum_{(F, \chi) \in \Gamma_n} p(F)\chi(F, \chi) = \frac{1}{n!} \sum_{\sigma \in S_{n-1}} [x_{\sigma(2)}, [x_{\sigma(3)}, \cdots, [x_{\sigma(n)}, x_1]]] .
\]

Proof. Given \(F = T\), let \(v_1, \cdots, v_k\) be all the children of the root. By Lemma 3.36
\[
p(T) = \frac{p(T_{v_1}) \cdots p(T_{v_k})}{n}.
\]

Introduce new variables \(y_1, \cdots, y_{n-1}\). Consider a word in \(y\) defined the same as that for \(x\)
\[
\sum_{(F', \chi') \in \Gamma_{n-1}} p(F')\chi(F', \chi') .
\]

Let \(ad_{x_i}\) denote the operator \(ad_{x_i} = [x_i, -]\). Let \(\Xi\) denote the substitution
\[
y_1 \mapsto ad_{x_2}, \quad y_2 \mapsto ad_{x_3}, \quad \cdots, y_{n-1} \mapsto ad_{x_n} .
\]

Using \([ad_{x_i}, ad_{x_j}] = ad_{[x_i, x_j]}\), it is not hard to see that
\[
\sum_{(F, \chi) \in \Gamma_n} p(F)\chi(F, \chi) = \frac{1}{n!} \frac{1}{n} \sum_{(F', \chi') \in \Gamma_{n-1}} p(F')\chi(F', \chi') x_1 .
\]
Assume Lemma 3.40 holds for \( n - 1 \). Then

\[
\sum_{(F', \chi') \in \Gamma_{n-1}} p(F') y(F', \chi') = \frac{1}{(n-1)!} \sum_{\sigma \in S_{n-1}} y_{\sigma(1)} y_{\sigma(2)} \cdots y_{\sigma(n)}.
\]

This proves the lemma. \( \square \)

We can now proceed to prove Lemma 3.40.

Let \( \sigma \in S_n \) be a permutation of \( \{1, 2, \cdots, n\} \). \( \sigma \) can be expressed in cyclic notation by

\[
\sigma = (a_1 \cdots a_i) (a_{i+1} \cdots a_z) \cdots (a_{k-1} + 1 \cdots a_1), \quad i_k = n.
\]

We say the above cyclic expression is ordered if

\[
a_1 = \min\{a_1, \cdots, a_i\}, \quad a_{i+1} = \min\{a_{i+1}, \cdots, a_z\}, \quad \cdots, a_{k-1} + 1 = \min\{a_{k-1} + 1, \cdots, a_n\}
\]

and

\[
a_1 < a_{i+1} < a_{i+1} + 1 < \cdots < a_{k-1} + 1.
\]

Each \( \sigma \) has a unique ordered cyclic expression. We denote the following number

\[
|\sigma| := i_1! (i_2 - i_1)! \cdots (i_k - i_{k-1})!.
\]

Given \((j_1 j_2 \cdots j_k)\), we denote

\[
x_{(j_i)} := x_{j_i}, \quad x_{(j_1 \cdots j_k)} := [x_{j_k}, [x_{j_{k-1}}, \cdots, [x_{j_2}, x_{j_1}]]] .
\]

We also denote

\[
x_\sigma := x_{(a_1 \cdots a_i)} x_{(a_{i+1} \cdots a_z)} \cdots x_{(a_{k-1} + 1 \cdots a_1)} \in R_n.
\]

Let \( \Omega_k \) denote the set of partitions of \( \{1, \cdots, n\} \) into \( k \) subsets. We write each \( \omega \in \Omega_k \) as

\[
\omega = I_1 \cup I_2 \cup \cdots \cup I_k
\]

where the index is ordered in such a way that

\[
\min i < \min i \cdots < \min i.
\]

For such partition \( \omega = I_1 \cup I_2 \cup \cdots \cup I_k \), let

\[
\Gamma^\omega_n \subset \Gamma_n
\]

be those normally marked forest \((F, \chi)\) that consists of \( k \) rooted trees \( T_1, \cdots, T_k \) and

\[
\chi : V(T_1) \mapsto I_1, \quad V(T_2) \mapsto I_2, \quad \cdots, V(T_k) \mapsto I_k.
\]

Then we have

\[
\sum_{(F, \chi) \in \Gamma_n} p(F) x_{(F, \chi)} = \sum_k \sum_{\omega \in \Omega_k} \sum_{(F, \chi) \in \Gamma^\omega_n} \sum_{F = \{T_1, \cdots, T_k\}} (p(T_1) x_{T_1}) (p(T_2) x_{T_2}) \cdots (p(T_k) x_{T_k}) .
\]

Apply Lemma 3.36 and Lemma 3.42 to each partition in \( \omega \), we find

\[
\sum_k \sum_{\omega \in \Omega_k} \sum_{(F, \chi) \in \Gamma^\omega_n} \sum_{F = \{T_1, \cdots, T_k\}} (p(T_1) x_{T_1}) (p(T_2) x_{T_2}) \cdots (p(T_k) x_{T_k}) = \sum_{\sigma \in S_n} \frac{1}{|\sigma|} x_{\sigma}.
\]
By Proposition B.2 in Appendix B, we have the combinatorial formula
\[
\sum_{\sigma \in S_n} \frac{1}{|\sigma|} x_{\sigma} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.
\]
It follows that
\[
\sum_{(F, \chi) \in \Gamma_n p(F)} x_{(F, \chi)} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.
\]
This finishes the induction step for Lemma 3.40, hence completes the proof of Theorem 3.4.

## A Modular forms and elliptic functions

### Modular forms

Modular forms are functions on the upper-half plane \( \mathbb{H} \) that enjoy nice transform properties under the action of \( \text{SL}_2(\mathbb{Z}) < \text{SL}_2(\mathbb{R}) = \text{Aut} \mathbb{H} \). Quasi-modular forms and almost-holomorphic modular forms are generalizations of modular forms. Readers who are not familiar with these notions are referred to [KZ95, Zag08] for details. Here we only collect some basic definitions and facts that are frequently used in this paper.

**Definition A.1.** Let \( f \) be a meromorphic function on \( \mathbb{H} \).

1. \( f \) is said to be **modular** of weight \( k \) if
   \[
f\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).\]

2. \( f \) is said to be **quasi-modular** of weight \( k \) (and depth \( \ell \)) if there exist holomorphic functions \( f_1, \ldots, f_\ell \) on \( \mathbb{H} \) such that
   \[
f\left( \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau) + \sum_{i=1}^{\ell} c^{k-i} (c\tau + d)^{k-i} f_i(\tau), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}).\]

**Definition A.2.** A function \( f \) on \( \mathbb{H} \) is called a **holomorphic modular form** of weight \( k \) if

1. \( f \) is holomorphic on \( \mathbb{H} \).
2. \( f \) is modular of weight \( k \).
3. \( f \) has sub-exponential growth at \( \tau = i\infty \) in the sense that \( f(\tau) = O(e^{C\text{im}\tau}) \) as \( \text{im} \tau \to \infty \), for any \( C > 0 \).

**Definition A.3.** If \( f \) satisfies \( \text{(i)} \) \( \text{(iii)} \) above, and

\( \text{(ii')} \ f \) is quasi-modular of weight \( k \) (and depth \( \ell \)).

Then it is called a **holomorphic quasi-modular form** of weight \( k \) (and depth \( \ell \)). If \( f \) only satisfies \( \text{(i)} \) and \( \text{(ii')} \) then it is called a **weakly holomorphic quasi-modular form** of weight \( k \) (and depth \( \ell \)).

**Definition A.4.** If \( f \) satisfies \( \text{(iii)} \) above, and
(i”) \( f \) is almost-holomorphic on \( H \): \( f \in \mathcal{O}_H[\frac{1}{\text{im}\, \tau}] \) (Definition [3.1]).

(ii”) \( f \) is modular in the sense that

\[
f \left( \frac{a\tau + b}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^k f(\tau, \tau), \quad \forall \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}).
\]

Then it is called an almost-holomorphic modular form of weight \( k \).

Of central importance are the Eisenstein series that are defined by

\[
E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m\tau + n)^{2k}}, \quad k \geq 2,
\]

\[
E_2(\tau) = \frac{1}{2\zeta(2)} \left( \sum_{n \neq 0} \frac{1}{n^2} + \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^2} \right),
\]

where \( \zeta(2k), k \geq 1 \) are the zeta-values. Define also

\[
\hat{E}_2(\tau, \overline{\tau}) = E_2(\tau) - \frac{3}{\pi \text{im} \, \tau}.
\]

The Eisenstein series admit Fourier expansions in \( q = \exp(2\pi i\tau) \) given by

\[
E_{2k}(\tau) = 1 - \frac{4k}{B_{2k}} \sum_{d \geq 1} \sigma_{2k-1}(d) q^d = 1 - \frac{4k}{B_{2k}} \sum_{m \geq 1} \frac{m^{2k-1}q^m}{1 - q^m}, \quad q = e^{2\pi i\tau}, \quad k \geq 1. \quad (A.1)
\]

They have the following transformations under the action of \( \gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}_2(\mathbb{Z}) \)

\[
E_{2k}(\gamma \tau) = (c\tau + d)^2 E_{2k}(\tau), \quad k \geq 2
\]

\[
E_2(\gamma \tau) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau + d),
\]

\[
\hat{E}_2(\gamma \tau, \overline{\gamma \tau}) = (c\tau + d)^2 \hat{E}_2(\tau, \overline{\tau}). \quad (A.2)
\]

The spaces of holomorphic modular, holomorphic quasi-modular, almost-holomorphic modular forms for \( \text{SL}_2(\mathbb{Z}) \) form graded rings. Denote these rings by \( M, \tilde{M}, \hat{M} \) respectively, then

\[
M \cong \mathbb{C}[E_4, E_6], \quad \tilde{M} \cong \mathbb{C}[E_2, E_4, E_6], \quad \hat{M} \cong \mathbb{C}[\hat{E}_2, E_4, E_6]. \quad (A.3)
\]

The ring \( \hat{M} = \mathbb{C}[E_2, E_4, E_6] \) is furthermore a differential ring under \( \frac{1}{2\pi i} \partial_\tau = q \frac{d}{dq} \). The following relations are known as the Ramanujan identities

\[
\frac{1}{2\pi i} \partial_\tau E_2 = \frac{1}{12}(E_2^2 - E_4), \quad \frac{1}{2\pi i} \partial_\tau E_4 = \frac{1}{3}(E_2 E_4 - E_6), \quad \frac{1}{2\pi i} \partial_\tau E_6 = \frac{1}{2}(E_2 E_6 - E_4^2). \quad (A.4)
\]

See [KZ95] for details on these.

Since the generator \( 1/ \text{im} \, \tau \) is algebraically independent over the ring \( \mathcal{O}_H \), the notion of holomorphic limit in Definition [3.1] is well-defined. This notion plays an important role in discussing the relation between quasi-modular and almost-holomorphic modular forms.
where $u$

where $\wp$

Theorem A.5 (Kaneko-Zagier [KZ95]). The holomorphic limit

$$
\lim_{\tau \to \infty} : \tilde{M} \to \hat{M}
$$

induces a graded ring isomorphism between $\tilde{M}$ and $\hat{M}$. The inverse is called modular completion.

It is straightforward to check that Theorem A.5 can be generalized to give an isomorphism (again by the holomorphic limit $\lim_{\tau \to \infty}$) between the space of modular functions in $\mathcal{M}_H[\frac{1}{\im \tau}]$ and the space of quasi-modular, meromorphic functions on $H$.

Elliptic functions

Elliptic functions with respect to the lattice $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z} \tau$ are meromorphic functions on $\mathbb{C}$ that are invariant under the translation by the lattice. They are pull-backs of meromorphic functions on $E_\tau = \mathbb{C}/\Lambda_\tau$. It is a classical fact that the functional field of $E_\tau$ is given by

$$
k(E_\tau) = \mathbb{C}(\wp(z), \wp'(z))/((\wp'(z))^2 = 4\wp(z)^3 - g_2(\tau)\wp(z) - g_3(\tau)), \quad (A.5)
$$

where $\wp(z)$ is the Weierstrass $\wp$-function, $\wp' = \partial_z \wp$, and

$$
g_2(\tau) = \frac{4}{3}\pi^4 E_4(\tau), \quad g_3 = \frac{8}{27}\pi^6 E_6(\tau) \quad (A.6)
$$

are holomorphic modular forms of weight 4, 6 respectively.

The elliptic function $\wp(z; \tau)$ is even in $z$, with order 2 poles along $\Lambda_\tau$ on the universal cover $\mathbb{C}$. Explicitly one has

$$
\wp(z; \tau) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda_\tau \setminus \{0,0\}} \left( \frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right) = \frac{1}{z^2} + \frac{g_2(\tau)}{20} z^2 + \frac{g_3(\tau)}{28} z^4 + \ldots \quad (A.7)
$$

The elliptic functions $\wp, \wp'$ are also modular under the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$

$$
\wp\left( \frac{az + b}{ct + d} \right) = (ct + d)^2 \wp(z; \tau), \quad \wp'\left( \frac{az + b}{ct + d} \right) = (ct + d)^3 \wp'(z; \tau).
$$

The Fourier expansion of the meromorphic function $\wp(z; \tau)$ is given by

$$
\wp(u; q) = (2\pi i)^2 \sum_{k \geq 1} \frac{ku^k}{1 - q^k} + (2\pi i)^2 \sum_{k \geq 1} \frac{kq^k u^{-k}}{1 - q^k} - \frac{\pi^2}{3} E_2, \quad |q| < |u| < 1, \quad (A.8)
$$

where $u = e^{2\pi i \frac{z}{\tau}}, q = e^{2\pi i \frac{u}{\tau}}$. See the textbooks [Lan85, Siv09] for more details.

Throughout this work we often suppress the arguments $z, \tau, u, q$ etc. in the functions when no confusion should arise.

When integrating elliptic functions, one naturally encounters the so-called quasi-elliptic functions [Zag91, Lib09, GM20]. This notion is derived from the notion of quasi-Jacobi forms of index zero [EZ85, Lib09, DMZ12, GM20]. In this work we only need the special case, namely the Weierstrass zeta function given by

$$
\zeta(z) = \frac{1}{z} + \sum_{\lambda \in \Lambda_\tau \setminus \{0,0\}} \left( \frac{1}{z + \lambda} - \frac{1}{\lambda} + \frac{z}{\lambda^2} \right).
$$
It satisfies $\partial_z \zeta = -\varphi$. Under the elliptic transformations $z \mapsto z + 1, z \mapsto z + \tau$ one has

$$
\zeta(z + 1) - \zeta(z) = \eta_1, \quad \zeta(z + \tau) - \zeta(z) = \eta_2, \quad \forall z \neq 0.
$$

Here $\eta_1, \eta_2$ are given by

$$
\eta_1(\tau) = \frac{\pi^2}{3} E_2(\tau), \quad \eta_2(\tau) = \frac{\pi^2}{3} \tau E_2(-\tau).
$$

From the transformation of $E_2$ given in (A.2), one sees that $Z := \zeta - z\eta_1$ satisfies

$$
Z(z + 1) - Z(z) = 0, \quad Z(z + \tau) - Z(z) = -2\pi i.
$$

The de Rham cohomology $H^1_{dR}(E_\tau, \mathbb{C})$ is generated by the cohomology classes of the Abelian differentials $dz, \varphi(z)dz$. With respect to the canonical representatives $\{A, B\}$ mentioned earlier, the integrals of the former are given by the periods

$$
\int_A dz = 1, \quad \int_B dz = \tau.
$$

While the integrals of the latter are called quasi-periods and are given by

$$
\int_A \varphi(z; \tau)dz = -\eta_1(\tau), \quad \int_B \varphi(z; \tau)dz = -\eta_2(\tau). \tag{A.9}
$$

They satisfy the Legendre period relation (i.e., the 1st Riemann-Hodge bilinear relation)

$$
-\eta_2(\tau) + \eta_1(\tau)\tau = 2\pi i.
$$

which is equivalent to the quasi-modularity of $E_2(\tau)$ given in (A.2). See [Kat76] for a nice account on this.

B  An algebraic identity

In this appendix, we present an algebraic identity that is used in the proof of our main Theorem

3.4 We first recall the following notations in the proof of Theorem 3.4.

Let $\sigma \in S_n$ be a permutation of $\{1, 2, \ldots, n\}$. Then $\sigma$ can be expressed in cyclic notation by

$$
\sigma = (a_1 \cdots a_i)(a_{i+1} \cdots a_k) \cdots (a_{i-k+1} \cdots a_i), \quad i_k = n.
$$

We say the above cyclic expression is ordered if

$$
a_1 = \min\{a_1, \cdots, a_i\}, \quad a_{i+1} = \min\{a_{i+1}, \cdots, a_k\}, \quad \cdots, a_{i-k+1} = \min\{a_{i-k+1}, \cdots, a_n\}
$$

and

$$
a_1 < a_{i+1} < a_{i+1} < \cdots < a_{i-k+1}.
$$

Each $\sigma$ has a unique ordered cyclic expression. We denote the following number

$$
|\sigma| := i_1!(i_2 - i_1)! \cdots (i_k - i_{k-1})!.
$$

Let $R_n = \mathbb{C} \langle x_1, \cdots, x_n \rangle$ be the tensor algebra in $n$ variables $x_1, \cdots, x_n$. The generators $x_i$’s do not commute with each other, and each element in $R_n$ is expressed in terms of a linear combination of words in $x_i$’s. For any $a, b \in R_n$, we denote

$$
[a, b] := ab - ba.
$$
Given \((j_1 j_2 \cdots j_k)\), we denote

\[ x_{(j)} := x_j, \quad x_{(j_1 \cdots j_k)} := [x_{j_1}, [x_{j_1-1}, \cdots, [x_{j_1}, x_{j_1}]]]. \]

Given \(\sigma\) with its ordered cyclic expression as above, we write

\[ x_\sigma := x_{(a_1 \cdots a_i)} x_{(a_{i+1} \cdots a_2)} \cdots x_{(a_{i+1} \cdots a_k)} \in R_n. \]

Example B.1. Let \(n = 3\). We are interested in the sum \(\sum_{\sigma \in S_3} \frac{x_\sigma}{|\sigma|}\) which is given by

\[
\sum_{\sigma \in S_3} \frac{x_\sigma}{|\sigma|} = x_{123} + \frac{1}{2} x_{132} + \frac{1}{2} x_{213} + \frac{1}{2} x_{231} + \frac{1}{6} x_{312} + \frac{1}{6} x_{321}.
\]

\[
= \frac{1}{3!} \left( x_{123} + x_{132} + x_{213} + x_{231} + x_{312} + x_{321} \right) = \frac{1}{3!} \sum_{\sigma \in S_3} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}.
\]

The combinatorial identity that we find in this example actually holds in general. The next proposition might be known to experts. Since we couldn’t locate a precise reference, in what follows we supply a proof for completeness.

**Proposition B.2.** The following identity holds in \(R_n\)

\[
\sum_{\sigma \in S_n} \frac{x_\sigma}{|\sigma|} = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}.
\]

**Proof.** We prove by induction on \(n\).

Let \(I\) be the ideal in \(R_n\) generated by elements of the form

\[ \cdots x_i \cdots x_i \cdots \]

i.e., those words where some variable \(x_i\) has appeared at least twice. Let

\[ G_n = R_n/I \]

be the quotient ring. Let \(\bar{x}_i\) be the corresponding generator in \(G_n\). We only need to prove

\[
\sum_{\sigma \in S_n} \frac{\bar{x}_\sigma}{|\sigma|} = \frac{1}{n!} \sum_{\sigma \in S_n} \bar{x}_{\sigma(1)} \bar{x}_{\sigma(2)} \cdots \bar{x}_{\sigma(n)} \quad \text{holds in } G_n.
\]

Each element \(f \in G_n\) can be written as

\[ f = f_{(0)} + f_{(1)} + \cdots + f_{(n)} \]

where \(f_{(k)}\) is homogeneous of degree \(k\) in \(\bar{x}_i\)'s. We first observe that

\[ \frac{1}{n!} \sum_{\sigma \in S_n} \bar{x}_{\sigma(1)} \bar{x}_{\sigma(2)} \cdots \bar{x}_{\sigma(n)} = \left( e^{\bar{x}_1 + \cdots + \bar{x}_n} \right)_{(n)}. \]

Introduce a variable \(t\). Then in \(G_n\), we have

\[ \left( e^{\bar{x}_1 + \cdots + \bar{x}_n} \right)_{(n)} = \frac{\partial}{\partial t} \left( e^{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_n} \right)_{(n)}. \]
By Duhamel’s formula, we have

\[
\frac{\partial}{\partial t} (e^{tx_1 + x_2 + \cdots + x_n}) = \int_0^1 ds e^{s(tx_1 + x_2 + \cdots + x_n)} x_1 e^{-s(tx_1 + x_2 + \cdots + x_n)} e^{tx_1 + x_2 + \cdots + x_n}.
\]

Let us write \(Y = \bar{x}_2 + \cdots + \bar{x}_n\). Using the quotient relation in \(G_n\), we find

\[
\frac{\partial}{\partial t} (e^{tx_1 + x_2 + \cdots + x_n}) = \int_0^1 ds e^{sY} x_1 e^{-sY} = \sum_{k \geq 0} \int_0^1 ds \frac{s^k}{k!} [Y, \ldots, [Y, x_1]] e^{sY}
\]

\[
= \sum_{k \geq 0} \frac{1}{(k+1)!} [Y, \ldots, [Y, x_1]] e^{sY}
\]

\[
= \sum_{k \geq 0} \frac{1}{(k+1)!} \sum_{i_1, \ldots, i_k \in \{2, \ldots, n\}} [x_{i_k}, [x_{i_{k-1}}, \ldots, [x_{i_1}, x_1]]] \exp(\sum_{j \in \{2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}} x_j).
\]

We can view \([x_{i_k}, [x_{i_{k-1}}, \ldots, [x_{i_1}, x_1]]]\) as coming from an ordered cyclic expression with

\((i_1 \cdots i_k)(\ldots) (\ldots)\).

Then the equality

\[
\frac{1}{n!} \sum_{\sigma \in S_n} \bar{x}_{\sigma(1)} \bar{x}_{\sigma(2)} \cdots \bar{x}_{\sigma(n)} = \sum_{\sigma \in S_n} \frac{\bar{x}_{\sigma}}{|\sigma|}
\]

follows from the above expression and the induction applied to \(\exp(\sum_{j \in \{2, \ldots, n\} \setminus \{i_1, \ldots, i_k\}} x_j). \)

\[\square\]

### C  Examples on evaluation of integrals

In this part, as a double check we offer an alternative direct computation of the Feynman graph integral in Example 3.24

\[
\hat{I}_p = \int_{E_1} \left( \prod_{i=1}^3 \frac{d^2 z_i}{\text{im } \tau} \right) \Phi_{p}, \quad \Phi_{p} = \tilde{P}(z_1, z_2) \tilde{P}(z_2, z_3) \tilde{P}(z_3, z_1).
\]

We apply Proposition 2.18 successively for the evaluation of the iterated integral. The details are given as follows.

The first integration on \(z_3\) gives

\[
\int_{E_1} \Phi d z_3 = \int_{A_3} \Phi d z_3 - \frac{-\pi}{\text{im } \tau} \cdot \text{Res}_{z_3 = z_3} (z_3 \Phi d z_3) - \frac{-\pi}{\text{im } \tau} \cdot \text{Res}_{z_3 = z_1} (z_3 \Phi d z_3)
\]

\[
+ \frac{-\pi}{\text{im } \tau} \cdot \text{Res}_{z_3 = z_2} (\Phi d z_3) - \frac{-\pi}{\text{im } \tau} \cdot \text{Res}_{z_3 = z_1} (\Phi d z_3) \cdot \bar{z}_3 |_{p_0}^2
\]

\[
= \int_{A_3} \Phi d z_3 + \frac{-\pi}{\text{im } \tau} \cdot (2 \tilde{P}^2 (z_2 - z_1) + \frac{-\pi}{\text{im } \tau} \cdot \tilde{P} (z_2 - z_1) \tilde{P} (z_2 - z_1) (z_2 - z_1)).
\]

The previous computations in Example 3.24 show that

\[
\int_{A_3} \Phi d z_3 = \tilde{P} (z_2 - z_1) \left( (2 \pi i)^4 \sum_{k \neq 0} \frac{k^2 q^k}{(1 - q^k)^2} \left( \frac{u_1}{u_2} \right)^k + \left( \frac{-\pi}{\text{im } \tau} \right)^2 \right).
\]

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This is not elliptic anymore but only quasi-elliptic. However, in the integration domain for the iterated regularized integral we can compute directly
\[
\text{Res}_{z_2 = z_1}(d z_2 \int_{A_3} \Phi_1 d z_3) = 0. 
\]
Similarly, by using the following identity in Remark 2.25
\[
(2\pi i)^4 \cdot 2 \sum_{k \geq 1} \frac{k^2 q^k}{(1 - q^k)^2} = \frac{1}{9} \pi^4 (E_4 - E_2^2), 
\]
we have
\[
\text{Res}_{z_2 = z_1}(z_2 d z_2 \int_{A_3} \Phi_1 d z_3) = (2\pi i)^4 \sum_{k \neq 0} \frac{k^2 q^k}{(1 - q^k)^2} + \left( -\frac{\pi}{\text{im} \, \tau} \right)^2 = \frac{1}{9} \pi^4 (E_4 - E_2^2) + \left( -\frac{\pi}{\text{im} \, \tau} \right)^2.
\]
It follows that
\[
\int_{E_i} \frac{d^2 z_2}{\text{im} \, \tau} \int_{A_3} \Phi_1 d z_3 
= \int_{A_2} \frac{d z_2}{\text{im} \, \tau} \int_{A_3} \Phi_1 d z_3 - \frac{\pi}{\text{im} \, \tau} \cdot \text{Res}_{z_2 = z_1} \left( z_2 d z_2 \left( \int_{A_3} \Phi_1 d z_3 \right) \right)
+ \frac{\pi}{\text{im} \, \tau} \cdot \text{Res}_{z_2 = 1} \left( z_2 d z_2 \left( \int_{A_3} \Phi_1 d z_3 \right) \right) = \frac{1}{128} (2\pi i)^6 (-E_3^3 + 3E_2E_4 - 2E_6) - \left( -\frac{\pi}{\text{im} \, \tau} \right)^2 \frac{1}{9} \pi^4 (E_4 - E_2^2).
\]
We also have
\[
\int_{E_i} \frac{d^2 z_2}{\text{im} \, \tau} \frac{\pi}{\text{im} \, \tau} \cdot (-2) \hat{P}^2(z_2 - z_1) = \frac{1}{2\pi i} \int_{E_i} \frac{d^2 z_2}{\text{im} \, \tau} \hat{P}^2(z_2 - z_1) = \frac{1}{2\pi i} \int_{E_i} \frac{d^2 z_2}{\text{im} \, \tau} \hat{P}(z) \hat{P}'(z)(z - z)dz \wedge dz
= \frac{1}{2\pi i} \left( -\frac{\pi}{\text{im} \, \tau} \right)^2 \int_{E_i} \hat{P}(z) \hat{P}'(z)(z - z)dz \wedge dz.
\]
Applying Proposition 2.18 and (A.7), one has
\[
\frac{1}{2\pi i} \left( -\frac{\pi}{\text{im} \, \tau} \right)^2 \int_{E_i} \hat{P}(z) \hat{P}'(z)(z - z)dz \wedge dz
= \frac{1}{2\pi i} \left( -\frac{\pi}{\text{im} \, \tau} \right)^2 \left( \frac{1}{2}(\tau - \tau)^2 \int_{A} d(\frac{1}{2} \hat{P}^2(z)) - 2\pi i \text{Res}_{z=0}(\hat{P}(z) \hat{P}'(z)(z) \cdot \frac{1}{2} (z - z)^2) \right)
= -(\frac{\pi}{\text{im} \, \tau} \cdot \tau)^2 \text{Res}_{z=0}(\hat{P}(z) \hat{P}'(z)(z) \cdot \frac{1}{2} (z - z)^2)
= \frac{1}{2\pi i} \left( -\frac{\pi}{\text{im} \, \tau} \right)^2 \hat{E}_2.
\]
Putting all these together, we obtain the desired result in Example 3.24
\[
\hat{I}_T = \int_{E_i} \left( \prod_{j=1}^{3} \frac{d^2 z_i}{\text{im} \, \tau} \right) \Phi_1 = \int_{E_i} \left( \prod_{j=1}^{2} \frac{d^2 z_i}{\text{im} \, \tau} \right) \Phi_1 = \frac{1}{128} (2\pi i)^6 (-E_3^3 + 3E_2E_4 - 2E_6).
\]
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