ON THE NEIGHBORHOOD OF A TORUS LEAF AND DYNAMICS OF HOLOMORPHIC FOLIATIONS

TAKAYUKI KOIKE\textsuperscript{1} AND NOBORU OGAWA\textsuperscript{2}

Abstract. Let $X$ be a complex surface and $Y$ be an elliptic curve embedded in $X$. Assume that there exists a non-singular holomorphic foliation $\mathcal{F}$ with $Y$ as a compact leaf, defined on a neighborhood of $Y$ in $X$. We investigate the relation between Ueda’s classification of the complex analytic structure of a neighborhood of $Y$ and complex dynamics of the holonomy of $\mathcal{F}$ along $Y$.

1. Introduction

Let $X$ be a complex manifold and $Y$ be a compact complex submanifold of $X$. Assume that there exists a non-singular holomorphic foliation $\mathcal{F}$ which has $Y$ as a compact leaf and is defined on a neighborhood $V$ of $Y$ in $X$. Our interest is the relation between complex-analytic properties of small neighborhoods of $Y$ and complex dynamics of the holonomy of $\mathcal{F}$ along $Y$.

In this paper, we consider the case where $X$ is a complex surface and $Y$ is an elliptic curve as a compact leaf of $\mathcal{F}$. It follows from the fundamental results in foliation theory (\cite{B}, \cite{M2}) that the normal bundle $N_{Y/X}$ of $Y$ is topologically trivial. On the other hand, a pair $(Y, X)$ with topologically trivial normal bundle can be classified into three types $(\alpha)$, $(\beta)$, and $(\gamma)$ in accordance with Ueda’s classification (\cite{U}, see also \S\textsuperscript{2.2}). The pair $(Y, X)$ is said to be of type $(\beta)$ if there exists a non-singular holomorphic foliation $\mathcal{G}$ defined on a neighborhood of $Y$ which has $Y$ as a leaf and has $U(1)$-linear holonomy along $Y$ (i.e. the image of the holonomy homomorphism $\text{Hol}_{\mathcal{G}, Y} : \pi_1(Y, *) \to \text{Diff}(\mathbb{C}, 0)$ is a subgroup of $U(1) = \{ t \in \mathbb{C} \mid |t| = 1 \}$). We denote by $\text{Diff}(\mathbb{C}, 0)$ the group of germs at 0 of holomorphic functions $f \in \mathcal{O}_{\mathbb{C}, 0}$ such that $f(0) = 0$ and $f'(0) \neq 0$. In this case, $Y$ admits a \textit{pseudoflat} neighborhoods system. The pair $(Y, X)$ is said to be of type $(\alpha)$ if, roughly speaking, it is different from the case of type $(\beta)$ in $n$-jet along $Y$ for some positive integer $n$ (see \S\textsuperscript{2.2} for the precise definition). Ueda showed that $Y$ admits a \textit{strongly pseudoconcave} neighborhoods system in this case [U, Theorem 1]. The remaining case is called type $(\gamma)$. The first example of the pair $(Y, X)$ of type $(\gamma)$ is constructed by Ueda [U, \S5.4]. Note that this Ueda’s example satisfies our assumption on the existence of $\mathcal{F}$ on $V$ as above. In \cite{K3}, the first author investigated the complex-analytic properties of small neighborhoods of $Y$ for this Ueda’s example $(Y, X)$, whose generalization is one of the biggest motivations of the present paper.

We choose a closed curve $c$ in a leaf $Y$ through a base point $p$ and a transversal $T$ for $\mathcal{F}$ at $p$. Here, we identify $(T, p)$ with a domain $(U, 0)$ in $\mathbb{C}$. As moving $T$ in foliation charts around $c$, the return map determines an element $f$ of $\mathcal{O}_{\mathbb{C}, 0}$ such that $f(0) = 0$ and $f'(0) \neq 0$.
(equivalently, a germ at 0 of local holomorphic diffeomorphisms on $T$ fixing 0). The germ $f$ is called the holonomy (germ) of $\mathcal{F}$ along $c$, which depends only on the homotopy class $\gamma$ of $c$ relative the base point $p$. The induced homomorphism $\text{Hol}_{\mathcal{F}, Y} : \pi_1(Y, p) \to \text{Diff}(\mathbb{C}, 0)$ is called the holonomy homomorphism of $\mathcal{F}$ along $Y$. In our setting, for any $\gamma, \gamma' \in \pi_1(Y, p)$, $f = \text{Hol}_{\mathcal{F}, Y}(\gamma)$ and $g = \text{Hol}_{\mathcal{F}, Y}(\gamma')$ commute. Note that, when the transversal is replaced, the holonomy varies in the conjugacy class. Therefore, in what follows, we change the functions $f$ and $g$ to simultaneous conjugATIONS of them as necessary.

Our purpose is to determine the types of $(Y, X)$ from the information of holonomy of $\mathcal{F}$ along $Y$. According to the observation in [CLPT, Remark 2.2], it is natural to focus on the case where both $\lambda = f'(0)$ and $\mu = g'(0)$ are elements of $U(1)$ (see Remark 2.9). In this case, once we choose a basis $(\gamma, \gamma')$ of $\pi_1(Y, p)$, the triple $(Y, X, \mathcal{F})$ can be classified into ten cases I, II, ..., X below in accordance with the (non-) torsionness of $\lambda$ and $\mu$ and the (non-) linearizability of $f$ and $g$ (i.e. for example, whether there exists an element $\varphi \in \text{Diff}(\mathbb{C}, 0)$ such that $\varphi^{-1} \circ f \circ \varphi(w) = \lambda \cdot w$).

|                | $\mu \in \text{U}(1)$: torsion | $\mu \in \text{U}(1)$: non-torsion |
|----------------|--------------------------------|----------------------------------|
| $\lambda \in \text{U}(1)$ | $f$: linearizable | $g$: non-linearizable | $f$: linearizable | $g$: non-linearizable |
| : torsion      |                             | $I$     | $II$  | $III$ | $IV$  |
| $\lambda \in \text{U}(1)$ | $f$: non-linearizable | $V$     | $VI$  | $VII$ |       |
| : non-torsion  |                             |         |       |       |       |

Case I: both $\lambda$ and $\mu$ are torsion elements in $U(1)$ and both $f$ and $g$ are linearizable.  
Case II: both $\lambda$ and $\mu$ are torsion, $f$ is linearizable, and $g$ is non-linearizable  
Case III: $\lambda$ is torsion, $\mu$ is non-torsion, and both $f$ and $g$ are linearizable 
Case IV: $\lambda$ is torsion, $\mu$ is non-torsion, $f$ is linearizable, and $g$ is non-linearizable  
Case V: both $\lambda$ and $\mu$ are torsion, and both $f$ and $g$ are non-linearizable  
Case VI: $\lambda$ is torsion, $\mu$ is non-torsion, $f$ is non-linearizable, and $g$ is linearizable  
Case VII: $\lambda$ is torsion, $\mu$ is non-torsion, and both $f$ and $g$ are non-linearizable  
Case VIII: both $\lambda$ and $\mu$ are non-torsion, and both $f$ and $g$ are linearizable  
Case IX: both $\lambda$ and $\mu$ are non-torsion, and $f$ is linearizable, and $g$ is non-linearizable  
Case X: both $\lambda$ and $\mu$ are non-torsion and both $f$ and $g$ are non-linearizable

It suffices to consider only these ten cases from the symmetry. Each case is invariant under conjugations, in particular, it does not depend on the choice of transversals. On the other hand, it does depend on the choice of generators of $\pi_1(Y, p)$.

As a typical example, a foliated manifold $(Y, X, \mathcal{F})$ given by the standard suspension construction realizes each of ten cases above and satisfies the condition (*) below. The main result is the following theorem.

**Theorem 1.1.**

(i) In Case I, the pair $(Y, X)$ is of type $(\beta)$.
(ii) In Case II, the pair $(Y, X)$ is of type $(\alpha)$.
(iii) In Case III, the pair $(Y, X)$ is of type $(\beta)$.

\footnote{For example in Case I, III and VIII, just the existence of the linearization functions $\varphi, \psi \in \text{Diff}(\mathbb{C}, 0)$ such that $\varphi^{-1} \circ f \circ \varphi(w) = \lambda \cdot w$ and $\psi^{-1} \circ g \circ \psi(w) = \mu \cdot w$ is assumed and nothing on the relationship between $\varphi$ and $\psi$ is assumed literally. However in reality, it turns out that $f$ and $g$ can be linearized simultaneously in these cases (i.e. we have that $\varphi = \psi$, see §2).}

2
(iv) In Case IV, the pair \((Y, X)\) is of type \((\gamma)\).
(v) In Case V, the pair \((Y, X)\) is of type \((\alpha)\) or \((\beta)\). Both pairs of types \((\alpha)\) and \((\beta)\) exist.
(vi) No pairs \((Y, X)\) is in Case VI.
(vii) No pairs \((Y, X)\) is in Case VII.
(viii) In Case VIII, the pair \((Y, X)\) is of type \((\beta)\).
(ix) No pairs \((Y, X)\) is in Case IX.
(x) In Case X, under the condition (*) below, the pair \((Y, X)\) is of type \((\gamma)\).

Condition (*) there exist a neighborhood \(V\) of \(Y\) and a holomorphic sub-

mersion \(\pi : V \to Y\) whose restriction to \(Y\) is the identity.

First note that, we use the condition (*) to show only the statement \((x)\), more precisely, to show about Case X in Theorem 1.2 below. Also note that we can easily reword Theorem

1.3 constructed by Pérez-Marco. See

we will focus on the property of hedgehogs

small cycle property, called

that the pair \((Y, X)\) is a generalization of his result. In his example, \(f_{1.1}\) gives a generalization of his result. In his example, \(f\) is the identity and \(g\) has a special property, called small cycle property. By using this, he determined the type. Whereas, we will focus on the property of hedgehogs (a completely invariant set \(K\) as in Theorem

1.3) constructed by Pérez-Marco. See §2.1 and §3.

The main contribution is the statements \((iv)\) and \((x)\) of Theorem 1.1 The others can be proved from relatively simple arguments based on Ueda’s results and some fundamental results of complex dynamics around indifferent fixed points. We here describe the outline

of the proof of \((iv)\) and \((x)\). It follows from standard observation of the normal bundle of \(Y\) that the pair \((Y, X)\) is not of type \((\alpha)\) in Case IV and \(X\). If the pair \((Y, X)\) is of type \((\beta)\), there exists a pluriharmonic function \(\Phi\) on \(V \setminus Y\) for a sufficiently small neighborhood \(V\) of \(Y\) such that \(\Phi(p) = O(- \log \text{dist}(p, Y))\) as \(p \to Y\), where “dist” is the local Euclidean distance. Therefore \((x)\) in Theorem 1.1 is deduced from Theorem 1.2 Similarly, \((iv)\) is deduced from Theorem 1.4

THEOREM 1.2. Assume that the triple \((Y, X, \mathcal{F})\) is in Case IV or \(X\) with the condition (*). Let \(V\) be a neighborhood of \(Y\) in \(X\) and \(\Phi\) be a continuous function from \(V\) to \(\mathbb{R} \cup \{\infty\}\), where \(\mathbb{R} \cup \{\infty\}\) is homeomorphic to the standard \((0, 1] \subset \mathbb{R}\). Assume that \(\Phi|_{V \setminus Y}\) is pluriharmonic. Then \(\Phi\) is bounded from above on a neighborhood of \(Y\).

Theorem 1.2 is shown by leading a contradiction to the maximum principle for the restriction of \(\Phi\) on a leaf of \(\mathcal{F}\) (see [11] for the detail). For this purpose, we find a dense leaf \(L\) in an invariant set containing \(Y\). The existence of such \(L\) is guaranteed by the following fact in complex dynamics:

THEOREM 1.3 ([24], [24], [25]). Let \(f\) and \(g\) be elements of \(\mathcal{O}_{C,0}\) such that \(f(0) = g(0) = 0\) and that \(\lambda = f'(0)\) and \(\mu = g'(0)\) satisfy \(|\lambda| = |\mu| = 1\). Assume that \(\mu\) is a non-torsion element of \(U(1)\), and that \(f\) and \(g\) commute. Then, the following hold:

(i) For any small neighborhood \(U\) in \(C\) of the origin, there exists a connected compact subset \(K\) of \(\overline{U}\) such that \(0 \in K\), \(K \neq \{0\}\), \(\mathbb{C} \setminus K\) is connected, and that \(K\) is a completely invariant set of both \(f\) and \(g\): i.e. \(f(K) = f^{-1}(K) = g(K) = g^{-1}(K) = K\) holds.
(ii) If \(g\) is linearizable, then \(f\) is also linearizable by the same linearization function. In
this case, $K$ can be chosen as the closure of a domain included in $U$ which has the origin as an interior point.

(iii) If $g$ is non-linearizable, then $K$ has the origin as a boundary point. In this case, there exists a point $w$ of $K$ such that the orbit $\{g^n(w) \mid n \in \mathbb{Z}\}$ is dense in $K$.

Theorem 1.3 is based on Pérez-Marco’s theorem [P5, Thm.III.14]. For the convenience of the readers, we restate (a weaker version of) his theorem as Theorem 2.4 and give a slightly modified proof in §3.

In Case IV, Theorem 1.2 can be generalized as follows. Note that we need not assume the condition (*) in this statement.

**Theorem 1.4.** Assume that the triple $(Y, X, F)$ is in Case IV (for some $(\gamma, \gamma')$). Let $V$ be a neighborhood of $Y$ in $X$ and $\Phi$ be a continuous function from $V$ to $\mathbb{R} \cup \{\infty\}$, where $\mathbb{R} \cup \{\infty\}$ is homeomorphic to the standard $(0, 1] \subset \mathbb{R}$. Assume that $\Phi|_{V\setminus Y}$ is plurisubharmonic. Then $\Phi$ is a plurisubharmonic function on $V$. Especially, it is bounded from above on a neighborhood of $Y$.

The above theorem can be regarded as a weak analogue of Ueda’s theorem [U, Theorem 2] on the constraint of the increasing degree of plurisubharmonic functions defined on $V \setminus Y$, where $V$ is a neighborhood of $Y$. In [K2], the first author applied [U, Theorem 2] to show the non-semipositivity (i.e. non-existence of a $C^\infty$ Hermitian metric with semi-positive curvature) of a line bundle $L$ on $X$ which corresponds to the divisor $Y$ when $(Y, X)$ is of type $(\alpha)$. As an application of Theorem 1.4 we have the following:

**Corollary 1.5.** Let $L$ be the line bundle on $X$ which corresponds to the divisor $Y$. Assume that the triple $(Y, X, F)$ is in Case I, II, ..., VIII, or IX. Then $L$ is semi-positive (i.e. $L$ admits a $C^\infty$ Hermitian metric with semi-positive curvature) if and only if the pair $(Y, X)$ is of type $(\beta)$.

**Question 1.6.** Does $L$ admit a $C^\infty$ Hermitian metric with semi-positive curvature when the pair $(Y, X)$ is in Case X?

The organization of the paper is as follows. In §2, we review some fundamental facts on linearization theorems, Siegel compacta, and Ueda theory. In §3, we give a slightly modified proof of [P5, Thm.III.14]. We prove Theorem 1.2 and 1.4 in §4, Theorem 1.1 in §5, and Corollary 1.5 in §6.

**Acknowledgment.** The authors are grateful to Professor Eric Bedford for informing us about Pérez-Marco’s theory on Siegel compacta. We also would like to thank Professor Tetsuo Ueda for valuable comments and suggestions. The first author is supported by Leading Initiative for Excellent Young Researchers (No. J171000201).

2. Preliminaries

2.1. **Linearization theorems and Siegel compacta.** In this section, we review some basic properties of linearizability of $f \in \mathcal{O}_{\mathbb{C},0}$ which satisfies $f(0) = 0$ and $f'(0) \neq 0$, equivalently, of local holomorphic diffeomorphisms $f$ fixing the origin 0.
Let \( f(z) = \lambda z + O(z^2) \) be a local holomorphic diffeomorphism fixing 0. We say that \( f \) is linearizable around 0 if there exist open neighborhoods \( U, V \) of 0, and a biholomorphism \( h: U \to V \) such that \((h \circ f \circ h^{-1})(z) = \lambda z\). It is classically known that the linearizability of \( f \) depends on the choice of \( \lambda \). If \( |\lambda| \neq 0, 1 \), then \( f \) is linearizable (Koenigs’ linearization theorem). On the other hand, there are some obstructions for the case that \( |\lambda| = 1 \). The fixed point 0 is said to be rationally indifferent (resp. irrationally indifferent) if \( |\lambda| = 1 \) and \( \lambda \) is torsion (resp. non-torsion). A local holomorphic diffeomorphism with rationally indifferent fixed point 0 is not linearizable around 0 if it is not a linear one. In contrast, the linearizability problem for the irrationally indifferent case is much more difficult. In this case, \( f \) is linearizable if \((\log \lambda)/2\pi i\) satisfies the Diophantine condition (Siegel’s linearization theorem [Sie]). Further, this condition can be sharpened to the Bruno condition. The maximal linearization domain is biholomorphic to the unit disk, on which \( f \) is analytically conjugate to a rotation. This is called the Siegel disk of \( f \). Obviously, the domain is invariant under \( f \) and \( f^{-1} \), which is said to be completely invariant under \( f \).

On the other hand, in [P4], Pérez-Marco showed the existence of completely invariant sets (not necessarily linearizable domains) around indifferent fixed points. A Jordan domain \( U \) with \( C^1 \)-boundary is said to be admissible for \( f \) if \( f \) and \( f^{-1} \) are defined and univalent on an open neighborhood of the closure \( \overline{U} \) of \( U \).

**Theorem 2.1 ([P4])** Let \( f(z) = \lambda z + O(z^2) \) be a local holomorphic diffeomorphism with the indifferent fixed point 0. Let \( U \) be an admissible neighborhood of 0. Then there exists a subset \( K \) in \( \mathbb{C} \) which satisfies the following conditions:

(i) \( K \) is compact, connected and full (i.e., \( \mathbb{C} \setminus K \) is connected),
(ii) \( 0 \in K \subset \overline{U} \),
(iii) \( K \cap \partial U \neq \emptyset \),
(iv) \( f(K) = f^{-1}(K) = K \).

Moreover, assume that \( f \) is not of finite order. Then, \( f \) is linearizable if and only if \( 0 \in \text{Int} K \).

![Figure 1. A hedgehog \( K \) of \((U, f)\).](image)

Pérez-Marco called this completely invariant set a Siegel compactum, which can be regarded as a generalization of Siegel disk. The Siegel compactum of the pair \((U, f)\) is
denoted by $K(U,f)$. In the irrationally indifferent case, such an invariant set $K$ is called a hedgehog. In general, it is topologically very complicated. According to [P1, Theorem 5], sufficiently near the irrationally indifferent point 0, non-linearizable hedgehogs have no interior points and are not locally connected at any point different from 0. By considering the associated analytic circle diffeomorphisms, he studied some properties of Siegel compacta and found several applications ([P4]). In particular, with regards to the dynamics on Siegel compacta, the following fact is remarkable.

**Theorem 2.2** ([P4, Theorem IV.2.3]). Let $f$ be as in Theorem 2.1. For $\mu_K$-a.e. point $z$ in $K$, the orbit of $z$ is dense in $\partial K$. In particular, if $f$ is non-linearizable, the orbit is dense in $K$. Here $\mu_K$ is the harmonic measure at $\infty$ of $K$ in $\mathbb{C}P^1$.

To prove Theorem 1.1 we need to understand the linearizability of commuting local holomorphic diffeomorphisms fixing the origin 0.

**Proposition 2.3** (e.g. [P2, §4]). Let $f(z) = \lambda z + O(z^2)$ and $g(z) = \mu z + O(z^2)$ be local holomorphic diffeomorphisms which commute, i.e., $f \circ g = g \circ f$ holds. Assume that $\mu$ is non-torsion in $\mathbb{C}^*$. If $g$ is linearizable around 0, then $f$ is also linearizable by the same linearization map. If $g$ is non-linearizable, then $\lambda$ is contained in $U(1)$. In addition to the latter case, if $\lambda$ is torsion (resp. non-torsion), then $f$ is linearizable (resp. non-linearizable).

The first part of the statement follows by comparing the coefficients of power series of $f$ and $g$. The second part is obtained by using Koenigs’ linearization theorem. The rest part also follows by the standard arguments. For more details, see e.g. [P2, §4].

To deal with Case X, we focus on some properties of hedgehogs of $f$ and $g$. In [P2] and [P5], Pérez-Marco studied hedgehogs for commuting local holomorphic diffeomorphisms. In particular, a part of Thm.III.14 in [P5] is useful for our purpose. We state a slightly weaker version of his result.

**Theorem 2.4** (a part of [P5, Thm.III.14]). Let $f(z) = \lambda z + O(z^2)$ and $g(z) = \mu z + O(z^2)$ be commuting local holomorphic diffeomorphisms with the irrationally indifferent fixed point 0. Assume that $g$ is non-linearizable. Then, for any open neighborhood $W$ of 0, there exists a compact subset $K$ in $W$ which is a common hedgehog of $f$ and $g$. More precisely, there exist admissible neighborhoods $U$ and $V$ in $W$ for $f$ and $g$ respectively such that $K(U,f) = K(V,g)$ holds.

Note that $f$ is automatically non-linearizable from Proposition 2.3. This theorem will be proved in §3. At the end of this section, we give a proof of Theorem 1.3.

**Proof of Theorem 1.3**. Let us check the assertions in each case showed in §1. Before that, by the assumption of $\mu$, Case I, II, and V are excluded. Moreover, it follows from Proposition 2.3 that Case VI, VII, and IX do not occur. If $\lambda$ is torsion, then we may assume that $f$ is the identity after taking a finite covering.

Take a small neighborhood $U$ of 0. The set $K$ in Theorem 1.3 is obtained by considering the Siegel disk of $(U,g)$ in Case III and the hedgehog of $(U,g)$ in Case IV. Further, in Case VIII, consider a common Siegel disk $K$ of $f$ and $g$, which exists by Proposition 2.3.\[6\]
Finally, in Case X, consider a common hedgehog $K$ of $f$ and $g$, which exists by Theorem 2.3. Therefore the assertion (i) follows. The assertion (ii) and (iii) are the first statement of Proposition 2.3 and Theorem 2.2 respectively.

\[ \square \]

2.2. **Review of Ueda’s neighborhood theory.** Let $X$ be a complex surface and $Y$ a compact curve with the topologically trivial normal bundle $N_{Y/X}$. Fix a finite open covering $\{U_j\}$ of $Y$. Since $Y$ is compact and Kähler, $N_{Y/X}$ is U(1)-flat, i.e., the transition functions on $\{U_{jk}\}$ can be represented by U(1)-valued constant functions $\{t_{jk}\}$. Here $U_{jk} = U_j \cap U_k$. Take an open neighborhood $V_j$ of $U_j$ in $X$ and set $V := \bigcup_j V_j$. As shrinking $V_j$, we can choose the defining function $w_j$ of $U_j$ in $V_j$ such that $(w_j/w_k)|_{U_{jk}} \equiv t_{jk}$.

For a system of such defining functions, the expansion of $t_{jk}w_k|_{V_{jk}}$ in the variable $w_j$ is written as

$$ t_{jk}w_k = w_j + f_{jk}^{(n+1)}(z_j) \cdot w_j^{n+1} + O(w_j^{n+2}) $$

for $n \geq 1$. Such a system is said to be of type $n$. Then it follows that $\{(U_{jk}, f_{jk}^{(n+1)})\}$ satisfies the cocycle conditions (see [U, §2]). Denote the cohomology class by

$$ u_n(Y, X) := \{(U_{jk}, f_{jk}^{(n+1)})\} \in H^1(Y, N_{Y/X}^{-n}), $$

which is called the $n$-th Ueda class of $(Y, X)$. The $n$-th Ueda class is an obstruction to existence of a system of type $(n + 1)$. Indeed, it is not difficult to see that a type $n$ system can be refined to be of type $(n + 1)$ if (and only if) $u_n(Y, X) = 0$. Therefore the following two cases occur:

- There exists a positive integer $n$ such that the following holds:
  - For any $m \leq n$, there is a defining system of type $m$ such that $u_m(Y, X) = 0$ for $m < n$ and $u_n(Y, X) \neq 0$.
- For any positive integer $n$, there exists a defining system of type $n$ such that $u_n(Y, X) = 0$.

In the former case, the pair $(Y, X)$ is said to be of finite type or of type $(\alpha)$ (more precisely, of type $n$). The latter case, we say, the pair $(Y, X)$ is infinite type. For example, if $Y$ admits a holomorphic tubular neighborhood in $X$, then $(Y, X)$ is infinite type. Here a holomorphic tubular neighborhood means a neighborhood of $Y$ in $X$ which is biholomorphic to that of the zero section of the normal bundle $N_{Y/X}$. More generally, we consider the case that $Y$ admits a pseudoflat neighborhood system in $X$, that is, a neighborhoods system with Levi-flat boundary. In such a case, $(Y, X)$ is said to be of type $(\beta)$. Then $(Y, X)$ is infinite type.

In terms of defining function systems of $Y$, there exists such a system $\{w_j\}$ as

$$ t_{jk}w_k = w_j. $$

Namely, the U(1)-flat structure on the normal bundle $N_{Y/X}$ can be extended to $[Y]$ around $Y$, where $[Y]$ is the line bundle which corresponds to the divisor $Y$, i.e., there exists a neighborhood $V$ of $Y$ in $X$ such that $[Y]|_V$ is U(1)-flat.

**Remark 2.5.** It does not change whether the type is finite or infinite after finite covering procedures, though the smallest number $n$ of non-vanishing Ueda classes varies.

Ueda showed the following result:

Ueda showed the following result:
Theorem 2.6. ([U Theorem 3]) Suppose that the pair \((Y, X)\) is infinite type. If the normal bundle \(N_{Y/X} \in \text{Pic}^0(Y)\) is torsion or satisfies the Diophantine condition, then \(Y\) is of type \((\beta)\), that is, it admits a pseudoflat neighborhood system in \(X\).

As previous results, Arnol’d first studied neighborhoods of elliptic curves embedded in a surface with topologically trivial normal bundle (see [A]). By regarding it as a kind of linearization problem, he applied the technique of Siegel’s linearization theorem to this problem. Ueda’s theorem is a partial generalization of Arnol’d theorem.

When \((Y, X)\) is of type \((\alpha)\), there are some results about the existence of strictly plurisubharmonic functions on a neighborhood of \(Y\) and the constraint of its increasing degree.

Theorem 2.7. ([U Theorem 1, 2]) Suppose that the pair \((Y, X)\) is of type \(n\). Then the following hold:

(i) For any real number \(a > n\), there exist a neighborhood \(V\) of \(Y\) and a strictly plurisubharmonic function \(\Phi\) defined on \(V \setminus Y\) such that \(\Phi(p) = O(\text{dist}(p, Y)^{-a})\) as \(p \to Y\).

(ii) Let \(V\) be a neighborhood of \(Y\). For any positive real number \(a < n\) and any strictly plurisubharmonic function \(\Psi\) defined on \(V \setminus Y\) such that \(\Psi(p) = o(\text{dist}(p, Y)^{-a})\) as \(p \to Y\), there is a neighborhood \(W\) of \(Y\) in \(V\) such that \(\Psi|_{W \setminus Y}\) is constant.

Remark 2.8. In contrast, by definition, the curve \(Y\) is of type \((\beta)\) admits a holomorphic foliation defined on an open neighborhood of \(Y\) and the holonomy along the compact leaf \(Y\) is \(U(1)\)-linear. Thus, there is a pluriharmonic function defined on \(V \setminus Y\) which diverges logarithmically toward \(Y\).

Let us explain how to specify the types of compact leaves in holomorphic foliations. Recall our setting in the present paper: an elliptic curve \(Y\) equips a holomorphic foliation \(F\) as the compact leaf and the linear part of the holonomy of \(F\) along \(Y\) is contained in \(U(1)\). The strategy is the following: First, by the Serre duality, we have

\[ H^1(Y, N_{Y/V}^{-m}) \cong \begin{cases} \mathbb{C} & (\text{if } N_{Y/V}^{-m} = 1_Y) \\ 0 & (\text{if } N_{Y/V}^{-m} \neq 1_Y) \end{cases}. \]

Thus, we can roughly determine the type of \(Y\). If \(N_{Y/V}\) is torsion, then \(Y\) is of type \((\alpha)\) or \((\beta)\). Whereas, if \(N_{Y/V}\) is non-torsion, then \(Y\) is of type \((\beta)\) or \((\gamma)\). Next, we deal with them on a case by case basis. The foliation chart of \(F\) is often effective to decide whether the type is \((\alpha)\) or not. However, there are exceptions as Case V. See §5 and Remark 2.9 below. On the other hand, to decide whether the type is \((\beta)\) or \((\gamma)\), we focus on pluriharmonic functions which diverge toward \(Y\). See Remark 2.8.

Remark 2.9. The ueda type can not be specified without the unitarity condition for the linear part of the holonomy, even if the holonomy is linearizable. In [CLPT] Remark 2.2, they constructed examples in the case III or VIII in §1 without the unitarity condition, although they are of type \((\alpha)\).

3. HEDGEHOGS FOR COMMUTING HOLOMORPHIC Diffeomorphisms

As we mentioned in §2.1, Theorem 2.4 (a part of [P5 Thm.III.14]) is useful for our purpose. In this section, we give a slightly modified proof, based on [P5]. To do this,
we need several propositions. Proposition 3.1 below is obtained in [P3] and [P5], which is a key proposition. See the references and also [Y] for the proof. Theorem 3.3 is also shown, which is one of the main application of Proposition 3.1. We give a sketch of the proof. Furthermore, we prepare Lemma 3.2 and Lemma 3.4. Finally, Theorem 2.4 will be proved by using Lemma 3.4 and Theorem 3.3. Note that this can be proved without Theorem III.4 in [P5]. In this section, we consider $\mathbb{C}P^1$ with the Fubini-Study metric $g_{FS}$ as the ambient space.

**Proposition 3.1 ([P3, Proposition 1], [P5, Proposition II.3]).** Let $g(z) = \mu z + O(z^2)$ be a local holomorphic diffeomorphism with the irrationally indifferent fixed point 0. Assume that $g$ is non-linearizable around 0. Let $U$ be an admissible domain of $g$ and $K$ a hedgehog of $(U, g)$. Then, for each $n \in \mathbb{N}$, there exists a quintuple $(\Omega_n, B_n, \eta_n, R_n, A_n)$ associated with $K(U,g)$, where $\Omega_n$ is an open neighborhood of $K$ in $\mathbb{C}P^1$, $B_n$ is a closed annulus in $\Omega_n \setminus K$ separating $\partial \Omega_n$ from $K$, $\eta_n$ is a Jordan closed curve in the interior of $B_n$ separating two boundary components of $B_n$, $R_n$ is a closed quadrilateral in $B_n$, and $A_n$ is a closed annulus in $\Omega_n \setminus (K \cup R_n)$ separating $R_n$ from $K$ whose modulus tends to $\infty$ as $n \to \infty$. The quintuple satisfies the following conditions:

1. For each $q_j$ ($j = 0, \ldots, n$), the iterations $g^{\pm q_j}$ are defined on $\Omega_n$, where $q_n$ is given by the continued fractional approximation $(p_n/q_n)_{n \in \mathbb{N}}$ of the irrational number $\alpha = (\log \mu)/2\pi \sqrt{-1}$,
2. for any point $z$ in $\eta_n$, there exists an iteration $g^{m_n}(z)$ which is contained in $R_n$,
3. for any point $z$ in the component of $\mathbb{C}P^1 \setminus B_n$, if there is an integer $k$ such that $g^k(z)$ is contained in the other component of $\mathbb{C}P^1 \setminus B_n$, then there exists an iteration $g^{k_n}(z)$ which is contained in $R_n$.

![Figure 2](image-url) The separating annulus $B_n$, the meridian curve $\eta_n$ of $B_n$, and the quadrilateral $R_n$ associated with a hedgehog $K$ of $g$: The trapped subsequence $(g^{k_n}(z))_{n \in \mathbb{N}}$ in the statement (3) is depicted in the right figure. The quadrilateral $R_n$ converges to a point $z_0$ in $K \cap \partial U$. See Lemma 3.2.

This is a rewrite of [P3, Proposition 1] as the statement for the hedgehog $K$ through a uniformization map $\psi: \mathbb{D} \to \mathbb{C}P^1 \setminus K$. In fact, he used this version in [P3, §3 and §4]. Let us denote the corresponding quintuple in $\mathbb{D}$ by $(\tilde{\Omega}_n, \tilde{B}_n, \tilde{\eta}_n, \tilde{R}_n, \tilde{A}_n)_{n \in \mathbb{N}}$. Note the symbols which we use. Compare with the original statement of [P3, Proposition 1], the
two boundary components of $\hat{B}_n$ correspond to curves $\gamma_0(n)$ and $\gamma_1(n)$, $\hat{\eta}_n$ corresponds to $\gamma(n)$. Also, $\hat{R}_n$ and $\hat{A}_n$ correspond to $R_n$ and $A_n$ respectively.

The statement (2) and (3) imply that the dynamics around the hedgehog behaves as “quasi-rotation”. The closed curve $\eta_n$ is called a quasi-invariant curve, that is, after some iterations, the curve returns to near the initial position in the sense of Hausdorff distance with respect to the Poincaré metric on $\mathbb{C}P^1 \setminus K$. For more details, see [P5] and [Y].

**Lemma 3.2.** Let $(U, g)$ and $(\Omega_n, B_n, \eta_n, R_n, A_n)_{n \in \mathbb{N}}$ associated with $K_{(U,g)}$ be as in Proposition 3.1. Then the annulus $B_n$ converges to $K$ in the sense of Hausdorff convergence with respect to the Fubini-Study metric. Moreover, for any point $z_0$ in $K \cap \partial U$, we can take the quadrilateral $R_n$ so as to converge to the point $z_0$.

**Proof of Lemma 3.2.** It is clear from the construction of $\hat{B}_n$ in the proof of [P5], Proposition II.3] or [Y] that the annulus $\hat{B}_n$ converges to the boundary $\partial \mathbb{D}$ and the corresponding annulus $B_n$ converges to the hedgehog $K$. Therefore, we show only the latter statement. Choose any point $z_0$ in $K \cap \partial U$. Since $\partial U$ is of class $C^1$, there is a path $\gamma: [0, 1) \to \mathbb{C}P^1 \setminus \overline{U}$ which lands at $z_0$, that is, the limit $\lim_{t \to 0} \gamma(t)$ exists and is $z_0$. For a uniformization map $\psi: \mathbb{D} \to \mathbb{C}P^1 \setminus K$, it follows from [M Corollary 17.10] that the path $\gamma$ maps under $\psi^{-1}$ to a path $\tilde{\gamma}$ in $\mathbb{D}$ which lands at some point on $\partial \mathbb{D}$. The landing point is denoted by $p$. For sufficiently large $n$, the closed annulus $\hat{B}_n$ intersects with the path $\tilde{\gamma}$. We can choose a point $q_n = \tilde{\gamma}(t_n)$ in $\hat{B}_n \cap \tilde{\gamma}$ for each $n$ such that $(t_n)_{n \in \mathbb{N}}$ is an increasing sequence.

According to the construction of $\hat{R}_n$ in [P5] or [Y], we can construct a quadrilateral $R_n$ so as to include the point $q_n$. Note that $q_n$ converges to $p$ as $n \to \infty$. Let us return to $\mathbb{C}P^1 \setminus K$ under $\psi$. The quadrilaterals $R_n$ and the path $\tilde{\gamma}$ map to $R_n$ and $\gamma$. See Figure 3.

Then, we apply the modulus inequality (cf. [LV] §6.4])

$$\text{Mod}(A_n) \leq \frac{2\pi^2}{\ell^2}$$

to the separating annulus $A_n$ surrounding $R_n$, where $\ell$ is the infimum of the length of closed curves separating two boundary components of $A_n$ with respect to the Fubini-Study metric $g_{FS}$ on $\mathbb{C}P^1$. Since the modulus $\text{Mod}(A_n)$ tends to $\infty$ from Proposition 3.1, it follows from the standard argument that the boundary beside $R_n$ degenerates to a single point. Hence, so does the quadrilateral $R_n$. By the choice of $R_n$'s, it converges to the point $z_0$. \qed

**Theorem 3.3** ([P3, Theorem 1], [P5, Theorem III.12.]). Let $g(z) = \mu z + O(z^2)$ be a local holomorphic diffeomorphism with the irrationally indifferent fixed point $0$. Assume that $g$ is non-linearizable. Then, the sequence $(g^n(z))_{n \in \mathbb{N}}$ does not converge to 0 as $n \to \infty$ for any point $z$ distinct from 0.

Here, we give a sketch of the proof of Theorem 3.3 according to [P5]. Let $U$ be an admissible domain for $g$ and $K$ a hedgehog of $(U, g)$. First, we take a point $z \in K \setminus \{0\}$. It is known that any orbits in $K$ are recurrent ([P3, Corollaire 1]), so that the sequence $(g^n(z))_{n \in \mathbb{N}}$ does not converge to 0. Second, take a point $z \notin K$ where $g$ is defined and a point $z_0 \in K \cap \partial U \neq \emptyset$. Let $B_n$ and $R_n$ be as in Proposition 3.1 and Lemma 3.2. After enlarging $n$, we may assume that the point $z$ belongs to the component of $\mathbb{C}P^1 \setminus B_n$ which does not contain $K$. If the sequence $(g^k(z))_{k \in \mathbb{N}}$ converges to 0, then it follows from
There is a (sequence of) closed quadrilateral \( R_n \) intersecting with the path \( \gamma \), which is surrounded by a closed annulus \( A_n \) whose modulus tends to \( \infty \) as \( n \to \infty \). The construction in \( \bar{D} \) (under a uniformization map \( \psi : \bar{D} \to \mathbb{CP}^1 \setminus K \)) is depicted in the left figure.

Proposition 3.1 (3) that there exists a subsequence \( (g^{k_n}(z))_{n \in \mathbb{N}} \) such that \( g^{k_n}(z) \in R_n \) for each \( n \). See Figure 2. Therefore, by Lemma 3.2, the sequence \( (g^k(z))_{k \in \mathbb{N}} \) accumulates at \( z_0 \). This contradicts the assumption, so that the theorem follows.

To show Theorem 2.4, we prepare the following lemma.

**Lemma 3.4.** Let \( g \) be as Proposition 3.1. Let \( K \) and \( K' \) be two hedgehogs of \( (U, g) \) and \( (U', g) \), where \( U \) and \( U' \) are admissible domains of \( g \). Then, \( K \subset K' \) or \( K' \subset K \) hold.

**Proof of Lemma 3.4.** We prove this by contradiction. Assume that \( K \not\subset K' \) and \( K' \not\subset K \) hold. Set \( D_r = \{ z \in \mathbb{C} : |z| < r \} \), \( D = D_1 \), and \( A_R = \{ z \in \mathbb{C} : R < |z| < 1 \} \). First, we take a uniformization map

\[
\varphi : \overline{D} \to \mathbb{CP}^1 \setminus K'.
\]

Consider an open neighborhood of \( K' \) as

\[
V_{\epsilon} = \varphi(A_{1-\epsilon}) \cup K'.
\]

for \( \epsilon > 0 \) and a compact exhaustion \( (V_{\epsilon})_{\epsilon>0} \). Since \( K \not\subset K' \) and \( K \) is compact, there is \( \delta > 0 \) such that \( K \subset \overline{V_{\delta}} \) and \( K \cap \partial V_{\delta} \neq \emptyset \). Choose a point \( z_0 \in K \cap \partial V_{\delta} \). Note that \( z_0 \not\in V_{\delta/2} \). We take the connected component of \( U \cap V_{\delta} \) containing \( K \). After a suitable smoothing the boundary, the domain is an admissible domain of \( g \), denoted by \( V \). Note that \( K \subset \overline{V} \) and \( z_0 \in K \cap \partial V \neq \emptyset \) still hold. Therefore, \( K \) can be regarded as a hedgehog of \( (V, g) \). Now apply Proposition 3.1 and Lemma 3.2 to the pair \( (K = K_{(V, g)}, \{z_0\}) \). For each \( n \in \mathbb{N} \), take \( \eta_n \) and \( R_n \) as in the proposition.

Let us show the following assertion: there exists a large integer \( N \) such that, for any \( n \geq N \), \( R_n \cap V_{\delta/2} = \emptyset \) and \( \eta_n \cap K' \neq \emptyset \) hold. The former statement follows directly from Lemma 3.2. On the other hand, the latter statement can be shown as follows.
Assume that there is a subsequence \((n_k)_{k \in \mathbb{N}}\) so that \(\eta_{n_k} \cap K' = \emptyset\). Similar to the previous paragraph, take a uniformization map
\[
\psi : \mathbb{D} \to \mathbb{C}P^1 \setminus K,
\]
an open neighborhoods of \(K\) as
\[
V'_k = \psi(A_{1-\epsilon}) \cup K
\]
for \(\epsilon > 0\), and choose \(\delta' > 0\) such that \(K' \subset V'_{\delta'}\) and \(K' \cap \partial V'_{\delta'} \neq \emptyset\) hold. Here we used the assumption \(K' \not\subset K\) and the compactness of \(K'\). Also, we choose a point \(z'_0 \in K' \cap V'_{\delta'}\). By Lemma 3.2, \(\eta_{n_k}\) is contained in \(V'_{\delta'/2}\) for large \(k\). Jordan closed curve theorem shows that \(\eta_{n_k}\) decomposes \(\mathbb{C}P^1\) into two domains \(W_0\) and \(W_\infty\) such that \(\partial W_0 = \partial W_\infty = \eta_{n_k}\), where \(0 \in W_0\) and \(\infty \in W_\infty\) hold. Note that \(z'_0\) belongs to \(W_\infty\). By the assumption above, \(K'\) is contained in \(W_0 \cup W_\infty\). However, \(0 \in K' \cap W_0 \neq \emptyset\) and \(z'_0 \in K' \cap W_\infty \neq \emptyset\) hold, so that this contradicts the connectivity of \(K'\).

For sufficiently large \(n\) which satisfies the assertion above, take a point \(z_1 \in \eta_{n} \cap K' \neq \emptyset\). It follows from Proposition 3.1 (2) that there exists an iteration \(g^{m_n}(z_1)\) contained in \(R_n\). On the other hand, since \(R_n \cap V_{\delta/2} = \emptyset\), the point \(g^{m_n}(z_1)\) lies outside of \(V_{\delta/2}\). This contradicts the invariance of \(K'\) under \(g\).

\[\square\]

**Proof of Theorem 2.4.** For any open neighborhood \(W\) of 0, there is a small \(R > 0\) such that \(\mathbb{D}_R = \{z : |z| < R\} \subset W\) and \(f, g, g \circ f, f \circ g\) are defined on \(\mathbb{D}_R\), further, \(f\) and \(g\) are univalent on an open neighborhood of \(\overline{\mathbb{D}_R}\). For sufficiently small \(\epsilon > 0\) and any \(r \in (0, \epsilon)\), \(f(\mathbb{D}_r) \subset \mathbb{D}_R\) and \(f^{-1}(\mathbb{D}_r) \subset \mathbb{D}_R\). By Theorem 2.1, for each \(r \in (0, \epsilon)\), there is a hedgehog \(K_r\) of \((\mathbb{D}_r, g)\). Since \(f\) and \(g\) commute, \(f(K_r)\) is also a hedgehog of \((f(\mathbb{D}_r), g)\). As applying Lemma 3.4 to these hedgehogs, we have \(K_r \subset f(K_r)\) or \(f(K_r) \subset K_r\). Also, we may assume that \(f(K_r) \subset K_r\) holds by exchanging \(f\) and \(f^{-1}\).

The rest part is the same as the proof of [P5, Thm.III.14]. Iterating by \(f\), which is well-defined, we have the nested sequence
\[
K_r \supset f(K_r) \supset f^2(K_r) \supset \cdots
\]
The set \(L = \bigcap_{n \geq 0} f^n(K_r)\) satisfies the following properties:

1. \(L\) is compact, connected, and \(\mathbb{C} \setminus L\) is connected,
2. \(0 \in L\),
3. \(L \neq \{0\}\), and
4. \(L\) is invariant under \(f, f^{-1}, g,\) and \(g^{-1}\).

It is not difficult to show (1),(2), and (4). Let us show (3). If \(L = \{0\}\), then, for any \(z \in K_r \setminus \{0\}\), the sequence \((f^n(z))_{n \in \mathbb{N}}\) converges to 0 as \(n \to \infty\). However, \(f\) is also non-linearizable from Proposition 2.3 so that this contradicts Theorem 3.3. Therefore \(L\) is a common hedgehog of \(f\) and \(g\).

\[\square\]

### 4. Proof of Theorem 1.2 and 1.4

#### 4.1. Proof of Theorem 1.2
Fix a pair \((\gamma, \gamma')\) so that \((Y, X, \mathcal{F})\) is in Case IV or X. As taking a sufficiently small neighborhood \(V\) of \(Y\), we may assume that \(\Phi\) is bounded
from below on $V$. Choose a transversal $(T,p)$ identified with a domain $(U,0)$ in $\mathbb{C}$. The pair $(f = \text{Hol}(\gamma), g = \text{Hol}(\gamma'))$ defined on $U$ satisfies the condition of Case IV or X. Let $\hat{K}$ be a complete invariant set in Theorem \ref{Theorem1.3} (i), which is a (resp. common) hedgehog of $f$ in Case IV (resp. of $f$ and $g$ in Case X). If necessary, as retaking a smaller $U$, we may assume that the saturated set $\hat{K}$ of $K$ is contained in $V$. Note that $Y$ is included in $\hat{K}$. By Theorem \ref{Theorem1.3} (iii), there is a dense leaf $L$ in $\hat{K}$. The leaf $L$ is biholomorphic to $\mathbb{C}^*$ in Case IV and to $\mathbb{C}$ in Case X. It follows from the holonomy conditions and the condition $(\ast)$ which we mentioned in \S 1. Denote by $i: L \rightarrow \hat{K} \subset V$ the holomorphic immersion. The function $i^* \Phi$ is harmonic on $L \cong \mathbb{C}^*$ or $\mathbb{C}$, and is bounded from below. By Liouville’s theorem, it takes a constant value $C$. Hence, it follows from the continuity that the restriction $\Phi|_{\hat{K}}$ is the constant function $C$. □

In the proof above, we used the condition $(\ast)$ only for assuring that the leaf $L$ is biholomorphic to $\mathbb{C}^*$ or $\mathbb{C}$. The authors could not drop this “technical” condition.

**Question 4.1.** In Case IV or X, what can we say about complex structures on the dense leaf $L$ without the condition $(\ast)$?

According to [P1, Theorem 5], $K$ is not locally connected at any point distinct from 0. By using this, we can improve Theorem \ref{Theorem1.2} as follows:

**Theorem 4.2.** Assume that the pair $(Y,X)$ is in Case IV or X with the condition $(\ast)$. Let $V$ be a neighborhood of $Y$ in $X$ and $\Phi$ be a pluriharmonic function defined on $V \setminus Y$ bounded from below. Then $\Phi$ is a constant function.

**Proof of Theorem 4.2.** Under the assumption that $\Phi$ is bounded from below on $V$ instead of the continuity along $Y$, $\Phi|_{\hat{K} \setminus Y} \equiv C$ follows from the proof of Theorem \ref{Theorem1.2}. Without loss of generality, we may assume that $C = 0$, that is, $\hat{K} \setminus Y \subset \Phi^{-1}(0)$. We prove the theorem by contradiction assuming that $\Phi$ is not a constant function.

Let $w$ be a holomorphic coordinate of the transversal $U$ and $(x,y)$ be the real coordinate such that $w = x + \sqrt{-1}y$. Let $p$ be a point in $Z := \Phi^{-1}(0) \cap (U \setminus Y)$. When the partial differentials

$$
\Phi_x(p) := \frac{\partial}{\partial x}(\Phi|_U)(p)
$$

or

$$
\Phi_y(p) := \frac{\partial}{\partial y}(\Phi|_U)(p)
$$

are not equal to zero, it follows from the implicit function theorem that $U_p \cap Z$ is diffeomorphic to an interval for a small neighborhood $U_p$ of $p$ in $U$. As

$$
\frac{\partial}{\partial w}((\Phi + \sqrt{-1}\Phi^*)|_U) = \frac{1}{2}(\Phi_x - \sqrt{-1}\Phi_y + \sqrt{-1}\Phi_x^* + \Phi_y^*) = \Phi_x - \sqrt{-1}\Phi_y
$$

where $\Phi^*$ is the harmonic conjugate function, we can conclude that $U_p \cap Z$ is either an isolated point or diffeomorphic to an interval.

Fix a point $p \in K \setminus \{0\} \subset Z$. By the connectedness of $K$ and the argument above, we have that $I := U_p \cap Z$ is diffeomorphic to an interval for a small $U_p$. Denote by $J$ the set $I \cap K$ and by $J_p$ the connected component of $J$ which contains $p$. First, we show that the open interval $I_{p,q} \subset I$ with boundary $\partial I_{p,q} = \{p,q\}$ is contained in $J_p$ if there exists
a point \( q \in J_p \setminus \{ p \} \). If not, there exists a point \( r \in I_{p,q} \setminus J_p \). Denote by \( I_1 \) and \( I_2 \) the connected components of \( I \setminus \{ r \} \). The decomposition \( J_p = (J_p \cap I_1) \cup (J_p \cap I_2) \) leads to the contradiction to the connectedness of \( J_p \), so that the claim follows. Next, according to [P1, Theorem 5], \( K \) is not locally connected at any point distinct from 0. Thus, it follows from the above fact that \( J_p = \{ p \} \) holds, which contradicts to the connectedness of \( K \).

\[ \square \]

4.2. Proof of Theorem 1.4. First, we show the existence of a point \( p \in Y \) such that \( \Phi(p) < \infty \). For this, it is sufficient to lead to the contradiction by assuming that \( \Phi|_Y \equiv \infty \). Take a sufficiently small tubular neighborhood \( V_0 \) of \( Y \) which is relatively compact in \( V \). By the assumption, one can take a sufficiently large constant \( M > 0 \) so that \( V_M = \{ \Phi > M \} \) is a neighborhood of \( Y \) which is relatively compact in \( V_0 \): \( Y \subset V_M \subset V_0 \subset \mathbb{C}^n \).

Take any point \( p \in Y \) and a transversal \( T \) through \( p \) with a local coordinate \( w \) such that \( T \cap Y = \{ w = 0 \} \) holds. The pair \((f = \text{Hol}(\gamma), g = \text{Hol}(\gamma'))\) satisfies the condition of Case IV and we may assume that \( f \) is the identity after a finite covering. Let \( K \subset T \) be a hedgehog of \( g \) for an admissible domain \( U \) such that \( V_M \cap T \subset \{ w = 0 \} \cap T \). We can take such a domain \( U \) by enlarging \( M \) if necessary. By the construction, \( K \not\subset V_M \) holds. There exists a point \( x \) of \( K \) such that the forward orbit \( O_+(x) \) is dense in \( K \). It follows from Theorem [E2] and the recurrence property of orbits in \( K \) ([P3 Corollaire 1]). Denote by \( L \) the leaf of \( \mathcal{F} \) through \( x \). In this case, \( L \) is biholomorphic to an annulus \( A = \{ r < |z| < R \} \) for \( 0 \leq r < R \leq \infty \). Fix a point \( y \in K \setminus \overline{V_M} \), for example, \( y \in K \setminus \partial U \). Take a sufficiently small neighborhood \( D \) of \( y \) in \( T \) such that \( \hat{D} = \{ (z,w) \mid z \in \gamma, w \in D \} \subset V_0 \) and \( \hat{D} \cap \overline{V_M} = \emptyset \). Here we are using a coordinates \((z,w)\) of a neighborhood of \( \gamma \) in \( X \) such that \( z \) can be regarded as a coordinate of a neighborhood of \( \gamma \) in \( Y \) and that each leaf of \( \mathcal{F} \) is defined by \( \{ w = \text{constant} \} \), which actually exists by Siu's theorem [Siu].

As 0 and \( y \) is contained in \( \overline{O_+(x)} \), we can take positive integers \( n_1, n_2, \) and \( n_3 \) such that \( n_1 < n_2 < n_3, g^{n_1}(x), g^{n_2}(x) \in U, \) and that \( g^{n_3}(x) \in V_M \). Denote by \( i: A \cong L \to V_0 \) the holomorphic immersion. Let us denote by \( \ell_j \) the loop in \( A \) which corresponds to the loop defined by \( \{ (z,w) \mid z \in \gamma, w \in D \} \subset \hat{D} \) for \( j = 1, 3 \). Consider the annulus \( A \subset A \) with \( \partial A = \ell_1 \cup \ell_3 \). Then the function \( i^* \Phi|_A \) is subharmonic by the assumption, and satisfies the condition \( i^* \Phi|_{\partial A} < M \) by the construction. As \( g^{n_3}(x) \in V_M \), we also have that \( \sup_A i^* \Phi|_A > M \), which contradicts to the maximum principle.

Denote by \( \Omega \) the set \( \{ p \in Y \mid \Phi(p) < \infty \} \). It follows from the argument above that \( \Omega \neq \emptyset \). All we have to do here is to show that \( \Omega = Y \), which is also done by slightly improving this argument actually. However, here we directly deduce the conclusion from the fact \( \Omega \neq \emptyset \) by using rather simple arguments on the analysis of plurisubharmonic functions. By [D, Theorem 5.24], \( \Phi \) is plurisubharmonic on a neighborhood of \( p \) in \( V \) for each \( p \in \Omega \). As \( \Omega \) is an open subset of \( Y \) and \( Y \) is connected, it is sufficient to show that \( \Omega \) is closed. Take a point \( q \in \overline{\Omega} \) and a coordinate \((z,w)\) such that \((z,w) = (0,0)\) at \( q \) and that \( Y = \{ w = 0 \} \) on this locus. Take also a sequence \((q_\nu = (z_\nu,0))_{\nu} \subset \Omega \) with \( q_\nu \to q \) as \( \nu \to \infty \). Then it holds for a sufficiently small \( \varepsilon > 0 \) that

\[
\Phi(q) = \Phi(0,0) = \lim_{\nu \to \infty} \Phi(z_\nu,0) \leq \lim_{\nu \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \Phi(z_\nu, \varepsilon e^{i\theta}) d\theta.
\]
As it is clear that \( \{\sup_{0 \leq \theta < 2\pi} \Phi(z_{\nu}, e^{\sqrt{-1}\theta})\}_\nu \) is bounded from above, it holds that \( q \in \Omega. \) \hfill \Box

5. Proof of Theorem 1.1

First we show the assertion (i). Assume that the triple \( (Y, X, \mathcal{F}) \) is in Case I. According to Remark 2.5 it turns out that it is sufficient to show the theorem by assuming that \( f \) is the identity (by taking a finite covering of \( X \) and changing the transversal coordinate). It is easily observed that \( g \) is also the identity in this case. By considering the foliation chart corresponding to this, we obtain a system \( \{w_j\} \) of local defining functions of \( Y. \) Thus the pair \( (Y, X) \) is of type \( (\beta). \)

Next we show the assertion (ii). Assume that the triple \( (Y, X, \mathcal{F}) \) is in Case II. By the same argument as above, we may assume that \( f \) is the identity. We may also assume that \( \mu = 1 \) by considering a finite covering of \( X. \) Note that \( N_{Y/X} \) is holomorphically trivial in this case. Let

\[
g(w) = w + \sum_{\nu=2}^{\infty} b_{\nu} w^\nu
\]

be the expansion of \( g. \) Denote by \( n \) the minimum element of the set \( \{\nu \in \mathbb{Z} \mid \nu \geq 2, \ b_{\nu} \neq 0\}. \) Then, the foliation chart of \( \mathcal{F} \) gives a system \( \{w_j\} \) of local defining functions of type \( n, \) which means that type\( (Y, X) \geq n. \) By definition, the \( (n - 1) \)-th Ueda class corresponds to (the conjugate class of) the representation \( \rho: \Gamma \rightarrow \mathbb{C} \) defined by \( \rho(1) = 0 \) and \( \rho(\tau) = b_n \) under the natural identification \( H^1(Y, N_{Y/X}^{-n+1}) = H^1(Y, O_Y) = H^{0,1}(Y, \mathbb{C}) \) and the injection \( H^{0,1}(Y, \mathbb{C}) \rightarrow H^1(Y, \mathbb{C}). \) Thus we have that \( u_{n-1}(Y, X) \neq 0, \) which means that the pair \( (Y, X) \) is of type \( n. \) Therefore it is of type \( (\alpha). \)

The assertion (iii) is shown by the same argument as the proof of (i) above.

Next, assume that the triple \( (Y, X, \mathcal{F}) \) is in Case IV or \( X. \) As \( N_{Y/X} \) is non-torsion in these cases, the pair \( (Y, X) \) is of infinite type (see \[2.2\]). As there exists a pluriharmonic function \( \Phi: W \setminus Y \rightarrow \mathbb{R} \) with \( \Phi(p) = O(-\log \text{dist}(p, Y)) \) as \( p \rightarrow Y \) by shrinking \( W \) if the pair is of type \( (\beta), \) the assertions (iv) and (x) follows from Theorem 1.4 and Theorem 1.2 respectively. Thus the pair \( (Y, X) \) is of type \( (\gamma). \)

Next we give the proof of the assertion (v). As \( N_{Y/X} \) is torsion in this case, we can apply Theorem 2.6 to conclude that the type of the pair is whether \( (\alpha) \) or \( (\beta). \) By considering the example by [CLPT] with general choice of the representation \( \alpha, \) we have an example of the pair of type \( (\alpha) \) in Case V. In what follows, we construct a pair of type \( (\beta) \) in Case V. Define an affine bundle \( V \rightarrow Y \) over \( Y = \mathbb{C}/\Gamma \) by

\[
V = \mathbb{C}^2 / \left\langle \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} \tau \\ \tau \end{array} \right) \right\rangle,
\]

or equivalently, \( V \) is the quotient \( \mathbb{C}^2 / \sim \) of \( \mathbb{C}^2 \) with a coordinates \( (x, \xi) \) by the relation generated by \( (x, \xi) \sim (x + 1, \xi + 1) \sim (x + \tau, \xi + \tau). \) The projection to \( Y \) is the one induced by the first projection \( (x, \xi) \mapsto x. \) Let \( X \) be the ruled surface over \( Y \) which is a compactification of \( V \) by adding the infinity section. We will identify the infinity section with \( Y \) by the natural manner and also denote it by \( Y: X = V \cup Y. \) Denote by
the foliation on $X$ whose leaves are locally defined by the equation $\xi = (\text{constant})$. By regarding $w = 1/\xi$ as a local defining function of $Y$, we have that

$$f(w) = \frac{1}{\frac{1}{w} + 1} = \frac{w}{w + 1}$$

and

$$g(w) = \frac{1}{\frac{1}{w} + \tau} = \frac{w}{w + \tau}$$

hold, by which one can see that this example $(Y, X, F)$ is in Case V. On the other hand, by considering another coordinate $(\hat{x}, \hat{\xi})$ of $C^2$ defined by $\hat{x} = x$ and $\hat{\xi} = \xi - x$, we can easily see that $X$ is biholomorphic to $X \cong Y \times \mathbb{C}P^1$ and that $Y$ corresponds to the subvariety $Y \times \{\infty\}$ of $Y \times \mathbb{C}P^1$. Therefore we have that the pair is of type $(\beta)$.

The assertions $(vi), (vii)$, and $(ix)$ follow from Lemma 2.3 (see also the proof of Theorem 1.3). The assertion $(viii)$ follows from Lemma 2.3. □

**Remark 5.1.** In this section, we proved Theorem 1.1 for given quintuple $(Y, X, F, \gamma, \gamma')$. However, one can easily reword Theorem 1.1 to the statement on the relation between the type of the pair $(Y, X)$ and dynamical properties of the foliation $F$ which does not depend on the choice of $\gamma$ and $\gamma'$. For this purpose, one may replace the definition of “Case I” with “There exists a basis $(\gamma, \gamma')$ of $\pi_1(Y, \ast)$ such that both $\lambda$ and $\mu$ are torsion elements in $U(1)$ and both $f$ and $g$ are linearizable”, “Case II” with “For a suitable choice of a basis $(\gamma, \gamma')$, both $\lambda$ and $\mu$ are torsion and $f$ is linearizable. For any choice of such a basis, $g$ is non-linearizable”, and “Case III” with “For a suitable choice of a basis $(\gamma, \gamma')$, $\lambda$ is torsion and both $f$ and $g$ are linearizable. For any choice of such a basis, $\mu$ is non-torsion”, and so on.

6. **Proof of Corollary 1.5**

By [K2], $L$ is not semi-positive when the pair $(Y, X)$ is of type $(\alpha)$. Assume that the pair $(Y, X)$ is of type $(\beta)$. Then there exists a neighborhood $V$ of $Y$ such that $L$ admits a unitary flat metric $h_V$ on a neighborhood $V$ of $Y$ (i.e. $h_V$ is a $C^\infty$ Hermitian metric on $L|_V$ whose Chern curvature is 0, see [2.2]). On the other hand, $L$ admits a singular Hermitian metric $h_{\text{sing}}$ such that $h_{\text{sing}}|_{X \setminus Y}$ is a $C^\infty$ Hermitian metric on $L|_{X \setminus Y}$, $h_{\text{sing}} \to \infty$ holds when a point approaches to $Y$, and that the Chern curvature of $h_{\text{sing}}|_{X \setminus Y}$ is 0. Indeed, the singular Hermitian metric defined by $|f_Y|^2_{h_{\text{sing}}} \equiv 1$ satisfies this property, where $f_Y \in H^0(X, L)$ is a section with $\text{div}(f_Y) = Y$. A $C^\infty$ Hermitian metric $h$ on $L$ with semi-positive curvature can be constructed by using the regularized minimum construction for these two metrics $h_V$ and $h_{\text{sing}}$, which is the same construction as we used for proving [K2 Corollary 3.4]. This proves the semi-positivity of $L$ when the pair $(Y, X)$ is of type $(\beta)$.

Therefore all we have to do is to show that $L$ is not semi-positive assuming that the triple $(Y, X, F)$ is in Case IV, which is done by the same manner as in the proof of the main theorem in [K2] by using Theorem 1.4 instead of [U Theorem 2]. □
References

[A] V. I. Arnol’d, Bifurcations of invariant manifolds of differential equations and normal forms in neighborhoods of elliptic curves, Funkcional. Anal. i Priložen., 10 (1976), No.4, 1–12. (English translation: Functional Anal. Appl., 10 (1977), No.4, 249–257).

[B] R. Bott, Lectures on characteristic classes and foliations, Springer Lecture Notes in Math., 279 (1972), 1–94.

[CLPT] B. Claudon, F. Loray, J. V. Pereira, F. Touzet, Compact leaves of codimension one holomorphic foliations on projective manifolds, arXiv:math/1512.06623.

[D] J.-P. Demailly, Complex analytic and differential geometry, https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[K1] T. Koike, On minimal singular metrics of certain class of line bundles whose section ring is not finitely generated, Ann. Inst. Fourier, 65 (2015), No.5, 1953–1967.

[K2] T. Koike, On the minimality of canonically attached singular Hermitian metrics on certain nef line bundles, Kyoto J. Math., 55 (2015), No.3, 607–616.

[K3] T. Koike, On a neighborhood of a torus leaf of a certain class of holomorphic foliations on complex surfaces, arXiv:1510.02287.

[LV] O. Lehto, K. I. Virtanen, Quasiconformal Mappings in the Plane, 2nd edition. Grundlehren Math. Wiss., 126 (1973), Springer-Verlag, New York-Heidelberg.

[M] J.W. Milnor, Dynamics in one complex variable. Third edition, Annals of Mathematics Studies, 160 (2006), Princeton University Press, Princeton, NJ.

[M2] J.W. Milnor, On the existence of a connection with curvature zero, Comment. Math. Helv., 32 (1958), 215–223.

[P1] R. Pérez-Marco, Topology of Julia sets and hedgehogs. Preprint, Université de Paris-Sud (1994), 94–48.

[P2] R. Pérez-Marco, Non linearizable holomorphic dynamics having an uncountable number of symmetries. Invent. Math., 119 (1995), No.1, 67–127.

[P3] R. Pérez-Marco, Sur une question de Dulac et Fatou, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), No.8, 1045–1048.

[P4] R. Pérez-Marco, Fixed points and circle maps, Acta Mathematica, 179 (1997), No.2, 243–294.

[P5] R. Pérez-Marco, Hedgehog dynamics, manuscript.

[Sie] C. L. Siegel, Iterations of analytic functions, Ann. of Math., 43 (1942), 607–612.

[Siu] Y. T. Siu, Every Stein subvariety admits a Stein neighborhood, Invent. Math., 38 (1976), 89–100.

[U] T. Ueda, On the neighborhood of a compact complex curve with topologically trivial normal bundle, Math. Kyoto Univ., 22 (1983), 583–607.

[Y] J.C. Yoccoz, Analytic linearization of circle diffeomorphisms, Dynamical systems and small divisors, Springer Lecture Notes in Math., 1784 (2002), 125–173.

1 Department of Mathematics, Graduate School of Science, Osaka City University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan

E-mail address: tkoike@sci.osaka-cu.ac.jp

2 Department of Mathematics, Tokai University, 4-1-1 Kitakaname, Hiratsuka-shi, Kanagawa 259-1292, Japan

E-mail address: nogawa@tsc.u-tokai.ac.jp