A practice problem for Schrödinger transmission

Zafar Ahmed
Nuclear Physics Division, Bhabha Atomic Research Centre
Trombay, Bombay 400 085, India
zahmed@apsara.barc.ernet.in
(April 1, 2022)

Abstract

We present the case of quantal transmission through a smooth, single-piece exponential potential, \( V(x) = -V_0 e^{x/a} \), in contrast to the piece-wise continuous potentials, as a pedagogic model to demonstrate the analytic extraction of transmission (reflection) co-efficient.

The Schrödinger transmission through a one-dimensional potential well/barrier constitutes an essential topic in the courses and text-books [1,2] on quantum mechanics. The main interest is to find the probability of transmission, \( T(E) \), and that of reflection, \( R(E) \), when a particle is incident with an energy \( E \) at the potential from one side. For a real potential these two probabilities are such that \( T(E) + R(E) = 1 \), ensuring the conservation of the incident flux of the particles. Due to simplicity of treatment, usually a rectangular well/barrier or a sharp step-potentials are included as examples. These potentials are both piece-wise constant and piece-wise continuous, the former is a three-piece and the latter is a two-piece function.

For the case of three-piece potentials : \( V(|x| \geq a) = 0 \) and \( V(-a \leq x \leq a) = f(x) \), where the Schrödinger equation does not possess analytic form, the numerical computation of \( T(E) \) is performed by letting \( \Psi(x \leq -a) = Ae^{ikx} + Be^{-ikx} \), \( \Psi(-a \leq x \leq a) = Cu(x) + Dv(x) \) and \( \Psi(x \geq a) = Fe^{ikx} \) [3]. The functions \( u(x) \) and \( v(x) \) denote two linearly independent solutions of the Schrödinger equation with initial values as \( u(0) = 1, u'(0) = 0; v(0) = 0, v'(0) = 1 \), for numerical integration of the Schrödinger from \( x = 0 \) upto \( x = \pm a \). Here the prime denotes differentiation with respect to \( x \).

For the case of a rectangular well/barrier of depth/height, \( V_0 \), these functions are \( \cos \kappa x \) and \( \frac{\sin \kappa x}{\kappa} \), where \( \kappa = \sqrt{2m(E \pm V_0)}/\hbar \). The
FIG. 1. Depiction of the exponential barrier Eq. (1), when $V_0 = 1$ and $a = 1$ in arbitrary units. Here i,t,r denote the direction of incident, reflected and transmitted waves: (a) when the particle is incident from left, (b) when the particle is incident from right. In part (b), the filled circles represent two bodies A and B separated by a long-ranged interaction which is presently hypothesized as exponential potential as in Eq. (1).

Matching of the wavefunctions and their derivatives at $x = \pm a$, leads to the elimination of $C$ and $D$ and we can get the complex transmission amplitude as $t(E) = C/A = \sqrt{T(E)e^{i\theta(E)}}$ and complex reflection amplitude as $r(E) = B/A = \sqrt{R(E)e^{i\phi(E)}}$. The energy-dependent quantities $\phi(E)$ and $\theta(E)$ are called reflection phase-shift and transmission phase-shift, respectively. The energy derivative of the phase-shift is interpreted as the time spent by a particle in these processes [1,2].

However, in practical situations the potentials are smooth (continuous and differentiable everywhere) and single piece which may or may not vanish asymptotically. These potentials are the parabolic barrier [4], the Eckart barrier [3,4,5], the Fermi-step barrier[3,5], the Rosen-Morse barrier [6], the Ginocchio barrier [7], the Scarf barrier [8], the Morse barrier [9] and a potential which interpolates between Morse and Eckart barriers [10]. The transmission co-efficients for these models have been obtained analytically. In all such cases [3-10], a
single function is obtained as the solution of the Schrödinger equation which behaves as a transmitted wave on one side of the potential at asymptotically large distances (say, \( x \to \infty \)). On the other side of the potential the same wavefunction behaves asymptotically (say, \( x \to -\infty \)) as a combination of oppositely travelling incident and reflected waves. Moreover, these asymptotic forms of the wavefunctions may or may not be plane-waves depending on whether the potential converges to zero or not at the asymptotic distance. This dual asymptotic behaviour of a single wave-solution on two sides of the potential barrier which is absent in the piece-wise constant/continuous potentials is worthwhile to demonstrate through a simple do-able example. It will be even more valuable if one can extract simple forms of \( T(E) \) and \( R(E) \) analytically.

In this paper, we present a smooth single-piece potential (see Fig.1): 

\[
V(x) = -V_0 e^{x/a},
\]

(1)

which vanishes asymptotically as \( x \to -\infty \) but it diverges to \(-\infty\) as \( x \to \infty \). We feel that the attractive exponential potential (1) presents a simple instance containing the essential features of analytic extraction of transmission co-efficient for a one piece smooth potential.

In the following we proceed to find \( T(E) \) and discuss various aspects of Schrödinger transmission arising thereby. The time-independent Schrödinger equation with (1) is written as

\[
\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} [E + V_0 e^{x/a}] \psi(x) = 0,
\]

(2)

where \( m, E \) and \( \hbar (= \frac{\hbar}{2\pi}) \) denote the mass and energy of the incident particle and \( \hbar \) is the Planck-constant. The time-dependent wave function is written as

\[
\Psi(x,t) = \psi(x)e^{iEt/\hbar}.
\]

(3)

Let us substitute \( z = pe^{\frac{2\pi}{\hbar}} \) in Eq. (2) and write

\[
\frac{d^2 \psi}{dz^2} + \frac{1}{z} \frac{d\psi}{dz} + \left( 1 - \frac{q^2}{z^2} \right) \psi = 0,
\]

(4)

where \( p = \sqrt{\frac{8mV_0a^2}{\hbar^2}} \) and \( q = \sqrt{\frac{8mEa^2}{\hbar^2}} = 2ka \). The transformed equation is the standard cylindrical Bessel equation [11] whose linearly independent solutions for any positive energy are the Bessel functions : \( J_{iq}(z) \) and \( J_{-iq}(z) \) or the Henkel functions : \( H^{(1)}_{iq}(z) \) and \( H^{(2)}_{iq}(z) \). Noting the asymptotic property that

\[
H^{(1),(2)}_{iq}(z) \sim \sqrt{2/(\pi z)} e^{\pm i(z-\frac{iq\pi}{2}-\frac{\pi}{4})}, \quad \text{when } |z| \to \infty,
\]

(5)
as \( z(= pe^{\frac{2\pi}{a}}) \) is large for large positive values of \( x \), we admit
\[
\Psi(x, t) = H^{(1)}_{iq}(pe^{\frac{2\pi}{a}}) e^{iEt/h}, \tag{6a}
\]
\[
\approx \sqrt{\frac{2}{\pi p}} e^{-i\pi/4} e^{iq\pi/2} e^{-\frac{\pi}{8\hbar}} \exp[i(pe^{\frac{2\pi}{a}} - Et/h)], \text{ when } x \to \infty, \tag{6b}
\]
\[
\equiv \sqrt{\frac{2}{\pi p}} e^{-i\pi/4} e^{iq\pi/2} e^{-\frac{\pi}{8\hbar}} \exp[i\Phi(x)]. \tag{6c}
\]
This represents a wave which is exponentially decaying yet travelling towards \( x \to \infty \). The condition of constancy of the phase i.e., \( \frac{d\Phi}{dt} = 0 \) yields positive phase velocity:
\[
\frac{dx}{dt} = \frac{2Ea}{\hbar p} e^{-x^2/a} \exp[i(kx - Et/\hbar)],
\]
which is positive definite and does not change sign and hence the direction of the wave is unchanged. Thus Eq. (6a) represents a transmitted wavefunction, \( \Psi_t \). The other degenerate and linearly independent solution of the Schrödinger equation (2), \( H^{(2)}_{iq}(z) \) has been dropped.

It is instructive to note that a linear combination of these solutions can not produce the boundary condition of a transmitted wave, traveling towards \( x \to \infty \).

Next, we note valuable identities:
\[
H^{(1)}_{\nu}(z) = i [e^{-i\pi \nu} J_{\nu}(z) - J_{-\nu}(z)]/ \sin(\nu \pi), \text{ and } H^{(2)}_{\nu}(z) = i [J_{\nu}(z) - e^{i\pi \nu} J_{-\nu}(z)]/ \sin(\nu \pi). \tag{7}
\]
We can also write Eq. (6a) as
\[
\Psi(x, t) = \frac{1}{\sinh q\pi} \left[ e^{q\pi} J_{iq}(\frac{p}{2} e^{\frac{2\pi}{a}}) - J_{-iq}(\frac{p}{2} e^{\frac{2\pi}{a}}) \right] e^{iEt/\hbar}. \tag{8}
\]

The asymptotic behaviour of Bessel function for small \( z \)-values is given by [11]
\[
J_{\pm \nu}(z) \approx \frac{(z/2)^{\pm \nu}}{\Gamma[1 \pm \nu]} \tag{9}
\]
Consequently, we can write by combining Eqs. (8) and (9)
\[
\Psi(x, t) \approx \frac{e^{q\pi}(p/2)^{iq}}{\sinh q\pi \Gamma[1 + iq]} e^{i(kx - Et/\hbar)} - \frac{(p/2)^{-iq}}{\sinh q\pi \Gamma[1 - iq]} e^{-i(kx + Et/\hbar)}, \quad x \to -\infty. \tag{10}
\]
When \( x \to -\infty \), \( z \) tends to a very small value, for the result (9) to hold. The right side of the above equation represents a linear combination of two oppositely travelling plane waves. The phase-velocities of the first and second part are \( 2E/(\hbar k) \) and \(-2E/(\hbar k)\), respectively. Since the transmitted wave has been identified to have positive phase-velocity travelling from left to right so as to leave the barrier, the first part represents the incident wave, \( \Psi_i \) travelling from left to right impinging upon the barrier. Similarly, the second part represents a reflected
wave, $\Psi_r$, travelling from right to left. It is instructive to note here that the asymptotic form of $\Psi(x,t)$ as $x \to -\infty$ is a linear combination of the plane-waves since the potential (1) actually vanishes at a large distance on the left side of the barrier (see Fig. 1).

The wave-function $\Psi(x,t)$ in Eqs. (8) and (10) are nothing but exact and asymptotic forms of the sum of incident wave, $\Psi_i$, and reflected wave, $\Psi_r$, wavefunctions, respectively. Having found the incident, reflected and transmitted wavefunctions, we now find the flux,

$$J = \frac{\hbar}{2im} \left( \Psi^*(x,t) \frac{d\Psi(x,t)}{dx} - \Psi(x,t) \frac{d\Psi^*(x,t)}{dx} \right)$$

arising due to $\Psi_i, \Psi_r, \Psi_t$. Yet another instructive point is in order here. The flux can also be written as a Wronskian:

$$J = \frac{\hbar}{2im} W[\Psi^*, \Psi].$$

Since $\Psi$ and $\Psi^*$ are the linearly independent solution of (2), the Wronskian, $W[\Psi^*, \Psi]$, will be constant (independent of $x$) of motion (see e.g., p. 48 in [2]). Therefore, if the (approximate) asymptotic forms of $\Psi_i, \Psi_r, \Psi_t$ in Eqs. (6b) and (10) are used they will yield the same value of flux $S$ as the exact wavefunctions will do. One can check this point for the present problem by recalling the Wronskians [11],

$$W[J_{+\nu}(z), J_{-\nu}(z)] = -\frac{2\sin \nu \pi}{\pi z}, \quad W[H^{(1)}_{\nu}(z), H^{(2)}_{\nu}(z)] = -\frac{4i}{\pi z}$$

and

$$[H^{(1)}_{iq}(z)]^* = -e^{\pi q} H^{(2)}_{iq}(z).$$

Thus, the various fluxes work out to be

$$J_i = \frac{\hbar k e^{2\pi q}}{\pi mq \sinh \pi q}, \quad J_r = \frac{\hbar k}{\pi mq \sinh \pi q}, \quad J_t = \frac{\hbar e^{\pi q}}{\pi ma}.$$ 

In obtaining the above results we have used the formula: $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$ [11]. Finally, the transmission and reflection co-efficients are found as

$$T(E) = \frac{J_t}{J_i} = [1 - e^{-2\pi q}], \quad R(E) = \frac{J_r}{J_i} = e^{-2\pi q}, \quad q = \sqrt{\frac{E}{\Delta}}, \quad \Delta = \frac{\hbar^2}{8ma^2}.$$ 

These co-efficients satisfy $T(E) + R(E) = 1$. Let us find the special values of $T(E)$ and $R(E)$ for consistency. We get $T(\infty) = 1, R(\infty) = 0; T(0) = 0, R(0) = 1$ which are physically feasible. Interestingly, these co-efficients (14) do not depend up on the parameter, $V_0$, and rightly so. Since any finite positive value of $V_0$ is expressible as $V_0 = e^{\pm b/a}$, and it would merely result in shifting the whole potential on the $x-$axis by a distance $\pm b$: $V(x) = -e^{(x\pm b)/a}$. 

5
In mathematical terms, when $\hbar \to 0 \Rightarrow q \to \infty$, the classical limits are obtained as $T_{\text{classical}}(E) = 1$, and $R_{\text{classical}}(E) = 0$ from Eq. (14). However, practically the classical limit is reached when an effective parameter, for example $\Delta$ (14), becomes extremely small. This in turn is materialized when the tunneling system becomes massive or/and a length scale found in the interaction (e.g., $a$ here) becomes extremely large. In the cases [3-10], where a potential offers a barrier of a height, $V_0$, the classical limit of $T(E)$ is a step function of energy: $T_{\text{classical}}(E < V_0) = 0, T_{\text{classical}}(E \geq V_0) = 1$. Since the exponential potential (1) does not offer any barrier to an incident particle in a classical sense, we have got $T_{\text{classical}}(E) = 1$.

Let us divide Eqs. (10) and (6b) by the coefficient of $e^{i(kx - Et/\hbar)}$ in (10) to re-write $\Psi(x, t)$ respectively as

$$\Psi(x, t) \approx e^{i(kx - Et/\hbar)} - \left( e^{-\pi q}(p/2)^{-2iq} \frac{\Gamma[1 + iq]}{\Gamma[1 - iq]} \right) e^{-i(kx + Et/\hbar)}, \quad x \to -\infty, \quad(15a)$$

$$\Psi(x, t) \approx \left( \frac{2}{\pi p} e^{-\pi/4}(p/2)^{-iq} \sinh q \pi \Gamma[1 + iq] e^{-q \pi/2} \right) e^{-\frac{\pi}{2} x} \exp\left[i\left(p e^{\frac{x}{a}} - Et/\hbar\right)\right], \quad \text{when } x \to \infty. \quad(15b)$$

The multiplicative factor of $e^{-i(kx + Et/\hbar)}$ in Eq. (15a) is called reflection amplitude, $r(E)$ and the factor written in big brackets in Eq. (15b) is called transmission amplitude.

$$r(E) = e^{-q \pi}(p/2)^{-2iq} \frac{\Gamma[1 + iq]}{\Gamma[1 - iq]}, \quad t(E) = \frac{2}{\pi p} e^{-\pi/4} e^{-\pi q/2}(p/2)^{-iq} \sinh \pi q \Gamma(1 + iq). \quad(16)$$

Generally, $R(E) = |r(E)|^2$, however, $T(E) = |t(E)|^2$ holds only if the potential converges to the same asymptotic value on both the sides of the barrier. Since here it is not the case (see Fig. 1), the expression of $T(E)$ as obtained in Eq. (14) by using the ratio of transmitted to incident flux, is more fundamental and one finds that $|t(E)|^2$ is proportional to (if not identical) to $T(E)$. Let us define $\text{Arg}(p/2)^{iq} = \alpha$ and $\text{Arg}(1 + iq) = \beta$. For the exponential potential barrier, we get

$$\phi(E) = -2\alpha + 2\beta \pm \pi \quad \text{and} \quad \theta(E) = -\alpha + \beta - \pi/4. \quad(17)$$

When a particle is transmitted through a potential from one side to the other, it sees the total potential in every detail. Eventually, detailed balancing takes place to give rise to reciprocity of transmission: it does not matter if the particle is incident from left or right the transmission amplitude remains the same (invariant), i.e., $t_{\text{left}}(E) = t_{\text{right}}(E)[1, 2]$. 6
This implies the invariance of both transmission probability i.e., \( T_{\text{left}}(E) = T_{\text{right}}(E) \) and transmission phase-shift i.e., \( \theta_{\text{left}}(E) = \theta_{\text{right}}(E) \). Further, due to the conservation of flux invariance of reflection probability also takes place, namely, \( R_{\text{left}}(E) = R_{\text{right}}(E) \).

The behaviour of \( \phi - \theta \) with respect to the side of the incidence the particle is known (p. 109 [1]) as.

\[
[\phi_{\text{left}}(E) - \theta_{\text{left}}(E)] = (2n + 1)\pi - [\phi_{\text{right}}(E) - \theta_{\text{right}}(E)], \quad n = 0, 1, 2, ...
\]  

Interestingly, it turns out that if the potential is real and symmetric, we have \( \phi_{\text{left}} = \phi_{\text{right}} \) [12], and hence \( \phi - \theta = \pi/2 \). The present potential being non-symmetric provides an opportunity for a simple pedagogic demonstration of the result in Eq. (18) and the invariance of \( T(E), \theta(E) \) and \( R(E) \) with respect to the side from which the particle is incident on the potential.

In the following, we proceed to repeat the calculations when the side of the incidence has been changed from left to right (see Fig. 1(b)). Using the standard relations (7) : we find that

\[
2e^{-i\pi \nu} J_{\nu}(z) = e^{-2i\pi \nu} H_{\nu}^{(2)}(z) + H_{\nu}^{(1)}(z).
\]  

This identity suggests that

\[
\Psi(x, t) = 2e^{-q\pi} J_{-i\nu}(\frac{p}{2} e^{\frac{\pi i}{4}}) e^{-iEt/\hbar},
\]  

which in turn has an analytic behaviour for \( x \to -\infty \),

\[
\Psi(x, t) \approx 2e^{-q\pi} \frac{(p/2)^{-i\nu}}{\Gamma(1 - i\nu)} e^{-i(kx + Et/\hbar)}.
\]  

This represents a wave travelling to the left after being transmitted from the potential (see Fig. 1(b)). Using Eq.(9) and the asymptotic properties of \( H_{\nu}^{(1),(2)} \) as in Eq. (5), we find that

\[
\Psi(x, t) \approx 2\sqrt{\frac{2}{\pi p}} e^{i\pi/4} e^{-3\nu q/2} e^{-i(kx + Et/\hbar)} \left( e^{-i[\frac{\nu}{2} e^{\frac{\pi i}{4}} + Et/\hbar]} - i e^{-\nu q} e^{-i[\frac{\nu}{2} e^{\frac{\pi i}{4}} - Et/\hbar]} \right), \quad \text{when} \quad x \to \infty.
\]  

In the asymptotic form of the wave function (21), the first part denotes the incident wave, \( \Psi_i \), entering from the right side of the potential (see Fig.1(b)) and travelling to the left. The second part denotes oppositely travelling reflected wave, \( \Psi_r \). The transmitted wave, \( \Psi_t \), is represented by Eq.(19). One can recover, the results found in Eq. (14) by evaluating the flux arising due to these wave-forms when the side of incidence has been changed. The recovery of \( T(E) \) and \( R(E) \) despite the change of the side of the incidence of the particle
on the potential (see Fig. 1(b)) demonstrates their invariance with respect to the side of incidence of the particle whether it is left or right.

Once again, let us divide Eqs. (19b) and (20) by the co-efficient written outside the big bracket in Eq. (20) to get

$$\Psi(x,t) \approx \sqrt{\pi p^2 e^{-i\pi/4} e^{q\pi/2} \left(\frac{p/2}{\Gamma[1 - iq]}\right)^{-i(kx + Et/h)}}, \quad \text{when } x \to -\infty,$$

$$\Psi(x,t) \approx e^{-x^4 a} \left(e^{-i\left(\frac{p}{2} e^{\frac{q}{2} x^2} + \frac{Et}{\bar{h}}\right)} - ie^{-pq} e^{i\left(\frac{p}{2} e^{\frac{q}{2} x^2} - \frac{Et}{\bar{h}}\right)}\right), \quad \text{when } x \to \infty.$$  

(22a)  

(22b)

The amplitudes can be extracted as earlier:

$$r_{\text{right}}(E) = -ie^{-pq} \quad \text{and} \quad t(E) = \sqrt{\pi p^2 e^{-i\pi/4} e^{q\pi/2} \left(\frac{p/2}{\Gamma[1 - iq]}\right)}.$$  

(23)

Further, we find the phase-shifts as

$$\phi_{\text{right}}(E) = 3\pi/2 \quad \text{and} \quad \theta_{\text{right}}(E) = -\alpha + \beta - \pi/4$$  

(24)

to compare with phase-shifts given in Eq. (17), and this completes the demonstration of the general results given in Eq. (18).

It is further relevant to know that when the potential is complex, the reciprocity of the transmission amplitude with respect to the side of the incident particle still holds. However, reflection and absorption amplitudes and co-efficients depend on the side (left/right) of incident particle, if only, the total complex potential is non-symmetric, namely $V_{\text{complex}}(-x) \neq V_{\text{complex}}(x)$ [13].

The utility of the Schrödinger transmission in many physical processes can be appreciated as follows. Look at one body, $A$, at rest in Fig. 1(b) and let other body, $B$, come from a side with an energy, $E$. Assume that the interactions between $A$ and $B$ are of two types: short-ranged and long-ranged. Thus, the interaction barrier (e.g. Eq.1(b)) due to long-ranged interaction has to be overcome by the body $B$ in order to appear on the other side to have any worthwhile contact/reaction with $A$, characterized by their short-ranged interaction. The probability of the appearance of the body, $B$, on the other side is clearly proportional to the probability of transmission through the potential barrier. This suggests that the reaction rates of $A$ and $B$, would depend on the transmission probability, $T(E)$, apart from other factors.

In fact, barrier penetration method of determining the fusion rates (see [10] and Refs. therein) of nuclei is one such instance where fusion rates are determined by finding the
penetrability ($T(E)$) of the inter-nuclear potential barrier formed due to short-ranged nuclear attraction and Coulomb plus centrifugal repulsion.

In other kind of instances, a bound system (e.g. $\alpha$-particles in a nucleus [14] or electrons of a metal in the field emission [15]) faces a barrier and the emission rates are then determined by the transmission co-efficient of the barrier.

The analytically solvable, smooth and single-piece potential barriers mentioned in the introduction find applications in several physical processes. The extraction of transmission co-efficient for these barriers involves higher order functions. In this regard, the exponential barrier (1) presented here, despite being simple allows one to have the actual experience of extracting transmission (reflection) amplitude/co-efficient from a smooth, single-piece potential analytically. This problem may be practised right after doing rectangular well/barrier and before attempting the “higher-order” problems found in Refs. [3-10]. The exponential potential model also illustrates, with simplicity, several other aspects of Schrödinger transmission from one-dimensional, real potentials.

ACKNOWLEDGEMENTS

I would like to thank Dr Sudhir. R. Jain and Dr Abhas Mitra for critical reading of the manuscript.

REFERENCES

1. A. Messiah, *Quantum Mechanics* (North-Holland Publishing. Co., Amsterdam, 1962) vol. I, pp. 91-114.

2. E. Merzbacher, *Quantum Mechanics* (John Wiley & Sons, New York, 1970) Ch. 6, pp. 80-114.

3. S. Flügge, *Practical Quantum Mechanics* (Springer-Verlag, Berlin, 1971) vol. 1, Prob. No. 37 & 39.

4. D. Rapp, *Quantum Mechanics* (Holt,Rinhart & Wilson Inc., New York, 1970), Part III, Ch. 8, pp. 125-200; G. Barton, “Quantum mechanics of inverted harmonic oscillator potential”, Ann. Phys. (N.Y.) 166, 322 (1986); Z. Ahmed, “Tunneling through asymmetric parabolic barriers”, J. Phys. A: Math. Gen. (1997)) 3115-3116.
5. L.D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Pergamon Press, London, 1958).

6. P.M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Ltd., New York, 1953) pp. 1650-1660.

7. J.N. Ginocchio, “A class of exactly solvable 1: one-dimensional Schrödinger equation”, Ann. Phys. 152, 203 (1984); B.Sahu, S.K. Agarwalla and C.S. Shastry, “Above-barrier resonances : Analytic expression for energy and width”, J. Phys. A: Math. Gen. 35, 4349-4358 (2002).

8. A. Khare and U. P. Sukhatme, “Scattering amplitudes for supersymmetric shape-invariant potential by operator method”, J. Phys. A: Math. Gen. 21 L501-L508 (1988).

9. Z. Ahmed, “Tunneling through the Morse barrier”, Phys. Lett. A 157, 1-5, (1991).

10. Z. Ahmed, “Tunneling through a one dimensional potential barrier”, Phys. Rev. A, 47 4761-4757 (1993).

11. G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, (academic Press Inc., San Diego, 1995) pp. 627-664; M. Abramowitz and A. S. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1970) pp. 358-365.

12. B.F. Buxton and M.V. Berry, “Bloch wave degeneracies in systematic high energy electron diffraction”, Phil. Trans. R. Soc. London. Ser. A 282, 485-525 (1976) (see p.534); G. Barton, “Levinson theorem in one dimension : Heuristics”, J. Phys. A 18 497 (1985); Y. Nogami and C.Y. Ross, “Sacattering from a non-symmetric potential in one dimension : as a coupled channel problem”, Am. J. Phys., 64, 923 (1996).

13. Z. Ahmed, “Schrödinger transmission through one-dimensional complex potentials”, Phys. Rev. A 64 042716-(1-4) (2001).

14. e.g., H. A. Enge,*Introduction to Nuclear Physics* (Addison Wesley Publishing Co., Reading, Mass., 1966), pp. 274-295.

15. e.g., A.S. Davydov, *Quantum Mechanics* (Pergamon Press, New York, 1965).