ANALYTIC SURGERY AND GLUING OF THE BISMUT-LOTT TORSION FORM AND ETA FORM

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Abstract. Given a fiber bundle with closed connected fibers, and a family of separating hypersurfaces, we study the behavior of the Bismut-Lott analytic torsion form, and the eta form for a duality bundle, under analytic surgery in the sense of Hassell, Mazzeo and Melrose. We find that under the surgery limit, the rescaled heat kernel is non-singular, while both the Bismut-Lott analytic torsion form and eta form can be written as the sum of a logarithmic term, which satisfies the Igusa additivity property, the b- Bismut-Lott analytic torsion form (respectively the b- eta form), and an error term coming from the reduced normal operator. Hence we obtain a gluing formula for these invariants.

1. Introduction

The gluing problem, according to Mazzeo and Piazza [19], is the study of the behavior of some global invariants, with respect to decomposition of their underlying (family of) compact Riemannian manifolds. Early results in this direction include the gluing formula for the eta invariant by Bunke [6], and the analytic torsion by Vishik [30] (which is in turn used to prove the Cheeger-Müller theorem). Hassell, Mazzeo and Melrose, in [20,15,14] study the behavior of the spectrum and resolvent of the Laplacian using analytic surgery (see below), which as a by-product also prove gluing formula for eta invariant [15] and analytic torsion [14].

In this paper we consider the gluing problem of some higher invariants over a fiber bundle $Z \to M \xrightarrow{\pi} B$ with compact fibers without boundary, namely, the Bismut-Lott torsion form and eta form over a duality bundle.

1.1. Bismut-Lott torsion form. Throughout this paper, for each $b \in B$, we denote the fiber $\pi^{-1}(b)$ by $Z_b$ or simply $Z$.

For simplicity we shall assume here that $Z$ is odd dimensional and oriented, and denote the vertical Hodge star operator by $\varsigma_Z$. Let $D := d_V + L + \i_\Theta$ be the flat Bismut super-connection with respect to some splitting

$$TM = H \oplus V,$$

where $V := \text{Ker}(d\pi)$, and $D' := d_V^* + L' + \wedge_\Theta$ be its adjoint. To define the Bismut-Lott analytic torsion form one considers the rescaled super-connection

$$\overline{\partial}(t) := \frac{1}{2} t^{\frac{1}{2}} - \frac{N_H}{2} (D + D') t^{\frac{N_H}{2}}.$$

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and the rescaled heat kernel
\[ e^{-\bar{\partial}(t)^2} := t^{-\frac{N_H}{2}} e^{-\frac{t}{4} (D + D')^2} t^\frac{N_H}{2}. \]

**Definition 1.1.** The Bismut-Lott analytic torsion form \( T(M, g) \) is the even form on \( B \) defined by the explicit formula
\[
T(M, g)(b) := (2\pi i)^{-\frac{N_H}{2}} \int_0^\infty \int_{Z_b} \text{str} \left( \frac{N_H}{2} \left( 1 - 2\bar{\partial}(t)^2 \right) e^{-\bar{\partial}(t)^2} (p, p) \right) dp \, dt
\]
for horizontal degree \( \geq 1 \) (the Bismut-Lott torsion coincides with the Ray-Singer torsion at horizontal degree 0 and we shall only consider the horizontal degree \( \geq 1 \) case).

1.2. The eta form. We shall follow the setting of Bunke-Ma \[7\], simplified by Azzali-Goette-Schick \[1\]. Again let \( Z \to M \to B \) be a fiber bundle, for simplicity we shall assume that \( Z \) is even dimensional and oriented, and denote the vertical Hodge star operator by \( \varsigma_Z \). Let \( E \) be a vector bundle over \( M \), endowed with a flat connection \( \nabla_E \). We suppose \( E \) is further endowed with a duality bundle structure, namely, let \( \varepsilon \in \{1, -1\} \) be fixed, there exists a non-degenerate bilinear form \( Q \) on \( E \) such that:
1. \( Q \) is \( \varepsilon \)-symmetric: \( Q(v, w) = \varepsilon Q(w, v) \), \( \forall v, w \in E_p \);
2. \( Q \) is parallel under \( \nabla_E \): \( \nabla_E Q = 0 \).

Then there exists a bundle map \( J^E : E \to E \) (known as a metric structure) such that
1. \( (J^E)^2 = \varepsilon \);
2. \( Q(J^E v, J^E w) = Q(v, w) \), \( \forall (v, w) \);
3. \( g^E(v, w) := Q(v, J^E w) \) is an inner product.

We denote by \( V' \) the dual bundle of \( V \). On \( C^\infty(M; \wedge^* V' \otimes E) \), regarded as sections of the infinite dimensional Bismut bundle, the bi-linear form on
\[ Q_b(\omega \otimes u, \omega' \otimes u') := \int_M \omega \wedge \omega' Q( u, u') \]
is \( (-1)^{\frac{\dim Z (\dim Z + 1)}{2}} \varepsilon \)-symmetric. Define for any \( \xi \in V \)
\[ c(\xi) := \iota_\xi - \wedge_\xi : \wedge^* V' \to \wedge^* V', \]
and the chirality operator
\[ J^Z := c(\xi_1)c(\xi_2) \cdots c(\xi_{\dim Z}) \]
for any positively oriented orthonormal basis \( \{\xi_1, \cdots, \xi_{\dim Z}\} \), and let
\[ J := J^Z \otimes J^E. \]

Then \( Q_b(\cdot, J \cdot) \) is just the standard inner product of \( C^\infty(M; \wedge^* V' \otimes E) \).

Let \( D := d_V + L + \iota_\Theta \) be the flat Bismut super-connection as in the previous section, then
\[ D' = -J^{-1}d_V J + J^{-1}L J - J^{-1} \iota_\Theta J, \]
is the adjoint super-connection of \( D \) with respect to the inner product \( Q_b(\cdot, J \cdot) \). Again let
\[ \bar{\partial}(t) := \frac{1}{2} t^\frac{1}{2} \sqrt{-N_H} (D + D') t^\frac{N_H}{2}. \]
be the rescaled super-connection and define
$$X(t) := \frac{1}{2} t^2 \frac{N}{2} (D - D') t^{\frac{N}{2}}.$$

**Definition 1.2.** The eta form is defined by the formula

$$\eta(M, g)(b) := (2\pi i)\frac{-N}{2} \int_0^\infty \int_Z \text{tr} (J(\frac{d}{dt} \delta(t))) e^{-\delta(t)^2 (p, p)} dp dt.$$

Recall that
$$\frac{d}{dt} \delta(t) = \frac{1}{2} [N \nabla, X(t)],$$
it follows that

$$\eta(M, g)(b) = (2\pi i)\frac{-N}{2} \int_0^\infty \int_Z \frac{1}{2} \text{tr} (J[N \nabla, X(t)] e^{-\delta(t)^2 (p, p)}) dp dt.$$

1.3. **Motivation: the higher Cheeger-Müller conjecture and Igusa’s axiomatic description of torsion.** As an extension of the Cheeger-Müller theorem, Wagoner [31] conjectured that Reidemeister-Franz topological torsion (RF-torsion) [11, 27, 28] and Ray-Singer torsion (RS-torsion) [26] can be extended to invariants of a fiber bundle $Z \to M \to B$ with odd dimensional fibers endowed with a flat vector bundle $E$ over $M$. Bismut and Lott [5] confirmed the analytic part of Wagoner’s conjecture by constructing the Bismut-Lott analytic torsion form, as described in (1). When the dimension of the fiber is odd, one has

$$dT(M, g) = \text{Ch}^0(H^\bullet(Z, E), g)\nabla H^\bullet(Z, E)).$$

In particular, when the bundle is acyclic (i.e. $H^\bullet(Z, E) = 0$), $T(M, g)$ is closed, and its class in $H(B)$ is independent of the metric or connection [5, Corollary 3.25]. More generally, if $M \to B$ is unipotent and $E$ unitary, then there exists another covariantly flat metric on $H^\bullet(Z, E)$ and one can construct a form $T(H^\bullet(Z, E))$ such that

$$dT(H^\bullet(Z, E)) = \text{Ch}^0(H^\bullet(Z, E), g)\nabla H^\bullet(Z, E)).$$

The class

$$[T(M, g) - T(H^\bullet(Z, E))] \in H^\bullet(B)$$

[16, Definition 2.8], [25, Definition 2.1] is then an invariant known as the Bismut-Lott torsion class (BL-torsion) of $M \to B$.

On extending the RF-torsion, Igusa [16] constructed the Igusa-Klein torsion (IK-torsion), and Dwyer, Weiss and Williams [8] constructed another torsion form (DWW-torsion), using different topological methods. The IK-torsion and DWW-torsion are always closed topological invariants. It is therefore natural to conjecture that in the acyclic case the BL-torsion, IK-torsion and DWW-torsion are equal in $H^\bullet(B)$, thus extending the Cheeger-Müller theorem. Such conjecture was proposed as an open problem in a conference on higher torsion invariants in 2003 and is known as the higher Cheeger-Müller conjecture.

One useful tool in proving the higher Cheeger-Müller conjecture is the axiomatic characterization of torsion by Igusa [17]:

**Proposition 1.3.** Let $Z \to M \to B$ be an unipotent fiber bundle with orientable fiber $Z$. Any natural cohomology class $T(M \to B)$ that satisfies
(1) The transfer axiom: For any oriented sphere bundle $S^n \to \tilde{M} \to M$ one has
\[ T(\tilde{M} \to B) = \chi(S^n)T(M \to B) + \pi_*^{M \to B}(T(\tilde{M} \to M) \cup e(M)), \]
where $\pi_*^{M \to B}$ denotes the push-forward, i.e. fiber-wise integration and $e(M)$ denotes the Euler class);

(2) The additivity axiom: For any double $\hat{M}_\pm$ across any family of separating hypersurface,
\[ 2T(M \to B) = T(\hat{M}_+ \to B) + T(\hat{M}_- \to B); \]
is a sum of the Igusa-Klein torsion and the higher Miller-Morita-Mumford class.

Ohrt [22] generalizes Igusa’s axiom to case with bundle coefficients.
Badzioch, Dorabiala, Klein and Williams [2] showed that the DWW-torsion satisfies Igusa’s axioms, therefore the DWW-torsion equals to the IK-torsion; and the twisted version of DWW-torsion is constructed by Ohrt [23] and also shown to satisfy Ohrt’s axioms. As for the Bismut-Lott torsion, it follows from Ma [18] that it satisfies the transfer axiom. Hence the work on establishing the higher Cheeger-Müller conjecture focuses on proving the additivity axiom for the BL-torsion.

There has been some attempts along this line of research. Under the additional assumption that there exists a fiber-wise Morse function, Bismut and Goette [4] (see also [11, 12, 29]) prove a gluing formula that is equivalent to the additivity axiom. Zhu [32] considers some form of analytic surgery. That is, Zhu considers a family of separating hypersurface $Y \to X \to B$ in $Z \to M \to B$ that is a product near $X$, replaces a neighborhood of $X$ by a neck $(-R,R) \times X$ with length $2R$, and then computes the limit of the Bismut-Lott torsion form as $R \to \infty$. Under the additional assumption that $H^*(Y) = 0$, a gluing formula is proved. The assumption is used to prove that the Laplacian has a uniform spectral gap, which in turn implies the heat kernel part in convergence rapidly as $R \to \infty$. In the general case, some non-zero eigenvalues of the Laplacian converges to 0. Subsequently Puchol, Zhang and Zhu [24, 25] (see also [33, 10]) improve the method, and prove the general case, without the assumption $H^*(Y) = 0$. They do so by considering Witten deformation simultaneously with analytic surgery. After some sophisticated computation involving cutoffs and partition of unity, they prove that as both the neck length $R$ and the Witten deformation parameter $T$ goes to infinity in some compatible way, the Laplacian has a uniform spectral gap, and moreover the Bismut-Lott torsion form converges to a ‘model case’ they construct.

1.4. Analytic surgery and the content of this paper. One of the main motivations of this paper is to study the method of [24], using the aforementioned machinery of Hassell, Mazzeo and Melrose [15].

Let $Y \to X \to B$ be a family of separating hyper-surfaces of $Z \to M \to B$.

At first sight, one would need to generalize the construction of [15] to the fiber bundle case. However, both [11] and [3] are expressions of the square of the rescaled Bismut super-connection. Moreover, following the observation of [5] and [1], the square of the (rescaled) Bismut super-connection $\partial(t)^2$ is a fiber-wise operator, because one has
\[ (D + D')^2 = -(D - D')^2 \]
(in the DeRham grading, see [1, Remark 3.12]). Therefore it suffices to restrict to each fiber $Z$ and consider analytic surgery with separating hyper-surface $Y$, and we are back to [15].

Following [20, 15], one identifies $Y \times (-1,1)$ with a neighborhood of $Y$ in $Z$ and considers a family of metrics near $Y$ of the form

$$g_\epsilon = g_Y + \frac{dx^2}{\sqrt{x^2 + \epsilon^2}}, \quad \epsilon \in (0, \epsilon_0).$$

and studies the corresponding family of Laplacian operators $\Delta_\epsilon$. By considering kernels as sections ($D$-densities) on certain blowups of $Z \times Z \times [0, \epsilon_0)$, the surgery double space $Z^2_{Ls}$, one proves that the resolvent of $\Delta_\epsilon$ is then given by a smooth kernel [15]. Moreover, (see [15, Theroem 1.1]) it can be shown that the small eigenvalues of $\Delta_\epsilon$ converges to 0 as

$$(\text{ias } \epsilon)^2 z^2,$$

where ias $\epsilon := 1/\sinh^{-1}(1/\epsilon)$ and $z$ is one of the eigenvalues of the reduced normal operator $\text{RN}(d_\epsilon + d_\epsilon^*)$, and the collection of these $z$ is a discrete, periodic subset of $\mathbb{R} \subset \mathbb{C}$. Hence after blowing up the resolvent space one gets a spectral gap at 0, and the heat kernel can be constructed using holomorphic functional calculus. The leading terms of the resolvent and heat kernel are also computed in [15]. In Section 2 we shall briefly review these results and define our notations.

In Section 3 we turn to the rescaled Bismut super-connection. Since $(D + D')^2$ is a fiber-wise operator, its analytic surgery on each fiber, $(D_\epsilon + D'_\epsilon)^2$, is well defined. Moreover $(D_\epsilon + D'_\epsilon)^2$ differs from $\Delta_\epsilon$ by terms of horizontal degree $\geq 1$, therefore its resolvent can be written down using the resolvent of $\Delta_\epsilon$ and the Neumann series. In particular, the spectrum of $(D_\epsilon + D'_\epsilon)^2$ is the same as $\Delta_\epsilon$. After explaining the geometric settings in Sections 3.1 and 3.2, we consider the Neumann series in Section 3.3, expressing it as a sum of products of $(\Delta_\epsilon - \lambda^2)^{-1}$. Then in Sections 3.4 and 3.5, we compute explicitly the leading behavior of the resolvent of $(D_\epsilon + D'_\epsilon)^2$ at various faces, using the resolvent of $\Delta_\epsilon$. We study the rescaled heat kernel in Sections 3.6 and 3.7. It turns out that in the regime ias $\epsilon \to 0$, $t \to \infty$, $\tau = t(\text{ias } \epsilon)^2$ finite, the rescaling gives an extra factor of (at least) ias $\epsilon$ for each horizontal order, canceling the (ias $\epsilon)^{-1}$ factor arising from composition in the Neumann series, and the resulting limit is finite! So in this sense analytic surgery is compatible with adiabatic limits.

In Section 4 we apply analytic surgery to the Bismut-Lott torsion form. Hence we shall consider

$$T(M, g_\epsilon) := (2\pi i)^{-N/4} \int_0^\infty \int_Z \text{str} \left( \frac{N_V}{2}(1 - 2\partial_\epsilon(t)^2)e^{-\partial_\epsilon(t)^2}(p, p) \right) dp dt,$$

where the integrals here denote push-forward. Because the leading order of the integrand (5) is the same as that of the eta invariant [15, Section 9], the computation is a straightforward adaption of the arguments there. We summarize our computation with the following theorem.

**Theorem 1.4.** As ias $\epsilon \to 0$

$$T(M, g_\epsilon) = a_{0,1} \log(\text{ias } \epsilon) + a_{0,0}$$
modulo terms going to 0 as $\epsilon \to 0$, where the coefficient $a_{0,1}$ satisfies the additivity axiom, therefore

$$2T(M, g_\epsilon) - T(\hat{M}_+, g_\epsilon) - T(\hat{M}_-, g_\epsilon)$$

converges. Moreover, the constant term $a_{0,0}$ is the sum of the $(b\cdot)$ Bismut-Lott torsion form of $M_\pm$

$$(2\pi i)^{-\frac{N_p}{2}} \int_0^\infty \int_Z \text{tr} \left( \frac{N_p}{2}(1 - 2\delta_Z(t)^2)e^{-\delta_Z(t)^2(p, p)} \right) dt \frac{dt}{t}$$

(hence obviously satisfying the additivity axiom), and an error term

$$(2\pi i)^{-\frac{N_p}{2}} \int_0^\infty \int_{-1}^1 \text{tr} \left( \frac{N_p}{2}(-\tau)^{-K} \frac{1}{2} (\text{RN}(\Delta_\epsilon) - \tilde{\Omega}) (\text{RN}(\Delta_\epsilon) - \tilde{\Omega}) e^{-\frac{\tau}{2} (\text{RN}(\Delta_\epsilon) - \tilde{\Omega})} (s, s) \right) ds d\tau,$$

at horizontal degree $2K$.

Here $\text{RN}(\Delta_\epsilon)$ is the reduced normal operator constructed by Hassell, Mazzeo and Melrose [15]. The error terms is closely related to the ‘one dimensional model’ of Pu-chol, Zhang and Zhu [24], and seems to be computable using Witten deformation, thus simplifying their method.

In Section 5, as another application of analytic surgery, we consider analytic surgery of the eta form (3), i.e.,

$$\eta(M, g_\epsilon) := (2\pi i)^{-\frac{N_p}{2}} \int_0^\infty \int Z \text{tr} \left( J[N_V, X_\epsilon(t)] e^{-\delta_z(t)^2(p, p)} \right) dt \frac{dt}{t}$$

and compute its leading behavior as $\epsilon \to 0$. The computation is similar to the torsion form case.

**Theorem 1.5.** As $\epsilon \to 0$,

$$\eta(M, g_\epsilon) = b_{0,1} \log(\epsilon) + b_{0,0}$$

modulo terms going to 0 as $\epsilon \to 0$, where the coefficient $b_{0,1}$ satisfies the additivity axiom, therefore

$$2\eta(M, g_\epsilon) - \eta(\hat{M}_+, g_\epsilon) - \eta(\hat{M}_-, g_\epsilon)$$

converges. Moreover, the constant term $b_{0,0}$ is the sum of the $(b\cdot)$ higher form of $M_\pm$

$$(2\pi i)^{-\frac{N_p}{2}} \int_0^\infty \int Z \frac{1}{4} \text{tr} \left( J[N_V, t^\frac{1}{2} \frac{N_p}{2} (D_Z - D_Z') t^\frac{N_p}{2} e^{-\delta_Z(t)^2(p, p)} \right) dt \frac{dt}{t}$$

(hence obviously satisfying the additivity axiom), and an error term

$$(2\pi i)^{-\frac{N_p}{2}} \int_0^\infty \int_{-1}^1 \frac{1}{4} \text{tr} \left( JN_V [ - \Omega_1, (-\tau)^{-K} (\text{RN}(\Delta_\epsilon) - \tilde{\Omega}) e^{-\frac{\tau}{2} (\text{RN}(\Delta_\epsilon) - \tilde{\Omega})}] ds d\tau,$$

at horizontal degree $2K + 1$. 
2. A brief review of analytic surgery

2.1. Manifold with boundary. In this sub-section we recall some results in [21, Section 6]. Let \( Z \) be a manifold with embedded boundary. Denote the boundary of \( Z \) by \( Y \). Fix a collar neighborhood \( Y \times [0,1) \subset Z \) of \( Y \), and denote the corresponding boundary defining function by \( x \). We endow the interior of \( Z \) with the product metric

\[
g_Z = g_Y + \frac{dx^2}{x^2}
\]
on \( Y \times [0,1) \).

For simplicity, we assume that \( Z \) is oriented. This induces an orientation on \( Y \), and the Riemannian volume form on \( Y \times [0,1) \subset Z \) is given by

\[
\mu_Z = \frac{dx}{x} \wedge \mu_Y,
\]

where \( \mu_Y \) is the Riemannian volume form of \( Y \). We denote the Hodge star operator on \( Z \) and \( Y \) by \( \varsigma_Z \) and \( \varsigma_Y \) respectively.

The set \( \mathcal{V}_b \) of all smooth vector fields on \( Z \) tangential to \( Y \) is closed under Lie bracket. Indeed, these vector fields can be regarded as sections of a vector bundle, namely, the \( b \)-tangent bundle \( bTZ \) which is the vector bundle on \( Z \) given by

\[
bTZ = \text{Span} \{ x \frac{\partial}{\partial x} \} \oplus TY
\]
on \( Y \times [0,1) \) and isomorphic to the tangent bundle in the interior. The dual bundle of \( bTZ \) is the \( b \)-cotangent bundle, which we shall denote by \( bT^*Z \). Hence on \( Y \times [0,1) \)

\[
bT^*Z = \text{Span} \{ \frac{dx}{x} \} \oplus TY.
\]

2.2. Hodge theory for manifolds with boundary. Let \( E \) be a vector bundle over \( Z \) endowed with a flat connection \( \nabla^E \). On \( Y \times [0,1) \subset Z \) of \( Y \), one naturally identifies \( E \big|_{Y \times [0,1)} \) with \( \pi_Y^*E \big|_Y \) (where \( \pi_Y : Y \times [0,1) \to Y \) denotes projection to \( Y \) component) by parallel transport with respect to \( \nabla^E \) along \( [0,1) \). For simplicity we also endow \( E \) with a Hermitian metric \( h \), such that \( h \big|_{Y \times [0,1)} = \pi_Y^*h \big|_Y \). Here we recall the \( b \)-version of Hodge theory, which computes the \( E \)-valued absolute and relative DeRham cohomology. Define the (infinite) Sobolev space

\[
W^\infty(Z; E) := \left\{ u \in C^\infty(Z \setminus Y; E) : \nabla^E_{X_1} \cdots \nabla^E_{X_m} u \in L^2(Z; E), \forall X_1, \cdots, X_m \in C^\infty(Z; b^*TZ) \right\}.
\]

Set

\[
\mathcal{H}^k_Y := \{ u \in C^\infty(Y; \wedge^k T^*Y \otimes E) : \Delta_Y u = 0 \}
\]

\[
\tilde{\mathcal{H}}^k_Z := \bigcup_{\varepsilon > 0} \{ u \in x^{-\varepsilon} W^\infty(Z; \wedge^k bT^*Z \otimes E) : \Delta_Z u = 0 \}.
\]

By [21, Equation (6.56)], on \( Y \times [0,1) \) any elements in \( \tilde{\mathcal{H}}^k_Z \) can be written in the form

\[
u = u_{11} \log x + u_{12} + \frac{dx}{x} \wedge (u_{12} \log x + u_{22}) + u'
\]

for some \( u_{11}, u_{21} \in \mathcal{H}^k_Y, u_{12}, u_{22} \in \mathcal{H}^{k-1}_Y, u' \in x^\varepsilon W^\infty(Z; \wedge^k bT^*Z \otimes E), \varepsilon > 0 \). Hence one gets a well defined boundary data map

\[
\text{BD} : \tilde{\mathcal{H}}^k_Z \to (\mathcal{H}^k_Y \oplus \mathcal{H}^{k-1}_Y) \oplus (\mathcal{H}^k_Y \oplus \mathcal{H}^{k-1}_Y), \quad u \mapsto (u_{11}, u_{12}, u_{21}, u_{22}).
\]
The image of BD is a Lagrangian subspace of $(\mathcal{H}_Y^k \oplus \mathcal{H}_Y^{k-1}) \oplus (\mathcal{H}_Y^k \oplus \mathcal{H}_Y^{k-1})$. Denote by $\Lambda^D_k \subset \mathcal{H}_Y^k \oplus \mathcal{H}_Y^{k-1}$ (resp. $\Lambda^N_k$) the image of projection of the image of BD to the second (resp. first) $\mathcal{H}_Y^k \oplus \mathcal{H}_Y^{k-1}$ component. Hence $\Lambda^D_k$ and $\Lambda^N_k$ is the orthogonal complement of each other.

Recall that $\Lambda^D_k$ splits into

$$\Lambda^D_k = (\Lambda^D_k \cap (\mathcal{H}_Y^k \oplus \{0\})) \oplus (\Lambda^D_k \cap (\{0\} \oplus \mathcal{H}_Y^{k-1})).$$

We shall denote

$$A^k := \Lambda^D_k \cap \mathcal{H}_Y^k, \quad R^k := \Lambda^D_k \cap \mathcal{H}_Y^{k-1}$$

and the $L^2$, absolute and relative Hodge spaces by

$$\tilde{\mathcal{H}}^k_{L^2 Z} := \{u \in \tilde{\mathcal{H}}^k_Z : \text{BD} \ u = 0\}$$
$$\tilde{\mathcal{H}}^k_{\text{Abs} Z} := \{u \in \tilde{\mathcal{H}}^k_Z : \text{BD} \ u \in A^k\}$$
$$\tilde{\mathcal{H}}^k_{\text{Rel} Z} := \{u \in \tilde{\mathcal{H}}^k_Z : \text{BD} \ u \in R^k\}$$

respectively.

**Remark 2.1.** Note that the boundary data map and the Hodge spaces are well defined for any vector bundle $E$ endowed with a connection $\nabla^E$, not necessarily flat (c.f. [15 (3.23), (3.25)]). The generalized inverse construction in [21 Section 5] still holds, and one still has the Hodge decomposition [21 Equation (6.52)] for the $E$-valued case. In particular, if $E$ is flat then the arguments leading to the proof of [21 Proposition 6.14] generalizes trivially and one shows that the Hodge spaces are naturally isomorphic to the $E$-valued $L^2$, absolute and relative cohomology respectively.

### 2.3. Surgery blowup of a single manifold.

Let $Z$ be a closed, connected manifold without boundary, $Y \subset Z$ be an embedded hyper-surface, such that $Z \setminus Y$ has two connected components $Z = Z_+ \cup Z_-$. We identify $Y \times (-1, 1)$ with a neighborhood of $Y$ in $Z$, and we are interested in the family of metrics near $Y$

$$g_\epsilon = g_Y + \frac{dx^2}{\sqrt{x^2 + \epsilon^2}}, \quad \epsilon \in [0, \epsilon_0).$$

Thus one considers the manifold with corners $Z_s$ obtained by blowing up $Z \times [0, \epsilon_0)$ along $Y \times \{0\}$. Recall that blowing up is compatible with Cartesian product, hence near $Y$, $Z_s$ is $Y \times (-1, 1)_s$, where $(-1, 1)_s$ is the blow up of $(-1, 1) \times [0, \epsilon_0)$ along $\{(0, 0)\}$, and has boundary defining functions

$$\epsilon/x \quad (x > 0), \quad \sqrt{x^2 + \epsilon^2}, \quad -\epsilon/x \quad (x < 0).$$

Thus $Z_s$ has three faces. The new face added by blowing up $Y$ is (canonically) isometric to

$$\bar{Y} := Y \times [-1, 1],$$
endowed with the $b$-metric on both ends (i.e. $Y \times \mathbb{R}$ in the interior); and the original face becomes

$$\bar{Z} := \bar{Z}_+ \cup \bar{Z}_-$$

with each component endowed with the $b$-metric.
The logarithmic blowup $Z_{\log}$ is defined by changing the boundary defining functions of $(-1, 1)_s$ (in $Z_s$) to

\[ \text{ilg}(\pm \epsilon/x), \quad \text{ilg}(\sqrt{x^2 + \epsilon^2}), \]

or equivalently

\[ \text{ias}(\pm \epsilon/x), \quad \text{ias}(\sqrt{x^2 + \epsilon^2}), \]

where ilg and ias denote respectively the real functions $\epsilon \mapsto \frac{-1}{\log \epsilon}$ and $\epsilon \mapsto \frac{1}{\sinh^{-1}(1/\epsilon)}$.

Note that these two functions are smooth functions of each other and equal to the first degree.

**Definition 2.2.** The total boundary blowup $Z_{L_s}$ is defined by blowup of the corners of $Z_{\log}$.

Again, because blowing up is compatible with Cartesian product, near $Y$, $Z_{L_s}$ is just $Y \times (-1, 1)_{L_s}$.

**Notation 2.3.** We shall denote the boundary faces of $Z_{L_s}$ lifted from $\bar{Z}$ and $\bar{Y}$ by $F_0$ and $F_1$ respectively; And denote the new boundary added by blowing up $Z_{\log}$ to $Z_{L_s}$ by $F_2$ ($[15]$ denotes these surfaces by $B_0, B_1, B_2$ respectively).

Hence one has canonical isometries

\[ F_0 = Z_{\log}, \quad F_1 = \bar{Y}_{\log} \]

(with the cusp metric), and by $[15]$ (3.5) $F_2$ is a canonically trivial fiber bundle over $Y$.

Similar to the $b$-tangent bundle, the set $\mathcal{V}_{L_s}$ of all vector fields tangential to Ker$(d(\text{ilg} \epsilon))$ and tangential to $Y$ on $F_2 = Y \times [-1, 1]$ is closed under Lie bracket, and can be regarded as sections of a vector bundle, the $L_s$-tangent bundle $L_s TX$. Explicitly, using coordinates $\rho_2 = \xi, \rho_0 = \frac{\kappa}{\kappa + \xi}$ and $\rho_2 = \kappa, \rho_1 = \frac{\xi}{\kappa + \xi}$ to cover $F_2$, where $\xi := \text{ilg} x, \kappa := \text{ilg}(\epsilon/x)$, and let

\[ V_0 := (x^2 + \epsilon^2)^\frac{1}{2} \frac{\partial}{\partial x} \]

\[ = \left(1 + e^{-\frac{2(1-\kappa \rho_0)}{\rho_2}}\right)^\frac{1}{4} \left(\rho_2 \frac{\partial}{\partial \rho_2} - \rho_0 \rho_2 \frac{\partial}{\partial \rho_0}\right) = \left(1 + e^{-\frac{\rho_2}{\rho_1}}\right)^\frac{1}{4} \left(\rho_2 \frac{\partial}{\partial \rho_2} - \rho_1 \rho_2 \frac{\partial}{\partial \rho_1}\right). \]

Then $L_s TX$ is the vector bundle on $Z$ given by

\[ L_s TZ = TY \oplus \text{Span} \{V_0\} \]

near $Y \times [0, 1]_{L_s}$.

**Definition 2.4.** Following $[15]$ differential operators that are sums and compositions of vector fields in $\mathcal{V}_{L_s}$ and tensors are called $L_s$-differential operators.

Recall that the DeRham, Euler and Laplacian operators lift to $L_s$-differential operators $d_\epsilon, (d_\epsilon + d_\epsilon^\dagger), \Delta_\epsilon$ respectively. Here we recall the explicit form of $d_\epsilon$. 
Lemma 2.5. [15] Lemma 3.5] Near $F_1$ using coordinates $y$ for $Y$ and $\text{ias } \epsilon, r := \sinh^{-1} |x/\epsilon|$ for $[0, \pm 1]_{Ls}$,

\begin{align}
(13) & \quad d_e = dr \frac{\partial}{\partial r} + d_y; \\
(14) & \quad d_e = (\text{ias } \epsilon) ds \frac{\partial}{\partial s} + d_y.
\end{align}

Near $F_2$ using coordinates $y$ for $Y$ and $\text{ias } \epsilon, s := \text{ias } \epsilon/\text{ias } |x/\epsilon|$ for $[0, \pm 1]_{Ls}$.

2.4. Surgery double spaces. To carry the kernels of pseudo-differential operators, one considers the surgery double space.

Definition 2.6. Define $Z^2_{\log}$ to be the logarithmic blowup of

\[ Y \times Y \times \{0\}, Y \times Z \times \{0\}, Z \times Y \times \{0\} \subset Z \times Z \times [0, \epsilon_0) \]

(in that order) in $Z^2 \times [0, \epsilon_0)_\epsilon$. We shall denote these new faces by $F_{11}, F_{10}, F_{01}$ respectively. Then $Z^2_{Ls}$ is defined to be the blowup of all corners of $Z^2_{\log}$ with co-dimension $\geq 2$, in order of decreasing co-dimension.

Notation 2.7. We shall denote the new faces of $Z^2_{Ls}$ as follows: $F_{22}$ denotes the blowup of $F_{11} \cap F_{00} \cap (F_{10} \cup F_{01}); F_{21}, F_{12}, F_{20}, F_{02}, F_{33}$ respectively denote the blowup of $F_{11} \cap F_{01}, F_{11} \cap F_{10}, F_{10} \cap F_{00}, F_{01} \cap F_{00}, F_{11} \cap F_{00}$.

The corresponding boundary defining function shall be denoted by $\rho_{mn}$.

Recall that $F_{mn}$ is canonical diffeomorphic to some blow up of $F_m \times F_n$ (see [15] Proposition 4.1) for details.

Definition 2.8. The projections

\[ (p_1, p_2) \mapsto p_1, (p_1, p_2) \mapsto p_2, \quad p_1, p_2 \in Z \]

lifts to maps from $Z^2_{Ls}$ to $Z_{Ls}$ (see [15] Lemma 2.5, Corollary 2.6), which we shall denote by $\pi_L, \pi_R$ respectively. The lifted diagonal is defined to be

\[ \Delta_{Ls} := (\pi_L, \pi_R)^{-1}\{ (p, p) : p \in Z_{Ls} \}. \]

It can be shown [15] that $\Delta_{Ls}$ is canonically diffeomorphic to $Z_{Ls}$, and has faces

\[ \Delta_{00} := \Delta_{Ls} \cap F_{00} \cong F_0, \Delta_{11} := \Delta_{Ls} \cap F_{11} \cong F_1, \Delta_{33} := \Delta_{Ls} \cap F_{33} \cong F_2. \]

2.5. Degree and $D$-densities. Recall that $Z_{Ls}$ (resp. $Z^2_{Ls}$) is defined by blowing up the corners of $Z_{\log}$ (resp. $Z^2_{\log}$).

Definition 2.9. For each face of $Z^2_{Ls}$, $F_{mn}$, we define the degree $\deg(F_{mn})$ to be the co-dimension of the corner where $F_{mn}$ originates. Thus,

\[ \deg(F_{00}), \deg(F_{11}), \deg(F_{10}), \deg(F_{01}) = 1 \]
\[ \deg(F_{21}), \deg(F_{12}), \deg(F_{20}), \deg(F_{02}), \deg(F_{33}) = 2 \]
\[ \deg(F_{22}) = 3. \]

Similarly for $Z_{Ls}$, define

\[ \deg(F_0) = \deg(F_1) = 1, \quad \deg(F_2) = 2. \]
Let $N$ be a manifold with corners, $\Omega(N)$ be the density bundle of $N$. Hence $\Omega(N) = \text{Span}_{\mathbb{R}}(\nu)$, where $\nu$ is the Riemannian volume of $N$. Denote the faces of $N$ by $N_i$, with degree $\text{deg}(N_i)$ and (fixed) boundary defining function $\rho_i$. Recall that the $b$-density bundle of $N$ is $\Omega_b(N) := \prod_{N_i} \rho_i^{-\text{deg}(N_i)} \Omega(N)$ = $\prod_{N_i} \rho_i^{-\text{deg}(N_i)-1} \Omega(N)$.

**Definition 2.10.** The $D$-density bundle of $N$ is defined to be $\Omega_D(N) := \prod_{N_i} \rho_i^{-\text{deg}(N_i)} \Omega_b(N) = \prod_{N_i} \rho_i^{-\text{deg}(N_i)} \nu$.

Specializing to the case of $Z_{Ls}$ and $Z_{Ls}^2$, one has [15, Lemma 2.10] $\Omega_D(Z_{Ls}) = \beta^* \Omega_b(Z_s)$, $\Omega_D(Z_{Ls}^2) = \beta^* \Omega_b(Z_s^2)$.

**Definition 2.11.** Denote by $\pi_L, \pi_R : Z_{Ls}^2 \to Z_{Ls}$ the lift of the projections $(p_1, p_2) \mapsto p_1, (p_1, p_2) \mapsto p_2$ respectively. Let $E_1, E_2$ be vector bundles over $Z$. The kernel density bundle is the bundle $K(E_1, E_2) := \pi_L^* (F \otimes \Omega_D(Z_{Ls})^{-\frac{1}{2}}) \otimes \pi_R^* (E \otimes \Omega_D(Z_{Ls})^{-\frac{1}{2}})^t \otimes \Omega_D(Z_{Ls}^2)^{\frac{1}{2}}$

over $Z_{Ls}^2$.

**Definition 2.12.** [15 (4.18)] Let $K$ be an index set for $Z_{Ls}^2$, $k \in \mathbb{R}$. Define $\Psi^K(E_1, E_2)$ to be the space of all sections of $K(E_1, E_2)$ that is smooth on the interior and polyhomogeneous all boundaries with index $K$;

$\Psi^k(E_1, E_2)$ to be the space of (distributional) sections of $K(E_1, E_2)$ that is classical co-normal of order $k - \frac{1}{2}$ to the diagonal $\Delta_{Ls}$ and vanishes rapidly on all boundaries disjoint from $\Delta_{Ls}$. Elements of

$\Psi^k(E_1, E_2) + \Psi^K(E_1, E_2)$

are called $Ls$-pseudodifferential operators of order $k$ and index $K$.

2.6. **Surgery triple space and composition.** To analyze the composition of $Ls$-pseudodifferential operators one considers the $Ls$-triple space, which is defined similarly to $Z_{Ls}^2$. That is, one first defines $Z_{log}^3$ to be the logarithmic blowup

$Z_{log}^3 := [Z^3 \times [0, \epsilon_0]; Y \times Y \times Y \times \{0\},
Y \times Y \times Z \times \{0\}, Y \times Z \times Y \times \{0\}, Z \times Y \times Y \times \{0\},
Y \times Z \times Z \times \{0\}, Z \times Y \times Z \times \{0\}, Z \times Z \times Y \times \{0\}]_{log}$

and then $Z_{Ls}^3$ is defined to be the total logarithmic blowup of all corners of $Z_{log}^3$ with co-dimension $\geq 2$, in order of decreasing co-dimension.
Define maps \( \pi_S, \pi_C, \pi_F : Z^3 \to Z^2 \)

(15) \( \pi_S(p_1, p_2, p_3) := (p_2, p_3), \quad \pi_C(p_1, p_2, p_3) := (p_1, p_3), \quad \pi_F(p_1, p_2, p_3) := (p_1, p_2). \)

These maps lift to maps from \( Z^3_{1s} \) to \( Z^2_{1s} \), which we shall still denote by \( \pi_S, \pi_C, \pi_F. \)

The composition is then given by \([15\text{ Equation (4.26)}]\)

(16) \[ AB\nu = (\pi_C)_* \left( (\pi_C^* \nu)(\pi_F^* A)(\pi_S^* B) \right) \left| \frac{d\log \epsilon}{(\log \epsilon)^{\frac{1}{2}}} \right|^{rac{1}{2}} \]

for any section \( \nu \) of \( \Omega^1_{Ls} \).

For any kernel operator \( \Psi \in \Psi^k(E_1, E_2) + \Psi^K(E_1, E_2) \) with \( K \) minimal, denote the degree of \( \Psi \) at the face \( F \) by

\[ \deg_F(\Psi) := \min \{ z : (z, q) \in K(F) \}. \]

Recall that the degree of the composition of two operators at each face is described by the push forward formula.

**Lemma 2.13.** \([15\text{ Theorem 4.2}]\) For each face \( F' \) of \( Z^2_{1s} \)

\[ \deg_{F'}(\Psi_1 \circ \Psi_2) = \min_{\{ F \text{ face of } Z^3_{1s} : \pi_C(F) = F' \}} \left( \deg_{\pi_F(F)}(\Psi_1) + \deg_{\pi_S(F)}(\Psi_2) + \deg(F') - \deg(F) \right). \]

Observe that each face \( F \subset Z^3_{1s} \) is determined by the triplet \( (\pi_F(F), \pi_C(F), \pi_S(F)) \).

Computing \( \deg(F') - \deg(F) \) explicitly, case by case, one concludes with the following corollary.

**Corollary 2.14.** Consider \( \mathcal{F} := \{ (\pi_F(F), \pi_C(F), \pi_S(F)) : F \text{ face of } Z^3_{1s} \} \). The only triplets in \( \mathcal{F} \) containing \( F_{33} \) are

\( (F_{33}, F_{33}, F_{33}), (F_{22}, F_{33}, F_{22}), (F_{m2}, F_{m2}, F_{33}), (F_{33}, F_{2m}, F_{2m}), \quad m = 0, 1, 2; \)

and all triplets of the form

\( (F_{mi}, F_{mn}, F_{in}), \quad m, i, n = 0, 1, 2 \)

are in \( \mathcal{F} \). That exhausts the elements in \( \mathcal{F} \). Correspondingly

\[ \deg(\pi_C(F)) - \deg(F) \]

\[ = \left\{ \begin{array}{ll} -1 & \text{if } (\pi_F(F), \pi_C(F), \pi_S(F)) = (F_{m2}, F_{mn}, F_{2n}) \text{ or } (F_{22}, F_{33}, F_{22}) \\ 0 & \text{otherwise.} \end{array} \right. \]

2.7. **The reduced normal operator of the Dirac and Laplacian operator.** In this section we recall the reduced normal operator of the Dirac and Laplacian operator. Note that the dual of \( V_0 \) in \([12]\) is \( (x^2 + \epsilon^2)^{-\frac{1}{2}} dx \). Write

\[ \gamma := cl((x^2 + \epsilon^2)^{-\frac{1}{2}} dx), \]

Then \( \gamma \) acts on \( \mathcal{H}_Y \oplus \frac{dx}{\sqrt{x^2 + \epsilon^2}} \mathcal{H}_Y \) as \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

We refer to \([15]\) Propositions 3.7, 3.8 for details and simply take the result as our definition.
Definition 2.15. The reduced normal operators $\text{RN}(d_c + d_c^*)$, $\text{RN}(d_c - d_c^*)$ and $\text{RN}(\Delta_c)$ acts on $C^\infty([-1, 1], \mathcal{H}_Y \oplus \frac{dx}{\sqrt{x^2 + \epsilon}}\mathcal{H}_Y)$, and is given by

$$\begin{align*}
\text{RN}(d_c + d_c^*)u & = \gamma \frac{du}{ds} \quad \text{with boundary condition } u(\pm 1) \in \Lambda^D_+ \\
\text{RN}(\Delta_c)u & = -\frac{d^2 u}{ds^2} \quad \text{with boundary condition } u(\pm 1) \in \Lambda^D_+ \frac{du}{ds}(\pm 1) \in \Lambda^N_+.
\end{align*}$$

Recall that the eigenvalues of $\text{RN}(d_c + d_c^*)$ is periodic with period $\frac{\pi}{2}$, and there are $\dim \mathcal{H}_Y$ eigenvectors with eigenvalue in $(0, \frac{\pi}{2}]$. The eigenvectors of $\text{RN}(d_c + d_c^*)$ with eigenvalue $\alpha$ are of the form

$$e^{i\alpha s} \phi + e^{-i\alpha s} \psi,$$

where $\phi, \psi \in \mathcal{H}_Y \oplus \frac{dx}{\sqrt{x^2 + \epsilon}}\mathcal{H}_Y$ are such that

$$\gamma \phi = -i\phi \quad \gamma \psi = i\psi \quad S_+ S_- \phi = e^{4i\alpha} \phi \quad S_+ S_- \psi = e^{-4i\alpha} \psi.$$

Because $\Delta_c = (d_c + d_c^*)^2$, $\text{RN}(\Delta_c)$ has the same eigenvectors, but corresponding to eigenvalues $\alpha^2$.

Next, we recall the transfer of the reduced normal operators. Recall [15] Proposition 4.1 that for $(m, n) \neq (3, 3)$, $F_{mn}$ is diffeomorphic to a blow up of $F_m \times F_n$; and $F_{33}$ is diffeomorphic to $(Y \times [-1, 1]) \times (Y \times [-1, 1])$. We regard

$$e^{i\alpha s} \phi(y) + e^{-i\alpha s} \psi(y)$$

as a section on $Y \times [-1, 1]$, then projection is just the kernel on $F_{22}$:

$$(e^{i\alpha s'} \phi(y') + e^{-i\alpha s'} \psi(y'))(e^{i\alpha s} \phi(y) + e^{-i\alpha s} \psi(y)).$$

Definition 2.16. [15] p.183 The transfer of $\text{RN}(d_c + d_c^*)$ to $F_{22}$ is the kernel

$$\mathcal{T}_{22}(\text{RN}(d_c + d_c^*)) := \sum_\alpha \alpha(e^{i\alpha s'} \phi(y') + e^{-i\alpha s'} \psi(y'))(e^{i\alpha s} \phi(y) + e^{-i\alpha s} \psi(y)).$$

The transfer of the Laplacian $\mathcal{T}_{22}(\text{RN}(\Delta_c))$, its resolvent $\mathcal{T}_{22}(\text{RN}(\Delta_c - z)^{-1})$ and heat kernel $e^{-\tau \mathcal{T}_{22}(\text{RN}(\Delta))}$ are defined similarly. The transfer to other faces $F_{mn}$ are defined similarly.

2.8. The resolvent and heat kernel of the Laplacian. In this section we recall the resolvent of the Laplacian constructed in [15]. Loosely speaking, the resolvent of the Laplacian is a family of kernels parameterized by $\lambda \in \mathbb{C}$.

In [15] Section 6, it was shown by construction explicit by parametrix that the small eigenvalues of $\Delta_c$ are of the form $(i\alpha \epsilon)^2 x^2$, where $z$ are the eigenvalues of the reduced normal operator $\text{RN}(d_c + d_c^*)$ in the last section. Thus the spectrum set of $\Delta_c$, as $i\alpha \epsilon$ varies, lies in

$$Z^0_{\lambda \epsilon R} := \left[0, e_0 \log \times \mathbb{C}; \{0, 0\}\right].$$

As for the resolvent, one considers the blowup $F_{mn} \times \{0\}_\lambda$ inside $Z^2_{\lambda \epsilon} \times \mathbb{C}$ in order of decreasing codimension, using the parameter

$$z := \lambda / i\alpha \epsilon.$$


Notation 2.17. We shall denote this blow up space by $Z^2_{LsR}$. Its faces are either of the form

$$F^C_{mn} := (\mathbb{C} \setminus \{0\}) \times F_{mn}$$

or blow up of $\{0\} \times F_{mn}$, which we shall denote by $F^0_{mn}$. We define the degree of $F^C_{mn}$ and $F^0_{mn}$ to be the same as degree of $F_{mn}$.

With these blow up spaces [15] describes the resolvent. Recall [15, Lemma 4.7] that $\Delta_\varepsilon$ acts on kernels as

$$\Delta_\bar{Z}$$ on $F_{0m}$

$$\Delta_Y$$ on $F_{1m}$

$$\Delta_\bar{Y}$$ on $F_{2m}$

$$\Delta_Y$$ on $F_{3m}$,

where $\Delta_\bar{Z}$ and $\Delta_Y$ denote respectively the (cusp) Laplacian on $\bar{Z}_{log}$ and $\bar{Y}_{log}$. With these notations one has:

Proposition 2.18. [15, Theorem 7.2] The resolvent $(\Delta_\varepsilon - \lambda^2)^{-1}$ lifts to an $Ls$-pseudodifferential operator in $\Psi^{-2,0}(Z^2_{LsR}) + \lambda^{-2}\Psi^{-\infty,0}(Z^2_{LsR})$, with leading behavior

$$(17) \quad e^{-t\Delta_\varepsilon} = \frac{1}{2\pi i} \int_{\Gamma_0} e^{-t\lambda^2} (\Delta_\varepsilon - \lambda^2)^{-1} 2\lambda d\lambda + \frac{1}{2\pi i} \int_{\Gamma} e^{-t\lambda^2} (\Delta_\varepsilon - \lambda^2)^{-1} 2\lambda d\lambda,$$

where $\Gamma \subset Z^0_{LsR}$ is a codimension one $p$-submanifold, a contour for each ias $\varepsilon$, satisfying [15 Section 8.2 (i)-(iv)]:

(1) $\Gamma$ lies in the right half plane;
Through the Hodge isomorphism, the image of $\pi$ is isomorphic to sections extending to smooth sections on $H$ while the image of $\phi$ is isomorphic to sections extending to smooth sections on $H$. It follows from exactness that $\Pi$ is disjoint from the poles of $(\Delta_\epsilon - \lambda^2)^{-1}$.

On the other hand, $\Gamma_0$ is an arbitrary family of small circles centered at 0, enclosing only the very small eigenvalues of $\Delta_\epsilon$ for $\iota \epsilon > 0$ (c.f. [15, Section 6.10]) and the zero eigenvalue of $\text{RN}(\Delta_\epsilon)$.

Remark 2.20. Here we describe the very small eigenvalues when $\Delta_\epsilon = (d_\epsilon + d_\epsilon^* + d_\epsilon^* g)^2$. Recall that $Y$ divides $Z$ into $Z_\pm$. Denote by $A_\pm^*, R_\pm^*$ corresponding to $Z_\pm$ as in [10]. Following [14, Section 4], one observes that the multiplicity of the zero eigenvalues of $\text{RN}(\Delta_\epsilon)$ equals

$$\dim(A_+^* \cap A_-^*) + \dim(R_+^* \cap R_-^*) + \dim(\mathcal{H}_{L^2Z^+}) + \dim(\mathcal{H}_{L^2Z^-}).$$

For each $\epsilon \geq 0$, Hodge decomposition identifies the absolute Hodge space [11] with the absolute DeRham cohomology

$$\mathcal{H}_{Abs}^* (Z_\pm, E) \cong H_{Abs}^* (Z_\pm, E), \quad (\epsilon = 0),$$

$$\text{Ker}(\Delta_\epsilon) \cong H^* (Z, E), \quad (\epsilon > 0).$$

On the other hand one has the Mayer-Vietoris sequence

$$\to H^{*-1} (Y, E) \to H^* (Z, E) \to H_{Abs}^* (Z_+^*, E) \oplus H_{Abs}^* (Z_-^*, E) \to H^* (Y, E) \to .$$

Through the Hodge isomorphism, the image of $H^* (Z, E)$ in $H_{Abs}^* (Z_+^*, E) \oplus H_{Abs}^* (Z_-^*, E)$ is isomorphic to

$$(A_+^* \cap A_-^*) \oplus \mathcal{H}_{L^2Z^+} \oplus \mathcal{H}_{L^2Z^-},$$

while the image of $H_{Abs}^* (Z_+^*, E) \oplus H_{Abs}^* (Z_-^*, E)$ in $H^* (Y, E)$ is

$$A_+^* + A_-^* = (R_+^* \cap R_-^*)^\perp.$$

It follows from exactness that

$$\dim(A_+^* \cap A_-^*) + \dim(R_+^* \cap R_-^*) + \dim(\mathcal{H}_{L^2Z^+}) + \dim(\mathcal{H}_{L^2Z^-}) = \dim H^* (Z, E),$$

which is independent of $\epsilon$. Hence we conclude that the very small eigenvalues are identically zero and the projector onto zero eigenspace, $\Pi_\epsilon$, has constant rank and therefore is a smooth kernel.

To describe $\Pi_\epsilon$ explicitly, observe that every element $\psi^0 \in A_+^* \cap A_-^*$ extends to a smooth section on $Z$ by $\mathcal{H}_{Abs}^* (Z_\pm)$, and similar for $\varphi^0 \in R_+^* \cap R_-^*$. In turn, these sections extend to smooth sections on $Z_{L^2}$ annihilated by $\Delta_\epsilon$. Hence one can construct $\{\phi_i, \psi_j, \varphi_k\}$ such that

- For any $\epsilon > 0$, $\{\phi_i, (\iota \epsilon)^{\frac{1}{2}} \psi_j, (\iota \epsilon)^{\frac{1}{2}} \varphi_k\}$ is an orthonormal basis of the null space of $\Delta_\epsilon$;
- At $\epsilon = 0$, $\{\phi_i\}, \{\psi_j|_{Z_\pm}\}, \{\varphi_k|_{Z_\pm}\}$ are orthonormal bases of $\mathcal{H}_{L^2Z^+} \oplus \mathcal{H}_{L^2Z^-}$.

$\{u \in \mathcal{H}_{Abs}^* (Z_\pm) : BD u \in A_+^* \cap A_-^*\}$ and $\{u \in \mathcal{H}_{Rel}^* (Z_\pm) : BD u \in R_+^* \cap R_-^*\}$ respectively.
Then \( \Pi_\epsilon \) is the pull back to \( Z^2_{Ls} \) of
\[
\sum_i \phi_i(y)\phi_i(y') + (\text{ias } \epsilon) \sum_j \psi_i(y)\psi_j(y') + (\text{ias } \epsilon) \sum_k \varphi_k(y)\varphi_k(y') .
\]
We shall also denote
\[
\Pi_{L2} := \sum_i \phi_i(y)\phi_i(y'), \quad \Pi_0 := \sum_j \psi_i(y)\psi_j(y') + \sum_k \varphi_k(y)\varphi_k(y').
\]

We turn to construct an appropriate blowup space to carry the integrand of (17).

**Definition 2.21.** The heat-resolvent (double) space \( Z^2_{LsHR} \) is defined by blowing up \( \{0\} \times F^\infty_{mn} \) and then \( \{0\} \times F^t_{mn} \) in \( C \times Z^2_{LsH} \), in order of decreasing degree.

We denote the faces of \( Z^2_{LsHR} \) lifted from the old faces \( C \setminus \{0\} \times F^\infty_{mn} \) and \( C \setminus \{0\} \times F^t_{mn} \) by
\[
F^\infty_{mn}, F^t_{mn}, \quad F^\infty_{mn}, F^t_{mn},
\]
and the new faces by
\[
F^\infty_{mn}, F^t_{mn}, \quad F^\infty_{mn}, F^t_{mn}
\]
respectively.

The projections
\[
C \times Z^2_{Ls} \times R \to C \times Z^2_{Ls} \text{ and } C \times Z^2_{Ls} \times R \to C \times Z^2_{Ls} \times R
\]
lift to projections
\[
\pi_{\text{Res}} : Z^2_{LsHR} \to Z^2_{LsR}, \text{ and } \pi_{\text{Heat}} : Z^2_{LsHR} \to Z^2_{LsH}
\]
respectively. Also regarding \( e^{-\frac{\lambda^2}{4}} \) as a function on \( Z^0_{LsHR} \), then (17) is defined as the push forward
\[
(\pi_{\text{Heat}})_* \left[ (\pi_{\text{Res}}^*((\Delta_c - \lambda^2)^{-1} ) \left( 2\lambda^2 e^{-\frac{\lambda^2}{4}} \frac{dt}{(\log \epsilon)^{\frac{1}{2}}} \right)^{\frac{|d\lambda|}{(\log \epsilon)^{\frac{1}{2}}} } \right] = (\frac{d\lambda^2}{(\log \epsilon)^{\frac{1}{2}}} )^\frac{1}{2}
\]
Replacing \( Z^2_{Ls} \) by \( Z^3_{Ls} \) in Definition 2.21 one defines the heat-resolvent triple space \( Z^3_{LsHR} \). Clearly the projections \( \pi_S, \pi_C, \pi_F : Z^3 \to Z^2 \) in Equation (15) lifts to maps
\[
\pi_S, \pi_C, \pi_F : Z^3_{LsHR} \to Z^2_{LsHR},
\]
and composition (along fibers \( t, \lambda = \text{constant} \)) is well defined by (16). Moreover Lemma 2.13 still holds (because \( t, \lambda \) are just fixed parameters). In particular
\[
F := \{ (\pi_F(F), \pi_C(F), \pi_S(F)) : F \text{ face of } Z^3_{LsHR} \}
\]
contains
\[
(F^{I}_{33}, F^{I}_{33}, F^{I}_{33}), (F^{I}_{22}, F^{I}_{33}, F^{I}_{22}), (F^{I}_{m2}, F^{I}_{m2}, F^{I}_{22}), (F^{I}_{m3}, F^{I}_{2m}, F^{I}_{2m}), \quad m = 0, 1, 2;
\]
all triplets of the form
\[
(F^{I}_{mi}, F^{I}_{mn}, F^{I}_{in}), \quad m, i, n = 0, 1, 2,
\]
where \( I \) is any one of \( (\infty, 0), (\infty, C), (t, 0), (t, C) ; \) and also
\[
(F^\infty_{\infty}, F^\infty_{\infty}, F^\infty_{\infty}).
\]
Correspondingly
\[\text{deg}(\pi_C(F)) - \text{deg}(F) = \begin{cases} -1 & \text{if } (\pi_F(F), \pi_C(F), \pi_S(F)) = (F_{m2}, F_{mn}, F_{2n}) \text{ or } (F_{22}, F_{33}, F_{22}) \\ 0 & \text{otherwise.} \end{cases}\]

3. Analytic surgery of the rescaled heat kernel

3.1. The surgery blowup of a fiber bundle. Let \( Z \to M \xrightarrow{\pi} B \), \( Y \to X \to B \) be fiber bundles such that \( Y \) and \( X \) are respectively closed separating hyper-surfaces of \( Z \) and \( M \), and the following diagram commutes
\[
\begin{array}{c}
X \longrightarrow M \\
\downarrow \quad \downarrow \\
B \longrightarrow B
\end{array}
\]

We assume all these manifolds above are closed and connected. The hyper-surface \( X \) separates \( M \) into two connected components \( M_{\pm} \), and their closures are respectively given by \( M_{\pm} = M_{\pm} \cup X \). As in the last section, we fix a collar neighborhood \( X \times (-1, 1) \subset M \) that is compatible with the submersion above, i.e.,
\[
\begin{array}{c}
X \times \{0\} \longrightarrow X \times (-1, 1) \longrightarrow M \\
\downarrow \quad \downarrow \quad \downarrow \\
B \longrightarrow B \longrightarrow B
\end{array}
\]

We regard \( X \subset M \) as separating hyper-surface and perform blowup as in the last section. This process of blowing up is fiber-wise in the following sense. Observe that on \( X \times (-1, 1) \subset M \) the blowup is
\[ (X \times (-1, 1))_{\text{Ls}} = X \times (-1, 1)_{\text{Ls}} \to B \times [0, \epsilon_0) \]
and the blowup
\[ M_{\text{Ls}} \to B \times [0, \epsilon_0) \]
is obtained by gluing back \( M \setminus X \) away from \( X \). In particular, for any \( b \in B \), \( \pi^{-1}(\{b\} \times [0, \epsilon_0)) = Z_{\text{Ls}} \) is just the blowup of \( Z = \pi^{-1}(b) \).

3.2. The Bismut super-connection under surgery. On \( Z \to M \xrightarrow{\pi} B \), denote the vertical bundle by
\[ V := \text{Ker}(d\pi) \subset TM. \]

We fix a splitting
\[ TM = V \oplus H. \]

We shall without loss of generality further assume that in a collar neighborhood \( X \times (-1, 1) \subset M \), \( V \) further splits orthogonally into
\[ V = V_X \oplus T(-1, 1), \tag{20} \]
where \( V_X := V \cap TX = \text{Ker}(d\pi|_{TX}) \), and one has splitting
\[ TX = V_X \oplus H. \]
One identifies $H$ with the pullback bundle $\pi^*TB$, and endows $H$ with a metric $g_H$ by pulling back a Riemannian metric on $B$, then one fixes a Riemannian metric on $M$ of the form $g_M = g_V \oplus g_H$, where $g_V$ is a metric on $V$.

Let $E$ be a vector bundle over $M$, endowed with a flat connection $\nabla^E$. We shall denote the dual of a vector bundle by adding a prime. We shall denote by $N_V$ and $N_H$ respectively the vertical and horizontal degree on $\wedge^\bullet T^*M = \wedge^\bullet V' \otimes \wedge^\bullet H'$. The flat Bismut super-connection on $M$ is the de Rham exterior differential operator on $C^\infty(M, \wedge^\bullet T^*M \otimes E)$, split into degrees according to the identification $\wedge^\bullet V' \otimes \wedge^\bullet H' \otimes E \cong \wedge^\bullet T^*M \otimes E$:

$$D := d_V + L + \iota_\Theta,$$

where $d_V$ is the vertical de Rham differential, $L$ is the Bott connection, and $\iota_\Theta$ is the contraction with the $V$-valued horizontal 2-form $\Theta$ defined by

$$\Theta(X_1^H, X_2^H) := -P^V[X_1^H, X_2^H], \quad \forall X_1, X_2 \in \Gamma^\infty(TB).$$

Recall that in the Bismut-Lott case, the adjoint connection of the flat Bismut super-connection with respect to the standard inner product is

$$D' := d_V^* + L' + \wedge_\Theta,$$

where

$$d_V^* = -Z^{-1}d_V Z,$$

is the adjoint operator of $d_V$. In the eta form case the adjoint connection of the flat Bismut super-connection with respect to the inner product $Q_b(\cdot, J\cdot)$ is

$$D' := -J^{-1}d_V J + J^{-1}LJ - J^{-1}\iota_\Theta J$$

where

$$d_V^* = -J^{-1}d_V J,$$

is the adjoint operator of $d_V$ with respect to $Q_b(\cdot, J\cdot)$.

Consider the curvature of the Bismut super-connection

$$\frac{1}{2}(D + D').$$

It is straightforward to compute

$$(D + D')^2 = -(D - D')^2$$

$$= \Delta - (d_V - d_V^*)(L - L' + \iota_\Theta - \wedge_\Theta) - (L - L' + \iota_\Theta - \wedge_\Theta)(d_V - d_V^*)$$

$$- (L - L' + \iota_\Theta - \wedge_\Theta)^2.$$

Observe that $L - L'$ is a tensor, therefore $(D + D')^2$ is a fiber-wise operator and it suffices to consider each $Z$ and its lift to $Z_{Ls}$.

Remark 3.1. On $M_{Ls}$, $H$ is still a valid horizontal distribution. Moreover, because $V_X \oplus H$ is an integral distribution on $X \times (-1, 1)$, the image of $\Theta|_{X \times (-1, 1)}$ lies in $V_X$. Thus $L + \iota_\Theta$ is unaffected by the blowup; while $d_V$ lifts to the $Ls$-differential operator $d_{V, \epsilon}$ as described in Lemma 2.5. Therefore the corresponding family of flat Bismut super-connections is indeed (for each $\epsilon \geq 0$)

$$D_\epsilon = d_{V, \epsilon} + L + \iota_\Theta,$$
and its adjoint is
\[ D' = d_{V,\epsilon} + L + \wedge_\Theta. \]

It follows that
\[ D_{\epsilon} - D' = d_{V,\epsilon} - d'_{V,\epsilon} + L - L' + \iota_\Theta - \wedge_\Theta, \]
which is equivalent to the lift of the fiber-wise operator \( D - D' \) to \( Z_{LS} \).

Returning to \((D + D')^2 = -(D - D')^2\), it lifts to the \( Ls \)-differential operator
\[ (D_{\epsilon} + D'_{\epsilon})^2 = -(D_{\epsilon} - D'_{\epsilon})^2 \]
\[ = \Delta_{\epsilon} - (d_{V,\epsilon} - d'_{V,\epsilon})(L - L' + \iota_\Theta - \wedge_\Theta) - (L - L' + \iota_\Theta - \wedge_\Theta)(d_{V,\epsilon} - d'_{V,\epsilon}) \]
\[ - (L - L' + \iota_\Theta - \wedge_\Theta)^2, \]
where \( \Delta_{\epsilon} := (d_{V,\epsilon} + d'_{V,\epsilon})^2 \) is just the surgery Laplacian on each fiber.

Denote the number operators, \( N_V, N_H \), on \( \wedge^* H' \otimes \wedge^* V' \otimes E \) by
\[ N_V |_{\wedge^* H' \otimes \wedge^* V' \otimes E} := q, \quad N_H |_{\wedge^* H' \otimes \wedge^* V' \otimes E} := q'. \]
Then the rescaled operator,
\[ \partial(t)^2 := -t^{-\frac{NH}{2}} (D - D')^2 t^{\frac{NH}{2}}, \]
also lifts to the \( Ls \)-differential operator
\[ \partial_{\epsilon}(t)^2 := -t^{-\frac{NH}{2}} (D_{\epsilon} - D'_{\epsilon})^2 t^{\frac{NH}{2}}. \]

3.3. **The resolvent.** Since \(-(D_{\epsilon} - D'_{\epsilon})^2 \) differs from \( \Delta_{\epsilon} \) by terms of horizontal degree \( \geq 1 \), the resolvent of the former can be readily described by the resolvent of the later, using the Neumann series. Namely, set
\[ \Omega := L - L' + \iota_\Theta - \wedge_\Theta. \]

Then one has:

**Lemma 3.2.** The inverse
\[ (-(D_{\epsilon} - D'_{\epsilon})^2 - \lambda^2)^{-1} = (\Delta_{\epsilon} - (d_{V,\epsilon} - d'_{V,\epsilon})\Omega - \Omega(d_{V,\epsilon} - d'_{V,\epsilon}) - \Omega^2 - \lambda^2)^{-1} \]
exists as a kernel in \( Z_{LS}^{2} \) whenever \( (\Delta_{\epsilon} - \lambda^2)^{-1} \) exist, and is given by
\[ (\Delta_{\epsilon} - \lambda^2)^{-1} \]
\[ + (\Delta_{\epsilon} - \lambda^2)^{-1} \sum_{k=1}^{\dim B} \left( (d_{V,\epsilon} - d'_{V,\epsilon})\Omega + \Omega(d_{V,\epsilon} - d'_{V,\epsilon}) + \Omega^2 \right) (\Delta_{\epsilon} - \lambda^2)^{-1} \]

Next, we simplify the formula (22). Denote
\[ R_0 := \Omega^2, R_1 := \Omega(d_{V,\epsilon} - d'_{V,\epsilon}), R_2 := (d_{V,\epsilon} - d'_{V,\epsilon})\Omega. \]

Then the right hand side of (22) becomes
\[ \sum_k (\Delta_{\epsilon} - \lambda^2)^{-1} \left( (d_{V,\epsilon} - d'_{V,\epsilon})\Omega + \Omega(d_{V,\epsilon} - d'_{V,\epsilon}) + \Omega^2 \right) (\Delta_{\epsilon} - \lambda^2)^{-1} \]

Then for any $k$

$$\sum_{i_1, \ldots, i_k=0,1,2} (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_1} (\Delta_{\varepsilon} - \lambda^2)^{-1} \cdots (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1}.$$ 

Consider a term in (23) with the factor

$$\cdots R_1 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_2 \cdots.$$ 

We have

$$R_1 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_2 = \Omega(d_{V;\varepsilon} - d_{V;\varepsilon}^*) (\Delta_{\varepsilon} - \lambda^2)^{-1} (d_{V;\varepsilon} - d_{V;\varepsilon}^*) \Omega$$

$$= \Omega(\Delta_{\varepsilon} - \lambda^2)^{-1} (d_{V;\varepsilon} - d_{V;\varepsilon}^*)^2 \Omega$$

$$= \Omega(-1 - \lambda^2 (\Delta_{\varepsilon} - \lambda^2)^{-1}) \Omega$$

$$= -R_0 - \Omega(\lambda^2 (\Delta_{\varepsilon} - \lambda^2)^{-1}) \Omega.$$ 

Observe that for each term of the form

$$(\Delta_{\varepsilon} - \lambda^2)^{-1} R_2 \cdots R_2 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_1 \cdots R_1 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_{j+1}} \cdots R_{i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1}$$

in (23), there exists exactly one term in (23) of the form

$$(\Delta_{\varepsilon} - \lambda^2)^{-1} R_2 \cdots R_2 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_1 \cdots R_1 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_{j+1}} \cdots R_{i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1},$$

i.e., with the first occurrence of $R_0$ replaced by $R_1 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_2$, and vice versa. Adding the two one gets

$$(\Delta_{\varepsilon} - \lambda^2)^{-1} R_2 \cdots R_2 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_1 \cdots R_1 (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_{j+1}} \cdots R_{i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1},$$

where we denote

$$R_3 := -\Omega(\lambda^2 (\Delta_{\varepsilon} - \lambda^2)^{-1}) \Omega.$$ 

Repeating the same arguments for the sum over

$$(\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_{j+1}} \cdots R_{i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1},$$

one eliminates all terms involving $R_0$ and replaces them by $R_3$. It follows that (23) equals

$$\sum_{k=0}^{\infty} \sum_{i_1, \ldots, i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_1} (\Delta_{\varepsilon} - \lambda^2)^{-1} \cdots (\Delta_{\varepsilon} - \lambda^2)^{-1} R_{i_k} (\Delta_{\varepsilon} - \lambda^2)^{-1},$$

where the summation is over $i_1, \ldots, i_k = 0,1,2,3$ and such that if $i_j = 1, i_{j+1} \neq 2$.

### 3.4. Leading behavior at $P_{mn}^0$. In this sub-section we compute the leading term of (24) at $P_{mn}^0$. Note that (22) has no term in positive power of $\log(ias \varepsilon)$. Suppose we write such log term as

$$\sum_{j,k} H_{j,k}(ias \varepsilon)^j (\log(ias \varepsilon))^k.$$ 

Then for any $k \geq 1$

$$(\Delta_{\varepsilon} - (d_{V;\varepsilon} - d_{V;\varepsilon}^*) \Omega - (d_{V;\varepsilon} - d_{V;\varepsilon}^*) - \Omega^2 - \lambda^2) \sum_j H_{j,k}(ias \varepsilon)^j (\log(ias \varepsilon))^k = 0.$$
Applying \((\Delta_{\epsilon} - \lambda^2)^{-1}\) to the lowest horizontal degree component of the above equation one sees inductively that

\[ H_{j,k} = 0. \]

We begin with considering the term

\[(\Delta_{\epsilon} - \lambda^2)^{-1} R_3(\Delta_{\epsilon} - \lambda^2)^{-1} = -(\Delta_{\epsilon} - \lambda^2)^{-1} \Omega \lambda^2 (\Delta_{\epsilon} - \lambda^2)^{-1} \Omega (\Delta_{\epsilon} - \lambda^2)^{-1}. \]

Recall that near \(F^{0}_{mn}\), one uses the rescaled variable \(\lambda = z \text{ias} \epsilon\), and replaces \((\Delta_{\epsilon} - \lambda^2)^{-1}\) by its Taylor series as described in Lemma 2.13. Hence at \(F^{0}_{mn}\), (25) is given by the composition

\[-z^2 (\text{ias} \epsilon)^2 G_m \Omega G_{ii'} \Omega G_{\nu n}.\]

For \((m,n) \neq (0,0)\), the leading order term necessarily comes from \(i, i' = 2\), because the leading order of \((\Delta_{\epsilon} - z^2 (\text{ias} \epsilon)^2)^{-1}\) is \((\text{ias} \epsilon)^{-1}\), while by Lemma 2.13 each multiplication by \(G_{m2}, G_{22}\) adds one order of \((\text{ias} \epsilon)^{-1}\). Hence one gets the leading term

\[ z^2 (\text{ias} \epsilon)^2 G_{m2}^{-1} \Omega (\text{ias} \epsilon)^{-1} G_{22}^{-1} \Omega (\text{ias} \epsilon)^{-1} G_{2n}^{-1} \]
\[ = (\text{ias} \epsilon)^{-3} z^2 \mathcal{T}_{m2}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}) \Omega \mathcal{T}_{22}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}) \Omega \mathcal{T}_{2n}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}) \]
\[ = (\text{ias} \epsilon)^{-3} z^2 \mathcal{T}_{mn}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1} \Omega (\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}). \]

Similarly, when \((m,n) = (0,0)\), because the leading term of \(G_{00}\) is \((\text{ias} \epsilon)^{-2} z^{-2} \Pi_{L^2}\), the leading order term comes from

\[ z^2 (\text{ias} \epsilon)^2 G_{00} \Omega G_{00} \Omega G_{00}, \]

is of order \((\text{ias} \epsilon)^{-4}\), and one has

\[ z^2 (\text{ias} \epsilon)^2 (\text{ias} \epsilon)^{-2} G_{00}^{-2} \Omega (\text{ias} \epsilon)^{-2} G_{00}^{-2} \Omega (\text{ias} \epsilon)^{-2} G_{00}^{-2} \]
\[ = (\text{ias} \epsilon)^{-4} z^{-4} \Pi_{L^2} \Omega \Pi_{L^2} \Omega \Pi_{L^2}. \]

The arguments above can be repeated. We conclude with the following lemma.

**Lemma 3.3.** Near the face \(F^{0}_{mn}\), \((m,n) \neq (0,0),\)

\[(\Delta_{\epsilon} - \lambda^2)^{-1} (R_3(\Delta_{\epsilon} - \lambda^2)^{-1})^k \]
\[ = (-1)^k \lambda^{2k} (\Delta_{\epsilon} - \lambda^2)^{-1} (\Omega (\Delta_{\epsilon} - \lambda^2)^{-1})^{2k} \]
\[ = (-1)^k (\text{ias} \epsilon)^{-1 - 2k} z^{2k} \mathcal{T}_{mn}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1} (\Omega (\text{RN}(\Delta_{\epsilon}) - z^2)^{-1})^{2k}) \]

modulo lower order terms. Near the face \(F^{0}_{00},\)

\[(\Delta_{\epsilon} - \lambda^2)^{-1} (R_3(\Delta_{\epsilon} - \lambda^2)^{-1})^k \]
\[ = (-1)^k (\text{ias} \epsilon)^{-2 - 2k} z^{2k} \Pi_{L^2} (\Omega \Pi_{L^2})^{2k} \]

modulo lower order terms.

We take the sum of terms in Lemma 3.3 over all \(k\), and would like to consider the leading term for each fixed horizontal degree. Observe that the lowest horizontal degree component of

\[(\text{ias} \epsilon)^{-1 - 2k} z^{2k} \mathcal{T}_{mn}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1} (\Omega (\text{RN}(\Delta_{\epsilon}) - z^2)^{-1})^{2k}) \]
is of horizontal degree \(2k\), obtained by taking the degree 1 part of \(\Omega\), i.e.,

\[(\text{ias} \epsilon)^{-1 - 2k} z^{2k} \mathcal{T}_{mn}((\text{RN}(\Delta_{\epsilon}) - z^2)^{-1} (\Omega_1 (\text{RN}(\Delta_{\epsilon}) - z^2)^{-1})^{2k}), \]
where \( \Omega_1 := L - L' \) is the horizontal degree 1 part of \( \Omega \).

**Corollary 3.4.** For any \( K = 1, 2 \cdots \) fixed, the leading term of

\[
\sum_{k=1} (-1)^k (\text{ias } \epsilon)^{-1 - 2k} z^{2k} T_{mn} \left( (\text{RN}(\Delta_\epsilon - z^2)^{-1}(\Omega(\text{RN}(\Delta_\epsilon - z^2)^{-1} z^{2k}) \right),
\]

of horizontal degree \( 2K \) is

\[
(-1)^K (\text{ias } \epsilon)^{-1 - 2K} z^{2K} T_{mn} \left( (\text{RN}(\Delta_\epsilon - z^2)^{-1}(\Omega(\text{RN}(\Delta_\epsilon - z^2)^{-1} z^{2K}) \right); \tag{26}
\]

similarly the leading term of

\[
(-1)^k \sum_{k=1} (\text{ias } \epsilon)^{-2 - 2k} z^{-2 - 2k} II L^2(\Omega II L^2)^{2k}
\]

of horizontal degree \( 2K \) is

\[
(-1)^K (\text{ias } \epsilon)^{-2 - 2K} z^{-2 - 2K} II L^2(\Omega II L^2)^{2K}. \tag{27}
\]

We turn to terms in \( (24) \) involving \( R_1 \) or \( R_2 \). Here the simplest term is of the form

\[
(\Delta_\epsilon - \lambda^2)^{-1} R_1(\Delta_\epsilon - \lambda^2)^{-1}, \quad (\Delta_\epsilon - \lambda^2)^{-1} R_2(\Delta_\epsilon - \lambda^2)^{-1}.
\]

Consider

\[
(\Delta_\epsilon - \lambda^2)^{-1} R_1(\Delta_\epsilon - \lambda^2)^{-1} = (\Delta_\epsilon - \lambda^2)^{-1} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*)(\Delta_\epsilon - \lambda^2)^{-1}.
\]

Again we write \( (\Delta_\epsilon - (\text{ias } \epsilon)^2 z^2)^{-1} = \sum_{j=-2} (\text{ias } \epsilon)^j G^{(j)}_{mn} \) at \( F^0_{mn} \). Since the resolvent is also a parametrix, by Lemma 2.18 and 15 [Equation (6.2)], \( d_{V,\epsilon} - d_{V,\epsilon}^* \) annihilates \( G^{(-2)}_{00}, G^{(-1)}_{mn} \) for all \((m, n) \) and \( G^{(0)}_{mn} \) for \((m, n) \neq (0, 0), (1, 1), (3, 3) \). It follows that for \((m, n) \neq (0, 0) \) the top term comes from either

\[
(\text{ias } \epsilon)^{-1} G^{(-1)}_{m_2 n_2} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*)(\text{ias } \epsilon) G^{(1)}_{2n_2},
\]

\[
(\text{ias } \epsilon)^{-1} G^{(-1)}_{m_0 n_0} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*) G^{(0)}_{0n_0},
\]

\[
(\text{ias } \epsilon)^{-1} G^{(-1)}_{m_1 n_1} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*) G^{(0)}_{11},
\]

\[
(\text{ias } \epsilon)^{-1} G^{(-1)}_{m_2 n_2} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*) G^{(0)}_{33},
\]

\[
(\text{ias } \epsilon)^{-2} G^{(-2)}_{00} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*)(\text{ias } \epsilon) G^{(1)}_{0n_0}.
\]

Observe that all of these terms are of order \((\text{ias } \epsilon)^{-1}\). Similarly, at \( F^0_{00} \) the leading term is

\[
(\text{ias } \epsilon)^{-2} G^{(-2)}_{00} \Omega(d_{V,\epsilon} - d_{V,\epsilon}^*) G^{(0)}_{00}.
\]

Similar results hold for \((\Delta_\epsilon - \lambda^2)^{-1} R_2(\Delta_\epsilon - \lambda^2)^{-1} \). Repeating the arguments one observes that multiplying by \( R_1(\Delta_\epsilon - \lambda^2)^{-1} \) from the right or \((\Delta_\epsilon - \lambda^2)^{-1} R_2 \) from the left does not increase order of \((\text{ias } \epsilon)^{-1}\), hence one concludes with the following lemma.

**Lemma 3.5.** For any expression of the form

\[
(\Delta_\epsilon - \lambda^2)^{-1} R_2 \cdots R_2(\Delta_\epsilon - \lambda^2)^{-1} R_1 \cdots R_1(\Delta_\epsilon - \lambda^2)^{-1},
\]

the leading order is \((\text{ias } \epsilon)^{-2}\) on \( F^0_{00} \) and \((\text{ias } \epsilon)^{-1}\) on other faces; and its horizontal order is greater than or equal to the sum of number of \( R_1 \) and \( R_2 \) appearing in the expression.
It follows that for any fixed horizontal degree $2K, K = 1, 2, 3, \cdots$, terms in Lemma 3.3 is always of higher order that of (26), and therefore does not contribute to the leading term in the sum (21).

Since terms in (21) is a product of factors appearing in Lemmas 3.3 and 3.5, combing the proofs of these lemmas results in the following theorem.

**Theorem 3.6.** For any $K = 1, 2, 3, \cdots$, the leading term of the horizontal degree $2K$ part of (21) is given by Corollary 3.3, i.e.,

\[
(28) \quad (1)^{K}(\text{ias } \epsilon)^{-1-2K}z^{2K}\mathcal{T}_{mn}((\text{RN}(\Delta_{e}) - z^{2})^{-1}(\Omega_{1}(\text{RN}(\Delta_{e}) - z^{2})^{-1})^{2K})
\]

at $F_{mn}^{0}, (m, n) \neq (0, 0)$, and

\[
(29) \quad (1)^{K}(\text{ias } \epsilon)^{-2-2K}z^{-2-2K}\Xi_{1}(\Omega_{1}\Xi_{1})^{2K}
\]

at $F_{00}^{0}$.

### 3.5. Leading behavior at $F_{mn}^{c}$.

Recall Proposition 2.18 that the resolvent

\[
\Delta_{e} = -(d_{e} - d_{e}^{*})^{2},
\]

restricted to the faces $F_{mn}^{c}$ is just the resolvent of the restriction of $\Delta_{e}$ to the logarithmic blowup. The same result holds for $-(D_{e} - D_{e}^{*})^{2}$.

Denote by

\[
D_{\hat{Z}} := d_{\hat{Z}} + L + \iota\theta
\]

the flat Bismut super-connection on $\hat{Z} \to M_{+} \sqcup M_{-} \to B$ with respect to the splitting $TM = V \oplus H$ restricted to $M_{\pm}$, and by

\[
D_{\hat{Y}} := d_{\hat{Y}} + L + \iota\theta = drd_{m} + L + \iota\theta
\]

the flat Bismut super-connection on $Y \times \mathbb{R} \to X \times \mathbb{R} \to B$ with respect to the splitting $T(\mathbb{R} \times X) = (T\mathbb{R} \oplus V_{X}) \oplus H$.

Denote respectively by $D_{\hat{Z}}', D_{\hat{Y}}'$ their adjoint connections.

**Lemma 3.7.** The resolvent $-(D_{e} - D_{e}^{*})^{2} - \lambda^{2})^{-1}$ has leading behavior

\[
\begin{align*}
(\Delta_{\hat{Z}}^{c} - \lambda^{2} - \lambda^{2})^{-1} & + O(\text{ias } \epsilon) & \quad \text{at } F_{00}^{c} \\
(\Delta_{\hat{Y}}^{c} - \lambda^{2} - \lambda^{2})^{-1} & + O(\text{ias } \epsilon) & \quad \text{at } F_{11}^{c} \\
\Delta_{\hat{Y}}^{c} - \lambda^{2} - \lambda^{2})^{-1} & + O(\text{ias } \epsilon) & \quad \text{at other } F_{33}^{c}, F_{mn}^{c}
\end{align*}
\]

to infinite order of ias $\epsilon$.

**Proof.** By Proposition 2.18 $(\Delta_{e} - \lambda^{2})^{-1}$ vanishes to infinite orders on $F_{mn}^{c}$ for $(m, n) \neq (0, 0), (1, 1)$ or $(3, 3)$. Applying (22), it follows that the only possibly non-vanishing terms are

\[
\begin{align*}
\Delta_{\hat{Z}}^{c} - \lambda^{2} & \sum_{k=0}^{\dim B} \left( ((d_{\hat{Z}} - d_{\hat{Z}}^{*})\Omega + \Omega(d_{\hat{Z}} - d_{\hat{Z}}^{*}) + \Omega^{2})(\Delta_{\hat{Z}} - \lambda^{2})^{-1} \right)^{k} \\
\Delta_{\hat{Y}}^{c} - \lambda^{2} & \sum_{k=0}^{\dim B} \left( ((d_{\hat{Y}} - d_{\hat{Y}}^{*})\Omega + \Omega(d_{\hat{Y}} - d_{\hat{Y}}^{*}) + \Omega^{2})(\Delta_{\hat{Y}} - \lambda^{2})^{-1} \right)^{k}
\end{align*}
\]
\[ (\Delta \gamma - \lambda^2)^{-1} \sum_{k=0}^{\dim B} \left( (dY - d_Y^*)\Omega + \Omega(dY - d_Y^*) + \Omega^2 \right) (\Delta \gamma - \lambda^2)^{-1} \]

respectively on \( F_{00}^C, F_{11}^C, F_{33}^C \). Restricted to \( F_{00}^C \),

\[ -(D_\epsilon - D_\epsilon')^2 |_{F_{00}^C} = \Delta_Z + (D_Z - D_Z^*)\Omega + \Omega(D_Z - D_Z^*) + \Omega^2 = (D_Z + D_Z')^2. \]

Hence the first equation for \( F_{00}^C \) follows from the Neumann series. For \( F_{11}^C, F_{33}^C \) the argument is the same. \( \square \)

3.6. The rescaled heat kernel. Recall that the rescaled heat kernel is defined to be

\[ e^{-\frac{1}{t} \left(-\Delta - (D_\epsilon - D_\epsilon')^2\right)^{\frac{N\mu}{2}} \sum_{k=0}^{\dim B} \left( (dY - d_Y^*)\Omega + \Omega(dY - d_Y^*) + \Omega^2 \right) (\Delta \gamma - \lambda^2)^{-1} \]

where \( \Gamma_0 \) and \( \Gamma \) are the same contours as defined in (17). The leading behavior of the (rescaled) heat kernel can be easily read from that of the resolvent.

Consider the second term on the right hand side of (30) on \( F_{mn}^\infty, (m, n) \neq (0, 0) \). Since on \( \text{ias} \epsilon = 0 \), \( \Gamma \) intersects \( F_{mn}^\infty, F_{mn}^\infty \) of \( Z_{\text{LHHR}}^2 \), the non-rescaled heat kernel

\[ e^{-\frac{1}{4t} \left(-\Delta - (D_\epsilon - D_\epsilon')^2\right)^2} \]

is the sum of pushforward of these faces.

On \( F_{mn}^\infty, e^{-\frac{1}{4t} \left(-\Delta - (D_\epsilon - D_\epsilon')^2\right)^2} \) vanishes to infinite order of \( \text{ias} \epsilon \), therefore this face has no contribution.

On \( F_{mn}^\infty \), using the reparameterization \( \tau := (\text{ias} \epsilon)^2 t, z := \lambda / (\text{ias} \epsilon) \) and applying Theorem 3.6, the leading term of \( e^{-\frac{1}{4t} \left(-\Delta - (D_\epsilon - D_\epsilon')^2\right)^2} \) of horizontal order \( 2K \) thus becomes

\[ T_{mn} \left( \int_{\Gamma \cap F_{mn}^\infty} e^{-\frac{1}{4t} \left(-\Delta - (D_\epsilon - D_\epsilon')^2\right)^2} \right) \]

Then one rescales the heat kernel. For a term of horizontal degree \( k \), rescaling has the effect of multiplying a factor

\[ (\text{ias} \epsilon)^k \tau^{-\frac{k}{2}}. \]

Hence the leading term of the second term of (30) is

\[ \text{ias} \epsilon \sum_{K=0}^{\dim B} \left( \int_{\Gamma \cap F_{mn}^\infty} e^{-\frac{1}{4t} \left(-\Delta - (D_\epsilon - D_\epsilon')^2\right)^2} \right) \]

As for the case \( (m, n) = (0, 0) \), the contour integral

\[ \int_{\Gamma \cap F_{00}^\infty} (\text{ias} \epsilon)^{-2} (-\tau)^{-K} z^{-2-2K} \Pi_L^2 (\Omega_1 \Pi_L^2)^{2K} (\text{ias} \epsilon)^2 2zdz \]

vanishes because the contour does not enclose any pole.
We turn to the first term of (30). On ias \( e = 0 \), \( \Gamma_0 \) only intersects \( F_{mn}^{\infty,0} \) of \( Z_2^{\text{LHS}} \).

On \( F_{mn}^{\infty} \), \( (m,n) \neq (0,0) \), the leading term of \( e^{-\partial_z(t)^2} \) is similar to that of (32), but with a different contour,

\[
\lim_{K \to \infty} \sum_{K=0} e^{-\frac{t^2}{4}(\partial_z(t)^2)}(\Delta - \partial_z(t)^2)^{-1}(\Omega_1(\Delta - \partial_z(t)^2))^{2K}2zdz.
\]

As for the case \( (m,n) = (0,0) \), one computes directly

\[
\int_{0} e^{-\frac{t^2}{4}(\partial_z(t)^2)}(\partial_z(t)^2)^{-1}(\Omega_1(\partial_z(t)^2))^{2K}2zdz
\]

\[
=2(\partial_z(t)^2)(\partial_z(t)^2)^{-1}(\Omega_1(\partial_z(t)^2))^{2K}2zdz
\]

\[
=2(2\pi i)(\partial_z(t)^2)(\partial_z(t)^2)^{-1}(\Omega_1(\partial_z(t)^2))^{2K}2zdz
\]

\[
=2(2\pi i)(\partial_z(t)^2)(\partial_z(t)^2)^{-1}(\Omega_1(\partial_z(t)^2))^{2K}2zdz
\]

\[
=\frac{1}{K!}2(2\pi i)(\partial_z(t)^2)(\partial_z(t)^2)^{-1}(\Omega_1(\partial_z(t)^2))^{2K}2zdz
\]

We turn to the leading behavior of the rescaled heat kernel on \( F_{mn}^t \).

**Lemma 3.8.** Restricted to each fiber \( Z \), the heat kernel \( e^{-\frac{t^2}{4}(D_z-D_{\tau z}^2)} \) is a \( D \)-density on \( Z_{\text{LHS}}^2 \) that is \( t^{\frac{\dim Z}{2}} \times \) smooth near \( t = 0 \). It is smooth near other faces and has leading behavior

\[
e^{-\frac{t^2}{4}(D_z-D_{\tau z}^2)^2} \text{ at } F_{00}^t
\]

\[
e^{-\frac{t^2}{4}(D_{\tau z}^2-D_{\tau z}^2)^2} \text{ at } F_{11}^t
\]

\[
e^{-\frac{t^2}{4}(D_{\tau z}^2-D_{\tau z}^2)^2} \text{ at } F_{33}^t
\]

\[0 \text{ at other } F_{mn}^t\]

to infinite order.

**Proof.** The principal symbol of \( -(D - D^t)^2 \) is the same as that of \( \Delta \), therefore for finite \( t \) (i.e., away from \( F_{mn}^{\infty} \) and \( F_{\infty}^{\infty} \)), the Levi parametrix construction, as described in [20], is valid for \( -(D - D^t)^2 \), and one obtains a heat kernel on \( Z_2^2 \). Moreover the heat kernel is unique, it follows that the heat kernel \( e^{-\frac{t^2}{4}(D_z-D_{\tau z}^2)^2} \) is the pull-back of the heat kernel on \( Z_2^2 \). Furthermore we knew that the heat kernel on \( Z_2^2 \) has leading behavior

\[
e^{-\frac{t^2}{4}(D_z-D_{\tau z}^2)^2} \text{ at } F_{00}^t
\]

\[
e^{-\frac{t^2}{4}(D_{\tau z}^2-D_{\tau z}^2)^2} \text{ at } F_{11}^t
\]

\[0 \text{ at other faces}\]

up to an error of \( O(t) \). Hence (35) follows by pulling back. \( \square \)

Lastly, we consider the face \( F_{\infty} \) (i.e. \( t = \infty \)).

**Lemma 3.9.** Near \( F_{\infty} \), the rescaled heat kernel \( e^{-\partial_z(t)^2} \) is smooth and has leading term

\[
e^{-\frac{t^2}{4}(D_z^2-D_{\tau z}^2)^2}.
\]
Proof. Near this face $e^{-\frac{\tau z^2}{4}}$ is rapidly decreasing whenever the real part of $\lambda$ is greater than 0. Therefore the second term of (30) is of order (ias $\epsilon$)$^\infty$. The first contour integral of (30) is the residue of its integrand

$$
\sum_{k=0}^{\dim B} (\Delta_\epsilon - \lambda^2)^{-1}\left( (dV_{\epsilon, c} - dV_{\epsilon, c}^*)\Omega + \Omega (dV_{\epsilon, c} - dV_{\epsilon, c}^*) + \Omega^2) (\Delta_\epsilon - \lambda^2)^{-1}\right)^k
$$

at $z = \frac{1}{\text{ias } \epsilon} = 0$. One writes the resolvent of $\Delta_\epsilon$ as:

$$
(\Delta_\epsilon - \lambda^2)^{-1} = \frac{U}{\lambda} + ((1 - \Pi_\epsilon)\Delta_\epsilon (1 - \Pi_\epsilon) - \lambda^2)^{-1},
$$

and notes that $((1 - \Pi_\epsilon)\Delta_\epsilon (1 - \Pi_\epsilon) - \lambda^2)^{-1}$ is holomorphic at $\lambda = 0$.

Here, observe that $\Delta_\epsilon = (dV_{\epsilon, c} + dV_{\epsilon, c}^*)^2$, where $dV_{\epsilon, c}$ is the adjoint operator of $dV_{\epsilon, c}$ with respect to the standard inner product in the Bismut-Lott case, and $Q_b(\cdot, J\cdot)$ in the eta case for each $\epsilon \in [0, \epsilon_0)$. Again, Hodge decomposition identifies the absolute Hodge space with the absolute DeRham cohomology (with coefficients in $\wedge^*H^\prime \otimes E$). In particular the Mayer-Vietoris arguments as in Remark 2.20 leading to (18) still applies, and it follows that both terms on the right hand side of (38) are smooth up to ias $\epsilon = 0$. Therefore the first term of (30) is also smooth up to ias $\epsilon = 0$.

The limit of the rescaled heat kernel can furthermore be computed explicitly. Recall [1] Theorem 4.1] that for any ias $\epsilon > 0$, the rescaled heat kernel converges to

$$
e^{(\frac{1}{2} U_{\Omega, \Pi_0})^2}.
$$

Therefore by smoothness the same formula holds for ias $\epsilon = 0$. □

Remark 3.10. One can also directly compute the limit of the rescaled heat kernel at $\tau \to \infty$ using (33). Denote by $\Pi_0^{\text{RN}}$ the projection to the zero eigenspace of $\text{RN}(\Delta_\epsilon)$. Hence $\Pi_0 = \Pi_{mn}(\Pi_0^{\text{RN}})$ at $F_{mn}^\infty$ for $(m, n) \neq (0, 0)$. Then near $z = 0$ one has Laurent expansion

$$(\text{RN}(\Delta_\epsilon) - \lambda^2)^{-1} = z^{-2} \Pi_0^{\text{RN}} + \cdots$$

modulo terms holomorphic at $z = 0$. It follows that

$$(\text{RN}(\Delta_\epsilon) - z^2)^{-1}(\Omega_1(\text{RN}(\Delta_\epsilon) - z^2)^{-1})^{2K} = z^{-4K-2} \Pi_0^{\text{RN}} (\Omega_1 \Pi_0^{\text{RN}})^{2K}$$

plus terms with less negative power of $z$, hence for any $K' \geq 4K + 2$ one has

$$
\int_{|z| = c} z^{K'} (\text{RN}(\Delta_\epsilon) - z^2)^{-1}(\Omega_1(\text{RN}(\Delta_\epsilon) - z^2)^{-1})^{2K} dz = 0
$$

for any sufficiently small $c$. We expand

$$
\tau^{-K} z^{2K} (2z) e^{-\frac{z^2}{4}} = 2z \sum_{i=0}^{\infty} \frac{\tau^{i-K} z^{2i+2K}}{(-4)^i i!}.
$$

Combining with (39), the contour integral in (33) becomes

$$
\int_{\Gamma_0^{\infty} \cap F_{mn}^\infty} e^{-\frac{z^2}{4}} (-\tau)^{-K} z^{2K} (\text{RN}(\Delta_\epsilon) - z^2)^{-1}(\Omega_1(\text{RN}(\Delta_\epsilon) - z^2)^{-1})^{2K} 2z dz
$$
\[
\sum_{i=0}^{K} (-1)^K \tau^{-K} \int_{\Gamma_0 \cap F_{mn,0}} \frac{z^{2i+2K}}{(4\pi i)^N} (\text{RN}(\Delta_e) - z^2)^{-1} (\Omega_1(\text{RN}(\Delta_e) - z^2)^{-1})^{2K} 2zd\tau.
\]

As \( \tau \to \infty \), all but the \( i = K \) term vanishes, and when \( i = K \), the integrand is
\[
\frac{z^{4K}}{(4\pi i)^N} (\text{RN}(\Delta_e) - z^2)^{-1} (\Omega_1(\text{RN}(\Delta_e) - z^2)^{-1})^{2K} 2zd\tau = \frac{z^{4K}}{(4\pi i)^N} z^{-4K-2} \Pi_0^{\text{RN}}(\Omega_1 \Pi_0^{\text{RN}})^{2K} 2zd\tau
\]
modulo holomorphic terms, which after applying the transfer map \( T_{mn} \) is just the horizontal degree \( 2K \) part of \( e^{\frac{1}{2} \Pi(\Omega_1 \Pi_0)^2} \).

3.7. **The heat kernel of the reduced normal operator.** Equations (32), (33) suggest some holomorphic functional calculus involving \( \text{RN}(\Delta) \). In this sub-section we study these terms in detail.

Recall that near \( X \subset M \), the flat Bismut super-connection is
\[
D = dY + dx \frac{\partial}{\partial x} + L + \iota_{\Theta} = D_Y + dx \frac{\partial}{\partial x},
\]
where
\[
D_X := dY + L + \iota_{\Theta}
\]
is the flat Bismut super-connection of the fiber bundle \( Y \to X \to M \). In particular, observe that the \( L \) and \( \iota_{\Theta} \) components of \( D_X \) are the same as that of \( D \), but restricted to \( X \). The adjoint connection of \( D_X \) is
\[
D'_X := d^*_Y + L' + \wedge_{\Theta}.
\]
Again \( D_Y - D'_Y \) is a fiber-wise operator and one has
\[
(D_Y + D'_Y)^2 = -(D_Y - D'_Y)^2.
\]

The null space \( \mathcal{H}_Y^{\bullet \bullet} \) over each \( b \in B \) forms a vector bundle over \( B \). Denote by \( \Pi_Y \) the orthogonal projection from \( C^\infty(Y; \wedge^* V' \otimes \wedge^* H' \otimes E) \) onto \( \mathcal{H}_Y^{\bullet \bullet} \). Then \( \Pi_Y L \Pi_Y \) and its adjoint \( \Pi_Y L' \Pi_Y \) are flat connections on \( \mathcal{H}_Y^{\bullet \bullet} \). Define the tensor
\[
\tilde{\Omega} := \Pi_Y (L - L') \Pi_Y = \Pi_Y \Omega_1 \Pi_Y \in C^\infty(B; T^* B \otimes \text{Hom}(\wedge^* V' \otimes \wedge^* H' \otimes E)).
\]

Also recall that the reduced normal operator \( \text{RN}(\Delta) \) acts on \( \mathcal{H}_Y^{\bullet \bullet} \)-valued functions on \([-1, 1]\). Suppose that \( z^2 \) is not an eigenvalue of \( \text{RN}(\Delta) \), i.e., there is an unique solution of the boundary value problem
\[
(-\frac{d^2}{ds^2} - z^2)u = f, \quad u(\pm 1) \in \Lambda_D^{\pm}, \quad \frac{du(\pm 1)}{ds} \in \Lambda_N^{\pm}
\]
for any \( f \), and the solution is given by the resolvent \( \text{RN}(\Delta_e) - \lambda^2)^{-1}f \). Then by the same arguments of Lemma 3.2, the resolvent \( \text{RN}(\Delta_e) - \tilde{\Omega} - z^2)^{-1} \) exists whenever \( \text{RN}(\Delta_e) - z^2)^{-1} \) exists and is given by
\[
(\text{RN}(\Delta_e) - \tilde{\Omega} - z^2)^{-1} = \sum_{k=0} (\text{RN}(\Delta_e) - z^2)^{-1} (\tilde{\Omega}(\text{RN}(\Delta_e) - z^2)^{-1})^k.
\]

Because
\[
(\text{RN}(\Delta_e - z^2))^{-1} = \Pi_Y(\text{RN}(\Delta_e - z^2))^{-1} = (\text{RN}(\Delta_e - z^2))^{-1} \Pi_Y,
\]
comparing the horizontal degree \(2K\) part of \((30)\) with that of the integrand of \((32)\) and \((33)\), one sees that

\[
(41)\quad e^{-\frac{\tau^2}{4}}(-\tau)^{-K}z^{2K}(R\mathcal{N}(\Delta_\epsilon) - z^2)^{-1}(\Omega_1(R\mathcal{N}(\Delta_\epsilon) - z^2)^{-1})^{2K} \\
= (-\tau)^{-K}e^{-\frac{\tau^2}{4}}z^{2K}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega} - z^2)^{-1} \\
= 4^K\tau^{-K}\frac{d^K}{d\tau^K}(e^{-\frac{\tau^2}{4}})(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega} - z^2)^{-1}
\]
at horizontal degree \(2K\).

**Theorem 3.11.** On \(F^\infty_{mn}, (m, n) \neq 0\), the rescaled heat kernel has leading behavior

\[
e^{-\tilde{\Omega}(t)^2} = (\text{ias } e)^T \mathcal{N}_m \left((-\tau)^{-K}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega})K e^{-\frac{\tau}{4}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega})}\right)
\]
at order \(2K\).

**Proof.** The claim follows by integrating \((31)\) over \((\Gamma \cup \Gamma_0) \cap F^\infty_{mn}\) and then using \((32), (33)\). Because \((\Gamma \cup \Gamma_0) \cap F^\infty_{mn}\) encloses the spectrum of \(R\mathcal{N}(\Delta_\epsilon)\), and hence \(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega}\), the claim follows. \(\square\)

Theorem 3.11 leads us to study the asymptotic behavior of the heat kernel \(e^{-\tilde{\Omega}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega})}\) and its \(\tau\) derivatives. To begin with, recall \([15, (5.7)]\) that the heat kernel \(e^{-\frac{\tau}{4}R\mathcal{N}(\Delta_\epsilon)}\) is described by the reflection principle:

\[
(42)\quad e^{-\frac{\tau}{4}R\mathcal{N}(\Delta_\epsilon)}(s, s') = \frac{1}{\sqrt{\pi\tau}}e^{-\frac{(s-s')^2}{\tau}} \\
+ \frac{1}{\sqrt{\pi\tau}} \sum_{l=0}^{n} e^{-\frac{(4l+2-s-s')^2}{\tau}}(S_+S_-)^l \bigg| S_+ \\
+ \frac{1}{\sqrt{\pi\tau}} \sum_{l=0}^{n} e^{-\frac{(4l-2-s-s')^2}{\tau}}(S_-S_+)^l \bigg| S_- \\
+ \frac{1}{\sqrt{\pi\tau}} \sum_{l=1}^{n} e^{-\frac{(4l+s-s')^2}{\tau}}(S_+S_-)^l \\
+ \frac{1}{\sqrt{\pi\tau}} \sum_{l=1}^{n} e^{-\frac{(4l+s-s')^2}{\tau}}(S_-S_+)^l,
\]

where \(S_\pm: \mathcal{H}^\pm(Y) \rightarrow \mathcal{H}^\pm(Y)\) equals 1 in \(\Lambda^D_\pm\) and \(-1\) on \(\Lambda^N_\pm\) are the scattering matrices. Hence \(e^{-\frac{\tau}{4}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega})}\) can be obtained by Duhamel expansion \([3, \text{Theorem 9.48}]\)

\[
e^{-\frac{\tau}{4}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega})} = e^{-\frac{\tau}{4}R\mathcal{N}(\Delta_\epsilon)} \\
+ \sum_{k=1}^{n} \left(\frac{\tau}{4}\right)^k \int_{(r_0, \ldots, r_k) \in \Delta_k} e^{-\frac{\tau_0}{4}R\mathcal{N}(\Delta_\epsilon)} \tilde{\Omega} e^{-\frac{\tau_1}{4}R\mathcal{N}(\Delta_\epsilon)} \cdots \tilde{\Omega} e^{-\frac{\tau_k}{4}R\mathcal{N}(\Delta_\epsilon)} d\sigma_k,
\]

where \(\Delta_k := \{(r_0, \ldots, r_k) \in \mathbb{R}^{k+1} : r_i \geq 0, r_0 + \cdots + r_k \leq 0\}\) denotes the standard simplex. Since \(\tilde{\Omega}\) is of horizontal degree 1, the horizontal degree \(2K\) component of \(e^{-\frac{\tau}{4}(R\mathcal{N}(\Delta_\epsilon) - \tilde{\Omega})}\) is just the \(2K\)-th term of the Duhamel expansion above, i.e.,

\[
(43)\quad \left(\frac{\tau}{4}\right)^{2K} \int_{(r_0, \ldots, r_K) \in \Delta_K} e^{-\frac{\tau_0}{4}R\mathcal{N}(\Delta_\epsilon)} \tilde{\Omega} e^{-\frac{\tau_1}{4}R\mathcal{N}(\Delta_\epsilon)} \cdots \tilde{\Omega} e^{-\frac{\tau_K}{4}R\mathcal{N}(\Delta_\epsilon)} d\sigma_K.
\]
Hence, using \[ (41) \], it follows that the horizontal degree 2 component of
\[ (-\tau)^{-K}(RN(\Delta_\epsilon) - \tilde{\Omega})K e^{-\tilde{\tau}(RN(\Delta_\epsilon) - \tilde{\Omega})} \]
in Theorem 3.11 is \( O(\tau^{-\frac{2}{4}}) \) as \( \tau \to 0 \) for all \( K \).

4. Analytic Surgery of the Bismut-Lott Torsion Form

The analytic surgery limit of the Bismut-Lott torsion form is just the limit of the lift
\[ T(M, g_e) := (2\pi i)^{-\frac{N\mu}{2}} \int_0^\infty \int_{Z_b} \text{str} \left( \frac{N\mu}{2}(1 - \frac{1}{2}\tilde{\partial}_\epsilon(t)^2)e^{-\tilde{\partial}_\epsilon(t)^2}(p, p) \right) dp dt \]
as \( \text{ias} \) \( \epsilon \to 0 \). In other words, we restrict \( \frac{N\mu}{2}(1 - \frac{1}{2}\tilde{\partial}_\epsilon(t)^2)e^{-\tilde{\partial}_\epsilon(t)^2} \) to the lifted diagonal \( \Delta_{LHS} \), take point-wise trace and then integrate (push-forward) following [13] Section 9.

Next, we restrict to the lifted diagonal \( \Delta_{LHS} = \Delta_{Ls} \times [0, \infty]_\tau; (F_{33} \cap \Delta_{Ls}) \times \{ \infty \}; (F_{11} \cap \Delta_{Ls}) \times \{ \infty \}; (F_{00} \cap \Delta_{Ls}) \times \{ \infty \} \).

Hence \( \Delta_{LHS} \) has six boundary faces at \( \text{ias} \) \( \epsilon = 0 \): \( \Delta_{mm}^t, m = 0, 1, 3 \) are the old faces of \( \Delta_{Ls} \times [0, \infty) \), while \( \Delta_{mm}^\infty \) are the blowup of \( \{ \infty \} \times (F_{mm} \cap \Delta_{Ls}) \). By [13] p. 218, \( \Delta_{33}^t, \Delta_{33}^\infty \), \( \Delta_{00}^t, \Delta_{00}^\infty \) are of degree 2 and \( \Delta_{11}^t, \Delta_{11}^\infty \) are of degree 1. We consider their behavior under the push forward to \( [0, \text{ias} \epsilon_0] \) (note that the face \( \{ \text{ias} \epsilon = 0 \} \) is of degree 1).

We first consider the behavior of \( \text{str} \left( \frac{N\mu}{2}(1 - \frac{1}{2}\tilde{\partial}_\epsilon(t)^2)e^{-\tilde{\partial}_\epsilon(t)^2}(p, p) \right) \) at \( \Delta_{00}^t, \Delta_{11}^t, \Delta_{33}^t \) at \( t \to 0 \). To shorten notation, we denote
\[ \tilde{\partial}_Z(t) := t^{\frac{1}{2}} - \frac{N\mu}{2}(D_Z + D'_Z) t^{\frac{N\mu}{2}} \]
\[ \tilde{\partial}_Y(t) := t^{\frac{1}{2}} - \frac{N\mu}{2}(D_Y + D'_Y) t^{\frac{N\mu}{2}}. \]

These operators are the restriction of \( \tilde{\partial}_\epsilon(t) \) to the faces \( F_{00}^t \) and respectively \( F_{11}^t, F_{33}^t \). On \( F_{mm}^t \), applying the restriction of \( (1 - \frac{1}{2}(D_\epsilon + D'_\epsilon)^2) \) to Lemma 3.8 and rescale, one gets that the leading term of \( (1 - 2\tilde{\partial}_\epsilon(t)^2)e^{-\tilde{\partial}_\epsilon(t)^2} \) is
\[ (1 - 2\tilde{\partial}_Z(t)^2)e^{-\tilde{\partial}_Z(t)^2} \text{ at } F_{00}^t \]
\[ (1 - 2\tilde{\partial}_Y(t)^2)e^{-\tilde{\partial}_Y(t)^2} \text{ at } F_{11}^t \]
\[ (1 - 2\tilde{\partial}_Y(t)^2)e^{-\tilde{\partial}_Y(t)^2} \text{ at } F_{33}^t \]
and 0 at other \( F_{mn}^t \) to infinite order.

Next, we restrict to the lifted diagonal \( \Delta_{LHS} \) and take super-trace. First, consider \( \text{str} \left( \frac{N\mu}{2}(1 - 2\tilde{\partial}_\epsilon(t)^2)e^{-\tilde{\partial}_\epsilon(t)^2}(p, p) \right) \) at \( \Delta_{11}^t \) and \( \Delta_{33}^t \).

**Lemma 4.1.** One has
\[ \text{str} \left( \frac{N\mu}{2}(1 - 2\tilde{\partial}_\epsilon(t)^2)e^{-\tilde{\partial}_\epsilon(t)^2}(p, p) \right) = 0. \]

**Proof.** Observe that the interior of \( \bar{Y} \) is \( \mathbb{R} \times Y \), where one has
\[ -(D_Y - D'_Y)^2 = -\frac{d^2}{dt^2} - (D_Y - D'_Y)^2. \]
Then we use the arguments of \[5\] Proposition 3.28. The heat kernel splits into

\[
e^{-\frac{t}{4}(-(D_Y - D_Y')^2)} = e^{-\frac{t}{4}(-(D_Y - D_Y')^2)} e^{-\frac{t}{4} \Delta_R},
\]

where \(e^{-\frac{t}{4} \Delta_R}\) is the Gaussian heat kernel on each fiber \(\mathbb{R}\). It follows that

\[
(47) \quad (1 - \frac{t}{2}(D_Y + D_Y')^2)e^{-\frac{t}{4}(-(D_Y - D_Y')^2)} = (1 - \frac{t}{2}(D_Y + D_Y')^2)e^{-\frac{t}{4}(-(D_Y - D_Y')^2)}(1 - \frac{t}{2} \Delta_R)e^{-\frac{t}{4} \Delta_R} - (\frac{t}{2}(D_Y + D_Y')^2)e^{-\frac{t}{4}(-(D_Y - D_Y')^2)}(\frac{t}{2} \Delta_R)e^{-\frac{t}{4} \Delta_R}.
\]

By direct computation,

\[
(1 - \frac{t}{2} \Delta_R)e^{-\frac{t}{4} \Delta_R}(r, r') = \frac{1}{\sqrt{\pi t}} e^{-\frac{4(r - r')^2}{t}} + 2t \frac{d}{dt} \left( \frac{1}{\sqrt{\pi t}} e^{-\frac{4(r - r')^2}{t}} \right),
\]

which vanishes when \(r = r'\), and so does the first term on the right of (47). Also write

\[
N_Y = N_Y + N_R
\]

where \(N_Y, N_R\) are respectively the number operator of the \(V_Y\) and \(V_R\) in

\[
\wedge^\bullet(T^*(\mathbb{R} \times Y)) = \wedge^\bullet(T^*Y \oplus T^*\mathbb{R}).
\]

Then (46) becomes

\[
- \text{str}(\frac{N_Y + N_R}{2} t^{-\frac{N_Y}{2}} (\frac{t}{2}(D_Y + D_Y')^2)e^{-\frac{t}{4}(-(D_Y - D_Y')^2)} t \frac{N_Y}{2} (\frac{t}{2} \Delta_R)e^{-\frac{t}{4} \Delta_R}) - \text{str}(\frac{t}{2} \Delta_R e^{-\frac{t}{4} \Delta_R}) \text{str}(\frac{t}{2} \Delta_R e^{-\frac{t}{4} \Delta_R}) = 0.
\]

Since \(D_Y + D_Y'\) is odd,

\[
\text{str}(t^{-\frac{N_Y}{2}} (\frac{t}{2}(D_Y + D_Y')^2)e^{-\frac{t}{4}(-(D_Y - D_Y')^2)} t \frac{N_Y}{2}) = 0.
\]

Similarly the super-trace of \(\Delta_R e^{-\frac{t}{4} \Delta_R}\) equals 0, hence the lemma. \(\square\)

It follows from Lemma 4.11 that the \(b\)-integral

\[
\int_Z \text{str}(\frac{N_Y}{2} (1 - 2\delta Z(t)^2)) e^{-\bar{\delta}_Z(t)^2} (p, p) \, dp
\]

is actually convergent for each \(t\), because the integrand vanishes at the boundary \(\Delta_{33}^t \cap \Delta_{00}\). Moreover, by [3] Theorem 3.21], as \(t \to 0\),

\[
\text{str}(\frac{N_Y}{2} (1 - 2\delta_Y(t)^2)) e^{-\bar{\delta}_Y(t)^2} (p, p)
\]

is \(O(t^{\frac{1}{2}})\). It follows that for any \(C > 0\),

\[
\int_0^C \int_Z \text{str}(\frac{N_Y}{2} (1 - 2\delta_Y(t)^2)) e^{-\bar{\delta}_Y(t)^2} (p, p) \, dp \frac{dt}{t}
\]

converges for all \(\epsilon \geq 0\).

We turn to the \(t \to \infty\) limit of expression inside the super-trace \(\text{str}(\cdots)\) of (44), i.e.,

\[
(48) \quad \frac{N_Y}{2} (1 - 2\delta_Y(t)^2) e^{-\bar{\delta}_Y(t)^2}.
\]
Using the same arguments as the proof of Theorem 3.11 but replacing \( e^{-\frac{t^2}{4}} \) with \((1 - \frac{t^2}{2}) e^{-\frac{t^2}{4}}, \) \((1 - \frac{s^2}{2}) e^{-\frac{s^2}{4}} \) after re-parameterizing, one gets that at \( F_{\infty}^m, (m, n) \neq (0, 0), \) the leading term of (48) at horizontal degree 2\( K \) is

\[
(49) \quad \frac{N_v}{\tau} \int_{(\Gamma \cup \Gamma_0) \cap F_{\infty}^m} \tau^{-2Kz^2} (1 - \frac{\tau^2}{2}) e^{-\frac{\tau^2}{4}} T_{mn} \left( (\text{RN}(\Delta_e) - \tilde{\Omega} - \frac{s^2}{2})^{-1} \right) 2szdz = \frac{N_v}{\tau} (\text{ias} \, \epsilon)^{-2} T_{mn} \left( (\text{RN}(\Delta_e) - \tilde{\Omega}) K (1 - \frac{\tau^2}{2} (\text{RN}(\Delta_e) - \tilde{\Omega})) e^{-\frac{\tau^2}{4} (\text{RN}(\Delta_e) - \tilde{\Omega})} \right).
\]

At \( F_{\infty}^0, \) by similar arguments as (49), one computes the leading term

\[
(50) \quad \int_{\Gamma_0 \cap F_{\infty}^m} (1 - \frac{\tau^2}{2}) e^{-\frac{\tau^2}{4}} (\text{ias} \, \epsilon)^{-2} \tau^{-2Kz^2} 2K \Pi_{L^2} (\Omega_1 \Pi_{L^2})^2 (\text{ias} \, \epsilon)^{-2} 2szdz = \left( \frac{1}{K^2} - \frac{1}{2(K - 1)} \right) (2(\Pi_{L^2} \Omega_1 \Pi_{L^2})^2 K (\text{ias} \, \epsilon)^{-2} 2szdz
\]

Since \((\Pi_{L^2} \Omega_1 \Pi_{L^2})\) is of horizontal degree 1 and preserves the vertical \( Z \)-grading, after taking super-trace one sees that the leading term of (48) vanishes for \( K \geq 1. \)

4.1. The \((\text{ias} \, \epsilon)^{-1} \) term. This term is the push-forward of the constant term at \( \Delta_{33}^i. \) It is just the renormalized Bismut-Lott torsion on \( \mathbb{R} \times Y: \)

\[
(2\pi i)^{-\frac{NH}{2}} \int_0^\infty \int_{\mathbb{R} \times Y} \text{str} \left( \frac{N_v}{\tau} (1 - 2\partial_Y (t)^2) e^{-\frac{\partial_Y (t)^2}{4}} \right) \frac{dt}{t}.
\]

By Lemma 4.1 again the integrand vanishes, therefore

\[
(52) \quad a_{-1,0} = 0.
\]

4.2. The log term. In the last section, we saw that on \( F_{\infty}^m, (m, n) \neq (0, 0), \) the leading term of \( e^{-\frac{\partial_Y (t)^2}{4}} \) is \((\text{ias} \, \epsilon) \Omega (\tau - \frac{t}{2}). \) Hence we are in the same situation as \[13] Section 9. \]

The log term comes from \( F_{00} \cap F_{33}^\infty \) and \( F_{11} \cap F_{33}^\infty, \) where \( \tau^2 = t = 0 \) and \( \tau^2 = t = \frac{1}{2} \) is \( \text{ias} \, \epsilon. \)

We consider the asymptotic behavior of the heat kernel (12) as \( \tau \to 0. \) When \( s = s' \in [-1, 1], \) the only terms in (12) that does not go to 0 as \( \tau \to 0 \) are

\[
\frac{1}{\sqrt{\pi \tau}} e^{-\frac{(s-s')^2}{\tau}}, \quad \frac{1}{\sqrt{\pi \tau}} e^{-\frac{(2s-s')^2}{\tau}} S_+, \quad \frac{1}{\sqrt{\pi \tau}} e^{-\frac{(2s-s')^2}{\tau}} S_-
\]

and moreover away from \( s = s' = 1 \) (respectively \( s = s' = -1 \)) the second (respectively third) term also goes to 0. Therefore, at \( F_{00} \cap F_{33}^\infty, \) the Duhamel expansion becomes

\[
(53) \quad e^{-\frac{\tau}{4} (\text{RN}(\Delta_e) - \tilde{\Omega})} (s, s') = \frac{1}{\sqrt{\pi \tau}} e^{-\frac{(s-s')^2}{\tau}}
\]

\[+ \sum_{k=1}^{k=1} \frac{\tau^k}{4} \sum_{i_0, \ldots, i_k = 0, 1} \int_{(r_0 \cdots r_k) \in \Delta_k} A_{i_0}(r_0 \tau) \tilde{\Omega} A_{i_1}(r_1 \tau) \cdots \tilde{\Omega} A_{i_k}(r_k \tau) d\sigma_k\]
near $s = s' = 1$, and
\begin{equation}
\frac{t}{4}(\RN(\Delta) - \tilde{\Omega})(s, s') = \frac{1}{\sqrt{\pi t}} e^{-\frac{4(s-s')^2}{t}} + \sum_{k=1}^{+\infty} \sum_{i_0, \ldots, i_k=0,1} \int_{(r_0, \ldots, r_k) \in \Delta_k} B_{i_0}(r_0\tau) \tilde{\Omega} B_{i_1}(r_1\tau) \cdots \tilde{\Omega} B_{i_k}(r_k\tau) d\sigma_k
\end{equation}

near $s = s' = -1$ modulo terms going to 0 at infinite order, where
\begin{align*}
A_0(\tau) &:= B_0(\tau) := \frac{1}{\sqrt{\pi t}} e^{-\frac{4(s-s')^2}{t}}, A_1(\tau) := \frac{1}{\sqrt{\pi t}} e^{-\frac{4(2-s-s')^2}{t}} S_+, B_1(\tau) := \frac{1}{\sqrt{\pi t}} e^{-\frac{4(2-s-s')^2}{t}} S_-
\end{align*}

As for $\Delta_{11} \cap \Delta_{33}^\infty$ one evaluates the Duhamel expansion at $s = s' = 0$ and gets
\begin{equation}
\frac{t}{4}(\RN(\Delta) - \tilde{\Omega})(s, s') = \frac{1}{\sqrt{\pi t}} e^{-\frac{4(s-s')^2}{t}} + \sum_{k=0}^{+\infty} \int_{(r_0, \ldots, r_k) \in \Delta_k} A_{i_0}(r_0\tau) \tilde{\Omega} A_{i_1}(r_1\tau) \cdots \tilde{\Omega} A_{i_k}(r_k\tau) d\sigma_k
\end{equation}

For even horizontal degree $\geq 2$ its super-trace clearly vanishes by parity.

Consider the doubles $\hat{M}_{\pm}$. Denote the corresponding Laplacian operators by $\Delta^\pm_D$. These are special cases of the above, with both boundary conditions at $s = \pm 1$ equal to $\Lambda^D_{\pm}$. Therefore one has
\begin{equation}
\frac{t}{4}(\RN(\Delta^\pm_D) - \tilde{\Omega})(s, s') = \frac{1}{\sqrt{\pi t}} e^{-\frac{4(s-s')^2}{t}} + \sum_{k=1}^{+\infty} \sum_{i_0, \ldots, i_k=0,1} \int_{(r_0, \ldots, r_k) \in \Delta_k} A_{i_0}(r_0\tau) \tilde{\Omega} A_{i_1}(r_1\tau) \cdots \tilde{\Omega} A_{i_k}(r_k\tau) d\sigma_k
\end{equation}

near both $s = s' = 1$ and
\begin{equation}
\frac{t}{4}(\RN(\Delta^\pm_D) - \tilde{\Omega})(s, s') = \frac{1}{\sqrt{\pi t}} e^{-\frac{4(s-s')^2}{t}} + \sum_{k=1}^{+\infty} \sum_{i_0, \ldots, i_k=0,1} \int_{(r_0, \ldots, r_k) \in \Delta_k} B_{i_0}(r_0\tau) \tilde{\Omega} B_{i_1}(r_1\tau) \cdots \tilde{\Omega} B_{i_k}(r_k\tau) d\sigma_k
\end{equation}

near both $s = s' = 1$. Furthermore in these two cases the heat kernels are invariant under $s \mapsto -s$, $s' \mapsto -s'$.

From (49), the coefficient of the log term $a_{0,1}$ is the sum of $\tau$ derivatives of (53) and (54), evaluated at $s = s' = \pm 1$ respectively. One can use the same method to compute the log terms $a^\pm_{0,1}$ of the analytic surgery
\[ T(\hat{M}^\pm, g_\epsilon)(b), \]

simply replacing (53) and (54) by (56) for $a^+_0$ and (57) by $a^-_{0,1}$. It follows that one has
\[ 2a_{0,1} - a^+_0 - a^-_{0,1} = 0. \]

Hence we conclude with the following lemma.

**Lemma 4.2.** All divergent terms of
\[ 2T(M, g_\epsilon)(b) - T(\hat{M}^+, g_\epsilon)(b) - T(\hat{M}^-, g_\epsilon)(b) \]

vanishes and the expression converges for all $b \in B$. 

4.3. The constant term. In this section we turn to the $a_{0,0}$ term.

**Theorem 4.3.** The horizontal order $2K$ component of the $O((\text{ias } \epsilon)^0)$ term in the expansion of $T(M, g_b)(b), b \in B$ is

$$(2\pi i)^{-\frac{N_b}{2}} \int_{0}^{\infty} \int_{Z_b} \text{str} \left( \frac{N_b}{2} (1 - 2\tilde{g}_Z(t)^2) e^{-\tilde{g}_Z(t)^2} (p, p) \right) dp dt$$

$$+ \int_{0}^{\infty} \int_{-1}^{1} \text{str} \left( \frac{N_b}{2} (-\tau)^{-K}(RN(\Delta_e) - \tilde{\Omega})^K (1 - \frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})) e^{-\frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})} (s, s) \right) ds \frac{d\tau}{\tau}.$$

**Proof.** The $a_{0,0}$ term is the sum of push-forwards of the super-trace of the constant term at $\Delta_{00}^\infty$, which by (50) vanishes for horizontal degree $\geq 1$, the push-forward of the (trace of the constant term at) $\Delta_{11}^\infty$ which is the same as $a_{-1,0}$ and also vanishes, the constant term on $\Delta_{00}^\infty$, and the (ias $\epsilon$) term of $\Delta_{00}^\infty$.

From (46), pushing forward the constant term on $\Delta_{00}^\infty \cong Z_b \times \mathbb{R}$ one gets

$$\int_{0}^{\infty} \int_{Z_b} \text{str} \left( \frac{N_b}{2} (1 - 2\tilde{g}_Z(t)^2) e^{-\tilde{g}_Z(t)^2} (p, p) \right) dp \frac{dt}{t}.$$  

From (49), the push-forward of the (ias $\epsilon$) term at $\Delta_{33}^\infty = Y \times [-1, 1]$ gives at horizontal order $2K$

$$\int_{0}^{\infty} \int_{\Delta_{33}^\infty} \text{str} \left( \frac{N_b}{2} (-\tau)^{-K} T_{33}((RN(\Delta_e) - \tilde{\Omega})^K (1 - \frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})) e^{-\frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})}) \right) \bigg|_{\Delta_{33}^\infty} dp \frac{d\tau}{\tau}$$

Recall [15] (4.13) that $F_{33}^\infty$ is diffeomorphic to $F_2 \times F_2$, therefore

$$\int_{\Delta_{33}^\infty} \text{str} \left( \frac{N_b}{2} T_{33}((RN(\Delta_e) - \tilde{\Omega})^K (1 - \frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})) e^{-\frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})}) \right) \bigg|_{\Delta_{33}^\infty} dp$$

is just the operator super-trace of

$$\frac{N_b}{2} (RN(\Delta_e) - \tilde{\Omega})^K (1 - \frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})) e^{-\frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})}$$

as a kernel operator for each $\tau$. Regarding (59) as a linear map from $H_{1, \epsilon} \otimes E$ to itself for each $s, s' \in [-1, 1]$, (58) can be rewritten as

$$\int_{-1}^{1} \text{str} \left( \frac{N_b}{2} (RN(\Delta_e) - \tilde{\Omega})^K (1 - \frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})) e^{-\frac{\tau}{2}(RN(\Delta_e) - \tilde{\Omega})} (s, s') \right) ds,$$

where str now denotes the super-trace as a linear map for each $(s, s')$. The theorem follows after multiplying the last expression by $(2\pi i)^{-\frac{N_b}{2} (-\tau)^{-K}}$ and then integrating over $\tau$. 

Combining (51), (52), Lemma 4.2 and Theorem 4.3 one clearly obtains Theorem 1.4.

The first term in Theorem 4.3 is just the $b$-version of the Bismut-Lott torsion form over $\bar{Z} \to M \to B$. Because $M = M^+ \cup M^-$ it can also be written as the sum

$$(2\pi i)^{-\frac{N_b}{2}} \int_{0}^{\infty} \int_{Z} \text{str} \left( \frac{N_b}{2} (1 - 2\tilde{g}_Z(t)^2) e^{-\tilde{g}_Z(t)^2} (p, p) \right) dp \frac{dt}{t}$$

$$= (2\pi i)^{-\frac{N_b}{2}} \int_{0}^{\infty} \int_{Z^+} \text{str} \left( \frac{N_b}{2} (1 - 2\tilde{g}_Z(t)^2) e^{-\tilde{g}_Z(t)^2} (p, p) \right) dp \frac{dt}{t}.$$
+ (2\pi i)^{-\frac{N\mu}{2}} \int_0^\infty \int_{Z^-} \text{str} \left( \frac{N\mu}{2} (1 - 2\partial\bar{\partial}Z(t)^2) e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} (p,p) \right) dp \frac{dt}{t},

and therefore satisfies the additivity property. Not much is known about second, error term but it seems to be closely related to the ‘one dimensional model’ of Puchol, Zhang and Zhu [21]. It seems such term can be computed using Witten deformation on the one dimensional fiber bundle \([-1,1] \times B \rightarrow B\).

5. ANALYTIC SURGERY OF THE ETA FORM

In this section we turn to the analytic surgery of the eta form. For simplicity, we assume that near \(Y\), the vector bundle \(E\), the flat vector bundle \(\nabla^E\) and the duality structure \(Q\) is the pull back of the corresponding object from \(Y\). Our method closely follows that of the Bismut-Lott torsion form of the last section.

To begin with, writing

\[ D = d_V + L + \iota_\Theta, \quad D' = -J^{-1}d_VJ + J^{-1}LJ - J^{-1}\iota_\Theta J, \]

one observes that the integrand of (3) becomes

\[
(60) \quad \text{tr} \left( J[N_V, \mathcal{X}(t)] e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} \right) \\
= \frac{1}{2} \text{tr} \left( J[N_V, t^{\frac{1}{2}} (d_V + J^{-1}d_VJ) + t^{-\frac{1}{2}}(\iota_\Theta + J^{-1}\iota_\Theta J)] e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} \right) \\
= \frac{1}{2} \text{tr} \left( JN_V t^{\frac{1}{2}} - \frac{N\mu}{2} \right) ((d_V - d^*_V + \iota_\Theta - \iota^*_\Theta) e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} \\
- e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}}(d_V - d^*_V + \iota_\Theta - \iota^*_\Theta) t^{\frac{N\mu}{2}}),
\]

which is a fiber-wise operator. Hence analytic surgery of the eta form is well defined:

\[
(61) \quad \eta(M, g_\epsilon) := (2\pi i)^{-\frac{N\mu}{2}} \int_0^\infty \int_{Z^-} \frac{1}{2} \text{tr} \left( J[N_V, \mathcal{X}_\epsilon(t)] e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} (p,p) \right) dp \frac{dt}{t} \\
= (2\pi i)^{-\frac{N\mu}{2}} \int_0^\infty \int_{Z^-} \frac{1}{4} \text{tr} \left( JN_V t^{\frac{1}{2}} - \frac{N\mu}{2} \right) ((d_V, - d^*_V, + \iota_\Theta - \iota^*_\Theta) e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} \\
- e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}}(d_V - d^*_V + \iota_\Theta - \iota^*_\Theta) t^{\frac{N\mu}{2}}) dp \frac{dt}{t},
\]

where

\[ \mathcal{X}_\epsilon := t^{\frac{1}{2}} - \frac{N\mu}{2} \frac{1}{2} (D_\epsilon - D'_\epsilon) t^{\frac{N\mu}{2}}. \]

To compute the limit of \((61)\) as \(\epsilon \rightarrow 0\), we consider the leading behavior of

\[ (d_{V,\epsilon} - d^*_{V,\epsilon} + \iota_\Theta - \iota^*_\Theta) e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} \text{ and } e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}}(d_{V,\epsilon} - d^*_{V,\epsilon} + \iota_\Theta - \iota^*_\Theta). \]

As in (30), the heat kernel is given by holomorphic functional calculus

\[
e^{-\frac{\partial\bar{\partial}Z(t)^2}{2}} = \frac{1}{2\pi i} \int_{\Gamma_0} e^{-\frac{2\lambda^2}{4}} \left( -(D_\epsilon - D'_\epsilon)^2 - \lambda^2 \right)^{-1} 2\lambda d\lambda \\
+ \frac{1}{2\pi i} \int_{\Gamma} e^{-\frac{2\lambda^2}{4}} \left( -(D_\epsilon - D'_\epsilon)^2 - \lambda^2 \right)^{-1} 2\lambda d\lambda.
\]
It follows that
\[
(d_{\epsilon} - d_{\epsilon}^\ast + i\Theta - i\Theta^\ast)e^{-\frac{1}{4}(-(D_\epsilon - D_\epsilon')^2)}
= \frac{1}{2\pi i} \int_{\Gamma_0} e^{-\frac{1}{4}\lambda^2} (dV_\epsilon - dV_\epsilon^\ast + i\Theta - i\Theta^\ast)\left(-(D_\epsilon - D_\epsilon')^2 - \lambda^2\right)^{-1} 2\lambda d\lambda
+ \frac{1}{2\pi i} \int_{\Gamma} e^{-\frac{1}{4}\lambda^2} (dV_\epsilon - dV_\epsilon^\ast + i\Theta - i\Theta^\ast)\left(-(D_\epsilon - D_\epsilon')^2 - \lambda^2\right)^{-1} 2\lambda d\lambda.
\]

Next, we follow the same method of Section 3.4. We begin with expanding the integrand in the above expression, applying Lemma 3.2 for \(-(D_\epsilon - D_\epsilon')^2 - \lambda^2\)^{-1} and then using (24) to get
\[
\tag{62}
(dV_\epsilon - dV_\epsilon^\ast)\sum_k \sum_{i_1, \ldots, i_k} (\Delta_\epsilon - \lambda^2)^{-1} R_{i_1} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1}
+ (i\Theta - i\Theta^\ast)\sum_k \sum_{i_1, \ldots, i_k} (\Delta_\epsilon - \lambda^2)^{-1} R_{i_1} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1},
\]
where the summation is over \(i_1, \ldots, i_k = 1, 2, 3\) and such that if \(i_j = 1, i_{j+1} \neq 2\).

Consider the leading behavior of the first summation on the right hand side of (62). One uses the same arguments leading to Lemma 3.6 to obtain the following lemma.

**Lemma 5.1.** For any \(K = 1, 2, 3, \ldots\), the leading term of
\[
(dV_\epsilon - dV_\epsilon^\ast)\sum_k \sum_{i_1, \ldots, i_k} (\Delta_\epsilon - \lambda^2)^{-1} R_{i_1} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1}
\]
of horizontal degree \(2K + 1\) is
\[
\tag{63}
(i\epsilon)^{-1-2K} z^{2K} T_{mn}((-1 - z^2 (\text{RN}(\Delta_\epsilon) - z^2)^{-1})(\Omega_1 (\text{RN}(\Delta_\epsilon) - z^2)^{-1})^{2K+1})
\]
near \(F_{mn}^0\), \(m, n \neq (0, 0)\), and
\[
(i\epsilon)^{-1-2K} z^{2K} (-1 - z^2 \Pi L^2)(\Omega_1 \Pi L^2)^{2K+1}
\]
near \(F_{00}^0\).

**Proof.** First suppose \(i_1 = 2\), then
\[
(dV_\epsilon - dV_\epsilon^\ast)(\Delta_\epsilon - \lambda^2)^{-1} R_{i_1} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1}
= (-1 - \lambda^2 (\Delta_\epsilon - \lambda^2)^{-1})\Omega (\Delta_\epsilon - \lambda^2)^{-1} R_{i_2} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1}.
\]
Multiplication by \(\Omega\) and also \(\lambda^2 (\Delta_\epsilon - \lambda^2)^{-1}\) (which equals \((i\epsilon) z^2 (\text{RN}(\Delta_\epsilon) - z^2)^{-1} + \cdots\) near \(F_{mn}^0\)) does not increase order of \((i\epsilon)^{-1}\). Hence applying Lemma 3.6 to
\[
(\Delta_\epsilon - \lambda^2)^{-1} R_{i_2} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1}
\]
one obtains (63).

Next, suppose \(i_1 = 1\), so one considers
\[
\tag{64}
(dV_\epsilon - dV_\epsilon^\ast)(\Delta_\epsilon - \lambda^2)^{-1} \Omega (dV_\epsilon - dV_\epsilon^\ast)(\Delta_\epsilon - \lambda^2)^{-1} R_{i_2} \cdots R_{i_k} (\Delta_\epsilon - \lambda^2)^{-1}.
\]
By the same arguments leading to Lemma 3.5, the first \((d_{V,\epsilon} - d_{V,\epsilon}^*)(\Delta_{\epsilon} - \lambda^2)^{-1}\Omega\) factor does not increase order of \((\text{ias} \epsilon)^{-1}\), and its horizontal order is \(\geq 1\). By Lemma 3.6, the leading term of

\[(\Delta_{\epsilon} - \lambda^2)^{-1}R_{i_2} \cdots R_{i_k}(\Delta_{\epsilon} - \lambda^2)^{-1}\]

of horizontal order at most 2\(K\) is

\[(\text{ias} \epsilon)^{-1 - 2K}z^{2K}T_{mn}\left(\left(\text{RN}(\Delta_{\epsilon}) - z^2\right)^{-1}\Omega_1(\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}\right)^{2K}\] at \(F_{mn}^0, (m, n) \neq (0, 0)\)

\[(\text{ias} \epsilon)^{-2 - 2K}z^{-2 - 2K}P_{Lz}^2(\Omega_1 P_{Lz}^2)^{2K}\] at \(F_{00}^0\).

Both factors are annihilated by the second \(d_{V,\epsilon} - d_{V,\epsilon}^*\) in (64). It follows that terms of the form (64) do not contribute to the leading term in our lemma.

The case when \(i_1 = 3\) is similar to the case \(i_1 = 1\). One considers

\[(d_{V,\epsilon} - d_{V,\epsilon}^*)(\Delta_{\epsilon} - \lambda^2)^{-1}(-\Omega)(\lambda^2(\Delta_{\epsilon} - \lambda^2)^{-1}\Omega(\Delta_{\epsilon} - \lambda^2)^{-1}R_{i_2} \cdots R_{i_k}(\Delta_{\epsilon} - \lambda^2)^{-1})\]

Note that

\[(\text{ias} \epsilon)^{-1 - 2K}z^{2K}T_{mn}\left(\left(\text{RN}(\Delta_{\epsilon}) - z^2\right)^{-1}\Omega_1(\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}\right)^{2K}\] (ias \(\epsilon)^{-2 - 2K}z^{-2 - 2K}P_{Lz}^2(\Omega_1 P_{Lz}^2)^{2K}\)

have even horizontal order, therefore the leading term of

\[(\Delta_{\epsilon} - \lambda^2)^{-1}R_{i_2} \cdots R_{i_k}(\Delta_{\epsilon} - \lambda^2)^{-1}\]

of horizontal order at most 2\(K - 1\) is of order \((\text{ias} \epsilon)^{-1 - 2(K - 1)}\) at \(F_{mn}^0, (m, n) \neq (0, 0)\) and \((\text{ias} \epsilon)^{-2 - 2(K - 1)}\) at \(F_{00}^0\). Again since the \((d_{V,\epsilon} - d_{V,\epsilon}^*)(\Delta_{\epsilon} - \lambda^2)^{-1}(-\Omega)(\lambda^2(\Delta_{\epsilon} - \lambda^2)^{-1}\Omega\)

factor does not increase order of \((\text{ias} \epsilon)^{-1}\) we conclude that terms of the form (65) do not contribute to the leading term in our lemma.

Because the factor \(\iota_\Theta - \iota_\Theta^*\) increases horizontal degree by 2 and does not change the order of \(\text{ias} \epsilon\), it does not contribute to the leading term of (62). It follows that the leading term of

\[(d_{V,\epsilon} - d_{V,\epsilon}^* + \iota_\Theta - \iota_\Theta^*)(- (D_{\epsilon} - D_{\epsilon}^*)^2 - \lambda^2)^{-1}\]

of horizontal order 2\(K + 1\) is given by

\[(\text{ias} \epsilon)^{-1 - 2K}z^{2K}T_{mn}\left((-1 - z^2(\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}\Omega_1(\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}\right)^{2K + 1}\]

near \(F_{mn}^0, (m, n) \neq (0, 0)\), and

\[(\text{ias} \epsilon)^{-1 - 2K}z^{2K}(-1 - z^2 P_{Lz}^2)(\Omega_1 P_{Lz}^2)^{2K + 1}\]

near \(F_{00}^0\). Similarly, the leading term of

\[(- (D_{\epsilon} - D_{\epsilon}^*)^2 - \lambda^2)^{-1}(d_{\epsilon} - d_{\epsilon}^* + \iota_\Theta - \iota_\Theta^*)\]

of horizontal order 2\(K + 1\) is given by

\[(-1)^K(\text{ias} \epsilon)^{-1 - 2K}z^{2K}T_{mn}\left((-\Omega_1)^{2K + 1}(-1 - z^2(\text{RN}(\Delta_{\epsilon}) - z^2)^{-1}\right)\]

at \(F_{mn}^0, (m, n) \neq (0, 0)\) and

\[(-1)^K(\text{ias} \epsilon)^{-1 - 2K}z^{2K}(\Omega_1 P_{Lz}^2)^{2K + 1}(-1 - z^2 P_{Lz}^2)\]
Analytic Surgery and Gluing of the Bismut-Lott Torsion Form and η Form

at $F_{00}^0$. Taking the difference of the two, it follows that leading term of

$$(d_\varepsilon - d_w^s + \iota_\theta - \iota_0^s)(- (D_\varepsilon - D_w')^2 - \lambda^2)^{-1} - (- (D_\varepsilon - D_w')^2 - \lambda^2)^{-1}(d_\varepsilon - d_w^s + \iota_\theta - \iota_0^s)$$

of horizontal order $2K + 1$ is given by

$$(- \Omega_1, (-1)^{K} (\text{ias } \epsilon)^{-1-2K} z^{2K} \mathcal{T}_{mn} ((\text{RN}(\Delta_e) - z^2)^{-1} (\Omega_1 (\text{RN}(\Delta_e) - z^2)^{-1})^{2K})$$

at $F_{mn}^0, (m,n) \neq (0,0)$ and

$$(- \Omega_1, (-1)^{K} (\text{ias } \epsilon)^{-1-2K} z^{2K} \Pi_{L2} (\Omega_1 \Pi_{L2})^{2K})$$

at $F_{00}^0$.

Observe that the expressions (68) and (69) are respectively the commutator of $-\Omega_1$ and (28) and (29). Hence the arguments in Section 3.6 can be applied without any changes to compute the leading term of the rescaled heat kernel.

**Theorem 5.2.** The leading term of

$$t^\frac{1}{2} - \frac{N_H}{2} ((d_{V,\varepsilon} - d_{V,\varepsilon}^s + \iota_{\theta} - \iota_0^s) e^{-\frac{s}{2}((D_\varepsilon - D'_{\varepsilon})^2) - e^{-\frac{s}{2}((D_\varepsilon - D'_{\varepsilon})^2)} (d_{V,\varepsilon} - d_{V,\varepsilon}^s + \iota_{\theta} - \iota_0^s)) t^{\frac{N_H}{2}}$$

of horizontal order $2K + 1$ is

$$(\text{ias } \epsilon) \left[ - \Omega_1, (1 - K) (\text{RN}(\Delta_e) - \Theta) F_{mn} (\Omega_1 (\text{RN}(\Delta_e) - \Theta))\right]$$

at $F_{mn}^0, (m,n) \neq (0,0)$,

$$\left[ - \Omega_1, \frac{1}{K^2} (2\pi i) (\Pi_{L2} \Omega_1 \Pi_{L2})^{2K}\right]$$

at $F_{00}^0$.

Similar to (66), on $F_{mn}^t$, by applying the restriction of $(d_{V,\varepsilon} - d_{V,\varepsilon}^s + \iota_{\theta} - \iota_0^s)$ to Lemma 3.8 the leading term of

$$t^\frac{1}{2} - \frac{N_H}{2} ((d_{V,\varepsilon} - d_{V,\varepsilon}^s + \iota_{\theta} - \iota_0^s) e^{-\frac{s}{2}((D_\varepsilon - D'_{\varepsilon})^2) - e^{-\frac{s}{2}((D_\varepsilon - D'_{\varepsilon})^2)} (d_{V,\varepsilon} - d_{V,\varepsilon}^s + \iota_{\theta} - \iota_0^s)) t^{\frac{N_H}{2}}$$

is

$$\left[ t^\frac{1}{2} (d_Z - d_Z^s) + t^\frac{1}{2} (\iota_{\theta} - \iota_0^s), e^{-\delta_2(t^2)} \right]$$

at $F_{00}^t$,

$$\left[ t^\frac{1}{2} (d_Y - d_Y^s) + t^\frac{1}{2} (\iota_{\theta} - \iota_0^s), e^{-\delta_1(t^2)} \right]$$

at $F_{11}^t$,

$$\left[ t^\frac{1}{2} (d_Y - d_Y^s) + t^\frac{1}{2} (\iota_{\theta} - \iota_0^s), e^{-\delta_1(t^2)} \right]$$

at $F_{33}^t$,

and 0 at other $F_{mn}^t$ to infinite order.

Next, we restrict

$$JN_{V} \left[ t^\frac{1}{2} (d_{V,\varepsilon} - d_{V,\varepsilon}^s) + t^\frac{1}{2} (\iota_{\theta} - \iota_0^s), e^{-\delta(t^2)} \right]$$

to the diagonal, take point-wise trace and then integrate as in the previous section.

For the small $t$ limit, using [11, Remark 3.11], and local index technique (see, for example, [6, Theorem 3.16], one proves that the integral (61) is convergent near $t = 0$.

For the large $t$ limit, observe that the leading terms (70), (71), (72) have exactly the same order as the Bismut-Lott torsion form case, respectively (49), (50), and (46) and therefore the same arguments apply. The leading terms of the push-forward of (61), i.e.,

$$\left(2\pi i\right)^{-\frac{N_H}{2}} \int_0^\infty \int_{Z_b} \frac{1}{4} \text{tr} \left( JN_{V} \left[ t^\frac{1}{2} (d_{V,\varepsilon} - d_{V,\varepsilon}^s) + t^\frac{1}{2} (\iota_{\theta} - \iota_0^s), e^{-\delta(t^2)} \right] (p,p) \right) d\rho dt$$


is of the form
\[ b_{-1,0}(\text{i} \text{as } \epsilon)^{-1} + b_{0,1} \log(\text{i} \text{as } \epsilon) + b_{0,0} \]
plus terms that goes to 0 as \( \text{i} \text{as } \epsilon \) goes to 0.

5.1. The \((\text{i} \text{as } \epsilon)^{-1}\) term. The \(b_{-1,0}(\text{i} \text{as } \epsilon)^{-1}\) term is the push-forward of the constant term at \( \Delta_{33}^4 \approx \mathbb{R} \times Y:\)
\[
(2\pi i)^{-\frac{NH}{2}} \int_0^\infty \int_{\mathbb{R} \times Y} \frac{1}{t} \text{tr} \left( J N_V \left[ t^\frac{1}{2} (d_Y - d_Y^*) + t^{-\frac{1}{2}} (t_\Theta - t_\Theta^*) , e^{-\partial_Y(t)^2} \right] (p,p) \right) dp \frac{dt}{t}.
\]

Similar to Lemma 4.1, we have the following lemma.

**Lemma 5.3.** One has
\[
\text{tr} \left( J N_V \left[ t^\frac{1}{2} (d_Y - d_Y^*) + t^{-\frac{1}{2}} (t_\Theta - t_\Theta^*) , e^{-\partial_Y(t)^2} \right] (p,p) \right) = 0.
\]

**Proof.** Again the (rescaled) heat kernel splits into
\[
e^{-\frac{1}{4}(-(D_Y - D_Y^*)^2)} = e^{-\frac{1}{4}(-(D_Y - D_Y^*)^2)} e^{-\frac{i}{2} \Delta}\]
\[
e^{-\partial_Y(t)^2} = t^{-\frac{NH}{2}} e^{-\frac{1}{4}(-(D_Y - D_Y^*)^2)} t^{\frac{NH}{2}} e^{-\frac{i}{2} \Delta}.
\]

Likewise
\[
D_Y + D_Y^* = D_Y + D_Y^* + c(\frac{\partial}{\partial r}) \frac{\partial}{\partial r},
\]
and by choosing a positively oriented orthonormal basis of \( V, \{ \xi_1, \cdots, \xi_{\dim Z} \} \), such that \( \xi_1 = \frac{\partial}{\partial r} \), one has
\[
J = c(\frac{\partial}{\partial r}) \circ (\frac{1}{\sqrt{\epsilon}} J Y \otimes J E),
\]
where
\[
J Y := c(\xi_2) \cdots c(\xi_{\dim Z}).
\]

Using (60) one has
\[
\text{tr} \left( J N_V \left[ t^\frac{1}{2} (d_Y - d_Y^*) + t^{-\frac{1}{2}} (t_\Theta - t_\Theta^*) , e^{-\partial_Y(t)^2} \right] (p,p) \right)
= 2t \text{tr} \left( J \left( \frac{\partial}{\partial r} \partial_Y \right) e^{-\partial_Y(t)^2} \right)
= \text{tr} \left( J \left( \frac{1}{\sqrt{\epsilon}} J Y \otimes J E \right) t^{-\frac{NH}{2}} e^{-\frac{i}{4}(-(D_Y - D_Y^*)^2)} t^{\frac{NH}{2}} e^{-\frac{i}{2} \Delta} \right)
\]
\[
+ \text{tr} \left( \left( \frac{1}{\sqrt{\epsilon}} J Y \otimes J E \right) \left( \frac{1}{2} (d_Y - d_Y^*) - \frac{1}{2} (t_\Theta + t_\Theta^*) \right) t^{-\frac{NH}{2}} e^{-\frac{i}{4}(-(D_Y - D_Y^*)^2)} t^{\frac{NH}{2}} e^{-\frac{i}{2} \Delta} \right)
\]
\[
\times \text{tr} \left( c(\frac{\partial}{\partial r}) e^{-\frac{i}{2} \Delta} \right).
\]

Hence the Lemma follows from the observations that \( \frac{\partial}{\partial r} e^{-\frac{i}{4} \Delta} (r, r') = 0 \)
at \( r = r' \), and
\[
\text{tr} \left( c(\frac{\partial}{\partial r}) e^{-\frac{i}{2} \Delta} \right) = 0. \quad \Box
\]

From Lemma 5.3 follows that
\[
b_{-1,0} = 0.
\]
5.2. The log term. As in Section 4.2 the log term comes from $\Delta_0 \cap \Delta^\infty_{33}$ and $\Delta_1 \cap \Delta^\infty_{33}$, and the calculation is similar.

Lemma 5.4. Let

$$\eta(M^\pm, g_c)$$

be the surgery eta invariant of the surgery of the double spaces $M^\pm$, and $b^+_0$ be the coefficient of the log term of $\eta(M^\pm, g_c)$. Then one has

$$2b_0 - b^+_0 - b^-_0 = 0.$$

Hence all divergent terms of

$$2\eta(M, g_c)(b) - \eta(M^+, g_c)(b) - \eta(M^-, g_c)(b)$$

vanishes and the expression converges for all $b \in B$.

Proof. It suffices to substitute the heat kernel $e^{-\frac{t}{4}(RN(\Delta_c)-\Omega)}$ in (70) by its Duhamel expansion (53) and (54). In place of (55), one observes that $\tilde{\Omega}$ commutes with $N_V$ and anti-commutes with $J$, which implies

$$\text{tr}(JN_V[\Omega_1, T_{33}(e^{-\frac{t}{4}RN(\Delta_c)})(s, s)\tilde{\Omega}^2K])$$

$$= \text{tr}(JN_V[\Omega_1, T_{33}(e^{-\frac{t}{4}RN(\Delta_c)})(s, s)\Pi_Y\tilde{\Omega}^2K])$$

$$= 2\text{tr}(JN_VT_{33}(e^{-\frac{t}{4}RN(\Delta_c)})(s, s)\tilde{\Omega}^{2K+1})$$

$$= -2\text{tr}(T_{33}(e^{-\frac{t}{4}RN(\Delta_c)})(s, s)\tilde{\Omega}^{2K+1}JN_V) = 0.$$

The rest of the proof is same as that of Lemma 4.2

5.3. The constant term. The $b_{0,0}$ term again is the sum of four push-forwards, namely, the constant term of $\Delta_{0,0}$, $\Delta^\infty_{0,0}$, $\Delta^\infty_{1,1}$ and the (ias $\epsilon$) term of $\Delta^\infty_{33}$.

Theorem 5.5. The horizontal order $2K + 1$ component of the $O((\text{ias } \epsilon)^0)$ term in the expansion of $\eta(M, g_c)(b), b \in B$ is

$$\left(2\pi i\right)^{-\frac{N_\mu}{2}} \left(\int_0^\infty \int_{Z_0}^\infty \frac{1}{4} \text{tr} \left(J[N_V, \left(t^{-\frac{N_\mu}{4}}(D_Z - D_Z')t^{\frac{N_\mu}{4}}\right)e^{-\frac{3}{2}(t)^{2}}(p, p)]\right) \right) dt \frac{dp}{t}$$

$$+ \int_0^\infty \int_{-1}^1 \frac{1}{4} \text{tr} \left(JN_V\left[ - \Omega_1, (\tau)^{-K}(RN(\Delta_c) - \tilde{\Omega})K e^{-\frac{3}{2}(RN(\Delta_c) - \tilde{\Omega})}\right)\right) ds \frac{d\tau}{\tau}.$$
It follows from (71) that the contribution from $\Delta_{\infty}^\infty$ is the push-forward of
\[
\text{tr} \left( JN_V \left[ -\Omega_1, \left( \frac{-1}{K} \right)^K 2(2\pi i)(\Pi L_2 \Omega_1 \Pi L_2)^2K \right] \right)
\]
\[= (-1)^{2K+1} \text{tr} \left( [ -\Omega_1, \left( \frac{-1}{K} \right)^K 2(2\pi i)(\Pi L_2 \Omega_1 \Pi L_2)^2K ] JN_V \right)
\]
\[= 0.
\]
Finally, one takes the (ias $\epsilon$) term of $F_{33}^\infty$, i.e. (70), restricts to $\Delta_{33}^\infty (s = s')$, multiplies by $JN_V$, takes the trace, and then performs push-forward to obtain the second term of (75):
\[
(2\pi i)^{-N_{K}} \int_0^\infty \int_{-1}^1 \frac{1}{\tau} \text{tr} \left( JN_V \left[ -\Omega_1, (-\tau)^{-K} (\text{RN}(\Delta_\epsilon) - \tilde{\Omega})^K e^{-\frac{\tau}{2}(\text{RN}(\Delta_\epsilon) - \tilde{\Omega})} \right] (s,s) \right) ds d\tau.
\]
That completes the proof of the theorem. \(\square\)

Clearly, Theorem 1.5 Follows from (74), Lemma 5.3, Lemma 5.4 and Theorem 75.

Even less is known about second, error term of (75) to the author. It seems one can also set up some one dimensional model, and consider Witten deformation similar to the Busmut-Lott torsion case.

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