Research Article

Alpha Power Transformed Inverse Lomax Distribution with Different Methods of Estimation and Applications

Ramadan A. ZeinEldin,1,2 Muhammad Ahsan ul Haq,3,4 Sharqa Hashmi,4,5 and Mahmoud Elsehety6

1Deanship of Scientific Research, King Abdulaziz University, Jeddah, Saudi Arabia
2Faculty of Graduate Studies for Statistical Research, Cairo University, Giza, Egypt
3Quality Enhancement Cell, National College of Arts, Lahore, Pakistan
4College of Statistical & Actuarial Sciences, University of the Punjab, Lahore, Pakistan
5Lahore College for Women University (LCWU), Lahore, Pakistan
6King Abdulaziz University, Jeddah, Saudi Arabia

Correspondence should be addressed to Muhammad Ahsan ul Haq; ahsanshani36@gmail.com

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In this paper, a new three-parameter lifetime distribution, alpha power transformed inverse Lomax (APTIL) distribution, is proposed. The APTIL distribution is more flexible than inverse Lomax distribution. We derived some mathematical properties including moments, moment generating function, quantile function, mode, stress strength reliability, and order statistics. Characterization related to hazard rate function is also derived. The model parameters are estimated using eight estimation methods including maximum likelihood, least squares, weighted least squares, percentile, Cramer–von Mises, maximum product of spacing, Anderson–Darling, and right-tail Anderson–Darling. Numerical results are calculated to compare the performance of these estimation methods. Finally, we used three real-life datasets to show the flexibility of the APTIL distribution.

1. Introduction

The inverse Lomax (IL) is originally developed as a lifetime distribution. The inverse Lomax is member of family of generalized beta distribution. Kleiber and Kotz [1] showed that IL distribution can be used in economics. The IL distribution has many applications in modeling of different trends of hazard rate function (hrf), i.e., decreasing or upside-down bathtub failure rate of life testing of components. The IL distribution is used in [2] to get Lorenz ordering relationship among ordered statistics. McKenzie et al. [3] used the IL model and applied it to geophysical databases. Singh et al. [4] investigated the reliability estimators of IL distribution under Type II censoring. Bayesian estimation of mixture of the IL model under the censoring scheme was studied in [5]. Inverse power Lomax distribution was studied in [6] and Weibull IL distribution in [7].

The probability density (pdf) and cumulative distribution function (cdf) of the IL distribution are as follows:

\[ g(x; a, b) = \frac{ab}{x^2} \left(1 + \frac{b}{x}\right)^{-a-1}, \quad x, a, b > 0, \quad (1) \]

\[ G(x; a, b) = \left(1 + \frac{b}{x}\right)^{-a}, \quad x, a, b > 0. \quad (2) \]

Mahdavi and Kundu [8] introduced the alpha power transformation (APT) method to add an additional parameter to a family of distributions to increase flexibility in given family. The cdf and pdf of APT-G family are
and the corresponding pdf is
\[
f(x; \alpha) = \begin{cases} 
\log(\alpha) \alpha^{G(x)} - 1 \alpha - 1, & \text{if } \alpha > 0, \alpha \neq 1, \\
g(x) \alpha^{G(x)}, & \text{if } \alpha = 1, 
\end{cases}
\]
and the corresponding pdf is
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f(x; \alpha) = \begin{cases} 
\log(\alpha) \alpha^{G(x)} - 1 \alpha - 1, & \text{if } \alpha > 0, \alpha \neq 1, \\
g(x) \alpha^{G(x)}, & \text{if } \alpha = 1. 
\end{cases}
\]

In the literature, many probability distributions are generalized using this approach; for example, alpha power transformed Weibull (APT-W) distribution in [9]. APT generalized exponential distribution in [10], APT Lindley distribution in [11], APT extended exponential distribution in [12], alpha power inverse-Weibull distribution in [14], APT inverse-Lindley distribution in [15], APT power Lindley studied in [16], and APT Pareto distribution proposed in [17].

The main goal of this research article is to introduce a simpler and more flexible model called APTIL (APT inverse Lindley) distribution. Furthermore, the key motivations of using APTIL distribution in the practice are as follows:

(i) To improve the flexibility of the existing distributions by using APT-G

\[
F(x; \alpha) = \begin{cases} 
\log(\alpha) \alpha^{G(x)} - 1 \alpha - 1, & \text{if } \alpha > 0, \alpha \neq 1, \\
G(x), & \text{if } \alpha = 1, 
\end{cases}
\]

and the corresponding pdf is
\[
f(x; \alpha) = \begin{cases} 
\log(\alpha) \alpha^{G(x)} - 1 \alpha - 1, & \text{if } \alpha > 0, \alpha \neq 1, \\
g(x) \alpha^{G(x)}, & \text{if } \alpha = 1. 
\end{cases}
\]

The survival function (sf) and the hrf for APTIL distribution for \( x > 0 \) are in the following forms:

\[
S(x) = \frac{\alpha - a^{(1+b/x)^a}}{\alpha - 1}, \quad \text{if } \alpha \neq 1, \alpha > 0,
\]

and \( b \). Clearly, the pdf of APTIL distribution is the function for \( \alpha \neq 1 \) and \( a < 1 \) and unimodal for \( \alpha \neq 1 \) and \( a > 1 \). The hrf of the APTIL model can be decreasing or upside-down bathtub for \( \alpha \neq 1 \) and \( a < 1 \) and \( a > 1 \), respectively.

Figure 1 demonstrates the graphs of pdf and hazard function of APTIL distribution for different values of \( a, \alpha \), and \( b \).

2. APTIL Distribution

The random variable (rv) \( X \) is said to have APTIL distribution denoted by APTIL(\( a, \alpha \), and \( b \)) with two shape parameters and one scale as \( a, \alpha \), and \( b \), respectively. The pdf of \( X \) for \( x \geq 0 \) is

\[
f(x) = \begin{cases} 
\log(\alpha) \alpha^{G(x)} - 1 \alpha - 1, & \text{if } \alpha > 0, \alpha \neq 1, \\
g(x) \alpha^{G(x)}, & \text{if } \alpha = 1. 
\end{cases}
\]

and \( b \). Clearly, the pdf of APTIL distribution is the function for \( \alpha \neq 1 \) and \( a < 1 \) and unimodal for \( \alpha \neq 1 \) and \( a > 1 \). The hrf of the APTIL model can be decreasing or upside-down bathtub for \( \alpha \neq 1 \) and \( a < 1 \) and \( a > 1 \), respectively.

2.1. Useful Expansions. Here, an explicit expression for APTIL pdf is given. By using the series representation of exponential function, equation (5) can be written as
3. Properties of APTIL Distribution

This section deals with some statistical properties of APTIL distribution.

3.1. Mode. The mode of APTIL distribution is derived by

\[ f'(x) = 0; \]

Figure 1: Plots of the pdf and hrf of the APTIL distribution for different values of parameters.
The mode of APTIL distribution cannot be expressed in the closed form. Computer software, e.g., Mathematica or R, can be used to compute the mode of APTIL distribution for specific values of parameters.

For $\alpha = 1$, the mode of IL distribution can be easily calculated from the following equation:

$$f'(x) = \begin{cases} 
  \frac{ab((b + x)/x)^{-2\alpha} \alpha ((b + x)/x)^{\alpha} \log(a)((a - 1)b - 2x)((b + x)/x)^{\alpha} + ab \log(a)}{x^2 (b + x)^2 (\alpha - 1)} & \text{if } \alpha \neq 1, \\
  \frac{\alpha ((b + x)/x)^{\alpha} - a^2}{x^2 (b + x)^2} & \text{if } \alpha = 1.
\end{cases}$$

(16)

$$\mu'_r = ab^{r+1} \sum_{j=0}^{\infty} \frac{(\log(a))^{r+1}}{(a-1)!} \frac{\Gamma(r + a(i+1)) \Gamma(1-r)}{\Gamma(a(i+1) + 1)}.$$  

(20)

**Proof.** Let $X$ be a r.v. with pdf given in equation (5). For any real number $a > 0, b > 0, \alpha > 0$, and $r \geq 0$, the $r^{th}$ moments of APTIL distribution are obtained as

$$\mu'_r = \int_0^\infty x^r f(x; a, a, b) \, dx.$$

(21)

Using expression from equation (10), we get

$$\mu'_r = \int_0^\infty x^r \sum_{i=0}^{\infty} w_i g_{a(i+1), b}(x) \, dx, \\
\mu'_r = \sum_{i=0}^{\infty} ab^{r+1} \frac{\Gamma(r + a(i+1)) \Gamma(1-r)}{\Gamma(a(i+1) + 1)}, \\
r = 1, 2, 3, \ldots.$$

(22)

where $\Gamma(0) = -\gamma, \Gamma(-r) = ((-1)^r/r!) \varphi(r) - ((-1)^r/r!) \gamma$ for $r = 1, 2, \ldots, \gamma$ denotes Euler’s constant and $\varphi(r) = \sum_{k=1}^{r} 1/k$ [18]. The mean of $X$ can be obtained using equation (20) by putting $r = 1$:

$$\mu = \mu'_1 = E(X) = ab^2 \sum_{i=0}^{\infty} \frac{(\log(a))^{i+1}}{(a-1)!} \frac{\Gamma(1 + a(i+1)) \gamma}{\Gamma(a(i+1) + 1)}.$$ 

(23)

where $\gamma$ denotes Euler’s constant. The $r^{th}$ central moment $\mu_r$ of $X$ is derived as

$$\mu_r = E(X - \mu)^r = \sum_{k=0}^{r} \binom{r}{k} (-1)^{r-k} \mu_r^k.$$ 

(24)

The variance of APTIL distribution is easily obtained as

$$\sigma^2 = \mu'_2 - (\mu'_1)^2 = ab^2 \sum_{i=0}^{\infty} \frac{(\log(a))^{i+1}}{(a-1)!} \frac{\Gamma(2 + a(i+1)) \Gamma(1-r)}{\Gamma(a(i+1) + 1)} - (\mu'_1)^2.$$ 

(25)

**Lemma 1.** It can be shown that, for APTIL, $\lim_{x \to 0} F(x) = 0$ and $\lim_{x \to \infty} F(x) = 1$

**Proof.**

$$\lim_{x \to 0} F(x) = \lim_{x \to 0} \frac{\alpha(1 + (b/x))^\alpha - 1}{\alpha - 1} = \frac{\alpha^{(1+\alpha)} - 1}{\alpha - 1} = \frac{1 - 1}{\alpha - 1} = 0,$$

$$\lim_{x \to \infty} F(x) = \lim_{x \to \infty} \frac{\alpha(1 + (b/x))^\alpha - 1}{\alpha - 1} = \frac{\alpha^{(1+\alpha)} - 1}{\alpha - 1} = 1.$$

(19)

3.3. Moments

**Theorem 1.** Let $X$ be a r.v. from APTIL distribution, then its $r^{th}$ moment is

$$\tau(a, b, \alpha, r, \delta) = \int_0^\infty x^{r-2} \left(1 + \frac{b}{x}\right)^{\alpha - 1} \alpha^{(1+\alpha)/x} \delta^\delta \, dx,$$

(26)
is calculated as
\[
\tau(a, b, a, r, \delta) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\delta)^j}{j!} \frac{(\log(a))^i}{i!} (b)^{r+j-1} \Gamma(r + a(i + 1) + j) \Gamma(1 - r - j) \Gamma(a(i + 1) + 1),
\]
(27)

Proof. The moment generating function can be derived by
\[
M_x(t) = \frac{ab \log(a)}{\alpha - 1} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(bt)^j}{j!} \frac{(\log(a))^{i+1}}{i!} \Gamma(a(i + 1) + j) \Gamma(1 - r - j) \Gamma(a(i + 1) + 1).
\]
(28)

Using Lemma 2, the moments, moment generating function, characteristic generating function, and raw moment can be easily obtained by
\[
M_X(t) = \frac{ab \log(a)}{\alpha - 1} \tau(a, b, a, 0, t),
\]
\[
\Phi_X(t) = \frac{ab \log(a)}{\alpha - 1} \tau(a, b, a, 0, it),
\]
(30)
\[
E(X^r) = \frac{ab \log(a)}{\alpha - 1} \tau(a, b, a, r, 0).
\]

3.5. Order Statistics. Let \(X_1, X_2, \ldots, X_n\) be a sample from the APTIL\((\alpha, a, b)\) distribution with order statistics \(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\). The pdf of rv \(X_{(r)}\) \((r = 1, 2, \ldots, n)\) is obtained as
\[
f_{X_{(r)}}(x) = \frac{n! \log(a)}{(r - 1)!(n - r)!} x^{r-1} (1 - F^{(r)}(x))^{n-r} f(x), \quad f(x) = \frac{ab \log(a)}{\alpha - 1} \frac{(\log(a))^{i+1}}{i!} \Gamma(a(i + 1) + j) \Gamma(1 - r - j) \Gamma(a(i + 1) + 1).
\]
(31)

The pdf of \(X_{(r)}\) can be expressed as
\[
f_{X_{(r)}}(x) = \frac{n!ab \log(a)}{(r - 1)!(n - r)!} x^{r-1} (1 - \frac{a}{\alpha} x^{1/(\alpha a)})^{n-r} \left(1 - \frac{a}{\alpha} x^{1/(\alpha a)}\right)^{r-1}.
\]
(32)

where \((1 + (b/x))^{-\alpha} = \xi\). Particularly, pdf of the first and \(n^{th}\) order statistics can be easily derived from equation (32) as
\[
f_{X_{(1)}}(x) = \frac{nab \log(a)}{\alpha - 1} (\xi^{1/(\alpha a)} - 1) \left(1 - \frac{a}{\alpha} x^{1/(\alpha a)}\right)^{n-1},
\]
(33)
\[
f_{X_{(n)}}(x) = \frac{nab \log(a)}{\alpha - 1} (\xi^{1/(\alpha a)} - 1) \left(1 - \frac{a}{\alpha} x^{1/(\alpha a)}\right)^{n-1},
\]
(34)
respectively.

3.6. Stress-Strength Model. Let \(X_1\) and \(X_2\) be two independent random variables with APTIL\((\alpha_1, a_1, b_1)\) and APTIL\((\alpha_2, a_2, b_2)\) distributions, respectively. If \(X_1\) represents “stress” and \(X_2\) represents “strength,” the reliability is defined by
\[
R = \int_0^\infty f_2(x; a_2, a_2, b_2) f_1(x; a_1, a_1, b_1) dx.
\]
Then, we can write
\[
R = \frac{a_2 b_2 \log(a_2)}{\alpha_2 - 1} \int_0^{\infty} x^{r-2} \left(1 + \frac{b_2}{x}\right)^{1-a_2} a_2^{(b_2/\alpha_2) - a_2} \left(1 + \frac{a_1}{\alpha_1} (b_1/x)^{1-a_1}\right)^{a_1 (b_1/\alpha_1) - a_1} dx
\]
(35)

Using Lemma 2 from equation (27), we have
\[
\begin{align*}
&= \frac{a_2 b_2 \log(a_2)}{(a_1 - 1)(a_2 - 1)} \int_0^\infty a_1^{1+(b/x)} \left( x^{-2} \left( 1 + \frac{b}{x} \right)^{-1-a_2} a_2^{1+(b/x)} \right) dx
\end{align*}
\]

\[
= \frac{1}{(a_1 - 1)} \left[ \frac{a_2 b_2 \log(a_2)}{(a_2 - 1)} \int_0^\infty a_1^{1+(b/x)} \left( x^{-2} \left( 1 + \frac{b}{x} \right)^{-1-a_2} a_2^{1+(b/x)} \right) dx \right]
\]

\[
= \frac{a_1 b_1 \log(a_1)}{(a_1 - 1)(a_2 - 1)} \tau(a_2, b_2, a_2, 0, 0)
\]

\[
R = \frac{1}{(a_1 - 1)} E \left[ a_1^{1-(b/x)} \right] - \frac{a_2 b_2 \log(a_2)}{(a_1 - 1)(a_2 - 1)} \tau(a_2, b_2, a_2, 0, 0).
\]

The effect of parameters \(a\), \(b\), and \(\alpha\) on mean, variance, skewness, and kurtosis is displayed in Figures 2 and 3, respectively.

**Remark 1**

(i) The mean and variance of APTIL distribution increase as “\(a\)” or “\(b\)” increase for fixed value of \(\alpha\)

(ii) For increasing \(\alpha\), the mean of distribution decreases as “\(b\)” decreases for higher value of “\(a\)” and increases for lower value of “\(a\)”

(iii) For increasing \(\alpha\), the variance of distribution increases as “\(b\)” increases for lower value of “\(a\)”

(iv) The skewness and kurtosis of APTIL distribution decrease as “\(a\)” increases or “\(b\)” decreases for fixed value of \(\alpha\)

3.7. Characterization Based on Hazard Function. In this section, characterizations of APTIL distribution based on the hazard function are presented. It is known that hazard function \(h(x)\) satisfies the following differential equation:

\[
\frac{f'(x)}{f(x)} = \frac{h'(x)}{h(x)} - h(x).
\]

**Theorem 2.** Let \(X: (0, \infty)\) be a continuous r. v. with pdf (5) iff its hrf \(h(x)\) satisfies the differential equation:

\[
h'(x) - (a + 1)x^{-2} \left( 1 + \frac{b}{x} \right)^{-1} h(x) = \frac{ab((b+x)/x)^{-2a} a((b+x)/x)^{-1} \log(a)}{x^2 (b+x)^2 (a - a((b+x)/x)^{-1})^2 \left( -\alpha + a((b+x)/x)^{-1} \right) + ab \ln(a)},
\]

under the boundary conditions \(h(0) \geq 0\).

\[
h'(x) = \frac{ab((b+x)/x)^{-2a} a((b+x)/x)^{-1} \log(a)}{x^2 (b+x)^2 (a - a((b+x)/x)^{-1})^2 \left( -\alpha + a((b+x)/x)^{-1} \right) + ab \ln(a)},
\]

\[
h'(x) - (a + 1)x^{-2} \left( 1 + \frac{b}{x} \right)^{-1} h(x) = \frac{a(1+a)b^2 (1 + (b/x))^{-a-2} a((b+x)/x)^{-1} \log(a)}{x^4 (a-1) (1 - (a((b+x)/x)^{-1}/a - 1))}
\]

\[
+ \left\{ \frac{ab((b+x)/x)^{-2a} a((b+x)/x)^{-1} \log(a)}{x^2 (b+x)^2 (a - a((b+x)/x)^{-1})^2 \left( -\alpha + a((b+x)/x)^{-1} \right) + ab \ln(a)} \right\}.
\]

Proof. If rv \(X\) has the hrf given in (8), then
Now the result follows.
Conversely if equation (38) holds, then
\[
\frac{d}{dx} \left[ (1 + b/x)^{a+1} h(x) \right] = \frac{d}{dx} \left[ \frac{ab(1 + (b/x))^{a+1} ((b + x)/x)^{-a} \alpha^{-\alpha} \log(\alpha)}{x^2 [\alpha - \alpha ((b + x)/x)^{-\alpha}]} \right],
\]
which implies \(C = 0\).

4. Estimation of Parameters

The parameters of the APTIL distribution are estimated using various methods including maximum likelihood (ML), least squares (LS), weighted least squares (WLS), percentile (PC), Cramer–von Mises (CV), maximum product of spacing (MPS), Anderson–Darling (AD), and right-tail Anderson–Darling (RTAD) methods of estimation.

4.1. ML Estimation. Let \(X_1, \ldots, X_n\) have the observed values from APTIL distribution. The MLs of the proposed model parameters \(\alpha, a,\) and \(b\) are derived using the log-likelihood function say \(\ell\) which is given by
\[
\ell = n \ln(\log(\alpha)) - n \ln(\alpha - 1) + n \ln(\alpha) + n \ln(b)
\]

\[
= -2 \sum_{i=1}^{n} \log(x_i) - (a + 1) \sum_{i=1}^{n} \ln \left(1 + \frac{b}{x_i}\right) + \ln(\alpha) \sum_{i=1}^{n} \left(1 + \frac{b}{x_i}\right)^{-\alpha}.
\]

(41)
The ML equations of the APTIL distribution are given by
\[
\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} \log(\alpha) - \frac{n}{\alpha - 1} + \frac{1}{\alpha} \sum_{i=1}^{n} \left(1 + \frac{b}{x_i}\right)^{-\alpha},
\]
\[
\frac{\partial \ell}{\partial a} = \frac{n}{\alpha} \sum_{i=1}^{n} \ln \left(1 + \frac{b}{x_i}\right) - \ln(\alpha) \sum_{i=1}^{n} \left(1 + \frac{b}{x_i}\right)^{-\alpha} \ln \left(1 + \frac{b}{x_i}\right),
\]
\[
\frac{\partial \ell}{\partial b} = \frac{n}{b} - (a + 1) \sum_{i=1}^{n} \frac{1}{x_i + b} + a \ln(\alpha) \left(1 + \frac{b}{x_i}\right)^{-\alpha - 1} \frac{1}{x_i}.
\]

(42)
Equating \(\partial \ell/\partial \alpha, \partial \ell/\partial a,\) and \(\partial \ell/\partial b\) with zeros and solving simultaneously, we obtain the ML estimators of \(\alpha, a,\) and \(b\).

4.2. Ordinary and Weighted LS Estimators. Suppose \(X_1, X_2, \ldots, X_n\) is a random sample from APTIL distribution with corresponding ordered sample of \(X_{(1)}, X_{(2)}, \ldots, X_{(n)}\). The mean and variance of APTIL are independent of unknown parameter and are as follows:

\[
\begin{aligned}
\mu &= \frac{\alpha - 1}{\alpha - \frac{a}{n}} \left(1 + \frac{b}{x_{(1)}}\right)^{a-1}, \\
\sigma^2 &= \frac{\alpha - \frac{a}{n}}{\alpha - \frac{a}{n}} \left(1 + \frac{b}{x_{(1)}}\right)^{a-2},
\end{aligned}
\]

\[
E(F(X_{(i)})) = \frac{i}{n + 1},
\]

(43)
\[
\text{var}(F(X_{(i)})) = \frac{i(n - i + 1)}{(n + 1)^2 (n + 2)},
\]

(44)
where \(F(X_{(i)})\) is the cdf of APTIL distribution with \(X_{(i)}\) being the \(i\)th order statistic. Then, LS estimators ([19]) are obtained by minimizing the SSE:

\[
\sum_{i=1}^{n} \left[ F_{(i)}(x) - \frac{i}{n + 1} \right]^2,
\]

(45)
with respect to \(\alpha, a,\) and \(b\). So, the LS estimators (LSEs) of the parameters \(\alpha, a,\) and \(b\) of the APTIL are obtained by minimizing the following:

\[
\sum_{i=1}^{n} \left[ a(1 + (b/x_{(i)}))^\alpha - 1 - \frac{i}{n + 1} \right]^2,
\]

(46)
with respect to \(\alpha, a,\) and \(b\).

4.3. PC Estimator (PCE). Let \(X_{(1)} < X_{(2)} < \cdots < X_{(n)}\) be order statistics. Using on PC method of estimation ([20, 21]), the estimators of \(\alpha, a,\) and \(b\) are derived by minimizing the following:

\[
\sum_{i=1}^{n} \left[ \ln \left(\frac{i}{n + 1}\right) - \ln \left(\frac{a(1 + (b/x_{(i)}))^\alpha - 1 - \frac{i}{n + 1}}{\alpha - 1}\right) \right]^2,
\]

(47)
with respect to \(\alpha, a,\) and \(b\).

4.4. The Cramer-von Mises Minimum Distance Estimators. The CV method is based on the distance between the estimated cdf and the empirical cdf ([22, 23]). The CV estimators are obtained by minimizing

\[
\text{CV} = \frac{1}{12n} + \sum_{i=1}^{n} \left[ \frac{a(1 + (b/x_{(i)}))^\alpha - 1 - \frac{i}{n + 1}}{\alpha - 1} - \frac{2i - 1}{2n} \right]^2.
\]

(48)
Macdonald [24] stated about the CV method that it depends on minimum distance estimators providing empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators.

4.5. Maximum Product of Spacing Estimators. The MPS method is a powerful alternative to the ML method for estimating the population parameters of continuous distributions ([25]). Let

\[ D_i(\alpha, a, b) = F(x_{(i)} | \alpha, a, b) - F(x_{(i-1)} | \alpha, a, b), \]

\[ i = 1, 2, \ldots, n + 1, \tag{49} \]

be the uniform spacings of a random sample from the APTIL distribution, where

\[ F(x_{(0)} | \alpha, a, b) = 0, \]
\[ F(x_{(n+1)} | \alpha, a, b) = 1, \]
\[ \sum_{i=1}^{n+1} D_i(\alpha, a, b) = 1. \tag{50} \]

The MPS estimator is obtained by maximizing the geometric mean (GM) of the spacings:

\[ \text{GM}(\alpha, a, b) = \left\{ \prod_{i=1}^{n+1} D_i(\alpha, a, b) \right\}^{1/(n+1)}, \tag{51} \]

w.r.t. \( \alpha, a, \) and \( b \). The MPS estimator of \( \alpha, a, \) and \( b \) can be obtained by maximizing the logarithm of the GM of sample spacing’s equation (51). No closed solution exists, so the numerical method is used to find the estimates.

4.6. Anderson-Darling and Right-Tail Anderson-Darling Estimators. The method of Anderson–Darling estimation was introduced by [26] in the context of statistical tests. By adapting it to the APTIL model, the Anderson–Darling estimates (ADEs) of \( \alpha, a, \) and \( b \) can be obtained by minimizing, with respect to \( \alpha, a, \) and \( b \), the function given by

\[ \text{AD} = -n - \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left\{ \log[F(x_{(i)}; \alpha, a, b)] + \log[1 - F(x_{(n+1-i)}; \alpha, a, b)] \right\}. \tag{52} \]
### Table 1: Estimates and MSEs of APTIL distribution for ML, LS, WLS, CV, PC, MPS, AD, and RTAD estimates for set 1.

| n   | MLEs | LSEs | WLSEs | CVEs | PCEs | MPSEs | ADEs | RTADEs |
|-----|------|------|-------|------|------|-------|------|--------|
| 0.5 | 0.16  | 0.54  | 0.18  | 0.17  | 0.15  | 0.12  | 0.22  | 0.31   |
| 50  | 0.679 | 0.921 | 0.827 | 0.197 | 0.813 | 0.084 | 0.030 | 0.882  |
| 100 | 2.151 | 1.561 | 1.695 | 0.345 | 1.693 | 0.243 | 1.789 | 0.009  |
| 200 | 1.983 | 1.684 | 1.803 | 0.274 | 1.737 | 0.278 | 1.798 | 0.029  |

### Table 2: Estimates and MSEs of APTIL distribution for ML, LS, WLS, CV, PC, MPS, AD, and RTAD estimates for set 2.

| n   | MLEs | LSEs | WLSEs | CVEs | PCEs | MPSEs | ADEs | RTADEs |
|-----|------|------|-------|------|------|-------|------|--------|
| 0.5 | 0.16  | 0.54  | 0.18  | 0.17  | 0.15  | 0.12  | 0.22  | 0.31   |
| 50  | 0.679 | 0.921 | 0.827 | 0.197 | 0.813 | 0.084 | 0.030 | 0.882  |
| 100 | 2.151 | 1.561 | 1.695 | 0.345 | 1.693 | 0.243 | 1.789 | 0.009  |
| 200 | 1.983 | 1.684 | 1.803 | 0.274 | 1.737 | 0.278 | 1.798 | 0.029  |

### Table 3: Estimates and MSEs of APTIL distribution for ML, LS, WLS, CV, PC, MPS, AD, and RTAD estimates for set 3.

| n   | MLEs | LSEs | WLSEs | CVEs | PCEs | MPSEs | ADEs | RTADEs |
|-----|------|------|-------|------|------|-------|------|--------|
| 0.5 | 0.16  | 0.54  | 0.18  | 0.17  | 0.15  | 0.12  | 0.22  | 0.31   |
| 50  | 0.679 | 0.921 | 0.827 | 0.197 | 0.813 | 0.084 | 0.030 | 0.882  |
| 100 | 2.151 | 1.561 | 1.695 | 0.345 | 1.693 | 0.243 | 1.789 | 0.009  |
| 200 | 1.983 | 1.684 | 1.803 | 0.274 | 1.737 | 0.278 | 1.798 | 0.029  |

### Table 4: Estimates and MSEs of APTIL distribution for ML, LS, WLS, CV, PC, MPS, AD, and RTAD estimates for set 4.

| n   | MLEs | LSEs | WLSEs | CVEs | PCEs | MPSEs | ADEs | RTADEs |
|-----|------|------|-------|------|------|-------|------|--------|
| 0.5 | 0.16  | 0.54  | 0.18  | 0.17  | 0.15  | 0.12  | 0.22  | 0.31   |
| 50  | 0.679 | 0.921 | 0.827 | 0.197 | 0.813 | 0.084 | 0.030 | 0.882  |
| 100 | 2.151 | 1.561 | 1.695 | 0.345 | 1.693 | 0.243 | 1.789 | 0.009  |
| 200 | 1.983 | 1.684 | 1.803 | 0.274 | 1.737 | 0.278 | 1.798 | 0.029  |
Similarly, the right-tail Anderson–Darling estimates (RTADEs) of $\alpha$, $a$, and $b$ can be obtained by minimizing, with respect to $\alpha$, $a$, and $b$, the function given by

$$\text{RTAD} = \frac{n}{2} - 2 \sum_{i=1}^{n} \left\{ \log \left[ F \left( x_{(i)} \mid \alpha, a, b \right) \right] \right\}$$

$$- \frac{1}{n} \sum_{i=1}^{n} (2i - 1) \left\{ \log \left[ 1 - F \left( x_{(n+1-i)} \mid \alpha, a, b \right) \right] \right\}.$$  

(53)

5. Simulation Study

Here, we come up with a numerical study to compare the behavior of different estimates. We generate 1000 random samples of size $n = 50, 100, \text{and} 200$ from the APTIL distribution. Four sets of the parameters are assigned as follows: set1 = $(\alpha = 0.5, b = 0.5, a = 2)$, set2 = $(\alpha = 0.5, b = 0.5, a = 3)$, set3 = $(\alpha = 0.3, b = 0.5, a = 2)$, and set4 = $(\alpha = 0.3, b = 0.5, a = 3)$. The MLE, LSE, WLS, CVE, PCE, MPSE, ADE, and RTADE of $\alpha$, $a$, and $b$ are determined. Then, the
estimates of all methods and their mean square errors (MSEs) are documented in Tables 1–4.

Form Table 5, for the parameter combinations, we can conclude that the ML estimation method outperforms all the other estimation methods (overall score of 20.5). Therefore, depending on our study, we can consider the ML estimation method is the best for APTIL distribution.

### 6. Applications

In this section, we utilized three data sets to show that APTIL can be a better life testing distribution compared with some known probability distributions such as APT Weibull (APW) distribution [9], alpha power transformed inverse exponential (APTIE) distribution [10],

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**Table 7: Goodness of fit measures for the data sets.**

| Model  | ℓ   | AIC  | BIC  | A*    | W*    |
|--------|-----|------|------|-------|-------|
| APTIL  | −216.492 | 438.984 | 444.597 | 0.27257 | 0.038885 |
| APW    | −225.272 | 456.544 | 462.158 | 1.52944 | 0.207808 |
| APIE   | −236.609 | 477.217 | 480.960 | 6.46851 | 1.221870 |
| APP    | −263.177 | 530.354 | 534.097 | 10.7680 | 2.139800 |
| MOLBE  | −224.746 | 453.492 | 457.235 | 1.55599 | 0.216626 |
| IL     | −235.826 | 475.652 | 479.394 | 6.48532 | 1.236730 |

| Model  | ℓ   | AIC  | BIC  | A*    | W*    |
|--------|-----|------|------|-------|-------|
| APTIL  | −197.035 | 400.070 | 405.684 | 0.26554 | 0.0345284 |
| APW    | −198.597 | 403.195 | 408.871 | 0.857503 | 0.1104240 |
| APIE   | −200.564 | 437.154 | 440.896 | 5.353100 | 0.9553700 |
| APP    | −197.201 | 398.403 | 402.145 | 0.372636 | 0.0614331 |
| IL     | −205.056 | 414.112 | 417.854 | 2.530860 | 0.4221020 |

| Model  | ℓ   | AIC  | BIC  | A*    | W*    |
|--------|-----|------|------|-------|-------|
| APTIL  | −151.910 | 309.819 | 314.023 | 0.430672 | 0.056662 |
| APW    | −153.147 | 312.293 | 316.497 | 0.491873 | 0.069053 |
| APIE   | −153.372 | 310.744 | 313.546 | 0.596184 | 0.089069 |
| APP    | −156.025 | 314.169 | 316.972 | 0.761996 | 0.108866 |
| MOLBE  | −155.336 | 314.673 | 317.475 | 2.259080 | 0.323858 |
| IL     | −153.639 | 310.279 | 315.081 | 0.458900 | 0.066987 |

**Figure 4:** TTT Plots for all three data sets.
Marshall Olkin length biased exponential (MOLBE) distribution [27], APT inverted Weibull (APIW) distribution [14], APT Pareto (APTP) distribution [17], and inverse Lomax (IL) distribution [4]. The corresponding probability density functions of competitor models are given below:

![Figure 5: Estimated pdf and cdf of the APTIL model and other competing models for first data.](image)

![Figure 6: Estimated pdf and cdf of APTIL model and other competing models for second data.](image)
The first dataset was originally reported by [28] which represents the maximum annual flood discharges of the North Saskatchewan in unit of 1000 cubic feet per second, of the North Saskatchewan River at Edmonton, over a period of 47 years. The data are as follows: 19.885, 20.940, 21.820, 23.700, 24.888, 25.460, 25.760, 26.720, 27.500, 28.100, 28.600, 30.200, 30.380, 31.500, 32.600, 32.680, 34.400, 35.347, 35.700, 38.100, 39.020, 39.200, 40.000, 40.400, 42.250, 44.020, 44.730, 44.900, 46.300, 46.300, 50.330, 51.442, 57.220, 58.700, 58.800, 61.200, 61.740, 65.440, 65.597, 66.000, 74.100, 75.800, 84.100, 106.600, 109.700, 121.970, 121.970, and 185.560.

The second dataset represents the marks in Mathematics for 48 students in the slow pace programme in the year 2013 [29]. The data are as follows: 29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23,

\[

\begin{align*}
  f_{APIE}(x) &= \frac{\lambda \log(\alpha)}{\alpha - 1} x^{\frac{\alpha - 1}{\alpha}} e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}} \\
  f_{APW}(x) &= \frac{\log(\alpha) \beta}{\alpha - 1} x^{\frac{\alpha - 1}{\alpha}} e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}} \\
  f_{APIW}(x) &= \frac{\log(\alpha) \beta}{\alpha - 1} x^{\frac{\alpha - 1}{\alpha}} e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}} \\
  f_{APW}(x) &= \frac{\beta \log(\alpha)}{\alpha - 1} x^{\frac{\alpha - 1}{\alpha}} e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}} \\
  f_{MOLBE}(x) &= \frac{\left(\frac{x}{\beta}\right)^\alpha x e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}} + \left(1 - (1 - \alpha) + (\frac{x}{\beta}) e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}}\right) x}{\left(1 - (1 - \alpha) + (\frac{x}{\beta}) e^{-(\lambda x) x^{\frac{\alpha}{\alpha - 1}}}\right)^2}.
\end{align*}
\]
58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, and 31.

The third data consists of a sample of 30 failure times of air-conditioned system of an airplane [30]. The data are as follows: 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, and 95.

For the selection of best fit model, we used the following criteria: Akaike information criterion (AIC), Bayesian information criterion (BIC), Anderson–darling (A*), and Cramer–von Mises (W*) test. The maximum likelihood estimates are presented in Table 6, and the goodness of fit measures are presented in Table 7.

We can use the likelihood ratio (LR) test to compare the fit of the ALTIL distribution with other models for given data sets. The form of the test is suggested its name

\[
LRT = 2 \log \left( \frac{L_0(\hat{\theta})}{L_0(\hat{\theta})} \right),
\]

where the LR is the ratio of two likelihood functions; the simpler model \( s \) has fewer parameters than the general \( g \) model. The LR test rejects the null hypothesis if \( \chi^2 > \chi^2_{0.05} \),

where \( \chi^2 \) denotes the upper 100% point of the \( \chi^2 \) distribution.

The shape hazard function for modeling can be analyzed using graphical technique called total time in the test (TTT) plot (for more details, see [31]). From Figure 4, for the first and second data, the TTT plot is concave and provides evidence of the monotonic hazard rate. For the third data set, the TTT plot is convex and according to [31], it provides evidence that the hazard rate is decreasing.

The ALTIL distribution gives the lowest values of AIC, BIC, A*, and W* tests among all the fitted models to these data sets. So, it could be selected as the best model among them. The fitted pdf and estimated cumulative distribution function of the ALTIL are displayed in Figures 5–7 for the three data sets, respectively.

The empirical data and estimated density plot show closeness which depict that the ALTIL model fits all three data sets well. The ALTIL model is compared with other competitive models. The estimated cdf curve of ALTIL model also confirms the above results.

The likelihood ratio test is performed to compare the ALTIL distribution with other fitted models to test \( H_0 \) against \( H_1 \) discussed above, and results are shown in Table 8. Low values of the likelihood ratio mean that the observed result was much less likely to occur under the null hypothesis as compared with the alternative. We conclude that ALTIL distribution provides better fit than all other competitive models at the level of significance \( \leq 0.05 \).

7. Conclusions

In this research, we proposed and studied the ALTIL distribution. Some structural characteristics of the ALTIL distribution are derived. The asymptotic behavior of its density function is studied. Characterization related to hazard rate function is also obtained. Estimation of the population parameters is achieved using eight various procedures. Simulation results are carried out to assess the performance of estimators. Real data sets are used for the applications to show the flexibility of the ALTIL model.

Appendix

A. Proof of Lemma 2

By using the series representation of exponential function and Taylor’s series expansion of the function \( e^{ax} \) in equation (27), we have

\[
\tau(a, b, r, \delta) = \sum_{i=0}^{\infty} \frac{(\delta)^i}{i!} n \cdot p \cdot q^{n+j-1} \Gamma(a, b + x) \cdot \Gamma(a + j + 1) 
\]

We use the following relation given by [30],

\[
\int_{0}^{\infty} x^{n-1} (p + qx)^{-(n+1)} dx = \frac{1}{\Gamma(1+n)} \frac{1}{\Gamma(1+n)} \Gamma(1+n-1) 
\]

\[
\frac{1}{\Gamma(1+n-1)} \frac{1}{\Gamma(1+n-1)} \Gamma(1+n-1) 
\]

B. Code for Simulation

\[
u^{(j)} := \text{runif}(n, 0, 1),
\]

\[
x_{i,j} := \left[ \frac{b1}{\sqrt{\log(\alpha1) \cdot (\log(\text{ratio} + (\alpha1 - 1) + 1))}} - 1 \right].
\]

\[
t^{(j)} := \text{sort}(x^{(j)}).
\]

For MLE and MPSE,

\[
SA1(j) := \text{Maximize} \left( \text{log } L1, a, \alpha, b \right).
\]

For other methods,

\[
YA2(j) := \text{Minimize} \left( L2, a, \alpha, b \right).
\]

Data Availability

Data are given within the manuscript, and they are available in Section 6.
Conflicts of Interest

The authors declare that they have no conflicts of interest.

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