Timescape realized

Herbert Balasin

Institut
für
Theoretische Physik
TU-Wien

Wiedner Hauptstraße 8-10
1040 Wien
AUSTRIA

Abstract

We discuss a concrete proposal to realize the observer dependence of expansion redshift employing a map that relates static to conformally static geometries. This provides a manifest realization of Wiltshire’s proposal for observer selection which serves as a model to explain accelerated expansion
Introduction

Recently Wiltshire [1, 2] has advocated a re-interpretation of “dark”-energy (or in a more pristine language a cosmological constant) as an observer selection effect purely compatible with standard General Relativity (GR). Although his approach incorporates the general strategy that averaging Einstein’s equations may not coincide with the Einstein-tensor of the averaged geometries – the difference accounting for dark energy – an approach usually called solving the back-reaction problem [3, 4, 5], Wiltshire’s work contains an important additional observation connected with a genuine generally-relativistic effect – the non-uniqueness of time.

In particular his proposal focuses on the selection of observers due to the fact that matter in the present epoch of the evolution of the universe is concentrated in web-like structures that bound huge, nearly matterless voids. His proposal relies on the fact that according to GR regions with higher matter density slow down clocks locally, whereas time in void-regions ticks relatively faster, neither of which needs to coincide with cosmic time of a corresponding averaged FLRW-model. In his original article [1] Wiltshire provides an interesting static example, namely the so-called Majumdar-Papapetrou metric [6], an electro-vacuum solution of Einstein’s equations consisting of an arbitrary number of extremely-charged black-holes. On a large scale this may be taken as constant density $k = 0$ “Einstein-cylinder”, whereas “microscopically”, within one of the bound regions, time between two homogeneous slices in general ticks slower. As Wiltshire himself points out this model captures only part of the idea namely the gravitational slowing of clocks, whereas the cosmological redshift is missing.

In the following we propose a geometrical model that incorporates this very effect and thus may provide a simple and exact setting for Wiltshire’s idea. We will first focus on basic geometrical ideas namely similarities between the standard gravitational redshift and its analogue due to expansion. In the following this observation will provide a generalization of this similar-
ity to a map between static and conformally static geometries, i.e. geometries admitting a timelike, hypersurface-orthogonal Killing vector and a timelike, hypersurface-orthogonal conformal Killing vector respectively. Thereafter we apply this setting to an arbitrary static geometry and calculate the corresponding redshift relation which provides an exact expression describing the mixing between cosmological and stationary effects. This result holds in particular true for extension of Wiltshire’s original example the Majumdar-Papapetrou geometry. However, in order to account for an initial homogeneity a simple, rather obvious further generalization is needed. This observation entails further that the conformal class of our models is Friedmann or equivalently $k$-Einstein.

1 Redshifts

In order to motivate our approach let us consider the following simple observation that unifies the calculation of redshift in static and cosmological spacetimes. In order to derive the redshift in static (stationary) geometries in a geometrical manner relative to static (Killing-) observers [7] one uses the geometric fact that the inner product between the Killing $\xi^a$ and the tangent $k^a$ to an arbitrary autoparallel remains constant along the latter, i.e.

$$ (k \nabla)(\xi_a k^a) = k^a k^b \nabla_b \xi_a = 0 $$

since $(k \nabla)k^a = 0$ and $\nabla_v (a \xi_b) = 0$. (1)

Normalizing $\xi^a$ turns it into the four-velocity $u^a$ of the static observer and therefore the frequency of a light-signal relative to this observer is given by

$$ \omega = -u_0 k^a = -\frac{\xi_0 k^a}{\sqrt{-\xi^2}}. $$
In adapted coordinates the static geometry and its Killing become

\[ ds^2 = -V^2(x^k)dt^2 + h_{ij}(x^k)dx^idx^j \]
\[ \xi^a = \partial^a_t. \]

Tagging the emitter with \( E \) and a possible observer along another Killing trajectory with \( O \), we then have

\[
\omega_E = -\frac{\xi^a k^a}{\sqrt{-\xi^2}} \bigg|_E = -\frac{\xi^a k^a}{\sqrt{-\xi^2}} \bigg|_O = \\
= \omega_O \frac{\sqrt{-\xi^2}}{\sqrt{-\xi^2}} \bigg|_E = \omega_O \frac{V(x^k_O)}{V(x^k_E)},
\]

which is nothing but the redshift between different static observers.

Let us now consider the analogous cosmological situation. Usually a different kind of argument is put forward relying on the spatial symmetries of the models (cf. [7], p.103). However, we will show that the situation is completely analogous to the static case if we take into account that (I) remains intact if we replace \( \xi^a \) by a timelike, conformal-Killing as long as \( k^a \) remains null, i.e. a light-ray (which we did assume before anyways). In particular all the FLRW-geometries admit a conformal Killing-vector \( \xi^a \) as can be seen from

\[
ds^2 = -dt^2 + a^2(t)\left( \frac{dr^2}{1 - kr^2} + r^2d\Omega^2 \right) = a^2(t)\left( -\frac{dt^2}{a^2(t)} + \frac{dr^2}{1 - kr^2} + d\Omega^2 \right) = a^2(\eta)(-d\eta^2 + \frac{dr^2}{1 - kr^2} + d\Omega^2) \]
\[ \xi^a = \partial^a_\eta = a(t)\partial^a_t, \]

where – by a slight abuse of notation – we have used the same symbol \( a \) for both the scale factor as function of \( t \) as well as \( \eta \) respectively. Now the
redshift formula is actually completely identical, since
\[(k \nabla)(k^a \xi_a) = k^a k^b \nabla_a \xi_b = \omega k^a k^b g_{ab}\]
since \((k \nabla) k^a = 0\), \(k^a k^b g_{ab} = 0\) and \(\nabla_a \xi_b = \omega g_{ab}\)

The only change with respect to the static situation comes about from the norm of the conformal Killing vector
\[
\omega_E = -u^a_E k_a = -\frac{\xi^a k_a}{\sqrt{-\xi^2}} \bigg|_{E} = -\frac{\xi^a k_a}{\sqrt{-\xi^2}} \bigg|_{O}
= \frac{\sqrt{-\xi^2}|_O}{\sqrt{-\xi^2}|_E} \omega_O = \frac{a(t_O)}{a(t_E)} \omega_O.
\] (2)

Up to now these are just calculations with respect to different spacetime-families which exhibit some similarities. However we will show that they lend themselves to providing a map between this very spacetimes. This can already be seen in the FLRW-context, where the \(k\)-“Einstein-cylinder”
\[
ds^2 = -d\eta^2 + dr^2/(1 - kr^2) + r^2 d\Omega^2
\]
maps onto the Friedmann metric
\[
ds^2 = a^2(\eta)(-d\eta^2 + dr^2/(1 - kr^2) + r^2 d\Omega^2)
= -dt^2 + a^2(t)(dr^2/(1 - kr^2) + r^2 d\Omega^2)
\]
and the Killing \(\partial^a_\eta\) maps to the conformal Killing \(\partial^a_\eta = a(t) \partial^a_t\) respectively.
2 Mapping static spacetimes to spacetimes with a timelike hypersurface-orthogonal conformal Killing vector

Motivated by the previous analogy, we will now consider the general situation of mapping a static spacetime $\hat{g}_{ab}$ to a spacetime $g_{ab}$ admitting a conformal, hyperspace-orthogonal Killing vector. Let us begin with the static spacetime, which may be written in adapted coordinates as

$$d\hat{s}^2 = -V^2(x^k)dt^2 + h_{ij}(x^k)dx^i dx^j$$

$$\xi^a = \partial^a_t, \quad \xi^2 = -V^2,$$

which explicitly displays the Killing property $L_{\xi} \hat{g}_{ab} = 0$. Denoting Killing-time $t$ by $\eta$ and mapping $d\hat{s}^2$ onto

$$ds^2 = a^2(\eta)d\hat{s}^2 = -V^2(x^k)a^2(\eta)d\eta^2 + a^2(\eta)h_{ij}(x^k)dx^i dx^j$$

$$= -V^2(x^k)dt^2 + a^2(t)h_{ij}(x^k)dx^i dx^j,$$

we clearly see that $\partial^a_\eta = a(t)\partial^a_t$ is now a hypersurface-orthogonal, conformal Killing of the mapped $g_{ab} = a^2(t)\hat{g}_{ab}$.

Let us now turn to the converse: given a geometry $g_{ab}$ admitting a hypersurface-orthogonal conformal Killing $\xi^a$ such that $\nabla_a \eta \in < \xi_a >$ is there a conformally related metric $\hat{g}_{ab}$ such that $\xi^a$ is a static Killing?

In general we have

$$L_{\xi} \hat{g}_{ab} = 2\omega g_{ab}. \quad (3)$$

We require the existence of a conformal factor $\Omega$ such that $\xi^a$ is Killing relative to the rescaled metric, i.e.

$$0 = L_{\xi}(\Omega^{-2}g_{ab}) = -2\Omega^{-3}L_{\xi}\Omega g_{ab} + \Omega^{-2}2\omega g_{ab}.$$
which entails \( L_\xi \Omega/\Omega = \omega \), which in turn may be solved by a simple integration thereby establishing \( \tilde{g}_{ab} = \Omega^{-2}g_{ab} \) with the required properties. More explicitly, i.e. in local coordinates adapted to the conformal Killing vector the metric becomes

\[
ds^2 = -w(\eta, x^k)d\eta^2 + w_{ij}(\eta, x^k)dx^idx^j \quad \xi^a = \partial_{\eta}^a.
\]

Condition (3) is turned into

\[
-w'd\eta^2 + w'_{ij}dx^idx^j = -2\omega w d\eta^2 + 2\omega w_{ij}dx^idx^j, \tag{4}
\]

where the prime denotes the derivative with respect to \( \eta \). We thus obtain from (4) two conditions, the first of which \((\log w)' = 2\omega\) is readily integrated to give

\[
w(\eta, x^k) = \eval{w(x^k)}_{\eta_0} \frac{\int_{\eta_{0}}^{\eta} \omega'(\eta')}{\omega(\eta)}
\]

and the second becomes \( w'_{ij} = (\log w)' w_{ij} \) and thus

\[
w_{ij}(\eta, x^k) = \eval{w_{ij}(x^k)}_{\eta_0} \frac{\int_{\eta_{0}}^{\eta} \omega'(\eta')}{\omega(\eta)}
\]

Therefore we find for the static counterpart \( \tilde{g}_{ab} \) of the metric \( g_{ab} \)

\[
\begin{align*}
\tilde{ds}^2 &= -\tilde{w}e^{\frac{\int_{\eta_{0}}^{\eta} \omega'(\eta')}}{d\eta^2} + \tilde{w}_{ij}e^{\frac{\int_{\eta_{0}}^{\eta} \omega'(\eta')}} dx^idx^j \\
&= -\tilde{w}dt^2 + a^2(t)w_{ij}dx^idx^j, \\
\tilde{d}s^2 &= -\tilde{w}(x^k)d\eta^2 + w_{ij}(x^k)dx^idx^j,
\end{align*}
\tag{5}
\]

where we used \( dt = e^{\frac{\int_{\eta_{0}}^{\eta} \omega'(\eta')}}{d\eta} \). This is precisely the form that we started from

\[
g_{ab} = a^2(\eta)\tilde{g}_{ab}
\]
which shows that our map between geometries is invertible.

3 Redshift for geometries with a static, conformal Killing vector

Let us now consider the change in the redshift formula for our mapped geometry. Starting from

\[ ds^2 = -w(x^k)dt^2 + a^2(t)w_{ij}(x^k)dx^i dx^j \]  

(6)

with conformal Killing vector \( \xi^a = a(t)\partial^a_i \) we obtain for the frequencies of two observers following conformal Killing trajectories

\[ \omega_E = \frac{\sqrt{-\xi^2}}{\sqrt{-\xi^2}} \omega_O = \left( \xi^2 = -a^2(t)w(x^k) \right) = \frac{a(t_O)}{a(t_E)} \frac{w(x^k)}{w(x^k)} \omega_O. \]  

(7)

In particular for the Majumdar-Papapetrou geometry

\[ ds^2 = -\frac{1}{V^2} dt^2 + V^2 dx^i dx^i \quad V = 1 + \sum_i \frac{2M_i}{|x^m - x^m_i|} \]

we obtain

\[ \omega_E = \frac{a(t_O)}{V(x^k)} \frac{V(x^k)}{a(t_E)} \omega_O, \]  

(8)

which depends now on spatial position as well as time.

These simple relations clearly display Wiltshire’s idea that gravitational redshift is modified by the observer position. Only in an averaged sense does the original FLRW-formula \(^2\) hold, i.e. if \( V \) (or \( w \)) becomes constant.
Einstein tensor for geometries with a static, conformal Killing vector

Up to now our considerations have been purely kinematical. In order to incorporate dynamics we have to calculate the Einstein-tensor for conformally related geometries. The metric and difference tensor between the corresponding Levi-Civita derivatives are given by

\[ g_{ab} = \Omega^2 \delta_{ab} \]
\[ C^a_{bc} = \nabla_b \log \Omega \delta^a_c + \nabla_c \log \Omega \delta^a_b - \nabla^a \log \Omega \delta_{bc} \]

From this and the expression of the Riemann-tensor

\[ R^a_{bcd} = \delta^a_{bc} + \nabla_b \nabla_c \log \Omega - \nabla_c \nabla_b \log \Omega - \nabla^m \nabla^m \log \Omega g_{ab} \]

we obtain the Einstein-tensor

\[ G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} = \delta^a_{bc} + \nabla_b \nabla_c \log \Omega - \nabla_c \nabla_b \log \Omega - \nabla^m \nabla^m \log \Omega g_{ab} \]

In our particular case \( \Omega = \alpha \) and for the metric (6) the above becomes

\[ R_{ab} - \frac{1}{2} R g_{ab} = \delta^a_{bc} + \nabla_b \nabla_c \log \Omega - \nabla_c \nabla_b \log \Omega - \nabla^m \nabla^m \log \Omega g_{ab} \]

which display a structure similar to the FLRW-models due to the appearance of the containing \( \dot{a}/a \) and its derivative. The static Einstein-tensor plays the role of the spatial curvature. This is actually no co-incidence since
for the “Einstein-cylinder” as static geometry our transformation yields the corresponding FLRW-expression.

5 A possible generalization - conformal Friedmann

Although the previous sections show the main features of the timescape proposal a closer look at the derivation of the map between static geometries and geometries admitting a timelike, hypersurface-orthogonal, conformal Killing vector-field allows a rather natural generalization. To this end let us reconsider the derivation of the map and begin with an arbitrary geometry admitting a timelike, hypersurface-orthogonal, conformal Killing vector-field $\xi^a$. In adapted coordinates $\xi^a$ and the metric $g_{ab}$ become

\[ ds^2 = -w(x^k, \eta) d\eta^2 + w_{ij}(x^k, \eta) dx^i dx^j \]
\[ \xi^a = \partial^a \eta \]

which yields via the conformal Killing equation $L_\xi g_{ab} = 2 \omega g_{ab}$

\[ w'(x^k, \eta) = 2\omega(x^k, \eta) w(x^k, \eta) \]
\[ w'_{ij}(x^k, \eta) = 2\omega(x^k, \eta) w_{ij}(x^k, \eta). \]

These relations are solved by

\[ w(x^k, \eta) = \tilde{w}(x^k) e^{\int_0^\eta \omega(x^k, \eta') d\eta'} , \]
\[ w'_{ij}(x^k, \eta) = \frac{w'(x^k, \eta)}{w(x^k, \eta)} w_{ij}(x^k, \eta) , \]
\[ w_{ij}(x^k, \eta) = w(x^k, \eta) w_{ij}(x^k) . \]
Therefore we have

\[ ds^2 = w(x^k, \eta)(-d\eta^2 + \hat{w}_{ij}(x^k)dx^i dx^j) \]

or if we denote \( w(x^k, \eta) = a^2(\eta, x^k) \)

\[ g_{ab} = a^2(\eta, x^k)\hat{g}_{ab} \quad (10) \]

where \( \hat{g}_{ab} = -d\eta_a d\eta_b + \hat{w}_{ij}(x^k)dx^i_a dx^j_b \) is the corresponding static geometry with \( \xi^a = \partial^a_\eta \) as Killing-vector\(^1\). Moreover, relative to the static geometry \( \xi^a \) is covariantly constant, since \( \nabla^a \hat{\xi}_b = \nabla^a [a \hat{\xi}_b] + \nabla^a (a \hat{\xi}_b) \) and the first term vanishes due to exactness of \( \hat{\xi}_a = -d\eta_a \) and the second by \( \xi^a \) being Killing.

At first glance this might seem strange since in the previous section the static Killing was not covariantly constant. However the previous results may easily be recovered as the special case of factorizing conformal factor, i.e. \( a^2(\eta, x^k) = a^2(\eta)w(x^k) \), which allows to bring the metric into the “preferred” form

\[ ds^2 = a^2(\eta, x^k)(-d\eta^2 + \hat{w}_{ij}(x^k)dx^i dx^j) \]
\[ = a^2(\eta)(-w(x^k)d\eta^2 + w_{ij}(x^k)dx^i dx^j) \]
\[ = -w(x^k)dt^2 + a(t)w_{ij}(x^k)dx^i dx^j. \]

This is precisely the result obtained earlier. However, by admitting arbitrary (i.e. non-factorizing) conformal factors the static part is only defined up to a static, (i.e. \( x^k \)-dependent) conformal transformation.

The redshift relation \((8)\) becomes

\[ \omega_E = \frac{a(\eta_O, x^k_O)}{a(\eta_E, x^k_E)} \omega_O. \quad (11) \]

\(^1\)Actually the last gives rise to an even simpler, geometrical proof. Namely taking the conformal Killing norm we have \( \hat{g}_{ab} = g_{ab}/(-\xi^2) \) and if we take \( (\xi \nabla)\xi^2 = 1/2(\nabla \xi)\xi^2 \) into account \( L_{\xi}\hat{g}_{ab} = 0 \) follows.
Taking the emission time $\eta_E$ to be (close to) decoupling and the corresponding conformal factor position-independent, we easily see that the redshift expression

$$\omega_E = \frac{a(\eta_O, x_O^k)}{a(\eta_E)} \omega_O$$

depends only on the observer position. This is actually an important prerequisite that would have been impossible with a factorizing conformal factor. In particular each “conformal” observer sees an isotropic red-shift but with different temperatures depending on the observer location, as pointed out by Wiltshire.

Finally the Einstein tensor with respect to (9) and (10) becomes

\[
G_{ab} = \frac{\xi_a \xi_b}{2} \left( \frac{\dot{a}'}{a} \right)^2 - 2 \dot{D}^2 \log a - \dot{D}_m \log a \dot{D}^m \log a + 2 \xi_b \left( \frac{\dot{D}_a}{a} \right) + 2 \xi_a \left( \frac{\dot{D}_b}{a} \right) - \left( \frac{\dot{a}'}{a} \right)^2 \dot{D}_m \log a + \left[ 3 \dot{G}_{ab} - 2 \dot{D}_a \dot{D}_b \log a + 2 \dot{D}_a \log a \dot{D}_b \log a + \dot{w}_{ab} \left( -2 (\frac{\dot{a}'}{a})^2 + 2 \dot{D}^2 \log a + \dot{D}_m \log a \dot{D}^m \log a \right) \right].
\]
In particular the terms proportional to the tensor-square of $\xi^a$ exhibit an even greater similarity to the Friedmann-equation than in the case of factorizing conformal factor. This is mainly due to the fact that corresponding static geometry in the general case is closer to the $k$-“Einstein-cylinder”.

In order to provide a model for the right-hand side of the Einstein-equations we consider coupling to standard, i.e. matter-dominated perfect-fluids

$$G_{ab} = 8\pi T_{ab} \quad T_{ab} = \rho u_a u_b,$$

Covariant conservation of $T_{ab}$, i.e. $\nabla^a T_{ab} = 0$, which follows from the Einstein-equations, requires

$$(u \nabla)u_b \rho + \nabla_a (\rho u^a) u_b = 0,$$

which in particular entails the geodeticity of matter, i.e. $(u \nabla)u^a = 0$. However, for the observer comoving with the expansion, i.e. the conformal Killing $\xi^a$ has $\dot{\xi}^a = \xi^a / \sqrt{-\xi^2}$. Therefore

$$(\dot{u} \nabla)\dot{u}^a = \frac{1}{-\xi^2} (\xi \nabla)\xi^a + \frac{\xi^a}{(-\xi^2)^2} \xi^c (\xi \nabla)\xi^c = \frac{1}{-\xi^2} h^a_{\ c} (\xi \nabla)\xi^c.$$

The conformal Killing-property of $\xi^a$ entails

$$(\xi \nabla)\xi_a = -\xi^c \nabla_a \xi_c + \frac{1}{2} (\nabla \xi) \xi_a$$

$$= -\frac{1}{2} \nabla_a \xi^2 + \frac{1}{2} (\nabla \xi) \xi_a,$$

and therefore the acceleration becomes

$$(\dot{u} \nabla)\dot{u}^a = \frac{1}{2\xi^2} h^a_{\ bc} \nabla_c \xi^2,$$

which shows that the latter is in general non-vanishing if the conformal factor has a spatial variation as required by the timescape model. Therefore we find
a relative velocity between the matter and the expansion frame as seems to be suggested by observations like “dark flow” [1, 2].

Summary and Conclusion

In the present note we put forward, based on a simple geometrical observation (from the redshift relations) that static and cosmological spacetimes can actually be treated on the same footing. More precisely this idea gives rise to a map between these geometries. In particular this map should provide a concrete realization of Wiltshire’s proposal of observer dependence for the redshift as can be seen from (7), (11). Taking a closer look at the expression we see that it becomes the standard FLRW-relation if we consider an averaged, i.e. spatially-constant, homogeneous Killing norm. Although our proposal was originally motivated by a “factorizing” conformal factor, as in (6), it is important to note that our results hold even in the more general case discussed in the previous section. This generalization exhibits the possibility of an isotropic, albeit position-dependent, red-shift. Moreover the local conformal relation $ds^2 = \Omega^2 (-d\eta^2 + d\sigma_k^2)$ shows that the expansion vs. proper time relation is position independent, i.e. $\Delta \tau = \Omega \Delta \eta$ and $\Delta l = \Omega \Delta \sigma$ imply $\Delta l/\Delta \tau = \Delta \sigma/\Delta \eta$. It is clear that our observation is just a first preliminary step, but we do hope that it provides a starting point for geometries that can be turned in homogeneous and isotropic models upon averaging and still provides some of the “microscopic” effects that should account for accelerated expansion in Wiltshire’s timescape proposal.

References

[1] Wiltshire D, NewJ.Phys.9 377, (2007)
[2] Wiltshire D, Int.J.Mod.Phys.D18 212, (2009)
[3] Ellis G F R, *General Relativity and Gravitation*, eds. Bertotti, de Felice and Pascolini, Reidel Doderecht 215, (1984)

[4] Buchert T, *Gen.Rel.Grav.* **32**, 105 (2000)

[5] Korzynski M, *Class.Quant.Grav.* **27**, 105015 (2010)

[6] Majumdar S D, *Phys. Rev.* **72** 390, (1947); Papapetrou A, *Proc.Roy.Soc. (London)* **A51** 191, (1947)

[7] Wald R, General Relativity, Univ.Chic.press (1984)

[1] Kashlinsky A, Atrio-Barandela F, Kocevski D and Ebeling H, *Astrophys.J.* 691:1479-1493, (2009)

[2] Wiltshire D, Smale P, Mattesson T and Watkins R, *arXiv*:1201.5371 (2012)