Update of the phase-shift analysis of the low-energy $\pi N$ data

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Abstract

This paper presents a new phase-shift analysis of the low-energy (pion laboratory kinetic energy $T \leq 100$ MeV) pion-nucleon ($\pi N$) data; this solution will be identified as ‘ZRH17’ in future works. In this research programme, the modelling of the $s$- and $p$-wave $K$-matrix elements is achieved by following either of two methods: a) suitable low-energy parameterisations or b) the analytical expressions obtained from a hadronic model (i.e., the ETH model) based on meson-exchange $t$-channel Feynman graphs, as well as on $s$- and $u$-channel graphs with $N$ and $\Delta(1232)$ as intermediate states. Included analytically in the former case are the important direct ($s$-channel) contributions from nearby baryon resonances, e.g., from the $\Delta(1232)$ in the partial wave $P_{33}$ and from the $N(1440)$ (known as Roper resonance) in the partial wave $P_{11}$. In regard to the ETH model, included analytically are the small $s$- and $u$-channel contributions from all well-established (four-star) $s$ and $p$ higher baryon resonances with masses up to 2 GeV. The analysis with the $K$-matrix parameterisations does not impose theoretical constraints and leads to the reliable identification of the outliers in the database, thus enabling the creation of self-consistent input for further analysis. After the removal of the discrepant measurements, the two elastic-scattering databases are jointly submitted to the analysis with the ETH model; the theoretical constraints of crossing symmetry and isospin invariance are imposed at this stage. The optimal values of the model parameters and the corresponding Hessian matrices are obtained from the fits of the ETH model and yield Monte-Carlo predictions for the low-energy constants of the $\pi N$ system, for the $\pi N$ phase shifts, and for the standard observables which are subjected to experimentation. The combined $\pi^+ p$ and $\pi^- p$ charge-exchange (CX) databases are also analysed following the same procedure. Various minimisation techniques have been employed in this programme over time; the Arndt-Roper formula is recently used, enabling the (controlled, i.e., regulated by the reported or assigned normalisation uncertainty of each data set) floating of the input data sets.

This study reveals a number of serious discrepancies in the low-energy $\pi N$ interaction. First, the results of the fits to the two combined databases (e.g., to the $\pi^\pm p$ elastic-scattering databases, and to the combined $\pi^+ p$ and $\pi^- p$ CX databases) significantly differ in regard to two model parameters, resulting in significant differ-
ences in the model predictions for two of the phase shifts, the $s$-wave phase shift $\delta_{0^+}^{1/2}$ (S11) and the $p$-wave phase shift $\delta_{1^+}^{3/2}$ (P33). Second, there is a sizeable difference between the model-predicted $\pi^-p$ $s$-wave scattering length $a^{cc}$ and the experimental results obtained at the $\pi N$ threshold from the strong shift of the $1s$ state in pionic hydrogen. Third, the analysis of the various $\chi^2_{\text{min}}$ values of the fits manifests a difficulty to account for the $\pi^-p$ CX database when imposing the theoretical constraint of isospin invariance. Fourth, the analysis of the scale factors of the fits of the ETH model to the combined $\pi^+p$ and $\pi^-p$ CX databases reveals that the overall tendency of the modelling is to generate overestimated fitted differential cross sections for the $\pi^+p$ reaction and underestimated ones for the $\pi^-p$ CX reaction at low energy. Fifth, significant effects are seen in the reproduction of the absolute normalisation of the experimental data of the $\pi^-p$ CX reaction on the basis of predictions obtained from the results of the fits of the ETH model to the $\pi^+p$ elastic-scattering databases; these effects are increasing with decreasing beam energy. Assuming the correctness of the absolute normalisation of the bulk of the low-energy $\pi N$ data and the negligibility of residual electromagnetic effects, the aforementioned discrepancies may be interpreted as evidence that the isospin invariance does not hold in the hadronic part of the $\pi N$ interaction. The results of this work agree well with the previous studies in this programme, in which the BERTIN76 $\pi^+p$ differential cross sections had not been used.

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Key words: $\pi N$ elastic scattering; $\pi N$ charge exchange; $\pi N$ phase shifts; $\pi N$ coupling constant; low-energy constants of the $\pi N$ system; isospin breaking

1 Introduction

This paper summarises the knowledge acquired in more than two decades of study of the pion-nucleon ($\pi N$) interaction at low energy, in regard to both the theoretical framework, used in the modelling of the experimental data, as well as to the techniques developed in the statistical analysis. The backbone of this research programme is the ETH model, established in its current form shortly after the mid 1990s. After a short introduction to the development of the theoretical basis and of the analysis technique, this paper reports the results of a new data analysis and the predictions (for the low-energy constants of the $\pi N$ system, for the $\pi N$ phase shifts, etc.) emanating thereof.

The low-energy $\pi N$ database (DB) is comprised of measurements of the differential cross section (DCS), analysing power (AP), partial-total cross section (PTCS), and total cross section (TCS) for the two elastic-scattering (ES) processes ($\pi^\pm p \rightarrow \pi^\pm p$) and for the $\pi^-p$ charge-exchange (CX) reaction ($\pi^-p \rightarrow \pi^0n$). In two of the experiments in the $\pi^-p$ CX DB, only the first three of the Legendre-expansion coefficients (LECs) of the DCS are known.
Included in the DB are also the measurements of the strong shift and width of the 1s state in pionic hydrogen, which yield the $\pi N$ s-wave scattering lengths; however, only the measurement of the width is used in the optimisation. Given their accuracy, the measurements of the strong shift are used as the means for assessing the self-consistency of the analysis in terms of the correctness of the absolute normalisation of the available measurements, of the negligibility of residual electromagnetic (EM) effects in the corrections applied to the experimental data, and of the fulfilment of the theoretical constraints of crossing symmetry and isospin invariance (which the partial-wave amplitudes of the ETH model obey).

The term ‘low-energy’ implies the restriction of the pion laboratory kinetic energy $T$ up to 100 MeV; the recent phase-shift analyses (PSAs) in this programme have been performed exclusively in this energy range. There are four important reasons why one should restrict the analysis to low energy.

- The low-energy DB is extensive enough to enable exclusive analyses.
- The parameters of the ETH model are (related to) coupling constants, suitable for analyses at small values of the 4-momentum transfer. The use of the model above the $\Delta(1232)$ resonance would probably necessitate the introduction of form factors.
- It is unlikely that the theoretical constraints, which are valid in the high-energy region, also hold at low energy; one such constraint relates to the isospin invariance in the hadronic part of the $\pi N$ interaction. To estimate the dispersion integrals, dispersion-relation analyses rely (by and large) on high-energy data; therefore, the analysis of the low-energy measurements in an unbiased way (i.e., eliminating high-energy influences) is not possible in such schemes. It was recently shown that one such commonly used partial-wave analysis (PWA) [1] is biased in the low-energy region, in that it yields a solution which does not reflect the behaviour of the low-energy data [2].
- Interest in the low-energy $\pi N$ interaction was maintained for many years by the possibility of extracting the value of the $\pi N \Sigma$ term using the low-energy phase shifts as input; the extrapolation of the $\pi N$ scattering amplitude in the unphysical region is more reliable when exclusively based on ‘close-by’ measurements, avoiding high-energy influences. This possibility was realised in Ref. [3].

Before advancing, a notation which largely facilitates the repetitive referencing to the DBs will be introduced.

- $\text{DB}_+$ for the $\pi^+ p$ DB,
- $\text{DB}_-$ for the $\pi^- p$ ES DB,

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1 In the following, ‘isospin invariance in the $\pi N$ interaction’ will be used as the short form of ‘isospin invariance in the hadronic part of the $\pi N$ interaction’. It is known that isospin invariance is broken in the EM part of the interaction.
• DB0 for the DB relating to the $\pi^- p$ CX reaction,
• DB+/- for the combined DB+ and DB-, and
• DB+/0 for the combined DB+ and DB0.

In addition, the prefix ‘t’ (as, for instance, in tDB+) denotes a ‘truncated’ DB, i.e., the resulting DB after the removal of the outliers. Finally, DOF stands for ‘degree of freedom’ and NDF for ‘number of DOFs’.

The structure of this paper is as follows.

• In Section 2, the two ways of modelling the s- and p-wave $K$-matrix elements are described. a) The physical model contains $\sigma$- and $\rho$-exchange $t$-channel Feynman graphs (simply graphs henceforth), as well as $N$ and $\Delta$s-and $u$-channel graphs. The model is isospin-invariant and also incorporates crossing symmetry. The data description is currently achieved on the basis of seven real parameters. As this model was developed at the ETH (Zurich) during the 1990s, it is conveniently called ‘ETH model’. b) The second model employs simple low-energy parameterisations of the $s$- and $p$-wave $K$-matrix elements; to distinguish it from the ETH model, it will be called ‘phenomenological’ whenever needed. As the forms used in the low-energy parameterisations of the $s$- and $p$-wave $K$-matrix elements do not impose theoretical constraints onto the analysis of the data, the phenomenological model is suitable for the identification of the discrepant measurements (outliers) in the fitted DB and for assessing the self-consistency of the data prior to their submission to further analysis with the ETH model. The data description with the phenomenological model is achieved with seven real parameters for each value of the total isospin $I$ (i.e., for $I = 1/2, 3/2$). At the end of Section 2, a few important details are given regarding the minimisation function and other quantities which are necessary in quantifying the goodness of the reproduction of entire data sets.

• Section 3 presents a new PSA of the low-energy data. Results for the parameters of the ETH model are given, as well as the corresponding predictions for the low-energy constants of the $\pi N$ system and for the phase shifts. A thorough analysis of the scale factors of the experiments contained in the fitted DBs follows, along with the details on the reproduction of data sets which have not been included in the fitted DBs. Finally, the reproduction of the DB0 is investigated on the basis of predictions obtained from the fits of the ETH model to the tDB+/−.

• Section 4 attempts the identification of the origin of the discrepancies observed in the analysis of the low-energy DBs. Three possibilities are discussed. The first is a trivial one, laying the blame for the discrepancies on experimental mismatches. The second attributes the effects (at least, in part) to the incompleteness of the EM corrections applied to the hadronic part of the $\pi N$ amplitude on the way to the evaluation of the observables. In Physics terms, the third possibility is the most compelling one. It posits
the thesis that the discrepancies suggest a departure from the triangle identity, thus attributing the effects to the violation of isospin invariance in the $\pi N$ interaction.

- In the last section, the results are discussed and put into perspective, the understanding of the dynamics of the $\pi N$ system at low energy is summarised, and a number of possibilities for future developments are addressed.

It will be assumed that the physical quantities appearing in this paper, be they the model parameters, the scattering lengths/volumes, the phase shifts, etc., are not purely hadronic as they might contain residual EM effects. Such effects are predominantly associated with the use of the physical (instead of the unknown hadronic) masses of the proton, of the neutron, and of the charged and neutral pions (in the hadronic part of the interaction) in the determination of the EM corrections. Although a part of these effects might have already been captured by the procedure put forward in the determination of the EM corrections in Refs. [4,5], it remains unknown how large any residual effects might be. Evidently, the importance of the residual EM corrections must be assessed prior to advancing to definite conclusions in regard to the level of the isospin breaking in the $\pi N$ interaction. At this time, one cannot but retain the cautious attitude of considering all hadronic quantities in all analyses of the $\pi N$ data as ‘EM-modified’. However, as the repetitive use of this term is tedious, it will be implied but omitted henceforth.

The proper references to the experimental works in the low-energy $\pi N$ interaction may be found in earlier papers; only those of the experimental reports, which attract particular attention in parts of this work, will be explicitly referred to.

2 Assuming that isospin invariance holds in the $\pi N$ interaction, only two (of course, complex) amplitudes enter the physical description of the three $\pi N$ reactions: the $I = 3/2$ amplitude ($f_3$) and the $I = 1/2$ amplitude ($f_1$). Neglecting the EM effects, the $\pi^+ p$ reaction is described by $f_3$, the $\pi^- p$ ES reaction by $(2f_1 + f_3)/3$, and the $\pi^- p$ CX reaction by $\sqrt{2}(f_3 - f_1)/3$. From these relations, the triangle identity is obtained between the three corresponding amplitudes $f_{\pi^+ p}$, $f_{\pi^- p}$, and $f_{CX}$:

$$f_{\pi^+ p} - f_{\pi^- p} = \sqrt{2}f_{CX} .$$

3 In some earlier works in this programme, all such quantities were marked by the symbol ‘~’. In view of the fact that all hadronic quantities obtained from all PSAs/PWAs of the data (be they parameters used in the modelling or the predictions derived from the results of optimisations) are unavoidably affected, there is no such need. Owing to the presence of these residual EM effects, there is no purely hadronic quantity in any of the PSAs/PWAs of the $\pi N$ data, not only in those conducted in the context of this programme.
Modelling of the $s$- and $p$-wave $K$-matrix elements

2.1 The history of the ETH model

The ETH model may be considered as a product of the study of the properties of pionic-atom data of isoscalar nuclei [6,7,8] within the framework of the relativistic mean-field theory of the 1980s; the objective in that investigation was an explanation for the $s$-wave repulsion in the $\pi$-nucleus interaction. In its original form, the model did not contain the off-shell contributions in the $s$- and $u$-channel graphs with a $\Delta(1232)$ intermediate state. Owing to the interest of Leisi’s group at the ETH to measure the strong shift and width of the $1s$ states in pionic hydrogen and deuterium, the emphasis in the early 1990s was placed on the extraction of estimates for the scattering lengths, i.e., for quantities characterising the interaction at the $\pi N$ threshold ($T = 0$ MeV). The first attempts to account for the energy dependence of the then-available phase shifts (up to the $\Delta(1232)$ resonance) turned out to be successful after the inclusion of the spin-1/2 contributions in the $\Delta(1232)$ propagator [9,10,11].

Of paramount importance in the theoretical foundation of the model was Ref. [12]. The analytical contributions from the main graphs of the model to the partial-wave amplitudes are detailed in Appendix A of that paper. Also extracted in Ref. [12] were estimates for the model parameters, obtained from fits to three commonly used (at that time) phase-shift solutions. A subsequent paper [13] investigated (for the first time in this programme) the reproduction of the modern (meson-factory) low-energy measurements: the inevitable conclusion was that, on several occasions, the predictions (obtained from the then-available phase-shift solutions) and the modern data did not agree.

In order to validate the results obtained with the model, a novel way of parameterising the $s$- and $p$-wave $K$-matrix elements was developed and applied to the $\pi^+p$ DCSs in Ref. [14]. The method, put forward in that paper, became indispensable in subsequent works. On one hand, the modelling of the hadronic part of the $\pi N$ interaction via these parameterisations is free of theoretical constraints, other than the expected low-energy behaviour of the $K$-matrix elements. This property makes the approach suitable for assessing the self-consistency of the input DB and for reliably identifying the outliers, i.e., the measurements in the DB which do not match the expectations set by the bulk of each input DB. On the other hand, deployed for the first time in Ref. [14] was an alternative (compared to the earlier works) plan of action which was pursued in all subsequent works, namely the extraction of the important information by fitting directly to the low-energy $\pi N$ measurements; before Ref. [14], an indirect approach was used, by fitting the ETH model to ‘fitted results’, i.e., to the phase shifts, extracted via dispersion relations in
analyses by other groups. This modification rendered this programme almost self-sufficient\textsuperscript{4}, i.e., to a large extent, independent of external results.

After the compatibility of the results, obtained from the phenomenological model and those extracted from the fits of the ETH model to the (same) data, had been verified, the investigation turned upon the development of the methodology aiming at the extraction of a self-consistent set of low-energy $\pi N$ measurements. The first attempt towards this objective was made with the implementation of robust statistics [15,16]. In those works, the isospin invariance in the $\pi N$ interaction was also addressed. Although the robust fit to the data is reliable, more conventional statistical approaches were followed in subsequent papers.

An important step was next taken. The EM corrections (which must be applied to the phase shifts and to the partial-wave amplitudes on the way to the evaluation of the observables) were obtained in an iterative procedure involving two stages: a) fits of the ETH model to the modern data and b) numerical solution of the relativised Schrödinger equations containing the sum of an EM and an effective hadronic potential. A few iteration steps sufficed in achieving the convergence of the EM corrections. The newly obtained EM corrections [4,5] replaced (in all subsequent PSAs) those of the NORDITA works [17,18,19], which had been used in this programme throughout the 1990s. The PSA of the DB$^+_{+/-}$ with the new EM corrections, also featuring a minimisation function which allows for the (controlled, i.e., regulated by the reported or assigned normalisation uncertainty of each data set) floating (see Section 2.4) of the input data sets, was performed in Ref. [20].

A number of issues were addressed in a series of more recent papers. A new PSA was performed in Ref. [21], using updated values of the physical constants and improving on Ref. [20] in two ways: a) only one test for the acceptance of each input data set was performed in Ref. [21] (on the basis of the contribution of that data set to the overall $\chi^2_{\text{min}}$) and b) a more stringent acceptance criterion was adopted in the statistical tests, namely the use of the p-value\textsuperscript{5} threshold $p_{\text{min}}$ which is associated with $2.5\sigma$ effects in the normal distribution. The new $p_{\text{min}}$ value is approximately equal to $1.24 \cdot 10^{-2}$, i.e., slightly exceeding $1.00 \cdot 10^{-2}$, the threshold regarded by most statisticians as the outset of statistical significance. The isospin invariance in the $\pi N$ interaction was revisited in Ref. [22]; the results were found compatible with those reported in earlier works [15,20].

In two subsequent papers [23,24], exclusive analyses of the extensive low-

\textsuperscript{4} The small $d$ and $f$ waves are fixed in this programme from the SAID ‘current-solution’ results [1], at present from the WI08 solution.

\textsuperscript{5} It is casual to refer to p-values in the statistical hypothesis testing in most domains of basic or applied research in Economics, Psychology, Biology, Medical Physics, etc.
energy $\pi^\pm p$ DCSs of the CHAOS Collaboration [25] (labelled in the DBs with the identifier ‘DENZ04’) were conducted; the data had appeared around the time when Ref. [20] was submitted for reviewing and could not have been used in that paper. The important conclusion of both works was that the angular distribution of the DENZ04 $\pi^+ p$ data sets is incompatible with the rest of the modern DB (which had been established as self-consistent in Refs. [15,20,21]). Unable to provide an explanation for this mismatch, we decided to refrain from using any part of the DENZ04 DCSs in this programme.

Finally, the derivation of the contributions from all graphs of the ETH model to the partial-wave amplitudes and the extraction of the $\Sigma$ term within the framework of the model were addressed in a recent paper [26].

2.2 The physical content of the ETH model

The ETH model is based on graphs of scalar-isoscalar ($I = J = 0$) and vector-isovector ($I = J = 1$) $t$-channel exchanges, as well as the $N$ and $\Delta(1232)$ $s$- and $u$-channel contributions (see Fig. 1). The small contributions from the well-established (four-star) $s$ and $p$ higher baryon resonances with masses up to 2 GeV are also analytically included in the model. The tensor component of the $I = J = 0$ $t$-channel graph was added (for the sake of completeness) in Ref. [15].

Information on the model parameters may be obtained from Ref. [26]. In regard to the scalar-isoscalar $t$-channel graph, the recommendation of the Particle-Data Group (PDG) is to make use of a Breit-Wigner mass between 400 and 550 MeV (see properties of $f_0(500)$ in Ref. [27]). As a result, the recent fits of the ETH model to the data are performed at seven (equally weighted) $m_\sigma$ values between 400 and 550 MeV (using an increment of 25 MeV). All uncertainties in this work (i.e., in the values of the model parameters, in the predictions for the low-energy constants of the $\pi N$ system, in the phase shifts, and in the observables) contain the effects of the $m_\sigma$ variation, as well as the Birge factor $\sqrt{\chi^2_{min}/NDF}$ taking account of the goodness of each fit.

When a fit to the data is made using all eight parameters of the ETH model, it turns out that there is a strong correlation between $G_\sigma$, $G_\rho$, and $x$. To suppress the correlations, one needs to fix one of these parameters. The recent fits of the ETH model have been performed using $x = 0$, in accordance with current effective-field theoretical models of the low-energy $\pi N$ scattering. Therefore, each fit of the ETH model (at fixed $m_\sigma$) is performed with the following seven parameters.

- Scalar-isoscalar $t$-channel graph: $G_\sigma$ and $\kappa_\sigma$;
Fig. 1. The main Feynman graphs of the ETH model: scalar-isoscalar ($I = J = 0$) and vector-isovector ($I = J = 1$) $t$-channel graphs (upper part), and $N$ and $\Delta(1232)$ $s$- and $u$-channel graphs (lower part). The small contributions from the well-established $s$ and $p$ higher baryon resonances with masses up to 2 GeV (not shown) are also analytically included in the model.

- Vector-isovector $t$-channel graph: $G_\rho$ and $\kappa_\rho$;
- $N$ $s$- and $u$-channel graphs: $g_{\pi NN}$;
- $\Delta(1232)$ $s$- and $u$-channel graphs: $g_{\pi N\Delta}$ and $Z$.

The $s$ and $p$ higher baryon resonances do not introduce any additional free parameters [26].

A number of physical constants need to be fixed in our PSAs; the values of all these quantities, fixed from the most recent PDG compilation [27], are detailed in Table 1.

2.3 The phenomenological model

The assumptions in analyses employing the $K$-matrix parameterisations of this section relate to: a) the number of terms which one retains from the infinite power series (expansion of the hadronic $K$-matrix elements in terms of a suitable variable, e.g., of the pion kinetic energy $\epsilon$ in the center-of-momentum (CM) system) and b) the forms used in the modelling of the resonant contributions. Past experience shows that the low-energy parameterisations of the $s$- and $p$-wave $K$-matrix elements of this programme successfully capture the dynamics of the $\pi N$ system. A variety of tests have been carried out, demon-
Table 1

The values of the physical constants used in this work, obtained from the most recent PDG compilation [27]. The value of $\Gamma_\Delta$ is the product of the $\Delta(1232)$ total Breit-Wigner width and the resonance’s branching fraction to $\pi N$ decay modes (nearly 100%).

| Physical quantity (unit)            | Value                              |
|-------------------------------------|------------------------------------|
| Inverse of the fine-structure constant $\alpha^{-1}$ | 137.035999139                     |
| $\hbar c$ (MeV fm)                  | 197.3269788                        |
| Electron mass $m_e$ (MeV)           | 0.5109989461                       |
| Charged-pion mass $m_c$ (MeV)       | 139.57018                          |
| Neutral-pion mass $m_0$ (MeV)       | 134.9766                           |
| $\rho$ mass $m_\rho$ (MeV)         | 775.26                             |
| Proton mass $m_p$ (MeV)             | 938.2720813                        |
| Neutron mass $m_n$ (MeV)            | 939.5654133                        |
| Pion charge radius $R_c$ (fm)       | 0.672                              |
| Proton magnetic moment $\kappa_p + 1$ ($\mu_N$) | 2.7928473508                     |
| Pion-decay constant $F_\pi$ (MeV)   | 130.50                             |
| $\Delta(1232)$ Breit-Wigner mass $M_\Delta$ (MeV) | 1232                              |
| $\Gamma_\Delta$ (MeV)              | $117 \cdot 0.994$                  |

...strating beyond doubt that the outliers are flanked by measurements which can be successfully accounted for by the phenomenological model. Therefore, the detection of outliers in the fits cannot be attributed to the inadequacy of the parametric forms of this section to account for the energy dependence of the phase shifts; it is indicative of experimental discrepancies.

In the analysis of the measurements with the phenomenological model, terms up to (and including) $\epsilon^2$ are retained. Owing to the current uncertainties in the measurements, the coefficients of higher orders in the expansion of the $K$-matrix elements cannot be determined reliably from the available data at low energy.

### 2.3.1 Fits of the phenomenological model to the low-energy $DB_+$

For the $\pi^+p$ reaction, the $s$-wave phase shift is parameterised as

$$ q \cot \delta_{0^+}^{3/2} = (a_{0^+}^{3/2})^{-1} + b_3 \epsilon + c_3 \epsilon^2, $$ (1)
where $q$ denotes (the magnitude of) the CM 3-momentum. The $p_{1/2}$-wave phase shift is parameterised according to the form

$$\tan \delta_{1/2}^{3/2} / q = d_{31} \epsilon + e_{31} \epsilon^2 .$$  (2)

As the $p_{3/2}$ wave contains the $\Delta(1232)$ resonance, a singular (at $W = M_\Delta$) term must be added to the background term, leading to the expression

$$\tan \delta_{3/2}^{3/2} / q = d_{33} \epsilon + e_{33} \epsilon^2 + \frac{\Gamma_\Delta M_\Delta}{2 q^2_\Delta (p_{0\Delta} + m_p) W (M_\Delta - W)} (p_0 + m_p)^2 q^2 W (M_\Delta - W),$$  (3)

where $p_0$ is the proton CM energy and $W$ is the total CM energy; the quantities $M_\Delta$ and $\Gamma_\Delta$ are defined in Table 3. The quantities $q_\Delta$ and $p_{0\Delta}$ denote the values of the variables $q$ and $p_0$, respectively, at the position of the $\Delta(1232)$ resonance ($W = M_\Delta$). The singular term in Eq. (3) was obtained from Ref. [26], see $K_{1+}$ in Eqs. (39) and the corresponding $K_{3/2}^{1+}$ element (after the isospin decomposition of $K_{1+}$ is taken into account), as well as footnote 10 therein.

2.3.2 Fits of the phenomenological model to the low-energy $DB_-$ and $DB_0$

The isospin $I = 3/2$ amplitudes, obtained from the final fit of the phenomenological model to the tDB$_+$, are imported into the analysis of the low-energy $DB_-$ and $DB_0$. In this part, another seven parameters (different for these two DBs) are introduced, to parameterise the $I = 1/2$ amplitudes. The new parametric forms are similar to those given by Eqs. (1)-(3), with the parameters $a_{01/2}^{1/2}$, $b_1$, $c_1$, $d_{13}$, $e_{13}$, $d_{11}$, and $e_{11}$. Of course, it is necessary to explicitly include one additional contribution in $\delta_{1/2}^{3/2}$. This contribution originates from a $P_{11}$ higher baryon resonance, the Roper resonance, and is of importance due to the proximity of that state to the low-energy region. Given that this contribution must be taken into account, the second $P_{11}$ state of Table 2 has also been included in $\delta_{1/2}^{3/2}$, despite the fact that this state is too distant to have any noticeable influence at low energy.

$$\tan \delta_{1/2}^{1/2} / q = d_{11} \epsilon + e_{11} \epsilon^2 + \sum_{i=1}^2 \frac{(\Gamma_R)_i (M_R)_i ((p_{0R})_i + m_p)}{2 (q^2_R)_i ((M_R)_i + m_p)^2 W ((M_R)_i - W) (p_0 + m_p) W (M_R)_i (p_0 + m_p)} (W + m_p)^2 q^2 W (M_R)_i (p_0 + m_p),$$  (4)

where $(\Gamma_R)_i$ is the partial width of each contributing $P_{11}$ state to $\pi N$ decay modes and $(M_R)_i$ the Breit-Wigner mass of that state. The quantities $q_R$ and $p_{0R}$ denote the $q$ and $p_0$ values at each resonance position ($W = (M_R)_i$). The singular terms in Eq. (4) were obtained from Ref. [26], see Section 3.5.1 therein, in particular, Eq. (54) for $K_{1-}$.

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2.4 Minimisation function

In our recent PSAs of the $\pi N$ data, use has been made of the minimisation function given by the Arndt-Roper formula [28], i.e., of the minimisation function which the SAID group also use in their analyses of the $\pi N$ data. The contribution of the $j$-th data set to the overall $\chi^2$ is of the form

$$\chi^2_j = \sum_{i=1}^{N_j} \left( \frac{y_{ij}^{\text{exp}} - y_{ij}^{\text{th}}}{\delta y_{ij}^{\text{exp}}} \right)^2 + \left( \frac{z_j - 1}{\delta z_j} \right)^2,$$

where $y_{ij}^{\text{exp}}$ denotes the $i$-th data point of the $j$-th data set, $y_{ij}^{\text{th}}$ the corresponding fitted ('theoretical') value, $\delta y_{ij}^{\text{exp}}$ the statistical uncertainty of $y_{ij}^{\text{exp}}$, $z_j$ a scale factor (applied to the entire data set), $\delta z_j$ the normalisation uncertainty (reported by the experimental group or assigned prior to the data analysis [20]), and $N_j$ the number of the data points in the data set which had not been identified as outliers in previous iteration steps. The fitted values $y_{ij}^{\text{th}}$ are obtained by means of the parameterised forms of the $s$- and $p$-wave amplitudes of Sections 2.2 and 2.3. The optimal value of the scale factor $z_j$ minimises $\chi^2_j$; for each data set, a unique solution for $z_j$ is obtained via the relation

$$z_j = \frac{\sum_{i=1}^{N_j} y_{ij}^{\text{exp}} y_{ij}^{\text{th}} / (\delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}},$$

which leads to

$$(\chi^2_j)_{\text{min}} = \sum_{i=1}^{N_j} \left( \frac{y_{ij}^{\text{exp}} - y_{ij}^{\text{th}}}{\delta y_{ij}^{\text{exp}}} \right)^2 - \frac{\left( \sum_{i=1}^{N_j} (y_{ij}^{\text{exp}} - y_{ij}^{\text{th}}) y_{ij}^{\text{th}} / (\delta y_{ij}^{\text{exp}})^2 \right)^2}{\sum_{i=1}^{N_j} (y_{ij}^{\text{th}} / \delta y_{ij}^{\text{exp}})^2 + (\delta z_j)^{-2}}.$$

The overall $\chi^2 = \sum_{j=1}^{N} (\chi^2_j)_{\text{min}}$ (where $N$ denotes the number of the accepted data sets in the fit) is a function of the parameters entering the modelling of the $s$- and $p$-wave amplitudes. These parameters are varied until $\chi^2$ is minimised, yielding $\chi^2_{\text{min}}$.

For the purpose of the optimisation, the standard MINUIT package [29] of the CERN library (FORTRAN version) has always been used in this programme. Each optimisation is achieved with the (robust) sequence: SIMPLEX, MINIMIZE, MIGRAD, and MINOS.

- SIMPLEX is a function-minimisation method, using the simplex method of Nelder and Mead. Being a stepping method, SIMPLEX does not produce a Hessian matrix.
- MINIMIZE minimises the user-defined function by calling MIGRAD and reverts to SIMPLEX in case that the MIGRAD call fails to converge.
- MIGRAD is the workhorse of the MINUIT software library. It is a variable-metric method, also checking for the positive-definiteness of the Hessian.
matrix.

- MINOS performs a detailed error analysis for each of the model parameters separately. It may be time consuming, but its results (i.e., the asymmetric uncertainties of the model parameters) are reliable as they take the non-linearities in the problem into account, as well as the correlations among the model parameters.

All the aforementioned methods admit an optional argument, fixing the maximal number of calls of each particular method; if this number is reached, the corresponding method is terminated (by MINUIT, internally) irrespective of whether the specific method has converged or not. To ensure the successful termination of the MINUIT application and the convergence of its methods, the MINUIT output has always been routinely inspected in this programme; failures were observed only when analysing the CHAOS DCSs, see Refs. [23,24].

2.5 Assessment of the goodness of the reproduction of a data set by a baseline solution

The methodology for quantifying the goodness of the reproduction of entire data sets by a reference or baseline solution (BLS) was thoroughly established in Ref. [24]; the details are repeated here for the sake of self-sufficiency. A few definitions will first be given.

- A BLS is a set of values and associated uncertainties \((y_{ij}^{\text{th}}, \delta y_{ij}^{\text{th}}, i \in [1, N_j])\), corresponding to the values of the kinematical variables, i.e., of the CM scattering angle \(\theta\) and of \(T\), at which the measurements \((y_{ij}^{\text{exp}}, \delta y_{ij}^{\text{exp}}, i \in [1, N_j])\) have been acquired.

- A BLS comprises predictions obtained via Monte-Carlo simulation, taking into account the results of the optimisation (i.e., the fitted values and the uncertainties of the model parameters, as well as the Hessian matrix of each fit) of a PWA of \(\pi N\) data.

Being a sum of independent normalised residuals, each following the normal distribution, the test-statistic - to be introduced later on, namely by Eq. (10) - is expected to follow the \(\chi^2\) distribution. As the objective is the identification of data sets which are poorly reproduced, the expressions of this section are tailored to one-sided tests (right-tail events).

Let the background process, underlying the phenomenon under investigation, be a stochastic one, described by the probability density function \(f(x) \geq 0\), where \(x \in [0, \infty)\) is a numerical result obtained via a measurement conducted.
on the system. Kolmogorov’s second axiom dictates that
\[ \int_0^\infty f(x)dx = 1. \]
The so-called p-value is defined as the upper tail of the corresponding cumulative distribution function
\[ p(x_0) = \int_{x_0}^\infty f(x)dx. \] (7)
The p-value represents the probability that a new measurement on the same system (under identical conditions) yield a result \( x \) which is more statistically significant than \( x_0 \) (in this case, the result \( x > x_0 \)). Assuming the validity of the null hypothesis, the p-value may therefore be interpreted as the measure of the result \( x_0 \) being due to chance: ‘small’ p-values attest to the statistical significance of the measurement which yielded the result \( x_0 \).

Evidently, before assessing the statistical significance of a measurement, one must define what is meant by ‘small’ p-values. The \( p_{\text{min}} \) value, signifying the outset of statistical significance\(^6\), is the only subjective notion in Statistics. In reality, the fixation of \( p_{\text{min}} \) rests on a delicate trade-off between two risks: a) of accepting the alternative hypothesis (of an effect not being due to statistical contrivance) when it is false and b) of rejecting the alternative hypothesis when it is true. Of relevance in the choice of the \( p_{\text{min}} \) value is a decision as to which of these two risks is being assigned greater importance. For instance, if the implications of risk (b) are considered to be more serious compared to those of risk (a), an increase in \( p_{\text{min}} \) is tenable.

Most statisticians accept \( p_{\text{min}} = 1.00 \cdot 10^{-2} \) as the outset of statistical significance and \( p_{\text{min}} = 5.00 \cdot 10^{-2} \) as the threshold indicating probable significance. An interesting recent paper [30] interprets the lack of reproducibility of the results in various scientific disciplines as evidence that the currently accepted \( p_{\text{min}} \) values are rather ‘optimistic’. To compensate for this ‘optimism’, the author recommends the reduction of these thresholds by one order of magnitude\(^7\). Although Johnson’s paper stimulated an arduous debate, in particular in 2014, it is unlikely that the currently accepted thresholds of statistical significance will be revised in the near future. Presumably, such important changes to the established norms do not occur overnight. For the time being, the 2.5\( \sigma \)

---

\(^6\) The threshold of statistical significance is usually denoted as \( \alpha \).

\(^7\) Although Ref. [30] states that “nonreproducibility in scientific studies can be attributed to a number of factors, including poor research designs, flawed statistical analyses, and scientific misconduct”, it is more likely that, at least as far as the research in the \( \pi N \) domain is concerned, the main reason is the ‘excessive optimism’ when assessing the systematic effects in the experiments; in all probability, these uncertainties are frequently underestimated.
threshold will continue to signify the outset of statistical significance in this programme.

The probability density function of the $\chi^2$ distribution with $\nu > 0$ DOFs is

$$f(x, \nu) = \begin{cases} \frac{1}{2^{\nu/2}\Gamma(\nu/2)} x^{\nu/2-1} \exp(-x/2), & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

(8)

where $\Gamma(y)$ is the standard gamma function

$$\Gamma(y) = \int_0^\infty t^{y-1} \exp(-t) dt .$$

For a quantity $x$ following the $\chi^2$ distribution, the expectation value $E[x]$ is simply equal to $\nu$ and the variance $E[x^2] - (E[x])^2$ is equal to $2\nu$. The relation $E[x] = \nu$ has led most physicists to the use of the reduced $\chi^2$ value (i.e., of the ratio $\chi^2/\nu$) as a measure of the goodness of the data description in the modelling or in the reproduction of measurements; provided that $\chi^2/\nu \approx 1$, the outcome of the test is claimed to be satisfactory. At this point, the following two remarks need to be made.

- The statistical hypothesis testing formally relies on the use of the p-value. The use of the reduced $\chi^2$ to quantify the statistical significance is an approximate ‘rule of thumb’, an informal one.
- The interesting question in the statistical hypothesis testing relates to the value of $\chi^2/\nu$ at which the results start appearing unsatisfactory; evidently, a threshold value for $\chi^2/\nu$ may be extracted from $p_{\text{min}}$, yet it is (of course) $\nu$-dependent, hence cumbersome to use.

Evidently, such a departure from simplicity is counterproductive. To assess the statistical significance of a result, one simply needs to compare the corresponding p-value, associated with the estimated $\chi^2$ for $\nu$ DOFs, to $p_{\text{min}}$. This is achieved by simply inserting $f(x, \nu)$ of Eq. (8) into Eq. (7), along with $x_0 = \chi^2$, and evaluating the integral; several software implementations of dedicated algorithms are available, e.g., see Refs. [31] (Chapter on ‘Gamma Function and Related Functions’) and [32], the routine PROB of the CERN software library, the functions CHIDIST/CHISQ.DIST.RT of Microsoft Excel, etc.

Various definitions of the ‘data set’ have been in use, involving different choices of the experimental conditions which must remain stable/constant during the data-acquisition session. The properties of the incident beam and the (physical, geometrical) properties of the target were used in the past in order to distinguish the data sets of experiments performed at one place (i.e., at a meson factory) over a short period of time (typically, a few weeks). However, data sets have appeared in experimental reports relevant to the $\pi N$ interac-
tion, which did not only involve different beam energies, but also contained measurements of different reactions (e.g., mixing $\pi^+p$ and $\pi^-p$ ES measurements). As a result, the prerequisite for accepting measurements as comprising one data set is that they share the same measurement of the absolute normalisation (and, consequently, normalisation uncertainty $\delta z_j$). Of course, this is a prerequisite, hence a necessary, not a sufficient, condition. The decision regarding the acceptance of a set of measurements as comprising one data set cannot be made without an investigation of the stability of the experimental conditions at which the measurements had been acquired (which may be difficult for ‘outsiders’ to assess), as well as of their (on/off-line) processing on the way to the extraction of the final experimental results.

After this introduction, it is time to enter the details of the reproduction of data sets. Described in the remaining part of this section are tests of the overall reproduction, of the shape, and of the absolute normalisation. It is assumed that the absolute normalisation of the $j$-th data set is known up to a relative uncertainty $\delta z_j$ and that none of the important quantities, appearing in the denominators of the expressions of this section, vanishes.

As mentioned in the beginning of the section, the methodology for assessing the goodness of the reproduction of the $j$-th data set was put forward in Ref. [24]; it involves the determination of the amount of floating to be applied to the BLS ($y_{ij}^{th}, \delta y_{ij}^{th}, i \in [1, N_j]$) in order that it ‘best’ accounts for the entire data set ($y_{ij}^{exp}, \delta y_{ij}^{exp}, i \in [1, N_j]$).

First, the ratios $r_{ij} = y_{ij}^{exp}/y_{ij}^{th}$ are evaluated; if the quantities $y_{ij}^{exp}$ and $y_{ij}^{th}$ are independent (which is certainly true in this programme, given that the tested data sets have not been used in the determination of the BLS), the uncertainties $\delta r_{ij}$ are obtained via the application of Gauss’s error-propagation formula

$$\delta r_{ij} = r_{ij} \sqrt{\left(\frac{\delta y_{ij}^{exp}}{y_{ij}^{exp}}\right)^2 + \left(\frac{\delta y_{ij}^{th}}{y_{ij}^{th}}\right)^2}.$$  \hspace{1cm} (9)

The goodness of the reproduction is judged on the basis of the function $\chi^2_j(z_j)$ defined as

$$\chi^2_j(z_j) = \sum_{i=1}^{N_j} \left(\frac{r_{ij} - z_j}{\delta r_{ij}}\right)^2 + \left(\frac{z_j - 1}{\delta z_j}\right)^2.$$ \hspace{1cm} (10)

It is convenient to introduce the weights $w_{ij}$ via the relation $w_{ij} = (\delta r_{ij})^{-2}$.

The second term on the right-hand side of Eq. (10) takes account of the floating of the BLS. This contribution depends on how well the absolute normalisation of the $j$-th data set is known. For a poorly-known absolute normalisation, $\delta z_j$ is large and the resulting contribution of the floating of the data set is small; the opposite is true in case of a well-known absolute normalisation. Evidently, the ‘best’ reproduction of the $j$-th data set is achieved when, by varying $z_j$,
the function $\chi_j^2(z_j)$ is minimised, resulting in the condition
\[
\frac{\partial \chi_j^2(z_j)}{\partial z_j} = 0.
\]
The solution of this equation is
\[
 z_j = \frac{\sum_{i=1}^{N_j} w_{ij} r_{ij} + (\delta z_j)^{-2}}{\sum_{i=1}^{N_j} w_{ij} + (\delta z_j)^{-2}}. \tag{11}
\]
Inserting this expression for $z_j$ into Eq. (10), one obtains
\[
(\chi_j^2)_{\text{min}} = \left( \sum_{i=1}^{N_j} w_{ij} + (\delta z_j)^{-2} \right)^{-1} \left( \sum_{i=1}^{N_j} w_{ij} \sum_{i=1}^{N_j} w_{ij} r_{ij}^2 - (\sum_{i=1}^{N_j} w_{ij} r_{ij})^2 \right)
+ (\delta z_j)^{-2} \sum_{i=1}^{N_j} w_{ij} (r_{ij} - 1)^2. \tag{12}
\]
Expression (12) yields the minimal $\chi^2$ value for the reproduction of the $j$-th data set, containing $N_j$ data points. In fact, one additional measurement had been made on this data set, namely the one fixing its absolute normalisation, which is known with relative uncertainty $\delta z_j$. Therefore, the NDF for this data set is equal to $N_j + 1 - 1 = N_j$; the subtraction of one unit is due to the use of Eq. (11) as a constraint, fixing the value of the scale factor $z_j$. Therefore, the quantity $(\chi_j^2)_{\text{min}}$ of Eq. (12) is expected to follow the $\chi^2$ distribution with $\nu = N_j$ DOFs. To obtain the p-value of the overall reproduction of the $j$-th data set, one uses Eq. (7) with $f(x) = f(x, \nu)$ of Eq. (8), along with $x_0 = (\chi_j^2)_{\text{min}}$ and $\nu = N_j$.

Two additional tests on each data set are possible. These tests are useful in case that the overall reproduction of a data set is poor; they enable a decision on whether the poor reproduction is to be blamed on the shape or on the absolute normalisation of the data set.

- To examine the shape of the $j$-th data set (with respect to that of the BLS), one needs to allow the BLS to reproduce the data set regardless of the floating contribution in Eq. (10). This is equivalent to setting $\delta z_j \to \infty$ or $(\delta z_j)^{-2} = 0$ in Eqs. (11) and (12). The corresponding quantities will be denoted as $\hat{z}_j$ and $(\chi_j^2)_{\text{stat}}$, respectively; the quantity $(\chi_j^2)_{\text{stat}}$ represents the fluctuation in the $j$-th data set which (assuming the correctness of the shape of the data set) is of pure statistical nature.

\[
\hat{z}_j = \frac{\sum_{i=1}^{N_j} w_{ij} r_{ij}}{\sum_{i=1}^{N_j} w_{ij}}. \tag{13}
\]
\[(\chi_j^2)_{\text{stat}} = \left( \sum_{i=1}^{N_j} w_{ij} \right)^{-1} \left( \sum_{i=1}^{N_j} w_{ij} \sum_{i=1}^{N_j} w_{ij} r_{ij}^2 - \left( \sum_{i=1}^{N_j} w_{ij} r_{ij} \right)^2 \right) \quad (14)\]

As expected, both expressions are identical to those derived for the weighted average of a set of independent measurements and for the corresponding \(\chi^2\) value for constancy. Owing to the fact that the normalisation uncertainty is not used in Eq. (14), the quantity \((\chi_j^2)_{\text{stat}}\) is expected to follow the \(\chi^2\) distribution with \(\nu = N_j - 1\) DOFs. The p-value, obtained from Eq. (7) with \(x_0 = (\chi_j^2)_{\text{stat}}\) and \(\nu = N_j - 1\), may be used in order to assess the constancy of the input values \(r_{ij}\) or, equivalently in our case, to examine the shape of the \(j\)-th data set with respect to the BLS\(^8\).

- To assess the compatibility of the absolute normalisations of the \(j\)-th data set and of the BLS, one first estimates the floating contribution to \((\chi_j^2)_{\text{min}}\) via the relation

\[\begin{align*}
(\chi_j^2)_{\text{sc}} &= (\chi_j^2)_{\text{min}} - (\chi_j^2)_{\text{stat}} \\
&= \frac{(\delta z_j)^{-2} \left( \sum_{i=1}^{N_j} w_{ij} (r_{ij} - 1) \right)^2}{\left( \sum_{i=1}^{N_j} w_{ij} + (\delta z_j)^{-2} \right) \sum_{i=1}^{N_j} w_{ij}},
\end{align*}\]

where use of Eqs. (12) and (14) has been made. The quantity \((\chi_j^2)_{\text{sc}}\) is expected to follow the \(\chi^2\) distribution with 1 DOF (which, of course, is the normal distribution).

The three tests of the goodness of the reproduction of the \(j\)-th data set by a BLS are summarised as follows.

- The overall reproduction can be tested using \((\chi_j^2)_{\text{min}}\) of Eq. (12) as \(x_0\) in Eq. (7) and \(\nu = N_j\) DOFs.
- The shape (statistical fluctuation) can be tested using \((\chi_j^2)_{\text{stat}}\) of Eq. (14) as \(x_0\) in Eq. (7) and \(\nu = N_j - 1\) DOFs.
- The absolute normalisation can be tested using \((\chi_j^2)_{\text{sc}}\) of Eq. (15) as \(x_0\) in Eq. (7) and \(\nu = 1\) DOF.

Evidently, the only subjective aspect in the technique, employed in the data analysis, pertains to the choice of the \(p_{\text{min}}\) value signifying the statistical significance.

\(^8\) In fact, the test simply assesses how well the input data could be represented by their average value. A failure suggests either a bad shape (e.g., a slope being present in the input data) or ‘scattered’ input values with small uncertainties.
The 2017 PSAs

The solution presented in this paper will be labelled ‘ZRH17’. The current status of the data analysis can be found in Refs. [2,22,26]. There is one difference to these earlier PSAs, namely the inclusion of the BERTIN76 $\pi^+p$ DCSs [33] in the DB$_+$. The arguments for excluding these measurements became increasingly more tenuous with time, in particular after the appropriate tests of the self-consistency of the input DBs had been implemented in the processing software. The BERTIN76 data sets are generally omitted from low-energy analyses on the basis of reasoning which the methodology in this programme can tackle, namely the presence of a small number of outliers in the measurements and a discrepancy in the absolute normalisation of some of these data sets (with respect to the bulk of the modern low-energy $\pi N$ data). A closer look at the reproduction of the BERTIN76 data with earlier results convinced us that the analysis procedure can handle both subjects in cases of small additions to the low-energy DB. The argument that the BERTIN76 data sets should be excluded because ‘it is not clear from the publication that the normalisation effects were investigated in that experiment at all’ is not firm, as the DB$_+$ does contain data sets with unknown (not reported) normalisation uncertainty, e.g., the AULD79 DCSs, the FRIEDMAN99 PTCSs, as well as the CARTER71 and PEDRONI78 TCSs. The strongest argument in favour of the inclusion of the BERTIN76 data sets in the DB$_+$ is that the exclusive analysis of these data with the methodology of Section 2.3.1 suggests only the elimination of the 67.40 MeV data set (due to its three obvious outliers at large backward angles). The remaining data may be accounted for reasonably well with the phenomenological model, yielding a final $\chi^2_{\text{min}}$ value of about 50.6 for 53 DOFs and an acceptable $a^{3/2}_{0+}$ fitted value. As a result, the BERTIN76 data sets will be included in the DB$_+$ from now on, with large assigned normalisation uncertainties [20], as the case has been with those of the data sets whose normalisation uncertainties are unknown.

A minor difference to earlier analyses is the inclusion in the set of the $s$ and $p$ higher baryon resonances of one $P_{11}$ state which moved from the three-star to the four-star status in the most recent PDG compilation [27]: the state is known as $N(1710)$. As that resonance may be thought of as a heavy Roper resonance, the formalism of Section 3.5.1 of Ref. [26] applies (of course, with the appropriate Breit-Wigner mass and the partial decay width to $\pi N$ decay modes). The contributing $s$ and $p$ higher baryon resonances are detailed in Table 2.
Table 2

The physical constants relating to the well-established $s$ and $p$ higher baryon resonances with masses up to 2 GeV. The quantities $M_R$ and $\Gamma$ respectively denote the Breit-Wigner mass and total Breit-Wigner width of the resonance, whereas $BF$ is the branching fraction to $\pi N$ decay modes; evidently, $\Gamma_R = \Gamma \cdot BF$ is the width applicable to the contributions from each resonance to $\pi N$ scattering. Upper part: $N$ states ($I = 1/2$); lower part: $\Delta$ states ($I = 3/2$). The state $N(1710)$ moved from the three-star to four-star status in the recent PDG compilation [27].

| State       | $M_R$ (MeV) | $\Gamma$ (MeV) | BF (%) |
|-------------|-------------|----------------|--------|
| $N(1440)$ ($P_{11}$) | 1430        | 350            | 65.0   |
| $N(1535)$ ($S_{11}$) | 1535        | 150            | 45.0   |
| $N(1650)$ ($S_{11}$) | 1655        | 140            | 60.0   |
| $N(1710)$ ($P_{11}$) | 1710        | 100            | 12.5   |
| $N(1720)$ ($P_{13}$) | 1720        | 250            | 11.0   |
| $\Delta(1620)$ ($S_{31}$) | 1630        | 140            | 25.0   |
| $\Delta(1910)$ ($P_{31}$) | 1890        | 280            | 22.5   |

3.1 Fits of the phenomenological model

The first fit to the DB$_+$ of 452 DOFs resulted in $\chi^2_{\text{min}} \approx 913.3$. Following the procedure of eliminating one DOF at each iteration step, i.e., the one with the largest contribution to $(\chi^2_j)_{\text{min}}$ of the data set which is worst described (i.e., corresponding to the lowest p-value, if smaller than $p_{\text{min}}$), one obtains the tDB$_+$ with $\chi^2_{\text{min}} \approx 537.4$ and 414 DOFs. Three data sets, identified as problematic already in Ref. [14], stick out of the DB$_+$ in a dramatic manner: the BRACK90 data set at 66.80 MeV (with eleven data points), the BERTIN76 data set at 67.40 MeV (with ten data points), and the JORAM95 data set at 32.70 MeV (with seven data points). Of the remaining ten outliers, two relate to the absolute normalisation (belonging to the BRACK86 data sets and being accompanied by normalisation uncertainties as low as 1.2 and 1.4%). In summary, the elimination of 38 DOFs results in the reduction of the $\chi^2_{\text{min}}$ by about 375.9, i.e., by about 9.9 units per excluded DOF. The tDB$_+$ is detailed in Table 8. Apart from some experimental details, the table also contains the contribution $(\chi^2_j)_{\text{min}}$ of each data set to $\chi^2_{\text{min}}$, the p-value associated with the goodness of the description of each data set in the final fit to the tDB$_+$, and the scale factor $z_j$ obtained with Eq. (5).

The final results of the optimisation of the DB$_-$ and DB$_0$ are given in Tables 9 and 10, respectively. In the former case, the elimination of 8 DOFs of the 332
initial DOFs of the DB_ results in the reduction of the $\chi_{\text{min}}^2$ by 150.5 (from about 519.6 to about 369.1), i.e., by about 18.8 units per excluded DOF. Only one data set with 5 data points (the BRACK90 data set at 66.80 MeV) needs to be removed. In the case of the BD_0, the removal of the absolute normalisation of three (out of seven) FITZGERALD86 data sets is noticeable. Although this fact may provide arguments for calling into question the absolute normalisation in all FITZGERALD86 data sets, the absolute normalisation of the four remaining data sets was retained (as the elimination of the absolute normalisation of these four data sets was not requested when strictly applying our rejection criteria). The elimination of 4 DOFs (the BREITSCHOPF06 measurement of the TCS at 75.10 MeV also needed to be removed) of the 326 initial DOFs of the DB_0 results in the reduction of the $\chi_{\text{min}}^2$ by 71.9 units (from about 390.2 to about 318.3), i.e., by about 18.0 per excluded DOF.

Judged solely on the basis of the description of the measurements they contain, it appears that the tDB_+, the tDB_−, and the tDB_0 are not of the same quality (in terms of self-consistency). However, the formal statistical test does exist in order to support or refute such a thesis: in order to prove that the description of two DBs a and b is different, the ratio

$$F_{a/b} = \frac{\chi_{a}^2/NDF_a}{\chi_{b}^2/NDF_b}$$

must significantly differ from 1. The ratio $F_{a/b}$ follows Fisher’s ($F$) distribution with NDF$_a$ and NDF$_b$ DOFs. From the two final fits of the phenomenological model to the tDB_+ and to the tDB_−, one obtains $F_{+/−} \approx 1.14$ for 414 and 324 DOFs, which is translated into the p-value of $1.08 \cdot 10^{-1}$. Therefore, the frequently expressed claim about the dissimilarity of the DB_+ and the DB_− cannot be formally sustained for the $p_{\text{min}}$ value of this work. On the other hand, the two final fits to the tDB_+ and to the tDB_0 yield $F_{+/0} \approx 1.31$ for 414 and 322 DOFs, which is translated into the p-value of $5.11 \cdot 10^{-3}$. Therefore, a significant difference in quality between the fitted tDB_+ and the tDB_0 can be formally sustained for the $p_{\text{min}}$ value of this work. This remark must be borne in mind when analysing these two reactions in a joint optimisation scheme.

Prior to submitting the data to further analysis, two additional fits were performed, using the 14 parameters of the phenomenological model of Section 2.3. The fit to the tDB_+− yielded $\chi_{\text{min}}^2 \approx 897.4$ for 738 DOFs. The fit to the tDB_+/0 yielded $\chi_{\text{min}}^2 \approx 840.8$ for 736 DOFs. No additional data points had to be removed; therefore, the tDB_+/− and the tDB_+/0, comprised of the subsets detailed in Tables 8-10, may be submitted to further analysis.
Table 3

The values of the seven parameters of the ETH model obtained from the fits to the $tDB_{+/−}$ and $tDB_{+/0}$. To facilitate the comparison with other works, the value of the constant $g_{πNN}$ is also given in the form $f_π^{2}_{πNN}$, see Eq. (17).

| Parameter | $tDB_{+/−}$ | $tDB_{+/0}$ |
|-----------|-------------|-------------|
| $G_σ$ (GeV$^{-2}$) | 24.1 ± 1.3 | 25.5 ± 2.3 |
| $κ_σ$ | $−0.095 ± 0.069$ | 0.057 ± 0.097 |
| $G_ρ$ (GeV$^{-2}$) | 55.37 ± 0.61 | 59.66 ± 0.52 |
| $κ_ρ$ | 0.73 ± 0.37 | 1.47 ± 0.17 |
| $g_{πNN}$ | 13.01 ± 0.12 | 13.522 ± 0.095 |
| $g_{πNΔ}$ | 29.55 ± 0.26 | 28.99 ± 0.25 |
| $Z$ | $−0.509 ± 0.060$ | $−0.295 ± 0.098$ |
| $f_π^{2}_{πNN}$ | $(74.5 ± 1.4) · 10^{-3}$ | $(80.5 ± 1.1) · 10^{-3}$ |

3.2 Fits of the ETH model

The modelling of the $πN$ s- and $p$-wave scattering amplitude with the phenomenological model of Section 2.3 is suitable as a test of the self-consistency of the input DBs and as an unbiased method for the identification of the outliers. However, neither does it provide insight into the underlying physical processes nor can it easily incorporate the theoretical constraint of crossing symmetry. To this end, the ETH model is employed at the next stage of each PSA.

3.2.1 Model parameters

The optimal values of the seven model parameters, obtained from the fits to the $tDB_{+/−}$ and $tDB_{+/0}$, are listed in Table 3. These sets of values enable the determination of predictions for the low-energy constants of the $πN$ system, for the phase shifts and amplitudes, and for the observables.

The differences between the results for the model parameters $G_ρ$ and $g_{πNN}$ are noticeable. To facilitate the comparison with other works, the constant $g_{πNN}$ is also given in Table 3 after it has been converted into the standard

---

9 The scattering amplitudes of the two ES processes are linked via the interchange $s ↔ u$ in the two invariant amplitudes $A_±(s, t, u)$ and $B_±(s, t, u)$, where $s$, $t$, and $u$ are the standard Mandelstam variables.
pseudovector coupling constant

\[
f_{\pi NN}^2 = \left( \frac{m_e}{2m_p} \right)^2 \frac{g_{\pi NN}^2}{4\pi}.
\]  

(17)

The same analysis was performed for \( p_{\text{min}} \) values corresponding to 2\( \sigma \) and 3\( \sigma \) effects in the normal distribution. Tests of the constancy of the values of all seven model parameters across the three relevant solutions (i.e., those obtained at the three \( p_{\text{min}} \) values) were carried out, yielding no evidence for the rejection of the null hypothesis (constancy of the values). Therefore, for reasonable \( p_{\text{min}} \) values assumed in the statistical tests, the results of the fits of the ETH model to the data are independent of \( p_{\text{min}} \).

It is time to reflect on the final \( \chi^2_{\text{min}} \) values obtained thus far (see Table 4). The results for \( p_{\text{min}} \approx 1.24 \cdot 10^{-2} \) will be examined first. The separate fits of the phenomenological model of Section 2.3 to the data yielded the \( \chi^2_{\text{min}} \) values of 537.4, 369.1, and 318.3 for the tDB\(_+\), for the tDB\(_-\), and for the tDB\(_0\), with 414, 324, and 322 DOFs, respectively. The \( \chi^2_{\text{min}} \) values obtained with the phenomenological model in the two analyses of the combined truncated DBs (i.e., tDB\(_+/-\) and tDB\(_+\)/0) come out close to the sum of the corresponding results for the separate fits: 897.4 (to be compared to the sum of 906.5) for the tDB\(_+/-\) and 840.8 (to be compared to the sum of 855.7) for the tDB\(_+\)/0. (As the NDF in the tDB\(_-\) and in the tDB\(_0\) are close, the results are, for all practical purposes, comparable.) Therefore, one observes that, in the case of the fits of the phenomenological model, the difference of the two \( \chi^2_{\text{min}} \) values (which is about 56.6) mainly reflects the difference of the \( \chi^2_{\text{min}} \) values in the separate fits to the tDB\(_-\) and to the tDB\(_0\) (which is equal to 50.8, the smaller \( \chi^2_{\text{min}} \) value corresponding to the modelling of the tDB\(_0\)).

The increase in the \( \chi^2_{\text{min}} \) values when the ETH model is used (over the result obtained with the phenomenological model) is due to the imposition of theoretical constraints (e.g., of crossing symmetry and isospin invariance in the \( \pi N \) interaction). One would expect that the difference in the \( \chi^2_{\text{min}} \) values between the ETH-model fit and the phenomenological-model fit to the data would approximately be the same for the two fitted DBs, i.e., for the tDB\(_+/-\) and the tDB\(_+\)/0. However, this is far from the truth. The value of +56.6 for the difference \( \chi^2_{\text{min}}(\text{tDB}\(_+/-\)) - \chi^2_{\text{min}}(\text{tDB}\(_+\)/0) \) in the fits of the phenomenological model turns into −96.3 when using the ETH model. This is the result of the considerably larger increase in the \( \chi^2_{\text{min}} \) for the tDB\(_+\)/0 fits from the phenomenological to the ETH model: this increase amounts to 212.1, to be compared to 59.2 for the tDB\(_+/-\) fits. Evidently, the replacement of the tDB\(_-\) by the tDB\(_0\) results in a significant deterioration of the overall description of the data in the fits of the ETH model. This deterioration serves as an indication of a difficulty in the description of the tDB\(_+\)/0 in terms of one set of values of the parameters of the ETH model. This result could be explained if
The various $\chi^2_{\text{min}}$ values obtained in the analysis of the low-energy $\pi N$ data, along with the NDF in each fit, for three values of $p_{\text{min}}$ (the confidence threshold used in the statistical tests). In regard to the $K$-matrix results, two rows are given for each $p_{\text{min}}$ value: the first row corresponds to the first fit to the specific input DB, whereas the second row to the final fit, after all the outliers (at that $p_{\text{min}}$ value) are removed from the DB. Measurements are removed from the DBs only when analysing the data with the phenomenological model of Section 2.3; this enables the determination of outliers on the basis of comparisons with measurements of the same $\pi N$ reaction.

All fits of the ETH model have been performed using $m_\sigma = 475$ MeV; $\chi^2_{\text{min}}$ varies little with $m_\sigma$: between 956.1 and 957.0 for the fits to the tDB$^+/-$, between 1051.8 and 1054.5 for those to the tDB$^+/0$ (these results correspond to $p_{\text{min}} \approx 1.24 \cdot 10^{-2}$ and assume the $m_\sigma$ variation between 400 and 550 MeV [27]). Separate fits of the ETH model to the tDB$^+$, tDB$^-$, and tDB$^0$ had been attempted in the past, but (due to the largeness of the correlations among the model parameters) the results are not considered reliable.

| Parametric model | tDB$^+$ | tDB$^-$ | tDB$^0$ | tDB$^+/-$ | tDB$^+/0$ |
|------------------|---------|---------|---------|-----------|-----------|
| $p_{\text{min}} \approx 2.70 \cdot 10^{-3}$ (3σ) | | | | | |
| $K$-matrix (first) | 913.3/452 | 520.6/332 | 394.1/326 | 974.6/748 | 900.7/743 |
| $K$-matrix (final) | 591.8/420 | 395.4/328 | 328.9/323 | 974.6/748 | 886.1/742 |
| ETH model | - | - | - | 1033.8/755 | 1096.6/749 |
| $p_{\text{min}} \approx 1.24 \cdot 10^{-2}$ (2.5σ) | | | | | |
| $K$-matrix (first) | 591.8/420 | 395.2/328 | 326.8/323 | 897.4/738 | 840.8/736 |
| $K$-matrix (final) | 537.4/414 | 369.1/324 | 318.3/322 | 897.4/738 | 840.8/736 |
| ETH model | - | - | - | 956.6/745 | 1052.9/743 |
| $p_{\text{min}} \approx 4.55 \cdot 10^{-2}$ (2σ) | | | | | |
| $K$-matrix (first) | 537.4/414 | 369.2/324 | 318.2/322 | 806.3/720 | 771.0/720 |
| $K$-matrix (final) | 480.9/400 | 333.8/320 | 301.2/320 | 806.3/720 | 756.0/712 |
| ETH model | - | - | - | 861.9/727 | 954.0/719 |

the theoretical basis on which the data analysis rests (e.g., isospin invariance in the $\pi N$ interaction) is disturbed. Inspection of Table 4 reveals that the information obtained from the results for the two other $p_{\text{min}}$ values (i.e., those corresponding to a $2\sigma$ and $3\sigma$ effect in the normal distribution) matches the results for $p_{\text{min}} \approx 1.24 \cdot 10^{-2}$. 

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3.2.2 Phase shifts

The predictions for the $s$- and $p$-wave phase shifts from the fits of the ETH model to the tDB$_{+/−}$ and to the tDB$_{+/0}$ are given in Tables 5 and 6, respectively. These phase shifts are also shown in Figs. 2-7, along with the SAID solution WI08 [1] and their five single-energy values (wherever available). Noticeable differences between the two predictions of this work are observed in the case of the phase shifts $\delta_{0+}^{1/2}$ (S31) and $\delta_{1+}^{3/2}$ (P33). In the former case, the difference is statistically significant throughout the energy range of this work; in the latter, up to about 80 MeV. In both cases, the corresponding WI08 solutions lie in between the two predictions of this work. In the $p$-wave phase shift $\delta_{1−}^{1/2}$, the WI08 solution is less negative and does not seem to agree with either of the predictions of this work.
3.2.3 The low-energy constants of the $\pi N$ system

The $\pi N$ scattering amplitude $\mathcal{F}(\vec{q}', \vec{q})$ at low energy may be confined to $s$- and $p$-wave contributions (e.g., see Ref. [34], pp. 17–18). Introducing the isospin of the pion as $\hat{t}$ and that of the nucleon as $\hat{t}/2$, one may write

$$
\mathcal{F}(\vec{q}', \vec{q}) = b_0 + b_1 \hat{t} \cdot \hat{t} + \left( c_0 + c_1 \hat{t} \cdot \hat{t} \right) \vec{q}' \cdot \vec{q} + i \left( d_0 + d_1 \hat{t} \cdot \hat{t} \right) \hat{\sigma} \cdot (\vec{q}' \times \vec{q}), \quad (18)
$$

where $\hat{\sigma}$ is (twice) the spin of the nucleon and $\vec{q}'$ is the 3-momentum of the outgoing pion in the CM system.

Equation (18) defines the isoscalar and isovector $s$-wave scattering lengths ($b_0$ and $b_1$) and the isoscalar(isovector)-scalar(vector) $p$-wave scattering volumes.
Table 6

The values of the six $s$- and $p$-wave phase shifts (in degrees), obtained from the results of Table 3 and the corresponding Hessian matrices (not given in this work) for the fits of the ETH model to the tDB$_{+/0}$.

| $T$ (MeV) | $\delta^{3/2}_{0+}$ (S31) | $\delta^{1/2}_{0+}$ (S11) | $\delta^{3/2}_{1+}$ (P33) | $\delta^{3/2}_{1-}$ (P31) | $\delta^{1/2}_{1+}$ (P13) | $\delta^{1/2}_{1-}$ (P11) |
|-----------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 20        | 2.536(52)       | 4.476(63)       | 1.3384(91)      | 0.2432(79)      | 0.1739(71)      | 0.412(12)       |
| 25        | 2.944(55)       | 4.994(67)       | 1.897(12)       | 0.335(11)       | 0.2360(98)      | 0.543(16)       |
| 30        | 3.345(58)       | 5.457(70)       | 2.531(15)       | 0.434(15)       | 0.301(13)       | 0.673(22)       |
| 35        | 3.741(60)       | 5.876(73)       | 3.241(17)       | 0.539(19)       | 0.369(16)       | 0.797(27)       |
| 40        | 4.135(61)       | 6.259(76)       | 4.030(20)       | 0.650(23)       | 0.439(20)       | 0.915(33)       |
| 45        | 4.529(64)       | 6.613(80)       | 4.899(22)       | 0.766(28)       | 0.509(24)       | 1.024(39)       |
| 50        | 4.924(66)       | 6.939(84)       | 5.852(23)       | 0.886(33)       | 0.580(28)       | 1.122(46)       |
| 55        | 5.319(70)       | 7.242(89)       | 6.895(25)       | 1.010(38)       | 0.652(32)       | 1.208(53)       |
| 60        | 5.716(74)       | 7.524(95)       | 8.033(27)       | 1.137(43)       | 0.723(37)       | 1.281(61)       |
| 65        | 6.115(80)       | 7.79(10)        | 9.273(30)       | 1.267(49)       | 0.795(41)       | 1.340(68)       |
| 70        | 6.515(87)       | 8.03(11)        | 10.622(34)      | 1.401(56)       | 0.865(46)       | 1.383(77)       |
| 75        | 6.916(95)       | 8.26(12)        | 12.090(40)      | 1.537(62)       | 0.936(52)       | 1.411(85)       |
| 80        | 7.32(10)        | 8.47(13)        | 13.686(47)      | 1.675(69)       | 1.005(57)       | 1.422(94)       |
| 85        | 7.72(11)        | 8.66(14)        | 15.421(58)      | 1.815(76)       | 1.074(63)       | 1.42(10)        |
| 90        | 8.13(13)        | 8.84(15)        | 17.307(70)      | 1.958(84)       | 1.141(68)       | 1.39(11)        |
| 95        | 8.54(14)        | 9.01(17)        | 19.355(86)      | 2.102(91)       | 1.207(74)       | 1.35(12)        |
| 100       | 8.95(15)        | 9.16(18)        | 21.58(10)       | 2.248(99)       | 1.272(80)       | 1.29(13)        |

($c_0$, $c_1$, $d_0$, and $d_1$), which may be projected onto the isospin-spin basis using standard expressions. Table 7 contains the predictions of the ETH model for all these quantities, corresponding to the tDB$_{+/-}$ and tDB$_{+/0}$ solutions. The results for the scattering lengths $a^{3/2}_{0+}$ and $a^{1/2}_{0+}$ are similar to those obtained in this programme during the last two decades.

The disagreement of the $a^{cc}$ value obtained from the analysis of the tDB$_{+/-}$ with the results extracted from the pionic-hydrogen experiments at the Paul Scherrer Institut [35,36] has been known for a long time [20,21,23,37]. Corrected for EM effects according to Ref. [37], the $a^{cc}$ value of Ref. [35] is equal to 0.08576(80) $m^{-1}$, the one of Ref. [36] to 0.08549(59); the statistical and systematic uncertainties in both experiments have been linearly combined. The
disagreement with the result obtained from the scattering data (via the extrapolation of the model amplitudes to threshold) currently amounts to about a $3.8-3.9\sigma$ effect in the normal distribution (see also remarks in Section 3.2.4). The closest to the experimental result which the model predictions have ever reached was in Refs. [15,16], when robust statistics was employed in the optimisation and the floating of the experimental data sets was not allowed; the $a^{cc}$ value of Ref. [16] (p. 3022) was: $0.0826(20) \, m_e^{-1}$. Assuming the correctness of the absolute normalisation of the bulk of the tDB$_{+/−}$ above (the $\pi N$) threshold and the negligibility of the residual effects in the EM corrections of Refs. [4,5], this discrepancy may be explicable in terms of one of the following two reasons (or their combination).

- A departure from isospin symmetry in the low-energy $\pi N$ interaction. The ETH model is isospin-invariant (hence also isospin-symmetric). Close to threshold, the $\pi^+p$ scattering amplitude is dominated by the scattering-length combination $b_0 + b_1$, whereas the $\pi^-p$ ES amplitude by the combination $b_0 - b_1$. Visual inspection of Table 4 of Ref. [26] demonstrates that the scattering length $b_1$ is almost exclusively accounted for by the $\rho$-exchange graph. Consequently, if the $\rho$-exchange graph is affected by isospin-breaking
Fig. 4. Same as Fig. 3 for the phase shift $\frac{3}{2}^{1+}$ (P33). To facilitate the comparison of the values contained in this figure, the energy-dependent quantity $\delta_R$ ($= (0.20 \cdot T + 1.54) \cdot T \cdot 10^{-2}$, with $T$ in MeV and $\delta_R$ in degrees) has been subtracted from all data. The SAID solution WI08 [1] is given by the dashed curve; the five points shown (at $T = 20, 30, 47, 66,$ and $90$ MeV) are the WI08 single-energy values.

effects, the model-predicted $b_1$ would deviate from the experimentally obtained value as it does not include isospin-breaking effects (at least, as direct contributions to the model amplitudes), whereas the experimentally obtained value does. As a matter of fact, a mechanism for the violation of isospin symmetry in ES, involving the $\rho^0 - \omega$ mixing, has been known for a long time. Further comments on this possibility may be found in the last section of this work.

- A problem with the experimental values of the strong shift of the $1s$ state in pionic hydrogen. It is frequently assumed that the pionic-hydrogen data (i.e., the measured strong shift and width of the $1s$ state in pionic hydrogen) suffer from fewer problems in comparison with the experiments aiming at measuring the DCS above threshold; this assumption might be incorrect. For the sake of example, one may consider the determination of the width of the $1s$ state in pionic hydrogen, as emerging from the most recent experimental activity of the Pionic-Hydrogen Collaboration at the Paul Scherrer Institut;
Fig. 5. The energy dependence of the phase shift $\delta_{3/2}^{3/2}$ (P31) obtained from the fits of the ETH model to the tDB$_{+-}$ (blue band) and tDB$_{+}/0$ (yellow band). The bands represent 1σ uncertainties around the average values. The SAID solution WI08 [1] is given by the dashed curve.

the final value for the strong shift from that experiment was reported about one decade after the experiment was completed [36]. Preliminary results for the width of the 1s state were announced at a conference around the same time [38]. Referring to Fig. 3 of Ref. [38], no concrete picture emerges for the width of the 1s state because the corresponding probability distributions, obtained from three transitions ($2, 3, 4p \to 1s$), each measured twice, hardly overlap. In a perfect world, and assuming that all effects have correctly been taken into account, these six probability distributions should agree, as they all represent the width of one specific state. Unfortunately, this is not the case.

The mismatch between the two $a^{cc}$ results cannot be blamed on the modelling of the hadronic part of the $\pi N$ interaction by the ETH model. To demonstrate that one obtains similar results by other means of modelling of the hadronic part of the $\pi N$ interaction, the description of the available data sets in tDB$_{-}$ closest to (but above) threshold may be examined; as shown in Fig. 9 and in
Fig. 6. Same as Fig. 5 for the phase shift $\delta_{1+}^{1/2}$ (P13).

Table 9, these data sets have been taken around 30 MeV. The description of these data sets with the ETH model is very satisfactory, yielding an average scale factor of $1.007 \pm 0.013$. Therefore, the model results reflect the absolute normalisation of these experimental data sets almost perfectly. It is interesting to investigate the goodness of the reproduction of these data by another phase-shift solution. In Refs. [39,40], the authors compared their $\pi^-p$ DCSs to the Karlsruhe-Helsinki KH80 solution [41]. In their Table III of Ref. [40], they give ratios of their DCSs to the corresponding KH80 predictions, also including in this comparison the FRANK83 [42] and BRACK90 [43] data sets around 30 MeV. The extracted ratios between the various data sets and the KH80 predictions read as: 0.989, 0.952, 0.937, and 1.000 for the FRANK83 [42], BRACK90 [43], JORAM95 [39], and JORAM95 [40] data sets, respectively. It thus follows that the DCSs around 30 MeV lie between 0.937 and 1.000 of the KH80 predictions, i.e., their absolute normalisation is somewhat below the KH80 prediction. On the other hand, the KH80 prediction for the scattering length $a_{cc}$ is $0.0817(24) \ m_c^{-1}$ (see Table 4 in Ref. [41] and the relations in Table 7 in this work), in agreement with the prediction from the fits of the ETH model to the tDB$_{+/\ -}$ (Table 7). However, both experimental results of Refs. [35,36], corrected for EM effects according to Ref. [37], exceed the KH80
prediction by about $(5.0 \pm 3.2)\%$, which converted into a cross section would correspond to a discrepancy at the level of $(10.3 \pm 6.8)\%$. Therefore, the data sets above threshold (but reasonably close to it) set the level of the absolute normalisation of the $\pi^-p$ ES reaction to about $0.97 \pm 0.03$ of the KH80 prediction, whereas the experimentally obtained information at threshold would rather set the normalisation level to $1.103 \pm 0.068$ of the KH80 prediction. (Owing to the more precise - compared to the KH80 results - prediction of the ETH model for the scattering lengths, this discrepancy appears to be more statistically significant in the case of the model.) This PSA relies on the correctness of the absolute normalisation of the bulk of the low-energy $\pi N$ data. The $\pi^-p$ ES DCSs at 30 MeV are reproduced very well; because of this, the model-predicted $a^{cc}$ is bound to be smaller than the experimentally obtained values at threshold. (For the sake of completeness, the $a^{cc}$ result, obtained from the fits of the phenomenological model to the tDB$_{+/−}$, comes out equal to $0.082(4)\ m^{-1}$. ) To conclude, the discrepancy between the model-predicted $a^{cc}$ and the values obtained from the experimental results of Refs. [35,36] does not seem to be due to the use of the ETH model in the analysis. In fact, it is debatable whether any reasonable model could generate discrepancies at the observed level within an energy interval of 30 MeV. Therefore, in all
Table 7

Upper part: The isoscalar and isovector s-wave scattering lengths (in \( m_c^{-1} \)) and the isoscalar(isovector)-scalar(vector) p-wave scattering volumes (in \( m_c^{-3} \)) based on the fits of the ETH model to the tDB\(_{+/−} \) and tDB\(_{+/0} \); these quantities are defined by Eq. (18). Middle part: the results for the standard spin-isospin quantities. Lower part: the predictions for the \( π^- p \) ES and CX scattering lengths.

| Scattering length/volume | tDB\(_{+/−} \) | tDB\(_{+/0} \) |
|--------------------------|---------------|---------------|
| \( b_0 = +\frac{2}{3} a_{0+}^{3/2} + \frac{1}{3} a_{0+}^{1/2} \) | 0.0021(12) | 0.0026(25) |
| \( b_1 = +\frac{1}{3} a_{0+}^{3/2} - \frac{1}{3} a_{0+}^{1/2} \) | -0.0777(60) | -0.08279(51) |
| \( c_0 = +\frac{2}{3} a_{1+}^{3/2} + \frac{1}{3} a_{1+}^{1/2} \) | 0.2064(24) | 0.2125(35) |
| \( c_1 = +\frac{2}{3} a_{1+}^{3/2} + \frac{1}{3} a_{1−}^{3/2} - \frac{2}{3} a_{1+}^{1/2} - \frac{1}{3} a_{1−}^{1/2} \) | 0.1755(18) | 0.1847(15) |
| \( d_0 = -\frac{2}{3} a_{1+}^{3/2} + \frac{2}{3} a_{1−}^{3/2} - \frac{1}{3} a_{1+}^{1/2} + \frac{1}{3} a_{1−}^{1/2} \) | -0.1856(19) | -0.1955(16) |
| \( d_1 = -\frac{1}{3} a_{1+}^{3/2} + \frac{1}{3} a_{1−}^{3/2} + \frac{1}{3} a_{1+}^{1/2} - \frac{1}{3} a_{1−}^{1/2} \) | -0.06786(81) | -0.0703(10) |
| \( a_{0+}^{3/2} \) | -0.0756(15) | -0.0802(25) |
| \( a_{0+}^{1/2} \) | 0.1576(14) | 0.1682(29) |
| \( a_{1+}^{3/2} \) | 0.2118(21) | 0.2210(19) |
| \( a_{1−}^{3/2} \) | -0.04165(75) | -0.0448(13) |
| \( a_{1+}^{1/2} \) | -0.03157(64) | -0.0339(12) |
| \( a_{1−}^{1/2} \) | -0.0815(14) | -0.0889(21) |
| \( a^{cc} = +\frac{1}{3} a_{0+}^{3/2} + \frac{2}{3} a_{0+}^{1/2} \) | 0.0799(11) | 0.0854(27) |
| \( a^{c0} = +\sqrt{\frac{3}{5}} a_{0+}^{3/2} - \sqrt{\frac{2}{5}} a_{0+}^{1/2} \) | -0.10989(85) | -0.11708(72) |

probability, the problem lies elsewhere.

Unlike the tDB\(_{+/−} \) \( a^{cc} \) prediction, the prediction obtained from the results of the fits of the ETH model to the tDB\(_{+/0} \) matches the experimental result. Additionally, the same prediction for \( a^{c0} \) comes closer to the result of Ref. [35] for the width of the 1s state in pionic hydrogen, which (corrected for EM effects according to Ref. [37]) is equal to \(-0.1276(57) \, m_c^{-1} \).

Within the context of the tree-level approximation of the ETH model, the \( Σ \) term is obtained via Eq. (61) of Ref. [26]; using the results of the fits of the ETH model to the tDB\(_{+/−} \), that equation yields: \( Σ = 70.3 ± 3.0 \, \text{MeV} \). Following Olsson’s method [44] (see Section 4.2.2 of Ref. [26] for a discussion), one obtains the isoscalar effective range \( C^+ = -0.1053 ± 0.0042 \, m_c^{-3} \) and, from Eq. (63) of Ref. [26], \( Σ = 68.0 ± 2.4(\text{stat.}) ± 1.7(\text{syst.}) \, \text{MeV} \). Therefore, the \( Σ \)-term values, obtained with these two methods, are compatible with one
another and are in good agreement with the corresponding results of Ref. [26].

For the sake of completeness, the Σ-term prediction, obtained from the results of the fits of the ETH model to the tDB_{+0}, reads as: Σ = 74.3 ± 6.1 MeV. The agreement with the result obtained from the fits of the ETH model to the tDB_{+/−} implies that the isoscalar part of the πN interaction is practically unaffected by the change of the DB. Evidently, if the discrepancy in a^{xc} is to be blamed on the violation of isospin symmetry, then only the isovector part of the interaction appears to be affected to a significant degree (see also the results for b_{0} and b_{1} in Table 7). This conclusion would match the earlier remark regarding the attribution of the departure from isospin symmetry to the ρ-exchange graph (ρ^{0} − ω mixing).

3.2.4 Scale factors

Investigated in this section is the distribution of the scale factors z_{j} obtained from the fits of the ETH model. When the Arndt-Roper formula is used in the optimisation, the expectation is that the data sets which must be scaled ‘upwards’ balance (on average) those which must be scaled ‘downwards’. Additionally, the energy dependence of the scale factors must not be significant. If these prerequisites are not fulfilled, the modelling of the input measurements cannot be considered as satisfactory. As demonstrated in Ref. [2], the fulfilment of these conditions does not only involve the entire set of the scale factors z_{j} in a fit, but also those of arbitrary subsets of the fitted DB, consistent with the basic principles of the Sampling Theory (adequate population, representative sampling). One obvious comparison is dictated by the particularity of the input DBs of this work: it must be verified that the scale factors, corresponding to the two distinct subsets of the fitted DBs, i.e.,

- to the tDB_{+} and to the tDB_{−} in the case of the fits of the ETH model to the tDB_{+/−}
- to the tDB_{+} and to the tDB_{0} in the case of the fits of the ETH model to the tDB_{+/0}

are centred on 1 and show no significant energy dependence.

For both the π^{+}p (Fig. 8) and π^{−}p ES (Fig. 9) data sets, the z_{j} values above and below 1 roughly balance and their energy dependence is not statistically significant. The weighted linear fit (T being the independent variable) to the scale factors for the π^{+}p reaction yields the intercept of 1.007 ± 0.019 and the slope of (−0.4 ± 2.5) · 10^{-4} MeV^{-1}. The weighted linear fit to the scale factors for the π^{−}p ES reaction yields the intercept of 1.030 ± 0.018 and the slope of (−4.1 ± 2.3) · 10^{-4} MeV^{-1}. Therefore, the departure from the expectation for an unbiased outcome of the optimisation (intercept 1 and vanishing slope) is not statistically significant in both cases.
That being said, one nonetheless notices that the effects in the $\pi^{-}p$ ES intercept (and slope) appear to be somewhat more pronounced than in the $\pi^{+}p$ case. This is not a new result; a similar tendency has been observed in our past PSAs. Taken at face value, the $\pi^{-}p$ ES result for the weighted linear fit to the $z_j$ values suggests that (when using the ETH model) the joint optimisation scheme of the tDB$_{+/−}$ enforces the underestimation of the $\pi^{-}p$ ES DCS in the vicinity of the threshold, by about $(3.0 \pm 1.8)\%$. If one takes this bias literally, one could consider the application of a correction of about $(1.47 \pm 0.88)\%$ to the model-predicted $a^{ee}$ value (when the results of the fits of the ETH model to the tDB$_{+/−}$ are used for the predictions). Even in this case, the corrected model-predicted $a^{ee}$ $(0.08157 \pm 0.0014 \, m_{c}^{-1})$ would still disagree with the results of Refs. [35,36], corrected for EM effects according to Ref. [37]; however,
Fig. 9. The scale factors $z_j$ of the data sets in the tDB$_-$, obtained from the fits of the ETH model to the tDB$_+/-$: solid points: DCS, crosses: AP. The value, corresponding to the data set which was freely floated (see Table 9), has not been included in this plot. Also not included are the entries for the two data sets of Ref. [45]; as mentioned in the caption of Fig. 8, $T$ was not kept constant within each of these data sets. The dashed straight line represents the optimal, weighted linear fit to the data shown and the shaded band 1σ uncertainties around the fitted values. The red line represents the optimal, unbiased outcome of the optimisation.

The discrepancy between the model-predicted $a^{\infty}$ and the $a^{\infty}$ values obtained from the experimental results of Refs. [35,36] would appear milder (equivalent to an effect at the level of about 2.6σ in the normal distribution).

Two weighted linear fits to the optimal $z_j$ values of Figs. 10 (for the $\pi^+p$) and 11 (for the $\pi^-p$ CX reaction) were also carried out in the case of the tDB$_+/0$ solution. The results of these two fits do not match well the expectations for an unbiased outcome of the optimisation $^{10}$. The two values of the intercept

$^{10}$ The fitted values of the intercept and slope are almost left intact in case the scale factors of the (four) FITZGERALD86 data sets are not included in the weighted linear fit; the resulting values are: $1.051 \pm 0.011$ for the new intercept and $(-6.1 \pm 2.0) \cdot 10^{-4}$ MeV$^{-1}$ for the new slope.
were: $0.964 \pm 0.020$ in the case of the $\pi^+p$ reaction and $1.054 \pm 0.013$ for the $\pi^-p$ CX reaction. The slope was found to be compatible with 0 in the former case: $(3.4 \pm 2.6) \cdot 10^{-4}$ MeV$^{-1}$. On the other hand, significant effects were observed in the analysis of the scale factors corresponding to the $\pi^-p$ CX reaction; the slope came out equal to $(-6.0 \pm 2.2) \cdot 10^{-4}$ MeV$^{-1}$. These results are indicative of a problematic situation: when forcing the data of these two reactions into a joint optimisation scheme with the ETH model, the overall tendency of the modelling is to generate overestimated fitted DCS values for the $\pi^+p$ reaction and underestimated ones for the $\pi^-p$ CX reaction at low energy. Evidently, the optimisation of the description of the input data is achieved at the expense of introducing biases in the description of both subsets of the tDB$^{+}/0$. Equivalently, one might claim that the $I = 3/2$ amplitudes, obtained with the model, have a difficulty to simultaneously account for the $\pi^+p$ and $\pi^-p$ CX reactions. As such difficulties were not experienced in the PSA of the DB$^{+/-}$ (at least, to a significant degree), it is justified to raise the question whether the inclusion of the tDB$^0$ into a joint optimisation scheme, along with data from any of the two ES DBs, is meaningful. It is also clear that the results obtained from the analysis of the tDB$^{+}/0$ (shown in Tables 3, 6, and 7) must be taken with caution.

Interestingly, the effects observed in Figs. 10 and 11 match very well those obtained in the analysis of the output of the SAID solution WI08, see Figs. 4 and 6 of Ref. [2]! The inevitable conclusion is that, when forcing the $\pi^-p$ CX reaction data into a joint optimisation scheme, the analysis becomes biased at low energies, in the sense of an unmistakable departure from the statistical expectation.

The results of this section demonstrate the importance of the steps which must be taken in a data analysis, towards the understanding of the outcome of the optimisation. The goodness of the data description of the entire input DB must be investigated; however, the same also applies to the parts of the DB which comprise autonomous subsets (in this case, the measurement sets for the different reactions). Unless these steps are successfully carried out, there can be no confidence in the results of an analysis in terms of presence of a bias in the optimisation.

3.2.5 Reproduction of the DENZ04 DCSs

To address the reproduction of the ES data which have not been included in the DB$^{+/-}$ of this work, one may follow the methodology of Section 2.5. There is one main block of measurements which, though analysed separately in the recent past, has not been included in the DB$^{+/-}$, namely the extensive DENZ04 [25] DCSs. The DENZ04 data sets were thoroughly compared to the rest of the modern data in Refs. [23,24].
The reproduction of these data by the predictions obtained from the fits of the ETH model to the tDB_{+/-} is displayed in Figs. 12 and 13. The p-values corresponding to the tests of the overall reproduction, of the shape, and of the absolute normalisation, as outlined at the end of Section 2.5, are shown in Table 11. It is evident from this table that the reproduction of all the DENZ04 π^+p DCSs is very poor and that, in all six cases, the problems are related to the shape of the angular distribution of the cross section; the shape of the 25.80 MeV π^-p ES DCSs is also surprising. It is true that part of the data could have been used in our recent PSAs. However, unless the peculiar shape of the π^+p DCSs is understood, we will refrain from using the entirety of the DENZ04 data. Following the recommendation of the CHAOS Collaboration and using their unsplit data, one obtains (as a sum over the \( (\chi^2_j)_{\text{min}} \) contributions of Eq. (12)) a reproduction \( \chi^2 \) of 1027.1 for 275 π^+p measurements and 256.8 for 271 π^-p ES measurements. The problem with the π^+p contribution is obvious.

The SAID group have segmented the data into forward-, medium-, and backward-angle measurements (which is a good idea for several reasons [23,24]) and have included the segmented data sets into their DB. As expected, compared to the
Fig. 11. The scale factors $z_j$ of the data sets in the DB$_0$, obtained from the fits of the ETH model to the tDB$_+/0$: solid points: DCS, diamonds: PTCS/TCS, crosses: AP, triangles: LECs; the square represents the $b_1$ value obtained from the measurement of the width of the 1$s$ state in pionic hydrogen [35]. The values, corresponding to the three data sets which were freely floated (see Table 10), have not been included. The dashed straight line represents the optimal, weighted linear fit to the data shown and the shaded band $1\sigma$ uncertainties around the fitted values. The red line represents the optimal, unbiased outcome of the optimisation.

original DCSs, the segmented data sets yield more reasonable results. As a contribution of the DENZ04 DCSs to the $\chi^2_{\text{min}}$ of their fit, the SAID group report a total of 1102.7 for 545 data points (626.9/274 for the $\pi^p$, 475.8/271 for the $\pi^-p$ ES data sets). Therefore, even the segmented data sets are poorly described by the WI08 solution. The following $\pi^p$ data sets cannot be reproduced by the results of the fits of the ETH model to the tDB$_+/-$ of this work: 19.90 and 43.30(rot.) MeV data sets at forward angles, the 43.30(rot.) MeV data set at medium angles, and the 25.80 MeV data set at backward angles; the problems with the first and third data sets pertain to shape, whereas with the other two data sets to absolute normalisation. From the $\pi^-p$ ES data sets, only the 25.80 MeV data set at forward angles is poorly reproduced (due to shape). Using the split data sets, reproduction $\chi^2$'s (sums over the $(\chi^2_j)_{\text{min}}$
contributions of Eq. (12)) of 330.5 for 275 $\pi^+p$ measurements and 185.8 for 271 $\pi^-p$ ES measurements are obtained. From these $\chi^2$ contributions, one may not draw the conclusion that the ZRH17 solution (obtained from the tDB$_{+/−}$) accounts for the DENZ04 data better than the WI08 solution. The comparison is unfair as the values of this work represent sums of reproduction $\chi^2$ values and contain the theoretical uncertainties; on the other hand, the values of the SAID group represent sums of description $\chi^2$ values and can only contain the experimental uncertainties.

3.2.6 Reproduction of the $\pi^-p$ PTCSs and TCSs

In Section 3 of Ref. [20], it was argued that, as the $\pi^-p$ PTCSs contain a large component from $\pi^-p$ CX scattering, they should not be used in this programme; in spite of the scarcity of these measurements, their inclusion in any part of the analysis could provide ground for criticism and cast doubt on the interpretation of the results in terms of the isospin invariance in the $\pi N$ interaction. The results of the reproduction of the three available $\pi^-p$ PTCSs at low energy are shown in Table 12. The contributions from the $\pi^-p$ CX reaction are large and substantiate the line of argument of Ref. [20] to avoid including these data in the DB$_-$. The measurements of the $\pi^-p$ PTCS may be compared to the predictions obtained after summing up the contributions of the $\pi^-p$ ES PTCS and that of the entire $\pi^-p$ CX TCS; this operation is dictated by the experimental technique employed in the acquisition of the $\pi^-p$ PTCS measurements, namely, the detection only of the $\pi^-$'s (interacting or passing through) downstream of the target, within a cone of aperture $2\theta_L$ with its apex at the geometrical centre of the target, where $\theta_L$ is the laboratory-angle cut associated with the measurement. Regarding the component of the $\pi^-p$ CX TCS, one may use predictions obtained either from the fits of the ETH model to the tDB$_{+/−}$ or from those to the tDB$_{+\,0}$. Although a slight preference for the tDB$_{+\,0}$ predictions may be observed, the experimental uncertainties are too large to enable definite conclusions.

In addition, there are six ‘measurements’ of the so-called $\pi^-p$ ES ‘total-nuclear’ cross section, obtained from measurements of the PTCS after the EM contributions of the Coulomb peak (as well as the contributions of the Coulomb-nuclear interference) are removed. The proper treatment of these data is described in Ref. [18] (see Section ‘The total cross section’ therein); in regard to the estimates for this quantity, the suggestion of Ref. [18] is to use the corrected (i.e., not the hadronic) phase shifts in the definitions of the partial-wave amplitudes and omit the Coulomb phase shift (and, of course, the direct Coulomb amplitude). The measurements and the predictions, based on the fits to both the tDB$_{+/−}$ and to the tDB$_{+\,0}$, are given in Table 13. Despite the fact that
the uncertainties in the measurements are too large to enable safe conclusions, the prediction based on the fits of the ETH model to the tDB_{+/-} again comes closer to the measurements.

3.2.7 Reproduction of the DB_{0} by the results of the fits of the ETH model to the tDB_{+/-}

This is the first time that the methodology developed in Ref. [24] will be applied to the reproduction of the DB_{0}. Investigated here is the absolute normalisation of the DB_{0} using the prediction based on the fits of the ETH model to the tDB_{+/-} as reference (BLS). To this end, one must determine the amount at which the reference predictions for each data set in the DB_{0} (i.e., the \( y_{ij}^{th} \) values) must be floated in order to optimally reproduce the data of the specific data set (i.e., the \( y_{ij}^{exp} \) values). Therefore, relevant in this part of the analysis are the scale factors for free floating, obtained from Eq. (13).

The extracted values of the scale factors \( \hat{z}_j \) and their total uncertainties, i.e., the purely statistical uncertainties

\[
\delta \hat{z}_j = \frac{1}{\sqrt{\sum_{i=1}^{N_j} w_{ij}}}
\]

(where the weights \( w_{ij} \) were defined in Section 2.5) combined (quadratically) with the normalisation uncertainty of the data set \( \delta z_j \), may be found in Table 14 and, plotted separately for the DCS, TCS, and LECs measurements, in Fig. 14. Not included in the figure (but given in the table) are the entries for the three FITZGERALD86 data sets which had been freely floated in the optimisation (see Table 10), the entry for the BREITSCHOPF06 one-point data set which had been eliminated, the entry for the scattering length \( b_1 \) of Ref. [35], and the APs (being ratios of cross sections, the APs are not suitable for investigating isospin-breaking effects). (In principle, after squaring the result of Ref. [35], one could assign it to the DCS set, in which case the outcome would have been consistent with the scale factors \( \hat{z}_j \) of the ISENHOWER99 10.60 MeV data; however, it was decided to show (and fit) genuine DCS measurements in Fig. 14.) An interesting feature of Table 14 is that all 18 (out of a total of 54) data sets, which are poorly reproduced, have problematic absolute normalisation; there is no data set with problematic shape.

Inspection of Fig. 14 leaves no doubt that, when using as BLS the prediction based on the fits of the ETH model to the tDB_{+/-}, the scale factors \( \hat{z}_j \) of the data sets in the DB_{0} contain a large amount of fluctuation. As the tDB_{+/-} prediction is smooth, this fluctuation reflects the variation of the absolute normalisation of the data sets in the DB_{0}. For the sake of example, the \( \hat{z}_j \) value for the FRLEZ98 data set comes out equal to 1.425(99). This data set lies in between three data sets with considerably smaller \( \hat{z}_j \) values, i.e., between
An exponential function of the form
\[ \dot{\hat{z}} = \alpha \exp(-\beta T) + 1 \]
may be fitted to these \( \dot{\hat{z}}_j \) values. Using the data shown in Fig. 14, one obtains a questionable fit with \( \chi^2_{\text{min}} \approx 72.3 \) for 45 DOFs, p-value \( \approx 6.10 \cdot 10^{-3} \), and the optimal values of the parameters \( \alpha \) and \( \beta \) equal to 0.361(59) and 0.0221(39) MeV\(^{-1} \), respectively. These values suggest sizeable isospin-breaking effects in the low-energy region, increasing with decreasing beam energy.

4 Possible causes of the discrepancies in the low-energy \( \pi N \) interaction

The discrepancies, revealed by the study of the \( \pi N \) interaction at low energy, suggest that at least one of the following assumptions is not fulfilled.

- The absolute normalisation of the bulk of the low-energy \( \pi N \) data is reliable.
- The residual effects in the EM corrections of Refs. [4,5] are negligible.
- The isospin invariance holds in the hadronic part of the \( \pi N \) interaction.

The three possibilities, arising from the non-fulfilment of these assumptions, will now be explored further.

4.1 Experimental mismatches

The first explanation for the discrepancies involves a trivial effect, namely the systematic incorrectness of the absolute normalisation of the modern low-energy \( \pi N \) data. For the sake of example, such a situation would arise if a part of the final-state pions evaded detection or if the flux of the incident beam were overestimated in the modern low-energy \( \pi N \) experiments; in both cases, the DCSs at low energy would turn out to be systematically underestimated.

The first point concerns the absolute normalisation of some low-energy experiments. It is not understood how it is possible that, in some cases, the absolute normalisation is as much ‘off’ as it appears to be. For instance, the absolute normalisation of the FITZGERALD86 data sets at the three lowest
energies in that experiment appears to exceed the absolute normalisation of
the bulk of the tDB\(_0\) by an average of almost 70\% (see Table 10); the reported
normalisation uncertainty in the experiment was 7.8\%. Such an effect may be
due to one of the following reasons (or their combination): a) The energy of
the incoming beam had not been what the experimental group expected. b)
The effects of the contamination of the incoming beam were underestimated in
the experiment. c) The normalisation uncertainty in the experiment had been
grossly underestimated. d) The determination of the absolute normalisation
in the experiment had been erroneous.

The second point concerns the smallness of the normalisation uncertainty
reported in several low-energy experiments. Arguments were presented in
Ref. [20], to substantiate the suggestion that the reported uncertainties in
the \(\pi N\) experiments at low energy must have been underestimated. In 23 (out
of 52) DCS data sets (with known normalisation uncertainty) in the initial
DB\(_{+/-}\) of this work, normalisation uncertainties below 3\% had been reported.
The smallest normalisation uncertainty in the tDB\(_{+/-}\) is equal to a mere 1.2\%,
which (given the experience gained after two decades of relevant experimen-
tation) is highly suspicious. While pondering over these issues, one frequently
comes to the opinion that it would make sense to disregard all claims of such
exaggerated and unrealistic accuracy, and assign to all the relevant data sets
a normalisation uncertainty of 3\%. However, such an approach would appear
arbitrary, providing ground for criticism. Such revisions must be instigated by
the original experimental groups, not by analysts. Consequently, one is left
with no other choice than to rely on an approach placing importance on the
bulk of the low-energy \(\pi N\) data and to apply a reasonable statistical procedure
for the elimination of the outliers.

There is one disturbing discrepancy in the analysis of the tDB\(_{+/-}\), namely the
disagreement between the \(a^{cc}\) value obtained as an extrapolation from the data
above threshold and the \(a^{cc}\) values extracted from the experimental results of
Refs. [35,36]. To advance the next point, let us assume that

- the isospin symmetry holds in the \(\pi N\) interaction,
- the absolute normalisation of the bulk of the tDB\(_{+/-}\) is reliable, and
- the \(\epsilon_{1s}\) measurements [35,36] are reliable.

Under these assumptions, the two extracted \(a^{cc}\) values should be compatible, if
consistent EM corrections have been applied to the data, i.e., corrections which
have been obtained within the same framework (as the case was with the EM
corrections of Refs. [4,5,37]) and are complete (they contain the contributions
from all relevant physical effects). In case of sizeable residual EM effects (see
next section), the question surfaces as to their energy dependence. On the
other hand, it is not easy to advance a straightforward explanation for the
results of Table 4 on the basis of the systematic incorrectness of the absolute
normalisation of the bulk of the low-energy $\pi N$ data.

4.2 Residual EM effects

Although the attribution of the results of Table 4 to residual EM effects does not seem likely (any residual EM corrections would equally affect the description of the data with the phenomenological model and with the ETH model), the completeness of the EM corrections in the $\pi N$ interaction at low energy is an important issue which must be properly addressed. In Refs. [20,37], some information had been given on the effects which the residual EM corrections contain; the missing contributions are predominantly related to the use of the physical (instead of the unknown hadronic) masses for the proton, for the neutron, and for the charged and neutral pions in the determination of the EM corrections. On the other hand, provided that the inclusion of the residual EM effects should result in an improved description of the input data, the iterative procedure which had been put forward in the determination of the EM corrections in Refs. [4,5] might have already captured a part of these effects.

The EM corrections are identified as the changes (in the phase shifts and in the partial-wave amplitudes) in the results of a calculation based on an isospin-invariant hadronic model, due to taking into account also the EM interaction. There are important issues which need to be addressed (and resolved) in the re-assessment of the EM effects in the $\pi N$ interaction.

- Clear definitions of what is meant by ‘EM part’ and by ‘hadronic part’ of the $\pi N$ interaction.
- Suitable methodology for dealing with the hadrons after their constituents (quarks and antiquarks) have been deprived of their EM properties.
- Suitable methodology for dealing with the altered kinematics, after the interacting hadrons have been deprived of the EM contributions to their rest masses. In regard to this point, it cannot be excluded that the only practical approach is to assume that the hadronic masses of the various particles are fixed to the physical ones, an assumption underlying both the NORDITA work and Refs. [4,5].

The clear definitions and the methodology for the treatment of the EM effects in the $\pi N$ interaction necessitate theoretical advances and perhaps also involve an optimisation of the description of the available data. Such a project goes well beyond the narrow scope and the short-term goals of a doctoral thesis. It should better be thought of as a five-year project involving a team of experienced scientists. In addition, it requires the close collaboration of theorists and experimentalists. It is unclear to us how knowledge of the hadronic part of the $\pi N$ interaction may be improved by new low-energy experiments,
while the subject of the EM corrections remains unresolved. Even in the case of perfect data and no discrepancies in the low-energy database, one would still need to address the extraction of the important (hadronic) information from those perfect measurements.

The new EM corrections must be obtained in a wide energy range and must be easy to implement and use, as straightforward as the corrections of the NORDITA group had been. It would be convenient to have the results in tabulated form (perhaps containing more energies than the NORDITA tables). Of course, from the practical point of view, it would be optimal if one dedicated group undertook the responsibility not only of the development, but also of the implementation of the EM corrections: the optimal solution would be to place the derivation of the corrections in one software library and make them available to the user via suitable methods with simple interfaces. In principle, the only input is the beam energy, the (model-dependent) \( \pi N \) phase shifts, and the reaction type, whereas the output would be the corrected phase shifts and (complex) partial-wave amplitudes, or even the corresponding values of the observables. Such a scheme would disperse the current dubiousness in regard to the comparison of the results obtained in the various PWAs of the \( \pi N \) data: at this time, one cannot be certain whether the differences revealed by the comparisons originate in the dissimilarity of the modelling schemes of the hadronic interaction, of the analysis techniques, or of the treatment of the EM effects. It is unfortunate that, as it rests on the modelling of the hadronic interaction via effective potentials, the methodology of Refs. [4,5] is suitable for providing EM corrections only at low energy; there seems to be no promising way to extend this approach above 100 MeV. In all probability, the NORDITA work must be repeated, benefiting from the experience gained from the experimentation and analysis of the \( \pi N \) data during the last 30 years, as well as from the facilitation of the dissemination of the information which the world-wide web offers nowadays.

\[ \text{4.3 The violation of isospin invariance in the hadronic part of the } \pi N \text{ interaction} \]

The effects, described at the end of Section 3.2.1, at the end of Section 3.2.4, and in Section 3.2.7, may suggest that the isospin invariance in the \( \pi N \) interaction is broken at a level exceeding by far the theoretical expectations. Such effects are, of course, not new: they had been investigated in a series of papers [15,16,20,22], as well as (a few years before Refs. [15,16] appeared) in the pioneering work of Gibbs, Ai, and Kaufmann [46].

This is the last of the possibilities which may be put forward in an attempt to explain the discrepancies and, admittedly, the most interesting one in Physics.
terms; this possibility may account for the results of Table 4. Although the conclusions of Refs. [15,16,46] might not be generally accepted, one is tempted to raise the question: ‘Why should the isospin invariance hold in the first place?’ After all, the hadronic masses of the \(u\) and \(d\) quarks are different; similarly, the masses of the nucleons differ (beyond ‘trivial’ EM effects), and so do those of the \(\Delta(1232)\) isospin states. It appears, therefore, that the right question to ask is not whether isospin invariance in the \(\pi N\) interaction is broken, but at which level it is. The isospin-breaking effects in the \(\pi N\) interaction have been treated within the framework of the heavy-baryon Chiral-Perturbation Theory and found to be small, at the percent level at most [47].

On the other hand, one may compare the level of the isospin breaking in the \(\pi N\) interaction to what has long been known for the \(NN\) system [48]. The hadronic part of the low-energy \(NN\) interaction is characterised by three scattering lengths, corresponding to the three \(^1S_0\) states \(pp\), \(nn\), and \(np\). If charge independence (which is used in the \(NN\) domain as a synonym for isospin invariance) would hold, these three scattering lengths would come out equal. In reality, after the removal of the EM effects, their values are [48]:

\[
a_{pp} = -17.3(4) \text{ fm}, \quad a_{nn} = -18.8(3) \text{ fm}, \quad a_{np} = -23.77(9) \text{ fm}.
\]  

(In Ref. [48], these scattering lengths carry the superscript ‘N’, indicating that they are nuclear ones, i.e., obtained after the EM corrections had been applied.) Obviously, these values violate charge independence and, to a lesser extent, charge symmetry, as

\[
\Delta a_{CD} = (a_{pp} + a_{nn})/2 - a_{np} = 5.7(3)\text{fm}
\]  

and

\[
\Delta a_{CSD} = a_{pp} - a_{nn} = 1.5(5)\text{fm}
\]

are significantly non-zero. These values correspond to the violation of charge independence in the low-energy \(s\)-wave part of the \(NN\) scattering amplitude by about 27% and of charge symmetry by about 8%. The level of the charge-independence breaking in the \(NN\) interaction is to be compared to the 5 to 10% effects which were reported in Refs. [2,15,16,20,22,46] for the low-energy \(\pi N\) system. If the \(\pi N\) interaction is regarded as the basis for the description of the \(NN\) interaction (as the case is in meson-exchange theories of the strong interaction), it is logical to expect that the isospin-breaking effects cascade from the \(NN\) interaction down to the \(\pi N\) interaction. Under the assumption that the \(NN\) force is modelled at low energy via the one-pion exchange mechanism, Babenko and Petrov [49] recently obtained a sizeable splitting in the \(\pi N\) coupling constant, i.e., significantly different values for the couplings of the charged pion and of the neutral pion to the nucleon. One should not forget that, aiming at providing an explanation for the unexpected result of Ref. [46], Piekarewicz had (already in 1995) attributed the isospin-breaking effects to changes in the coupling constant due to the mass difference
between the $u$ and the $d$ quarks [50]. In this context, changes in the fitted value of the $\pi N$ coupling constant when involving the $\pi^- p$ CX data in the optimisation have been reported in this programme, e.g., see Table 3 in this work and Table 4 in Ref. [22].

Two mechanisms had been proposed in the past, to account for the violation of the isospin invariance in the $\pi N$ interaction: the first mechanism affects the ES processes ($\rho^0 - \omega$ mixing [51]), the second the $\pi^- p$ CX reaction ($\pi^0 - \eta$ mixing [52]). As both the $\omega$ and the $\eta$ states are singlets, the coupling of the former to the $\rho^0$ and of the latter to the $\pi^0$ explicitly violate the isospin invariance in the $\pi N$ interaction. Given that only one graph (i.e., the $t$-channel $\rho$-exchange graph) is affected in the case of the ES (see Fig. 15), whereas all graphs are affected in the case of the $\pi^- p$ CX reaction (see Fig. 16), one would naively expect that the isospin-breaking effects are more significant in the latter case.

Table 3 demonstrates that the coupling constant $g_{\pi NN}$ is significantly impacted on when replacing the tDB by the tDB0 in the analysis. Of course, if isospin invariance in the $\pi N$ interaction is broken, there is no such thing as one coupling constant $g_{\pi NN}$; one must distinguish between $g_{\pi^0 pp}$, $g_{\pi^0 nn}$, and $g_{\pi^\pm pn}$. In this case, the fits of the ETH model to the DB+/− essentially determine $g_{\pi^\pm pn}$, whereas those involving the $\pi^- p$ CX reaction also contain contributions from $g_{\pi^0 pp}$ and $g_{\pi^0 nn}$. As a result, the value of the coupling constant $g_{\pi NN}$ extracted from the fits of the ETH model to the tDB+/0 represents a weighted average of these three quantities. The difference in the two $g_{\pi NN}$ entries of Table 3 implies that at least one of the two $g_{\pi^0 NN}$ coupling constants differs from $g_{\pi^\pm pn}$. The isospin-breaking effects on $g_{\pi NN}$ have been studied theoretically, e.g., see Ref. [53], where the authors report that $g_{\pi^\pm pn}$ should be equal to the average of the two $g_{\pi^0 NN}$ values and provide an estimate for the splitting between 1.2 and 3.7%. The difference between the two $g_{\pi NN}$ values of Table 3 is about 3.9%.

### 5 Summary and perspective

This paper serves three main purposes.

- It provides details on the development of the theoretical and analysis frameworks, employed in the study of the pion-nucleon ($\pi N$) interaction at low energy (up to pion laboratory kinetic energy $T$ of 100 MeV) since the early 1990s.
- It presents a new phase-shift analysis (PSA) of the low-energy $\pi N$ data, after the database (DB) was enhanced with measurements which had been omitted in most of the earlier PSAs of this research programme.
- It provides a permanent link to future results obtained within the framework
of this programme. The material under this link will be updated whenever new results become available.

In the current analysis framework, the separate analysis of the DBs for the two elastic-scattering (ES) processes, as well as for the \(\pi^-p\) charge-exchange (CX) reaction, is enabled via suitable low-energy parameterisations of the s- and p-wave \(K\)-matrix elements (representing the ‘phenomenological model’ in this work). The analysis with the phenomenological model is free of theoretical constraints, other than the expected low-energy behaviour of the \(K\)-matrix elements. Deployed at the first stage of each new PSA, the processing of the data with the phenomenological model enables the unbiased identification of the outliers in the DB and the creation of self-consistent input for the analysis conducted at the second stage.

The hadronic model of this programme, the so-called ‘ETH model’, is based on meson-exchange t-channel graphs, as well as on the s- and u-channel \(N\) and \(\Delta(1232)\) contributions (see Fig. 1); the model also contains the contributions from all well-established (four-star) \(s\) and \(p\) higher baryon resonances with masses up to 2 GeV. The analysis of the measurements with the ETH model is performed at the second stage in each PSA; imposed at this stage are the theoretical constraints of crossing symmetry and isospin invariance, which the hadronic part of the \(\pi N\) scattering amplitude of the ETH model obeys.

To ensure that both isospin amplitudes (in each partial wave) are reliably determined from the measurements, joint analyses of the \(\pi^+p\) DB (tDB\(_+\) for short, where the prefix ‘t’ indicates the removal of the outliers from the initial DB) with either of the two \(\pi^-p\) tDBs - i.e., with the DB of the \(\pi^-p\) ES process (tDB\(_-\)) or with the one of the \(\pi^-p\) CX reaction (tDB\(_0\)) - are performed. The optimal values of the model parameters and their Hessian matrix are obtained from these fits and are subsequently used in the generation of Monte-Carlo predictions for the low-energy constants of the \(\pi N\) system, for the phase shifts, and for the standard observables. Representative examples of observables are: the differential cross section (DCS), the analysing power (AP), the partial-total cross section (PTCS), and the total cross section (TCS). Predictions for these observables may be extracted for any of the three low-energy processes, at any value of the relevant kinematical variables, i.e., energy and scattering angle for the DCS and AP, energy and laboratory-angle cut for the PTCS, and energy for the TCS. In regard to the exchanged state in the \(I = J = 0\) t-channel graph of the ETH model, the recent recommendation of the Particle-Data Group (PDG) [27] is to make use of a Breit-Wigner mass between 400 and 550 MeV. As a result, fits of the ETH model to the data are recently performed at seven (equally weighted) \(m_\sigma\) values between 400 and 550 MeV. All uncertainties in this work contain the effects of the \(m_\sigma\) variation.

Given that the electromagnetic (EM) corrections (which are applied to the \(\pi N\) phase shifts and to the partial-wave amplitudes on the way to the evaluation
of the observables) of Refs. [4,5] were obtained by using the physical (instead of the unknown hadronic) masses for the proton, for the neutron, and for the charged and neutral pions, a cautious attitude is assumed, by considering the physical quantities of the analysis (i.e., the fit parameters, the low-energy constants of the $\pi N$ system, the phase shifts, etc.) not purely hadronic, but ‘EM-modified’, i.e., containing residual EM effects. Although a part of these effects might have already been captured by the procedure put forward in the determination of the EM corrections in Refs. [4,5], it remains unknown how important any residual effects might be.

A number of serious discrepancies have been established herein, either during the data analysis or after comparing the predictions, obtained from the results of the fits to the low-energy $\pi N$ data, to the experimental information obtained at the $\pi N$ threshold.

- The results for the parameters of the ETH model, obtained from the joint optimisation of the a) $\pi^+p$ and $\pi^-p$ ES tDBs (tDB$_{+/−}$ for short), and b) $\pi^+p$ and $\pi^-p$ CX tDBs (tDB$_{+0}$ for short) are contained in Table 3. Different results are extracted from these two analyses for the model parameters $g_{\pi NN}$ and $G_\rho$. The model predictions, obtained from the two solutions, significantly differ in the case of the phase shifts $\delta_{0+}^{1/2}$ (S11) and $\delta_{1+}^{3/2}$ (P33) (see Figs. 3 and 4, respectively).
- The predictions for the low-energy constants of the $\pi N$ system, based on the fits of the ETH model to the tDB$_{+/−}$ and on those to the tDB$_{+0}$, are shown in Table 7. The discrepancy between the model-predicted $a_{cc}^{cc}$ (fits to the tDB$_{+/−}$) and the $a_{cc}^{cc}$ results of the measurement of the strong shift of the $1s$ state in pionic hydrogen [35,36] is sizeable (see Sections 3.2.3 and 4.1).
- Table 4 contains the various $\chi^2_{\text{min}}$ values obtained in the analysis of the three low-energy $\pi N$ DBs or the two combinations of DBs of this work, for three $p_{\text{min}}$ values, corresponding to setting the level of statistical significance to the equivalent of 2$\sigma$, 2.5$\sigma$, and 3$\sigma$ effects in the normal distribution. The table demonstrates that the replacement of the tDB$_{−}$ by the tDB$_{0}$ results in a considerable deterioration of the quality of the data description in the fits of the ETH model. The strikingly different behaviour of the results of the fits of the phenomenological model and those of the ETH model manifest a difficulty to account for the DB$_{0}$ when imposing the theoretical constraint of isospin invariance in the analysis.
- The analysis of the scale factors of the fits of the ETH model to the tDB$_{+/−}$ and of those to the tDB$_{+0}$ is performed in Section 3.2.4. The scale factors in the former case (see Figs. 8 and 9) are centred on 1 and do not show a significant energy dependence. Sizeable effects are observed in Figs. 10 and 11, when combining the tDB$_{+}$ with the tDB$_{0}$. A very similar effect was observed in Ref. [2] in the output of the SAID solution WI08 [1]. It appears that, when forcing the tDB$_{0}$ into a joint optimisation scheme, regardless of
whether one uses the ETH model or dispersion relations [1] in the analysis, the overall tendency of the modelling is to generate overestimated fitted DCS values for the \( \pi^+p \) reaction and underestimated ones for the \( \pi^-p \) CX reaction at low energy. Evidently, the optimisation (of the description of the input data) is achieved at the expense of introducing biases in the description of both subsets (reactions) comprising the set of the input measurements.

- The reproduction of the \( \pi^-p \) CX DCSs by the results of the fits of the ETH model to the tDB\(_{+/-}\) has been investigated in Section 3.2.7. The optimal scale factors, shown in Fig. 14, represent the expected level of the absolute normalisation of the experimental data on the basis of predictions obtained from the tDB\(_{+/-}\). Assuming that the three assumptions, listed in the beginning of Section 4, are fulfilled, these quantities should come out centred on 1 and should show no statistically significant energy dependence. On the contrary, significant effects are observed in the low-energy region, increasing with decreasing beam energy.

Assuming the correctness of the absolute normalisation of the bulk of the low-energy \( \pi N \) data and the negligibility of the residual effects in the EM corrections of Refs. [4,5], the aforementioned discrepancies can only be blamed on the violation of isospin invariance in the \( \pi N \) interaction at low energy. The findings of this study agree well with those reported since the mid 1990s, when isospin invariance in the \( \pi N \) interaction was first tested by using the then-available experimental information. This agreement is notable given the changes of the DBs (e.g., the DB\(_0\) has been enlarged by a factor of seven in the meantime), the analysis methods, and the EM corrections applied to the input data. These findings strongly disagree with predictions obtained within the framework of the heavy-baryon Chiral-Perturbation Theory [47], according to which the expected maximal isospin-breaking effects in the low-energy \( \pi N \) interaction should be of order of 1%.

In Section 4.2, we put forward arguments to substantiate our recommendation to re-assess the EM corrections in the \( \pi N \) interaction. New low-energy experiments cannot advance the knowledge of the hadronic part of the \( \pi N \) interaction, while the subject of the EM corrections remains unresolved. We outlined some of the issues which need to be addressed; it appears to us that the most important of these issues relates to the use of the physical (instead of the unknown hadronic) masses in the determination of the EM corrections of Refs. [4,5] and [17,18,19]. Assuming that this subject is addressed and resolved, there are a number of directions in which further progress could be made in this programme.

- Feynman graphs have been proposed, based on the \( \rho^0 - \omega \) (Fig. 15) and \( \pi^0 - \eta \) (Fig. 16) mixing mechanisms, to account for the isospin-breaking effects in \( \pi N \) scattering. The \( \rho^0 - \omega \) mixing mechanism could account for the discrepancy between the \( a^{cc} \) values obtained from the experimental results
of Refs. [35,36] and the prediction obtained from the extrapolation of the model amplitudes from the energies corresponding to the scattering data to threshold. The $\pi^0 - \eta$ mixing mechanism could make significant contributions in relation to the reproduction of the DB$_0$ by the predictions based on the results of the fits of the ETH model to the tDB$_{+/-}$.

- The discrepancies between the DENZ04 data and the bulk of the remaining modern low-energy $\pi N$ data must be understood. The DENZ04 data cover low $T$ values, a ‘corner’ of the phase space in which the $K$-matrix parameterisations, which have been used in this programme for almost two decades, should have worked best. It is perplexing that both methods of this work can successfully analyse all the rest of the $\pi N$ data (taken by different groups, with different detectors, at different meson-factory facilities and times) up to 100 MeV, but are unable to yield any meaningful results from the DENZ04 measurements, which extend only up to 43.30 MeV. Given the characteristics of the DENZ04 data, there is no room for questioning the theoretical background on which this work relies; meaningful results should have been obtained from these data both with the phenomenological model of Section 2.3, as well as with the ETH model.

The problematic nature of the DENZ04 DCSs may also be revealed after examining the reports of the CHAOS Collaboration. For instance, upon inspection of the results of the single-energy phase-shift solution obtained from these data (see Table 6.1 of Denz’s dissertation and Fig. 4 of the main publication of the CHAOS Collaboration [25]), one cannot but feel uneasy about the quoted values. Of course, it is true that the single-energy phase-shift solutions cannot exhibit the smoothness of the results obtained when the energy dependence of the phase shifts is modelled via suitable functions, and measurements taken at a number of energy values are fitted to. That being said, the value of the P31 phase shift (i.e., $+0.65^\circ$, no uncertainty was quoted) in Table 6.1 of the dissertation, at 19.90 MeV, is wrong by about $0.9^\circ$; the largest of the $p$-wave phase shifts (P33) is itself about $1^\circ$ at that energy! At 20 MeV, the WI08 result [1] for P31 ($-0.22^\circ$) agrees with the typical values obtained in this programme. This discrepancy alone provides good reason for the re-examination of the results of Ref. [25]. As the beam energy in the CHAOS data sets was sufficiently low, an investigation of the description of their DCSs within the framework of the Chiral-Perturbation Theory (e.g., with the method of Ref. [54]) should be possible. This would be an interesting subject to pursue.

- The extrapolation of the model amplitudes into the unphysical region, towards the Cheng-Dashen point, taking proper account of the analyticity constraint, would lead to the extraction of the $\pi N \Sigma$ term beyond the tree-level approximation. Although a comparison with the literature (see end of Section 3.2.3, as well as Ref. [26] for details) has demonstrated that the corrections to the value obtained with the tree-level approximation must be small, the proper extraction of the $\Sigma$ term in the context of the ETH model is worth pursuing.
Our predictions for the standard observables (i.e., DCS, AP, PTCS, and TCS) for the three low-energy $\pi N$ reactions, obtained from the two solutions of this work, are simple to obtain, free of charge, and available within a few days of the request.

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Table 8

The data sets comprising the tDB+. The columns correspond to details on each data set as follows: the identifier of the data set, the pion laboratory kinetic energy $T_j$ of the data set (in MeV), the NDF after the removal of any outliers from the original data, the scale factor $z_j$ of Eq. (5), the normalisation uncertainty $\delta z_j$ (reported or assigned), the value of $(\chi^2_j)^{\text{min}}$ of Eq. (6), and the p-value of the fit. In the case of free floating, $z_j$ is obtained from Eq. (5) in the limit $\delta z_j \to \infty$, equivalent to setting $(\delta z_j)^{-2}$ to 0 in Eq. (5). The entries of this table are taken from the final fit of the phenomenological model of Section 2.3.1 for the $p_{\text{min}}$ value which is associated with 2.5$\sigma$ effects in the normal distribution.

| Identifier     | $T_j$ | $N_j$ | $z_j$     | $\delta z_j$ | $(\chi^2_j)^{\text{min}}$ | p-value | Comments                     |
|----------------|-------|-------|-----------|--------------|--------------------------|---------|------------------------------|
| DCSs           |       |       |           |              |                          |         |                              |
| BERTIN76       | 20.80 | 10    | 1.3891    | 0.1220       | 17.5787                  | $6.25 \cdot 10^{-2}$ |                   |
| BERTIN76       | 30.50 | 10    | 1.2382    | 0.1176       | 8.8172                   | $5.50 \cdot 10^{-1}$ |                   |
| BERTIN76       | 39.50 | 8     | 1.1712    | 0.1135       | 16.3735                  | $3.73 \cdot 10^{-2}$ |                   |
| BERTIN76       | 51.50 | 10    | 1.1227    | 0.1080       | 4.3009                   | $9.33 \cdot 10^{-1}$ |                   |
| BERTIN76       | 81.70 | 10    | 1.1045    | 0.0942       | 13.7206                  | $1.86 \cdot 10^{-1}$ |                   |
| BERTIN76       | 95.90 | 9     | 1.0217    | 0.0878       | 17.2011                  | $4.57 \cdot 10^{-2}$ |                   |
| AULD79         | 47.90 | 11    | 1.0038    | 0.1097       | 14.9554                  | $1.85 \cdot 10^{-1}$ |                   |
| RITCHIE83      | 65.00 | 8     | 1.0457    | 0.0240       | 17.9256                  | $2.18 \cdot 10^{-2}$ |                   |
| RITCHIE83      | 72.50 | 10    | 1.0086    | 0.0200       | 4.8830                   | $8.99 \cdot 10^{-1}$ |                   |
| RITCHIE83      | 80.00 | 10    | 1.0329    | 0.0140       | 20.9360                  | $2.15 \cdot 10^{-2}$ |                   |
| RITCHIE83      | 95.00 | 10    | 1.0339    | 0.0150       | 14.0011                  | $1.73 \cdot 10^{-1}$ |                   |
| FRANK83        | 29.40 | 28    | 0.9712    | 0.0370       | 19.3485                  | $8.87 \cdot 10^{-1}$ |                   |
| FRANK83        | 49.50 | 28    | 1.0369    | 0.2030       | 33.2171                  | $2.28 \cdot 10^{-1}$ |                   |
| FRANK83        | 69.60 | 27    | 0.9294    | 0.0950       | 22.7717                  | $6.97 \cdot 10^{-1}$ |                   |
| FRANK83        | 89.60 | 27    | 0.8599    | 0.0470       | 31.8922                  | $2.36 \cdot 10^{-1}$ |                   |
| BRACK86        | 66.80 | 4     | 0.8940    | 0.0120       | 2.3847                   | $6.65 \cdot 10^{-1}$ | freely floated   |
| BRACK86        | 86.80 | 8     | 0.9410    | 0.0140       | 12.1838                  | $1.43 \cdot 10^{-1}$ | freely floated   |
| BRACK86        | 91.70 | 5     | 0.9741    | 0.0120       | 10.5015                  | $6.22 \cdot 10^{-2}$ |                   |
| BRACK86        | 97.90 | 5     | 0.9710    | 0.0150       | 8.5597                   | $1.28 \cdot 10^{-1}$ |                   |
| BRACK88        | 66.80 | 6     | 0.9486    | 0.0210       | 10.7499                  | $9.64 \cdot 10^{-2}$ |                   |
| BRACK88        | 66.80 | 6     | 0.9573    | 0.0210       | 9.4298                   | $1.51 \cdot 10^{-1}$ |                   |
| WIEDNER89      | 54.30 | 19    | 0.9901    | 0.0304       | 15.2229                  | $7.08 \cdot 10^{-1}$ |                   |
Table 8 continued

| Identifier  | $T_j$ | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{\text{min}}$ | p-value | Comments                         |
|-------------|------|------|------|--------------|-----------------------------|---------|----------------------------------|
| BRACK90     | 30.00| 6    | 1.0748| 0.0360       | 14.6308                     | 2.33 · 10^{-2} |
| BRACK90     | 45.00| 8    | 0.9996| 0.0220       | 8.1624                      | 4.18 · 10^{-1} |
| BRACK95     | 87.10| 8    | 0.9651| 0.0220       | 13.6820                     | 9.04 · 10^{-2} |
| BRACK95     | 98.10| 8    | 0.9749| 0.0200       | 16.6371                     | 3.41 · 10^{-2} |
| JORAM95     | 45.10| 8    | 0.9592| 0.0330       | 15.5740                     | 4.89 · 10^{-2} | 124.42°, 131.69° removed |
| JORAM95     | 68.60| 9    | 1.0451| 0.0440       | 11.4362                     | 2.47 · 10^{-1} |
| JORAM95     | 32.20| 19   | 0.9950| 0.0340       | 26.9604                     | 1.06 · 10^{-1} | 37.40° removed |
| JORAM95     | 44.60| 18   | 0.9494| 0.0340       | 31.2853                     | 2.67 · 10^{-2} | 30.74°, 35.40° removed |

| APs         |      |      |      |              |                             |         |                                  |
| SEVIOR89    | 98.00| 6    | 1.0302| 0.0740       | 5.1000                      | 5.31 · 10^{-1} |
| WIESER96    | 68.34| 3    | 0.9194| 0.0500       | 3.6914                      | 2.97 · 10^{-1} |
| WIESER96    | 68.34| 4    | 0.9426| 0.0500       | 4.1110                      | 3.91 · 10^{-1} |
| MEIER04     | 57.20-87.20| 12 | 0.9887| 0.0350       | 14.0208                     | 2.99 · 10^{-1} |
| MEIER04     | 45.20, 51.20| 6  | 0.9675| 0.0350       | 8.4382                      | 2.08 · 10^{-1} |
| MEIER04     | 57.30-87.20| 7  | 1.0126| 0.0350       | 11.7316                     | 1.10 · 10^{-1} |

| PTCSs       |      |      |      |              |                             |         |                                  |
| KRISS97     | 39.80| 1    | 1.0109| 0.0300       | 1.3641                      | 2.43 · 10^{-1} |
| KRISS97     | 40.50| 1    | 1.0014| 0.0300       | 0.0944                      | 7.59 · 10^{-1} |
| KRISS97     | 44.70| 1    | 1.0011| 0.0300       | 0.0123                      | 9.12 · 10^{-1} |
| KRISS97     | 45.30| 1    | 1.0014| 0.0300       | 0.0162                      | 8.99 · 10^{-1} |
| KRISS97     | 51.10| 1    | 1.0232| 0.0300       | 3.1035                      | 7.81 · 10^{-2} |
| KRISS97     | 51.70| 1    | 1.0018| 0.0300       | 0.0225                      | 8.81 · 10^{-1} |
| KRISS97     | 54.80| 1    | 1.0061| 0.0300       | 0.1079                      | 7.42 · 10^{-1} |
| KRISS97     | 59.30| 1    | 1.0248| 0.0300       | 1.2163                      | 2.70 · 10^{-1} |
| KRISS97     | 66.30| 2    | 1.0507| 0.0300       | 4.1799                      | 1.24 · 10^{-1} |
| KRISS97     | 66.80| 2    | 1.0075| 0.0300       | 0.6015                      | 7.40 · 10^{-1} |
| KRISS97     | 80.00| 1    | 1.0148| 0.0300       | 0.4035                      | 5.25 · 10^{-1} |
| KRISS97     | 89.30| 1    | 1.0079| 0.0300       | 0.2825                      | 5.95 · 10^{-1} |
| KRISS97     | 99.20| 1    | 1.0542| 0.0300       | 3.9853                      | 4.59 · 10^{-2} |
Table 8 continued

| Identifier   | $T_j$ | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{min}$ | p-value    | Comments       |
|--------------|-------|-------|-------|--------------|---------------------|------------|----------------|
| FRIEDMAN99   | 45.00 | 1     | 1.0405| 0.0600       | 1.9500              | 1.63 · 10^{-1} |               |
| FRIEDMAN99   | 52.10 | 1     | 1.0161| 0.0600       | 0.2137              | 6.44 · 10^{-1} |               |
| FRIEDMAN99   | 63.10 | 1     | 1.0366| 0.0600       | 0.4990              | 4.80 · 10^{-1} |               |
| FRIEDMAN99   | 67.45 | 2     | 1.0531| 0.0600       | 1.2804              | 5.27 · 10^{-1} |               |
| FRIEDMAN99   | 71.50 | 2     | 1.0510| 0.0600       | 0.8803              | 6.44 · 10^{-1} |               |
| FRIEDMAN99   | 92.50 | 2     | 1.0424| 0.0600       | 0.5743              | 7.50 · 10^{-1} |               |

| Identifier   | $T_j$ | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{min}$ | p-value    | Comments       |
|--------------|-------|-------|-------|--------------|---------------------|------------|----------------|
| CARTER71     | 71.60 | 1     | 1.0933| 0.0600       | 2.7454              | 9.75 · 10^{-2} |               |
| CARTER71     | 97.40 | 1     | 1.0492| 0.0600       | 0.6780              | 4.10 · 10^{-1} |               |
| PEDRONI78    | 72.50 | 1     | 1.0125| 0.0600       | 0.1421              | 7.06 · 10^{-1} |               |
| PEDRONI78    | 84.80 | 1     | 1.0318| 0.0600       | 0.3409              | 5.59 · 10^{-1} |               |
| PEDRONI78    | 95.10 | 1     | 1.0228| 0.0600       | 0.1983              | 6.56 · 10^{-1} |               |
| PEDRONI78    | 96.90 | 1     | 1.0164| 0.0600       | 0.1279              | 7.21 · 10^{-1} |               |
Table 9
The equivalent of Table 8 for the tDB. The entries of this table are taken from the final fit of the phenomenological model of Section 2.3.2.

| Identifier     | $T_j$ | $N_j$ | $z_j$  | $\delta z_j$ | $(\chi^2_j)_{\text{min}}$ | p-value | Comments                      |
|----------------|-------|-------|--------|---------------|----------------------------|---------|-------------------------------|
| **DCSs**       |       |       |        |               |                            |         |                               |
| FRANK83        | 29.40 | 28    | 0.9828 | 0.0350        | 30.9473                    | 3.19 $\cdot$ 10^{-1} |                  |
| FRANK83        | 49.50 | 28    | 1.0959 | 0.0780        | 28.9886                    | 4.13 $\cdot$ 10^{-1} |                  |
| FRANK83        | 69.60 | 27    | 1.0845 | 0.2530        | 23.9222                    | 6.35 $\cdot$ 10^{-1} |                  |
| FRANK83        | 89.60 | 27    | 0.9428 | 0.1390        | 24.5018                    | 6.02 $\cdot$ 10^{-1} |                  |
| BRACK86        | 66.80 | 5     | 0.9972 | 0.0130        | 14.1709                    | 1.46 $\cdot$ 10^{-2} |                  |
| BRACK86        | 86.80 | 5     | 1.0030 | 0.0120        | 1.2841                     | 9.37 $\cdot$ 10^{-1} |                  |
| BRACK86        | 91.70 | 5     | 0.9962 | 0.0120        | 3.0671                     | 6.90 $\cdot$ 10^{-1} |                  |
| BRACK86        | 97.90 | 5     | 0.9996 | 0.0120        | 5.9755                     | 3.09 $\cdot$ 10^{-1} |                  |
| WIEDNER89      | 54.30 | 18    | 1.1545 | 0.0304        | 23.4289                    | 1.75 $\cdot$ 10^{-1} | 15.55° removed, freely floated |
| BRACK90        | 30.00 | 5     | 1.0200 | 0.0200        | 4.9541                     | 4.22 $\cdot$ 10^{-1} |                  |
| BRACK90        | 45.00 | 9     | 1.0505 | 0.0220        | 12.2067                    | 2.02 $\cdot$ 10^{-1} |                  |
| BRACK95        | 87.50 | 6     | 0.9781 | 0.0220        | 11.2079                    | 8.22 $\cdot$ 10^{-2} |                  |
| BRACK95        | 98.10 | 7     | 1.0065 | 0.0210        | 8.6249                     | 2.81 $\cdot$ 10^{-1} | 36.70° removed    |
| JORAM95        | 32.70 | 4     | 0.9937 | 0.0330        | 3.7658                     | 4.39 $\cdot$ 10^{-1} |                  |
| JORAM95        | 32.70 | 2     | 0.9538 | 0.0330        | 5.5263                     | 6.31 $\cdot$ 10^{-2} |                  |
| JORAM95        | 45.10 | 4     | 0.9531 | 0.0330        | 12.3720                    | 1.48 $\cdot$ 10^{-2} |                  |
| JORAM95        | 45.10 | 3     | 0.9453 | 0.0330        | 9.4129                     | 2.43 $\cdot$ 10^{-2} |                  |
| JORAM95        | 68.60 | 7     | 1.0757 | 0.0440        | 13.8222                    | 5.44 $\cdot$ 10^{-2} |                  |
| JORAM95        | 68.60 | 3     | 1.0306 | 0.0440        | 2.0983                     | 5.52 $\cdot$ 10^{-1} |                  |
| JORAM95        | 32.20 | 20    | 1.0607 | 0.0340        | 21.2248                    | 3.84 $\cdot$ 10^{-1} |                  |
| JORAM95        | 44.60 | 20    | 0.9422 | 0.0340        | 30.5867                    | 6.09 $\cdot$ 10^{-2} |                  |
| JANOUSCH97     | 43.60 | 1     | 1.0458 | 0.1500        | 0.2017                     | 6.53 $\cdot$ 10^{-1} |                  |
| JANOUSCH97     | 50.30 | 1     | 1.0370 | 0.1500        | 0.1613                     | 6.88 $\cdot$ 10^{-1} |                  |
| JANOUSCH97     | 57.30 | 1     | 1.0829 | 0.1500        | 4.7490                     | 2.93 $\cdot$ 10^{-2} |                  |
| JANOUSCH97     | 64.50 | 1     | 1.0327 | 0.1500        | 0.0719                     | 7.89 $\cdot$ 10^{-1} |                  |
| JANOUSCH97     | 72.00 | 1     | 1.3138 | 0.1500        | 5.1473                     | 2.33 $\cdot$ 10^{-2} |                  |
Table 9 continued

| Identifier  | $T_j$ | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{min}$ | p-value          | Comments       |
|-------------|------|------|------|-------------|----------------|-----------------|---------------|
| ALDER83     | 98.00| 6    | 1.0107 | 0.0400     | 5.4390          | $4.89 \cdot 10^{-1}$ |               |
| SEVIOR89    | 98.00| 5    | 0.9905 | 0.0740     | 1.5211          | $9.11 \cdot 10^{-1}$ |               |
| HOFMAN98    | 86.80| 11   | 1.0007 | 0.0300     | 6.3554          | $8.49 \cdot 10^{-1}$ |               |
| PATTERSON02 | 57.20| 10   | 0.9463 | 0.0370     | 10.6309         | $3.87 \cdot 10^{-1}$ |               |
| PATTERSON02 | 66.90| 9    | 0.9993 | 0.0370     | 5.3473          | $8.03 \cdot 10^{-1}$ |               |
| PATTERSON02 | 66.90| 10   | 0.9595 | 0.0370     | 14.0581         | $1.70 \cdot 10^{-1}$ |               |
| PATTERSON02 | 87.20| 11   | 0.9816 | 0.0370     | 8.4860          | $6.69 \cdot 10^{-1}$ |               |
| PATTERSON02 | 87.20| 11   | 0.9915 | 0.0370     | 5.0987          | $9.26 \cdot 10^{-1}$ |               |
| PATTERSON02 | 98.00| 12   | 1.0043 | 0.0370     | 6.7490          | $8.74 \cdot 10^{-1}$ |               |
| MEIER04     | 67.30, 87.20 | 3   | 0.9936 | 0.0350     | 2.9874          | $3.94 \cdot 10^{-1}$ |               |
Table 10

The equivalent of Table 8 for DBq. The entries of this table are taken from the final fit of the phenomenological model of Section 2.3.2.

| Identifier      | $T_j$  | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{\mathrm{min}}$ | p-value          | Comments   |
|-----------------|--------|-------|-------|--------------|-----------------------------|------------------|------------|
| DUCLOS73        | 22.60  | 1     | 0.9411| 0.0800       | 1.2569                      | $2.62 \cdot 10^{-1}$ |
| DUCLOS73        | 32.90  | 1     | 0.9716| 0.0800       | 0.2717                      | $6.02 \cdot 10^{-1}$ |
| DUCLOS73        | 42.60  | 1     | 0.9114| 0.0800       | 2.3067                      | $1.29 \cdot 10^{-1}$ |
| FITZGERALD86    | 32.48  | 2     | 1.5065| 0.0780       | 2.5236                      | $2.83 \cdot 10^{-1}$  | freely floated |
| FITZGERALD86    | 36.11  | 2     | 1.7285| 0.0780       | 1.4510                      | $4.84 \cdot 10^{-1}$  | freely floated |
| FITZGERALD86    | 40.26  | 2     | 1.8344| 0.0780       | 7.5596                      | $2.28 \cdot 10^{-2}$  |
| FITZGERALD86    | 47.93  | 3     | 1.1595| 0.0780       | 10.5775                     | $1.42 \cdot 10^{-2}$  |
| FITZGERALD86    | 51.78  | 3     | 1.0828| 0.0780       | 5.0129                      | $1.71 \cdot 10^{-1}$  |
| FITZGERALD86    | 55.58  | 3     | 1.0582| 0.0780       | 1.2627                      | $7.38 \cdot 10^{-1}$  |
| FITZGERALD86    | 63.21  | 3     | 1.0303| 0.0780       | 0.8705                      | $8.33 \cdot 10^{-1}$  |
| FRLEŽ98         | 27.50  | 6     | 1.0851| 0.0870       | 10.6500                     | $9.98 \cdot 10^{-2}$  |
| ISENHOWER99     | 10.60  | 4     | 1.0155| 0.0600       | 2.0567                      | $7.25 \cdot 10^{-1}$  |
| ISENHOWER99     | 10.60  | 5     | 1.0020| 0.0400       | 1.4165                      | $9.23 \cdot 10^{-1}$  |
| ISENHOWER99     | 10.60  | 6     | 1.0129| 0.0400       | 7.9153                      | $2.44 \cdot 10^{-1}$  |
| ISENHOWER99     | 20.60  | 5     | 0.9758| 0.0400       | 1.7915                      | $8.77 \cdot 10^{-1}$  |
| ISENHOWER99     | 20.60  | 6     | 1.0089| 0.0400       | 8.1554                      | $2.27 \cdot 10^{-1}$  |
| ISENHOWER99     | 39.40  | 4     | 1.0699| 0.0600       | 7.5624                      | $1.09 \cdot 10^{-1}$  |
| ISENHOWER99     | 39.40  | 5     | 1.0575| 0.0400       | 8.5836                      | $1.27 \cdot 10^{-1}$  |
| ISENHOWER99     | 39.40  | 5     | 0.9536| 0.0400       | 5.0752                      | $4.07 \cdot 10^{-1}$  |
| SADLER04        | 63.86  | 20    | 0.9574| 0.0650       | 15.5490                     | $7.44 \cdot 10^{-1}$  |
| SADLER04        | 83.49  | 20    | 0.9882| 0.0520       | 11.7418                     | $9.25 \cdot 10^{-1}$  |
| SADLER04        | 94.57  | 20    | 1.0278| 0.0450       | 6.5310                      | $9.98 \cdot 10^{-1}$  |
| JIA08           | 34.37  | 4     | 0.8424| 0.1000       | 4.8421                      | $3.04 \cdot 10^{-1}$  |
| JIA08           | 39.95  | 4     | 0.8629| 0.1000       | 3.3753                      | $4.97 \cdot 10^{-1}$  |
| JIA08           | 43.39  | 4     | 0.8688| 0.1000       | 2.8491                      | $5.83 \cdot 10^{-1}$  |
| JIA08           | 46.99  | 4     | 0.9665| 0.1000       | 4.6574                      | $3.24 \cdot 10^{-1}$  |
Table 10 continued

| Identifier   | $T_j$ | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{\text{min}}$ | p-value | Comments                        |
|--------------|-------|-------|-------|--------------|--------------------------|---------|--------------------------------|
|              |       |       |       |              |                          |         |                                |
| **DCSs**     |       |       |       |              |                          |         |                                |
| JIA08        | 54.19 | 4     | 0.8807| 0.1000      | 2.7134                   | 6.07 · 10$^{-1}$ |
| JIA08        | 59.68 | 4     | 0.9088| 0.1000      | 3.7250                   | 4.44 · 10$^{-1}$ |
| MEKTEROVIĆ09 | 33.89 | 20    | 1.0220| 0.0340      | 16.7913                  | 6.66 · 10$^{-1}$ |
| MEKTEROVIĆ09 | 39.38 | 20    | 1.0137| 0.0260      | 14.8620                  | 7.84 · 10$^{-1}$ |
| MEKTEROVIĆ09 | 44.49 | 20    | 1.0102| 0.0270      | 33.0364                  | 3.34 · 10$^{-2}$ |
| MEKTEROVIĆ09 | 51.16 | 20    | 1.0369| 0.0290      | 14.4697                  | 8.06 · 10$^{-1}$ |
| MEKTEROVIĆ09 | 57.41 | 20    | 1.0412| 0.0290      | 20.4457                  | 4.30 · 10$^{-1}$ |
| MEKTEROVIĆ09 | 66.79 | 20    | 1.0252| 0.0300      | 20.5292                  | 4.25 · 10$^{-1}$ |
| MEKTEROVIĆ09 | 86.62 | 20    | 1.0016| 0.0290      | 30.5906                  | 6.08 · 10$^{-2}$ |
|              |       |       |       |              |                          |         |                                |
| **LECs**     |       |       |       |              |                          |         |                                |
| SALOMON84    | 27.40 | 3     | 0.9714| 0.0310      | 2.7999                   | 4.24 · 10$^{-1}$ |
| SALOMON84    | 39.30 | 3     | 0.9942| 0.0310      | 1.1563                   | 7.64 · 10$^{-1}$ |
| BAGHERI88    | 45.60 | 3     | 1.0068| 0.0310      | 0.1090                   | 9.91 · 10$^{-1}$ |
| BAGHERI88    | 62.20 | 3     | 0.9616| 0.0310      | 3.0615                   | 3.82 · 10$^{-1}$ |
| BAGHERI88    | 76.40 | 3     | 0.9749| 0.0310      | 3.0227                   | 3.88 · 10$^{-1}$ |
| BAGHERI88    | 91.70 | 3     | 1.0137| 0.0310      | 2.9532                   | 3.99 · 10$^{-1}$ |
|              |       |       |       |              |                          |         |                                |
| **Width of the 1s state in pionic hydrogen** | | | | | | | |
| SCHROEDER01  | 0.00  | 1     | 0.9834| 0.0219      | 1.1696                   | 2.79 · 10$^{-1}$ |
|              |       |       |       |              |                          |         |                                |
| **APs**      |       |       |       |              |                          |         |                                |
| STAŠKO93     | 100.00| 4     | 0.9936| 0.0440      | 1.4684                   | 8.32 · 10$^{-1}$ |
| GAULARD99    | 98.10 | 6     | 1.0191| 0.0450      | 0.8495                   | 9.91 · 10$^{-1}$ |
## Table 10 continued

| Identifier        | $T_j$ | $N_j$ | $z_j$ | $\delta z_j$ | $(\chi^2_j)_{\text{min}}$ | p-value   | Comments          |
|-------------------|-------|-------|-------|--------------|-----------------------------|-----------|-------------------|
| BUGG71            | 90.90 | 1     | 1.0213| 0.0600       | 0.1322                      | $7.16 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 38.90 | 1     | 0.9959| 0.0300       | 0.1721                      | $6.78 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 43.00 | 1     | 1.0011| 0.0300       | 0.0261                      | $8.72 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 47.10 | 1     | 0.9981| 0.0300       | 0.0535                      | $8.17 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 55.60 | 1     | 0.9954| 0.0300       | 0.1870                      | $6.65 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 64.30 | 1     | 0.9731| 0.0300       | 3.6387                      | $5.65 \cdot 10^{-2}$ |                   |
| BREITSCHOPF06     | 65.90 | 1     | 0.9784| 0.0300       | 2.2366                      | $1.35 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 76.10 | 1     | 0.9818| 0.0300       | 1.5582                      | $2.12 \cdot 10^{-1}$ |                   |
| BREITSCHOPF06     | 96.50 | 1     | 0.9803| 0.0300       | 0.6997                      | $4.03 \cdot 10^{-1}$ |                   |
Table 11

The details of the reproduction of the DENZ04 DCSs [25] by the BLS obtained from the results of the fits of the ETH model to the tDB_{+/−}. The uncertainties $\delta y_{ij}$ of the BLS have been taken into account, see Eq. (9). The columns correspond to details on each data set as follows: the pion laboratory kinetic energy $T_j$ of the data set (in MeV), the number of the data points $N_j$ of the $j$-th data set, and the three p-values associated with a) the overall reproduction of the data set, b) the reproduction of its shape, and c) the reproduction of its absolute normalisation, as explained in Section 2.5. The table corresponds to the unsplit (original) data sets of the CHAOS Collaboration.

| $T_j$   | $N_j$ | Overall       | Shape       | Abs. norm.   |
|---------|-------|---------------|-------------|--------------|
| $\pi^+ p$ scattering |       |               |             |              |
| 19.90   | 33    | 2.29 $\cdot$ 10$^{-8}$ | 1.33 $\cdot$ 10$^{-8}$ | 7.49 $\cdot$ 10$^{-1}$ |
| 25.80   | 43    | 9.12 $\cdot$ 10$^{-74}$ | 2.91 $\cdot$ 10$^{-74}$ | 6.81 $\cdot$ 10$^{-1}$ |
| 32.00   | 46    | 2.56 $\cdot$ 10$^{-4}$  | 1.82 $\cdot$ 10$^{-4}$  | 8.32 $\cdot$ 10$^{-1}$ |
| 37.10   | 49    | 4.04 $\cdot$ 10$^{-3}$  | 5.14 $\cdot$ 10$^{-3}$  | 1.23 $\cdot$ 10$^{-1}$ |
| 43.30   | 53    | 9.66 $\cdot$ 10$^{-10}$ | 5.81 $\cdot$ 10$^{-10}$ | 9.51 $\cdot$ 10$^{-1}$ |
| 43.30(rot.) | 51  | 9.93 $\cdot$ 10$^{-12}$ | 5.99 $\cdot$ 10$^{-12}$ | 6.78 $\cdot$ 10$^{-1}$ |
| $\pi^- p$ ES |     |               |             |              |
| 19.90   | 31    | 1.00 $\cdot$ 10$^0$   | 1.00 $\cdot$ 10$^0$   | 6.45 $\cdot$ 10$^{-1}$ |
| 25.80   | 45    | 2.33 $\cdot$ 10$^{-5}$ | 1.84 $\cdot$ 10$^{-5}$ | 4.43 $\cdot$ 10$^{-1}$ |
| 32.00   | 45    | 6.11 $\cdot$ 10$^{-1}$ | 7.66 $\cdot$ 10$^{-1}$ | 2.85 $\cdot$ 10$^{-2}$ |
| 37.10   | 50    | 9.88 $\cdot$ 10$^{-1}$ | 9.90 $\cdot$ 10$^{-1}$ | 2.77 $\cdot$ 10$^{-1}$ |
| 43.30   | 51    | 7.59 $\cdot$ 10$^{-1}$ | 8.35 $\cdot$ 10$^{-1}$ | 6.84 $\cdot$ 10$^{-2}$ |
| 43.30(rot.) | 49 | 9.14 $\cdot$ 10$^{-1}$ | 9.75 $\cdot$ 10$^{-1}$ | 1.99 $\cdot$ 10$^{-2}$ |
Table 12

Reproduction of the $\pi^- p$ PTCSs which have not been included in the DB$_-$. The first four columns correspond to the details of each data point as follows: the identifier of the data point, the pion laboratory kinetic energy $T_j$ of the data point (in MeV), the laboratory-angle cut $\theta_L$ (in degrees), and the experimental result. The next three columns contain predictions obtained from the two PSAs of this work. The quantity $\sigma_{ES}$ is the $\pi^- p$ ES PTCS corresponding to the quoted $\theta_L$ value. The quantity $\sigma_{+/-}$ represents the sum of $\sigma_{ES}$ (corresponding entry in the fifth column) and of the $\pi^- p$ CX TCS (i.e., the DCS integrated over the entire solid angle) predicted from the fits of the ETH model to the DB$_+/\_$. Similarly, the quantity $\sigma_{+0}$ represents the sum of $\sigma_{ES}$ (corresponding entry in the fifth column) and of the $\pi^- p$ CX TCS predicted from the fits of the ETH model to the DB$_+\_$. All cross sections are expressed in mb. The normalisation uncertainties of the measurements are not contained in the table: a normalisation uncertainty of 6% had been assigned to the corresponding FRIEDMAN99 $\pi^+ p$ PTCSs (see Table 8). (In Ref. [56], Friedman et al. reported $\pi^+ p$ and $\pi^- p$ ES PTCSs, originally identified as FRIEDMAN90. Corrections to the published values appeared a few years later for the low-energy $\pi^+ p$ PTCSs [57], taking into account the revision in the energy calibration of the M11 pion channel at TRIUMF (which occurred in the early 1990s); in Ref. [57], there is no mention of corrections for the low-energy $\pi^- p$ PTCS of Ref. [56]. As a result, despite the fact that both the $\pi^+ p$ and the $\pi^- p$ ES PTCSs originally appeared in the same paper, they are assigned different identifiers in the DB: the corrected low-energy $\pi^+ p$ PTCSs are identified as FRIEDMAN99, whereas the original low-energy $\pi^- p$ ES PTCS as FRIEDMAN90.) A normalisation uncertainty of 3% had been reported in the KRISS97 $\pi^+ p$ PTCSs.

| Identifier       | $T_j$ | $\theta_L$ | $\sigma$   | $\sigma_{ES}$ | $\sigma_{+/-}$ | $\sigma_{+0}$ |
|------------------|------|------------|------------|---------------|---------------|---------------|
| FRIEDMAN90       | 50.00| 30         | 8.50 ± 0.60| 1.85 ± 0.16   | 8.00 ± 0.18   | 8.52 ± 0.16   |
| KRISS97          | 80.00| 30         | 14.60 ± 0.60| 2.782 ± 0.081| 14.107 ± 0.095| 14.67 ± 0.12  |
| KRISS97          | 99.20| 30         | 23.4 ± 1.1 | 4.495 ± 0.041| 21.81 ± 0.15  | 22.31 ± 0.15  |
Table 13

Reproduction of the so-called $\pi^-p$ ES total-nuclear cross sections which have not been included in the DB. The first three columns correspond to the details of each data point as follows: the identifier of the data point, the pion laboratory kinetic energy $T_j$ of the data point (in MeV), and to the measurement itself. The next two columns contain the predictions obtained from the two PSAs of this work: the quantity $\sigma_{+/-}$ represents the prediction based on the fits of the ETH model to the DB$_{+/-}$, whereas the quantity $\sigma_{+/0}$ the one based on the fits of the ETH model to the DB$_{+/0}$. All cross sections are expressed in mb. The normalisation uncertainties of the measurements are not contained in this table: a normalisation uncertainty of 6% had been assigned to both experiments in the corresponding $\pi^+p$ PTCSs (see Table 8).

| Identifier  | $T_j$  | $\sigma$      | $\sigma_{+/-}$ | $\sigma_{+/0}$ |
|------------|--------|---------------|----------------|---------------|
| CARTER71   | 76.70  | $15.80 \pm 0.20$ | $13.99 \pm 0.11$ | $15.11 \pm 0.25$ |
| CARTER71   | 96.00  | $23.12 \pm 0.15$ | $21.72 \pm 0.13$ | $22.63 \pm 0.17$ |
| PEDRONI78  | 72.50  | $13.6 \pm 1.9$   | $12.79 \pm 0.12$ | $13.93 \pm 0.27$ |
| PEDRONI78  | 84.80  | $17.70 \pm 0.90$ | $16.770 \pm 0.095$ | $17.82 \pm 0.21$ |
| PEDRONI78  | 95.10  | $21.2 \pm 1.0$   | $21.27 \pm 0.12$ | $22.20 \pm 0.17$ |
| PEDRONI78  | 96.90  | $22.2 \pm 2.0$   | $22.18 \pm 0.13$ | $23.08 \pm 0.17$ |
Table 14

The scale factors $\hat{z}_j$ of the data sets in the DB$_0$ using as BLS the predictions based on the fits of the ETH model to the tDB$_{\pm/\pm}$. The first three columns correspond to the details of each data set as follows: the identifier of the data set in the DB$_0$, the pion laboratory kinetic energy $T_j$ of the data set (in MeV), and the number of data points $N_j$ of the $j$-th data set. The columns ‘Overall’, ‘Shape’, and ‘Abs. norm.’ contain the p-values corresponding a) to the overall reproduction of the data set, b) to the reproduction of its shape, and c) to the reproduction of its absolute normalisation, as explained in Section 2.5. The quantity $\delta\hat{z}_j$ is the total uncertainty of the optimal scale factor $\hat{z}_j$ (see beginning of Section 3.2.7).

| Identifier       | $T_j$ | $N_j$ | Overall   | Shape    | Abs. norm. | $\hat{z}_j$ | $\delta\hat{z}_j$ |
|------------------|------|------|----------|----------|------------|-------------|-----------------|
| DUCLOS73         | 22.60| 1    | 8.78 · 10$^{-1}$ | −        | 8.78 · 10$^{-1}$ | 1.0208      | 0.1353          |
| DUCLOS73         | 32.90| 1    | 5.45 · 10$^{-1}$ | −        | 5.45 · 10$^{-1}$ | 1.0772      | 0.1276          |
| DUCLOS73         | 42.60| 1    | 5.99 · 10$^{-1}$ | −        | 5.99 · 10$^{-1}$ | 0.9386      | 0.1167          |
| FITZGERALD86     | 32.48| 3    | 3.18 · 10$^{-11}$| 1.88 · 10$^{-1}$ | 3.25 · 10$^{-12}$ | 2.0246      | 0.1471          |
| FITZGERALD86     | 36.11| 3    | 7.01 · 10$^{-14}$| 3.15 · 10$^{-1}$ | 3.42 · 10$^{-15}$ | 2.2815      | 0.1627          |
| FITZGERALD86     | 40.26| 3    | 1.97 · 10$^{-16}$| 2.05 · 10$^{-2}$ | 1.30 · 10$^{-16}$ | 2.2737      | 0.1539          |
| FITZGERALD86     | 47.93| 3    | 2.77 · 10$^{-3}$ | 8.93 · 10$^{-1}$ | 1.95 · 10$^{-4}$  | 1.4676      | 0.1255          |
| FITZGERALD86     | 51.78| 3    | 4.18 · 10$^{-2}$ | 2.32 · 10$^{-1}$ | 2.15 · 10$^{-2}$  | 1.2420      | 0.1052          |
| FITZGERALD86     | 55.58| 3    | 1.62 · 10$^{-1}$ | 8.10 · 10$^{-1}$ | 2.99 · 10$^{-2}$  | 1.2181      | 0.1005          |
| FITZGERALD86     | 63.21| 3    | 2.10 · 10$^{-1}$ | 7.66 · 10$^{-1}$ | 4.57 · 10$^{-2}$  | 1.1876      | 0.0939          |
| FRLEZ98          | 27.50| 6    | 1.33 · 10$^{-5}$ | 1.53 · 10$^{-2}$ | 1.77 · 10$^{-5}$  | 1.4250      | 0.0990          |
| ISENhower99      | 10.60| 4    | 5.42 · 10$^{-3}$ | 6.08 · 10$^{-1}$ | 3.38 · 10$^{-4}$  | 1.4320      | 0.1205          |
| ISENhower99      | 10.60| 5    | 5.70 · 10$^{-3}$ | 8.43 · 10$^{-1}$ | 1.06 · 10$^{-4}$  | 1.3126      | 0.0806          |
| ISENhower99      | 10.60| 6    | 1.41 · 10$^{-5}$ | 1.77 · 10$^{-1}$ | 6.74 · 10$^{-7}$  | 1.2845      | 0.0573          |
| ISENhower99      | 20.60| 5    | 1.90 · 10$^{-2}$ | 9.64 · 10$^{-1}$ | 3.24 · 10$^{-4}$  | 1.1880      | 0.0523          |
| ISENhower99      | 20.60| 6    | 1.23 · 10$^{-4}$ | 1.90 · 10$^{-1}$ | 7.99 · 10$^{-6}$  | 1.2089      | 0.0468          |
| ISENhower99      | 39.40| 4    | 2.05 · 10$^{-4}$ | 2.05 · 10$^{-1}$ | 3.09 · 10$^{-5}$  | 1.4842      | 0.1162          |
| ISENhower99      | 39.40| 5    | 9.71 · 10$^{-6}$ | 3.21 · 10$^{-1}$ | 3.02 · 10$^{-7}$  | 1.2310      | 0.0451          |
| ISENhower99      | 39.40| 5    | 2.75 · 10$^{-1}$ | 5.40 · 10$^{-1}$ | 7.26 · 10$^{-2}$  | 1.0763      | 0.0425          |
| SADLER04         | 63.86| 20   | 7.27 · 10$^{-1}$ | 6.94 · 10$^{-1}$ | 5.30 · 10$^{-1}$  | 1.0428      | 0.0680          |
| SADLER04         | 83.49| 20   | 8.60 · 10$^{-1}$ | 8.50 · 10$^{-1}$ | 4.32 · 10$^{-1}$  | 1.0418      | 0.0532          |
| SADLER04         | 94.57| 20   | 9.87 · 10$^{-1}$ | 9.95 · 10$^{-1}$ | 1.82 · 10$^{-1}$  | 1.0626      | 0.0469          |
Table 14 continued

| Identifier   | $T_j$  | $N_j$ | Overall   | Shape          | Abs. norm. | $\hat{z}_j$ | $\delta \hat{z}_j$ |
|--------------|-------|-------|-----------|----------------|------------|-------------|---------------------|
| JIA08        | 34.37 | 4     | 7.49 \cdot 10^{-1} | 6.17 \cdot 10^{-1} | 7.10 \cdot 10^{-1} | 1.0430 | 0.1158     |
| JIA08        | 39.95 | 4     | 8.91 \cdot 10^{-1} | 7.73 \cdot 10^{-1} | 9.58 \cdot 10^{-1} | 0.9935 | 0.1247     |
| JIA08        | 43.39 | 4     | 9.89 \cdot 10^{-1} | 9.95 \cdot 10^{-1} | 6.29 \cdot 10^{-1} | 0.9349 | 0.1348     |
| JIA08        | 46.99 | 4     | 4.64 \cdot 10^{-1} | 3.54 \cdot 10^{-1} | 5.62 \cdot 10^{-1} | 1.0816 | 0.1409     |
| JIA08        | 54.19 | 4     | 9.27 \cdot 10^{-1} | 9.30 \cdot 10^{-1} | 5.10 \cdot 10^{-1} | 0.9145 | 0.1296     |
| JIA08        | 59.68 | 4     | 6.66 \cdot 10^{-1} | 5.01 \cdot 10^{-1} | 8.85 \cdot 10^{-1} | 0.9821 | 0.1235     |
| MEKTEROVIĆ09 | 33.89 | 20    | 2.61 \cdot 10^{-3} | 5.62 \cdot 10^{-1} | 6.42 \cdot 10^{-7} | 1.1985 | 0.0399     |
| MEKTEROVIĆ09 | 39.38 | 20    | 2.65 \cdot 10^{-3} | 7.84 \cdot 10^{-1} | 1.13 \cdot 10^{-7} | 1.1687 | 0.0318     |
| MEKTEROVIĆ09 | 44.49 | 20    | 4.45 \cdot 10^{-5} | 2.63 \cdot 10^{-2} | 2.57 \cdot 10^{-6} | 1.1478 | 0.0314     |
| MEKTEROVIĆ09 | 51.16 | 20    | 6.96 \cdot 10^{-3} | 8.35 \cdot 10^{-1} | 3.81 \cdot 10^{-7} | 1.1702 | 0.0335     |
| MEKTEROVIĆ09 | 57.41 | 20    | 2.92 \cdot 10^{-3} | 6.02 \cdot 10^{-1} | 5.76 \cdot 10^{-7} | 1.1577 | 0.0315     |
| MEKTEROVIĆ09 | 66.79 | 20    | 3.42 \cdot 10^{-2} | 4.35 \cdot 10^{-1} | 2.26 \cdot 10^{-4} | 1.1192 | 0.0323     |
| MEKTEROVIĆ09 | 86.62 | 20    | 5.35 \cdot 10^{-2} | 8.32 \cdot 10^{-2} | 7.74 \cdot 10^{-2} | 1.0535 | 0.0303     |

**LECs**

| Identifier   | $T_j$  | $N_j$ | Overall   | Shape          | Abs. norm. | $\hat{z}_j$ | $\delta \hat{z}_j$ |
|--------------|-------|-------|-----------|----------------|------------|-------------|---------------------|
| SALOMON84    | 27.40 | 3     | 3.45 \cdot 10^{-1} | 5.65 \cdot 10^{-1} | 1.40 \cdot 10^{-1} | 1.0827 | 0.0561     |
| SALOMON84    | 39.30 | 3     | 1.94 \cdot 10^{-1} | 6.96 \cdot 10^{-1} | 4.58 \cdot 10^{-2} | 1.1179 | 0.0590     |
| BAGHERI88    | 45.60 | 3     | 2.78 \cdot 10^{-3} | 9.07 \cdot 10^{-1} | 1.93 \cdot 10^{-4} | 1.1364 | 0.0366     |
| BAGHERI88    | 62.20 | 3     | 6.46 \cdot 10^{-1} | 6.45 \cdot 10^{-1} | 3.76 \cdot 10^{-1} | 1.0343 | 0.0388     |
| BAGHERI88    | 76.40 | 3     | 3.97 \cdot 10^{-1} | 3.60 \cdot 10^{-1} | 3.37 \cdot 10^{-1} | 1.0344 | 0.0359     |
| BAGHERI88    | 91.70 | 3     | 1.54 \cdot 10^{-1} | 2.39 \cdot 10^{-1} | 1.22 \cdot 10^{-1} | 1.0613 | 0.0396     |

**Width of the 1s state in pionic hydrogen**

| Identifier    | $T_j$  | $N_j$ | Overall   | Shape          | Abs. norm. | $\hat{z}_j$ | $\delta \hat{z}_j$ |
|---------------|-------|-------|-----------|----------------|------------|-------------|---------------------|
| SCHROEDER01   | 0.00  | 1     | 6.10 \cdot 10^{-6} | – | 6.10 \cdot 10^{-6} | 1.1611 | 0.0356     |

**APs**

| Identifier    | $T_j$  | $N_j$ | Overall   | Shape          | Abs. norm. | $\hat{z}_j$ | $\delta \hat{z}_j$ |
|---------------|-------|-------|-----------|----------------|------------|-------------|---------------------|
| STAŠKO93     | 100.00 | 4     | 7.74 \cdot 10^{-1} | 7.34 \cdot 10^{-1} | 4.73 \cdot 10^{-1} | 0.9217 | 0.1090     |
| GAULARD99    | 98.10 | 6     | 9.54 \cdot 10^{-1} | 9.43 \cdot 10^{-1} | 5.46 \cdot 10^{-1} | 0.9647 | 0.0585     |
Table 14 continued

| Identifier         | $T_j$ | $N_j$ | Overall | Shape | Abs. norm. | $\hat{z}_j$ | $\delta \hat{z}_j$ |
|--------------------|-------|-------|---------|-------|------------|-------------|-----------------|
| BUGG71             | 90.90 | 1     | 2.84·10^{-1} | –     | 2.84·10^{-1} | 1.0664      | 0.0620          |
| BREITSCHOPF06      | 38.90 | 1     | 3.13·10^{-1} | –     | 3.13·10^{-1} | 1.1046      | 0.1036          |
| BREITSCHOPF06      | 43.00 | 1     | 2.86·10^{-1} | –     | 2.86·10^{-1} | 1.1607      | 0.1508          |
| BREITSCHOPF06      | 47.10 | 1     | 4.32·10^{-1} | –     | 4.32·10^{-1} | 1.0974      | 0.1241          |
| BREITSCHOPF06      | 55.60 | 1     | 4.64·10^{-1} | –     | 4.64·10^{-1} | 1.0681      | 0.0931          |
| BREITSCHOPF06      | 64.30 | 1     | 5.60·10^{-1} | –     | 5.60·10^{-1} | 0.9599      | 0.0687          |
| BREITSCHOPF06      | 65.90 | 1     | 8.63·10^{-1} | –     | 8.63·10^{-1} | 0.9885      | 0.0670          |
| BREITSCHOPF06      | 75.10 | 1     | 1.19·10^{-1} | –     | 1.19·10^{-1} | 0.9224      | 0.0497          |
| BREITSCHOPF06      | 76.10 | 1     | 8.70·10^{-1} | –     | 8.70·10^{-1} | 0.9893      | 0.0653          |
| BREITSCHOPF06      | 96.50 | 1     | 9.77·10^{-1} | –     | 9.77·10^{-1} | 0.9989      | 0.0396          |
$y_{ij}^{\exp}/y_{ij}^{\text{th}}$ vs $\theta$ (deg)

- $\pi^+$ p: 19.90 MeV
- $\pi^+$ p: 25.80 MeV
- $\pi^+$ p: 32.00 MeV
Fig. 12. The DENZ04 $\pi^+ p$ DCSs [25] ($y_{ij}^{\text{exp}}$), normalised to the corresponding predictions ($y_{ij}^{\text{th}}$) obtained from the results of the fits of the ETH model to the tDB_{++/--}. The normalisation uncertainties of the experimental data sets (see Refs. [21,25] for details) are not shown.
$\theta$ (deg)

$y_{exp} / y_{th}$

$\pi^- p$ elastic:
- 19.90 MeV
- 25.80 MeV
- 32.00 MeV

$y_{ij}^{exp} / y_{ij}^{th}$
Fig. 13. The DENZ04 $\pi^- p$ ES DCSs [25] ($y_{ij}^{\text{exp}}$), normalised to the corresponding predictions ($y_{ij}^{\text{th}}$) obtained from the results of the fits of the ETH model to the tDB$_{+/-}$. The normalisation uncertainties of the experimental data sets (see Refs. [21,25] for details) are not shown.
Fig. 14. The scale factors $\hat{z}_j$ for free floating, evaluated with Eq. (13), for those of the data sets in the $DB_0$ which represent measurements of the $\pi^- p$ CX DCS. The BLS has been obtained from the fits of the ETH model to the $tDB_{+/−}$; solid points: DCS, diamonds: TCS, inverse triangles: LECs. Not included in the plot are the three freely-floated FITZGERALD86 data sets (see Table 10), as well as the BREITSCHOPF06 75.10 MeV entry. The shaded band represents 1σ uncertainties around the fitted values using a simple exponential form (see Section 3.2.7).
Fig. 15. Feynman graphs involving the $\rho^0 - \omega$ mixing, a potential mechanism for the violation of isospin invariance in the hadronic part of the $\pi N$ interaction in the case of the $\pi^+ p$ and $\pi^- p$ ES reactions.

Fig. 16. Feynman graphs involving the $\pi^0 - \eta$ mixing, a potential mechanism for the violation of isospin invariance in the hadronic part of the $\pi N$ interaction in the case of the $\pi^- p$ CX reaction.