A PERCOLATION FORMULA

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Abstract
Let $A$ be an arc on the boundary of the unit disk $U$. We prove an asymptotic formula for the
probability that there is a percolation cluster $K$ for critical site percolation on the triangular
grid in $U$ which intersects $A$ and such that 0 is surrounded by $K \cup A$.

Motivated by questions of Langlands et al [LPSA94] and M. Aizenman, J. Cardy [Car92, Car]
derived a formula for the asymptotic probability for the existence of a crossing of a rectangle
by a critical percolation cluster. Recently, S. Smirnov [Sm1] proved Cardy’s formula and
established the conformal invariance of critical site percolation on the triangular grid. The
paper [LSW] has a generalization of Cardy’s formula. Another percolation formula, which is
still unproven, was derived by G. M. T. Watts [Wat96]. The current paper will state and
prove yet another such formula. A short discussion elaborating on the general context of these
results appears at the end of the paper.

Consider site percolation on the triangular lattice in $\mathbb{C}$ with small mesh $\delta > 0$, where each
site is declared open with probability $1/2$, independently. (See [Gri89, Kes82] for background
on percolation.) It is convenient to represent a percolation configuration by coloring the
corresponding hexagonal faces of the dual grid; black for an open site, white for a closed site.
(The faces are taken to be topologically closed. Some edges are colored by both colors, but
that has no significance.) Let $\mathcal{B}$ denote the union of the black hexagons, intersected with the
closed unit disk $\overline{U}$, and for $\theta \in (0, 2\pi)$ let $\mathcal{A} = \mathcal{A}(\theta)$ be the event that there is a connected
component $K$ of $\mathcal{B}$ which intersects the arc

$$A_\theta := \{ e^{i s} : s \in [0, \theta] \} \subset \partial U$$

and such that 0 is surrounded by $K \cup A_\theta$. The latter means that 0 is in a bounded component
of $\mathbb{C} \setminus (A_\theta \cup K)$ or $0 \in K$. Figure 1 shows the two distinct topological ways in which this could
happen.

**Theorem 1.**

$$\lim_{\delta \downarrow 0} \frac{1}{2} - \frac{\Gamma(2/3)}{\sqrt{\pi} \Gamma(1/6)} F_{2,1} \left( \frac{1}{2}, \frac{2}{3}, \frac{3}{2}; -\cot^2 \frac{\theta}{2}, \cot \frac{\theta}{2} \right) \cot \frac{\theta}{2}.$$
Here, $F_{2,1}$ is the hypergeometric function. See [EMOT53, Chap. 2] for background on hypergeometric functions.

There is a second interpretation of the Theorem. Let $C_1$ be the cluster of either black or white hexagons which contains 0. (If 0 is on the boundary of two clusters of different colors, let $C_1$ be the black cluster containing 0, say.) Let $C_2$ be the (unique) cluster which surrounds $C_1$ and is adjacent to it. Inductively, let $C_{n+1}$ be the cluster surrounding and adjacent to $C_n$, and of the opposite color. Let $m$ be the least integer such that $C_m$ is not contained in $U$, and let $C_0$ be the component of $U \setminus C_m$ which surrounds 0. Let $X := 1$ if $C_0 \notin \partial U$, let $X := 0$ if $C_m \cap A_0 = \emptyset$, and otherwise set $X := 1/2$. Then \( \lim_{\delta \downarrow 0} E[X] = \lim_{\delta \downarrow 0} P[A] \). This is so because

\[ A = \{ X = 1 \} \cup \{ X = 1/2 \text{ and } C_m \text{ is black} \} \cup \{ m = 1, X > 0 \text{ and } 0 \in \partial C_m \} , \]

and the probability that $m = 1$ goes to zero as $\delta \downarrow 0$ (since a.s. there is no infinite cluster).

Theorem 1 will be proved by utilizing the relation between the scaling limit of percolation and Stochastic Loewner evolution with parameter $\kappa = 6$ (a.k.a. SLE$_6$), which was conjectured in [Sch00] and proven by S. Smirnov [Smii].

We now very briefly review the definition and the relevant properties of chordal SLE. For a thorough treatment, see [RS]. Let $\kappa \geq 0$, let $B(t)$ be Brownian motion on $\mathbb{R}$ starting from $B(0) = 0$, and set $W(t) = \sqrt{\kappa} B(t)$. For $z$ in the upper half plane $\mathbb{H}$ consider the time flow $g_t(z)$ given by

\[ \partial_t g_t(z) = \frac{2}{g_t(z) - W(t)}, \quad g_0(z) = z. \quad (1) \]

Then $g_t(z)$ is well defined up to the first time $\tau = \tau(z)$ such that $\lim_{t \uparrow \tau} g_t(z) - W(t) = 0$. For all $t > 0$, the map $g_t$ is a conformal map from the domain $H_t := \{ z \in \mathbb{H} : \tau(z) > t \}$ onto $\mathbb{H}$. The process $t \mapsto g_t$ is called Stochastic Loewner evolution with parameter $\kappa$, or SLE$_\kappa$.

In [RS] it was proven that at least for $\kappa \neq 8$ a.s. there is a uniquely defined continuous path $\gamma : [0, \infty) \to \mathbb{H}$ with $\gamma(0) = 0$, called the trace of the SLE, such that for every $t \geq 0$ the set $H_t$ is equal to the unbounded component of $\mathbb{H} \setminus \gamma[0,t]$. In fact, a.s.

\[ \forall t \geq 0, \quad \gamma(t) = \lim_{z \to W(t)} g_t^{-1}(z), \]
where \( z \) tends to \( W(t) \) from within \( \mathbb{H} \). Additionally, it was shown that \( \gamma \) is a.s. transient, namely \( \lim_{t \to \infty} |\gamma(t)| = \infty \), and that when \( \kappa \in (0, 8) \) we have for every \( z_0 \in \mathbb{H} \) that \( P[ z_0 \in \gamma[0, \infty) ] = 0 \).

It was also shown [RS] that \( \gamma \) is a simple path a.s. iff \( \kappa \leq 4 \). When \( \kappa > 4, \kappa \neq 8 \), although not a simple path, \( \gamma \) does not cross itself; that is, a.s. for every \( t_0 > 0 \) there is a continuous homotopy \( H : [0, 1] \times (t_0, \infty) \to \mathbb{H} \) such that \( H(0, t) = \gamma(t) \) and \( H((0, 1] \times (t_0, \infty)) \cap \gamma[0, t_0] = \emptyset \).

This property easily follows from the fact that \( \gamma(t_0, \infty) \) is the image of a continuous path in \( \mathbb{H} \) (which, by the way, has essentially the same law as \( \gamma \)) under the continuous extension of \( g_{t_0}^{-1} : \mathbb{H} \to \mathbb{H} \setminus [0, t_0] \), to \( \mathbb{H} \). (See, for example, [RS, Proposition 2.1(ii), Theorem 5.2].)

Fix some \( z_0 = x_0 + i y_0 \in \mathbb{H} \). Then we may ask if \( \gamma \) passes to the right or to the left of \( z_0 \), topologically. (Formally, this should be defined in terms of winding numbers, as follows. Let \( \beta_t \) be the path from \( \gamma(t) \) to 0 which follows the arc \( |\gamma(t)| \partial \mathbb{U} \) clockwise from \( \gamma(t) \) to \( |\gamma(t)| \) and then takes the straight line segment in \( \mathbb{R} \) to 0. Then \( \gamma \) passes to the left of \( z_0 \) if the winding number of \( \gamma[0, t] \cup \beta_t \) around \( z_0 \) is 1 for all large \( t \). As noted above, \( \gamma \) is a.s. transient, and therefore there is some random time \( t_0 \) such that the winding number is constant for \( t \in (t_0, \infty) \). This constant is either 0 or 1, since \( \gamma \) does not cross itself, as discussed above.) Theorem 1 will be established by applying the following with \( \kappa = 6 \):

**Theorem 2.** Let \( \kappa \in [0, 8) \), and let \( z_0 = x_0 + i y_0 \in \mathbb{H} \). Then the trace \( \gamma \) of chordal SLE\( \kappa \) satisfies

\[
P[ \gamma \text{ passes to the left of } z_0 ] = \frac{1}{2} + \frac{\Gamma(4/\kappa)}{\sqrt{\pi} \Gamma(2 \kappa/\kappa)} \frac{x_0}{y_0} F_{x_0 y_0} \left( \frac{1}{2}, \frac{3}{2}, \frac{1}{2} - \frac{x_0^2}{y_0^2} \right).
\]

When \( \kappa = 2, 8/3, 4 \) and 8 the right hand side simplifies to \( 1 + \frac{x_0 y_0}{\pi |z_0|^2} - \frac{\arg z_0}{\pi} \), \( 1 + \frac{x_0}{2 |z_0|^2}, 1 - \frac{2 \arg z_0}{\pi} \), and \( 1/2 \), respectively.

Let \( x_t := \text{Re} g_t(z_0) - W(t), y_t := \text{Im} g_t(z_0), \) and \( w_t := x_t/y_t \).

**Lemma 3.** Almost surely, \( \gamma \) is to the left of \( z_0 \) iff \( \lim_{t \uparrow \tau(z_0)} w_t = \infty \) and a.s. \( \gamma \) is to the right of \( z_0 \) iff \( \lim_{t \downarrow \tau(z_0)} w_t = -\infty \).

**Proof.** Suppose first that \( \kappa \in [0, 4] \). In that case, a.s. \( \gamma \) is a simple path and \( \tau(z_0) = \infty \), by [RS]. Given \( \gamma \), we start a planar Brownian motion \( B \) from \( z_0 \). Suppose that \( \gamma \) is to the left of \( z_0 \). This implies that \( B \) will first hit \( \mathbb{R} \cup \gamma[0, \infty) \) in \( [0, \infty) \) or from the right hand side of \( \gamma \).

Since \( \gamma \) is transient, as \( t \uparrow \infty \) the probability that \( B \) first hits \( \gamma[0, t] \cup \mathbb{R} \) from the right hand side of \( \gamma \) or in \( [0, \infty) \) tends to 1. By conformal invariance of harmonic measure, this means that the harmonic measure in \( \mathbb{H} \) of \( W(t, \infty) \) from \( g_t(z_0) \) tends to 1. Therefore, \( \lim_{t \uparrow \tau} w_t = \infty \).

The argument in the case where \( \gamma \) is to the right of \( z_0 \) is entirely similar. Since \( \gamma \) must be either to the left or to the right of \( z_0 \), this proves the lemma in the case \( \kappa \in [0, 4] \).

For \( \kappa \in (4, 8) \), the analysis is similar. The difference is that a.s. \( \gamma \) is not a simple path, \( \tau(z_0) < \infty \), and \( z_0 \) is in a bounded component of \( \mathbb{H} \setminus \gamma[0, \tau(z_0)] \) (see [RS]). Clearly, \( z_0 \) is not in a bounded component of \( \mathbb{H} \setminus \gamma[0, t] \) when \( t < \tau(z_0) \). Hence, at time \( \tau(z_0) \) the path \( \gamma \) closes a loop around \( z_0 \). Since \( \gamma \) does not cross itself, the issue then is whether this is a clockwise or counter-clockwise loop. As above, if the loop is clockwise, then as \( t \uparrow \tau(z_0) \) the harmonic measure from \( z_0 \) in \( \mathbb{R} \cup \gamma[0, t] \) is predominantly on \( [0, \infty) \) and the right side of \( \gamma[0, t] \). This implies that \( w_t \uparrow \infty \). If the loop is counter-clockwise, we get \( w_t \downarrow -\infty \), by the same reasoning. This completes the proof.

\[\Box\]
Proof of Theorem 2. Writing (1) in terms of the real and imaginary parts gives,

\[ dx_t = \frac{2}{x_t^2 + y_t^2} \, dx(t), \quad dy_t = -\frac{2}{x_t^2 + y_t^2} \, dy(t). \]

Itō’s formula then gives,

\[ dw_t = -\frac{dW(t)}{y_t} + 4 \frac{w_t \, dt}{x_t^2 + y_t^2}. \] (2)

Make the time change

\[ u(t) = \int_0^t \frac{dt}{y_t}, \]

and set

\[ \tilde{W}(t) = \int_0^t \frac{dW(t)}{y_t}. \]

Note that \( \tilde{W}/\sqrt{\kappa} \) is Brownian motion as a function of \( u \). From (2), we now get

\[ dw = -d\tilde{W} + \frac{4 \, w \, du}{w^2 + 1}. \] (3)

We got rid of \( x_t \) and \( y_t \), and are left with a single variable diffusion process \( w(u) \). (This is no mystery, but a simple consequence of scale invariance.) An immediate consequence of this and the lemma is that a.s. \( \lim_{t \to \tau(z_0)} u = \infty \), because the diffusion (3) a.s. does not hit \( \pm \infty \) in finite time. It is clear that \( u(t) < \infty \) when \( t < \tau(z_0) \), because \( y_t \) is monotone decreasing and positive for \( t \in [0, \tau(z_0)] \).

Given a starting point \( \dot{w} \in \mathbb{R} \) for the diffusion (3), and given \( a, b \in \mathbb{R} \) with \( a < \dot{w} < b \), we are interested in the probability \( h(\dot{w}) = h_{a,b}(\dot{w}) \) that \( w \) will hit \( b \) before hitting \( a \). Note that \( h(w(u)) \) is a local martingale. Therefore, assuming for the moment that \( h \) is smooth, by Itō’s formula, \( h \) satisfies

\[ \frac{\kappa}{2} h''(w) + \frac{4 \, w}{w^2 + 1} h'(w) = 0, \quad h(a) = 0, \quad h(b) = 1. \]

By the maximum principle, these equations have a unique solution, and therefore we find that

\[ h(w) = \frac{f(w) - f(a)}{f(b) - f(a)}, \] (4)

where

\[ f(w) := F_{2,1}(1/2, 4/\kappa, 3/2, -w^2) \, w. \]

We may now dispose of the assumption that \( h \) is smooth, because Itō’s formula implies that the right hand side in (4) is a martingale, and it easily follows that it must equal \( h \). By [EMOT53, 2.10.(3)] and our assumption \( \kappa < 8 \) it follows that

\[ \lim_{w \to -\infty} f(w) = \pm \frac{\sqrt{\pi} \Gamma((8 - \kappa)/(2\kappa))}{2 \Gamma(4/\kappa)}. \] (5)

In particular, the limit is finite, which shows that \( \lim_{w \to -\infty} h_{a,b}(w) > 0 \) for all \( w > a \). Hence, the diffusion process (3) is transient. Moreover,

\[ P \left[ \lim_{u \to -\infty} w(u) = +\infty \right] = \frac{f(\dot{w}) - f(-\infty)}{f(\infty) - f(-\infty)}. \]

An appeal to the lemma now completes the proof. \( \square \)
Proof of Theorem 1. As above, let $\mathcal{B}$ be the intersection of the union of the black hexagons with $\overline{U}$, and let $\mathfrak{B}$ be the union of $\mathcal{B}$ and the set $S := \{ re^{is} : r \geq 1, s \in [0, \theta] \}$. Let $\beta$ be the intersection of $\overline{U}$ with the outer boundary of the connected component of $\mathfrak{B}$ containing $S$. Then $\beta$ is a path in $\overline{U}$ from 1 to $e^{i\theta}$. It is immediate that the event $\mathcal{A}$ is equivalent to the event that 0 appears to the right of the path $\beta$; that is, that the winding number of the concatenation of $\beta$ with the arc $A_\theta$ with the clockwise orientation around 0 is 1.

S. Smirnov [Smi1] has shown that as $\delta \downarrow 0$ the law of $\beta$ tends weakly to the law of the image of the chordal SLE$_6$ trace $\gamma$ under any fixed conformal map $\phi : \mathbb{H} \to \overline{U}$ satisfying $\phi(0) = 1$ and $\phi(\infty) = e^{i\theta}$. (See also [Smi2].) We may take

$$\phi(z) = e^{i\theta} \frac{z + \cot \frac{\theta}{2} - i}{z + \cot \frac{\theta}{2} + i}.$$ 

The theorem now follows by setting $\kappa = 6$ in Theorem 2.

Discussion. According to J. Cardy (private communication, 2001), presently, the conformal field theory methods used by him to derive his formula do not seem to supply even a heuristic derivation of Theorem 1. On the other hand, it seems that, in principle, probabilities for “reasonable” events involving critical percolation can be expressed as solutions of boundary-value PDE problems, via SLE$_6$. But this is not always easy. In particular, it would be nice to obtain a proof of Watts’ formula [Wat96]. The event $\mathcal{A}$ studied here was chosen because the corresponding proof is particularly simple, and because the PDE can be solved explicitly.

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