BOSE-EINSTEIN CONDENSATES AND SPECTRAL PROPERTIES OF MULTICOMPONENT NONLINEAR SCHRODINGER EQUATIONS

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ABSTRACT. We analyze the properties of the soliton solutions of a class of models describing one-dimensional BEC with spin $F$. We describe the minimal sets of scattering data which determine uniquely both the corresponding potential of the Lax operator and its scattering matrix. Next we give several reductions of these MNLS, derive their $N$-soliton solutions and analyze the soliton interactions. Finally we prove an important theorem proving that if the initial conditions satisfy the reduction then one gets a solution of the reduced MNLS.

1. INTRODUCTION. It is well known that Bose-Einstein condensate (BEC) of alkali atoms in the $F = 1$ hyperfine state, elongated in $x$ direction and confined in the transverse directions $y, z$ by purely optical means are described by a 3-component normalized spinor wave vector $\Phi(x, t) = (\Phi_1, \Phi_0, \Phi_{-1})^T(x, t)$. Considering dimensionless units and using special choices for the scattering lengths one can show that $\Phi(x, t)$ satisfies the multicomponent nonlinear Schrödinger (MNLS) equation [14], see also [15, 19, 26, 2, 22]:

$$i \partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 + 2\Phi_{-1}^* \Phi_0^2 = 0,$$

$$i \partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(|\Phi_{-1}|^2 + |\Phi_0|^2)\Phi_0 + 2\Phi_1^* \Phi_{-1} = 0,$$

$$i \partial_t \Phi_{-1} + \partial_x^2 \Phi_{-1} + 2(|\Phi_{-1}|^2 + 2|\Phi_0|^2)\Phi_{-1} + 2\Phi_1^* \Phi_0^2 = 0. \quad (1)$$

Similarly spinor BEC with $F = 2$ is described by a 5-component normalized spinor wave vector $\Phi(x, t) = (\Phi_2, \Phi_1, \Phi_0, \Phi_{-1}, \Phi_{-2})^T(x, t)$. For specific choices of the scattering lengths in dimensionless coordinates the corresponding set of equations for $\Phi(x, t)$ take the form [27]:

$$i \partial_t \Phi_{\pm 2} + \partial_x^2 \Phi_{\pm 2} + 2(\Phi, \Phi^*)\Phi_{\pm 2} - (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2)\Phi_{\pm 2}^* = 0,$$

$$i \partial_t \Phi_{\pm 1} + \partial_x^2 \Phi_{\pm 1} + 2(\Phi, \Phi^*)\Phi_{\pm 1} + (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2)\Phi_{\pm 1}^* = 0,$$

$$i \partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(\Phi, \Phi^*)\Phi_0 - (2\Phi_2 \Phi_{-2} - 2\Phi_1 \Phi_{-1} + \Phi_0^2)\Phi_0^* = 0. \quad (2)$$

Both models have natural Lie algebraic interpretation and are related to the symmetric spaces $\text{BD}1 \simeq \text{SO}(n + 2)/\text{SO}(n) \times \text{SO}(2)$ with $n = 3$ and $n = 5$ respectively. They are integrable by means of inverse scattering transform method [4, 25, 12].

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Using a modification of the Zakharov-Shabat ‘dressing method’ we describe the soliton solutions [14, 17] and the effects of the reductions on them.

Sections 2 contains the basic details on the direct and inverse scattering problems for the Lax operator. Section 3 is devoted to the construction of their soliton solutions. In Section 4 we formulate the minimal sets of scattering data \( L \) which determine uniquely both the scattering matrix and the potential \( Q(x, t) \). Section 5 gives a few important examples of algebraic reductions of the MNLS. In Section 6 we analyze the soliton interactions of the MNLS. To this end we evaluate the limits of the generic two-soliton solution for \( t \to \pm \infty \). As a result we establish that the effect of the interactions on the soliton parameters is analogous to the one for the scalar NLS equation and consists in shifts of the ‘center of mass’ and shift in the phase. In Section 7 we prove an important theorem proving that if the initial conditions satisfy the reduction then one gets a solution of the reduced MNLS.

2. The method for solving MNLS for any \( F \).

2.1. The Lax representation. The above MNLS equations (1) and (2) are the first two members of a series of MNLS equations related to the BD.I-type symmetric spaces. They allow Lax representation as follows [4, 10, 12]

\[
\begin{align*}
L \psi(x, t, \lambda) &\equiv i \partial_x \psi + U(x, t, \lambda) \psi(x, t, \lambda) = 0, \\
M \psi(x, t, \lambda) &\equiv i \partial_t \psi + V(x, t, \lambda) \psi(x, t, \lambda) = 0,
\end{align*}
\]

(3)

where

\[
\begin{align*}
U(x, t, \lambda) &= Q(x, t) - \lambda J, \\
V(x, t, \lambda) &= V_0(x, t) + \lambda V_1(x, t) - \lambda^2 J, \\
V_1(x, t) &= Q(x, t), \\
V_0(x, t) &= i \text{ad}^{-1} \frac{dQ}{dx} + \frac{1}{2} \left[ \text{ad}^{-1} Q, Q(x, t) \right].
\end{align*}
\]

(4)

For those familiar with Lie algebras I remind that, as usual, \( Q(x, t) \) and \( J \) are elements of the corresponding Lie algebra, which in our case is \( g \simeq \text{so}(n+2) \). The choice of the Cartan subalgebra element \( J \) determines the co-adjoint orbit of \( g \); in our case \( J \) is dual to \( e_1 \), see [13]. It introduces grading in \( g = g^{(0)} \oplus g^{(1)} \) where \( g^{(0)} \simeq \text{so}(n) \). The root system of \( g^{(0)} \) consists of all roots of \( \text{so}(n+2) \) which are orthogonal to \( e_1 \); the linear subspace \( g^{(1)} \) is spanned by the Weyl generators \( E_\alpha \) and \( E_{-\alpha} \) for which the roots \( \alpha \in \Delta^+_1 \) are such that their scalar products \( (\alpha, e_1) = 1 \). Thus the potential

\[
Q(x, t) = \sum_{\alpha \in \Delta^+_1} (q_\alpha(x, t) E_\alpha + p_\alpha(x, t) E_{-\alpha})
\]

(5)

may be viewed as local coordinate of the above mentioned symmetric space. The linear operator \( \text{ad} J X = [J, X] \) and \( \text{ad}^{-1} \) is well defined on the image of \( \text{ad} J \) in \( g \).

In what follows we will use the typical representation of \( \text{so}(n+2) \) with \( n = 2r-1 \) in which \( Q \) and \( J \) take the following block-matrix structure:

\[
Q(x, t) = \begin{pmatrix} 0 & \bar{q}^T & 0 \\ \bar{p}^T & 0 & s_0 \bar{q} \\ 0 & \bar{p}^T s_0 & 0 \end{pmatrix}, \quad J = \text{diag}(1, 0, \ldots, 0, -1).
\]

(6)

For physical applications one uses mostly potentials satisfying the typical reduction, i.e. \( \bar{p}(x, t) = \bar{q}^*(x, t) \). The vector \( \bar{q}(x, t) \) for integer \( F = r \) has \( 2r + 1 \) components

\[
\bar{q}(x, t) = (\Phi_{r-1}, \ldots, \Phi_0, \ldots, \Phi_{-r+1})^T(x, t),
\]

(7)
and the corresponding matrices $s_0$ enter in the definition of the orthogonal algebras \( so(2r - 1) \); namely $X \in so(2r + 1)$ if

$$X + S_0X^TS_0 = 0, \quad S_0 = \sum_{s=1}^{2r+1} (-1)^{s+1}E_{s,n+1-s}, \quad S_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -s_0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (8)$$

By $E_{sp}$ above we mean $2r + 1 \times 2r + 1$ matrix with matrix elements $(E_{sp})_{ij} = \delta_{si}\delta_{pj}$. With the definition of orthogonality used in (8) the Cartan generators $H_k = E_k,k - E_{2r+2-k,2r+2-k}$ are represented by diagonal matrices.

If we make use of the typical reduction $Q = Q^\dagger$ (or $\bar{p}^* = \bar{q}$) the generic MNLS type equations related to $BD_2$ references therein. The Jost solutions of The Hamiltonians for the MNLS equations (8) are given by

$$H_{MNLS} = \int_{-\infty}^{\infty} dx \left( (\partial_x \bar{q}^\dagger, \partial_x \bar{q}) - (\bar{q}^\dagger, \bar{q})^2 + \frac{1}{2} (\bar{q}^T, s_0 \bar{q})^2 \right). \quad (10)$$

2.2. The Direct and the Inverse scattering problem for $L$. We remind some basic features of the scattering theory for the Lax operators $L$, see \cite{10, 12}. There we have made use of the general theory developed in \cite{32, 33, 29, 3, 6} and the references therein. The Jost solutions of $L$ are defined by:

$$\lim_{x \to -\infty} \phi(x, t, \lambda)e^{i\lambda Jx} = 1, \quad \lim_{x \to \infty} \psi(x, t, \lambda)e^{i\lambda Jx} = 1 \quad (11)$$

and the scattering matrix $T(\lambda, t) \equiv \psi^{-1}\phi(x, t, \lambda)$. The special choice of $J$ and the fact that the Jost solutions and the scattering matrix take values in the group $SO(2r + 1)$ we can use the following block-matrix structure of $T(\lambda, t)$

$$T(\lambda, t) = \begin{pmatrix} m_1^+ & -\bar{b}^T \bar{c}^1_1 \\ \bar{b}^+ & T_{22} \end{pmatrix}, \quad \hat{T}(\lambda, t) = \begin{pmatrix} m_1^- & -\bar{b}^T \bar{c}^1_1 \\ \bar{b}^+ & T_{22} \end{pmatrix}, \quad (12)$$

where $\bar{b}^\pm(\lambda, t)$ and $\bar{b}^\pm(\lambda, t)$ are $2r - 1$-component vectors, $T_{22}(\lambda)$ is $2r - 1 \times 2r - 1$ block matrix, and $m_1^+(\lambda)$, and $c_1^+(\lambda)$ are scalar functions. Below we often use $\hat{X}$ to denote the matrix inverse to $X$.

Remark 1. The typical reduction $\bar{p}(x, t) = \bar{q}^*(x, t)$ mentioned above imposes on $T(\lambda, t)$ the constraint $T(\lambda, t) = \hat{T}(\lambda, t)$ for real values of $\lambda \in \mathbb{R}$, i.e.

$$m_1^+(\lambda) = m_1^+(-\lambda), \quad \bar{B}_1^-(-\lambda) = \bar{B}_1^+(-\lambda),$$

$$c_1^+(\lambda) = c_1^+(-\lambda), \quad \bar{B}_1^+(-\lambda) = \bar{B}_1^+(-\lambda). \quad (13)$$

The Lax representation (1) allows one to prove that if $\bar{q}(x, t)$ satisfies the MNLS (9) then the scattering matrix $T(\lambda, t)$ satisfies the linear evolution equation [12]:

$$\frac{dT}{dt} - \lambda^2 [J, T(\lambda, t)] = 0, \quad (14)$$
or in components:

\[
\begin{align*}
\frac{d\tilde{b}_1^\pm}{dt} & \pm \lambda^2 \tilde{b}_1^\pm(t, \lambda) = 0, \\
\frac{d\tilde{B}_1^\pm}{dt} & \pm \lambda^2 \tilde{B}_1^\pm(t, \lambda) = 0, \\
\frac{dm_1^\pm}{dt} & = 0, \\
\frac{d\mathbf{m}_2^\pm}{dt} & = 0.
\end{align*}
\]  

(15)

Thus the block-diagonal matrices \(D^\pm(\lambda)\) can be considered as generating functionals of the integrals of motion. Thus the problem of solving the MNLS eq. is based on the effective analysis of the mapping between the potential \(Q(x, t)\) of \(L\) and the scattering matrix \(T(\lambda, t)\).

2.3. The fundamental analytic solution and the Riemann-Hilbert problem. The most effective method for the above mentioned analysis consists in constructing the fundamental analytic solution (FAS) of \(L\)-operators of type (3) and reducing the inverse scattering problem to an equivalent Riemann-Hilbert problem (RHP). Skipping the details (see [11]) we just outline the construction of FAS for \(L\). Obviously the FAS, like any other fundamental solutions of \(L\) must be linearly related to the Jost solutions. For the class of potentials \(Q(x, t)\) with vanishing boundary conditions there exist two FAS \(\chi^\pm(x, t, \lambda)\) which allow analytic extension for \(\lambda \in \mathbb{C}_\pm\) respectively. For real \(\lambda\) they are related to the Jost solutions by

\[
\chi^\pm(x, t, \lambda) = \phi(x, t, \lambda)S^+_J(t, \lambda) = \psi(x, t, \lambda)T^+_J(t, \lambda)D^+_J(\lambda),
\]  

(16)

where \(T^+_J(t, \lambda)\), \(D^+_J(\lambda)\) and \(T^+_J(t, \lambda)\) are the generalized Gauss factors of \(T(\lambda, t)\), see [29, 5, 7]:

\[
\begin{align*}
T(\lambda, t) &= T^+_J(t, \lambda)S^+_J(t, \lambda), & T(\lambda, t) &= T^+_J(t, \lambda)D^+_J(t, \lambda), \\
T^+_J(t, \lambda) &= e^{\pm(\rho^+, E^+_1)}, & S^+_J(t, \lambda) &= e^{\pm(\tau^-, E^+_1)}, \\
D^+_J(\lambda) &= \text{diag} ( (m^+_1)^{\mp 1}, m^+_2, (m^+_1)^{\mp 1} ).
\end{align*}
\]  

Here

\[
\begin{align*}
\tau^\pm(\lambda, t) &= (\tau^\pm_{r-1}, \ldots, \tau^\pm_0, \ldots, \tau^\pm_{r+1})^T (\lambda, t), \\
(\bar{\rho}^+, \bar{E}^+_1) &= \sum_{k=1}^{r-1} (\tau^+_k E_{e_1-e_k+1} + \tau^-_k E_{e_1+e_k+1}) + \tau^+_0 E_{e_1}, \\
(\bar{\tau}^+, \bar{E}^+_1) &= \sum_{k=1}^{r-1} (\tau^-_k E_{-e_1+e_k+1} + \tau^-_k E_{-e_1-e_k+1}) + \tau^-_0 E_{-e_1},
\end{align*}
\]  

(18)

and similar expressions for \((\bar{\rho}^-, \bar{E}^-_1)\). Above we have made use of the fact that \(\Delta^+_1\) consists of the roots \(\{ e_1 - e_k, e_1 + e_k \}_{k=1}^{r-1} \). The functions \(m^+_1\) and \(n \times n\) matrix-valued functions \(m^+_2\) are analytic for \(\lambda \in \mathbb{C}_\pm\). One can check, that the analogs of the reflection coefficients \(\bar{\rho}^\pm\) and \(\bar{\tau}^\pm\) are expressed by:

\[
\bar{\rho}^\pm = \frac{\tilde{B}^-}{m_1^\pm}, \quad \bar{\tau}^- = \frac{\tilde{B}^+}{m_1^\pm}, \quad \bar{\rho}^\mp = \frac{\tilde{B}^+}{m_1^\pm}, \quad \bar{\tau}^\mp = \frac{\tilde{B}^-}{m_1^\pm}.
\]

Remark 2. The typical reduction means that for \(\lambda \in \mathbb{R}\) the reflection coefficients are constrained by (see remark 1 above):

\[
\bar{\rho}^+(\lambda) = \bar{\tau}^-(\lambda), \quad \bar{\tau}^+(\lambda) = \bar{\tau}^-(\lambda), \quad \lambda \in \mathbb{R}.
\]  

(19)
Remark 3. An immediate consequence of the analyticity of \( \xi^\pm(x, t, \lambda) \) is that \( D^\pm(\lambda) \) are analytic functions for \( \lambda \in \mathbb{C}_\pm \). This fact follows from the relation 
\[ \lim_{x \to \infty} \xi^\pm(x, \lambda) = D^\pm(\lambda). \]
Zakharov and Shabat proved a theorem \cite{32,33} which states that if \(G_J(x,\lambda,t)\) satisfies:
\[
\frac{dG}{dx} - \lambda [J,G(x,\lambda,t)] = 0,
\]
\[
\frac{i dG}{dt} - \lambda^2 [J,G(x,\lambda,t)] = 0,
\]
then the corresponding solutions of the RHP allow one to construct \(\chi^\pm(x,\lambda) = \xi^\pm(x,\lambda)e^{-i\lambda Jx}\) as a fundamental solution of the Lax pair eq. (1).

We will say that \(\xi^\pm_0(x,\lambda)\) is a regular solution to the RHP (23) if the block-diagonal part of it has neither zeroes nor poles in its whole region of analyticity.

If we have solved the RHP’s and know the FAS \(\xi^+_0(x,t,\lambda)\) then the formula
\[
Q(x,t) = \lim_{\lambda \to \infty} \lambda \left( J - \xi^+_0(x,t,\lambda).J\xi^+_0(x,t,\lambda) \right),
\]
allows us to recover the corresponding potential of \(L\).

3. Singular solutions of RHP and soliton solutions of MNLS. Zakharov-Shabat’s theorem ensures that if a given RHP allows regular solution, then this solution is unique. However the RHP may have many singular solutions. The construction of such singular solutions starting from a given regular one is known as the dressing Zakharov-Shabat method \cite{32,33}. Indeed, if \(\xi^\pm_0(x,t,\lambda)\) are regular solutions to the RHP, then
\[
\xi^\pm(x,t,\lambda) = u(x,t,\lambda)\xi^\pm_0(x,t,\lambda)
\]
with conveniently chosen dressing factor \(u(x,t,\lambda)\) may again be a solution of the RHP \cite{32,33}. Obviously this factor must be analytic (with the exception of finite number of singular points) in the whole complex \(\lambda\)-plane and can explicitly be constructed using only the solution of the regular RHP.

In order to obtain \(N\)-soliton solutions one has to apply the dressing procedure to the trivial solution of the RHP \(\xi_0(x,t,\lambda) = 1\). We choose a dressing factor with \(2N\)-poles \cite{11}:
\[
u(x,t,\lambda) = 1 + \sum_{k=1}^N \left( \frac{A_k(x,t)}{\lambda - \lambda_k^+} + \frac{B_k(x,t)}{\lambda - \lambda_k^-} \right).
\]
The \(N\)-soliton solution itself can be generated via the following formula
\[
Q_{N,s}(x,t) = \sum_{k=1}^N [J,A_k(x,t) + B_k(x,t)].
\]
The dressing factor \(u(x,\lambda)\) must satisfy the equation
\[
it \frac{\partial u}{\partial x} + Q_{N,s}(x,t)u(x,t,\lambda) - \lambda [J, u(x,t,\lambda)] = 0
\]
and the normalization condition \(\lim_{\lambda \to \infty} u(x,\lambda) = 1\).

The residues of \(u(x,\lambda)\) admit the following decomposition
\[
A_k(x,t) = X_k(x,t)F^T_k(x,t), \quad B_k(x,t) = Y_k(x,t)G^T_k(x,t),
\]
where all matrices involved are supposed to be rectangular and of maximal rank \(s\) \cite{30,9}. By comparing the coefficients before the same powers of \(\lambda - \lambda_k^\pm\) in (31) we convince ourselves that the factors \(F_k\) and \(G_k\) can be expressed by the fundamental analytic solutions \(\chi^\pm_0(x,t,\lambda) = e^{-\lambda(x+\lambda t)}J\) as follows
\[
F^T_k(x,t) = F^T_{k,0}[\chi^+_0(x,t,\lambda_k^+)]^{-1}, \quad G^T_k(x,t) = G^T_{k,0}[\chi^-_0(x,t,\lambda_k^-)]^{-1}.
\]
The constant rectangular matrices $F_{k,0}$ and $G_{k,0}$ obey the algebraic relations

$$F_{k,0}^{T}S_{0}F_{k,0} = 0, \quad G_{k,0}^{T}S_{0}G_{k,0} = 0.$$  

The other two types of factors $X_{k}(x,t)$ and $Y_{k}(x,t)$ are solutions to the algebraic system

$$S_{0}F_{k} = X_{k}\alpha_{k} + \sum_{l \neq k} \frac{X_{l}F_{l}^{T}S_{0}F_{k}}{\lambda_{l}^{+} - \lambda_{k}^{+}} + \sum_{l} \frac{Y_{l}G_{l}^{T}S_{0}F_{k}}{\lambda_{l}^{-} - \lambda_{k}^{-}},$$

$$S_{0}G_{k} = \sum_{l} \frac{X_{l}F_{l}^{T}S_{0}G_{k}}{\lambda_{l}^{+} - \lambda_{k}^{+}} + Y_{k}\beta_{k} + \sum_{l \neq k} \frac{Y_{l}G_{l}^{T}S_{0}G_{k}}{\lambda_{l}^{-} - \lambda_{k}^{-}}.$$  

The square $s \times s$ matrices $\alpha_{k}(x,t)$ and $\beta_{k}(x,t)$ introduced above depend on $\chi_{0}^{+}$ and $\chi_{0}^{-}$ and their derivatives by $\lambda$ as follows

$$\alpha_{k}(x,t) = -F_{0,k}^{T}[\chi_{0}^{+}(x,t,\lambda_{k}^{+})]^{-1}\partial_{\lambda} \chi_{0}^{+}(x,t,\lambda_{k}^{+})S_{0}F_{0,k} + \alpha_{0,k},$$

$$\beta_{k}(x,t) = -G_{0,k}^{T}[\chi_{0}^{-}(x,t,\lambda_{k}^{-})]^{-1}\partial_{\lambda} \chi_{0}^{-}(x,t,\lambda_{k}^{-})S_{0}G_{0,k} + \beta_{0,k}.$$  

Below for simplicity we will choose $F_{k}$ and $G_{k}$ to be $2r + 1$-component vectors. Then one can show that $\alpha_{k} = \beta_{k} = 0$ which simplifies the system (32). We also introduce the following more convenient parametrization for $F_{k}$ and $G_{k}$, namely (see eq. (35)):

$$F_{k}(x,t) = S_{0}|n_{k}(x,t)\rangle = \begin{pmatrix} e^{-z_{k}t + i\phi_{k}} \\ -\sqrt{2}s_{0}\tilde{v}_{0,k} \\ e^{z_{k}t - i\phi_{k}} \end{pmatrix},$$

$$G_{k}(x,t) = |n_{k}^{*}(x,t)\rangle = \begin{pmatrix} e^{z_{k}t + i\phi_{k}} \\ \sqrt{2}s_{k}^{*}\tilde{v}_{0,k} \\ e^{-z_{k}t - i\phi_{k}} \end{pmatrix},$$

where $\tilde{v}_{0,k}$ are constant $2r - 1$-component polarization vectors and

$$z_{j} = \nu_{j}(x + 2\mu_{j}t + \xi_{0,j}), \quad \phi_{j} = \mu_{j}x + (\mu_{j}^{2} - \nu_{j}^{2})t + \delta_{0,j},$$

$$\langle n_{j}^{T}(x,t)|S_{0}|n_{j}(x,t)\rangle = 0, \quad \text{or} \quad (\tilde{v}_{0,j}s_{0}\tilde{v}_{0,j}) = 1.$$  

With this notations the polarization vectors automatically satisfy the condition $\langle n_{j}(x,t)|S_{0}|n_{j}(x,t)\rangle = 0$. Thus for $N = 1$ we get the system:

$$|Y_{1}\rangle = \frac{(\lambda_{1}^{+} - \lambda_{1}^{-})|n_{1}\rangle}{\langle n_{1}^{T}|n_{1}\rangle}, \quad |X_{1}\rangle = \frac{(\lambda_{1}^{+} - \lambda_{1}^{-})S_{0}|n_{1}\rangle}{\langle n_{1}^{T}|n_{1}\rangle},$$

which is easily solved. As a result for the one-soliton solution we get:

$$\tilde{q}_{1s} = -i\sqrt{2}(\lambda_{1}^{+} - \lambda_{1}^{-})e^{-i\phi_{1}}\frac{e^{-z_{1}s_{0}|\tilde{v}_{01}\rangle + e^{z_{1}}|\tilde{v}_{01}\rangle}}{\Delta_{1}} \quad \text{or} \quad \Delta_{1} = \cosh(2z_{1}) + |\tilde{v}_{01}|^{2}.$$  

(37)
For $n = 3$ we put $\nu_{0k} = |\nu_{0k}|e^{\alpha_{0k}}$ and get:

$$
\Phi_{1s;\pm 1} = -\frac{\sqrt{2} |\nu_{01;1}| |\nu_{01;3}| (\lambda^+_1 - \lambda^-_1)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{13}} \times (\cosh(z_1 \mp \zeta_{01}) \cos(\alpha_{13}) - i \sinh(z_1 \pm \zeta_{01}) \sin(\alpha_{13})) ,
$$

$$
\Phi_{1s;0} = -\frac{\sqrt{2} |\nu_{01;2}| (\lambda^+_1 - \lambda^-_1)}{\Delta_1} e^{-i\phi_1} (\sinh(z_1 \cos(\alpha_{02}) + i \cosh(z_1 \sin(\alpha_{02}))) ,
$$

$$
\beta_{13} = \frac{1}{2}(\alpha_{03} - \alpha_{01}), \quad \zeta_{01} = \frac{1}{2} \ln \frac{|\nu_{01;3}|}{|\nu_{01;1}|}, \quad \alpha_{13} = \frac{1}{2}(\alpha_{03} + \alpha_{01}).
$$

Note that the ‘center of mass’ of $\Phi_{1s;1}$ (resp. of $\Phi_{1s;-1}$) is shifted with respect to the one of $\Phi_{1s;0}$ by $\zeta_{01}$ to the right (resp to the left); besides $|\Phi_{1s;1}| = |\Phi_{1s;-1}|$, i.e. they have the same amplitudes.

For $n = 5$ we put $\nu_{0k} = |\nu_{0k}|e^{\alpha_{0k}}$ and get analogously:

$$
\Phi_{1s;\pm 2} = -\frac{\sqrt{2} |\nu_{01;1}| |\nu_{01;5}| (\lambda^+_1 - \lambda^-_1)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{15}} \times (\cosh(z_1 \mp \zeta_{01}) \cos(\alpha_{15}) - i \sinh(z_1 \pm \zeta_{01}) \sin(\alpha_{15})) ,
$$

$$
\Phi_{1s;\pm 1} = \frac{\sqrt{2} |\nu_{01;2}| |\nu_{01;4}| (\lambda^+_1 - \lambda^-_1)}{\Delta_1} e^{-i\phi_1 \pm i\beta_{24}} \times (\cosh(z_1 \mp \zeta_{02}) \cos(\alpha_{24}) - i \sinh(z_1 \pm \zeta_{02}) \sin(\alpha_{24})) ,
$$

$$
\Phi_{1s;0} = -\frac{\sqrt{2} |\nu_{01;3}| (\lambda^+_1 - \lambda^-_1)}{\Delta_1} e^{-i\phi_1} (\cosh(z_1 \cos(\alpha_{03}) - i \sinh(z_1 \sin(\alpha_{03}))) ,
$$

$$
\beta_{15} = \frac{1}{2}(\alpha_{05} - \alpha_{01}), \quad \zeta_{01} = \frac{1}{2} \ln \frac{|\nu_{01;5}|}{|\nu_{01;1}|}, \quad \alpha_{15} = \frac{1}{2}(\alpha_{05} + \alpha_{01}),
$$

$$
\beta_{24} = \frac{1}{2}(\alpha_{04} - \alpha_{02}), \quad \zeta_{02} = \frac{1}{2} \ln \frac{|\nu_{01;4}|}{|\nu_{01;2}|}, \quad \alpha_{24} = \frac{1}{2}(\alpha_{04} + \alpha_{02}).
$$

Similarly the ‘center of mass’ of $\Phi_{1s;2}$ and $\Phi_{1s;1}$ (resp. of $\Phi_{1s;-2}$ and $\Phi_{1s;-1}$) are shifted with respect to the one of $\Phi_{1s;0}$ by $\zeta_{01}$ and $\zeta_{02}$ to the right (resp to the left); besides $|\Phi_{1s;2}| = |\Phi_{1s;-2}|$ and $|\Phi_{1s;1}| = |\Phi_{1s;-1}|$.

For $N = 2$ we get:

$$
|n_1(x,t)| = \frac{X_2(x,t)f_{21}}{\lambda^+_2 - \lambda^-_1} + \frac{Y_1(x,t)\kappa_{11}}{\lambda_1 - \lambda^-_1} + \frac{Y_2(x,t)\kappa_{21}}{\lambda^+_2 - \lambda^-_1},
$$

$$
|n_2(x,t)| = \frac{X_1(x,t)f_{12}}{\lambda^+_1 - \lambda^-_2} + \frac{Y_1(x,t)\kappa_{12}}{\lambda^-_1 - \lambda^+_2} + \frac{Y_2(x,t)\kappa_{22}}{\lambda^+_1 - \lambda^-_2},
$$

$$
S_0|n_1^*(x,t)| = \frac{X_1(x,t)\kappa_{11}}{\lambda^+_2 - \lambda^-_1} + \frac{X_2(x,t)\kappa_{11}}{\lambda^-_2 - \lambda^-_1} + \frac{Y_2(x,t)f_{21}}{\lambda^+_2 - \lambda^-_1},
$$

$$
S_0|n_2^*(x,t)| = \frac{X_1(x,t)\kappa_{21}}{\lambda^-_1 - \lambda^-_2} + \frac{X_2(x,t)\kappa_{22}}{\lambda^-_2 - \lambda^-_1} + \frac{Y_1(x,t)f_{12}}{\lambda^-_1 - \lambda^-_2},
$$

where

$$
\kappa_{kj}(x,t) = e^{z_k + z_j + i(\phi_k - \phi_j)} + e^{-z_k - z_j - i(\phi_k - \phi_j)} + 2 (\bar{\nu}_{0k} \sigma_{0j}) ,
$$

$$
f_{kj}(x,t) = e^{z_k - z_j - i(\phi_k - \phi_j)} + e^{z_j - z_k + i(\phi_k - \phi_j)} - 2 (\bar{\nu}_{0k}^* \sigma_{0j}),
$$
In other words:

\[
M \hat{x} = \begin{pmatrix}
0 & \frac{f_{12}}{\lambda_1 - \lambda_2} & \frac{\kappa_{11}}{\lambda_1 - \lambda_2} & \frac{\kappa_{12}}{\lambda_1 - \lambda_2} \\
\frac{f_{12}}{\lambda_1 - \lambda_2} & 0 & \frac{\kappa_{12}}{\lambda_2 - \lambda_1} & \frac{\kappa_{22}}{\lambda_2 - \lambda_1} \\
\frac{\kappa_{11}}{\lambda_1 - \lambda_2} & \frac{\kappa_{12}}{\lambda_2 - \lambda_1} & 0 & \frac{f_{12}}{\lambda_2 - \lambda_1} \\
\frac{\kappa_{21}}{\lambda_1 - \lambda_2} & \frac{\kappa_{22}}{\lambda_2 - \lambda_1} & \frac{f_{12}}{\lambda_2 - \lambda_1} & 0
\end{pmatrix}
\begin{pmatrix}
X_1 \\
X_2 \\
Y_1 \\
Y_2
\end{pmatrix} = \begin{pmatrix}
|n_1\rangle \\
|n_2\rangle \\
S_0|n_1\rangle \\
S_0|n_2\rangle
\end{pmatrix}.
\]

(42)

We can rewrite \(M\) in block-matrix form:

\[
M = \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}, \quad M_{22} = M_{11}^*, \quad M_{21} = -M_{12}^T,
\]

\[
M_{11} = \frac{f_{12}}{\lambda_1 - \lambda_2} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, \quad M_{12} = \begin{pmatrix}
\frac{\kappa_{11}}{\lambda_1 - \lambda_2} & \frac{\kappa_{12}}{\lambda_2 - \lambda_1} \\
\frac{\kappa_{21}}{\lambda_1 - \lambda_2} & \frac{\kappa_{22}}{\lambda_2 - \lambda_1}
\end{pmatrix}.
\]

(43)

The inverse of \(M\) is given by:

\[
M^{-1} = \begin{pmatrix}
N_{11}^{-1} & -N_{11}^{-1} M_{12} M_{11}^* \\
-N_{22}^{-1} M_{21} M_{11}^* & N_{22}^{-1}
\end{pmatrix},
\]

\[
N_1 = M_{11} - M_{12} M_{11}^* M_{21}, \quad N_2 = M_{11} - M_{21} M_{11}^* M_{12},
\]

(44)

From eqs. (42) and (44) we obtain [12]:

\[
|X_1\rangle = \frac{1}{Z} \left( \frac{f_{12}^*}{\lambda_1 - \lambda_2} |n_2\rangle - \frac{\kappa_{22}}{\lambda_2 - \lambda_1} S_0|n_1\rangle + \frac{\kappa_{12}}{\lambda_1 - \lambda_2} S_0|n_2\rangle \right),
\]

\[
|X_2\rangle = \frac{1}{Z} \left( -\frac{f_{12}}{\lambda_1 - \lambda_2} |n_1\rangle + \frac{\kappa_{21}}{\lambda_1 - \lambda_2} S_0|n_1\rangle - \frac{\kappa_{11}}{\lambda_1 - \lambda_2} S_0|n_2\rangle \right),
\]

\[
|Y_1\rangle = \frac{1}{Z} \left( \frac{\kappa_{22}}{\lambda_2 - \lambda_1} |n_1\rangle - \frac{\kappa_{21}}{\lambda_1 - \lambda_1} |n_2\rangle - \frac{f_{12}}{\lambda_1 - \lambda_2} S_0|n_2\rangle \right),
\]

\[
|Y_2\rangle = \frac{1}{Z} \left( -\frac{\kappa_{12}}{\lambda_2 - \lambda_1} |n_1\rangle + \frac{\kappa_{11}}{\lambda_1 - \lambda_2} |n_2\rangle + \frac{f_{12}}{\lambda_1 - \lambda_2} S_0|n_1\rangle \right),
\]

(45)

where

\[
Z = \left( \frac{|f_{12}|^2}{|\lambda_2 - \lambda_1|^2} + \frac{\kappa_{12}\kappa_{21}}{4\nu_1\nu_2} \right).
\]

(46)

Inserting this result into eq. (30) we obtain the following expression for the 2-soliton solution of the MNLS:

\[
Q_{2s}(x, t) = [J, A_1 + B_1 + A_2 + B_2] = \frac{1}{Z} [J, C(x, t) - S_0 C^T(x, t) S_0],
\]

\[
C(x, t) = \frac{\kappa_{22}}{\lambda_2 - \lambda_1} |n_1\rangle \langle n_1^*| - \frac{\kappa_{12}}{\lambda_1 - \lambda_2} |n_1\rangle \langle n_2| - \frac{\kappa_{21}}{\lambda_1 - \lambda_2} |n_2\rangle \langle n_1| + \frac{\kappa_{11}}{\lambda_1 - \lambda_1} |n_2\rangle \langle n_2^*| - \frac{f_{12}^*}{\lambda_1 - \lambda_2} |n_1\rangle \langle n_2| S_0 - \frac{f_{12}}{\lambda_1 - \lambda_2} S_0 |n_2^*\rangle \langle n_1|.
\]

(47)

4. The minimal sets of scattering data. It is well known that the locations of the singularities of the RHP \(\lambda_k^\pm \in \mathbb{C}_+\) are zeroes of the functions \(m_k^\pm(\lambda)\) and discrete eigenvalues of the Lax operator \(L\). We will say that these eigenvalues are simple if the corresponding eigensubspaces are one dimensional. This corresponds to our choice of \(F_k(x, t)\) and \(G_k(x, t)\) as vectors. Eigenspaces of higher multiplicities \(s > 1\) can be obtained choosing \(F_k(x, t)\) and \(G_k(x, t)\) as \(s \times (2r + 1)\) matrices of rank \(s\).
Theorem 4.1. Let the potential $Q(x, t)$ be such that the corresponding Lax operator $L$ has finite number of simple discrete eigenvalues located at the points $λ_k^± ∈ \mathbb{C}_±$ respectively, $k = 1, \ldots, N$. Then as minimal sets of scattering data uniquely determine both the scattering matrix $T(λ, t)$ and the corresponding potential $Q(x, t)$ one can consider the sets

$$\mathcal{T}_1 = \{ 𝜏^+(λ, 0), \ λ ∈ \mathbb{R} \}; \quad \mathcal{T}_2 = \{ 𝜏_k^+, λ_k^± ∈ \mathbb{C}_+, \ k = 1, \ldots, N \}, \quad \text{where the constant vectors } 𝜏_k^+ \text{ and } 𝜌_k^± \text{ determine the corresponding eigenfunction of } L.$$  

where the constant vectors $𝜏_k^+$ and $𝜌_k^±$ satisfy the normalization condition $(𝜏_k^+, T_{0k}) = 0$ and $( JT_{0k}, 𝜌_k^±) = 0$.

Remark 4. The data $λ_k^+$ and $λ_k^- = (λ_k^+)^*$ characterize the discrete eigenvalues of $L$. The vectors $𝜏_k^+$ and $𝜏_k^-$ (resp. $𝜌_k^+$ and $𝜌_k^-$) determine the corresponding eigenfunction of $L$. Note also that by definition these vectors satisfy $( JT_{0k}, 𝜏_k^+) = 0$ and $( JT_{0k}, 𝜌_k^±) = 0$.

Outline of the proof. Let us be given $T_0$. Using $𝜏^+(λ, t)$ and $𝜏^-(λ, t) = (𝜏^+(λ, t))^*$ we construct $S_{0IJ}^Q(λ, t)$ and $S_{0J}^Q(λ, t)$ and therefore obtain also the sewing function $G_0(λ, t) = S_{0J}^1(λ, t)S_{0J}^3(λ, t)$ for a regular RHP. According to the Zakharov-Shabat theorem it has unique solution $ξ_0^±(x, t, λ)$. The corresponding regular potential is obtained by:

$$Q_0(x, t) = \lim_{λ → ∞} \lambda \left( J - ξ_0^±(x, t, λ)Jξ_0^±(x, t, λ) \right) = [J, ξ_0^±(x, t)],$$  

where $ξ_0^+(x, t) = \lim_{λ → ∞} λ(ξ_0^+(x, t, λ) - 1)$.

Next we use the dressing method to dress the regular solution $ξ_0^±(x, t, λ)$ with the dressing factor $u(x, t, λ)$ of the form (29). In order to do it we make use of the set of eigenvalues $λ_k^+$ and $λ_k^- = (λ_k^+)^*$ and instead of the polarization vectors (34) we use:

$$|n_k(x, t)| = ξ_0^+(x, t, λ_k^+e^{-iλ_k^+(x+λ_k^+t)J}τ_0^+, \quad |n_k^-(x, t)| = ξ_0^-(x, t, λ_k^-e^{-iλ_k^-(x+λ_k^-t)J}τ_0^-.$$

After solving the algebraic equations for $|X_k(x, t)|$ and $|Y_k(x, t)|$ we find explicitly the dressed potential

$$Q(x, t) = Q_0(x, t) + \sum_{k=1}^N [J, A_k(x, t) + B_k(x, t)],$$

which proves the first part of the theorem.

Let us now show how one can recover $T(λ, t)$ from $T_0$. Given the regular solution $ξ_0^±(x, t, λ)$ we can find

$$D_{0J}^±(λ) = \lim_{x → ∞} ξ_0^±(x, t, λ),$$

and also

$$T_{0J}^±(λ)D_{0J}^±(λ) = \lim_{x → ∞} e^{i(λx+λ^2t)J}ξ_0^±(x, t, λ)e^{-i(λx+λ^2t)J}.$$
Thus we have recovered all Gauss factors $T_{0,j}^\pm (\lambda)$, $D_{0,j}^\pm (\lambda)$ and $S_{0,j}^\pm (\lambda)$ of the `undressed’ scattering matrix $T_0(\lambda, t)$, so

$$T_{0,j}(\lambda, t) = T_{0,j}^\pm (\lambda, t) D_{0,j}^\pm (\lambda) S_{0,j}^\pm (\lambda, t).$$  \hfill (55)

In order to take into account the effect of dressing we make use of the relations between the dressed and undressed Jost solutions:

$$\psi(x, t, \lambda) = u(x, t, \lambda)\psi_0(x, t, \lambda)\tilde{u}_+(\lambda),$$
$$\phi(x, t, \lambda) = u(x, t, \lambda)\phi_0(x, t, \lambda)\tilde{u}_-(\lambda),$$  \hfill (56)

where $u_\pm(\lambda) = \lim_{x \to \pm \infty} u(x, t, \lambda)$. As a result we get:

$$T(\lambda, t) = \psi(x, t, \lambda)\phi(x, t, \lambda)$$
$$= u_+(\lambda)\psi_0(x, t, \lambda)\phi_0(x, t, \lambda)\tilde{u}_-(\lambda)$$
$$= u_+(\lambda)T_0(\lambda, t)\tilde{u}_-(\lambda).$$

Skipping the details we state the result:

$$u_+(\lambda) = \begin{pmatrix} c(\lambda) & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & 1/c(\lambda) \end{pmatrix}, \quad u_-(\lambda) = \begin{pmatrix} 1/c(\lambda) & 0 & 0 \\ 0 & \mathbb{I} & 0 \\ 0 & 0 & c(\lambda) \end{pmatrix},$$  \hfill (58)

where $c(\lambda) = \prod_{j=1}^N \frac{\lambda - \lambda_j^+}{\lambda - \lambda_j^-}$.

The fact that the set $\mathcal{T}_2$ is also a minimal set of scattering data is proved analogously.

5. Reductions of MNLS. Along with the typical reduction $Q = Q^\dagger$ mentioned above one can impose additional reductions using the reduction group proposed by Mikhailov [21]. They are automatically compatible with the Lax representation of the corresponding MNLS eq. Below we make use of two types of $\mathbb{Z}_2$-reductions[8]:

1) $C_1 U^\dagger(x, t, \lambda^*)C_1^{-1} = U(x, t, \lambda), \quad C_1 V^\dagger(x, t, \lambda^*)C_1^{-1} = V(x, t, \lambda),$
2) $C_2 U^T(x, t, \lambda)C_2^{-1} = -U(x, t, \lambda), \quad C_2 V^T(x, t, \lambda)C_2^{-1} = -V(x, t, \lambda),$  \hfill (59)

where $C_1$ and $C_2$ are involutions of the Lie algebra $so(2r+1)$, i.e. $C_2^2 = \mathbb{I}$. They can be chosen to be either diagonal (i.e., elements of the Cartan subgroup of $SO(2r+1)$) or elements of the Weyl group.

The typical reductions of the MNLS eqs. is a class 1) reduction obtained by specifying $C_1$ to be the identity automorphism of $\mathfrak{g}$; below we list several choices for $C_1$ leading to inequivalent reductions:

1a) $C_1 = \mathbb{I}, \quad \bar{p}(x) = \bar{q}^*(x), \quad 1b) \quad C_1 = K_1, \quad \bar{p}(x) = K_{01}\bar{q}^*(x),$ 
1c) $C_1 = S_{e_2}, \quad \bar{p}(x) = K_{02}\bar{q}^*(x), \quad 1d) \quad C_1 = S_{e_2}S_{e_3}, \quad \bar{p}(x) = K_{03}\bar{q}^*(x).$

We also make use of type 2) reductions:

2e) $C_2 = K_4, \quad \bar{q}(x) = -K_{04}s_0\bar{q}(x), \quad \bar{p}(x) = -K_{04}s_0\bar{p}(x),$ 
2f) $C_2 = K_5, \quad \bar{q}(x) = K_{05}\bar{q}(x), \quad \bar{p}(x) = K_{05}\bar{p}(x),$  \hfill (61)

where

$$K_j = \text{block-diag}(1, K_{0j}, 1), \quad K_{01} = \text{diag}(\epsilon_1, \ldots, \epsilon_{r-1}, 1, \epsilon_{r-1}, \ldots, \epsilon_1).$$  \hfill (62)
for $j = 1, 2, 3, 5$ and $e_j = \pm 1$. The matrices $K_{02}$, $K_{03}$ and $K_4$ are not diagonal and may take the form:

$$K_{02} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad K_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & K_{04} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$K_{02} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \quad (63)$$

Each of the above reductions impose constraints on the FAS, on the scattering matrix $T(\lambda)$ and on its Gauss factors $S_j^\pm(\lambda)$, $T_j^\pm(\lambda)$ and $D_j^\pm(\lambda)$. For the type 1 reductions (cases 1a) – 1d)) these have the form:

$$(S^+(\lambda^*))^\dagger = K_j^{-1} \hat{S}^-(\lambda) K_j \quad (T^+(\lambda^*))^\dagger = K_j^{-1} \hat{T}^-(\lambda) K_j \quad (D^+(\lambda^*))^\dagger = K_j^{-1} \hat{D}^-(\lambda) K_j$$

where the matrices $K_j$ are specific for each choice of the automorphisms $C_1$, see eqs. (60). In particular, from the last line of (64) and (61) we get:

$$(m_1^+(\lambda^*))^* = m_1^-(\lambda),$$

and consequently, if $m_1^+(\lambda)$ has zeroes at the points $\lambda^+_k$, then $m_1^-(\lambda)$ has zeroes at:

$$\lambda^-_k = (\lambda^+_k)^*, \quad k = 1, \ldots, N. \quad (66)$$

For the type 2) reductions we obtain:

$$\begin{cases} 
2e) & (S^+(\lambda))^T = K_4^{-1} \hat{S}^+(\lambda) K_4 \quad (T^+(\lambda))^T = K_4^{-1} \hat{T}^+(\lambda) K_4 \\
 & (D^+(\lambda))^T = K_4^{-1} \hat{D}^+(\lambda) K_4 \\
 & \tilde{\sigma}^\pm = - K_{0450} \tilde{\tau}^\pm, \quad \tilde{\rho}^\pm = - K_{0450} \tilde{\rho}^\pm,
\end{cases} \quad (67)$$

and

$$\begin{cases} 
2f) & (S^+(\lambda))^T = K_5^{-1} \hat{S}^-(\lambda) K_5 \quad (T^+(\lambda))^T = K_5^{-1} \hat{T}^-(\lambda) K_5 \\
 & (D^+(\lambda))^T = K_5^{-1} \hat{D}^-(\lambda) K_5 \\
 & \tilde{\tau}^+(\lambda) = K_{05} \tilde{\tau}^-(\lambda), \quad \tilde{\rho}^\pm(\lambda) = - K_{05} \tilde{\rho}^\pm(\lambda),
\end{cases} \quad (68)$$

For the 2e) reduction with $n = 3$ we may choose $K_4$ to corresponds to the Weyl group element $S_{e_1}$, so $K_{04} = \mathbb{1}$. As a result we get:

$$\Phi_1 = - \Phi_{-1} \quad (69)$$

and $\Phi_0$ arbitrary. This reduction of eq. (1) is also important for the BEC [22]. From (67) we find $\nu_{01} = \nu_{03}$. The effect of this constraint is that for the one-soliton solution we get $\Phi_{1e_1} = - \Phi_{1e_1}$. Our next remark following [23] is that this reduction applied to the $F = 1$ MNLS (1) leads to a 2-component MNLS which after the change of variables

$$\Phi_1 = \frac{1}{2}(w_1 + iw_2), \quad \Phi_0 = \frac{i}{\sqrt{2}}(w_1 - iw_2), \quad (70)$$

leads to two disjoint NLS equations for $w_1$ and $w_2$ respectively.
It is only logical that applying the constraint $\nu_{01} = \nu_{03}$ the explicit expression for the one-soliton solution (38) simplifies and reduces to the standard soliton solutions of the scalar NLS.

For the other two examples of type 2) reductions we choose $n = 5$ and $K_4$, and $K_5$ correspond to the Weyl group elements $S_{e_2}S_{e_3}$ and $S_{e_2-e_3}$ respectively. Then $K_{04} = -s_0$ and

$$K_{05} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad (71)$$

For these choices of $K_4, K_5$ we obtain:

2c) $\Phi_2 = \Phi_{-2}, \quad \Phi_1 = \Phi_{-1},$

2f) $\Phi_{\pm 2} = \pm \frac{c}{\sqrt{1 + c^2}} \Phi'_{\pm 1}, \quad \Phi_{\pm 1} = \frac{1}{\sqrt{1 + c^2}} \Phi'_{\pm 1}, \quad (72)$

It reduces the $F = 2$ spin BEC model into the $F = 1$ model.

The corresponding relations for the Gauss factors and for the polarization vectors are given by:

$$\Phi_{\pm 2} = \pm \frac{c}{\sqrt{1 + c^2}} \Phi'_{\pm 1}, \quad \Phi_{\pm 1} = \frac{1}{\sqrt{1 + c^2}} \Phi'_{\pm 1}, \quad (73)$$

6. Two Soliton interactions. In this section we generalize the classical results of Zakharov and Shabat about soliton interactions [31] to the class of MNLS equations related to BD.I symmetric spaces. For detailed exposition see the monographs [29, 3]. These results were generalized for the vector nonlinear Schrödinger equation by Manakov [20], see also [1, 16, 28]. The Zakharov Shabat approach consisted in calculating the asymptotics of generic $N$-soliton solution of NLS for $t \to \pm \infty$ and establishing the pure elastic character of the generic soliton interactions. By generic here we mean $N$-soliton solution whose parameters $\lambda_k^\pm = \mu_k \pm i\nu_k$ are such that $\mu_k \neq \mu_j$ for $k \neq j$. The pure elastic character of the soliton interactions is demonstrated by the fact that for $t \to \pm \infty$ the generic $N$-soliton solution splits into sum of $N$ one soliton solutions each preserving its amplitude $2\nu_k$ and velocity $\mu_k$. The only effect of the interaction consists in shifting the center of mass and the initial phase of the solitons. These shifts can be expressed in terms of $\lambda_k^\pm$ only; for detailed exposition see [3].

Let us apply these ideas to the MNLS equations studied above. Namely we use the 2-soliton solution (47) derived above and calculate its asymptotics along the trajectory of the first soliton. To this end we keep $z_1(x,t)$ fixed and let $\tau = z_2 - z_1$ tend to $\pm \infty$. Therefore it will be enough to insert the asymptotic values of the matrix elements of $M$ for $\tau \to \pm \infty$ and keep only the leading terms. For $\tau \to \infty$ that gives:

$$\kappa_{22} \sim e^{2\tau} \exp(\nu_2 z_1/\nu_1) + 2C_1,$$
$$\kappa_{12} = e^{\tau} \exp((1 + \nu_2/\nu_1) z_1 + i(\phi_1 - \phi_2)) + O(1),$$
$$\kappa_{21} = e^{\tau} \exp((1 + \nu_2/\nu_1) z_1 - i(\phi_1 - \phi_2)) + O(1),$$
$$f_{12} = e^{\tau} \exp(-(1 - \nu_2/\nu_1) z_1 + i(\phi_1 - \phi_2)) + O(1), \quad (74)$$
The next consequence is that both Gauss factors initial data will satisfy the corresponding reduction 2e) \((75)\). The above MNLS eqs. is purely elastic. The solitons preserve their shapes and phase. From this point of view the interaction is the same like for the scalar NLS eq.

Let the minimal set of scattering data \((76)\), say \(T\), with the same sewing function and the same canonical normalization. Therefore we get:

\[
\lim_{\tau \to -\infty} \hat{q}_{2s}(x,t) = -\frac{i\sqrt{2}v_1 e^{-i(\phi_1-\alpha_+)} (e^{-z_1+r_+} s_0 |\vec{v}_{01}| + e^{z_1+r_+} |\vec{v}_{01}^*|)}{\cosh(2(z_1 + r_+)) + (\vec{v}_{01}^* , \vec{v}_{01})},
\]

\[
\lim_{\tau \to -\infty} \hat{q}_{2s}(x,t) = \frac{i\sqrt{2}v_1 e^{-i(\phi_1+\alpha_+)} (e^{-z_1+r_+} s_0 |\vec{v}_{01}| + e^{z_1-r_+} |\vec{v}_{01}^*|)}{\cosh(2(z_1 - r_+)) + (\vec{v}_{01}^* , \vec{v}_{01})}
\]

where

\[
\alpha_+ = \arg \frac{\lambda_+^1 - \lambda_2^+}{\lambda_1^1 - \lambda_2^+}.
\]

For \(n = 3\) and \(n = 5\) the right hand sides of \((76)\) coincide with the one-soliton solutions \((38)\) and \((39)\) respectively. This means that the 2-soliton interaction for the above MNLS eqs. is purely elastic. The solitons preserve their shapes and velocities and the only effect of the interaction consist in shifts of the center of mass and the phase. From this point of view the interaction is the same like for the scalar NLS eq.

It is important to check whether the \(N\)-soliton interactions consist of sequence of elementary 2-soliton interactions and the shifts are additive.

7. Effects of reductions and initial conditions on MNLS.

**Theorem 7.1.** Let the minimal set of scattering data \(T_j, j = 1,2\) for \(t = 0\) satisfy the reduction conditions \((67)\). Then the solution \(q(x,t)\) of the MNLS with such initial data will satisfy the corresponding reduction 2e) \((61)\).

**Proof.** Let the minimal sets of scattering data, say \(T_1\) satisfy the reduction conditions \((67)\) for \(t = 0\). It is easy to check that their evolution law \((15)\) is compatible with the reduction, so \((67)\) will hold for all \(t > 0\). As a result the corresponding Gauss factors \(S^\pm, T^\pm\) and \(D^\pm\), and consequently, the sewing function in the RHP \(G(x,t,\lambda)\) will satisfy

\[
G(x,t,\lambda) = K^{-1}_4 G^T(x,t,\lambda) K_4.
\]

The next consequence is that both \(\xi^\pm\) and \(K_4^{-1} \xi^{\pm,T} K_4\) are solutions of the RHP \((23)\) with the same sewing function and the same canonical normalization. Therefore from the uniqueness of the solution of RHP we get that the regular solutions of this RHP satisfy:

\[
\xi_0^{\pm}(x,t,\lambda) = K_4^{-1} \xi_0^{\pm,T} (x,t,\lambda) K_4.
\]

Next we note that the scattering data related to the discrete spectrum also satisfy the reduction conditions. This means that the dressing factor \(u(x,t,\lambda)\) and the singular solutions \(\xi^\pm(x,t,\lambda) = u(x,t,\lambda) \xi_0^{\pm}(x,t,\lambda) \tilde{u}_-(\lambda)\) also satisfy:

\[
u(x,t,\lambda) = K_4^{-1} \tilde{u}^T(x,t,\lambda) K_4, \quad \xi^\pm(x,t,\lambda) = K_4^{-1} \xi^{\pm,T}(x,t,\lambda) K_4.
\]

It remains to check that from equations \((27)\) and \((79)\) there follows:

\[
Q(x,t) = -K_4^{-1} Q^T(x,t) K_4.
\]
Remark 5. Note that the above arguments are not specific for the choice of $K_4$. The above theorem can be proved along the same lines for any reduction of type 1 and type 2.

A simple consequence of the above theorem is the following. Consider $n = 3$ and choose $\tau_1^+ = \tau_3^+$ for $t = 0$. Then the corresponding solution of $F = 1$ BEC (1) will also satisfy $\Phi_1 = -\Phi_{-1}$ for all $t > 0$, i.e. will be a solution to

\begin{align*}
    i\partial_t \Phi_1 + \partial_x^2 \Phi_1 + 2(|\Phi_1|^2 + 2|\Phi_0|^2)\Phi_1 - 2\Phi_1^* \Phi_0 = 0, \\
    i\partial_t \Phi_0 + \partial_x^2 \Phi_0 + 2(2|\Phi_1|^2 + |\Phi_0|^2)\Phi_0 - 2\Phi_0^* \Phi_1^2 = 0,
\end{align*}

(81)

If we insert eq. (70) into (81) we obtain

\begin{align*}
    i\partial_t w_1 + \partial_x^2 w_1 + 2|w_1|^2 w_1 = 0, \\
    i\partial_t w_2 + \partial_x^2 w_2 + 2|w_2|^2 w_2 = 0,
\end{align*}

(82)

Therefore, if we want to analyze the specific features of $F = 1$ BEC we have to avoid such initial conditions.

Similarly, if for $n = 5$ we choose in (39) $\nu_{01,1} = \nu_{01,5}$, $\nu_{01,2} = -\nu_{01,4}$ we will obtain in fact a solution to $F = 1$ BEC.

8. Conclusions and discussion. Using the Zakharov-Shabat dressing method we have obtained the two-soliton solution and have used it to analyze the soliton interactions of the MNLS equation. The conclusion is that after the interactions the solitons recover their polarization vectors $\nu_0$, velocities and frequency velocities. The effect of the interaction is, like in for the scalar NLS equation, shift of the center of mass $z_1 \rightarrow z_1 + r_+ + \alpha_+$ and shift of the phase $\phi_1 \rightarrow \phi_1 + \alpha_+$. Both shifts are expressed through the related eigenvalues $\lambda^+_j$ only.

The next step would be to analyze multi-soliton interactions. Our hypothesis is that each soliton will acquire a total shift of the center of mass that is sum of all elementary shifts from each two soliton interactions. Similar result is expected for the total phase shift of the soliton.

Finally we have proved a theorem, stating that a symmetry imposed on the minimal set of scattering data leads to a symmetry of the corresponding solution. So if we want to analyze the specific features of a given MNLS we have to avoid such initial conditions.

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