On the convergence rate of the “out-of-order”
block Gibbs sampler

Zhumengmeng Jin and James P. Hobert
Department of Statistics
University of Florida
October 2021

Abstract

It is shown that a seemingly harmless reordering of the steps in a block Gibbs sampler can actually invalidate the algorithm. In particular, the Markov chain that is simulated by the “out-of-order” block Gibbs sampler does not have the correct invariant probability distribution. However, despite having the wrong invariant distribution, the Markov chain converges at the same rate as the original block Gibbs Markov chain. More specifically, it is shown that either both Markov chains are geometrically ergodic (with the same geometric rate of convergence), or neither one is. These results are important from a practical standpoint because the (invalid) out-of-order algorithm may be easier to analyze than the (valid) block Gibbs sampler (see, e.g., Yang and Rosenthal [2019]).

1 Introduction

Let \(\Pi(dx, dy, dz)\) be a joint probability distribution having support \(X \times Y \times Z\), and suppose we wish to construct a Gibbs sampler for this distribution. (Additional mathematical rigor will be introduced in Section 2.) The simplest version of the Gibbs sampler is that which cycles through the so-called full conditional distributions, \(\Pi_{X|Y,Z}(dx|y, z)\), \(\Pi_{Y|X,Z}(dy|x, z)\), and \(\Pi_{Z|X,Y}(dz|x, y)\). Let \(\Gamma = \{(X_n, Y_n, Z_n)\}_{n=0}^{\infty}\) denote the corresponding Markov chain. If the current state of the chain \(\Gamma\) is \((X_n, Y_n, Z_n) = (x, y, z)\), then we move to the next state, \((X_{n+1}, Y_{n+1}, Z_{n+1})\), using the following well-known three step procedure.

---

Key words and phrases. Geometric ergodicity, geometric rate of convergence, Markov chain, Markov chain Monte Carlo, mixing time, total variation distance
Iteration $n + 1$ of the Gibbs sampler:

1. Draw $X_{n+1} \sim \Pi_{X|YZ}(\cdot|y, z)$, and call the observed value $x'$.

2. Draw $Y_{n+1} \sim \Pi_{Y|XZ}(\cdot|x', z)$, and call the observed value $y'$.

3. Draw $Z_{n+1} \sim \Pi_{Z|XY}(\cdot|x', y')$.

Now suppose that we have the ability to draw from the distribution $\Pi_{X|Z}(dx|z)$. This leads to a sequential method of making draws from $\Pi_{X|Z}(dx|z)$, and then from $\Pi_{Y|XZ}(dy|x, z)$. Hence, we can consider an alternative algorithm, the so-called block Gibbs sampler, in which $X$ and $Y$ are sampled jointly given $Z$. It is well established that sampling a set of variables jointly in this way often leads to a more efficient algorithm. Denote the block Gibbs Markov chain by $\tilde{\Gamma} = \{(\tilde{X}_n, \tilde{Y}_n, \tilde{Z}_n)\}_{n=0}^{\infty}$. If the current state is $((\tilde{X}_n, \tilde{Y}_n, \tilde{Z}_n) = ((x, y), z)$, then we move to the next state $((\tilde{X}_{n+1}, \tilde{Y}_{n+1}, \tilde{Z}_{n+1})$ using the following three step procedure.

Iteration $n + 1$ of the block Gibbs sampler:

1. Draw $\tilde{X}_{n+1} \sim \Pi_{X|Z}(\cdot|z)$, and call the observed value $x'$.

2. Draw $\tilde{Y}_{n+1} \sim \Pi_{Y|XZ}(\cdot|x', z)$, and call the observed value $y'$.

3. Draw $\tilde{Z}_{n+1} \sim \Pi_{Z|XY}(\cdot|x', y')$.

The target distribution, $\Pi(dx, dy, dz)$, is the invariant probability distribution for both $\Gamma$ and $\tilde{\Gamma}$. (The Markov chains considered in this section are assumed to satisfy standard regularity conditions - spelled out in Section 2 - that imply, among other things, a unique invariant distribution.) Therefore, the Gibbs sampler and the block Gibbs sampler are both valid Markov chain Monte Carlo (MCMC) algorithms for exploring $\Pi(dx, dy, dz)$. There is, however, a very important difference between these two algorithms regarding the ordering of the steps. The three steps of the regular Gibbs sampler are interchangeable in the sense that they can be reordered without affecting the validity of the algorithm. In other words, instead of using the order $X \rightarrow Y \rightarrow Z$, any of the six possible orderings could be used, and all will result in a Markov chain with invariant distribution $\Pi(dx, dy, dz)$. This is not the case, however, with the block Gibbs sampler. Indeed, the first two steps of the block Gibbs sampler must be done in order because the same value of $Z$ must be conditioned on in both. So, we could actually move the third step to the top, and still have a valid algorithm, but any other change in the order of the steps will invalidate the algorithm. To be more specific, let
\( \Gamma^* = \{(Y_n^*, Z_n^*, X_n^*)\}_{n=0}^\infty \) denote the Markov chain associated with what we call the “out-of-order” block Gibbs sampler. If the current state is \((Y_n^*, Z_n^*, X_n^*) = (y, z, x)\), then we move to the next state \((Y_{n+1}^*, Z_{n+1}^*, X_{n+1}^*)\) using the following three step procedure.

### Iteration \( n + 1 \) of the out-of-order block Gibbs sampler:

1. Draw \( Y_{n+1}^* \sim \Pi_{Y|XZ}(\cdot|x, z) \), and call the observed value \( y^* \).
2. Draw \( Z_{n+1}^* \sim \Pi_{Z|XY}(\cdot|x, y^*) \), and call the observed value \( z^* \).
3. Draw \( X_{n+1}^* \sim \Pi_{X|Z}(\cdot|z^*) \).

At first glance, this reordering may seem perfectly harmless, but, in reality, it invalidates the algorithm. Indeed, \( \Gamma^* \) does not have the correct invariant distribution. Again, the reason is that the two steps in the block Gibbs sampler that together yield a draw from \( \Pi_{XY|Z}(dx, dy|z) \) have been separated, so we are no longer simulating from \( \Pi_{XY|Z}(dx, dy|z) \). To see that the invariant distribution has changed, first note that the Markov transition kernel (Mtk) of \( \Gamma^* \) is given by

\[
K^*((y, z, x), (dy', dz', dx')) = \Pi_{X|Z}(dx'|z') \Pi_{Z|XY}(dz'|x, y') \Pi_{Y|XZ}(dy'|x, z) .
\]

**Remark 1.** Note that \( K^*((y, z, x), (dy', dz', dx')) \) does not actually depend on \( y \). In the sequel, we drop such variables from the notation when it’s convenient.

Now defining \( \Pi^*(dx, dy, dz) = \Pi_{X|Z}(dx|z) \Pi_{Y|Z}(dy, dz) \), we have

\[
\int_X \int_Y \int_Z K^*((y, z, x), (dy', dz', dx')) \Pi^*(dx, dy, dz) \\
= \Pi_{X|Z}(dx'|z') \int_X \int_Y \int_Z \Pi_{Z|XY}(dz'|x, y') \Pi_{Y|XZ}(dy'|x, z) \Pi_{X|Z}(dz|z) \Pi_{Y|Z}(dy, dz) \\
= \Pi_{X|Z}(dx'|z') \int_X \int_Z \Pi_{Z|XY}(dz'|x, y') \Pi_{Y|XZ}(dy'|x, z) \Pi_{X|Z}(dz|z) \Pi_{Y|Z}(dy, dz) \\
= \Pi_{X|Z}(dx'|z') \Pi_{Y|Z}(dy', dz') \\
= \Pi^*(dx', dy', dz') .
\]

Hence, the invariant distribution of \( \Gamma^* \) is \( \Pi^*(dx, dy, dz) \), not the target distribution \( \Pi(dx, dy, dz) \).

A little thought reveals that basing an MCMC algorithm on \( \Gamma^* \) is effectively the same as simulating the correct chain, \( \tilde{\Gamma} = \{((\tilde{X}_n, \tilde{Y}_n), \tilde{Z}_n)\}_{n=0}^\infty \), but then basing inferences on the *shifted* version \( \{(\tilde{Y}_n, \tilde{Z}_n, \tilde{X}_{n+1})\}_{n=0}^\infty \). Since \( \Pi \) and \( \Pi^* \) share many of the same marginal and conditional distributions, some (MCMC) estimators based on \( \Gamma^* \) will be valid, but others will not. For example, since
(in general) \(\Pi^*_X(dX, dY) = \int_Z \Pi_{X|Z}(dX|z)\Pi_{Y|Z}(dy, dz) \neq \Pi_{XY}(dx, dy)\), the \(\Gamma^*\)-based estimator of the expectation of \(g(X, Y)\) with respect to \(\Pi\), that is, \(n^{-1} \sum_{i=0}^{n-1} g(X^*_i, Y^*_i)\), will be inconsistent (assuming that \(g(X, Y)\) actually depends on both \(X\) and \(Y\)).

We now explain how our work on the out-of-order block Gibbs sampler was motivated by an analysis in Yang and Rosenthal [2019]. Consider the following simple random effects model:

\[
y_i = \theta_i + e_i,
\]

for \(i = 1, \ldots, m\), where the components of \(\theta = (\theta_1, \ldots, \theta_m)^T\) are iid \(N(\mu, A)\), the components of \(e = (e_1, \ldots, e_m)^T\) are iid \(N(0, V)\), and \(\theta\) and \(e\) are independent. The error variance, \(V\), is assumed known. The basic idea is that we will observe the \(y_i\)s, and then attempt to make inferences about the unknown parameters, \(A\) and \(\mu\). Consider a Bayesian statistical model in which \(A\) and \(\mu\) are taken to be \textit{a priori} independent with

\[
\pi(\mu) \propto 1 \quad \text{and} \quad A \sim IG(a, b),
\]

where \(a, b > 0\), and we say \(W \sim IG(a, b)\) if its density is proportional to \(w^{a-1}e^{-b/w}I(0, \infty)(w)\).

Denote the resulting posterior distribution as \(\Pi(dA, d\mu, d\theta|y)\), where \(y = (y_1, \ldots, y_m)^T\) represents the observed data.

Rosenthal [1996] derived and analyzed a block Gibbs sampler for \(\Pi(dA, d\mu, d\theta|y)\). In terms of the general block Gibbs sampler described above, \((A, \mu)\) plays the role of \((X, Y)\), and \(\theta\) plays the role of \(Z\). So \(X = (0, \infty), Y = \mathbb{R}\) and \(Z = \mathbb{R}^m\). Denote the Markov chain by \(\Lambda = \{(A_n, \mu_n, \theta_n)\}_{n=0}^\infty\). If the current state is \((A_n, \mu_n, \theta_n) = ((A, \mu), \theta)\), then we move to the next state \((A_{n+1}, \mu_{n+1}, \theta_{n+1})\) using the following three step procedure.

\underline{Iteration} \(n + 1\) of Rosenthal’s block Gibbs sampler:

1. Draw \(A_{n+1} \sim IG\left(a + \frac{m-1}{2}, b + \frac{1}{2} \sum_{i=1}^{m} (\theta_i - \bar{\theta})^2\right)\).

2. Draw \(\mu_{n+1} \sim N\left(\bar{\theta}, \frac{A_{n+1}}{m}\right)\).

3. For \(i = 1, 2, \ldots, m\), draw the \(i\)th component of \(\theta_{n+1}\) from \(N\left(\frac{V\mu_{n+1} + \mu_{n+1} y_i}{A_{n+1} + V}, \frac{A_{n+1}}{A_{n+1} + V}\right)\).

The first two steps of this algorithm constitute a joint draw from the distribution of \((A, \mu)\) given \((\theta, y)\), which we write as \(\Pi_1(dA, d\mu|\theta, y)\). In particular, if we factor \(\Pi_1\) as follows

\[
\Pi_1(dA, d\mu|\theta, y) = \Pi_{21}(dA|\theta, y)\Pi_{22}(d\mu|A, \theta, y),
\]

4
then in the first step we draw \( A_{n+1} \sim \Pi_{21}(\cdot | \theta, y) \), and in the second step we draw \( \mu_{n+1} \sim \Pi_{22}(\cdot | A_{n+1}, \theta, y) \). In the posterior distribution, the components of \( \theta \) are conditionally independent given \((\mu, A, y)\), which is why the third step of the algorithm consists of \( m \) univariate draws.

Recently, Yang and Rosenthal [2019] set out to perform a more nuanced analysis of Rosenthal’s [1996] block Gibbs sampler, but ended up analyzing the out-of-order version of Rosenthal’s [1996] algorithm instead. Let \( \Lambda^* = \{(\mu^*_n, \theta^*_n, A^*_n)\}_{n=0}^{\infty} \) denote their Markov chain. If the current state is \((\mu^*_n, \theta^*_n, A^*_n) = (\mu, \theta, A)\), then we move to the next state \((\mu^*_{n+1}, \theta^*_{n+1}, A^*_{n+1})\) using the following three step procedure.

**Iteration \( n + 1 \) of Yang & Rosenthal’s algorithm:**

1. Draw \( \mu^*_{n+1} \sim N(\bar{\theta}, A/m) \).

2. For \( i = 1, 2, \ldots, m \), draw the \( i \)th component of \( \theta^*_{n+1} \) from \( N\left(\frac{V\mu^*_n + Ay_i}{A+V}, \frac{AV}{A+V}\right) \).

3. Draw \( A^*_{n+1} \sim IG\left(a + \frac{m-1}{2}, b + \frac{1}{2} \sum_{i=1}^m (\theta^*_{n+1,i} - \bar{\theta}^*_{n+1})^2\right) \).

It appears that Yang and Rosenthal [2019] changed the order of the steps in Rosenthal’s [1996] block Gibbs sampler because the resulting Markov chain was easier to analyze. However, they were apparently unaware that the modified algorithm they analyzed is not a valid MCMC algorithm for the target posterior distribution \( \Pi(dA, d\mu, d\theta | y) \). Thus, without supplementary results relating \( \tilde{\Lambda} \) and \( \Lambda^* \), the convergence results developed by Yang and Rosenthal [2019] are not entirely germane to the exploration of \( \Pi(dA, d\mu, d\theta | y) \).

In this note, we develop general results relating the convergence behaviors of the (valid) block Gibbs Markov chain, \( \tilde{\Gamma} \), and the (invalid) out-of-order block Gibbs Markov chain, \( \Gamma^* \). In particular, we develop inequalities that relate the total variation distance to stationarity for one Markov chain to that of the other, and these bounds imply that the two chains mix at essentially the same rate. (Keep in mind that these two Markov chains have different invariant distributions.) We also show that \( \tilde{\Gamma} \) is geometrically ergodic if and only if \( \Gamma^* \) is geometrically ergodic, and that their convergence rates are the same. Our results demonstrate that one can perform convergence analysis of the block Gibbs sampler *indirectly* by analyzing the out-of-order block Gibbs sampler, which may be easier to handle, as was apparently the case for Yang and Rosenthal [2019].

## 2 Main Results

We begin with some general state space Markov chain theory. Let \( E \) be a set, and let \( \mathcal{E} \) be a countably generated \( \sigma \)-algebra of subsets of \( E \). Let \( P : E \times \mathcal{E} \rightarrow [0,1] \) be a Mtk, and assume
that the corresponding Markov chain is irreducible, aperiodic and positive Harris recurrent. (See [Meyn and Tweedie 2009] for definitions.) Let $\Pi$ denote the unique invariant probability measure, and let $P^n$ denote the $n$-step Mtk (so, as usual, $P \equiv P^1$). If $\mu$ is a probability measure on $E$, then we use $\mu P^n(\cdot)$ to denote the probability measure $\int_E P^n(u, \cdot) \mu(du)$. Following Nummelin [1984] and Tierney [1994], we say that the Markov chain is geometrically ergodic if there exist a non-negative extended real-valued function $M$ with $\int_E M(u) \Pi(du) < \infty$ and a $\rho \in [0, 1)$ such that

$$\|P^n(u, \cdot) - \Pi(\cdot)\| \leq M(u) \rho^n$$

for all $n \in \mathbb{N}$ and all $u \in E$. Here, $\|\cdot\|$ denotes the usual total variation norm for signed measures.

We define the geometric convergence rate of the chain, $\rho_*$, to be the infimum over $\rho \in [0, 1]$ such that (3) holds for some non-negative extended real-valued function $M$, integrable with respect to $\Pi$. Clearly, the chain is geometrically ergodic if and only if $\rho_* < 1$.

Remark 2. This definition of geometric ergodicity is slightly less standard than an alternative definition in which $M$ is not assumed to be integrable with respect to $\Pi$, but is assumed to be finite $\Pi$-almost everywhere. While Roberts and Rosenthal’s [1997] Proposition 2.1 shows that the two definitions are actually equivalent, the geometric convergence rates under the two definitions are not necessarily equal.

A standard result concerning total variation distance is as follows [see, e.g., Roberts and Rosenthal, 2004, Proposition 3]. If $\mu$ and $\nu$ are probability measures on $E$, then

$$\|\mu(\cdot) - \nu(\cdot)\| = \sup_{f: E \rightarrow [0,1]} \left| \int f \, d\mu - \int f \, d\nu \right|.$$

This fact will be used repeatedly in our proofs.

Now let $K$ denote the Mtk of the block Gibbs Markov chain, $\tilde{\Gamma} = \{(\tilde{X}_n, \tilde{Y}_n, \tilde{Z}_n)\}_{n=0}^{\infty}$, described in the Introduction. So we have

$$K((x, y, z), (dx', dy', dz')) = \Pi_{Z|XY}(dz' \mid x', y') \Pi_{XY|Z}(dx', dy' \mid z).$$

(We assume that all Markov chains considered in this section satisfy the basic regularity conditions laid out at the start of this section.) It is well known that the sequences $\{(\tilde{X}_n, \tilde{Y}_n)\}_{n=0}^{\infty}$ and $\{\tilde{Z}_n\}_{n=0}^{\infty}$ are themselves (reversible) Markov chains with invariant distributions given by $\Pi_{XY}(dx, dy)$ and $\Pi_{Z}(dz)$, respectively (see, e.g., Liu et al. [1994]). Let $K_{XY}$ denote the Mtk of the $(X, Y)$-marginal chain, and define $K_Z$ similarly. For example,

$$K_{XY}((x, y), (dx', dy')) = \int_z \Pi_{XY|Z}(dx', dy' \mid z) \Pi_{Z|XY}(dz \mid x, y).$$
As mentioned in the Introduction, there is one way to reorder the steps in the block Gibbs sampler without invalidating the algorithm. The corresponding Markov chain has \( M_{tk} \) given by

\[
K^\dagger((z, x, y), (dz', dx', dy')) = \Pi_{XY|Z}(dz', dy'| z')\Pi_{Z|XY}(dz'| x, y).
\]

All four of these Markov chains \((K, K^\dagger, K_{XY}, K_Z)\) converge at the same rate [Roberts and Rosenthal, 2001, Diaconis et al., 2008]. As in the Introduction, let \( K^* \) denote the \( M_{tk} \) of the out-of-order chain, so, again,

\[
K^*((y, z, x), (dy', dz', dx')) = \Pi_{X|Z}(dx'| z')\Pi_{Z|XY}(dz'| x, y')\Pi_{Y|XZ}(dy'| x, z).
\]

Here is our first result.

**Proposition 3.** Fix \( z \in \mathcal{Z} \) and let \( \nu_z \) denote the probability measure of the random vector \((X, Z)\) such that \( Z = z \) w.p. 1, and the conditional distribution of \( X \) given \( Z \) is \( \Pi_{X|Z}(\cdot| z) \). Then

\[
\|K^n((x, y, z), \cdot) - \Pi(\cdot)\| \leq \|K^n_{Z}(z, \cdot) - \Pi(\cdot)\| \leq \|\nu_zK^{n-2}(\cdot) - \Pi(\cdot)\|.
\]

Fix \((x, z) \in \mathcal{X} \times \mathcal{Z}\) and let \( \nu_{(x,z)} \) denote the probability measure of the random vector \((X, Y)\) such that \( X = x \) w.p. 1, and the conditional distribution of \( Y \) given \( X \) is \( \Pi_{Y|XZ}(\cdot|x, z) \). Then

\[
\|K^*n((y, z, x), \cdot) - \Pi^*(\cdot)\| \leq \nu_{(x,z)}K^{n-1}_{XY}(\cdot) - \Pi_{XY}(\cdot)\| \leq \nu_{(x,z)}K^{(n-1)}(\cdot) - \Pi(\cdot)\|.
\]

**Proof:** We begin with (4). The first inequality follows immediately from Lemma 2.4 of Diaconis et al. [2008]. Now, it is straightforward to show that

\[
K^n_{Z}(z, dz') = \int_X \int_Y \int_Z \Pi_{Z|XY}(dz'| x', y')\Pi_{Y|XZ}(dy'| x', z'')K^{n-1}(x'', z), (dy'', dz'', dx'))\Pi_{X|Z}(dx'| z)
\]

\[
= \int_Y \int_Z \int_X \int_Y \Pi_{Z|XY}(dz'| x', y')\Pi_{Y|XZ}(dy'| x', z'')\int_X K^{n-1}(x'', z), (dy'', dz'', dx'))\Pi_{X|Z}(dx'| z)
\]

\[
= \int_Y \int_Z \int_X \int_Y \Pi_{Z|XY}(dz'| x', y')\Pi_{Y|XZ}(dy'| x', z'')\nu_zK^{n-1}(dy'', dz'', dx').
\]

Also,

\[
\Pi_{Z}(dz') = \int_X \int_Y \int_Z \Pi_{Z|XY}(dz'| x', y')\Pi_{Y|XZ}(dy'| x', z'')\Pi_{Y|Z}(dy''| z'')\Pi_{X|Z}(dx'| z'')
\]

\[
= \int_X \int_Y \int_Z \Pi_{Z|XY}(dz'| x', y')\Pi_{Y|XZ}(dy'| x', z'')\Pi^*(dx', dy'', dz'').
\]
Hence for the second inequality in (4),

\[ \left\| K^n_z(z, \cdot) - \Pi_{\mathcal{Z}}(\cdot) \right\| \]

\[ = \sup_{0 \leq f \leq 1} \left| \int_{\mathcal{Z}} f(z') \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{Y}} \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x', y') \Pi_{\mathcal{Y}|\mathcal{X}Z}(dy'|x', z'') \nu_x K^{(n-1)}_z(dy'', dz'', dx') \right. \]
\[ \left. - \int_{\mathcal{Z}} f(z') \int_{\mathcal{Y}} \int_{\mathcal{X}} \int_{\mathcal{Y}} \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x', y') \Pi_{\mathcal{Y}|\mathcal{X}Z}(dy'|x', z'') \Pi^*(dx', dy'', dz'') \right| \]

\[ = \sup_{0 \leq f \leq 1} \left| \int_{\mathcal{Z}} \int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Y}} f(z') \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x', y') \Pi_{\mathcal{Y}|\mathcal{X}Z}(dy'|x', z'') \nu_x K^{(n-1)}_z(dy'', dz'', dx') \right. \]
\[ \left. - \int_{\mathcal{Z}} \int_{\mathcal{X}} \int_{\mathcal{Y}} \int_{\mathcal{Y}} f(z') \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x', y') \Pi_{\mathcal{Y}|\mathcal{X}Z}(dy'|x', z'') \Pi^*(dx', dy'', dz'') \right| \]

\[ \leq \sup_{0 \leq g \leq 1} \left| \int_{\mathcal{Z}} \int_{\mathcal{X}} \int_{\mathcal{Y}} g(x', y'', z'') \left[ \nu_x K^{(n-1)}_z(dy'', dz'', dx') - \Pi^*(dx', dy'', dz'') \right] \right| \]

\[ = \left\| \nu_x K^{(n-1)}_z(\cdot) - \Pi^*(\cdot) \right\| . \]

Now for (5). It is straightforward to show that

\[ K^{*n}((y, z, x), (dy', dz', dx')) \]
\[ = \int_{\mathcal{Y}} \int_{\mathcal{X}} \Pi_{\mathcal{X}|\mathcal{Z}}(dx'|z') \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x'', y') K^{(n-1)}_{\mathcal{X}Y}(x, y'') (dy'', dx') \Pi_{\mathcal{Y}|\mathcal{X}Z}(dy'|x, z) \]
\[ = \int_{\mathcal{X}} \Pi_{\mathcal{X}|\mathcal{Z}}(dx'|z') \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x'', y') \int_{\mathcal{Y}} K^{(n-1)}_{\mathcal{X}Y}(x, y'') (dy'', dx') \Pi_{\mathcal{Y}|\mathcal{X}Z}(dy'|x, z) \]
\[ = \int_{\mathcal{X}} \Pi_{\mathcal{X}|\mathcal{Z}}(dx'|z') \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x'', y') \nu_{(x,z)} K^{(n-1)}_{\mathcal{X}Y}(dx'', dy') . \]

Also,

\[ \Pi^*(dy', dz', dx') = \Pi_{\mathcal{X}|\mathcal{Z}}(dx'|z') \Pi_{\mathcal{Z}|\mathcal{X}Y}(dz'|x'', y') \Pi_{\mathcal{X}Y}(dx'', dy') . \]
Hence for the first inequality of (5),

\[
\|K^n((y, z, x), \cdot) - \Pi^*(\cdot)\| \\
= \sup_{0 \leq f \leq 1} \left| \int_X \int_Y \int_Z f(x', y', z') \int_X \Pi_{X|Z}(dx'|z') \Pi_{Z|XY}(dz'|x'', y') \nu_{(x, z)} K_{XY}^{n-1}(dx'', dy') \\
- \int_X \int_Y \int_Z f(x', y', z') \int_X \Pi_{X|Z}(dx'|z') \Pi_{Z|XY}(dz'|x'', y') \Pi_{XY}(dx'', dy') \right| \\
= \sup_{0 \leq f \leq 1} \left| \int_X \int_Y \int_Z \left[ f(x', y', z') \Pi_{X|Z}(dx'|z') \Pi_{Z|XY}(dz'|x'', y') \right] \nu_{(x, z)} K_{XY}^{n-1}(dx'', dy') \\
- \int_X \int_Y \int_Z \left[ f(x', y', z') \Pi_{X|Z}(dx'|z') \Pi_{Z|XY}(dz'|x'', y') \right] \Pi_{XY}(dx'', dy') \right| \\
\leq \sup_{0 \leq f \leq 1} \left| \int_X \int_Y g(x'', y') \left[ \nu_{(x, z)} K_{XY}^{n-1}(dx'', dy') - \Pi_{XY}(dx'', dy') \right] \right| \\
= \| \nu_{(x, z)} K_{XY}^{n-1}(\cdot) - \Pi_{XY}(\cdot) \|,
\]

The second inequality follows from the fact that

\[
\nu_{(x, z)} K_{XY}^n(dx', dy') = \int_Z \nu_{(x, z)} K_1^n(\cdot, (dx', dy')).
\]

\[\square\]

In light of Proposition 3, it should not be surprising that the block Gibbs Markov chain is geometrically ergodic if and only if the out-of-order block Gibbs Markov chain is geometrically ergodic, and that they have the same rate of convergence. This is the subject of our next result.

**Proposition 4.** There are only two possibilities: (i) \(\tilde{\Gamma}\) and \(\Gamma^*\) are both geometrically ergodic with the same geometric convergence rate, or (ii) neither \(\Gamma\) is geometrically ergodic.

**Proof.** First, assume that \(\Gamma^*\) is geometrically ergodic. That is, for all \(n \in \mathbb{N}\) and all \((y, z, x) \in Y \times Z \times X\),

\[
\|K^n((y, z, x), \cdot) - \Pi^*(\cdot)\| \leq M_1(x, z) \rho_1^n
\]

where \(\rho_1 \in [0, 1)\) and \(\int_X \int_Y \int_Z M_1(x, z) \Pi^*(dx, dy, dz) < \infty\). We now show that this implies the geometric ergodicity of the \(Z\)-marginal. It’s easy to see that

\[
\Pi_Z(dz') = \int_X \int_Y \int_Z \Pi_{Z|XY}(dz'|x', y') \Pi_{Y|XZ}(dy'|x', z'') \Pi_{YZ}(dy'', dz'') \Pi_{X|Z}(dx'|z'') \Pi_{X|Z}(dx''|z)
\]

\[
= \int_X \int_Y \int_Z \Pi_{Z|XY}(dz'|x', y') \Pi_{Y|XZ}(dy'|x', z'') \Pi^*(dx', dy'', dz'') \Pi_{X|Z}(dx''|z).
\]
Combining this with (6), we have

\[
\|K^*_Z(z, \cdot) - \Pi_Z(\cdot)\| = \sup_{0 \leq f \leq 1} \left| \int_Z f(z') \int_X \int_X \int_Y \int_Y \Pi_{Z|XY}(dz'|x', y') \Pi_Y|XZ(dy'|x', z'') \right.
\]

\[
K^{*(n-1)}((x'', z), (dy'', dz'', dx')) \Pi_X|Z(dx''|z)
\]

\[- \int_Z f(z') \int_X \int_X \int_Y \int_Y \Pi_{Z|XY}(dz'|x', y') \Pi_Y|XZ(dy'|x', z'') \Pi^*(dx', dy'', dz'') \Pi_X|Z(dx''|z) \right|
\]

\[
= \sup_{0 \leq f \leq 1} \left| \int_X \int_Y \int_Z f(x', y'', z'') \left[ K^{*(n-1)}((x'', z), (dy'', dz'', dx')) - \Pi^*(dx', dy'', dz'') \right] \Pi_X|Z(dx''|z) \right|
\]

\[
\leq \int_X \left\{ \sup_{0 \leq f \leq 1} \left| \int_Y \int_Z g(x', y'', z'') \left[ K^{*(n-1)}((x'', z), (dy'', dz'', dx')) - \Pi^*(dx', dy'', dz'') \right] \Pi_X|Z(dx''|z) \right\} \right| \Pi_X|Z(dx''|z)
\]

\[
= \int_X \left| K^{*(n-1)}((x'', z), \cdot) - \Pi^*(\cdot) \right| \Pi_X|Z(dx''|z) .
\]

Now using the assumed geometric ergodicity of \(K^*\), we have

\[
\|K^*_Z(z, \cdot) - \Pi_Z(\cdot)\| \leq \rho_1^{n-1} \int_X M_1(x, z) \Pi_X|Z(dx|z) = \frac{M_1^*(z)}{\rho_1} \rho_1^n ,
\]

where we have defined \(M_1^*(z) := \int_X M_1(x, z) \Pi_X|Z(dx|z)\). Now using the integrability of \(M_1(x, z)\) with respect to \(\Pi^*(dx, dy, dz)\), we have

\[
\int_X \int_Y \int_Z M_1(x, z) \Pi^*(dx, dy, dz) = \int_X \int_Z M_1(x, z) \Pi_X|Z(dx, dz)
\]

\[
= \int_Z \left[ \int_X M_1(x, z) \Pi_X|Z(dx|z) \right] \Pi_Z(dz)
\]

\[
= \int_Z M_1^*(z) \Pi_Z(dz)
\]

\[
< \infty .
\]

It follows immediately that the \(Z\)-marginal chain is geometrically ergodic, and that its geometric rate of convergence is no larger than \(\rho_1\). Again, the \(Z\)-marginal chain and the block Gibbs chain share the same geometric rate of convergence.
Now assume that the block Gibbs sampler is geometrically ergodic. This implies that the 
\((X, Y)\)-marginal chain is also geometrically ergodic with the same rate, so, for all \(n \in \mathbb{N}\) and all 
\((x, y) \in X \times Y\),
\[
\|K^n_{XY}((x, y), \cdot) - \Pi_{XY}(\cdot)\| \leq M_2(x, y)\rho_2^n
\]
where \(\rho_2 \in [0, 1)\) and \(\int_X \int_Y M_2(x, y)\Pi(dx, dy) < \infty\). It’s easy to see that
\[
\Pi^*(dy', dz', dx') = \Pi_{X|Z}(dz'|z')\Pi_{Y|Z}(dy', dz')
\]
\[
= \int_Y \int_X \Pi_{X|Z}(dz'|z')\Pi_{Z|XY}(dx'|x'', y')\Pi_{XY}(dx'', dy)\Pi_{Y|XZ}(dy'|x, z).
\]
Combining this with (7), we have
\[
\|K^n((y, z, x), \cdot) - \Pi^*(\cdot)\|
\]
\[
= \sup_{0 \leq f \leq 1} \left| \int_X \int_Y \int_Z f(x', y', z') \int_Y \Pi_{X|Z}(dz'|z')\Pi_{Z|XY}(dx'|x'', y')
\]
\[
\quad \left[ K^n_{XY}((x, y''), (dx'', dy'))\Pi_{Y|XZ}(dy'|x, z) - \int_X \int_Y \int_Z f(x', y', z') \int_Y \Pi_{X|Z}(dz'|z')\Pi_{Z|XY}(dx'|x'', y')\Pi_{XY}(dx'', dy)\Pi_{Y|XZ}(dy'|x, z) \right]
\]
\[
= \sup_{0 \leq f \leq 1} \left| \int_Y \int_X \int_Y \left[ \int_Z \int_X f(x', y', z') \Pi_{X|Z}(dz'|z')\Pi_{Z|XY}(dx'|x'', y') \right] \Pi_{XY}(dx'', dy)\Pi_{Y|XZ}(dy'|x, z) \right|
\]
\[
\leq \sup_{0 \leq g \leq 1} \left| \int_Y \int_X \int_Y g(x'', y') \left[ K^n_{XY}((x, y''), (dx'', dy')) - \Pi_{XY}(dx'', dy) \right] \Pi_{Y|XZ}(dy'|x, z) \right|
\]
\[
\leq \int_Y \left\{ \sup_{0 \leq g \leq 1} \left| \int_Y \int_X g(x'', y') \left[ K^n_{XY}((x, y''), (dx'', dy')) - \Pi_{XY}(dx'', dy) \right] \right| \right\} \Pi_{Y|XZ}(dy'|x, z)
\]
\[
= \int_Y \left\| K^n_{XY}((x, y''), \cdot) - \Pi_{XY}(\cdot)\| \Pi_{Y|XZ}(dy'|x, z) \right\|
\]
Now using the assumed geometric ergodicity of \(K_{XY}\), we have
\[
\|K^n((y, z, x), \cdot) - \Pi^*(\cdot)\| \leq \rho_2^{n-1} \int_Y M_2(x, y')\Pi_{Y|XZ}(dy'|x, z) = \frac{M_2'(x, z)}{\rho_2} \rho_2^n,
\]
where we have defined \(M_2'(x, z) := \int_Y M_2(x, y')\Pi_{Y|XZ}(dy'|x, z)\). Using the integrability of \(M_2(x, y)\)
with respect to $\Pi_{XY}(dx, dy)$, we have

$$
\int_X \int_Y M_2(x, y) \Pi_{XY}(dx, dy) = \int_X \int_Y \int_Z M_2(x, y) \Pi(dx, dy, dz) \\
= \int_X \int_Z \left[ \int_Y M_2(x, y) \Pi_{Y|XZ}(dy|x, z) \right] \Pi_{XZ}(dx, dz) \\
= \int_X \int_Z M'_2(x, z) \Pi_{XZ}(dx, dz) \\
< \infty.
$$

It follows immediately that the out-of-order chain is geometrically ergodic, and that its geometric rate of convergence is no larger than $\rho_2$.

\[\square\]

**Acknowledgment.** The authors are grateful to Qian Qin for helpful conversations.

**References**

Persi Diaconis, Kshitij Khare, and Laurent Saloff-Coste. Gibbs sampling, exponential families and orthogonal polynomials (with discussion). *Statistical Science*, 23:151–200, 2008.

J. S. Liu, W. H. Wong, and A. Kong. Covariance structure of the Gibbs sampler with applications to comparisons of estimators and augmentation schemes. *Biometrika*, 81:27–40, 1994.

Sean P. Meyn and Richard L. Tweedie. *Markov Chains and Stochastic Stability*. Cambridge University Press, Cambridge, U.K., 2 edition, 2009.

Esa Nummelin. *General Irreducible Markov Chains and Non-negative Operators*. Cambridge, London, 1984.

Gareth O. Roberts and Jeffrey S. Rosenthal. Geometric ergodicity and hybrid Markov chains. *Electronic Communications in Probability*, 2:13–25, 1997.

Gareth O. Roberts and Jeffrey S. Rosenthal. Markov chains and de-initializing processes. *Scandinavian Journal of Statistics*, 28:489–504, 2001.

Gareth O. Roberts and Jeffrey S. Rosenthal. General state space Markov chains and MCMC algorithms. *Probability Surveys*, pages 20–71, 2004.
Jeffrey S. Rosenthal. Analysis of the Gibbs sampler for a model related to James-Stein estimators. *Statistics and Computing*, 6:269–275, 1996.

Luke Tierney. Markov chains for exploring posterior distributions (with discussion). *Annals of Statistics*, 22:1701–1728, 1994.

Jun Yang and Jeffrey S. Rosenthal. Complexity results for MCMC derived from quantitative bounds. *arXiv:1708.00829*, 2019.