THE SIGMA FUNCTION FOR TRIGONAL CYCLIC CURVES

JIRYO KOMEDA, SHIGEKI MATSUTANI, AND EMMA PREVIATO

Abstract. A recent generalization of the “Kleinian sigma function” involves the choice of a point \( P \) of a Riemann surface \( X \), namely a “pointed curve” \((X,P)\). This paper concludes our explicit calculation of the sigma function for curves cyclic trigonal at \( P \). We exhibit the Riemann constant for a Weierstrass semigroup at \( P \) with minimal set of generators \( \{3, 2r + s, 2s + r\}, r < s \), equivalently, non-symmetric; we construct a basis of \( H^1(X, \mathbb{C}) \) and a fundamental 2-differential on \( X \times X \); we give the order of vanishing for sigma on Wirtinger strata of the Jacobian of \( X \), and a solution to the Jacobi inversion problem.

1. Introduction

The Weierstrass \( \wp \) and \( \sigma \) function, related by the identity \( \wp(u) = -\frac{d^2}{du^2} \ln \sigma(u) \), are defined for elliptic curves. Sigma in turn is related to Jacobi’s theta functions [WW 20-421]. Theta functions were then defined by Riemann for any Abelian variety, while an equivalent (in the sense of [La], cf. Section 4) function was defined for hyperelliptic Jacobians by Klein [Kl1, Kl2]. In fact, as pointed out in [KS], Klein defined it also for all Jacobians of curves of genus three [K3], and in loc. cit., the authors generalize it to any Jacobian by requiring a modular invariance under the action of \( \text{SL}(2g, \mathbb{Z}) \) (up to a root of unity). A different approach, which we follow in this paper, originally proposed by Buchstaber, Enolski˘ı and Le˘ıkin [BEL1] and Eilbeck, Enolski˘ı and Le˘ıkin [EEL] is based on Baker’s results [B2] that connect the (transcendental) sigma function with the (algebraic) functions and differentials of the curve. In this approach involves the choice of a point \( P \) on the curve \( X \), since the relevant objects are written in terms of \( H^0(X, \mathcal{O}(*P)) \). The representation of the affine curve \( X \setminus P \) is therefore also relevant, and so is the Weierstrass semigroup (W-semigroup) at \( P \). Until recently, the only explicit formulas for sigma were produced for \((n,s)\) curves (cf. Section 2), which are plane affine curves that have a smooth compactification by one point ‘at infinity’, playing the role of \( P \), and 2-generator W-semigroup. Ayano [Ay1, Ay2] was able to follow this approach and construct sigma for “telescopic” (cf. Section 2) W-semigroups; at the same time, the first two authors gave a construction for a new case, neither of \((n,s)\) nor of telescopic type [MK], pursued in [KMP1, KMP2]. Those papers covered curves cyclic at \( P \) with W-semigroup that has

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minimal set of generators \( \{3, 2r + s, 2s + r\} \), where \( r, s \) are natural numbers with \( r < s \).

Slow but steady (as in the ancient Zwahili proverb “Pole pole ndio mwen do”), in this paper we complete the explicit results for all curves trigonal cyclic at \( P \) (in the sense of \([Ac]\), cf. Subsection 2). Our methods do not cover the non-cyclic case; for example, Klein’s quartic \( X^3Y + Y^3Z + Z^3X \) in projective coordinates \([X, Y, Z]\), the genus-three curve with maximum number of automorphisms, has W-semigroup \( \langle 3, 5, 7 \rangle \) at \( P = [0, 0, 1] \) but is not cyclic since all the automorphisms of order three are conjugate and the quotient curves have genus one. Indeed, our strategy, introduced by Komeda and Matsutani \([MK]\), is to piece together local coordinates at \( P \) by using a suitable set of singular \((n, s)\) curves which are images of \( X \): the images of their affine rings generate the affine ring of \( X \backslash P \). These singular curves can be represented as monomial curves (cf. Subsection 2.1); in fact, singular \( Z_3 \) curves in the sense of \([FZ]\) because of the action of \( Z/3Z \). On the plus side, the value of our approach is that it works for any \( k \)-gonal cyclic cover of \( \mathbb{P}^1 \); although we only work out the trigonal case because the general \( k \) would necessitate very large formulas, we provide general statements to the fullest extent possible. We clarify that the words “trigonal” and \( k \)-gonal are used only as an indication of the fact that the pointed curve \((X, \infty)\) has W-semigroup of type three, five, respectively; a different convention requires that \( k \)-gonal curves not be \( j \)-gonal for \( j < k \) (as in trigonal curves, which by definition are not hyperelliptic); this cannot be guaranteed in our examples, as noticed in \([B1, \text{Ch. V, §70}]\). Cyclic covers of \( \mathbb{P}^1 \) have been connected with vanishing properties of the theta function (for recent work cf. \([Ac, FZ]\)), since there is an induced action on the Jacobian.

We emphasize that the main effort in the theory is directed toward explicitness. In the \([BEL1, BLE, EEL]\) approach, the differentials that satisfy the generalized Legendre relations (cf. Subsection 4.4) are built out of algebraic functions on the curve, and this is where the cyclic action on our local coordinates at \( P \) is crucial. The two versions of the Kleinian sigma function (modular vs. algebraic) have not been fully compared yet; in fact, much remains to be done before the algebraic approach can be exploited to its full extent in applications to physics and dynamics. This is important for two reasons: PDEs and integrable systems can be solved explicitly—for example, our formulas for the order of vanishing of sigma on strata of the Jacobian give information on the qualitative behavior of the solutions; and the numerical properties of the Weierstrass points of the curve come into play, which opens vistas toward applications to open questions in Weierstrass-semigroup theory (cf. \([KMP1, \text{Section 2}]\)).

We use the word “curve” for a compact Riemann surface: on occasion, we use a singular representation of the curve; since there is a unique smooth curve with the same field of meromorphic function, this should not cause confusion. By “natural number” we mean a positive integer, excluding zero from \( \mathbb{N} \). The contents of this paper are organized as follows: in Section 2 we collect definitions, properties, and representation-theoretic interpretations of W-semigroups, cyclic covers of \( \mathbb{P}^1 \), monomial curves. Section 3 contains our main constructions for the general trigonal cyclic case. In Section 4 we introduce the transcendental aspect and the sigma function. Our explicit expressions for the trigonal cyclic curves use classical and new theorems for the sigma function of \((n, s)\) curves. In
Section 5, we collect the vanishing theorems for sigma and the solution to the Jacobi inversion problem. In the Appendix we provide technical proofs for constructing our basis of \( H^1(X, \mathbb{C}) \), the first cohomology group of the constant sheaf of the curve.

### 2. Numerical and Weierstrass semigroups

We set up the notation for W-semigroups, and recall the results we use.

A pointed curve is a pair \((X, P)\), with \(P\) a point of a curve \(X\); the W-semigroup for \(X\) at \(P\), which we denote by \(H(X, P)\), is the complement in \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\) of the Weierstrass gap sequence (W-gaps) \(L\), namely the set of natural numbers \(\{\ell_0 < \ell_1 < \cdots < \ell_{g-1}\}\) such that \(H^0(K_X - \ell_i P) \neq H^0(K_X - (\ell_i - 1) P)\), for \(K_X\) a representative of the canonical divisor (we identify divisors and the corresponding sheaves; \(K_X\) corresponds to the sheaf of holomorphic differentials \(\Omega_X\)). By the Riemann-Roch theorem, \(H(X, P)\) is a numerical semigroup. In general, a numerical semigroup \(H\) has a unique (finite) minimal set of generators, \(M(H)\), say, and the cardinality \(g\) of \(\mathbb{N}_0 \setminus H\) is called the genus of \(H\) (for terminology, history and references cf. \([B-A]\)). The Schubert index of the set \(L\) is

\[
\alpha(L) := \{\alpha_0(L), \alpha_1(L), \ldots, \alpha_{g-1}(L)\},
\]

where \(\alpha_i(L) := \ell_i - i - 1\) \([EH]\). For the smallest element \(\alpha_{\text{min}}(H)\) in \(M(H)\) we call a semigroup \(H\) an \(\alpha_{\text{min}}(H)\)-semigroup, therefore \(H(X, P)\) for a trigonal cyclic pointed curve \((X, P)\) is a 3-semigroup.

If we let the row lengths be \(\Lambda_i = \alpha_{g-i} + 1, i \leq g\), then \(\Lambda := (\Lambda_1, ..., \Lambda_g)\) is the Young diagram of the semigroup. A W-semigroup is called symmetric when \(2g - 1\) occurs in the gap sequence; thus, \(H(X, P)\) is symmetric if and only if its Young diagram is symmetric, i.e. invariant under reflection across the main diagonal.

Telescopic numerical semigroups were introduced in Information Theory (Goppa codes), because the parameters of the corresponding code, hence also the answer to the Diophantine Frobenius Problem \([R, \text{Ch. 7}]\), can be computed efficiently. They are defined to have generators \(\{\alpha_0(L), \alpha_1(L), \ldots, \alpha_{g-1}(L)\}\) that, up to reordering, are a telescopic sequence:

**Definition 2.1.** The sequence \(\{w_1, ..., w_s\}\) is called telescopic if \(\gcd(w_1, ..., w_s) = 1\) and \(w_i/d_i \in S_{i-1}\) \((i = 2, ..., s)\), where \(d_i = \gcd(w_1, ..., w_i)\) and \(S_{i}\) is the semigroup generated by \(\{w_1/d_i, ..., w_i/d_i\}\), \(1 \leq i \leq s\).

Telescopic semigroups are symmetric \([R, \text{Lemma 7.1.7}]\), but not *vice versa*, as the generators \(\alpha(L) = \{6, 13, 14, 15, 16\}\) show, since the Young diagram is symmetric \([KMP1]\) but the sequence is not telescopic even after reordering.

Miura analyzed, in particular, an affine model for the curves with W-semigroup at a point which is telescopic, for applications to coding theory \([Min]\). His results appeared only in Japanese, and for that reason Ayano included the complete proofs in his paper \([Ay1]\) where he constructed the sigma function; he used sigma to obtain Jacobi inversion formulas and vanishing theorems over Wirtinger strata \([Ay2]\).
2.1. The monomial Ring $B_H$. A numerical semigroup $H$ has an associated ring $B_H \simeq k[Z_1, Z_2, ..., Z_\ell]/\ker \varphi$, where for a minimal set of generators $M = \{m_1, ..., m_\ell\}$, $\varphi$ is the epimorphism from the polynomial ring $\mathbb{C}[Z] := \mathbb{C}[Z_1, Z_2, ..., Z_\ell]$ to $B_H := \mathbb{C}[t^{m_1}, t^{m_2}, ..., t^{m_\ell}]$, given by $Z_a \mapsto t^{ma}$; Herzog shows that the kernel is a monomial ideal $[He$, Prop. 1.4].

There is an action of $\mathbb{C}^*$ on the monomial ring $B_H$, given as $Z_j \mapsto a^m Z_j$ for $a \in \mathbb{C}^*$. Accordingly, we say that $Z_a$ has weight $a$.

2.2. Weierstrass normal form. We now consider the “Weierstrass normal form” (W-normal form): this gives a (possibly singular) model for any pointed curve in affine space of the same dimension as a minimal set of generators of the W-semigroup. Baker [B1, Ch. V, §§60-79] gives a complete review, proof and examples of the theory (he calls it “Weierstrass canonical form”), which is a generalization of Weierstrass’ equation for elliptic curves. We refer to Kato [Ka], who also produces this representation, with proof that it exists.

Proposition 2.2. A pointed curve $\langle X, \infty \rangle$ with W-semigroup $H(\infty, \infty)$ for which $a_{\min}(H(X, \infty)) = m$ can be viewed as an $m : 1$ cover of $\mathbb{P}^1$; we denote by $x$ an affine coordinate on $\mathbb{P}^1$ such that the point $\infty$ on $X$ is mapped to $x = \infty$; we let $m_i := \min\{h \in H(\infty, \infty) \setminus \{0\} \mid i \equiv h \mod m\}, i = 0, 1, 2, \ldots, m - 1, m_0 = m$ and $n = \min\{m_j \mid (m, j) = 1\}$. Then, $(X, \infty)$ is defined by an irreducible equation,

\begin{equation}
(2.2) \quad f(x, y) = 0,
\end{equation}

for a polynomial $f \in \mathbb{C}[x, y]$ of type,

\begin{equation}
(2.3) \quad f(x, y) := y^m + A_1(x)y^{m-1} + A_2(x)y^{m-2} + \cdots + A_{m-1}(x)y + A_m(x),
\end{equation}

where the $A_i(x)$’s are polynomials in $x$, of degree $\leq jn/m$, with equality being attained only for $j = m$:

\begin{equation}
A_i = \sum_{j=0}^{\lfloor in/m \rfloor} \lambda_{i,j} x^j, \quad \lambda_{i,j} \in \mathbb{C}, \quad \lambda_{m,m} = 1.
\end{equation}

The affine curve $\text{Spec} \mathbb{C}[x, y]/f(x, y)$ may be singular and we denote by $X$ its unique normalization. Kato’s proof shows that the affine ring of the curve $X \setminus \infty$ is generated by functions $y_{m_i}$ whose only pole is $\infty$ with order the $g$ non-gaps $m_i$: let $I_m := \{m_1, m_2, \ldots, m_{m-1}\} \setminus \{m_{i_0}\}$, where $i_0$ is such that $n = m_{i_0}$, take $x = y_m$ and $y = y_n$; then the affine ring of $X \setminus \infty$ can be presented as $\mathbb{C}[x, y_n, y_{m_3}, \ldots, y_{m_l}]$ for $i_j \in M_g$, where $M_g := \{m_1, m_2, \ldots, m_\ell\} \subset \mathbb{N}^\ell$ with $(m, m_j) = 1$ for $i \neq j$, $m_1 = m, m_2 = n$, is a minimal set of generators for $H(\infty, \infty)$.

The curve in W-normal form, namely (2.3), admits a local $\mathbb{Z}/m\mathbb{Z}$-action at $\infty$, in the following sense. Sending $Z_1$ to $1/x$ and $Z_i$ to $1/y_{m_i}$ for $m_i \in \alpha(L)$, $i > 1$, we have the same kernel as under the homomorphism to the semigroup ring. The action is defined by sending $Z_i$ to $\zeta_m Z_i$, where $\zeta_m$ is a primitive $m$-th root of unity.

Our results on the sigma function pertain to the case in which $A_1, \ldots, A_{m-1}$ vanish, i.e., $y_m^m = A_1(x)$; this is the case if and only if $(x, y) \mapsto x$ is a cyclic cover with Galois
We call this case “cyclic W-normal form”, and in this paper, we obtain vanishing and inversion theorems for \( m = 3 \).

We treat as distinct the case when the curve has W-semigroup \( H(X, \infty) \) generated by two elements \((m, n)\) and the W-normal form defines a non-singular plane affine curve. This is known as \((n, s)\)-curve [BEL1, EEL] and much work on the sigma function already appeared, both in the cyclic and non-cyclic case. Our results for the cyclic case go through for \((3, s)\) curves (for \( r = 0 \)) although some statements need to be slightly modified and will be specified in the rest of the paper.

3. Trigonal cyclic curves

3.1. 3-semigroups. The minimal set of generators of a 3-semigroup can have two or three elements because 3 and the smallest element that is not a multiple of 3 generate all larger numbers with the same residue mod 3. We record the easy:

**Proposition 3.1.** A numerical semigroup \( \langle 3, p, q \rangle \) which is not generated by two elements, has minimal set of generators (not necessarily in increasing order) of type \( \{3, 2r+s, r+2s\} \) for some positive integers \( r, s, s > r \), and has genus \( g = r + s - 1 \).

**Proof.** Of the three possibilities, (1) \( p + q = 0 \mod 3 \), (2) \( p + q = 1 \mod 3 \), (3) \( p + q = 2 \mod 3 \), (2) would imply that both \( p \) and \( q \) are 2 mod 3, whereas the set of generators is assumed minimal. We similarly exclude (3), so we can write \( p = 3\ell + a \), \( q = 3\ell' + a' \), where \( a \cdot a' = 2 \), so that \( 2a - a' = 0 \mod 3 \); finally, \( r = (2p - q)/3 \) and \( s = (2q - p)/3 \) are positive integers because if \( q > 2p - 1 \), again 3 and \( p \) generate. The genus is the number of gaps, \( r + s - 1 \). \( \square \)

**Remark 3.2.** For \( r = 0 \) in Proposition 3.1 \( \langle 3, 2r+s, r+2s \rangle \) is reduced to the case of two generators, \( (3, s) \) and \( X \) is the compactification of a plane smooth curve: it is the \( r = 3 \) case of a cyclic \((n, s) = (r, s)\) curve in [MP1, MP2]; in this paper we generalize those results.

For brevity, henceforth we let \( \widehat{r} := 2r + s \) and \( \widehat{s} = 2s + r \), so that for \( r < s \), \( \widehat{r} < \widehat{s} \). Unless \( r = 0 \), we assume that \( 3, \widehat{r}, \widehat{s} \) are a minimal set of generators of the W-semigroup. When \( s > r > 0 \), the triple \( (3, \widehat{r}, \widehat{s}) \) is not telescopic in any order (because as we demonstrate, the corresponding Young diagram is not symmetric) whereas a semigroup with two generators is clearly telescopic.

We give more detail on the monomial ring (defined in Subsection 2.1) of a 3-semigroup. Again by [He] Th. 3.7],

**Proposition 3.3.** For the \( \mathbb{C} \)-algebra homomorphism,

\[
\varphi : k[Z] := k[Z_3, Z_{\widehat{r}}, Z_\widehat{s}] \to k[t^a]_{a \in \{3, \widehat{r}, \widehat{s}\}}, \quad Z_a \mapsto t^a,
\]

the kernel of \( \varphi_{\varphi} \) is generated by \( f_b = 0, b = 2\widehat{r}, \widehat{s} + \widehat{r}, g \), where

\[
f_{2\widehat{r}} = Z_{\widehat{r}}^2 - Z_3^s Z_\widehat{s}, \quad f_{\widehat{s} + \widehat{r}} = Z_{\widehat{r}} Z_\widehat{s} - Z_3^{s+r}, \quad f_g = Z_\widehat{s}^2 - Z_3^s Z_{\widehat{r}}.
\]
3.2. Singular trigonal cyclic curves. Let

\[
k_s(x) := (x - b_1) \cdots (x - b_s) \equiv x^s + \sum_{i=1}^{s} \lambda_i^{(s)} x^{s-i},
\]

\[
k_r(x) := (x - b_{s+1}) \cdots (x - b_{s+r}) \equiv x^r + \sum_{i=1}^{r} \lambda_i^{(r)} x^{r-i},
\]

\[
k_{s+r}(x) := (x - b_1) \cdots (x - b_s) \cdots (x - b_{s+r}) \equiv x^{r+s} + \sum_{i=1}^{r+s} \lambda_i^{(s+r)} x^{s+r-i},
\]

and \( \lambda_0^{(r)} = \lambda_0^{(s)} = \lambda_0^{(s+r)} = 1. \) We define \( f_{2\hat{r}}, f_{\hat{s}+\hat{r}}, f_{2\hat{s}} \in \mathbb{C}[x, y_\hat{r}, y_\hat{s}] \) by

\[
f_{2\hat{r}} = y_\hat{r}^2 - y_\hat{s}k_s(x), \quad f_{\hat{s}+\hat{r}} = y_\hat{r}y_\hat{s} - k_r(x)k_s(x), \quad f_{2\hat{s}} = y_\hat{s}^2 - y_\hat{r}k_s(x),
\]

cf. 3.1. We obtain cyclic W-normal forms,

\[
y_\hat{r}^3 = k_r^2(x)k_s(x), \quad y_\hat{s}^2 = k_s^2(x)k_r(x).
\]

The monomial-ring action trivially extends to the ring

\[
R_\lambda := \mathbb{Q}[x, y_\hat{r}, y_\hat{s}, \lambda_1^{(s)}, \ldots, \lambda_s^{(s)}, \lambda_1^{(r)}, \ldots, \lambda_r^{(r)}]/(f_{2\hat{r}}, f_{\hat{s}+\hat{r}}, f_{2\hat{s}}),
\]

graded by \( \text{wt}_\lambda : R_\lambda \to \mathbb{Z}, \lambda_i^{(s)} \to 3i \) and \( \lambda_i^{(r)} \to 3i, \) so that the equations that define the curve are homogeneous. This ring parametrizes the moduli (in a loose sense) of the curves over the rationals, and it can be used to address number-theoretic issues, although we do not do so in this paper.

We consider the curve \( X^{(3,\hat{r},\hat{s})} = \text{Spec } R^{(3,\hat{r},\hat{s})}, \) where

\[
R^{(3,\hat{r},\hat{s})} := \mathbb{C}[x, y_\hat{r}, y_\hat{s}]/(f_{2\hat{r}}, f_{\hat{s}+\hat{r}}, f_{2\hat{s}}),
\]

so that \( X^{(3,\hat{r},\hat{s})} \) is a curve in affine 3-space with coordinates \((x, y_\hat{r}, y_\hat{s}).\) For brevity, we write \( R \) for \( R^{(3,\hat{r},\hat{s})} \) and \( X, \) a \((3, \hat{r}, \hat{s})\)-type curve, for \( X^{(3,\hat{r},\hat{s})}. \) Equation (3.2), shows that there is a global \( \mathbb{Z}/3\mathbb{Z} \) action on \( X \) given by:

\[
\hat{\zeta}_3(x, y_\hat{r}, y_\hat{s}) = (x, \zeta_3 y_\hat{r}, \zeta_3^2 y_\hat{s}),
\]

where \( \zeta_3 \) is a primitive 3rd root of unity.

We denote the branch points of \( X \) viewed as a cover of the \( x \)-line by

\[
B_i := (x = b_i, y_\hat{r} = 0, y_\hat{s} = 0), \quad (i = 1, 2, 3, \ldots, s + r),
\]

and we consider the divisor,

\[
\mathfrak{B}_0 := B_1 + B_2 + \cdots + B_{r+s}.
\]

When \( r = 0, \) (3.2) corresponds to the plane curves of genus \( g = s - 1, \)

\[
y_\hat{r}^3 = k_s(x), \quad y_\hat{s} = y_\hat{r}^2,
\]

and our results clearly hold for that case also; assume henceforth \( r > 0. \)
Nagata’s Jacobian criterion can be used as in [KMP1, Prop.3.2] to show that Spec $R$ is non-singular, by checking that the $3 \times 3$ matrix of the partial derivatives in $(x, y, z)$ of $f_2, f_{s+2}$ and $f_2$ has rank 2 at each point of the curve.

3.3. The semigroup sequence. We define monic monomials $\phi_i$ in the ring $R$, whose poles at $\infty$ are the elements of the W-semigroup. The $\phi_i$’s are determined by requiring that the ordered set $(R_\phi := \{\phi_n \mid n \in \mathbb{N}\}, <)$, satisfies $N(n) < N(n + 1)$, where $N(n) := N(3, x, y)(n) = \text{wt}(\phi_n)$ (the order of pole at $\infty$), and

$$R = R^{(3, x, y)} = H^0(X, \mathcal{O}_X) = \bigoplus_{n=0} \mathbb{C}\phi_n = \bigoplus_{n \in R_\phi} \mathbb{C}\eta.$$ 

3.4. Differentials of the first kind. Using (3.3), define a subspace of $R$: $\tilde{R} := \{f \in R \mid \text{there exists } \ell \text{ such that } (f - \mathfrak{M}_0 + \ell \infty > 0) = \bigoplus_{n=0} \mathbb{C}\hat{\phi}_n\}$, with basis an ordered (by weight) subset $(\tilde{R}_\hat{\phi} := \{\hat{\phi}_i \mid i = 0, \ldots, <\}$ of $(R_\phi, <)$. Let $\tilde{R}_{\hat{\phi}, n} := \{\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_n\}$. We denote the weight of $\hat{\phi}_n$ by $\tilde{N}(n) := \tilde{N}(3, x, y)(n) := \text{wt}(\hat{\phi}_n)$, consistent with the $\mathbb{Z}/3\mathbb{Z}$-action.

Since $\text{wt}(y_1y_2) = 3(r + s)$ and $\text{wt}(x^i) = 3i (i = 0, 1, \ldots, g - 1)$ is less than $3(r + s)$, we obtain the following statement:

Lemma 3.4. (1) $\hat{\phi}_i (i = 0, 1, 2, \ldots)$ is divisible by $y_1$ or $y_2$.

(2) $\hat{\phi}_{g+2} = y_1y_2$

(3) For $i < g + 2$, there is a non-negative integer $a$ such that $\hat{\phi}_i = x^ay_1$ or $\hat{\phi}_i = x^ay_2$.

(4) $\hat{\phi}_{g-1} = x^{g-1}y_3y_1$.

(5) $\hat{\phi}_i \neq x$ if $\tilde{r} + 3 < \tilde{s}$ or $s - r > 3$.

(6) By letting $g_1 := \left\lfloor \frac{\tilde{s} - 1}{3} \right\rfloor$ and $g_2 := \left\lfloor \frac{\tilde{r} - 1}{3} \right\rfloor$, we have $g = g_1 + g_2$ and $R_{\hat{\phi}, g-1} = \tilde{R}_1 \bigoplus \tilde{R}_2$, where $\tilde{R}_1 := \{x^iy_{1} \mid i = 0, \ldots, g_1 - 1\}$, $\tilde{R}_2 := \{x^iy_{2} \mid i = 0, \ldots, g_2 - 1\}$.

Proof. (1) and (5) are obvious. It is obvious that $y_1y_2$ belongs to $\tilde{R}$, so $\hat{\phi}_q = y_1y_2$ for $q$ such that $\tilde{N}(q) = 3r + 3s$. Since $y_1y_2 = k_1(x)k_2(x)$, $y_1y_2/(x - b_1)$ belongs to $R$, but not to $\tilde{R}$. We let $i_r$ and $i_s$ such that $\hat{\phi}(i_r) = y_1$ and $\hat{\phi}(i_s) = y_2$, which implies $i_r < i_s < q - 2$.

Hence clearly, for $i < q$, $\hat{\phi}_i = \left\{x^{j_i} y_{1}^{k_i} \mid j_i, k_i \geq 0 \right\}$, where $j_i$ and $k_i$ are non-negative integers. Hence $\tilde{N}(q - a) = \tilde{N}(q) - a$ for $a = 1, 2$ and $\tilde{N}(q - 3) = \tilde{N}(q) - 4$. If $i > q$, it is obvious that $\tilde{N}(i) = \tilde{N}(i - 1) + 1$. Noting that $\tilde{N}(0) = \tilde{r} = 2r + s$, $q = g_1 + g_2 + 2 = r + s + 1 = g + 2$, (6) is proved and thus (3) and (2) are also proved. Hence $\tilde{N}(g - 1) = 2g - 2 + r + s$, and $\hat{\phi}_{g-1}$ is equal to $x^{g_2-1} y_1 \cdot x^{g_2} y_2$ or $x^{g_3-1} y_1 \cdot x^{g_3} y_2$ and both are equal to $x^{g_1-1} y_2 y_1$. This proves (4).

In the course of the proof we showed the following:
Remark 3.9. When $\nu$s, so that
\[ \frac{\prod_{i=0}^{\nu} k_i(x)k_s(x)}{y_i y_s} \equiv \sum_{i=0}^{\nu} a_i \frac{x^i dx}{k_r(x)}, \]
is holomorphic over $X$, then each $a_i$ vanishes.

Lemma 3.6. For a non-negative integer $n < s + r$, if the differential
\[ \sum_{i=0}^{n} a_i \frac{x^i dx}{y_i y_s} \equiv \sum_{i=0}^{n} a_i \frac{x^i dx}{k_r(x)}, \]
is a basis of $H^0(X \setminus \infty, \Omega_X)$ as a $\mathbb{C}$-vector space, in
particular a basis of $H^0(X, \Omega_X)$ is given by \( \left\{ \phi_i : \frac{dx}{3y_i y_s}, \mid \phi_i \in \hat{R}_{g} \right\}, i = 1, 2, \ldots, g \).

Corollary 3.8. Coordinates for the canonical embedding of $X$ into $\mathbb{P}^{g-1}$ can be given by
[\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_{g-2}, \hat{\phi}_{g-1}].

Remark 3.9. When $r = 0$, $\hat{R}$ is given by $\hat{R} = y_i R$, i.e., $\hat{\phi}_i = y_i \phi_i$ for every $i = 0, 1, 2, \ldots,$
so that $\nu^i = \frac{\phi_i dx}{3y_i y_s} = \frac{\phi_{i+1} dx}{3y_i y_s}$.

3.5. The canonical divisor.

Remark 3.10. The divisors of our basis of one-forms are given by:
\begin{align*}
(\nu^1) &= B_{s+1} + \cdots + B_{s+r} + (2s + r)\infty, \\
(\nu^i) &= B_1 + B_2 + \cdots + B_s + (2r+s)\infty, \quad 1 \leq i < s \\
(\nu^j) &= \sum_{a=0}^{2} (0, c_a^3 \sqrt{k_r(0)}, c_a^{2a} \sqrt{k_s(0)}) + B_{s+1} + \cdots + B_{s+r} + (2s + r - 3)\infty, \quad 1 \leq j < r,
\end{align*}
where $B_a := (b_a, 0, 0) (a = 1, 2, \ldots, s + r)$. Noting $\nu^{s+r} = g + 1$, we have:
\[ \mathcal{K}_X \sim 2(g - 1)\infty - 2(B_{s+1} + \cdots + B_{s+r} - (r)\infty) \]
\begin{align*}
&\sim 2((g - 1)\infty - 2(B_1 + B_2 + \cdots + B_s - (s)\infty)) \\
&\sim 2(g - 1)\infty - (B_1 + B_2 + \cdots + B_s + B_{s+1} + \cdots + B_{s+r} - (s + r)\infty).
\end{align*}

In fact, any positive divisor linearly equivalent to $\mathcal{K}_X$ must include points of $X \setminus \infty$, because $H(X, \infty)$ is non-symmetric, whereas in the symmetric case, which includes $(n, s)$-curves, $(2g - 2)\infty$ is a canonical divisor.
3.6. Differentials of the second and third kind. We produce a differential form which, up to a tensor of holomorphic one-forms, is the normalized fundamental differential of the second kind in \[\mathcal{E}\] Corollary 2.6, namely, a two-form \(\Omega(P_1, P_2)\) on \(X \times X\) which is symmetric,

\[
\Omega(P_1, P_2) = \Omega(P_2, P_1),
\]

has its only pole (of second order) along the diagonal of \(X \times X\), and in the vicinity of each point \((P_1, P_2)\) is expanded in power series as

\[
\Omega(P_1, P_2) = \left(\frac{1}{(t_{P_1} - t_{P_2})^2} + d_\geq(1)\right) dt_{P_1} \otimes dt_{P_2} \quad \text{(as } P_1 \to P_2\text{)}.
\]

We follow the work done in \[\text{BEL1 BLE EEL}\] for \((n, s)\) curves, adapted in \[\text{MK KMP1}\] to \((3, 4, 5)\)- and \((3, 7, 8)\)-curves. The explicit form of \(\Omega\) enables the algebraic (as opposed to modular) construction of sigma: we call it “EEL-construction”, as it is given theoretically in \[\text{EEL}\]. A computation shows:

**Proposition 3.11.** Let \(\Sigma(P, Q)\) be the following form,

\[
\Sigma(P, Q) := \frac{y_{f, P} y_{s, P} + y_{f, P} y_{s, Q} + y_{f, Q} y_{s, P}}{3(x_P - x_Q) y_{f, P} y_{s, P}} dx_P.
\]

Then \(\Sigma(P, Q)\) has the properties

1. \(\Sigma(P, Q)\) as a function of \(P\) is singular at \(Q = (x_Q, y_{f, Q}, y_{s, Q})\) and \(\infty\), and vanishes at \(\hat{\zeta}_3(Q) = (x_Q, \zeta_3 y_{f, Q}, \zeta_3^2 y_{s, Q})\).
2. \(\Sigma(P, Q)\) as a function of \(Q\) is singular at \(P\) and \(\infty\).

**Remark 3.12.**

1. When \(r = 0\) \((3.6)\) reduces to

\[
\Sigma(P, Q) = \frac{y_{f, P}^2 + y_{f, P} y_{f, Q} + y_{f, Q}^2}{3(x_P - x_Q) y_{f, P}^2} dx_P.
\]

2. The Galois group \(\mathbb{Z}/3\mathbb{Z}\) of the covering \(X \to (\mathbb{P}^1, P \mapsto x)\), acts on the numerator of \(\Sigma\) in Proposition 3.11. This is what enables our technique for \(k\)-cyclic pointed curves \((X, P)\): finding the explicit expression in affine coordinates for the normalized fundamental differential of the second kind is the focus of current research even for \((n, s)\) curves (2-generator \(W\)-semigroup).

**Definition 3.13.**

1. For \(f \in \mathbb{C}[x_P, x_Q, y_{s, P} y_{f, P}, y_{s, Q} y_{f, Q}]\), \(\hat{\sigma}_{Q, i}(f)\) is the \(y_{f, Q}\)-coefficient of \(f\) if \(\hat{\phi}_i \in \hat{R}\) or the \(y_{s, Q}\)-coefficient if \(\hat{\phi}_i \in \hat{R}\).
2. \(\hat{D}_{r,s}(P, Q) \in \mathbb{Z}[x_P, x_Q, y_{s, P} y_{f, P}, y_{s, Q} y_{f, Q}, y_{f, P}, \lambda^{(r)}_1, \ldots, \lambda^{(r)}_r, \lambda^{(s)}_1, \ldots, \lambda^{(s)}_s]\) is defined as:

\[
\hat{D}_{r,s}(P, Q) := y_{s, P} y_{f, Q} D^{(+)}_{s,r}(P, Q) + y_{s, Q} y_{f, P} D^{(-)}_{s,r}(P, Q),
\]

where

\[
D^{(\pm)}_{s,r}(P, Q) \in \mathbb{Z}[x_P, x_Q, \lambda^{(r)}_1, \ldots, \lambda^{(r)}_r, \lambda^{(s)}_1, \ldots, \lambda^{(s)}_s].
\]
where second kind such that they have a simple pole at \( \infty \).

Appendix A. The Corollary, which gives the expression for the normalized fundamental differential, is a straightforward verification.

\[ D_{s,r}^{(+)}(P,Q) := \sum_{j=0}^{s+y-2} \sum_{i=0}^{s-j-2} (i+1)\lambda_j^{(s+r)}x_{P}^{r+2-s-j-i}x_{Q}^{s-j-i-2} \]

\[ + \sum_{j=0}^{s-2} \sum_{i=0}^{s-j-2} \sum_{k=0}^{r} (i+1)\lambda_j^{(s)}\lambda_{r-k}x_{P}^{s-j-i-2}x_{Q}^{k+i} \]

\[ D_{s,r}^{(-)}(P,Q) := \sum_{j=0}^{s+y-2} \sum_{i=0}^{s-j-2} (i+1)\lambda_j^{(s+r)}x_{Q}^{r+2-s-j-i}x_{P}^{s-j-i-2} \]

\[ + \sum_{j=0}^{r-2} \sum_{i=0}^{s-j} \sum_{k=0}^{s} (i+1)\lambda_j^{(r)}\lambda_{s-k}x_{Q}^{r-j-i-2}x_{P}^{k+i} \]

Note that \( \tilde{D}_{r,s}(P,Q) \) is homogeneous with respect to the extended weight \( w_t \).

On the non-singular curve \( X \), the following Proposition holds: the proof is given in Appendix A. The Corollary, which gives the expression for the normalized fundamental differential, is a straightforward verification.

**Proposition 3.14.** There exist differentials \( \nu^I_{j} = \nu^I_{j}(x,y) \) \( (j = 1,2,\ldots,g) \) of the second kind such that they have a simple pole at \( \infty \) and satisfy the relation,

\[ d_Q \Sigma(P,Q) - d_P \Sigma(Q,P) = \sum_{i=1}^{g(\tilde{r},\tilde{s})} \left( \nu^I_{i}(Q) \otimes \nu^I_{i}(P) - \nu^I_{i}(P) \otimes \nu^I_{i}(Q) \right) \]

where

\[ d_Q \Sigma(P,Q) := dx_P \otimes dx_Q \frac{\partial}{\partial x_Q} \frac{y_{s,P,y_{s,P}} + y_{s,P,s,y_{s,P}} + y_{s,P}y_{s,P}dx_P}{(x_P - x_Q)3y_{s,P}y_{s,P}} dx_P. \]

The set of differentials \( \{ \nu^I_{1}, \nu^I_{2}, \nu^I_{3}, \ldots, \nu^I_{g} \} \) is determined modulo the linear space spanned by \( \nu^I_{j} \) \( j = 1,\ldots,g \) and it has representatives

\[ \nu^I_{i}(P) = \begin{cases} \frac{(\tilde{\phi}_i(\tilde{D}_{r,s}(P,Q))/y_{s,P})dx_P}{3y_{s,P}} & \text{if } \tilde{\phi}_i \in \tilde{R}_{\tilde{s}}; \\ \frac{(\tilde{\phi}_i(\tilde{D}_{r,s}(P,Q))/y_{s,P})dx_P}{3y_{s,P}} & \text{if } \tilde{\phi}_i \in \tilde{R}_{\tilde{s}}; \end{cases} \]

where \( \tilde{R}_{\tilde{s}} \) and \( \tilde{R}_{\tilde{s}} \) are defined in Definition 3.7.

**Corollary 3.15.** (1) The one-form \( \Pi^{P}_{P}(P) := \Sigma(P,P_1)dx - \Sigma(P,P_2)dx \) is a differential of the third kind, whose only (first-order) poles are \( P = P_1 \) and \( P = P_2 \), with residues +1 and -1 respectively.
(2) The fundamental differential of the second kind \( \Omega(P_1, P_2) \) is given by
\[
\Omega(P_1, P_2) = d_{P_2} \Sigma(P_1, P_2) + \sum_{i=1}^{g} \nu^{I_i}(P_1) \otimes \nu^{II_i}(P_2)
\]
\[
= \frac{F(P_1, P_2)dx_1 \otimes dx_2}{9(x_1 - x_2)^2y_{\tilde{r},P_1}y_{\tilde{s},P_1}y_{\tilde{r},P_2}y_{\tilde{s},P_2}}, \quad F \in R \otimes R.
\]

Lemma 3.16. We have
\[
\lim_{P_1 \to \infty} \frac{F(P_1, P_2)}{\phi_{\tilde{g}}^{-1}(P_1)(x_1 - x_2)^2} = \phi_{\tilde{g}}(P_2),
\]
and \( \phi_{\tilde{g}}(P_2) \) is equal to \( x_{\tilde{g}r}^{g}y_{\tilde{r},P_2} \) or \( x_{\tilde{g}s}^{g}y_{\tilde{s},P_2} \).

Proof. Lemma 3.4 applied to \( D_{r,s} := D_{s,r}^{(+)} - D_{s,r}^{(-)} \) in Definition 3.13

Remark 3.17. Corollary 3.15 and Lemma 3.16 hold for \( r = 0 \), with
\[
\lim_{P_1 \to \infty} \frac{F(P_1, P_2)}{\phi_{\tilde{g}}^{-1}(P_1)(x_1 - x_2)^2} = \phi_{\tilde{g}}(P_2).
\]

For \( P = (x, y_{\tilde{r}}, y_{\tilde{s}}) \), we let \( h_i(P) = 3y_{\tilde{r}}y_{\tilde{s}}\nu^{I_i}(P)/dx, \ 1 \leq i \leq g \) (a representative in \( R \) of the corresponding differential).

In Section 4 (Proposition 4.14) we will use:
\[
\Omega_{P_1, P_2}^{Q_1, Q_2} := \int_{P_2}^{P_1} \int_{Q_2}^{Q_1} \Omega(P, Q)
\]
\[
= \int_{P_2}^{P_1} (\Sigma(P, Q_1) - \Sigma(P, Q_2)) + \sum_{i=1}^{g} \int_{P_2}^{P_1} \nu^{I_i}(P) \int_{Q_2}^{Q_1} \nu^{II_i}(P).
\]

4. Transcendental aspects

In this section we set up the notation for the Jacobian of the curve and its invariant vector fields, briefly recall properties of the sigma function, definitions for Wirtinger strata and the Jacobi inversion problem.

A pointed curve \( (X, P) \) has a natural embedding in the normalized Jacobian \( J^{\circ} := \mathbb{C}^{g}/\Gamma^{\circ} \), where \( \Gamma^{\circ} \) is a normalized period lattice \( \mathbb{Z}^{g} + \tau \mathbb{Z}^{g} \), and let \( \kappa^{\circ} : \mathbb{C}^{g} \to \mathbb{C}/\Gamma^{\circ} \) be the projection. We refer to [F, Ch. I] for the definition of the Riemann theta function with characteristics \( \theta \left[ \begin{smallmatrix} a \\ b \end{smallmatrix} \right] (z, \tau), \ z \in \mathbb{C}^{g} \).

Let \( \tilde{X} \) be the fundamental covering space of \( X \), \( \varpi : \tilde{X} \to X \) the projection from the path space to the orbit space, and \( \iota : X \to \tilde{X} \) the natural embedding \( X \) into \( \tilde{X} \) such that \( \varpi \circ \iota = id \). For the \( k \)-symmetric product of \( \tilde{X} \), \( S^{k}\tilde{X} \), we define the Abel map \( \tilde{v} \) from \( S^{k}\tilde{X} \) to \( \mathbb{C}^{g} \), normalized at \( P \in X \), by taking the sum of the integrals of a normalized basis of differentials of the first kind, from the point \( \iota P \in \tilde{X} \), to the \( k \)-tuple of points \( P_j \), through
any paths that join \( iP \) to each \( P_j \) (these become identified in the orbit space). Using the embedding \( i \), we also have a map \( v \) from the symmetric product of the curve to \( \mathcal{J}^0 \), 
\[ v = \kappa^o \circ \tilde{v} \circ i : S^kX \to \mathcal{J}^0. \]
We denote by \( \Theta^o \) the divisor of \( \mathcal{J}^0 \) defined by \( \theta(z, \tau) \) and recall the following classical result, for which we choose to quote Theorem 7 and Theorem 11 in [Le]:

**Proposition 4.1.**  
1. The “canonical” theta divisor \( v(S^kX) \) is a translate of \( \Theta^o \) by 
\[ \kappa^o(\Delta), \text{ where } \Delta \text{ is the “Riemann constant” (cf. [F] Ch. I, (13))}. \]
2. An effective divisor \( D \) of degree is \( 2g - 2 \) satisfies \( v(D) - v((2g - 2)P) + 2\Delta = 0 \) modulo \( \Gamma^o \) if and only if \( D \) is the divisor of a holomorphic differential.
3. When the canonical divisor equals \( (2g - 2)P \), the image \( \kappa^o(\Delta) \) of Riemann constant is a point of order two on the Jacobian.

We recall that \( K_X = (2g - 2)P \) exactly when the W-semigroup \( H(X, P) \) is symmetric. To streamline the theory, in [KMP2] we introduced a positive divisor \( \mathfrak{B} \), of degree \( d_0 := \deg(\mathfrak{B}) \), such that:

**Proposition 4.2.** For \( \mathfrak{B} \) given by the equality (in the sense of linear equivalence) \( K_X = 2D_0 = 2(g - 1 + d_0)P - 2\mathfrak{B} \), the “shifted Abel maps” defined by \( \tilde{v}_k(P_1, \ldots, P_k) = \tilde{v}(P_1, \ldots, P_k) + \tilde{v}(\mathfrak{B}) \), for \( P_1, \ldots, P_k \in \tilde{X} \), and \( v_s := \kappa^o \tilde{v}_s \), and for the “shifted Riemann constant” defined by \( \Delta_s := \Delta - \tilde{v}(\mathfrak{B}) \in \mathbb{C}^g \),

1. \( \Delta_s \) belongs to \( \frac{1}{2} \Gamma \),
2. The difference between the “shifted canonical theta divisor” \( v_s(S^{g-1}X) \) and \( \Theta^o \) is given by the shifted Riemann constant \( \Delta_s \in \mathbb{C}^g \), i.e., as sets,
\[ v_s(S^{g-1}X) + \Delta_s = \Theta^o \mod \Gamma^o. \]

We note that \( K_X = 2D_0 \) defines \( D_0 \), or rather its image in the Jacobian, which is a divisible Abelian group. We note also that Proposition 4.2 is trivially true for \( \mathfrak{B} = 0 \) when \( H(X, P) \) is symmetric.

**Corollary 4.3.** There is a theta characteristic,

\[ \delta := \begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} \in \left( \frac{1}{2} \mathbb{Z} \right)^{2g}, \]
which is equal to the shifted Riemann constant \( \Delta_s \), namely, for every \( (P_1, P_2, \ldots, P_{g-1}) \in S^{g-1} \tilde{X}, \theta(\tilde{v}_s(P_1, \ldots, P_{g-1}) + \Delta_s, \tau) = \theta[\delta](\tilde{v}_s(P_1, \ldots, P_{g-1}), \tau) = 0. \]

4.1. The trigonal cyclic case. Recalling the calculation of \( K_X \) in Subsection 3.6 the divisor of Proposition 4.2 is \( \mathfrak{B} = B_{s+1} + B_{s+2} + \cdots + B_{r+s} \), thus \( K_X \sim 2(g - 1 + r) \infty - 2\mathfrak{B}. \)

We introduce notation convenient for stating our results on the sigma function, its vanishing order, and Jacobi inversion. For a standard symplectic basis \( \alpha_i, \beta_j (1 \leq i, j \leq g) \), of \( H_1(X, \mathbb{Z}) \) we denote the period matrices by

\[ [\omega' \omega''] = \frac{1}{2} \left[ \int_{\alpha_i} \nu^i_j \int_{\beta_i} \nu^i_j \right]_{i,j=1,2,\ldots,g}. \]
so the matrix \( \tau = \omega'^{-1} \omega'' \), and \( \Gamma \) is the lattice generated by \( \omega' \) and \( \omega'' \), equivalently, by \( (I_g, \tau) \). The ‘unnormalized’ Jacobian is given by \( J := \mathbb{C}^g / \Gamma \), \( \kappa : \mathbb{C}^g \to J \). Since the basis \( \{ \nu_i^j \} \ (i = 1, \ldots, g) \) differs from the standard basis of \( H^0(X, \Omega) \) normalized with respect to the \( \alpha, \beta \)-cycles, we redefine the Abel map \( \tilde{w} \) and attendant shifted Abel map \( w_s : X \to \mathbb{C}^g \), so that

\[
\begin{align*}
\tilde{w} & = (2\omega')^{-1} \tilde{w}, & \tilde{w}_s & = (2\omega')^{-1} \tilde{w}_s, & w & = \kappa \circ \tilde{w} \circ \iota, & w_s & = \kappa \circ \tilde{w}_s \circ \iota.
\end{align*}
\]

Since \( \mathfrak{B} = B_{s+1} + B_{s+2} + \cdots + B_{r+s} \), the shifted Abel map is given by:

\[
(4.3) \quad \tilde{w}_s(P_1, \ldots, P_k) := \tilde{w}(P_1, \ldots, P_k) + \tilde{w}(\nu B_{s+1}, \cdots, \nu B_{s+r}).
\]

For each component \( u_i \) of a vector \( u \in \mathbb{C}^g \), we extend the weight \( \text{wt} \) that we introduced for \( R_\phi \) and \( R_\phi \) and the basis of holomorphic differentials that we introduced in Proposition 3.7, \( \text{wt}(u_i) = \Lambda_{g-i+1} + (g - i) \). It should be noted that the image of the normalized Abel map does not have such a natural weight, where the word “natural” refers to the order of poles at infinity of functions on the affine part of the curve (and, in consequence, to the cyclic action). In particular, there is a \( \mathbb{Z}/3\mathbb{Z} \)-action on \( J_g \), defined by: \( \tilde{\zeta}_3(x, y_r, y_z) = (x, \zeta_3 y_r, \zeta_3^2 y_z) \) and the holomorphic differentials in Definition 3.7.

We define the subvarieties \( \mathcal{W}_k \) and \( \mathcal{W}_s^k \)

\[
(4.4) \quad \mathcal{W}_k := w(\mathcal{S}_k X), \quad \mathcal{W}_s^k := w_s(\mathcal{S}_k X),
\]

We call them “Wirtinger strata” because their images under the \(|2\Theta| \) divisor map are the classical Wirtinger varieties. By the Abel-Jacobi theorem, \( \mathcal{W}_g = \mathcal{W}_g^g = J \).

To state vanishing theorems in Section 4, we also define the strata: \( \mathcal{W}_{s,1}^k := w(\mathcal{S}_1^k X) \ (\mathcal{W}_k := (\mathcal{S}_k^k X)) \), where \( \mathcal{S}_m^g(X) := \{ D \in \mathcal{S}^g(X) \mid \dim |D| \geq m \} \).

### 4.2. Affine functions on \( X \) and linear equivalence.

Let \( n \) be a positive integer and \( P_1, \ldots, P_n \) be in \( X \setminus \infty \); define the Frobenius-Stickelberger (FS) matrices of \( R \) and \( \tilde{R} \),

\[
(4.5) \quad \Psi_n(P_1, P_2, \ldots, P_n) := (\phi_i(P_j))_{i=0, \ldots, n-1, j=1, \ldots, n}, \quad \tilde{\Psi}_n(P_1, P_2, \ldots, P_n) := (\tilde{\phi}_i(P_j))_{i=0, \ldots, n-1, j=1, \ldots, n}.
\]

The Frobenius-Stickelberger (FS) determinant is

\[
\psi_n(P_1, \ldots, P_n) := \det(\Psi_n(P_1, \ldots, P_n)), \quad \tilde{\psi}_n(P_1, \ldots, P_n) := \det(\tilde{\Psi}_n(P_1, \ldots, P_n)).
\]

**Definition 4.4.** For \( P, P_1, \ldots, P_n \in (X \setminus \infty) \times \mathcal{S}^n(X \setminus \infty) \), we define \( \mu_n(P) \) by

\[
\begin{align*}
\mu_n(P) & := \mu_n(P; P_1, \ldots, P_n) := \lim_{P_i \to P} \frac{1}{\psi_n(P_1', \ldots, P_n')} \psi_{n+1}(P_1', \ldots, P_n', P), \\
\tilde{\mu}_n(P) & := \tilde{\mu}_n(P; P_1, \ldots, P_n) := \lim_{P_i \to P} \frac{1}{\tilde{\psi}_n(P_1', \ldots, P_n')} \tilde{\psi}_{n+1}(P_1', \ldots, P_n', P),
\end{align*}
\]

where \( \psi_n \) and \( \tilde{\psi}_n \) are the determinants of the matrices \( \Psi_n \) and \( \tilde{\Psi}_n \) respectively.
where the $P_i'$ are generic, the limit is taken (irrespective of the order) for each $i$; and
$\mu_n,k(P_1,\ldots,P_n)$ by and $\widehat{\mu}_n,k(P_1,\ldots,P_n)$ by

$$
\mu_n(P) = \phi_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \mu_n,k(P_1,\ldots,P_n)\phi_k(P),
$$

$$
\widehat{\mu}_n(P) = \widehat{\phi}_n(P) + \sum_{k=0}^{n-1} (-1)^{n-k} \widehat{\mu}_n,k(P_1,\ldots,P_n)\widehat{\phi}_k(P),
$$

with the convention $\mu_{n,n}(P_1,\ldots,P_n) \equiv \widehat{\mu}_{n,n}(P_1,\ldots,P_n) \equiv 1$.

**Remark 4.5.** When $r = 0$, $\widehat{\mu}_n(P)$ is equal to $y_{\widehat{r}}(P)\mu_n(P)$ because of Remark 3.3.

The meromorphic functions we introduced enable us to express the addition structure of Pic $X$ in terms of FS-matrices. We developed this theory for $(n,s)$ curves and here we adapt it to curves $(X,P)$ that have a non-symmetric semigroup.

For $n$ points $(P_i)_{i=1}^{n} \in X \setminus \infty$, we find an element of $R$ associated with any point $P = (x,y)$ in $(X \setminus \infty)$, $\alpha_n(P) := \alpha_n(P_1,\ldots,P_n) = \sum_{i=0}^{n} a_i \phi_i(P)$, $a_i \in \mathbb{C}$ and $a_n = 1$, which has a zero at each point $P_i$ (with multiplicity, if the $P_i$ are repeated) and has smallest possible order of pole at $\infty$ with this property. Then $\alpha_n$ can be identified with $\mu_n(P)$. We have the following lemma.

**Lemma 4.6.** Let $n$ be a positive integer. For $(P_i)_{i=1}^{n} \in S^n(X \setminus \infty)$, the function $\alpha_n$ over $X$ induces the map $\alpha_n : S^n(X \setminus \infty) \to S^{n(n)-n}(X)$, sending $(P_i)_{i=1}^{n} \in S^n(X \setminus \infty)$ to an element $(Q_i)_{i=1}^{n} \in S^{N(n)-n}(X)$, such that

$$
\sum_{i=1}^{n} P_i - n\infty \sim \sum_{i=1}^{n} Q_i + (N(n) - n)\infty.
$$

Note that the function $\widehat{\alpha}_n := \frac{\widehat{\mu}_n dx}{3y_{\widehat{r}}y_{\widehat{s}}}$ is non-singular over $X \setminus \infty$, unlike $\frac{\mu_n dx}{3y_{\widehat{r}}y_{\widehat{s}}}$. The divisor of $\widehat{\mu}_n(x)$ contains $\sum_{i=1}^{s+r} B_a - (s + r)\infty \sim 2(B_{s+1} + \cdots + B_{s+r}) - 2r\infty$ for any $n$.

The following is proved as in [KMP1], Lemma 6.2.

**Lemma 4.7.** Let $n$ be a positive integer. For $(P_i)_{i=1}^{n} \in S^n(X \setminus \infty)$, the function $\widehat{\alpha}_n$ over $X$ induces a map $\widehat{\alpha}_n : S^n(X \setminus \infty) \to S^{\widehat{N}(n)-n-g-1}(X)$, that sends $(P_i)_{i=1}^{n}$ to a $(Q_i)_{i=1}^{n}, \widehat{N}(n)-n-g-1$ such that

$$
\sum_{i=1}^{n} P_i + B_{s+1} + \cdots + B_{s+r} - (n + r)\infty
\sim \left(\sum_{i=1}^{\widehat{N}(n)-n-g-1} Q_i + B_{s+1} + \cdots + B_{s+r} - (\widehat{N}(n) - n - s)\infty\right).
$$
In order for the preimage of $\alpha_n$ and $\widehat{\alpha}_n$ to include the base point $\infty$, we extend the maps as follows: for an effective divisor $D$ in $S^n(X)$ of degree $n$, let $D'$ be the maximal subdivisor of $D$ which does not contain $\infty$, $D = D' + (n - m)\infty$ where $\deg D' = m(\leq n)$ and $D' \in S^m(X \setminus \infty)$, and define $\overline{\alpha}_n$ by $\overline{\alpha}_n(D) = \alpha_m(D') + [N(n) - n - (N(m) - m)]\infty$. We modify the derivation of the Abel-Jacobi theorem and Serre duality given in [KMP1], based on Remark 3.10 (assuming $\alpha_n$ and $\widehat{\alpha}_n$ behave in analogous ways to each other). By linear equivalence (cf. Lemmas 4.6 and 4.7):

**Proposition 4.8.** For a positive integer $n$, the Abel map composed with $\alpha_n$ (given in terms of $\mu_n$) and $\widehat{\alpha}_n$ (in terms of $\widehat{\mu}_n$) induce $$\iota_n : \mathcal{W}^n \to \mathcal{W}^n_{\infty^{n-g-1}}, \quad \widehat{\iota}_n : \mathcal{W}^n_s \to \mathcal{W}^n_{\infty^{n-g-1}}.$$ 

**Remark 4.9.** As per Proposition 4.8 under the shifted Abel map we have, for any $P_1, P_2, \ldots, P_g$ and appropriate $Q$’s in $X$, 

$$-w_s(P_1, P_2, \ldots, P_{g-1}) = w_s(Q_1, Q_2, \ldots, Q_{g-1}), \mod \Gamma, \quad (4.6)$$

$$-w_s(P_1, P_2, P_3, \ldots, P_g) = w_s(Q_1, Q_2, Q_3, \ldots, Q_g) \mod \Gamma.$$ 

The first relation, 

$$-w(P_1, P_2, \ldots, P_{g-1}) = w(Q_1, Q_2, \ldots, Q_{g-1}) + 2w(B_{s+1}, \ldots, B_{s+r}), \quad (4.7)$$

shows that Serre duality on $X$ is given as, $\widehat{\iota}_{g-1} : \mathcal{W}^{g-1}_s \to \mathcal{W}^{g-1}_s$ by 

$$P_1 + P_2 + \cdots + P_{g-1} + B_{s+1} + \cdots + B_{s+r} - (2r + s)\infty \sim -(Q_1 + Q_2 + \cdots + Q_{g-1} + B_{s+1} + \cdots + B_{s+r} - (2r + s)\infty).$$

We therefore denote image($\iota_n$) by $[-1], \mathcal{W}^n_s$, especially $\iota_g : \mathcal{W}^g_s \to [-1], \mathcal{W}^g_s$. For $n \geq g$, $\iota_g \circ \iota_n$ gives addition in the Picard group, 

$$\mathcal{W}^n_s \xrightarrow{\iota_n} \mathcal{W}^g_s \xrightarrow{\iota_g} \mathcal{W}^g_s, \quad (w_s(P_1, \ldots, P_n) \equiv w_s(Q_1, \ldots, Q_g) \mod \Gamma.$$ 

In particular, the addition law on the Jacobian is given by $\iota_g \circ \iota_{2g}$ 

$$\mathcal{W}^{2g}_s \xrightarrow{\iota_{2g}} \mathcal{W}^g_s \xrightarrow{\iota_g} \mathcal{W}^g_s, \quad (w_s(P_1, \ldots, P_g, P'_1, \ldots, P'_g) \equiv w_s(Q_1, \ldots, Q_g) \mod \Gamma.$$ 

The above arguments and Lemma 3.5 give the following corollary (Serre duality and the Abel-Jacobi theorem):

**Corollary 4.10.** $-\mathcal{W}^{g-1}_s = \mathcal{W}^{g-1}_s, \quad -\mathcal{W}^g_s = \mathcal{W}^g_s.$

It is the ‘shift’ that allows us to conclude:

**Proposition 4.11.** For some $(P_1, \ldots, P_{g-1}) \in \mathcal{S}^{g-1}X$, $w_s(P_1, \ldots, P_{g-1}) = 0$. 

**Proof.** Noting $g = r + s - 1$, we set $(P_1, \ldots, P_{g-1}) = (B_1, \ldots, B_{s+r})$; then, $(y_r) = B_1 + \cdots + B_s + 2(B_{s+1}, \ldots, B_{s+r}) - (2r + s)\infty \sim 0.$ 

$\square$
4.3. **Vector fields.** We give expressions for differential operators on $S^kX$ or invariant vector fields on the Jacobian using the coordinates of the Abel map. We use the convention that for $P_a \in X$, $P_a$ is expressed by $(x_a, y_{\tau,a}, y_{\sigma,a})$ or $(x_{P_a}, y_{\tau,P_a}, y_{\sigma,P_a})$. By letting $(u_1, \ldots, u_g) := w(P_1, \ldots, P_g)$, we have

\[
\begin{pmatrix}
\frac{\partial}{\partial u_1} \\
\frac{\partial}{\partial u_2} \\
\frac{\partial}{\partial u_3} \\
\vdots \\
\frac{\partial}{\partial u_g}
\end{pmatrix}
= \Psi^{-1}_g(P_1, P_2, \ldots, P_g) \begin{pmatrix}
3y_{\tau,1}y_{\sigma,1}\frac{\partial}{\partial x_1} \\
3y_{\tau,2}y_{\sigma,2}\frac{\partial}{\partial x_2} \\
3y_{\tau,3}y_{\sigma,3}\frac{\partial}{\partial x_3} \\
\vdots \\
3y_{\tau,g}y_{\sigma,g}\frac{\partial}{\partial x_g}
\end{pmatrix}.
\]

These relations hold for the image of the shifted Abel map, thus similar relations involving submatrices of $\Psi_g$ hold over the strata $w_s(S^kX)$ for $k < g$; also:

\[
\sum_{i,j=1}^{g} \hat{\phi}_{i-1}(P_1)\hat{\phi}_{j-1}(P_2) \frac{\partial^2}{\partial w_i(P_1) \partial w_j(P_2)} = 9y_{\tau,1}y_{\sigma,1}y_{\tau,2}y_{\sigma,2} \frac{\partial^2}{\partial x_1 \partial x_2}.
\]

4.4. **The sigma function.** In genus one, the Weierstrass sigma function [WWW, 21–43]

\[
\sigma(u) = \frac{2\omega_1}{\theta'} \exp \left\{ \frac{\eta_1}{2\omega_1} u^2 \right\} \theta_1 \left( \frac{u}{2\omega_1} \omega_1 \right)
\]

where $\theta_1$ is the theta function with characteristics $[1/2, 1/2]$, $\theta_1' := \theta_1'(0, \omega_1)$ is “equivalent” in the sense of [La, Ch. VI] to a first-order theta function with characteristics, and the complete integrals of first and second kind, $\omega_1, \omega_2$ and $\eta_1, \eta_2$ satisfy the Legendre relation: $\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2$. Theta functions satisfy a given relation with respect to the period lattice, they are called equivalent when they differ by a “trivial” theta function, and [La, Ch. X Th. 1.1] shows that there is a unique normalized entire theta function representing on $\mathbb{C}$ the inverse image under $\kappa$ of a positive divisor on $\mathbb{C}/\Gamma$. Unlike theta, $\sigma$ has modular invariance (under a different choice of basis of $\Gamma$), up to a root of unity. To generalize Weierstrass’ sigma to higher genus, the authors of [BEL1, BLE] use the “fundamental 2-differential of the second kind” [BEL1, 2, 2, 3], to write a suitable basis of $H^1(X, \mathbb{C})$ which satisfies the “generalized Legendre relations”.

We obtained explicit expressions for the trigonal cyclic curve $(X, P)$ and wrote sigma in [KMP1], for a $(3,7,8)$ curve; for the general trigonal cyclic case, the proofs follow the same lines as those in [KMP1, Section 5], and we omit them.

We write the periods:

\[
[\eta', \eta''] := \frac{1}{2} \left[ \int_{\alpha_i} \nu_{II} \int_{\beta_i} \nu_{II} \right]_{i,j=1,2,\ldots,g}.
\]

**Proposition 4.12.** The following matrix satisfies the generalized Legendre relation:

\[
M := \begin{bmatrix}
2\omega' & 2\omega'' \\
2\eta' & 2\eta''
\end{bmatrix}, \quad M \begin{bmatrix}
I_g & -I_g \\
I_g & -I_g
\end{bmatrix}^t M = 2\pi \sqrt{-1} \begin{bmatrix}
I_g & -I_g \\
I_g & -I_g
\end{bmatrix},
\]

where $\omega', \omega''$ and $\eta', \eta''$ are the periods.
where $I_g$ is the $g \times g$ unit matrix.

Using theta characteristics $\delta$ as in Corollary 4.14 we define $\sigma$ as an entire function of (a column-vector) $u = (u_1, u_2, \ldots, u_g) \in \mathbb{C}^g$,

$$
\sigma(u) = c \exp(-\frac{1}{2} \Im \eta' \omega^{-1} u) \vartheta[\delta](\frac{1}{2} \omega^{-1} u; \omega^{-1} \omega')
$$

(4.12) $$
= c \exp(-\frac{1}{2} \Im \eta' \omega^{-1} u) \times \sum_{n \in \mathbb{Z}^g} \exp \left\{ \pi \sqrt{-1} \left\{ \Re (n+\delta') \omega^{-1} \omega'u(n+\delta') + \Re (n+\delta'')(\omega^{-1} u + 2\delta') \right\} \right\}
$$

where $c$ is a certain constant, in fact a rational function of the $b$’s in the equation of the curve (Subsection 3.2). This function has the required modular invariance and is homogeneous with respect to the extended weight of $\text{wt}_\lambda$.

**Proposition 4.13.** For $u, v \in \mathbb{C}^g$, and $\ell = (2 \omega' \ell' + 2 \omega'' \ell'') \in \Gamma$, if we define $L(u, v) := 2 \Im (\eta' v' + \eta'' v'')$, $\chi(\ell) := \exp \left\{ \pi \sqrt{-1} \left( 2 \Re (\ell' \delta'' - \ell'' \delta') + \Re (\ell' \ell'') \right) \right\}$, the following holds

$$
\sigma(u + \ell) = \sigma(u) \exp(L(u + \frac{1}{2} \ell, \ell)) \chi(\ell). 
$$

4.5. **The Riemann fundamental relation.** Using (4.17) to detect the divisors corresponding to the minus-sign operation on $\mathcal{J}_g$, we review a relation which we call the Riemann fundamental relation [13, §196]:

**Proposition 4.14.** For $(P, Q, P_i, P_i') \in \tilde{X}^2 \times (\mathcal{S}_g(\tilde{X}) \setminus \mathcal{S}_1^g(\tilde{X})) \times (\mathcal{S}_g(\tilde{X}) \setminus \mathcal{S}_1^g(\tilde{X}))$, $u := \tilde{w}_s(P_1, \ldots, P_g)$, $u' := \tilde{w}_s(P_1', \ldots, P_g')$,

$$
\exp \left( \sum_{i,j=1}^g \Omega_{\infty P_i, \infty P_j} \right) = \frac{\sigma(\tilde{w}(P) - u)\sigma(\tilde{w}(Q) - u')}{\sigma(\tilde{w}(Q) - u)\sigma(\tilde{w}(P) - u')}
$$

**Proposition 4.15.** For $(P, P_1, \ldots, P_g) \in X \times \mathcal{S}_g(X) \setminus \mathcal{S}_1^g(X)$ and $u \in \kappa^{-1} w_s(P_1, \ldots, P_g)$, the equality,

$$
\sum_{i,j=1}^g \varphi_{i,j} \frac{1}{x-a} \frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u)
$$

holds for every $a = 1, 2, \ldots, g$, where we set

$$
\varphi_{ij}(u) := -\frac{\sigma_i(u)\sigma_j(u)}{\sigma(u)^2} - \frac{\partial^2}{\partial u_i \partial u_j} \log \sigma(u),
$$

$$
\sigma_{i_1, i_2, \ldots, i_n}(u) := \prod_{j=1}^{n} \frac{1}{\partial u_{i_j}} \sigma(u).
$$
Proof. First we note that the above \( \varphi \) functions are defined over the Jacobian \( J \), namely \( \varphi_{ij}(u) = \varphi_{ij}(u + \ell) \) for \( \ell \in \Gamma \), because of the functional relation satisfied by \( \sigma \). Using the property of the vector fields in (4.9) and taking logarithm of both sides in the Riemann fundamental relation and differentiating along \( P_1 = P \) and \( P_2 = P_a \), we obtain the claim. \( \square \)

By translation-invariance of the \( \varphi \)-function, we may view the domain as subset of \( J \); more precisely, \( \varphi_{ij}(u) := \varphi_{ij}(u') \) for \( u' \in \mathbb{C}^g \) and \( u := \kappa u \in J \).

5. Vanishing on Wirtinger strata and Jacobi inversion

Using the subvarieties (4.4), we let (5.1)
\[
\Theta^k_s := W_s^k \cup [-1]W_s^k,
\]
and we define (5.2)
\[
\Theta_{s,1} := w_s(S^k(X)) \cup [-1]w_s(S^k(X)).
\]

For a Young diagram \( \lambda = (\lambda_1, \ldots, \lambda_n) \), the Schur function \( s_\lambda \) are defined by the ratio of determinants of \( n \times n \) matrices [Mac],
\[
s_\lambda(T) = \frac{|t_j^{\lambda_j+n-j}|}{|t_j^{j-1}|}
\]
where \( t =^t(t_1, \ldots, t_n) \) are the transpose of the rows. We also regard it as a function of \( T =^t(T_1, \ldots, T_n) \), \( T_k := \frac{1}{k} \sum_{i=1}^g t_j^k \), denoted by \( S_\lambda(T) = s_\lambda(t) \).

Following a formula proven for the rational/polynomial case [BEL2], Nakayashiki showed the following for the case of an \((n,s)\) curve [N1] and a general Riemann surface [N3 Theorem 10]:

**Proposition 5.1.** With the notation of Proposition 4.1, for a general non-singular curve \( X \), we have the following results: For \( e + \Delta \in (\kappa^s)^{-1}\Theta_s^{g-1} \) and for \( u \in \mathbb{C}^g \),
\[
C\theta((2\omega')^{-1}u + e, \tau) = S_\lambda(T)|_{r_\lambda_{i+g-i} = u_i} + \text{higher-weight terms},
\]
where \( C \) is a suitable constant.

We note that for \( e \) in Proposition 5.1, \( 2\omega' e + \tilde{w}(\nu \mathfrak{B}) \in \kappa^{-1}\Theta_{s-1}^{g-1} \) and thus
\[
C\theta \left[ \frac{\delta'}{\delta u} \right] ((2\omega')^{-1}u + e + \tilde{v}(\nu \mathfrak{B}), \tau) = S_\lambda(T)|_{r_\lambda_{i+g-i} = u_i} + \text{higher-weight terms},
\]
for \( \delta' \) in the definition of \( \sigma \) (4.12) from Corollary 4.3. We also note that \( u + 2\omega' e + \tilde{w}(\nu \mathfrak{B}) \in \tilde{w}_s(S^{g-1}X) = \mathbb{C}^g \). Further since \( \kappa(2\omega' e) \) belongs to the canonical theta divisor \( w(S^{g-1}X) \), by letting \( e = -\tilde{v}(\nu (\mathfrak{B})) \) from Proposition 4.11 (with \( v(B_1, \ldots, B_{s+r}) = -v(\mathfrak{B}) \) and \( r + s = g - 1 \)), we have the following result:
Proposition 5.2. The leading term in the Taylor expansion of the $\sigma$ function associated with $X$, with normalized constant factor $c$, is expressed by a Schur function
\[
\sigma(u) = S_{\Lambda}(T)|_{T_{\Lambda+g} = u^\ell} + \sum_{\alpha} a_{\alpha} u^\alpha,
\]
where $a_{\alpha} \in \mathbb{Q}[b_1, \ldots, b_{s+r}]$, $\alpha = (\alpha_1, \ldots, \alpha_g)$ and $u^\alpha = u_1^{\alpha_1} \cdots u_g^{\alpha_g}$.

Remark 5.3. In [MP2], we investigated the Riemann-Kempf theory for a $C_{rs}$ curve (another name, coding-theory terminology, for an $(n,s)$ curve), obtaining the expansion of sigma in terms of the Schur functions for Young diagrams of symmetric semigroups. However, the arguments we gave never used the symmetry of the Young diagrams. Thus the results are applicable to the non-symmetric case, and can be obtained from [MP2] directly, alternative to Nakayashiki’s results [N3].

We introduce the truncated Young diagrams of $\Lambda = (\Lambda_1, \Lambda_2, \ldots, \Lambda_k, \ldots, \Lambda_g)$: $\Lambda^{(k)} = (\Lambda_1, \Lambda_2, \ldots, \Lambda_k)$ and $\Lambda^{[k]} = (\Lambda_{k+1}, \ldots, \Lambda_g)$.

For a given truncated Young diagram $\Lambda^{[k]}$ (viewed as a partition of the total number of its boxes), with rank $n_k$ (the length of the diagonal), its “Frobenius characteristics” [FH §4.1] is $(a_1, a_2, a_3, \ldots, a_{n_k}; b_1, b_2, b_3, \ldots, b_{n_k})$, $a_i \geq a_j$ and $b_i \geq b_j$ for $i < j$, with $a_i$ and $b_i$ the number of boxes below and to the right of the $i$-th box of the diagonal. Define:

$$N_k := \sum_{i=1}^{n_k} (a_i + b_i + 1).$$

Proposition 5.4. Let us consider a truncated Young diagram $\Lambda^{[k]}$ with the Frobenius characteristics of the partition $(a_1, a_2, a_3, \ldots, a_{n_k}; b_1, b_2, b_3, \ldots, b_{n_k})$. For each pair $(a_i, b_i)$, there is $\ell_i \in \{k+1, \ldots, g\}$, with $\ell_1 = k+1$ for $i = 1$, such that

$$\Lambda_{\ell_i} + g - \ell_i = a_i + b_i + 1.$$

Let us denote such $\ell_i$ by $L^{[k]}(a_i, b_i)$ and

$$\mu_k := \{L^{[k]}(a_1, b_1), L^{[k]}(a_2, b_2), \ldots, L^{[k]}(a_{n_k}, b_{n_k})\}.$$

and for $i \leq k + 1$,

$$\mu_k^{(i)} := \{i, L^{[k]}(a_2, b_2), \ldots, L^{[k]}(a_{n_k}, b_{n_k})\}.$$

Note that $\mu_k = \mu_k^{(k+1)}$. We state Riemann’s singularity theorem and its version in [N3 Th. 2] as follows:

Proposition 5.5. Let $(P_1, \ldots, P_k)$ belong to $S^k(X \setminus \infty) \setminus (S^k(X) \cap S^k(X \setminus \infty))$ and $u \in \kappa^{-1} w_X(P_1, \ldots, P_k)$.

1. For every multi-index $(\alpha_1, \ldots, \alpha_m)$ with $\alpha_i \in \{1, \ldots, g\}$ and $m < n_k$,

$$\frac{\partial^m}{\partial u_{\alpha_1} \cdots \partial u_{\alpha_m}} \sigma(u) = 0.$$
(5.8) \( \text{Theorem 5.8. (Vanishing Theorems)} \)

(2) For \( (\beta_1, \ldots, \beta_{n_k}) = \hat{\mu}^{(i)}_k \) \( (i = 1, 2, \ldots, k + 1) \),

\[
\left( \frac{\partial^n u_k}{\partial u_{\beta_1} \cdots \partial u_{\beta_{n_k}}} \right) \sigma(u) \neq 0.
\]

**Proposition 5.6.** For all \( 1 \leq k \leq g - 1 \) \( u^{[k]} \in k^{-1}(\Theta^k_s \setminus (\Theta^k_{s,1} \cup \Theta^k_{s,-1})) \), \( u \in \mathbb{C}^g \), \( v \in \mathcal{W}^1 \)

\[
\left. \frac{\partial^\ell}{\partial u_v^\ell} \sigma(u) \right|_{u = u^{[k]}} = 0, \quad \ell < N_k; \quad \left. \frac{\partial^{N_k}}{\partial u_v^{N_k}} \sigma(u) \right|_{u = u^{[k]}} \neq 0.
\]

5.1. **Jacobi inversion formulae over \( \Theta^k \).** For any cyclic trigonal curve, of \( (3, \tilde{r}, \tilde{s}) \)-type (including \( r = 0 \)), we have the Jacobi inversion formulae:

**Theorem 5.7. (Jacobi inversion formula)**

For \( (P, P_1, \ldots, P_g) \in X \times \mathcal{S}^g(X) \setminus \mathcal{S}_1^g(X) \), we have

(a) \( \mu_\hat{g}(P; P_1, \ldots, P_g) = \begin{aligned} \hat{\phi}_g(P) - \sum_{i=1}^g \varphi_{g,i}(w_1(P_1, \ldots, P_g)) \hat{\phi}_{g,i-1}(P). \end{aligned} \)

(b) \( \varphi_{g,k+1}(w_1(P_1, \ldots, P_g)) = (-1)^{g-k} \mu_{g,k}(P_1, \ldots, P_g), \quad (k = 0, \ldots, g - 1). \)

**Proof.** Same as in Prop. 4.6 of MP1. \( \square \)

Bearing in mind Proposition 5.5 we have the following theorem, in which both sides are obtained by taking a limit by \( P_{k+1} \rightarrow \infty \) as mentioned in MP1 Th. 5.1; indeed, some of the left-hand sides are given by zero over zero but they are well-defined in the limit.

**Theorem 5.8. (Vanishing Theorems)** The following relations hold for the \( \hat{\mu} \) functions:

(1) \( \Theta^g \) case: for \( (P_1, \ldots, P_g) \in \mathcal{S}^g \setminus \mathcal{S}_1^g \) and \( u \in \kappa^{-1}(\Theta^g) = \mathbb{C}^g \),

\[
\frac{\sigma_i(u)\sigma_i(u) - \sigma_{g,i}(u)\sigma(u)}{\sigma^2(u)} = (-1)^{g+r-i} \hat{\mu}_{g,i-1}(P_1, \ldots, P_g), \quad \text{for } 1 \leq i \leq g.
\]

(2) \( \Theta^k \) case: for \( (P_1, \ldots, P_k) \in \mathcal{S}^k \setminus \mathcal{S}_1^k \) and \( u \in \kappa^{-1}(\Theta^k) \), \( (k = 1, 2, \ldots, g - 1), \)

\[
\frac{\sigma_i(u)}{\sigma_k(u)} = \frac{\sigma_i^{(1)}(u)}{\sigma_k^{(1)}(u)} = \begin{cases} (-1)^{k-i+1} \hat{\mu}_{k,i-1}(P_1, \ldots, P_k) & \text{for } 0 < i \leq k, \\ 1 & \text{for } i = k + 1, \\ 0 & \text{for } k + 1 < i \leq g. \end{cases}
\]

(3) \( \Theta^1 \) case: \( (P_1) \in \mathcal{S}^1 \) and \( u \in \kappa^{-1}(\Theta^1) \),

\[
\frac{\sigma_i(u)}{\sigma_1(u)} = \frac{\sigma_i^{(1)}(u)}{\sigma_1^{(1)}(u)} = \frac{\hat{\phi}_i}{\hat{\phi}_0}.
\]

and if \( r < s - 3 \), the right hand side is equal to \( x(P_1) \).

**Proof.** Essentially the same as in Th. 5.1 of MP2. \( \square \)
Remark 5.9. Every curve in Weierstrass normal form with $W$-semigroup $\langle 3, \widehat{r}, \widehat{s} \rangle$ corresponds to the same monomial ring as the cyclic case (cf. Proposition 3.3). Though the structure ring $R$ differs, it is expected that the same $R_\phi$ and $\hat{R}_\phi$, bases and subsets, also play similar roles as in the cyclic case. Therefore, we expect to be able to adapt the results of this paper to any curve in Weierstrass normal form with $W$-semigroup $\langle 3, \widehat{r}, \widehat{s} \rangle$. 

A. Appendix: Proof of Proposition 4.5

A computation shows:

**Lemma A.1.** The function $h(t, s) := \frac{t^\ell - s^\ell}{t - s} - \frac{d}{dt} t^\ell$ satisfies the following relations:

1. $h(t, t) = 0$,
2. $\frac{h(t, s)}{t - s} \in \mathbb{Q}[t, s]$, and
3. $\frac{h(t, s)}{t - s} = -\sum_{a=0}^{\ell-2} (a + 1)s^{\ell-2}a$ for $\ell > 1$, $\frac{h(t, s)}{t - s} = 0$ for $\ell = 0, 1$.

We now compute the differentials of $y_{\hat{r}}$ and $y_{\hat{s}}$:

$$
\frac{d}{dx} y_{\hat{s}} = \frac{1}{3y_{\hat{s}}^2} (2k_{s,p}k_{s} + k_{s}^2k_{s}'),
\frac{d}{dx} y_{\hat{r}} = \frac{1}{3y_{\hat{r}}^2} (2k_{s,p}k_{r} + k_{s}^2k_{r}'),
$$

where we set $k_{a,p} = k_a(x_P)$ and $k_{a,p}' = dk_a(x_P)/dx_P$. The result follows from the equalities:

$$
\frac{\partial}{\partial x_Q} \left( y_{\hat{r},p}y_{\hat{s}} + y_{\hat{r},p}y_{\hat{s},Q} + y_{\hat{r},Q}y_{\hat{s},p} + y_{\hat{r},Q}y_{\hat{s},Q} \right) = \frac{\partial}{\partial x_P} \left( y_{\hat{r},Q}y_{\hat{s},Q} + y_{\hat{r},Q}y_{\hat{s},p} + y_{\hat{r},p}y_{\hat{s},Q} \right)
$$

We show that the above is equal to

$$
\frac{1}{y_{\hat{r},p}y_{\hat{s},Q}D_{r,s}(P, Q) - y_{\hat{r},Q}y_{\hat{s},p}D_{r,s}(Q, P)},
$$

where $D_{r,s}(P, Q) = D_{s,r}^{(+)}(P, Q) - D_{s,r}^{(-)}(Q, P)$ from Definition 3.13. Indeed, by Lemma A.1

$$
\frac{1}{x_Q - x_P} \left( \frac{k_{s+r,Q} - k_{s+r,P}}{x_Q - x_P} - k_{s+r,Q}' \right) = -\sum_{j=0}^{s+r-2} \sum_{i=0}^{r-j-2} (i + 1)(s+r)_{i} x_P^{r+s-j-i-2} x_Q^i,
$$

$$
\frac{1}{x_Q - x_P} \left( \frac{k_{s+r,Q} - k_{s+r,P}}{x_Q - x_P} - k_{s+r,Q}' \right) = \sum_{j=0}^{r-2} \sum_{i=0}^{s-j-2} (i + 1)\lambda_{j}^{(s+r)} x_Q^{r+s-j-i-2} x_P^i,
$$

$$
\frac{1}{x_Q - x_P} \left( \frac{k_{s+r,Q} - k_{s+r,P}}{x_Q - x_P} - k_{s,p}k_{r,p} - k_{s,Q}k_{r,Q}' \right) = \sum_{j=0}^{s-j-2} \sum_{k=0}^{r} (i + 1)\lambda_{j}^{(s)}\lambda_{k}^{(r)} x_Q^{r+s-j-i-2} x_P^i + k,
$$

$$
-\sum_{j=0}^{s-j-2} \sum_{k=0}^{r} (i + 1)\lambda_{j}^{(s)}\lambda_{k}^{(r)} x_Q^{r+s-j-i-2} x_P^i + k,$$
noting that \( k_{s+r,Q} - k_{s+r,P} = k_{s,Q}(k_{r,Q} - k_{r,P}) + k_{r,P}(k_{s,Q} - k_{r,P}) \). Thus we obtained the expression of \( D_{r,s} \).

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Department of Mathematics, Center for Basic Education and Integrated Learning, Kanagawa Institute of Technology, Atsugi, 243-0292, JAPAN
E-mail address: komeda@gen.kanagawa-it.ac.jp

Industrial Mathematics, National Institute of Technology, Sasebo College, 1-1 OkiShin-machi, Sasebo, Nagasaki, 857-1193, JAPAN,
E-mail address: smatsu@sasebo.ac.jp

Department of Mathematics and Statistics, Boston University, Boston, MA 02215-2411, U.S.A.
E-mail address: ep@bu.edu