Whitehead torsion of inertial $h$-cobordisms*

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Abstract

We study the Whitehead torsions of inertial $h$-cobordisms, and identify various types representing a nested sequence of subsets of the Whitehead group. A number of examples are given to show that these subsets are all different in general.

1 Introduction

The $h$-cobordism theorem plays a crucial role in modern geometric topology, providing the essential link between homotopy and geometry. Indeed, comparing manifolds of the same homotopy type, one can often use surgery methods to produce $h$-cobordisms between them, and then hope to be able to show that the Whitehead torsion $\tau(W^{n+1},M^n)$ in $\text{Wh}(\pi_1(M^n))$ is trivial. By the $s$-cobordism theorem, the two manifolds will then be isomorphic (homeomorphic or diffeomorphic, according to which category we work in).

The last step, however, is in general very difficult, and what makes the problem even more complicated, but at the same time more interesting, is that there exist $h$-cobordisms with non-zero torsion, but were the ends still are isomorphic (cf. [10], [11], [18], [12]). Such $h$-cobordisms we call inertial. The central problem is then to determine the subsets of elements of the Whitehead group $\text{Wh}(\pi_1(M^n))$ which can be realized as Whitehead torsion of inertial $h$-cobordisms. This is in general very difficult, and only partial results in this direction are known ([10], [11], [18]).

The purpose of this note is to shed some light on this important problem.

2 Inertial $h$-cobordisms

In this section we recall basic notions and constructions concerning various types of $h$-cobordisms. We will follow the notation and terminology of [12]. For convenience we choose to formulate everything in the category of topological manifolds, but for most of what we are going to say, this does not make

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much difference. See Section 5 for more on the relations between the different categories.

An $h$-cobordism $(W;M,M')$ is a manifold $W$ with two boundary components $M$ and $M'$, each of which is a deformation retract of $W$.

We will think of this as an $h$-cobordism from $M$ to $M'$, thus distinguishing it from the dual $h$-cobordism $(W;M',M)$. Since the pair $(W;M)$ determines $M'$, we will often use the notation $(W;M)$ for $(W;M,M')$. We denote by $\mathcal{H}(M)$ the set of homeomorphism classes relative $M$ of $h$-cobordisms from $M$.

If $X$ is a path connected space, we denote by $\text{Wh}(X)$ the Whitehead group $\text{Wh}(\pi_1(X))$. Note that this is independent of choice of base point of $X$, up to unique isomorphism.

The $s$-cobordism theorem (cf. [19], [21]) says that if $M$ is compact connected (closed) and of dimension at least 5 there is a one-to-one correspondence between $\mathcal{H}(M)$ and $\text{Wh}(M)$ associating to the $h$-cobordism $(W;M,M')$ its Whitehead torsion $\tau(W;M) \in \text{Wh}(M)$. Given an element $(W;M,M') \in \mathcal{H}(M)$ the restriction of a retraction $r: W \to M$ to $M'$ is a homotopy equivalence $h: M' \to M$, uniquely determined up to homotopy. By a slight abuse of language, any such $h$ will be referred to as “the natural homotopy equivalence”. It induces a unique isomorphism

$$h_\ast : \text{Wh}(M') \to \text{Wh}(M).$$

Recall also that there is an involution $\tau \to \bar{\tau}$ on $\text{Wh}(M)$ induced by transposition of matrices and inversion of group elements (cf. [21], [22]). If $M$ is non-orientable, the involution is also twisted by the orientation character $\omega: \pi_1(M) \to \{-1\}$, i.e. inversion of group elements is replaced by $\tau \to \omega(\tau)\tau^{-1}$.

Let $(W;M,M')$ and $(W';M',M)$ be dual $h$-cobordisms with $M,M'$ of dimension $n$. Then $\tau(W;M)$ and $\tau(W';M')$ are related by the basic duality formula (cf. [21], [12])

$$h_\ast(\tau(W;M')) = (-1)^n \tau(W;M).$$

We refer to Section 5 for further discussion of Whitehead torsion.

**Definition 2.1.** The inertial set of a manifold $M$ is defined as

$$I(M) = \{(W;M,M') \in \mathcal{H}(M) | M \cong M' \},$$

or the corresponding subset of $\text{Wh}(M)$.

There are many ways to construct inertial $h$-cobordisms. Here we recall three of these.

**A.** Let $G$ be an arbitrary (finitely presented) group. Then there is a 2-dimensional simplicial complex $K$ (finite) with $\pi_1(K) \cong G$. Let $\tau_0 \in \text{Wh}(G)$ be an element with the property that $\tau_0 = \tau(f)$ for some homotopy self-equivalence $f: K \to K$. Denote by $N(K)$ a regular neighborhood of $K$ in a high-dimensional Euclidean space $\mathbb{R}^n (n \geq 5$ will do). Approximate $f: K \to K \subseteq N(K)$ by an embedding whose image has neighborhood $N'(K) \subset \text{int} N(K)$. By uniqueness of neighborhoods, $N'(K) \approx N(K)$. Then $W = N(K) - \text{int} N'(K)$ is an inertial
$h$-cobordism whose torsion $\tau(W; \partial N(K))$ can be identified with $\tau_0$ via the $\pi_1$-isomorphisms $\partial N(K) \subset N(K) \supset K$. (cf. \[10\], \[11\]).

**B.** Let $f : M \to M$ be a homotopy self-equivalence of a closed manifold and let $\tau_0 = \tau(f) \in \text{Wh}(M)$. Approximate $f : M \to M \subset M \times D^n$ by an embedding (cf. \[25\]), where $D^n$ is the $n$-dimensional disk, $n$ big. In the same way as in A, this will lead to an inertial $h$-cobordism between two copies of $M \times S^{n-1}$, with torsion $\tau_0$ (cf. \[12\]).

**C.** Let $(W; M, M')$ be an $h$-cobordism with torsion $\tau_0 = \tau(W; M)$. Form the double (cf. \[21\], \[12\]):

$$\tilde{W}(W; M, M) := \left( W \cup_{M'} W; M, M \right)$$

Then $\tau(\tilde{W}; M) = \tau_0 + (-1)^n \bar{\tau}_0$ and this again often leads to a nontrivial inertial $h$-cobordism; for example if $n$ is odd and the involution $- : \text{Wh}(M) \to \text{Wh}(M)$ is nontrivial.

It will be convenient to introduce the notation $D(M)$ for the subgroup $\{ \tau + (-1)^n \bar{\tau} \mid \tau \in \text{Wh}(M) \}$ of $\text{Wh}(M)$. Note that $D(M)$ depends only on $\pi_1(M)$, orientation and the dimension of $M$.

The construction in C leads to $h$-cobordisms that are particularly simple and have special properties: not only do they come with canonical identifications of the two ends, but they are also strongly inertial.

**Definition 2.2.** (Cf. \[12\]). The $h$-cobordism $(W; M, M')$ is called strongly inertial, if the natural homotopy equivalence $h : M' \to M$ is homotopic to a homeomorphism.

The set of (Whitehead torsions of) strongly inertial $h$-cobordism will be denoted by $SI(M)$. It was observed in \[12\] that $SI(M) \subseteq \text{Wh}(M)$ is a subgroup.

Obviously $SI(M) \subseteq I(M)$ and there are many examples of inertial but not strongly inertial $h$-cobordism, for example constructed using the methods in A or B. In fact, for any manifold $M$ of dimension $n \geq 5$, we have

$$I(M \#_k(S^p \times S^{n-p})) = \text{Wh}(M \#_k(S^p \times S^{n-p})),$$

for $2 \leq p \leq n - 2$ and $k$ big enough \[9\]. (If $\pi_1(M)$ is finite, $k = 2$ suffices.)

However, for $SI(M)$ there are restrictions. For example, since the natural homotopy equivalence $h$ is homotopic to a homeomorphism, its Whitehead torsion $\tau(h)$ must vanish. But we have (equation (3) in Section \[9\]

$$\tau(h) = -\tau(W; M) + (-1)^n \bar{\tau}(W; M),$$

so $\tau(W; M)$ must satisfy the formula $\tau(W; M) = (-1)^n \bar{\tau}(W; M)$, i.e.

$$SI(M) \subseteq A(M) := \{ \tau \in \text{Wh}(M) \mid \tau = (-1)^n \bar{\tau} \}.$$  

In special cases we have even stronger restrictions, as in the following result (Theorem 1.3 in \[12\]):
Theorem 2.3. Suppose $M$ is a closed oriented manifold of odd dimension with finite abelian fundamental group. Then every strongly inertial $h$-cobordism from $M$ is trivial.

This result motivated us to look more closely at strongly invertible $h$-cobordisms with finite fundamental groups. Our main interest is the following:

**Problem:** Let $M^n$ be a closed $n$-dimensional (oriented) manifold with $n \geq 5$ and with finite fundamental group $\pi_1(M^n)$. Determine the subset $SI(M^n)$ of $Wh(M^n)$. In particular, is $SI(M^n) = D(M)$?

Note that if $G$ is a finite abelian group, then the involution $- : Wh(G) \to Wh(G)$ is trivial (cf. [22]), and consequently $D(M^n) = \{0\}$ for $n$ odd. Hence, in this case $SI(M) = D(M)$ by Theorem 2.3.

Our first new observation is that $SI(M^n) = \{0\}$ also for odd dimensional manifolds $M^n$ with $\pi_1(M^n)$ finite periodic, namely:

**Theorem 2.4.** Let $(W^{n+1}, M^n, N^n)$ be a strongly inertial $h$-cobordism with $M$ orientable, $n$ odd and $\pi = \pi_1(M^n)$ finite periodic. Then $W^{n+1} = M^n \times I$ for $n \geq 5$. Hence $SI(M^n) = \{0\}$.

The class of finite periodic fundamental groups has attracted a lot of attention in topology of manifolds and transformation groups (cf. [20], [17]). The most extensive classification results for manifolds with finite fundamental groups involve this class of groups.

Let $M^n$ be a closed, oriented manifold with $\pi_1(M^n)$ finite abelian. If $n$ is odd, then, as we observed, $SI(M^n) = \{0\}$. In the even dimensional case the situation is quite different.

**Theorem 2.5.** For every $n \geq 3$ there are oriented manifolds $M^{2n}$ with $\pi_1(M^{2n})$ finite cyclic and with $\{0\} \neq D(M) \neq SI(M^{2n})$.

The following result shows that orientability is essential in Theorem 2.3.

**Theorem 2.6.** In every odd dimension $n \geq 5$ there are closed nonorientable manifolds with finite, cyclic fundamental groups and strongly inertial $h$-cobordisms from $M$ with nontrivial Whitehead torsion.

Note that in this case $D(M)$ is trivial.

**Remarks 2.7.** (i). There are obvious inclusions $\{0\} \subset D(M) \subset SI(M) \subset I(M) \subset Wh(M)$. In addition it is proved in [6] that $A(M) \subset I(M)$, such that combined with (1) we have a sequence of subsets

$$\{0\} \subset D(M) \subset SI(M) \subset A(M) \subset I(M) \subset Wh(M).$$

Clearly each of these inclusions can be an equality for some $M$, but for each pair of subsets we now have examples of manifolds where the inclusion is proper. (For $SI(M) \neq A(M)$, see e.g. [12], Example 6.4.)

(ii) $D(M)$ and $A(M)$ depend only on the fundamental group, and Khan [14] has shown that $SI(M)$ is homotopy invariant. It would be interesting to know
if \( SI(M) \) also only depends on the fundamental group. If so, it is a functorial, algebraically defined subgroup of \( Wh(M) \) between \( D(M) \) and \( A(M) \). What could it be?

Observe also that the quotient \( A(M)/D(M) \) is equal to the Tate cohomology group \( \hat{H}^n(\mathbb{Z}_2; Wh(M)) \), where \( n = \dim M \), and therefore \( SI(M)/D(M) \) is a subgroup. Another description of this subgroup is given in the beginning of Section 3.

Note that Hausmann has shown that \( I(M) \) is not homotopy invariant, and in general is not a subgroup of \( Wh(M) \) [11]. However, it is preserved by the involution \( \tau \mapsto (-1)^{n+1}\bar{\tau} \) [11, Lemma 5.6].

There is one more piece of structure that we should mention: the group \( \pi_0(\text{Top}(M)) \) of isotopy classes of homeomorphisms of \( M \) acts on \( Wh(M) \) via the isomorphisms induced on the fundamental group. (Recall that \( Wh(M) \) is independent of choice of base point.) Geometrically, this corresponds to changing an \( h \)-cobordism \( (W; M) \) by the way \( M \) is identified with part of the boundary of \( W \). Hence the orbits represent equivalence classes under homeomorphisms preserving boundary components, but not necessary the identity on any of them. A simple example to illustrate this is the case where \( M = P_1 \# P_2 \), where \( P_1 \) and \( P_2 \) are copies of the same manifold. Since \( Wh(M) \cong Wh(P_1) \oplus Wh(P_2) \) [24], this means that every \( h \)-cobordism from \( M \) is a band-connected sum \( W_1 \#_{S^{n-1} \times I} W_2 \) of \( h \)-cobordisms from \( P_1 \) and \( P_2 \), and the homeomorphisms interchanging \( P_1 \) and \( P_2 \) just interchanges \( W_1 \) and \( W_2 \).

The observation now is that the action of \( \pi_0(\text{Top}(M)) \) clearly preserves the filtration (2).

Note that on \( Wh(M) \) this action factors through an action of the group \( \pi_0(\text{Aut}(M)) \) of homotopy classes of homotopy equivalences of \( M \). Since the action of \( \pi_0(\text{Aut}(M)) \) is defined algebraically, it must also preserve the functorial subgroups \( D(M) \) and \( A(M) \).

This action does not have an easy geometric interpretation, but \( SI(M) \) is still preserved, by the more subtle functoriality of [14, Theorem 3.1], as explained in Corollary 3.2 below. However, it is an easy consequence of [11, Theorem 6.1] that it does not in general preserve \( I(M) \).

### 3 Proofs

In this section all manifolds have dimension at least five. The proofs are based on the following commutative diagram, which is part of the braid (6) in Section 6. The rows are the Wall-Sullivan exact sequences for topological surgery (cf. [25, 23]), and the columns are part of the Rothenberg sequences for \( L \)-groups.
and structure sets.

\[
\begin{array}{cccc}
L^n_{n+2}(M) & \xrightarrow{\gamma^n} & S^n(M \times I) & \xrightarrow{\eta^n} \ N(M \times I) & \xrightarrow{\theta^n} \ L^n_{n+1}(M) \\
\downarrow l_1 & & \downarrow t & & \downarrow l_0 \\
L^h_{n+2}(M) & \xrightarrow{\gamma^h} & S^h(M \times I) & \xrightarrow{\eta^h} \ N(M \times I) & \xrightarrow{\theta^h} \ L^h_{n+1}(M) \\
\downarrow \delta_L & & \downarrow \delta_S & & \downarrow \delta_S \\
\hat{H}^n(\mathbb{Z}_2; \text{Wh}(M)) & \equiv & \hat{H}^n(\mathbb{Z}_2; \text{Wh}(M))
\end{array}
\]

We want to understand the quotient group \(SI(M)/D(M)\), and the clue is the following observation:

**Lemma 3.1.** \(SI(M)/D(M) = \text{im} \delta_S \subset \hat{H}^n(\mathbb{Z}_2; \text{Wh}(M)) \subset \text{Wh}(M)/D(M)\).

**Proof.** (See also [14].) Recall that an element of \(S^h(M \times I)\) is represented by a homotopy equivalence \(f : W \rightarrow M \times I\) which is a homeomorphism on the boundary. Hence we can think of \(W\) as an \(h\)-cobordism from \(M\), and as such it is clearly strongly inertial. Since the map \(\delta_S\) is as induced by \((f : W \rightarrow M \times I) \mapsto \tau(W, M)\), the inclusion \(\subset\) follows.

To prove the opposite inclusion, let \((W; M, N)\) be a strongly inertial \(h\)-cobordism representing an element \(z\) in \(SI(M)/D(M)\), and let \(H : N \times I \rightarrow M\) be a homotopy from the natural homotopy equivalence \(h_W = r_M|N\) to a homeomorphism. Define a map \(W \rightarrow M\) as the composite \(W \xrightarrow{r_M} W \cup_N N \times I \rightarrow M\), where the last map is \(H\) on the collar \(N \times I\) and the retraction \(r_M\) on \(W\). Combined with any map \((W; M, N) \rightarrow (I; 0, 1)\) this defines an element of \(S^h(M \times I)\) which image \(z \in SI(M)/D(M)\). \(\square\)

We include the following corollary, which is our way of understanding Theorem 3.1 in [14] and its proof.

**Corollary 3.2.** Let \(f : M \rightarrow M'\) be a homotopy equivalence of closed manifolds. Then the induced isomorphism \(f_* : \text{Wh}(M) \rightarrow \text{Wh}(M')\) restricts to an isomorphism \(f_* : SI(M) \rightarrow SI(M')\).

**Proof.** We need to verify that \(f_*(SI(M)) \subseteq SI(M')\).

Lemma 3.1 and functoriality of the surgery exact sequence imply that the induced homomorphism \(f_* : \text{Wh}(M)/D(M) \rightarrow \text{Wh}(M')/D(M')\) restricts to a homomorphism \(f_* : SI(M)/D(M) \rightarrow SI(M')/D(M')\). In other words, if \(x \in SI(M)\), then \(f_*(x) = y + d\), where \(y \in SI(M')\) and \(d \in D(M')\). But then obviously also \(f_*(x) \in SI(M')\). \(\square\)

The most obvious way to try to prove Theorems 2.5 and 2.6 will now be to show that in these cases the homomorphism \(l_1\) in the diagram above is not onto.

In the case of Theorem 2.5 we need to study the map of even \(L\)-groups: \(l_1 : L^2_2(\pi) \rightarrow L^2_2(\pi)\), where \(\pi = \pi_1(M)\) and \(2m = \dim M + 2\). Now assume
that $\pi = \mathbb{Z}_k$ is a cyclic group of odd order $k$. Then $l_1$ is injective. In fact, its image splits off as the free part plus a $\mathbb{Z}_2$ (Arf invariant) if $m$ is odd. Hence, any other torsion in $L^h_{2m}(\mathbb{Z}_k)$ maps nontrivially by $\delta_L$.

The extra torsion is computed from the Rothenberg sequence relating $L^h_s$ and $L^L_s$-groups:

$$\cdots \longrightarrow L^p_{2m+1}(\mathbb{Z}_k) \longrightarrow H^2(\mathbb{Z}_2; \tilde{K}_0\mathbb{Z}[\mathbb{Z}_k]) \longrightarrow L^h_{2m}(\mathbb{Z}_k) \longrightarrow L^p_{2m}(\mathbb{Z}_k),$$

where the groups $L^p_{2m+1}(\mathbb{Z}_k)$ vanish by [4] Corollary 4.3, p.58).

An example where $H^2(\mathbb{Z}_2; \tilde{K}_0\mathbb{Z}[\mathbb{Z}_k])$ is nontrivial is provided by [13] Theorem 7.1, p.449], where it is shown that $\tilde{K}_0(\mathbb{Z}[\mathbb{Z}_{15}]) \approx \mathbb{Z}_2$. Hence, if we choose $M$ to be any orientable, closed manifold of even dimension and fundamental group $Z_{15}$, then $D(M) \neq SI(M)$.

To see that $D(M) \neq 0$, recall that $\text{Wh}(\mathbb{Z}_{15}) \approx \mathbb{Z}^4$ (see e.g. [2] 11.5)), and that the involution is trivial for abelian groups. Then $D(M) = 2 \text{Wh}(\mathbb{Z}_{15}) \approx \mathbb{Z}^4$.

For Theorem 2.6 consider the cyclic 2-group $\mathbb{Z}_{2^k}$, $k \geq 4$, with the nontrivial orientation character $\omega : \mathbb{Z}_{2^k} \to \{\pm 1\}$. Computations in [27] Theorem 3.4.5 and [4] Theorem B and formula p.44 give

$$L^h_{2m+1}(\mathbb{Z}_{2^k}, \omega) \xrightarrow{\delta_L} H^1(\mathbb{Z}_2; \text{Wh}(\mathbb{Z}_{2^k})) \approx (\mathbb{Z}_{2^k})^{k-3},$$

where the cohomology is with respect to the involution twisted by $\omega$.

The proof of Theorem 2.4 goes by an argument similar to the proof of Theorem 1.3 in [12] (Theorem 2.3 above). We need the following facts:

**FACT 1:** The involution $- : \text{Wh}(\pi) \to \text{Wh}(\pi)$ is trivial.

This is Claim 3 in [KS3] p.1527.

**FACT 2:** The homomorphism $l_1$ is surjective.

This is Claim 1 in [KS3] p.1527.

**FACT 3:** The homomorphism $l_0$ is injective on the image of $\theta^s$.

*Proof of FACT 3.* Since $\text{im} \theta^s \subseteq L^s_{n+1}(\pi_2)$, where $\pi_2$ is the Sylow 2-subgroup of $\pi$ (cf. [Wa]) it is enough to show that restriction $l_0 : L^s_{n+1}(\pi_2) \to L^s_{n+1}(\pi_2)$ is injective. To this end note that $\text{SK}_1(\pi_2) = 0$ (cf. [22]), where

$$\text{SK}_1(\pi) := \text{Ker}(K_1(\mathbb{Z}[\pi]) \to K_1(\mathbb{Q}[\pi])).$$

Indeed $\pi_2$ is either generalized quaternionic or cyclic!

As a consequence $L^s_{n+1}(\pi_2) \cong L'_{n+1}(\pi_2)$ where $L'(-)$ are the weakly simple $L$-groups of C.T.C. Wall from [27]. Now, there is an exact sequence (cf. [27] p. 78])

$$0 \to L^L_{2n}(\pi_2) \to L^h_{2n}(\pi_2) \to \text{Wh}(\pi_2) \otimes \mathbb{Z}_2 \to L^s_{2n-1}(\pi_2) \to L^h_{2n-1}(\pi_2) \to 0$$

and hence $l_0|_{\text{im} \theta^s}$ is injective as claimed.

Given the Facts (1-3) the proof of Theorem 2.4 is just a repetition of the argument in [12].
4 Further remarks

(1) Let \(\pi\) be a finite group with \(SK_1(\pi) = 0\), for example any dihedral group, or many nonabelian metacyclic groups, etc. (see [22] for more such groups). Then \(Wh(\pi) \cong Wh'(\pi)\) is torsion free and the involution \(\bar{\tau}: Wh(\pi) \to Wh(\pi)\) is trivial (cf. [22]). This is enough to extend Theorem 2 to this class of fundamental groups.

Indeed, let \((W^{n+1}; M^n, N^n)\) be a strongly inertial \(h\)-cobordism, \(n\) odd. We can assume \(n \geq 5\). Let \(h: N^n \to M^n\) be the natural homotopy equivalence. Since \(h\) is homotopic to a homeomorphism, then \(\tau(h) = 0\). On the other hand, \(\tau(h) = 2\tau(W^{n+1}, M^n)\). This implies \(\tau(W^{n+1}, N^n) = 0\) and \(W^{n+1} = M^n \times I\), i.e. \(SI(M^n) = \{0\}\).

(2) There are periodic groups \(\pi\) with \(SK_1(\pi) \neq 0\). For example groups containing \(\mathbb{Z}_p \times Q(8)\), where \(p \geq 3\) is prime and \(Q(8)\) is the quaternionic group of order 8. (cf. [22]).

(3) There exist strongly inertial \(h\)-cobordisms with nontrivial Whitehead torsion \(\tau(W^{n+1}, M^n, N^n)\) with \(n\) odd \(n \geq 5\).

To be more specific, let \(p\) be an odd prime and let \(G\) be a \(p\)-group such that \(SK_1(G)_{(p)}\) is non-trivial, for example the group given in Example 8.11 of [22], p. 201. Then the argument on page 323 of [22] shows that that the involution \(\bar{\tau}: Wh(G) \to Wh(G)\) is nontrivial. Now let \(M^n, n\) odd, \(n \geq 5\) be a manifold with \(\pi_1(M^n) \cong G\). Then the doubling construction gives a strongly inertial \(h\)-cobordism \((W^{n+1}, M^n, N^n)\) with \(\tau(W^{n+1}, M^n)\) of the form \(\tau_0 - \bar{\tau}_0\) for \(\tau_0 \in Wh(G)\). Choosing \(\tau_0 \in Wh(G)\) with \(\tau_0 \neq \bar{\tau}_0\) gives the desired inertial \(h\)-cobordism.

(4) Let \(G\) be a finite group and \(M^n, n \geq 5, n\) odd, a closed manifold with \(\pi_1(M^n) \cong G\). The following is a curious restatement of a special case of our problem.

**Question:** Is \(SI(M^n) = \{0\}\) if the involution \(\bar{\tau}: Wh(G) \to Wh(G)\) is the identity? (“Only if” is trivial in this case, since \(\{\tau - \bar{\tau} | \tau \in Wh(G)\} \subset SI(M^n)\).)

**Comments:** (a) The answer is yes for \(G\)-finite abelian or periodic.

(b) Suppose \(SI(M^n) = \{0\}\), and let \(\tau_0 \in Wh(G)\) be given. Again the doubling construction gives a strongly inertial \(h\)-cobordism \((W^{n+1}, M^n, N^n)\) with torsion \(\tau = \tau_0 - \bar{\tau}_0\). Since \(SI(M^n) = \{0\}\) then \(\tau_0 = \bar{\tau}_0\), i.e. the involution is trivial. On the other hand suppose the involution \(\bar{\tau}: Wh(G) \to Wh(G)\) is trivial and let \((W^{n+1}, M^n, N^n)\) be a strongly inertial \(h\)-cobordism. Since the natural homotopy equivalence \(h: M^n \to N^n\) is homotopic to a homeomorphism, then \(0 = \tau(h) = -2\tau(W^{n+1}, M^n)\). In particular, if \(Wh(G)\) is torsion free, then the involution \(\bar{\tau}: Wh(G) \to Wh(G)\) is always trivial. Hence, for all such groups \(SI(M^n) = \{0\}\).
(5) There exist 4-dimensional inertial $s$-cobordisms which are not products! (cf. [3], [17]).

5 Addendum 1: On topological invariance

It is a consequence of the $s$-cobordism theorem and smoothing theory that if $M$ is a compact manifold and $\dim M \geq 5$, then the classification of $h$-cobordisms from $M$ up to isomorphism relative to $M$ is the same in the three categories $TOP$, $PL$ and $DIFF$. For example, if $M$ is smooth and $(W, M)$ is a topological $h$-cobordism, then $W$ has a smooth structure, unique up to concordance, extending that of $M$, and if two such $h$-cobordisms are homeomorphic rel $M$, then they are also diffeomorphic rel $M$.

However, the following question is more subtle:

**Question 5.1.** Suppose $(W; M, N)$ is a smooth $h$-cobordism which is inertial in $TOP$, does it follow that it is also inertial in $DIFF$?

In other words: if $M$ and $N$ are homeomorphic, are they then also diffeomorphic? (Similar questions can of course be asked for the pairs of categories $(DIFF, PL)$ and $(PL, TOP)$.)

Note that this indeed holds for the examples provided by the general results and constructions above: for example $D(M)$, $A(M)$ and those obtained by connected sum with products of spheres, and in Lemma 8.1 of [12] we claimed that the answer is always yes. However, this was based on a too optimistic application of the product structure theorem for smoothings, and it does not hold as it stands. We have, unfortunately, not been able to correct this in general, but here is a proof in the case of strongly inertial $h$-cobordisms.

**Proposition 5.2.** Let $M$ be a smooth, compact manifold. If $W$ is a $PL$ $h$-cobordism from $M$, then $W$ has a smooth structure compatible with the given structure on $M$, unique up to concordance. If $W$ is strongly inertial in $PL$, then it is also strongly inertial in $DIFF$.

Replacing the pair of categories $(DIFF, PL)$ by $(DIFF, TOP)$ or $(PL, TOP)$, a similar result is true, provided $M$ has dimension at least 5.

**Proof.** Denote by $\Gamma(M)$ the set of concordance classes of smoothings of the underlying $PL$ manifold $M$. By smoothing theory, this is a homotopy functor. In particular, if $(W; M, N)$ is an $h$-cobordism, the inclusions $M \subset_{JM} W$ and $N \subset_{JN} W$ induce restriction isomorphisms

$$\Gamma(M) \xrightarrow{\partial M} \Gamma(W) \xrightarrow{\partial N} \Gamma(N).$$

This proves the first part of the Lemma and also defines a unique concordance class of structures on $N$.

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1 We would like to thank Jean-Claude Hausmann for pointing out the error in [12].
Now let $M_\alpha$ be the given structure on $M$, $W_\alpha$ a structure on $W$ restricting to $M_\alpha$, and $N_\alpha$ the restriction of this again to $N$, such that $(W_\alpha; M_\alpha, N_\alpha)$ is a smooth $h$-cobordism. Observe that since $j_M$ has a homotopy inverse $r_M$, the composite isomorphism $\Gamma(M) \to \Gamma(N)$ is induced by $r_M \circ j_N$, i.e. the natural homotopy equivalence $h_W$. But if the $h$-cobordism is (PL) strongly inertial, the isomorphism is also induced by a PL homeomorphism $f$. This means that $N_\alpha$ is concordant to the smoothing $N_{f^*}\alpha$ on $N$ transported from $M_\alpha$ by $f$ in such a way that $f$ becomes a diffeomorphism between $N_{f^*}\alpha$ and $M_\alpha$.

Let $(N \times I)_\beta$ be a concordance between $N_\alpha$ on $N \times \{0\}$ and $N_{f^*}\alpha$ on $N \times \{1\}$. By the product structure theorem ([7, part I]) there is a diffeomorphism $H : (N \times I)_\beta \to N_\alpha \times I$ restricting to the identity on $N \times \{0\}$. Then $F(x,t) = H(f(x),t)$ defines a homotopy (in fact PL isotopy) between $f$ and a diffeomorphism between $M_\alpha$ and $N_\alpha$. But $f$ was homotopic to $h_W$.

The proofs in the other cases are analogous, but one now needs the triangulation theory of [15], which is only valid in dimensions $\geq 5$.

Remark 5.3. If $\dim M = 4$, Question 5.1 has a negative answer, even in the strongly inertial case. In fact, the first counterexamples to the $h$-cobordism theorem given by Donaldson in [5] are even strongly inertial, so even Proposition 5.2 (in case (DIFF, TOP)) fails in this dimension.

6 Addendum 2: Comments on torsion

We collect here some useful observations concerning the Whitehead torsions of homotopy equivalences of manifolds and relations with $h$-cobordisms.

Recall that to a homotopy equivalence $f : K \to L$ of finite complexes is associated a Whitehead torsion $\tau(f) = f_* \tau(M_f, K) \in Wh(L)$ [2]. Then the torsion of an $h$-cobordism $(W, M)$ can be expressed as

$$\tau(W, M) = r_* \tau(i) = -\tau(r),$$

where $i$ is the inclusion $M \subset W$ and $r$ is a retraction $W \to M$. If $j : N \hookrightarrow W$ is the inclusion of the other end of $W$, we can express the torsion of the natural homotopy equivalence $h = r \circ j$ as

$$\tau(h) = \tau(r) + r_* (\tau(j)) = -\tau(W; M) + r_* j_*(\tau(W; N)) = -\tau(W; M) + (-1)^n \tau(W; M). \quad (3)$$

The following observation shows that torsions of homotopy equivalences of manifolds can not be arbitrary, unlike for $h$-cobordisms.

Lemma 6.1. Let $f : (N, \partial N) \to (M, \partial M)$ be a homotopy equivalence between compact, oriented and connected manifolds of dimension $n$, such that $f$ is a homeomorphism on the boundary, and let $\tau \in Wh(M)$ be its torsion. Then

$$\tau + (-1)^n \tau^* = 0.$$
Proof. There is a commutative diagram

\[
\begin{array}{ccc}
C_*(N) & \xrightarrow{f_*} & C_*(M) \\
\downarrow & & \downarrow \\
C_*(N, \partial N) & \xrightarrow{f^*_\text{rel}} & C_*(M, \partial M) \\
D_N & \xrightarrow{\partial_\text{M}} & D_M \\
C^*(N) & \xrightarrow{f^*} & C^*(M)
\end{array}
\]

of finitely generated \(\mathbb{Z}\pi_1(M)\)-modules, where the lower vertical maps are given by Poincaré duality. (Everything with coefficients in \(\mathbb{Z}\pi_1(M)\)). Then

\[
\tau(D_M) = \tau(f^\text{rel}) + f_*\tau(D_N) + f_*\tau(f^\#).
\]

(Here \(h_*\) is the map induces on Whitehead groups.) The result now follows, since the Poincaré duality maps have vanishing torsion, \(\tau(f^\text{rel}) = \tau(f^\#) = \tau\) and \(h_*(\tau(f^\#)) = (-1)^n(\tau(f^\#))^*\).

\[\square\]

Remark. More generally, without the assumption that \(f|\partial N\) is a homeomorphism (or at least a simple homotopy equivalence), we get the formula

\[
\tau(f) - \tau(f|\partial M) + (-1)^n(\tau(f))^* = 0.
\]

Example. Many finite groups have free Whitehead groups, and then it is known that the involution is trivial (Wall). For an even–dimensional closed manifold with one of these groups as fundamental group, it follows that all homotopy equivalences are simple.

The lemma is used in the following geometric proof of the Rothenberg sequence for structure sets. We use the convention that \(S^h(M) (\beta(M))\) denotes the structure set of maps which are homeomorphisms on the boundary.

**Theorem 6.2.** Let \(M\) be a compact, oriented and connected manifold of dimension \(n\). Then there is an exact sequence of based sets (groups, in the topological category)

\[
\rightarrow S^h(M \times I) \xrightarrow{\theta} \tilde{H}^n(\mathbb{Z}/2; Wh(M)) \xrightarrow{\psi} S^s(M) \xrightarrow{\iota} S^h(M) \xrightarrow{\theta} \tilde{H}^{n-1}(\mathbb{Z}/2; Wh(M)).
\]

Proof. The map \(\iota\) is the obvious forgetful map; \(\psi\) and \(\theta\) will be defined below.

We start with \(\theta\). Recall that

\[
\tilde{H}^{n-1}(\mathbb{Z}/2; Wh(M)) = \{\tau \in Wh(M)|\tau = (-1)^{n-1}\tau^*\}/\{\tau + (-1)^{n-1}\tau^*\},
\]

\[
= \{\tau \in Wh(M)|\tau + (-1)^n\tau^* = 0\}/\{\tau - (-1)^n\tau^*\}.
\]
If \( f : N \rightarrow M \) represents an element of \( S^h(M) \), it then follows from the lemma above that \( \tau(f) \) represents an element of \( \tilde{H}^{n-1}(\mathbb{Z}/2; Wh(M)) \). We have to show that this element is well-defined.

Let \( f' : N' \rightarrow M \) represent the same element of \( S^h(M) \) as \( f \). Then there is an \( h \)-cobordism \( W \) from \( N \) to \( N' \) and a map \( F : W \rightarrow M \) restricting to \( f \) and \( f' \) at the ends.

![Diagram](4)

Let \( \sigma = \tau(W, N) \) be the torsion of the \( h \)-cobordism. By equation (3) above we have \( \tau(h) = -\sigma + (-1)^n \sigma^* \), where \( h : N' \rightarrow N \) satisfies \( f \circ h \simeq f' \). But then

\[
\tau(f') = f_\ast \tau(h) + \tau(f),
\]

and \( f_\ast \tau(h) \) is trivial in \( \tilde{H}^{n-1}(\mathbb{Z}/2; Wh(M)) \).

Trivially \( \theta \circ \iota = 0 \). Suppose now that \( f \in S^h(M) \) satisfies \( \theta(f) = 0 \), i.e. \( \tau(f) = \sigma - (-1)^n \sigma^* \), for some \( \sigma \in Wh(M) \). Let \( W \) be an \( h \)-cobordism with one end equal to \( N \) and with Whitehead torsion \( \tau(W, N) = f_\ast^{-1}(\sigma) \). Extension of the map \( f \) yields a diagram like (4). Then \( f' \) and \( f \) represent the same element of \( S^h(M) \), and \( \tau(f') = 0 \).

The last construction is also used to define \( \psi \): let \( \tau \in Wh(M) \) represent an element of \( \tilde{H}^n(\mathbb{Z}/2; Wh(M)) \), i.e. \( \tau = (-1)^n \tau^* \), and let \( W \) be an \( h \)-cobordism with one end equal to \( M \) and torsion \( \tau \). If the other end is \( N \), the natural homotopy equivalence \( h : N \rightarrow M \) has torsion \( \tau(h) = -\tau + (-1)^n \tau^* = 0 \) and we set \( \psi(\tau) = h \).

If \( \tau = \sigma + (-1)^n \sigma^* \) we can choose \( W \) to be the “double” of an \( h \)-cobordism with torsion \( \sigma \) (Milnor, Lemma 11.4), and the construction gives \( h = \text{id}_M \). Hence \( \psi \) is well-defined.

The construction of \( \psi \) is illustrated by the following special case of diagram (4):

![Diagram](5)

Exactness at \( \beta(M) \) follows when we observe that this diagram also expresses precisely that \( h \) is equivalent to \( \text{id}_M \) in \( S^h(M) \).

\[ \square \]
Remarks. (1) The sequence can be continued to a long exact sequence of groups to the left by hooking it up in the obvious way with the sequences for $M \times I$, $M \times I^2$ and so on.

(2) The maps $L^*_n(M) \to S^*(M)$ in the surgery sequences ($*=s, h$) give an obvious map from the $L$-theory Rothenberg sequence, and an interlocking braid (continuing to the left)

\begin{equation}
\begin{array}{cccc}
N(M \times I) & L^h_{n+1}(M) & \tilde{H}^{n-1}(\mathbb{Z}/2; Wh(M)) \\
L^s_{n+1}(M) & S^h(M) & L^n_s(M) \\
\tilde{H}^{n}(\mathbb{Z}/2; Wh(M)) & S(M) & N(M) & L^n_h(M)
\end{array}
\end{equation}

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