Differential Operators on the Weighted Densities on the Supercircle $S^1|n$

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Abstract

Over the $(1,n)$-dimensional real supercircle, we consider the $\mathcal{K}(n)$-modules $D_{\lambda,\mu}^{n,k}$ of linear differential operators of order $k$ acting on the superspaces of weighted densities, where $\mathcal{K}(n)$ is the Lie superalgebra of contact vector fields. We give, in contrast to the classical setting, a classification of these modules for $n = 1$. We also prove that $D_{\lambda,\mu}^{n,k}$ and $D_{\rho,\nu}^{n,k}$ are isomorphic for $\rho = \frac{2n}{2} - \mu$ and $\nu = \frac{2n}{2} - \lambda$. This work is the simplest superization of a result by Gargoubi and Ovsienko [Modules of Differential Operators on the Real Line, Functional Analysis and Its Applications, Vol. 35, No. 1, pp. 13–18, 2001.]

1 Introduction.

Let $\text{Vect}(S^1)$ be the Lie algebra of vector fields on $S^1$. Consider the 1-parameter deformation of the natural $\text{Vect}(S^1)$-action on $C^\infty(S^1)$:

$$L^\lambda_{X_h}(f) = L_{X_h}(f) + \lambda h f,$$

where $X_h = h \frac{d}{dx}$. The $\text{Vect}(S^1)$-module so defined is the space $\mathcal{F}_\lambda$ of weighted densities of weight $\lambda \in \mathbb{R}$:

$$\mathcal{F}_\lambda = \left\{ f dx^\lambda \mid f \in C^\infty(S^1) \right\}.$$

We denote $D_{\lambda,\mu}$ the space of linear differential operators from $\mathcal{F}_\lambda$ to $\mathcal{F}_\mu$ and $D_{\lambda,\mu}^k$ the space of linear differential operators of order $k$ which are naturally endowed with a $\text{Vect}(S^1)$-module structure defined by the action $L^\lambda_{X_h}$:

$$L^\lambda_{X_h}(A) := L^\mu_{X_h} \circ A - A \circ L^\lambda_{X_h},$$

where $A \in D_{\lambda,\mu}^k$. Gargoubi and Ovsienko [16] classified these modules and gave a complete list of isomorphisms between distinct modules $D_{\lambda,\mu}^k$. The classification problem for modules of differential operators on a smooth manifold was posed for $\lambda = \mu$ and solved for modules of second-order operators.

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The modules $D_{\lambda,\mu}^k$ on $\mathbb{R}$ were classified in [15]. In the multidimensional case, this classification problem was solved in [21, 22].

In this paper we study the simplest super analog of the problem solved in [16], namely, we consider the supercircle $S^{1|n}$ equipped with the contact structure determined by a 1-form $\alpha_n$, and the Lie superalgebra $K(n)$ of contact vector fields on $S^{1|n}$. We introduce the $K(n)$-module $\mathfrak{D}_{\lambda}^n$ of $\lambda$-densities on $S^{1|n}$ and the $K(n)$-module of linear differential operators, $\mathfrak{D}_{\lambda,\mu}^n := \text{Hom}_{\text{diff}}(\mathfrak{F}_{\lambda}^n, \mathfrak{F}_{\mu}^n)$, which are super analogs of the spaces $\mathfrak{F}_{\lambda}$ and $\mathfrak{D}_{\lambda,\mu}$ respectively. The $K(n)$-module $\mathfrak{D}_{\lambda,\mu}^n$ is filtered:

$$\mathfrak{D}_{\lambda,\mu}^n \subset \mathfrak{D}_{\lambda,\mu}^{n+1} \subset \mathfrak{D}_{\lambda,\mu}^{n+2} \subset \cdots \subset \mathfrak{D}_{\lambda,\mu}^{n-k+1} \subset \mathfrak{D}_{\lambda,\mu}^{n-k} \subset \cdots$$

For $n = 1$, we omit the subscript $n$, that is, $\mathfrak{D}_{\lambda}^n$ and $\mathfrak{D}_{\lambda,\mu}^n$ will be simply denoted $\mathfrak{D}_{\lambda}$ and $\mathfrak{D}_{\lambda,\mu}$.

The aim of this paper is to classify these modules $\mathfrak{D}_{\lambda,\mu}^n$. For $n = 1$ we shall give a complete list of isomorphisms between distinct modules $\mathfrak{D}_{\lambda,\mu}^n$. Moreover, we prove that $\mathfrak{D}_{\lambda,\mu}^n$ and $\mathfrak{D}_{\rho,\nu}^n$ are $K(n)$-isomorphic for $\nu = 2 - n - \lambda$ and $\rho = 2 - n - \mu$. The complete classification of modules $\mathfrak{D}_{\lambda,\mu}^n$ for $n \geq 2$, needs an other study. We mention that a similar problem was considered in [11] for the case of pseudodifferential operators instead of differential operators.

2 The main definitions.

In this section, we recall the main definitions and facts related to the geometry of the supercircle $S^{1|1}$; for more details, see [10, 13, 18, 19].

2.1 The Lie superalgebra of contact vector fields on $S^{1|1}$

Let $S^{1|1}$ be the supercircle with local coordinates $(x, \theta)$, where $\theta$ is an odd indeterminate: $\theta^2 = 0$. We introduce the vector fields $\eta = \partial_\theta + \theta \partial_x$ and $\eta = \partial_\theta - \theta \partial_x$. The supercircle $S^{1|1}$ is equipped with the standard contact structure given by the distribution $\langle \eta \rangle$. That is, the distribution $\langle \eta \rangle$ is the kernel of the following 1-form:

$$\alpha = dx + \theta d\theta.$$

On $C^\infty(S^{1|1})$, we consider the contact bracket

$$\{F, G\} = FG' - F'G - \frac{1}{2}(\alpha|F)\eta(F)\eta(G),$$

where the subscript $'$ stands for $\frac{df}{dx}$ and $|F|$ is the parity of an homogeneous function $F$. Let $\text{Vect}(S^{1|1})$ be the superspace of vector fields on $S^{1|1}$:

$$\text{Vect}(S^{1|1}) = \left\{ F_0 \partial_x + F_1 \partial_\theta \mid F_0, F_1 \in C^\infty(S^{1|1}) \right\},$$

and consider the superspace $K(1)$ of contact vector fields on $S^{1|1}$ (also known as the Neveu-Schwartz superalgebra without central charge, cf. [13, 24]). That is, $K(1)$ is the superspace of vector fields on $S^{1|1}$ preserving the distribution $\langle \eta \rangle$:

$$K(1) = \left\{ X \in \text{Vect}(S^{1|1}) \mid [X, \eta] = F_\eta \eta \right\} \text{ for some } F_\eta \in C^\infty(S^{1|1}).$$
The Lie superalgebra $\mathcal{K}(1)$ is spanned by the vector fields of the form:

$$X_F = F \partial_x - \frac{1}{2} (-1)^{|F|} \overline{\eta}(F) \overline{\eta}, \quad \text{where} \quad F \in C^\infty(S^{1|1}).$$

Of course, $\mathcal{K}(1)$ is a subalgebra of $\text{Vect}(S^{1|1})$, and $\mathcal{K}(1)$ acts on $C^\infty(S^{1|1})$ through:

$$\mathfrak{L}_{X_F}(G) = FG' - \frac{1}{2} (-1)^{|F|} \overline{\eta}(F) \cdot \overline{\eta}(G).$$

The bracket in $\mathcal{K}(1)$ can be written as: $[X_F, X_G] = X_{\{F, G\}}$.

### 2.2 The space of weighted densities on $S^{1|1}$

In the super setting, by replacing $dx$ by the 1-form $\alpha$, we get analogous definition for weighted densities i.e. we define the space of $\lambda$-densities as

$$\mathfrak{F}_\lambda = \left\{ F(x, \theta) \alpha^\lambda \mid F(x, \theta) \in C^\infty(S^{1|1}) \right\}.$$

As a vector space, $\mathfrak{F}_\lambda$ is isomorphic to $C^\infty(S^{1|1})$, but the Lie derivative of the density $G\alpha^\lambda$ along the vector field $X_F$ in $\mathcal{K}(1)$ is:

$$\mathfrak{L}_{X_F}^\lambda (G\alpha^\lambda) = (\mathfrak{L}_{X_F}(G) + \lambda F') \alpha^\lambda.$$

Obviously $\mathcal{K}(1)$ and $\mathfrak{F}_{-1}$ are isomorphic as $\mathcal{K}(1)$-modules. Naturally $\mathcal{K}(1)$ acts on the super-space $\mathfrak{D}_{\lambda, \mu} := \text{Hom}_{\text{diff}}(\mathfrak{F}_\lambda, \mathfrak{F}_\mu)$ through:

$$\mathfrak{L}_{X_F}^{\lambda, \mu}(A) = \mathfrak{L}_{X_F}^{\mu} \circ A - (-1)^{|A||F|} A \circ \mathfrak{L}_{X_F}^{\lambda}. \quad (2.1)$$

Since $\overline{\eta}^2 = -\partial_x$, any differential operator $A \in \mathfrak{D}_{\lambda, \mu}$ can be expressed in the form:

$$A(F\alpha^\lambda) = \sum_{i=0}^\ell a_i \overline{\eta}^i(F) \alpha^\mu, \quad (2.2)$$

where the coefficients $a_i \in C^\infty(S^{1|1})$ and $\ell \in \mathbb{N}$. For $k \in \frac{1}{2} \mathbb{N}$, the space of differential operators of the form (2.2) with $\ell = 2k$ is denoted by $\mathfrak{D}_{\lambda, \mu}^k$, and called the space of differential operators of order $k$. Thus, we have a $\mathcal{K}(1)$-invariant filtration:

$$\mathfrak{D}_{\lambda, \mu}^0 \subset \mathfrak{D}_{\lambda, \mu}^{\frac{1}{2}} \subset \mathfrak{D}_{\lambda, \mu}^1 \subset \mathfrak{D}_{\lambda, \mu}^{\frac{3}{2}} \subset \cdots \subset \mathfrak{D}_{\lambda, \mu}^{k} \subset \cdots \quad (2.3)$$

The quotient module $\mathfrak{D}_{\lambda, \mu}^i / \mathfrak{D}_{\lambda, \mu}^{i-\frac{1}{2}}$ is isomorphic to the module of weighted densities $\Pi^{2i} (\mathfrak{F}_{\mu-\lambda-i})$ (see, e.g., [13]), where $\Pi$ is the change of parity map. Thus, the graded $\mathcal{K}(1)$-module $\text{gr}\mathfrak{D}_{\lambda, \mu}$ associated with the filtration (2.3) is a direct sum of density modules:

$$\text{gr}\mathfrak{D}_{\lambda, \mu} = \bigoplus_{i=0}^\infty \Pi^i \left( \mathfrak{F}_{\mu-\lambda-\frac{i}{2}} \right).$$

Note that this module depends only on the shift, $\mu - \lambda$, of the weights and not on $\mu$ and $\lambda$ independently. We call this $\mathcal{K}(1)$-module the space of symbols of differential operators and denote it $\mathfrak{S}_{\mu-\lambda}$. The space of symbols of order $\leq k$ is

$$\mathfrak{S}_{\mu-\lambda}^k = \bigoplus_{i=0}^{2k} \Pi^i \left( \mathfrak{F}_{\mu-\lambda-\frac{i}{2}} \right).$$
3 Classification results

We now give a complete classification of the modules $\mathcal{D}_{\lambda,\mu}^k$. First note that the difference $\delta = \mu - \lambda$ of weight is an invariant: the condition $\mathcal{D}_{\lambda,\mu}^k \simeq \mathcal{D}_{\nu,\rho}^k$ implies that $\mu - \lambda = \nu - \rho$. This is a consequence of the equivariance with respect to the vector field $X_{\lambda}$. Moreover, recall that, for every $k \in \frac{1}{2}\mathbb{N}$, there exists a $K(1)$-invariant conjugate map from $\mathcal{D}_{\lambda,\mu}^k$ to $\mathcal{D}_{\lambda,\mu}^{-k}$ defined by:

$$a\eta^i \mapsto (1 - |\frac{k+1}{2}| - |x|\eta^i) \circ a.$$  

Clearly, this map is a $K(1)$-isomorphism. The module $\mathcal{D}_{\lambda,\mu}^{-k}$ is called the adjoint module of $\mathcal{D}_{\lambda,\mu}^k$. A module with $\lambda + \mu = \frac{1}{2}$ is said to be self-adjoint. We say that a module $\mathcal{D}_{\lambda,\mu}^k$ is singular if it is only isomorphic to its adjoint module.

Our main result of this paper is the following:

**Theorem 3.1.** i) For $k \leq 2$, the $K(1)$-modules $\mathcal{D}_{\lambda,\mu}^k$ and $\mathcal{D}_{\nu,\rho}^k$ are isomorphic if and only if $\mu - \lambda = \nu - \rho$ except for the modules listed in the following table and their adjoint modules which are all singular.

| $k$          | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ | 2 |
|--------------|---------------|---|---------------|---|
| $(\lambda, \mu)$ | $(0, \frac{1}{2})$ | $(0, \frac{1}{2})$ | $(0, \mu), (-\frac{1}{2}, 1)$ | $(0, \mu), (\lambda, \frac{1}{2} - \lambda), (\lambda, \lambda + 2)$ |

ii) For $k > 2$, the $K(1)$-modules $\mathcal{D}_{\lambda,\mu}^k$ are all singular.

The proof of Theorem 3.1 will be the subject of sections 6 and 8. In fact, we need first to study the action of $K(1)$ on $\mathcal{D}_{\lambda,\mu}^k$ in terms of $\mathfrak{osp}(1|2)$-equivariant symbols.

4 Modules of differential operators over $\mathfrak{osp}(1|2)$

Consider the Lie superalgebra $\mathfrak{osp}(1|2) \subset K(1)$ generated by the functions: $1, x, x^2, \theta, x\theta$. The Lie superalgebra $\mathfrak{osp}(1|2)$ plays a special role and allows one to identify $\mathcal{D}_{\lambda,\mu}^k$ with $\mathfrak{S}_{\mu-\lambda}^k$, in a canonical way. The following result (see [13]) shows that, for generic values of $\lambda$ and $\mu$, $\mathcal{D}_{\lambda,\mu}^k$ and $\mathfrak{S}_{\mu-\lambda}^k$ are isomorphic as $\mathfrak{osp}(1|2)$-modules.

**Theorem 4.1.** [13]. (i) If $\mu - \lambda$ is nonresonant, i.e., $\mu - \lambda \notin \frac{1}{2}\mathbb{N} \setminus \{0\}$, then $\mathcal{D}_{\lambda,\mu}^k$ and $\mathfrak{S}_{\mu-\lambda}^k$ are $\mathfrak{osp}(1|2)$-isomorphic through the unique $\mathfrak{osp}(1|2)$-invariant symbol map $\sigma_{\lambda,\mu}$ defined by:

$$\sigma_{\lambda,\mu}(a\eta^k) = \sum_{n=0}^{k} \Pi^{k-n} \left( \gamma_n^k \eta^n(a) \alpha^{\mu-\lambda - \frac{k-n}{2}} \right), \quad (4.1)$$

where

$$\gamma_n^k = (-1)^{|\frac{n+1}{2}|} \left( \frac{\left[ \frac{k+1}{2} \right]}{2(n+1+(-1)^{n+k})} \right) \left( \frac{\left[ \frac{k-1}{2} \right] + 2\lambda}{2(n+1+(-1)^{n+k})} \right) \left( \frac{2(\mu - \lambda) + n - k - 1}{\lfloor \frac{n+1}{2} \rfloor} \right) \quad (4.2)$$

with $\binom{\nu}{i} = \frac{\nu(\nu-1)\cdots(\nu-i+1)}{i!}$ and $[x]$ denotes the integer part of a real number $x$. 


(ii) In the resonant cases the \( \mathfrak{osp}(1|2) \)-modules \( \mathcal{D}_{\lambda,\mu} \) and \( \mathcal{S}_{\mu-\lambda} \) are not isomorphic, except for \( (\lambda, \mu) = (\frac{1-m}{2}, \frac{1+m}{2}) \), where \( m \) is an odd integer.

The main idea of proof of Theorem 3.1 is to use the \( \mathfrak{osp}(1|2) \)-equivariant symbol mapping \( \sigma_{\lambda,\mu} \) to reduce the action of \( \mathcal{K}(1) \) on \( \mathcal{D}_{\lambda,\mu}^k \) to a canonical form. In other words, we shall use the following diagram
\[
\begin{array}{cccc}
\mathcal{D}_{\lambda,\mu}^k & \xrightarrow{\eta_{X_F}} & \mathcal{D}_{\lambda,\mu}^k \\
\sigma_{\lambda,\mu} & & & \sigma_{\lambda,\mu} \\
\mathcal{S}_{\mu-\lambda}^k & \xrightarrow{\tilde{\eta}_{X_F}} & \mathcal{S}_{\mu-\lambda}^k
\end{array}
\]
and compare the action \( \mathcal{K}(1 \circ \eta_{X_F} \circ \sigma_{\lambda,\mu}^{-1} ) \) with the standard action of \( \mathcal{K}(1) \) on \( \mathcal{S}_{\mu-\lambda}^k \).

5 The action of \( \mathcal{K}(1) \) in the \( \mathfrak{osp}(1|2) \)-invariant form.

The action of \( \mathcal{K}(1) \) on \( \mathcal{D}_{\lambda,\mu}^k \) in terms of \( \mathfrak{osp}(1|2) \)-equivariant symbols is closely related to the space \( \mathcal{S}_{\lambda}^k \) of \( \mathfrak{osp}(1|2) \)-invariant linear operators from \( \mathcal{K}(1) \) to \( \mathcal{D}_{\lambda,\lambda+k-1} \) vanishing on \( \mathfrak{osp}(1|2) \). For \( k > 2 \), the space \( \mathcal{S}_{\lambda}^k \) is one dimensional, spanned by the maps:
\[
X_F \mapsto \left( G\alpha \mapsto \mathcal{J}_{\lambda}^k(F,G)\alpha^{\lambda+k-1} \right)
\]
where \( \mathcal{J}_{\lambda}^k \) is the supertransvectant \( \mathcal{J}_{\lambda}^{1,\lambda} \) defined in \[3\] (see also \[17, 14\]). The operators \( \mathcal{J}_{\lambda}^k \) labeled by semi-integer \( k \) are odd and they are given by:
\[
\mathcal{J}_{\lambda}^k(F,G) = \sum_{i+j=[k], i\geq 2} \Gamma_{i,j,k}^\lambda \left( (-1)^{|F|}([k]-j-2)F^{(i)}G^{(j)} - (2\lambda + [k] - i)\pi(F^{(i)})G^{(j)} \right).
\]

The operators \( \mathcal{J}_{\lambda}^k \), where \( k \in \mathbb{N} \), are even and they are given by:
\[
\mathcal{J}_{\lambda}^k(F,G) = \sum_{i+j=k-1, i\geq 2} (-1)^{|F|} \Gamma_{i,j,k}^\lambda \pi(F^{(i)})\pi(G^{(j)}) - \sum_{i+j=k, i\geq 3} \Gamma_{i,j,k}^\lambda F^{(i)}G^{(j)},
\]
where \( \binom{x}{i} = \frac{x(x-1)...(x-i+1)}{i!} \) and \( [k] \) denotes the integer part of \( k \), \( k > 0 \), and
\[
\Gamma_{i,j,k}^\lambda = (-1)^{j} \binom{[k]-2}{j} \binom{2\lambda + [k]}{i}.
\]

We will need the expressions of \( \mathcal{J}_{\lambda}^k \), \( \mathcal{J}_{\lambda}^k \) and \( \mathcal{J}_{\lambda}^k \):
\[
\begin{align*}
\mathcal{J}_{\lambda}^k(F,G) &= \pi(F^{''})G, \\
\mathcal{J}_{\lambda}^k(F,G) &= \left( 4\lambda F^{(3)}G - (-1)^{|F|}\pi(F^{''})\pi(G) \right), \\
\mathcal{J}_{\lambda}^k(F,G) &= \left( 2\lambda\pi(F^{(3)})G - 3\pi(F^{''})G' - (-1)^{|F|}F^{(3)}\pi(G) \right),
\end{align*}
\]

Now, using formula \[5.1\] and the graded Leibniz formula:
\[
\pi^j \circ F = \sum_{i=0}^{j} \binom{j}{i} (-1)^{|F|(j-i)}\pi(F)^{\pi^{-i}} \quad \text{where} \quad \binom{j}{i} = \begin{cases} \left( \frac{j}{2} \right) & \text{if } i \text{ is even or } j \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}
\]
we easily check the following result:
Lemma 5.1. The natural action of $\mathcal{K}(1)$ on $\mathcal{O}^k_{\lambda,\mu}$ is given by $\mathcal{L}^{\lambda,\mu}(A) := \sum_{i=0}^{2k} a_i^X \pi_i^F$, where

$$a_i^X = \mathcal{L}^{\mu-\nu,\frac{1}{2}}_{X \pi} (a_i) - \sum_{j \geq i+1} (-1)^{(i+j)} (-1)^{j} \zeta_{i,j,\lambda} \pi_j^{-i} (F') a_j$$

(5.3)

with

$$\zeta_{i,j,\lambda} = \lambda \binom{j}{j-i} \frac{(-1)^j}{2} \binom{j}{j-i+1} \frac{j}{s} + \binom{j}{j-i-2} \frac{j}{s}.$$  

(5.4)

Now, we need to study the action of $\mathcal{K}(1)$ over $\mathcal{O}^k_{\lambda,\mu}$ in terms of $\mathfrak{osp}(1|2)$-equivariant symbols, thus, let

$$\mathcal{L}^{\lambda,\mu}_{X \pi} \left( \sum_{p=0}^{2k} \prod (P_p \alpha^{\beta-\frac{p}{2}}) \right) = \sum_{p=0}^{2k} \prod (P_p^{X \pi} \alpha^{\beta-\frac{p}{2}}), \text{ where } \beta = \mu - \lambda,$$

(5.5)

then, we need to compute the terms $P_p^{X \pi}$.

Proposition 5.2. (i) The terms $P_p^{X \pi}$ are given by:

$$P_p^{X \pi} = \mathcal{L}^{\lambda-\mu,\frac{p}{2}}_{X \pi} (P_p) + \sum_{j=p+3}^{2k} \beta_j p^{j-p} \circ \mathcal{L}^{\lambda-\mu,\frac{p}{2}}_{X \pi} (X_F, P_j),$$

(5.6)

where $\pi(F) = (-1)^{|F|} F$ and the coefficients $\beta_j$ are some functions of $\lambda$ and $\mu$.

(ii) For $j \leq 5$, the coefficients $\beta_j$ of formula (5.6) are given by:

$$\beta_0^3 = \beta_0^3 (\lambda, \mu) = -\frac{\lambda(2\delta+2\lambda-1)}{2\delta-2},$$

$$\beta_0^4 = \beta_0^4 (\lambda, \mu) = -\frac{3\lambda(2\delta+2\lambda-1)}{2\delta-2} \frac{2\delta+4\lambda-1}{2\delta-4},$$

$$\beta_0^1 = \beta_0^1 (\lambda, \mu) = -\frac{2\delta+4\lambda-1}{2\delta-4},$$

$$\beta_0^5 = \beta_0^5 (\lambda, \mu) = -\frac{2\delta+4\lambda-1}{2\delta-4},$$

$$\beta_1^1 = \beta_1^1 (\lambda, \mu) = -\frac{2\delta+4\lambda-1}{2\delta-4},$$

$$\beta_2^2 = \beta_2^2 (\lambda, \mu) = -\frac{2\delta+4\lambda-1}{2\delta-4}.$$  

(5.7)

Proof. (i) According to Lemma 5.1 and formula (4.1), we prove that $P_p^{X \pi}$ can be expressed as follows

$$P_p^{X \pi} = \mathcal{L}^{\lambda-\mu,\frac{p}{2}}_{X \pi} (P_p) + \sum_{j=0}^{2k} f_j (X_F, P_j),$$

(5.8)

where the $f_j$ are bilinear maps from $\mathcal{K}(1) \times \mathfrak{g}^{\delta-\frac{p}{2}}$ to $\mathfrak{g}^{\delta-\frac{p}{2}}$ vanishing on $\mathfrak{osp}(1|2)$. Since $\mathcal{L}^{\lambda,\mu}$ is a $\mathcal{K}(1)$-action on $\mathfrak{g}^k$, then $f_j$ has the same parity as the integer $j-p$. Moreover, for $X_G \in \mathfrak{osp}(1|2)$, we have

$$\mathcal{L}_{X_G} \circ \mathcal{L}^{\lambda,\mu}_{X \pi} = \mathcal{L}^{\lambda,\mu}_{[X_G, X \pi]} + (-1)^{|F|} |G| \mathcal{L}^{\lambda,\mu}_{X \pi} \circ \mathcal{L}_{X_G}.$$  

Thus, from (5.8), we deduce that

$$\mathcal{L}^{\delta-\frac{p}{2}}_{X_G} f_j (X_F, P_j) = f_j ([X_G, X_F], P_j) + (-1)^{|F||G|} f_j (X_F, \mathcal{L}^{\delta-\frac{p}{2}}_{X \pi} (P_j)),$$
Therefore, the map $\pi^{j-p} \circ f_j$ is a supertranvectant vanishing on $\mathfrak{osp}(1|2)$. Thus, up to a scalar factor, we have $f_j = \pi^{j-p} \circ \mathfrak{J}^{\delta-\frac{j}{2}}_{\frac{p}{2}+1}$ for $j \geq p + 3$; otherwise $f_j = 0$.

(ii) By a direct computation, using formulas (5.3) and (4.1), we get expression (5.7). □

6 Proof of Theorem 3.1 in the generic case

In this section we prove Theorem 3.1 for the nonresonant values of $\delta = \mu - \lambda$.

Proposition 6.1. For $k \in \frac{1}{2}\mathbb{N}$, let $T : \mathfrak{D}^k_{\lambda,\mu} \to \mathfrak{D}^k_{\rho,\nu}$ be an isomorphism of $\mathcal{K}(1)$-modules. Then the linear mapping $\sigma_{\rho,\nu} \circ T \circ \sigma^{-1}_{\lambda,\mu}$ on $\mathfrak{S}^k_{\delta}$ is diagonal and the $\Pi^i \left( \mathfrak{F}^{\delta-\frac{i}{2}} \right)$ are eigenspaces:

$$\Pi^i \left( P_{1}^T \alpha^{\delta-\frac{i}{2}} \right) := \sigma_{\rho,\nu} \circ T \circ \sigma^{-1}_{\lambda,\mu} \left( \Pi^i \left( P_{1} \alpha^{\delta-\frac{i}{2}} \right) \right) = \Pi^i \left( \tau_{1} P_{1} \alpha^{\delta-\frac{i}{2}} \right), \quad \tau_{i} \in \mathbb{R} \setminus \{0\}.$$

Proof. Since $T$ is an isomorphism of $\mathcal{K}(1)$-modules, it is also an isomorphism of $\mathfrak{osp}(1|2)$-modules. The uniqueness of the $\mathfrak{osp}(1|2)$-equivariant symbols mapping shows that the linear mapping $\sigma_{\rho,\nu} \circ T \circ \sigma^{-1}_{\lambda,\mu}$ on $\mathfrak{S}^k_{\delta}$ is diagonal and the $\Pi^i \left( \mathfrak{F}^{\delta-\frac{i}{2}} \right)$ are eigenspaces. □

6.1 The construction of isomorphisms

To prove Theorem 3.1 we construct the desired isomorphism explicitly in terms of projectively equivariant symbols using Proposition 5.2 and Proposition 6.1

i) For $k = \frac{1}{2}$, formula (6.1) defines an isomorphism $T : \mathfrak{D}^{\frac{1}{2}}_{\lambda,\mu} \to \mathfrak{D}^{\frac{1}{2}}_{\rho,\nu}$ for all $\tau_0, \tau_1 \neq 0$:

$$\left( P_{0}^{X_{P}}, P_{1}^{X_{P}} \right) = \left( \mathfrak{A}_{X_{P}}(P_{0}), \mathfrak{L}^{\frac{1}{2}}_{X_{P}}(P_{1}) \right).$$

Since the action (5.6) is

$$\left( P_{0}^{X_{P}}, P_{1}^{X_{P}}, P_{2}^{X_{P}} \right) = \left( \mathfrak{A}_{X_{P}}(P_{0}), \mathfrak{L}^{\frac{1}{2}}_{X_{P}}(P_{1}), \mathfrak{L}^{\frac{3}{2}}_{X_{P}}(P_{2}) \right).$$

ii) For $k = 1$, formula (6.1) defines an isomorphism $T : \mathfrak{D}^{1}_{\lambda,\mu} \to \mathfrak{D}^{1}_{\rho,\nu}$ for all $\tau_0, \tau_1, \tau_2 \neq 0$:

$$\left( P_{0}^{X_{P}}, P_{1}^{X_{P}}, P_{2}^{X_{P}} \right) = \left( \mathfrak{A}_{X_{P}}(P_{0}), \mathfrak{L}^{\frac{1}{2}}_{X_{P}}(P_{1}), \mathfrak{L}^{\frac{3}{2}}_{X_{P}}(P_{2}) \right).$$

Since the action (5.6) is

$$\left( P_{0}^{X_{P}}, P_{1}^{X_{P}}, P_{2}^{X_{P}} \right) = \left( \mathfrak{A}_{X_{P}}(P_{0}), \mathfrak{L}^{\frac{1}{2}}_{X_{P}}(P_{1}), \mathfrak{L}^{\frac{3}{2}}_{X_{P}}(P_{2}) \right).$$

iii) For $k = \frac{3}{2}$, let $T : \mathfrak{D}^{\frac{3}{2}}_{\lambda,\mu} \to \mathfrak{D}^{\frac{3}{2}}_{\rho,\nu}$ be any $\mathcal{K}(1)$-isomorphism defined by:

$$\Pi^i \left( P_{1}^T \alpha^{\delta-\frac{i}{2}} \right) = \Pi^i \left( \tau_{1} P_{1} \alpha^{\delta-\frac{i}{2}} \right), \quad i = 0, 1, 2, 3, \tau_{i} \in \mathbb{R} \setminus \{0\}.$$

Since the action (5.6) is defined, in this case, by:

$$P_{0}^{X_{P}} = \mathfrak{A}_{X_{P}}(P_{0}), \quad P_{1}^{X_{P}} = \mathfrak{L}^{\frac{1}{2}}_{X_{P}}(P_{1}), \quad P_{2}^{X_{P}} = \mathfrak{L}^{\frac{3}{2}}_{X_{P}}(P_{2}).$$
Theorem 3.1 is now completely proved for nonresonant values of 
subsection, using the approach of the deformation theory (see, e.g., [1, 3, 7, 8, 23]), we will

iv) For $k > 5$, we have

$$P_0^{X, L} = \mathcal{D}_X^\ast(P_0) + \beta_0^3 \otimes \frac{3^k - 2}{2} (F, P_0),$$

and then, as in the previous case, we have to distinguish two cases:

(1) If $\beta_0^3(\lambda, \mu) \neq 0$, then we get a family of isomorphisms $T : \mathcal{D}_{\lambda, \mu}^\ast \rightarrow \mathcal{D}_{\rho, \nu}^\ast$ given by

(6.1) with

$$\tau_0, \tau_1, \tau_2, \tau_3 \in \mathbb{R} \setminus \{0\} \quad \text{and} \quad \tau_0 = \tau_0 \beta_0^3(\rho, \nu) \beta_0^3(\lambda, \mu).$$

(2) If $\beta_0^3(\lambda, \mu) = 0$, that is, $\lambda = 0$ or $\mu = \frac{1}{2}$, then, we have $\beta_0^3(\rho, \nu) = 0$, so, the modules

$\mathcal{D}_{\lambda, \mu}^\ast$ and $\mathcal{D}_{\rho, \nu}^\ast$ are equal or conjugate. Thus $\mathcal{D}_{\lambda, \mu}^\ast$ is singular.

Thus, we get the following conditions:

$$\tau_0 \beta_0^3(\lambda, \mu) = \tau_3 \beta_0^3(\rho, \nu), \quad \tau_0 \beta_0^3(\lambda, \mu) = \tau_4 \beta_0^3(\rho, \nu), \quad \tau_1 \beta_0^3(\lambda, \mu) = \tau_4 \beta_1^4(\rho, \nu)$$

and then, as in the previous case, we have to distinguish two cases:

(1) If $\beta_0^3, \beta_0^4, \beta_1^4 \neq 0$, then we get a family of isomorphisms $T : \mathcal{D}_{\lambda, \mu}^\ast \rightarrow \mathcal{D}_{\rho, \nu}^\ast$ given by

(6.1) with

$$\tau_2, \tau_3 \in \mathbb{R} \setminus \{0\}, \quad \tau_4 = \tau_3,$$

$$\tau_0 = \tau_0 \beta_0^3(\rho, \nu) \beta_0^3(\lambda, \mu), \quad \text{and} \quad \tau_1 = \tau_3 \beta_0^4(\rho, \nu) \beta_1^4(\lambda, \mu).$$

(2) If $\beta_0^3 = 0$ or $\beta_0^4 = 0$ or $\beta_1^4 = 0$, then, as in the previous case, we prove that the modules $\mathcal{D}_{\lambda, \mu}^\ast$ and $\mathcal{D}_{\lambda, \mu}^\ast$ are singular.

v) For $k = 5$, any isomorphism $T : \mathcal{D}_{\lambda, \mu}^5 \rightarrow \mathcal{D}_{\rho, \nu}^5$ has a diagonal form by Proposition 6.1

The equivariant conditions of $T$ lead to the following system

$$\tau_0 \beta_0^3(\lambda, \mu) = \tau_3 \beta_0^3(\rho, \nu),$$

$$\tau_0 \beta_0^4(\lambda, \mu) = \tau_4 \beta_0^4(\rho, \nu),$$

$$\tau_0 \beta_0^5(\lambda, \mu) = \tau_5 \beta_0^5(\rho, \nu),$$

$$\tau_1 \beta_0^4(\lambda, \mu) = \tau_4 \beta_1^4(\rho, \nu),$$

$$\tau_1 \beta_0^5(\lambda, \mu) = \tau_5 \beta_1^5(\rho, \nu),$$

$$\tau_2 \beta_2^5(\lambda, \mu) = \tau_5 \beta_2^5(\rho, \nu).$$

One can readily check that this system has solutions only if $\lambda = \rho$ or $\rho + \mu = \frac{1}{2}$. The first isomorphism is tautological, and the second is just the passage to the adjoint module.

vi) For $k > 5$, let $T : \mathcal{D}_{\lambda, \mu}^k \rightarrow \mathcal{D}_{\rho, \nu}^k$ be an isomorphism of $\mathcal{K}(1)$-modules. The restriction of $T$ to $\mathcal{D}_{\lambda, \mu}^5 \subset \mathcal{D}_{\lambda, \mu}^k$ must be an isomorphism onto $\mathcal{D}_{\rho, \nu}^5$. So, we must have $\lambda = \rho$ or $\rho + \mu = \frac{1}{2}$. Theorem 3.1 is now completely proved for nonresonant values of $\delta = \mu - \lambda$. In the next subsection, using the approach of the deformation theory (see, e.g., [1, 3, 7, 8, 23]), we will give the relationship between singular modules and cohomology for nonresonant values of $\delta$. 
6.2 Cohomological interpretation of singularity of $O^k_{\lambda,\mu}$, $k \leq 2$

Of course, the actions of $K(1)$ on $G_\delta$ defined by $\hat{D}^{\lambda,\mu}$ and $\hat{D}^{\rho,\nu}$ are two $osp(1|2)$-trivial deformations of the natural action $L$. These deformations become trivial when restricted to $osp(1|2)$. So, they are related to the the $osp(1|2)$-relative cohomology space $[3]:$

$$H^1_{diff} (K(1), osp(1|2); End_{diff}(G_\delta)) = \bigoplus_{p \leq j} H^1_{diff} (K(1), osp(1|2); \Pi^{j-p} \left( D_{\delta-\frac{j}{2},\delta-\frac{j}{2}} \right)),$$

where $H^1_{diff}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. This $osp(1|2)$-relative cohomology space is spanned by the nontrivial 1-cocycles $\Pi^{j-p} \left( \pi^{j-p} \circ 3^{\delta-\frac{j}{2}} \left( \cdot, \cdot \right) \right)$ where $j-p \in \{3, 4, 5, 6, 8\}$ (see, e.g., [3,10]). So, for $k \leq 2$, by fundamental arguments of the theory of deformation [23], we can see that, for $j-p = 3$ or 4, if $\beta^j_p(\lambda,\mu) = 0$ and $\beta^j_p(\rho,\nu) \neq 0$, then the $K(1)$-modules $O^k_{\lambda,\mu}$ and $O^k_{\rho,\nu}$ are not isomorphic. Thus, singular modules appear whenever at least one of the coefficients $\beta^j_p(\lambda,\mu)$ in (5.6) vanishes.

**Remark 6.2.** Clearly, the $osp(1|2)$-trivial deformation of the action of $K(1)$ on the space of symbols $G^{\frac{3}{2}}_{\frac{3}{2}-\lambda}$ is trivial. So, as a $K(1)$-modules, we have $O^{\frac{3}{2}}_{\lambda,\frac{3}{2}} \simeq G^{\frac{3}{2}}_{\frac{3}{2}-\lambda}$.

7 Obstructions to the existence of $osp(1|2)$-equivariant symbol mappings

For the resonant values $\delta$, there exist a series of cohomology classes of $osp(1|2)$ that are obstructions for the isomorphism in Theorem [3,1]. More precisely, consider the linear mappings $\Upsilon_n : osp(1|2) \to O^{\frac{3}{2}}_{\lambda,\frac{3}{2}}$ given by

$$\Upsilon_n(X_F) = (-1)^{|F|} \left( (n-1)\eta^4(F)\eta^{2n-3} + \eta^3(F)\eta^{2n-2} \right).$$

(7.1)

We can check (see [1]) that these mappings are nontrivial 1-cocycles on $osp(1|2)$ for any $n \in \mathbb{N} \setminus \{0\}$. Theses cocycles arises in the action (2.1) of $osp(1|2)$ on $O^k_{\lambda,\mu}$. We can nevertheless define a canonical symbol mapping in the resonant case such that its deviation from $osp(1|2)$-equivariance is measured by the corresponding cocycle (7.1).

7.1 $osp(1|2)$-modules deformation

From now on, $\delta \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots, k\}$. Here, we will construct a nontrivial deformation of the natural action of the Lie superalgebra $osp(1|2)$ on $G^{\delta}_\delta$, generated by the cocycles (7.1).

**Proposition 7.1.** The map $L : osp(1|2) \to \text{End}(G^{\delta}_\delta)$ defined by

$$L_{X_F} \left( \sum_{i=0}^{2k} \Pi^i \left( \hat{P}_i \alpha^{\delta-i} \right) \right) = \sum_{i=0}^{2k} \Pi^i \left( \hat{P}_i^{X_F} \alpha^{\delta-i} \right)$$

(7.2)

with

$$\begin{cases}
\hat{P}_i^{X_F} = L_{X_F}^{-\frac{i}{2}}(P_i) & \text{if } i < 4\delta - 2k - 1 \text{ or } i > 2\delta - 1, \\
\hat{P}_i^{X_F} = L_{X_F}^{-\frac{i}{2}}(P_i) - \varepsilon_i^{\delta-\frac{i}{2}}(-1)^{|P_i|} \left( \frac{(4i-1)}{2} \eta^4(F)\eta^{i-2}(P_\delta) + \eta^3(F)\eta^{i-1}(P_\delta) \right) & \text{if } 4\delta - 2k - 1 \leq i \leq 2\delta - 1,
\end{cases}$$

where $\varepsilon_i^{\delta-\frac{i}{2}}(-1)^{|P_i|} \left( \frac{(4i-1)}{2} \eta^4(F)\eta^{i-2}(P_\delta) + \eta^3(F)\eta^{i-1}(P_\delta) \right)$.
where $s = 4\delta - i - 1$ and
\[
\varepsilon^s_i = \begin{cases} 
(-1)^{2\delta} \left( \lambda + \frac{1}{2} i \right) \gamma^s_{s-1-i} & \text{if } i \text{ is even} \\
-(-1)^{2\delta} \frac{1}{4} \gamma^s_{s-1-i} & \text{if } i \text{ is odd},
\end{cases}
\] (7.3)
is an action of the Lie superalgebra $\mathfrak{osp}(1|2)$ on the superspace of symbols $\mathfrak{S}^k_0$ of order $\leq k$.

Proof. First, it is easy to see that the map $\Gamma : \mathfrak{D}_{\lambda,\mu} \to \Pi(\mathfrak{D}_{\lambda,\mu})$ defined by $\Gamma(A) = \Pi(\pi \circ A)$ satisfies
\[
\mathfrak{D}^\lambda_{X_F} \circ \chi = (-1)^{|F|} \Gamma \circ \mathfrak{D}^\lambda_{X_F} \quad \text{for all } X_F \in \mathfrak{osp}(1|2).
\]
Thus, we deduce the structure of the first cohomology space $H^1(\mathfrak{osp}(1|2); \Pi(\mathfrak{D}_{\lambda,\mu}))$ from $H^1(\mathfrak{osp}(1|2); \mathfrak{D}_{\lambda,\mu})$. Indeed, to any 1-cocycle $\Upsilon$ on $\mathfrak{osp}(1|2)$ with values in $\mathfrak{D}_{\lambda,\mu}$ corresponds an 1-cocycle $\Gamma \circ \Upsilon$ on $\mathfrak{osp}(1|2)$ with values in $\Pi(\mathfrak{D}_{\lambda,\mu})$. Obviously, $\Upsilon$ is a couboundary if and only if $\Gamma \circ \Upsilon$ is a couboundary. Second, we can readily check that the map $\mathcal{L}$ satisfies the homomorphism condition
\[
\mathcal{L}_{[X_F, X_G]} = [\mathcal{L}_{X_F}, \mathcal{L}_{X_G}] \quad \text{for all } X_F, X_G \in \mathfrak{osp}(1|2).
\]
So, the map $\mathcal{L}$ is the nontrivial deformation of the natural action of $\mathfrak{osp}(1|2)$ on $\mathfrak{S}^k_0$ generated by the cocycles (7.1), up to the map $\Gamma$. \hfill \square

Denote by $\mathcal{M}^k_{\lambda,\mu}$ the $\mathfrak{osp}(1|2)$-module structure on $\mathfrak{S}^k_0$ defined by $\mathcal{L}$ for a fixed $\lambda$ and $\mu$.

Remark 7.2. Note that the map $\mathcal{L}$ given in Proposition 7.1 define an action of $\mathfrak{osp}(1|2)$ on $\mathfrak{S}^k_0$ for any scalars replacing those in (7.3).

7.2 Normal symbol

Here, we prove existence and uniqueness (up to normalization) of $\mathfrak{osp}(1|2)$-isomorphism between $\mathfrak{D}^k_{\lambda,\mu}$ and $\mathcal{M}^k_{\lambda,\mu}$ providing a “total symbol” of differential operators in the resonant cases. The following Proposition gives the existence of such an isomorphism.

Proposition 7.3. There exists an $\mathfrak{osp}(1|2)$-invariant symbol map called a normal symbol map
\[
\bar{\sigma}_{\lambda,\mu} : \mathfrak{D}^k_{\lambda,\mu} \xrightarrow{\sim} \mathcal{M}^k_{\lambda,\mu}.
\] (7.4)
It sends a differential operator $A = \sum_{i=0}^{2k} a_i(x, \theta) \eta^i$ to the tensor density
\[
\bar{\sigma}_{\lambda,\mu}(A) = \sum_{j=0}^{2k} \Pi^j \left( \bar{a}_j \alpha^{-\frac{j}{2}} \right),
\] (7.5)
where $\bar{a}_j = \sum_{i \geq j} \xi^i_j \eta^{-j} (a_i)$ with
\[
\begin{align*}
\xi^i_j &= \omega^i_{j,s} \xi^s_j + \kappa^i_{j,s} \quad \text{if } 4\delta - 2k - 1 \leq j \leq 2\delta - 1 \text{ and } i > s, \\
\xi^i_j &= \gamma^i_{i-j} \quad \text{otherwise},
\end{align*}
\] (7.6)
where
\[
\omega^i_{j,s} = \left( -1 \right)^{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i}{2}\right]}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i-1}{2}\right] + 2\lambda}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i+k}{2}\right]}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i+s}{2}\right]}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i+k+s}{2}\right]}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i+s}{2}\right]}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i+k+s}{2}\right]}{\left[\frac{i+j}{2}\right]} \frac{\left[\frac{i+s}{2}\right]}{\left[\frac{i+j}{2}\right]}
\] (7.7)
and
\[ \kappa_{j,s}^i = (-1)^{2s} \epsilon_j \sum_{\ell=s+1}^{i} \frac{\omega_j^i}{\vartheta_j^i \gamma_{s-\ell}} \]
with \( \gamma_{s-\ell} \) as in (4.12), \( s = 4\delta - j - 1 \), and

\[ \vartheta_j^i \ell = \begin{cases} \frac{1}{2} \left[ \frac{\ell - j}{2} \right] + (\delta - \frac{j}{2}) & \text{if } \ell - j \text{ is odd} \\ \frac{\ell - j}{2} & \text{if } \ell - j \text{ is even.} \end{cases} \]

Proof. For \( A \in \mathcal{D}_{\lambda,\mu} \) and \( X_F \in \mathfrak{osp}(1|2) \), we have

\[ \tilde{\alpha}_{\lambda,\mu}(\mathfrak{osp}^i(A)) = \sum_{j=0}^{2k} \Pi^j \left( \tilde{a}_{\lambda,\mu}^{X_F} \alpha^{-j} \right) \quad \text{with} \quad \tilde{a}_{\lambda,\mu}^{X_F} = \sum_{i>j}^{2k} \xi_j^{i-j}(a_i^{X_F}). \quad (7.8) \]

Substituting expression (5.3) for \( a_i^{X_F} \) in (7.8), we get

\[ \tilde{a}_{\lambda,\mu}^{X_F} = \mathfrak{osp}^{\delta-j}(a_j) - \sum_{i>j+1}^{2k} \rho^j_i \eta(F^i)\eta^{i-j-1}(a_i) - \sum_{i>j+2}^{2k} \overline{\rho}^j_i F^i \eta^{i-j-2}(a_i), \quad (7.9) \]

where

\[ \rho^j_i = (-1)^{(i-j)(\delta+1)} \left( \Lambda^i_{j-1,i,j} + \Lambda^i_{j+1,i,j} \right), \]

\[ \overline{\rho}^j_i = (-1)^{(i-j)(\delta+1)} \left( \Lambda^i_{j-1,i,j} + \Lambda^i_{j+1,i,j} \right). \]

with \( \Lambda^i_{j} = (-1)^{(i-j)} \) and \( \zeta_{i,j,\lambda} \) as in (5.4). So, we can see that, for \( j < 4\delta - 2k - 1 \) or \( j > 2\delta - 1 \), the symbol map (7.4) commutes with the action of \( \mathfrak{osp}(1|2) \) if and only if the following system is satisfied:

\[ \frac{(i-j)}{2} \xi^i_j = \frac{1}{2} \xi^{-1}_j \text{ if } i \text{ and } j \text{ are even,} \]

\[ \frac{(i-j)}{2} \xi^i_j = (2\lambda + \frac{1}{2}) \xi^{-1}_j \text{ if } i \text{ and } j \text{ are odd,} \]

\[ (i - 2\delta - \frac{1}{2}) \xi^i_j = (2\lambda + \frac{1}{2}) \xi^{-1}_j \text{ if } i \text{ is odd and } j \text{ is even,} \]

\[ (i - 2\delta - \frac{1}{2}) \xi^i_j = \frac{1}{2} \xi^{-1}_j \text{ if } i \text{ is even and } j \text{ is odd.} \]

Hence, the solution of the system (7.10) with the initial condition \( \xi^i_j = 1 \) is unique and given by \( \xi^i_j = \gamma^i_{j-1} \). For \( 4\delta - 2k - 1 \leq j \leq 2\delta - 1 \), the \( \mathfrak{osp}(1|2) \)-equivariance condition reads

\[ \tilde{a}_{\lambda,\mu}^{X_F} = \mathfrak{osp}^{\delta-j}(a_j) - \epsilon_j (-1)^{\frac{s-j-1}{2}} \eta^{i-j-2}(a_i) + \eta^{i-j-1}(a_i). \quad (7.11) \]

Thus, it is easy to see that the solutions of equation (7.11) with indeterminate \( \xi^i_j \) are given by (7.13).

To study the uniqueness (up to normalization) for the symbol map given by (7.4), we need the following result.

\[ \square \]
Proposition 7.4. The action of \( \mathcal{K}(1) \) over \( \mathcal{M}_{\lambda,\mu}^{k} \) in terms of \( \mathfrak{osp}(1|2) \)-equivariant normal symbols is given by:

\[
\begin{align*}
\tilde{a}_{p}^{X} & = \xi_{X,p}^{\frac{k-2}{2}}(\tilde{a}_{p}) + \sum_{j=p+3}^{2k} \chi_{p}^{j} \eta^{j-p} \circ \tilde{J}_{X,p}^{\frac{k-2}{2}}(X_{F}, \tilde{a}_{j}) & \text{if } p < 4\delta - 2k - 1 \\
& \quad - \varepsilon_{p}^{s}(-1)^{p} \sum_{j=p+3}^{2k} \chi_{p}^{j} \eta^{j-p} \circ \tilde{J}_{X,p}^{\frac{k-2}{2}}(X_{F}, \tilde{a}_{j}) & \text{if } 4\delta - 2k - 1 \leq p \leq 2\delta - 1 \\
& \quad + \sum_{j=p+3}^{2k} \chi_{p}^{j} \eta^{j-p} \circ \tilde{J}_{X,p}^{\frac{k-2}{2}}(X_{F}, \tilde{a}_{j}) & \text{if } 4\delta - 2k - 1 \leq p \leq 2\delta - 1
\end{align*}
\]

(7.12)

where \( \chi_{p}^{j} \) and \( \Xi_{p}^{s} \) are functions of \( (\lambda, \mu) \) and \( \varepsilon_{p}^{s} \) is as in (7.3) with \( s = 4\delta - p - 1 \).

To prove Proposition 7.4, we need the following classical fact:

Lemma 7.5. Consider a linear differential operator \( b : \mathcal{K}(1) \rightarrow \mathfrak{D}_{\lambda,\mu} \).

(i) If \( b \) is an 1-cocycle vanishing on \( \mathfrak{osp}(1|2) \), then \( b \) is a supertransvectant.

(ii) If \( b \) satisfies

\[ \Delta(b)(X, Y) = b(X) = 0 \]

for all \( X \in \mathfrak{osp}(1|2) \),

where \( \Delta \) stands for differential of cochains on \( \mathcal{K}(1) \) with values in \( \mathfrak{D}_{\lambda,\mu} \) (see, e.g., [3, 5, 10]), then \( b \) is a supertransvectant.

We also need the following

Theorem 7.1. \( \mathfrak{B}[10] \)

\[
\dim_{\text{diff}}^{1}(\mathcal{K}(1); \mathfrak{D}_{\lambda,\mu}) = 1 \text{ if }
\begin{align*}
\mu - \lambda & = 0 \quad \text{for all } \lambda, \\
\mu - \lambda & = \frac{3}{2} \quad \text{for all } \lambda, \\
\mu - \lambda & = 2 \quad \text{for all } \lambda, \\
\mu - \lambda & = \frac{4}{2} \quad \text{for all } \lambda, \\
\mu - \lambda & = 3 \quad \text{and } \lambda \in \{0, -\frac{5}{2}\}, \\
\mu - \lambda & = 4 \quad \text{and } \lambda = \frac{-7 + \sqrt{21}}{4},
\end{align*}
\]

\[
\dim_{\text{diff}}^{1}(\mathcal{K}(1); \mathfrak{D}_{0,\frac{1}{2}}) = 2. \text{ Otherwise, } \mathcal{H}_{\text{diff}}^{1}(\mathcal{K}(1); \mathfrak{D}_{\lambda,\mu}) = 0.
\]

The spaces \( \mathcal{H}_{\text{diff}}^{1}(\mathcal{K}(1); \mathfrak{D}_{\lambda,\mu}) \) are spanned by the cohomology classes of \( \Upsilon_{\lambda,\lambda + \frac{1}{2}} = 3^{\lambda}_{\frac{3}{2}+1} \).
where $k \in \{3, 4, 5, 6, 8\}$, and by the cohomology classes of the following 1-cocycles:

$$
\begin{align*}
\eta_{\lambda, \lambda}(X_F)(G \alpha^\lambda) &= F'G\alpha^\lambda, \\
\eta_{0, \frac{1}{2}}(X_F)(G) &= \bar{\eta}(F')G\alpha^{\frac{1}{2}}, \\
\bar{\eta}_{0, \frac{1}{2}}(X_F)(G) &= \bar{\eta}(F')G\alpha^{\frac{1}{2}}, \\
\eta_{-\frac{1}{2}, 1}(X_F)(G\alpha^{-\frac{1}{2}}) &= \left(\bar{\eta}(F')G' + (-1)^{|F'|}\bar{\eta}(G)\right)\alpha^{\frac{3}{2}}, \\
\eta_{-1, \frac{2}{3}}(X_F)(G\alpha^{-1}) &= \left((-1)^{|F'|}(F''\bar{\eta}(G) + 2F''\bar{\eta}(G')) + 2\bar{\eta}(F'')G' + \bar{\eta}(F'')G''\right)\alpha^{\frac{3}{2}}.
\end{align*}
$$

Proof (Proposition 7.4). By direct computation, using formula (5.3), we can see that the action of $\mathcal{K}(1)$ on $M^k_{\lambda, \mu}$ in terms of $\mathfrak{osp}(1|2)$-equivariant normal symbols is given by:

$$
\tilde{a}_p^{\chi_F} = \mathbb{S}_{\chi_F}^{\delta - \frac{2}{3}}(\tilde{a}_p) + (\text{terms in } \bar{\eta}_n(F), n \geq 3).
$$

So, it is a deformation of the natural action of $\mathcal{K}(1)$ over $\bar{\mathcal{S}}_k^\lambda$ such that its restriction to $\mathfrak{osp}(1|2)$ coincides with the map $\mathcal{L}$ given by (7.2). Therefore, according to Theorem 7.1 and Lemma 7.3, we deduce that it is given by:

For $p < 4\delta - 2k - 1$ or $p > 2\delta - 1$,

$$
\tilde{a}_p^{\chi_F} = \mathbb{S}_{\chi_F}^{\delta - \frac{2}{3}}(\tilde{a}_p) + \sum_{j=p+3}^{2k} h^j_p \pi^{j-p} \circ \mathbb{J}_{\chi_F}^{\delta - \frac{2}{3}}(X_F, \tilde{a}_j) + \sum_{j=0}^{2k} C_{j, p}(X_F, \tilde{a}_j). \quad (7.13)
$$

For $4\delta - 2k - 1 \leq p \leq 2\delta - 1$ and $s \neq p + 5$,

$$
\tilde{a}_p^{\chi_F} = \mathbb{S}_{\chi_F}^{\delta - \frac{2}{3}}(\tilde{a}_p) + \sum_{j=p+3}^{2k} h^j_p \pi^{j-p} \circ \mathbb{J}_{\chi_F}^{\delta - \frac{2}{3}}(X_F, \tilde{a}_j) + \sum_{j=0}^{2k} C_{j, p}(X_F, \tilde{a}_j) - \epsilon^s_p(-1)^{|\tilde{a}_s|} \left(\frac{(p-s-1)}{2} \eta^4(F)\bar{\eta}^{-p-2}(\tilde{a}_s) + \eta^3(F)\bar{\eta}^{-p-1}(\tilde{a}_s)\right). \quad (7.14)
$$

For $4\delta - 2k - 1 \leq p \leq 2\delta - 1$ and $s = p + 5$,

$$
\tilde{a}_p^{\chi_F} = \mathbb{S}_{\chi_F}^{\delta - \frac{2}{3}}(\tilde{a}_p) + \sum_{j=p+3}^{2k} h^j_p \pi^{j-p} \circ \mathbb{J}_{\chi_F}^{\delta - \frac{2}{3}}(X_F, \tilde{a}_j) + \sum_{j=0}^{2k} C_{j, p}(X_F, \tilde{a}_j) + \mathcal{R}_p^s(-1)^{|\tilde{a}_s|} \left(2F''\eta^3(\tilde{a}_s) + \eta^4(F)\bar{\eta}(\tilde{a}_s) - F''\bar{\eta}(\tilde{a}_s) - 2\eta(F'')\bar{\eta}(\tilde{a}_s)\right) \quad (7.15)
$$

where $h^j_p, R^s_p, S^s_p$ are functions of $(\lambda, \mu)$ and $C_{j, p} : \mathcal{K}(1) \to \mathcal{D}_{4, \delta - \frac{2}{3}, \delta - \frac{2}{3}}$ are linear maps vanishing on $\mathfrak{osp}(1|2)$ with the same parity as the integer $j - p$. The homomorphism condition of the action of $\mathcal{K}(1)$ on $M^k_{\lambda, \mu}$ in terms of $\mathfrak{osp}(1|2)$-equivariant normal symbols implies that $\Delta C_{j, p}$ vanish on $\mathfrak{osp}(1|2)$. So, by Lemma 7.5, we can see that the maps $C_{j, p}$ are supertransvectants vanishing on $\mathfrak{osp}(1|2)$. Therefore, we deduce that formulas (7.13) and (7.14) can be expressed as in Proposition 7.4. Moreover, for $s = p + 5$, up to a scalar factor, the supertransvectant $\mathbb{J}_{\chi_F}^{-1}$ is given by

$$
\mathbb{J}_{\chi_F}^{-1}(X_F, \tilde{a}_s) = (-1)^{|F'|} \left(2\eta(F'')\bar{\eta}(\tilde{a}_s) + 3\eta(F'')\bar{\eta}(\tilde{a}_s) - F''\bar{\eta}(\tilde{a}_s)\right). \quad (7.16)
$$

Using formula (7.16), we deduce that expression (7.15) can be expressed as in Proposition 7.4. This completes the proof.
Corollary 7.6. The constants $\chi_p$ given by (7.12) and $\xi_p$ given by (7.6) satisfy the following relations

$$
\Theta_p^j \chi_p^j \xi_p^j = \zeta_p^j - \sum_{r=1}^{j-1} (-1)^j \zeta_p^{j-r} \Lambda_p^j \Theta_p^r \chi_p^r \xi_p^r \quad \text{if } j \geq p + 4,
$$

(7.17)

where

$$
(-1)^{j-p} \zeta_p^j = -\Lambda_p^j \zeta_p^{j-p} - \Lambda_p^j \zeta_p^{j-1} \left( \frac{j-p}{2} \right) \zeta_p^{j-1} + \Lambda_p^j \zeta_p^{j-p-2} - \Lambda_p^j \zeta_p^{j-3} \zeta_p^{j-3},
$$

(7.18)

$\zeta_i, j, \lambda$ is as in (5.4) and $\Theta_p^j$ is the coefficient of

$$
(-1)^{|F| + (j-p)} \tau_j \eta_j \zeta_p^{j-p-3} (a_j) \quad \text{in} \quad \pi_j \circ \tau_j \eta_j \zeta_p^{j-p-3} (a_j).
$$

Proof. First, using the fact that $\tilde{a}_r = \sum_{i=p}^{2k} \xi_i \eta_i^{j-r} (a_i)$ and formula (7.12), we can check that the coefficient of $(-1)^{|F| + (j-p)} \tau_j \eta_j \zeta_p^{j-p-3} (a_j)$ in $\tilde{a}_p \chi_p^F$ for $j \geq p + 3$, is

$$
\Theta_p^j \chi_p^j \xi_p^j + \sum_{r=p+3}^{j} (-1)^{j-r} \Lambda_p^j \Theta_p^r \chi_p^r \xi_p^r.
$$

On the other hand, we have $\tilde{a}_p \chi_p^F = \sum_{i=p}^{2k} \xi_i \eta_i^{j-p} (a_i \chi_p^F)$, where $a_i \chi_p^F$ is given by (7.13). Thus, by direct computation, we can see that the coefficient of $(-1)^{|F| + (j-p)} \tau_j \eta_j \zeta_p^{j-p-3} (a_j)$ in $\tilde{a}_p \chi_p^F$ is $\zeta_p^j$ given by (7.18). Corollary 7.6 is proved.

The normal symbol map depends on the choice of $\xi_p^s$, which play a role arbitrary. Clearly, $s - p$ is odd. Moreover, we can readily check that the coefficient of $\xi_p^s$ in $\chi_p^{s+2}$ vanishes for $s = p + 1$. So, in the following, we will use the normal symbol map uniquely defined, up to a scalar factor, by imposing the following condition to $\xi_p^s$:

(i) if $s \geq p + 3$ we choose $\xi_p^s$ such that $\chi_p^s = 0$,

(ii) if $s = p + 1$ we choose $\xi_p^s$ so as to cancel the first term of the following sequence, where the coefficient of $\xi_p^s$ is nonzero:

$$
\chi_p^{s+3}, \chi_p^{s-3}, \chi_p^{s-4}, \ldots, \chi_p^0.
$$

(7.19)

Note that this choice is possible thanks to Corollary 7.6.

8 Proof of Theorem 3.1 in the resonant case

The existence and uniqueness of the normal symbol allow us, by a similar process to that used in section 3 to complete the proof of Theorem 3.1.

Proposition 8.1. Let $T : \mathcal{D}_{X, \mu}^k \rightarrow \mathcal{D}_{Y, \nu}^k$ be an isomorphism of $K(1)$-modules. Then $T$ is diagonal in terms of normal symbols.
Proof. Similar to that of Proposition 6.1
Now, let $A \in \mathcal{D}^k_{\lambda,\mu}$. The normal symbol of $T(A)$ is

$$\tilde{\sigma}_{\lambda,\mu}(T(A)) = \sum_{j=0}^{2k} \Pi^j \left( \tilde{a}^T_j \alpha^j \right).$$

Proposition 8.1 implies that there exist a constants $\tau_0, \ldots, \tau_{2k}$ depending on $\lambda, \mu, \rho$ and $\nu$, such that $\tilde{a}^T_j = \tau_j \tilde{a}_j$ for all $j = 0, \ldots, 2k$. The condition of $\mathfrak{osp}(1|2)$-equivariance of $T$ in terms of normal symbol, leads to the following system:

\begin{align*}
\tau_p \chi_p^j(\lambda, \mu) &= \tau_j \chi_p^j(\rho, \nu) \quad \text{for} \quad p = 0, \ldots, 2k \quad \text{and} \quad j \geq p + 3, \\
\tau_p \epsilon_p^s(\lambda, \mu) &= \tau_s \epsilon_p^s(\rho, \nu) \quad \text{for} \quad 4\delta - 2k - 1 \leq p \leq 2\delta - 1, \\
\tau_p \Xi_p^s(\lambda, \mu) &= \tau_s \Xi_p^s(\rho, \nu) \quad \text{for} \quad 4\delta - 2k - 1 \leq p \leq 2\delta - 1 \quad \text{and} \quad s = p + 5,
\end{align*}

where the $\chi_p^j$ are given by (7.12) for $j \geq p + 3$, and $\epsilon_p^s$ is as in (7.3) with $s = 4\delta - p - 1$.

### 8.1 Isomorphisms of $\mathcal{K}(1)$-modules in terms of normal symbol

The resolution of the system (8.1) shows that the isomorphisms of $\mathcal{K}(1)$-modules in terms of normal symbol, in the resonant case, are an extension, except for $(k, \delta) = (2, 2)$, of isomorphisms in terms of $\mathfrak{osp}(1|2)$-equivariant symbols in the nonresonant case. Indeed, by solving the system (8.1), using formulas (7.17) and (7.6) with the help of condition (7.19), we get:

i) For $k = \frac{1}{2}$, an isomorphism $T : \mathcal{D}^{\frac{1}{2}}_{\lambda,\mu} \rightarrow \mathcal{D}^{\frac{1}{2}}_{\rho,\nu}$ is obtained by taking

$$\left( \tilde{a}^T_0, \tilde{a}^T_1 \right) = \left( \frac{\rho}{\lambda} \tilde{a}_0, \tilde{a}_1 \right).$$

ii) For $k = 1$, an isomorphism $T : \mathcal{D}^{1}_{\lambda,\mu} \rightarrow \mathcal{D}^{1}_{\rho,\nu}$ is obtained by taking (with $\tau \neq 0$)

$$\begin{cases}
(\tilde{a}^T_0, \tilde{a}^T_1, \tilde{a}^T_2) = (\frac{\rho}{\lambda} \tilde{a}_0, \tilde{a}_1, \tau \tilde{a}_2) & \text{for} \quad \delta = \frac{1}{2}, \\
(\tilde{a}^T_0, \tilde{a}^T_1, \tilde{a}^T_2) = (\tau \tilde{a}_0, \tilde{a}_1, \tilde{a}_2) & \text{for} \quad \delta = 1.
\end{cases}$$

iii) For $k = \frac{3}{2}$, we get an isomorphism $T : \mathcal{D}^\frac{3}{2}_{\lambda,\mu} \rightarrow \mathcal{D}^\frac{3}{2}_{\rho,\nu}$ by taking in (8.1):

$$\tau_0 = \frac{\rho^2}{2\lambda}, \quad \tau_1 = \frac{\rho}{\lambda}, \quad \tau_3 = 1, \quad \tau_2 \neq 0 \quad \text{for} \quad \delta = \frac{1}{2},$$

$$\tau_0 = \frac{\rho(2\rho + 1)}{\lambda(2\lambda + 1)}, \quad \tau_1 = \tau_2 \neq 0, \quad \tau_3 = 1, \quad \tau_1 \neq 0 \quad \text{for} \quad \delta = 1,$$

$$\tau_0 = \frac{\rho(\rho + 1)}{\lambda(\lambda + 1)}, \quad \tau_2 = \frac{2\rho + 1}{2\lambda + 1}, \quad \tau_3 = 1, \quad \tau_1 \neq 0 \quad \text{for} \quad \delta = \frac{3}{2}.$$

iv) For $k = 2$, we get an isomorphism $T : \mathcal{D}^2_{\lambda,\mu} \rightarrow \mathcal{D}^2_{\rho,\nu}$ by taking in (8.1):

$$\tau_0 = \frac{\rho^2}{2\lambda}, \quad \tau_1 = \frac{\rho}{\lambda}, \quad \tau_3 = \tau_4 = 1, \quad \tau_2 \neq 0 \quad \text{for} \quad \delta = \frac{1}{2},$$

$$\tau_0 = \frac{2\rho + 1}{2\lambda + 1}, \quad \tau_1 = \tau_2 = \frac{4\rho + 1}{4\lambda + 1}, \quad \tau_3 = \frac{\lambda}{\rho}, \quad \tau_4 = 1 \quad \text{for} \quad \delta = 1,$$

$$\tau_0 = \frac{\rho(\rho + 1)}{\lambda(\lambda + 1)}, \quad \tau_2 = \frac{2\rho + 1}{2\lambda + 1}, \quad \tau_1 = \tau_3 = \tau_4 = 1, \quad \text{for} \quad \delta = \frac{3}{2}.$$

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The case $\delta = 2$ is particularly because the isomorphisms of nonresonant case do not extend to the resonant case. Indeed, the equivariance condition of an isomorphism $T : \mathcal{D}^2_{\lambda,\mu} \to \mathcal{D}^2_{\rho,\nu}$ implies

$$
\gamma_3 = \frac{\lambda_0^3(\rho,\nu)}{\lambda_0^3(\lambda,\mu)} = \frac{\rho(2\rho+3)}{\lambda(2\lambda+3)},
\gamma_4 = \frac{\lambda_0^3(\rho,\nu)}{\lambda_0^3(\lambda,\mu)} = \frac{4\rho+3}{4\lambda+3},
\gamma_5 = \frac{\lambda_0^3(\rho,\nu)}{\lambda_3^3(\lambda,\mu)} = 1,
$$

This system has a solution if $\lambda = \rho$ or $\rho + \lambda = 1/2$.

v) For $k = 5/2$, the system (8.1) has a solution if $\lambda = \rho$ or $\rho + \lambda = 1/2$. The first isomorphism is tautological, and the second is just the passage to the adjoint module. The case $k > 5/2$ is deduced from the case $k = 5/2$.

Now, for fixed $k \leq 2$, the same arguments as in subsection 6.2 together with Theorem 7.1 show that singular modules appear whenever at least one of the coefficients $\chi_j^i$ or $\varepsilon_p^i$ in (7.12) vanishes. Theorem 8.1 is proved for resonant case.

**Remark 8.2.** For $k = 2$, the resonant case $\delta = 2$ seems to be particularly interesting.

### 9 Differential Operators on $S^{1|n}$

In this section, we consider the supercircle $S^{1|n}$ instead of $S^{1|1}$. That is, we consider the supercircle $S^{1|n}$ for $n \geq 2$ with local coordinates $(x, \theta_1, \ldots, \theta_n)$, where $\theta = (\theta_1, \ldots, \theta_n)$ are odd variables. Any contact structure on $S^{1|n}$ can be reduced to a canonical one, given by the following 1-form:

$$\alpha_n = dx + \sum_{i=1}^n \theta_i d\theta_i.$$

The space of $\lambda$-densities will be denoted

$$\mathfrak{F}^n_\lambda = \left\{ F(x, \theta)\alpha_n^\lambda \mid F(x, \theta) \in \mathcal{C}^{\infty}(S^{1|n}) \right\}.$$  

(9.1)

We denote by $\mathcal{D}^n_{\lambda,\mu}$ the space of differential operators from $\mathfrak{F}^n_\lambda$ to $\mathfrak{F}^n_\mu$ for any $\lambda, \mu \in \mathbb{R}$. The Lie superalgebra $\mathcal{K}(n)$ of contact vector fields on $S^{1|n}$ is spanned by the vector fields of the form (see, e.g., [2, 18]):

$$X_F = F \partial_x - \frac{1}{2}(-1)^{|F|} \sum_{i} \eta_i(F) \overline{\eta}_i,$$

where $\eta_i = \partial_{\theta_i} - \theta_i \partial_x$. Since $-\eta_i^2 = \partial_x$, and $\partial_i = \eta_i - \theta_i \eta_i^2$, every differential operator $A \in \mathcal{D}^n_{\lambda,\mu}$ can be expressed in the form

$$A(F\alpha_n^\lambda) = \sum_{\ell=(\ell_1,\ldots,\ell_n)} a_\ell(x, \theta) \eta_1^{\ell_1} \ldots \eta_n^{\ell_n}(F) \alpha_n^\mu,$$

(9.2)

where the coefficients $a_\ell(x, \theta) \in \mathcal{C}^{\infty}(S^{1|n})$ (see [3]). For $k \in \frac{1}{2}\mathbb{N}$, we denote by $\mathcal{D}^n_{\lambda,\mu}$ the subspace of $\mathcal{D}^n_{\lambda,\mu}$ of the form

$$A(F\alpha_n^\lambda) = \sum_{\ell_1+\ldots+\ell_n \leq 2k} a_{\ell_1,\ldots,\ell_n}(x, \theta) \eta_1^{\ell_1} \ldots \eta_n^{\ell_n}(F) \alpha_n^\mu.$$  

(9.3)
Thus, we have a filtration:
\[ \mathcal{D}_{n,0}^{n,0} \subset \mathcal{D}_{n,0}^{n,1} \subset \mathcal{D}_{n,0}^{n,2} \subset \cdots \subset \mathcal{D}_{n,0}^{n,\ell} \subset \mathcal{D}_{n,0}^{n,\ell} \ldots \] \hspace{1cm} (9.4)

Now, let us consider the density space \( \mathcal{D}_n^{\frac{2-n}{2}} \) over the supercircle \( S^{1,|n]} \). The Berizin integral \( \mathcal{B}_n : \mathcal{D}_n^{\frac{2-n}{2}} \rightarrow \mathbb{R} \) can be given, for any \( \varphi = \sum f_{1,\ldots,n}(x)\theta_1^{i_1} \ldots \theta_n^{i_n} \), by the formula
\[ \mathcal{B}_n(\varphi) = \int_{S^1} f_{1,\ldots,n} dx. \]

**Proposition 9.1.** The Berizin integral \( \mathcal{B}_n \) is \( \mathcal{K}(n) \)-invariant. That is, for any \( \varphi \in \mathcal{D}_n^{\frac{2-n}{2}} \) and for any \( H \in C^\infty(S^{1,|n]} \), we have \( \mathcal{B}_n \left( \mathcal{L}_{X_H}^{\frac{2-n}{2}}(\varphi) \right) = 0 \). The product of densities composed with \( \mathcal{B}_n \) yields a bilinear \( \mathcal{K}(n) \)-invariant form:
\[ \langle \cdot, \cdot, \cdot \rangle : \mathcal{D}_n \otimes \mathcal{D}_n \rightarrow \mathbb{R}, \quad \lambda + \mu = \frac{2-n}{2}. \]

Proof. Note that \( C^\infty(S) \) is assumed to be \( \{ f \in C^\infty(\mathbb{R}) \mid f \) is 2\pi-periodic\}. For \( n = 0 \), we have \( \mathcal{L}_{X_H}^1(F) = HF' + H'F = (HF)' \), therefore, \( \mathcal{B}_0(\mathcal{L}_{X_H}^1(F\alpha_n^1)) = 0 \). For \( n = 1 \), using equation \( (2.1) \) for \( \lambda = \frac{1}{2} \), we easily show that \( \mathcal{B}_1(\mathcal{L}_{X_H}^\frac{1}{2}(F\alpha_n^1)) = 0 \).

Let us consider \( F = F_1 + F_2\theta_n \in C^\infty(S^{1,|n]} \) and \( H \in C^\infty(S^{1,|n]} \), where \( \partial_n F_1 = \partial_n F_2 = \partial_n H = 0 \) with \( \partial_i := \frac{\partial}{\partial \theta_i} \). We easily prove that
\[ \mathcal{L}_{X_H}^{\frac{2-n}{2}} F = \mathcal{L}_{X_H}^{\frac{2-n}{2}} F_1 + \left( \mathcal{L}_{X_H}^{\frac{2-(n-1)}{2}} F_2 \right) \theta_n. \]
So, we have \( \mathcal{B}_n \left( \mathcal{L}_{X_H}^{\frac{2-n}{2}} \left( F\alpha_n^{\frac{2-n}{2}} \right) \right) = \mathcal{B}_{n-1} \left( \mathcal{L}_{X_H}^{\frac{2-(n-1)}{2}} \left( F_1\alpha_n^{\frac{2-(n-1)}{2}} \right) \right) = 0. \)

On the other hand, by a direct computation, we show that
\[ \mathcal{L}_{X_H^{\theta_n}}^{\frac{2-n}{2}}(F) = (-1)^{|H|} \theta_n \left( \mathcal{L}_{X_H}^{\frac{2-(n-1)}{2}}(F_1) - \frac{1}{2}(H'F_1 + HF_1) \right) + \frac{1}{2}(-1)^{|F_2|}HF_2. \]
So, it is clear that \( \mathcal{B}_n \left( \mathcal{L}_{X_H^{\theta_n}}^{\frac{2-n}{2}} \left( F\alpha_n^{\frac{2-n}{2}} \right) \right) = 0. \) This completes the proof. \( \square \)

**Corollary 9.1.** There exists a \( \mathcal{K}(n) \)-invariant conjugation map:
\[ * : \mathcal{D}_{\lambda,\mu}^{n,k} \rightarrow \mathcal{D}_{\frac{n}{2} - \mu, \frac{n}{2} - \lambda}^{n,k} \quad \text{defined by} \quad \langle A\varphi, \psi \rangle = (-1)^{|A||\varphi|} \langle \varphi, A^*\psi \rangle. \]
Moreover, \( * \) is \( \mathcal{K}(n) \)-isomorphism \( \mathcal{D}_{\lambda,\mu}^{n,k} \cong \mathcal{D}_{\frac{n}{2} - \mu, \frac{n}{2} - \lambda}^{n,k} \) for every \( k \in \frac{1}{2}\mathbb{N} \).
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