INEQUALITIES FOR ORTHONORMAL FAMILIES OF VECTORS IN INNER PRODUCT SPACES RELATED TO BUZANO’S, RICHARD’S AND KUREPA’S RESULTS

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Abstract. Some inequalities for families of orthornormal vectors in inner product spaces that are related with Buzano’s, Richard’s and Kurepa’s results are given.

1. Introduction

In [3], M. L. Buzano obtained the following extension of the celebrated Schwarz’s inequality in a real or complex inner product space \((H; \langle \cdot, \cdot \rangle)\):

\[
|\langle a, x \rangle \langle x, b \rangle| \leq \frac{1}{2} \|a\| \|b\| + |\langle a, b \rangle| \|x\|^2,
\]

(1.1)

for any \(a, b, x \in H\).

It is clear that the above inequality becomes, for \(a = b\), the Schwarz’s inequality

\[
|\langle a, x \rangle|^2 \leq \|a\|^2 \|x\|^2, \quad a, x \in H;
\]

(1.2)

in which the equality holds if and only if there exists a scalar \(\lambda \in \mathbb{K} (\mathbb{R}, \mathbb{C})\) so that \(x = \lambda a\).

As noted by M. Fujii and I. Kubo in [5], where they provided a simple proof of (1.1) on using orthogonal projection arguments, the case of equality holds in (1.1) if

\[
x = \begin{cases} 
\alpha \left( \frac{\langle a, x \rangle}{\|a\|} + \frac{\langle a, b \rangle}{\|a, b\|} \cdot \frac{b}{\|b\|} \right), & \text{when } \langle a, b \rangle \neq 0 \\
\alpha \left( \frac{\langle a, x \rangle}{\|a\|} + \beta \cdot \frac{b}{\|b\|} \right), & \text{when } \langle a, b \rangle = 0,
\end{cases}
\]

(1.3)

where \(\alpha, \beta \in \mathbb{K}\).

As noted by T. Precupanu in [8], independently of Buzano, U. Richard [9] obtained the following similar inequality holding in real inner product spaces:

\[
\frac{1}{2} \|x\|^2 \left[\|a\| \|b\| \right] \leq \langle a, x \rangle \langle x, b \rangle \leq \frac{1}{2} \|x\|^2 \left[\|a\| + \|b\| \right].
\]

(1.4)

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In [7], J. Peˇ cari´ c gave a simple direct proof of (1.4) without mentioning the work of either Buzano or Richard, but tracked down the result, in a particular form, to an earlier paper of C. Blatter [1].

In [4], the author has improved Buzano’s inequality by showing that, in fact, one has:

\[
|\langle a, x \rangle \langle x, b \rangle| \leq \frac{1}{2} \|\|a\|\|b\| + |\langle a, b \rangle| \|x\|^2.
\]

In the same paper, the author has also improved the celebrated Kurepa inequality for the complexification of a real inner product space, namely, the inequality [6]

\[
|\langle z, x \rangle_C|^2 \leq \frac{1}{2} \|x\|^2 \left[ \|z\|^2_C + |\langle z, \bar{z} \rangle_C| \right] \leq \|x\|^2 \|z\|^2_C,
\]

where \(x \in H\), \((H; \langle \cdot, \cdot \rangle)\) is a real space, \(z \in H_C\), \((H_C; \langle \cdot, \cdot \rangle_C)\) is the complexification of \((H; \langle \cdot, \cdot \rangle)\), and \(\bar{z}\) is the conjugate vector of \(z\).

The refinement of Kurepa’s result (1.6) obtained in [4] is as follows:

\[
|\langle z, x \rangle_C|^2 \leq \left| \langle z, x \rangle_C - \frac{1}{2} \langle z, \bar{z} \rangle_C \|x\|^2 \right| + \frac{1}{2} |\langle z, \bar{z} \rangle_C| \|x\|^2
\]

\[
\leq \frac{1}{2} \|x\|^2 \left[ \|z\|^2_C + |\langle z, \bar{z} \rangle_C| \right]
\]

\[
\leq \|x\|^2 \|z\|^2_C,
\]

with the same assumptions as above.

The main aim of the present paper is to obtain similar results for families of orthonormal vectors in \((H; \langle \cdot, \cdot \rangle)\), real or complex space, that are naturally connected with the celebrated Bessel inequality.

2. A Generalisation for Orthonormal Families

We say that the finite family \(\{e_i\}_{i \in I}\) (\(I\) is finite) of vectors is orthonormal if \(\langle e_i, e_j \rangle = 0\) if \(i, j \in I\) with \(i \neq j\) and \(\|e_i\| = 1\) for each \(i \in I\). The following result may be stated:

**Theorem 1.** Let \((H; \langle \cdot, \cdot \rangle)\) be an inner product space over the real or complex number field \(\mathbb{K}\) and \(\{e_i\}_{i \in I}\) a finite orthonormal family in \(H\). Then for any \(a, b \in H\), one has the inequality:

\[
\left| \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle - \frac{1}{2} \langle a, b \rangle \right| \leq \frac{1}{2} \|a\| \|b\|.
\]

The case of equality holds in (2.1) if and only if

\[
\sum_{i \in I} \langle a, e_i \rangle e_i = \frac{1}{2} a + \left( \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle - \frac{1}{2} \langle a, b \rangle \right) \cdot \frac{b}{\|b\|^2}.
\]
Proof. It is well known that, for \( e \neq 0 \) and \( f \in H \), the following identity holds:

\[
\|f\|^2 \|e\|^2 - |\langle f, e \rangle|^2 = \left\| f - \frac{\langle f, e \rangle}{\|e\|^2} e \right\|^2.
\]

(2.3)

Therefore, in Schwarz’s inequality

\[
|\langle f, e \rangle|^2 \leq \|f\|^2 \|e\|^2, \quad f, e \in H;
\]

(2.4)

the case of equality, for \( e \neq 0 \), holds if and only if

\[
f = \frac{\langle f, e \rangle}{\|e\|^2} e.
\]

Let \( f := 2 \sum_{i \in I} \langle a, e_i \rangle e_i - a \) and \( e := b \). Then, by Schwarz’s inequality (2.4), we may state that

\[
\left\| \sum_{i \in I} \langle a, e_i \rangle e_i - a \right\|^2 \leq \left\| \sum_{i \in I} \langle a, e_i \rangle e_i - a \right\| \|b\|^2
\]

(2.5)

with equality, for \( b \neq 0 \), if and only if

\[
2 \sum_{i \in I} \langle a, e_i \rangle e_i - a = \left( \sum_{i \in I} \langle a, e_i \rangle e_i - a, b \right) \frac{b}{\|b\|^2}.
\]

(2.6)

Since

\[
\left< \sum_{i \in I} \langle a, e_i \rangle e_i - a, b \right> = 2 \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle - \langle a, b \rangle
\]

and

\[
\left\| \sum_{i \in I} \langle a, e_i \rangle e_i - a \right\|^2 = 4 \left\| \sum_{i \in I} \langle a, e_i \rangle e_i \right\|^2 - 4 \text{Re} \left< \sum_{i \in I} \langle a, e_i \rangle e_i, a \right> + \|a\|^2
\]

\[
= 4 \sum_{i \in I} |\langle a, e_i \rangle|^2 - 4 \sum_{i \in I} |\langle a, e_i \rangle|^2 + \|a\|^2
\]

\[
= \|a\|^2,
\]

hence by (2.5) we deduce the desired inequality (2.1).

Finally, as (2.2) is equivalent to

\[
\sum_{i \in I} \langle a, e_i \rangle e_i - a = \left( \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle - \frac{1}{2} \langle a, b \rangle \right) \frac{b}{\|b\|^2},
\]

hence the equality holds in (2.1) if and only if (2.2) is valid.
The following result is well known in the literature as Bessel’s inequality
\[
\sum_{i \in I} |\langle x, e_i \rangle|^2 \leq \|x\|^2, \quad x \in H,
\] (2.7)
where, as above, \( \{e_i\}_{i \in I} \) is a finite orthonormal family in the inner product space \((H; \langle \cdot, \cdot \rangle)\).

If one chooses \(a = b = x\) in (2.1), then one gets the inequality
\[
\left| \sum_{i \in I} |\langle x, e_i \rangle|^2 - \frac{1}{2} \|x\|^2 \right| \leq \frac{1}{2} \|x\|^2,
\]
which is obviously equivalent to Bessel’s inequality (2.7). Therefore, the inequality (2.1) may be regarded as a generalisation of Bessel’s inequality as well.

Utilising the Bessel and Cauchy-Bunyakovsky-Schwarz inequalities, one may state that
\[
\left| \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle \right| \leq \left[ \sum_{i \in I} |\langle a, e_i \rangle|^2 \sum_{i \in I} |\langle b, e_i \rangle|^2 \right]^{\frac{1}{2}} \leq \|a\| \|b\|\] (2.8)
A different refinement of the inequality between the first and the last term in (2.8) is incorporated in the following:

**Corollary 1.** With the assumption of Theorem 1, we have
\[
\left| \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle \right| \leq \sum_{i \in I} |\langle a, e_i \rangle|^2 \sum_{i \in I} |\langle b, e_i \rangle|^2 \leq \frac{1}{2} \|a\| \|b\| + \frac{1}{2} |\langle a, b \rangle|.
\]

**Remark 1.** If the space \((H; \langle \cdot, \cdot \rangle)\) is real, then, obviously, (2.1) is equivalent to:
\[
\frac{1}{2} (\langle a, b \rangle - \|a\| \|b\|) \leq \sum_{i \in I} \langle a, e_i \rangle \langle e_i, b \rangle \leq \frac{1}{2} (\|a\| \|b\| + \langle a, b \rangle).
\] (2.9)

**Remark 2.** It is obvious that if the family comprises of only a single element \(e = \frac{x}{\|x\|}, \quad x \in H, \ x \neq 0\), then from (2.9) we recapture the refinement of Buzano’s inequality incorporated in (1.5) while from (2.10) we deduce Richard’s result from (1.4).

The following corollary of Theorem 1 is of interest as well:

**Corollary 2.** Let \( \{e_i\}_{i \in I} \) be a finite orthonormal family in \((H; \langle \cdot, \cdot \rangle)\). If \(x, y \in H \setminus \{0\}\) are such that there exists the constants \(m_i, n_i, M_i, N_i \in \mathbb{R}, \ i \in I\) such that:
\[
-1 \leq m_i \leq \frac{\text{Re} \langle x, e_i \rangle}{\|x\|}, \quad \frac{\text{Re} \langle y, e_i \rangle}{\|y\|} \leq M_i \leq 1, \quad i \in I
\] (2.11)
and
\[-1 \leq n_i \leq \frac{\text{Im} \langle x, e_i \rangle}{\|x\|}, \quad \frac{\text{Im} \langle y, e_i \rangle}{\|y\|} \leq N_i \leq 1, \quad i \in I \tag{2.12}\]

then
\[2 \sum_{i \in I} (m_i + n_i) - 1 \leq \frac{\text{Re} \langle x, y \rangle}{\|x\| \|y\|} \leq 1 + 2 \sum_{i \in I} (M_i + N_i). \tag{2.13}\]

**Proof.** Using Theorem 1 and the fact that for any complex number \(z\), \(|z| \geq |\text{Re} z|\), we have
\[
\left| \sum_{i \in I} \text{Re} [\langle x, e_i \rangle \langle e_i, y \rangle] - \frac{1}{2} \text{Re} \langle x, y \rangle \right| \leq \left| \sum_{i \in I} \langle x, e_i \rangle \langle e_i, y \rangle - \frac{1}{2} \langle x, y \rangle \right| \leq \frac{1}{2} \|x\| \|y\|. \tag{2.14}\]

Since
\[
\text{Re} [\langle x, e_i \rangle \langle e_i, y \rangle] = \text{Re} \langle x, e_i \rangle \text{Re} \langle y, e_i \rangle + \text{Im} \langle x, e_i \rangle \text{Im} \langle y, e_i \rangle,
\]

hence by (2.14) we have:
\[
- \frac{1}{2} \|x\| \|y\| + \frac{1}{2} \text{Re} \langle x, y \rangle \leq \sum_{i \in I} \text{Re} \langle x, e_i \rangle \text{Re} \langle y, e_i \rangle + \sum_{i \in I} \text{Im} \langle x, e_i \rangle \text{Im} \langle y, e_i \rangle \leq \frac{1}{2} \|x\| \|y\| + \frac{1}{2} \text{Re} \langle x, y \rangle. \tag{2.15}\]

Utilising the assumptions (2.11) and (2.12), we have
\[
\sum_{i \in I} m_i \leq \sum_{i \in I} \frac{\text{Re} \langle x, e_i \rangle \text{Re} \langle y, e_i \rangle}{\|x\| \|y\|} \leq \sum_{i \in I} M_i \tag{2.16}\]

and
\[
\sum_{i \in I} n_i \leq \sum_{i \in I} \frac{\text{Im} \langle x, e_i \rangle \text{Im} \langle y, e_i \rangle}{\|x\| \|y\|} \leq \sum_{i \in I} N_i. \tag{2.17}\]

Finally, on making use of (2.15) – (2.17), we deduce the desired result (2.13).

**Remark 3.** By Schwarz’s inequality, it is obvious that, in general,
\[-1 \leq \frac{\text{Re} \langle x, y \rangle}{\|x\| \|y\|} \leq 1.
\]

Consequently, the left inequality in (2.13) is of interest when \(\sum_{i \in I} (m_i + n_i) > 0\), while the right inequality in (2.13) is of interest when \(\sum_{i \in I} (M_i + N_i) < 0\).
3. Refinements of Kurepa’s Inequality

Let \((H; \langle \cdot, \cdot \rangle)\) be a real inner product space generating the norm \(\|\cdot\|\). The complexification \(H_C\) of \(H\) is defined as a complex linear space \(H \times H\) of all ordered pairs \((x, y)\), \(x, y \in H\) endowed with the operations:

\[
(x, y) + (x', y') := (x + x', y + y'), \quad x, x', y, y' \in H;
\]

\[
(\sigma + i\tau) \cdot (x, y) := (\sigma x - \tau y, \tau x + \sigma y), \quad x, y \in H \quad \text{and} \quad \sigma, \tau \in \mathbb{R}.
\]

On \(H_C := H \times H\), endowed with the above operations, one can now canonically define the scalar product \(\langle \cdot, \cdot \rangle_C\) by:

\[
\langle z, z' \rangle_C := \langle x, x' \rangle + \langle y, y' \rangle + i \langle x', y' \rangle - \langle x, y' \rangle
\]

where \(z = (x, y)\), \(z' = (x', y') \in H_C\). Obviously,

\[
\|z\|^2 = \|x\|^2 + \|y\|^2, \quad z = (x, y) \in H_C.
\]

One can also define the conjugate of a vector \(z = (x, y)\) by \(\bar{z} := (x, -y)\). It is easy to see that, the elements of \(H_C\), under defined operations, behave as formal “complex” combinations \(x + iy\) with \(x, y \in H\). Because of this, we may write \(z = x + iy\) instead of \(z = (x, y)\). Thus, \(\bar{z} = x - iy\). Under this setting, S. Kurepa [6] proved the following refinement of Schwarz’s inequality:

\[
|\langle a, z \rangle_C|^2 \leq \frac{1}{2} \|a\|^2 \left[\|z\|^2 + |\langle z, \bar{z} \rangle_C|\right] \leq \|a\|^2 \|z\|^2,
\]

for any \(a \in H\) and \(z \in H_C\).

This was motivated by generalising the de Bruijn result for sequences of real and complex numbers obtained in [2].

The following result holds.

**Theorem 2.** Let \(\{e_j\}_{j \in I}\) be a finite orthonormal family in the real inner product space \((H; \langle \cdot, \cdot \rangle)\). Then for any \(w \in H_C\), where \((H_C; \langle \cdot, \cdot \rangle_C)\) is the complexification of \((H; \langle \cdot, \cdot \rangle)\), one has the following Bessel’s type inequality:

\[
\left| \sum_{j \in I} \langle w, e_j \rangle_C^2 \right| \leq \sum_{j \in I} |\langle w, e_j \rangle_C|^2 = \frac{1}{2} |\langle w, \bar{w} \rangle_C| + \frac{1}{2} |\langle w, \bar{w} \rangle_C|\]

\[
\leq \frac{1}{2} \left[\|w\|^2 + |\langle w, \bar{w} \rangle_C|\right] \leq \|w\|^2.
\]

**Proof.** Define \(f_j \in H_C, f_j := (e_j, 0)\), \(j \in I\). For any \(k, j \in I\) we have

\[
\langle f_k, f_j \rangle_C = (\langle e_k, 0 \rangle, \langle e_j, 0 \rangle)_C = \langle e_k, e_j \rangle = \delta_{kj},
\]

therefore \(\{f_j\}_{j \in I}\) is an orthonormal family in \((H_C; \langle \cdot, \cdot \rangle_C)\).
If we apply Theorem 1 for $(H; \langle \cdot, \cdot \rangle_\mathbb{C})$, $a = w, b = \bar{w}$, we may write:

$$\left| \sum_{j \in I} \langle w, e_j \rangle_\mathbb{C} (e_j, \bar{w})_\mathbb{C} - \frac{1}{2} \langle w, \bar{w} \rangle_\mathbb{C} \right| \leq \frac{1}{2} \| w \|_\mathbb{C} \| \bar{w} \|_\mathbb{C}.$$  \hspace{1cm} (3.4)

However, for $w := (x, y) \in H_\mathbb{C}$, we have $\bar{w} = (x, -y)$ and

$$\langle e_j, \bar{w} \rangle_\mathbb{C} = \langle (e_j, 0), (x, -y) \rangle_\mathbb{C} = \langle e_j, x \rangle - i \langle e_j, -y \rangle = \langle e_j, x \rangle + i \langle e_j, y \rangle$$

and

$$\langle w, e_j \rangle_\mathbb{C} = \langle (x, y), (e_j, 0) \rangle_\mathbb{C} = \langle e_j, x \rangle - i \langle e_j, -y \rangle = \langle x, e_j \rangle + i \langle y, e_j \rangle$$

for any $j \in I$. Thus $\langle e_j, \bar{w} \rangle = \langle w, e_j \rangle$ for each $j \in I$ and since

$$\| w \|_\mathbb{C} = \| \bar{w} \|_\mathbb{C} = \left( \| x \|^2 + \| y \|^2 \right)^{\frac{1}{2}},$$

we get from (3.4) that

$$\left| \sum_{j \in I} \langle w, e_j \rangle_\mathbb{C}^2 - \frac{1}{2} \langle w, \bar{w} \rangle_\mathbb{C} \right| \leq \frac{1}{2} \| w \|_\mathbb{C}^2.$$  \hspace{1cm} (3.5)

Now, observe that the first inequality in (3.3) follows by the triangle inequality, the second is an obvious consequence of (3.5) and the last one is derived from Schwarz’s result.

**Remark 4.** If the family $\{e_j\}_{j \in I}$ contains only a single element $e = \frac{x}{\|x\|}, x \in H, x \neq 0$, then from (3.3) we deduce (1.7), which, in its turn, provides a refinement of Kurepa’s inequality (3.2).

### 4. An Application for $L_2 [-\pi, \pi]$

It is well known that in the Hilbert space $L_2 [-\pi, \pi]$ of all functions $f : [-\pi, \pi] \to \mathbb{C}$ with the property that $f$ is Lebesgue measurable on $[-\pi, \pi]$ and $\int_{-\pi}^{\pi} |f(t)|^2 dt < \infty$, the set of functions

$$\left\{ \frac{1}{\sqrt{2 \pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \ldots, \frac{1}{\sqrt{\pi}} \cos nt, \frac{1}{\sqrt{\pi}} \sin nt, \ldots \right\}$$

is orthonormal.

If by trig $t$, we denote either $\sin t$ or $\cos t, t \in [-\pi, \pi]$, then on using the results from Sections 2 and 3, we may state the following inequality:

$$\left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \text{trig}(kt) dt \cdot \int_{-\pi}^{\pi} \overline{g(t)} \text{trig}(kt) dt - \frac{1}{2} \int_{-\pi}^{\pi} f(t) g(t) dt \right|^2 \leq \frac{1}{4} \int_{-\pi}^{\pi} |f(t)|^2 dt \int_{-\pi}^{\pi} |g(t)|^2 dt,$$  \hspace{1cm} (4.1)
where all trig (kt) is either sin kt or cos kt, \( k \in \{1, \ldots, n\} \) and \( f \in L_2[-\pi, \pi] \).

This follows by Theorem 1.

If one uses Corollary 1, then one can state the following chain of inequalities

\[
\left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \text{trig}(kt) \, dt \cdot \int_{-\pi}^{\pi} g(t) \text{trig}(kt) \, dt \right| \\
\leq \left| \frac{1}{\pi} \sum_{k=1}^{n} \int_{-\pi}^{\pi} f(t) \text{trig}(kt) \, dt \cdot \int_{-\pi}^{\pi} g(t) \text{trig}(kt) \, dt - \frac{1}{2} \int_{-\pi}^{\pi} f(t) \frac{g(t)}{\text{trig}(kt)} \, dt \right| \\
+ \frac{1}{2} \left| \int_{-\pi}^{\pi} f(t) g(t) \, dt \right| \\
\leq \frac{1}{2} \left[ \left( \int_{-\pi}^{\pi} |f(t)|^2 \, dt \right) \int_{-\pi}^{\pi} |g(t)|^2 \, dt \right]^{\frac{1}{2}} + \left| \int_{-\pi}^{\pi} f(t) \frac{g(t)}{\text{trig}(kt)} \, dt \right| \\
\leq \left( \int_{-\pi}^{\pi} |f(t)|^2 \, dt \right) \int_{-\pi}^{\pi} |g(t)|^2 \, dt \right)^{\frac{1}{2}},
\]

(4.2)

where \( f \in L_2[-\pi, \pi] \).

Finally, by employing Theorem 2, we may state:

\[
\left| \frac{1}{\pi} \sum_{k=1}^{n} \left[ \int_{-\pi}^{\pi} f(t) \text{trig}(kt) \, dt \right]^2 \right| \\
\leq \left| \frac{1}{\pi} \sum_{k=1}^{n} \left[ \int_{-\pi}^{\pi} f(t) \text{trig}(kt) \, dt \right]^2 - \frac{1}{2} \int_{-\pi}^{\pi} f^2(t) \, dt \right| + \frac{1}{2} \left| \int_{-\pi}^{\pi} f^2(t) \, dt \right| \\
\leq \frac{1}{2} \left[ \int_{-\pi}^{\pi} |f(t)|^2 \, dt + \left| \int_{-\pi}^{\pi} f^2(t) \, dt \right| \right] \leq \int_{-\pi}^{\pi} |f(t)|^2 \, dt,
\]

where \( f \in L_2[-\pi, \pi] \).

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