Thermodynamics of the two–dimensional O(3) \( \sigma \)–model with fixed–point lattice action

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Abstract

The free energy density of the two–dimensional O(3) non–linear \( \sigma \)–model is calculated at finite temperature and finite spatial extent. We make both an analytic calculation in the perturbative regime and a Monte–Carlo study at low temperatures. We show that using the fixed–point action instead of the standard Wilson action leads to a great reduction of the cut–off effects.

Key words: Cut–off effects. Fixed–point action. Cluster Monte–Carlo study
1 Introduction

The calculation of thermodynamic quantities in Yang–Mills gauge theories using lattice regularization suffers from huge cut–off effects. Decreasing the lattice spacing $a$, the cost of such calculations grows with $\left[ \frac{1}{a} \right]^\epsilon$, where in SU(3) gauge theory $\epsilon$ is around 10. This requires a very careful analysis to remove the cut–off effects. Therefore in field theory one looks for improved actions which give good results even on lattices with bad resolution. Wilson’s idea of the renormalization group (RG) [1,2] aims at that direction: one integrates out the short–distance degrees of freedom and replaces them by a smaller set of effective degrees of freedom, which describe long–distance physics. One possibility is to use a blocking procedure introduced originally by Kadanoff. Theoretically there exists a perfect lattice action, that predictions are absolutely free of any cut–off effects. We use here the fixed–point (FP) action, which is the classical perfect action and is expected to be a good approximation for the perfect action. For the O(3) non–linear $\sigma$–model, the model we are considering here, the FP action was developed in [3]. Recently in a nice work [4], Alessandro Papa pointed out the great advantage of the FP action for SU(3) gauge theory: The free energy density near the phase transition point showed an impressive reduction of cut–off effects compared to earlier results derived by other lattice actions [5]. However the few points calculated in that paper do not allow a complete view of the behaviour in the decrease of cut–off effects. In the case of the $\sigma$–model we can continue this project and give the free energy density over a large temperature region. Hence we check the cut–off effects arising from different lattice actions both in the high temperature regime (perturbation theory) and at low temperatures using Monte–Carlo (MC) simulations.

The Euclidean continuum version of the partition function for the O(3) non–linear $\sigma$–model in two dimensions reads as follows:

$$Z = \int DS(x) \, e^{-\beta A(S)}$$

where $S$ is a three component vector with $S^2(x) = 1$ and measure

$$DS(x) = \prod_x d^3S(x) \, \delta(S^2(x) - 1).$$

The term $\beta A(S)$ is the continuum action with periodic boundary conditions for the field $S$ at finite temperature $T = 1/L_t$ and spatial extent $L_s$.

$$\beta A_{\text{cont}}(S) = \frac{\beta}{2} \int_0^{L_t} \int_0^{L_s} d\mu \, \partial_\mu S(x) \partial_\mu S(x)$$
The free energy density \( f \) is given by the relation

\[
Z = e^{-fV}
\]  

where \( V \) is \( L_t \times L_s \) and \( f \) depends on both \( L_t \) and \( L_s \). To get the temperature and spatial size dependent, physical part of this quantity we subtract the contribution resulting from the square box, in which we set \( L_t \) equal to \( L_s \). In the perturbative regime and for infinite spatial extent we get \(-f/T^2 = \pi/3\) which is the two–dimensional case of an ideal gas of massless scalar particles.

This letter consists of the following parts: in section 2 we give a short review of the ideas which lead to the FP action in the \( \sigma \)–model. The next section presents the perturbative calculation of the free energy density and a comparison between different lattice actions. Section 4 treats the cluster–MC analysis necessary for the calculation in the non–perturbative, low temperature regime and shows the corresponding results.

## 2 FP action

We summarize now some important results about the FP action of the \( O(3) \) \( \sigma \)–model which we use in this letter. For a detailed discussion we refer the reader to ref. [3].

We divide our original lattice into \( 2 \times 2 \) blocks labeled by \( n_B \). The four spins \( S_n \) inside a block form, by taking some mean, a new spin variable \( R_{n_B} \) on a lattice with a lattice unit twice as large as the original one. The new action \( A'(R) \) is defined by integration over the original lattice:

\[
e^{-\beta A'(R)} = \int DS(x) \ e^{-\beta [A(S)+\mathcal{T}(R,S)]} \]  

where \( \mathcal{T} \) is the kernel of the RG transformation with the normalization that ensures the invariance of the partition function

\[
\int DR(x) \ e^{-\beta \mathcal{T}(R,S)} = 1 .
\]  

In the limit \( \beta \to \infty \) the blocking kernel \( \mathcal{T} \) has the form

\[
\mathcal{T}(R,S) = \kappa \sum_{n_B} \left( \left| \sum_{n \in n_B} S_n \right| - R_{n_B} \cdot \sum_{n \in n_B} S_n \right)
\]
where $\kappa$ (fixed here to 2) can be tuned to find a short–ranged FP–action. In the limit mentioned above equation (5) reduces to a saddle point problem, giving

$$A'(R) = \min_{\{S\}} \left\{ A(S) + T(R, S) \right\}, \quad (8)$$

and for the FP action

$$A_{FP}(R) = \min_{\{S\}} \left\{ A_{FP}(S) + T(R, S) \right\}. \quad (9)$$

For parametrizing it, one can write in general

$$A_{FP}(S) = -\frac{1}{2} \sum_{n,r} \rho(r)(1 - S_{n}S_{n+r}) + \sum_{n_1,n_2,n_3,n_4} c(n_1, n_2, n_3, n_4)(1 - S_{n_1}S_{n_2})(1 - S_{n_3}S_{n_4}) + \ldots, \quad (10)$$

where it is a significant help, that the first two functions $\rho$ and $c$ can be calculated analytically. The Fourier transform of $\rho$ has the form

$$\rho(q) = \sum_{l=-\infty}^{+\infty} \frac{1}{(q + 2\pi l)^2} \prod_{i=0}^{1} \frac{\sin^2(q_i/2)}{(q_i/2 + \pi l_i)^2} + \frac{1}{3\kappa}. \quad (11)$$

The summation is over the integer vector $l = (l_0, l_1)$ and $(q + 2\pi l)^2 = (q_0 + 2\pi l_0)^2 + (q_1 + 2\pi l_1)^2$. Furthermore one can show that $\rho$ fulfills the FP equation for a free scalar theory with a gaussian block–transformation.

Beside the two functions $\rho$ and $c$ the resulting couplings were found in [3] by a numerical fitting procedure and recently improved in [6] to control the topological effects. We use the parametrization obtained in [6]. It consists of 24 different couplings which can be put on a $2 \times 2$ lattice. When working with such an action to measure the free energy density one expects to see cut–off effects, due to the following three points:

- The FP action is not the perfect action.
- In the simulation a parametrized form of the FP action is used which introduces some parametrization errors.
- Finally, on extremely small lattices ($N_t = 2$) the action is disturbed by the small size, which causes cut–off effects that decrease exponentially with $N_t$. 
3 Perturbative results

In a normal perturbative treatment of eq. (1), one introduces a Faddeev–Popov term to avoid the zero mode problem [7]. One arrives finally, when changing to Fourier space, at the following form of the partition function:

$$e^{-\sum'_{k} \ln k^2 + \ln V}$$  \hspace{1cm} (12)

where the sum here just leaves away the unwanted zero mode $k = (0,0)$, $\sum'_{k} = \sum_{k,k \neq (0,0)}$. As a remainder of the Faddeev–Popov term we have to keep the logarithm of the volume in the partition function. Due to equation (3) the Fourier sum above is discreet and one can use any regularization scheme to extract the continuum value $f_{\text{cont}}$. When using lattice regularization one must replace the continuum propagator $k^2$ in eq. (12) with the corresponding lattice propagator, for example with the function $\rho$ of eq. (11) in the FP case. We then check the cut–off effects by considering the ratios $f_{W}/f_{\text{cont}}, f_{S}/f_{\text{cont}}, f_{i}/f_{\text{cont}}$ and $f_{\text{FP}}/f_{\text{cont}}$, where the abbreviations "W, S" denote the use of Wilson and Symanzik action, whereas "fi" means an action with a nearest neighbour and a diagonal coupling, whose ratio is the same as in the FP action. Although there is no problem to give $f$ as a function of $\zeta$, $\zeta = L \cdot T = L_s / L_t$, we present the cut–off effects for $\zeta = 3$. Table 1 gives the results $f_{\text{cont}}/f_{\text{latt}}$ for different lattice schemes, a different number of points $N_t$ in the time direction (i.e. for different resolutions) and $N_s = 3 \times N_t$. An approximation of the decrease of the cut–off effects can then be given through a function of the form:

$$1 + a \cdot \frac{1}{N_t^2} + b \cdot \frac{1}{N_t^4} + c \cdot \frac{1}{N_t^6} + \ldots$$ \hspace{1cm} (13)

The Wilson action shows the expected quadratic decrease (fig. 2a), whereas in the Symanzik case (fig. 2b) the fit is consistent with $a = 0$. Figure 2c shows the behaviour of $f_{i}$ and figure 2d finally presents a perfect exponential decrease of the cut–off effects for the FP action. The smooth curve is given by $1 + A \cdot \exp(-BN_t)$, where $A$ and $B$ are constants.

4 MC methods

In a MC simulation the partition function cannot be measured directly, on the other hand, using equations (1,4), the derivative of the logarithm of $Z$ with respect to $\beta$ gives:

$$\frac{\partial}{\partial \beta} \ln Z = -Z^{-1} \int DS \mathcal{A}(S) e^{-\beta \mathcal{A}(S)}.$$  \hspace{1cm} (14)
Table 1
Numerical values for different lattice actions with $N_s = 3 \times N_t$

| # | $N_t$ | $f_W/f_{\text{cont}}$ | $f_{\text{cont}}/f_S$ | $f_{\text{cont}}/f_{\text{fit}}$ | $f_{\text{cont}}/f_{\text{FP}}$ |
|---|---|---|---|---|---|
| 2 | 1.1606 | 0.964058 | 1.016701 | 1.006314 |
| 4 | 1.0455 | 1.001507 | 1.004622 | 1.000016 |
| 6 | 1.0182 | 1.000155 | 1.001773 | 1.000000 |
| 8 | 1.0098 | 1.000024 | 1.000729 | 1.000000 |
| 10 | 1.0062 | 1.000007 | 1.000240 | 1.000000 |

One obtains

$$-rac{f}{T^2} = \int_{\beta_0}^{\beta} d\beta' \left[ \frac{\langle A_{\text{vac}} \rangle}{\zeta^2} - \frac{\langle A_c \rangle}{\zeta} \right]$$

(15)

where $\langle A_c \rangle$ denotes the expectation value of the action on the lattice $N_t \times N_s$ and $\langle A_{\text{vac}} \rangle$ the same on the square lattice which we subtract as vacuum contribution.

In our simulations we used a cluster algorithm [8], which almost completely eliminates the critical slowing–down, which was especially important for the measurement of the mass gap. For the measurement of the action we performed $8 \times 10^5$ sweeps per $\beta$–value and estimated the error by the method of bunching. The size of the error is of the order of the circles in figure 2, where one finds the signal of the integrand in eq. (15). The smooth curve is a spline fit interpolation, which we used for the integration, where $\beta_0$ had to be chosen so small that the signal for that value disappears. To estimate the error of this integration we compared the value resulting from the spline curve with the value one gets, joining the single points in fig. 2 by straight lines. This error is not bigger than half the size of the symbols in figures 3, 4 and 5. To compare the cut–off effects we measured the action for Wilson and FP parametrization.

The above procedure gives just the free energy density as a function of $\beta$. To extract physical results we must relate every $\beta$–value to a temperature. This can be done by measuring another physical quantity, the mass gap. In a lattice simulation one can only get the dimensionless quantity $m \cdot a$, where the lattice spacing $a$ is a function of $\beta$. Using the relation $T = 1/N_t a$ we find:

$$\frac{T}{m} = \frac{1}{maN_t}$$

(16)
Fig. 1. Figures a–d show the disappearance of the cut–off effects for different lattice actions. The smooth curves are fitted functions explained in the text.

where $N_t$ is the number of points in the time direction of the lattice used for the measurement of $\langle A_t \rangle$. We then extracted the quantity $m \cdot a(\beta)$ for some $\beta$–values by taking a square lattice which was about seven times larger than the correlation length to avoid finite size effects. Inserting these values in eq. (16) we got a finite number of points $T/m(\beta)$. To our data we added further results of the mass gap in the FP case from ref. [6], while for the Wilson action we used the results from [9].

We present first (fig. 3) a comparison between Wilson action and FP action. Fig. 4 gives the detailed results obtained with the FP action and the last figure (fig. 5) shows a spline interpolation of the FP data.
Fig. 2. The signal of the integrand in eq. (15) for a $4 \times 12$ lattice is plotted against the values of $\beta$. The smooth curve is the spline interpolation we used for the integration.

5 Summary

The most important result of this work is the great reduction of cut–off effects with the FP action compared to other lattice actions. In the perturbative regime the cut–off effects disappear exponentially fast and are nearly lost on a lattice as small as $N_t = 4$. Concerning the cut–off effects, our results of MC simulations show within the range of statistical error a similar behaviour in the low–temperature region. Although we are working with a relatively small statistical error we are not able to detect any power–law in the decrease of the cut–off effects.

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Fig. 3. The discreet points give the free energy density $-f/T^2$ for different lattice resolutions with Wilson action, whereas the smooth curve is a spline interpolation of the data resulting from a $3 \times 9$ lattice with FP action. The error of the free energy density is smaller than the size of the symbols. The horizontal line shows the perturbative value for $\zeta = 3$ in the high temperature limit.

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Fig. 4. The free energy density for different lattice resolutions using FP action. The error of these values is smaller than the size of the symbols.

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Fig. 5. The smooth curves show a spline interpolation of the data for $2 \times 6$, $3 \times 9$ and $4 \times 12$ lattices in the FP case (fig. 4). Within statistical error the values for the $5 \times 15$ lattice lie on the curve of the $4 \times 12$ lattice.