The Holevo capacity of infinite dimensional channels and the additivity problem.

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Abstract

The Holevo capacity of arbitrarily constrained infinite dimensional quantum channel is considered and its properties are discussed. The notions of input and output optimal average states are introduced. The continuity properties of the Holevo capacity with respect to constraint and to channel are explored.

The main result of this paper is the statement that additivity of the Holevo capacity for all finite dimensional channels implies its additivity for all infinite dimensional channels with arbitrary constraints.

Keywords: quantum channel, \( \chi \)-capacity, additivity problem

1 Introduction

The Holevo capacity (in what follows, \( \chi \)-capacity) of a quantum channel is an important characteristic defining amount of classical information, which can be transmitted by this channel using nonentangled encoding and entangled decoding, see e.g. [8], [10], [22]. For additive channels the \( \chi \)-capacity coincides with the full classical capacity of a quantum channel. At present the main interest is focused on quantum channels between finite dimensional quantum systems. But having in mind possible applications it is necessary to deal with infinite dimensional quantum channels, in particular, Gaussian channels.

In this paper the \( \chi \)-capacity for arbitrarily constrained infinite dimensional quantum channel is considered. It is shown that despite nonexistence of an optimal ensemble in this case it is possible to define the notion of

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the output optimal average state for such a channel, inheriting important properties of the image of the average state of an optimal ensemble for finite dimensional channels (proposition 1). A "minimax" expression for the $\chi$-capacity is obtained and an alternative characterization of the output optimal average state as a minimum point of a lower semicontinuous function on a compact set is given (proposition 2).

The notion of the $\chi$-function of an infinite dimensional quantum channel is introduced. It is shown that the $\chi$-function of an arbitrary channel is a concave lower semicontinuous function with natural chain properties, having continuous restriction to any set of continuity of the output entropy (propositions 3-4). This and the result in [12] imply continuity of the $\chi$-function for Gaussian channels with power constraint (example 1). For the $\chi$-function the analog of Simon’s dominated convergence theorem for quantum entropy is also obtained (corollary 4).

The question of continuity of the $\chi$-capacity as a function of channel is considered. It is shown that the $\chi$-capacity is continuous function of channel in the finite dimensional case while in general it is only lower semicontinuous (theorem 1, example 2).

The above results make it possible to obtain the infinite dimensional version of theorem 1 in [11], which shows equivalence of several formulations of the additivity conjecture (theorem 2).

The main result of this paper is the statement that additivity of the $\chi$-capacity for all finite dimensional channels implies its additivity for all infinite dimensional channels with arbitrary constraints (theorem 3). This is done in two steps by using several results (lemma 5, proposition 5 and 6). These results are also applicable to analysis of individual pairs of channels as it is demonstrated in the proof of additivity of the $\chi$-capacity for two arbitrarily constrained infinite dimensional channels with one of them noiseless or entanglement breaking (proposition 7).

2 Basic quantities

Let $\mathcal{H}$ be a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ be the set of all bounded operators on $\mathcal{H}$ with the cone $\mathcal{B}_+(\mathcal{H})$ of all positive operators, $\mathcal{I}(\mathcal{H})$ be the Banach space of all trace-class operators with the trace norm $\| \cdot \|_1$ and $\mathcal{S}(\mathcal{H})$ be the closed convex subset of $\mathcal{I}(\mathcal{H})$ consisting of all density operators on $\mathcal{H}$, which is complete separable metric space with the metric defined by the trace
norm. Each density operator uniquely defines a normal state on $\mathfrak{B}(\mathcal{H})$ \[1\], so, in what follows we will also for brevity use the term "state". Note that convergence of a sequence of states to a state in the weak operator topology is equivalent to convergence of this sequence to this state in the trace norm \[3\].

In what follows log denotes the function on $[0, +\infty)$, which coincides with the logarithm on $(0, +\infty)$ and vanishes at zero. Let $A$ and $B$ be positive trace class operators. Let $\{\ket{i}\}$ be a complete orthonormal set of eigenvectors of $A$. The entropy is defined by $H(A) = -\sum_i \bra{i} A \log \bra{i} A \ket{i}$ while the relative entropy – as $H(A \parallel B) = \sum_i \bra{i} (A \log A - A \log B + B - A) \ket{i}$, provided $\text{ran} A \subseteq \text{ran} B$,\(^1\) and $H(A \parallel B) = +\infty$ otherwise (see [16], [17] for more detailed definition). The entropy and the relative entropy are nonnegative lower semicontinuous (in the trace-norm topology) concave and convex functions of their arguments correspondingly [17],[19],[29]. We will use the following inequality
\[
H(\rho \parallel \sigma) \geq \frac{1}{2} \| \rho - \sigma \|_1^2, \tag{1}
\]
which holds for arbitrary states $\rho$ and $\sigma$ in $\mathfrak{B}(\mathcal{H})$ \[19\].

Arbitrary finite collection $\{\rho_i\}$ of states in $\mathfrak{B}(\mathcal{H})$ with corresponding set of probabilities $\{\pi_i\}$ is called ensemble and is denoted by $\Sigma = \{\pi_i, \rho_i\}$. The state $\bar{\rho} = \sum_i \pi_i \rho_i$ is called the average state of the above ensemble. Following [12] we treat an arbitrary Borel probability measure $\pi$ on $\mathfrak{B}(\mathcal{H})$ as generalized ensemble and the barycenter of the measure $\pi$ defined by the Pettis integral
\[
\bar{\rho}(\pi) = \int_{\mathfrak{B}(\mathcal{H})} \rho \pi(d\rho)
\]
as the average state of this ensemble. In this notations the conventional ensembles correspond to measures with finite support. For arbitrary subset $\mathcal{A}$ of $\mathfrak{B}(\mathcal{H})$ we denote by $\mathcal{P}_\mathcal{A}$ the set of all probability measures with barycenters contained in $\mathcal{A}$.

For arbitrary finite set of ensembles $\{\{\pi_i^k, \rho_i^k\}_{i=1}^{n(k)}\}_{k=1}^m$ and arbitrary probability distribution $\{\lambda_k\}_{k=1}^m$ consider the ensemble consisting of $\sum_{k=1}^m n(k)$ states $\{\rho_i^k\}_{ki}$ with the corresponding probabilities $\{\lambda_k \pi_i^k\}_{ki}$. We will call this ensemble convex combination of the above ensembles and denote it by $\sum_{i=1}^{m} \lambda_k \{\pi_i^k, \rho_i^k\}_{i=1}^{n(k)}$. By using the relation between the notion of an ensemble and the notion of a probability measure one can say that a convex

\(^1\)ran denotes the closure of the range of an operator in $\mathcal{H}$
A combination of ensembles corresponds to a convex combination of measures corresponding to these ensembles.

In analysis of the $\chi$-capacity we shall use Donald’s identity [6], [19]

$$\sum_{i=1}^{n} \pi_i H(\rho_i \parallel \hat{\rho}) = \sum_{i=1}^{n} \pi_i H(\rho_i \parallel \check{\rho}) + H(\check{\rho} \parallel \hat{\rho}), \quad (2)$$

which holds for arbitrary ensemble $\{\pi_i, \rho_i\}$ of $n$ states with the average state $\check{\rho}$ and arbitrary state $\hat{\rho}$.

Let $H, H'$ be a pair of separable Hilbert spaces which we shall call correspondingly input and output space. A channel $\Phi$ is a linear positive trace preserving map from $\mathfrak{F}(H)$ to $\mathfrak{F}(H')$ such that the dual map $\Phi^\ast : \mathfrak{B}(H') \rightarrow \mathfrak{B}(H)$ (which exists since $\Phi$ is bounded [5]) is completely positive. Let $\mathcal{A}$ be an arbitrary closed subset of $\mathfrak{S}(H)$. We consider constraint on input ensemble $\{\pi_i, \rho_i\}$, defined by the requirement $\check{\rho} \in \mathcal{A}$. The channel $\Phi$ with this constraint is called the $\mathcal{A}$-constrained channel. We define the $\chi$-capacity of the $\mathcal{A}$-constrained channel $\Phi$ as (cf. [9], [10], [11])

$$\bar{C}(\Phi; \mathcal{A}) = \sup_{\rho \in \mathcal{A}} \chi_{\Phi}(\{\pi_i, \rho_i\}), \quad (3)$$

where

$$\chi_{\Phi}(\{\pi_i, \rho_i\}) = \sum_i \pi_i H(\Phi(\rho_i) \parallel \Phi(\check{\rho})).$$

In [12] it is shown that the $\chi$-capacity of the $\mathcal{A}$-constrained channel $\Phi$ can be also defined by

$$\bar{C}(\Phi; \mathcal{A}) = \sup_{\pi \in \mathcal{P}_{\mathcal{A}}} \int_{\mathfrak{S}(H)} H(\Phi(\rho) \parallel \Phi(\check{\rho}))(\pi(\check{\rho})) d\rho, \quad (4)$$

which means coincidence of the above supremum over all measures in $\mathcal{P}_{\mathcal{A}}$ with the supremum over all measures in $\mathcal{P}_{\mathcal{A}}$ with finite support.

The $\chi$-capacity $\bar{C}(\Phi; \mathfrak{S}(H))$ of the unconstrained channel $\Phi$ is also denoted by $\bar{C}(\Phi)$.

The $\chi$-function of the channel $\Phi$ is defined by

$$\chi_{\Phi}(\rho) = \bar{C}(\Phi; \{\rho\}) = \sup_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i) \parallel \Phi(\rho)). \quad (5)$$

The $\chi$-function of finite dimensional channel $\Phi$ is a continuous concave function on $\mathfrak{S}(H)$ [11]. The properties of the $\chi$-function of arbitrary infinite dimensional channel $\Phi$ are considered in section 4.

4
3 The optimal average state

It is known fact that for an arbitrary finite dimensional channel $\Phi$ and arbitrary closed set $\mathcal{A}$ there exists an optimal ensemble $\{\pi_i, \rho_i\}$ on which the supremum in the definition (3) of the $\chi$-capacity is achieved [4], [23]. The image of the average state of this optimal ensemble plays an important role in the analysis of finite dimensional channels [11].

For general infinite dimensional constrained channel there are no reasons for existence of an optimal ensemble (with finite number of states). In this case it is natural to introduce the notion of optimal generalized ensemble (optimal measure) on which the supremum in the definition (4) of the $\chi$-capacity is achieved. In [12] the sufficient condition for existence of an optimal measure for infinite dimensional constrained channel is obtained and the example of the channel with no optimal measure is given.

The aim of this section is to show that even in the case of nonexistence of optimal generalized ensemble we can define the notion of output optimal average state, inheriting the basic properties of the image of the average state of an optimal ensemble for a finite dimensional constrained channel. Using this notion we can generalize some results of [11] to the infinite dimensional case.

Note first that Donald’s identity (2) implies the following observation.

**Lemma 1.** Let $\{\{\pi_{i,k}^n, \rho_{i,k}^n\}_{i=1}^n\}_{k=1}^m$ be a finite set of ensembles and $\{\lambda_k\}_{k=1}^m$ be a probability distribution. Then

$$
\chi_\Phi \left( \sum_{k=1}^m \lambda_k \{\pi_{i,k}^n, \rho_{i,k}^n\}_{i=1}^n \right) = \sum_{k=1}^m \lambda_k \chi_\Phi \left( \{\pi_{i,k}^n, \rho_{i,k}^n\}_{i=1}^n \right) + \chi_\Phi \left( \{\lambda_k, \bar{\rho}_k\}_{k=1}^m \right),
$$

where $\bar{\rho} = \sum_{k=1}^m \lambda_k \bar{\rho}_k$ is the average state of the ensemble $\sum_{k=1}^m \lambda_k \{\pi_{i,k}^n, \rho_{i,k}^n\}_{i=1}^n$.

In the case $m = 2$ for arbitrary $\lambda \in [0; 1]$ the following inequality holds

$$
\chi_\Phi \left( \lambda\{\pi_{i,1}^n, \rho_{i,1}^n\}_{i=1}^n + (1 - \lambda)\{\pi_{i,2}^n, \rho_{i,2}^n\}_{i=1}^n \right) \geq \lambda \chi_\Phi \left( \{\pi_{i,1}^n, \rho_{i,1}^n\}_{i=1}^n \right) + (1 - \lambda) \chi_\Phi \left( \{\pi_{i,2}^n, \rho_{i,2}^n\}_{i=1}^n \right) + \lambda(1 - \lambda) \|\Phi(\bar{\rho}_2) - \Phi(\bar{\rho}_1)\|_1^2.
$$

**Proof.** By definition

$$
\chi_\Phi \left( \sum_{k=1}^m \lambda_k \{\pi_{i,k}^n, \rho_{i,k}^n\}_{i=1}^n \right) = \sum_{k=1}^m \lambda_k \sum_{i=1}^n \pi_{i,k}^n H \left( \Phi(\rho_{i,k}^n) \| \Phi(\bar{\rho}) \right).
$$
Applying Donald’s identity to the each inner sum in the right side of the above expression we obtain the main identity of the lemma.

To prove the inequality in the case \( m = 2 \) it is sufficient to apply inequality for the below estimation of the relative entropies in the main identity of the lemma:

\[
\lambda H(\Phi(\bar{\rho}_1)\|\Phi(\lambda \bar{\rho}_1 + (1 - \lambda)\bar{\rho}_2)) + (1 - \lambda)H(\Phi(\bar{\rho}_2)\|\Phi(\lambda \bar{\rho}_1 + (1 - \lambda)\bar{\rho}_2)) \geq \frac{1}{2}\lambda \|\Phi(\lambda \bar{\rho}_1 + (1 - \lambda)\bar{\rho}_2) - \Phi(\bar{\rho}_1)\|^2_1.
\]

\( \square \)

Despite possible nonexistence of optimal ensemble for the \( \mathcal{A} \)-constrained channel \( \Phi \) the definition of the \( \chi \)-capacity implies existence of a sequence of ensembles with the following properties.

**Definition 1.** A sequence of ensembles \( \{\pi_i^k, \rho_i^k\} \) with the average \( \bar{\rho}^k \in \mathcal{A} \) such that

\[
\lim_{k \to +\infty} \chi_{\Phi}(\{\pi_i^k, \rho_i^k\}) = \bar{C}(\Phi; \mathcal{A})
\]

is called approximating sequence for the \( \mathcal{A} \)-constrained channel \( \Phi \).

A state \( \bar{\rho} \) is called input optimal average state for the \( \mathcal{A} \)-constrained channel \( \Phi \) if this state \( \bar{\rho} \) is a limit of the sequence of the average states of some approximating sequence of ensembles for the \( \mathcal{A} \)-constrained channel \( \Phi \).

This definition admits that input optimal average state may not exist or may not be unique. If there exists an optimal measure for the \( \mathcal{A} \)-constrained channel \( \Phi \) then its barycenter is an input optimal average state. It follows from lemma 1 and proposition 1 in \[12\]. Existence of an input optimal average state is also a sufficient condition for existence of an optimal measure for the \( \mathcal{A} \)-constrained channel \( \Phi \) if the restriction of the output entropy to the set \( \mathcal{A} \) is continuous at this state \[12\].

Despite possible nonexistence of partial limits of the sequence of the average states of a particular approximating sequence the following proposition guarantees convergence of the sequence of their images.

**Proposition 1.** Let \( \mathcal{A} \) be convex subset of \( \mathcal{S}(\mathcal{H}) \) such that \( \bar{C}(\Phi; \mathcal{A}) < +\infty \). Then there exists the unique state \( \Omega(\Phi, \mathcal{A}) \) in \( \mathcal{S}(\mathcal{H}') \) such that

\[
\sup_{\sum \mu_j \sigma_j \in \mathcal{A}} \sum \mu_j H(\Phi(\sigma_j)\|\Omega(\Phi, \mathcal{A})) = \bar{C}(\Phi; \mathcal{A}),
\]

(the supremum is over all ensembles \( \{\mu_j, \sigma_j\} \) with the average state \( \bar{\sigma} \in \mathcal{A} \)).
For arbitrary approximating sequence of ensembles \( \{ \pi^k_i, \rho^k_i \} \) for the \( \mathcal{A} \)-constrained channel \( \Phi \) there exists
\[
\lim_{k \to +\infty} \Phi(\bar{\rho}_k) = \Omega(\Phi, \mathcal{A}).
\]

**Proof.** Show first that for arbitrary approximating sequence of ensemble \( \{ \Sigma_k = \{ \pi^k_i, \rho^k_i \}_{i=1}^{n(k)} \} \) for the \( \mathcal{A} \)-constrained channel \( \Phi \) the sequence \( \{ \Phi(\bar{\rho}_k) \} \) converges to a particular state in \( \mathcal{S}(\mathcal{H}') \). By definition of an approximating sequence for arbitrary \( \varepsilon > 0 \) there exists \( N_\varepsilon \) such that
\[
\chi_\Phi(\Sigma_k) > \bar{C}(\Phi; \mathcal{A}) - \varepsilon
\]
for all \( k \geq N_\varepsilon \). By lemma 1 (with \( m = 2 \) and \( \lambda = 1/2 \)) for all \( k_1 \geq N_\varepsilon \) and \( k_2 \geq N_\varepsilon \) we have
\[
\bar{C}(\Phi; \mathcal{A}) - \varepsilon \leq \frac{1}{2} \chi_\Phi(\Sigma_k) + \frac{1}{2} \chi_\Phi(\Sigma_k)
\]
and hence \( \| \Phi(\bar{\rho}_{k_2}) - \Phi(\bar{\rho}_{k_1}) \| < \sqrt{8\varepsilon} \). Thus the sequence \( \{ \Phi(\bar{\rho}_k) \} \) is a Cauchy sequence and hence it converges to a particular state \( \rho' \) in \( \mathcal{S}(\mathcal{H}') \).

Let \( \{ \mu_j, \sigma_j \}_{j=1}^m \) be an arbitrary ensemble with the average \( \bar{\sigma} \in \mathcal{A} \). Consider the family of ensembles
\[
\Sigma^\eta_k = (1 - \eta)\{ \pi^k_i, \rho^k_i \}_{i=1}^{n(k)} + \eta \{ \mu_j, \sigma_j \}_{j=1}^m, \quad \eta \in [0, 1], k \in \mathbb{N}
\]
with the average states \( \bar{\rho}^\eta_k \). By convexity of \( \mathcal{A} \) we have \( \bar{\rho}^\eta_k \in \mathcal{A} \) for all \( \eta \in [0, 1] \) and \( k \in \mathbb{N} \). By the above observation
\[
\lim_{k \to +\infty} \Phi(\bar{\rho}^\eta_k) = (1 - \eta)\rho' + \eta \Phi(\bar{\sigma}). \quad (6)
\]

By definition
\[
\chi_\Phi(\Sigma^\eta_k) = (1 - \eta) \sum_{i=1}^{n(k)} \pi^k_i H(\Phi(\rho^k_i)\|\Phi(\bar{\rho}^\eta_k)) + \eta \sum_{j=1}^m \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}^\eta_k)). \quad (7)
\]
Since \( C(\Phi; \mathcal{A}) < +\infty \) the both sums in the right side of this expression are finite. Applying Donald’s identity \([2]\) to the first sum we obtain
\[
\sum_{i=1}^{n(k)} \pi^k_i H(\Phi(\rho^k_i)\|\Phi(\bar{\rho}^\eta_k)) = \chi_\Phi(\Sigma^0_k) + H(\Phi(\bar{\rho}_k)\|\Phi(\bar{\rho}^\eta_k)).
\]
Substitution of the above expression into (7) gives

\[ \chi_{\Phi}(\Sigma^\eta_k) = \chi_{\Phi}(\Sigma^0_k) + (1 - \eta)H(\Phi(\bar{\rho}_k)\|\Phi(\bar{\rho}_k^0)) \]

\[ + \eta \left[ \sum_{j=1}^m \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_k^0)) - \chi_{\Phi}(\Sigma^0_k) \right]. \]

Due to nonnegativity of the relative entropy it follows that

\[ \sum_{j=1}^m \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_k^0)) \leq \eta^{-1} \left[ \chi_{\Phi}(\Sigma^\eta_k) - \chi_{\Phi}(\Sigma^0_k) \right] + \chi_{\Phi}(\Sigma^0_k), \quad \eta \neq 0. \tag{8} \]

By definition of the approximating sequence we have

\[ \lim_{k \to +\infty} \chi_{\Phi}(\Sigma^0_k) = \bar{C}(\Phi; A) \geq \chi_{\Phi}(\Sigma^\eta_k) \tag{9} \]

for all \( k \). It follows that

\[ \liminf_{\eta \to +0} \liminf_{k \to +\infty} \eta^{-1} \left[ \chi_{\Phi}(\Sigma^\eta_k) - \chi_{\Phi}(\Sigma^0_k) \right] \leq 0 \tag{10} \]

By lower semicontinuity of the relative entropy (6), (8), (9) and (10) imply

\[ \sum_{j=1}^m \mu_j H(\Phi(\sigma_j)\|\rho') \leq \liminf_{\eta \to +0} \liminf_{k \to +\infty} \sum_{j=1}^m \mu_j H(\Phi(\sigma_j)\|\Phi(\bar{\rho}_k^0)) \leq \bar{C}(\Phi; A). \]

This proves that

\[ \sup_{\Sigma_j, \mu_j, \sigma_j \in A} \sum_j \mu_j H(\Phi(\sigma_j)\|\rho') \leq \bar{C}(\Phi; A), \tag{11} \]

To prove the converse inequality consider an approximating sequence \( \{\pi_i^k, \bar{\rho}_i^k\} \). Applying Donald’s identity (2) we obtain

\[ \sum_i \pi_i^k H(\Phi(\rho_i^k)\|\rho') = \sum_i \pi_i^k H(\Phi(\rho_i^k)\|\Phi(\bar{\rho}_i^k)) + H(\Phi(\bar{\rho}_i^k)\|\rho'). \]

By the approximating property of the sequence \( \{\pi_i^k, \bar{\rho}_i^k\} \) the first term in the right side tends to \( \bar{C}(\Phi; A) \) as \( k \to +\infty \), while the second is nonnegative. This proves “\( \geq \)” and, hence, “\( = \)” in (11).
By inequality (1) and the below lemma 2 inequality (11) implies that for arbitrary approximating sequence of ensembles \( \{ \mu_j^k, \sigma_j^k \} \) for the \( \mathcal{A} \)-constrained channel \( \Phi \) the corresponding sequence \( \Phi(\bar{\sigma}_k) \) converges to the state \( \rho' \). Thus this state \( \rho' \) does not depend on the choice of an approximating sequence, so, it is determined only by the channel \( \Phi \) and by the constraint set \( \mathcal{A} \). Denote this state by \( \Omega(\Phi, \mathcal{A}) \). Lemma 2 implies also that \( \rho' = \Omega(\Phi, \mathcal{A}) \) is the unique state for which equality in (11) holds. □

**Lemma 2.** Let \( \mathcal{A} \) be a set such that \( \bar{C}(\Phi; \mathcal{A}) < +\infty \) and \( \rho' \) be a state in \( \mathcal{S}(\mathcal{H}') \) such that
\[
\sum_j \mu_j H(\Phi(\sigma_j) \| \rho') \leq \bar{C}(\Phi; \mathcal{A})
\]
for arbitrary ensemble \( \{ \mu_j, \sigma_j \} \) with the average \( \bar{\sigma} \in \mathcal{A} \). Then for arbitrary approximating sequence \( \{ \pi_i^k, \rho_i^k \} \) of ensembles for the \( \mathcal{A} \)-constrained channel \( \Phi \) with the corresponding sequence of average states \( \bar{\rho}_k \) there exists \( \lim_{k \to +\infty} H(\Phi(\bar{\rho}_k) \| \rho') = 0 \).

**Proof.** Let \( \{ \pi_i^k, \rho_i^k \} \) an approximating sequence of ensembles with the corresponding sequence of the average states \( \bar{\rho}_k \). By assumption we have
\[
\sum_i \pi_i^k H(\Phi(\rho_i^k) \| \rho') \leq \bar{C}(\Phi; \mathcal{A}).
\]
Applying Donald’s identity (2) to the left side we obtain
\[
\sum_i \pi_i^k H(\Phi(\rho_i^k) \| \rho') = \sum_i \pi_i^k H(\Phi(\rho_i^k) \| \Phi(\bar{\rho}_k)) + H(\Phi(\bar{\rho}_k) \| \rho') \tag{12}
\]
From the two above expressions we have
\[
H(\Phi(\bar{\rho}_k) \| \rho') \leq \bar{C}(\Phi; \mathcal{A}) - \sum_i \pi_i^k H(\Phi(\rho_i^k) \| \Phi(\bar{\rho}_k))
\]
But the right side tends to zero as \( k \) tends to infinity due to the approximating property of the sequence \( \{ \pi_i^k, \rho_i^k \} \). □

Proposition 1 shows in particular that if the set of input average states for the \( \mathcal{A} \)-constrained channel \( \Phi \) is nonempty then it maps by the channel \( \Phi \) into a single state.

**Corollary 1.** If there exists input optimal average state \( \bar{\rho} \) for the \( \mathcal{A} \)-constrained channel \( \Phi \) then \( \Phi(\bar{\rho}) = \Omega(\Phi, \mathcal{A}) \).

Note that compactness of the set \( \mathcal{A} \) guarantees existence of at least one input average state.
This corollary justifies the following definition.

**Definition 1’.** The state \( \Omega(\Phi, A) \) is called output optimal average state for the \( A \)-constrained channel \( \Phi \).

There exist examples of constrained channels with finite \( \chi \)-capacity but with no input optimal average state, for which the output optimal average state is explicitly determined and plays an important role in studying of these channels.

**Corollary 2.** Let \( A \) be a convex set. Then

\[
\bar{C}(\Phi; A) \geq \chi_{\Phi}(\rho) + H(\Phi(\rho)\|\Omega(\Phi, A)) \quad \text{for arbitrary state } \rho \text{ in } A.
\]

**Proof.** It is sufficient to consider the case \( \bar{C}(\Phi; A) < +\infty \). Let \( \{\pi_i, \rho_i\} \) be an arbitrary ensemble such that \( \sum_i \pi_i \rho_i = \rho \in A \). By proposition 1

\[
\sum_i \pi_i H(\Phi(\rho_i)\|\Omega(\Phi, A)) \leq \bar{C}(\Phi; A).
\]

This inequality and Donald’s identity

\[
\sum_i \pi_i H(\Phi(\rho_i)\|\Omega(\Phi, A)) = \chi_{\Phi}(\{\pi_i, \rho_i\}) + H(\Phi(\rho)\|\Omega(\Phi, A))
\]

complete the proof. □

There exists another approach to the definition of the state \( \Omega(\Phi, A) \). It is possible to show that finiteness of the \( \chi \)-capacity of the \( A \)-constrained channel \( \Phi \) implies compactness of the set \( \overline{\Phi(A)} \).\(^2\) For arbitrary ensemble \( \{\mu_j, \sigma_j\} \) with the average \( \overline{\sigma} \in A \) consider the lower semicontinuous function

\[
F_{\{\mu_j, \sigma_j\}}(\rho') = \sum_j \mu_j H(\Phi(\sigma_j)\|\rho') \quad \text{on the set } \overline{\Phi(A)}.
\]

The function \( F(\rho') = \sup_{\sum_j \mu_j \sigma_j \in A} F_{\{\mu_j, \sigma_j\}}(\rho') \) is also lower semicontinuous on the compact set \( \overline{\Phi(A)} \) and, hence, achieves its minimum on this set. The following proposition asserts, in particular, that the state \( \Omega(\Phi, A) \) can be defined as the unique minimal point of the function \( F(\rho') \).

**Proposition 2.** Let \( A \) be a convex set. The \( \chi \)-capacity of the \( A \)-constrained channel \( \Phi \) can be expressed as

\[
\bar{C}(\Phi; A) = \inf_{\rho' \in \overline{\Phi(A)}} \left[ \sup_{\sum_j \mu_j \sigma_j \in A} \sum_j \mu_j H(\Phi(\sigma_j)\|\rho') \right].
\]

\(^2\)We give this assertion without proof since it will not be used in the strong arguments.
If $\bar{C}(\Phi; A) < +\infty$ then $\Omega(\Phi, A)$ is the only state on which the infinum in the right side is achieved.

Proof. If $\bar{C}(\Phi; A) < +\infty$ then $F(\Omega(\Phi, A)) = \bar{C}(\Phi; A)$ due to proposition 1. Let $\rho'$ be a state such that

$$\sup_{\sum_j \mu_j \sigma_j \in A} \sum_j \mu_j H(\Phi(\sigma_j) \| \rho') = F(\rho') \leq F(\Omega(\Phi, A)) = \bar{C}(\Phi; A).$$

then by proposition 1 $\rho' = \Omega(\Phi, A)$.

If $\bar{C}(\Phi; A) = +\infty$ then the right side of the expression in proposition 2 is also equal to $+\infty$. Indeed, if $\rho'$ is a state in $\mathfrak{S}(H')$ such that

$$\sup_{\sum_j \mu_j \sigma_j \in A} \sum_j \mu_j H(\Phi(\sigma_j) \| \rho') < +\infty$$

then equality (12) valid for arbitrary approximating sequence of ensembles $\{\hat{\pi}^k, \hat{\rho}^k\}$ for the $A$-constrained channel $\Phi$ implies $\bar{C}(\Phi; A) < +\infty$. □

Note that the expression for the $\chi$-capacity in the above proposition can be considered as a generalization of the "mini-max formula for $\chi^*$" in [23] to the case of an infinite dimensional constrained channel.

Remark 1. Propositions 1-2 and corollaries 1-2 does not hold without assumption of convexity of the set $A$. To show this it is sufficient to consider the noiseless channel $\Phi = \text{Id}$ and the compact set $A$, consisting of two states $\rho_1$ and $\rho_2$ such that $H(\rho_1) = H(\rho_2) < +\infty$ and $H(\rho_1 \| \rho_2) = +\infty$. In this case $\bar{C}(\Phi; A) = H(\rho_1) = H(\rho_2)$, the states $\rho_1$ and $\rho_2$ are input optimal average states in the sense of definition 1 with the different images $\Phi(\rho_1) = \rho_1$ and $\Phi(\rho_2) = \rho_2$.

4 The $\chi$-function

The function $\chi(\Phi)(\rho)$ on $\mathfrak{S}(H)$ is defined by (5). It is shown in [12] that

$$\chi(\Phi)(\rho) = \sup_{\pi \in \mathcal{P}(\rho)} \int_{\mathfrak{S}(H)} H(\Phi(\sigma) \| \Phi(\rho)) \pi(d\sigma),$$

(13)

where $\mathcal{P}(\rho)$ is the set of all probability measures on $\mathfrak{S}(H)$ with the barycenter $\rho$, and that under the condition $H(\Phi(\rho)) < +\infty$ the supremum in (13) is achieved on some measure supported by pure states.
Note that $H(\Phi(\rho)) = +\infty$ does not imply $\chi(\rho) = +\infty$. Indeed, it is easy to construct a channel $\Phi$ from a finite dimensional system into infinite dimensional one such that $H(\Phi(\rho)) = +\infty$ for any $\rho \in \mathcal{S}(\mathcal{H})$. On the other hand, by the monotonicity property of the relative entropy [18]

$$\sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\rho)) \leq \sum_i \pi_i H(\rho_i \| \rho) \leq \log \dim \mathcal{H} < +\infty$$

for arbitrary ensemble $\{\pi_i, \rho_i\}$, and hence $\chi(\rho) \leq \log \dim \mathcal{H} < +\infty$ for any $\rho \in \mathcal{S}(\mathcal{H})$.

For arbitrary state $\rho$ such that $H(\Phi(\rho)) < +\infty$ the $\chi$-function has the following representation

$$\chi(\rho) = H(\Phi(\rho)) - \hat{H}(\rho),$$

where

$$\hat{H}(\rho) = \inf_{\pi \in P(\rho)} \int_{\mathcal{S}(\mathcal{H})} H(\Phi(\rho)) \pi(d\rho) = \inf \sum_i \pi_i \rho_i = \rho \sum_i \pi_i H(\Phi(\rho_i))$$

is a convex closure of the output entropy $H(\Phi(\rho))$ (this is proved in [25]).

Note that the notion of the convex closure of the output entropy is widely used in the quantum information theory in connection with the notion of the entanglement of formation (EoF). Namely, in the finite dimensional case EoF was defined in [2] as the convex hull (=convex closure) of the output entropy of a partial trace channel from the state space of a bipartite system onto the state space of its single subsystem. In the infinite dimensional case the definition of EoF as the $\sigma$-convex hull of the output entropy of a partial trace channel is proposed in [7] while some advantages of the definition of EoF as the convex closure of the output entropy of a partial trace channel are considered in [25]. It is shown that the two above definitions coincides on the set of states with finite entropy of partial trace, but their coincidence for arbitrary state remains an open problem.

In the finite dimensional case the output entropy $H(\Phi(\rho))$ and its convex closure (=convex hull) $\hat{H}(\rho)$ are continuous concave and convex functions.

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3For example, the channel $\Phi : \rho \mapsto \frac{1}{2} \rho \oplus \frac{1}{2} \text{Tr}(\rho) \tau$, where $\tau$ is a fixed state with infinite entropy.

4Note that the second equality in (15) holds under the condition $H(\Phi(\rho)) < +\infty$, it is not valid in general (see lemma 2 in [25] and the notes below).
on $\mathcal{S}(\mathcal{H})$ correspondingly and the representation (14) is valid for all states. It follows that in this case the function $\chi_{\Phi}(\rho)$ is continuous and concave on $\mathcal{S}(\mathcal{H})$

In the infinite dimensional case the output entropy $H(\Phi(\rho))$ is only lower semicontinuous and, hence, the function $\chi_{\Phi}(\rho)$ is not continuous even in the case of the noiseless channel $\Phi$, for which $\chi_{\Phi}(\rho) = H(\Phi(\rho))$. But it turns out that the function $\chi_{\Phi}(\rho)$ for arbitrary channel $\Phi$ has properties similar to the properties of the output entropy $H(\Phi(\rho))$.

**Proposition 3.** The function $\chi_{\Phi}(\rho)$ is a nonnegative concave and lower semicontinuous function on $\mathcal{S}(\mathcal{H})$ such that

$$\chi_{\Phi}(\bar{\rho}) - \sum_{i=1}^{n} \pi_i \chi_{\Phi}(\rho_i) \geq \sum_{i=1}^{n} \pi_i H(\Phi(\bar{\rho}_i)\|\Phi(\bar{\rho}))$$

(16)

for arbitrary ensemble $\{\pi_i, \rho_i\}_{i=1}^{n}$ with the average state $\bar{\rho}$.

If the restriction of the output entropy $H(\Phi(\rho))$ to a particular subset $\mathcal{A} \subseteq \mathcal{S}(\mathcal{H})$ is continuous then the restriction of the function $\chi_{\Phi}(\rho)$ to this subset $\mathcal{A}$ is continuous as well.

**Proof.** Nonnegativity of the $\chi$-function is obvious. Let us show first its concavity. Note that for a convex set of states with finite output entropy this concavity easily follows from (14). But to prove concavity on the whole state space we will use the approach based on lemma 1 and providing inequality (16). Let $\varepsilon > 0$ be arbitrary. By definition of the $\chi$-function for each $i = 1, n$ there exists ensemble $\{\mu^i_j, \sigma^i_j\}_{j=1}^{m(i)}$ with the average $\rho_i$ such that $\chi_{\Phi}(\{\mu^i_j, \sigma^i_j\}) > \chi_{\Phi}(\rho_i) - \varepsilon$. Since the average state of the ensemble $\sum_{i=1}^{n} \pi_i \{\mu^i_j, \sigma^i_j\}$ coincides with $\bar{\rho}$, by using lemma 1 we have

$$\chi_{\Phi}(\bar{\rho}) \geq \chi_{\Phi}(\sum_{i=1}^{n} \pi_i \{\mu^i_j, \sigma^i_j\}) \geq \sum_{i=1}^{n} \pi_i \chi_{\Phi}(\{\mu^i_j, \sigma^i_j\})$$

$$\sum_{i=1}^{n} \pi_i H(\Phi(\bar{\rho}_i)\|\Phi(\bar{\rho})) \geq \sum_{i=1}^{n} \pi_i H(\Phi(\bar{\rho}_i)\|\Phi(\bar{\rho})) - \varepsilon.$$

Since $\varepsilon$ can be arbitrary small inequality (16) is established. It obviously implies concavity of the $\chi$-function.

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5This inequality can be considered as a generalization to the case of the $\chi$-function of the following well known identity for quantum entropy

$$H(\bar{\rho}) - \sum_{i=1}^{n} \pi_i H(\rho_i) = \sum_{i=1}^{n} \pi_i H(\rho_i\|\bar{\rho}).$$
To prove lower semicontinuity of the $\chi$-function we have to show
\[
\lim \inf_{n \to +\infty} \chi_{\Phi}(\rho_n) \geq \chi_{\Phi}(\rho_0).
\] (17)
for arbitrary state $\rho_0$ and arbitrary sequence $\rho_n$ converging to this state $\rho_0$.

For arbitrary $\varepsilon > 0$ let $\{\pi_i, \rho_i\}$ be an ensemble with the average $\rho_0$ such that
\[
\sum_i \pi_i H(\Phi(\rho_i)||\Phi(\rho_0)) \geq \chi_{\Phi}(\rho_0) - \varepsilon.
\]
By lemma 3 below there exists the sequence of ensembles $\{\pi_i^n, \rho_i^n\}$ of fixed size such that
\[
\lim_{n \to +\infty} \pi_i^n = \pi_i, \quad \lim_{n \to +\infty} \rho_i^n = \rho_i, \quad \text{and} \quad \rho_n = \sum_{i=1}^m \pi_i^n \rho_i^n.
\]
By definition we have
\[
\lim \inf_{n \to +\infty} \chi_{\Phi}(\rho_n) \geq \lim \inf_{n \to +\infty} \sum_i \pi_i^n H(\Phi(\rho_i^n)||\Phi(\rho_n)) \geq \sum_i \pi_i H(\Phi(\rho_i)||\Phi(\rho_0)) \geq \chi_{\Phi}(\rho_0) - \varepsilon,
\]
where lower semicontinuity of the relative entropy was used. This implies (17) (due to the freedom of the choice of $\varepsilon$).

The last assertion of proposition 3 follows from the representation (14) and from lower semicontinuity of the function $\hat{H}_{\Phi}(\rho)$ established in [23]. □

**Lemma 3.** Let $\{\pi_i, \rho_i\}$ be an arbitrary ensemble of $m$ states with the average state $\rho$ and let $\{\rho_n\}$ be an arbitrary sequence of states converging to the state $\rho$. There exists the sequence $\{\pi_i^n, \rho_i^n\}$ of ensembles of $m$ states such that
\[
\lim_{n \to +\infty} \pi_i^n = \pi_i, \quad \lim_{n \to +\infty} \rho_i^n = \rho_i, \quad \text{and} \quad \rho_n = \sum_{i=1}^m \pi_i^n \rho_i^n.
\]
**Proof.** Without loss of generality we may assume that $\pi_i > 0$ for all $i$. Let $\mathcal{D} \subseteq \mathcal{H}$ be the support of $\rho = \sum_{i=1}^m \pi_i \rho_i$ and $P$ be the projector onto $\mathcal{D}$. Since $\rho_i \leq \pi_i^{-1} \rho$ we have
\[
0 \leq A_i \equiv \rho^{-1/2} \rho_i \rho^{-1/2} \leq \pi_i^{-1} I,
\]
where we denote by $\rho^{-1/2}$ the generalized (Moore-Penrose) inverse of the operator $\rho^{1/2}$ (equal 0 on the orthogonal complement to $\mathcal{D}$).
Consider the sequence $B_i^n = \rho_i^{1/2} A_i \rho_i^{1/2} + \rho_i^{1/2} (I^H - P) \rho_i^{1/2}$ of operators in $\mathcal{B}(H)$. Since $\lim_{n \to +\infty} \rho_n = \rho = P \rho$ in the trace norm, we have

$$\lim_{n \to +\infty} B_i^n = \rho_i^{1/2} A_i \rho_i^{1/2} = \rho_i$$

in the weak operator topology. The last equality implies $A_i \neq 0$. Note that $\text{Tr} B_i^n = \text{Tr} A_i \rho_n + \text{Tr} (I^H - P) \rho_n < +\infty$ and hence

$$\lim_{n \to +\infty} \text{Tr} B_i^n = \text{Tr} A_i \rho = \text{Tr} \rho_i = 1.$$  

Denote by $\rho_i^n = (\text{Tr} B_i^n)^{-1} B_i^n$ a state and by $\pi_i^n = \pi_i \text{Tr} B_i^n$ a positive number for each $i$, then $\lim_{n \to +\infty} \pi_i^n = \pi_i$ and $\lim_{n \to +\infty} \rho_i^n = \rho_i$ in the weak operator topology and hence, by the result in [3], in the trace norm. Moreover,

$$\sum_{i=1}^m \pi_i^n \rho_i^n = \sum_{i=1}^m \pi_i B_i^n = \rho_i^{1/2} \rho^{-1/2} \sum_{i=1}^m \pi_i \rho_i \rho^{-1/2} \rho_i^{1/2} + \rho_i^{1/2} (I^H - P) \rho_i^{1/2} = \rho_n. \quad \square$$

In the modern convex analysis the notion of strong convexity (concavity) plays an essential role [20]. By using inequality (1) and proposition 3 we obtain the following observation.

**Corollary 3.** $\chi_\Phi(\rho)$ is a strongly concave function on $\mathcal{S}(H)$ in the following sense

$$\chi_\Phi(\lambda \rho_1 + (1 - \lambda) \rho_2) \geq \lambda \chi_\Phi(\rho_1) + (1 - \lambda) \chi_\Phi(\rho_2) + \frac{1}{2} \lambda (1 - \lambda) \|\Phi(\rho_2) - \Phi(\rho_1)\|^2,$$

for arbitrary $\rho_1$ and $\rho_2$ in $\mathcal{S}(H)$.

The similarity of the properties of the functions $\chi_\Phi(\rho)$ and $H(\Phi(\rho))$ is stressed by the following analog of Simon’s dominated convergence theorem for quantum entropy [28], which will be used later.

**Corollary 4.** Let $\rho_n$ be a sequence of states in $\mathcal{S}(H)$, converging to the state $\rho$ and such that $\lambda_n \rho_n \leq \rho$ for some sequence $\lambda_n$ of positive numbers, converging to 1. Then

$$\lim_{n \to +\infty} \chi_\Phi(\rho_n) = \chi_\Phi(\rho).$$

**Proof.** The condition $\lambda_n \rho_n \leq \rho$ implies decomposition $\rho = \lambda_n \rho_n + (1 - \lambda_n) \rho_n'$, where $\rho_n' = (1 - \lambda_n)^{-1} (\rho - \lambda_n \rho_n)$ is a state. By concavity of the $\chi$-function we have

$$\chi_\Phi(\rho) \geq \lambda_n \chi_\Phi(\rho_n) + (1 - \lambda_n) \chi_\Phi(\rho_n') \geq \lambda_n \chi_\Phi(\rho_n),$$

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which implies \( \limsup_{n \to +\infty} \chi_{\Phi}(\rho_n) \leq \chi_{\Phi}(\rho) \). This and lower semicontinuity of the \( \chi \)-function completes the proof. \( \square \)

**Example 1.** Let \( H' \) be a positive unbounded operator on the space \( \mathcal{H}' \) such that \( \operatorname{Tr} \exp(-\beta H') < +\infty \) for all \( \beta > 0 \) and \( h' \) be a positive number. In the proof of proposition 3 in [12] continuity of the output entropy \( H(\Phi(\rho)) \) to the subset \( A_{h'} = \{ \rho \in \mathcal{S}(\mathcal{H}) | \operatorname{Tr} \Phi(\rho)H' \leq h' \} \) was established.\(^6\) By proposition 3 the restriction of the \( \chi \)-function to the set \( A_{h'} \) is continuous. As it is mentioned in [12], the above continuity condition is fulfilled for Gaussian channels with the power constraint of the form \( \operatorname{Tr} \rho H \leq h \), where \( H = R^T \epsilon R \) is the many-mode oscillator Hamiltonian with nondegenerate energy matrix \( \epsilon \) and \( R \) are the canonical variables of the system.

We shall use the following chain properties of the \( \chi \)-function.

**Proposition 4.** Let \( \Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}') \) and \( \Psi : \mathcal{S}(\mathcal{H}') \mapsto \mathcal{S}(\mathcal{H}'') \) be two channels. Then

\[
\chi_{\Psi \Phi}(\rho) \leq \chi_{\Phi}(\rho) \quad \text{and} \quad \chi_{\Psi \Phi}(\rho) \leq \chi_{\Psi}(\Phi(\rho)) \quad \text{for arbitrary} \ \rho \ \text{in} \ \mathcal{S}(\mathcal{H}).
\]

**Proof.** The first inequality follows from the monotonicity property of the relative entropy [18] and (5), while the second one is a direct corollary of the definition (5) of the \( \chi \)-function. \( \square \)

## 5 On continuity of the \( \chi \)-capacity

In this section the question of continuity of the \( \chi \)-capacity as a function of channel is considered. Dealing with this question we must choose a topology on the set \( \mathcal{C}(\mathcal{H}, \mathcal{H}') \) of all quantum channels from \( \mathcal{S}(\mathcal{H}) \) into \( \mathcal{S}(\mathcal{H}') \). This choice is essential only in the infinite dimensional case because all locally convex Hausdorff topologies on a finite dimensional space are equivalent.

Let \( \mathcal{L}(\mathcal{H}, \mathcal{H}') \) be the linear space of all continuous linear mapping from \( \mathcal{S}(\mathcal{H}) \) into \( \mathcal{S}(\mathcal{H}') \). We will use the topology on \( \mathcal{C}(\mathcal{H}, \mathcal{H}') \subset \mathcal{L}(\mathcal{H}, \mathcal{H}') \) generated by the topology of strong convergence on \( \mathcal{L}(\mathcal{H}, \mathcal{H}') \).

**Definition 2.** The topology on the linear space \( \mathcal{L}(\mathcal{H}, \mathcal{H}') \) defined by the family of seminorms \( \{ \| \Phi \|_\rho = \| \Phi(\rho) \|_1 \}_\rho \in \mathcal{S}(\mathcal{H}) \) is called the topology of strong convergence.

\(^6\)The value \( \operatorname{Tr} \Phi(\rho)H' \) is defined as a limit of nondecreasing sequence \( \operatorname{Tr} \Phi(\rho)Q'_n H' \), where \( Q'_n \) is the spectral projector of \( H' \) corresponding to the lowest \( n \) eigenvalues [10].
Since an arbitrary operator in $\mathfrak{T}(\mathcal{H})$ can be represented as a linear combination of operators in $\mathfrak{S}(\mathcal{H})$ it is possible to consider only seminorms $\| \cdot \|_\rho$ corresponding to $\rho \in \mathfrak{S}(\mathcal{H})$ in the above definition.

Note that a sequence $\Phi_n$ of channels in $\mathcal{C}(\mathcal{H}, \mathcal{H}')$ strongly converges to a channel $\Phi \in \mathcal{C}(\mathcal{H}, \mathcal{H}')$ if and only if $\lim_{n \to +\infty} \Phi_n(\rho) = \Phi(\rho)$ for all $\rho \in \mathfrak{S}(\mathcal{H})$. Due to the result in [3] the above limit may be in the weak operator topology.

**Theorem 1.** Let $\mathcal{A}$ be an arbitrary closed and convex subset of $\mathfrak{S}(\mathcal{H})$.

In the case of finite dimensional spaces $\mathcal{H}$ and $\mathcal{H}'$ the $\chi$-capacity $\bar{C}(\Phi, \mathcal{A})$ is a continuous function on the set $\mathcal{C}(\mathcal{H}, \mathcal{H}')$. If $\Phi_n$ is an arbitrary sequence of channels in $\mathcal{C}(\mathcal{H}, \mathcal{H}')$, converging to some channel $\Phi$ in $\mathcal{C}(\mathcal{H}, \mathcal{H}')$, then there exists

$$\lim_{n \to \infty} \Omega(\Phi_n, \mathcal{A}) = \Omega(\Phi, \mathcal{A}).$$

(18)

In general the $\chi$-capacity $\bar{C}(\Phi, \mathcal{A})$ is a lower semicontinuous function on the set $\mathcal{C}(\mathcal{H}, \mathcal{H}')$ equipped with the topology of strong convergence.

**Proof.** Let us first show lower semicontinuity of the $\chi$-capacity. Let $\varepsilon > 0$ and $\Phi_\lambda$ be an arbitrary net of channels, strongly converging to the channel $\Phi$, and $\{\pi_i, \rho_i\}$ be an ensemble with the average $\bar{\rho}$ such that $\chi_{\Phi}(\{\pi_i, \rho_i\}) > \bar{C}(\Phi, \mathcal{A}) - \varepsilon$. By lower semicontinuity of the relative entropy [29]

$$\liminf_{\lambda} \sum_i \pi_i H(\Phi_\lambda(\rho_i) \| \Phi_\lambda(\bar{\rho})) \geq \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})) > \bar{C}(\Phi, \mathcal{A}) - \varepsilon$$

This implies

$$\liminf_{\lambda} \bar{C}(\Phi_\lambda, \mathcal{A}) \geq \bar{C}(\Phi, \mathcal{A}).$$

It follows that

$$\liminf_{n \to +\infty} \bar{C}(\Phi_n, \mathcal{A}) \geq \bar{C}(\Phi, \mathcal{A})$$

(19)

for arbitrary sequence $\Phi_n$ of channels strongly converging to a channel $\Phi$.

Now to prove the continuity of the $\chi$-capacity in the finite dimensional case it is sufficient to show that for the above sequence of channels

$$\limsup_{n \to +\infty} \bar{C}(\Phi_n, \mathcal{A}) \leq \bar{C}(\Phi, \mathcal{A}).$$

(20)

For an arbitrary $\mathcal{A}$-constrained channel from $\mathcal{C}(\mathcal{H}, \mathcal{H}')$ there exists optimal ensemble consisting of $m = (\dim \mathcal{H})^2$ states (probably, some states with zero

\footnote{Convexity of $\mathcal{A}$ is used only in the proof of [18].}
Let $\mathcal{P}$ be the compact space of all probability distributions with $m$ outcomes. Consider the compact space $\mathcal{P}C^m = \mathcal{P} \times \mathcal{S}(\mathcal{H}) \times \cdots \times \mathcal{S}(\mathcal{H})$, consisting of sequences $\{\pi_i^m, \rho_1, \ldots, \rho_m\}$, corresponding to arbitrary input ensemble $\{\pi_i, \rho_i\}_{i=1}^m$ of $m$ states.

Suppose (20) is not true. Without loss of generality we may assume that

$$\lim_{n \to +\infty} \bar{C}(\Phi_n, \mathcal{A}) > \bar{C}(\Phi, \mathcal{A}).$$

Let $\{\pi_i^n, \rho_i^n\}_{i=1}^m$ be an optimal ensemble for the $\mathcal{A}$-constrained channel $\Phi_n$. By compactness of $\mathcal{P}C^m$ we can choose a subsequence $\{\pi_i^{nk}, \rho_i^{nk}\}_{i=1}^m$ converging to some element $\{\pi_i^*, \rho_i^*\}_{i=1}^m$ of the space $\mathcal{P}C^m$. By definition of the product topology on $\mathcal{P}C^m$ it means that

$$\lim_{k \to \infty} \pi_i^{nk} = \pi_i^*, \quad \lim_{k \to \infty} \rho_i^{nk} = \rho_i^*.$$  

The average state of the ensemble $\{\pi_i^*, \rho_i^*\}_{i=1}^m$ is a limit of the sequence of average states of the ensembles $\{\pi_i^{nk}, \rho_i^{nk}\}_{i=1}^m$ and hence lies in $\mathcal{A}$ (which is closed by the assumption).

By continuity of the quantum entropy in finite dimensional case we have

$$\lim_{k \to +\infty} \bar{C}(\Phi_n, \mathcal{A}) = \lim_{k \to +\infty} \chi_{\Phi_n}(\{\pi_i^{nk}, \rho_i^{nk}\}) = \chi_{\Phi}(\{\pi_i^*, \rho_i^*\}) \leq \bar{C}(\Phi, \mathcal{A}),$$

which contradicts to (21).

Comparing (19) and (20) we see that

$$\lim_{n \to +\infty} \bar{C}(\Phi_n, \mathcal{A}) = \bar{C}(\Phi, \mathcal{A}).$$

It follows that the above ensemble $\{\pi_i^*, \rho_i^*\}_{i=1}^m$ is optimal for the $\mathcal{A}$-constrained channel $\Phi$. Hence, there exists the input optimal average state $\check{\rho}^*$ for the $\mathcal{A}$-constrained channel $\Phi$ which is a partial limit of the sequence $\{\check{\rho}^n\}$ of the input optimal average states for the $\mathcal{A}$-constrained channels $\Phi_n$.

Suppose (18) is not true. Without loss of generality we may (by compactness argument) assume that there exists $\lim_{n \to +\infty} \Omega(\Phi_n, \mathcal{A}) \neq \Omega(\Phi, \mathcal{A})$.

By proposition 1 this contradicts to the previous observation.  \(\square\)
The assumption of finite dimensionality in the first part of theorem 1 is essential. The following example shows that generally the $\chi$-capacity is not continuous function of a channel even in the stronger trace norm topology on the space of all channels. The example is a purely classical channel which has a standard extension to a quantum one.

**Example 2.** Consider Abelian von Neumann algebra $l_\infty$ and its predual $l_1$. Let $\{\Phi^q_n; n = 1, 2, \ldots; q \in (0, 1)\}$ be the family of classical unconstrained channels defined by the formula

$$\Phi^q_n(\{x_1, x_2, \ldots, x_n, \ldots\}) = \{(1 - q) \sum_{i=1}^\infty x_i, q \sum_{i=n+1}^\infty x_i, q x_1, \ldots, q x_n, 0, 0, \ldots\}$$

for $\{x_1, x_2, \ldots, x_n, \ldots\} \in l_1$. Defining $\Phi^0(\{x_1, x_2, \ldots, x_n, \ldots\}) = \{\sum_{i=1}^\infty x_i, 0, 0, \ldots\}$ we have

$$\|\Phi^q_n - \Phi^0(\{x_i\}_{i=1}^\infty)\|_1 = q \|\{\sum_{i=1}^\infty x_i, \sum_{i=n+1}^\infty x_i, x_1, \ldots, x_n, 0, 0, \ldots\}\|_1$$

$$= q (|\sum_{i=1}^\infty x_i| + |\sum_{i=n+1}^\infty x_i| + |x_1| + \ldots + |x_n|) \leq 3q \|\{x_i\}_{i=1}^\infty\|_1,$$

hence $\|\Phi^q_n - \Phi^0\| \to 0$ as $q \to 0$ uniformly in $n$.

To evaluate the $\chi$-capacity of the channel $\Phi^q_n$ it is sufficient to note that

$$H(\Phi^q_n(\text{any pure state})) = h_2(q) = -q \log q - (1 - q) \log(1 - q)$$

and

$$H(\Phi^q_n(\text{any state})) \leq H(\Phi^q_n(\{\frac{1}{n+1}, \frac{1}{n+1}, \ldots, \frac{1}{n+1}, 0, 0, \ldots\})) = q \log(n + 1) + h_2(q).$$

It follows by definition that $\bar{C}(\Phi^q_n) = q \log(n + 1)$, $q \in (0, 1)$, $n \in \mathbb{N}$.

Take arbitrary $C$ such that $0 < C \leq +\infty$ and choose a sequence $q(n)$ such that $\lim_{n \to \infty} q(n) = 0$ while $\lim_{n \to \infty} q(n) \log(n + 1) = C$. Then we have $\lim_{n \to \infty} ||\Phi^q_n - \Phi^0|| = 0$ but $\lim_{n \to \infty} \bar{C}(\Phi^q_n) = C > 0 = \bar{C}(\Phi^0)$. □

**Remark 2.** The above example demonstrates harsh discontinuity of the $\chi$-capacity in the infinite dimensional case. One can see that a similar discontinuity underlies Shor’s construction [27] allowing to prove equivalence of different additivity properties by using channel extension and a limiting procedure.

### 6 Additivity for constrained channels

Let $\Phi : \mathcal{G}(\mathcal{H}) \mapsto \mathcal{G}(\mathcal{H}')$ and $\Psi : \mathcal{G}(\mathcal{K}) \mapsto \mathcal{G}(\mathcal{K}')$ be two channels with the constraints, defined by closed subsets $\mathcal{A} \subset \mathcal{G}(\mathcal{H})$ and $\mathcal{B} \subset \mathcal{G}(\mathcal{K})$ correspondingly. For the channel $\Phi \otimes \Psi$ we consider the constraint defined by the
requirements \( \bar{\omega}^H := \text{Tr}_K \bar{\omega} \in A \) and \( \bar{\omega}^K := \text{Tr}_H \bar{\omega} \in B \), where \( \bar{\omega} \) is the average state of an input ensemble \( \{\mu_i, \omega_i\} \). The subset of \( \mathcal{S}(H \otimes K) \) consisting of states \( \omega \) such that \( \text{Tr}_K \omega \in A \) and \( \text{Tr}_H \omega \in B \) will be denoted \( A \otimes B \).

**Lemma 4.** The set \( A \otimes B \) is a convex subset of \( \mathcal{S}(H \otimes K) \) if and only if the sets \( A \) and \( B \) are convex subsets of \( \mathcal{S}(H) \) and of \( \mathcal{S}(K) \) correspondingly.

The set \( A \otimes B \) is a compact subset of \( \mathcal{S}(H \otimes K) \) if and only if the sets \( A \) and \( B \) are compact subsets of \( \mathcal{S}(H) \) and of \( \mathcal{S}(K) \) correspondingly.

**Proof.** The first statement of this lemma is trivial. To prove the second note that compactness of the set \( A \otimes B \) implies compactness of the sets \( A \) and \( B \) due to continuity of partial trace.

The proof of the converse implication is based on the following characterization of a compact set of states: a closed subset \( A \) of \( \mathcal{S}(H) \) is compact if and only if for any \( \varepsilon > 0 \) there exists finite dimensional projector \( P_\varepsilon \) such that \( \text{Tr} P_\varepsilon \rho > 1 - \varepsilon \) for all \( \rho \in A \). This characterization can be deduced by combining results of [21] and [3] (see the proof of the lemma in [10]). Its proof is also presented in the Appendix of [12].

Let \( A \) and \( B \) be compact. By the above characterization for arbitrary \( \varepsilon > 0 \) there exist finite rank projectors \( P_\varepsilon \) and \( Q_\varepsilon \) such that
\[
\text{Tr} P_\varepsilon \rho > 1 - \varepsilon, \quad \forall \rho \in A \quad \text{and} \quad \text{Tr} Q_\varepsilon \sigma > 1 - \varepsilon, \quad \forall \sigma \in B.
\]
Since \( \omega^H \in A \) and \( \omega^K \in B \) for arbitrary \( \omega \in A \otimes B \) we have
\[
\text{Tr}((P_\varepsilon \otimes Q_\varepsilon) \cdot \omega) = \text{Tr}((P_\varepsilon \otimes I_K) \cdot \omega) - \text{Tr}(P_\varepsilon \otimes (I_K - Q_\varepsilon)) \cdot \omega) \\
\geq \text{Tr} P_\varepsilon \omega^H - \text{Tr}(I_K - Q_\varepsilon) \omega^K > 1 - 2\varepsilon.
\]
The above characterization implies compactness of the set \( A \otimes B \). □

The conjecture of additivity of the \( \chi \)-capacity for the \( A \)-constrained channel \( \Phi \) and the \( B \)-constrained channel \( \Psi \) is [11], [12]

\[
\bar{C}(\Phi \otimes \Psi; A \otimes B) = \bar{C}(\Phi; A) + \bar{C}(\Psi; B).
\] (22)

**Remark 3.** Let \( \Omega(\Phi; A) \) and \( \Omega(\Psi; B) \) be the output optimal average states for the \( A \)-constrained channel \( \Phi \) and the \( B \)-constrained channel \( \Psi \) correspondingly. Additivity of the \( \chi \)-capacity [22] implies that \( \Omega(\Phi; A) \otimes \Omega(\Psi; B) \) is the output optimal average state for the \( A \otimes B \)-constrained channel \( \Phi \otimes \Psi \).

Indeed, let \( \{\pi_i^k; \rho_i^k\} \) and \( \{\mu_j^k; \sigma_j^k\} \) be approximating sequences of ensembles for the \( A \)-constrained channel \( \Phi \) and the \( B \)-constrained channel \( \Psi \). By proposition 1 the sequences \( \{\Phi(\rho_i^k)\} \) and \( \{\Psi(\sigma_j^k)\} \) converge to \( \Omega(\Phi; A) \) and to

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Ω(Ψ; B) correspondingly. By (22) the sequence of ensembles \( \{ \pi^k_{ij} \mu^k_{ij}; \rho^k_i \otimes \sigma^k_j \} \) is an approximating sequence for the \( A \otimes B \)-constrained channel \( \Phi \otimes \Psi \). By proposition 1 the limit \( \Omega(\Phi, A) \otimes \Omega(\Psi, B) \) of the sequence \( \{ \Phi(\bar{\rho}^k) \otimes \Psi(\bar{\sigma}^k) \} \) is the output optimal average states for the \( A \otimes B \)-constrained channel \( \Phi \otimes \Psi \).

The results of the previous sections make possible to obtain the following infinite dimensional version of theorem 1 in [11].

**Theorem 2.** Let \( \Phi : \mathcal{S}(H) \mapsto \mathcal{S}(H') \) and \( \Psi : \mathcal{S}(K) \mapsto \mathcal{S}(K') \) be arbitrary channels. The following properties are equivalent:

(i) equality (22) holds for arbitrary subsets \( A \subseteq \mathcal{S}(H) \) and \( B \subseteq \mathcal{S}(K) \) such that \( H(\Phi(\rho)) < +\infty \) for all \( \rho \in A \) and \( H(\Psi(\sigma)) < +\infty \) for all \( \sigma \in B \);

(ii) inequality

\[
\chi_{\Phi \otimes \Psi}(\omega) \leq \chi_{\Phi}(\omega^H) + \chi_{\Psi}(\omega^K) \tag{23}
\]

holds for arbitrary state \( \omega \) such that \( H(\Phi(\omega^H)) < +\infty \) and \( H(\Psi(\omega^K)) < +\infty \);

(iii) inequality

\[
\hat{H}_{\Phi \otimes \Psi}(\omega) \geq \hat{H}_\Phi(\omega^H) + \hat{H}_\Psi(\omega^K) \tag{24}
\]

holds for arbitrary state \( \omega \) such that \( H(\Phi(\omega^H)) < +\infty \) and \( H(\Psi(\omega^K)) < +\infty \).

**Proof.** (i) \( \Rightarrow \) (iii). Let \( \omega \) be an arbitrary state with finite \( H(\Phi(\omega^H)) \) and \( H(\Psi(\omega^K)) \). The validity of (i) implies

\[
\bar{C}(\Phi \otimes \Psi; \{\omega^H\} \otimes \{\omega^K\}) = \bar{C}(\Phi; \{\omega^H\}) + \bar{C}(\Psi; \{\omega^K\}).
\]

By remark 3 the state \( \Phi(\omega^H) \otimes \Psi(\omega^K) \) is the output optimal average state for the \( \{\omega^H\} \otimes \{\omega^K\} \)-constrained channel \( \Phi \otimes \Psi \). Noting that \( \omega \in \{\omega^H\} \otimes \{\omega^K\} \) and applying corollary 1 we obtain

\[
\chi_{\Phi}(\omega^H) + \chi_{\Psi}(\omega^K) = \bar{C}(\Phi; \{\omega^H\}) + \bar{C}(\Psi; \{\omega^K\})
\]

\[
= \bar{C}(\Phi \otimes \Psi; \{\omega^H\} \otimes \{\omega^K\}) \tag{25}
\]

\[
\geq \chi_{\Phi \otimes \Psi}(\omega) + H(\Phi(\omega^H) \otimes \Psi(\omega^K)).
\]

Due to

\[
H(\Phi(\omega^H) \otimes \Psi(\omega^K)) = H(\Phi(\omega^H)) + H(\Psi(\omega^K)) - H((\Phi \otimes \Psi)(\omega))
\]

the inequality (25) together with (14) implies (24).
(iii) ⇒ (ii). It can be derived from expression (14) for the χ-function and subadditivity of the (output) entropy.

(ii) ⇒ (i). It follows from the definition of the χ-capacity [3] and inequality (23) that

$$\bar{C}(\Phi \otimes \Psi; A \otimes B) \leq \bar{C}(\Phi; A) + \bar{C}(\Psi; B).$$

Since the converse inequality is obvious, there is equality here. □

The validity of inequality (23) for arbitrary ω ∈ S(H ⊗ K) seems to be substantially stronger than the equivalent properties in theorem 2. This property is called subadditivity of the χ-function for the channels Φ and Ψ. By using arguments from the proof of theorem 2 it is easy to see that subadditivity of the χ-function for the channels Φ and Ψ is equivalent to validity of equality (22) for arbitrary subsets $A \subseteq S(H)$ and $B \subseteq S(K)$.

By using proposition 6 below it is possible to show that properties (i) – (iii) in the above theorem are equivalent to subadditivity of the χ-function for the channels Φ and Ψ having the following property: $H(\Phi(\rho)) < +\infty$ and $H(\Psi(\sigma)) < +\infty$ for arbitrary finite rank states $\rho \in S(H)$ and $\sigma \in S(K)$.

We see later (proposition 7) that the set of quantum infinite dimensional channels for which the subadditivity of the χ-function holds is nontrivial.

Remark 4. By theorem 1 in [11] the subadditivity of the χ-function for arbitrary finite dimensional channels Φ and Ψ is equivalent to validity of inequality (24) for arbitrary state $\omega \in S(H \otimes K)$, which implies additivity of the minimal output entropy

$$\inf_{\omega \in S(H \otimes K)} H(\Phi \otimes \Psi(\omega)) = \inf_{\rho \in S(H)} H(\Phi(\rho)) + \inf_{\sigma \in S(K)} H(\Psi(\sigma)) \quad (26)$$

for these channels. This follows from the inequality

$$H(\Phi(\otimes \Psi(\omega)) \geq \hat{H}_{\Phi}(\omega) \geq H(\omega_H^\otimes) + H(\omega_K)$$

$$\geq \inf_{\rho \in S(H)} H(\Phi(\rho)) + \inf_{\sigma \in S(K)} H(\Psi(\sigma)) \quad (27)$$

valid for arbitrary state $\omega \in S(H \otimes K)$ for which inequality (24) holds.

In contrast to this in the infinite dimensional case we can not prove the above implication (without some additional assumptions). The problem consists in existence of pure states in $S(H \otimes K)$ with infinite entropies of partial traces, which can be called superentangled. To show this note first that the
monotonicity property of the relative entropy [18] provides the following inequality

\[ H(\omega^H) + H(\omega^K) - H(\omega) = H(\omega^H \otimes \omega^K) \geq H(\Phi \otimes \Psi(\omega)) \]

which shows that \( H(\omega^H) = H(\omega^K) < +\infty \) implies \( H(\Phi(\omega^H)) < +\infty \) and \( H(\Psi(\omega^K)) < +\infty \) for arbitrary pure state \( \omega \in \mathcal{S}(H \otimes K) \) with finite output entropy \( H(\Phi \otimes \Psi(\omega)) \). By this and theorem 2 the subadditivity of the \( \chi \)-function for arbitrary infinite dimensional channels \( \Phi \) and \( \Psi \) implies validity of inequality (24) and hence validity of inequality (27) for all pure states \( \omega \) such that \( H(\omega^H) = H(\omega^K) < +\infty \) and \( H(\Phi \otimes \Psi(\omega)) < +\infty \). So, if we considered only such states \( \omega \) in the calculation of the minimal output entropy for the channel \( \Phi \otimes \Psi \) we would obtain that it is equal to the sum of \( \inf_{\rho \in \mathcal{S}(H)} H(\Phi(\rho)) \) and \( \inf_{\sigma \in \mathcal{S}(K)} H(\Psi(\sigma)) \), but this additivity can be (probably) broken by taking into account superentangled states.

7 Generalization of the additivity conjecture

The main aim of this section is to show that the conjecture of additivity of the \( \chi \)-capacity for arbitrary finite dimensional channels \( \Phi \) and \( \Psi \) implies the additivity of the \( \chi \)-capacity for arbitrary infinite dimensional channels with arbitrary constraints.

It is convenient to introduce the following notation. The channel \( \Phi \) is

- FF-channel if \( \dim H < +\infty \) and \( \dim H' < +\infty \);
- FI-channel if \( \dim H < +\infty \) and \( \dim H' \leq +\infty \).

Speaking about quantum channel \( \Phi \) without reference to FF or FI we will assume that \( \dim H \leq +\infty \) and \( \dim H' \leq +\infty \).

Let \( \Phi : \mathcal{S}(H) \mapsto \mathcal{S}(H') \) be an arbitrary channel such that \( \dim H' = +\infty \) and \( P_n \) be a sequence of finite rank projectors in \( H' \) increasing to \( I_{H'} \) and \( H'_n = P_n(H') \). Consider the channel

\[ \Phi_n(\rho) = P'_n \Phi(\rho) P'_n + (\text{Tr}(I_{H'} - P'_n) \Phi(\rho)) \tau_n \] (28)
from $\mathcal{S}(\mathcal{H})$ into $\mathcal{S}(\mathcal{H}'_n \oplus \mathcal{H}''_n) \subset \mathcal{S}(\mathcal{H}')$, where $\tau_n$ is a pure state in some finite dimensional subspace $\mathcal{H}'_n$ of $\mathcal{H}' \ominus \mathcal{H}'_n$. If $\dim \mathcal{H}' < +\infty$ we will assume that $\Phi_n = \Phi$ for all $n$. Note that for arbitrary FI-channel $\Phi$ the corresponding channel $\Phi_n$ is a FF-channel for all $n$.

For arbitrary channel $\Psi : \mathcal{S}(\mathcal{K}) \mapsto \mathcal{S}(\mathcal{K}')$ we will consider the sequences $\Phi_n$ and $\Phi_n \otimes \Psi$ of channels as approximations for the channels $\Phi$ and $\Phi \otimes \Psi$ correspondingly. Despite the discontinuity of the $\chi$-capacity as a function of a channel in the infinite dimensional case the following result is valid.

**Lemma 5.** Let $\Phi$ and $\Psi$ be arbitrary channels. If subadditivity of the $\chi$-function holds for the channel $\Phi_n$ defined by (28) and the channel $\Psi$ for all $n$ then subadditivity of the $\chi$-function holds for the channels $\Phi$ and $\Psi$.

**Proof.** The channel $\Phi_n$ can be represented as the composition $\Pi_n \circ \Phi$ of the channel $\Phi$ with the channel $\Pi_n : \mathcal{S}(\mathcal{H}') \mapsto \mathcal{S}(\mathcal{H}'_n \oplus \mathcal{H}''_n)$ defined by

$$
\Pi_n(\rho') = P_n' \rho' P_n' + (\text{Tr}(I_{\mathcal{H}'_n} - P_n') \rho') \tau_n.
$$

Proposition 4 implies

$$
\chi_{\Phi_n}(\rho) = \chi_{\Pi_n \circ \Phi}(\rho) \leq \chi_{\Phi}(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}), \quad \forall n \in \mathbb{N}.
$$

Since

$$
\lim_{n \to +\infty} \Phi_n(\rho) = \Phi(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H})
$$

it follows from theorem 1 that

$$
\lim \inf_{n \to +\infty} \chi_{\Phi_n}(\rho) \geq \chi_{\Phi}(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}).
$$

The two above inequalities imply

$$
\lim_{n \to +\infty} \chi_{\Phi_n}(\rho) = \chi_{\Phi}(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}). \quad (29)
$$

It is easy to see that

$$
\Phi_n \otimes \Psi(\omega) = (P_n' \otimes I_{\mathcal{K}'}) \cdot (\Phi \otimes \Psi(\omega)) \cdot (P_n' \otimes I_{\mathcal{K}'})
$$

$$
+ \tau_n \otimes \text{Tr}_{\mathcal{H}'} \left( ((I_{\mathcal{H}'_n} - P_n') \otimes I_{\mathcal{K}'}) \cdot (\Phi \otimes \Psi(\omega)) \right), \quad \forall \omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}).
$$

Hence

$$
\lim_{n \to +\infty} \Phi_n \otimes \Psi(\omega) = \Phi \otimes \Psi(\omega), \quad \forall \omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K}),
$$

24
and by theorem 1 we have
\[ \liminf_{n \to +\infty} \chi_{\Phi_n \otimes \Psi}(\omega) \geq \chi_{\Phi} \otimes \Psi(\omega), \quad \forall \omega \in \mathcal{G}(\mathcal{H} \otimes \mathcal{K}). \] (30)

By the assumption
\[ \chi_{\Phi_n \otimes \Psi}(\omega) \leq \chi_{\Phi_n}(\omega^\mathcal{H}) + \chi_{\Psi}(\omega^\mathcal{K}), \quad \forall \omega \in \mathcal{G}(\mathcal{H} \otimes \mathcal{K}), \quad \forall n \in \mathbb{N}. \]

This, (29) and (30) imply
\[ \chi_{\Phi \otimes \Psi}(\omega) \leq \chi_{\Phi}(\omega^\mathcal{H}) + \chi_{\Psi}(\omega^\mathcal{K}), \quad \forall \omega \in \mathcal{G}(\mathcal{H} \otimes \mathcal{K}). \square \]

**Proposition 5.** Subadditivity of the $\chi$-function for all FF-channels implies subadditivity of the $\chi$-function for all FI-channels.

**Proof.** This can be proved by double application of lemma 5. First, we prove the subadditivity of the $\chi$-function for any two channels, when one of them is of FI-type while another is of FF-type. Second, we remove FF restriction from the last channel. $\square$

Now we will turn to channels with infinite dimensional input quantum system. We will use the following notion of subchannel.

**Definition 3.** The restriction of a channel $\Phi : \mathcal{G}(\mathcal{H}) \mapsto \mathcal{G}(\mathcal{H}')$ to the set of states with support contained in a subspace $\mathcal{H}_0$ of the space $\mathcal{H}$ is called subchannel $\Phi_0$ of the channel $\Phi$, corresponding to the subspace $\mathcal{H}_0$.

It is easy to see that subadditivity of the $\chi$-function for the channels $\Phi$ and $\Psi$ implies subadditivity of the $\chi$-function for arbitrary subchannels $\Phi_0$ and $\Psi_0$ of the channels $\Phi$ and $\Psi$. The properties of the $\chi$-function established in section 4 make possible to prove the following important result.

**Proposition 6.** Let $\Phi : \mathcal{G}(\mathcal{H}) \mapsto \mathcal{G}(\mathcal{H}')$ and $\Psi : \mathcal{G}(\mathcal{K}) \mapsto \mathcal{G}(\mathcal{K}')$ be arbitrary channels. Subadditivity of the $\chi$-function for any two FI-subchannels of the channels $\Phi$ and $\Psi$ implies subadditivity of the $\chi$-function for the channels $\Phi$ and $\Psi$.

**Proof.** It is sufficient to consider the case $\dim \mathcal{H} = +\infty$, $\dim \mathcal{K} \leq +\infty$. Let $\omega$ be an arbitrary state in $\mathcal{G}(\mathcal{H} \otimes \mathcal{K})$. Let $\{|\varphi_k\rangle\}_{k=1}^{\infty}$ and $\{|\psi_k\rangle\}_{k=1}^{\dim \mathcal{K}}$ be ONB of eigenvectors of the compact positive operators $\omega^\mathcal{H}$ and $\omega^\mathcal{K}$ such that the corresponding sequences of eigenvalues are nonincreasing. Let $P_n = \sum_{k=1}^{n} |\varphi_k\rangle\langle \varphi_k|$ and $Q_n = \sum_{k=1}^{n} |\psi_k\rangle\langle \psi_k|$. In the case $\dim \mathcal{K} < +\infty$ we will assume $Q_n = I_\mathcal{K}$ for all $n \geq \dim \mathcal{K}$. The nondecreasing sequences $\{P_n\}$ and $\{Q_n\}$ of finite rank projectors converge to $I_\mathcal{H}$ and to $I_\mathcal{K}$ correspondingly in the strong operator topology. Let $\mathcal{H}_n = P_n(\mathcal{H})$ and $\mathcal{K}_n = Q_n(\mathcal{K})$. 25
Consider the sequence of states
\[ \omega_n = (\text{Tr} \left( (P_n \otimes Q_n) \cdot \omega \right))^{-1} (P_n \otimes Q_n) \cdot \omega \cdot (P_n \otimes Q_n), \]
which are well defined for all \( n \) by the choice of the projectors \( P_n \) and \( Q_n \).

Since obviously
\[ \lim_{n \to +\infty} \omega_n = \omega \] (31)
proposition 3 implies
\[ \liminf_{n \to +\infty} \chi_{\Phi \otimes \Psi} (\omega_n) \geq \chi_{\Phi \otimes \Psi} (\omega). \] (32)

The next part of the proof is based on the following operator inequalities
\[ \lambda_n \omega_n^H \leq \omega^H, \quad \lambda_n \omega_n^K \leq \omega^K, \quad \text{where} \quad \lambda_n = \text{Tr} \left( (P_n \otimes Q_n) \cdot \omega \right). \] (33)

Let us prove the first inequality. By the choice of \( P_n \) and due to \( \text{supp} \omega_n^H \subseteq \mathcal{H}_n \) it is sufficient to show that \( \lambda_n \omega_n^H \leq P_n \omega^H \). Let \( \varphi \in \mathcal{H}_n \). By definition of partial trace
\[ \langle \varphi | \lambda_n \omega_n^H | \varphi \rangle = \sum_{k=1}^{\dim \mathcal{K}} \langle \varphi \otimes \psi_k | P_n \otimes Q_n \cdot \omega \cdot P_n \otimes Q_n | \varphi \otimes \psi_k \rangle \]
\[ = \sum_{k=1}^{m} \langle \varphi \otimes \psi_k | \omega | \varphi \otimes \psi_k \rangle \leq \sum_{k=1}^{\dim \mathcal{K}} \langle \varphi \otimes \psi_k | \omega | \varphi \otimes \psi_k \rangle = \langle \varphi | \omega^H | \varphi \rangle, \]
where \( m = \min \{ n, \dim \mathcal{K} \} \). The second inequality is proved by the same way.

By using (31) and applying corollary 4 due to (33) we obtain
\[ \lim_{n \to +\infty} \chi_\Phi (\omega_n^H) = \chi_\Phi (\omega^H) \quad \text{and} \quad \lim_{n \to +\infty} \chi_\Psi (\omega_n^K) = \chi_\Psi (\omega^K). \] (34)

For each \( n \) the \( \{ \omega_n^H \} \)-constrained channel \( \Phi \) and the \( \{ \omega_n^K \} \)-constrained channel \( \Psi \) can be considered as FI-subchannels of the channels \( \Phi \) and \( \Psi \) corresponding to the subspaces \( \mathcal{H}_n \) and \( \mathcal{K}_n \). Hence by the assumption
\[ \chi_{\Phi \otimes \Psi} (\omega_n) \leq \chi_\Phi (\omega_n^H) + \chi_\Psi (\omega_n^K), \quad \forall n \in \mathbb{N}. \]

This, (32) and (34) imply
\[ \chi_{\Phi \otimes \Psi} (\omega) \leq \chi_\Phi (\omega^H) + \chi_\Psi (\omega^K). \square \]
It is known, that additivity of the $\chi$-capacity for all unconstrained FF-channels is equivalent to subadditivity of the $\chi$-function for all FF-channels [11], [27]. By combining this with proposition 5 and proposition 6 we obtain the following extension of the additivity conjecture.

**Theorem 3.** The additivity of the $\chi$-capacity for all FF-channels implies additivity of the $\chi$-capacity for all channels with arbitrary constraints.

This theorem and theorem 2 implies the following result concerning superadditivity of the convex closure of the output entropy for infinite dimensional channels. Note that in the case of partial trace channel the convex closure of the output entropy coincides with the entanglement of formation (EoF).

**Corollary 5.** If inequality (24) holds for all FF-channels $\Phi$ and $\Psi$ and all states $\omega$ then inequality (24) holds for all channels $\Phi$ and $\Psi$ and all states $\omega$ such that $H(\Phi(\omega)) < +\infty$ and $H(\Psi(\omega)) < +\infty$.

**Proof.** The validity of inequality (24) for two FF-channels $\Phi$ and $\Psi$ and for all states $\omega$ is equivalent to subadditivity the $\chi$-function for these channels [11]. Hence the assumption of the corollary and theorem 3 imply subadditivity of the $\chi$-function for any channels, which, by theorem 2, implies the validity of inequality (24) for all channels $\Phi$ and $\Psi$ and all states $\omega$ such that $H(\Phi(\omega)) < +\infty$ and $H(\Psi(\omega)) < +\infty$.

**Remark 5.** By combining Shor’s theorem in [27] and theorem 3 we obtain that additivity of the minimal output entropy (26) for all FF-channels implies additivity of the $\chi$-capacity (22) for all channels with arbitrary constraints. But due to existence of superentangled states (see remark 4) we can not show that it implies additivity of minimal output entropy for all channels. So, in the infinite dimensional case the conjecture of additivity of the minimal output entropy for all channels seems to be substantially stronger that the conjecture of additivity of the $\chi$-capacity for all channels with arbitrary constraints.

Note that in contrast to proposition 5, proposition 6 relates the subadditivity of the $\chi$-function for the initial channels with the subadditivity of the $\chi$-function for its FI-subchannels (not any FI-channels!). This makes it applicable for analysis of individual channels as it is illustrated in the proof of proposition 7 below.

We will use the following natural generalization of the notion of entanglement breaking finite dimensional channel [15].

**Definition 4.** A channel $\Phi : \mathcal{G}(\mathcal{H}) \mapsto \mathcal{G}(\mathcal{H}')$ is called entanglement breaking if for an arbitrary Hilbert space $\mathcal{K}$ and for an arbitrary state $\omega$ in $\mathcal{G}(\mathcal{H} \otimes \mathcal{K})$ the state $\Phi \otimes \text{Id}(\omega)$ lies in the closure of the convex hull of all
product states in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$, where $\text{Id}$ is the identity channel from $\mathcal{S}(\mathcal{K})$ onto itself.

Generalizing the result in [15] it is possible to show that a channel $\Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}')$ is entanglement-breaking if and only if it admits representation

$$\Phi(\rho) = \int_X \rho'(x) \mu_\rho(dx)$$

where $X$ is a complete separable metric space, $\rho'(x)$ is a Borel $\mathcal{S}(\mathcal{H}')$-valued function on $X$ and $\mu_\rho(A) = \text{Tr}(\rho M(A))$ for any Borel $A \subset X$, with $M$ positive operator valued measure on $X$ [14].

The following proposition is a generalization of proposition 2 in [11].

**Proposition 7.** Let $\Psi$ be an arbitrary channel. The subadditivity of the $\chi$-function holds in each of the following cases:

(i) $\Phi$ is a noiseless channel;
(ii) $\Phi$ is an entanglement breaking channel;
(iii) $\Phi$ is a direct sum mixture (cf.[11]) of a noiseless channel and a channel $\Phi_0$ such that the subadditivity of the $\chi$-function holds for $\Phi_0$ and $\Psi$ (in particular, an entanglement breaking channel).

Proof. In the proof of each point of this proposition for FF-channels the finite dimensionality of the underlying Hilbert spaces was used (cf.[26],[11]). The idea of this proof consists in using our extension results (proposition 6 and lemma 5).

(i) Note that any FI-subchannel of an arbitrary noiseless channel is a noiseless FF-channel. Hence by proposition 6 it is sufficient to prove the subadditivity of the $\chi$-function for arbitrary noiseless FF-channel $\Phi$ and arbitrary FI-channel $\Psi$. But this can be done with the help of lemma 5. Indeed, using this lemma with the noiseless FF-channel in the role of the fixed channel $\Psi$ we can deduce the above assertion from the subadditivity of the $\chi$-function for arbitrary two FF-channels with one of them is a noiseless (proposition 2 in [11]).

(ii) Note that any FI-subchannel of an arbitrary entanglement breaking channel is entanglement breaking. Hence by proposition 6 it is sufficient to prove the subadditivity of the $\chi$-function for arbitrary entanglement breaking FI-channel $\Phi$ and arbitrary FI-channel $\Psi$. Similar to the proof of (i) this can be done with the help of lemma 5, but in this case it is necessary to apply this lemma twice. First we prove the subadditivity of the $\chi$-function for arbitrary entanglement breaking FI-channel $\Phi$ and arbitrary FF-channel $\Psi$ by noting
that any FF-channel $\Phi\_n$, involved in lemma 5, inherits the entanglement breaking property from the channel $\Phi$ and using the subadditivity of the $\chi$-function for arbitrary two FF-channels with one of them is an entanglement breaking \[26\]. Second, by using the result of the first step we remove the FF restriction from another channel $\Psi$.

(iii) Note that any FI-subchannel of the channel $\Phi\_q = q\mathrm{Id} \oplus (1-q)\Phi\_0$ has the same structure with FF-channel $\mathrm{Id}$ and FI-channel $\Phi\_0$. By the remark before proposition 6 subadditivity of the $\chi$-function for the channels $\Phi\_0$ and $\Psi$ implies subadditivity of the $\chi$-function for arbitrary their subchannels. Hence by proposition 6 it is sufficient to prove (iii) for FI-channel $\Phi\_q$ and FI-channel $\Psi$.

Let $\omega$ be a state in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$ with $\dim \mathcal{H} < +\infty$ and $\dim \mathcal{K} < +\infty$. It follows that $\chi_{\mathrm{Id}}(\omega^\mathcal{H}) = H(\omega^{\mathcal{H}}) < +\infty$. By the established subadditivity of the $\chi$-function for FF-channel $\mathrm{Id}$ and the FI-channel $\Psi$ and by the assumed subadditivity of the $\chi$-function for FI-channel $\Phi\_0$ and the FI-channel $\Psi$ we have

$$\chi_{\mathrm{Id} \otimes \Psi}(\omega) \leq \chi_{\mathrm{Id}}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}) \quad \text{and} \quad \chi_{\Phi\_0 \otimes \Psi}(\omega) \leq \chi_{\Phi\_0}(\omega^{\mathcal{H}}) + \chi_{\Psi}(\omega^{\mathcal{K}}).$$

Using this and lemma 3 in \[11\] we obtain

$$\chi_{q\Phi\_1 \oplus (1-q)\Phi\_2}(\{\pi_i, \rho_i\}) = q\chi_{\Phi\_1}(\{\pi_i, \rho_i\}) + (1-q)\chi_{\Phi\_2}(\{\pi_i, \rho_i\})$$

for arbitrary ensemble $\{\pi_i, \rho_i\}$ of states in $\mathcal{S}(\mathcal{H})$ and arbitrary $q \in [0; 1]$.

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\[9\]This lemma implies that for arbitrary channels $\Phi\_1$ and $\Phi\_2$ from $\mathcal{S}(\mathcal{H})$ to $\mathcal{S}(\mathcal{H}'\_1)$ and to $\mathcal{S}(\mathcal{H}'\_2)$ correspondingly one has

$$\chi_{q\Phi\_1 \oplus (1-q)\Phi\_2}(\{\pi_i, \rho_i\}) = q\chi_{\Phi\_1}(\{\pi_i, \rho_i\}) + (1-q)\chi_{\Phi\_2}(\{\pi_i, \rho_i\})$$
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