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Cite as: AIP Advances 11, 025204 (2021); https://doi.org/10.1063/5.0034753
Submitted: 22 October 2020 . Accepted: 22 December 2020 . Published Online: 01 February 2021

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ABSTRACT
We present a simple description of pion–nucleon (πN) scattering taking into account the full complexity of pion absorption and creation on the nucleon. To do this, we solve Dyson–Schwinger equations within the framework of time-ordered perturbation theory. This enables us to construct partial wave separable πN t matrices that can be useful in models of nuclear processes involving fully dressed nucleons. At the same time, our approach demonstrates features of quantum field theory, such as particle dressing, renormalization, and the use of Dyson–Schwinger equations, in a non-relativistic context that is maximally close to that of quantum mechanics. For this reason, this article may also be of pedagogical interest.

I. INTRODUCTION
From the late 1970s through to the early 1990s, much effort was devoted to modeling the coupled πNN – NN system. For practical reasons, almost all these works used the non-relativistic framework of Time-Ordered Perturbation Theory (TOPT) and implemented the so-called one-pion approximation (OPA) whereby all states containing two or more pions were neglected. The OPA allowed for the use of Faddeev-like equations for the coupled πNN – NN system but contained an inconsistency in that the nucleons in NN states were only partially dressed. In this respect, we note that in the OPA, nucleons in πNN states contain no explicit dressing at all, while the πN input to these equations (consisting of the dressed πNN vertex and a “background” πN t matrix) allows for a fully dressed nucleon in the pole-term. This inconsistency was finally resolved by relaxing the OPA and reformulating the πNN – NN equations in terms of convolution integrals, which resulted in all nucleons in the model (including the nucleons in the πN and NN input to the model) being fully dressed. Unfortunately, no numerical work on the resulting consistent formulation has been carried out so far. Here, we would like to make a modest contribution to the future solution of the convolution equations by constructing the πN t matrices that describe πN scattering data, and because all nucleons are fully dressed, it necessarily involves the solution of Dyson–Schwinger equations.

In the process of carrying out this work, it became apparent that the constructed πN equations and their solution can also be useful from a pedagogical point of view. In particular, we have in mind that the study of Relativistic Quantum Field Theory (RQFT) forms one of the most difficult challenges in a physics student’s education. Yet some of the most fundamental concepts encountered in RQFT, such as particle creation and annihilation, self-energy, renormalization, and associated equations such as the Dyson–Schwinger equations, can be easily demonstrated in the non-relativistic framework of Time-Ordered Perturbation Theory (TOPT) where little
more than a knowledge of standard quantum mechanics is needed and, in particular, where the substantial extra machinery of a relativistically covariant approach is completely avoided. The current paper is able to provide just such a demonstration on the example of a simple yet field-theoretically complete model of pion–nucleon (πN) scattering where phenomenological separable potentials and a phenomenological bare πNN vertex are used to fit experimental πN phase shifts. The equations of the model are conveniently represented in a diagrammatic form in Fig. 2 of Sec. IV and have their origin in the works that described the coupled NN – πNN system using few-body equations;[11–14] however, all these works involved the OPA where all states of two or more pions are neglected. Our innovation, therefore, is to implement a complete dressing of all nucleon propagators appearing in the model, a task that involves the numerical solution of the Dyson–Schwinger equations [as represented by Figs. 2(b)–2(d)] and which, in the process, demonstrates some of the main features of RQFT presented in a simple non-relativistic setting. To enhance the pedagogical aspects of this paper, we endeavor to give a more detailed presentation of our work than may normally be expected.

II. PION–NUCLEON MODEL

The pion–nucleon model we consider involves only pion and nucleon degrees of freedom. To take into account that pions can get created and absorbed by the nucleon, we shall use a Hilbert space \( \mathcal{H} \) that is a direct sum of subspaces, each consisting of states with a different number of pions; thus,

\[
\mathcal{H} = \mathcal{H}_N \oplus \mathcal{H}_{\pi N} \oplus \mathcal{H}_{\pi NN} \oplus \ldots .
\]

Each of these subspaces contains free particle states that shall be labeled by the momenta of the particles (to save on notation, we suppress spin and isospin labels); thus, \( |\mathbf{p}\rangle \in \mathcal{H}_N, |\mathbf{k} p\rangle \in \mathcal{H}_{\pi N}, |\mathbf{k}_1 k_2 p\rangle \in \mathcal{H}_{\pi NN}, \ldots \), where \( \mathbf{p} \) is the momentum of the nucleon and \( \mathbf{k}, \mathbf{k}_1, \mathbf{k}_2, \ldots \) are pion momenta. This multi-pion plus one-nucleon system is then described within a second quantization approach[4] where the Hamiltonian \( H \) is expressed in terms of annihilation and creation operators, \( a_\pi, a_\pi^\dagger \) for pions and \( a_N, a_N^\dagger \) for nucleons, defined such that

\[
|\mathbf{k}\rangle = a_\pi^\dagger (\mathbf{k}) |0\rangle, \quad |\mathbf{p}\rangle = a_N^\dagger (\mathbf{p}) |0\rangle,
\]

where |0\rangle is the vacuum state, and satisfying the commutation and anti-commutation relations

\[
\{a_\pi (\mathbf{k}), a_\pi^\dagger (\mathbf{k'})\} = \delta (\mathbf{k} - \mathbf{k'}), \quad \{a_N (\mathbf{k}), a_N^\dagger (\mathbf{k'})\} = 0,
\]

\[
\{a_N (\mathbf{p}), a_N^\dagger (\mathbf{p'})\} = \delta (\mathbf{p} - \mathbf{p'}), \quad \{a_\pi (\mathbf{p}), a_\pi^\dagger (\mathbf{p'})\} = 0.
\]

In particular, the free Hamiltonians for pions and nucleons are given by

\[
H_\pi^0 = \int \omega_\pi (\mathbf{k}) \, a_\pi^\dagger (\mathbf{k}) a_\pi (\mathbf{k}) \, d\mathbf{k},
\]

\[
H_N^0 = \int (E_N (\mathbf{p}) + m_0) \, a_N^\dagger (\mathbf{p}) a_N (\mathbf{p}) \, d\mathbf{p},
\]

where

\[
\omega_\pi (\mathbf{k}) = \left( \mathbf{k}^2 + m_{\pi}^2 \right)^{1/2}, \quad E_N (\mathbf{p}) = \frac{\mathbf{p}^2}{2m},
\]

with \( m_\pi \) being the mass of the pion and \( m \) being the mass of the nucleon. Note that for simplicity of presentation, we have used non-relativistic kinematics for the nucleon and relativistic kinematics for the pion (necessitated by the relatively small mass of the pion). Such a "semi-relativistic" choice of kinematics should be sufficiently accurate to describe πN scattering for the center of mass (c.m.) energies below \( m + 3m_\pi \), being approximately the experimental onset of inelasticity, and will provide a convenient connection to related works using such kinematics.\[12,13\]

Because the interactions that we shall use contribute to the physical mass \( m \) of a nucleon, the rest mass used for the nucleon in Eq. (4b), the so-called bare mass \( m_0 \), needs to be chosen appropriately (by contrast, our interactions do not contribute to the physical mass of the pion). We note that the free particle states are eigenstates of the total free Hamiltonian \( H_0 \),

\[
H_0 = H_\pi^0 + H_N^0
\]

so that

\[
H_0 |\mathbf{p}\rangle = \omega_\pi (\mathbf{p}) |\mathbf{p}\rangle, \quad H_0 |\mathbf{p}\rangle = (E_N (\mathbf{p}) + m_0) |\mathbf{p}\rangle,
\]

\[
H_0 |\mathbf{k}_1 k_2 \ldots p\rangle = [\omega_\pi (\mathbf{k}_1) + \omega_\pi (\mathbf{k}_2) + \ldots + E_N (\mathbf{p}) + m_0] |\mathbf{p}\rangle.
\]

The full Hamiltonian is then specified as

\[
H = H_0 + H_I,
\]

where \( H_I \) is the interaction Hamiltonian. We assume that \( H_I \) is due to strong interactions only; thus, parity is conserved in our model. In this work, we shall assume a model interaction Hamiltonian of the form

\[
H_I = \int J_N (\mathbf{k}) \, a_\pi^\dagger (\mathbf{k}) \, d\mathbf{k} + \int J_N^\dagger (\mathbf{k}) \, a_\pi (\mathbf{k}) \, d\mathbf{k},
\]

where

\[
J_N (\mathbf{k}) = \int \delta (\mathbf{p} - \mathbf{k}) \, \frac{1}{\sqrt{2m}} \, F_\pi (\mathbf{p}, \mathbf{p'}) a_\pi (\mathbf{p}) a_\pi^\dagger (\mathbf{p'}) \, d\mathbf{p} \, d\mathbf{p'},
\]

\[
J_N^\dagger (\mathbf{k}) = \int \delta (\mathbf{p} - \mathbf{k}) \, \frac{1}{\sqrt{2m}} \, F_\pi (\mathbf{p'}, \mathbf{p}) a_\pi^\dagger (\mathbf{p'}) a_\pi (\mathbf{p}) \, d\mathbf{p} \, d\mathbf{p'}.
\]

In the above (and below), \( \omega_\pi \) is to be understood as meaning \( \omega_\pi (\mathbf{k}) \). This interaction Hamiltonian describes the elementary processes \( \pi N \leftrightarrow N \) whose amplitude is given by the real function \( F_\pi (\mathbf{p}, \mathbf{p'}) = F^\ast (\mathbf{p}, \mathbf{p'}) \) and which we shall refer to as the bare \( \pi NN \) vertex. It will be assumed that \( F_\pi (\mathbf{p}, \mathbf{p'}) \) is invariant under rotations and under space inversion so that angular momentum and parity are conserved in the resulting description of \( \pi N \) scattering.

With the model thus defined, we can now define the Green function \( G \) that shall be used to describe pion–nucleon scattering as

\[
\delta (\mathbf{k} + \mathbf{p} - \mathbf{k'} - \mathbf{p'}) G (\mathbf{p'}, \mathbf{k}, \mathbf{p}, E) = \left( \mathbf{k'}/\sqrt{E + m_\pi} \right) \delta (\mathbf{k} + \mathbf{p} - \mathbf{k'} - \mathbf{p'}).\n\]
In a similar way, we define the single nucleon Green function $g$, also called the dressed nucleon propagator, as

$$
\delta(p - p')g(E, p) = \langle p' \bigg| \frac{1}{E' - H} \bigg| p \rangle.
$$

(12)

the bare nucleon propagator $g_0$ as

$$
\delta(p - p')g_0(E, p) = \langle p' \bigg| \frac{1}{E' - H_0} \bigg| p \rangle.
$$

(13)

and the Green function for the process $N \to \pi N, \tilde{f}$, as

$$
\delta(p' + k' - p) \tilde{f}(p', p, E) = \langle k' p' \bigg| \frac{1}{E' + H} \bigg| p \rangle.
$$

(14)

To help evaluate these Green functions, it is useful to note the commutation relations

$$
[a_\pi(k), H_0] = w_\pi(k) a_\pi(k),
$$

(15a)

$$
[a_\pi(k), H_1] = J_N(k),
$$

(15b)

which follow from Eqs. (3), (4), and (9). In a similar way, it follows that

$$
I_N(k) |p'\rangle = \frac{1}{\sqrt{\omega_k}} F_0(p, p') |p\rangle,
$$

(16a)

$$
|p'\rangle I_N^\dagger(k) = \langle p | \frac{1}{\sqrt{\omega_k}} F_0(p', p),
$$

(16b)

where $p + k = p'$. Thus,

$$
H_1 |p'\rangle = \int \delta(p + k - p') \frac{1}{\sqrt{\omega_k}} F_0(p, p') |k p\rangle \, dk \, dp,
$$

(17a)

$$
|p'\rangle H_1 = \int \delta(p + k - p') |k p\rangle \frac{1}{\sqrt{\omega_k}} F_0(p', p) \, dk \, dp.
$$

(17b)

III. TIME-ORDERED PERTURBATION THEORY

Despite the relative simplicity of the quantum field theoretic pion–nucleon model introduced above, there is no known way to calculate any of the Green functions of Eqs. (11) and (12), or (14) exactly. We therefore follow the usual path taken in both quantum mechanics and quantum field theory and introduce a perturbation expansion. In our case, this means expanding the Green function operator $1/(E' - H_1)$ into a series around the free Green function operator $1/(E' - H_0)$,

$$
\frac{1}{E' - H} = \frac{1}{E' - H_0} + \frac{1}{E' - H_0} H_1 \frac{1}{E' - H_0} + \frac{1}{E' - H_0} H_1 \frac{1}{E' - H_0} H_1 \frac{1}{E' - H_0} + \ldots.
$$

(18)

This expansion defines time-ordered perturbation theory (TOPT) and has the feature that momentum matrix elements of each term of the expansion can be represented graphically by a "perturbation diagram." For example, if one evaluates this expansion for the matrix element defining the dressed nucleon propagator $g(E)$ of Eq. (12), one obtains a series whose graphical representation is given in Fig. 1.

The diagrammatic rules that relate these diagrams to mathematical expressions can be found by evaluating the momentum matrix elements of just the first three terms of Eq. (18). With the help of Eq. (17), one obtains

$$
\delta(p - p')g(E, p) = \langle p' \bigg| \frac{1}{E' - H_0} \bigg| p \rangle + \langle p' \bigg| \frac{1}{E' - H_0} H_1 \frac{1}{E' - H_0} \bigg| p \rangle + \ldots
$$

$$
= \delta(p - p')g_0(E - E_p) + \delta(p - p')g_0(E, p') \int \frac{1 \times f_0(p', k')}{E' - E_N(k'')} \frac{1 \times f_0(k''', p)}{E' - E_N(k''')} \frac{1 \times f_0(k''''', p)}{E' - E_N(k''''')} \, dk \, dp + \ldots,
$$

(19)

where $p'' = p - k$ so that

$$
g(E, p) = g_0(E, p) + g_0(E, p) \int \tilde{f}_0(p, k) \, dk \, dp
$$

$$
\times \frac{1}{E' - E_N(k''')} f_0(k''', p) g_0(E, p) \, dk + \ldots,
$$

(20)

where

$$
f_0(kp'', p) = \frac{1}{\sqrt{\omega_k}} F_0(kp'', p),
$$

(21a)

$$
\tilde{f}_0(kp', p''') = \frac{1}{\sqrt{\omega_k}} \tilde{F}_0(kp', p''').
$$

(21b)

Comparing Fig. 1 and Eq. (20), it is evident that a solitary solid line corresponds to the bare nucleon propagator $g_0$, every open-circle $N \to \pi N (\pi N \to N)$ vertex in Fig. 1 corresponds to the bare vertex function $f_0 (\tilde{f}_0)$ defined in Eq. (21), every intermediate state of solid (nucleon) and dashed (pion) lines corresponds to a propagator whose mathematical expression is given by the appropriate matrix elements of $1/(E' - H_0)$, and every closed loop corresponds to an integral over the pion momentum.
IV. PION–NUCLEON EQUATIONS

As the Green function $G(p',k,p,E)$ describing $\pi N$ scattering can be equated to a complete sum of perturbation diagrams, this provides the opportunity to rearrange this sum so as to express it completely in terms of useful quantities such as potentials, $t$ matrices, and other Green functions, thereby generating scattering equations of a similar nature to those found in quantum mechanics (e.g., the Lippmann–Schwinger equation). Such a rearrangement leads to a set of coupled equations for $\pi N$ scattering illustrated in Fig. 2. First derived by Mizutani and Koltun using Feshbach projection operators, these equations were later derived in the same context of TOPT as used here and also in the context of RQFT. Here, we shall give a brief derivation following the arguments used in Ref. 4.

We start by writing the Green function $G(p',k,p,E)$ in the operator form as

$$G(E) = G_0(E) + G_0(E)t(E)G_0(E),$$

where $G_0(E)$ is the “dressed $\pi N$ propagator” consisting of all the disconnected diagrams of $G(E)$ and $t(E)$ is the $\pi N \rightarrow \pi N$ $t$ matrix, which is defined by this equation. Note that each quantity in Eq. (22) is an operator acting in subspace $H_{\pi N}$, with $G(E)$ being specifically defined such that

$$(k'p'\mid G(E)\mid kp) = \left(k'p'\right)\frac{1}{E' - H}\left(kp\right).$$

It is evident that the term $G_0(E)t(E)G_0(E)$ in Eq. (22) consists of all possible connected $\pi N \rightarrow \pi N$ diagrams and that $t(E)$ consists of the same diagrams but with no attached initial- and final-state $\pi N$ propagators, colloquially referred to as diagrams with “chopped legs.”

Further progress can be made by defining a “background” $\pi N$ $t$ matrix $t^{(1)}(E)$ as the sum of all diagrams of $t(E)$ that have one or

$$t(E) = f(E)g(E)f'(E) + t^{(1)}(E),$$

where $f(E) (\tilde{f}(E))$ is the “dressed vertex” consisting of all possible $N \rightarrow \pi N (\pi N \rightarrow N)$ chopped-leg diagrams with at least one pion in every intermediate state. Similarly, one can define $t$ matrix $t^{(2)}(E)$ as the sum of all diagrams of $t(E)$ that have two or more pions in every intermediate state, in which case we can write Lippmann–Schwinger-like equations

$$t^{(1)}(E) = v(E) + v(E)G_0(E)t^{(1)}(E)$$
$$= v(E) + tf_0(E)v(E),$$

where $v(E) \equiv t^{(2)}(E)$ plays the role of a $\pi N \rightarrow \pi N$ potential. Using a similar argument, one can obtain the equations

$$f(E) = f_0(E) + t^{(1)}(E)G_0(E)f_0(E),$$
$$\tilde{f}(E) = \tilde{f}_0(E) + \tilde{f}_0(E)G_0(E)t^{(1)}(E),$$

where $f_0(E) \equiv f^{(2)}(E)$ and $\tilde{f}_0(E) \equiv \tilde{f}^{(2)}(E)$ is the “bare vertex” consisting of all possible $N \rightarrow \pi N (\pi N \rightarrow N)$ chopped-leg diagrams with at least two pions in every intermediate state. Finally, one can similarly write

$$g(E) = g_0(E) + g_0(E)\Sigma(E)g(E),$$

where

$$\Sigma(E) = \tilde{f}_0(E)G_0(E)f(E)$$
$$= \tilde{f}(E)G_0(E)f_0(E)$$

is the nucleon “self-energy” or “dressing” term consisting of all diagrams of $g(E)$ with at least one pion in every intermediate state but...
with chopped legs. The set of equations consisting of Eqs. (24)–(28) are illustrated in Fig. 2 and provide an exact and useful way of expressing the \( \pi N \) t matrix \( t(E) \).

In the context of RQFT, Eq. (27) [illustrated in Fig. 2(d)] is known as the Dyson equation, while the coupled set of equations [Eqs. (25a), Eq. (26a), and Eq. (27)], illustrated in Figs. 2(b)–2(d), are known as the Dyson–Schwinger (DS) equations, and that is how we shall refer to the TOPT versions of these equations here. A feature of the Dyson equation is the fact that the dressed nucleon propagator \( g \) is expressed in terms of the self-energy term \( \Sigma \), which itself is expressed in terms of \( g \) via the \( \pi N \) propagator \( G_0 \). Such self-referencing also occurs for the background \( t \) matrix \( t^{(1)} \) and the dressed vertex \( f \) in the DS equations, a feature that makes these equations embody a lot of physics even in the case where the bare vertex \( f_0 \), isotopic invariant \( \nu \) and background potential \( v \) are modeled phenomenologically, as will be the case in Sec. V.

V. SOLVING THE DYSON–SCHWINGER EQUATIONS

Here, we shall follow an often used procedure where the bare \( \pi NN \) vertex \( f_0 \) and the “background” \( \pi N \) potential \( v \) are modeled by energy-independent parameterized phenomenological functions; however, unlike in all such previous models,\(^{1,2,4,6,8,13}\) we shall not be using the approximation where the exact \( \pi N \) propagator \( G_0(E,k,p) \), defined as

\[
\delta(k'-k)\delta(p'-p)G_0(E,k,p) = \left(k'p\right)\frac{1}{E^+ - H}\left(kp\right)_{\text{disc}},
\]

is modeled by the pole term \( 1/[E^+ - E_0(p) - m - \omega_1(k)] \); rather, we shall retain its full exact form, which in the model specified by the Hamiltonian of Eqs. (4) and (9) is given by

\[
G_0(E,k,p) = g[E - E_0(p) - \omega_1(k)].
\]

As mentioned above, it is this exact form for \( G_0 \) that gives the DS equations the property of retaining a rich amount of physics despite what may be lost by taking phenomenological forms for \( f_0 \) and \( v \).

A. Partial wave equations

It is convenient to solve the Dyson–Schwinger set of equations, given in the operator form in Eqs. (25)–(28), in the center of mass (c.m.) of the \( \pi N \) system so that \( k + p = k' + p' = 0 \), in which case Eq. (30) can be expressed as

\[
G_0(E,k) = g(E - \omega_1),
\]

where

\[
\omega_1 = E_0(k) + \omega_1(k).
\]

In order to reduce the dimension of these equations from 3 to 1, we shall work in the partial wave representation using the basis states

\[
|s_\nu j_\nu m_\nu k_\nu \rangle = \sum_{m_1 m_2 \cdots m_t} \langle l m_\nu m_1 m_2 \cdots m_t | j_\nu m_\nu \rangle t_1 m_1 t_2 m_2 | \cdots | t_m m_t k_\nu \rangle 
\]

\[
\times \int dk_\nu Y_{l_\nu m_\nu}(k_\nu) |t_1 m_1 t_2 m_2 | \cdots | t_m m_t k_\nu \rangle,
\]

where \( t_1 = 1 \) is the isospin of the pion, \( t_2 = 1/2 \) is the isospin of the nucleon, \( s = 1/2 \) is the spin of the nucleon, \( l \) specifies the \( \pi N \) relative orbital angular momentum, \( t \) is the total isospin, and \( j \) is the total angular momentum. By construction, the model Hamiltonian of Eq. (8) is invariant under rotations, which implies that all matrix elements using the above partial wave basis states will not depend on the magnetic quantum numbers \( m_l \) and \( m_t \). Similarly, the model Hamiltonian is chosen to be invariant under space inversion, thus ensuring parity is conserved in our model, which, in turn, leads to \( \pi N \) partial wave matrix elements that preserve the value of \( l \). We shall refer to the partial wave specified by the quantum numbers \( \{ j\ell \} \) using the usual (for \( \pi N \) scattering) spectroscopic notation of the form “\( L_{j\ell} \).” Because the nucleon has quantum numbers \( t = j = 1/2 \) and the pion has an intrinsic parity of \(-1\), it follows that the first term on the right hand side (RHS) of Eq. (24), the so-called nucleon pole term, contributes only in the \( P_{11} \) partial wave. Likewise, the background \( \pi N \) t matrix \( t^{(1)} \) appearing in the expression for the dressed \( \pi NN \) vertices of Eq. (26) is the one in the \( P_{11} \) partial wave. Thus, restricting the discussion to \( \pi N \) scattering in the \( P_{11} \) partial wave, we can write the numerical form of the partial wave equations corresponding to Fig. 2 as

\[
t(k', k, E) = f(k, E)g(E)f(k, E) + t^{(1)}(k', k, E),
\]

\[
f^{(1)}(k', k, E) = v(k', k) + \int_0^\infty dq q^2 v(k', q)g(E - \omega_q)t^{(1)}(q, k, E),
\]

\[
f(k, E) = f_0(k) + \int_0^\infty dq q^2 f_0(q)g(E - \omega_q)f(q, E),
\]

\[
g(E) = \frac{1}{E^+ - m_0 - \Sigma(E)},
\]

\[
\Sigma(E) = \int_0^\infty dq q^2 f_0(q)g(E - \omega_q)f(q, E),
\]

where partial wave labels have been omitted to save on notation. For scattering in partial waves other than \( P_{11} \), essentially the same equations would apply, the only differences being that the pole term in Eq. (34a) would not appear, and one would need to distinguish the \( t^{(1)} \) appearing in Eq. (34a) from the \( P_{11} \) partial wave \( t^{(1)} \) appearing in Eq. (34c).

In order to help solve these equations numerically, it is useful to know the analytic structure of the dressed nucleon propagator \( g(E) \). As a basic requirement of the theory, \( g(E) \) must have a pole at the physical nucleon mass \( m \). To ensure this, the bare mass is set to the value \( m_0 = m - \Sigma(m) \) so that

\[
g(E) = \frac{1}{E^+ - m - \Sigma(E) + \Sigma(m)},
\]

and therefore,

\[
g(E) \sim \frac{Z}{E^+ - m},
\]

where \( Z \) is the wave function renormalization constant given by

\[
Z = \frac{1}{1 - \Sigma(m)}.
\]
In 1967, Varma used extensively to investigate effects of pion absorption, π production, and dibaryons, 12,17–20 three-body forces, 21 pion photoproduction, 22,23 and dibaryons, 24 and also to facilitate solutions of relativistic equations. 12,13,15–20 The motivation for their widespread use is their convenience, often leading to significant simplifications in both analytic and numerical work. Although short-range interactions are naturally separable at energies close to resonance poles, they can also be well approximated through the use of low-rank separable potentials at energies away from scattering poles, as long as the underlying realistic potential is energy independent. 25–33 For πN scattering at the considered energies (0 < Tπ < 390 MeV), much of the low-energy energy dependence of the underlying interaction is due to the nucleon pole term, which only contributes to the P11 partial wave. As we treat the pole term separately, it is reasonable to assume that the P11 background term and the rest of the partial wave interactions can be well approximated with separable potentials. 34 Nevertheless, the neglect of residual energy dependence and the lack of crossing symmetry in our model suggests that the separable potentials developed in this work would be most useful as input to calculations that do not rely critically on the accuracy of the off-shell behavior of the πN interaction.

In the case of a separable potential, Eq. (34b) can be solved algebraically, giving also a separable form for the background πN t matrix,

$$t^{(1)}(k', k, E) = h(k') \tau(E) h(k),$$

where

$$\tau(E) = \left[ 1 - \lambda \int_0^\infty dq q^2 h(q) g(E - \omega_q) h(q) \right]^{-1}.$$

Defining the four dressing terms

$$\Sigma_{ij}(E) = \int_0^\infty dq q^2 \phi_i(q) g(E - \omega_q) \phi_j(q) \quad (i, j = 1, 2),$$

where $g(E)$ and its imaginary part will prove very useful in the numerical solution of the DS equations. This expression for $g(E)$ is also convenient for the evaluation of Eq. (38) as

$$g(E) = \frac{Z}{E^+ - m} - \frac{1}{\pi} \int_{m_+ m_-}^{\infty} \frac{\text{Im}(\omega)}{(E^+ - \omega)^2} d\omega.$$

As we shall see, this relationship between $g(E)$ and its imaginary part will prove very useful in the numerical solution of the DS equations. This expression for $g(E)$ is also convenient for the evaluation of Eq. (38) as

$$g'(E) = -\frac{Z}{(E^+ - m)^2} + \frac{1}{\pi} \int_{m_+ m_-}^{\infty} \frac{\text{Im}(\omega)}{(E^+ - \omega)^2} d\omega.$$

### B. Separable potential model

To keep this model as simple as possible, we choose a separable form for the partial wave potential $v$,

$$v(k', k) = h(k') \lambda h(k),$$

where $h(k)$ is a phenomenological form factor and $\lambda$ specifies the sign of the potential (for the $P_{11}$ partial wave under consideration here, $\lambda = -1$). The use of separable potentials to describe the strong interactions of a pion and a nucleon has a long and rich history. In 1967, Varma used such potentials to perform the first three-body calculation of the πNN system; since then, they have been used extensively to investigate effects of pion absorption, 12,13,15–20 three-body forces, 21 pion photoproduction, 22,23 and dibaryons, 24 and also to facilitate solutions of relativistic equations. 25–26 The motivation for their widespread use is their convenience, often leading to significant simplifications in both analytic and numerical work. Although short-range interactions are naturally separable at energies close to resonance poles, they can also be well approximated through the use of low-rank separable potentials at energies away from scattering poles, as long as the underlying realistic potential is energy independent. 25–33 For πN scattering at the considered energies (0 < Tπ < 390 MeV), much of the low-energy energy dependence of the underlying interaction is due to the nucleon pole term, which only contributes to the $P_{11}$ partial wave. As we treat the pole term separately, it is reasonable to assume that the $P_{11}$ background term and the rest of the partial wave interactions can be well approximated with separable potentials. 34 Nevertheless, the neglect of residual energy dependence and the lack of crossing symmetry in our model 35 suggests that the separable potentials

![FIG. 3. (a) Real and (b) imaginary parts of $g^k(E)(E - m)$ where $g^k(E) = g(E)/Z$ is the renormalized dressed nucleon propagator. The solid curves are for the Dyson propagator (resulting from the solution of the DS equations), while the dashed curves are for the non-Dyson propagator (resulting from the use of the one-pion approximation).](image-url)
where \( \phi_1(q) \equiv f_0(q) \) and \( \phi_2(q) \equiv h(q) \), the DS equations can be conveniently expressed as the set of three equations,

\[
\begin{align}
    g(E) &= \frac{1}{E^2 - m^2 - \Sigma(E) + \Sigma(m)}, \quad (45a) \\
    \Sigma(E) &= \Sigma_{11}(E) + \Sigma_{12}(E) \tau(E) \Sigma_{21}(E), \quad (45b) \\
    \tau(E) &= [1 - \lambda \Sigma_{22}(E)]^{-1}, \quad (45c)
\end{align}
\]

which determine the dressed nucleon propagator \( g(E) \), together with the additional two equations

\[
\begin{align}
    t(k', k) &= f(k, E) g(E) \tilde{f}(k, E) + h(k') \tau(E) h(k), \quad (46a) \\
    f(k, E) &= f_0(k) + h(k) \tau(E) \Sigma_{21}(E), \quad (46b)
\end{align}
\]

that determine the consequent dressed \( \pi NN \) vertex \( f \) and \( \pi N \) t matrix \( t \).

**C. Numerical procedure**

In our approach, modeling \( \pi N \) scattering with Eq. (34) begins by choosing parametrized analytic functions for the form factors \( f_0(k) \) and \( h(k) \). These form factors need to fulfill the requirement of providing a momentum cutoff that ensures finite values for the integrals defining the \( \Sigma \) functions of Eq. (44), and they must behave linearly with \( k \) in the limit of low momenta in order to be consistent with the \( l = 1 \) nature of a \( P_{11} \) partial wave amplitude. We shall follow a previous work where separable potentials were used to model \( \pi N \) scattering and choose the following analytic forms:\(^{13}\)

\[
\begin{align}
    f_0(k) &= \frac{k C_0}{\sqrt{\omega_{h}(k)} (k^2 + \Lambda^2)^{\beta_0}}, \quad (47a) \\
    h(k) &= \frac{k C_1}{\sqrt{\omega_{h}(k)} \left[ (k^2 + \beta_1^2)^{\eta_1} + \frac{C_2 k^{2\beta_2}}{(k^2 + \beta_2^2)^{\eta_2}} \right]}, \quad (47b)
\end{align}
\]

where \( C_0, C_1, C_2, \beta_0, \beta_1, \beta_2, \Lambda \) are free parameters and the powers \( \eta_0, \eta_1, \eta_2, \eta_3 \) are integers that can be chosen to change the functional form of the form factors.

For any given set of parameters, the first task is to solve the DS equations in the form of Eq. (45) for the dressed nucleon propagator \( g(E) \). We do this by following an iterative procedure where an approximation to \( g(E) \) is used in Eq. (44) to calculate all the \( \Sigma \) functions, which are then used to calculate a new (and hopefully more accurate) version of \( g(E) \) using Eq. (45). The process is repeated until convergence for \( g(E) \) is achieved. By construction, the resulting propagator \( g(E) \) satisfied the DS equations and can then be used to generate the dressed vertex \( f \) and the \( \pi N t \) matrix \( t \) using Eq. (46).

**1. Technical aspects**

To carry out the integral in Eq. (44) numerically, we use Gaussian quadratures, and to avoid the singularity coming from the pole of \( g(E) \), we rotate the integration contour from the positive real axis into the fourth quadrant of the complex \( q \) plane. However, a practical problem remains in carrying out these integrals because in order to generate a propagator \( g(E) \) at any iteration, one needs to

---

**TABLE I.** Parameters of four fits (labeled as models M9, M8, M7, and M6) to the \( P_{11} \pi N \) phase shifts through the solution of the Dyson–Schwinger equations of Eq. (34). The first nine parameters refer to the form factors of Eq. (47), while \( m_0 \) is the bare nucleon mass and \( Z \) is the nucleon wave function renormalization constant.

| Potential | \( n_0 \) | \( n_2 \) | \( n_3 \) | \( \Lambda \) (fm\(^{-1}\)) | \( \beta_0 \) (fm\(^{-1}\)) | \( \beta_1 \) (fm\(^{-1}\)) | \( C_0 \) | \( C_1 \) | \( C_2 \) | \( m_0 \) (fm\(^{-1}\)) | \( Z \) |
|-----------|--------|--------|--------|----------------|----------------|----------------|--------|--------|--------|----------------|--------|
| M9        | 1      | 2      | 3      | 1.719 91  | 1.141 62     | 1.881 54       | 0.64225 | 0.238 08 | 8.9632 | 5.002 89       | 0.900 54 |
| M8        | 1      | 2      | 3      | 2.723 29  | 1.268 78     | 1.783 28       | 1.1778  | 0.329 23 | 6.2150 | 5.335 79       | 0.793 69 |
| M7        | 1      | 2      | 3      | 4.049 94  | 1.417 6      | 1.772 46       | 0.422 71 | 4.8618  | 5.692 6 | 0.694 81       |        |
| M6        | 1      | 2      | 3      | 10.802 3  | 1.627 18     | 1.868 4        | 4.7174  | 0.582 64 | 3.5638 | 6.295 40       | 0.600 27 |

**FIG. 4.** (a) Real and (b) imaginary parts of \( g^0(E)(E - m) \) where \( g^0(E) \) is the renormalized dressed Dyson nucleon propagator for the four models specified in Table I.
know the previous iteration’s propagators \(g(E - \omega_q)\) for each of the rotated quadrature points \(q_i\). Thus, the number of energies at which one needs to know \(g\) quickly escalates as the iteration proceeds. To get around this problem, we make use of the fact that the dressed nucleon propagator \(g(E)\) at each step of the iteration, satisfies the dispersion relation of Eq. (39). This allows us to evaluate Eq. (44) as

\[
\Sigma_{ij}(E) = \int_0^{\infty} dq \frac{q^2 \phi_i(q) \phi_j(q)}{E^+ - \omega_q - m} \left(1 - \frac{1}{\pi} \int_{m+\omega_q}^{\infty} d\omega \text{Im}(g(\omega)) \int_0^{\infty} dq q^2 \phi_i(q) \phi_j(q)ight),
\]

which requires knowledge of \(g(E)\) only at a number of fixed values of \(E\) corresponding to the Gaussian integration points \(\omega_i\) used to evaluate the \(\omega\) integral in Eq. (48). The iterative process thus proceeds according to the following steps:

1. For any given set of form factor parameters, begin the iteration by generating the “non-Dyson” dressed nucleon propagator \(g^{(0)}(E)\) defined by Eq. (45) but where the nucleon in the \(\pi N\) propagator used in the dressing terms \(\Sigma\) is not explicitly dressed, that is, by using

\[
\Sigma_{ij}(E) \to \int_0^{\infty} dq \frac{q^2 \phi_i(q) \phi_j(q)}{E^+ - \omega_q - m}.
\]

It is just this \(g^{(0)}(E)\) that has been used in previous works\(^{13,36}\) to model \(\pi N\) scattering.

**FIG. 5.** Fits to the \(s\) - and \(p\) -wave \(\pi N\) phase shifts resulting from the solution of the DS equations [Eq. (34)] where the dressed Dyson nucleon propagator \(g(E)\) corresponds to the \(M7\) model of Table I. The \(P_{11}\) phase shift fit is the one using the \(M7\) model, while all non-\(P_{11}\) phase shift fits are specified by the separable form factor of Eq. (50) with the corresponding parameters given in Table II. The phase shift data correspond to the WI08 fit of Ref. 37.
2. Having constructed the “zeroth iteration” of $g(E)$ as above, we now use this $g$ in Eq. (48) to generate new dressing terms $\Sigma_i(E)$.

3. Using these newly constructed $\Sigma_i(E)$’s in Eq. (45) generates the next iteration of the dressed nucleon propagator $g(E)$.

D. Numerical results

After constructing the dressed nucleon propagator $g(E)$ as prescribed by the 3 steps outlined above, we now repeat steps 2 and 3 over and over, thus generating successive iterations of $g(E)$, denoted as $g^{(1)}(E), g^{(2)}(E), g^{(3)}(E), \ldots$, until the values of $g^{(r)}(E)$ converge according to the criterion $\left|\frac{g^{(r)}(E) - g^{(r-1)}(E)}{g^{(r)}(E)}\right| < \varepsilon$ for all $\omega$ integration points $\omega_i$, where $\varepsilon$ is some chosen tolerance value. We have found that the iterated dressed propagators $g^{(r)}(E)$ converge for all considered models using a convergence tolerance of $\varepsilon = 10^{-4}$ and that, correspondingly, the resulting numerical values of the converged $g(E)$ functions are stable to at least 5 significant figures with respect to variations in the number of quadrature points used for all integrals and in the contour rotation angle used for all the $q$ momentum integrals. We note that the convergence of $g(E)$ also provides a self-consistency check that no spurious resonance poles have been generated by the assumed separable potentials.

With the Dyson–Schwinger equations of Eq. (34) solved in this way, it is interesting to compare the resulting fully dressed “Dyson” nucleon propagator $g(E)$ with the “non-Dyson” one where the coupled equations of Fig. 2 are solved in the “one-pion approximation” where nucleon dressing in $\pi N$ states is neglected.

We present this comparison in Fig. 3 for the case where the parameters of the input bare $\pi NN$ vertex and background $\pi N$ potential are those of model M1 in Ref. 13. For ease of comparison, we have plotted the corresponding real and imaginary values of $(E - m)g(E)/Z$, being the renormalized nucleon propagators with the nucleon pole term factored out. As can be seen, there is a substantial difference between the two propagators, suggesting the importance of retaining nucleon dressing in $\pi N$ states.

To obtain a variety of models of nucleon dressing, we have carried out fits to the WI08 $P_1\pi N$ phase shifts (for pion laboratory energies up to 390 MeV) by using the functional forms of Eq. (47) for a number of choices of the integers $n_0$–$n_3$ and for a range of cutoff values $\Lambda$ for the bare $\pi NN$ vertex function $f_0(k)$. Each such fit was constrained to reproduce the $\pi NN$ coupling constant $f_{\pi NN}^2 = 0.079$ in the way described in Ref. 13. The parameters of four such fits are given in Table I with the corresponding values of $(E - m)g(E)/Z$ plotted in Fig. 4. Unsurprisingly, the large number of parameters in this model allows one to fit $\pi N$ data equally well for a wide range of cutoff parameters $\Lambda$. Although this flexibility of the model can be viewed as one of its weaknesses, it does allow one to accommodate the wide variety of $\pi NN$ vertex cutoffs, in the range $300 < \Lambda < 2200$ MeV, used in the literature.

Finally, we show that Eq. (34), which use the fully dressed Dyson propagators, are able to be used to fit all $s$- and $p$-wave $\pi N$ phase shifts for pion laboratory kinetic energies in the range $0 < T_{lab} < 390$ MeV. To demonstrate this explicitly, we have chosen the $M7$ model of Table I whose cutoff parameter is $\Lambda = 800$ MeV, a value suggested by an investigation of quantum chromodynamic

| $\pi N$ Partial wave | $\lambda$ | $P_{11}$ model | $S_1$ | $\alpha_1$ (fm$^{-1}$) | $S_2$ | $\alpha_2$ (fm$^{-1}$) |
|---------------------|---------|----------------|-------|----------------------|-------|----------------------|
| $S_{11}$            | $-1$    | M9             | -11.557 | 10.057               | -0.1248 | 0.85466              |
|                     |         | M8             | -11.715 | 9.9173               | -0.1335 | 0.85973              |
|                     |         | M7             | -11.818 | 9.7449               | -0.1433 | 0.86536              |
|                     |         | M6             | -12.046 | 9.6221               | -0.1551 | 0.87236              |
|                     |         | M9             | -3.8017 | 1.9930               | -1.0897 | 1.3266               |
| $S_{31}$            | $+1$    | M8             | 4.2003  | 2.0120               | -1.1815 | 1.3292               |
|                     |         | M7             | 4.7214  | 2.0350               | -1.3090 | 1.3365               |
|                     |         | M6             | 5.4998  | 2.0668               | -1.5227 | 1.3574               |
|                     |         | M9             | 9.5578  | 2.5488               |         |                      |
| $P_{31}$            | $+1$    | M8             | 10.339  | 2.5549               |         |                      |
|                     |         | M7             | 11.250  | 2.5620               |         |                      |
|                     |         | M6             | 12.370  | 2.5704               |         |                      |
|                     |         | M9             | 3.6208  | 1.8244               | 1.5209  | 3.2380               |
| $P_{13}$            | $+1$    | M8             | 3.5755  | 1.8022               | 1.4249  | 3.0757               |
|                     |         | M7             | 3.5051  | 1.7801               | 1.3111  | 2.9082               |
|                     |         | M6             | 3.3856  | 1.7492               | 1.1932  | 2.7126               |
|                     |         | M9             | 0.7487  | 1.7551               | 1.2295  | 4.9512               |
| $P_{33}$            | $-1$    | M8             | 0.80133 | 1.7535               | 1.2212  | 4.8276               |
|                     |         | M7             | 0.8606  | 1.7519               | 1.2126  | 4.7019               |
|                     |         | M6             | 0.93047 | 1.7501               | 1.2041  | 4.5665               |
For the non-$P_{11}$ partial wave $\pi N$ potentials, we use the separable forms of Thomas\textsuperscript{12} whose form factors are parameterized as

$$h(k) = \frac{S_1}{a_1^2 + k^2} + \frac{S_2}{a_2^2 + k^2} \quad (50a)$$

for $s$-waves and

$$h(k) = \frac{S_1 k}{(a_1^2 + k^2)^2} + \frac{S_2 k^3}{(a_2^2 + k^2)^2} \quad (50b)$$

for $p$-waves. These form factors were used by Thomas to describe pion–deuteron scattering in a calculation using semi-relativistic kinematics.\textsuperscript{12}

Our partial wave phase shift fits using the $M_7$ model for the Dyson propagator $g(E)$ are shown in Fig. 5 with the corresponding parameters listed (under the rows labeled $M_7$) in Table II. Equally good fits to all the phase shifts can be obtained using the other models for $g(E)$ ($M_9$, $M_8$, and $M_6$) with the corresponding parameters being given in Table I for the $P_{11}$ and Table II for the other partial waves. It should be noted, however, that equally good fits can also be obtained using non-Dyson propagators. Thus, even though we are able to demonstrate the importance of nucleon dressing, the large number of parameters in our model does not allow us to identify any features of the phase-shift data that may prefer the "Dyson" over the "non-Dyson" dressed nucleon propagators.

In summary, we have presented a simple but field-theoretically complete model of $\pi N$ scattering that demonstrates a number of basic features of QFT such as particle absorption and creation, self-energy, renormalization, and the use of Dyson–Schwinger equations but without the inherent complexity of a relativistically covariant approach. To do this, we used the non-relativistic framework of TOPT and exploited the simplicity of separable potentials to demonstrate how the DS equations can be solved in order to describe experimental data. It is worth noting that the model presented, together with the resulting fits to $\pi N$ phase shifts as recorded in Fig. 5 and Tables I and II, can be of direct practical use in models of pion–multi–nucleon systems, such as the convolution model of the $\pi NN$ system,\textsuperscript{49} where a complete dressing of all nucleons is required.

ACKNOWLEDGMENTS

A.N.K. was supported by the Shota Rustaveli National Science Foundation (Grant No. FR17-354).

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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