MONODROMY OF SUBVARIETIES OF PEL-SHIMURA VARIETIES

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Abstract. The aim of this paper is to generalize results of C.-L. Chai about the monodromy of Hecke invariant subvarieties in [4] to Shimura varieties of PEL-type.

1. Introduction

This paper deals with a generalization of the following results of C.-L. Chai about Hecke invariant subvarieties. Let $\mathcal{A}_{g,N}$ denote the moduli space over $\mathbb{F}_p$ of $g$-dimensional principally polarized abelian varieties in characteristic $p$ with symplectic level-$N$ structure. Let $l \neq p$ be a prime number and let $N > 3$ be a natural number relatively prime to $pl$. Let $Z$ be a smooth locally closed subvariety of $\mathcal{A}_{g,N}$. Assume that $Z$ is stable under all $l$-adic Hecke correspondences coming from $\text{Sp}_{2g}(\mathbb{Q}_l)$ and that the $l$-adic Hecke correspondences operate transitively on the set of connected components of $Z$. Furthermore, let $Z^0$ be a connected component of $Z$ with generic point $z$ and denote by $\mathcal{A} \to Z^0$ the universal abelian scheme restricted to $Z^0$. Then there are the following Propositions:

Proposition 1.1. [4, Proposition 4.1] Assume that the $l$-adic monodromy representation $\rho_l : \pi_1(Z^0, z) \to \text{Sp}_{2g}(\mathbb{Q}_l)$ attached to $\mathcal{A} \to Z^0$ has infinite image. Then its image contains an open subgroup of $\text{Sp}_{2g}(\mathbb{Q}_l)$.

One can show that the conditions of this Proposition imply that $Z$ is connected. This can be applied to subvarieties which are not contained in the supersingular locus of $\mathcal{A}_{g,N}$.

Proposition 1.2. [4, Proposition 4.4] Assume that $\mathcal{A}$ is not supersingular. Then $Z = Z^0$.

One can use these results for example to show that non-supersingular Newton strata are irreducible ([5, Theorem 3.1]). In this paper, we generalize the first result as described in the following Proposition. We denote by $\text{Sh}_{K^p}$ the geometric special fiber of a moduli space $\text{Sh}_{K^p}$ corresponding to an integral Shimura PEL-datum constructed by R. E. Kottwitz in [7]. The associated reductive group over $\mathbb{Q}$ is denoted by $\mathbf{G}$, with derived group $\mathbf{G}_1$. Let $P$ be the finite set of primes of $\mathbb{Q}$ containing $p$ and the primes $l$ such that some simple component of $\mathbf{G}_1$ is $\mathbb{Q}_l$-anisotropic.

Proposition 1.3. Let $\mathcal{D}$ be an integral Shimura PEL-datum, unramified at a prime $p$. Let $\mathbf{H}_1, \ldots, \mathbf{H}_n$ be the simple components of the derived group $\mathbf{G}_1$. Let $Z \subseteq \text{Sh}_{K^p}$ be a smooth, locally closed subscheme.
(1) Suppose that for a prime $l \notin P$ the $l$-Hecke correspondences of the simply connected covering of $G_1$ act transitively on the set of connected components of $Z$. If for all $i = 1, \ldots, n$ the set $\text{im}(\rho_l) \cap H_i(\mathbb{Q}_l)$ is not finite, then $\text{im}(\rho_l) = K_{0,l}$.

(2) Suppose that the prime-to-$P$ Hecke correspondences of the simply connected covering of $G_1$ act transitively on the set of connected components of $Z$. If for all $i = 1, \ldots, n$ and all $l \notin P$ the set $\text{im}(\rho_l) \cap H_i(\mathbb{Q}_l)$ is not finite, then $\text{im}(\rho_P) = K_{P,0}$.

Finally, we use recent results of E. Viehmann and T. Wedhorn about the Ekedahl-Oort stratification of PEL Shimura varieties to generalize the second Proposition. Here, $N_{b,0}$ denotes the basic Newton stratum of $\text{Sh}^p_{K_0}$.

Theorem 1.4. Let $D$ be an integral Shimura PEL-datum of type $A$ or $C$, unramified at a prime $p$. Suppose that the prime-to-$P$ Hecke correspondences of $G_1$ act transitively on the set of connected components of $Z$. If $z \notin N_{b,0}$, then $Z$ is connected.

One might hope to use this Theorem to show the irreducibility of non-basic Newton strata for PEL Shimura varieties similar to the Siegel case.

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2. Preliminaries

2.1. Shimura PEL-data and their moduli problems. In this section we recall the definition of a Shimura PEL-datum and the associated moduli space of abelian varieties with additional structures. The main references are [7] and [9].

A tuple $D = (B,*, V, \langle , \rangle, \mathcal{O}_B, \Lambda, h)$ is called an integral Shimura PEL-datum, unramified at a positive prime $p \in \mathbb{Z}$, if it consists of the following data:

- $B$ is a finite-dimensional simple $\mathbb{Q}$-algebra, such that there is an isomorphism $B_{\mathbb{Q}_p} \cong \bigoplus_{p \in S} M_d(F_p)$, where $F$ denotes the center of $B$ and $S$ the set of primes of $F$ over $p$, with $F_p/\mathbb{Q}_p$ unramified for all $p \in S$.
- $*$ is a $\mathbb{Q}$-linear positive involution on $B$.
- $V$ is a finitely generated left $B$-module.
- $\langle , \rangle : V \times V \rightarrow \mathbb{Q}$ is a symmetric form on $V$ with $\langle bv, w \rangle = \langle v, b^* w \rangle$ for all $v, w \in V$ and all $b \in B$.
- $\mathcal{O}_B$ is a $*$-invariant $\mathbb{Z}_{(p)}$-order of $B$ such that
  $$\mathcal{O}_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong \bigoplus_{p \in S} M_d(\mathcal{O}_{F_p}),$$
  where $\mathcal{O}_F$ denotes the ring of integers of $F$ and the isomorphism is induced by the corresponding isomorphism for $B$ described above.
- $\Lambda$ is an $\mathcal{O}_F$-invariant $\mathbb{Z}_{(p)}$-lattice in $V$, such that $\langle , \rangle$ induces a perfect pairing $\Lambda_{\mathbb{Z}_{(p)}} \times \Lambda_{\mathbb{Z}_{(p)}} \rightarrow \mathbb{Z}_p$.
- $h : C \rightarrow \text{End}_B(V) \otimes_\mathbb{Q} \mathbb{R}$ is a homomorphism such that, if $\iota$ is the involution on $\text{End}_B(V)$ coming from $\langle , \rangle$, then $h(\iota) = h(z)^*$ and the form $(v, w) \mapsto \langle v, h(\iota)w \rangle$ on $V_{\mathbb{R}}$ is positive definite.
There are algebraic $\mathbb{Q}$-groups $\mathbf{G}$ and $\mathbf{U}$ defined by
\[
\mathbf{G}(R) := \{ g \in \text{GL}_B(V \otimes \mathbb{Q} \ R) \mid gg^* \in R^\times \}
\]
and
\[
\mathbf{U}(R) := \{ g \in \text{GL}_B(V \otimes \mathbb{Q} \ R) \mid gg^t = 1 \}.
\]
The derived group of $\mathbf{G}$ is denoted by $\mathbf{G}_1$, hence we have an inclusion
\[
\mathbf{G}_1 \subset \mathbf{U} \subset \mathbf{G}.
\]

For each Shimura PEL-datum as above there is the following associated moduli problem due to R. E. Kottwitz ([7, §5]). Let $\mathbb{A}_f^p$ denote the ring of finite adeles with trivial $p$-component and let $K^p \subseteq \mathbf{G}(\mathbb{A}_f^p)$ be an open compact subgroup. Furthermore, we denote by $E$ the reflex field associated to the Shimura PEL-datum $D$ and by $\mathcal{O}_E$ its ring of integers. Consider the functor
\[
\mathcal{A} := \mathcal{A}_{D,K^p} = (\text{Sch}/\mathcal{O}_{E,(p)}) \rightarrow (\text{Sets})
\]
from locally noetherian $\mathcal{O}_{E,(p)}$-Schemes to the category of sets that maps an $\mathcal{O}_{E,(p)}$-scheme $S$ to the set of isomorphism classes of tuples $(A, \lambda, i, \eta)$, where:

- $A$ is an abelian scheme over $S$.
- $\lambda : A \rightarrow A^\vee$ is a $\mathbb{Z}_{(p)}$-polarization.
- $i : \mathcal{O}_B \rightarrow \text{End}(A) \otimes \mathbb{Z}_{(p)}$ is a $\mathbb{Z}_{(p)}$-homomorphism satisfying
  \[
  \lambda \circ i(\alpha^*) = i(\alpha)^\vee \circ \lambda
  \]
  for all $\alpha \in \mathcal{O}_B$.
- $\eta$ is a prime-to-$p$ level $K^p$-structure, see the next section for details.

An isomorphism $(A, \lambda, i, \eta) \cong (A', \lambda', i', \eta')$ between two tuples is given by a $\mathbb{Z}_{(p)}^\times$-isogeny $f : A \rightarrow A'$, such that $\lambda = rf^\vee \circ \lambda' \circ f$ for some positive $r \in \mathbb{Z}_{(p)}^\times$, $f \circ i = i' \circ f$ and $\eta = f \circ \eta'$.

Furthermore, we assume that the determinant condition of Kottwitz is fulfilled, see [7, §5]. Then, if the group $K^p$ is sufficiently small, the functor $\mathcal{A}$ is representable by a quasi-projective smooth scheme $\text{Sh}_{K^p}$ over $\mathcal{O}_{E,(p)}$, see for instance [9, Section 2.3.3]. We fix a prime $\nu$ of $E$ over $p$ with residue field $\kappa$ and denote by $\text{Sh}_{K^p} := \text{Sh}_{K^p} \otimes \kappa$ the special fiber of $\text{Sh}_{K^p}$ over $\nu$.

2.2. **Level $K^p$-structures and Hecke correspondences.** Let $(A, \lambda, i, \eta)$ be a tuple as above over a base scheme $S$ and $\mathfrak{s} \in S$ a geometric point with residue field $\kappa(\mathfrak{s})$. We denote by $A_{\mathfrak{s}}$ the fiber of $A$ over $\mathfrak{s}$ and by $T(A_{\mathfrak{s}})$ its Tate module. Define $V^p(A_{\mathfrak{s}}) := T(A_{\mathfrak{s}}) \otimes \mathbb{Z}_p$. A prime-to-$p$ level structure is an $\mathcal{O}_B$-linear isomorphism
\[
\eta : V^p_{A_{\mathfrak{s}}} \cong V^p(A_{\mathfrak{s}})
\]
such that the following holds: the Weil pairing $e : T(A_{\mathfrak{s}}) \times T(A_{\mathfrak{s}}) \rightarrow \mathbb{A}_f^p(1)$ gives a pairing $\langle , \rangle : V^p(A_{\mathfrak{s}}) \times V^p(A_{\mathfrak{s}}) \rightarrow \mathbb{A}_f^p(1)$ using the polarization $\lambda : A \rightarrow A^\vee$, and if we identify $\mathbb{A}_f^p(1)$ with $\mathbb{A}_f^p$ we require that $\eta$ maps the
We now describe the prime-to-all connected components of $\text{Sh}^{p}$ of $\text{S}$. An étale covering pair $(A, \lambda, i, \pi)$ is surjective, where $\eta$ is a point of $\text{Sh}^{p}$ for each inclusion $K_{0}^{p} \subseteq K_{0}^{p}$ of open compact subgroups of $G(A_{f})$. The group $G(A_{f})$ acts on $(\text{Sh}_{K^{p}})_{K^{p} \subseteq G(A_{f})}$ defined by

$$\text{Sh}_{K^{p}} \to \text{Sh}_{\eta^{-1}K^{p}g}, \quad (A, \lambda, i, \eta) \mapsto (A, \lambda, i, \eta g)$$

for $g \in G(A_{f})$. The tower $(\text{Sh}_{K^{p}})_{K^{p} \subseteq G(A_{f})}$ over all open normal subgroups $N \subseteq K^{p}$ is a pro-étale Galois covering with Galois group $\text{lim}_{\rightarrow} K^{p}/N = K^{p}$. Let $k \supseteq \kappa$ be a field, $x \in \text{Sh}_{K^{p}}(k)$ a point and $\tilde{x} \in (\text{Sh}_{K^{p}}(k))_{K^{p} \subseteq G(A_{f})}$ over $x$. We define the prime-to-$p$ Hecke orbit $H^{p}(x)$ of $x$ as the projection of $G(A_{f}) \cdot \tilde{x}$ to $\text{Sh}_{K^{p}}(k)$ and the $l$-Hecke orbit $H_{l}(x)$ for a prime $l \neq p$ as the projection of $G(Q_{l}) \cdot \tilde{x}$ to $\text{Sh}_{K^{p}}(k)$ under the canonical morphism $G(Q_{l}) \to G(A_{f})$. One easily sees that the Hecke correspondences are independent from the choice of the point $\tilde{x}$. Furthermore, $H^{p}(x)$ and $H_{l}(x)$ are countable sets due to the surjective map $G(A_{f})/K_{0}^{p} \to H^{p}(x)$.

3. Monodromy groups

Let $Z \subseteq \text{Sh}_{K^{p}}$ be a smooth, locally closed subscheme. We assume that $Z(k)$ is closed under prime-to-$p$ Hecke correspondences, that is $H^{p}(Z(k)) \subseteq Z(k)$. Let $z \in Z$ denote the generic point of an irreducible component $Z^{0}$ of $Z$, which is also a connected component of $Z$. We say that the prime-to-$p$ Hecke correspondences act transitively on the set of connected components of $Z$ if

$$\Pi_{0}(\overline{H^{p}(Z^{0}(k)) \cap Z}) \to \Pi_{0}(Z)$$

is surjective, where $\overline{H^{p}(Z^{0}(k))}$ denotes the Zariski closure of the Hecke correspondences. We also consider an analogue definition for the $l$-Hecke correspondences. Furthermore, let

$$\overline{\rho}^{p} : \pi_{1}(Z^{0}, \pi) \to K_{0}^{p}$$

respectively

$$\overline{\rho} : \pi_{1}(Z^{0}, \pi) \to K_{0,l} := \text{im}(K_{0}^{p} \to G(Q_{l}))$$

denote the $K_{0}^{p}$ respectively $K_{0,l}$-conjugacy classes of representations with respect to the tuple $(A, \lambda, i, \overline{\eta})$ corresponding to the morphism

$$Z^{0} \hookrightarrow \text{Sh}_{K_{0}^{p}} \to \text{Sh}_{K_{0}^{p}}.$$
We denote by $P$ the finite set of primes of $\mathbb{Q}$ containing $p$ and the primes $l$ such that some simple component of $G_1$ is $\mathbb{Q}_l$-anisotropic. Let $A_f^P$ be the finite adeles with trivial $P$ components. There is the following proposition, proven by C.-L. Chai in the Siegel case, [4, Proposition 4.1 and 4.5.4]:

**Proposition 3.1.** Let $D$ be an integral Shimura PEL-datum, unramified at a prime $p$. Let $H_1, \ldots, H_n$ be the simple components of the derived group $G_1$. Let $Z \subseteq \text{Sh}_{K_0^p}$ be a smooth, locally closed subscheme.

1. Suppose that for a prime $l \notin P$ the $l$-Hecke correspondences of the simply connected covering of $G_1$ act transitively on the set of connected components of $Z$. If for all $i = 1, \ldots, n$ the set $\text{im}(\rho_l) \cap H_i(\mathbb{Q}_l)$ is not finite, then $\text{im}(\rho_l) = K_{0,l}$.

2. Suppose that the prime-to-$P$ Hecke correspondences of the simply connected covering of $G_1$ act transitively on the set of connected components of $Z$. If for all $i = 1, \ldots, n$ and all $l \notin P$ the set $\text{im}(\rho_l) \cap H_i(\mathbb{Q}_l)$ is not finite, then $\text{im}(\rho^P) = K_0^P$.

**Proof.** Of course (1) implies (2). So let us show (1). Choose a prime $l \notin P$. Let $N \subseteq K_0^p$ denote an open normal subgroup such that $K_0^p / N$ is trivial outside of $l$, and consider the pro-étale covering

$$(\text{Sh}_N)_{N \subseteq G(A_f^p)} \to \text{Sh}_{K_0^p}.$$

The group $K_{0,l}$ acts continuously on this covering via $l$-Hecke correspondences. Consider the pro-étale covering of $Z^0$ obtained by base change with $Z^0 \hookrightarrow \text{Sh}_{K_0^p}$,

$$\widetilde{Z}^0 := (\text{Sh}_N \times_{\text{Sh}_{K_0^p}} Z^0)_{N \subseteq G(A_f^p)} \to Z^0,$$

together with the induced action of $K_{0,l}$. We choose a point $\tilde{z} \in \widetilde{Z}^0$ over $\overline{z}$ and thus obtain a connected component $\overline{Z}^0$. The algebraic fundamental group of $Z^0$ acts on $\overline{Z}^0$ through the obvious morphism

$$\pi_1(Z^0, \overline{z}) \to \pi_1(\text{Sh}_{K_0^p}, \overline{z}) \to \text{Aut}_{\text{Sh}_{K_0^p}}((\text{Sh}_N)_{N \subseteq G(A_f^p)}) = K_0^p,$$

where $\text{Sh}_{K_0^p}$ is the connected component containing $Z^0$. One easily sees that this morphism coincides with the morphism

$$\rho^p : \pi_1(Z^0, \overline{z}) \to K_0^p$$

coming from the prime-to-$p$ level $K_0^p$-structure. It follows from the general theory of the fundamental group that $M := \text{im}(\rho_l)$ is the stabilizer of the connected component $\overline{Z}^0$ with respect to the $K_{0,l}$-action. We get a homeomorphism $K_{0,l} / M \cong \pi_0(\overline{Z}^0)$ of profinite sets. For the pro-étale covering

$$\widetilde{Z} := (\text{Sh}_N \times_{\text{Sh}_{K_0^p}} Z)_{N \subseteq G(A_f^p)} \to Z$$

we get an analogous continuous bijection

$$G_1(\mathbb{Q}_l) / \text{Stab}_{G_1}(\overline{Z}^0) \cong \pi_0(\widetilde{Z})$$

since the $l$-Hecke correspondences act transitively on the connected components of $Z$ and hence also on the connected components of $\overline{Z}$. It follows
with the same arguments as in [4, Lemma 2.8] that this isomorphism is a homeomorphism.

As explained in the proof of [4, Proposition 4.1], the algebraic group

\[ M := \operatorname{im}(\rho_l) \subseteq G_{\mathbb{Q}_l} \]

is semisimple, hence it lies in \( G_{1,\mathbb{Q}_l} \), and we have

\[ \operatorname{Stab}_{G_1}(Z_0) \subseteq N_{G_1}(M)(\mathbb{Q}_l) \]

Since there is a continuous surjection

\[ G_1(\mathbb{Q}_l)/\operatorname{Stab}_{G_1}(Z_0) \twoheadrightarrow G_1(\mathbb{Q}_l)/N_{G_1}(M)(\mathbb{Q}_l) \]

and the set on the left is profinite, the quotient \( G_1(\mathbb{Q}_l)/N(\mathbb{Q}_l) \) is compact, where we set

\[ N := N_{G_1}(M)^0. \]

Then [2, Proposition 9.3] implies that \( N \) contains a maximal \( \mathbb{Q}_l \)-split solvable subgroup \( A \) of \( G_{1,\mathbb{Q}_l} \), and because \( G_{1,\mathbb{Q}_l} \) is isotropic, [2, Proposition 8.4,8.5] shows that \( A \neq 0 \) is the unipotent radical of a minimal parabolic subgroup of \( G_{1,\mathbb{Q}_l} \). Finally, we use [2, Proposition 8.6] and conclude that \( N \) contains the smallest normal subgroup of \( N_{G_1}(\operatorname{rad}_u(N)) \) containing \( A \). But \( N \) is reductive since \( G_{1,\mathbb{Q}_l} \) and \( M \) are semisimple (see [4, Lemma 3.3]), hence we showed that it contains a nontrivial normal connected subgroup of \( G_{1,\mathbb{Q}_l} \). The assumptions on \( \rho_l \) then yield \( N = G_{1,\mathbb{Q}_l} \), so \( M \) itself is a nontrivial normal subgroup of \( G_{1,\mathbb{Q}_l} \) intersecting all simple subgroups. This implies \( N = G_{1,\mathbb{Q}_l} \).

We now know that the Lie algebras of the \( l \)-adic Lie group \( M \) and its Zariski closure coincide, because \( M \) is semisimple (see [11, Corollary 7.9]), so \( M \subseteq K_{0,l} \) contains an open subgroup. It follows that

\[ K_{0,l}/M \cong \Pi_0(Z_0) \]

is finite. Because \( Z \) is quasi-projective, it has only finitely many connected components, which implies that \( \Pi_0(Z) \) is also finite. If \( \#\Pi_0(Z) \neq 1 \), then the simply connected group \( G_{1}^c(\mathbb{Q}_l) = H_{1}^c(\mathbb{Q}_l) \times \cdots \times H_{n}^c(\mathbb{Q}_l) \) contains a nontrivial subgroup of finite index. But the Kneser-Tits conjecture for simple and simply connected \( \mathbb{Q}_l \)-isotropic groups implies that all the groups \( H_{i}^c(\mathbb{Q}_l) \) have no nontrivial noncentral normal subgroups, see [11, Theorem 7.1, 7.6]. It follows that they do not contain any nontrivial subgroup of finite index, and the same holds for \( G_{1}^c(\mathbb{Q}_l) \). This shows

\[ K_{0,l}/M \cong \Pi_0(Z_0) = \{1\}. \]

\[ \square \]

**Corollary 3.2.** If the conditions of Proposition 3.1 are fulfilled, then \( Z \) is connected.

**Proof.** This follows at once from the proof above. \[ \square \]
4. Hecke correspondences and Newton strata

From now on, we assume that the Shimura PEL-datum $\mathcal{D}$ is of type $A$ or $C$, which implies that the group $G$ is connected. Let $x \in Sh_{K_0^p}(k)$ be a geometric point, given by a tuple $(A, \lambda, i, \mathfrak{m})$. We define a functor by

$$I_x(R) := \{g \in \text{End}_B(A) \otimes_{\mathbb{Z}(p)} R \mid gg^* \in R^x\},$$

where $R$ is a $\mathbb{Z}(p)$-algebra and $^*$ is the Rosati involution with respect to the polarization $\lambda$. This defines a reductive group over $\mathbb{Q}$. We fix an element in the prime-to-$p$ level $K_0^p$-structure $\mathfrak{m}$, which gives together with the injection

$$\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{A}_f^p \hookrightarrow \text{End}(V^p(A))$$

proved by J. Tate an injection

$$I_{x,A_f^p} \hookrightarrow G_{A_f^p}$$

of algebraic groups. One easily sees that the tuple $(A, \lambda, i, \mathfrak{m})$ is isomorphic to $(A, \lambda, i, \mathfrak{m})^{\eta}$ if and only if $g$ lies in the set $I_x(\mathbb{Z}(p))K_0^p$, where we consider $I_x(\mathbb{Z}(p)) \subset G(\mathbb{A}_f^p)$ with respect to the chosen element of the level structure.

If $g \in G(\mathbb{Q}_l) \subset I_x(\mathbb{Z}(p))K_0^p = I_{x,l}(\mathbb{Z}(p))K_{0,l}$,

where $I_{x,l}(\mathbb{Z}(p))$ denotes the group of $\mathbb{Z}(p)$-isogenies in $I_x(\mathbb{Z}(p)) \subset G(\mathbb{A}_f^p)$ whose $l'$-components lie in $K_{0,l'}$ for all primes $l' \neq l$. It follows that there are bijections

$$I_x(\mathbb{Z}(p)) \backslash G(\mathbb{A}_f^p)/K_0^p \cong \mathcal{H}^p(x)$$

$$I_{x,l}(\mathbb{Z}(p)) \backslash G(\mathbb{Q}_l)/K_{0,l} \cong \mathcal{H}_l(x)$$

We are interested in points with infinite Hecke orbit. It turns out that $\mathcal{H}^p(x)$ is infinite if $x$ does not lie in the basic locus of the Newton stratification of $Sh_{K_0^p}$. For an overview of the Newton stratification see for instance [13, 7.2] and the references therein. Here we give a short description of the properties that are important for the purpose of this paper.

Let $L$ denote the quotient field of the Witt ring of $k$ with Frobenius morphism $\sigma$. Let $B(G)$ be the set of $G(L)$- $\sigma$-conjugacy classes $\{g^{-1}b\sigma(g) \mid g \in G(L)\}$ of elements $b \in G(L)$. There is a partial ordering on $B(G)$ and a finite subset $B(G, \mu) \subset B(G)$ containing a unique maximal element $b_\mu$ and a unique minimal element $b_0$, see [5, §6]. Here $\mu$ denotes a dominant cocharacter of a maximal torus of $G$ defined by the Shimura PEL-datum $\mathcal{D}$. A $\sigma$-conjugacy class $\mathfrak{m}$ is called basic if it contains an element lying in $T(L)$, where $T$ is an elliptic maximal torus of $G$. There is a stratification

$$Sh_{K_0^p} = \bigcup_{b \in B(G, \mu)} N_b$$

with locally closed subsets $N_b \subset Sh_{K_0^p}$ such that $N_b \subseteq \bigcup_{b \leq b'} N_{b'}$. For each $b \in G(L)$ there is an affine algebraic $\mathbb{Q}_p$-group $J_b$ defined by

$$J_b(R) := \{g \in G(R \otimes_{\mathbb{Q}_p} L) \mid g(b\sigma) = (b\sigma)g\},$$

see [13, Proposition 1.12]. Observe that there is an injection $I_{x,\mathbb{Q}_p} \hookrightarrow J_b$ of algebraic groups. Recall that a geometric point $x = (A, \lambda, i, \mathfrak{m})$ is called
hypsersymmetric if $I_{x, Q^p} \cong J_b$. We have the following lemma, which was observed by C.-L. Chai in the Siegel case in [3, Proposition 1].

**Lemma 4.1.** Let $x \in Sh_{K_0^p}(k)$ be a geometric point.

1. If $x \notin N_{b_0}(k)$, then $H^l(x)$ is not finite for all primes $l$ such that $G_{Q_l}$ is split. In particular, $H^p(x)$ is not finite.
2. If $x \in N_{b_0}(k)$, then $H^p(x)$ is finite.

**Proof.** It follows from [12, Proposition 2.4] that $b_0$ is the only basic point in $B(G, \mu)$. Let us consider the first statement of the lemma. It suffices to show that $I_{x}(Q_l) \backslash G(Q_l)$ is not compact, which is equivalent to $I_{x}(Q_l)$ not containing a maximal $Q_l$-split connected solvable subgroup, which is a Borel subgroup since $G_{Q_l}$ is split. It follows that this quotient is compact if and only if $I_{x, Q_l}$ is parabolic, but the only parabolic and reductive subgroup is $G_{Q_l}$ itself. Furthermore, [13, Corollary 1.14, Remark 1.15] implies that the reductive group $J_b$ is a form of $G_{Q_l}$ if and only if $b$ is basic. Then the first statement of the Lemma holds because $I_x$ cannot be a form of $G$.

A geometric point $x$ in the basic locus of $Sh_{K_0^p}$ is hypersymmetric, see [13, Theorem 6.30], hence $I_{x, Q_p} \cong J_b$ and $I_{x, \lambda^p} \cong G_{\lambda^p}$. Then (2) follows at once from the finiteness of class numbers of algebraic groups, see for instance [11, Theorem 5.1].

We are now ready to prove the main result of this paper. We consider a smooth, locally closed subscheme $Z \subseteq Sh_{K_0^p}$ as above and fix a connected component $Z^0$ with generic point $z$.

**Theorem 4.2.** Suppose that the prime-to-$P$ Hecke correspondences of $G_1$ act transitively on the set of connected components of $Z$. If $z \notin N_{b_0}$, then $Z$ is connected.

**Proof.** It suffices to show that $\text{im}(\rho_l)$ is not finite for all $l \neq p$. Then Corollary [12, Theorem 3.2] implies that $Z$ is connected, because the image of $\rho_l$ lies in the simply connected semisimple group $G_1$. So suppose $\text{im}(\rho_l)$ is a finite group for some prime $l \neq p$ and let $(A, \lambda, i, \eta)|_{Z^0}$ correspond to the morphism $Z^0 \rightarrow Sh_{K_0^p}$. Then there is a finite field extension $L$ of $\kappa(z)$ such that for the normalization $Z'$ of $Z^0$ in $L$ the abelian scheme $A' := A \times_{Z^0} Z'$ has the trivial monodromy representation $\rho'_l$. A successive application of the result [10, Theorem 2.1] of F. Oort then implies that $A'_l$ is isogenous to an abelian variety $A_0$ defined over $\overline{F}_p$, which extends to an isogeny $A' \rightarrow A_0 \times_{\overline{F}_p} Z'$ using [9, Proposition 1.2.7].

We claim that the Zariski closure of $Z^0$ in $Sh_{K_0^p}$ is a proper scheme over $\overline{F}_p$.

To show this, consider the Zariski closure

$$\overline{Z^0} \subseteq Sh_{K_0^p}$$

in the toroidal compactification $\overline{Sh}_{K_0^p}$ of the Shimura variety, which is a proper and smooth scheme over $\overline{F}_p$ containing $Sh_{K_0^p}$ as an open dense subscheme, see [9, Theorem 6.4.1.1]. Furthermore, there is a degenerating family $(G, \lambda_G, i_G, \eta)$ in the sense of [9, Definition 5.4.2.1], where $G$ is a semiabelian
scheme over $\overline{Sh}_{K_0}$ with PEL-structure that restricts to the universal abelian scheme with PEL-structure over $Sh_{K_0}$. Denote by 
$$Z' \rightarrow \overline{Z}$$
the normalization of $\overline{Z^0}$ in $L$. Since $Z' \subseteq \overline{Z}$ is an open subscheme, we can again use [6, Proposition I.2.7] and extend the isogeny $A_0 \times_{\mathbb{F}_p} Z' \rightarrow A'$ uniquely to a morphism 
$$A_0 \times_{\mathbb{F}_p} \overline{Z}' \rightarrow G_{\overline{Z}'}$$
of semiabelian schemes. It is easy to see that then $G_{\overline{Z}'}$ is an abelian scheme over $\overline{Z}'$. This implies that the image of the morphism $\overline{Z}' \rightarrow \overline{Sh}_{K_0}$ lies in the open subset where $G$ is an abelian variety, hence $\overline{Z^0}$ lies in $Sh_{K_0}$, which shows the claim.

In [14], E. Viehmann and T. Wedhorn generalize the Ekedahl-Oort stratification to special fibers of PEL-Shimura varieties of type $A$ and $C$. That is, there is a finite and partially ordered set $(J^W, \preceq)$ containing a unique minimal and maximal element, such that there is a stratification 
$$Sh_{K_0} = \bigcup_{\omega \in J^W} S^\omega$$
of locally closed and quasi-affine subschemes $S^\omega$, see [14, §2, §4 and Theorem 9.6]. Furthermore, 
$$\overline{S^0} = \bigcup_{\omega' \preceq \omega} S^{\omega'}$$
see [14, Theorem 6.1], and the minimal stratum is closed, nonempty, and lies in $N_{b_0}$ [14, Proposition 8.16].

We know that $\overline{Z^0}$ is a proper scheme over $\mathbb{F}_p$ and that it is closed under $H^p$ since $Z^0$ is closed under Hecke correspondences. It follows directly from the definition of the Ekedahl-Oort stratification that each stratum is also closed under $H^p$. This implies together with Lemma [11] that $\overline{Z^0} \cap N_{b_0} \neq \emptyset$. For if there is a point of $\overline{Z^0}$ not lying in the basic Newton stratum, it cannot lie in the smallest Ekedahl-Oort stratum. Furthermore, the closure of its prime-to-$p$ Hecke orbit is proper and of dimension $>0$, so it has to meet a smaller Ekedahl-Oort stratum. Then repeat this argument. But this statement is a contradiction to the isogeny $A' \times_{\overline{Z}'} \overline{Z} \rightarrow A_0 \times_{\mathbb{F}_p} \overline{Z}'$, which says that $\overline{Z^0}$ lies in a single Newton stratum. □

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