Infinitely many homoclinic solutions for a class of subquadratic second-order Hamiltonian systems

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Abstract:

In this paper, we consider the existence of infinitely many homoclinic solutions for a class of subquadratic second-order Hamiltonian systems \( \ddot{u} - L(t)u + W_u(t,u) = 0 \), where \( L(t) \) is unnecessarily positive definite. Using the variant fountain theorem, we obtain a new result of infinitely many homoclinic solutions for the second-order Hamiltonian systems.

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1 Introduction and main results

In this paper, we consider the following second-order Hamiltonian systems

\[
\ddot{u} - L(t)u + W_u(t,u) = 0, \quad \forall \ t \in \mathbb{R}
\]  

(HS)

where \( u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N \), \( W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R}) \) and \( L \in C(\mathbb{R}, \mathbb{R}^{N \times N}) \) is a symmetric matrix-valued function. As usual we say that a solution \( u \) of (HS) is nontrivial homoclinic (to 0) if \( u \in C^2(\mathbb{R}, \mathbb{R}^N) \), \( u \neq 0 \), \( u(t) \to 0 \) and \( \dot{u}(t) \to 0 \) as \( t \to \pm \infty \).

In the applied sciences, Hamiltonian systems play a key role in practical problems concerning gas dynamics, fluid mechanics, relativistic mechanics and nuclear physics. The existence of homoclinic solutions is one of the most important problems in the theory of Hamiltonian systems. Recently, the existence and multiplicity of homoclinic orbits for Hamiltonian systems have been extensively and intensively studied in many papers via critical theory, see [1-21] and the references therein.

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The case where $L(t)$ and $W(t, x)$ are either independent of $t$ or periodic in $t$ is studied by several authors, see [1–3, 7, 8, 12–16]. More precisely, in the paper [16], Rabinowitz has shown the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of (HS). Later, using the same method, several results for general Hamiltonian systems were obtained by Izydorek and Janczewska [8], Lv et al. [12].

When $L(t)$ and $W(t, x)$ are not periodic with respect to $t$, the problem of existence of homoclinic orbits for (HS) is quite different from the case just described, due to the lack of compactness of the Sobolev embedding. In [17], Rabinowitz and Tanaka studied system (HS) without a periodicity assumption, both for $L$ and $W$. More precisely, they assumed that the smallest eigenvalue of $L(t)$ tends to $+\infty$ as $|t| \to \infty$, using a variant of the Mountain Pass theorem without the Palais-Smale condition, and proved that system (HS) possesses a homoclinic orbit. Inspired by the work of Rabinowitz and Tanaka [17], many results [4, 6, 10, 11, 14, 15, 18, 20, 21] were obtained for the case of aperiodicity. Among of them, most of the results were obtained by the following assumption that $L(t)$ is positive definite for all $t \in \mathbb{R}$,

$$(L(t)u, u) > 0, \quad \forall t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \setminus \{0\}.$$  

Motivated by [6, 20], in this paper we will study the existence of infinitely many homoclinic solutions for (HS), where $L(t)$ is not necessarily positive definite for all $t \in \mathbb{R}$. Our main tool is the variant fountain theorem established in [22]. Our main results are the following theorems.

**Theorem 1.1.** Assume that $L$ and $W$ satisfy the following conditions:

- **(L1)** There exists an $\alpha < 1$ such that
  $$l(t)|t|^\alpha \to \infty \quad \text{as } |t| \to \infty$$
  where $l(t) := \inf_{|u|=1, u \in \mathbb{R}^N} (L(t)u, u)$ is the smallest eigenvalue of $L(t)$;

- **(L2)** There exist constants $\bar{a} > 0$ and $\bar{r} > 0$ such that
  (i) $L \in C^1(\mathbb{R}, \mathbb{R}^{N \times N})$ and $|L'(t)u| \leq \bar{a}|L(t)u|, \quad \forall |t| > \bar{r} \text{ and } u \in \mathbb{R}^N$, or
  (ii) $L \in C^2(\mathbb{R}, \mathbb{R}^{N \times N})$ and $(L''(t) - \bar{a}L(t))u, u) \leq 0, \quad \forall |t| > \bar{r} \text{ and } u \in \mathbb{R}^N$, where $L'(t) = (d/dt)L(t)$ and $L''(t) = (d^2/dt^2)L(t)$;

- **(W)** $W(t, u) = a(t)|u|^{\nu}$ where $a : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $a \in L^\mu(\mathbb{R}, \mathbb{R})$, $1 < \nu < 2$ is a constant, $2 \leq \mu \leq \hat{\nu}$ and
  $$\hat{\nu} = \begin{cases} 
  \frac{2}{3 - 2\nu}, & 1 < \nu < \frac{3}{2} \\
  \infty, & \frac{3}{2} \leq \nu < 2
  \end{cases}$$

Then (HS) possesses infinitely many homoclinic solutions.

**Remark 1.2.** When we choose $\nu \in (1, \frac{3}{2})$, it is easy to see that $W$ satisfies the condition (W) of Theorem 1.1 but does not satisfy the corresponding conditions in [6, 20]. Furthermore, the constant $\mu$ can be change in $[2, \hat{\nu}]$.  

\[ \]
2 Preliminaries

In this section, we will establish the variational setting for (HS). In what follows it will always be assumed that $L(t)$ satisfies (L1). We denote by $\mathcal{A}$ the selfadjoint extension of the operator $-(d^2/dt^2) + L(t)$ with domain $\mathcal{D}(\mathcal{A}) \subset L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^N)$. Let $\{E(\lambda) : -\infty < \lambda < +\infty\}$ and $|A|$ be the spectral resolution and the absolute value of $\mathcal{A}$ respectively, and $|A|^{1/2}$ be the square root of $|A|$. Set $U = I - E(0) - E(-0)$. Then $U$ commutes with $\mathcal{A}$, $|A|$ and $|A|^{1/2}$, and $\mathcal{A} = U|A|$ is the polar decomposition of $\mathcal{A}$ (see [9]). Let $E = \mathcal{D}(|A|^{1/2})$ and define on $E$ the inner product

$$(u, v)_0 = (|A|^{1/2}u, |A|^{1/2}v)_2 + (u, v)_2$$

and norm

$$\|u\|_0 = (u, u)_0^{1/2},$$

where $(\cdot, \cdot)_2$ denotes the usual $L^2$-inner product. Then $E$ is a Hilbert space. Since $C_c^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in $E$, we can see that $E$ is continuous embedded in $H^1(\mathbb{R}, \mathbb{R}^N)$ (see [6]). Furthermore, we have the following lemma by [6].

Lemma 2.1. If $L$ satisfies (L1), then $E$ is compactly embedded in $L^p \equiv L^p(\mathbb{R}, \mathbb{R}^N)$ for all $1 \leq p \in (2/(3 - \alpha), \infty]$.

Lemma 2.2. Let $L$ satisfies (L1) and (L2), then $\mathcal{D}(\mathcal{A})$ is continuously embedded in $W^{2,2}(\mathbb{R}, \mathbb{R}^N)$, and consequently, we have

$$|u(t)| \to 0 \text{ and } |\dot{u}(t)| \to 0 \text{ as } |t| \to \infty, \quad \forall u \in \mathcal{D}(\mathcal{A}).$$

From [6], by (L1) and Lemma 2.1, we can know that $\mathcal{A}$ possesses a compact resolvent. Therefore, the spectrum $\sigma(\mathcal{A})$ consists of eigenvalues numbered in $\lambda_1 \leq \lambda_2 \leq \cdots \to \infty$ (counted with multiplicity), and a corresponding system of eigenfunctions $\{e_n : n \in \mathbb{N}\}$, $\mathcal{A}e_n = \lambda_n e_n$, forms an orthogonal basis in $L^2$. We let

$$n^- = \sharp \{i|\lambda_i < 0\}, \quad n^0 = \sharp \{i|\lambda_i = 0\}, \quad \bar{n} = n^- + n^0$$

and

$$E^- = \text{span}\{e_1, \cdots, e_{n^-}\}, \quad E^0 = \text{span}\{e_{n^- + 1}, \cdots, e_{\bar{n}}\} = \text{Ker}\mathcal{A}, \quad E^+ = \text{span}\{e_{\bar{n} + 1}, \cdots\},$$

where the closure is taken in $E$ with respect to the norm $\| \cdot \|_0$. Then

$$E = E^- \oplus E^0 \oplus E^+.$$

Next, we introduce on $E$ the following inner product

$$(u, v) = (|A|^{1/2}u, |A|^{1/2}v)_2 + (u^0, v^0)_2,$$
and norm

\[ |u|^2 = (u, u) = \| \mathcal{A}^{1/2} u \|^2_2 + \| u^0 \|^2_2, \]

where \( u = u^- + u^0 + u^+ \) and \( v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+ \). Furthermore, the norms \( \| \cdot \|_0 \) and \( \| \cdot \| \) are equivalent by \( \mathcal{H} \). From now on we will take \((E, \| \cdot \|)\) instead of \((E, \| \cdot \|_0)\) as the working space without loss of generality.

**Remark 2.3.** We note that the decomposition \( E = E^- \oplus E^0 \oplus E^+ \) is also orthogonal with respect to inner products \((\cdot, \cdot)_1\) and \((\cdot, \cdot)_2\). Moreover, we will denote by \( E = E^- \oplus E^0 \oplus E^+ \) the orthogonal decomposition with respect to the inner products \((\cdot, \cdot)\) unless otherwise specified.

**Remark 2.4.** Note that the norms \( \| \cdot \|_0 \) and \( \| \cdot \| \) are equivalent, then by Lemma 2.1, for any \( 1 \leq p \in (2/(3 - \alpha), \infty] \), there exists constant \( \beta_p > 0 \) such that

\[ \| u \|_p \leq \beta_p \| u \|, \quad \forall u \in E, \quad (2.1) \]

where \( \| u \|_p \) denotes the usual norm of \( L^p \) and \( \beta_p \) is independent of \( u \).

Let

\[ \mathcal{O}(u, v) = (|\mathcal{A}|^{1/2} U u, |\mathcal{A}|^{1/2} v), \quad \forall u, v \in E \]

be the quadratic form associated with \( \mathcal{A} \), where \( U \) is the polar decomposition of \( \mathcal{A} \). For any \( u \in \mathcal{D}(\mathcal{A}) \) and \( v \in E \), we have

\[ \mathcal{O}(u, v) = \int_{\mathbb{R}} ((\dot{u}, \dot{v}) + (L(t) u, v)) dt. \quad (2.2) \]

Since \( \mathcal{D}(\mathcal{A}) \) is dense in \( E \), then (2.2) holds for all \( u, v \in E \). Furthermore, by definition, we can have

\[ \mathcal{O}(u, v) = ((P^+ - P^-) u, v) = \| u^+ \|^2 - \| u^- \|^2 \quad (2.3) \]

for all \( u = u^- + u^0 + u^+ \in E \), where \( P^\pm : E \to E^\pm \) are the respective orthogonal projections.

By (2.2) and (2.3), we define the functional \( \Phi \) on \( E \) by

\[ \Phi(u) = \frac{1}{2} \int_{\mathbb{R}} ((\dot{u})^2 + (L(t) u, u)) dt - \int_{\mathbb{R}} W(t, u) dt \]

\[ = \frac{1}{2} \| u^+ \|^2 - \frac{1}{2} \| u^- \|^2 - \int_{\mathbb{R}} W(t, u) dt \]

\[ = \frac{1}{2} \| u^+ \|^2 - \frac{1}{2} \| u^- \|^2 - \Psi(u), \quad (2.4) \]

where \( \Psi(u) = \int_{\mathbb{R}} W(t, u) dt = \int_{\mathbb{R}} a(t) |u|^2 dt \) for all \( u = u^- + u^0 + u^+ \in E = E^- \oplus E^0 \oplus E^+ \).

**Remark 2.5.** Combining (W) with Lemma 2.1, we can easily see that \( \Phi \) and \( \Psi \) are well-defined. We will consider two cases as follows.
Case (i) If $2 \leq \mu < \infty$, then
\[
|\Psi(u)| = \left| \int_{\mathbb{R}} W(t, u) dt \right| = \left| \int_{\mathbb{R}} a(t)|u|^\nu dt \right|
\leq \left( \int_{\mathbb{R}} |a(t)|^\mu dt \right)^{\frac{1}{\mu}} \left( \int_{\mathbb{R}} |u|^{\nu^*} dt \right)^{\frac{1}{\nu^*}}
= \|a\|_\mu \|u\|_{\nu^*}^{\nu^*} < \infty
\]
where $\frac{1}{\mu} + \frac{1}{\nu^*} = 1$, $\nu^* \geq 1$.

Case (ii) If $\mu = \infty$, then $|\Psi(u)| \leq \|a\|_\infty \|u\|_{\nu^*}^{\nu} < \infty$.

Lemma 2.6. Let $(L1)$, $(L2)$ and $(W)$ hold. Then $\Psi \in C^1(E, \mathbb{R})$ and $\Psi' : E \to E^*$ is compact, and consequently $\Phi \in C^1(E, \mathbb{R})$. Moreover,
\[
\Psi'(u)v = \int_{\mathbb{R}} (W_u(t, u), v) dt = \int_{\mathbb{R}} (\nu a(t)|u|^\nu \cdot u, v) dt
\]
(2.5)
\[
\Phi'(u)v = (u^+, v^+) - (u^-, v^-) - \Psi'(u)v
\]
(2.6)
for all $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+$, and any critical points of $\Phi$ on $E$ are homoclinic solutions of $(HS)$ satisfying $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

Proof. We first show that (2.5) holds by definition. If $2 \leq \mu < \infty$, then $1 < \mu^* \leq 2$, where $\frac{1}{\mu} + \frac{1}{\mu^*} = 1$. For any given $u, v \in E$, by the Mean Value Theorem and the Hölder inequality, we have
\[
\left| \int_{\mathbb{R}} [W(t, u + v) - W(t, u) - (W_u(t, u), v)] dt \right|
= \left| \int_{\mathbb{R}} \left[ \int_0^1 (W_u(t, u + \theta v) - W_u(t, u), v) d\theta \right] dt \right|
\leq 2\nu \int_{\mathbb{R}} |a(t)|(|u| + |v|)^{\nu - 1}|v| dt
\leq 2\nu \int_{\mathbb{R}} |a(t)|(|u|^{\nu - 1} + |v|^{\nu - 1})|v| dt
\]
\[\leq 2\nu \left( \int_{\mathbb{R}} |a(t)|^{\mu} dt \right)^{\frac{1}{\mu}} \left( \int_{\mathbb{R}} |u|^{\nu (\nu - 1)} |v|^{\nu^*} dt \right)^{\frac{1}{\nu^*}} + 2\nu |a|_\mu \|v\|_{\nu^*}^{\nu^*} \]
\[= 2\nu |a|_\mu \|u\|_{\nu^*}^{\nu^* - 1} \|v\| + 2\nu \beta_{\mu, \nu^*} |a|_\mu \|v\|_{\nu^*}^{\nu^*} \]
\[\leq 2\nu \beta_{\frac{2\mu^*}{2+\mu^*}, \frac{\nu^*}{\mu^*}} |a|_\mu \|u\|^{\nu^* - 1} \|v\| + 2\nu \beta_{\mu, \nu^*} |a|_\mu \|v\| \to 0, \quad \text{as } v \to 0 \text{ in } E
\]
where $\frac{2\mu^*}{2+\mu^*} \geq 1$ and the second inequality holds by the fact that if $0 < p < 1$, then $(|a| + |b|)^p \leq |a|^p + |b|^p$.\]
∀ a, b ∈ ℜ. If µ = ∞, then similar to the proof of (2.7), we can obtain
\[
\left| \int_\mathbb{R} [W(t, u + v) - W(t, u) - (W_u(t, u), v)] \, dt \right| \\
\leq 2ν\|a\|_\infty (\|u\|_\infty^{\nu-1} + \|v\|_\infty^{\nu-1}) \int |v| \, dt \\
\leq 2ν\|a\|_\infty^{\nu-1} \beta_1 (\|u\|_\infty^{\nu-1} + \|v\|_\infty^{\nu-1}) \|v\| \to 0, \quad \text{as} \quad v \to 0 \in E
\] (2.8)
where the last inequality holds by (2.1) and \(\beta_\infty, \beta_1\) are constants there. Combining (2.7) and (2.8), (2.5) holds immediately by the definition of Fréchet derivatives. Consequently, (2.6) also holds due to the definition of Φ.

Next, we verify that \(Ψ' : E \to E^*\) is compact. Let \(u_n \rightharpoonup u_0\) (weakly) in \(E\), by Lemma 2.1, we have \(u_n \to u_0\) in \(L^p\) for all \(1 \leq p < 2/(3 - \alpha), \infty\). If \(2 \leq \mu < \infty\), using the Hölder inequality, we can obtain
\[
\|Ψ'(u_n) - Ψ'(u_0)\|_{E^*} = \sup_{\|v\| = 1} \|\langle Ψ'(u_n) - Ψ'(u_0), v \rangle\|
\leq \sup_{\|v\| = 1} \left| \int_\mathbb{R} (W_u(t, u_n) - W_u(t, u_0), v) \, dt \right|
\leq \sup_{\|v\| = 1} \left[ \left( \int_\mathbb{R} |W_u(t, u_n) - W_u(t, u_0)|^\mu \, dt \right)^{\frac{1}{\mu}} \|v\|_{\mu} \right]
\leq \beta_\mu^* \left( \int_\mathbb{R} |W_u(t, u_n) - W_u(t, u_0)|^\mu \, dt \right)^{\frac{1}{\mu}}, \quad \forall \, n \in \mathbb{N}
\] (2.9)
where the last inequality holds by (2.1) and \(\beta_\mu^*\) is the constant there. \(\frac{1}{\mu} + \frac{1}{\mu^*} = 1\). Next, we will prove that \(W_u(t, u_n) \to W_u(t, u_0)\) in \(L^\mu(\mathbb{R}, \mathbb{R}^N)\). Observing that \(u_n\) is bounded in \(L^\infty\), then by the Jensen inequality, we have
\[
\int_\mathbb{R} |W_u(t, u_n) - W_u(t, u_0)|^\mu \, dt \\
\leq 2^{\mu-1} \nu^\mu \int_\mathbb{R} |a(t)|^\mu (|u_n|^\mu + |u_0|^\mu) \, dt \\
\leq 2^{\mu-1} \nu^\mu \int_\mathbb{R} |a(t)|^\mu (\|u_n\|_\infty^\mu + \|u_0\|_\infty^\mu) \, dt \\
\leq 2^{\mu-1} \nu^\mu M \int_\mathbb{R} |a(t)|^\mu \, dt
\]
where \(M = 2 \max\{\|u_0\|_\infty^\mu, \|u_n\|_\infty^\mu, \forall \, n \in \mathbb{N}\}\). Combining the fact that \(u_n \to u_0\) in \(L^\infty\) and the Lebesgue’s Dominated Convergence Theorem,
\[
\left( \int_\mathbb{R} |W_u(t, u_n) - W_u(t, u_0)|^\mu \, dt \right)^{\frac{1}{\mu}} \to 0, \quad \text{as} \quad n \to \infty.
\]
Next, we will deal with the case of \(\mu = \infty\) (i.e. \(\nu > \frac{3}{2}\)), this part is mainly motivated by the proof of Lemma 2 in [14]. By the Hölder inequality, we have
\[
\|Ψ'(u_n) - Ψ'(u_0)\|_{E^*} \leq \sup_{\|v\| = 1} \left[ \left( \int_\mathbb{R} |W_u(t, u_n) - W_u(t, u_0)|^2 \, dt \right)^{\frac{1}{2}} \|v\|_2 \right]
\leq \beta_2 \left( \int_\mathbb{R} |W_u(t, u_n) - W_u(t, u_0)|^2 \, dt \right)^{\frac{1}{2}}, \quad \forall \, n \in \mathbb{N}
\] (2.10)
We note that by Lemma 2.1, $u_n \to u_0$ in $L^{2(\nu-1)}$ for $\nu > \frac{3}{2}$, passing to a subsequence if necessary, it can be assumed that
\[
\sum_{n=1}^{\infty} \|u_n - u_0\|_{2(\nu-1)} < +\infty,
\]
which implies that
\[
\sum_{n=1}^{\infty} |u_n(t) - u_0(t)| = g(t) \in L^{2(\nu-1)}(\mathbb{R}, \mathbb{R}).
\]
Since $\nu > \frac{3}{2}$, then
\[
\int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u_0)|^2 \, dt \\
\leq \int_{\mathbb{R}} 2\nu^2 |a(t)|^2 (|u_n|^{2(\nu-1)} + |u_0|^{2(\nu-1)}) \, dt \\
\leq \int_{\mathbb{R}} 2\nu^2 |a(t)|^2 (2^{2\nu-3}|u_n - u_0|^{2(\nu-1)} + (2^{2\nu-3} + 1)|u_0|^{2(\nu-1)}) \, dt \\
\leq 2^{2\nu-1}\nu^2\|a\|_\infty^2 \int_{\mathbb{R}} (|g(t)|^{2(\nu-1)} + |u_0|^{2(\nu-1)}) \, dt \\
\leq 2^{2\nu-1}\nu^2\|a\|_\infty^2 (\|g\|_{2(\nu-1)} + \beta_2^{2(\nu-1)} \|u_0\|_{2(\nu-1)})
\]
Using the Lebesgue’s Dominated Convergence Theorem, we have
\[
\left( \int_{\mathbb{R}} |W_u(t, u_n) - W_u(t, u_0)|^2 \, dt \right)^{\frac{1}{2}} \to 0, \quad \text{as} \quad n \to \infty.
\]
Consequently, $\Psi'$ is weakly continuous, and then $\Psi'$ is continuous. Therefore $\Psi \in C^1(E, \mathbb{R})$ and hence $\Phi \in C^1(E, \mathbb{R})$. Moreover, $\Psi'$ is compact by the weak continuity of $\Psi'$ since $E$ is a Hilbert Space.

Finally, we will show that the critical points of $\Phi$ on $E$ are homoclinic solutions of (HS). By standard arguments, we can see that any critical points of $\Phi$ on $E$ satisfy (HS) and $u \in C^2(\mathbb{R}, \mathbb{R}^N)$. We note that if $1 < \nu < \frac{4}{3}$, then $2 \leq \mu \leq \frac{2}{\frac{3}{2} - 2\nu}$. For $\mu = 2$, by (HS), we have
\[
\|Au\|_2^2 = \int_{\mathbb{R}} |W_u(t, u)|^2 \, dt \\
\leq \nu^2 \|u\|_{2(\nu-1)}^{2(\nu-1)} \int_{\mathbb{R}} |a(t)|^2 \, dt \\
\leq \nu^2 \beta_2^{2(\nu-1)} \|u\|_{2(\nu-1)}^{2(\nu-1)} \int_{\mathbb{R}} |a(t)|^\mu \, dt < \infty. \tag{2.11}
\]
In the case of $2 < \mu \leq \frac{2}{\frac{3}{2} - 2\nu}$, then
\[
\|Au\|_2^2 = \int_{\mathbb{R}} |W_u(t, u)|^2 \, dt \\
\leq \nu^2 \left( \int_{\mathbb{R}} |a(t)|^\mu \, dt \right)^{\frac{2}{\mu}} \left( \int_{\mathbb{R}} |u|^{2\mu(\nu-1)} \, dt \right)^{\frac{1}{\mu}} \\
\leq \nu^2 \beta_2^{2(\nu-1)} \|u\|_{2(\nu-1)}^{2(\nu-1)} \left( \int_{\mathbb{R}} |a(t)|^\mu \, dt \right)^{\frac{2}{\mu}} \\
\leq \nu^2 \beta_2^{2(\nu-1)} \|u\|_{2(\nu-1)}^{2(\nu-1)} \left( \int_{\mathbb{R}} |a(t)|^\mu \, dt \right)^{\frac{2}{\mu}} < \infty, \tag{2.12}
\]
Then there exist
\[ \lambda, u \rightarrow n \]
In particular, if critical points \( f_2 \)
\( \Phi \) defined above satisfies
(1) \( \Phi \) maps bounded sets to bounded sets uniformly for \( \lambda \in [1, 2] \). Furthermore, \( \Phi(u) = \Phi(u) \) for all
(2) \( B(u) \geq 0; B(u) \rightarrow \infty \) as \( \|u\| \rightarrow \infty \) on any finite dimensional subspace of \( E \);
(3) There exist \( \rho_k > r_k > 0 \) such that
\[ a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi(u) \geq 0 > b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi(u), \quad \forall \lambda \in [1, 2] \]
and
\[ d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi(u) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \text{ uniformly for} \lambda \in [1, 2]. \]

Then there exist \( \lambda_n \rightarrow 1, u_{\lambda_n} \in Y_n \) such that
\[ \Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \quad \Phi_{\lambda_n}(u_{\lambda_n}) \rightarrow c_k \in [d_k(2), b_k(1)] \quad \text{as} \quad n \rightarrow \infty. \]

In particular, if \( \{u_{\lambda_n}\} \) has a convergent subsequence for every \( k \), then \( \Phi_1 \) has infinitely many nontrivial critical points \( \{u_k\} \in E \setminus \{0\} \) satisfying \( \Phi_1(u_k) \rightarrow 0^{-} \) as \( k \rightarrow \infty \).

In order to apply Theorem 2.7, we define the functionals \( A, B \) and \( \Phi \) on the working space \( E = \mathcal{D}(|A|^{1/2}) \) by
\[ A(u) = \frac{1}{2}\|u^+\|^2, \quad B(u) = \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} W(t, u)dt, \quad (2.13) \]
and
\[ \Phi(u) = A(u) - \lambda B(u) = \frac{1}{2}\|u^+\|^2 - \lambda \left( \frac{1}{2}\|u^-\|^2 + \int_{\mathbb{R}} W(t, u)dt \right) \quad (2.14) \]
for all \( u = u^- + u^0 + u^+ \in E \) and \( \lambda \in [1, 2] \). By Lemma 2.6, we can see that \( \Phi \) is \( C^1(E, \mathbb{R}) \) for all \( \lambda \in [1, 2] \).

Let \( X_j := \text{span}\{e_j\}, j \in \mathbb{N}, \) where \( \{e_j, j \in \mathbb{N}\} \) is the system of eigenfunctions and the orthogonal basis in \( L^2 \) below Lemma 2.2. Furthermore, we note that \( \Phi_1 = \Phi \), where \( \Phi \) is the functional defined in (2.4).
3 Proof of theorems

Lemma 3.1. Let (L1), (L2) and (W) hold, then $B(u) \geq 0$. Moreover, $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $E$.

Proof. From the definition of the functional $B$ and (W), $B(u) \geq 0$ holds obviously. Next we will prove that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $E$. Now we claim that for any finite dimensional subspace $F \subset E$, there exists $\varepsilon > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : a(t)|u(t)|^{\nu} \geq \varepsilon \|u\|^{\nu} \} \geq \varepsilon, \quad \forall u \in F \setminus \{0\}. \quad (3.1)$$

If not, for any $n \in \mathbb{N}$, there exists $u_n \in F \setminus \{0\}$ such that

$$\text{meas}\{t \in \mathbb{R} : a(t)|u_n(t)|^{\nu} \geq \frac{1}{n} \|u_n\|^{\nu} \} < \frac{1}{n}. \quad (3.2)$$

Set $v_n = \frac{u_n}{\|u_n\|} \in F \setminus \{0\}$, then $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\text{meas}\{t \in \mathbb{R} : a(t)|v_n(t)|^{\nu} \geq \frac{1}{n} \} < \frac{1}{n}. \quad (3.2)$$

Since $\dim F < \infty$, passing to a subsequence if necessary, we can assume that $v_n$ converges to some $v_0$ in $F$. Obviously, $\|v_0\| = 1$. Due to the equivalence of any two norms on $F$, by Lemma 2.1, we have $v_n \to v_0$ in $L^p$, $p \in [1, \infty)$, i.e.

$$\int_{\mathbb{R}} |v_n - v_0|^p dt \to 0, \quad \text{as } n \to \infty. \quad (3.3)$$

If $2 \leq \mu < \infty$, then by (3.3) and the Hölder inequality, we have

$$\int_{\mathbb{R}} a(t)|v_n - v_0|^{\nu} dt \leq \left( \int_{\mathbb{R}} |a(t)|^{\mu} \right)^{\frac{1}{\mu}} \left( \int_{\mathbb{R}} |v_n - v_0|^{\nu + \mu} \right)^{\frac{1}{\nu + \mu}} = \|a\|_{L^\mu} \left( \int_{\mathbb{R}} |v_n - v_0|^{\nu + \mu} \right)^{\frac{1}{\nu + \mu}} \to 0, \quad \text{as } n \to \infty, \quad (3.4)$$

where $\frac{1}{\mu} + \frac{1}{\nu + \mu} = 1$, $\nu \mu^* \geq 1$. Consequently, there exists $\delta > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : a(t)|v_0(t)|^{\nu} \geq \delta \} \geq \delta. \quad (3.5)$$

Otherwise, for any $n \in \mathbb{N}$, we have

$$\text{meas}\{t \in \mathbb{R} : a(t)|v_0(t)|^{\nu} \geq \frac{1}{n} \} = 0,$$

which implies that

$$0 \leq \int_{\mathbb{R}} a(t)|v_0(t)|^{\nu + \mu} dt \leq \frac{1}{n} \|v_0\|_p^{\nu + \mu} \to 0, \quad \text{as } n \to \infty.$$ 

Thus, $v_0 = 0$ which is in contradiction to the fact that $\|v_0\| = 1$. Next, for all $n \in \mathbb{N}$, we let

$$\Omega_0 = \{t \in \mathbb{R} : a(t)|v_0(t)|^{\nu} \geq \delta\}, \quad \Omega_n = \{t \in \mathbb{R} : a(t)|v_n(t)|^{\nu} < \frac{1}{n}\}$$

and
and $\Omega^c_n = \mathbb{R} \setminus \Omega_n = \{ t \in \mathbb{R} : a(t)|v_n(t)|^{\nu} \geq \frac{1}{n} \}$. Therefore, for $n$ large enough, from (3.2) and (3.5), we have

$$\text{meas}(\Omega_n \cap \Omega_0) \geq \text{meas}(\Omega_0) - \text{meas}(\Omega^c_n) \geq \delta - \frac{1}{n} \geq \frac{\delta}{2} \quad \text{and} \quad \frac{1}{2^{\nu-1}} \delta - \frac{1}{n} \geq \frac{1}{2^{\nu}} \delta.$$  \hspace{1cm} (3.6)

Then, for $n$ large enough, we have

$$\int_{\mathbb{R}} a(t)|v_n - v_0|^\nu dt \geq \int_{\Omega_n \cap \Omega_0} a(t)|v_n - v_0|^\nu dt \geq \frac{1}{2^{\nu-1}} \int_{\Omega_n \cap \Omega_0} a(t)|v_0|^\nu dt - \int_{\Omega_n \cap \Omega_0} a(t)|v_n|^\nu dt \geq \left( \frac{1}{2^{\nu-1}} \delta - \frac{1}{n} \right) \text{meas}(\Omega_n \cap \Omega_0) \geq \frac{1}{2^{\nu+1}} \delta^2 > 0,$$

which contradicts to (3.4), so (3.1) holds. Now, we let

$$\Omega_u = \{ t \in \mathbb{R} : a(t)|u(t)|^{\nu} \geq \varepsilon \|u\|^\nu \}, \quad \forall \ u \in F \setminus \{0\},$$  \hspace{1cm} (3.7)

where $\varepsilon$ is given in (3.1). Then by (3.1), we have

$$\text{meas}(\Omega_u) \geq \varepsilon, \quad \forall \ u \in F \setminus \{0\},$$  \hspace{1cm} (3.8)

Combining (W) and (3.8), for all $u \in F \setminus \{0\}$, we can see that

$$B(u) = \frac{1}{2}\|u^+\|^2 + \int_{\mathbb{R}} W(t, u)dt \geq \int_{\Omega_u} a(t)|u(t)|^{\nu} dt \geq \varepsilon \|u\|^\nu \text{meas}(\Omega_u) \geq \varepsilon^2 \|u\|^\nu,$$

which implies $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $E$. If $\mu = \infty$, similar to the case of $2 \leq \mu < \infty$, by the standard arguments we can prove that there exists $\varepsilon_1 > 0$ such that

$$\text{meas}\{ t \in \mathbb{R} : a(t)|u(t)|^{\nu} \geq \varepsilon_1 \|u\|^\nu \} \geq \varepsilon_1, \quad \forall \ u \in F \setminus \{0\}. \hspace{1cm} (3.10)$$

Therefore, by (3.9), we can conclude that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of $E$. The proof is complete. \hfill \Box

**Lemma 3.2.** Under the conditions in Theorem 1.1, then there exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that

$$a_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \Phi_\lambda(u) \geq 0, \quad \forall \ \lambda \in [1, 2], \ k \geq \bar{n} + 1,$$

and

$$d_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad \text{as} \ k \to \infty \quad \text{uniformly for} \ \lambda \in [1, 2],$$

where $Z_k = \bigoplus_{j=k}^{\infty} X_j$ for all $k \in \mathbb{N}$.
**Proof.** By the definition of $\bar{n}$ below the Lemma 2.2, we can know that $Z_k \subset E^+$ for all $k \geq \bar{n} + 1$. Therefore, for all $k \geq \bar{n} + 1$, by (W) and (2.14), we have

$$
\Phi_\lambda(u) = \frac{1}{2}||u||^2 - \lambda \int_\mathbb{R} W(t, u)dt
\geq \frac{1}{2}||u||^2 - 2 \int_\mathbb{R} W(t, u)dt
= \frac{1}{2}||u||^2 - 2 \int_\mathbb{R} a(t)||u||^\nu dt, \quad \forall (\lambda, u) \in [1, 2] \times Z_k.
$$

(3.11)

If $2 \leq \mu < \infty$, we let $\eta_k := \sup_{u \in Z_k, ||u|| = 1} ||u||_\nu^{\nu^*}$, where $\frac{1}{\mu} + \frac{1}{\nu^*} = 1$. By Lemma 2.1, we can conclude that $\eta_k \to 0$ as $k \to \infty$. Therefore, combining (3.11) with (W), we have

$$
\Phi_\lambda(u) \geq \frac{1}{2}||u||^2 - 2 ||a||_\mu ||u||_{\nu^*}^{\nu^*} \geq \frac{1}{2}||u||^2 - 2 \eta_k^\nu ||a||_\mu ||u||^{\nu^*}, \quad \forall (\lambda, u) \in [1, 2] \times Z_k.
$$

(3.12)

Let $\rho_k := (8\eta_k^\nu ||a||_\mu)^{1/(2-\nu)}$, due to the convergence of $\eta_k$, it is clearly that $\rho_k \to 0$ as $k \to \infty$. Consequently, by the definition of $\rho_k$ and (3.12), we have

$$
a_k(\lambda) := \inf_{u \in Z_k, ||u|| = \rho_k} \Phi_\lambda(u) \geq \frac{1}{4}\rho_k^2 > 0, \quad \forall k \geq \bar{n} + 1.
$$

(3.13)

Moreover, by (3.11), for all $k \geq \bar{n} + 1$ and $u \in Z_k$ with $||u|| \leq \rho_k$, we obtain that

$$
\Phi_\lambda(u) \geq -2\eta_k^\nu ||a||_\mu ||u||^{\nu^*} \geq -2\eta_k^\nu ||a||_\mu \rho_k^{\nu^*}.
$$

We notice that $1 < \nu < 2$, then

$$
0 \geq \inf_{u \in Z_k, ||u|| \leq \rho_k} \Phi_\lambda(u) \geq -2\eta_k^\nu ||a||_\mu \rho_k^{\nu^*}, \quad \forall k \geq \bar{n} + 1.
$$

Since $\eta_k, \rho_k \downarrow 0$ as $k \to \infty$, we have

$$
d_k(\lambda) := \inf_{u \in Z_k, ||u|| \leq \rho_k} \Phi_\lambda(u) \to 0 \quad as \ k \to \infty \ uniformly \ for \ \lambda \in [1, 2].
$$

For the case of $\mu = \infty$, similar to the above procedure, we can get the same result. We omit it here. The proof is complete. \hfill \Box

**Lemma 3.3.** Assume that (L1), (L2) and (W) hold, then for the sequence $\{\rho_k\}_{k \in \mathbb{N}}$ obtained in Lemma 3.2, there exists a sequence $\{r_k\}_{k \in \mathbb{N}}$ such that $\rho_k > r_k > 0$ for $\forall k \in \mathbb{N}$ and

$$
b_k(\lambda) := \max_{u \in Y_k, ||u|| = r_k} \Phi_\lambda(u) < 0, \quad \forall \lambda \in [1, 2].
$$

(3.14)

where $Y_k = \bigoplus_{j=1}^k X_j = \text{span}\{e_1, \ldots, e_k\}$ for $\forall k \in \mathbb{N}$. 

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Proof. For \( \forall \, k \in \mathbb{N} \), we notice that \( Y_k \) is a finite dimensional subspace of \( E \). Therefore, for \( \forall \, \lambda \in [1, 2] \), from (W), (3.7), (3.8) and (3.10), let \( \varepsilon_0 = \min\{\varepsilon, \varepsilon_1\} \), we have

\[
\Phi_{\lambda}(u) = \frac{1}{2} \|u^+\|^2 - \lambda \left( \frac{1}{2} \|u^-\|^2 + \int_{\Omega} W(t, u) dt \right)
\]

\[
\leq \frac{1}{2} \|u\|^2 - \int_{\Omega} W(t, u) dt
\]

\[
\leq \frac{1}{2} \|u\|^2 - \int_{\Omega} a(t)|u|^\nu dt
\]

\[
\leq \frac{1}{2} \|u\|^2 - \varepsilon_0\|u\|^\nu \text{meas}(\Omega_u)
\]

\[
\leq \frac{1}{2} \|u\|^2 - \varepsilon_0^2\|u\|^\nu, \quad \forall \, u \in Y_k, k \in \mathbb{N}.
\]

For \( \forall \, k \in \mathbb{N} \), we choose \( 0 < r_k < \min\{\rho_k, \varepsilon_0^{\frac{2}{\nu-1}}\} \). From (3.15), straightforward computation shows that

\[
b_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \Phi_{\lambda}(u) \leq -\frac{r_k^2}{2} < 0, \quad \forall \, k \in \mathbb{N}.
\]

The proof is complete. \( \Box \)

Next we will give the proof of our main result.

Proof of Theorem 1.1. Combining Remark 2.5 and (2.14), it is clearly that the condition (F1) in Theorem 2.7 holds. By Lemma 3.1, 3.2 and 3.3, we can see that conditions (F2) and (F3) in Theorem 2.7 hold for all \( k \geq \bar{n} + 1 \). Consequently, from Theorem 2.7, for all \( k \geq \bar{n} + 1 \), there exist \( \lambda_n \to 1 \), \( u_{\lambda_n} \in Y_n \) such that

\[
\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0, \, \Phi_{\lambda_n}(u_{\lambda_n}) \to c_k \in [d_k(2), b_k(1)] \quad \text{as} \quad n \to \infty.
\]

In the following the fist step is to show that \( \{u_{\lambda_n}\} \) is bounded in \( E \). For the case of \( 2 \leq \mu < \infty \), since \( \Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0 \), by (2.6) and (2.14), we have

\[
\Phi'_{\lambda_n}|_{Y_n}(u_{\lambda_n})u^+_{\lambda_n} = \|u^+_{\lambda_n}\|^2 - \lambda_n \int_{\Omega} (W_n(t, u_{\lambda_n}), u^+_{\lambda_n}) dt = 0.
\]

Therefore, by (W) and the Hölder inequality, we can obtain that

\[
\|u^+_{\lambda_n}\|^2 = \lambda_n \int_{\Omega} (W_n(t, u_{\lambda_n}), u^+_{\lambda_n}) dt
\]

\[
\leq 2 \int_{\Omega} |a(t)||u_{\lambda_n}|^{\nu-1}|u^+_{\lambda_n}| dt
\]

\[
\leq 2 \left( \int_{\Omega} |a(t)|^\mu dt \right)^{\frac{1}{\mu}} \left( \int_{\Omega} |u_{\lambda_n}|^{\mu(\nu-1)}|u^+_{\lambda_n}|^{\mu^*} dt \right)^{\frac{1}{\mu^*}}
\]

\[
\leq 2 \nu |a||u_{\lambda_n}|^{\nu-1} \left( \int_{\Omega} |u^+_{\lambda_n}|^{2^*} dt \right)^{\frac{\nu-1}{2^*}} \left( \int_{\Omega} \|u^+_{\lambda_n}\|^{2\nu} dt \right)^{\frac{2}{2\nu}}
\]

\[
= 2 \nu |a||u_{\lambda_n}|^{\nu-1} \|u^+_{\lambda_n}\|^{\frac{2\nu}{2\nu-1}} \|u^+_{\lambda_n}\|^{\frac{2\nu}{2\nu-1}}
\]

\[
\leq M_1 |a||u_{\lambda_n}|^{\nu}
\]

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for some $M_1 > 0$, where \( \frac{1}{\mu} + \frac{1}{\nu} = 1 \), \( \frac{2\nu^2}{2 + 2\mu^2 - \mu^2} \geq 1 \) and the last inequality holds by (2.1). Furthermore, combing (2.6) with (3.16) and the Hölder inequality, we have
\[
-\Phi_{\lambda_n}(u_{\lambda_n}) - \frac{1}{2} \Phi_{\lambda_n}'(y_n(u_{\lambda_n}))y_n(u_{\lambda_n}) - \Phi_{\lambda_n}(u_{\lambda_n}) = \lambda_n(1 - \nu) \int \frac{|a(t)||u_{\lambda_n}|^\nu}{dt}
\geq \frac{1}{2\nu^2 + \lambda_n(1 - \nu) \int \frac{|a(t)||u_{\lambda_n} + u_{\lambda_n}^0|^\nu}{dt} - \lambda_n(1 - \nu) \int \frac{|a(t)||u_{\lambda_n}^+|^\nu}{dt}
\geq \frac{e^2}{2\nu^2 + \lambda_n(1 - \nu)} ||u_{\lambda_n} + u_{\lambda_n}^0||^\nu - \lambda_n(1 - \nu)||a||_\mu ||u_{\lambda_n}^+||^\nu \]
where the last inequality holds by the fact that \( \dim(E \oplus E^0) < \infty \) and (3.1). We notice that \( 1 < \nu < 2 \), then (3.18) and (3.19) implies that \( \{|u_{\lambda_n}^+|^2\} \) is bounded. Next, we just have to show that \( \{|u_{\lambda_n}^- + u_{\lambda_n}^0|^2\} \) is also bounded. Consequently, from (3.19) and (2.1), we get
\[
||u_{\lambda_n}^- + u_{\lambda_n}^0||^\nu \leq -M_2 \Phi_{\lambda_n}(u_{\lambda_n}) + M_3 ||u_{\lambda_n}^-||^\nu \mu \leq -M_2 \Phi_{\lambda_n}(u_{\lambda_n}) + M_4 ||u_{\lambda_n}^-||^\nu \]
for some positive constants \( M_2, M_3 \) and \( M_4 \). Notice that \( \{|u_{\lambda_n}^-|^2\} \) is bounded, by (3.16), we can conclude that \( \{|u_{\lambda_n}^- + u_{\lambda_n}^0|^2\} \) is also bounded. Therefore, there exists \( M_5 > 0 \) such that \( ||u_{\lambda_n}^-||^2 + ||u_{\lambda_n}^- + u_{\lambda_n}^0||^2 \leq M_5 \), i.e. \( \{u_{\lambda_n}\} \) is bounded in \( E \).

Finally, we prove that \( \{u_{\lambda_n}\} \) has a strong convergent subsequence in \( E \). In fact, due to the boundedness of \( \{u_{\lambda_n}\} \) and \( \dim(E \oplus E^0) < \infty \), without loss of generality, we can assume that
\[
u_{\lambda_n} \rightarrow u_0 \text{ and } u_{\lambda_n}^- \rightarrow u_0^- , \quad u_{\lambda_n}^0 \rightarrow u_0^0 , \quad u_{\lambda_n}^+ \rightarrow u_0^+ , \quad \text{as } n \rightarrow \infty ,
\]
for some \( u_0 = u_0^- + u_0^0 + u_0^+ \in E^- \oplus E^0 \oplus E^+ \). As a result of the Riesz Representation Theorem, \( \Phi_{\lambda_n}'|_{Y_n} : Y_n \rightarrow Y_n^* \) and \( \Psi' : E \rightarrow E^* \) can be viewed as \( \Phi_{\lambda_n}'|_{Y_n} : Y_n \rightarrow Y_n \) and \( \Psi' : E \rightarrow E \) respectively, where \( Y_n^* \) and \( E^* \) are the corresponding dual spaces. Notice that
\[
0 = \Phi_{\lambda_n}'|_{Y_n}(u_{\lambda_n}) = u_{\lambda_n}^+ - \lambda_n(u_{\lambda_n}^- + P_n \Psi'(u_{\lambda_n})) , \quad \forall \ n \in \mathbb{N} ,
\]
where \( P_n : E \rightarrow Y_n \) is the orthogonal projection for \( \forall \ n \in \mathbb{N} \). That is to say
\[
u_{\lambda_n}^+ = \lambda_n(u_{\lambda_n}^- + P_n \Psi'(u_{\lambda_n})) \quad \forall \ n \in \mathbb{N} ,
\]
By the proof of Lemma 2.6, we can see that \( \Psi' : E \rightarrow E \) is weakly continuous. Combining the weakly continuity of \( \Psi' \) and (3.21), we have \( \lambda_n(u_{\lambda_n}^- + P_n \Psi'(u_{\lambda_n})) \) converges strongly in \( E \) and hence \( u_{\lambda_n}^+ \rightarrow u_0^+ \) in \( E \), which together with (3.21) implies \( u_{\lambda_n} \rightarrow u_0 \) in \( E \). If \( \mu = \infty \), the proof is similar to the above arguments, we omit it here.

Now by the last conclusion of Theorem 2.7, we obtain that \( \Phi = \Phi_1 \) has infinitely many nontrivial critical points. Consequently, \( \text{HS} \) possesses infinitely many homoclinic solutions by Lemma 2.6. The proof of Theorem 1.1 is complete. \( \square \)
Remark 3.4. In this paper, we have considered the existence of infinitely many homoclinic solutions for a class of subquadratic second-order Hamiltonian systems, where \(1 < \nu < \frac{3}{2}\) is allowed. We view this result as merely one first step in the theory for the case of \(1 < \nu < \frac{3}{2}\), there are still many problems to pursue. For example, when \(1 < \nu < \frac{3}{2}\), the upper bound of \(\mu\) whether can be \(\infty\), what we will discuss in the future study.

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