THE UNIFORM SPREADING SPEED IN COOPERATIVE SYSTEMS WITH NON-UNIFORM INITIAL DATA

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Abstract. This paper considers the spreading speed of cooperative nonlocal dispersal system with irreducible reaction functions and non-uniform initial data. Here the non-uniformity means that all components of initial data decay exponentially but their decay rates are different. It is well-known that in a monostable reaction-diffusion or nonlocal dispersal equation, different decay rates of initial data yield different spreading speeds. In this paper, we show that due to the cooperation and irreducibility of reaction functions, all components of the solution with non-uniform initial data will possess a uniform spreading speed which decreasingly depends only on the smallest decay rate of initial data. The decreasing property of the uniform spreading speed in the smallest decay rate further implies that the component with the smallest decay rate can accelerate the spatial propagation of other components. In addition, all the methods in this paper can be carried over to the cooperative system with classical diffusion (i.e. random diffusion).

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1. Introduction

The long-range dispersal, such as the spread of infectious disease across countries and continents by the travel of infected humans \[19\], has increasingly become an important phenomenon nowadays, and it has attracted extensive attention of researchers (see \[7, 32, 36\]). Mathematically the long-range dispersal can be modelled by a nonlocal dispersal operator that describes the movements between not only adjacent but also nonadjacent spatial locations. A typical nonlocal dispersal equation with reaction is given by

\[
\frac{du}{dt} = k \ast u - u + f(u), \quad t > 0, \quad x \in \mathbb{R},
\]

where \(u(t, x)\) stands for the population density at location \(x\) and time \(t\), \(f(u)\) is a reaction function, and the nonlocal dispersal operator is represented by

\[
k \ast u(t, x) - u(t, x) = \int_{\mathbb{R}} k(x - y)u(t, y)dy - u(t, x).
\]

Here \(k : \mathbb{R} \to \mathbb{R}\) is a nonnegative and continuous function with \(\int_{\mathbb{R}} k(x)dx = 1\). As stated in \[15\], \(k(x - y)\) can be viewed as the probability for individuals to move from location \(y\) to location \(x\), \(k \ast u(t, x) = \int_{\mathbb{R}} k(x - y)u(t, y)dy\) stands for the rate at which individuals arrive at location \(x\) from other locations, and \(-u(t, x) = -\int_{\mathbb{R}} k(y - x)u(t, x)dy\) is the rate at which individuals

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leave location \( x \) and move to other locations. One of the most significant research topics in the literature for (1.1) is the wave propagation phenomena which are associated with the studies of traveling wave solutions, entire solutions and spreading speeds. These results can be used to describe the spreading process of populations, such as the spatial spread of infectious diseases and the invasion of species. For the traveling wave solutions of (1.1), we refer to the classical works by Bates et al. [7], Carr and Chmaj [8], Chen [9], Chen and Guo [10], Coville, Dávila and Martínez [12], Schumacher [39], Yagisita [51], etc. For the entire solutions of (1.1), we refer to, for example, Li, Sun and Wang [24]. For the spreading speeds of (1.1), we refer to the works by Lutscher, Pachepsky and Lewis [30], Shen and Zhang [40], Zhang, Li and Wang [53], Rawal, etc. For the entire solutions of (1.1), we refer to, the works by Bates et al. [5], Carr and Chmaj [8], Chen [9], Chen and Guo [10], Coville, Dávila and Martínez [12], Schumacher [39], Yagisita [51], etc. For the traveling wave solutions of (1.1), we refer to the classical traveling wave solutions, entire solutions and spreading speeds. These results can be used to literature for (1.1) is the wave propagation phenomena which are associated with the studies of invasion of species. For the traveling wave solutions of (1.1), we refer to, for example, Li, Sun and Wang [24]. For the spreading speeds of (1.1), we refer to the works by Lutscher, Pachepsky and Lewis [30], Shen and Zhang [40], Zhang, Li and Wang [53], Rawal, etc. For the entire solutions of (1.1), we refer to, the works by Bates et al. [5], Carr and Chmaj [8], Chen [9], Chen and Guo [10], Coville, Dávila and Martínez [12], Schumacher [39], Yagisita [51], etc. For the entire solutions of (1.1), we refer to, for example, Li, Sun and Wang [24].

In this paper, we are concerned with the spreading speed of the following \( m \)-component nonlocal dispersal system

\[
\begin{align*}
U_t &= D(K * U - U) + F(U), & t > 0, \ x \in \mathbb{R}, \\
U(0, x) &= U_0(x) = (u_{1,0}(x), \ldots, u_{m,0}(x)), & x \in \mathbb{R},
\end{align*}
\]

(1.2)

where \( U = (u_1, \ldots, u_m) \), \( K = (k_1, \ldots, k_m) \), \( F = (f_1, \ldots, f_m) \), \( D = \text{diag}(d_1, \ldots, d_m) \) with \( d_j > 0 \), and \( 2 \leq m \in \mathbb{Z}^+ \). The nonlocal dispersal is represented by

\[ K * U(t, x) - U(t, x) \overset{\Delta}{=} (k_1 * u_1(t, x) - u_1(t, x), \ldots, k_m * u_m(t, x) - u_m(t, x)). \]

We assume that \( F(U) \) is cooperative (namely \( \frac{\partial}{\partial x} f_j(U) \geq 0 \) for any \( j \neq i \)) and monostable with an unstable equilibrium \( U \equiv 0 \in \mathbb{R}^m \) and a stable equilibrium \( U \equiv P \in (\mathbb{R}^+)^m \). Assume that

\[ U_0(\cdot) \not\equiv 0, \ 0 \leq U_0(x) \leq P \text{ for all } x \in \mathbb{R}. \]

The kernel \( K \in C(\mathbb{R}, \mathbb{R}^m) \) is symmetric on \( \mathbb{R} \) and satisfies the Mollison condition (see [12, 35, 36]), in the sense that, there exists \( \Lambda > 0 \) such that

\[ \int_{\mathbb{R}} k_j(x) e^{\Lambda |x|} dx < +\infty, \ j \in \{1, \ldots, m\}. \]

The local dispersal system, as a counterpart of (1.2), is called the reaction-diffusion system which reads as

\[
\begin{align*}
U_t &= D \Delta U + F(U), & t > 0, \ x \in \mathbb{R}, \\
U(0, x) &= U_0(x), & x \in \mathbb{R}.
\end{align*}
\]

(1.3)

When \( m = 2 \), traveling wave solutions and entire solutions were obtained for (1.2) by Li, Xu and Zhang [25], Meng, Yu and Hsu [31], and for (1.3) by Hsu and Yang [20], Zhao and Wang [54], Xu and Zhao [46], Wu and Hsu [45]. When the initial data \( U_0 \) are compactly supported (or equivalently \( U_0(x) \equiv 0 \) for large \( x > 0 \)), there are numerous results on the spreading spread of (1.2) and (1.3). For the nonlocal dispersal system (1.2), we refer to Bao et al. [3], Bao, Shen and Shen [4], Hu et al. [21]. For the local dispersal system (1.3) and its discrete-time counterpart, we refer to Kolmogorov, Petrovsky and Piskunov [22] and Aronson and Weinberger [12] for the case \( m = 1 \) (i.e. classical reaction-diffusion equation), and Weinberger [43], Lui [31], Weinberger, Lewis and Li [44], Li, Weinberger and Lewis [26], Liang and Zhao [27, 28], Fang and Zhao [14], and Wang [12] for the case \( m \geq 2 \).

Note that the aforementioned existing results on the spreading speeds of (1.2) and (1.3) essentially assume that the initial data \( U_0(x) \) are compactly supported. However, when the
initial data \( U_0(x) \) are not compactly supported, the results of spreading speed are much fewer. Especially, when the initial value function decays exponentially, namely

\begin{equation}
(1.4) \quad u(0, x) \sim C e^{-\sigma|x|} \text{ as } |x| \to +\infty \text{ with } \sigma > 0, \ C > 0,
\end{equation}

the system (1.2) with \( m = 1 \), namely (1.1), has a spreading speed

\begin{equation}
(1.5) \quad s(\sigma) = \frac{1}{\sigma} \left\{ \int_{\mathbb{R}} k(x)e^{\sigma x} dx - 1 + f'(0) \right\} \text{ for } \sigma \in (0, \sigma^*),
\end{equation}

where \( \sigma^* = \min\{\sigma > 0 \mid s(\sigma) = \min\{s(\sigma); \sigma > 0\}\} \), see e.g. \([13, 41, 50]\). Similar results for (1.3) with \( m = 1 \) (i.e. reaction-diffusion equation) and exponentially decaying initial data were previously obtained by Booty, Haberman and Minzon \([6]\), Hamel and Nadin \([18]\), McKeen \([33]\), and Sattinger \([35]\), etc. When \( m = 2 \), a recent work by Xu, Li and Ruan \([48]\) studied the spreading speed of (1.2) for initial data \( u_{1,0}(x) \) and \( u_{2,0}(x) \) decaying exponentially with the same decay rate.

The purpose of this paper is to study the spreading speed of (1.2) where \( m \geq 2 \) and all components of initial data \( U_0 \) decay exponentially but their decay rates may be different. That is we assume that each component of \( U_0(x) \) has its own decay rate, namely

\begin{equation}
(1.6) \quad u_{j,0}(x) \sim C_j e^{-\lambda_j|x|} \text{ as } |x| \to +\infty \text{ with } C_j > 0 \text{ for any } j \in J \triangleq \{1, \ldots, m\}.
\end{equation}

We call the initial data \( U_0(x) \) are non-uniform if there exist some \( i, j \in J \) with \( i \neq j \) such that \( \lambda_i \neq \lambda_j \). The case of non-uniform initial data considered in this paper is essentially different from the case in \([48]\) where \( m = 2 \) and \( \lambda_1 = \lambda_2 \). From (1.5) and other results mentioned above, we conclude that the spreading speed of scalar dispersal equations essentially depends on the decay rate of exponentially decaying initial data. For the dispersal system, if all components of initial data \( U_0 \) have the same decay rate (i.e. uniform initial data), the spreading speed can still be determined by this single decay rate as shown in \([48]\) for \( m = 2 \). But now if the initial data are non-uniform, an immediate question is whether all components of (1.2) have the same spreading speed, and if so, which component will play a prevailing role in determining this spreading speed. To proceed, we give the definition of uniform spreading speed of (1.2).

**Definition 1.1** (Uniform spreading speed). Given initial data \( U_0 \) satisfying (1.6), a positive constant \( c_0 \) is called the uniform spreading speed of the solution of (1.2), if for any \( j \in J \) and \( \varepsilon \in (0, c_0) \), there is a constant \( \nu > 0 \) such that

\[
\left\{ \begin{array}{l}
\lim_{t \to +\infty} \sup_{|x| > (c_0 + \varepsilon)t} u_j(t, x) = 0, \\
\lim_{t \to +\infty} \inf_{|x| \leq (c_0 - \varepsilon)t} u_j(t, x) \geq \nu.
\end{array} \right.
\]

We will show that when the reaction function \( F \) is cooperative and \( F'(0) \) is irreducible, all components of the solution \( U \) of (1.2) with non-uniform initial data (different decay rates) satisfying (1.6) have a uniform spreading speed (the same spreading speed), see Theorem 2.2. Furthermore, this uniform spreading speed depends only on the smallest decay rate \( \lambda_0 \triangleq \min\{\lambda_j, j \in J\} \) and is decreasing with respect to \( \lambda_0 \), which implies that the component with the smallest decay rate can accelerate the spatial propagation of other components of \( U \) (see details in Section 2). We also refer to a recent work by Xu, Li, and Ruan \([47]\) where the acceleration propagation of (1.3) was obtained for non-uniform non-exponentially decaying initial data, and other works by Coulon and Yangari \([11]\), Yangari \([52]\), and Xu, Li and Lin \([49]\) for the acceleration propagation with non-uniform nonlocal dispersal kernels and compactly supported initial data.
The rest of this paper is organized as follows. In Section 2, we present the main assumptions
and results. In Section 3, we study a special case where all components of initial data have
the same decay rate $\lambda$, and prove that $\text{(1.2)}$ has a uniform spreading speed dependent on $\lambda$. In
Section 4, we focus on the general case that the initial data satisfy $\text{(1.6)}$ and complete the proof
of our main result.

2. MAIN ASSUMPTIONS AND RESULTS

In this section, we give the main assumptions and results. Let us introduce some notations
first. For $U = (u_1, \ldots, u_m) \in \mathbb{R}^m$, $V = (v_1, \ldots, v_m) \in \mathbb{R}^m$, we write $U \geq V$ if $u_j \geq v_j$ for any
$j \in J$; $U \succ V$ if $u_j > v_j$ for any $j \in J$. Denote
\[
[U, V] = \{ \phi \in \mathbb{R}^m; U \leq \phi \leq V \}.
\]
Let $\|U\| = \sqrt{u_1^2 + \ldots + u_m^2}$ denote the norm of $\mathbb{R}^m$. We write $0 = (0, \ldots, 0) \in \mathbb{R}^m$ and $1 = (1, \ldots, 1) \in \mathbb{R}^m$. Assume that
\begin{enumerate}[(A1)]
\item[(a):] there is a strictly positive equilibrium $P = (p_1, p_2, \ldots, p_m)$ such that $F(0) = F(P) = 0$ and $F \in C^1([0, P], \mathbb{R}^m)$; there is no other equilibrium $\phi$ in $[0, P]$ such that $F(\phi) = 0$.
\item[(b):] $F$ is cooperative in $[0, P]$, namely $\frac{\partial}{\partial u_i} f_j(U) \geq 0$ for any $U \in [0, P]$ and $j \neq i$.
\item[(c):] $F'(0)$ is an irreducible matrix satisfying
\[
\max \{ \Re \lambda \mid \det(\lambda I - F'(0)) = 0 \} > 0.
\]
\item[(d):] for any $j \in J$, the function $k_j$ is nonnegative, continuous, symmetric on $\mathbb{R}$, and
decreasing on $\mathbb{R}^+$. Moreover, $\int_{\mathbb{R}} k_j(x)dx = 1$ and there exists $\Lambda > 0$ such that
\begin{equation}
(2.1) \quad \int_{\mathbb{R}} k_j(x)e^{\Lambda |x|}dx < +\infty.
\end{equation}
\end{enumerate}
Note that $\text{(1.2)}$ is monostable on $[0, P]$ under (A1)(a) and (c); namely, the equilibrium $U \equiv 0$ is
unstable and $U \equiv P$ is stable. From (A1)(b), the matrix $F'(0)$ is essentially nonnegative. Note
that a matrix $A = (a_{ij})_{m \times m}$ is called essentially nonnegative if all coefficients of the matrix
$(A - \min_{x \in \{1, \ldots, m\}} \{a_{ji}\}I_m)$ are nonnegative.

We define
\[
\Lambda = \sup \left\{ \lambda > 0 \mid \int_{\mathbb{R}} k_j(x)e^{\lambda x}dx < +\infty \text{ for all } j \in \{1, \ldots, m\} \right\} \in (0, +\infty) \cup \{+\infty\}.
\]
For $\lambda \in (0, \Lambda)$, let $K(\lambda)$ denote the $m \times m$ matrix as follows
\[
K(\lambda) = D \cdot \text{diag} \left\{ \int_{\mathbb{R}} k_1(y)e^{\lambda y}dy, \ldots, \int_{\mathbb{R}} k_m(y)e^{\lambda y}dy \right\} - D + F'(0).
\]
Since $F'(0)$ is irreducible, so is $K(\lambda)$. By the Perron-Frobenius theorem (see \cite{23}), $K(\lambda)$ has
an eigenvalue $\gamma(\lambda)$ with algebraic multiplicity one, and we denote by $V(\lambda)$ the positive unit
eigenvector corresponding to $\gamma(\lambda)$, namely $K(\lambda)V(\lambda) = \gamma(\lambda)V(\lambda)$ and
\begin{equation}
(2.2) \quad V(\lambda) \gg 0 \quad \text{for } \lambda \in (0, \Lambda).
\end{equation}
From the symmetry of $k_j$, it follows that $\int_{\mathbb{R}} k_j(y)e^{\lambda y}dy \geq 1$ for any $\lambda \in (0, \Lambda)$. Then (A1)-(c)
implies that $\gamma(\lambda) > 0$. For $\lambda \in (0, \Lambda)$, denote
\begin{equation}
(2.3) \quad c(\lambda) = \gamma(\lambda)/\lambda > 0.
\end{equation}
Obviously, $c(\lambda)$ is continuous on $(0, \bar{\lambda})$ and
\begin{equation}
(2.4) \quad c(\lambda)\lambda V(\lambda) - K(\lambda)V(\lambda) = 0 \text{ for any } \lambda \in (0, \bar{\lambda}).
\end{equation}
Define
\begin{equation}
(2.5) \quad c^* \triangleq \inf_{\lambda \in (0, \bar{\lambda})} \{c(\lambda)\} < +\infty.
\end{equation}

It was shown in [21, Lemma 2.4] that $\lambda^* < +\infty$, where $\lambda^*$ is the smallest positive number at which the above infimum is attained, namely
\[ c^* = c(\lambda^*) = \gamma(\lambda^*)/\lambda^* > 0. \]

**Remark 2.1.** The function $c(\cdot)$ defined by (2.5) is strictly decreasing on $(0, \lambda^*)$. Indeed, by Lemma 6.5 and (6.5) in Lui [31], $c(\lambda)$ is twice continuously differentiable and decreasing (i.e. $c'(\lambda) \leq 0$) on $(0, \lambda^*)$, and it satisfies
\[ (\lambda^2 c')' = 2\lambda c' + \lambda^2 c'' \geq 0. \]

Then $c''(\lambda) \geq 0$ for $\lambda \in (0, \lambda^*)$. Suppose that $c(\lambda)$ is decreasing but not strictly decreasing on $(0, \lambda^*)$. Then there exists $\mu \in (0, \lambda^*)$ such that $c'(\mu) = 0$. From $c'(\lambda) \leq 0$ and $c''(\lambda) \geq 0$ on $(0, \lambda^*)$, we get that $c'(\lambda) = 0$ for any $\lambda \in [\mu, \lambda^*)$, which implies by the continuity of $c(\lambda)$ that $c(\lambda) = c(\lambda^*)$ for $\lambda \in [\mu, \lambda^*)$. On the other hand, recall that $\lambda^*$ is the smallest positive number at which $\inf_{\lambda>0}\{c(\lambda)\}$ is attained, which means $c(\lambda) > c(\lambda^*)$ for $\lambda \in (0, \lambda^*)$. It is a contradiction.

There are some additional assumptions on $F$. 

(A2): for $\lambda \in (0, \lambda^*], F(\min\{P, qV(\lambda)\}) \leq qF'(0)V(\lambda)$ for any $q > 0$.

(A3): there are positive numbers $q_0$, $\delta_0$, and $M$ such that
\[ F(U) \geq F'(0)U - MU^{1+\delta_0} \text{ for any } U \in [0, P] \text{ with } \|U\| \leq q_0, \]
where $U^{1+\delta_0} = (u_1^{1+\delta_0}, \ldots, u_m^{1+\delta_0}) \in \mathbb{R}^m$.

The assumptions (A2) and (A3) correspond to the Fisher-KPP assumption in the scalar case, that is the assumption $f'(0)u - Mu^{1+\delta_0} \leq f(u) \leq f'(0)u$. The assumption (A3) can be easily satisfied, for example, when $F \in C^{1+\delta_0}[0, q_01]$. As stated in [21], under (A1)-(A3), $c^*$ is the spreading speed of (1.2) with compactly supported initial data. Denote
\[ \lambda_0 \triangleq \min\{\lambda_j \mid j \in J\}. \]

The following theorem about the uniform spreading speed for non-uniform initial data is the main result of this paper.

**Theorem 2.2.** Assume (A1), (A2), and (A3) hold. For the non-uniform initial data $U_0(x)$ satisfying (1.6) with $\lambda_0 \in (0, \lambda^*)$, the solution of (1.2) has a uniform spreading speed $c(\lambda_0)$, which is independent of the decay rate $\lambda_j$ satisfying $\lambda_j > \lambda_0$. Moreover, $c(\lambda_0)$ is strictly decreasing with respect to $\lambda_0 \in (0, \lambda^*)$.

From Theorem 2.2, the cooperation and irreducibility of reaction functions can ensure that all components of the solution of (1.2) with non-uniform initial data have a uniform spreading speed. In fact, if $F \in C^1[0, P]$ and $\frac{\partial}{\partial u_i}f_j(0) > 0$ with $i \neq j$, then as seen from the $j$th equation of (1.2), namely
\[ \frac{\partial}{\partial t}u_j = d_j(k_j \ast u_j - u_j) + f_j(U), \]
the component \( u_i \) of \( U \) has a direct positive effect on the growth of the component \( u_j \), when \( u_j \) is small enough. We say \( u_i \) has an indirect positive effect on the growth of \( u_j \), if \( u_i \) does not directly affect the growth of \( u_j \) (i.e. \( \partial_{x_{jp}} f_j(0) = 0 \)), but through other components of \( U \), in the sense that there exists a set \( \{j_1, j_2, \ldots, j_k\} \) with \( j_1 = i \) and \( j_k = j \) such that \( \partial_{x_{jp}} f_{j_p}(0) > 0 \) for any \( p = 2, \ldots, k \). The irreducibility of \( F'(0) = (\partial_{u_i} f_j(0))_{m \times m} \) means that a direct or indirect positive effect exists between any two components of \( U \), and hence, all components of the solution with non-uniform initial data can have a uniform spreading speed.

Theorem 2.2 shows that the uniform spreading speed depends only on the smallest decay rate \( \lambda_0 \). This conclusion, along with the fact that the spreading speed \( c(\lambda_0) \) is strictly decreasing on \( (0, \lambda^*) \) in Remark 2.1 means that the component with the smallest decay rate can accelerate the spatial propagation of other components. To understand this, we assume the \( j_0 \)th component of initial data \( U_0 \) has the smallest decay rate, namely \( \lambda_{j_0} = \lambda_0 \in (0, \lambda^*) \). Let the decay rate of the \( j_0 \)th component \( u_{j_0,0} \) become smaller and fix the decay rates of other components of initial data \( U_0 \). We denote the new decay rate of \( u_{j_0,0} \) by \( \lambda' \in (0, \lambda_0) \). Then the uniform spreading speed becomes \( c(\lambda') \) from \( c(\lambda_0) \). Since \( c(\cdot) \) is strictly decreasing on \( (0, \lambda^*) \), we have that \( c(\lambda') > c(\lambda_0) \), which means that the decrease of the smallest decay rate in the initial data can increase the spreading speed of other components of the solution.

Our idea to prove Theorem 2.2 consists of two steps. First, we focus on the special case that all components of initial data \( U_0 \) have the same decay rate \( \lambda \in (0, \lambda^*) \) (namely \( \lambda_j = \lambda \) for any \( j \in J \)) and prove that the solution has a uniform spreading speed \( c(\lambda) \) in Section 3. Second, the general case that \( U_0 \) satisfies (1.6) with \( \lambda_0 \in (0, \lambda^*) \) is considered in Section 4. By constructing a lower solution, we show that after a period of time \( T > 0 \), all components of \( U(T, \cdot) \) are larger than an exponentially decaying function with the decay rate \( \lambda_0 \). This case is then transformed into the special case considered in Section 3 as long as \( u_j(T, x) \) is set as the new initial data.

Moreover, from Theorem 2.2 and its proof in Section 4, the components whose decay rates are not \( \lambda_0 \) affect neither the result of uniform spreading speed nor its proof method. Therefore, Theorem 2.2 also holds if (1.6) is changed by the following assumption

\[
(H): \text{there exist } j_0 \in \{1, 2, \ldots, m\} \text{ and } \lambda_0 > 0 \text{ such that} \\
u_{j_0,0}(x) \sim C e^{-\lambda_0 |x|}, \quad u_{j,0}(x) \leq e^{-\lambda_0 |x|} \text{ for } j \neq j_0 \text{ and } |x| \text{ large enough.}
\]

In this assumption, the component \( u_{j,0} \) of \( U_0 \) with \( j \neq j_0 \) is not restricted to exponentially decaying functions, but any function that is smaller than \( e^{-\lambda_0 |x|} \) when \( |x| \) is large enough.

Remark 2.3. The methods in this paper are also applicable to the reaction-diffusion cooperative system (1.3). Therefore, no matter whether we consider a nonlocal or local dispersal system, the cooperation and irreducibility of \( F \) can ensure that the solution has a uniform spreading speed and the component of \( U \) with the smallest decay rate can accelerate the spatial propagation of other components.

3. Case of the same decay rate

In this section, we consider the case that all components of initial data have the same decay rate \( \lambda \in (0, \lambda^*) \). First, we state two important lemmas that are proved in [18, Theorem 4.1] (for Lemma 3.1) and [21, Theorem 4.5] (for Lemma 3.2).

Lemma 3.1. (Symmetry and monotone property) If the functions \( k_j(\cdot) \) and \( u_{j,0}(\cdot) \) are symmetric on \( \mathbb{R} \) and decreasing on \( \mathbb{R}^+ \) for any \( j \in J \), so is \( u_j(t, \cdot) \) for any \( t > 0 \) and \( j \in J \).
Lemma 3.2. (Comparison principle) Assume that $\bar{U}$ is an upper solution and $\underline{U}$ is a lower solution of (1.2); namely $\frac{\partial}{\partial t} \bar{U}(t, x)$ and $\frac{\partial}{\partial t} \underline{U}(t, x)$ exist and
\[
\frac{\partial}{\partial t} \bar{U} - DK * \bar{U} + D \bar{U} - F(\bar{U}) \geq 0 \text{ for } t > 0, \ x \in \mathbb{R},
\]
\[
\frac{\partial}{\partial t} \underline{U} - DK * \underline{U} + D \underline{U} - F(\underline{U}) \leq 0 \text{ for } t > 0, \ x \in \mathbb{R}.
\]
If $\bar{U}(0, x) \geq \underline{U}(0, x)$ for $x \in \mathbb{R}$, then $\bar{U}(t, x) \geq \underline{U}(t, x)$ for any $t \geq 0$ and $x \in \mathbb{R}$.

The following result is a special case of Theorem 2.2 where all components of $U_0$ have the same decay rate $\lambda \in (0, \lambda^*)$.

Proposition 3.3. Assume (A1), (A2), and (A3) hold. Let $U_0(x)$ satisfy (1.6) with $\lambda_j = \lambda \in (0, \lambda^*)$ for any $j \in J$. Then the solution of (1.2) has a uniform spreading speed $c(\lambda)$.

Proof. Let $U = (u_1, \ldots, u_m)$ be the solution of (1.2) with initial data $U_0$. By (1.6) and (2.2), there is a constant $\Gamma > 0$ large enough such that
\[
U_0(x) \ll \Gamma e^{-\lambda|x|} V(\lambda).
\]
For $\lambda \in (0, \lambda^*)$, define
\[
(3.1) \quad \bar{U}(t, x) = \min \left\{ P, \Gamma e^{-\lambda z} V(\lambda) \right\} \quad \text{with } z = |x| - c(\lambda) t, \ t \geq 0, \ x \in \mathbb{R}.
\]
Now we check that $\bar{U} = (\bar{u}_1, \ldots, \bar{u}_m)$ is an upper solution. Let $v_j(\lambda)$ denote the $j$th component of $V(\lambda)$, namely $V(\lambda) = (v_1(\lambda), \ldots, v_m(\lambda))$. For any $j \in J$, when $z < \lambda^{-1} \ln(\Gamma v_j(\lambda)/p_j)$, we have that $\bar{u}_j(t, x) = p_j$. Then by (A1)-(b), from $\bar{u}_i(t, x) \leq p_i$ for any $i \in J$ we can get that
\[
\frac{\partial}{\partial t} \bar{u}_j - d_j k_j * \bar{u}_j + d_j \bar{u}_j - f_j(\bar{U}) \geq -f_j(P) = 0.
\]
When $z \geq \lambda^{-1} \ln(\Gamma v_j(\lambda)/p_j)$, it holds that $\bar{u}_j(t, x) = \Gamma e^{-\lambda z} v_j(\lambda)$. We denote $f_{j,i} = \frac{\partial}{\partial u_i} f_j(0)$ and (A1)-(b) implies $f_{j,i} \geq 0$ for $i \neq j$. By (A2) and (2.4), we have that
\[
\frac{\partial}{\partial t} \bar{u}_j - d_j k_j * \bar{u}_j + d_j \bar{u}_j - f_j(\bar{U})
\]\n\[
\geq \Gamma e^{-\lambda z} \left[ \left( c(\lambda) \lambda - d_j \int_{\mathbb{R}} k_j(y) e^{\lambda y} dy + d_j \right) v_j(\lambda) - \sum_{i=1}^{m} f_{j,i} v_i(\lambda) \right] = 0.
\]
Thus $\bar{U} = (\bar{u}_1, \ldots, \bar{u}_m)$ is an upper solution of (1.2). Lemma 3.2 implies that
\[
U(t, x) \leq \bar{U}(t, x) \leq \Gamma e^{-\lambda z} V(\lambda) \quad \text{for any } t \geq 0 \text{ and } x \in \mathbb{R}.
\]
Then for any $\varepsilon > 0$ and $j \in J$, we have that
\[
\lim_{t \to +\infty} \sup_{|x| \geq (c(\lambda) + \varepsilon) t} u_j(t, x) \leq \lim_{t \to +\infty} \sup_{|x| \geq (c(\lambda) + \varepsilon) t} \Gamma e^{-\lambda(|x| - c(\lambda) t) v_j(\lambda)} \leq \lim_{t \to +\infty} \Gamma e^{-\lambda \varepsilon t} v_j(\lambda) = 0.
\]
Now we just need to prove that for any $\varepsilon \in (0, c(\lambda))$ and $j \in J$, there exists $\nu > 0$ such that
\[
(3.2) \quad \lim_{t \to +\infty} \inf_{|x| \leq (c(\lambda) - \varepsilon) t} u_j(t, x) \geq \nu.
\]
The proof of (3.2) consists of the following two steps.

First, we prove that there exist two positive constants $\gamma$ and $y_0$ such that
\[
(3.3) \quad U(1, x) \geq \gamma \min \left\{ e^{-\lambda x}, e^{-\lambda y_0} \right\} V(\lambda), \ x \in \mathbb{R}.
\]
From (1.6) it follows that $U_0(x) \gg 0$ for sufficiently large $|x|$. Then by (A1)-(d), there exists $N_0 \in \mathbb{N}^+$ such that

$$K \ast K \ast \cdots \ast K \ast U_0(x) \gg 0 \quad \text{for any } x \in \mathbb{R}. \quad (3.4)$$

For $j \in J$, let $\pi_j : \mathbb{R}^m \to \mathbb{R}^m$ denote the function

$$\pi_j : (u_1, \ldots, u_m) \mapsto (0, \ldots, u_j, \ldots, 0);$$

namely the $j$th component of $\pi_j(U)$ is $u_j$ while others are zero. We define

$$b_j = \inf_{u_j \in [0,p_j]} \{f_j(\pi_j(U))/u_j\}. \quad (3.5)$$

Let $n \in \mathbb{N}^+$ and we divide equally the time period of $[0, \tau]$ into $n$ parts, namely $[0, \tau/n]$, $[\tau/n, 2\tau/n]$, $\ldots$, and $[(n-1)\tau/n, \tau]$. In $[0, \tau/n]$, we consider

$$W(t,x) = (w_1(t,x), \ldots, w_n(t,x)), \quad t \in [0, \tau/n], \quad x \in \mathbb{R},$$

where

$$w_j(t,x) = M_j[u_j,0(x) + td_j k_j \ast u_j,0(x)]e^{(b_j-d_j)t}, \quad j \in J \quad (3.6)$$

and

$$M_j = (1 + d_j \tau/n)^{-1}(1 + e^{(b_j-d_j)\tau/n})^{-1}, \quad j \in J.$$ 

It is easy to check that

$$\partial_t w_j - d_j k_j \ast w_j + d_j w_j - b_j w_j \leq 0 \quad \text{for } j \in J.$$ 

For $t \in [0, \tau/n]$, by $u_{j,0}(x) \leq p_j$ we have that

$$w_j(t,x) \leq M_j p_j [1 + d_j \tau/n]e^{(b_j-d_j)\tau/n} \leq p_j \quad \text{for } x \in \mathbb{R}.$$ 

From (A1)-(b) and (3.5), it follows that

$$W_t - DK \ast W + DW - F(W)$$

$$\leq W_t - DK \ast W + DW - (f_1(\pi_1(W)), \ldots, f_m(\pi_m(W)))$$

$$\leq W_t - DK \ast W + DW - \text{diag}(b_1, \ldots, b_m)W \leq 0.$$ 

By $W(0,x) \leq U_0(x)$ for $x \in \mathbb{R}$, from Lemma 3.2 we get that

$$U(\tau/n, x) \geq W(\tau/n, x). \quad (3.7)$$

Denote $C_1 \triangleq \min_{j \in J} \{M_j d_j \tau/n \}$ and then

$$U(\tau/n, x) \geq C_1 K \ast U_0(x).$$

Repeat this argument for $t \in [\tau/n, 2\tau/n]$ and substitute $K \ast U_0(x)$ for $U_0(x)$. We can find a constant $C_2 > 0$ such that

$$U(2\tau/n, x) \geq C_2 K \ast U_0(x).$$

Similarly, there exists $C_n > 0$ such that

$$U(\tau, x) \geq C_n K \ast K \ast \cdots \ast K \ast U_0(x) \quad \text{for } x \in \mathbb{R}. \quad (n)$$

When $n = N_0$, it follows from (3.4) that $U(\tau, x) \gg 0$. When $n = 1$, we get from (3.6) and (3.7) that

$$U(\tau, x) \geq W(\tau, x) \geq C_\tau U_0(x) \quad \text{with } C_\tau = \min_{j \in J} \{M_j e^{(b_j-d_j)\tau}\}. \quad (3.8)$$
Then for any \( \tau > 0 \) there exists \( C_\tau > 0 \) such that

\[
U(\tau, x) \geq 0 \text{ and } U(\tau, x) \geq C_\tau U_0(x) \text{ for } x \in \mathbb{R}.
\]

When \( \tau = 1 \), by (1.6) with \( \lambda_j = \lambda \) we can find \( \gamma > 0 \) and \( y_0 > 0 \) satisfying (3.9).

Let \( \gamma \) be smaller (if necessary) such that \( \gamma e^{-\lambda y_0} \leq q_0 \), where \( q_0 \) is given by assumption (A3). Define \( W_0(x) = \gamma \min \{ e^{-\lambda |x|}, e^{-\lambda y_0} \} V(\lambda), x \in \mathbb{R} \). Then \( \| W_0(x) \| \leq q_0 \) for \( x \in \mathbb{R} \) and

\[
U(1, x) \geq W_0(x) = \begin{cases} 
\gamma e^{-\lambda |x|} V(\lambda) & \text{for } |x| \geq y_0, \\
\gamma e^{-\lambda y_0} V(\lambda) & \text{for } |x| \leq y_0.
\end{cases}
\]

Let \( W(t, x) \) be the solution of (1.2) with initial data \( W(0, x) = W_0(x) \). Then we get from Lemma 3.2 that

\[
U(t + 1, x) \geq W(t, x) \text{ for } t \geq 0, \ x \in \mathbb{R}.
\]

Since \( W_0(\cdot) \) is symmetric and decreasing on \( \mathbb{R}^+ \), so is \( W(t, \cdot) \) by Lemma 3.1

Second, we construct a lower solution and prove (3.2). Now define some nations. By Remark 2.1 for any \( \lambda \in (0, \lambda^*) \), there is a constant \( \delta_\lambda = \lambda^*/\lambda - 1 > 0 \) such that

\[
c(\lambda + \lambda s) < c(\lambda) \text{ for any } s \in (0, \delta_\lambda).
\]

Denote

\[
\mu = \lambda (1 + \delta) > 0 \quad \text{with} \quad \delta \triangleq \min \{ \delta_0, \delta_\lambda/2 \} > 0,
\]

where the positive constant \( \delta_0 \) is given by (A3). Then it follows that

\[
c(\mu) < c(\lambda).
\]

For \( j \in J \), let \( G(c, \lambda; j) \) be the \( j \)th component of the vector \( c\lambda V(\lambda) - K(\lambda)V(\lambda) \); namely

\[
G(c, \lambda; j) \triangleq \left( c\lambda - d_j \int_{\mathbb{R}} k_j(y)e^{\lambda y} dy + d_j \right) v_j(\lambda) - \sum_{i=1}^{m} f_{j,i} v_i(\lambda), \ c > 0, \ \lambda > 0,
\]

where \( f_{j,i} = \frac{\partial}{\partial x_i} f_j(0) \) and \( v_j(\lambda) \) is the \( j \)th component of \( V(\lambda) \gg 0 \). For \( \lambda \in (0, \lambda^*) \), it follows from (2.4) that

\[
G(\lambda, \lambda; j) = \left( c(\lambda) \lambda - d_j \int_{\mathbb{R}} k_j(y)e^{\lambda y} dy + d_j \right) v_j(\lambda) - \sum_{i=1}^{m} f_{j,i} v_i(\lambda) = 0.
\]

By (3.11) we get that

\[
G(c(\lambda), \mu; j) = \left( c(\lambda) \mu - d_j \int_{\mathbb{R}} k_j(y)e^{\mu y} dy + d_j \right) v_j(\mu) - \sum_{i=1}^{m} f_{j,i} v_i(\mu) > G(c(\mu), \mu; j) = 0.
\]

For \( \lambda \in (0, \lambda^*) \), we define \( U = (u_1, \ldots, u_m) \) as follows

\[
U(t, x) = \max \left\{ 0, \ \gamma e^{-\lambda z} V(\lambda) - Le^{-\mu z} V(\mu) \right\} \text{ with } z = |x| - c(\lambda)t, \ t \geq 0, \ x \in \mathbb{R},
\]

where \( L \) is a positive constant large enough such that

\[
L \geq \max \left\{ \frac{\gamma e^{-\mu y_0}}{1 + \delta} \max_{j \in J} \left( \frac{v_j(\lambda)}{v_j(\mu)} \right), \ M\gamma^{1+\delta} \max_{j \in J} \left( \frac{v_j^{1+\delta}(\lambda)}{G(c(\lambda), \mu; j)} \right) \right\}.
\]

Denote

\[
y_j = \lambda^{-1} \delta^{-1} \ln \left( \frac{L(1+\delta)v_j(\mu)}{\gamma v_j(\lambda)} \right) \quad \text{and} \quad z_j = \lambda^{-1} \delta^{-1} \ln \left( \frac{Lv_j(\mu)}{\gamma v_j(\lambda)} \right) \text{ for } j \in J.
\]
Then \( y_j > z_j \) for any \( j \in J \). Note that \( y_j \) and \( z_j \) correspond respectively to the maximum point of \( z \mapsto \gamma e^{-\lambda z} v_j(\lambda) - Le^{-\mu z} v_j(\mu) \) and the root of \( \gamma e^{-\lambda z} v_j(\lambda) - Le^{-\mu z} v_j(\mu) = 0 \), that is

\[
\mathbf{u}_j(t, x) = \begin{cases} 
0, & \text{when } z < z_j, \\
\gamma e^{-\lambda z} v_j(\lambda) - Le^{-\mu z} v_j(\mu) = 0, & \text{when } z = z_j, \\
\gamma e^{-\lambda z} v_j(\lambda) - Le^{-\mu z} v_j(\mu) > 0, & \text{when } z > z_j,
\end{cases}
\]  

(3.15)

and

\[
\max_{z \in \mathbb{R}} \{ \gamma e^{-\lambda z} v_j(\lambda) - Le^{-\mu z} v_j(\mu) \} = \gamma e^{-\lambda y_j} v_j(\lambda) - Le^{-\mu y_j} v_j(\mu) > 0.
\]

From (3.14), it follows that \( y_j \geq y_0 \) for any \( j \in J \). Then we have that

\[
(3.16) \quad \sup_{t \geq 0, x \in \mathbb{R}} \mathbf{u}_j(t, x) = \mathbf{u}_j(t, c(\lambda) t + y_j) = \gamma e^{-\lambda y_j} v_j(\lambda) - Le^{-\mu y_j} v_j(\mu) \leq \gamma e^{-\lambda y_0} v_j(\lambda).
\]

Since \( V(\lambda) \) is a unit vector, it holds that

\[
(3.17) \quad \| \mathbf{U}(t, x) \| \leq \gamma e^{-\lambda y_0} q_0 \quad \text{for any } t \geq 0, \ x \in \mathbb{R}.
\]

Particularly, when \( t = 0 \), it follows from (3.16) that \( \mathbf{U}(0, x) \leq \gamma e^{-\lambda y_0} V(\lambda) \) for any \( x \in \mathbb{R} \). The definition of \( \mathbf{U}(t, x) \) implies that \( \mathbf{U}(0, x) \leq \gamma e^{-\lambda|x|} V(\lambda) \) for \( x \in \mathbb{R} \). Then we get from (3.9) that

\[
(3.18) \quad \mathbf{U}(0, x) \leq W_0(x) \quad \text{for } x \in \mathbb{R}.
\]

In order to verify \( \mathbf{U}(t, x) \) is a lower solution, namely

\[
\mathbf{U}_j - DK * \mathbf{U} + D \mathbf{U} - F(\mathbf{U}) \leq 0,
\]

we check it holds for each component. For any \( j \in J \), when \( z < z_j \), since \( \mathbf{u}_j(t, x) = 0 \), it is easy to check that

\[
\frac{\partial}{\partial t} \mathbf{u}_j - d_jk_j * \mathbf{u}_j + d_j \mathbf{u}_j - f_j(\mathbf{U}) \leq 0.
\]

When \( z \geq z_j \), we get that

\[
\mathbf{u}_j(t, x) = \gamma e^{-\lambda z} v_j(\lambda) - Le^{-\mu z} v_j(\mu),
\]

\[
\mathbf{u}_j(t, x) \geq \gamma e^{-\lambda z} v_i(\lambda) - Le^{-\mu z} v_i(\mu) \quad \text{for } i \neq j.
\]

From (A3) and (3.17), it follows that

\[
f_j(\mathbf{U}) \geq \sum_{i=1}^{m} f_{j,i} \mathbf{u}_i(t, x) - M \mathbf{u}^{1+\delta}(t, x)
\]

\[
\geq \sum_{i=1}^{m} f_{j,i} \left[ \gamma e^{-\lambda z} v_i(\lambda) - Le^{-\mu z} v_i(\mu) \right] - M \gamma^{1+\delta} e^{-\mu z} v_j^{1+\delta}(\lambda).
\]

Then some calculations show that

\[
\frac{\partial}{\partial t} \mathbf{u}_j - d_jk_j * \mathbf{u}_j + d_j \mathbf{u}_j - f_j(\mathbf{U}) \leq \gamma e^{-\lambda z} \left[ (c(\lambda)\lambda - d_j \int_{\mathbb{R}} k_j(y)e^{\lambda y} dy + d_j) v_j(\lambda) - \sum_{i=1}^{m} f_{j,i} v_i(\lambda) \right]
\]

\[
- Le^{-\mu z} \left[ (c(\lambda)\mu - d_j \int_{\mathbb{R}} k_j(y)e^{\mu y} dy + d_j) v_j(\mu) - \sum_{i=1}^{m} f_{j,i} v_i(\mu) \right] + M \gamma^{1+\delta} e^{-\mu z} v_j^{1+\delta}(\lambda)
\]

\[
= \gamma e^{-\lambda z} G(c(\lambda), \lambda; j) - e^{-\mu z} \left[ LG(c(\lambda), \mu; j) - M \gamma^{1+\delta} v_j^{1+\delta}(\lambda) \right].
\]
By (3.12), (3.13), and (3.14), for \( z \geq z_j \), we have
\[
\frac{\partial}{\partial t} u_j - d_j k_j u_j + d_j u_j - f_j(U) \leq 0.
\]
Therefore, \( U(t, x) \) is a lower solution.

Lemma 3.2 and (3.18) imply that \( W(t, x) \geq U(t, x) \) for any \( t \geq 0, x \in \mathbb{R} \).

Let \( y_{\text{max}} \triangleq \max_{j \in J} \{ y_j \} \). It follows from \( y_j > z_j \) that \( y_{\text{max}} > \max_{j \in J} \{ z_j \} \), which implies by (3.15) that
\[
\nu \triangleq \min_{j \in J} \{ u_j(t, c(\lambda)t + y_{\text{max}}) \} > 0.
\]
We denote \( W(t, x) \) by \( (w_1(t, x), \ldots, w_m(t, x)) \). Then it follows that
\[
w_j(t, x) \geq \nu \text{ for any } |x| \leq c(\lambda)t + y_{\text{max}} \text{ and } j \in J.
\]
By (3.10) we get that
\[
u \text{ for any } |x| \leq c(\lambda)t + y_{\text{max}} \text{ and } j \in J,
\]
which implies (3.2). This completes the proof of Proposition 3.3.

\[\Box\]

4. General case

In this section, we give the proof of Theorem 2.2. By constructing a lower solution, we transform the proof for the general case where \( U_0 \) satisfies (1.6) into the special cases in Section 3, where all components of the initial data have the same decay rate.

Proof of Theorem 2.2. The strictly decreasing property of \( c(\lambda_0) \) with respect to \( \lambda_0 \in (0, \lambda^*) \) has been obtained in Remark 2.1. By (1.6) and \( \lambda_0 \leq \lambda_j \) for \( j \in J \), there exists \( C > 0 \) such that \( u_{j,0}(x) \leq Ce^{-\lambda_0|x|} \) for \( j \in J \) and large \( |x| \). Then the proof of
\[
\lim_{t \to +\infty} \sup_{|x| \geq (c(\lambda_0)+\varepsilon)t} u_j(t, x) = 0 \text{ for } j \in J
\]
is similar to the counterpart in the proof of Propositions 3.3 and we only need to substitute \( \lambda_0 \) for \( \lambda \).

Now prove that for any \( \varepsilon \in (0, c(\lambda_0)) \), there is a constant \( \nu > 0 \) such that
\[
\lim_{t \to +\infty} \inf_{|x| \leq (c(\lambda_0)-\varepsilon)t} u_j(t, x) \geq \nu \text{ for any } j \in J.
\]
From the proof of Propositions 3.3, we only need to prove that there exist \( T > 0 \) and \( M_0 > 0 \) such that
\[
u \text{ for any } j \in J,
\]
where
\[
p(x) = e^{-\lambda_0|x|}.
\]
Now we reorder the equations in the system (1.2) (namely, reorder the components of \( U \)). Define
\[
f_{j,i} = \frac{\partial}{\partial u_i} f_j(0).
\]
Choose the component who has the smallest decay rate as the first component $u_1$ of $U$, and then $\lambda_1 = \lambda_0 = \min\{\lambda_j, j \in J\}$. Since $F'(0)$ is irreducible, we can choose the second component $u_2$ such that $f_{2,1} > 0$. Similarly, we can choose the third component $u_3$ satisfying $f_{3,1} > 1$ or $f_{3,2} > 0$. Repeat this process, we reorder the components of $U$ satisfying that for any $i \in \{2, 3, \ldots, m\}$ there exists $j \in \{1, 2, \ldots, i - 1\}$ such that $f_{i,j} > 0$.

We give an important inequality. Since $F \in C^1([0, P])$, by (A1)-(b), we can find a constant $q_3 > 0$ such that

\[
(4.2) \quad f_j(U) \geq (f_{j,j} - 1)u_j + \frac{1}{2} \sum_{i \neq j} f_{j,i}u_i \text{ for any } j \in J \text{ and } U \in [0, q_3].
\]

In order to prove (4.1), we need to construct a lower solution

\[
W(t, x) = (w_1(t, x), \ldots, w_m(t, x)) \in [0, q_3], \quad t \geq 1, \ x \in \mathbb{R}.
\]

The form of $W(t, x)$ will be given for every component. First, we construct the first component $w_1(t, x)$ of $W(t, x)$. By (3.8) and (1.6) with $\lambda_1 = \lambda_0$, there is a constant $C_0 \in (0, q_3]$ such that

\[
u_1(1, x) \geq C_0 p(x) \text{ for } x \in \mathbb{R}.
\]

Let

\[
w_1(t, x) = M_1 e^{-\alpha(t-1)} p(x) \text{ for } t \geq 1, \ x \in \mathbb{R},
\]

where $M_1$ is a constant in $(0, C_0]$ and

\[
\alpha \geq \max_{j \in J} \{d_j + |f_{j,j}|\} + 2.
\]

Note that $M_1$ will be reselect as a smaller constant later. It is easy to check that

\[
w_1(t, x) \leq M_1 < C_0 \leq q_3 \text{ for } t \geq 1, \ x \in \mathbb{R}
\]

and

\[
(4.3) \quad w_1(1, x) \leq M_1 p(x) \leq u_1(1, x) \text{ for } x \in \mathbb{R}.
\]

From $p(x) \geq 0$, it follows that $k_1 \ast w_1 \geq 0$. By the cooperation of $F$ and $[122]$, we have that $f_1(W) \geq (f_{1,1} - 1)w_1$ for $W \in [0, q_3]$. Then some calculations show that

\[
\frac{\partial}{\partial t} w_1 - d_1 k_1 \ast w_1 + d_1 w_1 - f_1(W) \leq M_1 (-\alpha + d_1 - f_{1,1} + 1)e^{-\alpha(t-1)} p(x) \leq 0.
\]

Second, we construct the second component $w_2(t, x)$ of $W(t, x)$ under the condition $f_{2,1} > 0$. Define

\[
w_2(t, x) = M_2 \left( e^{-\beta_2(t-1)} - e^{-\alpha(t-1)} \right) p(x) \text{ for } t \geq 1, \ x \in \mathbb{R},
\]

where

\[
\beta_2 = d_2 + |f_{2,2}| + 1, \quad M_2 \triangleq \frac{f_{2,1} M_1}{2(\alpha - d_2 + f_{2,2} - 1)}.
\]

By $\alpha \geq \beta_2$, we get that $w_2 \geq 0$ for $t \geq 1$, which implies that $k_2 \ast w_2 \geq 0$. Let $M_1$ be smaller (if necessary) satisfying $M_1 \leq 2q_3/f_{2,1}$. From $\alpha \geq d_2 - f_{2,2} + 2$, it follows that

\[
w_2(t, x) \leq M_2 \leq \frac{1}{2} f_{2,1} M_1 \leq q_3 \text{ for } t \geq 1, \ x \in \mathbb{R}.
\]
Assumption (A1)-(b) and (4.2) show that \( f_2(W) \geq (f_{2,2} - 1)w_2 + \frac{1}{2} f_{2,1} w_1 \) for \( W \in [0, q_3 1] \). Then we have that

\[
\frac{\partial}{\partial t} w_2 - d_2 k_2 \ast w_2 + d_2 w_2 - f_2(W) \\
\leq \frac{\partial}{\partial t} w_2 - d_2 k_2 \ast w_2 + d_2 w_2 - (f_{2,2} - 1)w_2 - \frac{1}{2} f_{2,1} w_1 \\
\leq M_2 \left[ (-\beta_2 + d_2 - f_{2,2} + 1)e^{-\beta_2(t-1)} + (\alpha - d_2 + f_{2,2} - 1)e^{-\alpha(t-1)} \right] p(x) \\
- \frac{1}{2} M_1 f_{2,1} e^{-\alpha(t-1)} p(x) \\
\leq \left[ M_2(\alpha - d_2 + f_{2,2} - 1) - \frac{1}{2} M_1 f_{2,1} \right] e^{-\alpha(t-1)} p(x) = 0.
\]

Moreover, it is easy to check that

\( w_2(1, x) = 0 \) for \( x \in \mathbb{R} \).

Note that \( e^{-\beta_2 s} \geq 2e^{-\alpha s} \) for \( s \geq \tau \triangleq \ln 2 \) and then

\[
(4.4) \quad w_2(t, x) \geq M_2 e^{-\alpha(t-1)} p(x) \quad \text{for} \quad t \geq 1 + \tau, \; x \in \mathbb{R},
\]

which is a key inequality for the construction of \( w_j \) with \( j > 2 \) when \( f_{j, 2} > 0 \).

Third, we construct the third component \( w_3(t, x) \) of \( W(t, x) \) under the condition \( f_{3, 1} > 0 \) or \( f_{3, 2} > 0 \). For the case \( f_{3, 1} > 0 \), we can construct \( w_3(t, x) \) by the same method as \( w_2(t, x) \). For the case \( f_{3, 2} > 0 \), we define

\[
w_3(t, x) = \begin{cases} 
0, & 1 \leq t \leq 1 + \tau, \\
M_3 \left( e^{-\beta_3(t-1-\tau)} - e^{-\alpha(t-1-\tau)} \right) p(x), & t \geq 1 + \tau,
\end{cases}
\]

where

\[
\beta_3 = d_3 + |f_{3, 3}| + 1,
\]

\[
M_3 \triangleq \frac{f_{3, 2} M_2}{2e^{\alpha \tau}(\alpha - d_3 + f_{3, 3} - 1)}.
\]

Let \( M_1 \) be smaller (if necessary) such that \( M_3 \leq q_3 \), and then \( 0 \leq w_3(t, x) \leq q_3 \) for \( t \geq 1, \; x \in \mathbb{R} \).

By (4.3) and (4.4), we have that

\[
f_3(W) \geq (f_{3, 3} - 1)w_3 + \frac{f_{3, 2}}{2} w_2 \geq (f_{3, 3} - 1)w_3 + \frac{f_{3, 2}}{2} M_2 e^{-\alpha(t-1)} p(x) \quad \text{for} \quad W \in [0, q_3 1].
\]

Following similar calculations to these for \( w_2 \), we can prove that

\[
\frac{\partial}{\partial t} w_3 - d_3 k_3 \ast w_3 + d_3 w_3 - f_3(W) \leq 0.
\]

We also have that

\[
w_3(t, x) \geq M_3 e^{-\alpha(t-1-\tau)} p(x) \quad \text{for} \quad t \geq 1 + 2\tau, \; x \in \mathbb{R},
\]

which provide the key inequality for the construction of \( w_j \) with \( j > 3 \) when \( f_{j, 3} > 0 \).

To construct \( w_j \) for \( j \in \{4, 5, \ldots, m\} \), when \( f_{j, 1} > 0 \), we apply the construction method for \( w_2 \), and when \( f_{j, i} > 0 \) for some \( i \in \{2, \ldots, j - 1\} \), we use the construction method for \( w_3 \) in the case \( f_{3, 2} > 0 \). Then we can define every component of \( W(t, x) \) satisfying

\[
\frac{\partial}{\partial t} W - DK \ast W + DW - F(W) \leq 0 \quad \text{for} \quad t \geq 1, \; x \in \mathbb{R},
\]

and

\[
w_i(t, x) \geq M_i e^{-\alpha(t-(i-2)\tau)} p(x) \quad \text{for} \quad t \geq 1 + (i-1)\tau, \; x \in \mathbb{R}, \; i = 2, \ldots, m.
\]
We obtain two constants 
\[ T = 1 + (m - 1)\tau \quad \text{and} \quad M_0 = \left\{ M_i e^{-\alpha(T-1-(i-2)\tau)} \right\}_{i \in \{2, \ldots, m\}} \]
such that
\[ w_i(T, x) \geq M_0 p(x), \quad x \in \mathbb{R} \quad \text{for any } i = 1, \ldots, m. \tag{4.5} \]
The definition of $W$ also shows that $w_i(1, x) = 0$ for any $i = 2, \ldots, m$. We get from (4.3) that
\[ U(1, x) \geq W(1, x), \quad x \in \mathbb{R}. \]
It follows from Lemma 3.2 that
\[ U(t, x) \geq W(t, x) \quad \text{for } t \geq 1, \quad x \in \mathbb{R}. \]
Then we have $U(T, x) \geq W(T, x)$ for $x \in \mathbb{R}$, which implies (4.1) by (4.5). It completes the proof of Theorem 2.2. \qed

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