Like light, gravitational waves can be gravitationally lensed by massive objects along their travel path. Strong lensing produces several images from the same binary coalescence and is forecasted to have a promising rate in ground-based gravitational detectors. To search for this effect in the data, one would, in principle, have to analyze all the possible combinations of the individual detected events, whose number will be ever-increasing. To keep up with the rising computational cost, we propose a fast and precise methodology to analyze strongly-lensed gravitational wave events. The method works by effectively using the posterior of the first image as prior for the second image. Thanks to its increased speed tractability, this method enables the joint analysis of more than two images. In addition, it opens the door to new strong lensing studies where a large number of injections is required.

1 Introduction

When gravitational waves (GWs) travel from their source to the Earth, they can pass by massive objects (e.g. galaxies or galaxy clusters) and get gravitationally lensed\(^1\). For strong lensing, the GW wavelength is larger than the typical size of the object, so that we are in the geometric-optics limit and the frequency evolution of the GW is unaffected. Therefore, strong lensing leads to several magnified images with the same frequency evolution, separated in time. These would appear in the data as events originating from the same sky location, with the same parameters, except for the time of arrival, the apparent luminosity distance, and an overall phase shift, which are modified as a consequences of lensing.

Hence, the main idea when searching for strongly lensed GWs is to look for events that have the same intrinsic parameters and are coming from the same sky location. However, in principle one should investigate all the possible pairs one could make from the individual events, leading to an increasing list as the number of detections increases. At design sensitivity, the LIGO\(^2\) and Virgo\(^3\) detectors could detect up to \(O(1000)\) events per year, leading to \(5 \times 10^5\) pairs to analyze when searching for strong lensing\(^4\). Currently, two parameter-estimation-based methods exist to search for lensed pairs. The first one looks for the consistency between a subset of parameters after having performed a Gaussian kernel density estimation on them\(^5\). This leads to the possibility of performing tests in a few minutes, but comes with
a relatively large false alarm probability. The second method consists in analyzing the two data streams jointly, sampling the full joint likelihood. This leads to a high accuracy but is significantly slower.

In this work, we present a new method that offers both speed and precision. We detail the Bayesian framework and show an example analysis. Our method is coded in a package called GOLUM.

2 Strongly lensed gravitational waves

When a GW gets strongly lensed, it is split in multiple images, each of them being potentially observable. These images are categorized into different types, corresponding to a different overall phase shift. The modification of the lensed waveform in the frequency domain takes the form

\[ \tilde{h}_L(f; \theta, \mu_j, t_j, n_j) = \sqrt{\mu_j} e^{i(2\pi f t_j - i\pi n_j \text{sign}(f))} \tilde{h}_U(f; \theta), \]

where \( \tilde{h}_U(f; \theta) \) is the unlensed waveform. Assuming GWs from coalescing binary black holes (BBHs), \( \theta \) is the usual set of parameters associated with such an event, while \( \mu_j \) is the magnification of the \( j^{th} \) image and \( t_j \) its arrival time. The discrete parameter \( n_j \), called the Morse factor, encodes the different image types. If \( n_j = 0 \), the image is of type I, corresponding to a minimum of the Fermat potential; in this case the overall shape of the wave is unaffected. If \( n_j = 0.5 \), we have a type II image, corresponding to a saddle point of the potential, in which case the GW is Hilbert-transformed. Finally, if \( n_j = 1 \), we have a type III image, corresponding to a maximum of the Fermat potential; here the wave is inverted. The values for the time delays and magnifications depend on the nature of the lens. For a galaxy lens, the images are separated from minutes to months, while for galaxy cluster lenses, typical time delays can be months. The time delay can be absorbed into an observed coalescence time \( t_{c,\text{obs},j} = t_c + t_j \), and the magnification into an observed luminosity distance \( d^\text{obs,j}_L = d_L/\sqrt{\mu_j} \). However, considering higher-order modes in the signal, the Morse factor \( n_j \) can not just be absorbed into an observed coalescence phase.

3 Bayesian framework

* Under the hypothesis \( \mathcal{H}_I \), where \( I = U \) if unlensed and \( I = L \) if lensed, the interferometer data for a pair of events are, in terms of noise and signals,

\[ d_j(t) = n_j(t) + \tilde{h}_I(t; \vartheta_{1,j}), \]

where for the unlensed hypothesis \( \vartheta_{U,j} = \theta_j \), the usual BBH parameters, which can take different values for the two events. For the lensed hypothesis the parameters of the first event are \( \vartheta_{L,1} = \Phi = \{ \theta_1, n_1 \} \), and defining relative lensing parameters \( \Phi = \{ \mu_{rel}, t_{21}, n_{21} \} \) with \( \mu_{rel} = \mu_2/\mu_1 \), \( t_{21} = t_2 - t_1 \), and \( n_{21} = n_2 - n_1 \), for the second event one has \( \vartheta_{L,2} = \{ \Theta, \Phi \} \).

For each pair of events, under the lensing hypothesis one can compute a joint evidence

\[ p(d_1, d_2|\mathcal{H}_L) = \int p(d_1|\Theta)p(d_2|\Theta, \Phi)p(\Theta,p(\Phi)d\Theta d\Phi), \]

where \( p(d_1|\Theta), p(d_2|\Theta, \Phi) \) are the likelihoods for the two images, and \( p(\Theta), p(\Phi) \) are priors. With this we can compare the two hypotheses through the coherence ratio

\[ C_U^L = \frac{p(d_1,d_2|\mathcal{H}_L)}{p(d_1,d_2|\mathcal{H}_U)} = \frac{p(d_2|d_1,\mathcal{H}_L)p(d_1|\mathcal{H}_L)}{p(d_2|\mathcal{H}_U)p(d_1|\mathcal{H}_U)}, \]

where we used that \( p(d_1, d_2|\mathcal{H}_L) = p(d_2|d_1,\mathcal{H}_L)p(d_1|\mathcal{H}_L) \) for the lensed hypothesis, and independence between the two data streams for the unlensed hypothesis.

The key point of GOLUM is now that one can approximately identify

\[ p(d_2|d_1, \mathcal{H}_L) = \int L(\Phi)p(\Phi)d\Phi, \]

*A more detailed description can be found in Ref.8.
where the conditioned likelihood \( L(\Phi) = \langle p(d_2|\Theta, \Phi) \rangle_{p(\Theta|d_1)} \) is the likelihood of the second event averaged over the posterior samples of the first one. Recasting the problem in this way is what enables a significant computational speed-up of the analysis problem.

Using the conditioned likelihood, one can evaluate the ratio of evidences \( C^E_{U} \). However, we also want to make use of the improvement in parameter determination offered by lensing\(^6\). The conditioned evidence gives access to the posterior distribution of the lensing parameters \( p(\Phi|d_1, d_2) \), and by reweighing,

\[
p(\Theta, \Phi|d_1, d_2) \propto \frac{p(d_2|\Theta, \Phi)}{p(d_1, d_2|\Phi)} p(\Theta|d_1) p(\Phi|d_1, d_2)
\]

This enables us to recover joint parameters similar to those obtained by joint parameter estimation frameworks. Once the first image has been analyzed, the second image analysis and the reweighing process take \( O(30) \) CPU minutes, hence the computational time is essentially that of a single event analysis. In addition, these calculations are not restricted to pairs and can be extended to any number of images\(^8\).

### 4 Example analysis for a quadruply lensed gravitational wave event

As an illustration, we inject a signal from a precessing BBH merger generated with IMRPHENOMPV2\(^{13}\) into synthetic stationary Gaussian noise for a network of detectors made of the two LIGO detectors and Virgo, at design sensitivity\(^{14,3}\). The event has masses of 36 \( M_\odot \) and 29.2 \( M_\odot \) and spin magnitudes 0.4 and 0.3, with tilt angles of 0.5 rad and 1 rad, respectively. The spin vectors’ azimuthal angle is 1.7 rad, while the precession angle about the orbital angular momentum is 0.3 rad. The apparent luminosity distance for the first image is 1500 Mpc. The inclination angle is 0.4 rad, the polarization angle is 2.66 rad, and the unlensed coalescence phase is 1.3 rad. For the sky position of the event, the right ascension is 1.375 rad, and the declination is \(-1.21\) rad. The first image is a type II image, with a Morse factor of 0.5. For the first image analysis, we use a uniform prior in chirp mass and mass ratio, time of arrival. The priors specified above, and we use the posterior samples in a conditioned likelihood for the second analysis. For this image, and all the others, the prior for the relative magnification is uniform between 0 and 10, the prior for the time delay is uniform in \([t_{j1} - 0.1, t_{j1} + 0.1]\), and the prior on the Morse factor difference is discrete uniform in \([0, 0.5, 1, 1.5]\). After its analysis, we get the posterior for the second image. Reweighed samples from the image 1 and 2 analyses can then be used in a conditioned likelihood to analyze the third image, and similarly for the fourth image. The process is thus made of a succession of GOLUM runs and reweighing, one for each additional image, reducing significantly the cost of multiple image analyses.

To illustrate the benefit of lensing on the posterior distribution, we focus on the evolution of the sky localization as a function of the number of images. The evolution of these posteriors can be seen in Fig. 1, where with each additional image the sky location area is reduced, going from \(\sim 20 \text{deg}^2\) for one image to \(\sim 2 \text{deg}^2\) for four images at 90\% confidence. This is in agreement with the expected improvement in sky localization for lensed BBHs when multiple images are detected\(^{15}\).

### 5 Conclusions

We have introduced GOLUM, a fast and precise methodology to analyze strongly lensed gravitational waves, and have shown its use in an example analysis. The method enables us to have fast analyses with good accuracy, helping to tackle the issue of an increasing number of event pairs to be analyzed.
Figure 1 – The 90% credible region of the sky location of a strongly lensed gravitational-wave singlet (black), pair (green), triplet (blue), and quadruplet (red). There is a clear improvement in the sky localization with the addition of every gravitational-wave image. The final 90% confidence sky area is ~ 2 deg² in this example, down from ~ 20 deg² for the singlet. An improved sky localization would be particularly useful for lensed host galaxy localization.  

In addition, it enables the analysis of more than two images. These benefits also make the methodology useful for science case studies related to strong lensing, where one wants to inject a large number of signals.

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