Perturbed Operators on Banach Spaces

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Abstract

Let \(X\) be a Banach space over \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\), and let \(f := F + C\) be a weakly coercive operator from \(X\) onto \(X\), where \(F\) is a \(C^1\)–proper operator, and \(C\) a \(C^1\)–compact operator. Sufficient conditions are provided to assert that the perturbed operator \(f\) is a \(C^1\)–diffeomorphism. Three corollaries are given. The first one, when \(F\) is a linear homeomorphism. The second one, when \(F\) is a \(k\)–contractive perturbation of the identity. The third one, when \(X\) is a Hilbert space and \(F\) a particular linear operator. The proof of our results is based on properties of Fredholm operators, as well as on local and global inverse mapping theorems, and the Banach fixed point theorem. As an application two examples are given.

Key words: compact operator, Fredholm mapping, weakly coercive operator, index, local \(C^r\)–diffeomorphism, global inverse theorem.

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1. INTRODUCTION AND PRELIMINARIES

The existence of \(C^r\)–diffeomorphisms, \(0 \leq r \leq \infty\), are used in applications of functional analysis. For example, given an operator equation \(fu = v\) between Banach spaces \(X, Y\), the knowledge that the operator \(f : X \to Y\) is a \(C^r\)–diffeomorphism naturally implies the existence and uniqueness of solution \(u\) in the initial space \(X\) for any fixed \(v\) of the final space \(Y\). In practice, operators can be perturbed by the addition of other operators. The goal of our paper is to study the behavior of some these perturbed operators. We show that when \(Y = X\), and a \(C^1\)–proper operator \(F\) is perturbed by a \(C^1\)–compact operator \(C\), the resultant perturbed operator \(f := F + C\) is a global \(C^1\)–diffeomorphism, if some regularity conditions are verified. Hence existence and uniqueness of solution are verified for any fixed value \(v\) of the final space. When the hypotheses required to the operators are weakened but we suppose that a fixed \(v\) is a

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regular value for the perturbed operator, we only obtain that this particular equation have at most finitely many solutions.

A first corollary of our main result shows that if $F$ is a linear homeomorphism and $C$ a linear compact operator with $\|F\| \neq \|C\|$ that $f$ is a global $C^1$–diffeomorphism. A second corollary shows that a linear $k$–contractive plus a $C^1$–compact perturbation of the identity is a global $C^1$–diffeomorphism, if some regularity conditions are verified. A third corollary is given for $C^1$–compact perturbations of particular linear operators on Hilbert spaces.

The tools used are local and global inverse mapping theorems, and a corollary of the Banach fixed point theorem [6], as well as properties of Fredholm mappings [8].

We could have used degree theory [2] or continuation methods [1] trying to prove the existence of finite number of solutions, but this does not guarantee the desired uniqueness.

As an application of our results, two examples are furnished.

Other existence theorems have been given by the first author in finite dimensional setting [4] and in infinite dimensional setting [5], and by both authors in [3].

**Definitions** [6-8].

The operator $T : M \subseteq X \to X$, where $(X, d)$ is a metric space and $M$ is a closed nonempty subset of $X$, is called $k$–contractive iff the following holds: $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in M$ and for a fixed $k$, $0 \leq k < 1$. We call to this constant $k$, the contractive constant of the operator $T$.

In the following definitions, $X$ and $Y$ are Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, if nothing else is specified.

Given two operators $f, g : X \to Y$, then $f$ is called a compact perturbation of $g$ (a $k$–contractive perturbation) provided that there is a compact operator (a $k$–contractive operator), $h : X \to Y$ such that, $f = g + h$.

An operator $A : X \to Y$ is called weakly coercive whenever $\|A(u)\| \to \infty$ if and only if $\|u\| \to \infty$.

The symbol dim means dimension, codim means codimension, ker means kernel, and $\text{R}(L)$ stands for range of the operator $L$.

An operator $A : D(A) \subset X \to Y$ is said to be compact whenever it is continuous and the image $A(B)$ is relatively compact (i.e. its closure $\overline{A(B)}$ is compact in $Y$) for every bounded subset $B \subset D(A)$. Obviously this property is equivalent to the following: If $(u_n)_{n \geq 1}$ is a bounded sequence in $D(A)$, then there is a subsequence $(u_{n_k})_{k \geq 1}$ such that the sequence $(A(u_{n_k}))_{k \geq 1}$ is convergent in $Y$.

The operator $A$ is said to be proper whenever the pre-image $A^{-1}(K)$ of every compact subset $K \subset Y$ is also a compact subset of $D(A)$.

The operator $A$ is said a submersion at the point $u$ if it is a $C^1$–mapping on a neighborhood of $u$, $A'(u) : X \to Y$ is surjective, and $\ker(A(u))$ has topological complement in $X$. The point $u \in X$ is called a regular point of $A$ if and only if $A$ is a submersion at $u$. The point $v \in Y$ is called a regular value if and only if $A^{-1}(v)$ is empty or consists solely of regular points.
That $L : X \to Y$ is a linear Fredholm operator means that $L$ is linear and continuous and both numbers $\dim(\ker(L))$ and $\mathrm{codim}(\mathrm{R}(L))$ are finite, and therefore $\ker(L) =: X_1$ is a Banach space and has topological complement $X_2$, since $\dim(X_1)$ is finite. The integer number $\text{Ind}(L) = \dim(\ker(L)) - \mathrm{codim}(\mathrm{R}(L))$ is called the index of $L$.

Let $A : D(A) \subseteq X \to Y$. If $D(A)$ is open, then $A$ is said to be a Fredholm operator whenever both $A$ is a $C^1$-operator and the Fréchet derivative of the operator $A$ at the point $x$, $A'(x) : X \to Y$, is a Fredholm linear operator for all $x \in D(A)$. If $\text{Ind}(A'(x))$ is constant with respect to $x \in D(A)$, then we call this number the index of $A$ and write it as $\text{Ind}(A)$.

Two Banach spaces $X, Y$ are called isomorphic if and only if there is a linear homeomorphism (isomorphism) $L : X \to Y$.

By $\mathcal{F}(X,Y)$, $\mathcal{L}(X,Y)$ and $\text{Isom}(X,Y)$ we denote, respectively, the set of all linear Fredholm operators $A : X \to Y$, the set of all linear continuous operators $L : X \to Y$, and the set of all isomorphisms $L : X \to Y$.

Let $f : M \subseteq X \to Y$ with $M$ an arbitrary set. Then $f$ is called a $C^{r}$-operator when it can be extended locally to a $C^{r}$-operator in the usual sense, which means that, for each $x \in M$, there exists an open neighborhood $U(x)$ such that $f$ can be extended to a $C^{r}$-operator on $U(x)$.

Let $M$ and $N$ be arbitrary sets in $X$ and $Y$, respectively. Let $r$ be either a nonnegative integer or $\infty$. Then a mapping $f : M \to N$ is called a $C^{r}$-diffeomorphism if and only if $f$ is bijective and both $f$ and $f^{-1}$ are $C^{r}$-operators. A local $C^{r}$-diffeomorphism at $x_0$ is a $C^{r}$-diffeomorphism from some neighborhood $U(x_0)$ in $X$ onto some neighborhood $U(f(x_0))$ in $Y$.

For a map $r : U(0) \subseteq X \to Y$, we will write $r(x) = o(|x|)$, $x \to 0$, provided that $\frac{r(x)}{|x|} \to 0$ as $x \to 0$.

By $(\cdot, \cdot)$ we denote the inner product on a Hilbert space $X$. By $\text{span}(\Omega)$ we denote the Lebesgue measure of the set $\Omega$. By $f_o(x, y)$ is denoted the partial derivative of $f$ at $(x, y)$ with respect $y$.

**Theorem 1.** (Global inverse mapping theorem of Banach and Mazur) [6]. Assume that $0 \leq r \leq \infty$. Let $f : X \to Y$ be a local $C^{r}$-diffeomorphism, at every point of $X$. Then $f$ is a $C^{r}$-diffeomorphism if and only if $f$ is proper.

**Theorem 2.** (Local inverse mapping theorem) [6]. Let $f : U(x_0) \subseteq X \to Y$ be a $C^1$-mapping. Then $f$ is a local $C^1$-diffeomorphism at $x_0$ if and only if $f'(x_0) : X \to Y$ is bijective.

**Theorem 3.** (Banach’s open mapping theorem) [6]. If $A : X \to Y$ is linear, continuous, and surjective, then: (i) $A$ maps open sets into open sets. (ii) If the inverse operator $A^{-1} : Y \to X$ exists, then it is continuous.

**Theorem 4.** [6]. Let $g : D(g) \subset X \to Y$ be a compact operator. Let $a \in D(g)$. If the derivative $g'(a)$ exists, then $g'(a)$ is also a compact operator.

**Theorem 5.** [8]. Let $S \in \mathcal{F}(X,Y)$. The perturbed operator $S + C$ satisfies $S + C \in \mathcal{F}(X,Y)$ and $\text{Ind}(S + C) = \text{Ind}(S)$ if $C \in \mathcal{L}(X,Y)$ and $C$ is a compact operator.
Theorem 6. [8]. Let $S \in \mathcal{F}(X,Y)$. Then there exists a number $\xi > 0$ such that $T \in \mathcal{F}(X,Y)$ and $\text{Ind}(T) = \text{Ind}(S)$ for all operators $T \in \mathcal{L}(X,Y)$ with $\|T - S\| < \xi$.

Theorem 7. (Banach fixed point theorem) [6]. Let $T : M \subseteq X \rightarrow M$ be a contractive operator, where $(X,d)$ is a complete metric space and $M \subseteq X$ is a closed nonempty set. Then the following holds: (a) Existence and uniqueness. The equation $x = Tx$, $x \in M$, has exactly one solution, i.e., $T$ has exactly a fixed point $x$ on $M$. (b) Convergence of the iteration. For any fixed $x_0 \in M$, the sequence $(x_n)_{n \geq 0}$ of successive approximations, where $x_{n+1} = Tx_n$, $n = 0, 1, 2, ...$, converges to the fixed point $x$.

Theorem 8. (Corollary of the Banach fixed point theorem) [6]. Let $X$ be a complete metric space, $M \subseteq X$ be a closed nonempty set of $X$, and let $P$ be a metric space, called the parameter space. Suppose the following conditions are satisfied: (i) For each $p \in P$, the operator $T_p : M \subseteq X \rightarrow M$ is $k$-contractive, but with the same contractive constant $k$ for all the operators $T_p$. (ii) for a fixed $p_0 \in P$ and for all $x \in M$, $\lim_{p \rightarrow p_0} T_p x = T_{p_0} x$. Then: (a) For each $p \in P$, the equation $x = T_p x$ has exactly one solution $x_p \in M$. (b) $\lim_{p \rightarrow p_0} x_p = x_{p_0}$.

Theorem 9. (Lax-Milgram Theorem) [8]. Let $B : X \rightarrow X$ be a linear continuous operator on the Hilbert space $X$. Suppose there is $c > 0$ such that $|\langle Bu|u \rangle| \geq c\|u\|^2$ for all $u \in X$. Then, for each given $f \in X$, the operator equation $Bu = f$, $u \in X$ has a unique solution.

2. PERTURBED OPERATORS ON BANACH SPACES

Next, we establish our main result.

Lemma 1. Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Suppose that the following conditions are verified:

(i) The operator $K : X \rightarrow X$ is $k$-contractive with $k \in [0,1)$.

(ii) The operator $I : X \rightarrow X$ is the identity.

Then:

(a) The operator $F := I + K$ is a homeomorphism.

Proof. Ad (a). Equation

$$F(x) = y \quad \text{for any fixed } \ y \in X,$$  \hspace{1cm} (1)

is equivalent to equation,

$$x = y - K(x).$$

Define, the operator

$$T_y : X \rightarrow X, \quad T_y(x) := y - K(x).$$

Operators $\{T_y\}_{y \in X}$ fulfill the hypotheses of Theorem 8. In fact: $X$ is the closed nonempty set of the complete metric space $X$. The parameter space is also $X$. 
Furthermore (i) For fixed parameter \( y \in X \), operator \( T_y \) is \( k \)-contractive and the contraction constant \( k \) is the same for all operators \( \{ T_y \}_{y \in X} \), which is the contraction constant of operator \( K \). (ii) For any fixed parameter \( y_0 \in X \) and for all \( x \in X \)

\[
\lim_{y \to y_0} T_y(x) = T_{y_0}(x).
\]

Then the conclusion of Theorem 8 are verified, i.e.:

Conclusion (a). For each fixed \( y \) in the parameter space \( X \), there exists a unique \( x_y \) in the Banach space \( X \), such that \( T_y(x_y) = x_y \), or written in other way

\[
y - K(x_y) = x_y.
\]

Hence for any fixed \( y \in X \), there is a unique \( x_y \in X \) which is solution of equation (1), i.e.,

\[
F(x_y) = y.
\]

Hence \( F \) is a bijection, where \( F^{-1}(y) = x_y \).

Conclusion (b). For any fixed parameter \( y_0 \in X \), we have

\[
\lim_{y \to y_0} x_y = x_{y_0} \Rightarrow \lim_{y \to y_0} F^{-1}(y) = F^{-1}(y_0).
\]

Therefore the operator \( F^{-1} \) is continuous.

Since the operator \( K \) is continuous, being \( k \)-contractive, and since operator \( I \) is continuous, then the operator \( F \) is continuous. Since \( F \) is a bijection, and since \( F \) and \( F^{-1} \) are continuous operators, therefore \( F \) is a homeomorphism.

**Lemma 2.** Let \( X \) be a Banach space over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), and let \( f : X \to X \), \( f := F + C \) be a weakly coercive operator. Suppose that:

(i) \( F : X \to X \) is a proper continuous operator.

(ii) \( C : X \to X \) is a compact operator.

Therefore:

(a) Operator \( f \) is proper.

**Proof.** Ad (a). Since \( X \) is a Banach space, a set in \( X \) is compact if, and only if, it is sequentially compact. In the following we do not distinguish the notation between sequences and their subsequences. Let \( A \) be any fixed compact set of \( X \); so \( A \) is a bounded and closed set of \( X \). Let fix any sequence \( (y_n)_{n \geq 1} \) in \( A \),

\[
(y_n)_{n \geq 1} \subset A, \quad \text{where} \quad y_n = f(x_n) \quad \text{for all} \quad n.
\]

Since \( A \) is a bounded set, and since \( f \), by hypothesis, is a weakly coercive operator, then \( f^{-1}(A) \) is a bounded set and the sequence

\[
(x_n)_{n \geq 1} \subset f^{-1}(A)
\]

is a bounded set.
Since $A$ is a compact set and $(f(x_n))_{n \geq 1} \subset A$, there is a subsequence $(x_n)_{n \geq 1}$ such that the subsequence $(f(x_n))_{n \geq 1}$ converges to a point $u$ of $A$, i.e.,

$$\lim_{n \to \infty} f(x_n) = u \in A.$$ 

Since the subsequence $(x_n)_{n \geq 1}$ is a bounded set and $C$ is a compact operator, there is a subsequence $(x_n)_{n \geq 1}$ such that $(C(x_n))_{n \geq 1}$ converges to a point $v$ of $X$, i.e.,

$$\lim_{n \to \infty} C(x_n) = v \in X.$$ 

Then there is a subsequence $(x_n)_{n \geq 1}$ such that

$$\lim_{n \to \infty} (f - C)(x_n) = u - v \in X$$

or equivalently

$$\lim_{n \to \infty} F(x_n) = u - v.$$ 

Since $F$ is a proper operator and since $\{(F(x_n))_{n \geq 1} \cup \{u - v\}\}$ is a compact set, then

$$F^{-1}\{(F(x_n))_{n \geq 1} \cup \{u - v\}\}$$

is a compact set. Hence there is a convergent subsequence of the set

$$\left\{(x_n)_{n \geq 1} \cup F^{-1}(u - v)\right\}$$

that we call again $(x_n)_{n \geq 1}$, i.e.,

$$\lim_{n \to \infty} x_n := w.$$ 

Furthermore, since the operator $f$ is continuous as addition of continuous operators, and since the set $A$ is closed, therefore $f^{-1}(A)$ is also a closed set. Now

$$\lim_{n \to \infty} x_n = w \quad \text{and} \quad (x_n)_{n \geq 1} \subset f^{-1}(A),$$

then $w \in f^{-1}(A)$. Therefore the sequence

$$(x_n)_{n \geq 1} \subset f^{-1}(A)$$

has a subsequence $(x_n)_{n \geq 1}$ which converges to a point $w \in f^{-1}(A)$, and hence $f^{-1}(A)$ is a compact set. If no points in the compact set $A$ belong to $f(X)$, then $f^{-1}(A) = \emptyset$, which is a compact set. If there is only a finite number of points $f(x_i), i = 1, \ldots, N$ in $A$, trivially, its preimage by $f$, $x_i, i = 1, \ldots, N$, is a compact set. Therefore the preimage of any compact set $A$ for $f$ is also a compact set. Hence $f$ is a proper operator.

**Lemma 3.** Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let be $f : X \to X$, $f := f + C$. Suppose that:

(i) $f : X \to X$ is a $C^1$-operator, and there is $x_0 \in X$ such that,
\( F'(x_0) \) is bijective.

(ii) \( C : X \to X \) is a \( C^1 \)-compact operator

Then:

(a) \( f \) is a \( C^1 \)-Fredholm operator of index zero, i.e.,
\( f'(x) \in \mathcal{F}(X, X) \), and \( \text{Ind}(f'(x)) = 0 \) for all \( x \in X \).

**Proof. Ad (a).** Since the linear continuous operator \( F'(x_0) \) is a bijection
\[ \text{ker}(F'(x_0)) = \{0\}, \quad \text{and} \quad \text{R}(F'(x_0)) = X; \]
and so
\[ \dim(\text{ker}(F'(x_0))) = 0, \quad \text{and} \quad \text{codim}(\text{R}(F'(x_0))) = 0. \]
Therefore,
\[ F'(x_0) \in \mathcal{F}(X, Y), \quad \text{and} \quad \text{Ind}(F'(x_0)) = 0. \]

Since \( C \) is a \( C^1 \)-compact operator, Theorem 4 implies that \( C'(x) \) is also a compact operator. Since \( F'(x_0) \in \mathcal{F}(X, Y) \), with \( \text{Ind}(F'(x_0)) = 0 \) and since \( C'(x_0) \in \mathcal{L}(X, Y) \) is a compact operator, Theorem 5 implies that the operator
\[ f'(x_0) = F'(x_0) + C'(x_0) \in \mathcal{F}(X, Y) \] and \( \text{Ind}(f'(x_0)) = 0. \) (2)

Since \( f \) is a \( C^1 \)-operator as addition of \( C^1 \)-operators, then the operator
\[ f' : X \to \mathcal{L}(X, X), \quad x \mapsto f'(x), \]
is continuous. Since the mapping \( x \mapsto f'(x) \) is continuous, Theorem 6 implies that, the index \( \text{Ind}(f'(x)) \) is locally constant. Hence, since domain of \( f \) is \( X \), the index is independent of \( x \in X \). Therefore, equation (2) implies that
\[ \text{Ind}(f) = \text{Ind}(f'(x_0)) = 0. \]

Hence \( f \) is a \( C^1 \)-Fredholm operator of index zero.

**Theorem 10.** Let \( X \) be a Banach space over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), and let \( f : X \to X, \quad f := F + C \) be a weakly coercive operator. Suppose that:

(i) \( F \) is a \( C^1 \)-proper operator, and there is \( x_0 \in X \) such that \( F'(x_0) \) is a linear homeomorphism.

(ii) \( C \) is a \( C^1 \)-compact operator.

**Therefore:**

(a) If for any fixed \( x \in X \), \( f'(x) \) is weakly coercive, then \( f \) is a global \( C^1 \)-diffeomorphism.

(b) If \( y \in X \) is a regular value of \( f \), then the equation \( f(x) = y \) has at most finitely many solutions

**Proof.** Since \( F \) is a proper continuous operator, \( C \) a compact operator, and \( f \) a weakly coercive operator, then Lemma 2 implies that \( f \) is a proper operator.
Hypothesis (i), (ii), and Lemma 3 imply that is a \( C^1 \)-Fredholm operator of index zero, i.e.

\[
f'(x) \in \mathcal{F}(X, X), \text{ and } \text{Ind}(f'(x)) = 0 \text{ for any fixed } x \in X. \tag{3}
\]

**Ad (a).** The statement that for any fixed \( x \in X \), the operator \( f'(x) \) is weakly coercive, will drive us to

\[
\ker(f'(x)) = 0 \text{ for all } x \in X.
\]

In fact if \( \ker(f'(x)) \neq \{0\} \), there would be an \( y \in X \) with \( y \neq 0 \) such that

\[
(f'(x))(y) = 0.
\]

Let \( \lambda \in (0, +\infty) \), then

\[
\|\lambda y\| = |\lambda| \|y\| \to \infty \text{ when } \lambda \to \infty,
\]

and

\[
\|(f'(x))(\lambda y)\| = |\lambda| \|(f'(x))(y)\| = 0 \text{ when } \lambda \to \infty,
\]

since \(|\lambda| \|f'(x)y\| = 0\) for all \( \lambda \), because of \( y \in \ker(f'(x)) \backslash \{0\} \). This is a contradiction, because \( f'(x) \) is weakly coercive. Therefore

\[
\ker(f'(x)) = \{0\} \text{ for all } x \in X. \tag{4}
\]

Hence \( f'(x) \) is an injective operator for any fixed \( x \in X \).

Equations (3) and (4) imply that

\[
\text{codim } R(f'(X)) = X, \text{ and then } R(f'(X)) = X.
\]

Hence \( f'(x) \) is a bijective linear continuous operator for any fixed \( x \in X \). Theorem 2 implies that \( f \) is a local \( C^1 \)-diffeomorphism at any \( x \in X \).

Due to \( f \) being a proper operator and a local \( C^1 \)-diffeomorphism at any \( x \in X \), Theorem 1 implies that \( f \) is a global \( C^1 \)-diffeomorphism.

**Ad (b).** Since \( y \) is a regular value for \( f \), then \( f^{-1}(y) := G \) is empty or consists solely of regular points \( G := \{x_i\} \). Then if \( G \neq \emptyset \), there is \( x_i \), such that \( f \) has a submersion at \( x_i \), that is: \( f(x_i) = y \), \( \ker(f'(x_i)) \) has topological complement in \( X \) and \( f'(x_i) \) is surjective. Since \( f'(x_i) \) is a surjective operator, \( R(f'(x_i)) = X \), and then \( \text{codim } R(f'(x_i)) = 0 \). Equation (3) implies that \( f'(x) \in \mathcal{F}(X, X), \text{ and } \text{Ind}(f'(x)) = 0 \) for any fixed \( x \in X \). Since \( \text{Ind}(f'(x_i)) = 0 \) and since \( \text{codim } R(f'(x_i)) = 0 \), therefore \( \dim(\ker(f'(x_i))) = 0 \). Hence \( \ker(f'(x_i)) = \{0\} \) and \( f'(x_i) \) is injective. Thus \( f'(x_i) \) is a bijection. Since \( f'(x_i) \) is a bijection, then the local inverse mapping theorem 2 implies that \( f \) is a local \( C^1 \)-diffeomorphism at \( x_i \). Therefore there is a neighborhood \( U(x_i) \) of \( x_i \), and a neighborhood \( V(y) \) of \( y \), such that, \( f(x) = y \) with \( x \in U(x_i) \) if and only if \( x = x_i \).

Since \( f \) is a proper operator and \( y \) is a compact set, then \( G = f^{-1}(y) \) is a compact set. If \( G \neq \emptyset \), since there is a finite subcovering \( \bigcup_{1 \leq i \leq N} U(x_i) \) of \( G \) for
the open covering $\bigcup_{x_i \in G} U(x_i)$ of the compact set $G$, therefore there is at most a finite number of points $x$ such that $f(x) = y$.

**Corollary 1.** Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $f : X \to X$, $f := F + C$, where:

(i) $F$ is a linear homeomorphism
(ii) $C$ is a linear compact operator.
(iii) $\|F\| \neq \|C\|

Then:

(a) $f$ is a global $C^1$–diffeomorphism.

**Proof. Ad (a).** Since $f$, $F$, $C$ are linear continuous operators, then they are $C^1$–operators, and for any fixed $x \in X$

$$(F'(x))(y) = F(y), \quad (C'(x))(y) = C(y), \text{ and } (f'(x))(y) = f(y) \text{ for all } y \in X.$$ 

Therefore

$$\left\| F\left( \frac{y}{\|y\|} \right) \right\| - \left\| C\left( \frac{y}{\|y\|} \right) \right\| \|y\| = \|F(y)\| - \|C(y)\| \leq \|f(y)\|$$

$$\leq \left( \left\| F\left( \frac{y}{\|y\|} \right) \right\| + \left\| C\left( \frac{y}{\|y\|} \right) \right\| \right) \|y\|$$

for all $y \neq 0$.

Hence

$$\|F\| - \|C\| \|y\| \leq \|f(y)\| \leq (\|F\| + \|C\|) \|y\|,$$

and so if $\|F\| \neq \|C\|$, 

$$\|f(y)\| \to \infty \text{ if and only if } \|y\| \to \infty.$$ 

Thus $f$ is weakly coercive, and furthermore $f'(x)$ is also weakly coercive for any fixed $x \in X$.

Since $F$ is a homeomorphism and since continuous operators leave invariant compact sets, $F$ is a proper operator.

Since $F$ is a linear homeomorphism, $F'(x) = F$ and $(F^{-1})' (x) = F^{-1}$ for all $x \in X$. Therefore $F'(x)$ is a linear homeomorphism for any fixed $x \in X$.

Hypotheses of Theorem 10 (a) are fulfilled, then $f$ is a global $C^1$–diffeomorphism.

**Corollary 2.** Let $X$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and let $f : X \to X$, $f = I + K + C$ be a weakly coercive operator, where:

(i) $I$ is the identity operator.
(ii) $K$ is a linear $k$–contractive operator, with $k \in [0, 1)$.
(iii) $C$ is a $C^1$–compact operator.
(iv) $I + K + C'(x)$ is a weakly coercive operator for all $x \in X$.

Therefore:

(a) $f$ is a global $C^1$–diffeomorphism.
(b) If operator $K_1 = -K$, and operator $C_1 = -C$, then $K_1 + C_1$ has
a unique fixed point.

**Proof. Ad (a).** Set the operator

\[ F : X \to X, \quad F(x) := I(x) + K(x). \]

Therefore the weakly coercive operator \( f \) can be written as

\[ f = F + C. \]

Lemma 1 implies that \( F \) is a homeomorphism, and then a proper operator, because of \( F^{-1} \) being continuous, and since continuous operators map compact sets into compact sets.

Since \( I \) and \( K \) are linear, the homeomorphism \( F \) is linear. Therefore \( F'(x) = F \) and \( (F^{-1})'(x) = F^{-1} \) for all \( x \in X \). Hence \( F'(x) \) is a linear homeomorphism for any fixed \( x \in X \).

The hypothesis (iv) implies that \( f'(x) \) is a weakly coercive operator for any fixed \( x \in X \).

Since hypotheses of Theorem 10 (a) are fulfilled, \( f \) is a global \( C^1 \)-diffeomorphism.

**Ad (b).** Since \( f \) is a global diffeomorphism, there is a unique \( x \in X \) with \( f(x) = 0 \), therefore \( (K_1 + C_1)(x) = x \).

**Corollary 3.** Let \( X \) be a Hilbert space space over \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), and let \( f : X \to X, \quad f = F + C \) be a weakly coercive operator.

Suppose that:

(i) \( F \) is a linear continuous operator and there is a constant \( k > 0 \) such that \( \|Fx \mid x\| \geq k \|x\|^2 \) for all \( x \in X \).

(ii) \( C \) is a \( C^1 \)-compact operator, for all \( x \in X \), \( \|F \| \neq \|C'(x)\| \).

Therefore:

(a) \( f \) is a global \( C^1 \)-diffeomorphism.

**Proof. Ad (a).** Theorem 9 implies that \( F \) is a linear continuous bijection. Then the open mapping Theorem 3 (ii) implies that the linear operator \( F^{-1} \) is also continuous. Therefore \( F \) is a linear homeomorphism, and then a linear \( C^1 \)-proper operator.

Since \( F \) is linear and continuous, \((F'(x))(y) = F(y)\) for any fixed \( x \in X \) and all \( y \in X \). Therefore for any fixed \( x \in X \), \( F'(x) \) is a linear homeomorphism.

For any fixed \( x \in X \), operator \( f'(x) \) is weakly coercive. In fact:

\[
\left(\|F\left(\frac{y}{\|y\|}\right)\| + \left\|\left(C'(x)\right)\left(\frac{y}{\|y\|}\right)\right\|\right) \|y\| \geq \|f'(x)(y)\| = \|F(y) + (C'(x))(y)\|
\]

\[
= \left\|F\left(\frac{y}{\|y\|}\right)\|y\| + \left(\left(C'(x)\right)\left(\frac{y}{\|y\|}\right)\right) \|y\|\right\| \geq \left\|F\left(\frac{y}{\|y\|}\right)\right\| - \left\|\left(C'(x)\right)\left(\frac{y}{\|y\|}\right)\right\| \|y\|. \] (10)

for all \( y \in Y \). Then

\[
\|F\| + \|C'(x)\| \leq \|f'(x)(y)\| \geq \|F\| - \|C'(x)\|. \]

(10)
Therefore

\[ \| (f'(x))(y) \| \to \infty \quad \text{if and only if} \quad y \to \infty. \]

Hence for any fixed \( x \in X \), \( f'(x) \) is weakly coercive.

Theorem 10 (a) implies that \( f \) is a global \( C^1 \)-diffeomorphism.

3. EXAMPLES.

Let \( G \) be a nonempty bounded open set in \( \mathbb{R}^n \), and \( \Omega := \overline{G} \). In the two following examples let \( X \) be the Banach space \( X = C(\Omega) \) of continuous functions on the compact set \( \Omega \) with the norm \( \| u \| = \max_{x \in \Omega} |u(x)| \), and let the operator \( f : X \to X \) be the perturbed operator \( f := F + C \), where operators \( F, C : X \to X \) will be defined below. We want to know if \( f \) is a \( C^1 \)-diffeomorphism and then if for any \( v \in X \) there is one and only one \( u \in X \) such that \( fu = v \).

**Example 1.** Let be \( F := I + K \), with \( I \) the identity operator on the space \( X \), and \( K, C : X \to X \) a \( k \)-contractive and a compact operator respectively to be defined below. We define these particular operators \( K, C \) to be consider.

Operator \( K \) is defined in the following way:

for any \( u \in X \), \( u \mapsto Ku \), where \( Ku(x) := \int_{\Omega} k(x,y) u(y) dy \) for all \( x \) in \( \Omega \).

being

\[ k(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R} \]

a continuous function with \( \max_{(x,y) \in \Omega \times \Omega} |k(x,y)| \) \( \text{meas} (\Omega) < 1 \).

Since \( \Omega \times \Omega \subset \mathbb{R}^n \times \mathbb{R}^n \) is a compact set and \( k(\cdot, \cdot) \) a continuous function, there is \( \max_{(x,y) \in \Omega \times \Omega} |k(x,y)| := M \), and furthermore \( k(\cdot, \cdot) \) is uniformly continuous on \( \Omega \times \Omega \). Trivially \( K \) maps \( X \) into \( X \), and it is a linear operator. Since for \( u \in X \),

\[ \| Ku \| = \max_{x \in \Omega} \left| \int_{\Omega} k(x,y) u(y) dy \right| \leq \max_{x \in \Omega} \int_{\Omega} |k(x,y) u(y)| dy \leq M \text{meas}(\Omega) \| u \|, \]

then the operator \( K \) is bounded. Since \( K \) is a linear and bounded operator, it is a \( C^1 \)-operator with \( K(u) = K \) for all \( u \in X \).

\( K \) is a contractive operator since \( M \text{meas}(\Omega) := k < 1 \). In fact, for

\[ u, v \in X, \| Ku - Kv \| = \| K(u - v) \| = \max_{x \in \Omega} \left| \int_{\Omega} (k(x,y) (u - v)(y)) dy \right| \]

\[ \leq M \text{meas}(\Omega) \| u - v \| = k \| u - v \|. \]

Operator \( C \) is defined in the following way:

for any \( u \in X \), \( u \mapsto Cu \), where \( Cu(x) := \int_{\Omega} g(x,y) u(y) dy \) for all \( x \) in \( \Omega \).
being
\[ g(\cdot, \cdot) : \Omega \times \Omega \to \mathbb{R}. \]
a continuous function. The functions \( k(\cdot, \cdot), g(\cdot, \cdot) \) can be selected when linear operator \( K + C \) verify the condition \( \|K + C\| \neq 1 \), which is required so that the operator \( I + K + C \) to be weakly coercive.

By using the same reasoning that in the previous operator, we obtain that \( C : X \to X \) is a linear \( C^1(X) \) operator with \( C(u) = C \) for all \( u \) in \( X \), and that there is a real number \( N \) such that \( \max_{(x,y)\in\Omega\times\Omega} |g(x, y)| \operatorname{meas}(\Omega) := N \), and furthermore \( g(\cdot, \cdot) \) is uniformly continuous on \( \Omega \times \Omega \).

Since \( C \) is a continuous operator, to show that \( C \) is a compact operator, we have to prove that \( C \) maps bounded sets into relatively compact sets. To this end, let \( L \) be the bounded set \( L := \{u \in X : \|u\| \leq r\} \). Since \( \|Cu\| \leq N \operatorname{meas}(\Omega)r \) for all \( u \in L \), the set \( C(L) \) is a bounded set. Then, by the Arzelá-Ascoli theorem [6-7], to show that \( C(L) \) is a relatively compact set, we only have to prove that \( C(L) \) is equicontinuous. Let \( Cu \in C(L) \) and \( x, z \in \Omega \), we have
\[
|Cu(x) - Cu(z)| = \left| \int_{\Omega} g(x, y)u(y)dy - \int_{\Omega} g(z, y)u(y)dy \right|
\]
\[
= \left| \int_{\Omega} u(y)(g(x, y) - g(z, y))dy \right| \leq r \int_{\Omega} |g(x, y) - g(z, y)|dy.
\]
Since \( g(\cdot, \cdot) \) is uniformly continuous, for any \( \varepsilon > 0 \) there is \( \delta(\varepsilon) > 0 \) such that
\[
\|x - z\| < \delta(\varepsilon) \Rightarrow |g(x, y) - g(z, y)| < \varepsilon,
\]
and then
\[
|Cu(x) - Cu(z)| \leq r\varepsilon \operatorname{meas}(\Omega) \quad \text{for all } u \in L,
\]
and all \( x, z \in \Omega \) with \( \|x - z\| < \delta(\varepsilon) \).

Therefore \( C(L) \) is equicontinuous. Hence \( C \) is a compact operator.

By Lemma 1 the operator \( F \) is a homeomorphism. Trivially \( F \) is linear as addition of linear operators. Therefore \( F \) is a \( C^1 \)-linear homeomorphism, and
\[
F'(u) = F \quad \text{for all } u \in X.
\]
Hence operator \( F'(u) \) is a linear homeomorphism for all \( u \in X \).

Operator \( f = F + C = I + (K + C) : X \to X \) is a linear continuous operator as addition of linear continuous operators, then
\[
f'(u) = f \quad \text{for all } u \in X. \tag{5}
\]

\[
\|fu\| \leq \|f\| \|u\|,
\]
which implies \( \|fu\| \) is bounded if \( \|u\| \) is bounded.

Since \( \|K + C\| \neq 1 \),
\[
\|f(u)\| = \|(K + C + I)u\| \geq \left\| (K + C) \frac{(u)}{\|u\|} \right\| \|u\| - \|u\| \|u\|, \quad \text{for all } u \neq 0,
\]
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then, 
\[ \|f(u)\| \geq |\|K + C\| - 1\|u\| \]

Therefore \( \|f(u)\| \to \infty \) as \( \|u\| \to \infty \). Hence \( f \) is weakly coercive. Formula (5) implies that for any fixed \( u \in X \), operator \( f'(u) \) is also weakly coercive.

Since the hypotheses of Theorem 10 (a) are verified, \( f \) is a \( C^1 \)-diffeomorphism.

**Example 2.** In this example the operator \( F \) is the identity \( I \) on the space \( X \), which is a linear homeomorphism, and then a \( C^1 \)-proper operator with \( f'(u) = I \) for all \( u \) in \( X \).

Compact operator \( C \) is defined in the following way: Let \( Q := \Omega \times \Omega \times \mathbb{R} \), and let

\[ h : Q \to \mathbb{R} \]

be a \( C^1 \)-function with \( \sup_{(x,y,u) \in Q} |h(x,y,u)| = \mathcal{L} \).

Now define operator \( C : X \to X \), \( u \to Cu \) as \( Cu(x) := \int_{x \in \Omega} h(x,y,u(y))dy \) for all \( x \in \Omega \). Indeed if \( u \in X \), \( Cu \in X \).

Since \( h \) is continuous \( \forall \varepsilon > 0 \) there is \( \delta(\varepsilon, u) \) such that \( \|x - z\| + |u - v| < \delta(\varepsilon, u) \Rightarrow |h(x,y,u) - h(z,y,v)| < \varepsilon \), and then

\[ |Cu(x) - Cu(z)| = \left| \int_{x \in \Omega} h(x,y,u(y)) - h(z,y,u(y))dy \right| \leq \varepsilon \text{ meas}(\Omega). \]

Therefore \( Cu \in X \).

We use the technique of linearization of \( C(u + m) \) with respect to \( m \) to study the existence of first Fréchet derivative at the point \( u \in X \). This means that for fixed \( u \in X \), we search for a decomposition of the form

\[ C(u + m) = Cu + dC(u; m) + o(\|m\|) \quad \text{for all} \quad m \in X \]

in a neighborhood of zero, where \( dC(u; m) \) represents the linear part with respect to \( m \). This linear part should be continuous. Indeed,

\[
C(u + m)(x) = \int_{\Omega} h(x,y,u(y) + m(y))\,dy
= \int_{\Omega} \{h(x,y,u(y) + h_u(x,y,u(y))m(y) + r_2\} \,dy
= Cu(x) + \int_{\Omega} h_u(x,y,u(y))m(y)\,dy + R_2,
\]

where \( \|R_2\| = o(\|m\|), \, m \to 0 \), for all \( m \) in a neighborhood of zero. Therefore the linear part is

\[ dC(u; m)(x) = \int_{\Omega} h_u(x,y,u(y))m(y)dy, \]

which certainly is continuous for fixed \( u \in X \). Hence the operator \( C \) is Fréchet differentiable at \( u \), and the Fréchet derivative of \( C \) at \( u \) is \( C'(u)m = dC(u; m) \). Furthermore

\[ \|C'u(m)\| \leq \mathcal{L} \text{ meas}(\Omega) \|m\|, \quad \text{and} \quad \|C'(u)\| \leq \mathcal{L} \text{ meas}(\Omega) \quad \text{for all} \quad u \in X. \]
Since the Fréchet derivative $C'(u)$ exists for all $u \in X$, then the Fréchet derivative of $C$ at $X$, $C' : X \rightarrow \mathcal{L}(X, X)$, also exists. To show that $C \in C^1(X, X)$, we have to prove that $C'$ is continuous at any point $u \in X$. Let $u, v, m \in X$. Since $C'(u), C'(v) \in \mathcal{L}(X, X)$, then $C'(u) - C'(v) \in \mathcal{L}(X, X)$ and

$$\|C'(u) - C'(v)\| = \sup_{m \in X, \|m\| \leq 1} \|(C'(u) - C'(v))m\|.$$ 

Since $h_u$ is continuous, because of being $h \in C^1(Q)$, $\forall \varepsilon > 0$ there is $\delta(\varepsilon, u)$ such that $|u(y) - v(y)| < \delta(\varepsilon, u) \Rightarrow |h_u(x, y, u(y)) - h_u(x, y, v(y))| < \varepsilon$. Now, observe that if $\|u - v\| < \delta(\varepsilon, u)$ then

$$\|C'(u)m - C'(v)m\| = \max_{x \in \Omega} \left| \int_{\Omega} (h_u(x, y, u(y)) - h_u(x, y, v(y))m(y)dy \right|$$

$$\leq \varepsilon \text{meas}(\Omega) \|m\|.$$ 

Hence the operator $C'$ is continuous, and thus $C$ is a $C^1$-operator.

Operator $C$ is compact: In fact, it is continuous since it is a $C^1$-operator, and furthermore maps bounded sets into relatively compact sets as we will show.

Let $P$ be any bounded subset of $X$, then there is $R > 0$ such that $u \in P$ implies that $u \in X$ and $\|u\| \leq R$. Let $Q_1$ be the following compact set

$$Q_1 := \{x, y, u \in \Omega \times \Omega \times \mathbb{R} : |u| \leq R\}.$$ 

To apply the Ascoli–Arzela theorem [6-7], we have to show that $C(P)$ is bounded and equicontinuous so that $C(P)$ is a relatively compact set. Since $h$ is continuous and $Q_1$ is a compact set, then $h$ is uniformly continuous on $Q_1$ and there is $M = \max_{(x,y,u)\in Q_1} |h(x, y, u)|$. Therefore for any fixed $u \in P, \|u\| \leq R \Rightarrow |u(y)| < R$ for all $y \in \Omega$, and

$$\max_{(x,y,u)\in Q_1} \left| \int_{\Omega} h(x, y, u(y))dy \right| \leq M \text{meas}(\Omega).$$ 

Hence $C(P)$ is bounded. Since $h$ is uniformly continuous on $Q_1$,

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ such that } \|x - z\| < \delta(\varepsilon), x, z \in \Omega \Rightarrow |Cu(x) - Cu(z)|$$

$$\leq \int_{\Omega} |h(x, y, u(y) - h(z, y, u(y))| < \varepsilon \text{meas}(\Omega).$$ 

Therefore $C(P)$ is equicontinuous. Hence $C$ maps bounded sets in relatively compact sets. Thus $C$ is a compact operator.

Operator $f = I + C$ is weakly coercive. In fact. Since

$$\|f(u)\| \leq \|u\| + \|Cu\| \leq \|u\| + M \text{meas}(\Omega),$$ 

$f$ is bounded if $u$ is bounded. Since

$$\|f(u)\| \geq \|Cu\| - \|u\| \rightarrow \infty \text{ if } u \rightarrow \infty.$$
i.e., \( \|f(u)\| \to \infty \) if and only if \( \|u\| \to \infty \).

For any fixed \( u \in X \), the operator \( C'u \) is weakly coercive:

Since \( \|(f'(u))v\| \leq (1 + \|C'(u)\|) \|v\| \), then \( \|(f'(u))v\| \) is bounded if \( \|u\| \) is bounded. Since \( \|(f'(u))v\| \geq \left|1 - \|C'(u)\|\|v\|\right| \|v\| \), then

\[ \|(f'(u))v\| \to \infty \] if \( \|v\| \to \infty \).

Therefore the hypotheses of Theorem 10 (a) are fulfilled. Hence \( f \) is a global \( C^1 \)-diffeomorphism.

References

[1] E.L. Allgower and K. Georg, Numerical Continuation Methods, Springer Series in Computational Mathematics 13, Springer-Verlag, New York, 1990.

[2] J. Fonseca and W. Gangbo, Degree Theory in Analysis and Applications, Clarendon Press, Oxford Lecture Series in Mathematics and its Applications 2, Oxford, 1995.

[3] J.M. Soriano and M. Ordoñez Cabrera, Continuation Methods and Condensing Mappings, Nonlinear Anal.-Theory Methods Appl. 102, 84–90 (2014).

[4] J.M. Soriano, Global minimum point of a convex function, Appl. Math. Comput. 55 (2-3), 213–218 (1993).

[5] J.M. Soriano, Continuous embeddings and continuation methods, Nonlinear Anal.-Theory Methods Appl. 70 (11) 4118–4121(2009).

[6] E. Zeidler, Nonlinear functional analysis and its applications I, Springer-Verlag, New York, 1992.

[7] E. Zeidler, Applied Functional Analysis, Springer-Verlag, Applied Mathematical Sciences 108, New York, 1995.

[8] E. Zeidler, Applied Functional Analysis, Springer-Verlag, Applied Mathematical Sciences 109, New York, 1995.

[9] E. Zeidler, Nonlinear functional analysis and its applications II A, Springer-Verlag, New York, 1992.