Pseudogap in 1d revisited

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Two decades ago, Sadovskii found an exact solution of a model describing a pseudogap in electron energy spectrum (first introduced by Lee, Rice and Anderson). The discovery of a pseudogap in high-$T_c$ superconductors has revived the interest to his exact solution. I review the model with the emphasis on physical content, point out an error in the original Sadovskii’s solution and explain which problem he actually solved. A recent incorporation of Sadovskii’s ideas into a description of “hot spots” on the Fermi surface in cuprate superconductors (Schmalian, Pines and Stojković) is briefly discussed.

I. INTRODUCTION

A model of electrons with a pseudogap from fluctuations of an order parameter was introduced in 1973 by P. A. Lee, T. M. Rice and P. W. Anderson [1]. A few years later, M. V. Sadovskii showed that it admits an exact solution [2,3]. The model describes a Peierls system (a metallic chain with a charge density wave instability) above the phase transition temperature $T_P$. The exact solvability comes at a price: (a) The solution is specifically tailored for one dimension. (b) It is assumed that Peierls-Kohn phonons are described by a non-self-interacting boson field. These two limitations of the Sadovskii’s solution have been known since its publication.

Recently, however, I discovered an unfortunate error in the original paper by Sadovskii and now I am convinced that he actually solved a completely different, rather unphysical problem. This and the fact that Sadovskii’s work is often regarded as the one and only exact model of the pseudogap [4,5] has prompted me to review this model. While its mathematical side has been discussed quite thoroughly by Sadovskii himself, the physical content deserves further comment.

The plan of the paper is as follows. After a brief description of the Peierls instability in a one-dimensional conductor (Sec. I A), a suitable mathematical formalism will be presented in Sec. I B. It will be shown that assessment of multi-phonon contributions to the fermion energy spectrum requires a knowledge of statistical properties of the phonon ensemble. The model of Sadovskii, which postulates Gaussian statistics for the phonons, is introduced, interpreted and thoroughly illustrated in Sec. II. This is done in order to demystify its well-known yet strange-looking electron spectrum in the limit of long-range phonon correlation length $\xi$. I will then point out a previously unnoticed error in Sadovskii’s “exact” solution for a finite $\xi$ (Sec. III) and explain which problem Sadovskii has actually solved. Finally, a recent extension of the Sadovskii model to “higher” dimensions by Schmalian et al. [6] in the context of high-$T_c$ superconductivity will be discussed in Sec. IV.

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A. Peierls instability and fluctuations in 1 dimension

An excellent introduction to the Peierls effect can be found in G. Grüner’s book [6]. See also an article by G. A. Toombs [7], which reviews in detail theoretical and experimental developments prior to 1984. In a one-dimensional electron gas (with Fermi momentum $p_F$) fluctuations of electron density are particularly strong near the wavevectors $\pm 2p_F$. This happens because creating a hole and an electron with momenta near $\pm p_F$ costs little energy. Therefore, when electron-phonon interaction couples lattice vibrations to these fluctuations, phonon modes with momenta near $2p_F$ become “soft” and a static charge-density wave (CDW) appears below a transition temperature $T_P$.

In the context of the mean-field theory of a Peierls transition, an energy gap opens at the Fermi points exactly at $T = T_P$. However, some remnant of the gap can be created by fluctuations even above $T_P$. Lee, Rice and Anderson calculated the electron self-energy induced by the emission and reabsorption of a (dressed) phonon (Fig. 1). Approximating the two-point phonon correlation by two Lorentzian lines peaked at $\pm 2p_F$, one obtains

$$\Sigma(\omega, p_F + p) = \frac{\delta^2}{\omega + pv + iv\xi^{-1}},$$

where $\delta^2 = 4\pi^3 T_P^2 \xi / 7\zeta(3)v$ can be regarded as the average fluctuation of the order parameter, $\xi$ is the phonon correlation length and $v$ the Fermi velocity.

FIG. 1. One-phonon correction to the fermion propagator. A phonon with momentum $2p_F + k$ (dashed line) connects electron states with momenta $p_F + p$ and $-p_F + p - k$, respectively energies $pv$ and $-(p - k)v$.

B. Continuum formulation

In the vicinity of the two Fermi points, the equations of motion for the fermion field $c_k(t)$ in the presence of longitudinal phonons is

$$\frac{id}{dt} - vq)c_{p_F+q}(t) = gL^{-1/2} \int \frac{dp}{2\pi} \delta x_{2p_F+q-p} c_{p_F+p}(t),$$

$$\frac{id}{dt} + vq)c_{-p_F+q}(t) = g^*L^{-1/2} \int \frac{dp}{2\pi} \delta x_{-2p_F+q-p} c_{p_F+p}(t).$$

Here $\delta x$ is the atomic displacement along the chain direction, $g$ is the electron-phonon coupling and $L$ is the chain length. The electron energy spectrum near $\pm p_F$ has been linearized, $\epsilon_{\pm p_F+q} \approx \pm vq$.

It is convenient to combine right and left-moving fermion fields into a column
\[ \psi_q = \begin{pmatrix} \psi_{Rq} \\ \psi_{Lq} \end{pmatrix} = \begin{pmatrix} c_{pF+q} \\ c_{-pF+q} \end{pmatrix}. \] (3)

Phonons can be described as a complex gap field \( \Delta(t,x) \) defined in terms of the Fourier transform of the displacement

\[ \Delta_k(t) = g \delta x_{2pF+k}(t), \quad \Delta_k^*(t) = g^* \delta x_{2pF+k}^*(t) = g^* x_{-2pF-k}(t). \] (4)

The last equality is a statement that atomic displacements are real. In contrast, \( \Delta_k^* \neq \Delta_{-k} \), whereby \( \Delta^*(t, x) \neq \Delta(t, x) \), i.e., the gap field \( \Delta(t, x) \) is genuinely complex, except when the CDW is commensurate with the lattice, \( 2p_F = \pi/a \) or \( 2\pi/a \).

In the new notation, Eqs. (2) can be written as

\[
\begin{align*}
&i\left( \partial/\partial t + v \partial/\partial x \right) \psi_R(t,x) = \Delta(t,x) \psi_L(t,x), \\
&i\left( \partial/\partial t - v \partial/\partial x \right) \psi_L(t,x) = \Delta^*(t,x) \psi_R(t,x).
\end{align*}
\] (5)

In what follows, units in which \( \hbar = v = 1 \) will often be employed to simplify the notation.

The fermion propagator for the ground state \( |0\rangle \) can be defined as a \( 2 \times 2 \) matrix \( \hat{G} \) with matrix elements

\[
G_{\sigma\sigma'}(t - t', x - x') = -i\langle 0 | [\psi_{\sigma'}(t', x') \psi_{\sigma}^\dagger(t, x)] | 0 \rangle,
\] (6)

\( \sigma = 1 \) for right and \( -1 \) for left fermions. Thermal Green’s functions can be defined in a similar way. The propagator matrix satisfies the equation

\[
\left[ i \partial/\partial t + i \sigma_3 \partial/\partial x - \hat{\Delta}(t,x) \right] \hat{G}(t - t', x - x') = \delta(t - t') \delta(x - x'),
\] (7)

where \( \hat{\Delta}(t,x) \) is the off-diagonal matrix \( \Delta(t,x) \sigma_+ + \Delta^*(t,x) \sigma_- \) and \( \sigma_i \) are the Pauli matrices.

The free \( |\Delta(t,x) = 0\rangle \) propagator \( \hat{G}^{(0)} \) is diagonal in the basis of left and right-moving fermions, where \( \sigma_3 = \sigma = \pm 1 \):

\[
G^{(0)}_{\sigma\sigma}(t,x) = -1/2\pi \frac{1}{vt - \sigma x - i0 \text{sign}(t)}.
\] (8)

We will use extensively its Fourier transforms,

\[
G^{(0)}_{\sigma\sigma}(\omega, p) = \frac{1}{\omega - \sigma p + i0 \text{sign}(\omega)},
\] (9)

\[
G^{(0)}_{\sigma\sigma}(\omega, x) = -i \text{sign}(\omega) \theta(\sigma \omega x) e^{i\sigma \omega x},
\] (10)

where \( \theta(x) \) is the unit step-function. Eq. (10) indicates that fermions can only propagate in a single direction. Unless specified otherwise, it will be assumed throughout the paper that \( \omega > 0 \).

The gap field \( \Delta(t,x) \) is considered to be static, \( \Delta(x) \). In thermal field theory, this corresponds to a classical approximation, in which the typical frequency of a boson is much less than the temperature (and the occupation number of that mode greatly exceeds 1).
As long as this does not lead to an ultraviolet catastrophe, it appears to be a reasonable approximation.

All we need now to determine the properties of the fermions are the correlation functions for the gap field. In the symmetric phase (above $T_P$),

$$\langle \Delta(x) \rangle = \langle \Delta^*(x) \rangle = 0, \quad \langle \Delta(x) \Delta(x') \rangle = \langle \Delta^*(x) \Delta^*(x') \rangle = 0.$$  \hfill (11)

The two-point correlation function and its Fourier transform are

$$D(x - x') \equiv \langle \Delta(x) \Delta^*(x') \rangle = \delta^2 e^{-|x - x'|/\xi}$$  \hfill (12)

$$D(k) = \delta^2 \frac{2\xi^{-1}}{k^2 + \xi^2}.$$  \hfill (13)

C. Fermion spectrum to order $\delta^2$

The free ($\Delta = 0$) fermion density of states $\mathcal{N}^{(0)}(\omega)$ can be read off directly from the propagator $\hat{G}^{(0)}(\omega, x)$:

$$\mathcal{N}^{(0)}(\omega) = -\frac{1}{\pi} \operatorname{Tr} \operatorname{Im} \hat{G}^{(0)}(\omega, 0) \equiv -\frac{1}{\pi} \sum_{\sigma = \pm 1} \operatorname{Im} \hat{G}^{(0)}_{\sigma\sigma}(\omega, x) \bigg|_{x=0}. \hfill (14)$$

While the value of the Green’s function (10) is not defined at $x = 0$, we can either take the limit $x \to 0^+$ or integrate over momenta

$$-\pi^{-1} \operatorname{Im} \hat{G}^{(0)}_{\sigma\sigma}(\omega, p) = \delta(\omega - \sigma p). \hfill (15)$$

Either way, the free density of states (per spin) is

$$\mathcal{N}^{(0)}(\omega) = 1/\pi = 1/\pi v,$$  \hfill (16)

as one expects in one dimension.

The fermion Green’s function in the presence of a gap field can be obtained by starting with the free propagator and iterating Eq. (4). This procedure gives an expansion of $\hat{G}$ in powers of the gap field,

$$\langle \hat{G} \rangle = \hat{G}^{(0)} + \langle \hat{G}^{(2)} \rangle + \langle \hat{G}^{(4)} \rangle + \ldots \hfill (17)$$

The brackets signify averaging over configurations of the phonon field. The lowest-order correction $\langle \hat{G}^{(2)} \rangle$ is a diagonal matrix. E.g., for right-moving fermions,

$$G^{(2)}_{RR}(x', x) = \int d\zeta d\zeta' \ G^{(0)}_{RR}(x' - \zeta) \Delta(\zeta) \ G^{(0)}_{LL}(\zeta - \zeta') \Delta^*(\zeta') \ G^{(0)}_{RR}(\zeta' - x). \hfill (18)$$

Averaging over a phonon ensemble with a mean fluctuation (12) brings out $\langle \Delta(\zeta) \Delta^*(\zeta') \rangle = \delta^2 e^{-(\zeta - \zeta')/\xi}$. Then

$$\langle G^{(2)}_{RR}(x', x) \rangle = \delta^2 \int d\zeta d\zeta' \ e^{-(\zeta - \zeta')/\xi} \ G^{(0)}_{RR}(x' - \zeta) \ G^{(0)}_{LL}(\zeta - \zeta') \ G^{(0)}_{RR}(\zeta' - x), \hfill (19)$$
which is translationally invariant.

Surely this correction can be computed in an easier way, by working directly with Fourier transformed quantities, Fig. [Fig. 1] and Eq. (1). Coordinate representation, nevertheless, is also useful. After all, the local density of states is given by

$$
-\frac{1}{\pi} \text{Tr} \text{Im} \hat{G}(\omega; x, x).
$$

As a bonus, we will see where and why Sadovskii’s exact solution actually works — see Sec. [Sec. III C].

**FIG. 2.** Second-order correction to the fermion propagator $G^{(2)}(\omega, x', x)$. Solid lines: free fermion propagator $-ie^{i\omega l_n}$. Dashed line: two-point phonon correlation $\delta^2 e^{-|\zeta - \zeta'|/\xi} = \delta^2 e^{-l_1/\xi}$. The “time” direction is added to split apart fermion lines.

Thanks to the presence of a step-function in the free propagators (10), it is more convenient to integrate over path lengths $l_1, l_2, l_3$ than over intermediate coordinates $\zeta_1, \zeta_2$ (Fig. 2). When $\omega > 0$, the free fermion propagator is

$$
G^{(0)}_{\sigma\sigma} = -ie^{i\omega l_n}, \quad 0 \leq l_n < \infty.
$$

The lengths of fermion legs are not completely independent as the total displacement $x' - x$ is fixed. This constraint is implemented by inserting

$$
\delta(x - x' - l_0 + l_1 - l_2) = \int \frac{dp}{2\pi} e^{ip(x-x' - l_0 + l_1 - l_2)}
$$

in the integrand.

In particular, when we are interested in the local density of states, it makes sense to evaluate

$$
\langle G^{(2)}_{RR}(x, x) \rangle = i\delta^2 \int \frac{dp}{2\pi} \int_0^\infty dl_2 e^{i(\omega-p)l_2} \int_0^\infty dl_1 e^{i(\omega+p+\xi)l_1} \int_0^\infty dl_0 e^{i(\omega-p)l_0} 
$$

$$
= \delta^2 \int \frac{dp}{2\pi} \frac{1}{\omega - p + i\Delta} \frac{1}{\omega + p + i\xi^{-1}} \frac{1}{\omega - p + i\Delta}. \tag{22}
$$

The integrand on the last line is the familiar second-order self-energy (11) with the external legs reattached. After the integration, we find the density of states to order $\delta^2$:

$$
\mathcal{N}^{(0)}(\omega) + \mathcal{N}^{(2)}(\omega) = \frac{1}{\pi v} \left( 1 + \text{Re} \frac{2\delta^2}{(2\omega + iv\xi^{-1})^2} \right) \tag{23}
$$

(the factor of 2 comes from adding the contribution of left-moving fermions). The density of states is reduced in the range $|\omega| < v\xi^{-1}/2$, which can be called a pseudogap. This approximation is valid only when the fluctuations are fast enough, $\delta \ll v\xi^{-1}/2$. 

5
D. Beyond $\delta^2$

Correction to the fermion propagator of order $\delta^{2n}$ reads

$$\langle G^{(2n)}_{RR}(x', x) \rangle = \int d\zeta_n d\zeta'_n \ldots d\zeta_1 d\zeta'_1 \ G^{(0)}_{RR}(x' - \zeta_n) \ldots G^{(0)}_{RR}(\zeta'_1 - x)$$

$$\times D(\zeta_n, \zeta'_n, \ldots, \zeta_1, \zeta'_1),$$

(24)

where the $2n$-point correlation functions is

$$D(\zeta_n, \zeta'_n, \ldots, \zeta_1, \zeta'_1) = \langle \Delta(\zeta_n) \Delta^*(\zeta'_n) \ldots \Delta(\zeta_1) \Delta^*(\zeta'_1) \rangle$$

(25)

In principle, the $2n$-point correlation function must be determined in a microscopic theory. In most cases, however, evaluation of higher-order phonon correlation functions is a rather difficult job. Alternatively, one can try to see what comes out of (24) given a certain statistics of the phonon field (25).

A rather trivial example would be that of the mean-field approximation, in which the displacement amplitude is uniform throughout the chain, i.e., $\Delta(\zeta) = \delta$ with certainty for any $\zeta$. In this case, (27) reduces to

$$D(\zeta_n, \zeta'_n, \ldots, \zeta_1, \zeta'_1) = \delta^{2n}. \quad (26)$$

Then, the Fourier transform of the right-moving propagator is

$$\langle G_{RR}(p) \rangle = \sum_{n=0}^{\infty} \langle G^{(2n)}_{RR}(p) \rangle = \sum_{n=0}^{\infty} \frac{\delta^{2n}}{\omega - pv)^n \omega + pv + \delta^2}$$

(27)

$$= \frac{\omega + pv}{\omega^2 - p^2 v^2 - \delta^2}. \quad (28)$$

The spectral function contains two narrow peaks,

$$A(\omega, p) = -\pi^{-1} \text{Im} G_{RR}(\omega + i0, p)$$

$$= \frac{\tilde{\epsilon}_p + pv}{2\tilde{\epsilon}_p} \delta(\omega - \tilde{\epsilon}_p) + \frac{\tilde{\epsilon}_p - pv}{2\tilde{\epsilon}_p} \delta(\omega + \tilde{\epsilon}_p). \quad (29)$$

where

$$\tilde{\epsilon}_p = \sqrt{p^2 v^2 + \delta^2} \quad (30)$$

(recall that $p$ is the distance to the Fermi momentum $\pm p_F$). The density of states vanishes for $|\omega| < \delta$ and exhibits a pile-up near $\omega = \pm \delta$:

$$N(\omega) = \frac{\theta(\omega^2 - \delta^2)}{\pi v} \frac{|\omega|}{\sqrt{\omega^2 - \delta^2}}. \quad (31)$$

To describe a state without a long-range order, one can consider a phonon statistics with a fixed gap amplitude $\delta$ and a fluctuating phase (the nonlinear $O(2)\sigma$ model). The model is characterized by a single parameter, a temperature-independent phase stiffness $\alpha$. The two-point phonon correlation function has been calculated, e.g., in [6]:

6
\[ D(x, x') = \delta^2 e^{-|x-x'|/\xi}, \quad \xi = \alpha/T. \] (32)

Higher-order correlations can be computed in a similar way:

\[
D(x_n, x'_n, \ldots, x_1, x'_1) \equiv \langle \Delta(x_n) \Delta^*(x'_n) \ldots \Delta(x_1) \Delta^*(x'_1) \rangle \\
= \delta^{2n} \exp \left( -\sum_{i,j=1}^{n} \frac{|x_i-x'_j| + |x'_i-x_j| - |x_i-x_j| - |x'_i-x'_j|}{2\xi} \right). \tag{33}
\]

As the temperature approaches zero, \( \xi \to \infty \) and one recovers the statistics of the mean-field theory (32). Accordingly, the fermion energy spectrum in the limit of a long correlation length approaches the BCS form (31), something one rightfully expects.

II. MODEL OF SADOVSKII

A phonon system with different statistics was considered in the 1970’s by M. V. Sadovskii. Instead of a fixed gap amplitude and Gaussian phase fluctuations (as in Sec. I D), his model is concerned with independent Gaussian fluctuations of real and imaginary parts of \( \Delta(x) \). In other words, both phase and amplitude of the gap are allowed to fluctuate. This feature leads to a very different fermion spectrum in the limit of slow fluctuations.

A. Phonons with Gaussian statistics

Statistical properties of a Gaussian random variable \( \Delta(x) \) are completely determined once the mean value and two-point correlations are specified:

\[
\langle \Delta(x) \rangle = \langle \Delta^*(x) \rangle = 0, \tag{34}
\]
\[
\langle \Delta(x) \Delta(x') \rangle = \langle \Delta^*(x) \Delta^*(x') \rangle = 0, \tag{35}
\]
\[
\langle \Delta(x) \Delta^*(x') \rangle = D(x-x') \equiv \delta^2 e^{-|x-x'|/\xi}. \tag{36}
\]

All higher-order correlations (35) are then given by Wick’s theorem,

\[
D(x_n, x'_n, \ldots, x_1, x'_1) = D(x_n - x'_n) \ldots D(x_2 - x'_2) D(x_1 - x'_1) + \text{permutations of primed coordinates}. \tag{37}
\]

The right-hand side includes \( n! \) terms, e.g.,

\[
D(x_2, x'_2, x_1, x'_1) = \delta^4 e^{-|x_2-x'_2|/\xi} e^{-|x_1-x'_1|/\xi} + \delta^4 e^{-|x_2-x'_1|/\xi} e^{-|x_1-x'_2|/\xi}. \tag{38}
\]

B. Solution for \( \xi \to \infty \)

As first noted by Sadovskii [2], the determination of the electron energy spectrum simplifies in the limit of long-range (slow) fluctuations of the order parameter, \( \xi \gg \delta/v \). The electron energy spectrum in this limit is strikingly different from the spectrum with a sharp
gap discussed in Sec. I D. Instead, one finds a broadly smeared gap, or pseudogap, which is caused by fluctuations of the gap amplitude, absent in the previous model.

Taking $\xi \to \infty$ reduces (36) to a constant and thus (37) becomes coordinate-independent as well:

$$D(x_n, x'_n, \ldots, x_1, x'_1) = n! \delta^{2n}. \quad (39)$$

By comparing this result to what we had for the state with a fixed $\Delta$ (26), we can write the following expression for the fermion propagator

$$\langle G_{RR}(p) \rangle = \sum_{n=0}^{\infty} \langle G^{(2n)}_{RR}(p) \rangle = \sum_{n=0}^{\infty} n! \frac{\delta^{2n}}{(\omega - pv)^{n+1}(\omega + pv)^n}. \quad (40)$$

The only difference from (27) is the factor $n!$, which makes the sum divergent for any frequency and momentum.

This difficulty can be circumvented [2] if we recognize that the divergent sum is an asymptotic expansion of the Stiltjes integral:

$$\int_0^{\infty} \frac{e^{-t} dt}{1 - tx} = \sum_{n=0}^{\infty} x^n. \quad (41)$$

This is precisely our series. The left-hand side is perfectly finite for any $x$ away from the positive real axis. If $x$ approaches the real axis from the complex plane, $x \pm i0$, the integral (41) has a non-zero imaginary part, not reproducible by a sum of positive numbers on the right-hand side, hence a divergence.

Rather than trying to resum a divergent series, it is more useful to remove the divergence all together. For that purpose, we will go back to the original assumption about the Gaussian statistics, which is the source of $n!$. In the limit $\xi \to \infty$, instead of a random field $\Delta(x)$, we have a single random variable $\Delta$ describing the value of the gap field everywhere on the chain. Its Gaussian character, postulated above, is realized by considering an ensemble of chains, each with a different but fixed $\Delta$, with the distribution (“density of chains”)

$$\rho(\Delta) = e^{-|\Delta|^2/\delta^2} / \pi \delta^2, \quad (42)$$

which gives, as required,

$$\langle \Delta^n \Delta^* \rangle \equiv \int |\Delta|^{2n} \rho(\Delta) d^2 \Delta = n! \delta^{2n}. \quad (43)$$

On every single chain, there is a perfect Peierls gap of size $|\Delta|$, which, however, varies from chain to chain.

To obtain, e.g., the density of states in such an ensemble, one can average the result for a single gap (31) over the distribution of gaps (42), which gives a smeared-out gap [2]. The density of states vanishes as $\omega^2$ at low frequencies. The fermion spectral function can be obtained in a similar way, by integrating the BCS spectral weight with two $\delta$-functions (29) over the gap distribution (42). As a result of a varying gap amplitude $|\Delta|$, one finds [2] peaks that are significantly broad, especially near the Fermi points ($p = 0$), where the spectral line shape is

$$A(\omega, 0) = |\omega| \delta^{-2} e^{-\omega^2/\delta^2}. \quad (44)$$

A large linewidth, of order $\delta$, reflects not a scattering rate, but rather an inhomogeneous broadening due to a varying gap size.
C. What do Sadovskii’s chains look like?

A typical Sadovskii’s chain is shown in Fig. 3. The gap field, or the complex amplitude of atomic displacements, remains approximately constant over distances smaller than the correlation length $\xi$. Even though this chain looks very rough, its two-point correlation function $D(x, x')$ is quite smooth. In fact,

$$D(x, x') = \delta^2 e^{-|x-x'|/\xi}$$

(45)

exactly for this particular chain.

But wait. How is a correlation function defined for a single chain? And why is the correlation function translation invariant,

$$D(x + \zeta, x' + \zeta) = D(x, x'),$$

(46)

whereas the chain is not? Answer:

$$D(x, x') \overset{\text{def}}{=} \frac{1}{L} \int_0^L d\zeta \Delta(x + \zeta) \Delta^*(x' + \zeta).$$

(47)

This is obviously translation invariant for periodic boundary conditions.

![Typical Sadovskii’s chain](image)

FIG. 3. Half of a typical Sadovskii chain. The correlation length (the two-head arrow) is $\xi = 1/8$, the mean value of the gap is $\delta = 1$. Solid line: $\text{Re}\Delta(x)$, dashed line: $\text{Im}\Delta(x)$. The actual atomic displacement at a point $x$ is given by $\text{Re} \left[ \Delta(x)e^{2\pi k_F x} \right]$.

To see how a chain with the right correlation function can be constructed, rewrite (47) in terms of Fourier components

$$D(x, x') = L^{-1} \int_0^L d\zeta \left[ L^{-1/2} \sum_k \Delta_k e^{ik(x+\zeta)} \right] \left[ L^{-1/2} \sum_{k'} \Delta^*_{k'} e^{-ik'(x'+\zeta)} \right]$$

$$= L^{-1} \sum_k |\Delta_k|^2 e^{ik(x-x')} \to \int \frac{dk}{2\pi} |\Delta_k|^2 e^{ik(x-x')}$$

(48)
in the limit \( L \to \infty \). Choosing now
\[
\Delta_k = \delta e^{i\theta_k} \sqrt{\frac{2\xi^{-1}}{k^2 + \xi^{-2}}}
\]
with an arbitrary phase \( \theta_k \) immediately yields (43). This is approximately how the chain in Fig. 3 has been simulated.

Furthermore, it can now be seen that an ensemble of such chains exhibits the required Gaussian statistics (37). For instance, the four-point correlation function for a single chain is
\[
D(x_2, x_2', x_1, x_1') = \frac{1}{L} \int_0^L d\zeta \, \Delta(x_2 + \zeta) \Delta(x_1 + \zeta) \Delta^*(x_2' + \zeta) \Delta^*(x_1' + \zeta)
\]
\[
= L^{-2} \sum_{\{k\}} \Delta_{k_2} e^{ik_2 x_2} \Delta^*_{k_2'} e^{-ik_2' x_2'} \Delta_{k_1} e^{ik_1 x_1} \Delta^*_{k_1'} e^{-ik_1' x_1'},
\]
where momenta satisfy the constraint \( k_1 + k_2 = k_1' + k_2' \).

The crucial step is to average (50) over the arbitrary phases \( \theta_k \), which enter this expression in the form of the factor
\[
\exp \left[ i(\theta_{k_1} + \theta_{k_2} - \theta_{k_1'} - \theta_{k_2'}) \right].
\]

Phases \( \theta_k \) are independent random variables uniformly distributed in the interval \( 0 < \theta_k < 2\pi \). Averaging over them makes (51) vanish, unless the phase factors cancel one another pairwise:
\[
k_1 = k_1', \quad k_2 = k_2', \quad \text{or} \quad k_1 = k_2, \quad k_2 = k_1'.
\]

The so averaged four-point correlation function reads
\[
\langle D(x_2, x_2', x_1, x_1') \rangle = L^{-1} \sum_{k_2} |\Delta_{k_2}|^2 e^{ik_2 (x_2 - x_2')} L^{-1} \sum_{k_1} |\Delta_{k_1}|^2 e^{ik_1 (x_1 - x_1')}
\]
\[
+ L^{-1} \sum_{k_2} |\Delta_{k_2}|^2 e^{ik_2 (x_2 - x_1')} L^{-1} \sum_{k_1} |\Delta_{k_1}|^2 e^{ik_1 (x_1 - x_2')}
\]
\[
- L^{-2} \sum_k |\Delta|^4 e^{ik (x_2 - x_2' + x_1 - x_1')}.
\]

The third line in (53) is needed to adjust for the overcounting of the terms with \( k_1 = k_1' = k_2 = k_2' \) in (52). Comparing these to (13) reveals (almost) Gaussian statistics:
\[
\langle D(x_2, x_2', x_1, x_1') \rangle = D(x_2, x_2') D(x_1, x_1') + D(x_2, x_1') D(x_1, x_2')
\]
\[
- L^{-1} \int \frac{dk}{2\pi} |\Delta_k|^4 e^{ik (x_2 - x_2' + x_1 - x_1')}.
\]

In the limit \( L \to \infty \), the extra term vanishes as \( \xi/L \) or faster. This procedure can evidently be extended to arbitrarily high orders.
Wick’s theorem with 32 chains

\[
\langle D(x,x,0,0) \rangle = D(x,x)D(0,0) + D(x,0)D(0,x)
\]

\[
\langle D(x,0,0,0) \rangle = 2D(x,0)D(0,0)
\]

FIG. 4. Four-point correlation functions \( \langle D(x,x,0,0) \rangle \) (crosses) and \( \langle D(x,0,x,0) \rangle \) (diamonds) averaged over 32 different chains. Solid lines are the corresponding Gaussian curves. \( \xi = L/64, \delta = 1. \)

We thus have found an efficient way to simulate the ensemble considered by Sadovskii. Namely, by using the known Fourier amplitudes (49), we can generate a sufficiently large number of chains with different choices of random phases \( \theta_k \). That this number does not have to be too large is demonstrated in Fig. 4. Average four-point correlations in an ensemble of just 32 chains agree quite well with their Gaussian expectation values. Note a systematic downward shift for \( \langle D(x,x,0,0) \rangle \), which is expected to equal \( \xi/L \).

As a matter of fact, it is not necessary to consider an ensemble of chains. Instead, one can regard a single long enough chain as an ensemble of its segments (of length \( L \)). Doing so resolves the apparent paradox of strongly broadened electron states in the limit \( \xi \to \infty \). As long as one studies properties of electrons within a single segment, this broadening will exist until \( \xi \) exceeds the segment length \( L \). In the limit \( \xi \gg L \), electrons will see a well-defined gap \( \Delta \) (within this segment) and their spectral function will exhibit two sharp Bogoliubov peaks at energies \( \pm \sqrt{p^2v^2 + |\Delta|^2} \). However, if we now pack our instruments and go to another segment of the chain, far enough away, we could find there a different value of the gap. Averaging over an infinitely long chain, on the other hand, will always give the Gaussian broadening as \( \xi < L = \infty \) always in that case.

Remark. The possibility of gap-amplitude fluctuations is largely ignored in the literature. Common-sense wisdom suggests that variations of the energy gap cost too much energy and therefore only the phase of the order parameter \( \Delta \) is expected to fluctuate at low temperatures. There are two counterarguments here. First, a qualitative one, that gap-size fluctuations further increase the entropy and thus reduce the free energy. Another, quantitative argument appeals to the well-known solution [8] for the fermion spectrum in the Luttinger model with attraction, truly an exactly solvable model with a pseudogap. Cooper pairs, the dominating fluctuations in this model [8], produce a broadly smeared energy gap in the fermion spectrum, even at zero temperature. In the limit of a large correlation length, phase fluctuations alone cannot account for a strong smearing. Thus, amplitude fluctuations
are inevitably present in this model and may be more commonplace than usually thought.

**III. SADOVSKII’S SOLUTION**

We now return to the theoretical analysis of the problem, this time for a finite correlation length $\xi$.

**A. Sadovskii’s conjecture**

As previously noted, evaluation of Feynman diagrams is more convenient in momentum space, where the two-point phonon correlation is a Lorentzian [13],

$$D(k) = \delta^2 \frac{2\xi^{-1}}{k^2 + \xi^{-2}}$$  \hspace{1cm} (55)

As we have seen before, the order-$\delta^2$ correction to the electron Green’s function is (Fig. II)

$$G^{(2)}_{RR}(p) = \delta^2 \frac{1}{\omega - p + i0} \frac{1}{\omega + p + i\xi^{-1}} \frac{1}{\omega - p + i\xi^{-1}}.$$  \hspace{1cm} (56)

This line is rather transparent: integration over momentum $k$ transferred to the phonon simply shifts the imaginary in the denominator of the intermediate electron propagator from $+i0$ to $+i\xi^{-1}$.

![FIG. 5. Two-phonon contributions to the fermion propagator. (a) is generated by the first-order self-energy, while (b) contains two-phonon self-energy.](image)

Such a simple form of the second-order correction,

$$\Sigma^{(2)}_{RR}(\omega, p) \propto G^{(0)}_{LL}(\omega + i\xi^{-1}, p),$$  \hspace{1cm} (57)

has prompted Sadovskii to *conjecture* that contributions of higher-order graphs to $G_{\sigma\sigma}(p)$ are given by the following simple rules:

- A phonon line contributes $\delta^2$,  \hspace{1cm} (58)
- An electron line contributes $\frac{1}{\omega \pm p + i\nu\xi^{-1}}$,  \hspace{1cm} (59)

where $\nu$ is the number of phonon lines above a given electron line. The sign in front of $p$ alternates as the fermion propagates left and right.

For instance, according to this rule, corrections of order $\delta^4$ to the electron Green’s function (Fig. III) should read
\[
(a) = \delta^4 \left( \frac{1}{\omega - p + i0} \right)^3 \left( \frac{1}{\omega + p + i\xi^{-1}} \right)^2, \tag{60}
\]
\[
(b) = \delta^4 \left( \frac{1}{\omega - p + i0} \right)^2 \left( \frac{1}{\omega - p + 2i\xi^{-1} + \omega + p + i\xi^{-1}} \right)^2. \tag{61}
\]

Basing on this Ansatz, Sadovskii was able to derive and solve a recursion relation for the self-energy of order \(\delta^{2n} \) following a method due to Elyutin \cite{9}. The exact Green’s function was then obtained in a continued fraction representation. This remarkable derivation is getting quite popular these days \cite{4,5}.

**B. Failure in order \(\delta^4\)**

Unfortunately, Ansatz \cite{59} works only for a limited class of diagrams [e.g., Fig. \(5(a)\)] and is simply incorrect for others [Fig. \(5(b)\)]. The problem, quite mundane, is in sloppy handling of the imaginary part — equal to \(+i0\) \(\text{sign}(\omega)\) or \(i\omega_n\) depending on the formalism — in the denominator of \(G^{(0)}(\omega, p)\).

Recall that, to order \(\delta^2\), we integrated
\[
\int \frac{dk}{2\pi} \frac{1}{\omega + p - k + i0} \frac{2\xi^{-1}}{k^2 + \xi^{-2}};
\]
which has two poles above the real \(k\) axis and only one pole below. If we complete the integration contour in the lower half of the plane, only one pole is inside and the resulting expression is simple.

In evaluating graph (b) in Fig. \(5\), the integral over momentum \(q\) of the external phonon line reads
\[
\int \frac{dq}{2\pi} \frac{1}{[\omega + (p - q) + i0]^2} \frac{1}{\omega - (p - q - k) + i0} \frac{2\xi^{-1}}{q^2 + \xi^{-2}}, \tag{63}
\]
which has two poles on either side of the real \(q\) axis, so that, whichever way the contour is completed at infinity, the result contains two terms, rather than one. Integrating over \(k\) first does not help either:
\[
\int \frac{dq}{2\pi} \frac{1}{[\omega + (p - q) + i0]^2} \frac{1}{\omega - (p - q) + i\xi^{-1}} \frac{2\xi^{-1}}{q^2 + \xi^{-2}} \tag{64}
\]
is plagued by the same problem. The integral over \(q\), completed above the real axis, yields the result conjectured by Sadovskii \cite{61} plus a non-zero contribution from the pole at \(q = \omega + p + i0\).

The situation does not change when one uses thermal Green’s functions, in which case \(\omega + i0\) \(\text{sign}(\omega)\) is replaced with \(i\omega_n\), and the same problem arises. In higher orders, expressions for \(G^{(2n)}(p)\) becomes progressively more complicated by the presence of diagrams with a phonon line running over many electron propagators.
C. Which problem did Sadovskii solve, exactly?

![Diagram](image)

FIG. 6. Correction to the fermion propagator $G^{(4)}(x', x)$. Solid lines: free fermion propagator $-ie^{i\omega l_n}$. Dashed lines: two-point phonon correlation $\delta^2 e^{-|\zeta_i - \zeta'_i|/\xi}$. Vertical dimension is added for clarity.

The fact that the trouble is caused by infinitesimal imaginary numbers in fermion propagators may create an illusion that the problem can be somehow fixed. It is more instructive to look at it in configuration space. We will now see exactly which problem Sadovskii solved.

Using conventions of Sec. I C, we write out the expression for the first of the two diagrams for $\langle G^{(4)}_{RR}(p) \rangle$, Fig. 6(a):

$$-i \int_0^\infty dl_4 \ldots dl_0 e^{i(\omega - p)l_4} e^{i(\omega + p)l_3} e^{i(\omega - p)l_2} e^{i(\omega + p)l_1} e^{i(\omega - p)l_0} \times \delta^4 e^{-|\zeta_2 - \zeta'_2|/\xi} e^{-|\zeta_1 - \zeta'_1|/\xi}$$

(65)

[cf. Eq. (22)]. As $|\zeta_2 - \zeta'_2| = l_3$ and $|\zeta_1 - \zeta'_1| = l_1$, the integrals over lengths $\{l_n\}$ can be immediately carried out and one obtains (66).

The other diagram, Fig. 6(b), differs by a permutation of $\zeta'_1$ and $\zeta'_2$, so that only the second line of (66) changes and now reads

$$\times \delta^4 e^{-|\zeta_2 - \zeta'_1|/\xi} e^{-|\zeta_1 - \zeta'_2|/\xi}$$

(66)

While $|\zeta_1 - \zeta'_2| = l_2$, the other distance, $|\zeta_2 - \zeta'_1|$, cannot be simply expressed as a sum of some path lengths, which is what causes the problem. Note, however, that, had we replaced the physical distance $|\zeta_2 - \zeta'_1|$ with the sum of path lengths $l_1 + l_2 + l_3$, the previous expression would have read

$$\times \delta^4 e^{-l_1/\xi} e^{-2l_2/\xi} e^{-l_3/\xi},$$

(67)

which could be easily integrated over lengths yielding Eq. (61), precisely what Sadovskii wanted.
Once the physical distance between two points $|\zeta_i - \zeta'_j|$ in the phonon correlation function $D(\zeta_i, \zeta'_j)$ has been replaced with the length of the fermion path between these points, Sadovskii’s conjecture (56) is valid in all orders of perturbation theory. Indeed, define $\nu_m$ to be the number of phonon lines above the fermion leg $l_m$, which can be done unambiguously by straightening out the fermion trajectory (i.e., by using Fig. 6 instead of Fig. 5). The contribution of a given diagram to $G_{RR}^{2n}(p)$ will then be a product of independent factors

$$\delta^{2n} \prod_{m=0}^{2n} (-i) \int_0^\infty e^{i\omega l_m} e^{-i(-1)^m p l_m} e^{-\nu_m l_m / \xi} dl_m = \delta^{2n} \prod_{m=0}^{2n} \frac{1}{\omega - (-1)^m p + i\nu_m \xi^{-1}},$$

(68)

precisely as required by (56).

It is thus clear that the original Ansatz of Sadovskii solves a rather unphysical problem, in which phonon correlations $\langle \Delta(x)\Delta^*(x') \rangle$ depend not on the geometrical distance $|x - x'|$, but rather on the length of the path the fermion traveled between points $x$ and $x'$. This point is further illustrated using a two-dimensional example in Sec. IV.

**IV. EXTENSION TO HIGHER DIMENSIONS?**

It has already been mentioned that the calculation of Sadovskii is tailored to one spatial dimension. This limitation stems from the fact that the order-$\delta^2$ correction to the fermion self-energy in the presence of classical fluctuations of an order parameter,

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{\omega + (p - k) \cdot v + i0} \frac{2\xi^{-1}}{k^2 + \xi^{-2}},$$

(69)

has a simple form in $d = 1$ dimension only. Not having a simple, “self-replicating” form for the lowest-order correction possibly indicates that there is little hope of finding a general recipe for higher orders.

**A. Model of “hot spots” in the cuprates**

Recently, however, J. Schmalian, D. Pines and B. Stojković applied the ideas of Sadovskii to a two-dimensional system, high-$T_P$ cuprate superconductors, to investigate the nearly antiferromagnetic Fermi liquid (NAFL). This development further illustrates in what context the solution of Sadovskii is applicable. It turns out that the dimensionality of the system is not important. A really necessary ingredient is the peculiar form of order-parameter correlations, which should decay exponentially with the “distance” measured along the fermion path. (There is also a technical, but very important, requirement that the spectrum of free electrons be flat, i.e., $\epsilon_{pF+p} = p \cdot v$, where $v$ is a constant vector.)
FIG. 7. (a) A sketch of the Fermi surface (solid line) in the cuprates. \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) are Fermi velocities at two hot spots (filled circles) connected by the antiferromagnetic wave vector \( \mathbf{Q} = (\pi, \pi) \). The dashed line is the locus of states most strongly affected by the AFM scattering, \( \epsilon_{\mathbf{p} + \mathbf{Q}} = \epsilon_{\mathbf{p}} \). (b) A fermion initially in the vicinity of the hot spot \( \mathbf{p}_1 \) travels in a zigzag manner in real space switching between non-collinear velocities \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \) as it is scattered by spin fluctuations.

In the NAFL approach, electrons, considered to be ideal fermions, interact with antiferromagnetic (AFM) spin fluctuations, whose static susceptibility is peaked near wavenumber \( \mathbf{Q} = (\pi, \pi) \) in reciprocal lattice units:

\[
\chi(\mathbf{Q} + \mathbf{q}) \approx \frac{\chi(\mathbf{Q})}{1 + q^2 \xi^2}.
\]  

(70)

The strongest effect of AFM fluctuations on the fermion energy spectrum is expected when scattering by wave vector \( \mathbf{Q} \) connects states of the same energy, \( \epsilon_{\mathbf{p} + \mathbf{Q}} = \epsilon_{\mathbf{p}} \). Such points in the Brillouin zone form a line shown in Fig. 7(a) for a tight-binding fermion energy spectrum (nearest and next-nearest neighbor hopping). Places where this line intersects the Fermi surface have been termed “hot spots”. Low-energy fermionic excitations in these spots are presumably fried by spin fluctuations and are short-lived, hence the name. This must be true, at least to some extent, as photoemission shows extremely broad peaks (hundreds of meV) in the electron spectral weight \( A(\omega, \mathbf{p}) \) at these momenta [11].

Consider the lowest-order fermion self-energy from one-magnon exchange (the same diagram as in Fig. 1). After linearizing the free fermion spectrum near the hot spots

\[
\epsilon_{\mathbf{p}_n + \mathbf{p}} \approx \mathbf{v}_n \cdot \mathbf{p},
\]  

(71)

the self-energy for a fermion near hot spot \( \mathbf{p}_1 \) reads

\[
\Sigma(\omega, \mathbf{p}_1 + \mathbf{p}) \approx \int \frac{d^2\mathbf{q}}{(2\pi)^2} \frac{\chi(\mathbf{Q})}{1 + q^2 \xi^2} \frac{1}{\omega + \mathbf{v}_2 \cdot (\mathbf{p} - \mathbf{q}) + i\delta}.
\]  

(72)

Here \( \mathbf{Q} + \mathbf{q} \) is the momentum transferred to the magnon. Note that the intermediate electron is near the other hot spot \( \mathbf{p}_2 \). This has precisely the form of Eq. (69) and one cannot get a simple expression out of it, to say nothing of higher-order corrections.
Schmalian et al. noted that Fermi velocities at conjugated hot spots (e.g., $v_1$ and $v_2$) are almost perpendicular to each other. If then one replaces the susceptibility (70) with a product

$$\chi(Q + q) \approx \delta^2 \frac{2\xi^{-1}}{q_1^2 + \xi^{-2}} \frac{2\xi^{-1}}{q_2^2 + \xi^{-2}},$$

where $q_n$ is the component of $q$ along $v_n$, a very simple self-energy results:

$$\Sigma(\omega, p_1 + p) \approx \frac{\delta^2}{\omega + v_2 \cdot p + iv\xi^{-1}} = G^{(0)}(\omega + iv\xi^{-1}, p_2 + p),$$

where $v = |v_1| = |v_2|$. Moreover, higher-order diagrams can be evaluated in a similar manner yielding simple expressions in the form conjectured by Sadovskii (56). Lo and behold, the problem becomes tractable to arbitrary order and the electron Green’s function can be obtained in the continued fraction representation [5], as discussed by Sadovskii.

### B. What makes it solvable

One should not be surprised that the trick with factorization (73) makes the problem solvable. The factorization amounts to taking spin-spin correlations in real space in the form

$$\chi(r - r') \propto \langle s^+(r)s^-(r') \rangle \propto e^{-|x-x'|/\xi}e^{-|y-y'|/\xi},$$

where $x$ and $y$ are Cartesian components of the electron in the plane (along the directions of $v_1$ and $v_2$, i.e., approximately along the crystal axes). This is precisely the same as to say that order parameter correlations decay with the “distance” measured along the fermion path, Fig. 7(b), as I noted in the beginning of this Section. That said, it is not even necessary to require that $v_1$ and $v_2$ be orthogonal.

Technically, the similarity with the one-dimensional problem of Sadovskii arises because the electron energy spectrum has been linearized, whatever the actual number of dimensions is. Indeed, according to (74), the energy as a function of momentum varies only in the direction of $v_n$. The fermion spectrum is exactly dispersionless in all directions perpendicular to $v_n$. This means, literally, that a free fermion propagates along a straight line, Fig. 7(b). In this approximation, $d^2\epsilon/dp_i dp_j = 0$, a localized wave packet does not disperse as it propagates with velocity $v_i = d\epsilon/dp_i$. It would not be a stretch to say that this is essentially a one-dimensional problem.

### C. No pseudogap in the DOS

Despite great similarities, there is one important aspect in which this two-dimensional problem differs from the purely one-dimensional case of Sadovskii. In plain English, a fermion never returns to a starting point: it zigzags away, Fig. 7(b). In contrast, $v_1 = -v_2$ in one dimension and a fermion does return to the starting point “once in a while”. An important consequence of this innocuous observation is that, with $v_1 \neq -v_2$, the local fermion propagator is unaffected by fluctuations to all orders.
\[ G(\omega, x, x) = G^{(0)}(\omega, x, x), \]  
\[ \text{(76)} \]
because \( G^{(n)}(\omega, x, x) = 0 \) for any order \( n > 0 \). Therefore, the local density of electron states is exactly the same as for free, noninteracting fermions:

\[ N(\omega) = -\pi^{-1}\text{Im}G(\omega, x, x) = -\pi^{-1}\text{Im}G^{(0)}(\omega, x, x) = N^{(0)}(\omega). \]
\[ \text{(77)} \]
In particular, this means that a local probe, such as tunneling microscopy [12] or NMR [13], should not observe any pseudogap behavior in the “hot spots” scenario!

The use of the coordinate representation makes a proof of this statement almost trivial: the step function in the free electron propagator (10) makes all corrections to the free propagator vanish for fermion trajectories with points outside the cone formed by the vectors \( v_1 \) and \( v_2 \). Thereby trajectories returning to the starting point consist of a single point, have zero integration measure (in the case of at least one intermediate point) and therefore do not contribute to the propagator.

Of course, this can be seen in momentum space as well. For simplicity, take the velocities \( v_1 \) and \( v_2 \) to be orthogonal to each other and choose a pair of coordinate axes along them. A generic correction of order \( \delta^{2n} \) to the propagator of a fermion near Hot Spot 1 contains two momentum-dependent factors:

\[ \delta^{2n} \prod_{a=0}^{n} \frac{1}{\omega - p_1 + i\nu_{1a}\xi^{-1}} \prod_{b=1}^{n} \frac{1}{\omega - p_2 + i\nu_{2b}\xi^{-1}}, \]
\[ \text{(78)} \]
where \( \nu_{1a}, \nu_{2b} > 0 \) (strict inequality!). Its contribution to the local DOS is obtained by integrating over the momentum components \( p_1 \) and \( p_2 \) and taking the imaginary part. This expression is an analytical function of \( p_1 \) below the real axis. If \( n > 0 \), the integration contour can be completed at infinity in the lower half of the complex \( p_1 \) plane (the integrand vanishes there fast enough). Since no singularities are encircled, the integral vanishes for any \( n > 0 \). The exceptional case \( n = 0 \) (free propagator) has been dealt with in Sec. [14].

It is worth stressing that Sadovskii’s solution should be considered as a long-wavelength approximation only (as it is based on a linearized electron spectrum). In practical terms, one should not attempt to draw conclusions about the detailed band structure basing on a solution of this type. Things like a Brillouin zone or a van Hove singularity simply do not belong in this theory. While integrating the spectral weight over a Brillouin zone may show a slight reduction in the density of states near the Fermi level [14], such an extrapolation of an effective field theory to real-life details is not warranted. The only conclusion that can be drawn safely is that, as the lattice spacing is taken to zero, any trace of the pseudogap disappears. Thus, the pseudogap is not natural in this model.
FIG. 8. Splitting of the free electron band (dashed line) into upper and lower bands (solid lines) in the presence of long-range AFM order with wavevector $\mathbf{Q} = (\pi/a, \pi/a)$. When the band splitting $2\delta$ is smaller than the bandwidth, there is no gap in the density of states.

While it may appear paradoxical that the DOS is unaffected, it is, in fact, a direct consequence of the assumptions that made the calculation of Schmalian et al. possible. It is also directly related to an observation by Randeria [15] that the NAFL pseudogap is not tied to the Fermi surface (the dotted line and the solid line in Fig. 7(a), respectively). In the antiferromagnetic scenario for the pseudogap, the spectral weight of the fermion states on the dotted line is moved from $\epsilon_p$ to higher and lower energies in the range $\epsilon_p \pm \delta$. Since, however, the energy $\epsilon_p$ varies along the dotted line (by the amount equal to $4t_2$, where $t_2$ is the next-nearest neighbor hopping amplitude), the pseudogap will be completely washed out if $4t_2$ exceeds $\delta$. Linearization of spectrum (71) is equivalent to assuming $\delta \ll 4t_2$ (no local pseudogap). As Monthoux and Pines suggested, $4t_2 = 0.45$ eV [17], so that any pseudogap of a lesser width will be washed out in the density of states. This situation is illustrated schematically in Fig. 8.

In contrast, there is no washing out of a pseudogap created by Cooper pair fluctuations. In that case, fermion states coupled by emission or absorption of a Cooper pair are electrons and holes of equal momenta and spin. Therefore, their velocities are equal and opposite,

$$v_h = \frac{d(-\epsilon_p)}{dp} = -\frac{d\epsilon_p}{dp} = -v_e,$$

as long as time reversal is a good symmetry of the system ($\epsilon_{-p} = \epsilon_p$). In the problem with a linearized dispersion, a fermion moves along a straight line back and forth alternating between an electron and a hole. There are non-zero corrections to the local propagator $G(\omega, x, x)$ in all orders, which means that scattering by Cooper pair fluctuations does affect the local DOS. Put simply, a pseudogap created by pairing fluctuations is tied to a Fermi surface.
V. SUMMARY

In this paper, it has been demonstrated that the issue of phonon statistics is quite impor-
tant for the properties of electrons in the pseudogap regime above the ordering tempera-
ture. Knowledge of the two-point correlation function $\langle \Delta(x) \Delta^*(x') \rangle$ allows one to compute the electron Green’s function or self-energy to the second order in the gap size $\delta$ only. When the correlation length of the fluctuations increases beyond the point $\xi > v/\delta$, higher-order phonon contributions become important, which is why multi-phonon correlation functions are needed. It has been shown explicitly that different choices of phonon statistics lead to widely different results for the fermion spectrum in the particularly interesting limit of slow fluctuations, $\xi \gg v/\delta$.

A model of phonons with Gaussian statistics [2] has been revisited and thoroughly dis-
cussed, both in momentum and coordinate domains. It has been shown that its “exact” solution for a finite correlation length [3] contains an error and, in fact, solves another, rather unphysical problem.

The physical reason why the gap in the density of states remains smeared even for very long correlation lengths in the model with Gaussian phonons resides with the fluctuations of the gap amplitude inherent in the model. This smearing should not be interpreted as a presence of a large (of order $\delta$) scattering rate. Rather, it should be regarded as an inhomogeneous broadening of energy levels, which, being a reversible process, can be distinguished from relaxational broadening. To do so, one may attempt to study the fermion lineshape using time-resolved spectroscopy rather than frequency-domain methods (cf. NMR).

Finally, I have discussed a few aspects of the newly proposed scenario for the behavior of electrons at “hot spots” in cuprate superconductors [5]. In particular, it appears that the antiferromagnetic fluctuations alone cannot explain the presence of a strong pseudogap seen by local probes of the density of states, such as tunneling spectroscopy and NMR. As has been noted before [15], pairing fluctuations seem to be a necessary ingredient to explain the pseudogap at low frequencies.

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