POST-HOPF ALGEBRAS, RELATIVE ROTA-BAXTER OPERATORS AND SOLUTIONS OF THE YANG-BAXTER EQUATION

YUNNAN LI, YUNHE SHENG, AND RONG TANG

Abstract. In this paper, first we introduce the notion of a post-Hopf algebra, which gives rise to a post-Lie algebra on the space of primitive elements and there is naturally a post-Hopf algebra structure on the universal enveloping algebra of a post-Lie algebra. A novel property is that a cocommutative post-Hopf algebra gives rise to a generalized Grossman-Larsson product, which leads to a subadjacent Hopf algebra and can be used to construct solutions of the Yang-Baxter equation. Then we introduce the notion of relative Rota-Baxter operators on Hopf algebras. A cocommutative post-Hopf algebra gives rise to a relative Rota-Baxter operator on its subadjacent Hopf algebra. Conversely, a relative Rota-Baxter operator also induces a post-Hopf algebra. Then we show that relative Rota-Baxter operators give rise to matched pairs of Hopf algebras. Consequently, post-Hopf algebras and relative Rota-Baxter operators give solutions of the Yang-Baxter equation in certain cocommutative Hopf algebras. Finally we characterize relative Rota-Baxter operators on Hopf algebras using relative Rota-Baxter operators on the Lie algebra of primitive elements, graphs and module bialgebra structures.

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Key words and phrases. post-Hopf algebra, Hopf algebra, relative Rota-Baxter operator, Yang-Baxter equation, matched pair

2020 Mathematics Subject Classification. 16T05, 16T25, 17B38.
1. Introduction

The Yang-Baxter equation is an important subject in mathematical physics \([44]\). Drinfeld highlighted the importance of the study of set-theoretical solutions of the Yang-Baxter equation in \([12]\). The pioneer works on set-theoretical solutions are those of Etingof-Schedler-Soloviev \([15]\), Lu-Yan-Zhu \([29]\) and Gateva-Ivanova-Vanden Bergh \([19]\). To understand the structure of set-theoretical solutions, Rump introduced braces in \([36]\) for abelian groups, which provide involutive nondegenerate solutions. See also \([3, 6]\) for more details about the finite simple solutions of the Yang-Baxter equation. Later Guarnieri and Vendramin generalized braces to the nonabelian case and introduced skew braces in \([24]\), which provide nondegenerate set-theoretical solutions of the Yang-Baxter equation. Recently, Gateva-Ivanova \([18]\) used braided groups and braces to study set-theoretical solutions of the Yang-Baxter equation. In \([1]\), Angiono, Galindo and Vendramin introduced the notion of Hopf braces, generalizing Rump’s braces and Guarnieri-Vendramin’s skew-braces. Any Hopf brace produces a solution of the Yang-Baxter equation.

In this paper, we provide another approach to understand the structure of set-theoretical solutions of the Yang-Baxter equation in certain Hopf algebras. In particular, we introduce the notion of post-Hopf algebras, which naturally provide solutions of the Yang-Baxter equation in the underlying vector spaces. We also introduce the notion of relative Rota-Baxter operators on Hopf algebras, which naturally give rise to post-Hopf algebras, and thus to solutions of the Yang-Baxter equation. The whole theory is based on the fact that a cocommutative post-Hopf algebra gives rise to a generalized Grossman-Larsson product, which leads to a subadjacent Hopf algebra. Note that the classical Grossman-Larsson product was defined in the context of polynomials of ordered rooted trees \([34]\), and have important applications in the studies of Magnus expansions \([8, 13]\) and Lie-Butcher series \([33, 34]\).

A post-Hopf algebra is a Hopf algebra \(H\) equipped with a coalgebra homomorphism from \(H \otimes H\) to \(H\) satisfying some compatibility conditions (see Definition \([2.1]\)). Magma algebras, in particular ordered rooted trees, provide a class of examples of post-Hopf algebras. A cocommutative post-Hopf algebra gives rise to a new subadjacent Hopf algebra and a module bialgebra structure on itself. The terminology of post-Hopf algebras is justified by the fact that a post-Hopf algebra gives rise to a post-Lie algebra on the space of primitive elements. The notion of post-Lie algebras was introduced in \([34]\), and have important applications in geometric numerical integration \([10, 11]\). In \([44]\), Ebrahimi-Fard, Lundervold and Munthe-Kaas studied the Lie enveloping algebra of a post-Lie algebra, which turns out to be a post-Hopf algebra. They also find that there is a new Hopf algebra structure (the subadjacent Hopf algebra) on the Lie enveloping algebra of a post-Lie algebra, by which the Magnus expansions and Lie-Butcher series can be constructed. The subadjacent Hopf algebra is also the main ingredient in our construction of solutions of the Yang-Baxter equation. Moreover, we show that cocommutative post-Hopf algebras and cocommutative Hopf braces are equivalent. As a byproduct, we obtain the notion of pre-Hopf algebras as commutative post-Hopf algebras.

Rota-Baxter operators on Lie algebras and associative algebras have important applications in various fields, such as Connes-Kreimer’s algebraic approach to renormalization of quantum field theory \([7]\), the classical Yang-Baxter equation and integrable systems \([4, 28, 38]\), splitting
of operads [3], double Lie algebras [20] and etc. See the book [25] for more details. Recently, the notion of Rota-Baxter operators on groups was introduced in [26], and further studied in [4]. One can obtain Rota-Baxter operators of weight 1 on Lie algebras from that on Lie groups by differentiation. Then in the remarkable work [21], Goncharov succeeded in defining Rota-Baxter operators on cocommutative Hopf algebras such that many classical results still hold in the Hopf algebra level. In this paper, we introduce a more general notion of relative Rota-Baxter operators on Hopf algebras containing Goncharov’s Rota-Baxter operators as special cases. A cocommutative post-Hopf algebra naturally gives rise to a relative Rota-Baxter operator on its subadjacent Hopf algebra, and conversely, a relative Rota-Baxter operator also induces a post-Hopf algebra.

Remarkably, a relative Rota-Baxter operator on a cocommutative Hopf algebra naturally gives rise to a matched pair of Hopf algebras. In particular, for a cocommutative post-Hopf algebra, the original Hopf algebra and the subadjacent Hopf algebra form a matched pair of Hopf algebras satisfying certain good properties. Based on this fact, we construct solutions of the Yang-Baxter equation in a Hopf algebra using post-Hopf algebras as well as relative Rota-Baxter operators, and give explicit formulas of solutions for the post-Hopf algebras coming from ordered rooted trees. We further characterize relative Rota-Baxter operators using graphs in the smash product Hopf algebra and module structures.

The paper is organized as follows. In Section 2, first we introduce the notion of post-Hopf algebras and show that a cocommutative post-Hopf algebra gives rise to a subadjacent Hopf algebra together with a module bialgebra structure on itself. Then we show that there is a one-to-one correspondence between cocommutative post-Hopf algebras and cocommutative Hopf braces. In Section 3, we introduce the notion of relative Rota-Baxter operators and show that post-Hopf algebras are the underlying structures, and give rise to relative Rota-Baxter operators on the subadjacent Hopf algebras. In Section 4, we show that a relative Rota-Baxter operator gives rise to a matched pair of Hopf algebras. In particular, a cocommutative post-Hopf algebra gives rise to a matched pair of Hopf algebras. Consequently, one can construct solutions of the Yang-Baxter equation using post-Hopf algebras and relative Rota-Baxter operators. In Section 5, we give some alternative characterizations of relative Rota-Baxter operators using relative Rota-Baxter operators on the Lie algebra of primitive elements, graphs and module bialgebra structures.

**Convention.** In this paper, we fix an algebraically closed ground field k of characteristic 0. For any coalgebra (C, Δ, ε), we compress the Sweedler notation of the comultiplication Δ as

\[ Δ(x) = x_1 \otimes x_2 \]

for simplicity. Furthermore, for \( n \geq 1 \) we write

\[ Δ^{(n)}(x) = (Δ \otimes \text{id}^{0(n-1)}) \cdots (Δ \otimes \text{id})Δ(x) = x_1 \otimes \cdots \otimes x_{n+1}. \]

Let \((H, \cdot, 1, Δ, ε, S)\) be a Hopf algebra. Denote by \(G(H)\) the set of group-like elements in \(H\), which is a group. Denote by \(P_{g,h}(H)\) the subspace of \((g,h)\)-primitive elements in \(H\) for \(g, h \in G(H)\). Denote by \(P(H)\) the subspace of primitive elements in \(H\), which is a Lie algebra. For other basic notions of Hopf algebras, we follow the textbooks [32].
2. Post-Hopf Algebras

In this section, first we introduce the notion of a post-Hopf algebra, and show that a cocommutative post-Hopf algebra gives rise to a subadjacent Hopf algebra together with a module bialgebra structure on itself. A post-Hopf algebra induces a post-Lie algebra structure on the space of primitive elements and conversely, there is naturally a post-Hopf algebra structure on the universal enveloping algebra of a post-Lie algebra. Then we show that cocommutative post-Hopf algebras and cocommutative Hopf braces are equivalent. Finally, we introduce the notion of a pre-Hopf algebra which is a commutative post-Hopf algebra.

Recall from [17, 33] that a post-Lie algebra \((h, [\cdot, \cdot]_h, \triangleright)\) consists of a Lie algebra \((h, [\cdot, \cdot]_h)\) and a binary product \(\triangleright : h \otimes h \to h\) such that

\[
\begin{align*}
(1) & \quad x \triangleright [y, z]_h = [x \triangleright y, z]_h + [y, x \triangleright z]_h, \\
(2) & \quad [(x, y)_h + x \triangleright y - y \triangleright x] \triangleright z = x \triangleright (y \triangleright z) - y \triangleright (x \triangleright z).
\end{align*}
\]

Any post-Lie algebra \((h, [\cdot, \cdot]_h, \triangleright)\) has a subadjacent Lie algebra \(h' := (h, [\cdot, \cdot]_h)\) defined by

\[
[x, y]_h := x \triangleright y - y \triangleright x + [x, y]_h, \quad \forall x, y \in h,
\]

and Eqs. (1)-(2) equivalently mean that the linear map \(L_\triangleright : h \to gl(h)\) defined by \(L_\triangleright x y = x \triangleright y\) is an action of the Lie algebra \((h, [\cdot, \cdot]_h)\) on \((h, [\cdot, \cdot]_h)\).

2.1. Post-Hopf algebras and their basic properties.

**Definition 2.1.** A post-Hopf algebra is a pair \((H, \triangleright)\), where \(H\) is a Hopf algebra and \(\triangleright : H \otimes H \to H\) is a coalgebra homomorphism satisfying the following equalities:

\[
\begin{align*}
(3) & \quad x \triangleright (y \cdot z) = (x_1 \triangleright y) \cdot (x_2 \triangleright z), \\
(4) & \quad x \triangleright (y \triangleright z) = (x_1 \cdot (x_2 \triangleright y)) \triangleright z,
\end{align*}
\]

for any \(x, y, z \in H\), and the left multiplication \(\alpha_\triangleright : H \to \text{End}(H)\) defined by

\[
\alpha_\triangleright x y = x \triangleright y, \quad \forall x, y \in H,
\]

is convolution invertible in \(\text{Hom}(H, \text{End}(H))\). Namely, there exists unique \(\beta_\triangleright : H \to \text{End}(H)\) such that

\[
\alpha_\triangleright x_1 \beta_\triangleright x_2 = \beta_\triangleright x_1 \alpha_\triangleright x_2 = \varepsilon(x)\text{id}_H, \quad \forall x \in H.
\]

A homomorphism from a post-Hopf algebra \((H, \triangleright)\) to \((H', \triangleright')\) is a Hopf algebra homomorphism \(g : H \to H'\) satisfying

\[
g(x \triangleright y) = g(x) \triangleright' g(y), \quad \forall x, y \in H.
\]

It is obvious that post-Hopf algebras and homomorphisms between post-Hopf algebras form a category, which is denoted by PH. We denote by \(\text{cocPH}\) the subcategory of PH consisting of cocommutative post-Hopf algebras and homomorphisms between them.

**Remark 2.2.** Similar axioms in the definition of a post-Hopf algebra also appeared in the definition of \(D\)-algebras [13, 54] and \(D\)-bialgebras [51] with motivations from the studies of numerical Lie group integrators and the algebraic structure on the universal enveloping algebra of a post-Lie algebra.
Moreover, we have the following properties.

**Lemma 2.3.** Let \((H, \triangleright)\) be a post-Hopf algebra. Then for all \(x, y \in H\), we have

1. \(x \triangleright 1 = \varepsilon(x)1\),
2. \(1 \triangleright x = x\),
3. \(S(x \triangleright y) = x \triangleright S(y)\).

**Proof.** Since \(\triangleright\) is a coalgebra homomorphism, we have

\[
\begin{aligned}
x \triangleright 1 &= (x_1 \triangleright 1)\varepsilon(x_2 \triangleright 1) = (x_1 \triangleright 1) \cdot (x_2 \triangleright 1) \cdot S(x_3 \triangleright 1) \\
&= (x_1 \triangleright 1) \cdot S(x_2 \triangleright 1) = \varepsilon(x \triangleright 1)1 = \varepsilon(x)1.
\end{aligned}
\]

By Eq. (6), we have \(\alpha_{p,1} \beta_{p,1} = \beta_{p,1} \alpha_{p,1} = \text{id}_H\), which means that \(\alpha_{p,1}\) is a linear automorphism of \(H\). On the other hand, we have

\[
\begin{aligned}
\alpha_{p,1}^2 x &= 1 \triangleright (1 \triangleright x) \triangleright (1 \triangleright 1) \triangleright x \\
&= 1 \triangleright x = \alpha_{p,1} x.
\end{aligned}
\]

Hence, \(1 \triangleright x = \alpha_{p,1} x = x\).

Finally we have

\[
\begin{aligned}
S(x \triangleright y) &= S(x_1 \triangleright y_1)\varepsilon(x_2)\varepsilon(y_2) = S(x_1 \triangleright y_1) \cdot (x_2 \triangleright \varepsilon(y_2)1) \\
&= S(x_1 \triangleright y_1) \cdot (x_2 \triangleright (y_2 \cdot S(y_3))) = S(x_1 \triangleright y_1) \cdot (x_2 \triangleright y_2) \cdot (x_3 \triangleright S(y_3)) \\
&= \varepsilon(x_1 \triangleright y_1)(x_2 \triangleright S(y_2)) = \varepsilon(x_1)\varepsilon(y_1)(x_2 \triangleright S(y_2)) = x \triangleright S(y).
\end{aligned}
\]

Now we give the main result in this section.

**Theorem 2.4.** Let \((H, \triangleright)\) be a cocommutative post-Hopf algebra. Then

\[H_p := (H, \ast_p, 1, \Delta, \varepsilon, S_p)\]

is a Hopf algebra, which is called the **subadjacent Hopf algebra**, where for all \(x, y \in H\),

\[
\begin{aligned}
x \ast_p y &:= x_1 \cdot (x_2 \triangleright y), \\
S_p(x) &:= \beta_{p,x_1}(S(x_2)).
\end{aligned}
\]

Furthermore, \((H, \cdot, 1, \Delta, \varepsilon, S)\) is a left \(H_p\)-module bialgebra via the action \(\triangleright\).

**Proof.** Since \(\triangleright\) is a coalgebra homomorphism and \(H\) is cocommutative, we have

\[
\begin{aligned}
\Delta(x \ast_p y) &= \Delta(x_1 \cdot (x_2 \triangleright y)) \\
&= \Delta(x_1) \cdot \Delta(x_2 \triangleright y) \\
&= (x_1 \otimes x_2) \otimes ((x_3 \triangleright y_1) \otimes (x_4 \triangleright y_2)) \\
&= (x_1 \cdot (x_3 \triangleright y_1)) \otimes (x_2 \cdot (x_4 \triangleright y_2)) \\
&= (x_1 \cdot (x_2 \triangleright y_1)) \otimes (x_3 \cdot (x_4 \triangleright y_2)) \\
&= (x_1 \ast_p y_1) \otimes (x_2 \ast_p y_2)
\end{aligned}
\]
for all \(x, y \in H\), which implies that the comultiplication \(\Delta\) is an algebra homomorphism with respect to the multiplication \(*\). Moreover, we have
\[
\varepsilon(x *_p y) = \varepsilon(x_1 \cdot (x_2 \triangleright y)) = \varepsilon(x_1)\varepsilon(x_2 \triangleright y) = \varepsilon(x)\varepsilon(y),
\]
which implies that the counit \(\varepsilon\) is also an algebra homomorphism with respect to the multiplication \(*_p\). Since the comultiplication \(\Delta\) is an algebra homomorphism with respect to the multiplication \(\cdot\), for all \(x, y, z \in H\), we have
\[
(x *_p y) *_p z = (x_1 *_p y_1) \cdot ((x_2 *_p y_2) \triangleright z) = (x_1 \cdot (x_2 \triangleright y_1)) \cdot ((x_3 \cdot (x_4 \triangleright y_2)) \triangleright z).
\]
Moreover, we have
\[
(x \triangleright y) \triangleright z = x \cdot (x \triangleright y) = x \cdot (x \triangleright y_1) \cdot (x_3 \triangleright (y_2 \triangleright z)) = x *_p (y *_p z),
\]
which implies that the multiplication \(*_p\) is associative. For any \(x \in H\), by (1) and (2), we have
\[
x *_p 1 = x \cdot (x \triangleright 1) = x \cdot \varepsilon(x_2) = x,
\]
\[
1 *_p x = 1 \cdot (1 \triangleright x) = x.
\]
Thus, \((H, *_p, 1, \Delta, \varepsilon)\) is a cocommutative bialgebra. Since \(\triangleright\) is a coalgebra homomorphism and \(H\) is cocommutative, we know that
\[
\Delta \beta_{\triangleright, x} = (\beta_{\triangleright, x_1} \otimes \beta_{\triangleright, x_2}) \Delta,
\]
and \(S_\triangleright\) is a coalgebra homomorphism. Also, note that
\[
x_1 *_p S_\triangleright(x_2) = x_1 \cdot (x_2 \triangleright S_\triangleright(x_2)) = x_1 \cdot (\alpha_{\triangleright, x_2}(S_\triangleright(S(x_4)))) = x_1 \cdot (\varepsilon(x_2)S_\triangleright(x_3)) = \varepsilon(x_1),
\]
and it means that
\[
S^2_\triangleright(x) = \varepsilon(x_1)S^2_\triangleright(x_2) = (x_1 *_p S_\triangleright(x_2)) *_p S^2_\triangleright(x_3) = x_1 *_p (S_\triangleright(x_2)) *_p S_\triangleright(S_\triangleright(x_3))) = x_1 *_p \varepsilon(S_\triangleright(x_2))1 = x.
\]
\[
S_\triangleright(x_1) *_p x_2 = S_\triangleright(x_1) *_p S^2_\triangleright(x_2) = \varepsilon(S_\triangleright(x))1 = \varepsilon(x)1.
\]
Therefore, \((H, *_p, 1, \Delta, \varepsilon, S_\triangleright)\) is a cocommutative Hopf algebra.
Moreover, we have
\[
(x *_p y) \triangleright z = (x_1 \cdot (x_2 \triangleright y)) \triangleright z = x \triangleright (y \triangleright z).
\]
Then by (3) and (4), \((H, \cdot, 1)\) is a left \(H_\triangleright\)-module algebra. Since \(\triangleright\) is also a coalgebra homomorphism, \((H, \cdot, 1, \Delta, \varepsilon, S)\) is a left \(H_\triangleright\)-module bialgebra via the action \(\triangleright\). \(\square\)
Remark 2.5. (i) The product (3) generalizes the Grossman-Larsson product [22, 34, 35] defined in the context of (noncommutative) polynomials of (ordered) rooted trees. The Grossman-Larsson product plays important roles in the theories of Magnus expansions [3] and Lie-Butcher series [33, 34].

(ii) By (7) and (1), we have
\[ \alpha_{\p, x_1} \alpha_{\p, S_\p(x_2)} = \alpha_{\p, x_1} \alpha_{\p, y_2} = \alpha_{\p, x_1} | = \varepsilon(x) \text{id}, \quad \text{i.e.} \quad \beta_{\p, x} = \alpha_{\p, S_\p(x)}, \]
as \beta_{\p} is also the convolution inverse of \( \alpha_{\p} \). Then we can rewrite Eq. (11) as
\[ S_{\p}(x) = S_{\p}(x_1) \triangleright S(x_2), \quad \forall x \in H. \]

Example 2.6. Any Hopf algebra \( H \) has at least the following trivial post-Hopf algebra structure,
\[ x \triangleright y = \varepsilon(x)y, \quad \forall x, y \in H. \]

In the sequel, we study the relation between post-Hopf algebras and post-Lie algebras.

Theorem 2.7. Let \((H, \triangleright)\) be a post-Hopf algebra. Then its subspace \( P(H) \) of primitive elements is a post-Lie algebra.

Proof. Since \( \triangleright \) is a coalgebra homomorphism, for all \( x, y \in P(H) \), we have
\[
\Delta(x \triangleright y) = (x_1 \triangleright y_1) \otimes (x_2 \triangleright y_2) = (1 \triangleright 1) \otimes (x \triangleright y) + (1 \triangleright y) \otimes (x \triangleright 1) + (x \triangleright 1) \otimes (y \triangleright 1) + (1 \triangleright y) \otimes (1 \triangleright 1) \\
\]
Thus, we obtain a linear map \( \triangleright : P(H) \otimes P(H) \to P(H) \). By (3), for all \( x, y \in P(H) \), we have
\[ x \triangleright (y \cdot z) = (1 \triangleright y) \cdot (x \triangleright z) + (x \triangleright y) \cdot (1 \triangleright z) \]
\[
y \cdot (x \triangleright z) + (x \triangleright y) \cdot z. \]
Thus, we have
\[ x \triangleright [y, z] = x \triangleright (y \cdot z) - x \triangleright (z \cdot y) = y \cdot (x \triangleright z) + (x \triangleright y) \cdot z - z \cdot (x \triangleright y) - (x \triangleright z) \cdot y = [x \triangleright y, z] + [y, x \triangleright z]. \]
By (3), we have
\[ x \triangleright (y \triangleright z) = (1 \cdot (x \triangleright y)) \triangleright z + (x \cdot (1 \triangleright y)) \triangleright z = (x \triangleright y) \triangleright z + (x \cdot y) \triangleright z. \]
Thus, we have
\[ [x, y] \triangleright z = (x \cdot y) \triangleright z - (y \cdot x) \triangleright z = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z - y \triangleright (x \triangleright z) + (y \triangleright x) \triangleright z. \]
Therefore, \((P(H), [\cdot, \cdot], \triangleright)\) is a post-Lie algebra. \( \square \)
In [14, 55] the authors studied the universal enveloping algebra of a pre-Lie algebra and also of a post-Lie algebra. By [14, Proposition 3.1, Theorem 3.4], the binary product $\triangleright$ in a post-Lie algebra $(h, [\cdot, \cdot], \triangleright)$ can be extended to its universal enveloping algebra and induces a subadjacent Hopf algebra structure isomorphic to the universal enveloping algebra $U(h_\triangleright)$ of the subadjacent Lie algebra $h_\triangleright$.

We summarize their result in the setting of post-Hopf algebras as follows. We do not claim any originality (see [14, 55] for details).

**Theorem 2.8.** Let $(h, [\cdot, \cdot], \triangleright)$ be a post-Lie algebra with its subadjacent Lie algebra $h_\triangleright$. Then

$$(U(h), \tilde{\triangleright})$$

is a post-Hopf algebra, where $\tilde{\triangleright}$ is the extension of $\triangleright$ determined by

$$1 \triangleright u = u, \quad x_1 \cdots x_r \triangleright u = x_1 \tilde{\triangleright} (x_2 \cdots x_r \triangleright u) - (x_1 \triangleright x_2 \cdots x_r) \triangleright u$$

for all $x_1, \ldots, x_r \in h$ and $u \in U(h)$ with $r \geq 1$.

Moreover, the subadjacent Hopf algebra $U(h_\triangleright)$ is isomorphic to the universal enveloping algebra $U(h_\triangleright)$ of the subadjacent Lie algebra $h_\triangleright$.

In a recent work [16], Foissy extended any magma operation on a vector space $V$, i.e. an arbitrary bilinear map $\ast : V \otimes V \to V$, to the coshuffe Hopf algebra $(TV, \cdot, \Delta^{\cosh})$ as follows:

$$1 \triangleright a = a, \quad x \triangleright a = x \ast a,$$

$$(x \otimes x_1) \triangleright a = x \ast (x_1 \triangleright a) - (x \ast x_1) \triangleright a,$$

$$
\vdots
$$

$$(x \otimes x_1 \otimes \cdots \otimes x_n) \triangleright a = x \ast (x_1 \otimes \cdots \otimes x_n \triangleright a) - \sum_{i=1}^{n} (x_1 \otimes \cdots \otimes x_{i-1} \otimes (x \ast x_i) \otimes x_{i+1} \otimes \cdots \otimes x_n) \triangleright a,$$

and

$$1 \triangleright 1 = 1, \quad x \triangleright 1 = 0,$$

$$X \triangleright (a_1 \otimes \cdots \otimes a_m) = (X_1 \triangleright a_1) \otimes \cdots \otimes (X_m \triangleright a_m),$$

where $x, x_1, \ldots, x_n, a, a_1, \ldots, a_m \in V$ and $X \in TV, \Delta^{\cosh(m-1)} X = X_1 \otimes \cdots \otimes X_m$.

According to the discussion in [16], it is straightforward to obtain the following result.

**Theorem 2.9.** Let $(V, \ast)$ be a magma algebra. Then $(TV, \cdot, \Delta^{\cosh}, \triangleright)$ is a post-Hopf algebra.

**Example 2.10.** Let $OT$ be the set of isomorphism classes of ordered rooted trees, which is denoted by

$$OT = \{ \ldots, \ldots, \ldots, \ldots, \ldots \}.$$
Let $k\{OT\}$ be the free $k$-vector space generated by $OT$. The left grafting operator $\rhd: k\{OT\} \otimes k\{OT\} \to k\{OT\}$ is defined by

$$\tau \rhd \omega = \sum_{s \in \text{Nodes}(\omega)} \tau \circ_s \omega, \quad \forall \tau, \omega \in OT,$$

where $\tau \circ_s \omega$ is the ordered rooted tree resulting from attaching the root of $\tau$ to the node $s$ of the tree $\omega$ from the left. For example, we have

$$\rhd = \rhd + \rhd + \rhd + \rhd.$$

It is obvious that $(k\{OT\}, \rhd)$ is a magma algebra. By Theorem 2.9, $(T k\{OT\}, \cdot, \Delta, \mu, \varepsilon)$ is a post-Hopf algebra, where the underlying coshuffle Hopf algebra $(T k\{OT\}, \cdot, \Delta_cosh)$ has the linear basis consisting of all ordered rooted forests and its antipode $S$ is given by

$$S(\tau_1 \tau_2 \cdots \tau_m) = (-1)^m \tau_m \tau_{m-1} \cdots \tau_1, \quad \forall \tau_1, \tau_2, \ldots, \tau_m \in OT.$$

Moreover, it is the universal enveloping algebra of the free post-Lie algebra on one generator $\{\}$.

Let $B^+: T k\{OT\} \to k\{OT\}$ be the linear map producing an ordered tree $\tau$ from any ordered rooted forest $\tau_1 \cdots \tau_m$ by grafting the $m$ trees $\tau_1, \ldots, \tau_m$ on a new root $*$ in order. For example, we have

$$B^+(\bigsqcup) = \bigsqcup.$$

Let $B^-: k\{OT\} \to T k\{OT\}$ be the linear map producing an ordered forest from any ordered rooted tree $\tau$ by removing its root. For example, we have

$$B^- \left( \bigtriangleup \right) = \bigsqcup.$$

Moreover, the operation $B^-$ extends to $T k\{OT\}$ by

$$B^- (\tau_1 \cdots \tau_m) = B^- (\tau_1) \cdots B^- (\tau_m), \quad \forall \tau_1, \ldots, \tau_m \in OT.$$

Note that the subadjacent Hopf algebra $(T k\{OT\}, \ast_\triangledown, \Delta_cosh, S_\triangledown)$ is isomorphic to the Grossman-Larson Hopf algebra of ordered rooted forests defined in [22]. Using the left grafting operation, the multiplication $\ast_\triangledown$ is given by

$$X \ast_\triangledown Y = B^- (X \triangledown B^+(Y))$$

for all ordered rooted forests $X, Y$, and the antipode $S_\triangledown$ can be recursively defined by

$$S_\triangledown (1) = 1, \quad S_\triangledown (X) \triangledown S (X) + (\text{Id} - \mu \varepsilon)(S_\triangledown (X_1)) \triangledown S (X_2),$$

where $\mu$ is the unit map and $\varepsilon$ is the counit map.

**Example 2.11.** We classify all post-Hopf algebra structures on the smallest noncommutative and non-cocommutative Hopf algebra, namely, Sweedler’s 4-dimensional Hopf algebra

$$H_4 = k\langle 1, g, x, gx \mid g^2 = 1, x^2 = 0, gx = -xg \rangle,$$
with its coalgebra structure and its antipode given by
\[
\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \quad \varepsilon(g) = 1, \quad \varepsilon(x) = 0, \quad S(g) = g, \quad S(x) = -gx.
\]
Further \( G(H_4) = \{1, g\} \), \( P_{1,g}(H_4) = kx \) and \( P_{g,1}(H_4) = kgx \).

Let \((H_4, \triangleright)\) be a post-Hopf algebra structure on \( H_4 \). Then
\[
\Delta(x \triangleright g) = (x \triangleright g) \otimes (g \triangleright g),
\]
and
\[
\Delta(g \triangleright x) = (g \triangleright x) \otimes (g \triangleright 1) + (g \triangleright g) \otimes (g \triangleright x)
\]
Namely, \( g \triangleright g \in G(H_4) \) and \( g \triangleright x \in P_{1,g}(H_4) \). Since \( g \in G(H_4) \) implies that \( \alpha_{g \triangleright g} \) is invertible by Eq. (5), we know that \( g \triangleright g = g \) and \( g \triangleright x \in P_{1,g}(H_4) \). Also,
\[
g \triangleright (g \triangleright x) = (g \triangleright g) \triangleright x = g^2 \triangleright x = 1 \triangleright x = x.
\]
Therefore, \( g \triangleright x = x \) or \( -x \). On the other hand,
\[
\Delta(x \triangleright g) = (x \triangleright g) \otimes (1 \triangleright g) + (g \triangleright g) \otimes (x \triangleright g) = (x \triangleright g) \otimes g + g \otimes (x \triangleright g).
\]
Then \( x \triangleright g \in P_{g,1}(H_4) \), and thus \( x \triangleright g = 0 \). So
\[
\Delta(x \triangleright x) = (x \triangleright x) \otimes (1 \triangleright 1) + (g \triangleright x) \otimes (x \triangleright 1) + (x \triangleright g) \otimes (1 \triangleright x) + (g \triangleright g) \otimes (x \triangleright x)
\]
That is, \( x \triangleright x \in P_{1,g}(H_4) \), and we can set \( x \triangleright x = ax \) for some \( a \in k \). Then
\[
a(g \triangleright x) = x \triangleright (g \triangleright x) = (x(1 \triangleright g) + g(x \triangleright g)) \triangleright x = xg \triangleright x = -gx \triangleright x
\]
It implies that \( g \triangleright x = -x \) unless \( a = 0 \).

In summary, one can easily check that there is the post-Hopf algebra structure \((H_4, \triangleright_a)\) for any \( a \in k \) illustrated as below, such that \( \alpha_{g \triangleright a} \) has the convolution inverse \( \alpha_{a \triangleright g} \).

```
\[\begin{array}{c|cccc}
\triangleright_a & 1 & g & x & gx \\
\hline
1 & 1 & g & x & gx \\
g & 1 & g & -x & -gx \\
x & 0 & 0 & ax & agx \\
gx & 0 & 0 & ax & agx \\
\end{array}\]
```

Moreover, if \( a \neq 0 \), there is the post-Hopf algebra isomorphism from \((H_4, \triangleright_a)\) to \((H_4, \triangleright_1)\) mapping \( g \) to \( g \) and \( x \) to \( ax \). Hence, the Sweedler 4-dimensional Hopf algebra has three non-isomorphic post-Hopf algebra structures \((H_4, \varepsilon \otimes \text{id})\), \((H_4, \triangleright_0)\) and \((H_4, \triangleright_1)\).

2.2. Post-Hopf algebras and Hopf braces. In this subsection, we establish the relation between Hopf braces and post-Hopf algebras.

In [19], Angiono, Galindo and Vendramin introduced the notion of Hopf braces, generalizing Rump’s braces and their nonabelian generalizations, skew left braces [24, 27, 37, 37], which are stemmed from group theory. All these algebraic objects have deep relations with the Yang-Baxter equation.
Definition 2.12 ([1], Definition 1.1)]. Let \((A, \Delta, \varepsilon)\) be a coalgebra. A Hopf brace over \(A\) consist of two Hopf algebra structures \((A, \cdot, 1, \Delta, \varepsilon, S)\) and \((A, \circ, 1, \Delta, \varepsilon, S)\) satisfying the following compatibility condition,

\[(a \circ (b \cdot c)) = (a_1 \circ b) \cdot (S(a_2) \cdot (a_3 \circ c)), \quad \forall a, b, c \in A.\]

We will simply denote a Hopf brace by \((A, \cdot, \circ)\).

Theorem 2.13. Let \((H, \triangleright)\) be a cocommutative post-Hopf algebra. Then the Hopf algebra \((H, \cdot, 1, \Delta, \varepsilon, S)\) and the subadjacent Hopf algebra \((H, \ast_\triangleright, 1, \Delta, \varepsilon, S_\triangleright)\) form a Hopf brace. Conversely, any cocommutative Hopf brace \((H, \cdot, \circ)\) gives a post-Hopf algebra \((H, \triangleright)\) with \(\triangleright\) defined by \(x \triangleright y = S(x_1) \cdot (x_2 \circ y), \forall x, y \in H.\)

Proof. Let \((H, \triangleright)\) be a cocommutative post-Hopf algebra. We only need to show that the multiplications \(\cdot\) and \(\ast_\triangleright\) satisfy the compatibility condition (12), which follows from

\[x \ast_\triangleright (y \cdot z) = x_1 \cdot (x_2 \triangleright (y \cdot z))\]
\[= x_1 \cdot ((x_2 \triangleright y) \cdot (x_3 \triangleright z))\]
\[= x_1 \cdot (x_2 \triangleright y) \cdot S(x_3) \cdot x_4 \cdot (x_5 \triangleright z)\]
\[= (x_1 \ast_\triangleright y) \cdot S(x_2) \cdot (x_3 \ast_\triangleright z),\]

for any \(x, y, z \in H.\)

Conversely, it is straightforward but tedious to check that a cocommutative Hopf brace \((H, \cdot, \circ)\) induces a post-Hopf algebra \((H, \triangleright)\).

2.3. Pre-Hopf algebras. A post-Lie algebra \((\mathfrak{h}, [\cdot, \cdot], \triangleright)\) reduces to a pre-Lie algebra if the Lie bracket \([\cdot, \cdot]_\mathfrak{h}\) is abelian. More precisely, a pre-Lie algebra \((\mathfrak{h}, \triangleright)\) is a vector space \(\mathfrak{h}\) equipped with a binary product \(\triangleright : \mathfrak{h} \otimes \mathfrak{h} \to \mathfrak{h}\) such that

\[(x \triangleright y) \triangleright z - x \triangleright (y \triangleright z) = (y \triangleright x) \triangleright z - y \triangleright (x \triangleright z), \quad \forall x, y, z \in \mathfrak{h}.\]

From this perspective, we introduce the notion of pre-Hopf algebras as special post-Hopf algebras.

Definition 2.14. A post-Hopf algebra \((H, \triangleright)\) is called a pre-Hopf algebra if \(H\) is a commutative Hopf algebra.

The above properties for post-Hopf algebras are still valid for pre-Hopf algebras.

Corollary 2.15. Let \((H, \triangleright)\) be a cocommutative pre-Hopf algebra. Then

\[H_\triangleright := (H, \ast_\triangleright, 1, \Delta, \varepsilon, S_\triangleright)\]

is a Hopf algebra, which is called the subadjacent Hopf algebra, where the multiplication \(\ast_\triangleright\) and the antipode \(S_\triangleright\) are given by (13) and (14) respectively.

Moreover, \(H\) is a left \(H_\triangleright\)-module bialgebra via the action \(\triangleright\).

Corollary 2.16. Let \((H, \triangleright)\) be a pre-Hopf algebra. Then its subspace \(P(H)\) of primitive elements is a pre-Lie algebra.
Recall that a pre-Lie algebra \((\mathfrak{h}, \rhd)\) also gives rise to a subadjacent Lie algebra \(\mathfrak{h} \rhd\) in which the Lie bracket is defined by

\[
[x, y]_{\mathfrak{h} \rhd} = x \rhd y - y \rhd x, \quad \forall x, y \in \mathfrak{h}.
\]

**Corollary 2.17.** Let \((\mathfrak{h}, \rhd)\) be a pre-Lie algebra with its subadjacent Lie algebra \(\mathfrak{h} \rhd\). Then the product \(\rhd\) can be extended to the one \(\bar{\rhd}\) on the symmetric algebra \(\text{Sym}(\mathfrak{h})\), making it a pre-Hopf algebra. Moreover, the subadjacent Hopf algebra \(\text{Sym}(\mathfrak{h}) \bar{\rhd}\) is isomorphic to the universal enveloping algebra \(U(\mathfrak{h} \rhd)\) of the subadjacent Lie algebra \(\mathfrak{h} \rhd\).

**Example 2.18.** Let \(\mathcal{T}\) be the set of isomorphism classes of rooted trees, which is denoted by

\[
\mathcal{T} = \{ \ldots, \overrightarrow{v}, \overrightarrow{u}, \overrightarrow{v} = \overrightarrow{v}, \overrightarrow{u}, \overrightarrow{v} = \overrightarrow{v}, \overrightarrow{u} = \overrightarrow{v}, \overrightarrow{u} = \overrightarrow{v}, \ldots \}\).
\]

Let \(k[\mathcal{T}]\) be the free \(k\)-vector space generated by \(\mathcal{T}\). The grafting operator \(\rightharpoonup : k[\mathcal{T}] \otimes k[\mathcal{T}] \to k[\mathcal{T}]\) is defined by

\[
\tau \rightharpoonup \omega = \sum_{s \in \text{Nodes}(\omega)} \tau \circ_s \omega, \quad \forall \tau, \omega \in \mathcal{T},
\]

where \(\tau \circ_s \omega\) is the rooted tree resulting from attaching the root of \(\tau\) to the node \(s\) of the tree \(\omega\). For example, we have

\[
\tau \rightharpoonup \omega = \tau \rightharpoonup \omega = \tau \rightharpoonup \omega = \tau \rightharpoonup \omega = \tau \rightharpoonup \omega.
\]

Moreover, Chapoton and Livernet \([7]\) have shown that \((k[\mathcal{T}], \rightharpoonup)\) is the free pre-Lie algebra generated by \(\{\ast\}\). By Theorem 2.9, we deduce that \((T k[\mathcal{T}], \cdot, \Delta^{\cosh}, \rhd)\) is a post-Hopf algebra. Since \((k[\mathcal{T}], \rightharpoonup)\) is a pre-Lie algebra, the post-Hopf algebra structure reduces to the symmetric algebra \(S k[\mathcal{T}]\). Thus, we deduce that \((S k[\mathcal{T}], \cdot, \Delta^{\cosh}, \rhd)\) is a pre-Hopf algebra. Furthermore, it is the universal enveloping algebra of the free pre-Lie algebra \((k[\mathcal{T}], \rightharpoonup)\), and its subadjacent Hopf algebra \((S k[\mathcal{T}], \ast_{\bar{\rhd}}, \Delta^{\cosh}, S_{\bar{\rhd}})\) is dual to the Connes-Kreimer Hopf algebra of rooted trees.

### 3. Relative Rota-Baxter operators on Hopf algebras

In this section, first we recall relative Rota-Baxter operators on Lie algebras and groups, and Rota-Baxter operators on cocommutative Hopf algebras. Then we introduce a more general notion of relative Rota-Baxter operators of weight 1 on cocommutative Hopf algebras with respect to module bialgebras. We establish the relation between the category of relative Rota-Baxter operators of weight 1 on cocommutative Hopf algebras and the category of post-Hopf algebras.

Let \(\phi : \mathfrak{h} \to \text{Der}(\mathfrak{t})\) be an action of a Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_{\mathfrak{h}})\) on a Lie algebra \((\mathfrak{t}, [\cdot, \cdot]_{\mathfrak{t}})\). A linear map \(T : \mathfrak{t} \to \mathfrak{h}\) is called a **relative Rota-Baxter operator (of weight 1)** on \(\mathfrak{h}\) with respect to \((\mathfrak{t}; \phi)\) if

\[
[T(u), T(v)]_{\mathfrak{h}} = T\left(\phi(T(u))v - \phi(T(v))u + [u, v]_{\mathfrak{h}}\right), \quad \forall u, v \in \mathfrak{t}.
\]
Let $\Phi : \mathcal{H} \to \text{Aut}(\mathcal{K})$ be an action of a group $\mathcal{H}$ on a group $\mathcal{K}$. A map $T : \mathcal{K} \to \mathcal{H}$ is called a relative Rota-Baxter operator (of weight 1) if

$$T(h) \cdot T(k) = T(h \cdot \Phi(T(h)) k), \quad \forall h, k \in \mathcal{K}. \quad (15)$$

Given any Hopf algebra $(H, \Delta, \varepsilon, S)$, define the adjoint action of $H$ on itself by $\text{ad}_y x = x_1 y S(x_2)$. A Rota-Baxter operator (of weight 1) on a cocommutative Hopf algebra $H$ was defined by Goncharov in [21], which is a coalgebra homomorphism $B$ satisfying

$$B(x)B(y) = B(x_1 \text{ad}_B(x_2) y) = B(x_1 B(x_2) y S(B(x_3))), \quad \forall x, y \in H. \quad (16)$$

In the sequel, all the (relative) Rota-Baxter operators under consideration are of weight 1, so we will not emphasize it anymore.

Now we generalize the above adjoint action to arbitrary actions and introduce the notion of relative Rota-Baxter operators on Hopf algebras.

Definition 3.1. Let $H$ and $K$ be two Hopf algebras such that $K$ is a left $H$-module bialgebra via an action $\cdot$. A coalgebra homomorphism $T : K \to H$ is called a relative Rota-Baxter operator with respect to the left $H$-module bialgebra $(K, \cdot)$ if the following equality holds:

$$T(a)T(b) = T(a_1 (T(a_2) \cdot b)), \quad \forall a, b \in K. \quad (17)$$

A homomorphism between two relative Rota-Baxter operators $T : K \to H$ and $T' : K' \to H'$ is a pair of Hopf algebra homomorphisms $f : H \to H'$ and $g : K \to K'$ such that

$$fT = T'g, \quad g(x \cdot a) = f(x) \cdot g(a), \quad \forall x \in H, a \in K. \quad (18)$$

It is obvious that relative Rota-Baxter operators on Hopf algebras and homomorphisms between them form a category, which is denoted by $\text{rRB}$. We denote by $\text{cocrRB}$ the subcategory of $\text{rRB}$ consisting of relative Rota-Baxter operators with respect to cocommutative left module bialgebras and homomorphisms between them.

A cocommutative post-Hopf algebra naturally gives rise to a relative Rota-Baxter operator.

Proposition 3.2. Let $(H, \triangleright)$ be a cocommutative post-Hopf algebra and $H_{\triangleright}$ the subadjacent Hopf algebra. Then the identity map $\text{id}_H : H \to H_{\triangleright}$ is a relative Rota-Baxter operator with respect to the left $H_{\triangleright}$-module bialgebra $(H, \triangleright)$.

Moreover, if $g : H \to H'$ is a post-Hopf algebra homomorphism from $(H, \triangleright)$ to $(H', \triangleright')$, then $(g, g)$ is a homomorphism from the relative Rota-Baxter operator $\text{id}_H : H \to H_{\triangleright}$ to $H' \to H'_{\triangleright}$. Consequently, we obtain a functor $T : \text{cocPH} \to \text{cocrRB}$ from the category of cocommutative post-Hopf algebras to the category of relative Rota-Baxter operators with respect to cocommutative left module bialgebras.

Proof. For any $x, y, z \in H$, we have

$$\text{id}_H(x) \triangleright \text{id}_H(y) = x \triangleright y = x_1 \cdot (x_2 \triangleright y) = \text{id}_H(x_1 \cdot (\text{id}_H(x_2) \triangleright y)), \quad \text{so id}_H : H \to H_{\triangleright}$$

is a relative Rota-Baxter operator with respect to the left $H_{\triangleright}$-module bialgebra $(H, \triangleright)$.

Let $g : H \to H'$ be a post-Hopf algebra homomorphism from $(H, \triangleright)$ to $(H', \triangleright')$. Then $(g, g)$ obviously satisfy Eq. (13). Since $g$ is a coalgebra homomorphism and

$$g(x \triangleright y) = g(x_1 \cdot (x_2 \triangleright y)) = g(x_1) \cdot' (g(x_2) \triangleright' g(y)) = g(x) \triangleright' g(y),$$
we deduce that $g$ is a homomorphism from the Hopf algebra $H_v$ to $H'_v$. Therefore, $(g, g)$ is a homomorphism from the relative Rota-Baxter operator $id_H : H \to H_v$ to $id_{H'} : H' \to H'_v$. It is straightforward to check that this is indeed a functor.

It is well-known that a relative Rota-Baxter operator $T : \mathfrak{t} \to \mathfrak{h}$ on a Lie algebra $\mathfrak{h}$ with respect to an action $(\mathfrak{t}; \phi)$ endows $\mathfrak{t}$ with the following post-Lie algebra structure $\triangleright_T$.

\begin{equation}
\forall u, v \in \mathfrak{t},
\end{equation}

\begin{equation}
\phi(T(u))v = \phi(T(u)v),
\end{equation}

\begin{theorem}
Let $T : K \to H$ be a relative Rota-Baxter operator with respect to a left $H$-module bialgebra $(K, \to)$. Then there exists a post-Hopf algebra structure $\triangleright_T : K \otimes K \to K$ on $K$ given by
\begin{equation}
da \triangleright_T b = T(a) \to b.
\end{equation}

Let $T : K \to H$ and $T' : K' \to H'$ be two relative Rota-Baxter operators and $(f, g)$ a homomorphism between them. Then $g$ is a homomorphism from the post-Hopf algebra $(K, \triangleright_T)$ to $(K', \triangleright_{T'})$. Consequently, we obtain a functor $\Xi : \mathfrak{rRB} \to \mathfrak{PH}$ from the category of relative Rota-Baxter operators on Hopf algebras to the category of post-Hopf algebras.

Moreover, the functor $\Xi|_{\mathfrak{crrRB}}$ is right adjoint to the functor $\mathcal{Y}$ given in Proposition 3.2.

\begin{proof}
Since $T$ is a coalgebra homomorphism and $\to$ is the left module bialgebra action, we have
\begin{align*}
\Delta_K(a \triangleright_T b) &= \Delta_K(T(a) \to b) = (T(a_1) \to b_1) \otimes (T(a_2) \to b_2) = (a_1 \triangleright_T b_1) \otimes (a_2 \triangleright_T b_2), \\
\varepsilon_K(a \triangleright_T b) &= \varepsilon_K(T(a) \to b) = \varepsilon_H(T(a))\varepsilon_K(b) = \varepsilon_K(a)\varepsilon_K(b),
\end{align*}

which implies that $\triangleright_T$ is a coalgebra homomorphism. Similarly, we have
\begin{align*}
a \triangleright_T (bc) &= T(a) \to (bc) = (T(a_1) \to b)(T(a_2) \to c) = (a_1 \triangleright_T b)(a_2 \triangleright_T c).
\end{align*}

Then by (17), we obtain
\begin{align*}
(a_1(a_2 \triangleright_T b)) \triangleright_T c &= T(a_1(T(a_2) \to b)) \to c = (T(a)T(b)) \to c \\
&= T(a) \to (T(b) \to c) = a \triangleright_T (b \triangleright_T c).
\end{align*}

Define linear map $S_T : K \to K$ by
\begin{equation}
S_T(a) = S_H(T(a_1)) \to S_K(a_2).
\end{equation}

Then for all $a \in K$, we have
\begin{align*}
T(S_T(a)) &= \varepsilon_H(T(a_1))T(S_T(a_2)) \\
&= S_H(T(a_1))T(a_2)T(S_T(a_3)) \\
&= S_H(T(a_1))T(a_2T(a_3) \to S_T(a_4)) \\
&= S_H(T(a_1))T(a_2T(a_3) \to (S_H(T(a_4)) \to S_K(a_5))) \\
&= S_H(T(a_1))T(a_2T(a_3)S_H(T(a_4)) \to S_K(a_5)) \\
&= S_H(T(a_1))T(a_2S_K(a_3)) \\
&= S_H(T(a_1))T(\varepsilon_K(a_2))
\end{align*}
Then we have
\[(22) \quad \text{for all } a \in K, \text{ define } \beta_{\triangleright_T, a} \in \text{End}(K) \text{ by } \beta_{\triangleright_T, a} := \alpha_{\triangleright_T, S_T(a)}. \text{ That is,} \]
\[
\beta_{\triangleright_T, a} b = \alpha_{\triangleright_T, S_T(a)} b = S_T(a) \triangleright_T b.
\]
Then we have
\[
\alpha_{\triangleright_T, a_1} \beta_{\triangleright_T, a_2} b = T(a_1) \triangleright_T (T(S_T(a_2)) \triangleright_T b) = T(a_1)T(S_T(a_2)) \triangleright_T b = T(a_1)S_T(a_2) \triangleright_T b = T(a_1S_K(a_2)) \triangleright_T b = T((\varepsilon_K(a_1)) \triangleright_T b = \varepsilon_K(a)b,
\]
\[
\beta_{\triangleright_T, a_1} \alpha_{\triangleright_T, a_2} b = T(S_T(a_1)) \triangleright_T (T(a_2) \triangleright_T b) = T(S_T(a_1))T(a_2) \triangleright_T b = S_H(T(a_1))T(a_2) \triangleright_T b = \varepsilon_H(T(a))b = \varepsilon_K(a)b.
\]
Therefore, \(\alpha_{\triangleright_T}\) is convolution invertible. Hence, \((K, \triangleright_T)\) is a post-Hopf algebra.

Let \((f, g)\) be a homomorphism from the relative Rota-Baxter operator \(T\) to \(T'\). Then we have
\[
\varepsilon_H(T(a))b = \varepsilon_K(a)b,
\]
which implies that \(g\) is a homomorphism from the post-Hopf algebra \((K, \triangleright_T)\) to \((K', \triangleright_{T'})\). It is straightforward to see that this is indeed a functor.

Next we prove that \(\Xi_{\text{cocrRB}} : \text{cocrRB} \rightarrow \text{cocPH}\) is right adjoint to \(\Upsilon : \text{cocPH} \rightarrow \text{cocrRB}\). Namely,
\[
\text{Hom}_{\text{cocrRB}}(\text{id} : H' \rightarrow H', T : K \rightarrow H) \simeq \text{Hom}_{\text{cocPH}}((H', \triangleright_{T'}), (K, \triangleright_T)),
\]
where \(T : K \rightarrow H\) is a relative Rota-Baxter operator on a Hopf algebra \(H\) with respect to a cocommutative module bialgebra \((K, \rightarrow)\) and \((H', \triangleright_{T'})\) is a cocommutative post-Hopf algebra.

Let \(g : (H', \triangleright_{T'}) \rightarrow (K, \triangleright_T)\) be a post-Hopf algebra homomorphism. Let \(f = Tg\), which is obviously a coalgebra homomorphism. For all \(x, y \in H'\), we have
\[
f(x \triangleright_{T'} y) = T(g(x_1(x_2 \triangleright_{T'} y))) = T(g(x_1)(g(x_2) \triangleright_T g(y)))
= T(g(x_1)(T(g(x_2)) \triangleright_T g(y))) = T(g(x))T(g(y)) = f(x)f(y),
\]
which implies that \(f = Tg : H'_{\triangleright_T} \rightarrow H\) is a Hopf algebra homomorphism.

Moreover, it is straightforward to obtain
\[
g(x \triangleright_T y) = g(x) \triangleright_T g(y) = T(g(x)) \triangleright_T g(y) = f(x) \triangleright_T g(y).
\]
Hence, \((f, g)\) is a homomorphism between the relative Rota-Baxter operators \(\text{id} : H' \rightarrow H'_{\triangleright_T}\) and \(T : K \rightarrow H\).
Conversely, if \((f, g)\) is a homomorphism between the relative Rota-Baxter operators \(\text{id} : H' \to H'\), and \(T : K \to H\), we have \(f = Tg\) and \(g : (H', \triangleright') \to (K, \triangleright_T)\) is a post-Hopf algebra homomorphism. \qed

By Theorem 3.3 and Theorem 2.4, we immediately get the following result.

**Corollary 3.4.** Let \(T : K \to H\) be a relative Rota-Baxter operator with respect to a cocommutative \(H\)-module bialgebra \((K, \hookrightarrow)\). Then \((K, *_T, 1, \Delta, \varepsilon, S_T)\) is a Hopf algebra, which is called the **descendent Hopf algebra** and denoted by \(K_T\), where the antipode \(S_T\) is given by (24) and the multiplication \(*_T\) is given by

\[
a *_T b = a_1(T(a_2) \rightarrow b).
\]

Moreover, \(T : K_T \to H\) is a Hopf algebra homomorphism.

4. **Matched pairs of Hopf algebras and solutions of the Yang-Baxter equation**

In this section, we show that a relative Rota-Baxter operator on cocommutative Hopf algebras naturally gives rise to a matched pair of Hopf algebras. As applications, we construct solutions of the Yang-Baxter equation using post-Hopf algebras and relative Rota-Baxter operators on cocommutative Hopf algebras.

First we recall the smash product and matched pairs of Hopf algebras. Let \(H\) and \(K\) be two Hopf algebras such that \(K\) is a cocommutative \(H\)-module bialgebra via an action \(\hookrightarrow\). There is the following **smash product** on \(K \otimes H\),

\[
(a \# x)(a' \# x') = a(x_1 \rightarrow a') \# x_2 x'
\]

for any \(x, x' \in H\), \(a, a' \in K\), where \(a \otimes x \in K \otimes H\) is rewritten as \(a \# x\) to emphasize this smash product. We denote such a smash product algebra by \(K \rtimes H\). In particular, if \(H\) is also cocommutative, then \(K \rtimes H\) becomes a cocommutative Hopf algebra with the usual tensor product comultiplication and the antipode defined by \(S(a \# x) = (S_H(x_1) \rightarrow S_K(a)) \# S_H(x_2)\).

**Definition 4.1.** A **matched pair of Hopf algebras** is a 4-tuple \((H, K, \hookrightarrow, \leftrightarrow)\), where \(H\) and \(K\) are Hopf algebras, \(\hookrightarrow : H \otimes K \to K\) and \(\leftrightarrow : H \otimes K \to H\) are linear maps such that \(K\) is a left \(H\)-module coalgebra and \(H\) is a right \(K\)-module coalgebra and the following compatibility conditions hold:

\[
\begin{align*}
(x \hookrightarrow (ab)) &= (x_1 \hookrightarrow a_1)((x_2 \leftrightarrow a_2) \rightarrow b) \quad (24) \\
x \hookrightarrow 1_K &= \varepsilon_H(x)1_K \quad (25) \\
(x_1 \leftrightarrow a) &= (x \leftrightarrow (y_1 \hookrightarrow a_1))(y_2 \leftrightarrow a_2) \quad (26) \\
1_H \leftrightarrow a &= \varepsilon_K(a)1_H \quad (27) \\
(x_1 \leftrightarrow a_1) \otimes (x_2 \leftrightarrow a_2) &= (x_2 \leftrightarrow a_2) \otimes (x_1 \hookrightarrow a_1) \quad (28)
\end{align*}
\]

for all \(x, y \in H\) and \(a, b \in K\).

Let \((H, K, \hookrightarrow, \leftrightarrow)\) be a matched pair of Hopf algebras. The **double crossproduct** \(K \rtimes H\) of \(K\) and \(H\) is the \(k\)-vector space \(K \otimes H\) with the unit \(1_K \otimes 1_H\), such that its product, coproduct,
counit and antipode are given by
\begin{align}
(a \otimes x)(b \otimes y) &= a(x_1 \rightarrow b_1) \otimes (x_2 \leftarrow b_2)y, \\
\Delta(a \otimes x) &= a_1 \otimes x_1 \otimes a_2 \otimes x_2, \\
\varepsilon(a \otimes x) &= \varepsilon_K(a)\varepsilon_H(x), \\
S(a \otimes x) &= (S_H(x_2) \rightarrow S_K(a_2)) \otimes (S_H(x_1) \leftarrow S_K(a_1)),
\end{align}
for all $a, b \in K$ and $x, y \in H$. See [50] for further details of the double crossproducts.

By [50, Proposition 21.6], we have

**Theorem 4.2.** With above notations, $(H, K, \rightarrow, \leftarrow)$ is a matched pair of Hopf algebras if and only if there exist a Hopf algebra $A$ and injective Hopf algebra homomorphisms $i_K : K \rightarrow A$, $i_H : H \rightarrow A$ such that the map
\[
\xi : K \otimes H \rightarrow A, a \otimes x \mapsto i_K(a)i_H(x)
\]
is a linear isomorphism.

Let $T : K \rightarrow H$ be a relative Rota-Baxter operator with respect to a cocommutative $H$-module bialgebra $(K, \rightarrow)$. Define a linear map $\leftarrow : H \otimes K \rightarrow H$ by
\[
x \leftarrow a = S_H(T(x_1 \rightarrow a_1))x_2T(a_2).
\]

**Theorem 4.3.** With the above notations, if $H$ is also cocommutative, then it is a right $K_T$-module coalgebra via the action $\leftarrow$ given in Eq. (33). Moreover, the 4-tuple $(H, K_T, \rightarrow, \leftarrow)$ is a matched pair of cocommutative Hopf algebras.

**Proof.** We define a linear map $\Phi_T : K \otimes H \rightarrow K \otimes H$ as following:
\[
\Phi_T(a \otimes x) = a_1 \otimes T(a_2)x, \forall x \in H, a \in K.
\]
Since $T$ is a coalgebra homomorphism, the linear map $\Phi_T$ is invertible. Moreover, we have
\[
\Phi_T^{-1}(a \otimes x) = a_1 \otimes S_H(T(a_2))x, \forall x \in H, a \in K.
\]
Transfer the smash product Hopf algebra structure $K \rtimes H$ to $K \otimes H$ via the linear isomorphism $\Phi_T : K \otimes H \rightarrow K \rtimes H$, we obtain a Hopf algebra $(K \otimes H, \cdot_T, 1_T, \Delta_T, \varepsilon_T, \Xi_T)$. Denote elements in $K \otimes H$ by $a \bowtie x, b \bowtie y$ for $x, y \in H, a, b \in K$, by the cocommutativity of $K$, we have
\[
(a \bowtie x) \cdot_T (b \bowtie y) = (\Phi_T^{-1}(\Phi_T(a \bowtie x)\Phi_T(b \bowtie y))
\]
\[
= \Phi_T^{-1}(a_1(T(a_2)x_1 \rightarrow b_1)#T(a_2)x_2T(b_2)y)
\]
\[
= a_1(T(a_2)x_1 \rightarrow b_1) \bowtie S_H(T(a_3(T(a_4)x_2 \rightarrow b_2)))T(a_5)x_3T(b_3)y
\]
\[
= a_1(T(a_2) \rightarrow (x_1 \rightarrow b_1)) \bowtie S_H(T(a_3(T(a_4) \rightarrow (x_2 \rightarrow b_2)))T(a_5)x_3T(b_3)y
\]
\[
= a_1 \ast_T (x_1 \rightarrow b_1) \bowtie S_H(T(a_2)x_2 \rightarrow b_2)T(a_3)x_3T(b_3)y
\]
\[
= a_1 \ast_T (x_1 \rightarrow b_1) \bowtie S_H(T(x_2 \rightarrow b_2))S_H(T(a_2))T(a_3)x_3T(b_3)y
\]
\[
= a \ast_T (x_1 \rightarrow b_1) \bowtie S_H(T(x_2 \rightarrow b_2))x_3T(b_3)y
\]
\[
= a \ast_T (x_1 \rightarrow b_1) \bowtie (x_2 \leftarrow b_2)y,
\]
\[
\Delta_T(a \bowtie x) = (\Phi_T^{-1} \otimes \Phi_T^{-1})(\Delta\Phi_T(a \bowtie x))
\]
isomorphism $\nabla T$ is a relative Rota-Baxter operator with respect to the $H$-module bialgebra Proposition 4.4. If $K \otimes 1$ is a subalgebra of the Hopf algebra $(K \otimes H, \cdot_T, 1_T, \Delta_T, \epsilon_T, \nabla_T)$, then $T$ is a relative Rota-Baxter operator with respect to the $H$-module bialgebra $(K, \nabla)$. 

Proof. Since $K \otimes 1$ is a subalgebra of $(K \otimes H, \cdot_T, 1_T, \Delta_T, \epsilon_T, \nabla_T)$, for any $a, b \in K$ we have $(a \nabla 1) \cdot_T (b \nabla 1) = \Phi_T^{-1}(\Phi_T(a \nabla 1)\Phi_T(b \nabla 1))$.
Proof. Let $(H, \triangleright)$ be a cocommutative post-Hopf algebra and $H^\triangleright := (H, *, _\triangleright, 1, \Delta, \varepsilon, S^\triangleright)$ the subadjacent Hopf algebra given in Theorem 2.4. By Proposition 3.2, the identity map $id : H \to H^\triangleright$ is given by

\[ a \triangleright b = S^\triangleright(a_1 \triangleright b_1) *_{\triangleright} a_2 *_{\triangleright} b_2. \tag{36} \]

Moreover, we have the compatibility condition

\[ a *_{\triangleright} b = (a_1 \triangleright b_1) *_{\triangleright} (a_2 \triangleright b_2). \tag{37} \]

Proof. We only need to check the stated compatibility condition, which follows from

\[
(a_1 \triangleright b_1) *_{\triangleright} (a_2 < b_2) = (a_1 \triangleright b_1) *_{\triangleright} (S^\triangleright(a_2 \triangleright b_2) *_{\triangleright} a_3 *_{\triangleright} b_4)
\]
\[
= ((a_1 \triangleright b_1) *_{\triangleright} S^\triangleright(a_2 \triangleright b_2)) *_{\triangleright} a_3 *_{\triangleright} b_4
\]
\[
= \varepsilon(a_1 \triangleright b_1)a_2 *_{\triangleright} b_2
\]
\[
= \varepsilon(a_1)\varepsilon(b_1)a_2 *_{\triangleright} b_2
\]
\[
= a *_{\triangleright} b. \quad \Box
\]

At the end of this section, we show that post-Hopf algebras and relative Rota-Baxter operators on cocommutative Hopf algebras give rise to solutions of the Yang-Baxter equation.

**Definition 4.6.** A solution of the Yang-Baxter equation on a vector space $V$ is an invertible linear endomorphism $R : V \otimes V \to V \otimes V$ such that

\[(R \otimes id)(id \otimes R)(R \otimes id) = (id \otimes R)(R \otimes id)(id \otimes R).\]

**Theorem 4.7.** Let $(H, \triangleright)$ be a cocommutative post-Hopf algebra. Then $R : H \otimes H \to H \otimes H$ defined by

\[ R(x \otimes y) = (x_1 \triangleright y_1) \otimes (x_2 < y_2), \]

where $<$ is defined by $[\triangleright]$, is a coalgebra isomorphism and a solution of the Yang-Baxter equation on the vector space $H$. 

Proof. Denote by $H^I_\triangleright$ and $H^I_\triangleleft$ two copies of the Hopf algebra $H_\triangleright$. By Corollary 4.5, $(H^I_\triangleright, H^I_\triangleleft, \triangleright, \triangleleft)$ is a matched pair of cocommutative Hopf algebras. Thus, $A = H^I_\triangleright \triangleright H^I_\triangleleft$ is a Hopf algebra that factorized into Hopf algebras $H^I_\triangleright$ and $H^I_\triangleleft$. By Theorem 4.2, there is a coalgebra isomorphism

$$\xi : H^I_\triangleright \otimes H^I_\triangleleft \xrightarrow{i_{H^I_\triangleright} \otimes i_{H^I_\triangleleft}} A \otimes A \xrightarrow{m_{i_{H^I_\triangleright} \otimes i_{H^I_\triangleleft}}} A = H^I_\triangleright \triangleright H^I_\triangleleft.$$  

We consider the coalgebra homomorphism

$$\Psi = \xi^{-1} \circ m_{i_{H^I_\triangleright} \otimes i_{H^I_\triangleleft}} \circ (i_{H^I_\triangleright} \otimes i_{H^I_\triangleleft}) : H^I_\triangleright \otimes H^I_\triangleleft \rightarrow H^I_\triangleright \otimes H^I_\triangleleft.$$  

Since $H^I_\triangleright \triangleright H^I_\triangleleft$ is a cocommutative Hopf algebra, we deduce that $\Psi$ is a coalgebra isomorphism. Moreover, $\Psi$ satisfies the following equations:

$$\Psi \circ (m_{H_\triangleright} \otimes \text{Id}) = (\text{Id} \otimes m_{H_\triangleright}) \circ (\Psi \otimes \text{Id}) \circ (\text{Id} \otimes \Psi),$$  

$$\Psi \circ (\text{Id} \otimes m_{H_\triangleright}) = (m_{H_\triangleright} \otimes \text{Id}) \circ (\text{Id} \otimes \Psi) \circ (\Psi \otimes \text{Id}),$$  

$$\Psi(1 \otimes x) = x \otimes 1,$$  

$$\Psi(x \otimes 1) = 1 \otimes x.$$  

For all $x, y \in H$, we have

$$\Psi(x \otimes y) = (x_1 \triangleright y_1) \otimes (x_2 \triangleleft y_2) = R(x \otimes y).$$  

By (37), we have $m_{H_\triangleright} = m_{H_\triangleright} \circ \Psi$. Thus, we deduce that $R = \Psi$ is a braiding operator on the cocommutative Hopf algebra $H_\triangleright := (H, \ast_\triangleright, 1, \Delta, e, S_\triangleright)$. By [23, Theorem 4.11], we obtain that $R$ is a solution of the Yang-Baxter equation on the vector space $H$. \qed

Example 4.8. Consider the post-Hopf algebra $(T_k(\mathcal{OT}), \Delta^{\cosh}, \triangleright)$ given in Example 2.10. Then $R : T_k(\mathcal{OT}) \otimes T_k(\mathcal{OT}) \rightarrow T_k(\mathcal{OT}) \otimes T_k(\mathcal{OT})$ defined by

$$R(X \otimes Y) = (X_1 \triangleright Y_1) \otimes (X_2 \triangleleft Y_2), \quad X, Y \in T_k(\mathcal{OT})$$  

is a coalgebra isomorphism and a solution of the Yang-Baxter equation on the vector space $T_k(\mathcal{OT})$. More precisely, we have

$$R(X \otimes Y) = (X_1 \triangleright Y_1) \otimes B^{-}(S_\triangleright(X_2 \triangleright Y_2) \triangleright (X_3 \triangleright B^{+}(Y_3))),$$  

where $\Delta^{\cosh(2)}X = X_1 \otimes X_2 \otimes X_3$ and $\Delta^{\cosh(2)}Y = Y_1 \otimes Y_2 \otimes Y_3$.

Example 4.9. Consider the pre-Hopf algebra $(S_k(\mathcal{T}), \Delta^{\cosh}, \triangleright)$ given in Example 2.18. Then $R : S_k(\mathcal{T}) \otimes S_k(\mathcal{T}) \rightarrow S_k(\mathcal{T}) \otimes S_k(\mathcal{T})$ defined by

$$R(X \otimes Y) = (X_1 \triangleright Y_1) \otimes (X_2 \triangleleft Y_2), \quad X, Y \in S_k(\mathcal{T})$$  

is a coalgebra isomorphism and a solution of the Yang-Baxter equation on the vector space $S_k(\mathcal{T})$. More precisely, for forests $X, Y \in S_k(\mathcal{T})$, we have

$$R(X \otimes Y) = (X_1 \triangleright Y_1) \otimes B^{-}(S_\triangleright(X_2 \triangleright Y_2) \triangleright (X_3 \triangleright B^{+}(Y_3))),$$  

where $\Delta^{\cosh(2)}X = X_1 \otimes X_2 \otimes X_3$ and $\Delta^{\cosh(2)}Y = Y_1 \otimes Y_2 \otimes Y_3$. 

\[ \]
Let $T : K \to H$ be a relative Rota-Baxter operator on $H$ with respect to a commutative $H$-module bialgebra $(K, \rightarrow)$. By Theorem 3.3, $(K, \triangleright_T)$ is a commutative post-Hopf algebra. By Corollary 3.4, there is a descendent Hopf algebra $K_T = (K, \ast_T, \Delta, \epsilon, S_T)$, such that $K$ is a $K_T$-module bialgebra via the action $\triangleright_T$ defined in (20). By Corollary 3.4, we have

Corollary 4.8. The 4-tuples $(K_T, K_T, \triangleright_T, \triangleleft_T)$ is a matched pair of cocommutative Hopf algebras, here $\triangleleft_T$ is given by (20) and $\triangleright_T$ is given by

$$a \triangleleft_T b = S_T(a_1 \triangleright_T b_1) \ast_T a_2 \ast_T b_2.$$ (38)

Moreover, we have the compatibility condition

$$a \ast_T b = (a_1 \triangleright_T b_1) \ast_T (a_2 \triangleleft_T b_2).$$ (39)

By Theorem 4.7, we have

Corollary 4.9. Let $T : K \to H$ be a relative Rota-Baxter operator with respect to a commutative $H$-module bialgebra $(K, \rightarrow)$. Then $R : K \otimes K \to K \otimes K$ defined by

$$R(a \otimes b) = (a_1 \triangleright_T b_1) \otimes (a_2 \triangleleft_T b_2)$$

is a coalgebra isomorphism and a solution of the Yang-Baxter equation on the vector space $K$, where $\triangleright_T$ and $\triangleleft_T$ are defined by (20) and (38) respectively.

5. Equivalent characterizations of relative Rota-Baxter operators

In this section, we give some alternative characterizations of relative Rota-Baxter operators using relative Rota-Baxter operators on the Lie algebra of primitive elements, graphs and module bialgebra structures.

5.1. Restrictions and extensions of relative Rota-Baxter operators. Let $K$ be a cocommutative $H$-module bialgebra via an action $\rightarrow$. It is obvious that via the restrictions of the action $\rightarrow$, we obtain actions of $G(H)$ on $G(K)$ and of $P(H)$ on $P(K)$, for which we use the same notations. As expected, a relative Rota-Baxter operator with respect to a cocommutative $H$-module bialgebra $(K, \rightarrow)$ will naturally induces a relative Rota-Baxter operator on the group $G(H)$ and on the Lie algebra $P(H)$ respectively.

Theorem 5.1. Let $T : K \to H$ be a relative Rota-Baxter operator with respect to a cocommutative $H$-module bialgebra $(K, \rightarrow)$.

(i) $T|_{G(K)}$ is a relative Rota-Baxter operator on the group $G(H)$ with respect to the action $(G(K), \rightarrow)$;

(ii) $T|_{P(K)}$ is a relative Rota-Baxter operator on the Lie algebra $P(H)$ with respect to the action $(P(K), \rightarrow)$.

Proof. Since $T$ is a coalgebra homomorphism, it follows that $T|_{G(K)}$ is a map from $G(K)$ to $G(H)$, and $T|_{P(K)}$ is a map from $P(K)$ to $P(H)$.

For any $a, b \in G(K)$, we have

$$T(a)T(b) = T(a(T(a) \rightarrow b)).$$
which implies that \( T|_{G(H)} \) is a relative Rota-Baxter operator on the group \( G(H) \) with respect to the action \((G(H), \rightarrow)\).

For any \( a, b \in P(K) \), we have \( T(a)T(b) = T(ab) + T(T(a) \rightarrow b) \), and thus
\[
[T(a), T(b)] = T(T(a) \rightarrow b) - T(T(b) \rightarrow a) + T([a, b]).
\]
Hence, \( T|_{P(K)} \) is a relative Rota-Baxter operator on the Lie algebra \( P(H) \) with respect to the action \((P(K), \rightarrow)\). \( \square \)

Let \( \phi : \mathfrak{h} \to \text{Der}(\mathfrak{t}) \) be an action of a Lie algebra \((\mathfrak{h}, [\cdot, \cdot]_\mathfrak{h})\) on \((\mathfrak{t}, [\cdot, \cdot]_\mathfrak{t})\). Then \( \phi \) can be extended to a module bialgebra action \( \bar{\phi} : U(\mathfrak{h}) \to \text{End}(T_k(\mathfrak{t})) \) by
\[
\bar{\phi}(x)(1) = 0, \quad \bar{\phi}(x)(y_1 \cdots y_r) = \sum_{i=1}^r y_1 \cdots y_{i-1} \phi(x)(y_i)y_{i+1} \cdots y_r,
\]
where \( T_k(\mathfrak{t}) \) is the tensor \( k \)-algebra of \( \mathfrak{t} \), \( x \in \mathfrak{h} \) and \( y_1, \ldots, y_r \in \mathfrak{t} \), \( r \geq 1 \). As \( \mathfrak{h} \) acts on \( \mathfrak{t} \) by derivations, it induces a module bialgebra action \( \bar{\phi} \) of \( U(\mathfrak{h}) \) on \( U(\mathfrak{t}) \).

The following extension theorem of relative Rota-Baxter operators from Lie algebras to their universal enveloping algebras generalizes the case of Rota-Baxter operators given in [31, Theorem 2].

**Theorem 5.2.** Any relative Rota-Baxter operator \( T : \mathfrak{t} \to \mathfrak{h} \) on a Lie algebra \( \mathfrak{h} \) with respect to an action \((\mathfrak{t}, \phi)\) can be extended to a unique relative Rota-Baxter operator \( \bar{T} : U(\mathfrak{t}) \to U(\mathfrak{h}) \) with respect to the extended \( U(\mathfrak{h}) \)-module bialgebra \((U(\mathfrak{t}), \bar{\phi})\) by
\[
\bar{T}(y_1 \cdots y_n) = (T(y_1)\bar{T} - \bar{T}\phi(T(y_1))) \cdots (T(y_n)\bar{T} - \bar{T}\phi(T(y_n)))(1), \quad \forall y_1, \ldots, y_n \in \mathfrak{t}, \; n \geq 1,
\]
where those \( T(y_k) \)'s left to \( \bar{T} \) are interpreted as the left multiplication by them.

Furthermore, the post-Hopf algebra \((U(\mathfrak{t}), \triangleright_\mathfrak{t})\) induced by the relative Rota-Baxter operator \( \bar{T} : U(\mathfrak{t}) \to U(\mathfrak{h}) \) as in Theorem 3.3 coincides with the extended post-Hopf algebra \((U(\mathfrak{t}), \triangleright_\mathfrak{t})\) from \((\mathfrak{t}, \triangleright_\mathfrak{t})\) given in Theorem 2.3. Namely, we have the following diagram

\[
\begin{array}{ccc}
(1, \triangleright_\mathfrak{t}) & \xrightarrow{\text{extension}} & (U(\mathfrak{t}), \triangleright_\mathfrak{t}) \\
\text{Rota-Baxter action} & & \text{Rota-Baxter action} \\
\mathfrak{t} \xrightarrow{T} \mathfrak{h} & \xrightarrow{\text{extension}} & U(\mathfrak{t}) \xrightarrow{T} U(\mathfrak{h}).
\end{array}
\]

**Proof.** Let \( J_t = (yz - zy - [y, z]_t | y, z \in \mathfrak{t}) \) be the ideal of \( T_k(\mathfrak{t}) \) such that \( U(\mathfrak{t}) \simeq T_k(\mathfrak{t})/J_t \). We recursively define a linear map \( \bar{T} : T_k(\mathfrak{t}) \to U(\mathfrak{h}) \) by
\[
\bar{T}(1) = 1, \quad \bar{T}(yu) = T(y)\bar{T}(u) - \bar{T}(\phi(T(y))u), \quad \forall y \in \mathfrak{t}, \; u \in \mathfrak{t}^n, \; n \geq 0.
\]
Then it is straightforward to deduce that \( \bar{T}(J_t) = 0 \) and we have the induced linear map \( \bar{T} : U(\mathfrak{t}) \to U(\mathfrak{h}) \).

Next we prove that \( \bar{T} : U(\mathfrak{t}) \to U(\mathfrak{h}) \) is a relative Rota-Baxter operator. Namely,
\[
\bar{T}(u)\bar{T}(v) = \bar{T}(u_1\phi(\bar{T}(u_2)v))
\]
for any \( u \in U(\mathfrak{t})_m, v \in U(\mathfrak{t})_n \). It can be done by induction on \( m \). The case when \( m = 1 \) is due to the recursive definition \((41)\) of \( \tilde{T} \). For \( yu \in U(\mathfrak{t})_{m+1} \), since \( \phi \) is a module bialgebra action, we have

\[
\tilde{T}(yu)\tilde{T}(v) = T(y)\tilde{T}(u)\tilde{T}(v) - \tilde{T}(\phi(T(y))u)\tilde{T}(v) \\
= T(y)\tilde{T}(u_1\phi(\tilde{T}(u_2))v) - \tilde{T}((\phi(T(y))u_1)(\phi(\tilde{T}(u_2)))v) - \tilde{T}(u_1(\phi(\tilde{T}(\phi(T(y))u_2))v)) \\
= \tilde{T}(yu_1\phi(\tilde{T}(u_2))v) + \tilde{T}(\phi(T(y))(u_1\phi(\tilde{T}(u_2)))v) \\
- \tilde{T}((\phi(T(y))u_1)(\phi(\tilde{T}(u_2)))v) - \tilde{T}(u_1(\phi(\tilde{T}(\phi(T(y))u_2))v)) \\
= \tilde{T}(yu_1\phi(\tilde{T}(u_2))v) + \tilde{T}(u_1\phi(\tilde{T}(yu_2))v) \\
= \tilde{T}((yu_1)\phi(\tilde{T}((yu_2))v),
\]

which implies that \( \tilde{T} : U(\mathfrak{t}) \to U(\mathfrak{h}) \) is a relative Rota-Baxter operator. The above procedure also implies that the extension from \( T : \mathfrak{t} \to \mathfrak{h} \) to \( \tilde{T} : U(\mathfrak{t}) \to U(\mathfrak{h}) \) is unique.

By \((19)\), the induced post-Lie product \( \triangleright_T \) on \( \mathfrak{t} \) is given by

\[
y \triangleleft_T z = \phi(T(y))z, \quad \forall y, z \in \mathfrak{t}.
\]

Then by Theorem \((2.8)\) the extended post-Hopf product \( \bowtie_T \) on \( U(\mathfrak{t}) \) is recursively defined by

\[
y \bowtie_T 1 = 0, \quad y \bowtie_T z = (y \triangleright_T z)v + z(y \bowtie_T v), \\
1 \bowtie_T v = v, \quad yu \bowtie_T v = y \bowtie_T (u \bowtie_T v) - (y \bowtie_T u) \bowtie_T v,
\]

for any \( y, z \in \mathfrak{t}, u, v \in U(\mathfrak{t}) \). On the other hand, by \((20)\), we know that

\[
u \triangleright_T v = \bar{\phi}(\bar{T}(u))v, \quad \forall u, v \in U(\mathfrak{t}).
\]

In particular, \( y \triangleright_T 1 = 0, 1 \triangleright_T v = v \) and

\[
y \triangleright_T zv = \bar{\phi}(\bar{T}(y))(zv) = (\phi(T(y))z)v + z\phi(T(y))v \\
= (y \triangleright_T z)v + z(y \triangleright_T v), \\
yu \triangleright_T v = \bar{\phi}(\bar{T}(yu))v = \bar{\phi}(T(y)\bar{T}(u) - \bar{T}(\phi(T(y))u))v \\
= \phi(T(y))(\bar{T}(u)v) - \bar{T}(\phi(T(y))u)v \\
= y \triangleright_T (u \triangleright_T v) - (y \triangleright_T u) \triangleright_T v.
\]

Therefore, the two post-Hopf products on \( U(\mathfrak{t}) \) coincide, and we get the desired diagram. \( \square \)

5.2. Graph characterization. Now we use graphs to characterize relative Rota-Baxter operators on Hopf algebras.

**Definition 5.3.** Given any coalgebra homomorphism \( f : K \to H \), we define the graph of \( f \), which is denoted by \( \text{Gr}_f \), as the subspace \( \text{im}((id \otimes f)\Delta_K) \) of \( K \otimes H \), namely,

\[
\text{Gr}_f = \{a_1 \otimes f(a_2) \mid a \in K\}.
\]

**Theorem 5.4.** A coalgebra homomorphism \( T : K \to H \) is a relative Rota-Baxter operator with respect to a cocommutative \( H \)-module bialgebra \( (K, \rightarrow) \) if and only if the graph \( \text{Gr}_T \) is a Hopf subalgebra of the smash product Hopf algebra \( K \rtimes H \) and isomorphic to \( K_T \).
Proof. Let $T : K \to H$ be a relative Rota-Baxter operator. Then for all $a, b \in K$, we have
\[
(a_1 \# T(a_2))(b_1 \# T(b_2)) = a_1(T(a_2) \to b_1)\# T(a_3)T(b_2)
\]
\[
= a_1 \ast_T b_1 \# T(a_2)T(b_2)
\]
\[
= a_1 \ast_T b_1 \# T(a_2 \ast_T b_2)
\]
\[
= (a \ast_T b_1) \# T((a \ast_T b_2)) \in \text{Gr}_T,
\]
as the binary operation $\ast_T$ on $K$ defined in (23) is a coalgebra homomorphism by the cocommutativity of $K$, which implies that $\text{Gr}_T$ is a subalgebra of $K \rtimes H$ with unit $1\# 1 = 1\# T(1)$.

Also, as $T$ is a coalgebra homomorphism and $K$ is cocommutative,
\[
\begin{align*}
\Delta(a_1 \# T(a_2)) &= (a_1 \# T(a_3)) \otimes (a_2 \# T(a_4)) \\
&= (a_1 \# T(a_2)) \otimes (a_3 \# T(a_4)) \in \text{Gr}_T \otimes \text{Gr}_T,
\end{align*}
\]
\[
\begin{align*}
S(a_1 \# T(a_2)) &= (S_H(T(a_1)) \to S_K(a_2)) \# S_H(T(a_3)) \\
&= S_T(a_1) \# T(S_T(a_2)) \\
&= S_T(a_1) \# T(S_T(a_2)) \in \text{Gr}_T,
\end{align*}
\]
where the antipode formula above is due to the property of $S_T$ shown in Proposition 3.24. Hence, $\text{Gr}_T$ is a Hopf algebra inside $K \rtimes H$. Furthermore, define linear map
\[
\Psi : K_T \to \text{Gr}_T, \ a \mapsto a_1 \# T(a_2).
\]
By the calculation above, we know that $\Psi$ is a Hopf algebra isomorphism with its inverse $\Psi^{-1} = \text{id} \otimes \varepsilon_H$.

Conversely, assume that $\text{Gr}_T$ is a subalgebra of $K \rtimes H$. For any $a, b \in K$, there exists $w \in K$ such that
\[
a_1 \ast_T b_1 \# T(a_2)T(b_2) = (a_1 \# T(a_2))(b_1 \# T(b_2)) = w_1 \# T(w_2).
\]
In particular, since $\ast_T$ is bilinear, we have
\[
w = w_1 \varepsilon_H(T(w_2)) = a_1 \ast_T b_1 \varepsilon_H(T(a_2)T(b_2)) = a \ast_T b,
\]
\[
T(w) = \varepsilon_K(w_1)T(w_2) = \varepsilon_K(a_1 \ast_T b_1)T(a_2)T(b_2) = T(a)T(b),
\]
which indicate that $T$ is a relative Rota-Baxter operator. \hfill \Box

Let $\phi : \mathfrak{h} \to \text{Der}(\mathfrak{t})$ be an action of a Lie algebra $(\mathfrak{h}, [\cdot, \cdot]_\mathfrak{h})$ on a Lie algebra $(\mathfrak{t}, [\cdot, \cdot]_\mathfrak{t})$. Denote by $\mathfrak{t} \rtimes \mathfrak{h}$ the semi-direct product Lie algebra of $\mathfrak{t}$ and $\mathfrak{h}$ with respect to the action $(\mathfrak{t}, \phi)$. More precisely, the Lie bracket $[\cdot, \cdot]_\mathfrak{a} : \wedge^2 (\mathfrak{t} \oplus \mathfrak{h}) \to \mathfrak{t} \oplus \mathfrak{h}$ is given by
\[
[u, x], (v, y)]_\mathfrak{a} = ([u, v]_\mathfrak{t} + \phi(x)(v) - \phi(y)(u), [x, y]_\mathfrak{h}), \quad \forall x, y \in \mathfrak{h}, u, v \in \mathfrak{t}.
\]
It is well known that $T : \mathfrak{t} \to \mathfrak{h}$ is a relative Rota-Baxter operator if and only if the graph of $T$, $\text{Gr}_T := \{(u, T(u)) | u \in \mathfrak{t}\}$ is a subalgebra of $\mathfrak{t} \rtimes \mathfrak{h}$.

Now we consider the lifted relative Rota-Baxter operator $\bar{T} : U(\mathfrak{t}) \to U(\mathfrak{h})$ of the relative Rota-Baxter operator $T : \mathfrak{t} \to \mathfrak{h}$. It turns out that the Hopf algebra $\text{Gr}_T$ can serve as the universal enveloping algebra of the Lie algebra $\text{Gr}_T$. 


Proposition 5.5. Let $\tilde{T} : U(\mathfrak{t}) \to U(\mathfrak{h})$ be the lifted relative Rota-Baxter operator of the relative Rota-Baxter operator $T : \mathfrak{t} \to \mathfrak{h}$. Then $Gr_T \simeq U(Gr_T)$, i.e. the universal enveloping algebra of the graph $Gr_T$ of the relative Rota-Baxter operator $T : \mathfrak{t} \to \mathfrak{h}$ is isomorphic to the Hopf algebra $Gr_T$, which is the graph of the relative Rota-Baxter operator $\bar{T} : U(\mathfrak{t}) \to U(\mathfrak{h})$.

Proof. First it is straightforward to obtain the following Hopf algebra isomorphism

$$\bar{\varphi} : U(\mathfrak{t} \rtimes \mathfrak{h}) \to U(\mathfrak{t} \rtimes \mathfrak{h}), \quad (u, x) \mapsto u \# 1 + 1 \# x, \quad \forall x \in \mathfrak{h}, \ u \in \mathfrak{t}. $$

By Theorem 5.3, $Gr_T$ is a Hopf subalgebra of $U(\mathfrak{t}) \# U(\mathfrak{h})$ isomorphic to $U(\mathfrak{t}) T$. Let $\psi := \bar{\varphi}|_{Gr_T}$, then $\psi$ is injective and

$$\psi(u, T(u)) = u \# 1 + 1 \# T(u) = (id \otimes \bar{T}) \Delta(u) \in Gr_T, \quad \forall u \in \mathfrak{t}. $$

Also, note that $\text{Im} \psi$ generates $Gr_T$ as an algebra. Hence, $\psi$ induces a Hopf algebra isomorphism $\tilde{\psi} : U(Gr_T) \to Gr_T$ by the Theorem of Heyneman and Radford [52, Theorem 5.3.1], as $U(Gr_T)_1 = k \oplus Gr_T$ and $\tilde{\psi}|_{U(Gr_T)_1}$ is also injective. $\square$

5.3. Module and module bialgebra characterization. Next we give another characterization of relative Rota-Baxter operators on Hopf algebras using new module structures and new module bialgebra structures. Let $H$ and $K$ be Hopf algebras such that $K$ is a cocommutative $H$-module bialgebra via an action $\rightarrow$.

Theorem 5.6. A coalgebra homomorphism $T : K \to H$ is a relative Rota-Baxter operator if and only if $K$ endowed with the binary operation $\star_T$ in (2.2) is an algebra, denoted by $K_T = (K, \star_T)$, and $H$ is a $K_T$-module via the action $\star_T$ defined by

$$a \star_T x := T(a)x, \quad \forall x \in H, a \in K.$$

Proof. If $T : K \to H$ is a relative Rota-Baxter operator, then by Corollary 5.4, $K_T = (K, \star_T)$ is an algebra with unit 1. Also,

$$1 \star_T x = T(1)x = 1x = x,$$

$$(a \star_T b) \star_T x = T(a \star_T b)x = T(a)T(b)x = a \star_T (b \star_T x),$$

for any $x \in H, \ a, b \in K$. That is, $H$ is a $K_T$-module.

Conversely, if $K_T = (K, \star_T)$ is an algebra and $H$ is a $K_T$-module via the stated action $\star_T$, then particularly

$$T(a_1(T(a_2 \rightarrow b))) = T(a \star_T b) = T(a \star_T b)1 = (a \star_T b) \star_T 1 = a \star_T (b \star_T 1) = T(a)(T(b)1) = T(a)T(b).$$

Namely, (17) holds, and $T : K \to H$ is a relative Rota-Baxter operator. $\square$

The following result is straightforward to obtain.

Lemma 5.7. Let $H$ and $K$ be two cocommutative Hopf algebras such that $K$ is an $H$-module bialgebra via an action $\rightarrow$. Then $K$ is a $K \rtimes H$-module bialgebra defined by

$$(a \# x).b := \text{ad}_a(x \to b), \quad \forall x \in H, \ a, b \in K.$$
Proposition 5.8. Let $T : K \to H$ be a relative Rota-Baxter operator. Then $K$ has a cocommutative $K_T$-module bialgebra structure via the following action,
\[
\text{ad}_{T,a}b := \text{ad}_{a_1}(T(a_2) \hookrightarrow b), \quad \forall a, b \in K.
\]

Proof. Let $T : K \to H$ be a relative Rota-Baxter operator. By Theorem 5.4, the graph $\text{Gr}_T$ is a Hopf algebra inside $K \rtimes H$. Therefore $K$ becomes a $\text{Gr}_T$-module bialgebra by Lemma 5.7. Furthermore, pulled back by the Hopf algebra isomorphism $\Psi : K_T \to \text{Gr}_T$ given in (43), $K$ becomes a $K_T$-module bialgebra via the desired action $\text{ad}_T$. \qed

Acknowledgements. This research is supported by NSFC (Grant No. 11922110, 12071094, 12001228).

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**School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China**

*Email address: ynl1@gzhu.edu.cn*

**Department of Mathematics, Jilin University, Changchun 130012, Jilin, China**

*Email address: shengyh@jlu.edu.cn*

**Department of Mathematics, Jilin University, Changchun 130012, Jilin, China**

*Email address: tangrong@jlu.edu.cn*