Some remarks on traces on the infinite-dimensional Iwahori–Hecke algebra

Yury A. Neretin

The infinite-dimensional Iwahori–Hecke algebras $H_{\infty}(q)$ are direct limits of the usual finite-dimensional Iwahori–Hecke algebras. They arise in a natural way as convolution algebras of bi-invariant functions on groups $GL_{\infty}(\mathbb{F}_q)$ of infinite-dimensional matrices over finite fields having only finite number of non-zero matrix elements under the diagonal. In 1988 Vershik and Kerov classified all indecomposable positive traces on $H_{\infty}(q)$. Any such trace generates a representation of the double $H_{\infty}(q) \otimes H_{\infty}(q)$ and of the double $GL_{\infty}(\mathbb{F}_q) \times GL_{\infty}(\mathbb{F}_q)$. We present constructions of such representations; the traces are some distinguished matrix elements. We also obtain some (simple) general statements on relations between unitary representations of groups and representations of convolution algebras of measures bi-invariant with respect to compact subgroups.

1 Introduction

1.1. Algebras of bi-invariant measures. Let $G$ be a topological group. Denote by $\mathcal{M}(G)$ the algebra of finite compactly supported complex-valued measures on $G$. The addition in $\mathcal{M}(G)$ is the addition of measures, the multiplication is the convolution. Namely, let $\mu_1$, $\mu_2$ be measures supported by compact sets $L_1$, $L_2$ respectively. Consider the measure $\mu_1 \times \mu_2$ on $L_1 \times L_2$. The convolution $\mu_1 * \mu_2$ is the pushforward of $\mu_1 \times \mu_2$ under the map $(g_1, g_2) \mapsto g_1g_2$ from $L_1 \times L_2$ to $G$. We also define an involution $\mu \mapsto \mu^*$ in $\mathcal{M}(G)$. Namely, $\mu^*$ is the pushforward of the the complex conjugate measure $\mu$ under the map $g \mapsto g^{-1}$,

$$(\mu_1 * \mu_2)^* = \mu_2^* * \mu_1^*.$$

We also define a transposition that send $\mu$ to its image under the map $g \mapsto g^{-1}$. We have

$$(\mu_1 * \mu_2)^t = \mu_2^t * \mu_1^t,$$

the involution is anti-linear map and the transposition is linear.

Remark. If a group $G$ is finite, then $\mathcal{M}(G)$ is the group algebra of $G$. $\Box$

Let $\rho$ be a unitary representation of $G$ in a Hilbert space $H$ (Hilbert spaces assumed to be separable). For any measure $\mu \in \mathcal{M}(G)$ denote by $\rho(\mu)$ the operator

$$\rho(\mu) = \int_G \rho(g) d\mu(g).$$

This determines a $*$-representation of the algebra $\mathcal{M}(G)$,

$$\rho(\mu_1) \rho(\mu_2) = \rho(\mu_1 \ast \mu_2), \quad \rho(\mu) + \rho(\nu) = \rho(\mu + \nu), \quad \rho(\mu^*) = \rho(\mu)^*.$$

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Let $K \subset G$ be a compact subgroup. By $\delta_K$ we denote the probabilistic Haar measure on $K$ regarded as an element of $\mathcal{M}(G)$. Denote by $\mathcal{M}(G//K)$ the subalgebra of $\mathcal{M}(G)$ consisting of measures, which are invariant with respect to left and right shifts by elements of $K$. Clearly, for any $\mu \in \mathcal{M}(G)$ we have $\delta_K * \mu * \delta_K \in \mathcal{M}(G//K)$ and for an element $\nu \in \mathcal{M}(G)$ the following identity holds:

$$\nu \in \mathcal{M}(G//K) \iff \delta_K * \nu * \delta_K = \nu.$$  

Clearly, the subalgebra $\mathcal{M}(G//K)$ is closed with respect to the involution.

Let $\rho$ be a unitary representation of $G$ in a Hilbert space $H$. Denote by $H^K \subset H$ the subspace consisting of all $K$-fixed vectors, denote by $P^K$ the operator of orthogonal projection to $H^K$. Clearly,

$$\rho(\nu) = \begin{pmatrix} \bar{\rho}(\nu) & 0 \\ 0 & 0 \end{pmatrix}.$$  

(1.1)

So for any unitary representation $\rho$ of $G$ we get a $*$-representation $\bar{\rho}(\cdot)$ of the algebra $\mathcal{M}(G//K)$ in the Hilbert space $H^K$.

Denote by $\text{URep}(G)_K$ the set of equivalence classes of unitary representations of the group $G$ such that $H^K$ is a cyclic subspace\(^2\). Denote by $\text{Rep}(G//K)$ the set of $*$-representations of the algebra $\mathcal{M}(G//K)$. The following statement is obvious (see below Proposition 2.1)

The map $\text{Res}_{G//K} : \rho \mapsto \bar{\rho}$ from $\text{URep}(G)_K$ to $\text{Rep}(\mathcal{M}(G//K))$ is injective.

An inverse construction $\bar{\rho} \mapsto \rho$ (if $\bar{\rho}$ is contained in the image of $\text{Res}_{G//K}$) is semi-explicit: having a representation of $\mathcal{M}(G//K)$ one can define a reproducing kernel space and a representation of $G$ in this space (see Proposition 2.2). Generally speaking, a positive-definiteness of a kernel is a non-trivial question and a description of the image of $\text{Res}_{G//K}$ also is non-trivial\(^3\). For our purposes the following statement is sufficient:

If $G$ is a compact group or a direct limit of compact groups, then the map $\text{Res}_{G//K}$ is a bijection, see Propositions 2.3–2.5.

Recall also two variations of these definitions. Let $G$ be a unimodular locally compact group\(^4\), let $\lambda$ be a Haar measure on $G$. Let $K \subset G$ be a compact

\(^2\)I.e., linear combinations of vectors $\rho(g)\xi$, where $\xi$ ranges in $H^K$ and $g$ ranges in $G$, are dense in $H$.

\(^3\)For instance, if $G$ is a semisimple Lie group and $K$ is its maximal compact subgroup.

\(^4\)A locally compact group is called unimodular if its Haar measure is two-side invariant, see, e.g., [17], Subsect. 9.1.
subgroup. Denote by \( \mathcal{C}(G) \) the convolution algebra of all compactly supported continuous functions on \( G \). This algebra is a subalgebra in \( \mathcal{M}(G) \), namely, for any function \( \varphi \in \mathcal{C}(G) \) we assign the measure \( \varphi(g) \mu(g) \). By \( \mathcal{C}(G/\!/K) \subset \mathcal{C}(G) \) we denote the subalgebra of \( K \)-bi-invariant functions, \( \varphi(k_1 g k_2) = \varphi(g) \) for \( k_1, k_2 \in K \).

Next, let \( G \) be a unimodular totally disconnected locally compact group, \( k \) be an open compact subgroup. Denote by \( \mathcal{A}(G/\!/K) \) the convolution algebra consisting of \( K \)-bi-invariant compactly supported locally constant functions. Clearly, in this case
\[
\mathcal{A}(G/\!/K) = \mathcal{C}(G/\!/K) = \mathcal{M}(G/\!/K).
\]

Convolution algebras \( \mathcal{M}(G/\!/K) \) are a usual tool of representation theory, see, e.g., [19], [9], [1], [12], [13], [5], [2], [18], [20], Chapter 5.

1.2. Iwahori-Hecke algebras. Consider a finite field \( \mathbb{F}_q \) with \( q \) elements.

Let \( G \) be the group \( \text{GL}(n, \mathbb{F}_q) \) of invertible matrices of order \( n \) over \( \mathbb{F}_q \). Let \( K = B(n) \) be the subgroup of upper triangular matrices. The algebra \( \mathcal{M}(G/\!/K) \) was described by Iwahori [12]. Denote by \( s_m \in \text{GL}(n, \mathbb{F}_q) \) the operator that transposes basis vectors \( e_m \) and \( e_{m+1} \) in \( \mathbb{F}_q^n \) and fixes other \( e_j \). So \( m = 1, 2, \ldots, n-1 \). Set \( \sigma_m := \delta_K \ast s_m \ast \delta_K \). Then the elements \( \sigma_m \) generate the algebra
\[
\mathcal{H}_n(q) := \mathcal{M}(\text{GL}(n, \mathbb{F}_q)/B(n))
\]
and relations are
\[
\begin{align*}
\sigma_m^2 &= (q-1)\sigma_m + q \quad \text{or} \quad (\sigma_m + 1)(\sigma_m - q) = 0; \quad (1.2) \\
\sigma_m \sigma_{m+1} \sigma_m &= \sigma_{m+1} \sigma_m \sigma_{m+1} \quad (1.3) \\
\sigma_m \sigma_l &= \sigma_l \sigma_m \quad \text{if } |m-l| > 1. \quad (1.4)
\end{align*}
\]
The dimension of \( \mathcal{H}_q = n! \). The involution and the transposition are determined by
\[
\sigma_m^* = \sigma_m, \quad \sigma_m^r = \sigma_m.
\]

By the construction \( q \) is a power of a prime. However, the algebra with relations (1.2)–(1.4) makes sense for any \( q \in \mathbb{C} \), for \( q = 1 \) we get the group algebra of the symmetric group \( S_n \). For all \( q \) that are not roots of units algebras \( \mathcal{H}_n(q) \) are isomorphic. Therefore they have the same dimensions of representations.

1.3. The group of almost triangular matrices and its Iwahori–Hecke algebra. Following Vershik and Kerov [35], denote by \( \text{GLB}(\mathbb{F}_q) \) the group of all infinite invertible matrices over \( \mathbb{F}_q \) having only finite number of nonzero matrix elements under the diagonal.\(^5\)

Denote by \( B(\infty) \) the subgroup of \( \text{GLB}(\mathbb{F}_q) \) consisting of upper triangular matrices. For a matrix \( g \in K \) its diagonal elements \( g_{jj} \) are contained in \( \mathbb{F}_q^\times := \mathbb{F}_q \setminus 0 \) and elements \( g_{ij} \), where \( i < j \) are contained in \( \mathbb{F}_q \). So the set \( B(\infty) \) is a

\(^5\)There are several approaches to 'representation theory of infinite-dimensional groups over finite fields', see a discussion of different works in [27], Subsect. 1.11.
product of a countable number of copies if $F_q^\infty$ and a countable number of copies of $F_q$. We equip $K$ with the product topology and get a structure of a compact topological group. We take uniform probabilistic measures on the sets $F_q^\infty$, $F_q$ and equip the group $B(\infty)$ with the product measure. It is easy to verify that we get the Haar measure on $B(\infty)$.

The space $GLB(F_q)$ is a disjoint union of a countable number of cosets $gB(\infty)$. We equip $GLB(F_q)$ with the topology of disjoint union, this determines a structure of a unimodular locally compact topological group on $GLB(F_q)$. Equivalently, a sequence $g^{(\alpha)} \in GLB(F_q)$ converges to $g$ if the following two conditions hold:

— for each $i, j$ we have $g_{ij}^{(\alpha)} = g_{ij}$ for sufficiently large $\alpha$;

— there exists $n$ such that for all $i, j$ such that $i < j$ and $j \geq n$ we have $g_{ij}^{(\alpha)} = 0$ for all $\alpha$.

Denote by $GLB(n, F_q) \subset GLB(F_q)$ the subgroup consisting of matrices $g$ satisfying the condition: $g_{ij} = 0$ for all pairs $(i, j)$ such that $j > n$, $i < j$. In other words this group is generated by $GL(n, F_q)$ and $B(\infty)$. We get an increasing family of compact subgroups in $GLB(F_q)$, and $GLB(F_q)$ is the direct limit $GLB(F_q) = \lim_{n \to \infty} GLB(n, F_q)$.

The Iwahori–Hecke algebra $H_\infty(q) := M(GLB(F_q)/B(\infty)) = A(GLB(F_q)/B(\infty))$ is the algebra with generators $\sigma_1, \sigma_2, \sigma_3, \ldots$ and the same relations (1.2)–(1.4). Also, it is the direct limit $H_\infty(q) = \lim_{n \to \infty} H_n(q)$.

Again, algebras $H_\infty(q)$ are well defined for all $q \in \mathbb{C}$. By default we assume $q > 0$.

1.4. Traces on $H_\infty(q)$. A trace $\chi$ on an associative algebra $A$ with involution is a linear functional $A \to \mathbb{C}$ such that

a) $\chi(AB) = \chi(BA)$ for any $A, B \in A$;

b) $\chi(A^*) = \overline{\chi(A)}$.

c) $\chi(A^* A) \geq 0$ for any $A \in A(q)$.

The set of all traces is a convex cone: if $\chi_1, \chi_2$ are traces, then for any $a_1, a_2 \geq 0$ the expression $a_1 \chi_1 + a_2 \chi_2$ is a trace. A trace $\chi$ is normalized if $\chi(1) = 1$. A trace $\nu$ is indecomposable if it does not admit a representation of the form $\nu = a_1 \chi_1 + a_2 \chi_2$, where $\chi_1, \chi_2$ are traces non-proportional to $\nu$ and $a_1, a_2 > 0$.

In [34], [15], [35], [36] Vershik and Kerov classified all indecomposable traces on the infinite-dimensional Iwahori–Hecke algebras $H_\infty(q)$. See also continuations of this work in [21], [4], [10].

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*Kerov died in 2020, the work was not completely published, the text [36] was based on his posthumous notes.*
Let us formulate the classification theorem. For \( \lambda \leq \nu \) we denote
\[
\zeta_{\lambda, \mu} := \sigma_{\mu-1} \sigma_{\mu-2} \cdots \sigma_{\lambda}.
\]
Consider a partition \( \nu \) of \( n \),
\[
n = \nu_1 + \nu_2 + \cdots, \quad \nu_1 \geq \nu_2 \geq \cdots > 0.
\]
We set
\[
\lambda_j := \nu_1 + \cdots + \nu_j,
\]
and consider the following elements of the Iwahori–Hecke algebra
\[
\zeta_{\lambda} := \cdots \zeta_{\lambda_3, \lambda_4} \zeta_{\lambda_2, \lambda_3} \zeta_{\lambda_1, \lambda_2}
\]
(all factors commute). For the following two statements, see [30], [8], Sect.8.2.

— The elements \( \zeta_{\lambda} \) form a basis of the space
\[
\mathcal{H}_n(q)/[\mathcal{H}_n(q), \mathcal{H}_n(q)],
\]
where \([\mathcal{H}_n(q), \mathcal{H}_n(q)] \subset \mathcal{H}_n(q)\) is the subspace generated by all commutators
\( ab - ba \), where \( a, b \) range in \( \mathcal{H}_n(q) \). Therefore,

— a trace on \( \mathcal{H}_n(q) \) is determined by its values on elements \( \zeta_{\lambda} \).

This implies that the same statement is valid for \( \mathcal{H}_\infty(q) \). Next,

— By [34], for any indecomposable trace \( \chi \) on \( \mathcal{H}_\infty(q) \) we have
\[
\chi(\zeta_{\lambda}) = \prod_j \zeta_{\lambda_j}, \quad (1.5)
\]
and for any \( m > 0 \)
\[
\chi(\zeta_{\lambda_1+m, \lambda_2+m}) = \chi(\zeta_{\lambda_1, \lambda_2}). \quad (1.6)
\]
Therefore, any trace \( \chi \) on \( \mathcal{H}_\infty(q) \) is determined by its values on elements
\[
\zeta_m := \zeta_{[1,m]}.
\]
Notice, that \( \zeta_1 = 1 \).

**Theorem 1.1** (Vershik, Kerov [34]) Let \( q > 0 \). Indecomposable traces \( \chi^{\alpha, \beta, \gamma} \)
on \( \mathcal{H}_\infty(q) \) are enumerated by collections of parameters
\[
\alpha_1 \geq \alpha_2 \geq \cdots \geq 0, \quad \beta_1 \geq \beta_2 \geq \cdots \geq 0, \quad \gamma \geq 0;
\]
\[
\sum \alpha_i + \sum \beta_j + \gamma = 1.
\]
The value of \( \chi^{\alpha, \beta, \gamma} \) on \( \zeta_m \) is given by the formula
\[
\chi^{\alpha, \beta, \gamma}(\zeta_m) = \frac{1}{q - 1} \sum_{\mu_1 \geq 0, \mu_2 \geq 0, \ldots} \prod_{k \geq 1} \left( \frac{q^k - 1}{k^{\mu_k}} \right) \prod_{k \geq 2} p_k(\alpha, \beta)^{\mu_k}, \quad (1.7)
\]

\( \sum k^{\mu_k} = m \).
where
\[ p_k(\alpha, \beta) := \sum_i \alpha_i^k + (-1)^{k+1} \sum_i \beta_i^k \]  
(1.8)
denote the super-Newton sums.

**Remark.** Recall that symmetric functions \( [20] \) can be represented as polynomials of the Newton sums \( p_k(\alpha) := \sum \alpha_i^k \). Supersymmetric functions are polynomials in super-Newton sums \( p_k(\alpha, \beta) \), see \([16], [32], [3] \), Chapter 2.

**Remark.** For \( q = 1 \) (i.e., for the infinite symmetric group) the expression (1.7) has a removable singularity,
\[ \chi^{\alpha, \beta, \gamma}(\zeta_m) = p_m(\alpha, \beta) \text{ for } m \geq 2. \]
This special case of Theorem \( [1.1] \) is the Thoma theorem \([33] \) (see, also \([3] \), Chapter 4): irreducible normalized characters of the infinite symmetric group have the following form
\[ \chi_{\alpha, \beta, \gamma}(g) = \prod_{m \geq 2} p_m(\alpha, \beta)^{r_m(g)}, \]
where \( r_m(g) \) is the number of cycles of \( g \) of order \( m \).

### 1.5. Traces and representations of the double \( H_\infty(q) \otimes H_\infty(q) \)

Let \( \mathcal{A} \) be a \(*\)-algebra with unit, let \( \chi \) be a trace on \( \mathcal{A} \). Denote by \( \mathcal{A}^\circ \) the algebra anti-isomorphic to \( \mathcal{A} \), it coincides with \( \mathcal{A} \) as a linear space, multiplication is given by \( A \circ B := BA \). The following formula determines an inner product on \( \mathcal{A} \):
\[ \langle A, B \rangle_\chi = \chi(B^* A). \]

Denote by \( \overline{\mathcal{A}}_\chi \) the corresponding Hilbert space. Assume that operators of left multiplication are bounded\(^7\) in \( \overline{\mathcal{A}}_\chi \). Then we have an action of \( \mathcal{A} \) on \( \overline{\mathcal{A}}_\chi \) by left multiplications and the action of \( \mathcal{A}^\circ \) by right multiplications. These actions commute, so we get a representation of the tensor product \( \mathcal{A} \otimes \mathcal{A}^\circ \),
\[ \tau_\chi(A \otimes B)X := AXB, \quad \text{where } X \in \overline{\mathcal{A}}_\chi, A \in \mathcal{A}, B \in \mathcal{A}^\circ. \]

**Remark.** Representations \( \tau_\chi \) of \( \mathcal{A} \otimes \mathcal{A}^\circ \) are not arbitrary representations, they satisfy the condition:
\[ \tau_\chi(A \otimes 1) 1 = \tau_\chi(1 \otimes A) 1. \]

Notice, that
\[ \chi(A) = \langle \tau_\chi(A \otimes 1) 1, 1 \rangle_\chi. \quad (1.9) \]
So a trace on \( \mathcal{A} \) is a certain matrix element of a certain \( (\mathcal{A}, \mathcal{A}^\circ) \)-bimodule.

Let us return to a discussion the Iwahori-Hecke algebras. Notice that a transposition in \( H_\infty(q) \) is an anti-isomorphism, so we can regard \( H_\infty(q) \) as a \( (H_\infty(q) \otimes H_\infty(q)) \)-module.

\(^7\)So have a structure of a Hilbert algebra in the sense of Dixmier (see \([6], \S I.6, [7], A.54).
Lemma 1.2 We have an isomorphism of algebras
\[ \mathcal{M}\left( \text{GLB}(\mathbb{F}_q) \times \text{GLB}(\mathbb{F}_q) \mathbb{F} B(\infty) \times B(\infty) \right) \cong \mathcal{M}\left( \text{GLB}(\mathbb{F}_q) \mathbb{F} B(\infty) \right) \otimes \mathcal{M}\left( \text{GLB}(\mathbb{F}_q) \mathbb{F} B(\infty) \right) = \mathcal{H}_\infty(q) \otimes \mathcal{H}_\infty(q). \]

More generally, for any locally compact group \( G \) with a countable base of topology and an open compact subgroup \( K \), the following algebras are isomorphic
\[ \mathcal{M}(G \times G / K \times K) \cong \mathcal{M}(G / K) \otimes \mathcal{M}(G / K). \]

Lemma 1.3 For any indecomposable trace \( \chi^{\alpha,\beta,\gamma} \) on \( \mathcal{H}_\infty(q) \) the operators of left-right multiplication
\[ \tau_\chi(A \otimes B) X := AXB \]
are bounded in \( \mathcal{H}_\infty(q)_\chi \).

The first statement is obvious, the second statement is clear from the condition (1.2). Indeed, the spectrum of a self-adjoint operator of multiplication by a generator \( \sigma_m \) consists of points 1 and \( q \), so its norm is \( \max(1,q) \).

Our Proposition 2.5 implies the following corollary:

Corollary 1.4 (see, [35]) Let \( q = p^l \), where \( p \) is a prime. Then for any indecomposable trace \( \chi^{\alpha,\beta,\gamma} \) on \( \mathcal{H}_\infty(q) \) there exists an irreducible representation \( \rho \) of the double \( \text{GLB}(\mathbb{F}_q) \times \text{GLB}(\mathbb{F}_q) \), for which \( \text{Res}(\rho) \) is isomorphic to the representation \( \tau_\chi \) of \( \mathcal{H}_\infty(q) \otimes \mathcal{H}_\infty(q) \).

Remark. To avoid a misleading, we must say some remarks. For a type I topological group \( G \) any irreducible representation of \( G \times G \) is a tensor product \( \pi_1 \otimes \pi_2 \) of irreducible representations of two copies of \( G \) (see, e.g., [7], 13.1.8). The group \( \text{GLB}(\mathbb{F}_q) \) is not of type I and a similar implication is false. According [34] for an indecomposable trace \( \chi \), the operator algebra generated by the representation of \( \mathcal{H}_\infty(q) \otimes 1 \) in \( V = \mathcal{H}_\infty(q) \) is a finite Murray–von Neumann factor. The representation of \( 1 \otimes \mathcal{H}_\infty(q) \) also generates a factor, which is the commutant of the first factor. The representation of the double \( \mathcal{H}_\infty(q) \otimes \mathcal{H}_\infty(q) \) is irreducible. The trace of the unit operator is finite (= 1), so it is a factor of a type \( \text{II}_1 \) or \( \text{II}_2 \). But \( \mathcal{H}_\infty(q) \) has only two irreducible finite dimensional representations, both are one-dimensional [8]. The first is the trivial representation \( (\alpha_1 = 1, \text{other } \alpha \text{'s and } \beta \text{'s are zero}) \), the second is the representation sending all \( \sigma_j \) to \(-1 \) \( (\beta_1 = 1, \text{other } \beta \text{'s and } \alpha \text{'s are zero}) \). In the remaining cases we have \( \text{II}_1 \)-factors. The representations of \( \text{GLB}(\mathbb{F}_q) \) corresponding to the one-dimensional characters are the trivial representation and the Steinberg representation (see, [8]) Indeed, the algebra \( \mathcal{H}_\infty(q) \) is isomorphic to the group algebra of the symmetric group \( S_n \), so dimensions of their irreducible representations coincide. So they have two one-dimensional representations, all other representations have dimensions \( \geq n - 1 \). Therefore finite-dimensional irreducible representations of \( \mathcal{H}_\infty(q) \) are one-dimensional.
The remaining characters $\chi^{\alpha,\beta,\gamma}$ correspond to representations of $\text{GL}_B(F_q)$ generating $\mathbb{II}_\infty$-factors.

Our next purpose is to construct explicitly representations $\tau_\chi$ of $\mathcal{H}_\infty(q) \otimes \mathcal{H}_\infty(q)$. We consider only the case $\gamma = 0$.

1.6. $R$-matrix. Let $V$ be a Hilbert space equipped with an orthonormal basis $v_i$, where $j$ ranges in non-zero integers. Consider $R$-matrix given by the formula

$$R := - \sum_{i < 0} e_{ii} \otimes f_{ii} + q \sum_{i > 0} e_{ii} \otimes f_{ii} - \sqrt{q} \sum_{i \neq j, i \neq 0, j \neq 0} e_{ji} \otimes f_{ij} + (q - 1) \sum_{i < j, i \neq 0, j \neq 0} e_{ij} \otimes f_{ij}, \quad (1.10)$$

see Kerov [15]. Namely, $R$ determines an operator in $V \otimes V$, here $e_{\alpha\beta}$ denote matrix units in the first copy of $V$,

$$e_{\alpha\beta}v_i = \begin{cases} 0, & \text{if } i \neq \alpha; \\ v_\beta, & \text{if } i \neq \alpha, \end{cases}$$

$f_{\alpha\beta}$ denote matrix units in the second copy of $V$. Then

$$R^2 = (q - 1)R + q1 \otimes 1.$$ 

Consider the space $V_{\otimes n} = V \otimes V \otimes \ldots$. Denote by $R_{i(i+1)}$ the operator $R$ acting on the $i$-th and $(i + 1)$-th factors. Then the map

$$\sigma_j \mapsto R_{j(j+1)} \quad (1.11)$$

determines a representation $\mathcal{H}_n(q)$ in $V_{\otimes n}$.

1.7. Representations of $\mathcal{H}_\infty(q) \otimes \mathcal{H}_\infty(q)$. Here we imitate the construction of representations of a double of an infinite symmetric group proposed by Wassermann [37] and Olshanski [29] (see, also, [3], Chapter 10).

Let $\gamma = 0$. Let $W$ be a copy of the space $V$ defined in the previous subsection, let $w_j \in W$ be the basis corresponding to $v_j$. Consider the tensor product $V \otimes W$ and the following unit vector

$$\xi := \sum_{j > 0} \sqrt{\beta_j} v_{-j} \otimes w_{-j} + \sum_{j > 0} \sqrt{\alpha_j} v_j \otimes w_j \in V \otimes W.$$ 

Since $\gamma = 0$, we have $\|\xi\|^2 = \sum \alpha_j + \sum \beta_j = 1$. Consider the infinite tensor product of Hilbert spaces

$$X := (V \otimes W, \xi) \otimes (V \otimes W, \xi) \otimes \ldots$$

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9For positive indices $i$ the $R$ is a standard $R$-matrix (see, e.g., [14]), the first summand in formula (1.10) gives a kind of superization. For $q = 1$ the matrix $R$ produces a structure equivalent to a supersymmetric (or $\mathbb{Z}_2$-graded) tensor product, see [29], [3], Chapter 10.

10Recall that the definition of infinite tensor products of Hilbert spaces requires a choosing of distinguished vectors, see, e.g., [11], Addendum A.
Denote 
\[ \Xi := \xi \otimes \xi \otimes \cdots \in \mathcal{X}. \]

For \( j \in \mathbb{N} \) define an operator
\[ R_{j(j+1)}^{\text{left}} : \mathcal{X} \to \mathcal{X} \]
in the following way. We represent \( \mathcal{X} \) as
\[ \mathcal{X} = (V \otimes W, \xi) \otimes (V \otimes W, \xi) \otimes \cdots \in \mathcal{X}. \]

The middle factor is
\[ V \otimes W \otimes V \otimes W \simeq (V \otimes V \otimes W \otimes W) \]
(for finite tensor products distinguished vectors have no matter). Then \( R_{j(j+1)}^{\text{left}} \) is a tensor product of the following operators:
- \( 1 \otimes (j-1) \) in the first factor;
- \( R \otimes (1 \otimes 1) \) in the middle factor \( (V \otimes V) \otimes (W \otimes W) \);
- \( 1 \otimes (\infty - j-1) \) in the last factor.

We get a representation of \( \mathcal{H}_\infty (q) \) in \( \mathcal{X} \), denote it by
\[ A \mapsto A^{(l)}. \]

In the same way we define operators \( R_{j(j+1)}^{\text{right}} \) acting by twisted permutations of factors \( W \) and a representation of the second copy of the algebra \( \mathcal{H}_\infty (q) \).

**Theorem 1.5** a) For any \( A \in \mathcal{H}_\infty (q) \),
\[ \langle A^{(l)} \Xi, \Xi \rangle_\mathcal{X} = \chi^{\alpha,\beta,0}(A). \]

b) Moreover, the representation of \( \mathcal{H}_\infty (q) \otimes \mathcal{H}_\infty (q) \) in the cyclic span of \( \Xi \) is equivalent to the representation in \( \mathcal{H}_\infty (q) \).

**1.8. The further structure of the paper.** In Section 2 we show that a unitary representation \( \rho \in \text{URep}(G)_{K} \) is uniquely determined by the corresponding representation \( \tilde{\rho} \) of the algebra \( \mathcal{M}(G/K) \) (Proposition 2.1) and describe a quasi-explicit way of realization of \( \rho \) (Proposition 2.2) in a reproducing kernel space\(^{11}\) constructed by \( \tilde{\rho} \). Next, we show that for a direct limit of compact groups any \( * \)-representation of \( \mathcal{M}(L/K) \) corresponds to a unitary representation of \( L \) (Proposition 2.3). All these statements are obvious or very simple, however I could not find a source for formal references.

In Section 3 we prove Theorem 1.5, this proof does not depend on Section 2.

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\(^{11}\)On reproducing kernel spaces, see, e.g., 
[23], Sect. 7.1 and Subsect. 7.5.15.
1.9. Some comments. The group GLB($F_q$) is a locally compact group, whose properties are partially similar to properties of 'infinite dimensional' groups. The construction of Subsect. 1.7 looks like a relatively usual construction related to infinite dimensional groups. However, a behavior of double coset algebras is unusual.

As an example of the usual behavior, consider $G_\alpha(n) := GL(\alpha + n, \mathbb{R})$, $K_\alpha(n) := O(n)$. For $\alpha = 0$ the algebra $C(G_0(n)/K_0(n))$ is commutative and according Gelfand [9] this implies sphericity of the subgroup $O(n)$ in the group $GL(n, \mathbb{R})$. However, even the algebra $C(SL(2, \mathbb{R})/SO(2))$ is a nontrivial object, a multiplication is defined in terms of a hypergeometric kernel, see, e.g., [18]. For $\alpha > 0$ algebras $C(G_\alpha(n)/K_\alpha(n))$ at the present time seem completely incomprehensible.

We have a natural map of double coset spaces

$$K_\alpha(n) \setminus G_\alpha(n)/K_\alpha(n) \rightarrow K_\alpha(n + 1) \setminus G_\alpha(n + 1)/K_\alpha(n + 1).$$

but $K_\alpha(n + 1)$ is strictly larger than $K_\alpha(n)$ and this map does not induce a homomorphism from $C(G_\alpha(n)/K_\alpha(n))$ to $C(G_\alpha(n + 1)/K_\alpha(n + 1))$. However, a limit object exists (see [28], [24]) and it has a structure of a semigroup. Namely, there is a natural multiplication on the space of double cosets

$$\Gamma_\alpha(\infty) = K_\alpha(\infty) \setminus G_\alpha(\infty)/K_\alpha(\infty),$$

it admits a reasonable description (see [22], Section IX.4), and the semigroup $\Gamma_\alpha(\infty)$ acts in subspaces of $K_\alpha(\infty)$-fixed vectors of unitary representations of $G_\alpha(\infty) = GL(\infty, \mathbb{R})$. Moreover, this situation is typical for infinite dimensional groups and allows to produce unconventional algebraic structures (see, e.g., [26]).

In the context of the present paper, we have increasing groups $G(n)$ and constant compact subgroup $K$, for this reason we have embeddings $\mathcal{M}(G(n)/K) \rightarrow \mathcal{M}(G(n + 1)/K)$ and a direct limit $\mathcal{M}(G(\infty)/K)$. Moreover, we have comprehensive prelimit algebras $\mathcal{M}(G(n)/K)$. This situation is unusual among infinite-dimensional groups whose representation theories were topic of considerations of mathematicians. Certainly, additionally there is a group GL of two-side-infinite almost triangular matrices over $F_q$ and its symplectic, orthogonal, and 'unitary' subgroups (in all cases the subgroup $K$ consists of upper triangular matrices), the author does not see a way to extend this list.

2 Algebras of bi-invariant measures. Generalities

2.1. Reconstruction of a representation of $G$. To avoid an appearance of exotic measures let us fix a class of topological groups under considerations. Recall that a topological space is Polish if it is homeomorphic to a complete metric space. A topological group is Polish if it is a Polish topological space.
Consider a sequence
\[ G_1 \subset G_2 \subset \ldots \]
of Polish groups. We say that the group
\[ G := \bigcup G_j =: \lim_{\to} G_j \]
is their direct limit. We equip \( G \) with the topology of a direct limit, a set \( U \subset G \) is open if all intersections \( U \cap G_j \) are open in \( G_j \). Generally speaking such topologies are not metrizable. If a subset \( K \subset G \) is compact, then \( K \subset G_j \) for some \( j \).

**Remark.** A Polish group \( G \) is a direct limit of Polish groups, \( G_j = G \).

Let \( G = \lim G_j \) be a direct limit of Polish groups. Denote by \( M(G) \) the algebra of all compactly supported Borel complex-valued measures on \( G \), this algebra is a direct limit of algebras
\[ M(G) = \lim_{\to} M(G_j), \]

Notice that such measures are objects of classical measure theory on compact metrizable spaces (see, e.g. [33], Chapter 4). We say that a sequence \( \mu_j \in M(G) \) converges to \( \mu \) if there is a compact subset \( L \subset G \) such that \( \mu_j(G \setminus L) = 0 \) for all \( j \) and \( \mu_j \) weakly converge to \( \mu \) on \( L \).

Let \( K \subset G \) be a compact subgroup, without loss of generality we can assume \( K \subset G_1 \). Denote by \( M(G//K) \subset M(G) \) the subalgebra of \( K \)-bi-invariant compactly supported measures.

**Proposition 2.1** Let \( G \) be a direct limit of Polish groups. If a \( \ast \)-representation \( \tau \) of the algebra \( M(G//K) \) in a Hilbert space \( V \) can be represented as \( \text{Res}_{G/K}(\rho) \) and \( \text{Res}_{G/K}(\rho') \), then \( \rho \simeq \rho' \).

We formulate a stronger version of the statement including a way of a reconstruction of \( \rho \). Consider the homogeneous space \( K \setminus G \), denote by \( z_0 \) the base point of this space, i.e., the coset \( K \cdot 1 \). Define a kernel \( \mathcal{L}(z,u) \) on \( K \setminus G \times K \setminus G \) taking values in bounded operators \( V \to V \) by
\[ \mathcal{L}(z, u) = \tau(\delta_K \ast h g^{-1} \ast \delta_K), \]
where \( z_0 g = x, z_0 h = y \) (the result does not depend on a choice of \( g, h \)). Notice that the kernel is \( G \)-invariant,
\[ \mathcal{L}(xr, yr) = \mathcal{L}(x, y), \quad \text{for } r \in G. \]

Consider the space \( \Delta(K \setminus G, \tau) \) of finitely supported functions \( K \setminus G \to V \). For a vector \( v \in V \) denote by \( v \delta_z(x) \) the function, which equals \( v \) at the point \( z \) and 0 at over points. So, our space \( \Delta(K\setminus G) \) consists of finite linear combinations
\[ \sum_{j=1}^{n} v_j \delta_{z_j}(x). \]
The group $G$ acts in the space $\Delta(K \backslash G, \tau)$ by shifts of the argument. Define a sesquilinear linear form on $\Delta(K \backslash G, \tau)$ setting

$$\left\langle \sum v_i \delta_{a_i}, \sum w_j \delta_{b_j} \right\rangle = \sum_{i,j} \langle L(a_i, b_j) v_i, w_j \rangle_V.$$  \hfill (2.1)

If for all vectors $\sum v_i \delta_{a_i}$ we have

$$\left\langle \sum v_i \delta_{a_i}, \sum v_i \delta_{a_i} \right\rangle = \sum_{i,j} \langle L(a_i, a_j) v_i, v_j \rangle_V \geq 0,$$  \hfill (2.2)

then $\langle \cdot, \cdot \rangle$ is an inner product, we get a structure of a pre-Hilbert space and take the corresponding Hilbert space $\Delta(K \backslash G, \tau)$. Since the kernel is $G$-invariant, shifts on $K \backslash G$ induce unitary operators in $\Delta(K \backslash G, \tau)$.

**Proposition 2.2** Let $G$ be a direct limit of Polish groups. Let $\tau = \text{Res}_{G/\!\!/K}(\rho)$. Then (2.2) holds and the representation of $G$ in $\Delta(K \backslash G, \tau)$ is equivalent to $\rho$.

**Proof.** Consider a unitary representation $\rho \in \text{URep}(G)_{K}$ in a Hilbert space $H$ and the corresponding representation $\tau$ of $\mathcal{M}(G/\!\!/K)$ in $V := H^K$. We consider the map $J : \sum_i v_i \delta_{z_0 g_i} \mapsto \rho(g_i^{-1})v_i$.  \hfill (2.3)

We have

$$\langle \rho(g_i^{-1})v, \rho(h_j^{-1})w \rangle_H = \langle \rho(g_i^{-1}) \rho(\delta_K)v, \rho(h_j^{-1}) \rho(\delta_K)w \rangle_H =$$

$$= \langle \rho(\delta_K) \rho(h) \rho(g_i^{-1}) \rho(\delta_K)v, w \rangle_H = \langle \rho(\delta_K \ast hg^{-1} \ast \delta_K)v, w \rangle_H =$$

$$= \langle \rho(\delta_K \ast hg^{-1} \ast \delta_K)v, w \rangle_{H^K}.$$

Therefore

$$\left\langle J \left( \sum_i v_i \delta_{z_0 g_i} \right), J \left( \sum_j w_j \delta_{z_0 g_j} \right) \right\rangle_H = \left\langle \sum_i \rho(g_i^{-1})v_i, \sum_j \rho(h_j^{-1})w_j \right\rangle_H =$$

$$= \sum_{i,j} \langle \rho(\delta_K \ast h_j g_i^{-1} \ast \delta_K)v_i, w_j \rangle_{H^K} =$$

$$= \sum_{i,j} \langle \mathcal{L}(z_0 g_i, z_0 h_j)v_i, w_j \rangle_{H^K}.$$

So the map $J$ induces an inner product on the space $\Delta(K \backslash G, \tau)$, and this inner product coincides with the sesquilinear form (2.1). So (2.2) is positive and the map (2.3) determines a unitary operator $\Delta(K \backslash G, \tau) \to H$. \hfill $\square$

**Remark.** If $\tau$ is an arbitrary $*$-representation of $\mathcal{M}(G/\!\!/K)$, then we can repeat the definition of the space $\Delta(K \backslash G, \tau)$, but positivity (2.2) of an operator-valued reproducing kernel $\mathcal{L}(x, y)$ can be a heavy problem.

### 2.2. The case of compact groups.
Proposition 2.3 For a compact group \( L \) with a countable base any \( \ast \)-representation \( \tau \) of the algebra \( \mathcal{C}(L/\!/K) \) has the form \( \text{Res}_{L/\!/K}(\rho) \).

Corollary 2.4 The same statement holds for the algebra \( \mathcal{M}(L/\!/K) \).

Proof. Consider the set \( \hat{L} \) of all irreducible representations \( \rho_\alpha \) of the group \( L \) defined up to equivalence\(^{12} \) denote by \( H_\alpha \) (finite-dimensional) spaces of these representations, by \( \text{Mat}(H_\alpha) \) the algebras of all operators in \( H_\alpha \).

Consider the Fourier transform on \( L \), see, e.g., [17], Subsect. 12.2. Namely, for \( f \in \mathcal{C}(L) \) the operator-valued function

\[
\mathcal{F} f(\alpha) := \rho_\alpha(f) \in \text{Mat}(H_\alpha), \quad \text{where } \alpha \in \hat{L},
\]

is called the Fourier transform \( \mathcal{F} f \) of \( f \). By the definition, the Fourier transform sends convolutions to point-wise products. The \( \mathcal{F} \) is injective, a convenient descriptions of the Fourier-image of \( \mathcal{C}(G) \) and of the induced convergence in the Fourier-image are unknown. In any case, functions \( \alpha \mapsto \|\rho_\alpha(f)\| \) are bounded; the image of the Fourier transform contains the set of all finitary functions (i.e., functions whose elements are zeros for all but a finite number of \( \alpha \)). Moreover, finitary functions are dense in the Fourier-image. The uniform convergence of functions \( f_j \to f \) implies convergences \( \rho_\alpha(f_j) \to \rho_\alpha(f) \).

Denote by \( \chi_\beta \) the character of \( \rho_\beta \). Then

\[
\mathcal{F} \chi_\beta(\alpha) = \rho_\alpha(\chi_\beta) = \begin{cases} (\dim H_\beta)^{-1} \cdot 1, & \text{if } \alpha = \beta; \\ 0 & \text{otherwise}. \end{cases}
\]

Next, consider the subset \( \hat{L}_K \subset \hat{L} \) consisting of representations having non-zero \( K \)-fixed vectors. Define the Fourier transform \( \mathcal{F}_K \) on \( \mathcal{C}(L/\!/K) \) as operator-valued function

\[
\mathcal{F}_K f(\alpha) := \tilde{\rho}_\alpha(f) \in \text{Mat}(H^K_\alpha).
\]

These operators are left upper blocks of matrices (1.1). Properties of \( \mathcal{F} \) can easily translated to the corresponding properties of \( \mathcal{F}_K \). In particular, the Fourier-image \( \mathcal{F}_K(\mathcal{C}(L/\!/K)) \) contains a copy of each algebra \( \text{Mat}(H^K_\alpha) \), it consist of functions supported by one point \( \alpha \). Notice that elements

\[
\zeta_\beta := \dim(H_\beta) \cdot \chi_\beta \ast \delta_K \in \mathcal{C}(G)
\]

are commuting idempotents,

\[
\zeta_\beta \ast \zeta_\gamma = \begin{cases} \zeta_\beta, & \text{if } \beta = \gamma; \\ 0 & \text{otherwise}. \end{cases}
\]

Their Fourier-images are

\[
\mathcal{F}_K \zeta_\beta(\alpha) = \begin{cases} 1, & \text{if } \alpha = \beta; \\ 0 & \text{otherwise}. \end{cases}
\]

\(^{12}\)This set is finite if the group \( L \) is finite, otherwise it is countable.
Now consider a $\ast$-representation $\tau$ of the algebra $\mathcal{C}(G/K) \simeq \mathcal{F}_K(\mathcal{C}(G/K))$ in a Hilbert space $V$. Then operators $\tau(\zeta_\alpha)$ are commuting projectors, denote by $V_\alpha$ their images, these subspaces are pairwise orthogonal. Moreover, $V = \bigoplus \alpha V_\alpha$.

Indeed, let $v \in (\bigoplus \alpha V_\alpha)^\perp$. Then it is annihilated by all subalgebras $\text{Mat}(H^K_\alpha)$ in $\mathcal{F}_K(\mathcal{C}(G/K))$. But their sum is dense in the Fourier-image, therefore it is annihilated by the whole algebra $\mathcal{F}_K(\mathcal{C}(G/K))$. Thus, $v = 0$.

Thus, it is sufficient to construct a desired extension for each summand $V_\alpha$. So, without loss of generality we can assume $V_\alpha = V_\alpha$. Then for all $\beta \neq \alpha$ operators $\tau(\zeta_\beta)$ are zero and $\tau$ is zero on each $\text{Mat}(H^K_\beta)$. So $\mathcal{C}(L/K)/\ker \tau \simeq \text{Mat}(H^K_\alpha)$.

Any representation of a matrix algebra is a direct sum of irreducible tautological representations. By definition, each summand arises from the representation $\rho_\alpha$ of $L$.

**2.3. The case of direct limits of compact groups.**

**Proposition 2.5** Let $L$ be a direct limit $L := \lim \rightarrow L_j$ of compact groups, let all $L_j$ have countable bases of topology. Then any $\ast$-representation $\tau$ of the algebra $\mathcal{M}(L/K)$ has the form $\text{Res}_{L/K}(\rho)$.

**Proof.** It is sufficient to check the positivity of (2.2). Fix an expression $\sum a_i \delta_{a_i}$. Since the summation is finite, we have only finite collection $\{a_j\} \subset K \setminus L$. Therefore the subset is contained in some $K \setminus L_j$. But the prelimit group $L_j$ is compact, and we can apply Proposition 2.3.

**3 Characters of $\mathcal{H}_\infty(q)$**

**3.1. An algebra of operators in a tensor power.** Consider the Hilbert space $V$ as in Subsect. 1.7 equipped with the basis $v_j$, the and the copy $W$ of $V$. Consider the tensor product $V \otimes W$ and a unit vector

$$\xi := \sum_{j \neq 0} a_j^{1/2} v_j \otimes w_j,$$

where $a_j \geq 0$, $\sum a_j = 1$.

Consider the tensor power

$$(V \otimes W)^\otimes n \simeq V^\otimes n \otimes W^\otimes n$$

and the vector $\Xi := \xi^\otimes n \in (V \otimes W)^n$. Denote elements of the natural orthonormal basis in our space by

$$\eta \left[ \begin{array}{c} i_1 \\ \vdots \\ i_n \end{array} \right] := (v_{i_1} \otimes v_{j_1}) \otimes \cdots \otimes (v_{i_n} \otimes v_{j_n}),$$

in this notation

$$\Xi = \sum_{i_1, \ldots, i_n \in \mathbb{Z} \setminus 0} \prod_{k=1}^n a_k^{1/2} \cdot \eta \left[ \begin{array}{c} i_1 \\ \vdots \\ i_n \end{array} \right].$$
We say that an operator in $(V \otimes W)^{\otimes n}$ is $V$-diagonal if it has the form
\[ D^V(\Phi) \eta \left[ \begin{array}{cccc} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{array} \right] = \Phi(i_1, \ldots, i_n) \eta \left[ \begin{array}{cccc} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{array} \right], \]
where $\Phi(i_1, \ldots, i_n)$ is a bounded function $(\mathbb{Z} \setminus 0)^n \to \mathbb{C}$. For $\sigma \in S_n$ denote by $T^V(\sigma)$ the permutation of factors $V$ in $(V \oplus W)^n$ corresponding to $\sigma$,
\[ T(\sigma) \eta \left[ \begin{array}{cccc} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{array} \right] = \eta \left[ \begin{array}{cccc} i_{\sigma^{-1}(1)} & \cdots & i_{\sigma^{-1}(n)} \\ j_1 & \cdots & j_n \end{array} \right]. \]

Denote by $A^V$ the algebra of operators in $(V \otimes W)^{\otimes n}$ generated by permutations $T^V(\sigma)$ and operators $D^V(\Phi)$. Any element of this algebra can be represented as a linear combination of the form
\[ \sum_{\sigma \in S_n} T^V(\sigma)D^V_\sigma(\Phi) \text{ or } \sum_{\sigma \in S_n} D^V_\sigma(\tilde{\Phi})T^V(\sigma). \]

In the same way we define an algebra $A^W$, it consists of similar operators acting on the factors $W$. These two algebras commute. The map
\[ \sum_{\sigma \in S_n} T^V(\sigma)D^W_\sigma \to \sum_{\sigma \in S_n} T^W(\sigma)D^W_\sigma. \]

We define a linear anti-automorphism (‘transposition’) in $A^V$ by
\[ \left( \sum_{\sigma \in S_n} T^V(\sigma)D^V_\sigma(\Phi) \right)^t = \sum_{\sigma \in S_n} D^V_\sigma(\Phi)T^V(\sigma^{-1}). \]

**Proposition 3.1**

a) For any elements $A^V, B^V \in A^V$ we have
\[ \langle A^V B^V \Xi, \Xi \rangle = \langle B^V A^V \Xi, \Xi \rangle. \] (3.1)

b) For any element $B^W \in A^W$ we have
\[ B^W \Xi = (B^V)^t \Xi. \] (3.2)

c) For any elements $A^V \in A^V, B^W \in A^W$,
\[ \langle A^V B^W \Xi, \Xi \rangle = \langle (B^V)^t A^V \Xi, \Xi \rangle. \] (3.3)

So we come to the case of Hilbert algebras discussed in Subsect. 1.5. The function
\[ \chi(A^V) = \langle A^V \Xi, \Xi \rangle \]
is a trace on the algebra $A^V$ (an explicit formula is (3.4)). The $A^V$-cyclic span of $\Xi$ is identified with $A^V_n$, the map $A^V \to A^V_n$ is $A^V \to A^V \Xi$. The right action of our algebra on $A^V_n$ is given by $X \mapsto X(A^V)^t$.  

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The construction of Subsect. 1.7 gives an embedding
\[ \mathcal{H}_\infty(q) \to \mathcal{A}^V \]
and an action of \[ \mathcal{H}_\infty(q) \otimes \mathcal{H}_\infty(q) \] on the \[ \mathcal{A}_V^\chi \). In Subsect. 3.3 we will show that this embedding induces the Vershik–Kerov trace.

3.2. Proof of Proposition 3.1. We use notation \[ I := (i_1, \ldots, i_n), J := (j_1, \ldots, j_n) \]. For \[ \sigma \in S_n \] denote by \[ \Omega(\sigma) \] the set of all \[ I \] invariant with respect to \[ \sigma \]. In other words we decompose \[ \sigma \] into a product of independent cycles, and map \[ k \mapsto i_k \] is constant on cycles.

Lemma 3.2

\[
\left\langle T^V(\sigma)D^V(\Phi)\Xi, \Xi \right\rangle = \sum_{I = (i_1, \ldots, i_n) \in \Omega(\sigma)} \left( \prod_{k=1}^n a_{i_k} \cdot \Phi(I) \right) = \left\langle D^V(\Phi)T^V(\sigma)\Xi, \Xi \right\rangle.
\]

\[(3.4)\]

**Proof.** We have

\[
\left\langle T^V(\sigma)D^V(\Phi)\Xi, \Xi \right\rangle = \sum_I \sum_J \Phi(i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(n)}) \prod_{k=1}^n a_{i_k}^{1/2} \prod_{k=1}^n a_{j_k}^{1/2} \times \left\langle \eta^{i_{\sigma^{-1}(1)}} \cdots i_{\sigma^{-1}(n)}, \eta^{j_1} \cdots j_n \right\rangle.
\]

\[(3.5)\]

A summand can be non-zero only if two basis elements in the brackets \[ \left\langle \cdot, \cdot \right\rangle \] coincide. So \[ i_k = j_k, i_{\sigma^{-1}(k)} = j_k \]. Therefore \[ \sum_I \sum_J \] transforms to \[ \sum_{I \in \Omega(\sigma)} \]. Since \[ I \in \Omega(\sigma) \], we can replace \[ \Phi(\cdot) \] by \[ \Phi(I) \]. Since \[ J = I \], the product \[ \prod_k (\ldots) \prod_k (\ldots) \] replaces by \[ \prod_k a_{i_k} \].

The expression for \[ \left\langle D^V(\Phi)T^V(\sigma)\Xi, \Xi \right\rangle \] is similar to \[ (3.5) \], we only must replace the boxed \[ \Phi(\cdot) \] by \[ \Phi(i_1, \ldots, i_n) \]. \[ \Box \]

Lemma 3.3

\[
\left\langle (T^V(\sigma)D^V(\Phi))D^V(\Theta)\Xi, \Xi \right\rangle = \left\langle D^V(\Theta)(T^V(\sigma)D^V(\Phi))\Xi, \Xi \right\rangle.
\]

\[(3.6)\]

**Proof.** To obtain the left hand side of \[ (3.6) \], we must replace the boxed \[ \Phi(\cdot) \] in the calculation \[ (3.5) \] by

\[ \Phi(i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(n)}) \Theta(i_{\sigma^{-2}(1)}, \ldots, i_{\sigma^{-2}(n)}) \],

in the right hand side by

\[ \Theta(i_1, \ldots, i_n) \Phi(i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(n)}) \].

Since \[ I \in \Omega(\sigma) \], we will get the same result in both sides. \[ \Box \]
Lemma 3.4

\[ \left\langle T^\lambda (\lambda^{-1}) T^{\sigma}(\Phi) T^\lambda (\Xi), \Xi \right\rangle = \left\langle T^\lambda (\sigma) D^\lambda (\Phi) \Xi, \Xi \right\rangle. \]  

(3.7)

PROOF. The left-hand side equals to

\[ \sum_I \sum_J \Phi(i_{\lambda(1)}^{\sigma-1}, \ldots, i_{\lambda(n)}^{\sigma-1}) \prod_{k=1}^n a_{i_{\lambda-1\sigma-1\lambda(k)}} \prod_{k=1}^n a_j \times \]

\[ \times \left\langle \eta \left[ \begin{array}{cccc} i_{\lambda-1\sigma-1\lambda(1)} & \ldots & i_{\lambda-1\sigma-1\lambda(n)} \\ i_1 & \ldots & i_n \end{array} \right], \eta \left[ \begin{array}{cccc} j_1 & \ldots & j_n \end{array} \right] \right\rangle. \]

As in proof of Lemma (3.4), we come to

\[ \sum_{I \in \Omega(\lambda^{-1}\sigma-1\lambda)} \Phi(i_{\lambda-1\sigma-1\lambda(1)}, \ldots, i_{\lambda-1\sigma-1\lambda(n)}) \prod_{k=1}^n a_j. \]

(we applied Lemma 3.2). Next, we pass to a new index of summation

\[ I' := (i_{\lambda(1)}, \ldots, i_{\lambda(n)}), \quad I' \in \Omega(\sigma). \]

Keeping in mind Lemma 3.2 we get the desired expression in the right hand side. □

The statement a) of Lemma 3.1 follows from Lemmas 3.3 and 3.4.

PROOF OF THE STATEMENT b) OF PROPOSITION 3.1 We must verify the identity

\[ T^W (\lambda) D^W (\Psi) \Xi = D^V (\Psi) T^V (\lambda^{-1}) \Xi. \]

In the left hand-side we have

\[ \sum_I \sum_{j \in \Omega(\lambda^{-1}\sigma-1\lambda)} \Phi(i_{\lambda-1\sigma-1\lambda(1)}, \ldots, i_{\lambda-1\sigma-1\lambda(n)}) \prod_{k=1}^n a_j. \]

We pass to a new index of summation

\[ I' := (i_{\lambda-1\sigma-1\lambda(1)}, \ldots, i_{\lambda-1\sigma-1\lambda(n)}), \]

and get the right hand side. □

The statement c) of Proposition 3.1 follows from b).

3.3. The matrix element of \( \zeta_m \).

Lemma 3.5 Let \( H_n \) acts in \((V \otimes W)^\otimes n\) as in Subsect. 2.3. Then

\[ \langle \zeta_m \Xi, \Xi \rangle = \chi^{\alpha,\beta,\gamma}(\zeta). \]
Lemma 3.6 Decompose the $R$-matrix as

$$R := Q + D,$$

where

$$Q := -\sqrt{q} \sum_{i \neq j, i \neq 0, j \neq 0} e_{ji} \otimes f_{ij};$$

$$D := -\sum_{i < 0} e_{ii} \otimes f_{ii} + q \sum_{i > 0} e_{ii} \otimes f_{ii} + (q - 1) \sum_{i < j, i \neq 0, j \neq 0} e_{ij} \otimes f_{ij}.$$

Then

$$R_{(m-1)m} \cdots R_{23} R_{12} \Xi = D_{(m-1)m} \cdots D_{23} D_{12} \Xi.$$

Proof of Lemma 3.6 Denote

$$a_i := \begin{cases} \beta_i, & \text{for } i < 0; \\ \alpha_i, & \text{for } i > 0. \end{cases}$$

The operators $D_{j(j+1)}$ are $V$-diagonal. Next,

$$Q_{12} \Xi = -\sqrt{q} \sum_{I = (i_1, \ldots, i_m); i_1 \neq i_2} \prod_{k} a_{i_k}^{1/2} \eta \begin{bmatrix} i_2 & i_1 & i_3 & \cdots & i_n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{bmatrix}.$$

The operators $R_{23}, R_{34}, \ldots$ can not change $i_2$ in the first column of $\eta[\cdot]$, recall also that such operators do not act on the second row of $\eta[\cdot]$. Therefore all terms of $R_{(m-1)m} \cdots R_{23} Q_{12} \Xi$ have form

$$c, \eta \begin{bmatrix} \alpha_1^* & \cdots & \alpha_m^* \\ \beta_1^* & \cdots & \beta_m^* \end{bmatrix},$$

where $\alpha_1^* \neq \beta_1^*$. Inner product of such a term with $\Xi$ is 0, i.e.,

$$\langle R_{(m-1)m} \cdots R_{23} Q_{12} \Xi, \Xi \rangle = 0,$$

hence

$$\langle R_{(m-1)m} \cdots R_{23} R_{12} \Xi, \Xi \rangle = \langle R_{(m-1)m} \cdots R_{23} D_{12} \Xi, \Xi \rangle.$$

We repeat the same argument for $R_{23}, R_{34}, \ldots$ and get the desired statement.

$\square$

The operator $D_{(m-1)m} \cdots D_{23} D_{12}$ is $V$-diagonal. Denote by $\delta[\cdots]$ its eigenvalues

$$D_{(m-1)m} \cdots D_{23} D_{12} \eta \begin{bmatrix} i_1 & \cdots & i_m & \cdots & i_n \\ j_1 & \cdots & j_m & \cdots & j_n \end{bmatrix} = \delta(i_1, \ldots, i_m) \eta \begin{bmatrix} i_1 & \cdots & i_m & \cdots & i_n \\ j_1 & \cdots & j_m & \cdots & j_n \end{bmatrix}.$$
Lemma 3.7

\[
\langle \zeta_m \Xi, \Xi \rangle = \sum_{i_1, \ldots, i_m} \left( \delta(i_1, \ldots, i_m) \prod_{k=1}^{m} a_{i_k} \right). \tag{3.8}
\]

**Proof.** A straightforward summation gives

\[
\sum_{i_1, \ldots, i_n} \left( \delta(i_1, \ldots, i_m) \prod_{k=1}^{m} a_{i_k} \right)
\]

We transform this expression as

\[
\sum_{i_1, \ldots, i_m} \sum_{i_{m+1}, \ldots, i_n} \left( \delta(i_1, \ldots, i_m) \prod_{k=1}^{m} a_{i_k} \prod_{k=m+1}^{n} a_{i_k} \right) = \sum_{i_1, \ldots, i_m} \left( \delta(i_1, \ldots, i_m) \prod_{k=1}^{m} a_{i_k} \right) \times \left( \sum_{i} a_{i} \right)^{m-n}.
\]

But under our conditions \( \sum a_i = \sum \beta_p + \sum \alpha_q = 1. \)

Next, \( \delta(i_1, \ldots, i_n) \) is non-zero iff \( i_1 \leq \ldots \leq i_n \). The next statement also is obvious:

**Lemma 3.8** Let a tuple \( I: i_1 \leq \ldots \leq i_n \) consists of entries \((-\nu_1), \ldots, (-\nu_1) < 0\) with nonzero multiplicities \( \mu_1, \ldots, \mu_u \) and entries \( \nu_1, \ldots, \nu_v \) with nonzero multiplicities \( \nu_1, \ldots, \nu_v \). Then

\[
\delta(i_1, \ldots, i_n) = (-1)^{\sum \mu_i - 1} q^{\sum \nu_i - 1} (q - 1)^{u + v - 1}. \tag{3.9}
\]

**Corollary 3.9**

\[
\langle \zeta_m \Xi, \Xi \rangle = \frac{1}{q-1} \sum_{\varphi_1 \geq 0, \varphi_2 \geq 0, \ldots, \nu_1 \geq 0, \nu_2 \geq 0, \ldots} \prod_{i=1}^{\infty} \left( (-\beta_i)^{\varphi_i} (1 - q)^{\varepsilon(\varphi_i)} \right) \times
\]

\[
\prod_{j=1}^{\infty} \left( q_{\alpha_j} \psi_j (1 - q^{-1})^{\varepsilon(\psi_j)} \right), \tag{3.10}
\]

where

\[
\varepsilon(\theta) := \begin{cases} 0, & \text{if } \theta = 0; \\ 1, & \text{if } \theta > 0. \end{cases}
\]

**Proof.** We transform the expression (3.9) for \( \delta(\ldots) \) as

\[
\frac{1}{q-1} \prod_{k} \left( (-1)^{\mu_k} (1 - q) \right) \cdot \prod_{l} \left( q^{\nu_l} (1 - q^{-1}) \right).
\]
A straightforward summation in formula (3.8) gives an expression
\[
\langle \zeta_m \Xi, \Xi \rangle = \frac{1}{q-1} \sum_{u,v:u+v \geq 1} \sum_{\epsilon_1<\cdots<\epsilon_u, v_1<\cdots<v_v} \sum_{\mu_1,\ldots,\mu_u,\nu_1,\ldots,\nu_v} \prod_{k=1}^{u} \left( (-\beta_{\epsilon_k})^{\mu_k} (1-q) \right) \prod_{k=1}^{v} \left( (q\alpha_{\nu_l})^{\nu_l} (1-q^{-1}) \right).
\]

We change a parametrization of the set of summation assuming
\[
\varphi_{i_k} = \mu_k \text{ and } \varphi_i = 0 \text{ for all other } i;
\psi_{i_l} = \nu_l \text{ and } \psi_j = 0 \text{ for all other } l,
\]
and get (3.10), all new factors in the products in (3.10) are 1. □

**Lemma 3.10** The generating function
\[
G(z) := 1 + (q - 1) \left( z + \sum_{m \geq 2} \langle \zeta_m \Xi, \Xi \rangle z^m \right)
\]
is equal to
\[
G(z) := \prod_{i=1}^{\infty} \frac{1 + \beta_i qz}{1 + \beta_i z} \prod_{j=1}^{\infty} \frac{1 - \alpha_j z}{1 - \alpha_j qz} \tag{3.11}
\]

**Proof.** We represent the term \((q - 1)z\) as
\[
(q - 1)z = (q - 1) \left( \sum_j \beta_j + \sum_k \alpha_k \right) z = \sum_j (-\beta_j z)(1-q) + \sum_k (\alpha_k qz)(1-q^{-1}).
\]
Applying (3.10) we come to the following expression for the generating function:
\[
G(z) = \sum_{\varphi_1 \geq 0, \varphi_2 \geq 0, \ldots, \psi_1 \geq 0, \psi_2 \geq 0, \ldots} \prod_{i=1}^{\infty} \left( (-\beta_i z)^{\varphi_i} (1-q)^{\xi(\varphi_i)} \right) \times
\]
\[
\times \prod_{j=1}^{\infty} \left( (q\alpha_j z)^{\psi_j} (1-q^{-1})^{\xi(\psi_j)} \right).
\]
It decomposes into a product
\[
\prod_{i=1}^{\infty} \left( \sum_{\varphi_i} (-\beta_i z)^{\varphi_i} (1-q)^{\xi(\varphi_i)} \right) \prod_{j=1}^{\infty} \left( \sum_{\psi_j} (q\alpha_j z)^{\psi_j} (1-q^{-1})^{\xi(\psi_j)} \right) =
\]
\[
= \prod_{i=1}^{\infty} \left( 1 + (1-q)(1+\beta_i z)^{-1} - 1 \right) \prod_{j=1}^{\infty} \left( 1 + (1-q^{-1})(1-q\alpha_j z)^{-1} - 1 \right).
\]
This equals to (3.11). □
Proof of Lemma 3.5. We must verify the identity

\[ G(z) = 1 + (q - 1)z + \sum_{m \geq 2} z^m \chi_{\alpha, \beta}^0(z_m), \]

see (1.7). We have

\[ G(z) = \exp\{\ln G(z)\} = \exp\left\{ \sum_i \left( \ln(1 + \beta_i qz) - \ln(1 + \beta_i z) \right) + \sum_j \left( \ln(1 - \alpha_j z) - \ln(1 - \alpha_j qz) \right) \right\} = \exp\left\{ \sum_i \sum_k \frac{(-1)^{k-1}}{k} \beta_i^k (q^k - 1)z^k + \sum_j \sum_k \frac{1}{k} \alpha_j^k (q^k - 1)z^k \right\} \]

\[ = \prod_{k=1}^\infty \exp\left\{ \sum_j \left( \alpha_j^k + (-1)^{k-1} \sum_i \beta_i^k \right) \cdot \frac{1}{k} (q^k - 1)z^k \right\}. \]

It remains to decompose exponentials. \( \square \)

3.4. Proof of Theorem 1.5. Obviously, \( R_{i(i+1)}^{\text{left}} \) are contained in the algebra \( \mathcal{A}^V \) introduced in Subsect. 3.1. Therefore operators of the representation \( \mathcal{H}_n(q) \) are contained in \( \mathcal{A}^V \). By Proposition 3.1.a, the matrix element \( \langle A \Xi, \Xi \rangle \) is a trace on each \( \mathcal{H}_n \). Therefore it is a trace on \( \mathcal{H}_\infty(q) \). The properties (1.5)–(1.6) are obvious. After these remarks the statement a) of Theorem 1.5 follows from Lemma 3.5 and the statement b) from Proposition 3.1.b.

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Yury Neretin
Math. Dept., University of Vienna
&Institute for Theoretical and Experimental Physics (Moscow);
&MechMath Dept., Moscow State University;
&Institute for Information Transmission Problems;
yuri.neretin@math.univie.ac.at
URL: [http://mat.univie.ac.at/~neretin/](http://mat.univie.ac.at/~neretin/)