A conjectural Lefschetz formula for locally symmetric spaces

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Introduction

The theory of the Selberg Zeta Function is a vital part of the spectral geometry of locally symmetric spaces, see [2, 4, 5, 6, 7, 8, 9, 10, 11, 15, 16, 17, 18, 19, 22, 23, 25]. For higher rank spaces there is no zeta function. If the space is compact, the Lefschetz formula [12, 25] seems to be an appropriate replacement as applications show [13].

In this paper we suggest a Lefschetz formula for non-compact finite volume spaces and we prove it in the case of Riemann surfaces by exploiting the properties of the Selberg zeta function. This way of proof might be extended to rank one spaces, but for higher rank a new idea is required.

1 Global Lefschetz numbers

Let \( G \) denote a connected semisimple Lie group with finite center. Fix a maximal compact subgroup \( K \) with Cartan involution \( \theta \). So \( \theta \) is an automorphism of \( G \) with \( \theta^2 = \text{Id} \) and \( K \) is the set of all \( x \in G \) with \( \theta(x) = x \).

Let \( g_\mathbb{R}, k_\mathbb{R} \) denote the real Lie algebras of \( G \) and \( K \) and let \( g, k \) denote their complexifications. This will be a general rule: for a Lie group \( H \) we denote by \( h_\mathbb{R} \) the Lie algebra of \( H \) and by \( h = h_\mathbb{R} \otimes \mathbb{C} \) its complexification. Let \( b : g \times g \to \mathbb{C} \) be a positive multiple of the Killing form. On \( G, K \) and all parabolic subgroups as well as all Levi-components we install Haar measures given by the form \( b \) as in [20].

Let \( H \) be a non-compact Cartan subgroup of \( G \). Modulo conjugation we can assume that \( H = AB \) where \( A \) is a connected split torus and \( B \) is a closed subgroup of \( K \). Fix a parabolic \( P \) with split component \( A \). Then \( P \) has Langlands decomposition \( P = MAN \) and \( B \) is a Cartan subgroup of \( M \). Note that an arbitrary parabolic subgroup \( P' = M'A'N' \) of \( G \) occurs in this way if and only if the group \( M' \) has a compact Cartan subgroup. In this case we say that \( P' \) is a cuspidal parabolic.

The choice of the parabolic \( P \) amounts to the same as a choice of a set of positive roots \( \Phi^+(g, a) \) in the root system \( \Phi(g, a) \). The Lie algebra \( n \) of the unipotent radical \( N \) can be described as \( n = \bigoplus_{\alpha \in \Phi^+(g, a)} g_\alpha \), where \( g_\alpha \) is the root space attached to \( \alpha \), i.e., \( g_\alpha \) is the space of all \( X \in g \) such that \( \text{ad}(Y)X = \alpha(Y)X \) holds for every \( Y \in a \). Define \( \tilde{n} = \bigoplus_{\alpha \in \Phi^+(g, a)} g_{-\alpha} \). This is the opposite Lie algebra. Let \( \tilde{n}_\mathbb{R} = \tilde{n} \cap g_\mathbb{R} \) and \( \tilde{N} = \exp(\tilde{n}_\mathbb{R}) \). Then \( \tilde{P} = MAN \) is the opposite parabolic to \( P \).
Let $a^*$ denote the dual space of $a$. Since $A = \exp(a_0)$, every $\lambda \in a^*$ induces a continuous homomorphism from $A$ to $\mathbb{C}^*$ written $a \mapsto a^\lambda$ and given by $(\exp(H))^\lambda = e^{\lambda(H)}$. Let $\rho_p \in a^*$ be the half of the sum of all positive roots, each weighted with its multiplicity. So $a^{2\rho_p} = \det(a|n)$. Let $a_0^* \subset a_0$ be the negative Weyl chamber consisting of all $X \in a_0$ such that $\alpha(X) < 0$ for every $\alpha \in \Phi^+(g,a)$. Let $A^\tau = \exp(a_0^\tau)$ be the negative Weyl chamber in $A$. Further let $A^\tau$ be the closure of $A^\tau$ in $A$. This is a manifold with corners.

Let $K_M = M \cap K$. Then $K_M$ is a maximal compact subgroup of $M$ and it contains $B$. Fix an irreducible unitary representation $(\tau,V)$ of $K_M$. Then $V$ is finite dimensional. Let $\hat{\tau}$ be the dual representation to $\tau$.

Let $\hat{G}$ denote the unitary dual of $G$, i.e., it is the set of all isomorphy classes of irreducible unitary representations of $G$. Let $\hat{G}_{adm} \supset \hat{G}$ be the admissible dual. For $\pi \in \hat{G}_{adm}$ let $\pi_K$ denote the $(g,K)$-module of $K$-finite vectors in $\pi$ and let $\Lambda_\pi \in h^*$ be a representative of the infinitesimal character of $\pi$. Let $H^\bullet(n,\pi_K)$ be the Lie algebra cohomology with coefficients in $\pi_K$.

By [21] for each $q$ the $(a \oplus m, K_M)$-module $H^q(n,\pi_K)$ is admissible of finite length, i.e., a Harish-Chandra module.

For $\lambda \in a^*$ and an $A$-module $W$ let $W^\lambda$ denote the generalized $\lambda$-eigenspace, i.e., $W^\lambda$ is the set of all $w \in W$ such that there is $n \in \mathbb{N}$ with

$$(a - a^\lambda)^n w = 0$$

for every $a \in A$. Let $m = \mathfrak{g}_M \oplus \mathfrak{p}_M$ be the Cartan decomposition of the Lie algebra $m$ of $M$. For $\pi \in \hat{G}$ and $\lambda \in a^*$ let $L_\lambda^\pi(\pi)$ denote the representation-theoretic Lefschetz number given by

$$L_\lambda^\pi(\pi) \overset{\text{def}}{=} \sum_{p,q \geq 0} (-1)^{p+q+\dim N} \dim \left( H^q(n,\pi_K)^\lambda \otimes \wedge^p \mathfrak{p}_M \otimes \hat{\tau} \right)^{K_M}.$$

For a given smooth and compactly supported function $f \in C^\infty_c(G)$ we define its Fourier transform $\hat{f} : \hat{G} \to \mathbb{C}$ by

$$\hat{f}(\pi) \overset{\text{def}}{=} \text{tr} \pi(f).$$

**Proposition 1.1** (a) For every $\varphi \in C^\infty_c(A^-)$ there exists $f_\varphi \in C^\infty_c(G)$ such that for every $\pi \in \hat{G}$,

$$\hat{f}_\varphi(\pi) = \sum_{\lambda \in a^*} L_\lambda^\pi(\pi) \hat{\varphi}(\lambda),$$

where $\hat{\varphi}$ is the Fourier transform of $\varphi$, i.e., $\hat{\varphi}(\lambda) = \int_A \varphi(a) a^\lambda \, da$. 

(b) The sum in (a) is finite, more precisely, the Lefschetz number $L^\chi(\pi)$ is zero unless there is an element $w$ of the Weyl group of $(g, h)$ such that
\[
\lambda = (w\Lambda_\pi)_a - \rho_P.
\]

Proof: The proof of part (a) is contained in section 4 of [12], and (b) is a consequence of Corollary 3.32 in [21]. $\square$

2 Local Lefschetz numbers

Let $\Gamma \subset G$ be a discrete subgroup of finite covolume. Let $X = G/K$ be the symmetric space and $X_\Gamma = \Gamma\backslash X = \Gamma\backslash G/K$ be the corresponding locally symmetric quotient. The group $\Gamma$ is called neat if it is torsion-free and for every finite dimensional representation $\rho$ of $G$ and every $\gamma \in \Gamma$ the linear map $\rho(\gamma)$ does not have a root of unity other than 1 for an eigenvalue. Every arithmetic group has a finite index subgroup which is arithmetic and neat [3].

Let $L$ be a unimodular Lie group and $\Gamma$ a lattice if $L$. Let $H \subset L$ be a Lie subgroup such that the Weyl-group $W(L, H)$ is finite. Here $W(L, H)$ is the normalizer of $H$ modulo the centralizer of $H$. Let $b$ be a non-degenerate symmetric bilinear form on the Lie algebra of $L$ which is invariant under $H$. Suppose that there is a preferred Haar-measure $\mu_H$ on $H$. The form $b$ induces an $L$-invariant pseudo-Riemannian structure on $L/H$. The Gauß-Bonnet construction ([14], sect 24) extends to pseudo-Riemannian structures to give an Euler-Poincaré measure $\eta$ on $L/H$. Define a (signed) Haar-measure on $L$ by
\[
\mu_{b,H} = \eta \otimes \mu_H.
\]
Define the $H$-index by
\[
\text{Ind}_H(\Gamma \backslash L) \overset{\text{def}}{=} \frac{1}{|W(L, H)|} \mu_{b,H}(\Gamma \backslash L).
\]

Remarks

- If $L$ is reductive, $H$ a compact Cartan subgroup and $\Gamma$ cocompact and torsion-free, then the $H$-index equals the Euler-characteristic,
\[
\text{Ind}_H(\Gamma \backslash L) = \chi(\Gamma \backslash L / K_L),
\]
where $K_L$ is a maximal compact subgroup of $L$. 
• Assume $L$ reductive, $\Gamma$ neat and $H = AB$ a Cartan subgroup with $A$ central in $L$. Let $C$ be the center of $L$, then $C = AB_C$, where $B_C$ is compact. Let $\Gamma_C = \Gamma \cap C$ and $\Gamma_A = A \cap \Gamma_C B_C$ the projection of $\Gamma_C$ to $A$. Then $\Gamma_A$ is a discrete and cocompact subgroup of $A$. Under these circumstances,

$$\text{Ind}_H(\Gamma \backslash L) = \text{vol}(A/\Gamma_A) \chi(A\Gamma \backslash L/K_L).$$

• Let $G$ as before and let $H = AB$ be a non-compact Cartan subgroup of $G$. Let $\Gamma \subset G$ be neat and let $[\gamma] \in \mathcal{E}_P(\Gamma)$. Assume that $\gamma$ is regular, then

$$\text{Ind}_H(\Gamma_{\gamma} \backslash G_{\gamma}) = \text{vol}(\Gamma_{\gamma} \backslash G_{\gamma}).$$

Let $K_{\gamma} \subset G_{\gamma}$ be a maximal compact subgroup and let $X_{\gamma} = \Gamma \cap G_{\gamma} \backslash G_{\gamma}/K_{\gamma}$ the corresponding modular subvariety of $X_{\Gamma}$. Being semisimple, the element $\gamma$ lies in a Cartan subgroup $H_{\gamma} = A_{\gamma} B_{\gamma}$, where $A_{\gamma}$ is a split connected torus and $B$ is compact. Hence $\gamma = a_{\gamma} b_{\gamma}$. If we assume that $a_{\gamma}$ is a regular element of $A_{\gamma}$, then $A_{\gamma}$ is uniquely determined by $\gamma$.

Back to the notation of the first section let $\mathcal{E}_P(\Gamma)$ denote the set of all conjugacy classes $[\gamma]$ in $\Gamma$ such that $\gamma$ is in $G$ conjugate to an element $a_{\gamma} b_{\gamma}$ of $A^{-1} B$. Then there is a conjugate $H_{\gamma}$ of $H$ such that $\gamma \in H_{\gamma}$.

For $[\gamma] \in \mathcal{E}_P(\Gamma)$ we define the local Lefschetz number by

$$L^w(\gamma) \overset{\text{def}}{=} \text{Ind}_{H_{\gamma}}(\Gamma_{\gamma} \backslash G_{\gamma}) \frac{\text{tr} \tau(b_{\gamma})}{\text{det}(1 - a_{\gamma} b_{\gamma} | n)}.$$

### 3 The Lefschetz formula

The unitary $G$-representation on $L^2(\Gamma \backslash G)$ decomposes as

$$L^2(\Gamma \backslash G) = L^2_{\text{disc}} \oplus L^2_{\text{cont}},$$

where

$$L^2_{\text{disc}} = \bigoplus_{\pi \in \hat{G}} N_\Gamma(\pi) \pi$$

is a direct sum of irreducibles with finite multiplicities and $L^2_{\text{cont}}$ is a sum of continuous Hilbert integrals. In particular, $L^2_{\text{cont}}$ does not contain any irreducible subrepresentation.
Let $r$ be the dimension of $A$ and let $\alpha_1, \ldots, \alpha_r \in a^*_R$ be the primitive positive roots. Let $a^{*,+}_R = \{t_1\alpha_1 + \cdots + t_r\alpha_r : t_1, \ldots, t_r > 0\}$ be the positive dual cone and let $a^{*,-}_R$ be its closure in $a^*_R$.

For $\mu \in a^*$ and $j \in \mathbb{N}$ let $C^{\mu,j}(A^-)$ denote the space of all functions on $A$ which

- are $j$-times continuously differentiable on $A$,
- are zero outside $A^-$,
- satisfy $|D\varphi| \leq C|a^\mu|$ for every invariant differential operator $D$ on $A$ of degree $\leq j$, where $C > 0$ is a constant, which depends on $D$.

This space can be topologized with the seminorms

$$N_D(\varphi) = \sup_{a \in A} |a^{-\mu}D\varphi(a)|,$$

$D \in U(a)$, $\deg(D) \leq j$. Since the space of operators $D$ as above is finite dimensional, one can choose a basis $D_1, \ldots, D_n$ and set

$$\|\varphi\| = N_{D_1}(\varphi) + \cdots + N_{D_n}(\varphi).$$

The topology of $C^{\mu,j}(A^-)$ is given by this norm and thus $C^{\mu,j}(A^-)$ is a Banach space.

**Conjecture 3.1 (Lefschetz Formula)**

For $\lambda \in a^*$ and $\pi \in \hat{G}_{\text{adm}}$ there is an integer $N_{\Gamma,\text{cont}}(\pi, \lambda)$ which vanishes if $\text{Re}(\lambda) \notin a^{*,+}_R$ and there are $\mu \in A^*$ and $j \in \mathbb{N}$ such that for each $\varphi \in C^{\mu,j}(A^-)$ and with

$$m_\lambda(\pi) \overset{\text{def}}{=} N_{\Gamma}(\pi) + N_{\Gamma,\text{cont}}(\pi, \lambda)$$

we have

$$\sum_{\pi \in \hat{G}} \sum_{\lambda \in a^*} m_\lambda(\pi) L_\lambda^\pi(\pi) \int_A \varphi(a) a^\lambda da = \sum_{[\gamma] \in \hat{E}_F(\Gamma)} L^\gamma(\gamma) \varphi(a_\gamma).$$

Either side of this identity represents a continuous functional on $C^{\mu,j}(A^-)$.

In the following cases the conjecture is known.
(a) The conjecture holds if $\Gamma$ is cocompact. In that case the numbers $N_{\Gamma, \text{cont}}(\pi, \lambda)$ are all zero. This is shown in [12].

(b) In the next section we will prove the conjecture for $G = \text{PSL}_2(\mathbb{R})$.

We will now make the conjecture more precise for congruence subgroups. For this assume that $G = \mathcal{G}(\mathbb{R})$ for some semisimple linear group $\mathcal{G}$ defined over $\mathbb{Q}$. Let $\mathbb{A} = \mathbb{A}_{\text{fin}} \times \mathbb{R}$ be the adele ring over $\mathbb{Q}$. Assume that $\Gamma$ is a congruence subgroup, i.e., there exists a compact open subgroup $K_{\Gamma}$ of $\mathcal{G}(\mathbb{A}_{\text{fin}})$ such that $\Gamma = \mathcal{G}(\mathbb{Q}) \cap K_{\Gamma}$. To explain the conjectured nature of the number $N_{\Gamma, \text{cont}}(\pi)$ we will recall Arthur's trace formula. This formula is the equality of two distributions on $\mathcal{G}(\mathbb{A})$,

$$J_{\text{geom}} = J_{\text{spec}}.$$  

The geometric distribution $J_{\text{geom}}$ can be described in terms of weighted orbital integrals. Our interest however is focused on the spectral distribution $J_{\text{spec}}$. According to [1], Theorem 8.2, one has

$$J_{\text{spec}}(f) = \sum_{\chi} J_{\chi}(f),$$

where $\chi$ runs through conjugacy classes of pairs $(M_0, \pi_0)$ consisting of a $\mathbb{Q}$-rational Levi subgroup $M_0$ and its cuspidal automorphic representation $\pi_0$, the sum being absolutely convergent. The particular terms have expansions

$$J_{\chi}(f) = \sum_{M, \eta} J_{\chi, M, \eta}(f),$$

where the sum runs over all $\mathbb{Q}$-rational Levi subgroups $M$ of $\mathcal{G}$ containing a fixed minimal one (which we take to be the subgroup $A_0$ of diagonal matrices) and, for each $M$, over all automorphic representations $\eta$ of $\mathcal{M}(\mathbb{A})^1$. Explicitly,

$$J_{\chi, M, \eta}(f) = \sum_{s \in W_M} c_{M,s} \int_{i(a_M^\circ)} \sum_{P} \text{tr} (\mathcal{M}_L(P, \nu) M(P, s) \rho_{\chi, \eta}(P, \nu, f)) \, d\nu.$$  

Here, for a given element $s$ of the Weyl group of $M$ in $\mathcal{G}$, the Levi subgroup $L$ is determined by $a_M = (a_M)^s$, and $P$ runs through all parabolic subgroups of $\mathcal{G}$ having $M$ as a Levi component. Let us comment on the items in the
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Let $\rho(P, \nu)$ be the representation of $G(\mathbb{A})$ which is induced from the representation of $P(\mathbb{A})$ in

$$L^2(M(\mathbb{Q}) \backslash M(\mathbb{A})) \cong L^2(N(\mathbb{A})P(\mathbb{Q}) \backslash P(\mathbb{A}))$$

twisted by $\nu$. If one starts the induction with the subspace of the $\pi$-isotypical component spanned by certain residues of Eisenstein series coming from $\chi$, one gets a subrepresentation which is denoted by $\rho_{\chi, \eta}(P, \nu)$. We let $\rho_{\chi, \eta}(P, \nu, f)$ act in the space of $\rho(P, \nu)$ by composing it with the appropriate projector. Further, there is a meromorphic family of standard intertwining operators $M_{Q|P}(\nu)$ between dense subspaces of $\rho(P, \nu)$ and $\rho(Q, \nu)$ defined by an integral for Re $\nu$ in a certain chamber. The operator $M(P, s)$ is $M_{sP|P}(0)$ followed by translation with a representative of $s$ in $G(\mathbb{Q})$. And finally, $\mathcal{M}_{\mathcal{L}}(P, \nu)$ is obtained from such intertwining operators by a limiting process. The operator valued function

$$\nu \mapsto \mathcal{M}_{\mathcal{L}}(P, \nu) M(P, s)$$

extends to a meromorphic function on $(a_G^\mathbb{R})^*$. For $\nu \in (a_G^\mathbb{R})^*$ let $R_\nu$ denote the residue of this operator valued function at $\nu$. Arthur proved that the distribution

$$f \mapsto \text{tr} (R_\nu \rho_{\chi, \eta}(P, \nu, f)) = D(f)$$

is invariant. Let $1_{K_F}$ be the indicator function of $K_F$. We conjecture that the distribution on $G$,

$$D_\infty: \varphi \mapsto D \left( \frac{1}{\text{vol}(K_F)} 1_{K_F} \otimes \varphi \right)$$

is a finite linear combination of traces with integer coefficients, i.e.,

$$D_\infty(\varphi) = \sum_{\eta \in \hat{G}_{\text{adm}}} c(\chi, \eta, P, \nu, \pi) \text{ tr } \pi(\varphi)$$

for some $c(\chi, \eta, P, \nu, \pi) \in \mathbb{Z}$.

**Conjecture 3.2** Conjecture 3.1 holds with

$$N_{\Gamma, \text{cont}}(\pi, \lambda) = \frac{1}{a_\mathbb{R}^G}(\lambda) \sum_{\chi, \eta, P, \nu} c(\chi, \eta, P, \nu, \pi).$$
4 $\text{PSL}_2(\mathbb{R})$

In this section we will prove the conjecture in the simplest case, that of the group $G = \text{PSL}_2(\mathbb{R}) = \text{SL}_2(\mathbb{R})/\pm 1$. For this group there is, up to conjugation, only one choice for $H = AB$, namely $A = \left\{ \pm \left( \begin{array}{cc} t & 1/t \\ 1/t & t \end{array} \right) : t > 0 \right\}$ and $B = M = \{1\}$. We further choose $N = \left\{ \pm \left( \begin{array}{cc} 1 & x \\ x & 1 \end{array} \right) : x \in \mathbb{R} \right\}$ and set $P = MAN = AN$. Fix the maximal compact subgroup $K = \text{SO}_2(\mathbb{R})/\pm 1$.

Finally, we choose the form $b$ to be $b(X,Y) = \text{tr}(XY)$. Occasionally we will view a function $f$ on $G$ as a function on $\text{SL}_2(\mathbb{R})$ with $f(-x) = f(x)$.

Let us recall some facts from the representation theory of $\text{PSL}_2(\mathbb{R})$. Let $\lambda \in a^*$ and denote by $\pi_\lambda$ the corresponding principal series representation, so $\pi_\lambda$ lives on the space of measurable functions $f : G \to \mathbb{C}$ with $f(anx) = a^{\lambda+\rho}f(x)$ which are square integrable on $K$, modulo nullfunctions. The representation is the right regular representation, i.e., $\pi(y)f(x) = f(xy)$.

For each natural number $n$ there are exact sequences

$$0 \to D^+_1 \oplus D^-_1 \to \pi_{(2n-1)\rho} \to \delta_{2n-1} \to 0,$$

and

$$0 \to \delta_{2n-1} \to \pi_{(1-2n)\rho} \to D^+_1 \oplus D^-_1 \to 0,$$

where $D^\pm_{2n}$ are the discrete series, resp. limit of discrete series representations as in [20], and $\delta_{2n-1}$ is the unique irreducible representation of $G$ of dimension $2n-1$. In all other cases the representation $\pi_\lambda$ is irreducible. If $\lambda$ is purely imaginary, then $\pi_\lambda$ is isomorphic with $\pi_{-\lambda}$ and this is the only isomorphism between different principal series representations. The admissible dual $\hat{G}_{\text{adm}}$ consists of all irreducible principal series representations and all $D^\pm_{2n}$ and all $\delta_{2n-1}$. The unitary dual $\hat{G}$ comprises all irreducible $\pi_\lambda$, where $\lambda$ is purely imaginary, all $\pi_{t\rho}$ for $0 < t < 1$ and all $D_{2n}^\pm$. For $\lambda \in a^*$ we also write $\lambda$ for the quasi-character $a \mapsto a^\lambda$ of the group $A$.

**Proposition 4.1** 

(a) If $\pi \in \hat{G}_{\text{adm}}$ is a principal series representation, then $H^0(n, \pi_K) = 0$. More generally, we have $H^0(n, (\pi_\lambda)_K) = 0$ unless $\lambda = (1-2k)\rho$ for some $k \in \mathbb{N}$ in which case it is one-dimensional and $A$ acts via $(2k-2)\rho$.

(b) $H^0(n, \delta_{2n-1})$ is one dimensional for every $n \in \mathbb{N}$ and $A$ acts via the character $(1-2n)\rho$. 

(c) $H^0(n, \mathcal{D}^\pm_{2n}) = 0$ for every $n \in \mathbb{N}$.

**Proof:** Recall the Iwasawa decomposition $G = AN$. Explicitly, for $g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ we get

$$g = \pm \left( \frac{1}{\sqrt{c^2 + d^2}} \begin{pmatrix} ac + bd & 1 \\ c & d \end{pmatrix} \right) \frac{1}{\sqrt{c^2 + d^2}} \begin{pmatrix} d & -c \\ c & d \end{pmatrix}.$$  

Let $f \in H^0(n, \pi_K) = \pi^n_K$. Then for every $n \in \mathbb{N}$ we have $f(xn) = f(x)$. Let $w = \pm \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. If $f(w) = 0$ then $f = 0$. So we assume that $f(w) = 1$. Let $\lambda = sp$, $s \in \mathbb{C}$. Then for $x \in \mathbb{R}$,

$$1 = f \left( w \left( \begin{array}{c} 1 \\ x \\ 1 \end{array} \right) \right) = f \left( \begin{array}{cc} 0 & -1 \\ 1 & x \end{array} \right) = f \left( \left( \frac{1}{\sqrt{1+x^2}} \begin{array}{cc} 1 & -x \\ 1 & x \end{array} \right) \frac{1}{\sqrt{1+x^2}} \begin{array}{c} x \\ 1 \end{array} \right) = \sqrt{1+x^2}^{-s-1} f \left( \frac{1}{\sqrt{1+x^2}} \begin{array}{c} x \\ 1 \end{array} \right).$$

or

$$f \left( \pm \frac{1}{\sqrt{1+x^2}} \begin{array}{c} x \\ 1 \end{array} \right) = \sqrt{1+x^2}^{s+1}.$$  

Since $f \in \pi_K$, the function $f$ is continuous on $K$, so the limit as $x \to \infty$ must exist, which implies $s = -1$ or $\text{Re}(s) < -1$. In the case $s = -1$ the constant function $f(x) = 1$ indeed gives a basis for $H^0(n, \pi_K)$. If $\text{Re}(s) < -1$ then $f$ can only be a smooth function on $K$ if $s = 1 - 2k$ is integral and odd. In order to determine the $A$-actions we need to introduce some notation. For a $\mathbb{C}[A]$-module $V$ and $\lambda \in a^*$ we write $V^\lambda$ for the generalized $\lambda$-eigenspace in $V$. This means $v \in V^\lambda$ if and only if there is $n \in \mathbb{N}$ such that

$$(a - a^\lambda)^n v = 0$$  

for every $a \in A$. On $a^*$ we introduce a partial order $<$ as follows,

$$\nu < \mu \iff \mu - \nu \in \mathbb{N}\rho.$$
Lemma 4.2  (a) For $p = 0, 1$ and $\pi \in \hat{G}_{\text{adm}}$ we have
\[ H^p(n, \pi_K) = \bigoplus_{\nu = w \Lambda_\pi | a} H^p(n, \pi_K)^{\nu - \rho}, \]
where $\Lambda_\pi \in \mathfrak{h}^*$ is a representative of the infinitesimal character of $\pi$ and the sum runs over $w$ in the Weyl group $W(g, \mathfrak{h})$.

(b) If $H^0(n, \pi_K)^{\mu} \neq 0$, then there is $\nu \in a^*$ with $\nu < \mu$ such that $H^1(n, \pi_K)^{\nu} \neq 0$.

Proof: Part (a) is Corollary 3.32 of [21] and part (b) is Proposition 2.32 of [21].

Let $\pi = \pi_{(1-2k)\rho}$. According to part (a) of the Lemma, the group $A$ acts on $H^0(n, \pi_K)$ either via $(2k-2)\rho$ or $-2k\rho$. The second possibility is excluded by part (b) of the Lemma. This proves part (a) of the Proposition.

Part (b) of the Proposition follows from highest weight theory and part (c) follows from part (a) and the fact that $D_{2n}^+ \oplus D_{2n}^-$ is a subrepresentation of $\pi_{2n-1}$.

Proposition 4.3 For $\pi \in \hat{G}_{\text{adm}}$ the dimension of $H^1(n, \pi_K) = 1$ is one except if $\pi = \pi_\lambda$ where $\lambda$ is purely imaginary (unitary principal series) in which case the dimension is two.

If $\pi = \pi_\lambda$ is nonunitary principal series, then $A$ acts via $\lambda - \rho$.
If $\pi = \pi_\lambda$ is unitary principal series, then $A$ acts via $(\lambda - \rho) \oplus (-\lambda - \rho)$.
If $\pi = \delta_{2n-1}$, then $A$ acts via $-2n \rho$.
If $\pi = D_{2n}^\pm$, then $A$ acts via $(2n - 2) \rho$.

Proof: For any $MA$-module $U$ we have
\[ \text{Hom}_{MA}(H^1(n, \pi_K), U \otimes \mathbb{C}_{-\rho}) = \text{Hom}_G(\pi, \text{Ind}_P^G(U)), \]
(see Theorem 4.9 of [21]). The Proposition follows from this.

Since there is no choice for $\tau$ we leave this index out of the notation for the Lefschetz numbers.

Proposition 4.4  (a) Let $\pi_\mu \in \hat{G}_{\text{adm}}$ be a non-unitary principal series representation. Then $L_\lambda(\pi_\lambda) = 0$ unless $\lambda = \mu - \rho$. In that case,
\[ L_{\mu-\rho}(\pi_\mu) = 1. \]
(b) Let $\pi_\lambda$ be a unitary principal series. Then

$$L_{\mu-\rho}(\pi_\mu) = 1 = L_{-\mu-\rho}(\pi_\mu),$$

and $L_{\lambda}(\pi_\mu) = 0$ in all other cases.

(c) Let $n \in \mathbb{N}$. Then

$$L_{\lambda}(\delta^{2n-1}) = 0$$

except for

$$L_{(1-2n)\rho}(\delta^{2n-1}) = -1,$$

and

$$L_{-2n\rho}(\delta^{2n-1}) = 1.$$

(d) Let $n \in \mathbb{N}$. Then $L_{\tau}(\mathcal{D}^\pm_{2n}) = 0$ except for

$$L_{(2n-2)\rho}(\mathcal{D}^\pm_{2n}) = 1.$$

Next for the local Lefschetz numbers. If $[\gamma] \in \mathcal{E}_P(\Gamma)$, then $\gamma$ is $G$-conjugate to $\pm \left( N(\gamma)^{1/2} N(\gamma)^{-1/2} \right)$ for some $N(\gamma) > 1$. An element $\gamma$ of $\Gamma$ is called primitive if $\gamma = \sigma^n$ for $\sigma \in \Gamma$ and $n \in \mathbb{N}$ implies $n = 1$. Each $\gamma \in \mathcal{E}_P(\Gamma)$ is a power of a unique primitive $\gamma_0$ which will be called the primitive underlying $\gamma$.

We write $L(\gamma)$ for $L^\tau(\gamma)$ since $\tau$ is trivial anyway. Then

$$L(\gamma) = \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}}.$$

We will now recall some facts about the Selberg zeta function \[15, 24]. Let $\mathcal{E}_P^\#(\Gamma)$ denote the set of all primitive classes in $\mathcal{E}_P(\Gamma)$. The Selberg zeta function is given by the product

$$Z(s) = \prod_{\gamma \in \mathcal{E}_P^\#(\Gamma)} \prod_{k=0}^{\infty} (1 - N(\gamma)^{-s-k}).$$

The product converges locally uniformly for $\text{Re}(s) > 1$. The zeta function extends to a meromorphic function on the plane of finite order. It has a simple zero at $s = 1$ and zeros at $s = \frac{1}{2} \pm u$ of multiplicity $N_\Gamma(\pi_{u/2})$. These are all zeros or poles in $\text{Re}(s) \geq \frac{1}{2}$ except for $s = \frac{1}{2}$ where $Z(s)$ has a zero or pole of order $N_\Gamma(\pi_0)$ minus the number of cusps. The poles and zeros in
Re(s) < \frac{1}{2} can be described through the scattering matrix or intertwining operators [15, 18, 24].

Recall the inversion formula for the Mellin transform. Let the function \( \psi \) be integrable on \((0, \infty)\) with respect to the measure \( \frac{dt}{t} \), in other words, \( \psi \in L^1 \left( (0, \infty), \frac{dt}{t} \right) \). Then the Mellin transform of \( \psi \) is given by

\[
M\psi(s) \overset{\text{def}}{=} \int_0^\infty t^s \psi(t) \frac{dt}{t}, \quad s \in i\mathbb{R}.
\]

If \( \psi \) is continuously differentiable and \( \psi'(t)t, \psi''(t)t^2 \) are also in \( L^1 \left( (0, \infty), \frac{dt}{t} \right) \), then the following inversion formula holds,

\[
\psi(t) = \frac{1}{2\pi i} \int_{i\mathbb{R}} M\psi(s) t^{-s} \, ds.
\]

Now assume that \( \psi \) is supported in the interval \([1, \infty)\) and that for some \( \mu > 0 \) the functions \( \psi(t), \psi'(t)t, \psi''(t)t^2 \) all are \( O(t^{-\mu}) \). Then it follows that the integral \( M\psi(s) \) defines a function holomorphic in \( \text{Re}(s) < \mu \) and the integral in the inversion formula can be shifted,

\[
\psi(t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} M\psi(s) t^{-s} \, ds,
\]

for every \( C < \mu \).

Every \( \gamma \in \mathcal{E}_P(\Gamma) \) can be written as \( \gamma = \gamma_0^n \) for some uniquely determined \( \gamma_0 \in \mathcal{E}_P^0(\Gamma) \) and a unique \( n \in \mathbb{N} \). A computation yields for \( \text{Re}(s) > 1 \),

\[
\frac{Z'(s)}{Z(s)} = \sum_{\gamma \in \mathcal{E}_P^0(\Gamma)} \sum_{n=1}^\infty \frac{\log N(\gamma)}{1 - N(\gamma)^{-n}} N(\gamma)^{-ns} = \sum_{\gamma \in \mathcal{E}_P(\Gamma)} \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}} N(\gamma)^{-s}
\]

Let \( \psi \) be as above with \( \mu > 1 \) and let \( 1 < C < \mu \). Then, since \( \frac{Z'(s)}{Z(s)} \) is bounded in \( \text{Re}(s) = C \) we can interchange integration and summation to get

\[
\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{Z'(s)}{Z(s)} M\psi(s) \, ds = \sum_{\gamma \in \mathcal{E}_P(\Gamma)} \frac{\log N(\gamma_0)}{1 - N(\gamma)^{-1}} \psi(N(\gamma)).
\]
For $a \in A^{-} = \left\{ \begin{pmatrix} t & \cr & t^{-1} \end{pmatrix} : 0 < t < 1 \right\}$ set

$$\varphi(a) = \varphi \left( \begin{pmatrix} t & \\
 & t^{-1} \end{pmatrix} \right) \overset{\text{def}}{=} \psi \left( \frac{1}{t} \right).$$

Then $\varphi \in C^{2,2\mu}(A^{-})$ and

$$\frac{1}{2\pi i} \int_{C^{-}i\infty}^{C+i\infty} \frac{Z'(s)}{Z(s)} M\psi(s) \, ds = \sum_{\gamma \in \mathcal{E}_{\mathcal{P}}(\Gamma)} \frac{\log N(\gamma)}{1 - N(\gamma)^{-1}} \varphi(a_{\gamma}) \overset{\text{def}}{=} \sum_{\gamma \in \mathcal{E}_{\mathcal{P}}(\Gamma)} L(\gamma) \varphi(a_{\gamma}),$$

which is the right hand side of the Lefschetz formula.

Now suppose that $\varphi \in C^{j,2\mu}(A^{-})$ for some $j \in \mathbb{N}$ and some $\mu > 1$. Then the functions $\psi(t), \psi'(t)t, \ldots, \psi^{(j)}(t)t^{j}$ are all $O(t^{-\mu})$. Integration by parts shows that

$$M\psi(t) \frac{(-1)^j}{s(s+1)\cdots(s+j-1)} \int_{0}^{\infty} t^{s} \psi^{(j)}(t)t^{j} \frac{dt}{t}.$$ 

This implies that $M\psi(s) = O \left( (1+|s|)^{-j} \right)$ uniformly in $\{ \text{Re}(s) \leq \alpha \}$ for every $\alpha < \mu$.

For $R > 0$ and $a \in \mathbb{C}$ let $B_{r}(a)$ be the closed disk around $a$ of radius $r$. Let $g$ be a meromorphic function on $\mathbb{C}$ with poles $a_{1}, a_{2}, \ldots$. We say that $g$ is essentially of moderate growth, if there is a natural number $N$, a constant $C > 0$, and as sequence of real numbers $r_{n} > 0$ tending to zero, such that the disks $B_{r_{n}}(A_{n})$ are pairwise disjoint and that on the domain $D = \mathbb{C} \setminus \bigcup_{n} B_{r_{n}}(a_{n})$ it holds $|g(z)| \leq C|z|^{N}$. Every such $N$ is called a growth exponent of $g$.

**Lemma 4.5** Let $h$ be a meromorphic function on $\mathbb{C}$ of finite order and let $g = h'/h$ be its logarithmic derivative. Then $g$ is essentially of moderate growth with growth exponent equal to the order of $h$ plus two.

**Proof:** This is a direct consequence of Hadamard’s factorization Theorem applied to $h$. \qed

This Lemma together with the growth estimate for $M\psi$ implies that for $j$ large enough the contour integral over $C + i\mathbb{R}$ can be moved to the left,
deforming it slightly, so that one stays in the domain $D$, and gathering residues. Ultimately, the contour integral will tend to zero, leaving only the residues. One gets

$$\sum_{\gamma \in \mathcal{E}_P(\Gamma)} L(\gamma) \varphi(a_{\gamma}) = \sum_{s_0 \in \mathcal{C}} \left( \text{res}_{s=s_0} \frac{Z'}{Z}(s) \right) M\psi(s_0)$$

$$= \sum_{s_0 \in \mathcal{C}} \left( \text{res}_{s=s_0} \frac{Z'}{Z}(s) \right) \int_0^\infty \psi(t)t^{s_0} dt$$

$$= \sum_{s_0 \in \mathcal{C}} \left( \text{res}_{s=s_0} \frac{Z'}{Z}(s) \right) \int_{A^-} \varphi(a)a^{-s_0\rho} da.$$ 

This implies the conjecture in the case $G = \text{PSL}_2(\mathbb{R})$.

5 Applications

If we assume the Lefschetz formula in general, then most applications known in the compact case carry over to the non-compact case. In this section we will only highlight the prime geodesic theorem as an example. For this we assume that the parabolic $P = MAN$ is minimal, i.e., the group $M$ is compact. Let $r = \dim A$ and let $\alpha_1, \ldots, \alpha_r$ be positive multiples of simple roots such that for the modular shift $\rho_P$ we have

$$2\rho_P = \alpha_1 + \cdots + \alpha_r.$$

This fixes $\alpha_1, \ldots, \alpha_r$ up to order. We choose a Haar-measure (i.e., a form $b$) such that for the subset of $A$,

$$\{a \in A : 0 \leq \alpha_k(\log a) \leq 1, \ k = 1, \ldots, r\}$$

has volume 1.

Theorem 5.1 (Prime Geodesic Theorem)

For $T_1, \ldots, T_r > 0$ let

$$\Psi(T_1, \ldots, T_r) = \sum_{\gamma \in \mathcal{E}_P(\Gamma) \atop a_{\gamma}^{-\alpha} k \leq T_k, \ k=1,\ldots,r} \lambda_\gamma.$$
We assume that the Lefschetz formula holds. Then, as $T_k \to \infty$ for $k = 1, \ldots, r$ we have

$$\Psi(T_1, \ldots, T_r) \sim T_1 \cdots T_r.$$ 

**Proof:** The proof of the compact case \[13\] carries over. \[\square\]

We further note a consequence of the Prime Geodesic Theorem which comes about when one applies the Prime geodesic Theorem to $G = \text{SL}_d(\mathbb{R})$ and $\Gamma = \text{SL}_d(\mathbb{Z})$. Let $d$ be a prime number $\geq 3$. Let $\mathcal{C}$ be the set of all totally real number fields $F$ of degree $d$. Let $\mathcal{O}$ be the set of all orders $\mathcal{O}$ in number fields $F \in \mathcal{C}$. For an order $\mathcal{O} \in \mathcal{O}$ let $h(\mathcal{O})$ be its class number and $R(\mathcal{O})$ its regulator. For $\lambda \in \mathcal{O}^\times$ let $\sigma_1, \ldots, \sigma_d$ denote the real embeddings of $F$ ordere in a way that $|\sigma_k(\lambda)| \geq |\sigma_{k+1}(\lambda)|$ holds for $k = 1, \ldots, d-1$. For $k$ in the same range let

$$\alpha_k(\lambda) \overset{\text{def}}{=} k(d - k) \log \left( \frac{|\rho_k(\lambda)|}{|\rho_{k+1}(\lambda)|} \right).$$

Let

$$c = (\sqrt{2})^{1-d} \left( \prod_{k=1}^{d-1} \frac{2k(d - k)}{k} \right).$$

So $c > 0$ and it comes about as correctional factor between the Haar measure normalization used in the Prime Geodesic Theorem and the normalization used in the definition of the regulator.

**Theorem 5.2** For $T_1, \ldots, T_r > 0$ set

$$\vartheta(T) \overset{\text{def}}{=} \sum_{\substack{\lambda \in \mathcal{O}^\times/\pm 1, \mathcal{O} \in \mathcal{O} \\ 0 < \alpha_k(\lambda) \leq T_k \\ k = 1, \ldots, d-1}} R(\mathcal{O}) h(\mathcal{O}).$$

Then we have, as $T_1, \ldots, T_{d-1} \to \infty$,

$$\vartheta(T_1, \ldots, T_{d-1}) \sim \frac{c}{\sqrt{d}} T_1 \cdots T_{d-1}.$$
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