First and second sound in two-dimensional bosonic and fermionic superfluids

L. Salasnich\textsuperscript{1,2,3}, A. Cappellaro\textsuperscript{4}, K. Furutani\textsuperscript{1,3}, A. Tononi\textsuperscript{5}, G. Bighin\textsuperscript{6}

\textsuperscript{1}Dipartimento di Fisica e Astronomia “Galileo Galilei” and Padua QTech, Università di Padova, Via Marzolo 8, 35131 Padova, Italy
\textsuperscript{2}Istituto Nazionale di Ottica (INO) del Consiglio Nazionale delle Ricerche (CNR), Via Nello Carrara 1, 50019 Sesto Fiorentino, Italy
\textsuperscript{3}Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy
\textsuperscript{4}Institute of Science and Technology Austria (ISTA), Am Campus 1, 3400 Klosterneuburg, Austria
\textsuperscript{5}Université Paris-Saclay, CNRS, LPTMS, 91405 Orsay, France
\textsuperscript{6}Institut fur Theoretische Physik, Universität Heidelberg, Philosophenweg 19, 69120 Heidelberg, Germany

We review our theoretical results about the sound propagation in two-dimensional (2D) systems of ultracold fermionic and bosonic atoms. In the superfluid phase, characterized by the spontaneous symmetry breaking of the \(U(1)\) symmetry, there is the coexistence of first and second sound. In the case of weakly-interacting repulsive bosons, we model the recent measurements of the sound velocities of \(^{39}\text{K}\) atoms in 2D obtained in the weakly-interacting regime and around the Berezinskii-Kosterlitz-Thouless (BKT) superfluid-to-normal transition temperature. In particular, we perform a quite accurate computation of the superfluid density and show that it is reasonably consistent with the experiment. For superfluid attractive fermions, we calculate the first and second sound velocities across the whole BCS-BEC crossover. In the low-temperature regime we reproduce the recent measurements of first-sound speed with \(^{6}\text{Li}\) atoms. We also predict that only in the finite-temperature BEC regime there is mixing between sound modes.

I. INTRODUCTION

In this review paper the propagation of first and second sound in two-dimensional (2D) systems of ultracold fermionic and bosonic atoms is examined in light of our current theoretical findings. As well known, the second sound exists only in the \(U(1)\) symmetry-broken superfluid phase. We discuss a quite accurate determination of the superfluid density in the case of weakly-interacting repulsive bosons, finding a good agreement with the experiment to model the recent measurements \[1\] of the sound velocities of \(^{39}\text{K}\) atoms in 2D obtained in the weakly-interacting regime and around the Berezinskii-Kosterlitz-Thouless (BKT) superfluid-to-normal transition temperature \[2\]. We also analyze the first and second sound velocities across the whole BCS-BEC crossover for superfluid attractive fermions. By considering \(^{6}\text{Li}\) atoms we simulate and analyze the most recent measurements \[3\] of the first sound velocity in the low-temperature regime. This velocity is the only one triggered by a density probe because the decoupling of density and entropy fluctuations makes it possible \[4\]. The main results discussed here have been presented at the International Workshop “Quantum Transport with ultracold atoms” (Dresden, 2022).

According to Landau’s two fluid theory \[2\] of superfluids the total number density \(n\) of a system in the superfluid phase can be written as

\[ n = n_s + n_n , \tag{1} \]

where \(n_s\) is the superfluid density and \(n_n\) is the normal density. At the critical temperature \(T_c\) one has \(n_n = n\) and, correspondingly, \(n_s = 0\). Following Landau, in a superfluid a local perturbation excites two wave-like modes - first and second sound - which propagate with velocities \(u_1\) and \(u_2\). These velocities are determined by the positive solutions of the algebraic biquadratic equation

\[ u^4 - (c_{10}^2 + c_{20}^2)u^2 + c_{10}^2c_{20}^2 = 0 \tag{2} \]

The first sound \(u_1\) is the largest of the two positive roots of Eq. \[2\] while the second sound \(u_2\) is the smallest positive one. In the biquadratic equation \[2\] there is the adiabatic sound velocity

\[ c_{10} = \sqrt{\frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_{S,V}} \tag{3} \]

with \(S = S/N\) the entropy per particle, \(V = L^2\) the 2D volume (area) of a square of size \(L\), and \(N\) the total number of identical particles. There is also the entropic sound (or Landau) velocity

\[ c_{20} = \sqrt{\frac{1}{m} \left( \frac{\partial S}{\partial n} \right)_{N,V}} \tag{4} \]

with \(n_s/n_n\) the ratio between superfluid and normal density and \(m\) the mass of each particle, and the isothermal sound velocity

\[ c_T = \sqrt{\frac{1}{m} \left( \frac{\partial P}{\partial n} \right)_{T,V}} \tag{5} \]

with \(P\) the pressure and \(T\) the temperature. Thus, having the equation of state and the superfluid fraction of the system under investigation one can determine the first sound velocity \(u_1\) and the second sound velocity \(u_2\) by solving Eq. \[2\], namely

\[ u_{1,2} = \sqrt{\frac{1}{2} (c_{10}^2 + c_{20}^2) \pm \frac{1}{2} \sqrt{(c_{10}^2 + c_{20}^2)^2 - 4c_{10}^2c_{20}^2}} \tag{6} \]
II. WEAKLY-INTERACTING 2D BOSE GAS

The Helmholtz free energy [4] of a weakly-interacting two-dimensional gas of purely repulsive identical bosons of mass $m$ can be written as ($\hbar = k_B = 1$)

$$F = F_0 + F_Q + F_T = \frac{g}{2} \frac{N^2}{L^2} + \frac{1}{2} \sum_p E_p + T \sum_p \ln \left[ 1 - e^{-E_p/T} \right],$$

where $F_0$ is the mean-field zero-temperature free energy with $g > 0$ is the Bose-Bose interaction strength. $F_T$ is the low-temperature free energy and

$$E_p = \sqrt{\frac{p^2}{2m} \left( \frac{p^2}{2m} + 2gn \right)},$$

is the familiar Bogoliubov spectrum of elementary excitations. For the sake of completeness, we emphasize some formal analogy [5] between this Bogoliubov spectrum of bosonic particles, which can be derived from the Gross-Pitaevskii equation [7,8], and the neural spectrum which can be deduced from the Amari equation of the brain [9]. Indeed, the neural field equation of Amari resembles an imaginary-time Gross-Pitaevskii equation with a nonlocal term. The elementary (linearized) excitations of the Amari equation around a uniform configuration are the analog of the Bogoliubov spectrum of the Gross-Pitaevskii equation. The quantum correction $F_Q$ in the free energy is obviously ultraviolet divergent and requires a regularization procedure. Dimensional regularization [10] leads to

$$F_Q = -L^2 \frac{m}{8\pi} \left[ \ln \left( \frac{\epsilon_A}{gn} \right) - \frac{2}{\eta} \right] (gn)^2,$$

where $\epsilon_A = 4e^{-2\gamma - 1/2 / (ma_{2D}^2)}$ is a cutoff energy, $\gamma = 0.577$ is the Euler-Mascheroni constant, $a_{2D}$ is the 2D s-wave scattering length, and

$$\eta = \frac{mg}{2\pi}$$

is the adimensional gas parameter [2]. Moreover, one also finds

$$\frac{\epsilon_A}{gn} = \frac{2\pi e^{-2\gamma - 1/2 / \eta}}{N} \frac{\eta}{\epsilon_A}.$$

All the thermodynamic quantities can be obtained from the Helmholtz free energy of Eq. (7). For instance, the pressure $P$ is given by

$$P = -\left( \frac{\partial F}{\partial L^2} \right)_{N,T},$$

while the entropy reads

$$S = \left( \frac{\partial F}{\partial T} \right)_{N,L^2}.$$

Figure 1: Typical behavior of superfluidy fraction $n_s / n$ vs dimensional temperature $T / T_c$ in three-dimensional (3D) and two-dimensional (2D) superfluid systems. Notice that only in the 2D case there is a jump of the superfluid fraction at the Berezinskii-Kosterlitz-Thouless critical temperature $T_{BKT}$ of the superfluid-to-normal phase transition.

Instead, the normal density $n_n$ can be extracted from the Landau formula

$$n_n = -\int \frac{d^2 p}{(2\pi)^2} \frac{p^2}{2m} \frac{df_B(E_p)}{dE_p},$$

where $f_B(E) = 1 / (e^{E/T} - 1)$ is the Bose-Einstein distribution.

Actually, the Landau formula for the normal density does not take into account the formation of quantized vortices and anti-vortices by increasing the temperature. These quantized vortices are crucial for the 2D Bose gas to obtain the phenomenology predicted by Berezinskii [11] and Kosterlitz-Thouless [12]. The presence of quantized vortices renormalize the superfluid density $n_s = n - n_n$. The renormalized superfluid density $n_s(t = +\infty)$ is obtained by solving the Nelson-Kosterlitz renormalization group equations [13]

$$\begin{align*}
\partial_t K^{-1}(t) &= 4\pi^3 y^2(t) \\
\partial_t y(t) &= [2 - \pi K(t)] y(t)
\end{align*}$$

where $K(t) = n_s(t) / T$, with $n_s(t)$ the superfluid density at the adimensional fictitious time $t$, and $y(t) = \exp \left[ -\mu_c(t) / T \right]$ is the fugacity, where $\mu_c(t)$ is the vortex chemical potential at fictitious time $t$. In particular, the initial superfluid density $n_s(0)$ of the flow is the one obtained from $n_n$ of Eq. (14) as $n_n(0) = n - n_n$. The initial vortex chemical potential is instead given by the expression $\mu_c(0) = \pi^2 n_s(0) / (2m)$. We emphasize that in the determination of pressure and entropy one should take into account also the vortex contribution. However, in order to make the theoretical scheme more tractable we have included the quantized vortices only for the renormalized
superfluid density. Another relevant issue is the fact in the experiments the superfluids have a finite size. To describe consistently the finite size of the system, we solve Eqs. (15) up to a maximum value \( t_{\text{max}} = \ln(A_{1/2}/\xi) \) of the adimensional fictitious time \( t \), where \( A \) is the area of the system and \( \xi \) is the healing length, which is practically the size of the vortex core.

For 3D superfluids the transition to the normal state is a BEC phase transition, while in 2D superfluids the transition to the normal state is something different: a topological phase transition. An important prediction of the Kosterlitz-Thouless transition is that, contrary to the 3D case, in 2D the superfluid fraction \( n_s/n \) jumps to zero above the Berezinskii-Kosterlitz-Thouless critical temperature \( T_{\text{BKT}} \). See Fig. 1

In Fig. 2 we report our theoretical first and second sound velocities as a function of the adimensional temperature in comparison with recent experimental data near \( T_{\text{BKT}} \). As shown by the figure, the agreement between our theory and the experimental results is quite good.

Another relevant phenomenon is the hybridization with quasi-crossing of first sound \( u_1 \) and second sound \( u_2 \) which appears at a characteristic temperature \( T_{\text{hyb}} \). In particular, when the hybridization temperature \( T_{\text{hyb}} \) is crossed, there is an inversion of the role of density and entropy oscillations in the propagation of sounds. As discussed in detail in Ref. [2], a density perturbation excites mainly \( u_1 \) below the \( T_{\text{hyb}} \) and instead probes mainly \( u_2 \) above \( T_{\text{hyb}} \). The opposite happens by imposing a temperature gradient in the superfluid. Numerically, we have found that \( T_{\text{hyb}} \) grows by increasing the repulsive Bose-Bose interaction strength and eventually \( T_{\text{hyb}} \) coincides with \( T_{\text{BKT}} \).

### III. 2D FERMI GAS IN THE BCS-BEC CROSSOVER

In 2004 the 3D BCS-BEC crossover has been observed with ultracold gases made of two-component attractive fermionic \(^{40}\)K or \(^{6}\)Li atoms [14–16]. This crossover is obtained using a Fano-Feshbach resonance to change the 3D s-wave scattering length \( a_F \) of the inter-atomic potential. More recently also the 2D BEC-BEC crossover has been achieved experimentally [19, 22] with a Fermi gas of two-component \(^6\)Li atoms.

Two dimensional realistic interatomic attractive potentials always have a bound state, in contrast to the 3D situation. In particular [23], the binding energy \( \epsilon_B > 0 \) of two fermions is related to the 2D scattering length \( a_F \) by

\[
\frac{1}{g} = \frac{1}{2L^2} \sum_k \frac{1}{2m + \frac{1}{2} \epsilon_B}.
\]

To study the 2D BCS-BEC crossover we adopt the formalism of functional integration [23]. The partition function \( Z \) of the uniform system with fermionic fields \( \psi_s(\mathbf{r}, \tau) \) at temperature \( T \), in a 2-dimensional volume \( V = L^2 \), and with chemical potential \( \mu \) reads

\[
Z = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \{-S\},
\]

where \( \beta \equiv 1/T \) and

\[
S = \int_0^\beta d\tau \int_{L^2} d^2 \mathbf{r} \mathcal{L}
\]

is the Euclidean action functional with Lagrangian density

\[
\mathcal{L} = \bar{\psi}_s \left[ \partial_\tau - \frac{1}{2m} \nabla^2 - \mu \right] \psi_s - g \bar{\psi}_\uparrow \bar{\psi}_\downarrow \psi_\downarrow \psi_\uparrow
\]

where \( g > 0 \) is the strength of the attractive the s-wave pairing between fermions with opposite spin.

In particular, we are interested in the grand potential \( \Omega \), given by

\[
\Omega = -\frac{1}{\beta} \ln (Z) \simeq -\frac{1}{\beta} \ln \left( Z_{mf} Z_g \right) = \Omega_{mf} + \Omega_g,
\]

where

\[
Z_{mf} = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \exp \{-S_c(\psi_s, \bar{\psi}_s, \Delta_0)\}
\]

Figure 2: Sound velocities vs adimensional temperature. Here \( v_B = gn \) is the Bogoliubov velocity, \( N = 2178 \) is the number of atoms and \( \eta = 0.102 \). The blue line is our first sound velocity \( u_1 \) while the green line is our second sound velocity \( u_2 \). The dots with error bars are the experimental data obtained by Christodoulou et al. [1]. Figure adapted from Ref. [2].
is the mean-field partition function and
\[
Z_g = \int \mathcal{D}[\psi_s, \bar{\psi}_s] \mathcal{D}[\eta, \bar{\eta}] \exp \{ -S_g(\psi_s, \bar{\psi}_s, \eta, \bar{\eta}, \Delta_0) \}
\]

(23)
is the partition function of Gaussian pairing fluctuations.

After functional integration over quadratic fields, one finds that the mean-field grand potential reads
\[
\Omega_{mf} = \frac{\Delta_0^2}{g} L^2 + \sum_k \left( \frac{k^2}{2m} - \mu - E_{sp}(k) - \frac{2}{\beta} \ln \left( 1 + e^{-\beta E_{sp}(k)} \right) \right)
\]

(24)
where
\[
E_{sp}(k) = \sqrt{\left( \frac{k^2}{2m} - \mu \right)^2 + \Delta_0^2}
\]

(25)
is the spectrum of fermionic single-particle excitations.

The Gaussian grand potential is instead given by
\[
\Omega_g = \frac{1}{2\beta} \sum_Q \ln \det(M(Q))
\]

(26)
where $M(Q)$ is the inverse propagator of Gaussian fluctuations of pairs and $Q = (q, \Omega_m)$ is the 4D wavevector with $\Omega_m = 2\pi n / \beta$ the Matsubara frequencies and $q$ the 3D wavevector [27].

The sum over Matsubara frequencies is quite complicated and it does not give a simple expression. An approximate formula [28] is
\[
\Omega_g \approx \frac{1}{2} \sum_q E_{col}(q) + \frac{1}{\beta} \sum_q \ln \left( 1 - e^{-\beta E_{col}(q)} \right)
\]

(27)
where
\[
E_{col}(q) = \omega(q)
\]

(28)
is the spectrum of bosonic collective excitations with $\omega(q)$ derived from
\[
\det(M(q, \omega)) = 0.
\]

(29)

It is important to stress that the zero-point energy of the collective excitations is divergent. However, by using the convergence factor renormalization procedure (see Ref. [10] for a review of renormalization methods for the zero-point energy of ultracold atoms) one extracts a reliable finite contribution. In Fig. 4 we plot the pressure $P = -\Omega / L^2$ of the 2D Fermi gas in the BCS-BEC crossover comparing our zero-temperature theoretical results [29] with the available experimental data [19]. The agreement between theory and experiment is extremely good only including Gaussian fluctuations. For the specific investigation of the Gaussian fluctuations in the 2D crossover with analytical and numerical techniques see also Refs. [17, 18]. Quite remarkably, our $T = 0$ results with Gaussian fluctuations are in good agreement also with auxiliary-field path integral calculations [20] and diffusion Monte Carlo simulations [21].

We are now interested on the temperature dependence of superfluid density $n_s(T)$ of the system. At the Gaussian level $n_s(T)$ depends only on fermionic single-particle excitations $E_{sp}(k)$ [31]. Beyond the Gaussian level also bosonic collective excitations $E_{col}(q)$ contribute [32]. Thus, we assume the following Landau-type formula for the superfluid density [29]
\[
n_s(T) = n - \beta \int \frac{q^2 k}{(2\pi)^2} \frac{e^{-\beta E_{sp}(k)}}{(e^{\beta E_{sp}(k)} + 1)^2} - \frac{\beta}{2} \int \frac{q^2 \omega^2}{(2\pi)^2} \frac{e^{-\beta E_{col}(q)}}{(e^{\beta E_{col}(q)} - 1)^2}.
\]

(30)

This bare superfluid density can be renormalized by using the flow equations [13] of Kosterlitz-Thouless-Nelson, which take into account the effect of quantized vortices and anti-vortices [12, 13], by using Eq. [30] as initial condition. In Fig. 3 we plot the BKT critical temperature obtained by using the Nelson-Kosterlitz criterion [13]:
\[
T_{BKT} = \frac{\pi}{8 m} n_s(T_{BKT})
\]

(31)
The figure clearly shows that the mean-field prediction (dashed line) is meaningful only in the deep BCS regime of the 2D crossover. Instead our beyond-mean-field results (solid line) which include Gaussian fluctuations, are in reasonable good agreement with the available experimental data [33] (filled circles). In Ref. [30, 34] we have found that $T_{BKT}$ derived with the Nelson-Kosterlitz criterion slightly overestimates the critical temperature calculated by solving the renormalization group equations [13].

Figure 3: Zero-temperature scaled pressure $P/P_d$ vs scaled binding energy $\epsilon_B/\epsilon_F$. Filled squares with error bars: experimental data of Makhalov et al. [19]. Solid line: our regularized Gaussian pair fluctuation (GPF) theory. Figure adapted from Ref. [30].
Having the equation of state and the superfluid density at finite temperature, we can calculate the first sound velocity $u_1$ and the second sound velocity $u_2$ in the 2D BCS-BEC crossover by using Eq. (32). We also analyze the amplitudes modes $W_1$ and $W_2$ of the response to a density perturbation $\delta n$, i.e.

$$\delta n(x, t) = W_1 \delta n_1(x \pm u_1 t) + W_2 \delta n_2(x \pm u_2 t)$$

(32)

where

$$\frac{W_1}{W_1 + W_2} = \frac{(u_1^2 - c_{20}^2) u_2^2}{(u_1^2 - u_2^2) c_{20}^2}$$

(33)

and

$$\frac{W_2}{W_1 + W_2} = \frac{(c_{20}^2 - u_2^2) u_1^2}{(u_1^2 - u_2^2) c_{20}^2}.$$ (34)

In Fig. 5 we show that our theoretical determination of the sound velocities and density responses (insets) as a function of the interaction strength (actually the logarithm of the adimensional binding energy $\epsilon_B/\epsilon_F$) at quite low temperature $T$. The comparison with the experimental measurements of the first sound velocity (filled diamonds) suggest that our theoretical framework is quite good. It is important to stress that in the BCS regime the speed of second sound is rapidly going to zero.

In Fig. 6 we report instead the sound velocities as a function of the temperature $T$ for three values of the interaction strength (the three panels correspond to increasing values of the adimensional binding energy $\epsilon_B/\epsilon_F$). The density responses shown in the insets strongly suggest that a mixing between the first sound and second sound occurs only in the finite-temperature BEC regime. Notice that, taking into account Eqs. (3), (4), (5), (6), (33) and (34), the presence of mixing means that the adiabatic sound velocity $c_{10}$ is quite different with respect to the isothermal sound velocity $c_T$. Conversely, if $c_{10} \simeq c_T$ then $u_1 \simeq c_{10}$, $u_2 \simeq c_{20}$ and consequently $W_2 \simeq 0$. Just for comparison, for the 3D unitary Fermi gas we have recently shown that, contrary to 3D liquid helium, near the critical temperature the mixing of first and second sound is quite strong.

**IV. CONCLUSIONS**

We have shown that first and second sound of bosonic and fermionic superfluids can be derived adopting the Landau’s two-fluid theory which requires the equation of state and the superfluid fraction of the system under investigation. In the case of a 2D weakly-interacting Bose gas, we have found that the comparison of our theory with recent measurement near $T_{BKT}$ is quite good. In the BCS-BEC crossover of the 2D Fermi gas, we have proved that to obtain a good agreement with experimental data for the equation of state, the critical temperature $T_{BKT}$, and the sound modes, both fermionic single-particle excitations and bosonic collective excitations are needed. In conclusion, it is important to stress that all the results discussed here are valid in the collisional regime, where
collisionless Bose gas of results of sound and sound damping in a two-dimensional 2D systems of ultracold atoms is not yet fully clarified. In Ref. [37] we have found that the experimental results of sound and sound damping in a two-dimensional collisionless Bose gas of $^{87}$Rb atoms [38] are better reproduced, below the critical temperature of the superfluid-to-normal phase transition, by the Andreev-Khalatnikov equations of a collisionless superfluid with respect to the finding of the Vlasov-Landau equation [39, 40]. Finally, for the sake of completeness, we suggest to read the very recent review paper of Hu, Yao, and Liu [41] which contains a detailed historical account of the second sound in ultracold atoms.

$$\omega \tau \ll 1$$ with $\omega$ the frequency of the sound mode and $\tau$ the collision time of quasi-particles. However, in the collisionless regime ($\omega \tau \gg 1$) the role of superfluidity in 2D systems of ultracold atoms is not yet fully clarified. In Ref. [37] we have found that the experimental results of sound and sound damping in a two-dimensional collisionless Bose gas of $^{87}$Rb atoms [38] are better reproduced, below the critical temperature of the superfluid-to-normal phase transition, by the Andreev-Khalatnikov equations of a collisionless superfluid with respect to the finding of the Vlasov-Landau equation [39, 40]. Finally, for the sake of completeness, we suggest to read the very recent review paper of Hu, Yao, and Liu [41] which contains a detailed historical account of the second sound in ultracold atoms.

LS thanks Herwig Ott and Sandro Winberger for their kind invitation to the International Workshop “Quantum Transport with ultracold atoms” (2022). KF acknowledges Fondazione CARIPARO for a PhD fellowship. AT acknowledges support from ANR Grant Droplets No. ANR-19-CE30-0003-02. LS and KF are partially supported by the BIRD Project ”Ultracold atoms in curved geometries” of the University of Padova.

[1] P. Christodoulou, M. Valka, N. Dogra, R. Lopes, J. Schmitt, and Z. Hadzibabic, Observation of first and second sound in a BKT superfluid, Nature 594, 191 (2021).
[2] K. Furutani, A. Tononi, and L. Salasnich, Sound modes in collisional superfluid Bose gases, New J. Phys. 23, 043043 (2021).
[3] M. Bohlen, L. Sobirey, N. Luick, H. Biss, T. Enss, T. Lompe, and H. Moritz, Sound Propagation and Quantum-Limited Damping in a Two-Dimensional Fermi Gas, Phys Rev. Lett. 24, 240403 (2020).
[4] A. Tononi, A. Cappellaro, G. Bighin, and L. Salasnich, Propagation of first and second sound in a two-dimensional Fermi superfluid, Phys. Rev. A 103, L061303 (2021).
[5] L.D. Landau, The theory of superfluidity of helium II, J. Phys. (USSR) 5, 71 (1941).
[6] L. Salasnich, Power spectrum and diffusion of the Amari neural field, Symmetry 11, 134 (2019).
[7] E. P. Gross, Structure of a quantized vortex in boson systems, Nuovo Cimento 20, 454 (1961).
[8] L. P. Pitaevskii, Vortex lines in an imperfect Bose gas, Sov. Phys. JETP 13, 451 (1961).
[9] S. Amari, Biol. Cyber. 27, 77 (1977).
[10] L. Salasnich and F. Toigo, Zero-point energy of ultracold atoms, Phys. Rep. 640, 1 (2016).
[11] V.L. Berezinskii, Destruction of Long-range Order in One-dimensional and Two-dimensional Systems Possessing a Continuous Symmetry Group. II. Quantum Systems, Sov. Phys. JETP 34, 610 (1972).
[12] J.M. Kosterlitz and D.J. Thouless, Long range order and metastability in two dimensional solids and superfluids. (Application of dislocation theory), J. Phys. C 5, L124 (1972).
[13] D.R. Nelson and J.M. Kosterlitz, Universal Jump in the Superfluid Density of Two-Dimensional Superfluids, Phys. Rev. Lett. 39, 1201 (1977).
[14] C.A. Regal, M. Greiner, and D. S. Jin, Observation of Resonance Condensation of Fermionic Atom Pairs, Phys. Rev. Lett. 92, 040403 (2004).
[15] M.W. Zwierlein, C. A. Stan, C. H. Schunck, S. M. F. Raupach, A. J. Kerman, and W. Ketterle, Condensation of Pairs of Fermionic Atoms near a Feshbach Resonance, Phys. Rev. Lett. 92, 120403 (2004).
[16] J. Kinast, J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas, Evidence for Superfluidity in a Resonantly Interacting Fermi Gas, Phys. Rev. Lett. 92, 150402 (2004).
[17] L. Salasnich and F. Toigo, Composite bosons in the two-dimensional BCS-BEC crossover from Gaussian fluctuations, Phys. Rev. A 91, 011604(R) (2015).
[18] L. He, H. Lu, G. Cao, H. Hu and X.-J. Liu, Quantum fluctuations in the BCS-BEC crossover of two-dimensional Fermi gases, Phys. Rev. A 92, 023620 (2015).
[19] V. Makhalov, K. Martiyanyov, and A. Turlapov, Ground-State Pressure of Quasi-2D Fermi and Bose Gases, Phys. Rev. Lett. 112, 045301 (2014).
[20] H. Shi, S. Chiesa, and S. Zhang, Ground-state properties of strongly interacting Fermi gases in two dimensions, Phys. Rev. A 92, 033603 (2015).

[21] A. Galea, H. Dawkins, S. Gandolfi, and A. Gezerlis, Diffusion Monte Carlo study of strongly interacting two-dimensional Fermi gases, Phys. Rev. A 93, 023602 (2016).

[22] M.G. Ries, A.N. Wenz, G. Zurn, L. Bayha, I. Boettcher, D. Kedar, P.A. Murthy, M. Neidig, T. Lompe, and S. Jochim, Observation of Pair Condensation in the Quasi-2D BEC-BCS Crossover, Phys. Rev. Lett. 114, 230401 (2015).

[23] C. Mora and Y. Castin, Extension of Bogoliubov theory to quasicondensates, Phys. Rev. A 67, 053615 (2003).

[24] M. Randeria, J-M. Duan, and L-Y. Shieh, Bound states, Cooper pairing, and Bose condensation in two dimensions, Phys. Rev. Lett. 62, 981 (1989).

[25] N. Nagaosa, Quantum Field Theory in Condensed Matter (Springer, 1999).

[26] A. Altland and B. Simons, Condensed Matter Field Theory (Cambridge Univ. Press, 2006).

[27] R.B. Diener, R. Sensarma, Quantum fluctuations in the superfluid state of the BCS-BEC crossover, M. Randeria, Phys. Rev. A 77, 023626 (2008).

[28] E. Taylor, A. Griffin, N. Fukushima, Y. Ohashi, Pairing fluctuations and the superfluid density through the BCS-BEC crossover, Phys. Rev. A 74, 063626 (2006).

[29] G Bighin and I. Boettcher, Finite-temperature quantum fluctuations in two-dimensional Fermi superfluids, Physical Review B 93, 014519 (2016).

[30] G Bighin and L. Salasnich, Renormalization of the superfluid density in the two-dimensional BCS-BEC crossover, Int. J. Mod. Phys. B 32, 1840022 (2018).

[31] E. Babaev and H.K. Kleinert, Nonperturbative XY-model approach to strong coupling superconductivity in two and three dimensions, Phys. Rev. B 59, 12083 (1999).

[32] L. Benfatto, A. Toschi, and S. Caprara, Low-energy phase-only action in a superconductor: A comparison with the XY model, Phys. Rev. B 69, 184510 (2004).

[33] P.A. Murthy, I. Boettcher, L. Bayha, M. Holzmann, D. Kedar, M. Neidig, M.G. Ries, A.N. Wenz, G. Zurn, and S. Jochim, Observation of the Berezinskii-Kosterlitz-Thouless Phase Transition in an Ultracold Fermi Gas, Phys. Rev. Lett. 115, 010401 (2015).

[34] G Bighin and L. Salasnich, Vortices and antivortices in two-dimensional ultracold Fermi gases, Sci. Rep. 7, 45702 (2017).

[35] T. Ozawa and S. Stringari, Discontinuities in the First and Second Sound Velocities at the Berezinskii-Kosterlitz-Thouless Transition, Phys. Rev. Lett. 112, 025302 (2014).

[36] G. Bighin, A. Cappellaro, and L. Salasnich, Unitary Fermi superfluid near the critical temperature: thermodynamics and sound modes from elementary excitations, Phys. Rev. A 105, 063329 (2022).

[37] F. Sattin and L. Salasnich, Collisionless sound of bosonic superfluids in lower dimensions, Phys. Rev. A 103, 043324 (2021).

[38] J.L. Ville, R. Saint-Jalm, E. Le Cerf, M. Aidelsburger, S. Nascimbene, J. Dalibard, and J. Beugnon, Sound Propagation in a Uniform Superfluid Two-Dimensional Bose Gas, Phys. Rev. Lett. 121, 145301 (2018).

[39] M. Ota, F. Larcher, F. Dalfovo, L. Pitaevskii, N.P. Proukakis, and S. Stringari, Collisionless Sound in a Uniform Two-Dimensional Bose Gas, Phys. Rev. Lett. 121, 145302 (2018).

[40] A. Cappellaro, F. Toigo, and L. Salasnich, Collisionless dynamics in two-dimensional bosonic gases, Phys. Rev. A 98, 043605 (2018).

[41] H. Hu, X.-C. Yao, X.-J. Liu, Second sound with ultracold atoms: A brief historical account, e-print arXiv:2206.05914.