Estimates for eigenvalues of the Neumann and Steklov problems

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Abstract

We prove Li-Yau-Kröger type bounds for Neumann-type eigenvalues of the poly-harmonic operator and of the biharmonic operator on bounded domains in a Euclidean space. We also prove sharp estimates for lower order eigenvalues of a biharmonic Steklov problem and of the Laplacian, which directly implies two sharp Reilly-type inequalities for the corresponding first nonzero eigenvalue.

1 Introduction

Throughout this paper, let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ in the Euclidean $n$-space $\mathbb{R}^n$. We consider the Neumann eigenvalue problem of the Laplacian $\Delta$

\[
\begin{cases}
-\Delta u = \mu u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative on the boundary $\partial \Omega$ w.r.t the outward unit normal vector $\nu$. The system (1.1) can be used to describe the vibration of membrane and is also called the free membrane problem. It is well known that this problem has discrete spectrum $\{\mu_i\}_{i=1}^\infty$ diverging to infinity and satisfying

\[0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots \uparrow +\infty.\]

In [1], Ashbaugh and Benguria conjectured that

\[
\sum_{i=1}^n \frac{1}{\mu_{i+1}(\Omega)} \geq \frac{n}{\mu_2(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball},
\]

where $B_\Omega$ is the ball of same volume as $\Omega$, $\mu_i(\Omega)$ is the $i$-th Neumann eigenvalue on $\Omega$, and $\mu_2(B_\Omega)$ is the first nonzero Neumann eigenvalue on $B_\Omega$. In [22], Wang and Xia proved that

\[
\sum_{i=1}^{n-1} \frac{1}{\mu_{i+1}(\Omega)} \geq \frac{n-1}{\mu_2(B_\Omega)}, \quad \text{with equality if and only if } \Omega \text{ is a ball},
\]

which supports the above conjecture of Ashbaugh and Benguria.

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On the other hand, corresponding to the Li-Yau’s classical result for Dirichlet eigenvalues of the Laplacian \cite{14}, Kröger \cite{12} obtained the following inequality for the sum of the Neumann eigenvalues

\[
\sum_{j=1}^{k} \mu_j(\Omega) \leq (2\pi)^2 \frac{n}{n+2} \frac{k^{n+2}}{\omega_n(\Omega)^{\frac{n}{2}}}, \quad k \geq 1,
\]

(1.4)

and the upper bound estimate for the \( (k+1) \)-th Neumann eigenvalue

\[
\mu_{k+1}(\Omega) \leq (2\pi)^2 \left( \frac{n+2}{4\omega_n(\Omega)} \right)^{\frac{n}{2}} k^\frac{n}{2}, \quad k \geq 0,
\]

(1.5)

where \( \omega_n \) denotes the volume of the unit ball in \( \mathbb{R}^n \) and \( |\Omega| \) represents the volume of \( \Omega \).

In this paper, we consider the following eigenvalue problem of the poly-harmonic operator

\[
\begin{cases}
(-\Delta)^p u = \Gamma u & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = \sum_{i=1}^{m} \Delta^{m-i} u = \cdots = \Delta^{2m-1} u = 0 & \text{on } \partial \Omega, \text{ when } p = 2m, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega, \text{ when } p = 1,
\end{cases}
\]

(1.6)

where \( p, m \in \mathbb{N} \) with \( \mathbb{N} \) the set of all positive integers.

**Remark 1.1** For the eigenvalue problem (1.6), we prefer to give some facts as follows:

(1) Clearly, if \( p = 1 \), then (1.6) degenerates into the classical Neumann eigenvalue problem of the Laplacian, i.e. (1.1). Based on this fact, by the abuse of terminology, we will call (1.6) a Neumann-type eigenvalue problem of the poly-harmonic operator \((-\Delta)^p\).

(2) In fact, when one considers the eigenvalue problem (1.6), the regularity assumption of \( \partial \Omega \) should be made such that the embedding \( H^p(\Omega) \subset L^2(\Omega) \) is compact. Our assumption on smoothness of \( \partial \Omega \) here is stronger enough to ensure the compactness of this embedding. Actually, for Neumann eigenvalue problem of the Laplacian, i.e. the \( p = 1 \) case of (1.6), the Lipschitz continuity assumption for \( \partial \Omega \) is enough such that the embedding \( H^1(\Omega) \subset L^2(\Omega) \) is compact. To avoid regularity argument, which is not so necessary for our main results of this paper, we have assumed that the boundary \( \partial \Omega \) is smooth.

(3) Let \( H^p(\Omega) \) denote the Sobolev space of functions in \( L^2(\Omega) \) with derivatives up to order \( p \) in \( L^2(\Omega) \). For any \( v, w \in H^p(\Omega) \), one can define an inner product \( \langle \cdot, \cdot \rangle \) as follows:

\[
\langle v, w \rangle = \begin{cases}
\int_{\Omega} \left[ \Delta^m v \cdot \Delta^m w \right] + \Gamma \int_{\Omega} w, & \text{if } p = 2m, \ m \in \mathbb{N}, \\
\int_{\Omega} \left[ \nabla(\Delta^{m-1} v) \cdot \nabla(\Delta^{m-1} w) \right] + \Gamma \int_{\Omega} w, & \text{if } p = 2m - 1, \ m \in \mathbb{N},
\end{cases}
\]

where \( \nabla \) is gradient operator on \( \Omega \), and volume elements in the above integrals have been dropped.\(^1\)

Here

\[
\Delta^m v = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} \frac{\partial^{2m} v}{\partial x_{i_1}^2 \partial x_{i_2}^2 \cdots \partial x_{i_m}^2},
\]

and \( \Delta^{m-1} \) can be defined similarly. The weak version of (1.6) is then the variational problem

\[
\int_{\Omega} (\Delta^m v) \cdot (\Delta^m w) = \Gamma \int_{\Omega} w, \quad \forall w \in H^p(\Omega) \quad (if \ p = 2m, \ m \in \mathbb{N})
\]

or

\[
\int_{\Omega} \nabla(\Delta^{m-1} v) \cdot \nabla(\Delta^{m-1} w) = \Gamma \int_{\Omega} w, \quad \forall w \in H^p(\Omega) \quad (if \ p = 2m - 1, \ m \in \mathbb{N})
\]

\(^1\) For convenience, in the sequel we will drop the integral measures for all integrals except it is necessary.
in unknowns \( u \in H^p(\Omega) \) and \( \Gamma \in \mathbb{R} \). It is not hard to verify that under the boundary conditions proposed in \((1.6)\), the poly-harmonic operator \( (-\Delta)^p \) is self-adjoint w.r.t. the inner product \( \langle \cdot, \cdot \rangle \) defined as above. Then the standard theory of self-adjoint compact operators tells us that the spectrum of the eigenvalue problem \((1.6)\) is real and discrete consisting in a non-decreasing sequence

\[
0 = \Gamma_1(\Omega) < \Gamma_2(\Omega) \leq \Gamma_3(\Omega) \leq \cdots \uparrow +\infty,
\]

where each eigenvalue (i.e., element in the discrete spectrum) is repeated with its multiplicity.

(4) By the min-max principle, together with the divergence theorem and the boundary conditions in \((1.6)\), it is not hard to know that the Rayleigh-Ritz type characterization of the \( k \)-th nonzero eigenvalue \( \Gamma_k(\Omega) \) is given as follows:

When \( p = 2m, m \in \mathbb{N} \),

\[
\Gamma_k(\Omega) = \inf \left\{ \frac{\int_\Omega (\Delta u)^2}{\int_\Omega u^2} \middle| u, \ldots, \Delta^{2m-1} u \in H^2(\Omega), u \neq 0, \int_\Omega uu_j = 0, j = 1, \ldots, k-1 \right\}; \quad (1.7)
\]

When \( p = 2m-1, m \in \mathbb{N} \),

\[
\Gamma_k(\Omega) = \inf \left\{ \frac{\int_\Omega (\nabla u)^2}{\int_\Omega u^2} \middle| u, \ldots, \nabla^{2m-2} u \in H^2(\Omega), u \neq 0, \int_\Omega uu_j = 0, j = 1, \ldots, k-1 \right\}; \quad (1.8)
\]

where \( u_j \) is the eigenfunction of the eigenvalue \( \Gamma_j(\Omega) \). Besides, the eigenfunction \( u_1 \) of \( \Gamma_1(\Omega) = 0 \) should be nonzero constant function.

(5) For \( p = 2 \), by the min-max principle, one can deduce that the \( k \)-th nonzero eigenvalue of \((1.6)\) is not larger than the square of the \( k \)-th nonzero Neumann eigenvalue of the Laplacian, and so by using Schwarz inequality and \((1.3)\), we infer that

\[
\sum_{i=1}^{n-1} \frac{1}{\Gamma_{i+1}(\Omega)} \geq \frac{n-1}{\Gamma_2(\Omega)}.
\]

(6) We have noticed that Provenzano [18] discussed the Dirichlet and the Neumann eigenvalues of the poly-harmonic operator \( (-\Delta)^p \) in the space \( H^p(\Omega) \), with \( \Omega \subset \mathbb{R}^n \) a bounded domain such that the embedding \( H^p(\Omega) \subset L^2(\Omega) \) is compact, and successfully gave a relation between Dirichlet eigenvalues and Neumann eigenvalues therein – the \((k+p)\)-th Neumann eigenvalue is strictly less than the \( k \)-th Dirichlet eigenvalue for all \( k, p \in \mathbb{N} \). However, the boundary conditions proposed therein are different from the ones we have used in \((1.6)\). In fact, Provenzano’s boundary conditions proposed in [18] are

\[
\partial u = \cdots = \partial^{p-1} u = 0 \quad \text{on } \partial \Omega \quad \text{(Dirichlet case)}
\]

and

\[
\mathcal{N}_1 u = \mathcal{N}_2 u = \cdots = \mathcal{N}_p u \quad \text{on } \partial \Omega \quad \text{(Neumann case)}, \quad (1.9)
\]

which leads to a truth that our eigenvalue problem \((1.6)\) of the operator \( (-\Delta)^p \) is different from those two investigated in [18].

We can prove the following Kröger-type estimates for the Neumann-type eigenvalue problem \((1.6)\).

\footnote{The Neumann boundary condition \((1.6)\) can be simplified as \( \frac{\partial u}{\partial \nu} \big|_{\partial \Omega} = 0 \) for \( p = 1 \), \( \frac{\partial^2 u}{\partial \nu^2} \big|_{\partial \Omega} = \text{div}_{\partial \Omega} \left( \nabla^2 u \cdot \nu \right) \big|_{\partial \Omega} = 0 \) on \( \partial \Omega \) for \( p = 2 \), where \( \text{div}_{\partial \Omega} \) is the surface divergence on \( \partial \Omega \), \( \nabla^2 u \) is the Hessian of \( u \), and \( \left( \nabla^2 u \cdot \nu \right) \big|_{\partial \Omega} \) stands for the projection of \( \nabla^2 u \cdot \nu \) to the tangent bundle of \( \partial \Omega \). As also mentioned in [18], generally it is “a quite involved task” to write \((1.6)\) explicitly for \( p \geq 3 \).}
Theorem 1.2 Let $\Omega$ be a bounded connected domain, with smooth boundary $\partial \Omega$, in the Euclidean $n$-space $\mathbb{R}^n$ and let $\Gamma_j(\Omega)$ be the $j$-th eigenvalue of the system $(1.6)$. Then we have
\[\sum_{j=1}^{k} \Gamma_j(\Omega) \leq (2\pi)^{2p} \frac{n}{n+2p} k^{\frac{n+2p}{n}} \left( \frac{1}{\omega_n} \right)^{\frac{2p}{n}}, \quad k \geq 1, \tag{1.10}\]
and
\[\Gamma_{k+1}(\Omega) \leq (2\pi)^{2p} \left( \frac{n+2p}{2p\omega_n(\Omega)} \right)^{\frac{2p}{n}} k^{\frac{2p}{n}}, \quad k \geq 0, \tag{1.11}\]
where, as before, $\omega_n$ denotes the volume of the unit ball in $\mathbb{R}^n$ and $|\Omega|$ denotes the volume of $\Omega$.

Remark 1.3 (1) Clearly, when $p = 1$, the eigenvalue problem $(1.6)$ reduces to $(1.1)$, and then our upper bound estimates $(1.10)$, $(1.11)$ become Kr"oger’s inequalities $(1.4)$ and $(1.5)$, respectively.
(2) Laptev [13, Sections 2 and 3] showed that Kr"oger’s estimates (1.4) and (1.5) are corollaries of general (sharp) trace inequalities for convex functions of (self-adjoint) operators. Besides, in [13, Theorem 3.2], Laptev gave a lower bound for the counting function of the spectrum of the Friedrich’s extension $B_\mathcal{N}$ of the differential operator $B(D) = A^*(D)A(D)$ provided $B_\mathcal{N}$ has discrete spectrum. Here the differential operator $A(D)$ is defined by
\[A(D)u(x) = \sum_{\beta \leq l} A_\beta D^\beta u(x), \quad u \in C^\infty(\overline{\Omega}, \mathbb{C}^m), \quad m \in \mathbb{N}, \quad l \in \mathbb{N},\]
where $\Omega \subset \mathbb{R}^n$ is an open set with its closure $\overline{\Omega}$, the coefficients $A_\beta$ are $m \times m$-matrices independent of $x \in \Omega$. Letting $m = 1$, $B(\xi) = |\xi|^2$, $l \in \mathbb{N}$, and then the operator $B_\mathcal{N}$ coincides with the operator of the Neumann boundary problem for the poly-harmonic operator $(-\Delta)^l$. Hence, [13, Theorem 3.2 and Corollary 3.3] imply a lower bound for the counting function of Neumann eigenvalues of $(-\Delta)^l$. Now, we claim that $(1.11)$ is equivalent to (3.5) in [13]. In fact, let $N(\Gamma) = k$, where
\[N(\Gamma) = \sum_{r_i(\Omega) \leq \Gamma} 1 = \sup_{r_i(\Omega) \leq \Gamma} i\]
is the counting function, and then $\Gamma_{k+1}(\Omega) \geq \Gamma$. Thus, we infer from $(1.11)$ that
\[\Gamma \leq (2\pi)^{2p} \left( \frac{n+2p}{2p\omega_n(\Omega)} \right)^{\frac{2p}{n}} k^{\frac{2p}{n}}, \quad k \geq 0,\]
which implies that
\[N(\Gamma) \geq \frac{2p}{n+2p} \frac{1}{2\pi} \omega_n |\Omega|^{\frac{2p}{n}}.\]

Let $\overline{\Delta}$ and $\Delta$ be the Laplace-Beltrami operators on $\Omega$ and $\partial \Omega$, respectively. Let $\nabla$ and $\nabla$ be the gradient operators on $\Omega$ and $\partial \Omega$ separately. Consider the following Neumann-type eigenvalue problem of the biharmonic operator
\[\begin{cases}
\overline{\Delta}^2 u - \tau \Delta u = \Lambda u & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial \nu} - (1 - \sigma) \text{div}_{\partial \Omega} \left( \nabla^2 u \cdot \nu \right)_{\partial \Omega} - \frac{\partial \Delta u}{\partial \nu} = 0 & \text{on } \partial \Omega, \tag{1.12}
\end{cases}\]
where $\tau \geq 0$ and $\sigma \in (-1/(n-1), 1)$ are two constants, and, as before, $\text{div}_{\partial \Omega}$ denotes the tangential divergence operator on $\partial \Omega$, $\nabla^2 u$ is the Hessian of $u$, $(\nabla^2 u \cdot \nu)_{\partial \Omega}$ stands for the projection of $\nabla^2 u \cdot \nu$ to the
tangent bundle of $\partial \Omega$. In this setting, the problem (1.12) has discrete spectrum and all eigenvalues in the spectrum can be listed non-decreasingly as follows (see, e.g., [7, Proposition 4.1])

$$0 = \Lambda_1(\Omega) \leq \Lambda_2(\Omega) \leq \Lambda_3(\Omega) \leq \cdots \leq +\infty.$$ 

This problem is called the eigenvalue problem of free plate under tension and with nonzero Poisson’s ratio, which for $n = 2$ can be used to describe the deformation of a planar material under compression, $\tau, \sigma$ denote a parameter related to the tension and a Poisson’s ratio of the material, respectively. By the Rayleigh-Ritz characterization, the Neumann-type eigenvalues (if exist and with the abuse of terminology) of (1.12) are given by (see, e.g., [7, 15] while [2] for the case $\sigma = 0$)

$$\Lambda_k(\Omega) = \inf_{0 \neq u \in H^2(\Omega)} \left\{ \int_{\Omega} \left[ (1 - \sigma)|\nabla^2 u|^2 + \sigma(\Delta u)^2 + \tau|\nabla u|^2 \right] \right\},$$

where $u_j$ is an eigenfunction corresponding to the eigenvalue $\Lambda_j(\Omega)$, and $|\nabla^2 u|^2 = \sum_{i,j=1}^{n} \left( \frac{\partial^2 u}{\partial x_i \partial x_j} \right)^2$.

**Remark 1.4**

(1) In [6, 7, 15], the authors therein used the operator $\text{Proj}_{B_1} \left( \nabla^2 u \right)$ to denote the projection of $\nabla^2 u$ onto the space tangent to $\partial \Omega$, which obviously has the same meaning as $(\nabla^2 u \cdot \nu)_{\partial \Omega}$ here.  

(2) As before, let $B_1 \subset \mathbb{R}^n$ be the ball of same volume as $\Omega$. When $\tau > 0$, $\sigma = 0$, Chasman [6] proved the following isoperimetric inequality

$$\Lambda_1(\Omega) \leq \Lambda_1(B_1), \quad \text{with equality if and only if} \quad \Omega \text{ is a ball.}$$

When $\tau > 0$, $0 \leq \sigma < 1$, for the eigenvalue problem (1.12), we can obtain the following:

**Theorem 1.5** Let $\Omega$, $|\Omega|$ and $\omega_n$ be as in Theorem 1.2 and let $\Lambda_j(\Omega)$ be the $j$-th eigenvalue of the system (1.12).

(i) When $\tau \geq 0$ and $0 \leq \sigma < 1$, we have

$$\sum_{j=1}^{k} \Lambda_j(\Omega) \leq (2\pi)^4 \frac{n}{(n + 4)} k^{\frac{4}{n-\sigma}} \left( \frac{1}{\omega_n |\Omega|} \right)^{\frac{4}{n}} + \tau (2\pi)^2 \frac{n}{(n + 2)} k^{\frac{4}{n-\sigma}} \left( \frac{1}{\omega_n |\Omega|} \right)^{\frac{4}{n}}, \quad k \geq 1; \quad (1.14)$$

(ii) When $\tau = 0$ and $0 \leq \sigma < 1$, it holds

$$\Lambda_{k+1}(\Omega) \leq (2\pi)^4 \left( \frac{n + 4}{4\omega_n |\Omega|} \right)^{\frac{4}{n}} k^{\frac{4}{n}}, \quad k \geq 0; \quad (1.15)$$

(iii) When $\tau > 0$ and $0 \leq \sigma < 1$, we have

$$\Lambda_{k+1}(\Omega) \leq \min_{r > 2\pi \left( \frac{k}{\omega_n |\Omega|} \right)^{\frac{4}{n}}} \frac{n \omega_n |\Omega| \left( \frac{n + 4}{n-\sigma} + \tau \frac{n + 3}{n+2} \right)}{\omega_n |\Omega|^{n} - k (2\pi)^n}, \quad k \geq 0. \quad (1.16)$$

**Remark 1.6**

(1) Recently, when $\tau \geq 0$, $\sigma = 0$, Brandolini, Chiacchio and Langford [2] have already obtained upper bounds for the sum of the first $k$ eigenvalues $\Lambda_j(\Omega)$ and for the $(k+1)$-th eigenvalue $\Lambda_{k+1}(\Omega)$. Inspired by this fact and our Theorem 1.5 here, together with the coercivity argument for the sesquilinear form shown in [7, Section 4], the corresponding author, Prof. J. Mao, and his another collaborator can also get the estimates (1.14)-1.16 under a more general setting that $\tau \geq 0$, $\sigma \in (-1/(n - 1), 1)$ – see [15] Theorem 1.1 and Corollary 1.2 for details. Although [15] has been published formally very recently, we still prefer to
remain Theorem 1.3 to emphasize and embody the origin and continuity of our thought.

(2) Clearly, if $\tau = 0$ and $\sigma = 1$, then

$$
\begin{align*}
\Delta^2 u &= \Lambda u & \text{in } \Omega, \\
\Delta u &= \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \Omega,
\end{align*}
$$

(1.17)

which corresponds to the $p = 2$ case of the eigenvalue problem (1.6) except the boundary condition $\frac{\partial u}{\partial \nu} = 0$ missed. At end of [7] Section 4, Chasman showed that for the eigenvalue problem (1.17), all $H^2(\Omega)$ harmonic functions are eigenfunctions with eigenvalue zero, and one has at least an eigenvalue of infinite multiplicity. Based on this fact, we need to expel $\tau = 0$, $\sigma = 1$ in Theorem 1.5 here and add the boundary condition $\frac{\partial u}{\partial \nu} = 0$ in the previous eigenvalue problem (1.6).

We also consider the following Steklov-type eigenvalue problem of the biharmonic operator

$$
\begin{align*}
\Delta^2 u - \tau \Delta u &= 0 & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u}{\partial \nu^2} + \sigma \Delta u &= 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u}{\partial \nu} - (1 - \sigma) \text{div}_{\partial \Omega} \left( \nabla^2 u \cdot \nu \right) - \frac{\partial \Delta u}{\partial \nu} &= \lambda u & \text{on } \partial \Omega,
\end{align*}
$$

(1.18)

where $\tau, \sigma \in \mathbb{R}$ and other same symbols have the same meanings as those in (1.12).

**Remark 1.7** (1) Li and Mao [16] Theorem 2.1] showed clearly that if $\tau > 0$ and $\sigma \in (-1/(n - 1), 1)$, the eigenvalue problem (1.18) has the discrete spectrum and its elements (i.e., eigenvalues) can be listed non-decreasingly as follows

$$
0 = \lambda_0(\Omega) < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_k(\Omega) \leq \cdots \uparrow \infty.
$$

By means of variational principle, the Rayleigh-Ritz type characterization of the $(k+1)$-th eigenvalue $\lambda_{k+1}(\Omega)$ is given by

$$
\lambda_{k+1}(\Omega) = \inf_{0 \neq u \in H^2(\Omega)} \left\{ \frac{\int_{\Omega} \left[ (1 - \sigma) |\nabla^2 u|^2 + \sigma |\Delta u|^2 + \tau |\nabla u|^2 \right] \Omega - \int_{\partial \Omega} uu_j = 0, \right. \\
\left. j = 0, 1, \cdots, k \right\},
$$

(1.19)

where $u_j$ is an eigenfunction corresponding to the eigenvalue $\lambda_j(\Omega)$. Besides, the eigenfunction $u_0$ of $\lambda_0(\Omega) = 0$ should be nonzero constant function.

(2) When $\tau > 0$, $\sigma = 0$, Buoso and Provenzano [5] proved an isoperimetric inequality for the fundamental tone $\lambda_1(\Omega)$ of the system (1.18) which states that

$$
\lambda_1(\Omega) \leq \lambda_1(B_\Omega),
$$

with equality if and only if $\Omega$ is a ball. Here, as before, $B_\Omega \subset \mathbb{R}^n$ is the ball of same volume as $\Omega$. Very recently, Li and Mao [16] Theorem 1.1] showed that the above isoperimetric inequality is still true for $\tau > 0$ and $\sigma \in (-1/(n - 1), 1)$, and moreover, the inequality can be achieved when $\Omega$ is the ball $B_\Omega$.

(3) For some other estimates for $\lambda_i$’s, see, e.g., [3, 4, 5, 10, 19].

Our next result is a sharp lower bound for the sum of the reciprocals of the first $n$ nonzero eigenvalues of the problem (1.18).

**Theorem 1.8** Let $\Omega$ and $|\Omega|$ be as in Theorem 1.3 and let $\lambda_j(\Omega)$ be the $j$-th eigenvalue of the system (1.18). When $\tau > 0$ and $\sigma \in (-1/(n - 1), 1)$, we have

$$
\sum_{j=1}^{n} \frac{1}{\lambda_j(\Omega)} \geq \frac{|\partial \Omega|^2}{\tau|\Omega| \int_{\partial \Omega} |u|^2},
$$

(1.20)
where $H$ is the mean curvature vector of $\partial \Omega$ in $\mathbb{R}^n$, $|\partial \Omega|$ denotes the area of $\partial \Omega$. Equality in (1.20) holds if and only if $\Omega$ is a ball.

Using the monotonicity of eigenvalues $\lambda_i$'s and Theorem 1.8 immediately, we get

$$\frac{n}{\lambda_1(\Omega)} \geq \sum_{j=1}^{n} \frac{1}{\lambda_j(\Omega)} \geq \frac{|\partial \Omega|^2}{\tau |\Omega| \int_{\partial \Omega} |H|^2},$$

which directly implies the following Reilly-type eigenvalue estimate.

**Corollary 1.9** Under the assumptions in Theorem 1.8, we have

$$\lambda_1(\Omega) \leq n \tau \frac{|\Omega|}{|\partial \Omega|^2} \int_{\partial \Omega} |H|^2,$$

with equality holds if and only if $\Omega$ is a ball.

**Remark 1.10** Clearly, when the Reilly-type eigenvalue estimate in Corollary 1.9 attains the equality case, one also has $\lambda_1(\Omega) = \lambda_2(\Omega) = \cdots = \lambda_n(\Omega)$.

Our final result is a sharp lower bound for the sum of the reciprocals of the first $n$ nonzero eigenvalues of the Laplacian on a closed submanifold immersed in a Euclidean space. Namely, we have:

**Theorem 1.11** Let $M$ be an $n$-dimensional compact submanifold without boundary isometrically immersed in $\mathbb{R}^N$ and let $\eta_j(M)$ be the $j$-th nonzero closed eigenvalue of the Laplacian on $M$. We have

$$\sum_{j=1}^{n} \frac{1}{\eta_j(M)} \geq \frac{|M|}{\int_M |H|^2},$$

where $H$ is the mean curvature vector of $M$ in $\mathbb{R}^N$. Moreover, when $n = N - 1$, equality holds in (1.21) if and only if $M$ is a hypersphere of $\mathbb{R}^N$, and when $n < N - 1$, if the equality holds in (1.21), then $M$ is a minimal submanifold of some hypersphere of $\mathbb{R}^N$.

Using the monotonicity of nonzero closed eigenvalues $\eta_i$’s of the Laplacian and Theorem 1.11 immediately, we get

$$\frac{n}{\eta_1(M)} \geq \sum_{j=1}^{n} \frac{1}{\eta_j(M)} \geq \frac{|M|}{\int_M |H|^2},$$

which directly implies the following Reilly’s eigenvalue estimate (i.e., the main result of the influential paper [19]).

**Corollary 1.12** Under the assumptions in Theorem 1.11, we have

$$\eta_1(M) \leq \frac{n}{|M|} \int_M |H|^2,$$

and moreover, the equality holds implies the rigidity described as in Theorem 1.11.

**Remark 1.13** (1) Clearly, when the Reilly-type eigenvalue estimate in Corollary 1.12 attains the equality case, one also has $\eta_1(M) = \eta_2(M) = \cdots = \eta_n(M)$, and if furthermore $n = N - 1$, then $\eta_{n+2}(M) > \eta_{n+1}(M) = \eta_i(M)$ for $i = 1, 2, \cdots, n$, since the multiplicity of the first nonzero closed eigenvalue of the Laplacian on any $n$-sphere in $\mathbb{R}^{n+1}$ is $n + 1$ and the corresponding eigenfunctions are the restrictions (to $n$-sphere) of $n + 1$ coordinate functions of $\mathbb{R}^{n+1}$ (see, e.g., [8, Chapter 2] for this fact).

(2) Except Reilly’s estimate for the first nonzero eigenvalue of the Laplacian (see [19] or Corollary 1.12 here) and our Reilly-type estimate for the first nonzero eigenvalue of (1.18) – the Steklov-type eigenvalue problem of the biharmonic operator (see Corollary 1.9), some interesting Reilly-type estimates for the first nonzero eigenvalue of other type have also been obtained. For instance, Ilias and Makhoul [11] have obtained
the Reilly-type estimate for the first nonzero Steklov eigenvalue of the Laplacian on compact submanifolds (with boundary) isometrically immersed in a Euclidean space; Du and Mao [9] have obtained the Reilly-type estimate for the first nonzero closed eigenvalue of the nonlinear $p$-Laplacian ($1 < p < +\infty$) on compact submanifolds (without boundary) isometrically immersed into a Euclidean space, a unit sphere, or even a projective space.

For convenience, in the sequel, we prefer to simplify the notations for four types eigenvalues discussed in this paper, that is, we separately write $\Gamma_i(\Omega)$, $\Lambda_i(\Omega)$, $\lambda_i(\Omega)$ and $\eta_i(M)$ as $\Gamma_i$, $\Lambda_i$, $\lambda_i$ and $\eta_i$. We also make an agreement that these notations would be written completely if necessary.

This paper is organized as follows. In Section 3 we will prove Li-Yau-Kröger type estimates for lower-order eigenvalues of the Neumann-type eigenvalue problem (1.6) of the poly-harmonic operator and the Neumann-type eigenvalue problem (1.12) of the biharmonic operator. Two sharp extrinsic lower bounds for the sum of the reciprocals of the first $n$ nonzero eigenvalues of the Steklov-type eigenvalue problem (1.18) and for the sum of the reciprocals of the first $n$ nonzero closed eigenvalues of the Laplacian will be separately proven in Section 4.

2 Li-Yau-Kröger type estimates

In this section, inspired by [2, 12, 14], using the method of Fourier transformation, together with the Rayleigh-Ritz type characterizations (1.7), (1.8) and (1.13), we can separately give the proofs of two Li-Yau-Kröger type estimates by appropriately constructing trial functions.

First, we have:

**Proof of Theorem 1.2** Let $\{\phi_j\}_{j=1}^\infty$ be a set of orthonormal eigenfunctions of the system (1.6), that is,

$$
\begin{aligned}
& (-\Delta)^p \phi_j = \Gamma_j \phi_j & \text{in } \Omega, \\
& \frac{\partial \phi_j}{\partial \nu} = \Delta^m \phi_j = \cdots = \Delta^{2m-1} \phi_j = \frac{\partial \Delta^{2m-1} \phi_j}{\partial \nu} = 0 & \text{on } \partial \Omega, \text{ when } p = 2m, \\
& \frac{\partial \phi_j}{\partial \nu} = 0 & \text{on } \partial \Omega, \text{ when } p = 1, \\
& \frac{\partial \phi_j}{\partial \nu} = \Delta^{m-1} \phi_j = \cdots = \Delta^{2m-2} \phi_j = \frac{\partial \Delta^{2m-2} \phi_j}{\partial \nu} = 0 & \text{on } \partial \Omega, \text{ when } p = 2m - 1, m > 1,
\end{aligned}
$$

Define

$$
\hat{\Phi}(x, y) = \sum_{j=1}^k \phi_j(x) \phi_j(y), \quad x, y \in \Omega,
$$

and let

$$
\hat{\phi}(z, y) = \frac{1}{(2\pi)^k} \int_\Omega \hat{\Phi}(x, y) e^{ix \cdot z} dx
$$

be the Fourier transform of $\Phi$ in the variable $x$.

Since

$$
(2\pi)^k \hat{\phi}(z, y) = \sum_{j=1}^k \phi_j(y) \int_\Omega \phi_j(x) e^{ix \cdot z} dx
$$

is the orthogonal projection of the function $h_z(x) = e^{ix \cdot z}$ onto the subspace of $L^2(\Omega)$ spanned by $\phi_1, \cdots, \phi_k$, $\rho(z, y) = h_z(y) - (2\pi)^k \hat{\phi}(z, y)$ can be used as a trial function for $\Gamma_{k+1}$. Thus, we have from (1.7) and (1.8) that

$$
\Gamma_{k+1} \int_\Omega |\rho(z, y)|^2 dydz \leq \begin{cases} 
\int_\Omega |\Delta^n \rho(z, y)|^2 dydz, & \text{if } p = 2m, \\
\int_\Omega |\nabla_y (\nabla_y^{n-1} \rho(z, y))|^2 dydz, & \text{if } p = 2m - 1.
\end{cases}
$$
Integrating both sides of the above inequality over $B_r = \{ z \in \mathbb{R}^n ||z| < r \}$ yields

$$
\Gamma_{k+1} \leq \begin{cases} 
\inf_{r} \int_{B_r} \int_{\Omega} |\nabla^m \rho(z,y)|^2 dydz, & \text{when } p = 2m, \\
\inf_{r} \int_{B_r} \int_{\Omega} |\nabla^m \rho(z,y)|^2 dydz, & \text{when } p = 2m - 1,
\end{cases}
(2.1)
$$

where $r > 2\pi \left( \frac{k}{2m+1} \right)^{\frac{1}{n}}$. Noticing $|h_z(y)| = 1$ and $\hat{\Phi}(z,y) = \sum_{j=1}^{k} \phi_j(y) \hat{\phi}_j(z)$, we have

$$
\int_{B_r} \int_{\Omega} |\rho(z,y)|^2 dydz = \int_{B_r} \int_{\Omega} |h_z(y) - (2\pi)^{\frac{n}{2}} \hat{\Phi}(z,y)|^2 dydz = |h_z(y)|^2 - 2(2\pi)^{\frac{n}{2}} \text{Re} \left( \int_{B_r} \int_{\Omega} h_z(y) \overline{\hat{\Phi}(z,y)} dydz \right) + (2\pi)^n ||\hat{\Phi}(z,y)||^2
$$

$$
= \omega_n |r|^n - 2(2\pi)^{\frac{n}{2}} \text{Re} \left( \sum_{j=1}^{k} \int_{B_r} \int_{\Omega} e^{iy \cdot z} \phi_j(y) \overline{\hat{\phi}_j(z)} dydz \right)
$$

$$
+ (2\pi)^n \sum_{j=1}^{k} \int_{B_r} \int_{\Omega} \phi_j(y) \phi_l(y) \hat{\phi}_j(z) \overline{\hat{\phi}_l(z)} dydz
$$

$$
= \omega_n |r|^n - (2\pi)^n \sum_{j=1}^{k} \int_{B_r} |\phi_j(z)|^2 dz,
(2.2)
$$

where $||f||^2 = \int_{B_r} \int_{\Omega} |f|^2 dydz$.

From $h_z(y) y_p = (e^{iy \cdot z}) y_p = iz_p e^{iy \cdot z} = iz_p h_z(y)$, we get

$$
\nabla_y h_z(y) = \sum_{p=1}^{n} h_z(y) y_p y_p = -|z|^2 h_z(y),
$$

which gives

$$
|\nabla^m h_z(y)|^2 = |z|^{4m},
$$

and

$$
|\nabla^m h_z(y)|^2 = |z|^{4m-2}.
$$

Using integration by parts, we infer from the boundary condition of $(1.6)$ for $p = 2m$ that

$$
\int_{\Omega} \nabla_y^{m} h_z(y) \overline{\nabla_y^{m} \hat{\Phi}(z,y)} dy = \int_{\Omega} \nabla_y^{m-1} h_z(y) \overline{\nabla_y^{m+1} \hat{\Phi}(z,y)} dy
$$

$$
- \int_{\partial \Omega} \left( \nabla_y^{m-1} h_z(y) \frac{\partial \nabla_y^{m} \hat{\Phi}(z,y)}{\partial \nu} - \nabla_y^{m} \hat{\Phi}(z,y) \frac{\partial \nabla_y^{m-1} h_z(y)}{\partial \nu} \right) dy
$$

$$
= \int_{\Omega} \nabla_y^{m-1} h_z(y) \overline{\nabla_y^{m+1} \hat{\Phi}(z,y)} dy
$$

$$
= \int_{\Omega} h_z(y) \nabla_y^{2m} \hat{\Phi}(z,y) dy
$$

$$
- \int_{\partial \Omega} \left( h_z(y) \frac{\partial \nabla_y^{m-1} \hat{\Phi}(z,y)}{\partial \nu} - \nabla_y^{2m-1} \hat{\Phi}(z,y) \frac{\partial h_z(y)}{\partial \nu} \right) dy
$$

$$
= \int_{\Omega} h_z(y) \nabla_y^{2m} \hat{\Phi}(z,y) dy.
$$
So, when \( p = 2m \), we have

\[
\int_{B_r} \int_{\Omega} |\Delta_y^m \rho(z, y)|^2 dydz - \int_{B_r} \int_{\Omega} \|\Delta_y^m h_z(y) - (2\pi)^\frac{p}{n} \Delta_y^m \Phi(z, y)\|^2 dydz \\
= \|\Delta_y^m h_z(y)\|^2 - 2(2\pi)^\frac{p}{n} \text{Re} \left( \int_{B_r} \int_{\Omega} \Delta_y^m h_z(y) \overline{\Delta_y^m \Phi(z, y)} dydz \right) + (2\pi)^p \|\Delta_y^m \Phi(z, y)\|^2 \\
= \frac{n \pi^{m+4m}}{n+4m} \omega_n |\Omega| - 2(2\pi)^\frac{p}{n} \text{Re} \left( \int_{B_r} \int_{\Omega} h_z(y) \overline{\Delta_y^{2m} \Phi(z, y)} dydz \right) \\
+ (2\pi)^n \int_{B_r} \int_{\Omega} \left( \sum_{l_1=1}^{k} \Delta_y^{m} \phi_{l_1}(y) \overline{\phi_{l_1}(z)} \right) \left( \sum_{l_2=1}^{k} \Delta_y^{m} \phi_{l_2}(y) \overline{\phi_{l_2}(z)} \right) dydz \\
= \frac{n \pi^{m+4m}}{n+4m} \omega_n |\Omega| - 2(2\pi)^n \sum_{j=1}^{k} \Gamma_j \int_{B_r} \int_{\Omega} e^{iy \cdot z} \phi_j(y) \overline{\phi_j(z)} dydz \\
+ (2\pi)^n \sum_{l_1, l_2=1}^{k} \Gamma_{l_1} \int_{B_r} \int_{\Omega} \phi_{l_1}(y) \overline{\phi_{l_1}(z)} \phi_{l_2}(y) \overline{\phi_{l_2}(z)} dydz \\
= \frac{n \pi^{m+4m}}{n+4m} \omega_n |\Omega| - (2\pi)^n \sum_{j=1}^{k} \Gamma_j \int_{B_r} |\phi_j(z)|^2 dz. \quad (2.3)
\]

Similarly, by using integration by parts, we can infer from the boundary condition of \([116]\) for \( p = 2m - 1 \) that

\[
\int_{\Omega} \nabla_y (\Delta_y^{m-1} h_z(y)) \cdot \nabla_y (\overline{\Delta_y^{m-1} \Phi(z, y)}) dy \\
= -\int_{\Omega} \Delta_y^{m-1} h_z(y) \overline{\Delta_y^{m-1} \Phi(z, y)} dy + \int_{\partial \Omega} \Delta_y^{m-1} h_z(y) \frac{\partial \overline{\Delta_y^{m-1} \Phi(z, y)}}{\partial \nu} dy \\
= -\int_{\Omega} \Delta_y^{m-1} h_z(y) \overline{\Delta_y^{m-1} \Phi(z, y)} dy \\
= \cdots \cdots \cdots \\
= -\int_{\Omega} h_z(y) \overline{\Delta_y^{2m-1} \Phi(z, y)} dy \\
- \int_{\partial \Omega} \left( h_z(y) \frac{\partial \overline{\Delta_y^{2m-2} \Phi(z, y)}}{\partial \nu} - \overline{\Delta_y^{2m-2} \Phi(z, y)} \frac{\partial h_z(y)}{\partial \nu} \right) dy \\
= -\int_{\Omega} h_z(y) \overline{\Delta_y^{2m-1} \Phi(z, y)} dy.
\]
Therefore,
\[
\int_{B_r} \int_{\Omega} \left| \nabla_y (\Delta_y^{m-1} \rho(z, y)) \right|^2 \, dy \, dz
\]
\[
= \int_{B_r} \int_{\Omega} \left| \nabla_y (\Delta_y^{m-1} h_z(y)) - (2\pi)^\frac{m}{2} \nabla_y (\Delta_y^{m-1} \hat{\Phi}(z, y)) \right|^2 \, dy \, dz
\]
\[
= \left\| \nabla_y (\Delta_y^{m-1} h_z(y)) \right\|^2 - 2(2\pi)^\frac{m}{2} \text{Re} \left( \int_{B_r} \int_{\Omega} \nabla_y (\Delta_y^{m-1} h_z(y)) \cdot \nabla_y (\Delta_y^{m-1} \hat{\Phi}(z, y)) \, dy \, dz \right)
\]
\[
= \frac{n^{m+4m-2}}{n + 4m - 2} \omega_n |\Omega| + 2(2\pi)^\frac{m}{2} \text{Re} \left( \int_{B_r} \int_{\Omega} h_z(y) \Delta_y^{m-1} \hat{\Phi}(z, y) \, dy \, dz \right)
\]
\[
- (2\pi)^n \int_{B_r} \int_{\Omega} \left( \sum_{l_1=1}^{k} \Delta_y^{m-1} \phi_{l_1}(y) \hat{\phi}_{l_1}(z) \right) \left( \sum_{l_2=1}^{k} \Delta_y^{m-1} \phi_{l_2}(y) \hat{\phi}_{l_2}(z) \right) \, dy \, dz
\]
\[
= \frac{n^{m+4m-2}}{n + 4m - 2} \omega_n |\Omega| - 2(2\pi)^n \sum_{j=1}^{k} \Gamma_j \int_{B_r} \int_{\Omega} e^{i|y-z| \phi_j(y) \hat{\phi}_j(z)} \, dy \, dz
\]
\[
- (2\pi)^n \sum_{l_1,l_2=1}^{k} \Gamma_{l_1} \int_{B_r} \int_{\Omega} \phi_{l_1}(y) \hat{\phi}_{l_1}(z) \phi_{l_2}(y) \hat{\phi}_{l_2}(z) \, dy \, dz
\]
\[
= \frac{n^{m+4m-2}}{n + 4m - 2} \omega_n |\Omega| - (2\pi)^n \sum_{j=1}^{k} \Gamma_j \int_{B_r} \left| \hat{\phi}_j(z) \right|^2 \, dz.
\]
(2.4)

Substituting (2.2) and (2.3) or (2.4) into (2.1) yields
\[
\Gamma_{k+1} \leq \inf \left\{ \frac{n^{m+2p} \omega_n |\Omega| - (2\pi)^n \sum_{j=1}^{k} \Gamma_j \int_{B_r} \left| \hat{\phi}_j(z) \right|^2 \, dz}{\omega_n |\Omega| r^n - (2\pi)^n \sum_{j=1}^{k} \int_{B_r} \left| \hat{\phi}_j(z) \right|^2 \, dz} \right\},
\]
where the infimum is taken over \( r > 2\pi \left( \frac{k}{\omega_n |\Omega|} \right)^\frac{1}{n} \). By Plancherel's Theorem,
\[
c_j = \int_{B_r} \left| \hat{\phi}_j(z) \right|^2 \, dz \leq 1 \quad \text{for} \quad j = 1, \ldots, k.
\]
(2.6)

Combining (2.5) and (2.6), one gets
\[
\Gamma_{k+1} \left( \omega_n |\Omega| r^n - (2\pi)^n \sum_{j=1}^{k} c_j \right) \leq \frac{n}{n + 2p} \omega_n |\Omega| r^{n+2p} - (2\pi)^n \sum_{j=1}^{k} \Gamma_j c_j,
\]
that is,
\[
\Gamma_{k+1} \omega_n |\Omega| r^n - \frac{n}{n + 2p} \omega_n |\Omega| r^{n+2p} \leq (2\pi)^n \sum_{j=1}^{k} c_j - (2\pi)^n \sum_{j=1}^{k} \Gamma_j c_j \leq (2\pi)^n \sum_{j=1}^{k} (\Gamma_{k+1} - \Gamma_j).
\]

Solving the above inequality for \( \sum_{j=1}^{k} \Gamma_j \), we have
\[
(2\pi)^n \sum_{j=1}^{k} \Gamma_j \leq \frac{n}{n + 2p} \omega_n |\Omega| r^{n+2p} + (k(2\pi)^n - \omega_n |\Omega| r^n) \Gamma_{k+1}.
\]
Since \( r > 2\pi \left( \frac{k}{\omega_n|\Omega|} \right)^{\frac{1}{n}} \), then \( k(2\pi)^n - \omega_n|\Omega|r^n < 0 \), and we infer from the above inequality that
\[
\sum_{j=1}^{k} \Gamma_j \leq \frac{n\omega_n|\Omega|r^{n+2p}}{(n+2p)(2\pi)^n}.
\]

Letting \( r \to 2\pi \left( \frac{k}{\omega_n|\Omega|} \right)^{\frac{1}{n}} \), (1.10) follows.

Combining (2.4) and (2.5), we have
\[
\Gamma_{k+1} \leq \frac{n\omega_n|\Omega|}{\omega_n|\Omega|r^n - k(2\pi)^n} = F(r), \quad r > 2\pi \left( \frac{k}{\omega_n|\Omega|} \right)^{\frac{1}{n}}.
\]

Solving \( F'(r) = 0 \), we get
\[
r = 2\pi \left( \frac{(n+2p)k}{2p\omega_n|\Omega|} \right)^{\frac{1}{n}}.
\]

Taking the above value of \( r \) into (2.7), we have (1.11).

At the end of this section, we also have:

**Proof of Theorem 1.5** Let \( \{\psi_j\}_{j=1}^{\infty} \) be the set of orthonormal eigenfunctions of the system (1.12), that is,
\[
\begin{cases}
\Delta^2 \psi_j - \tau \Delta \psi_j = \Lambda_j \psi_j & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 \psi_j}{\partial x^2} + \sigma \Delta \psi_j = 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial \psi_j}{\partial n} - (1 - \sigma) \text{div}_{\partial \Omega} \left( \nabla^2 \psi_j \cdot \nu \right) - \frac{\partial \psi_j}{\partial n} = 0 & \text{on } \partial \Omega,
\end{cases}
\]

As in the proof of Theorem 1.2 we know that
\[
(2\pi)^\frac{1}{n} \hat{\psi}(z, y) = \sum_{j=1}^{k} \psi_j(y) \int_{\Omega} \psi_j(x)e^{ix \cdot z} dx
\]
is the orthogonal projection of the function \( h(z) = e^{ix \cdot z} \) onto the subspace of \( L^2(\Omega) \) spanned by \( \psi_1, \cdots, \psi_k \).

Thus, we can use \( \varphi(z, y) = h(z) - (2\pi)^\frac{1}{n} \hat{\psi}(z, y) \) as a trial function for \( \Lambda_{k+1} \) to obtain
\[
\Lambda_{k+1} \int_{\Omega} |\varphi(z, y)|^2 dy dz \leq \int_{\Omega} \left[ (1 - \sigma)|\nabla_y \varphi(z, y)|^2 + \sigma|\overline{\Delta_y \varphi(z, y)}|^2 + \tau|\nabla_y \varphi(z, y)|^2 \right] dy dz.
\]

Integrating both sides of the above inequality over \( B_r = \{z \in \mathbb{R}^n | |z| < r \} \) yields
\[
\Lambda_{k+1} \leq \inf_r \left\{ \frac{\int_{B_r} \int_{\Omega} \left[ (1 - \sigma)|\nabla_y \varphi(z, y)|^2 + \sigma|\overline{\Delta_y \varphi(z, y)}|^2 + \tau|\nabla_y \varphi(z, y)|^2 \right] dy dz}{\int_{B_r} \int_{\Omega} |\varphi(z, y)|^2 dy dz} \right\},
\]
where \( r > 2\pi \left( \frac{k}{\omega_n|\Omega|} \right)^{\frac{1}{n}} \). By a similar calculation to (2.2), we have
\[
\int_{B_r} \int_{\Omega} |\varphi(z, y)|^2 dy dz = \omega_n|\Omega|r^n - (2\pi)^n \sum_{j=1}^{k} \int_{B_r} |\hat{\psi}_j(z)|^2 dz.
\]

Let
\[
P = \int_{B_r} \int_{\Omega} \left[ (1 - \sigma)|\nabla_y \varphi(z, y)|^2 + \sigma|\overline{\Delta_y \varphi(z, y)}|^2 + \tau|\nabla_y \varphi(z, y)|^2 \right] dy dz = P_1 + P_2 + P_3,
\]
where

\[ P_1 = \int_{B_r} \int_{\Omega} \left( (1 - \sigma) |\nabla_y^2 h_z(y)|^2 + \sigma |\Delta_y h_z(y)|^2 + \tau |\nabla_y h_z(y)|^2 \right) dydz, \]

\[ P_2 = -2(2\pi)^2 \text{Re} \left\{ \int_{B_r} \int_{\Omega} \left( (1 - \sigma) \nabla^2_y \hat{\Psi}(z, y) \cdot \nabla^2_y \hat{\Psi}(z, y) + \sigma \Delta_y \hat{\Psi}(z, y) \Delta_y \hat{\Psi}(z, y) \right) dydz \right\}, \]

\[ P_3 = \int_{B_r} \int_{\Omega} \left( (1 - \sigma) |\nabla^2_y \hat{\Psi}(z, y)|^2 + \sigma |\Delta_y \hat{\Psi}(z, y)|^2 + \tau |\nabla_y \hat{\Psi}(z, y)|^2 \right) dydz. \]

Since \( |h_z(y)_{y_p}| = |z_p| \) and \( |h_z(y)_{y_py_q}| = |z_p||z_q| \), then \( |\Delta_y h_z(y)| = |z|^2 \), \( \nabla_y h_z(y) = |z| \) and

\[ |\nabla^2_y h_z(y)|^2 = \sum_{p,q=1}^{n} |h_z(y)_{y_py_q}|^2 = \sum_{p,q=1}^{n} |z_p|^2 |z_q|^2 = |z|^4. \]

So, we have

\[ P_1 = n\omega_n|\Omega| \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right). \tag{2.10} \]

Integrating by parts and noticing \( \hat{\Psi}(z, y) = \sum_{j=1}^{k} \psi_j(y)\tilde{\psi}_j(z) \), it follows that

\[ P_2 = -2(2\pi)^2 \text{Re} \left\{ \int_{B_r} \int_{\Omega} \left( (1 - \sigma) h_z(y) \Delta^2_y \hat{\Psi}(z, y) + \sigma h_z(y) \Delta^2_y \hat{\Psi}(z, y) \right. \right. \]

\[ - \left. \left. \tau h_z(y) \Delta_y \hat{\Psi}(z, y) \right) dydz \right\} \]

\[ = -2(2\pi)^2 \sum_{j=1}^{n} \Lambda_j \int_{B_r} |\tilde{\psi}_j(z)|^2 dz \tag{2.11} \]

and

\[ P_3 = \int_{B_r} \int_{\Omega} \left( (1 - \sigma) |\nabla^2_y \hat{\Psi}(z, y)|^2 + \sigma |\Delta_y \hat{\Psi}(z, y)|^2 + \tau |\nabla_y \hat{\Psi}(z, y)|^2 \right) dydz \]

\[ = \int_{B_r} \int_{\Omega} \hat{\Psi}(z, y) \left( \Delta^2_y - \tau \Delta_y \right) \hat{\Psi}(z, y) dydz \]

\[ = (2\pi)^2 \sum_{j=1}^{k} \Lambda_j \int_{B_r} |\tilde{\psi}_j(z)|^2 dz. \tag{2.12} \]

Combining (2.8) (2.12), we have

\[ \Lambda_{k+1} \leq \inf_r \left\{ \frac{\omega_n|\Omega| \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right) - (2\pi)^2 \sum_{j=1}^{k} \Lambda_j \int_{B_r} |\tilde{\psi}_j(z)|^2 dz}{\omega_n|\Omega|r^n - (2\pi)^2 \sum_{j=1}^{k} \int_{B_r} |\psi_j(z)|^2 dz} \right\}. \tag{2.13} \]

Letting

\[ c_j = \int_{B_r} |\psi_j(z)|^2 dz \leq 1 \quad \text{for } j = 1, \ldots, k, \tag{2.14} \]

we deduce from (2.13) that

\[ \Lambda_{k+1} \left( \omega_n|\Omega|r^n - (2\pi)^2 \sum_{j=1}^{k} c_j \right) \leq n\omega_n|\Omega| \left( \frac{r^{n+4}}{n+4} + \tau \frac{r^{n+2}}{n+2} \right) - (2\pi)^2 \sum_{j=1}^{k} \Gamma_j c_j, \]
which implies that
\[
\Lambda_{k+1} \omega_n |\Omega|^n - n \omega_n |\Omega| \left( \frac{r^{n+4}}{n+4} + \frac{\tau r^{n+2}}{n+2} \right) \leq (2\pi)^n \sum_{j=1}^k \left( \Gamma_{k+1} - \Gamma_j \right).
\]
Hence,
\[
(2\pi)^n \sum_{j=1}^k \Lambda_j \leq n \omega_n |\Omega| \left( \frac{r^{n+4}}{n+4} + \frac{\tau r^{n+2}}{n+2} \right) + (k(2\pi)^n - \omega_n |\Omega|^n) \Gamma_{k+1}.
\]
Since \( r > 2\pi \left( \frac{k}{\omega_n |\Omega|} \right)^\frac{1}{n} \), we infer from the above inequality that
\[
\sum_{j=1}^k \Lambda_j \leq \left( \frac{r^{n+4}}{n+4} + \frac{\tau r^{n+2}}{n+2} \right) \frac{n \omega_n |\Omega|}{(2\pi)^n}.
\]
Letting \( r \to 2\pi \left( \frac{k}{\omega_n |\Omega|} \right)^\frac{1}{n} \), one gets (1.14).
Combining (2.13) and (2.14), we have
\[
\Lambda_{k+1} \leq \frac{r^{n+4}}{n+4} + \frac{\tau r^{n+2}}{n+2} \frac{n \omega_n |\Omega|}{(2\pi)^n}, \quad \forall r > 2\pi \left( \frac{k}{\omega_n |\Omega|} \right)^\frac{1}{n}.
\]
Consequently, we have
\[
\Lambda_{k+1}(\Omega) \leq \min_{r > 2\pi \left( \frac{k}{\omega_n |\Omega|} \right)^\frac{1}{n}} \frac{n \omega_n |\Omega|}{(2\pi)^n} \left( \frac{r^{n+4} + \tau r^{n+2}}{n+4} \right), \quad k \geq 0.
\]
For the case \( \tau = 0 \), solving \( F'(r) = 0 \) yields
\[
r = 2\pi \left( \frac{n+2p}{2p \omega_n |\Omega|} \right)^\frac{1}{n}.
\]
Taking the above value of \( r \) into (2.15), we have (1.15).

3 Reilly type estimates

In the last section, by using the QR-factorization theorem and the variational principle, we can give the proofs of two sharp extrinsic lower bounds for the sum of the reciprocals of the first \( n \) nonzero eigenvalues (given in Theorems 1.8 and 1.11) by constructing appropriately trial functions. In fact, we have already used the method of QR-factorization (together with other approaches) to try to get estimates for the sum of the reciprocals of the first \( n \) nonzero eigenvalues of prescribed eigenvalue problems (see, e.g., [17]).

First, we have:

**Proof of Theorem 1.8** Let \( x_1, \ldots, x_n \) be the coordinate functions in \( \mathbb{R}^n \). Since \( \Omega \) is a bounded domain in \( \mathbb{R}^n \), we can regard \( \partial \Omega \) as a closed hypersurface of \( \mathbb{R}^n \) without boundary.

Let \( u_j \) be an eigenfunction corresponding to the eigenvalue \( \lambda_j \) such that \( \{ u_j \}_{j=0}^\infty \) is an orthonormal basis of \( L^2(\partial \Omega) \), that is,
\[
\begin{aligned}
\bar{\Delta} u_j - \tau \Delta u_j &= 0 & \text{in } \Omega, \\
(1 - \sigma) \frac{\partial^2 u_j}{\partial \nu^2} + \sigma \Delta u_j &= 0 & \text{on } \partial \Omega, \\
\tau \frac{\partial u_j}{\partial \nu} - (1 - \sigma) \text{div}_{\partial \Omega} \left( \nabla^2 u_j \cdot \nu \right) - \frac{\partial u_j}{\partial \nu} &= -\lambda_j u_i & \text{on } \partial \Omega, \\
\int_{\partial \Omega} u_i u_j &= \delta_{ij}.
\end{aligned}
\]
Observe that \( u_0 = 1/\sqrt{|\partial \Omega|} \) is a constant. By translating the origin appropriately, we can assume that

\[
\int_{\partial \Omega} x_i = 0, \quad i = 1, \ldots, n,
\]

that is, \( x_i \perp u_0 \). Next, we will show that a suitable rotation of axes can be made so as to insure that

\[
\int_{\partial \Omega} x_j u_i = 0, \quad j = 2, 3, \ldots, n \text{ and } i = 1, \ldots, j - 1.
\]

To see this, define an \( n \times n \) matrix \( Q = (q_{ji}) \), where \( q_{ji} = \int_{\partial \Omega} x_j u_i \), for \( i, j = 1, 2, \ldots, n \). Using the orthogonalization of Gram and Schmidt (i.e., QR-factorization theorem), we know that there exist an upper triangle matrix \( T = (T_{ji}) \) and an orthogonal matrix \( U = (a_{ji}) \) such that

\[
T_{ji} = \sum_{\gamma=1}^{n} x_{\gamma} q_{\gamma i} = \int_{\partial \Omega} \sum_{k=1}^{n} a_{jk} x_k u_i = 0, \quad 1 \leq i < j \leq n.
\]

Letting \( y_j = \sum_{k=1}^{n} a_{jk} x_k \), we get

\[
\int_{\partial \Omega} y_j u_i = \int_{\partial \Omega} \sum_{k=1}^{n} a_{jk} x_k u_i = 0, \quad 1 \leq i < j \leq n.
\]

Since \( U \) is an orthogonal matrix, \( y_1, y_2, \ldots, y_n \) are also coordinate functions on \( \mathbb{R}^n \). Therefore, denoting these coordinate functions still by \( x_1, x_2, \ldots, x_n \), one can get (3.2). From (3.1) and (3.2), one sees that \( x_j \perp \{u_0, u_1, \ldots, u_{j-1}\} \) in \( L^2(\partial \Omega) \).

It follows from the variational characterization (1.19) that

\[
\lambda_j \int_{\partial \Omega} x_j^2 \leq \int_{\Omega} \left( |\nabla^2 x_j|^2 + \tau |\nabla x_j|^2 \right) = \tau |\Omega|, \quad j = 1, \ldots, n,
\]

which implies that

\[
\sum_{j=1}^{n} \frac{1}{\lambda_j} \tau |\Omega| \geq \sum_{j=1}^{n} \int_{\partial \Omega} x_j^2 = \int_{\partial \Omega} |x|^2.
\]

Multiplying both sides of the above inequality by \( \int_{\partial \Omega} |H|^2 \), and using the Schwarz inequality, we obtain

\[
\sum_{j=1}^{n} \frac{1}{\lambda_j} \tau |\Omega| \int_{\partial \Omega} |H|^2 \geq \int_{\partial \Omega} |x|^2 \int_{\partial \Omega} |H|^2 \geq \left( \int_{\partial \Omega} \langle x, H \rangle \right)^2 = |\partial \Omega|^2,
\]

which gives (1.20).

If equality holds in (1.20), then all the inequalities in (3.3) should be equalities, which implies that \( x = \kappa H \) holds on \( \partial \Omega \) for some constant \( \kappa \neq 0 \). Thus, for any tangent vector field \( V \) on \( \partial \Omega \), we have \( V(|x|^2) = 2\langle V, x \rangle = 0 \) and so \( |x| \) and \( |H| \) are constants on \( \partial \Omega \). Since \( \partial \Omega \) is a closed hypersurface of \( \mathbb{R}^n \), we conclude that \( \partial \Omega \) is a round sphere. This completes the proof of Theorem 1.8. \( \square \)

At the end, we also have:

**Proof of Theorem 1.11** Denote by \( \Delta \) and \( \nabla \) the Laplacian and the gradient operator on \( M \), respectively. Without loss of generality, we can assume that \( M \) does not lie in a hyperplane of \( \mathbb{R}^N \). Let \( x = (x_1, \ldots, x_N) \) be the position vector of \( M \) in \( \mathbb{R}^N \), and let \( u_j \) be the normalized eigenfunction corresponding to the \( j \)-th nonzero eigenvalue \( \mu_j \) of the Laplacian of \( M \). By a similar discussion as in the proof of Theorem 1.8, we can assume that \( x_j \perp \{u_0, u_1, \ldots, u_{j-1}\} \) in \( L^2(M) \). Then one has

\[
\eta_j \int_{M} x_j^2 \leq \int_{M} |\nabla x_j|^2, \quad j = 1, \ldots, N,
\]
which implies that
\[ \sum_{j=1}^{N} \frac{1}{\eta_j} \int_M |\nabla x_j|^2 \geq \sum_{j=1}^{N} \int_M x_j^2 = \int_M |x|^2. \]
Since
\[ |\nabla x_j|^2 \leq 1, \quad \sum_{j=1}^{N} |\nabla x_j|^2 = n, \]
we have
\[ \sum_{j=1}^{N} \frac{1}{\eta_j} |\nabla x_j|^2 \leq \sum_{j=1}^{n} \frac{1}{\eta_j} |\nabla x_j|^2 + \sum_{A=n+1}^{N} |\nabla x_A|^2 \]
\[ = \sum_{j=1}^{n} \frac{1}{\eta_j} |\nabla x_j|^2 + \sum_{A=n+1}^{N} \eta_A \]
\[ \leq \sum_{j=1}^{n} \frac{1}{\eta_j} |\nabla x_j|^2 + \sum_{i=1}^{n} \frac{1}{\eta_i} (1 - |\nabla x_i|^2) \]
\[ = \sum_{j=1}^{n} \frac{1}{\eta_j}, \]
which gives
\[ \sum_{j=1}^{n} \frac{1}{\eta_j} |M| \geq \int_M |x|^2. \]
Multiplying both sides of the above inequality by \( \int_M |H|^2 \), and using the Schwarz inequality, we have
\[ \sum_{j=1}^{n} \frac{1}{\eta_j} |M| \int_M |H|^2 \geq \int_M |x|^2 \int_M |H|^2 \geq \left( \int_M <x, H> \right)^2 = |M|^2, \]
which implies that (1.21) is true.

If equality holds in (1.21), then equalities hold in all of the above inequalities, which implies that
\[ \eta_1 = \cdots = \eta_N \equiv C, \]
\[ \overline{\Delta} x_j = -C x_j, j = 1, \cdots, N, \text{ on } M, \]
and \( x = \kappa \overline{H} \) hold on \( M \) for some constant \( \kappa \neq 0 \). From these facts, we know that \( |x| \) and \( |\overline{H}| \) are constants on \( M \). Therefore, when \( n = N - 1, M \) is a hypersphere, and when \( n < N - 1, M \) is a minimal submanifold of some hypersphere of \( \mathbb{R}^N \).

\[ \blacksquare \]

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