CHARACTERIZATION OF RIESZ SPACES WITH
TOPOLOGICALLY FULL CENTER

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Abstract. Let $E$ be a Riesz space and let $E^\sim$ denote its order dual. The orthomorphisms $\text{Orth}(E)$ on $E$, and the ideal center $Z(E)$ of $E$, are naturally embedded in $\text{Orth}(E^\sim)$ and $Z(E^\sim)$ respectively. We construct two unital algebra and order continuous Riesz homomorphisms

$$
\gamma : ((\text{Orth}(E))^\sim)^\sim \to \text{Orth}(E^\sim)
$$

and

$$
m : Z(E)^\prime\prime \to Z(E^\sim)
$$

that extend the above mentioned natural inclusions respectively. Then, the range of $\gamma$ is an order ideal in $\text{Orth}(E^\sim)$ if and only if $m$ is surjective. Furthermore, $m$ is surjective if and only if $E$ has a topologically full center. (That is, the $\sigma(E, E^\sim)$-closure of $Z(E)x$ contains the order ideal generated by $x$ for each $x \in E_+$. ) As a consequence, $E$ has a topologically full center $Z(E)$ if and only if $Z(E^\sim) = \pi \cdot Z(E)^\prime\prime$ for some idempotent $\pi \in Z(E)^\prime\prime$.

1. Introduction

Let $E$ be a Banach lattice and let $Z(E)$ be its (ideal ) center. In general $Z(E)$ is a subalgebra and sublattice of $Z(E')$, the center of the Banach dual $E'$ of $E$. It is possible to extend this embedding to a contractive algebra and lattice homomorphism of $Z(E)^\prime\prime$ into $Z(E')$. It is natural to ask when the homomorphism would be onto $Z(E')$. It is clear that $Z(E)$ has to be large, since $Z(E')$ is always large. It turns out that the concept of largeness best suiting the center $Z(E)$ in this problem is that $Z(E)$ should be topologically full in the sense of Wickstead [17]. Namely, for each $x \in E_+$, the closure of $Z(E)x$ is the closed ideal generated by $x$. Then it is shown that the homomorphism is onto $Z(E')$ if and only if $Z(E)$ is topologically full [11, Corollary 2].

Our purpose in this paper is to consider the corresponding problem for a Riesz space $E$ with point separating order dual $E^\sim$. In the case of Riesz spaces however, there are, in general, two different algebras that act on $E$

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which should be considered. Namely, $\text{Orth}(E)$, the algebra of the orthomorphisms on $E$, and its subalgebra and (order) ideal $Z(E)$, the center of $E$. In the case of Banach lattices $\text{Orth}(E) = Z(E)$, therefore this problem does not arise. Recall that an orthomorphism on $E$ is an order bounded operator on $E$ that preserves bands, and $Z(E)$ is the ideal generated by the identity operator on $E$ in the Riesz space $\text{Orth}(E)$. In fact $\text{Orth}(E)$ is an $f$-algebra. The ideal center $Z(E)$, on the other hand, is a normed $AM$-lattice where the identity operator is the order unit of the $AM$-lattice. Similar to the Banach lattice case, $\text{Orth}(E)$ is embedded in $\text{Orth}(E^\sim)$ as an $f$-subalgebra and sublattice. Also as before, $Z(E)$ is embedded in $Z(E^\sim)$ as a subalgebra and sublattice. We construct two unital algebra and order continuous lattice homomorphisms

$$\gamma : (\text{Orth}(E^\sim))^\sim_n \rightarrow \text{Orth}(E^\sim)$$

and

$$m : Z(E)^{''} \rightarrow Z(E^\sim)$$

that extend the two embeddings mentioned above respectively. Then the range of $\gamma$ is an order ideal in $\text{Orth}(E^\sim)$ if and only if $m$ is onto $Z(E^\sim)$ (Corollary 2). Also, $m$ is onto $Z(E^\sim)$ if and only if $E$ has a topologically full center (Proposition 3). That is for each $x \in E_+$, the $\sigma(E,E^\sim)$-closure of $Z(E)x$ in $E$ contains the ideal generated by $x$. It follows that $E$ has a topologically full center $Z(E)$ if and only if the ideal center of its order dual $E^\sim$ is given as $Z(E^\sim) = \pi \cdot Z(E)^{''}$ for some idempotent $\pi \in Z(E)^{''}$.

We point out that the proofs of the above mentioned results differ from those used in the Banach lattice case [11]. The method used in this paper owes a lot to the work and results of Huijsmans and de Pagter [8] on the bidual of an $f$-algebra.

In the Banach lattice case it is shown that if $Z(E)$ is topologically full then it is maximal abelian [11, Corollary 3]. Wickstead [19] showed that the converse is not true. He constructed an interesting $AM$-lattice with a center that is maximal abelian but is not topologically full. In the case of Riesz spaces, even when $Z(E)$ is topologically full, it need not be maximal abelian. In fact, when $Z(E)$ is topologically full, its commutant in the order bounded operators is $\text{Orth}(E)$ (Corollary 4).

If $E$ is a Riesz space, by $E^\sim$ we will denote the Riesz space of order bounded linear functionals on $E$. All Riesz spaces considered in this paper are assumed to have separating order duals. $E_+^\sim$ will denote the order continuous linear functionals in $E^\sim$. We let $L_0(E,F)$ denote the space of order bounded linear operators from the Riesz space $E$ into the Riesz space $F$. When $T : E \rightarrow F$ is an order bounded operator between two Riesz spaces, the adjoint of $T$ carries $F^\sim$ into $E^\sim$ and we will denote it by $T'$. The dual of a normed space will be denoted by $E'$. In all terminology concerning Riesz spaces we will adhere to the definitions in [1] and [20].

Let us recall that for any associative algebra $A$, a multiplication (called the Arens multiplication) can be introduced in the second algebraic dual
A** of $A$ [2]. This is accomplished in three steps: given $a, b \in A$, $f \in A^*$ and $F, G \in A^{**}$, one defines $f \cdot a \in A^*$, $F \cdot f \in A^*$, and $F \cdot G \in A^{**}$ by the equations

$$
(f \cdot a)(b) = f(ab) \\
(F \cdot f)(a) = F(f \cdot a) \\
(F \cdot G)(f) = F(G \cdot f)
$$

For any Archimedean $f$-algebra $A$, the space $(A^\sim)_n$ is an Archimedean $f$-algebra with respect to the Arens multiplication [3].

We denote for any $F \in (A^\sim)_n^\sim$ the mapping $f \rightarrow F \cdot f$ by $\nu_F$. The map $\nu_F$ is an orthomorphism on $(A^\sim)_n$. The mapping $\nu : (A^\sim)_n \rightarrow Orth(A^\sim)$, defined by $\nu(F) = \nu_F$ is an algebra and Riesz homomorphism for any Archimedean $f$-algebra $A$. Moreover $\nu$ is onto $Orth(A^\sim)$ if and only if $(A^\sim)_n$ has a unit element. In that case, $\nu$ is injective by Theorem 5.2 in [8]. The main purpose of this paper is to extend the latter result to an arbitrary Riesz space.

2. THE ARENS HOMOMORPHISM

Let $E$ be a Riesz space and consider the bilinear map

$$(1) \quad Orth(E) \times E \rightarrow E$$

defined by $(\pi, x) \rightarrow \pi(x)$ for each $\pi \in Orth(E)$ and $x \in E$. Related to (1), we define the following bilinear maps:

$$(2) \quad E \times E^\sim \rightarrow (Orth(E))^\sim : (x, f) \rightarrow \psi_{x,f} : \psi_{x,f}(\pi) = f(\pi x)$$

$$(3) \quad E^\sim \times (Orth(E))^{\sim\sim} \rightarrow E^\sim : (f, F) \rightarrow F \bullet f : F \bullet f(x) = F(\psi_{x,f})$$

where $x \in E$, $f \in E^\sim$, $\pi \in Orth(E)$ and $F \in Orth(E)^{\sim\sim}$. We call the map defined in (3) the Arens extension of the map in (1).

For an Archimedean unital $f$-algebra $A$, we have $A^{\sim\sim} = (A^\sim)_n^\sim$ by Corollary 3.4 in [8]. Since $Orth(E)$ is a unital $f$-algebra with point separating order dual, it is Archimedean. This enables us to use the identification $(Orth(E))^{\sim\sim} = ((Orth(E))^\sim)_n^\sim$ throughout this paper. It is straightforward to check that the Arens product on the $f$-algebra $((Orth(E))^\sim)_n^\sim$ is compatible with the Arens extension defined in (3), that is, $E^\sim$ is a unital module over $((Orth(E))^\sim)_n^\sim$.

We use (3) to define a linear operator

$$\gamma : ((Orth(E))^\sim)_n^\sim \rightarrow L_b(E^\sim)$$

by $\gamma(F)(f) = F \bullet f$

for each $F \in ((Orth(E))^\sim)_n^\sim$ and $f \in E^\sim$. We will call $\gamma$ the Arens homomorphism of the order bidual of $Orth(E)$.

**Proposition 1.** $\gamma$ is a unital algebra and order continuous Riesz homomorphism such that $\gamma(((Orth(E))^\sim)_n^\sim) \subset Orth(E^\sim)$. 
Proof. It follows from the definition of $\gamma$ that $\gamma$ is a positive order continuous unital algebra homomorphism. Also it is easily checked that $\gamma(\pi) = \pi'$ for each $\pi \in \text{Orth}(E)$. Let $F \in (\text{Orth}(E))^\cap \cap \cap$ such that $|F| \leq \pi$ for some $\pi \in \text{Orth}(E)$. Then $|\gamma(F)| \leq \gamma(|F|) \leq \gamma(\pi) = \pi'$ by the positivity of $\gamma$. Since $\pi' \in \text{Orth}(E^\cap \cap \cap)$, we see that $\gamma(F) \in \text{Orth}(E^\cap \cap \cap)$. Then, since $\gamma$ is order continuous and the ideal generated by $\text{Orth}(E)$ is strongly order dense in $(\text{Orth}(E)^\cap \cap \cap)^\cap \cap \cap$, the range of $\gamma$ is contained in $\text{Orth}(E^\cap \cap \cap)$. That $\gamma$ is a Riesz homomorphism follows from the fact that it is an algebra homomorphism by Corollary 5.5 in \[9\]. □

If $A$ is an $f$-algebra then the range of the homomorphism $\nu : (A^\cap \cap \cap)^\cap \cap \cap \rightarrow \text{Orth}(A^\cap \cap \cap)$ is contained in the range of the Arens homomorphism $\gamma$.

Proposition 2. Let $A$ be an $f$-algebra with point separating order dual. The range of the homomorphism $\nu : (A^\cap \cap \cap)^\cap \cap \cap \rightarrow \text{Orth}(A^\cap \cap \cap)$ is contained in the range of the Arens homomorphism $\gamma$.

Proof. Let $P : A \rightarrow \text{Orth}(A)$ be the canonical embedding of $A$ into $\text{Orth}(A)$ (i.e., $P(a)(b) = ab$ for all $a, b \in A$). It is well known that $P(A)$ is a sublattice and an algebra ideal in $\text{Orth}(A)$. For each $\mu \in (\text{Orth}(A))^\cap \cap \cap$, define $\hat{\mu} \in A^\star$ by $\hat{\mu}(a) = \mu(P(a))$. Since positivity is preserved, it is clear that $\hat{\mu} \in A^\cap \cap \cap$ for each $\mu \in (\text{Orth}(A))^\cap \cap \cap$. Given $a \in A$, $f \in A^\cap \cap \cap$, we have $\psi_{a,f} \in (\text{Orth}(A))^\cap \cap \cap$. Then

\[
\hat{\psi}_{a,f}(b) = \psi_{a,f}(P(b)) = f(P(b)(a)) = f(ab) = (f \cdot a)(b)
\]

for each $b \in A$. That is $\hat{\psi}_{a,f} = f \cdot a$ for all $a \in A, f \in A^\cap \cap \cap$. Now let $F \in (A^\cap \cap \cap)^\cap \cap \cap$. Define $\hat{F} \in ((\text{Orth}(A))^\cap \cap \cap)^\cap \cap \cap$ by $\hat{F}(\mu) = F(\hat{\mu})$ for each $\mu \in (\text{Orth}(A))^\cap \cap \cap$. Since $0 \leq \mu$ implies $0 \leq \hat{\mu}$, we have $0 \leq \hat{F}$ whenever $0 \leq F$. That is $\hat{F} \in ((\text{Orth}(A))^\cap \cap \cap)^\cap \cap \cap = (\text{Orth}(A))^\cap \cap \cap$. For each $F \in (A^\cap \cap \cap)^\cap \cap \cap$, $f \in A^\cap \cap \cap$ and $a \in A$, we have

\[
\nu_F(f)(a) = F \cdot f(a) = F(f \cdot a) = F(\hat{\psi}_{a,f}) = \hat{F}(\psi_{a,f})
\]

\[
= \hat{F} \cdot f(a) = \gamma(\hat{F})(f)(a).
\]

Hence $\nu_F = \gamma(\hat{F})$. □

Let us check the behavior of $\gamma$ in some specific examples.

Example 1. Let $\omega$ denote all sequences and let $A = l^1$ the $f$-subalgebra of $\omega$ consisting of absolutely summable sequences. Then $A^\cap \cap \cap = l^\infty$ and $\text{Orth}(A) = \text{Orth}(A^\cap \cap \cap) = l^\infty$. Since $(A^\cap \cap \cap)^\cap \cap \cap = l^1$, $\nu$ is the inclusion map $l^1 \rightarrow l^\infty$ so that $\nu$ is one-to-one and not onto. On the other hand $((\text{Orth}(A))^\cap \cap \cap)^\cap \cap \cap = (l^\infty)^\cap \cap \cap \cap$ and $\gamma$ is the band projection of $(l^\infty)^\cap \cap \cap \cap$ onto $l^\infty$. Thus $\gamma$ is onto and not one-to-one.

Example 2. Let $A = c_0$ be the $f$-subalgebra of $\omega$ consisting of the sequences convergent to zero. Then $A^\cap \cap \cap = l^1$ and $\text{Orth}(A) = \text{Orth}(A^\cap \cap \cap) = l^\infty$. Since $(A^\cap \cap \cap)^\cap \cap \cap = l^\infty$, $\nu$ is the identity map on $l^\infty$ so that $\nu$ is one-to-one and onto. On the other hand $((\text{Orth}(A))^\cap \cap \cap)^\cap \cap \cap = (l^\infty)^\cap \cap \cap \cap$ and $\gamma$ is the band projection of $(l^\infty)^\cap \cap \cap \cap$ onto $l^\infty$. Thus $\gamma$ is onto and not one-to-one.
Example 3. Consider $C[0,1]$ with the product $*$ defined by $a*b = iab$ with $i(x) = x$ for all $x \in [0,1]$. Then $A = (C[0,1],*)$ is an Archimedean $f$-algebra. As shown in [5], $(A^\sim)_n^\sim$ is not semi-prime. Therefore $\nu$ is not one-to-one and not onto [8]. On the other hand Orth($A$) = $Z(C[0,1]) = C[0,1]$ and Orth($A^\sim$) = $Z(C[0,1]) = C[0,1]^\sim$. Since $(C[0,1])_n^\sim = C[0,1]$, $\gamma$ is the identity map on $C[0,1]^\sim$. Therefore $\gamma$ is one-to-one and onto.

3. Riesz spaces with topologically full center

We start with the following definition that is due to Wickstead [17] in the case of Banach lattices.

Definition 1. Suppose $E$ is a Riesz space. Then $E$ is said to have a topologically full center if for each $x \in E_+$ the $\sigma(E,E^\sim)$-closure of $Z(E)x$ contains the ideal generated by $x$.

Banach lattices with topologically full center were initiated in [17]. The class of Riesz spaces and the class Banach lattices with topologically full center are quite large. For example, in a $\sigma$-Dedekind complete Riesz space $E$ each positive element generates a projection band. Therefore for each $x \in E_+$, $Z(E)x$ is an ideal and $Z(E)$ is topologically full. Also Banach lattices with a quasi-interior point or with a topological orthogonal system have topologically full center [19]. However not all Riesz spaces have topologically full center.

Example 4. (Zaaden [20, p.664]) Let $E$ be the Riesz space of piecewise affine continuous functions on $[0,1]$. Clearly the ideal generated by the constant 1 function equals $E$. But, as shown by Zaaden, $Z(E)$ is trivial, that is, it consists of the scalar multiples of the identity. Therefore $E$ does not have a topologically full center.

The first example of an AM-space that has trivial center was given in [5]. A thorough study of Banach lattices with trivial center was undertaken in [18]. We refer the reader to [18] for further examples of Banach lattices with trivial center as well as a careful treatment of the following example of Goullet de Rugy mentioned above.

Example 5. (5) Let $K$ be a compact Hausdorff space with a point $p \in K$ such that $\{p\}$ is not a $G_\delta$-set in $K$ (e.g., [12, Example 4, p.140]). Let $C_0(K)$ denote the elements of $C(K)$ that vanish at $p$. Let $H$ denote the positive unit ball of $C_0(K)'$ with the relative $\sigma(C_0(K)',C_0(K))$-topology. Let $E = \{f \in C(H) : f(r\mu) = r f(\mu) \text{ for all } r \in [0,1] \text{ and for each } \mu \in H \text{ with } \|\mu\| = 1\}$. Then $E$ is an AM-space (without order unit). As a sublattice of $l^\infty(H \backslash \{0\})$, one has $E^d = \{0\}$. Therefore $Z(E)$ is embedded in $l^\infty(H \backslash \{0\})$ [15]. Then one may compute that $Z(E)$ consists of continuous bounded functions on $H \backslash \{0\}$ that are constant on the rays (i.e., $g(r\mu) = g(\mu)$ for all $r \in (0,1)$, for each $\mu \in H$ with $\|\mu\| = 1$). It follows by an argument in [15, p.371] that if there is a non-constant function in $Z(E)$ then $\{0\}$ is a $G_\delta$-set in $H$. That
in turn implies that \(\{p\}\) would be a \(G_δ\)-set in \(K\). Therefore \(Z(E)\) is trivial and \(E\) does not have a topologically full center.

\(Z(E)\) is an Archimedean unital \(f\)-algebra with order unit. The order unit norm induced on \(Z(E)\) is an algebra and lattice norm. \(\hat{Z}(E)\), the norm completion of \(Z(E)\), is an AM-space and a partially ordered Banach algebra where the order unit and the algebra unit coincide. Therefore by the Stone algebra theorem \(\hat{Z}(E) \cong C(K)\) (isometric algebra and lattice homomorphism) for some compact Hausdorff space \(K\). (Here \(C(K)\) denotes the real valued continuous functions on \(K\).) Then \(Z(E)' = \hat{Z}(E)' = \hat{Z}(E)^{\sim} = Z(E)^{\sim}\) and \(Z(E)'' = Z(E)''\). \(Z(E)''\) is an AM-space and with the Arens product, it is a partially ordered Banach algebra (an Archimedean \(f\)-algebra with unit) where the order unit and the algebra unit coincide. Therefore \(Z(E)'' \cong C(S)\) for some hyperstonian space \(S\). That is the Arens product on \(Z(E)''\) coincides with the pointwise product on \(C(S)\) [2].

Given the bilinear map

\[
Z(E) \times E \to E
\]
defined by \((T, x) \to Tx\) for each \(T \in Z(E)\) and \(x \in E\), we define the following bilinear maps:

\[
E \times E^\sim \to Z(E)' : (x, f) \mapsto \mu_{x,f} = \psi_{x,f}^{-}\mid Z(E)
\]

(5)

\[
E^\sim \times Z(E)'' \to E^\sim : (f, F) \mapsto (F \circ f)(x) = F(\mu_{x,f})
\]

(6)

where \(x \in E\), \(f \in E^\sim\) and \(F \in Z(E)''\). The Arens product on \(Z(E)''\) is compatible with the bilinear map defined in (6). That is, \(E^\sim\) is a unital module over \(Z(E)''\). [6] allows us to define a linear operator \(m : Z(E)'' \to L_b(E^\sim)\) where \(m(F)(f) = F \circ f\) for all \(f \in E^\sim\) and \(F \in Z(E)''\). It is easily checked that \(m(T) = \gamma(T)\) whenever \(T \in Z(E)\).

We have the following analogue of Proposition [3] for the map \(m\).

**Proposition 3.** \(m\) is a unital algebra and order continuous lattice homomorphism such that \(m(Z(E)''') \subset Z(E^{\sim})\).

**Proof.** That \(m\) is a positive order continuous algebra homomorphism is immediate from the definition of \(m\). That it is a lattice homomorphism follows as in Proposition [3].

For each \(F \in Z(E)''\), there is a net \(\{T_\alpha\}\) in \(Z(E)\) such that \(\|T_\alpha\| \leq \|F\|\) and \(T_\alpha \to F\) in \(\sigma(Z(E)'', Z(E)'\))-topology. Let \(f \in E_+''\) and \(x \in E_+\). Then

\[ -\|F\|f(x) \leq F \circ f(x) = \lim_{\alpha} f(T_\alpha x) \leq \|F\|f(x). \]

So \(F \in Z(E^\sim)\). \[\square\]

We will call the map \(m : Z(E)'' \to Z(E^\sim)\) the Arens homomorphism of the bidual of \(Z(E)\) (into \(Z(E^\sim)\)).

**Proposition 4.** Let \(A\) be a unital \(f\)-algebra with point separating order dual. Then the Arens homomorphism of the bidual of \(Z(A)\) is onto \(Z(A^{\sim})\).
Proof. Let $F \in Z(A^\sim)$ with $0 \leq F \leq 1$. Since $A$ is unital, Theorem 5.2 in \cite{8} implies that the algebra and lattice homomorphism $\nu : (A^\sim)_n^\sim \to \text{Orth}(A^\sim)$ of \cite{8} is one-to-one and onto. Therefore with a slight abuse of notation we will identify $(A^\sim)_n^\sim$ with $\text{Orth}(A^\sim)$. In the duality $(A^\sim)_n^\sim$, $A$ is dense in $(A^\sim)_n^\sim$ with respect to the weak topology. Since the locally solid convex topology $|\sigma|((A^\sim)_n^\sim, A^\sim)$ is compatible with the duality $(A^\sim)_n^\sim, A^\sim$ \cite{11} Theorem 11.13, p.170], there is a net $T$ by the Alaoglu Theorem, there is $F$ to $A$. Therefore we may suppose that $0 \leq a_\alpha \leq 1$. Hence, by the Alaoglu Theorem, there is $T \in Z(A)^\prime$ with $0 \leq T \leq 1$ such that a subnet $\{a_\beta\}$ converges to $T$ in the $\sigma(Z(A)^\prime, Z(A)^\prime)$-topology. Then, when $f \in A^\sim$ and $b \in A$,

$$
T \circ f(b) = T(\mu_{b,f}) = \lim_{\beta} \mu_{b,f}(a_\beta) = \lim_{\beta} \psi_{b,f}(a_\beta) = \lim_{\beta} f(a_\beta b).
$$

On the other hand, since $\{a_\beta\}$ converges to $F$ in the $|\sigma|((A^\sim)_n^\sim, A^\sim)$-topology implies convergence also in the $\sigma((A^\sim)_n^\sim, A^\sim)$-topology, we have

$$
F \cdot f(b) = F(f \cdot b) = \lim_{\beta} f \cdot b(a_\beta) = \lim_{\beta} f(a_\beta b).
$$

Therefore $m(T) = F$. \qed

Note that if $E$ is a Riesz space and if we set $A = \text{Orth}(E)$, then $Z(A) = Z(E)$. Also, since then $A$ is a unital $f$-algebra with point separating order dual, Proposition \cite{4} will be true for $A$. In what follows we will use these facts repeatedly.

Corollary 1. Let $E$ be a Riesz space. Let $A = \text{Orth}(E)$ and $m_A$ denote the Arens homomorphism of the bidual of $Z(A)$ onto $Z(A^\sim)$ (where $Z(A) = Z(E)$). Then

(1) the following diagram is commutative

$$
\begin{array}{ccc}
Z(E)^\prime & \overset{m_A}{\longrightarrow} & Z(A^\sim) \\
\overset{i}{\downarrow} & & \uparrow \gamma \\
Z(A)^\prime & \overset{m_A}{\longrightarrow} & (A^\sim)_n^\sim
\end{array}
$$

where $i$ denotes the natural inclusion map;

(2) $\gamma(Z((A^\sim)_n^\sim)) = m(Z(E)^\prime)$.

Proof. (1) Suppose $F \in Z(A)^\prime$ with $0 \leq F \leq 1$. Then, as in the proof of Proposition \cite{4} let $0 \leq a_\alpha \leq 1$ be a net in $Z(A) \subset A$ such that $\{a_\alpha\}$ converges to $F$ in the $\sigma(Z(A)^\prime, Z(A)^\prime)$-topology and also to $m_A(F) \in (A^\sim)_n^\sim$ in the $\sigma((A^\sim)_n^\sim, A^\sim)$-topology. Since $(A^\sim)_n^\sim = (\text{Orth}(E)^\prime)_n^\sim$, for each $x \in E$ and $f \in E^\sim$, we have

$$
\gamma(m_A(F))(f)(x) = m_A(F) \bullet f(x) = m_A(F)(\psi_{x,f}) = \lim_{\alpha} \psi_{x,f}(a_\alpha) = \lim_{\alpha} \mu_{x,f}(a_\alpha) = F(\mu_{x,f}) = m(F)(f)(x).
$$

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$$
\gamma(m_A(F))(f)(x) = m_A(F) \bullet f(x) = m_A(F)(\psi_{x,f}) = \lim_{\alpha} \psi_{x,f}(a_\alpha) = \lim_{\alpha} \mu_{x,f}(a_\alpha) = F(\mu_{x,f}) = m(F)(f)(x).
$$

Therefore $m(T) = F$. \qed
(2) As in proof of Proposition 4 we may identify \((A^\sim)_{n}^\sim\) with \(Orth(A^\sim)\) via the homomorphism \(\nu\). Then \(Z(A^\sim) = Z((A^\sim)_{n}^\sim)\). Since, by Proposition 4 \(m\_A\) is onto \(Z(A^\sim)\), from part (1), we have \(\gamma(Z((A^\sim)_{n}^\sim)) = m(Z(E)''\). □

**Corollary 2.** Let \(E\) be a Riesz space. Then \(\gamma((Orth(E)^\sim)_{n}^\sim)\) is an order ideal in \(Orth(E^\sim)\) if and only if \(m(Z(E)''\) = \(Z(E^\sim)\).

**Proof.** Let \(A = Orth(E)\). Suppose \(\gamma((A^\sim)_{n}^\sim)\) is an order ideal in \(Orth(E^\sim)\).

Since \(\gamma\) is a order continuous lattice homomorphism, the kernel \(Ker(\gamma)\) is a band in \((A^\sim)_{n}^\sim\). Let \((1-e)\) be the band projection of \((A^\sim)_{n}^\sim\) onto \(Ker(\gamma)\). Then \(e \cdot (A^\sim)_{n}^\sim\) is algebra and lattice isomorphic to the order ideal \(\gamma((A^\sim)_{n}^\sim)\) with \(\gamma(e) = 1\). Let \(T \in Z(E^\sim)\) with \(0 \leq T \leq 1 = \gamma(e)\). So there is \(F \in (A^\sim)_{n}^\sim\) such that \(\gamma(F) = T\). Moreover, since \(\gamma\) is an algebra and lattice homomorphism, we may choose \(F\) so that \(0 \leq F \leq e \leq 1\). That is we may suppose that \(F \in Z((A^\sim)_{n}^\sim)\). Then Corollary 1 part (2) implies there is \(G \in Z(E)''\) such that \(m(G) = \gamma(F) = T\).

Conversely, suppose \(m(Z(E)''\) = \(Z(E^\sim)\). Suppose \(0 \leq T \leq \gamma(F)\) for some \(0 \leq F \in (A^\sim)_{n}^\sim\) and for some \(T \in Orth(E^\sim)\). Since \(Orth(E^\sim)\) is Dedekind complete, there is \(\hat{T} \in Z(Orth(E^\sim))\) such that \(0 \leq \hat{T} \leq 1\) and \(T = \hat{T} \gamma(F)\). By Corollary 1 part (2), there is \(G \in Z((A^\sim)_{n}^\sim)\) such that \(\gamma(G) = \hat{T}\). Then \(T = \gamma(G) \gamma(F) = \gamma(G \cdot F)\) where \(G \cdot F \in (A^\sim)_{n}^\sim\). □

Corollary 2 gives the first part of the result stated in the abstract. We will complete the result by showing that the only Riesz spaces that satisfy Corollary 2 are necessarily those with a topologically full center. For Banach lattices a proof of this was given in [11]. Initially we will give the proof of the sufficiency.

**Proposition 5.** Let \(E\) be a Riesz space. Then \(E\) has a topologically full center if and only if the Arens homomorphism \(m : Z(E)'' \to Z(E^\sim)\) is surjective. Then there exits an idempotent \(\pi \in Z(E)''\) such that \(Z(E^\sim) = \pi \cdot Z(E)''\) and \(Ker(m) = (1 - \pi) \cdot Z(E)''\).

**Proof.** (Sufficiency) Suppose \(m\) is surjective. To show that \(E\) has topologically full center it is sufficient to show that each \(\sigma(E, E^\sim)\)-closed \(Z(E)\) submodule of \(E\) is an ideal. This is equivalent to showing that each \(\sigma(E^\sim, E)\)-closed \(Z(E)\) submodule of \(E^\sim\) is an ideal. Let \(M\) be a \(\sigma(E^\sim, E)\)-closed \(Z(E)\) submodule of \(E^\sim\) and let \(T \in Z(E)''\). There is a net \(\{T_\alpha\}\) in \(Z(E)\) that converges to \(T\) in the \(\sigma(Z(E)''\, Z(E)'\)-topology. For each \(x \in E\) and \(f \in M\) we have \(T_\alpha \circ f(x) = \mu_{x,f}(T_\alpha) \to T(\mu_{x,f}) = T \circ f(x)\).

Hence \(M\) is a \(Z(E)''\)-submodule of \(E^\sim\). Since \(Z(E^\sim) = m(Z(E)''\), \(M\) is a \(Z(E^\sim)\)-submodule. \(E^\sim\) is Dedekind complete. It is well known that a subspace of \(E^\sim\) is an ideal in \(E^\sim\) if and only if it is a \(Z(E^\sim)\)-submodule of \(E^\sim\) (e.g., [19]). Hence \(M\) is an ideal in \(E^\sim\). Since \(m\) is order continuous (Proposition 5), \(Ker(m)\) is a band in \(Z(E)''\). Hence there exists a band
projection \( \pi \in Z(E)^\sigma \) such that \( \text{Ker}(m) = (1 - \pi) \cdot Z(E)^\sigma \) and \( Z(E^\sim) = \pi \cdot Z(E)^\sigma \).

The proof of the converse requires some preparatory results.

Given a Riesz space \( E \), let \( A \) be a unital subalgebra of \( Z(E) \). Let \( A^0 \) denote the polar of \( A \) in \( Z(E)^\sigma \) and let \( A^{00} \) denote the polar of \( A^0 \) in \( Z(E)^{00} \).

By standard duality theory, we have that \( A' = Z(E)^\sigma / A^0 \) and \( A'' = A^{00} \subset Z(E)^{00} \). Since \( A \) is a normed algebra, \( A'' \) is a Banach algebra with the Arens product [2]. In fact \( A'' \) is a subalgebra of \( Z(E)^{00} \) when \( Z(E)^{00} \) has its Arens product.

**Lemma 1.** Let \( A \) be a unital subalgebra of \( Z(E) \). Then \( A^{00} \) is a subalgebra of \( Z(E)^{00} \) and the algebra product on \( A^{00} \) is identical with the Arens product on \( A'' \) under the canonical isomorphism of \( A^{00} \) with \( A'' \).

**Proof.** Let \( f, g \in A^0 \) and \( a, b \in A \). It is easily checked that \( f \cdot a \in A^0 \). Let \( F, G \in A^{00} \). There is a net \( \{a_\alpha \} \) in \( A \) that converges to \( F \) in the \( \sigma(Z(E)^{00}, Z(E)^\sigma) \)-topology. Let \( f \in A^0 \). Then

\[
F \cdot G(f) = F(G \cdot f) = \lim_{\alpha} G \cdot f(a_\alpha) = \lim_{\alpha} G(f \cdot a_\alpha) = 0
\]

and \( F \cdot G \in A^{00} \).

On the other hand, denote the isomorphism of \( A'' \) onto \( A^{00} \) by \( F \to \hat{F} \). Given \( f \in A' \), let \( \hat{f} \in Z(E)^\sigma \) denote any extension of \( f \) on \( Z(E) \). When \( F, G \in A'' \), let \( \{a_\alpha \} \) and \( \{b_\beta \} \) be nets in \( A \) that converge to \( \hat{F} \) and \( \hat{G} \) respectively in the \( \sigma(Z(E)^{00}, Z(E)^\sigma) \)-topology. Then, evidently, the respective nets also converge to \( F \) and \( G \) respectively in the \( \sigma(A'', A') \)-topology. For any \( f \in A' \), we have

\[
F \cdot G(f) = \lim_{\alpha, \beta} f(a_\alpha b_\beta) = \lim_{\alpha, \beta} \hat{f}(a_\alpha b_\beta) = \hat{F} \cdot \hat{G}(\hat{f}).
\]

**Lemma 2.** Let \( I \) be a closed algebra ideal in \( Z(E) \) and consider \( A = Z(E)/I \) with the quotient norm. Then \( A \) is a normed algebra and with the Arens product \( A'' \) may be identified with the subalgebra of \( Z(E)^{00} \) given by

\[
(I^{00})^d = \{ F \in Z(E)^\sigma : |F| \wedge |G| = 0 \text{ for all } G \in I^{00} \}.
\]

**Proof.** Let \( \hat{Z(E)} = C(K') \) for some compact Hausdorff space \( K' \) and let \( \mathcal{T} \) denote the closure of \( I \) in \( C(K') \). There is a closed subset \( K \) of \( K' \) such that \( \mathcal{T} = \{ a \in C(K') : a(K) = \{0\} \} \) and \( C(K')/\mathcal{T} = C(K) \). Furthermore \( A \) is a subalgebra of \( C(K) \). In fact \( C(K) \) is the completion of \( A \). Since \( \mathcal{T} \) is an order ideal in \( Z(E) \), \( A' \cong I^0 = (\mathcal{T})^0 \) is a band in \( Z(E)^\sigma \). Then \( A'' \cong Z(E)^{00} / I^{00} = (I^{00})^d \), since \( I^{00} \) is a band in \( Z(E)^{00} \). It remains to check that the Arens product of \( A'' \) is identical with the product on the subalgebra \((I^{00})^d \). Let \( F, G \in Z(E)^{00} \) and \( \hat{F}, \hat{G} \in A'' \) such that \( \hat{F} = F|_{I^0} \), \( \hat{G} = G|_{I^0} \). Let \( \{a_\alpha \} \) and \( \{b_\beta \} \) be nets in \( Z(E) \) that converge to \( F \) and \( G \) respectively in the \( \sigma(Z(E)^{00}, Z(E)^\sigma) \)-topology. Let \( [a] = a + I \) for each \( a \in Z(E) \). Then,
it follows that, \{[a_\alpha]\} and \{[b_\beta]\} converge to \(\hat{F}\) and \(\hat{G}\) respectively in the \(\sigma(A'', A')\)-topology. Then, for \(f \in I^0\), we have

\[
\hat{F} \cdot \hat{G}(f) = \lim_{\alpha} \lim_{\beta} f([a_\alpha][b_\beta]) = \lim_{\alpha} \lim_{\beta} f(a_\alpha b_\beta) = F \cdot G(f).
\]

Lemma 3. Let \(E\) be a Riesz space and \(J\) be an ideal in \(E^\sim\) that separates the points of \(E\). Suppose \(E\) is cyclic with respect to a unital \(f\)-subalgebra \(A\) of \(Z(E)\) for the dual pair \(\langle E, J \rangle\). Then \(m(A')|_J = Z(J)\).

Proof. The norm completion \(\hat{A}\) of \(A\) is a unital closed subalgebra of \(\hat{Z}(E)\). Therefore \(\hat{A} = C(K)\) for some compact Hausdorff space \(K\). Furthermore \(A^\sim = A' = C(K)'\) and \(A'' = C(K)'' = C(S)\) for some hyperstonian space \(S\). (We again mention that the usual multiplication on \(C(S)\) is the Arens extension of the product on \(C(K)\) as shown in [2].) Also the usual \(C(S)\)-module structure of \(C(K)'\) via its ideal center is the Arens homomorphism of the bidual of \(C(K)\) onto the ideal center of \(C(K)'\) (e.g., Proposition [4].)

In the rest of the proof, by Lemma [1] we consider \(A''\) as a subalgebra of \(Z(E)''\).

Let \(u \in E_+\) be a cyclic vector and \(f \in J_+\). Then \(\mu_{u,f} \in Z(E)'_+\) and \(\mu_{u,f}|_A = \hat{\mu}_{u,f} \in C(K)'_+\). Let \(P_f\) be the band projection of \(C(K)'\) onto the band \(B(\hat{\mu}_{u,f})\) generated by \(\hat{\mu}_{u,f}\). By the Lebesgue Decomposition Theorem and the Radon-Nikodym Theorem, we have

\[
B(\hat{\mu}_{u,f}) = P_f \cdot C(K)' = L^1(\hat{\mu}_{u,f}) = \{\mu \in C(K)' : |\mu| < < \hat{\mu}_{u,f}\}.
\]

The first equality above follows by Proposition [4].

Suppose \(e \circ f = 0\) for some idempotent \(e \in A'' = C(S)\). Let \(\{a_\alpha\}\) be a net in \(A\) that converges to \(e\) in the \(\sigma(A'', A')\)-topology. Then for each \(a \in A\),

\[
e \cdot \hat{\mu}_{u,f}(a) = e(\hat{\mu}_{u,f} \cdot a) = \lim_{\alpha} \hat{\mu}_{u,f} \cdot a(a_\alpha) = \lim_{\alpha} \hat{\mu}_{u,f}(a a_\alpha) = \lim_{\alpha} \mu_{u,f}(a a_\alpha) = \lim_{\alpha} f(a a_\alpha a) = \lim_{\alpha} f(a a_\alpha u) = \lim_{\alpha} \mu_{a u,f}(a a_\alpha) = e(\mu_{a u,f}) = e \circ f(au) = \hat{\mu}_{u,e \circ f}(a) = 0.
\]

Hence \(0 \leq e \leq 1 - P_f\). Conversely, since \(u\) is a cyclic vector, \(\hat{\mu}_{u,(1 - P_f) \circ f} = (1 - P_f) \cdot \hat{\mu}_{u,f} = 0\) implies that \((1 - P_f) \circ f = 0\). Therefore \(1 - P_f = \sup\{e \in C(S) : e \circ f = 0\text{ and } e = e^2\}\). That is, \(a \circ f = 0\) for some \(a \in A''\) if and only if \(P_f \cdot a = 0\).
Let $T \in Z(E^\sim)$ with $0 \leq T \leq 1$. Let $f \in J_\pm$. Then $0 \leq \hat{\mu}_{u,T}f \leq \hat{\mu}_{u,f}$ in $A'$. Therefore, by the Radon-Nikodym Theorem, there is $a_f \in L^\infty(\hat{\mu}_{u,f}) \subset C(S) = A''$ such that $\hat{\mu}_{u,T}f = a_f \cdot \hat{\mu}_{u,f} = \hat{\mu}_{u,a_ff}$. Since $u$ is a cyclic vector it follows that $Tf = a_f \circ f$. Suppose $g \in J$ such that $0 \leq g \leq f$. $E^\sim$ is Dedekind complete, there is $G \in Z(E^\sim)$ with $0 \leq G \leq 1$ such that $g = Gf$. Therefore

$$m(a_g)(g) = Tg = T(Gf) = G(Tf) = Gm(a_f)(f) = m(a_f)(Gf) = m(a_f)(g).$$

That is $(a_g - a_f) \circ g = 0$. Therefore $(a_g - a_f) \cdot P_g = 0$. Now suppose $f, g \in J_\pm$ and $h = f \lor g$. Then $(a_f - a_h) \cdot P_{f} = (a_g - a_h) \cdot P_{g} = 0$. Hence $a_g \cdot P_f = a_f \cdot P_g$ for all $f, g \in J_\pm$. Since $S$ is Stonian and $0 \leq a_f \leq P_f \leq 1$ for all $f \in J_\pm$, there is a unique $a \in A''$ such that $P_f \cdot a = a_f$ and $(1 - \sup\{P_f : f \in J_\pm\}) \cdot a = 0$. Then

$$Tf = a_f \circ f = (a \cdot P_f) \circ f = a \circ (P_f \circ f) = a \circ f$$

for all $f \in J_\pm$. □

Now we are ready to complete the proof of Proposition 5.

Proof. (Necessity) Suppose $x \in E_\pm$. Let $I(x)$ denote the ideal generated by $x$ in $E$ and let $Ann(x) = \{ T \in Z(E) : Tx = 0 \}$. $Ann(x)$ is a closed order and algebra ideal in $Z(E)$. Consider the map from $Z(E)$ into $Z(I(x))$ defined by $T \mapsto T|_{I(x)}$. Clearly the map is a norm reducing positive algebra homomorphism. Since the kernel of this map is equal to $Ann(x)$, the map is also a lattice homomorphism. It induces a norm reducing lattice and algebra homomorphism of $Z(E)/Ann(x)$ into $Z(I(x))$ where $T + Ann(x) = [T] \rightarrow T|_{I(x)}$. The induced map is an isometry. For example, if $T \in Z(E)_+$ with norm $||T|_{I(x)}||$ in $Z(I(x))$, then $(T \land ||T|_{I(x)}||)_x = T|_{I(x)}$ and $||T|| \leq ||T|_{I(x)}||$. Hence let $A = Z(E)|_{I(x)}$ be the normed unital $f$-subalgebra of $Z(I(x))$. Then since $A \cong Z(E)/Ann(x)$ (isometric, lattice and algebra homomorphism), we may think of $A''$ as a subalgebra of $Z(E)''$ (c.f., Lemma 2). Equivalently, we may think of $A''$ as a subalgebra of $Z(I(x))''$ (c.f., Lemma 1).

Let $J = \{ f|_{I(x)} : f \in E^\sim \}$. Clearly $J$ is an ideal in $I(x)$ that separates the points of $I(x)$. Let $P_x \in Z(E^\sim)$ denote the band projection of $E^\sim$ onto $(I(x))''$. The map $f|_{I(x)} \rightarrow P_x(f)$ is a lattice homomorphism of $J$ onto the band $(I(x))''$. Let $m_x$ be the Arens homomorphism of $Z(I(x))''$ into $Z(I(x)^\sim)$ and let $m$ be the same for $Z(E)''$ into $Z(E^\sim)$. We claim that on elements of $A''$, $m_x$ restricted to $J$ agrees with $m$ restricted to $(I(x))''$. Namely, let $F \in A''$, $f \in E^\sim$ and $y \in I(x)$. Choose a net $\{a_\alpha\}$ in $A$ that converges to $F$ in the $\sigma(A'', A')$-topology. Then

$$m_x(F)(f|_{I(x)})(y) = F \circ f|_{I(x)}(y) = F(\mu_{y,f|_{I(x)}}) = F(\mu_{y,f|_{I(x)}})$$

$$= \lim_{\alpha}[\mu_{y,f|_{I(x)}}](a_\alpha) = \lim_{\alpha}[\mu_{y,f|_{I(x)}}](a_\alpha) = \lim_{\alpha}f(a_\alpha y)$$
where $[\mu_{y,f}|_{I(x)}] = \mu_{y,f}|_{I(x)} + A^0 \in A' = Z(I(x))'/A^0$. On the other hand
\[
m(F)(P_x(f))(y) = P_x(m(F)(f))(y) = m(F)(f)(y) = F \circ f(y) = F(\mu_{y,f}) = \lim_{\alpha} \mu_{y,f}(a_\alpha) = \lim_{\alpha} f(a_\alpha y).
\]
In the last string of equalities, the first equality follows because $Z(E')$ is commutative. The second equality follows because the range of $1 - P_x$ is $I(x)0$ in $E$. Finally the fifth equality follows because $\mu_{y,f} \in Ann(x)^0 = A'$ in $Z(E')$. Hence, the claim is verified. In what follows we will keep the notation that we established in this initial part of the proof.

Suppose that $E$ has topologically full center. This means that $I(x)$ is cyclic with respect to the unital $f$-subalgebra $A$ of $Z(I(x))$ for the duality $\langle I(x), J \rangle$. Given $T \in Z(E')$ with $0 \leq T \leq 1$, let $\hat{T} = T|_{I(x)0} \in Z(J)$ where $\hat{T}(f|_{I(x)})(y) = T(P_x(f))(y)$ for each $f \in E'$ and $y \in I(x)$. Then by Lemma 3 there is $a_x \in A''$ with $0 \leq a_x \leq 1$ such that $m_x(a_x) = \hat{T}$ on $J$. Then we have that
\[
m(a_x)(P_x(f))(y) = m_x(a_x)(f|_{I(x)})(y) = \hat{T}(f|_{I(x)})(y) = T(P_x(f))(y)
\]
for all $f \in E'$ and $y \in I(x)$. Since the range of $1 - P_x$ is $I(x)0$ in $E'$, we have that
\[
m(a_x)(f)(y) = T(f)(y)
\]
for all $f \in E'$ and $y \in I(x)$. Let $1 - e_x = \sup\{e \in Z(E)' : e = e^2, m(e)(f)(x) = 0 \text{ for all } f \in E'\}$. Since $m$ is order continuous, we have $m(1 - e_x)(f)(x) = 0$ for all $f \in E'$. Hence $F \circ f(x) = 0$ for all $f \in E'$ for some $F \in Z(E)'$ if and only if $e_x \cdot F = 0$. (Note that $\{\mu_{x,f} : f \in E'\}$ is an ideal in $Z(E)'$ and $1 - e_x$ is the band projection of $Z(E)'$ onto the band in $Z(E)'$ that annihilates this ideal.) Now repeating the argument in the proof of Lemma 3 we find a unique $a \in Z(E)'$ such that $0 \leq a \leq 1$, and $e_x \cdot a = a_x$ for each $x \in E_+$. Then, for each $f \in E'$ and $x \in E_+$, we have
\[
T(f)(x) = m(a_x)(f)(x) = m(a \cdot e_x)(f)(x) = m(e_x)m(a)(f)(x) = m(a)(f)(x).
\]
Therefore $m(a) = T$. \hfill \Box

We will conclude this paper by stating some immediate consequences of Proposition 5.

Corollary 3. Let $A$ be an $f$-algebra with point separating order dual such that $(A')_n$ has a unit. Then $A$ has topologically full center.

Proof. By Theorem 5.2 [8], the homomorphism $\nu$ of $(A')_n$ is onto $Orth(A')$. Then Proposition 2 implies that the Arens homomorphism $\gamma : (Orth(A')_n) \to Orth(A')$ is also onto. Hence, by Corollary 2 $m$ is onto $Z(A')$. Therefore, by Proposition 5 $A$ has topologically full center. \hfill \Box
Remark 1. Characterizations of $f$-algebras $A$ such that $(A^*)_n$ has unit are given in [3, 4, 10]. Related to Corollary 3, we mention that we do not know any examples of semi-prime $f$-algebras that do not have topologically full center.

Corollary 4. Let $E$ be a Riesz space with topologically full center. Suppose $T$ is an order bounded operator on $E$. Then $T$ commutes with $Z(E)$ if and only if $T$ is in $Orth(E)$.

Proof. $T$ is order bounded implies $T' : E^\sim \to E^\sim$. If $T$ commutes with $Z(E)$, then $T'$ commutes with $m(Z(E)^\prime\prime)$. Since $Z(E)$ is topologically full, $m(Z(E)^\prime\prime) = Z(E^\sim)$. Therefore $T'$ commutes with $Z(E^\sim)$. That is, $T'$ commutes with the band projections on $E^\sim$. Since $E^\sim$ is Dedekind complete, each band in $E^\sim$ is a projection band. So $T'$ is band preserving and therefore $T' \in Orth(E^\sim)$. By a result in [16, Theorem 3.3], $T \in Orth(E)$. □

In view of Examples 4 and 5, the corollary may fail if $Z(E)$ is not topologically full. On the other hand, the result may be true even when $Z(E)$ is not topologically full. The example constructed by Wickstead in [19] shows this. We refer the reader to the introduction for more detailed information. Corollary 4 shows that $Orth(E)$ is maximal abelian when $Z(E)$ is topologically full. On the other hand, Wickstead’s example in [19] shows that the converse is not true.

Corollary 5. Let $E$ be a Riesz space with topologically full center. Then the following are equivalent:

1. $E^\sim\sim = (E^\sim)_n^\sim$.
2. $m$ is continuous when its domain has the $\sigma(Z(E)^\prime\prime(Z(E)^\prime))$-topology and its range has the $\sigma(E^\sim, E^\sim\sim)$-operator topology.

We leave the straightforward proof of the corollary to the interested reader.

Before stating our final corollary, we want to discuss its content and fix some notation. Let $K$ be a hyperstonian space. That is $K$ is a Stonian compact Hausdorff space and $C(K)$ is a dual Banach space. Let $C(K)_a$ denote the predual of $C(K)$. Recall that $C(K)^\prime = C(K)_a^\prime$, the order continuous linear functionals on $C(K)$. Hence $C(K)_a$ is a band in the Dedekind complete Banach lattice $C(K)^\prime$. Since $Z(C(K)^\prime) = C(K)^\prime$, there is an idempotent $p \in C(K)^\prime$ such that $p$ is the band projection on $C(K)^\prime$ with range $C(K)_a$. That is

$$p \cdot C(K)^\prime = C(K)_a = C(K)_a.$$  

Let $E$ be a Riesz space. Its order dual $E^\sim$ is a Dedekind complete Riesz space. Therefore $E^\sim$ has a topologically full center $Z(E^\sim)$. Furthermore $Z(E^\sim)$ is itself Dedekind complete as a Banach lattice. In fact, it is familiar that $Z(E^\sim) = C(K)$ for some hyperstonian space $K$ (This will become clear in the proof of the corollary.) Let $m : Z(E^\sim)^\prime\prime \to Z(E^\sim\sim)$ be the Arens homomorphism of the bidual of $Z(E^\sim)$. Since $Z(E^\sim)$ is topologically
full, we have \( m(Z(E^\sim)^\prime\prime) = Z(E^{\sim \sim}) \) and \( \text{Ker}(m) = (1 - \pi) \cdot Z(E^\sim)^\prime\prime \) for some idempotent \( \pi \in Z(E^\sim)^\prime\prime = C(K)^\prime\prime \) (Proposition 5). We will show that \( p \circ E^{\sim \sim} = (E^\sim)^n \).

**Corollary 6.** Let \( E \) be a Riesz space with point separating order dual \( E^\sim \). Let \( m \) be the Arens homomorphism of the bidual of \( Z(E^\sim) \) in \( Z(E^{\sim \sim}) \). Then

1. \( Z(E^\sim) \) is topologically full and \( Z(E^\sim) = C(K) \) for some hyperstonian space \( K \).
2. There is an idempotent \( \pi \in C(K)^\prime\prime \) such that \( \pi \cdot C(K)^\prime\prime = (1 - \pi) \cdot C(K)^\prime\prime \).
3. There is an idempotent \( p \in C(K)^\prime\prime \) with \( p \leq \pi \) such that \( p \cdot C(K)^\prime = C(K)^\prime \), and \( p \circ E^{\sim \sim} = (E^\sim)^n \).
4. \( E^{\sim \sim} = (E^\sim)^n \) if and only if \( p = \pi \).

**Proof.**

1. Since \( E^\sim \) is Dedekind complete, \( Z(E^\sim) = C(K) \) is topologically full and \( K \) is a Stonian compact Hausdorff space. (It is well known that \( K \) is hyperstonian. We include a proof for the sake of completeness.) To show that \( K \) is hyperstonian, it is sufficient to see that the order continuous linear functionals on \( C(K) \) separate the points of \( C(K) \) [12]. Consider \( E \subset (E^\sim)^n \subset E^{\sim \sim} \). Take positive elements \( x \in E, f \in E^\sim \) and \( a_f, a \in C(K) \) such that \( \{ a_f \} \) is an increasing net with sup \( a_f = a \) in \( C(K) \). Then, since \( E^\sim \) is Dedekind complete, \( \text{sup} a_f = a f \) in \( E^\sim \). Therefore \( a_f f(x) = a f(x) \), since \( x \) is an order continuous linear functional on \( E^\sim \). Consider \( \mu_{f,x} \in C(K)^\prime \) in the definition process of \( m \), we have \( \mu_{f,x}(b) = (bf) = b f(x) \) for all \( b \in C(K) \). Hence it follows that \( \mu_{f,x} \in C(K)^n \) for all \( x \in E \) and \( f \in E^\sim \).

   Also it is clear that these linear functionals separate the points of the center \( Z(E^\sim) = C(K) \). Therefore \( K \) is hyperstonian and \( C(K)^\prime = (C(K)^n)^\prime \) is the predual of \( C(K) \).

2. The existence of \( \pi \) is clear from Proposition 8. An equivalent means of defining \( \pi \in C(K)^\prime\prime \) is by observing that \( \pi \) is the supremum of the band projections on \( C(K)^\prime \) obtained by considering the bands generated by each linear functional of the form \( \mu_{f,x} \in C(K)^\prime \) when \( f \in E^\sim \) and \( x \in E^{\sim \sim} \).

3. Let \( p \in C(K)^\prime\prime \) be the band projection onto the band \( C(K)^n = C(K)^\prime \). An equivalent means of defining \( p \) would be to observe that \( p \) is the supremum of the band projections on \( C(K)^\prime \) obtained by considering the bands generated by each linear functional of the form \( \mu_{f,x} \in C(K)^\prime \) when \( f \in E^\sim \) and \( x \in E^{\sim \sim} \). Hence we have \( p \leq \pi \). It remains to show that \( p \circ E^{\sim \sim} = (E^\sim)^n \). Note that for each \( f \in E^\sim \) and each \( x \in E^{\sim \sim} \), we have \( \mu_{f,p x} = p \cdot \mu_{f,x} \). (Namely, let \( \{ a_{\alpha} \} \) be a net in \( C(K) \) that converges to \( p \) in \( \sigma(C(K)^n,C(K)^\prime) \)-topology. Then

\[
\mu_{f,p x}(a) = p \circ x''(af) = \lim_{\alpha} \mu_{p f, x''}(a) = \lim_{\alpha} x''(a_{\alpha} f)
\]

and

\[
p \cdot \mu_{f,x}(a) = p \cdot \mu_{f,x} \cdot a = \lim_{\alpha} \mu_{p f, x'}(a) = \lim_{\alpha} \mu_{p f, x'}(a_{\alpha}) = \lim_{\alpha} x''(a_{\alpha} f)
\]
for each $a \in C(K)$. Here the second set of displayed equalities follow from the definition of the Arens product on the bidual of $C(K)$ when $C(K)$ is considered as a unital $f$-algebra [3].) But $p \cdot \mu_{f,x''} \in C(K)_+ = C(K)'_n$ for each $f \in E^\sim$. By reversing the process we used in part (1), it follows that $p \circ x'' \in (E^\sim)_n$. Conversely if $x'' \in (E^\sim)_n$, the process we used in part (1) shows that $\mu_{f,x''} \in C(K)_+$. Therefore

$$\mu_{f,x''} = p \cdot \mu_{f,x''} = \mu_{f,p \circ x''}$$

for each $f \in E^\sim$. That is, $p \circ x'' = x''$ for all $x'' \in (E^\sim)_n$. So $p \circ E^{\sim\sim} = (E^\sim)^\sim$.

Now (4) is clear from parts (2) and (3). □

**Remark 2.** The article [8] has initiated considerable research on Arens product on the biduals of lattice ordered algebras, we include a partial list [3], [6], [7], [13], [14].

**References**

[1] Aliprantis, C.D. and Burkinshaw, O., *Positive Operators*, Academic Press, London, 1985

[2] Arens, R., "Operations induced in function classes", *Monatsh. Math.*, 55, 1-19, 1951

[3] Bernau, S.J and Huijsmans, C.B., "The order bidual of almost $f$-algebras and $d$-algebras", *Trans. Amer. Math. Soc.*, 347, 4259-4275, 1995

[4] Boulabiar, K. and Jaber, J., "The order bidual of $f$-algebras revisited", *Positivity*, 15, 271-279, 2011

[5] Goullet de Rugy, A., "La structure idéale des $M$-espaces", *J. Math. Pures Appl.*, 55, 331-373, 1972

[6] Grobler, J.J., "Commutativity of the Arens product in lattice ordered algebras", *Positivity*, 3, 357-364, 1999

[7] Huijsmans, C.B., "The order bidual of lattice ordered algebras. II", *J. Operator Theory*, 22, 277-290, 1989

[8] Huijsmans, C.B. and de Pagter, B., "The order bidual of lattice ordered algebras", *J. Func. Anal.*, 59, 41-64, 1984

[9] Huijsmans, C.B. and de Pagter, B., "Subalgebras and Riesz subspaces of an $f$-algebra", *Proc. London Math. Soc.*, 48, 41-64, 1984

[10] Jaber, J., "$f$-algebras with a $\sigma$-bounded approximate unit", *Positivity*, 18, 161-170, 2014

[11] Orhon, M., "The ideal center of the dual of a Banach lattice", *Positivity*, 14, 841-847, 2010

[12] Schaefer, H.H., *Banach lattices and positive operators*, Springer, Berlin, 1974

[13] Scheffold, E., "Der Bidual von F-Banachverbandsalgebren", *Acta Sci. Math. (Szeged)*, 55, 167-179, 1991

[14] Scheffold, E., "Über symmetrische Operatoren auf Banachverbanden und Arens-Regularität", *Czechoslovak Math. J.*, 48(123), 747-753, 1998

[15] Wickstead, A.W., "The structure space of a Banach lattice", *J. Math. Pures Appl.*, 56, 39-54, 1977

[16] Wickstead, A.W., "Representation and duality of multiplication operators on archimedean Riesz spaces", *Comp. Math.*, 35, 225-238, 1977

[17] Wickstead, A.W., "Extremal structures of cones of operators", *Quart. J. Math. Oxford (2)*, 32, 239-253, 1981

[18] Wickstead, A.W., "Banach lattices with trivial centre", *Proc. R. Ir. Acad.*, 88A, 71-83, 1988
[19] Wickstead, A.W., "Banach lattices with topologically full centre", Vladikavkaz Mat. Zh., 11, 50-60, 2009
[20] Zaanen, A.C., Riesz Spaces II, North Holland, Amsterdam, 1983

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