In particle physics, the quantization of a real Dirac field results in a special type of fermion that is its own antiparticle [1]. It is called Majorana fermion, and has not been found to exist in nature as elementary particles. In recent years, there have been intensive efforts to explore the possible existence of Majorana fermion in condensed matter systems and cold atomic gases [2–12]. Observation of the Majorana zero modes has been reported in many experiments [13,15].

In this work we find that a Majorana fermion can hop dynamically only in one direction in the bulk of a spinless p-wave superconductor introduced by Kitaev [19]. More complex dynamics can be understood with this one-way dynamical motion as the building block. An electron is a superposition of a pair of Majorana fermions \( \gamma_1 \) and \( \gamma_2 \). We find that Majorana fermion \( \gamma_1 \) can dynamically hop only in one direction by one lattice site while its partner \( \gamma_2 \) can dynamically hop only in the opposite direction. At the end of the hopping, there is a role exchange: Majorana fermion \( \gamma_1 \) becomes \( \gamma_2 \) while Majorana fermion \( \gamma_2 \) switches to \( \gamma_1 \). This means that when an electron is injected into such a superconductor, its two Majorana fermions \( \gamma_1 \) and \( \gamma_2 \) will split spatially by dynamics while exchanging their roles. This elementary one-way motion can be exploited to generate dynamically isolated Majorana fermions. For example, with on-resonance manipulation of the tunneling parameter, an electron can permanently split into a pair of Majorana fermions, which are spatially separated as far as one wishes.

Although an electron can be mathematically written as a pair of Majorana fermions, the spatial splitting of an electron into two Majorana fermions is difficult. So far, Majorana fermions are found to exist in condensed matter systems only in the form of zero-modes at edges or vortex centers [20,29]. Our results show that an electron can be split spatially into a pair of Majorana fermions with dynamics and isolated Majorana fermions can exist in the bulk of a system without edges. Majorana fermions as zero-modes are topological invariants and protected from most types of decoherence due to the non-Abelian statistics, and thus good candidates for topological quantum computation [30,31]. Our results may lead to the development of new methods for preparing and manipulating Majorana fermions.

We consider the one dimensional Kitaev model, which describes a spinless p-wave superconductor. It Hamiltonian is given by [19,32].

\[
H = -\mu \sum_{j=1}^{L} c_{j}^\dagger c_{j} - \sum_{j=1}^{L-1} (t_{p} \, c_{j}^\dagger c_{j+1} + \Delta \, c_{j+1}^\dagger c_{j}^\dagger + \text{h.c.}) ,
\]

where \( h.c. \) is for hermitian conjugate, \( \mu \) is the chemical potential, \( c_{j} \) is the electron annihilation operator for site \( j \), \( L \) is the length of the chain. The tunneling \( t_{p} \) and superconducting gap \( \Delta \) are the same for all the sites. The Kitaev system can be realized experimentally by contacting a nanowire that has strong spin-orbit coupling (e.g., InSb and InAs nanowire) with a s-wave superconductor and in a Zeeman field [13,14,33].

Mathematically, an electron can be written as a superposition of a pair of Majorana fermions,

\[
c_{j}^\dagger = \frac{1}{2}(\gamma_{j,1}^\dagger - i\gamma_{j,2}^\dagger) , \quad c_{j} = \frac{1}{2}(\gamma_{j,1}^\dagger + i\gamma_{j,2}^\dagger).
\]

It is clear that \( \gamma_{j,1} = \gamma_{j,1}^\dagger \) and \( \gamma_{j,2} = \gamma_{j,2}^\dagger \). For convenience, we call \( \gamma_{j,1} \) Majorana fermion of mode 1 at site \( j \) and \( \gamma_{j,2} \) Majorana fermion of mode 2. These two types of Majorana fermions anti-commute as can be checked. We combine two Majorana fermion modes at neighboring sites to form two new operators,

\[
\hat{c}_{j}^\dagger = \frac{1}{2}(\gamma_{j,2}^\dagger - i\gamma_{j+1,1}^\dagger) , \quad \hat{c}_{j} = \frac{1}{2}(\gamma_{j,2}^\dagger + i\gamma_{j+1,1}^\dagger).
\]

One can verify that \( \hat{c}_{j}^\dagger \) and \( \hat{c}_{j} \) are ordinary fermionic creation and annihilation operators.

For simplicity we focus on the condition \( \mu = 0, t_{p} = \Delta \) [19,32], where the Kitaev Hamiltonian in Eq. [1]...
γ to the right becoming ranfermion. (1) is indicated by arrows. (c) The oscillatory spatial splitting of γ
is a superposition of two Majorana fermions at neighboring sites. There are two unpaired Majorana fermions, γ₁,1 and γ₁,2, at the two ends; they are the zero edge modes. (b) The elementary one-way dynamical motion of Majorana fermions. Upper panel: mode 1 Majorana fermion γ₁,1 (red circle) can hop dynamically only to the left by one lattice site becoming γ₁−₁,2 (dashed blue diamond). Lower panel: mode 2 Majorana fermion γ₁,2 (blue diamond) can hop dynamically only to the right becoming γ₁+₁,1 (dashed red circle). The hopping is indicated by arrows. (1) t = 0; an electron at a given site. (2) t = T: its Majorana fermion of mode 1 (red circle) hops to the left becoming mode 2 (blue diamond) while its Majorana fermion of mode 2 (blue diamond) hops to the right becoming mode 1 (red circle). (3) t = 2T: the Majorana fermions hop back to their original site and re-combine to be an electron. The whole process repeats and we have an oscillation. Here the hopping directions are dictated by the elementary motion in (b); no other hopping directions are possible. (T = πℏ/(2Δ).

The figure is drawn for μ = 0, ℏp = Δ.

becomes,

\[ H = i\Delta \sum_{j=1}^{L-1} \gamma_{j,2} \gamma_{j+1,1} = 2\Delta \sum_{j=1}^{L-1} \left( \gamma_j^+ \gamma_j - \frac{1}{2} \right) \].

(3)

This means that the energy eigenstates of this superconductor are composed of integer number of quasi-particles denoted by \( \gamma_j^\dagger \gamma_j \) instead of real electrons. Note that the Majorana fermion \( \gamma_{1,1} \) at the left end and the Majorana fermion \( \gamma_{L,2} \) at the right end are missing in the diagonalized Hamiltonian [3]. Physically, this means that there are two Majorana fermions \( \gamma_{1,1} \) and \( \gamma_{1,2} \), which are localized at the two ends with zero eigen-energy. The above discussion is schematically illustrated in Fig. 1(a).

In this work our focus is on the dynamics of Majorana fermions. For a given stationary superconducting state \( |g\rangle \), we introduce one Majorana fermion into the system at site \( j \). There are two possible states, \( |1\rangle = \gamma_{j,1}|g\rangle \) and \( |2\rangle = i\gamma_{j-1,2}|g\rangle \). From Eq. (3) we have

\[ H|1\rangle = \Delta|2\rangle, \quad H|2\rangle = \Delta|1\rangle \]

(4)

\[ H(|1\rangle + |2\rangle) = \Delta(|1\rangle + |2\rangle) \]

(5)

\[ H(|1\rangle - |2\rangle) = -\Delta(|1\rangle - |2\rangle). \]

(6)

Eq. (4) indicates that both \( |1\rangle \) and \( |2\rangle \) are not eigenstates and they will evolve dynamically. We first look at \( |1\rangle \). With Eqs. (5)(6), we can immediately write down its time evolution as

\[ |\psi_1(t)\rangle = \frac{1}{2} e^{-i\Delta t} (|1\rangle + |2\rangle) + \frac{1}{2} e^{i\Delta t} (|1\rangle - |2\rangle), \]

\[ = \left\{ \cos \left( \frac{\Delta t}{\hbar} \right) \gamma_{j,1} + \sin \left( \frac{\Delta t}{\hbar} \right) \gamma_{j-1,2} \right\} |g\rangle. \]

(7)

This shows that \( \gamma_{j,1} \), a Majorana fermion of mode 1 at site \( j \), will hop dynamically to the left site to site \((j - 1)\) after \( T = \pi\hbar/(2\Delta) \) while rotating into mode 2. After another \( T \), this mode 2 Majorana fermion will hop to the right and back to site \( j \) as mode 1. This oscillation will continue if the system is left undisturbed. Similarly, the state \( |2\rangle \) will evolve as

\[ |\psi_2(t)\rangle = \left\{ \cos \left( \frac{\Delta t}{\hbar} \right) \gamma_{j,2} - \sin \left( \frac{\Delta t}{\hbar} \right) \gamma_{j+1,1} \right\} |g\rangle. \]

(8)

A similar oscillation occurs: a Majorana fermion of mode 2 at site \( j \) hops dynamically to site \( j + 1 \) after \( T \) while rotating into mode 1; it hops back to site \( j \) after another \( T \).

A one-way dynamical motion has emerged: Majorana fermion \( \gamma_{j,1} \) can hop dynamically only to the left by one lattice site to become \( \gamma_{j-1,2} \); Majorana fermion \( \gamma_{j,2} \) can hop dynamically only to the right by one lattice site to become \( \gamma_{j+1,1} \). This is illustrated in Fig. 1(b).

This one-way hopping is elementary for two reasons. (1) \( |g\rangle \) can be any stationary state, and therefore the dynamics in Eqs. (7)(8) is generic. (2) As any state can be expressed as a composition of Majorana fermions, any dynamics is a combination of their respective motions. Even the famed edge modes can be understood with this one-way hopping. At the left end, \( \gamma_{1,1} \) can only hop to the left while there is no site to the left, so it is forever trapped becoming an edge mode. Similarly, \( \gamma_{L,2} \) at the right end is also trapped. The quasiparticle \( \gamma_j^\dagger = (\gamma_{j,2} - i\gamma_{j+1,1})/2 \) is composed of a Majorana fermion of mode 2 on the left site and a Majorana fermion of mode 1 on the right site. These two Majorana fermions hop towards each other and exchange their roles. At the end, nothing happens besides a trivial overall phase and we have an eigenstate.

We consider a more realistic situation where an electron is injected at site \( j \) (possibly with a STM tip) into this superconductor. As an electron is composed of a pair of Majorana fermions of different modes, it will
split spatially into two Majorana fermions periodically. This is shown schematically in Fig. 1(c): Dictated by the one-way hopping, the two Majorana fermions in the electron will hop in the opposite directions and become spatially separated at \( t = T \). However, as the modes have exchanged, the Majorana fermions can not hop further apart. At \( t = 2T \), they have to hop back to the original site to re-combine to be an electron. This intuitive picture can be confirmed if one chooses to solve the Schrödinger equation to find the dynamical evolution of an injected electron. The result is

\[
|\psi(t)\rangle = \left\{ \cos \left( \frac{\Delta t}{\hbar} \right) c_j^\dagger + \sin \left( \frac{\Delta t}{2\hbar} \right) \gamma_j \right\} |g\rangle.
\]  

(9)

It shows the same oscillating spatial splitting.

One may have noticed an interesting feature in the above splitting dynamics: the dynamics is localized and the wave function of an electron can never spread to infinity. This localized dynamics clearly persists for multiple electrons. It is quite peculiar as we know that the wave function of an electron in a real vacuum always diffuses and can spread to infinity. Localization in wave dynamics happens in rare occasions, such as Anderson localization in random potentials and solitons in nonlinear media.

Our discussion so far is done with the condition \( \mu = 0, t_p = d \). When this condition is slightly altered, the essential physics does not change. The one-way hopping and the dynamical splitting of an electron into a pair of Majorana fermions can still occur. The only difference is that the wave functions of the electron and Majorana fermions are broaden to spread over several lattice sites, instead of the ideal localization that we have with \( \mu = 0, t_p = d \).

Is it possible to break the localized dynamics and separate the Majorana fermion pair further apart in space? It is possible only when the system parameters are tuned. We have so far used the condition \( \mu = 0, t_p = \Delta \) in our discussion. We consider now a different condition \( \mu = 0, t_p = -\Delta \), that is, the sign of \( t_p \) is changed. This sign change is equivalent to the following transform

\[
c_j^\dagger \rightarrow ic_j^\dagger, \quad c_j \rightarrow -ic_j.
\]

(10)

with \( t_p = \Delta \). This transform exchanges the real and imaginary parts of an electron, or the modes in the Majorana fermion pair. Its consequence is that the one-way hopping dynamics is now reversed. For the condition \( \mu = 0, t_p = -\Delta \), Majorana fermion \( \gamma_{j,1} \) can only hop to the right by one lattice site to become \( \gamma_{j+1,2} \); Majorana fermion \( \gamma_{j,2} \) can only hop to the left by one lattice site to become \( \gamma_{j-1,1} \). The sign change of \( t_p \) is equivalent to the reversal of the one-way hopping.

We can now control the spatial separation of a pair of Majorana fermions. The scheme is shown in Fig. 2.

Consider again an electron injected at site \( j \). After \( T \), it splits into a pair of Majorana fermions. At this moment, if we change the sign of \( t_p \), the Majorana fermion on the right now has to continue hop to the right while the other has to hop to the left. After another \( T \), the two Majorana fermions are three-lattice-site apart. If we switch the sign of \( t_p \) again at this moment, the two Majorana fermions will hop further apart. If we keep the sign of \( t_p \), the two Majorana fermions are forever separated spatially as dictated by the one-way hopping as shown in the last panel of Fig. 2.

It is possible to achieve the sign change of \( t_p \) in experiment as both the phase and the magnitude of the tunneling parameter \( t_p \) can be modulated by introducing a high-frequency driving field [34].

Unlike the Majorana edge modes, our physical splitting of an electron into a pair of permanently separated Majorana fermions in the bulk is dynamic. As a result, it appears more difficult to control and manipulate these bulk Majorana fermions. However, this difficulty does not look too daunting if we look it from a different angle. The most promising application of these Majorana fermions is for quantum computing. We have to manipulate these Majorana fermions very fast to make the computing meaningful. Once we have developed these techniques, it would be likely that controlling the dy-
The oscillatory splitting seen in Eq. (9) and Fig. (c) is essentially a type of Zitterbewegung oscillation. To see this, we carry out a Fourier transformation to the momentum space, i.e., \( c_k = \frac{1}{\sqrt{N}} \sum_j c_j \exp(ik\alpha) \). Without loss of generality, we assume \( \alpha = 1 \). The Kitaev Hamiltonian in Eq. (1) then becomes
\[
H = \sum_k \left( c_k^\dagger c_{-k} \right) H_k \left( \begin{array}{c} c_k \\ c_{-k}^\dagger \end{array} \right),
\] (11)
where
\[
H_k = \begin{pmatrix} \xi(k) & \eta(k) \\ \eta(k)^* & -\xi(k) \end{pmatrix}.
\] (12)
with \( \xi(k) = -\mu - 2t_p \cos(k) \) and \( \eta(k) = i2\Delta \sin(k) \).
In the momentum space the Kitaev model is seen to assume a form identical to the BCS Hamiltonian. If we regard particle and hole as two components of a pseudo-spin, then the rotation of this pseudo-spin is governed by the Hamiltonian \( H_k \). Due to the dependence of \( H_k \) on \( k \), the pseudo-spin is coupled to momentum \( k \). We have an effective spin-orbit coupling; the oscillatory splitting in Eq. (9) is essentially a type of Zitterbewegung oscillation.

In summary, we have identified an elementary one-way hopping dynamics for Majorana fermions in Kitaev’s superconducting chain model. This elementary one-way hopping can be used to understand all the dynamics in the system, even including the edge modes. We have shown that it can also be exploited to split an electron to two spatially separated Majorana fermions in the bulk. Our work may stimulate further experimental works on preparing and manipulating Majorana fermions.

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