Acyclic coloring of special digraphs

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Abstract

An acyclic $r$-coloring of a directed graph $G = (V,E)$ is a partition of the vertex set $V$ into $r$ acyclic sets. The dichromatic number of a directed graph $G$ is the smallest $r$ such that $G$ allows an acyclic $r$-coloring. For symmetric digraphs the dichromatic number equals the well-known chromatic number of the underlying undirected graph. This allows us to carry over the W[1]-hardness and lower bounds for running times of the chromatic number problem parameterized by clique-width to the dichromatic number problem parameterized by directed clique-width. We introduce the first polynomial-time algorithm for the acyclic coloring problem on digraphs of constant directed clique-width. From a parameterized point of view our algorithm shows that the Dichromatic Number problem is in XP when parameterized by directed clique-width and extends the only known structural parameterization by directed modular width for this problem. For directed co-graphs, which is a class of digraphs of directed clique-width 2, and several generalizations we even show linear time solutions for computing the dichromatic number. Furthermore, we conclude that directed co-graphs and the considered generalizations lead to subclasses of perfect digraphs. For directed cactus forests, which is a set of digraphs of directed tree-width 1, we conclude an upper bound of 2 for the dichromatic number and we show that an optimal acyclic coloring can be computed in linear time.

Keywords: acyclic coloring; directed clique-width; directed co-graphs; polynomial time algorithms

1 Introduction

In this paper, we consider an approach for coloring the vertices of digraphs. An acyclic $r$-coloring of a digraph $G = (V,E)$ is a partition of the vertex set $V$ into $r$ sets such that all sets induce an acyclic subdigraph in $G$. The dichromatic number of $G$ is the smallest $r$ such that $G$ has an acyclic $r$-coloring. Acyclic colorings of digraphs received a lot of attention in [BFJ+04, Moh03, NL82] and also in recent works [LM17, MSW19, SW20]. The dichromatic number is one of two basic concepts for the class of perfect digraphs [AH15] and can be regarded as a natural counterpart of the well known chromatic number for undirected graphs.

In the Dichromatic Number problem (DCN) there is given a digraph $G$ and an integer $r$ and the question is whether $G$ has an acyclic $r$-coloring. If $r$ is constant and not part of the input, the corresponding problem is denoted by DCN$_r$. Even DCN$_2$ is NP-complete [FHM03], which motivates to consider the Dichromatic Number problem on special graph classes. Up to now, only few classes of digraphs are known for which the dichromatic number can be found in polynomial time. The set of DAGs is obviously equal to the set of digraphs of dichromatic number 1. Further, every odd-cycle free digraph [NL82] and every non-even digraph [MSW19] has dichromatic number at most 2.

The hardness of the Dichromatic Number problem remains true, even for inputs of bounded directed feedback vertex set size [MSW19]. This result implies that there are no XP-algorithms for the Dichromatic Number problem parameterized by directed width parameters such as directed path-width, directed tree-width, DAG-width or Kelly-width, since all of these are upper bounded in terms of the size of a smallest feedback vertex set. The first positive result concerning structural parameterizations of the Dichromatic

\[1\] XP is the class of all parameterized problems which can be solved by algorithms that are polynomial if the parameter is considered as a constant [DF13].
Number problem is the existence of an FPT-algorithm\footnote{FPT is the class of all parameterized problems which can be solved by algorithms that are exponential only in the size of a fixed parameter while polynomial in the size of the input size [DF13].} for the Dichromatic Number problem parameterized by directed modular width [SW19].

In this paper, we introduce the first polynomial-time algorithm for the Dichromatic Number problem on digraphs of constant directed clique-width. Therefore, we consider a directed clique-width expression $X$ of the input digraph $G$ of directed clique-width $k$. For each node $t$ of the corresponding rooted expression-tree $T$ we use label-based reachability information of the subgraph $G_t$ of the subtree rooted at $t$. For every partition of the vertex set of $G_t$ into acyclic sets $V_1, \ldots, V_s$ we compute the multi set $\langle \text{reach}(V_1), \ldots, \text{reach}(V_s) \rangle$, where $\text{reach}(V_i)$, $1 \leq i \leq s$, is the set of all label pairs $(a, b)$ such that the subgraph $G_t$ induced by $V_i$ contains a vertex labeled by $b$, which is reachable by a vertex labeled by $a$. By using bottom-up dynamic programming along expression-tree $T$, we obtain an algorithm for the Dichromatic Number problem of running time $n^{2^{O(k^2)}}$ where $n$ denotes the number of vertices of the input digraph. Since any algorithm with running in $n^{2^{O(k)}}$ would disprove the Exponential Time Hypothesis (ETH), the exponential dependence on $k$ in the degree of the polynomial cannot be avoided, unless ETH fails.

From a parameterized point of view our algorithm shows that the Dichromatic Number problem is in XP when parameterized by directed clique-width. Further, we show that the Dichromatic Number problem is W[1]-hard on symmetric digraphs when parameterized by directed clique-width under reasonable assumptions. The best parameterized complexity which can be achieved is given by an XP-algorithm. Furthermore, we apply definability within monadic second order logic (MSO) in order to show that for every integer $r$ it holds that $\text{DCN}_r$ is in FPT when parameterized by directed clique-width.

Since the directed clique-width of a digraph is at most its directed modular width [SW20], we reprove the existence of an XP-algorithm for $\text{DCN}$ and an FPT-algorithm for $\text{DCN}_r$ parameterized by directed modular width [SW19]. On the other hand, there exist several classes of digraphs of bounded directed clique-width and unbounded directed modular width, which implies that directed clique-width is the more powerful parameter and that the results of [SW19] does not imply any parameterized algorithm for directed clique-width.

In Table 1 we summarize the known results for $\text{DCN}$ and $\text{DCN}_r$ parameterized by width parameters.

| parameter                  | $\text{DCN}$ | $\text{DCN}_r$ |
|----------------------------|--------------|---------------|
| directed modular width     | FPT         | FPT           |
| directed clique-width      | W[1]-hard   | FPT           |
|                           | Theorem 4.5 | Theory 4.15   |
| directed tree-width        | NP-hard     | NP-hard       |
|                           | [MSW19]     | [MSW19]       |
| directed path-width        | NP-hard     | NP-hard       |
|                           | [MSW19]     | [MSW19]       |
| DAG-width                  | NP-hard     | NP-hard       |
|                           | [MSW19]     | [MSW19]       |
| Kelly-width                | NP-hard     | NP-hard       |
|                           | [MSW19]     | [MSW19]       |
| clique-width of $un(G)$    | not FPT     | ?             |

Table 1: Complexity of $\text{DCN}$ and $\text{DCN}_r$ parameterized by width parameters.

For directed co-graphs, which is a class of digraphs of directed clique-width 2 [GWY16], and several generalizations we even show a linear time solution for computing the dichromatic number and an optimal acyclic coloring. Furthermore, we conclude that directed co-graphs and the considered generalizations lead to subclasses of perfect digraphs [AH15]. For directed cactus forests, which is a set of digraphs of directed tree-width 1 [GR19b], the results of [Wie19] and [MSW19] lead to an upper bound of 2 for the dichromatic number and that an optimal acyclic coloring can be computed in polynomial time. We show that this even can be done in linear time.

2 Preliminaries

We use the notations of Bang-Jensen and Gutin [BJG18] for graphs and digraphs.

2 FPT is the class of all parameterized problems which can be solved by algorithms that are exponential only in the size of a fixed parameter while polynomial in the size of the input size [DF13].
2.1 Directed graphs

A directed graph or digraph is a pair $G = (V, E)$, where $V$ is a finite set of vertices and $E \subseteq \{(u, v) \mid u, v \in V, u \neq v\}$ is a finite set of ordered pairs of distinct vertices called arcs or directed edges. For a vertex $v \in V$, the sets $N^+(v) = \{u \in V \mid (v, u) \in E\}$ and $N^-(v) = \{u \in V \mid (u, v) \in E\}$ are called the set of successors and the set of all predecessors of $v$. The outdegree of $v$, $\text{outdegree}(v)$ for short, is the number of successors of $v$ and the indegree of $v$, $\text{indegree}(v)$ for short, is the number of predecessors of $v$.

A digraph $G' = (V', E')$ is a subdigraph of digraph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. If every arc of $E$ with both end vertices in $V'$ is in $E'$, we say that $G'$ is an induced subdigraph of digraph $G$ and we write $G' = G[V']$.

The out-degeneracy of a digraph $G$ is the least integer $d$ such that $G$ and all subdigraphs of $G$ contain a vertex of outdegree at most $d$.

For some given digraph $G = (V, E)$ we define its underlying undirected graph by ignoring the directions of the arcs, i.e. $\text{un}(G) = (V, \{(u, v) \mid (u, v) \in E, u, v \in V\})$. There are several ways to define a digraph $G = (V, E)$ from an undirected graph $G'$.

- both arcs $(u, v)$ and $(v, u)$, we refer to $G$ as a complete biorientation of $G'$. Since in this case $G$ is well defined by $G'$ we also denote it by $G\rightarrow\leftarrow G'$. Every digraph $G$ which can be obtained by a complete biorientation of some undirected graph $G'$ is called a complete bioriented graph or symmetric digraph.

- one of the arcs $(u, v)$ and $(v, u)$, we refer to $G$ as an orientation of $G'$. Every digraph $G$ which can be obtained by an orientation of some undirected graph $G'$ is called an oriented graph.

For a digraph $G = (V, E)$ an arc $(u, v) \in E$ is symmetric if $(v, u) \in E$. Thus, each bidirectional arc is symmetric. Further, an arc is asymmetric if it is not symmetric. We define the symmetric part of $G$ as $\text{sym}(G)$, which is the spanning subdigraph of $G$ that contains exactly the symmetric arcs of $G$. Analogously, we define the asymmetric part of $G$ as $\text{asym}(G)$, which is the spanning subdigraph with only asymmetric arcs.

By $P_n^\rightarrow = (\{v_1, \ldots, v_n\}, \{(v_1, v_2), \ldots, (v_{n-1}, v_n)\})$, $n \geq 2$, we denote the directed path on $n$ vertices, by $C_n^\rightarrow = (\{v_1, \ldots, v_n\}, \{(v_1, v_2), \ldots, (v_{n-1}, v_n), (v_n, v_1)\})$, $n \geq 2$, we denote the directed cycle on $n$ vertices.

A directed acyclic graph (DAG) is a digraph without any $C_n^\rightarrow$, for $n \geq 2$, as subdigraph. A vertex $v$ is reachable from a vertex $u$ in $G$ if $G$ contains a $P_n^\rightarrow$ as a subdigraph having start vertex $u$ and end vertex $v$. A digraph is odd cycle free if it does not contain a $C_n^\rightarrow$, for odd $n \geq 3$, as subdigraph. A digraph $G$ is planar if $\text{un}(G)$ is planar.

A digraph is even if for every 0-1-weighting of the edges it contains a directed cycle of even total weight.

2.2 Acyclic coloring of directed graphs

We consider the approach for coloring digraphs given in [NL82]. A set $V'$ of vertices of a digraph $G$ is called acyclic if $G[V']$ is acyclic.

Definition 2.1 (Acyclic graph coloring [NL82]) An acyclic $r$-coloring of a digraph $G = (V, E)$ is a mapping $c : V \rightarrow \{1, \ldots, r\}$, such that the color classes $c^{-1}(i)$ for $1 \leq i \leq r$ are acyclic. The dichromatic number of $G$, denoted by $\overline{\chi}(G)$, is the smallest $r$, such that $G$ has an acyclic $r$-coloring.

There are several works on acyclic graph coloring [BFJ+04, Mol03, NL82] including several recent works [LM17, MSW19, SW20]. The following observations support that the dichromatic number can regarded as a natural counterpart of the well known chromatic number $\chi(G)$ for undirected graphs $G$.

Observation 2.2 For every symmetric directed graph $G$ it holds that $\overline{\chi}(G) = \chi(\text{un}(G))$.

Observation 2.3 For every directed graph $G$ it holds that $\overline{\chi}(G) \leq \chi(\text{un}(G))$.

Observation 2.4 Let $G$ be a digraph and $H$ be a subdigraph of $G$, then $\overline{\chi}(H) \leq \overline{\chi}(G)$.

We consider the following problem.

Name Dichromatic Number (DCN)
Instance A digraph $G = (V, E)$ and a positive integer $r \leq |V|$.
Question Is there an acyclic $r$-coloring for $G$?
If \( r \) is constant and not part of the input, the corresponding problem is denoted by \( r \)-Dichromatic Number (DCN\(_r\)). Even DCN\(_2\) is NP-complete [FHM03].

3 Acyclic coloring of special digraph classes

As recently mentioned in [SW19], only few classes of digraphs for which the dichromatic number can be found in polynomial time are known. The set of DAGs is obviously equal to the set of digraphs of dichromatic number 1. Every odd-cycle free digraph [NL82] and every non-even digraph [MSW19] has dichromatic number at most 2. Thus, for DAGs, odd-cycle free digraphs, and non-even digraphs the dichromatic number can be computed in linear time.

3.1 Acyclic coloring of perfect digraphs

The dichromatic number and the clique number are the two basic concepts for the class of perfect digraphs [AH15]. The clique number of a digraph \( G \), denoted by \( \omega_d(G) \), is the number of vertices in a largest complete subdigraph of \( G \).

Definition 3.1 (Perfect digraphs [AH15]) A digraph \( G \) is perfect if, for every induced subdigraph \( H \) of \( G \), the dichromatic number \( \chi(H) \) equals the clique number \( \omega_d(H) \).

An undirected graph \( G \) is perfect if and only if its complete biorientation \( \overrightarrow{G} \) is a perfect digraph. Thus, Definition 3.1 is a generalization of perfectness to digraphs. While for undirected perfect graphs more than a hundred subclasses have been defined and studied [Hou06], for perfect digraphs there are only very few subclasses known. Obviously, DAGs and subclasses such as series-parallel digraphs, minimal series-parallel digraphs and series-parallel order digraphs [VTL82] are perfect digraphs.

By [AH15] the dichromatic number of a perfect digraph \( G \) can be found by the chromatic number of \( \text{un}(\text{sym}(G)) \), which is possible in polynomial time [GLS81].

Proposition 1 ([AH15]) For every perfect digraph the Dichromatic Number problem can be solved in polynomial time.

We show how to find an optimal acyclic coloring for directed co-graphs and special digraphs of directed tree-width one in linear time.

3.2 Acyclic coloring of directed co-graphs

Let \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) be two vertex-disjoint digraphs. The following operations have been considered by Bechet et al. in [BdGR97].

- The disjoint union of \( G_1 \) and \( G_2 \), denoted by \( G_1 \oplus G_2 \), is the digraph with vertex set \( V_1 \cup V_2 \) and arc set \( E_1 \cup E_2 \).
- The series composition of \( G_1 \) and \( G_2 \), denoted by \( G_1 \otimes G_2 \), is defined by their disjoint union plus all possible arcs between vertices of \( G_1 \) and \( G_2 \).
- The order composition of \( G_1 \) and \( G_2 \), denoted by \( G_1 \oslash G_2 \), is defined by their disjoint union plus all possible arcs from vertices of \( G_1 \) to vertices of \( G_2 \).

The following transformation has been considered by Johnson et al. in [JGST01] and generalizes the operations disjoint union and order composition.

- A graph \( G \) is obtained by a directed union of \( G_1 \) and \( G_2 \), denoted by \( G_1 \odot G_2 \) if \( G \) is a subdigraph of the order composition of \( G_1 \oslash G_2 \) and contains the disjoint union \( G_1 \oplus G_2 \) as a subdigraph.

We recall the definition of directed co-graphs from [CP06].

Definition 3.2 (Directed co-graphs [CP06]) The class of directed co-graphs is recursively defined as follows.

1. Every digraph with a single vertex (\( \{v\}, \emptyset \)), denoted by \( v \), is a directed co-graph.
2. If \( G_1 \) and \( G_2 \) are vertex-disjoint directed co-graphs, then

(a) the disjoint union \( G_1 \oplus G_2 \),
(b) the series composition \( G_1 \otimes G_2 \), and
(c) the order composition \( G_1 \oslash G_2 \) are directed co-graphs.

Every expression \( X \) using the four operations of Definition 3.2 is called a di-co-expression and \( \text{digraph}(X) \) is the defined digraph.

As undirected co-graphs can be characterized by forbidding the \( P_4 \), directed co-graphs can be characterized likewise by excluding the eight forbidden induced subdigraphs [CP06].

For every directed co-graph we can define a tree structure denoted as di-co-tree. It is an ordered rooted tree whose leaves represent the vertices of the digraph and whose inner nodes correspond to the operations applied on the subexpressions defined by the subtrees. For every directed co-graph one can construct a di-co-tree in linear time [CP06]. Directed co-graphs are interesting from an algorithmic point of view since several hard graph problems can be solved in polynomial time by dynamic programming along the tree structure of the input graph, see [BJM14, Gur17, GHK+20, GKR19b, GKR19c, GR18b, Ret98]. Moreover, directed co-graphs are very useful for the reconstruction of the evolutionary history of genes or species using genomic sequence data [HSW17, NEMM+18].

The set of extended directed co-graphs was introduced in [GR18a] by allowing the directed union and series composition of defined digraphs which leads to a superclass of directed co-graphs. Also for every extended directed co-graph we can define a tree structure, denoted as ex-di-co-tree. For the class of extended directed co-graphs it remains open how to compute an ex-di-co-tree.

**Lemma 3.3** Let \( G_1 \) and \( G_2 \) be two vertex-disjoint directed graphs. Then, the following equations hold:

1. \( \overline{\chi}(v) = 1 \)
2. \( \overline{\chi}(G_1 \oplus G_2) = \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)
3. \( \overline{\chi}(G_1 \otimes G_2) = \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)
4. \( \overline{\chi}(G_1 \oslash G_2) = \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)
5. \( \overline{\chi}(G_1 \ominus G_2) = \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)

**Proof**

1. \( \overline{\chi}(v) = 1 \) is obviously clear.

2. \( \overline{\chi}(G_1 \oplus G_2) \geq \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)
   
   Since the digraphs \( G_1 \) and \( G_2 \) are induced subdigraphs of digraph \( G_1 \oplus G_2 \), both values \( \overline{\chi}(G_1) \) and \( \overline{\chi}(G_2) \) lead to a lower bound for the number of necessary colors of the combined graph by Observation 2.4
   
   \[ \overline{\chi}(G_1 \oplus G_2) \leq \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \]
   
   Since the disjoint union operation does not create any new arcs, we can combine color classes of \( G_1 \) and \( G_2 \).

3. \( \overline{\chi}(G_1 \otimes G_2) \geq \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)
   
   By Observation 2.4
   
   \[ \overline{\chi}(G_1 \otimes G_2) \leq \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \]
   
   Since the order operation does not create any cycles, we can combine color classes of \( G_1 \) and \( G_2 \).

4. \( \overline{\chi}(G_1 \oslash G_2) \geq \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \)
   
   By Observation 2.4
   
   \[ \overline{\chi}(G_1 \oslash G_2) \leq \max(\overline{\chi}(G_1), \overline{\chi}(G_2)) \]
   
   Since the directed union does not create any cycles, we can combine color classes of \( G_1 \) and \( G_2 \).
5. $\overrightarrow{\chi}(G_1 \otimes G_2) \geq \overrightarrow{\chi}(G_1) + \overrightarrow{\chi}(G_2)$

Since every $G_1$ and $G_2$ are an induced subdigraphs of the combined graph, both values $\overrightarrow{\chi}(G_1)$ and $\overrightarrow{\chi}(G_2)$ lead to a lower bound for the number of necessary colors of the combined graph by Observation 2.4. Further, the series operations implies that every vertex in $G_1$ is on a cycle of length two with every vertex of $G_2$. Thus, no vertex in $G_1$ can be colored in the same way as a vertex in $G_2$. Thus, $\overrightarrow{\chi}(G_1) + \overrightarrow{\chi}(G_2)$ leads to a lower bound for the number of necessary colors of the combined graph.

For $1 \leq i \leq 2$ let $G_i = (V_i, E_i)$ and $c_i : V_i \to \{1, \ldots, \overrightarrow{\chi}(G_i)\}$ a coloring for $G_i$. For $G_1 \otimes G_2 = (V, E)$ we define a mapping $c : V \to \{1, \ldots, \overrightarrow{\chi}(G_1) + \overrightarrow{\chi}(G_2)\}$ as follows.

$$c(v) = \begin{cases} c_1(v) & \text{if } v \in V_1 \\ c_2(v) + \overrightarrow{\chi}(G_1) & \text{if } v \in V_2. \end{cases}$$

The mapping $c$ satisfies the definition of an acyclic coloring, because every color class $c^{-1}(j)$, $j \in \{1, \ldots, \overrightarrow{\chi}(G_1) + \overrightarrow{\chi}(G_2)\}$ is a subset of $V_1$ or of $V_2$, such that $c^{-1}(j)$ induces an acyclic digraph in $G_1$ or $G_2$ by assumption. Since the series operation does not insert any further arcs between two vertices of $G_1$ or $G_2$, vertex set $c^{-1}(j)$ induces also an acyclic digraph in $G$.

This shows the statements of the lemma.

Lemma 3.3 can be used to obtain the following result.

**Theorem 3.4** Let $G$ be a (-n extended) directed co-graph (given by an ex-di-co-tree). Then, an optimal acyclic coloring for $G$ and $\overrightarrow{\chi}(G)$ can be computed in linear time.

In order to state the next result, let $\omega(G)$ be the number of vertices in a largest clique in graph $G$. Since the results of Lemma 3.3 also hold for $\omega_d$ instead of $\overrightarrow{\chi}$ we obtain the following result.

**Proposition 2** Let $G$ be a (-n extended) directed co-graph (given by an ex-di-co-tree). Then, it holds that $\overrightarrow{\chi}(G) = \chi(un(sym(G))) = \omega(un(sym(G))) = \omega_d(G)$ and all values can be computed in linear time.

**Proposition 3** Every (extended) directed co-graph is a perfect digraph.

**Proof** We show the result by verifying Definition 3.1. Since every induced subdigraph of a (-n extended) directed co-graph is a (-n extended) directed co-graph, Proposition 2 implies that every (extended) directed co-graph is a perfect digraph.

Alternatively, the last result can be shown by the Strong Perfect Digraph Theorem [AH15] since for every (extended) directed co-graph $G$ the symmetric part $un(sym(G))$ is an undirected co-graph and thus a perfect graph. Furthermore (extended) directed co-graphs do not contain a directed cycle $C_n$, $n \geq 3$, as an induced subdigraph [CP06, GKR10a].

The results of Theorem 3.4 and Propositions 2 and 3 can be generalized to larger classes. Motivated by the idea of tree-co-graphs [Tim89] we replace in Definition 3.2 the single vertex graphs by a DAG, for which we know that the dichromatic number and also the clique number is equal to 1. Thus, Lemma 3.3 can be adapted to compute the dichromatic number in linear time. Furthermore following the proof of Proposition 3 we know that this class is perfect.

### 3.3 Acyclic coloring of directed cactus forests

We recall the definition of directed cactus forests, which was introduced in [GR19b] as a counterpart for undirected cactus forests.

**Definition 3.5 (Directed cactus forests [GR19b])** A directed cactus forest is a digraph, where any two directed cycles have at most one joint vertex.

Since directed cactus forests may contain a cycle they have dichromatic number at least 2. Further, the set of all cactus forests is a subclass of the digraphs of directed tree-width at most 1 [GR19b], which is a subclass of non-even digraphs [Wie19]. By MSW19 we can conclude that every cactus forest has dichromatic number at most 2 and that for every cactus forest an optimal acyclic coloring can be computed in polynomial time. In order to improve the running time, we show the following lemma.
Lemma 3.6 Let $G$ be a directed cactus forest. Then, an acyclic 2-coloring for $G$ and can be computed in linear time.

Proof Let $G = (V, E)$ be a directed cactus forest. In order to define an acyclic 2-coloring $c : V \to \{1, 2\}$ for $G$, we traverse a DAG $D_G$ defined as follows. $D_G$ has a vertex for every cycle $C$ on at least two vertices in $G$ and a vertex for every vertex of $G$ which is not involved in any such cycle of $G$.

Two vertices in $D_G$ which both represent a cycle in $G$ are adjacent by a single (arbitrary chosen) directed edge in $D_G$ if these cycles have a common vertex in $G$. A vertex in $D_G$ which represents a cycle $C$ in $G$ and a vertex in $D_G$ which represents a vertex $u$ in $G$ which is not involved in any cycle in $G$ are adjacent in the same way as the vertex of $C$ is adjacent to $u$ in $G$. Two vertices in $D_G$ which both represent a vertex in $G$ which is not involved in any cycle in $G$ are adjacent in $D_G$ in the same way as they are adjacent in $G$.

Then, we consider the vertices $v$ of $D_G$ by a topological ordering in order to define an acyclic 2-coloring for $G$.

- If $v$ represents a vertex in $G$ which is not involved in any cycle in $G$, we define $c(v) = 2$.
- If $v$ represents a cycle $C$ in $G$, then we distinguish the following cases.
  - If all vertices of $C$ are unlabeled up to now, we choose an arbitrary vertex $x \in C$ and define $c(x) = 1$ and $c(y) = 2$ for all $y \in C - \{x\}$.
  - If we have already labeled one vertex $x$ of $C$ by 1 then we define $c(y) = 2$ for all $y \in C - \{x\}$.
  - If we have already labeled one vertex $x$ of $C$ by 2 then we choose an arbitrary vertex $x' \in C - \{x\}$ and define $c(x') = 1$ and further $c(y) = 2$ for all $y \in C - \{x, x'\}$.

By this definitions, in every cycle $C$ of $G$ we color exactly one vertex of $C$ by 1 and all remaining vertices of $G$ by 2. Thus, $c$ leads to a legit acyclic 2-coloring for $G$. □

If $G$ is a DAG, then $c(x) = 1$ for $x \in V$ leads to an acyclic 1-coloring for $G$. Otherwise, Lemma 3.6 leads to an acyclic 2-coloring for $G$.

Theorem 3.7 Let $G$ be a directed cactus forest. Then, an optimal acyclic coloring for $G$ and $\chi(G)$ can be computed in linear time.

4 Parameterized algorithms for directed clique-width

For undirected graphs the clique-width [CO00] is one of the most important parameters. Clique-width measures how difficult it is to decompose the graph into a special tree-structure. Only tree-width [RS86] is a more studied graph parameter. Clique-width is more general than tree-width since graphs of bounded tree-width have also bounded clique-width [CR05]. The tree-width can only be bounded by the clique-width under certain conditions [GW00]. Many NP-hard graph problems admit polynomial-time solutions when restricted to graphs of bounded tree-width or graphs of bounded clique-width.

For directed graphs there are several attempts to generalize tree-width such as directed path-width, directed tree-width, DAG-width, or Kelly-width, which are representative for what people are working on, see e.g. the surveys [GHK+14] [GHK+16]. Unfortunately, none of these attempts allows polynomial-time algorithms for a large class of problems on digraphs of bounded width. This also holds for DCN$_r$ and DCN by the following theorem.

Theorem 4.1 ([MSW19]) For every $r \geq 2$ the $r$-Dichromatic Number problem is NP-hard even for input digraphs whose feedback vertex set number is at most $r + 4$ and whose out-degeneracy is at most $r + 1$.

Thus, even for bounded size of a directed feedback vertex set, deciding whether a directed graph has dichromatic number at most 2 is NP-complete. This result rules out XP-algorithms for DCN and DCN$_r$ by directed width parameters such as directed path-width, directed tree-width, DAG-width or Kelly-width, since all of these are upper bounded by the feedback vertex set number.

Next, we discuss parameters which allow XP-algorithms or even FPT-algorithms for DCN and DCN$_r$. The first positive result concerning structural parameterizations of DCN was given in [SW19] using the directed modular width (dmw).

Theorem 4.2 ([SW19]) The Dichromatic Number problem is in FPT when parameterized by directed modular width.
By [GHK+14], directed clique-width performs much better than directed path-width, directed tree-width, DAG-width, and Kelly-width from the parameterized complexity point of view. Hence, we consider the parameterized complexity of DCN parameterized by directed clique-width.

**Definition 4.3 (Directed clique-width [CO00])** The directed clique-width of a digraph $G$, $d$-$cw(G)$ for short, is the minimum number of labels needed to define $G$ using the following four operations:

1. Creation of a new vertex $v$ with label $a$ (denoted by $a(v)$).
2. Disjoint union of two labeled digraphs $G$ and $H$ (denoted by $G \oplus H$).
3. Inserting an arc from every vertex with label $a$ to every vertex with label $b$ ($a \neq b$, denoted by $\alpha_{a,b}$).
4. Change label $a$ into label $b$ (denoted by $\rho_{a\rightarrow b}$).

An expression $X$ built with the operations defined above using $k$ labels is called a directed clique-width $k$-expression. Let digraph$(X)$ be the digraph defined by $k$-expression $X$.

In [GWY16] the set of directed co-graphs is characterized by excluding two digraphs as a proper subset of the set of all graphs of directed clique-width 2, while for the undirected versions both classes are equal.

By the given definition every graph of directed clique-width at most $k$ can be represented by a tree structure, denoted as $k$-expression-tree. The leaves of the $k$-expression-tree represent the vertices of the digraph and the inner nodes of the $k$-expression-tree correspond to the operations applied to the subexpressions defined by the subtrees. Using the $k$-expression-tree many hard problems have been shown to be solvable in polynomial time when restricted to graphs of bounded directed clique-width [GWY16, GHK+14].

The relation of directed clique-width and directed modular width [SW20] is as follows.

**Lemma 4.4 ([SW20])** For every digraph $G$ it holds that $d$-$cw(G) \leq dmw(G)$.

On the other hand, there exist several classes of digraphs of bounded directed clique-width and unbounded directed modular width, e.g. the set of all directed paths $\{P_n|n \geq 1\}$, the set of all directed cycles $\{C_n|n \geq 1\}$, and the set of all minimal series-parallel digraphs [VTL82]. Thus, the result of [SW19] does not imply any XP-algorithm or FPT-algorithm for directed clique-width.

**Corollary 4.5** The Dichromatic Number problem is $W[1]$-hard on symmetric digraphs and thus on all digraphs when parameterized by directed clique-width.

**Proof** The Chromatic Number problem is $W[1]$-hard parameterized by clique-width [FGLS10]. An instance consisting of a graph $G = (V,E)$ and a positive integer $r$ for the Chromatic Number problem can be transformed into an instance for the Dichromatic Number problem on digraph $\overrightarrow{G}$ and integer $r$. Then, $G$ has an $r$-coloring if and only if $\overrightarrow{G}$ has an acyclic $r$-coloring by Observation 2.2. Since for every undirected graph $G$ its clique-width equals the directed clique-width of $\overrightarrow{G}$ [GWY16], we obtain a parameterized reduction.

Thus, under reasonable assumptions there is no FPT-algorithm the Dichromatic Number problem parameterized by directed clique-width and an XP-algorithm is the best that can be achieved. Next, we introduce such an XP-algorithm.

Let $G = (V,E)$ be a digraph which is given by some directed clique-width $k$-expression $X$. For some vertex set $V' \subseteq V$, we define reach$(V')$ as the set of all pairs $(a,b)$ such that there is a vertex $u \in V'$ labeled by $a$ and there is a vertex $v \in V'$ labeled by $b$ and $v$ is reachable from $u$ in $G[V']$.

**Example 4.6** We consider the digraph in Figure 4. The given partition into three acyclic sets $V = V_1 \cup V_2 \cup V_3$, where $V_1 = \{v_1,v_6,v_7\}$, $V_2 = \{v_2\}$, and $V_3 = \{v_3,v_4,v_5\}$ leads to the multi set $\mathcal{M} = \langle$reach$(V_1)$, reach$(V_2)$, reach$(V_3)$\rangle$, where reach$(V_1) = \{\{1,1\},\{2,2\},\{4,4\},\{1,2\},\{2,4\},\{1,4\}\}$ and reach$(V_2) = \{(3,3)\}$.

Within a construction of a digraph by directed clique-width operations only the edge insertion operation can change the reachability between the present vertices. Next, we show which acyclic sets remain acyclic when performing an edge insertion operation and how the reachability information of these sets have to be updated due to the edge insertion operation.
Lemma 4.7 Let $G = (V, E)$ be a vertex labeled digraph defined by some directed clique-width $k$-expression $X$, $a \neq b$, $a,b \in \{1, \ldots, k\}$, and $V' \subseteq V$ be an acyclic set in $G$. Then, vertex set $V'$ remains acyclic in digraph$(\alpha_{a,b}(X))$ if and only if $(b,a) \notin \text{reach}(V')$.

Proof If $(b,a) \in \text{reach}(V')$, then we know that in digraph$(X)$ there is a vertex $y$ labeled by $a$ which is reachable by a vertex $x$ labeled by $b$. That is, in digraph$(X)$ there is a directed path $P$ from $x$ to $y$. The edge insertion $\alpha_{a,b}$ leads to the arc $(y, x)$ which leads together with path $P$ to a cycle in digraph$((\alpha_{a,b}(X)))$.

If $(b,a) \notin \text{reach}(V')$ and $V' \subseteq V$ is an acyclic set in digraph$(X)$, then there is a topological ordering of digraph$(X)[V']$ such that every vertex labeled by $a$ is before every vertex labeled by $b$ in the ordering. The same ordering is a topological ordering for digraph$(\alpha_{a,b}(X))[V']$ which implies that $V'$ remains acyclic for digraph$(\alpha_{a,b}(X))$. $\square$

Lemma 4.8 Let $G = (V, E)$ be a vertex labeled digraph defined by some directed clique-width $k$-expression $X$, $a \neq b$, $a,b \in \{1, \ldots, k\}$, $V' \subseteq V$ be an acyclic set in $G$, and $(b,a) \notin \text{reach}(V')$. Then, reach$(V')$ for digraph$(\alpha_{a,b}(X))$ can be obtained from reach$(V')$ for digraph$(X)$ as follows:

- For every pair $(x,a) \in \text{reach}(V')$ and every pair $(b,y) \in \text{reach}(V')$, we extend reach$(V')$ by $(x,y)$.

Proof Let $R_1$ be the set reach$(V')$ for digraph$(X)$, $R_2$ be the set reach$(V')$ for digraph$(\alpha_{a,b}(X))$, and $R$ be the set of pairs constructed in the lemma starting with reach$(V')$ for digraph$(X)$. Then, it holds $R_1 \subseteq R$. Furthermore, the rule given in the lemma obviously puts feasible pairs into reach$(V')$ which implies $R \subseteq R_2$. It remains to show $R_2 \subseteq R$. Let $(c,d) \in R_2$. If $(c,d) \in R_1$ then $(c,d) \in R$ as mentioned above. Thus, let $(c,d) \notin R_1$. This implies that there is a vertex $u \in V'$ labeled by $c$ and a vertex $v \in V'$ labeled by $d$ and $v$ is reachable from $u$ in digraph$(\alpha_{a,b}(X))$). Since digraph$(X)$ is a spanning subdigraph of digraph$(\alpha_{a,b}(X))$ and the vertex labels are not changed by the performed edge insertion operation, there is a non-empty set $V_u \in V'$ of vertices labeled by $c$ and there is a non-empty set $V_v \in V'$ of vertices labeled by $d$ and no vertex of $V_v$ is reachable from a vertex of $V_u$ in digraph$(X)$. By the definition of the edge insertion operation we know that in digraph$(X)$ there is a vertex $u'$ labeled by $a$ and a vertex $v'$ labeled by $b$ such that $u'$ is reachable from $u$ and $v$ is reachable from $v'$. Thus, $(c,a) \in R_1$ and $(b,d) \in R_1$. Our rule given in the statement leads to $(c,d) \in R$. $\square$

For a disjoint partition of $V$ into acyclic sets $V_1, \ldots, V_4$, let $\mathcal{M}$ be the multi set$^{3}$ $(\text{reach}(V_1), \ldots, \text{reach}(V_4))$.

Let $F(X)$ be the set of all mutually different multi sets $\mathcal{M}$ for all disjoint partitions of vertex set $V$ into acyclic sets. Every multi set in $F(X)$ consists of nonempty subsets of $\{1, \ldots, k\} \times \{1, \ldots, k\}$. Each subset can occur 0 times and not more than $|V|$ times. Thus, $F(X)$ has at most

$$|(V| + 1)^{2k^2 - 1} \in |V|^{2O(k^2)}$$

mutually different multi sets and is polynomially bounded in the size of $X$.

In order to give a dynamic programming solution along the recursive structure of a directed clique-width $k$-expression, we show how to compute $F(a(v)), F(X \oplus Y)$ from $F(X)$ and $F(Y)$, and $F(\alpha_{a,b}(X))$ and $F(\rho_{a\rightarrow b}(X))$ from $F(X)$.

$^{3}$We use the notion of a multi set, i.e., a set that may have several equal elements. For a multi set with elements $x_1, \ldots, x_n$ we write $\mathcal{M} = (x_1, \ldots, x_n)$. There is no order on the elements of $\mathcal{M}$. The number how often an element $x$ occurs in $\mathcal{M}$ is denoted by $\psi(\mathcal{M}, x)$. Two multi sets $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal if for each element $x \in \mathcal{M}_1 \cup \mathcal{M}_2$, $\psi(\mathcal{M}_1, x) = \psi(\mathcal{M}_2, x)$, otherwise they are called different. The empty multi set is denoted by $\emptyset$. 

Figure 1: Digraph in Example [16]. The dashed lines indicate a partition of the vertex set into three acyclic sets. The numbers at the vertices represent their labels.
Lemma 4.9  Let $a, b \in \{1, \ldots, k\}$, $a \neq b$.

1. $F(a(v)) = \{\{(a, a)\}\}$

2. Starting with set $D = \{\}\times F(X) \times F(Y)$ extend $D$ by all triples that can be obtained from some triple $(M, M', M'') \in D$ by removing a set $L'$ from $M'$ or a set $L''$ from $M''$ and inserting it into $M$, or by removing both sets and inserting $L' \cup L''$ into $M$. Finally, we choose $F(X \oplus Y) = \{M \mid (M, (\cdot, \cdot), (\cdot)) \in D\}$.

3. $F(\alpha_{a,b}(X))$ can be obtained from $F(X)$ as follows. First, we remove from $F(X)$ all multi sets $(L_1, \ldots, L_s)$ such that $(b,a) \in L_t$ for some $1 \leq t \leq s$. Afterwards, we modify every remaining multi set $(L_1, \ldots, L_s)$ in $F(X)$ as follows:

   • For every $L_i$ which contains a pair $(x,a)$ and a pair $(b,y)$, we extend $L_i$ by $(x,y)$.

4. $F(\rho_{a\to b}(X)) = \{\{\rho_{a\to b}(L_1) , \ldots, \rho_{a\to b}(L_s)\} \mid (L_1, \ldots, L_s) \in F(X)\}$, where $\rho_{a\to b}(L_i) = \{(\rho_{a\to b}(c), \rho_{a\to b}(d)) \mid (c,d) \in L_i\}$ and $\rho_{a\to b}(c) = b$ if $c = a$ and $\rho_{a\to b}(c) = c$ if $c \neq a$.

Proof

1. In digraph$(a(v))$ there is exactly one vertex $v$ labeled by $a$ and thus the only partition of $V$ into one acyclic set of the vertex set of digraph$(a(v))$ is $\{v\}$. The corresponding multi set is $(\text{reach}(\{v\})) = \{\{(a,a)\}\}$.

2. $F(X \oplus Y) \subseteq \{M \mid (M, (\cdot, \cdot), (\cdot)) \in D\}$:

   Every acyclic set of digraph$(X \oplus Y)$ is either an acyclic set in digraph$(X)$, or an acyclic set in digraph$(Y)$, or is the union of two acyclic sets from digraph$(X)$ and digraph$(Y)$. All three possibilities are considered when computing $\{M \mid (M, (\cdot, \cdot), (\cdot)) \in D\}$ from $F(X)$ and $F(Y)$.

   $F(X \oplus Y) \supseteq \{M \mid (M, (\cdot, \cdot), (\cdot)) \in D\}$:

   Since the operation $\oplus$ does not create any new edges, the acyclic sets from digraph$(X)$, the acyclic sets from digraph$(Y)$, and the union of acyclic sets from digraph$(X)$ and digraph$(Y)$ remain acyclic sets for digraph$(X \oplus Y)$.

3. By Lemma 4.7 we have to remove all multi sets $(L_1, \ldots, L_s)$ from $F(X)$ for which holds that $(b,a) \in L_t$ for some $1 \leq t \leq s$. The remaining multi sets are updated correctly by Lemma 4.8.

4. In digraph$(X)$ there is a vertex labeled by $d$ which is reachable from a vertex labeled by $c$ if and only if in digraph$(\rho_{a\to b}(X))$ there is a vertex labeled by $\rho_{a\to b}(d)$ which is reachable from a vertex labeled by $\rho_{a\to b}(c)$.

This shows the statements of the lemma. □

Corollary 4.10  Let $G = (V,E)$ be a digraph given by a directed clique-width $k$-expression $X$. There is a partition of $V$ into $r$ acyclic sets if and only if there is some $M \in F(X)$ consisting of $r$ sets of label pairs.

Theorem 4.11  The Dichromatic Number problem on digraphs on $n$ vertices given by a directed clique-width $k$-expression can be solved in $n^{2^{O(k^2)}}$ time.

Proof  Let $G = (V,E)$ be a digraph of directed clique-width at most $k$ and $T$ be a $k$-expression-tree for $G$ with root $v$. For some vertex $u$ of $T$ we denote by $T_u$ the subtree rooted at $u$ and $X_u$ the $k$-expression defined by $T_u$. In order to solve the Dichromatic Number problem for $G$, we traverse $k$-expression-tree $T$ in a bottom-up order. For every vertex $u$ of $T$ we compute $F(X_u)$ following the rules given in Lemma 4.9.

By Corollary 4.10 we can solve our problem by $F(X_v) = F(X)$.

Our rules given Lemma 4.9 show the following running times. For every $v \in V$ and $a \in \{1, \ldots, k\}$ set $F(a(v))$ can be computed in $O(1)$. Set $F(X \oplus Y)$ can be computed in time $(n+1)^{2(k^2-1)} \in n^{2^{O(k^2)}}$ from $F(X)$ and $F(Y)$. Set $F(\alpha_{a,b}(X))$ and also $F(\rho_{a\to b}(X))$ can be computed in time $(n+1)^{2(k^2-1)} \in n^{2^{O(k^2)}}$ from $F(X)$.

In order to bound the number and order of operations within directed clique-width expressions, we can use the normal form for clique-width expressions defined in [EGW03]. The proof of Theorem 4.2 in [EGW03] shows that also for directed clique-width expression $X$, we can assume that for every subexpression, after
a disjoint union operation first there is a sequence of edge insertion operations followed by a sequence of relabeling operations, i.e. between two disjoint union operations there is no relabeling before an edge insertion. Since there are n leaves in T, we have n − 1 disjoint union operations, at most (n − 1) · (k − 1) relabeling operations, and at most (n − 1) · k(k−1)/2 edge insertion operations. This leads to an overall running time of \( n^{2^{O(k^2)}} \).

\[ \text{Example 4.12} \] We consider \( X = (\alpha_1, 2(\rho_2 \rightarrow 1(\alpha_2, 1(1(v_1) \oplus 2(v_2)))) \oplus 2(v_3)). \)

- \( F(1(v_1)) = \{ \{(1, 1)\} \} \)
- \( F(2(v_2)) = \{ \{(2, 2)\} \} \)
- \( F(1(v_1) \oplus 2(v_2)) = \{ \{(1, 1)\}, \{(2, 2)\}, \{(1, 2)\} \} \)
- \( F(\alpha_1, 2(1(v_1) \oplus 2(v_2))) = \{ \{(1, 1)\}, \{(2, 2)\}, \{(1, 2)\} \} \)
- \( F(\alpha_2, 1(1(v_1) \oplus 2(v_2))) = \{ \{(1, 1)\}, \{(2, 2)\} \} \)
- \( F(\rho_2 \rightarrow 1(\alpha_2, 1(1(v_1) \oplus 2(v_2)))) = \{ \{(1, 1)\}, \{(1, 2)\} \} \)
- \( F(\rho_2 \rightarrow 1(\alpha_2, 1(1(v_1) \oplus 2(v_2))) \oplus 2(v_3)) = \{ \{(1, 1)\}, \{(2, 2)\}, \{(1, 2)\} \} \)
- \( F(\alpha_1, 2(\rho_2 \rightarrow 1(\alpha_2, 1(1(v_1) \oplus 2(v_2))) \oplus 2(v_3)) \oplus 2(v_3)) = \{ \{(1, 1)\}, \{(2, 2)\}, \{(1, 2)\} \} \}

Thus, in \( F(X) \) there is one multi set consisting of two sets of label pairs and one multi set consisting of three sets of label pairs. This implies that \( \chi(\text{digraph}(X)) = 2 \).

The running time shown in Theorem 4.11 leads to the following result.

\[ \text{Corollary 4.13} \] The Dichromatic Number problem is in XP when parameterized by directed clique-width.

Up to now there are only very few digraph classes for which we can compute a directed clique-width expression in polynomial time. This holds for directed co-graphs, digraphs of bounded directed modular width, orientations of trees, and directed cactus forests. For such classes we can apply the result of Theorem 4.11. In order to find directed clique-width expressions for general digraphs one can use results on the related parameter bi-rank-width [KR13]. By [BJG18, Lemma 9.9.12] we can use approximate directed clique-width expressions obtained from rank-decomposition with the drawback of a single-exponential blow-up on the parameter.

Next, we give a lower bound for the running time of parameterized algorithms for Dichromatic Number problem parameterized by the directed clique-width.

\[ \text{Corollary 4.14} \] The Dichromatic Number problem on digraphs on \( n \) vertices parameterized by the directed clique-width \( k \) cannot be solved in time \( n^{2^{o(k)}} \), unless ETH fails.

\[ \text{Proof} \] In order to show the statement we apply the following lower bound for the Chromatic Number problem parameterized by clique-width given in [FGL+18]. Any algorithm for the Chromatic Number problem parameterized by clique-width with running in \( n^{2^{o(k)}} \) would disprove the Exponential Time Hypothesis. By Observation 4.2 and since for every undirected graph \( G \) its clique-width equals the directed clique-width of \( \vec{G} \) [GWY16], any algorithm for the Dichromatic Number problem parameterized by the directed clique-width can be used to solve the Chromatic Number problem parameterized by clique-width.

In order to show fixed parameter tractability for DCN, w.r.t. the parameter directed clique-width one can use its definability within monadic second order logic (MSO). We restrict to MSO1-logic, which allows propositional logic, variables for vertices and vertex sets of digraphs, the predicate arc(\( u, v \)) for arcs of digraphs, and quantifications over vertices and vertex sets [CF12]. In [GHK+14, Theorem 4.2] it has been shown that every MSO1-definable digraph problem is in FPT when parameterized by directed clique-width. Next, we will apply this result to DCN.

\[ \text{Theorem 4.15} \] For every integer \( r \) the \( r \)-Dichromatic Number problem is in FPT when parameterized by directed clique-width.
Proof Let \( G = (V, E) \) be a digraph. We can define \( \text{DCN}_r \) by an MSO\(_1\) formula

\[
\exists V_1, \ldots, V_r : \left( \text{Partition}(V, V_1, \ldots, V_r) \land \bigwedge_{1 \leq i \leq r} \text{Acyclic}(V_i) \right)
\]

with

\[
\text{Partition}(V, V_1, \ldots, V_r) = \forall v \in V : (\bigvee_{1 \leq i \leq r} v \in V_i) \land \#v \in V : (\bigvee_{i \neq j, 1 \leq i, j \leq r} v \in V_i \land v \in V_j)
\]

and

\[
\text{Acyclic}(V_i) = \forall V' \subseteq V, V' \neq \emptyset : \exists v \in V' : \text{outdegree}(v) = 0 \lor \text{outdegree}(v) \geq 2
\]

For the correctness we note the following. If there is some \( V' \subseteq V_i, V' \neq \emptyset, \) such that for every vertex \( v \in V' \) it holds that \( \text{outdegree}(v) = 1 \) in \( G \), then \( V' \) is a cycle in \( G \). Further, if there is a cycle \( V'' \) in \( G \), then there is a subset \( V'' \subseteq V'' \), such that \( G[V''] \) is a cycle and thus, for every vertex \( v \in V'' \) it holds that \( \text{outdegree}(v) = 1 \). Thus, the statement follows by the result of \( \text{GHK} + 14 \) Theorem 4.2 stated above.

5 Conclusions and outlook

The presented methods allow us to compute the dichromatic number on special classes of digraphs in polynomial time.

The shown parameterized solutions of Theorem 4.13 and 4.15 also hold for any parameter which is larger or equal than directed clique-width, such as the parameter directed modular width \( \text{SW20} \) (which even allows an FPT-algorithm by \( \text{SW19, SW20} \)) and directed linear clique-width \( \text{GR19a} \). Furthermore, restricted to semicomplete digraphs the shown parameterized solutions also hold for directed path-width \( \text{FP19} \) Lemma 2.14.

Further, the hardness result of Theorem 4.5 rules out FPT-algorithms for the Dichromatic Number problem parameterized by width parameters which can be bounded by directed clique-width. Among these are the clique-width and rank-width of the underlying undirected graph, which also have been considered in \( \text{Gan09} \) on the Oriented Chromatic Number problem.

From a parameterized point of view width parameters are so-called structural parameters, which are measuring the difficulty of decomposing a graph into a special tree-structure. Beside these, the standard parameter, i.e. the threshold value given in the instance, is well studied. Unfortunately, for the Dichromatic Number problem the standard parameter is the number of necessary colors \( r \) and does even not allow an XP-algorithm, since \( \text{DCN}_2 \) is NP-complete \( \text{MSW19} \). A positive result can be obtained for parameter "number of vertices" \( n \). Since integer linear programming is fixed-parameter tractable for the parameter "number of variables" \( \text{Len83} \), the existence of an integer program for \( \text{DCN} \) using \( O(n^2) \) variables implies an FPT-algorithm for parameter \( n \).

Remark 1 To formulate \( \text{DCN} \) for some directed graph \( G = (V, E) \) as an integer program, we introduce a binary variable \( y_j \in \{0, 1\}, j \in \{1, \ldots, n\}, \) such that \( y_j = 1 \) if and only if color \( j \) is used. Further, we use \( n^2 \) variables \( x_{i,j} \in \{0, 1\}, i, j \in \{1, \ldots, n\}, \) such that \( x_{i,j} = 1 \) if and only if vertex \( v_i \) receives color \( j \).

In order to ensure that every color class is acyclic, we will use the well known characterization that a digraph is acyclic if and only if it has a topological vertex ordering. A topological vertex ordering of a directed graph is a linear ordering of its vertices such that for every edge \((v_i, v_j)\) vertex \( v_i \) comes before vertex \( v_j \) in the ordering. The existence of a topological ordering will be verified by further integer valued variables \( t_i \in \{0, \ldots, n - 1\} \) realizing the ordering number of vertex \( v_i, i \in \{1, \ldots, n\} \). In order to define only feasible orderings for every color \( j \in \{1, \ldots, n\}, \) for edge \((v_i, v_j) \in A \) with \( x_{i,j} = 1 \) and \( x_{i', j} = 1 \) we verify \( t_{i'} \geq t_i + 1 \) in condition (4).

\[
\text{minimize } \sum_{j=1}^{n} y_j \tag{1}
\]
subject to

\[ \sum_{j=1}^{n} x_{i,j} = 1 \text{ for every } i \in \{1, \ldots, n\} \]  
\[ x_{i,j} \leq y_j \text{ for every } i, j \in \{1, \ldots, n\} \]  
\[ t_i' \geq t_i + 1 - n \cdot (1 - (x_{i,j} \land x_{i,j}')) \text{ for every } (v_i, v_i') \in E, j \in \{1, \ldots, n\} \]  
\[ y_j \in \{0, 1\} \text{ for every } j \in \{1, \ldots, n\} \]  
\[ t_i \in \{0, \ldots, n-1\} \text{ for every } i \in \{1, \ldots, n\} \]  
\[ x_{i,j} \in \{0, 1\} \text{ for every } i, j \in \{1, \ldots, n\} \]

Equations in (4) are not in propositional logic, but can be transformed for integer programming [Gar14].

It remains to verify whether the running time of our XP-algorithm for DCN can be improved to \(n^{2^{O(k)}}\), which is possible for the Chromatic Number problem by [EGW01, KR03]. Further, it remains open whether the hardness of Theorem 4.5 also holds for special digraph classes and for directed linear clique-width [GR19a]. Furthermore, the existence of an FPT-algorithm for DCN, w.r.t. parameter clique-width of the underlying undirected graph is open, see Table 1.

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