Bound states of the system of two fermions on the three-dimensional lattice

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Abstract. In this paper, the Hamiltonian of the system of two fermions on the three-dimensional lattice $\mathbb{Z}^3$ is considered. Under some conditions to the potential, it is proven that $H(k)$ has four invariant subspaces: in one of them $H(k)$ coincides with $H_0(k)$ and the restrictions of $H(k)$ to the rests three subspaces are unitarily equivalent to each other. Moreover, for the eigenvalues and eigenfunctions of these restricted operators explicit expressions are found.

1. Introduction

The nature of appearance the bound states of two-particle cluster operators for small values of parameters are investigated in detail by Minlos and Mamatov [1], and for more general case by Minlos and Mogilner [2].

The investigation of bound states of Hamiltonian $\hat{H}$ associated with a system of two particles moving on the $d-$ dimensional lattice $\mathbb{Z}^d$ is reduced to the study of the family of Schrödinger operators $H(k), k \in \mathbb{T}^d = (-\pi, \pi]^d$. The eigenfunctions and eigenvalues of $H(k)$ are treated as the bound states and their energies of $\hat{H}$, respectively (see [3] and references therein). Bound states of $\hat{H}$ of the system of two fermions and perturbations of eigenvalues of two-particle Schrödinger operators in one-dimensional lattice are studied in [4] and [5], respectively.

In this paper, we study the bound states of two fermions in the three-dimensional lattice $\mathbb{Z}^3$ (see (2.1)), i.e. the discrete spectra of the family of Schrödinger operators $H(k), k = (k_1, k_2, k_3) \in \mathbb{T}^3$ associated with the Hamiltonian (see (2.3)). If we assume that the potential $\hat{v}$ is of the form (4.1) and $k_1 = k_2 = k_3 = \pi$ then it can be shown that the spectrum of the operator $H(\pi, \pi, \pi) = 6I - V$ consists of only eigenvalues $6 - \alpha$ and $6$. Moreover, $6 - \alpha$ is threefold eigenvalue and $6$ is an eigenvalue of infinite multiplicity. In Theorems 4.2-4.4 we show that the results will be preserved under some perturbations.

2. Description of two-particle Hamiltonian

Let $\mathbb{Z}^3$ be the three-dimensional lattice, $\ell_2(\mathbb{Z}^3 \times \mathbb{Z}^3)$ be the Hilbert space of square-summable functions, defined on $\mathbb{Z}^3 \times \mathbb{Z}^3$, $\ell_2^s(\mathbb{Z}^3 \times \mathbb{Z}^3) := \{f \in \ell_2(\mathbb{Z}^3 \times \mathbb{Z}^3) : f(x, y) = -f(y, x)\}$ be the subspace of antisymmetric functions in $\ell_2(\mathbb{Z}^3 \times \mathbb{Z}^3)$.

In the coordinate representation the total Hamiltonian $\hat{H}$ of the system acts on the Hilbert space $\ell_2^s(\mathbb{Z}^3 \times \mathbb{Z}^3)$ and consists of the difference of free Hamiltonian $\hat{H}_0$ and interaction potential $\hat{V}_2$ of two-particles

$$\hat{H} = \hat{H}_0 - \hat{V}_2, \quad (2.1)$$
where free Hamiltonian $\hat{H}_0$ acts on $\ell^a_2(\mathbb{Z}^3 \times \mathbb{Z}^3)$ by

$$\hat{H}_0 = -\frac{1}{2m} \Delta \otimes I - \frac{1}{2m} I \otimes \Delta.$$ 

Here $m$ denotes the mass of the fermion, which we further assume to be equal to 1, $I$ is the identity operator in $\ell_2(\mathbb{Z}^3)$, the lattice Laplacian $\Delta$ is a difference operator describing the transfer of particle from one site of the lattice to the nearest neighbor site, i.e.

$$(\Delta \hat{\psi})(x) = \sum_{j=1}^{3} [\hat{\psi}(x + e_j) - 2\hat{\psi}(x) + \hat{\psi}(x - e_j)], \quad x \in \mathbb{Z}^3, \quad \hat{\psi} \in \ell_2(\mathbb{Z}^3),$$

where $\{e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)\}$ is the standard basis on $\mathbb{Z}^3$. The interaction $\hat{V}_2$ of two particles are described by

$$(\hat{V}_2 \hat{\psi})(x,y) = \hat{\psi}(x-y)\hat{\psi}(x,y), \quad \hat{\psi} \in \ell^a_2(\mathbb{Z}^3 \times \mathbb{Z}^3).$$

Further we assume that the function $\hat{v}$ satisfies

$$\hat{v} \in \ell^1(\mathbb{Z}^3) \quad \text{and} \quad \hat{v}(x) = \hat{v}(-x) \geq 0, \quad x \in \mathbb{Z}^3. \tag{2.2}$$

Under hypothesis (2.2) the operator $\hat{V}_2$ is a bounded, self-adjoint operator in $\ell^a_2(\mathbb{Z}^3 \times \mathbb{Z}^3)$. It then follows the boundedness and self-adjointness of the energy operator $\hat{H}$.

Let $\mathbb{T}^3$ be the three-dimensional torus, i.e. dual group to $\mathbb{Z}^3$, and $L^a_2(\mathbb{T}^3 \times \mathbb{T}^3)$ be the Hilbert space of square-integrable antisymmetric functions.

The momentum representation is obtained via the standard Fourier transform

$$F : \ell^a_2(\mathbb{Z}^3 \times \mathbb{Z}^3) \to L^a_2(\mathbb{T}^3 \times \mathbb{T}^3),$$

where $L^a_2(\mathbb{T}^3 \times \mathbb{T}^3) := \{ f \in L_2(\mathbb{T}^3 \times \mathbb{T}^3) : f(x, y) = -f(y, x) \}$. The Hamiltonian $H = H_0 - V = \hat{F} \hat{H} \hat{F}^{-1}$ in the momentum representation commutes with the Abelian group of unitary operators $U_s$ for $s \in \mathbb{Z}^3$ :

$$(U_s f)(k_1, k_2) = \exp (-i(s, k_1 + k_2)) f(k_1, k_2), \quad f \in L^a_2(\mathbb{T}^3 \times \mathbb{T}^3).$$

In [6], it is shown the existence of expansions of the Hilbert space $L^a_2(\mathbb{T}^3 \times \mathbb{T}^3)$ and the operators $U_s$, $H$ into the following direct integrals

$$L^a_2(\mathbb{T}^3 \times \mathbb{T}^3) = \int_{\mathbb{T}^3} \oplus L_2(F_k) \, dk, \quad U_s = \int_{\mathbb{T}^3} \oplus U_s(k) \, dk, \quad H = \int_{\mathbb{T}^3} \oplus \hat{H}(k) \, dk,$$

where

$$F_k = \{(k_1, k_2) \in \mathbb{T}^3 \times \mathbb{T}^3 : k_1 + k_2 = k\}, \quad k \in \mathbb{T}^3$$

and $U_s(k)$ is the multiplication operator by the function $\exp(-i(s, k))$ in $L_2(F_k)$. The fiber operator $\hat{H}(k)$ of $H$ acting in the Hilbert space $L_2(F_k)$ is unitarily equivalent to the Schrödinger operator $H(k) := H_0(k) - V$, acting on the Hilbert space

$$L^a_2(\mathbb{T}^3) := \{ f \in L_2(\mathbb{T}^3) : f(-q) = -f(q), \text{for almost all } q \in \mathbb{T}^3 \}$$

of odd square-integrable functions in $\mathbb{T}^3$, as follows

$$(H(k)f)(q) = \varepsilon_k(q) f(q) \quad \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{T}^3} v(q-s) f(s) \, ds. \tag{2.3}$$
Unperturbed operator $H_0(k)$ is the multiplication operator by the function

$$
\varepsilon_k(q) = \varepsilon(\frac{k}{2} + q) + \varepsilon(\frac{k}{2} - q) = 2 \sum_{j=1}^{3} \left[ 1 - \cos \frac{k_j}{2} \cos q_j \right].
$$

(2.4)

The perturbation operator $V$ is an integral operator in $L_2^0(T^3)$ with the kernel

$$
v(q - s) = \frac{1}{(2\pi)^{3/2}} \sum_{n \in \mathbb{Z}^3} \hat{v}(n) e^{i(q - s) \cdot n}, \quad q, s \in T^3
$$

and belongs to the trace class $\Sigma_1$.

3. Spectrum of the operator $H(k)$

For any $k \in T^3$ we denote by $M(k)$ and $m(k)$ the maximum and minimum of the function $\varepsilon_k(p)$:

$$
M(k) = \max_{k \in T^3} \varepsilon_k(p) = 2 \sum_{j=1}^{3} \left[ 1 + \cos \frac{k_j}{2} \right], \quad m(k) = \min_{p \in T^3} \varepsilon_k(p) = 2 \sum_{j=1}^{3} \left[ 1 - \cos \frac{k_j}{2} \right].
$$

According to the Weil Theorem [6] the continuous spectrum of $H(k)$ coincides with the essential spectrum of $H_0(k)$. The spectrum of unperturbed operator $H_0(k)$ is purely continuous and coincides with the segment $[m(k), M(k)]$. The width $\omega(k) = M(k) - m(k)$ of this segment is called the width of the continuous spectrum of $H(k)$ and it is equal to

$$
\omega(k) = 4 \cos \frac{k_1}{2} + 4 \cos \frac{k_2}{2} + 4 \cos \frac{k_3}{2}.
$$

Therefore,

$$
\min_{k \in T^3} \omega(k) = \omega(\pi, \pi, \pi) = 0, \quad \max_{k \in T^3} \omega(k) = \omega(0, 0, 0) = 12.
$$

The width of continuous spectrum of $H(k)$ in direction $e_j$, $j = 1, 2, 3$ is defined as

$$
\omega_j(k) = \max_{p_j \in T^3} \varepsilon_k(p) - \min_{p_j \in T^3} \varepsilon_k(p) = 4 \cos \frac{k_j}{2}.
$$

(3.1)

Note that

$$
\omega(k) = \omega_1(k) + \omega_2(k) + \omega_3(k).
$$

Moreover,

$$
\omega_j(k) \equiv 0 \quad \text{for} \quad k \in S(j) := \{k \in T^3 : k_j = \pi\}.
$$

(3.2)

If $k = (\pi, \pi, \pi)$, the width of the essential spectrum is 0, i.e. $\omega(\pi, \pi, \pi) = 0$ and the spectrum of the operator $H(\pi, \pi, \pi) = 6I - V$ consists of eigenvalues of the form $6 - \hat{v}(n)$, $n \in \mathbb{Z}^3$ and of the essential spectrum $\{6\}$. With decreasing of width of continuous spectrum the number of eigenvalues of the Schrödinger operator $H(k)$ increases (see [7]). If $k_j = \pi$ for some $j \in \{1, 2, 3\}$, then $\omega_j(k) = 0$ and there exists a potential $\hat{v}$ such that the operator $H(k)$ has infinitely many eigenvalues outside the essential spectrum [8].

Let us recall some well-known facts and notations from the Operator and Perturbation Theory. For any self-adjoint operator $B$ acting in Hilbert space $\mathcal{H}$ and not having essential spectrum above the point $\mu \in \mathbb{R}$, by $n(\mu, B)$ we denote the number of eigenvalues lying above $\mu$. 

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By \( N(k,z) \) we denote the number of eigenvalues of \( H(k) \), lying below \( z \leq m(k) \), i.e. \( N(k,z) = n(-z,-H(k)) \). In fact, the number \( N(k,m(k)) \) coincides with the number of eigenvalues of \( H(k) \) outside the essential spectrum (see [9], [10]). For any \( k \in \mathbb{T}^3 \) and \( z < m(k) \) we define an integral operator by

\[
G(k,z) = V^\frac{1}{2}r_0(k,z)V^\frac{1}{2},
\]

where \( r_0(k,z) \) is the resolvent of unperturbed operator \( H_0(k) \). Under the condition (2.2) the operator \( V \) is positive, and we denote by \( V^\frac{1}{2} \) the positive square-root of \( V \).

From the self-adjointness of \( H(k) = H_0(k) - V \) and from the positivity of \( V \) it follows that

\[
\sigma(H(k)) \cap (M(k),\infty) = \emptyset,
\]

hence, \( \sigma_{disc}(H(k)) \subset (-\infty,m(k)] \), which implies that it is enough to investigate the eigenvalues for \( z < m(k) \).

The solution \( f \) of the Schrödinger operator

\[
H(k)f = zf
\]

and fixed points \( \varphi \) of the operators \( G(k,z) \) are related as

\[
f = r_0(k,z)V^{\frac{1}{2}}\varphi, \quad \varphi = V^{\frac{1}{2}}f.
\]

Moreover, the Birman-Schwinger principle also holds [9],[10].

**Lemma 3.1.** If the limit operator \( \lim_{z \to m(k)} G(k,z) = G(k,m(k)) \) exists in operator topology, then it is compact and

\[
N(k,m(k)) = n(1,G(k,m(k))).
\]

Equality (3.5) means that the number of eigenvalues of \( H(k) \), lying below \( m(k) \) coincides with the eigenvalues of \( G(k,m(k)) \), greater than 1.

4. **Invariant subspaces of the operator \( H(k) \)**

In what follows, we consider the operator \( H(k) \) with the potential \( \hat{\nu} \) of the form:

\[
\hat{\nu}(x) = \begin{cases} 
1, & \text{if } x = 0 \\
\alpha, & \text{if } |x| = 1 \\
0, & \text{if } |x| \geq 2.
\end{cases}
\]

Here \( \alpha > 0 \) and \( x \in \mathbb{Z}^3 \), \( |x| = |x_1| + |x_2| + |x_3| \). The perturbation operator \( V \) is an integral operator \( L^2_0(\mathbb{T}^3) \) with the kernel

\[
\frac{1}{(2\pi)^3}v(p-q) = \frac{2\alpha}{(2\pi)^3} \left[ \sin p_1 \sin q_1 + \sin p_2 \sin q_2 + \sin p_3 \sin q_3 \right].
\]

If the potential \( \hat{\nu} \) has the form (4.1) and \( k_1 = k_2 = k_3 = \pi \), then the number \( z_1(\pi,\pi,\pi) = 6-\alpha \), is threefold eigenfunctions of \( H(\pi,\pi,\pi) \) with normed eigenfunctions

\[
\phi_{(1,0,0)}^{-++} = \frac{1}{2 \pi^{3/2}} \sin p_1, \quad \phi_{(0,1,0)}^{-++} = \frac{1}{2 \pi^{3/2}} \sin p_2, \quad \phi_{(0,0,1)}^{-++} = \frac{1}{2 \pi^{3/2}} \sin p_3.
\]
The number \( z_{\infty}(\pi, \pi, \pi) = 6 \) is an eigenvalue of infinite multiplicity, and associated eigenfunctions are

\[
\phi_{(n, m, l)}^{+++}(p) = \sin np_1 \cos mp_2 \cos lp_3, \quad \phi_{(n, m, l)}^{++-}(p) = \cos np_1 \sin mp_2 \cos lp_3, \\
\phi_{(n, m, l)}^{+-+}(p) = \cos np_1 \cos mp_2 \sin lp_3, \quad n + m + l \geq 2.
\]

Denote

\[
L_2^+(T) = \{ f \in L_2(T) : f(p) = f(-p) \}, \quad L_2^-(T) = \{ f \in L_2(T) : f(-p) = -f(p) \}
\]

which are the subspaces of even and odd functions, respectively. Moreover, by Riesz Lemma one gets

\[
L_2^0(T^3) = L_2^{+++}(T^3) \oplus L_2^{++-}(T^3) \oplus L_2^{+-+}(T^3) \oplus L_2^{--}(T^3), \tag{4.4}
\]

where

\[
L_2^{+++}(T^3) = L_2^+(T) \otimes L_2^+(T) \otimes L_2^+(T), \quad L_2^{++-}(T^3) = L_2^+(T) \otimes L_2^-(T) \otimes L_2^+(T)
\]

and

\[
L_2^{+-+}(T^3) = L_2^+(T) \otimes L_2^-(T) \otimes L_2^+(T), \quad L_2^{--}(T^3) = L_2^-(T) \otimes L_2^-(T) \otimes L_2^-(T).
\]

**Lemma 4.1.** The subspaces \( L_2^{+++}(T^3) \), \( L_2^{++-}(T^3) \), \( L_2^{+-+}(T^3) \) and \( L_2^{--}(T^3) \) are invariant with respect to \( H(k) \).

**Proof.** First, we prove the invariance of these spaces with respect to \( H_0(k) \), and then for \( V \). From the representation (2.4) it follows that \( \varepsilon_k \) belongs to the subspace

\[
L_2^{+++}(T^3) = \{ f \in L_2(T^3) : f(p_1, p_2, p_3) = f(-p_1, p_2, p_3) = f(-p_1, -p_2, p_3) = f(-p_1, -p_2, -p_3) \}.
\]

Hence from \( f \in L_2^{+++}(T^3) \) it follows that \( \varepsilon_k f \in L_2^{+++}(T^3) \). This implies that \( L_2^{+++}(T^3) \) is invariant with respect to \( H_0(k) \).

If \( f \in L_2^{++-}(T^3) \), it then follows from (4.2) that

\[
(Vf)(p_1, p_2, p_3) = \frac{1}{(2\pi)^{3/2}} \int_{T^3} v(p_1 - s_1, p_2 - s_2, p_3 - s_3) f(s_1, s_2, s_3) \, ds_1 \, ds_2 \, ds_3 =
\]

\[
\frac{2\alpha \sin p_1}{(2\pi)^3} \int_{T^3} \sin q_1 f(s_1, s_2, s_3) \, ds_1 \, ds_2 \, ds_3
\]

which belongs to \( L_2^{++-}(T^3) \). This proves the invariance of the subspace \( L_2^{++-}(T^3) \) with respect to \( V \), and hence with respect to operator \( H(k) = H_0(k) - V \). Analogously, one can prove the invariance of subspaces \( L_2^{+-+}(T^3) \), \( L_2^{++-}(T^3) \) and \( L_2^{--}(T^3) \) with respect to \( H(k) \). This completes the proof.

From the proof of Lemma 4.1 it follows that the restriction \( H^{+++}(k) = H_0(k) - V^{+++} \) of the operator \( H(k) \) acts as follows

\[
H^{+++}(k)f(p) = \varepsilon_k(p)f(p) - \frac{2\alpha}{(2\pi)^3} \int_{T^3} \sin p_1 \sin q_1 f(q) \, dq, \quad f \in L_2^{+++}(T^3). \tag{4.5}
\]
Denote by $H^{--}(k)$, $H^{++}(k)$ and $H^{+-}(k)$ the restrictions of $H(k)$ respectively in the Hilbert spaces $L^{--}(-\mathbb{T}^3)$, $L^{++}(-\mathbb{T}^3)$ and $L^{+-}(-\mathbb{T}^3)$. Note that

$$(H^{--}(k)f)(p) = H_0(k)f(p), \quad f \in L^{--}(-\mathbb{T}^2).$$

Notice that the spectrum of $H_0(k)$ is purely continuous and consists of the segment $[m(k), M(k)]$. Hence, it is enough to study $H^{++}(k)$, $H^{+-}(k)$ and $H^{+-}(k)$. Since they are unitarily equivalent to each other, i.e.

$$H^{++}(k) = U_2 H^{++}(k) U_2^{-1} = U_3 H^{++}(k) U_3^{-1},$$

where $(U_2 f)(p_1, p_2, p_3) = f(p_2, p_1, p_3)$, $(U_3 f)(p_1, p_2, p_3) = f(p_3, p_2, p_1)$, it suffices to consider just one of them, for example $H^{++}(k) = H^{++}(k_1, k_2, k_3)$.

**Theorem 4.2.** Let $k_1 = k_2 = \pi$. Then for any $k_3 \in (-\pi, \pi)$ the operator $H^{++}(\pi, \pi, k_3)$ has a unique nondegenerated eigenvalue

$$z_1^{++}(k_3) = 6 - \sqrt{\alpha^2 + 4 \cos^2 \frac{k_3}{2}}$$

with an associated eigenfunction

$$f_1^{++}(p) = \frac{C \sin p_1}{6 - 2 \cos \frac{k_3}{2} \cos p_3 - z_1^{++}(k_3)} \in L^{++}(\mathbb{T}^3),$$

here $C$ is a constant.

**Proof.** Since $H^{++}(\pi, \pi, k_3)$ is a self-adjoint operator, the spectrum is subset of real axis. The eigenvalue equation $H^{++}(\pi, \pi, k_3)f(p) = zf(p)$ is equivalent to the equation

$$(6 - 2 \cos \frac{k_3}{2} \cos q_3)f(p) - \frac{2\alpha}{(2\pi)^3} \int_{\mathbb{T}^3} \sin q_1 \sin s_1 f(s) \, ds = zf(p).$$

Introducing the notation

$$a = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \sin s_1 f(s) \, ds$$

from (4.7) we get the following representation for the eigenfunction $f(p)$:

$$f(p) = \frac{2\alpha a \sin p_1}{6 - 2 \cos \frac{k_3}{2} \cos p_3 - z}.$$  

Putting (4.9) and (4.8) we get the equation

$$a \left( 1 - \frac{2\alpha}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\sin^2 s_1 \, ds}{6 - 2 \cos \frac{k_3}{2} \cos s_3 - z} \right) = 0.$$

If $a = 0$, then it is clear that $f(p) \equiv 0$. This contradicts that $f$ is an eigenfunction. Thus,

$$\Delta(z) = 1 - \frac{2\alpha}{(2\pi)^3} \int_{\mathbb{T}^3} \frac{\sin^2 s_1 \, ds}{6 - 2 \cos \frac{k_3}{2} \cos s_3 - z} = 0.$$  

(4.10)
If \( z > 6 - 2 \cos \frac{k_3}{2} \) then \( \Delta(z) > 1 \). Hence we assume that \( z < 6 - 2 \cos \frac{k_3}{2} \). Let us calculate the following integral:

\[
2 \alpha \left( \frac{2 \pi}{3} \right)^3 \int_{T_3} \frac{\sin^2 s_1 \ ds}{6 - 2 \cos \frac{k_3}{2} \cos s - z} = \frac{\alpha}{2 \pi} \int_{T} \frac{ds_3}{6 - 2 \cos \frac{k_3}{2} \cos s_3 - z} \tag{4.11}
\]

Using the change of variables \( \xi = e^{is} \), the integral in the right-hand-side of (4.11) is represented as

\[
\frac{\alpha}{2 \pi} \int_{T} \frac{ds_3}{6 - 2 \cos \frac{k_3}{2} \cos s_3 - z} = \frac{-\alpha}{\cos \frac{k_3}{2}} \frac{1}{2 \pi i} \int_{|\xi|=1} \frac{d\xi}{\xi^2 - \frac{6 - z}{\cos \frac{k_3}{2}} \xi + 1}
\]

\[
= \frac{-\alpha}{\cos \frac{k_3}{2}} \frac{1}{2 \pi i} \int_{|\xi|=1} \frac{d\xi}{(\xi - \xi_1)(\xi - \xi_2)}. \tag{4.12}
\]

Hence we end up with Cauchy type integral, where

\[
\xi_1 = \frac{6 - z - \sqrt{(6 - z)^2 - 4 \cos^2 \frac{k_3}{2}}}{2 \cos \frac{k_3}{2}}, \quad \xi_2 = \frac{6 - z + \sqrt{(6 - z)^2 - 4 \cos^2 \frac{k_3}{2}}}{2 \cos \frac{k_3}{2}}
\]

are zeros of \( \xi^2 - \frac{6 - z}{\cos \frac{k_3}{2}} \xi + 1 \). If \( z < 6 - 2 \cos \frac{k_3}{2} \) then \( |\xi_1| < 1, |\xi_2| > 1 \). Then by Cauchy theorem on residues we get

\[
\frac{1}{2 \pi i} \int_{|\xi|=1} \frac{d\xi}{(\xi - \xi_1)(\xi - \xi_2)} = \frac{1}{\xi_1 - \xi_2} = -\frac{\cos \frac{k_3}{2}}{\sqrt{(6 - z)^2 - 4 \cos^2 \frac{k_3}{2}}} \tag{4.12}
\]

Inserting this into (4.10) we get the following equation

\[
1 - \frac{\alpha}{\sqrt{(6 - z)^2 - 4 \cos^2 \frac{k_3}{2}}} = 0.
\]

Solving it we get

\[
z_1 = 6 - \sqrt{\alpha^2 + 4 \cos^2 \frac{k_3}{2}}, \quad z_2 = 6 + \sqrt{\alpha^2 + 4 \cos^2 \frac{k_3}{2}}
\]

Since \( z_2 > 6 - 2 \cos \frac{k_3}{2} \), \( z_1 \) is a unique eigenvalue of \( H^{-++}(\pi, \pi, k_3) \).

**Theorem 4.3.** Let \( k_1 = k_3 = \pi \). Then for any \( k_2 \in (-\pi, \pi) \) the operator \( H^{-++}(\pi, k_2, \pi) \) has a unique nondegenerate eigenvalue

\[
z^{-++}_{1}(k_2) = 6 - \sqrt{\alpha^2 + 4 \cos^2 \frac{k_2}{2}} \tag{4.13}
\]
with associated eigenfunction

\[ f^{++}_{2}(p) = \frac{C \sin p}{6 - 2 \cos \frac{k_2}{2} \cos p_2 - z^{++}_{1}(k_2)} \in L^{++}_{2}(\mathbb{T}^3), \]

where \( C \) is a constant.

**Theorem 4.4.** Let \( k_2 = k_3 = \pi \) and \( k_1 \in \mathbb{T} \). Then for any \( \alpha > \cos \frac{k_1}{2} \) the operator \( H^{++}(k_1, \pi, \pi) \) has a unique nondegenerated eigenvalue

\[ z^{++}_{2}(k_1) = 6 - \alpha - \frac{1}{\alpha} \cos^2 \frac{k_1}{2} \]

with associated eigenfunction

\[ f^{++}_{3}(p) = \frac{C \sin p}{6 - 2 \cos \frac{k_1}{2} \cos p_1 - z^{++}_{2}(k_1)} \in L^{++}_{2}(\mathbb{T}^3), \]

where \( C \) is any constant.

**Proof.** The eigenvalue equation \( H^{++}(k_1, \pi, \pi)f(p) = zf(p) \) is equivalent to

\[ (6 - 2 \cos \frac{k_1}{2} \cos p_1)f(p) - \frac{2\alpha}{(2\pi)^2} \int_{\mathbb{T}^3} \sin p_1 \sin s_1 f(s) \, ds = zf(q), \]

which have nontrivial solutions if and only if \( z < 6 - 2 \cos \frac{k_1}{2} \) is the solution of the equation

\[ 1 - \frac{2\alpha}{(2\pi)^2} \int_{\mathbb{T}^3} \frac{\sin^2 s_1 \, ds}{6 - 2 \cos \frac{k_1}{2} \cos s_1 - z} = 1 - \frac{2\alpha}{2\pi} \int_{\mathbb{T}} \frac{\sin^2 s_1 \, ds}{6 - 2 \cos \frac{k_1}{2} \cos s_1 - z} = 0. \]

Using

\[
\frac{-\sin^2 s}{6 - 2 \cos \frac{k_1}{2} \cos s - z} = \frac{\cos^2 s - 1}{6 - 2 \cos \frac{k_1}{2} \cos s - z}
\]

\[
-\frac{\cos s}{2 \cos \frac{k_1}{2}} = -6 - z - \frac{6 - z}{4 \cos^2 \frac{k_1}{2}} + \frac{(6 - z)^2}{4 \cos^2 \frac{k_1}{2}} - 1 \frac{1}{6 - 2 \cos \frac{k_1}{2} \cos s - z}
\]

(4.16) is rewritten as

\[ 1 + \frac{\alpha(6 - z)}{2 \cos^2 \frac{k_1}{2}} - \frac{(6 - z)^2}{4 \cos^2 \frac{k_1}{2}} - 1 \frac{2\alpha}{2\pi} \int_{\mathbb{T}} \frac{ds}{6 - 2 \cos \frac{k_1}{2} \cos s_1 - z} = 0. \]

As in the proof of Theorem 4.3 it is easy to check

\[ \frac{2\alpha}{2\pi} \int_{\mathbb{T}} \frac{ds_1}{6 - 2 \cos \frac{k_1}{2} \cos s_1 - z} = \frac{2\alpha}{\sqrt{(6 - z)^2 - 4 \cos^2 \frac{k_1}{2}}}. \]
Now using (4.17) and last relation we get
\[ 1 + \frac{\alpha(6 - z)}{2\cos^2\frac{k_1}{2}} - \left[ \frac{(6 - z)^2}{4\cos^2\frac{k_1}{2}} - 1 \right] \frac{2\alpha}{\sqrt{(6 - z)^2 - 4\cos^2\frac{k_1}{2}}} = \]
\[ 1 + \frac{\alpha(6 - z)}{2\cos^2\frac{k_1}{2}} - \frac{\alpha\sqrt{(6 - z)^2 - 4\cos^2\frac{k_1}{2}}}{2\cos^2\frac{k_1}{2}} \]
Solving this equation with respect to \( z \) we get \( z = 6 - \alpha - \frac{1}{\alpha}\cos\frac{k_1}{2} \), which proves (4.14).

Now let us prove that for \( \alpha \leq \cos\frac{k_1}{2} \) the discrete spectrum of \( H^{-+}(k_1, \pi, \pi) \) is empty outside the essential spectrum. The limit Birman-Schwinger operator
\[ G^{-+}((k_1, \pi, \pi), m(k_1, \pi, \pi)) = (V^{-+})^{\frac{1}{2}} r_0((k_1, \pi, \pi), m(k_1, \pi, \pi))(V^{-+})^{\frac{1}{2}} \]
corresponding to \( H^{-+}(k_1, \pi, \pi) \) is of the form
\[ (G^{-+}((k_1, \pi, \pi), m(k_1, \pi, \pi))f)(p) = \frac{\alpha}{\cos(k_1/2)} \int_{-\pi}^{\pi} \sin p_1 \sin s_1 f(s) ds. \]
Thus,
\[ \|G^{-+}((k_1, \pi, \pi), m(k_1, \pi, \pi))\| = (G^{-+}((k_1, \pi, \pi), m(k_1, \pi, \pi))\varphi_1^{-}, \varphi_1^{-}) = \frac{\alpha}{\cos(k_1/2)}, \]
\[ (\varphi_1^{-}, \varphi_1^{-}) = 1, \quad \varphi_1^{-}(p) = \frac{\sin p_1}{2\pi\sqrt{\pi}}. \]
From here if \( \alpha \leq \cos\frac{k_1}{2} \) then \( \|G^{-+}((k_1, \pi, \pi), m(k_1, \pi, \pi))\| \leq 1 \). So, by Lemma 3.1 the operator \( H^{-+}(k_1, \pi, \pi) \) has empty discrete spectrum outside the essential spectrum. \( \square \)

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