The Foldy-Lax approximation of the scattered waves by many small bodies for the Lamé system

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We are concerned with the linearized, isotropic and homogeneous elastic scattering problem by many small rigid obstacles of arbitrary, Lipschitz regular, shapes in 3D case. We prove that there exist two constants \(a_0\) and \(c_0\), depending only on the Lipschitz character of the obstacles, such that under the conditions \(a \leq a_0\) and \(\sqrt{M - \frac{1}{2}} \leq c_0\) on the number \(M\) of the obstacles, their maximum diameter \(a\) and the minimum distance between them \(d\), the corresponding Foldy-Lax approximation of the farfields is valid. In addition, we provide the error of this approximation explicitly in terms of the three parameters \(M, a\) and \(d\). These approximations can be used, in particular, in the identification problems (i.e. inverse problems) and in the design problems (i.e. effective medium theory).

1 Introduction and statement of the results

Let \(B_1, B_2, \ldots, B_M\) be \(M\) open, bounded and simply connected sets in \(\mathbb{R}^3\) with Lipschitz boundaries, containing the origin. We assume that their sizes and Lipschitz constants are uniformly bounded. We set \(D_m := \epsilon B_m + z_m\) to be the small bodies characterized by the parameter \(\epsilon > 0\) and the locations \(z_m \in \mathbb{R}^3, m = 1, \ldots, M\).

Assume that the Lamé coefficients \(\lambda\) and \(\mu\) are constants satisfying \(\mu > 0\) and \(3\lambda + 2\mu > 0\). Let \(U^j\) be a solution of the Navier equation \((\Delta^e + \omega^2) U^j = 0\) in \(\mathbb{R}^3, \Delta^e := (\mu \Delta + \lambda + \mu) \nabla \text{div}\). We denote by \(U^s\) the elastic field scattered by the \(M\) small bodies \(D_m \subset \mathbb{R}^3\) due to the incident field \(U^i\). We restrict ourselves to the scattering by rigid bodies. Hence the total field \(U := U^i + U^s\) satisfies the following exterior Dirichlet problem of the elastic waves

\[
(\Delta^e + \omega^2) U^s = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \overline{D_m} \right), \\
U^s\big|_{\partial D_m} = 0, \quad 1 \leq m \leq M,
\]

with the Kupradze radiation conditions (K.R.C)

\[
\lim_{|x| \to \infty} |x| \left( \frac{\partial U_p}{\partial |x|} - ik_{p^r} U_p \right) = 0, \quad \text{and} \quad \lim_{|x| \to \infty} |x| \left( \frac{\partial U_s}{\partial |x|} - ik_{s^r} U_s \right) = 0,
\]

where the two limits are uniform in all the directions \(\hat{x} := \frac{x}{|x|} \in \mathbb{S}^2\). Also, we denote \(U_p := -\kappa_{p^r}^2 \nabla \cdot (\nabla \times U^s)\) to be the longitudinal (or the pressure or P) part of the field \(U^s\) and \(U_s := \kappa_{s^r}^2 \nabla \times (\nabla \times U^s)\) to be the transversal (or the shear or S) part of the field \(U^s\) corresponding to the Helmholtz decomposition \(U^s = U_p + U_s\). The constants \(\kappa_{p^r} := \frac{c_p}{c_s}\) and \(\kappa_{s^r} := \frac{c_s}{c_p}\) are known as the longitudinal and transversal wavenumbers, \(c_p := \sqrt{\lambda + 2\mu}\) and \(c_s := \sqrt{\mu}\) are the corresponding phase velocities, respectively and \(\omega\) is the frequency.

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The scattering problem (1.1)–(1.3) is well posed in the Hölder or Sobolev spaces, see [17], [18], [22], [23] for instance, and the scattered field $U^s$ has the following asymptotic expansion:

$$U^s(x) = \frac{e^{i \kappa x \cdot \hat{x}}}{|x|} U^p_\omega(\hat{x}) + \frac{e^{i \kappa x \cdot \hat{x}}}{|x|} U^s_\omega(\hat{x}) + O\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty,$$

(1.4)

uniformly in all directions $\hat{x} \in \mathbb{S}^2$. The longitudinal part of the far-field, i.e. $U^\infty_p(\hat{x})$ is normal to $\mathbb{S}^2$ while the transversal part $U^\infty_s(\hat{x})$ is tangential to $\mathbb{S}^2$. As usual in scattering problems we use plane incident waves in this work. For the Lamé system, the full plane incident wave is of the form $U^i(x, \theta) := \alpha \theta e^{i k \rho \cdot \rho} + \beta \theta^\perp e^{i k d \cdot \rho} x$, where $\theta^\perp$ is any direction in $\mathbb{S}^2$ perpendicular to the incident direction $\theta \in \mathbb{S}^2$, $\alpha, \beta$ are arbitrary constants. In particular, the pressure and shear incident waves are $U^{i,p}(x, \theta) := \theta e^{i k \rho \cdot \rho} x$ and $U^{i,s}(x, \theta) := \theta^\perp e^{i k d \cdot \rho} x$, respectively. Pressure incident waves propagate in the direction of $\theta$, whereas shear incident waves propagate in the direction of $\theta^\perp$. The functions $U^\infty_p(\hat{x}, \theta)$ and $U^\infty_s(\hat{x}, \theta)$ for $(\hat{x}, \theta) \in \mathbb{S}^2 \times \mathbb{S}^2$ are called the P part and the S part of the far-field pattern respectively.

**Definition 1.1** We define

1. $a$ as the maximum among the diameters, $diam$, of the small bodies $D_m$, i.e.

$$a := \max_{1 \leq m \leq M} diam(D_m) \quad \left[= \epsilon \max_{1 \leq m \leq M} diam(B_m)\right],$$

(1.5)

2. $d$ as the minimum distance between the small bodies $\{D_1, D_2, \ldots, D_m\}$, i.e.

$$d := \min_{m \neq j} d_{mj},$$

where $d_{mj} := dist(D_m, D_j)$. We assume that

$$0 < d \leq d_{max},$$

(1.6)

and $d_{max}$ is given.

3. $\omega_{max}$ as the upper bound of the used frequencies, i.e. $\omega \in [0, \omega_{max}]$.

4. $\Omega$ to be a bounded domain in $\mathbb{R}^3$ containing the small bodies $D_m$, $m = 1, \ldots, M$.

The main result of this paper is the following theorem.

**Theorem 1.2** There exist two positive constants $a_0$ and $c_0$ depending only on the size of $\Omega$, the Lipschitz character of $B_m$, $m = 1, \ldots, M$, $d_{max}$ and $\omega_{max}$ such that if

$$a \leq a_0 \quad \text{and} \quad \sqrt{M - \frac{a}{d}} \leq c_0$$

(1.7)

then the P-part, $U^\infty_p(\hat{x}, \theta)$, and the S-part, $U^\infty_s(\hat{x}, \theta)$, of the far-field pattern have the following asymptotic expressions

$$U^\infty_p(\hat{x}, \theta) = \frac{1}{4 \pi c_p^2} (\hat{x} \otimes \hat{x}) \left[\sum_{m=1}^M e^{-i \frac{\kappa}{c_p} z_m} Q_m + O\left(Ma^2 + M(M - 1)\frac{a^3}{d^2} + M(M - 1)^2 \frac{a^4}{d^3}\right)\right],$$

(1.8)

$$U^\infty_s(\hat{x}, \theta) = \frac{1}{4 \pi c_s^2} (I - \hat{x} \otimes \hat{x}) \left[\sum_{m=1}^M e^{-i \frac{\kappa}{c_s} z_m} Q_m + O\left(Ma^2 + M(M - 1)\frac{a^3}{d^2} + M(M - 1)^2 \frac{a^4}{d^3}\right)\right]$$

(1.9)

uniformly in $\hat{x}$ and $\theta$ in $\mathbb{S}^2$. The constant appearing in the estimate $O(\cdot)$ depends only on the size of $\Omega$, the Lipschitz character of the reference bodies $B_m$, $a_0$, $c_0$ and $\omega_{max}$. The vector coefficients $Q_m$, $m = 1, \ldots, M$, are the solutions of the following linear algebraic system

$$C_m^{-1} Q_m = -U^i(z_m, \theta) - \sum_{j=1}^M \Gamma_{j}^{\omega}(z_m, z_j) Q_j,$$

(1.10)
for \( m = 1, \ldots, M \), with \( \Gamma^\omega \) denoting the Kupradze matrix of the fundamental solution to the Navier equation with frequency \( \omega \). \( C_m := \int_{\partial D_m} \sigma_m(s) \, ds \) and \( \sigma_m \) is the solution matrix of the integral equation of the first kind

\[
\int_{\partial D_m} \Gamma^0(s, \omega) \sigma_m(s) \, ds = I, \quad s_m \in \partial D_m,
\]

(1.11)

with \( I \) the identity matrix of order 3. The algebraic system (1.10) is invertible under the condition:

\[
a/d \leq c_1 \tau^{-1}
\]

(1.12)

with

\[
t := \left[ \frac{1}{c_p} - 2 \text{diam} (\Omega) \frac{\omega}{c_0^2} \left( \frac{1}{1 - \left( \frac{1}{2} \kappa_p \text{diam} (\Omega) \right)^N \omega} + \frac{1}{2^{N-1}} \right) \right],
\]

which is assumed to be positive\(^1\) and \( N_2 := [2 \text{diam} (\Omega) \max[\kappa_p, \kappa_{pr} \omega e^2] \), where \([ \cdot ] \) denotes the integral part and \( \ln e = 1 \). The constant \( c_1 \) depends only on the Lipschitz character of the reference bodies \( B_m \), \( m = 1, \ldots, M \).

We call the expressions (1.8)–(1.9) the elastic Foldy-Lax approximation since the dominant terms are reminiscent to the exact form (called also the Foldy or the Foldy-Lax form) of the farfields derived in the scattering by finitely many point-like scatterers, see [20] for instance. These asymptotic expansions are useful for at least two reasons.

First, to estimate approximately the far-field, one needs only to compute the constant vectors \( Q_m \) which are solutions of a linear algebraic system, i.e. (1.10). This reduces considerably the computational effort comparing it to the methods based on integral equations, for instance, especially for a large number of obstacles. If the number of obstacles is actually very large then these asymptotics suggest the kind of an effective medium that can produce the same far-fields and provides the error rate between the fields generated by the obstacles and those generated by the effective medium.

Second, using formulas of the type (1.8) and (1.9), one can solve the inverse problems which consists of localizing the centers, \( z_m \), of the obstacles from the far-field measurements and also estimating their sizes from the computed capacitances \( C_m \), see [2], [15], [16], [19] for instance.

As a first reference on this topic, we mention the book by P. Martin [25] where the multiple scattering issue is well discussed and documented in its different scales. When the obstacles are distributed periodically in the whole domain, then homogenization techniques apply, see for instance [13], [21], [24]. As we see it in the previous theorem, we assume no periodicity. For such media, the type of result presented here are formally derived, for the acoustic and electromagnetic models, in a series of works by A. Ramm, see [35–37] and the references therein for his recent related results, where he used the (rough) condition \( \frac{1}{2} \ll 1 \). Recently, in [16], we derived such approximation errors under a quite general condition on the denseness of the scatterers (i.e. involving \( M \), \( a \) and \( d \)), i.e. of the form (1.7). The analysis is based on the use of integral equation methods and in particular the precise scaling of the surface layer potential operators. As it was mentioned in [16], the integral equation methods are also used in such a context, see for instance the series of works by H. Ammari and H. Kang and their collaborators, as [7] and the references therein. The difference between their asymptotic expansion and the one described in the previous theorem is that their polarization tensors are build up from densities which are solutions of a system of integral equations while in the previous theorem the approximating terms are build up from the linear algebraic system (1.10). It is obvious that solving an algebraic system is less expensive than solving a system of integral equations, especially when dealing with many scatterers. In addition, due to motivations coming from inverse problems, apart from few works as [8], they consider well separated scatterers and hence their asymptotic

\(^1\)If, in particular, \( \text{diam}(\Omega) \max[\kappa_p, \kappa_{pr} \omega e^2] < 1 \), then \( N_1 = 1 \) and hence \( t = \left[ \frac{1}{c_p} - 4 \text{diam}(\Omega) \frac{\omega}{c_0^2} \left( \frac{\omega}{c_0} + \frac{\omega}{c_p} \right) \right] \). Assuming the Lamé coefficient \( \lambda \) to be positive, then \( c_p > c_0 \). Hence, in this case, if \( \Omega \) is such that \( \text{diam}(\Omega) < \frac{\omega}{c_0} \min \left\{ \frac{1}{c_p}, \frac{c_0^2}{\omega c_p} \right\} \) then \( t > 0 \).
expansions are given only in terms of the diameters $a$ of the scatterers. We should, however, emphasize that they provide asymptotic expansions at all the higher orders and they are valid also for extended scatterers. This opens door for many interesting applications, see [6] for instance.

Let us mention the variational approach by V. Maz’ya, A. Movchan [27] and by V. Maz’ya, A. Movchan and M. Nieves [29] where they study the Poisson problem and obtain estimates in forms similar to the previous theorem with weaker conditions of the form $\frac{d}{2} \leq c$, or $\frac{d}{2} \leq c$, (here $d$ is the smallest distance between the centers of the scatterers). In their analysis, they rely on the maximum principle to treat the boundary estimates. To avoid the use of the maximum principle, which is not valid due to the presence of the wave number $\kappa$, we use boundary integral equation methods. The price to pay is the need of the stronger assumption $\sqrt{M} - 1/2 \leq c_0$. Another approach, based on the self-adjoint extensions of elliptic operator, is discussed in the works by S. A. Nazarov, and J. Sokolowski, see Section 4 of [34] for instance, where they derive the asymptotic expansions for the Poisson problem. Let us finally, mention that the particular case where the obstacles have circular shapes has been considered recently by M. Cassier and C. Hazard in [14] for the scalar acoustic model.

Regarding the Lamé system, we cite the works [4], [5], [9], [11] where, as we just mentioned, the asymptotics are given in terms of the size of the scatterers only. In these works, the authors considered transmission problems and showed that the corresponding moment tensors are in general anisotropic. If the inclusions are spherical, including the extreme cases of soft or rigid inclusions under certain conditions on the Lamé parameters, then these moment tensors are isotropic. Let us also mention the recent book [30], and the references therein, where an asymptotic expansion of the Green’s tensor corresponding to the Dirichlet-Lamé system, compare to [7] and [30]. The analysis is based on the use of the integral equation methods and the different scaling of the corresponding boundary integral operators. Due to the coupling of the two fundamental waves, i.e. the P-waves and the S-waves, at the boundaries of the obstacles, the analysis cannot be reduced to the one of our previous work [16]. Indeed, a considerable work is needed to derive explicitely these scaling, characterize the dominant parts of the elastic fields and justify the invertibility of corresponding algebraic system (1.10).

Before concluding the introduction, we state the following corollary where more precise estimates than those given in Theorem 1.2 are presented under some additional conditions on the scatterers.

**Corollary 1.3** Assume that the conditions of Theorem 1.2 are fulfilled.

1. We assume, in addition, that $D_m$ are balls with the same diameter $a$ for $m = 1, \ldots, M$, then we have the following asymptotic expansion for the P-part, $U_p^\infty(\hat{\chi}, \theta)$, and the S-part, $U_s^\infty(\hat{\chi}, \theta)$, of the far-field pattern:

$$U_p^\infty(\hat{\chi}, \theta) = \frac{1}{4\pi c_p^2}(\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{c_p}{2} z_m} Q_m + O \left( M \left[ a^2 + \frac{a^3}{d^3-3\alpha} + \frac{a^4}{d^4-6\alpha} \right] \right) \right. + M(M-1) \left[ \frac{a^3}{d^2\alpha} + \frac{a^4}{d^4-4\alpha} + \frac{a^4}{d^5-2\alpha} \right] + M(M-1)^2 \frac{a^4}{d^5-6\alpha} \right],$$

$$U_s^\infty(\hat{\chi}, \theta) = \frac{1}{4\pi c_s^2}(1 - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{c_s}{2} z_m} Q_m + O \left( M \left[ a^2 + \frac{a^3}{d^3-3\alpha} + \frac{a^4}{d^4-6\alpha} \right] \right) \right. + M(M-1) \left[ \frac{a^3}{d^2\alpha} + \frac{a^4}{d^4-4\alpha} + \frac{a^4}{d^5-2\alpha} \right] + M(M-1)^2 \frac{a^4}{d^5-6\alpha} \right],$$

where $0 < \alpha \leq 1$. 

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Consider now the special case $d = a^t$, $M = a^{-s}$ with $t, s > 0$. Then the asymptotic expansions (1.13)–(1.14) can be rewritten as

\[
U_p^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_p^2}(\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{\pi}{4} \frac{z_m}{a^t}} Q_m + O \left( a^{2-t} + a^{3-s-5t+3\alpha} + a^{4-s-9t+6\alpha} \right) + a^{3-2s-2\alpha} + a^{4-3s-3\alpha} + a^{4-2s-5t+2\alpha} \right].
\]

\[
U_c^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s^2}(1 - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{\pi}{4} \frac{z_m}{a^s}} Q_m + O \left( a^{2-t} + a^{3-s-5t+3\alpha} + a^{4-s-9t+6\alpha} \right) + a^{3-2s-2\alpha} + a^{4-3s-3\alpha} + a^{4-2s-5t+2\alpha} \right].
\]

As the diameter $a$ tends to zero the error term tends to zero for $t$ and $s$ such that $0 < t < 1$ and $0 < s < \min \{2(1-t), \frac{7-5t}{4}, \frac{12-9t}{9}, \frac{20-15t}{12}, \frac{4}{3} - t\alpha\}$. In particular for $t = \frac{1}{2}$, $s = 1$, we have

\[
U_p^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_p^2}(\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{\pi}{4} \frac{z_m}{a^t}} Q_m + O \left( a + a^{2\alpha} + a^{1-\alpha} + a^{1+2\alpha} \right) \right]
\]

\[
= \frac{1}{4\pi c_p^2}(\hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{\pi}{4} \frac{z_m}{a^t}} Q_m + O \left( a^{\frac{1}{2}} \right) \right], \quad \text{[obtained for } \alpha = \frac{1}{4}] \tag{1.15}
\]

\[
U_c^\infty(\hat{x}, \theta) = \frac{1}{4\pi c_s^2}(1 - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{\pi}{4} \frac{z_m}{a^s}} Q_m + O \left( a + a^{2\alpha} + a^{1-\alpha} + a^{1+2\alpha} \right) \right]
\]

\[
= \frac{1}{4\pi c_s^2}(1 - \hat{x} \otimes \hat{x}) \left[ \sum_{m=1}^{M} e^{-i \frac{\pi}{4} \frac{z_m}{a^s}} Q_m + O \left( a^{\frac{1}{2}} \right) \right], \quad \text{[obtained for } \alpha = \frac{1}{4}] \tag{1.16}
\]

2. Actually, the results (1.13)–(1.14) and (1.15)–(1.16) are valid for the non-flat Lipschitz obstacles $D_m = \epsilon B_m + z_m$, $m = 1, \ldots, M$, with the same diameter $a$, i.e., the $D_m$’s are Lipschitz obstacles and there exist constants $t_m \in (0, 1]$ such that

\[
B_{\alpha \frac{a}{2}}(z_m) \subset D_m \subset B_{\frac{a}{2}}(z_m), \quad \text{(1.17)}
\]

where $t_m$ are assumed to be uniformly bounded from below by a positive constant.

The results of this corollary can be used to derive the effective medium by perforation using many small bodies. Let us mention that in the literature, the derivation of the effective medium for the Lamé system can be found, see [10], for instance, and the references there in where the bodies are distributed periodically. Based on Corollary 1.3, this periodicity assumption can be avoided. In addition, (1.15) and (1.16) ensure the rate of the error in deriving such an effective medium. Details on this topic will be reported in a future work. We also refer to our recent work [1] where the effective medium for the acoustic model is derived with error estimates, avoiding the periodicity assumption.

The rest of the paper is organized as follows. In Section 2, we give the proof of the asymptotic expansion (1.8)–(1.9). In Section 3, we study the solvability of the linear algebraic system (1.10). Finally, in Section 4, as an appendix, we derive some needed properties of the layer potentials.

## 2 Proof of Theorem 1.2

We wish to kindly warn the reader that in our analysis we use sometimes the parameter $\epsilon$ and some other times the parameter $a$ as they appear naturally in the estimates. But we bear in mind the relation (1.5) between $a$ and $\epsilon$. 

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2.1 The fundamental solution

The Kupradze matrix \( \Gamma^\omega = (\Gamma^\omega_{ij})_{i,j=1}^3 \) of the fundamental solution to the Navier equation is given by

\[
\Gamma^\omega(x, y) = \frac{1}{\mu} \Phi_{x,\omega}(x, y) I + \frac{1}{\omega^2} \nabla_x \nabla_y [\Phi_{x,\omega}(x, y) - \Phi_{y,\omega}(x, y)],
\]

(2.1)

where \( \Phi_x(x, y) = \frac{1}{2\pi \sqrt{|\omega|^2 - \kappa^2}} \exp(i \kappa |x - y|) \) denotes the free space fundamental solution of the Helmholtz equation \( (\Delta + \kappa^2) u = 0 \) in \( \mathbb{R}^3 \). The asymptotic behavior of Kupradze tensor at infinity is given as follows

\[
\Gamma^\omega(x, y) = \frac{1}{4\pi^2} \hat{\chi} \otimes \hat{\chi} e^{i |\kappa| |x - y|} + \frac{1}{4\pi^2 c_p^2} (I - \hat{\chi} \otimes \hat{\chi}) e^{-i |\kappa| |x - y|} + O \left( |x|^{-2} \right)
\]

(2.2)

with \( \hat{\chi} = \frac{1}{|\chi|} \in \mathbb{S}^2 \) and \( I \) being the identity matrix in \( \mathbb{R}^3 \), see [3] for instance. As mentioned in [9], (2.1) can also be represented as

\[
\Gamma^\omega(x, y) = \frac{1}{4\pi} \sum_{l=0}^\infty \frac{l^l}{l!(l+2)} \left( (l+1) \kappa_{x,\omega}^{l+2} + \kappa_{y,\omega}^{l+2} \right) |x - y|^{-l-1} I
\]

\[
- \frac{1}{4\pi} \sum_{l=0}^\infty \frac{l^l}{l!(l+2)} \left( (l-1) \kappa_{x,\omega}^{l+2} - \kappa_{y,\omega}^{l+2} \right) |x - y|^{-l-3} (x - y) \otimes (x - y),
\]

(2.3)

from which we can get the gradient

\[
\nabla_x \Gamma^\omega(x, y) = - \frac{1}{4\pi} \sum_{l=0}^\infty \frac{l^l}{l!(l+2)} \left[ (l+1) \kappa_{x,\omega}^{l+2} + \kappa_{y,\omega}^{l+2} \right] |x - y|^{-l-3} (x - y) \otimes I
\]

\[
- \left( \kappa_{x,\omega}^{l+2} - \kappa_{y,\omega}^{l+2} \right) |x - y|^{-l-3} (l-3) |x - y|^{-2} \otimes (x - y)
\]

\[
+ I \otimes (x - y) + (x - y) \otimes I \right].
\]

(2.4)

2.2 The representation via double layer potential

We start with the following proposition on the solution of the problem (1.1)–(1.3) via the method of integral equations.

Proposition 2.1 There exists \( a_0 > 0 \) such that if \( a < a_0 \), the solution of the problem (1.1)–(1.3) is of the form

\[
U^i(x) = U^i(x) + \sum_{m=1}^M \int_{\partial D_m} \frac{\partial \Gamma^m(x, s)}{\partial v_m(x)} \sigma_m(s) \, ds, \quad x \in \mathbb{R}^3 \setminus \bigcup_{m=1}^M \bar{D}_m,
\]

(2.5)

where \( \sigma_m \in \mathcal{H}^r(\partial D_m) \), with \( r \in [0, 1] \), for \( m = 1, 2, \ldots, M \), and \( \frac{\partial}{\partial v_m} (\cdot) \) denotes the co-normal derivative on \( \partial D_m \) and is defined as

\[
\frac{\partial}{\partial v_m} (\cdot) := \lambda (\text{div} \cdot) N_m + \mu (\nabla \cdot + \nabla^T) N_m \quad \text{on} \quad \partial D_m,
\]

(2.6)

with \( N_m \) the outward unit normal vector of \( \partial D_m \).

Proof of Proposition 2.1 We look for the solution of the problem (1.1)–(1.3) of the form (2.5), then from the Dirichlet boundary condition (1.2) and the jumps of the double layer potentials, we obtain

\[
\frac{\sigma_j(s_j)}{2} + \int_{\partial D_j} \frac{\partial \Gamma^m(s_j, s)}{\partial v_j(s)} \sigma_j(s) \, ds + \sum_{m=1}^M \int_{\partial D_m} \frac{\partial \Gamma^m(s_j, s)}{\partial v_m(s)} \sigma_m(s) \, ds = -U^i(s_j),
\]

\[
\forall s_j \in \partial D_j, \quad j = 1, \ldots, M.
\]

(2.7)

The condition on \( a \) can be replaced by a condition on \( \omega \) as it can be seen from the proof.
One can write the system (2.7) in a compact form as \( \left( \frac{1}{2}I + DL + DK \right)\sigma = -U^{ln} \) with \( I := (I_{mj})_{m,j=1}^{M} \), \( DL := (DL_{mj})_{m,j=1}^{M} \) and \( DK := (DK_{mj})_{m,j=1}^{M} \), where

\[
I_{mj} = \begin{cases} 
I, & m = j, \\
0, & \text{else}, 
\end{cases} 
\]

\[
DL_{mj} = \begin{cases} 
D_{mj}, & m = j, \\
0, & \text{else}, 
\end{cases} 
\]

\[
DK_{mj} = \begin{cases} 
D_{mj}, & m \neq j, \\
0, & \text{else}, 
\end{cases} 
\]

(2.8)

\[
U^{ln} = U^{ln}(s_1, \ldots, s_M) := \left( U^i(s_1), \ldots, U^i(s_M) \right)^T
\]

and \( \sigma = (\sigma_1(s_1), \ldots, \sigma_M(s_M))^T \).

(2.9)

Here, for the indices \( m \) and \( j \) fixed, \( D_{mj} \) is the integral operator

\[
D_{mj}(\sigma_j(t)) := \int_{\partial D_j} \frac{\partial \Gamma^m(t,s)}{\partial \nu_j(s)} \sigma_j(s) \, ds.
\]

(2.11)

The operator \( \frac{1}{2}I + D_{mm} : H^r(\partial D_m) \rightarrow H^r(\partial D_m) \) is Fredholm with zero index and for \( m \neq j, D_{mj} : H^r(\partial D_j) \rightarrow H^r(\partial D_m) \) is compact for \( 0 \leq r \leq 1 \), when \( \partial D_m \) has a Lipschitz regularity, see [26], [31], [32]. So, \( \left( \frac{1}{2}I + DL + DK \right) := \prod_{m=1}^{M} \left( \frac{1}{2}I + DL + DK \right) : \prod_{m=1}^{M} H^r(\partial D_m) \rightarrow \prod_{m=1}^{M} H^r(\partial D_m) \) is Fredholm with zero index. We induce the product of spaces by the maximum of the norms of the space. To show that \( \left( \frac{1}{2}I + DL + DK \right) \sigma = 0 \) implies \( \sigma = 0 \).

Write,

\[
\tilde{U}(x) = \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma^m(s_j, s)}{\partial v_m(s)} \sigma_m(s) \, ds, \quad \text{in} \quad \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \bar{D}_m \right)
\]

and

\[
\tilde{U}(x) = \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma^m(s_j, s)}{\partial v_m(s)} \sigma_m(s) \, ds, \quad \text{in} \quad \bigcup_{m=1}^{M} D_m.
\]

Then \( \tilde{U} \) satisfies \( \Delta^s \tilde{U} + \omega^2 \tilde{U} = 0 \) for \( x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \bar{D}_m \right) \), with K.R.C. \( \tilde{U}(x) = 0 \) on \( \bigcup_{m=1}^{M} \partial D_m \). Similarly, \( \tilde{U} \) satisfies \( \Delta^s \tilde{U} + \omega^2 \tilde{U} = 0 \) for \( x \in \bigcup_{m=1}^{M} D_m \) with \( \tilde{U}(x) = 0 \) on \( \bigcup_{m=1}^{M} \partial D_m \). Taking the trace on \( \partial D_m, m = 1, \ldots, M \),

\[
\tilde{U}(s) = 0 \implies D_{mm}(\sigma_m(s) + \frac{\sigma_m(s)}{2}) + \sum_{j \neq m} D_{mj}(\sigma_j(s)) = 0
\]

(2.12)

and

\[
\tilde{U}(s) = 0 \implies D_{mm}(\sigma_m(s) - \frac{\sigma_m(s)}{2}) + \sum_{j \neq m} D_{mj}(\sigma_j(s)) = 0
\]

(2.13)

for \( s \in \partial D_m \) and for \( m = 1, \ldots, M \). Difference between (2.12) and (2.13) implies that, \( \sigma_m = 0 \) for all \( m \).

---

3In [26], [31], [32], this property is proved for the case \( \omega = 0 \). By a perturbation argument, we have the same results for every \( \omega \) in \( [0, \omega_{max}] \), assuming that \( \omega_{max} \) is smaller than the first eigenvalue \( w_{el} \) of the Dirichlet-Lamé operator in \( D_m \). By a comparison theorem, see [30], (6.131) in Lemma 6.3.6 for instance, we know that \( \mu w_{el} \leq w_{el} \) where \( w_{el} \) is the first eigenvalue of the Dirichlet-Laplacian operator in \( D_m \). Now, we know that \( (\frac{1}{2}\sqrt{\pi} J_{1/2,1})^2 \leq w_{el} \). Then, we need \( \omega_{max} < \frac{\pi}{\omega_{max}} \sqrt{\pi} J_{1/2,1} := a_0 \). Here \( J_{1/2,1} \) is the 1st positive zero of the Bessel function \( J_{1/2} \).
We conclude then that \( \frac{1}{2} I + DL + DK \) is invertible. □

2.3 An appropriate estimate of the densities \( \sigma_m, m = 1, \ldots, M \)

From the above theorem, we have the following representation of \( \sigma \):

\[
\sigma = \left( \frac{1}{2} I + DL + DK \right)^{-1} U^{ln}
\]

\[
= \left( \frac{1}{2} I + DL \right)^{-1} \left( I + \left( \frac{1}{2} I + DL \right)^{-1} DK \right)^{-1} U^{ln}
\]

\[
= \left( \frac{1}{2} I + DL \right)^{-1} \sum_{j=0}^{1} \left( - \left( \frac{1}{2} I + DL \right)^{-1} DK \right)^j U^{ln}, \quad \text{if} \quad \left\| \left( \frac{1}{2} I + DL \right)^{-1} DK \right\| < 1. \quad (2.14)
\]

The operator \( \frac{1}{2} I + DL \) is invertible since it is Fredholm of index zero and injective. This implies that

\[
\|\sigma\| \leq \frac{\|\left( \frac{1}{2} I + DL \right)^{-1}\|}{1 - \|\left( \frac{1}{2} I + DL \right)^{-1}\| \|DK\|} \|U^{ln}\|. \quad (2.15)
\]

Here,

\[
\|DK\| := \|DK\|_{L^\infty \left( \prod_{m=1}^{M} L^2(\partial D_m), \prod_{m=1}^{M} L^2(\partial D_m) \right)} \]

\[
= \max_{m=1}^{M} \sum_{j=1}^{M} \|DK_{mj}\|_{L^2(\partial D_j),L^2(\partial D_m))}
\]

\[
= \max_{m=1}^{M} \sum_{j=1}^{M} \|D_{mj}\|_{L^2(\partial D_j),L^2(\partial D_m))}, \quad (2.16)
\]

\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| := \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\|_{L^\infty \left( \prod_{m=1}^{M} L^2(\partial D_m), \prod_{m=1}^{M} L^2(\partial D_m) \right)} \]

\[
= \max_{m=1}^{M} \sum_{j=1}^{M} \left\| \left( \frac{1}{2} I + DL \right)^{-1}_{mj} \right\|_{L^2(\partial D_m),L^2(\partial D_j))}
\]

\[
= \max_{m=1}^{M} \left\| \left( \frac{1}{2} I + D_{mm} \right)^{-1} \right\|_{L^2(\partial D_m),L^2(\partial D_m))}, \quad (2.17)
\]

\[
\|\sigma\| := \|\sigma\|_{\prod_{m=1}^{M} L^2(\partial D_m)} = \max_{1 \leq m \leq M} \|\sigma_m\|_{L^2(\partial D_m)}, \quad (2.18)
\]

and

\[
\|U^{ln}\| := \|U^{ln}\|_{\prod_{m=1}^{M} L^2(\partial D_m)} = \max_{1 \leq m \leq M} \|U^{j}\|_{L^2(\partial D_m)} \cdot (2.19)
\]

In the following proposition, we provide conditions under which \( \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\| < 1 \) and then estimate \( \|\sigma\| \) via (2.15).
Proposition 2.2 There exists a constant \( \hat{c} \) depending only on the size of \( \Omega \), the Lipschitz character of \( B_m, m = 1, \ldots, M \), \( d_{\max} \), and \( \omega_{\max} \) such that if
\[
\sqrt{M - 1} \epsilon < \hat{c} \delta, \quad \text{then} \quad \| \sigma_m \|_{L^2(\partial D_m)} \leq c \epsilon
\]
where \( c \) is a positive constant depending only on the Lipschitz character of \( B_m \).

Proof of Proposition 2.2. For any functions \( f, g \) defined on \( \partial D_\epsilon \) and \( \partial B \) respectively, we define
\[
(f\gamma)(\xi) := \hat{f}(\xi) := f(\epsilon \xi + z) \quad \text{and} \quad (g\gamma)(x) := \hat{g}(x) := g\left(\frac{x - z}{\epsilon}\right). \tag{2.20}
\]
Let \( T_1 \) and \( T_2 \) be an orthonormal basis for the tangent plane to \( \partial D_\epsilon \) at \( x \) and let \( \partial / \partial T = \sum_{i=1}^2 \partial / \partial T_i T_i \) denote the tangential derivative on \( \partial D_\epsilon \). Then the space \( H^1(\partial D_\epsilon) \) is defined as
\[
H^1(\partial D_\epsilon) := \{ \phi \in L^2(\partial D_\epsilon); \partial \phi / \partial T \in L^2(\partial D_\epsilon) \}. \tag{2.21}
\]
We have the following lemma from [16].

Lemma 2.3 Suppose \( 0 < \epsilon \leq 1 \) and \( D_\epsilon := \epsilon B + z \subset \mathbb{R}^n \). Then for every \( \psi \in L^2(\partial D_\epsilon) \) and \( \phi \in H^1(\partial D_\epsilon) \), we have
\[
\| \psi \|_{L^2(\partial D_\epsilon)} = \epsilon^{\frac{1}{2}} \| \hat{\psi} \|_{L^2(\partial B)} \tag{2.22}
\]
and
\[
\epsilon^{\frac{1}{2}} \| \hat{\phi} \|_{H^1(\partial B)} \leq \| \phi \|_{H^1(\partial D_\epsilon)} \leq \epsilon^{\frac{1}{2}} \| \hat{\phi} \|_{H^1(\partial B)}. \tag{2.23}
\]
We divide the rest of the proof of Proposition 2.2 into two steps. In the first step, we assume we have a single obstacle and then in the second step we deal with the multiple obstacles case.

2.3.1 The case of a single obstacle
Let us consider a single obstacle \( D_\epsilon := \epsilon B + z \). Then define the operator \( \mathcal{D}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon) \) by
\[
(\mathcal{D}_{D_\epsilon} \psi)(s) = \int_{\partial D_\epsilon} \frac{\partial \Gamma^{(s,t)}(x,t)}{\partial \nu(t)} \psi(t) \, dt. \tag{2.24}
\]
Following the arguments in the proof of Proposition 2.1, the integral operator \( \frac{1}{2} I + \mathcal{D}_{D_\epsilon} : L^2(\partial D_\epsilon) \to L^2(\partial D_\epsilon) \) is invertible. If we consider the problem (1.1)–(1.3) in \( \mathbb{R}^3 \setminus \bar{D}_\epsilon \), we obtain
\[
\sigma = \left( \frac{1}{2} I + \mathcal{D}_{D_\epsilon} \right)^{-1} U^t, \quad \text{where} \quad DL + DK =: \mathcal{D}_{D_\epsilon}
\]
and then
\[
\| \sigma \|_{L^2(\partial D_\epsilon)} \leq \left\| \left( \frac{1}{2} I + \mathcal{D}_{D_\epsilon} \right)^{-1} U^t \right\|_{L^2(\partial D_\epsilon)}. \tag{2.25}
\]

Lemma 2.4 Let \( \phi, \psi \in L^2(\partial D_\epsilon) \). Then,
\[
\mathcal{D}_{D_\epsilon} \psi = (\mathcal{D}_{D_\epsilon}^t \hat{\psi})^\gamma, \tag{2.26}
\]
\[
\left( \frac{1}{2} I + \mathcal{D}_{D_\epsilon} \right) \psi = \left( \left( \frac{1}{2} I + \mathcal{D}_{D_\epsilon}^t \right) \hat{\psi} \right)^\gamma, \tag{2.27}
\]
\[
\left( \frac{1}{2} I + \mathcal{D}_{D_\epsilon} \right)^{-1} \phi = \left( \left( \frac{1}{2} I + \mathcal{D}_{D_\epsilon}^t \right)^{-1} \hat{\phi} \right)^\gamma. \tag{2.28}
\]
The following equalities
\begin{align}
\left\| \left( \frac{1}{2} I + D_{D_c} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_c), L^2(\partial D_c))} & = \left\| \left( \frac{1}{2} I + D_B^\varepsilon \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} \\
(2.29)
\end{align}
and
\begin{align}
\left\| \left( \frac{1}{2} I + D_{D_c} \right)^{-1} \right\|_{\mathcal{L}(H^1(\partial D_c), H^1(\partial D_c))} & \leq \epsilon^{-1} \left\| \left( \frac{1}{2} I + D_B^\varepsilon \right)^{-1} \right\|_{\mathcal{L}(H^1(\partial B), H^1(\partial B))} \\
(2.30)
\end{align}
with \( D_B^\varepsilon \psi(\xi) := \int_{\partial B} \frac{\partial \Gamma^{\varepsilon}(\xi, \eta)}{\partial \nu(t)} \psi(\eta) \, d\eta \).

**Proof of Lemma 2.4**

- We have,

\[
D_{D_c} \psi(s) = \int_{\partial D_c} \frac{\partial \Gamma^{\varepsilon}(s, t)}{\partial \nu(t)} \psi(t) \, dt = \int_{\partial D_c} \left[ \lambda (\div, \Gamma^{\varepsilon}(s, t)) N_t + \mu \left( \nabla, \Gamma^{\varepsilon}(s, t) + (\nabla, \Gamma^{\varepsilon}(s, t))^\top \right) N_t \right] \psi(t) \, dt
\]

- The above gives us (2.26). From (2.26), we can obtain (2.27).

  - The following equalities

\[
\left( \frac{1}{2} I + D_{D_c} \right) \left( \left( \frac{1}{2} I + D_B^\varepsilon \right)^{-1} \phi \right) \overset{\text{(2.27)}}{=} \left( \left( \frac{1}{2} I + D_B^\varepsilon \right)^{-1} \phi \right) \overset{\text{(2.27)}}{=} \phi^\wedge = \phi
\]

  provide us (2.28).

  - The following equalities

\[
\left\| \left( \frac{1}{2} I + D_{D_c} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_c), L^2(\partial D_c))} := \sup_{\phi \neq 0 \in L^2(\partial D_c)} \frac{\left\| \left( \frac{1}{2} I + D_{D_c} \right)^{-1} \phi \right\|_{L^2(\partial D_c)}}{||\phi||_{L^2(\partial D_c)}}
\]

\[
\overset{(2.22),(2.23)}{=} \sup_{\phi \neq 0 \in L^2(\partial D_c)} \frac{\epsilon \left\| \left( \frac{1}{2} I + D_{D_c} \right)^{-1} \phi \right\|_{L^2(\partial D_c)}}{\epsilon ||\phi||_{L^2(\partial B)}}
\]

\[
\overset{(2.28)}{=} \sup_{\phi \neq 0 \in L^2(\partial D_c)} \frac{\left\| \left( \frac{1}{2} I + D_B^\varepsilon \right)^{-1} \phi \right\|_{L^2(\partial B)}}{||\phi||_{L^2(\partial B)}}
\]

\[
= \left\| \left( \frac{1}{2} I + D_B^\varepsilon \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))}
\]

provide us (2.29). By proceeding in the similar manner we can obtain (2.30) as mentioned below,
\[
\left\| \left( \frac{1}{2} I + D_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1), L^2(\partial D_1))} := \left\| \frac{1}{2} I + D_{D_1} \right\|^{-1}_{\mathcal{H}^1(\partial D_1)} \leq \sup_{\phi \neq 0} \frac{\left\| \left( \frac{1}{2} I + D_{D_1} \right)^{-1} \phi \right\|_{\mathcal{H}^1(\partial D_1)}}{\|\phi\|_{\mathcal{H}^1(\partial D_1)}} \leq e^{-1} \sup_{\phi \neq 0} \frac{\left\| \left( \frac{1}{2} I + D_{D_1} \right)^{-1} \phi \right\|_{\mathcal{H}^1(\partial B)}}{\|\phi\|_{\mathcal{H}^1(\partial B)}} \leq e^{-1} \left\| \left( \frac{1}{2} I + D'_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1), L^2(\partial D_1))}.
\]

The next lemma provides us with an estimate of the left hand side of (2.29) by a constant \( C \) with a useful dependence of \( C \) in terms of \( B \) through its Lipschitz character and \( \omega \).

**Lemma 2.5** The operator norm of \( \left( \frac{1}{2} I + D_{D_1} \right)^{-1} : L^2(\partial D_1) \to L^2(\partial D_1) \) satisfies the estimate

\[
\left\| \left( \frac{1}{2} I + D_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial D_1), L^2(\partial D_1))} \leq \tilde{C}_0,
\]

with

\[
\tilde{C}_0 := \frac{4\pi}{4\pi - \frac{4\pi + 17\pi}{2\pi} + \frac{12\pi + 9\mu}{2\pi}} \omega^2 |\partial B| \left\| \left( \frac{1}{2} I + D'_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))}.
\]

Here, \( D'_{D_1} : L^2(\partial B) \to L^2(\partial B) \) is the double layer potential with the zero frequency.

Here we should mention that if \( \epsilon^2 \leq \frac{\pi}{\frac{4\pi + 17\pi}{2\pi} + \frac{12\pi + 9\mu}{2\pi}} \omega^2 |\partial B| \left\| \left( \frac{1}{2} I + D'_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))} \), then \( \tilde{C}_0 \) is bounded by

\[
\frac{4}{\pi} \left\| \left( \frac{1}{2} I + D'_{D_1} \right)^{-1} \right\|_{\mathcal{L}(L^2(\partial B), L^2(\partial B))},
\]

which is a universal constant depending only on \( \partial B \) through its Lipschitz character.

**Proof of Lemma 2.5** To estimate the operator norm of \( \left( \frac{1}{2} I + D_{D_1} \right)^{-1} \) we decompose \( D_{D_1} = D'_{D_1} = D'_{D_1} + D''_{D_1} \) into two parts \( D'_{D_1} \) (independent of \( \omega \)) and \( D''_{D_1} \) (dependent of \( \omega \)) given by

\[
D'_{D_1} \psi(s) := \int_{\partial D_1} \left( \frac{\partial}{\partial v(t)} \Gamma^0(s, t) \right) \psi(t) \, dt,
\]

\[
D''_{D_1} \psi(s) := \int_{\partial D_1} \left( \frac{\partial}{\partial v(t)} [\Gamma^0(s, t) - \Gamma^0(s, t)] \right) \psi(t) \, dt.
\]

With this definition, \( \frac{1}{2} I + D'_{D_1} : L^2(\partial D_1) \to L^2(\partial D_1) \) is invertible, see [26], [31], [32]. Hence, \( \frac{1}{2} I + D_{D_1} = \left( \frac{1}{2} I + D'_{D_1} \right) \left( I + \left( \frac{1}{2} I + D''_{D_1} \right)^{-1} D''_{D_1} \right) \) and so
\[
\left\| \left( \frac{1}{2} I + D_{D_{t}} \right)^{-1} \right\|_{L(L^2(\partial D_{t}),L^2(\partial D_{t}))} = \left\| \left( I + \left( \frac{1}{2} I + D_{D_{t}} \right)^{-1} D_{D_{t}}^{\mu} \right)^{-1} \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} \right\|_{L(L^2(\partial D_{t}),L^2(\partial D_{t}))} \leq \left\| \left( I + \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} D_{D_{t}}^{\mu} \right)^{-1} \right\|_{L(L^2(\partial D_{t}),L^2(\partial D_{t}))} \left\| \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} \right\|_{L(L^2(\partial D_{t}),L^2(\partial D_{t}))} .
\]

(2.34)

So, to estimate the operator norm of \( \left( \frac{1}{2} I + D_{D_{t}} \right)^{-1} \) one needs to estimate the operator norm of \( \left( I + \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} D_{D_{t}}^{\mu} \right)^{-1} \). In particular one needs to have the knowledge about the operator norms of \( \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} \) and \( D_{D_{t}}^{\mu} \) to apply the Neumann series. For that purpose, we can estimate the operator norm of \( \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} \) from (2.29) by

\[
\left\| \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} \right\|_{L(L^2(\partial D_{t}),L^2(\partial D_{t}))} = \left\| \left( \frac{1}{2} I + D_{D_{t}}^{\mu} \right)^{-1} \right\|_{L(L^2(\partial B),L^2(\partial B))} .
\]

(2.35)

Here \( D_{D_{t}}^{\mu} \tilde{\psi}(\xi) := \int_{\partial B} \left( \frac{3}{\overline{\omega}(\eta)} \Gamma^{0}(\xi, \eta) \right) \tilde{\psi}(\eta) \, d\eta \). From the definition of the operator \( D_{D_{t}}^{\mu} \) in (2.33), we deduce that

\[
D_{D_{t}}^{\mu} \psi(s) = \int_{\partial B} \left( \frac{\partial}{\partial \nu(\eta)} \left[ \Gamma^{\epsilon\omega}(\xi, \eta) - \Gamma^{0}(\xi, \eta) \right] \right) \tilde{\psi}(\eta) \, d\eta
\]

\[
= \int_{\partial B} \left[ \lambda \left( \text{div}_{\eta} \left[ \Gamma^{\epsilon\omega}(\xi, \eta) - \Gamma^{0}(\xi, \eta) \right] \right) N_{\eta} + \mu \left( \nabla_{\eta} \left[ \Gamma^{\epsilon\omega}(\xi, \eta) - \Gamma^{0}(\xi, \eta) \right] \right) + \left( \nabla_{\eta} \left[ \Gamma^{\epsilon\omega}(\xi, \eta) - \Gamma^{0}(\xi, \eta) \right] \right)_{T} N_{\eta} \right] \tilde{\psi}(\eta) \, d\eta
\]

\[
= \int_{\partial B} \left[ \lambda I_{1} \otimes N_{\eta} + \mu \left( I_{2} + I_{2} \right) N_{\eta} \right] \tilde{\psi}(\eta) \, d\eta ,
\]

(2.36)

where the vector \( I_{1} \) and the third order tensor \( I_{2} \) are estimated by using (2.3) and (2.4) as

\[
I_{1} = - \frac{\epsilon^{2}}{4\pi} \sum_{l=2}^{\infty} \frac{e^{l-2}i^{l}}{l(l+2)} \frac{(l-1)}{\omega^{2}} \left[ -2\kappa_{\mu}^{l+2} + (l+4)\kappa_{\mu}^{l+2} \right] |\xi - \eta|^{l-3}(\xi - \eta) ,
\]

(2.37)

\[
I_{2} = - \frac{\epsilon^{2}}{4\pi} \sum_{l=2}^{\infty} \frac{e^{l-2}i^{l}}{l(l+2)} \frac{(l-1)}{\omega^{2}} \left[ ((l+1)\kappa_{\mu}^{l+2} + \kappa_{\mu}^{l+2}) |\xi - \eta|^{l-3}(\xi - \eta) \otimes I \right. \\
- \left( \kappa_{\mu}^{l+2} - \kappa_{\mu}^{l+2} \right) |\xi - \eta|^{l-3} (l-3)|\xi - \eta|^{2} \otimes (\xi - \eta) + I \otimes (\xi - \eta) + (\xi - \eta) \otimes I \right] .
\]

(2.38)
Using the observation "\(|x|^p \|_{L^2(D)} \leq \|x^p\|_{L^2(D)} \|D\|^{-\frac{p}{2}}\)" we obtain

\[
\left| D^{\nu}_{D_\epsilon} \psi(s) \right| \\
\leq \lambda \frac{\epsilon^2}{4\pi} \sum_{l=0}^{\infty} \frac{e^{l-2}}{l!(l+2)} \frac{(l-1) \omega^2}{(l+4)\kappa^{l+2}_{\nu,\rho}} \int_{D_B} |\xi - \eta|^{l-2} |\hat{\psi}(\eta)| \, d\eta \\
+ 2\mu \frac{\epsilon^2}{4\pi} \left[ \sum_{l=0}^{\infty} \frac{e^{l-2}}{l!(l+2)} \frac{(l+4)\kappa^{l+2}_{\nu,\rho}}{(l+2)\kappa^{l+2}_{\nu,\rho}} \int_{D_B} |\xi - \eta|^{l-2} |\hat{\psi}(\eta)| \, d\eta \right] \\
+ \frac{(6\kappa^2_{\nu,\rho} + 4\kappa^4_{\nu,\rho})}{8\omega^2} \int_{D_B} |\hat{\psi}(\eta)| \, d\eta \\
\leq \lambda \frac{\epsilon^2}{4\pi} \|\hat{\psi}\|_{L^2(D_B)} |\partial B| \left( \sum_{l=0}^{\infty} \frac{e^{l-2}}{l!(l+2)} \frac{(l-1) \omega^2}{(l+4)\kappa^{l+2}_{\nu,\rho}} \|\xi - \eta\|^{l-2} \|\partial B\|^{\frac{2}{l}} \right) \\
+ 2\mu \frac{\epsilon^2}{4\pi} \|\hat{\psi}\|_{L^2(D_B)} |\partial B| \left( \sum_{l=0}^{\infty} \frac{e^{l-2}}{l!(l+2)} \frac{1}{\omega^2} \frac{(l+4)\kappa^{l+2}_{\nu,\rho}}{(l+2)\kappa^{l+2}_{\nu,\rho}} \|\xi - \eta\|^{l-2} \|\partial B\|^{\frac{2}{l}} \right) \\
+ \frac{(6\kappa^2_{\nu,\rho} + 4\kappa^4_{\nu,\rho})}{4\omega^2} \int_{D_B} |\hat{\psi}(\eta)| \, d\eta \\
\leq \omega^2 \frac{\epsilon^2}{4\pi} \|\hat{\psi}\|_{L^2(D_B)} |\partial B| \left( \frac{\mu}{c^2} \left[ \frac{\mu}{2} + (\lambda + 4\mu) \sum_{l=0}^{\infty} \left( \frac{1}{2} \epsilon \kappa_{\nu,\rho} \|\xi - \eta\| \|\partial B\|^{\frac{2}{l}} \right) \right] \right) \\
+ \frac{1}{c^2} \left[ \frac{\mu}{2} + (3\lambda + 2\mu) \sum_{l=0}^{\infty} \left( \frac{1}{2} \epsilon \kappa_{\nu,\rho} \|\xi - \eta\| \|\partial B\|^{\frac{2}{l}} \right) \right] \right) \right) \\
= \omega^2 \frac{\epsilon^2}{4\pi} \|\hat{\psi}\|_{L^2(D_B)} |\partial B| \left( \frac{\mu}{c^2} \left[ \frac{\mu}{2} + \frac{\lambda + 4\mu}{1 - \frac{1}{2} \epsilon \kappa_{\nu,\rho} \|\xi - \eta\| \|\partial B\|^{\frac{2}{l}} \right] \right) \\
+ \frac{1}{c^2} \left[ \frac{\mu}{2} + \frac{3\lambda + 2\mu}{1 - \frac{1}{2} \epsilon \kappa_{\nu,\rho} \|\xi - \eta\| \|\partial B\|^{\frac{2}{l}} \right] \right) \right) \right) \right), \text{ for } \epsilon < \frac{2 \min \{c_x, c_p\}}{\omega \max_m \text{diam}(B_m)} \\
\leq \hat{C}_1 \omega^2 \epsilon^2 \|\hat{\psi}\|_{L^2(D_B)}, \text{ for } \epsilon \leq \frac{\min \{c_x, c_p\}}{\omega \max_m \text{diam}(B_m)}, \tag{2.39}
\]

with \(\hat{C}_1 := \frac{1}{4\pi} \left[ \frac{4\lambda + 17\mu}{4\epsilon^2} + \frac{12\lambda + 9\mu}{2\epsilon^2} \right] \). From this we obtain,

\[
\left\| D^{\nu}_{D_\epsilon} \psi \right\|_{L^2(D_B)}^2 = \int_{D_B} \left| D^{\nu}_{D_\epsilon} \psi(s) \right|^2 \, ds \\
\leq (2.39) \int_{D_B} \left[ \hat{C}_1 \omega^2 \epsilon^2 \|\hat{\psi}\|_{L^2(D_B)} \right]^2 \, ds \\
= \hat{C}_1^2 \omega^4 \epsilon^6 |\partial B|^2 \|\hat{\psi}\|_{L^2(D_B)}^2.
\]

Hence

\[
\left\| D^{\nu}_{D_\epsilon} \psi \right\|_{L^2(D_B)} \leq \hat{C}_1 \omega^2 \epsilon^3 |\partial B|^\frac{3}{2} \|\hat{\psi}\|_{L^2(D_B)}. \tag{2.40}
\]
We estimate the norm of the operator $\mathcal{D}_{D_k}^{a_k}$ as
\[
\left\| \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right\|_{\mathcal{L}(L^{2}(\partial D_k), L^{2}(\partial D_k))}^{-1} \leq \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)^{-1} \mathcal{D}_{D_k}^{a_k} \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)_{\mathcal{L}(L^{2}(\partial D_k), L^{2}(\partial D_k))}^{-1} \leq C_2 \omega^2 e^2 |\partial B|^\frac{1}{2},
\]
where $C_2 := C_1 |\partial B|^\frac{1}{2} \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)^{-1} \mathcal{D}_{D_k}^{a_k} \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)_{\mathcal{L}(L^{2}(\partial D_k), L^{2}(\partial D_k))}^{-1}$. Assuming $\epsilon$ to satisfy the condition $\epsilon < \frac{1}{\sqrt{C_2 \omega^2 e^2}}$, then
\[
\left\| \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right\|_{\mathcal{L}(L^{2}(\partial D_k), L^{2}(\partial D_k))} < 1
\]
and hence by using the Neumann series we obtain the following bound
\[
\left\| \left( I + \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)^{-1} \mathcal{D}_{D_k}^{a_k} \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)_{\mathcal{L}(L^{2}(\partial D_k), L^{2}(\partial D_k))}^{-1} \right\| \leq \frac{1}{1 - \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)^{-1} \mathcal{D}_{D_k}^{a_k} \left( \frac{1}{2} I + \mathcal{D}_{D_k}^{a_k} \right)_{\mathcal{L}(L^{2}(\partial D_k), L^{2}(\partial D_k))}} \leq C_3 := \frac{1}{1 - C_2 \omega^2 e^2}.\]

By substituting the above and (2.35) in (2.34), we obtain the required result (2.31). \hfill $\square$

### 2.3.2 The multiple obstacles case

**Lemma 2.6** For each $k > 0$ and for every $n \in \mathbb{Z}^+$ with $n \geq ke^2 \ [:= N(k)]$ we have $n! \geq k^{n-1}$.

**Proof of Lemma 2.6** The result is true for $n = 1$. The proof goes as follows for $n > 1$:\footnote{Since, $\frac{n+1}{n-1} > 1$, $\frac{\ln \sqrt{2\pi} - n}{n-1} > 0$ and $0 < \frac{n+1}{n-1} < 2$ for $n > 1$.}

\[
\frac{n}{\sqrt{2\pi} n} \geq \frac{1}{2} \Rightarrow \ln k \leq \ln \sqrt{2\pi} - n + \frac{1}{2} \ln n \Rightarrow \ln k \leq \ln \sqrt{2\pi} + \left( n + \frac{1}{2} \right) \ln n \Rightarrow (n-1) \ln k \leq \sqrt{2\pi} + \left( n + \frac{1}{2} \right) \ln n \Rightarrow k^{n-1} \leq \sqrt{2\pi} \frac{\left( \frac{n}{e} \right)^n}{n-1}.\]
Now, we obtain the result using Stirlings approximation $n! \sim \sqrt{2\pi n \left(\frac{n}{e}\right)^n}$, precisely $\sqrt{2\pi n \left(\frac{n}{e}\right)^n} \leq n!$, see [33] for instance.

**Proposition 2.7**  For $m, j = 1, 2, \ldots, M$, the operator $D_{mj} : L^2(\partial D_j) \to L^2(\partial D_m)$ defined in Proposition 2.1, see (2.11), enjoys the following estimates:

- For $j = m$,
  \[
  \left\| \left( \frac{1}{2} I + D_{mm} \right)^{-1} \right\|_{L^2(\partial D_m), L^2(\partial D_m)} \leq \hat{C}_{om},
  \]  
  where
  \[
  \hat{C}_{om} := \frac{4\pi \left\| \left( \frac{1}{2} I + D_{mm}^\omega \right)^{-1} \right\|_{L^2(\partial B_m), L^2(\partial B_m)}}{4\pi - \frac{4\lambda + 17\mu}{2c_p^2} + \frac{12\lambda + 2\mu}{2c_p^2} \omega^2 \epsilon^2 |\partial B_m| \left\| \left( \frac{1}{2} I + D_{mm}^\omega \right)^{-1} \right\|_{L^2(\partial B_m), L^2(\partial B_m)}}.
  \]

- For $j \neq m$,
  \[
  \|D_{mj}\|_{L^2(\partial D_j), L^2(\partial D_m)} \leq \left[ \frac{\hat{C}_7}{d^2} + \hat{C}_8 \right] \frac{1}{4\pi |\partial B|} \epsilon^2,
  \]  
  where
  \[
  |\partial B| := \max_m |\partial B_m|,
  \]
  \[
  \hat{C}_7 := \left( \frac{\lambda + 6\mu}{c_s^2} + \frac{2\lambda + 6\mu}{c_p^2} \right) \text{ and }
  \]
  \[
  \hat{C}_8 := \frac{\omega^2}{c_s^2} \left( \frac{\mu}{2} + (\lambda + 4\mu) \frac{1 - \left(\frac{1}{2} \kappa_s \text{diam}(\Omega)\right)^{N_\Omega}}{1 - \frac{1}{2} \kappa_s \text{diam}(\Omega)} + (\lambda + 4\mu) \frac{1}{2^{N_\Omega - 1}} \right) + \frac{\omega^2}{c_p^2} \left( \frac{3\lambda + 4\mu}{2} \frac{1 - \left(\frac{1}{2} \kappa_p \text{diam}(\Omega)\right)^{N_\Omega}}{1 - \frac{1}{2} \kappa_p \text{diam}(\Omega)} + \frac{3\lambda + 4\mu}{2} \frac{1}{2^{N_\Omega - 1}} \right)
  \]
  with $N_\Omega = \left[2 \text{diam}(\Omega) \max [\kappa_s, \kappa_p] \epsilon^2 \right]$, where $[\cdot]$ denotes the integral part.

**Proof of Proposition 2.7**  The estimate (2.43) is nothing but (2.31) of Lemma 2.5, replacing $B$ by $B_m$, $z$ by $z_m$ and $D_j$ by $D_m$ respectively. It remains to prove the estimate (2.44). We have

\[
\|D_{mj}\|_{L^2(\partial D_j), L^2(\partial D_m)} = \sup_{\psi \neq 0} \frac{\|D_{mj}\psi\|_{L^2(\partial D_m)}}{\|\psi\|_{L^2(\partial D_j)}}.
\]  
  Let $\psi \in L^2(\partial D_j)$ then for $s \in \partial D_m$, we have

\[
D_{mj}\psi(s) = \int_{\partial D_j} \frac{\partial \Gamma^w(s, t)}{\partial v_j(t)} \psi(t) \, dt = \int_{\partial D_j} \left[ \lambda \left( \nabla_i \left[ \Gamma^w(s, t) \right] N_i + \mu \left( \nabla_i \left[ \Gamma^w(s, t) \right] + \left( \nabla_i \left[ \Gamma^w(s, t) \right] \right)^7 \right) N_i \right] \psi(t) \, dt
\]

where the vector $I_V$ and the third order tensor $I_{22}$ are given by

\[
I_V = -\frac{1}{4\pi} \sum_{l=0}^{\infty} \frac{l^2}{l! (l + 2)} \frac{(l - 1)}{\omega^2} \left[ -2\kappa_s^{l+2} + (l + 4)\kappa_p^{l+2} \right] |s - t|^{-3} (s - t),
\]
Then, by using Lemma 2.6, we estimate

\[ |D_{mj}\psi(s)| \leq \frac{\lambda}{4\pi} \left[ \frac{(\kappa_{\nu}^2 + 2\kappa_{\rho}^2)}{\omega^2} + \sum_{i=2}^{\infty} \frac{1}{l(l+2)} \frac{(l-1)}{\omega^2} \int_{D_j} |s - t|^{-2} \psi(t) \, dt \right] 
\]

\[ + \frac{2\mu}{4\pi} \left[ \frac{3(\kappa_{\nu}^2 + \kappa_{\rho}^2)}{\omega^2} \int_{D_j} |s - t|^{-2} \psi(t) \, dt + \frac{6\kappa_{\nu}^4 + 4\kappa_{\rho}^4}{8\omega^2} \int_{D_j} \psi(t) \, dt \right] 
\]

\[ + \sum_{i=2}^{\infty} \frac{1}{l(l+2)} \frac{(l-1)}{\omega^2} \left( 2\kappa_{\nu}^{l+2} + \kappa_{\rho}^{l+2} \right) \int_{D_j} |s - t|^{-2} \psi(t) \, dt \right] 
\]

\[ \leq \frac{\lambda}{4\pi} \|\psi\|_{L^2(D_j)} \|\partial D_j\|^{1/2} \]

\[ + \frac{2\mu}{4\pi} \|\psi\|_{L^2(D_j)} \|\partial D_j\|^{1/2} \]

\[ + \sum_{i=2}^{\infty} \frac{1}{l(l+2)} \frac{(l-1)}{\omega^2} \left( 2\kappa_{\nu}^{l+2} + \kappa_{\rho}^{l+2} \right) \int_{D_j} |s - t|^{-2} \psi(t) \, dt \right] 
\]

\[ \leq \frac{\lambda}{4\pi} \|\psi\|_{L^2(D_j)} \|\partial D_j\|^{1/2} \]

\[ + \frac{2\mu}{4\pi} \|\psi\|_{L^2(D_j)} \|\partial D_j\|^{1/2} \]

\[ + \sum_{i=2}^{\infty} \frac{1}{l(l+2)} \frac{(l-1)}{\omega^2} \left( 2\kappa_{\nu}^{l+2} + \kappa_{\rho}^{l+2} \right) \int_{D_j} |s - t|^{-2} \psi(t) \, dt \right] 
\]

\[ = \frac{1}{4\pi} \|\psi\|_{L^2(D_j)} \|\partial D_j\|^{1/2} \]

\[ + \frac{\mu}{2} \left( \lambda + 4\mu \right) \sum_{i=0}^{N_0-1} \left( \frac{1}{2} \kappa_{\nu}^i \text{diam}(\Omega) \right) + \left( \lambda + 4\mu \right) \sum_{i=0}^{N_0-1} \left( \frac{1}{2} \kappa_{\rho}^i \text{diam}(\Omega) \right) \]
\[
\begin{align*}
\leq & \frac{\epsilon \|\psi\|_{L^2(D_j)} |\partial B_j|^{1/2} }{4\pi} \left[ \frac{1}{d_{mj}^2} \left( \frac{\lambda + 6\mu}{c_j^2} + \frac{2\lambda + 6\mu}{c_p^2} \right) 
\right. \\
& + \frac{\omega^2}{c_j^2} \left( \frac{\mu}{2} + (\lambda + 4\mu) \frac{1 - \left( \frac{1}{2} \kappa_p \text{diam}(\Omega)_N \right)}{1 - \frac{1}{2} \kappa_p \text{diam}(\Omega)} \right) \\
& \left. + \frac{\omega^2}{c_p^2} \left( \frac{\mu}{2} + (3\lambda + 4\mu) \frac{1 - \left( \frac{1}{2} \kappa_p \text{diam}(\Omega)_N \right)}{1 - \frac{1}{2} \kappa_p \text{diam}(\Omega)} \right) \right]
\end{align*}
\]

form which, we get

\[
\|D_{mj}\psi\|_{L^2(\partial D_{mj})} = \left( \int_{\partial D_{mj}} |D_{mj}\psi(s)|^2 \, ds \right)^{1/2}
\leq \left( \int_{\partial D_{mj}} \frac{\tilde{C}_7}{d_{mj}^2} + \tilde{C}_8 \right) \frac{1}{4\pi} \epsilon |\partial B_j|^2 \|\psi\|_{L^2(\partial D_j)} \left( \int_{\partial D_{mj}} ds \right)^{1/2}
= \left[ \frac{\tilde{C}_7}{d_{mj}^2} + \tilde{C}_8 \right] \frac{1}{4\pi} \epsilon^2 |\partial B_j|^2 |\partial B_{mj}|^2 \|\psi\|_{L^2(\partial D_j)}. \tag{2.49}
\]

Substitution of (2.50) in (2.45) gives us

\[
\|D_{mj}\|_{L^2(\partial D_{mj}), L^2(\partial D_{mj})} \leq \left[ \frac{\tilde{C}_7}{d_{mj}^2} + \tilde{C}_8 \right] \frac{1}{4\pi} d_{mj}^2 |\partial B_j|^2 |\partial B_{mj}|^2
\leq \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] \frac{1}{4\pi} |\partial B_j| \epsilon^2.
\]

End of the proof of Proposition 2.2. By substituting (2.43) in (2.17) and (2.44) in (2.16), we obtain

\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \leq \max_{m=1}^{M} \tilde{C}_{6m} \tag{2.51}
\]

and

\[
\|DK\| \leq \frac{M - 1}{4\pi} \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] |\partial B| \epsilon^2. \tag{2.52}
\]

Hence, (2.52) and (2.51) jointly provide

\[
\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\| \leq \frac{M - 1}{4\pi} \left[ \frac{\tilde{C}_7}{d^2} + \tilde{C}_8 \right] \left( \max_{m=1}^{M} \tilde{C}_{6m} \right) |\partial B| \epsilon^2. \tag{2.53}
\]

By imposing the condition \(\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\| < 1\), we get the following from (2.15) and (2.18)–(2.19)

\[
\|\sigma_m\|_{L^2(\partial D_{mj})} \leq \|\sigma\| \leq \frac{\left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\|}{1 - \left\| \left( \frac{1}{2} I + DL \right)^{-1} \right\| \|DK\|} \|U^{ln}\|
\]
\[
\begin{align*}
\leq \hat{C}_p \left( \frac{1}{2} I + DL \right)^{-1} \| \sigma \|_{\max} & \leq \hat{C}_p \| U^I \|_{L^2(\partial D_m)} \left( \hat{C}_p \geq \frac{1}{1 - \hat{C}_p} \right)
\end{align*}
\]
for all \( m \in \{1, 2, \ldots, M\} \). But, for the plane incident wave of the Lamé system, \( U^I(x, \theta) := \alpha \theta e^{(i \omega \cdot \theta)/c_p} + \beta \theta e^{(i \omega \cdot \theta)/c_v} \), we have
\[
\| U^I \|_{L^2(\partial D_m)} \leq (|\alpha| + |\beta|) \epsilon |\partial B_m|^{\frac{1}{2}} \leq (|\alpha| + |\beta|) \epsilon |\partial B|^{\frac{1}{2}}, \quad \forall m = 1, 2, \ldots, M.
\]
Now by substituting (2.55) in (2.54), for each \( m = 1, \ldots, M \), we obtain
\[
\| \sigma \|_{L^2(\partial D_m)} \leq \hat{C}(\omega) \epsilon,
\]
where \( \hat{C}(\omega) := C \epsilon \partial B |(\alpha| + |\beta|) \).

The condition (2.57) reads as \( \sqrt{M - 1} \epsilon < \hat{c}d \) where we set
\[
\hat{c} := \left[ \frac{1}{4\pi} \left( \hat{C}_7 + \hat{C}_9 d_{\max}^2 \right) |\partial B| \max_{m=1}^{M} \hat{C}_{6m} \right]^{-\frac{1}{2}}
\]
and it serves our purpose in Proposition 2.2 and hence in Theorem 1.2.

\[\square\]

### 2.4 The single layer potential representation and the total charge

#### 2.4.1 The single layer potential representation

For \( m = 1, 2, \ldots, M \), let \( U^{\sigma_m} \) be the solution of the problem
\[
\begin{align*}
\{ (\Delta^e + \omega^2)U^{\sigma_m} = 0 \quad \text{in} & \quad D_m, \\
U^{\sigma_m} = \sigma_m \quad \text{on} & \quad \partial D_m.
\end{align*}
\]

The function \( \sigma_m \) is in \( H^1(\partial D_m) \), see Proposition 2.1. Hence \( U^{\sigma_m} \in H^2(D_m) \) and then \( \frac{\partial U^{\sigma_m}}{\partial n} \mid_{\partial D_m} \in L^2(\partial D_m) \).

From Proposition 2.1, the solution of the problem (1.1)–(1.3) has the form
\[
U^I(x) = U^I(x) + \sum_{m=1}^{M} \int_{\partial D_m} \frac{\partial \Gamma_m(x, s)}{\partial v_m(x)} \sigma_m(s) \, ds, \quad x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} D_m \right).
\]

It can be written in terms of single layer potential using Gauss’s theorem as
\[
U^I(x) = U^I(x) + \sum_{m=1}^{M} \int_{\partial D_m} \Gamma_m(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m(x)} \, ds, \quad x \in \mathbb{R}^3 \setminus \left( \bigcup_{m=1}^{M} \bar{D}_m \right).
\]

Indeed, by Betti’s third identity,
\[
\int_{\partial D_m} \frac{\partial \Gamma_m(x, s)}{\partial v_m(x)} \sigma_m(s) \, ds = \int_{\partial D_m} \Gamma_m(x, s) \frac{\partial U^{\sigma_m}(s)}{\partial v_m(x)} \, ds
\]
\[+ \int_{D_m} \left[ U^{\sigma_m}(y) \Delta^e \Gamma_m(x, y) - \Gamma_m(x, y) \Delta^e U^{\sigma_m}(y) \right] dy.
\]
Lemma 2.8 For $m = 1, 2, \ldots, M$, $U^{\sigma_n}$, the solution of the problem \((2.58)\), satisfies the estimate

$$\left\| \frac{\partial U^{\sigma_n}(s)}{\partial \nu_m(s)} \right\|_{H^{-1}(\partial D_m)} \leq C_7,$$

(2.62)

for some constant $C_7$ depending on $B_m$ through its Lipschitz character but it is independent of $\epsilon$.

Proof of Lemma 2.8 For $m = 1, 2, \ldots, M$, we write

$$U^m(x) := U^{\sigma_n}(\epsilon x + z_m), \quad \forall x \in B_m.$$ Then we obtain

$$\begin{cases}
(\Delta^\epsilon + \epsilon^2 \omega^2)U^m(x) = \epsilon^2 (\Delta^\epsilon + \omega^2) U^{\sigma_n}(\epsilon x + z_m) = 0, & \text{for } x \in B_m, \\
U^m(\xi) = U^{\sigma_n}(\epsilon \xi + z_m) = \sigma(\epsilon \xi + z_m), & \text{for } \xi \in \partial B_m,
\end{cases}$$

(2.63)

and also

$$\frac{\partial U^m(\xi)}{\partial \nu_m(\xi)} := \lambda (\text{div}_m U^m(\xi)) N_m(\xi) + \mu (\nabla_m U^m(\xi) + \nabla_m U^m(\xi)^\top) N_m(\xi)$$

$$= \epsilon \left[ \lambda (\text{div}_m U^{\sigma_n}(\epsilon \xi + z_m)) N_m(\epsilon \xi + z_m) \\
+ \mu (\nabla U^{\sigma_n}(\epsilon \xi + z_m) + \nabla U^{\sigma_n}(\epsilon \xi + z_m)^\top) N_m(\epsilon \xi + z_m) \right]$$

$$= \epsilon \frac{\partial U^{\sigma_n}}{\partial \nu_m}(\epsilon \xi + z_m).$$

Hence,

$$\left\| \frac{\partial U^m}{\partial \nu_m} \right\|_{L^2(\partial B_m)}^2 = \int_{\partial B_m} \left| \frac{\partial U^m(\eta)}{\partial \nu_m(\eta)} \right|^2 d\eta$$

$$= \int_{\partial D_m} \epsilon^2 \left| \frac{\partial U^{\sigma_n}(s)}{\partial \nu_m(s)} \right|^2 \epsilon^{-2} ds, \quad [s := \epsilon \eta + z_m]$$

$$= \left\| \frac{\partial U^{\sigma_n}}{\partial \nu_m} \right\|_{L^2(\partial D_m)}^2,$$

which gives us

$$\left\| \frac{\partial U^{\sigma_n}}{\partial \nu_m} \right\|_{L^2(\partial D_m)} \leq \epsilon \left\| \frac{\partial U^m}{\partial \nu_m} \right\|_{L^2(\partial B_m)} \leq C_7,$$ \quad \(\text{by } \epsilon \in H^1(\partial D_m), \quad \epsilon \in H^1(\partial B_m) \).

(2.64)

For every function $\epsilon_m \in H^1(\partial D_m)$, the corresponding $U^{\epsilon_m}$ exists in $D_m$ as mentioned in (2.58) and then the corresponding functions $U^m$ in $B_m$ and the inequality \((2.64)\) will be satisfied by these functions. Let $A_{D_m} : H^1(\partial D_m) \to L^2(\partial D_m)$ and $A_{B_m} : H^1(\partial B_m) \to L^2(\partial B_m)$ be the Dirichlet to Neumann maps. Then we get the following estimate from (2.64).

$$\left\| A_{D_m} \right\|_{L^2(\partial D_m), L^2(\partial D_m)} \leq \frac{1}{\epsilon} \left\| A_{B_m} \right\|_{L^2(\partial B_m), L^2(\partial B_m)}.$$ This implies that,

$$\left\| \frac{\partial U^{\sigma_n}}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} \leq \left\| A_{D_m} \right\|_{L^2(\partial D_m), H^{-1}(\partial D_m)}$$

$$\leq \left\| A_{B_m} \right\|_{L^2(\partial B_m), L^2(\partial B_m)}$$

$$\leq \frac{1}{\epsilon} \left\| A_{B_m} \right\|_{L^2(\partial B_m), L^2(\partial B_m)}.$$
Now, by (2.56) and (2.58),
\[
\left\| \frac{\partial U^\infty}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} \leq \hat{C}(\omega) \| A_{B_m} \|_{C(H^1(\partial B_m), L^2(\partial B_m))}.
\]
(2.66)
Hence the result is true as \( \| A_{B_m} \|_{C(H^1(\partial B_m), L^2(\partial B_m))} \) is bounded by a constant depending only on \( B_m \) through its size and Lipschitz character of \( B_m \).

**Definition 2.9** We call \( \sigma_m \in L^2(\partial D_m) \) satisfying (2.5), the solution of the problem (1.1)–(1.3), as elastic surface charge distributions (in short surface charge distributions). Using these surface charge distributions we define the total charge on each surface \( \partial D_m \) denoted by \( Q_m \) as
\[
Q_m := \int_{\partial D_m} \frac{\partial U^\infty(s)}{\partial \nu_m(s)} \, ds.
\]
(2.67)

2.4.2 Estimates on the total charge \( Q_m, m = 1, \ldots, M \)

In the following proposition, we provide an approximate of the far-fields in terms of the total charges \( Q_m \).

**Proposition 2.10** The \( P \)-part, \( U^\infty_p(\hat{x}, \theta) \), and the \( S \)-part, \( U^\infty_s(\hat{x}, \theta) \), of the far-field pattern of the problem (1.1)–(1.3) have the following asymptotic expansions respectively:
\[
U^\infty_p(\hat{x}, \theta) = \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \frac{\hat{x}}{c_p} \cdot \hat{z}_m} Q_m + O(a^2) \right],
\]
(2.68)
\[
U^\infty_s(\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (1 - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \frac{\hat{x}}{c_s} \cdot \hat{z}_m} Q_m + O(a^2) \right].
\]
(2.69)

if \( \kappa_p a < 1 \) and \( \kappa_s a < 1 \) where \( O(a^2) \leq \hat{C}_{sp} \omega a^2 \) with
\[
\hat{C}_{sp} := \frac{(|\alpha| + |\beta|)|\partial \beta| C \| A_{B_m} \|_{C(H^1(\partial B_m), L^2(\partial B_m))}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \frac{1}{\min\{c_s, c_p\}}.
\]

**Proof of Proposition 2.10** From (2.60), we have
\[
U^\infty(x) = \sum_{m=1}^{M} \int_{\partial D_m} \Gamma^\infty(x, s) \frac{\partial U^\infty(s)}{\partial \nu_m(s)} \, ds, \quad \text{for } x \in \mathbb{R}^3 \setminus \bigcup_{m=1}^{M} \tilde{D}_m.
\]
Substitution of the asymptotic behavior of the Kupradze tensor at infinity given in (2.2) in the above scattered field and comparing with (1.4), will allow us to write the \( P \)-part, \( U^\infty_p(\hat{x}, \theta) \), and the \( S \)-part, \( U^\infty_s(\hat{x}, \theta) \), of the far-field pattern of the problem (1.1)–(1.3) respectively as;
\[
U^\infty_p(\hat{x}, \theta) = \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \int_{S_m} e^{-i \kappa_p \hat{x} \cdot \hat{z}_m} \frac{\partial U^\infty(s)}{\partial \nu_m(s)} \, ds
\]
\[
= \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \kappa_p \hat{x} \cdot \hat{z}_m} Q_m + \int_{S_m} \left[ e^{-i \kappa_p \hat{x} \cdot \hat{z}_m} - \frac{\partial U^\infty(s)}{\partial \nu_m(s)} \right] \, ds \right],
\]
(2.70)
\[
U^\infty_s(\hat{x}, \theta) = \frac{1}{4\pi c_s^2} (1 - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \int_{S_m} e^{-i \kappa_s \hat{x} \cdot \hat{z}_m} \frac{\partial U^\infty(s)}{\partial \nu_m(s)} \, ds
\]
\[
= \frac{1}{4\pi c_s^2} (1 - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} \left[ e^{-i \kappa_s \hat{x} \cdot \hat{z}_m} Q_m + \int_{S_m} \left[ e^{-i \kappa_s \hat{x} \cdot \hat{z}_m} - \frac{\partial U^\infty(s)}{\partial \nu_m(s)} \right] \, ds \right].
\]
(2.71)
For every \( m = 1, 2, \ldots, M \), we have from Lemma 2.8:

\[
\left| \int_{\partial D_m} \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \right| \leq \| 1 \|_{H^1(\partial D_m)} \cdot \left\| \frac{\partial U^{m \omega}}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} \\
\leq \varepsilon |\partial \mathbf{B}|^{\frac{1}{2}} \cdot \left\| \frac{\partial U^{m \omega}}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} \\
\leq \left( 2.66 \right) \hat{C} a, \quad (2.72)
\]

with

\[
\hat{C} := \frac{\hat{C}(\omega)|\partial \mathbf{B}|^\frac{1}{2} \| A_{B_m} \|_{L^2(\partial B_m, L^2(\partial B_m))}}{\max_{1 \leq m \leq M} \text{diam}(B_m)} \quad (2.66).
\]

It gives us the following estimate for any \( \kappa \), i.e. \( \kappa = \kappa_p^r \) or \( \kappa_v^r \):

\[
\left| \int_{\partial D_m} \left[ e^{-i \mathbf{k} \cdot \mathbf{x}} - e^{-i \mathbf{k} \cdot \mathbf{z}_m} \right] \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \right| \leq \int_{\partial D_m} \left| e^{-i \mathbf{k} \cdot \mathbf{x}} - e^{-i \mathbf{k} \cdot \mathbf{z}_m} \right| \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \\
\leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa_l^{l} |s - z_m|^l \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \\
\leq \int_{\partial D_m} \sum_{l=1}^{\infty} \kappa_l^{l} \left\| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right\| \, ds \\
\leq \hat{C} a \sum_{l=1}^{\infty} \kappa_l^{l} \left\| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right\| \\
= \frac{1}{2} \hat{C} \kappa a^2 - \frac{1}{1 + \frac{2}{\kappa}} a, \quad \text{if} \quad a < \frac{2}{\kappa_{\text{max}}} \left( \leq \frac{2}{\kappa} \right). \quad (2.73)
\]

which means

\[
\int_{\partial D_m} \left[ e^{-i \mathbf{k} \cdot \mathbf{x}} - e^{-i \mathbf{k} \cdot \mathbf{z}_m} \right] \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \leq \hat{C} \kappa a^2, \quad \text{for} \quad a \leq \frac{1}{\kappa_{\text{max}}}. \quad (2.74)
\]

From (2.74), it follows that

\[
\int_{S_m} \left[ e^{-i \mathbf{k} \cdot \mathbf{x}} - e^{-i \mathbf{k} \cdot \mathbf{z}_m} \right] \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \leq \hat{C} \kappa_p^r a^2, \quad \text{if} \quad \varepsilon \leq \frac{\min(c_r, c_p)}{\omega_{\text{max}} \max_{1 \leq m \leq M} \text{diam}(B_m)}, \quad (2.75)
\]

\[
\int_{S_m} \left[ e^{-i \mathbf{k} \cdot \mathbf{x}} - e^{-i \mathbf{k} \cdot \mathbf{z}_m} \right] \left| \frac{\partial U^{m \omega}(s)}{\partial \nu_m(s)} \right| \, ds \leq \hat{C} \kappa_v^r a^2, \quad \text{if} \quad \varepsilon \leq \frac{\min(c_r, c_p)}{\omega_{\text{max}} \max_{1 \leq m \leq M} \text{diam}(B_m)}. \quad (2.76)
\]

Now substitution of (2.75) in (2.70) and (2.76) in (2.71) gives the required results (2.68), (2.69) respectively. \( \square \)

**Lemma 2.11** For \( m = 1, 2, \ldots, M \), the absolute value of the total charge \( Q_m \) on each surface \( \partial D_m \) is bounded by \( \varepsilon \), i.e.

\[
|Q_m| \leq \hat{c} \varepsilon, \quad (2.77)
\]

where \( \hat{c} := (|\alpha| + |\beta|)|\partial \mathbf{B}|\hat{C} \| A_{B_m} \|_{L^2(\partial B_m, L^2(\partial B_m))} \) with \( \partial \mathbf{B} \) and \( \hat{C} \) are defined in (2.44) and (2.54) respectively.
\textbf{Proof of Lemma 2.11} The proof follows as below:

\[
|Q_m| = \left| \int_{\partial D_m} \frac{\partial U^{\sigma_j}(s)}{\partial v_m(s)} \, ds \right|
\]

\[
\leq \|1\|_{H^1(\partial D_m)} \left\| \frac{\partial U^{\sigma_j}(s)}{\partial v_m(s)} \right\|_{H^{-1}(\partial D_m)}
\]

\[
\leq (2.66) \|1\|_{L^2(\partial D_m)} \hat{C}(\omega) \|A_{B_n}\|_{L(H^1(\partial B_n), L^2(\partial B_n))}
\]

\[
\leq (2.56) \epsilon |\partial \mathcal{B}| (|\alpha| + |\beta|) \hat{C} \|A_{B_n}\|_{L(H^1(\partial B_n), L^2(\partial B_n))}.
\]

\[
0 = U^j(s_m) = U^j(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds
\]

\[
= U^j(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds + \int_{\partial D_m} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_m(s)} \, ds
\]

\[
= U^j(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds + \int_{\partial D_m} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_m(s)} \, ds
\]

\[
= U^j(s_m) + \sum_{j=1}^{M} \int_{\partial D_j} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds + \int_{\partial D_m} \Gamma^{\omega}(s_m, s) \frac{\partial U^{\sigma_j}(s)}{\partial v_m(s)} \, ds.
\]  \tag{2.78}

To estimate \(\int_{\partial D_j} [\Gamma^{\omega}(s_m, s) - \Gamma^{\omega}(s_m, z_j)] \frac{\partial U^{\sigma_j}(s)}{\partial v_j(s)} \, ds\) for \(j \neq m\), we have from Taylor series that,

\[
\Gamma^{\omega}(s_m, s) - \Gamma^{\omega}(s_m, z_j) = (s - z_j) \cdot R(s_m, s), \quad \text{where}
\]

\[
R(s_m, s) = \int_0^1 \nabla \Gamma^{\omega}(s_m, s - \alpha(s - z_j)) \, d\alpha.
\]  \tag{2.79}

- From the definition of \(\Gamma^{\omega}(x, y)\) and by using the calculations made in (2.49), for \(s \in \bar{D}_j\), we obtain

\[
|R(s_m, s)| \leq \max_{y \in \bar{D}_j} |\nabla_y \Gamma^{\omega}(s_m, y)| < \frac{1}{4\pi} \left[ \frac{C_9}{d_m^2} + C_{10} \right]
\]  \tag{2.80}

with

\[
C_9 := 3 \left( \frac{1}{c_7^2} + \frac{1}{c_7^2} \right)
\]

and

\[
C_{10} := 2 \frac{\omega^2}{c_s^3} \left( \frac{1}{8} + \frac{1}{1 - \left( \frac{\omega}{2} \right)^\omega \cdot \text{diam}(\Omega)} N_2 \frac{N_2}{2N_2 - 1} \right)
\]

\[
+ \frac{\omega^2}{c_p^4} \left( \frac{1}{4} + \frac{1}{1 - \left( \frac{\omega}{2} \right)^\omega \cdot \text{diam}(\Omega)} N_2 \frac{N_2}{2N_2 - 1} \right).
\]
Indeed, for \( x \in \tilde{D}_m \) and \( s \in \tilde{D}_j \), we have from (2.4):

\[
\left| \nabla, \Gamma^\omega(x, s) \right| \leq \frac{1}{4\pi} \frac{1}{\omega^2} \left[ 3 \left( \kappa_{\omega}^2 + \kappa_{p^2}^2 \right) |x-s|^{-2} + \frac{1}{8} \left( 6\kappa_{\omega}^4 + 4\kappa_{p^2}^4 \right) \right]
\]

\[
+ \frac{1}{4\pi} \sum_{l=2}^{\infty} \frac{1}{(l-2)!(l+2)} \frac{1}{\omega^2} \left( 2\kappa_{\omega}^{l+2} + \kappa_{p^2}^{l+2} \right) |x-s|^{-l-2}
\]

\[
\leq \frac{1}{4\pi} \frac{1}{\omega^2} \left[ \frac{3}{d_{mj}^2} \left( \kappa_{\omega}^2 + \kappa_{p^2}^2 \right) + \frac{1}{4} \left( \kappa_{\omega}^4 + \kappa_{p^2}^4 \right) \right]
\]

\[
+ \sum_{l=2}^{\infty} \frac{1}{(l-2)!(l+2)} \left( 2\kappa_{\omega}^{l+2} + \kappa_{p^2}^{l+2} \right) diam(\Omega)^{-l-2}
\]

\[
\leq \frac{1}{4\pi} \left[ \frac{3}{d_{mj}^2} \left( \frac{1}{c_s^2} + \frac{1}{c_p^2} \right) + \frac{1}{4} \left( \frac{\omega^2}{c_s^4} + \frac{\omega^2}{c_p^4} \right) \right]
\]

\[
+ 2\frac{\omega^2}{c_s^4} \left( \sum_{l=0}^{N_\Omega-1} \left( \frac{1}{2} \kappa_{\omega} diam(\Omega) \right)^l + \sum_{l=N_\Omega}^{\infty} \left( \frac{1}{2} \kappa_{p} diam(\Omega) \right)^l \right)
\]

\[
+ \frac{\omega^2}{c_p^4} \left( \sum_{l=0}^{N_\Omega-1} \left( \frac{1}{2} \kappa_{p} diam(\Omega) \right)^l + \sum_{l=N_\Omega}^{\infty} \left( \frac{1}{2} \kappa_{p} diam(\Omega) \right)^l \right)
\]

\[
= \frac{1}{4\pi} \left[ \frac{C_9}{d_{mj}^2} + C_{10} \right]. \quad (2.81)
\]

For \( m, j = 1, \ldots, M, \) and \( j \neq m \), by making use of (2.80) and (2.72) we obtain the below:

\[
\left| \int_{\partial D_j} \left[ \Gamma^\omega(s_m, s) - \Gamma^\omega(s_m, z_j) \right] \frac{\partial U_{\gamma}^\omega(s)}{\partial \nu_j(s)} ds \right| = \left| \frac{\partial U_{\gamma}^\omega(s)}{\partial \nu_j(s)} \right| \int_{\partial D_j} \left| s - z_j \right| ds
\]

\[
\leq \frac{a}{4\pi} \left[ \frac{C_9}{d^2} + C_{10} \right] \int_{\partial D_j} \left| \frac{\partial U_{\gamma}^\omega(s)}{\partial \nu_j(s)} \right| ds
\]

\[
< \frac{a}{4\pi} \left[ \frac{C_9}{d^2} + C_{10} \right] a. \quad (2.82)
\]
Then (2.78) can be written as

$$
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \, ds + \int_{\partial D_m} \left[ \Gamma^{\infty}(s_m, s) - \Gamma^0(s_m, s) \right] \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \, ds
$$

$$
= -U^i(s_m) - \sum_{j=1}^M \Gamma^{\infty}(s_m, z_j) Q_j + O \left( (M - 1) \frac{a^2}{d^2} \right).
$$

(2.83)

By using the Taylor series expansions of the exponential term $e^{i\omega|s_m - s|}$, the above can also be written as,

$$
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \, ds + O(a)
$$

$$
= -U^i(s_m) - \sum_{j=1}^M \Gamma^{\infty}(s_m, z_j) Q_j + O \left( (M - 1) \frac{a^2}{d^2} \right).
$$

(2.84)

Indeed

- $\omega \leq \omega_{\text{max}}$ and for $m = 1, \ldots, M$, we have

$$
\left| \int_{\partial D_m} \left[ \Gamma^{\infty}(s_m, s) - \Gamma^0(s_m, s) \right] \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \, ds \right|
$$

$$
\leq \int_{\partial D_m} \left| \Gamma^{\infty}(s_m, s) - \Gamma^0(s_m, s) \right| \left| \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \right| \, ds
$$

$$
\leq \int_{\partial D_m} \frac{\omega}{4\pi} \left[ \frac{2}{c^2} \sum_{l=0}^{\infty} \left( \frac{1}{2} \right)^l \kappa_{l,\nu}(D_m) \right] + \frac{1}{c^3} \sum_{l=0}^{\infty} \left( \frac{1}{2} \right)^l \kappa_{l,\nu}(D_m) \right] \left| \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \right| \, ds
$$

$$
< \frac{\omega}{4\pi} \left[ \frac{2}{c^2} + \frac{1}{c^3} \right] \omega a, \text{ for } \epsilon \leq \frac{\min\{c_\nu, c_{\nu'}\}}{\omega_{\text{max}} \max_m \text{diam}(B_m)}.
$$

Define $U_m := \int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U^{\infty}(s)}{\partial v_m(s)} \, ds$, $s_m \in \partial D_m$. Then (2.84) can be written as

$$
U_m = -U^i(s_m) - \sum_{j=1}^M \Gamma^{\infty}(s_m, z_j) Q_j + O(a) + O \left( (M - 1) \frac{a^2}{d^2} \right).
$$

(2.85)

We set

$$
\tilde{U}_m := -U^i(z_m) - \sum_{j=1}^M \Gamma^{\infty}(z_m, z_j) Q_j, \text{ for } m = 1, \ldots, M.
$$

(2.86)

For $m = 1, \ldots, M$, let $\tilde{\sigma}_m \in L^2(\partial D_m)$ be the solutions of the following integral equation:

$$
\frac{\sigma_m(s)}{2} + \int_{\partial D_m} \frac{\partial \Gamma^0(x, s)}{\partial v_m(s)} \sigma_m(s) \, ds = \tilde{U}_m \text{ on } \partial D_m.
$$

(2.87)
Remark here that the left-hand side of (2.87) is the trace, on \( \partial D_m \), of the double layer potential \( \int_{\partial D_m} \frac{\partial \Gamma_0(x, s)}{\partial \nu_m(s)} \sigma_m(s) \, ds, \, x \in \mathbb{R}^3 \setminus D_m \). Dealing in the similar way as we derived (2.61), we obtain
\[
\int_{\partial D_m} \frac{\partial \Gamma_0(x, s)}{\partial \nu_m(s)} \sigma_m(s) \, ds = \int_{\partial D_m} \Gamma_0(x, s) \frac{\partial U_{\sigma_m}^\text{e}(s)}{\partial \nu_m(s)} \, ds,
\]
(2.88)
Where \( U_{\sigma_m}^\text{e} \) are the solutions of (2.58) replacing the frequency \( \omega \) by zero. As single layer potential is continuous up to the boundary, combining (2.87) and (2.88), we deduce that the constant potentials \( \bar{U}_m, \, m = 1, \ldots, M \), satisfy,
\[
\int_{\partial D_m} \Gamma_0(s_m, s) \frac{\partial U_{\sigma_m}^\text{e}(s)}{\partial \nu_m(s)} \, ds = \bar{U}_m, \, s_m \in \partial D_m.
\]
(2.89)
The total charge on the surface \( \partial D_m \) is given by
\[
\bar{Q}_m := \int_{\partial D_m} \frac{\partial U_{\sigma_m}^\text{e}(s)}{\partial \nu_m(s)} \, ds.
\]
For \( m = 1, \ldots, M \), and \( l = 1, 2, 3 \), [by proceeding in the similar manner as of (2.85)–(2.89)], let \( \bar{\sigma}_m^l \in L^2(\partial D_m) \) be the surface charge distributions which define,

- The constant potentials \( \bar{U}_m^l \in C^{3 \times 1} \) as
\[
\int_{\partial D_m} \Gamma_0(s_m, s) \frac{\partial U_{\sigma_m}^l(s)}{\partial \nu_m(s)} \, ds = \bar{U}_m^l := -(U^l(z_m))(l) e_l - \sum_{j=1}^{M} \Gamma_0(z_m, z_j) Q_j(l) e_l, \, s_m \in \partial D_m,
\]
(2.90)
with \( e_1 = (1, 0, 0)^\top, \, e_2 = (0, 1, 0)^\top \) and \( e_3 = (0, 0, 1)^\top \).
- The charge \( \bar{Q}_m^l \in C^{3 \times 1} \) on surface \( S_m \) as
\[
\bar{Q}_m^l := \int_{\partial D_m} \frac{\partial U_{\sigma_m}^\text{e}(s)}{\partial \nu_m(s)} \, ds,
\]
from which we can notice that \( \bar{U}_m = \sum_{l=1}^{3} \bar{U}_m^l, \, \bar{\sigma}_m = \sum_{l=1}^{3} \bar{\sigma}_m^l \) and \( \bar{Q}_m = \sum_{l=1}^{3} \bar{Q}_m^l \).

Now, we set the electrical capacitance \( \bar{C}_m \in C^{3 \times 3} \) for \( 1 \leq m \leq M \) through
\[
\bar{Q}_m^l = \bar{C}_m \bar{U}_m^l, \, l = 1, 2, 3, \text{ and hence } \bar{Q}_m = \bar{C}_m \bar{U}_m.
\]
(2.91)
We can write the above also as \( \bar{Q}_m = \bar{C}_m \bar{U}_m \) for each \( m = 1, 2, \ldots, M \).

Lemma 2.12 We have the following estimates for \( 1 \leq m \leq M \):
\[
\left\| \frac{\partial U_{\sigma_m}^\text{e}}{\partial \nu_m} - \frac{\partial U_{\sigma_m}^\text{e}}{\partial \nu_m} \right\|_{H^{-1}(\partial D_m)} = O \left( a + (M - 1) \frac{a^2}{d^2} \right),
\]
(2.92)
\[
Q_m - \bar{Q}_m = O \left( a^2 + (M - 1) \frac{a^3}{d^2} \right),
\]
(2.93)
where the constants appearing in \( O(\cdot) \) depend only on the Lipschitz character of \( B_m \).

Proof of Lemma 2.12 By taking the difference between (2.85) and (2.89), we obtain
\[
U_m - \bar{U}_m = \int_{\partial D_m} \Gamma_0(s_m, s) \left( \frac{\partial U_{\sigma_m}^\text{e}}{\partial \nu_m} - \frac{\partial U_{\sigma_m}^\text{e}}{\partial \nu_m} \right)(s) \, ds
= O(a) + O \left( (M - 1) \frac{a^2}{d^2} \right), \, s_m \in \partial D_m.
\]
(2.94)
Indeed by using Taylor series,

- \( U^i(s_m) - U^i(z_m) = O(a) \).
- \( \Gamma^\omega(s_m, z_j) - \Gamma^\omega(z_m, z_j) = O\left(\frac{z}{d^2}\right) \)
and the asymptoticity of \( \Gamma_j \).

In operator form we can write (2.94) as,

\[
(S^o_D)^* \left( \frac{\partial U_{\omega_m}}{\partial v_m} - \frac{\partial U_{\bar{\omega}_m}}{\partial v_m} \right) (s_m) := \int_{\delta D_m} \Gamma^0(s_m, s) \left( \frac{\partial U_{\omega_m}}{\partial v_m} - \frac{\partial U_{\bar{\omega}_m}}{\partial v_m} \right)(s) \, ds
\]

\[
= O(a) + O\left(\frac{(M - 1)a^2}{d^2}\right), \quad s_m \in \delta D_m.
\]

Here, \((S^o_D)^* : H^{-1}(\partial D_m) \rightarrow L^2(\partial D_m)\) is the adjoint of \(S^o_D : L^2(\partial D_m) \rightarrow H^1(\partial D_m)\). We know that,

\[
\left\| (S^o_D)^* \right\|_{L^2(\partial D_m), H^{-1}(\partial D_m)} = \left\| S^o_D \right\|_{L^2(\partial D_m), H^1(\partial D_m)}
\]

and

\[
\left\| (S^o_D)^* \right\|_{L^2(\partial D_m), H^{-1}(\partial D_m)} = \left\| (S^o_D)^* \right\|_{L^2(\partial D_m), H^1(\partial D_m)}
\]

then from (A.4) of Lemma A.2, we obtain \(\left\| (S^o_D)^* \right\|_{L^2(\partial D_m), H^{-1}(\partial D_m)} = O(a^{-1})\). Hence, we get the required results in the following manner.

- First,

\[
\left\| \frac{\partial U_{\omega_m}}{\partial v_m} - \frac{\partial U_{\bar{\omega}_m}}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \leq \left\| (S^o_D)^* \right\|_{L^2(\partial D_m), H^{-1}(\partial D_m)} \left\| O(a) + O\left(\frac{(M - 1)a^2}{d^2}\right) \right\|_{L^2(\partial D_m)}
\]

\[
= O\left(a + (M - 1)\frac{a^2}{d^2}\right).
\]

- Second,

\[
|Q_m - \bar{Q}_m| = \left| \int_{\delta D_m} \left( \frac{\partial U_{\omega_m}}{\partial v_m} - \frac{\partial U_{\bar{\omega}_m}}{\partial v_m} \right)(s) \, ds \right|
\]

\[
\leq \left\| \frac{\partial U_{\omega_m}}{\partial v_m} - \frac{\partial U_{\bar{\omega}_m}}{\partial v_m} \right\|_{H^{-1}(\partial D_m)} \left\| 1 \right\|_{H^1(\partial D_m)}
\]

\[
= O\left(a^2 + (M - 1)\frac{a^3}{d^2}\right).
\]

\[
\textbf{Lemma 2.13} \quad \text{For every} \ 1 \leq m \leq M, \text{the capacitance} \ C_m \text{and charge} \ \bar{Q}_m \text{are of the form:}
\]

\[
C_m = \max_{1 \leq m \leq M} \frac{\bar{C}_{B_m}}{\text{diam}(B_m)} a \quad \text{and} \quad \bar{Q}_m = \max_{1 \leq m \leq M} \frac{\bar{Q}_{B_m}}{\text{diam}(B_m)} a,
\]

\[
(2.95)
\]

where \(\bar{C}_{B_m}\) and \(\bar{Q}_{B_m}\) are the capacitance and the charge of \(B_m\), respectively.

\[
\text{Proof of Lemma 2.13} \quad \text{Take} \ 0 < \epsilon \leq 1, z \in \mathbb{R}^3 \text{and write} \ D_x := \epsilon B + z \subseteq \mathbb{R}^3. \text{For} \ \psi_x \in L^2(\partial D_x) \text{and} \ \psi \in L^2(\partial B), \text{define the operators} \ S^o_x : L^2(\partial D_x) \rightarrow H^1(\partial D_x) \text{and} \ S^o_B : L^2(\partial B) \rightarrow H^1(\partial B) \text{as:}
\]

\[
S^o_x \psi_x(x) := \int_{\delta D_x} \Gamma^0(x, y) \psi_x(y) \, dy, \quad \text{and} \quad S^o_B \psi(\xi) := \int_{\delta B} \Gamma^0(\xi, \eta) \psi(\eta) \, d\eta.
\]
Define \( U^{\psi_1} \) and \( U^\psi \) as the functions on \( \hat{D}_e \) and \( \hat{B} \) respectively in the similar way of (2.58). Then the operators

\[
\mathcal{S}_{\psi_e}^{=} U^{\psi_e}(x) := \int_{\partial D_e} \Gamma^0(x, y) \frac{\partial U^{\psi_e}}{\partial y}(y) \, dy \quad \text{and} \quad \mathcal{S}_{\psi}^{=} U^{\psi}(\xi) := \int_{\partial B} \Gamma^0(\xi, \eta) \frac{\partial U^{\psi}}{\partial \eta}(\eta) \, d\eta
\]

define the corresponding potentials \( \hat{U}_e, \hat{U}_B \) on the surfaces \( \partial D_e \) and \( \partial B \) w.r.t. the surface charge distributions \( \psi_e \) and \( \psi \) respectively. Let, these potentials be equal to some constant vector \( \mathcal{D} \in \mathbb{C}^{1 \times 1} \). Let the total charge of these conductors \( D_e, B \) are \( \hat{Q}_e \) and \( \hat{Q}_B \), and the capacitances are \( \hat{C}_e \) and \( \hat{C}_B \) respectively. Then we can write these as

\[
\hat{U}_e := \mathcal{S}_{\psi_e}^{=} U^{\psi_e}(x) = \mathcal{D}, \quad \hat{U}_B := \mathcal{S}_{\psi}^{=} U^{\psi}(\xi) = \mathcal{D}, \quad \forall x \in \partial D_e, \forall \xi \in \partial B.
\]

We have by definitions, \( \hat{Q}_e = \int_{\partial D_e} \frac{\partial U^{\psi_e}}{\partial y}(y) \, dy = \hat{Q}_B = \int_{\partial B} \frac{\partial U^{\psi}}{\partial \eta}(\eta) \, d\eta \), and \( \hat{C}_e \hat{U}_e = \hat{Q}_e, \hat{C}_B \hat{U}_B = \hat{Q}_B \).

Observe that,

\[
D = \mathcal{S}_{\psi_e}^{=} U^{\psi_e}(x) = \int_{\partial D_e} \Gamma^0(x, y) \frac{\partial U^{\psi_e}}{\partial y}(y) \, dy = \int_{\partial B} \Gamma^0(\xi, \eta) \frac{\partial U^{\psi}}{\partial \eta}(\eta) \, d\eta = D
\]

\[
\mathcal{S}_{\psi}^{=} U^{\psi}(\xi) := \int_{\partial B} \Gamma^0(\xi, \eta) \frac{\partial U^{\psi}}{\partial \eta}(\eta) \, d\eta
\]

Hence, \( U^{\psi_e} = \hat{U}_e^\psi \) and \( U^{\psi} = \hat{U}_B^\psi \). Now we have,

\[
\hat{Q}_e = \int_{\partial D_e} \frac{\partial U^{\psi_e}}{\partial y}(y) \, dy = \int_{\partial D_e} \frac{\partial \hat{U}_e^\psi}{\partial y}(y) \, dy = \int_{\partial B} \frac{\partial \hat{U}_B^\psi}{\partial \eta}(\eta) \, d\eta
\]

\[
= \int_{\partial B} \frac{\partial \hat{U}_B^\psi}{\partial \eta}(\eta) \, d\eta = \epsilon \int_{\partial B} \frac{\partial U^{\psi}}{\partial \eta}(\eta) \, d\eta = \epsilon \hat{Q}_B
\]

which gives us

\[
\hat{C}_e \hat{D} = \hat{C}_B \hat{U}_e = \hat{Q}_e = \epsilon \hat{Q}_B = \epsilon \hat{C}_B \hat{U}_B = \epsilon \hat{C}_B D.
\]

It is true for every constant vector \( D \) and hence \( \hat{C}_e = \epsilon \hat{C}_B \). As we have \( D_m = \epsilon B_m + z_m \) and \( a = \max_{1 \leq m \leq M} \text{diam } D_m = \epsilon \max_{1 \leq m \leq M} \text{diam } (B_m) \), we obtain

\[
\hat{Q}_m = \epsilon \hat{Q}_{B_m} = \max_{1 \leq m \leq M} \text{diam } (B_m) a \quad \text{and} \quad \hat{C}_m = \epsilon \hat{C}_{B_m} = \max_{1 \leq m \leq M} \text{diam } (B_m) a.
\]

**Lemma 2.14** For \( m = 1, 2, \ldots, M \), the elastic capacitances \( \hat{C}_m \in \mathbb{C}^{3 \times 3} \) defined through (2.91) are nonsingular.

**Proof of Lemma 2.14** As the capacitances \( \hat{C}_m \) depend only on the scatterers, let \( \sigma_m^l \in L^2(\partial D_m) \) be surface charge distributions which define the potentials \( \psi_l \) for \( l = 1, 2, 3 \), i.e.

\[
\int_{\partial D_m} \Gamma^0(s_m, s) \frac{\partial U_{\sigma_m^l}}{\partial \nu_m}(s) \, ds = \psi_l = U_{\sigma_m^l}, \quad \text{for } l = 1, 2, 3, m = 1, \ldots, M.
\]
We also have \[ \int_{\partial D_m} \bar{\frac{\partial U_{m_l}}{\partial n_m}} (s) \, ds, \int_{\partial D_m} \bar{\frac{\partial U_{m_2}}{\partial n_m}} (s) \, ds, \int_{\partial D_m} \bar{\frac{\partial U_{m_3}}{\partial n_m}} (s) \, ds \] = \bar{C}_m [U^1_m, U^2_m, U^3_m] = \bar{C}_m. Hence, it is enough if we show that the matrix \[ \left[ \int_{\partial D_m} \left( \frac{\partial U_{m_l}}{\partial n_m} \right) (s) \, ds \right]_{l, j = 1}^3 \] is invertible. In order to prove this, assume the linear combination \[ \sum_{l=1}^3 a_l \int_{\partial D_m} \bar{\frac{\partial U_{m_l}}{\partial n_m}} (s) \, ds = 0 \] for the scalars \( a_l \in \mathbb{C} \). From (2.96), we can deduce that
\[ \int_{\partial D_m} \int_{\partial D_m} \Gamma^0(m_1, m_2) \left( \sum_{l=1}^3 a_l \bar{\frac{\partial U_{m_l}}{\partial n_m}} (m_1) \right) \cdot \left( \sum_{l=1}^3 a_l \bar{\frac{\partial U_{m_l}}{\partial n_m}} (m_2) \right) \, ds_1 \, ds_2 = 0, \quad j = 1, 2, 3, \]
and hence
\[ \int_{\partial D_m} \int_{\partial D_m} \Gamma^0(m_1, m_2) \left( \sum_{l=1}^3 a_l \bar{\frac{\partial U_{m_l}}{\partial n_m}} (m_1) \right) \, ds_1 \, ds_2 = 0. \]
The positivity of the single layer operator implies, \[ \sum_{l=1}^3 a_l \bar{\frac{\partial U_{m_l}}{\partial n_m}} (s) = 0, \quad s \in \partial D_m. \]
Again now by making use of (2.96), we deduce
\[ \sum_{l=1}^3 a_l \bar{e_l} = \int_{\partial D_m} \Gamma^0(m, s) \left( \sum_{l=1}^3 a_l \bar{\frac{\partial U_{m_l}}{\partial n_m}} (s) \right) \, ds = 0, \quad s_m \in \partial D_m, \]
and hence \( a_l = 0 \) for \( l = 1, 2, 3 \). \( \square \)

**Proposition 2.15** For \( m = 1, 2, \ldots, M \), the total charge \( \bar{Q}_m \) on each surface \( \partial D_m \) of the small scatterer \( D_m \) can be calculated from the algebraic system
\[ \bar{C}_m^{-1} \bar{Q}_m = -U^i(z_m) - \sum_{j \neq m}^M \Gamma^{ao}(z_m, z_j) \bar{C}_j \left( \bar{C}_j^{-1} \bar{Q}_j \right), \quad (2.97) \]
with an error of order \( O \left( (M - 1)^2 \frac{a^2}{d^2} + (M - 1)^2 \frac{a^3}{d^3} \right) \).

**Proof of Proposition 2.15** We can rewrite (2.89) as
\[ \bar{C}_m^{-1} \bar{Q}_m = -U^i(z_m) - \sum_{j \neq m}^M \Gamma^{ao}(z_m, z_j) \bar{Q}_j \]
\[ = -U^i(z_m) - \sum_{j \neq m}^M \Gamma^{ao}(z_m, z_j) \bar{Q}_j - \sum_{j \neq m}^M \Gamma^{ao}(z_m, z_j) (\bar{Q}_j - \bar{Q}_j) \]
\[ = -U^i(z_m) - \sum_{j \neq m}^M \Gamma^{ao}(z_m, z_j) \bar{Q}_j + O \left( (M - 1) \frac{a^2}{d} + (M - 1)^2 \frac{a^3}{d^3} \right), \]
where we used (2.93) and the fact \( \Gamma^{ao}(z_m, z_j) = O \left( \frac{1}{d^2} + \omega \right), \omega \leq \omega_{\text{max}} \) and \( d \leq d_{\text{max}} \). Indeed,
\[ \left| \Gamma^{ao}(z_m, z_j) \right| \leq \frac{1}{4 \pi} \frac{1}{\omega^2} \left( \kappa^{1+2}_{m} + \kappa^{2+2}_{m} \right) |z_m - z_j|^{-1} \]
\[ + \frac{1}{4 \pi} \sum_{l=1}^\infty \frac{1}{(l - 1)! (l + 2)} \frac{1}{\omega^2} \left( 2 \kappa^{1+2}_{m} + \kappa^{2+2}_{m} \right) |z_m - z_j|^{-1}, \]
\[
\begin{aligned}
&\leq \frac{1}{4\pi} \sum_{l=1}^{\infty} \left( \frac{1}{d_{mj}} \left( \frac{k_{\nu}^2}{c_s^l} + \frac{k_p^2}{c_p^{l+2}} \right) + \frac{1}{(l-1)!l(l+2)} \left( 2k_{\nu}^{l+2} + k_p^{l+2} \right) \text{diam}(\Omega)^{l-1} \right) + \\
&\leq \frac{1}{4\pi} \sum_{l=1}^{\infty} \left( \frac{1}{d_{mj}} \left( \frac{1}{c_s^l} + \frac{1}{c_p^{l+2}} \right) + \frac{1}{(l-1)!l(l+2)} \left( 2\omega \frac{k_{\nu}}{c_s^{l-1}} \frac{k_p^{l-1}}{c_p^{l+2}} \right) \text{diam}(\Omega)^{l-1} \right] \\
&[\text{By recalling } N_{\text{ij}} = \left[2\text{diam}(\Omega) \max(k_{\nu}, k_p^2)e^3 \right] \text{ and using Lemma 2.6}] \\
&\leq \frac{1}{4\pi} \left[ \sum_{l=1}^{N_{\text{ij}}} \left( \frac{1}{2} k_{\nu} \text{diam}(\Omega)^{l-1} \right) \right] + \sum_{l=N_{\text{ij}}+1}^{\infty} \frac{1}{2(l-1)} \\
&= \frac{1}{4\pi} \left[ \frac{N_{\text{ij}}}{d_{mj}} \left( \frac{1}{c_s^l} + \frac{1}{c_p^{l+2}} \right) + 2k_p \frac{1}{c_s^{l-1}} \left( 1 - \frac{1}{2} k_{\nu} \text{diam}(\Omega)^{N_{\text{ij}}} \right) \right] \\
&= \frac{1}{4\pi} \left[ \frac{C_7}{d_{mj}} + C_8 \right] \\
&= \frac{1}{4\pi} \left[ \frac{1}{c_s^l} + \frac{2}{c_p^{l+2}} \right] \\
\end{aligned}
\]

with

\[
C_7 := \left[ \frac{1}{c_s^l} + \frac{2}{c_p^{l+2}} \right]
\]

and

\[
C_8 := 2 \frac{k_{\nu}}{c_s^l} \left( 1 - \frac{1}{2} k_{\nu} \text{diam}(\Omega)^{N_{\text{ij}}} \right) + \frac{k_p}{c_p^{l+2}} \left( \frac{1}{2} k_{\nu} \text{diam}(\Omega)^{N_{\text{ij}}} \right) + \frac{1}{2^{N_{\text{ij}}}}.
\]

2.4.3 The algebraic system

Define the algebraic system,

\[
\tilde{C}_m^{-1} \tilde{Q}_m := -U^T(z_m) - \sum_{j=1 \atop j \neq m}^{M} \Gamma_{\omega}(z_m, z_j) \tilde{C}_j \left( \tilde{C}_j^{-1} \right)^T \tilde{Q}_j,
\]

for all \( m = 1, 2, \ldots, M \). It can be written in a compact form as

\[
\mathbf{B} \tilde{Q} = U^T,
\]

where \( \tilde{Q}, U^T \in \mathbb{C}^{3M \times 1} \) and \( \mathbf{B} \in \mathbb{C}^{3M \times 3M} \) are defined as

\[
\mathbf{B} := \begin{pmatrix}
-\tilde{C}_1^{-1} & -\Gamma_{\omega}(z_1, z_2) & -\Gamma_{\omega}(z_1, z_3) & \cdots & -\Gamma_{\omega}(z_1, z_M) \\
-\Gamma_{\omega}(z_2, z_1) & -\tilde{C}_2^{-1} & -\Gamma_{\omega}(z_2, z_3) & \cdots & -\Gamma_{\omega}(z_2, z_M) \\
-\Gamma_{\omega}(z_3, z_1) & -\Gamma_{\omega}(z_3, z_2) & -\tilde{C}_3^{-1} & \cdots & -\Gamma_{\omega}(z_3, z_M) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-\Gamma_{\omega}(z_M, z_1) & -\Gamma_{\omega}(z_M, z_2) & -\Gamma_{\omega}(z_M, z_3) & \cdots & -\tilde{C}_M^{-1}
\end{pmatrix},
\]

\[
\tilde{Q} := \left( \tilde{Q}_1^T \tilde{Q}_2^T \cdots \tilde{Q}_M^T \right)^T \quad \text{and} \quad U^T := \left( U^T(z_1)^T U^T(z_2)^T \cdots U^T(z_M)^T \right)^T.
\]

The above linear algebraic system is solvable for the 3D vectors \( \tilde{Q}_j, 1 \leq j \leq M \), when the matrix \( \mathbf{B} \) is invertible. We discuss its invertibility in Section 3.
Now, the difference between (2.97) and (2.99) produce the following

\[
\tilde{C}^{-1}_m (\tilde{Q}_m - \tilde{Q}_m) = -\sum_{j=1}^{M} \Gamma_m(z, z, \tilde{\omega}) \left( \tilde{Q}_j - \tilde{Q}_j \right) + O \left( (M - 1) \frac{a^2}{d} + (M - 1)^2 \frac{a^3}{d^3} \right), \tag{2.101}
\]

for \( m = 1, \ldots, M \). Considering the above system of Equations (2.101) in the place of (2.99) and then by making use of the Corollary 3.3 and the fact that acoustic capacitances of the scatterers are bounded above and below by their diameters multiplied by constants which depend only on the Lipschitz character of the \( B_m \)'s, see [16, Lemma 2.11 and Remark 2.23], we obtain

\[
\sum_{m=1}^{M} (\tilde{Q}_m - \tilde{Q}_m) = O \left( M(M - 1) \frac{a^3}{d} + M(M - 1)^2 \frac{a^4}{d^3} \right). \tag{2.102}
\]

### 2.5 End of the proof of Theorem 1.2

The use of (2.93), (2.102) in (2.68) and (2.69) allows us to represent the asymptotic expansions of the P part, \( U_P^\infty(\hat{x}, \hat{\theta}) \), and the S part, \( U_S^\infty(\hat{x}, \hat{\theta}) \), of the far-field pattern of the problem (1.1)–(1.3) in terms of \( \tilde{Q}_m \) respectively as below:

\[
U_P^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} e^{-i \frac{\hat{z}_m}{2}} \tilde{z}_m \left[ Q_m + O \left( a^2 \right) \right]
\]

\[
= \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \sum_{m=1}^{M} e^{-i \frac{\hat{z}_m}{2}} \tilde{z}_m \left[ (\tilde{Q}_m + (Q_m - \tilde{Q}_m)) + (\tilde{Q}_m - \tilde{Q}_m) \right] + O \left( a^2 \right)
\]

\[
= \frac{1}{4\pi c_p^2} (\hat{x} \otimes \hat{x}) \left( \sum_{m=1}^{M} e^{-i \frac{\hat{z}_m}{2}} \tilde{z}_m \tilde{Q}_m \right)
\]

\[
+ O \left( Ma^2 + M(M - 1) \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^4}{d^3} \right), \tag{2.103}
\]

\[
U_S^\infty(\hat{x}, \hat{\theta}) = \frac{1}{4\pi c_p^2} (1 - \hat{x} \otimes \hat{x}) \sum_{m=1}^{M} e^{-i \frac{\hat{z}_m}{2}} \tilde{z}_m \left[ Q_m + O \left( a^2 \right) \right]
\]

\[
= \frac{1}{4\pi c_p^2} (1 - \hat{x} \otimes \hat{x}) \left( \sum_{m=1}^{M} e^{-i \frac{\hat{z}_m}{2}} \tilde{z}_m \tilde{Q}_m \right)
\]

\[
+ O \left( Ma^2 + M(M - 1) \frac{a^3}{d^2} + M(M - 1)^2 \frac{a^4}{d^3} \right). \tag{2.104}
\]

Hence, Theorem 1.2 is proved by setting \( \bar{\sigma}_m := \frac{\tilde{Q}_m}{\tilde{Q}_m} \) as the surface density which defines \( \tilde{Q}_m \). Finally, let us remark that
1. The constant \( \hat{c} := \left[ \frac{1}{2\pi} \left( \tilde{C}_1 + \tilde{C}_2 d_{\max}^2 \right) \right] [\partial B] \max_{1 \leq m \leq M} \hat{C}_m \left( \hat{C}_m \right)^{-1} \) appearing in Proposition 1.2 will serve our purpose in Theorem 1.2 by defining \( c_0 := \hat{c} \max_{1 \leq m \leq M} \text{diam} (B_m) \) respectively.

2. The coefficients \( \tilde{\sigma}_m, \tilde{\Omega}_m, \tilde{Q}_m, \tilde{C}_m \) play the roles of \( \sigma_m, Q_m, C_m \) respectively in Theorem 1.2.

3. The constant appearing in \( O \left( M \alpha^2 + M(M - 1) \alpha^3 + M(M - 1)^2 \alpha^4 \right \) is

\[
C^* \max \left\{ 1 + \frac{\max_{1 \leq m \leq M} \tilde{C}_m}{\max_{1 \leq m \leq M} \text{diam} (B_m)} C_T + C_4 d_{\max} \frac{2}{4\pi}, 1 + \frac{\hat{C}_m}{C^* \min \{ c_s, c_p \}} \right\}
\]

with

\[
C^* := \max_{1 \leq m \leq M} \frac{\max_{1 \leq m \leq M} \left\| S_{B_m}^{-1} \right\|_{L^2(H^1(\partial B_m), L^2(\partial B_m))}}{\text{diam} (B_m)} \times \max \left\{ \left[ \frac{\| \partial \|}{\min \{ c_s, c_p \}} + \frac{\hat{C}_m}{\pi} \left( \frac{2}{c_s^2} + \frac{1}{c_p^2} \right) \right] \omega, \frac{\hat{C}_m}{4\pi} \min \{ c_s, c_p \} \right\}.
\]

The constants \( |\partial B| \) and \( \hat{C} \) are defined in Proposition 2.7 and Proposition 2.10 respectively.

4. The constant \( a_0 \) appearing in (1.7) of Theorem 1.2 is the minimum among \( \frac{1}{\alpha_{\max}} \min \{ c_s, c_p \} \), and

\[
2\sqrt{\pi} \max_{1 \leq m \leq M} \text{diam} (B_m) \min_{1 \leq m \leq M} \left\| \left( \frac{1}{2} \right) \frac{\partial I + D_{B_m}^{-1}}{L^2(\partial B_m)} \right\|_{L^2(H^1(\partial B_m), L^2(\partial B_m))} \cdot \omega_{\max} \left( \frac{1}{2\pi} \right) \left( \frac{4\pi + 12\pi + 8\pi}{16\pi} \right)^{-1} |\partial B| \max_{1 \leq m \leq M} \left\| \left( \frac{1}{2} I + D_{B_m}^{-1} \right)^{-1} \right\|_{L^2(H^1(\partial B_m), L^2(\partial B_m))} \right\} \right)^{-1}.
\]

5. The constant \( c_1 \) appearing in (1.12) of Theorem 1.2 is

\[
\frac{2\pi}{\left( \frac{\omega_{\max} \min_{1 \leq m \leq M} C^*(B_m) \max_{1 \leq m \leq M} \text{diam} (B_m) \right)^{\frac{1}{2}} \left( \frac{1}{\alpha_{\max}} \min \{ c_s, c_p \} \right) - \frac{C^*(B_m)}{\max_{1 \leq m \leq M} C^*(B_m) \max_{1 \leq m \leq M} \text{diam} (B_m) \right)^{\frac{1}{2}}}
\]

The \( C^*(B_m) \) denoting the acoustic capacitance of the bodies \( B_m \) and it follows from Corollary 3.3 and from [16, Lemma 2.11].

From the last points, we see that the constants appearing in Theorem 1.2 depend only on \( d_{\max}, \omega_{\max}, \lambda, \mu \) and the \( B_m \)’s through their diameters, capacitances and the norms of the boundary operators \( S_{B_m}^{-1} : H^1(\partial B_m) \rightarrow L^2(\partial B_m), \left( \frac{1}{2} \right) \frac{\partial I + D_{B_m}^{-1}}{L^2(\partial B_m)} \rightarrow L^2(\partial B_m) \) and \( A_{B_m} : H^1(\partial B_m) \rightarrow L^2(\partial B_m) \). As was explained in the acoustic case in [16,Remark 2.23], the capacitances and the bounds of the operators \( S_{B_m}^{-1} \) and \( \left( \frac{1}{2} \right) \frac{\partial I + D_{B_m}^{-1}}{L^2(\partial B_m)} \) depend on the \( B_m \)’s actually only through their Lipschitz character. We also refer to [12] for additional information on the dependency of the layer potentials on the Lipschitz character of the bodies.

\[ \square \]

2.6 Proof of Corollary 1.3

For \( m = 1, \ldots, M \) fixed, we distinguish between the obstacles \( D_j, j \neq m \), which are near to \( D_m \) from the ones which are far from \( D_m \) as follows. Let \( \Omega_m, 1 \leq m \leq M \), be the balls of center \( z_m \) and of radius \( \left( \frac{\alpha}{2} + d^2 \right) \) with \( 0 < \alpha \leq 1 \). The bodies lying in \( \Omega_m \) will fall into the category, \( N_m \), of near by obstacles and the others into the category, \( F_m \), of far obstacles to \( D_m \). Since the obstacles \( D_m \) are balls with same diameter, the number of obstacles near by \( D_m \) will not exceed \( \left( \frac{\alpha + d}{\alpha + d^2} \right)^3 \left[ \frac{2\pi (\alpha + d^2/2)}{2\pi (\alpha + d/2)} \right] \).

With this observation, instead of (1.8)–(1.9), the P and the S parts of the far field will have the asymptotic expansions (1.13)–(1.14).

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• For the bodies $D_j \in N_m$, $j \neq m$, we have the estimate (2.80) but for the bodies $D_j \in F_m$, we obtain the following estimate
\[
|R(s_m, s)| \leq \max_{y \in D_j} |\nabla_y \Gamma(s_m, y)| < \frac{1}{4\pi} \left[ \frac{C_0}{\delta_m^2} + C_{10} \right].
\] (2.105)

• Due to the estimates (2.80) and (2.105), corresponding changes will take place in (2.82)–(2.84), (2.85), (2.92)–(2.93) and in (2.97)–(2.98) which inturn modify (2.101)–(2.102) and hence the asymptotic expansion (1.8) as follows
\[
U_{p}^{\infty}(\hat{x}, \theta) = \frac{1}{4\pi c_{p}^2} \left[ \sum_{m=1}^{M} \alpha_{m} \bar{z}_{m} Q_{m} + O \left( M \bar{a}^{2} + M(M-1) \frac{\alpha^{3}}{d^{2}a^{2}} + M \left( \frac{a + 2d^{a}}{a + d} \right)^{3} \frac{a^{3}}{d^{2}} \right) \right.
\]\[+ M(M-1) \frac{a^{4}}{d^{3}a^{2}} + M(-1) \left( \frac{a + 2d^{a}}{a + d} \right)^{3} \frac{a^{4}}{d^{2}a^{2}} + M \left( \frac{a + 2d^{a}}{a + d} \right)^{6} \frac{a^{4}}{d^{3}} \left] \right].
\] (2.106)

• Since $\kappa \leq \kappa_{\text{max}}, d \leq d^{a}, 0 < \alpha \leq 1$ and $\frac{d}{a} < \infty$, we have
\[
\left( \frac{a + 2d^{a}}{a + d} \right) = d^{a-1} \frac{ad^{-a} + 2}{ad^{-1} + 1} = O \left( d^{a-1} \right),
\]
which can be used to derive (1.13) from (2.106). In the similar way, we can obtain (1.14). Finally, it is easily seen that the above analysis applies also for non-flat Lipschitz domains $D_{m}$ by using the double inclusions (1.17) and the fact that the $t_{m}$’s are uniformly bounded from below by a positive constant. □

### 3 Solvability of the linear-algebraic system (2.100)

The main object of this section is to give a sufficient condition in order to get the invertibility of the linear algebraic system (2.100). To achieve this, first we state the following lemma which estimates the eigenvalues of the elastic capacitance matrix of each scatterer in terms of its acoustic capacitance.

**Lemma 3.1** Let $\lambda_{\text{min}}$ and $\lambda_{\text{max}}$ be the minimal and maximal eigenvalues of the elastic capacitance matrices $\mathcal{C}_{m}$, for $m = 1, 2, \ldots, M$. Denote by $C_{m}^{\alpha}$ the capacitance of each scatterer in the acoustic case, then we have the following estimate;
\[
\mu C_{m}^{\alpha} \leq \lambda_{\text{min}} \leq \lambda_{\text{max}} \leq (\lambda + 2\mu) C_{m}^{\alpha}, \quad \text{for} \quad m = 1, 2, \ldots, M.
\] (3.1)

**Proof of Lemma 3.1** The proof of this lemma follows as in [30, Lemma 6.3.6]. See also [28, Lemma 10]. □

Now, we prove the main lemma of this section.

**Lemma 3.2** The matrix $\mathbf{B}$ is invertible and the solution vector $\mathbf{Q}$ of (2.100) satisfies the estimate:
\[
\sum_{m=1}^{M} \| \mathbf{Q}^{m} \|_{2}^{2} \leq 4 \left( \frac{C_{m}^{\alpha}}{5\pi d} \sum_{m=1}^{M} \lambda_{\text{max}}^{\alpha} \right)^{-2} \left( \frac{C_{m}^{\alpha}}{5\pi d} \sum_{m=1}^{M} \lambda_{\text{max}}^{\alpha} \right)^{4} \sum_{m=1}^{M} \| U^{j} (z_{m}) \|_{2}^{2},
\] (3.2)

if we consider
\[
\left( \frac{\max_{1 \leq m \leq M} \lambda_{\text{max}}^{\alpha}}{\min_{1 \leq m \leq M} \lambda_{\text{max}}^{\alpha}} \right) < r^{-1} \left( \frac{\pi d}{\lambda_{\text{max}}^{\alpha}} \min_{1 \leq m \leq M} \lambda_{\text{max}}^{\alpha} \right) \quad \text{with the positively assumed value} \quad r := \left[ \frac{1}{d} - 2 \text{diam}(\Omega) \frac{\alpha}{\pi} \right] \left[ 1 - \left( \frac{1}{4\pi} \text{diam}(\Omega) \right)^{\alpha} \right] \frac{\alpha}{\pi} \left[ 1 - \left( \frac{1}{4\pi} \text{diam}(\Omega) \right)^{\alpha} \right]^{\alpha}.
\]

5Recall that, for $m = 1, \ldots, M$, $C_{m}^{\alpha} := \int_{D_{m}} \sigma_{m}(s) \, ds$ and $\sigma_{m}$ is the solution of the integral equation of the first kind $\int_{D_{m}} \sigma_{m}(s) \frac{d}{d} \, ds = 1, \quad t \in \partial D_{m}$, see [16].
Proof of Lemma 3.2 We can factorize \( B \) as
\[
B = -(I + B_n)C^{-1}
\]
where \( C := \text{Diag}(\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_M) \in \mathbb{R}^{3M \times 3M} \), \( I \) is the identity matrix and \( B_n := -C^{-1} - B \). Hence, the solvability of the system (2.100), depends on the existence of the inverse of \( (I + B_n, C) \). We have \( (I + B_n, C) : \mathbb{C}^{3M} \rightarrow \mathbb{C}^{3M} \), so it is enough to prove the injectivity in order to prove its invertibility. For this purpose, let \( X, Y \) are vectors in \( \mathbb{C}^M \) and consider the system
\[
(I + B_n, C)X = Y. \tag{3.3}
\]
Let \( \langle \cdot \rangle \) denotes the real and the imaginary parts of the corresponding complex number/vector/matrix. Now, the following can be written from (3.3):
\[
(I + B_n, C)X^{\text{real}} - B_n^{\text{img}}CX^{\text{img}} = Y^{\text{real}}, \tag{3.4}
\]
\[
(I + B_n, C)X^{\text{img}} + B_n^{\text{img}}CX^{\text{real}} = Y^{\text{img}}, \tag{3.5}
\]
which leads to
\[
\begin{align*}
\{(I + B_n, C)X^{\text{real}}, CX^{\text{real}}\} & - \{B_n^{\text{img}}CX^{\text{img}}, CX^{\text{real}}\} = \{Y^{\text{real}}, CX^{\text{real}}\}, \tag{3.6} \\
\{(I + B_n, C)X^{\text{img}}, CX^{\text{img}}\} + \{B_n^{\text{img}}CX^{\text{real}}, CX^{\text{img}}\} = \{Y^{\text{img}}, CX^{\text{img}}\}. \tag{3.7}
\end{align*}
\]
By summing up (3.6) and (3.7) will give
\[
\begin{align*}
\{X^{\text{real}}, CX^{\text{real}}\} + \{B_n^{\text{real}}CX^{\text{real}}, CX^{\text{real}}\} + \{X^{\text{img}}, CX^{\text{img}}\} + \{B_n^{\text{real}}CX^{\text{img}}, CX^{\text{img}}\}
&= \{Y^{\text{real}}, CX^{\text{real}}\} + \{Y^{\text{img}}, CX^{\text{img}}\}. \tag{3.8}
\end{align*}
\]
Indeed,
\[
\begin{align*}
\{B_n^{\text{img}}CX^{\text{img}}, CX^{\text{real}}\} &= \{CX^{\text{img}}, B_n^{\text{img}}CX^{\text{real}}\} \\
&= \{CX^{\text{img}}, B_n^{\text{img}}CX^{\text{real}}\} \\
&= \{B_n^{\text{img}}CX^{\text{real}}, CX^{\text{img}}\}.
\end{align*}
\]
We can observe that, the right-hand side in (3.8) does not exceed
\[
\begin{align*}
\{X^{\text{real}}, CX^{\text{real}}\}^{1/2} \{Y^{\text{real}}, CY^{\text{real}}\}^{1/2} + \{X^{\text{img}}, CX^{\text{img}}\}^{1/2} \{Y^{\text{img}}, CY^{\text{img}}\}^{1/2}
&\leq 2\{X^{\|\|}, (CX)^{\|\|}\}^{1/2} \{Y^{\|\|}, (CY)^{\|\|}\}^{1/2}. \tag{3.9}
\end{align*}
\]
Here \( W_m^{1/2} := \left[\|W_m^{\text{real}}\|^2 + \|W_m^{\text{img}}\|^2\right]^{1/2} = \|W_m\|_2 \), for \( W = X, Y \) and \( m = 1, \ldots, M \). Consider the second term in the left-hand side of (3.8). Using the mean value theorem for harmonic functions we deduce
\[
\begin{align*}
\{B_n^{\text{real}}CX^{\text{real}}, CX^{\text{real}}\} &= \sum_{1 \leq j, m \leq M, j \neq m} X_m^{\text{real}} \hat{C}_m \left[\Gamma_m(x_m, z_j)\right]^{\text{real}} \hat{C}_j X_j^{\text{real}} \\
&\geq t \sum_{1 \leq j, m \leq M, j \neq m} X_m^{\text{real}} \hat{C}_m \left(\frac{1}{|B(j)||B(m)|}\right) \int_{B(j)} \Phi_0(x, y)dy \, dxdy \hat{C}_j X_j^{\text{real}},
\end{align*}
\]
Similarly, if we consider the fourth term in the left-hand side of (3.8), we deduce
\[
\begin{align*}
\{B_n^{\text{real}}CX^{\text{img}}, CX^{\text{img}}\} &= \sum_{1 \leq j, m \leq M, j \neq m} X_m^{\text{img}} \hat{C}_m \left[\Gamma_m(x_m, z_j)\right]^{\text{real}} \hat{C}_j X_j^{\text{img}} \\
&\geq t \sum_{1 \leq j, m \leq M, j \neq m} X_m^{\text{img}} \hat{C}_m \left(\frac{1}{|B(j)||B(m)|}\right) \int_{B(j)} \Phi_0(x, y)dy \, dxdy \hat{C}_j X_j^{\text{img}},
\end{align*}
\]
where

$$t := \left[ \frac{1}{c_p^2} - 2\text{diam}(\Omega) \frac{\omega}{c_s^3} \left( 1 - \left( \frac{1}{2} \kappa \text{diam}(\Omega) \right)^{N_\Omega} \right) + \frac{1}{2^{N_\Omega + 1}} \right]$$

assumed to be positive. $\Phi_0(x, y) := 1/(4\pi|x - y|), x \neq y$ and $B^{(m)} := \{x : |x - z_m| < d/2\}, m = 1, \ldots, M$, are non-overlapping balls of radius $d/2$ with centers $z_m$, and $|B^{(m)}| = \pi d^3/6$ are the volumes of the balls. Also, we use the notation $B_d$ to denote the balls of radius $d/2$ with the center at the origin.

Indeed we can write $\Gamma^{\omega} (z_m, z_j)$ from (2.3) as,

$$\Gamma^{\omega} (z_m, z_j) = \frac{1}{4\pi |z_m - z_j|} \left[ \frac{1}{c_s^2} + \frac{1}{c_p^2} \right] \left( \frac{1}{c_s^2} - \frac{1}{c_p^2} \right) \left( \frac{1}{2} \frac{\omega^2 \text{diam}(\Omega)}{c_s^3} \right) (z_m - z_j) \otimes (z_m - z_j)$$

$$+ \sum_{l=1}^{\infty} \frac{i^l}{l!(l + 2) \omega^2} \left( (l + 1) \kappa_p^{l+2} + \kappa_p^{l+2} \right) |z_m - z_j|^{l+2} (z_m - z_j) \otimes (z_m - z_j)$$

$$- \sum_{l=1}^{\infty} \frac{i^l}{l!(l + 2) \omega^2} \left( (l - 1) \kappa_p^{l+2} - \kappa_p^{l+2} \right) |z_m - z_j|^{l+2} (z_m - z_j) \otimes (z_m - z_j)$$

from which, we get the required result by estimating $\Gamma^{\omega} (z_m, z_j)$. Notice that

$$|b_r| \leq \frac{1}{2} \left[ \frac{1}{c_s^2} \right]$$

and

$$|c_1 \Gamma + c_2 \Gamma| \leq \sum_{l=1}^{\infty} \frac{1}{(l - 1)!(l + 2) \omega^2} \left( 2\kappa_p^{l+2} + \kappa_p^{l+2} \right) |z_m - z_j|^{l+2}$$

[By recalling $N_\Omega = \left[ 2 \text{diam}(\Omega) \max \{\kappa, \kappa_p\} e^{2} \right]$ and using Lemma2.6]

$$\leq \text{diam}(\Omega) \left[ \frac{2\omega}{c_s^3} \left( \sum_{l=1}^{N_\Omega} \left( \frac{1}{2} \kappa \text{diam}(\Omega) \right)^{l-1} + \sum_{l=N_\Omega + 1}^{\infty} \frac{1}{2^{l-1}} \right) \right]$$

$$+ \frac{\omega}{c_p^3} \left( \sum_{l=1}^{N_\Omega} \left( \frac{1}{2} \kappa_p \text{diam}(\Omega) \right)^{l-1} + \sum_{l=N_\Omega + 1}^{\infty} \frac{1}{2^{l-1}} \right) \right]$$

$$= \text{diam}(\Omega) \left[ \frac{2\omega}{c_s^3} \left( \frac{1 - \left( \frac{1}{2} \kappa \text{diam}(\Omega) \right)^{N_\Omega}}{1 - \left( \frac{1}{2} \kappa \text{diam}(\Omega) \right) + \frac{1}{2^{N_\Omega + 1}}} \right) \right]$$

$$+ \frac{\omega}{c_p^3} \left( \frac{1 - \left( \frac{1}{2} \kappa_p \text{diam}(\Omega) \right)^{N_\Omega}}{1 - \left( \frac{1}{2} \kappa_p \text{diam}(\Omega) \right) + \frac{1}{2^{N_\Omega + 1}}} \right) \right] ,$$

Let $\Omega$ be a large ball with radius $R$. Also let $\Omega_s \subset \Omega$ be a ball with fixed radius $r (\leq R)$, which consists of all our small obstacles $D_m$ and also the balls $B^{(m)}$, for $m = 1, \ldots, M$. 

www.mn-journal.com
Let \( \gamma^{\text{real}}(x) \) and \( \gamma^{\text{img}}(x) \) be piecewise constant functions defined on \( \mathbb{R}^3 \) as

\[
\gamma^{\text{real}}(x) = \begin{cases} 
C_m X_m^{\text{real}}(x) & \text{in } B^{(m)}, \quad m = 1, \ldots, M, \\
0 & \text{otherwise.}
\end{cases}
\]  

(3.11)

Then

\[
\left\langle B_n^{\text{real}} C \gamma^{\text{real}}, C \gamma^{\text{real}} \right\rangle \geq \frac{36t}{\pi^2} \frac{d_0^6}{d_0^6} \left( \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(x) \gamma^{\text{real}}(y) \, dx \, dy \\
- \sum_{m=1}^{M} C_m X_m^{\text{real}} \int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x, y) \, dx \, dy \right),
\]  

(3.12)

\[
\left\langle B_n^{\text{img}} C \gamma^{\text{img}}, C \gamma^{\text{img}} \right\rangle \geq \frac{36t}{\pi^2} \frac{d_0^6}{d_0^6} \left( \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \gamma^{\text{img}}(x) \gamma^{\text{img}}(y) \, dx \, dy \\
- \sum_{m=1}^{M} C_m X_m^{\text{img}} \int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x, y) \, dx \, dy \right).
\]  

(3.13)

Applying the mean value theorem to the harmonic function \( \frac{1}{4\pi|x-y|} \), as done in [27], p. 109–110], we have the following estimate

\[
\int_{B^{(m)}} \int_{B^{(m)}} \Phi_0(x, y) \, dx \, dy = \frac{1}{4\pi} \int_{B_0} \int_{B_0} \frac{1}{|x-y|} \, dx \, dy \leq \frac{\pi d_0^6}{60}.
\]  

(3.14)

Consider the first term in the right-hand side of (3.12), denote it by \( A_R^{\text{real}} \), then by Green’s theorem

\[
A_R^{\text{real}} := \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(x) \gamma^{\text{real}}(y) \, dx \, dy \\
= \int_{\Omega} \left\{ \nabla_x \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right\}^T \left( \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right) \, dx \\
- \int_{\Omega} \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \left( \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right) \, dS_x.
\]  

(3.15)

We have

\[
C_R^{\text{real}} = \int_{\Omega} \left( \int_{\Omega} \frac{\partial}{\partial v_x} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right) \, dS_x \\
= \int_{\Omega} \left( \int_{\Omega} \frac{\partial}{\partial v_x} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \Phi_0(x, y) \gamma^{\text{real}}(y) \, dy \right) \, dS_x \\
= \int_{\Omega} \left( \int_{\Omega} \frac{-(x-y)}{4\pi|x-y|^3} \gamma^{\text{real}}(y) \, dy \right) \left( \int_{\Omega} \frac{1}{4\pi|x-y|^3} \gamma^{\text{real}}(y) \, dy \right) \, dS_x,
\]  

(3.16)

which gives the following estimate:

\[
|C_R^{\text{real}}| \leq \frac{1}{16\pi^2} \int_{\Omega} \int_{\Omega} \left| \gamma^{\text{real}}(y) \right| \left( \int_{\Omega} \left| \gamma^{\text{real}}(y) \right| \, dy \right)^2 \, dS_x
= \frac{1}{16\pi^2} \int_{\Omega} \left( \int_{\Omega} \left| \gamma^{\text{real}} \right|^2 \right) \, dS_x.
\]
Substitution of (3.17) in (3.15) gives
\[
\int_{\Omega} \int_{\Omega} \Phi_0(x, y) \Upsilon^{\operatorname{real}}(x) \Upsilon^{\operatorname{real}}(y) \, dx \, dy \\
\geq \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) \Upsilon^{\operatorname{real}}(y) \, dy \right|^2 \, dx - \frac{R^2 r^3}{3(R - r)^3} \sum_{m=1}^{M} \left| \tilde{C}_m X_m^{\operatorname{real}} \right|^2.
\] (3.18)

By considering the first term in the right-hand side of (3.13), and following the same procedure as mentioned in (3.15), (3.16) and (3.17), we obtain
\[
\int_{\Omega} \int_{\Omega} \Phi_0(x, y) \Upsilon^{\operatorname{img}}(x) \Upsilon^{\operatorname{img}}(y) \, dx \, dy \\
\geq \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) \Upsilon^{\operatorname{img}}(y) \, dy \right|^2 \, dx - \frac{R^2 r^3}{3(R - r)^3} \sum_{m=1}^{M} \left| \tilde{C}_m X_m^{\operatorname{img}} \right|^2.
\] (3.19)

Under our assumption \( t > 0 \), (3.12), (3.13), (3.14), (3.18) and (3.19) lead to
\[
\langle B_r^{\operatorname{real}} C X^{\operatorname{real}}, C X^{\operatorname{real}} \rangle \\
\geq \frac{36t}{\pi^3 \, d^3} \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) \Upsilon^{\operatorname{real}}(y) \, dy \right|^2 \, dx - \frac{R^2 r^3}{3(R - r)^3} + \frac{\pi \, d^5}{60} \sum_{m=1}^{M} \left| \tilde{C}_m X_m^{\operatorname{real}} \right|^2 \right),
\] (3.20)
\[
\langle B_r^{\operatorname{real}} C X^{\operatorname{img}}, C X^{\operatorname{img}} \rangle \\
\geq \frac{36t}{\pi^3 \, d^3} \left( \int_{\Omega} \left| \nabla_x \int_{\Omega} \Phi_0(x, y) \Upsilon^{\operatorname{img}}(y) \, dy \right|^2 \, dx - \frac{R^2 r^3}{3(R - r)^3} + \frac{\pi \, d^5}{60} \sum_{m=1}^{M} \left| \tilde{C}_m X_m^{\operatorname{img}} \right|^2 \right).
\] (3.21)

Then (3.8), (3.20) and (3.21) imply
\[
\left( \min_{m=1}^{M} \lambda_{m_{\text{el}}}^{\min} - \frac{36t}{\pi^2 \, d^5} \left[ \frac{R^2 r^3}{3(R - r)^3} + \frac{\pi \, d^5}{60} \right] \max_{m=1}^{M} \lambda_{m_{\text{el}}}^{\max} \right) \sum_{m=1}^{M} \left\| X_m \right\|_2^2 \\
\leq 2 \left( \max_{m=1}^{M} \lambda_{m_{\text{el}}}^{\max} \right) \left( \sum_{m=1}^{M} \left\| X_m \right\|_2^2 \right)^{1/2} \left( \sum_{m=1}^{M} \left\| Y_m \right\|_2^2 \right)^{1/2}.
\] (3.22)

As we have \( R \) arbitrary, by tending \( R \) to \( \infty \), we can write (3.22) as
\[
\left( \min_{m=1}^{M} \lambda_{m_{\text{el}}}^{\min} - \frac{3t}{5 \pi \, d} \max_{m=1}^{M} \lambda_{m_{\text{el}}}^{\max} \right) \sum_{m=1}^{M} \left\| X_m \right\|_2^2 \\
\leq 2 \left( \max_{m=1}^{M} \lambda_{m_{\text{el}}}^{\max} \right) \left[ \left( \sum_{m=1}^{M} \left\| X_m \right\|_2^2 \right)^{1/2} \left( \sum_{m=1}^{M} \left\| Y_m \right\|_2^2 \right)^{1/2} \right].
\] (3.23)

which yields
\[
\sum_{m=1}^{M} \left\| X_m \right\|_2^2 \leq 4 \left( \min_{m=1}^{M} \lambda_{m_{\text{el}}}^{\min} - \frac{3t}{5 \pi \, d} \max_{m=1}^{M} \lambda_{m_{\text{el}}}^{\max} \right)^{-2} \left( \max_{m=1}^{M} \lambda_{m_{\text{el}}}^{\max} \right)^2 \sum_{m=1}^{M} \left\| Y_m \right\|_2^2.
\] (3.24)
Thus, if \( \left( \max_{1 \leq m \leq M} \lambda_{\text{max}}^{a} \right) < t^{-1} \left( \frac{2\pi}{\mu} d \min_{1 \leq m \leq M} \lambda_{\text{min}}^{a} \right) \) then the matrix \( B \) in algebraic system (2.100) is invertible and the estimate (3.23) and so (3.2) holds.

Corollary 3.3 If \( (\lambda + 2\mu)^{2} \left( \max_{1 \leq m \leq M} C_{a}^{m} \right)^{2} < t^{-1} \left( \frac{2\pi}{\mu} d \min_{1 \leq m \leq M} C_{a}^{m} \right) \), then the matrix \( B \) is invertible and the solution vector \( \tilde{Q} \) of (2.100) satisfies the estimate:

\[
\sum_{m=1}^{M} \| \tilde{Q}_{m} \|_{2} \leq 2 \left( 1 - \frac{3t}{5\pi} \frac{\lambda_{\text{max}}^{a} - \frac{2}{\mu} \max_{m=1}^{M} C_{a}^{m}}{d \min_{m=1}^{M} C_{a}^{m}} \right)^{-1} \left( \frac{M}{\max_{m=1}^{M} C_{a}^{m}} \right) M \max_{m=1}^{M} \| U^{\dagger}(z_{m}) \|_{2} \lambda_{\text{max}}^{a} \max_{m=1}^{M} \| U^{\dagger}(z_{m}) \|_{2} .
\] (3.25)

Proof of Corollary 3.3 Let us assume the condition \( (\lambda + 2\mu)^{2} \left( \max_{1 \leq m \leq M} C_{a}^{m} \right)^{2} < t^{-1} \left( \frac{2\pi}{\mu} d \min_{1 \leq m \leq M} C_{a}^{m} \right) \), then from Lemma 3.1 the sufficient condition of Lemma 3.2 is satisfied and hence (3.2) holds. Now, by applying the norm inequalities to (3.2), we obtain

\[
\sum_{m=1}^{M} \| \tilde{Q}_{m} \|_{2} \leq 2 \left( \frac{\min_{m=1}^{M} \lambda_{\text{max}}^{a} - \frac{3t}{5\pi} \max_{m=1}^{M} \lambda_{\text{max}}^{a} }{d \min_{m=1}^{M} C_{a}^{m}} \right)^{-1} \left( \frac{M}{\max_{m=1}^{M} C_{a}^{m}} \right) M \max_{m=1}^{M} \| U^{\dagger}(z_{m}) \|_{2} \lambda_{\text{max}}^{a} \max_{m=1}^{M} \| U^{\dagger}(z_{m}) \|_{2} .
\] (3.26)

Now, again by applying Lemma 3.1 to the above inequality (3.26) gives the result (3.25).

A Appendix

The object of this section is to derive some used properties of the single layer operator \( S_{D_{\epsilon}} : L^{2}(\partial D_{\epsilon}) \rightarrow H^{1}(\partial D_{\epsilon}) \) defined by

\[
(S_{D_{\epsilon}} \psi)(x) := \int_{\partial D_{\epsilon}} \Gamma_{\epsilon_{\omega}}(x, y) \psi(y) \, dy .
\] (A.1)

Lemma A.1 There exists \( \epsilon_{0} \) such that if \( \epsilon < \epsilon_{0} \) then the operator \( S_{D_{\epsilon}} \) is invertible.

Proof of Lemma A.1 Proof of this Lemma follows as the one of Proposition 2.1.

Lemma A.2 Let \( \phi \in H^{1}(\partial D_{\epsilon}) \) and \( \psi \in L^{2}(\partial D_{\epsilon}) \). Then,

\[
S_{D_{\epsilon}} \psi = \epsilon \left( S_{D}^{\epsilon} \psi \right)^{\dagger} ,
\] (A.2)

\[
S_{D_{\epsilon}}^{-1} \phi = \epsilon^{-1} \left( S_{D}^{\epsilon^{-1}} \phi \right)^{\dagger}
\] (A.3)

and

\[
\| S_{D_{\epsilon}} \|_{L(H^{1}(\partial D_{\epsilon}), L^{2}(\partial D_{\epsilon}))} \leq \epsilon^{-1} \| S_{D}^{\epsilon^{-1}} \|_{L(H^{1}(\partial B), L^{2}(\partial B))}
\] (A.4)

with \( S_{D}^{\epsilon}_{\psi} \hat{\psi}(\xi) := \int_{\partial B} \Gamma_{\epsilon_{\omega}}(\xi, \eta) \hat{\psi}(\eta) \, d\eta \).

Proof of Lemma A.2

- We have,

\[
S_{D_{\epsilon}} \psi(x) = \int_{\partial D_{\epsilon}} \Gamma_{\epsilon_{\omega}}(x, y) \psi(y) \, dy
\]

\[
= \int_{\partial B} \frac{1}{\epsilon} \Gamma_{\epsilon_{\omega}}(\xi, \eta) \psi(\epsilon \eta + z) \epsilon^{2} d\eta
\]

\[
= \epsilon S_{D}^{\epsilon}_{\psi}(\xi).
\]

The above gives us (A.2).
• The following equalities, using (A.2),

\[ S_{D_1} \left( S_{B_1}^{-1} \phi \right) = \epsilon \left( S_{B_1} S_{B_1}^{-1} \phi \right) = \epsilon \phi \]

provides us (A.3).

• We have from the estimate,

\[ \| S_{D_1}^{-1} \|_{L^2(\partial D_1)} \leq \frac{\| S_{D_1}^{-1} \phi \|_{L^2(\partial B)}}{\| \phi \|_{H^1(\partial D_1)}} \leq \frac{\epsilon \| S_{D_1}^{-1} \phi \|_{L^2(\partial B)}}{\| \phi \|_{H^1(\partial B)}} \]

\[ = \epsilon^{-1} \| S_{B_1}^{-1} \|_{L^2(\partial B)} \cdot \]

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References

[1] B. Ahmad, D. P. Challa, M. Kirane, and M. Sini, The equivalent refraction index for the acoustic scattering by many small obstacles: with error estimates, J. Math. Anal. Appl. 424(1), 563–583 (2015).

[2] F. Al-Musallam, D. P. Challa, and M. Sini, Location and size estimation of small rigid bodies using elastic far-fields, Preprint, arxiv:1412.0785.

[3] C. J. S. Alves and R. Kress, On the far-field operator in elastic obstacle scattering, IMA J. Appl. Math. 67(1), 1–21 (2002).

[4] H. Ammari, E. Bretin, J. Garnier, W. Jing, H. Kang, and A. Wahab, Localization, stability, and resolution of topological derivative based imaging functionals in elasticity, SIAM J. Imaging Sci. (2012).

[5] H. Ammari, P. Calmon, and E. Iakovleva, Direct elastic imaging of a small inclusion, SIAM J. Imaging Sci. 1(2), 169–187, 2008.

[6] H. Ammari, J. Garnier, W. Jing, H. Kang, M. Lim, K. Sølna, and H. Wang, Mathematical and Statistical Methods for Multistatic Imaging, Lecture Notes in Mathematics 2098 (Springer, Cham, 2013).

[7] H. Ammari and H. Kang, Polarization and Moment Tensors, Applied Mathematical Sciences 162 (Springer, New York, 2007). With applications to inverse problems and effective medium theory.

[8] H. Ammari, H. Kang, E. Kim, and M. Lim, Reconstruction of closely spaced small inclusions, SIAM J. Numer. Anal. 42(6), 2408–2428 (electronic) (2005).

[9] H. Ammari, H. Kang, and H. Lee, Asymptotic expansions for eigenvalues of the Lamé system in the presence of small inclusions, Comm. Partial Differential Equations 32(10-12), 1715–1736 (2007).

[10] H. Ammari, H. Kang, and M. Lim, Effective parameters of elastic composites, Indiana Univ. Math. J. 55(3), 903–922 (2006).

[11] H. Ammari, H. Kang, G. Nakamura, and K. Tanuma, Complete asymptotic expansions of solutions of the system of elastostatics in the presence of an inclusion of small diameter and detection of an inclusion, J. Elasticity 67(2), 97–129 (2002).

[12] H. Ammari and J. K. Seo, An accurate formula for the reconstruction of conductivity inhomogeneities, Adv. in Appl. Math. 30(4), 679–705 (2003).

[13] A. Bensoussan, J.-L. Lions, and G. Papanicolaou, Asymptotic Analysis for Periodic Structures, Studies in Mathematics and its Applications 5 (North-Holland Publishing Co., Amsterdam, 1978).

[14] M. Cassier and C. Hazard, Multiple scattering of acoustic waves by small sound-soft obstacles in two dimensions: mathematical justification of the Foldy-Lax model, Wave Motion 50(1), 18–28 (2013).

[15] D. P. Challa and M. Sini, Inverse scattering by point-like scatterers in the Foldy regime, Inverse Problems 28(12), 125006 (2012).
[16] D. P. Challa and M. Sini, On the justification of the Foldy-Lax approximation for the acoustic scattering by small rigid bodies of arbitrary shapes, Multiscale Model. Simul. 12(1), 55–108 (2014).

[17] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Applied Mathematical Sciences 93 (Springer-Verlag, Berlin, second edition, 1998).

[18] D. L. Colton and R. Kress, Integral equation methods in scattering theory, Pure and Applied Mathematics (New York) (John Wiley & Sons Inc., New York, 1983). A Wiley-Interscience Publication.

[19] G. Hu, J. Li, H. Liu, and H. Sun, Inverse elastic scattering for multiscale rigid bodies with a single far-field pattern, SIAM J. Imaging Sci. 7(3), 1799–1825 (2014).

[20] G. Hu and M. Sini, Elastic scattering by finitely many point-like obstacles, J. Math. Phys. 54(4), 042901 (2013).

[21] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, Homogenization of Differential Operators and Integral Functionals, (Springer-Verlag, Berlin, 1994).

[22] V. D. Kupradze, Potential Methods in the Theory of Elasticity, Translated from the Russian by H. Gutfreund. Translation edited by I. Meroz (Israel Program for Scientific Translations, Jerusalem, 1965).

[23] V. D. Kupradze, T. G. Gegelia, M. O. Basheleishvili, and T. V. Burchuladze, Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, North-Holland Series in Applied Mathematics and Mechanics 25 (North-Holland Publishing Co., Amsterdam, russian edition, 1979). Edited by V. D. Kupradze.

[24] V. A. Marchenko and E. Y. Khruslov, Homogenization of Partial Differential Equations, Progress in Mathematical Physics 46 (Birkhäuser Boston Inc., Boston, MA, 2006).

[25] P. A. Martin, Multiple Scattering of Encyclopedia of Mathematics and its Applications 107 (Cambridge University Press, Cambridge, 2006). Interaction of time-harmonic waves with N obstacles.

[26] S. Mayboroda and M. Mitrea, The Poisson problem for the Lamé system on low-dimensional Lipschitz domains, In Integral Methods in Science and Engineering (Birkhäuser Boston, Boston, MA, 2006), pp. 137–160.

[27] V. Maz’ya and A. Movchan, Asymptotic treatment of perforated domains without homogenization, Math. Nachr. 283(1), 104–125 (2010).

[28] V. Maz’ya, A. Movchan, and M. Nieves, Uniform asymptotic formulae for Green’s tensors in elastic singularly perturbed domains, Asymptot. Anal. 52(3-4), 173–206 (2007).

[29] V. Maz’ya, A. Movchan, and M. Nieves, Mesoscale asymptotic approximations to solutions of mixed boundary value problems in perforated domains, Multiscale Model. Simul. 9(1), 424–448 (2011).

[30] V. Maz’ya, A. Movchan, and M. Nieves, Green’s Kernels and Meso-Scale Approximations in Perforated Domains, of Lecture Notes in Mathematics 2077 (Springer-Verlag, Berlin, 2013).

[31] O. Mendez and M. Mitrea, The Banach envelopes of Besov and Triebel-Lizorkin spaces and applications to partial differential equations, J. Fourier Anal. Appl. 6(5), 503–531 (2000).

[32] D. Mitrea, The method of layer potentials for non-smooth domains with arbitrary topology, Integral Equations Operator Theory 29(3), 320–338 (1997).

[33] V. Namias, A simple derivation of Stirling’s asymptotic series, Amer. Math. Monthly 93(1), 25–29 (1986).

[34] S. A. Nazarov and J. Sokolowski, Self-adjoint extensions for the Neumann Laplacian and applications, Acta Math. Sin. (Engl. Ser.) 22(3), 879–906 (2006).

[35] A. G. Ramm, Wave Scattering by Small Bodies of Arbitrary Shapes (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005).

[36] A. G. Ramm, Many-body wave scattering by small bodies and applications, J. Math. Phys. 48(10), 103511 29 (2007).

[37] A. G. Ramm, Wave scattering by small bodies and creating materials with a desired refraction coefficient, Afr. Mat. 22(1), 33–55 (2011).