INTRODUCTION

Let $G$ be a semisimple algebraic group with Lie algebra $\mathfrak{g}$. We consider generalisations of Lusztig’s $q$-analogue of weight multiplicity. Fix a maximal torus $T \subset G$. Let $m_\lambda^\mu$ be the multiplicity of weight $\mu$ in a simple $G$-module $V_\lambda$ with highest weight $\lambda$. Lusztig’s $q$-analogues $m_\lambda^\mu(q)$ (also known as Kostka-Foulkes polynomials for the root system of $G$) are certain polynomials in $q$ such that $m_\lambda^\mu(1) = m_\lambda^\mu$. A recent survey of their properties, with an eye towards combinatorics, is given in [19]. These polynomials arise in numerous problems of representation theory, geometry, and combinatorics. Work of Lusztig [16] and Kato [12] shows that, for $\lambda$ and $\mu$ dominant, $m_\lambda^\mu(q)$ are connected with certain Kazhdan-Lusztig polynomials for the affine Weyl group associated with $G$. To define $m_\lambda^\mu(q)$, one first considers a $q$-analogue of Kostant’s partition function, $P$. It is conceivable to replace the set of positive roots, $\Delta^+$, occurring in the definition of $P$ with an arbitrary finite multiset $\Psi$ in the character group $X$ of $T$. If the elements of $\Psi$ belong to an open half-space of $X \otimes \mathbb{Q}$ (this is our first hypothesis on $\Psi$), then we still obtain certain polynomials $m_{\lambda, \psi}^\mu(q)$. We always assume that $\lambda$ is dominant, whereas $\mu \in X$ can be arbitrary. In this article, we are interested in the non-negativity problem for the coefficients of $m_{\lambda, \psi}^\mu(q)$. For Lusztig’s $q$-analogues, this problem has been considered by Broer. He proved that $m_\lambda^\mu(q)$
has non-negative coefficients for any $\lambda \in \mathfrak{X}_+$ if and only if $(\mu, \alpha^\vee) \geq -1$ for all $\alpha \in \Delta^+$ (see [1, Theorem 2.4] and [4, Prop. 2(iii)]).

Our first goal is to provide sufficient conditions for $m_{\lambda, \Psi}^\mu(q)$ to have non-negative coefficients. Let $B$ be the Borel subgroup of $G$ corresponding to $\Delta^+$ (i.e., the roots of $B$ are positive!) and $\mathfrak{X}_+$ the set of dominant weights. The second hypothesis is that $\Psi$ is assumed to be the multiset of weights for a $B$-submodule $N$ of a $G$-module $V$. Then $m_{\lambda, \Psi}^\mu(q)$ is said to be a generalised Kostka-Foulkes polynomial. Let $P \supset B$ be any parabolic subgroup normalising $N$ and $G \times_P N$ the corresponding homogeneous vector bundle on $G/P$. We obtain a relation between the Euler characteristic of induced line bundles $L$ on the $G \times_P N$ and generalised Kostka-Foulkes polynomials. Using the collapsing $G \times_P N \to G \cdot N \subset V$, we get a vanishing result for $H^i(G \times_P N, L)$, $i \geq 1$, and conclude that $m_{\lambda, \Psi}^\mu(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$ if $\mu$ is sufficiently large. An explicit lower bound for $\mu$ is also given, see Section 3. This approach is based on the Grauert-Riemenschneider vanishing theorem. We also notice that Broer’s formula for $\frac{d}{dq} m_{\lambda}^\mu(q)$ [3] can be generalised to $m_{\lambda, \Psi}^\mu(q)$. The most natural examples of generalised Kostka-Foulkes polynomials occur if $\Psi \subset \Delta^+$. For instance, one can take $N$ to be a $B$-stable ideal in $\text{Lie}(B, B) \subset \mathfrak{g}$.

Our second goal is to study in details the special case in which $\Psi = \Delta^+_s$, the set of short positive roots. The required $B$-submodule, $V_\theta^+$, lies in $V_\theta$, where $\theta$ is the short dominant root. The polynomials $\overline{m}_{\lambda}^\mu(q) := m_{\lambda, \Delta^+_s}^\mu(q)$ are said to be short $q$-analogues. The numbers $\overline{m}_{\lambda}^\mu(1)$ appeared already in work of Heckman [8], and a geometric interpretation of $\overline{m}_{\lambda}^\mu(q)$ given in [24] shows that $\overline{m}_{\lambda}^\mu(q)$ have non-negative coefficients. Let $\Delta^+_l$ be the set of long positive roots, $W_l$ the (normal) subgroup of $W$ generated by all $s_\alpha$ ($\alpha \in \Delta^+_s$), and $\rho_l$ the half-sum of the long positive roots. Approach of Section 3 enables us to prove that $\overline{m}_{\lambda}^\mu(q)$ has nonnegative coefficients whenever $\mu + \rho_l \in \mathfrak{X}_+$ (Cor. 4.3). But to obtain exhaustive results, we take another path. We consider the shifted (= dot) action of $W_l$ on $\mathfrak{X}$, $(w, \mu) \mapsto w \circlearrowleft \mu = w(\mu + \rho_l) - \rho_l$, and show that $\overline{m}_{\lambda}^{\mu \circlearrowleft}(q) = (-1)^{\ell(w)} \overline{m}_{\lambda}^\mu(q)$. Therefore $\overline{m}_{\lambda}^\mu(q) \equiv 0$ if $\mu$ is not regular relative to the shifted $W_l$-action, and it suffices to consider $\overline{m}_{\lambda}^\mu(q)$ only for $\mu$ that are dominant with respect to $\Delta^+_l$. For a $\Delta^+_l$-dominant $\mu$, we prove that $\overline{m}_{\lambda}^\mu(q)$ has non-negative coefficients for all $\lambda \in \mathfrak{X}_+$ if and only if $(\mu, \alpha^\vee) \geq -1$ for all $\alpha \in \Delta^+_s$, see Theorem 4.10. This is an extension of Broer’s results in [1, Sect. 2]. Again, this stems from a careful study of cohomology of line bundles on $G \times_B V_\theta^+$. In these considerations, it is important that $W$ is a semi-direct product $W(\Pi_s) \ltimes W_l$, where the first group is generated by the short simple reflections. Modifying approach of R. Gupta [6], we define analogues of Hall-Littlewood polynomials (Section 5). These polynomials in $q$, denoted $\overline{T}_{\lambda}(q)$, are indexed by $\lambda \in \mathfrak{X}_+$ and form a $\mathbb{Z}$-basis for the $q$-extended character ring $\Lambda[q]$ of $G$. Let $\chi_{\lambda}$ be the character of $V_{\lambda}$ and $H$ the connected semisimple subgroup of $G$ whose root system is $\Delta_l$. The polynomials $\overline{T}_{\lambda}(q)$ interpolate between $\chi_{\lambda}$ (at $q = 0$) and a certain sum of irreducible characters of $H$ (at $q = 1$). We obtain some orthogonality relations
for $P_{\lambda}(q)$ and show that $\chi_\lambda = \sum_{\mu \in \mathcal{X}_+} m_{\lambda\mu}^\mu P_{\mu}(q)$. Moreover, the whole theory developed by R. Gupta in [6, 7] can be extended to this setting. For instance, we prove a version of Kato’s identity [12, 1.3] and point out a scalar product in $\Lambda[q]$ such that $\{P_{\lambda}(q)\}_{\lambda \in \mathcal{X}_+}$ to be an orthogonal basis. In a sense, the reason for such an extension is that $G \cdot V_{\theta} =: \mathcal{N}(V_{\theta})$ is the null-cone in $V_{\theta}$, and, as well as the nilpotent cone $\mathcal{N} \subset \mathfrak{g}$, this variety is an irreducible normal complete intersection. On the other hand, Theorem 4.10 yields vanishing of higher cohomology of the structure sheaf $\mathcal{O}_{G \times_B V_{\theta}^+}$, and, together with [15], this implies that $\mathcal{N}(V_{\theta})$ has only rational singularities.

We conjecture that if $\mu$ satisfies vanishing conditions of Theorem 4.10, then $m_{\lambda\mu}^\mu(q)$ can be interpreted as the ”jump polynomial” associated with a filtration of a subspace of $V_{\lambda}^\mu$, see Subsection 6.3. This is inspired by [5].

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1. Notation

Let $G$ be a connected semisimple algebraic group of rank $r$, with a fixed Borel subgroup $B$ and a maximal torus $T \subset B$. The corresponding triangular decomposition of $\mathfrak{g} = \text{Lie}(G)$ is $\mathfrak{g} = u^- \oplus \mathfrak{t} \oplus u$ and $\mathfrak{b} = \mathfrak{t} \oplus u$. The character group of $T$ is denoted by $\mathcal{X}$. Let $\Delta$ be the root system of $(G, T)$. Then $B$ determines the set of positive roots $\Delta^+$ and the monoid of dominant weights $\mathcal{X}_+$.

- $\Pi$ is the set of simple roots in $\Delta^+$;
- $\varphi_1, \ldots, \varphi_r$ are the fundamental weights in $\mathcal{X}_+$.

Write $W$ for the Weyl group and $s_\alpha$ for the reflection corresponding to $\alpha \in \Delta^+$. Set $N(w) = \{\alpha \in \Delta^+ \mid w\alpha \in -\Delta^+\}$ and $\varepsilon(w) = (-1)^{\ell(w)}$, where $\ell(w) = \#N(w)$ is the usual length function on $W$. For $\mu \in \mathcal{X}$, let $\mu^+$ denote the unique dominant element in $W\mu$. We fix a $W$-invariant scalar product $(\ ,\ )$ on $\mathcal{X} \otimes_\mathbb{Z} \mathbb{Q}$. As usual, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ for $\alpha \in \Delta$. For any $\lambda \in \mathcal{X}_+$, we choose a simple highest weight module $V_\lambda$; $V_{\lambda}^\mu$ is the $\mu$-weight space in $V_\lambda$ and $m_{\lambda\mu} = \text{dim } V_{\lambda}^\mu$.

We consider two partial orders in $\mathcal{X}$. For $\mu, \nu \in \mathcal{X}$,

- the root order is defined by letting $\mu \preceq \nu$ if and only if $\nu - \mu$ lies in the monoid generated by $\Delta^+$; notation $\mu \prec \nu$ means that $\mu \preceq \nu$ and $\mu \neq \nu$;
- the dominant order is defined by letting $\mu \preceq \nu$ if and only if $\nu - \mu \in \mathcal{X}_+$.

If $\Psi$ is a finite multiset in $\mathcal{X}$, then $|\Psi|$ is the sum of all elements of $\Psi$ (with respective multiplicities). Recall that $|\Delta^+|/2 = \varphi_1 + \ldots + \varphi_r$, and this quantity is denoted by $\rho$.

Let $P$ be a parabolic subgroup of $G$. For a $P$-module $N$, let $G \times_P N$ denote the homogeneous $G$-vector bundle on $G/P$ whose fibre over $\{P\} \in G/P$ is $N$; we write $L_{G/P}(V)$ for
the locally free $\mathcal{O}_{G/P}$-module of its sections. If $N$ is a submodule of a $G$-module, then the natural morphism $f : G \times_P N \to G \cdot N$ is projective and $G$-equivariant. It is a collapsing in the sense of Kempf [13]. Recall that $G \cdot N$ is a closed subvariety of $V$, since $N$ is $P$-stable. If $\dim G \times_P N = \dim G \cdot N$, then $f$ is said to be generically finite. If $N'$ is another $P$-module, then $G \times_P (N \oplus N')$ is a vector bundle on $G \times_P N$ with sheaf of sections $L_{G \times_P N}(N')$.

For any graded $G$-module $\mathcal{E} = \oplus_j \mathcal{E}_j$ with $\dim \mathcal{E}_j < \infty$, its $G$-Hilbert series is defined by

$$H_G(\mathcal{E}; q) = \sum_j \sum_{\lambda \in \mathcal{X}_+} \dim \text{Hom}_G(V_\lambda, \mathcal{E}_j) q^\lambda \in \mathbb{Z}[\mathcal{X}][[q]].$$

2. MAIN DEFINITIONS AND FIRST PROPERTIES

Let $V$ be a finite-dimensional rational $G$-module and $N$ a $P$-stable subspace of $V$. We assume that the $T$-weights occurring in $N$ lie in an open half-space of $\mathcal{X} \otimes \mathbb{Q}$. (This hypothesis implies that all $v \in N$ are unstable vectors in the sense of Geometric Invariant Theory.) Counting each $T$-weight according to its multiplicity in $N$, we get a finite multiset $\Psi$ in $\mathcal{X}$. The generalised partition function, $P_{\Psi}$, is defined by the series

$$\frac{1}{\prod_{\alpha \in \Psi} (1 - e^\alpha)} = \sum_{\nu} P_{\Psi}(\nu) e^\nu.$$

Accordingly, its $q$-analogue is defined by

$$\frac{1}{\prod_{\alpha \in \Psi} (1 - q e^\alpha)} = \sum_{\nu} P_{\Psi,q}(\nu) e^\nu.$$

In view of our assumption on $N$, the numbers $P_{\Psi}(\nu)$ are well-defined, and $P_{\Psi,q}(\nu)$ is a polynomial in $q$, with non-negative integer coefficients. Clearly, $P_{\Psi,q}(\nu)$ counts the “graded occurrences” of $\nu$ in the symmetric algebra $S^*(N)$. That is, $[q^\nu]P_{\Psi,q}(\nu) = \dim (S^j N)^\nu$.

For $\lambda \in \mathcal{X}_+$ and $\mu \in \mathcal{X}$, define the polynomials $m_{\lambda, \Psi}(q)$ by

$$m_{\lambda, \Psi}(q) = \sum_{w \in W} \varepsilon(w) P_{\Psi,q}(w(\lambda + \rho) - (\mu + \rho)).$$

This definition makes sense for any multiset $\Psi$. But we require that our $\Psi$ to be always the multiset of weights of a $P$-submodule of a $G$-module, since we are going to exploit geometric methods.

For $N = u \subset \mathfrak{g}$ and $\Psi = \Delta^+$, one obtains Lusztig’s $q$-analogue of weight multiplicity [16] (= Kostka-Foulkes polynomials for $\Delta$), and $m_{\lambda, \Delta^+}(1) = m_{\lambda}^\mu$. Therefore, $m_{\lambda, \Psi}(q)$ is said to be a $(\Psi, q)$-analogue of weight multiplicity or generalised Kostka-Foulkes polynomial. If $\Psi = \Delta^+$, we will omit the subscript $\Delta^+$ in previous formulae.

As $m_{\lambda, \Psi}(q)$ is a polynomial in $q$, one might be interested in its derivative. For $\Psi = \Delta^+$, a nice formula for $\frac{d}{dq} m_{\lambda}(q)$ is found by Broer [3, p.394]. We notice that his method works in general, and it is more natural to begin with a formula for the derivative of $P_{\Psi,q}(\nu)$. 
Theorem 2.1. \( \frac{d}{dq} \mathcal{P}_{\Psi,q}(\nu) = \sum_{\gamma \in \Psi} \sum_{n \geq 1} q^{n-1} \mathcal{P}_{\Psi,q}(\nu - n\gamma). \)

Proof. The derivative \( \frac{d}{dq} \mathcal{P}_{\Psi,q}(\nu) \) equals the coefficient of \( t \) in the expansion of \( \mathcal{P}_{\Psi,q+t}(\nu) \). Let the polynomials \( \mathcal{R}_{n,\mu}(q) \) be defined by the generating function

\[
\prod_{\alpha \in \Psi} \frac{1 - qe^{\alpha}}{1 - (q + t)e^{\alpha}} = \sum_{\nu} \mathcal{P}_{\Psi,q+t}(\nu)e^{\nu} = \sum_{\mu} \sum_{n \geq 0} \mathcal{R}_{n,\mu}(q)e^{\mu}t^n.
\]

It is easy to compute these polynomials for \( n = 0,1 \). First, taking \( t = 0 \), we obtain \( \sum_{\mu} \mathcal{R}_{0,\mu}(q)e^{\mu} = 1 \). Second, we have

\[
\sum_{\mu} \mathcal{R}_{1,\mu}(q)e^{\mu} = \left[ \prod_{\alpha \in \Psi} \frac{1 - qe^{\alpha}}{1 - (q + t)e^{\alpha}} \right]_{t=0} = \sum_{\alpha \in \Psi} e^{\alpha} = \sum_{\alpha \in \Psi} \sum_{n \geq 1} q^{n-1} e^{n\alpha}.
\]

Hence \( \mathcal{R}_{1,\mu}(q) = \begin{cases} q^{n-1} & \text{if } \mu = n\alpha, \alpha \in \Psi \\ 0, & \text{otherwise.} \end{cases} \)

Next, \( \sum_{\nu} \mathcal{P}_{\Psi,q+t}(\nu)e^{\nu} = \sum_{n,\mu,\gamma} \mathcal{R}_{n,\mu}(q)\mathcal{P}_{\Psi,q}(\gamma)e^{\mu+\gamma}t^n. \) Hence

\[
\mathcal{P}_{\Psi,q+t}(\nu)e^{\nu} = \sum_{n,\mu} \mathcal{R}_{n,\mu}(q)\mathcal{P}_{\Psi,q}(\nu - \mu)t^n,
\]

and extracting the coefficient of \( t \) we get the assertion. \( \square \)

Corollary 2.2. \( \frac{d}{dq} m_{\lambda,\Psi}^\mu(q) = \sum_{\gamma \in \Psi} \sum_{n \geq 1} q^{n-1} m_{\lambda,\Psi}^{\mu+n\gamma}(q). \)

It would be nice to have a formula for the degree of these polynomials and necessary conditions for \( m_{\lambda,\Psi}^\mu(q) \) to be nonzero. For Lusztig’s \( q \)-analogues, it is easily seen that \( m_{\lambda}^\mu(q) \neq 0 \) if and only if \( \mu \preceq \lambda \), and \( \deg m_{\lambda}^\mu(q) = \text{ht}(\lambda - \mu) \). However, if \( \Psi \) is arbitrary, i.e., there is no relation between \( \Delta^+ \) and \( \Psi \), then it is impossible to compare the degrees of different summands in Equation (2.1). The only general assertion we can prove concerns the case in which \( \Psi \subset \Delta^+ \).

Lemma 2.3. Suppose that \( \Psi \subset \Delta^+ \). Then \( m_{\lambda,\Psi}^\lambda(q) = 1 \) if and only if \( m_{\lambda,\Psi}^\mu(q) \neq 0 \), then \( \mu \preceq \lambda \).

Note that if \( m_{\lambda,\Psi}^\mu(q) \neq 0 \), then it is not necessarily true that \( \lambda - \mu \) lies in the monoid generated by \( \Psi \).

3. Cohomology of line bundles and generalised Kostka-Foulkes polynomials

3.1. Statement of main results. We assume that \( P \supset B \) and choose a Levi subgroup \( L \subset P \) such that \( L \supset T \). Write \( n \) for the nilpotent radical of \( \mathfrak{p} = \text{Lie}(P) \), and \( \Delta(n) \) for the roots of \( n \); hence \( \Delta(n) \subset \Delta^+ \). Let \( \mathfrak{X}^P \) denote the character group of \( P \). Obviously, \( \mathfrak{X}^P \) is
the character group of the central torus in $L$, and we may identify $X^P$ with a subgroup of $X$. Then $X^P_+ = X_+ \cap X^P$ is the monoid of $P$-dominant weights, i.e., the dominant weights $\lambda$ such that $P$ stabilises a nonzero line in $V_\lambda$. Let $\rho_P$ be the sum of those fundamental weights that belong to $X^P_+$.

In this section, we prove the following two theorems:

**Theorem 3.1.** Set $Z = G \times_P N$. For $\mu \in X^P$, let $L_{\mu}^\ast$ be the dual of the sheaf of sections of the line bundle $G \times_P (N \oplus \mathbb{C}_\mu) \to Z$. Then

(i) $H^i(Z, L_{Z}(\mu)^\ast) = 0$ for all $i \geq 1$ whenever $\mu \geq \rho_P + |\Psi| - |\Delta(n)|$.

(ii) If the collapsing $Z \to G \cdot N$ is generically finite, then $H^i(Z, L_{Z}(\mu)^\ast) = 0$ for all $i \geq 1$ whenever $\mu \geq |\Psi| - |\Delta(n)|$.

**Theorem 3.2.** Suppose $N$ is $P$-stable and $\mu \in X^P$.

(i) If $\mu \geq \rho_P + |\Psi| - |\Delta(n)|$, then $m_{\lambda, \Psi}^\mu(q)$ has non-negative coefficients for any $\lambda \in X_+$.

(ii) If the collapsing $G \times_P N \to G \cdot N$ is generically finite, then $m_{\lambda, \Psi}^\mu(q)$ has non-negative coefficients for any $\lambda \in X_+$ whenever $\mu \geq |\Psi| - |\Delta(n)|$.

(Note that $|\Psi|, |\Delta(n)| \in X^P$. Hence both inequalities concern weights lying in $X^P$.)

Actually, Theorem 3.2 follows from Theorem 3.1 and a relation between $(\Psi, q)$-analogues and cohomology of line bundles, see Theorem 3.9 below. Such an approach to $(\Psi, q)$-analogues is inspired by work of Broer [1, 2].

### 3.2. Algebraic-geometric facts

For future reference, we recall some standard results in the form that we need below. Let $U$ be the total space of a line bundle on an algebraic variety $Z$ and $\pi: U \to Z$ be the corresponding projection. If $E$ is a locally free $O_Z$-module, then $E^\ast$ is its dual.

**Lemma 3.3.** Let $F$ be the sheaf of sections of $\pi$.

(i) If $L$ is a locally free $O_Z$-module of finite type, then $\pi_+(\pi^* L) = \bigoplus_{n \geq 0} (L \otimes (F^\otimes n)^\ast)$.

(ii) If $G$ is a quasi-coherent sheaf on $U$, then $H^i(U, G) = H^i(Z, \pi_+ G)$ for all $i$.

**Proof.** (i) Use the “projection formula” and the equality $\pi_+(O_U) = \bigoplus_{n \geq 0} (F^\otimes n)^\ast$.

(ii) This is true because $\pi$ is an affine morphism. \qed

Thus, vanishing of higher cohomology for $\pi^* L$ will imply that for $L \otimes (F^\otimes n)^\ast$ for all $n \geq 0$. The following is a special case of the Grauert–Riemenschneider theorem in Kempf’s version ([13, Theorem 4]):

**Theorem 3.4.** Let $\omega_U$ denote the canonical bundle on $U$. Suppose there is a proper generically finite morphism $U \to X$ onto an affine variety $X$. Then $H^i(U, \omega_U) = 0$ for all $i \geq 1$. 
3.3. **Proof of Theorem 3.1.** Recall that $N$ is a $P$-submodule of a $G$-module $V$, $\Psi$ is the corresponding multiset of weights, and $\Psi$ belongs to an open half-space of $X \otimes_{\mathbb{Z}} \mathbb{Q}$. Our goal is to obtain a sufficient condition for vanishing of higher cohomology of line bundles on $Z := G \times_P N$.

For $\mu \in X^+_P$, let $C_\mu$ denote the corresponding one-dimensional $P$-module. Consider $U = G \times_P (N \oplus C_\mu)$ with projections $\pi : U \to G \times_P N$ and $\kappa : U \to G/P$. Then $\pi$ makes $U$ the total space of a line bundle on $Z$. For simplicity, the sheaf of sections of this bundle is often denoted by $L_Z(\mu)$ in place of $L_Z(C_\mu)$. Note that $L_Z(\mu)^\star = L_Z(-\mu)$. We regard $C_\mu$ as the highest weight space in the $G$-module $V_\mu$. Therefore $U$ admits the collapsing into $V \oplus V_\mu$.

Since $U$ is the total space of a $G$-linearised vector bundle on $G/P$, the canonical bundle $\omega_U$ is a pull-back of a line bundle on $G/P$. The top exterior power of the cotangent space at $e * \tilde{n} \in U$ ($e \in G$ is the identity and $\tilde{n} \in \mathbb{N} \oplus C_\mu$) is

$$\bigwedge^{\text{top}}(g/p)^* \otimes \bigwedge^{\text{top}}N^* \otimes (C_\mu)^* = \bigwedge^{\text{top}}n \otimes \bigwedge^{\text{top}}(N) \otimes (C_\mu)^*.$$  

The corresponding character of $P$ is $\gamma - \mu$, where $\gamma := |\Delta(n)| - |\Psi|$. Therefore

$$\omega_U \simeq \kappa^* \left( L_{G/P}(\mathbb{C}_{\gamma - \mu}) \right) \simeq \pi^* \left( L_Z(\gamma - \mu) \right).$$  

By Lemma 3.3, we obtain $\pi^*(\omega_U) = \bigoplus_{n \geq 0} L_Z(\gamma - \mu) \otimes L_Z(n\mu)^*$ and hence

$$H^i(U, \omega_U) = \bigoplus_{n \geq 0} H^i(Z, L_Z((n+1)\mu - \gamma)^*).$$  

In order to apply Theorem 3.4, we need sufficient conditions for the collapsing $f_\mu : U \to G \cdot (N \oplus C_\mu)$ to be generically finite. There are two possibilities now.

**A) The collapsing $f : Z \to G \cdot N$ is generically finite.**

It is then easily seen that $f_\mu$ is generically finite for any $\mu \in X^+_P$. This yields the following vanishing result:

**Proposition 3.5.** If $f_0 : Z \to G \cdot N$ is generically finite and $\gamma = |\Delta(n)| - |\Psi|$, then

$$H^i(Z, L_Z((n+1)\mu - \gamma)^*) = 0$$  

for any $\mu \in X^+_P$ and all $n \geq 0$, $i \geq 1$. In particular, taking $n = 0$ and letting $\nu = \mu - \gamma$, we obtain

$$H^i(Z, L_Z(\nu)^*) = 0$$  

for all $i \geq 1$ if $\nu \in X^P$ is such that $\nu \succ |\Psi| - |\Delta(n)|$.  

B) The collapsing \( f : Z \to G \cdot N \) is not generically finite.

Here we have to correct the situation, i.e., choose \( \mu \) such that \( f_\mu \) to be generically finite.

Looking at the collapsing \( f_\mu : G \times_P (N \oplus \mathbb{C}_\mu) \to G \cdot (N \oplus \mathbb{C}_\mu) \) the other way around, we notice that if \( \psi_\mu : G \times_P \mathbb{C}_\mu \to G \cdot \mathbb{C}_\mu \subset V_\mu \) is generically finite, then so is \( f_\mu \). However, \( \psi_\mu \) is generically finite (in fact, birational) if and only if \( \mu \in \mathcal{X}_+^P \) is a \( P \)-regular dominant weight, i.e., \( \mu \gg \rho_P \). Equivalently, \( \mu = \tilde{\mu} + \rho_P \) for some \( \tilde{\mu} \in \mathcal{X}_+^P \).

This provides a weaker vanishing result that applies to arbitrary \( P \)-submodules.

**Proposition 3.6.** Let \( N \) be an arbitrary \( P \)-submodule. If \( \mu \in \mathcal{X}_+^P \) and \( \mu \gg \rho_P \), then
\[
H^i(Z, L^Z((n+1)\mu - \gamma)^*) = 0
\]
for all \( n \geq 0 \), \( i \geq 1 \). In particular, taking \( n = 0 \) and letting \( \nu = \mu - \gamma \), we obtain
\[
H^i(Z, L^Z(\nu)^*) = 0 \quad \text{for all} \quad i \geq 1
\]
whenever \( \nu \in \mathcal{X}_P \) and \( \nu \gg \rho_P + |\Psi| - |\Delta(n)| \).

Combining Propositions 3.5 and 3.6, we obtain Theorem 3.1.

**Remark 3.7.** The estimate in part B) is not optimal, because we do not actually need generic finiteness for \( \psi_\mu \). It can happen that both \( f \) and \( \psi_\mu \) are not generically finite, while \( f_\mu \) is. (See e.g. Theorem 4.2 below.)

3.4. **Proof of Theorem 3.2.** The cohomology groups of \( L^Z(\mu) = L_{G \times_P N}(\mu) \) have a natural structure of a graded \( G \)-module by
\[
H^i(G \times_P N, L_{G \times_P N}(\mu)) \simeq \bigoplus_{j=0}^{\infty} H^i(G/P, L_{G/P}(S^j N^* \otimes \mathbb{C}_\mu)),
\]
where \( S^j N^* \) is the \( j \)-th symmetric power of the dual of \( N \). Set \( H^i(\mu) := H^i(Z, L^Z(\mu)^*) \). It is a graded \( G \)-module with
\[
(H^i(\mu))_j = H^i(G/P, L_{G/P}(S^j N \otimes \mathbb{C}_\mu)^*).
\]
As \( \dim(H^i(\mu))_j < \infty \), the \( G \)-Hilbert series of \( H^i(\mu) \) is well-defined:
\[
\mathcal{H}_G(H^i(\mu); q) = \sum_j \sum_{\lambda \in \mathcal{X}_+} \dim \text{Hom}_G(V_\lambda, (H^i(\mu))_j) e^\lambda q^j \in \mathbb{Z}[\mathcal{X}][[q]].
\]
We also need the non-graded version of functor \( \mathcal{H}_G \). If \( M \) is a finite-dimensional \( G \)-module, then
\[
\mathcal{H}_G(M) = \sum_{\lambda \in \mathcal{X}_+} \dim \text{Hom}_G(V_\lambda, M) e^\lambda \in \mathbb{Z}[\mathcal{X}].
\]
This extends to virtual \( G \)-modules by linearity.
Assume for a while that \( P = B \), i.e., \( Z = G \times_B N \). By the Borel-Weil-Bott theorem for \( G/B \), we have

\[
H^i(G/B, \mathcal{L}_{G/B}(\mu)^*) = \begin{cases} 
V^*_\nu, & \text{if } \nu = w(\mu + \rho) - \rho \in \mathfrak{X}_+ \text{ and } \ell(w) = i. \\
0, & \text{otherwise}.
\end{cases}
\]

Using the non-graded functor \( \mathcal{H}_G \), one can also write

\[
(3.1) \quad \mathcal{H}_G(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\mu)^*)) = \begin{cases} 
\varepsilon(w)e^{\nu*}, & \text{if } \nu = w(\mu + \rho) - \rho \in \mathfrak{X}_+. \\
0, & \text{otherwise}.
\end{cases}
\]

The following result is well known in case of Lusztig's \( q \)-analogues, see e.g. [5, Lemma 6.1]. For convenience of the reader, we provide a proof of the general statement.

**Theorem 3.8.** For any \( \mu \in \mathfrak{X} \), we have

\[
\sum_i (-1)^i \mathcal{H}_G(H^i(G \times_B N, \mathcal{L}_{G \times_B N}(\mu)^*); q) = \sum_{\lambda \in \mathfrak{X}_+} m^\mu_\lambda(q)e^{\lambda*}.
\]

**Proof.** Each finite-dimensional \( B \)-module \( M \) has a \( B \)-filtration such that the associated graded \( B \)-module, denoted \( \widetilde{M} \), is completely reducible. Then

\[
\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(M)^*) = \sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}((\widetilde{M})^*)).
\]

We will apply this to the \( B \)-modules \( S^j N \otimes C_\mu, j = 0, 1, \ldots \)

\[
\sum_i (-1)^i \mathcal{H}_G(H^i(G \times_B N, \mathcal{L}_{G \times_B N}(\mu)^*); q)
\]

\[
= \sum_{j=0}^{\infty} \mathcal{H}_G(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(S^j N \otimes C_\mu)^*); q)
\]

\[
= \sum_{j=0}^{\infty} \mathcal{H}_G(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\widetilde{S^j N} \otimes C_\mu)^*); q)
\]

\[
= \sum_{j=0}^{\infty} \sum_{\nu \vdash S^j N} \dim(S^j N)^\nu q^j \mathcal{H}_G(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\nu + \mu)^*))
\]

\[
= \sum_{\nu \vdash S^j N} \mathcal{P}_{\Psi,q}(\nu) \mathcal{H}_G(\sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\nu + \mu)^*),
\]

where notation \( \nu \vdash S^j N \) means that \( \nu \) is a weight of \( S^j N \). By the BWB-theorem, the weight \( \nu + \mu \) contributes to the last sum if and only if \( \nu + \mu + \rho \) is regular, i.e., \( w(\nu + \mu + \rho) - \rho = \)
\[ \lambda \in \mathfrak{X}_+ \] for a unique \( w \in W \). Therefore, using Eq. (3.1), we obtain

\[
\sum_{\nu} \mathcal{P}_{\nu, q}(\nu) \mathcal{H}_G \left( \sum_i (-1)^i H^i(G/B, \mathcal{L}_{G/B}(\nu + \mu)^*) = \sum_{\lambda \in \mathfrak{X}_+} \sum_{w \in W} \varepsilon(w) \mathcal{P}_{\nu, q}(w^{-1}(\lambda + \rho) - \mu - \rho) e^{\lambda^*} = \sum_{\lambda \in \mathfrak{X}_+} m_{\lambda, \Phi}(q) e^{\lambda^*},
\]
as required.

**Theorem 3.9.** For any \( \mu \in \mathfrak{X}_P \), we have

\[
\sum_i (-1)^i \mathcal{H}_G \left( H^i(G \times_P N, \mathcal{L}_{G \times_P N}(\mu)^*); q \right) = \sum_{\lambda \in \mathfrak{X}_+} m_{\lambda, \Phi}(q) e^{\lambda^*}.
\]

**Proof.** Using the Leray spectral sequence associated to the morphism \( G/B \rightarrow G/P \), one easily proves that, for any \( \mu \in \mathfrak{X}_P \), there is an isomorphism

\[
H^i(G/B, \mathcal{L}_{G/B}(S^1 N \otimes \mathbb{C}_\mu)^*) \simeq H^i(G/P, \mathcal{L}_{G/P}(S^1 N \otimes \mathbb{C}_\mu)^*).
\]

Thus, the assertion reduces to the previous theorem. \( \square \)

**Corollary 3.10.** If \( \mu \in \mathfrak{X}_P \) and \( H^i(G \times_P N, \mathcal{L}_{G \times_P N}(\mu)^*) = 0 \) for \( i \geq 1 \), then \( m_{\lambda, \Phi}(q) \) has non-negative coefficients for all \( \lambda \in \mathfrak{X}_+ \).

Now, combining this corollary and Propositions 3.5, 3.6, we obtain Theorem 3.2.

**Remark 3.11.** By Theorem 3.9, if higher cohomology of \( \mathcal{L}_\mathcal{Z}(\mu)^* \) vanishes, then the polynomial \( m_{\lambda, \Phi}(q) \) counts occurrences of \( V_{\lambda}^* \) in the graded \( G \)-module \( H^0(\mathcal{Z}, \mathcal{L}_\mathcal{Z}(\mu)^*) \). In particular, \( m_{\lambda, \Phi}(1) \) is the multiplicity of \( V_{\lambda}^* \) in \( H^0(\mathcal{Z}, \mathcal{L}_\mathcal{Z}(\mu)^*) \).

3.5. If we wish to get a generically finite collapsing for a \( B \)-stable \( N \subset V \), then \( P \) must be chosen as large as possible. That is, we have to take \( P = \text{Norm}_G(N) \), the normaliser of \( N \) in \( G \). However, even this does not guarantee the generic finiteness.

**Example 3.12.** Let \( \mathfrak{c} \) be a \( B \)-stable subspace of \( \mathfrak{u} \subset \mathfrak{g} \). Actually, \( \mathfrak{c} \) is a \( B \)-stable ideal of \( \mathfrak{u} \). Let \( P = \text{Norm}_G(\mathfrak{c}) \). The image of the collapsing \( G \times_P \mathfrak{c} \rightarrow G \cdot \mathfrak{c} \) is the closure of a nilpotent orbit. Hence \( \dim(G \cdot \mathfrak{c}) \) is even. However, \( \dim(G \times_P \mathfrak{c}) \) can be odd. For instance, take \( \mathfrak{c} = [\mathfrak{u}, \mathfrak{u}] \). If \( G \) is simple and \( G \neq SL_2 \), then \( \text{Norm}_G([\mathfrak{u}, \mathfrak{u}]) = B \). But \( \dim(G \times_B [\mathfrak{u}, \mathfrak{u}]) \) is even if and only if \( \text{rk}(G) \) is. It can be shown that the collapsing \( G \times_B [\mathfrak{u}, \mathfrak{u}] \rightarrow G \cdot [\mathfrak{u}, \mathfrak{u}] \) is generically finite if and only if \( \mathfrak{g} \in \{ A_{2n}, B_{2n}, C_{2n}, E_6, E_8, F_4, G_2 \} \).

\( B \)-stable (or “ad-nilpotent”) ideals of \( \mathfrak{u} \) provide the most natural class of examples of generalised Kostka-Foulkes polynomials. There is a rich combinatorial theory of these ideals. In particular, the normalisers of ad-nilpotent ideals has been studied in [21].
Example 3.13. a) For \( G = SL_{2n+1} \), consider \( \Psi = \{ \gamma \in \Delta^+ \mid \text{ht}(\gamma) \geq n+1 \} \). The corresponding ad-nilpotent ideal is \( u_n = \{ u \mid u_n \} \). By direct calculations, \( |\Psi| = \rho \). Therefore the normaliser of \( u_n \) equals \( B \) [21, Theorem 2.4(ii)]. Next, \( \dim(G \times_B u_n) = 2n^2 + 2n + \binom{n}{2} \) and the dense orbit in \( G \cdot u_n \) corresponds to the partition \((2, \ldots, 2, 1)\). Therefore \( \dim G \cdot u_n = 2n^2 + 2n \), and the collapsing is not generically finite unless \( n = 1 \). By Theorems 3.1(i) and 3.2(i) with \( P = B \), we obtain

- \( H^i(G \times_B u_n, L_{G \times_B u_n}(\mu)^*) = 0 \) for any \( \mu \in \mathfrak{X}_+ \) and \( i \geq 1 \);
- \( m^\mu_{\lambda, \Psi}(q) \) has non-negative coefficients for all \( \lambda, \mu \in \mathfrak{X}_+ \).

b) For \( G = SL_{2n} \), consider \( \Psi = \{ \gamma \in \Delta^+ \mid \text{ht}(\gamma) \geq n \} \). The corresponding ad-nilpotent ideal is \( u_{n-1} \). Since \( |\Psi| = \rho + \varphi_n \), the normaliser of \( u_{n-1} \) equals \( B \). Again, direct calculations show that \( \dim(G \times_B u_{n-1}) = \dim G \cdot u_{n-1} = \binom{n}{2} \). Here we have

- \( H^i(G \times_B u_n, L_{G \times_B u_n}(\mu)^*) = 0 \) for any \( \mu \triangleright \varphi_n \) and \( i \geq 1 \);
- \( m^\mu_{\lambda, \Psi}(q) \) has non-negative coefficients for all \( \lambda \in \mathfrak{X}_+ \) and \( \mu \triangleright \varphi_n \).

Remark 3.14. For an arbitrary \( B \)-stable subspace \( N \subset V \), the normaliser of \( N \) is fully determined by \( |\Psi| \). The proof of [21, Theorem 2.4(i),(ii)] goes through verbatim, and it shows that \( |\Psi| \) is dominant and

\[
\begin{cases}
\text{the root subspace } g_{-\alpha} & (\alpha \in \Pi) \\
\text{belong to Lie(Norm}_G(N)) \end{cases} \leftrightarrow \{ (\alpha, |\Psi|) = 0 \}.
\]

Equivalently, one can say that \( \text{Norm}_G(N) = \text{Norm}_G(\bigwedge^{\dim} N N) \), where \( \bigwedge^{\dim} N N \subset \bigwedge^{\dim} N V \).

4. The little adjoint module and short \( q \)-analogues

Let \( G \) be a simple algebraic group such that \( \Delta \) has two root lengths. There is a special interesting case in which \( \Psi = \Delta^+_s \) is the set of short positive roots. The subscripts ‘s’ and ‘l’ will be used to mark objects related to short and long roots, respectively. For instance, \( \Delta_l \) is the set of all long roots, \( \Delta^+_l = \Delta^+_s \cup \Delta^+_l \), and \( \Pi_s = \Pi \cap \Delta_s \). Let \( \bar{\theta} \) be the short dominant root. The \( G \)-module \( V_{\bar{\theta}} \) is said to be the 

Lemma 4.1. The set of nonzero weights of \( V_{\bar{\theta}} \) is \( \Delta_s; m^\nu_{\bar{\theta}} = 1 \) for \( \nu \in \Delta_s \) and \( m^\nu_{\bar{\theta}} = \# \Pi_s \).

The last equality is proved in [20, Prop. 2.8]; the rest is obvious. It follows that there is a unique \( B \)-stable subspace of \( V_{\bar{\theta}} \) whose set of weights is \( \Delta^+_s \). Write \( V_{\bar{\theta}}^+ \) for this subspace. In the rest of the article, we work with \( \Psi = \Delta^+_s \) and the \( B \)-stable subspace \( N = V_{\bar{\theta}}^+ \). In place of \( F_{\Delta^+_s}(\nu) \) and \( m^\mu_{\lambda, \Delta^+_l}(q) \), we write \( \overline{F}_q(\nu) \) and \( \overline{m}_q(\lambda) \), respectively. The polynomials \( \overline{m}_q(\lambda) \) are said to be short \( q \)-analogues (of weight multiplicities).

We have \( \mathfrak{X}_+ \cap \Delta^+_s = \{ \bar{\theta} \} \). Set \( \rho_s = \frac{1}{2} |\Delta^+_s| \) and \( \rho_l = \frac{1}{2} |\Delta^+_l| \). It is easily seen that \( \rho_s \) (resp. \( \rho_l \)) is the sum of fundamental weights corresponding to \( \Pi_s \) (resp. \( \Pi_l \)). Let \( H \) be the connected
semisimple subgroup of $G$ that contains $T$ and whose root system is $\Delta_t$. The Weyl group
of $H$ is the normal subgroup of $W$ generated by all “long” reflections. It is denoted by $W_l$. Let $G(\Pi_s)$ (resp. $g(\Pi_s)$) denote the simple subgroup of $G$ (subalgebra of $g$) whose set of
simple roots is $\Pi_s$. Then $rk g(\Pi_s) = \#\Pi_s$ and $B \cap G(\Pi_s) =: B(\Pi_s)$ is a Borel subgroup
of $G(\Pi_s)$. Clearly, $G(\Pi_s) \cdot T =: L$ is a standard Levi subgroup of $G$ and $G(\Pi_s) = (L, L)$.

The collapsing $G \times_B V^+_\theta \to G \cdot V^+_\theta$ is not generically finite, and Theorem 3.2(i) (with
$\rho_P = \rho$, $\Delta(n) = \Delta^+$, and $|\Delta^+_s| = 2 \rho_s$) yields the bound $\mu > 2 \rho_s - \rho = \rho_s - \rho_l$ for $m^\mu_\lambda(q)$. However, in this case there is a better bound, and our first goal is to obtain it. To this end,
we need some further properties of little adjoint modules.

The weight structure of $V^+_\theta$ shows that $V^+_\theta|_{G(\Pi_s)}$ contains the adjoint representation of
$G(\Pi_s)$. To distinguish the Lie algebra $g(\Pi_s)$ sitting in $g$ and the adjoint representation of
$G(\Pi_s)$ sitting in $V^+_\theta$, the latter will be denoted by $\hat{g}(\Pi_s)$. That is,

$$V^+_\theta|_{G(\Pi_s)} = \hat{g}(\Pi_s) \oplus R,$$

where $R$ is the complementary $G(\Pi_s)$-submodule. The above decomposition is $L$-stable
and hence $T$-stable. We have $R^T = 0$ and the weights of $R$ are those short roots that are
not $\mathbb{Z}$-linear combinations of short simple roots. Furthermore, $V^+_\theta = \hat{g}(\Pi_s)^+ \oplus R^+$, where
$R^+ \subset R$ and $g(\Pi_s)^+ = g(\Pi_s) \cap u$ is a maximal nilpotent subalgebra of $g(\Pi_s)$.

**Theorem 4.2.** If $\mu > \rho_l$, then the collapsing $f^\mu_s : G \times_B (V^+_\theta \oplus \mathbb{C}_\mu) \to G \cdot (V^+_\theta \oplus \mathbb{C}_\mu)$ is birational.

**Proof.** Recall that $\mathbb{C}_\mu$ is the line of $B$-highest weight vectors in $V^*_\mu$. Obviously, $f^\mu_s$ is birational if and only if the following property holds: for a generic point $(v, v_\mu) \in V^+_\theta \oplus \mathbb{C}_\mu$, if $g \cdot (v, v_\mu) \in V^+_\theta \oplus \mathbb{C}_\mu$ ($g \in G$), then $g \in B$. Let $\tilde{P}$ denote the standard parabolic subgroup of $G$ whose Levi subgroup is $L$. If $\mu > \rho_l$, then the normaliser in $G$ of the line $\langle v_\mu \rangle$ is contained in $\tilde{P}$. Consequently, if $g \cdot (v, v_\mu) \in V^+_\theta \oplus \mathbb{C}_\mu$, then $g \in \tilde{P}$.

Take $v = v' + r \in V^+_\theta$ ($r \in R$) such that $v'$ is a regular nilpotent element of $\hat{g}(\Pi_s)^+$. Write
$g = g_1 g_2 \in \tilde{P}$, where $g_1 \in G(\Pi_s)$ and $g_2$ lies in the radical of $\tilde{P}$, $\text{rad}(\tilde{P})$. It is easily seen that $\text{rad}(\tilde{P})$ preserves $R^+$ and acts trivially in $V^+_\theta / R^+$. Therefore $g_2$ does not change the $\hat{g}(\Pi_s)$-
component of $v$, i.e., $g_2 \cdot v = v' + r'$ ($r' \in R^+$). Hence $g \cdot v = g_1 \cdot v' + g_1 \cdot r'$, and $g_1 \cdot v' \in \hat{g}(\Pi_s)^+$ is still a regular nilpotent element of $\hat{g}(\Pi_s)$. But the latter is only possible if $g_1 \in B(\Pi_s)$ and hence $g \in B$. \hfill \Box

**Corollary 4.3.** If $\nu + \rho_l \in \mathcal{X}_+$, then

(i) $H^i(G \times_B V^+_\theta, L_{G \times_B V^+_\theta}(\nu^*)) = 0$ for $i \geq 1$;

(ii) $m^\mu_\lambda(q)$ has non-negative coefficients for all $\lambda \in \mathcal{X}_+$.

**Proof.** (i) Set $U = G \times_B (V^+_\theta \oplus \mathbb{C}_\mu)$ and $Z = G \times_B V^+_\theta$. Then $\omega_U = L_U(\gamma - \mu)$, where
$\gamma = |\Delta^+| - |\Delta^+_s| = 2 \rho_l$. By Theorems 3.4 and 4.2, $H^i(U, \omega_U) = 0$ for $i \geq 1$ whenever $\mu > \rho_l$. 

Hence $H^i(Z, L_Z((n+1)\mu - \gamma)^*) = 0$, see Section 3. In particular, $H^i(Z, L_Z(\nu)^*) = 0$, where $\nu = \mu - \gamma$. It remains to observe that $\nu > -\rho_i$.

(ii) This follows from (i) and Theorem 3.8.

\begin{proof}
(i) Since $\ell$ on reflections occurring in an expression for $w$ of $W$, then $w W$ is a conceptual proof. All the assertions can easily be verified using the classification, but our intention is to present a conceptual proof.

(ii) This follows from (i) and Theorem 3.8.
\end{proof}

Remark 4.4. The proof of Corollary 4.3(i) uses (a version of) the Grauert–Riemenschneider theorem. However, for $\nu = 0$ (at least) one can adapt Hesselink’s proof of [9, Theorem B], which does not refer to Grauert–Riemenschneider and goes through for any algebraically closed field $k$ of characteristic zero. Using this, one can prove the following: Let $\tilde{N}$ be any $B$-stable subspace of $V_{\theta}$ such that $\tilde{N} \supset V_{\theta}^+$. Then $H^i(G \times_B \tilde{N}, \mathcal{O}_{G \times_B \tilde{N}}) = 0$ for $i \geq 1$.

Let us describe a semi-direct product structure of $W$, which plays an important role below. Consider two subgroups of $W$:

- $W_i$ is generated by all “long” reflections in $W$. It is a normal subgroup of $W$.
- $W(\Pi_s)$ is generated by all simple “short” reflections, i.e., by $s_\alpha$ with $\alpha \in \Pi_s$.

**Lemma 4.5.**

(i) $W$ is a semi-direct product of $W_i$ and $W(\Pi_s)$: $W \simeq W(\Pi_s) \ltimes W_i$.

(ii) $W(\Pi_s) = \{ w \in W \mid w(\Delta_i^+) \subset \Delta_i^+ \}$.

\begin{proof}
(i) Since $W_i$ is a normal subgroup of $W$ and $W_i \cap W(\Pi_s) = \{1\}$, it suffices to prove that the natural mapping $W(\Pi_s) \times W_i \to W$ is onto. We argue by induction on the length of $w \in W$. Suppose $w \notin W(\Pi_s)$ and $w = w_1 s_\beta w_2 \in W_i, \beta \in \Pi_i$, is a reduced decomposition. Then $w = w_1 w_2 s_{\beta'}$, where $\beta' = w_2(\beta) \in \Delta_i$, and $\ell(w_1 w_2) < \ell(w)$. Thus, all long simple reflections occurring in an expression for $w$ can eventually be moved up to the right.

(ii) Since $s_\alpha(\Delta_j^+) \subset \Delta_j^+$ for $\alpha \in \Pi_s$, $W(\Pi_s) \subset \{ w \in W \mid w(\Delta_i^+) \subset \Delta_i^+ \}$. On the other hand, if $w(\Delta_i^+) \subset \Delta_i^+$ and $w = w's_\alpha$ is a reduced decomposition, then the equality $N(w) = s_\alpha(N(w')) \cup \{\alpha\}$ shows that $\alpha$ is necessarily short, so that we can argue by induction on $\ell(w)$.
\end{proof}

Recall that the null-cone of a $G$-module $V$, $\mathfrak{N}(V)$, is the zero set of all homogeneous $G$-invariant polynomials of positive degree. Next proposition summarises invariant-theoretic properties of $V_{\theta}$ and $\mathfrak{N}(V_{\theta})$ required below, which are of independent interest. All the assertions can easily be verified using the classification, but our intention is to present a conceptual proof.

**Proposition 4.6.**

(a) $\mathfrak{N}(V_{\theta}) = G \cdot V_{\theta}^+$. Hence it is irreducible;

(b) The restriction homomorphisms $\mathbb{C}[V_{\theta}] \to \mathbb{C}[\mathfrak{g}(\Pi_s)] \to \mathbb{C}[V_{\theta}^+]$ induce the isomorphisms $\mathbb{C}[V_{\theta}]^G \sim \mathbb{C}[\mathfrak{g}(\Pi_s)]^{G(\Pi_s)} \sim \mathbb{C}[V_{\theta}^+]^W(\Pi_s)$, and $\mathbb{C}[V_{\theta}]^G$ is a polynomial algebra.

(c) $\mathfrak{N}(V_{\theta})$ is a reduced normal complete intersection of codimension $\#(\Pi_s)$.
Outline of the proof. We refer to [22] for invariant-theoretic results mentioned below.

a) This follows from the Hilbert-Mumford criterion and the fact any maximal subset of weights of $V_{\theta}$, lying in an open half-space, is $W$-conjugate to $\Delta^+_s$.

b) The weight structure of $V_{\theta}$ shows that $V^0_\theta = V^H_\theta$. If $v \in V^0_\theta$ is generic, then $g \cdot v + V^0_\theta = V_\theta$. Therefore $G \cdot V^0_\theta$ is dense in $V_{\theta}$ and a generic stabiliser (= stabiliser in general position) for $G \cdot V_{\theta}$ contains $H$. Actually, it is not hard to prove that $H$ is a generic stabiliser for $G \cdot V_{\theta}$. By the Luna-Richardson theorem, we then have $\mathbb{C}[V_{\theta}]^G \cong \mathbb{C}[V_{\theta}^H]^{N_G(H)/H}$, and it is easily seen that $N_G(H)/H \cong W/W_1 \cong W(\Pi_s)$. Furthermore, the $W(\Pi_s)$-action on $V^0_\theta$ is nothing but the standard reflection representation on the Cartan subalgebra of $g$.\qed

c) Let $f_1, \ldots, f_m$ be basic invariants in $\mathbb{C}[V_{\theta}]^G \cong \mathbb{C}[\hat{g}(\Pi_s)]^{G(\Pi_s)}$, $m = \#(\Pi_s)$. Let $e \in \hat{g}(\Pi_s) \subset V_{\theta}$ be regular nilpotent. Then the differentials of the $f_i$’s are linearly independent at $e \in \mathfrak{N}(V_{\theta})$ [14]. Hence the ideal of $\mathfrak{N}(V_{\theta})$ is $(f_1, \ldots, f_m)$ and $\mathfrak{N}(V_{\theta})$ is a reduced complete intersection (cf. [14, Lemma 4]). Finally, $\mathfrak{N}(V_{\theta})$ contains a dense $G$-orbit whose complement is of codimension $\geq 2$. This yields the normality.

Our ultimate goal is to get a complete characterisation of weights $\mu \in \mathfrak{X}$ such that $\overline{m}_\mu^\lambda(q)$ has nonnegative coefficients for any $\lambda \in \mathfrak{X}_+$. To this end, we exploit a different approach that does not use vanishing theorems of Section 3.

A key observation is that short $q$-analogues obey certain symmetries with respect to the simple reflections $s_\alpha \in W$, $\alpha \in \Pi_l$. Clearly, $s_\alpha(\Delta^+_s) = \Delta^+_s$. Therefore $\overline{P}_q(\nu) = \overline{P}^l_q(s_\alpha \nu)$. Using this, we compute

\begin{equation}
(4.1) \quad \overline{m}_\mu^\lambda(q) = \sum_{w \in W} \varepsilon(w) \overline{P}_q(w(\lambda + \rho) - (\mu + \rho)) \nonumber
\end{equation}

\[= \sum_{w \in W} \varepsilon(w) \overline{P}_q(s_\alpha w(\lambda + \rho) - s_\alpha(\mu + \rho)) = - \sum_{w \in W} \varepsilon(w) \overline{P}_q(w(\lambda + \rho) - s_\alpha \mu - s_\alpha \rho)\]

\[= - \sum_{w \in W} \varepsilon(w) \overline{P}_q(w(\lambda + \rho) - (s_\alpha \mu - \alpha + \rho)) = -\overline{m}^\lambda_{s_\alpha(\mu + \alpha)}(q).
\]

The shifted action of $W_l$ on $\mathfrak{X}$ is defined by

\[w \circ \gamma = w(\gamma + \rho_l) - \rho_l.\]

For $\alpha \in \Pi_l$, one easily recognise $s_\alpha(\mu + \alpha)$ as $s_\alpha \circ \mu$ and hence Eq. (4.1) can be written as $\overline{m}^\lambda_{s_\alpha \circ \mu}(q) = -\overline{m}^\lambda_\mu(q)$. This readily implies the equality

\begin{equation}
(4.2) \quad \overline{m}^{w \circ \mu}_\lambda(q) = \varepsilon(w) \overline{m}^\mu_\lambda(q) \nonumber
\end{equation}

for any $w \in W_l$. Note that for $w \in W_l$, the length $\ell(w)$ depends on the choice of ambient group, $W$ or $W_l$, but the parity $\varepsilon(w)$ does not! (This is because $\varepsilon(w) = \det(w)$ for the reflection representation of $W$ in $\mathfrak{X} \otimes \mathbb{Q}$.)
Let $\mathcal{X}_{+,H}$ denote the monoid of $H$-dominant weights with respect to $\Delta^+_s$. From (4.2), we immediately deduce that

- it suffices to know $\overline{m}_\lambda(q)$ for $\mu \in \mathcal{X}_{+,H} - \rho$;
- if such a $\mu$ is not $H$-dominant, then it lies on a wall of the shifted dominant Weyl chamber for $H$, and hence $\overline{m}_\lambda(q) \equiv 0$;
- Thus, the problem is reduced to studying polynomials $\overline{m}_\lambda(q)$ for $\mu \in \mathcal{X}_{+,H}$.

Short $q$-analogues enjoy several good interpretations at $q = 1$. Write $\overline{m}_\lambda$ in place of $\overline{m}_\lambda'(1)$.

1) As already observed in Remark 3.11, if higher cohomology of $\mathcal{L}_{\mathbb{Z}}(\nu)^*$ vanish, then $\overline{m}_\lambda(1)$ is the multiplicity of $V_\lambda^*$ in $H^0(\mathbb{Z}, \mathcal{L}_{\mathbb{Z}}(\nu)^*)$.

2) If $\nu \in \mathcal{X}_{+,H}$ and $V_\nu^{(H)}$ is a simple $H$-module with highest weight $\nu$, then $\overline{m}_\lambda(1)$ is the multiplicity of $V_\nu^{(H)}$ in $V_\lambda|_H$, denoted $\text{mult}(V_\nu^{(H)}, V_\lambda|_H)$, see [8, Lemma 3.1].

(Our $\overline{m}_\lambda(1)$ is $m^{G,H}_\lambda(1)$ in the notation of [8]. In fact, Heckman works in a general situation, where $H \subset G$ is an arbitrary connected reductive group.) Furthermore, the numbers $\overline{m}_\lambda(1)$ are naturally defined for all $\lambda, \nu \in \mathfrak{X}$ and they satisfy the relation

$$\overline{m}_{w(\lambda + \rho) - \rho} = \overline{m}_\lambda(w) \overline{m}_{\nu - \rho} \quad w \in W, \; \bar{w} \in W_1.$$

(See Equation (3.7) in [8].) The semi-direct product structure of $W$ provides an extra symmetry to this picture that is absent in the general setting of [8]. Namely, if $\nu$ is $H$-dominant, then so is $w\nu$ for any $w \in W(\Pi_\nu)$. Using this one easily proves that $\overline{m}_\lambda = \overline{m}_{\nu - \rho}$ for all $\lambda \in \mathcal{X}_{+,H}$ and $w \in W(\Pi_\nu)$.

Recall that $\{\mu^+\} = W\mu \cap \mathcal{X}_{+,H}$. Let $w_\mu$ denote the unique element of minimal length such that $w_\mu(\mu) = \mu^+$.

**Lemma 4.7.** If $\mu \in \mathcal{X}_{+,H}$, then $w_\mu \in W(\Pi_\mu)$ and hence $\mu^+ - \mu$ is a nonnegative $\mathbb{Z}$-linear combination of short simple roots.

**Proof.** It is known that $N(w_\mu) = \{\gamma \in \Delta^+ \mid (\gamma, \mu) < 0\}$, see [4, Prop. 2(i)]. Since $\mu$ is $H$-dominant, $N(w_\mu) \subset \Delta^+_s$, and we conclude by Lemma 4.5(ii). \hfill $\square$

**Proposition 4.8.** Let $\mu \in \mathcal{X}_{+,H}$.

1) Suppose that there is $\nu \in \mathcal{X}_{+,H}$ such that $\mu \prec \nu \prec \mu^+$. Then $\overline{m}_\nu(q) \neq 0$ and $\overline{m}_\nu(1) = 0$. In particular, $\overline{m}_\nu(q)$ has both positive and negative coefficients.

2) If $V_{\mu^+}^*$ occurs in $H^0(G/B, \mathcal{L}_{G/B}(\mathfrak{s}^j(V_{\mu^+}^*) \otimes C_\mu)^*)$, then $j \geq \text{ht}(\mu^+ - \mu)$. Furthermore, for $j = \text{ht}(\mu^+ - \mu)$, $H^0(\ldots)$ contains a unique copy of $V_{\mu^+}^*$.

**Proof.** 1) Since $w_\mu \in W(\Pi_\mu)$, we have $\overline{m}_\nu = \overline{m}_\nu^+$, and the latter equals zero, because $\nu \prec \mu^+$. (Obviously, the $H$-module with highest weight $\mu^+$ cannot occur in $V_\nu|_H$.)

2) We have $\overline{m}_\nu(q) \neq 0$ and $\overline{m}_\nu(1) = 0$. (Obviously, the $H$-module with highest weight $\mu^+$ cannot occur in $V_\nu|_H$.)
Since $\mu \preceq \nu \prec \mu^+$ and $\mu^+ - \mu$ is a nonnegative $\mathbb{Z}$-linear combination of short simple roots, the latter holds for $\nu - \mu$ as well. Set $a = \text{ht}(\nu - \mu)$. By definition,

$$m^\nu_\rho(q) = \sum_{w \in W} \varepsilon(w) \mathcal{P}_q(w(\nu + \rho) - (\mu + \rho)).$$

As $\nu - \mu \in \text{Span}(\Pi_\alpha)$, the summand $\mathcal{P}_q(w(\nu + \rho) - (\mu + \rho))$ can be nonzero only if $w \in W(\Pi_\alpha)$. For $w = 1$, we have $\mathcal{P}_q(\nu - \mu) = q^a + (\text{lower terms})$. If $w \neq 1$, then $\deg \mathcal{P}_q(w(\nu + \rho) - (\mu + \rho)) < a$. Hence the highest term of $m^\nu_\rho(q)$ is $q^a$, and we are done.

2) This readily follows from the BWB-theorem and Lemma 4.7. 

Our main result on non-negativity for short $q$-analogues is a converse to the first claim of the previous proposition. For the proof of the main theorem, we need a technical lemma.

**Lemma 4.9.** 1) Suppose that $V^*_\gamma$ occurs in $H^i(G/B, \mathcal{L}_{G/B}(\wedge^j(V^{\mu+}_\delta/V^{\mu+}_\delta) \otimes \mathbb{C}_\mu)^*)$. Then $\nu \preceq \mu^+$. 2) (For $\nu = \mu^+$. If $V^*_{\mu^+}$ occurs in $H^i(G/B, \mathcal{L}_{G/B}(\wedge^j(V^{\mu+}_\delta/V^{\mu+}_\delta) \otimes \mathbb{C}_\mu)^*)$, then $j \geq i \geq \ell(w_\mu)$.)

**Proof.** Set $M_j = \wedge^j(V^{\mu+}_\delta/V^{\mu+}_\delta) \otimes \mathbb{C}_\mu$.

1) If $V^*_\gamma$ occurs in $H^i(G/B, \mathcal{L}_{G/B}(M_j)^*)$, then it also occurs in $H^i(G/B, \mathcal{L}_{G/B}(M_j^*)^*)$. By the BWB-theorem, there is then a weight $\gamma'$ of $M_j$ and $w \in W$ such that $\ell(w) = i$ and $w(\gamma' + \rho) - \rho = \nu$. All weights of $M_j$ are of the form $\mu - |A|$ for some $A \subset \Delta^+_s$, where $\#(A) \leq j$. Hence $w(\mu + \rho - |A|) = \rho + \nu$. Clearly, $w(\rho - |A|) = \rho - |C|$ for some $C \subset \Delta^+_s$ depending on $w$ and $A$. Thus, $w(\mu + \rho - |A|) \preceq w(\mu) + \rho$ and $\nu \preceq w(\mu) \preceq \mu^+$.

2) If $V^*_{\mu^+}$ occurs in $H^i(G/B, \mathcal{L}_{G/B}(\tilde{M}_j)^*)$, then, by the first part of the proof, we must have $w(\mu + \rho - |A|) = \rho + \mu^+$, where $A \subset \Delta^+_s$ and $\ell(w) = i$. Hence $w(\mu) = \mu^+$ and $w(\rho - |A|) = \rho$. Therefore $A = N(w)$ and $i = \ell(w) = \#(A) \geq \ell(w_\mu)$. Since $\#(A) \leq j$ as well, we are done.

The following is the main result of this section.

**Theorem 4.10.** For $\mu \in \mathcal{X}_{+H}$, the following conditions are equivalent:

(i) $H^i(G \times B V^{\mu+}_\delta, \mathcal{L}_{G \times B V^{\mu+}_\delta}(\mu)^*) = 0$ for all $i \geq 1$;

(ii) $m^\mu_\lambda(q)$ has nonnegative coefficients for any $\lambda \in \mathcal{X}_{+H}$;

(iii) If $\mu \preceq \nu \preceq \mu^+$ for $\nu \in \mathcal{X}_{+H}$, then $\nu = \mu^+$;

(iv) $(\mu, \alpha^\vee) \geq -1$ for all $\alpha \in \Delta^+_s$.

**Proof.** By Corollary 3.10, (i) implies (ii); and Proposition 4.8 shows that (ii) implies (iii). Since $\nu$ is already assumed to be $H$-dominant, (iii) and (iv) are equivalent in view of [4, Prop. 2(ii)].

It remains to prove the implication (iii) $\Rightarrow$ (i). Our argument is an adaptation of Broer’s proof of [1, Theorem 2.4]. We construct a similar Koszul complex and consider its spectral sequence of hypercohomology.
The pull-back vector bundle $G \times B (V_\theta^+ \oplus (V_\theta^+ / V_\theta^+))$ on $X := G \times B V_\theta$ has the global $G$-equivariant section $g \ast v \mapsto g \ast (v, \tilde{v})$ whose scheme of zeros is exactly $Z = G \times B V_\theta^+$. Here $\tilde{v}$ is the image of $v \in V_\theta$ in $V_\theta^+ / V_\theta^+$. Let $\iota : Z \to X$ denote the inclusion. The dual of this section gives rise to a locally free Koszul resolution of $O_Z$ regarded as $O_X$-module:

$$\cdots \to \mathcal{F}^{-1} \to \mathcal{F}^{0} \to \iota_* O_Z \to 0$$

with $\mathcal{F}^{-j} = L_X(\wedge^j (V_\theta^+ / V_\theta^+))^*[-j]$. Here the brackets ‘$[-j]$’ denote the degree shift of a graded module. (That is, if $M = \oplus M_i$, then $M[r] = M_{r+i}$.) Therefore the generators of the locally free $O_X$-module $\mathcal{F}^{-j}$ have degree $j$. Tensoring this complex with the invertible sheaf $L_X(C_\mu)^* = L_X(\mu)^*$, we get a locally free resolution of graded $O_X$-modules

$$\mathcal{F}(\mu)^* \to \iota_* L_Z(\mu)^* \to 0,$$

where $\mathcal{F}(\mu)^{-j} = L_X(\wedge^j (V_\theta^+ / V_\theta^+) \otimes C_\mu)^*[-j]$. Since $X \simeq G/B \times V_\theta$, we have the isomorphism

$$H^i(X, L_X(\wedge^j (V_\theta^+ / V_\theta^+) \otimes C_\mu)^*) \simeq \mathbb{C}[\theta] \otimes H^i(G/B, L_G/G_B(\wedge^j (V_\theta^+ / V_\theta^+) \otimes C_\mu)^*)$$

of graded $\mathbb{C}[\theta]$-modules. For the spectral sequence of hypercohomology associated to the Koszul complex (4.4), we have

$$'' E^{kl}_2 = H^k(X, \mathcal{H}(\mathcal{F}(\mu)^*)) = \begin{cases} H^k(X, \iota_* L_Z(\mu)^*) = H^k(Z, L_Z(\mu)^*), & \text{if } l = 0; \\ 0, & \text{if } l \neq 0. \end{cases}$$

and

$$' E^{kl}_1 = H^l(X, \mathcal{F}(\mu)^k) = \mathbb{C}[\theta][k] \otimes H^l(G/B, L_G/G_B(\wedge^k (V_\theta^+ / V_\theta^+) \otimes C_\mu)^*).$$

(See [25, 5.7] for basic facts on hypercohomology.) It follows that there is a spectral sequence of graded $\mathbb{C}[\theta]$-modules

(4.5) $$' E_{-j,i}^{1} = \mathbb{C}[\theta][-j] \otimes H^i(G/B, L_G/G_B(\wedge^j (V_\theta^+ / V_\theta^+) \otimes C_\mu)^*) \Rightarrow H^{i-j}(Z, L_Z(\mu)^*).$$

Let $i - j$ be maximal with $H^i(G/B, L_G/G_B(\wedge^j (V_\theta^+ / V_\theta^+) \otimes C_\mu)^*) \neq 0$. If $V_\nu^*$ occurs in this cohomology group, then $\nu \preceq \mu^+$, by Lemma 4.9(1). A basis for $V_\nu^*$ corresponds to some free generators of $\mathbb{C}[\theta]$-module $' E_{-j,i}^{1}$ of degree $j$. Since $i - j$ is maximal, these generators are in the kernel of $d_{-j,i}^{1}$. But they are not in the image of $d_{-j-1,i}^{1}$, as all elements of $' E_{-j-1,i}^{1}$ are of degree $> j$. Hence these generators correspond to nonzero generators of $' E_{-j,i}^{1}$. Likewise, their images in $' E_{k,i}^{1}$ do not vanish. In view of convergence of the above spectral sequence, this implies that the multiplicity of $V_\nu^*$ in $H^{i-j}(Z, L_Z(\mu)^*)$ is at least one. It follows that, for some $m \in \mathbb{N}$, the multiplicity of $V_\nu^*$ in $H^{i-j}(G/B, L_G/G_B(S^m(V_\theta^+) \otimes C_\mu)^*)$ is also at least one. Any weight of $S^m(V_\theta^+) \otimes C_\mu$ is of the form $\mu + \gamma$ with $\gamma \geq 0$. Hence $\nu + \rho = w(\mu + \gamma + \rho)$ for some $w \in W$ with $\ell(w) = i - j$. Consequently, $\nu \succeq \mu$ and altogether $\mu \preceq \nu \preceq \mu^+$. Hence $\nu = \mu^+$. Now, Lemma 4.9(2) yields $i = j \geq \ell(w_\mu)$. In particular, condition (i) holds. \qed
The following is an analogue of [1, Prop. 2.6].

**Proposition 4.11.** Suppose that $\mu \in \mathcal{X}_{+,H}$ satisfies vanishing conditions of Theorem 4.10. Then the graded $\mathbb{C}[V_{\theta}]$-module $H^0(\mathcal{Z}, \mathcal{L}_\mathcal{Z}(\mu)^*)$ is generated by the unique copy of $V_{\mu^+}^*$ sitting in degree $\text{ht}(\mu^+ - \mu)$.

**Proof.** Eq. (4.5) an the last part of the proof of Theorem 4.10 shows that

- The generators of the $\mathbb{C}[V_{\theta}]$-module $H^0(\mathcal{Z}, \mathcal{L}_\mathcal{Z}(\mu)^*)$ arise from $G$-modules sitting in $H^i(G/B, \mathcal{L}_{G/B}(\wedge(V_{\theta}/V_{\theta}^+) \otimes \mathbb{C}_\mu)^*)$, with $i \geq \ell(w_\mu)$;
- $H^i(G/B, \mathcal{L}_{G/B}(\wedge(V_{\theta}/V_{\theta}^+) \otimes \mathbb{C}_\mu)^*)$ only contains $G$-modules of type $V_{\mu^+}^*$.

It follows that the degree of generators of $H^0(\mathcal{Z}, \mathcal{L}_\mathcal{Z}(\mu)^*)$ is at least $\ell(w_\mu)$. On the other hand, if $H^0(G/B, \mathcal{L}_{G/B}(\wedge^j(V_{\theta}^+) \otimes \mathbb{C}_\mu)^*)$ contains a $G$-submodule of type $V_{\mu^+}^*$, then $j \leq \text{ht}(\mu^+ - \mu)$ by Proposition 4.8(2). Therefore, there cannot be generators of degree larger than $\text{ht}(\mu^+ - \mu)$. It only remains to prove that if $\mu \in \mathcal{X}_{+,H}$ satisfies the vanishing condition, then $\ell(w_\mu) = \text{ht}(\mu^+ - \mu)$. Clearly, $\ell(w_\mu) \leq \text{ht}(\mu^+ - \mu)$. Assume the inequality is strict. Then there is a $w \in W$ and a simple reflection $s_i$ such that $\mu \prec w(\mu) \prec s_iw(\mu) \prec \mu^+$ and $s_iw(\mu) = w(\mu) + k\alpha_i$ with $k \geq 2$. Then $\nu := w(\mu) + \alpha_i$ belongs to the convex hull of $w(\mu)$ and $s_iw(\mu)$; hence $\mu \prec \nu^+ \prec \mu^+$, which contradicts the vanishing condition. \hfill \Box

Finally, we mention that above two interpretations of numbers $m^X_\nu$ and Theorem 4.10 lead to an interesting equality.

**Proposition 4.12.** If $\nu \in \mathcal{X}_{+,H}$ and $(\nu, \alpha^\vee) \geq -1$ for all $\alpha \in \Delta^+_s$, then $H^0(G/H, \mathcal{L}_{G/H}(V^H_\nu)^*)$ and $H^0(G \times_B V_{\theta}^+, \mathcal{L}_{G \times_B V_{\theta}^+}(\nu)^*)$ are isomorphic $G$-modules. In particular, for $\nu = 0$, we obtain $\mathbb{C}[G/H] \simeq \mathbb{C}[G \times_B V_{\theta}^+]$ as $G$-modules.

**Proof.** By Frobenius reciprocity,

$$\text{mult}(V^X_\nu, H^0(G/H, \mathcal{L}_{G/H}(V^H_\nu)^*)) = \text{mult}(V^H_\nu, \mathcal{L}_{G/H}(V^H_\nu)^*) = \text{mult}(V^H_\nu, V_{\nu|H}).$$

Hence the multiplicity of $V^X_\nu$ in both spaces $H^0(\ldots)$ under consideration is equal to $m^X_\nu$. \hfill \Box

5. **Short Hall-Littlewood polynomials**

In this section, we define “short” analogues of Hall-Littlewood polynomials and establish their basic properties. Recall that $\Delta$ is a reduced irreducible root system, and $\Delta^+ = \Delta^+_s \cup \Delta^+_l$, $\Pi = \Pi_s \cup \Pi_l$, etc. It is convenient to assume that in the simply-laced case all roots are short and $\Pi_l = \emptyset$. Then the following can be regarded as a generalisation of Gupta’s theory [6, 7].

The character ring $A$ of finite-dimensional representations of $G$ is identified with $\mathbb{Z}[\mathcal{X}]^W$. For $\lambda \in \mathcal{X}_+$, let $\chi_\lambda$ denote the character of $V_\lambda$, i.e., $\chi_\lambda = \text{ch}(V_\lambda) = \sum_{\mu} m_\mu^\lambda e^\mu$. By Weyl’s character formula, $\chi_\lambda = J(e^{\lambda+\rho})/J(e^\rho)$, where $J = \sum_{w \in W} e(\lambda)w$ is the skew-symmetrisation
operator. Weyl’s denominator formula says that \( J(e^\rho) = e^\rho \prod_{\alpha > 0} (1 - e^{-\alpha}) \). The usual scalar product \( \langle \ , \rangle \) on \( \Lambda = \mathbb{Z}[\mathfrak{X}]^W \) is given by \( \langle \chi_\lambda, \chi_\nu \rangle = \delta_{\lambda, \nu} \).

The projection \( j : \mathbb{Z}[\mathfrak{X}] \to \mathbb{Z}[\mathfrak{X}]^W \) is given by \( j(f) := J(f)/J(e^\rho) \).

Set \( t^{(\Pi_\lambda)}(q) = \sum q^{\ell(w)} \), where the summation is over \( w \in W(\Pi_\lambda) \), the stabiliser of \( \lambda \) in \( W(\Pi_\lambda) \).

We will work in the \( q \)-extended character ring \( \Lambda[[q]] \) or its subring \( \Lambda[q] \) and agree to extend our operators and form \( q \)-linearly. We first put

\[
\Delta_q^{(s)} = \frac{e^\rho}{\prod_{\alpha \in \Delta_s^+} (1 - qe^\alpha)}, \quad \Delta_q^{(s)} = e^\rho \prod_{\alpha \in \Delta_s^+} (1 - qe^{-\alpha}).
\]

For \( \lambda, \mu \in \mathfrak{X}_+ \), define:

\[
\mathcal{E}_\mu(q) = j(e^{\mu} \cdot \Delta_q^{(s)}) , \quad \mathcal{P}_\lambda(q) = \frac{1}{t^{(\Pi_\lambda)}(q)} j(e^{\lambda} \cdot \Delta_q^{(s)}).
\]

Clearly, \( \mathcal{E}_\mu(q) \in \Lambda[[q]] \) and \( t^{(\Pi_\lambda)}(q) \cdot \mathcal{P}_\lambda(q) \in \Lambda[q] \). It will immediately be shown that \( \mathcal{P}_\lambda(q) \) is a well-defined element of \( \Lambda[q] \), i.e., \( t^{(\Pi_\lambda)}(q) \) divides \( j(e^\lambda \cdot \Delta_q^{(s)}) \) in \( \Lambda[q] \). We say that \( \mathcal{P}_\lambda(q) \) is a short Hall-Littlewood polynomial. (For, if \( \Delta_s^+ = \Delta^+ \) or if \( \Delta_s^+ \) and \( \Pi_\lambda \) are replaced with \( \Delta^+ \) and \( \Pi \) in the above definition, then one obtains the usual Hall-Littlewood polynomials \( P_\lambda(q) \) for \( \Delta \).)

**Proposition 5.1.**

\[
\mathcal{P}_\lambda(q) = J(e^{\lambda + \rho} \prod_{\alpha \in \Delta_s^+, (\alpha, \lambda) > 0} (1 - qe^\alpha)) J(\rho)^{-1}.
\]

**Proof.** 1) First consider the case in which \( \lambda = 0 \). Here

\[
J(e^\rho) \cdot j(e^0 \cdot \Delta_q^{(s)}) = J\left( \sum_{A \subset \Delta_s^+} (-q)^{\#A} e^{\rho - |A|} \right).
\]

It is known that \( \rho - |A| \) is regular if and only if \( A = N(w) \) for some \( w \in W \) [17]. Since \( A \subset \Delta_s^+ \), Lemma 4.5(ii) shows that actually \( w \in W(\Pi_\lambda) \). Hence

\[
J\left( \sum_{A \subset \Delta_s^+} (-q)^{\#A} e^{\rho - |A|} \right) = \sum_{w \in W(\Pi_\lambda)} (-q)^{\ell(w)} J(e^{\rho - |A|}) = \sum_{w \in W(\Pi_\lambda)} q^{\ell(w)} J(e^\rho) = t^{(\Pi_\lambda)}(q) J(e^\rho).
\]

This proves that \( \mathcal{P}_0(q) = 1 \).

2) For an arbitrary \( \lambda \in \mathfrak{X}_+ \), we notice that \( \sum_{w \in W_\lambda} \xi(w) w(e^{\lambda + \rho} \prod_{\alpha \in \Delta_s^+} (1 - qe^{-\alpha})) \) is divisible by \( t^{(\Pi_\lambda)}(q) \), by the first part of proof.

(One has to consider the splitting \( \prod_{\alpha \in \Delta_s^+} (1 - qe^{-\alpha}) = \prod_{\alpha : (\alpha, \lambda) = 0} (\ldots) \prod_{\alpha : (\alpha, \lambda) > 0} (\ldots) \), and use the fact that \( w(\prod_{\alpha : (\alpha, \lambda) = 0} (1 - qe^{-\alpha})) = \prod_{\alpha : (\alpha, \lambda) > 0} (1 - qe^{-\alpha}) \) for any \( w \in W_\lambda \).

This is already sufficient to conclude that \( \mathcal{P}_\lambda(q) \) belongs to \( \Lambda[q] \). Further easy calculations that require a splitting \( W \simeq W^\lambda \times W_\lambda \) are left to the reader. \( \square \)
Remark 5.2. Our proof is inspired by the remark in [6, p.70, last paragraph], where R. Gupta refers to Macdonald’s argument for the Hall-Littlewood symmetric functions.

Remark 5.3. The Hall-Littlewood polynomials $P_{\lambda}(q)$ interpolate between the irreducible characters $\chi_{\lambda}$ (if $q = 0$) and orbital sums $\frac{1}{\#(W)} \sum_{w \in W} e^{w\lambda}$ (if $q = 1$). For the short Hall-Littlewood polynomials $\overline{P}_{\lambda}(q)$, we still have $\overline{P}_{\lambda}(0) = \chi_{\lambda}$. At $q = 1$, we obtain a linear combination of irreducible characters for $H$. Namely, if $\chi_{\mu}^{(H)}$ denote the character of $V_{\mu}^{(H)}$, $\mu \in \mathfrak{X}_{+,H}$, then

$$\overline{P}_{\lambda}(1) = \frac{1}{\#(W(\Pi_{s}))} \sum_{w \in W(\Pi_{s})} \chi_{\mu}^{(H)}.$$  

An easy proof uses the semi-direct product structure of $W$ (Lemma 4.5) and Weyl’s character formula for $H$. (Note that if $\lambda \in \mathfrak{X}_{+}$, then $w\lambda \in \mathfrak{X}_{+,H}$ for any $w \in W(\Pi_{s})$.)

Theorem 5.4. In $\Lambda[[q]]$, the following relations hold:

1. $\langle E_{\mu}(q), \overline{P}_{\lambda}(q) \rangle = \delta_{\lambda,\mu}$
2. $\overline{E}_{\mu}(q) = \frac{j_{\mu}(\Pi_{s})}{\prod_{\alpha \in \Delta_{s}}(1 - qe^{\alpha})} \overline{P}_{\mu}(q)$ and $\overline{E}_{0}(q) = \frac{t_{0}(\Pi_{s})}{\prod_{\alpha \in \Delta_{s}}(1 - qe^{\alpha})}$.

Proof. (1) We mimic Gupta’s proof of [6, Theorem 2.5]. The plan is as follows:

(i) If $\chi_{\pi}$ occurs in $\overline{E}_{\mu}(q) = j(e^{\mu} \Delta_{q}^{(s)})$, then $\pi \succcurlyeq \mu$; and the coefficient of $\chi_{\mu}$ equals 1;
(ii) If $\chi_{\pi}$ occurs in $j(e^{\lambda} \Delta_{q}^{(s)})$, then $\pi \preccurlyeq \lambda$; and the coefficient of $\chi_{\lambda}$ equals $t_{\lambda}(\Pi_{s}^{})$;
(iii) Put $c_{\lambda,\mu} = \langle j(e^{\lambda} \Delta_{q}^{(s)}), j(e^{\mu} \Delta_{q}^{(s)}) \rangle$. Then $c_{\lambda,\mu} = c_{\mu,\lambda}$ and hence $t_{\mu}(\Pi_{s}^{}) \cdot \langle \overline{E}_{\lambda}(q), \overline{P}_{\mu}(q) \rangle = t_{\lambda}(\Pi_{s}^{}) \cdot \langle \overline{E}_{\mu}(q), \overline{P}_{\lambda}(q) \rangle$.

It will then follow that $c_{\lambda,\mu} = \delta_{\lambda,\mu} t_{\lambda}(\Pi_{s}^{})$ proving the assertion.

For (i): By Weyl’s character formula, the coefficient of $\chi_{\pi}$ in $j(e^{\mu} \Delta_{q}^{(s)})$ equals the coefficient of $e^{\pi + \rho}$ in (the expansion of)

$$J(e^{\rho}) \overline{E}_{\mu}(q) = \sum_{w \in W} \varepsilon(w) w \left( \frac{e^{\mu + \rho}}{\prod_{\alpha \in \Delta_{s}^{+}}(1 - qe^{\alpha})} \right).$$

This coefficient equals $\sum_{w,B} \varepsilon(w) q^{|B|}$, where the summation is over $w \in W$ and multi-sets $B$ of $\Delta_{s}^{+}$ such that $\pi + \rho = w(\mu + \rho + |B|)$. Then $\pi + \rho \succcurlyeq w^{-1}(\pi + \rho) = \mu + \rho + |B| \succcurlyeq \mu + \rho$. Hence $\pi \succcurlyeq \mu$. If $\pi = \mu$, then the only possibility is $w = 1$ and $B = \emptyset$.

For (ii): Now, we are interested in the coefficient of $e^{\pi + \rho}$ in

$$\sum_{w \in W} \varepsilon(w) w \left( e^{\lambda + \rho} \prod_{\alpha \in \Delta_{s}^{+}}(1 - qe^{-\alpha}) \right),$$

It is equal to $\sum_{w,A} \varepsilon(w) (-q)^{|A|}$, where the summation is over $w \in W$ and subsets $A \subset \Delta_{s}^{+}$ such that $\pi + \rho = w(\lambda + \rho - |A|)$. Since $w\lambda \preccurlyeq \lambda$ and $w(\rho - |A|) \preccurlyeq \rho$, we obtain $\pi + \rho \preccurlyeq \lambda + \rho$. Moreover, in case of equality we have $w\lambda = \lambda$ and $\rho - w^{-1}\rho = |A|$. This means that $w \in W_{\lambda}$.
and $N(w) = A \subset \Delta^+$. By Lemma 4.5(ii), we conclude that $w \in W(\Pi_\pi)$. Thus, $\# A = \ell(w)$ and the coefficient of $e^{\lambda+\rho}$ equals $\sum_{w \in W(\Pi_\pi)} q^{\ell(w)} = t^{(\Pi_\pi)}(q)$.

For (iii): Set $\bar{\xi} = \frac{1}{\prod_{a \in \Delta^+} (1 - q e^a)}$. It is a $W$-invariant element of $\Lambda[[q]]$ and $\Delta_q \bar{\xi} = \bar{\Delta}_q$. Hence $j(e^{\mu} \cdot \Delta_q \bar{\xi}) = j(e^{\mu} \cdot \bar{\Delta}_q)$. But $\bar{\xi}$ is also a self-dual character. Thus, we have

$$c_{\lambda, \mu} = \langle j(e^{\mu} \cdot \Delta_q \bar{\xi}), j(e^{\mu} \cdot \bar{\Delta}_q) \bar{\xi} \rangle = \langle j(e^{\mu} \cdot \Delta_q \bar{\xi}), j(e^{\mu} \cdot \bar{\Delta}_q) \rangle = c_{\mu, \lambda}.$$}

(2) The equality $P_\mu(q) = t^{(\Pi_\pi)}(q) \bar{\xi} \cdot P'_\mu(q)$ is essentially proved in (iii). Taking $\mu = 0$ yields the rest. $\square$

**Proposition 5.5.** $P_\mu(q) = \sum_{\lambda \in \mathcal{X}_+} \bar{m}_{\lambda}(q) \chi_{\lambda}$.

**Proof.** By definition, $J(e^\rho) P_\mu(q) = J\left(\frac{e^{\mu+\rho}}{\prod_{a \in \Delta^+} (1 - q e^a)}\right) = \sum_{\nu} P_q(\nu) J(e^{\mu+\nu+\rho})$.

The weight $\mu + \nu + \rho$ contributes to the last sum if and only if $\mu + \nu + \rho = w(\lambda + \rho)$ for some $\lambda \in \mathcal{X}_+$ and $w \in W$. Hence

$$\sum_{\nu} P_q(\nu) J(e^{\mu+\nu+\rho}) = \sum_{\lambda \in \mathcal{X}_+} \sum_{w \in W} P_q(w(\lambda + \rho) - (\mu + \rho)) J(e^{w(\lambda + \rho)})$$

$$= \sum_{\lambda \in \mathcal{X}_+} \sum_{w \in W} \varepsilon(w) P_q(w(\lambda + \rho) - (\mu + \rho)) J(e^{\lambda+\rho}) = \sum_{\lambda \in \mathcal{X}_+} \bar{m}_{\lambda}(q) J(e^{\lambda+\rho}).$$

$\square$

Part 1(ii) in the proof of Theorem 5.4 shows that $\{P_{\lambda}(q)\}_{\lambda \in \mathcal{X}_+}$ is a $\mathbb{Z}$-basis in $\Lambda[q]$. Furthermore, Theorem 5.4(1) and Proposition 5.5 readily imply that

$$\chi_{\pi} = \sum_{\lambda \in \mathcal{X}_+} \bar{m}_{\lambda}(q) P_{\lambda}(q).$$

Note that this sum is finite, since $\bar{m}_{\pi}(q) = 0$ unless $\lambda \preceq \pi$. Let us transform the expression for $P_{\lambda}(q)$ given by definition:

$$J(e^\rho) t^{(\Pi_\pi)}(q) \bar{P}_{\lambda}(q) = J(e^{\lambda+\rho} \prod_{a \in \Delta^+} (1 - q e^{-a}))$$

$$= J\left(\frac{e^\lambda \prod_{\alpha \in \Delta^+} (1 - q e^{-a})}{\prod_{\alpha > 0} (1 - e^{-a})}\right) \cdot e^\rho \prod_{\alpha > 0} (1 - e^{-a}) = \sum_{w} \left(\frac{e^\lambda \prod_{\alpha \in \Delta^+} (1 - q e^{-a})}{\prod_{\alpha > 0} (1 - e^{-a})}\right) \cdot J(e^\rho).$$

Hence $\bar{P}_{\lambda}(q) = \frac{1}{t^{(\Pi_\pi)}(q)} \sum_{w} \left(\frac{e^\lambda \prod_{\alpha \in \Delta^+} (1 - q e^{-a})}{\prod_{\alpha > 0} (1 - e^{-a})}\right)$, and substituting this in Equation (5.1) we obtain a generalisation of an identity of Kato (cf. [6, Theorem 3.9]):

$$\chi_{\pi} = \sum_{\lambda \in \mathcal{X}_+} \bar{m}_{\lambda}(q) \frac{1}{t^{(\Pi_\pi)}(q)} \sum_{w} \left(\frac{e^\lambda \prod_{\alpha \in \Delta^+} (1 - q e^{-a})}{\prod_{\alpha > 0} (1 - e^{-a})}\right).$$
Taking \( q = 1 \), we obtain
\[
\chi_\pi = \sum_{\lambda \in \chi_+} \frac{1}{\# W(\Pi_s)_{\lambda}} \sum_{w \in W} w \left( \frac{e^\lambda}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})} \right).
\]
Taking into account that \( W = W(\Pi_s) \times W_i \) and \( \overline{m}_{\pi}^\lambda = \overline{m}_{\pi}^{\lambda} \) for any \( w \in W(\Pi_s) \), this specialisation is equivalent to the formula \( \chi_\pi = \sum_{\lambda \in \chi_+, \mu} \overline{m}_{\pi}^{\lambda} \chi_\lambda \).

We introduce another bilinear form in \( \Lambda[ q ] \) such that \( \{ \overline{T}_\lambda(q) \} \) to be an orthogonal basis. To this end, the null-cone in \( V_\Phi \) plays the same role as the nilpotent cone \( \mathfrak{N} \subset \mathfrak{g} \) for the Hall-Littlewood polynomials \( P_\lambda(q) \), cf. [7, §2].

For a graded \( G \)-module \( M = \bigoplus_i M_i \) with \( \dim M_i < \infty \), the graded character of \( M \), \( \text{ch}_q(M) \), is the formal sum \( \sum_i \text{ch}(M_i)q^i \in \Lambda[ [ q ] ] \).

**Proposition 5.6.** The graded character of the graded \( G \)-algebra \( \mathbb{C}[\mathfrak{N}(V_\Phi)] \) equals
\[
\text{ch}_q(\mathbb{C}[\mathfrak{N}(V_\Phi)]) = \frac{\ell_0(\Pi_s)(q)}{\prod_{\alpha \in \Delta_+} (1 - qe^{\alpha})} = \ell_0(\Pi_s)(q) \overline{\xi} = \overline{F}_0(q).
\]

**Proof.** The weight structure of \( V_\Phi \) (Lemma 4.1) shows that the graded character of \( \mathbb{C}[V_\Phi] \) equals \( \text{ch}_q(\mathbb{C}[V_\Phi]) = \frac{1}{\prod_{\alpha \in \Delta_+} (1 - qe^{\alpha})} \). We know that \( \mathfrak{N}(V_\Phi) \) is a complete intersection of codimension \( m := \# \Pi_s \) and the ideal of \( \mathfrak{N}(V_\Phi) \) is generated by algebraically independent generators of \( \mathbb{C}[V_\Phi]^G \). Furthermore, if \( d_1, \ldots, d_m \) are the degrees of these generators, then \( d_1 - 1, \ldots, d_m - 1 \) are the exponents of \( W(\Pi_s) \) (Prop. 4.6). Thus,
\[
\text{ch}_q(\mathbb{C}[\mathfrak{N}(V_\Phi)]) = \frac{\prod_{i=1}^m (1 - q^{d_i})}{\prod_{\alpha \in \Delta_+} (1 - qe^{\alpha})} = \frac{\prod_{i=1}^m (1 + q + \cdots + q^{d_i-1})}{\prod_{\alpha \in \Delta_+} (1 - qe^{\alpha})},
\]
and it is well known that \( \ell_0(\Pi_s)(q) = \prod_{i=1}^m (1 + q + \cdots + q^{d_i-1}) \).

Combining Propositions 5.5 and 5.6 yields
\[
\text{ch}_q(\mathbb{C}[\mathfrak{N}(V_\Phi)]) = \sum_{\lambda \in \chi_+} \overline{m}_{\lambda}^q(q) \chi_\lambda,
\]
which is [24, Theorem 4]. In other words, \( \sum_{i \geq 0} \dim(\text{Hom}_G(V_\lambda, \mathbb{C}[\mathfrak{N}(V_\Phi)])i) q^i = \overline{m}_{\lambda}^q(q) \) for every \( \lambda \in \chi_+ \).

Define a new bilinear form in \( \Lambda[q] \) by letting
\[
\langle \chi_\lambda, \chi_\mu \rangle = \langle \chi_\lambda \chi_\mu^*, \ell_0(\Pi_s)(q) \overline{\xi} \rangle = \langle \chi_\lambda, \ell_0(\Pi_s)(q) \overline{\xi} \chi_\mu \rangle.
\]
In view of Proposition 5.6, \( \langle \chi_\lambda, \chi_\mu \rangle \) is a polynomial in \( q \) that counts graded occurrences of the \( G \)-module \( V_\lambda \otimes V_\mu^* \) in \( \mathbb{C}[\mathfrak{N}(V_\Phi)] \).

**Theorem 5.7.** \( \langle \overline{T}_\lambda(q), \overline{T}_\mu(q) \rangle = \frac{\ell_0(\Pi_s)(q)}{\ell_\mu(\Pi_s)(q)} \delta_{\lambda, \mu} \).

Proof. By definition and Theorem 5.4, we have
\[ \langle \overline{\mathcal{F}}_\lambda(q), \overline{\mathcal{F}}_\mu(q) \rangle = \overline{\mathcal{F}}_\lambda(q), \xi(t^{(H)})_\lambda(q) \xi(t^{(H)})_\mu(q) \rangle = \langle \overline{\mathcal{F}}_\lambda(q), \left( \frac{t^{(H)}_0(q)}{t^{(H)}_\mu(q)} \right) \mathcal{E}_\mu(q) \rangle = \frac{t^{(H)}_0(q)}{t^{(H)}_\mu(q)} \delta_{\lambda, \mu}. \]

Here we also use the fact that \( \Delta^{(s)}_q \xi = \Delta^{(s)}_q \) and hence \( t^{(H)}_\mu(q) \xi \overline{\mathcal{F}}_\mu(q) = \mathcal{E}_\mu(q). \) \( \square \)

Finally, using Eq. (5.1), we obtain
\[ \langle \chi_\lambda, \chi_\mu \rangle = \sum_{\pi \in \mathcal{X}_+} \overline{m}_\lambda^{\pi}(q) \overline{m}_\mu^{\pi}(q) \frac{t^{(H)}_0(q)}{t^{(H)}_\pi(q)}. \]

6. MISCELLANEOUS REMARKS

6.1. It is noticed in [6, 5.1] that Lusztig’s \( q \)-analogues \( m_\lambda^{\mu}(q) \) satisfy the identity
\[ \sum_{\mu \in \mathcal{X}} m_\lambda^{\mu}(q) e^\mu = \frac{J(e^\lambda + \rho)}{e^\rho \prod_{\alpha > 0} (1 - q e^{-\alpha})} = \chi_\lambda \cdot \prod_{\alpha > 0} \frac{(1 - e^{-\alpha})}{(1 - q e^{-\alpha})}. \]

This can be regarded as quantisation of the equality \( \chi_\lambda = \sum_\mu m_\lambda^\mu e^\mu \), which describes \( V_\lambda \) as \( T \)-module. In the context of short \( q \)-analogues, we wish to have a quantisation of the equality \( \chi_\lambda = \sum_{\mu \in \mathcal{X}_+, H} m_\lambda^{\mu}(H) \), which describes \( V_\lambda \) as \( H \)-module [8, §3]. The desired quantisation is

**Proposition 6.1.** \[ \sum_{\mu \in \mathcal{X}_+, H} \overline{m}_\lambda^{\mu}(q) \chi_\mu^{(H)} = \chi_\lambda \cdot \frac{1}{\#W_i} \sum_{w \in W_i} w \left( \prod_{\alpha \in \Delta^+_i} \frac{1 - e^{-\alpha}}{1 - q e^{-\alpha}} \right). \]

**Proof.** Using Weyl’s formula, the function \( (\mu \in \mathcal{X}_+, H) \mapsto \chi_\mu^{(H)} \) can be extended to the whole of \( \mathcal{X} \) such that it will satisfy the identity \( \chi_\mu^{(H)} = \varepsilon(w) \chi_\mu^{(H)}, \ w \in W_i. \) Recall that ‘\( \circ \)’ stands for the shifted action of \( W_i. \) Since the same identity holds for \( \overline{m}_\lambda^{\mu}(q) \), see Eq. (4.2), the left hand side can be replaced with \( \frac{1}{\#W_i} \sum_{\mu \in \mathcal{X}} \overline{m}_\lambda^{\mu}(q) \chi_\mu^{(H)}. \) The rest can by achieved via routine transformations of this sum, using the definition of \( \overline{m}_\lambda^{\mu}(q) \) and Weyl’s character formulae for \( H \) and \( G. \) \( \square \)

Yet another quantisation, which is easier to prove, is
\[ \sum_{\mu \in \mathcal{X}} \overline{m}_\lambda^{\mu}(q) e^\mu = \frac{J(e^\lambda + \rho)}{e^\rho \prod_{\alpha \in \Delta^+_i} (1 - q e^{-\alpha})} = \chi_\lambda \cdot \prod_{\alpha > 0} \frac{(1 - e^{-\alpha})}{\prod_{\alpha \in \Delta^+_i} (1 - q e^{-\alpha})}. \]

Comparing Equations (6.1) and (6.2), we obtain a relation between Lusztig’s and short \( q \)-analogues:
\[ \prod_{\alpha \in \Delta^+_i} (1 - q e^{-\alpha}) \sum_{\mu \in \mathcal{X}} m_\lambda^{\mu}(q) e^\mu = \sum_{\nu \in \mathcal{X}} \overline{m}_\lambda^{\nu}(q) e^\nu. \]
Whence $\mathfrak{m}_\mu(q) = \sum_{A \subseteq \Delta_1^+} (-q)^{#A} m_{\mu}^{\mu+A}(q)$. Or, conversely, $m_{\lambda}^\mu(q) = \sum_{B} q^{-B} \mathfrak{m}_\lambda^{A+B}(q)$, where $B$ ranges over the finite multisets in $\Delta_1^+$. In particular, taking $q = 1$ and $\mu = 0$, we obtain

$$\dim V_\lambda^H = \mathfrak{m}_\lambda^0 = \sum_{A \subseteq \Delta_1^+} (-1)^{#A} m_{\lambda}^{\mu+A}.$$ 

**Example.** If $G = Sp_{2n}$, then $H = (SL_2)^n$ and $\Delta_1^+ = \{2\varepsilon_1, \ldots, 2\varepsilon_n\}$. Here $\varepsilon_i + \ldots + \varepsilon_k$ is $W$-conjugate to $\varphi_k = \varepsilon_1 + \ldots + \varepsilon_k$ and the previous relation becomes

$$\dim V_\lambda^H = \sum_{k=0}^n (-1)^k \binom{n}{k} m_{\lambda}^{2\varphi_k}.$$

6.2. It is well known that, for $\lambda$ strictly dominant, the Hall-Littlewood polynomials $P_\lambda(q)$ have a nice specialisation at $q = -1$: If $\lambda \gg \rho$, then $P_\lambda(-1) = \chi_{\lambda - \rho} \chi_\rho$. (See [23, 7.4] for a generalisation to symmetrisable Kac-Moody algebras.) For $\Delta$ of type $A_n$, $P_\lambda(-1)$ is a classical Schur’s $Q$-function [18, III.8]. A similar phenomenon occurs for short Hall-Littlewood polynomials.

**Proposition 6.2.** Suppose $\lambda \gg \rho_s$ and $G$ is of type $B_n, C_n,$ or $F_4$. Then $P_\lambda(-1) = \chi_{\lambda - \rho_s} \chi_{\rho_s}$.

**Proof.** If $\lambda \gg \rho_s$, then $t^{(\Pi_\lambda)}(q) = 1$ and

$$P_\lambda(-1) = \sum_{w \in W} \varepsilon(w) w(e^{\lambda - \rho_s + \rho})(e^{\rho_s}) \prod_{\lambda \in \Delta_1^+} (1 + e^{-\alpha}) \cdot J(e^{\rho})^{-1} = \chi_{\lambda - \rho_s} \cdot \prod_{\lambda \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2}).$$

For $G$ is of type $B_n, C_n,$ or $F_4$, it is known that $\chi_{\rho_s} = \prod_{\lambda \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2})$ [20, Theorem 2.9].

**Remark 6.3.** The proof of equality $\chi_{\rho_s} = \prod_{\lambda \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2})$ in [20] is only based on the assumption that $\|\text{long}\|^2/\|\text{short}\|^2 = 2$, i.e., it does not refer to classification. For $G_2$, the true equality is $\prod_{\lambda \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2}) = \chi_{\rho_s} + 1$.

6.3. Ranee Brylinski proved that Lusztig’s $q$-analogues $m_{\lambda}^\mu(q)$ can be computed via a principal filtration on $V_\lambda^\mu$ whenever $H^i(G \times_B U, L_{G \times_B U}(C_\mu)^*) = 0$ for all $i \geqslant 1$. Namely, $m_{\lambda}^\mu(q)$ coincides with the “jump polynomial” of the principal filtration, see [5] for details. Another approach to her results can be found in [11].

I hope that a similar description exists for short $q$-analogues. First, we need a subspace of $V_\lambda$ whose dimension equals $\mathfrak{m}_\mu = \text{mult}(V_\mu^H, V_\lambda)$. Let $V_\lambda^{U(H)}$ be the subspace of $H$-highest vectors in $V_\lambda$ with respect to $\Delta_1^+$. Then $V_\lambda^{U(H),\mu} = V_\lambda^{U(H)} \cap V_\lambda^\mu$ has the required dimension. For $\alpha \in \Delta_1^+$, let $e_\alpha$ be a nonzero root vector of $g$. Brylinski’s principal filtration
is determined by the principal nilpotent element $e = \sum_{\alpha \in \Pi} e_\alpha$. In the context of short $q$-analogues, we consider $e_s = \sum_{\alpha \in \Pi_s} e_\alpha$ and the corresponding filtration of $V_{\lambda}^{U(H),\mu}$. That is, we set

$$J_{e_s}^p(V_{\lambda}^{U(H),\mu}) = \{ v \in V_{\lambda}^{U(H),\mu} \mid e_{s}^{p+1} \cdot v = 0 \}.$$  

The jump polynomial is defined to be

$$\tau_{\lambda}^\mu(q) = \sum_{p \geq 0} \dim(J_{e_s}^p(V_{\lambda}^{U(H),\mu})/J_{e_s}^{p-1}(V_{\lambda}^{U(H),\mu}))q^p.$$  

**Conjecture 6.4.** If $\mu \in \mathcal{X}_{+,H}$ satisfies vanishing conditions of Theorem 4.10, then $\tau_{\lambda}^\mu(q) = \overline{\tau}_{\lambda}^\mu(q)$.

6.4. Although the collapsing $f : Z = G \times_B V_\theta^+ \to \mathfrak{N}(V_\theta)$ is not generically finite, it can be used for deriving useful properties of the null-cone. Let $\varrho : \mathcal{O}_{\mathfrak{N}(V_\theta)} \to Rf_\ast \mathcal{O}_Z$ be the corresponding natural morphism. Since $f$ is projective, $H^0(Z, \mathcal{O}_Z)$ is a finite $\mathbb{C}[\mathfrak{N}(V_\theta)]$-module; and there is the trace map $H^0(Z, \mathcal{O}_Z) \to \mathbb{C}[\mathfrak{N}(V_\theta)]$ because $\mathfrak{N}(V_\theta)$ is normal. The trace map determines a morphism (in the derived category of $\mathcal{O}_{\mathfrak{N}(V_\theta)}$-modules) $\varrho' : Rf_\ast \mathcal{O}_Z \to \mathcal{O}_{\mathfrak{N}(V_\theta)}$. By Theorem 4.10, $H^i(Z, \mathcal{O}_Z) = 0$ for $i \geq 1$, i.e., $R^i f_\ast \mathcal{O}_Z = 0$ for $i \geq 1$. Hence $\varrho' \circ \varrho$ is a quasi-isomorphism of $\mathcal{O}_{\mathfrak{N}(V_\theta)}$ with itself. Therefore, by [15, Theorem 1], $\mathfrak{N}(V_\theta)$ has only rational singularities.

Clearly, this argument works in a more general context and yields the following:

**Proposition 6.5.** Let $N$ be a $P$-stable subspace in a $G$-module $V$. If $H^i(G \times_P N, \mathcal{O}_{G \times_P N}) = 0$ for all $i \geq 1$, then the normalisation of $G \cdot N$ has only rational singularities.

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