A Holographic model of the Kondo effect

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Outline

- Introduction
- Stringy model
- Bottom-up model
- Future directions
Kondo effect

Metals: Fermi liquid + phonons + impurities: \( \rho \sim \rho_0 + T^2 \)

But in some metals at low temperatures \( \rho \sim -\log(T) \)
Kondo effect

Scattering with magnetic impurities

\[ \rho \sim \rho_0 \left( 1 + \kappa \log \frac{T}{|\epsilon - \epsilon_F|} \right) \]

Antiferromagnetic coupling \( \kappa < 0 \)
Kondo effect

Perturbation theory breaks down at $T_K = |\epsilon - \epsilon_F| e^{1/\kappa}$

*High $T$ – weak coupling*

*Conduction Electron Spin*

*Non magnetic state*

*Low $T$ – strong coupling*

*Impurity Spin*
Kondo effect

- Single impurity problem solved
  Wilson’s RG, Nozières’ Fermi liquid description, the Bethe Ansatz, large-\(N\) limits, conformal field theory...

- Multiple impurities:
  Heavy fermion compounds with strange metal behaviour

\[ \rho \sim T \]

may be described by a Kondo lattice

- Holography: non-perturbative, large-\(N\)

Goal today: construct a holographic model for a single impurity
Kondo model \cite{Kondo, Affleck, Ludwig}:

\[
H = \frac{v_F}{2\pi} \psi_L^\dagger i \partial_x \psi_L + v_F \lambda_K \delta(x) \vec{S} \cdot \psi_L^\dagger \frac{1}{2} \vec{\tau} \psi_L,
\]

- s-wave reduction: 1+1 dimensions
- Kondo coupling marginal classically (CFT)
- Asymptotic freedom: UV fixed point
UV CFT

- Symmetries: Spin $SU(N)$, $k$ channels $SU(k)$, Charge $U(1)$
- Kac-Moody algebra: $SU(N)_k \times SU(k)_N \times U(1)$

$$[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + k \frac{n}{2} \delta^{ab} \delta_{n,-m}$$

- Finite number of highest weight states. $SU(2)_k$: spin $\leq k/2$
- Sugawara construction:

$$H = \frac{1}{2\pi(N+k)} J^a J^a + \frac{1}{2\pi(k+N)} J^A J^A + \frac{1}{4\pi Nk} J^2 + \lambda_K \delta(x) \vec{S} \cdot \vec{J}$$
CFT approach

IR CFT

- Redefinition of spin current:
  \[ \mathcal{J}^a \equiv J^a + \pi(N + k)\lambda_K \delta(x) S^a \]

- Critical coupling
  \[ \lambda_K = \frac{2}{N + k} \]

- Hamiltonian:
  \[ H = \frac{1}{2\pi(N + k)} \mathcal{J}^a \mathcal{J}^a + \frac{1}{2\pi(k + N)} J^A J^A + \frac{1}{4\pi Nk} J^2 \]

No impurity!

[Affleck & Ludwig '95]
CFT approach

- IR CFT = UV CFT with shifted spectrum
- IR spin representations = UV spin representations + impurity spin

Example: one channel, spin $N = 2$, $SU(2)_{\text{1}} \times U(1)$, $s_{\text{imp}} = 1/2$
  - UV: Neveu-Schwarz boundary conditions
    (spin, charge) = (0, 0), ($\pm 1/2$, 1)
  - IR: Ramond boundary condition
    (spin, charge) = ($\pm 1/2$, 2), (0, 1)
  - Phase of electron wavefunction on a circle changes by $\pi/2$
Possible phases in $SU(2)_k$:

- **Underscreening**: $2s_{\text{imp}} > k$
  Fermi liquid + impurity of spin $|s_{\text{imp}} - k/2|$

- **Critical screening**: $2s_{\text{imp}} = k$
  IR fixed point: $k$ free left-movers

- **Overscreening**: $2s_{\text{imp}} < k$
  Non-trivial IR fixed point: non-Fermi liquid behavior

Qualitatively similar for higher spin
Large-$N$

- $N \to \infty$, $\lambda_K \to 0$, $\lambda_K N$ fixed
- Spin of impurity: Young tableaux with $Q$ boxes
- Totally antisymmetric representation:
  \[ S^a = \chi^\dagger T^a \chi \]
  
  Slave fermions, dimension $[\chi] = 0$

  \[ \chi^\dagger \chi = Q \]

  Additional $U(N_f)$ symmetry

- Critical ($k = 1$) or overscreening ($k \geq 2$)
Kondo coupling as double-trace deformation:

\[ \lambda_K \delta(x) J^a S^a = \lambda_K \delta(x) \left( \psi_L^\dagger T^a \psi_L \right) \left( \chi^\dagger T^a \chi \right) \]

\[ = \frac{1}{2} \lambda_K \delta(x) \left( \psi_L^\dagger \chi \right) \left( \chi^\dagger \psi_L \right) \]

\[ = \frac{1}{2} \lambda_K \delta(x) \mathcal{O} \mathcal{O}^\dagger \]

- \( \mathcal{O} \) SU(N) singlet, charged under \( U(N_f) \times SU(k) \times U(1) \)
- Dimensions: \( [\psi_L] = [\mathcal{O}] = 1/2 \)
- Mean field transition:

\[ T > T_K, \quad \langle \mathcal{O} \rangle = 0, \quad SU(k) \times U(N_f) \times U(1) \]

\[ T < T_K, \quad \langle \mathcal{O} \rangle \neq 0, \quad SU(k) \times U(N_f) \times U(1) \rightarrow U(1)_D \]

[Senthil, Sachdev, Vojta]
Summary of Kondo effect at large $N$

- s-wave reduction: 1+1 chiral CFT + impurity
- double-trace coupling
- 0+1 superconductor

These will be the main ingredients to construct a holographic model
Supersymmetric defects with localized fermions

- D5/D3 $AdS_2 \subset AdS_5$
  [Kachru, Karch, Yaida] [Harrison, Kachru, Torroba]

- M2/D2 in ABJM $AdS_2 \subset AdS_4$
  [Jensen, Kachru, Karch, Polchinski, Silverstein]

- D6 in ABJM $AdS_2 \subset AdS_4$, [Benincasa, Ramallo]
  with backreaction [Itsios, Sfetsos, Zoakos]

- $D(8-p)$ in $Dp$ background $S^{7-p} \subset S^{8-p}$ [Benincasa, Ramallo]
  other sphere wrappings [Karaiskos, Sfetsos, Tsatis]

- Spectrum of Wilson loops
  [Mueck] [Faraggi, Pando Zayas] [Faraggi, Mueck, Pando Zayas]
### Stringy model

| | $x^0$ | $x^1$ | $x^2$ | $x^3$ | $x^4$ | $x^5$ | $x^6$ | $x^7$ | $x^8$ | $x^9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $N_c$ D3 | ● | ● | ● | ● | – | – | – | – | – | – |
| $N_7$ D7 | ● | ● | – | – | ● | ● | ● | ● | ● | ● |
| $N_5$ D5 | ● | – | – | – | ● | ● | ● | ● | ● | – |

- 3-7 strings = chiral fermions (current algebra)
- 3-5 strings = slave fermions
- 5-7 strings = bifundamental scalar (tachyon)

D5 branes become magnetic flux on D7 branes

$$S_{D7} \subset \int_{D7} P[C_6] \wedge F$$
Stringy model

- $N_c \rightarrow \infty : \text{AdS}_5 \times S^5$ background

$$\int_{S^5} F_5 = g_s (2\pi)^2 (2\pi \alpha')^2 N_c$$

- 7-7 strings: gauge field $A_\mu$ dual to $J_\mu$

$$S_{D7} \supset -\frac{N_c}{4\pi} \int_{\text{AdS}_3} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

Global symmetry:

$U(N_7)_{N_c}$
Stringy model

- 5-5 strings: gauge field $a_m$ dual to $Q$

\[ S_{D5} \supset N_5 \int P[C_4] \wedge f \]

- Dual to completely antisymmetric Wilson loop
  [Yamaguchi '06; Gomis, Passerini '06]

- Charge $Q = \text{number of fundamental strings}$

- Size of D5 on $S^5$ [Camino, Paredes, Ramallo '01]

\[ ds^2_{S^5} = d\theta^2 + \sin^2 \theta \ ds^2_{S^4}, \quad \theta_Q = \frac{Q}{N_c} \pi \]

Maximal charge $Q = N_c - 1$
Stringy model

- 5-7 strings: bifundamental scalar $\Phi$ dual to $\mathcal{O}$
- Double-trace coupling for $\mathcal{O}$ analogous to Kondo coupling
- Double-trace coupling can lead to condensation [Pomoni, Rastelli '08,'10]
- Holographic dual: boundary condition for $\Phi$ [Witten '01]
Bottom-up model

- s-wave reduction: $1+1$ CFT $\rightarrow \text{AdS}_3$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{z^2} \left( \frac{dz^2}{h(z)} - h(z) dt^2 + dx^2 \right), \quad h(z) = 1 - z^2/z_H^2,$$

Temperature: $T = 1/(2\pi z_H)$

- $SU(N)$-spin, $k = 1$ channel of chiral fermions, $U(1)$ charge

$$S_{CS} = -\frac{N}{4\pi} \int A \wedge dA$$

- Impurity $\text{AdS}_2$: $U(1)$ symmetry, operator $\mathcal{O}$

$$S_{\text{AdS}_2} = -\int d^3x \delta(x) \sqrt{-g} \left[ \frac{1}{4} f^{mn} f_{mn} + g^{mn} (D_m \Phi)^\dagger D_n \Phi + M^2 \Phi^\dagger \Phi \right]$$

$$D_m \Phi = \partial_m \Phi + iA_m \Phi - ia_m \Phi$$
Asymptotics

- Gauge field: charge $Q$ determines the spin representation of impurity

  \[ a_t(z) = \frac{Q}{z} + \mu \]

  Charge is an irrelevant operator: UV behavior is modified

  Scalar field effective mass fixed at the BF bound

  \[ M_{\text{eff}}^2 = M^2 - Q^2 = -\frac{1}{4} \]

- Scalar field at the BF bound:

  \[ \Phi = z^{1/2}(\alpha \log(z) + \beta) \]

  UV conformal dimensions $\Delta = \frac{1}{2}$

  We set $M^2 = 0$, $Q = -1/2$
Kondo coupling

- Double-trace deformation = boundary condition \[ \text{[Witten '01]} \]
  \[ \alpha = \kappa \beta \]

- Renormalization:
  \[ \Phi = z^{1/2} \beta_0 (\kappa_0 \log(\Lambda z) + 1) = z^{1/2} \beta (\kappa \log(\mu z) + 1) \]

- Running coupling
  \[ \kappa = \frac{\kappa_0}{1 + \kappa_0 \ln \left( \frac{\Lambda}{\mu} \right)} \]

  Dynamical scale: \( \Lambda_K = \Lambda e^{1/\kappa_0} \)

- \( \kappa < 0 \) “antiferromagnetic”: UV asymptotic freedom
Phases

Normal phase ($\Phi = 0$):

- Background charge $Q = -1/2$:

  $$a_t(z) = \frac{Q}{z} + \mu, \quad \mu = -\frac{Q}{z_H}$$

Broken phase ($\Phi \neq 0$):

- Background charge $Q = -1/2$:

  $$a_t(z) \simeq \frac{Q}{z} + \mu_T + O((\log z)^3)$$

- Background scalar field:

  $$\Phi \simeq \left(\frac{z}{z_H}\right)^{1/2} \beta_T (\kappa_T \log(z/z_H) + 1)$$

- Values of $\mu_T$, $\kappa_T$ and $\beta_T$ determined numerically
Kondo coupling

- Finite temperature solution:
  \[ \Phi = (z/z_H)^{1/2} \beta_T (\kappa_T \log(z/z_H) + 1) = \beta_0 (\kappa_0 \log(\Lambda z) + 1) \]

- Temperature-dependent coupling
  \[ \kappa_T = \frac{\kappa_0}{1 + \kappa_0 \ln \left( \frac{\Lambda}{2\pi T} \right)} \]

- High-temperatures \( T \gg \Lambda_K \):
  \[ \kappa_T \approx \frac{1}{\ln \left( \frac{\Lambda}{2\pi T} \right)} < 0 \]

- Low temperatures \( T \ll \Lambda_K \)
  \[ \kappa_T \approx \frac{1}{\ln \left( \frac{\Lambda}{2\pi T} \right)} > 0 \]
Instabilities in the normal phase

Scalar field $\Phi = e^{-i\omega t} \phi$:

$$h \partial_z (h \partial_z \phi) + \omega^2 \phi + \frac{1}{4z^2} \phi = 0$$

Tachyonic modes $\omega = -i\Omega$ exist when

$$\kappa = \frac{\alpha}{\beta} = \frac{1}{H_{\frac{1}{2}}(\sqrt{4\Omega^2 - 1} - 1) - \log 4}$$

This is possible $\forall \kappa$ except $\kappa_c < \kappa \leq 0$,

$$\kappa_c = \frac{1}{H_{-\frac{1}{2}} - \log 4} \approx -0.360674$$

Stable at high temperatures, unstable at low temperatures
Free energy

- Free energy = Euclidean action = - Action

Counterterms scalar field:

\[ S_\Phi = -\int dt \sqrt{-\gamma} \left( \frac{1}{2} + \frac{1}{\log \varepsilon} \right) \Phi^\dagger \Phi + \kappa \int dt \beta^2 \]

Counterterms gauge field:

\[ S_{at} = +\frac{1}{2} \int dt \sqrt{-\gamma} \gamma^{tt} a_t^2 \]

\[ F - F_0 = -\alpha \beta - \frac{1}{2} Q(\mu + Q) - 2Q^2 \alpha^2 + 2Q^2 \alpha \beta - Q^2 \beta^2 - \kappa \beta^2 \]

+ finite bulk integral
Thermodynamics of the broken phase

Free energy vs $T / T_K$

Condensate vs $T / T_K$
Chern-Simons field

\[ S = \int dz \int \frac{d\omega d\omega' dq dq'}{(2\pi)^2} \left[ A_\mu(\omega', q') D^{\mu\nu}(\omega, q) \delta(\omega - \omega') \delta(q - q') A_\nu(\omega', q') 
+ A_\mu(\omega', q') B^{\mu\nu}(\omega) \delta(\omega - \omega') A_\nu(\omega', q') + j^m(\omega) \delta(\omega - \omega') \delta(q - q') A_m(\omega, q') \right]. \]

\[ D^{\mu\nu}(\omega, q) = \frac{k}{2\pi} \left[ \epsilon^{\mu z \nu} \partial_z - i\omega \epsilon^{\mu t \nu} + iq \epsilon^{\mu x \nu} \right], \]

\[ B^{\mu\nu}(\omega) = \frac{1}{2\pi} \sqrt{-g} g^{mn} \delta^\mu_m \delta^n_\nu \Phi^\dagger \Phi = \frac{1}{2\pi} \phi^2 \sqrt{-g} g^{mn} \delta^\mu_m \delta^n_\nu. \]

\[ j^m(\omega) = \sqrt{-g} g^{mn} a_n \Phi^\dagger \Phi = -\frac{a_t \phi^2}{h(z)} \delta^m_t. \]
Solutions:

\[ F_{tx} = -\frac{2\pi}{k} j^z(\omega) + \delta F_{tx}, \quad F_{zx} = \frac{2\pi}{k} j^t(\omega) + \delta F_{zx} \]

Normal phase:

\[ F_{tx} = F_{zx} = 0 \]

Broken phase:

\[ F_{tx} = 0, \quad F_{zx} = \frac{2\pi}{k} \frac{a_t \phi^2}{h(z)} \delta(x) \]

\[ F_{zx} \] conjugate to \( A_t \): charge localized at the defect
Fluctuations

\[ 0 = \frac{k}{2\pi} \delta F_{tx}(\omega, q) + \phi^2 h(z) \int \frac{dq'}{2\pi} \delta A_z(\omega, q'), \]

\[ 0 = \frac{k}{2\pi} \delta F_{zx}(\omega, q) + \frac{\phi^2}{h(z)} \int \frac{dq'}{2\pi} \delta A_t(\omega, q'), \]

\[ 0 = \frac{k}{2\pi} \delta F_{zt}(\omega, q). \]

Zero-momentum fluctuations in the broken phase:

\[ \delta A_m = \partial_m \lambda, \quad \partial_x \lambda = 0, \quad \lambda(\omega, q) = 2\pi \delta(q)\tilde{\lambda}(\omega) \]

Fluctuations transverse to \( AdS_2 \) defect \( \delta A_x \) decouple from \( \delta A_m \)

\( \Delta = 1 \) scalar operator localized at the impurity
Equation of zero-momentum fluctuations

$$\partial_z (h(z) \phi^2 \partial_z \tilde{\lambda}(\omega)) + \frac{\omega^2 \phi^2}{h(z)} \tilde{\lambda}(\omega) = 0$$

Boundary:

$$\tilde{\lambda}(\omega) \sim \frac{\sqrt{z}}{\phi h(z)} (AY_0(\omega z) + BJ_0(\omega z))$$

Horizon:

$$\tilde{\lambda}(\omega) \sim \frac{\sqrt{z}}{\phi h(z)} \left( C_{out} (1 - z)^{1+i\omega/2} + C_{in} (1 - z)^{1-i\omega/2} \right)$$

Quasinormal modes: $A = 0$, $C_{out} = 0$
Correlators

- **Action of Chern-Simons field fluctuations**

\[
S = - \lim_{z \to 0} \int \frac{d\omega}{2\pi} \left[ \frac{ik\omega}{\pi} \tilde{\lambda}(\omega) \delta A_x(-\omega) + \omega^2 \phi^2 \tilde{\lambda}(\omega) \partial_z \tilde{\lambda}(-\omega) \right].
\]

- **Counterterms**

\[
S_{ct} = - \lim_{z \to 0} \frac{1}{2} \int dt dx \delta(x) \sqrt{-\gamma \gamma^{tt}} A_t^2 = - \lim_{z \to 0} \frac{1}{2} \int \frac{d\omega}{2\pi} \frac{\omega^2}{z} \tilde{\lambda}(\omega) \tilde{\lambda}(-\omega).
\]

- **Correlation function**

\[
\langle J^x J^x \rangle(\omega, q, q') \approx \frac{\delta^2}{\delta A^2} (S + S_{ct})
\]
**Correlators**

- **Two-point function:**
  \[
  \langle J^x J^x \rangle(\omega, q, q') \simeq \left[ \frac{8}{\pi^2 \kappa(\omega)} - \frac{4}{\pi} G(\omega) \right] \delta(q) \delta(q'),
  \]

- **Frequency-dependent coupling:**
  \[
  \kappa(\omega) = \frac{\kappa T}{1 - \kappa T \log \left( \frac{\omega e^{\gamma E}}{2} \right)}
  \]

- **Conductivity:**
  \[
  \sigma(\omega) \propto \frac{1}{\omega} \text{Im} G(\omega).
  \]
Conductivity & resistivity

conducivity vs $\omega/(2\pi T)$

DC resistivity vs $T/\mu$

Lowest quasinormal mode $\sim$ Kondo resonance $\frac{\omega_0}{2\pi T} \simeq -1.4i$

Resistivity grows at low temperatures!
Holographic toy model implements the large-$N$ and CFT approach:

- Current algebra in $1+1$ from s-wave reduction
- Wilson line as slave fermions on defect
- Kondo coupling = double-trace coupling

and captures main physical properties:

- Dynamical scale generation and asymptotic freedom
- Raise in the resistivity at low temperatures
Future directions

- Compute other properties: entropy, heat capacity, magnetic susceptibility, Wilson's ratio, spectrum of operators
- Multi-channel Kondo model
- Impurities in different representations of spin
- Several impurities with interactions
- Models with small spin?