Design of robust optimal regulator considering state and control nonlinearities

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ABSTRACT

Most real systems can be considered as a nonlinear and parameters of the systems may have uncertainties and external disturbances. Hence, in this paper, using the power series algorithm (PSA) based on State Dependent Riccati Equations (SDRE) and considering the proposed conditions that guarantee the asymptotic stability, the optimal regulator problem for a particular class of nonlinear systems is solved. Also, according to the specified pattern in the modified PSA (MPSA) method, weighting matrices are used as a function of state variables to achieve the better regulatory responses. Simulations are carried out on the Lorenz’s chaotic system with uncertainties and external disturbances. The efficiency of optimal regulators the PSA and the MPSA methods in eliminating the external disturbances and being robust to uncertainties are compared together. The results show that the values of the performance index and the control cost in the MPSA method are smaller than the PSA method. Then the regulatory response in the MPSA method is more efficient.

ARTICLE HISTORY

Received 22 November 2017
Accepted 29 April 2018

KEYWORDS

Nonlinear systems; robust control; optimal regulator; uncertainty; Lyapunov stability analysis; SDRE; power series algorithm (PSA)

1. Introduction

Due to the wide range of nonlinear systems, the problem of optimal control of nonlinear systems has attracted the attention of control engineers in recent years. An optimal control problem is defined as optimizing the value of a selected function as a performance index. Depending on the type of problem, this optimization can be performed by minimization or maximization of the problem.

The first work on the SDRE was done by Pearson (1962), Garrard, McClamroch, and Clark (1967), Burghart (1969), Wernli and Cook (1975), Krikelis and Kiriakidis (1992). Then the optimal control of the SDRE has been investigated by Cloutier, D’Souza, and Mrazek (1996), Khaloozadeh and Abdollahi (2002). Over the past two decades, SDRE has been used more extensively in optimizing the control of nonlinear systems (Abbaszadeh & Marquez, 2012; Aliyu, 2000; Amato, 2006; Amato, Colacino, Cosentino, & Merola, 2014; Amato, Cosentino, & Merola, 2010; Banks, Kwon, Toivanen, & Tran, 2006; Bernábé-Lorancá, Coello-Coello, & Osorio-Lama, 2012; Bouzaaouache & Braiek, 2008; Elloumi & Braiek, 2012; Mohan & Miller, 2008; Shamma & Cloutier, 2003; Strano & Terzo, 2015; Won & Biswas, 2007). The SDRE method can be used to control and stabilize a wide range of systems, such as satellites (Cyril, Jaar, & Misra, 1995; Flores-Abad & Ma, 2012; Inaba, Oda, & Asano, 2006; Tarabini et al., 2007), spacecraft (Kaiser, Sjöberg, Delcura, & Eilertsen, 2008; Xin & Pan, 2011), helicopter (Bogdanov et al., 2003; Dimitrov & Yoshida, 2004; Liu, Wu, & Lu, 2007; Yoshida, Dimitrov, & Nakanishi, 2006), robots (Abiko & Hirzinger, 2007; Huang, Wang, Meng, & Liu, 2015; Huang, Wang, Meng, Zhang, & Liu, 2016), chaotic system (Choi, 2012; Sinha, Henrichs, & Ravindra, 2000; Yuan, Liu, Lin, Hu, & Gong, 2017) and more.

The problem of eliminating disturbance in control systems is one of the most important issues that has always been considered, and in case of inappropriate attention to it, it can cause large damage to the control system. There are many ways to remove disturbance in the systems. Yu and Zhang explored the dynamic of disturbances and they are entered into the performance index and then zero it (Yu & Zhang, 2004). Kumar et al. designed an optimal controller for the ball and beam system considering the uncertainties of parameters (Kumar, Jerome, & Raaja, 2014).

The SDRE method can be used to synchronize control problem of complex systems. Li et al. investigated the event-triggered synchronization control problem for a class of complex networks with uncertain inner couplings (Shen, Tan, Wang, & Huang, 2017). Shen et al. established a unified framework to investigate both the quantized and the saturated control problems for a class of sampled-data systems under noisy sampling intervals (Li, Shen, Liang, & Shu, 2015). In all of these methods it is assumed that the nonlinearities of states are only linear dependent.
on the control input, such as:

\[ \dot{x} = f(x(t)) + g(x(t))u(t) \]  
\[ (1) \]

The uncertainty in the system model can negatively affect the performance of many systems. Therefore, it is important to design the optimal control law that not only stabilized the wide range of nonlinear systems but, robust to the uncertainties and external disturbances.

In this paper, the problem of the optimal regulator for a wider class of nonlinear systems is studied as follows:

\[ \dot{x} = f(x(t)) + g(x(t), u(t)) \]  
\[ (2) \]

Using the power series algorithm (PSA) based on SDRE equations and considering the proposed conditions that guarantee the asymptotic stability, the robust optimal regulator is designed, such that it can be used to cover a wide range of nonlinear systems, also it is robust to the uncertainties and external disturbances. Furthermore, according to the specified pattern (modified PSA (MPSA) method), weighting matrices are used as a function of state variables to achieve a better regulatory responses. Finally, the efficiencies of the proposed methods in the elimination of the uncertainties and disturbances for this class of nonlinear systems are represented and they are compared together.

This paper is organized as follows:
Section 1 presents the introduction.
Section 2 presents the PSA method in solving of optimal regulator problem.
Section 3 presents the MPSA method in solving of optimal regulator problem.
Section 4 presents the design of robust optimal regulator, based on the asymptotic stability conditions.
Section 5 presents the simulations results.
Section 6 presents the conclusions.

2. The PSA method in solving of optimal regulator problem

The considered a class of affine nonlinear system as (Equation (2)).

By substituting \( f(x(t)) = A(x(t)) \) and \( g(x(t), u(t)) = B(x(t), u(t))u(t) \), (Equation (2)) is rewritten as follows:

\[ \dot{x} = A(x(t)) + B(x(t), u(t))u(t) \]  
\[ (3) \]

where \( x \in \mathbb{R}^m, u : \mathbb{R}^m \rightarrow \mathbb{R}^k \).

The purpose is to find an optimal control law which minimizes the following cost function:

\[ J(x_0, u) = \frac{1}{2} \int_0^\infty (x^T(t)Qx(t) + u^T(t)Ru(t)) \, dt \]  
\[ (4) \]

where, the weighting matrices \( Q \) and \( R \) are semi-positive definite and positive definite respectively. The SDRE control law is obtained by defining the Hamiltonian function (Equation (5)) and applying the conditions of the problem to it for optimal control of the dynamical systems (Beeler and Cox, 2004):

\[ H(x(t), u(t), t) = \frac{1}{2}x^T(t)Qx(t) + \frac{1}{2}u^T(t)Ru(t) + \lambda(t)^T(A(x(t)) + B(x(t), u(t))u(t)). \]  
\[ (5) \]

The conditions are as follows:

\[ \dot{x}(t) = \frac{\partial H(x, u, t)}{\partial \lambda(t)} = A(x(t)) + B(x(t), u(t))u(t). \]  
\[ (6a) \]

\[ \dot{\lambda}(t) = -\frac{\partial H(x, u, t)}{\partial x(t)} = -Qx(t) - \frac{\partial A(x(t))}{\partial x(t)}\lambda(t) - \frac{\partial B(x(t), u(t))}{\partial x(t)}\lambda(t). \]  
\[ (6b) \]

\[ \frac{\partial H(x, u, t)}{\partial u(t)} = Ru(t) + B(x(t), u(t))^T\lambda(t) \]
\[ + \sum_{i=1}^{k} u_i(t) \left( \frac{\partial B(x(t), u(t))}{\partial u_i(t)} \right)^T \lambda(t) = 0. \]  
\[ (6c) \]

With respect to the change in \( \lambda(t) = P(x(t), u(t))x(t) \) as in (Equation (6)), the optimal control law \( u(t) \) is rewritten as follows:

\[ u(t) = -R^{-1}B(x(t), u(t))^TP(x(t), u(t))x(t) \]
\[ - R^{-1} \sum_{i=1}^{k} u_i(t) \left( \frac{\partial B(x(t), u(t))}{\partial u_i(t)} \right)^T P(x(t), u(t))x(t). \]  
\[ (7) \]

The following equations are derived from \( \dot{\lambda}(t) \):

\[ \dot{\lambda}(t) = P(x(t), u(t))\dot{x}(t) + \left[ \sum_{i=1}^{m} \frac{\partial P(x(t), u(t))}{\partial x_i(t)} \dot{x}_i(t) \right. \]
\[ + \left. \sum_{i=1}^{k} \frac{\partial P(x(t), u(t))}{\partial u_i(t)} \dot{u}_i(t) \right] x_i(t). \]  
\[ (8) \]

By replacing and simplifying relationships (Equations (6) and (8)), the following equation is obtained:

\[ [P(x, u)A(x) + A(x)^TP(x, u) - P(x, u)B(x, u)R^{-1}B(x, u)^TP(x, u) + Q] \]
By substituting (Equations (12) and (13)) in the SDRE equation (Equation (10)) and assuming $P(x, u)B(x, u)R^{-1} = P(x, u)$, the following control law is obtained:

$$u(t) = -R^{-1}B(x, u)^{T}\lambda(t)$$  \hspace{1cm} (11)

where $\lambda(t) = P(x, u)x(t)$.

In fact, in order to optimal control of nonlinear systems, (Equations (10) and (11)) must be solved and by obtaining $P(x, u)$ the SDRE control law $u(t)$ is obtained.

In the PSA method, $P(x(t), u(t))$ can be considered as power series:

$$P(x(t), u(t)) = L_0(x(t), u(t)) + \varepsilon L_1(x(t), u(t)) + \varepsilon^2 L_2(x(t), u(t)) + \cdots$$

\hspace{1cm} (12)

$$= \sum_{j=0}^{\infty} \varepsilon^j L_j(x(t), u(t)).$$

It is also possible to separate the matrices $A$ and $B$ into a constant part and a variable part as follows:

$$A(x(t), \varepsilon) = A_0 + \varepsilon \Delta A(x(t)) \quad \text{and} \quad B(x(t), u(t), \varepsilon) = B_0 + \varepsilon \Delta B(x(t), u(t)).$$

(13)

By substituting (Equations (12) and (13)) in the SDRE equation (Equation (10)) and assuming $\varepsilon = 1$, the equation is converted into several sub-equations as follows (Beeler and Cox, 2004):

$$L_0A_0 + A_0^T L_0 - L_0 B_0 R^{-1} B_0^T L_0 + Q = 0.$$  \hspace{1cm} (14)

and

$$L_1 (A_0 - B_0 R^{-1} B_0^T L_0) + (A_0^T - L_0^T B_0 R^{-1} B_0^T) L_1 + L_0 \Delta A + \Delta A^T L_0 - L_0 (B_0 R^{-1} \Delta B^T + \Delta B R^{-1} B_0^T) L_0 = 0.$$  \hspace{1cm} (15)

In general, the equations are written as:

$$L_j (A_0 - B_0 R^{-1} B_0^T L_0) + (A_0^T - L_0^T B_0 R^{-1} B_0^T) L_j + \sum_{i=0}^{j-1} L_i (B_0 R^{-1} \Delta B^T + \Delta B R^{-1} B_0^T) L_{j-i} - \sum_{i=0}^{j-2} L_i \Delta B R^{-1} \Delta B^T L_{j-2-i} = 0.$$  \hspace{1cm} (16)

Here, the matrix $B$ is assumed to be general and dependent on the states and control of the system. By determining the nonlinear term $k_p$ of the matrix $L_j$, the control law $u(t)$ can be obtained as follows (Beeler and Cox, 2004):

$$u((n+1)) = -R^{-1}B(x(t), u((n)))^T \times \sum_{j=0}^{k_p} L_j(x(t), u((n))) x(t).$$  \hspace{1cm} (17)

Remark 2.1: Equation (14) shows that if $A_0$ and $B_0$ are zero matrices, it is impossible to obtain the value of $L_0$. Since the other values of the power series are dependent on the value of $L_0$, the power series method cannot be used to design optimal control law.

3. The MPSA method in solving of optimal regulator problem

Weighting matrices can be chosen to be either constant or as function of state variables so as to obtain the desired response. In the modified PSA (MPSA) method, the weighting matrices $Q$ and $R$ are chosen as follows (Kumar et al., 2014):

$$Q = Q_0 + Q_1(x)$$  \hspace{1cm} (18)

$$R = R_0$$

and

$$Q_0 = \text{diag}(C_{10}, C_{20}, \ldots, C_{i0}), \quad C_{i0} > 0, i = 1, 2, \ldots, m$$

$$R_0 = \text{diag}(C_{10}, C_{20}, \ldots, C_{k0}), \quad C_{k0} > 0, i = 1, 2, \ldots, k$$  \hspace{1cm} (19)

$$Q_1(x) = \text{diag}(q_{11}(x_1), q_{12}(x_2), \ldots, q_m(x_m)), \quad i = 1, 2, \ldots, m$$

$$q_i(x) = c_{i1} x_1^2 + c_{i2} x_2^2 + \cdots + c_{i4} x_i^4, \quad c_j > 0, j = 2, 4, \ldots, s_i.$$  \hspace{1cm} (20)

Therefore, the modified power series (MPSA) are written in the form of the following equations:

$$L_0 A_0 + A_0^T L_0 - L_0 B_0 R^{-1} B_0^T L_0 + Q(x) = 0.$$  \hspace{1cm} (21)
In the equation above, $L_0$ is the first term of the Power Series, $Q$ is the weighting matrix, $A_0$ and $B_0$ are constant parts of the state matrix and input matrix, respectively.

The other terms of the power series are obtained in the Equations (15) and (16).

4. Design of robust optimal regulator based on asymptotic stability conditions

In this section, the purpose is to design an optimal regulator, so that the closed loop system of (Equation (22)) not only is robust to uncertainty and external disturbance but also, it minimized the cost function Equation (23) (Lin, 2007):

$$
\dot{x}(t) = (A(x(t)) + \alpha(\theta))x(t) + (B(x(t), u(t)) + b(\theta))u(x(t)) + B(x(t), u(t))w(t, x) \tag{22}
$$

where $A(x(t)) = A_0 + \Delta A(x(t))$ and $B(x(t), u(t)) = B_0 + \Delta B(x(t), u(t))$.

$$
v = J(x_0, u) = \min \frac{1}{2} \int_0^\infty (x^T(t)Fx(t) + x^T(t)Qx(t) + u^T(t)Ru(t) + w^T(t)w(t)) \, dt \tag{23}
$$

Theorem 4.1: Consider the (Equation (22)) and using the following assumptions, the robust optimal regulator is designed so that it can be used to the proposed system in the (Equation (22)) but is robust to the uncertainties and external disturbances.

**Assumption 4.1:** The matrices $A_0$ and $B_0$ should not be zero or at least they should include the constant uncertainties.

**Assumption 4.2:** The uncertainty in the state matrix $\varphi(\theta)$ is bounded and it is satisfy the matching condition as follows:

$$
\varphi^T(\theta)\varphi(\theta) \leq F \tag{24}
$$

in the equation $\alpha(\theta) = B\varphi(\theta)$ is matching condition. Also, $F$ is a semi-positive definite matrix.

**Assumption 4.3:** The uncertainty $b(\theta)$ is positive and bounded matrix as follows:

$$
0 \leq b(\theta) \leq N \tag{25}
$$

where $\theta = \gamma(x, u)$.

**Assumption 4.4:** There is a non-negative constant $w$, so that, the external disturbance $w(x, t)$ is bounded.

$$
|w(x, t)| \leq w. \tag{26}
$$

Proof: If the answer of the optimal control problem exists, then, this answer would be the answer of the robust control problem (Lin, 2007). In order to prove the stability of Lyapunov, consider $u(t) = u_{op}(t)$ as the answer of the optimal control problem. It is shown that for all uncertainties and disturbances, the system is asymptotically stable.

In fact, Equation (23) is considered as the Lyapunov function for the reference system. By definition, at least the control function $v(x)$ must satisfy in the Hamiltonian-Jacobian-Bellman (HJB) equation (Tan, Shu, & Lin, 2009). As a result, the following relation is obtained:

$$
\begin{align*}
\min (\frac{1}{2} x^T(t)Fx(t) + \frac{1}{2} x^T(t)Qx(t) + \frac{1}{2} u^T(t)Ru(t) + \frac{1}{2} w^T(t)w(t)) \\
+ v_x^T(A(x) + B(x, u)u(t))) = 0 & u_x = \frac{\partial v}{\partial x}, \quad u \in \mathbb{R}^m.
\end{align*} \tag{27}
$$

Since the optimal control law is $u_{op}(t)$, it should satisfy Equation (27):

$$
\begin{align*}
\frac{1}{2} x^T(t)Fx + \frac{1}{2} x^T(t)Qx + \frac{1}{2} u_{op}(t)Ru_{op}(t) + \frac{1}{2} w^T(t)w + v_x^T(A(x) + B(x, u)u_{op}(t)) &= 0 \tag{28} \\
R u_{op}(t) + B(x, u)P(x, u)u_{op}(t) + \sum_{i=1}^k u_{op}(t)(\frac{\partial B(x, u)}{\partial u_{op}})^T P(x, u)u(t) &= 0 \tag{29}
\end{align*}
$$

Using the (Equations (28) and (29)), it is shown that $v$ is a Lyapunov function for the Equation (22):

$$
\dot{v}(x) = v_x^T x = v_x^T(A(x)x + \alpha(\theta)x + B(x, u)u_{op}(t)) + b(\theta)u_{op}(t) + B(x, u)w(x, t). \tag{30}
$$

By simplifying, the equations are rewritten as follows:

$$
\begin{align*}
\dot{v}(x) &= v_x^T x = v_x^T(A(x)x + B(x, u)u_{op}(t)) + b(\theta)u_{op}(t) + v_x^T(B(x, u)w(x, t). \tag{31}
\end{align*}
$$

By replacing the (Equations (28) and (29)) in Equation (31), one could have:

$$
\begin{align*}
\dot{v}(x) &= -\frac{1}{2} x^T(t)Fx - \frac{1}{2} x^T(t)Qx - \frac{1}{2} u_{op}^TRu_{op} - \frac{1}{2} w^T(t)w \\
&- u_{op}^TRb(\theta)u_{op} - u_{op}^TRw(x, t).
\end{align*} \tag{32}
$$

Considering that uncertainty $b(\theta)$ is a positive and bounded matrix and all its elements are positive or zero,
the following relation is obtained:
\[ \dot{v}(x) \leq -\frac{1}{2}x^TFx - \frac{1}{2}x^TQx - \frac{1}{2}u_{op}^TR_{op} - \frac{1}{2}w^Tw - u_{op}^TB\phi(\theta)x - u_{op}^TBw(x, t). \] (33)

Given the assumptions \( \phi(\theta) = B\phi(\theta), F = \phi^T\phi \) and \( R = I \), the following relations are obtained:
\[ \dot{v}(x) \leq -\frac{1}{2}x^T\phi^T\phi x - \frac{1}{2}x^TQx - \frac{1}{2}u_{op}^Tu_{op} - \frac{1}{2}w^Tw - u_{op}^TB\phi(\theta) + u_{op}^TBw(x, t). \] (34)

By adding and decreasing the expression \( \frac{1}{2}w(x, t)^Tw(x, t) \) to (Equation (34)), one could have:
\[ \dot{v}(x) \leq -\frac{1}{2}x^TQx - \frac{1}{2}(\phi x + u_{op})(\phi x + u_{op}) - \frac{1}{2}u_{op}^Tu_{op} - \frac{1}{2}w^Tw + \frac{1}{2}w(x, t)^Tw(x, t) - \frac{1}{2}w(x, t)^T w(x, t). \] (35)

Given the symmetry of the matrices \( x^T\phi^T u_{op} \) and \( u_{op}^T \phi x \), the (Equation (35)) is rewritten as follows:
\[ \dot{v}(x) \leq -\frac{1}{2}x^TQx - \frac{1}{2}w^Tw + \frac{1}{2}w(x, t)^Tw(x, t) - \frac{1}{2}w(x, t)^T w(x, t). \] (36)

Because \( ||\phi x + u|| \geq ||u|| \), the following relation is obtained:
\[ \dot{v}(x) \leq -\frac{1}{2}x^TQx - \frac{1}{2}w^Tw + \frac{1}{2}w(x, t)^Tw(x, t) - \frac{1}{2}u_{op}^Tu_{op} - \frac{1}{2}w(x, t)^T w(x, t). \] (37)

In other words:
\[ \dot{v}(x) \leq -\frac{1}{2}x^TQx - \frac{1}{2}w^Tw + \frac{1}{2}w(x, t)^Tw(x, t) - \frac{1}{2}(u_{op} + w(x, t))^T (u_{op} + w(x, t)). \] (38)

With considering that the disturbances \( w(x, t) \) are bounded, one could have:
\[ ||w(x, t)|| \leq w \] (39)
\[ \dot{v}(x) \leq -\frac{1}{2}x^TQx \to \dot{v}(x) \leq 0. \] (40)

As a result, the system (Equation (22)) is asymptotically stable for all uncertainties and external disturbances. Therefore, with considering the following assumptions, the Lyapunov stability for this class of nonlinear systems is proved.

**Remark 4.1:** By inserting the constant non-zero uncertainties into the matrices \( A_0 \) and \( B_0 \), the first terms of the PSA and the MPSA methods can be obtained and the robust optimal regulator for this class of nonlinear system is designed.

To design optimal regulators, we have:
\[ A(x, \theta, \varepsilon) = (A_0 + \theta_0) + \varepsilon \Delta A(x, \theta) \quad \text{and} \quad B(x, u, \theta, \varepsilon) = (B_0 + \theta_0) + \varepsilon \Delta B(x, u, \theta). \] (41)

The first term of the PSA and the MPSA equations are obtained in the form of (Equations (42) and (43)), respectively:
\[ L_0(A_0 + \theta_0) + (A_0 + \theta_0)^T L_0 \]
\[ - L_0(B_0 + \theta_0) R^{-1}(B_0 + \theta_0)^T L_0 + Q + F = 0. \] (42)

\[ L_0(A_0 + \theta_0) + (A_0 + \theta_0)^T L_0 \]
\[ - L_0(B_0 + \theta_0) R^{-1}(B_0 + \theta_0)^T L_0 + Q(x) + F = 0. \] (43)

The other terms of the PSA and the MPSA equations are also, obtained using the following equation:
\[ L_j((A_0 + \theta_0) - (B_0 + \theta_0) R^{-1}(B_0 + \theta_0)^T L_0) + ((A_0 + \theta_0)^T \]
\[ - L_j^T(B_0 + \theta_0) R^{-1}(B_0 + \theta_0)^T) L_j + L_{j-1} \Delta A(x, \theta) \]
\[ + \Delta A^T(x, \theta) L_{j-1} - \sum_{j=1}^{j-1} (L_j(B_0 + \theta_0) R^{-1}(B_0 + \theta_0)^T L_{j-1}) \]
\[ - \sum_{i=0}^{j-1} L_i((B_0 + \theta_0) R^{-1} \Delta B^T(x, \theta) \]
\[ + \Delta B(x, \theta) R^{-1}(B_0 + \theta_0)^T L_{j-1-i} \]
\[ - \sum_{i=0}^{j-2} L_i \Delta B(x, \theta) R^{-1} \Delta B^T(x, \theta) L_{j-2-i} = 0. \] (44)

Also, the control law \( u(t) \) is obtained as follows:
\[ u_{(n+1)}(x(t)) = -R^{-1}(B(x(t), u_{(n)}(t), \theta))^T \]
\[ \times \sum_{j=0}^{k_p} L_j(x(t), u_{(n)}(t)) x(t). \] (45)

5. Simulation results

Consider the Lorenz system with initial condition \( x_0 \):
\[ \dot{x}_1 = \alpha(x_2 - x_1) \]
\[ \dot{x}_2 = \beta x_1 - x_2 - x_1 x_3 + (1 - \cos(x_1))e^\mu u \]
\[ \dot{x}_3 = -\gamma x_3 + x_1 x_2 \]
\[ x(0) = [2, 1, 0]^T. \]

In fact, \( \alpha, \beta \) and \( \gamma \) are constant parameters of the nonlinear system. And also by choosing the constants \( \alpha = 10, \beta = 28, \gamma = \frac{8}{3} \), the Lorenz system is chaotic (Shen & Wang, 2007). (Figures 1 and 2)
Figure 1. State variables in Lorenz’s chaotic system.

Figure 2. Three-dimensional phase diagram of Lorenz’s chaotic system.

In this section, it is assumed that there are uncertainties and disturbances in the system model. The uncertainties in this example are considered as follows:

\[ \theta_1 = 5 \sin^2 x_1, \quad \theta_2 = 0.1 e^{x_2} \quad \text{and} \quad \theta_3 = e^{x_2} + e^{-x_1} \]  

(47)

where \( \theta_1 \in [0, 5] \) and \( b(\theta) = [\theta_2 \theta_3]^T \).

Here \( \theta_1 \) is an uncertainty in the state matrix system. Also, the uncertainties in the input matrix system are \( \theta_2 \) and \( \theta_3 \). Moreover, the excited external disturbances \( w(x, t) \) are bounded and in the form of the following equation:

\[ w(x, t) = 2 \sin x_3 e^{-t}. \]  

(48)

First consider the nonlinear system as a whole:

\[ \dot{x} = A(x, \theta)x + B(x, u, \theta)u + B(x, u)w(x, t) \]  

(49)

where \( A(x, \theta) = A_0 + a(\theta) + \Delta A(x), \quad B(x, u, \theta) = B_0 + b(\theta) + \Delta B(x, u) \) and \( B(x, u) = B_0 + \Delta B(x, u) \). Therefore, the system equations will be considered as the following space-state model:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
-\alpha & \alpha & 0 \\
\beta & -1 & 0 \\
0 & 0 & -\gamma
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
-0.5x_3 & 0 & -0.5x_1 \\
0.5x_2 & 0.5x_1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
(1 - \cos(x_1))e^u \\
0
\end{bmatrix}
\begin{bmatrix}
\theta_2 \\
\theta_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
(1 - \cos(x_1))e^u \\
0
\end{bmatrix}
w(x, t) \tag{50}
\]

where \( e^u = 1 + u + u^2/2 + u^3/6, \ldots \). For this purpose, state matrix is analyzed as follows:

\[
\begin{bmatrix}
-\alpha & \alpha & 0 \\
\beta & -0.5x_3 & -1 + \theta_1 & -0.5x_1 \\
0.5x_2 & 0.5x_1 & -\gamma
\end{bmatrix}
- \begin{bmatrix}
-\alpha & \alpha & 0 \\
\beta & -0.5x_3 & -1 & -0.5x_1 \\
0.5x_2 & 0.5x_1 & -\gamma
\end{bmatrix}
= \begin{bmatrix}
0 \\
(1 - \cos(x_1))e^u \\
0
\end{bmatrix}
f(\theta) = Bf(\theta). \tag{51}
\]

Where matrix \( f(\theta) \) is limited and is composed as follows:

\[ f(\theta) = [0 \quad \theta_1 \quad 0] \quad \text{and} \quad \theta_1 \in [0, 5]. \tag{52} \]

The system (Equation (50)) is rewritten as follows:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
-\alpha & \alpha & 0 \\
\beta & -0.5x_3 & -1 & -0.5x_1 \\
0.5x_2 & 0.5x_1 & -\gamma
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ Bf(\theta)
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
(1 - \cos(x_1))e^u \\
0
\end{bmatrix}
\begin{bmatrix}
\theta_2 \\
\theta_3
\end{bmatrix}
+ \begin{bmatrix}
0 \\
(1 - \cos(x_1))e^u \\
0
\end{bmatrix}
w(x, t). \tag{53}
\]

The matrix \( F \) should be limited and external disturbance \( w(x, t) \) is bounded.

\[ f^T(\theta)f(\theta) \leq F \rightarrow \begin{bmatrix}
0 & 0 & 0 \\
0 & \theta_1^2 & 0 \\
0 & 0 & 0
\end{bmatrix} \leq \begin{bmatrix}
0 & 0 & 0 \\
0 & 25 & 0 \\
0 & 0 & 0
\end{bmatrix}. \tag{54} \]

In simulations, weighted matrices \( Q, R \) are chosen as follows:

\[ Q = I_{3 \times 3} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad \text{and} \quad R = I_{1 \times 1} = [1]. \tag{55} \]
Now, using a robust optimal control law for uncertainties \( \theta_1, \theta_2, \theta_3 \) and disturbances \( w(x, t) \), the Riccati equations are solved. In this section, the above method is simulated as follows: (Figures 3–5)

It is seen in the (Figure 4) the optimal regulator is robust to uncertainties in the system and can be used to eliminate the external disturbances. Using of the weighting matrices according to the (Equation (56)), the optimal regulator can be designed the MPSA method as follows. (Figures 6–8)

\[
Q(x) = I_{3 \times 3} = \begin{bmatrix}
1 + x_1^2 & 0 & 0 \\
0 & 1 + x_2^2 & 0 \\
0 & 0 & 0.95 + x_3^2
\end{bmatrix}
\]

and

\[
R = I_{1 \times 1} = [1].
\]  

(56)

It is clear that the designed optimal regulators are capable to control and stabilize the Lorenz’s chaotic system. In this section, the state variables and optimal regulator law obtained by the PSA and MPSA methods are compared with each other in (Figures 9–13).

As the simulation results show that the values of performance index and control cost in the MPSA method are lower than the PSA method. Then, this optimal regulator is more appropriate.
6. Conclusions

In this paper, the problem of optimal control is solved for a particular class of system in which nonlinearity in the state and control exists. In order to design an optimal control, we first used the PSA method to solve the SDRE equations, then the proposed some conditions that guarantee the asymptotic stability. Furthermore, to find the appropriate regulator response, weighting matrices are modified according to the specific pattern (MPSA method). Simulations are carried out on Choate's system that has nonlinearity in state and control. The simulation results show, these optimal regulators can be used to remove the uncertainties and disturbances in a wider range of nonlinear systems. Also, efficiency of optimal regulators the PSA and the MPSA methods in eliminating the disturbances and being robust to uncertainties are compared together. In comparison to the MPSA method has a higher convergence rate and lower control cost than the PSA method.

Disclosure statement

No potential conflict of interest was reported by the authors.

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