Semiclassical Defect Measures and the Observability Estimate for Schrödinger Operators with Homogeneous Potentials of Order Zero

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Abstract

We study the asymptotic behavior as $|x| \to \infty$ of Schrödinger operators with homogeneous potentials. For this purpose, we use methods from semiclassical analysis and investigate semiclassical defect measures. We prove their localization in direction which we apply in order to obtain a necessary condition of observability.

1 Introduction

Let $P = -\Delta + V$ be a Schrödinger operator on $\mathbb{R}^n$. We make the following assumption on the potential $V$.

Assumption A. (1) $V$ is a real valued smooth function.
(2) We can decompose $V$ as $V = V_\infty + V_s$; here $V_\infty$ is real-valued, and is homogeneous of order zero, i.e., $V_\infty(x) = V_\infty(\frac{x}{|x|})$ for $|x| \geq 1$, and $V_s(x) = o(x^{-1})$ as $|x| \to \infty$.

If $V_\infty$ is homogeneous of order zero, one can regard $V_\infty$ as a function on $S^{n-1}$. We use the same symbol $V_\infty$ for the original potential and restriction of this function to $S^{n-1}$.

Next we introduce a new semiclassical quantization.

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Theorem 1.1. \textit{(Existence of semiclassical defect measure)}

Let $f$ \textit{cutoff function} if $f$ \textit{admissible cutoff function}.

(1) For any $m \in \mathbb{N}$, there exists $C_m > 0$ such that $\sup_{r \in \mathbb{R}} |\partial_r^m f_h(r)| < C_m$ uniformly in $h$.

For $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$ and admissible cutoff function $f_h$, one can regard $f_h(r) a(\rho, \theta, \frac{n}{r})$ as an element of $C_0^\infty(T^*\mathbb{R}^n)$ for small $h$ with the natural \textit{diffeomorphism} $T_0^*(\rho, \theta) \times T^*S^{n-1} \simeq T^*\mathbb{R}^n \setminus \{(0, \xi)\}$ induced by polar coordinate. The function $\tilde{a}(x, \xi) = f_h(r) a(\rho, \theta, \frac{n}{r})$ on $T^*\mathbb{R}^n$ in the symbol class $S$, the symbol class with respect to the order function 1. In the other words for any $\alpha, \beta \in \mathbb{N}^n$, $\sup_{(x, \xi) \in T^*\mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \tilde{a}(x, \xi)| < \infty$ (See Section 2.1).

Now the \textit{Weyl quantization} $\tilde{a}_h^w(hx, D_x)$ of the symbol $\tilde{a}$ becomes a well-defined bounded linear operator on $L^2(\mathbb{R}^n)$, given as the extension of

$$\tilde{a}_h^w(hx, D_x)u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y, \xi)} a_h \left( \frac{h(x+y)}{2}, \xi \right) u(y)dyd\xi,$$

for $u \in S(\mathbb{R}^n)$. We write $\tilde{a}_h^w(hX, D_X) = \text{Op}_{f_h}(a)$.

Theorem 1.1. \textit{(Existence of semiclassical defect measure)}

Let $u_h \in L^2(\mathbb{R}^n)$ be a bounded family in $h$. There exists a sequence of positive numbers $h_m \to 0$ as $m \to \infty$ and a finite Radon measure $\mu_f$ on $\mathbb{R} \times T^*S^{n-1}$ and

$$\langle u_{h_m}, \text{Op}_{f_{h_m}}(a)u_{h_m} \rangle_{L^2(\mathbb{R}^n)} \to \int_{\mathbb{R} \times T^*S^{n-1}} a d\mu_f \text{ as } m \to \infty,$$

for all $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$. Furthermore, if $f_h$ is non negative, $\mu_f$ is also non negative.

Let $j \in C^\infty(\mathbb{R} : [0, 1])$ be such that $j(r) = 0$ if $r \leq \frac{1}{2}$ and $j(r) = 1$ if $1 \leq r$. Then this $j$ can be regarded as an admissible cut-off function.

Theorem 1.2. Under Assumption(A), let $u_h \in \mathcal{D}(P)$ be such that

\begin{equation}
\begin{cases}
(P - E)u_h = R_h \\
\|u_h\|_{L^2(\mathbb{R}^n)} = 1,
\end{cases}
\end{equation}

where $\|R_h\|_{L^2(\mathbb{R}^n)} = o(h)$ as $h \to 0$. We assume there exists $\chi \in C_0^\infty((1, \infty))$ such that $u_h(x) = \chi(h|x|)u(x) + R'_h$ with $\|R'_h\| = o(1)$ as $h \to 0$. Then we can prove the following:

(1) $E \in \text{Cr}(V)$, and

(2) $\text{supp}(\mu_j) \subset \{(0, \theta, 0) \in \mathbb{R} \times T^*S^{n-1} \mid \theta \in \text{Cr}(V) \cap V^{-1}(E)\}$. 

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The claim of (2) in Theorem 1.2 is a semiclassical version of the localization in direction proved in [6, 7, 8, 10]. One clear difference of their localization in direction from our version is the appearance of \( L^2 \) states which localizes to saddle point and the local minimum points. It is proved in [6] that there exists a distribution which localizes in saddle points or local minimum points. The appearance of \( L^2 \) states which localizes to saddle point and the local minimum points is essential in the sense that one can take \( u_h \) so that \( \mu_j \neq 0 \) and \( \mu_j \) is supported in the direction of local maxima or saddle point (See Section 4).

The statement of (1) in Theorem 1.2 implies intuitively that there are not so many \( o(h) \)-quasimodes whose support escapes from the origin with \( h^{-1} \) order. Actually, we can construct \( o(h) \)-quasimodes whose support escapes from the origin with \( h^{-2} \) order (See Section 4). We can also show some results on the relationship between quasimodes and its support.

Let \( c : [0, 1] \to [0, \infty) \) be a monotone increasing function such that \( c(h) = o(1) \) as \( h \to 0 \) and \( c(h)^{-1} = o(h^{-1}) \). We define an admissible cutoff function \( J_h \in C^\infty((0, \infty): [0, 1]) \) by \( J_h(r) = j((\log c(h)^{-1})^{-1} \log r) \). Then \( c(h)^{-\frac{1}{2}} \leq r \leq c(h)^{-\frac{1}{4}} \) if \( r \in \text{supp}[J_h] \). Let \( \tilde{J}_h(r) = j(4(\log c(h)^{-1})^{-1} \log r) \).

**Theorem 1.3.** Under the assumptions of Theorem 1.2 and the additional assumption \( E \notin \text{Cr}(V) \), there exists \( c(h) \) as required in the last paragraph such that if \( J_h u_h \to 0 \) as \( h \to 0 \), then \( u_h \to 0 \) on \( \{ x \in \mathbb{R}^n \mid |x| > h^{-1} c(h)^{-\epsilon} \} \) as \( h \to 0 \) for any \( \epsilon > 0 \).

The notion of semiclassical measure was first introduced in [15]. The study of partial differential equation using defect measure appeared in [13] and was refined in [5]. You can find several proofs of the existence of semiclassical measures in [2, 4, 14, 16]. You can find a good survey of this subject in [1].

In usual semiclassical analysis, we define a semiclassical defect measure as a measure on a cotangent bundle. Roughly speaking, this usual defect measure treats a point in the cotangent bundle whose orbits of the Hamiltonian flow generated by \( p \) are trapped. Actually, one can prove that if the Hamiltonian flow generated by \( p \) is non-trapping, \( \mu \) is identically zero. With some assumption, Schrödinger operators with homogeneous potentials are is non-trapping(See Section 2 of [8] for the detail). Thus we cannot apply usual semiclassical analysis.

One idea is to consider a point in the cotangent bundle whose orbits of the Hamiltonian flow generated by \( p \) scatters. We realize this idea by taking the position to infinity, instead of taking the energy to infinity. A non-semiclassical quantization similar to our new quantization can be found in [3].
We turn to the application of our semiclassical measure. We can prove an observability result. Let $\Omega \subset \mathbb{R}^n$, we say observability holds on $\Omega$ if for some $T > 0$ there exists $C_{\Omega,T} > 0$ such that

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_{\Omega,T} \int_0^T \int_{\Omega} |e^{-itP} u(x)|^2 dx dt$$

for any $u \in L^2(\mathbb{R}^n)$.

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^n$ be a domain which such that

$$\Omega \cap \{x \in \mathbb{R}^n \mid |x| > R\} \subset \mathbb{R}^n \setminus \{(r, \theta) \in \mathbb{R}^n \mid r > R, \text{dist}(\theta, \theta_0) < C T^{-k}\}$$

for some $R, C > 0$ and $\theta_0 \in S^{n-1}$ with $\partial_\theta^k V(\theta_0) = 0$ for any $k \leq k$, where $\ell(k)$ is a function on $\mathbb{N}$ such that $\ell(k) = k + 1$ if $k > 0$ and $\ell(0) = \frac{2}{3}$.

Then the observability on $\Omega$ fails for any $T > 0$, i.e., there exists $u_m \in L^2(\mathbb{R}^n)$ such that $\|u_m\|_{L^2(\mathbb{R}^n)} = 1$ and $\int_0^T \int_{\Omega} |e^{-itP} u_m(x)|^2 dx dt \to 0$ as $m \to \infty$.

It is known that observability is equivalent to the controllability in [12]. The controllability means the condition that for any $u_0 \in L^2(\mathbb{R}^n)$ there exists $f \in L^2((0,T) \times \Omega)$ such that the solution to the equation

$$\begin{cases}
  (i\partial_t + P)u(t, x) = f \chi(0,T) \times \Omega(t, x) \\
  u(0, x) = u_0(x).
\end{cases}$$

satisfies $u(t, x) \equiv 0$.

The relation between semiclassical defect measures and observability is shown in [11] that in the compact manifold case, if the geodesic satisfies geometric control condition, one can prove observability holds by using a semiclassical defect measure.

The plot of this paper is as follows. We first introduce a new semiclassical quantization and give some of its basic properties of to give a proof of Theorem 1.1 in section 2. We also prove some result in classical mechanics in Section 2. In section 3, we prove Theorem 1.2 and 1.3. The proof is essentially the same with that of te Hamiltonian flow invariance of usual semiclassical defect measures. We construct an example of $u_h$ such that the corresponding semiclassical defect measure $\mu$ is not identically zero in Section 4. Finally, we give a proof of Theorem 1.4 in Section 5.

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2 Preliminaries

2.1 Pseudodifferential Calculus for \( \text{Op}_{f_h}(a) \)

The aim of this subsection is to prove some properties of \( \text{Op}_{f_h}(a) \) as a bounded operator on \( L^2(\mathbb{R}^n) \) and to prove Theorem 1.1.

First we want to show \( \tilde{a}_h(x, \xi) = f_h(x) a(\rho, \theta, \frac{x}{|x|}) \in S \) for admissible cutoff function \( f_h \) and \( a \in C_0^\infty(\mathbb{R} \times T^* S^{n-1}) \), where

\[
S = \{ a \in C^\infty(T^* \mathbb{R}^n) \mid \forall \alpha, \beta \in \mathbb{N}^n, \sup_{(x, \xi) \in T^* \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \tilde{a}(x, \xi)| < \infty \}.
\]

Let \( v \in T_x (\mathbb{R}^n \setminus \{0\}) \). From the cartesian coordinates, we can write \( v = \sum_{m=1}^n v_i \partial_{x_i} \). Also, if we fix local coordinate \((U, \psi)\) of \( S^{n-1} \) with \( \frac{x}{|x|} \in U \), we can write \( v = v_r \partial_r + \sum_{m=1}^{n-1} v_\theta \partial_\theta \) using polar coordinate.

Let \( \tilde{\psi} \) be a map \((0, \infty) \times U \subset \mathbb{R}^n \rightarrow \psi(U) \) which takes \( x \) to \( \psi(\frac{x}{|x|}) \).

Then we can write \( v_r = \frac{x}{|x|} \cdot \tilde{v} \) and \( v_\theta = J(\tilde{\psi}) \frac{1}{|x|} (I_n - (\frac{x}{|x|})_{i,j}) \tilde{v} \) where \( \frac{x}{|x|} \cdot \tilde{v} = (v_1, v_2, \cdots, v_n) \) and \( J(\tilde{\psi}) \) denotes the Jacobi matrix of \( \tilde{\psi} \) at \( x \).

Let \( \xi \in T^* \mathbb{R}^n \). Then using dual coordinate of cartesian coordinates and polar coordinate, \( \xi \) can be written as \( \sum_{m=1}^n \xi_i dx_i + p dr + \sum_{m=1}^{n-1} \eta d\theta_i \).

Using cartesian coordinate, we see \( \xi(v) = \sum_{m=1}^n \xi_i v_i \). Using polar coordinate, we see \( \xi(v) = \rho \frac{x}{|x|} \cdot \tilde{v} + \eta \cdot J(\tilde{\psi}) \frac{1}{|x|} (I_n - (\frac{x}{|x|})_{i,j}) \tilde{v} \).

Substituting \( \tilde{v} = \frac{x}{|x|} \cdot \xi \), we see \( \rho = \frac{x}{|x|} \cdot \xi \). If \( \{I_n - (\frac{x}{|x|})_{i,j}\} \tilde{v} = \tilde{v} \) i.e. \( \frac{x}{|x|} \cdot \tilde{v} = 0 \), we see \( |x| \xi \cdot \tilde{v} = \eta \cdot J(\tilde{\psi}) \tilde{v} \), which means \( |x| \xi = t J(\tilde{\psi}) \eta \).

Thus we obtain

\[
\forall \alpha, \beta \in \mathbb{N}^n, \sup_{(x, \xi) \in T^* \mathbb{R}^n} |\partial_x^\alpha \partial_\xi^\beta \tilde{a}(x, \xi)| < \infty.
\]

\[
\iff \forall m, \ell \in \mathbb{N}, \tilde{\alpha}, \tilde{\beta} \in \mathbb{N}^{n-1}, \sup_{(x, \xi) \in T^* \mathbb{R}^n} |\partial_x^m \partial_\rho^\ell \partial_\theta^\tilde{\alpha} (r \partial_\eta^\tilde{\beta}) \tilde{a}(x, \xi)| < \infty.
\]

Then it is clear that \( \tilde{a}(x, \xi) \in S \).

We define dilation operator \( U_h \) by \( U_h u(x) = h^n u(\frac{x}{h}) \) for \( u \in L^2(\mathbb{R}^n) \). Then \( U_h \) is unitary and one can calculate

\[
\text{Op}_{f_h}(a) = U_h^{-1} \tilde{a}^w(X, h D_X) U_h.
\]

(2.1)
Thus we can apply an usual semiclassical analysis for $S$. Then one can use results in usual semiclassical analysis in [17] to obtain the following theorems.

**Theorem 2.1. (Calderon-Vaillancourt Theorem)**

For $a \in S$, there exists $C > 0$ such that

$$\|a^w(hX, D_X)\|_{L^2(\mathbb{R}^n)} \leq C \sup_{(x, \xi) \in \mathbb{R}^{2n}} |a(x, \xi)| + o(h^{\frac{1}{2}}) \quad \text{as} \quad h \to 0.$$ 

**Theorem 2.2. (Sharp Gårding inequality)**

Suppose $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$ is positive. Then there exist $C > 0$ and $h_0 > 0$ such that

$$\langle u, \text{Op}_h(a)u \rangle_{L^2(\mathbb{R}^n)} \geq -Ch\|u\|_{L^2(\mathbb{R}^n)}^2$$

for $u \in L^2(\mathbb{R}^n)$ and $0 < h < h_0$.

**Proof of Theorem 1.1.** The proof is essentially the same with that of the Theorem 5.2 in [17]. However we give the detail for the completeness.

Since $C_0(\mathbb{R} \times T^*S^{n-1})$ is separable with the topology defined by sup-norm and $C_0^\infty(\mathbb{R} \times T^*S^{n-1})$ is dense subspace, thus one can find $\{a_\ell\} \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$ which is dense in $C_0(\mathbb{R} \times T^*S^{n-1})$.

From Theorem 2.1, $\langle u_h, \text{Op}_{h,h}(a_1)u_h \rangle$ is bounded in $h$. Thus one can find sequence $h_m^{(1)}$ such that $h_m^{(1)} \to 0$ and $\langle u_{h_m^{(1)}}, \text{Op}_{h,h_{h_m^{(1)}}}(a_1)u_{h_m^{(1)}} \rangle \to F(a_1)$ as $m \to \infty$ for some $F(a_1)$.

Similarly, for $\ell = 2, 3, 4, \cdots$ one can find sequence $h_m^{(\ell)}$ which is subsequence of $h_m^{(\ell-1)}$ and $\langle u_{h_m^{(\ell)}}, \text{Op}_{h_{h_m^{(\ell)}}}(a_\ell)u_{h_m^{(\ell)}} \rangle \to F(a_\ell)$ as $m \to \infty$ for some $F(a_\ell)$. Then by diagonal argument one can find sequence $h_m$ such that $\langle u_{h_m}, \text{Op}_{h_{h_m}}(a)u_{h_m} \rangle \to F(a)$ as $m \to \infty$ for each $\ell$.

From Theorem 2.1, one can calculate as follows:

$$\langle u_{h_m}, \text{Op}_{h_{h_m}}(a)u_{h_m} \rangle$$

\begin{align*}
&\leq \|\text{Op}_{h_{h_m}}(a)\|_{L^2(\mathbb{R}^n)}\|u_{h_m}\|_{L^2(\mathbb{R}^n)} \\
&\leq C \sup_{(r,\rho,\theta,\eta) \in \mathbb{R}^{2n}} |f_{h_m}(r)a_\ell(\rho, \theta, \eta)| + o(h^{\frac{1}{2}}) \\
&\leq C \sup_{(\rho,\theta,\eta) \in \mathbb{R} \times T^*S^{n-1}} |a_\ell(\rho, \theta, \eta)| + o(h^{\frac{1}{2}}),
\end{align*}

where we have used the fact $f_h$ is uniformly bounded in the last line. Thus a functional $a_\ell \mapsto F(a_\ell)$ defines a bounded and linear functional $F$ on $C_0(\mathbb{R} \times T^*S^{n-1})$. Theorem 2.2 implies that $F$ is non-negative if $f_h$ is non-negative. Then Riesz-Markov-Kakutani theorem implies there exists a Radon measure $\mu_f$ such that $\langle u_{h_m}, \text{Op}_{h_{h_m}}(a)u_{h_m} \rangle \to \int_{\mathbb{R} \times T^*S^{n-1}} ad\mu_f$ as $m \to \infty$ for any $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$. 
Taking \( \chi_n \in C_0(\mathbb{R} \times T^*S^{n-1}) \) such that \( 0 \leq \chi_n \to 0 \) pointwise as \( n \to \infty \). Then one obtains \( \lim_{n \to \infty} \int_{\mathbb{R} \times T^*S^{n-1}} \chi_n d\mu_f = \mu_f(\mathbb{R} \times T^*S^{n-1}) \) from the monotone convergence theorem. Since \( f_\mu \) is uniformly bounded, Theorem 2.1 implies \( \lim_{n \to \infty} \int_{\mathbb{R} \times T^*S^{n-1}} \chi_n d\mu_f \leq C \). This means \( \mu_f(\mathbb{R} \times T^*S^{n-1}) \leq C \), which proves finiteness.

\[ \square \]

### 2.2 Induced Dynamical System

Here we consider the following dynamical system on \( \mathbb{R} \times T^*S^{n-1} \) which is induced by Hamiltonian flow of \( P \). Essentially, contents of this section is first done in [8] but we write here for the convince.

Let \( H \) be a vector field on \( T^*(\mathbb{R} \times T^*S^{n-1}) \) defined by

\[
H = q(\theta, \eta) \frac{\partial}{\partial \rho} + (\partial_q q)(\theta, \eta) \frac{\partial}{\partial \theta} - ((\partial_q q)(\theta, \eta) + (\partial_\theta V)(\theta) + 2\rho \eta) \frac{\partial}{\partial \eta},
\]

where \( q(\theta, \eta) = \frac{1}{2} \eta h(\theta) \eta \) is symbol of Laplacian on \( S^{n-1} \) and \( \Phi_t \) be a flow generated by \( H \).

The relation of this dynamical system and the Schrödinger operator with homogeneous potential is as follows:

Let \( \Phi_t \) be a Hamiltonian flow generated by the Hamiltonian of \( H \). For \( (r, \rho, \theta, \eta) \in T^*\mathbb{R}^n \) we write \( \Phi_t(r, \rho, \theta, \eta) = (\tilde{r}(t), \tilde{\rho}(t), \tilde{\theta}(t), \tilde{\eta}(t)) \). Then \( (\tilde{r}(t), \tilde{\rho}(t), \tilde{\theta}(t), \tilde{\eta}(t)) \) satisfy

\[
\frac{d}{dt} \tilde{r}(t) = 2\tilde{\rho}(t), \quad \frac{d}{dt} \tilde{\rho}(t) = q(\tilde{\theta}(t), \frac{\tilde{\eta}(t)}{\tilde{r}(t)}),
\]

\[
\frac{d}{dt} \tilde{\theta}(t) = (\partial_q q)(\tilde{\theta}(t), \frac{\tilde{\eta}(t)}{\tilde{r}(t)}), \quad \frac{d}{dt} \tilde{\eta}(t) = -((\partial_q q)(\tilde{\theta}(t), \frac{\tilde{\eta}(t)}{\tilde{r}(t)}) + (\partial_\theta V)(\tilde{\theta}(t))).
\]

If we take \( (\rho(t), \theta(t), \eta(t)) = (\tilde{r}(t), \tilde{\theta}(t), \frac{\tilde{\eta}(t)}{\tilde{r}(t)}) \), we obtain

\[
\frac{d}{dt} \rho(t) = \tilde{r}(t)^{-1}q(\theta(t), \eta(t)), \quad \frac{d}{dt} \theta(t) = \tilde{r}(t)^{-1}(\partial_q q)(\theta(t), \eta(t)),
\]

\[
\frac{d}{dt} \eta(t) = -\tilde{r}(t)^{-1}\{((\partial_q q)(\theta(t), \eta(t)) + (\partial_\theta V)(\theta(t))) + 2\rho(t) \eta(t)\}.
\]

We assume \( \tilde{r}(t) \neq 0 \) and \( \tilde{r}(t) \to \infty \) as \( t \to \infty \). We introduce new time \( \tau \) by \( \tau = \int_0^t \tilde{r}(s)^{-1}ds \). Then we see

\[
\frac{d}{d\tau} \rho(t) = q(\theta(t), \eta(t)), \quad \frac{d}{d\tau} \theta(t) = (\partial_q q)(\theta(t), \eta(t)),
\]

\[
\frac{d}{d\tau} \eta(t) = -((\partial_q q)(\theta(t), \eta(t)) + (\partial_\theta V)(\theta(t)) + 2\rho(t) \eta(t)).
\]
Thus considering the orbit of $\Phi_t$ corresponds to considering the orbit of Hamilton flow of $P$.

In the last of this section, we write $\Phi_t(\rho, \theta, \eta) = (\rho(t), \theta(t), \eta(t))$ for $(\rho, \theta, \eta) \in \mathbb{R} \times T^*S^{n-1}$.

**Lemma 2.3.** Total energy $\rho^2 + q(\theta, \eta) + V(\theta)$ is conserved.

**Proof.** Let $E(t) = \rho(t)^2 + q(\theta(t), \eta(t)) + V(\theta(t))$. Then we see

$$\frac{d}{dt} E(t) = 2\rho(t)q(\theta(t), \eta(t)) + \{(\partial_{\theta}q)(\theta(t), \eta(t)) + (\partial_{\theta}V)(\theta(t))\}(\partial_{\theta}q)(\theta(t), \eta(t))$$

$$- (\partial_{\theta}q)(\theta(t), \eta(t))\{(\partial_{\theta}q)(\theta(t), \eta(t)) + (\partial_{\theta}V)(\theta(t)) + 2\rho(t)\eta(t)\}$$

$$= 0.$$

\[\square\]

**Lemma 2.4.** $\lim_{t \to \infty} \rho(t)$ exists.

**Remark.** Since $\frac{d}{dt}\rho(t) = q(\theta(t), \eta(t))$, $q(\theta(t), \eta(t))$ is integrable on $(0, \infty)$.

**Proof.** Since $\frac{d}{dt}\rho(t) = q(\theta(t), \eta(t)) > 0$, $\rho(t)$ is monotone increasing. From Lem2.3, $\rho(t)^2 \leq \rho(t)^2 + q(\theta(t), \eta(t)) = E(0) - V(\theta(t))$. Since $V$ is bounded, so is $\rho(t)$. Thus $\rho(t)$ is monotone increasing and bounded, which concludes the proof. \[\square\]

**Lemma 2.5.** $q(\theta(t), (\partial_{\theta}V)(\theta(t)))$ is integrable on $(0, \infty)$ with respect to $t$.

**Proof.** Let $F(t) = -t(\partial_{\theta}V(\theta(t)))h(\theta(t))\eta(t)$. Then we obtain

$$\frac{d}{dt} F(t) = -t(Hess(V)(\theta(t))(\partial_{\theta}q)(\theta(t), \eta(t)))h(\theta(t))\eta(t)$$

$$- t(\partial_{\theta}V(\theta(t)))\{(\partial_{\theta}h(\theta(t)))(\partial_{\theta}q)(\theta(t), \eta(t))\}\eta(t)$$

$$+ t(\partial_{\theta}V(\theta(t)))h(\theta(t))\{(\partial_{\theta}q)(\theta(t), \eta(t)) + (\partial_{\theta}V)(\theta(t)) + 2\rho(t)\eta(t)\}$$

Thus there exists $C > 0$ such that

$$\frac{d}{dt} F(t) + Cq(\theta(t), \eta(t)) > Cq(\theta(t), (\partial_{\theta}V)(\theta(t))).$$

By integrating this inequality from $t = 0$ to $t = T$, we obtain

$$F(T) - F(0) + C \int_0^T q(\theta(t), \eta(t))dt > C \int_0^T q(\theta(t), (\partial_{\theta}V)(\theta(t)))dt.$$
Since $q(\theta(t), (\partial_\theta V)(\theta(t))) \geq 0$ it is sufficient to show there exists a sequence $T_j$ such that $T_j \to \infty$ as $j \to \infty$ and $\{F(T_j)\}$ has upper bound.

From the definition of $q$, we obtain

$$|F(t)| \leq q(\theta(t), \eta(t)) + q(\theta(t), (\partial_\theta V)(\theta(t))).$$

Since second term is bounded, we only have to show that first term is bounded for some $\{T_j\}$. Since first term is integrable, there exists a sequence $\{T_j\}$ such that there exist $C > 0$ such that $q(\theta(t), \eta(t)) < C$, which completes the proof. \qed

**Theorem 2.6.** $\lim_{t \to \infty} (\partial_\theta V)(\theta(t)) = \lim_{t \to \infty} \eta(t) = 0$.

**Proof.** Let $G(t) = q(\theta(t), (\partial_\theta V)(\theta(t)))$. Then we obtain

$$\frac{d}{dt}G(t) = 2^t(\text{Hess}(V)(\theta(t))(\partial_\theta q)(\theta(t), \eta(t)))h(\theta(t))(\partial_\theta V)(\theta(t)) + \frac{d}{d\theta}(\partial_\theta V)(\theta(t))(\partial_\theta q)(\theta(t), \eta(t))\eta(t))((\partial_\theta V)(\theta(t)).$$

Similarly to the proof of Lemma 2.5, one can prove that right hand side is integrable and $\lim_{t \to \infty} G(t)$ exists. Since $G(t)$ is integrable, this limit should be zero.

Since

$$\frac{d}{dt}V(\theta(t)) = \frac{d}{d\theta}(\partial_\theta V)(\theta(t))h(\theta(t))(\partial_\theta q)(\theta(t), \eta(t)) \leq q(\theta(t), (\partial_\theta V)(\theta(t)) + q(\theta(t), \eta(t))$$

$\lim_{t \to \infty} V(\theta(t))$ exists. From Lem 2.3, $q(\theta, \eta(t)) = E - \rho^2 - V(\theta(t))$ for some constant $E$. Since right hand side has limit as $t \to \infty$, $\lim_{t \to \infty} q(\theta, \eta(t))$ exists. Then integrability of $q(\theta, \eta(t))$ yields this limit is zero. \qed

### 3 Proof of Theorem 1.2 and Theorem 1.3

We first prepare a lemma and Theorem to prove of Theorem 1.2.

**Lemma 3.1.** Let $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1})$, then one obtains the following:

$$[\text{Op}_h(a), P] = \frac{h}{i}\{f_h(r)a(\rho, \theta, \frac{\eta}{r}), \rho^2 + q(\theta, \frac{\eta}{r}) + V(\theta)\}u(hX, DX) + E_h$$

as $h \to 0$, where $E_h$ is a family of pseudodifferential operator on $L^2(\mathbb{R}^n)$ depending on $h$ such that $\|E\|_{L^2(\mathbb{R}^n)} = o(h)$ as $h \to 0$. We note that $\{\cdot, \cdot\}$ denotes Poisson bracket.
Proof. From equality (2.1), one can directly obtain the assertion for $-\triangle$ from Theorem 4.18 in [17] i.e.

$$[\text{Op}_f h(a), -\triangle] = \frac{\hbar}{i} \left\{ f_h(r)a(\rho, \theta, \eta), \rho^2 + q(\theta, \eta) \right\} w(hX, D_X) + O(h^3).$$

Let $k \in C^\infty(\mathbb{R})$ $k(x) = 1$ if $x > \varepsilon$ and $k(x) = 0$ if $x < \frac{\varepsilon}{2}$, where $\varepsilon > 0$ is taken so that $f_h(x) = 0$ if $x < \varepsilon$. Then we can calculate as follows:

$$[\text{Op}_{h,c}(a), V] = [\text{Op}_{h,c}(a), k(hr)(V_\infty + V_s)] + [\text{Op}_{h,c}(a), \{1 - k(hr)\}V].$$

Since $V_\infty$ is homogeneous of order zero, $k(h|x|)V_\infty(x) = \tilde{V}(hx)$ is a smooth and bounded function on $C^\infty(\mathbb{R}^n)$. Then one can obtain the equality similarly to the case of $-\triangle$ from (2.1).

Concerning $V_s$, one can calculate $\|k(hr)V_s\|_{L^2(\mathbb{R}^n)} = O(h)$ as $h \to 0$ from the definition of $V_s$. Thus $[\text{Op}_{h,c}(a), j(2c(h)hr)V_s] = O(h)$ as $h \to 0$ from Theorem 2.1.

Next we claim that $[\text{Op}_{h,c}(a), \{1 - k(hr)\}] = O(h^3)$. Let $\tilde{k}(x) = 1 - k(|x|)$, then $\tilde{k} \in C^\infty(\mathbb{R}^n)$ from the definition of $k$. By conjugating semiclassical dilation $U_h$, one can calculate as follows:

$$\text{Op}_{f_h}(a)\tilde{k}(hx) = \{ f_h(r)a(\rho, \theta, \eta) \} w(hx, D_x) \tilde{k}(hX) = U_h^* \{ f_h(r)a(\rho, \theta, \eta) \} w(x, hD_x) \tilde{k}(x) U_h.$$

Since $\text{supp}(f_h(r)a(\rho, \theta, \eta)) \cap \text{supp}(\tilde{k}(x)) = \phi$, Theorem 4.18 in [17] implies the claim.

Since multiplication operator by $V$ is uniformly bounded in $h$, the claim implies $\[\text{Op}_{f_h}(a), \{1 - k(hr)\}\] = O(h^3)$, which concludes the proof.

\[\square\]

**Theorem 3.2. (Energy conservation)**

Assume Assumption A. Let $f_h$ be an admissible cutoff function and let $u_h \in \mathcal{D}(P)$ be such that

$$\begin{cases} (P - E)u_h = R_h \\ \|u_h\|_{L^2(\mathbb{R}^n)} = 1, \end{cases}$$

where $\|R_h\|_{L^2(\mathbb{R}^n)} = O(1)$ as $h \to 0$. Then support of $\mu_f$ is localized in energy surfaces in the following meaning:

$$\text{supp}(\mu_f) \subset \{(\rho, \theta, \eta) \in \mathbb{R} \times T^* S^{n-1} \mid \rho^2 + q(\theta, \eta) + V_\infty(\theta) = E\}.$$
Proof. Since \((P - E)u_h = o(1)\), one can calculate

\[
o(1) = \langle u_h, \text{Op}_{f_h}(a)(P - E)u_h \rangle_{L^2(\mathbb{R}^n)}
\]

\[
= \langle u_h, \{f_h(r)\rho^2 + q(\theta, \eta) + V(\theta - E)\}w(hX, D_x)u_h \rangle_{L^2(\mathbb{R}^n)} + o(1)
\]

as \(h \to 0\) where we have used the fact that \(\|\text{Op}_{f_h}(a)V_s\|_{L^2(\mathbb{R}^n)} = o(1)\) as \(h \to 0\).

Therefore, if we take a suitable subsequence \(h_m\) and \(m \to 0\), we obtain

\[
\int_{\mathbb{R} \times T^*S^{n-1}} a(\rho^2 + q(\theta, \eta) + V - E)d\mu_f = 0,
\]

which concludes the proof. 

Actually, it suffices to prove following Theorem to prove Theorem 1.3.

**Theorem 3.3.** Assume assumptions of Theorem 1.3. If \(E \notin \text{Cr}(V)\) and \(J_h u_h \to 0\) as \(h \to 0\), \(\tilde{J}_h u_h \to 0\) as \(h \to 0\).

**Proof of Theorem 1.3.** Let \(c_0(h) = h^{-1}\|R_h\|_{L^2(\mathbb{R}^n)}\).

We define \(c(h) = \max_{0 \leq \tilde{h} \leq h} \max\{\tilde{h}^\delta, c_0(\tilde{h})^\delta\}\) for some \(\delta \in (0, 1)\). Then \(c(h)\) is monotone increasing function on \((0, 1)\) and \(c(h)\) satisfies \(c(h) = o(1)\) and \(c(h)^{-1} = o(h^{-1})\) as \(h \to 0\). It is also clear that \(\|R_h\|_{L^2(\mathbb{R}^n)} = o(hc(h))\) as \(h \to 0\).

Then we can apply Theorem 3.3 and can prove Theorem 1.2 by iteration.

**Proof of Theorem 3.3.** Let \(\chi(r) = j(\frac{1}{2}r)(1 - j(\frac{1}{4}r))\). We define our cutoff function \(\chi_h\) by \(\chi_h(r) = \chi((\log(c(h)^{-1}))^{-1}\log r)\). Then we see that \(r \leq c(h)^{-1}\) on \(\text{supp}(\chi_h)\).

We assume \(\tilde{J}_h u_h \not\to 0\) as \(h \to 0\), which means semiclassical measure \(\mu_J\) is positive.

Since \(\|R_h\|_{L^2(\mathbb{R}^n)} = o(hc(h))\), we obtain the followings:

\[
o(hc(h)) = \langle u_h, \{c(h)r\chi_h(r)\rho^2 + q(\theta, \eta)\}w(hx, D_x), P \rangle_{L^2(\mathbb{R}^n)}
\]

\[
= h \langle u_h, \{c(h)r\chi_h(r)\rho^2 + q(\theta, \eta)\}, \rho^2 + q(\theta, \eta) + V(\theta) \rangle w(hx, D_x)u_h \rangle_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^3)
\]
We also see

\[
\{ r\chi_h(r)a(\rho, \theta, \frac{\eta}{r}), \rho^2 + q(\theta, \frac{\eta}{r}) + V(\theta) \}
\]

\[
= 2\rho\chi_h(r)a(\rho, \theta, \frac{\eta}{r}) + \chi_h(r)\{(\partial_\rho a)(\rho, \theta, \frac{\eta}{r})q(\theta, \frac{\eta}{r}) + (\partial_\theta a)(\rho, \theta, \frac{\eta}{r})(\partial_\eta q)(\theta, \frac{\eta}{r})
\]

\[
- (\partial_\eta a)(\rho, \theta, \frac{\eta}{r})((\partial_\theta q)(\theta, \frac{\eta}{r}) + (\partial_\theta V)(\theta) + 2\rho\frac{\eta}{r})\}
\]

\[
+ 2\rho(\log h^{-1})^{-1}(\partial_r \chi)((\log h^{-1})^{-1}\log r)a(\rho, \theta, \frac{\eta}{r}).
\]

Taking \( h \to 0 \), we see

\[
\int_{\mathbb{R} \times T^*S^{n-1}} 2pa(\rho, \theta, \frac{\eta}{r}) + \{(\partial_\rho a)(\rho, \theta, \frac{\eta}{r})q(\theta, \frac{\eta}{r}) + (\partial_\theta a)(\rho, \theta, \frac{\eta}{r})(\partial_\eta q)(\theta, \frac{\eta}{r})
\]

\[
- (\partial_\eta a)(\rho, \theta, \frac{\eta}{r})((\partial_\theta q)(\theta, \frac{\eta}{r}) + (\partial_\theta V)(\theta) + 2\rho\frac{\eta}{r})\}d\mu \]

where we have used that \( \mu_\chi = \mu_\tilde{J} \) since \( J\tilde{u}_h \to 0 \) as \( h \to 0 \).

Let \( H \) be a vector field on \( T^*(\mathbb{R} \times T^*S^{n-1}) \) defined by

\[
H = q(\theta, \frac{\eta}{r})\partial_\rho + (\partial_\eta q)(\theta, \frac{\eta}{r})\partial_\theta - ((\partial_\theta q)(\theta, \frac{\eta}{r}) + (\partial_\theta V)(\theta) + 2\rho\frac{\eta}{r})\partial_\eta,
\]

and \( \Phi_t \) be flow generated by \( H \). Using this \( \Phi_t \), (3.1) can be rewrite as

\[
\frac{d}{dt} \int_{\mathbb{R} \times T^*S^{n-1}} \Phi_t^*(ae^{2pt})d\mu = 0.
\]

If \( E \notin \text{Cv}(V) \), \( \lim_{t \to \infty} \rho(t) \neq 0 \), which means \( \int_{\mathbb{R} \times T^*S^{n-1}} \Phi_t^*(ae^{2pt})d\mu \) diverges if we take the limit \( t \to \infty \) or \( t \to -\infty \) since \( \mu \) is positive. This is contradiction and the assertion follows.

\[ \square \]

**Proof of Theorem 1.2.** Let \( C > 0 \) be such that \( \text{supp}(\chi) \subset (1, C) \). We define \( \tilde{\chi}(x) = j(x)(1 - j(\frac{1}{2C}x)) \). Then \( x\tilde{\chi}(x) \) is an admissible cutoff function.

For \( a \in C^\infty_0(\mathbb{R} \times T^*S^{n-1}) \), we calculate commutator of \( \text{Op}_{r\tilde{\chi}}(a) \) and \( P - E \) to obtain

\[
o(h) = \langle u_h, [\text{Op}_{r\tilde{\chi}}(a), P - E]u_h \rangle_{L^2(\mathbb{R}^n)}
\]

\[
= \frac{h}{\tilde{\chi}} \langle u_h, \text{Op}_j(\rho a + 2\rho a)u_h \rangle_{L^2(\mathbb{R}^n)} + o(h),
\]

similarly to the proof of Theorem 1.2.
Then we see $\mu_j = 0$ if $E \notin \text{Cv}(V)$, which is contradiction from Theorem 3.1 and the assumption on $u_h$. Thus $E \in \text{Cv}(V)$.

If $E \in \text{Cv}(V)$, similar to the above argument, we see

$$\text{supp}(\mu) \subset \{(\rho, \theta, \eta) \in \mathbb{R} \times T^*S^{n-1} \mid \lim_{t \to \infty} \rho(t) = 0\}. $$

Let $a \in C_0^\infty(\mathbb{R} \times T^*S^{n-1}; [0, \infty))$ be such that

$$\text{supp}(a) \cap \{q(\theta, \partial_\theta V(\theta)) + q(\theta, \eta) < \delta\} = \phi.$$ 

Then we see

$$\int_{\mathbb{R} \times T^*S^{n-1}} ad\mu_j = \lim_{t \to \infty} \int_{\mathbb{R} \times T^*S^{n-1}} \Phi_t^*(ae^{2\rho t})d\mu_j$$

$$\lim_{t \to \infty} \int_{\{\lim_{t \to \infty} \rho(t) = 0\}} \Phi_t^*(ae^{2\rho t})d\mu_j.$$ 

Since $\rho(t)$ is monotone increasing and $\lim_{t \to \infty} \rho(t) = 0$, $\rho(t) < 0$ which implies

$$\lim_{t \to \infty} \int_{\{\lim_{t \to \infty} \rho(t) = 0\}} \Phi_t^*(ae^{2\rho t})d\mu_j \leq \lim_{t \to \infty} \int_{\{\lim_{t \to \infty} \rho(t) = 0\}} \Phi_t^*(a)d\mu_j$$

from the fact $\lim_{t \to \infty} \Phi_t^*(a)(\rho, \theta, \eta) = 0$ pointwise from the definition of $a$, Theorem 2.6 and dominant convergence theorem,

$$\int_{\mathbb{R} \times T^*S^{n-1}} ad\mu_j = \lim_{t \to \infty} \int_{\{\lim_{t \to \infty} \rho(t) = 0\}} \Phi_t^*(ae^{2\rho t})d\mu_j = 0,$$

This means

$$\text{supp}(\mu) \subset \{(\rho, \theta, 0) \in \mathbb{R} \times T^*S^{n-1} \mid \theta \in \text{Cr}(V), \lim_{t \to \infty} \rho(t) = 0\}. $$

If $(\rho, \theta, \eta)$ is in the set of right hand side of the above line, $\rho(t) = \rho$ for any $t$ since the set of right hand side is fixed set of $\Phi_t$, which implies $\rho = 0$ and $\theta \in V^{-1}(E)$. Thus the assertion follows.

\[\square\]

4 Example of asymptotic eigenvectors whose defect measure does not vanish

In this section, we construct an example of $u_h$ such that corresponding semiclassical measure $\mu \neq 0$. We will show existence of the quasimodes with following support condition.
**Theorem 4.1.** Assume Assumption A.

(1) Let \( E \in [\min(V), \max(V)] \), \( \theta_0 \in V^{-1}(E) \subset S^{n-1} \) and \( k \in \mathbb{N} \cup \{0\} \) be such that \( \partial^k \theta V(\theta_0) = 0 \) for any \( k \leq k \). For any \( C > 0 \), there exists a solution \( u_h \) to the (1.1) which satisfies the following conditions:

1. \( u_h \in \mathcal{D}(P) \) and satisfies
   \[
   \begin{cases}
   (P - E)u_h = R_h \\
   \|u_h\|_{L^2(\mathbb{R}^n)} = 1,
   \end{cases}
   \]

2. \( \|R_h\|_{L^2(\mathbb{R}^n)} = o(h) \) if \( k > 1 \) and \( \|R_h\|_{L^2(\mathbb{R}^n)} = \Theta(h) \) if \( k = 0, 1 \) as \( h \to 0 \),

3. Let \( j \) be a function in Section 1. \( u_h \) satisfies \( j(hr)u_h(r, \theta) = u_h(r, \theta) \),

4. \( \text{supp}(u_h) \subset \{(r, \theta) \in \mathbb{R}^n \mid r > 1, \text{dist}(\theta, \theta_0) < C r^{-\ell(k)}\} \) for sufficiently small \( h > 0 \),

where \( \ell(k) \) is such that \( \ell(k) = k + 1 \) if \( k > 0 \) and \( \ell(0) = \frac{2}{3} \), and \( \text{dist}(\cdot, \cdot) \) denotes the distance defined by the metric on \( S^{n-1} \) induced by the Euclidean metric on \( \mathbb{R}^n \).

(2) Let \( \max(V) < E \), \( \theta_0 \in S^{n-1} \) and \( k \in \mathbb{N} \cup \{0\} \) be such that \( \partial^k \theta V(\theta_0) = 0 \) for any \( k \leq k \). For any \( C, \varepsilon > 0 \), there exists a solution \( u_h \) to the (1.1) which satisfies the following conditions:

1. \( u_h \in \mathcal{D}(P) \) and satisfies
   \[
   \begin{cases}
   (P - E)u_h = R_h \\
   \|u_h\|_{L^2(\mathbb{R}^n)} = 1,
   \end{cases}
   \]

2. \( \|R_h\|_{L^2(\mathbb{R}^n)} = \Theta(h) \) as \( h \to 0 \),

3. Let \( j \) be a function in Section 1. \( u_h \) satisfies \( j(hr)u_h(r, \theta) = u_h(r, \theta) \),

4. \( \text{supp}(u_h) \subset \{(r, \theta) \in \mathbb{R}^n \mid r > 1, \text{dist}(\theta, \theta_0) < C r^{-\ell(k)}\} \) for sufficiently small \( h > 0 \),

where \( \ell(k) \) is the same with (1).

**Remark.** From Theorem 3.1, condition 2. in both statements imply \( \mu_j \) do not vanish.

**Proof.** (1) We will construct \( u_h \) of form \( u_h(x) = f_h(r)g_h(\theta) \) by the polar coordinate which satisfies following conditions in addition to the conditions in Theorem 4.1:
1. \( \| \partial_r^2 + \frac{n-1}{r} \partial_r f_h \|_{L^2((0,\infty);r^{n-1} dr)} = o(h) \) as \( h \to 0 \).

2. If \( k > 1 \) (resp. \( k = 0, 1 \)), \( \| r^{-2} \Delta_{S^{n-1}} u_h \|_{L^2(\mathbb{R}^n)} = o(h) \) (resp. \( O(h) \)), where \( \Delta_{S^{n-1}} \) denotes Laplacian on \( S^{n-1} \).

3. There exists \( C > 0 \) such that \( |V(\theta) - E| \leq Ch^{\ell(k)} \) on supp(\( g_h \)).

We assume that \( E = 0 \). This does not lose generality since \( (V - E) \) is still homogeneous of order zero.

Let \( f \in C_0^\infty(1, \infty) \setminus \{0\} \). We define \( f_{h,0}(r) = Ch^{-n} f(hr) \) if \( k > 0 \) and \( f_{h,0}(r) = Ch^{-\frac{n+2}{2} f(h^{\frac{4}{3}} r)} \) if \( k = 0 \), where \( C > 0 \) is renormalizing constant. Then we see that \( r^{-1} \leq Ch \) on supp(\( f_h \)) for some \( C > 0 \) and one can easily calculate that \( f_h \) satisfies the condition 1. at the beginning of proof.

Since \( \partial_r^k V(\theta_0) = 0 \) for any \( \hat{k} \leq k \), from Taylor’s theorem, there exists a small neighbor \( U \) of \( \theta_0 \) such that \( V(\theta) = O(\text{dist}(\theta, \theta_0)^{k+1}) \) near \( \theta = \theta_0 \).

Let \( \phi \in C_0^\infty(\mathbb{R}) \) be such that supp(\( \phi \)) \( \subset (-1, 1) \) and \( \phi(x) = 1 \) if \( |x| \leq \frac{1}{2} \) and \( 0 \leq \phi \leq 1 \). We define \( \tilde{g}_h \) by

\[
\tilde{g}_h(\theta) = \phi \left( \frac{\text{dist}(\theta, \theta_0)}{h \frac{1+k}{k+1}} \right).
\]

Then we see that there exists \( C > 0 \) such that \( |V| \leq Ch^{1+k} \) on supp(\( \tilde{g}_h \)) for sufficiently small \( h \). Also, since \( \Delta_{S^{n-1}} \) is a second order differential operator, we obtain that \( \| \Delta_{S^{n-1}} \tilde{g}_h(\theta) \| = o(h^{-\frac{1+k}{k+1}}) \| \tilde{g}_h \| \). Let \( C_h^{(2)} = \| \tilde{g}_h \|_{L^2(S^{n-1})}^{-1} \) and \( g_h = C_h^{(2)} \tilde{g}_h \).

Since \( \Delta_{S^{n-1}} g_h(\theta) = o_{L^2(S^{n-1})}(h^{-\frac{1+k}{k+1}}) \) and \( r^{-2} f_h(r) = O_{L^2((0,\infty);r^{n-1} dr)}(h^{2}) \), we see \( r^{-2} \Delta_{S^{n-1}} u_h = o_{L^2(\mathbb{R}^n)}(h) \) if \( k > 0 \) the case \( k = 0 \) can be check easily. Combining with the conditions of \( f_h \) and \( g_h \), we see \( (P - E) u_h = o_{L^2(\mathbb{R}^n)}(h) \).

Actually, we can calculate \( \| u_h \|_{L^2(\mathbb{R}^n)} = 1 \) from the definition of \( f_h \) and \( g_h \). From the definition of \( f_h \), it is clear \( j(h_m|x|)u_{h_m}(x) = u_{h_m}(x) \to 0 \) for any sequence \( h_m \) such that \( h_m \to 0 \) as \( j \to \infty \). Thus the semiclassical defect measure \( \mu \) defined from \( u_h \) does not vanish.

Concerning about the proof of (2), let \( E = E_1 + E_2 \) where \( V(\theta_0) = E_2 \). Let \( f_h(r) = Ch^{-\frac{2}{3} n} f(h^{\frac{2}{3}} r)e^{\sqrt{E_1} r} \). Then one can obtain the conclusion similarly. \( \square \)

## 5 Proof of Theorem 1.4

In this section we prove observability result for Schrödinger operators with homogeneous potentials of order zero.
Proof of Theorem 1.4. We prove by constructing sequence of functions $u_m$ such that $\int_{0}^{T} \int_{\Omega} |e^{-itP} u_m(x)|^2 \, dx \, dt \to 0$ as $m \to \infty$.

Let $X = \{(r, \theta) \in \mathbb{R}^n \mid r > R, \text{dist}(\theta, \theta_0) < Cr^{-1/n}\}$ and $u_h$ be solution of (1.1) which constructed in Theorem 4.1. Then we can find $\bar{\chi} \in C_0^\infty(0, \infty)$ such that $\bar{\chi}(hr)f_h(r) = f_h(r)$.

From the assumption of $k$ and $R$, we can take $\varphi_h \in C^\infty(S^{n-1}; [0,1])$ so that $\text{supp}[\varphi] \cap \{\theta \in S^{n-1} \mid \text{dist}(\theta, \theta_0) < r^{-1/n}\} = \phi$ and $\bar{\chi}(hr)\varphi_h(\theta) = 1$ on $\Omega$ for sufficiently small $h > 0$. Then we see that $\text{supp}[\bar{\chi}(hr)\varphi(\theta)] \cap X = \phi$ for sufficiently small $h > 0$.

By the assumption on $u_{h_m}$ and $\varphi$, we see that

$$0 \leq \|u_{h_m}\|_{L^2(\Omega)} \leq \langle u_{h_m}, \chi_{\Omega} u_{h_m} \rangle \leq \langle u_{h_m}, \bar{\chi}(h_mr)\varphi_h(\theta)u_{h_m} \rangle,$$

where $\chi_{\Omega}(x)$ denotes characteristic function of $\Omega$. Then from Theorem 4.1 (2) and (3), $j(2hr\varphi_h(\theta))u_{h_m} = 0$ for sufficiently large $m$, which means $\|u_{h_m}\|_{L^2(\Omega)} = 0$ for sufficiently large $m$.

Next we claim $F_m(t) = \langle e^{-itP} u_{h_m}, \chi_{\Omega} e^{-itP} u_{h_m} \rangle_{L^2(\mathbb{R}^n)} \to 0$ as $m \to \infty$.

One can calculate as follows:

$$\frac{dF_m}{dt}(t) = -i\langle e^{-itP} Pu_{h_m}, \chi_{\Omega} e^{-itP} u_{h_m} \rangle_{L^2(\mathbb{R}^n)} + i\langle e^{-itP} u_{h_m}, \chi_{\Omega} e^{-itP} Pu_{h_m} \rangle_{L^2(\mathbb{R}^n)}.$$

Thus we see

$$\left| \frac{dF_m}{dt}(t) \right| \leq C\|u_{h_m}\|_{L^2(\mathbb{R}^n)} \|(P - E)u_{h_m}\|_{L^2(\mathbb{R}^n)} = C\|(P - E)u_{h_m}\|_{L^2(\mathbb{R}^n)},$$

where $C > 0$ is a constant independent of $t$ and we have used boundedness of $\chi_{\Omega}$ in the first inequality and uniform boundedness of $u_m$ in the second inequality.

Since $F_m(t) = F_m(0) + \int_{0}^{t} \frac{dF_m}{dt}(s) \, ds$, we see for $t \in [0, T]$,

$$|F_m(t)| \leq |F_m(0)| + \int_{0}^{t} \left| \frac{dF_m}{dt}(s) \right| \, ds \leq |F_m(0)| + C\|Pu_{h_m}\|_{L^2(\mathbb{R}^n)}.$$

Letting $m \to \infty$, we obtain the claim.

For any $\varepsilon > 0$, there exists sufficiently large $M > 0$ so that $m > M$ implies $\|\langle e^{-itP} u_{h_m}, \chi_{\Omega} e^{-itP} u_{h_m} \rangle_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{4}$. Then $\int_{0}^{T} \int_{\Omega} |e^{-itP} u_{m}(x)|^2 \, dx \, dt \leq \varepsilon$ for $m > M$, which concludes the proof. \qed
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