CONCENTRATION OF SOLUTIONS FOR THE FRACTIONAL NIRENBERG PROBLEM

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Abstract. The aim of this paper is to show the existence of infinitely many concentration solutions for the fractional Nirenberg problem under the condition that $Q_s$ curvature has a sequence of strictly local maximum points moving to infinity.

1. Introduction and main results. The classical Nirenberg problem is to ask whether, considering the standard sphere $(S^N, g_{S^N})$, $N \geq 2$, one can deform conformally the metric in such a way that the scalar curvature (Gauss curvature in $N = 2$) becomes a prescribed function $\tilde{K}$ on $S^N$. The equivalent problem is to consider the following equations

$$-\Delta_{g_{S^N}} w + 1 = \tilde{K} e^{2w} \text{ on } S^2$$

and

$$-\Delta_{g_{S^N}} v + C(N)R_0 v = C(N)\tilde{K} v^{N-2\frac{4}{N-4}} \text{ on } S^N \text{ for } N \geq 3,$$

where $C(N) =\frac{N-2}{4(N-4)}$, $R_0 = N(N-1)$ is the scalar curvature of $(S^N, g_{S^N})$ and $v = e^{\frac{N-2}{4}} w$.

It is well known that the linear operators in (1) and (2) are conformal Laplacians associated to the metric $g_{S^N}$ and denoted by $P^g_{S^N}$. This definition can also be extended to a general compact Riemannian manifold $(M, g)$ of dimension $N \geq 2$. There is another conformally covariant operator discovered by Paneitz defined as follows

$$P^g_2 = (-\Delta_g)^2 - \text{div}_g(A_N R_g g + B_N Ric_g) d + \frac{N-4}{2} Q^g_N,$$

where $Q^g_N$ is the Q-curvature and $Ric_g$ is the Ricci curvature of $g$, $A_N, B_N$ are suitable constants depending on $N$. See [16, 31] for more details. Graham, Jenne, Mason and Sparling [20] constructed a sequence of conformally covariant elliptic operators $\{P^g_k\}$ on Riemannian manifolds for all positive integers $k$ if $N$ is odd, and for $k \in \{1, 2, \ldots, N/2\}$ if $N$ is even. Moreover, $P^g_1$ is the conformal Laplacian $Lg := -\Delta_g + c(N)R_g$ and $P^g_2$ is the Paneitz operator. Up to positive constants, $P^g_1(1)$ is the scalar curvature of $g$ and $P^g_2(1)$ is the Q-curvature. In [26, 27], Li

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Yanyan et al gave a complete characterization for fully nonlinear conformally covariant differential operators of any integer order on \( \mathbb{R}^N \). Later on, Peterson \cite{peterson} constructed an intrinsically defined conformally covariant pseudo-differential operator of arbitrary real number order. Using a generalized Dirichlet to Neumann map, Graham and Zworski \cite{grahamzworski} introduced a meromorphic family of conformally invariant operators on the conformal infinity of asymptotically hyperbolic manifolds. Recently, Chang and González \cite{changgonzalez} took the way of Graham and Zworski to define conformally invariant operators \( P_s^g \) of noninteger order \( s \in (0, N/2) \) and the localization method of Caffarelli and Silvestre \cite{caffarellisilvestre} for the fractional Laplacian \((-\Delta)^s\) on the Euclidean space \( \mathbb{R}^N \). These lead naturally to a fractional order curvature problem \( R_s^g := P_s^g(1) \). The problem of fractional order curvature was studied extensively: see, e.g., \cite{boggan, caffarelli, christianson, dimant, lu} and the references therein.

The operator \( P_s^g \) of noninteger order \( s \in (0, N/2) \) is conformally covariant in the sense that if \( f \) is any smooth function and \( g = v^\frac{2}{N-2s} g_{\mathbb{R}^N} \) for some \( v > 0 \), then

\[
P_s^{g_{\mathbb{R}^N}}(vf) = v^{\frac{N+2s}{N-2s}} P_s^g(f).
\]

Similar to the formula for scalar curvature and the Paneitz-Branson \( Q \)-curvature, the \( Q \)-curvature for \( g \) of order \( 2s \) can be defined as \( Q_s^g = P_s^g(1) \). Thus, this raises a natural question: is there a metric \( \{g\} \) in the conformal class \( [g] \) such that \( Q_s^g \) equals to a prescribed function \( K \) on a smooth compact Riemannian manifold \((M,g)\) of dimension \( N \geq 2 \)? By \eqref{eq:2}, one needs to solve the following semilinear equation,

\[
P_s^g v = K v^{\frac{N+2s}{N-2s}}, \quad v > 0 \text{ on } M.
\]

If \((M,g)\) is the standard sphere \((\mathbb{S}^N, g_{\mathbb{S}^N})\), \eqref{eq:2} can be viewed as the fractional Nirenberg problem

\[
P_s^{g_{\mathbb{S}^N}} v = K v^{\frac{N+2s}{N-2s}}, \quad v > 0 \text{ on } \mathbb{S}^N,
\]

which has been studied in \cite{boggan, caffarelli, christianson, dimant, lu}.

The operator \( P_s^{g_{\mathbb{S}^N}} \) is the \( 2s \) order conformal Laplacian on \( \mathbb{S}^N \) and can be uniquely expressed as following

\[
P_s^{g_{\mathbb{S}^N}} = \frac{1}{\Gamma(B + \frac{1}{2} - s)} \frac{\Gamma(B + \frac{1}{2} + s)}{\Gamma(B + \frac{1}{2} - s)}, \quad B = \sqrt{-\Delta_{g_{\mathbb{S}^N}} + \left( \frac{N-1}{2} \right)^2},
\]

where \( \Gamma \) is the Gamma function and \( \Delta_{g_{\mathbb{S}^N}} \) is the Laplace-Beltrami operator on \((\mathbb{S}^N, g_{\mathbb{S}^N})\). The operator \( P_s \) can be seen more concretely on \( \mathbb{R}^N \) by using stereographic projection. Let \( N \) be the north pole of \( \mathbb{S}^N \) and \( \Phi: \mathbb{R}^N \to \mathbb{S}^N \setminus \{N\} \), \( x \mapsto \left( \frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{|x|^2 + 1} \right) \) be the inverse of stereographic projection operator from \( \mathbb{S}^N \setminus \{N\} \) to \( \mathbb{R}^N \). Then, by the conformal invariance of \( P_s \), one has the following relation

\[
P_s^{g_{\mathbb{S}^N}}(\varphi) \circ \Phi = |J_\Phi|^{-\frac{N+2s}{2s}} (-\Delta)^s \left( |J_\Phi|^{\frac{N+2s}{2s}} (\varphi \circ \Phi) \right), \quad \varphi \in C^\infty(\mathbb{S}^N),
\]

where \( |J_\Phi| = \left( \frac{2}{1+|x|^2} \right)^N \). Then, for a solution \( v \) of \eqref{eq:2}, \( u(x) = |J_\Phi|^{\frac{N-2s}{2s}} v(\Phi(x)) \) satisfies

\[
(-\Delta)^s u = (K \circ \Phi) u^{\frac{N+2s}{N-2s}}, \quad u > 0, \text{ on } \mathbb{R}^N.
\]
In this paper, we study the fractional Nirenberg problem (6) with \( s \in (0,1) \). We focus on concentration of solutions for (6). Actually, we consider the following problem

\[ (-\Delta)^s u = K(y)u^{\frac{N+2s}{N-2s}}, \quad u > 0 \text{ on } \mathbb{R}^N, \tag{7} \]

where \( s \in (0,1), 2s < N, K \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \).

For \( s \in (0,1), (-\Delta)^s \) is the fractional Laplacian operator with the following representation

\[ (-\Delta)^s u(x) = d_{s,N} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}, \]

where \( d_{s,N} \) is a positive constant depending only on \( s, N \). P.V. is in the sense of the principal value. Caffarelli and Silvestre [6] found another local representation

\[ (-\Delta)^s u(x) = -\lim_{z \to 0^+} d_{s,N} z^{1-2s} \partial_z \tilde{u}(x, z), \]

where \( d_{s,N} > 0 \) is a constant depending on \( s \) and \( \tilde{u} = \tilde{u}(x, z) \) is the solution of the following boundary value problem in the half space \( \mathbb{R}^{N+1}_+ = \{ (x, z) : x \in \mathbb{R}^N, z > 0 \} \):

\[
\begin{aligned}
&-\text{div}(z^{1-2s}\nabla \tilde{u}) = 0, \quad \text{in } \mathbb{R}^{N+1}_+,
&\tilde{u}(x, 0) = u(x), \quad \text{on } \mathbb{R}^N.
\end{aligned}
\]

The homogeneous fractional Sobolev space \( \dot{H}^s(\mathbb{R}^N) \) is defined by the completion of \( C_0^\infty(\mathbb{R}^N) \) with respect to the norm \( \|u\| = (\int_{\mathbb{R}^N} (\int_{\mathbb{R}^N} (\xi^{2s}\hat{\tilde{u}}(\xi)\hat{\tilde{v}}(\xi))^\frac{1}{2})^2 ) \), and \( \hat{\tilde{u}}(\xi) \) is the Fourier transformation of \( u \) defined by

\[ \hat{\tilde{u}}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} u(x)e^{-i\xi \cdot x}. \]

It follows from [19] that the above norm is equivalent to the Gagliardo seminorm \([u]_s\) of \( u \),

\[ [u]_s = \left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \right)^{\frac{1}{2}}. \]

Moreover,

\[ \dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{2^*_s}(\mathbb{R}^N), \text{ with } 2^*_s = \frac{2N}{N - 2s}. \]

Consider the equation

\[ (-\Delta)^s u = u^{\frac{N+2s}{N-2s}}, \quad u > 0 \text{ on } \mathbb{R}^N, \quad u \in \dot{H}^s(\mathbb{R}^N). \tag{8} \]

It has been proved in [12, 25] that the following function, for \( x \in \mathbb{R}^N \) and \( \lambda > 0 \),

\[ U_{x,\lambda}(y) = C_0 \left( \frac{\lambda}{1 + \lambda^2|y - x|^2} \right)^{\frac{N-2s}{2}}, \quad y \in \mathbb{R}^N, \]

where \( C_0 = C_0(N, s) > 0 \), solves (8) on \( \mathbb{R}^N \).

For \( x^j = (x^j_1, x^j_2, \cdots, x^j_N) \in \mathbb{R}^N, \lambda_j > 0, j = 1, 2. \) Define

\[ E_{x,\lambda} = \left\{ v \in \dot{H}^s(\mathbb{R}^N) : \left\langle \frac{\partial U_{x,\lambda}}{\partial \lambda_j}, v \right\rangle = \left\langle \frac{\partial U_{x,\lambda}}{\partial x^i_j}, v \right\rangle = 0, j = 1, 2, i = 1, \cdots, N \right\}. \]
Lemma 2.1. We begin with

\[ K(y) = K_j - Q_j(y - z^j) + R_j(y - z^j), \quad y \in B_\rho(z^j), \]

where \( K_j, j = 1, 2, \ldots \), satisfies \( K_0 \leq K_j \leq K'_0 \) for some constants \( K'_0 \geq K_0 > 0 \), \( Q_j(y) \) satisfies

\[ a_0 |y|^{\beta_j} \leq Q_j(y) \leq a_1 |y|^{\beta_j}, \quad j = 1, 2, \ldots, \]

for some constants \( a_1 \geq a_0 > 0 \) independent of \( j \) and \( R_j(y) \) satisfies \( R_j(y) = O \left( |y|^{\beta_j + \sigma} \right) \) for some \( \sigma > 0 \) independent of \( j \).

We now state our main result of the paper:

**Theorem 1.1.** Assume \((\mathcal{K})\) holds, then for \( \mu > 0 \) small enough and each strictly local maximum point \( z^1 \) of \( K(y) \), there exists another strictly local maximum point \( z^{2\ast} \) such that \((7)\) has a solution of the form

\[ u = 2 \sum_{j=1}^{2} K_{ij}^{2x+\frac{N}{2x}} U_{x, \lambda_j} + v, \]

where \( K_{ij} \) is the constant in \((\mathcal{K})\), \((x, \lambda, v)\) satisfies \( v \in E_{x, \lambda} \), \( x = (x^1, x^2) \), \( \lambda = (\lambda_1, \lambda_2) \) and

\[ \|v\| \leq \mu, \quad |x^1 - x^2| \geq \frac{1}{\mu}, \quad |x^j - z^{ij}| \leq \mu, \quad \lambda_j \geq \frac{1}{\mu}, \quad j = 1, 2. \]

The proof of our results is inspired by the methods of \([10, 32, 35]\). More precisely, we will use a reduction argument similar to \([32, 35]\) to prove Theorem 1.1, see also \([7, 8, 9, 29, 30, 34]\). It is worthwhile to point out that the solutions constructed in this paper have exactly two maximum points, the distance between which is very large. From the proof of the Theorem 1.1, we find that the interaction between two approximated solutions concentrated at two different strictly local maximum points of \( K(y) \), plays the important role in our construction of the solutions.

This paper is organized as follows. In Section 2, we give some basic estimates. In Section 3, we carry out the finite dimensional reduction procedure. The main results will be proved in Section 4.

2. Preliminaries: Basic estimates. In this section, we will give some basic estimates used in the later sections. Since the estimates are very similar to those in [2], we only give a sketch here. Set \( \varepsilon_{ij} = \left( \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j |x_i - x_j|^2 \right)^{\frac{2x+\frac{N}{2x}}{\alpha}} \), \( i \neq j \).

We begin with

**Lemma 2.1.** For \( i \neq j \), we have

\[
\int_{\mathbb{R}^N} U^{\frac{N+2x}{x}}_{x', \lambda_i} U_{x, \lambda_j} = C_0^{\frac{2x}{N+2x}} C_1 \varepsilon_{ij} + O \left( \varepsilon_{ij}^{\frac{N}{N+2x}} \right),
\]

\[
\int_{\mathbb{R}^N} U^{\frac{N}{x}}_{x', \lambda_i} U_{x, \lambda_j} = O \left( \varepsilon_{ij}^{\frac{N}{N+2x}} \log \varepsilon_{ij}^{-1} \right),
\]

\[
\int_{\mathbb{R}^N} U^{\alpha}_{x', \lambda_i} U_{x, \lambda_j} = O \left( \varepsilon_{ij} (\log \varepsilon_{ij}^{-1})^{\frac{N-2x}{N}} \right),
\]

where \( C_1 = \int_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\frac{N+2x}{x}}}, \alpha, \beta > 1 \) such that \( \alpha + \beta = 2^*_x \), \( \beta = \inf \{\alpha, \beta\} \).
The proof of this lemma can be found in [14].
Define
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{N - 2s}{4N} \int_{\mathbb{R}^N} K(y) |u|^{\frac{2N}{N-2s}}. \]
To solve the finite dimensional problem in Section 4, we need to expand the functional \( I(u) \) explicitly as follows.

**Lemma 2.2.**
\[
I \left( \sum_{j=1}^{2} K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right) = \sum_{j=1}^{2} I \left( K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right) - (K_1 K_2) \frac{2s-N}{4s} C_0 \frac{2N}{N-2s} C_1 \varepsilon_{12} \\
+ O \left( \varepsilon_{12} \right) + O \left( \sum_{j=1}^{2} \left( |x^j - z^j|^2 + \frac{1}{\lambda_j^2} \right) \varepsilon_{12} + \varepsilon_{12}^{1+\tau} \right),
\]
where \( \tau > 0 \) is some constant.

**Proof.**
\[
\left\| \sum_{j=1}^{2} K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right\|^2 = \sum_{j=1}^{2} \left\| K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right\|^2 + 2 (K_1 K_2) \frac{2s-N}{4s} \int_{\mathbb{R}^N} U_{x_1, \lambda_1}^N U_{x_2, \lambda_2}^{N+2s} C_1 \varepsilon_{12} + O \left( \varepsilon_{12}^{1+\tau} \right).
\]
Using the following inequality,
\[
|a + b|^p - a^p - b^p - p a^{p-1} b - p b^{p-1} | \leq \begin{cases} \left( C a^{p/2} b^{p/2} / 2, & 2 < p \leq 3, \\ C a^2 b^p - 2 C a^{p-2} b^2, & p > 3, \end{cases}
\]
we find
\[
\int_{\mathbb{R}^N} K(y) \left( \sum_{j=1}^{2} K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right)^{2s} \\
= \sum_{j=1}^{2} \int_{\mathbb{R}^N} K(y) \left( K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right)^{2s} + 2s \int_{\mathbb{R}^N} K(y) \left( K_1 \frac{2s-N}{4s} U_{x_1, \lambda_1} \right)^{2s-1} K_2 \frac{2s-N}{4s} U_{x_2, \lambda_2} \\
+ 2s \int_{\mathbb{R}^N} K(y) \left( K_2 \frac{2s-N}{4s} U_{x_2, \lambda_2} \right)^{2s-1} K_1 \frac{2s-N}{4s} U_{x_1, \lambda_1} + O \left( \varepsilon_{12}^{1+\tau} \right) \\
= \sum_{j=1}^{2} \int_{\mathbb{R}^N} K(y) \left( K_j \frac{2s-N}{4s} U_{x_j, \lambda_j} \right)^{2s} + 2s (K_1 K_2) \frac{2s-N}{4s} C_0 \frac{2N}{N-2s} C_1 \varepsilon_{12} \\
+ O \left( \sum_{j=1}^{2} \varepsilon_{12} / \lambda_j^{2s} \right) + O \left( \sum_{j=1}^{2} \left( |x^j - z^j|^{\frac{1}{\lambda_j^2}} + \frac{1}{\lambda_j^{2s}} \right) \varepsilon_{12} + \varepsilon_{12}^{1+\tau} \right).
\]
Combining the above estimates, we can obtain the desired expansion. \( \square \)
3. Finite Dimensional Reduction. In this section, we are ready to start the finite dimensional reduction. Define for \( \mu > 0 \) small, \( \{i_1, i_2\} \subset \mathbb{N}, i_1 \neq i_2, \)
\[
\mathcal{D}_\mu = \left\{ (x, \lambda) : x = (x^1, x^2) \in B_\mu(z^{i_1}) \times B_\mu(z^{i_2}), \lambda = (\lambda_1, \lambda_2) \in \left( \frac{1}{\mu}, +\infty \right) \times \left( \frac{1}{\mu}, +\infty \right) \right\}.
\]
The solutions we want to construct will be critical points of \( I \) of the form
\[
u = \sum_{j=1}^{2} K_{i_j}^{2+\nu} U_{x^j, \lambda_j} + v,
\]
where \( x^j \) is close to one of the local maximum points of \( K(y), v \in \dot{H}^s(\mathbb{R}^N) \) is sufficiently small.

Consider the following \( 2(N+1) \)-codimensional submanifold
\[
\mathcal{M}_\mu = \{(x, \lambda, v) : (x, \lambda) \in \mathcal{D}_\mu, v \in E_{x, \lambda}, \|v\| < \mu \},
\]
and the functional \( J : \mathcal{M}_\mu \to \mathbb{R}, \)
\[
J(x, \lambda, v) = I \left( \sum_{j=1}^{2} K_{i_j}^{2+\nu} U_{x^j, \lambda_j} + v \right).
\]
It is well-known that for \( \mu > 0 \) small enough,
\[
u = \sum_{j=1}^{2} K_{i_j}^{2+\nu} U_{x^j, \lambda_j} + v
\]
is a critical point of \( I(u) \) if and only if \( (x, \lambda, v) \in \mathcal{M}_\mu \) is a critical point of \( J(x, \lambda, v) \) in \( \mathcal{M}_\mu \). On the other hand, by the Lagrange multiplier rule, \( (x, \lambda, v) \) is a critical point of \( J(x, \lambda, v) \) constrained to \( \mathcal{M}_\mu \) if and only if there are \( A_j \in \mathbb{R}, B_{ji} \in \mathbb{R} (j = 1, 2, i = 1, 2, \cdots, N) \) such that for \( j = 1, 2, i = 1, 2, \cdots, N, \)
\[
\frac{\partial J(x, \lambda, v)}{\partial v} = \sum_{j=1}^{2} A_j \frac{\partial U_{x^j, \lambda_j}}{\partial \lambda_j} + \sum_{j=1}^{2} \sum_{i=1}^{N} B_{ji} \frac{\partial U_{x^j, \lambda_j}}{\partial x_i^j}, \quad (9)
\]
\[
\frac{\partial J(x, \lambda, v)}{\partial \lambda_j} = A_j \left( \frac{\partial^2 U_{x^j, \lambda_j}}{\partial \lambda_j^2}, v \right) + \sum_{i=1}^{N} B_{ji} \left( \frac{\partial^2 U_{x^j, \lambda_j}}{\partial \lambda_j \partial x_i^j}, v \right), \quad (10)
\]
\[
\frac{\partial J(x, \lambda, v)}{\partial x_i^j} = A_j \left( \frac{\partial^2 U_{x^j, \lambda_j}}{\partial x_i^j \partial \lambda_j}, v \right) + \sum_{i=1}^{N} B_{ji} \left( \frac{\partial^2 U_{x^j, \lambda_j}}{\partial x_i^j \partial x_i^j}, v \right), \quad (11)
\]

The following proposition is to reduce the problem of seeking a solution for (8) to that of finding a critical point for a function defined in a finite dimensional domain.

**Proposition 1.** Assume (K) holds. There exists \( \mu_0 > 0 \), such that for \( \mu \in (0, \mu_0) \) and \((x, \lambda) \in \mathcal{D}_\mu \), there exists a unique \( C^1 \)-map: \((x, \lambda) \in \mathcal{D}_\mu \to v(x, \lambda) \in E_{x, \lambda} \) such that \( v(x, \lambda) \) satisfies (9) for some \( A_j, B_{ji}, (i = 1, 2, \cdots, N, j = 1, 2) \). Moreover, \( v(x, \lambda) \) satisfies the following estimate
\[
\|v\| = O \left( \sum_{j=1}^{2} \left( \|x^j - z^j\| \beta_j + \frac{1}{\lambda_j^2} + \frac{1}{\lambda_j^{2+\nu}} \right) \right) + O \left( \varepsilon_{12}^{\frac{1+\tau}{12}} \right). \quad (12)
\]
where \( \tau > 0 \) is some constant.
Proof. We first expand $J(x, \lambda, v)$ near $v = 0$ as follows

$$J(x, \lambda, v) = J(x, \lambda, 0) + \langle f, v \rangle + \frac{1}{2} \langle Qv, v \rangle + R(v),$$

where $f \in E_{x, \lambda}$ is the linear form given by

$$\langle f, v \rangle = \left( \sum_{j=1}^{2} \frac{2^{N-1}}{x_j} U_{x_j, \lambda_j}, v \right) - \int_{\mathbb{R}^N} K(y) \left( \sum_{j=1}^{2} \frac{2^{N-1}}{x_j} U_{x_j, \lambda_j} \right) v, \quad \text{(13)}$$

and $R(v)$ is the higher order term satifying

$$D^i R(v) = O \left( \|v\|^{2+\theta-i} \right), \quad i = 1, 2.$$ 

where $\theta > 0$ is some constant.

By Proposition 2, we know that $Q$ is invertible and $\|Q^{-1}\| \leq C$ for some $C > 0$ independent of $x$ and $\lambda$. Now, following the arguments in [34], we find

$$\frac{\partial J(x, \lambda, v)}{\partial v} \bigg|_{E_{x, \lambda}} = f + Qv + DR(v).$$

There exists an equivalence between the existence of $v$ such that (9) holds for $(x, \lambda, v)$ and

$$f + Qv + DR(v) = 0, \quad v \in E_{x, \lambda}. \quad \text{(15)}$$

As in [34], by the implicit function theorem, there exists $\mu_0 > 0$ and a $C^1$-map $v : (x, \lambda) \in D_{\mu} \mapsto E_{x, \lambda}$ for $\mu \in (0, \mu_0)$ satisfying (15) and

$$\|v\| \leq C\|f\|. \quad \text{(16)}$$

Thus, we only need to estimate $\|f\|$.

By Hölder inequality and Lemma 2.1, we find

$$\langle f, v \rangle = \left( \sum_{j=1}^{2} K_{x_j}^{2^{N-1}} U_{x_j, \lambda_j}, v \right) - \int_{\mathbb{R}^N} K(y) \left( \sum_{j=1}^{2} K_{x_j}^{2^{N-1}} U_{x_j, \lambda_j} \right) v$$

$$= \sum_{j=1}^{2} \int_{\mathbb{R}^N} K_{x_j}^{2^{N-1}} U_{x_j, \lambda_j} v - \sum_{j=1}^{2} \int_{\mathbb{R}^N} K(y) \left( K_{x_j}^{2^{N-1}} U_{x_j, \lambda_j} \right) v + O \left( \frac{1}{\varepsilon_{12}} \right) \|v\|$$

$$= \sum_{j=1}^{2} \int_{\mathbb{R}^N} (K_{x_j} - K(y)) \left( K_{x_j}^{2^{N-1}} U_{x_j, \lambda_j} \right) v + O \left( \frac{1}{\varepsilon_{12}} \right) \|v\|$$

$$= O \left( \sum_{j=1}^{2} \left( \|x_j - z_j\|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}} + \frac{1}{\lambda_j^{2\beta_j}} \right) \right) \|v\| + O \left( \frac{1}{\varepsilon_{12}} \right) \|v\|.$$ 

Therefore, we complete the proof. 

\[ \square \]

**Proposition 2.** Assume (K) holds. Let $(x, \lambda) \in D_{\mu}$, then for $\mu > 0$ small enough, there exists a $\varrho > 0$ such that

$$\|Qv\| \geq \varrho \|v\|, \quad v \in E_{x, \lambda}.$$
Proof. We argue by contradiction. Suppose that there are \( \mu_n \to 0, (x^n, \lambda_n) \in D_\mu \) and \( v_n \in E_{x^n, \lambda_n} \) such that
\[
\|Q v_n\| = o(1)\|v_n\|,
\] where \( o(1) \to 0 \) as \( n \to \infty \). Without loss of generality, we may assume \( \|v_n\| = 1 \).

Let \( \tilde{v}_{j,n}(y) = \lambda_{j,n}^{\frac{2-s}{N}} v_n(\lambda_{j,n}^{-1} y + x^{j,n}) \), \( j = 1, 2 \). Then, \( \tilde{v}_{j,n}(y) \) is bounded in \( H^s(\mathbb{R}^N) \). Up to a subsequence, we may assume that there is \( v_j \in H^s(\mathbb{R}^N) \) such that
\[
\tilde{v}_{j,n}(y) \rightharpoonup v_j \,	ext{ weakly in } H^s(\mathbb{R}^N).
\]

We will show that \( v_j \equiv 0, \ j = 1, 2 \).

Define
\[
\tilde{U}_{j,n} = \lambda_{j,n}^{\frac{2-s}{N}} U_{x^n, \lambda_n}(\lambda_{j,n}^{-1} y + x^{j,n}),
\]
\[
W_{j,n} = \lambda_{j,n}^{\frac{N-2s}{2}} \frac{\partial U_{x^n, \lambda_n}}{\partial \lambda_n}(\lambda_{j,n}^{-1} y + x^{j,n}),
\]
\[
W_{j,n,i} = \lambda_{j,n}^{\frac{N-2s}{2}} \frac{\partial^2 U_{x^n, \lambda_n}}{\partial x_i^{j,n} \partial \lambda_n}(\lambda_{j,n}^{-1} y + x^{j,n}).
\]

Since \( v_n \in E_{x^n, \lambda_n} \), it is easy to see that
\[
\tilde{v}_{j,n}(y) \in \tilde{E}_n
\]
\[
:= \left\{ \phi \in \dot{H}^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} (-\Delta)^s W_{j,n}(-\Delta)^s \phi = \int_{\mathbb{R}^N} (-\Delta)^s W_{j,n,i}(-\Delta)^s \phi = 0 \right\}
\]
and
\[
\int_{\mathbb{R}^N} (-\Delta)^s \tilde{v}_{j,n}(-\Delta)^s \phi - \frac{N + 2s}{N - 2s} \int_{\mathbb{R}^N} \tilde{U}_{j,n} \tilde{v}_{j,n} \phi = o(1)\|\phi\|, \quad \phi \in \tilde{E}_n.
\] (18)

Now we claim that \( v_j \) satisfies
\[
(-\Delta)^s v_j - \frac{N + 2s}{N - 2s} U_{0,1}^{\frac{4s}{N-2s}} v_j = 0.
\] (19)

For any \( \phi \in C_0^\infty(\mathbb{R}^N) \), define \( \phi_n \in \tilde{E}_n \) as follows
\[
\phi_n = \phi - \sum_{j=1}^{2} \sum_{\ell=1}^{N} c_{j,i,n} W_{j,n,i} - \sum_{j=1}^{2} c_{j,n} W_{j,n}.
\]

Since \( \lim_{n \to \infty} \text{dist}(\text{supp}(W_{\ell,i,n}), \text{supp}(W_{\ell,n})) \to \infty, \ell \neq j \).

We find
\[
\lim_{n \to \infty} c_{\ell,i,n} = 0 \quad \text{and} \quad \lim_{n \to \infty} c_{\ell,n} = 0, \quad \ell \neq j
\]
and
\[
c_{j,i,n} \text{ and } c_{j,n} \text{ are bounded.}
\]

Inserting \( \phi_n \) into (18) and letting \( n \to \infty \), we get
\[
\int_{\mathbb{R}^N} (-\Delta)^s v_j(-\Delta)^s \phi - \frac{N + 2s}{N - 2s} \int_{\mathbb{R}^N} U_{0,1}^{\frac{4s}{N-2s}} v_j \phi
\]
\[
- c \int_{\mathbb{R}^N} \left[ (-\Delta)^s v_j(-\Delta)^s \left( \frac{\partial U_{0,1}}{\partial \lambda} \right)_{\lambda=1} \right] - \frac{N + 2s}{N - 2s} U_{0,1}^{\frac{4s}{N-2s}} \left( \frac{\partial U_{0,1}}{\partial \lambda} \right)_{\lambda=1}
\]
\[
- \sum_{j=1}^{2} c_{j} \int_{\mathbb{R}^N} \left[ (-\Delta)^s v_j(-\Delta)^s \left( \frac{\partial U_{0,1}}{\partial x_i} \right)_{x=0} \right] - \frac{N + 2s}{N - 2s} U_{0,1}^{\frac{4s}{N-2s}} \left( \frac{\partial U_{0,1}}{\partial x_i} \right)_{\lambda=1} = 0,
\] (20)
where \( c = \lim_{n \to \infty} c_{j,n} \) and \( c_i = \lim_{n \to \infty} c_{j,i,n} \).

Note that
\[
\int_{\mathbb{R}^N} (-\Delta)^{1/2} v_j (-\Delta)^{1/2} \left( \frac{\partial U_{0,1}}{\partial \lambda} \right)_{\lambda=1} - \frac{N+2s}{N-2s} U_{0,1}^{\frac{4s}{N-2s}} \left( \frac{\partial U_{0,1}}{\partial x_i} \right)_{x=0} = 0,
\]
\[
\int_{\mathbb{R}^N} (-\Delta)^{1/2} v_j (-\Delta)^{1/2} \left( \frac{\partial U_{0,1}}{\partial \lambda} \right)_{\lambda=1} = 0.
\]
Thus, we get
\[
\int_{\mathbb{R}^N} (-\Delta)^{1/2} v_j (-\Delta)^{1/2} \phi - \frac{N+2s}{N-2s} \int_{\mathbb{R}^N} U_{0,1}^{\frac{4s}{N-2s}} v_j \phi = 0.
\]
Since \( \tilde{v}_{j,n} \in \hat{E}_n \), we have
\[
\left( v_j, \frac{\partial U_{0,1}}{\partial \lambda} \right)_{\lambda=1} = \left( v_j, \frac{\partial U_{0,1}}{\partial x_i} \right)_{x=0} = 0, \quad i = 1, 2, \ldots, N.
\]
By the fact in [15] that \( U_{0,1} \) is nondegenerate, we find \( v_j = 0 \).

Therefore,
\[
\int_{\mathbb{R}^N} K(y) \left( \sum_{j=1}^{2} U_{x_j, \lambda_j, n} \right)^{\frac{N+4s}{N-2s}} v_n^2 \leq C \sum_{j=1}^{2} \int_{\mathbb{R}^N} U_{x_j, \lambda_j, n}^{\frac{4s}{N-2s}} v_n^2
\]
\[
\leq C \int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{2s}} \tilde{v}_{j,n}^2(z) \leq o_R(1) + o(1),
\]
where \( o_R(1) \to 0 \) as \( R \to \infty \).

It follows from (17), we obtain that \( \|v_n\| \to 0 \) as \( n \to \infty \). But this contradicts with \( \|v_n\| = 1 \). So we have completed the proof of the proposition.

4. **Proof of the main results.** In this section, we use a minimization procedure to prove Theorem 1.1.

**Proof of Theorem 1.1.** For notational brevity, we assume \( z^1 = z^{i_1}, \ K_j = K_{i_j}, \ j = 1, 2 \) and \( z^2 \) is another local maximum point of \( K(y) \) with \( d := |z^1 - z^2| \) large enough.

Define
\[
L_1 = d^{\beta_1 / (N-2) + \beta_2 / (N-2)}, \quad L_2 = d^{\beta_1 / (N-2) + \beta_2 / (N-2)},
\]
where \( \beta_1, \beta_2 \) are constants in (K).

Let \( v(x, \lambda) \) be the map obtained in Proposition 1. Now we consider the following minimization problem
\[
\inf \{ J(x, \lambda, v(x, \lambda)) : (x, \lambda) \in \mathcal{D}_{\mu, 2} \}, \quad (21)
\]
where
\[
\mathcal{D}_{\mu, 2} := \{(x, \lambda) : (x, \lambda) \in \mathcal{D}_{\mu, \lambda_j} \in [\gamma_1 L_1, \gamma_2 L_j], j = 1, 2\}, \quad \gamma_1, \gamma_2 > 0 \text{ are to be determined later.}
\]

It is clear that problem (21) has a minimizer \( (\bar{x}, \bar{\lambda}) \in \mathcal{D}_{\mu, 2} \). We will show that there is a \( \mu_0 > 0 \) such that for \( \mu \in (0, \mu_0) \), we can find a \( \mu_0 > 0 \) such that, if \( d := |z^1 - z^2| > \mu_0 \), the minimizer \( (\bar{x}, \bar{\lambda}) \) of (21) satisfies
\[
\bar{\lambda}_j \in (\gamma_1 L_j, \gamma_2 L_j), \ |\bar{x}^j - z^j| \leq \frac{\mu}{2}, \quad j = 1, 2. \quad (22)
\]
By Proposition 1 and Lemma 2.2, we can find that for \((x, \lambda) \in D_{\mu, 2},\)

\[
J(x, \lambda, v(x, \lambda)) = J(x, \lambda, 0) + O \left( \|f\| \|v\| + \|v\|^2 \right)
\]

\[
= J(x, \lambda, 0) + O \left( \sum_{j=1}^{2} \left( |x^j - z^j|^{2\beta_j} + \frac{1}{\lambda_j^{2\beta_j}} \right) + \varepsilon_{12}^{1+2\tau} \right)
\]

\[
= \sum_{j=1}^{2} I \left( K_j^{2s-N} U_{x^j, \lambda_j} \right) - (K_1 K_2)^{2s-N} D \varepsilon_{12}
\]

\[
+ O \left( \sum_{j=1}^{2} \frac{\varepsilon_{12}}{\lambda_j^{2\beta_j}} \right) + O \left( \sum_{j=1}^{2} \left( |x^j - z^j|^{2\beta_j} + \frac{1}{\lambda_j^{2\beta_j}} \right) \right)
\]

\[
+ O \left( \sum_{j=1}^{2} \left( |x^j - z^j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}} \right) \varepsilon_{12} + \varepsilon_{12}^{1+\tau} \right),
\]

where \(D = C_0^{2s-N} C_1.\)

On the other hand,

\[
I \left( K_j^{2s-N} U_{x^j, \lambda_j} \right)
\]

\[
= \frac{1}{2} \int_{R^N} \left| (-\Delta)^{\frac{\varepsilon}{2}} K_j^{2s-N} U_{x^j, \lambda_j} \right|^2 - \frac{1}{2s} \int_{R^N} K(y) \left( K_j^{2s-N} U_{x^j, \lambda_j} \right)^{2s}
\]

\[
= \frac{s}{N} K_j^{2s-N} \int_{R^N} U_{0,1}^{2s} - \frac{1}{2s} \int_{R^N} (K(y) - K_j) \left( K_j^{2s-N} U_{x^j, \lambda_j} \right)^{2s}
\]

\[
= \frac{s}{N} K_j^{2s-N} \int_{R^N} U_{0,1}^{2s} + \frac{1}{2s} K_j^{2s-N} \int_{R^N} Q(y-z^j) U_{x^j, \lambda_j}^{2s} + O \left( |x^j - z^j|^{\beta_j + \sigma} + \frac{1}{\lambda_j^{\beta_j + \sigma}} \right)
\]

\[
= \frac{s}{N} K_j^{2s-N} \int_{R^N} U_{0,1}^{2s} + \frac{1}{2s} K_j^{2s-N} \int_{R^N} Q \left( \frac{y}{\lambda_j} + x^j - z^j \right) U_{0,1}^{2s}
\]

\[
+ O \left( |x^j - z^j|^{\beta_j + \sigma} + \frac{1}{\lambda_j^{\beta_j + \sigma}} \right).
\]

Thus,

\[
I \left( K_j^{2s-N} U_{x^j, \lambda_j} \right)
\]

\[
= \frac{s}{N} K_j^{2s-N} \int_{R^N} U_{0,1}^{2s} + \frac{1}{2s} K_j^{2s-N} \int_{R^N} Q \left( \frac{y}{\lambda_j} \right) U_{0,1}^{2s} + O \left( \frac{1}{\lambda_j^{\beta_j + \sigma}} \right)
\]

\[
\leq \frac{s}{N} K_j^{2s-N} \int_{R^N} U_{0,1}^{2s} + \frac{\kappa}{\lambda_j^{\beta_j}} + O \left( \frac{1}{\lambda_j^{\beta_j + \sigma}} \right),
\]

where \(\kappa\) is a positive constant.

(1). We claim that \(|x^j - z^j| \leq \frac{C}{\lambda_j}|y|\).

Since

\[
J(x, \lambda, v(x, \lambda)) \leq J(z, \lambda, v(z, \lambda)),
\]
we find

\[
\sum_{j=1}^{2} \frac{s}{N} K_j 2^j \int_{\mathbb{R}^N} U_{0,1}^{2^j} - \frac{1}{2^j} \int_{\mathbb{R}^N} Q_j \left( \frac{|y|}{\lambda_j} + \bar{x}^j - z^j \right) U_{0,1}^{2^j} - (K_1 K_2) 2^j D \varepsilon_{12}
\]

\[
\leq \sum_{j=1}^{2} \frac{s}{N} K_j 2^j \int_{\mathbb{R}^N} U_{0,1}^{2^j} - \frac{2}{\lambda_j} + O \left( \sum_{j=1}^{2} \varepsilon_{12}^j \right)
\]

\[
+ O \left( \sum_{j=1}^{2} \left( |\bar{x}^j - z^j|^{\beta_j + \sigma} + \frac{1}{\lambda_j^{\beta_j + \sigma}} \right) \right) + O \left( \sum_{j=1}^{2} \left( |\bar{x}^j - z^j|^{\beta_j} + \frac{1}{\lambda_j^{\beta_j}} \right) \varepsilon_{12}^j \right) .
\]

Note that

\[
Q_j \left( \frac{|y|}{\lambda_j} + \bar{x}^j - z^j \right) \geq a_0 \left| \frac{|y|}{\lambda_j} + \bar{x}^j - z^j \right|^{\beta_j} \geq a_0 |\bar{x}^j - z^j|^{\beta_j} - C \frac{|y|^{\beta_j}}{\lambda_j^{\beta_j}} .
\]

Hence

\[
\sum_{j=1}^{2} |\bar{x}^j - z^j|^{\beta_j} \leq O \left( \sum_{j=1}^{2} \left( |\bar{x}^j - z^j|^{\beta_j + \sigma} + \frac{1}{\lambda_j^{\beta_j + \sigma}} \right) \right) + \varepsilon_{12}^j .
\]

Therefore our claim follows.

(2). We claim that \( \lambda_j \in (\gamma_1 L_j, \gamma_2 L_j) \), \( j = 1, 2 \).

Denote \( \lambda_j = t_j L_j \), \( j = 1, 2 \). Since \( \beta_j > N - 2s \), we can select \( (t_01, t_02) \in \mathbb{R}^2 \) with \( t_{0j} > 0 \) large enough such that

\[
\sum_{j=1}^{2} \frac{C'}{t_{0j}} - \frac{N-2s}{t_{01}} \frac{N-2s}{t_{01}} (K_1 K_2)^{N-2s} < -\delta_0 < 0 .
\]

Let \( \lambda_0 = (\lambda_{01}, \lambda_{02}) \), where \( \lambda_{0j} = t_{0j} L_j \), \( j = 1, 2 \), then we have

\[
J(z, \lambda_0, v(z, \lambda_0)) \leq \sum_{j=1}^{2} \frac{s}{N} K_j 2^j \int_{\mathbb{R}^N} U_{0,1}^{2^j} - c'_0 d^{-\frac{(N-2s)\beta_1}{(N-2s)\beta_1 + (N-2s)\beta_2}}
\]

for some constant \( c'_0 > 0 \).

Since \( |\lambda_j (\bar{x}^j - z^j)| \leq C \), we have that there exists a constant \( c''_0 > 0 \), such that

\[
\int_{\mathbb{R}^N} |\lambda_j (\bar{x}^j - z^j)|^{\beta_j} U_{0,1}^{2^j} \geq c''_0 .
\]
As a result, we get

\[
J(\bar{x}, \bar{\lambda}, v(\bar{x}, \bar{\lambda})) \geq \sum_{j=1}^{2} \frac{S}{N} K_j^{\frac{2s-N}{4}} \int_{\mathbb{R}^N} U_{0,1}^{2s} + \sum_{j=1}^{2} \left( \frac{1}{\lambda_j^{\frac{\beta_j}{2}}} + \varepsilon_{12}^{\frac{1}{2}} \right) + O \left( \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{\beta_j}{2}}} \varepsilon_{12} \right) 
\]

\[
\geq \sum_{j=1}^{2} \frac{S}{N} K_j^{\frac{2s-N}{4}} \int_{\mathbb{R}^N} U_{0,1}^{2s} + \sum_{j=1}^{2} \frac{\gamma_{12}^{\frac{1}{2}}}{\lambda_j^{\frac{\beta_j}{2}}} - (K_1 K_2)^{\frac{2s-N}{4}} D \varepsilon_{12} + O \left( \sum_{j=1}^{2} \frac{1}{\lambda_j^{\frac{\beta_j}{2}}} \varepsilon_{12} \right). 
\]

By the fact that \( J(\bar{x}, \bar{\lambda}, v(\bar{x}, \bar{\lambda})) \leq J(z, \lambda_0, v(z, \lambda_0)) \), we can obtain

\[
\varepsilon_{12} = \left( \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} \right)^{\frac{N-2s}{2}} = \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}} 
\]

\[
\leq \frac{2}{\gamma_1 \gamma_2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}}. 
\]

Hence

\[
\varepsilon_{12} = \left( \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} \right)^{\frac{N-2s}{2}} = \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}} 
\]

\[
\leq \frac{2}{\gamma_1 \gamma_2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}}. 
\]

Since \( \beta_1 > N-2s \), we can choose \( \gamma_1 > 0 \) small enough, such that

\[
\frac{\gamma_0}{\gamma_1} - \frac{2(K_1 K_2)^{\frac{2s-N}{4}} D}{\gamma_1^{N-2s}} \geq \delta' > 0. 
\]

Thus, we have

\[
\varepsilon_{12} = \left( \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} \right)^{\frac{N-2s}{2}} = \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}} 
\]

\[
\leq \frac{2}{\gamma_1 \gamma_2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}}. 
\]

This is contradiction to (25). Therefore, \( t_1 > \gamma_1 \).

Suppose that \( \bar{\lambda}_1 = \gamma_2 L_1 = \gamma_2 d^{\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}} \), then

\[
\varepsilon_{12} = \left( \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} \right)^{\frac{N-2s}{2}} = \frac{1 + o(1)}{\beta_1 \lambda_2 |\bar{x}| - \bar{x}^2 |^2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}} 
\]

\[
\leq \frac{2}{\gamma_1 \gamma_2} d^{-\frac{(N-2s)\beta_1 \beta_2}{(\beta_1 + \beta_2)(N-2s)/2}}. 
\]
Thus, we find
\[
\epsilon_0^2 \sum_{j=1}^{2} \frac{1}{\lambda_j^2} - (K_1 K_2)^{2s-N} D \varepsilon_{12} \geq -(K_1 K_2)^{2s-N} D \varepsilon_{12}
\]
\[
\geq -2(K_1 K_2)^{2s-N} D \frac{\gamma_2^{2s-N}}{\gamma_1^{s-N}} \sum_{j=1}^{2} \frac{(N-2s)^2 \beta_j}{(N-2(s-2)) \gamma_j^2}.
\]

Since
\[
\lim_{\gamma_2 \to \infty} \frac{2(K_1 K_2)^{2s-N} D \gamma_2^{2s-N}}{\gamma_1^{s-N}} = 0.
\]

So if we take \(\gamma_2 > 0\) large enough depending on \(\lambda_2\), we can also obtain a contradiction. Since the similar argument can also be applied to \(\lambda_2\), the claim follows.

As a result, the minimizer \((\bar{x}, \bar{\lambda})\) of \(I\) is an interior point of \(D_{\mu, 2}\). Thus, \(u = \sum_{j=1}^{2} K_{ij}^{2s-N} U_{\bar{x}_j, \bar{\lambda}_j} + v(\bar{x}, \bar{\lambda})\) is a critical point of \(I\). By Lemma A.6 in [14], we see that \(u\) is a solution. Therefore, we finish our proof.

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