A GENERALIZATION OF THE LODHA–MOORE GROUP

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Abstract. We generalized the Lodha–Moore group into \( n \)-adic and showed analogues in the Lodha–Moore group of properties between the Thompson group \( F \) and the generalized Thompson group \( F(n) \).

1. Introduction

In [15], Lodha and Moore introduced the group \( G_0 \) consisting of piecewise projective homeomorphisms of the real projective line. This group is a finitely presented torsion free counterexample to the von Neumann-Day problem [9,20], which asks whether every nonamenable group contains nonabelian free subgroups. Although counterexamples of the von Neumann-Day problem are known [1,2,13,17–19], it is still an open question whether the Thompson group \( F \) can be a new counterexample.

The Lodha–Moore group \( G_0 \) has similar properties to the Thompson group \( F \). Indeed, the Lodha–Moore group can also be defined as a finitely generated group consisting of homeomorphisms of the space of infinite binary sequences, whose generating set is obtained by adding an element to the well-known finite generating set of \( F \). Both have (small) finite presentations [4,15], normal forms with infinite presentations [5,14], simple commutator subgroups [7,8], trivial homotopy groups at infinity [5,21], no nonabelian free subgroups [3,15], and are of type \( F_\infty \) [5,14].

On the other hand, there exist various generalizations of the Thompson group \( F \). One of the most natural ones is the \( n \)-adic Thompson group \( F(n) \), which is obtained by replacing infinite binary sequences with infinite \( n \)-ary sequences. Even now, it is still being actively studied whether what is true for the group \( F \) is also true for the generalized group and what interesting properties can be obtained under the generalization.

In this paper, we generalize the Lodha–Moore group similarly and study its properties. Namely, we define the \( n \)-adic group \( G_0(n) \) of the Lodha–Moore group \( G_0 \) and show that several properties which hold for \( G_0 \) also hold for \( G_0(n) \). We remark that \( G_0(2) \) is isomorphic to \( G_0 \).

Let \( n, m \geq 2 \). We show the following:

Theorem 1.1. (1) The group \( G_0(n) \) admits an infinite presentation with a normal form of elements.

(2) The group \( G_0(n) \) is finitely presented.

Key words and phrases. amenable, finitely presented, Thompson group, torsion free.
(3) The group $G_0(n)$ is nonamenable.
(4) The group $G_0(n)$ has no free subgroups.
(5) The group $G_0(n)$ is torsion free.
(6) The groups $G_0(n)$ and $G_0(m)$ are isomorphic if and only if $n = m$ holds.
(7) The commutator subgroup of the group $G_0(n)$ is simple.
(8) The center of the group $G_0(n)$ is trivial.
(9) There does not exist any nontrivial direct product decomposition of the group $G_0(n)$.
(10) There does not exist any nontrivial free product decomposition of the group $G_0(n)$.

This paper is organized as follows. In Section 2, we generalize Dehornoy’s infinite presentation of $F$ to $F(n)$, which will be used to construct that of $G_0(n)$. To the best of the author’s knowledge, this is a new presentation of $F(n)$. In Section 3, we first recall the definition of the group $G_0$ and define the group $G_0(n)$. Then by using the infinite presentation of $F(n)$ constructed in Section 2, we generalize Lodha’s method to obtain that of $G_0(n)$ and its normal form. Finally, in Section 4, we study several properties of $G_0(n)$.

Let us mention some open problems which are known to hold in the case of $G_0$. First, it is an interesting question whether $G_0(n)$ can be realized as a subgroup of the group of piecewise projective homeomorphisms of the real projective line. The second problem is whether this group is of type $F_\infty$, and all homotopy groups are trivial at infinity. If it has these two properties, then $G_0(n)$ is an example of an (infinite) family of groups satisfying all Geoghegan’s conjectures for the Thompson group $F$.

Furthermore, we can consider some groups related to $G_0(n)$. The first one is constructed by using another definition of the map $y$ defined in Section 3.2. Although we define the map so that $G_0$ is naturally a subgroup of $G_0(n)$, we can consider several different generalizations.

We can also construct groups that contain $G_0(n)$. In [15], the group $G$ is defined, where $G$ contains $G_0$ as a subgroup. For our group, by adding some of the generators $y_0, y_{(n-1)1}$, $\ldots, y_{(n-1)(n-2)}, y_{(n-1)}$, we can define not only the group $G(n)$, which corresponds to $G$ but also the groups “between” $G_0(n)$ and $G(n)$.

2. The generalized Thompson group $F(n)$

2.1. Definition. Let $n \geq 2$. There exist several ways to define the generalized Thompson group $F(n)$. In this paper, we define it as a group of homeomorphisms on the $n$-adic Cantor set. We use tree diagrams to represent elements of the group visually.

We define $N$ to be the set $\{0, 1, \ldots, n-1\}$. We endow $N$ with the discrete topology and endow $N^\mathbb{N}$ with the product topology. Note that $N^\mathbb{N}$ and the Cantor set are homeomorphic. We also consider the set of all finite sequences on $N$ and write $N^{<\mathbb{N}}$ for it. For
$s \in N^{<N}$ and $t \in N^{<N}$ (or $N^N$), the concatenation is denoted by $st$. The group $F(n)$ is a finitely generated group that is generated by the following $n$ homeomorphisms on $N^N$:

$$x_0(\zeta) = \begin{cases} 
0\eta & (\zeta = 00\eta) \\
1\eta & (\zeta = 01\eta) \\
(n-2)\eta & (\zeta = 0(n-2)\eta) \\
(n-1)0\eta & (\zeta = 0(n-1)\eta) \\
(n-1)1\eta & (\zeta = 1\eta) \\
& \vdots \\
(n-1)(n-1)\eta & (\zeta = (n-1)\eta),
\end{cases}$$

$$x_1(\zeta) = \begin{cases} 
0\eta & (\zeta = 0\eta) \\
1\eta & (\zeta = 10\eta) \\
2\eta & (\zeta = 11\eta) \\
(n-2)\eta & (\zeta = 1(n-3)\eta) \\
(n-1)0\eta & (\zeta = 1(n-2)\eta) \\
(n-1)1\eta & (\zeta = 1(n-1)\eta) \\
(n-1)2\eta & (\zeta = 2\eta) \\
& \vdots \\
(n-1)(n-1)\eta & (\zeta = (n-1)\eta),
\end{cases}$$

$$x_{n-2}(\zeta) = \begin{cases} 
0\eta & (\zeta = 0\eta) \\
& \vdots \\
(n-3)\eta & (\zeta = (n-3)\eta) \\
(n-2)\eta & (\zeta = (n-2)0\eta) \\
(n-1)0\eta & (\zeta = (n-2)1\eta) \\
& \vdots \\
(n-1)(n-2)\eta & (\zeta = (n-2)(n-1)\eta) \\
(n-1)(n-1)\eta & (\zeta = (n-1)\eta),
\end{cases}$$
These maps are represented by tree diagrams as in Figure 1. Here, we briefly review the definition of tree diagrams. See [6] for details. An \( n \)-ary tree is a finite tree with a top vertex (root) with \( n \) edges, and all vertices except the root have degree only 1 (leaves) or \( n + 1 \). We define a caret to be an \( n \)-ary tree with no vertices whose degree is \( n + 1 \) (see Figure 2). Then, each \( n \)-ary tree is obtained by attaching carets to a leaf of a caret. We always assume that the root is the top and the others are descendants.

Each \( n \)-ary tree can be regarded as a finite subset of \( \mathbb{N}^\mathbb{N} \). To do this, we label each edge of each caret by \( 0, 1, \ldots, n - 1 \) from the left. Since every leaf corresponds to a unique path from the root to the leaf, we can regard it as an element in \( \mathbb{N}^\mathbb{N} \).

Let \( T_+ \) and \( T_- \) be \( n \)-ary trees with \( m \) leaves. Let \( a_1, \ldots, a_m \) be elements in \( \mathbb{N}^\mathbb{N} \) with lexicographic order corresponding to the leaves of \( T_+ \). For \( T_- \), define \( b_1, \ldots, b_m \) in the same way. Then, for every \( \zeta \in \mathbb{N}^\mathbb{N} \), there exists \( i \) uniquely such that \( \zeta = a_i \eta \) for some \( \eta \in \mathbb{N}^\mathbb{N} \). Thus we obtain a homeomorphism \( a_i \eta \mapsto b_i \eta \). It is known that every homeomorphism obtained from two \( n \)-ary trees with the same number of leaves in this way is generated by the composition of \( x_0, x_1, \ldots, x_{n-2}, x_{0[(n-1)]} \). See [16, Corollary 10.9] for the case \( n = 2 \).

We define \( \epsilon \) to be the empty word. Let \( i \in \{0, \ldots, n - 2\} \) and \( \alpha \in \mathbb{N}^\mathbb{N} \cup \{\epsilon\} \). We define the map \( x_{i[\alpha]} : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N} \) by

\[
x_{i[\alpha]}(\zeta) = \begin{cases} 
\alpha x_i(\eta) & (\zeta = \alpha \eta) \\
\zeta & (\zeta \neq \alpha \eta),
\end{cases}
\]

and we define

\[
X(n) := \{x_{i[\alpha]} \mid i = 0, \ldots, n - 2, \alpha \in \mathbb{N}^\mathbb{N} \cup \{\epsilon\}\}.
\]
This set contains the well-known infinite generating set of $F(n)$, which we describe below, and is the key generating set in Section 2.2.2.

Let $X'(n) := \{ x_0[s], \ldots, x_{n-2}[s] \mid s = \epsilon, n-1, (n-1)(n-1), \ldots \}$. We denote each element as $x_0 = X_0, \ldots, x_{n-2} = X_{n-2} x_0[(n-1)] = X_{n-1}, x_1[(n-1)] = X_n, \ldots$ only in this section for the sake of simplicity. For $i < j$, we have $X_i^{-1} X_j X_i = X_{j+n-1}$. This implies that every element in $F(n)$ has the following form:

$$X_{i_1}^{r_1} X_{i_2}^{r_2} \cdots X_{i_m}^{r_m} X_{j_k}^{-s_k} \cdots X_{j_2}^{-s_2} X_{j_1}^{-s_1}$$

where $i_1 < i_2 < \cdots < i_m \neq j_k > \cdots > j_2 > j_1$ and $r_1, \ldots, r_m, s_1, \ldots, s_k > 0$. We require that this form satisfies the following additional condition: if there exist $X_i$ and $X_i^{-1}$, then there also exists one of

$$X_{i+1}, X_{i+1}, X_{i+2}, X_{i+2}, \ldots, X_{i+n-1}, X_{i+n-1}^{-1}.$$

It is known that this form with the additional condition always uniquely exists. Thus we call this form normal form of elements of $F(n)$. The proof for the case $n = 2$ is in [5] Section 1.

2.2. A presentation of the generalized Thompson group. While the group $F(n)$ has the well-known infinite and finite presentations, we construct another presentation with respect to $X(n)$ for Section 3.3.

We list the relations of the elements in $F(n)$ as follows. Here, $A_i$s are defined at the beginning of Section 2.2.2 and the shift-map is defined in Definition 2.8.

$$
\begin{align*}
A_0 A_1 &= A_1[(n-1)] A_0, \\
A_1 A_2 &= A_2[(n-1)] A_1, \\
\vdots &
\end{align*}

\begin{align*}
A_0 A_2 &= A_2[(n-1)] A_0, \\
A_1 A_3 &= A_3[(n-1)] A_1, \\
\vdots &
\end{align*}

\begin{align*}
A_0 A_3 &= A_3[(n-1)] A_0, \\
A_1 A_4 &= A_4[(n-1)] A_1, \\
\vdots &
\end{align*}

\begin{align*}
A_0 A_n &= A_n[(n-1)] A_0, \\
A_0 A_0[(n-1)] &= A_0[(n-1)^2] A_0, \\
A_0 A_1[(n-1)] &= A_1[(n-1)^2] A_0, \\
\vdots &
\end{align*}

Table 1. Moving to a right column corresponds to the sift-map.

$$
\begin{align*}
A_0 A_0 &= A_0[(n-1)] A_0 A_0[0], \\
A_1 A_0 &= A_0[(n-1)] A_0 A_1[0], \\
\vdots &
\end{align*}

\begin{align*}
A_1 A_1 &= A_1[(n-1)] A_1 A_0[0], \\
A_2 A_0 &= A_0[(n-1)] A_0 A_2[0], \\
A_2 A_1 &= A_1[(n-1)] A_1 A_0[1], \\
\vdots &
\end{align*}

Table 2. Moving to a lower row corresponds to the sift-map.
Table 3. For each $\alpha$ in $\mathbb{N}^{<\mathbb{N}} \cup \{\epsilon\}$ and $k$ in $\{0, \ldots, n-2\}$, moving to a right column corresponds to the sift-map.

\[
\begin{align*}
A_{k[0\alpha]A_0} &= A_0A_{k[0\alpha]}, & A_{k[1\alpha]A_1} &= A_1A_{k[1\alpha]}, & \cdots \\
A_{k[0\alpha]A_2} &= A_2A_{k[0\alpha]}, & A_{k[1\alpha]A_3} &= A_3A_{k[1\alpha]}, & \cdots \\
A_{k[0\alpha]A_3} &= A_3A_{k[0\alpha]}, & \vdots & & \ddots \\
A_{k[(n-2)\alpha]A_0} &= A_0A_{k[0(n-2)\alpha]}, & A_{k[(n-1)\alpha]A_0} &= A_0A_{k[0(n-1)\alpha]}.
\end{align*}
\]

Table 4. For each $\alpha$ in $\mathbb{N}^{<\mathbb{N}} \cup \{\epsilon\}$ and $k$ in $\{0, \ldots, n-2\}$, moving to a right column corresponds to the sift-map.

2.2.1. Dehornoy’s results. For the “geometric” presentation of $F(n)$, we generalize Dehornoy’s method in [10]. Section 1 of [10] gives the way to find a presentation of a group. So we recall the general setting and the way.

**Definition 2.1.** [10, Section 1.1] Let $G$ be a group (monoid) and $T$ be a set. We define a partial (right) action to be a map $\phi$ from $G$ to the set of injections $\{f : T' \to T \mid T' \subset T\}$ such that the following are satisfied (in the following, we write $t \cdot g$ for the image of $t$ under $\phi(g)$ if it is defined):

(PA$_1$) For every $t \in T$, $t \cdot e = t$ holds;

(PA$_2$) For every $g, h \in G$, and $t \in T$, if $t \cdot g$ is defined, then $(t \cdot g) \cdot h$ is defined if and only if $t \cdot gh$ is defined, and if one of them is defined, we have $(t \cdot g) \cdot h = t \cdot gh$;

(PA$_3$) For every finite family $g_1, \ldots, g_m$, there exists $t \in T$ such that $t \cdot g_1, \ldots, t \cdot g_m$ are defined.

Dehornoy also introduced a stronger condition on partial actions.

**Definition 2.2.** [10, Section 1.3] We assume that $G$ has a partial action on $T$. Then we call a subset $S \subset T$ is discriminating if:

1. In (PA$_3$), we can take $t$ in the set $S \cdot G = \{s \cdot g \mid s \in S, g \in G \text{ such that } s \cdot g \text{ is defined}\}$;
2. Each $G$-orbit contains at most one element of $S$;
3. For every $s \in S$, its stabilizer is trivial.

**Remark 2.3.** In the above setting, for every $t \in S \cdot G$, there exist the unique $s \in S$ and $g \in G$ such that $t = s \cdot g$. 

\[
\begin{align*}
A_{k[0\alpha]A_1} &= A_1A_{k[0\alpha]}, & A_{k[1\alpha]A_2} &= A_2A_{k[1\alpha]}, & \cdots \\
A_{k[0\alpha]A_2} &= A_2A_{k[0\alpha]}, & A_{k[1\alpha]A_3} &= A_3A_{k[1\alpha]}, & \cdots \\
A_{k[0\alpha]A_3} &= A_3A_{k[0\alpha]}, & \vdots & & \ddots \\
A_{k[(n-2)\alpha]A_0} &= A_0A_{k[0(n-2)\alpha]}, & A_{k[(n-1)\alpha]A_0} &= A_0A_{k[0(n-1)\alpha]}.
\end{align*}
\]
When $R$ is a family of relations of a group, we write $w \equiv_R z$ if one can rewrite $w$ to $z$ by using elements in $R$. The following theorem holds.

**Theorem 2.4 ([10] Proposition 1.4).** Let $G$ be a group with a partial action on a set $T$. Let $X$ be a subset of $G$ and $R$ be a collection of relations satisfied in $G$ by the elements of $X$. Assume that $S$ is a discriminating subset of $T$ and that, for each $s$ in $S$ and $t$ in the $G$-orbit of $s$, a word $w_t$ on $X$ is chosen so that $t = s \cdot w_t$ holds. Then $\langle X \mid R \rangle$ is a presentation of $G$ if and only if for all $t, t'$ in $S \cdot G$ and $x$ in $X$,

$$t' = t \cdot x \implies w_{t'} \equiv_R w_t \times x,$$

where $\times$ denotes the concatenation of words in $X$.

2.2.2. A presentation of $F(n)$. In this section, we construct a partial action of $F(n)$ and give a presentation by using Theorem 2.4. Except for the construction of $w_t$, the discussions are almost the same as the case $n = 2$.

Let $T(n)$ be a set consisting of $n$-ary trees and the root (the graph with a single vertex). We first define partial actions of $n - 1$ elements $A_0, \ldots, A_{n-2}$ on $T(n)$ as illustrated in Figure 3. We note that the actions of $A_i$ ($i = 0, \ldots, n - 2$) on a single caret are not defined.

Let $\alpha$ in $N^{<N}$. Then, for $n$-ary trees which contain $\alpha$ as a subpath, we define a partial action of $A_{i[\alpha]}$ to be the partial mapping obtained by applying $A_i$ to the element of $T(n)$ positioned just below $\alpha$ (if it is defined). See Figure 4 and observe that the bottom caret is the only one moved by the action.

**Definition 2.5.** We define $G_n(\mathcal{A})$ to be the monoid generated by $X(n) := \{A_{i[\alpha]} \mid i \in \{0, \ldots, n - 2\}, \alpha \in N^{<N} \cup \{\epsilon\}\}$ and their inverse, where $\epsilon$ is the empty word and $A_{i[\epsilon]} := A_i$.

From the construction, the monoid $G_n(\mathcal{A})$ naturally has a partial action on $T(n)$. In order to make $G_n(\mathcal{A})$ into a group, we introduce a congruence on $G_n(\mathcal{A})$.

**Definition 2.6.** We assume that a group $G$ has a partial action on a set $T$. If there exists $t \in T$ such that $t \cdot g$ and $t \cdot g'$ are defined, and they coincide for all such $t$, then we define $g$ and $g'$ to be near-equal and write it as $g \approx g'$.

As in [10] Corollary 2.4, we can show that $G_n(\mathcal{A})$ and the set of all-right trees $S(n)$ satisfy the assumptions in [10] Lemma 2.2 (see Figure 7 as an example of an all-right tree). Thus, near-equality is a congruence on $G_n(\mathcal{A})$, and the quotient monoid $G_n(\mathcal{A})/\approx$ is a group. We write $G_n(\mathcal{A})$ for this group. Moreover, $S(n)$ is discriminating for the induced partial action on $G_n(\mathcal{A})$. We omit the proofs because they are the same for $n = 2$, but we recall the induced action for the reader’s convenience.

**Definition 2.7.** For $t, t'$ in $T(n)$ and $x$ in $G_n(\mathcal{A})$, we define $t \cdot x = t'$ if $t \cdot g = t'$ holds for some $g$ in $G_n(\mathcal{A})$ such that $g$ is a representative of $x$ (if there exists such $g$).
Figure 3. Partial actions of $A_1, \ldots, A_{n-2}$.

\[
\begin{array}{c}
\text{\textbullet} \\
A_1[2] = \text{\textbullet}
\end{array}
\]

Figure 4. Example of the action of $A_1[2]$ for $n = 4$. 

\[
\begin{array}{c}
\text{\textbullet} \\
\cdot A_1[2] = \text{\textbullet}
\end{array}
\]
By the definitions, it is easy to see that $G_n(\mathcal{A})$ and $F(n)$ are isomorphic. We simply write $g$ for the class of $g$ in $G_n(\mathcal{A})$ for the sake of simplicity.

In order to give a presentation of $G_n(\mathcal{A})$, we first define the “shift-map” on $G_n(\mathcal{A})$.

**Definition 2.8.** Define the map $[1]$ by the following:

$$
A_0 \xrightarrow{[1]} A_1 \xrightarrow{[1]} \cdots \xrightarrow{[1]} A_{n-2} \xrightarrow{[1]} A_0^{([n-1])} \xrightarrow{[1]} A_1^{([n-1])} \xrightarrow{[1]} \cdots
$$

and for each $k \in \{0, \ldots, n-2\}$ and $\alpha \in N^{<N} \cup \{\epsilon\}$,

$$
A_k[0\alpha] \xrightarrow{[1]} A_k[1\alpha] \xrightarrow{[1]} \cdots \xrightarrow{[1]} A_k^{([n-2]\alpha)} \xrightarrow{[1]} A_k^{([n-1]0\alpha)} \xrightarrow{[1]} A_k^{([n-1]1\alpha)} \xrightarrow{[1]} \cdots
$$

For the empty word $\epsilon$, define $[1](\epsilon) = \epsilon$. Furthermore, for an element $A_{k_1[\alpha_1]} \cdots A_{k_m[\alpha_m]}$ in $G_n(\mathcal{A})$, we define $[1](A_{k_1[\alpha_1]} \cdots A_{k_m[\alpha_m]}) = [1](A_{k_1[\alpha_1]}) \cdots [1](A_{k_m[\alpha_m]})$. A map $[i]$ denotes the composition of $[1]$ $i$ times.

**Remark 2.9.** By the definition, for each $A_k[\alpha]$ (even if $\alpha$ is empty), we have

$$
[n-1](A_k[\alpha]) = A_k^{([n-1]0\alpha)}.
$$

This fact is useful to check the relations.

Next, for an $n$-ary tree $t$, we define inductively two words $w_t$ and $w_t^*$ as follows:

**Definition 2.10.**

(1) If $t$ is a single vertex, we define

$$
w_t = w_t^* = \epsilon.
$$

(2) If $t$ is an $n$-ary tree, as in Figure 5, we define

$$
w_t = w_t^* \times [1](w_{t_1}^*) \times [2](w_{t_2}^*) \times \cdots \times [n-2](w_{t_{n-2}}^*) \times [n-1](w_{t_{n-1}}),
$$

$$
w_t^* = w_t^* \times [1](w_{t_1}^*) \times [2](w_{t_2}^*) \times \cdots \times [n-2](w_{t_{n-2}}^*) \times [n-1](w_{t_{n-1}}^*) \times A_0,
$$

where $\times$ denotes the concatenation.

Now, we show that $t = s \cdot w_t$ holds. We note that $s$ is an $n$-ary tree with the same number of carets of $t$. 

---

**Figure 5.** An $n$-ary tree $t$ with subtrees (or leaves) $t_0, \ldots, t_{n-1}$
\[
\begin{align*}
T_m \cdot w^*_t &= \begin{cases}
\begin{array}{c}
\cdots \\
T_{m} \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\end{cases} \\
&= \begin{cases}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\end{cases}
\end{align*}
\]

Figure 6. The second claim in Lemma 2.11.

\[
\begin{align*}
\begin{cases}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\end{cases} \\
n_0 + \cdots + n_{n-1} \text{ carets}
\end{align*}
\]

Figure 7. The \(n\)-ary tree \(T_m\).

\[
\begin{align*}
\begin{cases}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\end{cases} \\
n_0 \text{ carets} \\
n_1 + \cdots + n_{n-1} \text{ carets}
\end{align*}
\]

\[
\begin{cases}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\end{cases} \\
\cdot \begin{cases}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\begin{array}{c}
\cdots \\
\cdots \\
\cdots \\
\end{array}
\end{cases}
\end{align*}
\]

Figure 8. The action of \(w^*_t\).

Lemma 2.11. Let \(t\) in \(T(n)\) with \(m\) carets. We define \(T_m\) to be an all-right tree with \(m\) carets if \(m \geq 1\) and a single vertex if \(m = 0\). Then, for every \(t'\) in \(T(n)\), we have \(T_m \cdot w_t = t\), and the equality in Figure 6 holds.

Proof. We show by induction on \(m\). For the base case, since \(t, T_m\) are single vertices and, we have \(w_t = w'_t = \epsilon\), the equality is clear.

Let \(t\) be in Figure 5 and \(n_i\) be the number of carets of \(t_i\) for \(i = 0, \ldots, n - 1\). By the definitions,

\[
w_t = w^*_{t_0} \times [1](w^*_{t_1}) \times [2](w^*_{t_2}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w_{t_{n-1}})
\]

holds and \(T_m\) is in Figure 7. By the inductive hypothesis for \(w^*_{t_0}\), it acts on \(T_m\) as in Figure 8. Since \(w_t, w^*_t\) are the words on \(\{A_i, A_{i[(n-1)]}, A_{i[(n-1)](n-1)]}, \cdots | \ i = 0, \ldots, n-2\}\), the sift-map \([1]\) shifts indices just one. Thus, by applying the inductive hypothesis repeatedly, \(w^*_{t_0} \times [1](w^*_{t_1}) \times [2](w^*_{t_2}) \times \cdots \times [n-2](w^*_{t_{n-2}})\) acts on \(T_m\) as in Figure 9, and we write \(\tilde{t}\) for the resulting tree. Recall Remark 2.9 the action of \([n-1](w_{t_{n-1}})\) on \(\tilde{t}\) is obtained from that of \(w_{t_{n-1}}\) on the subtree \(T_{n-1}\) in \(\tilde{t}\). By the inductive hypothesis for \(w_{t_{n-1}}\), we have \(T_{n-1} \cdot w_{t_{n-1}} = t_{n-1}\). This completes the proof of the first claim that \(w_t\) satisfies the equality.

By the similar argument as in the case of the action of \(w_t\), that of \(w^*_t\) is calculated as shown in Figure 10. \(\square\)
It remains to prove that the condition (2.1) in Theorem 2.4 holds for a collection of relations. Let $R(n)$ be the set of the elements in Tables 1, 2, 3, and 4. It is easy to see that all elements in $R(n)$ are relations of $F(n)$.

**Lemma 2.12.** Let $t' = t \cdot A_{k[\alpha]}$. Then we have

\[ w_{t'} \equiv_{R(n)} w_{t} \cdot A_{k[\alpha]}, \]

\[ w_{t'}^* \equiv_{R(n)} w_{t}^* \cdot A_{k[\alpha]}. \]

**Proof.** When we rewrite a word $w$ to $z$ by applying a relation in Table $i$ ($1 \leq i \leq 4$), we denote by $w \equiv_{i} z$. We show by induction on the length of $\alpha$ as an $n$-ary sequence. For the base case, first, we assume $k = 0$. Then we can illustrate trees $t, t'$ as in Figure 11.
Let \( \tilde{t} \) be the subtree of \( t' \) consisting of \( t_0, \ldots, t_{n-1} \) (as in Figure 5). Then we have

\[
\begin{align*}
w_{t'} &= w_\tilde{t}^* \times [1](w_{t_0}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}}) \\
&= w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \times [n-1](w_{t_{n-1}}^*) \times A_0 \\
&\times [1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}}) \\
&\equiv_1 w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \\
&\times [n-1]([1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}})) \times A_0 \\
&= w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \\
&\times [n-1]((w_{t_{n-1}}^*) \times [1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}})) \times A_0 \\
&= w_{t} \cdot A_0,
\end{align*}
\]

and

\[
\begin{align*}
w_{t'}^* &= w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \times [n-1](w_{t_{n-1}}^*) \times A_0 \\
&\times [1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}}^*) \times A_0 \\
&\equiv_1 w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \\
&\times [n-1]((w_{t_{n-1}}^*) \times [1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}}^*)) \times A_0A_0 \\
&\equiv_2 w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \\
&\times [n-1]((w_{t_{n-1}}^*) \times [1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}}^*)) \\
&\times A_0_{[n-1]}A_0A_0A_0[0] \\
&= w_{t_0}^* \times [1](w_{t_1}^*) \times \cdots \\
&\times [n-1]((w_{t_{n-1}}^*) \times [1](w_{t_n}^*) \times \cdots \times [n-2](w_{t_{2n-3}}^*) \times [n-1](w_{t_{2n-2}}^*)) \times A_0 \\
&\times A_0A_0[0] \\
&= w_{t}^* \times A_0[0].
\end{align*}
\]

The case \( k \geq 1 \) can be proved in the same way.
Let $\alpha = 0/\beta$. That is, we consider the case when the condition $t'_0 = t_0 \cdot A_k[\beta]$ holds for $t$ and $t'$, as shown in Figure 12. When we rewrite a word by applying the inductive hypothesis, we use $\equiv_I$ to denote its equality. We note that

$$[1](w^*_t) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}})$$

and

$$[1](w^*_t) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}})$$

are words on the set

$$\{A_1, \ldots, A_{n-2}, A_{i[\alpha]} \mid i = 0, \ldots, n-2, \alpha = (n-1), (n-1)^2, \ldots \}.$$ 

Then we have

$$w'_t = w^*_{t'_0} \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}})$$

$$\equiv_I w^*_{t_0} \times A_k[\beta] \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}})$$

$$\equiv_3 w^*_{t_0} \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}}) \times A_k[\beta]$$

$$= w_t \times A_{k[\alpha]}.$$ 

and

$$w^*_{t'_0} = w^*_{t_0} \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}}) \times A_0$$

$$\equiv_I w^*_{t_0} \times A_k[\beta] \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}}) \times A_0$$

$$\equiv_3 w^*_{t_0} \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}}) \times A_k[\beta]A_0$$

$$\equiv_4 w^*_{t_0} \times [1](w^*_{t_1}) \times \cdots \times [n-2](w^*_{t_{n-2}}) \times [n-1](w^*_{t_{n-1}}) \times A_0A_k[\beta]$$

$$= w^*_t \times A_{k[0\alpha]}.$$ 

Since the collection of relations is closed under the shift-map, we can apply the inductive hypothesis when considering any other $i \beta$ ($1 \leq i \leq n-1$). Although the only case $w_t$ for $i = n-1$ is rewritten slightly differently (since $w_{t_{n-1}}$ appears in the word $w_t$ instead of $w^*_{t_{n-1}}$), all can be shown similarly. This completes the proof. \qed
3. **The Lodha–Moore group and its generalization**

3.1. **The Lodha–Moore group.** In this section, we briefly review the original Lodha–Moore group $G_0$. In some papers, this group is denoted by $G$. Let $x_0, x_{0[1]}$ be the maps defined in Section 2.1 for $n = 2$. The group $G_0$ is generated by these two maps and one more generator called $y_{10}$. To define this generator, we first define the homeomorphism called $y$.

**Definition 3.1.** The map $y$ and its inverse $y^{-1}$ is defined recursively based on the following rule:

\[
\begin{align*}
    y(00\zeta) &= 0y(\zeta) & y^{-1}(0\zeta) &= 00y^{-1}(\zeta) \\
    y(01\zeta) &= 10y^{-1}(\zeta) & y^{-1}(10\zeta) &= 01y(\zeta) \\
    y(1\zeta) &= 11y(\zeta) & y^{-1}(11\zeta) &= 1y^{-1}(\zeta).
\end{align*}
\]

For each $s$ in $2^{<\mathbb{N}}$, we also define the map $y_s$ by setting

\[
y_s(\xi) = \begin{cases} 
    s\xi(\eta), & \xi = s\eta \\
    \xi, & \text{otherwise}.
\end{cases}
\]

We give an example of a calculation of the $y_{001}$ on $00101101\cdots$:

\[
y_{001}(00101101\cdots) = 001y(01101\cdots) = 00110y^{-1}(101\cdots) = 0011001y(1\cdots).
\]

**Definition 3.2.** The group $G_0$ is a group generated by $x_0, x_{0[1]}$, and $y_{10}$.

The group $G_0$ is also realized as a group of piecewise projective homeomorphisms.

**Proposition 3.3 ([15 Proposition 3.1]).** The group $G_0$ is isomorphic to the group generated by the following three maps of $\mathbb{R}$:

\[
a(t) = t + 1, \quad b(t) = \begin{cases} 
    t & \text{if } t \leq 0 \\
    \frac{t}{t+1} & \text{if } 0 \leq t \leq \frac{1}{2} \\
    3 - \frac{1}{t} & \text{if } \frac{1}{2} \leq t \leq 1 \\
    t + 1 & \text{if } 1 \leq t,
\end{cases} \quad c(t) = \begin{cases} 
    \frac{2t}{1+t} & \text{if } 0 \leq t \leq 1 \\
    \frac{1}{t} & \text{otherwise}.
\end{cases}
\]

This proposition is shown by identifying $2^\mathbb{N}$ with $\mathbb{R}$ by the following maps:

\[
\begin{align*}
    \varphi : 2^\mathbb{N} &\to [0, \infty] \\
    \varphi(0\xi) &= \frac{1}{1 + \frac{1}{\varphi(\xi)}} \\
    \varphi(1\xi) &= 1 + \varphi(\xi) \\
    \Phi : 2^\mathbb{N} &\to \mathbb{R} \cup \{\infty\} \\
    \Phi(0\xi) &= -\varphi(\xi) \\
    \Phi(1\xi) &= \varphi(\xi),
\end{align*}
\]
where $\mathbb{R} \cup \{\infty\}$ denotes the real projective line. We note that for every $x \in \mathbb{R} \cup \{\infty\}$, the inverse image $\Phi^{-1}(\{x\})$ is either a one-point set or a two-point set.

3.2. $n$-adic Lodha–Moore group. In order to define a generalization of the Lodha–Moore group, we first define a homeomorphism on $N^N = \{0, \ldots, n-1\}^N$ corresponding to $y$ for the case of $n = 2$. We fix $n \geq 2$ and also denote this map by $y$ as in the case of $n = 2$.

**Definition 3.4.** The map $y$ and its inverse map $y^{-1}$ is defined recursively based on the following rule:

$$
\begin{align*}
y(00\zeta) &= 0y(\zeta) \\
y(01\zeta) &= 1\zeta \\
& \vdots \\
y(0(n-2)\zeta) &= (n-2)\zeta \\
y(0(n-1)\zeta) &= (n-1)0y^{-1}(\zeta) \\
y(1\zeta) &= (n-1)1\zeta \\
& \vdots \\
y((n-2)\zeta) &= (n-1)(n-2)\zeta \\
y((n-1)\zeta) &= (n-1)(n-1)y(\zeta)
\end{align*}
$$

**Remark 3.5.** Note that if we restrict the domain to $\{0, n-1\}^N$, $y$ is exactly the map defined in Definition 3.1 under the identification of $n-1$ with 1. We will use this fact to reduce the discussion to the case of $n = 2$.

For each $s$ in $N^N$, define the map $y_s$ by setting

$$
y_s(\xi) = \begin{cases} 
y(\eta), & \xi = s\eta \\
x, & \text{otherwise.} \end{cases}
$$

**Definition 3.6.** We define $G_0(n)$ to be the group generated by the $n+1$ elements $x_0, \ldots, x_{n-2}, x_0[n-1]$, and $y(n-1)0$. We call this group the $n$-adic Lodha–Moore group.

For the infinite presentation of $G_0(n)$ described in Section 3.3, we introduce an infinite generating set of this group. Let $i$ in $\{0, \ldots, n-2\}$ and $\alpha$ in $N^{<N} \cup \{\epsilon\}$. We recall that the map $x_{i[\alpha]}$ is defined as follows:

$$
x_{i[\alpha]}(\zeta) = \begin{cases} 
\alpha x_i(\eta), & (\zeta = \alpha\eta) \\
\zeta, & (\zeta \neq \alpha\eta).
\end{cases}
$$
We also recall that the group $F(n)$ is generated by the following infinite set:

$$X(n) = \left\{ x_{i[\alpha]} \mid i = 0, \ldots, n - 2, \alpha \in \mathbb{N}^\infty \cup \{\epsilon\} \right\}.$$ 

In addition, let

$$Y(n) := \left\{ y_{\alpha} \mid \begin{array}{l} \alpha \in \mathbb{N}^\infty, \\
\alpha \neq 0 \cdots 0, (n - 1) \cdots (n - 1), \epsilon, \\
\text{the sum of each number in } \alpha \text{ is equal to } 0 \mod n - 1 \end{array} \right\}.$$ 

We remark that for $\alpha = \alpha_1 \cdots \alpha_m \in \mathbb{N}^\infty$, the actions of $x_0, \ldots, x_{n-2}, x_{0[n-1]}$, and $y$ preserve the value of $\alpha_1 + \cdots + \alpha_m \mod n - 1$. Then the set $Z(n) := X(n) \cup Y(n)$ also generates the group $G_0(n)$.

3.3. Infinite presentation and normal form. In this section, we give the unique word with “good properties” for each element of $G_0(n)$ (Definition 3.32). In this process, we also give an infinite presentation of $G_0(n)$ (Corollary 3.40). Although almost all result in the rest of this section follows along the lines of the arguments in [14,15], we write them down for the convenience of the reader.

Let $s$ in $\mathbb{N}^\infty$ and $t$ in $\mathbb{N}^\infty$ or $\mathbb{N}^\infty$. We write $s \subset t$ if $s$ is a proper prefix of $t$ and write $s \subseteq t$ if $s \subset t$ or $s = t$. For $s, t$, we say that $s$ and $t$ are independent if one of the following holds:

- $s, t \in \mathbb{N}^\infty$ and neither $s \subseteq t$ and $t \subseteq s$ holds.
- $s \in \mathbb{N}^\infty, t \in \mathbb{N}^\infty$ and $s$ is not any prefixes of $t$.
- $s, t \in \mathbb{N}^\infty$ and $s \neq t$.

In all cases, we write $s \perp t$.

Let $s(i), t(i)$ denote the $i$-th number of $s, t$, respectively. Then we say $s < t$ if one of the following is true:

- (a) $t \subset s$;
- (b) $s \perp t$ and $s(i) < t(i)$, where $i$ is the smallest integer such that $s(i) \neq t(i)$.

We note that this order is transitive. For elements in $\mathbb{N}^\infty$, we use the same symbol to denote the lexicographical order.

We claim that the following collection of relations gives a presentation of $G_0(n)$ (Corollary 3.40):

1. the relations of $F(n)$ in Tables 1, 2, 3 and 4;
2. $y_i x_{i[s]} = x_{i[s]} y_{y_i[s]}(t)$ for all $i$ and $s, t \in \mathbb{N}^\infty$ such that $y_t \in Y(n)$ and $x_{i[s]}(t)$ is defined;
3. $y_s y_t = y_t y_s$ for all $s, t \in \mathbb{N}^\infty$ such that $y_s, y_t \in Y(n)$ and $s \perp t$;
4. $y_s = x_{0[s]} y_0 y_{y_s[0]} y_{s[0]}(n - 1)$ for all $s \in \mathbb{N}^\infty$ such that $y_s \in Y(n)$.

We note that $x_{0[n-1]}((n-1)0)$ is not defined, for example. All relations can be verified directly. We write $R(n)$ for the collection of these relations.
We first define a form, which is easy to compute the composition of maps.

**Definition 3.7.** A word Ω on Z(n) is in **standard form** if Ω is a word such as \( fy_{s_1}^t \cdots y_{s_m}^t \) where \( f \) is a word on X(n) and \( y_{s_1}^t \cdots y_{s_m}^t \) is a word on Y(n) with the condition that \( s_i < s_j \) if \( i < j \).

In some cases, it is helpful to use the following weaker form.

**Definition 3.8.** A word Ω on Z(n) is in **weak standard form** if Ω is a word such as \( fy_{s_1}^t \cdots y_{s_m}^t \) where \( f \) is a word on X(n) and \( y_{s_1}^t \cdots y_{s_m}^t \) is a word on Y(n) with the condition that \( s_j \subset s_i \), then \( i < j \).

We can always make a word in weak standard form into one in standard form.

**Lemma 3.9** ([14, Lemma 3.11] for \( n=2 \)). We can rewrite a weak standard form into a standard form with the same length by just switching the letters (i.e., relation (3)) finitely many times.

**Proof.** Let \( fy_{s_1}^t \cdots y_{s_m}^t \) be a word in weak standard form. We show this by induction on \( m \). It is obvious if \( m = 0, 1 \). For \( m \geq 2 \), by the induction hypothesis, we get a word \( fy_{s_1}^{t_1} \cdots y_{s_{m-1}}^{t_{m-1}} y_{s_m}^t \) where \( fy_{s_1}^{t_1} \cdots y_{s_{m-1}}^{t_{m-1}} \) is in standard form. Then one of the following holds:

1. \( s_{m-1}'t_{m-1} \supset s_t \);
2. \( s_{m-1}'t_{m-1} \subset s_t \) and \( s_{m-1}'(i) < s_t(i) \), where \( i \) is the smallest number such that \( s_{m-1}'(i) \neq s_t(i) \);
3. \( s_{m-1}'t_{m-1} \subset s_t \) and \( s_{m-1}'(i) > s_t(i) \), where \( i \) is the smallest number such that \( s_{m-1}'(i) \neq s_t(i) \).

If (1) or (2), \( fy_{s_1}^{t_1} \cdots y_{s_{m-1}}^{t_{m-1}} y_{s_m}^t \) is also in standard form. If (3), by applying the relation (3) to \( y_{s_{m-1}}^t y_{s_m}^t \) and using the induction hypothesis for the first \( m - 1 \) characters again, we get a word in standard form. \( \square \)

For an infinite \( n \)-ary word \( w \) in \( N^N \) and a word \( fy_{s_1}^t \cdots y_{s_m}^t \) in weak standard form, we define their **calculation** as follows: First, we apply \( f \) to \( w \). Then apply \( y_{s_1}^t, \ldots, y_{s_m}^t \) to \( f(w) \) in this order, where the latter term “apply” means to rewrite each \( y_{s_i} \) by using its definition in equation (3.1), and no rewriting can be done by the definition of the map \( y \).

**Example 3.10.** Let \( n = 4 \). For \( w = 3002 \cdots \) and \( y_{300}^{-1} y_{30}^{-1} y_1 \), we apply as follows:

\[
3002 \cdots \xrightarrow{y_{300}^{-1}} 300 y^{-1}(2 \cdots) \xrightarrow{y_{30}^{-1}} 30(y(0(y^{-1}(2 \cdots)))) \xrightarrow{y_1} 30(y(0(y^{-1}(2 \cdots))))
\]

Therefore the calculation of \( w = 3002 \cdots \) and \( y_{300}^{-1} y_{30}^{-1} y_1 \) is \( 30(y(0(y^{-1}(2 \cdots)))) \).

We write such the element as \( 30y0y^{-1}2 \cdots \) and sometimes regard it as a word on \( N \cup \{y, y^{-1}\} \). We also define calculations for finite words in \( N^{<N} \) and weak standard forms in the same way, although finite words are not in the domain of \( y \). We note that,
unlike infinite words, not all calculations of finite words can be defined. For example, in \( n = 4 \), the calculation of \( w = 3002 \) and \( y_{300}y_{30}y_1 \) is \( 30y_0y^{-1}2 \). However, that of \( w = 3 \) and \( y_{300}y_{30}y_1 \) is not defined.

We call the operation of rewriting a calculation by using the definition of \( y \) once as substitution and call the element in \( \mathbb{N}^N \) obtained by repeating the substitution output string.

**Example 3.11.** For \( w = 0^4(n-1)0^4(n-1) \cdots \) and \( y^2 \), we have the following substitutions:

\[
y^20^4(n-1)0^4(n-1) \cdots \rightarrow y0y0^2(n-1)0^4(n-1) \cdots \rightarrow y0^2y(n-1)0^4(n-1) \cdots .
\]

Their output string is \( 0(n-1)0^4(n-1)^4 \cdots \).

As described in [14, Section 3.6], the map \( y \) can be expressed as a finite state transducer. For the sake of simplicity, assume \( n \geq 3 \). Then our transducer is illustrated in Figure 13. In this setting, each edge is labeled by a form \( \sigma | \tau \), where \( \sigma \) and \( \tau \) are in \( \mathbb{N}^N \). The element \( \sigma \) represents the input string, and \( \tau \) represents the output string. The difference from the case of \( n = 2 \) is that there exists an accepting state (double circle mark). This difference corresponds to the fact that \( y \) vanishes in \( y\sigma \) by substitutions if \( \sigma \) is not in \( \{0, n-1\}^N \) or \( \{0, n-1\}^N \).

**Definition 3.12.** A calculation contains a potential cancellation if there exists a subword of the form

\[
y^{t_1} \sigma y^{t_2}, \sigma \in \mathbb{N}^N, t_i \in \{1, -1\}
\]
such that we get the word $\sigma'y^{-t_2}$ by substituting $y^i\sigma$ finitely many times.

**Example 3.13.** The subwords $y0(n - 1)y$ and $y00y^{-1}$ are potential cancellations. The subword $y0y$ is not a potential cancellation since we can not apply any substitutions. The subword $y^{-1}(n - 1)00(n - 1)y$ is also not a potential cancellation since we have $y^{-1}(n - 1)00(n - 1)y \rightarrow 0(n - 1)y00(n - 1)y \rightarrow 0(n - 1)(n - 1)0y^{-1}y$.

**Remark 3.14.** If a calculation contains a potential cancellation, then $\sigma$ is in $\{0, n - 1\}^{<N}$. Indeed, if not, $y$ vanishes.

The following holds.

**Lemma 3.15 (\cite{15} Lemma 5.9) for $n = 2$.** Let $\Lambda$ be a calculation that contains no potential cancellations. Then the word $\Lambda'$ obtained by substitution at any $y^\pm 1$ again contains no potential cancellations.

**Proof.** By Remark 3.14, it is sufficient to consider the case $\sigma \in \{0, n - 1\}^{<N}$. Then we can identify $\{0, n - 1\}$ with $\{0, 1\}$. By \cite{15} Lemma 5.9, we have the desired result. \qed

We generalize the notion of potential cancellation to the case of weak standard forms.

**Definition 3.16.** A weak standard form $fy_{s_1}^1 \cdots y_{s_m}^m$ has a potential cancellation if there exists $w$ in $N^N$ such that the calculation of $w$ by $fy_{s_1}^1 \cdots y_{s_m}^m$ contains a potential cancellation.

To construct unique words, we define the moves obtained from the relations of $G_0(n)$.

**Rearranging move** $y_{s_j}^i x_{j[s]}^{\pm 1} \rightarrow x_{j[s]}^{\pm 1} y_{s_j}^i$;

**Expansion move** $y_s \rightarrow x_0[s]y_0y_{s_0}^{-1}y_{s(n-1)}(n-1)$ and $y_{s}^{-1} \rightarrow x_0[s]y_{s(n-1)}^{-1}y_{s_0}y_{s(n-1)}^{-1}$;

**Commuting move** $y_uy_v \leftrightarrow y_vy_u$ (if $u \perp v$);

**Cancellation move** $y_{s_i}^{\pm i}y_{s_j}^{\pm j} \rightarrow \epsilon$;

**ER moves**

1. $f(y_{s_1}^1 \cdots y_{s_k}^k)y_u \rightarrow f(x_0[u](y_{x_0[u]}^{t_1}(s_1) \cdots y_{x_0[u]}^{t_k}(s_k))(y_{u_0}y_{u(n-1)}y_{u(n-1)}^{-1})$

2. $f(y_{s_1}^1 \cdots y_{s_k}^k)y_u^{-1} \rightarrow f(x_0[u](y_{x_0[u]}^{t_1}(s_1) \cdots y_{x_0[u]}^{t_k}(s_k))(y_{u_0}y_{u(n-1)}y_{u(n-1)}^{-1})$.

ER moves are a combination of an expansion move and rearranging moves.

By the definition of potential cancellations, we note that if we apply ER move to a standard form that contains a potential cancellation, then either the resulting word also contains a potential cancellation, or we can apply a cancellation move to the resulting word. Like the Thompson group $F$, each element of $G_0(n)$ can be represented by tree diagrams. Then the expansion move for $y_s$ is the replacement from the left diagram to the right diagram in Figure 14. See \cite{15} Section 4 for details of the case $n = 2$.

We introduce the following notion, which makes it easy to check whether the moves are defined.
Definition 3.18. For $s$ in $\mathbb{N}^{<\mathbb{N}}$, we write $\|s\|$ for the length of $s$. We define the depth of a weak standard form $fy_{s_1}^{y_1} \cdots y_{s_m}^{y_m}$ to be the integer $\min_{1 \leq i \leq m} \|s_i\|$. If a weak standard form is on $X(n)$, we define its depth to be $\infty$.

We show that an arbitrary word on $Z(n)$ can be rewritten in standard form. In order to do this, we prepare two lemmas.

Lemma 3.19 ([15] Lemma 5.2) for $n = 2$. Let $y_s$ in $Y(n)$ and $l$ in $\mathbb{N} \cup \{0\}$. Then there exists a standard form $\Omega$ obtained by applying expansion move and rearranging move on $y_s^{\pm 1}$ finitely many times that satisfies the following:

1. If there exists $x_{i[u]}$ in $\Omega$, then $s \subset u$ holds.
2. If there exists $y_u$ or $y_u^{-1}$, then it is the only one, and $s \subset u$ and $\|u\| \geq l$ hold.
3. If there exist $y_u, y_v$ with $u \neq v$, then $u \perp v$ holds.

Proof. We show this by induction on $l - \|s\|$. If $l - \|s\| \leq 0$, then $\Omega = y_s^{\pm 1}$ satisfies all the conditions.

We assume that $l - \|s\| > 0$ holds. Because we can discuss $y_s^{-1}$ in almost the same way, we consider only $y_s$. We apply expansion move on $y_s$ and get the word $x_0[s]y_0y_s^{-1}s_{(n-1)0}y_{s(n-1)(n-1)}$. Since we have

$$\max\{l - \|s0\|, l - \|s(n-1)0\|, l - \|s(n-1)(n-1)\|\} \leq l - \|s\|,$$

by the induction hypothesis, there exists three standard forms $\Omega_{s0}, \Omega_{s(n-1)0}, \Omega_{s(n-1)(n-1)}$ for $y_{s0}, y_{s(n-1)0}, y_{s(n-1)(n-1)}$, respectively, all of which satisfy the conditions. Thus the word is rewritten from $x_0[s]y_0y_{s(n-1)0}y_{s(n-1)(n-1)}$ to $x_0[s]\Omega_{s0}\Omega_{s(n-1)0}\Omega_{s(n-1)(n-1)}$. If there exists $x_{i[u]}$ in $\Omega_{s(n-1)0}$, then $u \supset s(n-1)0$ holds. If there exists $y_v^{\pm 1}$ in $\Omega_{s0}$, then $v \supseteq s0$ holds. Since $u \perp v$, $x_{i[u]}(v)$ is defined and equals to $v$. Thus, we can apply rearranging moves on this $x_{i[u]}$. A similar process can be done for the $x_{i[u]}$ in $\Omega_{s(n-1)(n-1)}$. This completes the proof.

Lemma 3.20 ([15] Lemma 5.3) for $n = 2$. Let $\Lambda$ be a word on $X(n)$, and $k$ be the length of $\Lambda$. Then there exists $l_0$ in $\mathbb{N}$ such that the following holds: If $\Omega$ is a standard form with depth $l \geq l_0$, then we have a standard form $\Omega'$ obtained from $\Omega\Lambda$ by rearranging moves, with the depth of $\Omega'$ being at least $l - k$.\[\square\]
Proof. We show this by induction on the length of $\Lambda$. For the base case, let $\Lambda = x_i^\pm$. If $x_i^\pm(t)$ is not defined, then $t = si$ holds. This means that if $t \in N^{<N}$ with $\|t\| \geq \|s\| + 2$, then $x_i^\pm(t)$ is defined. Hence let $l_0 = \|s\| + 2$. We note that

$$\|t\| - 1 \leq \|x_i^\pm(t)\| \leq \|t\| + 1$$

holds. Then we can apply rearranging moves on $x_i^\pm$ with every $y_t$ in $\Omega$. The depth of the resulting standard form $\Omega'$ is at least $l - 1$.

We assume that the claim holds for $k - 1$. Let $\Lambda = \Lambda' x_i^\pm$. By the induction hypothesis, there exists $l_0'$ for $\Lambda'$ such that we can apply moves $\Omega\Lambda' x_i^\pm \rightarrow \Omega'' x_i^\pm$ where the depth of $\Omega$ is $l \geq l_0'$ and that of $\Omega''$ is at least $l - (k - 1)$. Then, let $l_0 = \max\{l_0', k + \|s\| + 1\}$ and assume that the depth of $\Omega$ is $l \geq l_0$. Since we have $l - (k - 1) \geq \|s\| + 2$, for the same reason as in the base case, we can apply rearranging moves and obtain a standard form $\Omega'$ whose depth is at least $l - (k - 1) - 1 = l - k$. \hfill $\Box$

Now we make arbitrary words into standard forms.

Proposition 3.21 ([15] Lemma 5.4 for $n = 2$). Let $l$ be a natural number and $w$ be a word on $Z(n)$. Then we can rewrite $w$ into a standard form whose depth is at least $l$ by moves.

Proof. If $w$ is in $X(n)$, there is nothing to do since the depth of $w$ is $\infty$. Hence we assume that $w$ is not in $X(n)$.

We show the claim by induction on the length $m$ of $w$. For the base case, since $w$ is not in $X(n)$, we have $w = y_i^\pm$. By Lemma [3.19], the claim holds.

Let $m > 1$. By dividing $a^{x(k+1)}$ in $w$ as $a^{\pm k} a^{\pm 1}$ if necessary, we can decompose $w$ into two positive length words $\Omega_0 \Omega_1$. We assume that $\Omega_1$ is not on $X(n)$. By the induction hypothesis, we obtain a standard form $\Lambda \Upsilon$ from $\Omega_1$, where $\Lambda$ is on $X(n)$, and $\Upsilon$ is on $Y(n)$, with the depth being at least $l$.

Let $k$ be the length of $\Lambda$ and $r = \max\{\|u\| \mid y_i^\pm$ belongs to $\Upsilon\} + 1$. By the induction hypothesis, there exists a standard form $\Omega'_0$ obtained from $\Omega_0$, with the depth being at least $\max\{l_0, r + k\}$, where $l_0$ is a natural number for $\Lambda$ in Lemma [3.20]. Then we rewrite $\Omega'_0 \Lambda \rightarrow \Omega''_0$ where $\Omega''_0$ is in standard form with the depth being at least $r$.

The form $\Omega''_0 \Upsilon$ obtained through the previous transformations $w \rightarrow \Omega_0 \Omega_1 \rightarrow \Omega'_0 \Lambda \Upsilon \rightarrow \Omega''_0 \Upsilon$ is in weak standard form. Indeed, since $\Omega''_0$ and $\Upsilon$ (on $Y(n)$) are in standard form, by the definition of $r$, the claim holds. By Lemma [3.9] we get the standard form without changing the depth.

If $\Omega_1$ is in $X(n)$, then we set $\Lambda = \Omega$, $\Upsilon = \epsilon$, and $r = l$. Then we can apply the above argument. \hfill $\Box$

Next, we list three lemmas about ER moves. These are useful for getting words in standard form without potential cancellations.
Lemma 3.22 ([14] Lemma 3.21) for $n=2$. Let $f(y_{s_1}^{i_1} \cdots y_{s_k}^{i_k})y_u^{\pm 1}(y_{p_1}^{i_{p_1}} \cdots y_{p_m}^{i_{p_m}})$ be in weak standard form. Then the word obtained from $f(y_{s_1}^{i_1} \cdots y_{s_k}^{i_k})y_u^{\pm 1}(y_{p_1}^{i_{p_1}} \cdots y_{p_m}^{i_{p_m}})$ by the ER move on $y_u^{i=1}$ is also in weak standard form.

Sketch of proof. We show that only the case $y_u$. Let

$$fx_{0[u]}(y_{s_1}^{i_1} \cdots y_{s_k}^{i_k})(y_{u0}y_{u(n-1)}^{-1}y_{u(n-1)(-1)}(y_{p_1}^{i_{p_1}} \cdots y_{p_m}^{i_{p_m}}))$$

be the word obtained from the given word by ER move. We show this by considering whether each $s_i$ is independent with $u$ or not. If $s_i \perp u$ holds, then $x_{0[u]}(s_i) = s_i$, and there is nothing to do. If $s_i \supset u$ holds, since $x_{0[u]}(s_i)$ is defined, $x_{0[u]}(s_i) \supset u$ holds, and it can never be a proper prefix of $u0, u(n-1)0$, or $u(n-1)(-1)$. Since $p_j$ satisfies either $p_j \perp u$ or $p_j \subseteq u$, the claim holds in both cases. □

Lemma 3.23 ([14] Lemma 3.22) for $n=2$. Let $f\lambda_1$ be in weak standard form with no potential cancellations and $g\lambda_2$ be a word obtained from $f\lambda_1$ by an ER move. Then $g\lambda_2$ is in weak standard form with no potential cancellations.

Proof. Let $\lambda_1 = y_{s_1}^{i_1} \cdots y_{s_m}^{i_m}$ with every $t_i$ in $\{\pm 1\}$. We apply ER move on $y_{u_i}^{i_1}$. Since $f$ does not affect the existence of potential cancellations, we assume that $f$ is the empty word. We only consider the case $t_i = 1$ since the cases $t_i = 1$ and $t_i = -1$ are shown similarly. Let $s_j' := x_{0[s_i]}(s_j)$ ($1 \leq j \leq i - 1$). By the definition of ER move, we have

$$g\lambda_2 = x_{0[s_i]}(y_{s_1}^{i_1} \cdots y_{s_{i-1}}^{i_{i-1}})(y_{s_0}y_{s_{i-1}}^{-1}y_{s_{i-1}(n-1)}y_{s_{i-1}(n-1)(-1)}(y_{s_{i+1}}^{i_{i+1}} \cdots y_{s_m}^{i_m}).$$

Since it is clear by Lemma 3.22 that $g\lambda_2$ is in weak standard form, suppose that $g\lambda_2$ has a potential cancellation. Let $\tau \in \mathbb{N}^\infty$ be an element such that the calculation $\Lambda$ of $\tau$ by $(y_{s_1}^{i_1} \cdots y_{s_{m-1}}^{i_{m-1}})(y_{s_0}y_{s_{m-1}}^{-1}y_{s_{m-1}(n-1)}y_{s_{m-1}(n-1)(-1)}(y_{s_{m+1}}^{i_{m+1}} \cdots y_{s_m}^{i_m})$ contains a potential cancellation. Then, $s_i$ is a prefix of $\tau$. Indeed, we have the following:

1. If $s_j \perp s_i$, then $y_{s_j}^{i_j} = y_{s_j}^{i_j}$;
2. If $s_j \perp s_i$, then $y_{s_j}^{i_j}(\tau') \perp s_i$ where $\tau'$ is in $\mathbb{N}^\infty$ with $\tau' \perp s_i$;
3. If $s_j \supset s_i$, since $x_{0[s_i]}(s_j) \supset s_i$, we have $y_{s_j}^{i_j}(\tau') = y_{x_{0[s_i]}(s_j)}(\tau') = \tau'$ and $y_{s_i}(\tau') = \tau'$ where $\tau'$ is in $\mathbb{N}^\infty$ with $\tau' \perp s_i$;
4. $s_i, s_0, y_{s_0}^{-1}, y_{s_{n-1}^{-1}}0$, and $y_{s_{n-1}^{-1}(n-1)}$ all fix $\tau'$ where $\tau'$ is in $\mathbb{N}^\infty$ with $\tau' \perp s_i$.

Since ER move is defined, each $y_j$ satisfies either $s_j \perp s_i$ or $s_j \supset s_i$. If $s_i$ is not a prefix of $\tau$, $\tau \perp s_i$ holds. Then the calculations of $\tau$ with $f\lambda_1$ and with $g\lambda_2$ are the same. This contradicts the assumption that $f\lambda_1$ does not have a potential cancellation. Hence $s_i$ is a prefix of $\tau$.

Let $\Lambda'$ be the calculation of $x_{0[s_i]}^{-1}(\tau)$ by $y_{s_1}^{i_1} \cdots y_{s_m}^{i_m}$. By the assumption, this calculation does not contain a potential cancellation. On the other hand, $\Lambda$ is obtained from $\Lambda'$ by substituting once. However, this also contradicts Lemma 3.15.
Lemma 3.24. Let \( l \) be an arbitrary natural number. Then, by applying ER moves to the weak standard form \( fy_{s_1}^t \cdots y_{s_m}^t \) finitely many times, we obtain a weak standard form whose depth is at least \( l \).

Sketch of proof. If \( \|s_i\| < l \) holds, we apply ER moves on \( y_{s_i}^t \) (if \( t_i \neq \pm 1 \), we apply on the last one). We note that the move may not be defined. In that case, before we do that, we apply ER moves to \( y_{s_j}^t \) (\( j < i \)) which is the cause that the ER move on \( y_{s_i}^t \) is not defined. If it is also not defined, repeat the process. Since we can always apply ER moves on \( fy_{s_1}^t \), this process is finished. We do this process repeatedly until the depth of the obtained word is at least \( l \). By Lemma 3.22, it is also in weak standard form. □

Since subwords play an essential role in the notion of potential cancellations, we introduce the following definitions for simplicity.

Definition 3.25. Let \( fy_{s_1}^t \cdots y_{s_m}^t \) be in weak standard form. We say the pair \((y_{s_j}^t, y_{s_i}^t)\) is adjacent if the following two conditions hold:

1. \( s_i \subset s_j \),
2. if \( u \) in \( N^{<N} \) satisfies \( s_i \subset u \subset s_j \), then \( u \notin \{s_1, \ldots, s_m\} \).

We define an adjacent pair \((y_{s_j}^t, y_{s_i}^t)\) is a potential cancellation if \( y^t \sigma y^j \) is a potential cancellation, where \( \sigma \) is the word satisfying \( s_i \sigma = s_j \). It is clear from the definition that a weak standard form contains a potential cancellation if and only if there exists an adjacent pair such that it is a potential cancellation.

By the definition, for example, if \((y_{s_j}^t, y_{s_i}^t)\) is a potential cancellation, then either \( y_{s_j}^t = y_{s_0}^{-1}y_{0(n-1)} \), or \( y_{s_i}^{-1}y_{s(n-1)} \) holds, or \((y_{x_0(s)(j)}, y_{s_0})\), \((y_{x_0(s)(j)}, y_{s(n-1)}^{-1})\), or \((y_{x_0(s)(j)}, y_{s(n-1)(n-1)})\) is also a potential cancellation. As mentioned in Remark 3.14, we note that if an adjacent pair is a potential cancellation, then \( \sigma \) is in \( \{0, n-1\}^{<N} \).

We introduce the moves which are the “inverse” of ER moves. First, we define the conditions where the moves are defined.

Definition 3.26. We say that a weak standard form contains a potential contraction if it satisfies either of the following:

1. there exists a subword \( y_{s_0}y_{s(n-1)}^{-1}y_{s(n-1)(n-1)} \), but there does not exist \( y_{s(n-1)}^{\pm 1} \);
2. there exists a subword \( y_{s_0}y_{s(n-1)}y_{s(n-1)}^{-1} \), but there does not exist \( y_{s_0}^{\pm 1} \).

We also call the same when the condition above is satisfied by commuting moves.

Example 3.27. Let \( n = 4 \). A weak standard form \( y_{s_0}y_{s_0}^{-1}y_{s_0}y_{s_0}^{-1}y_{s_0} \) contains a potential contraction.

We now define moves.

Definition 3.28. Let \( f \lambda \) be in standard form. We assume that this form contains a potential contraction in the sense of (1). We apply commuting move to all \( y_{s_0}^n \) except the
subword in the standard form satisfying $0 < u \leq s(n-1)(n-1)$ to the left of $y_{s0}$ while preserving the order, and we replace $y_{s0}y_{s(n-1)}^{-1}y_{s(n-1)(n-1)}$ with $x_{0[s]}^{-1}y_{s}$. Then, move $x_{0[s]}^{-1}$ to just next to $f$ by rearranging moves. Finally, we apply cancellation moves or moves $x^a x^b \rightarrow x^{a+b}$ if necessary. The rearranging moves are defined, and the word obtained by the above sequence of moves is in standard form. We call this sequence of moves a contraction move. We also define the contraction move for (2) in the same way.

The moves and the claim in this definition are justified by the following lemma. We describe only case (1), but the similarity holds for case (2).

**Lemma 3.29.** In Definition 3.28 the rearranging moves are defined, and the resulting word is again in standard form.

**Proof.** By the definition of standard form, for $y_{s}^{n}$, we have either $s0 \perp u$ or $s0 \subseteq u$. In the former case, we have either $s \perp u$ or $s \subseteq u$. In the latter case, we have $s \subseteq u$. By the definition of potential contraction, there does not exist $y_{s(n-1)}^{\pm 1}$. Hence $x_{0[s]}^{-1}(u)$ is defined, and the rearranging moves can be done.

Suppose that we have finished applying commuting move. Let $s'_{j} := x_{0[s]}^{-1}(s_{j})$, and $x_{0[s]}^{-1}(y_{s_{1}}^{t_{1}} \cdots y_{s_{k}}^{t_{k}})y_{s}(y_{p_{1}}^{q_{1}} \cdots y_{p_{m}}^{q_{m}})$ be the resulting word. If $s'_{k} = s_{k}$, then $s_{k} \perp s$. Since the original word is in standard form, we have $s'_{k} < s$.

If $s'_{k} \neq s_{k}$, then $s'_{k} \supset s$. By the transitivity of this order, $x_{0[s]}^{-1}(y_{s_{1}}^{t_{1}} \cdots y_{s_{k}}^{t_{k}})y_{s}$ is in standard form.

We show that $y_{s}(y_{p_{1}}^{q_{1}} \cdots y_{p_{m}}^{q_{m}})$ is in standard form. We note that $s(n-1)(n-1) < p_{1}$ holds. If $s(n-1)(n-1) \supset p_{1}$, either $s = p_{1}$ or $s \supset p_{1}$ holds. In the former case, by $y_{s}y_{p_{1}} \rightarrow y_{s}^{1}1$ or $\rightarrow \epsilon$, we get the standard form. In the latter case, we have $s < p_{1}$. If $s(n-1)(n-1) \perp p_{1}$, since $(n-1)$ is the largest number, we have $s < p_{1}$. By the transitivity, $x_{0[s]}^{-1}(y_{s_{1}}^{t_{1}} \cdots y_{s_{k}}^{t_{k}})y_{s}(y_{p_{1}}^{q_{1}} \cdots y_{p_{m}}^{q_{m}})$ is in standard form. □

**Example 3.30.** In example 3.27 we obtain $x_{0[30]}^{-1}y_{301}y_{30}^{2}$ by applying a contraction move.

Contraction moves have the following property.

**Lemma 3.31.** Let $fy_{s_{1}}^{t_{1}} \cdots y_{s_{m}}^{t_{m}}$ be in standard form which contains a potential contraction and no potential cancellations. Then the word obtained by contraction move contains no potential cancellations.

**Proof.** We show only the case of potential contraction condition (1). We assume that we have produced a potential cancellation after applying a contraction move. If there exists such an adjacent pair, the pair must be $(y_{s}^{k}, y_{s'}^{k})$, where $s'$ satisfies $s' \subset s$. Indeed, if not, each adjacent pair is either $(y_{u_{1}}^{t_{1}}, y_{u_{2}}^{t_{2}})$ where $u_{1}, u_{2} \neq s$, or $(y_{u''}^{t_{1}}, y_{k}^{k})$ where $s''$ satisfies $s'' \subset s$. In the former case, since either $x_{0[s]}(u_{i}) \neq u_{i} (i = 1, 2)$ or $x_{0[s]}(u_{i}) = u_{i} (i = 1, 2)$ holds, this contradicts the assumption that $fy_{s_{1}}^{t_{1}} \cdots y_{s_{m}}^{t_{m}}$ contains no potential cancellations. In the latter case, we apply ER move on $y_{s}^{k}$. Since the contraction move is
the inverse of the ER move, if \( y'''_{s''} \) is \( y_{s0}^{-1} \), \( y_{s(n−1)0}^{-1} \), or \( y_{s(n−1)(n−1)}^{-1} \), then it contradicts the assumption that \( fy_{s1}^1 \cdots y_{sm}^m \) is in standard form. If \( y'''_{s''} \) is none of them, then it contradicts that \( fy_{s1}^1 \cdots y_{sm}^m \) contains no potential cancellations.

There does not exist \( y_{s}^{k_1} \) in \( fy_{s1}^1 \cdots y_{sm}^m \). Indeed, if there exists \( y_{s}^{k_1} \) \((k_1 < 0)\), since there exists \( y_{s0} \) by the assumption of potential contraction, \( fy_{s1}^1 \cdots y_{sm}^m \) contains a potential cancellation. If there exists \( y_{s}^{k_1} \) \((k_1 > 0)\), since \((y_{s}^{k_1}, y_{s}^{k_2})\) is a potential cancellation in the word after applying the contraction move, the adjacent pair \((y_{s}^{k_1}, y_{s}^{k_2})\) is a potential cancellation in \( fy_{s1}^1 \cdots y_{sm}^m \). This contradicts the assumption about no potential cancellations.

Now we consider the adjacent pair \((y_{s}, y_{s}^{k'})\). Let \( \sigma \) be the finite word such that \( s'\sigma = s \) holds. By the assumption, \( y\sigma y \) or \( y^{-1}\sigma y \) is potential cancellation. In the former case, by substitutions, we have \( y\sigma y = \sigma'y^{-1} \). Then, we have \( y\sigma 0 y = \sigma'y^{-1}0 y = \sigma'00 y^{-1} y \). This contradicts the assumption that the standard form contains no potential cancellations. Similarly, in the latter case, we have \( y^{-1}\sigma = \sigma'y^{-1} \), which also contradicts the assumption.

Our goal in the rest of this section is to give the unique word for each element in \( G_0(n) \), which satisfies the following:

**Definition 3.32.** Let \( fy_{s1}^1 \cdots y_{sm}^m \) be in standard form with no potential cancellations, no potential contractions, and \( f \) is in the normal form in the sense of \( F(n) \). Then we say that \( fy_{s1}^1 \cdots y_{sm}^m \) is in the normal form.

See Section 2.1 for the definition of the normal form of elements in \( F(n) \).

We obtain a normal form from an arbitrary word on \( Z(n) \) through the following four steps:

**Step 1** Convert an arbitrary word into a standard form (Proposition 3.21);

**Step 2** Convert a standard form into a standard form that contains no potential cancellations;

**Step 3** Convert a standard form that contains no potential cancellations into a standard form \( fy_{s1}^1 \cdots y_{sm}^m \) which contains no potential cancellations and no potential contractions;

**Step 4** Convert \( f \) into \( g \), where \( g \) is the unique normal form in \( F(n) \) (Section 2.1).

The remaining steps are only 2 and 3.

**Lemma 3.33** ([14] Lemma 4.5) for \( n=2 \). Let \( (y_{s1}^1 \cdots y_{sm}^m)y_{s}^t \) be in standard form such that the following hold:

1. \( t_1, \ldots, t_m, t \in \{1, -1\} \);
2. any two \( s_i, s_j \) are independent;
3. for any \( y_{s1}^1, y_{s1}^t \) and \( y_{s}^t \) is an adjacent pair with being potential cancellation.

Then by applying ER moves and cancellation moves, we obtain a standard form \( fy_{u1}^{v_1} \cdots y_{uk}^{v_k} \) such that any two \( u_i, u_j \) are independent and \( v_1, \ldots, v_k \in \{1, -1\} \).
Proof. We consider only the case $t = 1$. We claim that $x_{0[s]}(s_i)$ is defined for every $i$. Indeed, for $s_i$ such that $s \subset s_i$, the only case where $x_{0[s]}(s_i)$ is not defined is the case $s_i = s0$. Then since the adjacent pair $(y_{s0}, y_s)$ is not potential cancellation whether $t_i$ is 1 or $-1$, this contradicts the assumption (3).

Let $s_i' = x_{0[s]}(s_i)$. By applying ER move to $y_s$, we have

$$(y_{s1}^1 \cdots y_{sn}^m)y_s = x_{0[s]}((y_{s1}^1 \cdots y_{sn}^m)y_{s0}^{-1})y_{s(n-1)}y_{s(n-1)(n-1)}.$$ We note that $s0$, $s(n-1)0$, and $s(n-1)(n-1)$ are independent of each other. For every $y_{s1}^1$, one of $y_{s0}$, $y_{s(n-1)}^{-1}$, or $y_{s(n-1)(n-1)}$ corresponds to it, either as the inverse or as an adjacent pair. Let $y_{s1}$ be the corresponding one. If it is the inverse, we apply a cancellation move. If it forms an adjacent pair, the distance in an $n$-ary tree between $s_i'$ and $σ$ is smaller than the distance between $s_i$ and $s$. Thus, by iterating this process, we obtain the desired result. □

The following lemma completes step 2.

Lemma 3.34 ([14] Lemma 4.6 for $n=2$). By applying moves, any weak standard form can be rewritten into a standard form that contains no potential cancellations.

Proof. Let $fy_{s1}^1 \cdots y_{sn}^m \ (t_i \in \{-1, 1\})$ be a weak standard form. We show by induction on $m$. If $m \leq 1$, since there exists no adjacent pair, it is clear.

We assume that $m > 1$ holds. By applying the inductive hypothesis, Lemmas 3.23 and 3.24 to $fy_{s1}^1 \cdots y_{sn-1}^m$, we obtain a weak standard form $gy_{p1}^{n1} \cdots y_{pk}^{nk}$ with at least depth $\|s_m\| + 1$ and without potential cancellations. Since $p_i \subset s_m$ does not hold by its depth, $gy_{p1}^{n1} \cdots y_{pk}^{nk}y_{sm}^m$ is in weak standard form.

If there exists an adjacent pair that is a potential cancellation, it is only of the form $(y_{p1}^{n1}, y_{sm}^m)$. We record all such $y_{p1}^{n1}$. By applying commuting moves, we obtain a weak standard form $h(y_{v1}^{k1} \cdots y_{vl}^{kl})y_{uv}^{m1} \cdots y_{uo}^{m2}y_{sm}^m$ which satisfies the following:

1. $v_1, \ldots, v_o \in \{-1, 1\}$;
2. For $j = 1, \ldots, o$, $y_{uv}^{m1}$ and $y_{sm}^m$ is an adjacent pair which is a potential cancellation;
3. All other adjacent pairs are not potential cancellations.

Indeed, since $s_m$ is the shortest word, each adjacent element is the “second shortest.” Hence we can apply commuting moves to get a word.

We again note that Lemmas 3.23 and 3.24, i.e., we can increase the depth by ER moves while preserving in weak standard form without potential cancellations. We apply ER moves to $h(y_{v1}^{k1} \cdots y_{vl}^{kl})y_{uv}^{m1} \cdots y_{uo}^{m2}y_{sm}^m$ in two phases. First, we apply ER moves to the part of $h(y_{v1}^{k1} \cdots y_{vl}^{kl})$ so that we can apply ER moves to the word on $X(n)$ that appears when we apply Lemma 3.33 to $(y_{v1}^{k1} \cdots y_{uo}^{m2})y_{sm}^m$. Secondly, we apply Lemma 3.33 to $(y_{v1}^{k1} \cdots y_{uo}^{m2})y_{sm}^m$. By the same argument in Lemma 3.23, no new potential cancellations are produced in this process.

Finally, by Lemma 3.9, we have the desired result. □
The following lemma completes step 3.

Lemma 3.35 ([14] Lemma 4.7] for $n=2$). Any standard form which contains no potential cancellations can be rewritten into a standard form that contains no potential cancellations and no potential contractions.

Proof. We apply contraction moves repeatedly. By Lemmas 3.29 and 3.31, the applied word is again in standard form and contains no potential cancellations. Since this move makes the word on $Y(n)$ in the standard form strictly shorter, the process finishes. □

The only thing that remains to be proved is the uniqueness of the normal form. We will show this by contradiction. In order to do this, we define an “invariant” of the forms.

Definition 3.36. A calculation that contains no potential cancellations has exponent $m$ if $m$ is the number of $y^{\pm 1}$ satisfying the condition that the number appearing after it is only 0, $(n-1)$, or $y^{\pm 1}$.

Example 3.37. Let $n = 4$. The string $y0y02y^{-1}(n-1)y00\cdots$ has exponent 2. The string $y10\cdots$ has exponent 0.

For the proof of uniqueness, we prepare two lemmas.

Lemma 3.38 ([15] Lemma 5.10] for $n = 2$). Let $\Lambda$ be a finite word on $\mathbb{N} \cup \{y, y^{-1}\}$ that contains no potential cancellations. Let $m$ be the exponent of $\Lambda$. Then there exists a finite word $u$ on $\{0, n-1\}$ and $v$ in $\mathbb{N}^{< \mathbb{N}}$ such that $\Lambda u$ can be rewritten into $vy^m$ by substitutions.

Proof. By substitutions, we rewrite $\Lambda$ into $v'\Lambda'$, where $v'$ is in $\mathbb{N}^{< \mathbb{N}}$ and $\Lambda'$ is in $\{0, (n-1), y, y^{-1}\}$. By the definition of exponent, $\Lambda'$ also has exponent $m$, and by Lemma 3.15 this also contains no potential cancellations. Hence, by identifying $\{0, n-1\}$ with $\{0, 1\}$, the claim comes down to the case $n = 2$. □

For $u, s$ in $\mathbb{N}^{< \mathbb{N}}$, we say that $u$ dominates $s$ if the condition $u \perp s$ or $u \supset s$ is satisfied.

Lemma 3.39 ([14] Lemma 4.8] for $n = 2$). Let $fy^t_1 \cdots y^{t_i}_{s_i}$ and $gy^p_1 \cdots y^{p_m}_{p_m}$ be in standard form which represent the same element in $G_0(n)$. Let $u$ be in $\mathbb{N}^{< \mathbb{N}}$ such that the following hold:

1. $f(u) =: u_1$ and $g(u) =: u_2$ are defined;
2. $u_1$ dominates $s_1, \ldots, s_i$;
3. $u_2$ dominates $p_1, \ldots, p_m$.

Let $\Theta$ be the calculation of $u$ and $fy^t_1 \cdots y^{t_i}_{s_i}$, and let $\Lambda$ be the calculation of $u$ and $gy^p_1 \cdots y^{p_m}_{p_m}$. Assume that these calculations contain no potential cancellations. Then two exponents of the calculations are the same.
Let \( e(\Lambda) \) and \( e(\Theta) \) be the two corresponding exponents. We show this by contradiction. We can assume that \( e(\Lambda) > e(\Theta) \) without loss of generality. Since exponents are non-negative integer, let \( k := e(\Lambda) > 0 \). By Lemma 3.38, we have \( wy^k \) obtained from \( \Lambda v \) by substitutions, where \( v \in N^{<N} \). Since \( fy_{s_1} \cdots y_{s_l} \) and \( gy_{p_1} \cdots y_{p_m} \) are equal as elements of \( G_0(n) \), the output strings of \( \Theta v0^k(n-1)0^k(n-1)\cdots \) and \( \Lambda v0^k(n-1)0^k(n-1)\cdots \) are equal as elements of \( N^N \). Then the latter is \( \Lambda v0^k(n-1)0^k(n-1)\cdots = wy^k0^k(n-1)0^k(n-1)\cdots = w0(n-1)^2k0(n-1)^2k\cdots \).

Since \( e(\Lambda) > e(\Theta) \) holds, by the definition of \( y \), this implies a contradiction. \( \square \)

**Corollary 3.40.** The group obtained from the presentation \( (Z(n) \mid R(n)) \) is the \( n \)-adic Lodha–Moore group \( G_0(n) \).

**Proof.** We show that any standard form which contains no potential cancellations and represents an element of \( F(n) \) is always a word on \( X(n) \). Let \( fy_{s_1} \cdots y_{s_m} = g \) be standard forms, where \( f, g \in F(n) \), and \( m \geq 1 \). Assume that they contain no potential cancellations. Let \( u = s_100\cdots \) in \( N^N \). Then the exponent of the calculation of \( f^{-1}(u) \) and \( fy_{s_1} \cdots y_{s_m} \) is strictly greater than 0, and that of the calculation of \( f^{-1}(u) \) and \( g \) is 0. By Lemma 3.39, this is a contradiction.

In particular, for a word on \( Z(n) \) representing the identity element of \( G_0(n) \), we can rewrite it into a standard form that contains no potential cancellations by Proposition 3.21 and Lemma 3.34. Since this standard form is a word on \( X(n) \), as shown above, we can reduce it to the empty word by using the relations of \( F(n) \). \( \square \)

**Remark 3.41.** From the above argument, the uniqueness of the normal form of the elements of \( F(n) \) in \( G_0(n) \) follows.

The following completes the proof of the uniqueness.

**Theorem 3.42** ([14 Theorem 4.4] for \( n = 2 \)). For each element in \( G_0(n) \), its normal form is unique.

**Proof.** By Remark 3.3, we can assume that every normal form in the following argument is not in \( F(n) \).

We show this by contradiction. Let \( fy_{s_1} \cdots y_{s_l} \) and \( gy_{p_1} \cdots y_{p_m} \) be different normal forms representing the same element. We can assume that \( s_l \leq p_m \) without loss of generality. One of the following three holds:

1. \( s_l = p_m \) and \( t_l = q_m \);
(2) \( s_t = p_m \) and \( t_t \neq q_m \);

(3) \( s_t < p_m \).

First, we show that it is sufficient to consider only case (3).

In case (1), since \( y_{s_t}^{t_t} = y_{p_m}^{q_m} \), we start from \( fy_{s_t}^{t_t} \cdots y_{s_{t-1}}^{t_{t-1}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-1}}^{q_{m-1}} \). They are two different standard forms representing the same element. They contain no potential cancellations but possibly contain a potential contraction. Since we only cancel \( y_{s_t}^{t_t} = y_{p_m}^{q_m} \), the only case that contains a potential cancellation at this step is when \( y_{s_t}^{t_t} \) or \( y_{p_m}^{q_m} \) or both play the role of \( y_s^{±1} \) for some \( s \) (see Definition 3.26 (1)). By the assumption of the standard form, the case with \( y_s^{±1} \) (Definition 3.26 (2)) does not occur. We further divide case (1) into the following four subcases:

(1-1) Both \( fy_{s_t}^{t_t} \cdots y_{s_{t-1}}^{t_{t-1}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-1}}^{q_{m-1}} \) contain a potential contraction, and \( t_{t-1} = q_{m-1} \);

(1-2) Both \( fy_{s_t}^{t_t} \cdots y_{s_{t-1}}^{t_{t-1}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-1}}^{q_{m-1}} \) contain a potential contraction, and \( t_{t-1} \neq q_{m-1} \);

(1-3) \( fy_{s_t}^{t_t} \cdots y_{s_{t-1}}^{t_{t-1}} \) contains a potential contraction, and \( gy_{p_t}^{q_t} \cdots y_{p_{m-1}}^{q_{m-1}} \) contains no potential contractions, or vice versa;

(1-4) \( fy_{s_t}^{t_t} \cdots y_{s_{t-1}}^{t_{t-1}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-1}}^{q_{m-1}} \) contain no potential contractions.

In case (1-1), we consider \( fy_{s_t}^{t_t} \cdots y_{s_{t-2}}^{t_{t-2}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-2}}^{q_{m-2}} \) instead of the original words in order to eliminate the potential contraction part. Since the assumption of containing a potential contraction and being in standard form, \( s_t = p_m = s(n-1), s_{t-1} = p_{m-1} = s(n-1)(n-1) \) and \( t_{t-1} = q_{m-1} > 0 \) hold. Then \( fy_{s_t}^{t_t} \cdots y_{s_{t-2}}^{t_{t-2}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-2}}^{q_{m-2}} \) are different normal forms. Indeed, the only case where a potential contraction is generated by canceling \( y_{s_{t-1}}^{t_{t-1}} = y_{p_{m-1}}^{q_{m-1}} \) is when \( y_{s_{t-1}}^{t_{t-1}} = y_{p_{m-1}}^{q_{m-1}} \) plays the role of \( y_{s'}^{±1}(n-1) \) for \( s' = s(n-1) \), but this does not occur since \( y_{s_t}^{k_1} = y_{s(n-1)0}^{k_2} \) and \( y_{p_t}^{k_3} = y_{s(n-1)0}^{k_4} \) exist in two forms respectively and \( k_1, k_2 < 0 \) holds by the assumption of potential contractions. Thus, we get “shorter” normal forms that satisfy all the assumptions.

In case (1-2), we can assume that \( 0 < t_{t-1} < q_{m-1} \) without loss of generality. We consider \( fy_{s_t}^{t_t} \cdots y_{s_{t-2}}^{t_{t-2}} \) and \( gy_{p_t}^{q_t} \cdots y_{p_{m-2}}^{q_{m-2}} \) instead. For the same reason as for case (1-1), the former is in normal form. By the assumption of a potential contraction, \( y_{s_{t-2}}^{t_{t-2}} = y_{s(n-1)0}^{q_{m-2}} \) where \( i \) is in \( N \) and \( \sigma \) is a word in \( N^<N \cup \{ \epsilon \} \). Note that \( s(n-1)0 \leq s(n-1)0 \) holds. For the latter, by performing contraction moves as in step 3, the last letter is \( y_u^k \) where \( u \subseteq s \). This is the situation in case (3).

In case (1-3), we only consider the former situation. By the assumptions, \( s_t = p_m = s(n-1) \) holds for some \( s \). By applying contraction moves to \( fy_{s_t}^{t_t} \cdots y_{s_{t-1}}^{t_{t-1}} \) as in step 3, the last letter is \( y_u^k \) where \( u \subseteq s \subseteq s_t = p_m \). Since \( gy_{p_t}^{q_t} \cdots y_{p_{m-1}}^{q_{m-1}} \) is in normal form, and in particular in standard form, we have \( y_{p_{m-1}} \neq y_{s_t} \). This is the situation in case (3).

In case (1-4), both are “shorter” normal forms that satisfy all the assumptions. Therefore, any subcase of case (1) yields shorter words or in case (3).
In case (2), we can assume that neither $0 < q_m < t_l$ nor $t_l < q_m < 0$ without loss of generality. Consider $fy_{s_1}^{t_l} \cdots y_{s_{l-1}}^{t_{l-1}}$ and $gy_{p_1}^{q_1} \cdots y_{p_m}^{q_m-t_l}$ instead. The latter is in normal form, but as in case (1), the former may contain a potential contraction. Similarly, we divide case (2) into two subcases:

(2-1) $fy_{s_1}^{t_l} \cdots y_{s_{l-1}}^{t_{l-1}}$ contains a potential contraction;
(2-2) $fy_{s_1}^{t_l} \cdots y_{s_{l-1}}^{t_{l-1}}$ contains no potential contractions.

In case (2-1), the same argument as in (1-3) can be applied. In case (2-2), since $s_{l-1} \neq p_m$, this is the situation in case (3).

We note that case (1-1) or (1-4) happens only finitely many times due to the uniqueness of the normal form of $F(n)$. Thus, we consider case (3).

We only consider the case $q_m > 0$. We note that $fy_{s_1}^{t_l} \cdots y_{s_l} y_{p_m}^{-1}$ is in standard form since $s_l < p_m$. One of the following holds:

(i) there exists an infinite word $\sigma$ on $\{0, n-1\}$ such that for any finite word $\sigma_1 \subset \sigma$, $p_m \sigma_1$ is not in $\{s_1, \ldots, s_l\}$;
(ii) there exists an adjacent pair of the form $p_m$, $s_i$ where $p_m u = s_i$ for $u$ on $\{0, n-1\}$ which is not a potential cancellation.

Indeed, if both are false, it contradicts that $fy_{s_1}^{t_l} \cdots y_{s_l} y_{p_m}^{-1}$ contains no potential contractions. Then, in either case, there exists a finite word $w$ on $\{0, n-1\}$ such that the calculation $\Lambda$ of $fy_{s_1}^{t_l} \cdots y_{s_l} y_{p_m}^{-1}$ and $f^{-1}(p_m w)$ contains no potential cancellations. By expanding $w$ if necessary, we can consider the following three calculations:

(a) $\Theta$ is the calculation of $gy_{p_1}^{q_1} \cdots y_{p_m}^{q_m-t_l}$ and $f^{-1}(p_m w)$;
(b) $\Lambda'$ is the calculation of $fy_{s_1}^{t_l} \cdots y_{s_l} y_{p_m}^{-1}$ and $f^{-1}(p_m w)$;
(c) $\Theta'$ is the calculation of $gy_{p_1}^{q_1} \cdots y_{p_m}^{q_m}$ and $f^{-1}(p_m w)$.

All calculations contain no potential cancellations. Indeed, for $\Lambda'$ and $\Theta'$, it is clear from the assumption of normal form. For $\Theta$, either $q_m - 1$ is zero or strictly greater than zero since $q_m > 0$. In both cases, it is clear from the definitions and the assumption that $gy_{p_1}^{q_1} \cdots y_{p_m}^{q_m}$ contains no potential cancellations.

Let $e(\Theta)$, $e(\Theta')$, $e(\Lambda)$, and $e(\Lambda')$ be the its exponent, respectively. Since we have

$$fy_{s_1}^{t_l} \cdots y_{s_l} y_{p_m}^{-1} = gy_{p_1}^{q_1} \cdots y_{p_m}^{q_m-t_l}$$

as elements of $G_0(n)$, by Lemma 3.39, $e(\Lambda) = e(\Theta)$ holds. Similarly, we have $e(\Theta') = e(\Lambda')$.

By the construction of $\Lambda$ and $\Lambda'$, we have $e(\Lambda) > e(\Lambda')$. Similarly, we have $e(\Theta) \leq e(\Theta')$ since $q_m > 0$. Combining them, we obtain

$$e(\Theta) \leq e(\Theta') = e(\Lambda') < e(\Lambda),$$

which is a contradiction.

If $q_m < 0$, we can prove the same for $fy_{s_1}^{t_l} \cdots y_{s_l} y_{p_m}$ and $gy_{p_1}^{q_1} \cdots y_{p_m}^{q_m+1}$. \(\square\)
We show that \( G_0(n) \) has “expected” properties.

4. SEVERAL PROPERTIES OF \( G_0(n) \)

4.1. The finite presentation of \( G_0(n) \). Using the infinite presentation given in Section 3.3, we construct a finite presentation of \( G_0(n) \) with respect to the generating set \( \{x_0, \ldots, x_{n-2}, x_{0[n-1]}, y_{(n-1)0}\} \). As in [15, Theorem 3.3], we use two properties of \( F(n) \). Since \( G_0 = G_0(2) \) is already known to be finitely presented, we assume that \( n > 2 \).

For \( s \) in \( N^{\kappa N} \) that is neither 0\( \cdots \cdot 0 \) nor \((n-1)\cdots(n-1)\), we define a map \( f_s \) in \( F(n) \) as follows: Let \( a \) be the sum of each number in \( s \), and let \( d, k \) be the integers such that \( a = (n-1)d+k \) holds, where \( 0 \leq d \) and \( 0 \leq k < n-1 \). Then \( f_s \) is defined as in Figure 15. We fix a word on \( \{x_0, \ldots, x_{n-2}, x_{0[n-1]}\} \) representing \( f_s \). Since the leaf corresponding to \( s \) in the minimal \( n \)-ary tree containing \( s \) is the \( a \)-th leaf counting from the left (from 0), we note that \( f_s((n-1)k) = s \) holds.

**Lemma 4.1.** For the above \( f_s \), we have \( x_{i[(n-1)k]}f_s = f_sx_{i[s]} \) and \( y_{(n-1)0}f_s = f_sy_s \).

**Sketch of proof.** We note that we have \( f_s^{-1}x_{i[(n-1)k]}f_s = x_{i[f_s(n-1)k]} = x_{i[s]} \). Indeed, if \( w = sw' \in N^{N} \), the left hand side sends \( w \) to \( sx_i(w') \) via \((n-1)kw' \) and \((n-1)kx_i(w') \). If \( w \neq sw' \), since \( f_s^{-1}(w) \) does not contain \((n-1)k \) as a prefix, \( x_{i[(n-1)k]}(f_s^{-1}(w)) = f_s^{-1}(w) \).

Another statement can be shown in the same way. \( \square \)

**Lemma 4.2.** Let \( u, v \) be in \( N^{\kappa N} \) such that \( u < v \), \( u \perp v \), and \( y_u, y_v \) are in \( Y(n) \). Then there exists \( g \) in \( F(n) \) such that \( g(0(n-1)) = u \) and \( g((n-1)0) = v \) hold.

**Sketch of proof.** Like the construction of \( f_s \), by adding some carets, we can obtain two trees (may no be tree diagram) such that \( 0(n-1) \to u \). Since \( n > 2 \), we can add some carets between two leaves corresponding to \( 0(n-1) \) and \((n-1)0 \) so that \((n-1)0 \to v \). Finally, we add some carets to the rightmost if necessary, to make it a tree diagram. See Figure 16 for the sketch of the construction of \( g \). \( \square \)

**Theorem 4.3** ([15, Theorem 3.3] for \( n = 2 \)). \( G_0(n) \) admits a finite presentation.
Proof. Since \( F(n) \) is finitely presented \cite[Theorem 4.17]{1}, the relation (1) in \( R(n) \) is expressed by finite relations. Thus we only consider the other three relations (2), (3), and (4).

For the relation \( y_i x_i[s] = x_i[s] y_{x_i[a]}(t) \), we can rewrite as follows:

\[
y_i x_i[s] = f_i^{-1} y_{(n-1)0} f_i x_i[s],
\]

\[
x_i[s] y_{x_i[a]}(t) = x_i[s] f_i^{-1} y_{(n-1)0} f_i x_i[a](t).
\]

Thus, it is sufficient to show that

\[
y_{(n-1)0} f_i x_i[s] f_i^{-1} x_i[a](t) = f_i x_i[s] f_i^{-1} x_i[a](t) y_{(n-1)0},
\]

(4.1)

by using a finite number of relations. We note that \( f_i x_i[s] f_i^{-1} x_i[a](t) \) maps \((n-1)0\) to itself. Therefore, \( f_i x_i[s] f_i^{-1} x_i[a](t) \in F(n) \) is rewritten into a word on the finite set

\[
X_{(n-1)0}(n) := \{ x_0 x_i^{-1}, x_i([n-1]) \mid i = 1, \ldots, n-2 \}
\]

\[
\cup \{ x_i[0], x_i[1], \ldots, x_i[n-2], x_i([n-1]) \mid i = 0, \ldots, n-2 \}
\]

\[
\cup \{ x_0[0(n-1)], \ldots, x_0[(n-2)(n-1)], x_0([n-1]1(n-1)), \ldots, x_0([n-1](n-1)(n-1)) \},
\]

by finite relations of \( F(n) \). Since each element is commute with \( y_{(n-1)0} \), the collection of the finite relations \( h y_{(n-1)0} = y_{(n-1)0} h \) where \( h \) is in \( X_{(n-1)0}(n) \) implies equation (4.1).

For the relation \( y_s y_t = y_t y_s \) where \( s \perp t \), we note that we have \( y_{(n-1)0} y_{0(n-1)} = y_{0(n-1)} y_{(n-1)0} \). By using this one and relation (2), we have

\[
y_s y_t = f_s^{-1} y_{(n-1)0} f_s f_t^{-1} y_{(n-1)0} f_t = g^{-1} y_{0(n-1)} g g^{-1} y_{(n-1)0} g = g^{-1} y_{(n-1)0} g g^{-1} y_{(n-1)0} g = y_t y_s,
\]

where \( g \) is the element such that \( g(0(n-1)) = s \) and \( g((n-1)0) = t \).

Finally, for the relation \( y_s = x_0[s] y_{s0} y_{s(n-1)0} y_{s(n-1)(n-1)} \), we note that we have \( y_{(n-1)0} = x_0[(n-1)0] y_{(n-1)00} y_{(n-1)(n-1)0} y_{(n-1)0(n-1)(n-1)} \). Since \( f_s((n-1)0w) = sw \) for \( w \in N^{<N} \), by

\[
\text{Figure 16. The construction of } g \text{ in Lemma 4.2}
\]
using relation (2), we have

\[
y_s = f_s^{-1}y_{(n-1)0} f_s
\]

\[
= f_s^{-1}x_0[(n-1)0] y_{(n-1)0} y_{(n-1)0(n-1)0} y_{(n-1)0(n-1)(n-1)} f_s
\]

\[
= f_s^{-1}x_0[(n-1)0] f_s f_s^{-1}y_{(n-1)0} f_s f_s^{-1}y_{(n-1)0(n-1)0} f_s f_s^{-1}y_{(n-1)0(n-1)(n-1)} f_s
\]

\[
= x_0[s] y_{0}s_{(n-1)0} y_{s(n-1)(n-1)}.
\]

This completes the proof. \(\Box\)

4.2. **Nonamenability of** \(G_0(n)\). **In this section, we discuss embeddings of the groups** \(G_0(n)\). **For nonamenability, we only use the fact that a subgroup of an amenable group is also amenable \(^{11}\) Theorem 18.29 (1).** **The idea for the following proposition comes from** \(^6\).

**Theorem 4.4.** Let \(p, q \geq 2\) and assume that there exists \(d \in \mathbb{N}\) such that \(q-1 = d(p-1)\) holds. Then there exists an embedding \(I_{p,q} : G_0(p) \to G_0(q)\). Moreover, the equality \(I_{p,r} = I_{p,q}I_{q,r}\) holds for the maps \(I_{p,q}\), \(I_{q,r}\) and \(I_{p,r}\) defined for \(r \geq q \geq p \geq 2\) such that any two satisfy the condition.

**Proof.** For the sake of clarity, we label elements of \(G_0(p)\) with “tildes” and elements of \(G_0(q)\) with “hats.” We first recall the definition of the embedding \(F(p) \to F(q)\) given in \(^6\) Section 3, example (3)]. This homomorphism is defined by

\[
\tilde{x}_0 \mapsto \hat{x}_0 \quad \tilde{x}_1 \mapsto \hat{x}_d \quad \cdots \quad \tilde{x}_{p-2} \mapsto \hat{x}_{d(p-2)} \quad \tilde{x}_{0[(p-1)]} \mapsto \hat{x}_{0[(q-1)]},
\]

and extended to the (quasi-isometric) embedding \(^6\) Theorem 6]. By considering \(F(p)\) and \(F(q)\) as pairs of \(p\)-ary trees and \(q\)-ary trees, this embedding is regarded as a “caret replacement.” Indeed, for every pair of \(p\)-ary trees, inserting \(d-1\) edges between every pair of adjacent edges of each \(p\)-caret corresponds to the embedding. See Figure 17 for example.

Define the map \(i_{p,q} : P^{<N} = \{0, \ldots, p-1\}^{<N} \to Q^{<N} = \{0, \ldots, q-1\}^{N}\) by setting

\[
w_1w_2 \cdots w_k \mapsto (dw_1)(dw_2) \cdots (dw_k),
\]

namely, we multiply each number of a given word by \(d\). In addition, define the map \(Y(p) \to Y(q)\) by setting \(\tilde{y}_s \mapsto \hat{y}_{i_{p,q}(s)}\). By the definition of \(i_{p,q}\), we have that \(\hat{y}_{i_{p,q}(s)}\) is in \(Y(q)\).
By combining the two maps $F(p) \to F(q)$ and $Y(p) \to Y(q)$, we obtain the map
$I_{p,q} : F(p) \cup Y(p) \to F(q) \cup Y(q)$. This map can be extended to the homomorphism
$I_{p,q} : G_0(p) \to G_0(q)$. Indeed, since $i_{p,q} \circ (x) = y_{p,q} \circ (x)$ and $I_{p,q}(y) = y_{p,q}(y)$
hold, we can verify that the relations of the infinite presentation (Corollary 3.40) are
preserved under $I_{p,q}$ by directly calculations.

We claim that $I_{p,q} : G_0(p) \to G_0(q)$ is injective. We show this by contradiction. Assume
that $\ker(I_{p,q})$ is not trivial. By the construction, the restriction $I_{p,q}|_{F(p)}$ coincides with
the embedding $F(p) \to F(q)$ mentioned above. Hence if there exists $x$ in $\ker(I_{p,q})$ that is
not identity, then $x$ is not in $F(p)$. We note that the map $I_{p,q}$ preserves being in normal
form. In particular, if $x$ is not in $F(p)$, then $I_{p,q}(x)$ is not in $F(q)$ by the uniqueness
(Theorem 3.42). This implies that $I_{p,q}(x)$ is not identity, as desired.

Let $d_1$, $d_2$ and $d_3$ be natural numbers such that $q - 1 = d_1(p - 1)$, $r - 1 = d_2(q - 1)$
and $r - 1 = d_3(p - 1)$ holds respectively. We note that $d_3 = d_1d_2$ holds. By considering
the definitions of the map $i_{p,q}$ and the homomorphism from $F(p)$ to $F(q)$, the equality of
$I_{p,r}$ and $I_{p,q}I_{q,r}$ follow immediately.

\section*{Corollary 4.5} Let $n \geq 2$. Then $G_0(n)$ is nonamenable.

\begin{proof}
Since we have $n - 1 = (n - 1)(2 - 1)$ for every $n$, by Theorem 4.4, we have
an embedding from $G_0(2) = G_0$ into $G_0(n)$. This implies that $G_0(n)$ has a subgroup
that is isomorphic to $G_0$. Since $G_0$ is nonamenable [15, Theorem 1.1], $G_0(n)$ is also
nonamenable.
\end{proof}

\section*{Corollary 4.6} Let $s_i := 2^{i-1} + 1$. Then the sequence $G(s_1), G(s_2), \ldots$ forms an
inductive system of groups.

\section*{4.3. $G_0(n)$ has no free subgroups.} We assume $n \geq 3$. In order to show that $G_0(n)$
does not contain the free groups, we will follow [2, Section 3]. The difference in this paper
is that the domain and range set of the maps are $\mathbb{N}^\mathbb{N}$ instead of $\mathbb{R}$.

For an element $g$ in $G_0(n)$, we write the set-theoretic support $\text{supp}(g)$ for the set
$\{x \in \mathbb{N}^\mathbb{N} \mid g(x) \neq 0\}$. Although we equip $\mathbb{N}^\mathbb{N}$ with the topology that is homomorphic
to Cantor set, which is totally disconnected, we can consider “connected components” of
$\text{supp}(g)$, using the total order of $\mathbb{N}^\mathbb{N}$.

For $a, b \in \mathbb{N}^\mathbb{N}$ with $a < b$, set $(a, b) := \{x \in \mathbb{N}^\mathbb{N} \mid a < x < b\}$.

\section*{Lemma 4.7} For $g \in G_0(n)$, there exists a sequence $a_1 < a_2 < a_3 < a_4 \leq \cdots < a_{2m}$ in
$\mathbb{N}^\mathbb{N}$ such that we have

$$\text{supp}(g) = (a_1, a_2) \sqcup (a_3, a_4) \sqcup \cdots \sqcup (a_{2m-1}, a_{2m}).$$

We call each $(a_{2i}, a_{2i+1})$ a connected component of $\text{supp}(g)$.
Indeed, if $g \in N^N$ construct an element $w$ that
we restrict the domain set of $g$. Assume that
$N$ in the subgroup that is isomorphic to $G$. We recall that
Let $\eta_j \in N^N$ such that each $\eta_j \in N^N$ hold, we can assume that each $a_i$ and $b_i$ is in some set
$\{\eta_j \mid \eta_j \in N^N\}$ without loss of generality. Since $a_i < b_i < a_{i+1}$ holds, we can write each
$a_i$ and $b_i$ as $n_i a_i'$ and $n_i b_i'$, respectively.

We claim that each $a_i'$ and $b_i'$ can be replaced by words in $\{0, n-1\}^N$, respectively. Indeed, if $a_i'$ is not in $\{0, n-1\}^N$, then there exist $\hat{a}_i \in \{0, n-1\}^N \cup \{1\}$, $k \in \{1, \ldots, n-2\}$, and $a_i'' \in N^N$ such that $a_i' = \hat{a}_i k a_i''$ holds. Then, by the definition of $g$, $g(n_j \hat{a}_i k)$ is defined and equals to $n_j \hat{a}_i k a_i''$ is a fixed point of $g$. Moreover, for any other $k' \in \{1, \ldots, n-2\}$, we have $g(n_j \hat{a}_i k') = n_j \hat{a}_i k'$. This implies that we have
g(n_j \hat{a}_i 0(n-1)) = n_j \hat{a}_i 0(n-1), where $(n-1)$ denotes the element $(n-1)(n-1)(n-1) \cdots$
in $N^N$. By a similar argument for $b_i'$, we have $g(n_j \tilde{b}_i k) = n_j \tilde{b}_i k$, where $\tilde{b}_i \neq \hat{b}_i$. Since we have $g(n_j \tilde{b}_i k') \neq n_j \tilde{b}_i k'$ for any other $k'$, $n_j \tilde{b}_i 0(n-1)$ is in supp($g$). Even if each $a_i$ or $b_i$ is replaced by the above one, the order $a_i < b_i < \cdots$ is also preserved.

Let $w_1, \ldots, w_t$ be in $\{0, \ldots, n-1\}$ such that $f(n_j) = w_1 \cdots w_t$ holds. For $n_j \zeta \in N^N$ and $f y_{t_1}^{s_1} \cdots y_{t_m}^{s_m}$, let $w_1 y_{t_1}^{s_1} y_{t_2}^{s_2} \cdots y_{t_m}^{s_m} w_t y_{t_1}^{s_1}$ be their calculation. Then some $l_q$ may be zero. Assume that $f(n_j) = w_1 \cdots w_t$ is not in $\{0, n-1\}^N$ and let $z$ be the maximal number such that $w_z$ is in $\{1, \ldots, n-2\}$. By applying finitely many substitutions to all $y_e z'$ ($z' < z$), we obtain a word $w y_{t_1}^{s_1} w_{t_2}^{s_2} y_{t_3}^{s_3} \cdots w_{t_r}^{s_r} y_{t_1}^{s_1} \zeta$, where $w_{t_j} \in \{0, n-1\}$. Since $n_j a_i$ is the output string of $w y_{t_1}^{s_1} w_{t_2}^{s_2} \cdots w_{t_r}^{s_r} y_{t_1}^{s_1} a_i$, we have $w \leq n_j$. Indeed, it is clear that $w \perp n_j$ does not hold, and if $w > n_j$ holds, then it contradicts the facts that $a_i$ is in $\{0, n-1\}^N$, and the last number of $w$ is $w_j$. Let $n_j = w n_j'$. Then we note that $n_j'$ is in $\{0, n-1\}^N \cup \{\epsilon\}$ since $y_{t_1}^{s_1} w_{t_2}^{s_2} y_{t_3}^{s_3} \cdots w_{t_r}^{s_r} y_{t_1}^{s_1} a_i$ is in $\{0, n-1\}^N$ and its output string equals to $n_j' a_i$.

We recall that $G_0(n)$ has a subgroup that is isomorphic to $G_0$ (Theorem 4.4). We construct an element $g'$ which is in the subgroup and satisfies the assumption of $g$. Let $g'$ be the element represented by

$$f' y_{t_0}^{t_0'} y_{t_1}^{t_1'} y_{t_2}^{t_2'} \cdots y_{t_r-1}^{t_r-1'} y_{t_r}^{t_r'} w_{t_0}^{t_0'} w_{t_1}^{t_1'} w_{t_2}^{t_2'} \cdots w_{t_r-1}^{t_r-1'} w_{t_r}^{t_r'},$$

where $f'$ is a word on $\{x_0, x_0'(n-1)\}$ satisfying $f'((n-1)0 n_j') = (n-1)0 w_2' \cdots w_{t_r}'$. Since $n_j'$ and $w_2' \cdots w_{t_r}'$ are in $\{0, n-1\}^N$, there must exist such an element $f'$. By the construction, $g'$ is in the subgroup that is isomorphic to $G_0$. In addition, each $(n-1)0 n_j' a_i$ is a fixed point of $g'$, and $(n-1)0 n_j' b_i$ is in supp($g'$). By the construction of the map $J_{2,n}$ in Theorem 4.4 if we restrict the domain set of $g'$ to $\{0, n-1\}^N$ and identify $n-1$ with 1, then we obtain
the element of $G_0$ which satisfies all the assumptions about fixed points and elements of support. However, since this does not happen by Proposition 3.3, this is a contradiction.

Finally, assume that $f(n_j) = w_1 \cdots w_t$ is in $\{0, n - 1\}^N$. Since $w_1 y_1^t \cdots w_t y_1^t a_i$ is in $\{0, n - 1\}^N$, $n_j$ is also in $\{0, n - 1\}^N$. Therefore, $f$ is represented by a word on $\{x_0, x_0[(n-1)]\}$. We again note that some $l_q$ may be zero. Then $g' = f y_{w_1}^t \cdots w_t y_{w_1}^t$ is in $G_0(n)$, and an element of $G_0$ can be similarly constructed from $g'$. □

**Remark 4.8.** By the construction, the sequence $a_1, \ldots, a_{2m}$ in Lemma 4.7 is uniquely determined.

For a group $G$, $[G, G]$ denotes its commutator subgroup. For two elements of $x, y \in G$, $[x, y]$ denotes the commutator $xyx^{-1}y^{-1}$.

**Theorem 4.9.** The group $G_0(n)$ has no free subgroups.

**Proof.** Similarly to [3], we will show that if $G_1$ is a subgroup of $[G_0(n), G_0(n)]$, then either $\mathbb{Z}^2$ is a copy of a subgroup of $G_1$ or $[G_1, G_1]$ is trivial. Indeed, in that case, if $G$ is a subgroup of $G_0(n)$, then either $\mathbb{Z}^2$ is a copy of a subgroup of $G$ or $[G, G]$ is abelian, and the free group $F_2$ does not have such a property.

Let $G_1$ be a subgroup of $[G_0(n), G_0(n)]$, and assume that $[G_1, G_1]$ is not trivial. Then there exist elements $f$ and $g$ in $G_1$ such that $z = fgf^{-1}g^{-1} \neq Id$. For $f$ and $g$, let $a_1, \ldots, a_{2s}$ and $b_1, \ldots, b_{2t}$ be sequences given in Lemma 4.7 respectively. By choosing elements appropriately from $a_1, \ldots, a_{2s}, b_1, \ldots, b_{2t}$, we obtain $c_1, \ldots, c_{2m}$ such that

$$\text{supp}(f) \cup \text{supp}(g) = (c_1, c_2) \sqcup (c_3, c_4) \sqcup \cdots \sqcup (c_{2m-1}, c_{2m}).$$

Let $d_1, \ldots, d_{2t}$ be a sequence given by Lemma 4.7 for $[f, g]$. Then we have that for each $x \in \text{supp}([f, g])$, there exist $i$ and $i'$ such that

$$c_2i - 1 < d_{2i-1} < x < d_{2i} < c_{2i'}$$

(4.2) holds. Indeed, it is sufficient to show that for each common fixed point $x$ of $f$ and $g$, there exists a prefix $u$ such that $[f, g]$ fixes all the element of $\{u\xi \mid \xi \in N\}$. If $x$ is not in $\{0, n - 1\}^N$, then $y$ “vanishes.” Thus by regarding $f$ and $g$ as piecewise linear map of $F(n)$, it is clear. If $x$ is in $\{0, n - 1\}^N$, by identifying $\{0, n - 1\}$ with $\{0, 1\}$, it is also clear by Proposition 3.3.

We write $\text{supp}([f, g]) \subseteq \text{supp}(f) \cup \text{supp}(g)$ for the “proper” inclusion of each connected component of the support described above. Let

$$W := \{ h \in \langle f, g \rangle \mid \text{supp}(h) \subseteq \text{supp}(f) \cup \text{supp}(g), h \neq Id \},$$

where $\langle f, g \rangle$ is the subgroup generated by $f$ and $g$. The set $W$ is not empty since $z$ is in $W$.

Define the function $\kappa : W \to \mathbb{Z}_{\geq 0}$ by setting

$$\kappa(h) = \# \{ i \in \{1, 3, \ldots, 2m - 1\} \mid \text{supp}(h) \cap (c_i, c_{i+1}) \neq \emptyset \}.$$
Here $\#$ denotes the cardinality of the set. Let $z'$ be a minimizer of $\kappa$ and let $e_1, \ldots, e_{2p}$ denote a sequence given by Lemma 4.7 for $z'$. Since $z'$ is not identity map, there exist $i$ and $i'$ such that $c_{2i'-1} < e_{2i-1} < e_{2i} < c_{2i'}$ holds. We note in the following that each element of $G_0(n)$ preserves the order of $\mathbb{N}^{\mathbb{N}}$. Since there may exist more than one such $i$, we denote the smallest one by $i_-$ and the largest one by $i_+$. Then there exists $w \in \langle f, g \rangle$ such that $w(e_{2i_-}) > e_{2i_+}$ holds. By construction of $w$, we have

$$\text{supp}(z') \cap \text{supp}(w^{-1}z'w) \cap (c_{2i'-1}, c_{2i'}) = \emptyset.$$ 

This implies that we have

$$\text{supp}([z', w^{-1}z'w]) \cap (c_{2i'-1}, c_{2i'}) = \emptyset.$$ 

Indeed, if $x$ is in $\text{supp}(z')$, then $z'(x)$ is in $\text{supp}(z')$, thus $z'(x)$ is not in $\text{supp}(w^{-1}z'w)$ and we have $[z', w^{-1}z'w](x) = x$. If $x$ is in $\text{supp}(w^{-1}z'w)$, then the image of $x$ by $w^{-1}z'w$ is also in $\text{supp}(w^{-1}z'w)$. Since $z'(x) = x$ holds, we have $[z', w^{-1}z'w](x) = x$.

We note that

$$\text{supp}([z', w^{-1}z'w]) \subseteq \text{supp}(z') \cup \text{supp}(w^{-1}z'w) \subseteq \text{supp}(f) \cup \text{supp}(g).$$

However, $[z', w^{-1}z'w]$ is not in $W$. Indeed, if $\text{supp}(z') \cap (c_{2j-1}, c_{2j}) = \emptyset$ for any other $j$, then we have

$$\text{supp}([z', w^{-1}z'w]) \cap (c_{2j-1}, c_{2j}) = \emptyset,$$

since if $x$ is in $(c_{2j-1}, c_{2j})$ then $w(x)$ is also in $(c_{2j-1}, c_{2j})$. Thus if $[z', w^{-1}z'w]$ is in $W$, it contradicts the minimality of $\kappa(z')$. This implies that we have

$$z'(w^{-1}z'w) = (w^{-1}z'w)z'.$$

Therefore, these two maps $z'$ and $(w^{-1}z'w)$ generate the group which is isomorphic to $\mathbb{Z}^2$. We have the desired result. \qed

**Remark 4.10.** From this proof, we can see that $G_0(n)$ is torsion free. Indeed, let $g$ ($g \neq \text{Id}$) be in $G_0(n)$ and assume that $g^m = \text{Id}$ holds. Since $g$ preserves the order of $\mathbb{N}^{\mathbb{N}}$, for $x$ in $\text{supp}(g)$, we have either

$$x < g(x) < g^2(x) < \cdots < g^m(x) = x$$

or

$$x > g(x) > g^2(x) < \cdots < g^m(x) = x.$$ 

This is a contradiction.

**Remark 4.11.** If there exists an embedding from $G_0(n)$ into either $G_0$ or the Monod group $H$ [17], the theorem follows immediately because $G_0$ and $H$ contain no free subgroups.
4.4. The abelianization of $G_0(n)$ and simplicity of the commutator subgroup.

The idea for the theorems in this section comes from [4] and [7].

We note that the group $F(n)$ is called $F_{n,1}$ in [4, section 4D]. Thus there exists a surjective homomorphism $\phi : F(n) \to \mathbb{Z}^n$. We briefly recall its definition to compute the abelianization of $G_0(n)$.

Let $A_{n+1}$ be the free abelian group generated by $e_-, e_+$, and $e_i \ (i \in \mathbb{Z})$ and satisfying the relations $e_i = e_j$ if $i \equiv j \pmod{(n-1)}$. Then the rank of $A_{n+1}$ is $n + 1$. From an $n$-ary tree $Y$ and an $n$-ary tree $Z$ which contains $Y$ as a rooted subtree, we define the element $\delta(Z,Y)$ in $A_{n+1}$ as follows:

(1) Label the leftmost leaf of $Y$ as $-$, the rightmost as $+$, and the other leaves as $1, 2, \ldots$ from left to right.

(2) To construct $Z$ from $Y$, add a caret to some leaf of $Y$. Then record the label of the leaf.

(3) Regard the obtained tree as $Y$ again.

(4) Repeat the process (1) to (3) until the tree $Z$ is obtained.

(5) Add up all the $e_i$ that have the recorded labels as indices.

Since $e_i = e_j$ if $i \equiv j \pmod{(n-1)}$, we can add carets in any order in process (2).

For example, for the trees $Y$ and $Z$ in Figure 18 by adding carets from left to right, $\delta(Y,Z) = e_- + e_7 + e_{10} + e_+ = e_- + 2e_1 + e_+$. The process of this example is in Figure 19.

For the group $F(n)$, we define the homomorphism $\phi : F(n) \to A_{n+1}$ as follows: Let $x$ be in $F(n)$ and $(T_+, T_-)$ be a tree diagram that represents $x$. Then we set

$$\phi(x) := \delta(W,T_-) - \delta(W,T_+),$$
where $W$ is an $n$-ary tree that contains both $T_+$ and $T_-$. From the construction, this map is independent of the choice of $W$ and $(T_+, T_-)$, and this is a homomorphism. By calculating $\phi(x_0), \ldots, \phi(x_{n-2})$, and $\phi(x_0[\lfloor n-1 \rfloor])$, we have that $\text{Im} \phi = \{ \Sigma \lambda_i e_i \mid \Sigma \lambda_i = 0 \} \cong \mathbb{Z}^n$. Thus we obtain a surjective homomorphism $F(n) \to \mathbb{Z}^n$.

**Theorem 4.12 ([7, Lemma 2.1] for $n = 2$).** The abelianization of $G_0(n)$ is isomorphic to $\mathbb{Z}^{n+1}$.

**Proof.** We define the map $\pi : Z(n) \to \mathbb{Z}^n \bigoplus \mathbb{Z}$ by

$$x_{i[s]} \mapsto (\phi(x_{i[s]}), 0), \quad y_s \mapsto (0, 1).$$

Since $G_0(n)$ is $(n+1)$-generated group, its abelianization is a quotient of $\mathbb{Z}^{n+1}$. So if we obtain a surjective homomorphism $G_0(n) \to \mathbb{Z}^{n+1}$, then this map must be its abelianization map. Thus it is sufficient to show that $\pi$ extends to a homomorphism $G_0(n) \to \mathbb{Z}^{n+1}$. In order to do this, we only need to check that the relations in $R(n)$ are satisfied, which is clear except for (4). By the definition of $\phi$, we have $\phi(x_{0[s]}) = 0$ for each $x_{0[s]}$, where $s$ is in $\mathbb{N}^{<\mathbb{N}}$ such that $y_s$ is in $Y(n)$. Indeed, since $s$ is neither $0 \cdots 0$ nor $(n-1) \cdots (n-1)$, $\phi(x_{0[s]})$ is calculated as in Figure 20. Thus the relation (4) is also satisfied. \hfill $\square$

**Corollary 4.13.** Let $n, m \geq 2$. Then the groups $G_0(n)$ and $G_0(m)$ are isomorphic if and only if $n = m$ holds.

In the following, we show that the commutator subgroup $G_0(n)' = [G_0(n), G_0(n)]$ is simple. As in the case of $n = 2$, first we show that the second derived subgroup $G_0(n)'' = [G_0(n)', G_0(n)']$ is simple by using Higman’s Theorem, and then we show that $G_0(n)'' = G_0(n)'$. Let $\Gamma$ be a group of bijections of a set $E$. For $\alpha \in \Gamma$, we write the set-theoretic support $\text{supp}(\alpha)$ for the set $\{ x \in E \mid \alpha(x) \neq x \}$.

**Theorem 4.14 ([12, Theorem 1]).** Suppose that for every $\alpha, \beta, \gamma \in \Gamma \setminus \{ 1_{\Gamma} \}$, there exists $\rho$ such that

$$\gamma \left( \rho \left( \text{supp}(\alpha) \cup \text{supp}(\beta) \right) \right) \cap \rho \left( \text{supp}(\alpha) \cup \text{supp}(\beta) \right) = \emptyset$$

holds. Then the commutator subgroup $\Gamma'$ is simple.
Theorem 4.15 ([7] Theorem 2) for \( n = 2 \). The commutator subgroup of the group \( G_0(n) \) is simple.

The following two lemmas complete the proof.

Lemma 4.16. The second derived subgroup \( G_0(n)'' \) is simple.

Proof. Let \( \alpha, \beta, \gamma \in G_0(n)' \). Choose \( x \in \text{supp}(\gamma) \). If \( \gamma(x) > x \), then let \( I \) be the set \( \{ x' \in \mathbb{N}^n | x < x' < \gamma(x) \} \). If \( \gamma(x) < x \), then let \( I \) be the set \( \{ x' \in \mathbb{N}^n | \gamma(x) < x' < x \} \). Then \( \gamma(I) \cap I = \emptyset \) holds.

We note that \( \text{supp}(\alpha) \) and \( \text{supp}(\beta) \) are not \( \mathbb{N}^n \setminus \{00\cdots, (n-1)(n-1)\cdots\} \), by the argument in Theorem 4.9 (in particular, see the order (4.2)). Therefore there exists \( \rho \in F(n) \) such that

\[
\rho \left( \text{supp}(\alpha) \cup \text{supp}(\beta) \right) \subset I
\]

holds. Then we have

\[
\rho \left( \text{supp}(\alpha) \cup \text{supp}(\beta) \right) \cap \gamma \left( \rho \left( \text{supp}(\alpha) \cup \text{supp}(\beta) \right) \right) \subset I \cap \gamma(I) = \emptyset.
\]

By Theorem 4.14, \( G_0(n)'' \) is simple.

Lemma 4.17 ([7] Proposition 2.5) for \( n = 2 \). \( G_0(n)' = G_0(n)'' \).

Proof. We assume that \( n \geq 3 \).

Since \( G_0(n)'' \subset G_0(n)' \) holds, we show \( G_0(n)' \subset G_0(n)'' \). Let \( g \in G_0(n)' \) and \( fy_{t_1} \cdots y_{t_m} \) be its normal form. By the definition of the map \( \pi \) in Theorem 4.12, \( t_1 + \cdots + t_m = 0 \) holds and \( f \) is in \( F(n)' \). By [4] Theorem 4.13, we have \( F(n)' = F(n)'' \subset G_0(n)'' \). Therefore, it is sufficient to show that \( y_{t_1} \cdots y_{t_m} \) is in \( G_0(n)'' \).

We show this by induction on \( k = |t_1| + \cdots + |t_m| \) for words \( y_{t_1} \cdots y_{t_m} \) that satisfy \( t_1 + \cdots + t_m = 0 \) (which may not be in normal). We note that \( k \) is an even number. Since the base case \( k = 0 \) is clear, we assume \( k > 0 \). Then, since \( G_0(n)'' \) is a normal subgroup of \( G_0 \), by cyclically conjugating, we can assume that the word starts with a subword of the form \( y_s y_t^{-1} \). For example, we can do the following:

\[
y_{s_1}y_{s_2}y_{s_3}^{-1}y_{s_4}^{-1} \mapsto y_{s_1}^{-1}(y_{s_1}y_{s_2}y_{s_3}^{-1}y_{s_4}^{-1})y_{s_1} = y_{s_2}y_{s_3}^{-1}y_{s_4}^{-1}y_{s_1}.
\]

By the inductive hypothesis, it is sufficient to show that \( y_s y_t^{-1} \) is in \( G_0(n)'' \). We divide the proof into two cases:

Case (1): \( s \perp t \).

Since \( (y_s y_t^{-1})^{-1} = y_t y_s^{-1} \) holds, we can assume that \( s < t \) without loss of generality. By the definition of \( Y(n) \), \( s \) and \( t \) are neither \( 000 \cdots \) nor \( (n-1)(n-1)(n-1) \cdots \). Then since \( n \geq 3 \), there exists an element \( h' \) in \( F(n) \) such that

\[
h'((n-1)000) = s
\]
and
\[ h'( (n - 1)00(n - 1)0 ) = t \]
hold. Indeed, we can construct it similarly as in Figure 16. Then we have
\[ y_s y_t^{-1} = (h'^{-1} y_{(n-1)00} h')(h'^{-1} y_{(n-1)00}^{-1} h') = h'^{-1} y_{(n-1)00} y_{(n-1)00}^{-1} h'. \]
Thus it is sufficient to show that \( y_{(n-1)00} y_{(n-1)00}^{-1} \) is in \( G_0(n)'' \).

Let \( w = y_{(n-1)00} y_{(n-1)00}^{-1} \) \( G_0(n)' \). We note that there exists an element \( h \) in \( F(n)' \) such that
\[ h'((n - 1)00(n - 1)(n - 1)) = (n - 1)00 \]
and
\[ h((n - 1)00(n - 1)) = (n - 1)0(n - 1) \]
hold. Indeed, let \( h \) be as in Figure 21. Then since \( \phi(h) = 0 \) holds where \( \phi \) is the abelianization map of \( F(n) \), the element \( h \) is in \( F(n)' \).

Since \( w \) is in \( G_0(n)' \) and \( h \) is in \( F(n)' \), the commutator \( [w, h] = whw^{-1}h^{-1} \) is in \( G_0(n)'' \). By the construction of \( h \), we have \( hwh^{-1} = y_{(n-1)00(n-1)(n-1)} y_{(n-1)00(n-1)}^{-1} \). Thus we have
\[ [w, h] = w(whw^{-1})^{-1} \]
\[ = y_{(n-1)00} y_{(n-1)00}^{-1} \]
\[ = y_{(n-1)00} y_{(n-1)00}^{-1} y_{(n-1)00} y_{(n-1)00}^{-1} \]
By applying expansion move (in Definition 3.17) to \( y_{(n-1)00} \), we have
\[ y_{(n-1)00} y_{(n-1)00}^{-1} = x_0[[n-1]00] y_{(n-1)00} y_{(n-1)00}^{-1} \]
\[ = x_0[[n-1]00] y_{(n-1)00} y_{(n-1)00}^{-1} \]
Since \( x_0[[n-1]00] \) is in \( F(n)' \) (see Figure 20), we note that this element is in \( F(n)'' \). Therefore we have
\[ y_{(n-1)00} y_{(n-1)00}^{-1} = x_0^{-1}[[n-1]00] [w, h] \in G_0(n)'', \]
as required.
Case (2): $s$ is a prefix of $t$ or vice versa.

Since $(y_sy_n^{-1})^{-1} = y_ny_s^{-1}$ holds, we can assume that $s$ is a prefix of $t$ without loss of generality. Let $t = su$. By expansion move, we have $y_sy_n^{-1} = x_0[s]y_0y_n^{-1}s(n-1)_0y_{s(n-1)(n-1)}y_{su}^{-1}$. Then since $x_0[s]$ is in $F(n)' = F(n)'' \subset G_0(n)''$ (see Figure 20), it is enough to show that $y_0y_{s(n-1)}y_{s(n-1)(n-1)}y_{su}^{-1}$ is in $G_0(n)''$.

We further divide the proof of case (2) into two subcases.

Case (2-1): 0 is a prefix of $u$.

We note that $s0 \perp s(n-1)0$ and $s(n-1)(n-1) \perp su$ hold. Thus $y_0y_{s(n-1)0}$ and $y_{s(n-1)(n-1)}y_{su}^{-1}$ are in $G_0(n)''$, by case (1).

Case (2-2): 0 is not a prefix of $u$.

By cyclically conjugating, it is sufficient to show that $y_{s(n-1)0}y_{s(n-1)(n-1)}y_{su}^{-1}y_0$ is in $G_0(n)''$. Since $su \perp s0$ holds, $y_{s(n-1)0}y_{s(n-1)(n-1)}$ and $y_{su}^{-1}y_0$ are in $G_0(n)''$, by case (1). □

4.5. The center of $G_0(n)$. In this section, we show that the center of the group $G_0(n)$ is trivial. The idea for the theorem comes from [8, Section 4]. Let $D(n) := \{s\bar{u} \mid s \in \mathbb{N}^{<\mathbb{N}}\}$, where $\bar{u}$ denotes the element $000 \cdots$ in $\mathbb{N}^{\mathbb{N}}$. Then the following holds.

Lemma 4.18. For any $s\bar{u}$ in $D(n)$, there exists $x$ in $F(n)$ such that

$$\text{supp}(x) = (s\bar{u}, (n-1)) = \{\xi \in \mathbb{N}^{\mathbb{N}} \mid s\bar{u} < \xi < (n-1)\}$$

holds.

Proof. If $s = 0, \ldots, (n-2)$, then $x_0, \ldots, x_{n-2}$ satisfy the claim, respectively. By regarding $(n-1)\bar{u}$ as $(n-1)0\bar{u}$, we can assume that the length of $s$ is greater than or equal to 2.

Let $s = s'\bar{u}$ ($i \in \{0, \ldots, n-2\}$). If $s' = (n-1) \cdots (n-1)$, then $x_{i[s']}$ satisfies the claim. If $s' \neq (n-1) \cdots (n-1)$, by using $x_{i[s']}$, we can define an element in $F(n)$ as in Figure 22 which satisfies the claim. □
We note that $D(n)$ is a dense subset of $\mathbb{N}^n$.

**Theorem 4.19** ([7, Proposition 2.7] for $n = 2$). The center of $G_0(n)$ is trivial.

**Proof.** Let $f$ be an element of the center of $G_0(n)$. For $g \in G_0(n)$, assume that $\text{supp}(g) = (b_1, (n - 1))$ holds. Then we have $f(b_1) = b_1$. Indeed, if not, either $f(b_1) > b_1$ or $f^{-1}(b_1) > b_1$ holds. Since $g(b_1) = b_1$, in both cases, this contradicts that $fg = gf$ holds.

By Lemma 4.18, for every $s_0 \in D(n)$, there exists $g \in F(n) \subset G_0(n)$ such that $b_1 = s_0$ holds. Thus we have $f(s_0) = s_0$ for every $s_0 \in D(n)$. Since $D(n)$ is a dense subset of $\mathbb{N}^n$, we conclude that $f$ is the identity map. \hfill \Box

4.6. **Indecomposability with respect to direct products and free products.** In this section, we show that there exist no nontrivial “decompositions” using theorems of $G_0(n)$.

**Theorem 4.20.** There exists neither nontrivial direct product decompositions nor nontrivial free product decompositions.

**Proof.** Suppose that $G_0(n)$ is isomorphic to $K \times H$ for some groups $K$ and $H$. We first assume that $H$ (or $K$) is abelian. Then the center of $G_0(n)$ contains $\{1\} \times H$. Since the center of $G_0(n)$ is trivial (Theorem 4.19), $H$ must be the trivial group.

We assume that $K$ and $H$ are not abelian. We note that the commutator subgroup of $G_0(n) = K \times H$ is $[K, K] \times [H, H]$. Since $[K, K]$ and $[H, H]$ are not trivial, the group $\{1\} \times [H, H]$ is a nontrivial normal subgroup of $[K, K] \times [H, H]$. However, this contradicts that $[G_0(n), G_0(n)]$ is simple (Theorem 4.15).

Finally, we assume that $G_0(n) = K \star H$ for nontrivial groups $K$ and $H$. Let $k \in K \setminus \{1\}$ and $h \in H \setminus \{1\}$. Since $G_0(n)$ is torsion free (Remark 4.10), both $h$ and $k$ generate infinite cyclic groups $\langle k \rangle$ and $\langle h \rangle$, respectively. This implies that $G_0(n)$ has a subgroup $\langle k \rangle \star \langle h \rangle \cong \mathbb{Z} \star \mathbb{Z} = F_2$.

By Theorem 4.9, this is a contradiction. \hfill \Box

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