On Nonlocal Modified Gravity and its Cosmological Solutions

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Abstract During hundred years of General Relativity (GR), many significant gravitational phenomena have been predicted and discovered. General Relativity is still the best theory of gravity. Nevertheless, some (quantum) theoretical and (astrophysical and cosmological) phenomenological difficulties of modern gravity have been motivation to search more general theory of gravity than GR. As a result, many modifications of GR have been considered. One of promising recent investigations is Nonlocal Modified Gravity. In this article we present a brief review of some nonlocal gravity models with their cosmological solutions, in which nonlocality is expressed by an analytic function of the d’Alembert-Beltrami operator □. Some new results are also presented.
1 Introduction

General relativity (GR) was formulated one hundred years ago and is also known as Einstein theory of gravity. GR is regarded as one of the most profound and beautiful physical theories with great phenomenological achievements and nice theoretical properties. It has been tested and quite well confirmed in the Solar system, and it has been also used as a theoretical laboratory for gravitational investigations at other spacetime scales. GR has important astrophysical implications predicting existence of black holes, gravitational lensing and gravitational waves\(^1\). In cosmology, it predicts existence of about 95% of additional new kind of matter, which makes dark side of the universe. Namely, if GR is the gravity theory for the universe as a whole and if the universe is homogeneous and isotropic with the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric at the cosmic scale, then it contains about 68% of dark energy, 27% of dark matter, and only about 5% of visible matter [2].

Despite of some significant phenomenological successes and many nice theoretical properties, GR is not complete theory of gravity. For example, attempts to quantize GR lead to the problem of nonrenormalizability. GR also contains singularities like the Big Bang and black holes. At the galactic and large cosmic scales GR predicts new forms of matter, which are not verified in laboratory conditions and have not so far seen in particle physics. Hence, there are many attempts to modify General relativity. Motivations for its modification usually come from quantum gravity, string theory, astrophysics and cosmology (for a review, see [22, 60, 63]). We are mainly interested in cosmological reasons to modify Einstein theory of gravity, i.e. to find such extension of GR which will not contain the Big Bang singularity and offer another possible description of the universe acceleration and large velocities in galaxies instead of mysterious dark energy and dark matter. It is obvious that physical theory has to be modified when it contains a singularity. Even if it happened that dark energy and dark matter really exist it is still interesting to know is there a modified gravity which can imitate the same or similar effects. Hence, adequate gravity modification can reduce role and rate of the dark matter/energy in the universe.

Any well founded modification of the Einstein theory of gravity has to contain general relativity and to be verified at least on the dynamics of the Solar system. In other words, it has to be a generalization of the general theory of relativity. Mathematically, it should be formulated within the pseudo-Riemannian geometry in terms of covariant quantities and take into account equivalence of the inertial and gravitational mass. Consequently, the Ricci scalar \(R\) in gravity Lagrangian \(\mathcal{L}_g\) of the Einstein-Hilbert action should be replaced by an adequate function which, in general, may contain not only \(R\) but also some scalar covariant constructions which are possible in the pseudo-Riemannian geometry. However, we do not know what is here adequate function and there are infinitely many possibilities for its construction. Unfortunately, so far there is no guiding theoretical principle which could make appropriate choice between all possibilities. In this context the Einstein-Hilbert action

\(^1\) While we prepared this contribution, the discovery of gravitational waves was announced [1].
is the simplest one, i.e. it can be viewed as realization of the principle of simplicity in construction of $L_g$.

One of promising modern approaches towards more complete theory of gravity is its nonlocal modification. Motivation for nonlocal modification of general relativity can be found in string theory which is nonlocal theory and contains gravity. We present here a brief review and some new results of nonlocal gravity with related bounce cosmological solutions. In particular, we pay special attention to models in which nonlocality is expressed by an analytic function of the d'Alembert operator $\Box = \sqrt{-g} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$ like nonlocality in string theory. In these models, we are mainly interested in nonsingular bounce solutions for the cosmic scale factor $a(t)$.

In Sect. 2 we mention a few different approaches to nonlocal modified gravity. Section 3 contains rather general modified action with an analytic nonlocality and with corresponding equations of motion. Cosmological equations for the FLRW metric is presented in Sect. 4. Cosmological solutions for constant scalar curvature are considered separately in Sect. 5. Some new examples of nonlocal models and related Ansätze are introduced in Sect. 6. At the and a few remarks are also noticed.

2 Nonlocal Modified Gravity

We consider here nonlocal modified gravity. Usually a nonlocal modified gravity model contains an infinite number of spacetime derivatives in the form of a power series expansion with respect to the d'Alembert operator $\Box = \sqrt{-g} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$. In this article, we are mainly interested in nonlocality expressed in the form of an analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, where coefficients $f_n$ should be determined from various theoretical and phenomenological conditions. Some conditions are related to the absence of tachyons and ghosts.

Before to proceed with this analytic nonlocality it is worth to mention some other interesting nonlocal approaches. For approaches containing $\Box^{-1}$ one can see, e.g., [27, 26, 66, 61, 43, 67, 42, 43, 46, 47] and references therein. For nonlocal gravity with $\Box^{-1}$ see also [8, 58]. Many aspects of nonlocal gravity models have been considered, see e.g., [20, 16, 17, 59, 18, 36] and references therein.

Our motivation to modify gravity in an analytic nonlocal way comes mainly from string theory, in particular from string field theory (see the very original effort in this direction in [3]) and $p$-adic string theory [15, 38, 39, 40, 55]. Since strings are one-dimensional extended objects, their field theory description contains spacetime nonlocality expressed by some exponential functions of d'Alembert operator $\Box$.

At classical level analytic non-local gravity has proven to alleviate the singularity of the Black-hole type because the Newtonian potential appears regular (tending to a constant) on a universal basis at the origin [41, 11, 9]. Also there was significant success in constructing classically stable solution for the cosmological bounce [11, 13, 48, 51, 55].

Analysis of perturbations revealed a natural ability of analytic non-local gravities to accommodate inflationary models. In particular, the Starobinsky inflation was
studied in details and new predictions for the observable parameters were made\cite{24, 53}. Moreover, in the quantum sector infinite derivative gravity theories improve renormalization, see e.g. while the unitarity is still preserved\cite{56, 57, 53} (note that just a local quadratic curvature gravity was proven to be renormalizable while being non-unitary\cite{64}).

3 Modified GR with Analytical Nonlocality

To better understand nonlocal modified gravity itself, we investigate it here without presence of matter. Models of nonlocal gravity which we mainly investigate are given by the following action

$$S = \int d^4x \sqrt{-g} \left( \frac{M_P^2}{2} R - \Lambda + \frac{\lambda}{2} P(R) \mathcal{F}(\Box) Q(R) \right), \quad (1)$$

where $R$ is the scalar curvature, $\Lambda$ is the cosmological constant, $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ is an analytic function of the d’Alembert-Beltrami operator $\Box = \nabla^\mu \nabla_\mu$ where $\nabla_\mu$ is the covariant derivative. The Planck mass $M_P$ is related to the Newtonian constant $G$ as $M_P^2 = \frac{1}{8\pi G}$ and $P, Q$ are scalar functions of the scalar curvature. The spacetime dimensionality $D = 4$ and our signature is $(-, +, +, +)$. $\lambda$ is a constant and can be absorbed in the rescaling of $\mathcal{F}(\Box)$. However, it is convenient to remain $\lambda$ and recover GR in the limit $\lambda \to 0$.

Note that to have physically meaningful expressions one should introduce the scale of nonlocality using a new mass parameter $M$. Then the function $\mathcal{F}$ would be expanded in Taylor series as $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} \bar{f}_n \Box^n / M^{2n}$ with all barred constants dimensionless. For simplicity we shall keep $M^2 = 1$. We shall also see later that analytic function $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$, has to satisfy some conditions, in order to escape unphysical degrees of freedom like ghosts and tachyons, and to have good behavior in quantum sector (see\cite{2, 10, 31}).

Varying the action (1) by substituting

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu} \quad (2)$$

to the linear order in $h_{\mu\nu}$, removing the total derivatives and integrating from time to time by parts one gets

$$\delta S = \int d^4x \sqrt{-g} \frac{h_{\mu\nu}}{2} \left[ g_{\mu\nu} \right], \quad (3)$$

where
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\[ G_{\mu\nu} \equiv M_p^2 g_{\mu\nu} + g_{\mu\nu} \Lambda - \frac{\lambda}{2} g_{\mu\nu} P(\Box)Q + \lambda (R_{\mu\nu} - K_{\mu\nu}) V - \frac{\lambda}{2} \sum_{n=1}^{\infty} f_n \]

\times \sum_{l=0}^{n-1} \left( P_{\mu}^{(l)} Q_{\nu}^{(n-l-1)} + P_{\nu}^{(l)} Q_{\mu}^{(n-l-1)} - g_{\mu\nu} (g^{\alpha\sigma} P_{\rho}^{(l)} Q_{\sigma}^{(n-l-1)} + P_{\sigma}^{(l)} Q_{\rho}^{(n-l-1)}) \right) = 0

(4)
presents equations of motion for gravitational field \( g_{\mu\nu} \) in the vacuum. In (4) \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is the Einstein tensor, \( K_{\mu\nu} = \nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \Box \), \( V = P_R \mathcal{F}(\Box)Q + Q_R \mathcal{F}(\Box)P \),

where the subscript \( R \) indicates the derivative w.r.t. \( R \) (as many times as it is repeated) and

\[ P^{(l)} = \Box^l P, \quad P^{(l)}_{\rho} = \partial_{\rho} \Box^l P \text{ with the same for } Q, P_R, \ldots \]

In the case of gravity with matter, the full equations of motion are \( G_{\mu\nu} = T_{\mu\nu} \), where \( T_{\mu\nu} \) is the energy-momentum tensor. Thanks to the integration by parts there is always the symmetry of an exchange \( P \leftrightarrow Q \).

When \( \lambda = 0 \) in (4) we recognize the Einstein’s GR equation with the cosmological constant \( \Lambda \). If \( f_n = 0 \) for \( n \geq 1 \) then (4) corresponds to equations of motion of an \( f(R) \) theory.

4 Cosmological Equations for FLRW Metric

We use the FLRW metric

\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2 \right) \]

and look for some cosmological solutions. In the FLRW metric the Ricci scalar curvature is

\[ R = 6 \left( \frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2} \right) \]

and

\[ \Box = -\partial_t^2 - 3H \partial_t, \]

where \( H = \frac{\dot{a}}{a} \) is the Hubble parameter. We use natural system of units in which speed of light \( c = 1 \).

Due to symmetries of the FLRW spacetime, in (4) there are only two linearly independent equations. They are: trace and 00, i.e. when indices \( \mu = \nu = 0 \).

The trace equation and 00-equation, respectively, are
\[ M^2_p R - 4\Lambda + 2\lambda P \mathcal{F}(\Box)Q - \lambda (R + 3\Box V) = 0, \]  
\[ -\lambda \sum_{n=1}^{\infty} f_n \sum_{l=0}^{n-1} \left( g^\rho_\sigma \partial^\rho_\sigma \Box^l P \partial^\rho_\sigma \Box^{n-l-1} Q + 2\Box^l P \Box^{n-l} Q \right) = 0, \]  
\[ M^2_p G_{00} - \Lambda + \frac{\lambda}{2} P \mathcal{F}(\Box)Q + \lambda (R_{00} - \nabla_0 \nabla_0 - \Box) V - \frac{\lambda}{2} \sum_{n=1}^{\infty} f_n \]  
\[ \times \sum_{l=1}^{n-1} \left( 2\partial_\rho_\sigma \Box^l P \partial_\rho_\sigma \Box^{n-l-1} Q + g^\rho_\sigma \partial^\rho_\sigma \Box^l P \partial^\rho_\sigma \Box^{n-l-1} Q + \Box^l P \Box^{n-l} Q \right) = 0. \]  

5 Cosmological Solutions for Constant Scalar Curvature \( R \)

When \( R \) is a constant then \( P \) and \( Q \) are also some constants and we have that \( \Box R = 0 \), \( \mathcal{F}(\Box) = f_0 \). The corresponding equations of motion (5) and (6) contain solutions as in the local case. However, metric perturbations at the background \( R = \text{const.} \) can give nontrivial cosmic structure due to nonlocality.

Let \( R = R_0 = \text{constant} \neq 0 \). Then

\[ 6 \left( \ddot{a} \frac{\dot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right) = R_0. \]

The change of variable \( b(t) = a^2(t) \) transforms (7) into equation

\[ 3\ddot{b} - R_0 b = -6k. \]

Depending on the sign of \( R_0 \), the following solutions of equation (8) are

\[ b(t) = \frac{6k}{R_0} + \sigma e^{\sqrt{-\frac{R_0}{3}} t} + \tau e^{-\sqrt{-\frac{R_0}{3}} t}, \quad R_0 > 0, \]

\[ b(t) = \frac{6k}{R_0} + \sigma \cos \sqrt{-\frac{R_0}{3}} t + \tau \sin \sqrt{-\frac{R_0}{3}} t, \quad R_0 < 0, \]

where \( \sigma \) and \( \tau \) are some constant coefficients.

Substitution \( R = R_0 \) into equations of motion (5) and (6) yields, respectively,

\[ M^2_p R_0 - 4\Lambda + 2\lambda P f_0 Q - \lambda R_0 V_0 = 0, \]

\[ M^2_p G_{00} - \Lambda + \frac{\lambda}{2} P f_0 Q + \lambda R_{00} V_0 = 0, \]

where \( V_0 = f_0 (P_0 Q + Q_0 P)|_{R=R_0} \) and \( G_{00} = R_{00} + \frac{E_0}{2} \).

Combining equations (10) and (11) one obtains
\[ M_p^2 R_0 - 4\Lambda + 2\lambda P f_0 Q - \lambda R_0 V_0 = 0, \quad (12) \]
\[ 4R_{00} + R_0 = 0. \quad (13) \]

Equation (12) connects some parameters of the nonlocal model in the algebraic form with respect to \( R_0 \), while (13) implies a condition on the parameters \( \sigma, \tau, k \) and \( R_0 \) in solutions (9). Namely, \( R_{00} \) is related to function \( b(t) \) as

\[ R_{00} = \frac{-3\ddot{a}}{a} = \frac{3}{4} \frac{(b)^2 - 2bb}{b^2}. \quad (14) \]

Replacing \( R_{00} \) in (13) by (14) and using different solutions for \( b(t) \) in (9) we obtain

\[ 9k^2 = R_0^2 \sigma \tau, \quad R_0 > 0, \]
\[ 36k^2 = R_0^2(\sigma^2 + \tau^2), \quad R_0 < 0. \quad (15) \]

### 5.1 Case: \( R_0 > 0 \)

- Let \( k = 0 \). From \( 9k^2 = R_0^2 \sigma \tau \) follows that at least one of \( \sigma \) and \( \tau \) has to be zero. Thus there is possibility for an exponential solution for \( a(t) \) and \( a(t) = 0 \). Taking \( \tau = 0 \) and \( \sigma = a_0^2 \) one has

\[ b(t) = a_0^2 e^{\frac{R_0}{3}t}. \quad (16) \]

- If \( k = +1 \) one can find \( \varphi \) such that \( \sigma + \tau = \frac{6}{R_0} \cosh \varphi \) and \( \sigma - \tau = \frac{6}{R_0} \sinh \varphi \). Moreover, we obtain

\[ b(t) = \frac{12}{R_0} \cosh^2 \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right), \]
\[ a(t) = \sqrt{\frac{12}{R_0}} \cosh \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right). \quad (17) \]

- If \( k = -1 \) one can transform \( b(t) \) and \( a(t) \) to

\[ b(t) = \frac{12}{R_0} \sinh^2 \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right), \]
\[ a(t) = \sqrt{\frac{12}{R_0}} \sinh \left( \frac{1}{2} \sqrt{\frac{R_0}{3}} t + \varphi \right). \quad (18) \]
5.1.1 Case: $R = 12\gamma^2$

This is a special case of $R_0$, which simplifies the above expressions and yields de Sitter-like cosmological solutions.

- $k = 0$:
  \[ b(t) = a_0^2 e^{2\gamma t}, \quad a(t) = a_0 e^{\gamma t}. \]  
  \[ (19) \]

- $k = +1$:
  \[ b(t) = \frac{1}{\gamma} \cosh^2 \left( \gamma t + \frac{\varphi}{2} \right), \]
  \[ a(t) = \frac{1}{|\gamma|} \cosh \left( \gamma t + \frac{\varphi}{2} \right). \]
  \[ (20) \]

- $k = -1$:
  \[ b(t) = \frac{1}{\gamma} \sinh^2 \left( \gamma t + \frac{\varphi}{2} \right), \]
  \[ a(t) = \frac{1}{|\gamma|} \left| \sinh \left( \gamma t + \frac{\varphi}{2} \right) \right|. \]
  \[ (21) \]

5.2 Case: $R_0 < 0$

- When $k = 0$ then $\sigma = \tau = 0$, and consequently $b(t) = 0$.
- If $k = -1$ one can define $\varphi$ by $\sigma = \frac{6}{R_0} \cos \varphi$ and $\tau = \frac{6}{R_0} \sin \varphi$, and rewrite $b(t)$ and $a(t)$ as
  \[ b(t) = -\frac{12}{R_0} \cos \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right), \]
  \[ a(t) = \sqrt{-\frac{12}{R_0}} \cos \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right). \]
  \[ (22) \]

- In the last case $k = +1$, by the same procedure as for $k = -1$, one can transform $b(t)$ to expression
  \[ b(t) = \frac{12}{R_0} \sin^2 \frac{1}{2} \left( \sqrt{-\frac{R_0}{3}} t - \varphi \right), \]
  \[ (23) \]

which is not positive and hence yields no solution.

5.3 Case: $R_0 = 0$

The case $R_0 = 0$ can be considered as limit of $R_0 \to 0$ in both cases $R_0 > 0$ and $R_0 < 0$. When $R_0 > 0$ there is condition $9k^2 = R_0^2 \sigma \tau$ in [13]. From this condition,
$R_0 \rightarrow 0$ implies $k = 0$ and arbitrary values of constants $\sigma$ and $\tau$. The same conclusion obtains when $R_0 < 0$ with condition $36k^2 = R_0^2(\sigma^2 + \tau^2)$. In both these cases there is Minkowski solution with $b(t) = \text{constant} > 0$ and consequently $a(t) = \text{constant} > 0$, see \[9\].

6 Some Models and Related Ansätze for Cosmological Solutions

6.1 Nonlocal Gravity Model Quadratic in $R$

Nonlocal gravity model which is quadratic in $R$ was given by the action \[11, 12\]

$$S = \int d^4x \sqrt{-g} \left( \frac{R - 2\Lambda}{16\pi G} + R\mathcal{F}(\Box)R \right).$$

(24)

This model is important because it is ghost free and has some nonsingular bounce solutions, which can be regarded as a solution of the Big Bang cosmological singularity problem.

The corresponding equations of motion can be easily obtained from (5) and (6). To evaluate related equations of motion, the following Ansätze were used:

- Linear Ansatz: $\Box R = rR + s$, where $r$ and $s$ are constants.
- Quadratic Ansatz: $\Box R = qR^2$, where $q$ is a constant.
- Cubic Ansatz: $\Box R = CR^3$, where $C$ is a constant.
- Ansatz $\Box^n R = c_n R^{n+1}$, $n \geq 1$, where $c_n$ are constants.

These Ansätze make some constraints on possible solutions, but simplify formalism to find a particular solution (see \[29\] and references therein).

6.1.1 Linear Ansatz and Nonsingular Bounce Cosmological Solutions

Using Ansatz $\Box R = rR + s$ a few nonsingular bounce solutions for the scale factor are found: $a(t) = a_0 \cosh \left( \sqrt{\frac{\Lambda}{3}} t \right)$ (see \[11, 12\]), $a(t) = a_0 e^{\frac{1}{2}\sqrt{\frac{4}{3}} t^2}$ (see \[48, 49\]) and $a(t) = a_0 (\sigma e^{\lambda t} + \tau e^{-\lambda t})$ \[30\]. The first two consequences of this Ansatz are

$$\Box^n R = r^n (R + \frac{\Lambda}{3}), \quad n \geq 1, \quad \mathcal{F}(\Box)R = \mathcal{F}(r)R + \frac{s}{r} (\mathcal{F}(r) - f_0),$$

(25)

which considerably simplify nonlocal term.

Generalization of the above quadratic model in the form of nonlocal term $R^p \mathcal{F}(\Box) R^q$, where $p$ and $q$ are some natural numbers, was recently considered in \[28\]. Here cosmological solution for the scale factor has the form $a(t) = a_o e^{-\gamma t^2}$. 


6.2 Gravity Model with Nonlocal Term $R^{-1} \mathcal{F}(\Box)R$

This model was introduced in [31] and its action may be written in the form

$$S = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} + R^{-1} \mathcal{F}(\Box)R \right),$$

(26)

where $\mathcal{F}(\Box) = \sum_{n=0}^{\infty} f_n \Box^n$ and $f_0 = -\frac{\Lambda}{8\pi G}$ plays role of the cosmological constant.

The nonlocal term $R^{-1} \mathcal{F}(\Box)R$ in (26) is invariant under transformation $R \rightarrow CR$. This nonlocal term does not depend on the magnitude of scalar curvature $R$, but on its spacetime dependence, and in the FLRW case is relevant only dependence of $R$ on time $t$. Term $f_0 = -\frac{\Lambda}{8\pi G}$ is completely determined by the cosmological constant $\Lambda$, which according to $\Lambda CDM$ model is small and positive energy density of the vacuum. Coefficients $f_i, i \in \mathbb{N}$ can be estimated from other conditions, including agreement with dynamics the Solar system. In comparison to the model quadratic in $R$ (24), complete Lagrangian of this model remains to be linear in $R$ and in such sense is simpler nonlocal modification than (24).

In this model are also used the above Ansätze. Especially quadratic Ansatz $\Box R = qR^2$, where $q$ is a constant, is effective to consider power-law cosmological solutions, see [31, 32, 37, 33].

6.3 Some New Models and Ansätze

It is worth to consider some particular examples of action (1) when $P = Q = (R + R_0)^m$, i.e.

$$S = \int \left( \frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} (R+R_0)^m \mathcal{F}(\Box)(R+R_0)^m \right) \sqrt{-g} d^4x,$$

(27)

where $R_0 \in \mathbb{R}, m \in \mathbb{Q}$, and which have scale factor solution as

$$a(t) = A t^{n \gamma t^2}, \quad \gamma \in \mathbb{R}.$$

(28)

To this end we consider the Ansatz

$$\Box(R+R_0)^m = p(R+R_0)^m,$$

(29)

where $p$ is a constant and $\Box$ is the d’Alembert operator in FLRW metric.

From Ansatz (29) and scalar curvature $R$ for $k = 0$, we get the following system of equations:
In this case the Ansatz is

\[ \text{parameter. The scale factor is } a(t) = A e^{\gamma t}. \]

Here Ansatz is \( \Box \sqrt{R + R_0} = p \sqrt{R + R_0} \), where \( R_0 = -12\gamma \), \( p = -6\gamma \) and \( \gamma \) is a parameter. The scale factor is \( a(t) = A e^{\gamma t} \).

The first consequences of this Ansatz are

\[ \Box^\ell \sqrt{R + R_0} = p^\ell \sqrt{R + R_0}, \quad \ell \geq 0, \]
\[ \mathcal{F}(\Box) \sqrt{R + R_0} = \mathcal{F}(p) \sqrt{R + R_0}, \]
\[ R(t) = 12\gamma(1 + 4\gamma t^2). \]

Relevant action is

\[ S = \int \left( \frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} \sqrt{R - 12\gamma \mathcal{F}(\Box) \sqrt{R - 12\gamma}} \right) - g d^4 x. \]  \[ (31) \]

Equations of motion follow from \[ (29) \] and \[ (30) \], where \( P = O = \sqrt{R - 12\gamma} \). Straightforward calculation gives cosmological solution \( a(t) = A e^{\gamma t^2} \) with conditions:

\[ \mathcal{F}(p) = \frac{\gamma - 4\pi G A}{16\gamma \pi G \lambda}, \quad \mathcal{F}'(p) = \frac{4\pi G A - 3\gamma}{192\gamma^2 \pi G \lambda}, \quad p = -6\gamma. \]

6.3.2 Case 2: \( a(t) = A t^{2/3} e^{\gamma t^2}, m = \frac{1}{2} \)

In this case the Ansatz is \( \Box \sqrt{R + R_0} = p \sqrt{R + R_0} \), where \( R_0 \) and \( p \) are real constants.
The first consequences of this Ansatz are
\[ \Box \ell \sqrt{R + R_0} = p \ell \sqrt{R + R_0}, \quad \ell \geq 0, \]
\[ \mathcal{F}(\Box) \sqrt{R + R_0} = \mathcal{F}(p) \sqrt{R + R_0}. \]

For scale factor \( a(t) = \frac{t}{2} \) the Ansatz \( \Box \sqrt{R + R_0} = p \sqrt{R + R_0} \) is satisfied if and only if \( R_0 = -28\gamma \) and \( p = -6\gamma \).

Direct calculation shows that
\[ R(t) = 44\gamma + \frac{4}{3}t^{-2} + 48\gamma^2 t^2, \]
\[ \Box \ell \sqrt{R - 28\gamma} = (-6\gamma)^\ell \sqrt{R - 28\gamma}, \quad \ell \geq 0, \]
\[ \mathcal{F}(\Box) \sqrt{R - 28\gamma} = \mathcal{F}(-6\gamma) \sqrt{R - 28\gamma}, \]
\[ R = 96\gamma^2 t - \frac{8}{3}t^{-3}. \]

The related action is
\[ S = \int \left( \frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} \sqrt{R - 28\gamma} \mathcal{F}(\Box) \sqrt{R - 28\gamma} \right) \sqrt{-g} d^4x. \] (32)

The corresponding trace and 00 equations of motion are satisfied under conditions:
\[ \mathcal{F}(p) = -\frac{1}{8\pi GA}, \quad \mathcal{F}'(p) = 0, \quad \gamma = \frac{4}{7}\pi GA, \quad p = -6\gamma. \]

### 6.3.3 Case 3: \( a(t) = Ae^{\gamma t^2}, m = 1 \)

In this case \( \Box(R - 4\gamma) = -12\gamma(R - 4\gamma) \), what is an example of already above considered linear Ansatz. The corresponding action is
\[ S = \int \left( \frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} (R - 4\gamma) \mathcal{F}(\Box)(R - 4\gamma) \right) \sqrt{-g} d^4x. \] (33)

Equations of motion have cosmological solution \( a(t) = Ae^{\gamma t^2} \) under conditions:
\[ \mathcal{F}(p) = -\frac{1}{512\pi GA\gamma}, \quad \mathcal{F}'(p) = 0, \quad p = -12\gamma, \quad \gamma = 8\pi GA. \]

### 6.3.4 Case 4: \( a(t) = A \sqrt{e^{\gamma t^2}}, m = 1 \)

This case is quite similar to the previous one. Now Ansatz is \( \Box(R - 16\gamma) = -12\gamma(R - 16\gamma) \) and action
\[ S = \int \left( \frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} (R - 16\gamma) \mathcal{F}(\Box)(R - 16\gamma) \right) \sqrt{-g} \, d^4x. \]  

(34)

Scale factor \( a(t) = A \sqrt{t} e^{\gamma t^2} \) is solution of equations of motion if the following conditions are satisfied:

\[ \mathcal{F}(p) = -\frac{1}{320\pi G \Lambda \gamma}, \quad \mathcal{F}'(p) = 0, \quad p = -12\gamma, \quad \gamma = 8\pi G \Lambda. \]

6.3.5 Case 5: \( a(t) = A \sqrt{t} e^{n^2}, \quad m = -\frac{1}{4} \)

According to the Ansatz, in this case \( p = 3\gamma, \quad n = \frac{1}{2}, \quad R_0 = -36\gamma \). However the action

\[ S = \int \left( \frac{1}{16\pi G} R - \Lambda + \frac{\lambda}{2} \sqrt{R - 36\gamma} \mathcal{F}(\Box) \sqrt{R - 36\gamma} \right) \sqrt{-g} \, d^4x. \]  

(35)

has no solution \( a(t) = A \sqrt{t} e^{n^2} \) for the Ansatz \( \Box(R + R_0)^m = p(R + R_0)^m, \quad m = -\frac{1}{4} \).

7 Concluding Remarks

In this paper we presented a brief review of nonlocal modified gravity, where nonlocality is realized by an analytic function of the d’Alembert operator \( \Box \). Considered models are presented by actions, their equations of motion, related Ansätze and some cosmological solutions for the scale factor \( a(t) \). A few new models are introduced, and they deserve to be further investigated, especially Case 1 and Case 2 in section 6.

Many details on (1) and its extended versions can be found in [9, 10, 13, 49, 50, 51]. Perturbations and physical excitations of the equations of motion of action (24) around the de Sitter background are considered in [34] and [35], respectively. As some recent developments in nonlocal modified gravity, see [44, 21, 25, 41, 53, 68].

Notice that nonlocal cosmology is related also to cosmological models in which matter sector contains nonlocality (see, e.g. [4, 6, 19, 7, 52, 38, 39]). String field theory and \( p \)-adic string theory models have played significant role in motivation and construction of such models. One particular aspect in which non-local models prove important is the ability to resolve the Null Energy Condition obstacle [5] common to many models of generalized gravity. In short, that is an ability to construct a healthy model which has sum of energy and pressure of the matter positive and thereby avoids ghosts in the spectrum alongside with a nonsingular space-time structure [23].

Nonsingular bounce cosmological solutions are very important (as reviews on bouncing cosmology, see e.g. [62, 14]) and their progress in nonlocal gravity may be a further step towards cosmology of the cyclic universe [54].
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