Learning-Based Modular Indirect Adaptive Control for a Class of Nonlinear Systems

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Abstract

We study in this paper the problem of adaptive trajectory tracking control for a class of nonlinear systems with parametric uncertainties. We propose to use a modular approach, where we first design a robust nonlinear state feedback which renders the closed loop input-to-state stable (ISS), where the input is considered to be the estimation error of the uncertain parameters, and the state is considered to be the closed-loop output tracking error. Next, we augment this robust ISS controller with a model-free learning algorithm to estimate the model uncertainties. We implement this method with two different learning approaches. The first one is a model-free multi-parametric extremum seeking (MES) method and the second is a Bayesian optimization-based method called Gaussian Process Upper Confidence Bound (GP-UCB). The combination of the ISS feedback and the learning algorithms gives a learning-based modular indirect adaptive controller. We show the efficiency of this approach on a two-link robot manipulator example.

I. INTRODUCTION

Classical adaptive methods can be classified into two main approaches; ‘direct approaches’, where the controller is updated to adapt to the process, and ‘indirect approaches’, where the model is updated to better reflect the actual process. Many adaptive methods have been proposed over the years for linear and nonlinear systems, we could not possibly cite here all the design and analysis results that have been reported, instead we refer the reader to e.g., [1], [2] and the references therein for more details. Of particular interest to us is the indirect modular approach to adaptive nonlinear control, e.g., [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12]. In this approach, first the controller is designed by assuming that all the parameters are known and then an identifier is used to guarantee certain boundedness of the estimation error. The identifier is independent of the designed controller and thus the approach is called ‘modular’. For example, a modular approach has been proposed in [3] for adaptive neural control of pure-feedback nonlinear systems, where the input-to-state stability (ISS) modularity of the controller-estimator is achieved and the closed-loop stability is guaranteed by the small-gain theorem (see also [13], [14]).
In this work, we build upon this type of modular adaptive design and provide a framework which combines model-free learning methods and robust model-based nonlinear control to propose a learning-based modular indirect adaptive controller, where model-free learning algorithms are used to estimate, in closed-loop, the uncertain parameters of the model. The main difference with the existing model-based indirect adaptive control methods, is the fact that we do not use the model to design the uncertainty parameters estimation filters. Indeed, model-based indirect adaptive controllers are based on parameters estimators designed using the system’s model, e.g., the X-swapping methods presented in [2], where gradient descent filters obtained using the systems dynamics are designed to estimate the uncertain parameters. We argue that because we do not use the system’s dynamics to design uncertainties estimation filters we have less restrictions on the type of uncertainties that we can estimate, e.g., uncertainties appearing nonlinearly can be estimated with the proposed approach, see [5] for some earlier results on a mechatronics application. We also show here that with the proposed approach we can estimate at the same time a vector of linearly dependent uncertainties, a case which cannot be straightforwardly solved using model-based filters, e.g., refer to [15] where it is shown that the X-swapping model-based method fails to estimate a vector of linearly dependent model coefficients. In this work, we implement the proposed approach with two different model-free learning algorithms: The first one is a dither-based MES algorithm, and the second one is a Bayesian optimization-based method called GP-UCB. The latter solves the exploration-exploitation problem in the continuous armed bandit problem, which is a non-associative reinforcement learning (RL) setting. Indeed, MES is a model-free control approach with well known convergence properties, since it has been analyzed in many papers, e.g., [16], [17], [16], [18], [19]. This makes MES a good candidate for the model-free estimation part of our modular adaptive controller, as already shown in some of our preliminary results in [7], [8], [10]. However, one of the main limitations with dither-based MES is the convergence to local minima. To improve this part of the controller, we introduce here another model-free learning algorithm in the estimation part of the adaptive controller. Indeed, we propose in this paper to use a reinforcement learning algorithm based on Bayesian optimization methods, known as GP-UCB, e.g., [20], which contrary to the MES algorithm is guaranteed to reach the global minima in a finite search space.

One point worth mentioning at this stage is that comparatively to ‘pure’ model-free controllers, e.g., pure MES or model-free RL algorithms, the proposed control here has a different goal. Indeed, the available model-free controllers are meant for output or state regulation, i.e., solving a static optimization problem. In the contrary, here we propose to use model-free learning to complement a model-based nonlinear control to estimate the unknown parameters of the model, which means that the control goal, i.e., state or output trajectory tracking is handled by the model-based controller. The learning algorithm is used to improve the tracking performance of the model-based controller, and once the learning algorithm has converged, one can carry on using the nonlinear model-based feedback controller alone, i.e., without the need of the learning algorithm. Furthermore, due to the
fact that we are merging together a model-based control with a model-free learning algorithm, we believe that this type of controller can converge faster to an optimal performance, comparatively to the pure model-free controller, since by ‘partly’ using a model-based controller, we are taking advantage of the partial information given by the physics of the system, whereas the pure model-free algorithms assume no knowledge about the system, and thus start the search for an optimal control signal from scratch.

Similar ideas of merging model-based control and MES has been proposed in [12], [21], [22], [4], [5], [6], [7], [8], [10]. For instance in [12], [21] extremum seeking is used to complement a model-based controller, under linearity of the model assumption in [12] (in the direct adaptive control setting, where the controllers gains are estimated), or in the indirect adaptive control setting, under the assumption of linear parametrization of the control in terms of the uncertainties in [21]. The modular design idea of using a model-based controller with ISS guaranty, complemented with an MES-based module can be found in [5], [6], [7], [8], [10], where the MES was used to estimate the model parameters and in [4], [23], where feedback gains were tuned using MES algorithms.

The work of this paper falls in this class of ISS-based modular indirect adaptive controllers. The difference with other MES-based adaptive controllers is that, due to the ISS modular design we can use any model-free learning algorithm to estimate the model uncertainties, not necessarily extremum seeking-based. To emphasis this we show here the performance of the controller when using a type of RL-based learning algorithm as well.

The rest of the paper is organized as follows. In Section III we present some notations, and fundamental definitions that will be needed in the sequel. In Section III we formulate the problem. The nominal controller design are presented in Section IV. In Section IV-B a robust controller is designed which guarantees ISS from the estimation error input to the tracking error state. In Section IV-C the ISS controller is complemented with an MES algorithm to estimate the model parametric uncertainties. In section IV-D we introduce the RL GP-UCB algorithm as a model-free learning to complement the ISS controller. Section V is dedicated to an application example and the paper conclusion is given in Section VI.

II. Preliminaries

Throughout the paper, we use $\| \cdot \|$ to denote the Euclidean norm; i.e., for a vector $x \in \mathbb{R}^n$, we have $\|x\| \triangleq \|x\|_2 = \sqrt{x^T x}$, where $x^T$ denotes the transpose of the vector $x$. We denote by Card($S$) the size of a finite set $S$. The Frobenius norm of a matrix $A \in \mathbb{R}^{m \times n}$, with elements $a_{ij}$, is defined as $\|A\|_F \triangleq \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}$. Given $x \in \mathbb{R}^m$, the signum function is defined as $\text{sign}(x) \triangleq [\text{sign}(x_1), \text{sign}(x_2), \cdots, \text{sign}(x_m)]^T$, where $\text{sign}(.)$ denotes the classical signum function. We use $\dot{f}$ to denote the time derivative of $f$ and $f^{(r)}(t)$ for the $r$-th derivative of $f(t)$, i.e. $f^{(r)} \triangleq \frac{d^r f}{dt^r}$. We denote by $\mathbb{C}^k$, functions that are $k$ times differentiable and by $\mathbb{C}^\infty$, a smooth function. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$ [24]. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for a fixed $s$, the mapping $\beta(r, s)$ belongs to class
$\mathcal{K}$ with respect to $r$ and, for each fixed $r$, the mapping $\beta(r,s)$ is decreasing with respect to $s$ and $\beta(r,s) \to 0$ as $s \to \infty$ [24].

Next, we introduce some definitions that will be used in the sequel, e.g. [24]: Consider the system

$$\dot{x} = f(t,x,u)$$ (1)

where $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is piecewise continuous in $t$ and locally Lipschitz in $x$ and $u$, uniformly in $t$. The input $u(t)$ is piecewise continuous, bounded function of $t$ for all $t \geq 0$.

**Definition 1** ([24], [25]): The system (1) is said to be input-to-sate stable (ISS) if there exist a class $\mathcal{KL}$ function $\beta$ and a class $\mathcal{K}$ function $\gamma$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|).$$

**Theorem 1** ([24], [25]): Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -W(x), \quad \forall \|x\| \geq \rho(\|u\|) > 0$$ (2)

for all $(t,x,u) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$, where $\alpha_1$, $\alpha_2$ are class $\mathcal{K}_\infty$ functions, $\rho$ is a class $\mathcal{K}$ function, and $W(x)$ is a continuous positive definite function on $\mathbb{R}^n$. Then, the system (1) is input-to-state stable (ISS).

**Remark 1:** Note that other equivalent definitions for ISS have been given in [25, pp. 1974-1975]. For instance, Theorem 1 holds if inequality (2) is replaced by

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x,u) \leq -\mu(\|x\|) + \Omega(\|u\|)$$

where $\mu \in \mathcal{K}_\infty \cap C^1$ and $\Omega \in \mathcal{K}_\infty$.

III. PROBLEM FORMULATION

A. Nonlinear system model

We consider here affine uncertain nonlinear systems of the form

$$\dot{x} = f(x) + \Delta f(t,x) + g(x)u,$$
$$y = h(x),$$ (3)

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $y \in \mathbb{R}^m$ ($p \geq m$), represent the state, the input and the controlled output vectors, respectively. $\Delta f(t,x)$ is a vector field representing additive model uncertainties. The vector fields $f$, $\Delta f$, columns of $g$ and function $h$ satisfy the following assumptions.

**Assumption A1** The function $f : \mathbb{R}^n \to \mathbb{R}^n$ and the columns of $g : \mathbb{R}^n \to \mathbb{R}^p$ are $C^\infty$ vector fields on a bounded set $X$ of $\mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$ is a $C^\infty$ vector on $X$. The vector field $\Delta f(x)$ is $C^1$ on $X$. 

**Assumption A2** System (3) has a well-defined (vector) relative degree \( \{ r_1, r_2, \cdots, r_m \} \) at each point \( x^0 \in X \), and the system is linearizable, i.e., \( \sum_{i=1}^{m} r_i = n \).

**Assumption A3** The desired output trajectories \( y_{id} (1 \leq i \leq m) \) are smooth functions of time, relating desired initial points \( y_{id}(0) \) at \( t = 0 \) to desired final points \( y_{id}(t_f) \) at \( t = t_f \).

**B. Control objectives**

Our objective is to design a state feedback adaptive controller such that the output tracking error is uniformly bounded, whereas the tracking error upper-bound is function of the uncertain parameters estimation error, which can be decreased by the model-free learning. We stress here that the goal of learning algorithm is not stabilization but rather performance optimization, i.e., the learning improves the parameters estimation error, which in turn improves the output tracking error. To achieve this control objective, we proceed as follows: First, we design a robust controller which can guarantee input-to-state stability (ISS) of the tracking error dynamics w.r.t the estimation errors input. Then, we combine this controller with a model-free learning algorithm to iteratively estimate the uncertain parameters, by optimizing online a desired learning cost function.

**IV. Adaptive Controller Design**

**A. Nominal Controller**

Let us first consider the system under nominal conditions, i.e., when \( \Delta f(t, x) = 0 \). In this case, it is well know, e.g., [24], that system (3) can be written as

\[
y^{(r)}(t) = b(\xi(t)) + A(\xi(t))u(t),
\]

where

\[
y^{(r)}(t) = \begin{bmatrix} y^{(r_1)}(t) \\ y^{(r_2)}(t) \\ \vdots \\ y^{(r_m)}(t) \end{bmatrix}^T,
\]

\[
\xi(t) = \begin{bmatrix} \xi^1(t) \\ \vdots \\ \xi^m(t) \end{bmatrix},
\]

\[
\xi^i(t) = \begin{bmatrix} y_i(t) \\ \cdots \\ y_i^{(r_i-1)}(t) \end{bmatrix}, \quad 1 \leq i \leq m
\]

The functions \( b(\xi), A(\xi) \) can be written as functions of \( f, g \) and \( h \), and \( A(\xi) \) is non-singular in \( \tilde{X} \), where \( \tilde{X} \) is the image of the set of \( X \) by the diffeomorphism \( x \rightarrow \xi \) between the states of system (3) and the linearized model (4). Now, to deal with the uncertain model, we first need to introduce one more assumption on system (3).

**Assumption A4** The additive uncertainties \( \Delta f(t, x) \) in (3) appear as additive uncertainties in the input-output linearized model (4)-(5) as follows (see also [26])

\[
y^{(r)}(t) = b(\xi(t)) + A(\xi(t))u(t) + \Delta b(t, \xi(t)),
\]

where \( \Delta b(t, \xi) \) is \( C^1 \) w.r.t. the state vector \( \xi \in \tilde{X} \).
Remark 2: Assumption A4 can be ensured under the so-called matching conditions ([27], p. 146).

It is well known that the nominal model (4) can be easily transformed into a linear input-output mapping. Indeed, we can first define a virtual input vector \( v(t) \) as

\[
v(t) = b(\xi(t)) + A(\xi(t))u(t).
\]  

Combining (4) and (7), we can obtain the following input-output mapping

\[
y^{(r)}(t) = v(t).
\]  

Based on the linear system (8), it is straightforward to design a stabilizing controller for the nominal system (4) as

\[
u_n = A^{-1}(\xi) [v_s(t, \xi) - b(\xi)],
\]  

where \( v_s \) is a \( m \times 1 \) vector and the \( i \)-th (\( 1 \leq i \leq m \)) element \( v_{si} \) is given by

\[
v_{si} = y^{(r_i)}_{id} - K^{(r_i)}_{1}(y^{(r_i-1)}_{i} - y^{(r_i-1)}_{id}) - \cdots - K^{(r_i)}_{r_i}(y_{i} - y_{id}).
\]

If we denote the tracking error as \( e_i(t) \triangleq y_i(t) - y_{id}(t) \), we obtain the following tracking error dynamics

\[
e^{(r_i)}_{i}(t) + K^{(r_i)}_{1}e^{(r_i-1)}_{i}(t) + \cdots + K^{(r_i)}_{r_i}e_{i}(t) = 0,
\]  

where \( i \in \{1, 2, \ldots, m\} \). By properly selecting the gains \( K^{(r_i)}_{j} \) where \( i \in \{1, 2, \ldots, m\} \) and \( j \in \{1, 2, \ldots, r_i\} \), we can obtain global asymptotic stability of the tracking errors \( e_i(t) \). To formalize this condition, we add the following assumption.

**Assumption A5** There exists a non-empty set \( \mathcal{A} \) where \( K^{(r_i)}_{j} \in \mathcal{A} \) such that the polynomials in (11) are Hurwitz, where \( i \in \{1, 2, \ldots, m\} \) and \( j \in \{1, 2, \ldots, r_i\} \).

To this end, we define \( z = [z^1, z^2, \ldots, z^m]^T \), where \( z^i = [e_{i}, \dot{e}_{i}, \ldots, e^{(r_i-1)}_{i}] \) and \( i \in \{1, 2, \ldots, m\} \). Then, from (11), we can obtain

\[
\dot{z} = \hat{A}z,
\]

where \( \hat{A} \in \mathbb{R}^{n \times n} \) is a diagonal block matrix given by

\[
\hat{A} = \text{diag}\{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_m\},
\]

and \( \hat{A}_i \) (\( 1 \leq i \leq m \)) is a \( r_i \times r_i \) matrix given by

\[
\hat{A}_i = \begin{bmatrix}
0 & 1 \\
0 & 1 \\
0 & 1 \\
\vdots & 1 \\
-K^{(r_i)}_{1} & -K^{(r_i)}_{2} & \cdots & -K^{(r_i)}_{r_i}
\end{bmatrix}.
\]
As discussed above, the gains $K^j_k$ can be chosen such that the matrix $\tilde{A}$ is Hurwitz. Thus, there exists a positive definite matrix $P > 0$ such that (see e.g. [24])

$$\tilde{A}^TP + P\tilde{A} = -I.$$  \hfill(13)

In the next section, we build upon the nominal controller (9) to write a robust ISS controller.

### B. Lyapunov reconstruction-based ISS Controller

We now consider the uncertain model (3), i.e., when $\Delta f(t, x) \neq 0$. The corresponding exact linearized model is given by (6) where $\Delta b(t, \xi(t)) \neq 0$. The global asymptotic stability of the error dynamics (11) cannot be guaranteed anymore due to the additive uncertainty $\Delta b(t, \xi(t))$. We use Lyapunov reconstruction techniques to design a new controller so that the tracking error is guaranteed to be bounded given that the estimate error of $\Delta b(t, \xi(t))$ is bounded. The new controller for the uncertain model (6) is defined as

$$u_f = u_n + u_r,$$  \hfill(14)

where the nominal controller $u_n$ is given by (9) and the robust controller $u_r$ will be given later. By using the controller (14), and (6) we obtain

$$y^{(r)}(t) = b(\xi(t)) + A(\xi(t))u_f + \Delta b(t, \xi(t)),$$

$$= b(\xi(t)) + A(\xi(t))u_n + A(\xi(t))u_r + \Delta b(t, \xi(t)),$$

$$= v_s(t, \xi) + A(\xi(t))u_r + \Delta b(t, \xi(t)), \quad \hfill(15)$$

where (15) holds from (9). Which leads to the following error dynamics

$$\dot{z} = \hat{A}z + \hat{B}\delta,$$  \hfill(16)

where $\hat{A}$ is defined in (12), $\delta$ is a $m \times 1$ vector given by

$$\delta = A(\xi(t))u_r + \Delta b(t, \xi(t)),$$  \hfill(17)

and the matrix $\hat{B} \in \mathbb{R}^{n \times m}$ is given by

$$\hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_m \end{bmatrix},$$ \hfill(18)

where each $\hat{B}_i$ $(1 \leq i \leq m)$ is given by a $r_i \times m$ matrix such that

$$\hat{B}_i(l, q) = \begin{cases} 1 & \text{for } l = r_i, \ q = i \\ 0 & \text{otherwise.} \end{cases}$$
If we choose \( V(z) = z^T P z \) as a Lyapunov function for the dynamics (16), where \( P \) is the solution of the Lyapunov equation (13), we obtain

\[
\dot{V}(t) = \frac{\partial V}{\partial z} \dot{z},
\]

\[
= z^T (\dot{A}^T P + P \dot{A}) z + 2 z^T P \dot{B} \delta,
\]

\[
= -\|z\|^2 + 2 z^T P \dot{B} \delta,
\]

(19)

where \( \delta \) given by (17) depends on the robust controller \( u_r \).

Next, we design the controller \( u_r \) based on the form of the uncertainties \( \Delta b(t, \xi(t)) \). More specifically, we consider here the case when \( \Delta b(t, \xi(t)) \) is of the following form

\[
\Delta b(t, \xi(t)) = E Q(\xi, t),
\]

(20)

where \( E \in \mathbb{R}^{m \times m} \) is a matrix of unknown constant parameters, and \( Q(\xi, t): \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^m \) is a known bounded function of states and time variables. For notational convenience, we denote by \( \hat{E}(t) \) the estimate of \( E \), and by \( e_E = E - \hat{E} \), the estimate error. We define the unknown parameter vector \( \Delta = [E(1,1), ..., E(m,m)]^T \in \mathbb{R}^{m^2} \), i.e., concatenation of all elements of \( E \), its estimate is denoted by \( \hat{\Delta}(t) = [\hat{E}(1,1), ..., \hat{E}(m,m)]^T \), and the estimation error vector is given by \( e_\Delta(t) = \Delta - \hat{\Delta}(t) \).

Next, we propose the following robust controller

\[
u_r = -A^{-1}(\xi)[\hat{B}^T P z \|Q(\xi, t)\|^2 + \hat{E}(t) Q(\xi, t)].
\]

(21)

The closed-loop error dynamics can be written as

\[
\dot{z} = \tilde{f}(t, z, e_\Delta),
\]

(22)

where \( e_\Delta(t) \) is considered to be an input to the system (22).

**Theorem 2:** Consider the system (3), under Assumptions A1-A5, where \( \Delta b(t, \xi(t)) \) satisfies (20). If we apply to (3) the feedback controller (14), where \( u_n \) is given by (9) and \( u_r \) is given by (21). Then, the closed-loop system (22) is ISS from the estimation errors input \( e_\Delta(t) \in \mathbb{R}^{m^2} \) to the tracking errors state \( z(t) \in \mathbb{R}^n \).

**Proof:** By substitution (21) into (17), we obtain

\[
\delta = -\hat{B}^T P z \|Q(\xi, t)\|^2 - \hat{E}(t) Q(\xi, t) + \Delta b(t, \xi(t))
\]

\[
= -\hat{B}^T P z \|Q(\xi, t)\|^2 - \hat{E}(t) Q(\xi, t) + E Q(\xi, t),
\]

If we consider \( V(z) = z^T P z \) as a Lyapunov function for the error dynamics (16). Then, from (19), we obtain

\[
\dot{V} \leq -\|z\|^2 + 2 z^T P \hat{B} E Q(\xi, t) - 2 z^T P \hat{B} \hat{E}(t) Q(\xi, t)
\]

\[
- 2 \|z^T P \hat{B}\|^2 \|Q(\xi, t)\|^2,
\]
which leads to

\[
\dot{V} \leq -\|z\|^2 + 2z^TP\tilde{B}e_EQ(\xi, t) - 2\|z^TP\tilde{B}\|^2\|Q(\xi, t)\|^2.
\]

Since \(z^TP\tilde{B}e_EQ(\xi) \leq \|z^TP\tilde{B}\|\|e_EQ(\xi)\| = \|z^TP\tilde{B}\|\|e_\Delta\|\|Q(\xi)\|\), we obtain

\[
\dot{V} \leq -\|z\|^2 + 2\|z^TP\tilde{B}\|\|e_\Delta\|\|Q(\xi, t)\| - 2\|z^TP\tilde{B}\|^2\|Q(\xi, t)\|^2 \\
\leq -\|z\|^2 - 2\|z^TP\tilde{B}\|\|Q(\xi, t)\| \left( \frac{1}{2}\|e_\Delta\|^2 + \frac{1}{2}\|e_\Delta\|^2 \right) \\
\leq -\|z\|^2 + \frac{1}{2}\|e_\Delta\|^2.
\]

Thus, we have the following relation

\[
\dot{V} \leq -\frac{1}{2}\|z\|^2, \quad \forall \|z\| \geq \|e_\Delta\| > 0,
\]

Then from the Lyapunov direct theorem in [24], [25], we obtain that system (22) is ISS from input \(e_\Delta\) to state \(z\).

C. MES-based parametric uncertainties estimation

Let us define now the following cost function

\[
J(\hat{\Delta}) = F(z(\hat{\Delta})),
\]

where \(F : \mathbb{R}^n \to \mathbb{R}, \, F(0) = 0, \, F(z) > 0\) for \(z \in \mathbb{R}^n - \{0\}\). We need the following assumptions on \(J\).

Assumption A6 The cost function \(J\) has a local minimum at \(\hat{\Delta}^* = \Delta\).

Assumption A7 The initial error \(e_\Delta(t_0)\) is sufficiently small, i.e., the original parameter estimate vector \(\hat{\Delta}\) are close enough to the actual parameter vector \(\Delta\).

Assumption A8 The cost function \(J\) is analytic and its variation with respect to the uncertain parameters is bounded in the neighborhood of \(\hat{\Delta}^*\), i.e., \(\|\frac{\partial J}{\partial \Delta}(\hat{\Delta})\| \leq \xi_2, \xi_2 > 0\), \(\hat{\Delta} \in \mathcal{V}(\hat{\Delta}^*)\), where \(\mathcal{V}(\hat{\Delta}^*)\) denotes a compact neighborhood of \(\hat{\Delta}^*\).

We can now present the following result.

Lemma 3: Consider the system (23), under Assumptions A1-A8 where the uncertainty is given by (20). If we apply to (3) the feedback controller (14), where \(u_n\) is given by (9), \(u_r\) is given by (21), the cost function is given by (23), and \(\hat{\Delta}(t)\) are estimated through the ES algorithm

\[
\dot{x}_i = a_i \sin(\omega_it + \frac{\pi}{2})J(\hat{\Delta}), \quad a_i > 0, \\
\hat{\Delta}_i(t) = \dot{x}_i + a_i \sin(\omega_it - \frac{\pi}{2}), \quad i \in \{1, 2, \ldots, m^2\}
\]

(24)
with \( \omega_i \neq \omega_j, \omega_i + \omega_j \neq \omega_k \), \( i, j, k \in \{1, 2, \ldots, m^2\} \), and \( \omega_i > \omega^* \), \( \forall i \in \{1, 2, \ldots, m^2\} \), with \( \omega^* \) large enough. Then, the norm of the error vector \( z(t) \) admits the following bound

\[
\| z(t) \| \leq \beta(\| z(0) \|, t) + \gamma(\beta(\| e_\Delta(0) \|, t) + \| e_\Delta \|_{\text{max}}),
\]

where \( \| e_\Delta \|_{\text{max}} = \frac{\xi_1}{\omega_0} + \sqrt{\sum_{i=1}^{m^2} a_i^2} \), \( \xi_1 > 0, \omega_0 = \max_{i \in \{1, 2, \ldots, m^2\}} \omega_i \), \( \beta \in KL \), \( \tilde{\beta} \in KL \) and \( \gamma \in K \).

**Proof:** Based on Theorem [2], we know that the tracking error dynamics (22) is ISS from the input \( e_\Delta(t) \) to the state \( z(t) \). Thus, by Definition [1] there exist a class \( KL \) function \( \beta \) and a class \( K \) function \( \gamma \) such that for any initial state \( z(0) \), any bounded input \( e_\Delta(t) \) and any \( t \geq 0 \),

\[
\| z(t) \| \leq \beta(\| z(0) \|, t) + \gamma(\sup_{0 \leq \tau \leq t} \| e_\Delta(\tau) \|), \tag{25}
\]

Now, we need to evaluate the bound on the estimation vector \( \hat{\Delta}(t) \), to do so we use the results presented in [17]. First, based on Assumption \[A8\], the cost function is locally Lipschitz, i.e. there exists \( \eta_1 > 0 \) such that \( \| J(\Delta_1) - J(\Delta_2) \| \leq \eta_1 \| \Delta_1 - \Delta_2 \| \), for all \( \Delta_1, \Delta_2 \in \mathcal{V}(\hat{\Delta}^*) \). Furthermore, since \( J \) is analytic, it can be approximated locally in \( \mathcal{V}(\hat{\Delta}^*) \) by a quadratic function, e.g. Taylor series up to the second order. Based on this and on Assumptions \[A6\] and \[A7\], we can obtain the following bound ([17, p. 436-437],[28])

\[
\| e_\Delta(t) \| - \| d(t) \| \leq \| e_\Delta(t) - d(t) \| \leq \tilde{\beta}(\| e_\Delta(0) \|, t) + \frac{\xi_1}{\omega_0},
\]

where \( \tilde{\beta} \in KL \), \( \xi_1 > 0 \), \( t \geq 0 \), \( \omega_0 = \max_{i \in \{1, 2, \ldots, m^2\}} \omega_i \), and \( d(t) = [a_1 \sin(\omega_1 t + \frac{\pi}{2}), \ldots, a_{m^2} \sin(\omega_{m^2} t + \frac{\pi}{2})]^T \). We can further obtain that

\[
\| e_\Delta(t) \| \leq \tilde{\beta}(\| e_\Delta(0) \|, t) + \frac{\xi_1}{\omega_0} + \| d(t) \|
\]

\[
\leq \tilde{\beta}(\| e_\Delta(0) \|, t) + \frac{\xi_1}{\omega_0} + \sqrt{\sum_{i=1}^{m^2} a_i^2}.
\]

Together with (25) yields the desired result.

**Remark 3:** The adaptive controller of Lemma [3] uses the ES algorithm (24) to estimate the model parametric uncertainties. One might ask the question: where is the famous persistence of excitation (PE) condition here ? The answer can be found in the examination of equation (24). Indeed, the ES algorithm uses as ‘input’ the sinusoidal signals \( a_i \sin(\omega_i t + \frac{\pi}{2}) \) which clearly satisfy the PE condition. The main difference with classical adaptive control result, is that these excitation signals are not entering the system dynamics directly, but instead are applied as inputs to the ES algorithm, reflected on the ES estimations outputs and thus transmitted to the system through the feedback loop.

As we mentioned earlier, the dither-based MES has a problem of local minima, to improve this point we propose in the next section to use GP-UCB as the model-free learning algorithm for model uncertainties estimation.
D. GP-UCB based parametric uncertainties estimation

In this section we propose to use Gaussian Process Upper Confidence Bound (GP-UCB) algorithm to find the uncertain parameter \( \Delta \) vector [20], [29]. GP-UCB is a Bayesian optimization algorithm for stochastic optimization, i.e., the task of finding the global optimum of an unknown function when the evaluations are potentially contaminated with noise. The underlying working assumption for Bayesian optimization algorithms, including GP-UCB, is that the function evaluation is costly, so we would like to minimize the number of evaluations while having as accurate estimate of the minimizer (or maximizer) as possible [30]. For GP-UCB, this goal is guaranteed by having an upper bound on the regret of the algorithm – to be defined precisely later.

One difficulty of stochastic optimization is that since we only observe noisy samples from the function, we cannot really be sure about the exact value of a function at any given point. One may try to query a single point many times in order to have an accurate estimate of the function. This, however, may lead to excessive number of samples, and can be wasteful way of assigning samples when the true value of the function at that point is actually far from optimal. The Upper Confidence Bounds (UCB) family of algorithms provide a principled approach to guide the search [31]. These algorithms, which are not necessarily formulated in a Bayesian framework, automatically balance the exploration (i.e., finding regions of the parameter space that might be promising) and the exploration (i.e., focusing on the regions that are known to be the best based on the current available knowledge) using the principle of optimism in the face of uncertainty. These algorithms often come with strong theoretical guarantee about their performance. For more information about the UCB class of algorithms, refer to [32], [33], [34]. GP-UCB is a particular UCB algorithms that is suitable to deal with continuous domains. It uses a Gaussian Process (GP) to maintain the mean and confidence information about the unknown function.

We briefly discuss GP-UCB in our context following the discussion of the original papers [20], [29]. Consider the cost function \( J : D \to \mathbb{R} \) to be minimized. This function depends on the dynamics of the closed-loop system, which itself depends on the parameters \( \hat{\Delta} \) used in the controller design. So we may consider it as an unknown function of \( \hat{\Delta} \).

For the moment, let us assume that \( J \) is a function sampled from a Gaussian Process (GP) [35]. Recall that a GP is a stochastic process indexed by the set \( D \) that has the property that for any finite subset of the evaluation points, that is \( \{\hat{\Delta}_1, \hat{\Delta}_2, \ldots, \hat{\Delta}_t\} \subset D \), the joint distribution of \( \left( J(\hat{\Delta}_i) \right)_{i=1}^t \) is a multivariate Gaussian distribution. GP is defined by a mean function \( \mu(\hat{\Delta}) = \mathbb{E}[J(\hat{\Delta})] \) and its covariance function (or kernel) \( \kappa(\hat{\Delta}, \hat{\Delta}') = \text{Cov}(J(\hat{\Delta}), J(\hat{\Delta}')) = \mathbb{E}\left[\left( J(\hat{\Delta}) - \mu(\hat{\Delta}) \right) \left( J(\hat{\Delta}') - \mu(\hat{\Delta}') \right) \right] \). The kernel \( \kappa \) of a GP determines the behavior of a typical function sampled from the GP. For instance, if we choose \( \kappa(\hat{\Delta}, \hat{\Delta}') = \exp\left(-\frac{||\hat{\Delta} - \hat{\Delta}'||^2}{2l^2}\right) \), the squared exponential kernel with length scale \( l > 0 \), it implies that the the GP is mean square differentiable of all orders. We write \( J \sim \text{GP}(\mu, \kappa) \).
Let us first briefly describe how we can find the posterior distribution of a GP(0, K); a GP with zero prior mean. Suppose that for $\hat{\Delta}_{t-1} \triangleq \{\hat{\Delta}_1, \hat{\Delta}_2, \ldots, \hat{\Delta}_{t-1}\} \subset D$, we have observed the noisy evaluation $y_t = J(\hat{\Delta}_t) + \eta_t$ with $\eta_t \sim \mathcal{N}(0, \sigma^2)$ being i.i.d. Gaussian noise. We can find the posterior mean and variance for a new point $\hat{\Delta}^* \in D$ as follows: Denote the vector of observed values by $y_{t-1} = [y_1, \ldots, y_{t-1}]^T \in \mathbb{R}^{t-1}$, and define the Grammian matrix $K \in \mathbb{R}^{t-1 \times t-1}$ with $[K]_{i,j} = K(\hat{\Delta}_i, \hat{\Delta}_j)$, and the vector $K_s = [K(\hat{\Delta}_1, \hat{\Delta}^*), \ldots, K(\hat{\Delta}_{t-1}, \hat{\Delta}^*)]$. The expected mean $\mu_t(\hat{\Delta}^*)$ and the variance $\sigma_t(\hat{\Delta}^*)$ of the posterior of the GP evaluated at $\hat{\Delta}^*$ are (cf. Section 2.2 of [35])

$$
\mu_t(\hat{\Delta}^*) = K_s \left[ K + \sigma^2 I \right]^{-1} y_{t-1},
$$
$$
\sigma_t^2(\hat{\Delta}^*) = K(\hat{\Delta}^*, \hat{\Delta}^*) - K_s^\top \left[ K + \sigma^2 I \right]^{-1} K_s.
$$

At round $t$, the GP-UCB algorithm selects the next query point $\hat{\Delta}_t$ by solving the following optimization problem:

$$
\hat{\Delta}_t \leftarrow \arg\min_{\hat{\Delta} \in D} \mu_{t-1}(\hat{\Delta}) - \beta_t^{1/2} \sigma_{t-1}(\hat{\Delta}). \tag{26}
$$

Where $\beta_t$ depends on the choice of kernel among other parameters of the problem.

The optimization problem (26) is often nonlinear and non-convex. Nonetheless solving it only requires querying the GP, which in general is much faster than querying the original dynamical system. This is important when the dynamical system is a physical system and we would like to minimize the number of interactions with it before finding a $\hat{\Delta}$ with small $J(\hat{\Delta})$. One practically easy way to approximately solve (26) is to restrict the search to a finite subset $D'$ of $D$. The finite subset can be a uniform grid structure over $D$, or it might consist of randomly selected members of $D$.

The theoretical guarantee for GP-UCB is in the form of regret upper bound. Let us define $\Delta^* \leftarrow \arg\min_{\Delta \in D} J(\Delta)$, the global minimizer of the objective function. The regret at time $t$ is defined by $r_t = J(\hat{\Delta}_t) - J(\Delta^*)$. This is a measure of sub-optimality of the choice of $\hat{\Delta}_t$ according the cost function $J$. The cumulative regret at time $T$ is defined as $R_T = \sum_{t=1}^T r_t$. Ideally we would like $\lim_{T \to \infty} R_T = 0$.

The behavior of the cumulative regret $R_T$ depends on the set $D$ and the choice of kernel. If we fix the confidence parameter $\delta > 0$, for the squared exponential kernel, the asymptotic behavior of $R_T$ is

$$
O \left( \sqrt{T [\log (d+1)(T) + \log (1/\delta)]} \right),
$$

with probability at least $1 - \delta$ (cf. Theorem 3 of [20], [29]). This result does not even require the function $J$ to be a GP. It only requires the function to have a finite norm in the reproducing kernel Hilbert space (RKHS) $\mathcal{H}_K$ defined by the kernel $K$.

UCB algorithms are often formulated as maximization problems, so the “upper” confidence bound is calculated. Here we actually compute the “lower” confidence bound, but to keep the naming convection, we still GP-UCB instead of GP-LCB.
Remark 4: One main difference with some of the existing model-based adaptive controllers, is the fact that the learning-based estimation algorithm used here does not depend on the model of the system, i.e., the only information needed to compute the learning cost function (23) is the desired trajectory and the measured output of the system (please refer to Section V for an example). This makes the learning-based adaptive controllers suitable for the general case of nonlinear parametric uncertainties. For example in [36], a similar preliminary algorithm has been tested in the case of nonlinear models of electromagnetic actuators with a nonlinear parametric uncertainty. Another point worth mentioning here, is the fact that with the available modular model-based adaptive controllers, like the X-swapping modular algorithms, e.g., [37], it is not possible in some cases to estimate multiple uncertainties simultaneously. For instance, it is shown in [15] that the X-swapping adaptive control cannot estimate multiple uncertainties in the case of electromagnetic actuators, due to the linear dependency of the uncertain parameters, i.e., when we consider three parametric uncertainties affecting the same output acceleration, in which case the model-based estimation filters cannot distinguish between the uncertainties from this acceleration. However, when dealing with the same example, the MES-based modular indirect adaptive control approach was successful in estimating multiple uncertainties at the same time [28]. A similarly challenging case is considered in the example presented in the next section.

V. Two-link Manipulator Example

We consider here a two-link robot manipulator, with the following dynamics (see e.g. [38])

\[ H(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau, \]  

(27)

where \( q \triangleq [q_1, q_2]^T \) denotes the two joint angles and \( \tau \triangleq [\tau_1, \tau_2]^T \) denotes the two joint torques. The matrix \( H \in \mathbb{R}^{4 \times 4} \) is assumed to be non-singular and its elements are given by

\[
H_{11} = m_1 \ell_1^2 + I_1 + m_2 [\ell_1^2 + \ell_2^2 + 2\ell_1 \ell_2 \cos(q_2)] + I_2, \\
H_{12} = m_2 \ell_1 \ell_2 \cos(q_2) + m_2 \ell_2^2 + I_2, \\
H_{21} = H_{12}, \\
H_{22} = m_2 \ell_2^2 + I_2.
\]

(28)

The matrix \( C(q, \dot{q}) \) is given by

\[
C(q, \dot{q}) \triangleq \begin{bmatrix}
-h\dot{q_2} & -h\dot{q_1} - h\dot{q_2} \\
h\dot{q_1} & 0
\end{bmatrix},
\]

where \( h = m_2 \ell_1 \ell_2 \sin(q_2) \). The vector \( G = [G_1, G_2]^T \) is given by

\[
G_1 = m_1 \ell_1 g \cos(q_1) + m_2 g [\ell_2 \cos(q_1 + q_2) + \ell_1 \cos(q_1)], \\
G_2 = m_2 \ell_2 g \cos(q_1 + q_2),
\]

(29)

where, \( \ell_1, \ell_2 \) are the lengths of the first and second link, respectively, \( \ell_{c_1}, \ell_{c_2} \) are the distances between the rotation center and the center of mass of the first and second link respectively. \( m_1, m_2 \) are the masses of the
Where, 

\[ E \]

is solution of the Lyapunov equation (13), with \( P \) the moment of inertia of the first link and \( I_2 \) the moment of inertia of the second link, respectively, and \( g \) denotes the earth gravitational constant.

In our simulations, we assume that the parameters take the following values: \( I_2 = \frac{5}{12} \, kg \cdot m^2 \), \( m_1 = 10.5 \, kg \), \( m_2 = 5.5 \, kg \), \( \ell_1 = 1.1 \, m \), \( \ell_2 = 1.1 \, m \), \( \ell_{c_1} = 0.5 \, m \), \( \ell_{c_2} = 0.5 \, m \), \( I_1 = \frac{1}{12} \, kg \cdot m^2 \), \( g = 9.8 \, m/s^2 \). The system dynamics (27) can be rewritten as

\[
\dot{q} = H^{-1}(q)\tau - H^{-1}(q) [C(q, \dot{q})\dot{q} + G(q)].
\] (30)

Thus, the nominal controller is given by

\[
\tau_n = [C(q, \dot{q})\dot{q} + G(q)]
+ H(q) [\dot{q} - K_d(\dot{q} - \dot{q}_d) - K_p(q - q_d)],
\] (31)

where \( q_d = [q_{1d}, q_{2d}]^T \), denotes the desired trajectory and the diagonal gain matrices \( K_p > 0 \), \( K_d > 0 \), are chosen such that the linear error dynamics (as in (11)) are asymptotically stable. We choose as output references the 5th order polynomials \( \dot{q}_{1ref}(t) = \dot{q}_{2ref}(t) = \sum_{i=0}^{5} a_i(t/t_f)^i \), where the \( a_i \)'s have been computed to satisfy the boundary constraints \( q_{iref}(0) = 0, \dot{q}_{iref}(t_f) = q_f, \ddot{q}_{iref}(0) = \dot{q}_{iref}(t_f) = 0, \dot{q}_{iref}(0) = \dot{q}_{iref}(t_f) = 0, i = 1, 2 \), with \( t_f = 2 \, sec \), \( q_f = 1.5 \, rad \). In these tests, we assume that the nonlinear model (27) is uncertain. In particular, we assume that there exist additive uncertainties in the model (30), i.e.,

\[
\ddot{q} = H^{-1}(q)\tau - H^{-1}(q) [C(q, \dot{q})\dot{q} + G(q)] - E G(q).
\] (32)

Where, \( E \) is a matrix of constant uncertain parameters. Following (21), the robust-part of the control writes as

\[
\tau_r = -H(\tilde{B}^T P z\|G\|^2 - \hat{E} G(q)),
\] (33)

where

\[
\tilde{B}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

\[
P \text{ is solution of the Lyapunov equation (13), with }
\]

\[
\hat{A} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-K_p^1 & -K_d^1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -K_p^2 & -K_d^2
\end{bmatrix},
\]

\[
z = [q_1 - q_{1d}, \dot{q}_1 - \dot{q}_{1d}, q_2 - q_{2d}, \dot{q}_2 - \dot{q}_{2d}]^T, \text{ and } \hat{E} \text{ is the matrix of the parameters' estimates. Eventually, the final feedback controller writes as }
\]

\[
\tau = \tau_n + \tau_r.
\] (34)

We consider here the challenging case where the uncertain parameters are linearly dependent. In this case the uncertainties’ ‘effect’ is not observable from the measured output (see Remark 4). Indeed, in the case where the
uncertainties enter the model in a linearly dependent function, e.g. when the matric $\Delta$ has only one non-zero line, some of the classical available modular model-based adaptive controllers, like for instance X-swapping controllers, cannot be used to estimate all the uncertain parameters simultaneously. For example, it has been shown in [15], that the model-based gradient descent filters failed to estimate simultaneously multiple parameters in the case of the electromagnetic actuators example. For instance, in comparison with the ES-based indirect adaptive controller of [21], the modular approach does not rely on the parameters mutual exhaustive assumption, i.e., each element of the control vector needs to be linearly dependent on at least one element of the uncertainties vector. More specifically, we consider here the following case: $\Delta(1,1) = 0.3$, $\Delta(1,2) = 0.6$, and $\Delta(2,i) = 0$, $i = 1, 2$. In this case, the uncertainties’ effect on the acceleration $\ddot{q}_1$ cannot be differentiated, and thus the application of the model-based X-swapping method to estimate the actual values of both uncertainties at the same time is challenging. Similarly, the method of [21], cannot be readily applied because the second control $\tau_2$ is not linearly depend on the uncertainties, which only affects $\tau_1$. However, we show next that, by using the modular ISS-based controller, we manage to estimate the actual values of the uncertainties simultaneously and improve the tracking performance.

A. MES-based uncertainties estimation

The estimates of the two parameters $\hat{\Delta}_i$ ($i = 1, 2$) are computed using a discrete version of (24), given by

\begin{align}
x_i(k+1) &= x_i(k) + a_it_f \sin(\omega_it_f k + \frac{\pi}{2}) J(\hat{\Delta}), \\
\hat{\Delta}_i(k+1) &= x_i(k+1) + a_i \sin(\omega_it_f k - \frac{\pi}{2}), \quad i = 1, 2
\end{align}

where, $k \in \mathbb{N}$ denotes the iteration index, $x_i(0) = \hat{\Delta}_i(0) = 0$. We choose the following learning cost function

\begin{align}
J(\hat{\Delta}) &= \int_{t_f}^{0} (q(\hat{\Delta}) - q_d(t))^T Q_1 (q(\hat{\Delta}) - q_d(t)) dt \\
&+ \int_{t_f}^{0} (\dot{q}(\hat{\Delta}) - \dot{q}_d(t))^T Q_2 (\dot{q}(\hat{\Delta}) - \dot{q}_d(t)) dt,
\end{align}

where $Q_1 > 0$ and $Q_2 > 0$ denote the weight matrices. We implement the learning parameters: $a_1 = 0.1$, $a_2 = 0.05$, $\omega_1 = 7 \text{ rad/sec}$, $\omega_2 = 5 \text{ rad/sec}$. The obtained performance cost function is displayed on Figure 1(a), where we see that the performance improves over the learning iterations. The corresponding parameters estimation profiles are reported on Figures 1(b) and 1(c), which show a quick convergence of the first estimates $\hat{\Delta}_1$ to a neighborhood of the actual value. The convergence of the second estimates $\hat{\Delta}_2$ is slower, which is expected from the ES algorithms when many parameters are estimated at the same time. One has to underline here, however, that the convergence speed of the estimates and the excursion around their final mean values, can be directly fine-tuned by the proper choice of the learning coefficients $a_i$, $\omega_i$, $i = 1, 2$ in equation (35).

Finally, The tracking performance is shown on Figures 2(a) 2(b), where we can see that, after learning the actual values of the uncertainties, the tracking of the desired trajectories is recovered. We only show the first angular trajectories here, because the uncertainties affect directly only the acceleration $\ddot{q}_1$, and their effect on the tracking for the second angular variable is negligible.
B. GP UCB-based uncertainties estimation

In this section, to show that the modular ISS-based controller is independent of the choice of the learning algorithm, we apply the GP-UCB learning algorithm-based estimator to the same two-links manipulator example. We apply the algorithm \[ IV-D \] with the following parameters: \( \sigma = 0.1, l = 0.2, \) and \( \beta_t = 2\log\left(\frac{\text{card}(D')}{\delta}\right)^{\frac{1}{2}}\pi^2 \), with \( \delta = 0.05 \).

We test the GP-UCB algorithm under the same conditions as in the previous section. The obtained parameters and tracking results are reported on figures 3(a), 3(b), 3(c), 4(a), 4(b). We can see on these figures that similar
to the MES-based adaptive controller, the uncertainties are well estimated. One could argue that they are better estimated with the GP-UCB algorithm because there is no permanent dither signal, which leads to permanent oscillations in the MES-based learning. The tracking performance is clearly improved in this case as well, due to the precise estimation of the parameters.
(a) Cost function over the learning iterations (GP-UCB)

(b) Estimate of $\hat{\Delta}_1$ over the learning iterations (GP-UCB)

(c) Estimate of $\hat{\Delta}_2$ over the learning iterations (GP-UCB)

Fig. 3. Cost function and uncertainties estimates- (GP-UCB) algorithm
VI. CONCLUSION

We have studied the problem of adaptive control for nonlinear systems which are affine in the control with parametric uncertainties. For this class of systems, we have proposed the following controller: We use a modular approach, where we first design a robust nonlinear controller, designed based on the model (assuming knowledge of the uncertain parameters), and then complement this controller with an estimation module to estimate the actual values of the uncertain parameters. This type of modular approaches are certainly not new, e.g., the X-swapping methods. However, the novelty here is that the estimation module that we propose is based on model-free learning algorithms. Indeed, we propose to use two learning algorithms, namely, a multi-parametric extremum seeking algorithm, and a GP-UCB algorithm, to learn in realtime the uncertainties of the model. We call the learning approach ‘model-free’ for the simple reason that it only requires to measure an output signal from the system and compare it to a desired reference signal (independent of the model), to learn the best estimates of the
model uncertainties. We have guaranteed the stability (while learning) of the proposed approach, by ensuring that the model-based robust controller, leads to an ISS results, which guarantees boundedness of the states of the closed-loop system, even during the learning phase. The ISS result together with a convergent learning-algorithm eventually leads to a bounded output tracking error, which decreases with the decrease of the estimation error. We believe that, one of the main advantages of the proposed controller, comparatively to the existing model-based adaptive controllers, is that we can learn (estimate) multiple uncertainties at the same time even if they appear in the model equation in a challenging way, e.g., linearly dependent uncertainties affecting only one output, or uncertainties appearing in a nonlinear term of the model, which are well known limitations of the model-based estimation approaches. Another advantage of the proposed approach, is that due to its modular design, one could easily change the learning algorithm without having to change the model-based part of the controller. Indeed, as long as the first part of the controller, i.e., the model-based part, has been designed with a proper ISS property, one can ‘plug into it’ any convergent learning model-free algorithm, as demonstrated here by using two different learning approaches. We reported in this short paper some preliminary results about using GP-UCB in a modular adaptive control setting. In a longer journal version of the work, we will report more detailed comparisons between the MES-based adaptive controller, the GP UCB-based controller (for example in a more realistic noisy environment), and some existing model-based ‘classical’ adaptive controllers, e.g., as found in [1].

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