On Radical Property of Cross Polyomino Ideal

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Abstract. Roughly speaking, polyomino is a set of unit squares on the plane which is made by joining the squares side by side. The relation between polyomino and commutative algebra were established by Qureshi. Qureshi introduce the ideal from a polyomino. This ideal is called polyomino ideal. The cross polyomino is an example of polyomino whose ideal is not prime. In this article, we will discuss the radical property of cross polyomino ideal.

1. Introduction
Polyomino is a two dimensional object arising from combinatorics and recreational mathematics. It is related to the tiling problem on the plane. [1, 4, 5, 6]. Roughly speaking, polyomino is a set of unit squares on the plane which is made by joining the squares side by side.

The relation between polyomino and commutative algebra was introduced by Qureshi [11]. Qureshi introduced an ideal associated to polyominoes, which is called polyomino ideal [11]. Qureshi defined the convex polyomino and show that the associated polyomino ideal is prime ideal [11]. Simple polyominoes was also introduced on the same paper. Qureshi claimed that every simple polyomino has prime polyomino ideal. This claim has already proven in 2017 [12]. Qureshi and Hibi defined a specific class of non-simple polyomino and show that the associated polyomino ideal is prime ideal by localization technique [8].

The polyomino ideals is a generalization of ideals generated by the set of 2-minor of a matrix. Generally, the ideal of $t$-minor has an application in algebraic statistics [13, 10]. Motivated by that fact, there are a lot of research related to the ideal generated by the set of $t$—minor of a matrix [9, 7].

Another class of ideal which is related to prime ideal is radical ideal. Every prime ideal is radical, but the converse is not always true. Radical ideal is an important ingredient in algebraic geometry, for example the strong Nullstellensatz theorem [2]. Qureshi gave an example of non-simple polyomino which is not prime [12]. It can be checked that that example has radical polyomino ideal.

The organization of this paper is as follows. In the second section, we give the formal definition of polyominoes and some terminologies that related to polyominoes and will be used on the next section. In the third section, we will define the notion of cross polyominoes. We will prove a result related to a radical property of ideal associated to the cross polyomino.

2. Terminologies
We will explain some brief terminologies related to polyominoes and polyomino ideals from [11] that will be used in the whole paper. In this paper, all fields has characteristic of zero. Consider
the set \( \mathbb{N}^2 \) and define the partial order: \((i, j) \leq (k, \ell)\) if and only if \(i \leq k\) and \(j \leq \ell\).

(i) Let \(a, b \in \mathbb{N}^2\) with \(a \leq b\). The set \([a, b] = \{c \in \mathbb{N}^2 \mid a \leq c \leq b\}\) is called an interval.

(ii) Let \(a = (i, j)\) and \(b = (k, \ell)\). The elements \(a\) and \(b\) are called the diagonal corners of the interval \([a, b]\), and the elements \((i, \ell)\) and \((k, j)\) are called the antidiagonal corners of the interval \([a, b]\). Particularly, the elements \((i, j)\) and \((k, \ell)\) are called left lower corner and the right upper corner, respectively, of the interval \([a, b]\).

(iii) If \(b = a + (1, 1)\) then the interval \([a, b]\) is called a cell.

(iv) The edges of a cell \([a, a+(1,1)]\) are the intervals: \([a,a+(1,0)], [a,a+(0,1)], [a+(0,1), a+(1,1)], [a+(1,0), a+(1,1)]\).

(v) Let \(P\) be a finite collection of cells in \(\mathbb{N}^2\). The collection of all vertices of \(P\), denoted by \(V(P)\) is the union of all corners from each cell in \(P\).

(vi) Let \(a = (i, j), b = (k, \ell) \in \mathbb{N}^2\). The vertices \(a\) and \(b\) is called in a horizontal position if \(j = \ell\) and is called in a vertical position if \(i = k\).

(vii) Let \(P\) be a finite collection of cells in \(\mathbb{N}^2\). Let \(C\) and \(D\) be two cells in \(P\). The cells \(C\) and \(D\) are called connected if there exist a sequence of cells \(C = C_1, \ldots, C_m = D\) in \(P\) such that \(C_i \cap C_{i+1}\) is an edge of \(C_i\) for all \(i = 1, 2, \ldots, m - 1\).

(viii) The collection \(P\) is called a polyomino if any two cells in \(P\) are connected.

(ix) Let \(P\) be a polyomino in \(\mathbb{N}^2\). Let \((i,j), (k,\ell) \in V(P)\) such that \(i < k\) and \(j < \ell\). The interval \([i,j] = \{(r,s) \mid (r,s) \in [i,j]\}\) is called an inner interval of \(P\) if any cell in \([i,j]\) is an element in \(P\) for all \(i \leq r \leq k - 1\) and \(j \leq s \leq \ell - 1\).

(x) Let \(P\) be a polyomino and \(K\) be a field. Define the polynomial ring \(S\) over \(K\) with variables \(x_{ij}\) for all \((i, j) \in V(P)\). Each inner interval \([i,j]\) in \(P\) is associated to \(x_{ij}x_{k\ell} - x_{i\ell}x_{kj} \in S\), that is called the inner minor of \(P\).

(xi) Let \(P\) be a polyomino. The ideal \(I_P \subseteq S\) is an ideal generated by all inner minors of \(P\). The ideal \(I_P\) is called the polyomino ideal.

3. Cross Polyomino

We start the definition of cross polyomino with the notion of rectangle polyomino.

**Definition 3.1.** A polyomino \(P\) will be called rectangle polyomino if \(P = [a, b]\) for two distinct points \(a, b \in \mathbb{N}^2\).

This rectangle polyomino is basically an interval on \(\mathbb{N}^2\).

**Definition 3.2.** A polyomino \(P_{\mathcal{R}_1, \mathcal{R}_2}\) will be called cross polyomino if there exist two rectangle \(\mathcal{R}_1, \mathcal{R}_2\) that satisfy:

(i) Both \(\mathcal{R}_1 - \mathcal{R}_2\) and \(\mathcal{R}_2 - \mathcal{R}_1\) are not polyominoes.

(ii) if \(Q = \mathcal{R}_1 \cap \mathcal{R}_2 = [x, y]\) for some \(x, y \in \mathbb{N}^2\) and \(\mathcal{R} = [x - (1, 1), y + (1, 1)]\) then

\[
\mathcal{P}_{\mathcal{R}_1, \mathcal{R}_2} = (\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}) \setminus Q.
\]

Most of polyomino ideal of cross polyomino is not prime. For example, the following polyomino (see Figure 1) which can be found in [12]. There are 36 vertices in this polyomino, hence \(S = K[x_1, \ldots, x_{36}]\). Let \(I \subseteq S\) be the polyomino ideal. Note that \(x_1x_{10} - x_2x_9, x_6x_{12} - x_5x_9, x_{10}x_8 - x_7x_{11}\), and \(x_{12}x_3 - x_4x_{11}\) are elements of \(I\). Notice that

\[
x_1x_7x_{11} - x_2x_8x_9, x_4x_6x_{11} - x_3x_5x_9 \in I,
\]
because $x_1x_7x_{11} - x_2x_8x_9 = (x_1x_{10} - x_2x_9)x_8 - (x_{10}x_8 - x_7x_{11})x_1$ and $x_4x_6x_{11} - x_3x_5x_9 = (x_6x_{12} - x_5x_9)x_3 - (x_{12}x_3 - x_4x_{11})x_6$. By the same argument, we have

$$(x_1x_3x_5x_7 - x_2x_4x_6x_8)x_9 = x_1x_3x_5x_7x_9 - x_2x_4x_6x_8x_9 \in I.$$ 

If $I$ is a prime ideal, since $x_9 \notin I$ then $x_1x_3x_5x_7 - x_2x_4x_6x_8 \in I$. But this is impossible because there is no inner minor which has a term that divides neither $x_1x_3x_5x_7$ nor $x_2x_4x_6x_8$.

**Figure 1.** Non-prime polyomino ideal. The shaded region is not the part of polyomino.

Here is the main theorem of this article.

**Theorem 3.3.** Let $R_1, R_2$ be rectangle polyominoes such that $P_{R_1, R_2}$ is a cross polyomino. If $R_1 \cap R_2$ is a cell then the polyomino ideal associated to cross polyomino $P_{R_1, R_2}$ is radical.

The method for proving this theorem is based on the following lemma.

**Lemma 3.4.** [3] Let $S = K[x_1, x_2, \ldots, x_n]$ and $I \subseteq S$ be an ideal. If for some monomial orders $<$ in $S$, the initial ideal $in_{<}(I)$ is generated by square-free monomials then $I$ is a radical ideal.

Recall that, Gröbner bases of an ideal $I$ with respect to a monomial order $<$ is the generating set of $I$ such that the ideal generated by initial of each element of the Gröbner bases is the initial ideal $in_{<}(I)$. Thus, we need to set a monomial order on the polynomial ring, find the Gröbner bases of the polyomino ideal associated to cross polyomino ideal that is described in Theorem 3.3, and prove that the initial term of each element in the Gröbner bases is squarefree.

Let $P$ be a cross polyomino and $n = |V(P)|$. In this article, we use lexicographic order in polynomial ring $K[x_{ij} : (i, j) \in V(P)]$ based on total order $x_{ij} > x_{k\ell}$ if and only if one of the following holds

- $j < \ell$
- $j = \ell$ and $i < k$

We note some special cases of polyominoes. Consider the polyomino in Figure 2 with 4 cells labelled 1,2,3, and 4. We choose to omit or not to omit each of these cells. There are 16 different polyominoes that can be constructed from this polyominoes. We call them ”small” cross polynomials. For every small cross polyominoes, we can compute the Gröbner bases of each polyomino ideals by using some algebra softwares (we use singular) and confirm that every polyomino ideals are radical.

Let $P = P_{R_1, R_2}$ be a cross polyomino such that $R_1 \cap R_2$ is a cell. By translating the polyomino, we may assume that $(0, 0)$ is the right lower corner of $R_1 \cap R_2$. Let $R_1$ be the rectangle with height of 1 unit. We will use the previous convention throughout the rest of this article.
(vii) Let $f_1, f_2$ be elements of the Gröbner bases such that $x_{03}$ divides both $\text{in}_<(f_1)$ and $\text{in}_<(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$ for some $b_1, \ldots, b_r$ in the Gröbner bases, $\text{in}_<(b_i) \leq \text{in}_<(S(f_1, f_2))$, and $x_{21} \mid q_i$ for all $1 \leq i \leq r$.

(viii) Let $f$ be an element of the Gröbner bases, if $x_{0n}$ divides $\text{in}_<(f)$ for some positive integer $n$ then the integer $n$ is unique, moreover

(a) For $n > 2$, $x_{0n}$ divides $\text{in}_<(f)$ if and only if $x_{-1,n}$ divides to non-initial term of $f$
(b) If $n = -1$ then $f$ is an inner minor.
(c) There exist some $\alpha < (n, 1)$ such that $x_{\alpha} \mid \text{in}_<(f)$.

Observation 3.5. For small cross polyominoes, there is a Gröbner bases arising from Buchberger algorithm for their polyomino ideal satisfy the following properties:

(i) Each element is a binomial.
(ii) Both initial and non-initial terms are squarefree.
(iii) Let $f_1, f_2$ be elements of the Gröbner bases such that $x_{20}$ divides both $\text{in}_<(f_1)$ and $\text{in}_<(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$ for some $b_1, \ldots, b_r$ in the Gröbner bases, $\text{in}_<(b_i) \leq \text{in}_<(S(f_1, f_2))$, and $x_{21} \mid q_i$ for all $1 \leq i \leq r$.
(iv) Let $f$ be an element of the Gröbner bases, if $x_{n0}$ divides $\text{in}_<(f)$ for some nonnegative integer $n$ then the integer $n$ is unique, moreover

(a) For $n > 1$, $x_{n0}$ divides $\text{in}_<(f)$ if and only if $x_{n1}$ divides to non-initial term of $f$
(b) If $n = 0$ then $f$ is an inner minor.
(c) For $n > 0$, $f$ is an inner minor or for any integer $m > n$, $x_{m0}$ doesn’t divide the non-initial terms of $f$.
(d) $f$ is an inner minor or there exist some $\alpha < (n, 0)$ such that $x_{\alpha} \mid \text{in}_<(f)$.
(v) Let $f_1, f_2$ be elements of the Gröbner bases such that $x_{-3,1}$ divides both $\text{in}_<(f_1)$ and $\text{in}_<(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$ for some $b_1, \ldots, b_r$ in the Gröbner bases, $\text{in}_<(b_i) \leq \text{in}_<(S(f_1, f_2))$, and $x_{-3,0} \mid q_i$ for all $1 \leq i \leq r$.

Figure 2. Small cross polyomino.
Let $f_1, f_2$ be elements of the Gröbner bases such that $x_{-1,-2}$ divides both $\text{in}_{<}(f_1)$ and $\text{in}_{<}(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$ for some $b_1, \ldots, b_r$ in the Gröbner bases, $\text{in}_{<}(b_i) \leq \text{in}_{<}(S(f_1, f_2))$, and $x_{0,-2} \mid q_i$ for all $1 \leq i \leq r$.

Let $f$ be an element of the Gröbner bases, if $x_{-1,n}$ divides $\text{in}_{<}(f)$ for some negative integer $n$ then the integer $n$ is unique, moreover

(a) For $n < -1$, $x_{-1,n}$ divides $\text{in}_{<}(f)$ if and only if $x_{0n}$ divides to non-initial term of $f$

(b) If $n < -1$ then for every integer $m < n$, $x_{-1,m}$ doesn't divide the non-initial terms of $f$.

For any cross polyomino $P$ that meet the condition in the Theorem 3.3, there is a unique cross polyomino $P_0$ with the largest number of cells such that $P$ can be constructed by adding a cell to the leftmost, the rightmost, the topmost, and/or the bottommost cells of $P_0$ in any sequence until we get $P$. We need to argue that the new polyomino ideal is radical at each step. Consider the following four lemmas.

**Lemma 3.6** (Right addition). Let $P = P_{R_1,R_2}$ be a cross polyomino such that $R_1$ is the rectangle of height of 1, $R_1 \cap R_2 = [(-1,0),(0,1)]$ is a cell, and the right lower corner of $R_1$ is $(k,0)$ for some $k \geq 2$ then there exist a Gröbner bases of $I_P$ such that

(i) Each element is a binomial.

(ii) Both initial and non-initial terms are squarefree.

(iii) Let $s,t$ be the right lower, upper vertex of $R_1$, respectively. Let $f_1, f_2$ be elements of the Gröbner bases such that $x_s$ divides both $\text{in}_{<}(f_1)$ and $\text{in}_{<}(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$ for some $b_1, \ldots, b_r$ in the Gröbner bases, $\text{in}_{<}(b_i) \leq \text{in}_{<}(S(f_1, f_2))$, and $x_t \mid q_i$ for all $1 \leq i \leq r$.

(iv) Let $f$ be an element of the Gröbner bases, if $x_{0n}$ divides $\text{in}_{<}(f)$ for some nonnegative integer $n$ then the integer $n$ is unique, moreover

(a) For $n > 1$, $x_{0n}$ divides $\text{in}_{<}(f)$ if and only if $x_{n1}$ divides to non-initial term of $f$

(b) If $n = 0$ then $f$ is an inner minor.

(c) For $n > 0$, $f$ is an inner minor or for any integer $m > n$, $x_{0n}$ doesn't divide the non-initial terms of $f$.

(d) $f$ is an inner minor or there exist some $\alpha < (n,0)$ such that $x_\alpha \mid \text{in}_{<}(f)$.

**Proof.** Let $P = P_{R_1,R_2}$ be a cross polyomino such that $R_1 \cap R_2$ is a cell, $(0,0)$ is the right lower corner of $R_1 \cap R_2$, $R_1$ is the rectangle with height of 1 unit, and $a = (k,0)$ be the right lower corner of $R_1$. Let $R'_1 = R_1 \cup [a, a + (1,1)]$ and $P' = P_{R'_1,R_2}$. Let $b,c,d$ denote the vertices $a + (1,1), a + (1,0), a + (0,1)$, respectively. We will prove that the ideal $I_{P'}$ is radical.

Let $G$ be the Gröbner bases of $I_P$ that satisfy the above properties. Let

$$G_1 = \{x_{0n}x_b - x_{n1}x_c : 0 \leq n \leq k\}$$

and

$$G_2 = \left\{ \frac{1}{x_d} S(f, x_ax_b - x_cx_d) : f = x_af^+ - x_df^- \in G \text{ such that } \text{in}_{<}(f) = x_af^+ \right\}.$$ 

We claim that the set $G' = G \cup G_1 \cup G_2$ is the Gröbner bases of $I_{P'}$ that satisfying the same properties with $G$.

Define the mapping $\phi : K[x_{ij} : (i,j) \in V(P)] \to K[x_{ij} : (i,j) \in V(P')]$ by

$$\phi(x_{ij}) = \begin{cases} x_{i+1,j} & \text{if } i = k \\ x_{ij} & \text{otherwise.} \end{cases}$$
Note that, this mapping \( \phi \) is an injective mapping and maps \( G \) to \( G' \). Moreover if two element \( f, g \in K[x_{ij} : ij \in V(P')] \) has inverse images \( f', g' \) respectively and \( S(f', g') \) is reduced to zero by \( G \) then \( S(f, g) \) is reduced to zero by \( G' \) as well by the correspondence.

It’s clear that \( G' \) generates \( I_P \). We will check whether \( G' \) is the Gröbner bases by using the Buchberger criterion. Let \( f_1, f_2 \in G' \). We will compute the \( S \) polynomial \( S(f_1, f_2) \) and prove that the \( S \) polynomial will be reduced to zero by \( G' \). We divide the computation into 6 cases

(i) If \( f_1, f_2 \in G \) then \( S(f, g) \) will be reduced to zero by \( G \).
(ii) If \( f_1, f_2 \in G_1 \) then \( S(f, g) = x_c(x_{00}x_{m1} - x_{1n1}x_{m0}) \) for some \( 0 \leq m, n \leq k \). Therefore, \( S(f_1, f_2) \) is reduced to zero by \( G \).
(iii) If \( f_1, f_2 \in G_2 \) then assume \( f_i = f_i^+x_c - f_i^-x_b \) with \( \text{in}_<(f_i) = f_i^+c \) for \( i = 1, 2 \). Note that

\[
S(f_1, f_2) = \frac{x_b}{x_d} \left( \frac{x_a f_1^+ x_d f_2^-}{\gcd(x_a f_1^+, x_a f_2^+)} - \frac{x_a f_2^+ x_d f_1^-}{\gcd(x_a f_1^+, x_a f_2^+)} \right) = \frac{x_b}{x_d} S(x_a f_1^+ x_d f_2^- - x_a f_2^+ x_d f_1^-).
\]

By the assumption, \( S(x_a f_1^+ x_d f_2^- - x_a f_2^+ x_d f_1^-) = q_1 b_1 + \ldots + q_r b_r \) for some \( b_1 \ldots b_r \in G \) and \( x_d | q_i \) for all \( 1 \leq i \leq r \). This shows that \( S(f_1, f_2) \) is reduced to zero by \( G \).
(iv) If \( f_1 \in G_1 \) and \( f_2 \in G_2 \) then by the assumption we must have \( \gcd(\text{in}_<(f_1), \text{in}_<(f_2)) = 1 \). Thus \( S(f_1, f_2) \) is reduced to zero by \( G' \).
(v) If \( f_1 \in G \) and \( f_2 \in G_1 \) then we only need to compute \( S(f_1, f_2) \) for the case \( x_{(n0)} | \text{in}_<(f_1) \) and \( x_{(n0)} | \text{in}_<(f_2) \) for some \( 0 \leq n \leq k \).

- If \( n = 0 \) then \( f_1 \) is an inner minor of the form \( x_{00}x_{1c} - x_{00}x_{10} \). Thus

\[
S(f_1, f_2) = x_{01}(x_{11}x_c - x_{01}x_{10}).
\]

But \( x_{11}x_c - x_{10}x_{10} \in G_2 \) or equal to \( x_d x_c - x_a x_b \), therefore \( S(f_1, f_2) \) is reduced to zero by \( G' \).
- If \( n = k \) then \( f_1 = f_1^+x_a - f_1^-x_d \) and \( f_2 = x_a x_b - x_c x_d \). Therefore, \( S(f_1, f_2) \) is reduced to zero by \( G_2 \).
- If \( 0 < n < k \) then we divide this into two cases

  - If \( f_1 \) is an inner minor then \( f_1 \) has the form \( x_{n0}x_{1c} - x_{n1}x_{10} \) or \( x_{\ell - 1} x_{10} - x_{-1} x_{10} \).

    For the first case, by the same argument as above, we have \( S(f_1, f_2) \) is reduced to zero by \( G' \). For the second case, note that \( f_1 \) and \( f_2 \) have inverse image by \( \phi \) and the \( S \) polynomial of the inverse images is reduces to zero, therefore \( S(f_1, f_2) \) is reduced to zero.
Let $s, t$ be the left upper, lower vertex of $R_1$, respectively. Let $f_1, f_2$ be elements of the Gröbner bases such that $x_s$ divides both $in_{<}(f_1)$ and $in_{<}(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$, for some $b_1, \ldots, b_r$ in the Gröbner bases, $in_{<}(b_i) \leq in_{<}(S(f_1, f_2))$, and $x_t \mid q_i$ for all $1 \leq i \leq r$.

(iv) Let $f$ be an element of the Gröbner bases, if $x_{n1}$ divides $in_{<}(f)$ for some negative integer $n$ then the integer $n$ is unique, moreover

(a) For $n < -2$, $x_{n1}$ divides $in_{<}(f)$ if and only if $x_{n0}$ divides to non-initial term of $f$

(b) If $n = -1$ then $f$ is an inner minor.

(c) For $n < -1$, $f$ is an inner minor or for any integer $m < n$, $x_{m1}$ doesn’t divide the non-initial terms of $f$.

(d) There exist some $\alpha < (n, 1)$ such that $x_{\alpha} \mid in_{<}(f)$.

Lemma 3.8 (Top addition). Let $P = \mathcal{P}_{R_1, R_2}$ be a cross polyomino such that $R_1$ is the rectangle of height of 1, $R_1 \cap R_2 = [(-1, 0), (0, 1)]$ is a cell, and the left lower corner of $R_1$ is $(k, 0)$ for some $k \leq -3$ then there exists a Gröbner bases of $I_P$ such that

(i) Each element is a binomial.

(ii) Both initial and non-initial terms are squarefree.

(iii) Let $s, t$ be the right, left upper vertices of $R_2$, respectively. Let $f_1, f_2$ be elements of the Gröbner bases such that $x_s$ divides both $in_{<}(f_1)$ and $in_{<}(f_2)$ then $S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r$, for some $b_1, \ldots, b_r$ in the Gröbner bases, $in_{<}(b_i) \leq in_{<}(S(f_1, f_2))$, and $x_t \mid q_i$ for all $1 \leq i \leq r$. 

With the same technique, we have the following three lemmas.
(iv) Let \( f \) be an element of the Gröbner bases, if \( x_{0n} \) divides \( \in_s(f) \) for some positive integer \( n \) then the integer \( n \) is unique, moreover

(a) For \( n > 2 \), \( x_{0n} \) divides \( \in_s(f) \) if and only if \( x_{-1,n} \) divides to non-initial term of \( f \)

(b) For \( n > 1 \), \( f \) is an inner minor or for any integer \( m > n \), \( x_{0m} \) doesn’t divide the non-initial terms of \( f \).

(c) \( f \) is an inner minor or there exist some \( \alpha < (-1,n) \) such that \( x_\alpha | \in_s(f) \).

Lemma 3.9 (Bottom addition). Let \( P = P_{R_1,R_2} \) be a cross polyomino such that \( R_1 \) is the rectangle of height of 1, \( R_1 \cap R_2 = [(-1,0),(0,1)] \) is a cell, and the right lower corner of \( R_1 \) is \((0,k)\) for some \( k \leq -2 \) then there exist a Gröbner bases of \( I_P \) such that

(i) Each element is a binomial.

(ii) Both initial and non-initial terms are squarefree.

(iii) Let \( s,t \) be the left, right lower vertices of \( R_2 \), respectively. Let \( f_1, f_2 \) be elements of the Gröbner bases such that \( s \) divides both \( \in_s(f_1) \) and \( \in_s(f_2) \) then \( S(f_1, f_2) = q_1b_1 + \cdots + q_rb_r \) for some \( b_1, \ldots, b_r \) in the Gröbner bases, \( \in_s(b_i) \leq \in_s(S(f_1, f_2)) \), and \( x_t | q_i \) for all \( 1 \leq i \leq r \).

(iv) Let \( f \) be an element of the Gröbner bases, if \( x_{-1,n} \) divides \( \in_s(f) \) for some negative integer \( n \) then the integer \( n \) is unique, moreover

(a) For \( n < -1 \), \( x_{-1,n} \) divides \( \in_s(f) \) if and only if \( x_{0m} \) divides to non-initial term of \( f \)

(b) If \( n < -1 \) then for every integer \( m < n \), \( x_{-1,m} \) doesn’t divide the non-initial terms of \( f \).

Based on the previous lemmas, the proof of Theorem 3.3 is finished. Based on the observation on singular application, we claim that every cross polyominoes (even any polyominoes) has radical polyomino ideal. We have not have a proof for this claim.

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