C\textsubscript{T} for conformal higher spin fields from partition function on conically deformed sphere

Matteo Beccaria\textsuperscript{a} and Arkady A. Tseytlin\textsuperscript{b}

\textsuperscript{a}Dipartimento di Matematica e Fisica Ennio De Giorgi, Università del Salento & INFN, Via Arnesano, 73100 Lecce, Italy
\textsuperscript{b}The Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

E-mail: matteo.beccaria@le.infn.it, tseytlin@imperial.ac.uk

ABSTRACT: We consider the one-parameter generalization S\textsuperscript{4}\textsubscript{q} of 4-sphere with a conical singularity due to identification \( \tau = \tau + 2\pi q \) in one isometric angle. We compute the value of the spectral zeta-function at zero \( \tilde{\zeta}(q) = \zeta(0; q) \) that controls the coefficient of the logarithmic UV divergence of the one-loop partition function on \( S\textsuperscript{4}\textsubscript{q} \). While the value of the conformal anomaly a-coefficient is proportional to \( \tilde{\zeta}(1) \), we argue that in general the second \( c \sim C\textsubscript{T} \) anomaly coefficient is related to a particular combination of the second and first derivatives of \( \tilde{\zeta}(q) \) at \( q = 1 \). The universality of this relation for \( C\textsubscript{T} \) is supported also by examples in 6 and 2 dimensions. We use it to compute the c-coefficient for conformal higher spins finding that it coincides with the "\( r = -1 \)" value of the one-parameter Ansatz suggested in \texttt{arXiv:1309.0785}. Like the sums of \( a\textsubscript{s} \) and \( c\textsubscript{s} \) coefficients, the regularized sum of \( \tilde{\zeta}\textsubscript{s}(q) \) over the whole tower of conformal higher spins \( s = 1, 2, ... \) is found to vanish, implying UV finiteness on \( S\textsuperscript{4}\textsubscript{q} \) and thus also the vanishing of the associated Rényi entropy. Similar conclusions are found to apply to the standard 2-derivative massless higher spin tower. We also present an independent computation of the full set of conformal anomaly coefficients of the 6d Weyl graviton theory defined by a particular combination of the three 6d Weyl invariants that has a (2,0) supersymmetric extension.

\textsuperscript{1}Also at Lebedev Institute, Moscow.
1 Introduction

In this paper we revisit the question about conformal anomaly c-coefficients for conformal higher spin (CHS) fields previously addressed in [1, 2]. Conformal higher spins in 4d [3–8] have higher derivative kinetic terms \( h_s \partial^2 h_s \) (\( h_s \) are totally symmetric rank \( s \) tensors) and thus their generalization to a curved metric background is a non-trivial question (cf. [1, 9, 10, 2]). A curved space generalization is required in order to compute the corresponding conformal anomaly coefficients appearing in the (one-loop) effective action

\[
\Gamma = - \log Z = -B_4 \log \Lambda_{cv} + \text{finite}, \\
B_4 = \int d^4x \sqrt{g} b_4(x), \quad \bar{b}_4 = (4\pi)^2 b_4 = -a R^+ R^+ + c C^2 .
\]

Here \( R^+ R^+ \) is \( 32\pi^2 \) times the Euler density and \( C^2 \) is the square of the Weyl tensor. To compute the \( a_s \)-coefficient for spin \( s \) field it is enough to know the corresponding Weyl-covariant \( \nabla^{2s} + \ldots \) operator on a 4-sphere where it takes a simple factorized form of a
product of \( s \) "partially-massless" 2nd order Laplacians \([1, 11, 12, 9]\). As a result, one finds \([13, 1]\)

\[
a_s = \frac{1}{720} \nu \left( 14 \nu^2 + 3 \nu \right), \quad \nu \equiv s(s+1),
\]

where \( \nu \) is the number of dynamical degrees of freedom of a spin \( s \) CHS field. Remarkably, the total a-anomaly defined as the finite part of the regularized sum \( \sum_{s=1}^{\infty} e^{-\nu s} a_s \) vanishes \([13, 1]\). \(^1\) The factorization of the Weyl-covariant CHS kinetic operators applies for any conformally flat background, e.g., on \( S^4 \times S^3 \), where at large \( q = 2\pi \beta \) one finds that the corresponding free energy proportional to the Casimir energy \( E_c \) on \( \mathbb{R} \times S^3 \) is given by \([7]\) \(^2\)

\[
- \log Z(S^1_q \times S^3) \bigg|_{q \to \infty} = 2\pi q E_c + ... , \quad E_{c,s} = \frac{1}{720} \nu \left( 18\nu^2 - 14\nu - 11 \right) .
\]

To determine the value of the c-coefficient in (1.2) (which is proportional to the coefficient \( C_T \) in the 2-point function of the flat-space stress tensor) one is to consider a more general non conformally flat background. Assuming that \( c_s \) has a similar cubic \( \nu \)-polynomial structure as \( a_s \) in (1.3) and reproduce the known values for \( s = 1 \) \([16]\) and \( s = 2 \) \([17, 3]\), one is led to the following Ansatz \([1]\)

\[
c_s = \frac{1}{1080} \nu \left[ 43\nu^2 - 59\nu + r (\nu - 2)(\nu - 6) \right] ,
\]

where \( r \) is a free parameter. If one further assumes that all \( s > 2 \) CHS kinetic operators factorize into 2nd-order \( \nu \)-Lichnerowicz-type operators also on a Ricci flat background (like it happens for the \( s = 2 \) Weyl graviton) one then finds the expression (1.5) with \( r = \frac{1}{2} \) \([1]\).

However, the assumption of factorization on a generic Ricci-flat background is expected to fail in general \([9]\). Moreover, when the Weyl tensor is non-zero, different spins appear to mix in the kinetic term \([10]\) and the mixing terms lead to additional contributions to the total c-anomaly \([2]\). Also, the sum of \( c_s \) with \( r = \frac{1}{2} \) in (1.5) regularized with \( e^{-\nu s} \) does not vanish; it vanishes if instead one chooses \( r = -1 \) value \([1]\). \(^3\) The expression (1.5) with \( r = -1 \) was also shown to be selected by the consistency with the AdS/CFT - related correspondence between massless higher spin partition functions in (asymptotically) AdS\(_5\) space and the conformal higher spin partition functions at the 4d boundary \([18]\).

Below we will provide a strong independent evidence that the \( r = -1 \) value of (1.5)

\[
c_s = \frac{1}{1080} \nu \left( 14\nu^2 - 17\nu - 4 \right) ,
\]

\(^1\)This vanishing holds also in the more natural regularization \( \sum_{s=1}^{\infty} e^{-\nu (s+\frac{1}{2})} a_s \) consistent with AdS/CFT \([14]\). Besides, the regularized finite part of the total CHS partition function on \( S^4 \) is also trivial, i.e. \([15]\), i.e. \( \log Z_{\text{CHS}} = \sum_{s=1}^{\infty} \log Z_s = 0 \).

\(^2\)One finds again that the finite part of \( \sum_{s=1}^{\infty} e^{-\nu (s+\frac{1}{2})} E_{c,s} \) vanishes.

\(^3\)In fact, the finite part of \( \sum_{s=1}^{\infty} c_s e^{-\nu (s+\frac{1}{2})} \) vanishes for any value of \( r \). However, the special value \( r = -1 \) is still selected by demanding the consistency in the results for the total \( a \) and total \( c \) in the "minimal" case of a tower of even higher spins only where they should be opposite to the values for a complex 4d scalar for consistency with what happens for massless higher spins in AdS\(_5\) (see footnote 9 in \([14]\)).
is indeed the correct value of the c-coefficient for the conformal higher spin fields. The main idea will be to extract the value of $c_s$ from the CHS partition function on a 1-parameter deformation of the 4-sphere $S^4_q$ which is an Einstein space with a conical singularity

$$ds^2_q = d\theta^2 + \cos^2 \theta d\tau^2 + \sin^2 \theta d\Omega^2, \quad d\Omega^2 = da_1^2 + \sin^2 a_1 da_2^2,$$  \hspace{1cm} (1.7)

Here the deficit angle $2\pi(1-q)$ of the $\tau$ coordinate implies the presence of a conical singularity on a $S^2$ submanifold. For $q=\text{integer}$ this is a multiple cover of a sphere while for $q \equiv q^{-1}=\text{integer}$ this may be interpreted as an orbifold $S^4/\mathbb{Z}_q$.

The key observation is that since $S^4_q$ is locally conformally flat (away from conical singularity), one may assume that the CHS kinetic operator defined on $S^4_q$ still factorizes as it does on $S^4$, so that the expression for the partition function in terms of the contributions of determinants of 2nd order operators is then “inherited” from the $S^4$ case.\footnote{Alternatively, one may define the corresponding heat kernel in terms of the one on $S^4$ using Sommerfeld-type “orbifold” or “sum over images” construction.}

At the same time, having the Weyl tensor being non-zero at the singular subspace should allow one to extract the value of the c-coefficient from the $q$ dependence of the $B_4$ coefficient of the UV divergence in (1.1).

Note that the expression for $B_4$ in (1.2) applies to regular geometries while in the presence of conical singularities there will be additional “surface” terms \cite{19–21} with a non-trivial dependence on $q$ entering effectively through boundary conditions. As a result, the coefficient $B_4$ in (1.2) will become a non-trivial function of $q$. The log UV divergent contribution of the determinant of a single 2nd order Laplace-type operator can be computed as the value of the corresponding spectral zeta function at zero, i.e.\footnote{Note that $B_4$ as coefficient of the logarithmic divergence receives contribution from all (zero and non-zero) modes on the Laplacian, so that $\hat{\zeta}$ may need to be corrected by the contribution of the zero modes; here we formally assume that this correction is already taken into account.}

$$B_4(q) = \tilde{\zeta}(q), \quad \tilde{\zeta}(q) \equiv \zeta(0; q), \quad \zeta(z; q) = \sum_i d_i [\lambda_i(q)]^{-z}, \hspace{1cm} (1.8)$$

where $d_i$ are degeneracies of the eigenvalues $\lambda_i$. The general structure of $\tilde{\zeta}(q)$ for a bosonic field will be as follows

$$\tilde{\zeta}(q) = -\frac{\nu}{360q^3} + \frac{p_2}{q^2} + \frac{p_1}{q} + p_0 - 2E_c q, \hspace{1cm} (1.9)$$

where $\nu$ is the number of physical degrees of freedom (equal to 1 for a real $\partial^2$ scalar and $s(s+1)$ in the bosonic CHS field as in (1.3)) and $E_c$ is the corresponding Casimir energy on $S^3$.

Assuming one is able to compute $\tilde{\zeta}(q)$ it remains to extract the conformal anomaly $a$- and $c$-coefficients from it. Since the $q = 1$ case corresponds to the regular sphere $S^4$ when $B_4 = -2a\chi(S^4) = -4a$ we should have\footnote{This relation is true for the final expression for $\tilde{\zeta}$ taking into account possible zero modes arising from decomposition of fields into transverse and longitudinal parts, see footnote 14.}

$$a = -\frac{1}{4} \tilde{\zeta}(1). \hspace{1cm} (1.10)$$
We shall propose that the expression for $c$ in terms of $\hat{\zeta}(q)$ should read

$$c = -\frac{1}{4} \left[ q \hat{\zeta}(q) \right]'' \bigg|_{q=1} = -\frac{1}{4} \hat{\zeta}''(1) - \frac{1}{2} \hat{\zeta}'(1),$$  \hspace{1cm} (11.11)

where $\hat{\zeta}'(q) \equiv \frac{d}{dq} \hat{\zeta}(q)$. As we shall see, the same relation ($k_d$ is dimension-dependent normalization factor)

$$C_T = k_d \left[ q \hat{\zeta}(q) \right]'' \bigg|_{q=1},$$

is true also in $d = 2$ (C.4) and $d = 6$ (4.3) cases where $c$ is replaced by the corresponding $C_T$ coefficient (proportional to $a = \frac{1}{4} c$ in 2d and $c_3$ in 6d). This suggests its universal validity.

The relation (11.11) can be directly verified in all low-spin ($s \leq 1$) cases. Since $S_q^4$ is conformally related to $S_q^1 \times \mathbb{H}^3$ space, in the low-spin cases where $\hat{\zeta}(1) = 0$, eq. (11.11) becomes equivalent the relation derived in [22]. As it turns out, for higher-spin $s \geq 2$ cases $\hat{\zeta}(1) \neq 0$ and thus it is the relation (11.11) that should be applied.\(^7\)

Intuitively, the reason why $c$ should be related to $\hat{\zeta}''(1)$ can be understood from the fact that $c \sim C_T$ should be proportional to the 2-point function of the stress tensor which itself should be given by the second variation of the effective action over the metric, i.e. the second term in the expansion in the small deformation of the metric (1.7) away from the sphere $q = 1$ case.

One may be tempted to represent $B_4(q)$ as in (1.2), i.e. as a curvature integral with the $a$ and $c$ as coefficients of different geometrical invariants. However, $S_q^4$ has a singular curvature on $S^2$, and at best one may hope to get $B_4(q) = q \int_{S^4 \setminus \Sigma} b_4^{\text{bulk}} + \int_{\Sigma} b_4^{\text{surf}}(q)$, where $b_4^{\text{bulk}}$ is as in (1.2) and is evaluated on a smooth metric, while $b_4^{\text{surf}}(q)$ non-trivially depends on $q$ and invariants of $S^2$. Ref. [19] gave an explicit analysis of the conformal scalar operator on a singular manifold $\mathcal{M}_q = \mathcal{C}_q \times \Sigma$, where $\mathcal{C}_q$ is a flat cone with metric $ds^2 = dr^2 + r^2 d\phi^2$, $0 \leq \phi < 2\pi q$, while $\Sigma$ does not depend on $q$. The above splitting of $B_4(q)$ into "bulk" and "surface" parts can then be proved and also checked to be in agreement with the expression of $B_4(q)$ in terms of the spectral zeta-function [21] (see also Appendix F below). It is important to stress that in general the surface term $b_4^{\text{surf}}(q)$ has coefficients depending on $q$ in a non-universal way, i.e. its dependence on the spin of the field is only partly encoded in the values of the $a$ and $c$ coefficients. A major simplification occurs at first order in expansion in small $1 - q$ and for low spin $s = 0, \frac{1}{2}, 1$. In these cases it is possible to use the integral density in (1.2) and take the singular manifold into account by a delta-function contribution to the curvature [23, 20] (see also [24–26]). However, this is not true at higher orders in $1 - q$ (and even at leading order for bosons with spin $s \geq 2$ [27]) and this seems to prevent one from obtaining the general expression for the $c$-coefficient in this approach in a straightforward way.\(^8\)

\(^7\)One feature that distinguishes the cases with $s \leq 1$ from $s \geq 2$ conformal fields is that according to Appendix D of [2] for $s \geq 2$ CHS field in flat space it is necessary to use the equations of motion to prove gauge invariance of the improved symmetric traceless stress tensor. This may be related to a non zero value of the one-point function $\langle T_{\mu \nu} \rangle$ on $S_q^4$ or to $\hat{\zeta}(1) \neq 0$, suggesting a modification of the argument in [22] for $s \geq 2$.

\(^8\) For completeness, let us mention that if the expression of $\hat{\xi}(q)$ for a certain field were available as a
The rest of this paper is organized as follows. In section 2 we shall find \( \hat{\zeta}(q) \) (1.8) for the Laplace-type spin \( s \) operators on \( S^4_q \) that enter the partition function of CHS fields. We shall first explicitly determine the eigenvalues and their degeneracies for \( s = 0, 1, 2, 3, 4 \) cases (with generalization to \( s \geq 5 \) discussed in Appendix B). We shall then compute \( \hat{\zeta}(q) \) following the method discussed in Appendix A. The total expressions for the \( s = 1, 2, 3, 4, 5 \) CHS fields are presented in (2.24)–(2.27),(B.6). In subsection 2.3 we shall discuss the general structure (1.9) of \( \hat{\zeta}(q) \) relating it to the free energy on \( S^4_q \times \mathbb{H}^3 \).

In section 3 we shall use \( \hat{\zeta}(q) \) to determine the conformal anomaly coefficients corresponding to the CHS fields. We shall discuss the relations (1.10) and (1.11) and comment on similar relations following from Rényi entropy. We shall also determine the general expression for \( \hat{\zeta}_s(q) \) for any value of CHS spin \( s \) satisfying non-trivial consistency conditions. Like for the sums of \( a_s \) and \( c_s \) coefficients the regularized sum of \( \hat{\zeta}_s(q) \) over the whole tower of conformal higher spins is found to vanish, implying that the full CHS theory is one-loop UV finite on \( S^4_q \) space and thus implying as well the vanishing of the total Rényi entropy. Similar conclusions apply to the standard 2-derivative massless higher spin tower (see Appendix E).

In section 4 we shall discuss generalization to 6 dimensions. We shall compute the corresponding \( \hat{\zeta}(q) \) for low-spin \( s = 0, 1, 2 \) CHS fields on \( S^6_q \) space and show that the expected relation (4.3) for the \( c_3 \sim C_T \) conformal anomaly coefficient is fully consistent not only with the previously known 2-derivative scalar and 4-derivative vector results but also with the new result for the 6d Weyl graviton conformal anomaly computed independently from the Seeley-DeWitt coefficient in Appendix D.

The universality of the relation for \( C_T \) (1.12) is further supported by the discussion of the \( d = 2 \) case in Appendix C. In Appendix F we shall comment on the relation between the expressions for \( \hat{\zeta}(q) \) for spin \( s = 0, 1, 2, 3 \) Laplacians and some previous results [27] for the \( B_2 \) Seeley-DeWitt coefficient found in the "geometrical" approach.

2 Zeta-function of generalized spin \( s \) Laplacian on \( S^4_q \)

To compute the function \( \hat{\zeta}(q) \) in (1.8) for conformal higher spin fields on \( S^4_q \) and thus the corresponding \( a \) and \( c \) anomaly coefficients using (1.10) and (1.11) , our starting point will be the CHS partition function on \( S^4 \) [1]. It is expressed in terms of the determinants of generalized Laplace (or Lichnerowitzy-type) operators on unit-radius \( S^4 \) defined on a totally symmetric transverse traceless (TT) rank \( s \) tensor

\[
\Delta_s \equiv (-D^2 + M^2)s_{\perp},
\]

where \( M^2 \) is a constant parameter that need not be positive (the scalar curvature is \( R = 12 \)). For example, the one-loop \( S^4 \) partition functions of the standard 2-derivative massless

---

\footnote{function of the space dimension, then (1.11) could be cross checked against the representation of the surface contribution as a linear combination of specific conformal invariants on \( \Sigma \), see for instance [28].}
conformally coupled scalar,\(^9\) \(s = 1\) Maxwell vector, \(s = 2\) Weyl graviton and \(s = 3\) and \(s = 4\) CHS fields read

\[
Z_0 = \left[ \frac{1}{\det \Delta_0(2)} \right]^{1/2}, \quad Z_1 = \left[ \frac{\det \Delta_0(0)}{\det \Delta_{1\perp}(3)} \right]^{1/2}, \quad (2.2)
\]

\[
Z_2 = \left[ \begin{array}{cc} \det \Delta_{1\perp}(-3) & \det \Delta_0(-4) \\ \det \Delta_{2\perp}(4) & \det \Delta_{2\perp}(2) \end{array} \right]^{1/2}, \quad (2.3)
\]

\[
Z_3 = \left[ \begin{array}{cc} \det \Delta_{2\perp}(-8) & \det \Delta_{1\perp}(-9) & \det \Delta_0(-10) \\ \det \Delta_{3\perp}(5) & \det \Delta_{3\perp}(3) & \det \Delta_{3\perp}(-1) \end{array} \right]^{1/2}, \quad (2.4)
\]

\[
Z_4 = \left[ \begin{array}{cc} \det \Delta_{3\perp}(-15) & \det \Delta_{2\perp}(-16) & \det \Delta_{1\perp}(-17) & \det \Delta_0(-18) \\ \det \Delta_{4\perp}(6) & \det \Delta_{4\perp}(4) & \det \Delta_{4\perp}(0) & \det \Delta_{4\perp}(-6) \end{array} \right]^{1/2}. \quad (2.5)
\]

We will assume that these partition functions extended to \(S^4_q\) have the same product structure with each operator now defined on \(S^4_q\). Thus the problem of computing them reduces to finding the dependence of the spectrum of the operator (2.1) on the conical deformation parameter \(q\).

We will closely follow the approach of [28] where the scalar and vector operators were discussed, generalizing it to the \(s > 1\) case. We will first assume that \(q = 1/\gamma \leq 1\) where for an integer \(\gamma\) the space \(S^4_q\) becomes the \(\gamma\)-quotient of \(S^4\). For \(q = \gamma = 1\) the spectrum should reduce to the regular \(S^4\) one found in [29].

As explained later, for \(\gamma > 1\) the spectrum will in general be different in the intervals \(\gamma \in [n, n + 1)\). Starting from a certain \(n\), depending on the spin \(s\), the structure of the spectrum will be independent of \(\gamma\). The relevant range for us will be \(\gamma \in [1, 2)\) since to find the conformal anomaly coefficients in (1.10), (1.11) we will interested in the expansion near \(q \to 1\).\(^10\)

The eigenvalues \(\lambda_{n,m}\) of \(\Delta_{s\perp}(M^2)\) in (2.1) will be parametrized by the two integers \(n, m \geq 0\) as

\[
\lambda_{n,m}(\gamma) = (n + \gamma m)(n + \gamma m + 3) - s + M^2. \quad (2.6)
\]

The degeneracies \(d_{n,m}^{(s)}\) may be found using the correspondence between \(\Delta_{s\perp}(M^2)\) and the Laplacian on the ambient flat space with coordinates \((x^1, x^2, x^3, x^4, x^5)\) and the constraint \(|x|^2 \equiv x^a x^a = 1\) with the conical singularity implemented by the identification \(x^a(\tau) = x^a(\tau + 2\pi q)\) where \(\tau\) is the coordinate in (1.7) (see [28]).

The explicit spectrum for \(\gamma \neq 1\) can be constructed by starting with a suitable Ansatz for the eigenstates consistent with periodicity on \(S^4_{1/\gamma}\) generalizing to \(s > 1\) the discussion of the scalar and vector cases in [28].

---

\(^9\)The \(s = 0\) member of CHS tower in \(d = 4\) is non-dynamical, i.e. \(Z_0 = 1\), but it is useful to consider also the 2-derivative conformally coupled scalar to be able to compare with previous results on \(S^4_q\). In what follows the \(s = 0\) case will stand for the \(\tilde{\partial}^2\) scalar field.

\(^{10}\)If \(\gamma < 1\), there may be a finite number of normalizable eigenmodes that are, however, singular on some subspace. This has been noticed to happen already in the scalar case [28]. If only regular eigenmodes are considered, then our results extend to a neighbourhood of \(\gamma = q^{-1} = 1\).
2.1 Eigenvectors and degeneracies

In general, the eigenvectors of the Laplacian on the flat ambient space \((x^1, \ldots, x^5)\) will be tensors \((\Phi^{i_1 \ldots i_n})_{n,m}\) corresponding to the eigenvalues (cf. (2.6))

\[
\lambda_{n,m}(\gamma) = (n + \gamma m)(n + \gamma m + 3). \tag{2.7}
\]

They must be symmetric, traceless, and also tangential and transverse in the ambient space

\[
x^a \Phi_a^{i_1 \ldots i_n} = 0, \quad \partial_x x^a \Phi_a^{i_1 \ldots i_n} = 0. \tag{2.8}
\]

Here we split the coordinate indices as \(a = (i,+,\mp)\) where \(i = 1,2,3\) and \(x^\pm = \frac{1}{\sqrt{2}}(x^5 \pm ix^4)\). The ansatz for the tensor components with all indices from the 3-subspace reads \(^{11}\)

\[
(\Phi^{i_1 \ldots i_n})_{n,m} = |x|^{-n+\gamma m} (x^+)^{\gamma m} \sum_{p \geq 0} (B^{i_1 \ldots i_n})_{n,m} x^{i_1} \ldots x^{i_{n-2p}} (x^+ x^-)^p. \tag{2.9}
\]

The sum over \(p\) involves a finite number of terms, i.e. monomials of total degree \(s\) with some explicit power of \(x^+ x^-\). In the case of one \(\pm\) index we have instead

\[
(\Phi^{i_1 \ldots i_n \pm 1})_{n,m} = |x|^{-n+\gamma m} (x^+)^{\gamma m \pm 1} \sum_{p \geq 0} (B^{i_1 \ldots i_n \pm 1})_{n,m} x^{i_1} \ldots x^{i_{n-1-2p}} (x^+ x^-)^p, \tag{2.10}
\]

and similar expression is assumed when there are more indices of the \(+\) or \(\mp\) type. The regularity of the eigentensor components with one or more "\(\mp\)" indices, i.e. the absence of negative powers of \(x^+\), requires the sum over \(p\) to start at some positive value depending on the value of \(\gamma m\). This is the unique source of the \(\gamma\)-dependence of the spectrum. In practice, this is a feature that starts being relevant for \(s \geq 2\).

All eigenvectors of the form (2.9), (2.10) or with more \(\pm\) indices, appear together with a mirror copy where \(x^+ \leftrightarrow x^-\) when \(m > 0\). The solutions with \(m = 0\) are automatically symmetric under this exchange. By an explicit enumeration, we can then determine the degeneracies \(d^{(s)}_{n,m}\). For the \(s = 0\) and 1 cases we reproduce the results of [28] for \(S_q^d\) with any \(d\):

**Scalar:** \(n + m \geq 0\)

\[
d^{(0)}_{n,m>0} = 2 \binom{n+d-2}{d-2} \xrightarrow{d=4} (n+1)(n+2),
\]

\[
d^{(0)}_{n,0} = \binom{n+d-2}{d-2} \xrightarrow{d=4} \frac{1}{2} (n+1)(n+2). \tag{2.11}
\]

**Spin 1:** \(n + m \geq 1\)

\[
d^{(1)}_{n,m>0} = 2 (d-1) \binom{n+d-2}{d-2} \xrightarrow{d=4} 3 (n+1)(n+2),
\]

\[
d^{(1)}_{n,0} = \frac{1}{n+1} \left[ (n+d-3) \left[ d^2 + (n-4)d + 5 - n \right] \xrightarrow{d=4} \frac{1}{2} n (3n+5) \right]. \tag{2.12}
\]

\(^{11}\)The constraint \(|x| \equiv \sqrt{x^2x^r} = 1\) is imposed after taking the derivatives when imposing the transversality condition and applying the Laplacian [28].
For $s = 2,3,4$ and $\gamma \in [1,2)$ we find the following results for the degeneracies in $d = 4:^{12}$

**spin 2:** $n + m \geq 2$

\[
d_{n,0}^{(2)} = \frac{1}{2} (n - 1) (5n + 8), \quad d_{n,1}^{(2)} = n(5n + 11), \quad d_{n,m>1}^{(2)} = 5 (n + 1)(n + 2). \quad (2.13)
\]

**spin 3:** $n + m \geq 3$

\[
d_{n,0}^{(3)} = \frac{1}{2} (n - 2) (7n + 11), \quad d_{n,1}^{(3)} = (n - 1)(7n + 16),
\]

\[
d_{n,2}^{(3)} = n(7n + 17), \quad d_{n,m>2}^{(3)} = 7 (n + 1)(n + 2). \quad (2.14)
\]

**spin 4:** $n + m \geq 4$

\[
d_{n,0}^{(4)} = \frac{1}{2} (n - 3) (9n + 14), \quad d_{n,1}^{(4)} = 3 (n - 2)(3n + 7),
\]

\[
d_{n,2}^{(4)} = 3(n - 1)(3n + 8), \quad d_{n,3}^{(4)} = 9n (9n + 23),
\]

\[
d_{n,m>3}^{(4)} = 9 (n + 1)(n + 2). \quad (2.15)
\]

We suggest a generalization of these expressions for degeneracies to any integer $s > 4$ in Appendix B.

### 2.2 Computation of $\hat{\zeta}(q)$

To find the spectral zeta-function and thus $\hat{\zeta}(q)$ it remains to perform the sum in (1.8). Representing the eigenvalue (2.6) for particular $s$ and $M^2$ as

\[
\lambda_{n,m} = (n + \gamma m)(n + \gamma m + 3) - s + M^2 = (n + \gamma m + \mu)(n + \gamma m + \mu'), \quad (2.16)
\]

we thus need to compute

\[
\sum_{n \geq 0} \sum_{m \geq 0} d^{(s)}_{n,m} (n + \gamma m + \mu)(n + \gamma m + \mu')^{-z}. \quad (2.17)
\]

One possible approach is to follow [30] and expand $[\lambda_{n,m}]^{-z}$ in powers of the $n, m$ independent term $-\frac{1}{4} (\mu - \mu')^2.^{13}$ Doing the sum over $n$, one can then reduce the expression for the spectral $\zeta$ function of $\Delta_{\perp s}$ on $S^4_1/\gamma$ to a sum of terms with coefficients being the Hurwitz zeta functions $\zeta_R(a, b)$. Using the integral representation for $\zeta_R(a, b)$

\[
\zeta_R(a, b) = \frac{1}{\Gamma(a)} \int_0^\infty dy \frac{y^{a-1} e^{-b y}}{1 - e^{-y}}, \quad (2.18)
\]

one can then do the sum over $m$ in the integrand, expand in $\gamma$, integrate term by term in $y$, and finally send $z \to 0$ to obtain $\hat{\zeta}(q)$ in (1.8) with $q = 1/\gamma$. An alternative more straightforward approach (described in Appendix A in the $s = 0$ case) is to use the heat kernel

\footnote{\textsuperscript{12}It is useful to check the correspondence with the known results in the regular $S^4$ limit of $\gamma = 1$. For general spin $s$ and $\gamma = 1$, setting $N = n + m$ we have in $4d$: $d^{(s)}_{N} = \frac{1}{8} (2s + 1)(2N + 3)(N + s + 2)(N - s + 1)$. We have checked that indeed in all cases $\sum_{s \leq n + m \leq N} d^{(s)}_{n,m} = d^{(s)}_{N}$.}

\footnote{\textsuperscript{13}Only a finite number of terms will give a non-zero contribution in the limit $z \to 0$ so that the final result is stable for a sufficiently long expansion.}
representation taking into account the factorized form of the eigenvalues in (2.17). Applying this procedure, the general expressions for \( \hat{\xi}_s (q; M^2) \) corresponding to the operator (2.1) defined on \( S^4 \) with \( s = 0, 1, 2, 3, 4 \) are then found to be

\[
\begin{align*}
\hat{\xi}_0 (q; M^2) &= -\frac{1}{360 q^3} + \frac{2 - M^2}{12 q} + \frac{1}{120} (19 - 30 M^2 + 10 M^4) q, \\
\hat{\xi}_1 (q; M^2) &= -\frac{1}{120 q^3} + \frac{3 - M^2}{12 q} + M^2 - \frac{7}{3} + \frac{1}{40} (59 - 50 M^2 + 10 M^4) q, \\
\hat{\xi}_2 (q; M^2) &= -\frac{1}{72 q^3} - \frac{2}{q^2} + \frac{68 - 5 M^2}{12 q} + 5 M^2 - \frac{26}{3} + \frac{1}{24} (119 - 70 M^2 + 10 M^4) q, \\
\hat{\xi}_3 (q; M^2) &= -\frac{7}{360 q^3} - \frac{14}{q^2} - \frac{7 (M^2 - 53)}{12 q} + 14 M^2 - \frac{14}{3} + \frac{7}{120} (199 - 90 M^2 + 10 M^4) q, \\
\hat{\xi}_4 (q; M^2) &= -\frac{1}{40 q^3} - \frac{54}{q^2} - \frac{3 (M^2 - 150)}{4 q} + 30 M^2 + 56 + \frac{3}{40} (299 - 110 M^2 + 10 M^4) q.
\end{align*}
\]  

To obtain the total values of \( \hat{\xi} (q) \) for CHS fields it remains to sum up the contributions from different factors in the partition functions (2.2)–(2.5). When combining \( \hat{\xi} (q) \) (2.19) for the operators \( \Delta_s \) one needs to account for the contribution of the number \( n_z \) of the artificial zero modes introduced by the splitting of the fields into transverse parts, i.e. the corrected expression is

\[
\hat{\xi} (q) = \sum_{s'} \hat{\xi}_{s'} \perp (q) - n_z.
\]  

The number of zero modes associated to a TT spin \( s \) tensor is equal to the number of rank \( s - 1 \) conformal Killing tensors on \( S^4 \) [33][15]

\[
k_s = \frac{1}{12} s^2 (s + 1)^2 (2s + 1).
\]  

Explicitly, for the partition functions in (2.2)–(2.5) we get

\[
\begin{align*}
s &= 0 : & n_z &= 0, & s &= 1 : & n_z &= k_1 = 1, \\
s &= 2 : & n_z &= 2k_2 - k_1 = 29, & s &= 3 : & n_z &= 3k_3 - k_2 - k_1 = 236, \\
s &= 4 : & n_z &= 4k_4 - k_3 - k_2 - k_1 = 1100.
\end{align*}
\]  

As a result, we find the following expressions for \( \hat{\xi} (q) \) for the \( \partial^2 \) (\( M^2 = 2 \)) and \( \partial^4 \) (\( M^2 = 0, 2 \)) conformal scalars and \( s = 1, 2, 3, 4 \) CHS fields

\[
\begin{align*}
\hat{\xi}_q (q) &= -\frac{1}{360 q^3} - \frac{1}{120 q}, \\
\hat{\xi}_{q^{(4)}} (q) &= -\frac{1}{180 q^3} + \frac{1}{6 q} + \frac{3}{20} q.
\end{align*}
\]  

\[14\] As the original action and thus the partition function is expressed in terms of unconstrained fields one has to remove spurious zero modes related to splitting the rank \( s \) tensor into its transverse plus longitudinal parts (see [31, 32] for the case of \( s \leq 2 \) in \( d = 4 \)). This splitting introduces additional \( n_z \) zero modes of the Jacobian of the change of variables. These modes were not present in the original unconstrained operator and their number must be subtracted from \( \xi_{s'} \perp \) leading to \( B_q = \xi_{s'} = \xi_{s'} - n_z \).

\[15\] In general \( d \), this is the dimension of the \((s - 1, s - 1, 0, \ldots, 0)\) representation of \( SO(d + 1, 1) \).
The standard scalar and spin 1 cases were discussed in [34, 28]. The 4-derivative scalar expression was found in [35]. The vector expression in (2.24) agrees with the result of [36].

Similar analysis can be repeated in the case of the fermionic CHS fields with kinetic terms $\partial^2$ with $s = \frac{1}{2}, \frac{3}{2}, \ldots$ [1]. For $s = \frac{1}{2}$ fermion with the standard $\partial$-action or conformal $\partial^3$-action we find, using the results of [35]\footnote{One is to set $k = \frac{1}{2}$ and $\frac{3}{2}$ in eqs. (18),(19) in [35]. Note that $q$ in [35] corresponds to our $q = q^{-1}$.}.

\[
\begin{align*}
\hat{\zeta}_1(q) &= -\frac{1}{180 q^3} - \frac{1}{6 q} - \frac{11}{60} q, \\
\hat{\zeta}_2(q) &= -\frac{1}{60 q^3} - \frac{4}{q^2} + \frac{41}{6 q} - 11 - \frac{553}{60} q, \\
\hat{\zeta}_3(q) &= -\frac{1}{30 q^3} - \frac{40}{q^2} + \frac{227}{3 q} - 92 - \frac{2413}{30} q, \\
\hat{\zeta}_4(q) &= -\frac{1}{18 q^3} - \frac{200}{q^2} + \frac{1165}{3 q} - 1300 - \frac{2303}{6} q.
\end{align*}
\]

The leading small $q$ and large $q$ asymptotics of the $\hat{\zeta}(q)$-functions on $S^d_q$ in (2.23)–(2.28) have the universal structure (1.9), i.e.

\[
\hat{\zeta}(q) = -\frac{\nu}{360 q^3} + \ldots + (-2E_c) q,
\]

where $\nu$ is the number of dynamical degrees of freedom in bosonic case (rescaled by $\frac{7}{8}$ in the fermionic case) and $E_c$ is the Casimir energy on $\mathbb{R} \times S^3$. Indeed, the metric (1.7) effectively simplifies in these limits: for $q \to 0$ the $\tau$-direction shrinks to zero (or the transverse 3-space blows up) so we get effectively $S^d_q \times \mathbb{R}^3$, while for $q \to \infty$ the $\tau$-direction decompactifies and the space becomes similar to $\mathbb{R} \times S^3$. This suggests that log $Z$ should be related to the free energy on $S^d_q \times \mathbb{R}^3$ for $q \to 0$ and on $\mathbb{R} \times S^3$ for $q \to \infty$.\footnote{Let us mention in this connection a discussion [37] of an interesting duality between the $q \to 0$ and $q \to \infty$ limits of the partition function on $S^d_q \times S^3/\mathbb{Z}_n$.}
To make this connection more explicit we may use that the metric of $S^d_q$ is related by a conformal rescaling (by $\cos^2 \theta$ in (1.7)) to the metric of $S^1_q \times H^{d-1}$ where $H^{d-1}$ is a real hyperbolic space of unit curvature radius. The effective actions on $S^d_q$ and on $S^1_q \times H^{d-1}$ are then related by a finite integrated conformal anomaly term. This allows one to relate hyperbolic space of unit curvature radius.

Recalling that $\hat{\zeta}(q)$ is the coefficient of the log of the UV cutoff (cf. (1.1),(1.8)), restoring the dependence on the curvature radius $r$ and comparing the coefficients of log $r$ suggests a direct relation between $\hat{\zeta}(q)$ and $\mathcal{F}(q)$, or explicitly $\hat{\zeta}(q) = -\mathcal{F}(q)$. For $q \to \infty$ the free energy of $S^1_q \times H^3$ should approach the one on $\mathbb{R} \times H^3$. Since $H^3$ is related by the analytic continuation to $S^3$, that implies that $\mathcal{F}(q \gg 1) \to 2E_c q$ where $E_c$ is the Casimir energy on $S^3$.

In general, $\mathcal{F}(q)$ computed on $S^1_q \times S^3$ or $S^1_q \times H^3$ contains a non-universal (UV power-divergent) part proportional to the volume and thus linear in $q$ and a universal finite part. One may define $\mathcal{F}(q)$ in a particular scheme where all non-universal power UV divergences are subtracted out and the linear in $q$ part is the Casimir energy, i.e.

$$\hat{\zeta}(q) = -\mathcal{F}(q) = -\mathcal{F}(q) - 2E_c q \,,$$

(2.31)

where $\mathcal{F}(q)$ contains only non-positive powers of $q$. The function $\mathcal{F}(q)$ was computed for free conformal fields with spins $s \leq 1$, including higher derivative cases, in [38, 39].

In general, the definition of $E_c$ on $S^3$ is scheme-dependent – it depends on the definition of the stress tensor or the coefficient $g$ of the total derivative $D^2 R$ term in $\langle T_{\mu\nu} \rangle$, i.e. $E_c = \frac{1}{2}(a + \frac{1}{2}g)$ [40]. A natural scheme is the one when $E_c$ is determined from the single particle partition function of the corresponding CFT using the standard zeta-function definition (see, e.g., [7, 18]). It is this $E_c$ that appears as the $q$-coefficient in $\hat{\zeta}(q)$ (1.9),(2.29) ($\hat{\zeta}(q)$ itself is scheme-independent being the coefficient of log UV divergence on $S^d_q$).

One can indeed check that the order $q$ coefficients in (2.23)–(2.28) are the corresponding values of $E_c$ for the conformal $\partial^2$ and $\partial^4$ scalars, $\partial$ and $\partial^3$ fermions and $s = 1, 2, 3, 4$
CHS fields summarized below (see also (1.4))

| ϕ  | ϕ(4) | ψ  | ψ(3) | CHS₁ | CHS₂ | CHS₃ | CHS₄ |
|-----|------|-----|------|------|------|------|------|
| 1/240 | -3/40 | 17/960 | -29/960 | 11/120 | 553/120 | 2413/60 | 2303/12 |

(2.32)

In the opposite \( q \to 0 \) limit the free energy on \( S^1 \times \mathbb{R}^3 \) should approach the one on \( S^1 \times S^3 \) (see, e.g., [41]). Thus it should simply be proportional to the free energy of a single scalar or single fermion times the number of degrees of freedom. This pattern is indeed directly seen in (2.23)–(2.28). Note that this relation implies that in \( d \) dimensions the maximal power of \( q^{-1} \) in \( \tilde{\zeta}(q) \) in (2.29) should be \( d - 1 \).

3 Conformal anomaly coefficients from \( \tilde{\zeta}(q) \)

Having found \( \tilde{\zeta}(q) \) for a CFT on \( S^4_q \) one should be able to extract the information about the corresponding conformal anomaly coefficients \( a \) and \( c \) in (1.2). The \( a \)-coefficient is the one appearing in the log divergent part of the partition function on \( S^4 \). It is thus simply proportional to \( \tilde{\zeta} \) as in (1.9),

\[
a = -\frac{1}{4} \tilde{\zeta}(1) . \tag{3.1}
\]

Starting with (2.23)–(2.28) we indeed match the known values of the \( a \)-coefficient for the \( \partial^2 \) and \( \partial^4 \) conformal scalars, \( \partial \) and \( \partial^3 \) fermions and spin \( s = 1, 2, 3, 4 \) CHS fields from (1.3)

\[
| a | c |
|-----|-----|
| 1/360 | -7/90 | 11/720 | -3/80 | 31/180 | 87/20 | 171/5 | 1415/9 |
| 1/120 | -1/15 | 1/40 | -1/120 | 1/10 | 190/30 | 914/15 | 2r/3 | 890/3 + 14r/3 |

(3.2)

Here we included also the known values of the \( c \)-coefficient for the same \( s \leq 2 \) fields and also the \( s = 3, 4 \) values from (1.5) depending on the a priori unknown parameter \( r \).

While the value of the function \( \tilde{\zeta}(q) \) at \( q = 1 \) gives the \( a \)-coefficient, one observes from (2.23),(2.24),(2.28) that its first derivative vanishes at \( q = 1 \) for all low spin \( s = 0, \frac{1}{2}, 1 \) conformal fields \( \tilde{\zeta} \) (\( \tilde{\zeta}(q) = \frac{d}{dq} \tilde{\zeta}(q) \))

\[
\tilde{\zeta}_{\phi}^{-1}(1) = 0, \quad \tilde{\zeta}_{\psi(4)}^{-1}(1) = 0, \quad \tilde{\zeta}_{\phi}^{-1}(1) = 0, \quad \tilde{\zeta}_{\psi(3)}^{-1}(1) = 0, \quad \tilde{\zeta}_{1}^{-1}(1) = 0 . \tag{3.3}
\]

Surprisingly, this is no longer true for CHS fields with \( s \geq 2 \), i.e. \( \tilde{\zeta}_{\phi}^{-1}(1) \neq 0 \).

The second derivative of \( \tilde{\zeta}(q) \) at \( q = 1 \) is expected to be related to the conformal anomaly \( c \)-coefficient. We propose the following general expression for \( c \) (and also similar relation for \( C_T \) in other dimensions)

\[
c = -\frac{1}{4} \left. \frac{d^2}{dq^2} [q \tilde{\zeta}(q)] \right|_{q=1} = -\frac{1}{4} \tilde{\zeta}''(1) - \frac{1}{2} \tilde{\zeta}'(1) . \tag{3.4}
\]
In the low-spin cases when the first derivative vanishes (3.3), \( c \) is then given just by the second derivative term. Using the expressions for \( \hat{\zeta}(q) \) in (2.23)–(2.28) we indeed reproduce the known values of \( c \) for fields with \( s \leq 1 \) in (3.2).

For \( s = 2 \), i.e. the Weyl graviton, where \( \hat{\zeta}_2'(1) = -8 \) is no longer zero, we get from (3.4) precisely the known value \( c = \frac{199}{30} \) [17]. The same agreement is found in the case of \( d = 6 \) Weyl graviton as will be discussed in section 4 and Appendix D.

Note that as follows from (3.4) and the general form of \( \hat{\zeta}(q) \) in (1.9),(2.29) one has

\[
    c = E_c + \frac{1}{240} \nu - \frac{1}{2} p_2 ,
\]

where \( p_2 \) is the coefficient of the \( \frac{1}{q^2} \) term in \( \hat{\zeta}(q) \). Interestingly, \( p_2 = 0 \) for all lower-spin fields (see (2.23),(2.24),(2.28)) but is non-zero for higher spin CHS fields starting with Weyl graviton (cf. (2.25)-(2.27)).

In the case of CHS fields with \( s = 3 \) and 4 we find \( c_s \) in (3.2) corresponding to the value of the parameter \( r \) in (1.5) equal to -1, i.e.

\[
    r = -1 : \quad c_3 = \frac{904}{15} , \quad c_4 = 292 .
\]

This provides a strong evidence that the correct value of the \( c \)-coefficient of the CHS fields is given by (1.6).

Let us now compare (3.1) and (3.4) with similar relations for \( a \) and \( c \) expected from the free energy (2.30), (2.31) on \( S^1_q \times \mathbb{H}^3 \). Let us first recall the expression for the Rényi entropy in terms of the free energy on \( S^1_q \times \mathbb{H}^3 \)

\[
    S(q) = \frac{\mathcal{F}(q) - q \mathcal{F}(1)}{q - 1} .
\]

Then the expected expressions for the \( a \) and \( c \) anomaly coefficients are [22]

\[
    a = -\frac{1}{4} S(1) = \frac{1}{4} \mathcal{F}(1) - \frac{1}{4} \mathcal{F}'(1) ,
\]

(3.8)

\[
    c = \frac{1}{2} S'(1) = \frac{1}{4} \mathcal{F}''(1) .
\]

(3.9)

Using the relation (2.31) between \( \hat{\zeta}(q) \) for the conformal theory on \( S^1_q \) and \( \mathcal{F}(q) \) on \( S^1_q \times \mathbb{H}^3 \) we conclude that in all low-spin cases when \( \hat{\zeta}_2'(1) = -\mathcal{F}'(1) = 0 \) (3.3) the expressions (3.8) and (3.9) are indeed equivalent to (3.1) and (3.4). In particular, for the \( s = 1 \) case the conformal anomaly coefficients are reproduced correctly in both \( S^1_q \) and \( S^1 \times \mathbb{H}^3 \) approaches (see also [39]).

---

22 The expression for \( a \)-coefficient in terms of the entanglement entropy \( S(1) \) is assumed to incorporate the required edge mode contribution adding a constant term to \( \mathcal{F}(q) \) (see [42–44]) which is effectively included in the systematic computation of \( \hat{\zeta}(q) \) on \( S^1_q \). Recent work [36] extends this to the case of a conformally invariant \( p \)-form field in \( d = 2d - 2 \). As in the case of the Maxwell field in \( d = 4 \) the correct \( a \)-coefficient is found directly from the spectral computation on \( S^1_q \), while a constant shift is needed in the computation on \( S^1_q \times \mathbb{H}^{d-1} \). This shift is predicted [36] to be minus the entanglement entropy of a conformal \((p-1)\)-form field, in agreement with [43].
The first novel case is the $s = 2$ Weyl graviton when $\hat{\zeta}'(1) = - E'(1) \neq 0$ and the relation (3.9) is to be replaced by (3.4). The consistency of (3.4) for all three $s = 2, 3, 4$ CHS cases discussed explicitly above provides a strong evidence for its universal applicability. A similar expression is true also in $d = 6$ (where it leads to the correct $C_T \sim c_3$ coefficient for the 6d conformal graviton, see section 4 and Appendix D) and in $d = 2$ (see Appendix C). It would be important to derive (3.4) in general using the approach analogous to the one in [22], taking fully into account the special features of stress tensor for higher spin fields.

Using the expected general structure (2.29) of $\hat{\zeta}(q)$ with the expression (1.4) for $E_c$ for spin $s$ CHS field as well as the explicit results for $\hat{\zeta}_s(q)$ with $s = 1, 2, 3$ in (2.24)–(2.26) it is possible determine the general form of $\hat{\zeta}_s(q)$ for any value of $s$. Starting with an ansatz (with $v = s(s + 1)$)

$$\hat{\zeta}_s(q) = \frac{v}{360 q^3} + \frac{p_2(v)}{q^2} + \frac{p_1(v)}{q} + p_0(v) - \frac{v (18 v^2 - 14 v - 11)}{360} q, \quad (3.10)$$

where $p_i(v) = v(k_{i2}v^2 + k_{i1}v + k_{i0})$ are cubic polynomials in $v$ (so that $\hat{\zeta}_s$ is at most cubic in $v$ and vanishes for $v = 0$ as required to match the structure of conformal anomaly coefficients) one is able to fix the 9 unknown coefficients $k_{ij}$ by matching to the $s = 1, 2, 3$ expressions in (2.24)–(2.27). As a result,

$$\hat{\zeta}_s(q) = - \frac{v}{360 q^3} - \frac{v^2 (v - 2)}{36 q^2} + \frac{v (2 v^2 - 5 v - 1)}{36 q} - \frac{v^2 (2 v - 1)}{36} + \frac{v (18 v^2 - 14 v - 11)}{360} q. \quad (3.11)$$

Then a highly non-trivial consistency check is that for $s = 4$ and 5 this expression reproduces also $\hat{\zeta}_4(q)$ in (2.27) and $\hat{\zeta}_5(q)$ in (B.6). Furthermore, applying (3.1) we then match the known a-coefficient in (1.3), while applying (3.4) we get the $r = -1$ expression for the c-coefficient in (1.6).\footnote{The analog of (2.20) here contains $n_x = s k_s - \sum_{s' = 0}^{s-1} k_{s'} = 1/4 \pi v^2 (5 v - 1)$, generalizing the expressions in (2.22).}

Note also that

$$\hat{\zeta}'_s(1) = - \frac{1}{60} v (v - 2) (3 v + 2), \quad (3.12)$$

is a non-zero integer for all $s > 1$ CHS fields and thus contributes to $c$ in (3.4).

We observe also that not only the regularized sums of $a_s$ and $c_s$ but also the sum of the full $\hat{\zeta}_s(q)$ functions over all $s = 1, 2, \ldots$ vanishes, i.e.

$$\sum_{s=1}^{\infty} e^{-s(s+1)} \hat{\zeta}_s(q) \bigg|_{c \to 0, \text{ finite}} = 0, \quad (3.13)$$

so that the full CHS theory is one-loop UV finite on $S^4_q$ space.

This implies also the vanishing of the total free energy on $S^4_q \times \mathbb{H}^3$ (3.21) and thus of the associated Rényi entropy (3.7). This vanishing appears to be consistent with the "topological" nature of the CHS theory [15]. Similar conclusions are reached for the massless higher spin tower in Appendix E.
Let us note also that as the (one-loop) logarithmic UV divergences cancel in the full CHS theory, the finite part of the corresponding partition function $Z$ is scheme-independent. As was shown in [15], $Z = 1$ in flat space (assuming the same regularization as in (3.13), in which the total number of degrees of freedom vanishes) and also on $S^4$ (which could be expected given the cancellation of conformal a-anomalies). One may expect that since $S^4$ has a non-zero Weyl tensor it is likely that $Z(S^4)$ is a non-trivial function of $q$. It would be interesting to compute it using the heat-kernel method in Appendix A.

4 Generalization to six dimensions

Let us now demonstrate how similar computations of $\hat{\zeta}(q)$ and related conformal anomaly coefficients can be performed in six dimensions. In 6d for a classically Weyl invariant theory one gets instead of (1.2)

$$
\frac{1}{144\pi^2} \int d^6x \sqrt{\bar{g}} \bar{b}_6(x), \\
\bar{b}_6 = -a E_6 + c_1 I_1 + c_2 I_2 + c_3 I_3,
$$

where $E_6 = -\epsilon_6 \epsilon_6 R R R$ is proportional to the 6d Euler density and the 3 independent Weyl invariants are $I_1 = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}$, $I_2 = C_{\mu\nu}^{\rho} C_{\rho\sigma}^{\nu\sigma}$, and $I_3 = C_{\mu\nu\rho\gamma} D^{2} C^{\mu\nu\rho\gamma} + \ldots$ (see for details [45] and Appendix D).

The aim will be to consider the conical deformation $S^6_q$ of 6-sphere (with the metric as in (1.7) with $S^2$ singularity replaced by $S^4$ one), compute the spectral $\hat{\zeta}$-function at $z = 0$ or $\hat{\zeta}(q) = B_6$ as in (1.8) and then extract the values of the conformal anomaly coefficients from it. As the log divergent part of the free energy on $S^6$ should be proportional to $a$, we should have again $\hat{\zeta}(1) \sim a$. The $c_3$ coefficient proportional to $C_T$ in the 2-point function of stress tensors should be determined, as in 4d case, by the 2nd derivative of $\hat{\zeta}(q)$ at $q = 1$. The coefficients $c_3$ and $c_4$ related to 3-point functions of stress tensor may be possible to extract from the 3rd (or higher) derivative of $\hat{\zeta}(q)$ but we will not attempt this here.

Taking into account normalizations, the expected relations are then the direct analogs of (3.1) and (3.4) in 4d case:

$$
a = -\frac{1}{96} \hat{\zeta}(1),
$$

$$
c_3 = \frac{1}{12} \frac{\partial^2}{\partial q^2} [q \hat{\zeta}(q)] \bigg|_{q=1} = \frac{1}{12} \hat{\zeta}''(1) + \frac{1}{8} \hat{\zeta}'(1).
$$

The bosonic totally symmetric rank $s$ conformal higher spins in 6d have kinetic terms $h_s \Box^{s+\frac{2}{3}} h_s = h_s \Box^{s+1} h_s$. Below we shall consider only the lowest spin cases: $s = 0$ – the standard $\partial^3$ conformal scalar, $s = 1$ – the higher derivative $\partial^4$ vector [46, 47, 39] and $s = 2$ – the $\partial^6$ conformal graviton (see Appendix D). The corresponding partition functions on $S^6$ are [48] (cf. (2.2),(2.3))

$$
Z_0 = \left[ \frac{1}{\det \Delta_0(6)} \right]^{1/2}, \quad Z_1 = \left[ \frac{\det \Delta_0(0)}{\det \Delta_{1\perp}(7) \det \Delta_{1\perp}(5)} \right]^{1/2},
$$

$$
Z_2 = \left[ \frac{\det \Delta_{1\perp}(5) \det \Delta_0(-6)}{\det \Delta_{2\perp}(8) \det \Delta_{2\perp}(6) \det \Delta_{2\perp}(2)} \right]^{1/2},
$$

- 15 -
where \( \Delta_{\pm} (M^2) = (-D^2 + M^2)_{s, \pm} \) are defined on \( S^6 \). Assuming as in 4d case that the these partition functions have the same structure on \( S^6 \), their computation requires the knowledge of the spectrum of \( \Delta_{\pm} (M^2) \) on this space.

The analysis of the spectrum goes along the same lines as in Section 2. The analogs of the eigenvalues in (2.6) and (2.7) are obtained after the replacement \( n + \gamma m + 3 \to n + \gamma m + 5 \). The degeneracies of the spectrum for the conformal scalar and the 4-derivative spin 1 field are found from (2.11) and (2.12) where now \( d \to 6 \). The spin 2 degeneracies turn out to be \((n + m \geq 2)\)

\[
\begin{align*}
\tilde{d}^{(2)}_{n,0} &= \frac{1}{12} (n-1)(n+2)(n+3)(7n+22), \\
\tilde{d}^{(2)}_{n,1} &= \frac{1}{6} n(3+n)(4+n)(7n+17), \\
\tilde{d}^{(2)}_{n,m>1} &= \frac{1}{8} (n+1)(n+2)(n+3)(n+4).
\end{align*}
\]

As a result, \( \tilde{\xi} (q, M^2) \) in (1.8) for the operators \( \Delta_{\pm} (M^2) \) on \( S^6 \) are given by (cf. (2.19))

\[
\begin{align*}
\tilde{\xi}_0(q, M^2) &= \frac{1}{15120} q^3 + \frac{6 q^2 - 35}{4530} q^3 + \frac{24 - 10 M^2 + M^4}{144 q} + \frac{4315 - 3990 M^2 + 1050 M^4 - 84 M^6}{50240} q, \\
\tilde{\xi}_1(q, M^2) &= \frac{1}{3024} q^3 + \frac{5 q^2 - 41}{864} q^3 + \frac{5 (35 - 12 M^2 + M^4)}{144 q} \\
&\quad - \frac{553 - 210 M^2 + 15 M^4}{6048} q + \frac{9439 - 6342 M^2 + 1302 M^4 - 84 M^6}{84 q} q, \\
\tilde{\xi}_2(q, M^2) &= \frac{1}{1080} q^3 - \frac{1}{6 q^3} + \frac{1111 + 42 M^2}{2160} q^3 + \frac{6 M^2 - 55}{6 q^3} + \frac{1560 - 242 M^2 + 7 M^4}{72 q} q \\
&\quad - \frac{3166 - 1860 M^2 + 105 M^4}{2160} + \frac{17167 - 9198 M^2 + 1554 M^4 - 84 M^6}{1080} q.
\end{align*}
\]

Forming the combinations of these functions corresponding to the partition functions (4.4),(4.5) taking into account as in (2.20) the zero mode contributions\(^\text{24}\)

\[
s = 0 : \ n_z = 0; \quad s = 1 : \ n_z = 2 k_1 = 2; \quad s = 2 : \ n_z = 3 k_2 - k_1 = 83,
\]

we find that the total coefficients \( \tilde{\xi}(q) \) of the log UV divergence of the CHS partition functions (4.4),(4.5) on \( S^6 \) are given by (cf. (2.23)–(2.25))

\[
\begin{align*}
\tilde{\xi}_0(q) &= \frac{1}{15120} q^3 + \frac{1}{4320} q^3 + \frac{31}{30240} q, \\
\tilde{\xi}_1(q) &= \frac{1}{1680} q^3 - \frac{1}{288} q^3 - \frac{1}{6 q} - \frac{14}{15} - \frac{224}{280} q, \\
\tilde{\xi}_2(q) &= \frac{1}{420} q^3 - \frac{1}{2 q^3} + \frac{703}{360} q^3 - \frac{23}{2 q^3} + \frac{49}{3 q} - \frac{181}{9} - \frac{4143}{280} q.
\end{align*}
\]

As in the 4d case (2.29) the \( q \to 0 \) and \( q \to \infty \) asymptotics of \( \tilde{\xi}(q) \) are controlled by free energies on \( S^6 \times \mathbb{R}^5 \) and \( \mathbb{R} \times S^5 \) respectively, i.e.

\[
\tilde{\xi}_s(q) = \frac{v}{15120} q^3 + \ldots + (-2 E_c) q,
\]

\(^{24}\text{The number of zero modes associated with a transverse traceless totally symmetric rank } s \text{ field on } S^6 \text{ is (see [48] and refs. there) } k_s = \frac{1}{6}(2s + 3)s(s + 1)(s + 2)(s + 3), \text{ i.e. } k_0 = 0, k_1 = 1, k_2 = 28, \text{ etc.}\)
where the number of dynamical degrees of freedom $\nu$ [48] and the Casimir energy on $\mathbb{R} \times S^5$ [7] for 6d CHS fields are given by

\[
\nu = \frac{1}{4} (s + 1)^2 (s + 2)^2, \quad s = 0, 1, 2, \ldots \tag{4.13}
\]

\[
E_{c,s} = \frac{1}{605760} \nu (96 \nu^{3/2} - 232 \nu - 12 \nu^{1/2} + 117) . \tag{4.14}
\]

The values of $\tilde{\zeta}_s$ at $q = 1$ reproduce (4.2) the known a-coefficients [48] \(^{25}\)

\[
a_s = \frac{1}{1814400} \nu (88 \nu^{3/2} - 110 \nu - 4 \nu^{1/2} + 1) . \tag{4.15}
\]

We also observe that as in the 4d case (3.3),(3.12) the first derivative $\tilde{\zeta}_s'(1)$ vanishes for $s = 0$ and $s = 1$ but not for $s = 2$:

\[
\tilde{\zeta}_0'(1) = 0 , \quad \tilde{\zeta}_1'(1) = 0 , \quad \tilde{\zeta}_2'(1) = -12 . \tag{4.16}
\]

Using (4.3) we get the following values for the $c_{3,s}$ coefficients

\[
c_{3,0} = \frac{1}{2520} , \quad c_{3,1} = -\frac{5}{168} , \quad c_{3,2} = -\frac{1639}{420} . \tag{4.17}
\]

The $s = 0$ and $s = 1$ values match the known ones found earlier in [45, 39]. Remarkably, the $s = 2$ result for $c_{3,2}$ agrees with the direct computation of the corresponding 6d Seeley-DeWitt coefficient for the 6d Weyl graviton that we present in Appendix D where we also determine the values of the two other conformal anomaly coefficients $c_1$ and $c_2$ in (4.1) (see (D.10)). This provides a non-trivial check of the consistency of the relation (4.3) for the $C_T \sim c_3$ coefficient in 6d.

For completeness, let us summarize the values of the conformal anomalies for the 6d $s = 0, 1, 2$ CHS fields below:

\[
\begin{array}{cccc}
 s & a & c_1 & c_2 & c_3 \\
 0 & -\frac{5}{72 \times 7!} & -\frac{1}{540} & \frac{1}{3024} & \frac{1}{2520} \\
 1 & \frac{275}{8 \times 7!} & \frac{97}{180} & \frac{911}{5040} & -\frac{5}{168} \\
 2 & \frac{3005}{2 \times 7!} & \frac{1507}{45} & \frac{635}{126} & -\frac{1639}{420} \\
\end{array} \tag{4.18}
\]

**Acknowledgments**

We would like to thank S. Dowker and D. Fursaev for useful discussions. The work of AAT was supported by the ERC Advanced grant no. 290456, the STFC Consolidated grant ST/L00044X/1 and the Russian Science Foundation grant 14-42-00047 at Lebedev Institute.

\(^{25}\)With our normalization in the 6d conformal scalar case ($\nu = 1$) we get $a_0 = -\frac{1}{72576}$. 

\[-17-\]
A Details of computation of $\hat{\zeta}(q)$ for 4d massive scalar

Here we provide some details of the computation of $\hat{\zeta}(q)$ in section 2 on the example of 4d scalar operator $\Delta_0 (M^2) = -D^2 + M^2$. The case of the conformal coupling on unit-radius $S^4$ corresponds to $M^2 = 2$. Introducing the parameter $\mu$ related to $M^2$ by $M^2 = \frac{9-\mu^2}{4}$ and setting $\gamma = 1/\mu$ we then have for the corresponding spectral $\zeta$-function in (1.8)

$$\zeta(z; \gamma) = \sum_{n,m=0}^{\infty} d_{n,m} \left[ (\frac{3}{2} + n + m \gamma)^2 - \frac{\mu^2}{4} \right]^{-\frac{3}{2}}. \quad (A.1)$$

We used (2.6) and (2.11), i.e. $d_{n,0} = \frac{1}{2} (n + 1) (n + 2)$, $d_{n,m > 0} = (n + 1) (n + 2)$. The evaluation of (A.1) was first considered in [34] where it allowed to obtain the finite-temperature one-loop effective potential for a scalar field in de Sitter space-time. The result revealed an unexpected dependence of the logarithmic divergences on the temperature associated to the presence of a horizon which is directly related to $q$-dependence of $\hat{\zeta}(q) = \zeta(0,q)$ we discussed in section 2.\(^*\)

We will compute $\zeta(0; \gamma)$ in (A.1) by using a somewhat more direct method than employed in [34]. We first split the contribution from the $m = 0$ and $m > 0$ modes as

$$\zeta(z; \gamma) = \zeta^{(a)}(z; \gamma) + \zeta^{(b)}(z; \gamma), \quad \zeta^{(a)}(z; \gamma) = \sum_{n=0}^{\infty} \frac{1}{2} (n + 1) (n + 2) \left[ (\frac{3}{2} + n + m \gamma)^2 - \frac{\mu^2}{4} \right]^{-\frac{3}{2}},$$

$$\zeta^{(b)}(z; \gamma) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (n + 1) (n + 2) \left[ (n + m \gamma + \frac{3-\mu}{2}) (n + m \gamma + \frac{3+\mu}{2}) \right]^{-\frac{3}{2}}. \quad (A.2)$$

As we are interested only in the value at $z = 0$ each of the two terms $\zeta^{(a)}$ and $\zeta^{(b)}$ can be computed by expanding to quadratic order in $\mu^2$ only (higher order terms in $\mu$ will give vanishing contributions in the $z \to 0$ limit, cf. also footnote 13). For the first term we get

$$\zeta^{(a)}(z; q) = \frac{1}{8} [4 \zeta_R (2z - 2, \frac{3}{2}) - \zeta_R (2z, \frac{3}{2})] + \frac{1}{32} \mu^2 z \left[ 4 \zeta_R (2z, \frac{3}{2}) - \zeta_R (2z + 2, \frac{3}{2}) \right]$$

$$+ \frac{1}{256} \mu^4 z (z + 1) \left[ 4 \zeta_R (2z + 2, \frac{3}{2}) - \zeta_R (2z + 4, \frac{3}{2}) \right] + O(\mu^6), \quad (A.3)$$

where $\zeta_R(a,b)$ is the Hurwitz zeta-function (2.18). Dropping the contributions that manifestly vanish at $z \to 0$ (due to explicit factors of $z$ that multiply analytic terms) we find

$$\zeta^{(a)}(z; q) = \frac{1}{8} [4 \zeta_R (2z - 2, \frac{3}{2}) - \zeta_R (2z, \frac{3}{2})] + O(z). \quad (A.4)$$

This vanishes at $z = 0$

$$\zeta^{(a)}(0; q) = \frac{1}{8} [4 \times (-\frac{1}{4}) - (-1)] = 0. \quad (A.5)$$

Instead of following the same strategy in the case of $\zeta^{(b)}(0; q)$ we shall use a simpler approach by relating it to the $t^0$ coefficient in the expansion of the corresponding heat kernel

\(^*\)In the present context the parameter $\gamma = q^{-1}$ of the conical singularity corresponds to the ratio of the temperature and the Hawking temperature in [34].
replaced by the sum of the kernels corresponding to the "1-st order" factors in (A.2)\footnote{As the eigenvalues factorize, the same applies to the corresponding determinant, and thus the heat kernel can be replaced by a sum of heat kernels corresponding to the factors. The origin of the $\frac{1}{4}$ factor in the relation for $\zeta(0; q)$ may be understood by comparing dimensions of the proper-time cutoffs in the original heat kernel and its "factor" analogs, cf. also [30].}

$$K(t; \gamma) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} (n + 1)(n + 2) \left[ e^{-t(n+m\gamma+\frac{2q}{m})} + e^{-t(n+m\gamma+\frac{2q}{m})} \right]$$

$$K(t; \gamma) \approx \sum_{k=-4}^{t \approx 0} h_k t^k, \quad \zeta^{(b)}(0; q) = \frac{1}{2} h_0. \quad \text{(A.6)}$$

Computing the two sums, we readily obtain

$$K(t; \gamma) = \frac{e^{-\frac{1}{4}(\mu^2 - 1)} \left( e^{\mu t} + 1 \right)}{(e^t - 1)^3 (e^t - 1)} = \frac{2}{\gamma^2} - \frac{1}{t^3} + \left( \frac{\mu^2 - 1}{4\gamma} + \gamma \right) \frac{1}{t^2} + \frac{1}{8} \left( 1 - \mu^2 \right) \frac{1}{t}$$

$$- \frac{\gamma^3}{360} + \frac{\mu^2 - 1}{48\gamma} + \frac{5\mu^4 - 30\mu^2 + 17}{960\gamma} + O(t). \quad \text{(A.7)}$$

The $t^0$ term here gives (using (A.5) and $\gamma = 1/q$)

$$\zeta(0; q) = \zeta^{(a)}(0; q) + \zeta^{(b)}(0; q) = -\frac{1}{360q^2} + \frac{\mu^2 - 1}{48q} + \frac{5\mu^4 - 30\mu^2 + 17}{960q}, \quad \text{(A.8)}$$

which is equivalent to the expression in (2.19) after we recall that $\mu^2 = 9 - 4M^2$. For example, for the conformally coupled scalar with $\mu = 1$ we get, in agreement with (2.23),

$$\zeta(0; q) = -\frac{1}{360q^2} - \frac{q}{120}. \quad \text{(A.9)}$$

\section{Degeneracies of eigenvalues of bosonic spin $s$ Laplacian on $S^4_0$}

The degeneracies for $\gamma \in [1, 2]$ for spin $s \leq 4$ in (2.12)–(2.15) admit a natural generalization to all integer $s > 0$ (here $n + m \geq s$)

$$d_{n,0}^{(s)} = \frac{1}{2} (n - s + 1) [(2s + 1) n + K_0], \quad d_{n,1}^{(s)} = (n - s + 2) [(2s + 1) n + K_1],$$

$$d_{n,2}^{(s)} = (n - s + 3) [(2s + 1) n + K_2], \quad \ldots$$

$$d_{n,s-1}^{(s)} = n [(2s + 1) n + K_{s-1}], \quad d_{n,m+s-1}^{(s)} = (2s + 1) (n + 1)(n + 2), \quad \text{(B.1)}$$

where the integers $K_p$ are

$$K_p = 2 (1 + 2p) + (3 + 2p) (s - p), \quad p = 0, \ldots, s - 1. \quad \text{(B.2)}$$

One can check that the degeneracies in (B.1) are always non-negative and also even when $m > 0$ as follows from the expected symmetry of the spin $s$ Laplacian eigenstates under the exchange $x^+ \leftrightarrow x^-$ in this case. The total degeneracy

$$\sum_{m=0}^{s-1} d_{n,m}^{(s)} + \sum_{m=s}^{N} d_{n,m}^{(s)} + d_{N-m,m}^{(s)} = \frac{1}{6} (2s + 1)(2N + 3)(N + s + 2)(N - s + 1) \quad \text{(B.3)}$$
is equal as it should to the degeneracy of the level $N$ eigenvalue for the regular sphere $S^4$ (cf. footnote 12).

A further test of (B.1) is provided by the explicit calculation of the zeta function $\hat{\zeta}_s(q)$ for the CHS field with spin $s$. For instance, for $s = 5$ (B.1) gives ($n + m \geq 5$)

$$d_{n,0}^{(5)} = \frac{1}{2} (n - 4) (11 n + 17), \quad d_{n,1}^{(5)} = (n - 3) (11 n + 26),$$

$$d_{n,2}^{(5)} = (n - 2) (11 n + 31), \quad d_{n,3}^{(5)} = (n - 1) (11 n + 32),$$

$$d_{n,4}^{(5)} = n (11 n + 29), \quad d_{n,m>4}^{(5)} = 11 (n + 1) (n + 2). \quad (B.4)$$

Generalizing the spin 5 CHS partition function on $S^4$ \cite{1}

$$Z_5 = \left[ \frac{\det \Delta_{4\perp} (-24) \det \Delta_{3\perp} (-25) \det \Delta_{2\perp} (-26) \det \Delta_{1\perp} (-27) \det \Delta_0 (-28)}{\det \Delta_{5\perp} (7) \det \Delta_{5\perp} (5) \det \Delta_{5\perp} (1) \det \Delta_{5\perp} (-5) \det \Delta_{5\perp} (-13)} \right]^{1/2}, \quad (B.5)$$

to $S^4$ we find that the associated $\hat{\zeta}_5(q)$ function determined using (B.4) is given by

$$\hat{\zeta}_5(q) = -\frac{1}{12 q^3} - \frac{700}{q^2} + \frac{8245}{6 q} - 1475 - \frac{15769}{12} q. \quad (B.6)$$

This expression is in perfect agreement with our general proposal in (3.11) (here $\nu = 30$).

\section{Two-dimensional case}

In two dimensions there is just the “$a$” coefficient of the Weyl anomaly that also has the interpretation of the coefficient $C_T$ in the 2-point function of stress tensor, i.e. is the Virasoro central charge and thus is usually denoted as $c$. In standard normalization where a real $\partial^2$ scalar has $c = 1$ we have for the coefficient of the log UV divergence of the partition function (cf. (1.1))

$$B_2 = \frac{1}{4\pi} \int d^2 x \sqrt{g} \overline{b}_2, \quad \overline{b}_2 = a R, \quad a \equiv \frac{1}{5} c. \quad (C.1)$$

On $S^2$ one thus finds $B_2 = \frac{1}{5} c$. For a conformal field defined on a conical deformation $S^2_q$ of the 2-sphere we expect the corresponding spectral zeta-function at $z = 0$ to have a similar general form as in 4d (2.29) and in 6d (4.12), i.e.

$$\hat{\zeta}(q) = \frac{v}{6 q} + p_0 - 2 E_c q, \quad (C.2)$$

where the first and the last terms are fixed by the asymptotics corresponding to $S^4_q \times \mathbb{R}$ ($q \to 0$) and $\mathbb{R} \times S^1$ ($q \to \infty$). Here $v$ is the number of effective degrees of freedom with $v = 1$ for a real $\partial^2$ scalar and $v = -2$ for the 2d conformal higher spin fields with kinetic terms $h_s \square^{v-4} h_s = h_s \square^{v-1} h_s$ with $s = 1, 2, \ldots \cite{48}$. The Casimir energy $E_c$ on unit-radius $S^1$ should in general be related to the central charge $c$ by $\cite{49, 50} E_c = -\frac{1}{2 \nu} c$. Since $B_2(S^2) = \hat{\zeta}(1)$ we then conclude that

$$c = 3 \hat{\zeta}(1) = -12 E_c. \quad (C.3)$$
As follows from (C.2) and \( E_c = -\frac{1}{12} c \) we then have also the following representation for \( c \)

\[
c = 3 \frac{\phi}{d\sigma} [q \tilde{\zeta}(q)] \big|_{q=1} .
\]  

\[ (C.4) \]

Remarkably, this relation for \( c = C_T \) in 2d case is a direct counterpart of the similar 4d (3.4) and 6d (4.3) relations we proposed above. This supports their common origin and implies a universal applicability of \( C_T \sim \tilde{\zeta}''(1) + 2\tilde{\zeta}'(1) \) relation in any dimension.

In the case of the standard \( \partial^2 \) scalar field (C.2), (C.3) imply that \( \tilde{\zeta}(q) = \frac{1}{6q} + \frac{1}{6} q \).

\[ (C.5) \]

In the case of \( d = 2 \) CHS fields one finds [48, 13, 14]

\[
e_s = -2 \left[ 1 + 6s(s-1) \right] , \quad s \geq 2 .
\]  

\[ (C.6) \]

As a result, the function \( \tilde{\zeta}_s(q) \) for the 2d CHS fields consistent with the above relations (C.3), (C.6) turns out to be

\[
\tilde{\zeta}_s(q) = -\frac{1}{3q} - 2s(s-1) - \frac{1}{3} \left[ 1 + 6s(s-1) \right] q = -\frac{1}{3} \left( \frac{1}{q} - 1 \right) + \frac{c_s}{6} \left( q + 1 \right) .
\]  

\[ (C.7) \]

Note that while for a standard scalar (C.5) we find that \( \tilde{\zeta}'(1) = 0 \), for the CHS fields \( \tilde{\zeta}'(1) = -2s(s-1) \) so that for \( s \geq 2 \) it is again non-zero as in the 4d and 6d cases.

\section{Conformal anomaly coefficients for the Weyl graviton in six dimensions}

In 6d there are three dimension 6 non-trivial Weyl invariants \( I_1, I_2, I_3 \) that appear in (4.1)

\[
I_1 = C_{\alpha\mu\nu\beta} C_{\mu\rho\sigma} C_{\rho\sigma} \gamma, \quad I_2 = C_{\alpha\beta} \mu\nu C_{\rho\sigma} \mu\nu C_{\rho\sigma} \gamma, \\
I_3 = C_{\mu\alpha\beta\gamma} (D^2 \delta_\mu^\nu + 4 R_\mu^\nu - \frac{2}{5} R \delta_\mu^\nu) C^{\alpha\beta\gamma} + \text{total derivatives}.
\]  

\[ (D.1) \]

A candidate Weyl-invariant gravity action is then an integral of a linear combination of these 3 invariants.

There is a particular choice [51] \( W_6 = -I_3 + 3I_2 + 12I_1 \) that has special properties: (i) it vanishes on a Ricci flat background, and (ii) it admits \( (2, 0) \) locally superconformal extension [46, 52, 53]. Related to (i) and (ii) is that \( W_6 \) appears, respectively, as the coefficient of the logarithmic IR divergence of the Einstein action in AdS_7 evaluated on the solution of Dirichlet problem [54] and also as the log UV divergence of the \( (2, 0) \) tensor multiplet [45]. The resulting action may be written as

\[
S = \int d^6x \sqrt{g} \left[ R^{\mu\nu} D^2 R_{\mu\nu} - \frac{3}{11} R D^2 R - 2 R^{\mu\nu\rho\sigma} R_{\mu\rho} R_{\nu\sigma} - R^{\mu\nu} R_{\mu\nu} R + \frac{3}{35} R^3 \right] .
\]  

\[ (D.2) \]

The fact that it is expressed in terms of the Ricci tensor and is at most linear in the Weyl tensor implies that it can be rewritten as a 2nd derivative action involving several tensors of rank \( \leq 2 \) and it is uniquely selected by this requirement [55].
The quadratic part of (D.2) expanded around a curved space is governed by the 6th-order differential operator that factorizes, as it is easy to see, into the product of three 2nd order Lichnerowitz-type operators if the background is an Einstein one. Restricted to transverse traceless \( h_{\mu\nu} \) the kinetic operator in (D.2) is (cf. (4.5)) \(-D^6 + ... = \Delta_2 (8) \Delta_2 (6) \Delta_2 (2)\) with

\[
\Delta_2 (M^2) h_{\mu\nu} = (-D^2 + \frac{1}{30} M^2 R) h_{\mu\nu} - 2 C_{\mu\nu\rho} h^{\rho\sigma}, \tag{D.3}
\]

where \( R \) is the scalar curvature and \( C \) is the Weyl tensor. On a unit-radius \( S^6 \) where \( R = 30 \) and \( C = 0 \) we then get the one-loop partition function in (4.5).

This factorization implies that the one-loop conformal anomaly coefficients in (4.1) corresponding to (D.2) can be computed following [45] by using directly the general expression [56] for the \( b_6 \) Seeley-DeWitt coefficient of the corresponding 2nd order Laplace-type operator \( \Delta = -D^2 + X \) defined on \( k \leq 2 \) tensors that enter the generalization of the \( S^6 \) partition function in (4.5).

To simplify the computation one may use a shortcut and consider several special backgrounds. Considering \( S^6 \) case one can easily determine the value of \( \alpha \)-coefficient. For a symmetric-space Einstein background with a non-zero Weyl tensor (where \( i_i \) invariants satisfy one linear relation) one is able to find \( c_1 \) and \( c_3 \) in terms of \( c_2 \) [57]:

\[
a = \frac{3005}{3 \times 71}, \quad c_1 = \frac{5633}{105} - 4 c_2, \quad c_3 = -\frac{35543}{5040} + \frac{5}{8} c_2. \tag{D.4}
\]

The value of \( c_2 \) may be fixed by considering the case of a Ricci flat background where the partition function takes a simple form

\[
Z_2 = \left[ \frac{(\det \Delta_1)^4}{(\det \Delta_2)^3} \right]^{1/2}, \tag{D.5}
\]

with \( \Delta_1, \Delta_2 \) being the standard Laplacians acting on unconstrained vector and traceless tensor with \( R_{\mu\nu} = 0 \), \( \Delta_1 h_{\mu} = -D^2 h_{\mu}, \quad \Delta_2 h_{\mu\nu} = -D^2 h_{\mu\nu} - 2 C_{\mu\nu\rho} h^{\rho\sigma}. \) To find the corresponding \( b_6 \) coefficient for the vector Laplacian \( \Delta_1 \) from the general expressions in [56, 45] one is to use that here the covariant derivative contains an extra "internal" vector connection part with the curvature \( (F_{\mu\nu})_A^\beta = C_{\mu\nu\sigma}^\beta \). Then one finds

\[
7! b_6 [\Delta_1] = \frac{80}{7} C_{\alpha\beta\gamma\delta} C_{\beta\mu\nu} - \frac{164}{3} C_{\alpha\beta\mu}^\nu C_{\delta\gamma\rho}^\nu C_{\gamma\rho\mu} - 96 C_{\alpha\beta\gamma\delta} D^2 C_{\alpha\beta\gamma\delta} - 58 (D_{\mu} C_{\alpha\beta\gamma\delta})^2. \tag{D.6}
\]

In the case of spin 2 operator \(-D^2 + X \) one has \( (F_{\mu\nu})_A^\beta = \frac{1}{2} C_{\mu
u\rho\sigma}^\beta + ... \) (dots stand for 3 similar terms that symmetrize in \( (\alpha \beta) \) and \( (\rho \sigma) \)) and \( X_{\mu\nu,\rho\sigma} = -C_{\mu\nu\rho\sigma} - C_{\mu\nu\rho\sigma}. \) A straightforward computation then gives

\[
7! b_6 [\Delta_2] = \frac{49984}{9} C_{\alpha}^{\mu} C_{\beta\nu}^{\beta\gamma\delta} C_{\beta\mu\nu} - \frac{1388}{9} C_{\alpha\beta}^{\mu} C_{\delta\gamma\rho}^{\nu} C_{\gamma\rho\mu} + 1416 C_{\alpha\beta\gamma}^{\rho\delta} D^2 C_{\alpha\beta\gamma}^{\rho\delta} + 544 (D_{\mu} C_{\alpha\beta\gamma\delta})^2. \tag{D.7}
\]

\footnotetext{To compare, the usual Einstein theory partition function on a Ricci flat background is \( Z_{2,E} = \left[ \frac{(\det \Delta_1)}{(\det \Delta_2)} \right]^{1/2}. \)}
The full Seeley-DeWitt coefficient corresponding to (D.5) is given by the combination
\[ 3 b_6 \Delta_2 - 4 b_6 \Delta_1 \] and thus contains one of the total derivative terms discussed in [45].
Ignoring total derivative terms and using the relations between the invariants in (4.1) that exist in the Ricci flat case (\(E_6 = 32(2I_1 + I_2), I_3 = 4I_1 - I_2\)) one has in general
\[ \frac{\partial b_6}{\partial \rho} = - \left[ a - \frac{1}{192} (c_1 + 4c_2) \right] E_6 + (c_1 - 2c_2 + 6c_3) I_1 . \] (D.8)
As a result, we find
\[ a - \frac{1}{192} (c_1 + 4c_2) = \frac{377}{20160}, \quad c_1 - 2c_2 + 6c_3 = -\frac{1}{210} . \] (D.9)
Combining (D.9) with (D.4) we conclude that the first relation is satisfied identically while the second one determines \(c_2\). Thus finally
\[ c_1 = \frac{1507}{45}, \quad c_2 = \frac{635}{126}, \quad c_3 = -\frac{1639}{420} . \] (D.10)

### E Massless higher spins on \(S^4_q\)

One may define free massless higher spin (MHS) fields on \(S^4\) and consider, as in the case of AdS\(_4\) [58], the corresponding partition function built out of the determinants of the operators \(\Delta_{s,\perp}(M^2)\) on spin \(s\) TT tensors (2.1)
\[ Z_0 = \left[ \frac{1}{\det \Delta_0(2)} \right]^{1/2}, \quad Z_s = \left[ \frac{\det \Delta_{s-1,\perp}(1-s^2)}{\det \Delta_s(2+2s-s^2)} \right]^{1/2} . \] (E.1)
Then extending these MHS partition functions to \(S^4_q\) we may use the results in (2.19) and Appendix B to compute the corresponding total \(\hat{\xi}(q)\) function which is the coefficient of the log UV divergence. The expressions for the scalar and spin 1 field are the same as in (2.23) and (2.24), while for \(s = 2, 3, 4\) we obtain (cf. (2.24)–(2.27))
\[ \hat{\xi}_2(q) = -\frac{1}{180} q^3 - \frac{2}{q^2} + \frac{10}{3} - \frac{22}{3} - \frac{401}{60} q , \]
\[ \hat{\xi}_3(q) = -\frac{1}{180} q^3 - \frac{12}{q^2} + \frac{45}{2} q - \frac{39}{3} - \frac{2251}{60} q , \]
\[ \hat{\xi}_4(q) = -\frac{1}{180} q^3 - \frac{40}{q^2} + \frac{232}{3} q - \frac{376}{3} - \frac{7361}{60} q . \] (E.2)

Here we used (2.20) with the number of zero modes being (cf. (2.21))
\[ n_{z,s} = k_s - k_{s-1} = \frac{1}{6} s^2 (1 + 5 s^2) . \] (E.3)
A natural generalization to any \(s > 0\) is then (cf. (3.11))
\[ \hat{\xi}_s(q) = -\frac{1}{180} q^3 + \frac{s^2 (1 - s^2)}{6 q^2} - \frac{s^2 (3 - 2s^2)}{6 q} + \frac{s^2 (1 - 3s^2)}{6} - \frac{1}{60} (1 - 20 s^2 + 30 s^4) q . \] (E.4)

\(^{29}\)The analytic continuation from AdS\(_4\) to \(S^4\) corresponds to changing the sign of the square of the curvature radius or \(M^2 \rightarrow -M^2\). Note that the \(s = 0\) case is a special case of \(Z_s\) assuming one drops the ghost contribution. Here we set the radius of \(S^4\) to 1.
This has the expected general structure (2.29) with the number of degrees of freedom \( v = 2 \) and \( E_{c,s} \) being the Casimir energy of the MHS field on \( \mathbb{R} \times S^3 \) (see eq.(5.7) in [59]).

The coefficient of the total UV log divergence of the tower of massless higher spin fields on \( S^4 \) is given by \( \hat{\zeta}_0(1) + \sum_{s=1}^{\infty} \hat{\zeta}_s(1) \) and vanishes when regularized with an exponential cutoff or zeta-function [58, 60]. The same is true for the sum of the Casimir energies [59].

Similarly, on the conical \( S^4_q \) space we find that the total \( \hat{\zeta}(q) \) function also vanishes, i.e. the regularized sum

\[
\hat{\zeta}_0(q) + \sum_{s=1}^{\infty} e^{-\epsilon q} \hat{\zeta}_s(q) = \frac{-4}{q^2} - 12q + \frac{8}{q} + 8 + \frac{1}{e^3} + \frac{2q}{e^3} - \frac{q}{3} + \frac{2}{3} \epsilon^2 + \frac{1}{180}\pi^2 - \frac{q}{60} + \mathcal{O}(\epsilon),
\]

has zero finite part. Then the sum of free energies on \( S^4_q \times \mathbb{H}^3 \) (2.31) and thus the Rényi entropies (3.7) also vanish. Such formally defined Rényi entropy may be associated to the tower of massless higher spins in flat space and thus its vanishing is consistent with "topological" nature of such higher spin theory [15].

**F B2 Seeley-DeWitt coefficient for 4d spins \( s \leq 3 \)**

It is of interest to compare consequences of our expressions for the zeta-functions in section 2 with some previous results in [27]. Given a spectral zeta-function \( \zeta(z;q) \) for the operators \( \Delta_s(M^2) \) we can also extract the \( B_2 \) Seeley-DeWitt coefficient (of quadratic UV divergences) that appears in the \( t \to 0 \) expansion of the heat kernel in 4 dimensions, \( K(t) = B_0 t^{-2} + B_2 t^{-1} + B_4 + O(t) \). After a convenient rescaling we have

\[
\tilde{B}_2 \equiv \frac{(4\pi)^2}{\text{Vol}(S^4)} B_2 = 6 B_2 = 6 \lim_{z \to 1} (z - 1) \zeta(z;q). \tag{F.1}
\]

Reintroducing the factors of the scalar curvature (equal to 12 for a unit-radius \( S^4 \)) we have from (2.19)

\[
\begin{align*}
\tilde{B}_2^{(0)}(q,M^2) &= \frac{R}{24q} + q \left( \frac{R}{8} - M^2 \right), \\
\tilde{B}_2^{(1, -)}(q,M^2) &= -\frac{R}{2} + \frac{R}{8q} + q \left( \frac{5R}{8} - 3M^2 \right), \\
\tilde{B}_2^{(2, -)}(q,M^2) &= -\frac{R}{2} + \frac{5R}{24q} + q \left( \frac{5R}{24} - 5M^2 \right), \\
\tilde{B}_2^{(3, -)}(q,M^2) &= -7R + \frac{7R}{24q} + q \left( \frac{21R}{8} - 7M^2 \right).
\end{align*}
\]

Using the relations in Appendix A of [1], we then find for the coefficients in heat kernels of operators \( \Delta_s(M^2) \) defined on fields without the transversality condition

\[
\tilde{B}_2^{(1)}(q,M^2) = \tilde{B}_2^{(1, -)}(q,M^2) + \tilde{B}_2^{(0)}(q,M^2 - 3) = -\frac{R}{2} + \frac{R}{8q} + q \left( -4M^2 + R \right), \tag{F.3}
\]

\footnote{One can obtain \( E_{c,s} \) from the single particle partition function \( Z_s(x) = e^{-\frac{x}{(1-x)^{2s}} - x(2s-1)x} \) where \( x = e^{-\beta} \) using eq. (5.16) in [59].}

\footnote{Note that for massless higher spins in \( d \) dimensions the regularization prescription is with cutoff factor \( e^{-c(s+\frac{d-4}{2})} \) [14]. For conformal higher spins one has instead \( e^{-c(s+\frac{d+1}{2})} \) as they are effectively associated with the boundary, i.e. one is to replace \( d \to d - 1 \).}
\[ \tilde{B}_2^{(2)}(q, M^2) = \tilde{B}_2^{(2-1)}(q, M^2) + \tilde{B}_2^{(1-1)}(q, M^2 - 5) + \tilde{B}_2^{(0)}(q, M^2 - 8) = -3R + \frac{3R}{8q} + q \left( -9M^2 + \frac{33R}{8} \right), \]  
\[ \tilde{B}_2^{(3)}(q, M^2) = \tilde{B}_2^{(3-1)}(q, M^2) + \tilde{B}_2^{(2-1)}(q, M^2 - 7) + \tilde{B}_2^{(1-1)}(q, M^2 - 12) + \tilde{B}_2^{(0)}(q, M^2 - 15) = -10R + \frac{2R}{3q} + q \left( -16M^2 + 12R \right). \]  

It is convenient to split the coefficient \( \tilde{B}_2 \) into a regular "bulk" part and "surface" part coming from the conical singularity (cf. also discussion in Introduction)

\[
\tilde{B}_2(q, M^2) = \underbrace{q \tilde{B}_2(1, M^2)}_{\text{bulk}} + \underbrace{\tilde{B}_2(q, M^2) - q \tilde{B}_2(1, M^2)}_{\text{surface}}.
\]  

The "bulk" part is given by the usual Seeley-DeWitt coefficient evaluated on \( S_q^4 \) with the singular region excised: it is given by the standard \( S^4 \) (i.e. \( q = 1 \)) expression

\[
\tilde{B}_2(1, M^2) = N_s \left( \frac{R}{6} - M^2 \right), \quad N_s = (s + 1)^2,
\]

times the \( q \)-factor which accounts for the volume of \( S_q^4 \). The "surface" part in (F.6) vanishes for \( q = 1 \) by construction.\(^{32}\) The splitting (F.6) then takes the form

\[
\tilde{B}_2^{(0)}(q, M^2) = q \left( \frac{R}{6} - M^2 \right) + \frac{R}{4q} \left( \frac{1}{q} - 1 \right),
\]
\[
\tilde{B}_2^{(1)}(q, M^2) = q \left( \frac{2R}{6} - 4M^2 \right) + \left( -\frac{R}{2q} + \frac{R}{q} \right). \]
\[
\tilde{B}_2^{(2)}(q, M^2) = q \left( \frac{3R}{2} - 9M^2 \right) + \left( -3R + \frac{3R}{8q} + \frac{2R}{q} \right),
\]
\[
\tilde{B}_2^{(3)}(q, M^2) = q \left( \frac{8R}{3} - 16M^2 \right) + \left( -10R + \frac{2R}{3q} + \frac{8R}{3} \right). \]  

The spin 0,1,2 "surface" terms in (F.8) may be compared with the results of [27] (see eqs.(2.8), (2.11), (2.13) there with \( \beta = 2\pi q \))\(^{33}\)

\[
s = 0 : \frac{\beta}{6} \left( \frac{2\pi q}{R} \right)^2 - 1 \right] \frac{\text{Vol}(S^3)}{\text{Vol}(S^4)} = \frac{q}{2q} - \frac{q}{2},
\]
\[
s = 1 : \left[ N_1 \frac{\beta}{6} \left( \frac{2\pi q}{R} \right)^2 - 1 \right] + 2 \left( \beta - 2\pi \right) \frac{\text{Vol}(S^3)}{\text{Vol}(S^4)} = -6 + \frac{q}{2} + 4q,
\]
\[
s = 2 : \left[ N_2 \frac{\beta}{6} \left( \frac{2\pi q}{R} \right)^2 - 1 \right] + 12 \left( \beta - 2\pi \right) \frac{\text{Vol}(S^3)}{\text{Vol}(S^4)} = -36 + \frac{q}{2q} + \frac{63}{2} q. \]  

These match the "surface" terms in (F.8) after setting \( R = 12 \). The extension of the pattern in the l.h.s. of (F.9) to the spin 3 case that matches the \( s = 3 \) expression in (F.8) is

\[
s = 3 : \left[ N_3 \frac{\beta}{6} \left( \frac{2\pi q}{R} \right)^2 - 1 \right] + 40 \left( \beta - 2\pi \right) \frac{\text{Vol}(S^3)}{\text{Vol}(S^4)} = -120 + \frac{8}{q} + 112 q. \]  

\(^{32}\)In the approaches that represent the conical singularity in terms of a singular part in the curvature the "surface" term originates from an integral over \( S^2 \) near the cone singularity \( S^4_q \sim \epsilon_q \times S^2 \).

\(^{33}\)We factor out the volume ratio \( \frac{\text{Vol}(S^3)}{\text{Vol}(S^4)} \) that takes into account that the surface term in [27] is integrated over \( S^2 \). Notice also that the expression in eq. (2.13) of [27] has an additional \( 8\pi \) term that remains even in the smooth \( q \to 1 \) limit. In the above comparison we did not include this contribution. It is due to the dipole modes discussed in that paper. These are normalizable modes that exist for \( q < 1 \). They have a wave-function which is summable but singular at the cone’s apex. We do not see these modes because we constructed the spectral \( \xi \) function by considering as boundary conditions that the eigentensors of the Laplace-type operator are regular everywhere, i.e. the analogue of the Friedrich extension, see for instance section 1.5 of [61].

– 25 –
References

[1] A. A. Tseytlin, On partition function and Weyl anomaly of conformal higher spin fields, Nucl.Phys. B877 (2013) 598–631, [1309.0785].

[2] M. Beccaria and A. A. Tseytlin, On induced action for conformal higher spins in curved background, Nucl. Phys. B919 (2017) 359–383, [1702.00222].

[3] E. S. Fradkin and A. A. Tseytlin, Conformal supergravity, Phys.Rept. 119 (1985) 233–362.

[4] A. A. Tseytlin, On limits of superstring in AdS5 × S5, Theor.Math.Phys. 133 (2002) 1376–1389, [hep-th/0201112].

[5] A. Y. Segal, Conformal higher spin theory, Nucl.Phys. B664 (2003) 59–130, [hep-th/0207212].

[6] X. Bekaert, E. Joung and J. Mourad, Effective action in a higher-spin background, JHEP 1102 (2011) 048, [1012.2103].

[7] M. Beccaria, X. Bekaert and A. A. Tseytlin, Partition function of free conformal higher spin theory, JHEP 1408 (2014) 113, [1406.3542].

[8] M. Beccaria, S. Nakach and A. A. Tseytlin, On triviality of S-matrix in conformal higher spin theory, JHEP 09 (2016) 034, [1607.06379].

[9] T. Nutma and M. Taronna, On conformal higher spin wave operators, JHEP 1406 (2014) 066, [1404.7452].

[10] M. Grigoriev and A. A. Tseytlin, On conformal higher spins in curved background, 1609.09381.

[11] R. R. Metsaev, Ordinary-derivative formulation of conformal totally symmetric arbitrary spin bosonic fields, JHEP 1206 (2012) 062, [0709.4392].

[12] R. R. Metsaev, Arbitrary spin conformal fields in (A)dS, Nucl. Phys. B885 (2014) 734–771, [1404.3712].

[13] S. Giombi, I. R. Klebanov, S. S. Pufu and G. Tarnopolsky, AdS Description of Induced Higher-Spin Gauge Theory, JHEP 1310 (2013) 016, [1306.5242].

[14] S. Giombi, I. R. Klebanov and B. R. Safdi, Higher Spin AdSd+1/CFTd at One Loop, Phys.Rev. D89 (2014) 084004, [1401.0825].

[15] M. Beccaria and A. Tseytlin, On higher spin partition functions, J.Phys. A48 (2015) 257401, [1503.08143].

[16] M. Duff, Observations on Conformal Anomalies, Nucl.Phys. B125 (1977) 334.

[17] E. S. Fradkin and A. A. Tseytlin, Renormalizable asymptotically free quantum theory of gravity, Nucl. Phys. B201 (1982) 469–491.

[18] M. Beccaria and A. A. Tseytlin, Higher spins in AdS5 at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT, JHEP 1411 (2014) 114, [1410.3273].

[19] D. V. Fursaev, Spectral geometry and one loop divergences on manifolds with conical singularities, Phys. Lett. B334 (1994) 53–60, [hep-th/9405143].

[20] D. V. Fursaev and S. N. Solodukhin, On the description of the Riemannian geometry in the presence of conical defects, Phys. Rev. D52 (1995) 2133–2143, [hep-th/9501127].

[21] J. S. Apps and J. S. Dowker, The C(2) heat kernel coefficient in the presence of boundary discontinuities, Class. Quant. Grav. 15 (1998) 1121–1139, [hep-th/9712019].

[22] E. Perlmutter, A universal feature of CFT Renyi entropy, JHEP 03 (2014) 117, [1308.1083].
[23] D. V. Fursaev and S. N. Solodukhin, On one loop renormalization of black hole entropy, *Phys. Lett.* **B365** (1996) 51–55, [hep-th/9412020].

[24] S. N. Solodukhin, Entanglement entropy, conformal invariance and extrinsic geometry, *Phys. Lett.* **B665** (2008) 305–309, [0802.3117].

[25] A. F. Astaneh, A. Patrushev and S. N. Solodukhin, Entropy vs Gravitational Action: Do Total Derivatives Matter?, *1411.0926*.

[26] D. V. Fursaev, A. Patrushev and S. N. Solodukhin, Distributional Geometry of Squashed Cones, *Phys. Rev.* **D88** (2013) 044054, [1306.4000].

[27] D. V. Fursaev and G. Miele, Cones, spins and heat kernels, *Nucl. Phys.* **B484** (1997) 697–723, [hep-th/9605153].

[28] L. De Nardo, D. V. Fursaev and G. Miele, Heat kernel coefficients and spectra of the vector Laplacians on spherical domains with conical singularities, *Class. Quant. Grav.* **14** (1997) 1059–1078, [hep-th/9610011].

[29] M. A. Rubin and C. R. Ordóñez, Eigenvalues and degeneracies for n-dimensional tensor spherical harmonics, *Journal of Mathematical Physics* **25** (1984) 2888–2894.

[30] B. Allen, Phase Transitions in de Sitter Space, *Nucl. Phys.* **B226** (1983) 228.

[31] S. M. Christensen and M. J. Duff, Quantizing Gravity with a Cosmological Constant, *Nucl. Phys.* **B170** (1980) 480–506.

[32] E. S. Fradkin and A. A. Tseytlin, One Loop Effective Potential in Gauged O(4) Supergravity, *Nucl. Phys.* **B234** (1984) 472.

[33] M. G. Eastwood, Higher symmetries of the Laplacian, *Annals Math.* **161** (2005) 1645–1665, [hep-th/0206233].

[34] D. V. Fursaev and G. Miele, Finite temperature scalar field theory in static de Sitter space, *Phys. Rev.* **D49** (1994) 987–998, [hep-th/9302078].

[35] J. S. Dowker, Renyi entropies and $C_T$ for higher derivative free scalars and spinors on even spheres, *1706.01369*.

[36] J. S. Dowker, Renyi entropy and $C_T$ for p-forms on even spheres, *1706.04574*.

[37] E. Shaghoulian, Modular invariance on $S^1 \times S^3$ and circle fibrations, *1612.05267*.

[38] I. R. Klebanov, S. S. Pufu, S. Sachdev and B. R. Safdi, Renyi Entropies for Free Field Theories, *JHEP* **04** (2012) 074, [1111.6290].

[39] M. Beccaria and A. A. Tseytlin, $C_T$ for higher derivative conformal fields and anomalies of (1,0) superconformal 6d theories, *1705.00305*.

[40] A. Cappelli and A. Coste, On the Stress Tensor of Conformal Field Theories in Higher Dimensions, *Nucl. Phys.* **B314** (1989) 707.

[41] D. Kutasov and F. Larsen, Partition sums and entropy bounds in weakly coupled CFT, *JHEP* **0101** (2001) 001, [hep-th/0009244].

[42] K.-W. Huang, Central Charge and Entangled Gauge Fields, *Phys. Rev.* **D92** (2015) 025010, [1412.2730].

[43] W. Donnelly, B. Michel and A. Wall, Electromagnetic Duality and Entanglement Anomalies, *1611.05920*. 

– 27 –
[44] G. Wong, *A note on entanglement edge modes in Chern Simons theory*, 1706.04666.

[45] F. Bastianelli, S. Frolov and A. A. Tseytlin, *Conformal anomaly of (2,0) tensor multiplet in six-dimensions and AdS / CFT correspondence*, JHEP 0002 (2000) 013, [hep-th/0001041].

[46] M. Beccaria and A. A. Tseytlin, *Conformal a-anomaly of some non-unitary 6d superconformal theories*, JHEP 09 (2015) 017, [1506.08727].

[47] M. Beccaria and A. A. Tseytlin, *Conformal anomaly c-coefficients of superconformal 6d theories*, JHEP 01 (2016) 001, [1510.02685].

[48] A. A. Tseytlin, *Weyl anomaly of conformal higher spins on six-sphere*, Nucl.Phys. B877 (2013) 632–646, [1310.1795].

[49] H. W. J. Bloete, J. L. Cardy and M. P. Nightingale, *Conformal Invariance, the Central Charge, and Universal Finite Size Amplitudes at Criticality*, Phys. Rev. Lett. 56 (1986) 742–745.

[50] C. P. Herzog and K.-W. Huang, *Stress Tensors from Trace Anomalies in Conformal Field Theories*, Phys.Rev. D87 (2013) 081901, [1301.5002].

[51] L. Bonora, P. Pasti and M. Bregola, *Weyl cocycles*, Class.Quant.Grav. 3 (1986) 635.

[52] D. Butter, J. Novak and G. Tartaglino-Mazzucchelli, *The component structure of conformal supergravity invariants in six dimensions*, 1701.08163.

[53] D. Butter, S. M. Kuzenko, J. Novak and S. Theisen, *Invariants for minimal conformal supergravity in six dimensions*, JHEP 12 (2016) 072, [1606.02921].

[54] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, JHEP 9807 (1998) 023, [hep-th/9806087].

[55] R. Metsaev, *6d conformal gravity*, J.Phys. A44 (2011) 175402, [1012.2079].

[56] P. B. Gilkey, *The Spectral geometry of a Riemannian manifold*, J. Diff. Geom. 10 (1975) 601–618.

[57] Y. Pang, *One-Loop Divergences in 6D Conformal Gravity*, Phys.Rev. D86 (2012) 084039, [1208.0877].

[58] S. Giombi and I. R. Klebanov, *One Loop Tests of Higher Spin AdS/CFT*, JHEP 1312 (2013) 068, [1308.2337].

[59] S. Giombi, I. R. Klebanov and A. A. Tseytlin, *Partition Functions and Casimir Energies in Higher Spin AdS_{d+1}/CFT_d*, Phys. Rev. D90 (2014) 024048, [1402.5396].

[60] S. Giombi, *TASI Lectures on the Higher Spin - CFT duality*, in Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics: New Frontiers in Fields and Strings (TASI 2015): Boulder, CO, USA, June 1-26, 2015, pp. 137–214, 2017. 1607.02967.

[61] D. Iellici, *Aspects and applications of quantum field theory on spaces with conical singularities*. PhD thesis, Trento U., 1998. gr-qc/9805058.