Abstract

In this paper, we consider a class of convection-diffusion equations with memory effects. These equations arise as a result of homogenization or upscaling of linear transport equations in heterogeneous media and play an important role in many applications. First, we present a dememorization technique for these equations. We show that the convection-diffusion equations with memory effects can be written as a system of standard convection diffusion reaction equations. This allows removing the memory term and simplifying the computations. We consider a relation between dememorized equations and micro-scale equations, which do not contain memory terms. We note that dememorized equations differ from micro-scale equations and constitute a macroscopic model. Next, we consider both implicit and partially explicit methods. The latter is introduced for problems in multiscale media with high-contrast properties. Because of high-contrast, explicit methods are restrictive and require time steps that are very small (scales as the inverse of the contrast). We show that, by appropriately decomposing the space, we can treat only a few degrees of freedom implicitly and the remaining degrees of freedom explicitly. We present a stability analysis. Numerical results are presented that confirm our theoretical findings about partially explicit schemes applied to dememorized systems of equations.

1 Introduction

There are many problems that contain memory terms [1, 2, 12, 13]. One the well-known example is macro-dispersion due to small scales and reaction at small scales. It has the form

\[ u_t + a(x) \cdot \nabla u = \int_0^t \nabla \cdot (A(x, t, s) \nabla u(\tilde{x}(x, t, s), s)) \, ds. \]  

(1)

Here, \( u \) represents flow saturation in some porous medium, the term \( a(x) \) is a given velocity field at the macroscopic level, and \( \tilde{x} = \tilde{x}(x, t, s) \) is a trajectory that depends on fine-scale heterogeneities. The macro-dispersion (the term on the right hand side) is due to small-scale fluctuations of the velocity and the reaction at the micro-scale. The equation (1) occurs in many porous media related applications [9], which include groundwater, petroleum engineering, and...
biomedical applications. The velocity and macro-dispersion terms are, in general, heterogeneous as the velocity fluctuations are upscaled over the smallest scales. In this paper, our goal is to show how to dememorize these types of problems and its relation to homogenization, which does not contain memory related terms.

Solving (1) involves handling the memory terms and saving all previous time information. This can be difficult especially for multiscale and nonlinear problems. There have been several approaches that dememorize the problems of a different form [14]. In this paper, we follow similar concepts and dememorize (1) and consider its relation to equations at the micro-scales, which do not contain memory terms. In particular, we show that dememorized equations are, in some sense, homogenized equations, not similar to equations at the micro-scale. For example, the convection in dememorized problems occurs with averaged velocities. The diffusion is related to average quantity, which is represented by \( u \).

Dememorized equations constitute a system of coupled equations. In particular, the main equation can be written as

\[
 u_t + a \cdot \nabla u = v,
\]

where \( v \) is due to perturbation from the average state. The equations for \( v \) are convection-diffusion-reaction types, where convection and reaction effects occur with average rates. This equation contains a diffusion term, which depends on \( u \). Note that the equations at the micro-scales are purely convection-reaction types. We discuss the relation to microscale equations.

In this paper, we study dememorization and its discretization. We consider two types of discretizations, namely implicit and partially explicit. The latter is designed for multiscale problems based on a solution decomposition strategy [7, 8]. Because of the multiscale nature of the velocity and diffusion terms, one needs a very small time step when performing explicit discretization. The time step depends on the contrast. In [4], a partially explicit approach was first proposed for heterogeneous parabolic equations. The main idea of this approach is to handle some degrees of freedom implicitly, while the rest explicitly. As a result, we identify a few degrees of freedom on a coarse grid, that is much larger compared to spatial heterogeneities, and treat them implicitly. Our previous works (see also [10, 11]) show that the resulting approach is stable with appropriate decomposition of implicit and explicit components. In particular, implicit components account for fast flows, while explicit components account for slow flows. In this paper, we extend the partially explicit concept for the dememorized equations of (1). We use a spatial decomposition of the solution following the constraint energy minimizing generalized multiscale finite element method (CEM-GMsFEM) previously developed in [3]. In this decomposition, the fast and slow components of the solution are identified. Furthermore, we use implicit discretization for fast components and explicit discretization for slow components.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries for the model problem. In particular, we dememorize and derive the coupled system equivalent to the nonlocal equation with memory effects. In Section 3, we derive the numerical discretization schemes for the model problem. Numerical experiments are presented in Section 4. Concluding remarks are drawn in Section 5. In the Appendices, we present some remarks related to homogenization, constructing multiscale spaces, and stability estimates.

2 Preliminaries

In this section, we present some preliminaries for the nonlocal transport equations that arise in porous media applications. In particular, we introduce the model problem considered in this work, and derive the coupled system using the technique of dememorization.
2.1 Model Problem

Let $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) be a bounded domain and $T > 0$ be a given terminal time. We consider the following boundary value problem with memory effects: find $u(x, t)$ such that

$$u_t(x, t) + a(x) \cdot \nabla u(x, t) = \int_0^t \nabla \cdot (A(x, t, s) \nabla u(\tilde{x}(x, t, s), s)) \, ds \, \Omega \times (0, T],$$

$$u(x, 0) = u_0(x) \text{in } \Omega,$$

$$u(x, t) = 0 \text{ on } \partial \Omega,$$

where $a: \mathbb{R}^d \to \mathbb{R}^d$ is a vector-valued function, $A(x, t, s)$ represents a macro-dispersion coefficient, and $\tilde{x}(x, t, s)$ is a trajectory that satisfies

$$\frac{d}{ds} \tilde{x}(x, t, s) = \tilde{a}(\tilde{x}), \quad \tilde{x}(x, t, t) = x.$$  

That is, the trajectory with the velocity $\tilde{a}(\tilde{x})$ is such that at the time $t$ it reaches the point $x$. The trajectory has the following explicit expression $\tilde{x}(x, t, s) = x - (t - s)\tilde{a}$ when $\tilde{a}$ is a constant.

The kernel $A(x, t, s)$ is assumed to be in terms of the exponential term due to the reaction at the micro-scale. In this case, we assume that

$$A(x, t, s) = \kappa(x) e^{-\beta(t-s)}$$

for some permeability tensor $\kappa(x)$. Such reaction kernel can be derived from homogenization (see Appendix A). More generally, the kernel $A(x, t, s)$ has the form

$$A(x, t, s) = \sum_{i=1}^{M} \kappa_i(x) e^{-\beta_i(t-s)},$$

where $M$ is a positive integer. In this case, the functions $\kappa_i$’s are some heterogeneous fields.

The model problem (2) is well-known in the sense that it can be derived from upscaling (see Appendix A) of some micro-scale transport equations containing heterogeneous velocity fields [5, 15]. In this case, the solution of the macroscopic equation is an average of the microscopic solution, where microscopic equations do not contain memory terms. In this work, we show that one can re-write the macroscopic equations without memory. However, the resulting macroscopic equation is different from microscale equations (without memory). The form of macroscopic diffusion $A(x, t, s)$ is similar to the one obtained from upscaling.

2.2 Dememorization

In this section, we apply the technique of dememorization for the problem (2) with the kernel function having the form (5) and the trajectory (3). In particular, we introduce auxiliary variables $\{v_i\}_{i=1}^M$ and derive the coupled system for the main variable $u$ and the auxiliary variables.

The dememorization starts with the following auxiliary variables. For $i \in \{1, 2, \cdots, M\}$, we define

$$v_i(x, t) := \int_0^t e^{-\beta_i(t-s)} u(\tilde{x}(x, t, s), s) \, ds \quad \text{for any } (x, t) \in \Omega \times (0, T].$$

Note that, from the original equation (2), we have

$$u_t + a(x) \cdot \nabla u = \sum_{i=1}^{M} \nabla \cdot (\kappa_i(x) \nabla v_i).$$

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On the other hand, taking derivative (with respect to \( t \)) of \( v_i \), we obtain
\[
(v_i)_t + \beta_i v_i + \bar{a} \cdot \nabla v_i = u.
\]
Therefore, we obtain the following coupled system without memory effects:
\[
\begin{align*}
(v_i)_t + \beta_i v_i + \bar{a} \cdot \nabla v_i &= u & \text{in } \Omega \times (0, T], & i \in \{1, 2, \cdots, M\}, \\
u_t + a \cdot \nabla u - \sum_{i=1}^{M} \nabla \cdot (\kappa_i \nabla v_i) &= 0 & \text{in } \Omega \times (0, T], \\
u(x, 0) &= u_0(x) & \text{in } \Omega, \\
u(x, t) &= 0 & \text{in } \partial \Omega.
\end{align*}
\]
(7)

The boundary condition for each \( v_i(x, t) \) is defined via \( u \) using \([\Box] \) while assuming \( u \) outside \( \Omega \) is zero. We remark that if the permeability tensor has the form \( \kappa = \kappa_{11} (a \otimes a) \) for some bounded function \( \kappa_{11} \) and the conditions \( \beta \kappa_{11} + \nabla \kappa_{11} \cdot a \geq 0 \) and \( a \cdot n_{\partial \Omega} = 0 \) hold, where \( n_{\partial \Omega} \) is the unit outward normal vector to the boundary \( \partial \Omega \), then one can show that the continuous problem (7) is stable. See Appendix [\Box] for more details on the stability of the continuous problem (7). The stability analysis for the discretized convection-diffusion model with memory effects is challenging and will be one of our future works. In the following, we develop numerical discretization scheme for (7) and provide a stability estimate for the case when \( a = \tilde{a} \equiv 0 \).

For the numerical discretization, we introduce the variational formulation of the problem (7). To this aim, we define \( \Gamma \subset \partial \Omega \) such that \( \Gamma := \{ x \in \partial \Omega : \bar{a}(x) \cdot n_{\partial \Omega}(x) < 0 \} \) and we write \( H^1_0(\Gamma) := \{ v \in H^1(\Omega) : v|_{\Gamma} = 0 \} \). We assume that \( \Gamma \) has positive measure. The corresponding variational problem reads as follows: Find \( u(\cdot, t) \in H^1_0(\Omega) \) and \( \{ v_i(\cdot, t) \}_{i=1}^{M} \subset H^1_0(\Omega) \) such that
\[
\begin{align*}
((v_i)_t, \phi) + (\beta_i v_i, \phi) + (\bar{a} \cdot \nabla v_i, \phi) &= (u, \phi) & \forall \phi \in H^1_0(\Omega), & i \in \{1, 2, \cdots, M\}, \\
(u_t, \psi) + (a \cdot \nabla u, \psi) + \sum_{i=1}^{M} A_i(v_i, \psi) &= 0 & \forall \psi \in H^1_0(\Omega).
\end{align*}
\]
(8)

We denote \( (\cdot, \cdot) \) the inner product in \( L^2(\Omega) \) and \( A_i(\cdot, \cdot) \) is defined to be
\[
A_i(\phi, \psi) := \int_{\Omega} \kappa_i \nabla \phi \cdot \nabla \psi \, dx
\]
for any \( \phi \in H^1(\Omega) \) and \( \psi \in H^1(\Omega) \). We denote \( \| \cdot \| \) the \( L^2 \) norm induced by the inner product \( (\cdot, \cdot) \) and we write \( \| \cdot \|_{A_i} := \sqrt{A_i(\cdot, \cdot)} \). When \( M = 1 \), we simply write
\[
A(\phi, \psi) := \int_{\Omega} \kappa_1 \nabla \phi \cdot \nabla \psi \, dx
\]
for any \( \phi \in H^1(\Omega) \) and \( \psi \in H^1(\Omega) \) and the corresponding energy norm is written as \( \| \cdot \|_A := \sqrt{A(\cdot, \cdot)} \). For \( i \in \{1, 2, \cdots, M\} \), we assume that \( \beta_i \) is a positive constant and \( \kappa_i \in L^\infty(\Omega; \mathbb{R}^{d \times d}) \) is a permeability tensor that fulfils \( 0 < C_0 \leq \xi^T \kappa_i \xi \leq C_\infty < \infty \) for any \( \xi \in \mathbb{R}^d \) with \( |\xi| = 1 \) (with \( |\cdot| \) being the usual Euclidean norm in \( \mathbb{R}^d \)). We may define the operator \( R_i : H^1_0(\Omega) \rightarrow H^{-1}(\Omega) \) such that
\[
(R_i \phi, \psi) := A_i(\phi, \psi)
\]
for any \( \phi \in H^1(\Omega) \) and \( \psi \in H^1(\Omega) \).

Next, we derive an energy estimate for the solution in the absence of convection (see Appendix [\Box] for the case with convection). We remark that assuming \( a = \tilde{a} \equiv 0 \), if we take \( \psi = u \) in (8).
and integrate over \((0, T]\), we obtain
\[
0 = \int_0^T \left[ (u_t, u) + \sum_{i=1}^{M} (\mathcal{R}_i v, u) \right] dt \\
= \int_0^T \left[ (\partial_t u, u) + \sum_{i=1}^{M} (\mathcal{R}_i v_i, \beta v_i) + \sum_{i=1}^{M} (\mathcal{R}_i v_i, \partial_t v_i) \right] dt \\
= \frac{1}{2} \int_0^T \frac{d}{dt} \left( \|u\|^2 + \sum_{i=1}^{M} \|v_i\|^2_{\mathcal{A}_i} \right) dt + \sum_{i=1}^{M} \int_0^T A(v_i, \beta v_i) dt \\
= \frac{1}{2} \left( \|u(\cdot, T)\|^2 + \sum_{i=1}^{M} \|v_i(\cdot, T)\|^2_{\mathcal{A}_i} \right) - \|u_0\|^2 - \sum_{i=1}^{M} \|v_i(\cdot, 0)\|^2_{\mathcal{A}_i} + \beta \sum_{i=1}^{M} \int_0^T \|v_i(\cdot, t)\|^2_{\mathcal{A}_i} dt.
\]

Therefore, we have
\[
E(u, v; M; T) := \|u(\cdot, T)\|^2 + \sum_{i=1}^{M} \|v_i(\cdot, T)\|^2_{\mathcal{A}_i} \leq \|u_0\|^2 + \sum_{i=1}^{M} \|v_i(\cdot, 0)\|^2_{\mathcal{A}_i} := E(u, v; M; 0)
\]
for the function \(E(u, v; M; t) := \|u(\cdot, t)\|^2 + \sum_{i=1}^{M} \|v_i(\cdot, t)\|^2_{\mathcal{A}_i}\). It implies that the continuous problem (8) (when \(a = \tilde{a} \equiv 0\)) is stable with respect to the function \(E(\cdot, \cdot; M; t)\).

3 Numerical Discretizations

In this section, we set \(M = 1\) and develop the numerical discretization for (8) and provide stability estimate for the numerical schemes. We present the temporal discretizations using the implicit scheme and the recently developed partially explicit scheme based on a space decomposition strategy. We assume that some finite dimensional spaces \(V_H \subset H^1_0(\Omega)\) and \(W_H \subset H^1_0(\Omega)\) based on some (coarse-grid) partition for the domain are developed and we perform spatial discretization using Galerkin method with the ansatz spaces \(V_H\) and \(W_H\). The coarse-grid spaces \(V_H\) and \(W_H\) are defined via the recently developed CEM-GMsFEM for multiscale problems. See Appendix C for more details of the construction and definition.

3.1 Semi-implicit Scheme

In this section, we develop the implicit-in-time fully discretization for the problem (8). To this aim, we introduce a temporal partition \(\{t^n\}_{n=1}^{N_T} \) with \(t^n = n \Delta t\) \((k \in \{0, 1, \ldots, N_T\})\) and \(T = N_T \Delta t\); we also denote \(v^n = v(\cdot, t^n)\) and \(u^n = u(\cdot, t^n)\) for any \(n \in \{0, 1, \ldots, N_T\}\). We remark that the convection term is computed explicitly in our discretization. The (implicit) fully discretization reads as follows: find \(\{u_H^k\}_{k=1}^{N_T} \subset V_H\) and \(\{v_H^k\}_{k=1}^{N_T} \subset W_H\) such that the following system holds

\[
\begin{align*}
\frac{v_H^{n+1} - v_H^n}{\Delta t} + \beta (v_H^n, \phi) + (\tilde{a} \cdot \nabla v_H^n, \phi) - (u_H^{n+1}, \phi) &= 0 \quad \forall \phi \in W_H, \\
\frac{u_H^{n+1} - u_H^n}{\Delta t} + (a \cdot \nabla u_H^n, \psi) + A(v_H^{n+1}, \psi) &= 0 \quad \forall \psi \in V_H.
\end{align*}
\]

(9)

for any \(n \in \{0, 1, \ldots, N_T - 1\}\). The terms \(u_H^0\) and \(v_H^0\) are obtained from the initial conditions in the sense that
\[
(v_H^0, \phi) = (v_0, \phi) \quad \forall \phi \in W_H \quad \text{and} \quad (u_H^0, \psi) = (u_0, \psi) \quad \forall \psi \in V_H.
\]
We remark that assuming \( a = \tilde{a} \equiv 0 \), one can show the stability of the fully implicit scheme (9). Let \( \psi = u_H^{n+1} \) in (9) and we have

\[
0 = \frac{1}{\Delta t} (u_H^{n+1} - u_H^n, u_H^{n+1}) + (R v_H^{n+1}, u_H^{n+1})
\]

\[
= \frac{1}{2\Delta t} (\|u_H^{n+1}\|^2 - \|u_H^n\|^2 + \|u_H^{n+1} - u_H^n\|^2) + \frac{1}{\Delta t} (R v_H^{n+1}, v_H^{n+1} - v_H^n) + \beta(v_H^{n+1}, v_H^{n+1})
\]

\[
\geq \frac{1}{2\Delta t} (\|u_H^{n+1}\|^2 - \|u_H^n\|^2) + \beta\|v_H^{n+1}\|^2 + \frac{1}{2\Delta t} (\|v_H^{n+1}\|^2 - \|v_H^n\|^2 + \|v_H^{n+1} - v_H^n\|^2)
\]

Thus, we have shown that, for any \( n \in \{1, \ldots, N_T\} \),

\[
E^n(u_H, v_H) \leq E^0(u_H, v_H), \quad \text{where } E^n(u_H, v_H) := \|u_H^n\|^2 + \|v_H^n\|^2
\]

This shows the stability for the case with \( a = \tilde{a} \equiv 0 \). In Appendix [E] we give a stability proof for more general case.

### 3.2 Partially Explicit Splitting Scheme

In this section, we first briefly review the recently developed partially explicit splitting scheme and apply this scheme for discretizing (5). The partially explicit splitting scheme is based on a solution decomposition strategy for the coarse spaces \( V_H \) and \( W_H \). We assume each ansatz space can be written as a direct sum of two subspaces: \( V_H = V_H^1 \oplus V_H^2 \) and \( W_H = W_H^1 \oplus W_H^2 \); we seek approximations in these ansatz spaces. In particular, the component in the first subspace \( V_H^1 \) (resp. \( W_H^1 \)) will be treated implicitly during the evolution while the components in the second subspace \( V_H^2 \) (resp. \( W_H^2 \)) will be computed in an explicit manner. An enhancement in terms of computational efficiency can be achieved within this setting of implicit-explicit formulation.

With these ansatz spaces and the specific subspace decomposition, we can write \( u_H^n = u_H^{n,1} + u_H^{n,2} \) and \( v_H^n = v_H^{n,1} + v_H^{n,2} \) for any \( n \in \{0, 1, \ldots, N_T\} \). The partially explicit splitting scheme reads as follows: find \( \{u_H^{n,i}\}_{n=1}^{N_T} \subset V_H^i \) and \( \{v_H^{n,i}\}_{n=1}^{N_T} \subset W_H^i \) for \( i \in \{1, 2\} \) such that the following system holds

\[
\begin{align*}
\left( \frac{v_H^{n+1} - v_H^n}{\Delta t}, \phi_1 \right) + \beta(v_H^n, \phi_1) + (\tilde{a} \cdot \nabla v_H^n, \phi_1) - (u_H^{n+1}, \phi_1) &= 0 & \forall \phi_1 \in W_H^1, \\
\left( \frac{v_H^{n+1} - v_H^n}{\Delta t}, \phi_2 \right) + \beta(v_H^n, \phi_2) + (\tilde{a} \cdot \nabla v_H^n, \phi_2) - (u_H^{n+1}, \phi_2) &= 0 & \forall \phi_2 \in W_H^2, \\
\left( \frac{u_H^{n+1} - u_H^n}{\Delta t} + \frac{u_H^{n+1} - u_H^n}{\Delta t}, \psi_1 \right) + (a \cdot \nabla u_H^n, \psi_1) + A(v_H^{n+1}, \psi_1) &= 0 & \forall \psi_1 \in V_H^1, \\
\left( \frac{u_H^{n+1} - u_H^n}{\Delta t} + \frac{u_H^{n+1} - u_H^n}{\Delta t}, \psi_2 \right) + (a \cdot \nabla u_H^n, \psi_2) + A(v_H^{n+1}, \psi_2) &= 0 & \forall \psi_2 \in V_H^2,
\end{align*}
\]

for any \( n \in \{0, 1, \ldots, N_T - 1\} \). For the case of pure reaction (i.e., \( a = \tilde{a} = 0 \)), we can derive a stability estimate for the above-mentioned partially explicit splitting scheme (10). To this aim, we define a constant \( \gamma \in (0, 1) \) such that

\[
\gamma := \sup_{v_1 \in V_H^1, \ v_2 \in V_H^2} \frac{(v_1, v_2)}{\|v_1\| \|v_2\|}.
\]
For the pure reaction case, the partially explicit splitting scheme (10) is stable under appropriate assumptions on the subspaces \( V_1^H \) and \( V_2^H \). The stability estimate for the general convection-diffusion case is left as future work.

**Theorem 3.1** (Stability estimate of pure reaction case). Assume that \( a = \tilde{a} \equiv 0 \). Let \( \gamma \) be defined in (11). Suppose that the temporal step size \( \Delta t \) satisfies

\[
\Delta t \leq \beta (1 - \gamma) \inf_{v \in V_2^H} \frac{\|v\|^2}{\|v\|^2_A},
\]

(12) Then, the solutions \( u^n_H = u^n_{H,1} + u^n_{H,2} \) and \( v^n_H = v^n_{H,1} + v^n_{H,2} \) obtained from (10) satisfy the following stability estimate

\[
\tilde{E}^n(u_H,v_H) \leq \tilde{E}^0(u_H,v_H)
\]

for any \( n \in \{1, \ldots, N_T\} \), where \( \tilde{E}^n(u_H,v_H) := \|u_H^n\|^2 + \sum_{i=1}^2 \|v_{H,i}^n\|^2_A \) is the discrete energy function.

The proof of this result is given in Appendix D.

### 4 Numerical Experiments

In this section, we perform some numerical experiments using the discretization schemes discussed in the previous section. In all the experiments, we set the spatial domain to be \( \Omega = (0,1)^2 \). We set the velocity fields to be \( \tilde{a} = (0.05,0)^T \) and \( a = (0.1,0)^T \). The spatial domain is partitioned into uniform square elements with mesh size \( H = \sqrt{2}/10 \) to form a coarse grid. Next, for each coarse element from the coarse partition, we further divide it into \( 10 \times 10 \) uniform square elements so that the mesh size \( h \) of the fine grid is \( h = \sqrt{2}/100 \). We equip the variable \( u \) with the homogeneous Neumann boundary condition on the whole boundary \( \partial \Omega \).

For the ansatz space, we choose three local auxiliary functions (i.e., \( L_i = J_i = 3 \) for each \( i \in \{1, \ldots, N_e\} \) with \( N_e = 100 \); see Appendix C for more details) to form the local auxiliary space in each coarse element so that the dimensions of \( V_1^H \) and \( V_2^H \) are 300. We take the oversampling parameter to be \( m = 4 \). Based on the fine grid and implicit temporal discretization, we compute a numerical approximation which serves as a reference solution. In the following, we compute three different numerical approximations and compare them with the reference solution in terms of \( L^2 \) error:

1. The first approximation is obtained using only the first ansatz space \( V_1^H \) with the implicit temporal discretization (i.e., solving (9) with the space \( V_1^H \)). We refer to this approximation as implicit CEM.

2. Combining the additional ansatz space \( V_2^H \), we compute the second approximation over the space \( V_H = V_1^H \oplus V_2^H \) via implicit scheme (i.e., solving (9) with \( V_H \)). We refer to this approximation as implicit CEM with additional bases.

3. The third one is computed by solving the partially explicit splitting scheme (10). We refer to this approximation as partially explicit splitting CEM.

From these numerical examples, we find that the partially explicit scheme can achieve similar accuracy as the fully implicit scheme with less computing cost at each time level.
Example 4.1. In the first example, we set the initial condition to be $u_0(x) = \sin(\pi x_1) \sin(\pi x_2)$ for any $x = (x_1, x_2) \in \Omega$. Let $T = 0.05$ and the temporal step size is $\Delta t = T/100 = 5 \times 10^{-4}$. The permeability field $\kappa$ used in this example is depicted in Figure 1.

In Figure 2, we present the profiles of the three types of solutions at the terminal time - the reference solution (the implicit fine grid solution), the implicit CEM solution with additional bases, and the partially explicit splitting CEM solution. The relative $L^2$ error against time is presented in Figure 3. Despite the differences among these profiles of the numerical approximations, the relative $L^2$ error is around 8% using the partially explicit splitting scheme, which is relatively small and acceptable. Besides, from Figure 3, the error curves for the implicit CEM solution with additional bases and the partially explicit solution nearly coincide. This implies that one can achieve the same level of accuracy using the proposed partially explicit splitting scheme as the implicit CEM scheme with additional basis functions.

Figure 1: Permeability in Example 4.1

Figure 2: Solution profiles at terminal time in Example 4.1. Left: Reference solution. Middle: Implicit CEM solution with additional bases. Right: Partially explicit CEM solution.
Example 4.2. In the second example, the permeability field $\kappa$ is the same as the one in Example 4.1. We set the initial condition to be $u_0 = 0$. To avoid the solution $u$ being trivial, we add a constant-in-time source term $g_0(x) = g_0(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2)$ for any $(x_1, x_2) \in \Omega$ to the right-hand side of the third and forth equations in (10). Let $T = 0.05$ and the temporal step size is $\Delta t = T/100 = 5 \times 10^{-4}$.

The profiles of the numerical solutions at the terminal time are sketched in Figure 4. The $L^2$ error curves against time are shown in Figure 5. Similar to Example 4.1, from the plot of error curves, we find that there is a considerable decrease in terms of $L^2$ error when we include $V_H^2$. Moreover, we notice that the curves for the implicit CEM solution with additional bases and the partially explicit solution nearly coincide, which implies that they have similar accuracy. In those settings, the $L^2$ error at the terminal time is about 1.9%.
Example 4.3. In the third example, we take the problem setting from Example 4.2 except the permeability. The permeability field for this case contains more channels and it is depicted in Figure 6. The solutions profiles are plotted in Figure 7 and the relative $L^2$ error plot is shown in Figure 8. In this case, the $L^2$ error at the terminal time is about 2.2% using the partially explicit splitting scheme, which is comparable to the case using the implicit method with additional basis functions. This demonstrates the effectiveness and efficiency of the proposed partially explicit temporal discretization with additional basis functions from $V_H^2$.  

Figure 5: Plot of relative $L^2$ error in Example 4.2

Figure 6: Permeability in Example 4.3

Figure 7: Solution profiles at terminal time in Example 4.3. Left: Reference solution. Middle: Implicit CEM solution with additional bases. Right: Partially explicit CEM solution.
5 Conclusion

In this work, we propose dememorization technique for a class of convection-diffusion equations with memory effects. These macroscopic equations arise as a result of homogenization or upscaling of transport equations (without memory terms) in heterogeneous media. Because of transport at the microscales, the upscaled equations contain memory terms. The dememorization technique introduced in the paper allows removing the term with memory effect and simplifying the computations. The dememorized equations differ from the original micro-scale equations. For the numerical discretization, we consider both implicit and partially explicit splitting methods within the framework of CEM-GMsFEM. The latter scheme was previously introduced for problems in multiscale media with high-contrast properties, which had been shown to be effective for such category of problems. Numerical results were presented that demonstrate the effectiveness and efficiency of the partially explicit schemes applying to the dememorized system of equations.

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A Derivation of Macroscopic Equations

In this section, we derive the macro-scale equation with a nonlocal memory term from the micro-scale transport equation. We consider two cases. The first one is upscaling the transport equation with a perturbation perspective; while the second case deals with the flow transport in a special type of porous medium.

A.1 Perturbation Approach

Consider the following transport equation in a heterogeneous medium

\[ u_t^\varepsilon + a^\varepsilon \cdot \nabla u^\varepsilon = -ku^\varepsilon. \]  \hspace{1cm} (13)

In general, the velocity field \( a^\varepsilon : \mathbb{R}^d \to \mathbb{R}^d \) contains features at micro-scale driven by some hidden parameter \( \varepsilon \). Let \( a^\varepsilon = \bar{a} + \tilde{a} \) and \( u^\varepsilon = \bar{u} + \tilde{u} \), where \( \bar{a} \) and \( \bar{u} \) are some small perturbations from the mean values. Here, we can think of the case when \( a^\varepsilon \) (resp. \( u^\varepsilon \)) is a random velocity field with \( \bar{a} \) (resp. \( \bar{u} \)) being the expectation of the random field. Then, the equation \( (13) \) becomes

\[ \bar{u}_t + \bar{a} \cdot \nabla \bar{u} + \bar{a} \cdot \nabla \bar{u} = -k\bar{u}. \]  \hspace{1cm} (14)

Taking the expectation on both sides of \( (14) \), we have

\[ \bar{u}_t + \bar{a} \cdot \nabla \bar{u} = -k\bar{u}. \]  \hspace{1cm} (15)

Subtracting \( (15) \) from \( (14) \), we get

\[ \tilde{u}_t + \tilde{a} \cdot \nabla \tilde{u} + \tilde{a} \cdot \nabla \tilde{u} + \underbrace{\bar{a} \cdot \nabla \bar{u} - \bar{a} \cdot \nabla \bar{u}}_{\text{high order terms}} = -k\tilde{u} \]  \hspace{1cm} (16)

A.2 Flow Transport in a Special Type of Porous Medium

Consider the following transport equation in a heterogeneous medium where the velocity field is given by

\[ u_t + a \cdot \nabla u = -ku. \]  \hspace{1cm} (17)

In this case, the velocity field \( a : \mathbb{R}^d \to \mathbb{R}^d \) contains features at micro-scale driven by some hidden parameter \( \varepsilon \). Let \( a = \bar{a} + \tilde{a} \) and \( u = \bar{u} + \tilde{u} \), where \( \bar{a} \) and \( \bar{u} \) are some small perturbations from the mean values. Here, we can think of the case when \( a \) (resp. \( u \)) is a random velocity field with \( \bar{a} \) (resp. \( \bar{u} \)) being the expectation of the random field. Then, the equation \( (17) \) becomes

\[ \bar{u}_t + \bar{a} \cdot \nabla \bar{u} = -k\bar{u}. \]  \hspace{1cm} (18)

Taking the expectation on both sides of \( (18) \), we have

\[ \bar{u}_t + \bar{a} \cdot \nabla \bar{u} = -k\bar{u}. \]  \hspace{1cm} (19)

Subtracting \( (19) \) from \( (18) \), we get

\[ \tilde{u}_t + \tilde{a} \cdot \nabla \tilde{u} + \underbrace{\bar{a} \cdot \nabla \bar{u} - \bar{a} \cdot \nabla \bar{u}}_{\text{high order terms}} = -k\tilde{u} \]  \hspace{1cm} (20)

where \( \tilde{u}_t + \tilde{a} \cdot \nabla \tilde{u} \) represents the high order terms.
or equivalently,

\[ \ddot{u}_t + \ddot{a} \cdot \nabla \ddot{u} + k \ddot{u} = -\ddot{a} \cdot \nabla \ddot{u} + (\text{high order terms}). \tag{16} \]

In general, the fluctuation \( \ddot{u} \) is a function of \( x, t \), and some hidden random variables; thus, we simply write \( \ddot{u} = \ddot{u}(x, t) \). Notice that for any fixed \( t > 0 \), we define \( \varphi = \varphi(x, s; t) \) such that

\[ \varphi(x, s; t) = \ddot{u}(\tilde{x}(x, s; t), s) \quad \text{for any } s \in (0, t), \]

where \( \tilde{x} = \tilde{x}(x, s; t) \) is a trajectory which satisfies

\[ \frac{\partial \tilde{x}}{\partial s} = \ddot{a}, \quad \tilde{x}|_{s=t} = x. \]

In the case of \( \ddot{a} = \ddot{a}(x) \), we can rewrite the trajectory as \( \tilde{x} = x - \ddot{a}(t-s) \). Then, the equation (16) becomes

\[ \ddot{\varphi}_s + k \ddot{\varphi} = -\dddot{a} \cdot \nabla \dot{u} + (\text{high order terms}). \]

Assume \( \ddot{\varphi}|_{s=0} = 0 \). We integrate (with respect to \( s \)) the above equation over interval \((0, t)\), multiply the result by \( \ddot{a}_i(x) \) (where \( \dddot{a} = (\dddot{a}_i)_{i=1}^d \)), and take average with respect to the randomness. This gives

\[ \ddot{\varphi}_i = \dddot{a}_i \ddot{u} + k \int_0^t \dddot{a}_i \ddot{u}(\tilde{x}, s) ds = -\dddot{a}_i \int_0^t \dddot{a}_j(\tilde{x}) \frac{\partial \ddot{u}}{\partial x_j}(\tilde{x}, s) ds + (\text{high order terms}). \]

Here, the Einstein summation convention is used for the index \( j \). Neglecting the high order terms, this implies that \( q = \dddot{a}_i \ddot{u} \) satisfies the following ODE

\[ q_t + kq = -\dddot{a}_i \dddot{a}_j \frac{\partial \ddot{u}}{\partial x_j}. \tag{17} \]

One can solve for \( q \) and it implies that

\[ \dddot{a}_i \ddot{u} = -\dddot{a}_i \int_0^t e^{-k(t-s)} \dddot{a}_j(x - \dddot{a}(t-s)) \frac{\partial \ddot{u}}{\partial x_j}(x - \dddot{a}(t-s)) ds. \tag{17} \]

Assume that \( \dddot{a} \) is divergence-free. Substituting (17) into (15), we get

\[ \dddot{u}_t + \dddot{a} \cdot \nabla \dddot{u} + k \dddot{u} = \frac{\partial}{\partial x_i} \int_0^t e^{-k(t-s)} \dddot{a}_i(x) \dddot{a}_j(x - \dddot{a}(t-s)) \frac{\partial \ddot{u}}{\partial x_j}(x - \dddot{a}(t-s)) ds. \tag{18} \]

We remark that the Einstein summation convention is used for indices \( i \) and \( j \) in (18). This gives the macroscopic transport equation for \( \ddot{u} \) under the average velocity field \( \dddot{a} \) with a memory term on the right-hand side.

### A.2 Upscaling for Layered Media

In this case, we consider the transport equation in a layered medium and seek the solution

\[ u = u(x, t) \]

such that

\[ u_t + a \left( \frac{x_2}{\varepsilon} \right) \cdot \frac{\partial u}{\partial x_2} = 0, \]

with the initial condition \( u(x, 0) = H(x_1) \) for any \( x = (x_1, x_2) \) in a bounded domain, where \( H(\cdot) \) is the single variable Heaviside function. The \( a(\cdot) \) is a given scalar function describing the
velocity along the $x_2$ direction. We are interested in the homogenized solution given by the average along the $x_2$ direction as follows:

$$\bar{u} = \int u(x,t) dx_2.$$  

We remark that the above integral should be understood in the sense of average along the direction of $x_2$. We present a discrete case assuming that $a$ takes values $a_i$ in the $i$-th layer that has a width $m_i$ (with $n$ being total number of layers), i.e.,

$$a\left(\frac{x_2}{\varepsilon}\right) = a_i, \quad \text{if } y_{i-1} \leq \frac{x_2}{\varepsilon} < y_i, \quad m_i = y_i - y_{i-1}$$

for $i \in \{1, \cdots, n\}$. Then, the averaged solution can be written as

$$\bar{u}(x,t) = \sum_{i=1}^{n} m_i H(x - a_i t),$$

and the homogenized solution is given by

$$\frac{\partial \bar{u}}{\partial t} + \bar{a} \cdot \nabla \bar{u} + \beta \bar{u} = u,$$

$$\partial_t u + a \cdot \nabla u = \nabla \cdot \kappa \nabla u,$$

where $\bar{a}$, $\beta$, and $u$ ($i = 1, \cdots, n - 1$) satisfy

$$\sum_{k=1}^{n} m_k \frac{u_i - a_k}{u_i - a_k} = 0 \quad i = 1, \cdots, n - 1,$$

$$\sum_{i=1}^{n-1} \frac{\beta_i}{u_i - a_k} = \left(\bar{a} - a_k \right) \quad k = 1, \cdots, n,$$

$$\bar{a} = \sum_{i=1}^{n} m_i a_i.$$

See [6] for details. Note that one can show that $\beta_i$’s and $u_i$’s exist and are unique. Moreover, they have the following properties:

1. $a_1 \leq u_1 \leq a_2 \leq \cdots \leq u_{n-1} \leq a_n$;

2. $\sum_{i=1}^{n-1} \beta_i = var(a)$, where $var(a)$ denotes the variance of the velocity field and is given by $var(a) = \sum_{i=1}^{n} m_i a_i^2 - \left(\sum_{i=1}^{n} m_i a_i\right)^2$.

**B The Stability of Coupled System**

We consider the case of $M = 1$ in [7] for simplicity. The general case with $M > 1$ can be derived similarly. The equations become

$$\partial_t v - \bar{a} \cdot \nabla v + \beta v = u,$$

$$\partial_t u + a \cdot \nabla u = \nabla \cdot \kappa \nabla v,$$

with $u|_{\partial \Omega} = 0$. One can also consider other boundary conditions rather than the homogeneous Dirichlet type. In this case, one has to assume that $\bar{a} \cdot \mathbf{n}|_{\partial \Omega} = 0$. For simplicity, we assume that the spatial dimension is $d = 2$. 

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In the following, we consider the case when the permeability tensor \( \kappa \) has the form
\[
\kappa = \kappa_{11} \vec{a} \otimes \vec{a} + \kappa_{22} \vec{a}^\perp \otimes \vec{a}^\perp + \kappa_{12} (\vec{a}^\perp \otimes \vec{a} + \vec{a} \otimes \vec{a}^\perp),
\]
where \( \vec{a} \) and \( \vec{a}^\perp \) are divergence free and \( \vec{a}(x) \neq 0 \) for almost all \( x \in \Omega \). Here, \( \kappa_{11} \), \( \kappa_{22} \), and \( \kappa_{12} \) are some heterogeneous scalar functions. We also write \( \kappa_{21} = \kappa_{12} \). By applying \( \nabla \cdot \kappa \nabla \) to the first equation, we have
\[
\partial_t \nabla \cdot (\kappa \nabla v) - \nabla \cdot (\kappa \nabla (\vec{a} \cdot \nabla v)) + \beta \nabla \cdot (\kappa \nabla v) = \nabla \cdot (\kappa \nabla u),
\]
\[
\partial_t u + \vec{a} \cdot \nabla u = \nabla \cdot \kappa \nabla v.
\]
Testing the first equation with \( w \in H^1(\Omega) \) and using integration by parts, we obtain
\[
\int_{\Omega} \left( \partial_t (\kappa \cdot \nabla v) - \kappa \nabla (\vec{a} \cdot \nabla v) + \beta (\kappa \nabla v) \right) \cdot \nabla w \, dx = \int_{\Omega} (\kappa \nabla u) \cdot \nabla w \, dx \quad \forall w \in H^1(\Omega).
\]
Next, we define the following scalar functions such that
\[
\tilde{v}_1 := \vec{a} \cdot \nabla v, \quad \tilde{v}_2 := \vec{a}^\perp \cdot \nabla v, \quad \tilde{w}_1 := \vec{a} \cdot \nabla w, \quad \text{and} \quad \tilde{w}_2 := \vec{a}^\perp \cdot \nabla w.
\]
Thus, we have
\[
(\kappa \nabla v) \cdot \nabla w = \sum_{1 \leq i,j \leq 2} \kappa_{ij} \tilde{v}_i \tilde{w}_j,
\]
\[
(\kappa \nabla (\vec{a} \cdot \nabla v)) \cdot \nabla w = \sum_{i=1}^2 \tilde{w}_i \left[ \kappa_{11} \vec{a} + \kappa_{22} \vec{a}^\perp \right] \cdot \nabla \tilde{v}_1.
\]
We define a tensor \( C = (C_{ijk})_{1 \leq i,j,k \leq 2} \) by
\[
C_{ijk} := \begin{cases} 
\tilde{k}_{11} \tilde{a}_j + \tilde{k}_{22} \tilde{a}_j^\perp & \text{if } k = 1, \\
0 & \text{if } k = 2.
\end{cases}
\]
Then, we have
\[
(\kappa \nabla (\vec{a} \cdot \nabla v)) \cdot \nabla w = \sum_{i=1}^2 \tilde{w}_i \left[ \kappa_{11} \vec{a} + \kappa_{22} \vec{a}^\perp \right] \cdot \nabla \tilde{v}_1 = \sum_{1 \leq i,j,k \leq 2} \tilde{w}_i C_{ijk} \partial_j \tilde{v}_k.
\]
As a result, we have
\[
\sum_{1 \leq i,j,k \leq 2} \int_{\Omega} \tilde{w}_i C_{ijk} \partial_j \tilde{v}_k \, dx = \sum_{1 \leq i,j,k \leq 2} \int_{\Omega} \tilde{w}_i \left( \frac{C_{ijk} + C_{kji}}{2} \right) \partial_j \tilde{v}_k \, dx + \int_{\Omega} \tilde{w}_i \left( \frac{C_{ijk} - C_{kji}}{2} \right) \partial_j \tilde{v}_k \, dx
\]
\[
= \sum_{1 \leq i,j,k \leq 2} \int_{\Omega} \tilde{w}_i \left( \frac{C_{ijk} + C_{kji}}{2} \right) \partial_j \tilde{v}_k \, dx
\]
since
\[
\sum_{1 \leq i,j,k \leq 2} C_{ijk} - C_{kji} = 0.
\]
Moreover, we have
\[
\int_{\Omega} \tilde{v}_i \left( \frac{C_{ijk} + C_{kji}}{2} \right) \partial_j \tilde{v}_k \, dx
\]
\[
= \frac{1}{2} \int_{\Omega} \tilde{v}_i \frac{C_{ijk} + C_{kji}}{2} \partial_j \tilde{v}_k \, dx - \frac{1}{2} \left[ \int_{\Omega} (\partial_j \tilde{v}_i) \frac{C_{ijk} + C_{kji}}{2} \tilde{v}_k \, dx + \int_{\Omega} \tilde{v}_i \partial_j \left( \frac{C_{ijk} + C_{kji}}{2} \right) \tilde{v}_k \, dx \right]
\]
\[
= - \frac{1}{2} \int_{\Omega} \tilde{v}_i \partial_j \left( \frac{C_{ijk} + C_{kji}}{2} \right) \tilde{v}_k \, dx.
\]
Therefore, the system (19) is stable if
\[
\beta \sum_{1 \leq i, j \leq 2} \kappa_{ij} + \frac{1}{4} \sum_{1 \leq i, j, k \leq 2} (C_{ijk} + C_{kji}) \geq 0.
\]
In particular, if \( \kappa = \kappa_{11} \tilde{a} \otimes \tilde{a} \), we have
\[
\frac{1}{2} \left( \|u(T)\|^2 - \|u(0)\|^2 \right) = \int_0^T \int_{\Omega} u (\partial_t u + a \cdot \nabla u) \, dx dt = \int_0^T \int_{\Omega} u (\nabla \cdot \kappa \nabla v) \, dx dt
\]
and
\[
\int_{\Omega} (v (\partial_t \nabla \cdot \kappa \nabla v) - (\tilde{a} \cdot \nabla v) \nabla \cdot \kappa \nabla v + \beta v \nabla \cdot \kappa \nabla v) \, dx = \int_{\Omega} v \nabla \cdot \kappa \nabla u \, dx.
\]
Thus, integrating over \((0, T]\), we have
\[
\frac{1}{2} \left( \|\kappa_{11}^{1/2} (\tilde{a} \cdot \nabla v)(T)\|^2 - \|\kappa_{11}^{1/2} (\tilde{a} \cdot \nabla v)(0)\|^2 \right) + \int_0^T \int_{\Omega} (\beta \kappa_{11} + \tilde{a} \cdot \nabla \kappa_{11}) (\tilde{a} \cdot \nabla v)^2 \, dx dt = - \int_0^T \int_{\Omega} u (\nabla \cdot \kappa \nabla v) \, dx dt.
\]
Therefore, we have
\[
\frac{1}{2} \left( \|u(T)\|^2 + \|\kappa_{11}^{1/2} (\tilde{a} \cdot \nabla v)(T)\|^2 \right) \leq \frac{1}{2} \left( \|u(0)\|^2 + \|\kappa_{11}^{1/2} (\tilde{a} \cdot \nabla v)(0)\|^2 \right)
\]
if \( \beta \kappa_{11} + (\tilde{a} \cdot \nabla \kappa_{11}) \geq 0 \). In this case, the stability of (19) depends only on \( \beta \), \( \kappa_{11} \), and \( \tilde{a} \).

C Construction of the Ansatz Space

In this section, we present the construction of the ansatz space \( V_H = V_1^H \oplus V_2^H \) that will be used for the spatial discretization. This ansatz space is based on the framework of the recently developed CEM-GMsFEM. For the ansatz space \( W_H \), one can define it as the direct sum of \( V_H \) and the span of the degrees of freedom corresponding to the in-flow boundary \( \Gamma \). In the following, we define \( V(S) := H_0^1(S) \) for a (nonempty) proper subset \( S \subset \Omega \). In the following, we denote \( V = H_0^1(\Omega) \).

C.1 The Implicit Ansatz Space

In this section, we present the construction of the implicit ansatz space \( V_1^H \). The construction of this space starts by solving a class of constrained energy minimization problems. Let \( T_H \) be a coarse grid partition of \( \Omega \). Denote \( N_c \) the total number of coarse elements. For \( K_i \in T_H \), we first have to build a collection of auxiliary bases in \( V(K_i) \). Let \( \{\chi_i\}_{i=1}^{N_c} \) be a set of partition of unity functions corresponding to an overlapping partition of the domain. In each coarse element \( K_i \in T_H \), we solve the following eigenvalue problem:
\[
\int_{K_i} \kappa \nabla \psi_j^{(i)} \cdot \nabla v = \lambda_j^{(i)} s_i(\psi_j^{(i)}, v) \quad \forall v \in V(K_i),
\]
where
\[
s_i(u, v) = \int_{K_i} \tilde{\kappa} uv, \quad \tilde{\kappa} := \kappa H^{-2} \text{ or } \tilde{\kappa} := \kappa \sum_i |\nabla \chi_i|^2.
\]
We rearrange and gather the $L_i$ eigenfunctions corresponding to the first $L_i$ smallest eigenvalues. Define the auxiliary space

$$V_{aux} := \bigoplus_{i=1}^{N_e} V_{aux}^{(i)}, \quad V_{aux}^{(i)} := \text{span}\{ \psi_j^{(i)} : 1 \leq j \leq L_i \}$$

and the projection operator $\Pi : L^2(\Omega) \rightarrow V_{aux}$ such that

$$s(\Pi u, v) = s(u, v) \quad \forall v \in V_{aux}, \quad \text{where } s(u, v) := \sum_{i=1}^{N_e} s_i(u|_{K_i}, v|_{K_i}).$$

For an oversampling parameter $m \in \mathbb{N}$, we define $K_{i,m}$ to be an oversampling domain of $K_i$ as follows

$$K_{i,0} := K_i, \quad K_{i,m} := \bigcup\{ K \in T_H : K \cap K_{i,m-1} \neq \emptyset \} \quad \text{for } m \geq 1.$$ 

We simply denote $K_{i}^+ = K_{i,m}$ for some given oversampling parameter $m$. For each auxiliary basis $\psi_j^{(i)}$, we search for a local basis function $\phi_j^{(i)} \in V(K_i^+)$ such that

$$a(\phi_j^{(i)}, v) + s(\mu_j^{(i)}, v) = 0 \quad \forall v \in V(K_i^+),$$

$$s(\phi_j^{(i)}, \nu) = s(\psi_j^{(i)}, \nu) \quad \forall \nu \in V_{aux}(K_i^+),$$

for some $\mu_j^{(i)} \in V_{aux}$, where $V_{aux}(K_i^+) := \bigoplus_{K_j \subset K_i^+} V_{aux}^{(j)}$. The implicit ansatz space $V_H^1$ is defined to be

$$V_H^1 := \text{span}\{ \phi_j^{(i)} : 1 \leq i \leq N_e, 1 \leq j \leq L_i \}.$$ 

Let $\tilde{V} := \{ v \in V : \Pi(v) = 0 \}$. Based on the construction of $V_H^1$, we have the property that $V = V_H^1 \perp_{H^1} \tilde{V}$.

### C.2 The Explicit Ansatz Space

In this section, we construct the explicit ansatz space $V_H^2 \subset \tilde{V}$. For each coarse element $K_i$, we consider the following class of eigenvalue problems: find $\xi_j^{(i)} \in V(K_i) \cap \tilde{V}$ and $\gamma_j^{(i)} \in \mathbb{R}$ such that

$$\int_{K_i} k \nabla \xi_j^{(i)} \cdot \nabla v \, dx = \gamma_j^{(i)} \int_{K_i} \xi_j^{(i)} v \, dx \quad \forall v \in V(K_i) \cap \tilde{V}.$$ 

We define $V_{aux,2} := \text{span}\{ \xi_j^{(i)} : 1 \leq i \leq N_e, 1 \leq j \leq J_i \}$. For each $\xi_j^{(i)} \in V_{aux,2}$, we define $\zeta_j^{(i)} \in V(K_i^+)$ such that for some $\mu_j^{(i),1} \in V_{aux}$, $\mu_j^{(i),2} \in V_{aux,2}$, we have

$$a(\zeta_j^{(i)}, v) + s(\mu_j^{(i),1}, v) + (\mu_j^{(i),2}, v) = 0 \quad \forall v \in V(K_i^+),$$

$$s(\zeta_j^{(i)}, \nu) = 0 \quad \forall \nu \in V_{aux},$$

$$(\zeta_j^{(i)}, \nu) = (\xi_j^{(i)}, \nu) \quad \forall \nu \in V_{aux,2}.$$ 

We define $V_H^2 := \text{span}\{ \zeta_j^{(i)} : 1 \leq i \leq N_e, 1 \leq j \leq J_i \}$. Based on the construction, we have $\zeta_j^{(i)} \in \tilde{V}$ and thus $V_H^2 \subset \tilde{V}$. 

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\section{Proof of Theorem 3.1}

In the following, we omit the subscript $H$ and simply write $u = u_H$, $v = v_H$, $u^n_i = u^n_{H,i}$, and $v^n_i = v^n_{H,i}$ to simplify the notations. Note that the first two equations in (10) can be written as

\[
\frac{1}{\Delta t} (v^{n+1}_1 - v^n_1, \phi_1) + \beta(v^n_1, \phi_1) = (u^{n+1}_1, \phi_1) \quad \forall \phi_1 \in V^1_H,
\]
\[
\frac{1}{\Delta t} (v^{n+1}_2 - v^n_2, \phi_2) + \beta(v^n_2, \phi_2) = (u^{n+1}_2, \phi_2) \quad \forall \phi_2 \in V^2_H.
\]

Taking $\psi_1 = u^{n+1}_1 \in V^1_H$ and $\psi_2 = u^{n+1}_2 \in V^2_H$ in (10), we obtain

\[
\left( \frac{u^{n+1}_1 - u^n_1}{\Delta t} + \frac{u^{n+1}_2 - u^n_2}{\Delta t} , u^{n+1}_1 \right) + A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_1) = 0,
\]
\[
\left( \frac{v^{n+1}_1 - v^n_1}{\Delta t} + \frac{v^{n+1}_2 - v^n_2}{\Delta t} , u^{n+1}_2 \right) + A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_2) = 0,
\]

since $v^{n+1} = v^{n+1}_1 + v^{n+1}_2$. Due to the construction of $V^1_H$ and $V^2_H$, we have $A(v^{n+1}_j, u^{n+1}_k) = 0$ for any $j \neq k$. Note that, taking $\phi_1 = v^{n+1}_1$, $\phi_2 = v^{n+1}_2$, and making use of the operator $R$, we obtain

\[
A(u^{n+1}_1, v^{n+1}_1) = \frac{1}{\Delta t} A(v^{n+1}_1 - v^n_1, v^{n+1}_1) + \beta A(v^{n+1}_1, v^n_1),
\]
\[
A(u^{n+1}_2, v^{n+1}_2) = \frac{1}{\Delta t} A(v^{n+1}_2 - v^n_2, v^{n+1}_2) + \beta A(v^n_2, v^{n+1}_2).
\]

Then, we have

\[
A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_1) + A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_2) = A(v^{n+1}_1, u^{n+1}_1) + A(v^{n+1}_2, u^{n+1}_2) + A(v^{n+1}_1, u^{n+1}_2 - u^n_2)
\]
\[
= \sum_{i=1}^2 \left[ \left( \frac{1}{\Delta t} - \beta \right) A(v^{n+1}_i, v^{n+1}_i - v^n_1) + \beta A(v^{n+1}_i, v^{n+1}_i) \right] + A(v^{n+1}_2, u^{n+1}_2 - u^n_2).
\]

On the other hand, we have

\[
A(v^{n+1}_i, v^{n+1}_i - v^n_1) = \frac{1}{2} \left( \|v^{n+1}_i\|_A^2 - \|v^n_i\|_A^2 + \|v^{n+1}_i - v^n_i\|_A^2 \right) \quad \text{for any } i \in \{1, 2\}
\]

and

\[
|A(v^{n+1}_2, u^{n+1}_2 - u^n_2)| \leq \|v^{n+1}_2\|_A \cdot \|u^{n+1}_2 - u^n_2\|_A \leq \frac{\beta}{2} \|v^{n+1}_2\|_A^2 + \frac{1}{2\beta} \|u^{n+1}_2 - u^n_2\|_A^2.
\]

Therefore, we have

\[
A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_1) + A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_2) \geq \frac{1}{2\Delta t} \sum_{i=1}^2 \|v^{n+1}_i\|_A^2 - \frac{1}{2} \left( \frac{1}{\Delta t} - \beta \right) \sum_{i=1}^2 \left[ \|v^n_i\|_A^2 - \|v^{n+1}_i - v^n_i\|_A^2 \right] - \frac{1}{2\beta} \|u^{n+1}_2 - u^n_2\|_A^2.
\]

(20)
Moreover, we have
\[
\left( \frac{u^{n+1}_1 - u^n_1}{\Delta t} + \frac{u^{n+1}_2 - u^n_2}{\Delta t}, u^{n+1}_1 \right) + \left( \frac{u^{n+1}_1 - u^n_1}{\Delta t} + \frac{u^{n+1}_2 - u^n_2}{\Delta t}, u^{n+1}_2 \right)
\]
\[
= \frac{1}{\Delta t} (u^{n+1} - u^n, u^{n+1}) = \frac{1}{2\Delta t} (\|u^{n+1}\|^2 - \|u^n\|^2 + \|u^{n+1} - u^n\|^2)
\]
\[
\geq \frac{1}{2\Delta t} \left( \|u^{n+1}\|^2 - \|u^n\|^2 + \sum_{i=1}^2 \|u^{n+1}_i - u^n_i\|^2 - 2\gamma \|u^{n+1}_1 - u^n_1\| \|u^{n+1}_2 - u^n_2\| \right)
\]
\[
\geq \frac{1}{2\Delta t} \left( \|u^{n+1}\|^2 - \|u^n\|^2 + (1 - \gamma) \sum_{i=1}^2 \|u^{n+1}_i - u^n_i\|^2 \right). \quad (21)
\]

Adding (20) and (21), we obtain
\[
0 = \left( \frac{u^{n+1}_1 - u^n_1}{\Delta t} + \frac{u^{n+1}_2 - u^n_2}{\Delta t}, u^{n+1}_1 \right) + \left( \frac{u^{n+1}_1 - u^n_1}{\Delta t} + \frac{u^{n+1}_2 - u^n_2}{\Delta t}, u^{n+1}_2 \right)
\]
\[
+ A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_1) + A(v^{n+1}_1 + v^{n+1}_2, u^{n+1}_2)
\]
\[
\geq \frac{1}{2\Delta t} \left( \|u^{n+1}\|^2 - \|u^n\|^2 + \sum_{i=1}^2 (1 - \gamma) \|u^{n+1}_i - u^n_i\|^2 \right) + \frac{1}{2\Delta t} \sum_{i=1}^2 \|v^{n+1}_i\|^2
\]
\[
- \frac{1}{2} \left( \frac{1}{\Delta t} - \beta \right) \sum_{i=1}^2 \left[ \|v^n_i\|^2 - \|v^{n+1}_i - v^n_i\|^2 \right] - \frac{1}{2\beta} \|u^{n+1}_2 - u^n_2\|^2.
\]

If the stability condition (12) holds, then we obtain
\[
\tilde{E}^{n+1}(u, v) = \|u^{n+1}\|^2 + \sum_{i=1}^2 \|v^{n+1}_i\|^2 \|A\|
\]
\[
\leq \|u^n\|^2 + (1 - \beta \Delta t) \sum_{i=1}^2 \left[ \|v^n_i\|^2 - \|v^{n+1}_i - v^n_i\|^2 \right] - (1 - \gamma) \|u^{n+1}_1 - u^n_1\|^2
\]
\[
- \left[ (1 - \gamma) \|u^{n+1}_2 - u^n_2\|^2 - \frac{\Delta t}{\beta} \|u^{n+1}_2 - u^n_2\|^2 \right] \geq 0 \leq \|u^n\|^2 + \sum_{i=1}^2 \|v^n_i\|^2 = \tilde{E}^n(u, v)
\]

for any \( n \in \{0, 1, \cdots, N_T - 1\} \). This completes the proof.