LINEAR SOLAR MODELS

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ABSTRACT

We present a new approach to studying the properties of the Sun. We consider small variations of the physical and chemical properties of the Sun with respect to standard solar model predictions and we linearize the structure equations to relate them to the properties of the solar plasma. By assuming that the (variation of) present solar composition can be estimated from the (variation of) nuclear reaction rates and elemental diffusion efficiency in the present Sun, we obtain a linear system of ordinary differential equations which can be used to calculate the response of the Sun to an arbitrary modification of the input parameters (opacity, cross sections, etc.). This new approach is intended to be a complement to the traditional methods for solar model (SM) calculation and allows us to investigate in a more efficient and transparent way the role of parameters and assumptions in SM construction. We verify that these linear solar models recover the predictions of the traditional SMs with a high level of accuracy.

Key words: Sun: fundamental parameters – Sun: general – Sun: interior

Online-only material: color figures

1. INTRODUCTION

In the last three decades, there has been enormous progress in our understanding of the Sun. The predictions of the standard solar model (SSM), which is the fundamental theoretical tool to investigate the solar interior, have been tested by solar neutrino experiments and by helioseismology.

The deficit of the observed solar neutrino fluxes, reported initially by Homestake (Davis et al. 1968) and then confirmed by GALLEX (GALLEX Coll. 1999) and SAGE (SAGE Coll. 1999) which subsequently merged into GNO (GNO Coll. 2005), Kamiokande (Hirata et al. 1989), and Super-Kamiokande (Super-Kamiokande Coll. 2008), generated the so-called solar neutrino problem which stimulated a deep investigation of the solar structure, see, e.g., Bahcall (1989). The problem was solved in 2002 when the SNO experiment (SNO Coll. 2002) obtained a direct evidence for flavor oscillations of solar neutrinos and, moreover, confirmed the SSM prediction of the $^8$B neutrino flux with an accuracy which, according to the latest data (SNO Coll. 2008), is equal to about $\sim$6%, see, e.g., Serenelli (2010).$^3$

At the same time, helioseismic observations have allowed us to determine precisely several important properties of the Sun, such as the depth of the convective envelope which is known at the $\sim$0.2% level, the surface helium abundance which is obtained at the $\sim$1.5% level, and the sound speed profile which is determined with an accuracy equal to $\sim$0.1% in a large part of the Sun; see, e.g., Degl’Innocenti et al. (1997), Gough et al. (1996), Basu & Antia (2008), and references therein. As a result of these observations, the solar structure is now very well constrained, so that the Sun can be used as a solid benchmark for stellar evolution and as a “laboratory” for fundamental physics, see e.g., Ricci & Villante (2002), Fiorentini et al. (2001), and Bottino et al. (2002).

The future could be even more interesting. The KamLAND reactor (anti)neutrino experiment (KamLAND Coll. 2008) has confirmed the flavor oscillations hypothesis and has refined the determination of neutrino parameters. We now reliably know the solar neutrino oscillation probability and we can go back to the original program of solar neutrino studies, i.e., to probe nuclear reactions in the solar core. Present and future solar neutrino experiments, such as Borexino (Borexino Coll. 2008) and SNO+ (SNO Coll. 2006), have the potential to provide the first direct measurements of the CNO and pep neutrinos, thus probing dominant and sub-dominant energy generation mechanisms in the Sun.

At the same time, a new solar problem has emerged. Recent determinations of the photospheric heavy element abundances (Asplund et al. 2005, 2009) indicate that the Sun metallicity is lower than previously assumed (Grevesse & Sauval 1998). Solar models (SMs) that incorporate these lower abundances are no longer able to reproduce the helioseismic results. As an example, the sound speed predicted by SMs at the bottom of the convective envelope disagrees at the $\sim$1% level with the value inferred by helioseismic data, see e.g., Bahcall et al. (2005). Detailed studies have been done to resolve this controversy but a definitive solution of the ”solar composition problem” still has to be obtained; see, e.g., Basu & Antia (2008) and Asplund et al. (2009).

In this framework, it is important to analyze the role of physical inputs and the standard assumptions for SM calculations. This task is not always possible in simple and clear terms, since SSM construction relies on (time-consuming) numerical integration of a nonlinear system of partial differential equations. Several input parameters are necessary to fully describe the property of the solar plasma and some of them are not single numbers but complicated functions (e.g., the opacity of the solar interior) which, in principle, can be modified in a nontrivial way. Moreover, any modification of the Sun produces a variety

$^3$ For the sake of precision, the first model-independent evidence for solar neutrino oscillations and the first determination of the $^8$B solar neutrino flux was obtained in 2001 (see e.g., Fogli et al. 2001) by comparing the SNO charged-current result (SNO Coll. 2001) with the SK data with the method proposed by Villante et al. (1999). The year 2002 is, however, recognized as the “annus mirabilis” (Fogli et al. 2003) for solar neutrino physics. During 2002, in fact, the SNO neutral-current measurement (SNO Coll. 2002) and the first KamLAND results (KamLAND Coll. 2003) were released and, moreover, Raymond Davis, Jr. and Masatoshi Koshiba were awarded with the Nobel Prize.
of correlated effects which have to be taken into account all
together if we want to correctly extract information from the
comparison of theoretical predictions with observational data.
In order to overcome these difficulties, we provide a tool
which is at the same time simple and accurate enough to
describe the effects of a generic (small) modification of the
physical inputs. The starting point is the fact that, despite the
present disagreement with helioseismic data, the SSM is a rather
good approximation of the real Sun. We can thus assume small
variations of the physical and chemical properties of the Sun
with respect to the SSM predictions and use a linear theory
to relate them to the properties of the solar plasma. With the
additional assumption that the (variation of) present solar
composition can be estimated from the (variation of) nuclear
reaction rate and elemental diffusion efficiency in the present
Sun, we obtain a linear system of ordinary differential equations,
which can be easily integrated.

We believe that the proposed approach can be useful in
several respects. First, the construction of these linear solar
models (LSM) can complement the traditional methods for
SM calculations, allowing us to investigate in a more efficient
and transparent way the role of parameters and assumptions.
In a separate paper, we will use LSMs to discuss in general
terms the role of opacity and metals in the solar interior (F. L.
Villante et al. 2010, in preparation). Moreover, it can help to
introduce new effects and to understand their relevance, prior
to implementation into the more complicated SSM machinery.
Finally, this simplified approach could open the field of solar
physics beyond the small community of the SSM’s builders with
the result of making the Sun a more accessible “laboratory.”
Clearly, LSMs are not intended to be an alternative to SSMs,
which remain the fundamental theoretical tool to compare with
observations.

The plan of the paper is as follows. In the next section,
we briefly review SSM calculations. In Section 3, we expand
to linear order the structure equations of the present Sun. In
Section 4, we calculate the properties of the Sun and of the solar
plasma that are necessary to define LSMs. In Section 5, we
give the integration conditions for the linearized equations. In
Section 6, we discuss how to estimate the chemical composition
of the Sun. In Section 7, we give the equations that define LSMs
in their final form. Finally, in Section 8, we compare the results
of LSMs with those obtained by using the standard method
for SM calculations, showing the validity of our simplified
approach.

2. THE STANDARD SOLAR MODEL

In the assumption of spherical symmetry, the behavior of the
pressure (P), density (ρ), temperature (T), luminosity (l), and
mass (m) in the Sun is described by the structure equations
(Kippenhan & Weigert 1991):

\[
\frac{\partial m}{\partial r} = 4\pi r^2 \rho
\]

\[
\frac{\partial P}{\partial r} = -\frac{G_N m}{r^2} \rho
\]

\[
\frac{\partial l}{\partial r} = 4\pi r^2 \rho \epsilon(\rho, T, X_i)
\]

\[
\frac{\partial T}{\partial r} = -\frac{G_N m T \rho}{r^2 P} \frac{\partial}{\partial r} T
\]

\[
P = P(\rho, T, X_i),
\]

where \( r \) indicates the distance from the center, \( \epsilon(\rho, T, X_i) \) is
the energy produced per unit time and mass,\(^4\) the function
\( P = P(\rho, T, X_i) \) describes the equation of state (EOS) of
solar matter, and \( X_i(r) \) are the mass abundances of the various
chemical elements inside the Sun. In the equation which
describes the energy transport, the temperature gradient \( \nabla \)
\( \nabla \) is defined as \( \nabla \equiv \frac{\partial}{\partial r} \ln r \). If the energy transfer is due to
radiative processes (i.e., at the center of the Sun) one has

\[
\nabla = \nabla_{\text{rad}}
\]

where \( \nabla_{\text{rad}} \) is given by

\[
\nabla_{\text{rad}} = \frac{3}{16\pi ac G_N} \frac{\kappa(\rho, T, X_i)}{m T^4}
\]

\( \kappa(\rho, T, X_i) \) is the opacity of solar matter. In the presence
of convective motions (i.e., in the outer layer of the Sun), the
value of \( \nabla \) has to be calculated by taking into account
the convective energy transport which generally provides the
dominant contribution. In a large part of the Sun convective
envelope, one has

\[
\nabla \simeq \nabla_{\text{ad}} \simeq 0.4.
\]

This is expected when convection is very efficient and a
negligible excess of \( \nabla \) over the adiabatic value \( \nabla_{\text{ad}} \)
is sufficient to transport the whole luminosity. In the outermost layers of
the Sun, the situation is more complicated. The precise description
of convection in this regime is still an unsolved problem. One
uses a phenomenological model which predicts the efficiency
of convection as a function of the “mixing length parameter” \( \alpha \), which is related to the distances over which a moving
unit of gas can be identified before it mixes appreciably; see,
for example, Kippenhan & Weigert (1991). The transition between
the internal radiative core and the external convective envelope
occurs at the radius \( R_b \), where

\[
\nabla_{\text{rad}}(R_b) = \nabla_{\text{ad}},
\]

as is prescribed by the Schwarzschild criterion which states that
convective motions occur in the region of the star where
\( \nabla_{\text{rad}}(r) > \nabla_{\text{ad}} \).

The chemical composition of the present Sun is not known but
has to be calculated by coupling Equations (1), which describe
the mechanical and thermal structure, with the equations that
describe the chemical evolution of the Sun; see, e.g., Kippenhan
& Weigert (1991) for details. It is generally assumed that the
Sun was born chemically homogeneous and has modified its
classical composition due to nuclear reactions and elemental
diffusion. One assumes that relative heavy element abundances
in the photosphere have not been modified during the evolution
and uses the observationally determined photospheric composition
to fix the initial heavy element admixture. We remark that the
chemical abundances are uniform in the convective region
due to the very efficient mixing induced by convective motions.
The values \( X_{i, b} \), evaluated at the bottom of the convective region
are thus representative for the solar surface composition.

If complete information about the initial composition was available and if a complete theory of convection was known,

\footnote{We neglect, here and in the following, the contribution to energy balance
due to the heat released by the various shells of the Sun in their
thermodynamical transformations. This contribution, which depends on the
time derivative of pressure and temperature, is sub-dominant as can be
understood from the fact that the Sun evolves on time scales much larger than
the Kelvin–Helmholtz time.}
then there would be no free parameter. In practice, one has three free parameters, namely the initial metal abundance $Z_{ini}$, the initial helium abundance $Y_{ini}$, and the mixing length parameter $\alpha$. These are tuned in order to reproduce, at the solar age $t_0 = 4.57$ Gyr (Bahcall & Pinsonneault 1995), the observed solar luminosity $L_\odot = 3.8418 \times 10^{33}$ erg s$^{-1}$ (Bahcall et al. 2004), the observed solar radius $R_\odot = 6.9598 \times 10^{10}$ cm (Allen 1976), and the surface metal-to-hydrogen ratio $(\bar{Z}/X)_\odot$. In this paper we use the AS05 composition (Asplund et al. 2005) which corresponds to $(\bar{Z}/X)_\odot = 0.0165$.

An SSM, according to the definition of Castellani et al. (1997), is a solution of the above problem which reproduces, within uncertainties, the observed properties of the Sun, by adopting physical and chemical inputs chosen within their range of uncertainties. In this paper, we refer to the SSM obtained by using the FRANEC code (Chieffi & Straniero 1989; Ciacco et al. 1997), including Livermore 2006 EOS (Rogers et al. 1997); Livermore radiative opacity tables (OPAL; Iglesias & Rogers 1996) calculated for the AS05 chemical composition; molecular opacities from Ferguson et al. (2005) and conductive opacities from Cassisi et al. (2006), both calculated for AS05 composition; and nuclear reaction rates from the NACRE revision of the astrophysical factors (Martí et al. 2008; Costantini et al. 2008; Junghans et al. 2003) and the $^7$Be electron capture rate from Adelberger et al. (1998).

3. LINEAR EXPANSION OF THE STRUCTURE EQUATIONS

A variation of the input parameters (EOS, opacities, cross sections, etc.) with respect to the standard assumptions produces an SM which deviates from SSM predictions. The physical and chemical properties of the “perturbed” Sun can be described according to

$$h(r) = h(r)_0[1 + \delta h(r)]$$

$$X_i(r) = X_i(r)_0[1 + \delta X_i(r)]$$

$$Y(r) = Y(r)_0 + \Delta Y(r),$$

(6)

where $h = l$, $m$, $T$, $P$, $\rho$ and we use, here and in the following, the notation $\mathcal{Q}$ to indicate the prediction for the generic quantity $\mathcal{Q}$. For convenience, the deviations from SSM are expressed in terms of relative variations $\delta h(r)$ and $\delta X_i(r)$ for all quantities except the helium abundance, for which it is more natural to use the absolute variation $\Delta Y(r)$. In this paper, we will be mainly concerned with modifications of the radiative opacity, $\kappa(\rho, T, X_i)$, and of the energy generation coefficient, $\epsilon(\rho, T, X_i)$. We indicate with $\delta \xi$ the relative variations of these quantities along the SSM profile, i.e.:

$$I(T(r), \rho(r), Y(r), X_i(r)) = I_{SSM}(T(r), \rho(r), Y(r), X_i(r)) \times [1 + \delta I(r)],$$

(7)

where $I = \kappa$, $\epsilon$. When we consider the effect of a perturbation $\delta I(r)$ on the Sun, we have to take into account that the perturbed SM, due to that modification, has a different density, temperature, and chemical composition with respect to the SSM. The total difference $\delta I^\text{tot}(r)$ between the perturbed Sun and the SSM at a given point $r$ is defined by

$$I(T(r), \rho(r), Y(r), X_i(r)) = I_{SSM}(T(r), \rho(r), Y(r), X_i(r)) \times [1 + \delta I^\text{tot}(r)].$$

(8)

If we consider small perturbations ($\delta I \ll I$), we can expand to first order in $\delta T(r)$, $\delta \rho(r)$, $\delta X_i(r)$, and $\Delta Y(r)$, obtaining

$$\delta I^\text{tot}(r) = I_T(r) \delta T(r) + I_\rho(r) \delta \rho(r) + I_Y(r) \Delta Y(r) + \sum_i I_{X_i}(r) \delta X_i(r) + \delta I(r).$$

(9)

Here, the quantities $I$ describe the dependence of the properties of the stellar plasma on temperature, density, and chemical composition and are given by

$$I_T(r) = \frac{\partial \ln I}{\partial \ln T},$$

$$I_\rho(r) = \frac{\partial \ln I}{\partial \ln \rho},$$

$$I_Y(r) = \frac{\partial \ln I}{\partial \ln Y}$$

$$I_{X_i}(r) = \frac{\partial \ln I}{\partial X_i},$$

(10)

where $I = \kappa$, $\epsilon$, $P$. The symbol $|_{SSM}$ indicates that we calculate the derivatives $I$ along the density, temperature, and chemical composition profiles predicted by the SSM.

By using the above notations, we linearize the structure Equations (1), obtaining

$$\frac{\partial \delta m}{\partial r} = \frac{1}{l_m}[\delta \rho - \delta m],$$

$$\frac{\partial \delta P}{\partial r} = \frac{1}{l_P}[\delta m + \delta \rho - \delta P],$$

$$\delta P = \left[ P_\rho \delta \rho + P_T \delta T + P_Y \Delta Y + \sum_i P_{X_i} \delta X_i \right],$$

$$\frac{\partial \delta T}{\partial r} = \frac{1}{l_T} \left[ (1 + \epsilon_\rho) \delta \rho + \epsilon_T \delta T + \epsilon_Y \Delta Y + \sum_i \epsilon_{X_i} \delta X_i - \delta \epsilon T + \delta \epsilon \right],$$

$$\frac{\partial \delta \epsilon}{\partial r} = \frac{1}{l_\epsilon} \left[ \delta \epsilon + (\kappa_T - \delta \kappa_T) \delta T + (\kappa_{\rho} + 1) \delta \rho + \kappa_Y \Delta Y + \sum_i \kappa_{X_i} \delta X_i \right],$$

$$\frac{\partial \delta T}{\partial r} = \frac{1}{l_T} \left[ \delta m + \delta \rho - \delta P \right],$$

(11)

where $l_h = (\ln(h)/dr)^{-1}$ represents the scale height of the physical parameter $h$ in the SSM. The last two equations correspond to the linear expansion of the energy transport equation in the radiative and in the convective regime, respectively. In our approach, we use the radiative transport equation for $r \leq \bar{R}_b$ and the convective transport equation for $r > \bar{R}_b$, where

5 It is generally observed that the Sun responds linearly to relatively large modifications of the input parameters. This can be seen, e.g., from Tables 10 and 11 of Castellani et al. (1997) where quite substantial changes of the astrophysical factors of $p + p \rightarrow d + e^+ + \nu_e$ are considered and the effects of relatively large global rescaling of the opacity are discussed. The linearity of the response of the Sun to generic modifications of the opacity profile is discussed in Tripathy & Christensen-Dalsgaard (1998).

6 In the presence of convection, we assume that $\nabla \equiv \nabla_d$ and that the adiabatic gradient is not affected by the performed modifications of the input parameters.
4. PROPERTIES OF THE SUN AND OF THE SOLAR MATTER

In order to define the linearized structure equations, we have to calculate the functions \( l_b(r) \) and the logarithmic derivatives \( I_j(r) \). In Figure 1, we show the inverse scale heights \( 1/l_b(r) \) as a function of the solar radius. The plotted results have been obtained numerically and refer to our SSM. We see that the functions \( 1/l_p(r) \) and \( 1/l_f(r) \) vanish at the center of the Sun while they grow considerably (in modulus) in the outer regions, as a result of the fast decrease of pressure and temperature close to surface. The kink in the temperature scale height at \( R_b = 0.730 R_\odot \) marks the transition from the internal radiative region to the outer convective envelope. The functions \( 1/l_m(r) \) and \( 1/l_f(r) \) have the opposite behavior; they diverge at the solar center and vanish at large radii, as is expected by considering that most of the mass and the energy generation in the Sun is concentrated close to the solar center. To be more quantitative, the solar luminosity is produced in the inner radiative core \( (r \leq 0.3 R_\odot) \), which contains approximately 60% of the total solar mass. The radiative region \( (r \leq R_b) \) which covers about 40% of the total volume of the Sun, includes approximately 98% of the total solar mass.

In Figure 2, we show the logarithmic derivatives \( I_j(r) \) calculated numerically along the temperature, density, and chemical composition profiles predicted by our SSM. The left panel refers to the opacity derivatives \( \kappa_j(r) \) and shows that opacity is a decreasing function of temperature and helium abundance, while it is an increasing function of density. The coefficient \( \kappa_Z(r) \equiv \partial \ln \kappa / \partial \ln Z|_{SSM} \) quantifies the dependence of opacity on the total metallicity \( Z \). It has been calculated by rescaling all the heavy element abundances by a constant factor, so that the metal admixture remains fixed. Metals provide about ~40% of the opacity at the center of the Sun, while they account for about 80% of the total opacity at the bottom of the convective region; see also Basu & Antia (2008).

In the right panel of Figure 2, we present the logarithmic derivatives \( \epsilon_j(r) \) of the energy generation coefficient. These have been calculated by assuming that the abundances of

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**Figure 1.** Inverse scale heights \( 1/l_b(r) \) of temperature (black), pressure (red), mass (green), and luminosity (blue) as a function of the solar radius. (A color version of this figure is available in the online journal.)

**Figure 2.** Left panel: the logarithmic derivative \( \kappa_j(r) \) of the radiative opacity with respect to temperature (black), density (red), helium (green), and metals (blue) as a function of the solar radius. Right panel: the logarithmic derivative \( \epsilon_j(r) \) of the energy generation coefficient with respect to temperature (black), density (red), helium (green), and metals (blue: multiplied by a factor of 100) as a function of the solar radius. (A color version of this figure is available in the online journal.)
the secondary elements for PP-chain and CN-cycle, namely \(^3\)He, \(^{12}\)C, and \(^{14}\)N, can be estimated as it is described in Appendix A. The bumps at \(r \simeq 0.28 R_\odot\) are related to out-of-equilibrium behavior of the \(^3\)He abundance. The presence of these features has, however, a negligible influence on the solutions of Equations (11). The function \(1/l_1(r)\), which multiplies all the \(\epsilon_j(r)\) coefficients, drops, in fact, rapidly to zero at \(r \simeq 0.25 R_\odot\).

In the more internal region (where \(^3\)He assumes the equilibrium value), the displayed results can be understood by considering that energy is produced by the PP-chain (\(~99\%) with a small contribution (\(~1\%) by the CN-cycle. The slight increase of \(\epsilon_\phi(r)\) at \(r \lesssim 0.15 R_\odot\) reflects the fact that the CN-cycle, which strongly depends on temperature, gives a non-negligible contribution only at the center of the Sun. The coefficient \(\epsilon_\rho(r)\) is approximately equal to 1, as it is expected by considering that the rate of two-body reactions in the unit mass is proportional to the hydrogen abundance squared, i.e., to \(X^2 = (1 - Y - Z)^2\).

We have also evaluated numerically the logarithmic derivative of the pressure \(P_j(r)\) by using the Livermore 2006 EOS (Rogers et al. 1996). The obtained results can be approximated with good accuracy (about 1\%) by using the perfect gas EOS: \(P = (\rho k_B T)/(\mu m_\alpha)\), where \(m_\alpha\) is the atomic mass unit, \(k_B\) is the Boltzmann constant and \(\mu = (2 - 5/4 Y - 3/2 Z)^{-1}\) is the mean molecular weight. One obtains

\[
\begin{align*}
P_p(r) &= 1 \\
P_T(r) &= 1 \\
P_Y(r) &= -\frac{\partial \ln \mu}{\partial Y} = -\frac{5}{8 - 5Y - 6Z(r)}.
\end{align*}
\]

The derivative with respect to the metal abundance \(P_Z(r)\) is of the order of few percent and will be neglected in the following. By using the above relations, we obtain the following equation:

\[
d\delta \rho = \delta P(r) - \delta T(r) - P_T \Delta Y(r)
\]

that can be used to eliminate the quantity \(\delta \rho\) from the linear structure Equations (11). We arrive at

\[
\begin{align*}
d\delta m &= \frac{1}{l_m} \left[ \delta P - \delta T - \delta m - P_T \Delta Y \right] \\
d\delta P &= \frac{1}{l_P} \left[ -\delta T + \delta m - P_T \Delta Y \right] \\
d\delta \beta &= \frac{1}{l_\beta} \left[ \beta_P \delta P + \beta_T \delta T - \delta \beta + \beta_Y \Delta Y + \beta_Z \delta Z + \delta \epsilon \right] \\
d\delta T &= \frac{1}{l_T} \left[ \alpha_P \delta P + \alpha_T \delta T + \delta \alpha + \alpha_Y \Delta Y + \alpha_Z \delta Z + \delta \kappa \right] \text{ Rad.} \\
d\delta \delta T &= \frac{1}{l_T} \left[ -\delta T + \delta m - P_T \Delta Y \right] \text{ Conv.}
\end{align*}
\]

where the factors \(\alpha_i\) and \(\beta_i\) are given by

\[
\begin{align*}
\alpha_p &= \kappa_p + 1 \\
\alpha_T &= \kappa_T - \kappa_P - 5 \\
\alpha_Y &= - (\kappa_P + 1) P_T + \kappa_Y \\
\beta_P &= \epsilon_P + 1 \\
\beta_T &= \epsilon_T - \epsilon_P - 1 \\
\beta_Y &= - (\epsilon_P + 1) P_T + \epsilon_Y \\
\beta_Z &= \epsilon_Z.
\end{align*}
\]

In the above equations, we indicate with \(\delta Z\) the relative variation of the total metallicity, defined by

\[
Z(r) = \tilde{Z}(r) \left[ 1 + \delta Z(r) \right]
\]

and we implicitly assume that the heavy element admixture is unchanged.\(^8\)

5. BOUNDARY CONDITIONS

In order to solve the linear structure equations, one has to specify the integration conditions. This can be done quite easily at the center of the Sun. From the definitions of mass and luminosity, we know that \(m(0) = 0\) and \(l(0) = 0\). This implies that

\[
1 + \delta h(0) = \lim_{r \to 0} \frac{h(r)}{h(0)} = \frac{\delta h(0)}{\delta r} = \frac{\delta h(0)}{\delta r}
\]

for \(h = m, l\), so that one obtains

\[
\begin{align*}
\delta m(0) &= \delta P_0 - \delta T_0 - P_{Y,0} \Delta Y_0 \\
\delta P(0) &= \delta P_0 \\
\delta T(0) &= \delta T_0 \\
\delta l(0) &= \beta_{P,0} \delta P_0 + \beta_{T,0} \delta T_0 + \beta_{Y,0} \Delta Y_0 + \beta_{Z,0} \delta Z_0 + \delta \epsilon_0.
\end{align*}
\]

where the subscript “0” indicates that the various quantities are evaluated at the center of the Sun.

To implement surface conditions, we exploit the fact that the Sun has a convective envelope where the solution of the linearized structure equations can be explicitly calculated, as it is explained in the following.

First, we take advantage of the fact that there are no energy producing processes and that a negligible fraction of the solar mass is contained in the convective region. This implies that \(1/l_1(r) = 0\) and \(1/l_m(r) \simeq 0\), so that we can integrate the energy generation and the continuity equation. We obtain \(\delta l(r) \equiv 0\) and \(\delta m(r) \simeq 0\) where we clearly considered that only solutions that reproduce the observed solar luminosity and solar mass are acceptable.

Then, we consider that the chemical abundances are constant due to the very efficient convective mixing, so that \(\Delta Y(r) \equiv \Delta Y_b\) and \(P_T(r) \Delta Y_b \equiv -\delta \mu_b\), where \(\delta \mu_b\) is the relative variation of the mean molecular weight at the bottom of the convective region.

By taking this into account, we can rewrite the transport equation in the form

\[
\frac{\partial \delta u}{\partial r} = \frac{\delta u}{l_T}
\]

where \(\delta u\) is the fractional variation of squared isothermal sound speed \(u = P/\rho = k_B T/\mu \mu_m\). The general solution of this equation is \(\delta u(r) = \delta u_b \left[ T_b / T(r) \right]\) which shows that, in order to avoid \(\delta u(r)\) to “explode” at the surface of the Sun, it necessarily

\(^8\) The effects of changes in the heavy element admixture can be described in our approach by proper variations \(\delta h(r)\) and \(\delta \epsilon(r)\), see F. L. Villante et al. (2010, in preparation).
holds \( \delta u(r) = \delta T(r) - \delta \mu_b \simeq 0 \) in the internal layers of the convective envelope.

By using this result in the hydrostatic equilibrium equation, we obtain \( \delta P(r) = \delta \rho(r) \simeq \delta C \) where \( \delta C \) is an arbitrary constant. Finally, by considering that \( \delta \rho(r) \) is approximately constant in the internal layers of the convective region (where most of the mass of the convective envelope is contained), we use the continuity equation to improve the condition \( \delta m(r) \simeq 0 \), obtaining

\[ \delta m_{\text{conv}} = -\overline{m}_{\text{conv}} \delta C , \]

where \( \overline{m}_{\text{conv}} = \overline{M}_{\text{conv}} / M_\odot = 0.0192 \) is the fraction of solar mass contained in the convective region, as resulting from our SSM.

In conclusion, we have

\[
\begin{align*}
\delta m(\overline{R}_b) &= -\overline{m}_{\text{conv}} \delta C \\
\delta P(\overline{R}_b) &= \delta C \\
\delta T(\overline{R}_b) &= \delta \mu_b = -P_{\text{y,b}} \Delta Y_b \\
\delta l(\overline{R}_b) &= 0 .
\end{align*}
\]

The factor \( \delta C \) is a free parameter that cannot be fixed from first principles, since we cannot model exactly convection in the outermost super-adiabatic region. It has basically the same role as mixing length parameter in SSM construction.

6. THE CHEMICAL COMPOSITION OF THE SUN

The chemical composition of the perturbed Sun should be calculated by integrating the perturbed structure and chemical-evolution equations starting from an ad hoc chemical homogeneous ZAMS model. However, this would lead to several complications in our simplified approach. We prefer to use a simple approximate procedure that allows us to estimate with sufficient accuracy the helium and metal abundances of the modified Sun, without requiring us to follow explicitly its time evolution.

6.1. Notations

In order to quantify the relevance of the different mechanisms determining the present composition of the Sun, we express the helium and metal abundance according to

\[
\begin{align*}
Y(r) &= Y_{\text{ini}} [1 + D_Y(r)] + Y_{\text{nuc}}(r) \\
Z(r) &= Z_{\text{ini}} [1 + D_Z(r)].
\end{align*}
\]

Here, \( Y_{\text{ini}} \) and \( Z_{\text{ini}} \) are the initial values which, as explained previously, are free parameters in SM construction and are adjusted in order to reproduce the observed solar properties. In our SSM, we have \( Y_{\text{ini}} = 0.2611 \) and \( Z_{\text{ini}} = 0.0140 \). The terms \( D_Y(r) \) and \( D_Z(r) \) describe the effects of elemental diffusion. Finally, \( Y_{\text{nuc}}(r) \) represents the total amount of helium produced in the shell \( r \) by nuclear processes and can be calculated by integrating the rates of the helium-producing reactions during the Sun history.

In Figure 3, we show the quantities \( Y_{\text{nuc}}(r) \), \( \overline{Y}_{\text{nuc}}(r) \), and \( \overline{D}_{Z}(r) \) estimated from our SSM. Nuclear helium production is relevant in the internal radiative core \( (r \lesssim 0.3 R_\odot) \) where it is responsible for an enhancement of the helium abundance which can be as large as \( \overline{Y}_{\text{nuc},b} \simeq 0.35 \) at the center of the Sun. Elemental diffusion accounts for a \( \sim 10\% \) increase of helium and metals in the central regions with respect to external layers of the radiative region. The convective envelope, due to the very efficient convective mixing, is chemically homogeneous and can be fully described in terms of two numbers: the surface helium and metal abundances, indicated with \( Y_b \) and \( Z_b \), respectively.\(^9\)

These are related to initial values by

\[
\begin{align*}
Y_b &= Y_{\text{ini}} [1 + D_{Y,b}] \\
Z_b &= Z_{\text{ini}} [1 + D_{Z,b}],
\end{align*}
\]

where \( D_{Y,b} \) and \( D_{Z,b} \) parameterize the effects of elemental diffusion. In our SSM, we have \( Y_b = 0.2299 \), \( Z_b = 0.0125 \), \( D_{Y,b} = -0.121 \), and \( D_{Z,b} = -0.105 \).

We are interested in describing how the chemical composition is modified when we perturb the SSM. In the radiative core \((r \lesssim \overline{R}_b)\), we neglect the possible variations of the diffusion

\(^9\) We use the subscript “b” to emphasize that the surface abundances coincide with the values at the bottom of the convective region.
effects on the solar composition. A better accuracy is required in the convective region, because the surface helium abundance \( Y_b \) is an observable quantity. We, thus, discuss explicitly the role of diffusion and we write

\[
\Delta Y(r) = \Delta Y_{ini} [1 + \Delta Y_{conv}(r)] + \Delta Y_{nuc}(r)
\]

\[
\delta Z(r) = \delta Z_{ini},
\]

(24)

where \( \Delta Y_{nuc}(r) \) is the absolute variation of the amount of helium produced by nuclear reactions. A better accuracy is required in the convective region, because the surface helium abundance \( Y_b \) is an observable quantity. We, thus, discuss explicitly the role of diffusion and we write

\[
\Delta Y_b = (1 + \Delta T_{Y,b}) \Delta Y_{ini} + \Delta T_{ini} \Delta T_{Y,b} \delta D_{Y,b}
\]

\[
\delta Z_b = \delta Z_{ini} + \frac{\Delta T_{Z,b}}{1 + \Delta T_{Z,b}} \delta D_{Z,b},
\]

(25)

where \( \delta D_{Y,b} \) and \( \delta D_{Z,b} \) are the fractional variations of the diffusion terms.

It is important to remark that \( \Delta Y_b \) and \( \delta Z_b \) are related among each other, since the metals-to-hydrogen ratio at the surface of the Sun is observationally fixed. By imposing \( \delta(Z/X)_b = 0 \), we obtain

\[
\delta Z_b = -\frac{1}{1 - Y_b} \Delta Y_b
\]

(26)

where we considered that \( X_b \approx 1 - Y_b \). This relation can be rewritten in terms of the initial helium and metal abundances, obtaining

\[
\delta Z_{ini} = Q_0 \Delta Y_{ini} + Q_1 \delta D_{Y,c} + Q_2 \delta D_{Z,c}.
\]

(27)

where the coefficients \( Q_i \) are given by

\[
Q_0 = 1 - \frac{\Delta T_{Y,b}}{1 - Y_b} = -1.141
\]

\[
Q_1 = -\frac{\Delta T_{ini} \Delta T_{Y,b}}{1 - Y_b} = +0.041
\]

\[
Q_2 = 1 - \frac{\Delta T_{Z,b}}{1 + \Delta T_{Z,b}} = +0.118.
\]

(28)

6.2. Production of Helium by Nuclear Reactions

In order to predict the helium abundance in the radiative region, one has to estimate the variation of nuclear production of helium \( \Delta Y_{nuc}(r) \), see Equation (24). We note that the quantity \( \overline{\nu}_{nuc}(r) \) varies proportionally to \( \overline{\tau}(r) \) in the SSM, as is seen from Figure 3. This is natural since the helium production rate is directly proportional to the energy generation rate and, moreover, the energy generation profile \( \overline{\tau}(r) \) is nearly constant during the past history of the Sun. We assume that the observed proportionality holds true also in modified SMs, obtaining as a consequence the following relation:

\[
\Delta Y_{nuc}(r) = \overline{\nu}_{nuc}(r) \delta e^{tot}(r).
\]

(29)

If we expand \( \delta e^{tot}(r) \) according to relation (9) and solve with respect to \( \Delta Y(r) \), we can recast Equation (24) in the form\(^{11}\):

\[
\Delta Y(r) = \xi_Y(r) \Delta Y_{ini} + \xi_T(r) \delta T(r) + \xi_\rho(r) \delta \rho(r),
\]

(30)

where

\[
\xi_Y(r) = \frac{1 + \overline{\nu}_{nuc}(r)}{1 - \overline{\nu}_{nuc}(r) \overline{\tau}(r) \frac{\partial P}{\partial \overline{\rho}}},
\]

\[
\xi_T(r) = \frac{\overline{\nu}_{nuc}(r) \overline{\tau}(r) - \overline{\nu}_{nuc}(r) \frac{\partial P}{\partial \overline{\rho}}}{1 - \overline{\nu}_{nuc}(r) \overline{\tau}(r) \frac{\partial P}{\partial \overline{\rho}}},
\]

\[
\xi_\rho(r) = \frac{\overline{\nu}_{nuc}(r) \frac{\partial P}{\partial \overline{\rho}}}{1 - \overline{\nu}_{nuc}(r) \overline{\tau}(r) \frac{\partial P}{\partial \overline{\rho}}}.\]

(31)

The factors \( \xi_j(r) \) are shown in Figure 4.

6.3. Elemental Diffusion in the Convective Region

The total mass \( M_{conv,i} \) of the \( i \)th element contained in convective envelope evolves due to elemental diffusion according to relation (Aller & Chapman 1960; Thoul et al. 1994):

\[
\frac{1}{M_{conv,i}} \frac{dM_{conv,i}}{dt} = \omega_i H.
\]

(32)

Here, the parameter \( H \) is the “effective thickness” of the convective region defined by the relation \( 4\pi R_\odot \rho_b \omega_i = M_{conv,i} \). The quantity \( \omega_i \) is the diffusion velocity of the \( i \)th element at the bottom of the convective region that can be expressed as

\[
\omega_i = \frac{r_b \rho_b}{\rho_i} \left[ (A_{P,i} + \nabla a_i A_{T,i}) \frac{\partial \ln P(R_b)}{\partial r} \right],
\]

(33)

where \( A_{P,i}, A_{T,i} \) are the diffusion coefficients, see Thoul et al. (1994) for details. In the above relation, we neglected, for simplicity, the term proportional to the hydrogen concentration gradient\(^{12}\) and we took advantage of the fact that \( \partial \ln \rho \partial r = \nabla \partial \ln P/\partial r \) at the bottom of the convective region, as it is prescribed by the Schwarzschild criterion.

We have to estimate the effect of a generic modification of the SSM on elemental diffusion. In most cases, a good description is obtained by assuming that \( D_{Y,b} \) varies proportionally to the efficiency of diffusion in the present Sun, i.e., to the right-hand side of Equation (32) evaluated at the present time. This implies that

\[
\delta D_{Y,b} = \delta \omega_i - \delta H,
\]

(34)

where \( \delta H \) is the relative variation of the convective envelope effective thickness and \( \delta \omega_i \) is the relative variation of the diffusion velocity. By following the calculations described in Appendix B, one is able to show that

\[
\delta D_{Y,b} = \Gamma_Y \delta T_b + \Gamma_\rho \delta \rho_b,
\]

\[
\delta D_{Z,b} = \Gamma_Z \delta T_b + \Gamma_\rho \delta \rho_b,
\]

(35)

where \( \Gamma_Y = 2.05, \Gamma_Z = 2.73, \) and \( \Gamma_\rho = -1.10 \) and we assumed that all metals have the same diffusion velocity as iron.

By using the above relations and taking into account Equations (25) and (26) one obtains the surface abundances variations \( \Delta Y_b \) and \( \delta Z_b \) as a function of the free parameters \( \Delta Y_{ini} \) and \( \delta C \). Namely, we obtain

\[
\Delta Y_b = \Lambda_Y \Delta Y_{ini} + \Lambda_C \delta C
\]

\[
\Delta Z_b = \Lambda_Z \Delta Y_{ini} + \Lambda_C \delta C,
\]

(36)

\(^{10}\) The diffusion terms \( D_T(r) \) and \( D_Z(r) \) are at the few percent level in the radiative region. Their variations are, thus, expected to produce very small effects on the solar composition.

\(^{11}\) In the expansion of \( \delta e^{tot}(r) \) we neglected the term \( \epsilon_T(r) \delta Z(r) \). This term is expected to give a negligible contribution since the coefficient \( \epsilon_T(r) \) is at most equal to \(-0.05\), whereas the coefficients \( \epsilon_T(r), \epsilon_\rho(r), \) and \( \epsilon_T(r) \) are of order unity or larger; see the right panel of Figure 2.

\(^{12}\) This term gives a sub-dominant contribution at the present stage and, thus, is negligible also in the initial phases of the evolution when the Sun is essentially homogeneous.
where

\[ A_Y = \frac{1 + Y_{Y,b}}{1 + P_{Y,b} Y_{Y,b}} \Delta Y_{Y,b} \Gamma_Y = 0.838 \]

and

\[ A_C = \frac{Y_{Y,b} Y_{Z,b} \Delta Z_{Z,b} \Gamma_P}{1 + P_{Y,b} Y_{Y,b}} \Delta Y_{Y,b} \Gamma_Y = 0.033 \]  \hspace{1cm} (37) \]


Moreover, we can use relations (27) and (21) to calculate the variation of the initial metal abundance \( \delta Z_{\text{ini}} \) which, in our approach, coincides with the variation of metal abundance in the radiative region \( \delta Z(r) \). We obtain

\[ \delta Z(r) = \delta Z_{\text{ini}} = Q_Y \Delta Y_{\text{ini}} + Q_C \delta C, \]  \hspace{1cm} (39) \]

where

\[ Q_Y = Q_0 - P_{Y,b} A_Y (Q_1 \Gamma_Y + Q_2 \Delta Y_{Y,b}) = -0.887 \]

\[ Q_C = -P_{Y,b} A_C (Q_1 \Gamma_Y + Q_2 \Delta Y_{Y,b}) + \Gamma_P (Q_1 + Q_2) = -0.164. \]  \hspace{1cm} (40) \]

If we had neglected the effect of diffusion, we would have obtained \( A_Y = 0.879 \) and \( A_C = 0 \), \( B_Y = -1.141 \) and \( B_C = 0 \), and \( Q_Y = Q_0 = -1.14 \) and \( Q_C = 0 \).

7. THE FINAL SET OF EQUATIONS

We have now all the ingredients to formally define LSMs. The equations obtained in Section 6, namely Equations (30), (36), and (39), relate the chemical composition of the modified Sun to the present values of the structural parameters \( \delta P(r) \) and \( \delta T(r) \), to the energy generation coefficient \( \delta \epsilon(r) \) and to the free parameters \( \Delta Y_{\text{ini}} \) and \( \delta C \). These relations can be inserted in Equations (15) and in the integration conditions, Equations (19) and (21), obtaining a linear system of ordinary differential equations that completely determines the physical and chemical properties of the modified Sun. In this section, we give the equations in their final form.

The properties of the Sun in the radiative region \( (r \leq R_b) \) are described by

\[ \frac{d \delta m}{dr} = \frac{1}{l_m} [\gamma_P \delta P + \gamma_T \delta T - \delta m + \gamma_Y \Delta Y_{\text{ini}} + \gamma_C \delta C] \]

\[ \frac{d \delta P}{dr} = \frac{1}{l_P} [(\gamma_P - 1) \delta P + \gamma_T \delta T - \delta m + \gamma_Y \Delta Y_{\text{ini}} + \gamma_C \delta C] \]

\[ \frac{d \delta l}{dr} = \frac{1}{l_l} [\beta_P \delta P + \beta_T \delta T - \delta l + \beta_Y \Delta Y_{\text{ini}} + \beta_C \delta C + \beta_C \delta C] \]

\[ \frac{d \delta T}{dr} = \frac{1}{l_T} [\alpha_P \delta P + \alpha_T \delta T + \delta l + \alpha_Y \Delta Y_{\text{ini}} + \alpha_C \delta C + \delta \kappa + \alpha_C \delta C]. \]  \hspace{1cm} (41) \]

The coefficients \( \gamma_h \) which define the continuity and the hydrostatic equilibrium equation are given by

\[ \gamma_P = 1 - P_Y \xi_P \]  \hspace{1cm} \[ \gamma_T = -1 + P_Y \xi_T \]  \hspace{1cm} \[ \gamma_Y = -P_Y \xi_Y \]  \hspace{1cm} \[ \gamma_C = -P_Y \xi_C \]  \hspace{1cm} (42) \]

and are shown in Figure 5 as a function of the solar radius. The coefficients \( \alpha_h \) which define the transport and the energy generation equation are given by

\[ \beta_P = \beta_P + \beta_Y \xi_P \]  \hspace{1cm} \[ \beta_T = \beta_T + \beta_Y \xi_T \]  \hspace{1cm} \[ \beta_Y = \beta_Y \xi_Y + \beta_Z \xi_T \]  \hspace{1cm} \[ \beta_C = \beta_Z \xi_C \]  \hspace{1cm} \[ \beta_C = \beta_C \xi_C \]  \hspace{1cm} (43) \]

and are shown in Figure 6.
The integration conditions at the center of the Sun \((r = 0)\) are given by
\[
\begin{align*}
\delta m &= \gamma_{P,0} \delta P_0 + \gamma_{T,0} \delta T_0 + \gamma_{Y,0} \Delta Y_{\text{ini}} + \gamma_{\epsilon,0} \delta \epsilon_0 \\
\delta P &= \delta P_0 \\
\delta T &= \delta T_0 \\
\delta l &= \beta_{P,0} \delta P_0 + \beta_{T,0} \delta T_0 + \beta_{Y,0} \Delta Y_{\text{ini}} + \beta_{C,0} \delta C + \beta_{\epsilon,0} \delta \epsilon_0,
\end{align*}
\]
(44)
where \(\gamma_{P,0} = 1.167\), \(\gamma_{T,0} = -0.254\), \(\gamma_{Y,0} = 0.503\), \(\gamma_{\epsilon,0} = 0.161\) and \(\beta_{P,0} = 1.647\), \(\beta_{T,0} = 1.884\), \(\beta_{C,0} = 0.006\), \(\beta_{Y,0} = -1.142\), \(\beta_{\epsilon,0} = 0.624\). By imposing these conditions, the solution is obtained as a linear function of the four parameters \(\Delta Y_{\text{ini}}, \delta C, \delta T, \delta P\) which are fixed by requiring that
\[
\begin{align*}
\delta m &= -m_{\text{conv}} \delta C \\
\delta P &= \delta C \\
\delta T &= A'_Y \Delta Y_{\text{ini}} + A'_C \delta C \\
\delta l &= 0
\end{align*}
\]
(45)
at the convective boundary \((r = R_b)\), where \(A'_Y = -P_{Y,b} \alpha'_Y = 0.626\), \(A'_C = -P_{Y,b} \alpha'_C = 0.025\) and \(m_{\text{conv}} = 0.0192\). The solution is, thus, univocally determined.

The chemical composition of the radiative region can be calculated by using relations (30) and (39). The variation of the density profile \(\delta \rho(r)\) in the radiative region can be obtained by the relation:
\[
\delta \rho(r) = \gamma_P(r) \delta P(r) + \gamma_T(r) \delta T(r) + \gamma_Y(r) \Delta Y_{\text{ini}} + \gamma_\epsilon(r) \delta \epsilon(r).
\]
(46)
In the lower layers of the convective envelope the quantities \(\delta P(r), \delta T(r), \delta \rho(r)\) are nearly constant (see Section 5), while the quantity \(\delta m(r)\) gradually goes to zero, as is expected being the total mass of the Sun observationally determined. The surface chemical composition can be determined by using relation (36). We recall that the variations of surface helium and metal abundances are related among each other, in such a way that the hydrogen-to-metal surface ratio is unchanged with respect to the SSM.

Finally, the variation of the convective radius is calculated by using relation (12). We obtain
\[
\delta R_b = \Gamma_Y \Delta Y_{\text{ini}} + \Gamma_C \delta C + \Gamma_\epsilon \delta \epsilon_b,
\]
(47)
where
\[
\begin{align*}
\Gamma_Y &= -\frac{A_Y}{\xi_b} \left[ -P_{Y,b} \left( \kappa_{T,b} - 4 \right) + \kappa_{Y,b} - \frac{\kappa_{Z,b}}{1 - \xi_b} \right] = 0.449 \\
\Gamma_C &= -\frac{A_C}{\xi_b} \left[ -P_{Z,b} \left( \kappa_{T,b} - 4 \right) + \kappa_{Y,b} - \frac{\kappa_{Y,b}}{1 - \xi_b} \right] - 0.117 \\
\Gamma_\epsilon &= -\frac{1}{\xi_b} = -0.085.
\end{align*}
\]
(48)
In the derivation of the above relation, we considered that \(\delta l_b = 0, \delta m_b \simeq 0\) and we used Equations (21) and (36).

8. COMPARISON WITH FULL NONLINEAR CALCULATIONS

In order to show the validity of the proposed approach, we consider four possible modifications of the standard input and we compare the results obtained by LSMS with those obtained by the full nonlinear SM calculations. Three of the considered cases concern opacity modifications that are generally described as
\[
\kappa(\rho, T, X_i) = F(T) \kappa(\rho, T, X_i),
\]
(49)
where $\kappa(\rho, T, X_i)$ represent the standard value and $F(T)$ is a suitable function of the temperature. Namely, we consider

**OPA1**: overall rescaling of the opacity by a constant factor $F(T) \equiv 1.1$. This clearly corresponds to introduce the opacity variation:

$$\delta\kappa(r) \equiv 0.1$$

in Equations (41) for LSM calculations.

**OPA2**: smooth decrease of the opacity at the center of the Sun described by the function:

$$F(T) = 1 + \frac{A}{1 + \exp\left[\frac{T-T_c}{T_c}\right]}$$

where $A = -0.1$, $T_c = 9.4 \times 10^6$ K, $f = 0.1$. In our approach this corresponds to a variation $\delta\kappa(r)$ along the solar profile given by

$$\delta\kappa(r) = \frac{A}{1 + \exp\left[\frac{T_r-T_c}{T_c}\right]}$$

where $T(r)$ is the temperature profile predicted by the SSM (see the left panel of Figure 7).

**OPA3**: sharp increase of the opacity at the bottom of the convective envelope described by assuming $F(T) = 1.1$ for $T \leq 5 \times 10^6$ K (and $F(T) = 1$ otherwise). In our approach, this corresponds to

$$\delta\kappa(r) = 0.1 \quad \text{if} \quad T_r \leq 5 \times 10^6 \text{ K}$$

$$\delta\kappa(r) = 0 \quad \text{otherwise}.$$  

The above inequality can be rewritten in terms of a condition on the distance from the center of the Sun obtaining $\delta\kappa(r) = 0.1$ for $r \geq 0.4 R_\odot$ (and $\delta\kappa(r) = 0$ otherwise).

The fourth studied case concerns with modification of energy generation in the Sun. More precisely, we consider

**Spp**: increase of the astrophysical factor $S_{pp}$ of the $p + p \rightarrow d + e^+ + \nu_e$ reaction by +10%. In order to introduce this effect in LSM calculations, we consider the following variation of the energy generation profile:

$$\delta\epsilon(r) = \epsilon_{spp}(r) \delta S_{pp}, \quad (50)$$

where

$$\epsilon_{spp}(r) = \left. \frac{\partial \ln \epsilon}{\partial \ln S_{pp}} \right|_{\text{SSM}} \quad (51)$$

and $\delta S_{pp} = 0.1$. The function $\delta\epsilon(r)$ obtained in this way is shown in the right panel of Figure 7. The bump at $r \simeq 0.28 R_\odot$ is due to the out-of-equilibrium behavior of helium-3.

8.1. Physical and Helioseismic Properties of the Sun

In Figure 8, we show with solid lines the physical properties of LSmSs, obtained by solving the linear system of Equations (41), and with dotted lines the results obtained by the “standard” nonlinear SM calculations. The four panels correspond to the input modifications introduced in the previous section. We use different colors to show the variations of pressure (red), temperature (black), mass (blue), and luminosity (green) as a function of the solar radius.

We see that a very good agreement exists between LSmS and nonlinear SM results. To be more quantitative, the response of the Sun to the input modifications is reproduced at the 10% level or better, in all the considered cases. It is important to note that all the relevant features of the $\delta h(r)$ are obtained, indicating that the major effects are correctly implemented in our approach. The small differences between LSmSs and nonlinear SMs, typically more evident just below the convective region and/or at the center of the Sun, basically reflects the accuracy of the assumptions which have been used to estimate the variation of the chemical composition of the present Sun. In the OPA1 case, a disagreement exists in the $\delta l(r)$ behavior at the center of the Sun. This difference, which has no observable consequences, is mainly due to nonlinear effects. One should note, in this respect, that the constant 10% increase of the radiative opacity induces variations of temperature and pressure which are at the 1% level and a much smaller variation of $\delta l(r)$ (at the 0.1% level). In this situation, cancellations between different first-order competing contributions may occur.

13 The term “standard” is used here to refer to the conventional way of calculating SMs; see Section 2. For the sake of precision, we use the acronym “SM” to refer to these models. This is done to avoid confusion with the SSM which is intended as our best possible model of the Sun, i.e., the model calculated in the conventional way and, moreover, by using the best possible choice for the input parameters.
In Table 1, we present the variations of the initial abundances and of the photospheric chemical composition obtained by using relations (39) and (36). Moreover, we show the variation of the convective radius δRb calculated according to Equation (47). Finally, in Figure 9, we present the variations of the density profile δρ(r) calculated according to Equation (46) and the variation of the squared isothermal sound speed \( u = P/\rho \) given by

\[
\delta u(r) = \delta P(r) - \delta \rho(r).
\]  

We see that LSMs reproduce the results obtained by “standard” calculations with a very good accuracy. In particular, the variations of the helioseismic observables are obtained within 10% or better (unless they are extremely small). All this shows that our approach is sufficiently accurate to use LSMs as a tool to investigate the origin of the present discrepancy with helioseismic data. In a separate paper, we will use LSMs to analyze the role of opacity and metals in the Sun (F. L. Villante et al. 2010, in preparation).

8.2. Neutrino Fluxes

As a final application, we calculate the solar neutrino fluxes predicted by LSMs and we compare them with those obtained by using “standard” nonlinear SM calculations. The flux \( \Phi_\nu \) of solar neutrinos can be expressed as

\[
\Phi_\nu = \frac{1}{D^2} \int dr \frac{r^2 \rho(r) n_\nu(r)}{D^2},
\]

where \( D \) is the Sun-earth distance, \( n_\nu(r) \) is the total number of neutrinos produced per unit time and unit mass in the Sun, and the index \( \nu = \text{pp, Be, N, O} \) labels the neutrino producing reaction according to commonly adopted notations. If we expand to first order, the relative variation of the flux \( \delta \Phi_\nu \) can be expressed as

\[
\delta \Phi_\nu = \frac{1}{D^2} \int dr \frac{(r/D)^2 \rho(r) n_\nu(r)}{D^2} \left[ \delta n_\nu(r) + n_\nu T(r) \delta T(r) + n_\nu Y(r) \Delta Y(r) + n_\nu Z(r) \delta Z(r) + n_\nu Spp(r) \delta Spp \right],
\]
Figure 9. Comparison between the variations of density (red) and of squared isothermal sound speed (black) predicted by LSM (solid lines) and by “standard” nonlinear SM (dashed lines).

(A color version of this figure is available in the online journal.)

Table 1
Comparison between the Predictions of LSM and “Standard” Nonlinear SM for the Initial and Surface Chemical Abundances, the Convective Radius, and the Solar Neutrino Fluxes

|                  | OPA1 | OPA2 | OPA3 | Spp |
|------------------|------|------|------|-----|
| $\Delta Y_{\text{ini}}$ | 0.016 | 0.017 | -0.0056 | -0.0058 |
| $\delta Z_{\text{ini}}$ | -0.018 | -0.016 | 0.0000 | -0.001 |
| $\Delta Y_b$ | 0.014 | 0.014 | -0.0037 | -0.0036 |
| $\delta Z_b$ | -0.018 | -0.018 | 0.0049 | 0.0047 |
| $\delta R_b$ | -0.0020 | -0.0020 | -0.0067 | -0.0070 |
| $\delta \Phi_{\text{pp}}$ | -0.011 | -0.010 | 0.0045 | 0.0052 |
| $\delta \Phi_{\text{Be}}$ | 0.13 | 0.13 | -0.067 | -0.064 |
| $\delta \Phi_{\text{B}}$ | 0.27 | 0.27 | -0.17 | -0.17 |
| $\delta \Phi_{\text{N}}$ | 0.14 | 0.14 | -0.10 | -0.094 |
| $\delta \Phi_{\text{O}}$ | 0.21 | 0.22 | -0.14 | -0.14 |

Notes. Note that the absolute variations are reported for helium, whereas the relative variations are shown for all the other quantities.
where $\Phi_\nu$ is the SSM value and
\[
\begin{align*}
n_{\nu,\rho}(r) &= \left( \frac{\partial \ln n_\nu}{\partial \ln \rho} \right)_{SSM} + 1, \\
n_{\nu,T}(r) &= \left( \frac{\partial \ln n_\nu}{\partial \ln T} \right)_{SSM}, \\
n_{\nu,Z}(r) &= \left( \frac{\partial \ln n_\nu}{\partial \ln Z} \right)_{SSM}, \\
n_{\nu,Y}(r) &= \left( \frac{\partial \ln n_\nu}{\partial Y} \right)_{SSM}, \\
n_{\nu,Spp}(r) &= \left( \frac{\partial \ln n_\nu}{\partial \ln S_{pp}} \right)_{SSM}. 
\end{align*}
\]

The derivatives $n_{\nu,j}(r)$ have been calculated numerically by assuming that the abundances of secondary elements (helium-3, carbon-12, and nitrogen-14) can be estimated as described in Appendix A. The symbol $|_{SSM}$ indicates that we calculate the derivatives along the density, temperature, and chemical composition profiles predicted by the SSM.

The term $n_{\nu,Spp}(r) \delta S_{pp}$ is introduced to describe the effects of a variation of $S_{pp}$ on the rates of neutrino producing reactions. It clearly holds $n_{pp,Spp}(r) \equiv 1$, since the pp-neutrinos are produced by $p + p \rightarrow d + e^+ + \nu_e$. Boron and beryllium neutrinos are influenced by $S_{pp}$ because this parameter determines the helium-3 production rate (through deuterium) and, thus, also the rate of $^3$He + $^4$He $\rightarrow$ $^7$Be + $\gamma$ reaction. As a consequence, we have
\[
n_{Be,Spp}(r) = n_{B,Spp}(r) = \left( \frac{\partial \ln X_3}{\partial \ln S_{pp}} \right)_{SSM}, \tag{56}
\]
where $X_3$ is the helium-3 abundance. The CN-cycle efficiency, instead, does not depend on the value of $S_{pp}$ and, thus, we have $n_{N,Spp}(r) = n_{O,pp}(r) \equiv 0$.

It is useful to recast Equation (54) in the form
\[
\delta \Phi_\nu = \int dr \left[ \phi_{\nu,\rho}(r) \delta \rho(r) + \phi_{\nu,T}(r) \delta T(r) + \phi_{\nu,Y}(r) \Delta Y(r) + \phi_{\nu,Z}(r) \delta Z(r) + \phi_{\nu,Spp}(r) \delta S_{pp} \right], \tag{57}
\]
\[ \phi_{\nu,j}(r) = \frac{r^2 \bar{p}(r) \bar{n}_i(r) n_{\nu,j}(r)}{\int dr r^2 \bar{p}(r) \bar{n}_i(r)}. \] (58)

The functions \( \phi_{\nu,j}(r) \) are displayed in Figure 10. They show explicitly the well-known fact that the boron, beryllium, and CNO neutrinos are produced in a more internal region with respect to pp-neutrinos and that they more strongly depend on temperature. The complicated behavior of the functions \( \Phi_{\nu,j}(r) \) at \( r \approx 0.15 R_\odot \) is due to the out-of-equilibrium behavior of carbon-12 abundance. The N-neutrinos originate, in fact, from the decay of \(^{13}\text{N}\), which is produced by \(^{12}\text{C} + p \rightarrow ^{13}\text{N} + \gamma\) reaction. The Sun was born with a relatively large amount of carbon-12, which has been converted by CN-cycle into nitrogen-14 in the more internal regions. The \(^{12}\text{C}\) abundance is, thus, larger where CN-cycle is less effective and, as a consequence, a non-negligible production of N-neutrinos occurs at relatively large radii, where equilibrium conditions do not hold.

In Table 1, we compare the LSM results, i.e., the values \( \delta \Phi_{\nu} \), obtained by applying Equation (57) to the solutions of Equations (41), with the results obtained from “standard” calculations. We see that an excellent agreement exists, for all the cases considered in this paper.\(^{14}\) All this shows that our approach can be used as a tool to investigate the dependence of the solar neutrino fluxes on the input parameters adopted in SSM construction.

9. CONCLUSIONS

We have proposed a new approach to studying the properties of the Sun which is based on the following points:

1. We have considered small variations of the physical and chemical properties of the Sun with respect to SSM predictions and we have linearized the stellar equilibrium equations to relate them to the properties of the solar plasma (see Sections 3, 4, and 5).
2. We have derived simple relations which allow us to estimate the (variation of) present solar composition from the (variation of) nuclear reaction rates and elemental diffusion efficiency in the present Sun (see Section 6).

As a final result, we have obtained a linear system of ordinary differential equations (see Section 7) which can be easily solved and that completely determine the physical and chemical properties of the “perturbed” Sun.

In order to show the validity of our approach, we have considered four possible modifications of the input parameters (opacity and energy generation profiles) and we have compared the results of our LSMs with those obtained by “standard” methods for SM calculations (see Section 8). A very good agreement is achieved for all the structural parameters (mass, luminosity, temperature, pressure, etc.) and for all the helioseismic and solar neutrino observables.

We believe that our approach can complement the traditional method for SM calculations, allowing us to investigate in a more efficient and transparent way the role of the different parameters and assumptions. In particular, it could be useful to study the origin of the present discrepancy between SSM results and helioseismic data.

\(^{14}\) The fact that all the neutrino fluxes are correctly reproduced, even in the cases in which some compensations occur, is remarkable. See, e.g., the OPA3 case, in which a very small value for \( \delta \Phi_{\nu} \) flux is obtained (with respect to the other fluxes), as a consequence of the peculiar behavior of the \( \Phi_{\nu,j}(r) \) functions.

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APPENDIX A

THE ABUNDANCE OF SECONDARY ELEMENTS

Secondary elements are those elements which are both created and destroyed in a reaction chain or in a reaction cycle. The relevant secondary elements to understand energy and neutrino production in the Sun are \(^3\text{He}, ^{12}\text{C},\) and \(^{14}\text{N}\).

A.1. The \(^3\text{He}\) Abundance

The evolution of helium-3 abundance, \( X_3 \), is described by the equation\(^{15}\):

\[ \frac{dX_3}{dt} = \frac{3\rho}{m_u} \left[ \frac{X^2}{2} \langle \sigma v \rangle_{pp} - \frac{X^2}{9} \langle \sigma v \rangle_{33} - \frac{X_3 Y}{12} \langle \sigma v \rangle_{34} \right]. \] (A1)

where \( \langle \sigma v \rangle_{pp} \) is the reaction rate per particle pair of the \( p + p \rightarrow d + e^+ + \nu_e \) reaction, while \( \langle \sigma v \rangle_{33} \) and \( \langle \sigma v \rangle_{34} \) refer to \(^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2p\) and \(^3\text{He} + ^4\text{He} \rightarrow ^7\text{Be} + \gamma + \nu_e\) reactions, respectively. In the most internal region \( (r \leq 0.25 R_\odot) \), the rates of these reactions are fast with respect to the Sun evolutionary times and equilibrium is achieved:

\[ X_{3,eq} = \frac{3 Y}{8} \frac{\langle \sigma v \rangle_{34}}{\langle \sigma v \rangle_{33}} \left[ 1 + \sqrt{1 + 32 \frac{X^2}{Y} \frac{\langle \sigma v \rangle_{33}}{\langle \sigma v \rangle_{34}}} \right]. \] (A2)

In the outer regions \( (R \geq 0.2 R_\odot) \), we can neglect the contribution from \(^3\text{He} + ^4\text{He} \rightarrow ^7\text{Be} + \gamma\) and we obtain

\[ \frac{dX_3}{dt} = C_3 - D_3 X_3^2, \] (A3)

where \( C_3 = (3 X^2 \rho / \langle \sigma v \rangle_{pp})/(2 m_u) \) and \( D_3 = (\rho \langle \sigma v \rangle_{33})/(3 m_u) \). If we assume that the factors \( D_3 \) and \( C_3 \) are approximately constant over the time scale \( t_3 = C_3^{-1} \) during which \(^3\text{He}\) is produced, the above equation can be explicitly solved. We obtain

\[ X_3 = X_{3,eq} \tanh \left[ \frac{X_{3,max}}{X_{3,eq}} \right]. \] (A4)

where \( X_{3,max} = C_3 t \) represents the helium-3 abundance which would have been produced in the integration time \( t \) if we had neglected the \(^3\text{He}\) destruction processes (and we implicitly assumed that the helium-3 initial abundance is negligible).

In this work, we need to calculate the helium-3 response to a generic modification of the solar properties. We use the following approach. We calculate the \(^3\text{He}\) equilibrium value in a generic SM by using Equation (A2). We assume that Equation (A4) is valid at each point of the Sun and we estimate the value \( X_{3,max} \) in the SSM by inverting it (and by using the SSM-abundance \( X_3 \)). Then, we calculate the value of \( X_{3,max} \) in the modified SM by considering that \( X_{3,max} \) scales as

\(^{15}\) We assume that deuterium is at equilibrium, i.e., deuterium production and destruction rates are equal.
We assume that Equation (A4) is valid at each point of the Sun and calculate the values which would have been obtained in the integration time $t$.

\[
\eta = \eta_0 - \eta_0 - \eta_{\text{max}} \exp \left( \frac{\eta_{\text{max}}}{\eta_{\text{eq}}} \right),
\]

where $\eta_0$ is the initial value and $\eta_{\text{max}} = C N \tau$, with $C = N$ so that $\eta$ value which would have been obtained in the integration time $t$ if we had neglected the “destruction” term in Equation (A6).

In analogy to what was done for helium-3, we use the following approach to calculate the response of carbon-12 and nitrogen-14 to a generic modification of the solar properties. We calculate the $\eta_{\text{eq}}$ value in a generic SM by using Equation (A7). We assume that Equation (A8) is valid at each point of the Sun. We estimate the value $\eta_{\text{max}}$ in the SSM by inverting it (and using the SSM-values $\eta$ and $\eta_0$). Then, we calculate the value of $\eta_{\text{eq}}$ in the modified Sun by considering that it scales as $\eta_{\text{eq}} = N \rho X (\sigma v)_{1,12} - (\sigma v)_{1,14}$.

\[
\delta H = \delta M_{\text{conv}} - 2\delta R_b - \delta \rho_b - \frac{\partial \ln \rho(\bar{R}_b)}{\partial \ln r} \delta R_b.
\]

If we consider that $M_{\text{conv}} = \int_{R_b} dV 4\pi r^2 \rho(r)$ and that $\delta \rho(r)$ is approximately constant in the convective region, we obtain

\[
\delta M_{\text{conv}} = \delta \rho_b - \frac{\bar{R}_b}{H} \delta R_b.
\]

By using this relation and taking into account Equations (B5) and (B4), we arrive at the expression:

\[
\delta D_{i,b} = \frac{5}{2} \delta T_b - \delta P_b + \frac{d \ln A_{\text{tot},i}}{dY} \Delta Y_b + \left[ \frac{\partial \ln \rho(\bar{R}_b)}{\partial \ln r} + \frac{\bar{R}_b}{H} \right] \delta R_b.
\]
integration conditions at the bottom of the convective region expressed by Equations (21), we obtain the final relations:

\[
\begin{align*}
\delta D_Y,b &= \Gamma_Y \delta T_b + \Gamma_P \delta P_b + \Gamma_\kappa \delta \kappa_b \\
\delta D_Z,b &= \Gamma_Z \delta T_b + \Gamma_P \delta P_b + \Gamma_\kappa \delta \kappa_b,
\end{align*}
\]

(B8)

where \( \Gamma_Y = 2.05 \), \( \Gamma_Z = 2.73 \), \( \Gamma_P = -1.10 \), and \( \Gamma_\kappa = -0.06 \) and we assumed that all metals have the same diffusion velocity as iron. In the calculations presented in this paper, we neglect for simplicity the terms proportional to \( \delta \kappa_b \) which generally give a very small contribution.

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