Possible tests of neutrino maximal mixing and comments on matter effects

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Abstract

We show in a simple and general way that matter effects do not contribute to the averaged value of the probabilities of transition of solar $\nu_e$’s into other states in the case of maximal mixing of any number of massive neutrinos. We also show that future solar neutrino experiments (Super-Kamiokande and SNO) will allow to test the model with maximal mixing of three massive neutrinos in a way that does not depend on the initial solar neutrino flux.
The hypothesis of maximal neutrino mixing was proposed many years ago [1–3]. Recently the interest in this hypothesis is increased in connection with the atmospheric neutrino anomaly [4–6], whose explanation requires neutrino oscillations with large mixing. There are some theoretical arguments in favor of maximal mixing based on the seesaw mechanism [7,8] and on the hypothesis of permutation symmetry between generations at the unification scale [9]. In Refs. [10,11] the data of the experiments on the detection of solar and atmospheric neutrinos were analyzed under the assumption of maximal mixing of three neutrino fields with masses $m_1 \ll m_2 \ll m_3$. It was shown that the atmospheric neutrino data are well described in this model with $\Delta m_{31}^2 \simeq 10^{-2}$ eV$^2$. The data of all four solar neutrino experiments [12–15] cannot be described in this model with an acceptable value of $\chi^2$. If only the data of the Kamiokande, GALLEX and SAGE experiments [13–15] are taken into account, it is possible to obtain a good fit in the case of maximal mixing with $\Delta m_{31}^2 \lesssim 10^{-12}$ eV$^2$.

In the recent paper [16] it was shown that in the case of maximal mixing matter effects do not modify the vacuum oscillation probability of solar $\nu_e$'s to survive. This was shown with a numerical solution of the evolution equation of neutrinos in matter for the case of two and three generations. In the first part of this note we will show in a simple and general way that the averaged transition probabilities of solar $\nu_e$'s into any final state do not depend on the presence of matter in the case of the maximal mixing.

Let us consider first the very well known case of vacuum oscillations (see, for example, Ref. [17]). The averaged probability of the transition of solar $\nu_e$'s into any state $\nu_\alpha$ in the general case of $N$ massive neutrinos is given by

$$\langle P_{\nu_e \rightarrow \nu_\alpha} \rangle = \sum_{j=1}^{N} |U_{\alpha j}|^2 |U_{ej}|^2,$$

where $U_{\alpha j}$ are the elements of the mixing matrix. In Eq.(1) we have assumed that, due to averaging over the neutrino spectrum and over the regions where neutrinos are produced and detected, all the interference terms in the transition probabilities disappear. This takes place if all the neutrino mass-squared differences satisfy the following inequalities

$$|\Delta m_{jk}^2| \gg 10^{-10} \text{ eV}^2 \quad (j \neq k),$$

where $\Delta m_{jk}^2 \equiv m_j^2 - m_k^2$. In the case of maximal mixing [1–3]

$$|U_{\alpha j}|^2 = \frac{1}{N},$$

and we obtain the well known result [18]

$$\langle P_{\nu_e \rightarrow \nu_\alpha} \rangle = \frac{1}{N}.$$  

Let us notice that this is the minimum value for the averaged probability of solar $\nu_e$'s to survive. Using this expression (in the case $N = 2$) many years ago B. Pontecorvo [19] for the first time noticed that the detected flux of solar $\nu_e$'s can be as small as half of the expected flux.
Let us now consider solar $\nu_e$'s that are produced in the core of the sun with energy $E$. As a result of neutrino propagation in the interior of the sun the neutrino state on the surface of the sun has the following form

$$|\nu\rangle_\odot = \sum_\alpha A_\alpha^e(E) |\nu_\alpha\rangle ,$$

where $|\nu_\alpha\rangle$ is any neutrino state (active or sterile). The amplitudes $A_\alpha^e(E)$ are determined by the evolution equation of neutrinos in matter. The values of these amplitudes are not important here. We will use only the normalization condition

$$\sum_\alpha |A_\alpha^e(E)|^2 = 1 .$$

Let us stress that the amplitudes $A_\alpha^e$ can depend on the neutrino energy $E$. The neutrino state on the earth is given by

$$|\nu\rangle_\oplus = \sum_{\beta,j,\alpha} A_\beta^e U_{\beta j}^* e^{-iE_j T} U_{\alpha j} |\nu_\alpha\rangle ,$$

where $T \simeq R$, $R$ being the distance between the surface of the sun and the earth.

The probability of transitions of solar $\nu_e$'s into any state $\nu_\alpha$ is given by

$$P_{\nu_e \rightarrow \nu_\alpha} = \sum_{\beta,\rho,j,k} A_\beta^e A_\rho^e U_{\beta j}^* U_{\rho k} U_{\alpha k} \exp \left( -i \frac{\Delta m_{jk}^2}{2E} R \right) .$$

If we assume that the squared mass differences satisfy Eq.(2), all the interference terms disappear in the expression for the measurable averaged probability, which is given by

$$\langle P_{\nu_e \rightarrow \nu_\alpha} \rangle = \frac{1}{N} \sum_{j=1}^N |U_{\alpha j}|^2 \left| \sum_{\beta} A_\beta^e U_{\beta j}^* \right|^2 .$$

In the case of maximal mixing $|U_{\alpha j}|^2 = 1/N$ and from Eq.(3), using the unitarity of the mixing matrix and the normalization condition (3), we obtain

$$\langle P_{\nu_e \rightarrow \nu_\alpha} \rangle = \frac{1}{N} \sum_{j=1}^N \sum_{\beta} A_\beta^e U_{\beta j}^* \left| A_\beta^e \right|^2 = \frac{1}{N} \sum_{\beta} |A_\beta^e|^2 = \frac{1}{N} .$$

Thus, in the case of maximal mixing, if the condition (2) is satisfied, the probability of transitions of solar $\nu_e$'s into any state $\nu_\alpha$, active or sterile, is equal to $1/N$. This probability does not depend on the values of the amplitudes $A_\alpha^e$, which means that the matter effect is not observable in the case of maximal mixing. In Ref. [16] this result was obtained for the cases $N = 2$ and $N = 3$ with a numerical solution of the evolution equation of neutrinos in matter.

Let us notice that the result given in Eq.(10) can also be obtained by writing the neutrino state on the surface of the sun as a superposition of mass eigenstates.
\[ |\nu\rangle_\odot = \sum_{j=1}^{N} A_j^{e} |\nu_j\rangle, \quad \text{with} \quad A_j^{e} = \sum_\alpha A_\alpha^{e} U^*_{\alpha j}. \quad (11) \]

In general the amplitudes \( A_j^{e} \), which are normalized by \( \sum_j |A_j^{e}|^2 = 1 \), depend on the neutrino energy and there is more than one \( A_j^{e} \) different from zero. For example, in the case of three generations of neutrinos with \((\Delta m_{21}^2/E) \lesssim 10^{-5} \text{eV}^2 \text{MeV}^{-1}\) an electron neutrino produced in the core of the sun is the following superposition\(^1\) of effective mass eigenstates \( \nu_2^M \) and \( \nu_3^M \)

\[ |\nu_e^M\rangle = \sqrt{\frac{2}{3}} |\nu_2^M\rangle + \sqrt{\frac{1}{3}} |\nu_3^M\rangle. \quad (12) \]

In this case, if the propagation of the neutrino in the interior of the sun is adiabatic, only \( A_2^{e} \) and \( A_3^{e} \) are different from zero. If the propagation of the neutrino in the interior of the sun is non-adiabatic, transitions from \( \nu_2^M \) and \( \nu_3^M \) to \( \nu_1^M \) are possible and all three \( A_1^{e}, A_2^{e} \) and \( A_3^{e} \) are different from zero. From Eq.\(^{11}\), for the measurable averaged probability of \( \nu_e \to \nu_\alpha \) transitions, in which all the interference terms disappear, we have

\[ \langle P_{\nu_e \to \nu_\alpha} \rangle = \sum_{j=1}^{N} |A_j^{e}|^2 |U_{\alpha j}|^2 = \frac{1}{N}. \quad (13) \]

It is instructive to present another derivation of this result. If all the oscillating terms in the expression for the averaged probability of \( \nu_e \to \nu_\alpha \) transitions disappear, we have (see Ref.\(^{[20,21]}\))

\[ \langle P_{\nu_e \to \nu_\alpha} \rangle = \sum_{j,k} |U_{\alpha k}|^2 P_{kj} |U_{e j}^0|^2. \quad (14) \]

Here \( U_{e j}^0 \) are the elements of the mixing matrix at the point where the neutrino is produced and \( P_{kj} \) is the transition probability from the state with energy \( E_j \) in the production point to the state with energy \( E_k \) on the earth. Using the unitarity relations

\[ \sum_k P_{kj} = 1 \quad \text{and} \quad \sum_j |U_{e j}^0|^2 = 1, \quad (15) \]

in the case of maximal mixing we obtain Eq.\(^{[4]}\).

Let us notice that in the case of maximal mixing of two massive neutrino fields there is no MSW\(^{[22]}\) resonance in matter. In the case of maximal mixing of three massive neutrino fields, if \( \Delta m_{31}^2 \) is large (say, \( \Delta m_{31}^2 \approx 10^{-2} \text{eV}^2 \)) the resonance condition has the form

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\(^1\) This equation can be obtained from Eqs.(2.66) and (2.69) of Ref.\(^{[20]}\) with the mixing angles \( \omega, \psi, \varphi \) and the phase \( \delta \) such that \( \sin \omega = \sin \psi = 1/\sqrt{2}, \sin \varphi = 1/\sqrt{3} \) and \( \delta = \pi/4 \), which correspond to maximal mixing.
\[
2 \sqrt{2} G_F E_N e = \Delta m_{21}^2 \frac{\cos 2\omega}{\cos^2 \varphi},
\]

where \(N_e\) is the electron density and the Maiani parameterization of the mixing matrix has been used (see Ref. [21]). In the case of maximal mixing the right-hand side of Eq. (16) is equal to zero and there is no MSW resonance in matter.

In Refs. [10,11] it has been shown that the atmospheric neutrino data and the data of the Kamiokande, GALLEX and SAGE experiments [13–15] are well described in the case of maximal mixing of three neutrinos with \(\Delta m_{21}^2 \approx 10^{-2}\) eV\(^2\) and \(\Delta m_{31}^2 \lesssim 10^{-12}\) eV\(^2\). In this case the value of \(P_{\nu_e \rightarrow \nu_e}\) does not depend on energy and is equal to \(5/9\). We will discuss now the possibilities for the solar neutrino experiments of the next generation to check this model.

During 1996 two new solar neutrino experiments will start, Super-Kamiokande (S-K) [23]) and SNO [24]. In the S-K experiment solar neutrinos will be detected through the observation of elastic scattering (ES) of neutrinos on electrons,

\[
\nu + e^- \rightarrow \nu + e^-.
\]

In the SNO experiment solar neutrinos will be detected through the observation of the charged-current (CC) and neutral current (NC) processes

\[
\nu_e + d \rightarrow e^- + p + p,
\]

\[
\nu + d \rightarrow \nu + p + n,
\]

and also the ES process (17).

In the SNO experiment the spectrum of electrons in the CC process (18) will be measured and the flux of solar \(\nu_e\)'s on the earth as a function of neutrino energy \(E\) will be determined [24].

In both the S-K and SNO experiments, due to the high energy thresholds (about 5 MeV for CC and ES and 2.2 MeV for NC) only neutrinos coming from \(^8\)B decay will be detected. The energy spectrum of the initial \(^8\)B \(\nu_e\)'s can be written as

\[
\phi_B(E) = \Phi_B X(E).
\]

Here \(X(E)\) is a known normalized function determined mainly by the phase space factor of the decay \(^8\)B \(\rightarrow ^8\)Be + e\(^+\) + \(\nu_e\) (see Ref. [25]), and \(\Phi_B\) is the total flux of initial \(^8\)B solar \(\nu_e\)'s. In the following we will not make any assumption about the value of \(\Phi_B\).

In the maximal mixing model under consideration the survival probability of solar \(\nu_e\)'s has the constant value \(5/9\) and the shape of the spectrum of \(\nu_e\)'s on the earth is given by \(X(E)\). In the high-statistic S-K experiment the spectrum of recoil electrons will be measured with high accuracy. In the model under consideration the spectrum of the recoil electrons is given by

\[
n^{ES}(T) = \frac{5}{9} \Phi_B \int_{E_m(T)} \left( \frac{d\sigma_{\nu_ee}}{dT}(E, T) + \frac{4}{5} \frac{d\sigma_{\nu_\mu e}}{dT}(E, T) \right) X(E) dE,
\]

where \(T\) is the electron kinetic energy, \(d\sigma_{\nu_\ell e}/dT\) is the differential cross section of the ES process \(\nu_\ell e \rightarrow \nu_\ell e\) (with \(\ell = e, \mu\)) and \(E_m(T) = (1 + \sqrt{1 + 2 m_e/T}) T/2\). Let us notice that

\[
\sum_{\ell} \frac{d\sigma_{\nu_\ell e}}{dT}(E, T) X(E) dE \approx 1 \approx \frac{dN_e}{dT} X(E) dE.
\]
the main contribution to \( n^{ES}(T) \) comes from the first term in the integral. A comparison of the shape of the CC and ES spectra with the SNO and S-K data will be a test for the scheme with maximal mixing. Furthermore, this model allows to predict the ratios of different observables independently from the value of the total \( ^8B \) neutrino flux. In fact, for the ratio of the total ES and CC event rates \( R_{ES}^CC = N_{ES}^{CC}/N_{CC}^{CC} \) we have

\[
R_{ES}^{CC} = \frac{\int_{E_{th}^{ES}} E_{th}^{ES} \left( \sigma_{\nu_e e}(E) + \frac{4}{5} \sigma_{\nu_\mu e}(E) \right) X(E)dE}{\int_{E_{th}^{CC}} \sigma_{\nu_\mu d}(E)X(E)dE},
\]

where \( \sigma_{\nu_\mu d}(E) \) is the cross section for the CC process \( [28] \) and \( E_{th}^{ES} \) and \( E_{th}^{CC} \) are the neutrino energy thresholds. The result of our calculation of \( R_{ES}^{CC} \) is presented in Table I, where it is compared with the corresponding values calculated in the usual model with mixing of two massive neutrino fields and mixing parameters \( \Delta m^2, \sin^2 2\theta \) obtained from the analysis of the solar neutrino data (in Ref. [26] for MSW transitions and in Ref. [27] for vacuum oscillations). We used the cross section \( \sigma_{\nu_\mu d}(E) \) given in Ref. [28] and an electron kinetic energy threshold of 4.5 MeV.

Let us consider now the ratio of the recoil electron spectrum in the ES process and the total CC event rate \( r_{ES}^{CC}(T) = n_{ES}^{CC}(T)/N_{CC}^{CC} \). We have

\[
r_{ES}^{CC}(T) = \frac{\int_{E_{th}^{ES}} E_{th}^{ES} \left( \frac{d\sigma_{\nu_\mu e}}{dT}(E, T) + \frac{4}{5} \frac{d\sigma_{\nu_\mu e}}{dT}(E, T) \right) X(E)dE}{\int_{E_{th}^{CC}} \sigma_{\nu_\mu d}(E)X(E)dE},
\]

In Fig.1 we have plotted this ratio as a function of \( T \) in the interval \( 4.5 \text{ MeV} \leq T \leq 14 \text{ MeV} \).

In Fig.1 we have also plotted the ratio \( r_{ES}^{CC}(T) \) calculated in the usual model with mixing of two massive neutrinos for the cases of MSW transitions and vacuum oscillations. We have used the values of the mixing parameters \( \Delta m^2, \sin^2 2\theta \) given in Table I. It can be seen from Fig.1 that the investigation of the dependence of \( r_{ES}^{CC}(T) \) on \( T \) in the region of small \( T \) \( (T \lesssim 7 \text{ MeV}) \) will allow to distinguish the case of maximal mixing of three neutrinos from the usual large mixing angle MSW solution and from vacuum oscillations of two neutrinos.

For the ratio of the total CC and NC event rates \( R_{NC}^{CC} = N_{CC}^{NC}/N_{NC}^{NC} \) we have

\[
R_{NC}^{CC} = \frac{5}{9} \frac{\int_{E_{th}^{CC}} \sigma_{\nu_\mu d}(E)X(E)dE}{\int_{E_{th}^{NC}} \sigma_{\nu_\mu d}(E)X(E)dE},
\]

where \( \sigma_{\nu_\mu d}(E) \) and \( E_{th}^{NC} = 2.2 \text{ MeV} \) are the cross section and the energy threshold of the NC process \( [19] \). The result of our calculation of \( R_{NC}^{CC} \) is presented in Table I. We used the cross section \( \sigma_{\nu_\mu d}(E) \) given in Ref. [28].

Finally, for the ratio of the total ES and NC event rates \( R_{NC}^{ES} = N_{ES}^{NC}/N_{NC}^{NC} \) we have

\[
R_{NC}^{ES} = \frac{5}{9} \frac{\int_{E_{th}^{ES}} E_{th}^{ES} \left( \sigma_{\nu_\mu e}(E) + \frac{4}{5} \sigma_{\nu_\mu e}(E) \right) X(E)dE}{\int_{E_{th}^{NC}} \sigma_{\nu_\mu d}(E)X(E)dE}.
\]
The result of our calculation of $R_{ES}^{\text{NC}}$ is presented in Table I. It can be seen from Table I that the measurement of the ratios $R_{ES}^{\text{CC}}, R_{NC}^{\text{CC}}$ and $R_{NC}^{\text{ES}}$ will allow to distinguish the case of maximal mixing of three neutrinos from all the usual two-neutrinos solutions of the solar neutrino problem.

The comparison of the relations (21)–(25) with the data of the S-K and SNO experiments will be a crucial test of the maximal mixing model of Refs. [10,11].

In conclusion, we have presented simple and general arguments based on the unitarity of the mixing matrix which show that matter effects do not contribute to the values of the averaged transition probabilities of solar $\nu_e$’s into other states in the case of maximal mixing. We have also shown that the future solar neutrino experiments (Super-Kamiokande and SNO), in which solar $^8\text{B}$ neutrinos will be detected through CC and NC reactions, will allow to check the model with maximal mixing of three massive neutrino fields that describes the atmospheric neutrino anomaly and the data of the Kamiokande, GALLEX and SAGE experiments.

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TABLE I. Results of the calculation of the ratios \( R_{CC}^{ES}, R_{NC}^{CC} \) and \( R_{NC}^{ES} \) in the model with maximal mixing of three massive neutrino fields (see Eqs. (22), (24) and (25)). The values of these ratio calculated in the usual model with mixing of two massive neutrinos are also presented. The values of the mixing parameters \( \Delta m^2, \sin^2 2\theta \) have been obtained from the analysis of the solar neutrino data in Ref. [26] for the case of MSW transitions and in Ref. [27] for the case of vacuum oscillations.

| \( \nu_e \to \nu_\mu(\tau) \) | \( \Delta m^2 (\text{eV}^2) \) | \( \sin^2 2\theta \) | \( R_{CC}^{ES} \) | \( R_{NC}^{CC} \) | \( R_{NC}^{ES} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| MAXIMAL MIXING  |                  |                 | 2.1 \times 10^{-2} | 1.33 | 2.8 \times 10^2 |
| SMALL MIX. MSW  | 6.1 \times 10^{-6} | 6.5 \times 10^{-3} | 2.4 \times 10^{-2} | 0.79 | 1.9 \times 10^2 |
| LARGE MIX. MSW  | 9.4 \times 10^{-6} | 0.62            | 3.1 \times 10^{-2} | 0.46 | 1.4 \times 10^2 |
| VACUUM OSC.     | 8.0 \times 10^{-11} | 0.80          | 2.7 \times 10^{-2} | 0.68 | 1.9 \times 10^2 |
FIGURES

FIG. 1. Results of the calculation of the ratio \( r_{ES}^{CC}(T) \) as a function of the electron recoil kinetic energy \( T \) (see Eq. (23)). The curves of \( r_{ES}^{CC}(T) \) calculated in the usual model with mixing of two massive neutrinos are also presented. The values of the mixing parameters \( \Delta m^2, \sin^2 2\theta \) are given in Table I.
