Degree two approximate Boolean #CSPs with variable weights

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May 10, 2014

Abstract

A counting constraint satisfaction problem (#CSP) asks for the number of ways to satisfy a given list of constraints, drawn from a fixed constraint language \( \Gamma \). We study how hard it is to evaluate this number approximately. There is an interesting partial classification, due to Dyer, Goldberg, Jalsenius and Richerby [DGJR10], of Boolean constraint languages when the degree of instances is bounded by \( d \geq 3 \) - every variable appears in at most \( d \) constraints - under the assumption that “pinning” is allowed as part of the instance. We study the \( d = 2 \) case under the stronger assumption that “variable weights” are allowed as part of the instance. We give a dichotomy: in each case, either the #CSP is tractable, or one of two important open problems, \(#\text{BIS}\) or \(#\text{PM}\), reduces to the #CSP.

*Supported by an EPSRC doctoral training grant.
1 Introduction

A constraint satisfaction problem asks whether there is an assignment of values to some variables that satisfies given constraints. We will be looking at Boolean CSPs, where each variable takes the value 0 or 1. An example of a Boolean CSP is whether a graph has a perfect matching: whether each edge can be labelled 0 or 1 (these are the variables) such that (these are the constraints) at each vertex there is exactly one edge labelled 1.

Given a finite set of relations \( \Gamma \), the counting problem \( \#\text{CSP}(\Gamma) \) asks for the number of assignments that satisfy a conjunction of constraints of the form “\((v_1, \ldots, v_k) \in R\)” with \( R \in \Gamma \). The approximation complexity of \( \#\text{CSP}(\Gamma) \) is the complexity of the same problem but allowing a multiplicative error. Sometimes we will allow weighted constraints, called signatures, and in this case we write \( \mathcal{F} \) instead of \( \Gamma \).

An important feature of the perfect matchings example is that every variable is used twice: the degree of every variable is two. For larger degree bounds \( \#\text{CSP}(\Gamma) \) has been studied in [DGJR10]. The restriction of \( \#\text{CSP}(\mathcal{F}) \) to instances where each variable appears exactly twice has also been called a (non-bipartite) Holant problem [JLX11a].

To make progress on the degree two problem we allow instances to specify a weight for each of the two values each variable can take. The main result of the paper is a hardness result for degree two Boolean \#CSPs with these variable weights: in every case we show that that problem is either tractable or as hard as an important open problem. The core of the proof is that we can adapt the “fan-out” constructions of Feder [Fed01]; this does not work for delta matroids, but delta matroids can be handled specially. Along the way we give a generalisation of delta matroids to weighted constraints called “terraced signatures”. This definition directly describes when a constraint fails to give fan-out gadgets for degree-two \#CSPs.

We also give partial results for signatures and for some related problems.

1.1 Variable weights and degree bounds

We will consider the problem of approximately evaluating a \#CSP where the constraints, variables weights, and degrees are restricted. To discuss these problems it is useful to introduce some notation. For the main theorem we study the problems \( \#\text{CSP}^{\geq 0}_K(\Gamma) \) for a constraint language \( \Gamma \) of Boolean relations. The instances of \( \#\text{CSP}^{\geq 0}_K(\Gamma) \) consist of variable weights and constraints. Variable weights are arbitrary non-negative rationals, constraints are taken from \( \Gamma \), and every variable appears at most twice.

To discuss other results, and to put our results in a wider context, it is useful to generalise from \( \#\text{CSP}^{\geq 0}_K(\Gamma) \). Given a set of non-negative “variable weights” \( W \subseteq \mathbb{R} \times \mathbb{R} \) and a set of degree bounds \( K \subseteq \mathbb{N} \), we then have an approximate counting problem \( \#\text{CSP}^W_K(\Gamma) \): instances consist of a pair of variable weights from \( W \) for each variable, and a set of constraints from the set of relations \( \Gamma \), such that the degree of each variable is an integer in \( K \). To avoid clutter we will use the default values \( W = \{(1, 1)\} \) and \( K = \mathbb{N} \) when they are omitted, and abbreviate \( W = \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \) to \( \geq 0 \), and \( K = \{1, \ldots, d\} \) and \( K = \{d\} \) to \( = d \) and \( \leq d \) respectively. We will in fact generalise to sets of signatures \( \mathcal{F} \) and define \( \#\text{CSP}^W_K(\mathcal{F}) \). See Section 2.5 for more formal definitions.

For example, if we define \( \text{NAND} = \{(0, 0), (0, 1), (1, 0)\} \), then \( \#\text{CSP}(\{\text{NAND}\}) \) is equivalent to the problem of counting independent sets in a graph: the variables \( x_v \) of the CSP correspond to vertices of a (multi)graph, the constraints correspond to edges - there is a constraint \( \text{NAND}(x_u, x_v) \) for each edge \( uv \) of the graph - and the satisfying assignments of the CSP are the indicator functions of independent sets of this graph. As another example, if we define \( \text{PM}_3 = \{(0, 0, 1), (0, 1, 0), (1, 0, 0)\} \)
then \( \#\text{CSP} \equiv_2 (\{\text{PM}_3\}) \) is equivalent to counting perfect matchings of a graph in which every vertex has degree three (by the same encoding discussed previously for perfect matchings as a CSP), and \( \#\text{CSP} \equiv_2 (\{\text{PM}_3\}) \) is equivalent to counting weighted perfect matchings in a graph in which every vertex has degree three.

### 1.2 Main result

In approximation complexity a problem is considered tractable if it has a fully polynomial randomised approximation scheme (FPRAS) - see Section 2.6 for a definition. We will present results using the “AP-reduction” notation \( \leq_{\text{AP}} \) introduced in [DGGJ03]. \#X \leq_{\text{AP}} \#Y means that \#X has an FPRAS using an FPRAS for \#Y as an oracle. This also defines an equivalence relation \#X =_{\text{AP}} \#Y.

The main result states reductions from the problems \#SAT, \#BIS and \#PM to certain \#CSP problems. \#SAT is the problem of counting solutions to a SAT instance; it plays a similar role for approximation problems as NP plays for decision problems. \#BIS is the problem of counting the number of independent sets in a bipartite graph. We do not actually use this definition directly; \#BIS has been used in this way as a “hard” problem since it was introduced in [DGGJ03]. \#PM is the problem of counting perfect matchings in a graph. Finding an FPRAS for \#PM has been an important open research problem, certainly since the restriction of \#PM to bipartite graphs was shown to have an FPRAS [JSV01]. It is therefore a respectable “hard” problem for approximation.

We will give AP-reductions depending on whether \( \Gamma \) falls into certain classes of relations. Briefly, a relation is basically binary if it is a Cartesian product of relations of arity at most two, for example \( \{x \in \{0, 1\}^4 \mid x_1 x_2 = 1 \text{ and } x_3 \leq x_4\} \). A relation is in NEQ-conj if it is a conjunction of equalities, disequalities, and constants, for example \( \{x \in \{0, 1\}^6 \mid x_1 = x_2, x_2 \neq x_5, x_6 = 0\} \). A relation is in IM-conj if it is a conjunction of implications and constants, for example \( \{x \in \{0, 1\}^6 \mid x_1 \leq x_2 \leq x_3, x_6 = 0\} \).

A family \( C \) of subsets of a finite set is a delta matroid if for all \( X, Y \in C \) and \( i \in X \Delta Y \) there exists \( j \in X \Delta Y \) with \( X \Delta \{i, j\} \in C \), where the triangle operator means the symmetric difference. In this paper we will also call the corresponding relations \( R \subseteq \{0, 1\}^V \) delta matroids. For example, the set system \( \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\} \subseteq \{0, 1\}^3 \) is not a delta matroid: it contains \( X = \emptyset \) and \( Y = \{1, 2, 3\} \) but does not contain \( \{3, j\} \) for any \( j \in \{1, 2, 3\} \); hence the corresponding relation \( \{x \in \{0, 1\}^3 \mid x_3 \leq x_1, x_2\} \) is not a delta matroid. On the other hand \( \{x \in \{0, 1\}^3 \mid \sum x_i \in \{0, 2, 3\}\} \) is a delta matroid relation.

Our main theorem says:

**Theorem 1.** Let \( \Gamma \) be a finite set of relations. If \( \Gamma \subseteq \text{NEQ-conj} \) or every relation in \( \Gamma \) is basically binary then \( \#\text{CSP} \leq_2 (\Gamma) \) has an FPRAS. Otherwise,

- If \( \Gamma \subseteq \text{IM-conj} \) then \( \#\text{BIS} =_{\text{AP}} \#\text{CSP} \leq_2 (\Gamma) \).

- If \( \Gamma \not\subseteq \text{IM-conj} \) then \( \#\text{PM} \leq_{\text{AP}} \#\text{CSP} \leq_2 (\Gamma) \). If furthermore \( \Gamma \) is not a set of delta matroids then \( \#\text{SAT} =_{\text{AP}} \#\text{CSP} \leq_2 (\Gamma) \).

So in every case the problem is either tractable, or at least as hard as an important open problem. This is quite a different situation from the corresponding decision problems, considered in [DF03]. For degree-two decision CSP there is no known dichotomy, and there are many tractable problems using delta matroids.
1.3 Other results

These classes or relations, and the proof of Theorem 1 generalises to some extent to signatures. There is a similar notion of basically binary signatures. NEQ-conj generalises to Weighted-NEQ-conj, and IM-conj generalises to the class of logsupermodular signatures (these classes were used in the result of Bulatov et al. mentioned below). We will define a generalisation of delta matroids called “terraced” signatures. We establish the following results in Section 6.

**Theorem 2.** Let $F$ be a finite set of signatures. If every signature in $F$ is basically binary or every signature in $F$ is in Weighted-NEQ-conj, then $\#\text{CSP}^\geq_{\leq 2}(F)$ has an FPRAS. Otherwise assume furthermore that there is a signature in $F$ that is not terraced or that does not have basically binary support. Then:

- If every signature in $F$ is logsupermodular then $\#\text{BIS} \leq_{AP} \#\text{CSP}^\geq_{\leq 2}(F)$.
- If some signature in $F$ is not logsupermodular then $\#\text{PM} \leq_{AP} \#\text{CSP}^\geq_{\leq 2}(F)$. If furthermore some signature in $F$ is not terraced then $\#\text{SAT} =_{AP} \#\text{CSP}^\geq_{\leq 2}(F)$.

The case of terraced signatures whose support is basically binary is left as an open problem. Note that this theorem is stated for $\#\text{CSP}=2$ problems: every variable is used exactly twice and not just at most twice.

**Theorem 3.** Let $F$ be a finite set of signatures. Assume that not every signature in $F$ is in Weighted-NEQ-conj, and not every signature in $F$ is basically binary, and not every signature in $F$ is terraced. (This the same setting as the $\#\text{BIS}$ and $\#\text{SAT}$ reductions in Theorem 2.)

Unless all the following conditions hold, there is a finite set $W \subseteq \mathbb{R}^p \times \mathbb{R}^p$ such that $\#X \leq_{AP} \#\text{CSP}^W_{\leq 2}(F)$ where $\#X = \#\text{BIS}$ if every signature in $F$ is logsupermodular, and $\#X = \#\text{SAT}$ otherwise.

1. Every signature $F \in F$ is IM-terraced.

2. Either the support of every signature $F$ in $F$ is closed under meets ($x, y \in \text{supp}(F) \implies x \land y \in \text{supp}(F)$), or the support of every signature $F$ in $F$ is closed under joins ($x, y \in \text{supp}(F) \implies x \lor y \in \text{supp}(F)$).

3. No pinning of the support of a signature in $F$ is equivalent to $\text{EQ}_2$.

This situation is simpler for higher degrees, if $F$ contains a signature with non-degenerate support (a relation is degenerate if it is a product of arity 1 relations):

**Theorem 4.** Let $F$ be a finite set of signatures and assume that not every signature in $F$ has degenerate support. There exists a finite set of variable weights $W$ such that $\#\text{CSP}^W_{\leq 3}(F)$ has an FPRAS if and only if $\#\text{CSP}^W_{\leq 2}(F)$ has an FPRAS.

So under these assumptions, by the theorem of Bulatov et al. mentioned below, the tractable cases are just what can be computed exactly (unless $\#\text{BIS}$ has an FPRAS). On the other hand, we show that the tractable region has positive measure, loosely speaking, for all $d \geq 2$:

**Theorem 5.** Let $d, k \geq 2$. Let $F$ be a an arity $k$ signature with values in the range $[1, d^{(k-1)+1}/d^{(k-1)-1}]$. Then $\#\text{CSP}^\geq_{\leq d}(F)$ has an FPRAS.
### 1.4 Related work

The problem \( \#\text{CSP}_{\leq d}(\Gamma) \) for \( d \geq 3 \) was studied in [DGJR10]. In particular:

**Theorem.** [DGJR10, Theorem 24] Let \( \Gamma \) be a finite set of relations and let \( d \geq 6 \).

- If every \( R \in \Gamma \) is affine then \( \#\text{CSP}_{\leq d}(\Gamma \cup \Gamma_{\text{pin}}) \in \text{FP} \).
- Otherwise, if \( \Gamma \subseteq \text{IM-conj} \) then \( \#\text{CSP}_{d}(\Gamma \cup \Gamma_{\text{pin}}) =_{\text{AP}} \#\text{BIS} \).
- Otherwise, there is no FPRAS for \( \#\text{CSP}_{\leq d}(\Gamma \cup \Gamma_{\text{pin}}) \) unless \( \text{NP} = \text{RP} \).

Here \( \Gamma_{\text{pin}} = \{\{(0)\}, \{(1)\}\} \) and a relation is called affine if it is an affine subspace of \( \mathbb{F}_2^5 \).

[DGJR10, Theorem 24] can be seen as an extension of the following result of Bulatov et al [BDGJ12], which we also rely on in the proof:

**Lemma 6.** [BDGJ12, Theorem 16] Let \( \mathcal{F} \) be a finite set of signatures. If \( \mathcal{F} \) is not a subset of Weighted-\( \text{NEQ-conj} \) then for any finite subset \( S \) of arity-one signatures there is an FPRAS for \( \#\text{CSP}(\mathcal{F} \cup S) \). Otherwise,

- there is a finite subset \( S \) of arity-one signatures such that \( \#\text{BIS} \leq_{\text{AP}} \#\text{CSP}(\mathcal{F} \cup S) \), and
- if there is a function in \( \mathcal{F} \) that is not logsupermodular then there is a finite subset \( S \) of arity-one signatures such that \( \#\text{SAT} =_{\text{AP}} \#\text{CSP}(\mathcal{F} \cup S) \).

Note that arity one signatures are the same as variable weights for unbounded degree \( \#\text{CSPs} \). But when the degree is restricted, arity one signatures seem less powerful.

Feder [Fed01] showed that relations that are not delta matroids give “fan-out”: if \( \Gamma \) contains a relation that is not a delta matroid, and the decision problem \( \text{CSP}(\Gamma) \) is NP-complete, then the restriction of \( \text{CSP}(\Gamma) \) to degree two instances is also NP-complete. Theorems [1] and [2] use a similar kind of fan-out idea. The latest results on degree-two CSPs were given in [DF03].

There are some important results on two-state spin systems that are worth translating into the \( \#\text{CSP}_K \) notation. As mentioned earlier, Sly [Sly10] showed that for the hard-core model there is a computational transition at the “tree threshold” \( \lambda^*(d) \). It was known that the problem \( \#\text{CSP}_{\leq 6}(\Gamma_{\text{NAND}}) \) has a (deterministic) FPRAS for \( \lambda < \lambda^*(d) \). Sly showed that it does not have an FPRAS for \( \lambda > \lambda^*(d) \) unless \( \text{NP} = \text{RP} \) (with some technical restrictions on \( \lambda \)). This result has been extended recently [SST12] considering other models and removing the restrictions. On the other hand there are FPRASes for variants of \( \#\text{CSP}_K(B) \) for various symmetric binary signatures \( B \): see [SST11] for \( K = \{1, \ldots , d\} \) and [LLY12] for \( K = \mathbb{N} \).

To discuss other work it is useful to define some notation temporarily. Define \( \text{Holant}(\mathcal{F}) = \#\text{CSP}_{=2}(\mathcal{F}) \), and \( \text{Holant}^c \) is the same except that any arity one relation can be used, and \( \text{Holant}^* \) is the same except that any arity one complex-valued signature can be used. \( \text{Holant}^* \) was introduced in [ILX11a] to give results about the exact counting complexity (not allowing multiplicative error) of \( \text{Holant}^c \) problems. A dichotomy theorem for the exact counting complexity of \( \text{Holant}^* \) problems was given in [ILX11b], classifying each problem as polynomial-time computable or \#P-hard.

Yamakami [Yam11] studied the approximation complexity of \( \text{Holant}^*(\{F\}) \) (referring to it as \( \#\text{CSP}_\mathcal{F}^c \)) where \( F \) is in a certain set of arity three complex-valued signatures. It would be too much of a detour to present those results fully, but the conclusion is that these problems are either tractable or there is a certain approximation-preserving reduction from the problem \( \#\text{SAT}^c(K) \) (analogous to \( \#\text{SAT} \)) to \( \text{Holant}^*(\{F\}) \). Note that the node weight functions used to define \( \#\text{SAT}^c(K) \) are like variable weights, but the problems \( \#\text{CSP}^c \) and \( \text{Holant}^* \) defined in that paper do not use variable weights, but arity one signatures. In the same setting there are results for higher degree bounds [Yam10].
2 Definitions

$V$ will usually denote a finite set whose elements are called variables. Elements of $\{0,1\}^V$ will be called configurations of $V$. In this paper a relation $R$ on $V$ is a subset $R \subseteq \{0,1\}^V$. In this paper a signature $F$ on $V$ is a function $F : \{0,1\}^V \to \mathbb{R}_p$, where $\mathbb{R}_p$ is the set of non-negative polynomial-time computable reals, that is, non-negative reals $r$ for which there is a polynomial-time Turing machine that when given an integer $n$ in unary, outputs the first $n$ bits of the binary expansion of $r$. The set $V = V(R) = V(F)$ is called the variable set; the arity is $|V|$, and configurations in $\{0,1\}^k$ for integers $k$ are considered to have variable set $\{1, \ldots, k\}$.

We can rename the variables in an obvious way. (For any finite set $V'$, a bijection $\pi : V \to V'$ induces a bijection $\pi_*$ from relations (or signatures) on $V$ to relations (or signatures) on $V'$.) We will say that relations (or signatures) are equivalent if they are related by renaming variables. The difference between equivalent relations (or signatures) is never important in this paper, but keeping track of $V$ makes some arguments easier.

We will implicitly convert relations to signatures, so $R(x) = 1$ if $x \in R$ and $R(x) = 0$ otherwise. However, if $R$ is given in set notation we will instead use the more legible notation $1_R(x) = R(x)$.

It is useful to have special notation for inverting components of a configuration. For all $x \in \{0,1\}^V$ and all subsets $U \subseteq V$ define the flip $x^U \in \{0,1\}^V$ by $x^U_v = \bar{x}_v$ if and only if $v \in U$. A relation $R$ or signature $F$ can also be flipped: $x \in R^U$ if and only if $x^U \in R$, and $F^U(x) = F(x^U)$.

Also, by abuse of notation, for configurations $x, y \in \{0,1\}^V$, the set of elements on which $x$ and $y$ differ will be denoted $x \triangle y$.

We will use 0 and 1 to mean the all-zero and all-one configurations on some variable set. The complement $\overline{x}$ of a configuration $x$ is defined by $\overline{x}_i = 1 - x_i$. Define the meet $x \land y$ and join $x \lor y$ of configurations $x, y \in \{0,1\}^V$ by $(x \land y)_i = \min(x_i, y_i)$ and $(x \lor y)_i = \max(x_i, y_i)$.

2.1 Relations

Let $R \subseteq \{0,1\}^V$ be a relation. $R$ is an equality if it is of the form $\{x : x_i = x_j\}$. $R$ is a disequality if it is of the form $\{x : x_i \neq x_j\}$. $R$ is a pin if it is of the form $\{x : x_i = c\}$. $R$ is an implication if it is of the form $\{x : x_i \leq x_j\}$. Here $i, j \in V$ and $c \in \{0,1\}$.

Define $\text{NEQ-conj}$ to be the class of relations that are conjunctions of equalities, disequalities, and pins. Define $\text{IM-conj}$ to be the class of relations that are conjunctions of implications and pins; we will often use the characterisation that a relation is in IM-conj if and only if it is closed under meets and joins ([DGJ10, Corollary 18]). $R$ is a delta matroid if for all $x, y \in R$ and for all $i \in x \triangle y$ there exists $j \in x \triangle y$, not necessarily distinct from $i$, such that $x_{\{i,j\}} \in R$.

A non-empty relation $R$ on a non-empty variable set is decomposable if it is equivalent to the Cartesian product of at least two relations of arity at least one. Otherwise it is indecomposable. A relation is defined to be degenerate if it is equivalent to the Cartesian product of relations of arity at most one. A relation is defined to be basically binary if it is equivalent to the Cartesian product of relations of arity at most two.

We will use the following relations. $\text{EQ}_k = \{0,1\} \subseteq \{0,1\}^k$, $\text{NEQ} = \{(0,1),(1,0)\}$, $\text{PIN}_0 = \{(0)\}$, $\text{PIN}_1 = \{(1)\}$, $\text{NAND} = \{(0,0),(0,1),(1,0)\}$, $\text{OR} = \{(0,1),(1,0),(1,1)\}$, and $\text{IMP} = \{(0,0),(0,1),(1,1)\}$. Also $\text{PM}_k = \{x \in \{0,1\}^k : x_1 + \cdots + x_k = 1\}$.
2.2 Pinnings

A partial configuration $p$ of $V$ is defined to be an element of $\{0, 1\}^{\text{dom}(p)}$ for some subset $\text{dom}(p) \subseteq V$. If $x \in \{0, 1\}^{V \setminus \text{dom}(p)}$ then $(x, p)$ means the unique common extension of $x$ and $p$ to a configuration of $V$. Let $R \subseteq \{0, 1\}^V$ and let $p$ be a partial configuration of $V$. Define the (relation) pinning $R_p \subseteq \{0, 1\}^{V \setminus \text{dom}(p)}$ by $x \in R_p \iff (x, p) \in R$. Let $F : \{0, 1\}^V \to \mathbb{R}_p$ and let $p$ be a partial configuration of $V$. Define the (signature) pinning $F_p : \{0, 1\}^{V \setminus \text{dom}(p)} \to \mathbb{R}_p$ by $F_p(x) = F(x, p)$. In the delta matroid literature, the set system representation of a pinning is called a minor.

2.3 Signatures

Let $V$ and $V'$ be finite sets, and let $V \sqcup V'$ be their disjoint union. The tensor product $F \otimes G : \{0, 1\}^{V \sqcup V'} \to \mathbb{R}_p$ of two signatures $F : \{0, 1\}^V \to \mathbb{R}_p$ and $G : \{0, 1\}^{V'} \to \mathbb{R}_p$ is defined by $(F \otimes G)(x, x') = F(x)G(x')$ for all $x \in \{0, 1\}^V$ and $x' \in \{0, 1\}^{V'}$. We can define the tensor product of $m$ signatures $\otimes_{i=1}^m F_i = F_1 \otimes (F_2 \otimes \cdots \otimes (F_{m-1} \otimes F_m) \cdots)$. A signature is decomposable if it is equivalent to a tensor product of signatures of arity at least one. Otherwise it is indecomposable.

A signature is defined to be degenerate if it is equivalent to the tensor product of two signatures of arity one. A signature is defined to be basically binary if it is equivalent to the product of signatures of arity at most two.

$F'$ is a simple weighting of $F$ if $F'$ is the pointwise product $F'(x) = F(x)D(x)$ of $F$ with a degenerate signature $D$. Define Weighted-NEQ-conj to be the class of simple weightings of NEQ-conj relations - see Proposition 7 for how this related to Lemma 6. A signature $F : \{0, 1\}^V \to \mathbb{R}_p$ is logsupermodular if it satisfies $F(x \land y)F(x \lor y) \geq F(x)F(y)$ for all $x, y \in \{0, 1\}^V$.

We now come to the definition of terraced signatures, which are signatures such that the reductions in Section 6 (ultimately Lemma 27) fail. In Lemma 18 we will show that a relation is terraced if and only if it is a delta matroid, so we are defining a weighted generalisation of delta matroids.

A signature $F : \{0, 1\}^V \to \mathbb{R}_p$ is terraced if for all partial configurations $p$ of $V$ and all $i, j$ in the domain of $p$, if $F_p$ is identically zero then $F_p(i)$ and $F_p(j)$ are linearly dependent, that is, one is a scalar multiple of the other. The scalars can depend on $i$ and $j$. A signature $F : \{0, 1\}^V \to \mathbb{R}_p$ is IM-terraced if for all partial configurations $p$ of $V$ and all $i, j$ in the domain of $p$ such that $p_i \neq p_j$, if $F_p$ is identically zero then $F_p(i)$ and $F_p(j)$ are linearly dependent.

Let $V$ be a finite set, let $F : \{0, 1\}^V \to \mathbb{R}_p$ and let $h : V \to \mathbb{Z}$. Define the $h$-maximisation $F_{\text{h-max}} : \{0, 1\}^V \to \mathbb{R}_p$ by setting $F_{\text{h-max}}(x) = F(x)$ for all configurations $x$ of $V$ such that $\sum_i x_i h_i = \max_{y \in \text{supp}(F)} \sum y_i h_i$, and setting $F_{\text{h-max}}(x) = 0$ otherwise.

2.4 K-formulas

Our \#CSP instances will use a “primitive product summation (pps)” formula as in [BDGJ12]. These can be thought of as formal summations of products of function applications such as $\sum_y \text{NEQ}(x, y)\text{NEQ}(y, z)$.

For a set of signatures $\mathcal{F}$, a pps-formula $\phi$ over $\mathcal{F}$ consists of an external variable set $V = V^\phi$, an internal variable set $U = U^\phi$ disjoint from $V$, a set of atomic formula indices $I = I^\phi$, a signature $F_i = F_i^\phi \subseteq \mathcal{F}$ for each $i \in I$, and scope variables $\text{scope}(i, j) = \text{scope}^\phi(i, j) \in U \cup V$ for each $i \in I$ and $j \in V(F_i)$. The data associated to an index $i \in I$ $(F_i$ and $\text{scope}(i, j)$ for $j \in V(F_i))$ is called an atomic formula, denoted by a formal function application like $F_i(v_1, v_2, v_3)$. We will manipulate pps-formulas by inserting or deleting atomic formulas to obtain a new pps-formula.
Define $Z_\phi : \{0, 1\}^V \to \mathbb{R}_p$ as follows: for all configurations $x$ of $V$,

$$Z_\phi(x) = \sum \prod_{i \in I} F_i((x_{\text{scope}(i,j)})_{j \in V(F_i)})$$

The sum is over all extensions of $x$ to a configuration of $U \cup V$, and the notation $(x_{\text{scope}(i,j)})_{j \in V(F_i)}$ means the configuration in $\{0, 1\}^{V(F_i)}$ given by the composition $V(F_i) \xrightarrow{\text{scope}(i, \bullet)} V \xrightarrow{x} \{0, 1\}$.

This gives a quick way to specify all the data. The pps-formula given by

$$Z_\phi(x_1, \cdots, x_n) = \sum_{x_{n+1}, \cdots, x_{n+m} \in I} \prod_{i \in I} F_i(x_{\text{scope}(i,1)}, \cdots, x_{\text{scope}(i,a_i)})$$

for all $x_1, \cdots, x_n \in \{0, 1\}$, is the pps-formula with $V = \{1, \cdots, n\}$ and $U = n + 1, \cdots, n + m$ and the given $I, F_i$ and scope. (For this to make sense we must have $V(F_i) = \{1, \cdots, a_i\}$ for each $i \in I$, and the scope($i, j$) values must fall in $\{1, \cdots, n + m\}$.) We will say a signature $G$ is defined by a pps-formula over $\mathcal{F}$ if $G = Z_\phi$ for some pps-formula $\phi$ over $\mathcal{F}$. The variables do not have to be called $x_1, \cdots, x_{n+m}$; for example we could say that EQ$_2$ is defined by a pps-formula over $\{\text{NEQ}\}$ because

$$\text{EQ}_2(x, z) = \sum_y \text{NEQ}(x, y)\text{NEQ}(y, z)$$

for all $x, z \in \{0, 1\}$.

The degree $\deg_\phi(v)$ of an internal or external variable $v \in U \cup V$ is the number of times it occurs: the number of pairs $(i, j)$ such that scope$(i, j) = v$. For any subset $K$ of natural numbers, a $K$-formula is a pps-formula where if $K \not= \mathbb{N}$ then: the degree of every internal variable is in $K$, and the degree of every external variable is 1. ($\leq d$)-formulas and ($= d$)-formulas are $K$-formulas with $K = \{1, \cdots, d\}$ and $K = \{d\}$ respectively. As above we can say the $K$-formula given by some equation of the form (2,4), and we can say a signature is defined by a $K$-formula over $\mathcal{F}$.

**Proposition 7.** A signature $F : \{0, 1\}^k \to \mathbb{R}_p$ is in Weighted-NEQ-conj if and only if $F = Z_\phi$ for some pps-formula using EQ$_2$, NEQ and arity 1 signatures. Hence the version of Lemma 6 given in the introduction is a faithful translation.

**Proof.** For the forward direction it is easy to construct such a formula $\phi$. For the backward direction it will be convenient to first note a few properties of Weighted-NEQ-conj. In an indecomposable NEQ-conj relation $R$, every two variables are related by a chain of equalities and disequalities, so $R \subseteq \{x, \overline{x}\}$ for some $x$. An indecomposable signature in Weighted-NEQ-conj must have indecomposable support, so an indecomposable signature in Weighted-NEQ-conj has support of cardinality at most two.

Conversely, it is easy to check that any relation of cardinality at most two is in NEQ-conj, and any signature $F$ whose support has cardinality at most two is in Weighted-NEQ-conj. We can now check each stage of the expression for $Z_\phi$: (1.) If $F$ is in Weighted-NEQ-conj then so is $F''(x) = F((x_{\text{scope}(i,j)})_{j \in V(F)})$. (2.) If two signatures are in Weighted-NEQ-conj then so is their pointwise product. (3.) If $F(t, x)$ is in Weighted-NEQ-conj then so is $F''(x) = \sum F(t, x)$. The first two stages are obvious from the definition of Weighted-NEQ-conj. For the third stage, note that Weighted-NEQ-conj is closed under tensor products so we can assume that $F$ is indecomposable. Then $|\text{supp}(F')| \leq |\text{supp}(F)| \leq 2$ so $F'$ is in Weighted-NEQ-conj. 

---

1. if $K = \mathbb{N}$ then degrees do not matter, so we allow any pps-formula and do not insist that the external variables have degree 1
2. This is similar to “realizing” a signature in [JLX11a], and $T$-constructibility in [Yam11].
2.5 #CSPs

We will now formalise the definitions given in the introduction.

We will call $W$ a set of variable weights if one of the following conditions holds.

- $W \subseteq \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}$; elements of $W$ will be specified as binary fractions. The binary representation is important - see Section 9.

- $W$ is a finite subset of $\mathbb{R}_p \times \mathbb{R}_p$; elements of $W$ will be specified by their index in a fixed enumeration.

Let $\mathcal{F}$ be a finite set of signatures, let $W$ be a set of variable weights and let $K$ be a set of positive integers. A $\#\text{CSP}_K^W(\mathcal{F})$ instance $(w, \phi)$ consists of a function $w : V \to W$, and a $K$-formula $\phi$ with no external variables and with internal variables $V$, where $V = V^\phi$. The value of the instance is

$$Z^w_\phi = \sum_{x : V \to \{0,1\}} \left( \prod_{v \in V} w(v)x_v \right) \left( \prod_{i \in I} F_i((x_{\text{scope}(i,j)})_{j \in V(F_i)}) \right)$$

where the $I, F_i, \text{scope}$ are given by $\phi$. If $W = \{(1,1)\}$ we will omit $w$, so the instance is $\phi$ and the output is $Z_\phi$ (a slight abuse of notation - here $Z_\phi$ means the value of $Z_\phi$ applied to the arity zero configuration). It will occasionally be useful to refer to the contribution $\text{wt}_\phi^w(x)$ of a configuration $x$:

$$\text{wt}_\phi^w(x) = \left( \prod_{v \in V} w(v)x_v \right) \left( \prod_{i \in I} F_i((x_{\text{scope}(i,j)})_{j \in V(F_i)}) \right)$$

2.6 Approximation complexity

The paper [DGGJ03] introduced an analogue of Turing reductions for approximation problems, which we repeat here (except that we generalise by allowing $f$ to take non-integer values, as in [BDGJ12]).

A randomised approximation scheme for a function $f : \Sigma^* \to \mathbb{R}_p$ is a probabilistic Turing machine (TM) that takes as input a pair $(x, \epsilon) \in \Sigma^* \times (0,1)$ and produces as output an rational random variable $Y$ satisfying the condition $\Pr(\exp(-\epsilon)f(x) \leq Y \leq \exp(\epsilon)f(x)) \geq 3/4$. A randomised approximation scheme is said to be fully polynomial if it runs in time $\text{poly}(|x|, \epsilon^{-1})$. The phrase “fully polynomial randomised approximation scheme” is usually abbreviated to FPRAS.

Let $f, g : \Sigma^* \to \mathbb{R}_p$ be functions whose complexity (of approximation) we want to compare. An approximation-preserving reduction from $f$ to $g$ is a probabilistic oracle TM $M$ that takes as input a pair $(x, \epsilon) \in \Sigma^* \times (0,1)$, and satisfies the following three conditions: (i) every oracle call made by $M$ is of the form $(w, \delta)$, where $w \in \Sigma^*$ is an instance of $g$, and $0 < \delta < 1$ is an error bound satisfying $\delta^{-1} \leq \text{poly}(|x|, \epsilon^{-1})$; (ii) the TM $M$ meets the specification for being a randomised approximation scheme for $f$ whenever the oracle meets the specification for being a randomised approximation scheme for $g$; and (iii) the run-time of $M$ is polynomial in $|x|$ and $\epsilon^{-1}$. If an approximation-preserving reduction from $f$ to $g$ exists we write $f \leq_{\text{AP}} g$, and say that $f$ is AP-reducible to $g$. If $f \leq_{\text{AP}} g$ and $g \leq_{\text{AP}} f$ then we write $f =_{\text{AP}} g$.

3 Reductions

This section establishes some reductions between #CSPs.
We will often implicitly use the fact that \( \#\text{CSP}_K^W(F) \leq_{AP} \#\text{CSP}_K^{W'}(F) \) whenever \( K \subseteq K' \) and either \( W \subseteq W' \) or \( W' = \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \). The reduction is trivial except in the case where \( W \) consists of a finite set of polynomial-time computable variable weights and \( W' = \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \); in this case the reduction just needs to choose good enough approximations to the variable weights in \( W \).

\( K \)-formulas are designed to be used as gadgets in the following sense.

**Lemma 8.** Let \( F \) be a finite set of signatures. Let \( W \) be a set of variable weights containing \((1,1)\). Let \( K \subseteq \mathbb{N} \). Let \( \psi \) be a \( K \)-formula. Then \( \#\text{CSP}_K^W(F \cup \{ Z_{\psi} \}) \leq_{AP} \#\text{CSP}_K^W(F) \).

**Proof.** Given an instance \((w, \phi)\) of \( \#\text{CSP}(F \cup \{ Z_{\psi} \}) \), for each atomic formula \( Z_{\psi}(s) \), delete that atomic formula and insert a copy of each atomic formula in \( \psi \), renaming the external variables \( v \in V(\psi) \) to \( s(v) \) and renaming the internal variables of \( \psi \) to fresh variables. This process gives a new instance \((w', \phi')\) over \( F \) on a possibly larger variable set \( V' \), where we extend \( w \) to \( w' \) by setting \( w'(v, 0) = w'(v, 1) = 1 \) for all new variables \( v \).

In terms of \( Z_{\phi'}^w \), this process has the effect of replacing each use of \( Z_{\psi} \) by its summation-of-product definition and distributing out the sums over the internal variables. By distributivity \( Z_{\phi'}^w = Z_{\phi''}^{w'} \), and the degrees are all still in \( K \) so we can call the oracle on \( Z_{\phi''}^{w'} \) without changing the error parameter \( \epsilon \).

The following reduction is an important step in the proof of Theorem 1: it shows that we can get \( \text{PM}_3 \) from \( \{(x_1, x_2, x_3) \in \{0, 1\}^3 \mid x_1 + x_2 + x_3 \leq 1\} \) for example, unlike in the finite \( W \) setting of Section 3.

**Lemma 9.** Let \( F \) be a finite subset of signatures. Let \( G \in F \) and let \( h : V \to \mathbb{Z} \) where \( V \) is the variable set of \( G \). Let \( W = \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0} \) (we will also allow \( W = \{(2^a, 2^b) \mid a, b \in \mathbb{Z}\} \) for the proof of Theorem 3). Then \( \#\text{CSP}_K^W(F \cup \{ G_{h-\text{max}} \}) \leq_{AP} \#\text{CSP}_K^W(F) \).

**Proof.** The reduction is given an instance \((w, \phi)\) of \( \#\text{CSP}_K^W(F \cup \{ G_{h-\text{max}} \}) \) and error parameter \( \epsilon \) which we can assume is less than \( 1/2 \). We wish to compute a value \( Z \) such that \( \exp(-\epsilon)Z \leq Z_{\phi}^w \leq \exp(\epsilon)Z \).

Let \( s = |V| + |I^\phi| \) be the total number of variables and atomic formulas in \( \phi \). Let \( M \) be the maximum over: the values taken by signatures in \( F \), and the values \( w(v, i) \), and the value 1. Let \( m \) be the minimum over: the non-zero values taken by signatures in \( F \), and the non-zero values \( w(v, i) \), and the value 1. Let \( H \) be the maximum of \( \sum_i x_i h_i \) over \( x \in \text{supp}(F) \). Define \( G^{(n)} \) for all \( n \geq 0 \) by

\[
G^{(n)}(x) = G(x)2^n(\sum_i x_i h_i - H)
\]

Note that for all \( x \) either: \( \sum_i x_i h_i = H \) so \( G^{(n)}(x) = G_{h-\text{max}}(x) \), or \( \sum_i x_i h_i < H \) so \( G_{h-\text{max}}(x) = 0 \) and \( G^{(n)}(x) \leq M2^{-n} \).

Let \( n = \lceil |V| + s \log M - \log_2(m^s \epsilon/4) \rceil \). Note that \( 2^{|V|+s \log M - n} \leq (\epsilon/4)m^s \). Let \( I' \) be the set of atomic formula indices such that \( F_i^0 = G_{h-\text{max}} \). Let \( \phi' \) be the same as \( \phi \) except that \( F_i^{\phi'} = G^{(n)} \) for each \( i \in I' \).

Let \( Z = Z_{\phi}^w \) and \( Z' = Z_{\phi'}^{w'} \). We can approximate \( Z' \) using the oracle by replacing \( G^{(n)} \) by variable weights and \( G \). Specifically, let \( \phi'' \) be the same as \( \phi \) except that \( F_i^{\phi''} = G \) for each \( i \in I' \). For each variable \( v \) let \( h(v) \) be the sum of \( h_j \) over all \( i \in I' \) and \( j \in V(G) \) such that \( \text{scope}_{\phi'}(i, j) = v \). Let \( w''(v, 0) = w'(v, 0) \) and \( w''(v, 1) = w'(v, 1)2^{n h(v)} \). Then \( Z_{\phi''}^{w''} = Z_{\phi'}^{w'} 2^{n M |I'|} \). Call the oracle on \((w'', \phi'')\) with error parameter \( \epsilon/2 \) and divide the result by \( 2^{nM |I'|} \) to obtain a value \( Z'' \) such that \( \exp(-\epsilon/2)Z' \leq Z'' \leq \exp(\epsilon/2)Z' \) with probability at least \( 3/4 \).
For all configurations \( x \), if \( wt^w_\phi(x) \neq wt^w_\phi'(x) \) then \( wt^w_\phi(x) = 0 \) and \( wt^w_\phi(x) \leq M^s2^{-n} \). Hence

\[
|Z - Z'| \leq 2^{V[s]\log M - n} \leq M^s(\epsilon/4)
\]

If \( Z \neq 0 \) then \( Z'' > Z'/2 > Z/4 \geq M^s/4 \). The reduction can therefore output zero whenever \( Z'' \leq M^s/4 \). If \( Z = 0 \) then \( Z'' \leq 2Z' \leq M^s(\epsilon/2) < M^s/4 \) (for \( \epsilon < 1/2 \)). So if \( Z'' > M^s/4 \) then we can assume \( Z \neq 0 \). In this case we have \( |Z - Z'| \leq Z(\epsilon/4) \). Since \( e^{-\epsilon/2} \leq 1 - \epsilon/4 \) for \( \epsilon < 2 \),

\[
(1 - \epsilon/4)Z \leq Z' \leq (1 + \epsilon/4)Z
\]

\[
\exp(-\epsilon/2)Z \leq Z' \leq \exp(\epsilon/2)Z
\]

\[
\exp(-\epsilon)Z \leq Z'' \leq \exp(\epsilon)Z
\]

In this case the reduction can output \( Z'' \). \( \Box \)

Known polynomial-time algorithms can easily be modified to allow variable weights:

**Lemma 10.** [BDGJ12, Theorem 16], [JLX11b, Theorem 2.2] Let \( \mathcal{F} \) be a finite set of signatures. If \( \mathcal{F} \) is contained in Weighted-NEQ-conj then \( \#\text{CSP}^{\geq 0}(\mathcal{F}) \) has an FPRAS. If every signature in \( \mathcal{F} \) is basically binary, then \( \#\text{CSP}^{\geq 0}_{\leq 2}(\mathcal{F}) \) has an FPRAS. In fact these problems are in \( \text{FP} \), at least if the signatures in \( \mathcal{F} \) take rational values.

The following Lemma is useful for showing that a problem is AP-equivalent to \( \#\text{SAT} \).

**Lemma 11.** Let \( \mathcal{F} \) be a finite set of signatures. Then \( \#\text{CSP}^W_K(\mathcal{F}) \leq_{AP} \#\text{SAT} \).

**Proof.** We can approximate the values in the signatures and variables weights by rationals, and by scaling we can assume the values are in fact integers. The problem of evaluating a \#CSP, with explicit integer-valued signatures as part of the input, is in \#P and hence AP-reduces to \#SAT - see the remarks in Section 3 of [DGGJ03]. \( \Box \)

We will use pinning throughout. The following Lemma shows that we do not need to assume that \( \text{PIN}_0, \text{PIN}_1 \) are part of the constraint language.

**Lemma 12.** Let \( K \) be any non-empty set. Let \( \mathcal{F} \) be a finite set of signatures. Let \( W \) be a set of variable weights containing \((1,0)\) and \((0,1)\). Then

\[
\#\text{CSP}^W_K(\mathcal{F} \cup \{\text{PIN}_0, \text{PIN}_1\}) \leq_{AP} \#\text{CSP}^W_K(\mathcal{F})
\]

where \( \mathcal{F}' \) is the set of pinnings of signatures in \( \mathcal{F} \).

**Proof.** Let \( G_0 \in \mathcal{F} \) be a signature with \( \text{supp}(G_0) \not\subseteq \{1\} \) and let \( G_1 \in \mathcal{F} \) be a signature with \( \text{supp}(G_1) \not\subseteq \{0\} \). If these do not exist then \( \#\text{CSP}^W_K(\mathcal{F} \cup \{\text{PIN}_0, \text{PIN}_1\}) \) has an FPRAS.

First we will establish that there is a \( K \)-formula \( \psi \) over \( \mathcal{F} \), of some arity \( d \), such that \( 0 \in Z_\psi \). Indeed there exists \( z \in \text{supp}(G_0) \) and \( i \in V(G_0) \) such that \( z_i = 0 \). We may assume \( i = 1 \) and \( V(G_0) = \{1, \ldots, k\} \) for some \( k \). Then pick \( d \in K \) and let \( \psi \) be the \{(d)\}-formula defined by

\[
Z_\psi(x_1, \ldots, x_d) = \sum_{y_2, \ldots, y_k} \prod_{i=1}^d G(x_i, y_2, \ldots, y_k)
\]

By choice of \( z \) we have \( 0 \in Z_\psi \).
We will first show that
\[
\#\text{CSP}_K^W(\mathcal{F} \cup \{\text{PIN}_0\}) \leq_{AP} \#\text{CSP}_K^W(\mathcal{F} \cup \{Z_\psi\})
\]
The reduction is given an instance \((w, \phi)\) of \(#\text{CSP}_K^W(\mathcal{F} \cup \{\text{PIN}_0\})\). By scaling - keeping track of an overall multiplicative constant - we can assume that if there is an atomic formula \(\text{PIN}_0(v)\) in \(\phi\) then \(w(v, 0) = 1\) and \(w(v, 1) = 0\). Take \(d\) copies of this instance, but for each atomic formula \(\text{PIN}_0(v)\) in \(\phi\), rather than taking its \(d\) copies \(\text{PIN}_0(v_1) \cdots \text{PIN}_0(v_d)\), insert the atomic formula \(Z_\phi(v_1, \ldots, v_d)\) where the scope consists of the \(d\) copies of \(v\). This process gives an instance \((w', \phi')\) of \(#\text{CSP}_K^W(\mathcal{F} \cup \{Z_\psi\})\). Let \(s\) be the number of \(\text{PIN}_0\) atomic formulas in \(\phi\). Then \(Z^{w'}_{\phi'} = Z_\psi(0)^s (Z^w_\phi)^d\). So we get an approximation to \(Z^{w'}_{\phi'}\) within ratio \(e^c\) by asking the oracle for an approximation to \(Z^w_\phi\) to within ratio \(e^{d/e}\).

Using Lemma 8, and by a symmetric argument to get \(\text{PIN}_1\), we have
\[
\#\text{CSP}_K^W(\mathcal{F} \cup \{\text{PIN}_0, \text{PIN}_1\}) \leq_{AP} \#\text{CSP}_K^W(\mathcal{F})
\]
Pinnings can be expressed as \(K\)-formulas using \(\{\text{PIN}_0, \text{PIN}_1\}\), so again by Lemma 8
\[
\#\text{CSP}_K^W(\mathcal{F}' \cup \{\text{PIN}_0, \text{PIN}_1\}) \leq_{AP} \#\text{CSP}_K^W(\mathcal{F})
\]

When dealing with finite sets of variables weights in Theorem 8 it will be useful to be able to assume \(W = \{(1, 1)\}\). The following Lemma is not used in the proof of Theorem 1 however.

**Lemma 13.** Let \(K\) be a finite non-empty set of integers. Let \(\mathcal{F}\) be a finite set of signatures.

1. Let \(G\) be a finite set of simple weightings of signatures in \(\mathcal{F}\). There is a finite set of variable weights \(W\) such that \(#\text{CSP}_K^W(\mathcal{G}) \leq_{AP} \#\text{CSP}_K^W(\mathcal{F})\).

2. For all finite sets of variable weights \(W\) there is a finite set \(\mathcal{G}\) of simple weightings of signatures in \(\mathcal{F}\) such that \(#\text{CSP}_K^W(\mathcal{F}) \leq_{AP} \#\text{CSP}_K^W(\mathcal{G})\).

**Proof.** (1.) Each \(G \in \mathcal{G}\) can be expressed as \(G(x) = F_G(x) \prod_{x_j \in V(F)} U_{G,j}(x_j)\) for some \(F_G \in \mathcal{F}\) and arity 1 signatures \(U_{G,j}\). From now on we will let \(G\) range over \(\mathcal{G}\) and \(j\) range over \(V(G)\). Let \(W\) be the set consisting of variable weights \(\prod_{x_j \in V(G)} U_{G,j}(0)^{n(G,j)} \prod_{x_j \in V(G)} U(1)^{n(G,j)}\) for all choices of \(0 \leq n(G,j) \leq \max(K)\). Then \(|W| \leq (\max(K) + 1)^{\sum_{x \in V(G)} (|V(G)|)}\) so \(W\) is a finite set.

Given an instance \(\phi\) of \(#\text{CSP}_K^W(\mathcal{G})\), for each \(G \in \mathcal{G}\) and each atomic formula \(G(s)\), delete that atomic formula and insert an atomic formula \(F_G(s)\). Define \(w : V \rightarrow W\) by \(w(v) = \prod_{x_j \in V(G)} U_{G,j}(x_j)^{n(v,G,j)}\) for \(x = 0, 1\), where \(n(v,G,j)\) is the number of atomic formulas \(G(s)\) of \(\phi\) such that \(s(j) = v\). It follows that \(Z_\phi = Z_\phi\), so the reduction can just query the oracle with \((w', \phi')\), passing the instance’s error parameter to the oracle.

(2.) Let \(\mathcal{G}\) consist of all signatures of the form \(F(x) \prod_{x_i \in V(F)} w(i)_{x_i}\) with \(F \in \mathcal{F}\) and \(w : V(F) \rightarrow W \cup \{1, 1\}\).

Given an instance \((w, \phi)\) of \(#\text{CSP}_K^W(\mathcal{F})\), let \(V' = \{v \in V \mid \deg(v) > 0\}\). In terms of \(Z_\phi\), we will regroup the factors of \(w(v)_{x_v}\) into an existing atomic formula, for each \(v \in V'\). Specifically, let \(g : V' \rightarrow I^\phi\) be any map taking each variable \(v \in V'\) to the index of an atomic formula with \(v\) in its scope: \(\text{scope}^\phi(g(v), t) = v\) for some \(t\). Let \(\phi'\) be the \(K\)-formula with the same variables and scopes as \(\phi\), but for each \(i \in I^\phi\), define \(F^\phi_i\) by \(F^\phi_i(x) = F(x) \prod_{x_j \in V(F)} U_{i,j}(x_j)\) where \(U_{i,j}(y) = w(\text{scope}(i, j), y)\) if \(g(\text{scope}(i, j)) = i\), and \(U_{i,j}(y) = 1\) otherwise. Then \(w^\phi_i(x) = w^\phi_i(x)e\) for all configurations \(x\) of \(V\) where \(C = \prod_{v \in V(F)} (w(v, 0) + w(v, 1))\). Thus \(Z^w = CZ^\phi\); the reduction can call the \(#\text{CSP}_K(\mathcal{G})\) oracle to get an approximation to \(Z^\phi\) then multiply by \(C\).


4 Minimal pinnings

We will characterise various classes of signatures in terms of pinnings. This is in the same spirit as the ppp-definability studied in [DGJR10].

For a class $P$ of relations, we will say a relation $R$ is pinning-minimal $P$, or pinning-minimal subject to $P$, if $R$ is in $P$ and $R_p$ is not in $P$ for any non-trivial partial configuration $p$. Similarly we can say a signature is pinning-minimal $P$.

Define a signature pair to be a pair $(F,G)$ of signatures $F,G : \{0,1\}^V \to \mathbb{R}_p$, for some $V$. For a class $P$ of signature pairs we will say $(F,G)$ is pinning-minimal $P$ if $(F,G)$ is in $P$ and $(F_p,G_p)$ is not in $P$, for any non-trivial partial configuration $p$. A signature pair $(F,G)$ is defined to be linearly dependent if there exist $\lambda, \mu \in \mathbb{R}$, not both zero, such that $\lambda F = \mu G$.

**Lemma 14.** Let $(F,G)$ be a pinning-minimal linearly independent signature pair. Then $\text{supp}(F) \cup \text{supp}(G)$ for some configuration $x \in \text{supp}(F)$.

**Proof.** First we give another characterisation of linear independence of a signature pair. For any $F',G' : \{0,1\}^V \to \mathbb{R}_p$ consider the two-by-2$|V|$ matrix $M$, with columns indexed by $\{0,1\}^V$, defined by $M_{1,x} = F(x)$ and $M_{2,x} = G(x)$. The signature pair is linearly independent if and only if $M$ has row rank two, hence if and only if $M$ has column rank two, and hence if and only if there exist $x,y$ such that the two-by-two submatrix

$$M(x,y) = \begin{pmatrix} F(x) & F(y) \\ G(x) & G(y) \end{pmatrix}$$

has linearly independent rows.

Now let $(F,G)$ be a pinning-minimal linearly independent signature pair. For any $(x,y)$ such that $M(x,y)$ has linearly independent rows, let $p = \{i \mapsto x_i \mid x_i = y_i\}$. Then $(F_p,G_p)$ is a linearly independent signature pair. Hence $0 \neq y = \overline{x}$.

There exists some $x$ such that $M(x,\overline{x})$ has linearly independent rows. For all $z$ such that $F(z)$ or $G(z)$ is non-zero either $M(x,z)$ has linearly independent rows or $M(z,\overline{x})$ has linearly independent rows. By the previous paragraph, $z = \overline{x}$ or $z = \overline{\overline{x}} = x$. Hence $\text{supp}(F) \cup \text{supp}(G) \subseteq \{x,\overline{x}\}$. Finally, since $F$ is not identically zero, one of $x$ or $\overline{x}$ is in $\text{supp}(F)$. \hfill $\square$

**Lemma 15.** Let $F$ be a pinning-minimal non-logsupermodular signature. Then $\text{supp}(F) \subseteq \{0,x,\overline{x},1\}$ for some $x$.

**Proof.** For all $x,y$ such that $F(x \land y)F(x \lor y) < F(x)F(y)$, the pinning of $F$ by $\{i \mapsto x_i \mid x_i = y_i\}$ is not logsupermodular so $y = \overline{x}$. There exists such a tuple $x$. Also, taking the contrapositive, for all $y,z$ such that $z \neq y$ we have $F(z \land y)F(z \lor y) \geq F(z)F(y)$. Let $z \notin \{0,1,x,\overline{x}\}$. Then

$$F(x \land z)F(x \lor z) \geq F(x)F(z)$$

$$F(\overline{x} \land z)F(\overline{x} \lor z) \geq F(\overline{x})F(z)$$

$$F(0)F(z) \geq F(x \land z)F(\overline{x} \land z)$$

$$F(z)F(1) \geq F(x \lor z)F(\overline{x} \lor z)$$

In each case we have used the fact that the tuples on the right-hand-side are not complements, or, equivalently, the tuples on the left-hand-side are not 0 and 1.

Multiplying these four inequalities we get $F(0)F(1)C \geq F(x)F(\overline{x})C$ where

$$C = F(z)^2F(x \land z)F(x \lor z)F(\overline{x} \land z)F(\overline{x} \lor z)$$

In each case we have used the fact that the tuples on the right-hand-side are not complements, or, equivalently, the tuples on the left-hand-side are not 0 and 1.
The inequalities also imply that \( C \geq F(x)F(\overline{x})F(z)^4 \). But \( F(0)F(1) < F(x)F(\overline{x}) \) so \( C = 0 \) and hence \( F(z) = 0 \). \( \square \)

**Lemma 16.** Let \( R \) be a pinning-minimal relation subject to not being closed under joins (so there exists \( x, y \in R \) such that \( x \lor y \notin R \)). Then \( R = \{0, x, \overline{x}\} \) or \( R = \{x, \overline{x}\} \).

**Proof.** For all \( x, y \in R \) with \( x \lor y \notin R \), the pinning of \( R \) by \( \{i \mapsto x_i \mid x_i = y_i\} \) is not closed under joins so \( y = \overline{x} \). Hence there exists \( x \) with \( x, \overline{x} \in R \), and \( 1 \notin R \). Also, taking contrapositives, if \( y, z \in R \) and \( y \neq \overline{x} \) then \( y \lor z \in R \).

Let \( y \in R \setminus \{x, \overline{x}\} \). By the previous paragraph, \( x \lor y \in R \) and \( \overline{x} \lor y \in R \). But \( (x \lor y) \lor (\overline{x} \lor y) = 1 \notin R \), so \( x \lor y \) is the complement of \( \overline{x} \lor y \). Hence \( \max(x, y) = 1 - \max(1 - x_i, y_i) = \min(x_i, 1 - y_i) \) for all variables \( i \), which implies \( y = 0 \). \( \square \)

**Lemma 17.** Let \( F \) be a pinning-minimal non-degenerate signature. Then \( F \) has arity 2 or \( \text{supp}(F) = \{x, \overline{x}\} \) for some \( x \).

**Proof.** Pick some variable \( v \) in the variable set of \( F \). Let \( F_0 \) and \( F_1 \) be the pinnings of \( F \) by \( \{v \mapsto 0\} \) and \( \{v \mapsto 1\} \) respectively.

For any degenerate signature \( G \), the pinnings \( G_0 \) and \( G_1 \) defined in the same way are linearly dependent. So for all partial configurations \( p \) such that \( (F_0)_p \) and \( (F_1)_p \) are linearly independent, \( F_p \) is non-degenerate and hence \( \text{dom}(p) = \emptyset \). Furthermore if \( \lambda F_0 = \mu F_1 \) for some \( \lambda, \mu \) not both zero, then \( F \) is degenerate: by symmetry and scaling we can assume \( \mu = 1 \), so \( F_1 = \lambda F_0 \), and \( F \) is the tensor product of \( F_0 \) and the arity 1 signature \( U \) defined by \( U(0) = 1 \) and \( U(1) = \lambda \). Hence \( F_0 \) and \( F_1 \) form a pinning-minimal linearly independent signature pair. By Lemma 14, \( \text{supp}(F_0) \cup \text{supp}(F_1) = \{x, \overline{x}\} \) for some \( x \in \text{supp}(F_0) \).

If \( \text{supp}(F_0) \) and \( \text{supp}(F_1) \) are \( \{x\} \) and \( \{\overline{x}\} \) respectively (or vice versa) then \( \text{supp}(F) = \{(0, x), (0, \overline{x})\} \), so we are done. Otherwise \( \text{supp}(F_0) \) or \( \text{supp}(F_1) \) is therefore degenerate. But a degenerate relation is equivalent to \( \{0, 1\}^a \times \{0, 1\}^b \times \{1\}^c \) for some \( a, b, c \geq 0 \). Taking cardinalities we have \( 2 = 2^a \) so \( a = 1 \). The powers \( b \) and \( c \) must be zero because \( x_u \neq \overline{x}_u \) for all variables \( u \). Hence \( F \) has arity 2. \( \square \)

**Lemma 18.** A relation is a delta matroid if and only if its signature is terraced. (Recall that a relation \( R \) is a delta matroid if \( x, y \in R \) and for all \( i \in \Delta y \) there exists \( j \in \Delta x \) not necessarily distinct from \( i \), such that \( x^{(i,j)} \in R \). A signature \( F \) is terraced if for all partial configurations \( p \) of \( V \) and all \( i, j \) in the domain of \( p \), if \( F_p \) is identically zero then \( F_p^{(i,j)} \) is linearly independent.)

**Proof.** Let \( R \) be a delta matroid. Let \( p \) be a partial configuration such that \( R_p \) is empty and let \( i, j \) be variables on which \( p \) is defined and such that \( R_p^{(i,j)} \) are non-empty. We will show that \( R_p^{(i,j)} = R_p^{(j,i)} \). By symmetry it suffices to show that for all \( x \in R_p^{(i,j)} \) we have \( x \in R_p^{(j,i)} \). Pick \( y \in R_p^{(j,i)} \). By the delta matroid property applied to \( ((x, p^{(i)}) , (y, p^{(j)}) , i) \) there exists \( d \), such that \( x_d \neq y_d \) or \( d \in \{i, j\} \), and such that \( (x, p^{(i)})^{(i,d)} \) is in \( R \). Since \( R_p \) is empty we have \( d = j \) and hence \( x \in R_p^{(j,i)} \).

Conversely let \( R \) be a relation whose signature is terraced. For all \( x, y \in R \) and all \( d \in \Delta x \) we wish to show that \( x^{(d,d')} \in R \) for some \( d' \in \Delta y \). Let \( y' \in R \) satisfy \( \{d\} \subseteq \Delta y' \subseteq \Delta y \) with \( |\Delta y'| \) minimal. If \( \Delta y' = \{d\} \) we can take \( d' = d \). Otherwise pick \( d' \in (\Delta y') \setminus \{d\} \). Let \( p \) be the restriction of \( x^{(d,d')} \) to \( \{d, d'\} \cup \{i \mid x_i = y_i\} \). Configurations \( z \in R_p \) satisfy \( \{d\} \subseteq \Delta(p, z) \subseteq (\Delta y') \setminus \{d'\} \), but \( |\Delta(p, z)| < |\Delta y'| \) contradicts the choice of \( y' \); therefore \( R_p \) is empty. And \( R_p^{(d)} \) and \( R_p^{(d')} \) contain the restrictions of \( x \) and \( y \) respectively (to \( (\Delta y') \setminus \{d, d'\} \)). Since \( R \) has a terraced signature, \( R_p^{(d)} = R_p^{(d')} \) so \( x^{(d,d')} \in R \). \( \square \)
Lemma 19. For every pinning-minimal non-IM-terraced signature, there is an equivalent signature $F : \{0, 1\}^k \rightarrow \mathbb{R}_p$ such that:

- the pinning of $F$ by the partial configuration $p$ defined by $p(1) = 0$ and $p(2) = 1$ is identically zero, and

- there exists a configuration $z$ of $\{3, 4, \cdots , k\}$ and a non-degenerate signature $T : \{0, 1\}^2 \rightarrow \mathbb{R}_p$ such that for all $x, y_3, \cdots, y_k \in \{0, 1\}$ we have

$$F(x, x, y_3, \cdots, y_k) = \begin{cases} T(y_3, x) & \text{if } y = z \text{ or } y = \overline{z} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Consider an arbitrary pinning-minimal non-IM-terraced signature $F$. Since $F$ is not IM-terraced there exist $p, i, j$ such that $p_i = 0$ and $p_j = 1$ and $F_p$ is identically zero, but $F_{p(i)}$ and $F_{p(j)}$ are linearly independent. By renaming variables we can assume $i = 1$ and $j = 2$ and $V(F) = \{1, 2, \cdots, k\}$ for some $k$.

We will write 00 and 11 for the partial configurations $\{1 \mapsto 0, 2 \mapsto 0\}$ and $\{1 \mapsto 1, 2 \mapsto 1\}$ respectively, so $\{p^{(1)}, p^{(2)}\} = \{00, 11\}$. Let $p'$ be the restriction of $p$ to $\text{dom}(p) \setminus \{1, 2\}$. Then $F' = F_p'$ is also not IM-terraced: $F'_{00} = F_{p(i)}$ and $F'_{11} = F_{p(j)}$ are linearly independent. Hence $\text{dom}(p) = \{1, 2\}$ by minimality of $F$.

We will argue that $(F_{00}, F_{11})$ is a pinning-minimal linearly independent signature pair. We need to check that for any non-empty partial configuration $y$ of $\{3, 4, \cdots, k\}$ the pinnings $(F_{00})_y$ and $(F_{11})_y$ are linearly dependent. But $F_y$ is IM-terraced by minimality of $F$, and $(F_y)_p$ is identically zero, and $p^{(1)} = 00$ and $p^{(2)} = 11$, so $(F_{00})_y$ and $(F_{11})_y$ are indeed linearly dependent because the order in which pinnings are applied does not matter.

By Lemma 14 $\text{supp}(F_{00}) \cup \text{supp}(F_{11}) = \{z, \overline{z}\}$ for some configuration $z$ of $\{3, 4, \cdots, k\}$. Without loss of generality we may take $z_3 = 0$. Set $T(0, 0) = F_{00}(z)$ and $T(1, 0) = F_{00}(\overline{z})$ and $T(0, 1) = F_{11}(z)$ and $T(1, 1) = F_{11}(\overline{z})$. This $T$ satisfies the required expression for $F$. Furthermore the signatures $F_{00}$ and $F_{11}$ are linearly independent, hence so are the vectors $(T(0, 0), T(1, 0))$ and $(T(0, 1), T(1, 1))$, and hence $T$ is non-degenerate. □

Lemma 20. For every pinning-minimal non-terraced signature there is an equivalent signature $F : \{0, 1\}^k \rightarrow \mathbb{R}_p$ and a configuration $p : \{1, 2\} \rightarrow \{0, 1\}$ such that:

- $F_p$ is identically zero

- there exists a configuration $z : \{3, \cdots, k\} \rightarrow \{0, 1\}$ and a non-degenerate signature $T : \{0, 1\}^2 \rightarrow \mathbb{R}_p$ such that if $x$ is one of the flips $p^{(1)}$ or $p^{(2)}$ (either $(1 - p_1, p_2)$ or $(p_1, 1 - p_2)$ as elements of $\{0, 1\}^{\{1, 2\}}$), then for all $y_3, \cdots, y_k \in \{0, 1\}$ we have

$$F(x_1, x_2, y_3, \cdots, y_k) = \begin{cases} T(y_3, x_1) & \text{if } y = z \text{ or } y = \overline{z} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Given a pinning-minimal non-terraced signature $G$, there exists $p, i, j$ such that $G_p$ is identically zero, but $G_{p(i)}$ and $G_{p(j)}$ are linearly independent. Let $S$ be the set containing: $i$ if $p_i = 1$, and $j$ if $p_j = 0$. Then the flip $G^S$ is not IM-terraced: let $q = p^S$; then $G_q^S$ is identically zero but $G_{q(i)}^S = G_p(i)$ and $G_{q(j)}^S = G_p(j)$ are linearly independent.
Since $G$ is pinning-minimal non-terraced, every proper pinning of $G$ is terraced. Terracedness is preserved by flips, so every proper pinning of $G^S$ is terraced, and hence IM-terraced. So $G^S$ is pinning-minimal non-IM-terraced. The expression for a signature $F$ equivalent to $G^S$ is given by applying an arbitrary flip to the expression given by Lemma 19.

**Lemma 21.** Let $R$ be a delta matroid that is pinning-minimal subject to not being basically binary. There is an $h$-maximisation of $R$ equivalent to a flip of $PM_3$.

**Proof.** First note that any relation $R$ with an $h$-maximisation of $R$ equivalent to a flip of $PM_3$ is not basically binary: $h$-maximisation cannot make decomposable relations indecomposable, and an indecomposable arity 3 relation cannot be basically binary. Also, $R$ is indecomposable: if $R = R_1 \times R_2$ then since $R$ is not basically binary, either $R_1$ or $R_2$ is not basically binary, but $R_1$ and $R_2$ are pinnings of $R$.

We will in fact show that $R$ has the “sphere property” that there exists $x \in \{0, 1\}^3$ such that $x \in \{0, 1\}$ and $d = 1, 2$ such that $x^U \notin R$ for subsets $U$ of $\{1, 2, 3\}$ with $|U| < d$ and $x^U \in R$ for $|U| = d$. Then let $h(1) = 2x_1 - 1$ and $h(2) = 2x_2 - 1$ and $h(3) = 2x_3 - 1$. Observe that $S = R^{|h| \text{max}}$ consists precisely of the three configurations $x^U$ with $|U| = d$. In other words $S$ is a flip $PM_3^{U'}$, where $U' = U$ if $d = 1$ and $U' = \{1, 2, 3\} \setminus U$ if $d = 2$.

There exists a configuration not in $R$ (otherwise $R$ would be basically binary). So there is an arity zero pinning of $R$ that is the empty relation. Let $R_p$ be a maximal pinning subject to $R_p = \emptyset$. If $v \in \text{dom}(p)$ let $p'$ be the restriction of $p$ to $\text{dom}(p) \setminus \{v\}$; the pinning $R_{p'}$ is non-empty by maximality of $R_p$, and hence the relations $R_{p(v)}$ are non-empty. The signature of $R$ is terraced by Lemma 18 so $R_{p(v)} = R_{p(v')} \forall v, v' \in \text{dom}(p)$.

Recall that $R$ is indecomposable. But if $p$ has variable set $\{v\}$ for some $v$ then $R$ is the product of $\{p(v)\}$ with $R_{p(v)}$. So $p$ has arity at least 2.

If $p$ has arity at least 3, split $p$ as $(q, p')$ where $\text{dom}(q) = 3$. Pick $y \in R_{p(v)}$ (for any $v$). Let $R'$ be the pinning of $R$ by both $y$ and $p'$. Note that $q \notin R'$ but $q(v) \in R'$ for all variables $v \in \text{dom}(q)$. Hence $R = R'$ and $R'$ has the sphere property with $d = 1$.

The remaining case is that $p$ has variable set $\{i, j\}$ for some distinct variables $i, j$. Since $R$ is indecomposable, $R$ is not the product of $R_{p(i)}$ with an arity 2 relation on $\{i, j\}$. Hence $R_{p(i)}$ and $R_{p(i,j)}$ are linearly independent. Let $R'$ be a minimal pinning of $R$ such that $G = R_{p(i,j)}$ and $H = R_{p(i,j)}'$ are linearly independent.

By Lemma 14 we have $G \cup H = \{y, \overline{y}\}$ for some $y$, and without loss of generality, either $G = \{y\}$ or $H = \{\overline{y}\}$. Also, to recap: $R_p = \emptyset$ and $G = R_{p(i)} = R_{p(j)} \neq R_{p(i,j)} = H$. If $G = \{y\}$ then by the delta matroid property applied to $(p_{i,j}, \overline{y})$, $(p_{i,j}, y)$ and $j$ there exists $k \in \{j\} \cup \text{dom}(y)$ such that $(p_{i,j}, \overline{y}^{(j,k)})$ holds in $R$, but then $k$ must lie in $\text{dom}(y)$ and $y^{(k)} \in R_{p(j)} = G$. Hence $y$ has arity 1, $R$ has arity 3, and the sphere property holds with $x = (p_{i,j}, \overline{y})$ and $d = 2$. If $H = \{\overline{y}\}$ then by the delta matroid property applied to $(p_{i,j}, y)$, $(p_{i,j}, \overline{y})$ and $j$, there exists $k \in \{j\} \cup \text{dom}(y)$ such that $(p_{i,j}, y^{(j,k)})$ holds in $R$, but then $k$ must lie in $\text{dom}(y)$ and $y^{(k)} \in R_{p(i,j)} = H$. Hence $y$ has arity 1, $R$ has arity 3, and the sphere property holds with $x = (p_{i,j}, \overline{y})$ and $d = 1$.

**5 Main theorem**

**Theorem 11.** Let $\Gamma$ be a finite set of relations. If $\Gamma \subseteq \text{NEQ-conj}$ or every relation in $\Gamma$ is basically binary then $\#\text{CSP}_{\leq 2}^\geq 0(\Gamma)$ is in FP. Otherwise,

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• If $\Gamma \subseteq \text{IM-conj}$ then $\#\text{BIS} =_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$.

• If $\Gamma \not\subseteq \text{IM-conj}$ then $\#\text{PM} \leq_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$. If furthermore $\Gamma$ is not a set of delta matroids then $\#\text{SAT} =_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$.

Proof. The inclusion in FP is given by Lemma[10]. We will therefore assume that $\Gamma$ contains a relation that is not in NEQ-conj and a relation that is not basically binary. We will consider the four cases depending on whether $\Gamma \subseteq \text{IM-conj}$ and whether $\Gamma$ consists entirely of delta matroids:

| IM-conj | delta matroids |
|---------|----------------|
| yes     | yes            | impossible by Lemma[22] |
| no      | yes            | $\#\text{PM} \leq_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$ by Lemma[26] |
| yes     | no             | $\#\text{BIS} =_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$ by Lemmas[23, 6] and [24] |
| no      | no             | $\#\text{SAT} =_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$ by Lemmas[23, 6] and [11] |

Lemma 22. Let $R$ be a delta matroid in IM-conj. Then $R$ is basically binary.

Proof. We may assume that $R$ is indecomposable. Assume for contradiction that $R$ has arity at least three.

Let $V$ be the variable set of $R$. Note that no variables are pinned: if there exists $i \in V$ and $c \in \{0, 1\}$ such that $x_i = c$ for all $x \in R$, then $R$ is the product of $\{c\}$ with the pinning of $R$ by $\{i \mapsto c\}$, but this contradicts the assumption that $R$ is indecomposable. Since $R$ is in IM-conj and no variables are pinned, $R$ is a conjunction of implications of variables. Therefore there is a subset $P$ of $V \times V$ such that

$$R = \{x \mid x_i \leq x_j \text{ for all } (i, j) \in P\}$$

Consider the undirected graph $G$ on $V$ where $i$ and $j$ are adjacent if and only if $(i, j)$ or $(j, i)$ is in $P$. Then $G$ has at least three vertices, and since $R$ is indecomposable, $G$ is connected. Hence there is a vertex $i$ of degree at least two. There exist distinct variables $j, k \in V$ such that $(i, j), (i, k) \in P$, or $(j, i), (k, i) \in P$, or $(j, i), (i, k) \in P$. In the first case, there is no $\ell \in V$ such that $0^{(i, \ell)} \in R$. In the second case, there is no $\ell \in V$ such that $1^{(i, \ell)} \in R$. In the third case, there is no $\ell \in V$ such that $0^{(j, \ell)} \in R$. But the all-zero configuration 0 and the all-one configuration 1 are both in $R$. Hence the delta matroid property fails for $R$. □

Lemma 23. Let $\Gamma$ be a finite set of relations which are not all delta matroids. Then

$$\#\text{CSP}^{\geq 0}(\Gamma) \leq_{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma)$$

Proof. Let $R_1$ be a minimal non-terraced pinning of a relation in $\Gamma$. By Lemma[18], $R_1$ is pinning-minimal non-terraced, so by Lemma[27] possibly after renaming variables, there exist $p_1, p_2, z_3, \ldots, z_k \in \{0, 1\}$ and a non-degenerate signature $T : \{0, 1\}^2 \to \mathbb{R}_p$ such that for all $(x_1, x_2) \in \{(1-p_1, p_2), (p_1, 1-p_2)\}$ and all $y = (y_3, \ldots, y_k) \in \{0, 1\}^{(3, \ldots, k)}$ we have

$$R_1(x_1, x_2, y) = T(y_3, x_1)1_{\{z, x\}}(y)$$

Define

$$R_2(x_1, x_2, y) = 1_{\{(1-p_1, p_2), (p_1, 1-p_2)\}}(x_1, x_2)T(y_3, x_1)1_{\{z, x\}}(y)$$

for all $x_1, x_2, y_3, \ldots, y_k \in \{0, 1\}$. In other words, $R_1$ and $R_2$ agree except that the entries $R_2(1-p_1, 1-p_2, y)$ are zero. Hence $R_2 = (R_1)_{h_{\text{max}}}$ where $h(1) = 2p_1 - 1$ and $h(2) = 2p_2 - 1$ and $h(3) = \cdots = h(k) = 0$. □
Define
\[ R_3(x_1, x_2, y_3) = \sum_{y_4, \ldots, y_k} R_2(x_1, x_2, y) = 1 \{(1-p_1.p_2),(p_1,1-p_2)\}(x_1, x_2)T(y_3, x_1) \]

\[ T \text{ is necessarily 0, 1-valued, but } T \text{ is also non-degenerate. Hence } T \text{ has some zero, say } T(c, d) = 0. \]

Define
\[ R_4(x_1, x_2, y_3) = 1 \{(1-p_1.p_2),(p_1,1-p_2)\}(x_1, x_2)1 \{(1-c,d),(c,1-d)\}(y_3, x_1) \]

So \( R_3 \) and \( R_4 \) agree except that \( R_4(1-d, x_2, 1-c) \) is zero for \( x_2 = 0, 1 \). Hence \( R_4 = (R_3)_{h_{\text{max}}} \)

where \( h(1) = 2c - 1 \) and \( h(2) = 2d - 1 \) and \( h(3) = 0 \). By Lemma 9 and Lemma 8 we have
\[ \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma \cup \{ R_4 \}) \leq \#\text{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma) \]

Crucially \( R_4 \) is a conjunction of an equality or disequality on the first two variables, with an equality or disequality on the last two variables. This implies that \( R_4 \) consists of two complementary configurations and so \( R_4 \) is equivalent to \{\{(0, 0, 0), (1, 1, 1)\} or \{(0, 1, 1), (1, 0, 0)\} \}

The rest of the proof is what is called “2-simulating equality” in [DGJR10]. We are given an instance of \#\text{CSP}^{\geq 0}(\Gamma)\), which can be written as
\[ Z_{\phi}^w = \sum_{x: V \to \{0, 1\}} \left( \prod_{v \in V} w(v)_{x_v} \right) \left( \prod_{i \in I} F_i((x_{\text{scope}(i,j)})_{j \in V(F_i)}) \right) \]

We can assume that every variable has degree at least one. (Otherwise let \( \hat{V} = \{ v \in V \mid \deg(v) > 0 \} \) and let \( \hat{w} \) be the restriction of \( w \) to \( \hat{V} \); then \( Z_{\phi}^w = Z_{\phi}^\hat{w} \prod_{v \in V \setminus \hat{V}} (w(v)_0 + w(v)_1) \). Modify \( \phi \) as follows to produce a new (= 2)-formula \( \phi' \) on a variable set \( V' \). For each variable \( v \in V \), replace the \( d = \deg(v) \) uses of \( v \) by separate variables \( v_1, \ldots, v_d \) and insert new atomic formulas \( R_i(v_j, v_i, u_{i+1}) \) for \( i = 1, \ldots, d \), where \( u_{d+1} = u_1 \), to obtain a new formula \( \phi' \). Note that every variable in \( \phi' \) is used exactly twice. Set \( w'(v_1) = w(v) \) for all \( v \in V \), and \( w'(v_2) = \cdots = w'(v_d) = w(u_1) = \cdots = w(u_d) = 1 \). Then \( Z_{\phi}^w = Z_{\phi'}^w \): the contributions to \( Z_{\phi'}^w \) come from configurations where for each \( v \) the variables \( v_1, \ldots, v_d \) get the same value \( x_v \), and these configurations have the same weight as the corresponding configuration \( x \) in \( Z_{\phi}^w \). And we can just call the \#\text{CSP}^{\geq 0}_{\leq 2}(\Gamma \cup \{ R_4 \}) \) oracle to obtain \( Z_{\phi'}^w \).

\[ \square \]

Lemma 24. [BDGJ12 Proposition 25] Let \( F \) be a finite subset of \( \text{IM-conj} \). Then \#\text{CSP}^{\geq 0}(F) \leq \#\text{BIS}.

Proof. We have \#\text{CSP}(\{\text{IMP}\}) \leq \#\text{AP} \#\text{BIS} by [DGJ10 Theorem 3], so it suffices to show that \#\text{CSP}^{\geq 0}(\{\text{IMP}\}) \leq \#\text{AP} \#\text{CSP}(\{\text{IMP}\}) \). The construction in [BDGJ12 Proposition 25] simulates an arbitrary polynomial-time computable arity 1 signature using IMP, in polynomial time.

In [Fis66] it is shown that the problem of counting perfect matchings reduces to counting perfect matching of graphs of maximum degree three. Hence:

Lemma 25. [Fis66] \#PM \leq \#\text{CSP}^{=2}(\{PM_3\}).

Lemma 26. Let \( R \) be a delta matroid that is not basically binary. Then \#PM \leq \#\text{AP} \#\text{CSP}^{\geq 0}_{\leq 2}(\{R\}).
Proof. By Lemma \[21\] and Lemma \[12\] we can assume that \(R\) has arity 3 and there exists \(x \in \{0, 1\}\) and \(d = 1, 2\) such that \(x^U \notin R\) for subsets \(U\) of \(\{1, 2, 3\}\) with \(|U| < d\) and \(x^U \in R\) for \(|U| = d\). Let \(h(1) = 2x_1 - 1\) and \(h(2) = 2x_2 - 1\) and \(h(3) = 2x_3 - 1\). Then \(S = R_{h_{\text{max}}}\) consists precisely of the three configurations \(x^U\) with \(|U| = d\). In other words \(S\) is a flip \(PM_3^{U''}\). By Lemma \[9\] we have
\[
\#CSP_{\leq 2}(\{PM_3^{U''}\}) \leq AP \#CSP_{\leq 2}(\{R\})
\]
If \(|U'| \leq 1\) let \(U'' = U'\). Otherwise let \(U'' = \{1, 2, 3\} \setminus U'\); the complexity is not changed by exchanging the roles of 0 and 1:
\[
\#CSP_{\leq 2}(\{PM_3^{U''}\}) = AP \#CSP_{\leq 2}(\{U'\})
\]
In either case \(|U''| \leq 1\). If \(|U''| = 1\), reorder the variables if necessary we can assume \(U'' = \{1\}\). In this case \(PM_3\) can be expressed by a 2-formula over \(\{PM_3^{(1)}; PIN_1\}\):
\[
\begin{align*}
\text{NEQ}(y, z) &= \sum_x PM_3^{(1)}(x, y, z)PIN_1(x) \\
PM_3(x, y, z) &= \sum_{x'} \text{NEQ}(x, x')PM_3^{(1)}(x', y, z)
\end{align*}
\]
Hence by Lemma \[12\] and Lemma \[8\] we have \(\#CSP_{\leq 2}(\{PM_3\}) \leq AP \#CSP_{\leq 2}(\{PM_3^{(1)}\})\). In any case it suffices to show that \(\#PM \leq AP \#CSP_{\leq 2}(\{PM_3\})\), which is Lemma \[25\]. \(\square\)

6 An extension to signatures

In this section we will give the extensions of Theorem \[1\] mentioned in the introduction.

This section is quite technical, so here is a quick summary. We work in the setting of finite sets of variable weights as much as possible. We then collect all our results for arbitrary variable weights in Theorem \[2\] and collect all our results for finite sets of variable weights in Theorem \[3\]. First of all, Lemma \[27\] uses certain non-IM-terraced signatures to reduce a slightly different unbounded-degree problem “\(\#CSP(T^\otimes F^B)\)” defined below, to a degree-two problem \(\#CSP_{=2}(F)\), using an adaptation of the Holant theorem as used in [LX11a]. Lemma \[28\] provides unary signatures in this unbounded-degree problem. Lemma \[29\] ties the previous two Lemmas together and extends to any non-IM-terraced signature. Lemma \[30\] applies this to reducing \#BIS and \#SAT to certain \#CSPs. For infinite sets of variables weights, Lemma \[31\] reduces \#PM to certain \#CSPs, and Lemma \[32\] uses \(h\)-maximization to provide flips in some cases, which means non-terraced signatures are as useful as non-IM-terraced signatures in that setting.

Let \(T : \{0, 1\}^2 \to \mathbb{R}_p\) and let \(F\) be a signature. The following construction is used for holographic transformations of Holant problems (see for example [LX11a]), and is usually denoted \(T^\otimes kF\) if \(F : \{0, 1\}^k \to \mathbb{R}_p\). But it will be convenient not to include the arities \(k\). Define \(T^\otimes F : \{0, 1\}^V \to \mathbb{R}_p\) by
\[
(T^\otimes F)(x) = \sum_y \left( \prod_{i \in V(F)} T(x_i, y_i) \right) F(y)
\]

Let \(B = 1\) or \(B = 2\) and let \(T\) be a non-degenerate arity 2 signature. (To make the results stronger we will work with \(\#CSP_{=2}\) (or Holant) problems rather than \(\#CSP_{\leq 2}\). This is indirectly why we end up using the technical complication of the \(B = 2\) case.) In this section we will use the notation \(T^\otimes F^B\), where \(F\) is a signature, to denote \(T^\otimes F'\) where \(F'(x) = F(x)^B\). We will use the notation \(T^\otimes F^B\), where \(F\) is a set of signatures, to denote \(\{T^\otimes F^B \mid F \in F\}\).
Lemma 27. Let $B = 1$ or $B = 2$. Let $T : \{0, 1\}^2 \to \mathbb{R}_p$. Let $G : \{0, 1\}^{B+2} \to \mathbb{R}_p$ be a signature such that for all $x, y_1, \cdots, y_B \in \{0, 1\}$ we have $G(1, 0, y_1, \cdots, y_B) = 0$ and

$$G(x, x, y_1, \cdots, y_B) = EQ_B(y_1, \cdots, y_B)T(y_1)$$

Then

$$\#CSP(T \circ \mathcal{F}^B) \leq_{AP} \#CSP_{=2}(\mathcal{F})$$

Proof. Let $\phi$ be an instance of $\#CSP(T \circ \mathcal{F}^B)$. We may assume that every variable has non-zero degree.

We will enumerate each use of each variable in the following way. Let $V = V^\phi$, $I = I^\phi$, $F = F^\phi$ and $\text{scope} = \text{scope}^\phi$. Define $L = \{(v, d) \mid v \in V, 1 \leq d \leq \deg(v)\}$ and $R = \{(i, j) \mid i \in I, j \in V(F_i)\}$. There is a bijection use : $L \to R$ such that $\text{scope}(\text{use}(v, d)) = v$ for all $(v, d) \in L$. We wish to compute $Z = Z_\phi$, which is

$$\sum_{x \in \{0, 1\}^V} \prod_{i \in I} (T F_i)((z^{\text{scope}(i,j)})_{j \in V(F_i)})$$

For the rest of the proof, product indices $i, j, b, v, d$ will range over $i \in I$ and $j \in V(F_i)$ and $1 \leq b \leq B$ and $v \in V$ and $1 \leq d \leq \deg(v)$. The variables $x, y, z$ range over $x : L \to \{0, 1\}$ and $y : R \times \{1, \cdots, B\} \to \{0, 1\}$ and $z : V \to \{0, 1\}$, and $y_{b(i,j)}$ means $y_{(i,j),b}$ and $x^{(v, \deg(v))}$ means $(v, \deg(v))$. Define $\phi'$ to be the ($= 2$)-formula given by

$$Z_\phi = \sum_{x, y} \left( \prod_{v, d} G(x^{(v, d-1)}, x^{(v, d)}, y_1^{\text{use}(v, d)}, \cdots, y_B^{\text{use}(v, d)}) \right) \left( \prod_{i, b} F_i((y_{b(i,j)})_{j \in V(F_i)}) \right)$$

The reduction queries the $\#CSP_{=2}(\mathcal{F})$ oracle on $\phi'$, passing through the error parameter, and returns the result. To show that the reduction is correct we must show that $Z_\phi = Z_{\phi'}$. This is mostly algebraic manipulation with the products below.

$$Z\text{Terms}(z) = \prod_i (T \circ F_i^B)((z^{\text{scope}(i,j)})_{j \in V(F_i)})$$

$$Z\text{Trans}(y, z) = \prod_{i, j} EQ_B(y_{b(i,j)}, \cdots, y_{b(i,j)})T(y_1^{(i,j)}, z^{\text{scope}(i,j)})$$

$$Z\text{Terms}(y) = \prod_{i, b} F_i((y_{b(i,j)})_{j \in V(F_i)})$$

$$Z\text{Eq}(x) = \prod_{v} EQ_{\deg(v)}(x^{(v, 1)}, \cdots, x^{(v, \deg(v))})$$

$$Z\text{Trans}(x, y) = \prod_{v, d} EQ_B(y_1^{\text{use}(v, d)}, \cdots, y_B^{\text{use}(v, d)})T(y_1^{\text{use}(v, d)}, x^{(v, d)})$$

$$Z\text{GTrans}(x, y) = \prod_{v, d} G(x^{(v, d-1)}, x^{(v, d)}, y_1^{\text{use}(v, d)}, \cdots, y_B^{\text{use}(v, d)})$$

Note:

1. For fixed $z$ we have $Z\text{Terms}(z) = \sum_y Z\text{Trans}(y, z)Z\text{Terms}(y)$ by expanding the definition of $T F_i^B$. 

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2. Summing over $x$ with the factor $\text{XEq}(x)$ is the same as summing over $z$ and defining $x$ by $x^{(v,d)} = z_v$. Hence summing over $x$ with the factor $\text{XEq}(x)\text{XYTrans}(x,y)$ is the same as summing over $z$ with the factor $\text{YZTrans}(y,z)$.

3. Fix $x$ and $y$. If $\text{XEq}(x) = 1$ then $\text{XYTrans}(x,y) = \text{XYGTrans}(x,y)$ by definition of $G$. And if $\text{XEq}(x)$ is zero then so is $\text{XYGTrans}(x,y)$. Hence $\text{XEq}(x)\text{XYTrans}(x,y) = \text{XYGTrans}(x,y) = 0$.

Hence

\[
Z_\phi = \sum_z Z\text{Terms}(z) = \sum_{y,z} \text{YZTrans}(y,z)\text{YTerms}(y) = \sum_{x,y} \text{XEq}(x)\text{XYTrans}(x,y)\text{YTerms}(y) = \sum_{x,y} \text{XYGTrans}(x,y)\text{YTerms}(y) = Z_\phi'.
\]

\[
\text{Lemma 28. Let } B = 1 \text{ or } B = 2 \text{ and let } T: \{0,1\}^2 \rightarrow \mathbb{R}_p \text{ be a non-degenerate arity 2 signature. Assume that there exist } i, j \in \{0,1\} \text{ such that } T(0,i) > T(1,i) \text{ and } T(0,j) < T(1,j). \text{ Let } U(0), U(1) > 0 \text{ and let } F \text{ be any signature with } |\text{supp}(F)| > 1. \text{ There exists a simple weighting } G \text{ of } F \text{ such that } U \text{ is defined by a pps-formula over } T^\otimes G^B.
\]

\text{Proof. Let } x, x' \text{ be distinct tuples in supp}(F). \text{ By taking an equivalent signature if necessary we can assume that } V(F) = \{1, \cdots, n\} \text{ for some } n \text{ and that } x_n \neq x'_n. \text{ Let }

\[
H(y_n) = \sum_{x_1,y_1,\cdots,x_{n-1},y_{n-1}} T(x_1, y_1) \cdots T(x_{n-1}, y_{n-1}) F(y_1, \cdots, y_n)^B
\]

Note that $H(0), H(1) > 0$.

Let $\det T = T(0,0)T(1,1) - T(0,1)T(1,0)$. We will argue that there is an integer $m > 0$ and polynomial-time computable reals $W(0), W(1) > 0$ such that

\[
\begin{pmatrix}
H(0)W(0)^B \\
H(1)W(1)^B
\end{pmatrix} = \frac{1}{\det T} \begin{pmatrix}
T(1,1) & -T(0,1) \\
-T(1,0) & T(0,0)
\end{pmatrix} \begin{pmatrix}
U(0)^{1/m} \\
U(1)^{1/m}
\end{pmatrix} \tag{1}
\]

We just need to check that the right-hand-side of (1) has non-negative entries. There are two cases. If $T(1,1) > T(0,1)$ and $T(0,0) > T(1,0)$ then $\det T$ is positive and for sufficiently large $m$ we have $T(1,1)U(0)^{1/m} > T(0,1)U(1)^{1/m}$ and $T(1,0)U(0)^{1/m} < T(0,0)U(1)^{1/m}$. If $T(1,1) < T(0,1)$ and $T(0,0) < T(1,0)$ then $\det T$ is negative and for sufficiently large $m$ we have $T(1,1)U(0)^{1/m} < T(0,1)U(1)^{1/m}$ and $T(1,0)U(0)^{1/m} > T(0,0)U(1)^{1/m}$. In either case the right-hand-side of (1) has non-negative entries.

With these $m, W(0), W(1)$ we have

\[
\begin{pmatrix}
U(0)^{1/m} \\
U(1)^{1/m}
\end{pmatrix} = \begin{pmatrix}
T(0,0) & T(0,1) \\
T(1,0) & T(1,1)
\end{pmatrix} \begin{pmatrix}
H(0)W(0)^B \\
H(1)W(1)^B
\end{pmatrix}
\]
Define $G : \{0, 1\}^n \to \mathbb{R}_p$ by $G(x) = F(x)W(x_n)$. Then for all $x_1 \in \{0, 1\}$,

$$U(x_n) = \left( \sum_{y_n} T(x_n, y_n)H(y_n)W(y_n)^B \right)^m = \left( \sum_{x_1, \cdots, x_{n-1}} (T^\circ G^B)(x) \right)^m$$

By distributivity the right-hand-side can be written as a pps-formula over $T^\circ G^B$.

**Lemma 29.** Let $\mathcal{F}$ be a finite set of signatures containing a non-IM-terraced signature. There exists $B \in \{1,2\}$ and a non-degenerate arity 2 signature $T : \{0,1\}^2 \to \mathbb{R}_p$ such that for all finite sets of arity 1 signatures $S$ there is a finite set of variable weights $W$ such that

$$\#\text{CSP}(T^\circ \mathcal{F}^B \cup S) \leq \text{AP} \#\text{CSP}_{\leq 2}^W(\mathcal{F})$$

**Proof.** By Lemma 12 we can assume that $\mathcal{F}$ is closed under pinnings. Choose a pinning-minimal non-IM-terraced signature $F \in \mathcal{F}$. Renaming the variable set if necessary, $F$ has the form given by Lemma 19 and in particular there exists $T$ and $z \in \{0,1\}^B$, $B \geq 1$ such that $z_1 = 0$ and for all $x = 0,1$ and all $y \in \{0,1\}^B$ we have

$$F(x, x, y) = 1_{\{z, \overline{z}\}}(y)T(y_1, x_1)$$

If $B \geq 3$ there are $1 \leq i < j \leq B$ with $y_i = y_j$ and we can express the non-IM-terraced signature $F'$ defined by:

$$F'(x_1, x_2, y_1, \cdots, y_{i-1}, y_{i+1}, \cdots, y_{j-1}, y_{j+1}, \cdots, y_B) = \sum_y F(x_1, x_2, y_1, \cdots, y_{i-1}, y, y_{i+1}, \cdots, y_{j-1}, y, y_{j+1}, \cdots, y_B)$$

Repeating this and using Lemma 8 we can assume $B \leq 2$.

If $B = 2$ and $z_1 \neq z_2$, define $F'$ by

$$F'(x_1, x_2, y_1, y_2) = \sum_{t, y_2} F(x_1, x_2, y_1, y_2')F(t, t, y_2, y_2)$$

Then $F'(1, 0, y_1, y_2) = 0$ for all $y_1, y_2 \in \{0, 1\}$. Also, for all $x, y_1, y_2 \in \{0, 1\}$,

$$F'(x, x, y_1, 1 - y_2) = 1_{\{(0,0),(1,1)\}}(y_1, y_2) \left( \sum_t F(t, t, 1 - y_1, y_1) \right) T(y_1, x)$$

By Lemma 8 we can use $F'$ instead of $F$. Therefore we can assume that $z$ is either $(0,0)$ or $(0,0)$.

Furthermore by taking a simple weighting of $F$ and invoking Lemma 13 we can assume that there exist $i, j \in \{0, 1\}$ such that $T(0,i) > T(1,i)$ and $T(0,j) < T(1,j)$.

Indeed let $U(0) = T(1,0) + T(1,1)$ and $U(1) = T(0,0) + T(0,1)$. Replacing $F$ by the simple weighting $F'$ defined by

$$F'(x_1, x_2, y_1, y_2) = U(y_1)F(x_1, x_2, y_1, y_2)$$

has the effect of replacing $T(y, z)$ by $U(y)T(y, z)$. If $T(0,0)T(1,1) > T(0,1)T(1,0)$ then $U(0)T(0,0) > U(1)T(1,0)$ and $U(0)T(0,1) < U(1)T(1,1)$. Otherwise $T(0,0)T(1,1) < T(0,1)T(1,0)$ so $U(0)T(0,0) < U(1)T(1,0)$ and $U(0)T(0,1) > U(1)T(1,1)$. 

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Let \( S' \) be the set of permissive signatures in \( S \cup \{ U_0, U_1 \} \) where \( U_0(0) = 2, U_0(1) = 1 \) and \( U_1(0) = 1, U_1(1) = 2 \). For each \( U \in S' \), let \( F_U \) be the signature given by Lemma \ref{lem:exist} such that \( F_U \) is a simple weighting of \( F \) (or any other signature in \( F \) - we only use \( F \) for concreteness), and \( U \) can be expressed by a pps-formula over \( \{ T^\otimes F_U^B \} \). Let \( G = F \cup \{ F_U \mid U \in S' \} \). By Lemma \ref{lem:uniform} Lemma \ref{lem:uniform} and Lemma \ref{lem:uniform} we have
\[
\#\text{CSP}(T^\otimes F^B \cup S') \leq_{\text{AP}} \#\text{CSP}(T^\otimes G^B) \leq_{\text{AP}} \#\text{CSP}_{\leq_2}(G) \leq_{\text{AP}} \#\text{CSP}_{\leq_2}(F)
\]
for some finite set \( W \). Using \( U_0 \) and \( U_1 \) as variable weights we have:
\[
\#\text{CSP}\{(2^n,2^b); a,b \in \mathbb{Z}\}(T^\otimes F^B \cup S') \leq_{\text{AP}} \#\text{CSP}(T^\otimes F^B \cup S')
\]
But PIN\(_0 = (U_1)_{h-\text{max}} \) with \( h(1,0) = 1 \) and \( h(1,1) = 0 \), and similarly PIN\(_1 = (U_0)_{h-\text{max}} \) with \( h(1,0) = 0 \) and \( h(1,1) = 1 \), so by Lemma \ref{lem:uniform} we have
\[
\#\text{CSP}(T^\otimes F^B \cup S' \cup \{ \text{PIN}_0, \text{PIN}_1 \}) \leq_{\text{AP}} \#\text{CSP}\{(2^n,2^b); a,b \in \mathbb{Z}\}(T^\otimes F^B \cup S')
\]
The signatures in \( S \setminus S' \) are just scalar multiples of PIN\(_0 \) and PIN\(_1 \) so we have established that
\[
\#\text{CSP}(T^\otimes F^B \cup S) \leq_{\text{AP}} \#\text{CSP}_{\leq_2}(F).
\]

**Lemma 30.** Let \( F \) be a finite set of signatures. Assume that \( F \) contains a signature that is not in Weighted-NEQ-conj and a signature that is not IM-terraced. Let \#\( X = \#\text{SAT} \) otherwise. There is a finite set of variable weights \( W \) such that
\[
\#X \leq_{\text{AP}} \#\text{CSP}_{\leq_2}(F).
\]

**Proof.** By Lemma \ref{lem:uniform} we can assume \( F \) is closed under pinnings. Let \( G \) be a pinning-minimal signature subject to \( G \in F \) Weighted-NEQ-conj. In particular \( G \) is indecomposable. As in Proposition prop:wnicsisclone we will use the characterization that an indecomposable signature is in Weighted-NEQ-conj if and only if its support has order at most two.

Let \( B, T \) be as given by Lemma \ref{lem:exist} applied to \( F \). Either \( T(0,0)T(1,1) > 0 \) and \( \text{supp}(T^\otimes G^B) \supseteq \text{supp}(G) \), or \( T(1,1)T(1,0) > 0 \) and \( \text{supp}(T^\otimes G^B) \supseteq \{ \varphi \mid \varphi \in \text{supp}(G) \} \). In either case \( |\text{supp}(T^\otimes G^B)| \geq |\text{supp}(G)| \geq 2 \). If \( T^\otimes G^B = G_1 \otimes G_2 \) then \( G^B = (S^\otimes G_1) \otimes (S^\otimes G_2) \) where \( S \) is the matrix inverse of \( T \), that is, the unique solution to \( \sum_j T(i,j)S(j,k) = \text{EQ}_2(i,k) \) \((i,k) \in \{(0,1),(1,0)\})\). But \( G^B \) is indecomposable. Therefore \( T^\otimes G^B \) is indecomposable, and hence it is not in Weighted-NEQ-conj.

If \#\( X \) = \#\text{SAT} then let \( H \) be a pinning-minimal non-logsupermodular signature in \( F \). In particular by Lemma \ref{lem:nonlogsuper} \( \text{supp}(H) \subseteq \{ 0, \varphi, \phi, \varphi \} \) for some vector \( \varphi \) with \( a \) zeros and \( b \) ones for some \( a, b \geq 1 \). Hence
\[
\begin{pmatrix}
(T^\otimes H^B)(0) & (T^\otimes H^B)(\varphi) \\
(T^\otimes H^B)(\phi) & (T^\otimes H^B)(1)
\end{pmatrix} =
\begin{pmatrix}
T(0,0)^a & T(0,1)^a \\
T(1,0)^a & T(1,1)^a
\end{pmatrix}
\begin{pmatrix}
H(0)^B & H(\varphi)^B \\
H(\phi)^B & H(1)^B
\end{pmatrix}
\begin{pmatrix}
T(0,0)^b & T(0,1)^b \\
T(1,0)^b & T(1,1)^b
\end{pmatrix}
\]

Denote the latter expression by \( M_1M_2M_3 \). Since \( H(0)H(1) < H(\varphi)H(\phi) \), the middle matrix \( M_2 \) has a negative determinant. The determinants of the neighbouring matrices \( M_1 \) and \( M_3 \) have the same sign: if \( T(0,0)T(1,1) > T(0,1)T(1,0) \) they both have a positive determinant, otherwise they both have a negative determinant. Therefore the matrix on the left-hand-side has a negative determinant, and hence \( T^\otimes H^B \) is not logsupermodular.

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Let \( B, T \) be as given by Lemma \ref{lemma:im-conj-app} applied to \( \mathcal{F} \). By Lemma \ref{lemma:arith-conj} there is a finite set of arity 1 signatures \( S \) such that \( \#X \leq_{\mathcal{F}} \#\text{CSP}(T \circ \mathcal{F}^B \cup S) \). By the choice of \( B \) and \( T \) we have \( \#\text{CSP}(T \circ \mathcal{F}^B \cup S) \leq_{\mathcal{F}} \#\text{CSP}^W_{\mathcal{W}}(\mathcal{F}) \) for some finite set \( W \).

**Lemma 31.** Let \( \mathcal{F} \) be a terraced signature whose support is not basically binary. Then \( \#\text{PM} \leq_{\mathcal{F}} \#\text{CSP}_{\geq 0}(\{F\}) \).

**Proof.** By pinning and applying \( h \)-maximisation as in the proof of Lemma \ref{lemma:im-conj-app} we can assume \( \text{supp}(F) = \text{PM}^U \) for some \( U \subseteq \{1, 2, 3\} \).

We will show that \( \text{PM}^U \) is a simple weighting of \( F \). Let \( F' = F^U \) and \( U'_i(0) = U'_i(1) = 1 \) and \( U'_1(1) = 1/F'(1,0,0), U'_2(1) = 1/F'(0,1,0), U'_3(1) = 1/F'(0,0,1) \). Then \( F'(x_1, x_2, x_3) = \text{PM}_3(x_1, x_2, x_3) \) for all \( x_1, x_2, x_3 \in \{0,1\} \). For all \( x \in \{0,1\} \) define \( U_i(x) = U'_i(x) \) for \( i \in \{1, 2, 3\} \setminus U \) and \( U_i(x) = U'_i(1-x) \) for \( i \in U \). Then \( F(x_1, x_2, x_3) = \text{PM}_3(x_1, x_2, x_3) \) for all \( x_1, x_2, x_3 \in \{0,1\} \) as required.

By Lemma \ref{lemma:im-conj-app} there is an AP-reduction from \#PM to \#CSP_{\geq 0}(\{\text{PM}^U \}) \), and since \( \text{PM}^U \) is a simple weighting of \( F \), by Lemma \ref{lemma:im-conj-app} there is an AP-reduction from \#CSP_{\geq 0}(\{\text{PM}^U \}) \) to \#CSP_{\geq 0}(\{F\}) \).

**Lemma 32.** Let \( \mathcal{F} \) be a finite set of signatures, containing a signature whose support is not in IM-conj. Then \( \#\text{CSP}_{\geq 0}(\mathcal{F}^U) \leq_{\mathcal{F}} \#\text{CSP}_{\geq 0}(\mathcal{F}) \) where \( \mathcal{F}^U \) is the closure of \( \mathcal{F} \) under flips:

\[
\mathcal{F}^U = \{F^U \mid F \in \mathcal{F} \text{ and } U \subseteq V(F)\}
\]

**Proof.** It suffices to do one flip at a time: to show that for all \( G \in \mathcal{F} \) and all \( U \subseteq V(G) \) we have \#CSP(\mathcal{F} \cup \{G^U\}) \leq_{\mathcal{F}} \#\text{CSP}(\mathcal{F}) \). By Lemma \ref{lemma:im-conj-app} and Lemma \ref{lemma:im-conj-app} we can assume that \( \mathcal{F} \) is closed under pinnings and \( h \)-maximisations.

Pick a pinning-minimal signature \( F \in \mathcal{F} \) such that \( \text{supp}(F) \) is not IM-conj. By Lemma \ref{lemma:im-conj-app} \( \text{supp}(F) \) is a (proper) subset of \( \{0, x, \overline{x}, 1\} \) for some \( x \), and by taking an equivalent signature we can assume that there exists \( a, b \geq 1 \) such that \( x \) is an arity \( a+b \) vector with \( z_i = 0 \) for \( 1 \leq i \leq a \) and \( z_i = 1 \) for \( a+1 \leq i \leq a+b \). By taking a suitable \( h \)-maximisation (Lemma \ref{lemma:im-conj-app}) we may assume that \( 0, 1 \notin \text{supp}(F) \), and by simple weighting (Lemma \ref{lemma:im-conj-app}) we may assume that \( F \) is zero-one valued.

If the arity of \( F \) is two then \( F = \text{NEQ} \). But

\[
G^U(x, x') = \sum_{y \in U} G(y, x') \prod_{i \in U} \text{NEQ}(x_i, y_i)
\]

for all \( x \in \{0,1\}^U \) and \( x' \in \{0,1\}^{V(G) \setminus U} \). Hence \#CSP_{\geq 0}(\mathcal{F} \cup \{G^U\}) \leq_{\mathcal{F}} \#\text{CSP}_{\geq 0}(\mathcal{F}) \) by Lemma \ref{lemma:im-conj-app}.

If the arity of \( F \) is greater than two then for all \( x, x' \in \{0,1\}^a \) and \( y, y' \in \{0,1\}^b \) we have

\[
\text{EQ}_{2a}(x, x') = \sum_y F(x, y) F(y, x')
\]

\[
\text{EQ}_{2b}(y, y') = \sum_x F(x, y) F(y', x)
\]

One of these has arity at least three, so Lemma \ref{lemma:im-conj-app} can be applied. The Lemma is trivial in \( \mathcal{F} \) is contained in Weighted-NEQ-conj, and otherwise by Lemma \ref{lemma:im-conj-app} Lemma \ref{lemma:im-conj-app} and Lemma \ref{lemma:im-conj-app} we have

\[
\#\text{CSP}_{\geq 0}(\mathcal{F}^U) \leq_{\mathcal{F}} \#\text{SAT} \leq_{\mathcal{F}} \#\text{CSP}_{\geq 0}(\mathcal{F}) \leq_{\mathcal{F}} \#\text{CSP}_{\geq 0}(\mathcal{F})
\]

as required. And if \( \mathcal{F} \) is contained in Weighted-NEQ-conj then \#CSP_{\geq 0}(\mathcal{F}^U) \) already has an FPRAS again by Lemma \ref{lemma:im-conj-app}.
Theorem\[2\] Let $\mathcal{F}$ be a finite set of signatures. If every signature in $\mathcal{F}$ is logsupermodular then $\#\text{BIS} \leq_{\text{AP}} \#\text{CSP}_{\geq 0}^0(\mathcal{F})$ and $\mathcal{F}$ necessarily contains a signature that is not terraced.

Remark. These conditions describe when all the reductions in the following proof fail. They are certainly not exhaustive. For example the following relation $R$ is not in Weighted-NEQ-conj, is not basically binary, and is not terraced, but (the signature of) $R$ is IM-terraced, $R$ is closed under meets, and has no pinning equivalent to $\text{EQ}_2$.

Proof. The FPRAS is given by Lemma[10] If every signature in $\mathcal{F}$ is terraced, there is a signature $F$ in $\mathcal{F}$ that does not have basically binary support. By Lemma[22] $\mathcal{F}$ is not logsupermodular, and by Lemma[31] we have $\#\text{PM} \leq_{\text{AP}} \#\text{CSP}_{\geq 0}^0(\{F\})$ as required.

Theorem\[3\] Let $\mathcal{F}$ be a finite set of signatures. Assume that not every signature in $\mathcal{F}$ is Weighted-NEQ-conj, and not every signature in $\mathcal{F}$ is basically binary, and not every signature in $\mathcal{F}$ is terraced.

1. Every signature $F \in \mathcal{F}$ is IM-terraced.

2. Either the support of every signature $F$ in $\mathcal{F}$ is closed under meets $F(x, y) \in \text{supp}(F) \implies x \land y \in \text{supp}(F)$, or the support of every signature $F$ in $\mathcal{F}$ is closed under joins $F(x, y) \in \text{supp}(F) \implies x \lor y \in \text{supp}(F)$.

3. No pinning of the support of a signature in $F$ is equivalent to $\text{EQ}_2$.

Proof of Theorem. By Lemma[12] we can assume $\mathcal{F}$ is closed under pinning. We will consider each condition in turn.

1. Assume that $\mathcal{F}$ is not IM-terraced. The conclusion follows from Lemma[30].
2. Assume that condition 2 does not hold but condition 1 holds. Pick a non-terraced signature \( F' \in \mathcal{F} \). By Lemma 20 there is a signature \( F \) equivalent to a pinning of \( F' \) and satisfying certain conditions: \( V(F) = \{1, \ldots, |V(F)|\} \), and there are configurations \( p \in \{0, 1\}^{1,2} \) and \( z \in \{0, 1\}^{3,\cdots,|V(F)|} \) such that \( F_p \) is identically zero and for all \( x \in \{p^{(1)}, p^{(2)}\} \) and \( y \in \{0, 1\}^{3,\cdots,|V(F)|} \) we have

\[
F(x_1, x_2, y_3, \cdots, y_{|V(F)|}) = \begin{cases} 
T(y_3, x_1) & \text{if } y = z \text{ or } y = \overline{x} \\
0 & \text{otherwise}
\end{cases}
\]

We have assumed that condition 1 holds, so \( F' \) is IM-terraced, so \( p_1 = p_2 \). Permuting the domain \( \{0, 1\} \) if necessary we can assume \( p_1 = p_2 = 0 \) without loss of generality.

There is a signature \( G' \in \mathcal{F} \) such that \( \text{supp}(G') \) is not closed under joins; let \( G \) be a minimal pinning of \( G' \) such that \( \text{supp}(G) \) is not closed under joins.

By Lemma 16 there exists \( x \) such that \( \text{supp}(G) = \{0, x, \overline{x}\} \) or \( \text{supp}(G) = \{x, \overline{x}\} \). And \( \{0, 1\} \) is closed under joins, so \( x \neq 1 \) and there is a variable \( i \) such that \( x_i = 0 \). Since \( G \) is IM-terraced, \( \text{supp}(G) \) is a delta matroid (Lemma 18) and hence \( x^{(i, j)} \in \text{supp}(G) \) for some \( j \in V(G) \). But this can only mean that \( x^{(i, j)} = \overline{x} \), which implies \( V(G) = \{i, j\} \). Also, the arity of \( G \) is not 1. It will be harmless to take \( V(G) = \{1, 2\} \). With this assumption we have \( G(1, 1) = 0 \) and \( G(0, 1), G(1, 0) \neq 0 \).

Define \( H : \{0, 1\}^{V(F)} \rightarrow \mathbb{R}_p \) by

\[
H(x_1, x_2, y) = \sum_{t=0,1} G(x_1, t)F(t, x_2, y)
\]

If we shorten \( F_{1\rightarrow i, 2\rightarrow j} \) to \( F_{ij} \), and similarly define \( F'_{ij} \), and allow scalar multiplication of a signature by a constant, we have:

\[
\begin{align*}
H_{10} &= G(1, 0)F_{00} + G(1, 1)G_{10} \text{ which is identically zero} \\
H_{00} &= G(0, 0)F_{00} + G(0, 1)F_{10} = G(0, 1)F_{01} \\
H_{11} &= G(1, 0)F_{01} + G(1, 1)F_{11} = G(1, 0)F_{01}
\end{align*}
\]

Hence \( H \) is not IM-terraced. (A related trick, expressing IMP using OR and NAND, is used in [DGJRT0].)

We have shown (condition 1) that there is a finite set \( W \) such that \( \#X \leq \#\text{CSP}_{≤2}(\mathcal{F} \cup \{H\}) \); by Lemma 8 \( \#\text{CSP}_{≤2}(\mathcal{F} \cup \{H\}) \leq \#\text{CSP}_{≤2}(\mathcal{F} \cup \{F, G\}) \); and by Lemma 12 \( \#\text{CSP}_{≤2}(\mathcal{F} \cup \{F, G\}) \leq \#\text{CSP}_{≤2}(\mathcal{F}) \) where \( W' = W \cup \{(0, 1), (1, 0)\} \).

3. Assume that condition 3 does not holds but conditions 1 and 2 do hold. So there is a signature in \( \mathcal{F} \) whose support is not closed under joins, and a signature in \( \mathcal{F} \) whose support is not closed under meets. By permuting the domain \( \{0, 1\} \) if necessary we can assume without loss of generality that the support of every signature in \( \mathcal{F} \) is closed under meets.

Pick \( G \) satisfying \( \text{supp}(G) = \text{NAND} \) as follows. Let \( H \) be a minimal non-terraced pinning of a signature in \( \mathcal{F} \). Reordering the variables according to Lemma 20 there exist not necessarily distinct configurations \( y, y' \in \{0, 1\}^{3,\cdots,|V(H)|} \) such that \( (0, 1, y), (1, 0, y') \in \text{supp}(H) \), and \( (0, 0, y \land y'), (1, 1, y \lor y') \) are not both in \( \text{supp}(F) \). By assumption \( \text{supp}(G) \) is closed under
meets, so it must not be closed under joins. By the same argument used for condition 2, there is a pinning \( G \) of \( H \) of arity 2, and we can take \( V(G) = \{1, 2\} \) so \( \text{supp}(G) = \text{NAND} \).

Let \( h(1) = h(2) = 1 \) so \( \text{supp}(G_{h-\text{max}}) = \text{NEQ} \). Since \( G_{h-\text{max}} \) fails condition 2, there is a finite set \( W \) such that \( \#X \leq_{\text{AP}} \#\text{CSP}_{=2}^{W}(\mathcal{F} \cup \{G_{h-\text{max}}\}) \).

We will want to use variable weights that are arbitrary powers of two, so it is convenient to hide \( W \) at this point. By Lemma 13 there is a set of simple weightings \( \mathcal{G} \) of signatures in \( \mathcal{F} \), and a set of simple weightings \( \mathcal{G}' \) of \( G_{h-\text{max}} \), such that \( \#\text{CSP}_{=2}^{W}(\mathcal{F} \cup \{G_{h-\text{max}}\}) \leq_{\text{AP}} \#\text{CSP}_{=2}^{W}(\mathcal{G} \cup \mathcal{G}') \). Let

\[
P = \{(2^{p_0}, 2^{p_1}) \mid p_0, p_1 \in \mathbb{Z}\}
\]

Let \( \mathcal{G}'' \) be the set of simple weightings \( \mathcal{G}' \) of \( G \) satisfying \( G_{h-\text{max}}' \in \mathcal{G}' \). In other words, for all arity 1 signatures \( U, W \), if the signature defined by \( G_{h-\text{max}}(x, y)U(x)W(y) \) is in \( \mathcal{G}' \), then the signature defined by \( G(x, y)U(x)W(y) \) is in \( \mathcal{G}'' \). Note that \( |\mathcal{G}''| = |\mathcal{G}'| \) is finite. By Lemma 9

\[
\#\text{CSP}_{=2}^{W}(\mathcal{G} \cup \mathcal{G}') \leq_{\text{AP}} \#\text{CSP}_{=2}^{W}(\mathcal{G} \cup \mathcal{G}'')
\]

We will show that

\[
\#\text{CSP}_{=2}^{W}(\mathcal{G} \cup \mathcal{G}'') \leq_{\text{AP}} \#\text{CSP}_{=2}^{W}(\mathcal{G} \cup \mathcal{G}' \cup \{\text{EQ}_2\})
\]

(2)

We are given an instance \((w, \phi)\) of \( \#\text{CSP}_{=2}^{W}(\mathcal{G} \cup \mathcal{G}'') \). For each \( v \in V = V_{\phi} \) there exists \( p_v \) such that \( w(v, 1)/w(v, 0) = 2^{p_v} \) for \( i = 0, 1 \). Let \( V' = \{v_i \mid v \in V; 0 \leq i \leq |p_v|\} \). Define \( w' : V \rightarrow W \) by \( w'(v_0) = (1, 1) \) and for all \( i > 0 \),

\[
w'(v_i) = \begin{cases} (1, 2) & \text{if } p_v < 0 \\ (2, 1) & \text{if } p_v > 0 \end{cases}
\]

Modify \( \phi \) as follows to obtain a new formula \( \phi' \): for each \( v \in V \), insert atomic formulas \( \text{EQ}_2(v_0, v_1) \cdots \text{EQ}_2(v_{|p_v|-1}, v_{|p_v|}) \) and replace the two occurrences of \( v \) by \( v_0 \) and \( v_{|p_v|} \). Note that configurations \( x' \) of \( V' \) have zero weight in \((w', \phi')\) unless there exists \( x \in \{0, 1\}^V \) such that \( x_{v_i} = x_v \) for all \( v, i \), and in this case \( \text{wt}_{\phi'}(x') = \text{wt}_{\phi}(x)C \) where \( C = \prod_{v \in V} \min(w(v, 0), w(v, 1)) \).

Hence \( Z_{\phi} = Z_{\phi'}C \). And \( Z_{\phi'} \) can be approximated by the oracle. This establishes the AP-reduction (2).

To finish, let \( F \) be a pinning of a signature in \( \mathcal{F} \) such that \( \text{supp}(F) \) is equivalent to \( \text{EQ}_2 \). Then \( F(x, y) = \text{EQ}_2(x, y)F(x, x) \) for all \( x, y \in \{0, 1\} \) so \( F \) is a simple weighting of \( \text{EQ}_2 \). By Lemma 13 there is a finite set \( W' \) (which we can assume contains \((0, 1)\) and \((1, 0)\)) such that

\[
\#\text{CSP}_{=2}^{W'}(\mathcal{G} \cup \mathcal{G}' \cup \{\text{EQ}_2\}) \leq_{\text{AP}} \#\text{CSP}_{=2}^{W}(\mathcal{F} \cup \{\mathcal{F} \})
\]

and \( \#\text{CSP}_{=2}^{W'}(\mathcal{F} \cup \{\mathcal{F} \}) \leq_{\text{AP}} \#\text{CSP}_{=2}^{W}(\mathcal{F}) \) by Lemma 12.

\[\square\]

7 Degree three and higher

In this section we will study \( \#\text{CSP}_{=k}^{W}(\mathcal{F}) \) for \( k > 2 \). We will use a result of Slyph about the complexity of the partition function of the hardcore model on a graph. The partition function of the hardcore model with fugacity \( \lambda \), defined on a graph \( G \), is defined to be the sum of \( \lambda^{|I|} \) over independent sets \( I \) of \( G \).
Lemma 33 ([Sly10], Theorem 1). For every $d \geq 3$ there exists $\lambda_{c}(d), \epsilon(d) > 0$ such that when $\lambda_{c}(d) < \lambda < \lambda_{c}(d) + \epsilon(d)$, unless $\text{NP} = \text{RP}$, there does not exist an FPRAS for the partition function of the hardcore model with fugacity $\lambda$ for graphs of maximum degree at most $d$.

Lemma 34 is not stated as an AP reduction. We would like to present complexity-theoretic results that are not stated as AP reductions. Let $\#X$ and $\#Y$ be $\mathbb{R}^p$-valued function problems. The notation $\#X \leq_{\text{AP}} \#Y$ means: $\#X$ has an FPRAS if $\#Y$ has an FPRAS. The following Lemma is given as a remark in, for example, [DGGJ03] and [Jer03].

Lemma 34. If $\text{NP} = \text{RP}$ then $\#\text{SAT}$ has an FPRAS.

Lemma 35. Let $R = \text{NAND}$ or $R = \text{OR}$. There exists a finite set of variable weights $W$ such that $\#\text{SAT} \leq_{\text{AP}} \#\text{CSP}^W_{\leq 3}(\{R\})$.

Proof. It suffices to consider $R = \text{NAND}$; the definitions of $\#\text{CSP}$ are not affected by permuting the domain $\{0, 1\}$, so

$$\#\text{CSP}^W_{\leq 3}(\overline{\text{NAND}}) =_{\text{AP}} \#\text{CSP}^W_{\leq 3}(\text{NAND})$$

where $\overline{W} = \{(b, a) \mid (a, b) \in F\}$ and where $\overline{\text{NAND}}$ is defined by $\overline{\text{NAND}}(x) = \text{NAND}(\overline{x}) = \text{OR}(x)$.

Let $\lambda$ be a rational number such that $\lambda_{c}(d) < \lambda < \lambda_{c}(d) + \epsilon(d)$ where $\lambda_{c}$ and $\epsilon$ are given by Lemma 33. Let $W = \{(1, \lambda)\}$. We will show that $\#\text{HC}^W_{\leq 3}(\lambda) \leq_{\text{AP}} \#\text{HC}^W_{\leq 3}(\lambda)$ and the problem of computing the partition function of the hardcore model with fugacity $\lambda$ for graphs of maximum degree at most 3. Then $\#\text{SAT} \leq_{\text{AP}} \#\text{HC}^W_{\leq 3}(\lambda)$ by Lemma 33.

Given an instance $(V, E)$ of $\#\text{HC}^W_{\leq 3}(\lambda)$, we can query the $\#\text{CSP}^W_{\leq 3}(\text{NAND})$ oracle to approximate

$$Z^W_{\phi} = \sum_{x \in \{0, 1\}^V} \prod_{i \in W} \lambda^{x_i} \prod_{\{i, j\} \in E} \text{NAND}(x_i, x_j)$$

But this is just the sum of $\lambda^{|I|}$ over independent sets $I$ in $G$, which is the correct output of $\#\text{HC}^W_{\leq 3}(\lambda)$ on this instance. □

Theorem 4. Let $F$ be a finite set of signatures and assume that not every signature in $F$ has degenerate support. There exists a finite set of variable weights $W$ such that $\#\text{CSP}^{\leq_{0}}(F) \leq_{\text{AP}} \#\text{CSP}^W_{\leq_{0}}(F)$.

Proof. Let $F_1$ be a signature in $F$ whose support is non-degenerate. Let $F_2$ be a minimal non-degenerate pinning of $F_1$. Define $F(x_1, x_2) = \sum_{x_3, \ldots, x_k} F_2(x_1, \ldots, x_k)$. By Lemma 17 either the arity of $F_2$ is 2, or $\text{supp}(F_2)$ equals $\{x, \overline{x}\}$ for some tuple $x$. In either case $R = \text{supp}(F)$ is non-degenerate.

Now we claim that there are arity one signatures $U, V$, taking positive values, such that for all $x, y \in \{0, 1\}$ the value $F(x, y)U(x)V(y)$ is zero or one. Since $\text{supp}(F)$ is not degenerate there is a flip $F^S$ of $F$ with $F^S(0, 0) = 0$. We will find $U^S(0), U^S(1), V^S(0), V^S(1) > 0$ such that $F^S(x, y)U^S(x)V^S(y)$ is zero-one valued for all $x, y \in \{0, 1\}$; then $F(x, y)U(x)V(y)$ is also zero-one valued, where $U$ and $V$ are the flips $(U^S)^{S^0(1)}$ and $(V^S)^{S^0(1)}$, respectively, establishing the claim. Since $F^S$ is non-degenerate, $F^S(0, 1), F^S(1, 0) > 0$, so $\text{supp}(F^S) = \{(0, 1), (1, 0)\}$ or $\text{supp}(F^S) = \{(0, 1), (1, 0), (1, 1)\}$. In the first case take $U^S(x) = 1/F^S(x, 1-x)$ and $V^S(x) = 1$ for all $x = 0, 1$. In the second case set $U^S(1) = 1$ and $V^S(0) = 1/F^S(1, 0)$ and $V^S(1) = 1/F^S(1, 1)$ and $U^S(0) = F^S(1, 1)/F^S(0, 1)$.
We will show that there is a finite set $W$ such that
\[
\#CSP^{\geq 0}(\mathcal{F}) \leq_{AP}^* \#CSP^{W}_{\leq 3}(\mathcal{F} \cup \{R\})
\]  \hfill (3)

Then using both parts of Lemma \[13\] there is a finite set $W'$ such that
\[
\#CSP^{W'}_{\leq 3}(\mathcal{F} \cup \{R\}) \leq_{AP} \#CSP^{W'}_{\leq 3}(\mathcal{F} \cup \{F\})
\]

But $F_1 \in \mathcal{F}$, and $F_2$ is a pinning of $F_1$ (Lemma \[12\]), and $F$ is given by a $(\leq 3)$-formula over $F_2$ (Lemma \[8\]):
\[
\#CSP^{W'}_{\leq 3}(\mathcal{F} \cup \{F\}) \leq_{AP} \#CSP^{W'}_{\leq 3}(\mathcal{F} \cup \{F_2\}) \leq_{AP} \#CSP^{W'}_{\leq 3}(\mathcal{F})
\]

So we are done if we can show (3).

Up to equivalence,
\[
R \in \{\text{NAND, OR, EQ}_2, \text{NEQ, IMP}\}
\]

If $R = \text{NEQ}$ then $\sum_y R(x,y)R(y,z)$ is a $(\leq 3)$-formula expressing EQ_2. By Lemma \[8\] we have $\#CSP^{W}_{\leq 3}(\mathcal{F} \cup \{\text{EQ}_2\}) \leq_{AP} \#CSP^{W}_{\leq 3}(\mathcal{F})$ for all sets $W$ containing $(1,1)$. So we can ignore the case $R = \text{NEQ}$. If $R = \text{NAND}$ or $R = \text{OR}$ then $\#CSP^{\geq 0}(\mathcal{F}) \leq_{AP} \#\text{SAT} \leq_{AP} \#CSP^{W'}_{\leq 3}(\{R\})$ for some finite set $W$, by Lemma \[11\] and Lemma \[35\]. Otherwise $R = \text{EQ}_2$ or $R = \text{IMP}$. We will "simulate equality" as in \[DGJR10\].

By Lemma \[10\] we may assume that $\mathcal{F}$ is not contained in Weighted-NEQ-conj. It follows from \[BDGJ12\] Theorem 14, Proposition 25 that there is a finite set $S$ of arity 1 signatures such that $\#CSP(\mathcal{F} \cup S) =_{AP} \#CSP(\mathcal{F} \cup \{\text{IMP}\}) =_{AP} \#CSP^{\geq 0}(\mathcal{F})$. Let
\[
W = \{(U(0),U(1)) \mid U \text{ has arity 1, and } U \in \mathcal{F} \} \cup \{(1,1)\}
\]

We will show that $\#CSP(\mathcal{F} \cup S) \leq_{AP} \#CSP^{W}_{\leq 3}(\mathcal{F})$. Given an instance $(V,\phi)$ of $\#CSP(\mathcal{F} \cup S)$, for each variable $v$ replace all its occurrences by distinct variables $v_1, \ldots, v_d$ and insert new atomic formulas $R(v_1, v_2) \cdots R(v_d, v_1)$. This gives a new formula $\phi'$ on variables $V'$. Now replace any arity 1 atomic formula $U(v_i)$ by a variable weight on $v_i$; that is, delete these atomic formulas to obtain $\phi''$ and define $w : V' \rightarrow V$ by
\[
 w(v_i) = \begin{cases} (U(0),U(1)) & \text{if there is an atomic formula } U(v_i) \text{ in } \phi' \\ (1,1) & \text{otherwise} \end{cases}
\]

for all $v_i \in V'$. Then $\phi''$ is a $(\leq 3)$-formula with $Z'_{\phi''} = Z_{\phi}$, so we can just query the oracle.

\[\square\]

8 Tractable problems not in FP

In this section we will argue that there is a large tractable region for $\#CSP^{\geq 0}_{\leq d}$. The existence of these FPRASes contrasts with the unbounded problem $\#CSP^{\geq 0}_{\leq 3}$. Assuming that $\#BIS$ does not have an FPRAS, $\#CSP^{\geq 0}_{\leq 3}(F)$ has an FPRAS if and only if $\#CSP^{\geq 0}_{\leq 3}(F)$ is in FP, as least as long as $F$ is rational-valued (see Lemma \[6\] and \[DGJ10\]). But $\#CSP^{\geq 0}_{\leq d}(F)$ can have an FPRAS even when $\#CSP^{\geq 0}_{\leq d}(F)$ is $\#P$-hard.

**Proposition 36.** \[CLX09\] Theorem 5.3] If $\mathcal{F}$ is not a subset of Weighted-NEQ-conj then $\#CSP^{\geq 0}_{\leq 3}(\mathcal{F})$ (without any approximation) is $\#P$-hard.
Proof. Define $U$ by $U(0) = 1$ and $U(1) = 2$. Using variable weights instead of $U$, we have $\#\text{CSP}_{\leq 3}(\mathcal{F} \cup \{U\}) \leq_{\text{APF}} \#\text{CSP}_{\leq 3}^{\geq 0}(\mathcal{F})$. Now we can appeal to [CLX09 Theorem 5.3]. Their set “$\mathcal{A}$” does not contain $U$, and Weighted-NEQ-conj is contained in their “$\mathcal{P}$”. Hence $\#\text{CSP}_{\leq 3}(\{F, U\})$ is $\#\mathcal{P}$-hard.

This following argument is inspired by [ZLB11], and in particular we use the same quantity $J$.

**Theorem 5** Let $d, k \geq 2$. Let $F$ be a an arity $k$ signature with values in the range $[1, \frac{d(k-1)+1}{d(k-1)-1}]$. Then $\#\text{CSP}_{\leq d}^{\geq 0}(F)$ has an FPRAS.

**Proof.** We will use a path coupling argument on a Markov chain with Glauber dynamics. We will proceed by giving a FPAUS, which in this case is a randomised algorithm that, given an instance $(w, \phi)$ and $\epsilon > 0$, outputs a random configuration $\mu$ such that the total variation distance of $\mu$ from $\pi^w_\phi$ is at most $\epsilon$ where $\pi^w_\phi(\sigma) = \frac{\text{wt}^w_\phi(\sigma)}{Z^w_\phi}$; and the algorithm runs in time polynomial in the size of the input and $\log(1/\epsilon)$.

The FPAUS is to simulate a Markov chain of configurations $(X_t)_{t=0,1,...}$ and output $X_T$ for some $T$ to be determined later. For configurations $X$ and variables $v$ we will use the notation $X[v \mapsto j](u) = X(u)$ for $u \neq v$ and $X[v \mapsto j](v) = j$. Let $X_0 \in \{0, 1\}^V$ be any configuration. For each $t \geq 1$ let $v_t$ be distributed uniformly at random and let $X_t$ be distributed according to heat bath dynamics, that is, distributed according to $\pi^w_\phi$ conditioned on $X \in \{X_{t-1}[v_t \mapsto 0], X_{t-1}[v_t \mapsto 1]\}$. Thus

$$\mathbb{P}[X_t(i) = 1 | X_{t-1}, v_t] = \frac{\text{wt}^w_\phi(X_{t-1}[v \mapsto 1])}{\text{wt}^w_\phi(X_{t-1}[v \mapsto 0]) + \text{wt}^w_\phi(X_{t-1}[v \mapsto 1])}.$$ 

This probability is easy to compute exactly, so each step of the Markov chain can be simulated efficiently.

Consider another Markov chain $(Y_t)_{t \geq 0}$ distributed in the same way as $(X_t)_{t \geq 0}$, with the optimal coupling given that both chains choose the same variables $v_t$. So

$$\mathbb{P}[X_t(v_t) \neq Y_t(v_t) | X_{t-1}, Y_{t-1}, v_t] = |\mathbb{P}[X_t(v_t) = 1 | X_{t-1}, Y_{t-1}, v_t] - \mathbb{P}[Y_t(v_t) = 1 | X_{t-1}, Y_{t-1}, v_t]|$$

Define $\beta = \beta(w, \phi) = \max_{X_0, Y_0 : |X_0 \triangle Y_0| = 1} \mathbb{E}[d(X_1, Y_1)]$. Let $M$ be the maximum value taken by $F$. We will establish the bound

$$\beta \leq 1 - c|V|^{-1} \quad (4)$$

for some $c > 0$ depending only on the parameters $d, k, M$. Then by the General Path Coupling Theorem of [BD97] the total variation distance from the stationary distribution is at most $\epsilon$ as long as $T \geq \log(|V|\epsilon^{-1})/\log \beta^{-1} = \text{poly}(|V|, \log \epsilon^{-1})$. This gives the required FPAUS. Given the FPAUS, there is an FPRAS by [JVY86 Theorem 6.4] (the self-reducibility is Lemma [12]).

We will now bound $\beta$. Fix configurations $X_0$ and $Y_0$ that only differ on a single variable $u$. For all $v_1 \in V$ define

$$E(X_0, Y_0, v_1) = |\mathbb{P}[X_1(v_1) = 1 | X_0, Y_0, v_1] - \mathbb{P}[Y_1(v_1) = 1 | X_0, Y_0, v_1]|$$

30
Define $W_{ij} = W_{ij}(X_0, Y_0, v_1) = \text{wt}_w^w(X_0[u \mapsto i][v_1 \mapsto j])$ for all $i, j \in \{0, 1\}$. Then
\[
E(X_0, Y_0, v_1) = |\mathbb{P}[X_1(v_1) = 1|X_0, Y_0, v_1] - \mathbb{P}[Y_1(v_1) = 1|X_0, Y_0, v_1]| = \left| \frac{W_{01}}{W_{00} + W_{01}} - \frac{W_{11}}{W_{10} + W_{11}} \right| \\
= \frac{|W_{00}W_{11} - W_{01}W_{10}|}{|W_{00}W_{11} + W_{01}W_{10} + W_{00}W_{10} + W_{01}W_{11}|} \\
\leq \frac{|\sqrt{W_{00}W_{11}} - \sqrt{W_{01}W_{10}}|}{|\sqrt{W_{00}W_{11}} + \sqrt{W_{01}W_{10}}|}
\]

Let $v_1 \in V \\setminus \{u\}$. Denote by $I'(u, v_1) \subseteq I$ the set of indices of atomic formulas with $u$ and $v_1$ in their scope. For all $i \in I'(u, v_1)$ and all $j, k \in \{0, 1\}$, define
\[
F'_i(j, k) = F(x_{\text{scope}(i, 1)}, \cdots, x_{\text{scope}(i, k)})
\]
where $x_v = (X_0[u \mapsto i][v_1 \mapsto k])_v$. Define $W'(j, k) = \prod_{i \in I'} F'_i(j, k)$. The other weights depend on $u$ or $v_1$ alone, so $W'(0, 0)W'(1, 1)/W'(0, 1)W'(1, 0)$ equals $W_{00}/W_{01}W_{10}$ and
\[
E(X_0, Y_0, v_1) \leq \frac{|\sqrt{W'(0, 0)W'(1, 1)} - \sqrt{W'(0, 1)W'(1, 0)}|}{\sqrt{W'(0, 0)W'(1, 1)} + \sqrt{W'(0, 1)W'(1, 0)}}
\]

For all arity 2 signatures $G$ (taking strictly positive values), define $J(G) = \frac{1}{4} \log G(0, 0)G(1, 1)/G(0, 1)G(1, 0)$. Note that the functions $F'_i$ take values in the range $[1, M]$ so $|J(F'_i)| \leq \frac{1}{4} \log M$; also recall that tanh is non-decreasing and subadditive for positive reals, that is, tanh($x + y$) $\leq$ tanh($x$) + tanh($y$). Hence
\[
E(X_0, Y_0, v_1) \leq \frac{|\sqrt{W'(0, 0)W'(1, 1)} - \sqrt{W'(0, 1)W'(1, 0)}|}{\sqrt{W'(0, 0)W'(1, 1)} + \sqrt{W'(0, 1)W'(1, 0)}}
\]

\[
= \tanh |J(W')| \\
= \tanh \left| \sum_{i \in I'(u, v_1)} J(F'_i) \right| \\
\leq |I'(u, v_1)| \tanh \left( \frac{1}{2} \log M \right) \\
= |I'(u, v_1)| \frac{M - 1}{M + 1}
\]

The variable $u$ appears in at most $d$ atomic formulas, each of which contributes at most $k - 1$ to $\sum_{v_1} |I'(u, v_1)|$. Rearranging $M < \frac{d(k - 1) + 1}{d(k - 1) + 1}$ we get $d(k - 1) < M - 1 + 1 < 1$, so
\[
E[d(X_1, Y_1)] = 1 - \frac{1}{|V|} + \frac{1}{|V|} \sum_{v_1 \in V \setminus \{u\}} E(X_0, Y_0, v_1) \\
\leq 1 - \left( 1 - d(k - 1) \frac{M - 1}{M + 1} \right) /|V|
\]
giving the required bound $4$. \hfill \Box

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9 Infinite sets of variable weights are sometimes necessary

Theorem 3 gives some circumstances in which the set of variable weights in Theorem 2 can be taken to be finite. On the other hand, assuming that \#PM does not have an FPRAS, there is a situation where we cannot take the set of variable weights to be finite.

Let $G$ be a (simple) graph with a non-negative edge weight $\lambda(e)$ for each edge $e$ of $G$. Recall that a matching in $G$ is a subset $M$ of the edge set of $G$ such that no two edges in $M$ share a vertex. The partition function $Z_{MD}(G)$ of the monomer-dimer model on $G$ is the sum, over all matchings $M$ in $G$, of $\prod_{e \in M} \lambda(e)$.

Lemma 37 ([JS89], Corollary 3.7). There is an FPRAS for the partition function of the monomer-dimer model if the edge weights are given as integers in unary.

Proposition 38. Let $W$ be a finite set of integer-valued variable weights. Then $\#CSP_{\geq 2}(\{R\})$ has an FPRAS.

Proof. We can scale the variable weights to assume that $w(0) \in \{0, 1\}$ for all $w \in W$. We will give an AP-reduction from $\#CSP_{\geq 2}(\{R\})$ to the problem of computing the monomer-dimer partition function of a graph with positive edge weights specified in unary. Let $(w, \phi)$ be an instance of $\#CSP_{\geq 2}(\{R\})$.

Let $G$ be the edge-weighted multigraph whose vertices are atomic formula indices $I^\phi$ and with, for each $v \in V$ with $w(v, 0) = 1$, an edge with weight $w(v, 1)$ joining the two indices of the atomic formulas in which $v$ appears - and if a variable is used twice in the same atomic formula then we get a vertex with a loop. For each $v \in V$ with $w(v, 0) = 0$ delete the two vertices corresponding to the atomic formulas in which $v$ appears.

The definition of the partition function for the monomer-dimer model extends to multigraphs, and the value of the instance $(w, \phi)$ is $Z_{MD}(G)$: positive-weight configurations $\sigma : V \to \{0, 1\}$ of $Z_\phi$ correspond to subsets $M = \sigma^{-1}(1)$ of the edge set of $G$ that are matchings, and the weight $wt_\phi^w(x)$ is the weight $\prod_{e \in M} \lambda(e)$ of the corresponding matching $M$. We can transform this multigraph to a simple graph without changing the partition function: a set of parallel edges with weights $w_1, \cdots, w_k$ are equivalent to having a single edge with weight $w_1 + \cdots + w_k$, and any loop can be deleted.

The result of these transformations is a simple edge-weighted graph $G'$ such that $Z_{MD}(G) = Z_{\phi}^w$, and whose edge weights belong to $\sum_{w \in W'} w(1)$ for some subset $W'$ of $W$. But there are finitely many such edge weights, so we can write any such edge weight in unary in constant time, and use the oracle to approximate $Z_{MD}(G')$.

By Theorem 1 however, $\#PM \leq_{AP} \#CSP_{\geq 2}(\{R\})$.

9.1 Acknowledgements

The proof of Theorem 3 grew out of a study of degree-two #CSPs by Leslie Ann Goldberg and David Richerby, and their discussions with the author. The author wishes to thank Leslie Ann Goldberg and Russell Martin for their advice.
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