Purity and entropy evolution speed limits for open quantum systems

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We derive generic upper bounds on the rate of purity change and entropy increase for open quantum systems. These bounds depend solely on the generators of the nonunitary dynamics and are independent of the particular states of the systems. They are thus perfectly suited to investigate dephasing and thermalization processes of arbitrary systems. We apply these results to single and multiple dephasing channels, to a problem of quantum control in the presence of noise, and to cooling.

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Determining the maximal rate of evolution of an open system is of crucial importance in quantum physics. Any quantum system unavoidably couples to external degrees of freedom (the environment) that lead to loss of phase coherence and/or to thermalization [1]. In most applications, the main challenge is to minimize the effect of the environment. In quantum computing [2] and coherent control [3], for example, it is vital to achieve low dephasing rates in order to protect the system against decoherence. On the other hand, there are many instances where it is of advantage to maximize the influence of the surroundings. A case in point is the cooling of a quantum system and the preparation of pure states, where high cooling rates are sought after [4]. Additionally, in quantum thermodynamics, a bound on the rate of thermalization of a heat engine with its reservoirs will limit its cycle time and therefore put a restriction on maximal power output [5].

Bounds on the rate of quantum evolution are useful to assess if a process can be completed in a given time, without having to explicitly solve the (usually complicated) equations of motion [0,11]. Quantum speed limits, defined as the time derivative of a geometric quantity, for instance an angle between two states, are often introduced to characterize the maximal rate of evolution of a quantum system. Two prominent examples of quantum speed limits for closed systems are the Mandelstam-Tamm bound, \(|d\theta| \leq \Delta E_{\psi}/\hbar\) [12], and the Margolus-Levitin bound, \(|d\theta| \leq \langle E\rangle_{\psi}/\hbar\) [13]. Here \(\Delta E_{\psi}\) is the energy width of the initial pure state \(\psi(0)\), \(\langle E\rangle_{\psi}\) its mean energy above the ground state, and \(\theta = \arccos|\langle \psi(0)|\psi(t)\rangle|\) the geometric angle between initial and final states (we will set \(\hbar = 1\) in the sequel). In the last years, the Mandelstam-Tamm and Margolus-Levitin bounds have been extended to mixed [14] and driven [15, 17] closed quantum systems as well as to open quantum systems [18-20]. These speed limits depend explicitly on the state \(\rho\) of the system and can be written in the general form \(|d_iG(\rho)| \leq f(\rho, A)|\), where \(G(\rho)\) is a geometric quantity that characterizes the state of the system and \(f\) is a function of both the state and the generator \(A\) of the quantum evolution – the generator of unitary evolution is the Hamiltonian \(H\) of a closed system, while the generator of nonunitary Markovian evolution of an open system may be given by a Lindblad generator \(L\) [1]. The dependence of the bound on the system state \(\rho\) is especially useful when the evolution speed of different states are to be compared, as for example in quantum metrology when the optimal state for phase estimation is needed [18, 19].

Here we consider another class of quantum speed limits of the form \(|d_iG(\rho)| \leq f(A)|\), where the function \(f\) depends solely on the generator \(A\). Since these bounds do not refer to any particular state \(\rho\), they present a clear separation between the geometric part \(G(\rho)\) that describes the state of the system and the kinetic part \(f(A)\) that is controlled by the generator. These bounds are particularly well suited to investigate the impact of an external environment on generic quantum systems whose states either not known in detail, or are too complex to be determined exactly, e.g. those of an interacting many-body system. Bounds of this type have been studied for Hermitian [23] and non-Hermitian [24] Hamiltonians that often appear in systems with absorption as in optics. Our aim in this paper is to provide state-independent speed limits for the purity \(P = \text{tr}\rho^2\) and the von Neumann entropy \(S = -\text{tr}\rho \ln \rho\) of a generic open quantum system, two quantities that are commonly used to quantify the action of the environment [1]. In the following, we first illustrate our method in the usual Hilbert space of density matrices by deriving a bound for the purity speed in terms of the Hilbert-Schmidt norm of the Lindblad operators of the nonunitary dynamics. We then extend our approach to Liouville space (space of density "vectors" as described later on) and obtain a tighter inequality in terms of the spectral norm of the corresponding Hamilton superoperator. We further considerably improve the bound by introducing a purity deviation that is obtained by subtracting the steady-state dynamics of the time-dependent system. Remarkably, we show that this bound is tight, for all states and at all times, for a single qubit dephasing channel. We finally derive a speed limit for the rate of entropy decrease, and apply our formalism to
quantum control in the presence of noise and to cooling. Purity bound in Hilbert space. We begin by deriving an upper bound to the purity speed limit in the density matrix formulation. It will be convenient to consider an integral bound of the type,

$$
|G(\rho_f) - G(\rho_i)| \leq \int_{t_i}^{t_f} f(A_t) dt,
$$

(1)

which follows from the differential bound, $$|d_t G(\rho)| \leq f(A_t)$$, with the inequality $$|\int_{t_i}^{t_f} d_t G(\rho)| \leq \int_{t_i}^{t_f} |d_t G(\rho)|$$. This form is often more appealing as it relates the initial and final states of interest without reference to the intermediate dynamics. The right-hand side will typically have the form of an "action" integral.

We next exploit the fact that $$d_t G(\rho)$$ and using the triangle inequality, we have,

$$
\rho L_t = i[H, \rho] + \sum_k A_k \rho A_k^\dagger - \frac{1}{2} A_k^\dagger A_k \rho,
$$

(2)

where the operators $$A_k \in C^{N \times N}$$ describe the interaction with the external environment ($$L_t$$ is the time dependent Lindblad generator of the nonunitary dynamics). Master equations of the form (2) are the tool of choice to investigate the dynamics of systems weakly coupled to the external environment ($$L_t$$ accounts for the interaction with the environment). Thus, this condition provides useful means to separate between purely dephasing processes that can only reduce the purity and processes that have the capability reducing the entropy and cooling the system.

We use the cyclic property of the trace to write

$$
d_t \ln \langle r | \rho | r \rangle = 2 \langle r | L_t(\rho) | r \rangle / \langle r | \rho | r \rangle^2.
$$

Integrating over time and using the triangle inequality, we have,

$$
\left| \ln \frac{\mathcal{P}(t_f)}{\mathcal{P}(t_i)} \right| \leq \int t_i^{t_f} \frac{2 |\langle r | L_t(\rho) | r \rangle|}{|\langle r | \rho | r \rangle|^2} dt.
$$

(3)

We next exploit the fact that $$\mathcal{P}(t) = \ln \rho^2 = \|\rho\|_2^2$$, where $$\|\cdot\|_2$$ denotes the Hilbert-Schmidt norm. An upper bound to Eq. (3) can be derived with the help of elementary matrix algebra [27]. Combining the Cauchy-Schwarz inequality, $$|\langle r | L_t(\rho) | r \rangle| \leq \|r\|_2 \|L_t(\rho)\|_2$$, we find

$$
\|\rho\|_2 \|L_t(\rho)\|_2 \leq 2 \sum_k \|A_k\|_2^2 |\langle r | \rho | r \rangle|_2.
$$

Inserting this expression into Eq. (3), we obtain a "norm action" integral inequality of type \([1]\) for the logarithm of the purity:

$$
\left| \ln \frac{\mathcal{P}(t_f)}{\mathcal{P}(t_i)} \right| \leq 4 \int t_i^{t_f} \|A_k\|_2^2 dt.
$$

(4)

The quantity $$- \ln \mathcal{P}$$ is known as the Rényi entropy of order two or the collision entropy \([28]\). We note that the unitary evolution term, $$i[H, \rho]$$, can be omitted, since the norm is unitarily invariant. Equation (1) provides an upper bound to the speed of variation of the Rényi entropy from the more accessible Lindblad operators. Its practical usefulness stems from the fact that the operators $$A_k$$ can be determined via quantum process tomography \([29]\). This technique has been successfully demonstrated experimentally in NMR systems \([30]\), solid state qubits \([31]\), vibrational states of atoms in optical lattices \([32]\), quantum gate operations \([33,34]\) and quantum memory \([35]\), as well as to relaxing photon fields inside cavities \([36]\) and superconducting quantum circuits \([37]\). It is worth noticing that the purity has been measured directly in some cases, without having recourse to full quantum state tomography \([39,41]\). However, it is notoriously difficult to measure nonlinear functions of the density operator \([42]\) and hence the purity without having recourse to full quantum state tomography. This technique has been successfully demonstrated experimentally in NMR systems \([30]\), solid state qubits \([31]\), vibrational states of atoms in optical lattices \([32]\), quantum gate operations \([33,34]\) and quantum memory \([35]\), as well as to relaxing photon fields inside cavities \([36]\) and superconducting quantum circuits \([37]\). It is worth noticing that the purity has been measured directly in some cases, without having recourse to full quantum state tomography \([39,41]\). However, it is notoriously difficult to measure nonlinear functions of the density operator \([42]\) and hence the purity. It is worth noticing that the purity has been measured directly in some cases, without having recourse to full quantum state tomography.
We may now derive a purity bound in Liouville space by repeating the procedure previously used in Hilbert space. Starting from the Schrödinger-like equation \( (5) \), we first obtain the equality,

\[
\frac{\partial_t \langle r | r \rangle}{\langle r | r \rangle} = -i \frac{\langle r | H_r - H_{r}^\dagger | r \rangle}{\langle r | r \rangle}.
\]  

(7)

Integrating this expression over time and using the triangle inequality, we get,

\[
\left| \ln \frac{\mathcal{P}(t_f)}{\mathcal{P}(t_i)} \right| \leq \int_{t_i}^{t_f} \left| \frac{\langle r | H_r - H_{r}^\dagger | r \rangle}{\langle r | r \rangle} \right| dt.
\]  

(8)

The integrand may be further bounded by the spectral norm of the skew Hermitian part \( [27] \),

\[
\left| \frac{\langle r | H_r - H_{r}^\dagger | r \rangle}{\langle r | r \rangle} \right| \leq \| H_r - H_{r}^\dagger \|_{\text{sp}}.
\]  

(9)

Combining Eqs. (8) and (9), we eventually obtain a norm action bound for the Réyni entropy of the form \( (1) \):

\[
\left| \ln \frac{\mathcal{P}(t_f)}{\mathcal{P}(t_i)} \right| \leq \int_{t_i}^{t_f} \| H_r - H_{r}^\dagger \|_{\text{sp}} dt.
\]  

(10)

The advantage of the Liouville space is now apparent. In contrast to the Hilbert space bound \( (4) \), Eq. (10) was derived without using the triangle identity for the integrand and without the submultiplicativity property of the norm. We demonstrate in the Appendix that, as a result, the Liouville space bound is always better than the Hilbert space bound for the case of a pure dephasing channel, \( [A_k, A_j^\dagger] = 0 \). In the more general case, \( [A_k, A_j^\dagger] \neq 0 \), we have observed numerically that this property holds true for randomly generated operators \( A_k \), but were not able to show it analytically in full generality. Another argument in favor of the Liouville space bound can be found by looking at many-particle or many-level systems. Let us consider \( M \) independent particles subjected to the same dynamics. Since the system is in a product state at all times, the log-purity scales like \( \ln \mathcal{P}_M \sim M \ln \mathcal{P}_3 \). The Liouville space bound \( (10) \) then exhibits the correct \( M \) scaling, \( d\mathcal{P}_M/dt \leq M \| H_r - H_{r}^\dagger \|_{\text{sp}} \), in contrast to the Hilbert space bound \( (4) \), \( d\mathcal{P}_M/dt \leq M^{2M-1} \| A \|_2^2 \), which becomes worse with increasing \( M \). In addition, for a single \( N \)-level dephasing channel with eigenvalues \( \lambda_j(A) = \exp(i\zeta_j) \), we find that the purity speed in Liouville space is limited by \( \max |\lambda_i - \lambda_j|^2 \leq 4 \) (see Appendix), while it increases with \( N \), \( \max |\lambda_i - \lambda_j| \leq 4N \), in Hilbert space, thus badly overestimating the purity value for large \( N \).

The purity bound \( (10) \) may be further significantly improved in the following way. Let \( |r_s \rangle \) be a specific solution of the quantum evolution \( i\partial_t |r_s \rangle = H_r |r_s \rangle \). We define the deviation vector as \( |r_D \rangle = |r \rangle - |r_s \rangle \), and the corresponding purity deviation as \( \mathcal{P}_D = \langle r_D | r_D \rangle \). The purity deviation has a simple geometrical meaning as the square of the Euclidean distance, \( \| (\rho - \rho_s) \|_2^2 \), between the states \( \rho \) and \( \rho_s \) (the regular purity is the distance to the origin \( \rho_s = 0 \)). By taking the time derivative of \( \mathcal{P}_D \) and reiterating the above derivation, we readily find,

\[
\left| \ln \frac{\mathcal{P}_D(t_f)}{\mathcal{P}_D(t_i)} \right| \leq \int_{t_i}^{t_f} \| H_r - H_{r}^\dagger \|_{\text{sp}} dt,
\]  

(11)

where the purity \( \mathcal{P} \) has now been replaced by the purity deviation \( \mathcal{P}_D \). While Eq. (11) is valid for all vectors \( |r_s \rangle \), it becomes particularly useful when \( |r_s \rangle \) is given by the steady state, \( i\partial_t |r_s \rangle = 0 \). The benefit of the replacement \( \mathcal{P} \to \mathcal{P}_D \) is that only the part of the purity that changes in time is taken into account. The purity deviation bound \( (11) \) has the remarkable property that it may be tight at all times and all states for purely dephasing qubit channel (see below). We furthermore stress that the norm \( \| H_r - H_{r}^\dagger \|_{\text{sp}} \) can be determined from the measurable Lindblad operators \( A_k \).

Applications. We next apply our results to the case of a purely dephasing channel and to a problem of quantum coherent control in the presence of noise.

(a) Dephasing channel. We first consider, for simplicity, a single dephasing channel for a two-level system described by \( H_{\rho} = 0 \) and one nonzero Lindblad operator that satisfies \( [A, A_j^\dagger] = 0 \) \( [2] \). Without loss of generality, we assume that the operator \( A \) is traceless \( [24] \). In this situation, the Hilbert space bound \( (4) \) takes a minimal value that is exactly two times larger compared to the tighter Liouville space bound \( (10) \). For instance, for \( A = \sigma_x \) and an initial density matrix of the form \( \rho(t_i) = \{ |a, b \rangle, \{ b^*, 1 - a \} \} \), we find \( |\ln \mathcal{P}(t_f)/\mathcal{P}(t_i)| \leq 2(t_f - t_i) \) in Hilbert space and \( |\ln \mathcal{P}(t_f)/\mathcal{P}(t_i)| \leq t_f - t_i \) in Liouville space. Remarkably, the purity deviation bound

![Figure 1: Purity bounds for a qubit dephasing channel with \( H_{\rho} = \sigma_x \) and \( A = \sigma_z \) in a pure state at \( t_i = 0 \). The bound \( (12) \) (curve i) based on the purity deviation bound \( (11) \) is tighter than the purity bound in Liouville space \( (10) \) (curve ii) and the corresponding bound in Hilbert space \( (4) \) (curve iii). The bound \( (12) \) clearly delimits the region of allowed purities (blue lines obtained for various random initial conditions).](image-url)
is tight at all times in this case: We choose \( \rho_s \) to be the steady state given by the fully mixed state \( \rho_s = \{(a,0), (0,1-a)\} \) [45], and obtain the equality \( \ln|\mathcal{P}_D(t_f)/\mathcal{P}_D(t_i)| = \ln[2b^2 e^{-rt}/(2b^2 e^{-t_i})] = t_f - t_i \) which is exactly equal to the right hand side of (11). We are not aware of the existence of any other tight speed limit for open quantum systems. In a large Hilbert space and in the presence of multiple dephasing operators, the purity deviation bound (11) will be tight for initial conditions that populate the steady state and the fastest decaying mode exclusively.

Another merit of the purity deviation approach is that it can be used to get a better bound on the purity change for a general dephasing channel in a \( N \)-level systems. Setting \( \rho_s \) to be the fully mixed state (which always corresponds to a steady state in a dephasing dynamics) yields \( \text{tr}[(\rho - \rho_s)^2] = \text{tr}[\rho^2] - 1/N \). Using this in Eq. (11) we obtain,

\[
\mathcal{P}(t_f) \geq \frac{1}{N} + \left(\mathcal{P}(t_i) - \frac{1}{N}\right) \exp \left[ - \int_{t_i}^{t_f} \|H_r - H_r^\dagger\|_{\text{sp}} \, dt \right].
\]

This equation, valid for any dephasing channel, is always better compared to Eq. (10).

Let us illustrate the above results for a qubit dephasing channel with \( H_\rho = \sigma_x \) and \( A = \sigma_z \). Figure 1 shows the purity (blue lines) for various pure random initial conditions. We first observe that the speed limit in Liouville space (10) (curve ii) is tighter than the bound obtained in Hilbert space [4] (curve iii). We further note that the purity bound (13) derived from the purity deviation bound (11) is significantly better than the other two bounds. While it is not tight at all times for all initial states, it clearly delimits the regime of allowed purity values.

(b) Coherent control. Coherent control, in particular optimal control theory, is a powerful method for controlling dynamical processes in quantum mechanics [3]. The technique deals with finding the time-dependent Hamiltonian necessary to implement a certain state transformation under given restrictions (e.g. the available operators in a Hamiltonian) [46-49]. In Ref. 50 the effects of noise in the control amplitude, coming from external degrees of freedom, were explored. In particular, bounds on the minimal purity loss at the end of the evolution were derived. These bounds are important to determine whether a quantum system that is fully controllable when isolated [51-52], remains fully controllable when coupled to an environment. We shall use our approach to put an upper bound on the accumulated control dephasing noise. In a coherent control setup the evolution of the density matrix \( \rho \) is given by the Markovian master equation (2) where \( H_\rho = H_0 + \sum_{k=1}^M f_k(t) H_k \) is the noise-free control Hamiltonian and \( A_k = n_k(t)^{1/2} H_k \) are dephasing terms that arise from the noise in the control fields \( f_k \) (\( n_k \geq 0 \) without loss of generality). Since the Lindblad operators \( A_k \) are Hermitian in this case, Eq. (2) describes a (multichannel) dephasing problem, according to the Lidar-Shabani-Alicki criterion [26].

Employing Eq. (12) for an initial pure state, \( \mathcal{P}(t_i) = 1 \), we find,

\[
\mathcal{P}(t_f) \geq \frac{1}{N} + \frac{N-1}{N} \exp \left[ - \int_{t_i}^{t_f} \|H_r - H_r^\dagger\|_{\text{sp}} \, dt \right].
\]

Expression (13) gives a general lower bound on the final purity in coherent control problems in terms of the spectral norm of the skew-Hermitian part \( \|H_r - H_r^\dagger\|_{\text{sp}} \). It is always larger than \( 1/N \) (as it should) and larger than \( \exp[-\int_{t_i}^{t_f} \|H_r - H_r^\dagger\|_{\text{sp}} \, dt] \) predicted by the purity bound (10). It is sometimes possible to obtain even better results by not applying the triangle inequality. Let us consider the case of a single qubit with dephasing operators given by the Pauli matrices \( A_k = \sigma_y \). This example also illustrates the concrete evaluation of the spectral norm of the skew-Hermitian part as a function of the Lindblad operators. When using the triangle inequality in Liouville space, we obtain the inequality \( -\ln\mathcal{P}(t_f) \leq \int_{t_i}^{t_f} \sum_k |n_k(t)| \). However, the spectral norm can be here calculated analytically without the triangle inequality to read \( \|H_{r,k} - H_{r,k}^\dagger\|_{\text{sp}} = \sum_k |n_k(t)| - \min_k |n_k(t)| \). Assuming that the noise amplitudes are all equal to \( n_0 \), we find a bound, \( \mathcal{P}(t_f) \geq \exp(-3n_0(t_f - t_i)) \), with the triangle inequality that is worse than the bound, \( \mathcal{P}(t_f) \geq \exp(-2n_0(t_f - t_i)) \), obtained without. A similar calculation can be done for Eq. (12).

Bound on entropy reduction. We have so far treated purity decrease (pure dephasing) and purity increase (cooling) on the same footing. However, we may obtain better bounds for cooling processes by replacing \( \|H_r - H_r^\dagger\|_{\text{sp}} \) by \( \max(-i \text{eig}(H_r - H_r^\dagger)) \) in expression (10) for the speed limit in Liouville space. This is no longer
a norm and is valid only when purity is growing. Yet, when valid, it can be significantly better than the spectral norm bound, as shown in Fig. 2 for the simple decay channel characterized by $A = \sigma^- = (\sigma_x - i\sigma_y)/2$. We may moreover obtain a bound on the decay of the von Neumann entropy with the help of the Jensen inequality, $S = -\text{tr}\rho \ln \rho \geq -\ln \mathcal{P}(t)$, and Eq. (10).

$$S(t_f) \geq -\ln \mathcal{P}(t_i) - \int_{t_i}^{t_f} \max|\varepsilon\text{ig}(H_e - H_f)| dt. \quad (14)$$

For given initial and final entropies (or purities), expression (14) provides a bound on how fast (minimal time interval $t_f - t_i$) a quantum system can be cooled in terms of the generators of the open dynamics. The appearance of the entropy instead of the purity establishes an important link to thermodynamics and highlights its relevance for the investigation of e.g., quantum heat engines.

Conclusions. We have derived state-independent quantum speed limits for the purity and the entropy of Markovian open quantum systems. We have obtained increasingly tighter bounds by considering the Liouville space instead of the usual Hilbert space. We have additionally shown that these bounds can be significantly improved by introducing a purity deviation obtained by subtraction of the steady-state contribution. We have finally emphasized the usefulness of these results for the investigation of decoherence and thermalization processes in general, and applied them to concrete problems of dephasing, noisy coherent control and cooling.

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[53] The spectral gap in Eq. (13) is squared, since \( A \) appears in a quadratic form in the Lindblad master equation (2).

**Appendix**

We show in the following that the Liouville space bound (10) is tighter than the Hilbert space bound (4) for the case of a dephasing channel. We begin by considering a single dephasing channel described, without loss of generality, by \( H_\rho = 0 \) and only one nonzero Lindblad operator that satisfies \( [A, A^\dagger] = 0 \). The operator \( A \) is unitary diagonalizable and has orthogonal eigenvectors under the standard inner product. Hence, it uniquely defines an orthonormal basis through its eigenvectors. A density matrix that is diagonal in this basis will commute with \( A \) and \( A^\dagger \), and lead to \( L(\rho) = 0 \). Dephasing will therefore take place in the eigenbasis of \( A \). Without loss of generality this dephasing channel can then be studied by taking \( A \) to be a complex diagonal matrix.

Using the Hamilton superoperator definition (6), we find that \( H_r - H_r^\dagger \) is diagonal and therefore the eigenvalues are equal to the diagonal elements. By explicit calculation we find \( \text{eig}(H_r - H_r^\dagger) = -i\{\lambda_i - \lambda_j\}_{i,j=1}^N \) where \( \lambda_i \) are the eigenvalues of \( A \). For diagonal matrices the spectral norm is the largest matrix element (in absolute value) and therefore:

\[
\|H_r - H_r^\dagger\|_{sp} = \max |\lambda_i - \lambda_j| \leq 4 \|A\|_2^2,
\]

where we used \( |\lambda_j - \lambda_i| \leq 2\text{max}|\lambda_i| \leq 4 \|A\|_2^2 \). The speed limit in Liouville space (10) is hence always better (or equal) than the one in Hilbert space (4) for this type of dephasing evolution.

The above proof can be easily extended to a general dephasing channel with multiple dephasing operators \( [A_k, A_k^\dagger] = 0 \). We begin by applying the Hilbert space bound (4) to an initially pure state to obtain \( |\log \mathcal{P}(t_f)/\mathcal{P}(t_i)| \leq 4 \int_{t_i}^{t_f} \sum_k \|H_k\|_2 dt \). On the other hand, using the Liouville space bound (10), we have, \( |\log \mathcal{P}(t_f)/\mathcal{P}(t_i)| \leq \int_{t_i}^{t_f} \|\sum_k (H_{r,k} - H_{r,k}^\dagger)\|_{sp} dt \). Using the triangle inequality \( \|\sum_k (H_{r,k} - H_{r,k}^\dagger)\|_{sp} \leq \sum_k \|H_{r,k} - H_{r,k}^\dagger\|_{sp} \) and applying the inequality (15) for each \( H_{r,k} \), we find that the Liouville space bound is always tighter than the Hilbert space bound for any multichannel dephasing \( [A_k, A_k^\dagger] = 0 \).

We finally mention that in the special case where the operator \( A \) is Hermitian (e.g. in coherent control), the log-purity speed bound (10) simplifies to,

\[
|d_i(-\ln \mathcal{P})|_{A=A^\dagger} \leq \Delta_A^2,
\]

where \( \Delta_A = \max|\text{eig}(A)| - \min|\text{eig}(A)| \) is the spectral gap of the operator. Expression (13) is similar to the result, \( |d_i\theta| \leq (E_{\text{max}} - E_{\text{min}})/2 \), obtained in the unitary case (24) \( (E_{\text{max/min}} \text{ are the maximal/minimal instantaneous eigenvalues of the Hamiltonian) [53].}