Directional derivatives of the singular values of matrices depending on several real parameters

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Abstract

In this document I recapitulate some results by Hiriart-Urruty and Ye[2] concerning the properties of differentiability and the existence of directional derivatives of the multiple eigenvalues of a complex Hermitian matrix function of several real variables, where the eigenvalues are supposed in a decreasing order. Another version of these results was obtained by Ji-guang Sun[6–8].

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1 Differentiability of the eigenvalues of a complex Hermitian matrix

We will denote by Λ(C) the spectrum or set of eigenvalues of any complex square matrix C. Let Ω be an open subset of \( \mathbb{R}^p \) and let \( A : \Omega \to \mathbb{C}^{n \times n} \) be a matrix function of class \( C^1 \) such that for every \( x \in \Omega \) the matrix \( A(x) \) is Hermitian, i.e. \( A(x)^* = A(x) \) where * denotes the conjugate transpose. As it is well known the eigenvalues of \( A(x) \) are real numbers; thus, there exist \( n \) real functions defined on \( \Omega \), \( \lambda_1, \ldots, \lambda_n \), such that for all \( x \in \Omega \),

\[
\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_n(x)
\]

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are the eigenvalues of $A(x)$. Let $m \in \{1, \ldots, n\}$: it is easy to prove that the function $\lambda_m : \Omega \to \mathbb{R}$ is continuous. When the eigenvalue $\lambda_m(x_0)$ of $A(x_0)$ is simple, the function $\lambda_m$ is differentiable at $x_0 \in \Omega$. But in case of $\lambda_m(x_0)$ is a multiple eigenvalue of $A(x_0)$, $\lambda_m$ can be nondifferentiable at $x_0$. For example [4], let
\[
A(x_1, x_2) := \begin{pmatrix} x_1 & ix_2 \\
-ix_2 & -x_1 \end{pmatrix}
\]
be for $(x_1, x_2) \in \mathbb{R}^2$. It is obvious that for each $(x_1, x_2) \in \mathbb{R}^2$ the matrix $A(x_1, x_2)$ is Hermitian. Then
\[
\begin{vmatrix} \lambda - x_1 & -ix_2 \\
ix_2 & \lambda + x_1 \end{vmatrix} = \lambda^2 - x_1^2 - x_2^2;
\]
hence the eigenvalues of $A(x_1, x_2)$ are $\pm \sqrt{x_1^2 + x_2^2}$. Observe that the matrix
\[
A(0, 0) = \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix}
\]
has a double eigenvalue; but neither the function $\lambda_1(x_1, x_2) = \sqrt{x_1^2 + x_2^2}$, nor the function $\lambda_2(x_1, x_2) = -\sqrt{x_1^2 + x_2^2}$ are differentiable at $(0, 0)$.

Let $d \in \mathbb{R}^p$ be a unitary vector, i.e. $\|d\|_2 = 1$, where $\|\cdot\|_2$ denotes the Euclidean norm. The directional derivative of the function $\lambda_m$ at the point $x_0$ with respect to $d$ is defined as the limit
\[
\lambda'_m(x_0, d) := \lim_{t \to 0^+} \frac{\lambda_m(x_0 + td) - \lambda_m(x_0)}{t}
\]
whenever this limit exists.

Based on techniques and results from convex and nonsmooth analysis (in Clarke’s sense), Hiriart-Urruty and Ye proved Theorems 1 and 2. See [2, Theorem 4.5].

**Theorem 1** For all $x_0 \in \Omega$, for all unitary vector $d \in \mathbb{R}^p$, and for all $m \in \{1, \ldots, n\}$, there exists always
\[
\lambda'_m(x_0, d).
\]
Moreover, it can be proved that $\lambda'_m(x_0, d)$ is equal to a determined eigenvalue of a matrix constructed from $A(x_0)$ and $d$ in the following way: For each $x_0 \in \Omega$, there is a unitary matrix $U = [u_1, \ldots, u_n]$ such that
\[
U^* A(x_0) U = \text{diag}(\lambda_1(x_0), \ldots, \lambda_n(x_0)).
\]
Suppose that $\lambda_m(x_0)$ is a multiple eigenvalue of $A(x_0)$, of multiplicity $r_m$. Introduce two integers $i_m \geq 1, j_m \geq 0$ to precise the position that $\lambda_m(x_0)$ occupies among the $r_m$ repeated eigenvalues that are equal to it. Consider the detailed arrangement of the eigenvalues of $A(x_0)$:
\[
\lambda_1(x_0) \geq \cdots \geq \lambda_{m-i_m}(x_0) > \lambda_{m-i_m+1}(x_0) = \cdots = \lambda_m(x_0) = \lambda_{m+1}(x_0) = \cdots = \lambda_{m+j_m}(x_0) > \lambda_{m+j_m+1}(x_0) \geq \cdots \geq \lambda_n(x_0)
\]
That is to say, $j_m$ is the number of eigenvalues placed after the subscript $m$ that are equal to $\lambda_m(x_0)$; whereas $i_m$ is the number of eigenvalues placed before $m$ that are equal to $\lambda_m(x_0)$; plus one (we put $\lambda_m(x_0)$ in this list). Hence, $j_m$ may be zero, $i_m \geq 1$, and $i_m + j_m = r_m$. When $m = 1$, i.e. if we are considering $\lambda_1(x_0)$, we have $i_1 = 1$, $j_1 = r_1 - 1$. When $m = n$, i.e. for $\lambda_n(x_0)$, we have $i_n = r_n$, $j_n = 0$. In case $\lambda_m(x_0)$ is a simple eigenvalue, $i_m = 1$, $j_m = 0$. Although the notation does not indicate it, the numbers $i_m, j_m$ and $r_m$ depend on $x_0$.

Let $U_2$ be the $n \times r_m$ matrix formed by the $(m - i_m + 1)$th, \ldots, $(m + j_m)$th columns of the matrix $U$:

$$U_2 := \begin{bmatrix} u_{m-i_m+1}, \ldots, u_{m+j_m} \end{bmatrix};$$

i.e. $U_2$ is formed by $r_m$ orthonormal eigenvectors associated with the eigenvalue $\lambda_m(x_0)$ of $A(x_0)$. For each $j \in \{1, \ldots, p\}$ define

$$\frac{\partial A}{\partial x_j}(x_0) = \begin{bmatrix} \frac{\partial a_{ik}}{\partial x_j}(x_0) \end{bmatrix}$$

$a_{ik}(x)$ being the entries of $A(x)$. We will call $F'(d)$ to the $r_m \times r_m$ matrix

$$F'(d) := U_2^* \left( \sum_{j=1}^p d_j \frac{\partial A}{\partial x_j}(x_0) \right) U_2$$

for every unitary vector $d = (d_1, \ldots, d_p) \in \mathbb{R}^p$. Given that

$$\frac{\partial a_{ik}}{\partial x_j} = \frac{\partial a_{ik}}{\partial x_j} = \frac{\partial a_{ki}}{\partial x_j},$$

we have that the matrix $\frac{\partial A}{\partial x_j}$ is Hermitian, and so is $F'(d)$; indeed,

$$F'(d)^* = U_2^* \left( \sum_{j=1}^p d_j \frac{\partial A}{\partial x_j}(x_0) \right)^* U_2$$

$$= U_2^* \left( \sum_{j=1}^p d_j \left[ \frac{\partial A}{\partial x_j}(x_0) \right]^* \right) U_2$$

$$= U_2^* \left( \sum_{j=1}^p d_j \frac{\partial A}{\partial x_j}(x_0) \right) U_2 = F'(d).$$

Therefore, the eigenvalues of $F'(d)$ are real numbers.

**Theorem 2** The directional derivative $\lambda'_m(x_0, d)$ is given by

$$\lambda'_m(x_0, d) = \mu_{i_m}(F'(d))$$

where $\mu_{i_m}(F'(d))$ is the $i_m$th eigenvalue of $F'(d)$ when the eigenvalues are arranged in a decreasing order:

$$\mu_1(F'(d)) \geq \cdots \geq \mu_{r_m}(F'(d)).$$

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A theorem related to Theorem 2 was proved by Ji-guang Sun [8, Theorem 3.1] applying the implicit function theorem and the Rellich theorem.

**Theorem 3** The function
\[ t_m(x) := \lambda_{m-1} + \cdots + \lambda_{m} + \cdots + \lambda_{m+j} , \quad x \in \Omega \]
is differentiable at \( x_0 \).

See [2, Corollary 4.3] for a proof of this theorem.

**Corollary 4** There exists a neighborhood \( V \) of \( x_0 \), \( V \subset \Omega \), in which the function
\[ t_m(x) := \lambda_{m-1} + \cdots + \lambda_{m} + \cdots + \lambda_{m+j} , \quad x \in \Omega \]
is differentiable.

**Proof.** Let \( V \subset \Omega \) be a neighborhood of \( x_0 \), sufficiently small so that the inequalities
\[ \lambda_{m-1} > \lambda_{m-2} > \cdots > \lambda_{m+j} > \lambda_{m+j+1} \]
hold when \( x \in V \). Let \( x_1 \) be any point of \( V \). Then the arrangement of the eigenvalues of \( A(x_1) \)
\[ \lambda_{m-1} (x_1) \geq \cdots \geq \lambda_{m+j} (x_1) \]
may have groups of equalities. In view of Theorem 3, the sum of the functions \( \lambda_i \) corresponding to each one of these groups, is differentiable at \( x_1 \); therefore, as \( t_m \) is the sum of these sums, we deduce that \( t_m \) is differentiable at \( x_1 \).

\[ \blacksquare \]

## 2 Differentiability of the singular values of a complex matrix

**Notation.** Let \( m, n \) be positive integers and let \( q := \min(m, n) \). Given \( B \in \mathbb{C}^{m \times n} \), let
\[ \sigma_1(B) \geq \sigma_2(B) \geq \cdots \geq \sigma_q(B) \]
be the singular values of \( B \). For each \( k \in \{1, \ldots, q\} \), it is said that a pair of vectors of **unit length** \( y_k \in \mathbb{C}^{m \times 1} \), \( z_k \in \mathbb{C}^{n \times 1} \) are left and right **singular vectors** of \( B \) associated with the singular value \( \sigma_k(B) \) if \( B z_k = \sigma_k(B) y_k \) and \( B^* y_k = \sigma_k(B) z_k \).

Let \( A : \Omega \to \mathbb{C}^{m \times n} \) be a matrix function of class \( C^1 \). For each \( x \in \Omega \subset \mathbb{R}^p \), let
\[ s_1(x) \geq \cdots \geq s_q(x) , \quad \text{with} \quad q := \min(m, n) , \]
be the singular values of the matrix \( A(x) \) arranged in a decreasing order. Thus, we can define \( q \) functions \( s_i : \Omega \to \mathbb{R} \), \( i \in \{1, \ldots, q\} \). We are going to establish the properties of differentiability of these functions. By Wielandt’s lemma, the \( m + n \) eigenvalues of the Hermitian matrix
\[ M(x) := \begin{pmatrix} 0 & A(x) \\ A(x)^* & 0 \end{pmatrix} \in \mathbb{C}^{(m+n) \times (m+n)} \]
are
\[ s_1(x) \geq \cdots \geq s_q(x) \geq 0 = \cdots = 0 \geq -s_q(x) \geq \cdots \geq -s_1(x) \]
(it may have repeated intermediate zeros), for all \( x \in \Omega \). Hence, the analogous results to Theorems 1, 2 and 3 for Hermitian matrices are true.

**Theorem 5** Let \( k \in \{1, \ldots, q\} \), \( x_0 \in \Omega \), and \( d \in \mathbb{R}^p \) be a unitary vector. Then there exists the directional derivative
\[ s_k'(x_0, d). \]

Let \( u \in \mathbb{C}^{m \times 1}, v \in \mathbb{C}^{n \times 1} \), where \( u \neq 0 \) or \( v \neq 0 \). Then
\[
\begin{pmatrix} u \\ v \end{pmatrix}
\]
is an eigenvector of
\[
H := \begin{pmatrix} 0 & B \\ B^* & 0 \end{pmatrix}
\]
associated with its eigenvalue \( \sigma_k(B) \) if and only if
\[
Bv = \sigma_k(B)u, \quad (1)
\]
\[
B^*u = \sigma_k(B)v. \quad (2)
\]
So, if \( (y_k, z_k) \in \mathbb{C}^{m \times 1} \times \mathbb{C}^{n \times 1} \) is a pair of (left,right)-singular vectors of \( B \) associated with the singular value \( \sigma_k(B) \), then
\[
\begin{pmatrix} y_k \\ z_k \end{pmatrix}
\]
is an eigenvector of \( H \) corresponding to its eigenvalue \( \sigma_k(B) \).

Let \( x_0 \in \Omega \) be a fixed point, and let \( W \in \mathbb{C}^{(m+n) \times (m+n)} \) a unitary matrix that diagonalizes \( M(x_0) \):
\[
W^*M(x_0)W = \begin{pmatrix}
0 \\
\ddots \\
\ddots \\
0
\end{pmatrix}
\]

Suppose that
\[
s_1(x_0) \geq \cdots \geq s_{k-i_k}(x_0) > s_{k-i_k+1}(x_0) = \cdots = s_k(x_0) = s_{k+1}(x_0) = \cdots = s_{k+j_k}(x_0) > s_{k+j_k+1}(x_0) \geq \cdots \geq s_q(x_0) \geq \cdots \geq -s_1(x_0)
\]
are the eigenvalues of \( M(x_0) \), where \( s_k(x_0) \) is a multiple eigenvalue of multiplicity \( r_k = i_k + j_k \), \( i_k \) being the number of eigenvalues equal to \( s_k(x_0) \) placed
before the rank \( k + 1 \), and \( j_k \) is the number of eigenvalues equal to \( s_k(x_0) \) situate after the rank \( k \).

Call \( W_2 \) to the \((m + n) \times r_k \) matrix formed by the \((k - i_k + 1)\)th,...,(\(k + j_k\))th columns of the matrix \( W \). For each unitary vector \( d = (d_1, \ldots, d_p) \in \mathbb{R}^p \), define
\[
F'(d) := W_2^\prime \left[ \sum_{j=1}^{p} d_j \left[ \begin{array}{c} O \\ \left( \frac{\partial A}{\partial x_j}(x_0) \right)^* \\ O \end{array} \right] \right] W_2,
\]
which is an \( r_k \times r_k \) Hermitian matrix. Then, by Theorem 2, we have the next result.

**Theorem 6** For each unitary vector \( d = (d_1, \ldots, d_p) \in \mathbb{R}^p \)
\[
s_k'(x_0, d) = \mu_{i_k}(F'(d)),
\]
\( \mu_{i_k}(F'(d)) \) being the \( i_k \)th eigenvalue of the matrix \( F'(d) \) when we arrange the eigenvalues of this matrix in a decreasing order.

To facilitate the writing let \( W_2 \) be partitioned thus:
\[
W_2 = \begin{bmatrix} U_2 \\ V_2 \end{bmatrix}
\]
where
\[
U_2 := [u_{k-i_k+1}, \ldots, u_{k+j_k}], \quad V_2 := [v_{k-i_k+1}, \ldots, v_{k+j_k}], \quad U_2 \in \mathbb{C}^{m \times r_k}, V_2 \in \mathbb{C}^{n \times r_k}.
\]

**Corollary 7** For each unitary vector \( d = (d_1, \ldots, d_p) \in \mathbb{R}^p \) we have
\[
s_k'(x_0, d) = \mu_k,
\]
where \( \mu_k \) is the \( k \)th eigenvalue of
\[
U_2^* \left( \sum_{j=1}^{p} d_j \frac{\partial A}{\partial x_j}(x_0) \right) V_2 + \left( U_2^* \left( \sum_{j=1}^{p} d_j \frac{\partial A}{\partial x_j}(x_0) \right) V_2 \right)^*,
\]
when the eigenvalues are ranked in a decreasing order.

**Proof.** Given that
\[
W_2 := \begin{bmatrix} U_2 \\ V_2 \end{bmatrix},
\]
the matrix \( F'(d) \) is given by
\[
[U_2^*, V_2^*] \left( \sum_{j=1}^{p} d_j \left[ \begin{array}{c} O \\ \left( \frac{\partial A}{\partial x_j}(x_0) \right)^* \\ O \end{array} \right] \right) \left[ \begin{array}{c} U_2 \\ V_2 \end{array} \right] = \sum_{j=1}^{p} d_j \left[ V_2^* \frac{\partial A}{\partial x_j}(x_0) \right]^* U_2^* \frac{\partial A}{\partial x_j}(x_0) \left[ U_2 \right],
\]
\[
= V_2 \sum_{j=1}^{p} d_j \left[ \frac{\partial A}{\partial x_j}(x_0) \right]^* U_2 + U_2^* \sum_{j=1}^{p} d_j \frac{\partial A}{\partial x_j}(x_0) V_2 = U_2 \sum_{j=1}^{p} d_j \frac{\partial A}{\partial x_j}(x_0) V_2 + \left( U_2^* \sum_{j=1}^{p} d_j \frac{\partial A}{\partial x_j}(x_0) V_2 \right)^*. \quad (4)
\]
The sum of all singular values that coalesce with \( s_k(x_0) \) at \( x_0 \) is differentiable at \( x_0 \). Even more it is true as we can see in the next theorem.

**Theorem 8** The function 

\[
t_k(x) := s_{k-i_k+1}(x) + \cdots + s_k(x) + \cdots + s_{k+j_k}(x)
\]

is differentiable in a neighborhood \( V \subset \Omega \) of \( x_0 \).

The neighborhood \( V \) is determined by the \( x \in \Omega \) sufficient close to \( x_0 \) in order that the inequalities 

\[
s_{k-i_k}(x) > s_{k-i_k+1}(x) \quad \text{and} \quad s_{k+j_k}(x) > s_{k+j_k+1}(x)
\]

hold.

From Corollary 7 we can give another description of \( s'_k(x_0,d) \) in terms of singular vectors of \( A(x_0) \) associated with \( s_k(x_0) \).

**Theorem 9** With the previous notation, let 

\[
Y = [y_{k-i_k+1}, \ldots, y_{k+j_k}] \in \mathbb{C}^{m \times r_k}, \quad Z = [z_{k-i_k+1}, \ldots, z_{k+j_k}] \in \mathbb{C}^{n \times r_k}
\]

be matrices of orthonormal columns and such that \( (y_\ell, z_\ell) \in \mathbb{C}^{m \times 1} \times \mathbb{C}^{n \times 1} \) is a pair of (left,right)-singular vectors of \( A(x_0) \) associated with the singular value \( s_k(x_0) \) for \( \ell \in \{k-i_k+1, \ldots, k+j_k\} \). Then \( s'_k(x_0,d) \) is equal to the \( i_k \)th eigenvalue of the \( r_k \times r_k \) Hermitian matrix

\[
G := \frac{1}{2} \left[ Y^* \left( \sum_{j=1}^p d_j \frac{\partial A}{\partial x_j}(x_0) \right) Z + Z^* \left( \sum_{j=1}^p d_j \left[ \frac{\partial A}{\partial x_j}(x_0) \right]^* \right) Y \right],
\]

when the eigenvalues are ranked in a decreasing order.

### 3 Function of Ikramov-Nazari

With the notations of [3], let \((\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4, A \in \mathbb{C}^{n \times n}\). Define

\[
Q(\xi_1, \xi_2, \xi_3, \xi_4) := \begin{pmatrix} A & \xi_1 I & (\xi_3 + i \xi_4) I \\ 0 & A & \xi_2 I \\ 0 & 0 & A \end{pmatrix}, \quad n \geq 3.
\]

Set

\[
f(\xi) := s_{3n-2}(Q(\xi)).
\]

Suppose that the function \( f \) attains a local maximum at a given \( \xi^0 \in \mathbb{R}^4 \), say \( s_0 := s_{3n-2}(Q(\xi^0)) \). Let us also assume that \( s_0 > 0 \) and it is a multiple singular of \( Q(\xi^0) \). With the above notations, there are \( r_{3n-2} \) singular values before the place \( 3n - 2 + 1 \) and \( j_{3n-2} \) singular values after the place \( 3n - 2 \) equal to
s_{3n-2}(Q(\xi^0)). To shorten notation, we let \( p \) and \( q \) stand for \( i_{3n-2} \) and \( j_{3n-2} \), respectively. Thus, the multiplicity of \( s_0 \) is \( m = p + q \). Hence,

\[
s_1(Q(\xi^0)) \geq \cdots \geq s_{3n-2-p}(Q(\xi^0)) > s_{3n-2-p+1}(Q(\xi^0)) = \cdots = s_{3n-2}(Q(\xi^0)) = s_{3n-2+1}(Q(\xi^0)) = \cdots = s_{3n-2+q}(Q(\xi^0)) > s_{3n-2+q+1}(Q(\xi^0)) \geq \cdots \geq s_{3n}(Q(\xi^0)).
\]

Here \( p \geq 1 \) and \( q \geq 0 \). The function

\[
t(\xi) := s_{3n-2-p+1}(Q(\xi)) + \cdots + s_{3n-2+q}(Q(\xi))
\]

is differentiable in a neighborhood of \( \xi^0 \). Also for each \( k \in \{1, \ldots, 3n\} \) and each unitary vector \( d \in \mathbb{R}^4 \), the function

\[
g_k(\xi) := s_k(Q(\xi))
\]

admits the directional derivative

\[
g_k'(\xi^0, d).
\]

Observe that the used notation implies

\[
f(\xi) = g_{3n-2}(\xi), \quad \xi \in \mathbb{R}^4.
\]

Next, we determine the relationship between the directional derivatives \( f'(\xi^0, d) \) and \( f'(\xi^0, -d) \). Given that \( f \) has a local maximum at \( \xi^0 \), it follows that for all \( e \in \mathbb{R}^4 \),

\[
f'(\xi^0, e) := \lim_{h \to 0^+} \frac{f(\xi^0 + he) - f(\xi^0)}{h} \leq 0.
\]

Thus, \( f'(\xi^0, d) \leq 0 \) and \( f'(\xi^0, -d) \leq 0 \). What conditions must be satisfied in order for \( f'(\xi^0, d) = 0 \) to hold for all unit vector \( d \in \mathbb{R}^4 \)? By Theorem 6, \( f'(\xi^0, d) \) is equal to \( \mu_p(d) \), \( p \)th eigenvalue of the \( m \times m \) matrix

\[
F'(d) = [U_2^*, V_2^*] \left( \sum_{j=1}^4 d_j \begin{bmatrix} 0 & \frac{\partial Q}{\partial \xi_j}(\xi^0) \\ \frac{\partial Q}{\partial \xi_j}(\xi^0) & 0 \end{bmatrix} \right) \begin{bmatrix} U_2 \\ V_2 \end{bmatrix}
\]

with

\[
U_2 = [u_{3n-2-p+1}, \ldots, u_{3n-2+q}]
\]

\[
V_2 = [v_{3n-2-p+1}, \ldots, v_{3n-2+q}]
\]

where \( u_j \) and \( v_j \) are the left and right singular vectors

\[
Q(\xi^0) v_j = s_0 u_j, \quad j = 3n-2-p+1, \ldots, 3n-2+q,
\]

and the eigenvalues of \( F'(d) \) are arranged in this way

\[
\mu_1(d) \geq \cdots \geq \mu_p(d) \geq \cdots \geq \mu_m(d)
\]
Therefore,
\[ f' (\xi^0, d) = \mu_p (d). \]

Again by Theorem 6, we deduce that \( f' (\xi^0, -d) \) is equal to the \( p \)th eigenvalue of the Hermitian matrix \( F' (-d) \). But, it is worth noting that \( f' (\xi^0, -d) \) is not necessarily equal to \( \mu_p (-d) \). In fact, if

\[ \alpha_1 \geq \cdots \geq \alpha_m \]

are the eigenvalues of \( F' (-d) \), then

\[ f' (\xi^0, -d) = \alpha_p. \]

As \( F' (-d) = -F' (d) \), it follows

\[ -\mu_m (d) \geq \cdots \geq -\mu_p (d) \geq \cdots \geq -\mu_1 (d) \] (6)

are the eigenvalues of \( F' (-d) \); whence,

\[ f' (\xi^0, -d) = \alpha_p = -\mu_{m-(p-1)} (d). \] (7)

Now it is necessary to analyze the relative positions of the indices \( p \) and \( m-(p-1) \).

If \( p \leq m-(p-1) \), then \( \mu_p (d) \leq 0 \), and it follows that

\[ 0 \geq \mu_p (d) \geq \cdots \geq \mu_{m-(p-1)} (d) \geq \cdots \geq \mu_m (d). \]

Hence, \( 0 \geq \mu_{m-(p-1)} (d) \) and therefore \( \alpha_p = -\mu_{m-(p-1)} (d) \geq 0 \), but \( \alpha_p = f' (\xi^0, -d) \leq 0 \). Thus, \( \alpha_p = 0 \); i.e. \( f' (\xi^0, -d) = 0 \). Given that \( f \) has a local maximum at \( \xi^0 \), for all unit vector \( e \in \mathbb{R}^4 \), we have

\[ f' (\xi^0, e) = 0. \]

Doubtful case: If \( p > m-(p-1) \), then \( \mu_{m-(p-1)} (d) \geq \mu_p (d) \). But, although \( \mu_p (d) \leq 0 \), it is not guaranteed that the inequality \( \mu_{m-(p-1)} (d) \leq 0 \) holds.

4 Average of singular values

We know that the average of singular values of \( Q(\xi) \) that coalesce with the \( m \)-multiple singular value \( s_{3n-2} (Q(\xi^0)) \) at \( \xi = \xi^0 \), is a differentiable function in a neighborhood of \( \xi^0 \). Thus we consider the differentiable function

\[ H(\xi) := t(\xi) - ms_0; \]

obviously, \( H(\xi^0) = 0 \). Hence, the point \( \xi^0 \) belongs to the level hypersurface of level 0 of the function \( H(\xi) \). Let

\[ \nabla H (\xi^0) = \left( \frac{\partial H}{\partial \xi_1} (\xi^0), \frac{\partial H}{\partial \xi_2} (\xi^0), \frac{\partial H}{\partial \xi_3} (\xi^0), \frac{\partial H}{\partial \xi_4} (\xi^0) \right) \]

be the gradient of \( H(\xi) \) at \( \xi^0 \). Let \( d \in \mathbb{R}^4 \) such that

\[ \nabla H (\xi^0) \cdot d = 0, \]

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where \( \cdot \) denotes the ordinary scalar product in \( \mathbb{R}^4 \). Then, by the chain rule,

\[
H'(\xi^0, d) = \nabla H(\xi^0) \cdot d = 0.
\]

This implies

\[
0 = g'_{3n-2-p+1}(\xi^0, d) + \cdots + g'_{3n-2}(\xi^0, d) + \cdots + g'_{3n-2+q}(\xi^0, d);
\]

if we consider the \( m \times m \) Hermitian matrix \( F'(d) \), it means that the sum of its eigenvalues is zero:

\[
0 = \mu_1(d) + \cdots + \mu_p(d) + \cdots + \mu_m(d).
\]

When \( p = 1 \), this is equivalent to say that \( s_{3n-2}(Q(\xi^0)) \) is the first value of the chain of singular values equal to \( s_0 \), then all the functions

\[
g_{3n-2}(\xi), g_{3n-2+1}(\xi), \ldots, g_{3n-2+q}(\xi)
\]

take the same value at \( \xi^0 \), and it is equal to \( s_0 \). Moreover, all these functions have at \( \xi^0 \) a local maximum, because of

\[
f(\xi) := g_{3n-2}(\xi) \geq g_{3n-2+1}(\xi) \geq \cdots \geq g_{3n-2+q}(\xi).
\]

This implies that for all unitary \( d \in \mathbb{R}^4 \),

\[
\forall k = 3n - 2, \ldots, 3n - 2 + q, \quad g_k'(\xi^0, d) \leq 0;
\]

therefore, \( \mu_1(d) \leq 0, \ldots, \mu_m(d) \leq 0 \), and, given that \( t(\xi) \) has a local maximum at \( \xi^0 \) and is differentiable at \( \xi^0 \), we have

\[
\nabla t(\xi^0) = 0;
\]

whence \( \nabla H(\xi^0) = 0 \) and for all \( k = 3n - 2, \ldots, 3n - 2 + q \), \( g_k'(\xi^0, d) = 0 \); in particular, \( f'(\xi^0, d) = 0 \). This is proved because \( 0 = \mu_1(d) + \cdots + \mu_m(d) \); since \( \forall k, \mu_k(d) \leq 0 \), we obtain \( \forall k, g_k'(\xi^0, d) = 0 \); consequently, \( \forall k, g_k'(\xi^0, d) = 0 \).

From now on let \( p \) be any integer from the range we are considering. Furthermore, suppose that for all \( k = 3n - 2 - p + 1, \ldots, 3n - 2 + q \), all the functions \( g_k(\xi) \) have a local maximum at \( \xi^0 \). Then for all unitary \( d \in \mathbb{R}^4 \), \( g_k'(\xi^0, d) \leq 0 \). As \( t(\xi) \) has a local maximum at \( \xi^0 \), \( t'(\xi^0, d) = 0 \); but

\[
t'(\xi^0, d) = g'_{3n-2-p+1}(\xi^0, d) + \cdots + g'_{3n-2+q}(\xi^0, d);
\]

consequently, \( f'(\xi^0, d) = 0 \).

When some of the functions \( g_k(\xi) \) have a local maximum at \( \xi^0 \) and any others have a local minimum at \( \xi^0 \), the analysis becomes more complicated and I do not obtain any conclusion.

5 **Remark**

In January 31, 2005, I wrote an e-mail to J.B. Hiriart-Urruty asking him whether his results in [2] for real symmetric matrices could be generalized to complex Hermitian matrices. He forwarded my message to M. Torki [10], who answered affirmatively. Moreover, Torki told me that his results in [9] for second order
directional derivatives and real symmetric matrices were also true for the Hermitian case.

A particular case of Theorem 9 about the right derivative of the function $t \mapsto \sigma_k(A+tB)$ at $t = 0$, where $t$ is real and $A, B$ are $n \times n$ complex matrices, was obtained by Lippert [5, Lemma A.5] by a different method. See also Corollary 11 in [1].

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