Quantum decision making by social agents

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Abstract

Decision making of agents who are members of a society is analyzed from the point of view of quantum decision theory. This generalizes the approach, developed earlier by the authors for separate individuals, to decision making under the influence of social interactions. The generalized approach not only avoids paradoxes, typical of classical decision making based on utility theory, but also explains the error-attenuation effects observed for the paradoxes occurring when decision makers, who are members of a society, consult with each other increasing in this way the available mutual information.

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1 Introduction

Decision theory underlies essentially all the social sciences, including economics, finance, political sciences, psychology, and so on. It is also employed in studying the evolution of various social systems, where the evolution equations that describe population dynamics are constructed so as to provide the maximum of utility, or fitness, for the species of the considered social system.

The predominant theory, describing individual behavior under risk and uncertainty is nowadays the expected utility theory of preferences over uncertain prospects. This theory, first introduced by Bernoulli (1738) in his investigation of the St. Petersburg paradox, was axiomatized by von Neumann and Morgenstern (1953), and integrated with the theory of subjective probability by Savage (1954). The theory was shown to possess great analytical power by Arrow (1971) and Pratt (1964) in their work on risk aversion and by Rothschild and Stiglitz (1970, 1971) in their work on comparative risk. Friedman and Savage (1948) and Markowitz (1952) demonstrated its tremendous flexibility in representing decision makers attitudes toward risk. It is fair to state that expected utility theory has provided a solid foundation for the theory of games, the theory of investment and capital markets, the theory of search, and for other branches of economics, finance, and management (Lindgren, 1971; White, 1976; Hastings and Mello, 1978; Rivett, 1980; Buchanan, 1982; Berger, 1985; Marshall and Oliver, 1995; Bather, 2000; French and Insua, 2000; Raiffa and Schlaifer, 2000; Weirich, 2001; Gollier, 2001).

However, a number of economists and psychologists have uncovered a growing body of evidence showing that individuals do not always conform to prescriptions of expected utility theory. Moreover, human beings very often depart from the theory in predictable and systematic way. Actually, the possibility that problems could arise has already been discussed by Bernoulli (1738) himself. Then, many researchers, starting with the works by Allais (1953), Edwards (1955, 1962), and Ellsberg (1961), and continuing through the present, have experimentally confirmed pronounced and systematic deviations from the predictions of expected utility theory, leading to the appearance of many paradoxes. Neuroscience research suggests that the choice process used by human beings is systematically biased and suboptimal (Fehr and Rangel, 2011). Among the known paradoxes of classical utility making, we can list the Bernoulli St. Petersburg paradox (Bernoulli, 1738), the Allais paradox (Allais, 1953), the independence paradox (Allais, 1953), the Ellsberg paradox (Ellsberg, 1961), the Kahneman-Tversky paradox (Kahneman and Tversky, 1979), the Rabin paradox (Rabin, 2000), the Ariely paradox (Ariely, 2008), the disjunction effect (Tversky and Shafir, 1992), the conjunction fallacy (Tversky and Kahneman, 1983; Shafir et al., 1990), the isolation effects (McCaffery and Baron, 2006), the combined paradoxes (Yukalov and Sornette, 2009b, 2010b, 2011), the planning paradox (Kydland and Prescott, 1977), and dynamic inconsistency (Strotz, 1955; Frederick et al., 2002). A large literature on this topic can be found in the recent reviews (Camerer et al., 2003; Machina, 2008).

All paradoxes, which have been discovered in classical decision making, appear in decision problems that can be formulated as follows. One considers a set of outcome payoffs \( X \equiv \{x_i : i = 1, 2, \ldots \} \), on which a probability measure \( p : X \to [0, 1] \) is given. Over the payoff set, there are several lotteries, or prospects, \( \pi_j = \{x_i, p_j(x_i) : i = 1, 2, \ldots \} \), differing by
the outcome probabilities. The payoff set is the domain of a utility function \( u(x) \) that is a non-decreasing concave function. The expected utility of a lottery \( \pi_j \) is defined as \( U(\pi_j) = \sum_i u(x_i) p_j(x_i) \). A lottery \( \pi_1 \) is said to be preferable to \( \pi_2 \) if and only if \( U(\pi_1) > U(\pi_2) \). And the lotteries are indifferent, when \( U(\pi_1) = U(\pi_2) \). Suppose that the given data are such that, according to the classical decision making, a lottery \( \pi_1 \) is preferable or indifferent to \( \pi_2 \), that is, \( U(\pi_1) \geq U(\pi_2) \). However, decision makers, when deciding between several lotteries under uncertainty and in the presence of risk, often choose \( \pi_2 \), instead of \( \pi_1 \), thus, contradicting the prescription of utility theory.

Because of the large number of paradoxes associated with classical decision making, there have been many attempts to change the expected utility approach, which has been classified as non-expected utility theories. There exists a number of such non-expected utility theories, among which we may mention some of the best known: prospect theory (Edwards, 1955; Kahneman and Tversky, 1979), weighted-utility theory (Karmarkar, 1978, 1979; Chew, 1983), regret theory (Loomes and Sugden, 1982), optimism-pessimism theory (Hey, 1984), dual-utility theory (Yaari, 1987), ordinal-independence theory (Green and Jullien, 1988), and quadratic-probability theory (Chew et al., 1991). More detailed information can be found in the review by Machina (2008).

However, as has been shown by Safra and Sigal (2008), none of non-expected utility theories can explain all those paradoxes. The best that could be achieved is a kind of fitting for interpreting just one or, in the best case, a few paradoxes, while the other paradoxes remained unexplained. In addition, spoiling the structure of expected utility theory results in the appearance of complications and inconsistencies. As has been concluded in the detailed analysis of Al-Najjar and Weinstein (2009), any variation of the classical expected utility theory “ends up creating more paradoxes and inconsistencies than it resolves”.

The idea that the functioning of the human brain could be described by the techniques of quantum theory has been advanced by one of the founders of quantum theory (Bohr, 1933, 1958). Von Neumann, who is both a founding father of game theory and of expected utility theory on the one hand and the developer of the mathematical theory of quantum mechanics on the other hand, himself mentioned that the quantum theory of measurement can be interpreted as decision theory (von Neumann, 1955).

The main difference between the classical and quantum techniques is the way of calculating the probability of events. As soon as one accepts the quantum way of defining the concept of probability, the latter generally becomes non-additive. And one immediately meets such quantum effects as interference and entanglement. The possibility of employing the techniques of quantum theory in several branches of sciences, that previously have been analyzed by classical means, is nowadays widely considered. As examples, we can mention quantum game theory (Eisert and Wilkens, 2000; Landsburg, 2004; Guo et al., 2008), quantum information processing and quantum computing (Williams and Clewter, 1998; Nielsen and Chuang, 2000; Keyl, 2002).

After the works by Bohr (1933, 1958) and von Neumann (1955), there have been a number of discussions on the possibility of applying quantum rules for characterizing the process of human decision making (Aerts and Aerts, 1994; Segal and Segal, 1998; Baaquie, 2004, 2009; Busemeyer et al., 2006; Danilov and Lambert-Mogiliansky, 2008, 2010; Bagarello, 2009; Lambert-Mogiliansky et al., 2009; Kitto, 2009; Pothos and Busemeyer, 2010; Leaw
and Cheong, 2010; West and Grigolini, 2010; Zabaletta and Arizmendi, 2010). Many more references can be found in the recent review article (Yukalov and Sornette, 2009b). However, no general theory with quantitative predictive power has been suggested. This was the motivation for our introduction of a general quantum theory of decision making, based on the von Neumann theory of quantum measurements (von Neumann, 1955), that can be applied to any possible situations (Yukalov and Sornette, 2008, 2009a,b,c, 2010a,b, 2011). Our approach is, to the best of our knowledge, the first theory using the mathematical formulation of quantum theory that allows for the quantitative treatment of different classical paradoxes in the frame of a single general scheme. Indeed, practically all paradoxes of classical decision making find their natural explanation if the frame of the Quantum Decision Theory (QDT) (Yukalov and Sornette, 2008, 2009a,b,c, 2010a,b, 2011).

As has been stressed above, our framework does not assume that decision makers are quantum objects. The techniques of quantum theory are employed just as a convenient mathematical tool. Actually, the sole thing we need is the theory of Hilbert spaces. Generally, it is worth stressing that the use of quantum techniques requires that neither brain nor consciousness would have anything to do with genuinely quantum systems. The techniques of quantum theory are used solely as a convenient mathematical tool and language to capture the properties associated with decision making. It is known that the description of any quantum system could be done as if it was a classical system, via the introduction of the so-called contextual hidden variables. However, their number has to be infinite in order to capture the same level of elaboration as their quantum equivalent (Dakic et al., 2008), which makes unpractical the use of a classical equivalent description. Instead, quantum techniques are employed to describe systems in which interference and entanglement effects occur, because they are much simpler than to deal with a classical system having an infinite number of hidden unknown variables. Similarly, we use quantum techniques for decision theory in order to implicitly take into account the existence of many hidden variables in humans, such as emotions, subconscious feelings, and various biases. The existence of these hidden variables strongly influences decision making, as captured partially, for instance, by the notion of bounded rationality (Simon, 1955) and confirmed by numerous studies in Behavioral Economics, Behavioral Finance, Attention Economy, and Neuroeconomics (Cialdini, 2001; Loewenstein et al., 2008).

The standard setup displaying the paradoxes in classical decision making corresponds to individual decision makers that take decisions without consulting each other. However, in a number of experimental studies, it has been found that consultation sharply reduces errors in decision making. For example, Cooper and Kagel (2005) and Blinder and Morgan (2005) find that groups consistently play more strategically than do individuals and generate positive synergies in more difficult games. Charness et al. (2007a,b) show that group membership affects individual choices in strategic games. Charness and Rabin (2002) and Chen and Li (2009) investigate the minimal-group paradigm and find a substantial increase in charity concerns and social-welfare-maximizing actions when participants are matched with in-group members. It was found that the errors in the famous disjunction effect and conjunction fallacy strongly attenuate when group members get information by learning from their experience (Kühberger et al., 2001) or exchange information by consulting (Charness et al., 2010).
Explaining these attenuation effects, caused by information transfer through the interactions between decision makers, requires extending the theory from isolated individuals to human beings who are part of a society within which they interact and exchange information. It is the aim of the present paper to generalize the QDT approach to the case of decision makers who interact within a group or society. The information received from the society influences the decisions. This leads to a natural explanation of the error attenuation effect, as compared with the paradoxes existing for decisions without within-group consultations.

The plan of the paper is as follows. In Section 2, we present the generalization of QDT for a decision maker who is not a separate individual, but a member of a society. In Section 3, we show how the additional information, received by the decision maker through interactions with the surrounding society, leads to the decrease of errors compared with classical decision making. We discuss the experiment by Charness et al. (2010) and explain why the initial error in the conjunction fallacy diminishes with the received information. Section 4 concludes.

2 Social decision makers

Let us consider a society defined as a group of several agents. Each agent is a decision maker, whose decisions are influenced by other members of the society. A decision maker aims at choosing between several admissible choices, called lotteries or prospects. As in our previous papers (Yukalov and Sornette, 2008, 2009a,b,c, 2010a,b, 2011), for each prospect, we associate a vector in a Hilbert space. But now, in addition to the space of mind for a given separate decision maker, there exists the decision space of the society as a whole.

2.1 Decision spaces

Let an agent \( A \) be a member of a society. Assume that, for this agent, there exists a set of elementary prospects that are represented by a set of vectors \( \{ |n\rangle \} \). The elementary-prospect vectors are orthonormalized, so that the scalar product \( \langle m|n \rangle = \delta_{mn} \) is a Kronecker delta. The orthogonality of the elementary prospects means that they are independent and not compatible, so that only one of them can be realized. The space of mind of a decision maker is a closed linear envelope

\[
\mathcal{H}_A \equiv \text{Span}\{|n\rangle\} \quad (1)
\]

spanning all admissible elementary prospects. Similarly, such a space of mind can be constructed for each member of the society, the states of mind of two distinct individuals being in general different. Let the space of mind for all members of the society, except the agent \( A \), be denoted as \( \mathcal{H}_B \). Then the total decision space of the whole society is the tensor product

\[
\mathcal{H}_{AB} \equiv \mathcal{H}_A \otimes \mathcal{H}_B \quad (2)
\]

This is a Hilbert space, where a scalar product is defined. The space \( \mathcal{H}_B \) can also be presented as a tensor product of the individual spaces of all other society members.
The elementary prospects serve as a basis for constructing the Hilbert space of mind. But they are not necessarily the prospects a decision maker is evaluating. They just enumerate all admissible possibilities.

### 2.2 Prospect states

The decision maker $A$ considers a set of prospects

$$\mathcal{L} = \{\pi_j : j = 1, 2, \ldots N\}.$$  \hspace{1cm} (3)

Each prospect $\pi_j$ is put into correspondence to a vector $|\pi_j\rangle$, called the prospect state, in the space of mind $\mathcal{H}_A$. The prospects of $\mathcal{L}$ are, generally, composite objects composed of several elementary prospects. Many concrete examples are given in the published papers (Yukalov and Sornette, 2009a,b, 2010b, 2011).

Being an element of the space $\mathcal{H}_A$, a prospect state can be represented as an expansion over the elementary prospects,

$$|\pi_j\rangle = \sum_n \langle n | \pi_j \rangle |n\rangle.$$  \hspace{1cm} (4)

The prospect states are not assumed to be either orthogonal or normalized, so that the scalar product

$$\langle \pi_i | \pi_j \rangle = \sum_n \langle \pi_i | n \rangle \langle n | \pi_j \rangle$$  \hspace{1cm} (5)

is not a Kronecker delta. The prospects states are not orthogonal with each other, since they are not necessarily incompatible, but can interfere and entangle with each other. And the appropriate normalization condition will be imposed later.

The prospects are the targets of the decision maker. The set $\mathcal{L}$ of these prospects $\pi_j$ should be ordered, forming a complete transitive lattice. The ordering procedure will be given below. The aim of decision making is to find out which of the prospects is the most favorable.

There can exist two types of setups. One is when a number of agents choose between the given prospects. Another type is when a single decision maker takes decisions in a repetitive manner, for instance taking decisions several times. These two cases are treated similarly.

### 2.3 Prospect operators

To each prospect $\pi_j$, with a vector state $|\pi_j\rangle$ in the Hilbert space of mind $\mathcal{H}_A$, there corresponds the prospect operator

$$\hat{P}(\pi_j) \equiv |\pi_j\rangle\langle \pi_j|.$$  \hspace{1cm} (6)

By this definition, the prospect operators are self-adjoint. These operators, generally, are not projectors, as far as they are not necessarily idempotent,

$$\hat{P}^2(\pi_j) = \langle \pi_j | \pi_j \rangle \hat{P}(\pi_j).$$
which follows from the fact that the prospect states, generally, are not normalized. The prospect operators are not commutative, since the expressions

\[
\hat{P}(\pi_i)\hat{P}(\pi_j) = \langle \pi_i | \pi_j \rangle | \pi_i \rangle \langle \pi_i |, \quad \hat{P}(\pi_j)\hat{P}(\pi_i) = \langle \pi_j | \pi_i \rangle | \pi_j \rangle \langle \pi_i | ,
\]
differing by the order of operators, are not equivalent. The noncommutativity of the prospect operators represents the noncommutativity of decisions in real life (Yukalov and Sornette, 2009a,b, 2010b, 2011).

The collection \( \{ \hat{P}(\pi_j) \} \) of the prospect operators is analogous to the algebra of local observables in quantum theory. In the latter, as is known, not each product of local observables is, strictly speaking, an observable. But it is always possible to define symmetrized products so that the collection of local observables would form an algebra. In the same way as for the operators of local observables in quantum theory, we can consider the family \( \{ \hat{P}(\pi_j) \} \) of prospect operators as an algebra of observables in QDT.

### 2.4 Prospect probabilities

QDT is a probabilistic theory, with the prospect probabilities defined as the averages of the prospect operators. In that sense, the prospect probabilities play the role of the observable quantities (Yukalov and Sornette, 2008). In our previous papers, the averages were defined with respect to a given strategic state |\( \psi \rangle \) characterizing the decision maker. Such a procedure corresponds to the averaging over a prescribed pure state, which assumes that the considered decision maker is an individual, not interacting with any surrounding. But when considering a decision maker in a society, which he/she interacts with, such a decision maker cannot be characterized by a pure state.

The society as a whole could be described by a pure wave function, with the decision maker being a part of the society, which would then lead to the necessity of characterizing this decision maker by a statistical operator. This is in a direct analogy with treating subsystems of large systems by density matrices (Coleman and Yukalov, 2000).

Moreover, we could describe the society by a wave function only if we would assume that the society is completely isolated from its surrounding. But such an assumption is certainly unreasonable, since there are no absolutely isolated societies. Again, this is completely equivalent to the absence of absolutely isolated finite quantum systems (Yukalov, 2002, 2003a,b). Thus, the most general way of describing the society state is by a statistical operator.

In the present case, the society state, including the considered decision maker, is to be characterized by a statistical operator \( \hat{\rho}_{AB} \) that is a positive operator on \( \mathcal{H}_{AB} \), normalized as

\[
\text{Tr}_{AB}\hat{\rho}_{AB} = 1,
\]

with the trace operation being performed over \( \mathcal{H}_{AB} \). The observable quantities are to be defined by the expectation values over the statistical operator. Therefore the prospect probabilities are given by the averages

\[
p(\pi_j) \equiv \text{Tr}_{AB}\hat{\rho}_{AB}\hat{P}(\pi_j).
\]
The prospect operators act on the space of mind $\mathcal{H}_A$ of the decision maker. Hence the above average can be represented as

$$p(\pi_j) \equiv \text{Tr}_A \hat{\rho}_A \hat{P}(\pi_j) ,$$

(9)

where the trace is over $\mathcal{H}_A$ and the reduced statistical operator is

$$\hat{\rho}_A \equiv \text{Tr}_B \hat{\rho}_{AB} .$$

(10)

This operator characterizes the decision maker in the society. The reduction to the previous situation of a single separated decision maker, as considered in our previous papers, would correspond to the representation of the statistical operator $\hat{\rho}_A$ in the pure form $|\psi\rangle\langle\psi|$, with the state $|\psi\rangle$ being the decision maker strategic state. But, generally, the statistical operator $\hat{\rho}_A$ cannot be represented in such a factor form, since the decision maker state is entangled with that of the society.

Introducing the matrix elements over the elementary-prospect basis for the statistical operator

$$\rho_{mn} \equiv \langle m | \hat{\rho}_A | n \rangle$$

(11)

and for the prospect operators

$$P_{mn}(\pi_j) \equiv \langle m | \hat{P}(\pi_j) | n \rangle = \langle m | \pi_j \rangle \langle \pi_j | n \rangle$$

(12)

makes it possible to rewrite the prospect probabilities as

$$p(\pi_j) = \sum_{mn} \rho_{mn} P_{mn}(\pi_j) .$$

(13)

To really represent probabilities, the above quantities are to be normalized so that

$$\sum_{j=1}^N p(\pi_j) = 1 .$$

(14)

Since the statistical operator, by definition, is a positive operator, we have

$$0 \leq p(\pi_j) \leq 1 .$$

(15)

This defines the collection $\{p(\pi_j)\}$ as a probability measure. The most favorable prospect corresponds to the largest of the probabilities.

Let us introduce the utility factor

$$f(\pi_j) \equiv \sum_n \rho_{nn} P_{nn}(\pi_j)$$

(16)

and the attraction factor

$$q(\pi_j) \equiv \sum_{m \neq n} \rho_{mn} P_{mn}(\pi_j) ,$$

(17)
whose meanings will be explained below. Then, separating the diagonal and non-diagonal terms in the sum over \(m\) and \(n\), we obtain the probability of a prospect \(\pi_j\) as the sum

\[
p(\pi_j) = f(\pi_j) + q(\pi_j)
\]

of the above two factors.

Though some intermediate steps of the theory might look a bit complicated, the final result is rather simple and can be straightforwardly used in practice, provided the way of evaluating the utility and attraction factors are known.

### 2.5 Utility factors

As is known (Neumann, 1955), the expectation values of observables in quantum theory can be separated in two terms, one having a diagonal representation over the chosen basis and another being off-diagonal in this representation. The diagonal part corresponds to the classical value of the observable, while the off-diagonal part characterizes purely quantum effects caused by interference. The same holds in our case, where the prospect probability \(\pi\) is defined as the expectation value of the prospect operator. The diagonal part is the utility factor (16) describing the weight of the prospect calculated classically. To be defined as a weight, the set of these factors is to be normalized as

\[
\sum_{j=1}^{N} f(\pi_j) = 1 \, ,
\]

from where one has

\[
0 \leq f(\pi_j) \leq 1 \, ,
\]

since, by definition (16), the factor is non-negative.

In classical decision theory, the choice of a decision maker is based on the notion of expected utility. One considers a set of measurable payoffs \(\{x_i\}\) associated with the related probabilities \(p_j(x_i)\) whose family forms a probability measure with the standard properties

\[
\sum_i p_j(x_i) = 1 \, , \quad 0 \leq p_j(x_i) \leq 1 \, .
\]

A prospect \(\pi_j\) is represented by a lottery

\[
\pi_j \equiv \{x_i \, , p_j(x_i) : i = 1, 2, \ldots\} \, .
\]

Linear combinations of lotteries are defined as

\[
\sum_j \lambda_j \pi_j = \left\{x_i \, , \sum_j \lambda_j p_j(x_i)\right\} \, ,
\]

with the constants \(\lambda_j\) such that

\[
\sum_j \lambda_j = 1 \, , \quad 0 \leq \lambda_j \leq 1 \, .
\]
Introducing a utility function \( u(x) \), which is defined as a non-decreasing concave and positive function, one constructs the expected utility

\[
U(\pi_j) = \sum_i u(x_i)p_j(x_i) .
\]  

(22)

The utility factor is nothing but the expected utility reduced so as to satisfy the normalization condition (19), which gives

\[
f(\pi_j) = \frac{U(\pi_j)}{\sum_j U(\pi_j)} .
\]  

(23)

The same expression for the utility factor can be derived by maximizing the Shannon information under the given expected likelihood (Yukalov and Sornette, 2009b).

2.6 Attraction factors

The off-diagonal term in the expectation value (9) is the attraction factor (17) representing quantum interference, or coherence, effects. In QDT, the attraction factor is a contextual object describing subconscious feelings, emotions, and biases, playing the role of hidden variables. Despite their contextuality, the attraction factors satisfy some general properties that make possible their quantitative evaluation.

In view of normalizations (14) and (19), the attraction factors satisfy the \textit{alternation property}, such that the sum

\[
\sum_{j=1}^N q(\pi_j) = 0
\]  

(24)

over the prospect lattice \( \mathcal{L} \) is always zero, and the values of the attraction factor are in the range

\[-1 \leq q(\pi_j) \leq 1 .
\]  

(25)

In addition, the average absolute value of the attraction factor is estimated (Yukalov and Sornette, 2009b, 2011) by the \textit{quarter law}

\[
\frac{1}{N} \sum_{j=1}^N | q(\pi_j) | = \frac{1}{4} .
\]  

(26)

These properties allow us to \textit{quantitatively} define the prospect probabilities (18).

We may note that the attraction factor exists only for composite prospects, composed of several actions, while for elementary prospects this term is zero. This is easy to show as follows. Let \( e_j \) be an elementary prospect corresponding to a state \( |j\rangle \), hence \( \langle n|j \rangle = \delta_{nj} \). The related prospect operator \( \hat{P}(e_j) \) is defined in Eq. (6). Then the prospect probability (13) reduces to

\[
p(e_j) = \sum_{mn} \rho_{mn} \delta_{mj} \delta_{nj} = \rho_{jj} ,
\]

and the attraction factor is zero:

\[
q(e_j) = 0 .
\]
In this way, there exists a direct general relation between Quantum Decision Theory and classical decision theory, based on the maximization of expected utility. Classical decision theory is retrieved when the attraction factor is zero. The form of the utility factor (23) shows that, in this situation, maximizing the expected utility is equivalent to maximizing the utility factor. Thus, classical decision theory is a particular case of the more general QDT in the case when only objective information on the decision utility is taken into account, while subjective sides, such as biases, emotions, and subconscious feelings play no role. The latter variables do play a very important role in decision making performed in many important and practical situations. Our approach takes into account both the objective utility of considered prospects as well as their subjective attractiveness for the decision maker.

Let us briefly summarize. As we said, the attraction factor in QDT appears naturally in order to account for subconscious feelings, emotions, and biases. Despite the fact that the attraction factor is contextual, it satisfies three pivotal general properties: (i) an attraction factor varies in the interval \([-1, 1]\); (ii) the sum of all attraction factors over the lattice of considered prospects is zero; (iii) the average absolute value of an attraction factor is 0.25. These properties make it possible to give a quantitative evaluation of prospect probabilities and, thus, to develop a practical way of applying QDT to realistic problems of decision making.

2.7 Prospect ordering

Since the prospect probability (18) consists of two terms, we should consider both of them, when comparing the probabilities of different prospects. That is, we have to compare the usefulness as well as attractiveness of the prospects.

The usefulness of prospects is measured by the utility factor. The prospect \(\pi_1\) is more useful than \(\pi_2\), when

\[
f(\pi_1) > f(\pi_2).
\]

The prospects \(\pi_1\) and \(\pi_2\) are equally useful, if

\[
f(\pi_1) = f(\pi_2).
\]

And the prospect \(\pi_1\) is not less useful (more useful or equally useful) than \(\pi_2\), if

\[
f(\pi_1) \geq f(\pi_2).
\]

The attractiveness of prospects is characterized by their attraction factors. The prospect \(\pi_1\) is more attractive than \(\pi_2\), if

\[
q(\pi_1) > q(\pi_2).
\]

The prospects \(\pi_1\) and \(\pi_2\) are equally attractive, when

\[
q(\pi_1) = q(\pi_2).
\]

And the prospect \(\pi_1\) is not less attractive (more attractive or equally attractive) than \(\pi_2\), when

\[
q(\pi_1) \geq q(\pi_2).
\]
The comparison between the attractiveness of prospects can be done on the basis of the aversion to uncertainty and risk or ambiguity aversion (Rothschild and Stiglitz, 1970; Gollier, 2001; Sornette, 2003; Malvergne and Sornette, 2006; Abdellaoui et al., 2011a; 2011b; Yukalov and Sornette, 2011).

A prospect is more attractive when:

- (i) it provides more certain gain (more uncertain loss).
- (ii) it promotes to be active under certainty (passive under uncertainty).

The total evaluation of prospects that finally influences the decision maker choice is based on the prospect probabilities. The prospect \( \pi_1 \) is preferable to \( \pi_2 \), if

\[
p(\pi_1) > p(\pi_2).
\]  
(33)

The prospects \( \pi_1 \) and \( \pi_2 \) are indifferent, when

\[
p(\pi_1) = p(\pi_2).
\]  
(34)

And the prospect \( \pi_1 \) is preferable or indifferent to \( \pi_2 \), if

\[
p(\pi_1) \geq p(\pi_2).
\]  
(35)

The classification of prospects of a set \( \mathcal{L} \) as more or less preferable establishes an order in \( \mathcal{L} \) making this ordered set a lattice. Among all prospects, there exists the least preferable prospect with the minimal probability, and the most preferable prospect with the largest probability. Hence, the prospect lattice \( \mathcal{L} \) is complete. The lattice is also transitive since, if \( \pi_1 \) is preferable to \( \pi_2 \), with \( \pi_2 \) being preferable to \( \pi_3 \), then \( \pi_1 \) is preferable to \( \pi_3 \).

Decision makers choose the most preferable prospect, whose probability is the largest. Such a prospect is called optimal. The prospect \( \pi_* \) is optimal if and only if

\[
p(\pi_* ) = \max_j p(\pi_j).
\]  
(36)

In the presence of two criteria characterizing each prospect, a given prospect can be more useful, while being less attractive, or vice-versa. As a consequence, there are situations where the ordering of classical utility theory is inverted, so that the less useful though more attractive prospect is preferred, having the largest probability. This important fact can be formalized by the following statement.

**Proposition 1.** The prospect \( \pi_1 \) is preferable to \( \pi_2 \) if and only if

\[
f(\pi_1) - f(\pi_2) > q(\pi_2) - q(\pi_1).
\]  
(37)

**Proof:** It follows from the comparison of the prospect probabilities (18) for \( \pi_1 \) and \( \pi_2 \).

This inequality provides an explanation for the appearance of paradoxes in classical decision making as resulting from the role of the attraction factor representing the interference between prospects. It is remarkable that this simple idea seems to be sufficient to remove the empirical paradoxes and make QDT consistent with the decisions made by real human beings. The existence of the attraction factor is due to the presence of risk and uncertainty associated with the choices to be made.
2.8 Binary lattice

A situation that is very often considered in empirical research consists in choosing between two prospects, which corresponds to a binary lattice

\[ \mathcal{L} = \{\pi_1, \pi_2\} . \]  

(38)

This case is sufficient to clearly illustrate the above general considerations.

For a binary lattice, we have

\[ p(\pi_1) = f(\pi_1) + q(\pi_1) , \quad p(\pi_2) = f(\pi_2) + q(\pi_2) . \]  

(39)

The normalization (19) reads as

\[ f(\pi_1) + f(\pi_2) = 1 , \]  

(40)

and the alternation property (24) becomes

\[ q(\pi_1) + q(\pi_2) = 0 . \]  

(41)

If the considered two prospects are equally attractive, which implies \( q(\pi_1) = q(\pi_2) \), then, according to (41), we get \( q(\pi_1) = q(\pi_2) = 0 \). Therefore, the prospect probabilities coincide with their utility factors, \( p(\pi_1) = f(\pi_1) \) and \( p(\pi_2) = f(\pi_2) \). In such a situation, we return to the standard decision making recipe based on the comparison between the prospect utilities.

But when the prospects are not equally attractive, say \( \pi_1 \) is more attractive than \( \pi_2 \), that is, \( q(\pi_1) > q(\pi_2) \), then the alternation property (41) yields

\[ q(\pi_1) = -q(\pi_2) > 0 . \]

This allows one to make accurate predictions of the choice of real human being who have to choose an optimal prospect.

2.9 Individual decisions

Suppose that a decision maker has to choose between several prospects. Let he/she be assumed to make a decision sufficiently quickly, with no consultations with other members of society, and without getting additional information from other sources. This kind of decision making can be termed individual. Such a setup is typical of the majority of experimental observations, where different paradoxes have been documented.

In the case of this spontaneous decision making, it is possible to quantitatively predict typical decisions and, respectively, to explain the occurrence of characteristic paradoxes. This can be done as follows. Let us consider a binary lattice of prospects. Assume that, according to the risk-uncertainty aversion formulated above, the prospect \( \pi_1 \) is more attractive than \( \pi_2 \), hence

\[ q(\pi_1) > q(\pi_2) . \]
It is possible to estimate the attraction factors by their mean values, as explained above, evaluating \( q(\pi_1) \) as equal to \( 1/4 \) and \( q(\pi_2) \) as given by \( -1/4 \). At the same time, the probability belongs to the interval \([0, 1]\). To take this into account, it is convenient to invoke the function, called \textit{retract}, such that

\[
\text{Ret}_{[a,b]} \{z\} = \begin{cases} 
    a, & z \leq a \\
    z, & a < z < b \\
    b, & z \geq b
\end{cases}
\]

Then the prospect probabilities (39) can be represented as

\[
p(\pi_1) = \text{Ret}_{[0,1]} \left\{ f(\pi_1) + \frac{1}{4} \right\}, \\
p(\pi_2) = \text{Ret}_{[0,1]} \left\{ f(\pi_2) - \frac{1}{4} \right\}.
\]

(42)

Since, the utility factors are calculated by means of formula (23), one gets a quantitative estimate for the prospect probabilities, which makes it possible to choose the preferable prospect.

\textbf{Proposition 2.} Let the prospect \( \pi_1 \) from a binary prospect lattice be more attractive than \( \pi_2 \) and let the prospect probabilities be evaluated by expressions (42), then \( \pi_1 \) is preferable over \( \pi_2 \) when the utility factor of \( \pi_1 \) is such that

\[
f(\pi_1) > \frac{1}{4} \quad (\pi_1 > \pi_2).
\]

(43)

Respectively, the prospects are indifferent, if \( f(\pi_1) = 1/4 \) and the prospect \( \pi_2 \) is preferable, if \( f(\pi_1) < 1/4 \).

\textit{Proof:} It follows from expressions (42) and the condition that the prospect \( \pi_1 \) is more attractive than \( \pi_2 \), so that \( q(\pi_1) > q(\pi_2) \).

\subsection*{2.10 Comparison with experiment}

Strictly speaking, being defined to reflect subjective factors embodying subconscious feelings, emotions, and biases, the attraction factors are contextual. This means that their values can be different for different decision makers. Moreover, they can be different for the same decision maker at different times. These features seem to be natural when one keeps in mind that we are describing real humans, whose decisions are usually different, even under identical conditions. It is also known that the same decision maker can vary his/her decisions at different times and under different circumstances. However, focusing solely on the contextual character of the interference terms, gives the wrong impression of a lack of predictive power of the approach, which would make it rather meaningless.

Fortunately, there is a way around the problem of contextuality, based on the fact that QDT has been constructed as a probabilistic theory, with the probabilities interpreted in the frequentist sense. This is equivalent to saying that QDT is a theory of the aggregate behavior of a population. In other words, the predictions of the theory are statistical statements concerning the population of individualistic behaviors, namely, QDT provides the probability
for a given individual to take this or that decision, interpreted in the sense of the fraction of individuals taking these decisions.

The prospect probabilities, calculated in the frame of QDT, can be compared with the results of experimental tests. In experiments, one usually interrogates a pool of $M$ decision makers, asking them to choose a prospect from the given prospect set $\{\pi_j\}$. Different decision makers, of course, can classify as optimal different prospects. Since the utility factor is an objective quantity, we assume that it is the same for all decision makers. The difference between the decisions of the pool members happens because the attraction factors, being subjective quantities, can be different for different decision makers. Here, we thus do not need to invoke random utilities and heterogeneous expectations in the objective utility factor (Cohen, 1980; McFadden and Richter, 1991; Clark, 1995; Regenwetter, 2001). The heterogeneity or differences between different decision makers appear due to the presence of the attraction factor that embodies different states of minds among the human population, and as a function of context and time.

The experimental probability that a prospect $\pi_j$ is chosen can be defined as a frequency in the following way. Let $M_j$ agents from the total number $M$ of decision makers choose the prospect $\pi_j$. Then, assuming a large number of agents, the aggregate probability of this prospect is given by the frequency

$$p_{\text{exp}}(\pi_j) = \frac{M_j}{M}.$$  \hspace{1cm} (44)

This experimental probability is to be compared with the theoretical prospect probability $p(\pi_j)$, using the standard tools of statistical hypothesis testing.

It is also possible to define the aggregate value of the attraction factor by the equation

$$q(\pi_j) = p_{\text{exp}}(\pi_j) - f(\pi_j)$$  \hspace{1cm} (45)

and to compare this with the mean values $\pm 1/4$.

In this way, QDT provides a practical scheme that can be applied to realistic problems for various kinds of decision making in psychology, economics, and finance.

As an illustration, we have applied this theory to several examples in which the disjunction effect occurs. The latter is specified by Savage (1954) as a violation of the sure-thing principle. A typical setup for illustrating the disjunction effect is a two-step gamble (Tversky and Shafir, 1992). Suppose that a group of people accepted a gamble in which the player can either win an amount of money or lose a possibly different amount. After the first gamble, the participants are invited to gamble a second time, being free to either accept the second gamble or to refuse it. Experiments by Tversky and Shafir (1992) showed that the majority of people accept the second gamble when they know the result of the first gamble, whatever its result, whether they won or lost in the previous gamble, but only a minority accepted the second gamble when the outcome of the first gamble was unknown to them.

Another example, studied by Tversky and Shafir (1992), had to do with a group of students who reported their preferences about buying a nonrefundable vacation, following a tough university test. They could pass the exam or fail. The students had to decide whether they would go on vacation or abstain. It turned out that the majority of students purchased the vacation when they passed the exam as well as when they had failed. However, only a
minority of participants purchased the vacation when they did not know the results of the examination.

Another example of the disjunction effect concerns stock markets, as analyzed by Shafir and Tversky (1992). Consider the USA presidential election, when either a Republican or a Democrat wins. On the eve of the election, market players can either sell certain stocks from their portfolio or hold them. It is known that a majority of people would be inclined to sell their stocks, if they would know who wins, regardless of whether the Republican or Democrat candidate wins the upcoming election. This is because people expect the market to fall after the elections. At the same time, a great many people do not sell their stocks before knowing who really won the election, thus contradicting the sure-thing principle. Thus, investors could have sold their stocks before the election at a higher price, but, abiding to the disjunction effect, they were waiting until after the election to know its result, thereby selling sub-optimally at a lower price after stocks have already fallen.

We have presented a detailed analysis of the above experiments (Yuvalov and Sornette, 2009b; 2011). The absolute value of the aggregate attraction factor \( 15 \) was found, within the typical statistical error of the order of 20% characterizing these experiments, to coincide with the predicted value \( 0.25 \).

Another known paradox in classical decision making is the conjunction error. A typical situation is when people judge about a person, who can possess one characteristic and also some other characteristics, as in the often-cited example of Tversky and Kahneman (1980): Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which is more likely? (i) Linda is a bank teller; (ii) Linda is a bank teller and is active in the feminist movement. Most people answer (ii) which is an example of the conjunction fallacy.

There are many other examples of the conjunction fallacy. For a quantitative analysis, we have taken the data from Shafir et al. (1990), who present one of the most carefully accomplished and thoroughly discussed set of experiments on the conjunction fallacy. Again, we found (Yuvalov and Sornette, 2009b; 2011) that the value of the aggregate attraction factor, within the experimental accuracy of 20%, coincides with 0.25, in excellent agreement with the QDT quarter law.

The planning paradox has also found a natural explanation within QDT (Yuvalov and Sornette, 2009a). Moreover, it has been shown (Yuvalov and Sornette, 2010b) that QDT explains practically all typical paradoxes of classical decision making, arising when decisions are taken by separate individuals.

### 3 Influence of social interactions

The standard setup displaying the paradoxes in classical decision making corresponds to individual decision makers that take decisions without consulting each other. As has been mentioned in the Introduction, in a number of experimental studies, it has been found that exchange of information through consultations sharply reduces errors in decision making compared with the prescription of classical utility theory. For instance, the errors in the
disjunction effect and conjunction fallacy strongly decrease, when group members get information by learning from their experience (Kühberger et al., 2001) or exchange information by consulting (Charness et al., 2010).

The theory developed in the previous sections has been formulated for a decision maker that is a member of a society. An individual decision maker is just a particular instance for the application of the theory. The suggested general approach can also be applied to the case of a decision maker interacting with other members of the society and receiving information from them, which may change his/her preferences and decrease the errors typical of individual decision makers. A decision maker, receiving information from the surrounding members of his/her society, can be called a learning decision maker.

3.1 Learning decision maker

Let us denote by $\tau$ a measurable amount of information received by a decision maker from the surrounding society. The amount of information can be measured by invoking some of the known information measures (Khinchin, 1957; Arndt, 2004). The information can be received through direct interactions, that is, consultations with other members of the society. Or each member of the society can receive information by learning the results of other agents activity. For instance, the aggregate trades of agents in a market produce the data characterizing this market that is then available to all and mediates the indirect interactions between them. Learning these data gives information to each of the traders (Barber et al., 2009).

If each member of the society gets the amount of information $\tau$, the state of each member changes, hence the state of the society also varies depending on the amount of the exchanged information. The statistical operator, characterizing the society state, is now a function $\hat{\rho}_{AB}(\tau)$, which is normalized as

$$\text{Tr}_{AB}\hat{\rho}_{AB}(\tau) = 1.$$  \hfill (46)

We then follow a procedure similar to that described in Section 2. The prospect probability is defined as before by

$$p(\pi_j, \tau) \equiv \text{Tr}_{AB}\hat{\rho}_{AB}(\tau)\hat{P}(\pi_j),$$  \hfill (47)

with the difference that we have now an additional variable $\tau$. By convention, if the latter is set to zero, we return to the same formulas as those presented in Section 2. Since the prospect operators act on the space of mind $\mathcal{H}_A$, by defining the reduced statistical operator

$$\hat{\rho}_A(\tau) \equiv \text{Tr}_B\hat{\rho}_{AB}(\tau),$$  \hfill (48)

the prospect probability takes the form

$$p(\pi_j, \tau) \equiv \text{Tr}_A\hat{\rho}_A(\tau)\hat{P}(\pi_j).$$  \hfill (49)

And in the matrix representation, we get

$$p(\pi_j, \tau) = \sum_{mn} \rho_{mn}(\tau)P_{nm}(\pi_j),$$  \hfill (50)
with the notation
\[ \rho_{mn}(\tau) \equiv \langle m | \hat{\rho}_A(\tau) | n \rangle \] (51)
and the normalization condition
\[ \text{Tr}_A \hat{\rho}_A(\tau) = \sum_n \rho_{nn}(\tau) = 1 . \] (52)

Let us introduce the evolution operator \( \hat{U}(\tau) \) that describes the evolution of the system state under the varying amount of the exchanged information \( \tau \). The initial state, before the information exchange starts, is
\[ \hat{\rho}_{AB}(0) = \hat{1}_{AB} , \] (53)
and corresponds to the situation when decision makers were still separate non-interacting individuals.

The transformation resulting from the interactions between decision makers can be represented as
\[ \hat{\rho}_{AB}(\tau) = \hat{U}(\tau) \hat{\rho}_{AB} \hat{U}^+(\tau) . \] (54)
To satisfy the initial condition (53), it is necessary that the initial value of the evolution operator \( \hat{U}(0) \) be the identity operator \( \hat{1}_{AB} \) acting on \( \mathcal{H}_{AB} \):
\[ \hat{U}(0) = \hat{1}_{AB} . \] (55)
In order for the normalization condition (46) to be valid for all \( \tau \), the evolution operator has to be unitary such that
\[ \hat{U}^+(\tau) \hat{U}(\tau) = \hat{1}_{AB} . \] (56)
Assuming that \( \hat{U}(\tau) \) is continuous with respect to \( \tau \), differentiating condition (56), applying the operator \( \hat{U}(\tau) \) and using again (56) gives
\[ \frac{d\hat{U}(\tau)}{d\tau} + \hat{U}(\tau) \frac{d\hat{U}^+(\tau)}{d\tau} \hat{U}(\tau) = 0 . \] (57)
This equation for a unitary operator \( \hat{U}(\tau) \) can be rewritten as
\[ i \frac{d\hat{U}(\tau)}{d\tau} = \hat{H}_{AB} \hat{U}(\tau) , \] (58)
where \( H_{AB} \) is called the evolution generator, which is a self-adjoint operator on \( \mathcal{H}_{AB} \) assumed to be invariant with respect to \( \tau \). Equation (58) yields the evolution operator
\[ \hat{U}(\tau) = \exp \left( -i \hat{H}_{AB} \tau \right) . \] (59)
This evolution operator, in view of Eq. (54), defines the variation of the total state of the society under the varying amount of information \( \tau \).
### 3.2 Decision maker as an individual personality

The interaction of the decision maker with her social environment is supposed to ensure that she keeps her distinct identity and personality while, at the same time, possibly changing her state of mind. In other words, the surrounding society does influence the decision maker state, but does so in a way that does not suppress her as a person taking her own decisions. In modeling terms, this corresponds to the behavior of a subsystem that is part of a larger system that changes the subsystem properties, while the subsystem is not destroyed and retains its typical features. Such a subsystem is called *quasi-isolated* (Yukalov, 2011, 2012a). Another correspondence is the influence exerted on a finite system by an external measuring device that acts so as not to destroy the main system features, a situation referred to as *nondestructive measurements* (Yukalov, 2012b). In mathematical language, these properties are formulated as follows.

Reflecting the fact that the total system that is considered consists of the decision maker, her surrounding society, and their mutual interactions, the evolution generator \( \hat{H}_{AB} \) is represented as a sum of the corresponding three terms

\[
\hat{H}_{AB} = \hat{H}_A + \hat{H}_B + \hat{H}_{\text{int}}.
\]  

The first term characterizes the decision maker, which implies that the operator \( \hat{H}_A \) generates the space of mind \( \mathcal{H}_A \) by defining the basis of elementary prospects that are typical of the decision maker, through the eigenvalue problem

\[
\hat{H}_A \left| n \right\rangle = E_n \left| n \right\rangle ,
\]  

with the span over the basis yielding the space of mind \( I \). The second term, acting on the space \( \mathcal{H}_B \), describes the surrounding society. And the third term, acting on the total space \( \mathcal{H}_{AB} \), corresponds to the interaction of the decision maker with her social environment, associated with the process of information exchange.

As mentioned above, the interaction of the decision maker with her social environment is supposed to ensure that the decision maker keeps her identity and personality, although possibly changing her state of mind. In mathematical language, this is formulated as the following commutativity property

\[
\left[ \hat{H}_A, \hat{H}_{\text{int}} \right] = 0.
\]  

The latter, in combination with (60), is equivalent to the commutativity condition

\[
\left[ \hat{H}_A, \hat{H}_{AB} \right] = 0.
\]  

Actually, these general properties are sufficient for characterizing the decision maker as a distinct personality, and more detailed structure of the generators is not important.

Let the space \( \mathcal{H}_B \) be generated by the generator \( \hat{H}_B \) through the span over the basis formed by the eigenvectors given by the eigenproblem

\[
\hat{H}_B \left| k \right\rangle = B_k \left| k \right\rangle .
\]
In view of Eq. (62), there is a set of real numbers \( \{ \beta_{nk} \} \) such that the interaction term satisfies the equation
\[
\hat{H}_{\text{int}} | nk \rangle = \beta_{nk} | nk \rangle,
\]
in which \( |nk\rangle \equiv |n\rangle \otimes |k\rangle \) denotes the tensorial product between the eigenvectors \( |n\rangle \) and \( |k\rangle \). Then, the generator of the total system yields the eigenproblem
\[
\hat{H}_{AB} | nk \rangle = (E_n + B_k + \beta_{nk}) | nk \rangle.
\]

The above equations make it straightforward to derive the explicit expression for the prospect probability (50). For this purpose, let us introduce some convenient notations. We define the eigenvalue differences
\[
\omega_{mn} = E_m - E_n, \quad \varepsilon_{mnk} = \beta_{mk} - \beta_{nk}
\]
and the matrix elements
\[
\overline{\rho}_{mn}(\tau) = \rho_{mn}(0) \exp(-i\omega_{mn}\tau),
\]
in which
\[
\rho_{mn}(0) = \langle m \mid \hat{\rho}_A \mid n \rangle = \rho_{mn}.
\]

We introduce the effect density describing the distribution of the impacts of the surrounding environment affecting the considered decision maker during the process characterizing the transfer of information:
\[
g_{mn}(\varepsilon) \equiv \sum_k \frac{\langle mk \mid \hat{\rho}_{AB} \mid nk \rangle}{\langle m \mid \hat{\rho}_A \mid n \rangle} \delta(\varepsilon - \varepsilon_{mnk}).
\]
It is clear that the latter is normalized as
\[
\int_{-\infty}^{+\infty} g_{mn}(\varepsilon) \, d\varepsilon = 1.
\]
The Fourier transform of the effect density gives the decoherence factor
\[
D_{mn}(\tau) \equiv \int_{-\infty}^{+\infty} g_{mn}(\varepsilon)e^{-i\varepsilon\tau} \, d\varepsilon.
\]
Finally, we come to the prospect probability (50) represented as
\[
p(\pi_j, \tau) = f(\pi_j) + q(\pi_j, \tau).
\]
Here, the first term is the same utility factor as in Eq. (16). It does not depend on the received additional information, being assumed to be an objective invariant quantity. And the second term is the attraction factor as a function of the received information
\[
q(\pi_j, \tau) = \sum_{m \neq n} \overline{\rho}_{mn}(\tau) P_{nm}(\pi_j) D_{mn}(\tau).
\]
Generally, the decoherence factor can depend on the indices \(m, n\). For simplicity, it is possible, resorting to the theorem of average, to employ an averaged decoherence factor not depending on the indices, which reduces the attraction factor (73) to the form

\[ q(\pi_j, \tau) = \overline{q}(\pi_j, \tau)D(\tau), \]

where

\[ \overline{q}(\pi_j, \tau) \equiv \sum_{m \neq n} \rho_{mn}(\tau)P_{nm}(\tau). \]

Since the effect density is normalized as in Eq. (70), the decoherence factor \(D(\tau)\) derived from the above use of the theorem of average with (71) enjoys the property \(D(0) = 1\). Therefore, at zero information, the attraction factor

\[ q(\pi_j, 0) = q(\pi_j) \tag{74} \]

has the properties described in Section 2.6, and we return to the initial prospect probability

\[ p(\pi_j, 0) = p(\pi_j) \tag{75} \]

defined by Eq. (18).

Thus, we see that the absolute value of the attraction factor essentially depends on the value of the decoherence factors (71).

### 3.3 Attraction factor attenuation

If the surrounding society does not influence the decision maker, the effect density is given by the delta function \(\delta(\varepsilon)\). Then, the decoherence factor is constant: \(D(\tau) = 1\). That is, we always have the same expression of the prospect probability as in Eq. (18), which is quite clear, since getting no additional information does not change the preferences of the decision maker.

The nontrivial situation is when the decision maker consults with other members of the society, acquiring additional information. Interactions of the decision maker with the society can be of different types, which defines particular forms of the effect density.

If the number of the members in the society is large, and they act on the decision maker independently, then, by the central limit theorem, the effect density can be modeled by a Gaussian

\[ g(\varepsilon) = \frac{1}{\sqrt{2\pi} \gamma} \exp \left( -\frac{\varepsilon^2}{\gamma^2} \right), \tag{76} \]

where \(\gamma\) is the variance of the impacts from different members of the society. Respectively, the decoherence factor (71) is also a Gaussian

\[ D(\tau) = \exp \left( -\frac{\tau^2}{2\tau_c^2} \right) \quad \left( \tau_c \equiv \frac{1}{\gamma} \right), \tag{77} \]

diminishing with the increasing amount of information. Thence, the attraction factor (73) decreases with increasing \(\tau\), which implies the decrease of deviations from the classical decision...
making and the attenuation of the related paradoxes, as has been observed in experiments with social groups (Charness et al., 2010). The characteristic decoherence time $\tau_c$ is shorter for larger variance of the impacts, when there are many society members with different properties.

When the number of the society members is not large, the effect density can differ from the Gaussian form. For example, it can be given by the Lorentz distribution

$$g(\varepsilon) = \frac{\gamma}{\pi(\varepsilon^2 + \gamma^2)}.$$  \hspace{1cm} (78)

As a result, the decoherence factor is exponential:

$$D(\tau) = \exp(-\gamma \tau).$$  \hspace{1cm} (79)

If the effect density is represented by the Poisson distribution

$$g(\varepsilon) = \frac{1}{2\gamma} \exp\left(-\frac{|\varepsilon|}{\gamma}\right),$$  \hspace{1cm} (80)

the decoherence factor is of the power law form:

$$D(\tau) = \frac{1}{1 + (\gamma \tau)^2}.$$  \hspace{1cm} (81)

When the society influence is described by a uniform distribution on the bounded interval $[-\gamma, +\gamma]$,

$$g(\varepsilon) = \frac{1}{2\gamma} \Theta(\gamma - \varepsilon) \Theta(\gamma + \varepsilon),$$  \hspace{1cm} (82)

where $\Theta(\cdot)$ is a unit-step function, then the decoherence factor decays with oscillations as

$$D(\tau) = \frac{\sin(\gamma \tau)}{\gamma \tau}.$$  \hspace{1cm} (83)

These examples can be generalized by showing that, typically, the decoherence factor asymptotically diminishes with increasing information, which leads to a decreasing attraction factor and a convergence of the prospect probability to the classical form characterized by the utility factor. This is summarized by the following theorem.

**Proposition 3.** Let the effect density $g(\varepsilon)$ be a measurable function. Then, the prospect probability $p(\pi_j, \tau)$, under asymptotically large amount of information $\tau$, tends to the classical form represented by the utility factor $f(\pi_j)$:

$$\lim_{\tau \to \infty} p(\pi_j, \tau) = f(\pi_j).$$  \hspace{1cm} (84)

**Proof:** Suppose the effect density $g(\varepsilon)$ is measurable, hence being not of the delta-function type. By definition (70), it is $L^1$-integrable. Therefore, by the Riemann-Lebesgue lemma (Bochner and Chandrasekharan, 1949), the decoherence factor (71) tends to zero for asymptotically large $\tau$:

$$\lim_{\tau \to \infty} D(\tau) = 0.$$  \hspace{1cm} (85)
Consequently, because of relation (73), the attraction factor also tends to zero:

$$\lim_{\tau \to \infty} q(\pi_j, \tau) = 0.$$  \hspace{1cm} (86)

Then, from Eq. (72) it follows that the prospect probability reduces to the classical utility factor, as is stated in Eq. (84).

3.4 Conjunction fallacy disappearance

Charness et al. (2010) accomplished a series of experiments designed to test whether and to what extent individuals succumb to the conjunction fallacy. They used an experimental design of Tversky and Kahneman (1983) and found that, when subjects are allowed to consult with other subjects, the proportions of individuals who violate the conjunction principle fall dramatically, particularly when the size of the group rises. It has also been found that financial incentives for providing the correct answer are effective in inducing individuals to make efforts to find the correct answer. When individuals are forced to think, they recover in their minds additional information that has been forgotten or shadowed by emotions. The amount of received information increases with the size of the group. As a result, there is a substantially larger drop in the error rate when the group size is increased from two to three than when it is increased from one to two (Charness et al., 2007a; 2010). These findings confirm the earlier studies by Sutter (2005), who finds only a marginal difference between the choices of individuals and two-person groups, but a significant difference between the choices of two-person and four-person groups in an experimental guessing game. In any event, the effects of group interaction are not proportional to group size. In other words, the error attenuation decays faster than the inverse information, which is compatible with the decoherence factors of the Gaussian (71) or exponential (73) forms. In order to determine the exact form of the error attenuation, it would be necessary to perform a number of experiments in which the group size or the amount of received information would be varied over significant larger intervals than done until now.

In the experiment by Charness et al. (2010), with groups of three members, the fraction of individuals giving incorrect answers dropped to 0.17. It would be interesting to study how the errors would diminish with further increase of the number of the consulting decision makers. It is clear that the error should not disappear completely, since the amount of received information is never actually infinite. However, there exists a critical amount of information, when the error could be neglected. This critical value would also be interesting to find experimentally.

4 Conclusion

We have generalized the quantum decision theory (QDT), developed earlier for individual decision makers (Yukalov and Sornette, 2008, 2009a,b,c, 2010a,b, 2011), to the case of decision makers that are members of a society, in which agents interact with each other by exchanging information. Mathematically, this corresponds to replacing the description of strategic states of decision makers from wave functions to statistical operators. In QDT, a choice is
made by choosing the prospect that corresponds to the largest probability, each prospect probability consisting of two terms, a utility factor and an attraction factor. The utility factor characterizes the objective utility of prospects, while the attraction factor represents subjective feelings, emotions, and biases. Setting the attraction factor to zero reduces QDT to the classical decision making based on the maximization of expected utility. So, classical decision theory is a particular case of QDT. Without taking account of the attraction factor, classical decision makers depart from the predictions of classical utility theory, leading to a variety of paradoxes. But in QDT, all those paradoxes find simple and natural explanations, since the theory accounts for the decisions of real human beings.

At an initial stage, when the decision makers of a given society have not had yet sufficient time for mutual interactions to increase their information, the attraction factor, quantifying the deviations from classical decision theory, is crucially important. Its aggregate absolute value is about 0.25 on a maximum scale of 0 to 1 for the choice probabilities. It is therefore highly significant. The occurrence of the attraction factor is due to the interference of prospects in the decision maker brains. Since, in quantum theory, interference is necessarily connected with coherence, it is possible to say that the decision maker is in a coherent state.

However, the level of this coherence, and the value of the attraction factor, essentially depends on the amount of information available to a decision maker. If, in the process of mutual interactions between the members of the society, the amount of information of a decision maker increases, then the attraction factor diminishes. This can be called the decoherence process. Respectively, the prospect probabilities tend to their classical values represented by the utility factors. This rationalizes experimental findings showing that the deviations from classical decision making decrease when agents make decisions after consulting with each other (Charness et al., 2010).

It is possible to imagine a situation where a decision maker receives wrong, that is negative, information from the society members, for instance when cheating on this particular individual. In that case, the attraction factor should increase, hence, the deviations from classical decision making should rise. It would be interesting to perform such experiments with decision makers getting wrong or misleading information to calibrate better the effect density functions that are central to QDT for interacting individuals.

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