We calculate the finite volume effects for the pion and nucleon mass. For the pion mass we present the results of a full two-loop calculation in chiral perturbation theory. The outcome shows that the resummed version of the Lüscher formula we presented in an earlier work does indeed give an excellent approximation to the two-loop result. In view of this result we apply the same resummed formula to the nucleon mass. In the nucleon sector the extension of the chiral expansion to higher quark masses appears to be more problematic and it is therefore more difficult to make reliable predictions for the size of the finite volume effects. We present some preliminary numerical estimates.

1. Introduction

One of the many useful applications of effective field theories is to estimate the size of systematic effects in lattice calculations. This concerns both the unphysical size of the quark masses as well as the finite lattice spacing which, in the framework of the effective field theory, can be seen as a controlled modification of the underlying fundamental theory. Since one can keep track of this modification even at the level of the effective Lagrangian, one is able to calculate how the physics is changed at large distances. A conceptually different problem is to estimate the effects due to the finite volume — in this case one is modifying the physics at large distances but not touching the short-distance physics. Correspondingly, one can evaluate the distortion due to the finite volume by calculating any observable in infinite and in finite volume in the framework of the effective Lagrangian \( L \). This evaluation can only be reliable if the new scale introduced by the finite volume can be denoted as a “low energy scale”, i.e. if \( L^{-1} \ll \Lambda \), where the latter is a typical hadronic scale. In order to have a better numerical estimate of how large a volume should be, we further observe that the minimum nonzero momentum allowed in a box of size \( L \) is \( p = 2\pi/L \), and that if one identifies \( \Lambda \) with \( 4\pi F_\pi \) (as is usually done in chiral perturbation theory) one obtains

\[
2LF_\pi \gg 1 \quad \Rightarrow \quad L \gg 1 \text{fm} .
\]

In recent years several analytical calculations of finite volume effects within chiral perturbation theory (CHPT) have been performed by various groups (for a recent review, see \[2\]). These results may be used as a guideline by lattice groups when planning their runs and deciding on the size of their volumes. Since these effects are generally small (at the percent level) one could either try to estimate the volume for which these effects are negligible (below one percent, say), or alternatively to work in the smallest possible volume in which the effect can be reliably calculated and correct analytically for the finite volume. In the latter case it is of course important that the uncertainty in the analytical calculation be carefully estimated. Moreover, before blindly trusting some theoretical calculations, it is certainly a good practice to check them at the claimed level of accuracy — before one can save some computer time on the volume at least some should be devoted to checking the volume dependence.

In order to estimate the uncertainty in the finite volume calculations it is necessary to check the convergence of the chiral expansion. This requires evaluating at least two terms in the series for this specific effect. Since finite volume effects start only at the one loop level, a check of the convergence of the series requires a two-loop cal-
calculation. Until recently [24] no such calculations had ever been performed.

A shortcut which did not require a full two-loop calculation but relied on the Lüscher formula for the masses [5] (and on its extension to decay constants [6]), was proposed in [27]. Asymptotic formulae à la Lüscher express the finite volume shift as an integral over an infinite volume amplitude – e.g., the finite volume effect for the pion (proton) mass is expressed as an integral over the \( \pi \pi (\pi N) \) scattering amplitude. Inserting the tree level representation for the latter amplitude in the Lüscher formula yields the asymptotically dominating term of the one-loop finite volume calculation. But since the relevant infinite-volume scattering amplitudes are usually known well beyond the tree level, one can insert a better representation than the leading order one and easily go beyond the leading order calculation of the finite volume effect.

This has been done in [5,24] for the pion mass. The somewhat surprising results were: 1. the leading order in the chiral expansion receives very large corrections at next-to-leading, but reasonable ones at next-to-next-to leading order; 2. the leading exponential term (as given by the Lüscher formula) is numerically dominating only where the finite volume effect becomes negligible. The conclusions to be drawn from these results were that in order to reliably calculate these effects it was necessary to go beyond the leading term both in the chiral expansion as well as in the asymptotic expansion. The possibility to use the NNLO representation for the \( \pi \pi \) scattering amplitude had however allowed us to show that, apart from the anomalously large NLO correction (the reason for this large correction is well understood, cf. [9]), the series behaved as expected and for not too large pion masses was clearly converging. The shift from NLO to NNLO was typically smaller than the uncertainties in the NLO calculation due to the low energy constants.

These results lead us to formulate a simple recipe [7] to evaluate finite volume effects beyond LO. The recipe can be described as a resummed Lüscher formula: the integral over the infinite volume amplitude which appears in the latter is generalized to one with an index \( n \) such that the asymptotic behaviour becomes \( \exp(-\sqrt{n} M_\pi L) \), and the full result is given by a sum over all \( n \), with appropriate coefficients. This recipe yields the exact one-loop CHPT result if one inserts the LO amplitude in the integral – at NLO the correspondence is not exact anymore, but we argued that we expected the resummed asymptotic formula to give the largest part of the full two-loop calculation. Moreover we could show that algebraically the formula improved the accuracy of the asymptotic formula and that corrections were of the order \( O(e^{-M_L}) \) with \( M = (\sqrt{3}+1)/\sqrt{2}M_\pi \) instead of the \( O(e^{-M_L}) \) with \( M = \sqrt{3}/2M_\pi \) of the plain Lüscher formula.

Recently, we have completed the full two-loop calculation of the pion mass and could confirm [4] that the resummed asymptotic formula is an excellent approximation to the full two-loop results, and can therefore be used with confidence also in other cases. One interesting application is that of the nucleon mass: the latter has been calculated to one loop in CHPT [10] and the result has successfully described lattice data, despite the fact that these had been obtained for substantial pion masses and in rather small volumes. With the resummed formula we can go beyond the one-loop calculation and better estimate the uncertainties. These two new results (which will be published soon) will be briefly described here.

2. The pion mass to two loops

In Ref. [27] we have proposed a resummed version of the asymptotic formulae à la Lüscher as an efficient way to evaluate finite volume effects in CHPT beyond leading order. The Lüscher formula for the relative finite volume shift for the pion mass reads as follows

\[
R_{M_\pi} = \frac{M_\pi(L) - M_\pi}{M_\pi} = (2)
\]

\[
- \frac{3}{16\pi^2\lambda_\pi} \int_{-\infty}^{\infty} dy \ F_\pi(iy)e^{-\sqrt{1+y^2}\lambda_\pi} + O(e^{-M_L}) ,
\]

where \( F_\pi(\nu) \) denotes the infinite volume forward \( (t = 0) \) \( \pi \pi \) scattering amplitude in Minkowski
space, and

$$\lambda_{\pi} = M_{\pi} L$$

is the box length in pion mass units. As discussed in the introduction, $\lambda_{\pi}$ is assumed to be large, and the integral in (2) is only the dominating term in an asymptotic expansion in exponentials of multiples of $\lambda_{\pi}$. The estimated error in (2) is determined by $M \geq \sqrt{3/2}M_{\pi}$. In Lüscher’s proof of the formula the first step is to show that the dominating contribution to finite volume effects comes from diagrams where only one propagator is taken in finite volume. The propagator in finite volume is

$$G_L(x^0, x) = \sum_n G_{\infty}(x^0, x + nL),$$

where $G_{\infty}$ is the standard, infinite-volume propagator. The term with $n = 0$ in the sum is simply the infinite-volume propagator. According to the rules for doing CHPT calculations in finite volume [1], one has to proceed as in infinite volume and simply write all propagators as in (4). As shown by Lüscher the dominating term comes from graphs where one propagator at a time is taken in finite volume. He then showed that in this class of diagrams the dominating contribution comes from the terms with $|n| = 1$ in the sum in (4). Contributions from all other terms in the sum in (4), however, will also show up in the complete CHPT calculation and can be dealt with exactly as the first term in the sum. This is the origin of the resummation we suggested in [2] and leads to the resummed formula

$$R_{\pi} = -\frac{1}{32\pi^2\lambda_{\pi}} \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \times$$

$$\int_{-\infty}^{\infty} dy \, F_{\pi}(iy) e^{-\sqrt{n(1+y^2)}\lambda_{\pi}} + O(e^{-M_{\pi} L}),$$

where $m(n)$ is the number of integer vectors $z$ with $z^2 = n$. It is easy to verify that the term with $n = 1$ is identical to the original Lüscher formula [2]. Notice that although the formula contains subleading terms in the asymptotic expansion, the algebraic accuracy of the formula is in principle the same. On the other hand, an analysis of the contributions at the two-loop level not included in the resummed asymptotic formula shows that at this order in the chiral expansion [2] the formula receives corrections only of order $\exp(-M_{\pi} L)$ with $M = M_{\pi}(\sqrt{3}+1)/\sqrt{2}$, and so is also algebraically improved with respect to the original Lüscher formula.

On the basis of this and other arguments we have used this resummed formula to make a thorough numerical analysis of finite volume effects for masses and decay constants of the pseudoscalar mesons [7]. It is important, however, to check the claim that the resummed formula provides the main contribution of a full NLO calculation of the finite volume effects, at least in one concrete example. We have now done that by calculating the pion mass to two loops in CHPT.

The pion mass in finite volume to two loops can be written as

$$M_{\pi}(L)^2 = M_{\pi}^2 - \Sigma^{(1)} - \Sigma^{(2)}, \quad M_{\pi}^2 = M^2 - \Sigma^{(0)},$$

where

$$\Sigma^{(1)} = I_p + I_c + O(\xi^3),$$

$$I_p = \frac{M_{\pi}^2}{16\pi^2\lambda_{\pi}} \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \times$$

$$\int_{-\infty}^{\infty} dy \, F_{\pi}(iy) e^{-\sqrt{n(1+y^2)}\lambda_{\pi}},$$

$$I_c = -\frac{iM_{\pi}^2}{32\pi^3\lambda_{\pi}} \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}} \times$$

$$\int_{-\infty}^{\infty} d\bar{s} e^{-\sqrt{n(\xi^2+y^2)}\lambda_{\pi}} \text{disc}[F_{\pi}(\bar{s}, 1+iy)],$$

and introduced the abbreviation

$$\xi = \frac{M_{\pi}^2}{16\pi^2 F_{\pi}^2}.$$
Contributions from diagrams where more than one propagator is taken in finite volume are by definition not included in the resummed Lüscher formula. The whole $\Sigma^{(2)}$ contribution is a two-loop effect which goes beyond the latter and can be represented as

$$\Sigma^{(2)} = \frac{M_\pi^2 \xi^2}{8} \left[ 9\tilde{g}_1(\lambda_\pi)^2 - \lambda_\pi \tilde{g}_1(\lambda_\pi)\tilde{g}'_1(\lambda_\pi) \right] + M_\pi^2 \xi^2 \Delta + O(\xi^3),$$

where $\tilde{g}_1$ is the one-loop tadpole graph. The part from the sunset type diagrams which can not be written in terms of $\tilde{g}_1$ is denoted here with $\Delta$. It can be written as double integrals and can be evaluated numerically.

The numerical analysis of this formula is shown in Fig. 1. It is evident that the contributions which go beyond the resummed Lüscher formula are sizeable only in the region where $M_\pi L$ is not large, i.e. where the framework in which we are doing our calculations is not reliable anymore. The results show that the resummed Lüscher formula provides an efficient and reliable way to go beyond leading order in the evaluation of finite volume effects.

3. The nucleon mass with the resummed Lüscher formula

In this section we apply the resummed Lüscher formula to the nucleon mass and first provide the relevant analytical expression. The difference to the case of the pion mass is that now the $\pi N$ scattering amplitude, instead of the $\pi \pi$, has to be inserted in the integral.

The formula for the relative finite volume correction $R_N = (m_N(L)-m_N)/m_N$ reads as follows

$$R_N = \frac{3\varepsilon^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{m(n)}{\sqrt{n}\lambda_n} \left[ 2\pi\varepsilon_\pi g_{2N}^2 e^{-\sqrt{n(1-\varepsilon^2)}\lambda_\pi} - \int_{-\infty}^{\infty} dy e^{-\sqrt{n(1+y^2)}\lambda_\pi} \tilde{D}^+(y) \right],$$

where

$$\varepsilon_\pi = \frac{M_\pi}{2m_N}, \quad \tilde{D}^+(y) = m_N D^+(iM_\pi y, 0),$$

where the latter is one of the components of the elastic $\pi N$ scattering amplitude, which is defined as follows:

$$T(\pi^a(q)N(p) \rightarrow \pi^a'(q')N(p')) \equiv T_{a'a} = \delta_{a'a}T^+ + \frac{1}{2}[\tau_{a'}, \tau_a]T^-.$$  

Each of the two isospin components, $T^+$ and $T^-$ is then broken down into

$$T^\pm = \tilde{u}' \left[ D^\pm(\nu, t) - \frac{1}{4m_N} [\tilde{g}, \tilde{g}] B^\pm(\nu, t) \right] u$$

and each of the amplitudes depends on two kinematical variables:

$$t = (q - q')^2, \quad \nu = \frac{s-m}{4m_N}; s = (p+q)^2, \quad u = (p - q')^2.$$  

As seen in [9, 10], the $\pi N$ amplitude is needed here in a particular kinematical configuration, namely for $t=0$ and for $\nu$ purely imaginary and small: the contributions with large values of $\nu$ are suppressed by the exponential weight in the integral in [9]. It is therefore natural to make a Taylor expansion of the amplitude around $\nu=0$ after having subtracted the pole due to the one-nucleon exchange diagram (also called the Born term). Such an expansion is in fact already well
known in the phenomenology and is referred to as the subthreshold expansion. It reads as follows

\[ D^+(\nu, 0) = D_{pv}^+(\nu, 0) + D_p^+(\nu, 0) + D_{na}^+(s, u), \] (14)

where

\[ D_{pv}^+(\nu, 0) = \frac{g_{\pi N}^2}{m_N} \frac{\nu_B^2}{\nu_B^2 - \nu^2}, \]
\[ D_p^+(\nu, 0) = d_{10}^+ + d_{10}^+ \nu^2 + d_{20}^+ \nu^4, \] (15)

and \( \nu_B = -M_\pi^2/(2m_N) \). The function \( D_{na}^+(s, u) \) contains the analytically nontrivial part of the amplitude. Up to order \( g^4 \) in the chiral expansion this can be written as a sum of two single variable functions:

\[ D_{na}^+(s, u) = D_1^+(s) + D_1^+(u) + O(g^5) \] (16)

which admit the following dispersive representation:

\[ D_1^+(s) = \frac{\nu^5}{\pi} \int_{M_S}^{\infty} \frac{d\nu'}{\nu'^5}(\nu' - \nu - i\epsilon) \int_{M_S}^{\infty} \frac{d\nu}{\nu'^5}(\nu' - \nu - i\epsilon), \] (17)

where \( s' = 2m_N\nu' + m_N^2 + M_\pi^2 \). This representation shows that, due to the large number of subtractions, the function \( D_1^+ \) is small near \( \nu = 0 \). Its contribution to the finite size shift of the nucleon mass is negligible.

This observation leads us to the following expression for the relative finite volume shift \( R_N \):

\[ R_N = \frac{3m_0^2}{4\pi^2} \sum_{n=1}^{\infty} \frac{2\pi g_{\pi N}^2 \nu_B^2}{\nu_B^2 - \nu^2} \left[ 2\pi g_{\pi N}^2 \nu_B^2 e^{-\sqrt{n(1-\epsilon^2)}\lambda_s} - \frac{g_{\pi N}^2}{\nu_B^2} - \frac{d_{10}^+}{\nu_B^2} \right] + R_{N, na}, \] (18)

where

\[ I_{pv}(\lambda_s) = \int_{-\infty}^{\infty} dy \frac{e^{-\sqrt{y^2 + \lambda_s^2}}}{\epsilon^2 + y^2}, \]
\[ B_k(\lambda_s) = \int_{-\infty}^{\infty} dy \frac{y^k e^{-\sqrt{y^2 + \lambda_s^2}}}{\epsilon^2 + y^2}, \] (19)

are the relevant finite volume integrals, \( \bar{d}_0^+ = m_N M_\pi^2 d_{10}^+ \) and \( R_{N, na} \) is the remainder coming from the (subtracted) analytically nontrivial part of the amplitude.

If we neglect the contribution \( R_{N, na} \) the representation (18) is very simple and expresses the finite volume shift of the nucleon mass in terms of only a handful of physical observables: the pion and proton masses, \( M_\pi, m_N \), the pion-nucleon coupling constant \( g_{\pi N} \) and the three subthreshold parameters \( d_{10}^+ \) (the latter are not directly observable, but can be obtained from data with some theoretical treatment, cf. [11]). If one knows the low-energy constants (LEC) which appear in the chiral representation of these quantities, one can predict their quark mass dependence and therefore the finite volume shift \( R_N \) as a function of the quark masses.

An explicit representation for the quark mass dependence of the quantities which appear in Eq. (18) up to \( O(p^4) \) can be found in [12]. Unfortunately, however, our knowledge of the LEC which appear in there is much worse than that for the LEC of \( \pi \pi \) scattering. In short we can say that the quark mass dependence of the \( \pi N \) scattering amplitude, even in the subthreshold region, i.e. far away from the physical singularities of the scattering amplitude, is not very well known. We stress that for the physical value of the quark masses we do have reliable phenomenological information about the quantities which appear in Eq. (18), and that the finite volume effects can be therefore evaluated with rather small uncertainties, cf. [13]. The problem is the extrapolation to higher quark masses.

We have investigated this question and adopted the following strategy: we fit the physical values of the five quantities \( m_N, g_A \) and the three subthreshold parameters \( d_{10} \) (cf. [11]) as well as the lattice calculations of \( m_N \) (cf. [14,10]) and \( g_A \) [15] for \( M_\pi \sim 0.5 \) GeV. This gives us seven constraints and allows us to fix seven of the LEC appearing in the formulae for the mass dependence. On some of the other LEC there is phenomenological information coming from \( \pi N \) or \( \pi N \rightarrow \pi \pi N \) scattering – whenever possible we use this also. Some of the LEC, however, remain unconstrained and have to be estimated on a purely theoretical basis.

The details of this analysis, including a discussion of the numerical values of the various LEC, will
be discussed elsewhere [16].

An important point we wish to emphasize here is that our analysis indicates that the Born term is dominating even at higher values of quark masses. This means that the most important information we need, concerns the quark mass dependence of $m_N$ and $g_A$. For the former we rely on available lattice calculations (although at somewhat high pion masses), whereas for the latter we rely on the Goldberger–Treiman relation:

$$g_{\pi N} = \frac{g_A m_N}{F_\pi} \left(1 - \frac{2d_{16} M^2}{g} + \mathcal{O}(M^4)\right)$$  \hspace{1cm} (20)

and fix the LEC $d_{16}$ from the phenomenology. This gives a mild quark mass dependence in the relation – much more important is the quark mass dependence of $g_A$ and $m_N$. Also for $g_A$ we rely on available lattice calculations (again at somewhat high pion masses), which indicate a mild dependence of the axial charge on the quark mass, and try to interpolate the physical and the lattice value with the chiral representation. In this case, however, the phenomenological information about the LEC (which determine the quark mass dependence of $g_A$) would actually allow us to predict the latter [12]:

$$g_A = g + \left(4 \tilde{d}_{16} - \frac{g^3}{16 \pi^2 F^2}\right) M^2 + \frac{(1 + g^2) g M^3}{8 \pi M F^2} - \frac{(c_3 - 2c_4) g M^3}{6 \pi F^2} + \mathcal{O}(M^4)$$ \hspace{1cm} (21)

The constant $\tilde{d}_{16}$ has been determined from $\pi N \rightarrow \pi \pi N$ scattering measurements [17] to be between $-1$ and $-2$ GeV$^{-2}$, with about 1 GeV$^{-2}$ uncertainty$^2$, whereas the combination $c_3 - 2c_4$ can be determined from the subthreshold parameters of $\pi N$ scattering [12] and comes out to be also negative, with a value around $-9$ GeV$^{-2}$. Unfortunately the prediction indicates a very strong quark mass dependence already at very small quark masses and makes it difficult to extrapolate to the region where the lattice data are obtained: the $O(M^2)$ correction is large and negative and the $O(M^3)$ large and positive. Of course one can choose the value of $g_A$ in the chiral limit such that the value for the physical pion mass comes out right, but as soon as one goes to higher pion masses, $g_A$ tends to explode. The lattice data, on the other hand, indicate that at $M_\pi \sim 0.5$ GeV, $g_A$ is somewhat lower, but not by much, than the physical value. We have to conclude that either the phenomenological determination of the LEC appearing in $g_A$ is unreliable, or that the region of quark masses where the chiral expansion works for $g_A$ is very small.

The determination of $d_{16}$ is indeed subject to large uncertainties (of the order of $50$ to $80\%$ according to the estimates of Fettes, which do not seem to include any systematic effects), but those of $c_3$ and $c_4$ are more solid and indicate a small region of convergence of the chiral expansion for $g_A$. The presence of these large corrections has indeed been known since a long time (cf. [19]) and a possible solution of the discrepancy with the observed mass dependence (or the lack thereof) in lattice calculations has already been proposed by Hemmert, Procera and Weise [18]. In this paper the $\Delta$ resonance is included explicitly in the calculation and it is shown that if one stops at order $\epsilon^3$ in the small-scale expansion (SSE) one obtains a relatively steep dependence only close to the chiral limit, whereas at higher quark masses $g_A$ is a flat function of $M_\pi$. While this analysis identifies a mechanism that provides a change of behaviour between the very low and the middle pion mass region, it does not provide a fully satisfactory understanding of the mass dependence of $g_A$, in our opinion. The constant $d_{16}$ occurs also in the SSE as an independent LEC, in fact unrelated to the $\Delta$ resonance. In [18] the value of $d_{16}$ is also taken from the work of Fettes et al. [17]: this means that even in the SSE the term of order $M^2$ is a large correction already at the physical pion mass, and that the size of this correction grows fast with the pion mass. The contributions from the $\Delta$, which are also large, compensate these large corrections and give a rather flat behaviour up to $0.5$ GeV. However, since the $\Delta$ and the $d_{16}$ contributions are physically unrelated (also in the framework of the SSE), this compensation appears to be the result of a fine

$^2$The published values of $d_{16}$ have later been corrected in the final version of the PhD thesis of Nadia Fettes (see also the discussion in [18]). The numerical change, however, does not change the qualitative picture.
tuning between the value of $d_{16}$ and the $\Delta$ resonance couplings. The analysis in [18] shows that the $\Delta$ contributions could tame the strong dependence of $g_A$ on the quark mass, but does not yet explain why this happens.

This situation makes the numerical study of finite volume effects for the nucleon mass on the basis of Eq. (18) problematic. We proceed nonetheless and rely on a simple interpolation formula between the physical and the lattice value (at around $M_{\pi} \sim 0.5$ GeV) of $g_A$ – for all other quantities we rely on the chiral expansion. Our results are shown in Fig. 2 and indicate that although with large uncertainties, the finite volume corrections for the nucleon mass can be calculated with this method. The figure contains two curves (with the corresponding uncertainties): the dot-dashed curve indicates what one obtains if one stops the chiral expansion of the $\pi N$ scattering amplitude at $O(p^2)$, whereas the solid line gives the result obtained from the full $O(p^4)$ scattering amplitude. In the latter case in the amplitude there are more LEC to be determined and correlations among them are generated by the fits, but the net result is that the uncertainties become larger than if one stops at $O(p^2)$. The comparison clearly shows that the direct statistical estimate of the uncertainty at $O(p^2)$ is quite optimistic.

We conclude by comparing to the analysis in [10]. The algebraic relation between the two approaches is fully understood: if we expand the resummed Lüschner formula Eq. (18) to NLO we reproduce exactly the result of Ref. [10]. The partial inclusion of higher orders obtained if we do not expand Eq. (18) but insert the formulae of the $O(p^4)$ $\pi N$ scattering amplitude, on the other hand, allows us to check the convergence of the series. Fig. 2 shows a reasonable behaviour for $L = 2$ fm and $M_{\pi} \leq 0.5$ GeV. In Ref. [10] the comparison between lattice data and the CHPT calculation in finite volume has been performed even for smaller volumes and higher pion masses and has shown a surprisingly good agreement. The significance of this agreement can however only be assessed if one has a good estimate of the uncertainties, something which had not been attempted in [10]. We can do this now and compare our results to the same lattice data, which were obtained for $m_N$ at $M_{\pi} \simeq 0.55$ GeV and volumes $L < 2$ fm – a region where we would not trust our formulae from the start. The comparison is shown in Fig. 3 where one can see that if one stops at NLO one goes indeed through the data, in agreement with [10]. The inclusion of higher orders, however, spoils the good agreement, because for the central values of the LEC we have obtained from our fit we come closer to the LO curve. The band which gives our estimate of the uncertainty however shows that neither the agreement nor the disagreement are of any significance, because the uncertainties at these high quark masses and for such small volumes are simply too large.

4. Conclusions

We have discussed the results of a full two-loop calculation of the pion mass in finite volume in CHPT [4]. We have shown that contributions which are not included in the resummed asymptotic formula we proposed some time ago [27] are tiny and can be neglected in the region of pion masses and volumes where CHPT can be trusted. This check provides further evidence to our claim that the resummed asymptotic formulae are the most efficient way to evaluate these finite volume effects and in view of this there appears to be lit-
tle need to try and improve the analysis in [7] for masses and decay constants of the pseudoscalar mesons.

The resummed formula can also be applied to non pseudo-Goldstone bosons, and a particularly interesting application concerns the nucleon mass. This has been calculated in finite volume to one loop in CHPT [10] and compared to lattice data, although these had been obtained for rather heavy pion masses and in small volumes. Somewhat surprisingly, the CHPT calculation successfully described the volume dependence of these data [10]. We have described here a numerical analysis based on the resummed asymptotic formula and shown how higher order corrections become more and more sizeable as the pion mass increases. With the higher order corrections, also the uncertainties in the calculation increase. Indeed we have shown that in the region of pion masses and volumes where data are available the uncertainties are larger than the effect itself and concluded that the observed agreement between lattice data and the CHPT NLO calculation is accidental and of little significance.

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