Matter Lagrangians Coupled with Connections

by L. Fatibene, M. Francaviglia, S. Mercadante

Abstract: We shall here consider extended theories of gravitation in the metric-affine formalism with matter coupled directly to the connection. A sufficiently general procedure will be exhibited to solve the resulting field equation associated to the connection. As special cases one has the no-coupling case (which is standard in $f(R)$ literature) as well as the cases already analyzed in [1].

1. Introduction

Let $M$ be a connected paracompact spacetime manifold which allows global Lorentzian metrics. For the sake of simplicity let us assume that $\dim(M) = m \neq 2$.

Let us consider a metric $g_{\mu\nu}$ and a torsionless connection $\Gamma^\alpha_{\beta\mu}$ as fundamental fields. We shall hereafter consider a Lagrangian in the form

$$L = \sqrt{g} f(R) + L_m(g, \Gamma, \phi)$$

(1.1)

where $\phi$ is a set of matter fields, $g = |\det(g_{\mu\nu})|$ and $f$ is a generic (analytic) function. Similar situations have been considered before; see [2], [3]. However, to the best of our knowledge a general procedure to solve these field equations appears here for the first time. For physical applications see [4], [5], [6]. For physical interpretation see [7].

We can define $u^\lambda_{\alpha\beta} = \Gamma^\lambda_{\alpha\beta} - \delta^\lambda_{(\alpha} \Gamma^\beta_{)\epsilon} \epsilon$; this relation is invertible so that the field equations of (1.1) associated to $u^\lambda_{\alpha\beta}$ (as derived for example in [1]), are thence in the form

$$\nabla^\lambda (\sqrt{g} f' g^\alpha\beta) = \tilde{P}^{\alpha\beta}_\lambda$$

(1.2)

where $\tilde{P}^{\alpha\beta}_\lambda$ is a suitable tensor density of weight 1 which is a function of the matter fields $\phi$, the metric $g$ and possibly the connection $\Gamma$ itself. If the matter Lagrangian in (1.1) is linear in the connection then $\tilde{P}^{\alpha\beta}_\lambda$ does not in fact depend on the connection. In general the connection may appear in $\tilde{P}^{\alpha\beta}_\lambda$ through the covariant derivatives of matter field, in view of the covariance required.

When the connection enters linearly in $L_m$ then $\tilde{P}^{\alpha\beta}_\lambda = \tilde{P}^{\alpha\beta}_\lambda (g, \phi)$ is independent of the connection itself and one can always recast the matter Lagrangian under the form $L_m = \tilde{P}^{\alpha\beta}_\lambda (g, \phi) u^\lambda_{\alpha\beta} + Z(g, \phi)$ where now $Z(g, \phi)$ is the “standard” new matter pseudo-Lagrangian. We shall hereafter present the general solution of this equation in this particular case.

If instead the matter Lagrangian is not linear in the connection then the tensor density $\tilde{P}^{\alpha\beta}_\lambda$ does depend on the connection itself. In this case one can only try to “solve” the equation by considering $\tilde{P}^{\alpha\beta}_\lambda$ as an additional parameter. This is similar to what one is used to do in Hamiltonian mechanics in which one considers momenta as independent of the positions and solves the equation. At least in some cases one could obtain meaningful results, e.g. when

---

* This paper is published despite the effects of the Italian law 133/08 ([http://groups.google.it/group/scienceaction](http://groups.google.it/group/scienceaction)). This law drastically reduces public funds to public Italian universities, which is particularly dangerous for free scientific research, and it will prevent young researchers from getting a position, either temporary or tenured, in Italy. The authors are protesting against this law to obtain its cancellation.
the particular combination expressing Γ as a function of P happens to be independent of the connection despite the Ps may depend on Γ.

Let us stress that a similar technique is used in ordinary f(R) theories, in which the connection is “solved” as a function of the metric and the conformal factor f’ which (at least in the beginning, i.e. before using the master equation) is itself a function of the connection.

2. Existence and Uniqueness

First of all one defines as usual a conformal metric \( h_{\alpha\beta} = f' g_{\alpha\beta} \) so that the equation (1.2) can be recasted as

\[
\Gamma^\alpha_{\beta\lambda} \left( \sqrt{h} h^{\alpha\beta} \right) = \sqrt{h} P^\alpha_{\lambda} \tag{2.1}
\]

where we set \( \tilde{P}^\alpha_{\lambda} = \sqrt{h} P^\alpha_{\lambda} \).

Let us then consider the Levi-Civita connection \( \{ h \}^\alpha_{\mu
u} \) of the conformal metric \( h \). Accordingly the difference between the two connections \( K^\alpha_{\mu
u} := \Gamma^\alpha_{\mu
u} - \{ h \}^\alpha_{\mu
u} \) is a tensor. Equation (2.1) can be thence written as

\[
\nabla^h_{\lambda} \left( \sqrt{h} h^{\alpha\beta} \right) + K^\alpha_{\nu\lambda} \sqrt{h} h^{\nu\beta} + K^\alpha_{\beta\nu} \sqrt{h} h^{\alpha\nu} - K^\alpha_{\nu\lambda} \sqrt{h} h^{\alpha\beta} = P^\alpha_{\lambda} \tag{2.2}
\]

which in turn can be simplified to

\[
K^\alpha_{\nu\lambda} h^{\nu\beta} + K^\alpha_{\beta\nu} h^{\alpha\nu} - K^\alpha_{\nu\lambda} h^{\alpha\beta} = P^\alpha_{\lambda} \tag{2.3}
\]

Notice that this is an algebraic (in fact linear!) equation for the tensor \( K \).

The homogenous equation (obtained by setting \( P^\alpha_{\lambda} = 0 \)) has been already known in [8] to have a unique solution (i.e. \( K = 0 \)). Hence we just have to find a particular solution of (2.3); then that solution is unique in view of Rouché-Capelli theorem (see [9]).

As usual one can rely on a bit of luck, write down a number of tensors built with the metric \( h \) and the tensor \( P \) and search for a particular solution which is a linear combination of such basic tensors.

Let us then define \( P^\alpha_{\mu} := P^\alpha_{\mu\lambda} h_{\alpha\beta} \) and \( P^\alpha := P^\alpha_{\lambda} \). By noticing that the tensor \( K \) is defined to be symmetric in its lower indices, we can try with a linear combination

\[
K^\alpha_{\mu\nu} = a P^\alpha_{(\mu\nu\sigma)} + b P^\alpha_{(\mu\nu)\rho} + c P^\alpha_{\rho\mu\nu} + d h^{\alpha\lambda} P^\alpha_{\lambda} h_{\rho\mu} h_{\sigma\nu} \tag{2.4}
\]

By substituting back into (2.3) one gets

\[
\frac{b}{2} + d \right) P^\alpha_{\rho\sigma} h^{\rho\beta} + \left( \frac{a}{2} + c \right) P^\beta_{\mu\nu} h^{\mu\sigma} + \left( \frac{b}{2} + d \right) P^\nu_{\lambda\rho} h^{\nu\beta} + \left( \frac{a}{2} + c \right) P^\sigma_{\mu\lambda} h^{\sigma\alpha} + \left( \frac{b}{2} (m-1) + \frac{a}{2} + c \right) P^\alpha_{\nu\lambda} h^{\nu\beta} - \left( \frac{b}{2} \right) P^\nu_{\rho\mu} h^{\rho\alpha} = P^\alpha_{\lambda} \tag{2.5}
\]

Thus the ansatz (2.4) is a solution of equation (2.1) if the coefficients are

\[
b = 1 \quad d = \frac{1}{2} \quad c = \frac{1}{2(m-2)} \quad a = -\frac{1}{m-2} \tag{2.6}
\]

2
Since we have assumed that spacetime has dimension \( m \neq 2 \) this is a good solution. As usual two dimensional spacetimes are degenerate under many viewpoints and they must be treated separately. In this case, e.g., one can easily show that in dimension 2 there is no solution in the form (2.4) to equation (2.1).

3. EPS Compatible Connections

In [10] Ehlers-Pirani-Schild presented an axiomatic introduction to gravitational theories based on light rays and particles worldlines. In [8] and [1] we investigated a class of Lagrangians called Further Extended Theories of Gravitation (FETG) in which EPS compatibility is endowed by field equations.

We want here to show that in a FETG the matter Lagrangian is necessarily in the form

\[ L_m = \sqrt{g} g^\mu_\nu \nabla_\mu A_\nu \]  

where \( A \) is a vector density of weight \(-1\).

If the connection has to be EPS compatible with the metric structure one has to have

\[
-\frac{1}{2(m-2)} P_\mu \delta^\alpha_\mu - \frac{1}{2(m-2)} P_\nu \delta^\sigma_\nu + \frac{1}{2} P_\mu^{\sigma\alpha} h_{\alpha\sigma} + \frac{1}{2} P_\mu^{\sigma\alpha} h_{\mu\sigma} + \frac{1}{2(m-2)} P_\nu h^{\alpha\sigma} h_{\mu\nu} + \frac{1}{2} h^{\alpha\lambda} P_\lambda^{\sigma\alpha} h_{\mu\nu} h_{\sigma\nu} = \frac{1}{2} (h^{\alpha\sigma} h_{\mu\nu} - \delta^\alpha_\mu \delta^\sigma_\nu - \delta_\mu^\alpha \delta^\sigma_\nu) \alpha_e
\]

and we want to write \( P \) as a function of \( \alpha \).

By tracing this relation with \( h \) one has

\[ P_\mu = \frac{m(m-2)}{2} \alpha_\mu \]  

By tracing the same equation one gets

\[ P_\alpha \equiv P_\alpha^\lambda = \frac{m-2}{2} h^{\alpha\sigma} \alpha_e \]

By substituting back into equation (3.2) one obtains

\[
\frac{1}{2} P_\mu^{\sigma\alpha} h_{\alpha\sigma} + \frac{1}{2} P_\nu^{\sigma\alpha} h_{\mu\sigma} - \frac{1}{2} h^{\alpha\sigma} P_\sigma^{\alpha\nu} h_{\mu\sigma} = -\frac{m-2}{4} \left( h^{\alpha\sigma} h_{\mu\nu} - 2 \delta^\alpha_\mu \delta^\sigma_\nu \right) \alpha_e
\]

Let us now multiply by \( h^{\nu\lambda} \)

\[
\frac{1}{2} P_\mu^{\alpha\lambda} + h^{\nu[\lambda} P_\sigma^{\nu]\alpha} h_{\sigma\mu} = \frac{m-2}{2} \left( h^{\nu[\lambda} \delta^\alpha_\mu + \frac{1}{2} h^{\nu[\lambda} \delta^\alpha_\mu \right) \alpha_e
\]

One can now split this in the symmetric and skew part with respect to the indices \((\alpha\sigma)\) obtaining

\[
\begin{cases}
P_\mu^{\alpha\lambda} = \frac{m-2}{2} h^{\alpha\lambda} \delta_\mu \\
h^{\nu[\lambda} P_\sigma^{\nu]\alpha} h_{\sigma\mu} = \frac{m-2}{4} h^{\nu[\lambda} \delta^\alpha_\mu \alpha_e
\end{cases}
\]

These equations are not independent; in fact substituting the first into the second one, the second is identically satisfied.

Accordingly, in a FETG in dimension \( m = 4 \) the matter Lagrangian coupling with matter necessarily implies

\[ P_\mu^{\alpha\lambda} = h^{\alpha\lambda} \alpha_\mu \]  

If one wants the form \( \alpha \) to be independent of \( \Gamma \) then the matter Lagrangian is basically forced to be in the form (3.1) as it was assumed in [1].
4. Conclusions

We have shown how to solve field equations in Extended Theories of Gravitations in the metric-affine framework (à la Palatini) with an arbitrary coupling between matter and the connection. As an application we have shown that in order to have EPS compatibility the coupling should be necessarily in the form assumed in [8].

Acknowledgments

We wish to thank M.Ferraris for useful discussions. This work is partially supported by MIUR: PRIN 2005 on Leggi di conservazione e termodinamica in meccanica dei continui e teorie di campo. We also acknowledge the contribution of INFN (Iniziativa Specifica NA12) and the local research funds of Dipartimento di Matematica of Torino University.

References

[1] L. Fatibene, M.Ferraris, M.Francaviglia, S.Mercadante, Further Extended Theories of Gravitation: Part II, (submitted to IJGMMP); gr-qc/0911.2842
[2] T.P. Sotiriou, S. Liberati, Metric-affine f(R) theories of gravity, Annals Phys. 322 (2007) 935-966; gr-qc/0604006
[3] T.P. Sotiriou, f(R) gravity, torsion and non-metricity, Class. Quant. Grav. 26 (2009) 152001; gr-qc/0904.2774
[4] S. Capozziello, M. Francaviglia, Extended Theories of Gravity and their Cosmological and Astrophysical Applications, Journal of General Relativity and Gravitation 40 (2-3), (2008) 357-420.
[5] S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante, First Order Extended Gravity and the Dark Side of the Universe: the General Theory Proceedings of the Conference “Univers Invisibile”, Paris June 29 July 3, 2009 - to appear in 2010
[6] S. Capozziello, M. De Laurentis, M. Francaviglia, S. Mercadante, First Order Extended Gravity and the Dark Side of the Universe II: Matching Observational Data, Proceedings of the Conference “Univers Invisibile”, Paris June 29 July 3, 2009 - to appear in 2010
[7] S. Capozziello, M.F. De Laurentis, M. Francaviglia, S. Mercadante, From Dark Energy & Dark Matter to Dark Metric, Foundations of Physics 39 (2009) 1161-1176 gr-qc/0805.3642v4
[8] M. Di Mauro, L. Fatibene, M.Ferraris, M.Francaviglia, Further Extended Theories of Gravitation: Part I, (submitted to IJGMMP); gr-qc/0911.2841
[9] E. Sernesi, Linear Algebra: a Geometric Approach, Chapman & Hall Mathematics, 1993
[10] J.Ehlers, F.A.E.Pirani, A.Schild, The Geometry of Free Fall and Light Propagation, in General Relativity, ed. L.ORaifeartaigh (Clarendon, Oxford, 1972).