Replica Symmetry Breaking and the Renormalization Group Theory of the Weakly Disordered Ferromagnet

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Abstract

We study the critical properties of the weakly disordered $p$-component ferromagnet in terms of the renormalization group (RG) theory generalized to take into account the replica symmetry breaking (RSB) effects coming from the multiple local minima solutions of the mean-field equations. It is shown that for $p < 4$ the traditional RG flows at dimensions $D = 4 - \epsilon$, which are usually considered as describing the disorder-induced universal critical behavior, are unstable with respect to the RSB potentials as found in spin glasses. It is demonstrated that for a general type of the Parisi RSB structures there exists no stable fixed points, and the RG flows lead to the strong coupling regime at the finite scale $R_s \sim \exp(1/u)$, where $u$ is the small parameter describing the disorder. The physical consequences of the obtained RG solutions are discussed. In particular, we argue, that discovered RSB strong coupling phenomena indicate on the onset of a new spin glass type critical behaviour in the temperature interval $\tau < \tau_s \sim \exp(-1/u)$ near $T_c$. Possible relevance of the considered RSB effects for the Griffith phase is also discussed.
1 Introduction

In this paper we study the effects produced by weak quenched disorder on the critical phenomena in the ferromagnetic spin systems near the phase transition point. In the most general terms the traditional point of view on this problem could be summarized as follows.

According to the usual scaling theory near the critical temperature $T_c$ the only relevant scale that remains in the system is the correlation length $R_c$ which scales as $\sim \tau^{-\nu}$, where $\tau \equiv (T - T_c) / T_c << 1$ is the reduced temperature parameter and $\nu$ is the correlation length critical exponent.

If the disorder is weak (e.g. the concentration of impurities is small), its effect on the critical behavior in the vicinity of the phase transition point $T_c$ remains negligible so long as the correlation length $R_c$ is not too large, i.e. for temperatures $T$ not too close to $T_c$. In this regime the critical behavior will be essentially the same as in the pure system.

However, in the close vicinity of the critical point, at $\tau \equiv (T - T_c) / T_c \to 0$, the correlation length $R_c$ grows and becomes larger than the average distance between the impurities, so that the effective concentration of impurities, measured with respect to the correlation length, becomes large. The strength of disorder, described by small parameter $u$, affects only the width of the temperature region near $T_c$ in which the effective concentration gets large. If $uR_c^D \gg 1$, where $D$ is the spatial dimensionality, one has no grounds, in general, for believing that the effect of impurities will be small.

A very simple general criterion has been discovered, the so-called Harris criterion [1], which makes it possible to predict the effect of impurities qualitatively from only the critical exponents of the pure system. According to this criterion the impurities change the critical behavior only if $\alpha$, the specific heat exponent of the pure system, is greater than zero (i.e. the specific heat of the pure system is divergent at the critical point). According to the traditional point of view, when this criterion is satisfied, the disorder becomes relevant and a new universal critical behavior, with new critical exponents, is established sufficiently close to the phase transition point [2, 3]:

$$\tau < \tau_u \equiv u^{1/\alpha} \quad \text{(1.1)}$$

This argument identifies $1/\alpha$ as the cross-over exponent associated with randomness [2]. In contrast, when $\alpha < 0$ (the specific heat is finite), the disorder appears to be irrelevant, i.e. their presence does not affect the critical behavior.

Near the phase transition point the $D$-dimensional Ising-like systems can be described in terms of the scalar field Ginsburg-Landau Hamiltonian with a double-well potential:

$$H = \int d^Dx \left[ \frac{1}{2}(\nabla \phi(x))^2 + \frac{1}{2}(\tau - \delta \tau(x))|\phi(x)|^2 + \frac{1}{4}g|\phi(x)|^4 \right]. \quad \text{(1.2)}$$

Here the quenched disorder is described by random fluctuations of the effective transition temperature $\delta \tau(x)$ whose probability distribution is taken to be symmetric and Gaussian:
\[ P[\delta \tau] = p_0 \exp \left( -\frac{1}{4u} \int d^D x (\delta \tau(x))^2 \right) , \]  

where \( u \ll 1 \) is the small parameter which describes the disorder, and \( p_0 \) is the normalization constant. In Eq. (1.2) \( \tau \sim (T - T_c) \) and for notational simplicity, we define the sign of \( \delta \tau(x) \) so that positive fluctuations lead to locally ordered regions.

Now, if one is interested in the critical properties of the system, one has to integrate over all local field configurations up to the scale of the correlation length. This type of calculation is usually performed using a Renormalization Group (RG) scheme, which self-consistently takes into account all the fluctuations of the field on length scales up to \( R_c \).

To derive the traditional results for the critical properties of this system discussed above one can use the usual RG procedure developed for dimensions \( D = 4 - \epsilon \), where \( \epsilon \ll 1 \). Then one finds that in the presence of the quenched disorder the pure system fixed point becomes unstable, and the RG rescaling trajectories are arriving to another (universal) fixed point \( g_* \neq 0; u_* \neq 0 \), which yields the new critical exponents describing the critical properties of the system with disorder.

However, there exists an important point which missing in the traditional approach. Consider the ground state properties of the system described by the Hamiltonian (1.2). Configurations of the fields \( \phi(x) \) which correspond to local minima in \( H \) satisfy the saddle-point equation:

\[ -\Delta \phi(x) + (\tau - \delta \tau(x))\phi(x) + g\phi^3(x) = 0 . \]  

Clearly, the solutions of this equations depend on a particular configuration of the function \( \delta \tau(x) \) being inhomogeneous. The localized solutions with non-zero value of \( \phi \) exist in regions of space where \( \tau - \delta \tau(x) \) has negative values. Moreover, one finds a macroscopic number of local minimum solutions of the saddle-point equation (1.4). Indeed, for a given realization of the random function \( \delta \tau(x) \) there exists a macroscopic number of spatial "islands" where \( \tau - \delta \tau(x) \) is negative (so that the local effective temperature is below \( T_c \)), and in each of these "islands" one finds two local minimum configurations of the field: one which is "up", and another which is "down". These local minimal energy configurations are separated by finite energy barriers, whose heights become larger as the size of the "islands" are increased.

The problem is that the traditional RG approach is only a perturbative theory in which one treats the deviations of the field around the ground state configuration, and it can not take into account other local minimum configurations which are "beyond barriers". This problem does not arise in the pure systems, where the solution of the saddle-point equation is unique. However, in a situation like that discussed above, when one gets numerous local minimum configurations separated by finite barriers, the direct application of the traditional RG scheme may be questioned.

In a systematic approach one would like to integrate in an RG way over fluctuations around the local minima configurations. Furthermore, one also has to sum over all these local minima up to the scale of the correlation length. In view of the fact that the local minima configurations are defined by the random quenched function \( \delta \tau(x) \) in an...
essentially non-local way, the possibility to implement successfully such a systematic
approach seems rather hopeless.

On the other hand there exists another technique which has been developed specif-
cically for dealing with systems which exhibit numerous local minima states. It is the
Parisi Replica Symmetry Breaking (RSB) scheme which has proved to be crucial in the
mean-field theory of spin-glasses (see e.g. [3]). Recent studies show that in certain cases
the RSB approach can also be generalized for situations where one has to deal with
fluctuations as well [4], [7], [8].

It can be argued that the summation over multiple local minima configurations in
the present problem could provide additional non-trivial RSB interaction potentials for
the fluctuating fields [9]. Let us consider this point in some more details.

To carry out the appropriate average over quenched disorder we can use the standard
replica approach. To do this, we need to average the \( n \rightarrow 0 \) power of the partition
function. This is accomplished by introducing the replicated partition function,
\( Z_n \equiv Z_n[\delta \tau] \), where \( \langle ... \rangle \) denotes the averaging over \( \delta \tau(x) \) with the probability distribution
\eqref{eq:1.3}. Simple integration yields:

\[
Z_n \equiv Z_n(\delta \tau) = \int D\phi(x) \exp\left[ -\int d^Dx \left( \frac{1}{2} \sum_{a=1}^n (\nabla \phi_a(x))^2 + \tau \sum_{a=1}^n \phi_a^2(x) \\
+ \frac{1}{4} \sum_{a,b=1}^n g_{ab} \phi_a^2(x) \phi_b^2(x) \right) \right],
\]

where

\[
g_{ab} = g_{\delta ab} - u.
\]

is the replica symmetric (RS) interaction parameter. If one would start the usual RG
procedure for the above replica Hamiltonian (as it is done in the traditional approach),
then it would correspond to the perturbation theory around the homogeneous ground
state \( \phi = 0 \).

However, in the situation when there exist numerous local minima solutions of the
saddle-point equation \eqref{eq:1.4} one has to be more careful. Let us denote the local solutions
of the eq.\eqref{eq:1.4} by \( \psi(i)(x) \) where \( i = 1, 2, \ldots , N_0 \) labels the ”islands” where \( \delta \tau(x) > \tau \). If
the size \( L_0 \) of an ”island” where \( \langle \delta \tau(x) - \tau \rangle > 0 \) is not too small, then the value of \( \psi(i)(x) \)
in this ”island” should be \( \sim \pm \sqrt{\langle \delta \tau(x) - \tau \rangle / g} \), where \( \delta \tau(x) \) should now be interpreted
as the value of \( \delta \tau \) averaged over the region of size \( L_0 \). Such ”islands” occur at a certain
finite density per unit volume. Thus the value of \( N_0 \) is macroscopic: \( N_0 = \kappa V \), where
\( V \) is the volume of the system and \( \kappa \) is a constant. An approximate global extremal
solution \( \Phi(x) \) is constructed as the union of all these local solutions without regard for
interactions between ”islands.” Each local solution can occur with either sign, since we
are dealing with the disordered phase:

\[
\Phi_{(\alpha)}[x; \delta \tau(x)] = \sum_{i=1}^{\kappa V} \sigma_i \psi(i)(x),
\]

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where each $\sigma_i = \pm 1$. Accordingly, the total number of global solutions must be $2^\kappa V$. We denote these solutions by $\Phi_{(\alpha)}[x; \delta \tau(x)]$, where $\alpha = 1, 2, \ldots, K = 2^\kappa V$. As we mentioned, it seems unlikely that an integration over fluctuations around $\phi(x) = 0$ will include the contributions from the configurations of $\phi(x)$ which are near a $\Phi(x)$, since $\Phi(x)$ is "beyond a barrier," so to speak. Therefore, it seems appropriate to include separately the contributions from small fluctuations about each of the many $\Phi_{(\alpha)}[x; \delta \tau]$. Thus we have to sum over the $K$ global minimum solutions (non-perturbative degrees of freedom) $\Phi_{(\alpha)}[x; \delta \tau]$ and also to integrate over "smooth" fluctuations $\varphi(x)$ around them.

\[ Z[\delta \tau] = \int D\varphi(x) \sum_\alpha^K \exp \left( -H \left[ \Phi_{(\alpha)}[x; \delta \tau] + \varphi(x); \delta \tau \right] \right) \]

\[ = \int D\varphi(x) \exp \left( -H[\varphi; \delta \tau] \right) \times \tilde{Z}[\varphi; \delta \tau] , \]

where

\[ \tilde{Z}[\varphi; \delta \tau] = \sum_\alpha^K \exp \left( -H_\alpha - \int d^D x \left[ \frac{3}{2} g \Phi_{(\alpha)}^2 \varphi^2(x) + g \Phi_{(\alpha)} \varphi^3(x) \right] \right) , \]

and $H_\alpha$ is the energy of the $\alpha$-th solution.

Next we carry out the appropriate average over quenched disorder, and for the replica partition function, $Z_n$, we get:

\[ Z_n = \int D\delta \tau \int D\varphi_a \exp \left( -\frac{1}{4u} \int d^D x [\delta \tau(x)]^2 - \sum_{a=1}^n H[\varphi_a; \delta \tau] \right) \times \tilde{Z}_n[\varphi_a; \delta \tau] , \]

where the subscript $a$ is a replica index and

\[ \tilde{Z}_n[\varphi_a; \delta \tau] = \sum_{\alpha_1 \ldots \alpha_n}^K \exp \left( -\sum_a H_{\alpha_a} - \int d^D x \sum_a \left[ \frac{3}{2} g \Phi_{(\alpha_a)}^2 \varphi_a^2(x) + g \Phi_{(\alpha_a)} \varphi_a^3(x) \right] \right) , \]

(1.10)

It is clear that if the saddle-point solution is unique, then from the eq.\,(1.10), one would obtain the usual RS representation \((1.5),(1.6)\). However, in the case of the macroscopic number of the local minima solutions the problem is getting highly non-trivial. This situation is reminiscent of the (unsolved) problem of summing over the saddle-point solutions in the random-field Ising model, which is believed to provide the RSB phase near the phase transition point \([10]\).

It is obviously hopeless to try to make a systematic evaluation of the above replicated partition function. The global solutions $\Phi_{(\alpha)}$ are complicated implicit functions of $\delta \tau(x)$. These quantities have fluctuations of two different types. In the first instance, they depend on the stochastic variables $\delta \tau(x)$. But even when $\delta \tau(x)$ are completely fixed, $\Phi_{(\alpha)}(x)$ will depend on $\alpha$ (which labels the possible ways of constructing the global minimum out of the choices for the signs $\{\sigma\}$ of the local minima). A crude way of treating this situation is to regard the local solutions $\psi^{(i)}(x)$ as if they were random.
variables, even though $\delta \tau(x)$ has been specified. This randomness, which one can see is not all that different from that which exists in a spin glass, is the crucial one. It can be shown then, that due to the interaction of the fluctuating fields with the local minima configurations (the term $\phi^2_{\langle a \rangle} \phi^2_a$ in the eq.(1.11)), the summation over solutions in the replica partition function $\tilde{Z}_n[\phi_a]$, eq.(1.11), could provide the additional non-trivial RSB potential $\sum_{a,b} g_{ab} \phi^2_a \phi^2_b$ in which the matrix $g_{ab}$ has the Parisi RSB structure [9].

In the paper [9] due to several simplifying assumptions, the matrix $g_{ab}$ has been obtained to have explicit 1-step RSB structure, which, in general, may not be the case.

In this paper we are going to study the critical properties of weakly disordered systems in terms of the RG approach taking into account the possibility of a general type of the RSB potentials for the fluctuating fields. The idea is that hopefully, like in spin-glasses, this type of generalized RG scheme self-consistently takes into account relevant degrees of freedom coming from the numerous local minima. In particular, the instability of the traditional Replica Symmetric (RS) fixed points with respect to RSB indicates that the multiplicity of the local minima can be relevant for the critical properties in the fluctuation region.

It will be shown in the next Section that, whenever the disorder appears to be relevant for the critical behavior, the usual RS fixed points (which used to be considered as providing new universal disorder-induced critical exponents) are unstable with respect to “turning on” an RSB potential. Moreover, it will be shown that in the presence of a general type of the RSB potentials the RG flows actually lead to the strong coupling regime at the finite spatial scale $R_\ast \sim \exp(1/(u \nu))$ (which corresponds to the temperature scale $\tau_\ast \sim \exp(-\frac{1}{u})$). At this scale the renormalized matrix $g_{ab}$ develops strong RSB, and the values of the interaction parameters are getting non-small.

Usually the strong coupling situation indicates that certain essentially non-perturbative excitations have to be taken into account, and it could be argued that in the present model these are due to exponentially rare "instantons" in the spatial regions, where the value of $\delta \tau(x) \sim 1$, and the local value of the field $\varphi(x)$ must be $\sim \pm 1$. (Distant analog of this situation exists in the 2D Heisenberg model where the Poliakov renormalization develops into the strong coupling regime at a finite (exponentially large) scale which is known to be due to the non-linear localized instanton solutions [11]).

In Section 3 we discuss the physical consequences of the obtained RG solutions. In particular we show that due to the absence of fixed points at the disorder dominated scales $R >> u^{-\nu/\alpha}$ (or at the corresponding temperature scales $\tau << u^{1/\alpha}$) there must be no simple scaling of the correlation functions or of other physical quantities. Besides, we demonstrate, that the structure of the SG type two-points correlation functions is characterized by the strong RSB, indicating on the onset of a new type of the critical behaviour of the SG nature.

In Section 4 we consider the special case of systems with the number of spin components $p = 4$, in which the pure system specific heat critical exponent $\alpha = 0$. Here the disorder appears to be marginally irrelevant in a sense that it does not change the critical exponents. Nevertheless, the critical behaviour itself (described in terms of the logarithmic singularities) is effected by the disorder, and moreover, the RSB phenomena is demonstrated to be relevant in this case as well.
The remaining problems as well as future perspectives are discussed in the Conclusions. In particular, we discuss possible relevance of the considered RSB phenomena for the Griffith phase which is known to exist in a finite temperature interval near \( T_c \) \[12\].

## 2 Replica Symmetry Breaking in the Renormalization Group Theory

We consider the \( p \)-component ferromagnet with quenched random effective temperature fluctuations, which near the transition point can be described by the usual Ginzburg-Landau Hamiltonian:

\[
H[\delta \tau, \phi] = \int d^Dx \left[ \frac{1}{2} \sum_{i=1}^{p} (\nabla \phi_i(x))^2 \right. \\
+ \left. \frac{1}{2} (\tau - \delta \tau(x)) \sum_{i=1}^{p} \phi_i^2(x) + \frac{1}{4} g \sum_{i,j=1}^{p} \phi_i^2(x) \phi_j^2(x) \right],
\]

where the quenched random temperature \( \delta \tau(x) \) is described by the Gaussian distribution \( (1.3) \).

In terms of the standard replica approach after integration over \( \delta \tau(x) \) for the replica partition function one gets:

\[
Z_n \equiv Z^n(\delta \tau) = \\
\int D\phi^a_i(x) \exp \left[ - \int d^Dx \left( \frac{1}{2} \sum_{i=1}^{p} \sum_{a=1}^{n} (\nabla \phi^a_i(x))^2 + \frac{1}{2} \tau \sum_{i=1}^{p} \sum_{a=1}^{n} \phi^a_i(x)^2 + \frac{1}{4} \sum_{i,j=1}^{p} \sum_{a,b=1}^{n} g_{ab}[\phi^a_i(x)]^2[\phi^b_j(x)]^2 \right) \right],
\]

where

\[
g_{ab} = g_{\delta ab} - u.
\]

To study the critical properties of this system we use the standard RG procedure developed for dimensions \( D = 4 - \epsilon \), where \( \epsilon \ll 1 \). Along the lines of the usual rescaling scheme (see e.g. \[13\]) one gets the following (one-loop) RG equations for the interaction parameters \( g_{ab} \):

\[
\frac{dg_{ab}}{d\xi} = \epsilon g_{ab} - \frac{1}{8\pi^2} \left( 4g_{ab}^2 + 2(g_{aa} + g_{bb})g_{ab} + p \sum_{c=1}^{n} g_{ac}g_{cb} \right),
\]

where \( \xi \) is the standard rescaling parameter.

Changing \( g_{ab} \to 8\pi^2 g_{ab} \), and \( g_{a\neq b} \to -g_{a\neq b} \) (so that the off-diagonal elements would be positively defined), and introducing \( \tilde{g} \equiv g_{aa} \), we get the following RG equations:
\[ \frac{dg_{ab}}{d\xi} = \epsilon g_{ab} - (4 + 2p)\tilde{g}g_{ab} + 4g_{ab}^2 + p \sum_{c \neq a,b}^n g_{ac}g_{cb} \quad (a \neq b), \]  

\[ \frac{d}{d\xi} \tilde{g} = \epsilon \tilde{g} - (8 + p)\tilde{g}^2 - p \sum_{c \neq 1} g_{1c}^2 \]  

If one takes the matrix \( g_{ab} \) to be replica symmetric, as in the starting form of Eq. (2.3), then one would recover the usual RG equations for the parameters \( g \) and \( u \), and eventually one would obtain the well known results for the fixed points and the critical exponents \([2, 3]\). Here we leave apart the question as to how perturbations out of the RS subspace could arise (see discussion in \([4]\)) and formally consider the RG eqs. (2.5), (2.6) assuming that the matrix \( g_{ab} \) has a general Parisi RSB structure.

According to the standard technique of the Parisi RSB algebra (see e.g. \([5]\)), in the limit \( n \to 0 \) the matrix \( g_{ab} \) is parametrized in terms of its diagonal elements \( \tilde{g} \) and the off-diagonal function \( g(x) \) defined in the interval \( 0 < x < 1 \). All the operations with the matrices in this algebra can be performed according to the following simple rules (see e.g. \([6, 14]\)):

\[ g_{ab}^k \to (\tilde{g}^k; g^k(x)), \]  

\[ (g^2)_{ab} \equiv \sum_{c=1}^n g_{ac}g_{cb} \to (\tilde{c}; c(x)), \]

where

\[ \tilde{c} = \tilde{g}^2 - \int_0^1 dx g^2(x), \]  

\[ c(x) = 2(\tilde{g} - \int_0^1 dy g(y))g(x) - \int_0^x dy [g(x) - g(y)]^2. \]  

The RS situation corresponds to the case \( g(x) = const \) independent of \( x \).

Using the above rules from the eqs. (2.5), (2.6) one gets:

\[ \frac{dg(x)}{d\xi} = (\epsilon - (4 + 2p)\tilde{g})g(x) + 4g^2(x) - 2pg(x) \int_0^1 dy g(y) - p \int_0^x dy (g(x) - g(y))^2 \]  

\[ \frac{d}{d\xi} \tilde{g} = \epsilon \tilde{g} - (8 + p)\tilde{g}^2 + p\overline{g^2} \]

where \( \overline{g^2} \equiv \int_0^1 dx g^2(x) \).

Usually in the studies of the critical behaviour one is looking for the stable fixed-points solutions of the RG equations. The fixed-point values of the of the renormalized interaction parameters are believed to describe the asymptotic structure of the effective Hamiltonian which makes possible to calculate the singular part of the free energy, as well as the other thermodynamic quantities.

From the eq. (2.10) one can easily find out what should be the structure of the function \( g(x) \) at the fixed point, \( \frac{dg(x)}{d\xi} = 0, \frac{d}{d\xi} \tilde{g} = 0 \). Taking the derivative over \( x \) twice, one gets, from Eq. (2.10): \( g'(x) = 0 \). This means that either the function \( g(x) \) is constant (which is the RS situation), or it has the step-like structure. It is interesting to
note that the structure of fixed-point equations is similar to those for the Parisi function \( q(x) \) near \( T_c \) in the Potts spin-glasses \([13]\), and it is the term \( g^2(x) \) in Eq. (2.10) which is known to produce 1step RSB solution there. The numerical solution of the above RG equations convincingly demonstrates that whenever the triel function \( g(x) \) has the many-step RSB structure, it quickly develops into the 1-step one with the coordinate of the step being the most right step of the original many-step function.

Let us consider the 1-step RSB ansatz for the function \( g(x) \):

\[
g(x) = \begin{cases} 
g_0 & \text{for } 0 \leq x < x_0 \\
g_1 & \text{for } x_0 < x \leq 1 
\end{cases}
\]  

(2.12)

where \( 0 \leq x_0 \leq 1 \) is the coordinate of the step.

In terms of this ansatz the above fixed-point equations have several non-trivial solutions:

1) The RS fixed-point which corresponds to the pure system:

\[
g_0 = g_1 = 0; \quad \tilde{g} = \frac{1}{8 + p} \epsilon
\]  

(2.13)

This fixed point (in accordance with the Harris criterion) is stable for the number of spin components \( p > 4 \), and it is getting unstable for \( p < 4 \).

2) The disorder-induced RS fixed point (for \( p > 1 \)) \([2, 3]\):

\[
g_0 = g_1 = \epsilon \frac{4 - p}{16(p - 1)}; \quad \tilde{g} = \epsilon \frac{p}{16(p - 1)}.
\]  

(2.14)

It was usually considered to be the one which describes the new universal critical behaviour in systems with impurities. This fixed point has been shown to be stable (with respect to the RS deviations!) for \( p < 4 \), which is consistent with the Harris criterion. (For \( p = 1 \) this fixed point involves an expansion in powers of \( (\epsilon)^{1/2} \) and this structure is only revealed within a two-loop approximation). However, the stability analysis with respect to the RSB deviations shows that this fixed point is always unstable \([9]\). Therefore, whenever the disorder is relevant for the critical behaviour, the RSB perturbations must be getting the dominant factor in the asymptotic large scale limit.

3) The 1-step RSB fixed point \([4]\):

\[
g_0 = 0; \quad g_1 = \epsilon \frac{4 - p}{16(p - 1) - px_0(8 + p)};
\]  

\[
\tilde{g} = \epsilon \frac{p(1 - x_0)}{16(p - 1) - px_0(8 + p)}.
\]  

(2.15)

This fixed point can be shown to be stable (within 1-step RSB subspace!) for:

\[
1 < p < 4,
\]  

\[
0 < x_0 < x_c(p) \equiv \frac{16(p - 1)}{p(8 + p)}.
\]  

(2.16)
In particular, $x_c(p = 2) = 4/5$; $x_c(p = 3) = 32/33$, and $x_c(p = 4) = 1$. Using the result (2.13) one can easily obtain the corresponding critical exponents which are now getting to be non-universal being dependent on the starting parameter $x_0$ \textsuperscript{[9]} (see also next Section).

(Note, that in addition to the fixed points listed above there exist several other 1step RSB solutions which are either unstable or unphysical.)

The problem, however, is that if the parameter $x_0$ of the starting function $g(x; \xi = 0)$ (or, more generally, the coordinate of the most right step of the many-steps starting function) is beyond the stability interval, such that $x_c(p) < x_0 < 1$, then there exist no stable fixed points of the RG eqs.(2.10),(2.11). One faces the same situation, of course, in the case of a general continuous starting function $g(x; \xi = 0)$. Moreover, according to eq. (2.16) there exist no stable fixed points out of the RS subspace in the most interesting Ising case, $p = 1$.

Unlike the RS situation for $p = 1$, where one finds the stable $\sim \sqrt{\epsilon}$ fixed point in the two-loop RG equations \textsuperscript{[3]}, here adding next order terms in the RG equations doesn’t cure the problem. In the considered RSB case one finds that in the two-loops RG equations the values of the parameters in the fixed point are formally getting of the order of one, and it signals that we are entering the strong coupling regime where all the orders of the RG are getting relevant.

Nevertheless, to get at least some information about the physics behind this instability phenomena, one can proceed analyzing the actual evolution of the above one-loop RG equations. The scale evolution of the parameters of the Hamiltonian would still adequately describe the properties of the system until we reach a critical scale $\xi_*$, at which the strong coupling regime begins.

The evolution of the renormalized function $g(x; \xi)$ can be analyzed both numerically and analytically. It can be shown (see Appendix A) that in the case $p < 4$ for a general continuous starting function $g(x; \xi = 0) \equiv g_0(x)$ the renormalized function $g(x; \xi)$ tends to zero everywhere in the interval $0 \leq x < (1 - \Delta(\xi))$, while in the narrow (scale dependent) interval $\Delta(\xi)$ near $x = 1$ the values of of the function $g(x; \xi)$ grow:

$$g(x; \xi) \sim \begin{cases} a \frac{u}{1-u\xi}; & \text{at } (1-x) << \Delta(\xi) \\ 0; & \text{at } (1-x) >> \Delta(\xi) \end{cases}$$

$$\tilde{g}(\xi) \sim u \ln \frac{1}{1-u\xi}$$

where

$$\Delta(\xi) \approx (1-u\xi)$$

Here $a$ is a positive non-universal constant, and the critical scale $\xi_*$ is defined by the condition that the values of the renormalized parameters are getting of the order of one: $(1-u\xi_*) \sim u$, or $\xi_* \sim 1/u$. Correspondingly, the spatial scale at which the system is entering the strong coupling regime is:

$$R_* \sim \exp\left(\frac{1}{u}\right)$$

(2.20)
Note that the value of this scale is much bigger than the usual crossover scale \( \sim u^{-\alpha/\nu} \) (where \( \alpha \) and \( \nu \) are the pure system specific heat and the correlation length critical exponents), at which the disorder is getting relevant for the critical behaviour.

According to the above result, the value of the narrow band near \( x = 1 \) where the function \( g(x; \xi) \) is formally getting divergent is \( \Delta(\xi) \approx (1-ux) \rightarrow u \ll 1 \) as \( \xi \rightarrow \xi_* \).

Besides, it can also be shown (Appendix A) that the value of the integral \( \mathcal{F}(\xi) \equiv \int_0^1 g(x; \xi) \) is formally getting divergent logarithmically as \( \xi \rightarrow \xi_* \):

\[
\mathcal{F}(\xi) \sim u \ln \frac{1}{1-ux} \tag{2.21}
\]

Qualitatively similar asymptotic behaviour for \( g(x; \xi) \) is obtained for the case when the starting function \( g_0(x) \) has the 1-step RSB structure \( \langle 2.12 \rangle \), and the coordinate of the step \( x_0 \) is in the instability region (or for any \( x_0 \) in the Ising case \( p = 1 \)):

\[
g(x; \xi) \sim \begin{cases} 
g_1(0) & \text{at } x_0 < x < 1 \\
\frac{g_1(0)}{1-(4-2p+px_0)g_1(0)\xi} & \text{at } 0 < x < x_0 \\
0 & \text{at } 0 \leq x < x_0
\end{cases} \tag{2.22}
\]

Here \( g_1(0) \equiv g_1(\xi = 0) \sim u \), and the coefficient \( (4-2p+px_0) \) is always positive. In this case again, the system arrives into the strong coupling regime at scales \( \xi \sim 1/u \).

Note that the above asymptotics do not explicitly involve \( \epsilon \). Actually, the role of the parameter \( \epsilon > 0 \) is to "push" the RG trajectories out of the trivial Gaussian fixed point \( g = 0; \tilde{g} = 0 \). Thus, the value of \( \epsilon \), as well as the values of the starting parameters \( g_0(x) \), \( \tilde{g}_0 \), define a scale at which the solutions finally arrive to the above asymptotic regime.

In the case \( \epsilon < 0 \) (above dimensions 4) the Gaussian fixed point is stable; on the other hand, the strong coupling asymptotics still exists in this case as well, separated from the trivial one by a finite (depending on the value of \( \epsilon \)) barrier. Therefore, although \( \text{infinitely small} \) disorder remains irrelevant for the critical behaviour above dimensions 4, if the disorder is strong enough (bigger than certain depending on \( \epsilon \) threshold value) the RG trajectories could arrive to the above strong coupling regime again.

3 Scaling and Correlation functions

3.1 Temperature Scales

The renormalization of the mass term \( \tau(\xi) \sum_{a=1}^{n} \phi_a^2 \) is described by the following RG equation:

\[
\frac{d}{d\xi} \ln \tau = 2 - \frac{1}{8\pi^2} [(2+p)\tilde{g} + p \sum_{a \neq b}^n g_{1a}] \tag{3.1}
\]

Changing (as in the previous section) \( g_{ab} \rightarrow 8\pi^2 g_{ab} \), and \( g_{a \neq b} \rightarrow -g_{a \neq b} \), in the Parisi representation we get:

\[
\frac{d}{d\xi} \ln \tau = 2 - [(2+p)\tilde{g}(\xi) + p \int_0^1 g(x; \xi)] \tag{3.2}
\]
or

\[ \tau(\xi) = \tau_0 \exp\{2\xi - \int_0^\xi d\eta[(2 + p)\tilde{g}(\eta) + p\overline{f}(\eta)]\} \]  \hspace{1cm} (3.3)

where \( \tilde{g}(\eta) \) and \( \overline{f}(\eta) \equiv \int_0^\infty d\eta x g(x; \eta) \) are the solutions of the RG equations of the previous section.

Consider first what was the traditional (replica-symmetric) situation. The RS interaction parameters \( \tilde{g}(\xi) \) and \( g(\xi) \) are arriving to the fixed point values \( \tilde{g}_* \) and \( g_* \)(which are of the order of \( \epsilon \)), and then for the dependence of the renormalized mass \( \tau(\xi) \), according to (3.3), one gets:

\[ \tau(\xi) = \tau_0 \exp\{\Delta_\tau \xi\} \]  \hspace{1cm} (3.4)

where

\[ \Delta_\tau = 2 - [(2 + p)\tilde{g}_* + pg_*] \]  \hspace{1cm} (3.5)

At scale \( \xi_c \), such that \( \tau(\xi_c) \) is getting of the order of one, the system gets out of the scaling region. Since the RG parameter is by definition \( \xi = \ln R \), where \( R \) is the spatial scale, this defines the correlation length \( R_c \) as a function of the reduced temperature \( \tau_0 \). According to (3.4), one obtains:

\[ R_c(\tau_0) \sim \tau_0^{-\nu} \]  \hspace{1cm} (3.6)

where \( \nu = 1/\Delta_\tau \), eq.(3.5), is the critical exponent of the correlation length.

Actually, if the starting value of the disorder parameter \( g(\xi = 0) \equiv u \) is much smaller than starting value of the pure system interaction \( \tilde{g}(\xi = 0) \equiv g_0 \), the situation is a little bit more complicated. In this case the RG flow for \( \tilde{g}(\xi) \) first arrives to the pure system fixed point \( \tilde{g}_*^{(\text{pure})} \), as if the disorder perturbation does not exist. Then, since the pure system fixed point is unstable with respect to the disorder perturbations, at scales bigger than certain disorder dependent scale \( \xi_u \) the RG trajectories are eventually arriving to the stable (universal) disorder induced fixed point \( (\tilde{g}_*, g_*) \). According to the traditional theory \[2\] it is known that \( \xi_u \sim \frac{\nu}{\alpha} \ln \frac{1}{u} \). The corresponding spatial scale is \( R_u \sim u^{-\nu/\alpha} \), and it is big in terms of the small parameter \( u \).

Coming back to the scaling behaviour of the mass parameter \( \tau(\xi) \), eq.(3.4), we see that if the value of the temperature \( \tau_0 \) is such that \( \tau(\xi) \) is getting of the order of one before the crossover scale \( \xi_u \) is reached, then for the scaling behaviour of the correlation length (as well as for other thermodynamic quantities) one finds essentially the pure system result \( R_c(\tau_0) \sim \tau_0^{-\nu^{(\text{pure})}} \). However, the pure system critical behaviour is observed only until \( R_c \ll R_u \), which imposes the restriction on the temperature parameter: \( \tau_0 >> u^{1/\alpha} \equiv \tau_u \). In other words, at temperatures not too close to \( T_c \), \( \tau_u \ll \tau_0 \ll 1 \), the presence of disorder is irrelevant for the critical behaviour.

On the other hand, if \( \tau_0 \ll \tau_u \) (in the close vicinity of \( T_c \)), the RG trajectories for \( \tilde{g}(\xi) \) and \( g(\xi) \) are arriving (after crossover) into a new (universal) disorder induced fixed point \( (\tilde{g}_*, g_*) \), and the scaling of the correlation length (as well as other thermodynamic
quantities), according to eqs. (3.6)-(3.5), is getting to be controlled by a new universal critical exponent \( \nu \) which is defined by the RS fixed point \( (\tilde{g}_*, g_*) \) of the random system.

Consider now what is the situation if the RSB scenario takes place. Again, if the disorder parameter \( u \) is small, in the temperature interval \( \tau_u \ll \tau_0 \ll 1 \), the critical behaviour is essentially controlled by the pure system fixed point, and the presence of disorder is irrelevant. For the same reasons as discussed above, the system gets out of the scaling regime (\( \tau(\xi) \) is getting of the order of one) before the disorder parameters start ”pushing” the RG trajectories out of the pure system fixed point.

However, at temperatures \( \tau_0 \ll \tau_u \) the situation is getting completely different from the RS case. At scales \( \xi \gg \xi_u \) (although still \( \xi \ll \xi_* \sim \frac{1}{u} \)) according to the solutions (2.17), (2.22) the parameters \( \tilde{g}(\xi) \) and \( g(x; \xi) \), does not arrive to any fixed point, and they keep evolving as the scale \( \xi \) increases. Therefore, here, according to eq.(3.3), the correlation length (defined, as usual, by the condition that the renormalized \( \tau(\xi) \) is getting of the order of one) is getting to be defined by the following non-trivial equation:

\[
2 \ln R_c - \int_0^{\ln R_c} d\eta [(2 + p)\tilde{g}(\eta) + p\tilde{g}(\eta)] = \ln \frac{1}{\tau_0} \quad (3.7)
\]

Thus, as the temperature is getting sufficiently close to \( T_c \) (in the disorder dominated region \( \tau_0 \ll \tau_u \)) there will be no usual scaling dependence of the correlation length (as well as other thermodynamic quantities) like in the eq.(3.6).

Finally, as the temperature parameter \( \tau_0 \) is getting smaller and smaller, what happens is that at scale \( \xi_* \equiv \ln R_* \sim \frac{1}{u} \) we are entering into the strong coupling regime (such that the parameters \( \tilde{g}(\xi) \) and \( g(x; \xi) \) are getting non-small), while the renormalized mass \( \tau(\xi) \) remains still small.

According to the solution obtained in Appendix A, the integrals \( \int_0^{\xi_*} d\eta \tilde{g}(\eta) \) and \( \int_0^{\xi_*} d\eta \tilde{g}(\eta) \equiv G_c \) have a finite (depending on the initial conditions) value. Thus, according to eq.(3.7), for the crossover temperature we get:

\[
\tau_* \sim \exp(-\frac{\text{const}}{u}) \quad (3.8)
\]

In the close vicinity of \( T_c \) at \( \tau \ll \tau_* \) we are facing the situation that at large scales the interaction parameters of the asymptotic (zero-mass) Hamiltonian are getting non-small, and the properties of the system can not be analyzed in terms of simple one-loop RG approach. Nevertheless, the qualitative structure of the asymptotic Hamiltonian makes it possible to argue that in the temperature interval \( \tau \ll \tau_* \) near \( T_c \) the properties of the system should be essentially SG-like. The point is that it is the parameter describing the disorder, \( g(x; \xi) \), which is the most divergent.

In a sense, here the problem is qualitatively reduced back to the original one with strong disorder at the critical point. It doesn’t seem probable, however, that the state of the system will be described by non-zero true SG order parameter \( Q_{ab} = \langle \phi_a \phi_b \rangle \) (which would mean real SG freezing). Otherwise there must exist finite value of \( \tau \) at which real thermodynamic phase transition into the SG phase takes place, while we observe only the crossover temperature \( \tau_* \), at which change of critical regime occurs.
It seems more realistic to expect that at scales $\sim \xi^*$ the RG trajectories finally arrive to a fixed-point characterized by non-small values of the interaction parameters and strong RSB. Then, the SG-like behaviour of the system near $T_c$ will be characterized by its highly non-trivial critical properties exhibiting strong RSB phenomena.

### 3.2 Correlation Functions

Consider the scaling properties of the spin-glass type connected correlation function:

$$K(R) = \frac{\langle(\phi(0)\phi(R)) - \langle\phi(0)\rangle\langle\phi(R)\rangle\rangle^2}{\langle\phi(0)\phi(R)\rangle^2} \equiv \langle\langle\phi(0)\phi(R)\rangle\rangle^2$$

In terms of the replica formalism one gets:

$$K(R) = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{a \neq b} K_{ab}(R)$$

where

$$K_{ab}(R) = \langle\langle\phi_a(0)\phi_b(0)\phi_a(R)\phi_b(R)\rangle\rangle$$

In terms of the standard RG formalism for the replica correlation function $K_{ab}(R)$ one finds:

$$K_{ab}(R) \sim (G_0(R))^2(Z_{ab}(R))^2$$

where

$$G_0(R) = R^{-(D-2)}$$

is the free-field correlation function, and in the one-loop approximation the scaling of the mass-like object $Z_{ab}(R)$ (with $a \neq b$) is defined by the RG equation:

$$\frac{d}{d\xi} \ln Z_{ab}(\xi) = 2g_{ab}(\xi)$$

Here $g_{a\neq b}(\xi) > 0$ is the solution of the corresponding RG equations (2.5)-(2.6), $\xi = \ln R$, and $Z_{ab}(0) \equiv 1$.

For the correlation function (3.12) one finds:

$$K_{ab}(R) \sim (G_0(R))^2 \exp\{4 \int_0^{\ln R} d\xi g_{ab}(\xi)\}$$

Correspondingly, in the Parisi representation: $g_{a\neq b}(\xi) \to g(x;\xi)$ and $K_{a\neq b}(R) \to K(x;R)$, one gets:

$$K(x;R) \sim (G_0(R))^2 \exp\{4 \int_0^{\ln R} d\xi g(x;\xi)\}$$
To realize the effects of the RSB more clearly consider again what was the situation in the traditional RS case. Here (for \(p < 4\)) one finds that the interaction parameter \(g_{a\neq b}(\xi) \equiv u(\xi)\) arrives to the RS fixed point \(u_* = \epsilon \frac{4 - p}{16(p - 1)}\), and according to eqs. (3.13), (3.10) one obtains simple scaling:

\[
K_{rs}(R) \sim R^{-2(D-2)+\theta}
\]

with the universal disorder induced critical exponent

\[
\theta = \epsilon \frac{4 - p}{4(p - 1)}
\]

In the case of the 1-step RSB fixed point, eq.(2.15), the situation is getting somewhat more complicated. Here one finds that the correlation function \(K(x; R)\) also have 1RSB structure:

\[
K(x; R) \sim \begin{cases} 
K_0(R); & \text{for } 0 \leq x < x_0 \\
K_1(R); & \text{for } x_0 < x \leq 1
\end{cases}
\]

where (in the first order in \(\epsilon\))

\[
K_0(R) \sim R^{-2(D-2)} = G_0^2(R)
\]

\[
K_1(R) \sim R^{-2(D-2)+\theta_{1rsb}}
\]

with non-universal critical exponent \(\theta_{1rsb}\) explicitly depending on the coordinate of the step \(x_0\):

\[
\theta_{1rsb} = \epsilon \frac{4(4 - p)}{16(p - 1) - px_0(8 + p)}
\]

Since the critical exponent \(\theta_{1rsb}\) is positive, the leading contribution to the ”observable” quantity \(K(R) = \langle \langle \phi(0)\phi(R)\rangle \rangle^2\); eq.(3.10), is given by \(K_1(R)\):

\[
K(R) \sim (1 - x_0)K_1(R) + x_0K_0(R) \sim R^{-2(D-2)+\theta_{1rsb}}
\]

But the difference between the 1RSB the RS cases must be observed not only in the result that their critical exponents \(\theta\) of the correlation functions \(K(R)\) must be different. According to the traditional SG philosophy [5], the result that the scaling of the RSB correlation function \(K_{ab}(R)\) or \(K(x; R)\) does depend on the replica indices \((a, b)\) or the replica parameter \(x\), eq.(3.19), indicates that in different measurements of the correlation function for the same realization of the quenched disorder one is going to obtain different results, \(K_0(R)\) or \(K_1(R)\), with the probabilities defined by the value of \(x_0\).

In real experiments, however, one is dealing with the quantities averaged in space. In particular, for the two-point correlation functions the measurable quantity is obtained by integration over the two points, such that the distance \(R\) between them is fixed. Of course, the result obtained this way must be equivalent simply to \(K(R)\), eq.(3.22), found...
by formal averaging over different realizations of disorder, and different scalings \( K_0(R) \) and \( K_1(R) \) cannot be observed this way.

Nevertheless, for somewhat different scheme of the measurements the qualitative difference with the RS situation can be observed. In spin-glasses it is generally believed that RSB can be interpreted as factorization of the phase space into (ultrametric) hierarchy of "valleys", or local minima pure states separated by macroscopic barriers. Although in the present case the local minima configurations responsible for the RSB can not be separated by infinite barriers, it would be natural to interpret obtained phenomenon as effective factorization of the phase space into a hierarchy of valleys separated by finite barriers. Since the only relevant scale in the critical region is the correlation length the maximum energy barriers must be proportional to \( R^\alpha \) and they are getting divergent as the critical temperature is approached. In this situation one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley) qualitatively much bigger relaxation times would be required for overcoming barriers separated different valleys. Therefore, the traditional measurements of the observables in the "thermal equilibrium" can actually correspond to the equilibration within one valley only and not to the true thermal equilibrium. Then in different measurements (for the same sample) one could be effectively "trapped" in different valleys and thus the traditional spin-glass situation is restored.

To check whether the above speculations are correct or not, like in spin-glasses one can invent traditional "overlap" quantities which could hopefully reveal the existence of the multiple valley structures. For instance, one can introduce the spatially averaged quantity for pairs of different realizations of the disorder:

\[
K_{ij}(R) \equiv \frac{1}{V} \int d^D r \langle \phi(r) \phi(r + R) \rangle_i \langle \phi(r) \phi(r + R) \rangle_j
\]

where \( i \) and \( j \) label different realizations, and it is assumed that the measurable thermal average corresponds to a particular valley, and not to the true thermal average. If the RS situation takes place (so that only one global valley exists), then for different pairs of realizations one will be obtaining the same result \((3.17)\). On the other hand, in the case of the 1RSB, according to the general theory of the RSB \([5]\), after obtaining statistics over pairs of realizations for \( K_{ij}(R) \) one has to be getting the result \( K_0(R) \) with the probability \( x_0 \), and \( K_1(R) \) with the probability \( 1 - x_0 \).

Consider finally what would be the situation if a general type of the RSB takes place. According to the qualitative solution \((2.17)-(2.18)\), the function \( g(x; \xi) \) does not arrive to any fixed point at scales \( \xi >> \xi_u \sim \frac{u}{\alpha} \ln \frac{1}{u} \). Therefore, at the disorder dominated scales \( R >> R_u \sim u^{-\nu/\alpha} >> 1 \) there must be no scaling behaviour of the correlation function \( K(R) \). Near the critical scale \( \xi_* \sim 1/u \) the qualitative behaviour of the solution \( g(x; \xi) \) is shown in eq.\((2.17)\). Therefore, according to eq.\((3.16)\), near the critical scale \( R_* \sim \exp(1/u) \) for the correlation function \( K(x; R) \) one obtains:

\[
K(x; R) \sim \begin{cases} \frac{R^{-2(D-2)}(1 - u \ln R)^{-4a}}{4} \equiv K_1(R); & \text{for } (1 - x) << \Delta(R) \\ R^{-2(D-2)} = G^2_0(R) \equiv K_0; & \text{for } (1 - x) >> \Delta(R) \end{cases}
\]

\[(3.24)\]
where $\Delta(R) = (1 - u \ln R) \to u << 1$ as $R \to R_*$. 

At the critical scale one has $(1 - u \ln R_*) \sim u$, and according to eq.(3.24) the shape of the replica function $K(x; R)$ must be "quasi-1step":

$$K(x; R_*) \sim \begin{cases} 
    u^{-4a} \exp\left\{-\frac{2(D-2)}{u}\right\} \equiv K_1^*; & \text{for } (1 - x) < u \\
    \exp\left\{-\frac{2(D-2)}{u}\right\} \equiv K_0^*; & \text{for } (1 - x) >> u 
\end{cases} \quad (3.25)$$

According to the above discussion of the observable quantities for the 1-step RSB case, the result (3.25) could be measured for the spatially averaged overlaps of the correlation functions $K_{ij}(R)$, eq.(3.23), for the statistics of pairs of realizations of the disorder. Then, for the correlation function $K_{ij}(R)$ one is expected to be obtaining the value $K_1^*$ with the small probability $u$ and the value $K_0^*$ with the probability $(1 - u)$. Although both values $K_1^*$ and $K_0^*$ are expected to be exponentially small, their ratio $K_1^*/K_0^* \sim u^{-4a}$ must be big.

Finally, at scales $R >> R_*$ we are entering into the strong coupling regime, where simple one-loop RG approach can not be used any more.

### 3.3 Specific Heat

According to the standard procedure the leading singularity of the specific heat can be calculated as follows:

$$C \sim \int d^D R \left[ \langle \phi^2(0) \phi^2(R) \rangle - \langle \phi^2(0) \rangle \langle \phi^2(R) \rangle \right] \quad (3.26)$$

In terms of the RG scheme for the correlation function:

$$W(R) \equiv \langle \phi^2(0) \phi^2(R) \rangle - \langle \phi^2(0) \rangle \langle \phi^2(R) \rangle \quad (3.27)$$

one gets:

$$W(R) = (G_0(R))^2 m^2(R) \quad (3.28)$$

where $G_0(R) = R^{-(D-2)}$ is the free field two-point correlation function, and the mass-like object $m(R)$ is given by the solution of the following (one-loop) RG equation (c.f. eq.(3.2)):

$$\frac{d}{d\xi} \ln m(\xi) = -[(2 + p) \tilde{g}(\xi) - p \sum_{a \neq 1}^n g_{a1}(\xi)] \quad (3.29)$$

Here, as usual, $\xi = \ln R$, and the renormalized interaction parameters $\tilde{g}(\xi)$ and $g_{a\neq b}(\xi)$ are the solutions of the replica RG equations \(\text{eq.(2.5)-\text{(2.6)}}\). In the Parisi representation, $g_{a\neq b}(\xi) \to g(x; \xi)$, one gets:

$$m(R) = \exp\left\{- (2 + p) \int_0^{\ln R} d\xi \tilde{g}(\xi) - p \int_0^{\ln R} d\xi \int_0^1 dx g(x; \xi) \right\} \quad (3.30)$$

Then, after simple transformations for the singular part of the specific heat, eq.(3.24), one gets:
\[
C \sim \int_0^{\xi_{\text{max}}} d\xi \exp\{\epsilon \xi - 2(2 + p) \int_0^{\xi} d\eta \bar{g}(\eta) - 2p \int_0^{\xi} d\eta \tilde{g}(\eta)\} \tag{3.31}
\]

where \(\bar{g}(\eta) \equiv \int_0^{1} dx g(x; \eta)\). The infrared cut-off \(\xi_{\text{max}}\) in \(3.31\) is the scale at which the system gets out of the scaling regime.

Usually \(\xi_{\text{max}}\) is the scale at which the renormalized mass \(\tau(\xi)\), eq.\((3.3)\), is getting of the order of one, and if the traditional scaling situation takes place, one finds that \(\xi_{\text{max}} \sim \ln(1/\tau_0)\).

Again, consider first what was the situation in the traditional RS case. Here at scales \(\xi \gg \xi_u \sim \ln(1/u)\) (which correspond to the temperature region \(\tau_0 \ll \tau_u \sim u^{\nu/\alpha}\)) the renormalized parameters \(\bar{g}(\eta)\) and \(\bar{g}(\xi)\) are arriving into the universal fixed point \(\bar{g}_* = \epsilon \frac{p}{16(p-1)}; g_* = \epsilon \frac{4-p}{16(p-1)}\), (see Section 2, eq.\((2.14)\)) and according to \(3.31\) for the singular part of the specific heat one finds \([2],[3]\):

\[
C(\tau_0) \sim \int_{\ln(1/\tau_0)}^{\ln(1/\tau_0)} d\xi \exp\{\xi [\epsilon - 2(2 + p)\bar{g}_* - 2pg_*]\} \sim \tau_0^{\epsilon \frac{4-p}{4(p-1)}} \tag{3.32}
\]

So that in the close vicinity of \(T_c\) one would expect to observe new universal disorder induced critical behaviour with negative specific heat critical exponent \(\alpha = -\epsilon \frac{4-p}{4(p-1)}\) (unlike positive \(\alpha\) in the corresponding pure system).

Similarly, if the scenario with the stable 1-step RSB fixed points takes place, then one finds that the specific heat critical exponent \(\alpha(x_0)\) is getting to be non-universal, explicitly depending on the coordinate of the step \(x_0\) \([9]\):

\[
\alpha(x_0) = -\frac{1}{2} \frac{(4-p)(4-px_0)}{16(\nu-1) - px_0(p+8)}. \tag{3.33}
\]

In the general RSB case the situation is getting completely different. Here in the disorder dominated region \(\tau_* \ll \tau_0 \ll u^{\nu/\alpha}\) (which corresponds to scales \(\xi_u << \xi << \xi_*\)) the RG trajectories of the interaction parameters \(\bar{g}(\xi)\) and \(\bar{g}(\xi)\) does not arrive to any fixed point, and according to eq.\((3.32)\) one finds that the specific heat is getting to be a complicated function of the temperature parameter \(\tau_0\) which does not have the traditional scaling form.

Finally, in the SG-like region in the close vicinity of \(T_c\), where the interaction parameters \(g\) and \(\bar{g}\) are getting finite, one finds that the integral over \(\xi\) in eq.\((3.31)\) is getting converging (so that the upper cut-off scale \(\xi_{\text{max}}\) is getting irrelevant). Thus, in this case one obtains the result that the ”would be singular part” of the specific heat remains finite in the temperature interval \(\sim \tau_*\) around \(T_c\), so that the specific heat is getting non-singular at the phase transition point.

### 4 Marginal case \(p = 4\)

In the systems with the number of spin components \(p = 4\) (in which the pure system specific heat critical exponent \(\alpha = 0\)) the disorder appears to be marginally irrelevant in a sense that it does not change the critical exponents. Nevertheless, the critical behaviour
(described in terms of the logarithmic singularities) is effected by the disorder, and moreover, the RSB phenomena appear to relevant in this case as well.

Consider first the replica symmetric situation: $g(x;\xi) \equiv g(\xi)$. For the RG equations \(2.10\),\(2.11\) one gets:

\[
\frac{dg}{d\xi} = (\epsilon - 12\bar{g})g - 4g^2 \\
\frac{d\bar{g}}{d\xi} = (\epsilon - 12\bar{g})\bar{g} + 4g^2
\]

(4.1)

In the pure system ($g \equiv 0$) the fixed point is:

\[
\bar{g}_{\text{pure}} = \frac{1}{12}\epsilon
\]

(4.2)

Using eq.(3.32) for the singular part of the specific heat of the pure system one easily finds:

\[
C_{\text{pure}}(\tau) \sim \ln\left(\frac{1}{\tau}\right)
\]

(4.3)

Thus, although the specific heat critical exponent of the pure system is zero, the specific heat is still divergent in the critical point.

For the system with disorder the (replica symmetric) asymptotic solution of the eqs.(4.1) is:

\[
g(\xi) \simeq \frac{1}{4}\xi^{-1} \to 0 \\
\bar{g}(\xi) \simeq \frac{1}{12}\epsilon + q(\xi)
\]

(4.4)

where

\[
q(\xi) \sim \xi^{-2} \to 0
\]

(4.5)

In this case the renormalized parameters are asymptotically approaching the pure system fixed point $\bar{g} = \epsilon/12$, $g = 0$ (so that the disorder is marginally irrelevant). Nevertheless, due to slow power-law approach to the fixed point the logarithmic singularity of the specific heat is changing into another universal type. From the general expression (3.31) for the singular part of the specific heat one obtains:

\[
C \sim \int_{0}^{\ln(1/\tau)} d\xi \exp\left\{\int_{0}^{\xi} d\eta[\epsilon - 12\bar{g}(\eta) - 8g(\eta)]\right\}
\]

(4.6)

Using the result (4.4) one easily finds:

\[
C_{\text{rs}}(\tau) \sim \frac{1}{\ln(\frac{1}{\tau})}
\]

(4.7)

One can also easily check that (unlike the systems with $p < 4$) the crossover from the pure system critical behaviour, eq.(4.3), to the disorder induced one, eq.(4.7), takes place in the exponentially small temperature interval near $T_c$:

\[
\tau_u \sim \exp\left(-\frac{1}{u}\right)
\]

(4.8)
Consider now the effects of the RSB. The analytic solution of the RG equations (2.10), (2.11) (see Appendix B) shows that there is no strong coupling regime in the $p = 4$ case, and the asymptotic behaviour (at scales $\xi \gg 1$) of the renormalized parameters can be found exactly:

\[
\begin{aligned}
g(x; \xi) &\sim \begin{cases}
\xi^{-2}, & (1 - x) \gg \frac{1}{\sqrt{\gamma \xi}} \\
\frac{1}{\sqrt{\gamma \xi}}, & (1 - x) \ll \frac{1}{\sqrt{\gamma \xi}}
\end{cases} \\
\tilde{g}(\xi) &\sim \frac{\epsilon}{12} + q(\xi) \\
q(\xi) &\sim \xi^{-3/2} \to 0
\end{aligned}
\] (4.9)

Here $\gamma \equiv g_0'(x = 1) \sim u$ is the derivative of the starting RSB function $g_0(x)$ at $x = 1$.

Like in the RS case the renormalized parameters are asymptotically approaching the pure system fixed point $\tilde{g} = \epsilon/12$, $g(x) = 0$. Nevertheless, the structure of the asymptotic solution for the renormalized function $g(x; \xi)$ near this fixed point exhibits strong RSB.

However, the specific heat appears to be not affected by the RSB. According to eq.(3.31) the leading singularity of the specific heat is defined by the integral $\int_0^1 dx g(x; \xi) \equiv \bar{g}(\xi)$ and not the function $g(x; \xi)$ itself. It can be shown (see eq.(B.12)) that in the asymptotic regime the value of $\bar{g}(\xi)$ coincides with the RS asymptotics (4.4):

\[
\bar{g}(\xi) \sim \frac{1}{4} \xi^{-1}
\] (4.11)

Therefore, for the specific heat singularity one obtains the result coinciding with the RS one, eq.(4.7).

On the other hand, the asymptotic behaviour of the correlation functions, appears to be quite different from the results of the traditional RS solution. In the RS case, eq.(4.4), according to eq.(3.16) for the replica correlation function

\[
K(R) = \langle \langle \phi(0) \phi(R) \rangle \rangle^2
\] (4.12)

one easily finds the following result:

\[
K(R) \sim (G_0(R))^2 \exp\{4 \int_0^{\ln R} d\xi g(\xi)\} = (G_0(R))^2 \ln R
\] (4.13)

Therefore, in the RS case the disorder provides only the logarithmic correction to the correlation function.

In the case of the RSB solution, eq.(4.9), according to eq.(3.16) for the replica correlation function $K(x; R)$ one easily finds:

\[
K(x; R) \sim \begin{cases}
(G_0(R))^2 \exp\{\text{(const)} \sqrt{\gamma \ln R}\}, & (1 - x) \sqrt{\gamma \ln R} \gg 1 \\
(G_0(R))^2, & (1 - x) \sqrt{\gamma \ln R} \ll 1
\end{cases}
\] (4.14)

Correspondingly, for the "observable" correlation function, eq.(4.12), one eventually obtains:
\[ K(R) = \int_0^1 dx K(x; R) \sim (G_0(R))^2 \exp\{(\text{const}) \sqrt{\gamma \ln R}\} \] (4.15)

This result is essentially different from the RS one, eq.(4.13).

5 Discussion

In this section we summarize our conclusions concerning the random bond p-component Heisenberg ferromagnet and discuss the remaining issues.

Spontaneous replica symmetry breaking coming from the interaction of the fluctuations with the multiple local minima solutions of the mean-field equations has a dramatic effect on the renormalization group flows and on the critical properties. In the systems with the number of spin components \( p < 4 \) the traditional RG flows at dimensions \( D = 4 - \epsilon \), which are usually considered as describing the disorder-induced universal critical behavior, appear to be unstable with respect to the RSB potentials as found in spin glasses. For a general type of the Parisi RSB structures there exists no stable fixed points, and the RG flows lead to the strong coupling regime at the finite scale \( R_\ast \sim \exp(1/u) \), where \( u \) is the small parameter describing the disorder. Unlike the systems with \( 1 < p < 4 \), where there exist stable fixed points having 1-step RSB structures, eq.(2.15), in the Ising case, \( p = 1 \), there exist no stable fixed points, and any RSB interactions lead to the strong coupling regime.

If there is RSB in the fourth-order potential, one could identify a phase with a different symmetry than the conventional paramagnetic phase, and thus there would have to be a temperature \( T_{\text{RSB}} \) at which this change in symmetry occurs. Actually, the RSB situation is the property of the statistics of the saddle-point solutions only, and it is clear that for large enough \( \tau \) there must be no RSB. Therefore, one can try to solve the problem of summing over saddle-point solutions for arbitrary \( \tau \), aiming to find finite value of \( \tau_c \) at which the RSB solution for this problem disappears.

Of course, in general this problem is very difficult to solve, but one can easily obtain an estimate for the value of \( \tau_c \) (assuming that at \( \tau = 0 \) the RSB situation takes place). According to the qualitative study of this problem in the paper [3], the RSB solution can occur only when the effective interactions between the ”islands”, (where the system is effectively below \( T_c \)) are getting non-small. The islands are the regions where \( \delta \tau(r) > \tau \). According to the Gaussian distribution for \( \delta \tau(r) \), the average distance between them must be of the order of \( \exp[-\tau^2/u] \), so that the islands are getting distant at \( \tau > \sqrt{u} \). The interaction between the islands is exponentially small in their separation. Therefore at \( \tau > \sqrt{u} \) they must be getting weakly interacting, and there must be no RSB.

Note now that the shift of \( T_c \) with respect to the corresponding pure system is also of the order of \( \sqrt{u} \). On the other hand, the existence of local solutions to the mean-field equations remines the Griffith phase [12] which is claimed to be observed in the temperature interval between \( T_c \) of the disordered system and \( T_c \) of the corresponding pure system. On these grounds it is tempting to associate the (hypothetical) RSB
transition in the statistics of the saddle-point solutions with the Griffith transition. Correspondingly, it would also be natural to suggest that discovered RSB phenomena in the scaling properties of weakly disordered systems could be associated with the Griffith effects.

The other key question which remains unanswered, is whether or not the obtained strong coupling phenomena in the RG flows could be interpreted as the onset of a kind of the spin-glass phase near $T_c$. Since it is the RSB interaction parameter describing disorder, $g(x; \xi)$, which is the most divergent, it is tempting to argue that in the temperature interval $\tau << \tau^* \sim \exp(-1/u)$ near $T_c$ the properties of the system should be essentially SG-like.

It should be stressed, however, that in the present study we observe only the crossover temperature $\tau^*$, at which the change of the critical regime occurs, and it is hardly possible to associate this temperature with any kind of phase transition. Therefore, if the RSB effects could indeed provide any kind of true thermodynamic order parameter, then this must be true in a whole temperature interval where the RSB potentials exist.

The true spin-glass order (in the traditional sense) arises from the onset of non-zero order parameter $Q_{ab}(x) = \langle \phi_a(x)\phi_b(x) \rangle; a \neq b$, and, at least for the infinite-range model, $Q_{ab}$ develops the hierarchical dependence on replica indices obtained by Parisi [16]. In the present problem we only find that the coupling matrix $g_{ab}$ for the fluctuating fields develops strong RSB structure and its elements are getting non-small at the finite scale. Therefore, it seems more realistic to interpret discovered RSB strong coupling phenomena in the RG just as a new type of the critical behaviour characterized by strong SG-effects in the scaling properties rather then in the ground state.

In spin-glasses it is generally believed that RSB phenomenon can be interpreted as a factorization of the phase space into (ultrametric) hierarchy of ”valleys”, or local minima pure states, separated by macroscopic (infinite) barriers [5]. Although in the systems considered here the local minima configurations responsible for the RSB are not likely to be separated by infinite barriers (otherwise it would mean true SG freezing), it would be natural to interpret obtained phenomenon as effective factorization of the phase space into a hierarchy of valleys separated by finite barriers. Since the only relevant scale in the critical region is the correlation length the maximum energy barriers must be proportional to $R^D_c(\tau)$, and they are getting divergent as the critical temperature is approached. In this situation one could expect that besides the usual critical slowing down (corresponding to the relaxation inside one valley) qualitatively much bigger (exponentially large) relaxation times would be required for overcoming barriers separating different valleys. Therefore, the traditional measurements (made at finite equilibration times) can actually correspond to the equilibration within one valley only, and not to the true thermal equilibrium. Then in a close vicinity of the critical point different measurements of the critical properties of e.g. spatial correlation functions (in the same sample) would exhibit different results as if the state of the system is getting effectively ”trapped” in different valleys, and thus the traditional spin-glass situation will be observed.
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Appendix A: The asymptotic solution for \( p < 4 \) case

In this Appendix we derive the asymptotic solution of the RG eqs. (2.10), (2.11):

\[
\frac{d}{d\xi}g(x) = (\epsilon - (4 + 2p)\tilde{g})g(x) + 4g^2(x) - 2pg(x) \int_0^1 dyg(y) - p \int_0^x dy(g(x) - g(y))^2 \tag{A1}
\]

\[
\frac{d}{d\xi}\tilde{g} = \epsilon\tilde{g} - (8 + p)\tilde{g}^2 + p\tilde{g}^2 \tag{A2}
\]

(where \( \tilde{g}^2 \equiv \int_0^1 dxg^2(x) \)) for the number of components \( p < 4 \).

It can be shown aposteriori that the term \((\epsilon - (4 + 2p)\tilde{g})g(x)\) in the eq.(A1) is irrelevant in the asymptotic regime. So, consider the equation:

\[
\frac{d}{d\xi}g(x) = 4g^2(x) - 2pg(x) \int_0^1 dyg(y) - p \int_0^x dy(g(x) - g(y))^2 \tag{A3}
\]

After taking derivative over \( x \) and after simple transformations one gets:

\[
\frac{d}{d\xi}g'(x) = 2pg'(x)[(\lambda - 1)g(x) - \int_x^1 dy(1 - y)g'(y)] \tag{A4}
\]

where \( \lambda = 4/p > 1 \). Let us introduce:

\[
V(x) \equiv \int_x^1 dy(1 - y)g'(y) \tag{A5}
\]

According to this definition one has:

\[
g'(x) = -\frac{1}{1-x}V'(x) \tag{A6}
\]

\[
g(x) = \int_0^x dyg'(y) = -\int_0^x dy\frac{1}{1-y}V'(y) \tag{A6}
\]

Here for simplicity we consider the case \( g(x = 0) = 0 \) (the behaviour of the solution for \( g(x = 0) \neq 0 \) in the asymptotic regime can be shown to be qualitatively the same).

Then, for the eq.(A5) after simple transformations we get:

\[
\frac{d}{d\xi}V'(x) = -2pV'(x)[\int_0^x dy\frac{\lambda - y}{1 - y}V'(y) + \bar{g}(\xi)] \tag{A7}
\]
where \( \overline{g}(\xi) \equiv \int_0^1 dx g(x, \xi) = \int_0^1 dx (1 - x)g'(x) = V(x = 0; \xi) \).

Let us define now:

\[
W(x; \xi) = \int_0^x dy \frac{\lambda - y}{1 - y} V'(y)
\]

or

\[
V'(x) = \frac{1 - x}{\lambda - x} W'(x)
\]

From eq.(A7) one gets:

\[
\frac{d}{d\xi} W'(x) = -2p W'(x) [W(x) + \overline{g}(\xi)]
\]

Integrating over \( x \) yields:

\[
\frac{d}{d\xi} W(x) = -pW^2(x) - 2pW(x) \overline{g}(\xi)
\]

(Here the integration constant is zero because \( W(x = 0) \equiv 0 \).) This equation can be easily solved for any given function \( \overline{g}(\xi) \):

\[
W(x; \xi) = \frac{W_0(x) \exp[-2p \int_0^\xi d\eta \overline{g}(\eta)]}{1 + pW_0(x) \int_0^\xi dt \exp[-2p \int_0^t d\eta \overline{g}(\eta)]}
\]

where:

\[
W_0(x) \equiv W(x; \xi = 0) = -\int_0^x dy (\lambda - y)g'_0(y)
\]

and \( g_0(x) \equiv g(x; \xi = 0) \). Coming back through the definitions (A8) and (A5) for the function \( g(x; \xi) \) one gets:

\[
g(x; \xi) = \int_0^x dy \frac{g'_0(y)\Theta(\xi)}{[1 - p \int_0^\xi d\eta \Theta(\eta) \int_0^y dz (\lambda - z)g'_0(z)]^2}
\]

where:

\[
\Theta(\xi) = \exp[-2p \int_0^\xi d\eta \overline{g}(\eta)]
\]

Integrating \( \int_0^1 dx g(x; \xi) \equiv \overline{g}(\xi) \) one gets the equation for the unknown function \( \overline{g}(\xi) \):

\[
\overline{g}(\xi) = \int_0^1 dy \frac{(1 - y)g'_0(y)A(\xi)}{[1 - pA(\xi) \int_0^\xi d\eta \Theta(\eta) \int_0^y dz (\lambda - z)g'_0(z)]^2}
\]

Now the problem is to find the asymptotic behavior of \( \overline{g}(\xi) \).

Let us introduce:

\[
G(\xi) \equiv \int_0^\xi d\eta \overline{g}(\eta)
\]

Integrating (A16) we obtain:

\[
G(\xi) = \int_0^1 dy \frac{(1 - y)g'_0(y)A(\xi)}{[1 - pA(\xi) \int_0^\xi d\eta \Theta(\eta) \int_0^y dz (\lambda - z)g'_0(z)]}
\]
where:

\[ A(\xi) = \int_0^\xi d\eta \exp[-2pG(\eta)] \]  \hspace{1cm} (A19)

Let us redefine

\[ \psi(\xi) \equiv (A(\xi))^{-1} = \frac{1}{\int_0^\xi d\eta \exp[-2pG(\eta)]} \]  \hspace{1cm} (A20)

Then:

\[ G(\xi) = \int_0^1 dy \frac{(1 - y)g_0(y)}{\psi(\xi) - p \int_0^1 dz (\lambda - z)g_0(z)} \]  \hspace{1cm} (A21)

Now, let us redefine again:

\[ \psi(\xi) = p \int_0^1 dy (\lambda - y)g_0'(y) + \phi(\xi) \]  \hspace{1cm} (A22)

From eq.(A21) we get:

\[ G(\xi) = \int_0^1 dy \frac{(1 - y)g_0(y)}{p \int_0^1 dz (\lambda - z)g_0'(z) + \phi(\xi)} \]  \hspace{1cm} (A23)

Assuming that \( \phi(\xi) \) is small, (A23) can be estimated as follows:

\[ G(\xi) = G_c + \int_0^1 dy (1 - y)g_0'(y) \left[ \frac{1}{p \int_0^1 dz (\lambda - z)g_0(z) + \phi(\xi)} - \frac{1}{p \int_0^1 dz (\lambda - z)g_0'(z)} \right] \]  \hspace{1cm} (A24)

or

\[ G(\xi) = G_c - \phi(\xi) \int_0^1 dy \frac{(1 - y)g_0'(y)}{[p \int_0^1 dz (\lambda - z)g_0'(z) + \phi(\xi)][p \int_0^1 dz (\lambda - z)g_0'(z)]} \]  \hspace{1cm} (A25)

where

\[ G_c \equiv \int_0^1 dy \frac{(1 - y)g_0(y)}{p \int_0^1 dz (\lambda - z)g_0(z)} \]  \hspace{1cm} (A26)

For \( \phi(\xi) << 1 \) the leading contribution in the integral in (A25) comes from the vicinity of \( y = 1 \). Assuming that \( g_0'(y = 1) = \gamma \neq 0 \), this contribution can be estimated as follows:

\[ G(\xi) \approx G_c - \phi(\xi) \int_0^1 dy \frac{(1 - y)_\gamma}{[p \gamma (\lambda - 1)(1 - y) + \phi(\xi)]p \gamma (\lambda - 1)(1 - y)} \approx \]  \hspace{1cm} (A27)

\[ \approx G_c - \frac{\phi(\xi)}{\gamma p (\lambda - 1)^2} \ln \frac{1}{\phi(\xi)} \]

Such that, as \( \phi \to 0 \), the value of \( G(\xi) \) goes to finite value \( G_c \), but near this point the behavior of this function is non-analytic.
Now let us assume that there exists a certain scale $\xi_c$, such that $\phi(\xi \to \xi_c) \to 0$, and consider the behavior near $\xi_c$. Coming back to the definition (A20) we can estimate:

$$\psi(\xi) = \left[ \int_0^{\xi_c} d\eta \exp(-2pG(\eta)) \right]^{-1} =$$

$$= \left( \int_0^{\xi_c} d\eta \exp(-2pG(\eta)) - \int_0^{\xi_c} d\eta \exp(-2pG(\eta)) \right)^{-1} \simeq$$

$$\simeq \left( \int_0^{\xi_c} d\eta \exp(-2pG(\eta)) - \exp(-2pG_c)(\xi_c - \xi)^{-1} \right) \simeq$$

$$\simeq \frac{1}{\int_0^{\xi_c} d\eta \exp(-2pG(\eta))} + \frac{\exp(-2pG_c)}{[\int_0^{\xi_c} d\eta \exp(-2pG(\eta))]^2} (\xi_c - \xi)$$

(A28)

Comparing this result with (A22), we find that:

$$\phi(\xi) \simeq a(\xi_c - \xi)$$

(A29)

where the parameters $\xi_c$ and $a$ are defined by:

$$\frac{1}{\int_0^{\xi_c} d\eta \exp(-2pG(\eta))} = p \int_0^1 dy (\lambda - y) g_0'(y)$$

(A30)

and

$$a = \frac{\exp(-2pG_c)}{[\int_0^{\xi_c} d\eta \exp(-2pG(\eta))]^2} = [p \int_0^1 dy (\lambda - y) g_0'(y)]^2 \exp(-2pG_c)$$

(A31)

Let us estimate the parameters $\xi_c$ and $a$ by the order of magnitude. The characteristic value of the initial function $g_0(x)$ is of the order of $u << 1$, which is the characteristic value of the quenched disorder. If the initial function $g_0(x)$ does not have special anomaly near $x = 1$, then its derivative $\gamma$ must also be of the order of $u$. Then, the above integrals can be estimated as follows:

$$G_c = \int_0^1 dy \frac{(1 - y) g_0'(y)}{p \int_y^1 dz (\lambda - z) g_0'(z)} \sim 1$$

(A32)

$$\int_0^1 dy (\lambda - y) g_0'(y) \sim u$$

(A33)

$$\int_0^{\xi_c} d\eta \exp(-2pG(\eta)) \sim \xi_c$$

(A34)

Thus, from (A30) and (A31) for the parameters $\xi_c$ and $a$ we find:

$$\xi_c \sim \frac{1}{u}$$

(A35)

$$a \sim u^2$$

(A36)

Now we can describe the qualitative behavior of the asymptotic solution. According to (A27):
\[ \mathcal{G}(\xi) = \frac{d}{d\xi} G(\xi) \simeq \frac{a}{\gamma p^2 (\lambda - 1)^2} \ln \frac{1}{(\xi - \xi_c)} \sim \]

\[ \sim u \ln \frac{1}{1-u\xi} \]

Therefore the value of the integral \( \int_0^1 dx g(x; \xi) \equiv \mathcal{G}(\xi) \) is formally getting divergent at finite scale \( \xi_c \sim 1/u \).

Coming back to the result (A14) for the function \( g(x; \xi) \) we have:

\[ g(x; \xi) = \frac{\Theta(\xi)}{f_0^x dy \Theta(\eta)^2} \int_0^x dy \frac{g'_0(y)}{p \int_0^1 dz (\lambda - z) g'_0(z) + \phi(\xi)^2} \]

\[ = -\left[ \frac{d}{d\xi} \int_0^1 d\eta \Theta(\eta) \right] f_0^x dy \frac{g'_0(y)}{p \int_0^1 dz (\lambda - z) g'_0(z) + \phi(\xi)^2} \]

\[ = -\frac{d}{d\xi} \psi(\xi) \int_0^x dy \frac{g'_0(y)}{p \int_0^1 dz (\lambda - z) g'_0(z) + \phi(\xi)^2} \]

\[ \simeq a \int_0^x dy \frac{g'_0(y)}{p \int_0^1 dz (\lambda - z) g'_0(z) + a(\xi_c - \xi)^2} \]

(38)

Therefore, when approaching the critical scale \( \xi \rightarrow \xi_c \), the values of \( g(x; \xi) \) are formally getting big in the narrow interval \( (1-x) \ll \Delta(\xi) \), where:

\[ \Delta(\xi) \sim \frac{a}{\gamma} (\xi_c - \xi) \sim (1-u\xi) \]

(39)

In this interval:

\[ g(x; \xi) \simeq g(x = 1; \xi) \equiv g_1(\xi) \simeq \]

\[ \simeq a \int_1^1 dy \frac{1}{p(\lambda - 1) (1-y) + a(\xi_c - \xi)^2} \simeq \]

\[ \simeq \frac{1}{p(\lambda - 1)} \frac{1}{\xi_c - \xi} \]

\[ \sim a \frac{u}{1-u\xi} \]

Therefore, from the equation (A2) we see that \( \tilde{g} \) diverges as the logarithm:

\[ \tilde{g} = \int_0^1 dx g^2(x; \xi) \sim (1-u\xi) \frac{a^2 u^2}{(1-u\xi)^2} = a^2 \frac{u^2}{1-u\xi} \]

(41)

Therefore, from the equation (A2) we see that \( \tilde{g} \) diverges as the logarithm:

27
\[ g(\xi) \sim u \ln \frac{1}{1 - u\xi} \]  

(A42)

Thus, in the region near \( x = 1 \) where the value of \( g(x; \xi) \) was obtained to be of the order of \( u/(1 - u\xi) \) the first term \( (\epsilon - (4 + 2p)\tilde{g})g(x) \) in the eq.(A1) is much smaller than the other terms:

\[ \tilde{g}g(x) \sim \frac{u^2}{1 - u\xi} \ln \frac{1}{1 - u\xi} \ll \frac{u^2}{(1 - u\xi)^2} \sim g^2(x) \]  

(A43)

Of course, the above asymptotic solution does not make possible to obtain the behavior of the function \( g(x; \xi) \) in the whole interval \([0, 1]\) for all \( \xi \). Nevertheless, the numerical solution of the general RG equations (A1), (A2) clearly demonstrates that at large scales the function \( g(x; \xi) \) quickly goes to zero for all \( x \) not too close to 1, while in the narrow region near \( x = 1 \) the values of this function are getting divergent. Thus, the behaviour of the asymptotic solution for \( g(x; \xi) \) in the vicinity of the critical scale \( \xi_c \) could be qualitatively represented as follows:

\[
\begin{cases}
  a\frac{u}{1-u\xi}; & \text{for } (1-x) \ll \Delta(\xi) \\
  0; & \text{for } (1-x) >> \Delta(\xi)
\end{cases}
\]  

(A44)

where \( \Delta(\xi) = (1-u\xi) \to u \ll 1 \) as \( \xi \to \xi_c \), and \( a \) is a positive non-universal constant.

The obtained asymptotics can also be easily generalized for the situation when \( g(x = 0) \neq 0 \). One has to write: \( g(x; \xi) = (\text{the obtained solution}) + g(x = 0; \xi) \), then put it into the equation, obtain the equation for \( g(x = 0; \xi) \), and find the asymptotics for \( g(x = 0; \xi) \). It’s straightforward to check that qualitatively it doesn’t change the above results.

**Appendix B: The asymptotic solution for \( p = 4 \)**

In \( p = 4 \) case the asymptotic solution of the equations (2.10)-(2.11) can be obtained as follows.

Redefining the diagonal parameter \( \tilde{g}(\xi) \):

\[ \tilde{g}(\xi) = \frac{\epsilon}{12} + q(\xi) \]  

(B1)

we get:

\[ \frac{d}{d\xi}g(x) = -12q(\xi)g(x) + 4g^2(x) - 8g(x) \int_0^1 dyg(y) - 4 \int_0^x dy(g(x) - g(y))^2 \]  

(B2)

\[ \frac{d}{d\xi}q(\xi) = -\epsilon q - 12q^2 + 4\tilde{g}^2 \]  

(B3)

Then, proceeding like in the Appendix A, instead of eq.(A7) we obtain:
\[
\frac{d}{d\xi} V'(x; \xi) = -8V'(x; \xi)V(x; \xi) - 12V'(x; \xi)q(\xi) \quad (B4)
\]

Integration over \(x\) yields:
\[
\frac{d}{d\xi} V(x; \xi) = -4V^2(x; \xi) - 12q(\xi)V(x; \xi) \quad (B5)
\]

(the integration constant is zero, since \(V(x = 1) \equiv 0\)). The solution of this equation for any given function \(q(\xi)\) is:
\[
V(x; \xi) = \frac{V_0(x) \exp\{-12 \int_0^\xi d\eta q(\eta)\}}{1 + 4V_0(x) \int_0^\xi d\eta \exp\{-12 \int_0^\eta dtq(t)\}} \quad (B6)
\]

where \(V_0(x) \equiv V(x; \xi = 0) = \int_x^1 dy(1 - y)g_0'(x)\).

Coming back to the function \(g(x; \xi)\) we get:
\[
g(x; \xi) = \int_0^x dy g'(y) + g(x = 0; \xi) = -\int_0^x dy \frac{1}{1 - y} V'(y) + g(x = 0; \xi) \quad (B7)
\]

Using (B6) we find:
\[
g(x; \xi) = \int_0^x dy \frac{g_0'(y) \exp\{-12 \int_0^\xi d\eta q(\eta)\}}{[1 + 4 \int_y^1 dz(1 - z)g_0'(z) \int_0^\xi d\eta \exp\{-12 \int_0^\eta dtq(t)\}]^2} + g(x = 0; \xi) \quad (B8)
\]

Putting this result back into the original equation (B2) we get the equation for \(g(x = 0; \xi)\):
\[
\frac{d}{d\xi} g(x = 0; \xi) = -12q(\xi)g(x = 0; \xi) - 4g^2(x = 0; \xi) - 8g(x = 0; \xi)\eta(\xi) \quad (B9)
\]

where
\[
\eta(\xi) = \int_0^1 dx \int_0^x dy \frac{g_0'(y) \exp\{-12 \int_0^\xi d\eta q(\eta)\}}{[1 + 4 \int_y^1 dz(1 - z)g_0'(z) \int_0^\xi d\eta \exp\{-12 \int_0^\eta dtq(t)\}]^2} \quad (B10)
\]

Let us assume now that the parameter \(q(\xi)\) decays as \(\sim \xi^{-s}\) with \(s > 1\). Then the integral \(\int_0^\xi d\eta q(\eta)\) is converging at large \(\xi\), and for the exponent in \(\eta(\xi)\) we find that it is equal to a constant of the order of one: \(\exp\{-12 \int_0^\xi d\eta q(\eta)\} = A\).

Correspondingly, instead of eqs.\((B8),(B10)\) we get:
\[
g(x; \xi) \approx \int_0^x dy \frac{Ag_0'(y)}{[1 + 4A\xi \int_y^1 dz(1 - z)g_0'(z)]^2} + g(x = 0; \xi) \quad (B11)
\]

and
\[
\mathcal{g}(\xi) \simeq \int_0^1 dx \int_0^x dy \frac{Ag_0'(y)}{[1+4A\xi \int_y^1 dz (1-z)g_0'(z)]^2}
\]
\[= \frac{Ag_0'}{1+4A\xi g_0'}
\]
where \( \mathcal{g}_0 \equiv \int_0^1 dx g_0(x) = \int_0^1 dx (1-x)g_0'(x) \).

Simple analysis of the integral in eq.(B11) shows that actually it is the non-zero derivative \( g_0'(x) \) near the point \( x = 1 \) which is important in the asymptotic regime. Whatever the function \( g_0(x) \) is in the region \((1-x) \gg (\gamma \xi)^{-1/2}\), it is always decaying like \( \xi^{-2} \) there, while for \((1-x) \ll (\gamma \xi)^{-1/2}\) the decay is \( (\gamma \xi)^{-1/2} \), where \( \gamma = g_0'(x = 1) \):

\[
g(x; \xi) \sim \begin{cases} 
\xi^{-2}, & (1-x) \gg \frac{1}{\sqrt{\gamma \xi}} \\
\frac{1}{\sqrt{\gamma \xi}}, & (1-x) \ll \frac{1}{\sqrt{\gamma \xi}}
\end{cases}
\]

(B13)

Besides, using (B12), from eq.(B9) one finds that \( g(x = 0; \xi) \sim \xi^{-2} \).

Note now that according to the above asymptotic behaviour of the function \( g(x; \xi) \) at scales \( \xi \gg 1/u \) the leading contribution to the quantity \( \mathcal{g}^2 \equiv \int_0^1 g^2(x; \xi) \) comes from the region \((1-x) \ll \frac{1}{\sqrt{u \xi}} \):

\[
\mathcal{g}^2 \sim \frac{1}{\sqrt{\xi}} \left( \frac{1}{\sqrt{\xi}} \right)^2 = \xi^{-3/2}
\]

(B14)

Then, coming back to the eq.(B3) we find:

\[
q(\xi) \simeq \exp(-\epsilon \xi) \int_0^\xi dt t^{-3/2} \exp(+\epsilon t) \sim a_1 \xi^{-3/2} + a_2 \xi^{-5/2} + ... \sim \xi^{-3/2}
\]

(B15)

which is selfconsistent with the assumption \( q(\xi) \sim \xi^{-s} \) (with \( s > 1 \)) made above.

Therefore, the asymptotic behaviour of the solution for \( p = 4 \) at scales \( \xi \gg 1/u \) is given by eq.(B13).
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