SECOND MAIN THEOREM AND UNICITY OF MEROMORPHIC MAPPINGS FOR HYPER_SURFACES OF PROJECTIVE VARIETIES IN SUBGENERAL POSITION

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ABSTRACT. The purpose of this article is twofold. The first is to prove a second main theorem for meromorphic mappings of \( \mathbb{C}^m \) into a complex projective variety intersecting hypersurfaces in subgeneral position with truncated counting functions. The second is to show a uniqueness theorem for these mappings which share few hypersurfaces without counting multiplicity.

1. Introduction

Let \( f \) be a linearly nondegenerate meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \) and let \( \{H_j\}_{j=1}^q \) be \( q \) hyperplanes in \( N \)-subgeneral position in \( \mathbb{P}^n(\mathbb{C}) \). Then the Cartan-Nochka’s second main theorem for meromorphic mappings and hyperplanes (see [8], [9]) stated that

\[
|| (q - 2N + n - 1)T(r, f) \leq \sum_{i=1}^{q} N_{H_i(f)}^{[n]}(r) + o(T(r, f)).
\]

The above Cartan-Nochka’s second main theorem plays a very essential role in Nevanlinna theory, with many applications to Algebraic or Analytic geometry. One of the most interesting applications of the above theorem is to study the uniqueness problem of meromorphic mappings sharing hyperplanes. We state here the uniqueness theorem of L. Smiley, which is one of the most early results on this problem.

**Theorem A.** Let \( f, g \) be two meromorphic mappings of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Let \( H_1, ..., H_q \) be \( q \) \((q \geq 3n+2)\) hyperplanes of \( \mathbb{P}^n(\mathbb{C}) \) located in general position. Assume that \( f^{-1}(\bigcup_{i=1}^{q} H_i) = g^{-1}(\bigcup_{i=1}^{q} H_i) \) and

\[
\dim f^{-1}(H_i) \cap f^{-1}(H_j) \leq m - 2, \ \forall i \neq j.
\]

Then \( f = g \).

Many authors have generalized the above result to the case of meromorphic mappings and hypersurfaces.

In 2004, Min Ru [11] showed a second main theorem for algebraically nondegenerate meromorphic mappings and a family of hypersurfaces of a complex projective space \( \mathbb{P}^n(\mathbb{C}) \) in general position. With the same assumptions, T. T. H. An and H. T. Phuong [1]
improved the result of Min Ru by giving an explicit truncation level for counting functions. They proved the following.

**Theorem B** (An - Phuong [1]) Let \( f \) be an algebraically nondegenerate holomorphic map of \( \mathbb{C} \) into \( \mathbb{P}^n(\mathbb{C}) \). Let \( \{Q_i\}_{i=1}^q \) be \( q \) hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) in general position with \( \deg Q_i = d_i \quad (1 \leq i \leq q) \). Let \( d \) be the least common multiple of the \( d_i \)'s, \( d = \text{lcm}(d_1, \ldots, d_q) \). Let \( 0 < \epsilon < 1 \) and let

\[
L \geq 2d[2^n(n+1)n(d+1)\epsilon^{-1}]^n.
\]

Then,

\[
\| (q - n - 1 - \epsilon)T_f(r) \leq \sum_{i=1}^q \frac{1}{d_i} N_{Q_i(f)}^i(r) + o(T_f(r)).
\]

Using this result of An - Phuong, Dulock and Min Ru [2] gave a uniqueness theorem for meromorphic mappings sharing a family of hypersurfaces in general position. Then the natural question arise here: "How to generalize these results to the case where mappings take values in projective varieties and the family of hypersurfaces is in subgeneral position?"

Now, let \( V \) be a complex projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \) of dimension \( k \quad (k \leq n) \). Let \( Q_1, \ldots, Q_q \quad (q \geq k + 1) \) be \( q \) hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \). We say that the family \( \{Q_i\}_{i=1}^q \) is in general position in \( V \) if

\[
V \cap (\bigcap_{j=1}^{k+1} Q_i) = \emptyset \quad \forall 1 \leq i_1 < \cdots < i_{k+1} \leq q.
\]

In [5], G. Dethloff - D. D. Thai and T. V. Tan gave a concept of the notion "subgeneral position" for a family hypersurfaces as follows.

**Definition C.** \((N\text{-subgeneral position in the sense of Dethloff - Thai - Tan [5]})\). Let \( V \) be a projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \) of dimension \( k \quad (k \leq n) \). Let \( N \geq k \) and \( q \geq N + 1 \). Hypersurfaces \( Q_1, \ldots, Q_q \) in \( \mathbb{P}^n(\mathbb{C}) \) with \( V \nsubseteq Q_j \) for all \( j = 1, \ldots, q \) are said to be in \( N\text{-subgeneral position in} \ V \) if the two following conditions are satisfied:

(i) For every \( 1 \leq j_0 < \cdots < j_N \leq q, V \cap Q_{j_0} \cap \cdots \cap Q_{j_N} = \emptyset \).

(ii) For any subset \( J \subseteq \{1, \ldots, q\} \) such that \( 1 \leq |J| \leq k \) and \( \{Q_j, j \in J\} \) are in general position in \( V \) and \( V \cap (\bigcup_{j \in J} Q_j) \neq \emptyset \), there exists an irreducible component \( \sigma_J \) of \( V \cap (\bigcup_{j \in J} Q_j) \) with \( \dim \sigma_J = \dim (V \cap (\bigcup_{j \in J} Q_j)) \) such that for any \( i \in \{1, \ldots, q\} \setminus J \), if \( \dim (V \cap (\bigcup_{j \in J} Q_j)) = \dim (V \cap Q_i \cap (\bigcup_{j \in J} Q_j)) \), then \( Q_i \) contains \( \sigma_J \).

With this notion of \( N\)-subgeneral position, the above three authors proved the following second main theorem.

**Theorem D** (Dethloff - Thai - Tan [5]). Let \( V \) be a complex projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \) of dimension \( k \quad (k \leq n) \). Let \( \{Q_i\}_{i=1}^q \) be hypersurfaces of \( \mathbb{P}^n(\mathbb{C}) \) in \( N\text{-subgeneral...
position in $V$ in the sense of Definition C, with $\deg Q_i = d_i$ $(1 \leq i \leq q)$. Let $d$ be the least common multiple of $d_i$'s, i.e., $d = \text{lcm}(d_1, \ldots, d_q)$. Let $f$ be an algebraically nondegenerate meromorphic mapping of $\mathbb{C}^n$ into $V$. If $q > 2N - k + 1$ then for every $\epsilon > 0$, there exist positive integers $L_j$ $(1 \leq j \leq q)$ depending on $k, n, N, d_i$ $(1 \leq i \leq q), q, \epsilon$ in an explicit way such that
\[
|| (q - 2N + k - 1 - \epsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d_i} N_{Q_i(f)}^{[L_i]}(r) + o(T_f(r)).
\]

We would like to note that in Definition C, the second condition (ii) is not natural and it is very hard to examine this condition. Also the truncation levels $L_i$, as same as the truncation level $L$ in Theorem B, is very large and far from the sharp. Therefore, the application of them to truncated multiplicity problems will be restricted.

The first purpose in the present paper is to give a new second main theorem for meromorphic mappings into complex projective varieties, and a family of hypersurfaces in subgeneral position (in the sense of a natural definition as below) with a better truncation level for counting functions. Firstly, let us state the following.

Now, let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ ($k \leq n$). Let $d$ be a positive integer. We denote by $I(V)$ the ideal of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ defining $V$, $H_d$ the ring of all homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degree $d$ (which is also a vector space). We define
\[
I_d(V) := \frac{H_d}{I(V) \cap H_d} \quad \text{and} \quad H_V(d) := \dim I_d(V).
\]
Then $H_V(d)$ is called Hilbert function of $V$. Each element of $I_d(V)$ which is an equivalent class of an element $Q \in H_d$, will be denoted by $[Q]$.

Let $f : \mathbb{C}^n \rightarrow V$ be a meromorphic mapping. We said that $f$ is degenerate over $I_d(V)$ if there is $[Q] \in I_d(V) \setminus \{0\}$ so that $Q(f) \equiv 0$, otherwise we said that $f$ is nondegenerate over $I_d(V)$. It is clear that if $f$ is algebraically nondegenerate then $f$ is nondegenerate over $I_d(V)$ for every $d \geq 1$.

The family of hypersurfaces $\{Q_i\}_{i=1}^{q}$ is said to be in $N$-subgeneral position with respect to $V$ if for any $1 \leq i_1 < \cdots < i_{N+1}$,
\[
V \cap \left( \bigcap_{j=1}^{N+1} Q_{i_j} \right) = \emptyset.
\]
We will prove the following Second Main Theorem.

Theorem 1.1. Let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ ($k \leq n$). Let $\{Q_i\}_{i=1}^{q}$ be hypersurfaces of $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$, with $\deg Q_i = d_i$ $(1 \leq i \leq q)$. Let $d$ be the least common multiple of $d_i$'s, i.e., $d = \text{lcm}(d_1, \ldots, d_q)$. Let $f$ be a meromorphic mapping of $\mathbb{C}^n$ into $V$ which is nondegenerate over $I_d(V)$. If
\( q > \frac{(2N-k+1)H_V(d)}{k+1} \) then we have

\[
\left\| \left( q - \frac{(2N-k+1)H_V(d)}{k+1} \right) T_f(r) \right\| \leq \sum_{i=1}^{q} \frac{1}{d_i} \left( N^{[H_V(d)-1]}(r) + o(T_f(r)) \right).
\]

In the case where \( V \) is a linear space of dimension \( k \) and each \( H_i \) is a hyperplane, i.e., \( d_i = 1 \) (1 \( \leq i \leq q \)), then \( H_V(d) = k + 1 \) and Theorem 1 gives us the above second main theorem of Cartan - Nochka. We note that even the total defect given from the above Second Main Theorem is \( \frac{(2N-k+1)H_V(d)}{k+1} \geq n+1 \), but the truncated level \( (H_V(d) - 1) \) of the counting function, which is bounded from above by \( \left( \binom{n+d}{n} - 1 \right) \), is much smaller than that in any previous Second Main Theorem for hypersurfaces.

Also the notion of \( N \)-subgeneral position in our result is a natural generalization of the case of hyperplanes. Therefore, in order to prove the second main theorem in our situation we have to make a generalization of Nochka weights for the case of hypersurfaces in complex projective varieties.

In the last section of this paper, we prove a uniqueness theorem for meromorphic mappings sharing hypersurfaces in subgeneral position without counting multiplicity as follows.

**Theorem 1.2.** Let \( V \) be a complex projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \) of dimension \( k \) (\( k \leq n \)). Let \( \{Q_i\}_{i=1}^{q} \) be hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) in \( N \)-subgeneral position with respect to \( V \), \( \deg Q_i = d_i \) (1 \( \leq i \leq q \)). Let \( d \) be the least common multiple of \( d_i \)'s, i.e., \( d = \text{lcm}(d_1, \ldots, d_q) \). Let \( f \) and \( g \) be meromorphic mappings of \( \mathbb{C}^n \) into \( V \) which are nondegenerate over \( I_d(V) \).

Assume that:

(i) \( \dim(\text{Zero} Q_i(f) \cap \text{Zero} Q_i(g)) \leq m - 2 \) for every 1 \( \leq i < j \leq q \),

(ii) \( f = g \) on \( \bigcup_{i=1}^{q} (\text{Zero} Q_i(f) \cup \text{Zero} Q_i(g)) \).

If \( q > \frac{(2H_V(d)-1)}{d} + \frac{(2N-k+1)H_V(d)}{k+1} \) then \( f = g \).

We see that with the same assumption, the number of hypersurfaces in our result is smaller than that in the all previous results on uniqueness of meromorphic mappings sharing hypersurfaces. Also in the case of mapping into \( \mathbb{P}^n(\mathbb{C}) \) sharing hyperplanes in general position, i.e., \( V = \mathbb{P}^n(\mathbb{C}) \), \( H_V(d) = n + 1 \), \( N = n = k \); the above theorem gives us the uniqueness theorem of L. Smiley.

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2. Basic notions and auxiliary results from Nevanlinna theory

2.1. We set $||z|| = (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbb{C}^m$ and define $B(r) := \{z \in \mathbb{C}^m : ||z|| < r\}$, $S(r) := \{z \in \mathbb{C}^m : ||z|| = r\}$ $(0 < r < \infty)$.

Define

$$v_{m-1}(z) := (dd^c||z||^2)^{m-1}$$

and

$$\sigma_m(z) := d^c\log||z||^2 \wedge (dd^c\log||z||^2)^{m-1} \text{ on } \mathbb{C}^m \setminus \{0\}.$$

For a divisor $\nu$ on $\mathbb{C}^m$ and for a positive integer $M$ or $M = \infty$, we define the counting function of $\nu$ by

$$\nu^{[M]}(z) = \min\{M, \nu(z)\},$$

$$n(t) = \begin{cases} \int_{|\nu| \cap B(t)} \nu(z)v_{m-1} & \text{if } m \geq 2, \\ \sum_{|z| \leq t} \nu(z) & \text{if } m = 1. \end{cases}$$

Similarly, we define $n^{[M]}(t)$.

Define

$$N(r, \nu) = \int_1^r \frac{n(t)}{t^{2m-1}}dt \quad (1 < r < \infty).$$

Similarly, we define $N(r, \nu^{[M]})$ and denote it by $N^{[M]}(r, \nu)$.

Let $\varphi : \mathbb{C}^m \rightarrow \mathbb{C}$ be a meromorphic function. Denote by $\nu_{\varphi}$ the zero divisor of $\varphi$.

Define

$$N_{\varphi}(r) = N(r, \nu_{\varphi}), \quad N^{[M]}_{\varphi}(r) = N^{[M]}(r, \nu_{\varphi}).$$

For brevity we will omit the character $^{[M]}$ if $M = \infty$.

2.2. Let $f : \mathbb{C}^m \rightarrow \mathbb{P}^n(\mathbb{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{P}^n(\mathbb{C})$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each $f_i$ is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic subset $\{f_0 = \cdots = f_n = 0\}$ of codimension $\geq 2$. Set $\|f\| = (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$.

The characteristic function of $f$ is defined by

$$T_f(r) = \int_{S(r)} \log \|f\| \sigma_m - \int_{S(1)} \log \|f\| \sigma_m.$$

2.3. Let $\varphi$ be a nonzero meromorphic function on $\mathbb{C}^m$, which are occasionally regarded as a meromorphic map into $\mathbb{P}^1(\mathbb{C})$. The proximity function of $\varphi$ is defined by

$$m(r, \varphi) = \int_{S(r)} \log \max \{|\varphi|, 1\} \sigma_m.$$
The Nevanlinna’s characteristic function of \( \varphi \) is define as follows
\[
T(r, \varphi) = N_\varphi(r) + m(r, \varphi).
\]
Then
\[
T_\varphi(r) = T(r, \varphi) + O(1).
\]
The function \( \varphi \) is said to be small (with respect to \( f \)) if \( ||T_\varphi(r) = o(T_f(r)) \). Here, by the notation “\( || P \)” we mean the assertion \( P \) holds for all \( r \in [0, \infty) \) excluding a Borel subset \( E \) of the interval \([0, \infty)\) with \( \int_E dr < \infty \).

2.4. Lemma on logarithmic derivative (Lemma 3.11 \([12]\)). Let \( f \) be a nonzero meromorphic function on \( \mathbb{C}^m \). Then
\[
\left| m\left( r, \frac{D^\alpha(f)}{f} \right) \right| = O(\log^+ T(r, f)) \quad (\alpha \in \mathbb{Z}_+^m).
\]

Repeating the argument in (Prop. 4.5 \([6]\)), we have the following:

Proposition 2.5. Let \( \Phi_0, ... , \Phi_k \) be meromorphic functions on \( \mathbb{C}^m \) such that \( \{ \Phi_0, ... , \Phi_k \} \) are linearly independent over \( \mathbb{C} \). Then there exist an admissible set
\[
\{ \alpha \} = (\alpha_{i1}, ..., \alpha_{im})_{i=0}^k \subset \mathbb{Z}_+^m
\]
with \( |\alpha_i| = \sum_{j=1}^m |\alpha_{ij}| \leq k \) \( (0 \leq i \leq k) \) such that the following are satisfied:

(i) \( \{ D^{\alpha_i} \Phi_0, ... , D^{\alpha_i} \Phi_k \}_{i=0}^k \) is linearly independent over \( \mathcal{M} \), i.e., \( \det (D^{\alpha_i} \Phi_j) \neq 0 \).

(ii) \( \det (D^{\alpha_i} (h \Phi_j)) \) \( h^{k+1} \cdot \det (D^{\alpha_i} \Phi_j) \) for any nonzero meromorphic function \( h \) on \( \mathbb{C}^m \).

3. Generalization of Nochka weights

Let \( V \) be a complex projective subvariety of \( \mathbb{P}^n(\mathbb{C}) \) of dimension \( k \) \( (k \leq n) \). Let \( \{ Q_i \}_{i=1}^q \) be \( q \) hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of the common degree \( d \). Assume that each \( Q_i \) is defined by a homogeneous polynomial \( Q_i^* = C(x_0, ..., x_n) \). We regard \( I_d(V) = H_d / I(V) \cup H_d \) as a complex vector space and define
\[
\text{rank}\{ Q_i \}_{i \in R} = \text{rank}\{ [Q_i^*] \}_{i \in R}
\]
for every subset \( R \subset \{ 1, ... , q \} \). It is easy to see that
\[
\text{rank}\{ Q_i \}_{i \in R} \geq \dim V - \dim (\bigcap_{i \in R} Q_i \cap V).
\]

Definition 3.1. The family \( \{ Q_i \}_{i=1}^q \) is said to be in \( N\)-subgeneral position with respect to \( V \) if for any subset \( R \subset \{ 1, ... , q \} \) with \( |R| = N + 1 \) then \( \bigcap_{i \in R} Q_i \cap V = \emptyset \).

Hence, if \( \{ Q_i \}_{i=1}^q \) is in \( N\)-subgeneral position, by the above equality, we have
\[
\text{rank}\{ Q_i \}_{i \in R} \geq \dim V - \dim (\bigcap_{i \in R} Q_i \cap V) = k + 1
\]
(here we note that \( \dim(\emptyset) = -1 \)) for any subset \( R \subset \{ 1, ... , q \} \) with \( |R| = N + 1 \).
If \( \{Q_i\}_{i=1}^q \) is in \( n \)-subgeneral position with respect to \( V \) then we say that it is in general position with respect to \( V \).

Taking a \( \mathbb{C} \)-basis of \( I_d(V) \), we may consider \( I_d(V) \) as a \( \mathbb{C} \)-vector space \( \mathbb{C}^M \) with \( M = H_V(d) \).

Let \( \{H_i\}_{i=1}^q \) be \( q \) hyperplanes in \( \mathbb{C}^M \) passing through the coordinates origin. Assume that each \( H_i \) is defined by the linear equation

\[
a_{ij}z_j + \cdots + a_{iM}z_M = 0,
\]

where \( a_{ij} \in \mathbb{C} \) \((j = 1, \ldots, M)\), not all zeros. We define the vector associated with \( H_i \) by

\[
v_i = (a_{i1}, \ldots, a_{iM}) \in \mathbb{C}^M.
\]

For each subset \( R \subset \{1, \ldots, q\} \), the rank of \( \{H_i\}_{i \in R} \) is defined by

\[
\text{rank}\{H_i\}_{i \in R} = \text{rank}\{v_i\}_{i \in R}.
\]

The family \( \{H_i\}_{i=1}^q \) is said to be in \( N \)-subgeneral position if for any subset \( R \subset \{1, \ldots, q\} \) with \( \sharp R = N + 1 \), \( \bigcap_{i \in R} H_i = \{0\} \), i.e., \( \text{rank}\{H_i\}_{i \in R} = M \).

By Lemmas 3.3 and 3.4 in [9], we have the following.

**Lemma 3.2.** Let \( \{H_i\}_{i=1}^q \) be \( q \) hyperplanes in \( \mathbb{C}^{k+1} \) in \( N \)-subgeneral position, and assume that \( q > 2N - k + 1 \). Then there are positive rational constants \( \omega_i \) \((1 \leq i \leq q)\) satisfying the following:

i) \( 0 < \omega_j \leq 1 \), \( \forall i \in \{1, \ldots, q\} \),

ii) Setting \( \tilde{\omega} = \max_{j \in Q} \omega_j \), one gets

\[
\sum_{j=1}^q \omega_j = \tilde{\omega}(q - 2N + k - 1) + k + 1.
\]

iii) \( \frac{k+1}{2N-k+1} \leq \tilde{\omega} \leq \frac{k}{N} \).

iv) For \( R \subset Q \) with \( 0 < \sharp R \leq N + 1 \), then \( \sum_{i \in R} \omega_i \leq \text{rank}\{H_i\}_{i \in R} \).

v) Let \( E_i \geq 1 \) \((1 \leq i \leq q)\) be arbitrarily given numbers. For \( R \subset Q \) with \( 0 < \sharp R \leq N + 1 \), there is a subset \( R^o \subset R \) such that \( \sharp R^o = \text{rank}\{H_i\}_{i \in R^o} = \text{rank}\{H_i\}_{i \in R} \) and

\[
\prod_{i \in R} E_i^{\omega_i} \leq \prod_{i \in R^o} E_i.
\]

The above \( \omega_j \) are called Nochka weights, and \( \tilde{\omega} \) is called Nochka constant.

**Lemma 3.3.** Let \( H_1, \ldots, H_q \) be \( q \) hyperplanes in \( \mathbb{C}^M \), \( M \geq 2 \), passing through the coordinates origin. Let \( k \) be a positive integer, \( k \leq M \). Then there exists a linear subspace \( L \subset \mathbb{C}^M \) of dimension \( k \) such that \( L \not\subset H_i \) \((1 \leq i \leq q)\) and

\[
\text{rank}\{H_i \cap L, \ldots, H_i \cap L\} = \text{rank}\{H_i, \ldots, H_i\}
\]

for every \( 1 \leq l \leq k, 1 \leq i_1 < \cdots < i_l \leq q \).
Proof. We prove the lemma by induction on $M$ ($M \geq k$) as follows.

- If $M = k$, by choosing $L = C^M$ we get the desired conclusion of the lemma.
- If $M = M_0 \geq k + 1$. Assume that the lemma holds for every cases where $k \leq M \leq M_0 - 1$. Now we prove that the lemma also holds for the case where $M = M_0$.

Indeed, we assume that each hyperplane $H_i$ is given by the linear equation

$$a_{i1}x_1 + \cdots + a_{iM_0}x_{M_0} = 0,$$

where $a_{ij} \in C$, not all zeros, $(x_1, \ldots, x_{M_0})$ is an affine coordinates system of $C^{M_0}$. We denote the vector associated with $H_i$ by $v_i = (a_{i1}, \ldots, a_{iM_0}) \in C^{M_0} \setminus \{0\}$. For each subset $T$ of $\{v_1, \ldots, v_q\}$ satisfying $\sharp T \leq k$, we denote by $V_T$ the vector subspace of $C^{M_0}$ generated by $T$. Since $\dim V_T \leq \sharp T \leq k < M_0$, $V_T$ is a proper vector subspace of $C^{M_0}$. Then $\bigcup_T V_T$ is nowhere dense in $C^{M_0}$. Hence, there exists a nonzero vector $v = (a_1, \ldots, a_{M_0}) \in C^{M_0} \setminus \bigcup_T V_T$. Denote by $H$ the hyperplane of $C^{M_0}$ defined by

$$a_{11}x_1 + \cdots + a_{M_0}x_{M_0} = 0.$$

For each $v_i \in \{v_1, \ldots, v_{M_0}\}$, we have $v \not\in V_{\{v_i\}}$ then $\{v, v_i\}$ is linearly independent over $C$. It follows that $H_i \not\subset H$. Therefore, $H_i' = H_i \cap H$ is a hyperplane of $H$. Also we see that $\dim H = M_0 - 1$

By the assumption that the lemma holds for $M = M_0 - 1$, then there exists a linear subspace $L \subset H$ of dimension $k$ such that $L \not\subset H_i'$ ($1 \leq i \leq q$) and

$$\text{rank}\{H_i' \cap L, \ldots, H_i' \cap L\} = \text{rank}\{H_i', \ldots, H_i'\}$$

for every $1 \leq l \leq k, 1 \leq i_1 < \cdots < i_l \leq q$.

Since $L \not\subset H_i'$, it is easy to see that $L \not\subset H_i$ for each $i$ ($1 \leq i \leq q$). On the other hand, for every $1 \leq l \leq k, 1 \leq i_1 < \cdots < i_l \leq q$, we see that $v \not\in V_{\{v_{i_1}, \ldots, v_{i_l}\}}$. Then $\text{rank}\{v_{i_1}, \ldots, v_{i_l}, v\} = \text{rank}\{v_{i_1}, \ldots, v_{i_l}\} + 1$. This implies that

$$\text{rank}\{H_i', \ldots, H_i'\} = \dim H - \dim(\bigcap_{j=1}^{l} H_i') = M_0 - 1 - \dim(H \cap \bigcap_{j=1}^{l} H_i')$$

$$= \text{rank}\{H_{i_1}, \ldots, H_{i_l}, H\} - 1 = \text{rank}\{v_{i_1}, \ldots, v_{i_l}, v\} - 1$$

$$= \text{rank}\{v_{i_1}, \ldots, v_{i_l}\} = \text{rank}\{H_{i_1}, \ldots, H_{i_l}\}.$$

It follows that

$$\text{rank}\{H_i \cap L, \ldots, H_i \cap L\} = \dim L - \dim(L \cap \bigcap_{j=1}^{l} H_i) = \dim L - \dim(\bigcap_{j=1}^{l} (H_i' \cap L))$$

$$= \text{rank}\{H_i' \cap L, \ldots, H_i' \cap L\} = \text{rank}\{H_{i_1}, \ldots, H_{i_l}\}.$$

Then we get the desired linear subspace $L$ in this case.

- By the inductive principle, the lemma holds for every $M$. Hence we finish the proof of the lemma. □
Lemma 3.4. Let $V$ be a complex projective subvariety of $\mathbb{P}^n(\mathbb{C})$ of dimension $k$ ($k \leq n$). Let $Q_1, ..., Q_q$ be $q$ ($q > 2N - k + 1$) hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in $N-$subgeneral position with respect to $V$ of the common degree $d$. Then there are positive rational constants $\omega_i$ ($1 \leq i \leq q$) satisfying the following:

i) $0 < \omega_i \leq 1$, $\forall i \in \{1, ..., q\}$;

ii) Setting $\bar{\omega} = \max_{j \in Q} \omega_j$, one gets

$$\sum_{j=1}^{q} \omega_j = \bar{\omega}(q - 2N + k - 1) + k + 1.$$

iii) $\frac{k + 1}{2N - k + 1} \leq \bar{\omega} \leq \frac{k}{N}$.

iv) For $R \subset \{1, ..., q\}$ with $\sharp R = N + 1$, then $\sum_{i \in R} \omega_i \leq k + 1$.

v) Let $E_i \geq 1$ ($1 \leq i \leq q$) be arbitrarily given numbers. For $R \subset \{1, ..., q\}$ with $\sharp R = N + 1$, there is a subset $R^o \subset R$ such that $\sharp R^o = \text{rank}\{Q_i\}_{i \in R^o} = k + 1$ and

$$\prod_{i \in R^o} E_i^{|E_i|} \leq \prod_{i \in R} E_i.$$

Proof. We assume that each $Q_i$ is given by

$$\sum_{I \in \mathcal{I}_d} a_{iI} x^I = 0,$$

where $\mathcal{I}_d = \{(i_0, ..., i_n) \in \mathbb{N}^{n+1}_0 ; i_0 + \cdots + i_n = d\}$, $I = (i_0, ..., i_n) \in \mathcal{I}_d$, $x^I = x_0^{i_0} \cdots x_n^{i_n}$ and $a_{iI} \in \mathbb{C}$ ($1 \leq i \leq q$, $I \in \mathcal{I}_d$). Setting $Q_i^*(x) = \sum_{I \in \mathcal{I}_d} a_{iI} x^I$. Then $Q_i^* \in H_d$.

Taking a $\mathbb{C}-$basis of $I_d(V)$, we may identify $I_d(V)$ with $\mathbb{C}-$vector space $\mathbb{C}^M$ with $M = H_d(V)$. For each $Q_i$, we denote $v_i$ the vector in $\mathbb{C}^M$ which corresponds to $[Q_i^*]$ by this identification. We denote by $H_i$ the hyperplane in $\mathbb{C}^M$ associated with the vector $v_i$.

Then for each arbitrary subset $R \subset \{1, ..., q\}$ with $\sharp R = N + 1$, we have

$$\dim(\bigcap_{i \in R} Q_i \cap V) \geq \dim V - \text{rank}\{[Q_i]\}_{i \in R} = k - \text{rank}\{H_i\}_{i \in R}.$$

Hence

$$\text{rank}\{H_i\}_{i \in R} \geq k - \dim(\bigcap_{i \in R} Q_i \cap V) \geq k - (-1) = k + 1.$$

By Lemma 3.3, there exists a linear subspace $L \subset \mathbb{C}^M$ of dimension $k + 1$ such that $L \not\subset H_i$ ($1 \leq i \leq q$) and

$$\text{rank}\{H_i \cap L, \ldots, H_i \cap L\} = \text{rank}\{H_i, \ldots, H_i\}$$

for every $1 \leq l \leq k + 1, 1 \leq i_1 < \cdots < i_l \leq q$. Hence, for any subset $R \subset \{1, ..., q\}$ with $\sharp R = N + 1$, since $\text{rank}\{H_i\}_{i \in R} \geq k + 1$, there exists a subset $R' \subset R$ with $\sharp R' = k + 1$ and $\text{rank}\{H_i\}_{i \in R'} = k + 1$. It implies that

$$\text{rank}\{H_i \cap L\}_{i \in R} \geq \text{rank}\{H_i \cap L\}_{i \in R'} = \text{rank}\{H_i\}_{i \in R'} = k + 1.$$
This yields that rank \( \{ H_i \cap L \}_{i \in R} = k + 1 \), since \( \dim L = k + 1 \). Therefore, \( \{ H_i \cap L \}_{i=1}^{q} \) is a family of \( q \) hyperplanes in \( L \) in \( N \)-subgeneral position.

By Lemma 3.2, there exist Nochka weights \( \{ \omega_i \}_{i=1}^{q} \) for the family \( \{ H_i \cap L \}_{i=1}^{q} \) in \( L \). It is clear that assertions (i)-(iv) are automatically satisfied. Now for \( R \subset \{ 1, ..., q \} \) with \( \sharp R = N + 1 \), by Lemma 3.2(v) we have
\[
\sum_{i \in R} \omega_i \leq \text{rank} \{ H_i \cap L \}_{i \in R} = k + 1
\]
and there is a subset \( R^o \subset R \) such that:
\[
\sharp R^o = \text{rank} \{ H_i \cap L \}_{i \in R^o} = \text{rank} \{ H_i \cap L \}_{i \in R} = k + 1,
\]
\[
\prod_{i \in R} E_i^{\omega_i} \leq \prod_{i \in R^o} E_i, \quad \forall E_i \geq 1 \ (1 \leq i \leq q),
\]
\[
\text{rank} \{ Q_i \}_{i \in R^o} = \text{rank} \{ H_i \cap L \}_{i \in R^o} = k + 1.
\]
Hence the assertion (v) is also satisfied.

The lemma is proved. \( \square \)

4. SECOND MAIN THEOREMS FOR HYPERSURFACES

Let \( \{ Q_i \}_{i \in R} \) be a set of hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of the common degree \( d \). Assume that each \( Q_i \) is defined by
\[
\sum_{I \in \mathcal{I}_d} a_{iI} x^I = 0,
\]
where \( \mathcal{I}_d = \{(i_0, ..., i_n) \in \mathbb{N}_0^{n+1} ; i_0 + \cdots + i_n = d \} \), \( I = (i_0, ..., i_n) \in \mathcal{I}_d \), \( x^I = x_0^{i_0} \cdots x_n^{i_n} \) and \( (x_0 : \cdots : x_n) \) is homogeneous coordinates of \( \mathbb{P}^n(\mathbb{C}) \).

Let \( f : \mathbb{C}^m \rightarrow V \subset \mathbb{P}^n(\mathbb{C}) \) be an algebraically nondegenerate meromorphic mapping into \( V \) with a reduced representation \( f = (f_0 : \cdots : f_n) \). We define
\[
Q_i(f) = \sum_{I \in \mathcal{I}_d} a_{iI} f^I,
\]
where \( f^I = f_0^{i_0} \cdots f_n^{i_n} \) for \( I = (i_0, ..., i_n) \). Then we see that \( f^* Q_i = \nu_{Q_i}(f) \) as divisors.

**Lemma 4.1.** Let \( \{ Q_i \}_{i \in R} \) be a set of hypersurfaces in \( \mathbb{P}^n(\mathbb{C}) \) of the common degree \( d \) and let \( f \) be a meromorphic mapping of \( \mathbb{C}^m \) into \( \mathbb{P}^n(\mathbb{C}) \). Assume that \( \bigcap_{i=1}^{q} Q_i \cap V = \emptyset \). Then there exist positive constants \( \alpha \) and \( \beta \) such that
\[
\alpha ||f||^d \leq \max_{i \in R} |Q_i(f)| \leq \beta ||f||^d.
\]

**Proof.** Let \( (x_0 : \cdots : x_n) \) be homogeneous coordinates of \( \mathbb{P}^n(\mathbb{C}) \). Assume that each \( Q_i \) is defined by: \( \sum_{I \in \mathcal{I}_d} a_{iI} x^I = 0 \). Set \( Q_i(x) = \sum_{I \in \mathcal{I}_d} a_{iI} x^I \) and consider the following function
\[
h(x) = \frac{\max_{i \in R} |Q_i(x)|}{||x||^d}.
\]
where \( ||x|| = (\sum_{i=0}^{n} x_i^2)^{\frac{1}{2}} \).

We see that the function \( h \) a positive continuous function on \( V \). By the compactness of \( V \), there exist positive constants \( \alpha \) and \( \beta \) such that \( \alpha = \min_{x \in P^n(C)} h(x) \) and \( \beta = \max_{x \in P^n(C)} h(x) \). Therefore, we have

\[
\alpha ||f||^d \leq \max_{i \in R} |Q_i(f)| \leq \beta ||f||^d.
\]

The lemma is proved.

**Lemma 4.2.** Let \( \{Q_i\}_{i=1}^{q} \) be a set of \( q \) hypersurfaces in \( P^n(C) \) of the common degree \( d \). Then there exist \( (H_d(V) - k - 1) \) hypersurfaces \( \{T_i\}_{i=1}^{H_d(V) - k - 1} \) in \( P^n(C) \) such that for any subset \( R \subseteq \{1, \ldots, q\} \) with \( \sharp R = \text{rank}\{Q_i\}_{i \in R} = k + 1 \) then \( \text{rank}\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{M-k} = H_V(d) \).

**Proof.** For each \( i \) \((1 \leq i \leq q)\), take a homogeneous polynomial \( Q_i^* \in C[x_0, \ldots, x_n] \) of degree \( d \) defining \( Q_i \). We consider \( I_d(V) \) as a \( C \)–vector space of dimension \( H_d(V) \).

For each subset \( R \subseteq \{1, \ldots, q\} \) with \( \sharp R = \text{rank}\{Q_i^*\}_{i \in R} = k + 1 \), we denote by \( V_R \) the set of all vectors \( v = (v_1, \ldots, v_{H_d(V) - k - 1}) \in (I_d(V))^{H_d(V) - k - 1} \) such that \( \{[Q_i]_{i \in R}, v_1, \ldots, v_{H_d(V) - k - 1}\} \) is linearly dependent over \( C \). It is clear that \( V_R \) is an algebraic subset of \( (I_d(V))^{H_d(V) - k - 1} \).

Since \( \dim I_d(V) = H_d(V) \) and \( \text{rank}\{Q_i^*\}_{i \in R} = k + 1 \), there exists \( v = (v_1, \ldots, v_{H_d(V) - k - 1}) \in (I_d(V))^{H_d(V) - k - 1} \) such that \( \{[Q_i]_{i \in R}, v_1, \ldots, v_{H_d(V) - k - 1}\} \) is linearly independent over \( C \), i.e., \( v \notin V_R \). Therefore \( V_R \) is a proper algebraic subset of \( (I_d(V))^{H_d(V) - k - 1} \) for each \( R \).

This implies that

\[
(I_d(V))^{H_d(V) - k - 1} \setminus \bigcup_{R} V_R \neq \emptyset.
\]

Hence, there is \( (T_1^+, \ldots, T_{H_d(V) - k - 1}^+) \subseteq (I_d(V))^{H_d(V) - k - 1} \setminus \bigcup_{R} V_R \).

For each \( T_i^+ \), we take a representation \( T_i^* \in H_d \) of it and and take the hypersurface \( T_i^* \) in \( P^n(C) \), which is defined by the homogeneous polynomial \( T_i^* \) \((i = 1, \ldots, q)\). We have

\[
\text{rank}\{\{Q_i\}_{i \in R} \cup \{T_i\}_{i=1}^{H_d(V) - k - 1}\} = \text{rank}\{\{[Q_i]\}_{i \in R} \cup \{[T_i]\}_{i=1}^{H_d(V) - k - 1}\} = H_V(d)
\]

for every subset \( R \subseteq \{1, \ldots, q\} \) with \( \sharp R = \text{rank}\{Q_i\}_{i \in R} = k + 1 \).

The lemma is proved.

**Proof of Theorem 1.1.** We first prove the theorem for the case where all \( Q_i \) \((i = 1, \ldots, q)\) have the same degree \( d \).

It is easy to see that there is a positive constant \( \beta \) such that \( \beta ||f||^d \geq |Q_i(f)| \) for every \( 1 \leq i \leq q \). Set \( Q := \{1, \cdots, q\} \). Let \( \{\omega_i\}_{i=1}^{q} \) be as in Lemma 3.4 for the family \( \{Q_i\}_{i=1}^{q} \).

Let \( \{T_i\}_{i=1}^{M-k} \) be \((M-k)\) hypersurfaces in \( P^n(C) \), which satisfy Lemma 4.2.

Take a \( C \)–basis \( \{[A_i]\}_{i=1}^{H_{V}(d)} \) of \( I_d(V) \), where \( A_i \in H_d \). Since \( f \) is nondegenerate over \( I_d(V) \), \( \{A_i(f); 1 \leq i \leq H_{V}(d)\} \) is linearly independent over \( C \). Then there is an admissible set \( \{\alpha_1, \cdots, \alpha_{H_{V}(d)}\} \subseteq Z_n^+ \) such that

\[
W \equiv \det(D^{\alpha_j} A_i(f)(1 \leq i \leq H_{V}(d)))_{1 \leq j \leq H_{V}(d)} \neq 0
\]
and $|\alpha| \leq H_V(d) - 1, \forall 1 \leq j \leq H_V(d)$.

For each $R^o = \{r_1^0, ..., r_k^0\} \subset \{1, ..., q\}$ with rank$\{Q_i\}_{i \in R^o} = i R^o = k + 1$, set

\[ W_{R^o} \equiv \det(D^{\alpha_j}Q_{r_i^0}(f))(1 \leq v \leq k + 1, D^{\alpha_j}T_i(f))(1 \leq l \leq H_V(d) - k - 1))_{1 \leq j \leq H_V(d)} \]

Since rank$\{Q_{r_i^0}(1 \leq v \leq k + 1), T_i(1 \leq l \leq H_V(d) - k - 1)) = H_d(V)$, there exist a nonzero constant $C_{R^o}$ such that $W_{R^o} = C_{R^o} \cdot W$.

We denote by $\mathcal{R}^o$ the family of all subsets $R^o$ of $\{1, ..., q\}$ satisfying

\[ \text{rank} \{Q_i\}_{i \in R^o} = i R^o = k + 1. \]

Let $z$ be a fixed point. For each $R \subset Q$ with $\sharp R = N + 1$, we choose $R^o \subset R$ such that $R^o \in \mathcal{R}^o$ and $R^o$ satisfies Lemma 3.3(i) with respect to numbers $\{\beta_i ||f(z)||^d \}_{i=1}^q$. On the other hand, there exists $\bar{R} \subset Q$ with $\sharp \bar{R} = N + 1$ such that $|Q_i(f)(z)| \leq |Q_j(f)(z)|, \forall i \in \bar{R}, j \notin \bar{R}$. Since $\bigcap_{i \in R} Q_i = \emptyset$, by Lemma 4.1 there exists a positive constant $\alpha_{\bar{R}}$ such that $\alpha_{\bar{R}} ||f||^d(z) \leq \max_{i \in \bar{R}} |Q_i(f)(z)|$.

Then we see that

\[
\frac{||f(z)||^{d(\sum_{i=1}^q \omega_i)}|W(z)|}{|Q_1^{\omega_1}(f)(z) \cdots Q_q^{\omega_q}(f)(z)|} \leq \frac{|W(z)|}{\alpha_{\bar{R}}^{q-N-1} \beta^{N+1}} \prod_{i \in \bar{R}} \left( \frac{\beta_i ||f(z)||^d}{|Q_i(f)(z)|} \right)^{\omega_i} \\
\leq A_{\bar{R}} \frac{|W(\bar{R})(z)| \cdot ||f||^{d(k+1)}(z)}{\prod_{i \in \bar{R}^o} |Q_i(f)(z)|} \\
\leq B_{\bar{R}} \frac{|W_{\bar{R}^o}(z)| \cdot ||f||^{dH_V(d)}(z)}{\prod_{i \in \bar{R}^o} |Q_i(f)(z)| \prod_{i=1}^{H_V(d) - k - 1} |T_i(f)(z)|},
\]

where $A_{\bar{R}}, B_{\bar{R}}$ are positive constants.

Put $S_{\bar{R}} = B_{\bar{R}} \frac{|W_{\bar{R}^o}|}{\prod_{i \in \bar{R}^o} |Q_i(f)| \prod_{i=1}^{H_V(d) - k - 1} |T_i(f)|}$. By the lemma on logarithmic derivative, it is easy to see that

\[
|| \int_{S(r)} \log^+ S_{\bar{R}}(z) \sigma_m = o(T_f(r)).
\]

Therefore, for each $z \in C^m$, we have

\[
\log \left( \frac{||f(z)||^{d(\sum_{i=1}^q \omega_i)}|W(z)|}{|Q_1^{\omega_1}(f)(z) \cdots Q_q^{\omega_q}(f)(z)|} \right) \leq \log (||f||^{dH_V(d)}(z)) + \sum_{R \subset Q, \sharp R = N + 1} \log^+ S_{\bar{R}}.
\]

Integrating both sides of the above inequality over $S(r)$ with the note that: $\sum_{i=1}^q \omega_i = \bar{\omega}(q - 2N + k + 1) + k + 1$, we have

(4.3)

$$|| d(q - 2N + k - 1 - \frac{H_V(d) - k - 1}{\bar{\omega}})T_f(r) \leq \sum_{i=1}^q \frac{\omega_i}{\bar{\omega}} N_{Q_i(f)}(r) - \frac{1}{\bar{\omega}} N_W(r) + o(T_f(r)).$$

Claim. $\sum_{i=1}^q \omega_i N_{Q_i(f)}(r) - N_W(r) \leq \sum_{i=1}^q \omega_i N_{Q_i(f)}^{[H_V(d)-1]}(r)$. 

Indeed, let \( z \) be a zero of some \( Q_i(f)(z) \) and \( z \not\in I(f) = \{ f_0 = \cdots = f_n = 0 \} \). Since \( \{Q_i\}_{i=1}^{q} \) is in \( N \)-subgeneral position, \( z \) is not zero of more than \( N \) functions \( Q_i(f) \). Without loss of generality, we may assume that \( z \) is zero of \( Q_i(f) \) (\( 1 \leq i \leq k \leq N \)) and \( z \) is not zero of \( Q_i(f) \) with \( i > N \). Put \( R = \{1, \ldots, N+1\} \), choose \( R^1 \subset R \) with \( \dim R^1 = \operatorname{rank} \{ Q_i \}_{i \in R^1} = k + 1 \) and \( R^1 \) satisfies Lemma \( \ref{lemma:3.3} \) \( \nu \) with respect to numbers \( \left\{ e^{\max\{\nu_Q(z)-H_V(d)+1,0\}} \right\}_{i=1}^{q} \). Then we have

\[
\sum_{i \in R} \omega_i \max \{ \nu_{Q_i}(z) - H_V(d) + 1, 0 \} \leq \sum_{i \in R^1} \max \{ \nu_{Q_i}(z) - H_V(d) + 1, 0 \}.
\]

Then, it yields that

\[
\nu_W(z) = \nu_{W_q}(z) \geq \sum_{i \in R^1} \max \{ \nu_{Q_i}(z) - H_V(d) + 1, 0 \} \geq \sum_{i \in R} \omega_i \max \{ \nu_{Q_i}(z) - H_V(d) + 1, 0 \}.
\]

Thus

\[
\sum_{i=1}^{q} \omega_i \nu_{Q_i}(z) - \nu_W(z) = \sum_{i \in R} \omega_i \nu_{Q_i}(z) - \nu_W(z)
\]

\[
= \sum_{i \in R} \omega_i \min \{ \nu_{Q_i}(z), H_V(d) - 1 \} + \sum_{i \in R} \omega_i \max \{ \nu_{Q_i}(z) - H_V(d) + 1, 0 \} - \nu_W(z)
\]

\[
\leq \sum_{i \in R} \omega_i \min \{ \nu_{Q_i}(z), H_V(d) + 1 \} = \sum_{i=1}^{q} \omega_i \min \{ \nu_{Q_i}(z), M \}.
\]

Integrating both sides of this inequality, we get

\[
\sum_{i=1}^{q} \omega_i N_{Q_i}(r) - N_{W}(r) \leq \sum_{i=1}^{q} \omega_i N^{[H_V(d)-1]}_{Q_i}(r).
\]

This proves the claim.

Combining the claim and \( \ref{lemma:4.3} \), we obtain

\[
\| d(q - 2N + k - 1 - \frac{H_V(d) - k - 1}{\tilde{\omega}})T_f(r) \leq \sum_{i=1}^{q} \frac{\omega_i}{\tilde{\omega}} N^{[H_V(d)-1]}_{Q_i}(r) + o(T_f(r)) \leq \sum_{i=1}^{q} N^{[H_V(d)-1]}_{Q_i}(r) + o(T_f(r)).
\]

Since \( \tilde{\omega} \geq \frac{k + 1}{2N - k + 1} \), the above inequality implies that

\[
\| d \left( q - \frac{(2N - k + 1)H_V(d)}{k + 1} \right) T_f(r) \leq \sum_{i=1}^{q} N^{[H_V(d)-1]}_{Q_i}(r) + o(T_f(r)).
\]

Hence, the theorem is proved in the case where all \( Q_i \) have the same degree.
We now prove the theorem for the general case where $\deg Q_i = d_i$. Applying the above case for $f$ and the hypersurfaces $Q_i^a (i = 1, \ldots, q)$ of the common degree $d$, we have

$$|| (q - (2N - k + 1)H_V(d)) \frac{d}{k + 1} T_f(r) \leq \frac{1}{d} \sum_{i=1}^{q} \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r))$$

$$\leq \sum_{i=1}^{q} \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r))$$

$$= \sum_{i=1}^{q} \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r)).$$

The theorem is proved. \qed

5. Unicity of meromorphic mappings sharing hypersurfaces

**Lemma 5.1.** Let $f$ and $g$ be nonconstant meromorphic mappings of $\mathbb{C}^n$ into a complex projective subvariety $V$ of $\mathbb{P}^n(\mathbb{C})$, $\dim V = k$ ($k \leq n$). Let $Q_i$ ($i = 1, \ldots, q$) be moving hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ in $N$-subgeneral position with respect to $V$, $\deg Q_i = d_i$, $N \geq n$. Put $d = \text{lcm}(d_1, \ldots, d_q)$ and $M = \binom{n+d}{n} - 1$. Assume that both $f$ and $g$ are nondegenerate over $I_d(V)$. If $q > \frac{(2N - k + 1)H_V(d)}{k+1}$ then $|| T_f(r) = O(T_g(r))$ and $|| T_g(r) = O(T_f(r))$.

**Proof.** Using Theorem 1.1 for $f$, we have

$$|| (q - (2N - k + 1)H_V(d)) \frac{d}{k + 1} T_f(r) \leq \frac{1}{d} \sum_{i=1}^{q} \frac{1}{d_i} N_{Q_i(f)}^{[H_V(d)-1]}(r) + o(T_f(r))$$

$$\leq \sum_{i=1}^{q} \frac{H_V(d) - 1}{d_i} N_{Q_i(f)}^{[1]}(r) + o(T_f(r))$$

$$= \sum_{i=1}^{q} \frac{H_V(d) - 1}{d_i} N_{Q_i(f)}^{[1]}(r) + o(T_f(r))$$

$$\leq q(H_d(V) - 1) T_g(r) + o(T_f(r)).$$

Hence $|| T_f(r) = O(T_g(r))$. Similarly, we get $|| T_g(r) = O(T_f(r))$.

**Proof of Theorem 1.2.** We assume that $f$ and $g$ have reduced representations $f = (f_0 : \cdots : f_n)$ and $g = (g_0 : \cdots : g_n)$ respectively. Replacing $Q_i$ by $Q_i^a$ if necessary, without loss of generality, we may assume that $d_i = d$ for all $i = 1, \ldots, q$.

By Lemma 5.1, we have $|| T_f(r) = O(T_g(r))$ and $|| T_g(r) = O(T_f(r))$. Suppose that $f \neq g$. Then there exist two indices $s, t$ ($0 \leq s < t \leq n$) satisfying

$$H := f_sg_t - f_tg_s \neq 0.$$
By the assumption (ii) of the theorem, we have $H = 0$ on $\bigcup_{i=1}^{q} \text{Zero}Q_{i}(f) \cup \text{Zero}Q_{i}(g)$. Therefore, we have

$$\nu_{H}^{0} \geq \sum_{i=1}^{q} \min\{1, \nu_{Q_{i}(f)}^{0}\}$$

outside an analytic subset of codimension at least two. Then, it follows that

$$N_{H}(r) \geq \sum_{i=1}^{q} N^{[1]}_{Q_{i}(f)}(r). \tag{5.2}$$

On the other hand, by the definition of the characteristic function and Jensen formula, we have

$$N_{H}(r) = \int_{S(r)} \log |f_{s}g_{t} - f_{t}g_{s}| \sigma_{m}$$

$$\leq \int_{S(r)} \log ||f|| \sigma_{m} + \int_{S(r)} \log ||g|| \sigma_{m}$$

$$= T_{f}(r) + T_{g}(r).$$

Combining this and (4.2), we obtain

$$T_{f}(r) + T_{g}(r) \geq \sum_{i=1}^{q} N^{[1]}_{Q_{i}(f)}(r).$$

Similarly, we have

$$T_{f}(r) + T_{g}(r) \geq \sum_{i=1}^{q} N^{[1]}_{Q_{i}(g)}(r).$$

Summing-up both sides of the above two inequalities, we have

$$2(T_{f}(r) + T_{g}(r)) \geq \sum_{i=1}^{q} N^{[1]}_{Q_{i}(f)}(r) + \sum_{i=1}^{q} N^{[1]}_{Q_{i}(g)}(r). \tag{5.3}$$

From (5.3) and applying Theorem 1.1 for $f$ and $g$, we have

$$2(T_{f}(r) + T_{g}(r)) \geq \sum_{i=1}^{q} N^{[H_{V}(d)-1]}_{Q_{i}(f)}(r) + \sum_{i=1}^{q} N^{[H_{V}(d)-1]}_{Q_{i}(g)}(r)$$

$$\geq \frac{d}{H_{V}(d) - 1} \left( q - \frac{(2N - k + 1)H_{V}(d)}{k + 1} \right) (T_{f}(r) + T_{g}(r)) + o(T_{f}(r) + T_{g}(r)).$$

Letting $r \to +\infty$, we get

$$2 \geq \frac{d}{H_{V}(d) - 1} \left( q - \frac{(2N - k + 1)H_{V}(d)}{k + 1} \right) \Rightarrow q \leq \frac{2(H_{V}(d)-1)}{d} + \frac{(2N-k+1)H_{V}(d)}{k+1}.$$  

This is a contradiction.

Hence $f = g$. The theorem is proved. $\Box$
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