A new mathematical model of tunnelling and the Hartman effect puzzle

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Abstract. In this paper we develop and refine our recent model of a one-dimensional completed scattering, which gives an individual description of the transmission and reflection subprocesses at all stages of scattering. We show that the group, dwell and Larmor characteristic times, introduced in this model for the subprocesses, are in a full agreement with special relativity and allow us to solve the Hartman effect puzzle.

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1. Introduction

For a long time tunnelling a particle through a one-dimensional (1D) static potential barrier has been considered in quantum mechanics as a representative of well-understood phenomena. The quantum-mechanical model of this process, hereinafter referred to as "standard model" (SM), has been included in many textbooks on quantum mechanics. However, studying the temporal aspects of tunnelling (see reviews [11-2, 3-4, 5, 6, 7] and references therein), on the basis of the SM, shows that this model leads to anomalously short or even negative tunnelling times. A serious controversy raised by this result has not been overcome up to now.

Among a huge variety of proposals to solve the tunnelling time problem (TTP), the concepts of the group, dwell and Larmor times are the most prominent ones. They lie entirely within the framework of conventional quantum theory and complement each other in timing a scattering particle within the standard setting of this 1D scattering problem. However, even these concepts, being introduced within the framework of the SM, both in the case of the (nonrelativistic) Schrödinger equation (see, e.g., [8, 9, 10, 11, 12, 13]) and the (relativistic) Dirac equation (see, e.g., [14, 15, 16]), lead to the Hartman effect to be at variance with special relativity. As a result, at present there is no consensus in solving the TTP.

In our opinion such state of affairs is not occasional, for the TTP cannot be, in principle, solved within the framework of the SM. The main reason is that the quantum ensemble of particles, at the final stage of a 1D completed scattering, consists of two subensembles to occupy macroscopically distinct spatial regions. This fact implies performing two (infinite) identical series of independent measurements, separately for transmitted particles and separately for reflected ones.

The partition of the initial quantum ensemble of particles into two macroscopically distinct parts is crucial for understanding the nature of this quantum scattering process. By the probability theory (see [17] and references therein), experimental data obtained in two different sets of measurements (identical in either set) cannot be described by a single (Kolmogorov) probability space. This means that the only legitimate way of solving the TTP is an individual timing of either subensemble. Any averaging over the transmitted and reflected subensembles, contrary to probability theory, leads inevitably to nonphysical results.

It is evident that the SM does not support an individual timing of the subensembles in the barrier region. Indeed, such timing needs the knowledge of the whole time evolution of either subensemble. However, the SM does not obey this requirement. Thus, on the basis of this model, neither common characteristic times, nor individual ones can be introduced for the subensembles.

At the same time, as was shown in [18, 19], the Schrödinger equation, in reality, admits a separate description of the subensembles at all stages of scattering, and hence it is possible to introduce characteristic times for each of them. The problem, however, is that some aspects of the subensemble’s evolution have remained beyond the scope of
these papers. Our aim is just to complete the model \[18\] \[19\] and to resolve on this basis the controversy surrounding the Hartman effect. In doing so, we shall dwell shortly on the basic points of the model, to make the present paper all-sufficient.

2. Wave functions for the subprocesses of a 1D completed scattering

Remind that a 1D completed scattering was considered in \[18\] in the following setting. A particle impinges a symmetrical potential barrier \( V(x) = V(x_c - x) \) confined to the finite spatial interval \([a, b] \) \((a > 0)\); \( d = b - a \) is the barrier width, the point \( x_c \) is the centre of the barrier region. At the initial instant of time, long before the scattering event, the state of a particle \( \psi^{(0)}_{\text{full}}(x) \) approaches the in-asymptote \( \psi^{\text{in}}_{\text{full}}(x,t) \),

\[
\psi^{\text{in}}_{\text{full}}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A^{\text{in}}(k) \exp[i(kx - E(k)t/\hbar)]dk,
\]

which is supposed to be a normalized function to belong to the set \( S_\infty \) consisting from infinitely differentiable functions vanishing exponentially in the limit \( |x| \to \infty \); \( E(k) = \hbar^2 k^2/2m \). Without loss of generality, it is also supposed that

\[
<\psi^{(0)}_{\text{full}}|\hat{x}|\psi^{(0)}_{\text{full}}> = 0, \quad <\psi^{(0)}_{\text{full}}|\hat{p}|\psi^{(0)}_{\text{full}}> = \hbar k_0 > 0, \quad <\psi^{(0)}_{\text{full}}|\hat{x}^2|\psi^{(0)}_{\text{full}}>= l_0^2,
\]

where \( l_0 \) and \( k_0 \) are given parameters \((l_0 << a)\); \( \hat{x} \) and \( \hat{p} \) are the operators of the particle’s position and momentum, respectively. For the Gaussian wave packet \( A^{\text{in}}(k) = (l_0^2/\pi)^{1/4} \exp(-l_0^2(k-k_0)^2) \). For a completed scattering the average velocity, \( \hbar k_0/m \), is supposed to be much more than the rate of spreading the incident wave packet.

For each value of time \( t \) the state of a particle has the form

\[
\psi_{\text{full}}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A^{\text{in}}(k) \psi_{\text{full}}(x;k) \exp[-iE(k)t/\hbar];
\]

\( \psi_{\text{full}}(x;k) \) describes the stationary state of a particle, which is presented in \[18\] as follows

\[
\psi_{\text{full}}(x;k) = \begin{cases} \ e^{ikx} + b_{\text{out}}(k)e^{ik(2a-x)}, & \text{for } x \leq a; \\ a_{\text{full}} \cdot u(x-x_c;k) + b_{\text{full}} \cdot v(x-x_c;k), & \text{for } a \leq x \leq b; \\ a_{\text{out}}(k)e^{ik(x-d)}, & \text{for } x > b; \end{cases}
\]

\( u(x-x_c;k) \) and \( v(x-x_c;k) \) are such real solutions to the Schrödinger equation that \( u(x_c-x;k) = -u(x-x_c;k) \), \( v(x_c-x;k) = v(x-x_c;k) \) and \( \frac{du}{dx}v - \frac{dv}{dx}u = \kappa \) is a constant;

\[
a_{\text{out}} = \frac{1}{2} \left( \frac{Q}{Q^*} - \frac{P}{P^*} \right); \quad b_{\text{out}} = -\frac{1}{2} \left( \frac{Q}{Q^*} + \frac{P}{P^*} \right),
\]

\[
a_{\text{full}} = \frac{1}{\kappa} (P + P^* b_{\text{out}}) e^{ika} = -\frac{1}{\kappa} P^* a_{\text{out}} e^{ika}; \quad b_{\text{full}} = \frac{1}{\kappa} (Q + Q^* b_{\text{out}}) e^{ika} = \frac{1}{\kappa} Q^* a_{\text{out}} e^{ika};
\]

\[
Q = \left. \left( \frac{du(x-x_c)}{dx} + iku(x-x_c) \right) \right|_{x=b}; \quad P = \left. \left( \frac{dv(x-x_c)}{dx} + ikv(x-x_c) \right) \right|_{x=b}.
\]

As is shown in \[18\], the wave function \( \psi_{\text{full}}(x;k) \) to describe the stationary state of the whole ensemble of scattering particles can be uniquely presented as the superposition
of the functions $\psi_{\text{tr}}(x; k)$ and $\psi_{\text{ref}}(x; k)$ to describe the subensembles of transmitted and reflected particles, respectively - $\psi_{\text{full}}(x; k) = \psi_{\text{tr}}(x; k) + \psi_{\text{ref}}(x; k)$. Here

$$\psi_{\text{ref}}(x; k) = A_{\text{ref}}^\text{in} e^{ikx} + b_{\text{out}} e^{ik(2a-x)}, \quad \psi_{\text{tr}}(x; k) = A_{\text{tr}}^\text{in} e^{ikx} \quad \text{for} \quad x \leq a;$$

$$\psi_{\text{tr}}(x; k) = \kappa^{-1} \left( P A_{\text{ref}}^\text{in} + P^* b_{\text{out}} \right) e^{ikx} u(x-x_c; k) \quad \text{for} \quad a \leq x \leq x_c;$$

$$\psi_{\text{ref}}(x; k) \equiv 0, \quad \psi_{\text{tr}}(x; k) \equiv \psi_{\text{full}}(x; k) \quad \text{for} \quad x \geq x_c;$$

$$A_{\text{ref}}^\text{in} = b_{\text{out}} \left( b_{\text{out}}^* - a_{\text{out}}^* \right) \equiv b_{\text{out}}^* (b_{\text{out}} + a_{\text{out}}); \quad A_{\text{tr}}^\text{in} = a_{\text{out}}^* (a_{\text{out}} + b_{\text{out}}) \equiv a_{\text{out}} (a_{\text{out}}^* - b_{\text{out}}^*)$$

$$a_{\text{tr}} = \frac{P}{\kappa} A_{\text{tr}}^\text{in} e^{ika} = \frac{P Q^*}{P^* Q} \cdot a_{\text{full}}.$$  \hspace{1cm} (6)

We have to stress that not only $A_{\text{tr}}^\text{in} + A_{\text{ref}}^\text{in} = 1$ but also $|A_{\text{tr}}^\text{in}|^2 + |A_{\text{ref}}^\text{in}|^2 = 1$. The amplitudes $A_{\text{tr}}^\text{in}$ and $A_{\text{ref}}^\text{in}$ can also be presented in terms of the transmission and reflection coefficients - $A_{\text{ref}}^\text{in} = \sqrt{R} (\sqrt{T} \pm i \sqrt{R}) \equiv \sqrt{R} \exp(i\lambda)$, $A_{\text{tr}}^\text{in} = \sqrt{T} (\sqrt{T} \mp i \sqrt{R}) \equiv \sqrt{T} \exp \left[ i \left( \lambda + \text{sign}(\lambda) \frac{\pi}{2} \right) \right]; \lambda = \pm \arctan(\sqrt{T}/R); T = |a_{\text{out}}|^2, R = |b_{\text{out}}|^2$.

The main peculiarity of $\psi_{\text{tr}}(x; k)$ and $\psi_{\text{ref}}(x; k)$ is that each of them, unlike $\psi_{\text{full}}(x; k)$, contains one incoming and one outgoing wave. As is seen from (5), the unitary (Schrödinger’s) character of transmission and reflection is violated at the point $x_c$. At the same time both the functions as well as the corresponding probability current densities are continuous everywhere on the $O\overline{X}$-axis, including the point $x_c$.

By our approach, the point $x_c$ of any symmetrical potential barrier is a special one. In particular, reflected particles never cross this point in the course of scattering. This result agrees entirely with the fact that, for classical particles to impinge from the left a smooth symmetrical potential barrier, the middle of the barrier region is the extreme right turning point, irrespective of the particle’s mass and the barrier’s form and size.

For narrow in $k$-space wave packets $\psi_{\text{full}}(x, t)$ (see (2)) and corresponding ones $\psi_{\text{tr}}(x, t)$ and $\psi_{\text{ref}}(x, t)$ formed from $\psi_{\text{tr}}(x; k)$ and $\psi_{\text{ref}}(x; k)$, respectively, we have $\Re \langle \psi_{\text{tr}}(x, t) | \psi_{\text{ref}}(x, t) \rangle = 0$ for any value of $t$. Therefore, despite the existence of interference between $\psi_{\text{tr}}$ and $\psi_{\text{ref}}$, we have

$$\langle \psi_{\text{full}}(x, t) | \psi_{\text{full}}(x, t) \rangle = \mathbf{T} + \mathbf{R} = 1; \mathbf{T} = \langle \psi_{\text{tr}}(x, t) | \psi_{\text{tr}}(x, t) \rangle, \mathbf{R} = \langle \psi_{\text{ref}}(x, t) | \psi_{\text{ref}}(x, t) \rangle;$$

constants $\mathbf{T}$ and $\mathbf{R}$ are the transmission and reflection coefficients, respectively.

Note that the decomposition $\psi_{\text{full}}(x, t) = \psi_{\text{tr}}(x, t) + \psi_{\text{ref}}(x, t)$ holds for wave packets of any width. In this case $\mathbf{R}$ remains unchanged at all stages of scattering. However, $\mathbf{T}$ is now constant and equal to $1 - \mathbf{R}$ only long before and long after the scattering event. At the very stage of scattering this quantity is not now constant: as the wave packet $\psi_{\text{tr}}(x, t)$ does not obey the Schrödinger equation at the point $x_c$, the continuity, at this point, of the probability current density (PCD) of separate waves does not guarantee the continuity of the PCD for their superposition, for the continuity equation is nonlinear.

Thus, in the general case, in partitioning the whole ensemble of scattering particles into the to-be-transmitted and to-be-reflected subensembles at the stage of scattering,
the quantum mechanical formalism does not allow one to exclude entirely interference terms from $\psi_{tr}(x, t)$. However, as it follows from our numerical calculations, even for wave packets whose initial width is comparable with the barrier width, the relative deviation of the value of $T$ from $1 - R$ is small enough. This is a consequence of a large rate of spreading such packets. At the very stage of scattering the width of such packets becomes much larger than the barrier’s width, which results in weakening the effect of the violation of the continuity equation at the point $x_c$.

Note that the question of violating the unitary evolution of the subensembles at the point $x_c$ has remained the scope of the papers [18, 19]. To cover this gap, in the context of solving the Hartman effect puzzle, is the main goal of the present paper. Of interest here is the fact that due to non-unitarity the time derivative of the expectation values of observables involved in the timing procedures of the subensembles may contain extra terms, apart from the quantum Poisson brackets. To elucidate this question, it is sufficient, for the first time, to restrict oneself to the case of narrow in $k$-space wave packets when the variation of $T$ is negligible.

For example, it is easy to show that for such packets reflected electrons are affected, at the point $x_c$, by an extra (average) force to push particles out from the barrier region, backward into the left out-of-barrier one:

$$d < \hat{p} >_{ref} = \left\langle \frac{dV}{dx} \right\rangle_{ref} \Delta x - \frac{\hbar^2}{2m} \left| \frac{\partial \psi_{ref}}{\partial x} \right|_{x=x_c-0};$$

where angle brackets denote expected values of observables. For transmitted particles the second term in the analogous expression

$$d < \hat{p} >_{tr} = \left\langle \frac{dV}{dx} \right\rangle_{tr} + \frac{\hbar^2}{2m} \left( \left| \frac{\partial \psi_{tr}}{\partial x} \right|_{x=x_c+0} - \left| \frac{\partial \psi_{tr}}{\partial x} \right|_{x=x_c-0} \right)$$

equals to zero. Indeed, in the limit $l_0 \to \infty$

$$\left| \frac{\partial \psi_{tr}}{\partial x} \right|_{x=x_c+0} - \left| \frac{\partial \psi_{tr}}{\partial x} \right|_{x=x_c-0} = \kappa^2 \left( |a_{f,full}(k)|^2 - |a_{tr}(k)|^2 \right) = 0,$$

because the modules of the coefficients $a_{f,full}(k)$ and $a_{tr}(k)$ are equal (see (6)).

What is important is that the violation of the unitary subensemble’s evolution leads also to extra terms in the time derivatives for the $x$-th and $y$-th projections of the electron spin. They are these observables that are used for introducing the Larmor characteristic times for the subensembles (see [19]). In doing so, extra terms associated with the non-unitarity have not been considered in [19] because they do not describe the Larmor spin precession in a magnetic field confined to the barrier region - an effect to underlie the Larmor timing procedure. At the same time, as will be seen from the following, the appearance of such terms plays the key role in solving the old mystery associated with the Hartman effect.

As is known, the essence of this effect is that, for a particle tunnelling through a wide rectangular barrier, the phase (asymptotic group) time (see [20]) and the Larmor time to coincide with the dwell time (see [9]) saturate with increasing the barrier’s
width. Thus, in fact, to study all aspects of the Hartman effect, we have to dwell on all characteristic times introduced in [19] for transmission, taking now into account a non-unitary character of this subprocess.

3. The Hartman effect puzzle

3.1. The Hartman effect from the viewpoint of the group time concept

We begin our analysis of the Hartman effect with the group time concept. A new model implies introduction of two different group times - the exact group time and the asymptotic group time \( \tau_{tr}^{\text{exact}} \) and \( \tau_{tr}^{\text{as}} \). By [19], the former is introduced as the difference \( \tau_{tr}^{\text{exact}} = t_{tr}^2 - t_{tr}^1 \) where \( t_{tr}^2 \) and \( t_{tr}^1 \) are such moments of time that
\[
\frac{1}{T} \left( < \psi_{tr}(x, t_{tr}^1) | \dot{x} | \psi_{tr}(x, t_{tr}^1) > \right) = a; \quad \frac{1}{T} \left( < \psi_{tr}(x, t_{tr}^2) | \dot{x} | \psi_{tr}(x, t_{tr}^2) > \right) = b.
\]

As regards \( \tau_{tr}^{\text{as}} \), it describes the influence of the potential barrier on a particle within a wide enough interval \([a - L_1, b + L_2] \) where \( L_1, L_2 \gg l_0 \). In this case, instead of the exact wave functions for transmission, one may use the corresponding in- and out-asymptotes
\[
\psi_{tr}^{\text{in,out}}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A_{tr}(k) f_{tr}^{\text{in,out}}(k) \exp[i(kx - E(k)t/\hbar)]; \quad (7)
\]
\[
f_{tr}(k, t) = \sqrt{T} \exp \left[ i \left( \lambda + \text{sign}(\lambda) \frac{\pi}{2} \right) \right], \quad f_{tr}^{\text{out}}(k) = \sqrt{T} \exp[iJ(k) - kd]; \quad J = \text{arg}(a_{\text{out}}).
\]

Long before and long after the scattering event the motion of the centre of mass (CM) \( < \dot{x} >_{tr} \) of the wave packet \( \psi_{tr}(x, t) \) is described, respectively, by the expressions
\[
< \dot{x} >_{tr}^{\text{in}} = \frac{\hbar}{m} < k >_{tr} - < \lambda >_{tr}^{\text{in}}; \quad < \dot{x} >_{tr}^{\text{out}} = \frac{\hbar}{m} < k >_{tr} - < \lambda >_{tr}^{\text{out}} + d
\]

For the average starting point \( x_{tr}^{\text{start}} \) of transmitted particles we have \( x_{tr}^{\text{start}} = - < \lambda >_{tr}^{\text{in}} \), i.e., it differs from \( x_{tr}^{\text{full}} \) to characterize the whole ensemble of particles. This result distinguishes our approach from the standard wave-packet analysis based on the implicit assumption that transmitted (and reflected) particles start, on the average, from the point \( x_{tr}^{\text{start}} \) to coincide, by setting \( \hbar t_{tr} = \frac{m}{\hbar} \), with the origin of coordinates.

The time \( \tau_{tr}(L_1, L_2) \) spent by the CM \( < \dot{x} >_{tr} \) in the interval \([a - L_1, b + L_2]\) is
\[
\tau_{tr}(L_1, L_2) \equiv t_{tr}^{(2)} - t_{tr}^{(1)} = \frac{m}{\hbar} < k >_{tr} \left( < \lambda >_{tr}^{\text{out}} - < \lambda >_{tr}^{\text{in}} + L_1 + L_2 \right).
\]

The values of \( t_{tr}^{(2)} \) and \( t_{tr}^{(1)} \) obey the equations
\[
< \dot{x} >_{tr}^{\text{in}} (t_{tr}^{(1)}) = a - L_1; \quad < \dot{x} >_{tr}^{\text{out}} (t_{tr}^{(2)}) = b + L_2.
\]

The term \( \tau_{tr}^{\text{as}} (\tau_{tr} = \tau_{tr}(0, 0)) \) is just the asymptotic group transmission time,
\[
\tau_{tr}^{\text{as}} = \frac{md_{tr}^{gr}}{\hbar} < k >_{tr}; \quad d_{tr}^{gr} = < \lambda >_{tr}^{\text{out}} - < \lambda >_{tr}^{\text{in}}.
\]

For a particle tunnelling through the rectangular potential barrier of height \( V_0 \) \((E \leq V_0)\), with the notations \( \kappa = \sqrt{2m(V_0 - E)}/\hbar \) and \( \kappa_0 = \sqrt{2mV_0}/\hbar \), we have (see [19])
\[
d_{tr}^{gr}(k) = \frac{4}{\kappa} \left[ \kappa_0^2 \sinh^2(\kappa d/2) \left\{ \kappa_0^2 \sinh(\kappa d) - k^2 \kappa \right\} \right] \left[ \kappa_0^2 \sinh^2(\kappa d) + 2k^4 \kappa^2 \right].
\]
As is seen, like the phase time defined in the SM, this quantity saturates, too, with increasing the barrier’s width $d$. However, this fact does not at all mean that the effective velocity of a particle tunnelling through a wide rectangular barrier becomes superluminal. The figure 1 shows the function $<\hat{x}>_{tr}(t)$ to describe scattering the Gaussian wave packet ($l_0 = 10nm$, $E_0 = \hbar^2 k_0^2/2m = 0.05eV$) by the rectangular barrier ($a = 200nm$, $b = 215nm$, $V_0 = 0.2eV$). (Note, in this case the deviation of $T$ from $1 - R$ has not exceeded five percentages, though the wave-packet’s and barrier’s widths are of the same order.)

![Figure 1. The CM's positions of $\psi_{tr}(x,t)$ (circles) and the corresponding freely moving wave packet (dashed line) as functions of time $t$.](image)

This figure shows explicitly the principal difference between the exact and asymptotic group times. While the former gives the time spent by the wave packet’s CM just in the barrier region, the latter describes the influence of the barrier on the CM, in the course of a whole scattering process. More precisely, the quantity $\tau_{tr}^{as} - \tau_{free}$, where $\tau_{free} = md/\hbar k_0$, is the time delay in moving the CM of the transmitted wave packet in comparison with the motion of the CM of the corresponding freely moving one. In the case considered, $\tau_{tr}^{exact} \approx 0, 155ps$, $\tau_{tr}^{as} \approx 0, 01ps$, $\tau_{free} \approx 0, 025ps$.

As is seen, the influence of this opaque rectangular barrier on the transmitted wave packet has a complicated character. The exact group time says that the CM’s velocity becomes very small inside the barrier region. However, the asymptotic group time tells us that the total influence of the barrier on the wave packet has an accelerating character: the wave packet transmitted through the barrier moves ahead the corresponding freely moving one. However, this effect is related to the asymptotically large spatial interval. Therefore the saturation of the asymptotic group transmission time, with increasing the barrier’s width, does not at all mean that the CM of the transmitted wave packet crosses the barrier with a superluminal effective velocity.
3.2. The Hartman effect from the viewpoint of the dwell and Larmor time concepts

Now we have to analyze the Hartman effect on the basis of the dwell and Larmor time concepts which are closely connected with each other. Remind that the dwell time for transmission ($\tau_{\text{dwell}}$) is introduced in [19] for the stationary case -

$$
\tau_{\text{dwell}} = \frac{1}{I_{\text{tr}}(k)} \int_a^b |\psi_{\text{tr}}(x; k)|^2 dx = \frac{2}{I_{\text{full}}(k)} \int_{x_c}^b |\psi_{\text{full}}(x; k)|^2 dx
$$

(9)

where $I_{\text{tr}} = I_{\text{full}} = T(k)\hbar k/m$ is the probability current density; the second expression in (9) reflects the properties of symmetric potential barriers. For a particle tunnelling through the rectangular barrier we have (see [19])

$$
\tau_{\text{dwell}} = \frac{m}{2\hbar k \kappa^3} \left[ \left( \kappa^2 - k^2 \right) \kappa d + \kappa_0^2 \sinh(\kappa d) \right].
$$

(10)

In the limit $d \to \infty$, this quantity increases exponentially rather than saturates. Thus, both the exact group time and the dwell time tell us, contrary to the SM, that the opaque barrier strongly delays the motion of a particle when it enters the barrier region.

However, the intrigue is that the Larmor transmission time, being the average value of the dwell time, increases in this limit too [19]. This result seems to contradict the well-established fact to say that the final readings of the Larmor clock saturate in this case (see [9]). Of importance is that these (asymptotic) readings are introduced in terms of the transmitted wave packet and hence must be the same both in the standard and new models of a 1D completed scattering.

So that, finding the exponential increase of the dwell and Larmor transmission times, in the above limit, is an important but not definitive step in solving the Hartman effect puzzle. One has also to resolve the above discrepancy which remained unexplained in [19]. To complete solving this puzzle is just the main aim of the present paper, and, as will be seen from the following, the above non-unitary character of the time evolution of subprocesses plays a crucial role in this case. To show this, we reconsider some details of the Larmor-clock procedure introduced in [19].

By this procedure, there is a mixture of two electron’s ensembles - one of them consists from electrons with spin to be parallel to the OZ-axis, another is formed from particles with antiparallel spin - which impinge the potential barrier with a small constant uniform magnetic field $B$ switched on, in the barrier region, along the OZ-axis. At any instant of time $t$ the state of the mixture is described by the spinor $\Psi_{\text{full}}(x, t)$ to approach at $t = 0$ the in-asymptote $\Psi_{\text{full}}^{\in}(x)$ -

$$
\Psi_{\text{full}}(x, t) = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_{\text{full}}^{(t)}(x, t) \\ \psi_{\text{full}}^{(l)}(x, t) \end{pmatrix}, \quad \Psi_{\text{full}}^{\in}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \psi_{\text{full}}^{\in}(x); \quad (11)
$$

$\psi_{\text{full}}^{\in}(x)$ is a normalized wave function to satisfy conditions (1) where $l_0$ is large enough.

For electrons with spin up (down), the barrier’s height is effectively decreased (increased) by the value $\hbar \omega_L/2$; where $\omega_L = 2\mu B / \hbar$ is the frequency of the Larmor precession; $\mu$ is the magnetic moment. The corresponding Hamiltonian has the form

$$
\hat{H} = \frac{\vec{p}^2}{2m} + V(x) - \frac{\hbar \omega_L}{2} \sigma_z, \quad \text{if } x \in [a, b]; \quad \hat{H} = \frac{\vec{p}^2}{2m}, \quad \text{otherwise}
$$

(12)
hereinafter, \( \sigma_x, \sigma_y \) and \( \sigma_z \) are the Pauli matrices.

By [18], each component of the spinor \( \Psi_{\text{full}}(x,t) \) can be uniquely presented as a coherent superposition of two probability fields to describe transmission and reflection (we shall suppose that they are known) -

\[
\psi_{\text{full}}^{(1)}(x,t) = \psi_{\text{tr}}^{(1)}(x,t) + \psi_{\text{ref}}^{(1)}(x,t); \quad \psi_{\text{full}}^{(1)}(x,t) = \psi_{\text{tr}}^{(1)}(x,t) + \psi_{\text{ref}}^{(1)}(x,t).
\]  

(13)

As a consequence, the same decomposition takes place for spinor (11) -

\[
\Psi_{\text{full}}(x,t) = \Psi_{\text{tr}}(x,t) + \Psi_{\text{ref}}(x,t).
\]  

(14)

It is important to stress here that

\[
< \psi_{\text{full}}^{(1)}(x,t)|\psi_{\text{full}}^{(1)}(x,t) > = T^{(1)} + R^{(1)} = 1;
\]  

(15)

\[
T^{(1)} = < \psi_{\text{tr}}^{(1)}(x,t)|\psi_{\text{tr}}^{(1)}(x,t) >; \quad R^{(1)} = < \psi_{\text{ref}}^{(1)}(x,t)|\psi_{\text{ref}}^{(1)}(x,t) >;
\]  

(16)

\( T^{(1)} \) and \( R^{(1)} \) are the (constant) transmission and reflection coefficients for particles with spin up \( (\uparrow) \) and down \( (\downarrow) \), respectively.

Note that in-state (11) is the engine state of \( \sigma_x \) with the eigenvalue 1 (the average spin of the ensemble of incident particles is oriented along the \( x \)-direction). In the course of the scattering process the average spin of both transmitted and reflected particles will rotate in the plane orthogonal to the external magnetic field. However, since our main goal is interpreting the Hartman effect we will not be interested here in the spin’s dynamics of reflected particles.

To study the spin’s dynamics, it is convenient to present the average projections  \( \hat{S}_x, \hat{S}_y \) and  \( \hat{S}_z \) of the electron spin for the transmitted subensemble in the form

\[
\begin{align*}
< \hat{S}_x >_{\text{tr}} & \equiv \frac{\hbar}{2} \sin(\theta_{\text{tr}}) \cos(\phi_{\text{tr}}) = \frac{\hbar}{2\mathbf{T}} \Re(< \psi_{\text{tr}}^{(1)}|\psi_{\text{tr}}^{(1)} >), \\
< \hat{S}_y >_{\text{tr}} & \equiv \frac{\hbar}{2} \sin(\theta_{\text{tr}}) \sin(\phi_{\text{tr}}) = \frac{\hbar}{2\mathbf{T}} \Im(< \psi_{\text{tr}}^{(1)}|\psi_{\text{tr}}^{(1)} >), \\
< \hat{S}_z >_{\text{tr}} & \equiv \frac{\hbar}{2} \cos(\theta_{\text{tr}}) = \frac{\hbar}{4\mathbf{T}} \left( < \psi_{\text{tr}}^{(1)}|\psi_{\text{tr}}^{(1)} > - < \psi_{\text{tr}}^{(1)}|\psi_{\text{tr}}^{(1)} > \right);
\end{align*}
\]

\( \mathbf{T} = (T^{(1)} + T^{(1)})/2; \) analogous angles are also introduced for \( \Psi_{\text{full}} \) and \( \Psi_{\text{ref}} \).

For the initial condition (11) \( \theta_{\text{full}}(t) = \theta_{\text{full}}^{(0)} = \pi/2, \phi_{\text{full}}^{(0)} \equiv \phi_{\text{full}}(0) = 0 \), however

\[
\theta_{\text{tr}}^{(0)} = \arccos \left( \frac{T^{(1)} - T^{(1)}}{T^{(1)} + T^{(1)}} \right) \neq \frac{\pi}{2}; \quad \phi_{\text{tr}}^{(0)} = \arctan \left( \frac{\Im(< \psi_{\text{tr}}^{(1)}(x,0)|\psi_{\text{tr}}^{(1)}(x,0) >)}{\Re(< \psi_{\text{tr}}^{(1)}(x,0)|\psi_{\text{tr}}^{(1)}(x,0) >)} \right) \neq 0.
\]

Note, the norm of the narrow in \( k \)-space wave packets \( \psi_{\text{tr}}^{(1)}(x,t) \) is constant in time, therefore, despite a non-unitary evolution of transmission, \( \theta_{\text{tr}}(t) \equiv \theta_{\text{tr}}^{(0)} \). In this case

\[
< \hat{S}_z >_{\text{tr}} (t) = \frac{\hbar}{2} \cdot \frac{T^{(1)} - T^{(1)}}{T^{(1)} + T^{(1)}}.
\]

(18)

That is, this projection is constant, in a full agreement with the fact that the operator \( \hat{S}_z \) commutes with Hamiltonian (12). Thus, by our approach, unlike the SM (see [9]), the angle \( \theta_{\text{tr}}(t) \) cannot be used as a measure of the duration of dwelling an electron in
So that only the change of \( \phi_{tr}(t) \), due to the Larmor precession, can be used for measuring the time spent, on the average, by transmitted electrons in the barrier region. However, by our approach, apart from the Larmor precession there are other two reasons to influence the value of \( \phi_{tr}^{ad} \) - final readings of the Larmor clock. One of them has already known - the initial value of the angle \( \phi_{tr}^{(0)} \) is nonzero, unlike \( \phi_{full}^{(0)} \). Another reason, as will be seen from the following, is associated with a non-unitary character of the transmission subprocess. To study all peculiarities of the Larmor timing procedure for transmission, let us calculate the derivative \( d\phi_{tr}/dt \).

Since \( \phi_{tr} = \arctan (\langle \hat{S}_y \rangle_{tr} / \langle \hat{S}_x \rangle_{tr}) \), we have

\[
\frac{d\phi_{tr}}{dt} = \langle \hat{S}_x \rangle_{tr} \frac{d<\hat{S}_y>_{tr}}{dt} - \langle \hat{S}_y \rangle_{tr} \frac{d<\hat{S}_x>_{tr}}{dt}.
\] (19)

Calculations for the derivatives of the corresponding Pauli matrices show that

\[
\hat{T} \cdot \frac{d < \sigma_x >_{tr}}{dt} = \omega_L \int_a^b \Im[(\psi_{tr}^{(1)}(x,t))^* \psi_{tr}^{(1)}(x,t)] dx - \frac{\hbar}{2m} \Im\left[ \psi_{tr}^{(1)*}(x,t) \left( \frac{\partial \psi_{tr}^{(1)}(x+0,t)}{\partial x} - \frac{\partial \psi_{tr}^{(1)}(x-0,t)}{\partial x} \right) \right] - \psi_{tr}^{(1)*}(x,t) \left( \frac{\partial \psi_{tr}^{(1)}(x+0,t)}{\partial x} - \frac{\partial \psi_{tr}^{(1)}(x-0,t)}{\partial x} \right)
\] (20)

\[
\hat{T} \cdot \frac{d < \sigma_y >_{tr}}{dt} = -\omega_L \int_a^b \Re[(\psi_{tr}^{(1)}(x,t))^* \psi_{tr}^{(1)}(x,t)] dx + \frac{\hbar}{2m} \Re\left[ \psi_{tr}^{(1)}(x,t) \left( \frac{\partial \psi_{tr}^{(1)}(x+0,t)}{\partial x} - \frac{\partial \psi_{tr}^{(1)}(x-0,t)}{\partial x} \right) \right] + \psi_{tr}^{(1)*}(x,t) \left( \frac{\partial \psi_{tr}^{(1)}(x+0,t)}{\partial x} - \frac{\partial \psi_{tr}^{(1)}(x-0,t)}{\partial x} \right)
\]

(as \( \Psi_{ref}(x,t) = 0 \), for reflection, the second terms in similar expressions do not appear.)

Let now a magnetic field be infinitesimal. Then, considering (11), we have

\[
|<\hat{S}_y>_{tr}| \ll |<\hat{S}_x>_{tr}|, \quad \left| \frac{d < \sigma_x >_{tr}}{dt} \right| \ll \left| \frac{d < \sigma_y >_{tr}}{dt} \right| \sim \omega_L.
\]

So that Exp. (19) for \( d\phi_{tr}/dt \) becomes simpler - \( \frac{d\phi_{tr}}{dt} = \frac{d<\hat{S}_y>_{tr}}{dt} / <\hat{S}_x>_{tr} \).

As the value of \( \omega_L \) is small in (12), the functions \( \psi^{(1)}(x,t) \) and \( \psi^{(1)}(x,t) \) can be written in the form \( \psi^{(1)} \approx \psi - \omega_L \tilde{\psi}, \psi^{(1)} \approx \psi + \omega_L \tilde{\psi} \) where \( \psi \) and \( \tilde{\psi} \) do not depend on \( \omega_L \). Then, keeping in Exps. (17) and (20) only the main terms, we obtain

\[
\frac{d\phi_{tr}}{dt} = -\frac{\omega_L}{\hat{T}} \int_a^b |\psi_{tr}(x,t)|^2 dx - \frac{\omega_L}{m} \Re\left[ \psi_{tr}(x,t) \left( \frac{\partial \psi_{tr}^*(x+0,t)}{\partial x} - \frac{\partial \psi_{tr}^*(x-0,t)}{\partial x} \right) \right] - \psi_{tr}^*(x,t) \left( \frac{\partial \psi_{tr}(x+0,t)}{\partial x} - \frac{\partial \psi_{tr}(x-0,t)}{\partial x} \right).
\] (21)
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So, there are two reasons that lead to the change of the angle $\phi_{tr}$ in the course of scattering - the Larmor precession of the average spin of transmitted electrons in the magnetic field switched on in the barrier region and breaking the unitary evolution of this subensemble at the point $x = x_c$.

Note that both the terms in (21) are zero long before and long after the scattering event. So that, to simplify the definition of the total angle $\Delta \phi_{tr}$ of the spin rotation for a 1D completed scattering, one may shift the time of beginning and finishing this process to the minus and plus infinity, respectively. Thus,

$$\Delta \phi_{tr} = \int_{-\infty}^{\infty} \frac{d\phi_{tr}}{dt} dt = -\omega_L \left( \tau_{tr}^L + \tau_{int} \right),$$

(22)

where $\tau_{tr}^L$ is the Larmor transmission time. Considering Exp. (21) in (22), we obtain

$$\tau_{tr}^L = \frac{1}{T} \int_{-\infty}^{\infty} dt \int_a^b dx |\psi_{tr}(x, t)|^2.$$ 

(23)

So, the larger is the probability of finding a particle in the barrier region, the larger is the value of $\tau_{tr}^L$. The second term in $\Delta \phi_{tr}$, associated with a non-unitary evolution of $\psi_{tr}(x, t)$, has no relation to the average duration of staying a particle in the barrier region -

$$\tau_{int} = \frac{\hbar}{mT} \int_{-\infty}^{\infty} \Re \left[ \psi_{tr}(x_c, t) \left( \frac{\partial \tilde{\psi}_{tr}^*(x_c + 0, t)}{\partial x} - \frac{\partial \tilde{\psi}_{tr}^*(x_c - 0, t)}{\partial x} \right) - \tilde{\psi}_{tr}^*(x_c, t) \left( \frac{\partial \psi_{tr}(x_c + 0, t)}{\partial x} - \frac{\partial \psi_{tr}(x_c - 0, t)}{\partial x} \right) \right] dt.$$ 

(24)

For Gaussian-like wave packets, the integral over the time interval $(-\infty, \infty)$, in Exps. (23) and (24), can be calculated. Since

$$\psi_{tr}(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A^{in}(k)\psi_{tr}(x; k)e^{-iE(k)t/\hbar}dk,$$

(25)

where $\psi_{tr}(x; k)$ is defined by Exps. (5) and (6), we have (see [19])

$$\tau_{tr}^L = \frac{1}{T} \int_0^\infty \varpi(k)T(k)\tau_{dwell}(k)dk,$$

(26)

where $\varpi(k) = |A^{in}(k)|^2 - |A^{in}(-k)|^2$; note, for a completed scattering $|A^{in}(k_0)| \gg |A^{in}(-k_0)|$. Similarly, Exp. (24) for $\tau_{int}$ is reduced to the form

$$\tau_{int} = \frac{1}{T} \int_0^\infty \varpi(k)T(k)\tau_{int}(k)dk;$$

(27)

$$\tau_{int}(k) = \frac{1}{kT(k)} \Re \left[ \psi_{tr}(x_c; k) \left( \frac{\partial \tilde{\psi}_{tr}^*(x_c + 0; k)}{\partial x} - \frac{\partial \tilde{\psi}_{tr}^*(x_c - 0; k)}{\partial x} \right) - \tilde{\psi}_{tr}^*(x_c; k) \left( \frac{\partial \psi_{tr}(x_c + 0; k)}{\partial x} - \frac{\partial \psi_{tr}(x_c - 0; k)}{\partial x} \right) \right].$$

(27)

As regards the initial and final values of $\phi_{tr}$, then $\phi_{tr}^{(0)} = -2\omega_L \Im \psi_{tr}^{out} \psi_{tr}^{in} > \equiv -\omega_L T_{tr}^{(0)}, \ \phi_{tr}^{end} = -2\omega_L \Im \psi_{tr}^{out} \psi_{tr}^{out} > \equiv -\omega_L T_{tr}^{end}$. So that, in addition to the
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relationship (22), the following expressions must be also true: \( \Delta \phi_{tr} \equiv \phi_{tr}^{end} - \phi_{tr}^{(0)} = -\omega_L (\tau_{tr}^{end} - \tau_{tr}^{(0)}) \). Hence, finally, we have

\[
\tau_{tr}^{end} = \tau_{tr}^{(0)} + \tau_{tr}^{L} + \tau_{int},
\]  

(28)
i.e., the final readings \( \tau_{tr}^{end} \) of the Larmor clock do not give the time spent by transmitted particles in the barrier region. Thus, now we can explain the Hartman effect.

We have to stress once more that the time \( \tau_{end}^{tr} \), being defined in terms of the transmitted wave packet, is the same both in the standard and new models of a 1D completed scattering. For the most interesting case, namely for a particle with energy \( E \), which tunnels through the rectangular barrier (\( E < V_0 \)), we have

\[
\tau_{tr}^{end}(k) = \frac{mk}{\hbar \kappa} \cdot \frac{2\kappa d(k^2 - k^2) + \kappa_0^2 \sinh(2\kappa d)}{4k^2 \kappa^2 + \kappa_0^4 \sinh^2(\kappa d)}
\]  

(by the SM (see [9]), \( \tau_{tr}^{end} \) is equal to the dwell time). Besides, as is shown in [19],

\[
\tau_{tr}^{(0)}(k) = \frac{2mk}{\hbar \kappa} \cdot \frac{(\kappa^2 - k^2) \sinh(\kappa d) + \kappa_0^2 \kappa d \cosh(\kappa d)}{4k^2 \kappa^2 + \kappa_0^4 \sinh^2(\kappa d)}
\]  

(30)

(it is also useful to note that \( \tau_{ref}^{(0)}(k) = \tau_{ref}^{(0)}(k), \tau_{ref}^{end}(k) = \tau_{ref}^{end}(k) \)). As regards \( \tau_{tr}^{L} \), as it follows from Exp. (26), in the limit \( l_0 \to \infty \) considered here, it coincides with the dwell time \( \tau_{dwell}^{tr}(k) \).

So, as it follows from Exp. (29), \( \tau_{tr}^{end}(k) \) does saturate with increasing the barrier’s width. Within the SM where this quantity gives directly the time spent by a particle in the barrier region, this result leads to the contradiction with special relativity. However, in our model we meet a principally different situation. Now the tunnelling time increases exponentially with the increasing of \( d \), so that the effective velocity of electrons to enter the barrier region decreases exponentially rather than increases beyond all bounds, as it follows from the SM. As regards \( \tau_{tr}^{end}(k) \), for wide rectangular barriers this quantity is small due to the term \( \tau_{int} \) to be negative by value and comparable with \( \tau_{dwell}^{tr}(k) \). The role of the initial readings \( \tau_{tr}^{(0)}(k) \) is not so essential because, in the case considered, \( |\tau_{tr}^{(0)}| \ll \tau_{tr}^{end} \).

4. Conclusion

So, the fact of the saturation of the asymptotic group time and the final readings of the Larmor clock, with increasing the barrier’s width, for electrons tunnelling trough a wide rectangular barrier, does not at all mean that the effective velocity of tunnelling can be superluminal. As it follows from our model, none of these characteristic times gives the time spent, on the average, by transmitted electrons in the barrier region. In our model, the latter is described by the exact group transmission time and the dwell transmission time. Both these quantities show that the effective velocity of an electron decreases exponentially when it enters the region of a wide rectangular barrier.
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