A distributed (preconditioned) projected–reflected–gradient algorithm for stochastic generalized Nash equilibrium problems

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Abstract—We consider the stochastic generalized Nash equilibrium problem (SGNEP) with joint feasibility constraints and expected–value cost functions. We propose a distributed stochastic preconditioned projected reflected gradient algorithm and show its almost sure convergence when the pseudogradient mapping is cocoercive. The algorithm is based on monotone operator splitting methods for SGNEPs when the expected–value pseudogradient mapping is approximated at each iteration via an increasing number of samples of the random variable, an approach known as sample average approximation. Finally, we show that a non-preconditioned variant of our proposed algorithm has less restrictive convergence guarantees than state-of-the-art (preconditioned) forward-backward algorithms.

I. INTRODUCTION

Stochastic generalized Nash equilibrium problems (SGNEPs) have received some attention from the system and control community [1], [2], [3]. In a SGNEP, a set of agents interacts with the aim of minimizing their expected-value cost functions while subject to some joint feasibility constraints. The main feature is that both the cost function and the constraints are uncertain and depend on the strategies chosen by the other agents. Stochastic equilibrium problems arise when there is some uncertainty, modelled via a random variable with an unknown distribution. One main reason for the interest is related to their possible applications. For instance, any networked Cournot game with market capacity constraints and uncertainty in the demand can be modelled as a SGNEP [4], [5]. Other applications can be found in transportation systems, where the drivers perception of travel time is a possible source of uncertainty [6], and in electricity markets where companies schedule their energy production without fully knowing the demand [7].

Now, if the random variable is known, the expected–value formulation can be solved via a standard technique for deterministic GNEPs [8], [9]. Similarly to the deterministic case, one possible approach for SGNEPs is to recast the problem as a stochastic variational inequality (SVI) through the use of the Karush–Kuhn–Tucker conditions. Then, the problem can be rewritten as a monotone inclusion and solved via operator splitting techniques. Besides the fact that we use the algorithm to find a SGNE, the difficulty in the stochastic case is that the pseudogradient is usually not directly accessible, for instance because the expected value is hard or expensive to compute. For this reason, in many situations, the search for a solution of a SVI relies on samples of the random variable.

Two are the main approximation schemes used in the literature: the stochastic approximation (SA) and the sample average approximation (SAA). In the first case, the approximation is done by using only one realization of the random variable. SA was first presented in [10], it is computationally less expensive than SAA but usually it requires stronger assumptions on the pseudogradient mapping and on the parameters [2], [11], [12]. The second approximation scheme takes instead an increasing number of samples at every iteration. SAA has the disadvantage of being computationally costly but it requires weaker assumptions to ensure convergence [13], [14].

Depending on the monotonicity assumptions on the pseudogradient mapping or the affordable computational complexity, there are different algorithms that can be used to find a SGNE. Among others, one can consider the stochastic preconditioned forward–backward (SpFB) algorithm [15] which is guaranteed to converge to a Nash equilibrium under cocoercivity of the pseudogradient and by demanding one projection step per iteration. However, cocoercivity is not the weakest possible assumption, therefore one would like to have an algorithm that converges under mere monotonicity. For instance, the stochastic forward–backward–forward (SFBF) algorithm involves only one projection step per iteration but two costly evaluation of the pseudogradient mapping [13]. Another alternative is the stochastic extragradient (SEG) algorithm whose iterates are characterized by two projection steps and two evaluation of the pseudogradient mapping which may be expensive [14]. To summarize, having weaker assumptions comes at the price of implementing computationally expensive algorithms.

In this paper, we propose a stochastic preconditioned projected reflected gradient (SpPRG) algorithm for SGNEPs. The basic, deterministic version of this algorithm was first presented for variational inequalities by Malitsky in [16] and then extended to the stochastic case by Cui and Shanbhag in [11], [17]. Here, we consider the algorithm in [17] that uses the SAA scheme. The convergence of the algorithm is guaranteed when the pseudogradient mapping is monotone and “weak sharp”, a property that we discuss in Section VI. Unfortunately, the latter property is not trivial to check on the problem data. Therefore, to cope with SGNEPs, we assume that the pseudogradient mapping is cocoercive. Furthermore, in order to make our algorithm distributed, we exploit a suitable preconditioning. We also show that if
the equilibrium solution is unique, then mere monotonicity (as opposed to cocoercivity) is sufficient for convergence and preconditioning is not required. We emphasize that this is the first time that the PRG algorithm is designed for SGNEPs. Remarkably, under uniqueness of the solution, our algorithm has convergence guaranteed also for merely monotone pseudogradient mappings. This is a significant advantage compared to the SpPB which may not converge in that case. See Section VII for an example.

Notation: \( \langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) denotes the standard inner product and \( \| \cdot \| \) is the associated euclidean norm. We indicate that a matrix \( A \) is positive definite, i.e., \( x^T Ax > 0 \), with \( A > 0 \). \( A \otimes B \) indicates the Kronecker product between \( A \) and \( B \). Given a symmetric \( \Phi > 0 \), the \( \Phi \)-induced inner product is \( \langle x, y \rangle_\Phi = \langle \Phi x, y \rangle \) and the associated norm is defined as \( \| x \|_\Phi = \sqrt{\langle \Phi x, x \rangle} \). \( 0_m \) indicates the vector with \( m \) entries all equal to 0. Given \( x_1, \ldots, x_N \in \mathbb{R}^n, x := \text{col}(x_1, \ldots, x_N) = [x_1^T, \ldots, x_N^T]^T \), \( J_F = (I + 4F)^{-1} \) is the resolvent of the operator \( F \) where \( I \) is the identity operator. For a closed set \( C \subseteq \mathbb{R}^n \), the mapping \( \text{proj}_C : \mathbb{R}^n \rightarrow C \) denotes the projection onto \( C \), i.e., \( \text{proj}_C(x) = \text{argmin}_{v \in C} \| v - x \| \). \( \iota_C \) is the indicator function of the set \( C \), that is, \( \iota_C(x) = 1 \) if \( x \in C \) and \( \iota_C(x) = 0 \) otherwise. The set-valued mapping \( N_C : \mathbb{R}^n \rightarrow \mathbb{R}^n \) denotes the normal cone operator of the set \( C \), i.e., \( N_C(x) = \emptyset \) if \( x \notin C \); \( \{ v \in \mathbb{R}^n : \sup_{x \in C} u^T (x - z) \leq 0 \} \) otherwise.

For all \( x, y \in C \), a mapping \( F : C \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^n \) is:

- pseudomonotone if \( (x - y)^T F(y) - (x - y)^T F(x) \geq 0 \);
- monotone if \( (F(x) - F(y))^T (x - y) \geq 0 \); \( N \) -Lipschitz continuous if for \( L > 0 \) \( \| F(x) - F(y) \| \leq L \| x - y \| \).

The cost function should satisfy some assumptions, postulated next, to make our analysis possible. Such assumptions are standard for (stochastic) Nash equilibrium problems [1].

**Assumption 1:** For each \( i \in \mathcal{I} \) and \( x_{-i} \in \mathcal{X}_{-i} \), the function \( J_i(\cdot, x_{-i}) \) is convex and continuously differentiable. For each \( i \in \mathcal{I} \) and \( x \in \Xi \), the function \( J_i(x_{-i}, \xi) \) is convex, Lipschitz continuous, and continuously differentiable. The function \( J_i(x_{-i}, x_{-i}, \cdot) \) is measurable and for each \( x_{-i} \), its Lipschitz constant \( l_i(x_{-i}, \xi) \) is integrable in \( \xi \).

Since we consider a SGNEP, we introduce affine shared constraints, \( Ax \leq b \). Thus, we denote each agent \( i \in \mathcal{I} \) feasible decision set with the set-valued mapping

\[
\mathcal{X}_i(x_{-i}) := \left\{ y_i \in \Omega_i \mid A_i y_i \leq b - \sum_{j \neq i} A_j x_j \right\},
\]

where each \( A_i \in \mathbb{R}^{m_i \times n_i} \) indicates how agent \( i \) is involved in the coupling constraints and \( b \in \mathbb{R}^m \). The collective feasible set can be then written as

\[
\mathcal{X} = \Omega \cap \{ y \in \mathbb{R}^n | Ay - b \leq 0 \}
\]

where \( \Omega = \prod_{i=1}^N \Omega_i \) and \( A = [A_1, \ldots, A_N] \in \mathbb{R}^{m \times n} \). We suppose that there is no uncertainty in the constraints.

**Assumption 2:** For each \( i \in \mathcal{I} \), the set \( \Omega_i \) is nonempty, compact and convex. The set \( \mathcal{X} \) satisfies Slater’s constraint qualification.

The aim of each agent \( i \), given the decision variables of the other agents \( x_{-i} \), is to choose a decision \( x_i \), that solves its local optimization problem, i.e.,

\[
\forall i \in \mathcal{I} : \begin{align*}
\min_{x_i \in \Omega_i} & \quad J_i(x_i, x_{-i}) \\
\text{s.t.} & \quad A_i x_i \leq b - \sum_{j \neq i} A_j x_j.
\end{align*}
\]

From a game-theoretic perspective, we aim at computing a stochastic generalized Nash equilibrium (SGNE) [1].

**Definition 1:** A collective variable \( x^* \in \mathcal{X} \) is a stochastic generalized Nash equilibrium if, for all \( i \in \mathcal{I} \):

\[
J_i(x_i^*, x_{-i}^*) \leq \inf\{ J_i(y, x_{-i}^*) \mid y \in \mathcal{X}_i(x_{-i}^*) \}.
\]

In other words, a SGNE is a vector of strategies where none of the agents can decrease its cost function by unilaterally deviating from its decision variable.

When Assumptions[1] hold, existence of a SGNE of the game in (3) is guaranteed by [1, §3.1] but uniqueness does not hold in general [1, §3.2].

Among all the possible Nash equilibria, we focus on those that are also solutions of an associated (stochastic) variational inequality. First let us define the pseudogradient mapping as

\[
F(x) = \text{col} \left( E[\nabla_{x_i} J_i(x_i, x_{-i})]_{i \in \mathcal{I}} \right),
\]

where the exchange between the expected value and the gradient is possible because of Assumption[1] [1, Lem. 3.4]. Then, the stochastic variational inequality SVI(\( \mathcal{X}, F \)) is the problem of finding \( x^* \in \mathcal{X} \) such that

\[
\langle F(x^*), x - x^* \rangle \geq 0, \quad \text{for all} \quad x \in \mathcal{X}.
\]

with \( F(x) \) as in (4) and \( \mathcal{X} \) as in (1). We also note that any solution of SVI(\( \mathcal{X}, F \)) is a SGNE of the game in (3) while
the opposite does not hold in general. In fact, a game may have a Nash equilibrium while the associated VI may have no solution [21, Prop. 12.7].

Remark 1: Under Assumptions [12], the solution set of SVI(\mathcal{X}, \mathcal{F}) is non empty and compact, i.e. \text{SOL}(\mathcal{X}, \mathcal{F}) \neq \emptyset [18, Corollary 2.2.5].

B. Operator-theoretic characterization

In this section, we rewrite the SGNEP as a monotone inclusion, i.e., the problem of finding a zero of a set-valued monotone operator.

To this aim, we characterize the SGNE of the game in terms of the Karush–Kuhn–Tucker (KKT) conditions for the coupled optimization problems in (3). For each agent \(i\in \mathcal{I}\), let us denote with \(\lambda_i \in \mathbb{R}^{m_i}_{\geq 0}\) the dual variable associated with the coupling constraints. Then, the Lagrangian function, for every \(i \in \mathcal{I}\), is given by \(\mathcal{L}_i(x, \lambda_i) := J_i(x_i, x_{-i}) + \lambda_i^\top(x_i) + \lambda_i^\top(Ax - b).
\) It holds that the set of strategies \(x^*\) is a SGNE if and only if the following KKT conditions are satisfied [22, Th. 4.6]:

\[
\forall i \in \mathcal{I} : \begin{cases} 
0 \in \mathbb{E}[\nabla_{x_i} J_i(x_i^*, x_{-i}, \xi)] + N_{\Omega_i}(x_i^*) + A_i^\top \lambda_i \\
0 \in -(Ax^* - b) + N_{\mathbb{R}^m_{\geq 0}}(\lambda^*). 
\end{cases}
\]

Similarly, a variational SGNE (v-SGNE) can be characterized by using the KKT conditions associated to the SVI in (3) [22, Proposition 1.2.1]. The associated KKT optimality conditions reads as

\[
\forall i \in \mathcal{I} : \begin{cases} 
0 \in \mathbb{E}[\nabla_{x_i} J_i(x_i^*, x_{-i}, \xi)] + N_{\Omega_i}(x_i^*) + A_i^\top \lambda_i \\
0 \in -(Ax^* - b) + N_{\mathbb{R}^m_{\geq 0}}(\lambda^*). 
\end{cases}
\]

The connection between the KKT conditions in (7) and a v-SGNE is summarized next. From [20, Th. 3.1], it follows that if \(x^*\) is a solution of SVI(\mathcal{X}, \mathcal{F}) at which the KKT conditions (7) hold, then \(x^*\) is a solution of the SGNEP at which the KKT conditions (6) hold with \(\lambda_1 = \lambda_2 = \ldots = \lambda_N = \lambda^* \). Viceversa, if \(x^*\) is a solution of the SGNEP at which the KKT conditions (6) hold with \(\lambda_1 = \lambda_2 = \ldots = \lambda_N = \lambda^* \), \(x^*\) is a solution of SVI(\mathcal{X}, \mathcal{F}) in (3). In words, [20, Th. 3.1] says that variational equilibria are those such that the shared constraints have the same dual variables for all the agents.

III. DISTRIBUTED STOCHASTIC PRECONDITIONED PROJECTED REFLECTED GRADIENT ALGORITHM

In this section, we propose a distributed stochastic preconditioned projected reflected gradient (SpPRG) algorithm for finding a v-SGNE of the game in (3). The iterations are presented in Algorithm 1 which is inspired by [17], [16]. For each agent \(i\), the variables \(x_i^k, z_i^k \) and \(\lambda_i^k\) denote the local variables \(x_i, z_i \) and \(\lambda_i \) at the iteration time \(k\) while \(\alpha_i, \xi_i \) and \(\sigma_i \) are the step sizes. We note that agents can equivalently share the already computed dual variable \(\lambda = \text{col}(\lambda_1, \ldots, \lambda_N) \) or only \(\lambda = \text{col}(\lambda_1, \ldots, \lambda_N) \) and let the receiving agent do the computation. In any case, the number of times that the agents communicate is the same. Since we want the algorithm to be distributed, we assume that each agent \(i\) only knows its local data, i.e.,

\[
\Omega_i, A_i \text{ and } b_i.
\]

Moreover, each player is able to compute \(E[\nabla_{x_i} J_i(x_i, x_{-i}, \xi)]\), given the collective decision \(x\). Since the expected value can be hard to evaluate, users compute an approximation (we give more details later on). We assume therefore that each agent has access to all the decision variables that affect its pseudogradient. These information are collected, for each agent \(i\), in the set \(\mathcal{N}^i\), that is, the set of agents \(j\) whose decision \(x_j\) explicitly influences \(J_i\).

Since the v-SGNE requires consensus of the dual variables, we introduce an auxiliary variable \(z_i \in \mathbb{R}^m\) for all \(i \in \mathcal{I}\). The role of \(z = \text{col}(z_1, \ldots, z_N)\) is to help reaching consensus and it will be properly defined later in this section. The auxiliary variable \(z_i\) and a local copy of the dual variable \(\lambda_i\) are shared through the graph \(\mathcal{G}_\lambda = (\mathcal{I}, \mathcal{E}_\lambda)\). The set of edges \(\mathcal{E}_\lambda\) represents the exchange of the private information on the dual variables; \((i, j) \in \mathcal{E}_\lambda\) if agent \(i\) can receive \(\{\lambda_i, \lambda_j\}\) from agent \(j\). The set of neighbours of \(i\) in \(\mathcal{G}_\lambda\) is indicated with \(\mathcal{N}^\lambda_i = \{j|(j, i) \in \mathcal{E}_\lambda\}\) [8], [3]. Since each agent feasible set implicitly depends on all the other agents decisions (through the shared constraints), to reach consensus of the dual variables, all agents must coordinate and therefore, \(\mathcal{G}_\lambda\) must be connected.

Assumption 3: The dual-variable communication graph \(\mathcal{G}_\lambda\) is undirected and connected.

The weighted adjacency matrix of the dual variables graph is indicated with \(W \in \mathbb{R}^{N \times N}\). Let \(D = W - W^T\) be the Laplacian matrix associated to \(W\), where \(D = \text{diag}(d_1, \ldots, d_N)\) is the diagonal matrix of the agents degrees \(d_i = \sum_{j=1}^N w_{i,j}\). It follows from Assumption 3 [5] that the adjacency matrix \(W\) and the Laplacian \(L\) are both symmetric, i.e., \(W = W^T\) and \(L = L^T\).

To obtain the iterations presented in Algorithm 1, let us
From now on, we indicate

\[ \begin{align*}
0 \in \mathcal{T}(x, \lambda) := \left[ N_\Omega(x) + \mathbb{F}(x) + A^T \lambda \right] + \left[ \begin{array}{c} N_{R_{m \geq 0}}(\lambda) \end{array} \right],
\end{align*} \tag{8} \]

where \( \mathcal{T} : \mathcal{X} \times \mathbb{R}_{m \geq 0}^m \rightarrow \mathbb{R}^m \times \mathbb{R}^m \) is a set-valued mapping.

We note that the mapping \( \mathcal{T} \) can be written as the sum of two operators:

\[ \begin{align*}
A : [x] &\rightarrow [F(x)] \quad [x] \mapsto [N_\Omega(x)] + \begin{bmatrix} 0 & A^T \end{bmatrix} [x] \\
B : [x] &\rightarrow [N_{R_{m \geq 0}}(\lambda)] + \begin{bmatrix} -A & 0 \end{bmatrix} [x].
\end{align*} \tag{9} \]

Then, finding a solution of the game in (3) translates in finding a zero of the operator \( \mathcal{T} = A + B \) or equivalently, \( x^* \in \text{zer}(A + B) \).

Let \( \mathcal{L} \) be the Laplacian matrix of \( G_i \), and set \( L = L \otimes I_m \in \mathbb{R}^{N_m \times N_m} \). Following [8], to force consensus on the dual variables, we impose the Laplacian constraint \( \mathcal{L} x = 0 \). Then, to preserve monotonicity, we augment the two operators \( A \) and \( B \) introducing the auxiliary variable \( \lambda \). Let \( A = \text{diag}(A_1, \ldots, A_N) \in \mathbb{R}^{N \times N} \) and \( \lambda = \text{col}(A_1, \ldots, A_N) \in \mathbb{R}^{Nm} \) and similarly let us define \( b \) of suitable dimensions. Then, the extended operators read as

\[ \begin{align*}
\hat{A} : [x] &\rightarrow [F(x)] + \begin{bmatrix} 0 \end{bmatrix} \quad [x] \mapsto [N_\Omega(x)] + \begin{bmatrix} 0 & A^T \end{bmatrix} [x] \\
\hat{B} : [x] &\rightarrow [N_{R_{m \geq 0}}(\lambda)] + \begin{bmatrix} -A & 0 \end{bmatrix} [x].
\end{align*} \tag{10} \]

From now on, we indicate \( \omega = \text{col}(x, z, \lambda) \). The following lemma ensure that the zeros of \( A + B \) are v-SGNE.

**Lemma 1:** Let Assumptions 1-3 hold. Consider the operators \( \hat{A} \) and \( \hat{B} \) in (9) and \( A \) and \( B \) in (10). Then, the following statements hold.

(i) If \( \omega^* \in \text{zer}(\hat{A} + \hat{B}) \), then \( x^* \) is a v-SGNE of game in (3), i.e., \( x^* \) solves the SVI(\( \mathcal{X}, \mathbb{F} \)) in (3). Moreover \( \lambda^* = 1_N \otimes \lambda^* \), and \( (x^*, \lambda^*) \) satisfy the KKT condition in (7) i.e., \( \text{col}(x^*, \lambda^*) \in \text{zer}(A + B) \).

(ii) \( \text{zer}(A + B) \neq \emptyset \) and \( \text{zer}(\hat{A} + \hat{B}) \neq \emptyset \).

**Proof:** It follows from [8, Th. 2].

The properties of the operators \( \hat{A} \) and \( \hat{B} \) in (10) depends on the mappings \( \mathbb{F} \) in (4).

**Assumption 4:** \( \mathbb{F} \) is \( \beta \)-cocoercive for some \( \beta > 0 \).

**Remark 2:** If a function is \( \beta \)-cocoercive, it is also \( 1/\beta \)-Lipschitz continuous [19, Remark 4.15].

A technical discussion on this assumption is postponed to Section [V].

We can now show the necessary monotonicity properties of the extended operators.

**Lemma 2:** Let Assumptions 1 and 4 hold and let \( \Phi > 0 \). Then, \( \hat{A} \) and \( \hat{B} \) in (10) have the following properties.

(i) \( \hat{A} \) is \( \delta \)-cocoercive where \( 0 < \delta \leq \min \left\{ \frac{1}{\Phi \beta}, \beta \right\} \) and \( d^* \) is the maximum weighted degree of \( G_i \); 

(ii) \( \hat{B} \) is maximally monotone;

(iii) \( \Phi^{-1} \hat{A} \) is \( \theta \tau \)-cocoercive, with \( \tau = \frac{1}{\Phi - 1} \).

(iv) \( \Phi^{-1} \hat{B} \) is maximally monotone.

**Proof:** It follows from [8, Lem. 5] and [8, Lem. 7].

Since the expected value can be hard to compute, we take an approximation. At this stage, it is not important to specify if we use sample average or stochastic approximation, therefore, in what follows, we replace \( \mathbb{F} \) in (4) with an approximation \( \hat{F} \), given a vector sample of the random variable \( \xi_i \), and \( \hat{A} \) with

\[ \hat{A} : [x] \mapsto \left[ \begin{array}{c} \hat{F}(x, \xi) \end{array} \right] + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]

\[ \hat{B} : [x] \mapsto \left[ \begin{array}{c} 0 \\ b \end{array} \right] + \begin{bmatrix} \lambda \end{array} \quad (11) \tag{11} \]

Given \( \hat{A} \) in (11) and \( \hat{B} \) in (10), we can write Algorithm 1 in compact form as

\[ \omega^{k+1} = (I + \Phi^{-1} \hat{B})^{-1} (\omega^k - \Phi^{-1} \hat{A}(2\omega^k - \omega^{k-1})) \tag{12} \]

where \( \Phi \) is the preconditioning matrix. Specifically, let \( \alpha^{-1} = \text{diag}(\sigma_1^{-1}, \ldots, \sigma_N^{-1}) \in \mathbb{R}^{n \times n} \) and similarly \( \sigma^{-1} \) and \( \nu^{-1} \) of suitable dimensions. Then, we have

\[ \Phi = \begin{bmatrix} \alpha^{-1} & 0 & -\Lambda^T \\ 0 & \nu^{-1} & \mathcal{L} \\ -\Lambda & -\mathcal{L} & \sigma^{-1} \end{bmatrix}. \tag{13} \]

By expanding (12) with \( \hat{A} \) as in (11), \( \hat{B} \) as in (10) and \( \Phi \) as in (13), we obtain the iterations in Algorithm 1. We note that, since \( \hat{A} + \hat{B} \) is lower block triangular, the iterations of Algorithm 1 are sequential, i.e., \( \lambda_k \) depends on the last update \( x_{k+1} \) and \( z_{k+1} \) of the agents strategies and of the auxiliary variable respectively.

**IV. CONVERGENCE ANALYSIS WITH SAMPLE AVERAGE APPROXIMATION**

Since the distribution of the random variable is unknown, in the algorithm we have replaced the expected value with its approximation \( \hat{F} \). For the convergence analysis, we use the sample average approximation (SAA) scheme. We assume to have access to a pool of i.i.d. samples of the random variable collected, for all \( k \in \mathbb{N} \) and for each agent \( i \in \mathcal{I} \), in the vectors \( x_i^k \). At each time \( k \), the approximation is

\[ \hat{F}_i(x_i^k, x_{-i}^k, \xi_i^k) = \frac{1}{N_k} \sum_{h=1}^{N_k} \nabla x_i J_i(x_i^h, x_{-i}^h, \xi_i^h) \]

\[ = \mathbb{E}_{\xi_i} \left[ \frac{1}{N_k} \sum_{h=1}^{N_k} \nabla x_i J_i(x_i^h, x_{-i}^h, \xi_i^h) \right] \]

where \( N_k \) is the batch size, i.e., the number of sample to be taken. We define the distance between the expected value and its approximation as

\[ \epsilon_k = \mathbb{E}_{\xi_i} (x_i^\text{opt}, \xi_i^\text{opt}) - \mathbb{E}(x_i^k). \]

Since there is no uncertainty in the constraints, we have

\[ A_{\text{SAA}}(\omega^k, \xi^k) - A(\omega^k) = \epsilon_k = \text{col}(\epsilon_k, 0, 0), \]

where \( A_{\text{SAA}} \) is the operator \( A \) with approximation \( F_{\text{SAA}} \) as in (14). Let us introduce the filtration \( \mathcal{F} = \{ \mathcal{F}_k \} \), i.e., a family of \( \sigma \)-algebras such that \( \mathcal{F}_0 = \sigma (X_0) \), for all \( k \geq 1, \mathcal{F}_k = \sigma (X_0, \xi_1, \xi_2, \ldots, \xi_k) \) and \( \mathcal{F}_k \subseteq \mathcal{F}_{k+1} \) for all \( k \geq 0 \). The filtration \( \mathcal{F} \) collects the informations that each agent has
Lemma 2, \(\tau\) positive definite, we postulate the following assumption on Eq. \(\text{v-SPRG}\).

Concerning the batch size, the law in \(\text{(15)}\) is standard in the sample average approximation literature \([14, \text{Eq. 11}], [17, \text{Prop. 10}]\) since \(\text{Lemma 1}\).

For all \(k \in \mathbb{N}\),

\[
N_k \geq c(k + k_0)^{a+1}.
\]

\[
\text{for all } k \text{ and } C > 0, \text{ the stochastic error is such that}
\]

\[
\mathbb{E}[\{\epsilon^k\}|F_k] \leq \frac{C\sigma^2}{N_k} \text{ a.s.}.
\]

The bound for the stochastic error in \(\text{(16)}\) can be obtained as a consequence of some milder assumptions; we refer to \([13, \text{Lem. 4.2}], [14, \text{Lem. 3.12}], [15, \text{Lem. 6}]\) for more details. Concerning the batch size, the law in \(\text{(15)}\) is standard in the sample average approximation literature \([14, \text{Eq. 11}], [17, \text{Eq. v-SPRG}]\).

Further, since the preconditioning matrix must be positive definite, we postulate the following assumption on the parameters.

Assumption 7: Let \(\theta\) be the cocoercivity constant as in Lemma \([2]\) \(\tau = \frac{\theta}{\tau^0} \in (0, \frac{\theta}{c})\) and the step sizes \(\alpha, \nu\) and \(\sigma\) satisfy, for all \(i \in \mathcal{I}\),

\[
0 < \alpha_i \leq \left(\max_{j=1,\ldots,m} \left\{\frac{\sum_{k=1}^{\infty} (A_i^T)^{T} \|A_j^T\|}{\tau} + \tau\right\}\right)^{-1}
\]

\[
0 < \nu \leq \left(2d + \tau\right)^{-1}
\]

\[
0 < \sigma_i \leq \left(\max_{j=1,\ldots,m} \left\{\frac{\sum_{k=1}^{\infty} (A_i^T)^{T} \|A_j^T\|}{\sigma} + 2d + \tau\right\}\right)^{-1}
\]

where \(A_i\) indicates the entry \((j, k)\) of the matrix \(A_i^T\).

For example, we can obtain conditions \(\text{(17)}\) imposing that the preconditioning matrix \(\Phi\) to be diagonal dominant which, since it is symmetric, implies that \(\Phi\) is positive definite \([8]\).

We are now ready to state our convergence result.

Theorem 1: Let Assumptions \(\text{[1-7]}\) hold. Then, the sequence \((x_k)\) generated by Algorithm \([1]\) with \(F = F_{\text{SAA}}\) as in \(\text{(14)}\) converges a.s. to a v-SGNE of the game in \(\text{(4)}\).

Proof: The iterations of Algorithm \([1]\) are obtained by expanding \(\text{(13)}\) and solving for \(x_k, z_k\) and \(A_k\). Therefore, Algorithm \([1]\) is a SPRG iteration as in \(\text{(12)}\). The convergence of the sequence \((x_k, \lambda_k)\) to a v-GNE of the game in \(\text{(3)}\) then follows by \([17, \text{Prop. 10}]\) and Lemma \([1]\) since \(\Phi^{-1}A\) is cocoercive by Lemma \([2]\).

Remark 3: We note that adopting a SA scheme is not possible in this case because a vanishing step should be taken to control the stochastic error \([11]\). However, having a time-varying step implies using a variable metric, induced by the preconditioning matrix \(\Phi_k\) which depends on \(\alpha_k, \sigma_k\) and \(\nu, \rho\), for the convergence analysis. Although analysing a variable metric is possible, the matrix \(\Phi_k\) should satisfy additional assumptions that typically do not hold if the step size is vanishing \([23, \text{Prop. 3.4}]\).

V. TECHNICAL DISCUSSION ON WEAK SHARPNESS AND COCOERCIVITY

The original proof of the SPRG presented in \([17]\) for SVI shows convergence under the assumption of monotonicity and weak sharpness. The weak sharpness property was first introduced to characterize the minima of

\[
\min_{x \in \mathcal{X}} f(x)
\]

with \(f : \mathcal{X} \to \mathbb{R} [24]\). It was presented as an extension of the concept of strong (or sharp) solution, i.e., for all \(x^* \in \mathcal{X}^* = \text{SOL}(f, \mathcal{X})\)

\[
f(x) \geq f(x^*) + \rho \|x - x^*\|
\]

which holds if there is only one minimum. For generalizing non-unique solutions, the following definition was proposed in \([24]\): a set \(\mathcal{X}^*\) is a set of weak sharp minima for the function \(f\) if, for all \(x \in \mathcal{X}\) and \(x^* \in \mathcal{X}^*\),

\[
f(x) \geq f(x^*) + \rho \text{dist}(x, \mathcal{X}^*)
\]

where \(\text{dist}(x, \mathcal{X}^*) = \inf_{x^* \in \mathcal{X}^*} \|x - x^*\|\). We note that a strong solution is also a weak sharp minimum while the contrary holds only if the solution is unique \([24]\).

The concept was later extended to variational inequalities in \([25]\), using the formal definition

\[
- \mathcal{F}(x^*) \in \text{int} \left( \bigcap_{x \in \mathcal{X}} \left\{ T_{\mathcal{X}}(x) \cap N_{\mathcal{X}^*}(x) \right\}^\circ \right)
\]

which was already proved to be equivalent to \(\text{(19)}\) for the problem in \(\text{(18)}\) when \(\mathcal{F}(x^*) = \nabla f(x^*)\).

Unfortunately, the characterization in \(\text{(20)}\) is hard to use in a convergence proof. Therefore, more practical conditions have been proposed. The first one \([25]\) relies on the gap function and it reads as

\[
G(x) = \max_{y \in \mathcal{X}} \mathcal{F}(y, x - y) \geq \rho \text{dist}(x, \mathcal{X}^*)
\]

Another condition, used in the convergence proof of the SPRG \([17]\), was proposed in \([12]\):

\[
\langle \mathcal{F}(x^*), x - x^* \rangle \geq \rho \text{dist}(x, \mathcal{X}^*)
\]

for all \(x^* \in \mathcal{X}^*\) and \(x \in \mathcal{X}\). For the weak sharpness definition in \(\text{(20)}\) to be equivalent to \(\text{(21)}\) and \(\text{(22)}\), the (pseudo-dgradient) mapping should have the \(\text{F-unique}\) property, i.e., \(\mathcal{F}(\text{SOL}(\mathcal{F}, \mathcal{X}))\) should be at most a singleton \([18, \text{Section 2.3.1}]\). The class of operators that certainly have this property is that of \(\text{monotone}^\pm\) operators, namely, a monotone mapping \(F\) such that for all \(x, y \in \mathcal{X}\)

\[
\langle F(y) - F(x), y - x \rangle = 0 \Rightarrow F(y) = F(x)
\]

If a mapping is monotone\(^+\), then \(\text{(20)}\) is equivalent to \(\text{(21)}\) and \(\text{(22)}\) \([25, \text{Thm. 4.1}], [26, \text{Prop. 2}]\).

The monotone\(^+\) property does not necessarily hold for the extended operator \(A\) in \(\text{(10)}\), even if it holds for \(F\). However, it holds if the operator is cocoercive \([18, \text{Def. 2.3.9}]\). These considerations motivate our assumption. For more details on monotone\(^+\) operators and the weak sharpness property, we refer to \([27, 28, 25, 29]\).
We conclude this section with some examples showing that the condition in (22) may hold also if the mapping is not monotone$^+$ and that the domains are relevant for the validity of the assumption.

**Example 1:** Consider the variational inequality in (5) where $F(x) = \text{col}(-x_2, x_1)$ and $\mathcal{X} = [0, 1]^2$. The mapping $F$ is pseudomonotone but not monotone$^+$ on $\mathcal{X}$. The solution set is $\mathcal{X}^* = \{x \in \mathcal{X} : x_2 = 0\}$ and it holds that

$$G(x) = \max_{y \in \mathcal{X}} \langle F(y), x - y \rangle = \max_{y \in \mathcal{X}} -x_1 y_2 - y_1 y_2 + 2x_2 y_2 = 2x_2 = 2 \text{dist}(x, \mathcal{X}^*)$$

Therefore, $\mathcal{X}^*$ satisfies (21) with $\rho = 2$ but, for any $x^* \in \mathcal{X}^*$, $\{T_{\mathcal{X}}(x^*) \cap N_{\mathcal{X}}(x^*)\}^\circ = \{x_2 \leq 0\}$ and $-\tilde{F}(x^*) \notin \bigcap_{x^* \in \mathcal{X}^*} \{T_{\mathcal{X}}(x^*) \cap N_{\mathcal{X}}(x^*)\}^\circ$. Thus, the solution set $\mathcal{X}^*$ is not weakly sharp.

**Example 2:** Consider the mapping $F(x) = \text{col}(-x_2, x_1)$ and the associated variational inequality in (5) with $\mathcal{X} = [0, 1]^2$. Then the mapping $F$ is monotone but not monotone$^+$ on $\mathcal{X}$. The solution set is $\mathcal{X}^* = \{x \in \mathcal{X} : x_2 = 0\}$ and, similarly to (23), the conditions (21) and (22) hold.

Now, let $\mathcal{X} = \mathbb{R}^2$. In this case, there is only one solution and $\mathcal{X}^* = \{0\}$. However, (22) reads as

$$\langle F(0), x \rangle = 0 \geq \rho \text{ dist}(x, \mathcal{X}^*) = ||x||$$

which is false.

**VI. CONVERGENCE UNDER UNIQUENESS OF SOLUTION**

In light of the considerations in Section VI, we know that a unique solution is also a weak solution and that (22) may hold even if the mapping is not monotone$^+$. Therefore, here we consider the case of merely monotone operators but with unique solution and prove that the proposed (non-preconditioned) Algorithm 2 converges to a v-SGNE.

**Algorithm 2** Stochastic Relaxed Forward Backward (SRFB)

Initialization: $x_0^j \in \Omega_j$, $\lambda_0^j \in \mathbb{R}^m_{\geq 0}$, and $z_0^j \in \mathbb{R}^m$.

Iteration $k$: Agent $i$

1. Updates

$$\begin{align*}
\tilde{x}_i^k &= 2x_i^k - x_i^{k-1} \\
\tilde{z}_i^k &= 2z_i^k - z_i^{k-1} \\
\tilde{\lambda}_i^k &= 2\lambda_i^k - \lambda_i^{k-1}
\end{align*}$$

2. Updates:

$$\begin{align*}
x_i^{k+1} &= \text{prox}_{\gamma_1} [x_i^k - \alpha_i (\tilde{F}_i(\tilde{x}_i^k, \tilde{z}_i^k, \xi_i^k) + A_i^T \tilde{\lambda}_i^k)] \\
z_i^{k+1} &= z_i^k - \nu_i \sum_{j \in N_i^\lambda} w_{i,j} (\tilde{\lambda}_j^k - \lambda_j^k) \\
\lambda_i^{k+1} &= \text{proj}_{\mathbb{R}^m_{\geq 0}} [\lambda_i^k + \tau_i (A_i \tilde{x}_i^k - b_i) \\
&\quad - \tau \sum_{j \in N_i^\lambda} w_{i,j} [(z_j^{k+1} - z_j^k) - (\tilde{\lambda}_j^k - \lambda_j^k)]]
\end{align*}$$

Unlike Algorithm 1 in Algorithm 2 also $\tilde{z}$ should be updated, but only one communication round is required.

To obtain the iterates in Algorithm 2 a different splitting should be considered. Specifically, let

$$\begin{align*}
\tilde{C} : \begin{bmatrix} x \\ z \end{bmatrix} &\mapsto \begin{bmatrix} \tilde{F}(x) \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L & -L \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \\
\tilde{D} : \begin{bmatrix} x \\ z \end{bmatrix} &\mapsto \begin{bmatrix} N_{\Omega}(x) \\ 0 \end{bmatrix}
\end{align*}$$

Lemma 1 guarantees that the zeros of $\tilde{C} + \tilde{D}$ are the same as the zeros of $\mathcal{T}$ in (3).

Since the distribution of the random variable is unknown, we replace $\tilde{C}$ with

$$\tilde{C} : \begin{bmatrix} x, \xi \end{bmatrix} \mapsto \begin{bmatrix} \tilde{F}(x, \xi) \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ L & -L \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

where $\tilde{F}$ is an approximation of the expected value mapping $F$ in (4) given some realizations of the random vector $\xi$.

Then, Algorithm 2 in compact form reads similarly to Algorithm 1

$$\omega^{k+1} = (\text{Id} + \Phi^{-1} D)^{-1} (\omega_k - \Phi^{-1} \tilde{C}(2\omega_k - \omega^{k-1}))$$

where $\Phi > 0$ contains the inverse of step size sequences

$$\Phi = \text{diag}(\alpha, \nu, \sigma)$$

and $\alpha, \nu, \sigma$ are diagonal matrices. We note that $\Phi$ is not a preconditioning matrix in this case.

Now, to ensure that $\tilde{C}$ and $\tilde{D}$ have the properties that we use for the convergence result, we make the following assumption.

**Assumption 8:** $F$ as in (4) is monotone and $\ell_F$-Lipschitz continuous for some $\ell_F > 0$.

Then, the two operators $\tilde{C}$ and $\tilde{D}$ in (24) have the following properties.

**Lemma 3:** Let Assumptions 1 and 8 hold and let $\Phi > 0$. Then, $\tilde{C}$ and $\tilde{D}$ in (24) have the following properties.

1. $\tilde{C}$ is monotone and $\ell_2$-Lipschitz continuous.
2. The operator $\tilde{D}$ is maximally monotone.
3. $\Phi^{-1} \tilde{C}$ is monotone and $\ell_2$-Lipschitz continuous.
4. $\Phi^{-1} \tilde{D}$ is maximally monotone.

**Proof:** it follows from [8] and [15].

To guarantee that the weak sharpness property holds, we assume to have a strong solution.

**Assumption 9:** The SVI in (5) has a unique solution.

We can now state the convergence result.

**Theorem 2:** Let Assumptions 1, 8, and 9 hold. Then, the sequence $(x_k)_{k \in \mathbb{N}}$ generated by Algorithm 2 with $\tilde{F} = F^{\text{SRFA}}$ as in (14) converges a.s. to a v-SGNE of the game in (3).

**Proof:** The iterations of Algorithm 2 are obtained by expanding (26) and solving for $x_k, z_k$ and $\lambda_k$. Therefore, Algorithm 2 is a SPRG iteration as in (26). The convergence of the sequence $(x^k, \lambda^k)$ to a v-GNE of the game in (3) then follows by [17, Prop. 10] and Lemma 1 since $\Phi^{-1} \tilde{C}$ is monotone by Lemma 3 and has a unique solution.
VII. Numerical Simulations

Let us propose some numerical evaluations to validate the analysis: an illustrative example and a Nash-Cournot game. While the first comes from Example 2, the second is a realistic application to an electricity market with capacity constraints [8], [17].

All the simulations are performed on Matlab R2019b with a 2.3 GHz Intel Core i5 and 8 GB LPDDR3 RAM.

A. Illustrative example

We start with the stochastic counterpart of Example 2, that is, a monotone (non-cocoercive) stochastic Nash equilibrium problem with two players with strategies and respectively, and pseudogradient mapping

\[
F(x) = \begin{bmatrix}
0 & R_1(\xi) \\
-R_2(\xi) & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}.
\]

The random variables are sampled from a normal distribution with mean 1 and finite variance, following Assumption 6. The problem is unconstrained and the optimal solution is \((0, 0)\). The step sizes are taken to be the highest possible and we compare our SpPRG with the stochastic distributed preconditioned forward-backward (SpFB) which is guaranteed to converge under the same cocoercivity assumption with the SAA scheme [15].

Figure 1 shows that the SpFB does not converge while, due to the uniqueness of the solution, the SpPRG does.

![Fig. 1. Relative distance of the primal variable from the solution.](image)

B. Nash-Cournot game with market capacity constraints

Now, we consider an electricity market problem that can be casted as a network Cournot game with markets capacity constraints [8], [3], [17]. We consider a set of \(N = 20\) companies selling their product to a set of \(m = 7\) markets. Each generator decides the quantity of energy \(x_i\) to deliver to the \(m_i\) markets it is connected with. Each company \(i\) has a local constraint, i.e., a production limit, of the form \(0 < x_i < \gamma_i\) where each component of \(\gamma_i\) is randomly drawn from \([1, 1.5]\). Each company has a cost of production \(c_i(x_i) = 1.5x_i + q_i\), where \(q_i\) is a given constant, that is not uncertain. For simplicity, we assume the transportation costs are zero.

Each market \(j\) has a bounded capacity \(b_j\), randomly drawn from \([0.5, 1]\). The collective constraints are then given by \(Ax \leq b\) where \(A = [A_1, \ldots, A_N]\) and each \(A_i\) specifies in which market each company participates.

The prices of the markets are collected in \(P : \mathbb{R}^m \times \Xi \rightarrow \mathbb{R}^m\). The uncertainty variable, \(\xi\) which represents the demand uncertainty, appears in this functional. \(P\) is supposed to be a linear function and reads as \(P(\xi) = P(\xi) - DAx\). Each component of \(P = \text{col}(P_1(\xi), \ldots, P_5(\xi))\) is taken with a normal distribution with mean 3 and finite variance. The entries of \(D\) are randomly taken in \([0.5, 1]\).

The cost function of each agent is then given by

\[
J_i(x_i, x_{-i}, \xi) = c_i(x_i) - \mathbb{E}[P(\xi)^\top(Ax)A_ix_i].
\]

and it is influenced by the variables of the companies selling in the same market as in [8, Fig. 1]. The dual variables graph is a cycle graph with the addition of the edges \((2, 15)\) and \((6, 13)\) [8].

We simulate the SpFB, the forward-backward-forward (SFBF) and the extragradient (SEG) algorithms to make a comparison with our SPRG and SpPRG, using the SAA scheme. The parameters \(\alpha, \nu\) and \(\sigma\) are taken to be the highest possible that guarantee convergence.

As a measure of the distance from the solution, we consider the residual, \(\text{res}(x^k) = \lVert x^k - \text{proj}_{A}(x^k - F(x^k))\rVert\), which is equal zero if and only if \(x^k\) is a solution. The plots in Fig. 2 shows how the residual varies in the number of iterations while the plot in Fig. 3 shows the number of times that the pseudogradient mapping is computed. As one can see from the plots, the performances of SpPRG and SPRG are very similar. The difference in the trajectory is related to the different step sizes which depend on the Lipschitz constant of \(\mathcal{A}\) in [10] and \(\bar{C}\) in [24] respectively.

![Fig. 2. Relative distance of the primal variable from the solution.](image)

VIII. Conclusion

The stochastic projected reflected gradient algorithm can be applied to stochastic generalized Nash equilibrium seeking. To guarantee convergence to a solution and to obtain
a distributed algorithm, preconditioning should be used and the pseudogradient mapping should be cocoercive. However, should the equilibrium be unique, the cocoercivity assumption can be reduced to mere monotonicity.

REFERENCES

[1] U. Ravat and U. V. Shanbhag, “On the characterization of solution sets of smooth and nonsmooth convex stochastic Nash games,” SIAM Journal on Optimization, vol. 21, no. 3, pp. 1168–1199, 2011.
[2] J. Koshal, A. Nedic, and U. V. Shanbhag, “Regularized iterative stochastic approximation methods for stochastic variational inequality problems,” IEEE Transactions on Automatic Control, vol. 58, no. 3, pp. 594–609, 2013.
[3] C.-K. Yu, M. Van Der Schaar, and A. H. Sayed, “Distributed learning for stochastic generalized Nash equilibrium problems,” IEEE Transactions on Signal Processing, vol. 65, no. 15, pp. 3893–3908, 2017.
[4] V. DeMiguel and H. Xu, “A stochastic multiple-leader Stackelberg model: analysis, computation, and application,” Operations Research, vol. 57, no. 5, pp. 1220–1235, 2009.
[5] I. Abada, S. Gabriel, V. Briat, and O. Massol, “A generalized Nash–Cournot model for the northwestern european natural gas markets with a fuel substitution demand function: The gamnes model,” Networks and Spatial Economics, vol. 13, no. 1, pp. 1–42, 2013.
[6] D. Watling, “User equilibrium traffic network assignment with stochastic travel times and late arrival penalty,” European Journal of Operational Research, vol. 175, no. 3, pp. 1539–1556, 2006.
[7] R. Henrion and W. Römisch, “On m-stationary points for a stochastic equilibrium problem under equilibrium constraints in electricity spot market modeling,” Applications of Mathematics, vol. 52, no. 6, pp. 473–494, 2007.
[8] P. Yi and L. Pavel, “An operator splitting approach for distributed generalized Nash equilibria computation,” Automatica, vol. 102, pp. 111–121, 2019.
[9] G. Belgioioso and S. Grammatico, “Projected-gradient algorithms for generalized equilibrium seeking in aggregative games are preconditioned forward-backward methods,” in 2018 European Control Conference (ECC). IEEE, 2018, pp. 2188–2193.
[10] H. Robbins and S. Monro, “A stochastic approximation method,” The annals of mathematical statistics, pp. 400–407, 1951.
[11] [12] F. Yousefian, A. Nedíc, and U. V. Shanbhag, “Optimal robust smoothing extragradient algorithms for stochastic variational inequality problems,” in 53rd IEEE Conference on Decision and Control. IEEE, 2014, pp. 5831–5836.
[13] R. I. Bot, P. Mertikopoulos, M. Staudigl, and P. T. Vuong, “Forward-backward-forward methods with variance reduction for stochastic variational inequalities,” arXiv preprint arXiv:2002.03355, 2019.