Method of lower and upper functions and the existence of solutions to singular periodic problems for nonlinear differential equations of order two

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METHOD OF LOWER AND UPPER FUNCTIONS AND
THE EXISTENCE OF SOLUTIONS TO SINGULAR
PERIODIC PROBLEMS FOR NONLINEAR
DIFFERENTIAL EQUATIONS OF ORDER TWO

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Abstract. We construct nonconstant lower and upper functions for the periodic boundary
value problem $u'' = f(t, u), \ u(0) = u(2\pi), \ u'(0) = u'(2\pi)$ and find their estimates. By
means of these results we prove existence criteria for the problems $u'' \pm g(u) = e(t), \ u(0) = u(2\pi), \ u'(0) = u'(2\pi)$, where $\limsup_{x \to 0^+} g(x) = \infty$ is allowed and $e \in L[0, 2\pi]$ need not be
essentially bounded. We assume that $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ fulfills the Carathéodory conditions on
$[0, 2\pi] \times \mathbb{R}$, i.e. $f$ has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(., x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, .)$ is continuous on $\mathbb{R}$; (iii) for each compact set $K \subset \mathbb{R}$ the function $m_K(t) = \sup_{x \in K} |f(t, x)|$ is
Lebesgue integrable on $[0, 2\pi]$.

Keywords: Second order nonlinear ordinary differential equation, periodic solution, singular
problem, lower and upper functions, attractive and repulsive singularity, Duffing equation.

1. Introduction

In this paper we construct lower and upper functions to the periodic boundary value problem

\[ u'' = f(t, u), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi). \quad (1.1) \]

By means of these results, we prove existence criteria for the problems

\[ u'' \pm g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \]

where $\limsup_{x \to 0^+} g(x) = \infty$ is allowed and $e \in L[0, 2\pi]$ need not be essentially bounded. We assume that $f : [0, 2\pi] \times \mathbb{R} \mapsto \mathbb{R}$ fulfills the Carathéodory conditions on
$[0, 2\pi] \times \mathbb{R}$, i.e. $f$ has the following properties: (i) for each $x \in \mathbb{R}$ the function $f(., x)$ is measurable on $[0, 2\pi]$; (ii) for almost every $t \in [0, 2\pi]$ the function $f(t, .)$ is continuous on $\mathbb{R}$; (iii) for each compact set $K \subset \mathbb{R}$ the function $m_K(t) = \sup_{x \in K} |f(t, x)|$ is
Lebesgue integrable on $[0, 2\pi]$. 
For a given subinterval $J$ of $\mathbb{R}$ (possibly unbounded), $C(J)$ denotes the set of functions continuous on $J$. Furthermore, $L[0, 2\pi]$ stands for the set of functions Lebesgue integrable on $[0, 2\pi]$. $L^2[0, 2\pi]$ is the set of functions square Lebesgue integrable on $[0, 2\pi]$ and $AC[0, 2\pi]$ denotes the set of functions absolutely continuous on $[0, 2\pi]$. For $x$ bounded on $[0, 2\pi]$, $y \in L[0, 2\pi]$ and $z \in L^2[0, 2\pi]$ we denote

$$\|x\|_C = \sup_{t \in [0, 2\pi]} |x(t)|, \quad \|y\|_1 = \int_0^{2\pi} \|y\|_1, \quad \|y\|_1 = \int_0^{2\pi} \|y\|_1, \quad \|y\|_2 = \left( \int_0^{2\pi} \|z\|_2 dt \right)^{\frac{1}{2}}.$$

By a *solution* of (1.1) we mean a function $u : [0, 2\pi] \mapsto \mathbb{R}$ such that $u' \in AC[0, 2\pi]$, $u(0) = u(2\pi)$, $u'(0) = u'(2\pi)$ and

$$u''(t) = f(t, u(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi].$$

**Definition 1.** A function $\sigma_1$ is said to be a *lower function of the problem* (1.1) if $\sigma_1' \in AC[0, 2\pi]$,

$$\sigma_1''(t) \geq f(t, \sigma_1(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi],$$

$$\sigma_1(0) = \sigma_1(2\pi), \quad \sigma_1'(0) \geq \sigma_1'(2\pi).$$

Similarly, a function $\sigma_2$ is said to be an *upper function of the problem* (1.1) if $\sigma_2' \in AC[0, 2\pi]$,

$$\sigma_2''(t) \leq f(t, \sigma_2(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi],$$

$$\sigma_2(0) = \sigma_2(2\pi), \quad \sigma_2'(0) \leq \sigma_2'(2\pi).$$

The lower and upper functions approach we will use here is based on the following theorem which is contained in [8, Theorems 4.1 and 4.2].

**Theorem 2.** Let $\sigma_1$ and $\sigma_2$ be a lower and an upper function of the problem (1.1), respectively.

(I) Suppose $\sigma_1(t) \leq \sigma_2(t)$ on $[0, 2\pi]$. Then there is a solution $u$ of the problem (1.1) such that $\sigma_1(t) \leq u(t) \leq \sigma_2(t)$ on $[0, 2\pi]$.

(II) Suppose $\sigma_1(t) \geq \sigma_2(t)$ on $[0, 2\pi]$ and there is $m \in L[0, 2\pi]$ such that

$$f(t, x) \geq m(t) \quad \text{for a.e.} \quad t \in [0, 2\pi] \quad \text{and all}\quad x \in \mathbb{R},$$

$$f(t, x) \leq m(t) \quad \text{for a.e.} \quad t \in [0, 2\pi] \quad \text{and all}\quad x \in \mathbb{R}.$$ 

Then there is a solution $u$ of the problem (1.1) such that $\|u''\|_C \leq \|m\|_1$ and

$$\sigma_2(t_u) \leq u(t_u) \leq \sigma_1(t_u) \quad \text{for some}\quad t_u \in [0, 2\pi].$$
2. Construction of lower and upper functions

**Proposition 1.** Assume that there are $A \in \mathbb{R}$ and $b \in L[0,2\pi]$ such that

$$\bar{b} = 0,$$  \hspace{1cm} (2.1)

$$f(t, x) \leq b(t) \text{ for a.e. } t \in [0,2\pi] \text{ and all } x \in [A,B],$$  \hspace{1cm} (2.2)

where

$$B = A + \frac{\pi}{3} \|b\|_1.$$  \hspace{1cm} (2.3)

Then there exists a lower function $\sigma$ of the problem (1.1) such that

$$A \leq \sigma(t) \leq B \text{ on } [0,2\pi].$$  \hspace{1cm} (2.4)

**Proof.** Define

$$\sigma_0(t) = c_0 + \int_0^{2\pi} g(t, s)b(s)ds \text{ for } t \in [0,2\pi],$$

where

$$g(t, s) = \begin{cases} \frac{t(s - 2\pi)}{2\pi} & \text{if } 0 \leq t \leq s \leq 2\pi, \\ \frac{(t - 2\pi)s}{2\pi} & \text{if } 0 \leq s < t \leq 2\pi \end{cases}$$

is the Green function of the problem $v'' = 0$, $v(0) = v(2\pi) = 0$ and

$$c_0 = -\frac{1}{2\pi} \int_0^{2\pi} \left( \int_0^{2\pi} g(t, s)b(s)ds \right) dt.$$  

Then

$$\sigma''_0(t) = b(t) \text{ a.e. on } [0,2\pi]$$  \hspace{1cm} (2.5)

and

$$\sigma_0(0) = \sigma_0(2\pi).$$  \hspace{1cm} (2.6)

Furthermore, by virtue of (2.1) we have also

$$\sigma'_0(0) = \sigma'_0(2\pi).$$  \hspace{1cm} (2.7)

Multiplying the relation (2.5) by $\sigma_0$, integrating it over $[0,2\pi]$ and using the Hölder inequality we get

$$\|\sigma'_0\|_2^2 \leq \|b\|_1 \|\sigma_0\|_C.$$  

Further, as $\sigma_0 = 0$, the Sobolev inequality (see [5, Proposition 1.3]) yields

$$\|\sigma'_0\|_2^2 \leq \frac{\sqrt{\pi}}{\sqrt{6}} \|b\|_1 \|\sigma'_0\|_2,$$
and so
\[ \|\sigma_0\|_2 \leq \sqrt{\frac{\pi}{6}} \|b\|_1, \]
wherefrom using again the Sobolev inequality we get
\[ \|\sigma_0\|_C \leq \frac{\pi}{6} \|b\|_1. \]
Thus, the function \( \sigma \) given by
\[ \sigma(t) = \frac{\pi}{6} \|b\|_1 + A_0 + \sigma_0(t) \quad \text{for} \quad t \in [0, 2\pi] \quad (2.8) \]
satisfies (2.4). Furthermore, according to (2.1),(2.2) and (2.6)-(2.7) we have
\[ \sigma''(t) = \sigma_0''(t) = b(t) \geq f(t, \sigma(t)) \quad \text{for a.e.} \quad t \in [0, 2\pi] \quad (2.9) \]
and
\[ \sigma(0) = \sigma(2\pi), \quad \sigma'(0) = \sigma'(2\pi), \quad (2.10) \]
i.e. \( \sigma \) is a lower function of (1.1).

The following assertion is dual to Proposition 1 and its proof will be omitted.

**Proposition 2.** Assume that there are \( A \in \mathbb{R} \) and \( b \in L[0, 2\pi] \) such that
\[ b = 0 \]
and
\[ f(t, x) \geq b(t) \quad \text{for a.e.} \quad t \in [0, 2\pi] \quad \text{and all} \quad x \in [A, B] \]
where \( B \) is given by (2.3). Then there exists an upper function \( \sigma \) of the problem (1.1) with the property (2.4).

### 3. Applications to Lazer-Solimini singular problems

In this section we will consider possible singular problems of the attractive type
\[ u'' + g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \quad (3.1) \]
and of the repulsive type
\[ u'' - g(u) = e(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad (3.2) \]
where
\[ g \in C(0, \infty) \quad \text{and} \quad e \in L[0, 2\pi] \quad (3.3) \]
and it is allowed that \( \limsup_{x \to 0^+} g(x) = \infty. \)
The problem (3.1) has been studied by Lazer and Solimini in [6] for $e \in C[0, 2\pi]$ and $g$ positive. In [9, Corollary 3.3], their existence result has been extended to $e \in L[0, 2\pi]$ essentially bounded from above. Here, we bring conditions for the existence of solutions to (3.1) without assuming boundedness of $e$.

**Theorem 1.** Assume (3.3) and let there exist $A_1, A_2 \in (0, \infty)$ such that

\begin{align*}
g(x) \geq \tau & \text{ for all } x \in [A_1, B_1], \quad (3.4) \\
g(x) \leq \tau & \text{ for all } x \in [A_2, B_2], \quad (3.5)
\end{align*}

where

\[ B_1 - A_1 = B_2 - A_2 = \frac{\pi}{3} \| e - \tau \|_1 \quad (3.6) \]

and $A_2 \geq B_1$.

Then the problem (3.1) has a solution $u$ such that $A_1 \leq u(t) \leq B_2$ on $[0, 2\pi]$.

**Proof.** Define, for a.e. $t \in [0, 2\pi]$,

\[ f(t, x) = e(t) - \begin{cases} 
 g(A_1) & \text{if } x < A_1, \\
 g(x) & \text{if } x \geq A_1.
\end{cases} \]

Then $f$ satisfies the Carathéodory conditions on $[0, 2\pi] \times \mathbb{R}$. Furthermore, by (3.4) and (3.6), $f$ satisfies (2.1)-(2.3) with $b(t) = e(t) - \tau$ a.e. on $[0, 2\pi]$ and $[A, B] = [A_1, B_1]$. Hence, by Proposition 1 there exists a lower function $\sigma_1$ of (1.1) such that $\sigma_1(t) \in [A_1, B_1]$ for all $t \in [0, 2\pi]$. Similarly, (3.5), (3.6) and Proposition 2 yield the existence of an upper function $\sigma_2$ of (1.1) such that $\sigma_2(t) \in [A_2, B_2]$ on $[0, 2\pi]$. Now, since $A_2 \geq B_1$, we have $\sigma_1(t) \leq \sigma_2(t)$ on $[0, 2\pi]$ and the assertion (I) of Theorem 2 gives the existence of the desired solution $u$ to (1.1) which is naturally also a solution to (3.1). \hfill \Box

Classical Lazer and Solimini’s considerations [6] of the repulsive problem (3.2) have been extended by several authors (see e.g. [1]-[4], [7] and [10]). Here we present a related result, where $e$ need not be essentially bounded.

**Theorem 2.** Assume (3.3),

\[ \lim_{x \to 0^+} \int_x^1 g(\xi) d\xi = \infty, \quad (3.7) \]

and

\[ g_* := \inf_{x \in (0, \infty)} g(x) > -\infty. \quad (3.8) \]

Furthermore, let there exist $A_1, A_2 \in (0, \infty)$ such that

\begin{align*}
g(x) \leq -\tau & \text{ for all } x \in [A_1, B_1], \quad (3.9) \\
g(x) \geq -\tau & \text{ for all } x \in [A_2, B_2], \quad (3.10)
\end{align*}

where (3.6) is true and $A_1 \geq B_2$.

Then the problem (3.2) has a positive solution.
Proof. Denote
\[ K = \|e\|_1 + |g_*|, \quad B = B_1 + 2\pi K \] and \( K^* = K\|e\|_1 + \int_{A_2}^B |g(x)|\,dx. \)

It follows from (3.7) that \( \limsup_{x \to 0^+} g(x) = \infty \) and there exists \( \varepsilon \in (0, A_2) \) such that
\[ \int_{\varepsilon}^{A_2} g(x)\,dx > K^* \] and \( g(\varepsilon) > 0. \) \hfill (3.11)

Define
\[ \tilde{g}(x) = \begin{cases} g(x) & \text{if } x \geq \varepsilon, \\ g(\varepsilon) & \text{if } x < \varepsilon, \end{cases} \]
and
\[ f(t, x) = e(t) + \tilde{g}(x) \]
for a.e. \( t \in [0, 2\pi] \) and all \( x \in \mathbb{R}. \)

Now, we can argue as in the proof of Theorem 1 obtaining a lower function \( \sigma_1 \) and an upper function \( \sigma_2 \) of (1.1) such that \( \sigma_1(t) \geq \sigma_2(t) \) on \([0, 2\pi]\). The assertion (II) of Theorem 2 (with \( m(t) = g_* + e(t) \)) a.e. on \([0, 2\pi]\) implies that (1.1) has a solution \( u \) such that \( u(t_u) \in [A_2, B_1] \) for some \( t_u \in [0, 2\pi] \) and \( \|u''\|_{C} \leq K \). It remains to show that \( u(t) \geq \varepsilon \) holds on \([0, 2\pi]\).

Let \( t_0 \) and \( t_1 \in [0, 2\pi] \) be such that
\[ u(t_0) = \min_{t \in [0, 2\pi]} u(t) \] and \( u(t_1) = \max_{t \in [0, 2\pi]} u(t). \)

Clearly, \( A_2 \leq u(t_1) \leq B \). With respect to the periodic boundary conditions we have \( u'(t_0) = u'(t_1) = 0. \) Now, multiplying the differential relation \( u''(t) = e(t) + \tilde{g}(u(t)) \) by \( u'(t) \) and integrating over \([t_0, t_1]\) we get
\[
0 = \int_{t_0}^{t_1} u''(t)u'(t)\,dt = \int_{t_0}^{t_1} e(t)u'(t)\,dt + \int_{t_0}^{t_1} \tilde{g}(u(t))u'(t)\,dt,
\]
i.e.
\[
\int_{u(t_0)}^{u(t_1)} \tilde{g}(x)\,dx = -\int_{t_0}^{t_1} e(t)u'(t)\,dt \leq K\|e\|_1.
\]

Further,
\[
\int_{u(t_0)}^{A_2} \tilde{g}(x)\,dx \leq K\|e\|_1 + \int_{A_2}^{B} \tilde{g}(x)\,dx = K^*
\]
which, with respect to (3.11), is possible only if \( u(t_0) \geq \varepsilon \). Thus, \( u \) is a solution to (3.2). \hfill \Box

\textbf{Example 3.} Let \( g(x) = \frac{1}{x} \) on \((0, \infty)\). If \( \gamma > 0 \), then Theorem 1 ensures the existence of a positive solution to (3.1) for any \( e \in L[0, 2\pi] \) such that
\[
\sigma > 0 \quad \text{and} \quad \frac{\pi}{3} \sigma^{\frac{3}{2}} \|e - \sigma\|_{L} < 1. \hfill (3.12)
\]
The function $e(t) = c + \frac{1}{\sqrt{2\pi^2 t}} - \frac{1}{\pi}$ with $c \in \mathbb{R}$ is not essentially bounded from above on $[0, 2\pi]$. However, it satisfies (3.12) if

$$0 < c < \left(\frac{3}{\pi}\right)^\gamma.$$ 

We should mention that provided $e \in \mathbb{C}[0, 2\pi]$ or $e$ is essentially bounded from above, the condition $\tau > 0$ is sufficient for the existence of a solution to (3.1) (cf. [6] or [9], respectively).

**Example 4.** Let $e \in \mathbb{L}[0, 2\pi]$ be essentially unbounded from below and let

$$g(x) = \frac{1 + \sin \left(\frac{x}{\pi}\right)}{x} - \arctan(x), \quad x \in (0, \infty).$$

Then $g$ verifies the assumptions (3.3), (3.7) and (3.8) of Theorem 2. Let us suppose that $\tau = -5$. Then the equation $g(x) = 5$ has exactly 5 roots in the interval $[0.125, \infty)$. In particular, we have (see Figures 1 and 2)

$$x_1 \approx 0.126804, \quad x_2 \approx 0.141071, \quad x_3 \approx 0.167853, \quad x_4 \approx 0.200541, \quad x_5 \approx 0.244461,$$

$$g(x) > 5 \text{ on } (x_2, x_3) \cup (x_4, x_5) \text{ and } g(x) < 5 \text{ on } (x_1, x_2) \cup (x_3, x_4) \cup (x_5, \infty).$$

Therefore, by Theorem 2, if

$$\|e - \bar{e}\|_{L} \leq \frac{3}{\pi}(x_5 - x_4) \approx 0.0419392,$$

the problem

$$u'' = \frac{1 + \sin \left(\frac{u}{\pi}\right)}{u} - \arctan(u) + c(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi) \quad (3.13)$$

has a solution $u_1$ such that $u_1(t^*) \in [x_4, x_5 + d_1]$ for some $t^* \in [0, 2\pi]$, where $d_1 = x_5 - x_4$ (see Figure 3).

Similarly, by Theorems 1 and 2, if

$$\|e - \bar{e}\|_{L} < \frac{3}{2\pi}(x_5 - x_4) \approx 0.0209699,$$

the problem (3.13) has at least 2 different solutions $u_1$ and $u_2$, where $u_1(t^*) \in (x_5 - d_2, x_5 + d_2)$ for some $t^* \in [0, 2\pi]$ and $u_2(t) \in (x_4 - d_2, x_4 + d_2)$ for all $t \in [0, 2\pi]$, where $d_2 = \frac{1}{2}(x_5 - x_4)$ (see Figure 4).

Finally, if

$$\|e - \bar{e}\|_{L} \leq \frac{3}{\pi}(x_2 - x_1) \approx 0.0136238,$$

the problem (3.13) has at least 3 different solutions $u_1$, $u_2$ and $u_3$, where $u_1(t^*) \in [x_5 - d_3, x_5 + d_3]$ for some $t^* \in [0, 2\pi]$, $u_2(t) \in [x_4 - d_3, x_4 + d_3]$ for all $t \in [0, 2\pi]$ and $u_3(t) \in [x_1, x_2]$ for all $t \in [0, 2\pi]$, where $d_3 = x_2 - x_1$ (see Figure 5).
Existence of solutions to singular periodic problems

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