Can a charged ring levitate a neutral, polarizable object? Can Earnshaw’s theorem be extended to such objects?

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Abstract

Stable electrostatic levitation and trapping of a neutral, polarizable object by a charged ring is shown to be theoretically impossible. Earnshaw’s theorem precludes the existence of such a stable, neutral particle trap.

1 Introduction

In this tribute in honor of the memory of Prof. Dr. Herbert Walther, we consider the possibility of extending his famous work on the trapping of an ordered lattice of ions [1] in a Paul trap [2], to the trapping of neutral atoms, and more generally, to the possible levitation of a macroscopic neutral polarizable object, in a purely electrostatic trap, for example, in the DC electric field configuration of a charged ring. Earnshaw’s theorem will be extended to the case of such neutral objects, and we shall show below that the stable levitation and trapping of a neutral, polarizable object, which is a high-field seeker, is generally impossible in an arbitrary electrostatic field configuration. We shall do this first for the special case of the electrostatic configuration of a simple charged ring, and then for the general case of any DC electric field configuration.

Consider the charged-ring geometry shown in Figure 1. The region near the so-called “levitation” point \( L \) in this Figure is akin to the focal region of a lens in optics. Just as two converging rays of light emerging from a lens in physical optics cannot truly cross at a focus, but rather will undergo an

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Figure 1: A uniformly charged ring with radius $a$ lies on the horizontal $x$-$y$ plane, with its axis of symmetry pointing along the vertical $z$ axis. Can levitation and trapping of a neutral particle occur stably near point $L$, where there is a convergence of $E$-field lines?
“avoided crossing” near the focal point of this lens due to diffraction, so likewise two converging lines of the electric field cannot cross, and therefore they will also undergo an “avoided crossing” near $L$. There results a maximum in the $z$ component of the electric field along the vertical $z$ axis at point $L$. The resulting “avoided crossing” region of electric field lines in the vicinity of point $L$ is therefore similar to the Gaussian beam-waist region of a focused laser beam.

Ashkin and his colleagues $[3]$ showed that small dielectric particles, which are high-field seekers, are attracted to, and can be stably trapped at, such Gaussian beam waists in “optical tweezers”. Similarly here a neutral dielectric particle, which is a high-field seeker, will also be attracted to the region of the convergence of $E$-field lines in the neighborhood of $L$, where there is a local maximum in the electric field along the $z$ axis. The question arises: Can such a high-field seeker be stably levitated and trapped near $L$?

2 Calculation of the electric potential and field of a charged ring

The electric potential at the field point $P$ due to a charge element $dq'$ of the ring is given in by

$$d\Phi = \frac{dq'}{r}$$

(1)

where the distance $r$ from the source point, whose coordinates are $(x', y', 0)$, to the field point $P$, whose coordinates are $(x, 0, z)$, is

$$r = \sqrt{(x' - x)^2 + y'^2 + z^2}.$$  

(2)

(Primed quantities refer to the source point; unprimed ones to the field point). Since the charged ring forms a circle of radius $a$ which lies on the horizontal $x$-$y$ plane,

$$x'^2 + y'^2 = a^2.  \tag{3}$$

An infinitesimal charge element $dq'$ spanning an infinitesimal azimuthal angle of $d\phi'$ can be expressed as follows:

$$dq' = \left(\frac{Q}{2\pi a}\right) ad\phi' = -\left(\frac{Q}{2\pi a}\right) \frac{ad(x'/a)}{\sqrt{1 - (x'/a)^2}}$$

(4)

where $Q$ is the total charge of the ring. Let us introduce the dimensionless variables

$$\xi' \equiv \frac{x'}{a}, \quad \eta' \equiv \frac{y'}{a}, \quad \zeta \equiv \frac{z}{a}, \quad \varepsilon \equiv \frac{x}{a}.  \tag{5}$$

Thus

$$dq' = -\frac{Q}{2\pi} \frac{d\xi'}{\sqrt{1 - \xi'^2}}.$$
Due to the bilateral symmetry of the ring under the reflection $y' \to -y'$, it is useful to sum up in pairs the contribution to the electric potential from symmetric pairs of charge elements, such as $dq_1'$ and $dq_2'$ with coordinates $(x', +y', 0)$ and $(x', -y', 0)$, respectively, shown in Figure 1. These two charge elements contribute equally to the electric potential $\Phi$ if they span the same infinitesimal azimuthal angle $d\phi'$. Thus one obtains

$$\Phi (\varepsilon, \zeta) = \frac{Q}{\pi a} \int_{-1}^{+1} d\xi' \frac{1}{\sqrt{1 - \xi'^2}} \frac{1}{\sqrt{\varepsilon^2 - 2\varepsilon \xi' + 1 + \zeta^2}}.$$  

(6)

Along the $z$ axis, this reduces to the well-known result

$$\Phi (\varepsilon = 0, \zeta) = \frac{Q}{a} \frac{1}{\sqrt{1 + \zeta^2}} = \frac{Q}{\sqrt{z^2 + a^2}}.$$  

(7)

The $z$ component of the electric field, which is the dominant $E$-field component in the neighborhood of point $L$, is given by

$$E_z = -\frac{\partial \Phi}{\partial z} = \frac{Q}{\pi a^2} \int_{-1}^{+1} d\xi' \frac{1}{\sqrt{1 - \xi'^2}} \left( \frac{1}{\sqrt{\varepsilon^2 - 2\varepsilon \xi' + 1 + \zeta^2}} \right)^3.$$  

(8)

Along the $z$ axis, this also reduces to the well-known result

$$E_z = \frac{Qz}{(z^2 + a^2)^{3/2}},$$  

(9)

which has a maximum value at

$$z_0 = \frac{a}{\sqrt{2}} \text{ or } \zeta_0 = \frac{1}{\sqrt{2}}.$$  

(10)

The “levitation” point $L$ then has the coordinates

$$L \left( 0, 0, \frac{a}{\sqrt{2}} \right),$$  

(11)

neglecting for the moment the downwards displacement of a light particle due to gravity.

The potential energy $U$ for trapping a neutral particle with polarizability $\alpha$ in the presence of an electric field $(E_x, E_y, E_z)$ is given by

$$U = -\frac{1}{2} \alpha (E_x^2 + E_y^2 + E_z^2) \approx -\frac{1}{2} \alpha E_z^2,$$  

(12)

since the contributions to $U$ from the $x$ and $y$ components of the electric field, which vanish as $\varepsilon^4$ near the $z$ axis for small $\varepsilon$, can be neglected in a small neighborhood of $L$. 

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We now calculate the curvature at the bottom of the potential-energy well $U$ along the longitudinal $z$ axis, and also along the transverse $x$ axis. The force on the particle is given by

$$F = -\nabla U.$$  

(13)

Therefore the $z$ component of the force is, to a good approximation,

$$F_z = \alpha E_z \frac{\partial E_z}{\partial z},$$

(14)

and the Hooke’s law constant $k_z$ in the longitudinal $z$ direction is given by

$$k_z = -\frac{\partial F_z}{\partial z} = -\alpha \left\{ \left( \frac{\partial E_z}{\partial z} \right)^2 + E_z \frac{\partial^2 E_z}{\partial z^2} \right\},$$

(15)

where all quantities are to be evaluated at $L$ where $\varepsilon = 0$ and $\zeta_0 = 1/\sqrt{2}$. Taking the indicated derivatives and evaluating them at $L$, one obtains

$$k_z\big|_L = \frac{32 \alpha Q^2}{81 a^6},$$

(16)

where the positive sign indicates a longitudinal stability of the trap in the vertical $z$ direction.

The $x$ component of the force is, to the same approximation,

$$F_x = \alpha E_z \frac{\partial E_x}{\partial x},$$

(17)

and the Hooke’s law constant $k_x$ in the transverse $x$ direction is

$$k_x = -\frac{\partial F_x}{\partial x} = -\alpha \left\{ \left( \frac{\partial E_x}{\partial x} \right)^2 + E_z \frac{\partial^2 E_z}{\partial x^2} \right\},$$

(18)

where again all quantities are to be evaluated at $L$ where $\varepsilon = 0$ and $\zeta_0 = 1/\sqrt{2}$. Again taking the indicated derivatives and evaluating them at $L$, one obtains

$$k_x\big|_L = -\frac{16 \alpha Q^2}{81 a^6},$$

(19)

where the negative sign indicates a transverse instability in the horizontal $x$ direction.

Similarly, the Hooke’s law constant $k_y$ in the transverse $y$ direction is

$$k_y\big|_L = -\frac{16 \alpha Q^2}{81 a^6},$$

(20)

where the negative sign indicates a transverse instability in the horizontal $y$ direction. Note that the trap is azimuthally symmetric around the vertical axis, so that the $x$ and $y$ directions are equivalent to each other. Because of the negativity of two of the three Hooke’s constants $k_x$, $k_y$, and $k_z$, the trap will be unstable for small displacements in two of the three spatial dimensions near $L$, and hence $L$ is a saddle point. Note also that the sum of the three Hooke’s constants in Equations (16), (19), and (20) is zero, i.e.,

$$k_x + k_y + k_z = 0.$$  

(21)
3 Earnshaw’s theorem revisited

We shall see that Equation (21) can be derived from Earnshaw’s theorem when one generalizes this theorem from the case of a charged particle to the case of a neutral, polarizable particle in an arbitrary DC electrostatic field configuration. A quantitative consideration of the force on the particle due to the uniform gravitational field of the Earth, in conjunction with the force due to the DC electrostatic field configuration, does not change the general conclusion that the mechanical equilibrium for both charged and neutral polarizable particles is unstable.

4 Charged particle case

We shall first briefly review here Earnshaw’s theorem [1], which implies an instability of a charged particle placed into any configuration of electrostatic fields in a charge-free region of space in the absence of gravity. Suppose that there exist a point $L$ of mechanical equilibrium of a charged particle with charge $q$ in the presence of arbitrary DC electrostatic fields in empty space. The potential $\Phi$ for these fields obey Laplace’s equation

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0. \quad (22)$$

Now the force on the charged particle is given by

$$\mathbf{F} = -q \mathbf{\nabla} \Phi = -q \left\{ e_x \frac{\partial \Phi}{\partial x} + e_y \frac{\partial \Phi}{\partial y} + e_z \frac{\partial \Phi}{\partial z} \right\} = (F_x, F_y, F_z) . \quad (23)$$

where $e_x, e_y, e_z$ are the three unit vectors in the $x, y$, and $z$ directions, respectively. By hypothesis, at the point $L$ of mechanical equilibrium

$$\frac{\partial \Phi}{\partial x} \bigg|_L = \frac{\partial \Phi}{\partial y} \bigg|_L = \frac{\partial \Phi}{\partial z} \bigg|_L = 0 . \quad (24)$$

Stable equilibrium would require all three Hooke’s constants $k_x, k_y$, and $k_z$ at point $L$ to be positive definite, i.e.,

$$k_x = -\frac{\partial F_x}{\partial x} \bigg|_L = +\frac{\partial^2 \Phi}{\partial x^2} \bigg|_L > 0 \quad (25)$$

$$k_y = -\frac{\partial F_y}{\partial x} \bigg|_L = +\frac{\partial^2 \Phi}{\partial y^2} \bigg|_L > 0 \quad (26)$$

$$k_z = -\frac{\partial F_z}{\partial x} \bigg|_L = +\frac{\partial^2 \Phi}{\partial z^2} \bigg|_L > 0 . \quad (27)$$
However, Laplace’s equation, Equation (22), can be rewritten as follows:

\[ k_x + k_y + k_z = 0, \]  

(28)

i.e., the sum of the three components of Hooke’s constants for the charged particle must be exactly zero. The simultaneous positivity of all three Hooke’s constants is inconsistent with this, and hence at least one of the Hooke’s constants along one of the three spatial directions must be negative. Therefore the system is unstable.

The azimuthally symmetric field configurations like that of a charged ring is an important special case. Let \( z \) be the vertical symmetry axis of the ring. Suppose that there is stability in the longitudinal \( z \) direction (such as along the \( z \) axis above point \( L \)), so that

\[ k_z > 0. \]  

(29)

By symmetry

\[ k_x = k_y \equiv k_\perp \]  

(30)

so that Equation (28) implies that

\[ k_\perp = -\frac{1}{2}k_z < 0, \]  

(31)

implying instability in the two transverse \( x \) and \( y \) directions.

Conversely, suppose there is instability in the longitudinal \( z \) direction (such as along the \( z \) axis below point \( L \)), so that

\[ k_z < 0. \]  

(32)

Again, by symmetry

\[ k_x = k_y \equiv k_\perp \]  

(33)

so that Equation (28) implies that

\[ k_\perp = -\frac{1}{2}k_z > 0, \]  

(34)

implying stability in the two transverse \( x \) and \( y \) directions.

5 Adding a uniform gravitational field such as the Earth’s, in the case of a charged object

The potential energy of a charged, massive particle in a DC electrostatic field in the presence of Earth’s gravitational field is

\[ U_{\text{tot}} = q\Phi + mgz. \]  

(35)
Note that the term due to gravity, i.e., the $mgz$ term, is linear in $z$, and therefore will vanish upon taking the second partial derivatives of this term. Therefore the Hooke’s constants $k_x$, $k_y$, and $k_z$ will be unaffected by Earth’s gravity. The force on the particle is

$$F_{tot} = -\nabla U_{tot} = -q \nabla \Phi - mg e_z \quad (36)$$

where $e_z$ is the unit vector in the vertical $z$ direction. In equilibrium, $F_{tot} = 0$, but this equilibrium is again unstable, since upon taking another partial derivative of the term $mg e_z$ with respect to $z$ will yield zero, and therefore all of the above Hooke’s law constants are the same in the presence as in the absence of Earth’s gravity.

6 Generalization to the case of a neutral, polarizable particle

Now suppose that there exists a point $L$ of mechanical equilibrium of the neutral particle with positive polarizability $\alpha > 0$ somewhere within an arbitrary electrostatic field configuration. Such a particle is a high-field seeker, and hence point $L$ must be a point of high field strength. Choose the coordinate system so that the $z$ axis is aligned with respect to the local dominant electric field at point $L$. Thus the dominant electric field component at $L$ is thus $E_z$. The potential energy $U$ for a neutral particle with polarizability $\alpha$ in the presence of an electric field $(E_x, E_y, E_z)$ is given by

$$U = -\frac{1}{2} \alpha (E_x^2 + E_y^2 + E_z^2) \approx -\frac{1}{2} \alpha E_z^2, \quad (37)$$

since the contributions to $U$ from the $x$ and $y$ components of the electric field, which vanish as $\varepsilon^4$ near the $z$ axis for small $\varepsilon$, can be neglected in a small neighborhood of $L$. The force on the particle is

$$F = -\nabla U. \quad (38)$$

Therefore the $z$ component of the force is, to a good approximation,

$$F_z = \alpha E_z \frac{\partial E_z}{\partial z}, \quad (39)$$

and the Hooke’s law constant $k_z$ in the $z$ direction is given by

$$k_z = -\frac{\partial F_z}{\partial z} = -\alpha \left\{ \left( \frac{\partial E_z}{\partial z} \right)^2 + E_z \frac{\partial^2 E_z}{\partial z^2} \right\} \bigg|_L = -\alpha \left. E_z \frac{\partial^2 E_z}{\partial z^2} \right|_L, \quad (40)$$

where the last equality follows from the hypothesis of mechanical equilibrium at point $L$. 

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Similarly, the $x$ component of the force is, to the same approximation,

$$F_x = \alpha E_z \frac{\partial E_z}{\partial x}, \quad (41)$$

and the Hooke’s law constant $k_x$ in the $x$ direction is given by

$$k_x = -\frac{\partial F_x}{\partial x} = -\alpha \left\{ \left( \frac{\partial E_z}{\partial x} \right)^2 + E_z \frac{\partial^2 E_z}{\partial x^2} \right\} \bigg|_L = -\alpha E_z \frac{\partial^2 E_z}{\partial x^2} \bigg|_L, \quad (42)$$

where the last equality follows from the hypothesis of mechanical equilibrium at point $L$.

Similarly the $y$ component of the force is, to a good approximation,

$$F_y = \alpha E_z \frac{\partial E_z}{\partial y}, \quad (43)$$

and the Hooke’s law constant $k_y$ in the $y$ direction is given by

$$k_y = -\frac{\partial F_y}{\partial y} = -\alpha \left\{ \left( \frac{\partial E_z}{\partial y} \right)^2 + E_z \frac{\partial^2 E_z}{\partial y^2} \right\} \bigg|_L = -\alpha E_z \frac{\partial^2 E_z}{\partial y^2} \bigg|_L, \quad (44)$$

where again the last equality follows from the hypothesis of mechanical equilibrium at point $L$.

Thus the sum of the Hooke’s law constants along the $x$, $y$, and $z$ axes is given by

$$k_x + k_y + k_z = -\alpha \left\{ E_z \left( \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} \right) \right\} \bigg|_L$$

$$= -\alpha E_z \frac{\partial}{\partial z} \left( \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} \right) \bigg|_L = 0. \quad (45)$$

Therefore

$$(k_x + k_y + k_z)|_L = 0, \quad (46)$$

and again, the sum of the three Hooke’s law constants must be exactly zero according to Laplace’s equation.

Suppose that the system possesses axial symmetry around the $z$ axis with

$$k_z > 0, \quad (47)$$

i.e., with stability along the $z$ axis. Then by symmetry

$$k_x = k_y \equiv k_{\perp} \quad \text{(48)}$$

so that Equation (46) implies that

$$k_{\perp} = -\frac{1}{2} k_z < 0, \quad (49)$$

implying instability in both $x$ and $y$ directions. This is exactly what we found by explicit calculation for the case of a neutral, polarizable object near point $L$ of the charged ring.
7 Adding a uniform gravitational field such as the Earth’s, in the case of a neutral, polarizable object

The potential energy of a neutral, polarizable, massive particle in a DC electrostatic field plus Earth’s gravity is

\[ U_{tot} = U + mgz. \]  \hspace{1cm} (50)

Again, note that the term due to gravity, i.e., the \( mgz \) term, is linear in \( z \), and therefore will vanish upon taking the second partial derivatives of this term. Therefore again the Hooke’s constants \( k_x, k_y, \) and \( k_z \) will not be affected by Earth’s gravity. The force on the particle is

\[ \mathbf{F}_{tot} = -\nabla U_{tot} = -q \nabla U - mge_z \]  \hspace{1cm} (51)

where \( e_z \) is the unit vector in the vertical \( z \) direction. In equilibrium, \( \mathbf{F}_{tot} = 0 \), but this equilibrium is again unstable, since upon taking another partial derivative of the term \( mge_z \) with respect to \( z \) will yield zero, and therefore again all of the above Hooke’s law constants are the same in the presence as in the absence of Earth’s gravity.

8 Ways to evade Earnshaw’s theorem

Some known ways to evade Earnshaw’s theorem and thereby to construct a truly stable trap for charged or for neutral particles are (1) to use non-electrostatic fields such as a DC magnetic field (e.g., the Penning trap [5]) in conjunction with DC electric fields, or (2) to use time-varying, AC electric fields, rather than DC fields (e.g., the Paul trap [2]), or (3) to use active feedback to stabilize the neutral equilibrium of a charged particle in a uniform electric field, such as was done for a charged superfluid helium drop [6], or (4) to use the low-field seeking property of neutral, diamagnetic objects to levitate them in strong, inhomogeneous magnetic fields [7]. The latter two methods may be practical for levitating the superfluid helium “Millikan oil drops” in the experiment described in [8].

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