Phase Diagram of Vortices in High-\textit{T}_c Superconductors with a Melting Line in the deep \textit{H}_{c2} Region

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We use a simple elastic Hamiltonian for the vortex lattice in a weak impurity background which includes defects in the form of integer-valued fields to calculate the free energy of a vortex lattice in the deep \textit{H}_{c2} region. The phase diagram in this regime is obtained by applying the variational approach of Mézard and Parisi developed for random manifolds. We find a first-order line between the Bragg-glass and vortex-glass phase as a continuation of the melting line. In the liquid phase, we obtain an almost vertical third-order glass transition line near the critical temperature in the \textit{H} – \textit{T} plane. Furthermore, we find an almost vertical second-order phase transition line in the Bragg-glass as well as the vortex-glass phases which crosses the first-order Bragg-glass, vortex-glass transition line. We calculate the jump of the temperature derivative of the induction field across this second-order line as well as the entropy and magnetic field jumps across the first-order line.

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I. INTRODUCTION

The phase diagram of high-\textit{T}_c superconductors as a function of the magnetic field \textit{H} and temperature \textit{T} is mainly governed by the interplay of thermal fluctuations and quenched disorder [1, 2], leading to various different states of the vortex matter summarized in Fig. 1.

At low magnetic field the vortex solid melts into a vortex liquid (VL) via a first-order melting transition. Prominent examples of high-\textit{T}_c superconductors exhibiting a solid-liquid melting transition are the anisotropic compound YBa$_2$Cu$_3$O$_{7-\delta}$ (YBCO), and the strongly layered compound Bi$_2$Sr$_2$CaCu$_2$O$_8$ (BSCCO). The position of the melting line in the \textit{H} – \textit{T} plane is mainly influenced by the anisotropy of the superconductor [2]. In YBCO with its low anisotropy, most of the melting line \textit{B}_m lies in the vicinity of the upper critical field \textit{H}_{c2}, i.e. \textit{B}/\textit{H}_{c2} \gtrsim 0.5 where \textit{B} \approx \textit{H} is the induction field in the relevant regime. The phase diagram for superconductors with a melting line in this regime was discussed by us in Ref. [3]. This magnetic field regime consists of a vortex lattice, or vortex fluid, with overlapping vortex cores, if we use elastic constants calculated in Refs. [3, 6]. The phase diagram was derived on the basis of a defect melting model set up in Refs. [7, 8]. The model is Gaussian in the elastic strains and takes into account the defect degrees of freedom by integer-valued gauge fields. From this we derive effective Hamiltonians for the low-temperature solid and high-temperature liquid phases by summing over all defect fields. By further integrating out vortex degrees of freedom we obtain the partition functions of both phases. This is done with the help of the variational approach of Mézard and Parisi [9], originally developed for random manifolds and applied later to vortex lattices without defects in Refs. [10, 11]. A similar approach was used recently to calculate the glass transition line for YBCO via Ginzburg-Landau theory [12].

When including weak pinning, the solid phase becomes a quasi-long-range ordered Bragg-glass [1]. At higher magnetic fields, the quasi-long-range order is destroyed and there exist also a vortex-glass phase. The transition

![FIG. 1: Sketch of the phase diagram of BSCCO or similar high-\textit{T}_c superconductors where the phase transition lines lie far below \textit{H}_{c2}. The solid line denotes a first-order phase transition line being a first-order melting transition between the BG1-VL, BG1-VG1 phase and a disorder induced first-order transition between the BG2-VG2 phase. The (blue) dashed curve denoted by \textit{T}_d is found to be a second-order glass transition line. The (blue) dashed curve denoted by \textit{T}_d is found to be a second-order glass transition line. The intersection point with the first-order line is denoted by GP1. The red dashed-dotted curve is the \textit{T}_c line found by Fuchs et al. [20] using surface barrier experiments. It intersects the first-order line in the point GP1.](image-url)
is marked by the disappearance of Bragg peaks in scattering data. We obtain in Ref. [4] a phase diagram consisting of a unified first-order phase line between the Bragg-glass phase and the vortex-glass phase and the vortex-liquid which is sketched for BSCCO in Fig. 1. We point out that the first-order character of the transition line between the Bragg-glass phase and the vortex-glass phase is not experimentally established yet for YBCO. It is deduced from magnetic anomalies in response to the external magnetic field. For BSCCO, the first-order character of the Bragg-glass, vortex-glass line was measured by supercooling [13] and magnetic field shaking techniques [14].

Beside the unified first-order line found within our model for YBCO [4] (seen before within the Ginzburg-Landau model in Ref. [15]), a third-order glass transition line emanates near the critical point on the melting line as a phase boundary between the Bragg-glass and the vortex-liquid phase. We have shown in Ref. [4] by using hyperscaling relations that the higher-order character (more than second-order) of this line is in accordance with experiments and numerics which determines the scaling of the disorder phase correlation length [16, 17].

This glass transition line exists also for BSCCO. But beside this parallel of the glass transition lines of YBCO and BSCCO, Beidenkopf et al. [18] found for BSCCO an additional second-order glass transition line in the Bragg-glass phase by using also the magnetic shaking technique. The line showed up by plotting the derivative of the magnetic induction field with respect to the temperature. A jump was observed which also exist for the glass transition line in the vortex-glass phases. Thus in contrast to YBCO, they found for BSCCO that both lines are of second-order characteristics. Both lines meet in a single point within experimental uncertainties. This point is not the critical point found for general doping [15] which is characterized by a vanishing of the entropy jump [14] being the maximum of the unified first-order line. We label both second-order lines in Figure 1 by $T_d$. The intersection with the first-order line is denoted by $GP_2$. Both lines divide the vortex-glass phase named by VG2 from a phase named VG1 in Fig. 1 lying in the high-magnetic-field part above the first-order line. In the low-magnetic-field part the $T_d$ line divides two Bragg-glass phases denoted by BG2 and BG1 in Fig. 1.

Finally, we show in Fig. 1 a possible additional phase boundary labeled by $T_x$ which was found by Fuchs et al. [20] by measuring the vortex penetration through surface barriers. A similar line was also found by magnetization measurements [21]. This line divides the vortex liquid denoted by VL from the phase VG1 shown in Fig. 1. It meets the melting line in a point to be referred as GP1. Note that it is not experimentally clear whether the $T_x$-line has the characteristic of a phase transition. The $T_x$-line does not correspond to the irreversibility line where magnetization sweeps show hysteresis. The position of this line in the case of BSCCO is mostly influenced by surface barriers [22] in contrast to YBCO where the pinning mechanism is responsible for the irreversibility. This leads to a coincidence of the irreversibility line with the glass transition line between the vortex-glass and vortex-liquid [23]. It was shown in Ref. [21] via Josephson plasma experiments that the $T_x$-line is not accompanied by a Josephson decoupling between the layers ruling out the possibility of a transition from vortex lines to weakly coupled pancake vortices. That such a transition exist was proposed theoretically in Refs. [24, 25, 26]. So far we point out, that it is not experimentally clear yet what kind of phase VG1 is [21]. There are, for example, hints that this phase could be a disordered flux line liquid [27]. This is suggested by muon spin rotation experiments which, however are in contradiction to other experiments which reported Bragg-peaks in this phase [28]. In the interpretation of VG1 as a disordered flux line liquid, VG2 consists of a quasi-two-dimensional vortex solid.

It is the purpose of this paper to investigate the above phase transitions in the defect melting model mentioned earlier [3, 7, 8] used in Ref. [4] to calculate the phase diagram of YBCO. We will first review briefly the model. A more comprehensive discussion can be found in our former papers and the book [7]. In contrast to YBCO, BSCCO is a strongly layered material where the coupling between the layers is described by the Josephson coupling in the Hamiltonian of the system. For high magnetic fields beyond the first-order line, one obtains a suppression of the Josephson coupling between the layers [21, 29] with respect to the electromagnetic coupling. In our elastic model with defects, we can not get this decoupling. We shall accommodate it effectively via an appropriate modification of the elastic moduli of the vortex lattice system in this region of the magnetic field. With the help of the elastic moduli of Brandt [3] for BSCCO we show that one expects two thermal decouplings for the vortex strings in the liquid phase, corresponding to the two glass transition lines in Fig. 1, in contrast to the single line in YBCO [4]. On this way, we carry out the Mézard-Parisi analysis for the Hamiltonian of the vortex lattice system with pinning. It consists of a variational approach to fit the free energy of the replicated system with the free energy of a quadratic Hamiltonian. We obtain an almost vertical third-order depinning glass transition line in the liquid high-temperature phase located in the vicinity of the $T_x$ line in Fig. 1 separating a full replica symmetric saddle point solution at high temperatures and a full replica symmetry broken solution at lower temperatures. We show that the saddle point equation to the variational free energy has no solution for very low temperatures. This is also the case when going beyond the Mézard-Parisi theory within variational perturbation theory [30]. This is a systematic extension of the Mézard-Parisi theory to higher orders. It is well-known phenomenon of higher-order variational perturbation expansion of the quantum mechanical anharmonic oscillator, that variations of the trial free energy do not necessarily have to show a minimum or a maximum [30], where
the odd orders of variational perturbation theory possess a minimum, but even orders have no saddle points but only turning points. It is shown in Ref. [31] for the anharmonic oscillator that also turning points are acceptable. This is the principle of minimal sensitivity. Motivated by good results for the anharmonic oscillator we generalize the variational approach of M´ezard-Parisi by using the principle of minimal sensitivity for the calculation of the variational free energy. With the help of this extension we shall obtain a variational free energy in the whole interesting regime of the $H − T$ phase diagram for the vortex lattice. This phase diagram looks rather similar to the quantum critical phase diagram for the $\phi^4$-theory where saddle point solutions to the variational free energy where not solvable. Section VI goes beyond lowest order variational perturbation theory using generalized principle of minimal sensitivity. Section VII discusses observable consequences of our theory. In the Appendices A and B we supply additional material to Sections V and VI.

II. MODEL

The partition function to be used to describe the vortex lattice without disorder was proposed in Ref. [3]. It is motivated by similar melting models for two-dimensional square [7] and triangular [8] crystals. In Ref. [4] we derived from this partition function a low-temperature representation. This corresponds to the partition function of the vortex-lattice in the crystalline phase

$$Z_{\text{fl}} = \mathcal{N} \prod_{x,i} \left[ \int_{-\infty}^{\infty} \frac{u_i(x)}{a} \right] \exp \left[ -\frac{1}{k_B T} \left( H_0[u_i] + H_{\text{dis}}[u_i] \right) \right]$$

(1)

with the low-temperature Hamiltonian

$$H_0[u_i] = H_{T - 0}[u_i] = \frac{v}{2} \sum_x (\nabla_i u_i)(c_{11} - 2c_{66})(\nabla_i u_i)$$

$$+ \frac{1}{2} (\nabla_i u_j + \nabla_j u_i) c_{66} (\nabla_i u_j + \nabla_j u_i) + (\nabla_3 u_i) c_{44} (\nabla_3 u_i)$$

$$= \frac{v}{2} \sum_x (\nabla_i u_L) c_{11} (\nabla_i u_L) + (\nabla_3 u_L) c_{44} (\nabla_3 u_L)$$

$$+ (\nabla_i u_T) c_{66} (\nabla_i u_T) + (\nabla_3 u_T) c_{44} (\nabla_3 u_T).$$

(2)

Here $u_L = P_L u$ is the longitudinal part of the displacement where the projector $P_L$ is given by $(P_L)_{jk} = \frac{1}{\sqrt{\nabla_j}} (1/\sqrt{\nabla_j}) \nabla_k$. The transversal part of the displacements is then given by $u_T = P_T u \equiv u - u_L$. By using the dual representation in the form of stress fields we obtain a high-temperature representation of the partition function. This partition function describes the vortex system in the fluid phase. We obtain a partition function of the form (1) with Hamiltonian

$$H_0[u_i] = H_{T - \infty}[u_i] = \frac{v}{2} \sum_x (\nabla_i u_i)(c_{11} - c_{66})(\nabla_i u_i)$$

$$+ (\nabla_3 u_i) c_{44} (\nabla_3 u_i)$$

$$= \frac{v}{2} \sum_x (\nabla_i u_L) (c_{11} - c_{66})(\nabla_i u_L) + (\nabla_3 u_L) c_{44} (\nabla_3 u_L)$$

$$+ (\nabla_3 u_T) c_{44} (\nabla_3 u_T)$$

(3)

and $\mathcal{N} = 1/(4\pi \beta)^3$. In the following, the subscripts $i, j$ have values 1, 2, and $l, m, n$ have values 1, . . . , 3 where $N$ denotes the number of lattice sites. The parameter $\beta$ is proportional to the inverse temperature, $\beta = v c_{66}/k_B T(2\pi)^2$, where the volume $v$ of the fundamental cell is equal to $\sqrt{3}a^2a_3/2$ for the triangular lattice. Here $a$ is the transverse distance of neighboring vortex lines, and $a_3$ the persistence length of the dislocation lines introduced in Ref. [3]. Note that $a_3$ is assumed to be independent on the disorder potential in the average [3]. Its value is given by

$$a_3 \approx 4a \frac{2\lambda a_{ik}}{\pi \lambda_c}.$$

(4)

The lattice derivatives $\nabla_i$ are built from the link differences around a plaquette in the triangular lattice. These expressions can be found in Refs. [3, 13]. By analogy $\nabla_3$ is the lattice derivative in $z$-direction.

The second term in the exponent of (1)

$$H_{\text{dis}}[u_i] = \sum_x V(x + u),$$

(5)

accounts for disorder. We have suppressed the spatial arguments of the elastic parameters, which are functional matrices $c_{ij}(x, x') \equiv c_{ij}(x - x')$. Their precise forms were first calculated by Brandt [3] and generalized in Ref. [43] by taking into account thermal softening relevant for BSCCO. The elastic moduli $c_{44}$ and $c_{66}$ at low
reduced magnetic fields $B/H_{c2} < 0.25$ are given by
\[
c_{66} = \frac{B\phi_0}{(8\pi\lambda_{ab})^2},
\]
\[
c_{44} = \frac{B^2}{4\pi(1 + \lambda_{c2}^4 k^2 + \lambda_{b2}^4 k^2 k_3^2)} + \frac{B\phi_0}{32\pi^2\lambda_c^3} \ln \left(1 + \frac{2\lambda_0^2 k_3^2}{\lambda_b^2 k^2 \lambda_c^4}ight) + \frac{B\phi_0}{32\pi^2\lambda_c^3 k_3^2} \ln \left(1 + \frac{\lambda_b^2 k_3^2}{(1 + 2\lambda_0^2 k_3^2/(u^2))} \right)
\]
where $\lambda_c$ is the penetration depth in the $xy$-plane, and $K_{BZ}$ is the boundary of the circular Brillouin zone.

For BSCCO, we use the two-fluid model for the Josephson coupling cosine phase difference term in the Ginzburg-Landau model for BSCCO. This was shown in Refs. [4, 5] theoretically and in Refs. [21, 29] by determining Josephson plasma frequencies for BSCCO that one gets a suppression of the full Josephson energy between the layers when going from the vortex solid to the vortex liquid crossing the BG2-VG2, BG1-VG1 line. This leads effectively to a softening of the Josephson terms of $c_{44}$ in the VG1 and VG2 phases being the first two summands in (7). This justifies to use (8) as a good approximation for the full string tension in the whole interesting regime when also including Josephson decoupling.

The disorder potential $V(x)$ due to pinning is assumed to possess the Gaussian short-scale correlation function
\[
V(x)V(x') = \Delta(x_i - x_i') \delta_{x_i, x_i'}
\]
and
\[
d(T) a_{ij} \phi_0^3 \xi_{ab}^3 K(x_i - x_i') \delta_{x_i, x_i'}
\]
where $K(x_i - x_i') \approx 1/(\xi')^2$ for $|x - x'| < \xi'$, and zero elsewhere. The parameter $\phi_0$ is the magnetic flux quantum $\phi_0 = \hbar c/2e$, and parameter $\xi'$ is the correlation length of the impurity potential, which has a similar value as the coherence length $\xi_{ab}$ in the $xy$-plane.

In the following, we use an effective disorder correlation function with the Fourier transform
\[
\tilde{K}(q) = 2\pi \exp\left(-q^2 \xi_{ab}^2/2\right)
\]
which is restricted to the transversal fluctuations is given by
\[
c_{44} = \frac{\alpha_2^2}{a_{ij}^2} T V_{BZ} \int_{BZ} d^2k d\delta K_{ij} K_{ij} a_{ij}^2 K_{ij}^2 K_{ij}^3
\]
where the average is taken with respect to the low-temperature Hamiltonian (2) without disorder representing the elastic energy of the vortex lattice. The momentum integrations in (4) run over the Brillouin zone of the vortex lattice whose volume is $V_{BZ} = (2\pi)^3/v$, as indicated by the subscript BZ. $K_{ij}$ is the Fourier transform of $\nabla_j$. Approximation (5) for $c_{44}$ is correct in the regime $B T \lambda_{ab}^2 / 8 \phi_0 \ln(1/c_4^2) \lesssim 1$ which is valid in the vicinity of the melting line (6). In this regime we obtain that $c_{44}(k, k_3)$ is dominated by the last term in (7) for $|k_3| < \pi/\alpha_3$. For higher magnetic fields than the disorder induced first order BG2-VG2 line (see Fig. 1) we have $B T \lambda_{ab}^2 / 8 \phi_0 \ln(1/c_4^2) \gtrsim 1$ (6) meaning that the first term in (9) (7) is dominated over the third term in the region $k_3 \approx \pi/\alpha_3$. This implies that the approximation (8) would result in a wrong approximation for the magnetic field regime above the BG2-VG2 line. We can see from (2) and (3) that the string tension $c_{44}$ is not renormalized going from the vortex lattice to the vortex liquid. For deriving the full elastic constants (9), (10) one uses a quadratic approximation for the Josephson coupling cosine phase difference term in the Ginzburg-Landau model for BSCCO. It was shown in Refs. [4, 5] theoretically and in Refs. [21, 29] by determining Josephson plasma frequencies for BSCCO that one gets a suppression of the full Josephson energy between the layers when going from the vortex solid to the vortex liquid crossing the BG2-VG2, BG1-VG1 line. This leads effectively to a softening of the Josephson terms of $c_{44}$ in the VG1 and VG2 phases being the first two summands in (7). This justifies to use (8) as a good approximation for the full string tension in the whole interesting regime when also including Josephson decoupling.

The disorder potential $V(x)$ due to pinning is assumed to possess the Gaussian short-scale correlation function
\[
V(x)V(x') = \Delta(x_i - x_i') \delta_{x_i, x_i'}
\]

\[
d(T) a_{ij} \phi_0^3 \xi_{ab}^3 K(x_i - x_i') \delta_{x_i, x_i'}
\]

leading also to an exponentially vanishing of the disorder correlation function in real space. In Ref. [4] we have used this form for the correlation function in the solid phase for YBCO. In the present material BSCCO, this is even more justified because the disorder potential looks $\delta$-like for the vortices, due to the large thermal fluctuations of the vortices near the melting transition line (6).

The temperature dependence of the parameter $d(T)$ has two sources. One is the temperature dependence of the correlation length, the other is based on the pinning mechanism where we discuss in the following the $\delta T_c$-pinning or $\delta l$-pinning mechanisms (1).

\[
d(T) = d_0(1 - T/T_C)^{-1/2} \quad \text{for } \delta T_c - \text{pinning},
\]
\[
d(T) = d_0(1 - T/T_C)^{3/2} \quad \text{for } \delta l - \text{pinning}.
\]

III. MÉZARD-PARISI METHOD

We now carry out the calculation of the partition function (4) which is still complicated due to disorder. In
Ref. 3 we have done this for YBCO by using a quadratic approximation in the disorder strength. This leads to a reentrant behaviour of the melting line in the $H - T$ plane which did not agree with experimental results. By using the variational approach of Mégard-Parisi 30 to go beyond the quadratic approximation this reentrant behaviour is disappeared, leading to good results for the form of the melting line and agreement to the transition line between the Bragg-glass and vortex-glass. Here we use again the Mégard-Parisi theory to perform a similar calculation in the case of BSCCO. In order to go beyond second-order perturbation theory in the impurity potential, we use first the well known replica trick 30. The Mégard-Parisi theory consists in replacing the non-quadratic part of this replicated Hamiltonian as quadratic with a possible mixing of replica fields. By using the Bogoliubov variational principle we can find the best quadratic Hamiltonian so that its free energy named the Gibb’s measure of the trial Hamiltonian $F_{\text{var}}$ is as close as possible to the actual free energy of the system. This means that we have to search the minimum of

$$
F_{\text{var}} = F_{\text{trial}} + \langle H - H_{\text{trial}} \rangle_{\text{trial}}
$$

with the harmonic trial Hamiltonian

$$
H_{\text{trial}} = \frac{v}{2} \sum_{x} \sum_{\alpha, \beta} u^{\alpha}(x) G_{\alpha \beta}^{-1}(x - x') u^{\beta}(x').
$$

Here $\langle \cdot \rangle_{\text{trial}}$ stands for the average with respect of the Gibbs’s measure of the trial Hamiltonian $H_{\text{trial}}$, while $H$ denotes the replicated Hamiltonian. The indices $\alpha$, $\beta$ denotes the replicas.

In the general form, the search for an extremum is a complicated problem. A strong simplification for this was founded by Parisi for random-spin systems where he suggest to deal with a trial Hamiltonian within some subalgebra known as the Parisi algebra. This restriction can be motivated by physical arguments. 37. It will be clear soon for the solid as well as the fluid phase that the transverse part of $G_{\alpha \beta}$ can be chosen to have the form

$$
G_{\alpha \beta}^{-1} = G_0^{-1} \delta_{\alpha \beta} + \sigma_{\alpha \beta}
$$

where $G_0$ is the transverse part of the Green function of the Hamiltonian $H_0[u_i]$ 2 in the solid phase and 3 in the fluid phase.

Within the Parisi-algebra, the self-energy matrix $\sigma_{\alpha \beta}$ depends effectively only on one parameter $\lambda$ (see also App. B). In the general form it is allowed to be a continuous function $\sigma(s)$ with $0 < s < 1$. 9. Then the variational free energy has the form

$$
\Delta f_{\text{var}} = \Delta f_{\text{var}} / N \equiv f_{\text{var}}(B[\Delta]) - f_{\text{var}}(0)
$$

$$
= k_B T \int_0^1 ds \left[ \frac{1}{N} \int_{BZ} d^2k \, d \Delta \frac{d}{d \Delta} g(\Delta) + D(2B[\Delta(s)]) \right],
$$

$$
f_{\text{var}}(0) = -k_B T \left( \frac{1}{N} \ln N \right)
$$

$$
\frac{1}{2} \left( \frac{1}{V_{\text{BZ}}} \int_{BZ} d^2k dk_3 \ln \left[ \det \left( \frac{2 \pi k_B T}{v a^2} G_0 \right) \right] + D(0) \right)
$$

where

$$
g(\Delta) = \frac{1}{V_{\text{BZ}}} \int_{BZ} d^2k dk_3 \left( G_0^{-1} + \Delta \right)^{-1}.
$$

$N$ is the number of lattice sites, and $G_0^{-1}$ is given by

$$
G_0^{-1}(k, k_3) = \frac{c_{44}}{a^2} \left[ 2 - 2 \cos(k_3 a_3) \right]
$$

$$
+ \frac{c_{66}}{a^2} \left[ 1 - \frac{4}{3} \sum_{l=1}^3 \cos(k_l a_l) \right]
$$

in the solid low-temperature phase corresponding to 2, and

$$
G_0^{-1}(k, k_3) = \frac{c_{44}}{a^2} \left[ 2 - 2 \cos(k_3 a_3) \right]
$$

in the liquid high-temperature phase corresponding to 3. Here $e_l$ are the three unit link vectors around a plaquette in the triangular lattice. The gap function $\Delta(s)$ and the self-energy function $\sigma(s)$ are related by

$$
\Delta(s) = \int_0^s ds' s' \frac{d \sigma(s')}{ds'}.
$$

$B[\Delta(s)]$ is given by

$$
B[\Delta(s)] = \frac{k_B T}{v} g[\Delta(s)] - \frac{k_B T}{v} \int_{s}^1 ds' \frac{1}{s^2} g[\Delta(s')] = \frac{k_B T}{v} g[\Delta(1)] - \frac{k_B T}{v} \int_{s}^1 ds' \sigma(s') g'[\Delta(s')].
$$

In order to find a saddle point of $F_{\text{var}} / N$ we have to take the derivative of (17) with respect to $\Delta(s)$. This results in

$$
\sigma(s) = -2 \frac{k_B T}{v} D'(2B[\Delta(s)])
$$

where $D'(x)$ is the derivative $(d/dx) D(x)$. The disorder function $D$ is given by 4

$$
D(2(u^2)) = \frac{d(T)}{d(\xi^2)} \left( \frac{a_3}{(k_B T)^2} \lambda_{ab}^3 \int \frac{d^2q}{(2\pi)^2} K(q) e^{-\frac{1}{2}(u^2)} \right)
$$

$$
= \frac{d(T)}{\xi^2} \left( \frac{a_3}{(k_B T)^2} \lambda_{ab}^3 \frac{1}{\xi^2} \right)
$$

In the following, we discuss solutions of this equation in the cases that $\sigma(s)$ does not break the replica symmetry, possesses one-step replica symmetry breaking, or a continuous replica symmetry breaking.

In order to solve (24), we first have to calculate $g(\Delta)$ which we will denote by $g^{\Delta \rightarrow \Delta}(\Delta)$ with (20) in the solid phase, and by $g^{\Delta \rightarrow \infty}(\Delta)$ with (21) in the fluid phase. We shall use the elastic constants $c_{66}$ of Eq. 4 and the approximation 8 for $c_{44}$. In the liquid case, the result is

$$
g^{\Delta \rightarrow \infty}(\Delta) \approx \frac{1}{2} \left( \frac{a_3^2}{c_{44}} \Delta^{1/2} (1 + \Delta/4)^{1/2} \right)
$$

$$
+ 1.38 \frac{a_3^2}{c_{44}} \left( 1 + Z^0(\Delta) \right),
$$

where
and for the vortex solid
\[
g^{T-0}(\Delta) \approx \frac{0.098\pi a a_3}{\sqrt{c_{66}c_{44}^{(1)}}} \sqrt{\frac{\sqrt{3}}{2} \frac{a^2}{4\pi c_{66}}} \Delta^{1/2} + 1.38 \frac{a^2}{c_{44}^{(2)}} \frac{1}{(1 + Z_1^{(0)} \Delta/2)},
\]
where \(\Delta \equiv \Delta a^2/c_{44}^{(1)},\) and \(c_{44}^{(1)}\) denotes the function \(c_{44}(k, k_3 \to 0)\) of Eq. (28) for \(k_3 \lesssim 1/\lambda_{ab}\) and \(c_{44}^{(2)}\) denotes \(c_{44}\) of Eq. (28) in the region \(k_3 \gtrsim 1/\lambda_{ab}\) for \(k_3 = 1/a_3,\) i.e.
\[
c_{44}^{(1)} = \frac{B\phi_0}{32\pi L \lambda_{ab}^2 (1 + \lambda_{ab}^2 K_{BZ}^2)},
\]
\[
c_{44}^{(2)} = \frac{a^2 B\phi_0 ln(1+2B\lambda_{ab}/\phi_0 c_{44}^2)}{32\pi \lambda_{ab}^4}.
\]
For the derivation of (27) we have used the approximation \(a_3^2 c_{66}/a^2 c_{44}^{(2)} \ll 1\) valid in the vicinity of the melting line. We used further the abbreviation
\[
Z_1^{(0)} = 2 \cdot 1.38 \frac{a_3^{(1)}}{c_{44}^{(2)}} \sim \left(\frac{\lambda_{ab}}{a}\right)^2 \gg 1.
\]
In the solid phase the following abbreviations will be useful
\[
Z_1^{(0)} = \frac{1.38}{0.098\pi a} \frac{a_3^{(1)}}{c_{44}^{(2)}} \Delta = \frac{\lambda_{ab}}{a} \gg 1,
\]
\[
Z_3^{(1)} = 1.38 \frac{16\pi a^2}{\sqrt{\frac{3}{a^2}}} \left(\frac{c_{66}^{(1)}}{c_{44}^{(2)}}\right)^2 \sim \left(\frac{\lambda_{ab}}{a}\right)^2 \gg 1.
\]
Note that \(\lambda_{ab} \approx a\) in the vicinity of the critical point on the melting line where the disorder is most influential the shape of this line. In this regime we obtain large numbers on the right hand sides of (28), (30), and (31). In (26) and (27) the last terms have their origin in the integration over momenta \(1/\lambda_{ab} \lesssim |k_3| \lesssim \pi/a_3.\) The other terms come from the integration over small momenta. Expressions \(g^{T-0}(\Delta)\) (27) and \(g^{T-\infty}(\Delta)\) (28) are not exact results of the integration in (19). They are good approximations for \(g(\Delta)\) but also for \(g'(\Delta)\) and \(g''(\Delta)\) in the region \(\Delta \lesssim 1\) in the fluid phase and \(\Delta \lesssim ((\lambda_{ab}/a)^2/Z_1^{(0)})^{2/3}\) for the solid. It will be seen below that these are the relevant regimes for \(F_{var}\).
We now define the quantity
\[
A = \frac{4}{k_B T} \frac{a^{(1)} a^2}{a_3} \frac{\Delta}{\pi}.
\]
This is the regime where temperature fluctuations starts to dominate over disorder fluctuations for the coherently pinned vortex line pieces given by \(D(0)A \sim 1.\) The length of such line pieces are given the Larkin length \(L_c\) where disorder fluctuations grow to value \(\xi'.\)

One can now show by generalizing the calculation of the vortex fluctuations due to pinning and thermal fluctuations for YBCO in Ref. 35 that the additional last term in (28) causes a new length scale beyond the Larkin length. At this length scale, the vortex fluctuations are approximately constant forming a plateau. This length scales like \(L_T = a_3(Z_1^{(0)})\) for thermal fluctuations, and like \(L_D = a_3(Z_1^{(0)})^{2/3}\) for disorder fluctuations at low temperatures. Beyond these lengths, the displacement fluctuations starts to increase proportional to the cubic distance due to pinning, and proportional to the distance for thermal fluctuations. Both lengths, that of the coherently pinned vortex line pieces but also the vortex substring on the plateau can decouple due to thermal fluctuations.

Below we find two different depinning phase transition temperatures: One takes place when the temperature fluctuations exceed the disorder fluctuations over the coherently pinned vortex line pieces where the Larkin length fulfills \(L_c > L_T.\) This leads to the well-known depinning temperature of the coherently pinned vortex sub-string given by \(D(0)A \sim 1\) corresponding to the third-order phase transition (67) below. The second depinning transition takes place when the temperature fluctuations exceed the disorder fluctuations over the plateau in the regime where the Larkin length is given by \(L_c < L_D.\) This leads to the depinning temperature \(D(0)A \sim Z_1^{(0)}\) corresponding to the second-order phase transition (67) given below. Of course one can also see both depinning temperatures mentioned above in the temperature dependency of the Larkin length \(L_c(T)\) (30),

IV. SOLUTION OF THE MÉZARD-PARISI SADDLE POINT EQUATIONS

In the following, we discuss the solutions of the Mézard-Parisi Eqs. (22), (23) and (24) in the liquid and the solid phase.

A. Liquid Phase

In order to solve the Mézard-Parisi equations we transfer the analysis for YBCO of Ref. 4 to BSCCO. Note that by neglecting the second term in (27) we obtain a similar expression for \(g^{T-\infty}(\Delta)\) as for YBCO (4). This leads to the following results: The stable solution for \(\Delta(s)\) is replica symmetric for \(D(0)A \leq 2/\sqrt{3}\) and full replica symmetry broken in the case \(D(0)A > 2/\sqrt{3}.\) That for example the one-step replica symmetry breaking solution is not stable can be seen from the following fact: The
The one-step symmetry breaking solution of the saddle point equation \( \{24\} \) is stable when the replicon eigenvalue
\[
\lambda = 1 - \frac{8}{(\frac{\sqrt{2}}{2} D(0) A)^2} \left( \sqrt{\Delta_1} + 4 \left( \frac{Z^{(0)}_l}{1+Z^{(0)}_l} \right)^2 \right) \left[ 1 + \frac{2}{\sqrt{2}} D(0) \left( \frac{1}{\sqrt{A}} + \frac{Z^{(0)}_l}{1+Z^{(0)}_l} \right)^3 \right]^{-1}
\]
is larger than zero. By comparing \( \{33\} \) with \( \{34\} \) we obtain that \( \lambda < 0 \) when \( Z^{(0)}_l \Delta_1/2 \lesssim 13 \) meaning that the one-step replica symmetry breaking solution is unstable in this range. More generally one can show similarly as in Ref. \[3\] that all finite step replica symmetry breaking solutions are unstable for \( D(0) A \gtrsim 2/\sqrt{3} \).

Thus, we expect a continuous replica symmetry breaking solution in this parameter range. Note that continuous step replica symmetry breaking solutions of the saddle point equation \( \{24\} \) are stable in general \[4, 10\]. We now calculate this solution by using the methods given in Ref. \[4\]. First, the full replica symmetric solution for \( D(0) A \leq 2/\sqrt{3} \) is given by
\[
\hat{\Delta}(s) = 0 \quad \text{for} \quad \frac{\sqrt{3}}{2} D(0) A \leq 1.
\]
The continuous replica symmetry broken solution for \( D(0) A \gtrsim 2/\sqrt{3} \) can be derived from the saddle point equation \( \{24\} \) by differentiating both sides with respect to \( s \) resulting in
\[
\sigma'(s) = -\sigma'(s) 4 \left( \frac{k_B T}{v} \right)^2 g' [\Delta(s)] D'' (2B[\Delta(s)]) .
\]
This means that \( \sigma(s) \) is either constant or solves equation \( \{35\} \) divided by \( \sigma'(s) \). Dividing \( \{35\} \) by \( \sigma'(s) \) and forming the derive with respect to \( s \), we obtain with \( \{29\} \)
\[
2 \left( \frac{k_B T}{v} \right)^2 g''[\Delta(s)]^2 D'''' (2B[\Delta(s)])
\]
\[
= -s g''''[\Delta(s)] D'''' (2B[\Delta(s)]) .
\]
Equations \( \{36\} \) and \( \{37\} \) can be solved algebraically for the unknown functions \( \Delta(s) \) and \( B[\Delta(s)] \) leading to

\[
\hat{\Delta}(s) = \begin{cases} 0 & \text{for} \quad s \leq \frac{1}{(\frac{\sqrt{2}}{2} D(0) A)^{1/3}}, \\ \left( \frac{1+(Z^{(0)}_l)^2 \Delta^{3/2}}{1+4(Z^{(0)}_l)^2 \Delta^{5/2}} \right)^{5/3} = \left( \frac{\sqrt{2}}{2} D(0) A \right)^{1/3} s_c & \text{for} \quad \frac{1}{(\frac{\sqrt{2}}{2} D(0) A)^{1/3}} \leq s \leq s_c, \\ s & \text{for} \quad s \leq \frac{1}{s_c} \leq 1 \end{cases}
\]

where we used \( Z^{(0)}_l/A \approx c_l a^2/2 \xi^2 \gg 1 \) in the vicinity of the melting line \[3\] and the abbreviation \( Z^{(0)}_l = Z^{(0)}_l/(1+Z^{(0)}_l \Delta/2) \). This means that \( \hat{\Delta} \) is constant for \( s \geq s_c \). The constant \( s_c \) is given by the equation
\[
\frac{\left( 1+(Z^{(0)}_l)^2 \Delta^{3/2} \right)^{5/3}}{1+4(Z^{(0)}_l)^3 \Delta^{5/2}} = \left( \frac{\sqrt{2}}{2} D(0) A \right)^{1/3} s_c . \tag{39}
\]

Finally, we can calculate the disorder part of the variational free energy \( \Delta f_{\text{var}} \) \( \{17\} \) in the liquid phase \[3\]. With \( \Delta f_{\text{var}} = \Delta f_{\text{var}}^\text{kin} + \Delta f_{\text{var}}^\text{pot} \) where \( \Delta f_{\text{var}}^\text{kin} \) is the kinetic part represented by the first term in \( \{17\} \) of the disorder energy and \( \Delta f_{\text{var}}^\text{pot} \) is the potential energy part of the disorder energy (second term in \( \{17\} \)), we obtain
\[
\Delta f_{\text{var}}^\text{kin} = -\frac{k_B T}{4} \left[ \int ds \frac{1}{s^2} \left( \hat{\Delta}^{1/2}(s) + \frac{(Z^{(0)}_l)^2}{4} \hat{\Delta}^2(s) \right) \right. \\
- \left. \left( 1 - \frac{1}{s_c} \right) \left( \hat{\Delta}^{1/2}(s_c) + \frac{(Z^{(0)}_l)^2}{4} \hat{\Delta}^2(s_c) \right) \right] . \tag{40}
\]
\[ \Delta f_{\text{var}} = \frac{k_B T}{4} \left( \frac{\sqrt{3}}{2} D(0) A \right)^{2/3} \]

\[ \times \left[ \int_{D(i)}^{s_c} ds \frac{\Delta^{1/2}(s)}{\left(1 + (\tilde{Z}_i^{(0)})^2 \Delta^{3/2}(s)\right)^{1/3}} \right] \]

\[ + \left(1 - s_c\right) \left( \frac{2}{3} \Delta^{3/2}(s_c) + \frac{\tilde{Z}_i^{(0)}}{2} \Delta^2(s_c) \right) \quad \text{(41)} \]

By taking into account (38) and (39) we obtain that the glass transition line separating the phases VG1-VL at \( D(0) A = \sqrt{3}/2 \) is of third-order. We found the same order for the depinning transition in YBCO [4].

\[ \Delta f_{\text{var}} = \frac{k_B T}{4} \left( \frac{\sqrt{3}}{2} \right)^{1/3} (2\pi)^{1/3} \left( \frac{c_{66} a_3^2}{c_{44} a^2} \right)^{1/3} \left( D(0) A \right)^{1/3} \left( \frac{c_{66} a_3^2}{c_{44} a^2} \right) \]

\[ \times \left[ \int_{0}^{s_c} ds \frac{\Delta^{1/6}(s)}{\left(1 + \tilde{Z}_s^{(1)} \Delta^{1/2}(s)\right)^{1/3}} \right] \quad \text{(45)} \]

Finally, we can calculate the disorder part of the variational free energy \( \Delta f_{\text{var}} \). We obtain for this energy in the solid phase:

\[ \Delta f_{\text{var}}^{\text{kin}} = \frac{\sqrt{3}}{16 \pi} \left( \frac{c_{44} a^2}{c_{66} a_3^2} \right) \left( \frac{2}{3} \Delta^{3/2}(s) + \frac{\tilde{Z}_i^{(1)}}{2} \Delta^2(s) \right) \]

\[ \times \left[ \int_{0}^{s_c} ds \frac{1}{s^2} \left( \frac{2}{3} \Delta^{3/2}(s) + \frac{\tilde{Z}_i^{(1)}}{2} \Delta^2(s) \right) \right] \quad \text{(44)} \]

\[ \text{B. Solid Phase} \]

In the solid phase one can show that finite-step replica symmetry breaking solutions are unstable [4]. Similar as in the discussion of the fluid phase in the last subsection we obtain the following continuous replica symmetry broken solution of the saddle point equation [4].

\[ \Delta \approx \frac{3}{64\pi^2} \left( \frac{c_{66} a_3^2}{c_{44} a^2} \right) \left( D(0) A \right)^{1/3} \left( \frac{c_{66} a_3^2}{c_{44} a^2} \right)^{1/2} \quad \text{for } s_c \leq s \leq 1 \]

\[ \text{V. Existence and Stability of Saddle Point Solutions} \]

Trying to solve the implicit equation for \( \Delta(s_c) \) in the liquid phase (last line in (38)) and the solid phase (42) we obtain that in both cases a solution is not existing for very large \( D(0) A \gtrsim (D(0) A)_{\text{max}} \) corresponding to low temperatures or large disorder strengths according to (25, 32).

We obtain from Eq. (38) taken at \( s = s_c \) or directly from (39) for \( D(0) A \) at low temperatures

\[ (D(0) A)_{\text{max}} \approx \frac{2}{\sqrt{3}} Z_i^{(0)} \sim \left( \frac{\lambda_c}{a} \right)^2, \]

\[ \Delta_{\text{max}} \approx \frac{3^{2/5}}{(Z_i^{(0)})^{6/5}}, \quad \text{(47)} \]

\[ (s_c)_{\text{max}} \approx 1 \quad \text{(48)} \]
where $\tilde{\Delta} = \tilde{\Delta}_\text{max}$ and $s_c = (s_c)_\text{max}$ at $D(0)A = (D(0)A)_\text{max}$. In the solid phase we have from (42)

$$D(0)A_\text{max} \approx \frac{2}{\sqrt{3}} Z_v(0) \sim \left(\frac{\lambda}{\alpha}\right)^2,$$  \hfill (49)

$$\tilde{\Delta}_\text{max} \approx \frac{1}{(Z_v(0) Z_s(1))^{2/3}},$$ \hfill (50)

$$\approx \frac{9}{5}. \hfill (51)$$

The calculation was done by maximizing the left hand side of the implicit equation of (35) and (42) for $s_c \leq s \leq 1$ with respect to $\Delta$. This then gives the maximal $D(0)A$ value given by $(D(0)A)_\text{max}$ where we still get a solution for both implicit equations. Summarizing we obtain that the continuous replica symmetry broken solutions in the liquid as well as the solid phase stops to be stable solutions of the saddle point equations are infinite replica symmetry broken (where we have the restriction to $D(0)A \geq 2/\sqrt{3}$ in the liquid phase). More generally we obtain that every saddle point solution of (21) irrespective of its form is unstable for $\sqrt{3}D(0)/2 \geq Z_v(0)$ because the replicon eigenvalue $D$ is in fact a relevant variational perturbation theory we limit the $l$-sum to $l = m$. $F_{\text{var}}$ corresponds to the exact free energy of the system for $m \to \infty$ which means that $F_{\text{var}}$ does not depend on the choice of the trial Hamiltonian $H_{\text{trial}}$. The truncated sum depends on the choice of $H_{\text{trial}}$. Since the infinite sum is $H_{\text{trial}}$ independent, the best truncated sum should depend minimally on $H_{\text{trial}}$. A first approximation would be in taking a saddle point of $F_{\text{var}}$ with respect to the trial Hamiltonian $H_{\text{trial}}$ leading to (21) in the case $m = 1$.

To calculate $F_{\text{var}}$ beyond lowest order for a trial Hamiltonian $H_{\text{trial}}$ within the Parisi algebra is not an easy task. When going beyond lowest order we expect that the continuous replica symmetry breaking solution discussed above also the one-step replica symmetry breaking solution of (35) is no longer existing for $\sqrt{3}D(0)/2 \geq 4Z_v(0)$. This can be seen by Taylor expanding the left hand side of (21) with respect to $Z_v(0) \Delta_1/2$. Note the difference in the prefactor of $Z_v(0)$ compared to (10), (49). This leads us to the more generally assumption that there is no saddle point solution of (24) for $\sqrt{3}D(0)/2$ larger than $\approx Z_v(0)$. This is proved in Appendix A where it is shown that this is true for every finite-step replica symmetry broken solution of (27) by using results derived in Ref. [4].

We point out that $D(0)A \sim Z_v(0)$ is in fact a relevant parameter region for the glass transition line because we expect that the critical point is around $D(0)A \sim Z_v(0)$. Here we use (27) with the fact that the quadratic approximation to the disorder energy at the peak should be approximately $k_BT$ i.e. $D(c_j^2 a^2)k_BT \sim k_BT$. Note, as is shown in Fig. 1, the glass transition lines separating the phases BG2-BG1, VG2-VG1 crosses the first-order line BG2-VG2, BG1-VG1 in the vicinity of the critical point for optimal doping.

In the variational perturbation treatment of the anharmonic quantum mechanical oscillator we obtain a similar phenomenon. The even variational approximations to the free energy possess no extremum in the variational parameter $\phi_{\text{odd}}$. Only the odd perturbative orders where the M´ezard-Parisi theory belongs to the lowest order approximation within this perturbation theory, has a true minimum. In order to see whether we have a similar situation here, i.e. whether higher order variational approximations to the free energy possess a physical plausible extremum for $\Delta(s)$, we will calculate in the following section higher order variational approximations to the free energy.

VI. BEYOND LOWEST ORDER VARIATIONAL PERTURBATION THEORY

In this section, we will go beyond lowest order variational perturbation theory outlined in Section III. Starting from (14) we can immediately write down the next beyond lowest orders of the free energy $F$ within variational perturbation theory (30)

$$F_{\text{var}} = F_{\text{trial}} + (H - H_{\text{trial}})_{\text{trial,c}}$$

$$- \sum_{l=2}^m \frac{1}{l!} (\frac{k_B T}{l})^{l-1} ((H - H_{\text{trial}})^l)_{\text{trial,c}} \hfill (53)$$

$(H - H_{\text{trial}})^l$ is the averaging of $(H - H_{\text{trial}})^l$ with respect to the trial Hamiltonian $H_{\text{trial}}$ where we only take the connected expectation value part which means for example in second-order $((H - H_{\text{trial}})^2)_{\text{trial,c}} = ((H - H_{\text{trial}})^2)_{\text{trial}} - ((H - H_{\text{trial}})^2)_{\text{trial}}$. In order to calculate the free energy $F_{\text{var}}$ as in Section III within m'th order variational perturbation theory we limit the $l$-sum in (53) to $l = m$. $F_{\text{var}}$ corresponds to the exact free energy of the system for $m \to \infty$ which means that $F_{\text{var}}$ does not depend on the choice of the trial Hamiltonian $H_{\text{trial}}$. The truncated sum depends on the choice of $H_{\text{trial}}$. Since the infinite sum is $H_{\text{trial}}$ independent, the best truncated sum should depend minimally on $H_{\text{trial}}$. A first approximation would be in taking a saddle point of $F_{\text{var}}$ with respect to the trial Hamiltonian $H_{\text{trial}}$ leading to (21) in the case $m = 1$.

The calculation of the free energy in Appendix B within second-order variational perturbation theory ($m = 2$). We show that for $D(0)A > (D(0)A)_\text{max} \sim Z_v(0)$ there exist also in this case no continuous solutions of the saddle point equation in this second-order case. Thus, in contrast to the anharmonic oscillator where the variational perturbation theory leads to a solution of the saddle point equations for every odd order $\phi_{\text{odd}}$ a similar phenomenon is not existent in our case. Without explicit proof we now state the
conjecture that this is true for every finite order within variational perturbation theory. This means that there exist no saddle point of $F_{\text{var}}$ for large $\sqrt{3} D(0) A / 2 \approx \sqrt{3} \xi_c^{(0)}$ which is a relevant physical regime outlined at the end of the last section. One way out of this dilemma is to continue the continuous replica symmetry broken solutions given in [38] for the liquid and [42] for the solid to the regime $D(0) A > (D(0) A)_{\text{max}} \approx 2Z_c^{(0)}/\sqrt{3}$ by looking closer to the anharmonic oscillator problem solved in Ref. [31] via variational perturbation theory. As mentioned above, for equal orders within the variational perturbation expansion which means $m \in \mathbb{Z}$ in [53] one does not find a saddle point with respect to the trial harmonic Hamiltonian $H_{\text{trial}}$ being a quadratic potential in the anharmonic oscillator case. There it is shown that one gets good accordance with numerical solutions of the Schrödinger equation when interpreting the requirement of the minimally dependence of $F_{\text{var}}$ on $H_{\text{trial}}$ mentioned above by a vanishing of the second-order derivation of $F_{\text{var}}$ with respect to $H_{\text{trial}}$. This is equivalent to the demand that the first order variation of $F_{\text{var}}$ on $H_{\text{trial}}$ is minimal.

Transforming this general rule to our case by using [24], the self-energy function $\sigma(s)$ is given by the minimum of the functional

$$\text{Min}_{\sigma(s)} \left[ 1 + \frac{2}{\sigma(s)} \frac{k_B T}{v} D'(2B[\Delta(s)]) \right]$$

(54)

where we assume as a first approximation that this minimum is not dependent on $s$. This leads us to the following result for $D(0) A \geq (D(0) A)_{\text{max}}$:

The solutions $\Delta(s)$ of [54] where $\sigma(s)$ and $\Delta(s)$ related by [22] are given by [38], [42] with the substitution $D(0) A \rightarrow (D(0) A)_{\text{max}}$. The variational energies $\Delta f_{\text{var}}$ are given by [10], [11] in the liquid phase and [41], [45] in the solid phase with the same substitution. Furthermore, one has to multiply the potential part of the disorder energies [41] and [45] by a correction factor $(D(0) A) / (D(0) A)_{\text{max}}$ for $D(0) A > (D(0) A)_{\text{max}}$.

Summarizing, we obtain for BSCCO a third-order glass transition in the liquid phase having its reason in the breaking of the full replica symmetry across the transition line at $D(0) A / 2 / \sqrt{3}$. A similar transition was also found for YBCO [3]. Beside this transition we will show in the next section that [54] leads additionally to a second-order glass transition line at $D(0) A \approx 2Z_c^{(0)}/\sqrt{3}$ in both phases. We point out that this transition is not reasoned in the generalization of the saddle point criterium for the variational free energy to the more general principle of minimal sensitivity [54]. Up to now, we have only searched a saddle point of the variational free energy in the self-energy matrices $\sigma_{\alpha\beta}$ of the Parisi form (see the discussion below [15]) which could be motivated physically [27] as an Ansatz for the glassy-state self-energy matrices. Nevertheless, it could also be possible that the restriction to this subspace is the reason that we do not find a saddle point of the variational free energy for $D(0) A \gtrsim 2Z_c^{(0)} / \sqrt{3}$. On the other hand it is clear that also in this case the leaving of the stable saddle point solutions from the subspace of self-energy matrices of the Parisi form leads in general to a non-analytically of the free energy at the point $D(0) A \approx 2Z_c^{(0)} / \sqrt{3}$ and thus to a phase transition.

As we explained above the reason that the saddle point solutions of the variational free energy stops to exist within the Mélard-Parisi theory lies in the non-solvability of [39] for $s = s_c$. This follows further from the fact that $g(\Delta) \sim 1 / \Delta$ for large $\Delta$ and that $D''(2B[\Delta(s_c)]) \approx D''(0) \xi_c^{(0)}/B[\Delta(s_c)]$ for the relevant $\Delta(s_c)$ values where $g$ begin to show the behaviour $g(\Delta(s_c)) \sim 1 / \Delta(s_c)$. In deriving the approximation for $D''$ above we use $\xi_c^{(0)} a^2 \gg \xi^2$ for BSCCO (see the notes below [48] and [42]). In contrast to this we find for YBCO $\xi_c^{(0)} a^2 \ll \xi^2$ leading to the existence of the saddle point solutions of the variational free energy in the whole $H - T$ plane although we have also $g(\Delta) \sim 1 / \Delta$ for large $\Delta$ in this case [4]. This is the reason that one does not find the second-order glass transition line in YBCO.

Finally, we note that Giamarchi et al. in Ref. [10] only consider the small $\Delta$ behaviour of $g(\Delta)$ which is presumably the reason that they did not find the second-order glass transition line at least in the solid phase. The reason that we can compare only our low-temperature solid phase results with results in this paper lies in the fact that they did not consider defects as we do here being relevant in the high-temperature liquid phase. Note that they did not use temperature softened elastic constants in their calculation relevant for BSCCO [5].

### VII. OBSERVABLE CONSEQUENCES

In the following, we use the intersection criterium [8] with variational free energies [17] and [18] to get the first-order line separating the phases BG2-VG2, BG1-VG1, BG1-QL. This results in

$$B_m(T) = \frac{1}{12} \frac{1}{\sqrt{3} \pi^2} \frac{\lambda_{\alpha\beta}(0) \lambda_{\gamma\delta}(0)}{\lambda_{\alpha\beta}(0) \lambda_{\gamma\delta}(0)} \left( k_B T \right)^2$$

$$\times \exp \left[ - \frac{2}{k_B T} \left( f_{\text{var}}^{T=0} - f_{\text{var}}^{T=\infty} \right) \right]$$

(55)

where $f_{\text{var}}^{T=\infty}$ is given by the disorder part of the variational free energy which is the sum of [10] and [11] in the liquid case. $f_{\text{var}}^{T=0}$ corresponds to the disorder part of the variational free energy in the solid case given by [41] and [45].

Beside this first-order transition line we obtained a third-order glass transition line of the depinning form in the fluid phase separating the VL and VG1 phases

$$D(0) A = \frac{2}{\sqrt{3}}$$

(56)

and a second-order glass transition line separating BG2 in the solid phase and VG1 with VG2 in the
This means that we obtain within our analytical approximation a unified glass transition line in both phases in correspondence to the experimental findings shown in Figure 1.

In the following figures, we use parameter values for optimal doped BSSCO given by $\lambda_{ab}(0) \approx 2300\,\text{Å}$, $\xi_{ab}(0) \approx 30\,\text{Å}$, CuO$_2$ double layer spacing $a_s = 14\,\text{Å}$, $T_c = 90\,\text{K}$ and the anisotropy parameter $\gamma = \lambda_c/\lambda_{ab} \approx 250$. Due to the small coupling between the layers, the Josephson form of the interlayer coupling leads to a non-negligible softening of $\lambda_c$ or $\gamma = \lambda_c/\lambda_{ab}$, respectively, as a function of $B$ and $T$. In Ref. 29 it was found by Josephson plasma experiments that $\lambda_c(B, T)$ is nearly of the form $1/\lambda_c^2(B, T) \approx (1 + F(B/B_m)^{2})/\lambda_c^2(0, T)$ with some function $F$ which can be found in [29] and further that $\lambda_c^2(0, T)/\lambda_c^2(B_m, T) \approx 0.6$. This leads to $\gamma = \lambda_c/\lambda_{ab} \approx 250$ in the vicinity of the first-order line separating the phases BG2-VG2, BG1-VG1, BG1-VL. Here we used $\lambda_c(0, T)/\lambda_{ab}(T) \approx 200$ as in Ref. 3.

In Figure 2 we show (53) corresponding to the first order line separating phases BG2-VG2, BG1-VG1, BG1-VL for $\delta T_c$-pinning given by the correlation function (12) (upper figure) and $\delta l$ pinning (13) (lower figure) for various constants $d_0$. The square points in the figure denotes the experimentally determined first-order BG2-VG2, BG1-VG1, BG1-VL line of Beidenkopf et al. in Ref. 18. The $d_0$ values of the the straight (black) curves are chosen in such a way, that we reproduce in one of the best ways the experimentally given curves of Beidenkopf et al. and also the glass intersection point GP2. We obtain $d_0 = 2.5 \cdot 10^{-6}$ for $\delta T_c$-pinning case and $d_0 = 4 \cdot 10^{-6}$ for $\delta l$-pinning. The curves of representative variations of these almost best parameter values are given by the (red) dotted lines in Fig. 2. We obtain discrepancies in the form of the first-order line, BG2-VG2, BG1-VG1, BG1-VL line from the experiment. There are a large variety of the concrete forms of this line in the literature (see for example (14) for an almost horizontal BG2-VG2 line with a small kink near the intersection point GP2). The reason for the discrepancies comes mainly from the sensitivity of the curve on the disorder function (1) but also the negligence of the layerdness of BSSCO in our case could be one factor. Without taking into account dislocations, we expect a Josephson decoupling transition which is nearly temperature independent for low temperatures (37). The melting line and the decoupling line lies on top of each other when taking into account dislocations leading to the first-order line, BG2-VG2, BG1-VG1, BG1-VL transition line. Note that the Josephson decoupling is not complete over the transition line and further that the latent heat due to the Josephson degree of freedom is only 16% of the total latent heat over the first-order transition line (29). The competition between the temperature independent decoupling transition and the temperature dependent three dimensional first-order line should take into account the correct form of the whole first-order line for layered materials.

The small vertical marks on the curves in Fig. 2 denotes the glass intersection point GP2. We obtain especially for the $\delta T_c$-pinning case differences in the location of the glass intersection point GP2 with the experiment. In all shown three $\delta T_c$-pinning cases the glass transition point GP2 lies in the vicinity of the critical temperature $T_c$ where in both pinning mechanisms also the glass intersection point GP1 is located. Summarizing, we obtain as was also the case for YBCO (10) that the $\delta l$ pinning mechanism gives a better accordance to the experimental curves and glass intersection points than the $\delta T_c$ pinning mechanism.

In Figure 3 we show for $d_0 = 4 \cdot 10^{-6}$ with $\delta l$ pinning correlated impurities the whole phase diagram calculated with (55), (56) and (57) corresponding to the parameter values $d_0$ of the (black) solid line in the lower picture.
in Figure 2. Note that $H_c2$ cannot be resolved in this figure being almost vertical directed on the right boundary. Again we show for the experimentally determined phase diagram of Beidenkopf et al. \cite{9} where the square points denote the first-order BG2-VG2, BG1-VG1, BG1-VL transition line. The (blue) circle points denote the experimentally determined $T_d$ line BG2-BG1, VG2-VG1 of Beidenkopf et al. This line has to be compared with the (blue) dashed lines VG2-VG1, BG2-BG1, calculated with \cite{29}, where we get small discrepancies in the intersection point on the first-order line of the upper high magnetic curve VG2-VG1 in the liquid phase with the small magnetic field transition line BG2-BG1 in the solid phase. The reason is that we did not use the analytical approximation $\sqrt{3}(D(0)A)_{\text{max}}/2 \approx Z_{li}^{(0)}$ for $(D(0)A)_{\text{max}}$ valid in both phases but the numerical determined values calculated from the condition that \cite{29} stops to be solvable as described in Section V. As mentioned by Beidenkopf et al. in Ref. \cite{9} it could be experimentally possible that both lines do not intersect. From Fig. 3 we obtain that the point GP2 does not coincide with the maximum of the theoretical determined first-order BG2-VG2, BG1-VG1, BG1-VL transition line which coincides with the critical point \cite{4} (see also Fig. 4). This is possible for general doping \cite{19}. Nevertheless, we obtain a discrepancy between the position of our glass transition lines and the experimental findings. One reason comes from the approximations to the elastic moduli carried out in Section II but also corrections to \cite{26}, \cite{27} where we used $a_3c_{66}/a^2c_{14}^{(2)} \ll 1$. These approximations getting worse for higher magnetic fields \cite{8}. This leads to an additional bending of the first-order line in the direction to the temperature axis shown in Fig. 1 of Ref. \cite{8} without pinning. To get the same effective bending of this line as in the experiments we have to use a smaller $d_0$ value leading to BG2-BG1, VG2-VG1 lines located at smaller temperatures according to \cite{57}. Furthermore, a source of the additional bending can be also due to the decoupling transition between the Josephson layers as discussed above.

Beside these reasons also the restriction to the lowest order variational perturbation approximation could be a source for the difference of our theoretical finding of the glass transition line and the experimental ones. The calculation of the free energy within second-order variational perturbation theory is outlined in Appendix B. We did not carry out the calculation of the phase diagram within this order which is rather non-trivial being out of the scope of this work.

Finally, the (red) dashed-dotted line in Figure 3 shows the VG1-VL glass transition line calculated by the help of the depinning temperature formula \cite{50}. We do not
show for comparison the $T_c$ line of Fuchs et al. \textsuperscript{20} in the figure because they did not use an optimal doped crystal in the experiment.

In Fig. 4 we show in the upper picture the entropy jumps per double layer and vortex $\Delta S_l$ and in the lower picture the magnetic induction jumps $\Delta B$ over the first-order BG2-VG2, BG1-VG1, BG1-VL transition line. The (black) full line is calculated with $d_0 = 4 \cdot 10^{-6}$ in the $\delta l$ pinning case corresponding to parameter values of the phase diagram in Fig. 3. We used formulas derived in Ref. \textsuperscript{4} for the calculation. Note further, as was also the case in Ref. \textsuperscript{3}, that we did not use corrections for $\Delta B$ as a function of temperature across this transition.

The over all thick line (black) is the magnetic induction curve and its temperature derivative near the glass transition line in the BG2-BG1 solid phase than in the VG2-VG1 phase \textsuperscript{18, 19}. Nevertheless the displayed curves in their paper show a much smoother behaviour of the magnetic induction jumps $\Delta B$ and external magnetic field $H$. These differences are negligible in the interesting regime \textsuperscript{41}. The square points (blue) are experimentally determined values measured by Zeldov et al. \textsuperscript{41} for optimal doped BSCCO crystals. We note that there are other experiments in the literature for non-optimal doped crystals where $\Delta S$ and $\Delta B$ varies significantly \textsuperscript{14, 42}. The reason for this difference is not clear. The largest difference in Figure 4 between experimentally and theoretically determined curves is at high-temperatures near $T_c$.

As noted in \textsuperscript{3} this comes mainly from contributions of thermally activated vortex loops not inherent in our vortex lattice picture.

In the paper of Beidenkopf et al. \textsuperscript{18} the order of the glass transition lines VG2-VG1, BG2-BG1 was determined by measuring the magnetic induction jumps and its derivative with respect to the temperature across this line. They found a jump of $\partial B(H, T)/\partial T$ across the line leading to the conclusion that this transition is of second-order. They also deduced from their experiment that the jumps over the glass transition line are of almost the same magnitude in the BG1-BG2 phase and the VG1-VG2 phase \textsuperscript{18, 19}. Nevertheless the displayed curves in their paper show a much smoother behaviour of the magnetic induction curve and its temperature derivative near the glass transition line in the BG2-BG1 solid phase than in the VG2-VG1 phase. The problem of determining the order of the transition comes mainly from a large noise on the magnetic induction curves having their reason presumably in the spatial and temporal inhomogeneities of the system. This is the reason that Beidenkopf et al. in Ref. \textsuperscript{18} did not get a clear jump in the derivative in all measurements (while the induction itself bends sharply) to allow a systematic quantitative study of it \textsuperscript{42}.

The magnetic induction field $B$ is given by

$$B = H + \left[ 4\pi(k_BT)\frac{\partial}{\partial B} \frac{1}{Na} \ln(Z_{\text{nfl}}) + B \right]$$

$$+ 4\pi(k_BT)\frac{\partial}{\partial B} \frac{1}{Na} \ln(Z_{\text{nfl}})$$

(58)

where $Z_{\text{nfl}}$ is the partition function of the static non-fluctuating part of the vortex lattice. In Fig. 5 we show the disorder part of the magnetic induction field (black curves) $B$ given by

$$B_{\text{dis}} = -4\pi \frac{\partial}{\partial B} \frac{1}{Na} \Delta f_{\text{var}},$$

(59)

and also its derivative with respect to $T$ (dashed blue curves), i.e. $\partial B_{\text{dis}}/\partial T$ as a function of temperature for two different magnetic fields $B$ either in the VG2-VG1 phase (upper figure) or in the BG2-BG1 phase (lower figure). Here $\alpha$ is some subtraction parameter determined such that the $B_{\text{dis}}$ curve is symmetric around the glass intersection temperature. We used $\alpha = 0.0055 \text{Gauss}/\text{K}$ in the VG2-VG1 phase and $\alpha = 0.009 \text{Gauss}/\text{K}$ in the BG2-BG1 phase. The left-hand y-axis denotes the scale for $B_{\text{dis}} - \alpha T$, the right-hand y-axis for $\partial B_{\text{dis}}/\partial T - \alpha$.

![Figure 5: The disorder part $B_{\text{dis}} - \alpha T$ of the magnetic induction field defined by (59) (solid (black) curves), and its derivative with respect to $T$, i.e. $\partial B_{\text{dis}}/\partial T - \alpha$ (dashed blue curves) for two different magnetic fields $B$ either in the VG2-VG1 phase (upper figure) or in the BG2-BG1 phase (lower figure). Here $\alpha$ is some subtraction parameter determined such that the $B_{\text{dis}}$ curve is symmetric around the glass intersection temperature.](image-url)
for $B = 550$ Gauss. In the lower picture we show the disorder part of the magnetic induction field $B_{\text{dis}}$ in the solid phase for $B = 180$ Gauss. In correspondence to Beidenkopf et al. $^{18}$ we subtract to $B_{\text{dis}}$ a term linear in the temperature $T$ to get a symmetrical curve around the glass transition temperature. We obtain a negligible jump $\Delta \partial B_{\text{dis}}(B,T)/\partial T$ over the glass transition line in the solid BG2-BG1 phase. This is in contrast to the jump $\Delta \partial B_{\text{dis}}(B,T)/\partial T$ in the liquid high-temperature VG2-VG1 phase. By comparing the absolute values of this jump with the corresponding experimentally determined jump values $\Delta \partial B(H,T)/\partial T$ determined in Ref. $^{18}$ our values are about one order of magnitude smaller. Note that for our theory $\Delta \partial B_{\text{dis}}(B,T)/\partial T$ is about $10^{-1}$ smaller in the BG2-BG1 phase than in the VG2-VG1 phase. This could not resolved within our numerics in Fig. 5. That this is true can be seen from the following scaling consideration

$$
\Delta \frac{\partial}{\partial T} B_{\text{dis}}(B,T) = -\frac{4\pi}{v} \Delta \left[ \int ds \frac{\partial}{\partial B} \frac{\delta}{\delta \Delta(s)} \Delta f_{\varphi} \frac{\partial}{\partial T} \Delta(s) + \frac{\partial}{\partial T} \frac{\delta}{\delta \Delta(s)} \Delta f_{\varphi} \frac{\partial}{\partial B} \Delta(s) \right] \\
- \int dsds' \left( \frac{\delta}{\delta \Delta(s)} \frac{\delta}{\delta \Delta(s')} \Delta f_{\varphi} \right) \left( \frac{\partial}{\partial T} \Delta(s) \frac{\partial}{\partial T} \Delta(s') \right) \\
\sim -\frac{4\pi}{TBv} \left( \Delta f^{\text{kin}}_{\varphi} + \Delta f^{\text{pot}}_{\varphi} (Z_{\text{f}}^{(0)} \Delta(s_{c}))^{2} \right)
$$

where we used $^{53}$ in order to substitute terms containing $\Delta f^{\text{pot}}_{\varphi}$ to terms containing $\Delta f^{\text{kin}}_{\varphi}$. One can then see from analytic approximations but also numerical considerations that both terms in (60) are of almost equal value in the liquid phase VG2-VG1 but that the first kinetic term of the disorder free energy in (60) is much larger in the solid BG2-BG1 phase than the potential second part. Our numerics gives that the kinetic part of the disorder free energy $\Delta f_{\varphi}$ in the BG2-BG1 phase is one order of magnitude smaller than in the VG2-VG1 phase.

One source of the difference between the jump values of our theory and the experimental numbers could beside the approximations we used in our theory also the additional in-plane $ac$ equilibrizing magnetic shaking field in the experiment of Beidenkopf et al. $^{13}$. This shaking field is of the same magnitude as the $dc$ magnetic field in $z$-direction. It is immediately clear from the results in Ref. $^{18}$ as well as the theoretically and experimentally determined results for an additional in-plane $dc$ field instead of the $ac$ field $^{14, 12, 46}$ that the shaking field has only a small effect on the position of the first-order line and also the jump values $\Delta S_{l}$ and $\Delta B$. This can be understand by using the anisotropic scaling theory $^{11}$ leading to an attenuation of the in-plane field by a factor $\lambda_{ab}/\lambda_{c}$. In contrast to this we obtain from Fig. 5 that due to the smallness of the magnetic field $B_{\text{dis}}$ the shaking field can still have an effect on the jump value $\Delta \partial B/\partial T$ across the glass transition line BG2-BG1, VG2-VG1. Note that an in-plane magnetic $dc$ field can even put additional dislocations in the vortex lattice $^{46, 47}$.

**VIII. SUMMARY**

In this paper, we have derived the phase diagram for superconductors which have their phase transition lines at magnetic fields much smaller than $H_{c2}$, i.e. $B/H_{c2} < 0.25$ such as BSSCO. The model consists of the elastic degrees of freedom of the vortices with additional defect fields describing the defect degrees of freedom of the vortex lattice in the most simple way. For the impurity potential we have restricted ourselves to weak pinning $\delta c$ and $\delta l$-correlated impurities $^{1}$. This model was formerly used by us for describing the phase diagram of superconductors with a melting line near $H_{c2}$ $^{4}$. The layered structure of the superconductor, i.e. the Josephson coupling form between the layers, is not explicitly considered. We take this special coupling only into account via the elastic moduli of the lattice and an experimentally and analytically based decoupling scenario $^{24, 22, 26}$. In order to treat the impurity potential approximately we use a theory developed first by M´ezard and Parisi $^{3}$ for random-manifolds. This is based on a variational approach to the free energy via a quadratic trial Hamiltonian. After stating our model in Section II we have discussed the M´ezard-Parisi theory of the vortex lattice system in Section III. The minimum requirement for the trial free energy of the quadratic Hamiltonian leads to the saddle point equation $^{24}$ where the stable solutions are full replica symmetric for $D(0)A < \sqrt{3}/2$ in the fluid phase with $D(0)A$ is defined in (26) and (32). Everywhere else, the solutions are continuous replica symmetry broken. We expand these solutions to low temperatures $D(0)A > (D(0)A)_{\text{max}}$ with $^{37}$. That the saddle point equation $^{24}$ has no solution in general for $D(0)A \geq (D(0)A)_{\text{max}}$ is shown in Appendix A. The M´ezard-Parisi theory is the lowest-order approximation of a more general perturbation theory known as variational perturbation theory. In Appendix B we show how to go beyond the lowest-order approximation for the vortex lattice system up to second-order where also in this case a saddle point solution is not existent. Motivated by good results of the variational perturbation treatment for the anharmonic oscillator we generalize in Section VI the minimum requirement of the variational free energy determining the trial Hamiltonian to a more generalized principle of minimal sensitivity given in (54). This leads to a second-order phase transition line located at the points in the $H-T$ plane where the saddle point solutions cease to exist.

As was discussed by us at the end of Section VI, for YBCO in contrast to BSCCO the magnitude of the lattice fluctuations near the melting line is smaller than the correlation length of the impurity potential, i.e. $(\epsilon_{l}a_{0})^{2} \ll \xi^{2} A$. This is the reason that one does not find a similar non-existence of saddle point solutions to the variational free energy, in certain regions of the $H-T$
plane for YBCO as we obtain for BSCCO. This leads to
the absence of the second-order phase transition line in
YBCO. Further we note that Giamarchi and Doussal [10],
who calculated the physics of the vortex lattice with pin-
ing but without defects valid in the solid phase of real
systems, did not find in their work the ceasing of saddle
point solutions to the variational free energy in certain
regions in the solid phase. The reason lies in the fact that
they only consider small trial dimensionless gap functions
\( \Delta(s) \) in their calculation. Also they did not use temperature
softened elastic constants relevant for BSCCO [3].

The procedure described above leads to the following
physical consequences for BSCCO. Due to the form of the
elastic moduli in the deep \( H_{c2} \) region, we obtain two glass
phase transitions of the depinning form. The first transition
line of third-order is located in the fluid phase at high
temperatures not far from \( T_c \). It is given by \([10]\) identified
as the depinning temperature of a coherently pinned vortex substring. It separates the full replica symmetric solution to the variational energy at high temperatures
(VL phase) and a full replica symmetry broken solution
at lower temperatures (VG1 phase). This transition cor-
responds to the glass transition in YBCO. The transition line
is located in the vicinity of the experimentally found
\( T_x \) line [20]. The second transition is of second-order [67]
dividing the Bragg-glass and the vortex-glass phases in
four regions. It separates a full replica symmetry broken saddle point solution of the variational free energy (VG1,
BG1 phases) and a full replica symmetry broken turning point solution (VG2, BG2 phases). This transition line
is a temperature depinning transition where a substring
which is almost equally displaced due to disorder forming
a coherent vortex substring with respect to the temper-
ature fluctuations. We find that the derivative jump of the magnetic induction field with respect to the temper-
perature over this glass transition line in the Bragg-glass
phase is negligible in comparison to the jump in the liq-
uid phase. We compare this line with the experimentally
found second-order glass transition line by Beidenkopf et al. [18] located in the vicinity of our line. The jumps of the temperature derivate of the magnetic induction field in the vortex-glass phase of our theory is about one
order of magnitude lower in comparison to the experi-
mental values of Beidenkopf et al. [18]. They obtain a
similar value for the jumps in the Bragg-glass phase over
the glass transition line. In comparison to the glass trans-
ition line separating the VG2-VG1 phases they found a
softening of the jump in the vicinity of the glass transition line in the BG2-BG1 phase consistent with our findings.

We calculated the first-order melting transition line and
its disorder induced continuation dividing the Bragg-
glass phase BG2 and the vortex-glass phase VG2 by us-
ing an intersection criterium for the low and the high-
temperature expansion of the free energy. The whole
theoretical determined phase diagram and the experi-
mentally ones determined by Beidenkopf et al. [18] is
shown in Fig. 3. Finally we compared the entropy jumps
per layer and vortex, and also the magnetic induction
jump over the first-order line with the experimental find-
ings of Zeldov et al. [41]. This is shown in Fig. 4.

Summarizing, we have calculated the phase diagram of
a vortex lattice model stated in \([3, 4]\) for BSCCO with-
out taking explicitly into account the layered structure
of the material. Although we found certain quantita-
tive differences in the position of the experimental deter-
mined phase transition lines, the overall phase diagrams
looks rather similar. Discrepancies are maybe due to the
approximative evaluation of the theory and the layered
structure of BSCCO.

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**APPENDIX A: GENERAL PROOF OF THE NON-EXISTENT OF FINITE-STEP SADDLE POINT SOLUTIONS FOR LOW TEMPERATURES**

In this section we show that there exist no finite-step saddle point solution for the variational free energy \( \Delta f_{var} \) in the range \( D(0)A \gtrsim Z_l^{(0)} \) within first-order variational perturbation theory. This was shown in the continuous and additionally in the one-step case for the liquid phase in Section V. In order to derive this we use results derived in Section C in Ref. \([3]\). We obtain for an \( R \)-step replica symmetry breaking solution

\[
\sum_{i=1}^{R} \frac{1}{m_i} \left[ S(\Delta_{m_i}) - S(\Delta_{m_{i-1}}) \right] + Z \frac{D}{(2k_B T_g g[\Delta_{m_R}])^2} \left[ 2 k_B T_g g'[\Delta_{m_R}] \right]^{2}
\]

(\( A1 \))

where we used that \( \Delta_0 = 0, m_{R+1} = 1 \) and

\[
S(\Delta) = -\int_0^{\Delta(s)} d\Delta \Delta \frac{d}{d\Delta} g(\Delta)
\]

(\( A2 \))

It is shown in Ref. \([3]\) that \( Z > 0 \). We have

\[
S(\Delta) \approx \frac{1}{2} \frac{\Delta^{1/2}}{2} + \left[ \log \left( 1 + Z_l^{(0)} \frac{\Delta}{2} \right) - \frac{Z_l^{(0)} \Delta}{1 + Z_l^{(0)} \Delta} \right]
\]

(\( A3 \))

in the liquid case and

\[
S(\Delta) \approx \frac{\sqrt{3}}{2} \frac{1}{8 \pi} \left[ \frac{\epsilon_{44}^{(1)} \alpha^2}{c_{66} a^2} \right] \left( \frac{2}{3} \Delta^{3/2} + Z_l^{(1)} \frac{\Delta}{(Z_l^{(0)})^2} \log \left( 1 + Z_l^{(0)} \frac{\Delta}{2} \right) - \frac{Z_l^{(0)} \Delta}{1 + Z_l^{(0)} \Delta} \right)
\]

(\( A4 \))
for the solid. Next we use the inequalities \((\sum_{i=1}^{n} a_i)^2 \leq R \sum_{i=1}^{n} a_i^2\) for real number \(a_1, \ldots, a_R\) and further that \(\Delta_t \leq \Delta_{t+1}, \frac{\Delta_t^2}{\Delta_{t}} \leq S_{\Delta_t}/S_{\Delta_{t+1}}\), \((1/m_i - 1/m_{i+1})^2 \leq 1/m_i^2 - 1/m_{i+1}^2\) resulting in

\[
S(\Delta_{mR}) = \frac{D}{2 \frac{k_B T}{\sqrt{q}} \Delta_{mR}} \left( \frac{\Delta_{mR}}{D} \right)^2.
\]

This inequality can be only fulfilled for \(\sqrt{3}D(0)A/2 \lesssim 4RZ_{\beta}(0)\) which is a generalization of the one-step replica symmetry breaking case in the fluid phase discussed below Eq. (2).

**APPENDIX B: SECOND-ORDER VARIATIONAL PERTURBATION EXPANSION**

In this Section we calculate the second-order expansion terms within variational perturbation theory. The aim is to show that also to this order there are no saddle points of \(F_{\text{var}}\) with \(D(0)A \gtrsim Z^{(0)}\) corresponding to \(\xi_{ab} \sim \xi'_{ab} \ll a\). With this disorder part, we obtain for \(H - H_{\text{trial}}\)

\[
H - H_{\text{trial}} = H_{\text{dis}} - \frac{1}{2} \sum_{\alpha, \beta} \sum_{x, x'} \delta_{x, x'} \Delta(x_i + u^\alpha_i(x) - x_i - u^\beta_i(x')).
\]

We now classify terms of higher-order variational perturbation theory in two groups. When expanding \((H - H_{\text{trial}})^{\text{th}}\) in \(B\), we obtain first terms of the pure disorder Hamilton form \((H_{\text{dis}})^{\text{th}}\) which we denote by \((H - H_{\text{trial}})^{\text{th}}\) in \(B\). Second, there are monomials which contain at least one self-energy matrix factor \(\sigma_{\alpha \beta}\) in it denoted by \((H - H_{\text{trial}})^{\text{th}}\). These terms can be most easily treated by the square root trick. We now calculate first terms of the pure disorder Hamilton form.

1. **Pure disorder terms in Hamiltonian**

Within second-order variational perturbation theory, we obtain

\[
(H_{\text{dis}})^{\text{th}} = \frac{1}{4(2\pi)^4(k_BT)^2} \sum_{x, x', \alpha, \beta, \gamma, \delta} \int d^2q d^2q' \frac{D}{2 \frac{k_B T}{\sqrt{q}} \Delta(\Delta_q)} (\epsilon_{\alpha\beta}(u^{\alpha'}(x)-u^{\beta'}(x)) + \epsilon_{\alpha\beta}(u^{\alpha'}(x')-u^{\beta'}(x')))
\]

\[
\approx \frac{1}{4(2\pi)^4(k_BT)^2} \sum_{x, x', \alpha, \beta, \gamma, \delta} \int d^2q d^2q' \Delta(\Delta_q) (\Delta_q)
\]

\[
+ e^{-\frac{k_B T}{\sqrt{q}} \Delta^2 (G_{\alpha\alpha}(0) + G_{\beta\beta}(0) - G_{\alpha\beta}(0) - G_{\beta\alpha}(0))}
\]

\[
eq e^{-\frac{k_B T}{\sqrt{q}} \Delta^2 (G_{\gamma\gamma}(0) + G_{\delta\delta}(0) - G_{\gamma\delta}(0) - G_{\delta\gamma}(0))}
\]

As before, we restrict ourselves to the transversal part of the \(2 \times 2\) Green function \(G(x)\) defined by \(1/(2\pi)^3 \int d^2q d\xi \cdot (q_T \otimes q_T)^{-1} \cdot G(q) e^{iq.x + iq.x'}\). The replica sum in \(B\) is of the form \(\sum_{\alpha \beta \gamma \delta} F[G_{\alpha\beta}, G_{\gamma\delta}, G_{\alpha\gamma}, G_{\beta\delta}, G_{\alpha\delta}, G_{\beta\gamma}]\) where \(F[\cdot]\) is some functional of the various Green functions.

The fact that the Green function \(G_{\alpha\beta}\) is in the Parisi algebra implies that the Green function definitely depends on the overlap \(\alpha \cap \beta\) which we denote in the following by \(G_{\alpha \cap \beta}\). Furthermore, the operation \(\alpha \cap \beta\) on the replica indices has the ultrametric property. This means that whenever we choose three replicas \(\alpha, \beta, \gamma\), either all three of their overlaps are the same, i.e., \(\alpha \cap \beta = \alpha \cap \gamma = \beta \cap \gamma\), or one e.g. \(\alpha \cap \beta\) is larger than the other two. In the latter case the two are equal, i.e., \(\alpha \cap \beta > \alpha \cap \gamma = \beta \cap \gamma\). This means that of the three Green functions \(G_{\alpha\beta}, G_{\gamma\delta}, (0)\), only two are different. Similarly, of the six Green functions in \(F\) only three are different. The various possible Green function combinations can be most easily determined by mapping these six Green functions onto the edges of a tetrahedron where the Green functions on the adjacent edges of a face must fulfill the ultrametric property.

In the following, we restrict us to the leading term \(x = x'\) in \(B\). By carrying out the \(q, q'\) integral we obtain
\[
((H_{\text{dis}})^2)_{\text{trial,c}} \approx (k_BT)^2 \frac{N}{4} \sum_{\alpha,\beta,\gamma,\delta} \mathcal{D}^2(0) \\
\times \left( \frac{k_BT}{v} (g_{\alpha\alpha} - g_{\alpha\beta}) + \xi'^2 \right) \left( \frac{k_BT}{v} (g_{\gamma\gamma} - g_{\gamma\delta}) + \xi'^2 \right) - \frac{1}{4} \left( \frac{k_BT}{v} \right)^2 (g_{\alpha\gamma} + g_{\beta\delta} - g_{\beta\gamma} - g_{\alpha\delta})^2 \\
- \left( \frac{k_BT}{v} (g_{\alpha\alpha} - g_{\alpha\beta}) + \xi'^2 \right) \left( \frac{k_BT}{v} (g_{\gamma\gamma} - g_{\gamma\delta}) + \xi'^2 \right) \right). 
\] (B4)

The last subtracted term in (B4) is due to the connectedness of \(((H_{\text{dis}})^2)_{\text{trial,c}}\). From (B4) we obtain

\[
\sum_{\beta} \sigma_{\alpha\beta} = 0, 
\] (B5)

being the same equation as in the first-order variational perturbation theory case \([4, 9]\). We now restrict (B4) to the Parisi algebra by carrying out the program outlined above leading for \(n \to 0\) to

\[
((H_{\text{dis}})^2)_{\text{trial,c}} = n(k_BT)^2 N\mathcal{D}^2(0) \left\{ \right. \\
2 \int ds_1 ds_2 ds_3 (-\Theta_{0,1}(s_1) + \delta(s_1 - \sim) - \Theta_{1,1}(s_2) + \delta(s_2 - \sim) - s_2 \delta(s_2 - s_1)) (-\Theta_{0,1}(s_2) + \delta(s_2 - \sim) - s_2 \delta(s_2 - s_1)) (-\Theta_{0,1}(s_3) + \delta(s_3 - \sim) - s_3 \delta(s_3 - s_2)) \\
\left. \left( \frac{k_BT}{v} (\tilde{g} - g_{s_1}) + \xi'^2 \right) \left( \frac{k_BT}{v} (\tilde{g} - g_{s_2}) + \xi'^2 \right) - \frac{1}{4} \left( \frac{k_BT}{v} \right)^2 (g_{s_2} - g_{s_1})^2 \right) \\
+ \int ds_1 ds_2 ds_3 (-\Theta_{0,1}(s_1) + \delta(s_1 - \sim) - \Theta_{0,1}(s_2) + \delta(s_2 - s_1)) (-\Theta_{0,1}(s_2) + \delta(s_2 - s_1)) (-\Theta_{0,1}(s_3) + \delta(s_3 - s_2)) \\
\left( \frac{k_BT}{v} (\tilde{g} - g_{s_3}) + \xi'^2 \right) - \frac{1}{4} \left( \frac{k_BT}{v} \right)^2 (g_{s_2} - g_{s_1})^2 \right) \right\}. 
\] (B6)

Here \(g_s\) is the momentum integrated Green function of \(G_s\) according to Eq. (19). We define \(\delta(s_1 - \sim)\) by the functional \(\int ds_1 \delta(s_1 - \sim) H[g_{s_1}] = H(\tilde{g})\) where \(H\) is some functional of the integrated Green function \(g_{s_1}\) and \(\tilde{g} \equiv g_{0n}\).

For calculating the saddle point equation up to second order variational perturbation theory corresponding to (24) the derivative \((\delta/\delta g_s)((H_{\text{dis}})^2)_{\text{trial,c}}\) is relevant which should be added with an appropriate factor to the right hand side of Eq. (24). In order to derive this equation we first give the variational free energy \(F_{\text{var}}/N\) within second-order variational perturbation theory denoted by \(f_{\text{var,2}}\)

\[
f_{\text{var,2}} = f_{\text{var,1}} - \frac{1}{2(k_BT)}((H_{\text{dis}})^2)_{\text{trial,c}} \quad \text{(B7)}
\]

where \(f_{\text{var,1}}\) corresponds to the variational energy within first-order variational perturbation theory given in (17) and (18), i.e. \(f_{\text{var,1}} = f_{\text{var}}(0) + \Delta f_{\text{var,1}}\) where \(\Delta f_{\text{var,1}}\) is a modification of \(\Delta f_{\text{var}}\) specified in (17) according to

\[
\Delta f_{\text{var,1}} = P_1 \Delta f_{\text{var}}^{\text{kin}} + P_2 \Delta f_{\text{var}}^{\text{pot}}. 
\] (B8)

The additional prefactors \(P_i\) are modifications due to second-order perturbational expansion from terms proportional to \(\langle (H - H_{\text{trial}})^2)_{\text{trial,c}} \rangle\) containing at least one factor \(\sigma_{\alpha\beta}\). The constants \(P_i\) lying between 1/2 and 3/2 will be determined in the next subsection. Carrying out the variation of \(f_{\text{var,2}}\) with respect to \(g_s\) we obtain

\[
P_3 \sigma(s) = -2 \frac{k_BT}{v} P_4 \mathcal{D}'(2B(\Delta(s))) - \frac{\delta}{\delta g_s} \frac{\langle (H_{\text{dis}})^2)_{\text{trial,c}} \rangle}{(k_BT)^2}. 
\] (B9)

The calculation of \((\delta/\delta g_s)((H_{\text{dis}})^2)_{\text{trial,c}}\) is tedious but straightforward. Due to lack of space, we do not state the result here.

In order to discuss the sign of \(\langle (H_{\text{dis}})^2)_{\text{trial,c}} \rangle\) and \((\delta/\delta g_s)((H_{\text{dis}})^2)_{\text{trial,c}}\) we repeat the form of the Green functions in the Parisi algebra [9]

\[
\tilde{g} - g_s = \frac{1}{(2\pi)^3} \int d^2kdk_3 \left[ \frac{1}{G_0^{-1}(k,k_3) + \Delta(1)} \right. \left. + \int_1^1 ds' \frac{\sigma'(s)}{(G_0^{-1}(k,k_3) + \Delta(s))^2} \right]. \quad \text{(B10)}
\]
and
\[ \tilde{g} = \frac{1}{(2\pi)^3} \int d^2k dk_3 \, G_0(k, k_3) \times \left[ 1 + \int_0^1 ds \frac{1}{s^2} G_0^{-1}(k, k_3) + \Delta(s) \right]. \] (B11)

By using (B7), (B9) we obtain
\[ P_3 \sigma'(s) = -\sigma'(s) \left( \frac{2k_BT}{v} \right) g'(\Delta(s)) \] (B12)
\[ \times \left( \frac{2k_BT}{v} P_4 D''(2B[\Delta(s)]) + D_s \frac{\delta \left\langle (H_{\text{dis}})^2\right\rangle_{\text{trial},c}}{\delta g_s} (k_BT)^2 \right) \]

corresponding to (B3) in the first-order case. Here, we have used the differential operator
\[ D_s = \left( \sigma'(s) \left( \frac{2k_BT}{v} \right) g'(\Delta(s)) \right)^{-1} \frac{\partial}{\partial s}. \] (B13)

Dividing (B12) by \( \sigma'(s) \) and forming the derive with respect to \( s \), we obtain
\[ \left( \frac{2k_BT}{v} \right)^2 \frac{g'[(\Delta(s))^3]}{g'[(\Delta(s))] + \left( \frac{2k_BT}{v} P_4 D''(2B[\Delta(s)]) \right) + D_s \frac{\delta \left\langle (H_{\text{dis}})^2\right\rangle_{\text{trial},c}}{\delta g_s} (k_BT)^2 = s. \] (B14)

In contrast to the first-order results (B3), (B7) the second-order variational perturbation equations (B12), (B14) give no longer local algebraic equations for \( B'[\Delta(s)] \) and \( \Delta(s) \) but integral equations involving both quantities for different \( s \).

From (B10) we obtain that \( \delta/\delta g_s([H_{\text{dis}}]^2)_{\text{trial},c} \) depends through \( g_s \) on \( s \). One can show after a tedious but straightforward analysis that
\[ \left\langle (H_{\text{dis}})^2\right\rangle_{\text{trial},c} > 0, \] (B15)
\[ \frac{\delta}{\delta g_{sc}} \left\langle (H_{\text{dis}})^2\right\rangle_{\text{trial},c} > 0, \] (B16)
\[ D_s \frac{\delta}{\delta g_{sc}} \left\langle (H_{\text{dis}})^2\right\rangle_{\text{trial},c} > 0. \] (B17)

Here \( s_c \) is defined such that \( \sigma'(s) = 0 \) for \( s_c \leq s \leq 1 \). As in the first-order case, \( s_c \) can be determined by (B12) for \( s = s_c \) with \( B'[\Delta(s_c)] = g'[\Delta(s_c)] \). Then we obtain by the help of (B17) that equation (B12) is not solvable at \( s = s_c \) for small temperatures. More precisely we find that (B12) is not solvable for \( D(0)A \geq Z_i^{(0)} \) by using \( (\partial^2/\partial [\Delta(s_c)])^{-1} D_s(\delta/\delta g_{sc}) \left\langle (H_{\text{dis}})^2\right\rangle_{\text{trial},c} / (k_BT)^3 \sim D(0)A/Z_i^{(0)} \).

2. Terms containing at least one factor \( \sigma_{\alpha \beta} \)

Next, we consider contributions to second-order variational perturbation expansion \( \left\langle (H - H_{\text{trial}})^2\right\rangle_{\text{trial},c} \) containing at least one factor \( \sigma_{\alpha \beta} \). As described in the textbook Ref. [3] for the case of the anharmonic oscillator, these terms can be best derived with the help of the square root trick. In our system this trick consists in substituting \( \Delta(s) \) in \( f_{\text{var}} \) of Eq. (17) by \( (1-k)\Delta(s) \) denoted by \( \Delta f_{\text{var}}(k) \). The \( \left\langle (H - H_{\text{trial}})^2\right\rangle_{\text{trial},c,2} \) terms for \( k = 0 \) containing at least one factor \( \sigma_{\alpha \beta} \) are then given by
\[ \left\langle (H - H_{\text{trial}})^2\right\rangle_{\text{trial},c,2} = -2k_BT \frac{\partial}{\partial k^2} \Delta f_{\text{var}}(0) \] (B18)

This leads to the contributions in \( f_{\text{var},2} \) and the saddle point equation (41) which are a factor \( \Delta/\partial \Delta g(\Delta) / g(\Delta) \) or \( \Delta/\partial \Delta g''(\Delta) / g(\Delta) \) where \( m = 1, 2 \) smaller than the leading contributions. By using (B14), (B17) we obtain only non-negligible contributions to \( f_{\text{var},2} \) or the saddle point equation (B3), i.e. \( P_i \neq 1 \), for \( \Delta \ll 1/(Z_i^{(0)})^{1/3} \) in the fluid phase and \( \Delta \ll 1/(Z_i^{(1)})^{1/2} \) in the solid phase. We point out that \( \Delta_{\text{max}} \) in the fluid phase and \( \Delta_{\text{max}} \) in the solid phase in first-order variational perturbation theory is much larger than these \( \Delta \) values. Note that we obtain also corrections \( P_i \neq 1 \) in the regime \( \Delta > 1/Z_i^{(0)} \) much larger than \( \Delta_{\text{max}} \).

Thus, we consider the regime \( \Delta \ll 1/(Z_i^{(0)})^{1/3} \) in the fluid phase, and \( \Delta \ll 1/(Z_i^{(1)})^{1/2} \) in the solid phase. Here, we obtain prefactors \( P_i \) in (B7), (B9) which differ in general for both phases. We obtain
\[ P_1^{T \rightarrow 0} = 0, \quad P_1^{T \rightarrow 0} = 1, \quad P_2^{T \rightarrow 0} = \frac{1}{2}, \quad P_3^{T \rightarrow 0} = \frac{11}{8}, \quad P_3^{T \rightarrow 0} = \frac{7}{8}, \quad P_4^{T \rightarrow 0} = P_4^{T \rightarrow 0} = \frac{1}{2}. \] (B19)

[1] G. Blatter, M. V. Feigel’man, V. Geshkenbein, A. Larkin, and V. M. Vinokur, Rev. Mod. Phys. 66, 1125 (1994).
[2] T. Nattermann and S. Scheidl, Advances in Physics 49, 607 (2000).
[3] J. Dietel and H. Kleinert, Phys. Rev. B 74, 024515 (2006).
[4] J. Dietel and H. Kleinert, Phys. Rev. B 75, 144513 (2007).
[5] E. H. Brandt, Rep. Prog. Phys. 58, 1465 (1995).
[6] R. E. Hetzel, A. Sudbo, and D. A. Huse, Phys. Rev. Lett. 69, 518 (1992).
[7] H. Kleinert, Gauge Fields in Condensed Mat-
ter, Vol. II Stresses and Defects, World Scientific, Singapore, 1989. (readable online at
www.physik.fu-berlin.de/~kleinert/re.html#b2)
[8] J. Dietel and H. Kleinert, Phys. Rev. B 73, 024113 (2006).
[9] M. Mézard and G. Parisi, J. Phys. I 1, 809 (1991).
[10] T. Giamarchi and P. LeDoussal, Phys. Rev. Lett. 72, 1530 (1994); Phys. Rev. B 52, 1242 (1995).
[11] S. E. Korshunov, Phys. Rev. B 48, 3969 (1993).
[12] D. Li, B. Rosenstein, and V. Vinokur, J. Supercond. Novel Mag. 19, 369, (2006).
[13] C. J. van der Beek, S. Colson, M. V. Indenbom, and M. Konczykowski, Phys. Rev. Lett. 84, 4196 (2000).
[14] N. Avraham, B. Khaykovich, Y. Myasoedov, M. Rapaport, H. Shtrikman, D. E. Feldman, T. Tamegai, P. H. Kes, M. Li, M. Konczykowski, K. van der Beek, E. Zeldov, Nature (London) 411, 451 (2001).
[15] D. Li and B. Rosenstein, Phys. Rev. B 65, 220504(R) (2002); Phys. Rev. Lett. 90, 167004 (2003); Phys. Rev. B 70, 144521 (2004).
[16] M. P. A. Fisher, Phys. Rev. Lett. 62, 1415 (1989); D. S. Fisher, M. P. A. Fisher, and D. A. Huse, Phys. Rev. B 43, 130 (1991).
[17] P. L. Gammel, L. F. Schneemeyer, and D. J. Bishop, Phys. Rev. Lett. 66, 953 (1991).
[18] H. Beidenkopf, N. Avraham, Y. Myasoedov, H. Shtrikman, E. Zeldov, B. Rosenstein, E. H. Brandt and T. Tamegai, Phys. Rev. Lett. 95, 257004 (2005).
[19] H. Beidenkopf, T. Verdene, Y. Myasoedov, H. Shtrikman, E. Zeldov, B. Rosenstein, D. Li, and T. Tamegai, Phys. Rev. Lett. 98, 167004 (2007).
[20] D. T. Fuchs, E. Zeldov, T. Tamegai, S. Ooi, M. Rapaport, and H. Shtrikman, Phys. Rev. Lett. 80, 4971 (1998).
[21] T. Shiibauchi, T. Nakano, M. Sato, T. Kisu, N. Kameda, N. Okuda, S. Ooi, and T. Tamegai, Phys. Rev. Lett. 83, 1010 (1999).
[22] M. P. Raphael, M. E. Reeves, E. F. Skelton, and C. Kendziora, Phys. Rev. Lett. 84, 1587 (2000).
[23] T. Nishizaki, T. Naito, and N. Kobayashi, Phys. Rev. B 58, 11169 (1998).
[24] L. I. Glazman and A. E. Koshelev, Phys. Rev. B 43, 2835 (1991).
[25] L. L. Daemen, L. N. Bulaevskii, M. P. Maley, and J. Y. Coulter, Phys. Rev. B 47, 12191 (1993).
[26] R. Goldin, B. Horovitz, Phys. Rev. B 72, 024518 (2005).
[27] T. Blasius, Ch. Niedermayer, J. L. Tallon, D. M. Poole, A. Golnik, and C. Bernhard, Phys. Rev. Lett. 82, 4926 (1999).
[28] E. M. Forgan, M. T. Wylie, S. Lloyd, S. L. Lee, and R. Cubitt, Czech. J. Phys. 46, 1571 (1996).
[29] M. B. Gaifullin, Y. Matsuda, N. Chikumoto, J. Shimoyama, and K. Kishio, Phys. Rev. Lett. 84, 2945 (2000).
[30] H. Kleinert, Path Integrals in Quantum Mechanics, Statistics, Polymer Physics, and Financial Markets, World Scientific Publishing Co., Singapore 4th edition, 2006.
[31] W. Janke and H. Kleinert, Phys. Rev. Lett. 75, 2787 (1995).
[32] One can show that the quadratic disorder fluctuations in the single pinning regime $H$ are much smaller than $a^2$ in the interesting regime near the melting line, i.e. $u^2(0, a_3) \ll a^2$ with $a_3$ given by \[33\].
[33] M. Tinkham, Introduction to Superconductivity, McGraw-Hill, New York, 1996.
[34] A. E. Koshelev, L. I. Glazman, and A. I. Larkin, Phys. Rev. B 53, 2786 (1996).
[35] B. Horovitz, Phys. Rev. B 72, 024519 (2005).
[36] S. F. Edwards and P. W. Anderson, J. Phys. France 5, 965 (1975).
[37] V. Dotsenko, The theory of spin glasses and neural networks, Worlds Scientific, Singapore, 1989.
[38] This can be shown by using the methods of Section II B in Ref. \[34\] using a dynamical approach to the pinning of a single vortex.
[39] J. Kierfeld, Phys. Rev. B 69, 144513 (2004).
[40] D. M. Carlucci, C. De Dominicis, and T. Temesvari, J. Phys. I France 6, 1031 (1996).
[41] E. Zeldov, D. Majer, M. Konczykowski, V. B. Geshkenbein, V. M. Vinokur, and H. Shtrikman, Nature (London) 375, 791 (1995).
[42] K. Kadowaki and K. Kimura, Phys. Rev. B 57, 11674 (1998).
[43] H. Beidenkopf, private communication.
[44] B. Schmidt, M. Konczykowski, N. Morozov, and E. Zeldov, Phys. Rev. B 55, R8705 (1997).
[45] S. Ooi, T. Shiibauchi, N. Okuda, and T. Tamegai, Phys. Rev. Lett. 82, 4308 (1999).
[46] A. E. Koshelev, Phys. Rev. Lett. 83, 187 (1999).
[47] C. A. Bolle, P. L. Gammel, D. G. Grier, C. A. Murray, and D. J. Bishop, D. B. Mitzi, and A. Kapitulnik, Phys. Rev. Lett. 66, 112 (1991).
[48] T. Temesvari, C. De Dominicis, and I. Kondor, J. Phys. A: Math Gen. 27, 7569 (1994).