Classical Forces on Solitons in Finite and Infinite, Nonlinear, Planar Waveguides

J. I. Ramos and F. R. Villatoro
Departamento de Lenguajes y Ciencias de la Computación
E. T. S. Ingenieros Industriales
Universidad de Málaga
Plaza El Ejido, s/n
29013-Málaga
SPAIN

Abstract

Conservation equations for the mass, linear momentum and energy densities of solitons propagating in finite, infinite and periodic, nonlinear, planar waveguides and governed by the nonlinear Schrödinger equation are derived. These conservation equations are used to determine classical force densities which are compared with those derived by drawing a quantum mechanics analogy between the propagation of solitons and the motion of a quantum particle in a nonlinear potential well.

KEY TERMS: Spatial solitons, nonlinear Schrödinger equation, infinite planar waveguides, classical forces
1 Introduction

Soliton propagation has received a great deal of attention in recent years because of its ubiquity in many branches of physics, e.g., electrodynamics, fluid dynamics, plasma physics, etc., and because of its possible applications to distortionless signal transmission in ultra-high speed and long-distance transoceanic telecommunications by optical fibers, soliton lasers, all-optical soliton switches, nonlinear planar waveguides, nonlinear transmission lines, etc. [1]. For example, the envelope of the electric field in a weakly nonlinear, planar waveguide is governed by the nonlinear Schrödinger (NLS) equation which can be used to determine the forces acting on solitons [2,3].

Based on a quantum mechanics analogy between the NLS equation and the motion of a quantum particle in a potential well, Bian and Chan [2] and Ramos and Villatoro [3] determined the nonlinear and the diffraction force densities on solitons propagating in infinite and finite, respectively, weakly nonlinear, planar waveguides. In particular, Ramos and Villatoro [3] determined numerically the quantum force densities on solitons in finite, weakly nonlinear, planar waveguides subjected to periodic and homogeneous Dirichlet, Neumann and Robin boundary conditions at both boundaries of the waveguide.

In this paper, conservation equations for the mass, linear momentum and energy densities of solitons governed by the $n$th-dimensional NLS equation are derived. In particular, the linear momentum equation is used to obtain the classical force densities which are compared with the quantum force densities derived in Reference 3. Furthermore, these conservation equations are employed to assess the changes in linear momentum that a soliton under-
goes in a finite, weakly nonlinear, planar waveguide as it collides with the waveguide’s boundaries for periodic and homogeneous Dirichlet, Neumann and Robin boundary conditions.

2 Conservation Equations and Classical Force Densities on Solitons

As shown in References 1 and 2, the one-dimensional NLS equation in dimensionless Cartesian coordinates can be written as

\[ iu_t = -u_{xx} - |u|^2 u, \quad x \in D, \quad t \geq 0 \]  

(1)

where \( D \) is the spatial domain, the subscripts \( t \) and \( x \) denote partial differentiation with respect to time and the spatial coordinate, respectively, and \( u \) denotes the slowly varying amplitude of the electric field in a weakly, nonlinear slab waveguide.

Equation (1) can be written in dimensional form by transforming the dependent and independent variables as indicated in References 2 and 3.

In a recent work [3], the nondimensional NLS equation was used to draw an analogy between the propagation of a soliton in a weakly nonlinear, planar waveguide and the motion of a quantum particle in a nonlinear potential well. This analogy allowed to determine both the quantum energy density of and the force densities on solitons propagating in finite, nonlinear, planar waveguides. The main results obtained in [3] are repeated here for completeness and comparison with the classical mechanics ones derived in this paper,
and can be written as

\[ e_q(x, t) = \Re \{ iu^* u_t \} = -\Re \{ u^* u_{xx} + |u|^4 \} = e_k(x, t) + e_v(x, t), \] (2)

where \( e \) denotes energy density, the subscripts \( q, k \) and \( v \) stand for quantum, kinetic and potential, respectively, \( \Re \) denotes real part, and the asterisk denotes complex conjugation.

From the quantum energy density, one can easily obtain \[ f_n(x, t) = -\frac{\partial}{\partial x} e_v(x, t) = \frac{\partial |u|^4}{\partial x}, \quad f_d(x, t) = -\frac{\partial}{\partial x} e_k(x, t) = \frac{\partial}{\partial x} \Re \{ u^* u_{xx} \}, \] (3)

where \( f \) denotes a quantum force density, and the subscripts \( n \) and \( d \) stand for nonlinearity and diffraction, respectively.

Classical conservation equations and classical force densities on solitons may be obtained from the one-dimensional NLS equation as follows. The mass \( (m) \), linear momentum \( (M) \) and energy \( (E) \) densities of the soliton can be defined as

\[ m = u^* u, \quad M = i (u_x^* u - u^* u_x), \quad E = |u_x|^2 - \frac{1}{2} |u|^4. \] (4)

Using the NLS equation, i.e., Eq. (1), and the above densities, the following conservation equations can be easily obtained

\[ \frac{\partial m}{\partial t} + \frac{\partial M}{\partial x} = 0, \] (5)

\[ \frac{\partial M}{\partial t} = -\frac{\partial}{\partial x} \left( 4|u_x|^2 - \frac{\partial^2 m}{\partial x^2} - m^2 \right) = -\frac{\partial}{\partial x} \left( m^2 + 4E - \frac{\partial^2 m}{\partial x^2} \right), \] (6)

\[ \frac{\partial E}{\partial t} = \frac{\partial}{\partial x} \left( u_x^* \frac{\partial u}{\partial t} + u_x \frac{\partial u^*}{\partial t} \right) = i \frac{\partial}{\partial x} (u_x^* u_{xx} - u_x u_{x}^*) + \frac{\partial}{\partial x} (mM). \] (7)
It must be noted that the classical energy density can be written as the sum of potential and kinetic energies using action-angle variables [4].

For initial-value problems, i.e., $-\infty < x < \infty$, $t \geq 0$ and $|u| \to 0$ as $|x| \to \infty$, the integrals of the left-hand sides of the above three conservation equations over the whole spatial domain are zero. Therefore, the total mass or number of particles, the total linear momentum and the total energy are constant and coincide with the first, second and third invariants of the initial-value problem of the NLS equation [4]. The same result applies to periodic boundary-value problems. For finite line problems subject to homogeneous Dirichlet or Neumann boundary conditions at both boundaries, Eq. (5) indicates that the total mass is conserved, whereas the total linear momentum (cf. Eq. (6)) is not, in general, conserved.

Similar conservation equations to those derived above can be obtained in $n$th-dimensional space ($n=2,3$) where the NLS equation can be written in dimensionless form as

$$iu_t = -\nabla^2 u - |u|^2 u, \quad x \in D, \quad t \geq 0. \quad (8)$$

In $n$th-dimensional space, the mass, linear momentum and energy densities are defined as

$$m = u^* u, \quad \mathbf{M} = i (u \nabla u^* - u^* \nabla u), \quad E = \nabla u^* \cdot \nabla u - \frac{1}{2} |u|^4, \quad (9)$$

where the mass and energy densities are scalars and the momentum is a vector. Using Eq. (8), one can easily obtain the following conservation equations

$$\frac{\partial m}{\partial t} + \nabla \cdot \mathbf{M} = 0, \quad (10)$$

$$\frac{\partial \mathbf{M}}{\partial t} = -\nabla (4 \nabla u^* \cdot \nabla u - \nabla^2 m - m^2) = -\nabla (m^2 + 4E - \nabla^2 m), \quad (11)$$
\[
\frac{\partial E}{\partial t} = \nabla \cdot \left( \frac{\partial u}{\partial t} \nabla u^* + \frac{\partial u^*}{\partial t} \nabla u \right) = i \nabla \cdot (\nabla u^* \nabla^2 u - \nabla u \nabla^2 u^*) + \nabla \cdot (m \mathbf{M}), \tag{12}
\]

which indicate that the total mass, momentum and energy are conserved, i.e., they are invariant, for the initial-value problem of the \( n \)th-dimensional NLS equation if \(|u| \to 0\) as \(|\mathbf{x}| \to \infty\). Note that Eqs. (10)–(12) reduce to Eqs. (5)–(7) for one-dimensional problems.

From the conservation of linear momentum (cf. Eq. (11)), i.e., from Newton’s second law of classical mechanics, one can easily determine the classical potential and force densities as

\[
V_{cl} = m^2 + 4E - \nabla^2 m, \quad f_{cl} = -\nabla V_{cl}, \tag{13}
\]

where \( V_{cl} \) and \( f_{cl} \) denote the classical potential and classical force densities, respectively. Furthermore, one can easily deduce that the classical potential and force densities (cf. Eq. (13)) are related to the quantum energy and force densities (cf. Eqs (2) and (3) in \( n \)th-dimensional space) as

\[
e_q = \frac{1}{4}(V_{cl} - \nabla^2 m - 3m^2), \tag{14}
\]

\[
f_{cl} = 4f_q - \nabla(\nabla^2 m + 3m^2) = f_n + 2f_d - 2\nabla(\nabla u^* \cdot \nabla u). \tag{15}
\]

It is interesting to note that the above conservation equations for the mass, linear momentum and energy densities have been derived from the NLS equation, and have identical form to those of classical fluid dynamics [5] and electrodynamics [6,7]. Note that, in fluid mechanics, the mass conservation equation is also referred to as Lavoisier’s law or continuity equation, while, in electrodynamics, the conservation of charge is referred to as continuity equation.
It is also remarkable that the mass and linear momentum equations derived in this paper have the same form as those of a lossless, nonlinear transmission line [8], i.e.,

\[
\frac{\partial v(x,t)}{\partial x} = -L \frac{\partial I(x,t)}{\partial t}, \quad \frac{\partial I(x,t)}{\partial x} = -C(v) \frac{\partial v(x,t)}{\partial t}
\]

where \( v \) and \( I \) are the voltage and current, respectively, and \( L \) and \( C(v) \) denote the inductance and capacitance of the nonlinear transmission line per unit length, respectively. In this case, \( m, M \) and \( V_{cl} \) may be identified with the capacitance charge, \( Q, I \) and \( v/L \), respectively. Furthermore, nonlinear LC ladder networks [9] may result in evolution equations of the Korteweg-de Vries (KdV) type which have soliton solutions, and asymptotic methods permit to deduce the NLS equation from the KdV equation [10].

3 Force Densities on Solitons in Infinite, Nonlinear, Planar Waveguides

In this section, the classical and quantum energy and force densities acting on a soliton propagating in an infinite, weakly nonlinear, planar waveguide are determined analytically. The analytical solution to the initial-value problem of Eq. (1) can be written as

\[
u(x, t) = A \text{sech} \xi \exp i\eta
\]

where

\[
\xi = \frac{A}{\sqrt{2}} (x - x_0 - ct), \quad \eta = \frac{1}{2} \left[ c(x - x_0) + \left( A^2 - \frac{c^2}{2} \right) t + \phi_0 \right]
\]
where $A$, $c$, $x_0$ and $\phi_0$ are the soliton’s amplitude, speed, initial position and initial phase, respectively.

Using this solution, one can easily obtain the following quantum densities

$$e_q = \left(\frac{c^2}{4} - \frac{A^2}{2}\right) A^2 \text{sech}^2 \xi$$

(19)

$$f_n = -2\sqrt{2}A^5 \text{sech}^4 \xi \tanh \xi$$

(20)

$$f_d = \frac{A^3}{4} \text{sech}^2 \xi \tanh \xi \left[\sqrt{2} (c^2 - 2A^2) + 8\sqrt{2}A^2 \text{sech}^2 \xi\right]$$

(21)

$$f_q = f_n + f_d = \frac{\sqrt{2}}{4} A^3 \left(c^2 - 2A^2\right) \text{sech}^2 \xi \tanh \xi,$$

(22)

and the following classical potential and force densities

$$V_{cl} = A^2 c^2 \text{sech}^2 \xi$$

(23)

$$f_{cl} = \sqrt{2}A^3 c^2 \text{sech}^2 \xi \tanh \xi.$$ 

(24)

From Eqs. (19) and (23) and Eqs. (22) and (24), the following expression can be readily obtained

$$\frac{e_q}{V_{cl}} = \frac{f_q}{f_{cl}} = \frac{1}{4} - \frac{A^2}{2c^2},$$

(25)

which indicates that, for the initial-value problem of the NLS equation, the quantum and classical potential and force densities are proportional to their classical counterparts. In fact, the magnitude of the quantum force density is 25 per cent of that of the classical one for $A=c=1$ as shown in Figure 1. Note that the difference in sign between the classical and quantum force densities for $A=c=1$ is due to the definitions of the quantum energy and classical potential (cf. Eqs. (2) and (13)).
4 Force Densities on Solitons in Finite, Nonlinear, Planar Waveguides

In finite, nonlinear, planar waveguides, Eq. (1) was solved numerically by means of a second-order accurate, in both space and time, Crank-Nicolson method subject to

\[ \frac{\partial^n u}{\partial x^n}(x,t) = \frac{\partial^n u}{\partial x^n}(x + 2kH,t), \quad \forall n \geq 0, \quad k \in \mathbb{Z}, \quad x \in D, \quad t > 0, \quad (26) \]

for periodic waveguides, and

\[ u(-H,t) + \gamma u_x(-H,t) = 0, \quad u(H,t) + \gamma u_x(H,t) = 0, \quad t > 0, \quad (27) \]

where \( D=[-H,H] \), \( H = 50 \), and the values \( \gamma=0, \infty \) and \( \gamma \neq 0 \) correspond to homogeneous Dirichlet, Neumann and Robin boundary conditions, respectively, at both boundaries of the waveguide.

The time step and the grid size used in the calculations shown in the next section were 0.01 and 0.25, respectively, \( \phi_0=x_0=0 \) and \( A=c=1 \). The initial condition corresponded to the exact solution of the initial-value problem of the NLS equation, i.e., Eq. (17), translated in such a manner that there were not mathematical incompatibilities between the initial and boundary conditions.

5 Presentation of Results

Figures 1–3 show the classical and quantum force densities acting on solitons which propagate in weakly nonlinear, planar waveguides. The left side of
Figure 1 shows the force densities in an infinite waveguide, i.e., those corresponding to the initial-value problem of the NLS equation, and indicates that the magnitude of the classical force density is four times larger than the quantum one in agreement with the analytical results presented in Section 3. The left side of Figure 1 also shows the difference in sign between the classical and quantum force density which is due to the definitions of quantum energy and classical potential employed in this paper, and the $S$-shape of both force densities.

The results presented in the right side of Figure 1 correspond to a finite waveguide subject to periodic boundary conditions. Except for truncation errors, the classical and quantum force densities presented in the right side of Figure 1 coincide with those of the left side of the same figure. Furthermore, the linear momentum does not change as the soliton interacts with the right boundary in agreement with the analysis of Section 2.

For a finite waveguide subject to homogeneous Dirichlet boundary conditions at both boundaries, the left side of Figure 2 indicates that the classical force densities prior to and after the collision of the soliton with the right boundary are identical, and that, just during the soliton’s interaction with the boundary, the classical force density undergoes large changes at the right boundary. The maximum and minimum values of the classical force density are 0.5637 and -2.2249, respectively, whereas those of the quantum force density are 0.8059 and -0.7825, respectively. Note that, prior to and after the interaction of the soliton with the right boundary, the largest values of the classical and quantum force densities are 0.5443 and 0.1361, respectively, as shown in the upper left corner of Figure 1.
The left side of Figure 2 shows the classical and quantum force densities on solitons propagating in finite, planar waveguides subject to homogeneous Neumann boundary conditions at both boundaries. Due to the large changes in both the classical and the quantum force densities during the collision of the soliton with the right boundary, the $S$-shape of these force densities prior to and after the collision cannot be observed in the right side of Figure 2. The maximum and minimum values of the classical force density are 1.1283 and -133.2461, respectively, whereas those of the quantum force density are 6.9607 and -44.4007, respectively.

The results presented in Figure 2 clearly indicate that the magnitude of the classical force density is, in general, much larger and undergoes greater changes than the quantum one during the interaction of the soliton with the right boundary.

Since the one-dimensional NLS equation subject to homogeneous Dirichlet or Neumann boundary conditions is invariant under mirror reflections in $x$, the interaction of the soliton with the left boundary is expected to be identical to that with the right one. For this reason, only the interaction of the soliton with the right boundary was shown in Figure 2.

The results presented in Figure 3 correspond to a finite, planar waveguide subject to homogeneous Robin boundary conditions at both boundaries and $\gamma=1$ (cf. Eq. (27)). Since the one-dimensional NLS equation is invariant under reflections in $x$, while the homogeneous Robin boundary conditions are not, the interaction of the soliton with the right boundary is expected to be different from that with the left one as illustrated in the left and right, respectively, sides of Figure 3. This figure clearly indicates that, due to
the large increase that the classical force density undergoes as the soliton interacts with the right boundary, the S-shape of the classical force density can be barely seen. The maximum and minimum values of the classical force density during the interaction of the soliton with the right boundary are 21.7091 and -1.3303, respectively, whereas those of the quantum force density are 3.5286 and -0.7605, respectively.

Figure 3 also indicates that the classical force density undergoes larger variations than the quantum one and larger variations as the soliton interacts with the right boundary than when it interacts with the left one. The maximum and minimum values of the classical force density during the interaction of the soliton with the left boundary are 0.8590 and -6.1238, respectively, whereas those of the quantum force density are 0.5064 and -2.3679, respectively.

6 Conclusions

The nonlinear Schrödinger equation has been used to determine the conservation equations of mass, linear momentum and energy densities of solitons propagating in infinite or finite, weakly nonlinear, planar waveguides. It has been shown that these conservation equations have the same form as those of classical fluid dynamics and electrodynamics. It has also been shown that the classical force density is larger than the quantum one and that it undergoes large changes as the soliton interacts with boundaries subject to homogeneous boundary conditions. The largest changes in the classical force densities occur in planar waveguides subject to homogeneous Neumann
boundary conditions.

Acknowledgments

This research was supported by the Spanish D.G.I.C.Y.T. under Project no. PB91–0767. The second author (F.R.V.) has a fellowship from the Programa Sectorial de Formación de Profesorado Universitario y Personal Investigador, Subprograma de Formación de Investigadores ”Promoción General del Conocimiento”, from the Ministerio de Educación y Ciencia of Spain. The authors are extremely grateful to Dr. Isabel Prieto Barcia of the Departamento de Ingeniería de Comunicaciones of the Universidad de Málaga for her help and suggestions.

References

[1] L. F. Mollenauer, J. P. Gordon and M. N. Islam, “Soliton Propagation in Long Fibers with Periodically Compensated Loss”, IEEE Journal of Quantum Electronics, Vol. QE-22, 1986, pp. 157–173.

[2] J.-R. Bian and A. K. Chan, “Computations of the Diffraction Effect and the Nonlinear Effect on Spatial Solitons in Nonlinear Planar Waveguides”, Microwave and Optical Technology Letters, Vol. 4, 1991, pp. 184–191.

[3] J. I. Ramos and F. R. Villatoro, “Forces on Solitons in Finite, Nonlinear, Planar Waveguides”, Microwave and Optical Technology Letters, in press, 1994.
[4] S. Novikov, S. V. Manakov, L. P. Pitaevskii and V. E. Zakharov, “Theory of Solitons (The Inverse Scattering Method)”. Consultants Bureau, New York, 1984, p. 75.

[5] L. D. Landau and E. M. Lifshitz, “Fluid Mechanics”. 2nd. ed., Pergamon Press, New York, 1987, p. 9.

[6] L. D. Landau and E. M. Lifshitz, “Electrodynamics of Continuous Media”. Pergamon Press, New York, 1960, p. 301.

[7] D. J. Griffiths, “Introduction to Electrodynamics”. 2nd. ed., Prentice Hall, Englewood Cliffs, New Jersey, 1987, p. 320.

[8] A. N. Sharkovsky, “Chaos from a Time-Delayed Chua’s Circuit”, IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications, Vol. 40, 1993, pp. 781–783.

[9] M. M. Turner, G. Branch and P. W. Smith, “Methods of Theoretical Analysis and Computer Modeling of the Shaping of Electrical Pulses by Nonlinear Transmission Lines and Lumped-Element Delay Elements”, IEEE Transactions on Electronic Devices, Vol. 38, 1991, pp. 810–816.

[10] F. Calogero and W. Eckhaus, “Nonlinear Evolution Equations, Rescalings, Model PDEs and their Integrability. I”, Inverse Problems, Vol. 3, 1987, pp. 229–262.
Figure 1 Nonlinear ($f_n$), diffraction [$\text{Re}(f_d)$], and total ($f_{\text{total}}$) force densities and momentum density ($p$) for periodic boundary conditions.

Figure 2 Nonlinear ($f_n$), diffraction [$\text{Re}(f_d)$], and total ($f_{\text{total}}$) force densities and momentum density ($p$) for Dirichlet boundary conditions.

Figure 3 Nonlinear ($f_n$), diffraction [$\text{Re}(f_d)$], and total ($f_{\text{total}}$) force densities and momentum density ($p$) for Neumann boundary conditions.

Figure 4 Nonlinear ($f_n$), diffraction [$\text{Re}(f_d)$], and total ($f_{\text{total}}$) force densities and momentum density ($p$) for Robin boundary conditions and first collision with the right boundary.

Figure 5 Nonlinear ($f_n$), diffraction [$\text{Re}(f_d)$], and total ($f_{\text{total}}$) force densities and momentum density ($p$) for Robin boundary conditions and first collision with the left boundary.