‘Conformal Theories - AdS Branes’ Transform, or One More Face of AdS/CFT

E. Ivanov

Bogoliubov Laboratory of Theoretical Physics, JINR, 141 980 Dubna, Moscow Region, Russia

Abstract

The AdS/CFT transformation relates two nonlinear realizations of (super)conformal groups: their realization in the appropriate field theories in Minkowski space with a Goldstone dilaton field and their realization as (super)isometry groups of AdS (super)spaces. It exists already at the classical level and maps the field variables and space-time coordinates of the given (super)conformal field theory in $d$-dimensional Minkowski space $M_d$ on the variables of a scalar codimension one (super)brane in AdS$_{d+1}$ in a static gauge, the dilaton being mapped on the transverse AdS brane coordinate. We explain the origin of this coordinate map and describe some its implications, in particular, in $d = 1$ models of conformal and superconformal mechanics. We also give a suggestive geometric interpretation of this AdS/CFT transform in the pure bosonic case in the framework of an extended $2d + 1$-dimensional conformal space involving extra coordinates associated with the generators of dilatations and conformal boosts.

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1 Introduction

The cornerstone of AdS/CFT correspondence [1]-[3] is the assertion that the isometry group of an AdS$_n \times S^m$ background in which some type IIB string theory lives is identical to the standard conformal group (times the group of internal $R$ symmetry) of the appropriate conformal field theory living on the $(n-1)$-dimensional Minkowski space interpreted as a boundary of AdS$_n$. The supersymmetric version of this correspondence deals with the appropriate realizations of superconformal groups. Originally, the AdS/CFT correspondence was formulated as a duality between IIB string theory compactified on AdS$_5 \times S^5$ and $N = 4$ super Yang-Mills theory in $d = 4$ Minkowski space.

It was shown in [4]-[7], [1] that the invariance group of the worldvolume action of some probe brane in an AdS$_n \times S^m$ background (e.g. D3-brane in AdS$_5 \times S^5$) can be realized as a field-dependent modification of the standard (super)conformal transformations of the worldvolume. On the other hand, in the spirit of the AdS/CFT correspondence, the AdS superbrane worldvolume actions are expected to appear as the result of summing up some leading (in external momenta) terms in the low-energy quantum effective actions of the corresponding Minkowski space (super)conformal field theories in a phase with spontaneously broken (super)conformal symmetry (e.g. the effective action of $N = 4$ SYM theory in Coulomb branch) [1], [8]-[10]. It was argued in [11, 12] that the modified (super)conformal transformations can be understood as a quantum deformation of the standard (super)conformal transformations.

In this contribution, basically following the line of refs. [13, 14], we expound a different viewpoint on the interplay between the standard and modified (super)conformal transformations. The basic statement is that any conformal field theory in $d = p+1$-dimensional Minkowski space in a phase with spontaneously broken conformal symmetry and so with the dilaton Goldstone field can be brought, by an invertible change of variables, into the form in which it respects invariance just under the above mentioned field-dependent conformal transformations. Using this relation between the conformal and AdS bases (AdS/CFT map or transform), one can rewrite any conformal field theory with a dilaton among the involved fields in terms of the variables of the corresponding scalar AdS brane in a static gauge, and vice versa. The AdS images of the minimal conformally-invariant Lagrangians (i.e. those containing terms with no more than two derivatives) prove to necessarily include non-minimal terms composed of the first extrinsic curvature of the brane. On the other hand, the conformal field theory image of the minimal brane Nambu-Goto action is a non-polynomial and higher-derivative extension of the minimal Minkowski space conformal actions. We also discuss some further implications of the AdS/CFT transformation and its supersymmetric extension in the quantum-mechanical $d = 1$ systems [14, 15]. Also, a novel geometric interpretation of this transformation will be given. It highlights the relevance of some extended coset manifolds of (super)conformal groups as ambient manifolds for the above sort of the ‘conformal theories - AdS branes’ correspondence. In the bosonic case it is the $(2d + 1)$-dimensional coset manifold $CM^{2d+1} \sim SO(2, d)/SO(1, d - 1)$. It includes, besides the standard Minkowski coordinate, also independent coordinates associated with the generators of dilatations and conformal boosts.
2 Two nonlinear realizations of conformal group in $\mathcal{M}_d$

The group-theoretical origin of the AdS/CFT map to be discussed takes root in the existence of two different nonlinear realizations of the conformal group in $d$ dimensions.

The algebra of conformal group $SO(2,d)$ of $d = p + 1$-dimensional Minkowski space reads

\[
[M_{\mu\nu}, M^{\rho\sigma}] = 2\delta_\rho^{[\mu}M_{\nu]}^{\sigma}], \quad [P_\mu, M_{\nu\rho}] = -\eta_{[\mu\nu}P_{\rho]}, \quad [K_\mu, M_{\nu\rho}] = -\eta_{[\mu\nu}K_{\rho]},
\]

\[
[P_\mu, K_\nu] = 2(-\eta_{\mu\nu}D + 2M_{\mu\nu}), \quad [D, P_\mu] = P_\mu, \quad [D, K_\mu] = -K_\mu,
\]

(2.1)

where we antisymmetrize with the factor $1/2$. The standard nonlinear realization of this group (see e.g. [16]) is defined as left shifts of an element of the coset $SO(2,d)/SO(1,d-1)$:

\[
g = e^{y^\mu P_\mu}e^{\varphi D}e^{\Omega^\mu K_\mu}.
\]

(2.2)

The shifts with the parameters $a^\mu, b^\mu$ and $c$ related to the generators $P_\mu, K_\mu$ and $D$ induce the familiar conformal transformations of the coset coordinates

\[
\delta y^\mu = a^\mu + cy^\mu + 2(yb)y^\mu - y^2b^\mu, \quad \delta \varphi = c + 2yb.
\]

(2.3)

The left-covariant Cartan 1-forms are defined as follows

\[
g^{-1}dg = \omega_\mu^\nu P_\nu + \omega_D D + \omega_\mu^{\mu\nu} M_{\mu\nu} + \omega_K^\mu K_\mu
\]

\[
= e^{-\varphi} dy^\mu P_\mu + \left(d\varphi - 2e^{-\varphi}\Omega_\mu dy^\mu\right) D - 4e^{-\varphi} \Omega^\mu dy^\mu M_{\mu\nu}
\]

\[+ \left[d\Omega - \Omega^\mu dy^\mu + e^{-\varphi}\left(2\Omega_\nu dy^\nu\Omega^\mu - \Omega^2 dy^\mu\right)\right] K_\mu .
\]

(2.4)

The vector Goldstone field $\Omega^\mu(x)$ can be covariantly expressed through the dilaton $\varphi(x)$ [17]

\[
\omega_D = 0 \Rightarrow \Omega_\mu = \frac{1}{2} e^{\varphi} \partial_\mu \varphi, \quad \omega_P^\mu = e^{-\varphi} dy^\mu, \quad \omega_K^\mu = d\Omega^\mu - e^{-\varphi} \Omega^2 dy^\mu .
\]

(2.5)

The covariant derivative of $\Omega^\mu$ is defined by the relation

\[
\omega_K^\mu = \omega_P^\nu D_\nu \Omega^\mu \Rightarrow D_\nu \Omega^\mu = \frac{1}{2} e^{2\varphi} \left[\partial_\nu \partial^\mu \varphi + \partial_\nu \varphi \partial^\mu \varphi - \frac{1}{2} (\partial_\nu \varphi \partial_\nu \varphi) \delta^\mu_\nu\right].
\]

(2.6)

The conformally invariant measure of integration over $\{y^\mu\}$ is defined as the exterior product of $d$ 1-forms $\omega_P^\mu$

\[
S_1 = \int \mu(y) = \int d^{(p+1)}y \, e^{-(p+1)\varphi} .
\]

(2.7)

The covariant kinetic term of $\varphi$ can be constructed as

\[
S^\text{kin}_\varphi = \int d^{(p+1)}y \, e^{-(p+1)\varphi} \, D_\nu \Omega^\mu = \frac{1}{4} (p-1) \int d^{(p+1)}y \, e^{(1-p)\varphi} \, \partial_\nu \varphi \partial_\nu \varphi .
\]

(2.8)

In any field theory with spontaneously broken conformal symmetry, it is always possible to make a field redefinition which splits the full set of scalar fields of the theory into the dilaton $\varphi$ with the transformation law (2.3) and the subset of fields which are scalars of weight zero under conformal transformations. In this sense, the above nonlinear realization is universal.

While the standard nonlinear realization of $SO(2,d)$ describes a spontaneously broken phase of conformally invariant field theories, there is another sort of nonlinear realizations of the same
group [18] which proves to be relevant to the description of codimension one branes on AdS$_{d+1}$. In this realization, $SO(2, d)$ acts as the group of motion of AdS$_{d+1}$. It is related to the existence of the so-called AdS basis in the algebra (2.1).

In the AdS basis, we introduce the following generators

\[ \hat{K}_\mu = mK_\mu - \frac{1}{2m}P_\mu , \hat{D} = mD , \]

(2.9)

where $m$ is the inverse AdS radius. The basic relations of the $SO(2, d)$ algebra become

\[ [\hat{K}_\mu, \hat{K}_\nu] = 4M_{\mu\nu} , [P_\mu, \hat{K}_\nu] = 4mM_{\mu\nu} - 2\eta_{\mu\nu}\hat{D} , [\hat{D}, P_\mu] = mP_\mu , [\hat{K}_\mu, \hat{D}] = P_\mu + m\hat{K}_\mu . \]

(2.10)

The main difference between (2.10) and (2.1) is that the generators ($\hat{K}_\mu, M_{\rho\nu}$) generate the semi-simple subgroup $SO(1, d)$ of $SO(2, d)$, while the subgroup with ($\hat{K}_\mu, M_{\rho\nu}$) has the structure of a semi-direct product. As a result, in the coset element (2.2) rewritten in the new basis

\[ g = e^{x^\mu P_\mu} e^{q^D} e^{\Lambda^\nu K_\nu} , \]

(2.11)

$x^\mu$ and $q(x)$ are parameters of the coset manifold $SO(2, d)/SO(1, d)$ which is AdS$_{d+1}$ (this is the so called ‘solvable subgroup’ parametrization of AdS$_{d+1}$ [7]). Eq.(2.5) now yields

\[ \omega_D = 0 \Rightarrow \lambda_\mu = e^{mq} \frac{\partial_\mu q}{1 + \sqrt{1 - \frac{1}{2}e^{2mq}(\partial q \partial q)}} , \lambda^\nu \equiv \Lambda^\nu \frac{\tanh \sqrt{\frac{\lambda^2}{2}}}{\sqrt{\frac{\lambda^2}{2}}} . \]

(2.12)

The remaining coset space Cartan forms are then given by the expressions:

\[ \omega^\mu_p = e^{-mq} \left( \delta^\mu_\nu - \frac{\lambda_\mu \lambda_\nu}{1 + \frac{\lambda^2}{2}} \right) dx^\nu \equiv E^\mu_p dx^\nu = e^{-mq} \hat{E}^\mu_p dx^\nu , \omega^\mu_K = \frac{1}{1 - \frac{\lambda^2}{2}} \left( d\lambda^\mu - m\lambda^2 \omega^\mu_p \right) . \]

(2.13)

The covariant derivative of the Goldstone field $\lambda^\mu$ is defined by

\[ \omega^\nu_K = \omega^\mu_p D_\mu \lambda^\nu \Rightarrow D_\mu \lambda^\nu = \frac{1}{1 - \frac{\lambda^2}{2}} \left[ e^{mq} \left( \delta^\mu_\nu + \frac{\lambda_\mu \lambda_\nu}{1 - \frac{\lambda^2}{2}} \right) \partial_\mu \lambda^\nu - m\lambda^2 \delta^\nu_\mu \right] . \]

(2.14)

The transformation laws of $x^\mu, q(x)$ under the left shifts of (2.11) are as follows

\[ \delta x^\mu = a^\mu + c x^\mu + 2 (xb)x^\mu - x^2 b^\mu + \frac{1}{2m^2} e^{2mq} b^\mu , \delta q = \frac{1}{m} (c + 2xb) . \]

(2.15)

After a field redefinition, they are recognized as the field-dependent conformal transformations of refs. [1], [4]-[6] representing the AdS isometries in the solvable-subgroup parametrization [7].

The simplest invariant of the nonlinear realization considered is again the covariant volume of $x$-space obtained as an integral of wedge product of $(p + 1)$ 1-forms $\omega^\mu_p$. This invariant is basically the static-gauge Nambu-Goto (NG) action for $p$-brane in AdS$_{p+2}$

\[ S_{NG} = \int d^{(p+1)}x \left[ e^{-(p+1)mq} - \det E \right] = \int d^{(p+1)}x e^{-(p+1)mq} \left[ 1 - \sqrt{1 - \frac{1}{2}e^{2mq}(\partial q \partial q)} \right] , \]

(2.16)

where we have subtracted 1 to obey the standard requirement of absence of the vacuum energy [1]. The subtracted term is invariant under (2.15) on its own (up to a shift of the integrand
by a total derivative). The action (2.16) is universal, in the sense that it describes the radial (pure AdS) part of any \((n - 2)\)-brane action on \(\text{AdS}_n \times S^m\).

Note that the covariant derivative (2.14) which plays an important role in our construction is the tangent-space projection of the first extrinsic curvature \(K_{\mu \nu}\) of the brane in the static gauge [13] (for the definition of \(K_{\mu \nu}\), see e.g.[19]). In the flat \(m = 0\) case

\[
K_{\mu \nu} = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{1 - \frac{1}{2}(\partial q \partial q)}} \partial_{\mu} \partial_{\nu} q , \quad D_{\mu} \lambda_{\nu} = \frac{1}{\sqrt{2}} (E^{-1})_{\mu}^{\rho} (E^{-1})_{\nu}^{\omega} K_{\rho \omega} .
\]

The generalization to the AdS case is straightforward.

3 An equivalence relation between CFT and AdS bases

In both nonlinear realizations described above we deal with the same coset manifold, namely \(SO(2, d)/SO(1, d-1)\), in which the coset parameters are separated into the space-time coordinates and Goldstone fields in two different ways. Hence, there should exist a relation between these two parametrizations. It can be read off by comparing (2.2) and (2.11):

\[
y^{\mu} = x^{\mu} - \frac{e^{mq}}{2m} \lambda^{\mu} , \quad \varphi = mq + \ln \left(1 - \frac{\lambda^2}{2}\right) , \quad \Omega^{\mu} = m \lambda^{\mu} .
\]

It is invertible at any finite and non-zero \(m = 1/R\) and maps the Minkowski space conformal transformations (2.3) onto the field-dependent ones (2.15). This AdS/CFT transform can be defined only in the framework of extended coset manifolds \{\(y^{\mu}, \varphi, \Omega^{\mu}\)\} and \{\(x^{\mu}, q, \lambda^{\mu}\)\}. In Sect. 4 we shall see how (3.18) can be recovered in the setting with all coset parameters treated as independent coordinates.

Using (3.18), any Minkowski space conformal field theory with a dilaton among its basic fields can be projected onto the variables of AdS brane and vice versa. To find the precise form of various \(SO(2, d)\) invariants in both bases, let us define the transition matrix

\[
\frac{\partial y^{\nu}}{\partial x^{\mu}} \equiv A^{\nu}_{\mu} = \delta^{\nu}_{\mu} - \frac{\lambda_{\mu} \lambda^{\nu}}{1 + \frac{\lambda^2}{2}} - \frac{e^{mq}}{2m} \partial_{\mu} \lambda^{\nu} = \left(1 - \frac{\lambda^2}{2}\right) \hat{E}^{\nu}_{\rho} T^{\rho}_{\nu} , \quad T^{\nu}_{\rho} = \delta^{\nu}_{\rho} - \frac{1}{2m} D^{\nu}_{\rho} \lambda^{\mu} .
\]

The Jacobian of the change of space-time coordinates in (3.18) is

\[
J \equiv \det A = \left(1 - \frac{\lambda^2}{2}\right)^{p+1} \det \hat{E} \det T .
\]

Then, making the change of variables (3.18) in the invariant dilaton Lagrangians (2.7) and (2.8), we obtain, respectively,

\[
S_1 = \int d^{(p+1)} x \, e^{-(p+1)mq} \sqrt{1 - \frac{1}{2} e^{2mq} (\partial q \partial q)} \det T ,
\]

\[
S_{\text{kin}}^{\varphi} = m \int d^{(p+1)} x \, e^{-m(p+1)q} \sqrt{1 - \frac{1}{2} e^{2mq} (\partial q \partial q)} \left[\det T (T^{-1} D \lambda)^{\mu}_{\rho}\right] .
\]

A surprising fact is that the AdS image of the potential term of dilaton contains the NG part of the AdS \(p\)-brane action (2.16) modified by higher-derivative covariants collected in
\[ \det(I - \frac{1}{2m} D\lambda) = 1 - \frac{1}{2m} D_\mu \lambda^\mu + \ldots . \]  
As we saw, \( D_\mu \lambda^\mu \) is basically the \( p \)-brane extrinsic curvature. So, already the simplest conformal invariant in Minkowski space proves to produce, on the AdS side, an action which is the standard \( p \)-brane action in AdS\(_{p+2} \) plus corrections composed of the extrinsic curvature tensor. Note that only for the conformal actions containing no potential terms of dilaton the relations (3.18) can be treated as a genuine equivalence map taking the kinetic term of \( \varphi \) into that of \( q \) (plus higher order corrections).

Let us now see what the brane action (2.16) looks like in the conformal basis. Using

\[ D_\mu \Omega^\nu = m(T^{-1})^\omega_\mu D_\omega \lambda^\nu, \quad (T^{-1})^\mu_\nu = \delta^\nu_\mu + \frac{1}{2m^2} D_\mu \Omega^\nu. \]  

and making in (2.16) the change of variables inverse to (3.18), we find

\[ S_{NG} = \frac{1}{4m^2} \int d^{(p+1)} y e^{(1-p)\varphi} \frac{(\partial \varphi \partial \varphi)}{1 - \frac{1}{5m^2} e^{2\varphi} (\partial \varphi \partial \varphi)} \det \left( I + \frac{1}{2m^2} D\Omega \right). \]  

Thus we have found an equivalent representation of the static-gauge action (2.16) of \( p \)-brane in AdS\(_{p+2} \) as a non-linear extension of the conformally-invariant dilaton action in \( M_{p+1} \). In [13] the conformal field theory image of the full bosonic part of D3-brane action on AdS\(_5 \times S^5 \) was found.

### 4 Geometric interpretation of the AdS/CFT map in a conformal space with extra dimensions

As was already mentioned in the previous Section, the transformation (3.18) cannot be understood within the pure \{\( y, \varphi \)\}, or \{\( x, q \)\} geometries. Indeed, the extended manifold \{\( y^\mu, \varphi(y) \)\} has the topology of \( M_d \times R^1 \), while \{\( x^\mu, q(x) \)\} is a surface in AdS\(_{d+1} \). Clearly, no direct equivalence can be established between these two geometrically different manifolds.

The meaning of (3.18) can be clarified by embedding both these manifolds as subspaces into the extended conformal space \( CM^{2d+1} \equiv \{y^\mu, \varphi, \Omega^\nu\} = \{x^\mu, q, \lambda^\nu\} \). This is the coset \( SO(2,d)/SO(1,d-1) \) in which all parameters are treated as independent coordinates. The nonlinear realizations associated with the parametrizations (2.2) and (2.11) operate on \( d \)-dimensional hypersurfaces in this ambient space, parametrized, respectively, by \( y^\mu \) and \( x^\mu \).

The \( SO(2,d) \) transformation properties of the coordinates of \( CM^{2d+1} \) in the conformal and AdS parametrizations can be obtained as before by considering left action of \( SO(2,d) \) on the coset elements (2.2) and (2.11), in which all parameters are independent. The coset Cartan 1-forms in eq. (2.4) and their counterparts in the AdS basis define covariant differentials of the coordinates \{\( y^\mu, \varphi, \Omega^\nu \)\} and \{\( x^\mu, q, \lambda^\nu \)\}, respectively. The Lorentz Cartan form in (2.4) defines the \( SO(1,d-1) \) connection which enters the covariant differentials of those functions on \( CM^{2d+1} \) which have external indices with respect to the stability subgroup \( SO(1,d-1) \) and so carry non-trivial representations of the latter.

To clarify what is the meaning of the transformation (3.18) in this setting, let us consider a scalar function on \( CM^{2d+1} \), first in the conformal parametrization,

\[ F(y, \varphi, \Omega), \]  

and study the issue of existence of invariant subspaces in \( CM^{2d+1} \). This problem amounts to listing all possible covariant conditions which one can impose on \( F \) to effectively suppress the
dependence of $F$ on one or another coordinate of $CM^{2d+1}$. The covariant derivatives of the function $F$ are defined by the standard formula

$$
\begin{align*}
  dF(y, \varphi, \Omega) &= dy^\mu \partial_\mu F + d\Omega^\nu \partial_\nu F + d\varphi \partial^\varphi F = \omega_\mu^\nu \nabla^\nu y F + \omega_\mu^K \nabla^K y F + \omega_D \nabla^\varphi F ,
\end{align*}
$$

whence

$$
\begin{align*}
  \nabla^\mu y &= e^\varphi \partial/\partial y^\mu + 2\Omega_\mu \partial/\partial \varphi + \Omega^2 \partial/\partial \Omega^\mu , \quad \nabla^\varphi = \partial/\partial \varphi + \Omega^\nu \partial/\partial \Omega^\mu , \quad \nabla^\Omega = \partial/\partial \Omega^\mu .
\end{align*}
$$

While acting on a function with an external Lorentz $SO(1, d-1)$ index, the covariant derivative $\nabla^\mu y$ acquires a Lorentz connection which is determined by the Cartan form associated with the generators $M_{\mu\nu}$ in (2.4). E.g., the covariant derivative of a vector function $G_\mu$ is defined as

$$
\begin{align*}
  D^\mu_y G_\nu = \nabla^\mu_y G_\nu + 2(\Omega_\nu G_\mu - \eta_{\mu\nu} \Omega_\rho G_\rho). 
\end{align*}
$$

Other derivatives do not acquire any connections.

The evident chain of the subspaces closed under the action of $SO(2, d)$

$$
\{y, \varphi, \Omega\} \supset \{y, \varphi\} \supset \{y\}
$$

is in the one-to-one correspondence with the following $SO(2, d)$ covariant analyticity-type constraints on the generic function $F(y, \varphi, \Omega)$:

$$
\begin{align*}
(a) \; &\nabla^\Omega_\mu F^{(1)} = 0 \Rightarrow F^{(1)} = f(y, \varphi) , \quad (b) \; \nabla^\nu_\mu F^{(2)} = \nabla^\varphi F^{(2)} = 0 \Rightarrow F^{(2)} = f(y) .
\end{align*}
$$

The self-consistency of these constraints follows from the integrability conditions

$$
\nabla^\Omega_\mu \nabla^\nu_\nu - (\mu \leftrightarrow \nu) = 0 , \quad [\nabla^\varphi, \nabla^\Omega_\mu] = -\nabla^\Omega_\mu .
$$

Note that the field $f(y)$ in (4.30) has the conformal weight zero. In order to end up with a scalar field having the standard free field weight $(d-2)/2$, one should replace the second set of constraints in (4.30) by

$$
\nabla^\Omega_\mu F^{(2)} = 0 , \quad \nabla^\varphi F^{(2)} = \frac{1}{2}(d-2) F^{(2)} \Rightarrow F^{(2)} = e^{(\frac{d-1}{2})\varphi} \tilde{f}(y)
$$

($e^\varphi$ has the weight $-1$). This choice of covariant constraints is also self-consistent thanks to (4.31). In the considered parametrization all these constraints are easily solved just because the corresponding covariant derivatives are basically partial derivatives with respect to the appropriate coordinates and (4.30), (4.32) simply eliminate (or strictly fix) the dependence on the latter.

An important outcome of the above discussion is that the conformal parametrization $\{y, \varphi, \Omega\}$ manifests the embedding chain (4.29) which corresponds to splitting of $CM^{2d+1}$ into the product of the base Minkowski space $\mathcal{M}_d = \{y\}$ and the fiber $\{\varphi, \Omega\}$, with $SO(2, d)$ being realized in $\mathcal{M}_d$ as the standard conformal group. Let us now impose on the generic function $F$ a different type of the covariant constraint

$$
\left( \nabla^\Omega_\mu + \alpha \nabla^\nu_\mu \right) F^{(3)} = 0 ,
$$

where $\alpha$ is a constant. This constraint is again self-consistent due to the integrability conditions

$$
\left[ \nabla^\Omega + \alpha \nabla^\nu, \nabla^\Omega + \alpha \nabla^\nu \right] \sim \nabla^\Omega + \alpha \nabla^\nu ,
$$

(4.43)

(4.34)
or, equivalently,
\[(\nabla^\Omega_\mu + \alpha D^y_\mu)(\nabla^\Omega_\nu + \alpha \nabla^y_\nu) - (\mu \leftrightarrow \nu) = 0.\]

At \(\alpha = 0\) and \(\alpha = \infty\) this mixed constraint goes over to its counterpart from the set (4.30) and another admissible constraint related to (4.30) by conformal inversion. Note that at \(\alpha \neq 0, \infty\) one cannot impose on \(F\) any additional constraint involving \(\nabla \varphi\), since
\[
[\nabla \varphi, \nabla^\Omega_\mu + \alpha \nabla^y_\mu] \sim \nabla^\Omega_\mu - \alpha \nabla^y_\mu,
\]
and so \(\nabla \varphi\) does not constitute a closed subalgebra with the differential operator in (4.33). To understand the meaning of (4.33) one should pass to such a parametrization of \(CM^{2d+1}\) in which the differential operator in (4.33) is reduced to a partial derivative, suggesting that \(F\) subjected to (4.33) is independent of the relevant coordinate. Identifying
\[
\alpha = -1/2m^2
\]
and performing the coordinate change just according to (3.18), it is straightforward to find how the covariant derivatives look in the new coordinates \(\{x^\mu, q, \lambda^\mu\}\). In particular, we find
\[
\nabla^\Omega_\mu = \frac{1}{2m^2} \nabla^y_\mu = \frac{1}{m} \left(1 - \frac{\lambda^2}{2}\right) \frac{\partial}{\partial \lambda^\mu}.
\]

Thus the basis \(\{x, q, \lambda\}\) is just the one in which the differential constraint (4.33) takes the ‘short’ analyticity condition type form and so becomes explicitly solvable:
\[
F^{(3)}(y, \varphi, \Omega) \equiv \tilde{F}^{(3)}(x, q, \lambda) = \tilde{f}(x, q).
\]

We know that \(x^\mu\) and \(q\) provide a parametrization of AdS\(_{d+1}\), so the basis \(\{x, q, \lambda\}\) in \(CM^{2d+1}\) makes manifest the embedding
\[
CM^{2d+1} \supset \text{AdS}_{(d+1)},
\]
where the subspace AdS\(_{d+1}\) is again closed under the action of SO\((2, d)\) which is realized by the transformations (2.15). As distinct from the chain (4.29), one cannot extract any subspace in AdS\(_{d+1}\) which would be closed under SO\((2, d)\). In the conformal basis this property is rephrased as the impossibility to strengthen (4.33) by any additional constraint with \(\nabla \varphi\).

From the mathematical point of view, the embedding chains (4.29) and (4.39) (as well as some other possible ones \(^1\)) amount to different fiberings of the coset manifold \(CM^{2d+1} = SO(2, d)/SO(1, d - 1)\). The option (4.29) corresponds to the choice of \(\mathcal{M}_d = \{y^\mu\}\) as the base and \(\{q, \Omega^\mu\}\) as a fiber, while in the case (4.39) the base and fiber are AdS\(_{d+1}\) and the coset \(SO(1, d)/SO(1, d - 1) = \{\lambda^\mu\}\), respectively. Also notice that one could recover the variable change (3.18), up to an equivalence transformation \(q \rightarrow \tilde{q}(q), \lambda^\mu \rightarrow \tilde{\lambda}^\mu(\lambda, q)\), simply by requiring the covariant derivative in (4.37) to have the ‘short’ form
\[
\nabla^\Omega_\mu - \frac{1}{2m^2} \nabla^y_\mu = A^\nu_\mu(q, \lambda) \frac{\partial}{\partial \lambda^\nu}, \quad A^\nu_\mu = \delta^\nu_\mu + O(q, \lambda), \quad \det A|_{q=\lambda=0} \neq 0.
\]

\(^1\)One more interesting invariant subspace of \(CM^{2d+1}\) is the 2\(d\)-dimensional ‘bi-conformal space’ \([20]\) which is obtained by placing the generator \(D\) into the stability subgroup. It corresponds to imposing the single constraint \(\nabla \varphi F = 0\).
To summarize, the change of coordinates (3.18) defines passing from the parametrization \(\{y, \varphi, \Omega\}\) of \(CM^{2d+1}\) in which the Minkowski space geometry is manifest and \(SO(2, d)\) acts as the conformal group of \(\mathcal{M}_d\)\(^2\) to the parametrization \(\{x, q, \lambda\}\) where the AdS\(_{d+1}\) geometry is manifest and \(SO(2, d)\) is realized as the corresponding group of motion. In the original setting this transformation relates two different \(d\)-dimensional hypersurfaces in \(CM^{2d+1}\) which are parametrized, respectively, by the coordinates \(y^\mu\) and \(x^\mu\). The first hypersurface is pertinent to the standard nonlinear realization of \(SO(2, d)\) on \(y^\mu\) and dilaton field \(\varphi(y)\), while the second one is just the worldsurface of scalar \((d-1)\)-brane on AdS\(_{d+1}\) in a static gauge.

Besides relating Minkowski space conformal theories in spontaneously broken phase to AdS branes, the transformation (3.18) seems to have some interesting potential implications for the field theories on AdS spaces. This follows from the consideration of the present Section. Indeed, using (3.18) one can represent any unconstrained field on AdS\(_{d+1}\) in the solvable subgroup parametrization, \(f(x, q)\), as a constrained field on \(CM^{2d+1}\), i.e. \(F^{(3)}(y, \varphi, \Omega)\) subjected to the constraint (4.33) (with \(\alpha = -1/2m^2\)).\(^3\) Assuming for \(F(y, \varphi, \Omega)\) a series expansion in \(\Omega^\mu\),
\[
F(y, \varphi, \Omega) = F^0(y, \varphi) + F_\mu(y, \varphi)\Omega^\mu + \frac{1}{2}F_{\mu\nu}(y, \varphi)\Omega^\mu\Omega^\nu + O(\Omega^3),
\]
eq (4.33) expresses the whole infinite tower of symmetric tensor coefficients in such an expansion as multiple \(y\)- and \(\varphi\)-derivatives of \(F^0(y, \varphi)\):
\[
F_\mu = \frac{1}{2m^2} e^\varphi \partial_\mu F^0, \quad F_{\mu\nu} = \frac{1}{4m^4} e^{2\varphi} \partial_\mu \partial_\nu F^0 + \frac{1}{m^2} \eta_{\mu\nu} e^\varphi \partial_\varphi F^0,
\]
etc. Thus the AdS\(_{d+1}\) field \(f(x, q)\) proves to be equivalent to an infinite set of conformal fields on the Minkowski space \(\mathcal{M}_d = \{y^\mu\}\) emerging from the appropriate expansion of \(F_0(y, \varphi)\) in the dilaton-like coordiante \(\varphi\). Assuming that the correct expansion should be a general series in the positive and negative powers of \(z \equiv e^\varphi\) with the conformal dimension \(-1\), we conclude that these fields should carry all integer conformal dimensions from \(-\infty\) to \(\infty\) (some truncations are possible depending on the boundary conditions with respect to \(z\) or \(q\)). Conversely, some irreducible conformal field on \(\mathcal{M}_d\) can be represented as a constrained field on \(CM^{2d+1}\) in the AdS parametrization, and so it amounts to a set of fields on AdS\(_{d+1}\) with the properly restricted dependence on \(\{x^\mu, q\}\). It is worth noting that all these relationships are purely ‘kinematical’.

As for dynamics, the existence of the map (3.18) offers an interesting new opportunity in analysing the relationships between equations of motion for fields on AdS\(_{d+1}\) and conformally invariant equations in \(\mathcal{M}_d\), including those for higher spin fields. The covariant dynamical equations for fields of any spin on AdS were recently constructed in [22]. One more link with higher spins is suggested by the fact that general functions on \(CM^{2d+1}\) contain in their \(\Omega^\mu\)- or \(\lambda^\mu\)-expansions infinite sequences of symmetric Lorentz tensor fields which are basic ingredients of various versions of the higher integer spin theories (see e.g. review [23] and [24]). It would be interesting to study whether these theories admit a reformulation in \(CM^{2d+1}\) and what are possible implications of the transformation (3.18) in this context.

The above reasoning shows some important difference in the treatment of the relationship between \(d\)-dimensional Minkowski and AdS\(_{d+1}\) spaces in the conventional AdS/CFT approach and in the considered framework. While in the first approach the (compactified) \(\mathcal{M}_d\) is regarded as a boundary of AdS\(_{d+1}\), in the case under consideration these both manifolds coexist

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\(^2\)To be more rigorous, of the appropriate compactification of \(\mathcal{M}_d\).

\(^3\)This resembles e.g. the description of chiral superfields in \(N = 1, d = 4\) supersymmetry either as unconstrained functions on the chiral \(N = 1\) superspace or constrained functions on real \(N = 1\) superspace. Both these superspaces are subspaces of general complexified \(N = 1, d = 4\) superspace which palys a role similar to \(CM^{2d+1}\) in our case.
as different subspaces in the extended conformal space \( CM^{2d+1} \). The coordinate map (3.18) simply relates two different parametrisations of \( CM^{2d+1} \) which make manifest either \( \mathcal{M}_d \) geometry or AdS\(_{d+1} \) geometry. Note that one can equally relate the conformal parametrisation of \( CM^{2d+1} \) to the parametrisation which manifests the geometry associated with the subspace \( SO(2, d)/SO(2, d - 1) \subset CM^{2d+1} \). The corresponding invariant functions are singled out by the constraint (4.33) with \( \alpha \to -\alpha \). Obviously, there is also a change of coordinates from this new parametrisation to the AdS one. The subspace just mentioned has as its flat limit the ‘two-time’ \((d + 1)\)-dimensional space (with the signature \((++, --, \ldots)\)), so some interrelations with the ‘two-time’ physics [21] are expected to arise while exploring these maps and their consequences. One of such consequences is the possibility to relate AdS branes to those on this exotic manifold, and vice versa.

Finally, let us notice that the covariant derivatives applied to the functions of the type (4.30), (4.32) or (4.38) in general take them out of the subspaces on which they are defined. Only those covariant operators which commute with the analyticity conditions (4.30), (4.33) preserve the type of a given constrained function. A technically feasible way to construct such operators is to exploit invariance with respect to the appropriate right transformations of the coset parameters. An equivalent way to covariantly restrict general functions on \( CM^{2d+1} \) to the invariant subspaces is to require these functions to be invariant under right shifts of the coset elements (2.2) or (2.11) by the generators which enlarge the stability subgroup \( SO(1, d) \) to the stability subgroups of these subspaces viewed as coset manifolds. The differential operators appearing in the constraints (4.30), (4.32), (4.38) prove to be generators of these right shifts, and the constraints themselves admit a nice interpretation as the conditions of invariance under these shifts (or as a condition that the given field is an eigenfunction of some Cartan generator of the group of right shifts, as in (4.32)). Then the precise form of covariant differential operators preserving the given type of constrained function can also be found from the requirement of invariance with respect to the right shifts.

For instance, the first constraints in (4.30), (4.32) amount to invariance under the right transformations with the generator \( K^\mu \). Using (2.4), it is very easy to find how these right transformations are realized on Cartan forms, coset coordinates and covariant derivatives:

\[
\delta \Omega^\mu = \beta^\mu, \quad \delta \nabla^\mu_{\Omega} = 0, \quad \delta \nabla^\varphi = \beta^\mu \nabla^\mu_{\Omega}, \quad \delta \nabla^\varphi_{\mu} = 2\beta^\mu \nabla^\varphi + 2[(\Omega \cdot \beta)\delta^\varphi - \beta^\mu \Omega^\mu] \nabla^\Omega_{\mu},
\]

where \( \beta^\mu \) is the corresponding group parameter. Then it is easy to check that the covariant d’Alembertian \( \Box^{(d)}_{\text{cov}} \equiv \mathcal{D}^{\mu\nu} \nabla^{\nu}_{\mu} \), being applied to the functions subjected to the constraints (4.32), is invariant under these transformations and so cannot depend on \( \Omega^\mu \) (while such a dependence is present for generic functions on \( CM^{2d+1} \)). One finds

\[
\Box^{(d)}_{\text{cov}} F^{(2)} = \mathcal{D}^{\mu\nu} \nabla^{\nu}_{\mu} F^{(2)} = e^{(\frac{d}{2} - 1)\varphi} \Box^{(d)} \tilde{f}(y).
\]

The AdS\(_{d+1} \) case (4.33) is more complicated because the extra right transformations in this case are generated by \( \tilde{K}^\mu = mK^\mu - \frac{1}{2m}P^\mu \) which enlarges \( SO(1, d - 1) \) to the non-ableian stability subgroup \( SO(1, d) \) of AdS\(_{d+1} \). Nevertheless the corresponding analogs of (4.40) can be found in this case too, and an analog of the covariant d’Alembertian (4.41) can be uniquely determined from the condition of invariance under these transformations. It is constructed from the covariant derivatives as follows (with taking account of (4.33))

\[
\Box^{(d+1)}_{\text{cov}} = \frac{1}{2} \left( \mathcal{D}^{\mu\nu} + 2m^2 \nabla^\nu_{\mu} \right) \nabla^{\nu}_{\mu} - 2m^2 \nabla^\varphi \nabla^\varphi
\]
and in the basis \(\{x, q, \lambda^\mu\}\) it is independent of \(\lambda^\mu\) when acts on the functions subjected to (4.33)
\[
\Box^{(d+1)}_{\text{cov}} F^{(3)}(y, \varphi, \Omega) = \left( e^{2mq} \Box^{(d)} - 2 \frac{\partial^2}{\partial q^2} + 2md \frac{\partial}{\partial q} \right) f(x, q). \tag{4.43}
\]
It is just the covariant d’Alembertian of a scalar field on AdS\(_{d+1}\) in the considered parametrization. It is straightforward to check its invariance under the transformations (2.15).

5 The d=1 case: (super)conformal mechanics revisited

Conformal mechanics (CM) [25] and its superconformal extensions [26] are the simplest models of (super)conformal field theory. Recently, it was suggested [6] that the so-called ‘relativistic’ generalizations of these \(d = 1\) models are candidates for the conformal field theory dual to AdS\(_2\) (super)gravity in the AdS\(_2/\text{CFT}_1\) framework. The simplest model of this kind is a charged particle evolving on the AdS\(_2 \times S^2\) background (the Bertotti-Robinson metric). It describes a near-horizon geometry of the extreme \(d = 4\) Reissner-Nordström black hole. The action (or Hamiltonian) of the standard CM can be recovered from the worldline action (or Hamiltonian) of the ‘relativistic’ CM model in the ‘weak-field’ (or ‘small velocity’) approximation.

Both the ‘old’ and ‘new’ (super)conformal mechanics models respect the same (super) conformal symmetry, which suggests that these models can in fact be equivalent to each other. The \(d = 1\) version of the equivalence map (3.18) allows one to explicitly prove this conjecture.

The ‘old’ CM can be described in terms of nonlinear realization of the \(d = 1\) conformal group SO\((2,1)\) [27]. The so\((2,1)\) algebra is
\[
[P, D] = -P, \quad [K, D] = K, \quad [P, K] = -2D. \tag{5.44}
\]
One defines a nonlinear realization of SO\((1,2)\) as left shifts of the element
\[
g = e^{tP} e^{u(t)D} e^{\lambda(t)K}. \tag{5.45}
\]
The SO\((2,1)\) left shifts induce for \(t, u(t)\) and \(\lambda(t)\) the following transformations
\[
\delta t = a + bt + ct^2 \equiv a(t), \quad \delta u = \dot{a}(t) = b + 2ct, \quad \delta \lambda = ce^u. \tag{5.46}
\]
The left-invariant Cartan forms are defined by
\[
g^{-1}dg = \omega_P P + \omega_D D + \omega_K K = e^{-u} dt P + (du - 2e^{-u}\lambda dt) D + (d\lambda + e^{-u}\lambda^2 dt - \lambda du) K. \tag{5.47}
\]
The coset field \(\lambda(t)\) can be covariantly eliminated by the constraint
\[
\omega_D = 0 \Rightarrow \lambda = \frac{1}{2} e^u \dot{u}. \tag{5.48}
\]
Then the manifestly invariant worldline action
\[
S = -\frac{1}{2} \int (\mu \omega_k + \gamma \omega_P) = \frac{1}{2} \int dt \left( \frac{1}{4} \mu e^u \dot{u}^2 - \gamma e^{-u} \right), \tag{5.49}
\]
upon the identification \(x(t) = e^{1/2 u(t)}\) is just the ‘old’ conformal mechanics action [25],
\[
S = \frac{1}{2} \int dt \left( \mu \dot{x}^2 - \frac{\gamma}{x^2} \right). \tag{5.50}
\]
Let us now pass to the AdS$_2$ basis in (5.44), redefining $K$ and $D$ as
\[ \hat{K} = mK - \frac{1}{m} P , \quad \hat{D} = mD . \] (5.51)

An element of SO(2, 1) in the AdS basis is defined as
\[ g = e^{yP} e^{\phi(y)D} e^{\Omega(y)K} . \] (5.52)

The parameters $y, \phi$ represent AdS$_2 \sim$ SO(2, 1)/SO(1, 1) in the solvable subgroup parametrization:
\[ \delta y = a(y) + \frac{1}{m^2} c e^{2m\phi} , \quad \delta \phi = \frac{1}{m} \dot{a} = \frac{1}{m} (b + 2c y) , \quad \delta \Omega = \frac{1}{m} c e^{m\phi} . \] (5.53)

The Cartan forms in the AdS parametrization are related to (5.47) as
\[ \omega_K = m\hat{\omega}_K , \quad \omega_P = \hat{\omega}_P - \frac{1}{m} \hat{\omega}_K , \quad \omega_D = m\hat{\omega}_D . \] (5.54)

Like $\lambda(t)$ in eq. (5.48), the field $\Lambda(y) = \tanh \Omega(y)$ can be covariantly eliminated
\[ \hat{\omega}_D = 0 \Rightarrow \Lambda = \hat{\phi} e^{m\phi} \frac{1}{1 + \sqrt{1 - \hat{\phi}^2 e^{2m\phi}}} , \quad \hat{\omega}_P = e^{-m\phi} \sqrt{1 - e^{2m\phi} \hat{\phi}^2} dy , \quad \hat{\omega}_K = -\frac{m}{2} e^{-m\phi} \left( 1 - \sqrt{1 - e^{2m\phi} \hat{\phi}^2} \right) dy + \text{Total derivative} \times dy . \] (5.55)

The invariant action for $\phi(y)$ can now be easily constructed by substituting the expressions (5.56) for $\omega_P, \omega_K$ in (5.49) using the relation (5.54):
\[ S = \int [ (q - \tilde{\mu}) \hat{\omega}_P - (2/m)q \hat{\omega}_K ] = \int dy e^{-m\phi} \left( q - \tilde{\mu} \sqrt{1 - e^{2m\phi} \hat{\phi}^2} \right) , \] (5.57)

where
\[ q = \frac{1}{4}(m^2 \mu - \gamma) , \quad \tilde{\mu} = \frac{1}{4}(m^2 \mu + \gamma) . \] (5.58)

After a field redefinition, (5.57) is recognized as the radial-motion part of the ‘new’ CM action of ref. [6]. The same result could be equivalently obtained by performing in (5.49) the $d = 1$ AdS/CFT transformation obtained by comparison of (5.45) and (5.52).
\[ t = y - \frac{1}{m} e^{m\phi} \Lambda , \quad u = m \phi + \ln(1 - \Lambda^2) , \quad \lambda = m \Lambda . \] (5.59)

Let us briefly discuss (basically following [14]) how this correspondence can be generalized to SCM models. We consider $N = 2$ SCM as the simplest case.

The starting point is the $su(1,1|1)$ superalgebra which includes, apart from the $so(1,2)$ generators (2.1), those of Poincaré $\{Q, \overline{Q}\}$ and conformal $\{S, \overline{S}\}$ supersymmetries and the $U(1)$ generator $U$. In the conformal basis the non-vanishing (anti)commutators read:
\[ \{Q, \overline{Q}\} = 2iP , \quad \{Q, \overline{S}\} = 2iD - 2iU , \quad \{S, \overline{S}\} = 2iK , \quad \{S, \overline{Q}\} = 2iD + 2iU , \]
\[ \left[ P, \left( \begin{array}{c} S \\ \overline{S} \end{array} \right) \right] = - \left( \begin{array}{c} Q \\ \overline{Q} \end{array} \right) , \quad \left[ K, \left( \begin{array}{c} Q \\ \overline{Q} \end{array} \right) \right] = \left( \begin{array}{c} S \\ \overline{S} \end{array} \right) , \]
\[ \left[ D, \left( \begin{array}{c} Q \\ \overline{Q} \end{array} \right) \right] = \frac{1}{2} \left( \begin{array}{c} Q \\ \overline{Q} \end{array} \right) , \quad \left[ D, \left( \begin{array}{c} S \\ \overline{S} \end{array} \right) \right] = -\frac{1}{2} \left( \begin{array}{c} S \\ \overline{S} \end{array} \right) , \]
\[ \left[ U, \left( \begin{array}{c} Q \\ \overline{Q} \end{array} \right) \right] = \frac{1}{2} \left( \begin{array}{c} Q \\ -\overline{Q} \end{array} \right) , \quad \left[ U, \left( \begin{array}{c} S \\ \overline{S} \end{array} \right) \right] = \frac{1}{2} \left( \begin{array}{c} S \\ -\overline{S} \end{array} \right) . \] (5.60)
The standard nonlinear realization of $SU(1, 1|1)$ as the $d = 1, N = 2$ superconformal group is set up as left multiplications of the coset

$$g = e^{tP} \ e^{\theta Q + \bar{\theta} Q} \ e^{\psi S + \bar{\psi} \bar{S}},$$

(5.61)

where $(t, \theta, \bar{\theta}) \equiv z$ are coordinates of $d = 1, N = 2$ superspace and the remaining coset parameters are superfields given on this superspace. The transformation rules of the supercoset parameters and the structure of the related left-covariant Cartan superforms can be found in [14]. We only notice that on the $d = 1, N = 2$ superspace coordinates one recovers the standard $N = 2$ superconformal transformations, while all the superfield coset parameters are expressed through the only essential one $q(z)$ by the appropriate inverse Higgs constraints:

$$\lambda = \frac{1}{2} e^q \dot{q}, \quad \bar{\psi} = -\frac{i}{2} e^{\frac{1}{2}q} Dq, \quad \psi = -\frac{i}{2} e^{\frac{1}{2}q} \overline{D}q,$$

(5.62)

$$D = \frac{\partial}{\partial \theta} + i \bar{\theta} \partial_t, \quad \overline{D} = \frac{\partial}{\partial \bar{\theta}} + i \theta \partial_t, \quad \{ D, \overline{D} \} = 2 i \partial_t.$$

The invariant action of $N = 2$ SCM reads

$$S_{N=2} = \int dt d^2 \theta \left[ \frac{1}{2} D Y \overline{D} Y + \sqrt{\mu} \gamma \ln(Y) \right], \quad Y = e^{\frac{1}{2}q}.$$

(5.63)

Its bosonic core coincides with (5.49) upon identification $q|_{\theta = 0} = u$ and eliminating the auxiliary field $[D, \overline{D}] q|_{\theta = 0}$ by its equation of motion.

Now we shall consider a supersymmetric extension of the AdS basis (2.9). The only new thing we have to do is to make a rescaling of the superconformal generators as $\hat{S} = m S, \hat{\bar{S}} = m \bar{S}$. We define the realization of $SU(1, 1|1)$ in the AdS basis by its left action on the coset $SU(1, 1|1)/U(1)$ in the following parameterization:

$$g = e^{yP} \ e^{\theta Q + \bar{\theta} Q} \ e^{\Phi \hat{D}} \ e^{\Omega \hat{K}} \ e^{\xi \hat{S} + \bar{\xi} \bar{\bar{S}}},$$

(5.64)

Like in the case of standard nonlinear realization, one can directly find the transformation rules of the superspace coordinates and Goldstone superfields. As distinct from the standard case, the transformation laws of coordinates now essentially include Goldstone superfields, i.e. we deal with a field-dependent realization of $N = 2$ superconformal group which is a generalization of the bosonic realization (5.53). The only essential Goldstone superfield is $\Phi$, the remaining ones are expressed through $\Phi$ by the corresponding inverse Higgs constraints:

$$\Lambda = e^{m \phi} \partial_y \Phi \frac{1}{1 + \sqrt{1 - e^{2m \phi} (\partial_y \Phi)^2}}, \quad \xi = -\frac{i}{2} \frac{1 + \Lambda^2}{\sqrt{1 - \Lambda^2}} e^{m \phi} \overline{D}y \Phi.$$

(5.65)

By comparing two different parametrizations of the same coset $SU(1, 1|1)/U(1)$, eqs. (5.61) and (5.64), one can find $N = 2$ extension of the transformation (5.59)

$$t = y - \frac{1}{m} e^{m \phi} \Lambda, \quad q = m \Phi + \ln(1 - \Lambda^2), \quad \lambda = m \Lambda, \quad \psi = m \xi, \quad \bar{\psi} = m \bar{\xi}.$$

(5.66)

Now we can obtain the invariant superfield action which is pertinent to the above AdS realization of $d = 1, N = 2$ superconformal group and so is expected to describe $N = 2$
superextension of the bosonic BR particle action (5.57). One should perform the transformation (5.66) in the ‘old’ $N = 2$ SCM action (5.63). For simplicity, we choose $\gamma = 0$, which amounts to requiring zero vacuum energy. We obtain

$$S = \frac{1}{2} \int dt d^2 \theta \left( -\mu \bar{\psi} \psi \right) = \frac{\mu m^2}{8} \int dy d^2 \theta e^{m \phi} \left( \frac{1 - \Lambda^2}{1 + \Lambda^2} - \frac{1}{m} e^{m \phi} \partial_y \Lambda \right) \frac{(1 + \Lambda^2)^2}{1 - \Lambda^2} D_y \Phi \overline{D}_y \Phi (5.67)$$

where $\Lambda$ is expressed through $\Phi$ according to (5.65).

It is straightforward to pass to the component fields in (5.67) and to show that, when all fermions are discarded,

$$F = 0 \quad (5.68)$$
on shell. After substituting this into the pure bosonic part of the component action, $S_{\text{bos}}$, the latter, modulo a total derivative in the Lagrangian, becomes

$$S_{\text{bos}} = \frac{\mu m^2}{4} \int dy e^{-m \phi} \left( 1 - \sqrt{1 - e^{2m \phi} \left( \partial_y \phi \right)^2} \right), \quad (5.69)$$

which coincides with (5.57) upon the identification (5.58) (for $\gamma = 0$).

Thus (5.67) provides a manifestly $N = 2$ supersymmetric off-shell form of $N = 2$ superconformal extension of the ‘new’ conformal mechanics action (5.57) which describes the radial (AdS$_2$) motion of the charged particle in the BR AdS$_2 \times S^2$ background. By construction, it is related by the equivalence transformation (5.66) to the $\gamma = 0$ case of the ‘old’ $N = 2$ superconformal mechanics action (5.63).

The classical equivalence between the ‘old’ and ‘new’ (S)CM models can hopefully be extended to the quantum case and used to solve the quantum mechanics of the charged AdS$_2$ (super)particles in terms of (super)conformal quantum mechanics. In the classical hamiltonian approach, this equivalence, both for the radial motion and with the angular $S^2$ variables taken into account, was proved in a recent paper [15].

6 Conclusions

In this talk a new kind of the relation between field theories possessing spontaneously broken conformal symmetry in $d$-dimensional Minkowski space and the codimension-$(n + 1)$ branes in AdS$_{d+1} \times S^n$ type backgrounds in the static gauge was presented. This relation takes place already at the classical level and transforms the dilaton Goldstone field associated with the spontaneous breaking of scale invariance into the transverse (or radial) brane coordinate completing the $d$-dimensional brane worldvolume to the full AdS$_{d+1}$ manifold. The conformally invariant minimal actions in Minkowski space including the dilaton are transformed into nonlinear actions given on the AdS brane worldvolume and involving, as their essential part, couplings to the first extrinsic curvature of the brane. Conversely, the standard worldvolume AdS brane effective actions prove to be equivalent to some non-polynomial conformally invariant actions in the Minkowski space. The AdS/CFT map is one-to-one (at least, classically) for the conformal actions containing no dilaton potential and for brane actions with the vanishing vacuum energy. The geometric origin of it can be revealed most clearly within the nonlinear realization description of AdS branes [18] which generalizes the analogous description of branes in the flat backgrounds [28]-[30]. In particular, it turns out that the standard realization of the conformal group in the Minkowski space and its transverse brane coordinate-dependent realization
as the $\text{AdS}_{d+1}$ isometry group in the solvable-subgroup parametrization of $\text{AdS}_{d+1}$ are simply two alternative ways of presenting symmetry of the same system. Most interesting subjects for further study are the generalization of the above relationship to the case of AdS superbranes and, respectively, superconformal symmetries in dimensions $d > 1$, as well as the understanding of how it can be promoted to the quantum case. Possible uses of the transformation (3.18) for the further analysis of relationships between field theories on $\text{AdS}_{d+1}$ and $\mathcal{M}_d$ were already discussed in Sect. 4.

The existence of the coordinate map (3.18) suggests a novel view on the relationship between the conformal field theory actions and the worldvolume actions of AdS superbranes. As we saw, any conformal field theory action in the branch with spontaneously broken conformal symmetry, after singling out the dilaton field, can be rewritten in terms of the AdS brane variables, with the field-modified conformal transformations defining the relevant symmetry. This relationship exists at any finite and non-vanishing AdS radius $R = 1/m$. It is interesting to further explore this surprising ‘brane’ representation of (super)conformal field theories, especially in the quantum domain, and to better understand the role of couplings to extrinsic curvature which are unavoidable in this representation. Let us recall that a string with ‘rigidity’, i.e. with the extrinsic curvature terms added to the action, was considered as a candidate for the QCD string [19].

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