Classical antiparticles, quantum supersymmetry anomaly and constituent models

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Abstract

The two ways of constrained systems quantization are considered from the point of view of their self-consistency at the quantum level. With a transparent example of a particle in the external electromagnetic field we demonstrate that the procedure of gauge fixing turns out rather dangerous and may lead to a quantum anomaly in the operator algebra. We discuss additional classical symmetries as an essential element for tracing out this anomaly. The two cases of a spinning and a spinless particles in the external electromagnetic field are discussed to illustrate the situation.

Various classical and quantum mechanical aspects of Dirac equation continue attracting a lot of attention. One of the main reasons for it is that Dirac equation actually initiated the discussion of "negative energy states" which, in turn, gave rise to the concept of antiparticles. While the crucial role of antiparticles in quantum field theory cannot be overestimated, the relativistic mechanics of antiparticles is not understood well enough (for a brief review of the issue see [1]).

The problem is rooted in the ambiguity which one encounters describing the motion of relativistic particle in a Lorentz-invariant way: the measure of the length along the particle worldline can be defined only up to the sign, so that for the particle at rest one finds

\[ d\tau = \pm dt, \] (1)

where \( d\tau \) is the infinitesimal interval of the proper time, and \( dt \) is that in the given reference frame. These both solutions are equally valid, and it was realised many years ago [2] that

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the upper sign in (1) describes the “particle motion”, whereas the lower one corresponds to the “motion of antiparticle”.

The possibility of classical antiparticle motion is closely connected to the fundamental symmetries of the theory, as it was discussed in [1], and, in a more formal way, in [3]. It was shown in [3] that it was possible to perform the canonical quantization for a free particle in a manner which allows to remove the above-mentioned sign ambiguity and to describe the particle and the antiparticle together within the same theory both at the classical and quantum level. The procedure is easily generalized to the case of spinning Dirac particle, and ends up with the Dirac equation in the Foldy–Wouthuysen representation [4]. The latter representation not only decouples completely the positive and negative energy states, but also provides the Newton–Wigner position and spin operators [5] which, in contrast to the ones of the Dirac–Pauli representation, correspond to their classical counterparts and have clear physical meaning. The natural question, already asked in [1], is if it is worth bothering with the Dirac equation in any other representation? The answer is quite obvious: what one actually needs is a theory of interacting particles. Indeed, while there is no classical force which causes the particle–antiparticle mixing, it easily occurs at the quantum level. The aim of the present paper is to put this statement onto formal grounds. Namely, we demonstrate with a simple example that not all symmetries enjoyed by the classical spinning particle survive at the quantum level, if the quantization is performed in a manner which distinguishes between two signs in equation (1).

We start with the action describing the motion of a spinning particle in the external electromagnetic field $A_{\mu}$ (see e.g. [3]):

$$S = \int_{\tau_i}^{\tau_f} L d\tau,$$

(2)

$$L = -\frac{\mu}{2} \dot{x}_\nu (\dot{x}_\nu - i \chi \psi_\nu) - i \frac{\psi_\nu \dot{\psi}_\nu}{2} - \frac{m^2}{2\mu} + \frac{i}{2} (\psi_5 \dot{\psi}_5 + m \chi \psi_5) - g A_{\mu} \dot{x}_\mu + \frac{ig}{2\mu} \psi_\mu \psi_\nu F_{\mu\nu}. $$

(3)

Here $\tau$ is the proper time, $x_\mu$ and $\psi_\mu$ ($\mu = 0, 1, 2, 3$) are the position and Grassmannian spin variables, the fifth Grassmannian variable $\psi_5$ is introduced to consider massive particle, the dot means the derivative with respect to the proper time, and $\mu$ and $\chi$ are the einbein fields, $\mu$ being a commuting and $\chi$ an anticommuting variable.

Action (2,3) is invariant under reparametrization group transformations

$$\tau \rightarrow f(\tau), \quad \mu \rightarrow \frac{\mu}{f(\tau)}, \quad \chi \rightarrow \frac{\chi}{f(\tau)},$$

(4)

as well as under supergauge transformations generated (in the infinitesimal form) by the anticommuting quantity $\alpha(\tau)$:

$$\delta x_\nu = i \alpha \psi_\nu, \quad \delta \psi_\nu = -\alpha \mu (\dot{x}_\nu - \frac{i}{2} \chi \psi_\nu)$$

$$\delta \mu = i \alpha \mu^2 \chi, \quad \delta \chi = 2\dot{\alpha}, \quad \delta \psi_5 = m \alpha.$$ 

(5)
The presence of these invariances indicates that, in accordance with Dirac [7], there should be two primary first class constraints among the whole set of constraints for theory (2,3). Let us first briefly outline the standard way of dealing with such a situation.

The conjugated momenta are defined as

\[ p_\nu = \frac{\partial L}{\partial \dot{x}_\nu} = -\mu (\dot{x}_\nu - i\frac{1}{2} \chi \psi_\nu) - gA_\nu, \]

\[ p_\psi_\nu = \frac{\partial L}{\partial \dot{\psi}_\nu} = \frac{i}{2} \psi_\nu, \quad p_\psi_5 = \frac{\partial L}{\partial \dot{\psi}_5} = -\frac{i}{2} \psi_5, \]  

(6)

\[ \pi = \frac{\partial L}{\partial \dot{\mu}} = 0, \quad \pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0. \]

The constraints invoking momenta \( p_\psi_\mu \) and \( p_\psi_5 \) are of the second class, and these variables are eliminated from the theory with Dirac bracket

\[ \{ AB \}' = \frac{\partial A}{\partial x_\nu} \frac{\partial B}{\partial p_\nu} - \frac{\partial A}{\partial p_\nu} \frac{\partial B}{\partial x_\nu} + \frac{\partial A}{\partial \mu} \frac{\partial B}{\partial \pi} - \frac{\partial A}{\partial \pi} \frac{\partial B}{\partial \mu} - A \frac{\partial}{\partial x_\nu} \frac{\partial}{\partial \pi_\chi} B - A \frac{\partial}{\partial \pi_\chi} \frac{\partial}{\partial x_\nu} B + iA \frac{\partial}{\partial \psi_\mu} \frac{\partial}{\partial \psi_\nu} B - iA \frac{\partial}{\partial \psi_5} \frac{\partial}{\partial \psi_5} B \]  

(7)

Primary constraints \( \Phi_1 = \pi \approx 0 \) and \( \Phi_2 = \pi_\chi \approx 0 \) give rise to the secondary ones

\[ \Phi_3 = \{ \Phi_1 H \}' = \frac{1}{2\mu^2} \left( (p + gA)^2 - m^2 + ig\psi_\mu \psi_\nu F_{\mu\nu} \right), \]  

(8)

\[ \Phi_4 = \{ \Phi_2 H \}' = -\frac{i}{2} \left( (p + gA)\psi - m\psi_5 \right), \]  

(9)

where Hamiltonian \( H \) is given by the expression

\[ H = -\frac{1}{2\mu^2}[(p + gA)^2 - m^2 + igF_{\mu\nu} \psi_\mu \psi_\nu] + \frac{i}{2} \chi [(p + gA)\psi - m\psi_5] + \lambda_\chi \pi_\chi + \lambda \pi, \]  

(10)

with primary constraints \( \Phi_1 = \pi \) and \( \Phi_2 = \pi_\chi \) added with Lagrange multipliers \( \lambda \) and \( \lambda_\chi \).

The only nonzero brackets of the constraints

\[ \{ \Phi_1 \Phi_3 \}' = -\frac{1}{2\mu^2} \Phi_3, \]  

(11)

\[ \{ \Phi_4 \Phi_4 \}' = i\Phi_3, \]

vanish at the constraint surface together with the Hamiltonian, so that the set of constraints \( \{ \Phi_i \}, i = 1, 2, 3, 4 \), is the first class one.
The way to quantize a theory in the presence of the first class constraints was suggested by Dirac. In the given case the quantization \( a la \) Dirac is performed setting
\[
\hat{p}_\mu = i \frac{\partial}{\partial x_\mu}, \quad \psi_\mu = \frac{1}{\sqrt{2}} \gamma_5 \gamma_\mu, \quad \psi_5 = \frac{1}{\sqrt{2}} \gamma_5.
\] (12)

The physically relevant constraints \( \Phi_3 \) and \( \Phi_4 \) provide the following equations for the wave function:
\[
\hat{\Phi}_{KG} \Psi = \left( (p + gA)^2 - m^2 - \frac{i}{2} g \sigma_{\mu\nu} F_{\mu\nu} \right) \Psi = 0, \quad (13)
\]
\[
\hat{\Phi}_D \Psi = \gamma_5 (\gamma (p + gA) - m) \Psi = 0, \quad (14)
\]
yielding Hamiltonian in the Dirac–Pauli representation
\[
\hat{H} = \hat{a}(\vec{p} - gA) + \gamma_0 m + gA_0. \quad (15)
\]

The most important point here is that algebra of constraints (11) remains closed at the quantum level with operator realisation (12):
\[
[\hat{\Phi}_{KG}, \hat{\Phi}_{KG}] = 0, \quad [\hat{\Phi}_{KG}, \hat{\Phi}_D] = 0, \quad [\hat{\Phi}_D, \hat{\Phi}_D] = -\hat{\Phi}_{KG}, \quad (16)
\]
that makes equations (13) and (14) for the wave function compatible with one another.

Now we consider an alternative way of dealing with the constrained theory. One can impose additional constraints which fix the gauges; with these extra constraints the degeneracy of the theory is removed, all the constraints become the second class ones, and the resulting theory yields a Hamiltonian which is nonzero at the constraint surface. The consistent procedure for theory (2,3) is described in [8]; for our purposes it is enough to present a simplified version. To this end we fix only one gauge in reparametrization group (4) setting
\[
x_0 = \tau \quad (17)
\]
in Lagrangian (3) and anticipating the quantization at the time-like hyper-surface. In such a way the upper sign in equation (1) is chosen from the very beginning.

In what follows we shall consider the stationary problem with the four–potential \( A_\mu(x_0, \vec{r}) \) depending only on \( \vec{r} \).

With gauge fixing condition (17) the Lagrangian takes the form
\[
L = \frac{-\mu}{2} + \frac{\vec{r}^2}{2} + \frac{i}{2} \mu \chi \psi_0 - \frac{i}{2} \mu \chi (\vec{r} \vec{\psi}) - \frac{i}{2} \psi_\nu \dot{\psi}_\nu - \frac{m^2}{2\mu},
\]
\[
+ \frac{i}{2} (\psi_5 \dot{\psi}_5 + m \chi \psi_5) - gA_0 + g\vec{A} \vec{r},
\]
and the conjugated momenta are
\[
\vec{p} = \frac{\partial L}{\partial \vec{r}} = \mu (\vec{r} - \frac{i}{2} \chi \vec{\psi}) + g\vec{A},
\]
\[
\psi_5 = \frac{1}{\sqrt{2}} \gamma_5.
\]
\[ p_{\psi_\nu} = \frac{\partial L}{\partial \dot{\psi}_\nu} = \frac{i}{2} \psi_\nu, \quad p_{\psi_5} = \frac{\partial L}{\partial \dot{\psi}_5} = -\frac{i}{2} \psi_5, \]  
\[ \pi = \frac{\partial L}{\partial \dot{\mu}} = 0, \quad \pi_\chi = \frac{\partial L}{\partial \dot{\chi}} = 0. \]

Again we eliminate redundant variables \( p_{\psi_\mu}, p_{\psi_5} \) defining the Dirac brackets as

\[
\{AB\}' = \frac{\partial A}{\partial \bar{p}} \frac{\partial B}{\partial \bar{x}} - \frac{\partial A}{\partial \bar{x}} \frac{\partial B}{\partial \bar{p}} - \frac{\partial A}{\partial \bar{\mu}} \frac{\partial B}{\partial \bar{\mu}} + \frac{\partial A}{\partial \bar{\pi}} \frac{\partial B}{\partial \bar{\pi}} - \frac{1}{2} A \frac{\partial \bar{A}}{\partial \pi_\chi} B - A \frac{\partial \bar{A}}{\partial \pi_\chi} \frac{\partial B}{\partial \pi_\chi} + i A \frac{\partial \bar{A}}{\partial \psi_0} \frac{\partial B}{\partial \psi_0} - \frac{1}{2} i \chi \frac{\partial \bar{\psi}}{\partial \psi_0} + m A_0.
\]  

The Hamiltonian takes the form

\[
H = \frac{\mu}{2} + \left( \bar{p} - g \bar{A} \right)^2 + \frac{m^2}{2} \mu + \frac{2}{\mu} F_{0i} \psi_0 \psi_i - \frac{2}{\mu} F_{ik} \psi_i \psi_k + g A_0,
\]

and the remaining primary constraints

\[
\varphi_1 = \pi \\
\varphi_2 = \pi_\chi
\]

give rise to the secondary constraints

\[
\varphi_3 = \{ \varphi_1 H \}' = -\frac{1}{2} + \left( \frac{\bar{p} - g \bar{A}}{2} \right)^2 + \frac{m^2}{2} \chi \psi_0 + \frac{i}{\mu^2} F_{0i} \psi_0 \psi_i - \frac{i}{\mu^2} F_{ik} \psi_i \psi_k,
\]
\[
\varphi_4 = \{ \varphi_2 H \}' = \frac{i}{2} (\mu \psi_0 - (\bar{p} - g \bar{A}) \bar{\psi} + m \psi_5).
\]

Condition (17) fixes only the gauge in the reparametrization group, and Lagrangian (19) is still invariant under supergauge transformations. This means that constraint matrix \( C_{ij} = \{ \varphi_i \varphi_j \}' \) is still degenerate. We demonstrate it explicitly introducing the modified brackets

\[
\{ AB \}^* = \{ AB \}' - \{ A \varphi_1 \}' C_{13}^{-1} \{ \varphi_3 B \}' - \{ A \varphi_3 \}' C_{31}^{-1} \{ \varphi_1 B \}',
\]

where

\[
C_{13} = \{ \varphi_1 \varphi_3 \}' = \left( \frac{\bar{p} - g \bar{A}}{2} \right)^2 + \frac{m^2}{2} \chi \psi_0 + \frac{2 i}{\mu^2} F_{0i} \psi_0 \psi_i - \frac{i}{\mu^2} F_{ik} \psi_i \psi_k.
\]

Then the explicit calculation shows that the odd pair of constraints is of the first class,

\[
\{ \varphi_2 \varphi_2 \}^* = \{ \varphi_2 \varphi_4 \}^* = 0,
\]
\{\varphi_4 \varphi_4\}^* = 0, \quad (27)

and the physically relevant constraint \(\varphi_4\) commutes with the Hamiltonian:

\{\varphi_4 H\}^* = 0. \quad (28)

The physical Hamiltonian

\[ H_{ph} = \mu_0 + gA_0 + \frac{ig}{\mu_0} F_0 \psi_0 \psi_i \] \quad (29)

and the physical Dirac constraint

\[ \varphi_D = \mu_0 \psi_0 - (\vec{p} - g\vec{A}) \vec{\gamma} + m\psi_5 \] \quad (30)

are obtained on substituting the solution

\[ \mu_0 = \sqrt{(\vec{p} - g\vec{A})^2 + m^2 - igF_{ik}\psi_i \psi_k} \] \quad (31)

of the constraint equation \(\varphi_3 = 0\). Note that to arrive at algebra (27), (28) as well as at forms (29), (30) it is necessary to take into account the relations

\[ \psi_\mu \psi_\nu = -\psi_\nu \psi_\mu, \quad \psi_5 \psi_\mu = -\psi_\mu \psi_5, \quad \psi_5 \psi_5 = 0 \] \quad (32)

for the elements of the Grassmannian algebra.

The theory should be quantized with bracket (24) in the usual way, setting \(\hat{\vec{p}} = -i\frac{\partial}{\partial \vec{r}}\) and \(\psi_\mu = \frac{1}{\sqrt{2}} \gamma_5 \gamma_\mu, \quad \psi_5 = \frac{1}{\sqrt{2}} \gamma_5\). The wave function should not only satisfy the Schrödinger equation

\[ \hat{H}_{ph} \Psi = \left( \hat{\mu}_0 + gA_0 + \frac{ig}{2\mu_0} F_0 \sigma_0 i \right) \Psi = E \Psi, \] \quad (33)

but also the constraint equation

\[ \hat{\varphi}_D \Psi = \left( \hat{\mu}_0 \gamma_0 - (\hat{\vec{p}} - g\hat{\vec{A}}) \vec{\gamma} - m \right) \Psi = 0. \] \quad (34)

It is easy to see, however, that the quantum algebra of the Hamiltonian and the Dirac constraint is not closed,

\[ [\hat{H}_{ph},\hat{\varphi}_D]_\pm \neq 0, \] \quad (35)

so equations (33) and (34) are not compatible. We stress that it is not a problem of the operator ordering but it takes place because there is no relation for \(\gamma\)-matrices similar to (32) for Grassmannian variables.

\[ \text{1The solution for } \mu_0 \text{ does not contain part proportional to } \psi_0 \text{ as it is left explicitly in Hamiltonian (29), whereas in Dirac constraint (30) it would vanish due to the Grassmann nature of } \psi_0. \]
There are, of course, special types of the external field configurations for which the Hamiltonian commutes with the Dirac constraint at the quantum level (see e.g. [4]) but for the general case one has encountered a quantum supersymmetric anomaly which affects the physical results. In particular, the well–known Darwin term in the Hamiltonian is completely lost with such a kind of gauge fixing.

One can go further, and fix the gauge in the supergauge group (5) too, as it was done in [8]. Nevertheless, the resulting quantum theory [9] has not got the Darwin term restored. Moreover, as with the complete gauge fixing all the constraints are already of the second class, there is no additional equation for the wave function like (34), and one should not impose extra compatibility requirements.

We can see now that one is very lucky to be able to pin-point the source of troubles with spinning particle. Indeed, let us consider the case of scalar particle, where there are no spin variables and the only symmetry is the reparametrization one. Skipping the details we write out the ultimate Klein–Gordon equation for the wave function

\[(\hat{p} + gA)^2 - m^2)\Psi = 0 \quad (36)\]

for the case of no gauge fixing procedure a la Dirac, and the Schrödinger equation

\[\hat{H}_{ph}\Psi = \left(\sqrt{m^2 + (\hat{p} - e\vec{A})^2 + gA_0}\right)\Psi = E\Psi \quad (37)\]

for the case of gauge fixed by condition (17). Equations (36) and (37) yield different spectra, and the Darwin term, which also exists for the scalar particle (see e.g. [10]) is lost again in (37). In contrast to the Dirac particle case, there is no extra symmetry and no way to find out how it could happen.

The anomaly discussed is not an artifact of the time-like gauge fixing (17); the classical antiparticles do exist under any assumption on the evolution parameter \(\tau\). The gauge conditions which forbid the particle–antiparticle mixing at the quantum level exclude some physical phase space trajectories and are not admissible. For the time-like gauge fixing one truncates the phase space excluding the negative energy states, but, for example, for another popular light-cone gauge fixing the point \(p^+ = 0\) is excluded from the phase space.

While the situation is rather trivial in the transparent case of external field discussed above, in a more complicated cases of interacting particles one meets even more confusions. For example, the exactly solvable problem of a quark–antiquark pair in 1+1 space-time interacting via string was considered [11] with two different versions of the reparametrization group gauge fixing. The quantization in the proper-time and the light-cone gauges was performed yielding different quantum spectra. On the other hand, the only symmetry group for this theory at our disposal is the Poincaré one, and in both gauges the quantum Poincaré algebra appears to be closed.

Do our findings mean that as far as the uncontrolled deficiencies take place, one has to completely abandon the first quantization procedure? The answer is, of course, “no”.

\(^2\)Note that it is the electric field to be responsible for self-inconsistency (35).
The quantization a la Dirac, when the first class constraints are left in peace, is safe. One may develop the first-quantized field theory from the Feynman–Schwinger representation approach nicely reproducing the Feynman rules [12]. The technical simplicity and physical transparency of such a path integral formulation is obvious, as well as the advantage of being back to basic quantum mechanics.

We conclude with some phenomenological implications. The real particle–antiparticle problem difficulties start with the important case of QCD, where in the absence of exact solutions one relies upon models, all of which involving linearly rising force potentially dangerous from the point of view of the Klein paradox. The latter observation leads to the belief [13] that the proper quantum mechanical reduction of the underlying field theory should include the “no-pair” assumption. We do not share this belief as there is no way within the field theory to imply such an assumption in a self-consistent way. The phenomenological successes of constituent quark models tell us that the quark backward motion is suppressed, but this suppression should be dynamical one rather than imposed by hand-waving arguments. We are not able to prove this statement and refer to the example of 1+1 ’t Hooft model [14], where confinement does occur whereas the spectrum is conveniently bound from below without ad hoc “no-pair” assumption. Besides, numerical solutions for the ’t Hooft model in the mesonic rest frame exist [15], which explicitly exhibit the backward motion suppression. Moreover, the motion of a quark in the field of a static antiquark source was considered in this model [16], and the quark Hamiltonian was obtained both in Dirac–Pauli and Foldy–Wouthuysen representations demonstrating explicitly that the theory prevents itself from the Klein paradox.

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