Stability of inflating branes in a texture

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We investigate the stability of inflating branes embedded in an O(2) texture formed in one extra dimension. The model contains two 3-branes of nonzero tension, and the extra dimension is compact. When the gravitational perturbation is applied, the vacuum energy which is responsible for inflation on the branes stabilizes the branes if the symmetry-breaking scale of the texture is smaller than some critical value. This critical value is determined by the particle-hierarchy scale between the two branes, and is smaller than the 5D Planck-mass scale. The scale of the vacuum energy can be considerably low in providing the stability. This stability story is very different from the flat-brane case which always suffers from the instability due to the gravitational perturbation.

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I. INTRODUCTION

In “braneworld” models, our universe is represented by a (3 + 1)-dimensional brane floating in a higher-dimensional bulk spacetime [1–3]. The brane can be thought of as fundamental D-brane, or it can arise as a topological defect in a higher-dimensional field theory. In the simplest, codimension-one models, the brane can be pictured as a domain wall propagating in a 5D spacetime [1,4]. Higher codimensions with both gauge and global defects have also been considered. In particular, models have been discussed where the field configuration in the directions orthogonal to the brane is that of a cosmic string [5] and a monopole [6]. In these models, the bulk curvature produced by the defects plays a major role in localizing gravity on the brane and in solving the mass-hierarchy problem.

Recently, the author investigated localization of gravity in a different type of defect, the “texture” [7]. Global textures are produced when a continuous global symmetry $G$ is broken to a group $H$. The resulting homotopy group $\pi_D(G/H)$ is nontrivial, and the vacuum manifold is $S^D$; for $G = O(N)$ models, $H = O(N - 1)$ and $D = N - 1$. The mapping from the physical space to the vacuum manifold is $R^D \rightarrow S^D$ [8].

For textures, the scalar field takes only the vacuum-expectation value after the symmetry breaking, and thus nowhere in the physical space remains the scalar field in the unbroken-symmetry state. This is the key difference between textures and core defects.

Let us consider a global O(2) texture formed in five dimensions. In the mapping $R^1 \rightarrow S^1$, we identify the physical space $R^1$ with the extra dimensional space. After the scalar field completes its winding in the vacuum manifold, two spatial points on the boundaries of the physical space are mapped into the same point in the vacuum manifold. These two points are identified, and the extra dimension becomes compact.

The extra dimension is curved by the texture. The strength of the curvature depends on the symmetry-breaking scale of the texture. This curved geometry of the extra dimension sets the particle hierarchy as well as the Planck-mass hierarchy in the extra dimension. Thus, the role of the cosmological constant in the Randall-Sundrum scenario [3] is replaced by the texture.

In Ref. [7], we investigated “flat 3-branes” embedded in a texture where the 4D worldsheet is represented by a flat 4D metric. The model has two 3-branes of nonzero tension. The one has a positive, and the other has a negative tension. The particle-hierarchy scale between the two branes is determined solely by the symmetry-breaking scale. Gravity on the brane where TeV-scale particles are confined is essentially four dimensional. The higher dimensional effect of massive Kaluza-Klein gravitons is strongly suppressed on this brane. This result is similar to that of Ref. [9].

One crucial problem of this model was that there exists a tachyonic 4D graviton mode. Due to this mode, the direction tangent to the brane is unstable although in the end it is expected to relax to a final stable configuration after some dynamical evolution combined with the texture field along the orthogonal direction.

In this work, we are particularly interested in the mechanism to stabilize the branes in gravitational perturbations. As a recipe, we introduce “inflation” on 4D worldsheets.1 In doing so, we effectively add vacuum energy induced by

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1The inflating brane has been studied in different settings and contexts, for example, in Refs. [10,11].
the bulk curvature. When this energy compensates the energy deficit due to the tachyon, we expect that the branes stabilize.

The scale of required vacuum energy in stabilizing the branes is also of much concern. The expansion due to this energy should not be very large to be consistent with the acceleration of the late universe. We shall discuss this point also in this work.

The structure of this paper is as following. In Sec. II, we present the model, derive field equations, and obtain the solutions. In Sec. III, we introduce gravitational perturbations, and get the zero-mode graviton. In Sec. IV, we discuss the stability conditions in de Sitter space, and check the stability of the branes. In Sec. V, we evaluate the expansion parameter for stable branes, and we conclude in Sec. VI.

II. FIELD EQUATIONS AND SOLUTIONS

Let us consider an O(2) texture formed in one extra dimension. The vacuum manifold is $S^1$, and it is mapped to $R^1$ which we set to be the extra dimensional space. We shall consider a texture of one winding number. We assume also in this work.

The action describing the model of an O(2) texture and 3-branes in five dimensions is

$$S = \int d^5x \sqrt{-g} \left[ \frac{R}{16\pi G_N} - \frac{1}{2} \partial_\alpha \Phi^a \partial^\alpha \Phi^a - \frac{\lambda}{4} (\Phi^\alpha \Phi^a - \eta^2)^2 \right] - \int d^4x \sqrt{-h} \sigma_i(\Phi^a),$$

where $\Phi^a$ is the scalar field with $a = 1, 2$, $\eta$ is the symmetry-breaking scale, $\sigma_i$ is the tension of the $i$-th brane, and $g$ ($h$) is the 5D (4D) metric density. For later convenience, we define $\kappa^2 = 8\pi G_N = 1/M_*^2$, where $M_*$ is the fundamental 5D Planck mass.

We adopt a metric ansatz in a conformal form,

$$ds^2 = g_{MN}dx^M dx^N = B(y)(ds_4^2 + dy^2),$$

where $ds_4^2$ is the 4D world-volume metric.

The scalar field resides in the vacuum manifold, $\Phi^a \Phi^a = \eta^2$, and takes the form,

$$\Phi^a = \eta [\cos \chi(y), \sin \chi(y)],$$

where $\chi(y)$ is the phase factor in the field space. In this case, the nonlinear $\sigma$-model approximation is applied, and the coupling $\lambda$ appearing in the action is treated as a Lagrange multiplier. The scalar-field equation is then given by

$$\nabla^A \partial_A \Phi^a = - \frac{1}{\eta^2} \frac{\partial (\Phi^\phi)(\partial_A \Phi^\phi)}{\partial a} \Phi^a + \frac{\sqrt{-h}}{\sqrt{-g}} \frac{\partial \sigma_i(\Phi^a)}{\partial \Phi^a} \delta(y_i),$$

which reduces to

$$\chi'' + \frac{3B'}{2B} \chi' = \frac{2\sqrt{B}}{\eta^2} \left[ \frac{\partial \sigma_I}{\partial \chi} \delta(y) + \frac{\partial \sigma_{II}}{\partial \chi} \delta(y-y_*) \right].$$

Here, we assumed that there are two nonzero-tension branes located at $y = 0$ and $y = y_*$. The energy-momentum tensor of the texture and the branes is given by

$$T^M_N = \partial^M \Phi^a \partial_N \Phi^a - \frac{1}{2} \delta^M_N \partial_A \Phi^a \partial^A \Phi^a - \frac{\sqrt{-h}}{\sqrt{-g}} \sigma_i(\Phi^a) \delta(y_i) \text{diag}(1, 1, 1, 1, 0),$$

then the Einstein’s equation for the given metric and energy-momentum tensor leads to

$$-G^\mu_\mu = \frac{1}{B} \left[ \frac{3B''}{2B} + \frac{3}{4} \left( \frac{B'}{B} \right)^2 + \frac{\bar{R}(4)}{4} \right] = \kappa^2 \frac{\eta^2 \chi'^2}{2B} + \frac{\kappa^2}{\sqrt{B}} \left[ \sigma_I(\Phi^a) \delta(y) + \sigma_{II}(\Phi^a) \delta(y-y_*) \right],$$

$$-G^y_y = \frac{1}{B} \left[ -\frac{3}{2} \left( \frac{B'}{B} \right)^2 + \frac{\bar{R}(4)}{2} \right] = -\kappa^2 \frac{\eta^2 \chi'^2}{2B}. $$
Now, we consider inflation on 4D worldsheets, then the 4D world-volume metric can be
\[ ds_4^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + e^{2H(t)} dx^2 , \tag{9} \]
and the 4D curvature scalar is \( \bar{R}^{(4)} = 12H^2 \). (Note what is responsible for the expansion on the 4D worldsheet is not \( H \), but the “effective”-expansion parameter \( H_{eff} \) which will be discussed in Sec. V.)

The solutions to the Einstein and scalar-field equations are
\[ \begin{align*}
B(y) &= \left\{ \frac{B_0}{H} \sinh[3H(|y| + y_0)] \right\}^{2/3} , \tag{10} \\
\chi(y) &= \frac{\text{sign}(y)}{3b_0} \ln \left[ \frac{\tanh \frac{3H}{2} (|y| + y_0)}{\tanh \frac{3H y_0}{2}} \right] , \tag{11}
\end{align*} \]
where \( B_0 = b_0 \chi_0, b_0 = \kappa \eta / 2\sqrt{3}, \) and \( \chi_0 \) and \( y_0 \) are integration constants. We take \( B_0 > 0 \) and \( y_0 > 0 \) without changing physics.

The solutions are illustrated in Fig. 1. The phase factor \( \chi \) varies from \(-\pi \) to \( \pi \) for the scalar field to complete one winding in the vacuum manifold. We set \( \chi(y = 0) = 0 \) and \( \chi(y = \pm y_*) = \pm \pi \). The boundaries at \( y = \pm y_* \) are identified since their phase factors correspond to the same point in the vacuum manifold. Then this scalar-field configuration preserves \( Z_2 \)-symmetry about the branes.

From Eq. (11), setting boundary value \( \chi(\pm y_*) = \pm \pi \) gives a condition,
\[ \frac{\tanh \frac{3H y_*}{2} (y_* + y_0)}{\tanh \frac{3H y_0}{2}} = e^{3n} , \tag{12} \]
where we defined \( n = \pi b_0 = \pi \kappa \eta / 2\sqrt{3} \).

If we plug the solutions (10) and (11) into the Einstein equations, we can evaluate the tension of the brane,
\[ \sigma_1 = -\frac{6H}{\kappa^2} \frac{1}{\tanh[3H(y_0)]} \left[ \frac{b_0}{H} \sinh[3H y_0] \right]^{1/3} < 0 , \tag{13} \]
\[ \sigma_{11} = \frac{6H}{\kappa^2} \frac{1}{\tanh[3H(y_* + y_0)]} \left[ \frac{b_0}{H} \sinh[3H(y_* + y_0)] \right]^{1/3} > 0 . \tag{14} \]

The first brane at \( y = 0 \) has a negative tension, and the other has a positive tension. This result is very similar to the flat-brane case in Ref. [7] as well as to the Randall-Sundrum case [3]. In addition, from the scalar-field equation (5), we get
\[ \frac{d\sigma_1}{d\chi} = \frac{d\sigma_{11}}{d\chi} = 0 \tag{15} \]
at the location of the branes.

According to the warp factor \( B(y) \), the particle-mass scale flows along \( y \). The mass scale is largest where the warp factor acquires the largest value, i.e. on the second brane. We assume that the Planck-scale particles are confined on the second brane where the particle’s effective mass is the same with the bare mass.\(^{2}\) Then, on the brane located at \( y = y_i \), the effective-particle mass measured relatively to the second-brane particles is given by\(^{3}\)
\[ v_{eff} = \left\{ \frac{\sinh[3H(y_i + y_0)]}{\sinh[3H(y_* + y_0)]} \right\}^{1/3} v_i , \tag{16} \]
where \( v_i \) is the bare mass which we assumed Planck scale.\(^{4}\)

\(^{2}\)Throughout this paper we assume that the particles on the second brane are always of Planck scale while the particle scale on the first brane varies depending on the situation we consider.

\(^{3}\)For this derivation, please see Refs. [3,7].

\(^{4}\)Note that all the solutions and physical properties obtained in this section and in the following sections reduce to those of the flat-brane case in Ref. [7] in the limit of \( H \to 0 \).
III. ZERO-MODE GRAVITON

In this section, we investigate gravitational perturbations and the zero-mode graviton. Let us consider a small perturbation \( h_{\mu\nu} \) to the background metric,

\[
\text{(17)} \quad ds^2 = B(y) \left\{ \bar{g}_{\mu\nu} + \frac{h_{\mu\nu}}{B(y)} \right\} dx^\mu dx^\nu + dy^2.
\]

We assume that the only nonzero components are \( h_{\mu\nu} \) (\( h_{\mu y} = h_{y\mu} = 0 \); the massive graviphoton \( h_{\mu y} \) decouples, and we assume no radion mode \( h_{yy} \)), and apply a transverse-traceless gauge on \( h_{MN} \), \( h_{MN}[y] = 0 \) and \( h_{M}^N = 0 \), where the vertical bar denotes the covariant derivative with respect to the background metric. The Einstein’s equation for \( h_{MN} \) reads

\[
\text{(18)} \quad h_{MN}[y] A + 2R_{MANB}^{(B)} h^{AB} - 2R_{A(M}^{(B)} h_{N)}^{A} + \frac{2(6 - N)}{N - 2} \kappa^2 \sigma_I \delta(y_I) h_{MN} = 0.
\]

Here, the superscript \( (B) \) represents that the quantity is evaluated in the unperturbed background metric, and \( N = 5 \) is the number of the total dimensions (as distinguished from the index in \( h_{MN} \)). We search the solution of a form, \( h_{\mu\nu} = h(y) \epsilon_{\mu\nu}(x) \). In order to cast Eq. (18) into a nonrelativistic Schrödinger-type equation, we define a new function by \( \hat{h}(y) = B^{-1/4} h(y) \), then the equation leads to

\[
\text{(19)} \quad \left[ -\frac{1}{2} \frac{d^2}{dy^2} + V(y) \right] \hat{h}(y) = \frac{m_g^2}{2} \hat{h}(y),
\]

and the 4D part of the graviton satisfies the equation [11]

\[
\Box^{(4)} \hat{\epsilon}_{\mu\nu} = (m_g^2 + 2H^2) \hat{\epsilon}_{\mu\nu}, \tag{20}
\]

where \( m_g \) is the 4D graviton mass, and \( \Box^{(4)} \) is the 4D d’Alembertian evaluated in \( \bar{g}_{\mu\nu} \). The potential is given by

\[
V(y) = \frac{9H^2}{4} \left\{ 1 - \frac{1}{2} \coth^2[3H(|y| + y_0)] \right\} - \Sigma_I \delta(y), \tag{21}
\]

\[
= \frac{9H^2}{4} \left\{ 1 - \frac{1}{2} \coth^2[3H(-|y - y_*| + y_* + y_0)] \right\} + \Sigma_{II} \delta(y - y_*), \tag{22}
\]

viewed from the brane I and II respectively, and

\[
\Sigma_I = \frac{H}{2} \coth(3H y_0) \left\{ 1 - 4 \left[ \frac{B_0}{H} \sinh(3H y_0) \right]^{1/3} \right\}, \tag{23}
\]

\[
\Sigma_{II} = \frac{H}{2} \coth(3H(y_* + y_0)) \left\{ 1 - 4 \left[ \frac{B_0}{H} \sinh(3H(y_* + y_0)) \right]^{1/3} \right\}. \tag{24}
\]

For the zero mode, \( m_g = 0 \), the solution to Eq. (19) is

\[
\hat{h}_0^I(y) = \sqrt{\sinh[3H(|y| + y_0)]} \left\{ a_1 + a_2 \ln \left[ \tanh \frac{3H}{2} (|y| + y_0) \right] \right\}. \tag{26}
\]

Here, \( a_1 \) and \( a_2 \) are integration constants, and the superscript \( I \) means that the expression \( |y| \) is valid on the first brane. This zero mode represents a constant contribution plus the contribution of the scalar field to the gravitational perturbation in Eq. (17). On the second brane, the zero-mode wave function can be written as

\[
\hat{h}_0^{II}(y) = \sqrt{\sinh[3H(-|y - y_*| + y_* + y_0)]} \left\{ a_1 + a_2 \ln \left[ \tanh \frac{3H}{2} (-|y - y_*| + y_* + y_0) \right] \right\}. \tag{27}
\]

If we plug these solutions (26) and (27) in Eq. (19), we obtain two conditions at the boundaries, \( y = 0 \) and \( y = y_* \),
\[-\frac{a_1}{a_2} = -\frac{a_{12}}{2} = \frac{\frac{3}{2 \cosh(3Hy_0)} \left\{ 1 - \left[ \frac{B_0}{H} \sinh(3Hy_0) \right]^{1/3} \right\} + \ln \left[ \tanh \left( \frac{3Hy_0}{2} \right) \right]}{2 \cosh(3Hy_0) \left\{ 1 - \left[ \frac{B_0}{H} \sinh(3Hy_0) \right]^{1/3} \right\} + \ln \left[ \tanh \frac{3H}{2} (y_* + y_0) \right]} \quad (28)\]

These conditions are equivalent to imposing boundary conditions on the branes, which are Sturm-Liouville type,

\[\hat{h}_0(0) + \gamma_1 \hat{h}'_0(0) = 0,\]

\[\hat{h}_0(y_*) + \gamma_1 \hat{h}'_0(y_*) = 0,\]

where

\[\frac{1}{\gamma_i} = \frac{H}{2} \tanh[3Hy_i + y_0)] \left\{ 1 - 4 \left[ \frac{B_0}{H} \sinh(3Hy_i + y_0) \right]^{1/3} \right\}, \quad (32)\]

and \(y_i\) is the location of the brane, \(y_i = 0\) and \(y_{II} = y_*\).

We can further determine \(a_2\), from the normalization condition of the zero-mode wave function, \(\int_0^{y_*} |\hat{h}_0(y)|^2 dy = e_l/2\), where \(e_l\) is the unit length. To make the expression simpler, we introduce \(u_0 \equiv 3Hy_0\) and \(u_* \equiv 3Hy_0 + y_0\), then from the normalization,

\[\frac{e_l}{2a_2} = \frac{I_1 + I_2 + I_3}{H}, \quad (33)\]

where

\[\frac{3}{a_2} I_1 = \cosh u_* - \cosh u_0, \quad (34)\]

\[\frac{3}{2a_2} I_2 = \cosh u_* \ln \left( \frac{\tanh \frac{u_*}{2}}{1 - \tanh^2 \frac{u_*}{2}} \right) - \cosh u_0 \ln \left( \frac{\tanh \frac{u_0}{2}}{1 - \tanh^2 \frac{u_0}{2}} \right) + u_* - u_0 - \ln (e^{2u_*} - 1) + \ln (e^{2u_0} - 1), \quad (35)\]

\[\frac{3}{2} I_3 = \ln \left( \frac{\tanh \frac{u_*}{2}}{1 - \tanh^2 \frac{u_*}{2}} \right) \left( \frac{\tanh^2 \frac{u_*}{2}}{1 - \tanh^2 \frac{u_*}{2}} \ln \left( \frac{\tanh \frac{u_*}{2}}{1 + \tanh \frac{u_*}{2}} \right) \right) - \ln \left( \frac{\tanh \frac{u_0}{2}}{1 - \tanh^2 \frac{u_0}{2}} \right) \left( \frac{\tanh^2 \frac{u_0}{2}}{1 - \tanh^2 \frac{u_0}{2}} \ln \left( \frac{\tanh \frac{u_0}{2}}{1 + \tanh \frac{u_0}{2}} \right) \right) + \int_{\tanh \frac{u_0}{2}}^{1+\tanh \frac{u_0}{2}} \frac{\ln(t)}{1 - t} dt - \int_{\tanh \frac{u_0}{2}}^{1+\tanh \frac{u_0}{2}} \frac{\ln(t)}{1 - t} dt. \quad (36)\]

With the zero-mode wave function, the Newtonian gravitational potential between the test masses \(M_1\) and \(M_2\) separated by the distance \(r\) on the brane at \(y = y_i\), is given by

\[V_{N_{cut}}(r) = G_N \frac{|\hat{h}_0(y_i)|^2 M_1 M_2}{r}. \quad (37)\]

The 4D gravitational constant on the first brane is then given by

\[G_4(y = 0) = G_N \frac{|\hat{h}_0(0)|^2}{e_l} = G_N \frac{a_2^2}{e_l} \sinh u_0 \left\{ \frac{3}{2 \cosh u_0 \left[ 1 - \left( \frac{B_0}{H} \sinh u_0 \right)^{1/3} \right] } \right\}^2 \quad (38)\]

\[= G_N \frac{H \sinh u_0}{2(I_1 + I_2 + I_3)} \left\{ \frac{3}{2 \cosh u_0 \left[ 1 - \left( \frac{B_0}{H} \sinh u_0 \right)^{1/3} \right] } \right\}^2, \quad (40)\]

where \(G_N = 1/8\pi M^4_p\), and if we assume that our universe is at \(y = 0\) where we observe conventional 4D gravity, \(G_4(0) = 1/8\pi M^2_p\).
IV. STABILITY CHECK

In the previous section, we explored the gravitational perturbation and obtained the zero-mode graviton wave function. Remaining is the question whether the brane is stable under this perturbation, or not. In the flat-brane case [7], the model is unstable to the gravitational perturbation; there exists one tachyonic mode in 4D gravitons, which makes the direction tangent to the brane unstable. In the inflating-brane model, we add energy on the brane in the form of the cosmological constant. If this vacuum energy compensates the energy deficit by the tachyonic graviton, the story of the stability may be altered.

In flat space, the stability of the graviton is guaranteed if the 4D graviton mass squared is nonnegative, \( m_g^2 \geq 0 \). However, the stability condition in a de Sitter background is subtle and needs some caution. The stability of massive high-spin fields has been investigated very recently by Deser and Waldron [12]. According to their work, in particular for the spin 2 fields \((s = 2)\) which represent gravitons, the stability story is as following.

For the \( m_g^2 = 0 \) mode, the graviton possesses two helicity states, \( \pm s = \pm 2 \). This mode is always stable.

For \( 0 < m_g^2 < 2H^2 \), the graviton possesses full five helicity states \((\pm 2, \pm 1, 0)\), and the helicity-0 sector induces instability. Therefore, the graviton in de Sitter space is unstable in this regime, which is different from the flat-space case.

For \( m_g^2 = 2H^2 \), so called the “partially massless” mode, the dangerous helicity-0 sector disappears and the graviton is stable.

For \( m_g^2 > 2H^2 \), there exist full helicity states, but the helicity-0 sector is not harmful, and the graviton is stable.

As a whole, in de Sitter space the graviton is stable if \( m_g^2 = 0 \), or \( m_g^2 \geq 2H^2 \). Therefore, our policy in searching stable solutions is twofold. As the graviton zero-mode is always a solution satisfying the boundary conditions, first we need to check if this zero mode is the lowest mode. This means that there is no tachyonic mode, \( m_g^2 < 0 \), which is unstable. Second, provided that the zero mode is the lowest mode, we check if the first-excited massive graviton mode satisfies \( m_g^2(1) \geq 2H^2 \) to avoid the unstable regime \( 0 < m_g^2 < 2H^2 \).

A. Zero-mode check

One simple way to examine if the zero mode is the lowest mode of the wave equation, is to check the number of nodes of the zero-mode wave function. For the Sturm-Liouville boundary value problems, there exists a series of eigenvalues with a lower bound. The eigenfunction corresponding to the lowest eigenvalue possesses no node, and the number of nodes increases by one as the eigenvalue increases to the next. If the zero-mode wave function possesses no node between the boundaries, it will be the lowest mode, and there will be no tachyonic mode.

If we look at the zero-mode wave function (26), it is a monotonically increasing or decreasing function depending on the constants. However, it is not clear yet if this function possesses a node, or not. It depends on the value of constants in the wave function. There are three constants to be determined, \( y_0, y_*, \) and \( B_0 \) which we rescale as \( u_0 = 3Hy_0, u_* = 3Hy_* \) (as defined in the last section), and \( \beta = [(B_0/H)\ sinh\ u_*]^{1/3} \).

In order to determine the above three constants, we use three conditions introduced in the previous sections. First, we use the boundary value of the scalar field, Eq. (12),

\[
\frac{\tanh(u_*/2)}{\tanh(u_0/2)} = e^{3n} \equiv \nu ,
\]

where \( n = \pi \kappa \eta /2\sqrt{3} \) is a free parameter that we can fix by setting the symmetry-breaking scale.

The second condition comes from the particle hierarchy between the two branes, Eq. (16),

\[
\frac{v_{eff}(y = y_*)}{v_{eff}(y = 0)} = \left( \frac{\sinh u_*}{\sinh u_0} \right)^{1/3} \equiv 10^m \equiv \mu^{1/3} .
\]

Here, \( m \) is a free parameter that we can fix by requiring the particle hierarchy between the two branes. For example, if we assume that the effective-particle mass on the first brane is of TeV scale, we have \( m = 15 \) and \( \mu = 10^{45} \). (As it was mentioned earlier, the particle mass on the second brane is always set to Planck scale.)

The third condition comes from the boundary conditions for the zero-mode graviton, Eqs. (28) and (29),

\[
\frac{1}{\cosh u_*(1 - \beta)} - \frac{1}{\cosh u_0(1 - \mu^{-1/3} \beta)} + 2n = 0 ,
\]

which leads to a second-order polynomial equation for \( \beta \),

\[
\beta^2 - \frac{2n}{\mu^{1/3}} \beta + \left( \frac{1}{\cosh u_0} + \frac{1}{\cosh u_*} \right) = 0 .
\]
gravitons. In order to satisfy the stability condition addressed earlier, the first-excited massive mode should be

\[ 2n\mu^{-1/3}\beta^2 + \left[ -2n(1 + \mu^{-1/3}) - \frac{\mu^{-1/3}}{\cosh u_*} + \frac{1}{\cosh u_0} \right] \beta + 2n + \frac{1}{\cosh u_*} - \frac{1}{\cosh u_0} = 0. \] (C.3)

To determine the constants, we solve (C.1) and (C.2), and get

\[ \cosh u_0 = \frac{1 - 2\nu/\mu + \nu^2}{\nu^2 - 1}, \quad \cosh u_* = \frac{1 - 2\mu \nu + \nu^2}{1 - \nu^2}. \] (42)

As \( \cosh u_i > 1 \), we get \( \mu > \nu \). With these solutions for \( u_0 \) and \( u_* \), we can evaluate \( \beta \) from (C.3). Then all the constants are fully determined. In determining the constants, we have two free parameters, \( n \) and \( m \). As was mentioned earlier, \( n \) is free related with the symmetry-breaking scale, and \( m \) can be fixed by the requirement of the particle hierarchy between the two branes. We shall investigate the stability of the brane in a various range of these parameters.

Now let us explore under what condition the zero-mode wave function possesses no node in the region under consideration. Let us denote the location of the node as \( y = y_N \) if it existed. Then the wave function satisfies \( \hat{h}_0(y_N) = 0 \). There will be no node if \( y_N > y_* \). We introduce \( u_N \equiv 3H(y_N + y_0) \), then from Eqs. (26) and (28) the condition \( \hat{h}_0(u_N) = 0 \) reduces to

\[ \tanh\left(\frac{u_N}{2}\right) = \tanh\left(\frac{u_0}{2}\right) \exp\left[\frac{3}{2\cosh u_0(1 - \mu^{-1/3}\beta)}\right] \] (43)

\[ > \tanh\left(\frac{u_*}{2}\right) = \tanh\left(\frac{u_0}{2}\right) \exp(3n). \] (44)

Here, the inequality means that the node does not exist in the range, \( 0 \leq y \leq y_* \), i.e. \( u_N > u_* \), and (C.1) was used for Eq. (44). Keeping the no-node condition unchanged, the inequality also holds in the same way for the exponents of Eqs. (43) and (44), and with the aid of Eq. (41), reduces to a simple form,

\[ \cosh u_*(1 - \beta) > 0. \] (45)

As a result, the condition for no-tachyon is simply

\[ 0 < \beta < 1. \] (46)

We numerically search the values of the parameters \( m \) and \( n \) for which the condition (46) is satisfied. The resulting domain in the parameter space \( m \) vs. \( n \) is plotted in Fig. 2. For a given \( m \), there exists an upper bound on \( n \) under which the condition (46) is satisfied. For the most of the \( m \) values, this condition is satisfied if \( n \lesssim 0.425 \) (\( \kappa \eta \lesssim 0.47 \)). The parameters which give stable solutions will belong to a subset of this domain.

B. First-excited mode check

Since the extra dimension of our model is compact, there exists a discrete spectrum of massive Kaluza-Klein gravitons. In order to satisfy the stability condition addressed earlier, the first-excited massive mode should be \( m_0^2(1) \geq 2H^2 \). Let us investigate the massive modes in this section.

The massive-mode solution to the wave equation (19) is

\[ \hat{h}_{m_q}^I(y) = c_1 P_p^q[\coth[3H(|y| + y_0)] + c_2 Q_p^q[\coth[3H(|y| + y_0)]], \] (47)

where \( P \) \( (Q) \) is the associated Legendre function of the first \( (\) second \) kind, and

\[ p = -\frac{1}{2}, \quad q = \frac{1}{2} \sqrt{1 - \frac{4m_q^2}{9H^2}}. \] (48)

Similarly to the zero mode, \( \hat{h}_{m_q}^I(y) \) is obtained by making a change, \( |y| + y_0 \rightarrow -|y - y_*| + y_* + y_0 \), in \( \hat{h}_{m_q}^I(y) \), and from the boundary conditions at \( y = 0 \) and \( y = y_* \), we get

\[ \frac{c_1}{c_2} = \frac{s_1 Q_p^q(\coth u_0) - (p + q)(p - q + 1)Q_p^{q-1}(\coth u_0)}{s_1 P_p^q(\coth u_0) - (p + q)(p - q + 1)P_p^{q-1}(\coth u_0)} \] (49)

\[ = \frac{s_2 Q_p^q(\coth u_*) - (p + q)(p - q + 1)Q_p^{q-1}(\coth u_*)}{s_2 P_p^q(\coth u_*) - (p + q)(p - q + 1)P_p^{q-1}(\coth u_*)}. \] (50)
where

\[ s_1 = \cosh u_0 \left[ q + \frac{1}{6} (1 - 4 \mu^{-1/3} \beta) \right], \quad (51) \]

\[ s_2 = \cosh u_* \left[ q + \frac{1}{6} (1 - 4 \beta) \right]. \quad (52) \]

Equations (49) and (50) give a mass spectrum. It means that the massive mode exists only when the mass satisfies this equality, so the spectrum is discrete, \( m^2_g(i) \).

We numerically search the first-excited mode for given \( m \) and \( n \) which satisfies both the above mass-spectrum condition and \( m^2_g(1) \geq 2H^2 \). First, in the domain of the \( m-n \) plane obtained in the previous section, let us fix \( m \) and examine how the value of \( m^2_g(1) \) varies as \( n \) varies. The numerical result is given in Tab. I for \( m = 2 \). If \( n \) is large in the domain, the value of \( m^2_g(1) \) is smaller than \( 2H^2 \), which means unstable. As \( n \) decreases, this value increases. Below some critical value \( n_c \), \( m^2_g(1) > 2H^2 \) and the graviton becomes stable. This behavior is observed also for larger \( m \)’s.

Now we numerically search \( n_c \) for several values of \( m \). The result is shown in Tab. II and plotted in Fig. 3. Amazingly, from Fig. 3 we observe a very simple functional relation between \( m \) and \( \kappa \eta_c \equiv 2\sqrt{3} n_c / \pi \), in a very high accuracy,

\[ \kappa \eta_c = 10^{-m/2} = \sqrt{\frac{v_{\text{eff}}(y = 0)}{v_{\text{eff}}(y = y_*)}}. \quad (53) \]

As \( m \) increases, \( \eta_c \) decreases exponentially, and the stable domain in the \( m-n \) plane gets smaller and smaller. For \( m \gtrsim 8 \), the calculation goes beyond the present numerical-resolution limit because the involved numbers are too large, or small. However, we expect the above relation (53) continues for larger \( m \)’s.

We therefore conclude that there exists a domain in the \( m-n \) plane in which the parameters give stable solutions. The symmetry-breaking scale for stable solutions is very low, and the upper bound is given by Eq. (53). This result is very different from the flat-brane case which was always unstable [7].

| \( n \) | 0.4 | 0.3 | 0.2 | 0.1 | 0.01 | 0.001 |
|---|---|---|---|---|---|---|
| \( m^2_g(1)/H^2 \) | 0.48991 | 1.35916 | 1.79189 | 1.98834 | 2.04406 | 2.04460 |

TABLE I. The mass of the first-excited graviton for several values of \( n \), and for fixed \( m = 2 \).

| \( m \) | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|
| \( n_c \) | 0.08939 | 0.02793 | 0.00882 | 0.00279 | 0.00088 | 0.00028 |

TABLE II. The critical value \( n_c \) for several values of \( m \).
V. EFFECTIVE-EXPANSION PARAMETER $H_{\text{EFF}}$

In this section, we evaluate the effective-expansion parameter $H_{\text{eff}}$ for the stable branes. As we expected in the beginning, adding vacuum energy to the brane can stabilize the texture-brane system. The question is how much amount of the vacuum energy is necessary, in other words, how large $H_{\text{eff}}$ should be to stabilize the branes. Since $H_{\text{eff}}$ is responsible for the expansion of the universe, it should be very small to be consistent with the slow expansion of the late universe. The current acceleration of the universe constrains the expansion parameter as $H_{\text{obs}} \sim 10^{-44}\text{GeV}$.

From the 5D metric obtained in Sec. II, the induced 4D metric on the 4D hypersurface at $y = y_i = \text{const}$ is given by

$$ds^2_4 = \left\{ \frac{B_0}{H} \sinh[3H(\|y_i\| + y_0)] \right\}^{2/3} \left( -dt^2 + e^{2H}d\vec{x}^2 \right)$$

$$= -d\tau^2 + e^{2H} \left\{ \frac{B_0}{H} \sinh[3H(\|y_i\| + y_0)] \right\}^{-1/3} \tau d\vec{x}^2,$$

where $\vec{x}$ is the rescaled 3D coordinates $x$, and $\tau$ is the proper time measured on the hypersurface at $y = y_i$.

Then the 4D effective-expansion parameter is defined as

$$H_{\text{eff}} = \left( \frac{H}{B_0} \sinh[3H(\|y_i\| + y_0)] \right)^{-1/3}.$$

Here, for the given $m$ and $n$, $H$ is obtained from Eq. (40),

$$H = \left( \frac{M_{\ast}}{M_{\text{pl}}} \right)^2 \frac{2(I_1 + I_2 + I_3)}{\sinh u_0} \left[ \frac{2 \cosh u_0 (1 - \mu^{-1/3})}{3} \right]^2.$$

In determining $H_{\text{eff}}$ via $H$, we need to impose a constraint on the fundamental 5D Planck mass, i.e. $M_{\text{pl}}/M_{\ast}$ in Eq. (57). First, we may assume that the 5D Planck-mass scale is constrained in the same way as the particle scale on the brane. That means for the given particle hierarchy between the two branes, $v_{\text{eff}}(y_i)/v_{\text{eff}}(0) = 10^m$, the same hierarchy between the two Planck masses holds, $M_{\text{pl}}/M_{\ast} = 10^m$. According to this constraint, $M_{\ast} = \text{TeV}$ for $m = 15$, and $M_{\ast} = \text{GUT}$ scale ($= 10^{16}\text{GeV}$) for $m = 2$.

Second, regardless of the particle hierarchy, we fix the fundamental Planck mass to TeV scale, $M_{\ast} = \text{TeV}$, so $M_{\text{pl}}/M_{\ast} = 10^{15}$.

Another constraint that we can consider in evaluating $H_{\text{eff}}$ is the choice of the brane for residence. So far, we have assumed that our universe is on the first brane. In this case, the gravitational coupling is determined by, $G_4(y = 0) = 1/8\pi M_{\text{pl}}^2$, and from Eq. (56) the 4D effective-expansion parameter is given by

$$H_{\text{eff}} = \frac{\mu^{1/3}}{\beta} H.$$  

However, if we assume that we are living on the second brane,

$$H_{\text{eff}} = \frac{H}{\beta}.$$  

In this case, Eq. (38) is rewritten as $G_4(y = y_i) = 1/8\pi M_{\text{pl}}^2 = G_N |\hat{h}(y_i)|^2/e_i$, the resulting $H$ is a bit different from Eq. (57), and is given by

$$H = \left( \frac{M_{\ast}}{M_{\text{pl}}} \right)^2 \frac{2(I_1 + I_2 + I_3)}{\sinh u_0} \left[ \frac{2 \cosh u_0 (1 - \beta)}{3} \right]^2.$$

Depending on the choices of the Planck-mass hierarchy $M_{\text{pl}}/M_{\ast}$ and the location of our universe in 5D, we have very different $H_{\text{eff}}$ scales. The results are plotted in Fig. 4 and Fig. 5.

First, let us consider the case of $M_{\text{pl}}/M_{\ast} = 10^m$. 

(i) If we choose our universe is on the Brane I, i.e. \( G_4(y = 0) = 1/8\pi M_{pl}^2 \), the 4D effective-expansion parameter is very large \( H_{eff} \gtrsim 10^{19}\text{GeV} \) regardless of the particle-hierarchy scale \( m \).

(ii) If we assume that our universe is on the Brane II (Planck brane), i.e. \( G_4(y = y_s) = 1/8\pi M_{pl}^2 \), \( H_{eff} \) decreases as \( m \) increases, and for the TeV hierarchy \( (m = 15) \), the expansion parameter can be lowered to \( H_{eff} \approx 10^{-27}\text{GeV} \).

Second, let us consider that the fundamental 5D Planck mass is fixed to TeV scale, \( M_* = \text{TeV} \).

(i) For the Brane I case, \( H_{eff} \) decreases as \( m \) decreases, and the expanding parameter can be considerably lowered, \( H_{eff} \approx 10^{-26}\text{GeV} \), for the GUT hierarchy, \( m = 2 \).

(ii) For the Brane II case, \( H_{eff} \approx 10^{-27}\text{GeV} \) regardless of \( m \).

The numerical values of the results do not depend much on the scale of \( n \). In particular, for very small \( n \)'s, \( H_{eff} \) remains almost unchanged as \( n \) varies. The data in Fig. 4 and Fig. 5 were obtained for \( n = 10^{-8} \) which is supposed to give stable solutions for the whole range of \( m \).

VI. CONCLUSIONS

In this work, we investigated the gravitational perturbation on the inflating 3-branes embedded in an O(2) texture formed in the fifth dimension. In the previous work for the flat-brane case [7], the gravitational perturbation made the branes unstable. There existed one tachyonic mode in 4D gravitons. The main purpose of this work was to investigate if the branes can be stable to the perturbation when we allow the branes to inflate.

In a de Sitter background, the graviton is stable only if it is massless, or if the mass ranges as \( m^2_g \geq 2H^2 \). In the other ranges, the graviton becomes unstable [12]. In this work, we first checked if the zero-mode graviton of the texture-brane solutions is the lowest mode to avoid \( m^2_g < 0 \) modes. And then we checked if the mass of first-excited mode is within the stable range \( m^2_g(1) \geq 2H^2 \).

We found that there exist stable solutions if the symmetry-breaking scale satisfies

\[
\kappa\eta \lesssim \sqrt{\frac{v_{eff}(y = 0)}{v_{eff}(y = y_s)}},
\]

where \( v_{eff} \) is the effective-particle mass on the brane. (We assumed throughout this paper that the particles mass on the second brane is of Planck scale, \( v_{eff}(y = y_s) = 10^{18}\text{GeV} \).)

The scale of the effective-expansion parameter \( H_{eff} \) for the stable solutions varies depending on the constraints we impose. Those constrains are the scale of the fundamental 5D Planck mass \( M_* \), the choice of the brane for our universe, and the particle-hierarchy scale \( m \) between the two branes.

In order to explain the very slow acceleration of the late universe, we expect \( H_{eff} \) to be very small. The possibly low \( H_{eff} \) scales are achieved mainly when we constrain the 5D Planck mass to be \( M_* = \text{TeV} \). The resulting \( H_{eff} \) scales are, depending on the choices of the other constraints,

(i) \( H_{eff} \approx 10^{-26}\text{GeV} \) : for assuming that our universe is on the Brane I, and that the particle scale on the Brane I is GUT scale.

(ii) \( H_{eff} \approx 10^{-27}\text{GeV} \) : for assuming that our universe is on the Brane II regardless of the particle scale on the Brane I.

These \( H_{eff} \) scales are still larger than the current expansion parameter of the universe, \( H_{obs} \sim 10^{-42}\text{GeV} \). However, they are much smaller than the expansion rate during inflation, \( H_{inf} \sim 10^{10}\text{GeV} \). The stable texture-brane system can be formed and survive at the late stages in the evolution of the universe.

Once the symmetry-breaking scale \( \eta \) is in the stable regime in Eq. (61), \( H_{eff} \) is not very dependent on the magnitude of \( \eta \). Therefore, the values of \( H_{eff} \) shown above are globally valid for the whole stable texture-brane systems, and we do not need to finely tune \( \eta \) in order to have those small values of \( H_{eff} \).

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FIG. 1. A schematic picture of the model and solutions. Two branes are located at $y = 0$ and $y = y_*$. Two points, $y = y_*$ and $y = -y_*$ are identified, and the extra dimension is compact. The gravitational field $B$ is lowest on the first brane, and highest on the second brane. The scalar field $\chi$ is continuous across the second brane because the $2\pi$ shift makes no change in $\chi$. 

Brane II  Brane I  Brane II

$y = -y_*$  $y = 0$  $y = y_*$

$\chi$  $\pi$
FIG. 2. The parameter space of $m$ vs. $n$. The shaded region is where the graviton zero-mode is the lowest mode, and the tachyonic-graviton mode is absent. $m = 15$ corresponds to the TeV hierarchy between the two branes, and $m = 2$ corresponds to the GUT hierarchy.
FIG. 3. Plot of the critical value \( \eta_c \) under which there exist stable solutions. The numerical data (dots) fit very accurately the reference function \(-m/2\) (solid line), which states \( \kappa \eta_c \approx 10^{-m/2} \).
FIG. 4. Plot of $H_{\text{eff}}$ vs. $m$ for $n = 10^{-8}$ and $M_{\text{pl}}/M_* = 10^m$. If we assume that our universe is on the Brane I, $H_{\text{eff}}$ is very large $\gtrsim 10^{19}$GeV. If we assume that our universe is on the Brane II, $H_{\text{eff}}$ decreases as $m$ increases, and for the TeV hierarchy ($m = 15$), the expansion parameter can be lowered to $H_{\text{eff}} \approx 10^{-27}$GeV.
FIG. 5. A similar plot to Fig. 4, but for assuming $M_\ast = \text{TeV}$. For the Brane I case, $H_{\text{eff}}$ decreases as $m$ decreases, and the expansion parameter can be considerably lowered, $H_{\text{eff}} \approx 10^{-20}\text{GeV}$, for the GUT hierarchy ($m = 2$). For the Brane II case, $H_{\text{eff}} \approx 10^{-27}\text{GeV}$ regardless of $m$. 