FURTHER RESULTS ON THE EXISTENCE OF SUPER-SIMPLE PAIRWISE BALANCED DESIGNS WITH BLOCK SIZES 3 AND 4

GUANGZHOU CHEN* AND YUE GUO
School of Mathematics and Information Science
Henan Normal University, Xinxiang 453007, China

YONG ZHANG
School of Mathematics and Statistics, Yancheng Teachers University
Yancheng 224002, China

(Communicated by Mario Osvin Pavcevic)

Abstract. In statistical planning of experiments, super-simple designs are the ones providing samples with maximum intersection as small as possible. Super-simple pairwise balanced designs are useful in constructing other types of super-simple designs which can be applied to codes and designs. In this paper, the super-simple pairwise balanced designs with block sizes 3 and 4 are investigated and it is proved that the necessary conditions for the existence of a super-simple \((v, \{3, 4\}, \lambda)\)-PBD for \(\lambda = 7, 9\) and \(\lambda = 2k, k \geq 1\), are sufficient with seven possible exceptions. In the end, several optical orthogonal codes and superimposed codes are given.

1. Introduction

A pairwise balanced design (or PBD) is a pair \((\mathcal{X}, \mathcal{A})\) such that \(\mathcal{X}\) is a set of elements called points, and \(\mathcal{A}\) is a set of subsets (called blocks) of \(\mathcal{X}\), each of cardinality at least two, such that every pair of points occurs in exactly \(\lambda\) blocks of \(\mathcal{A}\). If \(v\) is a positive integer and \(K\) is a set of positive integers, each of which is greater than one, then we say that \((\mathcal{X}', \mathcal{A}')\) is a \((v, K, \lambda)\)-PBD if \(|\mathcal{X}'| = v\), and \(|A| \in K\) for every \(A \in \mathcal{A}\). We denote \(B(K, \lambda) = \{v : \text{there exists a } (v, K, \lambda)\text{-PBD}\}\). A set \(K\) is said to be PBD-closed if \(B(K, \lambda) = K\).

A PBD is resolvable if its blocks can be partitioned into parallel classes; a parallel class is a set of point-disjoint blocks whose union is the set of all points. The notation \((v, K, \lambda)\)-RPBD is used for a resolvable PBD. When \(K = \{k\}\), a \((v, K, \lambda)\)-PBD is a balanced incomplete block design, the notations \((v, k, \lambda)\)-BIBD and \((v, k, \lambda)\)-RBIBD are sometimes used in this case.

A design is said to be simple if it contains no repeated blocks. A design is said to be super-simple if the intersection of any two blocks has at most two elements. When \(k = 3\), a super-simple design is just a simple design. When \(\lambda = 1\), the designs are necessarily super-simple. A super-simple \((v, K_1, \lambda)\)-PBD is also a super-simple \((v, K_2, \lambda)\)-PBD if \(K_1 \subseteq K_2\).

2010 Mathematics Subject Classification: Primary: 05B05, 51E05; Secondary: 62K10, 94C30.
Key words and phrases: Super-simple designs, pairwise balanced designs, balanced incomplete block designs, group divisible designs, optical orthogonal code.

The first author is supported by NSF grant No. 11501181, Science Foundation for Youths (Grant No. 2014QK05) and Ph.D.(Grant No. qd14140) of Henan Normal University.
* Corresponding author: G. Chen(chenguangzhou0808@163.com).
A design is called cyclic if it admits an automorphism $\sigma$ of order $v = |\mathcal{X}|$, where $\mathcal{X}$ is the point set of the design, which can be identified with $\mathbb{Z}_v$. A cyclic $(v, K, \lambda)$-PBD is denoted by $(v, K, \lambda)$-PBD and a cyclic $(v, k, \lambda)$-BIBD is denoted by $(v, k, \lambda)$-BIBD in short.

The concept of super-simple designs was introduced by Gronau and Mullin [26]. The existence of super-simple designs is an interesting problem by itself, but there are also useful applications. For example, such designs are used in constructing perfect hash families [35] and coverings [7], in the construction of new designs [6] and in the construction of superimposed codes [32]. Super-simple pairwise balanced designs are powerful for the construction of other types of combinatorial structures [10] (such as super-simple designs). In statistical planning of experiments, super-simple designs are the ones providing samples with maximum intersection as small as possible. There are other useful applications in [4, 22, 29].

Quite recently, Chen [12] et al studied the existence of a super-simple $(v, K, \lambda)$-BIBD for $2 \leq \lambda \leq 6$ and showed the following result.

**Lemma 1.1.** ([24]) There exists a simple $(v, 3, \lambda)$-BIBD if and only if $v > \lambda + 2$, $\lambda(v-1) \equiv 0 \pmod{2}$ and $\lambda v(v-1) \equiv 0 \pmod{6}$.

The necessary conditions for the existence of a super-simple $(v, 4, \lambda)$-PBD are $v \geq 2\lambda + 2$, $\lambda(v-1) \equiv 0 \pmod{3}$ and $\lambda v(v-1) \equiv 0 \pmod{12}$. For the existence of super-simple $(v, 4, \lambda)$-BIBDs, the necessary conditions are known to be sufficient for $\lambda \in \{2, 6, 8, 9\}$ (see [3, 9, 13, 14, 15, 21, 23, 26, 31, 37]). Gronau and Mullin [26] solved the case for $\lambda = 2$, and the corrected proof appeared in [31]. The $\lambda = 3$ case was solved by Chen [13]. The $\lambda = 4$ case was solved independently by Adams, Bryant, and Khodkar [3] and Chen [14]. The case of $\lambda = 5$ was solved by Cao, Chen and Wei [9]. The case of $\lambda = 6$ was solved by Chen, Cao and Wei [15]. The case of $\lambda = 8$ was solved by Chen, Sun and Zhang [21]. The case of $\lambda = 9$ was solved by Zhang, Chen and Sun [37]. A survey on super-simple $(v, 4, \lambda)$-BIBDs with $v \leq 32$ appeared in [8]. We summarize these known results in the following result.

**Lemma 1.2.** ([3, 9, 13, 14, 15, 21, 23, 26, 31, 37]) The necessary conditions of a super-simple $(v, 4, \lambda)$-BIBD for $\lambda \in \{2, 3, 4, 5, 6, 8, 9\}$ are sufficient.

The necessary conditions for the existence of a super-simple $(v, 5, \lambda)$-BIBD are known to be sufficient for $\lambda \in \{2, 3, 4, 5\}$ (see [1, 16, 17, 18, 25, 36]). For more results on super-simple designs we refer the reader to [2, 5, 8, 11, 19, 20, 28, 34, 36] and references therein.

The existence of $(v, \{3, 4\}, 1)$-PBD was stated in [23, 27].

**Lemma 1.3.** ([23, 27]) There exists a $(v, \{3, 4\}, 1)$-PBD if and only if $v \equiv 0, 1 \pmod{3}$ and $v \geq 3$.

Quite recently, Chen [12] et al studied the existence of a super-simple $(v, \{3, 4\}, \lambda)$-PBD for $2 \leq \lambda \leq 6$ and showed the following result.

**Lemma 1.4.** ([12]) The necessary conditions of a super-simple $(v, \{3, 4\}, \lambda)$-PBD for $2 \leq \lambda \leq 6$ are sufficient except possibly for $(v, \lambda) \in \{(18, 5), (30, 5), (42, 5)\}$. 
In this paper, the existence of a super-simple \((v, \{3, 4\}, \lambda)\)-PBD for \(\lambda = 7, 9\) and \(\lambda = 2k, k \geq 4\), is investigated. The necessary conditions for the existence of such a super-simple design are \(v \geq \lambda + 2\) and \(\lambda(v - 1) \equiv 0 \pmod{3}\). We shall use direct and recursive constructions to show that the necessary conditions are also sufficient with some possible exceptions. Specifically, we shall prove the following theorem.

**Theorem 1.5.** The necessary conditions of a super-simple \((v, \{3, 4\}, \lambda)\)-PBD for \(\lambda = 7, 9\) and \(\lambda = 2k, k \geq 4\), are sufficient except possibly for \((v, \lambda) \in \{(10, 7), (12, 9), (12, 7), (16, 9), (18, 7), (30, 7), (42, 7)\} \).

The paper is organized as follows. Some recursive constructions are provided in Section 2. Some ingredient super-simple designs are given directly by computer search in Section 3. The proof of our main theorem is given in Section 4. Some applications in optical orthogonal codes and superimposed codes are mentioned in Section 5.

## 2. Recursive Constructions

In this section, the auxiliary design (group divisible design) is introduced and some known results stated for later use, and we also give some standard recursive constructions.

A **group divisible design** (or GDD) is a triple \((\mathcal{X}, \mathcal{G}, \mathcal{B})\) which satisfies the following properties:

(i) \(\mathcal{G}\) is a partition of a set \(\mathcal{X}\) (of points) into subsets called **groups**.

(ii) \(\mathcal{B}\) is a set of subsets of \(\mathcal{X}\) (called **blocks**) such that a group and a block contain at most one common point.

(iii) Every pair of points from distinct groups occurs in exactly \(\lambda\) blocks.

The **group type** (or **type**) of GDD is the multiset \(\{\{G\} : G \in \mathcal{G}\}\). We usually use an “exponential” notation to describe types: so type \(g_1^{u_1}g_2^{u_2} \cdots g_k^{u_k}\) denotes \(u_i\) occurrences of \(g_i, 1 \leq i \leq k\), in the multiset. A GDD with block sizes from a set of positive integers \(K\) is called a \((K, \lambda)\)-GDD. When \(\lambda = 1\), we simply write \(K\)-GDD. When \(K = \{k\}\), we simply write \(k\) for \(K\). Taking the groups of a GDD as blocks yields a PBD, and taking a parallel class of blocks of a PBD as groups also yields a GDD.

A \((k, \lambda)\)-GDD of group type \(v^k\) is called a **transversal design** and denoted by \(TD_\lambda(k, v)\) for short. The known result on super-simple \(TD_\lambda(4, v)\) is listed in the following which is used in Section 4.

**Lemma 2.1.** ([28]) A super-simple \(TD_\lambda(4, v)\) exists if and only if \(\lambda \leq v\) and \((\lambda, v)\) is neither \((1, 2)\) nor \((1, 6)\).

We shall use the following standard recursive constructions in the proof of Theorem 1.5. For details of these constructions, we refer the reader to [1, 3, 9, 12, 13, 14, 15, 16, 17, 18, 23, 26].

**Construction 1.** ([9, 16, 17, 18]) (Weighting) Let \((\mathcal{X}, \mathcal{G}, \mathcal{B})\) be a super-simple GDD with index \(\lambda_1\), and let \(w : \mathcal{X} \to \mathbb{Z}^+ \cup \{0\}\) be a weight function on \(\mathcal{X}\), where \(\mathbb{Z}^+\) is the set of positive integers. Suppose that for each block \(B \in \mathcal{B}\), there exists a super-simple \((k, \lambda_2)\)-GDD of type \(\{w(x) : x \in B\}\), then there exists a super-simple \((k, \lambda_1\lambda_2)\)-GDD of type \(\{\sum_{x \in G_i} w(x) : G_i \in \mathcal{G}\}\).

**Construction 2.** ([12]) (Breaking up groups) If there exists a super-simple \((K, \lambda)\)-GDD of type \(h_1^{u_1} \cdots h_t^{u_t}\) and a super-simple \((h_i + \eta, K, \lambda)\)-PBD for each \(i, 1 \leq i \leq t\), then there exists a super-simple \(\left(\sum_{i=1}^t h_iu_i + \eta, K, \lambda\right)\)-PBD, where \(\eta = 0\) or 1.
Construction 3. ([12]) If there exists a \((K, 1)\)-GDD of type \(h_1^{u_1} \cdots h_t^{u_t}\), then there exists a \((\sum_{i=1}^t h_iu_i + \eta, K, 1)\)-PBD, where \(K_1 = K \cup \{h_1, h_2, \cdots, h_t\}\) and \(\eta = 0\) or 1.

The following results are obvious but very useful. Their proofs are omitted here.

Lemma 2.2. If there exists a super-simple \((v, K_1, \lambda)\)-PBD, then there exists a super-simple \((v - m, K_2, \lambda)\)-PBD, where \(K_2 = \{k - i : k \in K_1\} - \{i \leq i \leq m\}\).

Lemma 2.3. If there exists a super-simple \((K_1, \lambda)\)-GDD of type \(h_1h_2h_3 \cdots h_{n-1}h_n\), then there exists a super-simple \((K_2, \lambda)\)-GDD of type \(h_1h_2h_3 \cdots (h_i - g_1)(h_{i+1} - g_2) \cdots (h_{i+m-1} - g_m) \cdots h_{n-1}h_n\), 1 \(\leq i \leq n - m + 1\), where \(K_2 = \{k - l : k \in K_1\}\) and \(1 \leq l \leq m\) and \(g_j < h_{i+j-1}, 1 \leq j \leq m\).

3. DIRECT CONSTRUCTIONS

In this section, direct constructions are used and some super-simple \((v, \{3, 4\}, \lambda)\)-PBDs for small values of \(v\) are obtained, which will be used as master designs or input designs in the recursive constructions. All these designs are obtained by computer.

Usually, it is difficult to find all the blocks of a design directly. So, a technique of \(
\text{\textquoteleft\textquoteleft} + d \text{\textquoteright\textquoteright} \mod {v}\) is used, which means that we try to find a subset \(S \subseteq B\) and an element \(d \in \mathbb{Z}_v\) such that \(\{B + kd : B \in S, k \in \mathbb{Z}\} = B\). The blocks of \(S\) are called base blocks. The \(\text{\textquoteleft\textquoteleft} + d \text{\textquoteright\textquoteright}\) is omitted when \(d = 1\) and then the design is cyclic.

Sometimes \(S\) is divided into two parts: \(\mathcal{P}\) and \(\mathcal{R}\), and we try to find an element \(m \in \mathbb{Z}_v\) and an integer \(s\) such that there is a subset \(\mathcal{P}_1 \subseteq \mathcal{P}\) satisfying \(\bigcup_{i=0}^{41} \{mB : B \in \mathcal{P}_1\} = \mathcal{P}\). Here \(m\) is a partial multiplier of order \(s\) of the design. In this article, \(m\) is taken to be some unit of the ring \(\mathbb{Z}_v\), i.e., \(m\) satisfies that gcd\((m, v) = 1\).

Further, the founded base blocks of \(S\) are shuffled when the program takes too much time to find a design. Most of these ideas come from the previous papers such as [16, 18, 20].

Lemma 3.1. There exists a super-simple \((v, \{3, 4\}, 7)\)-PBD for \(v \in \{22, 34, 46\}\).

Proof. For \(v \in \{22, 34, 46\}\), we take the point set \(X = \mathbb{Z}_v\), the base blocks are listed below and all the required blocks can be generated from them by \(+2\) (mod \(v\)).

\[
\begin{align*}
V = 22: & \quad \{0, 2, 6, 16\}, \{0, 1, 12, 15\}, \{0, 13, 17, 20\}, \{1, 8, 15, 19\}, \{0, 9, 11, 17\}, \{0, 5, 15, 16\}, \\
& \quad \{1, 11, 14, 21\}, \{0, 3, 5, 13\}, \{1, 14, 18, 19\}, \{0, 12, 14, 21\}, \{0, 1, 3, 8\}, \{0, 5, 11, 14\}, \\
& \quad \{0, 3, 19, 20\}, \{1, 4, 5, 14\}, \{0, 12, 16, 19\}, \{1, 10, 12, 14\}, \{0, 4, 8, 16\}, \{1, 7, 8, 17\}, \\
& \quad \{0, 5, 12, 20\}, \{1, 6, 9, 13\}, \{0, 6, 17, 19\}, \{1, 2, 3, 15\}, \{0, 1, 16, 17\}, \{0, 17, 18, 21\}, \\
& \quad \{1, 3, 17\}.
\end{align*}
\[
\begin{align*}
V = 34: & \quad \{0, 5, 10, 22\}, \{1, 4, 9, 12\}, \{0, 1, 30, 33\}, \{1, 5, 18, 27\}, \{1, 2, 11, 14\}, \{1, 18, 20, 21\}, \\
& \quad \{1, 12, 26, 29\}, \{1, 3, 5, 22\}, \{0, 7, 16, 27\}, \{1, 13, 21, 29\}, \{0, 4, 20, 28\}, \{0, 7, 19, 29\}, \\
& \quad \{1, 5, 20, 23\}, \{0, 8, 16, 21\}, \{1, 4, 17, 28\}, \{0, 11, 23, 28\}, \{1, 3, 8, 28\}, \{0, 11, 14, 21\}, \\
& \quad \{1, 16, 27, 29\}, \{1, 4, 20, 29\}, \{0, 8, 27, 33\}, \{1, 1, 13, 22\}, \{1, 18, 10, 14\}, \{0, 2, 26, 29\}, \\
& \quad \{1, 14, 19, 20\}, \{0, 5, 13, 20\}, \{0, 15, 23, 25\}, \{0, 2, 5, 32\}, \{0, 4, 18, 24\}, \{0, 2, 12, 31\}, \\
& \quad \{0, 9, 23, 27\}, \{0, 11, 12, 30\}, \{0, 1, 21, 25\}, \{0, 23, 29, 33\}, \{1, 5, 19, 25\}, \{0, 1, 2, 19\}, \\
& \quad \{0, 1, 3, 26\}, \{0, 6, 7, 30\}, \{0, 6, 18\}.
\end{align*}
\[
\begin{align*}
V = 46: & \quad \{0, 6, 33, 41\}, \{1, 19, 33, 42\}, \{1, 11, 18, 24\}, \{1, 2, 40, 42\}, \{1, 17, 32, 41\}, \\
& \quad \{0, 13, 14, 35\}, \{0, 12, 26, 36\}, \{1, 3, 13, 38\}, \{0, 2, 14, 17\}, \{0, 11, 15, 24\}, \\
& \quad \{0, 26, 34, 40\}, \{0, 36, 37, 43\}, \{1, 3, 36, 40\}, \{0, 8, 13, 22\}, \{1, 19, 30, 31\}, \\
& \quad \{0, 25, 28, 30\}, \{1, 13, 25, 44\}, \{0, 17, 18, 40\}, \{0, 37, 38, 41\}, \{1, 3, 5, 43\}, \\
& \quad \{1, 18, 20, 22\}, \{0, 11, 16, 39\}, \{1, 10, 11, 35\}, \{1, 10, 31, 45\}, \{1, 6, 8, 15\}.
\end{align*}
\]
Lemma 3.2. There exists a super-simple $(40, 4, 7)$-BIBD and a super-simple $(v, 4, 7)$ -CBIBD for $v \in \{37, 49, 73\}$.

Proof. For $v \in \{37, 49, 73\}$, we take the point set as $X = \mathbb{Z}_v$. With a computer program we found the required base blocks, which are divided into two parts, $P$ and $R$, where $P$ consists of some base blocks with a partial multiplier $m$ of order $s$ (i.e., each base block of $P$ has to be multiplied by $m^i$ for $0 \leq i \leq s-1$), and $R$ is the set of the remaining base blocks. We list $P, m, s$ and $R$ below. The desired super-simple design is generated by developing the base blocks (mod $v$).

For $v = 37$, the point set is $X = \mathbb{Z}_{37}$. The base blocks are also divided into two parts, $P$ and $R$, which are listed below and all the required blocks can be generated from them by $+2$ (mod $v$).

For $v = 40$, we take the point set as $X = \mathbb{Z}_{40}$. The base blocks are also divided into two parts, $P$ and $R$, which are listed below and all the required blocks can be generated from them by $+2$ (mod $v$).

For $v = 73$, the point set is $X = \mathbb{Z}_{73}$. The base blocks are also divided into two parts, $P$ and $R$, which are listed below and all the required blocks can be generated from them by $+2$ (mod $v$).

Lemma 3.3. There exists a super-simple $(v, 3, 4, 9)$-PBD for $v \in M = \{14, 18, 22, 26, 30, 34, 38, 42, 46, 54\}$.

Proof. For each $v \in M$, we take the point set $X = \mathbb{Z}_{v}$. With a computer program we found the required base blocks, which are divided into two parts, $P$ and $R$, where $P$ consists of some base blocks with a partial multiplier $m$ of order $s$ (i.e., each base block of $P$ has to be multiplied by $m^i$ for $0 \leq i \leq s-1$), and $R$ is the set of the remaining base blocks. We list $P, m, s$ and $R$ below. The desired super-simple design is generated by developing the base blocks $+2$ (mod $v$).
Lemma 4.4. There exists a super-simple \((v, 3, 2k)-BIBD\) for \(\lambda = 2k\) and \(k \geq 4\).  

In this section, we shall give the proof of Theorem 1.5. 

The necessary conditions for the existence of a super-simple \((v, 3, 2k)-BIBD\) for \(\lambda = 2k\), \(k \geq 4\), are the same as the necessary conditions for the existence of a simple \((v, 3, 2k)-BIBD\). So the existence of a super-simple \((v, 3, 2k)-BIBD\) had been solved already by Lemma 1.1.

Lemma 4.1. There exists a super-simple \((v, 3, 4), \lambda\)-PBD for \(\lambda = 2k\) and \(k \geq 4\).  

To prove Theorem 1.5, we shall divide it into two cases by the remaining value of \(\lambda = 7, 9\).

Case 1. \(\lambda = 7\)  

When \(\lambda = 7\), the necessary conditions for a super-simple \((v, 3, 4), 7\)-PBD become \(v \equiv 0, 1 \pmod{3}\) and \(v \geq 9\). We shall prove that there exists a super-simple \((v, 3, 4), 7\)-PBD for any \(v \equiv 0, 1 \pmod{3}\), \(v \geq 9\), except possibly for \(v \in \{10, 12, 18, 30, 42\}\).

Lemma 4.2. There exists a super-simple \((v, 3, 4), 7\)-PBD for any \(v \equiv 1, 3 \pmod{6}\) and \(v \geq 9\).  

Proof. For \(v \equiv 1, 3 \pmod{6}\) and \(v \geq 9\), there exists a super-simple \((v, 3, 4), 7\)-PBD since there exists a super-simple \((v, 3, 7\)-BIBD\) by Lemma 1.1.

Lemma 4.3. \((8)\) There exists a super-simple \((25, 4, 7\)-CBIBD and a super-simple \((v, 4, 7\)-BIBD for \(v \in \{16, 28\}\).

Lemma 4.4. There exists a super-simple \((v, 3, 4), 7\)-PBD for \(v \in \{24, 36, 48, 52, 60, 64, 72, 76, 84, 88, 94, 100\}\).

Proof. For \(v \in \{24, 36, 48, 72\}\), we can get a super-simple \((v, 3, 4), 7\)-PBD by deleting one point from the point set of a super-simple \((v + 1, 4, 7\)-BIBD\) from Lemma 4.3 and Lemma 3.2 respectively.

For \(v \in \{52, 60, 64, 76, 84, 88, 100\}\), it can be written as \(v = 4m\), where \(m \in \{13, 15, 16, 19, 21, 22, 25\}\). By Lemma 2.1 there exists a super-simple \(TD_{7}(4, m)\), then by applying Construction 2 with \(\eta = 0\) we can obtain a super-simple \((v, 3, 4), 7\)-PBD, where the input super-simple \((m, 3, 4), 7\)-PBD comes from Lemmas 4.2-4.3 and Lemma 3.1.

For \(v = 94\), we can get a super-simple \((\{3, 4\}, 7\)-GDD of type \((24)^{3}(22)^{1}\) by removing 2 points from the last group of a super-simple \((4, 7\)-GDD of type \((24)^{4}\) coming from Lemma 2.1. Then applying Construction 2 with \(\eta = 0\), we get a super-simple \((94, 3, 4), 7\)-PBD, where the input super-simple \((22, 3, 4), 7\)-PBD and \((24, 3, 4), 7\)-PBD come from Lemma 3.1 and Lemma 4.4 respectively.

Lemma 4.5. There exists a super-simple \((v, 3, 4), 7\)-PBD for any \(v \in M = \{54, 58, 66, 70, 78, 82\}\).

Proof. For \(v \in M\), let \(v = 3g + m + \eta\). The three parameters \(g, m\) and \(\eta\) are listed in the following table. We get a super-simple \((\{3, 4\}, 7\)-GDD of type \(g^{3} m^{1}\) by removing \(g - m\) points from the last group of a super-simple \(TD_{7}(4, g)\) coming
from Lemma 2.1. Then applying Construction 2 with $\eta = 0, 1$, we get a super-simple $(3g + m, \{3, 4\}, 7)$-PBD, where the input super-simple $(g + \eta, \{3, 4\}, 7)$-PBD and super-simple $(m + \eta, \{3, 4\}, 7)$-PBD come from Lemmas 4.2-4.4 and Lemma 3.1 respectively.

| $v = 3g + m$ | $g$ | $m$ | $\eta$ | $v = 4g + m$ | $g$ | $m$ | $\eta$ |
|-------------|-----|-----|--------|-------------|-----|-----|--------|
| 54          | 15  | 9   | 0      | 70          | 19  | 13  | 0      |
| 58          | 15  | 9   | 0      | 78          | 21  | 14  | 1      |
| 66          | 19  | 9   | 0      | 82          | 23  | 12  | 1      |

Lemma 4.6. There exists a super-simple $(v, \{3, 4\}, 7)$-PBD for $v \equiv 4 \pmod{6}$ and $v > 100$.

**Proof.** For $v \equiv 4 \pmod{6}$ and $v > 100$, it can be written as $v = 18t + m$, where $t \geq 5$ and $m \in \{16, 22, 28\}$. Removing $6t - (m - 1)$ points from the last group of a super-simple TD$_7(4, 6t)$ from Lemma 2.1, we get a super-simple $(\{3, 4\}, 7)$-GDD of group type $(6t)^3(m - 1)$. Applying Construction 2 with $\eta = 1$, a super-simple $(18t + m, \{3, 4\}, 7)$-PBD is obtained, where the input super-simple $(6t + 1, \{3, 4\}, 7)$-PBD and $(m, \{3, 4\}, 7)$-PBD come from Lemmas 4.2-4.3 and Lemma 3.1 respectively.

Lemma 4.7. There exists a super-simple $(v, \{3, 4\}, 7)$-PBD for $v \equiv 0 \pmod{6}$ and $v \geq 90$.

**Proof.** For $v \equiv 0 \pmod{6}$ and $v \geq 90$, we divided it into three cases: $v \equiv 0, 6, 12 \pmod{18}$ and $v \geq 90$.

For $v \equiv 0, 6 \pmod{18}$ and $v \geq 90$, it can be written as $v = 3(m + 5)$, where $m \equiv 1, 3 \pmod{6}$ and $m \geq 25$. We obtained a super-simple $(\{3, 4\}, 7)$-GDD of group type $m^3(15)^4$ by removing $m - 15$ points from the last group of a super-simple TD$_7(4, m)$ from Lemma 2.1. Then by applying Construction 2 with $\eta = 0$, we get a super-simple $(v, \{3, 4\}, 7)$-PBD, here, all these input super-simple $(m, \{3, 4\}, 7)$-PBD and $(15, \{3, 4\}, 7)$-PBD come from Lemma 4.2.

For $v \equiv 12 \pmod{18}$ and $v \geq 90$, it can be written as $v = 3(m + 7)$, where $m \equiv 3 \pmod{6}$ and $m \geq 27$. We can obtain a super-simple $(\{3, 4\}, 7)$-GDD of type $m^3(21)^4$ by deleting $m - 21$ points from the last group of a super-simple TD$_7(4, m)$ coming from Lemma 2.1. Then applying Construction 2 with $\eta = 0$, we get a super-simple $(v, \{3, 4\}, 7)$-PBD, here, all the input super-simple $(m, \{3, 4\}, 7)$-PBD and $(21, \{3, 4\}, 7)$-PBD come from Lemma 4.2. The proof is completed.

By Lemmas 4.2-4.7 and Lemmas 3.1-3.2, we have the following result.

**Theorem 4.8.** There exists a super-simple $(v, \{3, 4\}, 7)$-PBD for $v \equiv 0, 1 \pmod{3}$, $v \geq 9$, except possibly for $\{10, 12, 18, 30, 42\}$.

Case 2. $\lambda = 9$

When $\lambda = 9$, the necessary conditions for a super-simple $(v, \{3, 4\}, 9)$-PBD become $v \geq 11$. We shall prove that such a necessary condition is also sufficient except possibly for $v \in \{12, 16\}$.

**Lemma 4.9.** There exists a super-simple $(v, \{3, 4\}, 9)$-BIBD for $v \equiv 1 \pmod{2}$ and $v \geq 11$.

**Proof.** The conclusion immediately holds by Lemma 1.1.
Lemma 4.10. There exists a super-simple \((v, \{3, 4\}, 9)\)-BIBD for \(v \equiv 0 \pmod{4}\) and \(v \geq 20\).

Proof. The conclusion immediately holds by Lemma 1.2.

Lemma 4.11. There exists a super-simple \((v, \{3, 4\}, 9)\)-PBD for \(v \in M = \{50, 58, 62, 66, 70\}\).

Proof. For \(v \in M\), let \(v = 3g + m\). The two parameters \(g\) and \(m\) are listed in the following table. A super-simple \((\{3, 4\}, 9)\)-GDD of group type \(g^3m^1\) is obtained by deleting \(g - m\) points from the last group of the super-simple TD\(_9(4, g)\) from Lemma 2.1. Applying Construction 2 with \(\eta = 0\), we get a super-simple \((3g + m, \{3, 4\}, 9)\)-PBD. Here, all the input super-simple \((g, \{3, 4\}, 9)\)-PBD and \((m, \{3, 4\}, 9)\)-PBD come from Lemma 4.9.

| \(v = 3g + m\) | \(g\) | \(m\) | \(v = 3g + m\) | \(g\) | \(m\) |
|---------------|-----|-----|---------------|-----|-----|
| 50            | 13  | 11  | 66            | 17  | 15  |
| 58            | 15  | 13  | 70            | 19  | 13  |
| 62            | 17  | 11  |               |      |      |

Lemma 4.12. There exists a super-simple \((v, \{3, 4\}, 9)\)-PBD for \(v \equiv 2 \pmod{4}\) and \(v \geq 74\).

Proof. For \(v \equiv 2 \pmod{4}\) and \(v \geq 74\), it can be written as \(v = 12t + m\), where \(t \geq 5\) and \(m \in \{14, 18, 22\}\). Removing \((4t + 1) - (m - 3)\) points from the last group of a super-simple TD\(_9(4, 4t + 1)\) coming from Lemma 2.1, we get a super-simple \((\{3, 4\}, 9)\)-GDD of group type \((4t + 1)^3(m - 3)^1\). Applying Construction 2 with \(\eta = 0\), we obtain a super-simple \((v, \{3, 4\}, 9)\)-PBD. Here, all these input super-simple \((4t + 1, \{3, 4\}, 9)\)-PBD and \((m - 3, \{3, 4\}, 9)\)-PBD come from Lemma 4.9.

By Lemmas 4.9-4.12 and Lemma 3.3, we have the following result.

Theorem 4.13. There exists a super-simple \((v, \{3, 4\}, 9)\)-PBD for any \(v \geq 11\), except possibly for \(v \in \{12, 16\}\).

Proof of Theorem 1.5. Combine Lemma 4.1, Theorem 4.8 and Theorem 4.13, the proof is completed.

5. Concluding remarks

Super-simple cyclic designs with small values are believed to be useful not only in constructing new larger super-simple cyclic designs, but also in constructing optical orthogonal codes with index two and superimposed codes.

As defined in Chung [22]. A \((v, k, \rho)\) optical orthogonal code (OOC), \(C\), is a family of \((0, 1)\) sequences (called codewords) of length \(v\) and weight \(k\) which satisfy the following two properties (all subscripts are reduced modulo \(v\)).

1. (The Autocorrelation Property)

\[
\sum_{0 \leq t \leq v} x_t x_{t+i} \leq \rho
\]

for any \(x = (x_0, x_1, \ldots, x_{v-1}) \in C\) and any integer \(i \not\equiv 0 \pmod{v}\);
(2) (The Cross-Correlation Property)

\[ \sum_{0 \leq t \leq v} x_i y_{i+t} \leq \rho \]

for any \( x = (x_0, x_1, \ldots, x_{v-1}) \in \mathbb{C} \) and \( y = (y_0, y_1, \ldots, y_{v-1}) \in \mathbb{C} \) with \( x \neq y \), and any integer \( i \).

The parameter \( \rho \) is the index of the OOC. It is well known that the number of codewords of a \((v, k, \rho)\)-OOC can not exceed \( \frac{1}{\rho} \left( \frac{v+1}{2} \times \frac{v+2}{2} \times \cdots \times \frac{v+\rho}{2} \right) \) (Johnson Bound\[30\]). The OOC is said to be optimal when its size reaches this bound.

Suppose that there exists a super-simple \((v, k, \rho)\)-CBIBD. We construct a \((0, 1)\)-sequence of length \( v \) from each of the base blocks of the super-simple CBIBD such that the \( i \)-th position is 1 if and only if \( i \) is an element of the base block. According to the definitions of a super-simple CBIBD and an OOC, it is easy to see that the derived \((0, 1)\)-sequences constitute a \((v, 4, 2)\)-OOC with \( \frac{\lambda(v-1)}{k(k-1)} \) codewords. So we have the following by Lemma 3.2 and Lemma 4.3.

**Theorem 5.1.** There exists a \((v, 4, 2)\)-OOC with \( \frac{7(v-1)}{12} \) codewords for \( v \in \{25, 37, 49, 73\} \).

As stated in Kim and Lebedev\[32\], super-simple designs can also be used to construct superimposed codes. An \( N \times T(0, 1)\)-matrix \( C \) is called a \((w, r)\) superimposed code of size \( N \times T \), if for any pair of subsets \( I, J \subseteq [T] = \{1, 2, \ldots, T\} \) such that \( |I| = w, |J| = r \) and \( I \cap J = \emptyset \), there exists a coordinate \( x \in [N] = \{1, 2, \ldots, N\} \) such that \( c_{xp} = 1 \) for all \( p \in I \) and \( c_{xq} = 0 \) for all \( q \in J \).

The main problem in the study of superimposed codes is to find the minimal length \( N(T; w, r) \) of a \((w, r)\) superimposed code for a given cardinality \( T \). The following result can be found in [32].

**Lemma 5.2.** ([32]) A super-simple \((v, k, \lambda)\)-BIBD is a \((2, \lambda-1)\) superimposed code of size \( N \times v \), where \( N = \frac{\lambda v(v-1)}{k(k-1)} \).

Combining Lemma 3.2, Lemma 4.3 and Lemma 5.2, we have the following result.

**Theorem 5.3.** For any \( v \in \{16, 25, 28, 37, 40, 49, 73\} \), there exists a \((2, 6)\) superimposed code of size \( N \times v \), where \( N = \frac{7v(v-1)}{12} \).

**Acknowledgments**

The authors sincerely thank the editors and the referees for their many valuable comments which help us highly improving the paper.

**References**

[1] R. J. R. Abel and F. E. Bennett, Super-simple Steiner pentagon systems, Discrete Math., 156 (2008), 780–793.
[2] R. J. R. Abel, F. E. Bennett and G. Ge, Super-Simple Holey Steiner pentagon systems and related designs, J. Combin. Designs, 16 (2008), 301–328.
[3] P. Adams, D. Bryant and A. Khodkar, On the existence of super-simple designs with block size 4, Aequationes Math., 51 (1996), 230–246.
[4] T. L. Alderson and K. E. Mellinger, 2-dimensional optical orthogonal codes from singer groups, Discrete Appl. Math., 157 (2009), 3008–3019.
[5] F. Amizade and N. Soltankhah, Smallest defining sets of super-simple 2-(v, 4, 1) directed designs, Utilitas Mathematic, 96 (2015), 331–344.
[6] I. Bluskov, New designs, J. Combin. Math. Combin. Comput., 23 (1997), 212–220.
Further results on the existence of SSPBD with block sizes 3 and 4

Y. Zhang, K. Chen and Y. Sun, Super-simple balanced incomplete block designs with block size 4 and index 9, *J. Statist. Plann. Inference*, 139 (2009), 3612–3624.

Received for publication April 2017.

E-mail address: chenguangzhou0808@163.com
E-mail address: 1432123266@qq.com
E-mail address: zyyctc@126.com

**APPENDIX 1**

\( v = 22 : P : \{0,1,2,3\}, \{0,1,4,5\}, (m,s) = (7,5) \)

\[ R = \{1,2,11,12\}, \{1,11,17,19\}, \{0,10,17,20\}, \{0,6,8,19\}, \{0,1,9,17\}, \{0,1,11,15\}, \{1,3,5,19\}, \{1,12,14,16\}, \{0,3,14,17\}, \{0,2,14,18\}, \{0,8,16,17\}, \{1,3,6,13\}, \{1,2,5,6\}, \{1,3,9,12\}, \{0,10,11,18\}, \{0,13,15,20\}, \{0,15,16,18\}, \{0,5,16,21\}, \{0,10,14,19\}, \{1,3,5,16\}, \{0,9,15,19\}, \{0,6,12\}. \]

\( v = 26 : P : \{0,1,3,7\}, \{0,1,4,5\}, (m,s) = (7,12) \)

\[ R = \{0,6,8,9\}, \{0,12,20,24\}, \{0,11,22,24\}, \{0,4,14,19\}, \{1,6,14,22\}, \{0,2,12,18\}, \{0,4,11,16\}, \{0,16,18,22\}, \{0,9,13,22\}, \{0,12,13,25\}, \{1,2,4,10\}, \{1,8,14,20\}, \{0,3,13,16\}, \{1,3,9\}. \]

\( v = 30 : P : \{0,1,2,3\}, \{0,1,4,5\}, \{0,1,6,7\}, (m,s) = (7,4) \)

\[ R = \{1,12,16,22\}, \{1,7,11,14\}, \{0,2,14,24\}, \{1,14,26,29\}, \{1,5,19,26\}, \{0,1,11,15\}, \{0,11,12,29\}, \{0,10,21,26\}, \{1,4,14,19\}, \{0,2,12,16\}, \{0,3,10,13\}, \{0,9,15,18\}, \{0,11,17,25\}, \{0,4,16,27\}, \{1,17,21,22\}, \{1,15,23,29\}, \{0,6,9,21\}, \{1,8,10,14\}, \{0,16,21,24\}, \{0,9,17,19\}, \{0,18,24,28\}, \{1,11,25,26\}, \{0,2,22,29\}, \{1,7,8,16\}, \{1,3,5,13\}, \{0,8,9,28\}, \{1,2,5,29\}, \{1,6,8,21\}, \{0,3,8,16\}, \{0,11,21,23\}, \{0,3,4,11\}, \{1,5,23\}. \]

\( v = 34 : P : \{0,1,3,7\}, \{0,1,4,6\}, \{0,1,13,20\}, (m,s) = (3,11) \)

\[ R = \{0,1,12,16\}, \{1,12,15,31\}, \{0,12,23,33\}, \{0,2,4,21\}, \{1,18,19,27\}, \{0,12,17,26\}, \{1,13,25,27\}, \{1,3,13,18\}, \{0,17,23,28\}, \{0,10,17,20\}, \{1,9,15,28\}, \{0,8,10,16\}, \{1,5,9,25\}, \{0,9,13,26\}, \{1,2,6,18\}, \{0,3,20,27\}, \{0,4,10\}. \]

\( v = 38 : P : \{0,1,3,7\}, \{0,1,4,5\}, (m,s) = (3,18) \)

\[ R = \{0,4,20,26\}, \{1,6,13,32\}, \{0,14,19,22\}, \{1,10,29,35\}, \{0,10,12,34\}, \{0,4,19,28\}, \{0,23,34,36\}, \{0,17,19,33\}, \{0,4,12,13\}, \{0,10,23,30\}, \{0,4,18,24\}, \{1,8,20,30\}, \{0,2,30,32\}, \{0,12,18,26\}, \{0,16,19,27\}, \{0,1,2,21\}, \{0,17,18,36\}, \{0,10,15,32\}, \{0,3,6,30\}, \{0,4,10\}. \]

\( v = 42 : P : \{0,1,3,7\}, \{0,1,4,5\}, \{0,1,6,9\}, \{0,1,10,12\}, \{0,1,13,15\}, (m,s) = (5,5) \)

\[ R = \{0,2,13,21\}, \{1,5,24,26\}, \{0,23,24,36\}, \{0,18,21,39\}, \{1,15,21,27\}, \{1,19,23,35\}, \{0,14,23,27\}, \{0,10,24,28\}, \{0,2,18,26\}, \{0,9,16,23\}, \{0,22,26,28\}, \{0,10,22,38\}, \{1,3,9,22\}, \{0,15,26,38\}, \{1,11,32,36\}, \{0,15,28,36\}, \{1,11,26,27\}, \{1,15,26,39\}, \{0,10,14,20\}, \{1,8,20,27\}, \{0,6,13,22\}, \{1,12,20,40\}, \{1,13,26,37\}, \{1,14,23,34\}, \{0,9,21,30\}, \{1,5,12,23\}, \{1,3,26,32\}, \{0,13,29,35\}, \{0,4,15,31\}, \{0,3,17,26\}, \{0,8,19,22\}, \{1,7,31\}. \]

\( v = 46 : P : \{0,1,2,3\}, \{0,1,4,5\}, \{0,1,6,7\}, \{0,1,10,14\}, (m,s) = (3,11) \)

\[ R = \{1,11,15,19\}, \{1,13,39,41\}, \{1,9,33,41\}, \{0,5,28,32\}, \{1,10,27,43\}, \{1,11,21,39\}, \{1,23,26,45\}, \{1,13,19,35\}, \{0,7,23,37\}, \{1,4,24,26\}, \{0,10,26,34\}, \{0,12,36,40\}, \{1,14,24,32\}, \{1,24,35,45\}, \{0,5,6,11\}, \{0,14,20,37\}, \{1,23,37,43\}, \{0,23,33,45\}, \{1,7,27,30\}, \{0,12,42,44\}, \{1,28,36,43\}, \{1,31,33,35\}, \{0,7,30,45\}, \{1,3,19\}. \]

\( v = 54 : P : \{0,1,3,7\}, \{0,1,4,5\}, \{0,1,6,14\}, (m,s) = (5,15) \)
\[ R : \{0, 18, 23, 50\}, \{1, 27, 37, 46\}, \{0, 27, 36, 49\}, \{0, 2, 20, 46\}, \{1, 3, 30, 42\}, \{1, 8, 25, 44\}, \{1, 36, 45, 46\}, \{1, 18, 22, 31\}, \{1, 13, 16, 43\}, \{0, 3, 9, 39\}, \{1, 15, 24, 47\}, \{0, 9, 13, 36\}, \{0, 24, 36, 45\}, \{0, 32, 34, 50\}, \{0, 8, 11, 26\}, \{1, 17, 28, 39\}, \{1, 5, 23, 37\}, \{1, 4, 10, 37\}, \{0, 1, 9, 27\}, \{1, 20, 44, 46\}, \{0, 16, 37, 48\}, \{1, 11, 14, 21\}, \{1, 37, 47, 49\}, \{0, 8, 27, 53\}, \{0, 10, 20, 24\}, \{1, 13, 22, 52\}, \{0, 14, 32, 47\}, \{0, 13, 35, 41\}, \{1, 3, 5, 39\}, \{0, 1, 10, 21\}, \{1, 26, 37, 39\}, \{1, 15, 34, 40\}, \{0, 6, 8, 20\}, \{1, 4, 7, 16\}, \{0, 4, 20\}. \]