A gauge theoretical view of the charge concept in Einstein gravity *

Marc Toussaint
Institute for Theoretical Physics, University of Cologne
50923 Köln, Germany
www.thp.uni-koeln.de/~mt/

Abstract

We will discuss some analogies between internal gauge theories and gravity in order to better understand the charge concept in gravity. A dimensional analysis of gauge theories in general and a strict definition of elementary, monopole, and topological charges are applied to electromagnetism and to teleparallelism, a gauge theoretical formulation of Einstein gravity.

As a result we inevitably find that the gravitational coupling constant has dimension \( \hbar/\ell^2 \), the mass parameter of a particle dimension \( \hbar/\ell \), and the Schwarzschild mass parameter dimension \( \ell \) (where \( \ell \) means length). These dimensions confirm the meaning of mass as elementary and as monopole charge of the translation group, respectively. In detail, we find that the Schwarzschild mass parameter is a quasi-electric monopole charge of the time translation whereas the NUT parameter is a quasi-magnetic monopole charge of the time translation as well as a topological charge. The Kerr parameter and the electric and magnetic charges are interpreted similarly. We conclude that each elementary charge of a Casimir operator in the gauge group is the source of a (quasi-electric) monopole charge of the respective Killing vector.

Keywords: gauge theory of gravity, Kaluza-Klein, charge, monopole, mass, Taub-NUT.
1 Introduction

In the fifties, Yang and Mills [14] for the first time formulated the \( SU(2) \)-gauge theory by strictly keeping to the electromagnetic paradigm. At about the same time, Utiyama [13] formulated the general gauge theory of a semi-simple Lie group. These theories, as they explain the electro-weak and strong forces, were supplemented by the great success of particle physics to classify all leptons as representations of the electro-weak symmetry and all hadrons as representations of the flavor symmetry. O’Raifeartaigh [4] gives a more detailed insight into the history of gauge theories. The great success of such theories has also influenced modern formulations of gravity – one of the four fundamental forces which should also be representable in the framework of gauge theory. However, the obvious difference between the external spacetime symmetries and internal symmetries (as considered by Yang and Mills) causes some difficulties for a uniform formulation of all forces. Some ad-hoc assumption (the soldering) solves basic problems but perhaps diminishes the beauty of the theory. We refer to [10] (more introductory [9]) as a general formulation of gravity as a gauge theory (see table 1). For this work it is most important to understand teleparallelism as a gauge theory of translations with the anholonomic coframe \( \vartheta^a \) as gauge potential and torsion \( T^a \) as field strength. With a specific lagrangian, this theory is equivalent to Einstein gravity. This will enable us to reformulate standard Einsteinian solutions in the framework of teleparallelism and thus to interpret the solution parameters as translational charges.

Now, what is a charge? In general is seems plausible to define a charge to be a specific and


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| theory                          | gauge group                  | connection                        | field strength          |
|---------------------------------|------------------------------|-----------------------------------|-------------------------|
| general gauge theory            | semi-simple Lie group $G$    | $A \in \Lambda^1(M, G)$           | $F = D\Gamma \in \Lambda^2(M, G)$ |
| electrodynamics (non-physical)  | $U(1)$                       | $A$                               | $F = dA$                |
| affine gauge theory teleparallelism | soldered affine group | $\Gamma^\alpha_{\beta \rho}$, $\vartheta^\alpha$ | $R^\alpha_{\beta}, T^\alpha$ |
|                                | soldered translations        | $\vartheta^\alpha$                | $T^\alpha = d\vartheta^\alpha$ |

Table 1: Gravity may be described by formulating a gauge theory of the affine group. However, one has to ensure that the group, i.e. the Lie-algebra valued connection, applies to spacetime – is soldered to spacetime. This is done by splitting the connection into a linear part $\Gamma^\alpha_{\beta \rho}$ (with matrix indices $\alpha, \beta$ that work on the basis $e_\alpha$ of the local tangent space) and an inhomogeneous part $\vartheta^\alpha$ (that replaces the holonomic coframe $dx^\alpha$ and thereby realizes a translational gauge). The field strength splits into the curvature $R^\alpha_{\beta}$ and the torsion $T^\alpha$. Discarding the linear gauge ($\Gamma^\alpha_{\beta \rho} \equiv 0$), the theory reduces to teleparallelism.

The invariant property of a particle (usually given by one number, perhaps an integer). Since in gauge theories we take particles to be elements in a representation of the symmetry, we are directly led to the most basic notion of a charge, the elementary charge, classifying the representation of the symmetry the particle is an element of. But also a specific property of the gauge field which a particle necessarily induces may be considered as a charge. Such is, e.g., the monopole character of the electromagnetic field around an electron. This field is induced by the coupling of the electron’s elementary charge to the gauge field. Such could also be the magnetic monopole character of, say, the electromagnetic field around a Dirac monopole. But, since there exists no magnetic-type elementary charge, there is no reason a for particle induce such a field – except for topology. We will see that in the bundle formalism topological effects also motivate this third, topological kind of charge, including the quasi-magnetic monopole charge.

In the following we define these three kinds of charges and apply the definitions on Taub-NUT and Kerr-Newman type solutions of teleparallelism. It will be very satisfying to recognize the Schwarzschild mass parameter as a quasi-electric monopole charge of the time translation and the NUT parameter as a quasi-magnetic monopole charge of the time translation. The Kerr parameter is interpreted similarly. These results are in perfect analogy to monopoles in electromagnetism, they shed light on the dimensions of parameters, and they emphasize the analogy between internal and external gauge theories.

Before, in section 2, we insert a brief dimensional analysis of gauge theories in general. The Kaluza-Klein formulation of electromagnetism makes a comparison with gravity very simple.
2 Dimensional analysis of gauge theories

The essential fields involved in a gauge theory of a Lie group $G$ (with algebra $\mathcal{G}$) are the connection $A$, the field strength $F$, the excitation $H$, the lagrangian $\mathcal{L}$, and the Noether current $\Sigma$. From a geometrical point of view, the connection is introduced as a $\mathcal{G}$-valued 1-form on the principle bundle or, locally, as a $\mathcal{G}$-valued 1-form on spacetime, i.e. $A \in \Lambda^1(M, \mathcal{G})$. It yields the covariant exterior derivative $D = d + A$.

By its very definition, the exterior differentiation operator $d$ is dimensionless, $[d] = 1$. Hence we also require the connection to be dimensionless, $[A] = 1$. Now we need to give exactly two definitions in order to find all the remaining dimensions. First, we choose to define the dimension of a lagrangian $\mathcal{L}$ to be $[\mathcal{L}] = \hbar$. In the context of a classical gauge theory $\hbar$ is merely a name of a dimension as introduced here. However, thinking of Huygen’s principle and the path integral method, one may also call $\hbar$ a phase/2$\pi$ unit. And second, we define the basis elements $\lambda_a$ of the algebra $\mathcal{G}$ to have the dimension $[\lambda_a] = g/\hbar$. Again, so far $g$ is merely a name of a dimension introduced here. However, in the case of electromagnetism, it may be replaced by the unit $e$. Now it is easy to display the dimensions of the components of $A \equiv A^a \lambda_a \equiv A_i^a \lambda_a dx^i$ and $F \equiv F^a \lambda_a \equiv \frac{1}{2} F_{ij}^a \lambda_a dx^i \wedge dx^j$. You will find them in table 2.

In Yang-Mills theories a lagrangian typically describes propagating gauge fields, i.e. it is proportional to a square term of $F$. Here, for generality, we only assume $\mathcal{L} = \langle F \wedge H \rangle = F^a \wedge H_a$, where we introduced the excitation $H$, which is a $\mathcal{G}$-valued 2-form, and the metric $\langle , \rangle$. We read off the dimension of the excitation $H \equiv H^a \lambda_a \equiv \frac{1}{2} H_{ij}^a \lambda_a dx^i \wedge dx^j$ and of the Noether current $\Sigma_a := \delta \mathcal{L}/\delta A^a$. For consistency, the dimension of the metric has to be $\langle [ , ] \rangle = \hbar^2/g^2$. It follows $\langle [\lambda_a, \lambda_b] \rangle = 1$. The dimension of $[H]/[F] = g^2/\hbar$ may be interpreted as the dimension of the coupling constant $1/\kappa$ of a dynamical lagrangian with $H \approx 1/\kappa * F$.

In the case of electrodynamics, we only have one index $a = 0$ and we set $[\lambda_0] = e/\hbar$. We see that the algebra components $F^0, H^0$, and $\Sigma_a$ carry the conventional dimensions, whereas the dimensions of the fields $F, H, \Sigma$ are more unfamiliar. In the case of a translational gauge theory, we assign the dimension $1/\ell$ to the generators (where $\ell$ means length) and find that $[1/\kappa] = \hbar/\ell^2$. Since this dimensionality includes a length dimension perturbation theory does not work. When embedding electrodynamics in an extra dimension à la Kaluza-Klein, the $U(1)$ gauge is directly represented by the translation along the 5th dimension. We can introduce a length unit $\ell_5$ of this 5th dimension by identifying $e/\hbar = 1/\ell_5$. This is a geometrical interpretation of the electric unit $e$ as phase/2$\pi$ per length. Besides, if the 5th dimension is $U(1)$ with perimeter $L_5$, it seems natural that this ‘phase/2$\pi$ per length’-unit $e$ is quantized in quanta of $\hbar/L_5$. Thus, we may assume that the perimeter of $U(1)$ is $L_5 = \ell_5 = \hbar/e$.

We want to point out again that any dimensional system of a gauge theory (as long as
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Table 2: The table displays the dimensions of essential fields and objects involved in a gauge theory. In particular, it gives the SI-units in the case of electrodynamics and the dimensions for a translational gauge theory. We stress that the first two rows in this table are definitions, the third is an identity, and the rest is a consequence. The last block includes the dimensions of monopole and topological charges. The SI-units used in electrodynamics are C=Coulomb and Wb=Weber. We have the lagrangian \( \mathcal{L} \), group generators \( \lambda_a \), gauge potential \( A \), field strength \( F \), excitation \( H \), Noether current \( \Sigma \), algebra metric \( \langle , \rangle \), coupling constant \( 1/\kappa \), quasi-electric and -magnetic charge \( E \) and \( M \), and elementary charge \( I \).

| \( [\mathcal{L}] = F^a \wedge H_a \) | in general | in electrodynamics | in translational gauge theories |
|---------------------------------|-----------|------------------|-------------------------------|
| \( [\lambda_a] \) | \( h \) | \( \text{Wb} \text{ C} \) | \( h \) |
| \( A = A^a \lambda_a \) | \( g/h \) | \( 1/\text{Wb} \) | \( 1/\ell \) |
| \( F = F^a \lambda_a \) | 1 | 1 | 1 |
| \( H = H^a \lambda_a \) | \( g^2/h \) | \( \text{C}/\text{Wb} \) | \( h/\ell^2 \) |
| \( A^a = A_i^a \ dx^i \) | \( h/g \) | \( \text{Wb} \) | \( \ell \) |
| \( F^a = \frac{1}{2} F_{ij}^a \ dx^i \wedge dx^j \) | \( h/g \) | \( \text{Wb} \) | \( \ell \) |
| \( H^a = \frac{1}{2} H_{ij}^a \ dx^i \wedge dx^j \) | \( g \) | \( \text{C} \) | \( h/\ell \) |
| \( \Sigma_a = \delta \mathcal{L}/\delta A^a \) | \( g \) | \( \text{C} \) | \( h/\ell \) |
| \( \langle , \rangle = 1/[\lambda]^2 \) | \( h^2/g^2 \) | \( \text{Wb}^2 \) | \( \ell^2 \) |
| \( [F,H] \) | \( h \) | \( \text{Wb} \text{ C} \) | \( h \) |
| \( 1/\kappa = [H]/[F] = [H^a]/[F^a] \) | \( g^2/h \) | \( \text{C}/\text{Wb} \) | \( h/\ell^2 \) |
| \( \mathcal{E} = [\mathcal{M}] = [F] \) | 1 | 1 | 1 |
| \( \mathcal{E}^a = [\mathcal{M}^a] = [F^a] \) | \( h/g \) | \( \text{Wb} \) | \( \ell \) |
| \( \mathcal{I} \) | \( g \) | \( \text{C} \) | \( h/\ell \) |

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All generators have the same dimension) may be spanned by exactly two definitions, e.g. those for \([\mathcal{L}]\) and \([\lambda_a]\). This is the reason why every column in table 2 includes exactly two dimensions (or units).

Finally we note that the dimension of the hodge star * in \( n \) dimensions when applied on a \( p \)-form is \([*] = l^{n-2p}\).
3 Charge definitions

3.1 Monopole charges

We start by defining two types of monopole charges. These are properties of the gauge configuration given by the gauge field strength $F$:

\[ E := \lim_{r \to \infty} \frac{1}{4\pi} S^{n-2}(r) \oint S^2(r) F \] quasi-electric monopole charge, \hfill (1)

\[ M := \lim_{r \to \infty} \frac{1}{4\pi} S^2(r) \oint F \] quasi-magnetic monopole charge. \hfill (2)

The motivation for the definition of $E$ is obvious from the analogy to Maxwell’s inhomogeneous equation. The definition of $M$ may be motivated by including magnetic charges in Maxwell’s theory. Usually this is done by modifying the homogeneous Maxwell equation and introducing a source term on its rhs: $dF = \rho_{\text{mag}}$. But we think it is preferable to interpret $M$ as the topological invariant associated with the first Chern character class $[F]$ in the second cohomology (see below). With this we don’t need to introduce magnetic source terms into the homogeneous Maxwell equation but rather interpret magnetic monopoles as a topological feature – which one may visualize as a Dirac string [7] or rather accept as a feature of a $U(1)$-bundle (see figure 1). We choose the nomenclature quasi-electric and -magnetic to remind us of the analogies with electromagnetism. Since these definitions are general and not restricted to theories of gravitation, we do not choose the names gravi-electric and -magnetic.

3.2 Topological charges

One principle of topology is comparing manifolds by continuously deforming them. If two manifolds can continuously be deformed into each other, they are said be homeomorph. In topology one is mainly interested in the equivalence classes of homeomorphic manifolds. It turns out that there are three important ways of classifying manifolds: First, by identifying all homeomorphic manifolds with a set of simplices that are glued together (homology). Second, by considering those forms on the manifold that are closed but not exact (cohomology). In some way (recalling the Stokes theorem) it is not surprising that these two ways of classification are equivalent (de Rham theorem). And third, by considering maps merging a topologically well understood manifold (usually a $r$-sphere) into the manifold in question (homotopy). We take [3] as a reference for topology.

For us the second way, i.e. considering the cohomology group $H^r(M)$ of $r$-forms over $M$ that are closed but not exact, is very interesting. The Chern-Weil theorem enables to construct forms out of the gauge field on a fibre bundle that are closed and of which the exactness does
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Not depend on the gauge field. Such a form represents one element of the cohomology group independent of the gauge field. Hence, this element of the cohomology group indicates an invariant (under gauge transformations) topological feature of the bundle. One of these indicators, namely the first Chern character class, may be used to define magnetic monopoles. We will have a closer look on the Chern-Weil theorem below.

But also the third way of classifying the topology of $M$, i.e. considering the equivalence classes $\pi_r(M)$ of maps merging an $r$-sphere into $M$, is very helpful. A theorem proved by Steenrod and Pontrjagin (see, e.g., [2] page 75) states that all $G$-bundles over the base space $S^2$ can be classified by $\pi_1(G)$. Since the world-path of a monopole is a singularity, the topology of spacetime in the presence of a monopole is $\mathbb{R}^4 \setminus \{\text{world-path}\} \sim S^2 \times \mathbb{R}^+ \times \mathbb{R} \sim S^2$, i.e. spacetime has the topology of $S^2$. Hence, all $G$-bundles over spacetime can be classified by $\pi_1(G)$. In the case of electromagnetism we have $G = U(1)$ and $\pi_1(U(1)) = \mathbb{Z}$, and all gauge field configurations may be characterized by an integer number. This tells us that, in general, there do exist topologically non-trivial gauge configurations in electrodynamics.

Here, we define two topological charges:

$$\mathcal{C}_I := \lim_{r \to \infty} \frac{1}{4\pi} S^2(r) \times I \int (A \wedge F) \quad \text{Chern-Simons charge,} \quad (3)$$

$$\mathcal{P} := \frac{g^2}{4\pi \hbar} \mathbb{R}^4 \int (F \wedge F) \quad \text{Pontrjagin charge.} \quad (4)$$

Both charges are fruits of the Chern-Weil theorem which states that these are topological invariants. We remind the reader of the essential ideas of this theorem, for details see

Figure 1: The field strength of the Dirac monopole [7] $F = p\, d\Omega = p\, \sin\theta\, d\theta \wedge d\varphi$ has no global potential $A$ with $F = dA$. Dirac concluded that such a monopole must have a string (slice in spacetime) attached to it. If we slice spacetime along the negative $z$-axis, say, $F$ has a regular potential $A = p\, (1 - \cos\theta)\, d\varphi$. Alternatively, electromagnetism may be formulated as a gauge theory on a $U(1)$ bundle over spacetime. Topologically the spacetime around the (singular) monopole world path is $(\mathbb{R}^{3}_{\text{space}} \setminus \{o\}) \times \mathbb{R}_{\text{time}} \sim S^2$, where $o$ denotes the monopole’s location. Hence, all field configurations may be classified topologically by investigating $U(1)$ bundles over $S^2$. It turns out that an integer number (the magnetic charge) classifies all field configurations. The Moebius strip ($[0, 1]$ bundle over $S^1$) allows to visualize a topologically non-trivial bundle.
3. **Charge Definitions**

First, consider the curvature $F \in \Lambda^2(P, G)$ on a principle bundle $P$ over the base manifold $M$ and formulate polynomials $P(F)$ of this curvature. Then, search for such polynomials that are invariant under the adjoint action of the structure group $G$, i.e.

$$\forall g \in G : \ P(\text{Ad}_g F) = P(F).$$

Given such an invariant polynomial of $r$-th order, the Chern-Weil theorem states the following:

(i) $P(F)$ is closed, i.e. $dP(F) = 0$. Hence, we found an element of the $2r$-th cohomology group $[P(F)] \in H^{2r}(M)$. Here, $[P(F)]$ denotes the equivalence class of all $2r$-forms that differ from $P(F)$ only by an exact form. $[P(F)]$ is called characteristic class. Note that each monomial in this polynomial is also invariant.

(ii) If we have two curvatures $F$ and $F'$ on the same bundle it follows that $[P(F)] = [P(F')]$. This means that the characteristic class $[P(F)]$ is independent of $F$ and depends only on the topology of the bundle. It is a topological invariant.

(iii) Since $P(F)$ is closed, we find a local potential on a subset $U$ of $M$: $P(F) = dQ|_U$. It follows that $[Q]$ is an element of the $(2r-1)$-th cohomology $H^{2r-1}(\partial U)$ and is thus a topological invariant of $\partial U$. $Q$ is called Chern-Simons form.

In fact, we find the invariant polynomials (or monomials) $P_1(F) = F$ and $P_2(F) = \langle F \wedge F \rangle$, the first of which is called 1st Chern character term and the second 1st Pontrjagin term. We also find the Chern-Simons form $\langle A \wedge F \rangle$ of the 1st Pontrjagin term.

Hence, the 1st Chern character class $[F]$ is an element of the 2nd cohomology. The integration of $F$ over a closed 2-plane $S^2$, i.e. the quasi-magnetic monopole charge $\mathcal{M}$, thus leads to a number that specifies the cohomology class.

Similarly, the Chern-Simons form $\langle A \wedge F \rangle$ of the 1st Pontrjagin term is an element of the 3rd cohomology and we need a closed 3-plane for integration. In the case of a singular monopole world path in a $U(1)$ bundle, a natural choice for this 3-plane is $S^2 \times U(1)$, with $|U(1)| = \ell_5$. The integration $\mathcal{C}_{U(1)}$ of the Chern-Simons form over this plane thus leads to a finite number classifying the cohomology class. Analogously we have a second choice $I = \mathbb{R}_{\text{time}}$ to form a 3-plane $S^2 \times I$. However, this plane is not compact and will not lead to a finite number. We solve this problem by restricting $I$ to a finite time interval $I_T$ with $|I_T| = T$. Still, the 3-plane $S^2 \times I_T$ is not closed and, strictly speaking, $\mathcal{C}_{I_T}$ may not be considered a topological invariant. Thus we have to act with some caution.

The 1st Pontrjagin class $[(F \wedge F)]$ is determined by the integration of $\langle F \wedge F \rangle$ over a 4-plane – which we always consider to be spacetime. We will apply this definition in the context of a translational gauge theory, i.e. a geometry with torsion $T$. Thus, it is very instructive to note that the ‘translational Pontrjagin term’ $\langle T \wedge T \rangle$ is equivalent to the Nieh-Yan term

$$\mathcal{N} = T^\alpha \wedge T_\alpha - R_{\alpha\beta} \wedge \varphi^\alpha \wedge \varphi^\beta$$

in the case of vanishing curvature. The Nieh-Yan term may be produced by splitting the 5-dimensional Pontrjagin term of a deSitter-like $SO(5)$ gauge theory (via some inverse Inou-Wigner contraction) into the 4-dimensional $SO(4)$ Pontrjagin term and the rest. This
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rest refers to the translations and is, in fact, the Nieh-Yan term (5). This was illuminated by Chandia and Zanelli [6].

3.3 Elementary charges

One of the most beautiful things in physics is the success of particle physics in classifying particles with the help of representation theory for groups. This algebraic approach simply postulates that objects in nature must be an element of a representation of some symmetry. Objects (particles or states) that are inseparable are called elementary. This notion turns out to coincide with the mathematical notion of irreducibility. Both mean inseparable without losing the symmetry (or a faithful representation of it).

With elementary charge we denote those invariants that classify a particle, i.e. the irreducible representation the particle is an element of. Such a classification can be performed by finding all Casimir operators in the group algebra. These are polynomials of the group generators and commute with every group element. Hence, their eigenvalues, when applied on some particle field, are invariant under all symmetry transformations.

The Poincaré group, for example, has the Casimir operators

\[ C_1 := P_\alpha P^\alpha, \]
\[ C_2 := W_\alpha W^\alpha \quad \text{with} \quad W_\alpha := \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} L^{\beta\gamma} P^\delta. \]

Here, the translation operator \( P^\alpha \) represents the particle momentum, \( L^{\alpha\beta} \) are the generators of Lorentz rotations, and the so-called Pauli-Lubanski vector \( W^\alpha \) represents the particle spin. If nature incorporates the Poincaré symmetry, all particles can be classified by eigenvalues of \( C_1 \) (mass square) and \( C_2 \) (spin square). The classification with respect to their mass is guaranteed by the Dirac equation (for the Dirac spinor representation) or the Klein-Gordon equation (for the scalar representation). All these equations require the dimension \( \hbar/\ell \) for the mass parameter. (We take \( c = 1 \).)

In general, if the Casimir operator \( C \) is a polynomial of \( r \)-th order of the group generators and if \( T^r \) is an invariant eigenvalue of \( C \), i.e. \( (\hbar^r C - T^r) \phi = 0 \) for some eigenvector \( \phi \), then we call \( T \) an elementary charge. If we assume that \( C \) is built from generators with dimension \( [\lambda_a] \), the dimension of \( T \) is \( [T] = \hbar [\lambda_a] \). This leads to the remarkable relation between the dimension of an elementary charge and that of a monopole charge (cf. table 2):

\[ [T] = [1/\kappa] [E^a]. \]
Three further comments on mass and electromagnetic charges

(1) Electric charge may as well be understood as an elementary charge of the single $U(1)$-generator $P_5$, which is, of course, a Casimir operator. To see this, decompose the $u(1)$-valued connection 1-form into $A = A^5 P_5$ (with $A^5$ having the conventional dimension of Weber). The generator $P_5$ acts trivially on non-charged functions $P_5 \cdot \psi = 0$ but has any charged function as eigenstate $P_5 \cdot \psi = e \psi$ with the elementary charge $e$. The covariant derivative applied on the wave function of an electron, say, reads $D\psi = d\psi + A \cdot \psi = d\psi + A^5 P_5 \cdot \psi = d\psi + e A^5 \psi$, as we are used to write it. This coupling of the elementary charge to the gauge field induces the electric monopole character of the electromagnetic field. Hence, the electric charge density $\rho$ may be understood as elementary charge density.

A magnetic monopole character, though, cannot be induced by an elementary charge since there exists no second, magnetic-type Casimir operator. Hence, some $\rho_{\text{mag}}$ on the rhs of the inhomogeneous Maxwell equation may merely be understood as a density of topological defects, but not as elementary charge density.

(2) The dimension of the mass parameter $[m] = \hbar/\ell$ may be called phase/2\pi per length. In fact, the most obvious argument for this interpretation is the point particle action $\int m ds$. In this picture, if you identify a world path with a strap, then mass is the twist of this strap per length. Also note that $\lambda_c = \hbar/m$ is the Compton wave length of the particle.

(3) Since in 5D Kaluza-Klein space the electric charge $q$ is just as well an eigenvalue of the Casimir operator of the translation along the 5th dimension, electric charge is very similar to mass. Just as mass measures the horizontal (spacetime) momentum, the electric charge measures a vertical (fibre) momentum. In fact, Bleecker [1] defined electric charge as the ‘vertical velocity’ of a point particle path on a $U(1)$-bundle.

4 Translational monopole charges in gauge theories of gravity

We can now apply the charge definitions to analyze standard solutions of gauge theories of gravity for monopole charges. First, we concentrate on a subclass of the Plebanski-Demianski class of solutions including the Kerr-Newman and Taub-NUT solutions. For the monopole analysis we formulate them as a solution of a translational gauge theory of gravity, namely teleparallelism, and find quasi-electric and quasi-magnetic monopoles in the gauge of some translations, indeed. Later, we also investigate two solutions of the Poincaré gauge theory.

Before we start we should point out that the following would hardly have been possible without the use of the computer algebra system Reduce and its supplementary package Excalc. The calculations for the monopole analysis are rather straightforward but very
The notation might confuse at first. It is the direct analogue of the notation Plebanski and Demianski used in their paper [12]. It has a clear structure and can easily be modified into other solutions of the Plebanski-Demianski class. The metric solves the coupled Einstein-Maxwell equations if we choose the electromagnetic potential
\[ A = \frac{1}{\Delta^2} (q r d\tau + p \cos \theta d\sigma) . \]
This potential is the analog of the potential \( A = q/r dt + p \cos \theta d\varphi \) of an electric and magnetic charge in flat spacetime. For the monopole analysis, we translate this solution into a 5D Kaluza-Klein-type teleparallelism. This simply means that we add a 5th dimension that represents the electromagnetic part of the theory:
\[ g = \vartheta^0 \otimes \vartheta^0 - \vartheta^1 \otimes \vartheta^1 - \vartheta^2 \otimes \vartheta^2 - \vartheta^3 \otimes \vartheta^3 - \vartheta^5 \otimes \vartheta^5 , \]
\[ \vartheta^5 = dx^5 + \frac{1}{\Delta^2} (q r d\tau + p \cos \theta d\sigma) . \]

The 5th covector \( \vartheta^5 \) represents the gauge of the 5th translation, i.e. the electromagnetic gauge potential. The field strength of this gauge theory is the torsion \( T^a = d\vartheta^a \). The configuration solves the vacuum field equation \( dH^a = 0 \) of the teleparallelism theory. Here, \( H^a \) is the excitation of the translational gauge and is composed out of the three irreducible pieces of \( T^a \) such that the theory is equivalent to 5D Einstein gravity:
\[ H^a = \frac{1}{\kappa} \star (T^a - 3^{(2)} T^a + \frac{5}{2}^{(3)} T^a) \quad \text{or} \quad H^a = -\frac{1}{2} K^{\mu \nu} \wedge \eta_{\mu \nu} , \]
where \( K^{\mu \nu} \) is the contortion. For details see [10] or [9].

The following charges for this gauge configuration are calculated by the file kerrnut.exi [15] with parameters \((m, j, q, p)\):
\[ E = -m \partial_t - j^4 \partial_\varphi + q \partial_5 , \quad M = -p \partial_5 , \quad C_{U(1)} = -p \ell_5 , \quad C_{T} = 0 , \quad \mathcal{P} = 0 . \]
Consider $\mathcal{E}$ and note that we have a quasi-electric monopole charge $\mathcal{E}^t = -m$ in the time translation, a quasi-electric monopole charge $\mathcal{E}^\varphi = -j\frac{\pi}{4}$ in the translation along $\partial_\varphi$ (which is actually a rotation and the charge represents an angular momentum)\(^1\), and a (quasi-)electric monopole charge $\mathcal{E}^5 = q$ in the translation along $\partial_5$ (i.e. the $U(1)$ gauge of electrodynamics). In this solution all Killing vectors carry quasi-electric monopole charges. In fact, it seems quite plausible that the elementary charges of the three Casimir operators (momentum square, Pauli-Lubanski square, and the 5th translation) are the sources of the quasi-electric monopole charges of the Killing vectors that correspond to these Casimirs in a stationary geometry. As we are interested in dimensions, we find that the mass parameter has dimension $[m] = \ell$, the angular momentum per mass unit has dimension $[j] = 1$, and, if we measure the length along the 5th dimension in units of $\ell_5$, the electric charge has dimension $[q] = \ell_5$. In the previous dimensional discussion of electrodynamics, we defined $1/\ell_5 = e/\hbar$ and $[1/\kappa] = e^2/\hbar = h/\ell_5^2$. Hence, our results are consistent with eq. (8): The dimension of the elementary charge $[\mathcal{Z}] = e = h/\ell_5$ is equal to the coupling constant $[1/\kappa]$ times the dimension of the quasi-electric monopole charge $[\mathcal{E}^5] = [q] = \ell_5$. The same holds for the mass.

Considering $\mathcal{M}$ we are not surprised that $\mathcal{M}^5 = -p$ is a (quasi-)magnetic monopole charge of the 5th translation. The non-trivial Chern-Simons form $C_{U(1)}$ confirms the topological feature of magnetic monopoles in the $U(1)$-bundle.

### 4.2 The Taub-NUT solution

Let us turn to the Taub-NUT solution with mass parameter $m$, NUT parameter $n$, and electric charge $q$. Within the previous notation, i.e. with the coframe and metric defined in (14, 10, 15), the solution reads

\[
\begin{align*}
  d\tau &= dt - 2n \cos \theta \, d\varphi, \quad d\sigma = (r^2 + n^2) \, d\varphi, \\
  p &= 0, \quad \mathcal{Q}^2 = r^2 - 2mr - n^2 + q^2/4, \quad \mathcal{P} = \sin \theta, \quad \Delta^2 = r^2 + n^2. 
\end{align*}
\]

The result of the monopole analysis has been calculated with the program *kerrnut.exi* \[^1\] with parameters $(m, n, q)$:

\[
\begin{align*}
  \mathcal{E} &= -m \partial_t + q \partial_5, \quad \mathcal{M} = -2n \partial_t, \quad C_{U(1)} = 0, \quad C_{T^r} = -2n \, T, \quad \mathcal{P} = 4n - \frac{q^2}{2n}.
\end{align*}
\]

This clearly presents the NUT parameter $n$ as a quasi-magnetic monopole charge of the time translation. Table 3 gives another illustration of these results.

\[^1\] Usually, one associates a *grav-magnetic* or *gravito-magnetic* effect with the gravitational field of the Kerr solution. This is sensible since the rotating mass produces a field that is in analogy to the magnetic field produced by rotating electrons. However, rotating mass is not an analogue to a magnetic monopole. Instead, our calculation definitely proves that it is rather in analogy to an electric monopole – but with respect to the gauge of translations along the Killing vector $\partial_\varphi$. 

5 RELATING TO OTHER FORMALISMS

| Electric Monopole | Schwarzschild Solution |
|-------------------|------------------------|
| $A = -\frac{q}{r} dt$ | $Y^0 = \vartheta^0 - dt = \left(\sqrt{1 - \frac{2m}{r}} - 1\right) dt \to -\frac{m}{r} dt$ |
| $F = -\frac{q}{r^2} dt \wedge dr$ | $T^0 \to -\frac{m}{r} dt \wedge dr$ |

| Magnetic Monopole | Taub-NUT Solution |
|-------------------|-------------------|
| $A = p(1 - \cos \theta) d\varphi$ | $Y^0 = \vartheta^0 - dt \to 2n(1 - \cos \theta) d\varphi$ |
| $F = dA = pd\Omega$ | $T^0 \to 2n d\Omega$ |

Table 3: The table compares the electric monopole with the Schwarzschild solution and the Dirac monopole with the Taub-NUT solution. The gravitational solutions are presented in a teleparallel formalism. Arrows $\to$ mean the limit $r \to \infty$. The analogies between the electro-magnetic field strength $F$ and the field strength of time translation $T^0$ confirm our interpretation of the mass parameter $m$ and the NUT parameter $n$. The identification of $\vartheta^0 - dt$ with the gauge potential of time translation $Y^0$ takes soldering into account.

5 Relating to other formalisms

In this short section we will display the relation of our analysis to more conventional ones. Lynden-Bell et al. [11], e.g., wrote a detailed review on monopoles in gravity and also discussed the magnetic nature of NUT-space. Their considerations are based on the following definitions of the gravo-electric and -magnetic fields. They point out that a time-like Killing vector is necessary for this definition and hence they consider the general stationary metric (cf. [11] eq (3.1))

$$g = f^2 (dx^0 - A_idx^i)^2 - \gamma_{ij} dx^i dx^j ,$$

where $i = 1, 2, 3$ and $A_i$ and $\gamma_{ij}$ are arbitrary. Motivated by the expression of the force on a test particle with rest mass $m_0$ and velocity $\vec{v}$ in this geometry (cf. [11] (3.2))

$$\vec{f} = \frac{m_0}{\sqrt{1 - v^2/c^2}} \left[ \left( -\frac{1}{f} \nabla f \right) + \frac{v}{c} \times \left( f \text{ curl } A \right) \right] ,$$

and its formal analogy to the electromagnetic Lorentz force, they define the gravo-electric and -magnetic fields as (cf. [11] (3.3,3.4))

$$\vec{E} := -\frac{1}{f} \nabla f ,$$

$$\vec{B} := \text{curl } \vec{A} .$$

We can now give another interpretation of these definitions by reproducing them in our teleparallel formalism. The metric (21) is replaced by the coframe with

$$\vartheta^0 = f (dx^0 - A) ,$$
6  SUMMARY AND DISCUSSION

together with three spatial covectors \( \vartheta^i \) that are of no further interest. We introduced the space-like 1-form \( A = A_i \, dx^i \). Since in table 3 we notice a close relation between Newton’s force and the time component of torsion \( T^{\hat{0}} \), we calculate

\[
T^{\hat{0}} = d\vartheta^{\hat{0}} = df \wedge (dx^{\hat{0}} - A) - f \, dA = \frac{1}{f} df \wedge \vartheta^{\hat{0}} - f \, dA.
\]

(26)

Following the conventional space-time decomposition of the electromagnetic force we split this field strength of time translation into an electric and magnetic part:

\[
T^{\hat{0}} = -(E \wedge \vartheta^{\hat{0}} + B),
\]

(27)

\[
E := \frac{1}{f} \, df,
\]

(28)

\[
B := f \, dA.
\]

(29)

Thereby we reproduced the definitions (23,24) up to the factor \( f \) in \( B \). However, looking at the force (22) it seems more consistent to include this factor \( f \) in \( B \) in order to arrive at the conventional expression for the Lorentz force. We conclude that the conventional formalism presented by Lynden-Bell et al. is equivalent to our investigation in monopoles in the time-component of torsion \( T^{\hat{0}} \). However, their formalism is non-covariant at its very basis, it is insufficient to discuss monopole charges in other translations (e.g. the Kerr parameter as quasi-electric monopole charge in the translation along \( \partial_5 \)), and it does not allow to identify quasi-magnetic charges with Chern-Simons charges in the way we did. Finally, we cite the interesting statement of Rindler [5] section 8.12 according to which the minus sign in (27) – which is the only difference to the electromagnetic paradigm – is due to the attractive nature of the gravitational force.

6  Summary and discussion

The main results of this article are the dimensions summarized in table 2, the charge definitions (1-4), the dimensional relation (8) between elementary and monopole charges, and the explicit presentation of the charges (17) and (20) for the Kerr-Newman and Taub-NUT solution in the teleparallel formulation, respectively. Table 4 summarizes the interpretation of these charges. All this has only been possible because of the gauge theoretical formulation of gravity and stresses the analogies between internal and external gauge theories. Finally, we want to emphasize the following points:

(1) As we discussed in section 5, the (gravo-) electric and magnetic nature of the Schwarzschild and Taub-NUT solution, respectively, can also be pointed out in the Riemannian formulation of gravity. However, in the teleparallel formalism we arrived to recover the
Table 4: The correspondence between Casimir operators, Killing vectors, and monopole charges in the Plebanski-Demianski class of solutions. The three columns to the right refer to a stationary (and spherically symmetric) geometry. We have the Schwarzschild mass parameter $m$, Taub-NUT parameter $n$, Kerr parameter $j$, acceleration parameter $a$, electric charge $q$, and magnetic charge $p$.

Schwarzschild mass parameter and the NUT-parameter as monopole charges of the time-translation. First, this explains why Lynden-Bell et al. need to assume a time-like Killing vector for their definitions of gravo-electric and -magnetic fields, and second, this uncovers the analogy between those charges and charges of other translations, namely those along $\partial_\varphi$ and $\partial_5$. Furthermore, our definitions (1,2) have the advantage to be covariant.

(2) We proved that in the Plebanski-Demianski class of solutions [12] (when reformulated as teleparallel solutions) the five parameters $m$, $n$, $q$, $p$, and $j$ may be related to monopole charges. Unfortunately, we could not confirm the same for the acceleration parameter $a$. The reason might be the topologically non-trivial coordinate transformation eq (4.4) in [12]. However, for consistency we may expect that $a$ relates to a quasi-magnetic charge of the translation along $\partial_\varphi$. Assuming this, we agree with Plebanski and Demianski on their ordering of the parameters: The six parameters should be ordered as three pairs $(m, n)$, $(j, a)$, and $(q, p)$ each pair of which belongs to the time translation, the translation along $\partial_\varphi$, and the $U(1)$-translation, respectively. In each pair the first parameter denotes the quasi-electric charge and the second parameter the quasi-magnetic charge of these translations.

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References

[1] D. Bleecker: *Gauge theory and variational principles*. Addison-Wesley, London (1981). (Global Analysis, Pure and Applied, series A, no. 1)

[2] H. Chan, S.T. Tsou: *Some elementary gauge theory concepts*. World Scientific, Singapore (1993). (World Scientific Lecture Notes in Physics 47)
[3] M. Nakahara: *Geometry, topology, and physics*. Adam Hilger, Bristol (1990).

[4] L. O’Raifeartaigh: *The dawning of gauge theory*. Princeton Univ. Press (1997).

[5] W. Rindler: *Essential relativity*. Springer-Verlag, New York, 2nd edition (1997).

[6] O. Chandia, J. Zanelli: Topological invariants, instantons and chiral anomaly on spaces with torsion. Phys. Rev. **D55** (1997) 7580-7585, Los Alamos e-Print Archive hep-th/9702025.

[7] P.A.M. Dirac: Quantized singularities in the electromagnetic field. Proc. Roy. Soc. Lond. **A133** (1931) 60-72.

[8] P.A.M. Dirac: The theory of magnetic poles. Phys. Rev. **74** (1948) 817-830.

[9] F. Gronwald: Metric-affine gauge theory of gravity: I. Fundamental structure and field equations. Int. Jour. Mod. Phys. **D6** (1997) 263-303.

[10] F.W. Hehl, J.D. McCrea, E.W. Mielke, Y. Ne’eman: Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance. Phys. Rep. **258** (1995) 1-171.

[11] D. Lynden-Bell, M. Nouri-Zonoz: Classical monopoles: Newton, NUT space, gravomagnetic lensing, and atomic spectra. Rev. of Mod. Phys. **70** (1998) 427-445.

[12] J.F. Plebanski, M. Demianski: Rotating, charged, and uniformly accelerating mass in general relativity. Annals of Phys. **98** (1976) 98-127.

[13] R. Utiyama: Invariant theoretical interpretation of interaction. Phys. Rev. **101** (1956) 1597-1607.

[14] C.N. Yang, R.L. Mills: Conservation of isotopic spin and isotopic gauge invariance. Phys. Rev. **96** (1954) 191-195.

[15] Internet reference of this paper: http://www.thp.uni-koeln.de/~mt/work/1999charge/index.html