Quantization of spherically symmetric solution of SU(3) Yang-Mills theory

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Abstract

A recent investigation of the SU(3) Yang-Mills field equations found several classical solutions which exhibited a type of confinement due to gauge fields which increased without bound as \( r \to \infty \). This increase of the gauge fields gave these solutions an infinite field energy, raising questions about their physical significance. In this paper we apply some ideas of Heisenberg about the quantization of strongly interacting, non-linear fields to this classical solution and find that at large \( r \) this quantization procedure softens the unphysical behaviour of the classical solution, while the interesting short distance behaviour is still maintained. This quantization procedure may provide a general method for approximating the quantum corrections to certain classical field configurations.

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I. INTRODUCTION

Recently [1] several classical solutions to SU(3) Yang-Mills theory were discussed, which possessed either spherical or cylindrical symmetry. These solutions had gauge fields which tended toward $\infty$ at large distances, leading to a type of confining behaviour if one considered these solutions as background fields in which some test particle moved. These increasing gauge fields also led these solutions to the undesired property of having infinite field energy. One way in which these classical field configurations might nevertheless have some physical importance is if the quantization of these solutions reduced or eliminated the bad long distance behaviour. While perturbative quantization techniques work well for weakly interacting field theories such as $QED$ (or $QCD$ in the high energy limit), they are not useful when dealing with strongly interacting field theories. In Ref. [2] we applied some ideas of Heisenberg’s concerning the quantization of strongly interacting, non-linear fields to the cylindrical solution discussed in Ref. [1], and found that under certain assumptions the bad long distance behaviour of this solution was eliminated. Here we apply the same procedure to the spherical symmetric solution and show that again the bad long distance behaviour is eliminated. In addition to the specific benefit that the Heisenberg quantization method gives to the infinite energy solutions discussed here and in Ref. [2], it may provide some general procedure for approximating the quantum corrections to certain classical field configurations.

II. SPHERICALLY SYMMETRIC ANSATZ

We will briefly review the derivation and discuss some aspects of the spherically symmetric solution. The ansatz for the $SU(3)$ gauge field we take as in [3] [4] [5]:

$$A_0 = \frac{2\varphi(r)}{ir^2} \left( \lambda^a_0 x^a - \lambda^b x^b + \lambda^c x^c \right) + \frac{1}{2} \lambda^a \left( \lambda^b_{ij} + \lambda^b_{ji} \right) \frac{x^i x^j}{r^2} w(r), \quad (1a)$$

$$A_i^a = \left( \lambda^a_{ij} - \lambda^a_{ji} \right) \frac{x^j}{ir^2} (f(r) - 1) + \lambda^a_{jk} \left( \epsilon_{ikj} x^k + \epsilon_{ikj} x^j \right) \frac{x^l}{r^3} v(r), \quad (1b)$$
here $\lambda$ are the Gell-Mann matrices; $a = 1, 2, \ldots, 8$ is a color index; the Latin indices $i, j, k, l = 1, 2, 3$ are the space indices; $i^2 = -1$; $r, \theta, \phi$ are the usual spherically coordinates.

Substituting the ansatz of Eqs. (1) (with $f = \phi = 0$) into the Yang-Mills equations

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} F^{a\mu}_\nu \right) + f^{abc} F_{b\mu} A^c_\mu = 0,$$

(2)

yields the following complex set of coupled, non-linear differential equations

$$r^2 f'' = f^3 - f + 7 fv^2 + 2 vw \phi - f \left( w^2 + \phi^2 \right),$$

(3a)

$$r^2 v'' = v^3 - v + 7 vf^2 + 2 fw \phi - v \left( w^2 + \phi^2 \right),$$

(3b)

$$r^2 w'' = 6w \left( f^2 + v^2 \right) - 12fv \phi,$$

(3c)

$$r^2 \phi'' = 2\phi \left( f^2 + v^2 \right) - 4fvw.$$

(3d)

For the solution with increasing gauge fields we specialized by taking $f = \phi = 0$ (the case where $v = w = 0$ is similar) gives the following set of non-linear coupled equations

$$r^2 v'' = v^3 - v - vw^2,$$

(4a)

$$r^2 w'' = 6vw^2.$$  

(4b)

Near $r = 0$ we took the series expansion form for $v$ and $w$ as

$$v = 1 + v_2 \frac{r^2}{2!} + \ldots,$$

(5a)

$$w = w_3 \frac{r^3}{3!} + \ldots,$$

(5b)

where $v_2, w_3$ were constants which determined the initial conditions on $v$ and $w$ as in the last section. In the asymptotic limit $r \to \infty$ the form of the solutions to Eqs. (4) approaches the form

$$v \approx A \sin \left( x^\alpha + \phi_0 \right),$$

(6a)

$$w \approx \pm \left[ \alpha x^\alpha + \frac{\alpha - 1}{4} \cos \left( 2x^\alpha + 2\phi_0 \right) \right],$$

(6b)

$$3A^2 = \alpha(\alpha - 1).$$

(6c)
where \( x = r/r_0 \) is a dimensionless radius and \( r_0, \phi_0, \) and \( A \) are constants. The second, strongly oscillating term in \( w(r) \) is kept since it contributes to the asymptotic behaviour of \( w'' \). We did not find an analytical solution for the system of Eqs. (4), but it is straightforward to solve these equations numerically with any standard differential equation package such as that available in Mathematica [7]. Fig. 1 shows a representative solution to Eqs. (4).

The exponent of the power law increase of \( w \) (which is represented by \( \alpha \) in the asymptotic expressions) depended on the initial conditions, which were determined by the constants \( v_2, w_3 \). Generally the exponent \( \alpha \) would decrease from a value in the range \( 2 - 3 \) to a value in the range \( 1.2 - 1.8 \) for a wide range of initial conditions. This behaviour can be seen in the \( \log(w) - \log(x) \) plot in Fig. 2. Although, these classical gauge fields weakened slightly as \( r \) increased, they still diverged as \( r \to \infty \). Due to this feature of the ansatz function \( w \) the time part of the gauge field grew without bound as \( r \to \infty \), leading to both a classical type of confinement (a test particle placed in the background field of this solution would not be able to escape to \( \infty \)) and an undesired infinite field energy for this solution. Various phenomenological studies of quarkonia bound states use such increasing potentials to study the spectrum of the bound state [8] although usually the potential is taken to increase linearly. It should be mentioned that the asymptotic form of the classical solution given in Eqs. (4) are expected to be altered by the quantum corrections. The classical, short distance behaviour, as given in Fig. 1, should be roughly correct, since the pure gauge SU(3) theory that we are considering is asymptotically free.

This “bunker” solution has “magnetic” and “electric” fields associated with it. Using the ansatz for \( A_\mu \) from Eq. (1) these “magnetic” and “electric” fields have the following proportionalities

\[
\begin{align*}
H^a_r &\propto \frac{v^2 - 1}{r^2}, & H^a_\phi &\propto v', & H^a_\theta &\propto v', \\
E^a_r &\propto \frac{rw' - w}{r^2}, & E^a_\phi &\propto \frac{vw}{r}, & E^a_\theta &\propto \frac{vw}{r},
\end{align*}
\]

(7a)

(7b)

here for \( E^a_r, H^a_\theta \), and \( H^a_\phi \) the color index \( a = 1, 3, 4, 6, 8 \) and for \( H^a_\phi, E^a_\theta \) and \( E^a_\phi a = 2, 5, 7 \). The asymptotic behaviour of \( H^a_\phi, H^a_\theta \) and \( E^a_\phi, E^a_\theta \) is dominated by the strongly oscillating
function \( v(r) \). It may be postulated that quantum corrections to this strongly oscillating solution would tend to smooth it out so that it would not play a significant role in the large \( r \) limit. From Eqs. (7) and the asymptotic form of \( v(r) \), \( w(r) \) the radial components of the “magnetic” and “electric” have the following asymptotic behaviour

\[
H_r^a \propto \frac{1}{r^2}, \quad E_r^a \propto \frac{1}{r^{2-\alpha}}.
\]  

(8)

where the strongly oscillating portion of \( H_r^a \) is assumed not to contribute in the limit of large \( r \) due to smoothing by quantum corrections. The radial “electric” field falls off slower than \( 1/r^2 \) (since \( \alpha > 1 \)) indicating the presence of a confining potential. The \( 1/r^2 \) fall off of \( H_r^a \) indicates that this solution carries a “magnetic” charge. This was also true for the simple solutions discussed in Refs. [4] [5]. This leads to the result that if a test is placed in the background field of the bunker solution, the composite system will have unusual spin properties (i.e. if the test particle is a boson the system will behave as a fermion, and if the test particle is a fermion the system will behave as a boson). This is the spin from isospin mechanism [9].

By examining the classical SU(3) field equations of Eqs. (4) we have found field configurations which led to a classical confining behaviour, and which has some similarities with certain phenomenological models used to study heavy quark bound states. The most significant draw back of the present solutions is that it has infinite field energy. The asymptotic form of the energy density goes as

\[
\mathcal{E} \propto 4\frac{v^2}{r^2} + \frac{2}{3} \left( \frac{w'}{r} - \frac{w}{r^2} \right)^2 + 4\frac{v^2w^2}{r^4} + 2\frac{v^2 - 1}{r^4} \approx \frac{2\alpha^2(\alpha - 1)(3\alpha - 1)}{x^{4-2\alpha}}.
\]

(9)

Since we found \( \alpha > 1 \) this energy density will yield an infinite field energy when integrated over all space. This can be compared with the finite field energy monopole [10] and dyon solutions [11] [12].
III. QUANTIZATION OF THE “BUNKER” SOLUTION

Although the classical confining behaviour of this “bunker” solution may seem interesting due to its similarity with certain phenomenological potentials, the infinite field energy discussed at the end of the previous section strongly argues against the physical importance of this solution. One possible escape from this conclusion is if quantum effects weakened or removed the bad long distance behaviour of these solutions. However, strongly interacting, non-linear theories are notoriously hard to quantize. In order to take into account the quantum effects on the bunker solution we employ a method used by Heisenberg [3] in attempts to quantize the non-linear Dirac equation. By applying the dynamical equation of motion (in Heisenberg’s case the non-linear Dirac equation) to an n-point Green’s function, \( G_n \), one would arrive at an equation relating \( G_n \) to higher order Green’s functions (\( G_{n+1} \) for example). Then applying the dynamical equations of motion to the higher Green’s functions one would get equations relating these higher Green’s functions to even larger order Green’s function. Continuing in this way one arrived at an infinite set of differential equations relating Green’s functions of all orders. To handle this Heisenberg employed the Tamm-Dankoff method whereby he only considered Green’s functions up to some order thus cutting off the infinite set of equations. Here we will employ a similar method to the “bunker” solution in terms of the ansatz functions \( v \) and \( w \). Previously [2] we used this method on an infinite energy, string-like classical solution to the SU(3) equations. For more details on the application of the Heisenberg method to such classical solutions we refer the reader to this article.

In order to use Heisenberg’s quantization method on the nonlinear equations we make the following assumptions:

1. The degrees of freedom relevant for studying the “bunker” solution (both classically and also quantum mechanically) are given entirely by the two ansatz functions \( w, v \). No other degrees of freedom arise through the quantization process.

2. From Eq. (3F) and Fig. 1 \( w \) is a smoothly varying function for large \( x \), while \( v \) is
strongly oscillating. Thus we take $w(x)$ to be almost a classical degree of freedom while $v(x)$ is treated as a fully quantum mechanical degree of freedom. One might think that in this way only the behaviour of $v$ would change while $w$ stayed the same. However since $w$ and $v$ are coupled via the equations of motion we find that both functions are modified.

To begin we replace the ansatz functions by operators $\hat{w}(x), \hat{v}(x)$.

\begin{align*}
x^2\hat{v}'' &= \hat{v}^3 - \hat{v} - \hat{v}\hat{w}^2 \quad &\text{(10a)} \\
x^2\hat{w}'' &= 6\hat{w}\hat{v}^2 \quad &\text{(10b)}
\end{align*}

here the prime denotes a derivative with respect to $x$. Taking into account assumption (2) we let $\hat{w} \rightarrow w$ become just a classical function again, and replace $\hat{v}^2$ in Eq. (10b) by its expectation value to arrive at

\begin{align*}
x^2\hat{v}'' &= \hat{v}^3 - \hat{v} - \hat{v}w^2 \quad &\text{(11a)} \\
x^2\hat{w}'' &= 6w\langle v^2 \rangle \quad &\text{(11b)}
\end{align*}

where the expectation value $\langle \hat{v}^2 \rangle$ is taken with respect to fluctuations of $v$ around the classical solution of Eq. (3) or Fig. 1 (i.e. $\langle \hat{v}^2 \rangle = \int Dv e^{iS/\hbar} v^2$ where $S$ is the action and $Dv$ is a path integral measure over all possible configurations of $v$). If we took the expectation value of Eq. (11a) we would almost have a closed system of differential equations relating $w$ and $\langle \hat{v} \rangle$. The $\langle \hat{v}^2 \rangle$ term from Eq. (11b) and the $\langle \hat{v}^3 \rangle$ term from Eq. (11a) prevent the equations from being closed. Applying the operation $x^2\partial^2/\partial x^2$ to the operator $\hat{v}^2$ and using Eq. (11a) yields

\begin{align*}
x^2(\hat{v}^2)'' &= 2\hat{v}^2(\hat{v}^2 - 1 - w) + 2x^2(\hat{v}')^2 \quad &\text{(12)}
\end{align*}

If we took the expectation of the above equation with respect to fluctuations in the ansatz function operator $\hat{v}^2$, and combined this with Eq. (11b) we would almost have a closed system for determining $w$ and $\hat{v}^2$ except for the $\langle (\hat{v}')^2 \rangle$ term which comes from the last term on the right hand side of Eq. (12). Continuing in this way one could obtain an infinite set of equations for various powers of the ansatz function operator (i.e. $\hat{v}^n$). These higher
order equations never close. To deal with this problem we follow Heisenberg, and make some
assumption that effectively cuts off the system of equations at some finite order. By taking
the expectation of Eq. (12) and further by assuming that
\[
\langle (\hat{v}')^2 \rangle = \frac{\langle \hat{v}^2 \rangle - v_0^2}{x^2}
\]
we arrive at the closed system of equations from Eqs. (11b) (12)
\[
x^2 (\langle \hat{v}^2 \rangle '' = 2 \langle \hat{v}^2 \rangle - 2 \langle \hat{v}^2 \rangle w - 2v_0^2
\]
\[
x^2 w'' = 6w \langle \hat{v}^2 \rangle
\]
By making the assumption of Eq. (13) we have simplified Eqs. (11b) (12) to the closed
system given by Eqs. (14a) (14b). It is straightforward to show that in the limit 
\[ x \rightarrow \infty \]
\[
\langle \hat{v}^2 \rangle = v_0^2 + \frac{a}{x^\alpha}
\]
\[
w = \frac{b}{x^\alpha}
\]
solves Eqs. (14a) (14b) provided that \( v_0^2 = 1, b = -a \) and \( \alpha = 2, -3 \). In order for \( \langle \hat{v}^2 \rangle \)
and \( w \) to have acceptable behaviour at \( x \rightarrow \infty \) we take the \( \alpha = 2 \) solution. Substituting
the above expressions for \( \langle \hat{v}^2 \rangle \) with \( v_0 = +1 \) and \( w \) back into Eq. (11a), and assuming that
\( \langle \hat{v}^3 \rangle = \langle \hat{v} \rangle \langle \hat{v}^2 \rangle \) gives the following equations for \( \langle \hat{v} \rangle \) in the \( x \rightarrow \infty \) limit
\[
x^2 \langle \hat{v} \rangle '' = \langle \hat{v} \rangle (\langle \hat{v}^2 \rangle - 1) = \langle \hat{v} \rangle \frac{a}{x^2}
\]
Eq. (16) is solved by in the \( x \rightarrow \infty \) limit
\[
\langle \hat{v} \rangle = \pm \left( 1 + \frac{a}{6x^2} \right)
\]
Eq. (17) together with Eqs. (15a) (15b) provide information on the behaviour of the
“classical” ansatz function, \( w \), and the “quantum” ansatz function, \( v \), via \( \langle \hat{v} \rangle \) and \( \langle \hat{v}^2 \rangle \). The
main point of interest is that after applying the Heisenberg-like quantization procedure to
the classical solution of Fig. 1, the infinite increase of the ansatz function, \( w \), has changed
to an acceptable asymptotic behaviour \( (i.e. \) one that leads to a finite field energy). By
replacing the \(v^2\) and \((v')^2\) terms in Eq. (9) with \(\langle \dot{v}^2 \rangle\) and \((\langle \dot{v}' \rangle)^2\) - from Eqs. (15a) and (17) - respectively, and also using \(w\) from eq. (15b) we find that the field energy density of the quantized “bunker” solution takes the form

\[
\mathcal{E} \propto \frac{a^2}{x^8}
\]

in the limit in which quantum fluctuations become important (\(i.e.\) for non-Abelian theories which exhibit asymptotic freedom this means in the low energy or \(x \to \infty\) range) the energy density goes from the form given in Eq. (9) to that given in Eq. (18). This can be seen to give a finite field energy. In the high energy or short distance regime we assume that the fields approach the classical configuration of Figure 1 due to asymptotic freedom. This classical configuration is well behaved at \(x = 0\), but would yield an infinite field energy due to its divergence as \(x \to \infty\). In the long distance or low energy limit the energy density should go over into the form given in Eq. (18) which would then result in a finite field energy for this configuration, since the integral of Eq. (18) over the large \(x\) region, where it is valid, would give a finite field energy.

Finally by using \(w\), \(\langle \dot{v} \rangle\), and \(\langle \dot{v}^2 \rangle\) in the expressions for \(E\) and \(H\) given in Eq. (7) we find that the radial fields \((E_r, H_r)\) go like \(a/r^4\) while the angular fields \((E_{\theta,\phi}, H_{\theta,\phi})\) go like \(a/r^3\). Thus the quantization procedure outlined above modifies the undesirable long distance behaviour of the “electric” and “magnetic” fields as well as the energy density.

**IV. DISCUSSION**

In this paper we reviewed a certain classical field configuration for an SU(3) gauge theory. Near the origin the field configurations were finite, but as \(r \to \infty\) the fields diverged (see Figs. 1,2). This increasing field strength led to a classical type of confinement in that a test particle placed in the background field of this solution would not be able to escape to \(\infty\). Unfortunately, this diverging of the field as \(r \to \infty\) also led to this configuration having an infinite field energy. Previously it was suggested that quantum effects might
soften or eliminate this bad long distance behaviour. By applying a method similar to that Heisenberg used in quantizing the non-linear Dirac equation we find that the long distance behaviour is changed so as to give finite field energy. At short distances the fields should approach the classical configuration of Fig. 1 from the asymptotic freedom of the SU(3) gauge theory. This classical solution has the good features of not being divergent at $r = 0$ and in some limited region around $r = 0$ the fields increase in a way similar to that found in some phenomenological models of confinement. At long distances the fields should approach the configuration given by Eqs. (15a) (15b) (17) where the quantum effects have eliminated the divergence of the fields and field energy density as $r \to \infty$. 

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REFERENCES

[1] V. Dzhunushaliev, “Confining solutions of SU(3) Yang-Mills theory”, hep-th 9611096

[2] V. Dzhunushaliev and D. Singleton, “Quantization of strongly interacting fields”, hep-th/9806073

[3] W. Heisenberg, Nachr. Akad. Wiss. Göttingen, N8, 111 (1953); W. Heisenberg, Zs. Naturforsch., 9a, 292 (1954); W. Heisenberg, F. Kortel and H. Mütter, Zs. Naturforsch., 10a, 425 (1955); W. Heisenberg, Zs. für Phys., 144, 1 (1956); P. Askali and W. Heisenberg, Zs. Naturforsch., 12a, 177 (1957); W. Heisenberg, Nucl. Phys., 4, 532 (1957); W. Heisenberg, Rev. Mod. Phys., 29, 269 (1957)

[4] W.J. Marciano, H. Pagels, Phys. Rev. D12, 1093 (1975)

[5] Z. Horvath and L. Palla, Phys. Rev. D14, 1711 (1976)

[6] D.V. Gal’tsov and M.S.Volkov, Phys.Lett, B274, 173(1992).

[7] S. Wolfram, Mathematica 2nd Ed., (Addison-Wesley Publishing Company, 1991), p. 829

[8] E. Eichten et. al., Phys. Rev. D17, 3090 (1978)

[9] R. Jackiw and C. Rebbi, Phys. Rev. Lett. 36, 1116 (1976); P. Hasenfrantz and G. ’t Hooft, Phys. Rev. Lett. 36, 1119 (1976)

[10] G. ’t Hooft, Nucl. Phys. B 79, 276 (1974); A.M. Polyakov, Sov. Phys.-JETP 41, 988 (1975)

[11] M.K. Prasad and C.M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975); E.B. Bogomolnyi, Sov. J. Nucl. Phys. 24, 449 (1976)

[12] B. Julia and A. Zee, Phys. Rev. D11, 2227 (1975)
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Fig.1. The \( w(x) \) confining function, and the \( v(x) \) oscillating function of the \( SU(3) \) bunker solution. The initial conditions for this particular solution were \( v_2 = 0.1, \ w_3 = 2.0, \) and \( x_i = 0.001. \)

Fig.2. A plot of \( \log(w) - \log(x) \) of the solution from Fig. 2 showing the different power law behaviour in the small \( x \) and large \( x \) regions.