HAMILTON POWERS OF EULERIAN DIGRAPHS

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Abstract. In this note, we prove that the $\lceil \frac{1}{2} \sqrt{n} \log_2^2 n \rceil$th power of a connected $n$-vertex Eulerian digraph is Hamiltonian, and provide an infinite family of digraphs for which the $\lfloor \sqrt{n}/2 \rfloor$th power is not.

1. Preliminaries

The $k$th power of a (directed or undirected) graph $G$, denoted $G^k$, is the graph on the vertices of $G$ in which there is an edge from a vertex $u$ to a vertex $v$ if there exists a $uv$-path in $G$ of length at most $k$. It is well-known that the cube of any connected undirected graph is Hamiltonian (see [6, 11], also [3, Ex 10-14]). In 1974, Fleischner proved that the square of any two-connected undirected graph is Hamiltonian, solving the Plummer-Nash-Williams conjecture [4] (see [5] for a much simpler proof). Unfortunately, strongly-connected directed graphs (digraphs) may require the $\lceil n/2 \rceil$th power to be Hamiltonian; even $k$-strong connectedness is only sufficient for guaranteeing that the $\lceil n/(2k) \rceil$th power is Hamiltonian [10]. For a general survey on Hamilton cycles in digraphs, we refer the reader to [7]. Interestingly, results for Eulerian digraphs are not nearly so bleak. Through the study of minimally Eulerian digraphs (connected Eulerian digraphs with no proper connected Eulerian subgraph), we prove that

**Theorem 1.1.** The $\lceil \frac{1}{2} \sqrt{n} \log_2^2 n \rceil$th power of any $n$-vertex connected Eulerian digraph is Hamiltonian.

In fact, we prove an even stronger result (in Theorem 2.1) that, given a minimally Eulerian digraph $G = (V, A)$, specifies an ordering $v_1, \ldots, v_n$ of $V$ and an edge-disjoint directed path (dipath) decomposition $P_1, \ldots, P_n$ of $G$, such that each $P_i$ is a $v_iv_{i+1}$-dipath ($v_{n+1} := v_1$) of length at most $\lceil \frac{1}{2} \sqrt{n} \log_2^2 n \rceil$. In addition, we provide an infinite family of minimally Eulerian digraphs for which the $\lfloor \sqrt{n}/2 \rfloor$th power is not Hamiltonian (Example 2.2). For details regarding the importance of minimally Eulerian digraphs and their connection to the traveling salesman problem, we refer the reader to [2, 8].

1.1. Notation, Definitions, and Basic Results. Let $G = (V, A)$ be a simple digraph. If $G$ contains a spanning directed cycle (dicycle), then $G$ is Hamiltonian. If $G$ contains an Euler circuit (a circuit containing every edge), then $G$ is Eulerian. If $G$ is connected, this is equivalent to the condition that, for every vertex $v \in V$, the indegree $d^-(v)$ equals the outdegree $d^+(v)$. If $G$ is a connected Eulerian digraph and contains no proper connected Eulerian subgraph on the vertices of $G$, then $G$ is minimally Eulerian; equivalently, a connected Eulerian digraph $G$ is minimally Eulerian if, for any dicycle $C$ of $G$, the graph $G - C := (V, A - A(C))$ is disconnected. If $G$ contains no dicycle,
then \( G \) is acyclic. For more details regarding graph theoretic definitions and notation, we refer the reader to \[1\]. Let

\[
p_\#(G) := \frac{1}{2} \sum_{u \in V} |d^+(u) - d^-(u)|,
\]

a measure of how “close” to Eulerian a digraph is, and a key ingredient in our proof. The quantity \( p_\#(G) \) is exactly the minimal number of dipaths required in an edge-disjoint decomposition of \( G \) into dipaths and dicycles. That \( p_\#(G) \) dipaths are required follows immediately from the definition of \( p_\#(G) \) above. That \( p_\#(G) \) dipaths are sufficient follows from a simple greedy algorithm (iteratively perform walks from vertices \( u \) with \( d^+(u) > d^-(u) \), removing dicycles when they are formed, and only removing the dipath when a vertex \( v \) with \( d^+(v) = 0 \) is reached). The size of an acyclic digraph \( G \) is immediately bounded above by \( p_\#(G) (|V| - 1) \), and an even tighter estimate can be obtained relatively quickly:

**Proposition 1.2.** Let \( G = (V, A) \) be an acyclic digraph. Then \( |A| \leq \sqrt{2p_\#(G)} |V| \).

**Proof.** If \( p_\#(G) = 0, 1, 2 \), the result follows immediately, as \( |A| \leq p_\#(G) (|V| - 1) \). Now, let \( p_\#(G) > 2 \), \( V = \{v_1, ..., v_n\} \) be a topological sorting of \( G \) (i.e., \( v_i, v_j \in A \) implies that \( i < j \)), \( k \in \mathbb{N} \) be the smallest number such that \( p_\#(G) \leq \binom{k}{2} \), \( \ell = \lceil n/k \rceil \), and \( V_i = \{v_{(i-1)k+1}, ..., v_{ik}\} \), \( i = 1, ..., \ell - 1 \), \( V_\ell = \{v_{(\ell-1)k+1}, ..., v_n\} \). There are at most \( \binom{k}{2} \) edges within each of the subsets \( V_i \), \( i = 1, ..., \ell - 1 \), and at most \( \binom{n-k(\ell-1)}{2} \) within the subset \( V_\ell \). Our digraph \( G \) can be decomposed into \( p_\#(G) \) edge-disjoint dipaths, and, by the topological sorting of \( V \), each of the aforementioned \( p_\#(G) \) dipaths has at most \( \ell - 1 \) edges between the subsets \( V_1, ..., V_\ell \). Therefore, there are at most \( (\ell - 1)p_\#(G) \) total edges between the subsets \( V_1, ..., V_\ell \). Combining these estimates gives

\[
|A| \leq (\ell - 1) \left( \binom{k}{2} + p_\#(G) \right) + \binom{n-k(\ell-1)}{2}.
\]

Dividing by \( \sqrt{p_\#(G)} n \), we have

\[
\frac{|A|}{\sqrt{p_\#(G)} n} \leq \frac{\ell - 1}{n} \left( \frac{\binom{k}{2}}{\sqrt{p_\#(G)}} + \sqrt{p_\#(G)} \right) + \frac{\binom{n-k(\ell-1)}{2}}{\sqrt{p_\#(G)} n}.
\]

The right hand side is convex w.r.t. \( p_\#(G) \) and maximized when \( p_\#(G) \) is as small as possible. We note that, by the definition of \( k \), \( p_\#(G) > \binom{k-1}{2} \). So the right hand side can be bounded above by replacing \( p_\#(G) \) by \( \binom{k-1}{2} \), giving

\[
\frac{|A|}{\sqrt{p_\#(G)} n} \leq \frac{\ell - 1}{n} \left( \frac{(k-1)^2}{\binom{k-1}{2}} + \frac{(n - k(\ell - 1))(n - k(\ell - 1) - 1)}{2\binom{k-1}{2}} \right).
\]

The right hand side is a convex quadratic function in the term \( \ell \) (treating \( \ell \) as a variable independent of \( n \) and \( k \)), and therefore achieves its maximum at one of the endpoints of the interval \([n/k, n/k + 1]\). Setting \( \ell = n/k \) gives

\[
\frac{\ell - 1}{n} \left( \frac{(k-1)^2}{\binom{k-1}{2}} + \frac{(n - k(\ell - 1))(n - k(\ell - 1) - 1)}{2\binom{k-1}{2}} \right) = \frac{(k-1)^2}{k\binom{k-1}{2}} - \frac{k^2 - 3k + 2}{2n\binom{k-1}{2}}.
\]
and setting \( \ell = n/k + 1 \) gives
\[
\frac{\ell - 1}{n} \frac{(k-1)^2}{2^{(k-1)/2} n} + \frac{(n-k(\ell - 1))(n-k(\ell - 1) - 1)}{k^{(k-1)/2}} = \frac{(k-1)^2}{k^{(k-1)/2}}.
\]

Noting that \( k^2 - 3k + 2 \geq 0 \) for all \( k \in \mathbb{N} \), we conclude that the maximum over the interval \([n/k, n/k + 1]\) is obtained at \( \ell = n/k + 1 \). Replacing \( \ell \) by \( n/k + 1 \), we have
\[
|A| < \frac{(k-1)^2}{k^{(k-1)/2}} \sqrt{p_\#(G)} n = \frac{(k-1)^{3/2}}{k(k-2)^{1/2}} \sqrt{2p_\#(G)} n \leq \sqrt{2p_\#(G)} n,
\]
for \( k \geq 3 \) (recall, \( p_\#(G) > 2 \)).

From Proposition \ref{prop:lower_bound}, we immediately obtain a bound (tight up to a multiplicative constant; see Example \ref{ex:example}) on the maximum size of a minimally Eulerian digraph:

**Proposition 1.3.** Let \( G = (V, A) \) be a minimally Eulerian digraph. Then
\[
|A| \leq \sqrt{2|V| - 1} |V| + |V| - 1.
\]

**Proof.** \( G \) is a connected Eulerian digraph, so it admits a rooted, directed subgraph \( T \) of \( G \) in which there is a unique path (in \( T \)) from the root to any other vertex of \( G \). Every digycle of \( G \) must intersect an edge of \( T \), as the removal of any digycle from a minimally Eulerian graph disconnects it. Therefore, \( G - T \) is acyclic, and by Proposition \ref{prop:lower_bound}
\[
|A| \leq |A(G - T)| + |A(T)| \leq \sqrt{2|V| - 1} |V| + |V| - 1.
\]

\( \square \)

2. A **Proof of Theorem 1.1 and a Lower Bound**

To prove Theorem \ref{thm:main}, we show an even stronger statement regarding minimally Eulerian digraphs.

**Theorem 2.1.** Let \( G = (V, A) \), \(|V| = n > 1\), be a minimally Eulerian graph. Then there exists an ordering \( v_1, ..., v_n \) of \( V \) and an \( n \)-dipath edge-disjoint decomposition \( P_1, ..., P_n \) of \( G \) such that each \( P_i \) is a \( v_i v_{i+1} \)-dipath \((v_{n+1} := v_1)\) of length at most \( \lceil f(n) \sqrt{n} \rceil \), where
\[
f(n) = (\log_2 n)^{\log_3/2 + o(1)} \leq \frac{1}{2} \log_2^2 n.
\]

**Proof.** We first show that there exists an ordering \( v_1, ..., v_n \) of \( V(G) \) such that there is an \( n \)-dipath edge-disjoint decomposition \( P_1, ..., P_n \) of \( G \) such that each \( P_i \) is a \( v_i v_{i+1} \)-dipath. This ordering and decomposition can be constructed by picking a base vertex \( v_1 \in V(G) \) and considering an Eulerian circuit \( W \) of \( G \) starting at \( v_1 \), ordering the remaining vertices based on the order of first appearance in this circuit, and taking each dipath \( P_i \) to be the walk in \( W \) between the first appearance of \( v_i \) and the first appearance of \( v_{i+1} \). As \( G \) is minimally Eulerian, each such walk is a dipath. It suffices to consider \( n \geq 6388 \), as the length of a dipath is at most \( n - 1 \) and \( \lceil \frac{1}{2} \sqrt{n} \log_2^2 n \rceil \geq n - 1 \) for \( n = 1, ..., 6387 \).

Let \( v_1, ..., v_n \) be an ordering of \( V(G) \) and \( P_1, ..., P_n \) a decomposition of \( G \) into edge-disjoint \( v_i v_{i+1} \)-dipsaths \( P_i \). We choose this ordering and decomposition so that the elements of the set \( \{ |A(P_1)|, ..., |A(P_n)| \} \) are lexicographically minimized (i.e., minimizes the length of the longest dipath, minimizes the length of the \( 2^{nd} \) longest dipath conditional on the minimality of the longest dipath, etc). Let \( \hat{P} \) be the longest dipath in the set \( \{ P_1, ..., P_n \} \), with length \( |A(\hat{P})| = \alpha \sqrt{n} \) for some \( \alpha \geq \frac{1}{2} \left[ \log_2 n \right]^{\log_3/2} \). We aim
to build a sequence of subgraphs $H_0(\hat{\alpha}) \subset H_1 \subset H_2 \subset \ldots$, bound the order of each subgraph from below using the lexicographic minimality of path lengths, and conclude that if $\alpha$ is too large then some $H_i$ contains too many vertices, thus producing an upper bound on $\alpha$.

Let $H_\ell = \hat{\alpha}$. Let $H_\ell, \ell > 0$, be the union of all $P_i$ satisfying both $|A(P_i)| \geq \alpha \sqrt{n}/2^\ell$ and $\{v_i, v_{i+1}\} \cap V(H_{\ell-1}) \neq \emptyset$. Let $n_\ell, m_\ell$, and $k_\ell$ be the number of vertices, edges, and dipaths $P_i$ in $H_\ell$. We have $n_0 = \alpha \sqrt{n} + 1$, $m_0 = \alpha \sqrt{n}$, $k_0 = 1$ and, by construction, $m_\ell \geq k_\ell m_0/2^\ell$ for all $\ell \geq 0$.

We may produce a lower bound for the size of each $H_\ell$ by our lexicographic minimality condition. We claim that every vertex of $H_\ell$ is either the start- or end-vertex of a dipath $P_i$ of length at least $m_0/2^{\ell+1}$. Suppose, to the contrary, that some $v_i \in V(H_\ell)$ satisfies $|A(P_{i-1})|, |A(P_i)| < m_0/2^{\ell+1}$. Let $P_j$ be a dipath in $H_\ell$ containing $v_i$, and let us denote the $v_jv_i$ (resp. $v_jv_{i+1}$) portion of this path by $P_j^1$ (resp. $P_j^2$). By removing $P_j$, $P_{i-1}$, and $P_j$ from our set $\{P_1, \ldots, P_p\}$ and replacing them with $P_j^1$, $P_j^2$, and $P_j \cup P_{i+1}$, we have replaced a path of length $|A(P_j)| (|A(P_j)| \geq m_0/2^{\ell+1})$ with paths of all of length strictly less than $|A(P_j)|$, a contradiction. Therefore, $k_{\ell+1} \geq n_{\ell}/2$ for all $\ell \geq 0$, as every vertex in $V(H_\ell)$ is the start- or end-vertex of a dipath $P_i$ in $H_{\ell+1}$, and each dipath $P_i$ has only one start- and one end-vertex.

The graph $H_\ell$ can be decomposed into the edge-disjoint union of two graphs $H_{\ell,a}$ and $H_{\ell,e}$, where $H_{\ell,a}$ is acyclic with $p_\#(H_{\ell,a}) \leq k_\ell$ (as $H_\ell$ is the edge-disjoint union of $k_\ell$ paths) and $H_{\ell,e}$ is the vertex-disjoint union of minimally Eulerian graphs $H_{\ell,e}^{(1)}, \ldots, H_{\ell,e}^{(p_\ell)}$ for some $p_\ell$ (if the Eulerian graph $H_{\ell,e}^{(j)}$ is not minimal, neither is $G$). By Proposition 1.2 $H_{\ell,a}$ has at most $\sqrt{2k_\ell n_\ell} n_\ell$ edges. By Proposition 1.3 $H_{\ell,e}$ has at most

$$\sum_{j=1}^{p_\ell} \left( \sqrt{2(n_j^{(j)} - 1)n_j^{(j)} + n_j^{(j)} - 1} \right) \leq \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}$$

edges, where $n_j^{(j)} := |V(H_{\ell,e}^{(j)})|$, $j = 1, \ldots, p_\ell$. Therefore,

$$m_\ell \leq \sqrt{2k_\ell n_\ell} + \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}.$$%

Combining this inequality with the bound $m_\ell \geq k_\ell m_0/2^\ell$, we have

$$k_\ell m_0/2^\ell \leq \sqrt{2k_\ell n_\ell} + \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}.$$ (1)

Using Inequality (1), we produce a recursive lower bound on $n_\ell$ that gives an upper bound on $\alpha$. In particular, we aim to show that

$$n_\ell \geq \left( \frac{n_{\ell-1}m_0}{5 \times 2^\ell} \right)^{2/3} \quad \text{for all } \ell \leq \log_2(5^2 \alpha).$$ (2)

If $n_\ell \geq \sqrt{2k_\ell m_0/2^\ell}$, then Inequality (2) immediately holds, as

$$n_\ell = \frac{\sqrt{2k_\ell m_0/2^\ell}}{2^\ell} = \left( \left( \frac{n_{\ell-1}m_0}{5 \times 2^\ell} \right)^2 \left( \frac{2k_\ell}{5 \times 2^\ell} \right) \right)^{1/3} \geq \left( \frac{5^2 \alpha}{2^\ell} \right)^{1/3} \geq \left( \frac{n_{\ell-1}m_0}{5 \times 2^\ell} \right)^{2/3}$$

for $\alpha \geq 2^\ell/5^2$. Now, suppose that $n_\ell < \sqrt{2k_\ell m_0/2^\ell}$. Then $k_\ell m_0/2^\ell - \sqrt{2k_\ell n_\ell}$ is monotonically increasing with respect to $k_\ell$. Combining this fact with the bound $k_\ell \geq n_{\ell-1}/2$ and Inequality (1), we obtain

$$n_{\ell-1}m_0/2^{\ell+1} - \sqrt{n_{\ell-1}n_\ell} \leq k_\ell m_0/2^\ell - \sqrt{2k_\ell n_\ell} \leq \sqrt{2(n_\ell - 1)n_\ell + n_\ell - 1}.$$
This implies that
\[
\frac{n_{\ell} m_0}{2^{\ell+1}} \leq \sqrt{2(n_{\ell} - 1)} n_{\ell} + \sqrt{n_{\ell-1}} n_{\ell} - 1 < \frac{5}{2} n_{\ell}^{3/2},
\]
for \( n \geq 6388 \), as \( n_{\ell} \geq n_0 = \alpha \sqrt{n} + 1 \), and so the claim holds in this case as well.

Using the initial bound \( n_0 > m_0 \) and Inequality (2), we obtain
\[
n \geq n_{\ell} \geq n_0 \frac{(2/3)^\ell}{5 \times 2^{\ell+1}} = \frac{n_0 (2/3)^\ell}{16 m_0^{2/(25)}} < 16 m_0^{2/(25)} \times 2^{2\ell} = 16 \alpha^{2-(2/3)\ell} n^{1-\frac{1}{2}(2/3)\ell}
\]
for \( \ell \leq \log_2(5^2 \alpha) \). Taking the logarithm of both sides, we obtain the inequality
\[
\log_2 \alpha < \frac{1}{2 - (2/3)\ell} \left( \log_2(25/16) + 2\ell + \frac{1}{2}(2/3)\ell \log_2 n \right).
\] (3)

Setting \( \ell = \lceil \log_{3/2} \left( \frac{3}{11} \log_2 n \right) \rceil \), we have \( \ell < \log_2(5^2 \alpha) \), as
\[
\log_{3/2} \left( \frac{3}{11} \log_2 n \right) + 1 = \log_{3/2} \left( \frac{3}{11} \log_2 n + 1 \right) < (\log_{3/2} \log_2 n) + 2 \log_2(5) - 1 = \log_2 \left( \frac{5^2}{2} \log_{3/2}^2 \log_2 n \right).
\]

For \( \ell = \lceil \log_{3/2} \left( \frac{3}{11} \log_2 n \right) \rceil \), Inequality (3) implies that
\[
\log_2 \alpha < \frac{\log_2(25/16) + 2 \left( \log_{3/2} \left( \frac{3}{11} \log_2 n \right) + 1 \right) + \frac{1}{2} (2/3) \log_{3/2} \left( \frac{3}{11} \log_2 n \right) \log_2 n}{2 - (2/3) \log_{3/2} \left( \frac{3}{11} \log_2 n \right)}
\]
\[
= \frac{1}{1 - \frac{3}{6 \log_2 n}} \left[ \log_2(5/2) + \log_{3/2} \left( \frac{3}{11} \log_2 n \right) + \frac{11}{11} \right].
\]

Taking the (base two) exponential of both sides, we obtain
\[
\alpha < 2^{\log_2(5/2) - \log_{3/2}(11/3) + 11/12} \left[ \log_2 n \right]^{\log_{3/2}^2 n} \leq .46 \left[ \log_2 n \right]^{1.9995}.
\]

This completes the proof. \( \square \)

Finally, we give the following infinite class of digraphs to illustrate that Theorem 1.1 is tight up to a logarithmic factor.

**Example 2.2.** Let \( G_k = (V_k, A_k) \), \( k \in \mathbb{N}, k \geq 4 \), where \( V_k = \{ u_1, \ldots, u_{k-1}, v_1, \ldots, v_{\ell} \} \), \( \ell := k(k+1)/2 \), and \( u_i u_j \in A_k \) for \( 0 < j - i < k \), and \( u_\ell \in A_k \) for all \( i = 1, \ldots, \ell \), where \( \ell(i) \) is the smallest number \( p \in \mathbb{N} \) such that \( \sum_{j=1}^{p} (k+1-j) \geq i \). This digraph is minimally Eulerian, as every dicycle contains some vertex \( v_i \) and \( d^+(v_i) = d^-(v_i) = 1 \) for all \( i \). There are \( n = k^2 + k - 1 \) vertices and \( k(k^2 + 2k - 1)/2 \) edges (i.e., about \( n^{3/2}/2 \)). The distance between any pair \( v_i, v_j \) in the graph is at least
Figure 1. The minimally Eulerian graph $G_k$ from Example 2.2 for $k = 4$.

$\lceil (\ell + 1)/k \rceil = \lceil k/2 \rceil + 1 \geq \lfloor \sqrt{n}/2 \rfloor + 1$. In any Hamiltonian dicycle of a power of $G_k$, some pair $v_i, v_j$ must be adjacent, and so at least the $\lceil \sqrt{n}/2 \rceil + 1$th power is required. See Figure 1 for a visual example for $k = 4$.

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