Lie algebra of an $n$-Lie algebra

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Abstract

We construct the Lie algebra of an $n$-Lie algebra and we also define the notion of cohomology of an $n$-Lie algebra.

Key words: Lie algebra, $n$-Lie algebra, cohomology.

MSC (2010): 17B30, 17B56, 16W25.

1 Introduction

The notion of $n$-Lie algebra over a commutative field $K$ with characteristic zero, $n$ an integer $\geq 2$, introduced by Filippov [3], is a generalization of the notion of Lie algebra which corresponds with the usual case when $n = 2$.

When $n \geq 2$ is an integer and $K$ a commutative field, an $n$-Lie algebra structure on a $K$-vector space $G$ is due to the existence of a skew-symmetric $n$-multilinear map

$$\{,\ldots,\} : G^n = G \times G \times \ldots \times G \to G, (x_1, x_2, \ldots, x_n) \mapsto \{x_1, x_2, \ldots, x_n\},$$
such that
\[
\{ x_1, x_2, \ldots, x_{n-1}, \{ y_1, y_2, \ldots, y_n \} \} = \sum_{i=1}^{n} \{ y_1, y_2, \ldots, y_{i-1}, \{ x_1, x_2, \ldots, x_{n-1}, y_i \} \} \}
\]
for any \( x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n \) elements of \( \mathcal{G} \).

The above identity is called Jacobi identity of the \( n \)-Lie algebra \( \mathcal{G} \).

From this generalization, many authors, \cite{2}, \cite{4}, \cite{5}, extended the following notions: ideal of an \( n \)-Lie algebra, semi simple \( n \)-Lie algebra, nilpotent \( n \)-Lie algebra, solvable \( n \)-Lie algebra, Cartan subalgebra of an \( n \)-Lie algebra, etc.

The main goal of this paper is to construct the Lie algebra structure from an \( n \)-Lie algebra: so in this new context, we give definitions of ideal of an \( n \)-Lie algebra, semi simple \( n \)-Lie algebra, nilpotent \( n \)-Lie algebra, solvable \( n \)-Lie algebra, Cartan subalgebra of an \( n \)-Lie algebra. We also define the cohomology of an \( n \)-Lie algebra.

In what follows, \( K \) denotes a commutative field with characteristic zero, \( \mathcal{G} \) an \( n \)-Lie algebra over \( K \) with bracket \( \{ , \ldots, \} \) and finally \( n \geq 2 \) an integer.

\section{\( n \)-Lie algebra structure}

We recall that, \cite{3}, for \( n \geq 2 \), an \( n \)-Lie algebra structure on a \( K \)-vector space \( \mathcal{G} \) is due to the existence of a skew-symmetric \( n \)-multilinear map
\[
\{ , \ldots, \} : \mathcal{G}^n = \mathcal{G} \times \mathcal{G} \times \ldots \times \mathcal{G} \rightarrow \mathcal{G}, (x_1, x_2, \ldots, x_n) \mapsto \{ x_1, x_2, \ldots, x_n \},
\]
such that
\[
\{ x_1, x_2, \ldots, x_{n-1}, \{ y_1, y_2, \ldots, y_n \} \} = \sum_{i=1}^{n} \{ y_1, y_2, \ldots, y_{i-1}, \{ x_1, x_2, \ldots, x_{n-1}, y_i \} \} \}
\]
for any \( x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \ldots, y_n \) elements of \( \mathcal{G} \).

A derivation of an \( n \)-Lie algebra \( (\mathcal{G}, \{ , \ldots, \}) \), is a \( K \)-linear map
\[
D : \mathcal{G} \rightarrow \mathcal{G}
\]
such that
\[
D \{ x_1, x_2, \ldots, x_n \} = \sum_{i=1}^{n} \{ x_1, x_2, \ldots, D(x_i), \ldots, x_n \}
\]
for any \( x_1, x_2, \ldots, x_n \) elements of \( \mathcal{G} \).

We verify that the set of derivations of a \( n \)-Lie algebra \( \mathcal{G} \) is a Lie algebra over \( K \) which we denote \( \text{Der}_{K}(\mathcal{G}) \).
Proposition 1 If $(\mathcal{G}, \{\ldots,\})$ is an $n$-Lie algebra, then for any $x_1, x_2, \ldots, x_{n-1}$ elements of $\mathcal{G}$ the map

$$ad(x_1, x_2, \ldots, x_{n-1}) : \mathcal{G} \rightarrow \mathcal{G}, y \mapsto \{x_1, x_2, \ldots, x_{n-1}, y\},$$

is a derivation of $(\mathcal{G}, \{\ldots,\})$.

Proof. For any $x_1, x_2, \ldots, x_{n-1}$ elements of $\mathcal{G}$ and for any $y_1, y_2, \ldots, y_n$ elements of $\mathcal{G}$, we have

$$[ad(x_1, x_2, \ldots, x_{n-1})](\{y_1, y_2, \ldots, y_n\})$$

$$= \{x_1, x_2, \ldots, x_{n-1}, \{y_1, y_2, \ldots, y_n\}\}$$

$$= \sum_{i=1}^{n} \{y_1, y_2, \ldots, y_{i-1}, \{x_1, x_2, \ldots, x_{n-1}, y_i\}, y_{i+1}, \ldots, y_n\}$$

$$= \sum_{i=1}^{n} \{y_1, y_2, \ldots, y_{i-1}, [ad(x_1, x_2, \ldots, x_{n-1})](y_i), y_{i+1}, \ldots, y_n\}.$$

And that ends the proof. □

A morphism of an $n$-Lie algebra $\mathcal{G}$ into another $n$-Lie algebra $\mathcal{G}'$ is a $K$-linear map

$$\varphi : \mathcal{G} \rightarrow \mathcal{G}'$$

such that

$$\varphi(\{x_1, x_2, \ldots, x_n\}) = \{\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n)\}$$

for any $x_1, x_2, \ldots, x_n$ elements of $\mathcal{G}$.

We verify that the set of $n$-Lie algebras over $K$ is a category.

### 2.1 Lie algebra structure deduced from an $n$-Lie algebra

When $\mathcal{G}$ is an $n$-Lie algebra and when $\text{Der}_K(\mathcal{G})$ is the Lie algebra of $K$-derivations of $\mathcal{G}$, then the multilinear map

$$\mathcal{G}^{n-1} \rightarrow \text{Der}_K(\mathcal{G}), (x_1, x_2, \ldots, x_{n-1}) \mapsto ad(x_1, x_2, \ldots, x_{n-1}),$$

is skew-symmetric. If we denote $\Lambda_K^{n-1}(\mathcal{G})$, the $(n-1)$-exterior power of the $K$-vector space $\mathcal{G}$, there exists an unique $K$-linear map

$$ad_G : \Lambda_K^{n-1}(\mathcal{G}) \rightarrow \text{Der}_K(\mathcal{G})$$

such that

$$ad_G(x_1 \Lambda x_2 \Lambda \ldots \Lambda x_{n-1}) = ad(x_1, x_2, \ldots, x_{n-1})$$

for any $x_1, x_2, \ldots, x_{n-1}$ elements of $\mathcal{G}$.
We recall, [1], that when

\[ f : W \rightarrow W \]

is an endomorphism of a \( K \)-vector space \( W \) and when \( \Lambda_K(W) \) is the \( K \)-exterior algebra of \( W \), then there exists an unique derivation with degree zero

\[ D_f : \Lambda_K(W) \rightarrow \Lambda_K(W) \]

such that, for any \( p \in \mathbb{N} \),

\[ D_f(w_1 \Lambda w_2 \Lambda ... \Lambda w_p) = \sum_{i=1}^{p} w_1 \Lambda w_2 \Lambda ... \Lambda w_{i-1} \Lambda f(w_i) \Lambda w_{i+1} \Lambda ... \Lambda w_p \]

for any \( w_1, w_2, ..., w_p \) elements of \( W \).

When \( g : W \rightarrow W \)

is another endomorphism of the \( K \)-vector space \( W \), then

\[ [D_f, D_g] = D_{[f,g]} \]

where the bracket \([,]\) is the usual bracket of endomorphisms.

**Proposition 2** For any \( s_1, s_2 \) elements of \( \Lambda_{K}^{n-1}(G) \), then we have simultaneously

\[ [ad_{G}(s_1), ad_{G}(s_2)] = ad_{G} \left(D_{ad_{G}(s_1)}(s_2)\right) \]

and

\[ [ad_{G}(s_1), ad_{G}(s_2)] = ad_{G} \left(-D_{ad_{G}(s_2)}(s_1)\right) \]

**Proof.** We prove for indecomposable elements. Let \( s_1 = x_1 \Lambda x_2 \Lambda ... \Lambda x_{n-1} \) and \( s_2 = y_1 \Lambda y_2 \Lambda ... \Lambda y_{n-1} \). For any \( a \in G \), we get

\[ ([ad_{G}(s_1), ad_{G}(s_2)])(a) \]

\[ = \{x_1, x_2, ..., x_{n-1}, \{y_1, y_2, ..., y_{n-1}, a\}\} \]

\[ = \{y_1, y_2, ..., y_{n-1}, \{x_1, x_2, ..., x_{n-1}, a\}\} \]

\[ = \sum_{i=1}^{n-1} \{y_1, y_2, ..., y_{i-1}, \{x_1, x_2, ..., x_{n-1}, y_i\}, y_{i+1}, ..., y_{n-1}, a\} \]

\[ + \{y_1, y_2, ..., y_{n-1}, \{x_1, x_2, ..., x_{n-1}, a\}\} \]

\[ = \{y_1, y_2, ..., y_{n-1}, \{x_1, x_2, ..., x_{n-1}, a\}\} \]

\[ = \sum_{i=1}^{n-1} \{y_1, y_2, ..., y_{i-1}, \{x_1, x_2, ..., x_{n-1}, y_i\}, y_{i+1}, ..., y_{n-1}, a\} \]

\[ = ad_{G}(\sum_{i=1}^{n-1} y_1 \Lambda ... \Lambda y_{i-1} \Lambda [ad_{G}(x_1 \Lambda x_2 \Lambda ... \Lambda x_{n-1})] (y_i) \Lambda y_{i+1} \Lambda ... \Lambda y_{n-1}) \](a).
Thus we have

\[
[\text{ad}_G(s_1), \text{ad}_G(s_2)] \\
= ad_G\left(\sum_{i=1}^{n-1} y_i y_1 \Lambda y_{i-1} \Lambda \text{ad}_G(x_1 \Lambda x_2 \Lambda \ldots \Lambda x_{i-1}) \right)(y_i) y_{i+1} \Lambda \ldots \Lambda y_{n-1}
\]

\[
= ad_G \left( D_{\text{ad}_G(s_1)}(s_2) \right).
\]

On the other hand, we get

\[
([\text{ad}_G(s_1), \text{ad}_G(s_2)])(a) \\
= \{x_1, x_2, \ldots, x_{n-1}, \{y_1, y_2, \ldots, y_{n-1}, a\}\}
\]

\[
= \{x_1, x_2, \ldots, x_{n-1}, \{y_1, x_2, \ldots, x_{n-1}, a\}\}
\]

\[
- \sum_{i=1}^{n-1} \{x_1, x_2, \ldots, x_{i-1}, \{y_1, y_2, \ldots, y_{n-1}, x_i\}, x_{i+1}, \ldots, x_{n-1}, a\} \\
- \{x_1, x_2, \ldots, x_{n-1}, \{y_1, y_2, \ldots, y_{n-1}, a\}\}
\]

\[
- \left[ ad_G\left( - \sum_{i=1}^{n-1} x_1 \Lambda \ldots \Lambda x_{i-1} \Lambda [\text{ad}_G(y_1 \Lambda \ldots \Lambda y_{n-1})] (x_i) \Lambda x_{i+1} \Lambda \ldots \Lambda x_{n-1} \right) \right](a).
\]

Thus

\[
[\text{ad}_G(s_1), \text{ad}_G(s_2)] = ad_G \left( - D_{\text{ad}_G(s_1)}(s_2) \right).
\]

That ends the proofs. \(\blacksquare\)

We denote \(\mathcal{V}_K(\mathcal{G})\) the \(K\)-subvector space of \(\Lambda_k^{n-1}(\mathcal{G})\) generated by the elements of the form

\[
D_{\text{ad}_G(s_1)}(s_2) + D_{\text{ad}_G(s_2)}(s_1)
\]

where \(s_1\) and \(s_2\) describe \(\Lambda_k^{n-1}(\mathcal{G})\).

Let

\[
\Lambda_k^{n-1}(\mathcal{G}) \rightarrow \Lambda_k^{n-1}(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), s \mapsto \overline{s},
\]

be the canonical surjection. Considering that precedes, we deduce that

\[
\text{ad}_G[\mathcal{V}_K(\mathcal{G})] = 0.
\]

We denote

\[
\widetilde{\text{ad}}_G : \Lambda_k^{n-1}(\mathcal{G})/\mathcal{V}(\mathcal{G}) \rightarrow \text{Der}_K(\mathcal{G})
\]

the unique linear map such that

\[
\widetilde{\text{ad}}_G(\overline{s}) = \text{ad}_G(s)
\]
for any $s \in \Lambda^n_K(\mathcal{G})$.

For $s_1, s'_1, s_2, s'_2$ elements of $\Lambda^n_K(\mathcal{G})$, we have

$$D_{adG}(s_1)(s_2) + D_{adG}(s'_1)(s'_1) = D_{adG}(s_1-s'_1)(s_2) + D_{adG}(s_2-s'_2)(s'_1) + D_{adG}(s_1')(s_2) + D_{adG}(s_2')(s_1').$$

We deduce that when

$$\overline{s_1} = \overline{s'_1}$$

and

$$\overline{s_2} = \overline{s'_2},$$

then

$$D_{adG}(s_1)(s_2) + D_{adG}(s'_1)(s'_1) = D_{adG}(s'_1)(s_2) + D_{adG}(s_2)(s'_1).$$

Finally we get

$$\overline{D_{adG}(s_1)(s_2)} = \overline{D_{adG}(s'_1)(s'_2)}.$$

Thus the bracket

$$[\overline{s_1}, \overline{s_2}] = \overline{D_{adG}(s_1)(s_2)}$$

is well defined.

**Theorem 3** When $\mathcal{G}, \{\ldots, \} \) is an $n$-Lie algebra, then the map

$$[\cdot, \cdot] : \left[\Lambda^n_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})\right]^2 \rightarrow \Lambda^n_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), (\overline{s_1}, \overline{s_2}) \mapsto \overline{D_{adG}(s_1)(s_2)},$$

only depends on $\overline{s_1}$ and $\overline{s_2}$, and defines a Lie algebra structure on $\Lambda^n_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$.

Moreover the map

$$\widetilde{adG} : \Lambda^n_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \rightarrow Der_K(\mathcal{G}), \overline{s} \mapsto \overline{adG(s)},$$

is a morphism of $K$-Lie algebras.

**Proof.** The map $[\cdot, \cdot]$ is obviously bilinear. We have $[\overline{s_1}, \overline{s_2}] = \overline{D_{adG}(s_1)(s_2)}$. As $D_{adG}(s_1)(s_2) + D_{adG}(s_2)(s_1) \in \mathcal{V}_K(\mathcal{G})$, we immediately get

$$\overline{D_{adG}(s_1)(s_2)} = -\overline{D_{adG}(s_2)(s_1)}.$$

Thus $[\overline{s_1}, \overline{s_2}] = -[\overline{s_2}, \overline{s_1}]$.

For Jacobi identity, we write:

$$[\overline{s_1}, [\overline{s_2}, \overline{s_3}]] + [\overline{s_2}, [\overline{s_3}, \overline{s_1}]] + [\overline{s_3}, [\overline{s_1}, \overline{s_2}]]$$

$$= [\overline{s_1}, D_{adG}(s_2)(s_3)] - [\overline{s_2}, D_{adG}(s_1)(s_3)] + [\overline{s_3}, D_{adG}(s_1)(s_2)]$$

$$= [\overline{s_1}, D_{adG}(s_2)(s_3)] - [\overline{s_2}, D_{adG}(s_1)(s_3)] - [D_{adG}(s_1)(s_2), \overline{s_3}]$$

$$= D_{adG}(s_1) [D_{adG}(s_2)(s_3)] - D_{adG}(s_2) [D_{adG}(s_1)(s_3)] - D_{adG} [D_{adG}(s_1)(s_2)](s_3)$$

$$= D_{adG}(s_1) [D_{adG}(s_2)(s_3)] - D_{adG}(s_2) [D_{adG}(s_1)(s_3)] - D_{adG} [D_{adG}(s_1)(s_2)](s_3)$$

$$= 0.$$
Moreover, we get

\[
\begin{align*}
\left[ \widetilde{ad}_G(s_1), \widetilde{ad}_G(s_2) \right] &= \left[ ad_G(s_1), ad_G(s_2) \right] \\
&= \widetilde{ad}_G(D_{ad_G(s_1)}(s_2)) \\
&= \widetilde{ad}_G(D_{ad_G(s_1)}(s_2)) \\
&= \widetilde{ad}_G([s_1, s_2]).
\end{align*}
\]

That ends the proof of the two assertions. ■

Remark 1 Thus the map

\[
\widetilde{ad}_G : \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \mapsto ad_G(s),
\]

is a representation of \( \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \) into \( \mathcal{G} \).

Proposition 4 If \( \mathcal{G} \) is an \( n \)-Lie algebra, then the space of invariant elements of \( \mathcal{G} \) for the representation

\[
\widetilde{ad}_G : \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \mapsto ad_G(s),
\]

is the following set

\[
Inv(\mathcal{G}) = \{ x \in \mathcal{G} / \{ x, y_1, y_2, ..., y_{n-1} = 0 \} \}
\]

for any \( y_1, y_2, ..., y_{n-1} \in \mathcal{G} \).

Proof. Considering the representation

\[
\widetilde{ad}_G : \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \mapsto ad_G(s),
\]

we know that

\[
Inv(\mathcal{G}) = \left\{ x \in \mathcal{G} / \left[ \widetilde{ad}_G(\overline{s}) \right](x) = 0 \right\}
\]

for any \( \overline{s} \in \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \).

We verify that

\[
Inv(\mathcal{G}) = \{ x \in \mathcal{G} / \{ x, y_1, y_2, ..., y_{n-1} = 0 \} \}
\]

for any \( y_1, y_2, ..., y_{n-1} \in \mathcal{G} \). ■

Proposition 5 When \( \mathcal{G} \) is an \( n \)-Lie algebra, a subspace \( \mathcal{G}_0 \) of \( \mathcal{G} \) is stable for the representation

\[
\widetilde{ad}_G : \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow Der_K(\mathcal{G}), \overline{s} \mapsto ad_G(s),
\]

if and only if for any \( x \in \mathcal{G}_0 \) and for any \( y_1, y_2, ..., y_{n-1} \in \mathcal{G} \), we have

\[
\{ x, y_1, y_2, ..., y_{n-1} \} \in \mathcal{G}_0.
\]
Proof. It is obvious. ■

In what follows, we give the relation between the category of \( n \)-Lie algebras and the category of Lie algebras.

**Proposition 6** The correspondence

\[
\mathfrak{L}^n : \mathcal{G} \longrightarrow \mathfrak{L}^n(\mathcal{G}) = \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})
\]

is a covariant functor from the category of \( n \)-Lie algebras to the category of Lie algebras.

Proof. It is too quite obvious. ■

When \( F \) is a vector subspace of \( \mathcal{G} \), we denote \( \Lambda^{n-1}_K(F) \) the set of finite sums

\[
\left\{ \sum_{i_1 < i_2 < \ldots < i_{n-1}} x_{i_1} \Lambda x_{i_2} \Lambda \ldots \Lambda x_{i_{n-1}} / i_1, i_2, \ldots, i_{n-1} \in \mathbb{N}, x_{i_1}, x_{i_2}, \ldots, x_{i_{n-1}} \in F \right\}.
\]

We constructed a Lie algebra from an \( n \)-Lie algebra. Considering the functor

\[
\mathfrak{L}^n : \mathcal{G} \longrightarrow \mathfrak{L}^n(\mathcal{G}) = \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}),
\]

we will say that a subspace \( I \subset \mathcal{G} \) is an ideal of the \( n \)-Lie algebra \( \mathcal{G} \) if the image of the space \( \Lambda^{n-1}_K(I) \) by the canonical surjection

\[
\Lambda^{n-1}_K(\mathcal{G}) \longrightarrow \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), s \longmapsto \overline{s},
\]

is an ideal of the Lie algebra \( \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \).

We also will say that a subspace vectoriel \( C \subset \mathcal{G} \) is Cartan subalgebra of the \( n \)-Lie algebra \( \mathcal{G} \) if the image of the space \( \Lambda^{n-1}_K(C) \) by the canonical surjection

\[
\Lambda^{n-1}_K(\mathcal{G}) \longrightarrow \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), s \longmapsto \overline{s},
\]

is a Cartan subalgebra of the Lie algebra \( \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \).

We finally will say that an \( n \)-Lie algebra \( \mathcal{G} \) is a semi simple \( n \)-Lie algebra (nilpotent \( n \)-Lie algebra, solvable \( n \)-Lie algebra, commutative \( n \)-Lie algebra respectively) if the Lie algebra \( \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \) is semi simple (nilpotent, solvable, commutative respectively).

**Proposition 7** If \( \text{Inv}(\mathcal{G}) \) is the space of invariants elements of \( \mathcal{G} \) for the representation

\[
\widetilde{ad}_G : \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \longrightarrow \text{Der}_K(\mathcal{G}), \overline{s} \longmapsto \widetilde{ad}_G(s),
\]

then the image of \( \Lambda^{n-1}_K[\text{Inv}(\mathcal{G})] \) by the canonical surjection

\[
\Lambda^{n-1}_K(\mathcal{G}) \longrightarrow \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}),
\]

is contained in the center of the Lie algebra \( \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \).
Proposition 8 If a subspace $G_0$ of an $n$-Lie algebra $G$ is stable for the representation
\[
\widetilde{ad}_G : \Lambda_K^{n-1}(G)/\mathcal{V}_K(G) \longrightarrow \text{Der}_K(G), \pi \longmapsto ad_G(s),
\]
then $G_0$ is an ideal of the $n$-Lie algebra $G$, i.e. the image of $\Lambda_K^{n-1}(G_0)$ by the canonical surjection
\[
\Lambda_K^{n-1}(G) \longrightarrow \Lambda_K^{n-1}(G)/\mathcal{V}_K(G),
\]
is an ideal of the Lie algebra $\Lambda_K^{n-1}(G)/\mathcal{V}_K(G)$.

Proof. Here, we also reason with indecomposable elements. If we consider $x_1, x_2, \ldots, x_{n-1}$ elements of $G_0$ and $y_1, y_2, \ldots, y_{n-1}$ elements of $G$. We get
\[
\begin{align*}
[x_1 \Lambda x_2 \Lambda \ldots \Lambda x_{n-1}, y_1 \Lambda y_2 \Lambda \ldots \Lambda y_{n-1}] & = - \frac{1}{n-1} \sum_{i=1}^{n-1} x_1 \Lambda x_2 \Lambda \ldots \Lambda x_{i-1} \Lambda \{y_1, y_2, \ldots, y_{n-1}, x_i\} \Lambda x_{i+1} \Lambda \ldots \Lambda x_{n-1}.
\end{align*}
\]
As
\[
\{y_1, y_2, \ldots, y_{n-1}, x_i\} = 0
\]
for $i = 1, 2, \ldots, n - 1$, then
\[
[x_1 \Lambda x_2 \Lambda \ldots \Lambda x_{n-1}, y_1 \Lambda y_2 \Lambda \ldots \Lambda y_{n-1}] = 0.
\]
That ends the proof. \[\blacksquare\]
2.2 Cohomology of an $n$-Lie algebra

When $\mathcal{G}$ is an $n$-Lie algebra, we denote $d_n$ the cohomology operator associated with the representation

$$\tilde{ad}_G : \Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}) \rightarrow \text{Der}_K(\mathcal{G}), \sigma \mapsto \tilde{ad}_G(s).$$

For any $p \in \mathbb{N}$,

$$\mathcal{L}_{sks}^p(\Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G}), \mathcal{G})$$

denotes the $K$-vector space of skew-symmetric $p$-multilinear maps of

$$\Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})$$

into $\mathcal{G}$ and

$$\mathcal{L}_{sks}([\Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})], \mathcal{G}) = \bigoplus_{p \in \mathbb{N}} \mathcal{L}_{sks}^p([\Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})], \mathcal{G}).$$

We will say that the cohomology of the differential complex

$$(\mathcal{L}_{sks}([\Lambda^{n-1}_K(\mathcal{G})/\mathcal{V}_K(\mathcal{G})], \mathcal{G}), d_n)$$

is the cohomology of the $n$-Lie algebra $\mathcal{G}$.

We denote

$$H_n(\mathcal{G}) = \text{Ker}(d_n)/\text{Im}(d_n).$$

**Proposition 9** When $\mathcal{G}$ is an $n$-Lie algebra, then

$$H^0_n(\mathcal{G}) = \text{Inv}(\mathcal{G}).$$

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