Metastability of the proximal point algorithm with multi-parameters

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Abstract

In this article we use techniques of proof mining to analyse a result, due to Yonghong Yao and Muhammad Aslam Noor, concerning the strong convergence of a generalization of the proximal point algorithm which involves multiple parameters. Yao and Noor’s result ensures the strong convergence of the algorithm to the nearest projection point onto the set of zeros of the operator. Our quantitative analysis, guided by Fernando Ferreira and Paulo Oliva’s bounded functional interpretation, provides an effective bound on the metastability for the convergence of the algorithm, in the sense of Terence Tao.

1 Introduction

Proof mining is the term that describes the process of using proof-theoretical techniques to analyse mathematical proofs with the aim of extracting new information. The idea originated in the 1950’s when Georg Kreisel suggested to unwind proofs [25, 26, 7] and has subsequently been substantially developed, mainly by the works of Ulrich Kohlenbach and his collaborators, producing a vast number of results; analysing proofs from areas such as approximation theory, ergodic theory, fixed point theory, optimization theory and the theory of partial differential equations (see e.g. [19, 23, 17]). In the proof mining program proof interpretations are used as tools to extract constructive (i.e. computational) information from given proofs by recursion on the proofs. This information is implicit, hidden behind the use of quantifiers. Kreisel drew attention to the study of proofs of existential statements in particular, aiming at extracting realizers for the existential quantifiers as functions of parameters from the proof. The standard tool to carry out this analysis is Kohlenbach’s monotone functional interpretation which is a functional interpretation based on Kurt Gödel’s Dialectica interpretation [10, 1] that works with upper bounds for witnessing terms instead of precise witnesses. Recently, Fernando Ferreira and Paulo Oliva’s bounded functional interpretation (BFI) [9] has proven to be a valid alternative for the proof mining program, providing new insight to some theoretical questions concerning the elimination of sequential weak compactness [8].

Using the BFI, we analyse a result, due to Yonghong Yao and Muhammad Aslam Noor (Theorem 3), concerning the strong convergence of the proximal point algorithm with multi-parameters, a generalization of the well-known proximal point algorithm, to the nearest projection point onto the set of zeros of the operator [40]. Our main result (Theorem 25) provides an effective bound on the metastable version of the original result, in the sense of Terence Tao [38, 37], i.e. we obtain a function \( \phi : \mathbb{N} \times \mathbb{N}^N \rightarrow \mathbb{N} \) such that

\[
\forall k \in \mathbb{N} \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \phi(k, f) \forall i, j \in [n, n + f(n)] \left( \|z_i - z_j\| \leq \frac{1}{k+1} \right).
\]

Usually, the quantitative results, as well as their proofs, obtained by the proof mining program do not presuppose any particular knowledge of the logical tools because the latter are only used as an intermediate step and are not visible in the final product. Apart from Section 5, our work is no exception and so, knowledge of tools from logic in general and familiarity with the BFI in particular are not necessary to read this paper. Nevertheless, the complexity of the extracted information follows directly from the strength of the logical principles required for the proof. We would also like to point out that our work comes as a natural generalization of [27, 29, 28].

In Section 2 we introduce the proximal point algorithm with multi-parameters as well as associated notions concerning Hilbert spaces and operators. We also state the result by Yao and Noor for which we give a quantitative analysis in the subsequent sections and explain its original proof. In Section 3 we give quantitative versions of the central arguments of the original proof: sequential weak compactness, the projection argument

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and arithmetical comprehension. Our main result is proved in Section 4. We will start by giving quantitative versions of the lemmas used in the original proof and then obtain an effective bound on the metastability for the convergence of the algorithm. We leave some considerations of a logical nature that allow to better understand some of the aspects of the paper to Section 5.

2 Strong convergence of the mPPA

In this section we recall the proximal point algorithm and present the modified version with multi-parameters mPPA. We present a result by Yao and Noor concerning the strong convergence of the mPPA (Theorem 3) and give a detailed description of its proof. This will hopefully guide the reader through our quantitative analysis and the several steps that it requires.

2.1 The algorithm mPPA

We work in a real Hilbert space $H$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We recall that an operator $T : H \to 2^H$ in $H$ is said to be monotone if and only if whenever $(x, y)$ and $(x', y')$ are elements of the graph of $T$, it holds that $(x - x', y - y') \geq 0$. A monotone operator $T$ is maximal monotone if the graph of $T$ is not properly contained in the graph of any other monotone operator on $H$. We denote by $S$ the set of all zeros of $T$, i.e. $S = T^{-1}(0)$. For a comprehensive introduction to convex analysis and the theory of monotone operator in Hilbert spaces we refer to [2].

We fix $T$ a maximal monotone operator and assume henceforth $S$ to be nonempty. For $c > 0$, we use $J_c$ to denote the resolvent of $T$, i.e. the single-valued function defined by

$$J_c = (I + cT)^{-1}.$$

A mapping $f : H \to H$ is called nonexpansive if

$$\forall x, y \in H (\| f(x) - f(y) \| \leq \| x - y \|).$$

The set of fixed points of a mapping $f$ is the set $\text{Fix}(f) := \{ x \in H : f(x) = x \}$. The resolvent mapping $J_c$ is nonexpansive, and for every $c > 0$, the set of fixed points of $J_c$ is $S$. If $f$ is nonexpansive, then $\text{Fix}(f)$ is a closed and convex subset of $H$.

The following lemmas are well-known.

**Lemma 1** (Resolvent identity). For every $a, b > 0$ and every $x \in H$ it holds that

$$J_a(x) = J_b \left( \frac{b}{a} x + \left( 1 - \frac{b}{a} \right) J_a(x) \right).$$

**Lemma 2** ([30]). If $0 < a \leq b$, then for all $x \in H$ it holds that $\| J_a(x) - x \| \leq 2 \| J_b(x) - x \|$.

In the theory of maximal monotone operators, many problems, such as problems of minimization of a function, variational inequalities, etc., can be formulated as finding a zero of a maximal monotone operator (see e.g. [13] and the references therein). The proximal point algorithm PPA [33] is a powerful and successful algorithm in finding a solution of maximal monotone operators. Starting from any initial guess $x_0 \in H$, with an error sequence $(e_n)$, the PPA generates a sequence which approximates the solution.

$$x_{n+1} = J_{c_n}(x_n) + e_n. \quad \text{(PPA)}$$

Ralph Rockafellar showed that (PPA) converges weakly towards a zero provided that $(c_n)$ is bounded away from zero and $\|e_n\|$ is summable, i.e. $\sum_{n=0}^{\infty} \|e_n\| < \infty$. Osman Güler showed in [11], by providing a counter-example, that in general (PPA) does not converge strongly. For this reason, several modifications of the algorithm were proposed (see for example [11, 6, 12, 30, 34, 39]). We will focus on the following multi-parameter version of the proximal point algorithm. Let $z_0 \in H$ be an initial guess. We define

$$z_{n+1} = \lambda_n u + \gamma_n z_n + \delta_n J_{c_n}(z_n) + e_n. \quad \text{(mPPA)}$$

where $u \in H$ is given, and for all $n \in \mathbb{N}$ it holds that $c_n > 0, \lambda_n, \gamma_n, \delta_n \in (0, 1)$ and $\lambda_n + \gamma_n + \delta_n = 1$. 

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2.2 A result by Yao and Noor

Consider the following set of conditions.

\( (H_1) \lim_{n \to \infty} \lambda_n = 0. \)

\( (H_2) \sum_{n=0}^{\infty} \lambda_n = \infty. \)

\( (H_3) 0 < \liminf_{n \to \infty} \gamma_n \leq \limsup_{n \to \infty} \gamma_n < 1. \)

\( (H_4) c_n \geq c, \) where \( c \) is some positive constant.

\( (H_5) c_{n+1} - c_n \to 0. \)

\( (H_6) \sum_{n=1}^{\infty} \|c_n\| < \infty. \)

The result by Yao and Noor analysed in this paper is the following.

Theorem 3. ([40, Theorem 3.3]) Let \((z_n)\) be generated by \((\text{mPPA})\). Assume that \((H_1)-(H_6)\) hold. Then \((z_n)\) converges strongly to a point \(z \in S\), the nearest to \(u\).

Let us go through the structure of the proof of Theorem 3 and the auxiliary results that it requires.

The proof of Theorem 3 relies on Lemma 4 and Lemma 5 below, due to Tomonari Suzuki [36], for which we give quantitative versions in Section 4.1 (Lemma 18 and Lemma 19).1

Lemma 4. ([36, Lemma 2.1]) Let \((z_n)\) and \((w_n)\) be sequences in a Banach space \(E\) and let \((\alpha_n)\) be a sequence in \([0,1]\) such that \(\limsup \alpha_n < 1\). Suppose that \(z_{n+1} = \alpha_n w_n + (1-\alpha_n)z_n\) for all \(n \in \mathbb{N}\), \(\limsup\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq 0\) and \(d := \limsup\|w_n - z_n\| < \infty\). Then

\[ \forall k \in \mathbb{N} \left( \liminf_{n \to \infty} \|w_{n+k} - z_n\| - (1 + \alpha_n + \cdots + \alpha_{n+k-1})d = 0 \right). \]

Lemma 5. ([36, Lemma 2.2]) Let \((x_n)\) and \((z_n)\) be bounded sequences in a Banach space \(E\) and let \((\alpha_n)\) be a sequence in \([0,1]\) with \(0 < \liminf \alpha_n \leq \limsup \alpha_n < 1\). Suppose that \(x_{n+1} = \alpha_n x_n + (1 - \alpha_n)z_n\), for all \(n \in \mathbb{N}\) and \(\limsup\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq 0\). Then \(\lim \|z_n - x_n\| = 0\).

The proof of Theorem 3 is divided in the following steps:

1. Show that \((z_n)\) is bounded. This is just a simple proof by induction and some easy computations.

2. \(\lim \|z_{n+1} - z_n\| = 0\). Letting \(z_{n+1} = \gamma_n z_n + (1 - \gamma_n)y_n\), it is shown first that \(\limsup\|y_{n+1} - y_n\| - \|z_{n+1} - z_n\| \leq 0\). Using Lemma 5 one concludes that \(\lim \|y_n - y_{\ast}\| = 0\) which, by the definition of \(y_n\), is enough to conclude this step.

3. \(\lim \|J_{\alpha_n}(z_n) - z_n\| = 0\). From step (2) and the hypothesis of the theorem one concludes that \(\lim \|J_{\alpha_n}(z_n) - z_n\| = 0\). The conclusion follows from Lemma 2.

4. Projection argument. With \(p\) the projection point of \(u\) onto \(S\) it is shown that \(\forall q \in S\langle p - u, p - q \rangle \leq 0\). (We note that the existence of such point requires using countable choice.)

5. Sequential weak compactness and demiclosedness. Pick a subsequence \((z_{n_j})\) of \((z_n)\) such that \(\limsup (p - u, p - z_{n_j}) = \lim j (p - u, p - z_{n_j})\) and \((z_{n_j})\) converges weakly to some \(q \in S\). Here the following demiclosedness principle is used.

Lemma 6 (Demiclosedness principle [5]). Let \(C\) be a closed convex subset of \(H\) and let \(f : C \to C\) be a nonexpansive mapping such that \(\text{Fix}(f) \neq \emptyset\). Assume that \((x_n)\) is a sequence in \(C\) such that \((x_n)\) weakly converges to \(x \in C\) and \((1 - f)(x_n)\) converges strongly to \(y \in H\). Then \((I - f)(x) = y\).

By step (4) it follows that \(\limsup (p - u, p - z_n) \leq 0\).

6. Main combinatorial part. In this final step it is shown that the conditions of the following lemma are satisfied. This is enough to conclude the result.

Lemma 7. ([39, Lemma 2.5]) Let \((\alpha_n)\) be a sequence of nonnegative real numbers such that for all \(n \in \mathbb{N}\)

\[ a_{n+1} \leq (1 - s_n)a_n + s_n t_n + \delta_n, \]

where \((s_n) \subset [0,1]\) and \((t_n), (\delta_n)\) are such that \(\sum_{n=0}^{\infty} s_n = \infty\), \(\limsup_{n \to \infty} t_n \leq 0\) and \(\sum_{n=0}^{\infty} \delta_n < \infty\). Then \((\alpha_n)\) converges to zero.

\(^1\)In fact, Lemma 4 is only used to prove Lemma 5.
We will need the notion of monotone functional. For that matter we consider the notion of strong majorizability \( \leq^* \) from [4] in the following two particular cases. For functions \( f, g : \mathbb{N} \to \mathbb{N} \), we have

\[
g \leq^* f := \forall n \forall m \leq n (g(m) \leq f(n) \wedge f(m) \leq f(n)).
\]

For functionals \( \varphi, \psi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \), we have

\[
\varphi \leq^* \psi := \forall n \forall f : \mathbb{N} \to \mathbb{N} \forall m \leq n \forall g \leq^* f (\varphi(m, g) \leq \psi(n, f) \wedge \psi(m, g) \leq \psi(n, f)).
\]

We say that \( f \) is monotone if \( f \leq^* f \) and similarly for a functional \( \varphi \). For functions \( f : \mathbb{N} \to \mathbb{N} \), this monotone property coincides with the usual property \( \forall n \in \mathbb{N} (f(n) \leq f(n + 1)) \). Quantifications over monotone functions \( \forall f(f \leq^* f \to \cdots) \) will be abbreviated by \( \forall f(\cdots) \). Due to the particularities of the BFI, our quantitative results quantify over such monotone functions. Note however that for all \( f : \mathbb{N} \to \mathbb{N} \), one has \( f \leq^* f^M \), where \( f^M(n) := \max_{i \leq n} f(i) \). Hence there is no real restriction in working with monotone quantifications.

We start our quantitative analysis of Theorem 3 by giving quantitative versions of the hypothesis of the theorem. We assume that there exist \( a, c \in \mathbb{N} \setminus \{0\} \) and monotone functions \( \ell, L, \Gamma, E : \mathbb{N} \to \mathbb{N} \) such that

\[
(Q_1) \quad \forall k \in \mathbb{N} \forall n \geq \ell(k) \left( \lambda_n \leq \frac{1}{k+1} \right),
\]

\[
(Q_2) \quad \forall k \in \mathbb{N} \left( \sum_{i=1}^{L(k)} \lambda_i \geq k \right),
\]

\[
(Q_3) \quad \forall m \in \mathbb{N} \left( \frac{a}{k} \leq \gamma_m \leq 1 - \frac{a}{k} \right),
\]

\[
(Q_4) \quad \forall n \in \mathbb{N} (c_n \geq \frac{1}{k}).
\]

\[
(Q_5) \quad \forall k \in \mathbb{N} \forall n \geq \Gamma(k) \left( |e_{n+1} - c_n| \leq \frac{k}{k^2} \right).
\]

\[
(Q_6) \quad \forall k \in \mathbb{N} \forall n \in \mathbb{N} \left( \sum_{i=k}^{E(k)+n} \|e_i\| \leq \frac{1}{k+1} \right).
\]

The conditions \( Q_1 - Q_6 \) are quantitative versions of, respectively, the hypothesis \((H_1)-(H_6)\). Indeed, condition \( Q_1 \) states that \( \ell \) is a rate of convergence for the sequence \( (\lambda_n) \); condition \( Q_2 \) postulates that \( L \) is a rate of divergence for the series \( \sum_{i=1}^{\infty} \lambda_i \); condition \( Q_3 \) is the quantitative version of \((H_3)\) together with the fact that \( (\gamma_n) \subset (0,1) \); condition \( Q_4 \) expresses the fact that the terms of the sequence \((c_n)\) are above some positive quantity; condition \( Q_5 \) states that \( \Gamma \) is a rate of convergence for the difference of terms of the sequence \((c_n)\) and condition \( Q_6 \) expresses quantitatively that the sequence of the partial sums of the errors \( e_n \) is a Cauchy sequence with Cauchy rate \( E \).

Our main result (Theorem 25) is a quantitative version of Theorem 3, where we compute an explicit bound on the metastability of the iteration \( mPPA \) under the assumptions \( Q_1 - Q_6 \). The proof will be given progressively in order to emphasize some intermediate results.

3 Avoiding roadblocks

From the point of view of proof mining the analysis of the original proof of Theorem 3 presents some difficulties that prevent the extraction of simple bounds. The difficulties are in the projection argument in step (4), the weak compactness in step (5) and the assumed existence of the lim sup in Suzuki’s lemmas. In this section we make some small changes to the original argument in order to avoid such problems. A more detailed explanation to why these are problematic will be given in Section 5.

3.1 Projection argument

In this section we deal with the following projection argument which is used in the original proof

\[
\exists p \in S \forall k \in \mathbb{N} \forall q \in S \left( \|p - u\|^2 \leq \|q - u\|^2 + \frac{1}{k+1} \right),
\]

stating that there is a zero of \( T \) that is the nearest to \( u \). The squares are added here only for an easier connection to the inner product of the space that will be required below.

As noticed by Kohlenbach [18], instead of (2), it is enough for the quantitative analysis to consider the weaker statement

\[
\forall k \in \mathbb{N} \exists p \in S \forall q \in S \left( \|p - u\|^2 \leq \|q - u\|^2 + \frac{1}{k+1} \right).
\]
While the proof of (2) requires the use of countable choice, the statement (3) can be proved by a simple induction argument and this fact is reflected on the extracted bounds, which are recursively defined.

In [8], a detailed explanation of the analysis of this projection argument was shown, using the bounded functional interpretation. There, the assumption that one is working in a bounded set plays an important role in simplifying the interpretation of the projection. Although we do not have that assumption here, we can consider the weak projection statement in (3) restricted to a big enough ball containing some zero point \( s \in S \).

The idea is that since we are concerned with finding the zero of \( T \) which is closer to \( u \), we can restrict our search to zeros that are closer to \( u \) than \( s \). In fact, we can restrict the argument to a ball with radius greater than or equal to \( \| s - u \| + \| u \| + 1 \).

Let us elaborate on this. For \( n \in \mathbb{N} \), let \( B(n; 0) \) denote the (closed) ball centered at zero with radius \( n \), i.e. \( B(n; 0) := \{ x \in H : \| x \| \leq n \} \). Consider a natural number \( \tilde{N} \) such that \( \tilde{N} \geq \| s - u \| + \| u \| + 1 \). Consider

\[
\forall k \in \mathbb{N} \exists p \in S \cap B(\tilde{N}; 0) \forall q \in S \cap B(\tilde{N}; 0) \left( \| p - u \|^2 \leq \| q - u \|^2 + \frac{1}{k + 1} \right),
\]

which is the projection of \( u \) over the set \( S \cap B(\tilde{N}; 0) \). Since \( S \cap B(\tilde{N}; 0) \) is a nonempty set, (4) is clearly true.

We can replace the projection statement over \( S \) with this one over \( S \cap B(\tilde{N}; 0) \). To see that (3) implies (4), given \( k \in \mathbb{N} \), assume there is \( p \in S \) such that \( \forall y \in S \left( \| p - u \|^2 \leq \| y - u \|^2 + \frac{1}{(k+1)^2} \right) \).

Then, it suffices to see that such point \( p \) is in \( B(\tilde{N}; 0) \). From the assumption on \( p \), we have that for all \( q \in S \)

\[
\| p - u \| \leq \| q - u \| + \frac{1}{k + 1}.
\]

Now, since \( s \in S \),

\[
\| p \| \leq \| p - u \| + \| u \| \leq \| s - u \| + \frac{1}{k + 1} + \| u \| \leq \tilde{N},
\]

and \( p \in B(\tilde{N}; 0) \).

To show that (4) also implies (3), assume that for a given \( k \in \mathbb{N} \) we have \( p \in S \) such that

\[
\forall q \in S \cap B(\tilde{N}; 0) \left( \| p - u \|^2 \leq \| q - u \|^2 + \frac{1}{k + 1} \right).
\]

We just have to see that the above property extends to all \( S \). Take \( q \in S \setminus B(\tilde{N}; 0) \). Hence, \( \| q \| > \tilde{N} \) and we have

\[
0 \leq \| s - u \| + \| u \| - \| u \| \leq \tilde{N} - \| u \| < \| q \| - \| u \| \leq \| q - u \|.
\]

Hence

\[
\| p - u \|^2 \leq \| s - u \|^2 + \frac{1}{k + 1} \leq \| q - u \|^2 + \frac{1}{k + 1}.
\]

The arguments above show that it is innocuous to use (4) instead of (3). In fact, the only reason to do so is to deal with the bounded set \( S \cap B(\tilde{N}; 0) \) instead of the set \( S \), which significantly simplifies the analysis.

Since the zero set of \( T \) coincides with the fixed point set of its resolvent functions, we equivalently have

\[
\forall k \in \mathbb{N} \exists p \in F \cap B(\tilde{N}; 0) \forall q \in F \cap B(\tilde{N}; 0) \left( \| p - u \|^2 \leq \| q - u \|^2 + \frac{1}{k + 1} \right),
\]

where \( F := \text{Fix} \left( J_{\frac{1}{2}} \right) \).

\textbf{Notation 8.} From now on, we will write \( J \) instead of \( J_{\frac{1}{2}} \) and \( J_{\alpha} \) instead of \( J_{\alpha} \).

For each \( r \in \mathbb{N} \) and function \( f : \mathbb{N} \to \mathbb{N} \) we denote the \( r \)-fold iteration of \( f \) by \( f^{(r)} \). I.e. \( f^{(0)} \equiv \text{Id} \) and \( f^{(r+1)} = f(f^{(r)}) \).

The mining of (5) yields the following result.

\textbf{Proposition 9.} Let \( u \in H \) and \( \tilde{N} \in \mathbb{N} \) be such that \( \tilde{N} \geq \| s - u \| + \| u \| + 1 \), for some \( s \in S \). Define \( N := 2 \tilde{N} \). For any \( k \in \mathbb{N} \) and monotone function \( f : \mathbb{N} \to \mathbb{N} \), there are \( p \in B(\tilde{N}; 0) \) and \( n \leq f^{(r)}(0) \) such that \( \| J(p) - p \| \leq \frac{1}{f^{(n+1)}} \) and

\[
\forall q \in B(\tilde{N}; 0) \left( \| J(q) - q \| \leq \frac{1}{n + 1} \to \| p - u \|^2 \leq \| q - u \|^2 + \frac{1}{k + 1} \right),
\]

with \( r := N^2(k + 1) \).
We omit the proof of Proposition 9 and for details we refer the reader to [28] or [8]. In fact, the proof is essentially the same as in [8, Proposition 3.1], taking for \( x_0 \) the fixed point \( s \) and noticing that \( N \) is a bound on the diameter of \( B(\tilde{N}; 0) \). An analysis of the projection argument via the monotone functional interpretation was previously carried out by Kohlenbach in [18].

We considered only the fixed points of \( J \) since the fixed point set of all resolvent functions coincide. As a consequence, we have the following quantitative result.

**Lemma 10.** Let \((c_n) \subset \mathbb{R}^+ \) and \( c \in \mathbb{N} \setminus \{0\} \). For any \( k, n \in \mathbb{N} \) and any \( p \in H \),

\[
\|J(p) - p\| \leq \frac{1}{c_n c(k + 1)} \to \|J_n(p) - p\| \leq \frac{1}{k + 1}.
\]

**Proof.** Let \( e = J(p) - p \) and assume \( \|e\| \leq \frac{1}{c_n c(k + 1)} \). We have

\[
J(p) = p + e \leftrightarrow p \in p + e + \frac{1}{c} T(p + e)
\]

\[
\leftrightarrow -e \cdot c \in T(p + e)
\]

\[
\leftrightarrow p + (1 - c \cdot c_n)e \in p + e + c_n T(p + e)
\]

\[
\leftrightarrow J_n(p + (1 - c \cdot c_n)e) = p + e.
\]

Hence

\[
\|J_n(p) - p\| \leq \|J_n(p) - J_n(p + (1 - c \cdot c_n)e)\| + \|J_n(p + (1 - c \cdot c_n)e) - p\|
\]

\[
\leq (1 + |1 - c \cdot c_n|)\|e\| = c \cdot c_n\|e\| \leq \frac{1}{k + 1}.
\]

Let \( \zeta(k, n) := \max_{u \leq n} \{c_u\} \cdot c(k + 1) - 1 \). It follows from Lemma 10 that

\[
\|J(p) - p\| \leq \frac{1}{\zeta(k, n) + 1} \to \forall n' \leq n \left( \|J_{n'}(p) - p\| \leq \frac{1}{k + 1} \right).
\]

### 3.2 Sequential weak compactness

Sequential weak compactness, together with the demiclosedness principle, is used to show that \( \limsup (\tilde{p} - u, \tilde{p} - z_m) \leq 0 \), where \( \tilde{p} \) is the projection point of \( u \) onto \( S \). This means that there exists \( \tilde{p} \in S \) (the projection point) such that

\[
\forall k \in \mathbb{N} \exists n \in \mathbb{N} \forall m \geq n \left( \langle \tilde{p} - u, \tilde{p} - z_m \rangle \leq \frac{1}{k + 1} \right).
\]

We will see that, working with the weak projection argument (5), we have the weaker statement

\[
\forall k \in \mathbb{N} \exists p \in S \exists n \in \mathbb{N} \forall m \geq n \left( \langle p - u, p - z_m \rangle \leq \frac{1}{k + 1} \right),
\]

which is still enough to carry out the proof, avoiding the use of sequential weak compactness.

We start by relating the conclusion of Proposition 9 with the inner product.

The following results are essentially due to Kohlenbach and are an adaptation of [18, Lemmas 2.3 and 2.7] to the unbounded setting. For \( \gamma \in [0, 1] \), we will write \( w_1(p, q) \) as an abbreviation of \((1 - \gamma)p + \gamma q\).

**Lemma 11.** For all \( k, n \in \mathbb{N} \) and \( p_1, p_2 \in B(n; 0) \),

\[
\bigwedge_{j=1}^{2} \left( \|J(p_j) - p_j\| \leq \frac{1}{2k(k + 1)^2} \right) \to \forall \gamma \in [0, 1] \left( \|J(w_\gamma(p_1, p_2)) - w_\gamma(x_1, x_2)\| \leq \frac{1}{k + 1} \right).
\]

**Lemma 12.** For all \( k, n \in \mathbb{N} \) and \( p, q \in B(n; 0) \),

\[
\forall \gamma \in [0, 1] \left( \|p - u\|^2 \leq \|w_\gamma(p, q) - v_0\|^2 + \frac{1}{4n^2(k + 1)^2} \right) \to \langle p - u, p - q \rangle \leq \frac{1}{k + 1}.
\]

In the following we assume \( u \in H \) to be arbitrary and \( \tilde{N} \in \mathbb{N} \) to be such that \( \tilde{N} \geq \|s - u\| + \|u\| + 1 \), for some \( s \in S \). We define \( N := 2\tilde{N} \).
Proposition 13. For any \( k \in \mathbb{N} \) and monotone function \( f : \mathbb{N} \to \mathbb{N} \), there are \( n \leq 12N \left( \frac{f(R)(0) + 1}{k+1} \right)^2 \) and \( p \in B(\tilde{N};0) \) such that \( \|J(p) - p\| \leq \frac{1}{f(n) + 1} \) and
\[
\forall q \in B(\tilde{N};0) \left( \|J(q) - q\| \leq \frac{1}{n+1} \to \forall \gamma \in [0,1] \left( \|p - u\| \leq \|\varphi_t(p, q) - u\| + \frac{1}{k+1} \right) \right),
\]
with \( R := N^2(k + 1) \) and \( \hat{f}(m) := \max\{f(12N(m + 1)^2), 12N(m + 1)^2\} \).

The proof of Proposition 13 is as in [8, Corollary 3.5], using Proposition 9 and Lemma 11 applied to \( n = \tilde{N} \).

By Lemma 12 with \( n = \tilde{N} \) and Proposition 13 with \( k \) replaced with \( N^2(k + 1)^2 - 1 \), we obtain the following proposition.

Proposition 14. For any \( k \in \mathbb{N} \) and monotone function \( f : \mathbb{N} \to \mathbb{N} \), there are \( n \leq 12N \left( \frac{f(R)(0) + 1}{k+1} \right)^2 \) and \( p \in B(\tilde{N};0) \) such that \( \|J(p) - p\| \leq \frac{1}{f(n) + 1} \) and
\[
\forall q \in B(\tilde{N};0) \left( \|J(q) - q\| \leq \frac{1}{n+1} \to \langle p - u, p - q \rangle \leq \frac{1}{k+1} \right),
\]
with \( R := N^2(k + 1)^2 \) and \( \hat{f}(m) := \max\{f(12N(m + 1)^2), 12N(m + 1)^2\} \).

In Lemma 23 (iii) below, we compute a monotone function \( \xi \) satisfying
\[
\forall k \in \mathbb{N} \forall \xi : \mathbb{N} \to \mathbb{N} \exists n \leq \xi(k, f) \forall m \in [n, n + f(n)] \left( \|J(z_m) - z_m\| \leq \frac{1}{k+1} \right).
\]

Furthermore, in Lemma 22, we show that the sequence \((z_n)\) is bounded and compute an explicit natural number \( \tilde{N} \geq \|s - u\| + \|u\| + 1 \) such that \((z_n) \subset B(\tilde{N};0) \). We add this to our assumption about the natural number \( \tilde{N} \).

Proposition 15. Let \( \xi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) be a monotone function satisfying (7). For any \( k \in \mathbb{N} \) and monotone function \( f : \mathbb{N} \to \mathbb{N} \), there are \( n \leq \tilde{\xi}(k, f) \) and \( p \in B(\tilde{N};0) \) such that \( \|J(p) - p\| \leq \frac{1}{g(n) + 1} \) and
\[
\forall m \in [n, n + f(n)] \left( \langle p - u, p - z_{m+1} \rangle \leq \frac{1}{k+1} \right),
\]
where \( \tilde{\xi}(k, f) := \xi(12N(\hat{g}(R)(0) + 1)^2, f + 1) \) with \( R := N^4(k + 1)^2 \), \( g(m) := f(\xi(m, f + 1)) \) and \( \hat{g}(m) := \max\{g(12N(m + 1)^2), 12N(m + 1)^2\} \).

Proof. Let \( k \in \mathbb{N} \) and a monotone function \( f \) be given. By Proposition 14 applied to \( k \) and the function \( g \), we get \( n' \leq 12N(\hat{g}(R)(0) + 1)^2 \) and \( p \in B(\tilde{N};0) \) such that \( \|J(p) - p\| \leq \frac{1}{g(n') + 1} \) and
\[
\forall q \in B(\tilde{N};0) \left( \|J(q) - q\| \leq \frac{1}{n' + 1} \to \langle p - u, p - q \rangle \leq \frac{1}{k+1} \right).
\]
By (7), there is \( n \leq \xi(n', f + 1) \leq \xi(12N(\hat{g}(R)(0) + 1)^2, f + 1) \) such that
\[
\forall m \in [n, n + f(n) + 1] \left( \|J(z_m) - z_m\| \leq \frac{1}{n' + 1} \right).
\]
Hence, \( \forall m \in [n, n + f(n)] \left( \|J(z_{m+1}) - z_{m+1}\| \leq \frac{1}{n' + 1} \right) \). Since \((z_n) \subset B(\tilde{N};0) \), by (8) we conclude that
\[
\forall m \in [n, n + f(n)] \left( \langle p - u, p - z_{m+1} \rangle \leq \frac{1}{k+1} \right).
\]
Finally, by the monotonicity of the function \( f \),
\[
\|J(p) - p\| \leq \frac{1}{g(n') + 1} = \frac{1}{f(\xi(n', f + 1)) + 1} \leq \frac{1}{f(n) + 1}.
\]

Proposition 15 corresponds to the elimination of the sequential weak compactness argument. It can be seen as an application of the general principle in [8, Proposition 4.3] with \( \alpha(k, f) = \xi(k, f + 1) \), where the sequence being considered is \((z_{m+1})\), \( \beta \) is given by Proposition 14 and \( \varphi(x, y) = \langle x - u, y \rangle \).
3.3 Rational approximation of the lim sup

In this section we show that the assumption of the existence of the lim sup, as in Lemma 4, can be replaced by a rational approximation. A detailed explanation on the origin of these lemmas is given in Section 5.

The idea is that, by working with approximated notions, one can relax the properties of the lim sup to something which is already satisfied by a suitable rational number. We start with the following easy result.

**Lemma 16.** Let \( N \in \mathbb{N} \) and \((x_n)\) be a sequence of real numbers such that \( \forall n \in \mathbb{N}(0 \leq x_n \leq N) \). Then

\[
\forall k, n \in \mathbb{N} \exists f : \mathbb{N} \to \mathbb{N} \exists m \in [n, n + f(n)] \exists p < N(k + 1) \quad \left( x_m \geq \frac{p}{k + 1} \land \forall m' \in [n, n + f(n)] \left( x_{m'} \leq \frac{p + 1}{k + 1} \right) \right). \tag{9}
\]

**Proof.** Suppose towards a contradiction that (9) does not hold. Then there exist \( k, n \in \mathbb{N} \) and a monotone function \( f \) such that for all \( p < N(k + 1) \) it holds that

\[
\forall m \in [n, n + f(n)] \left( x_m < \frac{p}{k + 1} \right) \lor \exists m' \in [n, n + f(n)] \left( x_{m'} > \frac{p + 1}{k + 1} \right). \tag{10}
\]

This implies

\[
\forall p < N(k + 1) \left( A(p) \lor \neg A(p + 1) \right),
\]

where \( A(p) := \forall m \in [n, n + f(n)] \left( x_m < \frac{p}{k + 1} \right) \). One easily shows by induction on \( M \in \mathbb{N} \) that

\[
\forall M \left( \forall p < M \left( A(p) \lor \neg A(p + 1) \right) \rightarrow (A(0) \lor \neg A(M + 1)) \right).
\]

Hence, with \( M = N(k + 1) - 1 \) we conclude that

\[
\forall m \in [n, n + f(n)] (x_m < 0) \lor \exists m \in [n, n + f(n)] (x_m \geq N).
\]

Hence \( \exists m \in [n, n + f(n)] (x_m \geq N) \). Now, by (10), for \( p = N(k + 1) - 1 \) and the hypothesis, we have that for all \( m \in [n, n + f(n)] \) it holds that \( x_m < \frac{N(k + 1) - 1}{k + 1} = N - \frac{1}{k + 1} \), which gives a contradiction. We conclude that (9) holds. \( \square \)

Lemma 4 uses the following property of the lim sup.

\[
\forall k, M, J \in \mathbb{N} \exists m \geq M \forall n \geq m \left( x_{m + J} \geq \limsup x_n - \frac{1}{k + 1} \land x_n \leq \limsup x_n + \frac{1}{k + 1} \right). \tag{11}
\]

The next result is a quantitative version of (11).

**Lemma 17.** Let \( N \in \mathbb{N} \) and \((x_n)\) be a sequence of real numbers such that \( \forall n \in \mathbb{N}(0 \leq x_n \leq N) \). Let \( k, M, J \in \mathbb{N} \) and \( f : \mathbb{N} \to \mathbb{N} \) be monotone, let \( P := N(k + 1) \). For \( i \in \{0, \ldots, P\} \) define \( n_i = M + iJ \) and

\[
r_i := \begin{cases} 0, & i = P \\ J + r_{i+1} + f(n_{i+1} + r_{i+1}), & i < P. \end{cases}
\]

Then

\[
\exists p < P \exists m \in [M, \theta] \forall n \in [m, m + f(m)] \left( x_{m + J} \geq \frac{p}{k + 1} \land x_n \geq \frac{p + 1}{k + 1} \right), \tag{12}
\]

where \( \theta = \theta(k, M, N, f) := M + (P - 1)J + r_0 \).

**Proof.** Let \( k, J, M \in \mathbb{N} \) and \( f \) be a given monotone function. We define, for each \( i \leq P \), the monotone functions \( g_i := \lambda m.r_i \). We apply (9) with \( k = k, f = g_i \) and \( n = n_i \), for \( i \leq P \). Then, we find, for each \( i \leq P \), \( n_i \in [n_i, n_i + r_i] \) and \( p_i < P \) such that \( \forall n \in [n_i, n_i + r_i] \left( x_n \leq \frac{p_i + 1}{k + 1} \right) \) and \( x_{m_i} \geq \frac{p_i}{k + 1} \). Now, there exists \( i_0 \leq P \) such that \( p_{i_0} \leq p_{i_0 + 1} \), otherwise there would be a sequence of length \( P + 1 \) of natural numbers such that \( p_p < p_{p-1} < \cdots < p_1 < p_0 < P \), which is absurd. Define the natural numbers \( m := m_{i_0 + 1} - J \) and \( p := p_{i_0 + 1} \). Clearly \( m \in [M, \theta] \) and \( p < P \). We have that \( x_{m + J} \geq \frac{p}{k + 1} \). To conclude the result it is enough to show that \( [m, m + f(m)] \subseteq [n_{i_0}, n_{i_0} + r_{i_0}] \). Indeed, we would get, for \( n \in [m, m + f(m)] \) that

\[
x_n \leq \frac{p_{i_0 + 1} + 1}{k + 1} \leq \frac{p_{i_0} + 1}{k + 1} = \frac{p}{k + 1}.
\]

We have that \( m = m_{i_0 + 1} - J \geq n_{i_0 + 1} - J = n_{i_0} \) and, since \( f \) is monotone, \( m + f(m) \leq m_{i_0 + 1} + f(m_{i_0 + 1}) \leq n_{i_0 + 1} + r_{i_0 + 1} + f(n_{i_0 + 1} + r_{i_0 + 1}) = n_{i_0} + J + r_{i_0 + 1} + f(n_{i_0 + 1} + r_{i_0 + 1}) = n_{i_0} + r_{i_0} \). Hence \( [m, m + f(m)] \subseteq [n_{i_0}, n_{i_0} + r_{i_0}] \). \( \square \)
4 Quantitative analysis

In this section we carry out the quantitative analysis of Theorem 3. In Section 4.1 we state and prove quantitative versions of the main lemmas used in the original proof; i.e. quantitative versions Lemmas 4, 5 and 7. As for the latter we rely on a result from [32].

In Section 4.2 we turn to the proof of the theorem itself. The proof is divided into several steps in order to emphasize some intermediate results.

4.1 A quantitative version of Suzuki’s lemmas

We now turn to the two main lemmas required in the original proof. We present quantitative versions for each of these results. The first lemma is a partial quantitative version of Lemma 4, which is enough for what follows.

**Lemma 18** (First main lemma). Let \((z_n), (w_n)\) be sequences in a normed space \(X\). Let \(\alpha_n \in [0,1]\) and \(a \in \mathbb{N} \setminus \{0\}\) be such that
\[
\forall m \geq a \left( \alpha_m \leq 1 - \frac{1}{a} \right). \tag{13}
\]

Suppose that \(z_{n+1} = \alpha_n w_n + (1 - \alpha_n) z_n\), for all \(n \in \mathbb{N}\) and that there exists a monotone function \(\psi: \mathbb{N} \to \mathbb{N}\) satisfying
\[
\forall r \in \mathbb{N} \forall n \geq \psi(r) \left( \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r+1} \right). \tag{14}
\]

Let \(N \in \mathbb{N}\) be such that \(\forall n \in \mathbb{N} (\|w_n - z_n\| \leq N)\). Then
\[
\forall k, l \in \mathbb{N} \forall J \in \mathbb{N} \setminus \{0\} \exists f: \mathbb{N} \to \mathbb{N} \exists m \in [l, \varphi(k, f)] \exists p < NR(a, k, J)
\[
\left[ \left( \|w_{m+J} - z_m\| - \frac{1}{p} \prod_{i=0}^{J-1} \alpha_m + 1 \right) \frac{p + 1}{R(a, k, J)} \geq - \frac{1}{k+1} \right] \land
\[
\forall n \in [m, m + J + f(m)] \left( \|w_n - z_n\| \leq \frac{p + 1}{R(a, k, J)} \right),
\]
where \(R(a, k, J) = J(2J + 1)a^J(k + 1)\) and \(\varphi(k, f) = \varphi(f, l, J, a, \psi, N) := \theta(R(a, k, J) - 1, M(R(a, k, J) - 1), J, N, g)\), with \(M(r) = M(a, l, \psi, J, r) := \max\{a, l, \psi(J(r + 1))\}, g: \mathbb{N} \to \mathbb{N}\) is the monotone function defined by \(g(m) = J + f(m)\) and \(\theta\) is as in Lemma 17.

**Proof.** We have, for any \(n \in \mathbb{N}\),
\[
\|w_{n+1} - z_{n+1}\| - \|w_n - z_n\| \leq \left( \sum_{i=0}^{j-1} \|w_{n+i+1} - z_{n+i+1}\| - \|w_{n+i} - z_{n+i}\| \right)
\]
\[
\leq \left( \frac{1}{j(r+1) + 1} \right) \leq \frac{1}{r+1}.
\]

Hence
\[
\forall j \in \mathbb{N} \forall r \in \mathbb{N} \forall n \geq \psi(j(r + 1)) \left( \|w_{n+j} - z_{n+j}\| - \|w_n - z_n\| \leq \frac{1}{r+1} \right) \tag{15}
\]
Given \( l, r, J \in \mathbb{N} \), let \( M := M(r) = \max\{a, l, \psi(J(r + 1))\} \). We show that

\[
\forall r, l \in \mathbb{N} \forall J \in \mathbb{N} \forall n \geq M \forall j \leq J \left( \alpha_n \leq 1 - \frac{1}{a} \wedge \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r + 1} \wedge \|w_{n+j} - z_{n+j}\| - \|w_n - z_n\| \leq \frac{1}{r + 1} \right). \tag{16}
\]

By (13) we have \( \forall n \geq M \left( \alpha_n \leq 1 - \frac{1}{a} \right) \). The monotonicity of \( \psi \) entails that \( \psi(J(r + 1)) \geq \psi(r) \) and \( \psi(J(r + 1)) \geq \psi(j + r) \), for \( j \leq J \), because \( J(r + 1) \geq r \). Then

\[
\forall n \geq M \left( \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r + 1} \right),
\]

by (14) and

\[
\forall n \geq M \left( \|w_{n+j} - z_{n+j}\| - \|w_n - z_n\| \leq \frac{1}{r + 1} \right),
\]

by (15). Hence (16) holds. Let \( r, l \in \mathbb{N} \), \( J \in \mathbb{N} \) be arbitrary and \( f : \mathbb{N} \to \mathbb{N} \) be a monotone function. Applying Lemma 17 to \( r, M, J \) and \( g : \mathbb{N} \to \mathbb{N} \) defined by \( g(m) = J + f(m) \), we find \( p < N(r + 1) \) and \( m \in [M, \theta] \) such that \( x_{m+j} \geq \frac{p}{r+1} \) and \( \forall n \in [m, m + g(m)] \left( x_n \leq \frac{p + 1}{r + 1} \right) \), where \( x_n := \|w_n - z_n\| \) and \( \theta = \theta(r, M, J, N, g) \). Thus

\[
\forall r, l \in \mathbb{N} \forall J \in \mathbb{N} \setminus \{0\} \forall \theta : \mathbb{N} \to \mathbb{N} \exists m \in [l, \theta] \exists p < N(r + 1)
\]

\[
\left[ \forall n \geq m \left( \alpha_n \leq 1 - \frac{1}{a} \wedge \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r + 1} \right) \wedge \|w_{n+j} - z_{n+j}\| \geq \frac{p}{r + 1} \wedge \forall n \in [m, m + J + f(m)] \left( \|w_n - z_n\| \leq \frac{p + 1}{r + 1} \right) \right].
\]

We now argue that for all \( j \leq J - 1 \),

\[
\|w_{m+j} - z_{m+j}\| \geq \left( 1 + \sum_{i=j}^{J-1} \alpha_{m+i} \right) \frac{p + 1}{r + 1} \frac{(J - j)(2J + 1)a^{J-j}}{r + 1}. \tag{17}
\]

We have

\[
\frac{p}{r + 1} \leq \|w_{m+j} - z_{m+j}\|
\]

\[
= \|w_{m+j} - \alpha_{m+j-1}w_{m+j-1} - (1 - \alpha_{m+j-1})z_{m+j-1}\|
\]

\[
\leq \alpha_{m+j-1} \|w_{m+j} - w_{m+j-1}\| + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

\[
\leq \alpha_{m+j-1} \|w_{m+j} - z_{m+j-1}\| + \frac{1}{r + 1} + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

\[
= \alpha_{m+j-1} \|w_{m+j} - z_{m+j-1}\| + \frac{1}{r + 1} + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

\[
\leq \alpha_{m+j-1} \frac{p + 1}{r + 1} + \frac{1}{r + 1} + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|.
\]

Hence

\[
\|w_{m+j} - z_{m+j-1}\| \geq \frac{1 - \alpha_{m+j-1}^2}{1 - \alpha_{m+j-1}} \left( \frac{p + 1}{r + 1} + \frac{2}{r + 1} \right)
\]

\[
\left( 1 + \alpha_{m+j-1} \right) \frac{p + 1}{r + 1} - \frac{2a}{r + 1}
\]

\[
\geq \left( 1 + \alpha_{m+j-1} \right) \frac{p + 1}{r + 1} - \frac{2a}{r + 1}
\]

\[
\geq \left( 1 + \alpha_{m+j-1} \right) \frac{p + 1}{r + 1} - \frac{(2J + 1)a}{r + 1}.
\]

So, (17) holds for \( j = J - 1 \). To conclude we assume that (17) holds for some \( j \in \{1, \ldots, J - 1\} \) and want to
see that it holds for $j - 1$. Since

\[
\left(1 + \sum_{i=j}^{J-1} \alpha_{m+i}\right) \frac{p+1}{r+1} - \frac{(J-j)(2J+1)a^{J-j}}{(r+1)} \leq \|w_{m+j} - z_{m+j}\|
\]

\[
= \|w_{m+j} - \alpha_{m+j-1}w_{m+j-1} - (1 - \alpha_{m+j-1})z_{m+j-1}\|
\]

\[
\leq \alpha_{m+j-1} \|w_{m+j} - w_{m+j-1}\| + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

\[
\leq \alpha_{m+j-1} \sum_{i=j-1}^{J-1} \|w_{m+i} - w_{m+i+1}\| + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

\[
= \alpha_{m+j-1} \sum_{i=j-1}^{J-1} \alpha_{m+i} \|w_{m+i} - z_{m+i}\| + \frac{J}{r+1} + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

\[
\leq \alpha_{m+j-1} \sum_{i=j-1}^{J-1} \alpha_{m+i} \left(\frac{p+1}{r+1}\right) + \frac{J}{r+1} + (1 - \alpha_{m+j-1}) \|w_{m+j} - z_{m+j-1}\|
\]

we obtain

\[
\|w_{m+j} - z_{m+j-1}\| \geq \frac{p+1}{r+1} \sum_{i=j}^{J-1} \alpha_{m+i} \frac{(J-j)(2J+1)a^{j-j+1}}{(r+1)}.
\]

which implies that (17) holds for some $j \in \{1, \ldots, J-1\}$ as we wanted. Instantiating $j$ with 0 in (17) we obtain

\[
\|w_{m+j} - z_{m}\| \geq \left(1 + \sum_{i=0}^{J-1} \alpha_{m+i}\right) \frac{p+1}{r+1} \frac{(J)(2J+1)a^{j}}{r+1}.
\]

We have showed that

\[
\forall r, l \in \mathbb{N} \forall J \in \mathbb{N} \setminus \{0\} \forall f : \mathbb{N} \to \mathbb{N} \exists m \in \{l, \theta(r, M(r), J, N, g)\} \exists p < N(r+1) \quad \left[\begin{array}{l}
\|w_{m+j} - z_{m}\| \geq \left(1 + \sum_{i=0}^{J-1} \alpha_{m+i}\right) \frac{p+1}{r+1} \frac{(J)(2J+1)a^{j}}{r+1} \\
\|w_{m+j} - z_{m+j}\| \geq \frac{p}{r+1} \wedge \forall n \in [m, m + J + f(m)] \left(\|w_{n} - z_{n}\| \leq \frac{p+1}{r+1}\right) \quad \tag{18}
\end{array}\right].
\]

We conclude the result by putting $r = R(a, k, J) - 1$ in (18). \hfill \square

The second main lemma is a quantitative version of Lemma 5.

**Lemma 19 (Second main lemma).** Let $(z_n), (w_n)$ be sequences in a normed space $X$ such that $\|z_n\|, \|w_n\| \leq N$, for some $N \in \mathbb{N}$. Let $(\alpha_n) \subset [0, 1]$ and $a \in \mathbb{N} \setminus \{0\}$ be such that

\[
\forall m \geq a \left(\frac{1}{a} \leq \alpha_m \leq 1 - \frac{1}{a}\right).
\]

Suppose that $z_{n+1} = \alpha_n w_n + (1 - \alpha_n) z_n$, for all $n \in \mathbb{N}$ and that there exists a monotone function $\psi : \mathbb{N} \to \mathbb{N}$ such that

\[
\forall r \in \mathbb{N} \forall n \geq \psi(r) \left(\|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{r+1}\right). \tag{19}
\]
Then $$\forall k \in \mathbb{N} \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \chi(k, f) \forall m \in [n, n + f(n)] \left( \|w_m - z_m\| \leq \frac{1}{k + 1} \right),$$ with $$\chi(k, f) = \chi(k, f, a, \psi, N) = \varphi(k, f, a, J, \alpha, \psi, N),$$ where $$\varphi$$ is as in Lemma 18 and $$J := \max\{2Na(k+1), 1\}.$$ Proof. Suppose towards a contradiction that there exist $$k_0 \in \mathbb{N}$$ and a monotone function $$f_0$$ such that 

$$\forall m \leq \chi(k_0, f_0) \exists n \in [m, m + f_0(m)] \left( \|w_n - z_n\| > \frac{1}{k_0 + 1} \right).$$

We have that $$J \geq 1$$ and 

$$\left(1 + \frac{J}{a}\right) \frac{1}{k_0 + 1} \geq 2N + \frac{1}{k_0 + 1}.$$ Applying Lemma 18 with $$k = k_0, l = a, J = J$$ and $$f = f_0,$$ we find $$p \in \mathbb{N}$$ and $$m \in [a, \chi(k_0, f_0, a, J, \alpha, \psi, N)] = [a, \chi(k_0, f_0)]$$ such that

$$\|w_{m+j} - z_m\| \geq \left(1 + \sum_{i=0}^{J-1} \alpha_{m+i}\right) \frac{p + 1}{R(a, k_0, J)} \frac{1}{k_0 + 1} \wedge$$

$$\|w_{m+j} - z_{m+j}\| \geq \frac{p}{R(a, k_0, J)} \wedge \forall n \in [m, m + J + f_0(m)] \left( \|w_n - z_n\| \leq \frac{p + 1}{R(a, k_0, J)} \right).$$

We have $$[m, m + f_0(m)] \subseteq [m, m + J + f_0(m)],$$ and then $$\frac{1}{m+1} < \|w_n - z_n\| \leq \frac{p + 1}{R(a, k_0, J)},$$ for some $$n \in \mathbb{N}$$. Hence

$$2N + \frac{1}{k_0 + 1} \leq \left(1 + \frac{J}{a}\right) \frac{1}{k_0 + 1} \leq \left(1 + \sum_{i=0}^{J-1} \alpha_{m+i}\right) \frac{1}{k_0 + 1} \leq \frac{p + 1}{R(a, k_0, J)} \leq \|w_{m+j} - z_m\| + \frac{1}{k_0 + 1} \leq 2N + \frac{1}{k_0 + 1},$$

a contradiction. □

As mentioned in Section 2.2, the final step of the proof of Theorem 3 is an application of Lemma 7. There $$s_n = \|z_n - p\|,$$ with $$p$$ the projection point of $$u$$ onto $$S.$$ However, using approximations to the projection point instead of the projection point itself, the inequality (1) only holds with $$s_n + v_n$$ in place of $$s_n,$$ for $$(v_n)$$ a certain sequence of errors. The following result from [32] corresponds to a quantitative version of this statement.

Lemma 20 ([32]). Let $$(s_n)$$ be a bounded sequence of non-negative real numbers and $$D \in \mathbb{N}$$ a positive upper bound on $$(s_n).$$ Consider sequences of real numbers $$(\lambda_n) \subset (0, 1), (r_n), (v_n)$$ and $$(\gamma_n) \subset [0, +\infty)$$ and assume the existence of a monotone function $$L$$ satisfying condition (Q2). For natural numbers $$k, n$$ and $$p$$ assume

(i) $$\forall m \in [n, p] \left( v_m \leq \frac{1}{4(k + 1)(p + 1)} \wedge r_m \leq \frac{1}{4(k + 1)} \right).$$

(ii) $$\forall m \in \mathbb{N} \left( \sum_{i=n}^{n+m} \gamma_i \leq \frac{1}{4(k + 1)} \right).$$

(iii) $$\forall m \in \mathbb{N} (s_{m+1} \leq (1 - \lambda_m)(s_m + v_m) + \lambda_m r_m + \gamma_m).$$

Then

$$\forall m \in [\sigma(k, n), p] \left( s_m \leq \frac{1}{k + 1} \right),$$

with $$\sigma(k, n) := L (n + \lceil \ln(4D(k + 1)) \rceil) + 1.$$ □

A direct application of Lemma 20 gives the following result which is more suitable for our analysis.

Lemma 21. Let $$\Omega$$ be a bounded subset of $$H.$$ Let $$(\lambda_n) \subset (0, 1)$$ be given and, for each $$p \in \Omega,$$ consider the sequences of real numbers $$(s_{n,p}), (v_{n,p}), (r_{n,p})$$ and $$(\gamma_{n,p})$$ with $$(s_{n,p}), (\gamma_{n,p}) \subset [0, +\infty)$$ and such that, for all $$p \in \Omega,$$

$$\forall m \in \mathbb{N} (s_{m+1,p} \leq (1 - \lambda_m)(s_{m,p} + v_{m,p}) + \lambda_m r_{m,p} + \gamma_{m,p}).$$

For a natural number $$D \in \mathbb{N}$$ and monotone functions $$L, G : \mathbb{N} \rightarrow \mathbb{N}$$ and $$\Psi : \mathbb{N} \times \mathbb{N}^3 \rightarrow \mathbb{N},$$ suppose:

(i) $$L$$ satisfies condition (Q2).
(ii) For all \( p \in \Omega, D \) is a positive upper bound on \( (s_{n,p}) \).

(iii) For all \( p \in \Omega, G \) is a Cauchy rate on \( \sum \gamma_{n,p} \).

(iv) \( \forall k \in \mathbb{N}, \forall f : \mathbb{N} \to \mathbb{N} \exists p \in \Omega \exists n \leq \Psi(k, f) \forall m \in [n, f(n)] \left( v_{m,p} \leq \frac{1}{\sqrt{m+1}} \land r_{m,p} \leq \frac{1}{\sqrt{m+1}} \right) \).

Then, for any natural number \( k \) and monotone function \( f : \mathbb{N} \to \mathbb{N} \), there are \( p \in \Omega \) and \( n \leq \Theta(k, f) \) such that

\[ \forall m \in [n, f(n)] \left( s_{m,p} \leq \frac{1}{k+1} \right), \]

where \( \Theta(k, f) = \Theta(k, f, L, \Psi, G, D) := L(h(\Psi(4k+3, g))) + 1 \) with \( h(m) = \max\{m, G(4k+3) + 1\} + \lceil \ln(4D(k + 1)) \rceil \) and \( g(m) := 4(k+1) (f(L(h(m))) + 1) + 1 \).

**Proof.** Let \( k \in \mathbb{N} \) and a monotone function \( f \) be given. By Condition (iv), consider \( \hat{p} \in \Omega \) and \( n_1 \leq \Psi(4k+3, g) \) such that for \( m \in [n_1, g(n_1)] \)

\[ v_{m,\hat{p}} \leq \frac{1}{g(n_1)+1} \quad \text{and} \quad r_{m,\hat{p}} \leq \frac{1}{4(k+1)}. \]

Define \( n_2 := \max\{n_1, G(4k+3) + 1\} \). By Condition (iii), for all \( m \in \mathbb{N} \), \( \sum_{i=n_2}^{n_1} \gamma_{m,\hat{p}} \leq \frac{1}{\eta_{k+1}} \). We have \( n_1 \leq n_2 \) and

\[ g(n_1) \geq f(L(h(n_1))) + 1 = f(\sigma(k, n_2)), \]

where \( \sigma \) is as in Lemma 20. Hence, for \( m \in [n_2, f(\sigma(k, n_2))], \)

\[ v_{m,\hat{p}} \leq \frac{1}{4(k+1)}(f(\sigma(k, n_2)) + 1) \quad \text{and} \quad r_{m,\hat{p}} \leq \frac{1}{4(k+1)}. \]

We are in the conditions of Lemma 20 with \( n = n_2 \) and \( p = f(\sigma(k, n_2)) \), and so

\[ \forall m \in [\sigma(k, n_2), f(\sigma(k, n_2))] \left( s_{m,\hat{p}} \leq \frac{1}{k+1} \right). \]

Noticing that, by the monotonicity of \( L \), we have \( \sigma(k, n_2) \leq \Theta(k, f) \), we conclude the proof. \( \square \)

### 4.2 Metastability of the mPPA

We start by showing that the sequence \( (z_n) \) generated by \( \text{(mPPA)} \) is bounded. In fact, the next result ensures that we are in the conditions of Lemma 19.

**Lemma 22.** Let \( (z_n) \) be generated by \( \text{(mPPA)} \). Assume that there exist \( a, c \in \mathbb{N} \setminus \{0\} \) and \( s \in S \) and monotone functions \( \ell, L, \Gamma, E \) such that \( (Q_1) - (Q_6) \) hold. Then \( (z_n) \) is bounded. Indeed, \( \|z_n\| \leq \tilde{N} \), where

\[ \tilde{N} := \max \{\|u - s\|, \|z_0 - s\|\} + \max \{\|s\|, \|u\|\} + \mathcal{E}, \]

with \( \mathcal{E} := \sum_{i=0}^{\ell-1} \|e_i\| + 1 \). Moreover, with \( z_{n+1} = \gamma_n z_n + (1 - \gamma_n) w_n \), we have \( \|w_n\| \leq 2\tilde{N} a \) and

\[ \forall k \in \mathbb{N} \forall n \geq \psi(k) \left( \|w_{n+1} - w_n\| - \|z_{n+1} - z_n\| \leq \frac{1}{k+1} \right), \]

where \( \psi(k) := \max \{\ell(10a\tilde{N}(k+1)), \Gamma(10a\tilde{N}(k+1)), E(5a(k+1)) + 1\} \).

**Proof.** Observe that \( \sum_{i=0}^{\ell} \|e_i\| \leq \mathcal{E} \) for all \( n \in \mathbb{N} \). By the fact that \( \lambda_n + \gamma_n + \delta_n = 1 \), for all \( n \geq 0 \) and observing that each resonant \( J_e \) is nonexpansive, we have

\[ \|z_{n+1} - s\| = \|\lambda_n(u - s) + \gamma_n(z_n - s) + \delta_n(J_n(z_n) - s) + e_n\|
\leq \lambda_n \|u - s\| + \gamma_n \|z_n - s\| + \delta_n \|z_n - s\| + \|e_n\|
= \lambda_n \|u - s\| + (1 - \lambda_n) \|z_n - s\| + \|e_n\|. \]

One easily shows by induction on \( n \in \mathbb{N} \) that \( \|z_n - s\| \leq \max \{\|u - s\|, \|z_0 - s\|\} + \sum_{i=0}^{n-1} \|e_i\| \), from which we deduce that \( (z_n) \) is bounded, indeed

\[ \|z_n\| \leq \|z_0 - s\| + \|s\| \leq \tilde{N}. \]

We have that \( 0 \leq \|w_n\| \leq 2\tilde{N} a \). Indeed, by \( (Q_3) \) we have

\[ \|w_n\| = \frac{\|z_{n+1} - \gamma_n z_n\|}{1 - \gamma_n} \leq \frac{2\tilde{N}}{1 - \gamma_n} \leq 2\tilde{N} a. \]
We have
\[
\begin{align*}
    w_{m+1} - w_m &= \frac{\lambda_{m+1} u + \delta_{m+1} J_{m+1}(z_{m+1}) + e_{m+1}}{1 - \gamma_{m+1}} - \frac{\lambda_{m} u + \delta_{m} J_{m}(z_{m})}{1 - \gamma_{m}} \\
    &= \left( \frac{\lambda_{m+1}}{1 - \gamma_{m+1}} - \frac{\lambda_{m}}{1 - \gamma_{m}} \right) u + \frac{\delta_{m+1}}{1 - \gamma_{m+1}} (J_{m+1}(z_{m+1}) - J_{m}(z_{m})) \\
    &\quad + \left( \frac{\delta_{m+1}}{1 - \gamma_{m+1}} - \frac{\delta_{m}}{1 - \gamma_{m}} \right) J_{m}(z_{m}) + \frac{e_{m+1}}{1 - \gamma_{m+1}} - \frac{e_{m}}{1 - \gamma_{m}}.
\end{align*}
\]

We claim that
\[
\|J_{m+1}(z_{m+1}) - J_{m}(z_{m})\| \leq \|z_{m+1} - z_{m}\| + 2\tilde{N} c_{m+1} - c_{m}. \tag{22}
\]

To prove the claim observe that from (21) we derive that for every \( n, m \in \mathbb{N} \) it holds that \( \|J_{m}(z_{n})\| \leq \|J_{m}(z_{n}) - s\| + \|s\| \leq \|z_{n} - s\| + \|s\| \leq \tilde{N} \). If \( c_{m} \leq c_{m+1} \), by the resolvent identity we have
\[
\|J_{m+1}(z_{m+1}) - J_{m}(z_{m})\| = \|J_{m+1} \left( \frac{c_{m+1}}{c_{m}} z_{m+1} + \left( 1 - \frac{c_{m+1}}{c_{m}} \right) J_{m}(z_{m}) \right) - J_{m}(z_{m})\|
\]
\[
\leq \left| \frac{c_{m+1}}{c_{m}} - 1 \right| \|z_{m+1} - z_{m}\| + \left( 1 - \frac{c_{m+1}}{c_{m}} \right) \|J_{m+1}(z_{m+1}) - z_{m}\|
\]
\[
\leq \|z_{m+1} - z_{m}\| + c |c_{m+1} - c_{m}| \|J_{m+1}(z_{m+1}) - z_{m}\|
\]
\[
\leq \|z_{m+1} - z_{m}\| + 2\tilde{N} c_{m+1} - c_{m}.
\]

If \( c_{m+1} < c_{m} \), again by the resolvent identity we have
\[
\|J_{m}(z_{m}) - J_{m+1}(z_{m+1})\| = \|J_{m+1} \left( \frac{c_{m+1}}{c_{m}} z_{m} + \left( 1 - \frac{c_{m+1}}{c_{m}} \right) J_{m}(z_{m}) \right) - J_{m+1}(z_{m+1})\|
\]
\[
\leq \left| \frac{c_{m+1}}{c_{m}} - 1 \right| \|z_{m+1} - z_{m}\| + \left( 1 - \frac{c_{m+1}}{c_{m}} \right) \|J_{m}(z_{m}) - z_{m+1}\|
\]
\[
\leq \|z_{m+1} - z_{m}\| + c |c_{m+1} - c_{m}| \|J_{m}(z_{m}) - z_{m+1}\|
\]
\[
\leq \|z_{m+1} - z_{m}\| + 2\tilde{N} c_{m+1} - c_{m}.
\]

Hence (22) holds. Then
\[
\|w_{n+1} - w_n\| - \|z_{m+1} - z_{m}\| \leq \left| \frac{\lambda_{m+1}}{1 - \gamma_{m+1}} - \frac{\lambda_{m}}{1 - \gamma_{m}} \right| \|u\| + \left| \frac{\delta_{m+1}}{1 - \gamma_{m+1}} - 1 \right| \|z_{m+1} - z_{m}\|
\]
\[
+ \left| \frac{\delta_{m+1}}{1 - \gamma_{m+1}} - \frac{\delta_{m}}{1 - \gamma_{m}} \right| 2\tilde{N} c_{m+1} - c_{m}
\]
\[
+ \left| \frac{e_{m+1}}{1 - \gamma_{m+1}} - \frac{e_{m}}{1 - \gamma_{m}} \right|.
\tag{23}
\]

Let \( k \in \mathbb{N} \) and \( m \geq \psi(k) \). We will see that each of the terms in (23) is less than or equal to \( \frac{1}{5(k+1)} \).

Observe that \( \|u\| \leq \tilde{N} \). Since \( \ell \) is monotone, we have that \( \ell(10a\|u\|(k+1)) \leq \ell(10a\tilde{N}(k+1)) \leq \psi(k) \leq m \).

Then
\[
\left| \frac{\lambda_{m+1}}{1 - \gamma_{m+1}} - \frac{\lambda_{m}}{1 - \gamma_{m}} \right| \|u\| \leq \left( \frac{\lambda_{m+1}}{1 - \gamma_{m+1}} + \frac{\lambda_{m}}{1 - \gamma_{m}} \right) \|u\|
\]
\[
\leq (\lambda_{m+1} + \lambda_{m}) \|u\| a
\]
\[
\leq \frac{2a\|u\|}{10a\|u\|} = \frac{1}{5(k+1)}.
\tag{24}
\]

We have that
\[
\left| 1 - \frac{\delta_{m+1}}{1 - \gamma_{m+1}} \right| \|z_{m+1} - z_{m}\| \leq \left| \frac{1 - \gamma_{m+1} - \delta_{m+1}}{1 - \gamma_{m+1}} \right| 2\tilde{N} a
\]
\[
\leq \lambda_{m+1} 2\tilde{N} a
\]
\[
\leq \frac{2\tilde{N} a}{10Na(k+1)} = \frac{1}{5(k+1)}.
\tag{25}
\]
Since, \( \Gamma(10aNc(k+1)) \leq \psi(k) \) it holds that

\[
\left| \frac{\delta_{m+1}}{1 - \gamma_{m+1}} \right| 2\tilde{N}c|c_{m+1} - c_m| \leq \frac{2aN\tilde{N}c}{10aNc(k+1)} = \frac{1}{5(k+1)}.
\]

(26)

We also have

\[
\left| \frac{\delta_{m+1}}{1 - \gamma_{m+1}} - \frac{\delta_m}{1 - \gamma_m} \right| \|J_m(z_m)\| \leq \left( \frac{\lambda_{m+1}}{1 - \gamma_{m+1}} + \frac{\lambda_m}{1 - \gamma_m} \right) \tilde{N} \leq \frac{2aN\tilde{N}}{10aNc(k+1)} = \frac{1}{5(k+1)}.
\]

(27)

Observe that from \((Q_6)\) we have

\[
\forall k \in \mathbb{N} \forall n \geq E(k) + 1 \left( \sum_{i=E(k)+1}^{n} \|e_i\| \leq \frac{1}{k+1} \right).
\]

Since \(E(5a(k+1)) + 1 \leq \psi(k) \leq m\), we have

\[
\frac{\|e_m\|}{1 - \gamma_m} + \frac{\|e_m\|}{1 - \gamma_m} \leq a \left( \frac{\|e_m\|}{1 - \gamma_m} + \|e_{m+1}\| \right) \leq a \left( \frac{1}{5a(k+1) + 1} \right) \leq \frac{1}{5(k+1)}.
\]

(28)

Combining (23)-(28) we conclude that (20) holds.

We prove next some useful quantitative facts about the iteration \((z_n)\).

\textbf{Lemma 23.} Let \((z_n)\) be generated by \((\text{mPPA})\). Assume that there exist \(a, c \in \mathbb{N} \setminus \{0\}\) and \(s \in \mathbb{S}\) and monotone functions \(\ell, \Gamma, E\) such that \((Q_1) - (Q_6)\) hold. Then

(i) \( \forall k \in \mathbb{N} \forall f : N \rightarrow N \exists n \leq \tilde{\chi}(k, f) \forall m \in [n, n + f(n)] \left( \|z_{m+1} - z_m\| \leq \frac{1}{k+1} \right) \); 

(ii) \( \forall k \in \mathbb{N} \forall f : N \rightarrow N \exists n \leq \tilde{\chi}(2a(k+1), \bar{f}_k) \forall m \in [n, n + f(n)] \left( \|J_m(z_m) - z_m\| \leq \frac{1}{k+1} \right) \); 

(iii) \( \forall k \in \mathbb{N} \forall f : N \rightarrow N \exists n \leq \chi(k, f, a, \psi, 2a\tilde{N}) \forall m \in [n, n + f(n)] \left( \|J(z_m) - z_m\| \leq \frac{1}{k+1} \right) \);

where \( \tilde{\chi}(x, f) := \chi(k, f, a, \psi, 2a\tilde{N}) \) and \( \chi(k, f) := \chi(4a(k+1), \bar{f}_{2k+1}) \), with \( \bar{f}_k : N \rightarrow N \) the monotone function defined by \( \bar{f}_k(m) = \mu(k) + f(\text{max}\{\mu(k), m\}) \), \( \mu(k) := \text{max}\{\ell(4a(k+1)(\|u\|) + \tilde{N})\}, E(4a(k+1)+1\}, \psi, \tilde{N} \) are as in Lemma 22 and \( \chi \) is as in Lemma 19.

\textbf{Proof.} (i). By Lemma 22, we can apply Lemma 19 to conclude that

\[
\forall k \in \mathbb{N} \forall f : N \rightarrow N \exists n \leq \tilde{\chi}(k, f, a, \psi, 2a\tilde{N}) \forall m \in [n, n + f(n)] \left( \|w_m - z_m\| \leq \frac{1}{k+1} \right) .
\]

Since \( \|z_m - z_{m+1}\| = \|z_m - \gamma_m z_m - (1 - \gamma_m) w_m\| = (1 - \gamma_m) \|z_m - w_m\| \leq \|z_m - w_m\| \), we conclude that (i) holds.

(ii). Observe that

\[
\|z_m - J_m(z_m)\| \leq \|z_{m+1} - z_m\| + \lambda_m \|u - J_m(z_m)\| + \gamma_m \|z_m - J_m(z_m)\| + \|e_m\| .
\]

Then

\[
\|z_m - J_m(z_m)\| \leq \frac{\|z_{m+1} - z_m\|}{1 - \gamma_m} + \lambda_m \frac{\|u - J_m(z_m)\|}{1 - \gamma_m} + \|e_m\| .
\]

(29)

We have that

\[
\forall k \in \mathbb{N} \forall m \geq \mu(k) \left( \frac{\lambda_m \|u - J_m(z_m)\|}{1 - \gamma_m} + \|e_m\| \right) \leq \frac{1}{2(k+1)} .
\]

(30)

Indeed, for \( m \geq \mu(k) \) we have that

\[
\frac{\lambda_m \|u - J_m(z_m)\|}{1 - \gamma_m} + \|e_m\| \leq \frac{a \left( \|u\| + \tilde{N} \right)}{4a(k+1) \left( \|u\| + \tilde{N} \right)} + a \left( \sum_{i=E(4a(k+1)+1)}^{m} \|e_i\| \right) \leq \frac{1}{4(k+1)} + \frac{1}{4(k+1)} = \frac{1}{2(k+1)} .
\]
Applying Part (i) to $2a(k+1)$ and $\tilde{f}_k$ we find $n' \leq \tilde{\chi}(2a(k+1),\tilde{f}_k)$ such that
\begin{equation}
\forall m \in [n', n' + \tilde{f}_k(n')] \left( \frac{\|z_{m+1} - z_m\|}{1 - \gamma_m} \leq \frac{a}{(2a(k+1)) + 1} \leq \frac{1}{(2(k+1))} \right).
\end{equation}
(31)

Put $n = \max\{\mu(k), n'\}$. Now $[n, n + f(n)] \subseteq [n', n' + \tilde{f}_k(n')]$ because clearly $n' \leq n$ and $n + f(n) \leq n' + \mu(k) + f(\max\{\mu(k), n'\}) = n' + \tilde{f}_k(n')$. Then, from (30) and (31) we have that
\begin{equation}
\forall k \in \mathbb{N} \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists n \leq \max\{\mu(k), \tilde{\chi}(2a(k+1), \tilde{f}_k)\} \forall m \in [n, n + f(n)] \left( \|J_m(z_m) - z_m\| \leq \frac{1}{k+1} \right).
\end{equation}

By the definition of $\tilde{f}_k$ and $\tilde{\chi}$ it is easy to check that $\mu(k) \leq \tilde{\chi}(2a(k+1), \tilde{f}_k)$. We conclude that Part (ii) holds.

(iii). By Lemma 2 and $(Q_4)$ we have $\|J_m(z_m) - z_m\| \leq 2 \|J_m(z_m) - z_m\|$. Hence, Part (iii) follows from Part (ii). \hfill \Box

The next result ensures that Condition (iv) of Lemma 21 holds.

**Proposition 24.** For any $k \in \mathbb{N}$ and monotone function $f : \mathbb{N} \rightarrow \mathbb{N}$, there are $p \in B(\tilde{N};0)$ and $n \leq \psi(k, f)$ such that for all $m \in [n, f(n)]$,
\begin{equation}
v_{m,p} \leq \frac{1}{f(n)+1} \wedge r_{m,p} \leq \frac{1}{k+1},
\end{equation}
where $v_{m,p} = \|J_m(p) - p\| (\|J_m(p) - p\| + 2 \|z_m - p\|)$, $r_{m,p} = 2(u - p, z_{m+1} - p)$, $\Psi(k, f) := \tilde{\psi}(2k+1, h)$, where $\tilde{N}$ is as in Lemma 22, $\psi$ is the function defined in Proposition 15 and $h : \mathbb{N} \rightarrow \mathbb{N}$ is the monotone function defined by
\begin{equation}
h(m) := \zeta \left( (1 + 4\tilde{N})(f(m) + 1) - 1, f(m) \right),
\end{equation}
with $\zeta$ as in (6).

**Proof.** Let $k \in \mathbb{N}$ and monotone $f$ be given. Applying Proposition 15 to $2k+1$ and to the monotone function $h$ we obtain $p \in B(\tilde{N};0)$ and $n \leq \psi(2k+1, h)$ such that
\begin{equation}
\|J(p) - p\| \leq \frac{1}{h(n)+1}
\end{equation}
and
\begin{equation}
\forall m \in [n, n + f(n)] \left( \langle u - p, z_{m+1} - p \rangle \leq \frac{1}{2(k+1)} \right).
\end{equation}
(33)
Clearly (33) implies that for $m \in [n, f(n)]$ one has $r_{m,p} \leq \frac{1}{k+1}$.

Now, by (32)
\begin{equation}
\|J(p) - p\| \leq \frac{1}{\zeta \left( (1 + 4\tilde{N})(f(n) + 1) - 1, f(n) \right) + 1}.
\end{equation}
Hence, by (6), for $m \leq f(n)$,
\begin{equation}
\|J_m(p) - p\| \leq \frac{1}{(1 + 4\tilde{N})(f(n) + 1)}.
\end{equation}
Also $\|J_u(p) - p\| \leq 1$ so, for $m \leq f(n)$,
\begin{equation}
v_{m,p} = \|J_m(p) - p\| (\|J_m(p) - p\| + 2 \|z_m - p\|) \leq \frac{1}{(1 + 4\tilde{N})(f(n) + 1)} \leq \frac{1}{k+1}.
\end{equation}
which concludes the proof. \hfill \Box

We are now able to formalize and prove our main result.

**Theorem 25.** Let $(z_n)$ be generated by (mPPA). Assume that there exist $a, c \in \mathbb{N} \setminus \{0\}$ and $s \in S$ and monotone functions $\ell, L, \Gamma, E$ such that $(Q_1) - (Q_6)$ hold. Then
\begin{equation}
\forall k \in \mathbb{N} \forall f : \mathbb{N} \rightarrow \mathbb{N} \exists \phi(k, f) \forall i, j \in [n, n + f(n)] \left( \|z_i - z_j\| \leq \frac{1}{k+1} \right),
\end{equation}
where $\phi(k, f) := \Theta(4(k+1)^2 - 1, \lambda m, (m + f(m)), L, \Psi, G, 4\tilde{N}^2)$, where $G(k) = E((k+1)(M + 2(\tilde{N} + \|u\|)))$, $\Theta$ is as in Lemma 21, $\Psi$ as in Proposition 24, $M := 3E + 4\tilde{N}$, with $E$ and $\tilde{N}$ as in Lemma 22.
Proof. Let \( p \in B(\bar{N}; 0) \). Then

\[
\|z_{m+1} - p\|^2 \leq (\|z_{m+1} - p - e_m\|^2 + \|e_m\|)^2 \\
= \|z_{m+1} - p - e_m\|^2 + \|e_m\| \left( \|e_m\|^2 + 2 \|z_{m+1} - p - e_m\| \right) \\
\leq \|z_{m+1} - p - e_m\|^2 + M \|e_m\| \\
= \|z_{m+1} - p - \lambda_m(u - p) + \lambda_m u - p\|^2 + \lambda_m u - p\| + M \|e_m\| \\
\leq \|\gamma_m(z_m - p) + \delta_m(J_m(z_m) - p)\|^2 + 2\lambda_m(u - p, z_{m+1} - p) + \|e_m\| (M + 2\lambda_m \|u - p\|) \\
\leq (\gamma_m\|z_m - p\|) + \delta_m\|J_m(z_m) - J_m(p)\| + \|e_m\| (M + 2\lambda_m \|u - p\|) \\
\leq (1 - \lambda_m)\|z_m - p\|^2 + (1 - \lambda_m)\|J_m(p) - p\|^2 + 2\lambda_m(u - p, z_{m+1} - p) + \|e_m\| (M + 2\lambda_m \|u - p\|) \\
+ 2\lambda_m(u - p, z_{m+1} - p) + \|e_m\| (M + 2\lambda_m \|u - p\|).
\]

Then, for all \( m \in \mathbb{N} \)

\[
s_{m+1, p} \leq (1 - \lambda_m)s_{m, p} + (1 - \lambda_m)e_{m, p} + \lambda_m r_{m, p} + \gamma_{m, p},
\]

where \( s_{m, p} = \|z_m - p\|^2 \), \( e_{m, p} = \|J_m(p) - p\|^2 \), \( r_{m, p} = 2(u - p, z_{m+1} - p) \) and \( \gamma_{m, p} = \|e_m\| (M + 2\lambda_m \|u - p\|) \).

We verify that the conditions of Lemma 21 are satisfied with \( \Omega = B(\bar{N}; 0), D = 4\bar{N}^2, \Psi \) as in Proposition 24 and \( G(k) = E((k + 1)(M + 2(\bar{N} + [\|u\|]))) \).

The first condition holds by hypothesis. Since \( \|z_m - p\| \leq 2\bar{N} \), the second condition is true with \( D = (2\bar{N})^2 \).

For the third condition, using \((Q_6)\), we have

\[
G(k) + n \sum_{i = G(k) + 1}^{G(k) + n} \gamma_{i, u} = E((k + 1)(M + 2(\bar{N} + [\|u\|])) + n) \\
\leq \sum_{i = E((k + 1)(M + 2(\bar{N} + [\|u\|])) + n)}^{E((k + 1)(M + 2(\bar{N} + [\|u\|])) + 1)} \|e_m\| (M + 2\lambda_m \|u - p\|) \\
\leq \sum_{i = E((k + 1)(M + 2(\bar{N} + [\|u\|])) + n)}^{E((k + 1)(M + 2(\bar{N} + [\|u\|])) + 1)} \|e_m\| (M + 2\bar{N} + [\|u\|]) \\
\leq \frac{M + 2\bar{N} + [\|u\|]}{(k + 1)(M + 2\bar{N} + [\|u\|]) + 1} \leq \frac{1}{k + 1}.
\]

Finally, by Proposition 24 the fourth condition of Lemma 21 is also verified. By Lemma 21 we conclude that

\[
\forall k \in \mathbb{N} \forall f : N \to N \exists p \in B(\bar{N}; 0) \exists n \leq \Theta(k, f, L, \Psi, G, 4\bar{N}^2) \forall m \in [n, f(n)] \left( \|z_m - p\|^2 \leq \frac{1}{k + 1} \right).
\]

(34)

Let \( k \in \mathbb{N} \) and a monotone function \( f \) be given. By (34) applied to \( 4(k + 1)^2 - 1 \) and to the function \( \lambda_m (m + f(m)) \), we find \( p \in B(\bar{N}; 0) \) and \( n = \phi(k, f) \) such that for \( m \in [n, n + f(n)] \),

\[
\|z_m - p\|^2 \leq \frac{1}{4(k + 1)^2}.
\]

Hence, for \( m \in [n, n + f(n)] \), \( \|z_m - p\| \leq \frac{1}{2(k + 1)} \) and with \( i, j \in [n, n + f(n)] \), we have

\[
\|z_i - z_j\| \leq \|z_i - p\| + \|z_j - p\| \leq \frac{1}{k + 1},
\]

which concludes the proof. \( \square \)

As an application we consider the special case where the sequence \((c_n)\) is constant. We note that this case was also considered in Yao and Noor’s paper [40, Corollary 3.1].
Let $c_0$ be a positive real number. Consider the sequence $(c_n)$ constantly equal to $c_0$. Let $(z_n)$ be generated by $(\text{mPPA})$. Assume that there exist $a,c \in \mathbb{N} \setminus \{0\}$, $s \in S$ and monotone functions $\ell, L, E$ such that $(Q_1) - (Q_3)$ and $(Q_5)$ hold and that $c_0 \geq \frac{1}{2}$.

Clearly $(z_n)$ satisfies $(Q_4)$ and $(Q_5)$ with $\Gamma(k) \equiv 0$. The definition of $\psi$ in Lemma 22 simplifies to

$$
\psi(k) = \max\{\ell(8a\tilde{N}(k+1)), E(4a(k+1)) + 1\},
$$

which causes changes in $\xi$ in Lemma 23, $\tilde{\psi}$ in Proposition 15 and $\Psi$ in Proposition 24. This simplifies the bound $\phi$ obtained in Theorem 25.

It is well-known that for $(\lambda_n) \subset (0,1)$ the condition $\sum \lambda_n = \infty$ is equivalent to the condition $\prod(1 - \lambda_n) = 0$. Although we don’t do this here, one could have worked with a rate of convergence towards zero $L'$ for $\prod(1 - \lambda_n)$ instead of the rate of divergence $L$ for $\sum \lambda_n$. This is done for a similar quantitative analysis in [28]. In certain cases, that option may prove useful as a function $L'$ may be of lower complexity than that of a function $L$, e.g. the sequence $\lambda_n = \frac{1}{n^3}$ has a linear rate $L'$ but an exponential rate $L$.

5 Logical considerations

We finish with some considerations concerning the logical aspects of our analysis.

Let us start by pointing out that, in principle, the monotone functional interpretation could be used to analyse the results presented in this paper. However, our elimination of the sequential weak compactness argument can be seen as an application of the general method obtained in [8]. Also, using the BFI enables us to make use of Lemma 20 (shown in [32] using the BFI) which makes our analysis easier to carry out.

As already mentioned, the original proof of Theorem 3 requires strong principles. These principles are sequential weak compactness, countable choice as used in the projection argument and arithmetical comprehension which is used in Lemma 4. All of these principles imply that the original proof cannot be formalized in Gödel’s $\mathcal{T}$ [1, 10]. In fact, they require the use of a stronger form of recursion, called bar recursion [35, 3, 31], in order to be analysed. The changes to the original proof made in Section 3 allow us to avoid the use of bar recursion. Let us justify briefly why that is the case.

The elimination of countable choice required for the projection argument is carried out in Section 3.1. The key observation is that (2) can be replaced by (3). This is in line with earlier analyses (see for example [18, 8, 32]) and it is well-known that this allows for the extracted quantitative information to be expressed in terms of Gödel’s primitive recursive functionals.

The way to deal with sequential weak compactness is explained in full detail in [8]. The key point is that, in the context of the BFI, sequential weak compactness can be replaced by countable Heine/Borel compactness. The content of Section 3.2 shows that it can be adapted to our context in a similar way.

In Section 3.3 we made the modifications necessary to bypass arithmetical comprehension. The need to use arithmetical comprehension in the original proof arises from the assumption of the existence of the real number $d = \limsup \|w_n - z_n\|$ in Lemma 4. There are already many cases in the proof mining literature – using the monotone functional interpretation – of examples where it is possible to avoid the use of arithmetical comprehension\(^a\). For example, in [16, 20] the use of the existence of a limit point for a sequence in a compact geodesic space (which requires arithmetical comprehension) is replaced by a combinatorial argument. In [21, 22], a proof of an asymptotic regularity theorem that was based on countable nested uses of sequential compactness (and hence arithmetical comprehension) is analysed resulting (by elimination of sequential compactness) in a simple exponential bound. Moreover, in a series of papers, Kohlenbach showed how the monotone functional interpretation can be used to replace the use of arithmetical comprehension by optimal arithmetic substitutes (see e.g. [14, 15] and [17, Section 17.9]). More recently, in [24], Kohlenbach and Sipos give a rational approximation to the lim sup of a certain sequence by interpreting the approximation. As described in detail below, we must deal with a similar issue.

Let $N \in \mathbb{N}$ and let $(x_n)$ be a sequence of real numbers contained in the interval $[0,N]$. The existence of the lim sup $x_n$ can be stated as

$$
\exists d \in \mathbb{R} \forall k \in \mathbb{N} \left( \forall n \in \mathbb{N} \exists m \geq n \left( x_m \geq d - \frac{1}{k+1} \right) \land \exists m' \in \mathbb{N} \forall n' \geq n' \left( x_{m'} \leq d + \frac{1}{k+1} \right) \right).
$$

(35)

The main point is that we can weaken this statement by switching the outermost quantifiers

$$
\forall k \in \mathbb{N} \exists d \in \mathbb{R} \left( \forall n \in \mathbb{N} \exists m \geq n \left( x_m \geq d - \frac{1}{k+1} \right) \land \exists m' \in \mathbb{N} \forall n' \geq n' \left( x_{m'} \leq d + \frac{1}{k+1} \right) \right).
$$

(36)

\(^a\)We would like to thank Ulrich Kohlenbach for pointing this out to the first author and for providing the appropriate references.
In fact, we will show that such $d$ in (36) is already witnessed by a rational number by proving

$$\forall k \in \mathbb{N} \exists p < N(k + 1) \left( \forall n \in \mathbb{N} \exists m \geq n \left( x_m \geq \frac{p}{k+1} \right) \land \exists n' \in \mathbb{N} \forall m' \geq n' \left( x_{m'} \leq \frac{p+1}{k+1} \right) \right), \quad (37)$$

which implies that $d = \frac{p}{k+1}$ satisfies (36).

The idea behind (37) is the following. For each $k \in \mathbb{N}$, by dividing the interval $[0, N]$ into subintervals of length $\frac{p}{k+1}$, there exists $p < N(k+1)$ such that $\frac{p}{k+1} \leq \limsup x_n \leq \frac{p+1}{k+1}$. If we take $d = \frac{p+1}{2(k+1)}$, i.e. the middle point, then it should satisfy (36) for $2k+1$ (and hence for $k$). This results in statement (37).

Lemma 16, which is shown by $\Pi^0_1$-induction, can be seen to imply (37) (using a collection argument). First note that Lemma 16 implies

$$\forall k, n \in \mathbb{N} \exists f : \mathbb{N} \rightarrow \mathbb{N} \exists p < N(k + 1) \left( \exists m \geq n \left( x_m \geq \frac{p}{k+1} \right) \land \exists n' \in \mathbb{N} \forall m' \in [n', n' + f(n')] \left( x_{m'} \leq \frac{p+1}{k+1} \right) \right). \quad (38)$$

By a collection argument, we conclude

$$\forall k \in \mathbb{N} \exists p < N(k + 1) \forall n \in \mathbb{N} \exists f : \mathbb{N} \rightarrow \mathbb{N} \left( \exists m \geq n \left( x_m \geq \frac{p}{k+1} \right) \land \exists n' \in \mathbb{N} \forall m' \in [n', n' + f(n')] \left( x_{m'} \leq \frac{p+1}{k+1} \right) \right), \quad (39)$$

which by (monotone) choice axiom is equivalent to (37).

Let us elaborate on (39). Clearly it is a sufficient condition to (38). To see that (38) implies (39), assuming that (39) fails, we can prove

$$\forall r \leq N(k + 1) \exists n \exists f : \mathbb{N} \rightarrow \mathbb{N} \exists p < r \left( \forall m \geq n \left( x_m < \frac{p}{k+1} \right) \land \forall n' \in \mathbb{N} \exists m' \in [n', n' + f(n')] \left( x_{m'} > \frac{p+1}{k+1} \right) \right).$$

By instantiating $r = N(k + 1)$, one concludes that (38) must also fail.

Of course, this collection argument is fully justified by a form of induction. The reader can compare this way of proving (37), using $\Pi^0_1$-induction and a collection argument, to the similar [24, Proposition 4.2], where $\Pi^0_2$-induction was used.

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