Infinite-dimensional Grassmann-Banach algebras

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Abstract

A short review on infinite-dimensional Grassmann-Banach algebras (IDGBA) is presented. Starting with the simplest IDGBA over $K = \mathbb{R}$ with $l_1$-norm (suggested by A. Rogers), we define a more general IDGBA over complete normed field $K$ with $l_1$-norm and set of generators of arbitrary power. Any $l_1$-type IDGBA may be obtained by action of Grassmann-Banach functor of projective type on certain $l_1$-space. In non-Archimedean case there exists another possibility for constructing of IDGBA using the Grassmann-Banach functor of injective type.

Infinite-dimensional Grassmann-Banach algebras (IDGBA) and their modifications are key objects for infinite-dimensional versions of superanalysis (see [1]-[5] and references therein). They are generalizations of finite-dimensional Grassmann algebras to infinite-dimensional Banach case (for infinite-dimensional topological Grassmann algebras see also [6]).

Any IDGBA is an associative Banach algebra with unit over some complete normed field $K$ [7], whose linear space $G$ is a Banach space with the norm $||.||$ satisfying $||a \cdot b|| \leq ||a|| ||b||$ for all $a, b \in G$ and $||e|| = 1$, where $e$ is the unit. (For applications in superanalysis $K$ should be non-discrete, i.e. $0 < |v| < 1$ for some $v \in K$, where $|.|$ is the norm in $K$.) It contains an infinite subset of generators $\{e_\alpha, \alpha \in M\} \subset G$, satisfying

$$e_\alpha \cdot e_\beta + e_\beta \cdot e_\alpha = 0, \quad e_\alpha^2 = 0,$$

$\alpha, \beta \in M$, where $M$ is some infinite set. (The second relation in (2) follows from the first one if char$K \neq 2$, i.e. $1_K + 1_K \neq 0_K$.)

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The simplest IDGBA over $K = \mathbb{R}$ with $l_1$-norm was considered by A. Rogers in [2]. In this case $M = \mathbb{N}$ and any element of $a \in G$ can be represented in the form

$$a = a^0e + \sum_{k \in \mathbb{N}} \sum_{\alpha_1 < \ldots < \alpha_k} a^{\alpha_1 \ldots \alpha_k}e_{\alpha_1} \ldots e_{\alpha_k},$$  \hspace{1cm} (2)

where all $a^0, a^{\alpha_1 \ldots \alpha_k} \in K$ and

$$||a|| = |a^0| + \sum_{k \in \mathbb{N}} \sum_{\alpha_1 < \ldots < \alpha_k} |a^{\alpha_1 \ldots \alpha_k}| < +\infty.$$  \hspace{1cm} (3)

All series in (2) are absolutely convergent w.r.t. the norm (3).

In [8] a family of $l_1$-type IDGBA over a complete normed field $K$ was suggested. This family extends IDGBA from [2] to arbitrary $K$ and arbitrary infinite number of generators $\{e_{\alpha}, \alpha \in M\}$. For linearly ordered set $M$ the relations (2) and (3) survive, each sum in (2) and (3) contains not more than countable number of non-zero terms (AC) (here and below (AC) means that the axiom of choice [10] is used).

Here we outline an explicit construction of IDGBA from [8] for arbitrary (not obviously linearly ordered) index set $M$. Any element of this family $G(M, K, \langle \cdot \rangle)$ is defined by infinite set $M$ and an ordering mapping $\langle \cdot \rangle: P_0(M) \setminus \{\emptyset\} \rightarrow S_0(M)$, where $P_0(M)$ is the set of all finite subsets of $M$ and $S_0(M)$ the set of all ordered (non-empty) sets $(s_1, \ldots, s_k)$ of elements from $M$ ($k \in \mathbb{N}$). The ordering function $\langle \cdot \rangle$ obeys the relations

$$\langle \{\alpha_1, \ldots, \alpha_k\} \rangle = (\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(k)}),$$  \hspace{1cm} (4)

where $\sigma \in S_k$ is some permutation of $\{1, \ldots, k\}, k \in \mathbb{N}$. The mapping $\langle \cdot \rangle$ does exist (AC). For linearly ordered $M$ the canonical ordering function $\langle \cdot \rangle = \langle \cdot \rangle_0$ is defined by (3) with the inequalities $\alpha_{\sigma(1)} < \ldots < \alpha_{\sigma(k)}$ added.

The vector space of $G(M, K, \langle \cdot \rangle)$ is the Banach space $G = l_1(P_0(M), K)$ of absolutely summable functions $a: P_0(M) \rightarrow K$ with the norm

$$||a|| = \sum_{I \in P_0(M)} |a(I)| < +\infty.$$  \hspace{1cm} (5)

The operation of multiplication in $G$ is defined as follows

$$(a \cdot b)(I) = \sum_{I_1 \cup I_2 = I} \varepsilon(I_1, I_2)a(I_1)b(I_2),$$  \hspace{1cm} (6)
a, b ∈ G, I ∈ P₀(M), where ε : P₀(M) × P₀(M) → K is ε-symbol:

\[ \varepsilon(I_1, I_2) = \begin{cases} 0_K, & \text{if } I_1 \cap I_2 \neq \emptyset, \\ 1_K, & \text{if } I_1 = \emptyset, \text{ or } I_2 = \emptyset, \\ \varepsilon_\sigma, & \text{otherwise,} \end{cases} \quad (7) \]

where \( \varepsilon_\sigma = \pm 1_K \) is the parity of the permutation \( \sigma \): \( \langle I_1 \rangle \backslash \langle I_2 \rangle \rangle \mapsto \langle I_1 \cup I_2 \rangle \).

For any \( a \in G \) we get \( a = \sum_{I \in P_0(M)} a(I)e_I \), where \( \langle e_I, I \in P_0(M) \rangle \) is the Shauder basis in \( G \) defined by the relations: \( e_I(J) = \delta_{IJ}^J \) for \( I, J \in P_0(M) \).

The unit is \( e = e_\emptyset \) and generators are \( e_\alpha = e_{\{\alpha\}} \), \( \alpha \in M \). Decomposition (2) is valid for general ordering function \( \langle . . . \rangle \) if \( a^0 = a(\emptyset) \), \( a^{\alpha_1 \ldots \alpha_k} = a(\{\alpha_1, \ldots, \alpha_k\}) \) and relations \( \alpha_1 < \ldots < \alpha_k \) are understood as \( \langle a_1, \ldots, a_k \rangle \in \langle P_0(M) \rangle \), \( \langle P_0(M) \rangle \) depends essentially only upon the cardinal number \( |M| \) of the set \( M \), i.e. \( G(M_1, K, \langle . \rangle_1) \) and \( G(M_2, K, \langle . \rangle_2) \) are isomorphic (in the category of BA) if and only if \( |M_1| = |M_2| \) (AC) \( \S \). The Banach space of \( G(M, K, \langle . \rangle) \) may be decomposed into a sum of two closed subspaces

\[ G = G_0 \oplus G_1, \quad (8) \]

where \( G_i = \{ a \in G | a(I) = 0_K, I \in P_0(M), |I| \equiv i + 1 \text{ (mod 2)} \} \), \( i = 0, 1 \). (The subspace \( G_0 \) \( (G_1) \) consists of sums of even \( \text{(odd)} \) monoms in (2)). BA \( G(M, K, \langle . \rangle) \) with the decomposition (8) is a supercommutative (Banach) superalgebra

\[ a \cdot b = (-1_K)^{ij} b \cdot a, \quad a \in G_i, \quad b \in G_j, \quad (9) \]

\[ G_i \cdot G_j \subset G_{i+j} \text{ (mod 2)}, \quad (10) \]

\( i, j = 0, 1 \). The odd subspace \( G_1 \) has trivial (right) annihilator \( \S \)

\[ \text{Ann}(G_1) \equiv \{ a \in G | G_1 \cdot a = \{0\} \} = \{0\}. \quad (11) \]

This relation is an important one for applications in superanalysis, since it provides the definitions of all superderivatives as elements of \( G \). Note that any non-trivial (non-zero) associative supercommutative superalgebra over \( K \), \( \text{char } K \neq 2 \), is infinite-dimensional \( \S \) (for \( K = \mathbb{R}, \mathbb{C} \) see also \( \S \)).

Another important (e.g. for applications in superanalysis) proposition is the following one \( \S \): in \( G(M, K, \langle . \rangle) \) the element \( a \) is invertible if and only
if $a^0 = a(\emptyset) \neq 0_K$. (In [9] an explicit expression for inverse element $a^{-1}$ was obtained.)

IDGBA with $l_1$-norm forms a special subclass of more general family of IDGBA over $K$ [12], namely,

$$G(M, K, \langle , \rangle) \cong \hat{G}(l_1(M, K)),$$

where $\hat{G} = \hat{G}_K$ is the Grassmann-Banach functor of projective type [12]. Here

$$\hat{G}(E) = \hat{T}(E)/\hat{I},$$

where $\hat{T}(E)$ is a tensor BA of projective type corresponding to infinite-dimensional projectively proper Banach space $E$ over $K$ and $\hat{I}$ is a closed ideal generated by the subset $\{a^2, a \in E\}$. Banach space $E$ over $K$ is called projectively proper if all projective seminorms $p_k : E^{\otimes k} = E \otimes \ldots \otimes E$ ($k$-times) $\rightarrow R$, $k \geq 2$, are norms [12]. For $K = R, C$ any $E$ is projectively proper [11]. Tensor Banach functor $\hat{T} = \hat{T}_K$ was defined in [12] (for tensor BA without unit over $K = C$ see [13]). The Banach space of $\hat{T}(E)$ is a $l_1$-sum of projective tensor powers of $E$

$$\hat{T}(E) = \hat{T}_0(E) \oplus \sum_{i=0}^{\infty} \hat{T}_i(E),$$

where $\hat{T}_0(E) = K$, $\hat{T}_1(E) = E$ and $\hat{T}_k(E) = E^{\otimes} \ldots \otimes E$ ($k$-times) are projective tensor products, $k \geq 2$. The norm of $a = (a_0, a_1, \ldots) \in \hat{T}(E)$ is $\|a\| = ||a_0|| + ||a_1|| + \ldots$, where $a_i \in \hat{T}_i(E)$ and $\|\|_i$ is projective norm in $\hat{T}_i(E)$, $i = 0, 1, \ldots$.

For non-Archimedean field $K$ satisfying: $|x + y| \leq \max(|x|, |y|)$, $x, y \in K$, there exists another possibility for constructing of IDGBA [14]. The Grassmann-Banach functor of injective type $\hat{G} = \hat{G}_K$ is defined for certain subclass of injectively proper non-Archimedean Banach spaces over $K$. Banach space $E$ over $K$ is called injectively proper if the injective seminorms $w_k : E^{\otimes k} \rightarrow R$, $k \geq 2$, are norms [14]. In this case (13) is modified as follows

$$\hat{G}(E) = \hat{T}(E)/\hat{I},$$

where $\hat{T}(E)$ is tensor BA of injective type corresponding to $E$ and $\hat{I}$ is a closed ideal generated by the subset $\{a^2, a \in E\}$. The Banach space of $\hat{T}(E)$
is a $l_\infty$-sum of injective tensor powers of $E$

\[
\tilde{T}(E) = \bigoplus_{i=0}^{\infty} \tilde{T}_i(E),
\]  

(16)

where $\tilde{T}_0(E) = K$, $\tilde{T}_1(E) = E$ and $\tilde{T}_k(E) = E \hat{\otimes} \ldots \hat{\otimes} E$ ($k$-times) are injective tensor products, $k \geq 2$. The norm of $a = (a_0, a_1, \ldots) \in \tilde{T}(E)$ is $\|a\| = \sup(\|a_0\|_0, \|a_1\|_1, \ldots)$, where $a_i \in \tilde{T}_i(E)$ and $\|\cdot\|_i$ is injective norm in $\tilde{T}_i(E)$, $i = 0, 1, \ldots$. For $l_\infty$-spaces we have an isomorphism of BA

\[
G_\infty(M, K, \langle \cdot, \cdot \rangle) \cong \tilde{G}(l_\infty(M, K)),
\]  

(17)

where $G_\infty(M, K, \langle \cdot, \cdot \rangle)$ is the Grassmann-Banach algebra with the Banach space $l_\infty(P_0(M), K)$ and the multiplication defined in (13). Here $l_\infty(P_0(M), K)$ is the Banach space of bounded functions $a : P_0(M) \to K$ with the norm

\[
\|a\|_\infty = \sup(\|a(I)\|, I \in P_0(M)).
\]  

(18)

For applications in superanalysis the following supercommutative Banach superalgebras may be also used: $B \hat{\otimes} G$. Here $B$ is an associative commutative BA with unit over $K$, and $G$ is IDGBA. For $G = G(M, K, \langle \cdot, \cdot \rangle)$ we have an isomorphism of BA: $B \hat{\otimes} G(M, K, \langle \cdot, \cdot \rangle) \cong G(M, B, \langle \cdot, \cdot \rangle)$, where $G(M, B, \langle \cdot, \cdot \rangle)$ is obtained from $G(M, K, \langle \cdot, \cdot \rangle)$ by the replacement $K \mapsto B$ (for $M = \mathbb{N}$ see also [13]).

For non-Archimedean $B$, $G$ and $K$ another Banach superalgebra may be also considered: $B \hat{\otimes} G$. In this case $B \hat{\otimes} G_\infty(M, K, \langle \cdot, \cdot \rangle) \cong G_\infty(M, B, \langle \cdot, \cdot \rangle)$, where $G_\infty(M, B, \langle \cdot, \cdot \rangle)$ is obtained from $G_\infty(M, K, \langle \cdot, \cdot \rangle)$ by the replacement $K \mapsto B$.

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