Composition operators on Hardy-Sobolev spaces and BMO-quasiconformal mappings

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(Presented by V. Ryazanov)

Dedicated to the 80th anniversary of Professor Vladimir Gutlyanskii

Abstract. In this paper, we consider composition operators on Hardy-Sobolev spaces in connections with BMO-quasiconformal mappings. Using the duality of Hardy spaces and BMO-spaces, we prove that BMO-quasiconformal mappings generate bounded composition operators from Hardy–Sobolev spaces to Sobolev spaces.

Keywords. Sobolev spaces, Quasiconformal mappings.

1. Introduction

Composition operators on Sobolev spaces arise in the work by V. Maz’ya [38] in connection with the isoperimetric problem as operators generated by sub-areal mappings. In this pioneering work, a connection between the geometrical properties of mappings and the corresponding Sobolev spaces was established. In the present paper, we consider composition operators on Hardy–Sobolev spaces generated by BMO-quasiconformal mappings. The main result of the article states:

Let the Hardy–Sobolev spaces $H^{1,n}_r(\Omega)$ be defined in Lipschitz bounded domains in $\Omega \subset \mathbb{R}^n$, the Sobolev spaces $L^{1,n}_r(\Omega)$ be defined in bounded domains in $\Omega \subset \mathbb{R}^n$, and $\varphi : \Omega \rightarrow \Omega$ be a BMO-quasiconformal mapping. Then the inequality

$$\|f \circ \varphi^{-1} \|_{L^{1,n}_r(\Omega)} \leq \|Q \|_{\text{BMO}_2(\Omega)} \frac{1}{2} \|f \|_{H^{1,n}_r(\Omega)},$$

holds for any Lipschitz function $f \in \text{Lip}(\Omega)$, if a measurable function $Q : \Omega \rightarrow \mathbb{R}$ be such that a quasiconformal distortion $K_n(\varphi) \leq Q$ a.e. in $\Omega$.

BMO-quasiconformal mappings generalize the notion of quasiconformal mappings because $K$-quasiconformal mappings are BMO-quasiconformal mappings with $Q := K \in \text{BMO}(\Omega)$ [37]. Composition operators on Sobolev spaces in connection with quasiconformal mappings were considered in [54] in the frameworks of Reshetnyak’s problem (1968). Note that this problem arises for quasiconformal mappings and Royden algebras [33, 43]. In [54], it was proved that a homeomorphism $\varphi : \Omega \rightarrow \overline{\Omega}$, where $\Omega, \overline{\Omega}$ are domains in $\mathbb{R}^n$, generates, by the composition rule $\varphi^*(f) = f \circ \varphi$, the bounded operator on Sobolev spaces

$$\varphi^* : L^{1,n}_r(\overline{\Omega}) \rightarrow L^{1,n}_r(\Omega),$$

if and only if $\varphi$ is a quasiconformal mapping. In the case of Sobolev spaces $L^{1,p}_r(\overline{\Omega})$ and $L^{1,p}_r(\Omega)$, $p \neq n$, the analytic description was obtained in [52] using a notion of mappings of finite distortion introduced...
in [55]: a weakly differentiable mapping is called a mapping of finite distortion if $|D\varphi(x)| = 0$ a.e. on the set $Z = \{ x \in \Omega : J(x, \varphi) = 0 \}$. In [15], characterizations of composition operators in geometric terms for $n - 1 < p < \infty$ were obtained.

The case of Sobolev spaces $L^{1,q}(\bar{\Omega})$ and $L^{1,q}(\Omega), q < p$, is more complicated and, in this case, the composition operators theory is based on the countable-additive set functions, which are associated with the norms of composition operators and were introduced in [50] (see also [56]). The main result of [50] gives analytic and capacitary characterizations of composition operators on Sobolev spaces (see, also [56]) in terms of mappings of finite distortion [23, 55]. The multipliers theory has been applied to the change of variable problem in Sobolev spaces in [40].

In the last decade, the composition operators theory was considered on some generalizations of Sobolev spaces, such as Besov spaces and Triebel–Lizorkin spaces [22, 24, 25, 32, 44]. These types of composition operators have applications to the Calderón inverse conductivity problem [2]. Composition operators on Sobolev spaces over Banach function spaces (such as Orlicz, Lorentz, variable exponents, etc.) were considered in [26–30, 41, 42].

Remark that composition operators on Sobolev spaces have significant applications to the Sobolev embedding theory [14, 17] and to the spectral theory of elliptic operators (see, e.g., [16, 19, 20]). In some cases, the composition operators method allows one to obtain better estimates than the classical L. E. Payne and H. F. Weinberger estimates in convex domains [45].

The notion of $Q$-mappings was introduced in [34] (see also [35–37]). Recall that the homeomorphism $\varphi : \Omega \rightarrow \bar{\Omega}$ of domains $\Omega, \bar{\Omega} \subset \mathbb{R}^n$ is called a $Q$-homeomorphism with a non-negative measurable function $Q$, if

$$M(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^n(x) dx$$

for every family $\Gamma$ of rectifiable paths in $\Omega$ and every admissible function $\rho$ for $\Gamma$.

The $Q$-mappings with a function $Q$ belonging to the $A_n$-Muckenhoupt class are inverse to homeomorphisms generating bounded composition operators on the weighted Sobolev spaces [51] (see also [53]). In the case $Q \in \text{BMO}(\Omega)$, we have a class of BMO-quasiconformal mappings [37, 46]. Note that BMO-quasiconformal mappings have significant applications in the Beltrami equation theory [5].

The aim of the present article is to study $Q$-mappings with $Q \in \text{BMO}$ in connection with composition operators on Sobolev-type spaces. This leads us to consider composition operators on Hardy–Sobolev spaces.

The theory of Hardy spaces on the Euclidean space $\mathbb{R}^n$ arose in the work by E. M. Stein and G. Weiss [49]. Later, C. Fefferman and E. M. Stein [4] systematically developed the real-variable theory for the Hardy spaces $H^p(\mathbb{R}^n)$ with $p \in (0, 1)$, which plays an important role in various fields of analysis (see, for example, [47]). The Hardy and BMO-spaces on domains of $\mathbb{R}^n$ were considered in [6, 7]. The current state of the art and the references to applications of Hardy spaces on domains of $\mathbb{R}^n$ can be found in [13]. Composition operators on Hardy and Hardy–Sobolev spaces of analytic functions have been intensively studied for a long time and can be found, for example in [10, 48].
2. Hardy–Sobolev spaces

2.1 Sobolev spaces

Let $E$ be a measurable subset of $\mathbb{R}^n$, $n \geq 2$. The Lebesgue space $L^p(E)$, $1 \leq p < \infty$, is defined as a Banach space of $p$-summable functions $f : E \to \mathbb{R}$ equipped with the following norm:

$$
\|f \|_{L^p(E)} = \left( \int_E |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty.
$$

If $\Omega$ is an open subset of $\mathbb{R}^n$, the Sobolev space $W^{1,p}(\Omega)$, $1 \leq p < \infty$, is defined [39] as a Banach space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following norm:

$$
\|f \|_{W^{1,p}(\Omega)} = \|f \|_{L^p(\Omega)} + \|\nabla f \|_{L^p(\Omega)},
$$

where $\nabla f$ is the weak gradient of the function $f$, i.e. $\nabla f = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$.

The homogeneous seminormed Sobolev space $L^{1,p}(\Omega)$, $1 \leq p < \infty$, is defined as a space of locally integrable weakly differentiable functions $f : \Omega \to \mathbb{R}$ equipped with the following seminorm:

$$
\|f \|_{L^{1,p}(\Omega)} = \|\nabla f \|_{L^p(\Omega)}.
$$

2.2 Hardy and Hardy–Sobolev spaces

Let us recall the classical definition of Hardy spaces $H^1(\mathbb{R}^n)$ [47]. Let $\Phi$ be a function belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \Phi(x) \, dx = 1$. For all $t > 0$, define $\Phi_t(x) = t^{-n} \Phi(x/t)$ and the vertical maximal function

$$
\mathcal{M}f(x) = \sup_{t > 0} |\Phi_t * f(x)|.
$$

Let a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $f$ is said to be in $H^1(\mathbb{R}^n)$ if $\mathcal{M}f \in L^1(\mathbb{R}^n)$. The Hardy space $H^1(\mathbb{R}^n)$ is equipped with the norm

$$
\|f \|_{H^1(\mathbb{R}^n)} := \|\mathcal{M}f \|_{L^1(\mathbb{R}^n)}.
$$

There are several definitions of Hardy spaces [6,7,12] and Hardy–Sobolev spaces on domains $\Omega \subset \mathbb{R}^n$ (see, e.g. [1,13]). Following [1], we define two types of Hardy spaces on Lipschitz domains in $\mathbb{R}^n$. The Hardy space $H^1_1(\Omega)$ is defined as a space of functions $f \in H^1(\mathbb{R}^n)$ such that $\text{supp} f \subset \bar{\Omega}$. Endowed with the norm

$$
\|f \|_{H^1_1(\Omega)} := \|f \|_{H^1(\mathbb{R}^n)},
$$

it is a Banach space.

The Hardy space $H^1_r(\Omega)$ is defined as a space of functions $f$ that are restrictions to $\Omega$ of functions $F \in H^1(\mathbb{R}^n)$. If $f \in H^1_r(\Omega)$, then

$$
\|f \|_{H^1_r(\Omega)} := \inf \|F \|_{H^1(\mathbb{R}^n)},
$$

where the infimum is taken over all functions $F \in H^1(\mathbb{R}^n)$ such that $F|_{\Omega} = f$. The space $H^1_r(\Omega)$ equipped with this norm is a Banach space. In [12], it was shown that $H^1_r(\Omega)$ can be defined in terms of maximal function: $\|f \|_{H^1_r(\Omega)} = \|\mathcal{M}\Omega f \|_{L^1(\Omega)}$, where

$$
\mathcal{M}\Omega f(x) = \sup_{t \leq d(x,\partial\Omega)} |\Phi_t * f(x)|.
$$
We define the Hardy–Sobolev space \( H^{1,p}(\Omega) \) \((H^{1,p}_z(\Omega))\), \(1 \leq p < \infty\), as a space of weakly differentiable functions \( f \in L^p(\Omega) \) such that \(|\nabla f|^p \in H^1(\Omega)\) \((|\nabla f|^p \in H^1_z(\Omega))\) and equipped with the norms

\[
\| f \|_{H^{1,p}(\Omega)} := \| f \|_{L^p(\Omega)} + \| |\nabla f|^p \|_{H^1(\Omega)}^{\frac{1}{p}},
\]

\[
\| f \|_{H^{1,p}_z(\Omega)} := \| f \|_{L^p(\Omega)} + \| |\nabla f|^p \|_{H^1_z(\Omega)}^{\frac{1}{p}}.
\]

The homogeneous Hardy–Sobolev space \( H^{1,p}_r(\Omega) \) \((H^{1,p}_z(\Omega))\), \(1 \leq p < \infty\), are defined as a space of locally integrable weakly differentiable functions \( f : \Omega \to \mathbb{R} \) equipped with the following seminorms:

\[
\| f \|_{H^{1,p}_r(\Omega)} := \| |\nabla f|^p \|_{H^1(\Omega)}^{\frac{1}{p}} \text{ and }
\]

\[
\| f \|_{H^{1,p}_z(\Omega)} := \| |\nabla f|^p \|_{H^1_z(\Omega)}^{\frac{1}{p}}.
\]

Let us prove that a function

\[
\| \cdot \|_p : f \mapsto \| |\nabla f|^p \|_{H^1(\Omega)}^{\frac{1}{p}}
\]

is a seminorm (for the case of \( H^1_z(\Omega) \), the proof is similar).

1. **Nonnegativity:**

\[
\| f \|_{H^{1,p}_r(\Omega)} := \| |\nabla f|^p \|_{H^1(\Omega)}^{\frac{1}{p}} \geq 0 \text{ for all } f \in H^{1,p}_r(\Omega).
\]

2. **Absolute homogeneity:**

\[
\| kf \|_{H^{1,p}_r(\Omega)} := \| k |\nabla f|^p \|_{H^1(\Omega)}^{\frac{1}{p}} = \| k \| \| |\nabla f|^p \|_{H^1(\Omega)}^{\frac{1}{p}} = \| k \| \| f \|_{H^{1,p}_r(\Omega)}
\]

for any \( k \in \mathbb{R} \) and any \( f \in H^{1,p}_r(\Omega) \).

3. **Triangle inequality:** Let functions \( f, g \in H^{1,p}_r(\Omega) \). Then

\[
\|(f + g) \|_{H^{1,p}_r(\Omega)}^{\frac{1}{p}} = \| |\nabla (f + g)|^p \|_{H^1(\Omega)}^{\frac{1}{p}}
\]

\[
= \left( \int \sup_{t \leq d(x,\partial \Omega)} \left( \int_{B(x,t)} |\nabla f(y) + \nabla g(y)|^p \Phi_t(x-t) \, dy \right) \, dx \right)^{\frac{1}{p}}
\]

\[
\leq \left( \int \sup_{t \leq d(x,\partial \Omega)} \left( \int_{B(x,t)} (|\nabla f(y)| + |\nabla g(y)|)^p \Phi_t(x-t) \, dy \right) \, dx \right)^{\frac{1}{p}}
\]

\[
= \left( \int \sup_{t \leq d(x,\partial \Omega)} \left( \int_{B(x,t)} \left( \Phi_t(x-t) \right)^{\frac{1}{p}} |\nabla f(y)| + \left( \Phi_t(x-t) \right)^{\frac{1}{p}} |\nabla g(y)| \right)^p \, dy \right) \, dx \right)^{\frac{1}{p}}.
\]
Now, by using the Minkowski inequality, we have

\[
\left( \int_\Omega \left( \iint_{B(x,t)} \left[ (\Phi_t(x-t)) \frac{1}{p} |\nabla f(y)| + (\Phi_t(x-t)) \frac{1}{p} |\nabla g(y)| \right]^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
\leq \left( \int_\Omega \left( \iint_{B(x,t)} \Phi_t(x-t)|\nabla f(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
+ \left( \int_\Omega \left( \iint_{B(x,t)} \Phi_t(x-t)|\nabla g(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
\]

\[
= \left( \int_\Omega \left( \sup_{t \leq d(x,\partial \Omega)} \int_{B(x,t)} \Phi_t(x-t)|\nabla f(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
+ \left( \int_\Omega \left( \sup_{t \leq d(x,\partial \Omega)} \int_{B(x,t)} \Phi_t(x-t)|\nabla g(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
\]

Using the Minkowski inequality once more, we obtain

\[
\| (f + g) \|_{H_r^{1,p} (\Omega)}^{\frac{1}{p}} = \| \nabla f + \nabla g \|_{H_r^{1} (\Omega)}^{\frac{1}{p}} 
\leq \left( \int_\Omega \left( \sup_{t \leq d(x,\partial \Omega)} \int_{B(x,t)} \Phi_t(x-t)|\nabla f(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
+ \left( \int_\Omega \left( \sup_{t \leq d(x,\partial \Omega)} \int_{B(x,t)} \Phi_t(x-t)|\nabla g(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
\]

\[
= \left( \int_\Omega \left( \sup_{t \leq d(x,\partial \Omega)} \int_{B(x,t)} \Phi_t(x-t)|\nabla f(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
+ \left( \int_\Omega \left( \sup_{t \leq d(x,\partial \Omega)} \int_{B(x,t)} \Phi_t(x-t)|\nabla g(y)|^p \, dy \right) \, dx \right)^{\frac{1}{p}} 
\]

\[
= \| \nabla f \|_{H_r^{1} (\Omega)}^{\frac{1}{p}} + \| \nabla g \|_{H_r^{1} (\Omega)}^{\frac{1}{p}}.
\]
2.3 Duality of Hardy and BMO spaces

It is well-known that dual to the Hardy space $H^1(\mathbb{R}^n)$ is the space $\text{BMO}(\mathbb{R}^n)$, see, for example, [47]. Recall that a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ is a function of bounded mean oscillation ($f \in \text{BMO}(\mathbb{R}^n)$) [4] if

$$\|f\|_{\text{BMO}(\mathbb{R}^n)} := \sup_B \frac{1}{|B|} \int_B |f(x) - f_B| \, dx < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$ and $f_B = \frac{1}{|B|} \int_B f(x) \, dx$.

Since we consider the Hardy spaces defined on Lipschitz domains [1,7], we formulate the following version of duality (see [6, 8, 12]. Let $\Omega$ be a Lipschitz domain of $\mathbb{R}^n$. The space $\text{BMO}_z(\Omega)$ is defined as being the space of all functions in $\text{BMO}(\mathbb{R}^n)$ supported in $\Omega$ equipped with the norm

$$\|f\|_{\text{BMO}_z(\Omega)} := \|f\|_{\text{BMO}(\mathbb{R}^n)}.$$

Dual to the space $H^1_z(\Omega)$ is the space $\text{BMO}_z(\Omega)$.

The space $\text{BMO}_r(\Omega)$ is defined as the space of all restrictions to $\Omega$ of functions $\text{BMO}(\mathbb{R}^n)$. It is equipped with the norm

$$\|f\|_{\text{BMO}_r(\Omega)} := \inf \|F\|_{\text{BMO}(\mathbb{R}^n)},$$

where the infimum is taken over all functions $F \in \text{BMO}(\mathbb{R}^n)$ such that $F|_{\Omega} = f$. In [9], it was shown that $\text{BMO}_r(\Omega)$ can be described in another way, namely, as a space of locally integrable functions on $\Omega$ with

$$\|f\|_{\text{BMO}(\Omega)} := \sup_Q \frac{1}{|Q|} \int_B |f(x) - f_Q| \, dx < \infty,$$

where the supremum is taken over all cubes $Q \subset \Omega$ with the sides parallel to the axes. Then, the dual of the space $H^1_z(\Omega)$ is $\text{BMO}_r(\Omega)$.

3. $Q$-quasiconformal mappings

3.1 Modulus and capacity

The theory of $Q$-quasiconformal mappings has been extensively developed in recent decades, see, for example, [37]. Let us give the basic definitions.

The linear integral is denoted by

$$\int_\gamma \rho \, ds = \sup_{\gamma'} \int_{\gamma'} \rho \, ds = \sup_{0} \int_0^{l(\gamma')} \rho(\gamma'(s)) \, ds,$$

where the supremum is taken over all closed parts $\gamma'$ of $\gamma$, and $l(\gamma')$ is the length of $\gamma'$. Let $\Gamma$ be a family of curves in $\mathbb{R}^n$. Denote by $\text{adm}(\Gamma)$ the set of Borel functions (admissible functions) $\rho : \mathbb{R}^n \to [0, \infty]$ such that the inequality

$$\int_\gamma \rho \, ds \geq 1$$

holds for locally rectifiable curves $\gamma \in \Gamma$. 318
Let \( \Gamma \) be a family of curves in \( \mathbb{R}^n \), where \( \mathbb{R}^n \) is a one-point compactification of the Euclidean space \( \mathbb{R}^n \). The quantity 

\[
M(\Gamma) = \inf_{\mathbb{R}^n} \int \rho^\alpha \, dx
\]

is called the (conformal) module of the family of curves \( \Gamma \) [37]. The infimum is taken over all admissible functions \( \rho \in \text{adm}(\Gamma) \).

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), and \( F_0, F_1 \) disjoint non-empty compact sets in the closure of \( \Omega \). Let \( M(\Gamma(F_0, F_1; \Omega)) \) stand for the module of a family of curves connecting \( F_0 \) and \( F_1 \) in \( \Omega \). Then [37]

\[
M(\Gamma(F_0, F_1; \Omega)) = \text{cap}_n(F_0, F_1; \Omega),
\]

where \( \text{cap}_n(F_0, F_1; \Omega) \) is a conformal capacity of the condenser \( (F_0, F_1; \Omega) \) [39].

Recall that a homeomorphism \( \varphi : \Omega \to \Omega \) of domains \( \Omega, \Omega \subset \mathbb{R}^n \) is called a \( Q \)-homeomorphism [37] with a non-negative measurable function \( Q \), if

\[
M(\varphi \Gamma) \leq \int_{\Omega} Q(x) \cdot \rho^\alpha(x) \, dx
\]

for every family \( \Gamma \) of rectifiable paths in \( \Omega \) and every admissible function \( \rho \) for \( \Gamma \).

### 3.2 Mappings of finite distortion

Suppose a mapping \( \varphi : \Omega \to \mathbb{R}^n \) belonging to the class \( W_{\text{loc}}^{1,1}(\Omega) \). Then the formal Jacobi matrix \( D\varphi(x) \) and its determinant (Jacobian) \( J(x, \varphi) \) are well defined at almost all points \( x \in \Omega \). The norm \( |D\varphi(x)| \) is the operator norm of \( D\varphi(x) \), i.e. \( |D\varphi(x)| = \max\{|D\varphi(x) \cdot h| : h \in \mathbb{R}^n, |h| = 1\} \). We also let \( l(D\varphi(x)) = \min\{|D\varphi(x) \cdot h| : h \in \mathbb{R}^n, |h| = 1\} \).

Recall that a Sobolev mapping \( \varphi : \Omega \to \mathbb{R}^n \) is the mapping of finite distortion if \( D\varphi(x) = 0 \) for almost all \( x \) from \( Z = \{ x \in \Omega : J(x, \varphi) = 0 \} \) [55].

Let us define two \( p \)-distortion functions, \( 1 \leq p < \infty \), for Sobolev mappings of finite distortion \( \varphi : \Omega \to \bar{\Omega} \).

The outer \( p \)-dilatation

\[
K^O_p(x, \varphi) = \begin{cases} \frac{|D\varphi(x)|^p}{|J(x, \varphi)|}, & J(x, \varphi) \neq 0, \\ 0, & J(x, \varphi) = 0. \end{cases}
\]

The inner \( p \)-dilatation

\[
K^I_p(x, \varphi) = \begin{cases} \frac{|J(x, \varphi)|^p}{|D\varphi(x)|^p}, & J(x, \varphi) \neq 0, \\ 0, & J(x, \varphi) = 0. \end{cases}
\]

Note that \( K^I_n(x) \leq (K^O_n(x))^{n-1} \) and \( K^O_n(x) \leq (K^I_n(x))^{n-1} \).

The maximal dilatation (or, in short, the dilatation) of \( \varphi \) at \( x \) is defined by

\[
K_p(x) = K_p(x, \varphi) = \max(K^O_p(x, \varphi), K^I_p(x, \varphi)).
\]

Let us recall the weak inverse theorem for Sobolev homeomorphisms [18] (see also [11]).

**Theorem 3.1.** Let \( \varphi : \Omega \to \bar{\Omega} \), where \( \Omega, \bar{\Omega} \) are domains in \( \mathbb{R}^n \), be a homeomorphism of finite distortion which belongs to the class \( W_{\text{loc}}^{1,p}(\Omega) \), \( p \geq n-1 \), and possesses the Luzin \( N \)-property (an image of a set of measure zero has measure zero). Then the inverse mapping \( \varphi^{-1} : \bar{\Omega} \to \Omega \) is a mapping of finite distortion which belongs to the class \( W_{\text{loc}}^{1,1}(\bar{\Omega}) \).

Recall that homeomorphisms \( \varphi : \Omega \to \bar{\Omega} \) of the class \( W_{\text{loc}}^{1,n}(\Omega) \) possess the Luzin \( N \)-property (an image of a set of measure zero has measure zero) [55].
4. BMO-quasiconformal mappings and composition operator

Given a function $Q : \Omega \to [1, \infty]$, a sense-preserving homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$ is called to be $Q$-quasiconformal [34], if $\varphi \in W^{1, r}_{loc}(\Omega)$ and $K_n(x) \leq Q(x)$ for almost all $x \in \Omega$. If $\varphi$ is $Q$-quasiconformal with $Q \in \text{BMO}_r(\Omega)$, then $\varphi$ is said to be a BMO-quasiconformal mapping. In [37], it was proven that every BMO-quasiconformal mapping is a $Q$-homeomorphism with some $Q \in \text{BMO}_r$.

The first theorem represents a description of composition operators generated by BMO-quasiconformal homeomorphism.

**Theorem 4.1.** Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain and $\widetilde{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists BMO-quasiconformal homeomorphism $\varphi : \Omega \to \widetilde{\Omega}$. Then the inverse mapping $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ generates, by the composition rule $(\varphi^{-1})^* = f \circ \varphi^{-1}$, a bounded composition operator

$$(\varphi^{-1})^* : H^{1,n}_z(\Omega) \cap \text{Lip}(\Omega) \to L^{1,n}_z(\widetilde{\Omega}),$$

and the inequality

$$\|f \circ \varphi^{-1} | L^{1,n}_z(\widetilde{\Omega})\| \leq \|Q | \text{BMO}_r(\Omega)\|^{\frac{1}{n}} \|f | H^{1,n}_z(\Omega)\|$$

holds for any Lipschitz function $f \in \text{Lip}(\Omega)$.

**Proof.** Since $\varphi \in W^{1, r}_{loc}(\Omega)$, then $\varphi$ possesses the Luzin $N$-property, so the composition $f \circ \varphi^{-1}$ is well defined a. e. in $\widetilde{\Omega}$. Because $\varphi \in W^{1, r}_{loc}(\Omega)$ and has a finite distortion, then $\varphi^{-1} : \widetilde{\Omega} \to \Omega$ belongs to $W^{1, r}_{loc}(\widetilde{\Omega})$ [18].

Now, let there be given a Lipschitz function $g \in H^{1,n}_z(\widetilde{\Omega})$. Then $g \circ \varphi^{-1}$ is weakly differentiable in $\widetilde{\Omega}$, and as long as $\varphi$ has the Luzin $N$-property, the chain rule holds [23]. Hence,

$$\|g \circ \varphi^{-1} | L^{1,n}_z(\widetilde{\Omega})\|^n = \int_{\widetilde{\Omega}} |\nabla g \circ \varphi^{-1}(y)|^n \, dy$$

$$\leq \int_{\widetilde{\Omega}} |\nabla g|^n((\varphi^{-1}(y)) |D\varphi^{-1}(y)|^n \, dy.$$

By the definition of BMO-quasiconformal mappings, there exists a measurable function $Q \in \text{BMO}_r(\Omega)$ such that $K^I_n(x) \leq Q(x)$ for almost all $x \in \Omega$. Using the change of variables formula [3,21], we obtain

$$\int_{\Omega} |\nabla g|^n((\varphi^{-1}(y)) |D\varphi^{-1}(y)|^n \, dy = \int_{\Omega} |\nabla g|^n(y) |D\varphi^{-1}(\varphi(x))|^n |J(x, \varphi)| \, dx$$

$$= \int_{\Omega} |\nabla g|^n(y) \frac{|J(x, \varphi)|}{|D\varphi(x)|^n} \, dx \leq \int_{\Omega} |\nabla g|^n(x)Q(x) \, dx.$$

Now, by the duality of Hardy spaces $H^1_z$ and BMO$_r$-spaces [6], we have

$$\int_{\Omega} |\nabla g|^n(x)Q(x) \, dx \leq \|Q | \text{BMO}_r(\Omega)\| \cdot \|f | H^{1,n}_z(\Omega)\|^n.$$

Hence,

$$\|f \circ \varphi^{-1} | L^{1,n}_z(\Omega)\| \leq \|Q | \text{BMO}_r(\Omega)\|^{\frac{1}{n}} \|f | H^{1,n}_z(\Omega)\|$$

for any Lipschitz function $f \in H^{1,n}_z(\Omega)$.

\[\square\]
Let \( \varphi: \Omega \rightarrow \widetilde{\Omega} \) be a homeomorphism. Then \( \varphi \) is called to be a \( \text{BMO}_p \)-quasiconformal mapping, if \( \varphi \in W^{1,p}_{\text{loc}}(\Omega) \) and \( K_p(x) \leq Q(x) \) for almost all \( x \in \Omega \) and for some function \( Q \in \text{BMO}_r(\Omega) \).

In the case of \( \text{BMO}_p \)-quasiconformal mappings, we require an additional assumption of the Luzin \( N \)-property of a mapping \( \varphi \) if \( n-1 \leq p < n \).

**Theorem 4.2.** Let \( \Omega \subset \mathbb{R}^n \) be a Lipschitz bounded domain and \( \widetilde{\Omega} \subset \mathbb{R}^n \) be a bounded domain. Suppose there exists \( \text{BMO}_p \)-quasiconformal homeomorphism \( \varphi: \Omega \rightarrow \widetilde{\Omega} \), \( p \geq n-1 \) which possesses the Luzin \( N \)-property if \( n-1 \leq p < n \). Then the inverse mapping \( \varphi^{-1}: \widetilde{\Omega} \rightarrow \Omega \) generates, by the composition rule \((\varphi^{-1})^* = f \circ \varphi^{-1}\), a bounded composition operator

\[
(\varphi^{-1})^*: H^1_{\sharp}(\Omega) \cap \text{Lip}(\Omega) \rightarrow L^{1,p}(\widetilde{\Omega}),
\]

and the inequality

\[
\|f \circ \varphi^{-1}\|_{L^{1,p}(\widetilde{\Omega})} \leq \|Q\|_{\text{BMO}_r(\Omega)} \frac{1}{p} \|f\|_{H^1_{\sharp}(\Omega)}
\]

holds for any Lipschitz function \( f \in \text{Lip}(\Omega) \).

**Proof.** Since \( \varphi \) possesses the Luzin \( N \)-property, then the composition \( f \circ \varphi^{-1} \) is well defined a.e. in \( \Omega \). Because \( \varphi \in W^{1,p}_{\text{loc}}(\Omega) \), \( p \geq n-1 \), has a finite distortion and possess the Luzin \( N \)-property, \( \varphi^{-1}: \Omega \rightarrow \Omega \) belongs to \( W^{1,1}_{\text{loc}}(\Omega) \) [18].

Now, let there be given a Lipschitz function \( g \in H^1_{\sharp}(\Omega) \). Then \( g \circ \varphi^{-1} \) is weakly differentiable in \( \widetilde{\Omega} \), and as long as \( \varphi \) has the Luzin \( N \)-property, the chain rule holds [23]. Hence,

\[
\|g \circ \varphi^{-1}\|_{L^{1,p}(\widetilde{\Omega})} = \int_{\widetilde{\Omega}} |g \circ \varphi^{-1}(y)|^p \, dy
\]

\[
\leq \int_{\Omega} |g|^p(\varphi^{-1}(y))|D \varphi^{-1}(y)|^p \, dy.
\]

By the definition of \( \text{BMO} \)-quasiconformal mappings, there exists a measurable function \( Q \in \text{BMO}_r(\Omega) \) such that \( K_p(x) \leq Q(x) \) for almost all \( x \in \Omega \). Using the change of variables formula [3, 21], we obtain

\[
\int_{\Omega} |\nabla g|^p(\varphi^{-1}(y))|D \varphi^{-1}(y)|^p \, dy = \int_{\Omega} |\nabla g|^p(x)|D \varphi^{-1}(\varphi(x))|^p|J(x, \varphi)| \, dx
\]

\[
= \int_{\Omega} |\nabla g|^p(x) |J(x, \varphi)| \frac{|J(x, \varphi)|}{|D \varphi(x)|^p} \, dx \leq \int_{\Omega} |\nabla g|^p(x) Q(x) \, dx.
\]

Now, by the duality of Hardy spaces \( H^1_{\sharp} \) and \( \text{BMO}_r \)-spaces [6], we have

\[
\int_{\Omega} |\nabla g|^p(x) Q(x) \, dx \leq \|Q\|_{\text{BMO}_r(\Omega)} \cdot \|f\|_{H^1_{\sharp}(\Omega)}^p.
\]

Hence,

\[
\|f \circ \varphi^{-1}\|_{L^{1,p}(\widetilde{\Omega})} \leq \|Q\|_{\text{BMO}_r(\Omega)} \frac{1}{p} \|f\|_{H^1_{\sharp}(\Omega)}
\]

for any Lipschitz function \( f \in H^1_{\sharp}(\Omega) \). \( \square \)

Using the duality between \( H^1_{\sharp}(\Omega) \) and \( \text{BMO}_r(\Omega) \), in the same manner, we obtain the next two results:
Theorem 4.3. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain and $\overline{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists BMO-quasiconformal homeomorphism $\varphi : \Omega \to \overline{\Omega}$ with $Q \in \text{BMO}_z(\Omega)$. Then the inverse mapping $\varphi^{-1} : \overline{\Omega} \to \Omega$ generates, by the composition rule $(\varphi^{-1})^* = f \circ \varphi^{-1}$, a bounded composition operator

$$(\varphi^{-1})^* : H^1_r(\Omega) \cap \text{Lip}(\Omega) \to L^{1,n}(\overline{\Omega}),$$

and the inequality

$$\|f \circ \varphi^{-1} \|_{L^{1,n}(\overline{\Omega})} \leq \|Q \|_{\text{BMO}_z(\Omega)} \frac{1}{n} \|f \|_{H^1_r(\Omega)}$$

holds for any Lipschitz function $f \in \text{Lip}(\Omega)$.

Theorem 4.4. Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz bounded domain and $\overline{\Omega} \subset \mathbb{R}^n$ be a bounded domain. Suppose there exists BMO$_p$-quasiconformal homeomorphism $\varphi : \Omega \to \overline{\Omega}$, $p \geq n - 1$, with $Q \in \text{BMO}_z(\Omega)$, which possesses the Luzin N-property if $n - 1 \leq p < n$. Then the inverse mapping $\varphi^{-1} : \overline{\Omega} \to \Omega$ generates, by the composition rule $(\varphi^{-1})^* = f \circ \varphi^{-1}$, a bounded composition operator

$$(\varphi^{-1})^* : H^1_r(\Omega) \cap \text{Lip}(\Omega) \to L^{1,p}(\overline{\Omega}),$$

and the inequality

$$\|f \circ \varphi^{-1} \|_{L^{1,p}(\overline{\Omega})} \leq \|Q \|_{\text{BMO}_z(\Omega)} \frac{1}{p} \|f \|_{H^1_r(\Omega)}$$

holds for any Lipschitz function $f \in \text{Lip}(\Omega)$.

We also note the following regularity results:

Theorem 4.5. Given the mapping $\varphi : \Omega \to \overline{\Omega}$,

1. if the composition operator $\varphi^* : H^1_r(\overline{\Omega}) \to L^{1,p}(\Omega)$ is bounded, then $\varphi \in L^{1,p}(\Omega)$;
2. if the composition operator $\varphi^* : H^1_r(\overline{\Omega}) \to H^1_r(\Omega)$ is bounded, then $\varphi \in H^1_r(\Omega)$.

Proof. We prove the theorem only for the first case. The second one is proved in a similar way.

Due to the boundedness of $\varphi^*$,

$$\|f \circ \varphi \|_{L^{1,p}(\Omega)} \leq \|\varphi^*\| \|f \|_{H^1_r(\overline{\Omega})}.$$  

Substituting the coordinate functions $f_j = y_j$, $j = 1, \ldots, n$, we obtain

$$\|f_j \|_{H^1_r(\overline{\Omega})} = \int_{\overline{\Omega}} \sup_{0 < t \leq \text{dist}(x, \partial \overline{\Omega})} \left| \frac{1}{t^n} \int_{B(x, t)} \Phi \left( \frac{x - y}{t} \right) \right| dx = \int_{\overline{\Omega}} \sup_{0 < t \leq \text{dist}(x, \partial \overline{\Omega})} |1| dx = |\overline{\Omega}|.$$  

Hence,

$$\|f_j \circ \varphi \|_{L^{1,p}(\Omega)} = \|\varphi_j \|_{L^{1,p}(\Omega)} \leq |\overline{\Omega}| \|\varphi^*\|.$$  

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