Qubit quantum channels: A characteristic function approach

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A characterization of qubit quantum channels is introduced. In analogy to what happens in the context of Bosonic channels we exploit the possibility of representing the states of the system in terms of characteristic function. The latter are functions of non-commuting variables (Grassmann variables) and are defined in terms of generalized displacement operators. In this context we introduce the set of Gaussian channels and show that they share similar properties with the corresponding Bosonic counterpart.

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In quantum mechanics the transition from the initial state to the final state of a system is described in terms of quantum channels [1]. At a mathematical level these are linear maps operating on the set of bounded operators of the system, which preserve the trace and (if any) the positivity of the operators on which they act. Finally in order to represent a "physical" transformation, i.e., a transformation that could be implemented in a lab, a quantum channel must also possess the property of complete positivity (i.e., the positivity of any initial joint state to the final state of a system is described in terms of quantum channel degradability. An impressive achievement has been devoted in the last decades to study the properties of quantum channels. Indeed they play a fundamental role in many different branch of physics, specifically in all those sectors where one is interested in studying the decoherence and noise effects.

In the context of quantum information theory emphasis is put on characterizing the properties of quantum channels in terms of their information capacities [2, 3]. These figures of merit are the quantum counterpart of the Shannon capacity of a classical communication line [3], which "measure" the performances of the map in conveying classical or quantum information. Even though impressive achievement has been obtained in this field in the recent years, several open questions are still under investigation — we refer the reader to [3] and references therein for details.

The majority of the results obtained so far relate to two specific classes of channels, namely the qubit channels and the Bosonic Gaussian channels. The former are completely positive trace-preserving (CPT) transformations which act on the state of a single two-level quantum system (qubit). Due to the small size of the Hilbert space a simple parametrization of these channels has been obtained [6-8] while some additive issues [8, 9, 10] and several classical and quantum capacities [8, 9, 11, 12, 13] have successfully been solved (see also Ref. [3] for a review). Bosonic Gaussian channels [14, 15], on the contrary, are a specific subclass of CPT maps acting on a continuous variable system that preserve certain symmetries. These channels include a variety of physical transformations that are of fundamental interest in optics, including thermalization, loss and squeezing. As in the qubit channel case, additivity issues [16-17] and capacities [18, 19, 20, 21, 22] have been successfully solved for Bosonic Gaussian channels. Furthermore, they allow for a compact parametrization [17, 21, 23, 24] in terms of the characteristic function formalism [25, 26, 27].

In this paper we establish a parallelism among the qubit channels and the Bosonic Gaussian channels by introducing for the former a characteristic function representation. To do so we adapt the formalism introduced by Caldirola and Glauber in Ref. [28] for representing the density operators of Fermions to the case of two-level systems. In this context the channels are represented in terms of Green functions. Interestingly enough this allows us to define a set of Gaussian channels for qubit that share analogous properties with their continuous variable counterpart.

The paper is organized as follows. In Sec. I we briefly review the characteristic function formalism for Bosonic (and Fermionic) systems. In Sec. II we introduce the displacement operator and characteristic function for a qubit. To do so we introduce Grassmann variables and we use them to generalize the definition of coherent states for finite dimensional systems. We then present a Green function representation for qubit channels (Sec. III) and define the set of qubit Gaussian channels (Sec. IV) discussing their degradability properties. Conclusion and final remarks are presented in Sec. V. The paper includes also a couple of technical Appendixes: namely, in Appendix A we review some properties of Grassmann calculus, while in Appendix B we present a brief excursus on quantum channel degradability.

1. CHARACTERISTIC FUNCTION FOR BOSONS

In quantum optics a complete description of the state of a Bosonic mode characterized by annihilation and creation operators $a$ and $a^\dagger$, can be obtained in terms of its coherent states $|\mu\rangle$. These vectors possess various appealing properties. Specifically, they minimize the uncertainty relations of any couple of conjugate quadratures and they are eigenvectors of $a$. Most importantly, coherent states form an over-complete continuous set of vec-
tors parametrized by a single complex variable $\mu$. This allows us to expand any other state of the system as a superposition of the $|\mu\rangle$s with coefficients which define quasi-probability density functions. Exploiting this and the fact that the coherent states can be obtained by applying the displacement operator $D(\mu) \equiv \exp[\mu a^\dagger - \mu^* a]$ to the vacuum, one can also use the latter as an over-complete operator basis \cite{14,25,27}. In particular, given $\Theta$ any bounded operator of the system (e.g., a density matrix $\rho$), we can write

$$\Theta = \int d^2 \mu \, \chi(\mu) \, D(-\mu),$$

(1)

where $d^2 \mu \equiv d\text{Re}(\mu) d\text{Im}(\mu)$ and where

$$\chi(\mu) \equiv \text{Tr}[\Theta D(\mu)].$$

(2)

Equation (2) defines the characteristic function of the operator $\Theta$. This is a complex function of the variables $\mu$ and $\mu^*$ which provides us with a faithful description of the original operator thanks to the “orthogonality” relation

$$\text{Tr}[D(\mu)D(-\nu)] = \delta^{(2)}(\mu - \nu),$$

(3)

with $\delta^{(2)}(\mu - \nu)$ being the Dirac delta in the complex plane. To represent a density matrix $\rho$, the function (2) needs to possess certain properties \cite{14,25} including being continuously differentiable in $\mu$ and $\mu^*$ and verifying $\chi(0) = 1$. Within the characteristic function description, Gaussian states are defined as those $\rho$ whose $\chi(\mu)$ is a Gaussian function of the complex parameter $\mu$ (examples are thermal, coherent, and squeezed states).

Consider now the action of a linear super-operator $\mathcal{N}$ which transforms $\Theta$ into $\Theta' = \mathcal{N}(\Theta)$. Equation (1) allows us to represent this mapping in terms of a linear transformation of $\chi(\mu)$. Indeed the characteristic function of the output operator $\Theta'$ is

$$\chi'(\mu) = \text{Tr}[\Theta' D(\mu)] = \text{Tr}[\Theta \mathcal{N}_H[D(\mu)]] = \int d^2 \nu \, \chi(\nu) \, G(\nu, \mu),$$

(4)

with

$$G(\nu, \mu) \equiv \text{Tr}[D(-\nu)\mathcal{N}_H[D(\mu)]] = \text{Tr}[\mathcal{N}[D(-\nu)D(\mu)]] \equiv \text{Tr}[\mathcal{N}[D(-\nu)]D(\mu)].$$

(5)

In these expressions $\mathcal{N}_H$ is the dual of $\mathcal{N}$ which describes the channel in the Heisenberg picture and which is defined by the identity

$$\text{Tr}[\Theta_1 \mathcal{N}_H(\Theta_2)] = \text{Tr}[\mathcal{N}[\Theta_1(\Theta_2)],$$

(7)

for all $\Theta_{1,2}$ (see, for instance, \cite{21}). We call Eq. (5) the Green function of $\mathcal{N}$: according to the previous definitions it provides us with a complete characterization of the channel.

A special subset of CPT maps for Bosonic systems is the set of Gaussian channels \cite{15}. These are characterized by Green functions \cite{6} of the form

$$G(\nu, \mu) = \delta^{(2)}(\nu - \nu \mu - w \mu^*) \times \exp\left[-\frac{1}{2}(\mu^*, -\mu)\Gamma\left(\begin{array}{c} \mu \\ -\mu^* \end{array}\right)\right],$$

(8)

with $\Gamma$ being a real symmetric positive matrix (i.e., covariance matrix) and $\nu$ and $\nu$ being complex numbers — rigorously speaking, Eq. (8) defines one-mode Bosonic Gaussian channels. As can be directly verified from Eq. (1) such maps have the peculiar property of transforming input Gaussian states into output Gaussian states. An interesting fact about these channels is that, except for the additive classical noise channel \cite{21,23}, they admit a physical representation \cite{21} in terms of a single mode environment originally prepared in a Gaussian state. Indeed, the exceptional role of the additive classical noise channel corresponds to the fact that any one-mode Bosonic Gaussian channel can be represented as a unitary coupling with a single-mode environment plus an additive classical noise. Within such representation (without additive classical noise) one can show that the Bosonic Gaussian channels \cite{8} are either anti-degradable or weakly degradable \cite{20}. Moreover, in the case in which the single-mode representation is of Stinespring form (that is, if the environment state is pure) the channel is then anti-degradable or degradable in the sense of Ref. \cite{9} (for the sake of completeness explicit definitions of these properties are given in Appendix B).

A. Characteristic function for Fermions

The characteristic function formalism presented in the previous section can be generalized to describe Fermionic systems too \cite{28}. The main difference in this case is related to the fact that now the complex variables $\mu$ and $\mu^*$ are replaced by a couple of conjugate Grassmann variables $\xi$ and $\xi^*$, \cite{29} whose properties are reviewed in Appendix A. This is intrinsically related with the fact that the annihilation and creation operators of a Fermion obey anti-commutation rules instead of commutation rules \cite{30}. We will not review the analysis of Ref. \cite{28} since in the next section, when discussing the qubit case, we will rederive most of the results obtained in the Fermionic case.

II. REPRESENTATION OF A QUBIT

Various proposals for defining a (discrete) phase space for finite dimensional systems have been discussed so far by introducing generalized position and momentum operators (see, for instance, Ref. \cite{31} and references therein). Here we will not follow this line; instead we invoke the analogies between a qubit and a single Fermionic mode to
adapt the results of Ref. [28]. A similar approach was developed in Ref. [32] to solve non-Markovian master equations of a two-level atom interacting with an external field.

The starting point of our analysis is to observe that the lowering and raising operators of the qubit [i.e., \(\sigma_+ \equiv |1\rangle\langle 0|\) and \(\sigma_- \equiv (\sigma_+)^\dagger\)] satisfy anti-commutation rules similar to those of a Fermionic mode, i.e.,

\[
\{\sigma_-, \sigma_+\} = |0\rangle\langle 0| + |1\rangle\langle 1| \equiv 1, \\
\{\sigma_-, \sigma_-\} = \{\sigma_+, \sigma_-\} = 0. \tag{9}
\]

Identifying the qubit state \(|0\rangle\) with the Fermionic vacuum we can therefore treat \(\sigma_+\) and \(\sigma_-\) as Fermionic creation and annihilation operators, respectively. Following [28] we introduce then a couple of conjugate Grassmann variables \(\xi\) and \(\xi^\ast\) (see Appendix A) and impose standard anti-correlation with the annihilation and creation operators of the system, i.e.,

\[
\{\xi, \sigma_{\pm}\} = \{\xi^\ast, \sigma_{\pm}\} = 0. \tag{10}
\]

It is worth noticing that this implies that the projectors \(|0\rangle\langle 0| = \sigma_-\sigma_+ + \sigma_+\sigma_-\) as well as the Pauli matrix \(\sigma_z \equiv |0\rangle\langle 0| - |1\rangle\langle 1|\) commute with \(\xi\) and \(\xi^\ast\).

In the following we will also require that

\[
\xi \{j\} = (-1)^j \xi \xi^\ast \{j\}, \\
\xi^\ast \{j\} = (-1)^j \xi \xi^\ast \{j\}, \tag{11}
\]

for \(j = 0, 1\). This is not strictly necessary but it is consistent with Eq. (10) and allows us to simplify the calculations. For instance, given any collection of qubit operators \(\Theta_1, \Theta_2, \ldots, \Theta_{n+1}\) and the Grassmann numbers \(\xi_1, \xi_2, \ldots, \xi_n\) we can use Eq. (11) to verify that the following relation applies

\[
\text{Tr}[\Theta_1\xi_1\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1}] \tag{12}
\]

\[
= \xi_1\xi_2\cdots\xi_n \text{Tr}[\Theta_1\sigma_z\Theta_2\sigma_2\cdots\Theta_n\sigma_z\Theta_{n+1}]
\]

(an analogous expression holds also when replacing all, or part of, the \(\xi\)s with their complex conjugates — more details about the trace can be found in Appendix A).

The above definitions give us the possibility of operating with “hybrid” mathematical objects obtained by multiplying Grassmann variables and qubit operators. In this context we find it useful to define a generalized adjoint operation for these hybrid operators by arbitrarily imposing the conditions

\[
(\Theta_1\xi_1\Theta_2\xi_2\cdots\Theta_n\xi_n\Theta_{n+1})^\dagger = \Theta_{n+1}^\dagger \xi_n^\ast \Theta_n^\dagger \cdots \xi_2^\ast \Theta_2^\dagger \xi_1^\ast \Theta_1^\dagger, \tag{13}
\]

with \(\xi_i\) and \(\Theta_i\) as in Eq. (12).

### A. Qubit characteristic function

Qubit displacement operators can now be defined in analogy with [28] as

\[
D(\xi) \equiv \exp(\sigma_+\xi - \xi^\ast\sigma_-) \tag{14}
\]

\[
= 1 + \sigma_+\xi - \xi^\ast\sigma_- - \sigma_2 \xi^\ast\xi/2, \tag{15}
\]

where in the second line we used Eq. (10). As in the Bosonic case they satisfy the identity \(D^\dagger(\xi) = D(-\xi)\). Moreover the application of \(D(\xi)\) to the vacuum originates eigenvectors of the annihilation operator of the system (\(\sigma_-\)). These are the coherent states of our qubit, i.e.,

\[
|\xi\rangle = D(\xi)|0\rangle = \left(1 - \frac{\xi^\ast\xi}{2}\right)|0\rangle - \xi|1\rangle, \tag{16}
\]

whose norm is unity. These vectors are eigenvectors of \(\sigma_-\) in Grassmann sense (i.e., their eigenvalues are Grassmann variables; see Ref. [28] for details).

What is interesting for us is the fact that \(D(\xi)\) can be used to define a characteristic function for the operators of the system as in Eq. (2), i.e.,

\[
\chi(\xi) \equiv \text{Tr}[\Theta D(\xi)]. \tag{17}
\]

In particular, consider a \(\Theta\) which is characterized by the matrix

\[
\Theta \equiv \begin{pmatrix} \theta_{00} & \theta_{01} \\ \theta_{10} & \theta_{11} \end{pmatrix}, \tag{18}
\]

when expressed in the computational basis \{\(|0\rangle, |1\rangle\}\). In this case using the anti-commutation rules of Eq. (10) and the identity (12) we get

\[
\chi(\xi) = \text{Tr}[\Theta] + (\theta_{00} - \theta_{11}) \frac{\xi \xi^\ast}{2} + \theta_{01} \xi - \theta_{10} \xi^\ast. \tag{19}
\]

It is worth noticing that with respect to the analysis of Ref. [28] the characteristic functions analyzed here contain an extra term which is linear in \(\xi\) and \(\xi^\ast\). Indeed in the Fermionic case analyzed by Cahill and Glanuber the only allowed physical states are classical mixtures of \(|0\rangle/|0\rangle + |1\rangle/|1\rangle\) (this follows from the requirement of invariance under \(2\pi\) rotation with respect to an arbitrary axis). Consequently the off-diagonal terms associated with \(\theta_{01}\) and \(\theta_{10}\) do not need to be considered. When analyzing qubit systems, instead, quantum superpositions among \(|0\rangle\) and \(|1\rangle\) are allowed and we need to include also the linear contributions.

As in the Bosonic case, Eq. (10) can be inverted. In this case, however, Eq. (1) is replaced by

\[
\Theta = \int d^2\xi \chi(\xi) \tilde{E}(\xi), \tag{20}
\]

with \(\tilde{E}(\xi) \neq D(\xi)\) defined by

\[
\tilde{E}(\xi) \equiv \sigma_z - \xi \xi^\ast/2 + \sigma_+\xi - \xi^\ast\sigma_- \tag{21}
\]

The easiest way to verify this is by direct substitution of Eqs. (18) and (20) into Eq. (19) and by employing the integration rules (17).
1. Density operators

To represent a density operator

\[ \rho = \left( \begin{array}{cc} p & \gamma \\ \gamma^* & 1 - p \end{array} \right) \]  

(21)

the characteristic function needs to satisfy certain physical requirements. First of all, the Hermitianity of \( \rho \) and the normalization condition \( \text{Tr}[\rho] = 1 \) imply, respectively,

\[ \chi(\xi) = [\chi(-\xi)]^* , \]  

(22)

\[ \chi(0) = 1 , \]  

(23)

where complex conjugation is defined as in Eq. \((A4)\) to verify this simply use Eq. \((13)\) with \( \Theta = \rho \). The positivity of \( \rho \) imposes, instead, the following inequality to hold

\[ \left| \int d^2 \xi \chi(\xi)\xi^2 \right|^2 + \left| \int d^2 \xi \chi(\xi)\xi \right|^2 \leq \frac{1}{4} . \]  

(24)

This follows from the positivity condition \( |\gamma|^2 \leq p(1-p) \) and by the identity

\[ \gamma = \int d^2 \xi \chi(\xi)\xi^* , \]

\[ p = \int d^2 \xi \chi(\xi) + 1/2 . \]

Using similar arguments one can verify that Eqs. \((22)-(24)\) are also sufficient conditions for \( \chi(\xi) \) being a characteristic function of a density operator \( \rho \).

III. GREEN FUNCTION REPRESENTATION OF A QUBIT CHANNEL

Let us now consider the effect of a qubit quantum channel \( \mathcal{N} \) acting on the operator \( \Theta \) of the system. As in the Bosonic case we would like to derive its Green function representation \((\ref{eq:GreenFun})\). To do so we first evaluate the characteristic function \( \chi'(\xi) \) associated with \( A\Theta B \) with \( A \) and \( B \) being arbitrary qubit operators. This is

\[ \chi'(\xi) = \text{Tr}[A\Theta BD(\xi)] \]

\[ = \int d^2 \zeta \text{Tr}[A\chi(\zeta)\bar{E}(\zeta)BD(\xi)] , \]  

(25)

where we used Eq. \((12)\) with \( \chi(\xi) \) being the characteristic function of \( \Theta \) (from now on \( \zeta \) and \( \xi \) should be considered entries of the same Grassmann set). Our goal is to find a function \( G(\zeta, \xi) \) which gives

\[ \chi'(\xi) = \int d^2 \zeta \chi(\zeta) G(\zeta, \xi) , \]  

(26)

for all \( \chi(\xi) \). Notice that if \( \xi \) were a commuting variable (e.g., a complex variable) the problem could be solved by simply moving \( \chi(\xi) \) out of the trace operation of Eq. \((25)\) yielding \( G(\zeta, \xi) = \text{Tr}[A\bar{E}(\zeta)BD(\xi)] \). In the case under consideration, however, the situation is complicated by the fact that for moving out of trace the variables \( \xi \) or \( \xi^* \) we need to insert \( \sigma_z \)s as in Eq. \((12)\). Taking into account this fact, the solution becomes

\[ G(\zeta, \xi) = \text{Tr}[\sigma_z D(-\zeta)BD(\xi)] , \]  

(27)

as can be easily verified by direct integration of the Eqs. \((25)\) and \((26)\) for the most general characteristic function \((\ref{eq:GreenFun})\).

The Green function \((26)\) associated with a CPT map \( \mathcal{N} \) can then be obtained by using an operator sum representation \( [1, 2] \) of such channel and exploiting the linearity of the trace. Indeed, writing \( \mathcal{N}(\Theta) = \sum_k M_k \Theta M_k^\dagger \) with \( \{ M_k \} \) being Kraus operators of \( \mathcal{N} \), we get

\[ G(\zeta, \xi) = \sum_k \text{Tr}[M_k \sigma_z D(-\zeta)M_k^\dagger D(\xi)] \]

\[ = \text{Tr} \left[ \mathcal{N} \left( \sigma_z D(-\zeta) \right) D(\xi) \right] . \]  

(28)

Using Eq. \((A13)\) this can also be written as

\[ G(\zeta, \xi) = \text{Tr} \left[ \sigma_z D(-\zeta) \mathcal{N}_H \left( D(\xi) \right) \right] , \]  

(29)

with \( \mathcal{N}_H \) being the Heisenberg representation of the map \( \mathcal{N} \) defined in Eq. \((7)\). Equation \((29)\) shows that, as in the Bosonic case, a complete description of the channel is obtained by applying the dual map to the displacement operator — see Eq. \((\ref{eq:GreenFun})\). Exploiting the normalization condition \( \sum_k M_k^\dagger M_k = I \) we note that for \( \zeta = 0 \) the above expression yields

\[ G(\zeta, 0) = \text{Tr}[\sigma_z D(-\zeta)] = \zeta \zeta^* , \]  

(30)

which corresponds to the Grassmann delta function \( \delta^{(2)}(\zeta) \) defined in Eq. \((A3)\), in agreement with the requirement of channel being trace preserving — see Eqs. \((15)\) and \((16)\).

Finally, let \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) be two different qubit channels with Green functions \( G_1(\zeta, \xi) \) and \( G_2(\zeta, \xi) \), respectively. From the definition \((26)\) we then find that the Green function \( G_{12}(\zeta, \xi) \) of the composite map \( \mathcal{N}_2 \circ \mathcal{N}_1 \) in which we first operate with \( \mathcal{N}_1 \) and then with \( \mathcal{N}_2 \), can be expressed in terms of the following Grassmann convolution integral

\[ G_{12}(\zeta, \xi) = \int d^2 \xi' G_1(\zeta, \xi') G_2(\xi', \xi) , \]  

(31)

with \( \zeta, \xi, \) and \( \xi' \) Grassmann numbers.

A. Examples and canonical forms

As a particular case of Green function consider the identity map \( I \) which leaves all operators invariant, i.e., \( I(\Theta) = \Theta \). According to our definition we get

\[ G(\zeta, \xi) = \text{Tr}[\sigma_z D(-\zeta)D(\xi)] = (\zeta - \xi)(\zeta^* - \xi^*) . \]  

(32)
which, as expected, corresponds to the delta \( \delta^{(2)}(\zeta - \xi) \) of Eq. \( A.9 \). More generally from Ref. \( 8 \) we know that the most generic qubit quantum channel \( \mathcal{N} \) implements the following transformation,

\[
\mathcal{N}(\rho) = \mathcal{N}\left(\frac{1 + \vec{r} \cdot \vec{\sigma}}{2}\right) = \frac{1 + (\vec{r} + T \vec{r}) \cdot \vec{\sigma}}{2},
\]

(33)

where \( \vec{r} = (t_1, t_2, t_3) \) is a real vector, \( \vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\} \) is a vector containing the Pauli matrices, \( T \) is the Bloch vector describing the input state, and \( T \) is a real \( 3 \times 3 \) matrix. In Ref. \( 8 \) it is shown that \( T \) can be reduced, via changes of basis in \( \mathbb{C}^2 \) (i.e., via proper rotations of the input and output states), to the diagonal (canonical) form \( T = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \), with the real coefficients \( \lambda_{1,2,3} \) and \( t_{1,2,3} \) that need to satisfy certain conditions \( 8, 7 \) to guarantee the complete positivity of the map. In the Green function language such canonical form corresponds to have

\[
G(\zeta, \xi) = \delta^{(2)}(\zeta - \lambda_2 + \lambda_1 \xi - \lambda_2 - \lambda_1 \xi^*) \exp\left[-\frac{t_3}{2} \xi^* \xi\right] + (\lambda_3 - \lambda_1 \lambda_2) \xi \xi^* + \frac{t_1 - it_2}{2} \zeta \xi^* - \frac{t_1 + it_2}{2} \zeta^* \xi^*.
\]

(34)

\[\lambda_3 = \lambda_1 \lambda_2, \quad t_1 = t_2 = 0.\]

(37)

(38)

This in fact yields Gaussian Green functions with \( a = (\lambda_2 + \lambda_1)/2, \ b = (\lambda_2 - \lambda_1)/2 \) and \( c = t_3/2 \). We can then use \( 7, 6 \) to show that the corresponding transformation is CPT if and only if the following inequalities hold,

\[
|\lambda_k| \leq 1 \quad \text{for } k = 1, 2; \quad |t_3| \leq \sqrt{(1 - \lambda_1^2)(1 - \lambda_2^2)}.
\]

(39)

A. Canonical form for Gaussian channels

From Eq. \( 34 \) it is easy to verify that within the parametrization \( 6, 7 \) we can get Gaussian maps \( 35 \)

\[
\lambda_3 = \lambda_1 \lambda_2, \quad t_1 = t_2 = 0.
\]

(37)

(38)

Then we can parametrize \( t_3 \) by introducing the positive quantity \( q \in [0, 1] \) to write

\[
t_3 = (2q - 1) \frac{\cos(2\theta) - \cos(2\phi)}{2}.
\]

(41)

Replacing all this into Eq. \( 34 \) yields the following canonical form for the Green function of a qubit Gaussian channel, i.e.,

\[
G(\zeta, \xi) = \delta^{(2)}(\zeta - \cos \theta \cos \phi + \xi^* \sin \theta \sin \phi) \times \exp\left[\frac{2q - 1}{4} \frac{\cos(2\theta) - \cos(2\phi)}{\xi \xi^*}\right].
\]

(42)

We will see that the maps of this form have the peculiar property that they can always be described in terms of a unitary interaction of the form \( 31 \) with a single (not necessarily pure) qubit environment. For this reason we call them “qubit-qubit” channels. It is worth stressing that once again a similar property holds for the Bosonic case: there (almost) all the one-mode Bosonic Gaussian maps are in fact describable in terms of a single mode environment \( 20, 21 \).

B. Qubit-qubit maps: Pure environment case

An important subclass of the qubit-qubit channels of Eq. \( 42 \) is obtained for \( q = 1 \) and \( \theta \) and \( \phi \) generic, i.e.,

\[
G(\zeta, \xi) = \delta^{(2)}(\zeta - \xi \cos \theta \cos \phi + \xi^* \sin \theta \sin \phi) \times \exp\left[\frac{\cos(2\theta) - \cos(2\phi)}{4} \xi \xi^*\right].
\]

(43)

IV. GAUSSIAN CHANNELS FOR QUBITS

In analogy with the Bosonic case, in this section we introduce the definition of qubit Gaussian channels. We start noticing that in order to define these channels it does not make sense to focus on maps which transform Gaussian characteristic functions into Gaussian characteristic functions. Indeed, thanks to Eq. \( A.9 \), all characteristic functions of a qubit can be written in a Gaussian form \( 33 \). Therefore following Eq. \( 8 \) we say that a qubit map is Gaussian if its Green function has the form

\[
G(\zeta, \xi) = \delta^{(2)}(\zeta - a \xi - b \xi^*) \exp[-c \xi^* \xi],
\]

(35)

with \( a \) and \( b \) complex and \( c \) real \( 34 \) numbers, respectively, and with the exponential defined as in Eq. \( A.6 \). A trivial example is provided by the identity map \( I \) whose Green function \( 92 \) is of the form \( 35 \) for \( b = c = 0 \) and \( a = 1 \).

Generic mixtures of Gaussian channels do not necessarily have the form \( 35 \). Therefore the set of Gaussian channels is not convex. However, it has semi-group structure with respect to the channel composition rule \( \circ \). Indeed, given two Gaussian channels \( V_1 \) and \( V_2 \) characterized by parameters \( (a_1, b_1, c_1) \) and \( (a_2, b_2, c_2) \), respectively, from Eq. \( 31 \) it is easy to verify that the Green function of \( V_2 \circ V_1 \) is again of the form \( 35 \) with

\[
a = a_1 a_2 + b_1 b_2^*, \quad b = a_1 b_2 + b_1 a_2^*, \quad c = c_1 |a_2|^2 - |b_2|^2 + c_2.
\]

(36)

Both the semi-group property and the non-convexity property hold also in the Bosonic case.
According to Eq. (39) this corresponds to having $|t_3| = \sqrt{(1 - \lambda_x^2)(1 - \lambda_z^2)}$. As shown in Ref. [2] any CPT map which can be described in terms of an interaction with a single qubit environment originally prepared in a pure state can be expressed in this form by proper unitary rotation of the input and the output state. This implies that the maps (39) admit a Stinespring dilation (31) with a two-dimensional (qubit) environment $E$. Without loss of generality, we can assume an initial state of the environment of the form $\rho_E \equiv |0\rangle_E \langle 0|$. Following Ref. [1], one can then choose the unitary coupling $U$ to have the following block structure

$$ U = \begin{pmatrix} A_0 & -\sigma_x A_1 \sigma_x \\ A_1 & \sigma_x A_0 \sigma_x \end{pmatrix}, \quad (44) $$

with

$$ A_0 = \begin{pmatrix} \cos \theta & 0 \\ 0 & \cos \phi \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sin \phi \\ \sin \theta & 0 \end{pmatrix}, \quad (45) $$

being a Kraus set for the channel (the matrix (44) is expressed in the basis $\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}$ with $|jk\rangle \equiv |j\rangle \otimes |k\rangle_E$ for $j, k = 0, 1$.

The complementary channel $\mathcal{N}_\theta$ [3, 33, 36] can now be computed as in Eq. (32). Since it represents a qubit channel — it connects two two-dimensional Hilbert spaces (the input Hilbert space with the environmental one) — we can use Eq. (28) to evaluate its Green function obtaining

$$ \tilde{G}(\zeta, \xi) = \delta^{(2)}(\zeta - \xi \cos \theta \sin \phi + \xi^* \sin \theta \cos \phi) \times \exp \left[ \frac{\cos(2\theta) + \cos(2\phi)}{4} \xi \xi^* \right]. \quad (46) $$

It is still of the (pure-environment qubit-qubit) Gaussian form (46) and can be expressed in terms of the original Green function $G(\zeta, \xi)$ of $\mathcal{N}$ by simply shifting $\phi$ by $-\pi/2$ and by changing sign to $\theta$, i.e.,

$$ \tilde{G}(\zeta, \xi) = G(\zeta, \xi)\big|_{\theta \rightarrow -\theta, \phi \rightarrow -\phi - \pi/2}. \quad (47) $$

In Ref. [13] it has been shown that qubit-qubit channels with pure environment are degradable for $\cos(2\theta)/\cos(2\phi) \geq 0$, and anti-degradable otherwise. Here we will rederive this same result in the Green function formalism as a consequence of the Gaussianity of these maps, pointing out an interesting parallelism with their Bosonic counterpart.

In analogy with [20], we look for the intermediate map $T$ that should connect $\mathcal{N}$ with $\mathcal{N}_\theta$, in the class of qubit-qubit channels (with pure environment). Rewriting the degradability condition [13] in terms of the compositions rules [31] we can then recast the problem as follows

$$ \tilde{G}(\zeta, \xi) = \int d^2 \xi' G(\zeta, \xi') \tilde{G}(\xi', \xi), \quad (48) $$

where $G_x(\zeta, \xi)$ is the Green function (13) of the map $\mathcal{T}$ characterized by the parameters $\theta_x$ and $\phi_x$. By using Eq. (36) we find that, for $\cos(2\theta)/\cos(2\phi) \geq 0$, $\theta_x, \phi_x$ do exist such that Eq. (48) is satisfied. Specifically such parameters are defined by the relations,

$$ \cos(2\theta_x) = \frac{\cos(2\theta) - \cos(2\phi) + 2 \cos(2\theta) \cos(2\phi)}{\cos(2\theta) + \cos(2\phi)}, \quad \cos(2\phi_x) = \frac{\cos(2\theta) - \cos(2\phi) - 2 \cos(2\theta) \cos(2\phi)}{\cos(2\theta) + \cos(2\phi)}. \quad (49) $$

The case $\cos(2\theta)/\cos(2\phi) \leq 0$ can be treated analogously to show that the corresponding channels are anti-degradable. In fact, in the Green function formalism the anti-degradability condition (34) becomes

$$ G(\zeta, \xi) = \int d^2 \xi' \tilde{G}(\zeta, \xi') \tilde{G}(\xi', \xi), \quad (50) $$

where $\tilde{G}_x(\zeta, \xi)$ is the Green function of the connecting map $T$. We find that for $\cos(2\theta)/\cos(2\phi) \leq 0$, Eq. (50) is satisfied by choosing $\tilde{G}_x(\zeta, \xi)$ in the subclass of qubit-qubit channels with pure environment — i.e., Eq. (13) — with $\theta_x$ and $\phi_x$ determined by the expressions (19) after replacing $(\theta, \phi)$ by $(-\theta, \phi - \pi/2)$.

More directly this result can be established by using the correspondence (47) and the fact that the complementary channels of degradable maps are anti-degradable — see Appendix B. Consider, in fact, a (pure environment) qubit-qubit channel $\mathcal{N}$ with $\cos(2\theta)/\cos(2\phi) \leq 0$. According to Eq. (47) we know that its complementary $\mathcal{N}$ is still a (pure environment) qubit-qubit channel characterized by the parameters $(\theta', \phi') = (-\theta, \phi - \pi/2)$. Now it is easy to verify that $\cos(2\theta')/\cos(2\phi') = -\cos(2\theta)/\cos(2\phi) \geq 0$. Therefore from Eqs. (48) and (49) we can conclude that $\mathcal{N}$ is degradable while $\mathcal{N}$ is anti-degradable.

Note that, in the special case $\cos(2\theta) = \cos(2\phi) = 0$, both the degradability relations are satisfied. Therefore in this case the qubit-qubit channels with pure environment are both degradable and anti-degradable, with null quantum capacity.

### C. Qubit-qubit maps: Mixed environment case

Now let us consider the Gaussian channels (42) for $q \neq 1$. They can be represented in terms of a physical representation (31) with $U$ as in Eq. (44) and with $E$ being a single qubit environment initially prepared in the mixed state,

$$ \rho_E \equiv q|0\rangle_E \langle 0| + (1 - q)|1\rangle_E \langle 1|. \quad (51) $$

To verify this, we observe that with the above prescriptions Eq. (41) gives

$$ \mathcal{N}(\rho) = \text{Tr}_E[U \rho \otimes |q\rangle_E \langle 0| + (1 - q)|1\rangle_E \langle 1|] U^\dagger = q\mathcal{N}_0(\rho) + (1 - q)\mathcal{N}_1(\rho). \quad (52) $$
As in Sec. [V B] we prove that the maps \( N \) of Eq. (42) are weakly degradable for \( \cos(2\theta)/\cos(2\phi) \geq 0 \). In this regime in fact one can easily check that Eq. (43) can still be solved with \( G_{\theta,\phi}(\xi,\zeta) \) of the form (55) replacing \( \theta \) and \( \phi \) with \(-\theta_x,\phi_x+\pi/2\) where \( \theta_x,\phi_x \) satisfy the relations (49).

Proving anti-degradable for \( \cos(2\theta)/\cos(2\phi) \leq 0 \) is not simple because, in general, \( N \) is not in a Gaussian form — see Eq. (53). However, in this case we show that these channels cannot be used to transfer quantum information since their quantum capacity \( \mathcal{Q} [38] \) is null. To see this we notice that for \( \cos(2\theta)/\cos(2\phi) \leq 0 \), \( N \) is a mixture [52] of two channels (i.e., \( N_0 \) and \( N_1 \)) which are both anti-degradable and have hence null quantum capacity, i.e., \( Q(N_0) = Q(N_1) = 0 \) — see Appendix B.

Under these conditions it is easy to verify that also \( N \) must have a null \( Q \). Indeed let us consider a new CPT map,

\[
N'(\rho) = q \cdot N_0(\rho) \otimes |0\rangle_B \langle 0| + (1 - q) \cdot N_1(\rho) \otimes |1\rangle_B \langle 1|,
\]

where \( B \) is an ancillary system. We can now verify that the \( N' \) is isomorphic to \( E \circ N' \) with \( E(\ldots) = \text{Tr}_B[\ldots] \otimes |0\rangle_B \langle 0| \) being a CPT map which replaces all states of \( B \) with a fix given output \( |0\rangle_B \). Expressing \( Q \) in terms of the output coherent information [39] of the channel and using the quantum data processing inequality [1] we can verify that \( Q(N) \leq Q(N') \). Besides, by using the basic properties of von Neumann entropy [1] we can express the coherent information of \( N' \) as \( J(N',\rho) = qJ(N_0,\rho) + (1 - q)J(N_1,\rho) \). Putting all this together we get

\[
Q(N') = \lim_{N \to \infty} \max_{\rho} J(\rho) / N \leq qQ(N_0) + (1 - q)Q(N_1) = 0,
\]

and hence \( Q(N) = 0 \).

\[5 \]

V. CONCLUSIONS

In this work we introduce a characteristic function formalism for the qubit channels in terms of generalized displacement operators and Grassmann variables, inspired by a parallelism among these maps and the Bosonic Gaussian channels.

We then present a Green function representation of the quantum evolution that allows us to define the set of qubit Gaussian maps. In this context, we find that all the Gaussian channels are qubit-qubit, i.e., they can always be described in terms of a unitary interaction of a qubit system with a single (not necessarily pure) qubit environment. Similarly, it is known that in the Bosonic case (almost) all the one-mode Bosonic Gaussian maps are describable in terms of a single mode environment.

This formalism turns out to be elegant and powerful and, in particular, it can be used to study the weak-degradability properties of the qubit-qubit maps, for both pure and mixed qubit environments, in terms of Green functions.
On one hand, in the case of pure environment, the qubit-qubit maps are either degradable (i.e., additive coherent information) or anti-degradable (i.e., $Q=0$). Besides, the complementary maps are still qubit-qubit channels and so Gaussian. It is interesting to note that an equivalent property holds for one-mode Bosonic Gaussian channels. On the other hand, in the case of mixed environment, we show that the qubit-qubit maps are either weakly degradable or they cannot be used to transfer quantum information (i.e., $Q=0$). However, in this case the weakly complementary maps do not belong to the set of qubit-qubit channels and are not Gaussian.

It is important to stress that this Green function formalism shows clearly that the qubit Gaussian maps share analogous properties with their continuous variable counterpart, i.e., the Bosonic Gaussian channels.

Finally, we remark that the characteristic function approach, introduced in this paper for qubit systems, can be generalized to $d$-level quantum systems (qudit) in terms of generalized Grassmann variables [40].

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**APPENDIX A: GRASSMANN VARIABLES**

A Grassmann variable $\xi$ spans over a set of objects (the Grassmann numbers) $\xi_1$, $\xi_2$, $\ldots$, which anti-commute. Indeed, given any $\xi_i$ and $\xi_j$ elements of the set, they satisfy the relation

$$\xi_i \xi_j = -\xi_j \xi_i,$$  \hfill (A1)

while obeying ordinary commutation relations with respect to the multiplication by a complex number. In particular Eq. (A1) implies that a Grassmann variable is 2-nilpotent, i.e., $\xi^2 = 0$ (note that 0 is trivially included in the Grassmann variable set). At a mathematical level, the above conditions can be rigorously formalized by saying that Grassmann numbers are the generators of an algebra over the complex field which obey anti-commutation relations.

Complex conjugation of $\xi$ can be defined by introducing an extra Grassmann variable $\xi^*$ whose elements $\xi_1^*$, $\xi_2^*$, $\ldots$ obey the same relation (A1) and anti-commute with all the $\xi_i$s, i.e.,

$$\xi_i^* \xi_j = -\xi_j^* \xi_i^*, \quad (A2)$$

$$\xi_i^* \xi_j = -\xi_j \xi_i^*. \quad (A3)$$

To identify $\xi_i^*$ with the complex conjugate of $\xi_i$ we finally require the relations

$$(\xi_i^*)^* = \xi_i,$$  \hfill (A4)

$$(\xi_i^*)^* = x^* \xi_i^*$$

to be satisfied for any $x$ complex number or product of the $\xi_1, \xi_2, \ldots$ and $\xi_1^*, \xi_2^*, \ldots$.

Given the above properties it follows that the most general function $f(\xi, \xi^*)$ is linear both in $\xi$ and $\xi^*$, i.e.,

$$f(\xi, \xi^*) = A + B_1 \xi + B_2 \xi^* + C \xi^* \xi, \quad (A5)$$

with $A$, $B_1, B_2$, and $C$ independent from $\xi$ and $\xi^*$. In particular, the exponentials become

$$\exp(B_1 \xi + B_2 \xi^* + C \xi^* \xi) \equiv \sum_{n=0}^{\infty} (B_1 \xi + B_2 \xi^* + C \xi^* \xi)^n/n!$$

$$= 1 + B_1 \xi + B_2 \xi^* + C \xi^* \xi + B_1 \xi B_2 \xi^* / 2 + B_2 \xi^* B_1 \xi / 2. \quad (A6)$$

This expression can be used to verify that (apart from a global multiplicative term) any function $[A5]$ can be written as an exponential.

Integration over $\xi$ and $\xi^*$ can be defined by introducing the “differential” $d\xi$ and $d\xi^*$. These are assumed to obey the same anti-commutation relations obeyed by the variables $\xi$ and $\xi^*$, including Eqs. (A1), (10), and (11). The integrals are then defined according to the Berezin rules

$$\int d\xi = \int d\xi^* = 0,$$  \hfill (A7)

$$\int d\xi = \int d\xi^* \xi^* = 1.$$  \hfill (A7)

Joint integration with respect to $\xi$ and $\xi^*$ is finally defined by identifying the double differential $d^2\xi$ as follows,

$$d^2\xi \equiv d\xi^* d\xi = -d\xi d\xi^*. \quad (A8)$$

In this context one can identify an analogous of the Dirac delta function $\delta^{(2)}(\mu - \nu)$ in the complex plane. Such Grassmann delta is defined as

$$\delta^{(2)}(\xi - \zeta) = \int d^2 \kappa \exp[\kappa (\xi^* - \zeta^*) - (\xi - \zeta) \kappa^*]$$

$$= (\xi - \zeta) (\xi^* - \zeta^*), \quad (A9)$$

with $\xi$, $\zeta$, and $\kappa$ Grassmann variables. Indeed, from Eq. (A7) and from Eq. (A5) we have

$$\int d^2 \xi \delta^{(2)}(\xi - \zeta) f(\xi, \xi^*) = f(\zeta, \zeta^*), \quad (A10)$$

for all $f(\xi, \xi^*)$. Notice that the delta function (A9) commutes with any Grassmann numbers and satisfies the relation $\delta^{(2)}(\xi - \zeta) = \delta^{(2)}(\zeta - \xi) = -\delta^{(2)}(\xi^* - \zeta^*)$.

A useful property is the following. Given the function $f(\xi, \xi^*)$ one can define its even and odd parts, i.e.,

$$f_\pm(\xi, \xi^*) \equiv \frac{f(\xi, \xi^*) \pm f(-\xi, -\xi^*)}{2}. \quad (A11)$$

According to Eq. (A5) they are of the form $f_+(\xi, \xi^*) = A + C \xi^* \xi$ and $f_-(\xi, \xi^*) = B_1 \xi + B_2 \xi^*$, respectively. Now given $g(\xi, \xi^*)$ another function we can write

$$\int d^2 \xi f_\pm(\xi, \xi^*) g(\xi, \xi^*) = 0.$$
and thus
\[
\int d^2 \xi f(\xi, \xi^*) g(\xi, \xi^*) = \int d^2 \xi f_+(\xi, \xi^*) g_+(\xi, \xi^*) + \int d^2 \xi f_-(\xi, \xi^*) g_-(\xi, \xi^*). \tag{A12}
\]

1. More about trace

Equation \((\text{12})\) shows that the cyclicity of the trace needs to be modified when involving Grassmann terms. If we need to move only qubit operators, then the standard rule applies, i.e.,
\[
\text{Tr}[\Theta_1 \xi_1 \cdots \Theta_n \xi_n \Theta_{n+1}] = \text{Tr}[\Theta_{n+1} \Theta_1 \xi_1 \cdots \Theta_n \xi_n] = \text{Tr}[\xi_1 \cdots \Theta_n \xi_n \Theta_{n+1} \Theta_1]. \tag{A13}
\]

On the contrary, if we move also Grassmann variables, by exploiting the anti-commutation rules of the \(\xi_5\)s, we get
\[
\text{Tr}[\Theta_1 \xi_1 \Theta_2 \xi_2 \cdots \Theta_n \xi_n \Theta_{n+1}] = (-1)^{n-1} \text{Tr}[\xi_n \Theta_{n+1} \Theta_1 \xi_1 \cdots \Theta_n] = (-1)^{n-1} \text{Tr}[\Theta_2 \xi_2 \cdots \Theta_n \xi_n \Theta_{n+1} \Theta_1 \xi_1]. \tag{A14}
\]

Finally in conjunction with Eq. \((\text{13})\), Eq. \((\text{12})\) gives
\[
\left(\text{Tr}[\Theta_1 \xi_1 \Theta_2 \xi_2 \cdots \Theta_n \xi_n \Theta_{n+1}]\right)^* = \text{Tr}[\Theta_{n+1}^\dagger \xi_1^\dagger \Theta_1 \xi_1 \cdots \Theta_2^\dagger \xi_2^\dagger \Theta_1 \xi_1]. \tag{A15}
\]

APPENDIX B: WEAK-DEGRADABILITY VS. ANTI-DEGRADABILITY

It is a well known (see, e.g., \[21, 22\]) that any CPT map \(\mathcal{N}\) can be described by a unitary coupling between the system \(S\) with an external ancillary system \(E\) (describing the environment) prepared in some fixed pure state. This follows from the Stinespring dilation \[43\] of the map which is unique up to a partial isometry. More generally, one can describe \(\mathcal{N}\) as a coupling with an environment prepared in some mixed state \(\rho_E\), i.e.,
\[
\mathcal{N}(\rho) = \text{Tr}_E[U(\rho \otimes \rho_E)U^\dagger], \tag{B1}
\]
where \(\text{Tr}_E[\cdots]\) is the partial trace over the environment \(E\) and \(U\) is a unitary operator in the composite Hilbert space \(\mathcal{H}_S \otimes \mathcal{H}_E\). As proposed in Ref. \[21\] we call Eq. \((\text{B1})\) a “physical representation” of \(\mathcal{N}\) to distinguish it from the Stinespring dilation, and to stress its connection with the physical picture of the noisy evolution represented by \(\mathcal{N}\). Moreover, Eq. \((\text{B1})\) motivates the following definition \[20, 21\]. For any physical representation in Eq. \((\text{B1})\) of the quantum channel \(\mathcal{N}\) we define its weakly complementary as the map \(\tilde{\mathcal{N}}\) which takes the input state \(\rho\) into the state of the environment \(E\) after the interaction with \(S\), i.e.,
\[
\tilde{\mathcal{N}}(\rho) = \text{Tr}_S[U(\rho \otimes \rho_E)U^\dagger]. \tag{B2}
\]

The transformation \((\text{B2})\) is CPT and describes a quantum channel connecting systems \(S\) and \(E\). It is a generalization of the complementary channel \(\mathcal{N}_{\text{com}}\) defined in Refs. \[4, 35, 36\]. If some channel \(\mathcal{T}\) does exist such that
\[
(\mathcal{T} \circ \mathcal{N})(\rho) = \mathcal{N}(\rho), \tag{B3}
\]
for all density matrices \(\rho\), then \(\mathcal{N}\) is called weakly degradable and \(\tilde{\mathcal{N}}\) anti-degradable. Similarly if
\[
(\mathcal{T} \circ \tilde{\mathcal{N}})(\rho) = \mathcal{N}(\rho), \tag{B4}
\]
for some channel \(\mathcal{T}\) and all density matrices \(\rho\), then \(\mathcal{N}\) is anti-degradable while \(\tilde{\mathcal{N}}\) is weakly degradable (see \[20, 21\]). In Ref. \[9\] the channel \(\mathcal{N}\) is called degradable if one considers the environment in a pure state. Clearly any degradable channel \[9\] is weakly degradable but the opposite is not necessarily true.

Degradability and anti-degradability have been proved useful to analyze the quantum capacity \[38\] of the channel. On one hand, one can verify that anti-degradable channels (where this property is defined irrespectively from the purity of \(\rho_E\) associated with the physical representation) cannot be used to convey quantum messages in reliable fashion — i.e., their quantum capacity \(Q\) nullifies \[12, 20, 21\]. On the other hand, instead degradable channels \[9\] allows for a single letter formula expression for \(Q\) — i.e., the maximum of their output coherent information is additive.

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