Phase imaging from noisy interferometric observations via sparse coding of phase and amplitude

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Abstract. Phase-shifting interferometry is a coherent optical method that combines high accuracy with high measurement speeds. However, despite its advantageous properties, the inference of the object amplitude and the phase, herein termed wavefront reconstruction is not a trivial task owing to, namely, the Poissonian noise associated with the measurement process and to the 2π phase periodicity of the observation mechanism. In this paper, we formulate the wavefront reconstruction (phase and amplitude) as an inverse problem where the amplitude and the absolute phase are assumed to admit sparse linear representations in suitable sparsifying transforms. We introduce the sparse phase and amplitude reconstruction (SPAR) algorithm. SPAR takes into full consideration the Poissonian (photon counting) measurements and uses data adaptive BM3D frames as a sparse representation for amplitude and absolute phase.

1. Introduction

Many images (and signals) admit sparse representations in the sense that they are well approximated by linear combinations of a small number of functions taken from a know set. Fundamentally, this is a consequence of the self-similarity of these images: it is very likely to find in them many similar patches in different locations and at different scales [1].

Let \( c \in \mathbb{R}^n \) denote a vector representing an image, or a patch of it, and let us assume that it admits a sparse representation or sparse coding with respect to the columns of a given matrix \( \Psi \in \mathbb{R}^{n \times m} \); i.e., it is possible to write \( c = \Psi \theta \), where \( \theta \) is a vector containing only a few non-zero components. The matrix \( \Psi \) is termed a synthesis operator (or dictionary) because in the writing \( c = \sum_{j=1}^{m} \Psi_j \cdot \theta_j \), where \( \Psi_j \) are the columns of the matrix \( \Psi \), and \( \theta_j \) are the elements of the vector \( \theta \), \( c \) is synthesized as a linear combination of the columns of \( \Psi \) weighted by the elements of \( \theta \), often called the spectrum of \( c \). The synthesis based representations have a dual point of view in which, given an image \( c \in \mathbb{R}^m \), we compute its spectrum \( \theta \) by applying the so-called analysis operator (or dictionary) \( \Phi \) to \( c \), i.e. \( \theta = \Phi c \). It happens, that the synthesis dictionaries yielding sparse representations are often overcomplete, i.e., \( m > n \). The concept of frame is a generalization of the classical basis especially developed for overcomplete (synthesis and analysis) representations with linearly dependent approximating functions [1]. We extend this idea to complex-valued coherent wave fronts assuming that phase and amplitude of the wave front allow sparse representations.

Let \( o \in \mathbb{C}^n \) be a complex valued wavefront defined on a grid with \( n \) pixels. Denote \( B_o = \text{mod}(o) \) and \( \varphi_o = \text{angle}(o) \) and as, respectively, the corresponding images of amplitude
(modulus) and phase. Then we have \( o = B_o \exp(j \varphi_o) \). Herein, all functions applied to vectors are to be understood in the component-wise sense; the same applies to multiplication of vectors. With the objective of formulating treatable phase imaging inverse problems, most approaches follow a two-step procedure: in the first step, an estimate of the so-called principal (wrapped, interferometric) phase in the interval \([-\pi, \pi]\) is obtained. The latter procedure is known as phase unwrapping. In what follows we denote the principal phase as \( \varphi_o \) and the absolute phase as \( \varphi_{o,abs} \) We introduce the phase-wrap operator \( W \) linking the absolute and principal phase as \( \varphi_o = W(\varphi_{o,abs}) \), the inverse of this operator denotes the unwrapping, \( \varphi_{o,abs} = W^{-1}(\varphi_o) \).

Following to [2], we introduce the sparse wavefront modeling by the formulas:

\[
\begin{align*}
\text{mod}(o) &= \Psi_{a,o} \theta_{a,o}, \\
\varphi_{o,abs} &= \Psi_{\varphi,o} \theta_{\varphi,o}, \\
\theta_{a,o} &= \Phi_{a,o} \text{mod}(o), \\
\theta_{\varphi,o} &= \Phi_{\varphi,o} \varphi_{o,abs},
\end{align*}
\]

where \( \theta_{a,o} \) and \( \theta_{\varphi,o} \) are, respectively, the amplitude and the phase (absolute phase) spectra of the object \( o \). In (1), the amplitude (mod(\( o \))) and phase (angle (\( o \))) are synthesized from the amplitude and phase spectra \( \theta_{a,o} \) and \( \theta_{\varphi,o} \). On the other hand, the analysis Eqs. (2) give the spectra for amplitude and phase for the wavefront \( o \). Here \( \Phi \) and \( \Psi \) with the corresponding indexes are the analysis and synthesis matrices (dictionaries), respectively.

The \( l_0 \)-norm of \( \theta_{a,o} \) and \( \theta_{\varphi,o} \) denoted as \( ||\theta_{a,o}||_0 \) and \( ||\theta_{\varphi,o}||_0 \), counts the number of non-zero elements of these spectra and is used as the measure of the sparsity of approximation for amplitude and phase, respectively.

2. Optical setup and setting of the problem

We consider the two-beam interferometer shown in fig. 1. A coherent laser beam of the wavelength \( \lambda \) is divided by beam splitter (BS) into two paths, one of which contains a transmitting or reflecting object and the other, a phase shifter. The approach proposed in this paper takes into full consideration the Poissonian (photon counting) measurements. In this way we are targeting at optimal sparse reconstruction both phase and amplitude taking into consideration all details of the observation formation.

**Figure 1.** Phase-shift interferometry: on-axis setup. A single laser beam is splitted into two identical beams by a beam splitter (BS). A phase shifter introduces different phase shifts between the object and reference beams.

For the \( L \)-step phase-shifting interferometry the complex-valued wave field at the sensor plane is calculated as \( u_s = B_o \exp(j \varphi_o) + A_r \exp(-j \varphi_{rs}), \) \( s = 1, \ldots, L, \) where \( B_o \exp(j \varphi_o) \) and \( A_r \exp(-j \varphi_{rs}) \) are the object and reference wavefronts, respectively. Let us assume that our sensor takes measures on a rectangular grid with \( N \) digital elements and let \( Y_s = \{ Y_s[l], l = 1, \ldots, N \}, \) \( s = 1, \ldots, L, \) denote the \( L \) measured images with \( N \) elements each. The measurement process in optics amounts to count the photons hitting the sensor’s elements and is well modeled by independent Poisson random variables; that is \( p(Y_s[l] = k) = \exp(-I_s[l]) \frac{(I_s[l])^k}{k!} \), where \( k \) is an integer number of photons, \( I_s = |u_s|^2 \) is the intensity of the wavefront, thus given by \( I_s = B_o^2 + A_r^2 + 2B_oA_r \cos(\varphi_o + \varphi_{rs}) \).
Here $\chi$ and is a scaling parameter of the Poissonian flow which can be interpreted as an exposure time (and/or as a sensitivity of the sensor). The problem is to reconstruct the object (wavefront) phase $\varphi_o$ and the amplitude $B_o$ from the observations $\{Y_s, s = 1, ..., L\}$. The standard approach to this problem is as follows. The intensities $I_s$ are replaced by the observations $Y_s$ and the obtained redundant equations are solved with respect the unknowns $\varphi_o$ and the amplitude $B_o$ using the least square method. For the Poissonian observations this procedure is modified because $I_s$ should be replaced by $Y_s/\chi$. The scaling parameter $\chi$ appears here because $E\{Y_s\} = I_s\chi$. In particular, for $L = 4$ and $\varphi_r$ taking values $[0, \pi/2, \pi, 3\pi/2]$, phase and amplitude are defined by the equations: 

$$
\tan(\tilde{\varphi}_o) = \frac{Y_4 - Y_2}{Y_1 - Y_3}, \quad \tilde{B}_o = \sqrt{\sum_{s=1}^{4} Y_s/4\chi - A_r^2}.
$$

These estimates $(\tilde{\varphi}_o, \tilde{B}_o)$ usually are quite noisy due to the noisy observations $Y_s$. The straightforward idea to filter $(\tilde{\varphi}_o, \tilde{B}_o)$ in order to reconstruct $(\varphi_o, B_o)$ is not very productive because the noise in the estimates $(\tilde{\varphi}_o, \tilde{B}_o)$ of phase and amplitude is not additive, and, what is even worse, it is a signal dependent, and this dependence is of complex structure. The iterative algorithm based on the precise modelling of the observations, allows to get precise results even for very noisy data. Comparison of these results versus those obtained by the straightforward filtering of $(\tilde{\varphi}_o, \tilde{B}_o)$ is always in favor of our algorithm.

3. SPAR algorithm

Naturally, the success of the sparse imaging depends on how reach and redundant are the dictionaries $\Phi$ and $\Psi$ used for the analysis and synthesis of absolute phase and amplitude. For the analysis and synthesis operations, we use the BM3D frames, where BM3D is an abbreviation for Block-Matching and 3D filtering [2–4]. Herein, we adopt a multiobjective optimization approach to find an object. The main intention of the approach is simultaneous minimization of the minus log-likelihood function of observations (maximum likelihood approach) and the $l_0$-norms of amplitude and phase spectra supporting the sparsity of the estimates of these variables. The algorithm is derived from three successive stages minimizing three criteria. Overall the algorithm is named Sparse Phase and Amplitude Reconstruction (SPAR) algorithm. The details of mathematical formulation of the problems, algorithm implementation as well as the simulation results can be seen in [3,4].

4. Conclusion

Digital phase-shifting holography is considered. The variational maximum likelihood technique is discussed for phase and amplitude reconstruction of the coherent wavefront from Poissonian intensity measurements. Sparse modeling of amplitude and absolute phase of the wavefront is one of the key elements of the developed algorithm. The experiments show that the new algorithm yields the state-of-the-art accuracy for reconstruction both phase and amplitude of the object wavefront.

Acknowledgments

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