Hyperbolic Lagrangian coherent structures align
with contours of path-averaged scalars

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Abstract

We prove that, in area-preserving two-dimensional flows, hyperbolic Lagrangian Coherent Structures (LCS) align with the contours of path-averaged scalars, i.e. the time average of scalar fields along the trajectories of the dynamical system. The alignment is a consequence of the fact that the length of a repelling (respectively, attracting) LCS shrinks rapidly under advection in forward (respectively, backward) time. As a result, the points along a hyperbolic LCS sample similar values of the scalar field, leading to almost uniform distribution of the path-averaged scalar along the LCS. Our results illuminate the relation between the variational theory of hyperbolic LCSs and a significant subset of mixing diagnostics which are obtained as path-averaged scalars. We illustrate the theoretical results on a direct numerical simulation of two-dimensional Navier–Stokes equations. Furthermore, our results provide partial explanation for a recent observation that hyperbolic LCSs separate dynamically distinct regions in two-dimensional turbulence.

1 Introduction

The study of mixing and transport in fluid flow, where the velocity field is often a time-aperiodic vector field, has numerous engineering and environmental applications. Such unsteady vector fields are known to generate complex mixing patterns of passive tracers [1]. The tracer patterns are

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typically much more complicated than the instantaneous (or Eulerian) patterns of the generating vector field, rendering their Lagrangian treatment inevitable.

Today, a rich toolbox of Lagrangian methods is available for studying passive scalar mixing (see Ref. [2], for a review). While these methods share similar goals (namely, describing the mechanisms underlying chaotic advection), they are significantly different in specifics and span a wide range of approaches from geometric [3–5] to probabilistic [6, 7] to topological [8]. The relations between these methods are largely unknown.

The purpose of the present paper is to partially bridge this gap by establishing the relation between the variational method of hyperbolic Lagrangian coherent structures [3, 9] and a large class of Lagrangian indicators that are computed as the path average of some scalar field [10–12].

Hyperbolic Lagrangian Coherent Structures (LCS) refer to distinguished material surfaces (or material lines in the two-dimensional case) that facilitate mixing by their extensive stretching and shrinking effect on nearby material surfaces (see Ref. [13], for a review). In this sense, hyperbolic LCSs are the generalization of stable and unstable manifolds to unsteady dynamical systems. In Refs. [3, 9], hyperbolic LCSs are formulated as the solutions of appropriate variational problems, leading to necessary and sufficient conditions for unique identification of LCSs. Efficient and accurate numerical methods, based on this variational theory, are now available for computing hyperbolic LCSs [14–16].

On the other hand, many diagnostics [11, 12, 17–19] have been developed to describe Lagrangian mixing through readily implementable methods, avoiding the computational complexity of the more rigorous LCS theory. A significant subset of these diagnostics involve time-averaging a scalar field along the trajectories of the flow. The features of such path-averaged (or Lagrangian-averaged) scalars are argued to signal the main drivers of mixing. Mancho et al. [10], for instance, consider the path-average of $\mathcal{F}(u)$ where $\mathcal{F}$ is a functional of the vector field $u$. Pérez-Muñuzuri and Huhn [11] average instantaneous strain and Okubo–Weiss parameter [20] along the trajectories. Rypina et al. [12] consider the path-average of a Haar wavelet to separate mixing at various length scales.

A fair comparison shows that while the path-averaged diagnostics may return spurious structures, they do at the same time signal the true hyperbolic LCSs (see, for example, Fig.1). Most striking of all is perhaps the study of Kelley et al. [21] who investigated a path-averaged measure of scale-to-scale energy transfer in quasi-two-dimensional turbulence. They found a strong alignment between hyperbolic LCSs and the zero level curves
Figure 1: A path-averaged scalar field for the turbulent flow studied in Section 4. The averaging is taken over the time interval $t \in [0,50]$. The repelling LCSs (black curves), corresponding to the same time window, tend to align with the contours of path-averaged scalar. See Section 4.1 for details.

of the path-averaged energy flux, concluding that LCSs inhibit the local energy transfer between scales. Their study is of particular importance as it is one of the few that relate LCSs, which are obtained by purely kinematic considerations, to the dynamics of the flow (see Ref. [22] for a similar study).

Here we show that, in area-preserving two-dimensional flows, there is a close relation between path-averaged scalars and hyperbolic LCSs. In particular, we prove that, for reasonably long integration times, hyperbolic LCSs align with the contours of path-averaged scalars. The main idea underlying our results is the fact that repelling LCSs (respectively, attracting LCSs) rapidly shrink in length as advected forward (respectively, backward) in time. This implies that the initial conditions belonging to a hyperbolic LCS converge to one another under advection in the right time direction (Contrast this with typical chaotic trajectories that rapidly diverge from one another in either time direction). As a result, after some initial tran-
sients, all points belonging to an LCS probe similar values of the scalar field, leading to approximately uniform values of the path-averaged scalar along the LCS.

After reviewing the preliminaries in Section 2, we formulate and prove the above statement in Section 3. In Section 4, we illustrate these results on a direct numerical simulation of two-dimensional turbulence. Concluding remarks are presented in Section 5.

2 Preliminaries

2.1 Set-up

Consider the non-autonomous differential equation
\begin{equation}
\dot{x} = u(x, t), \quad x \in U \subset \mathbb{R}^2, \quad t \in I = [\alpha, \beta] \subset \mathbb{R}^+</equation>
defined on the open bounded phase space $U$ and the time interval $I = [\alpha, \beta]$. We assume that $u : U \times \mathbb{R}^+ \rightarrow \mathbb{R}^2$ is a sufficiently smooth vector field and denote the solution of (1) by $\phi_{t_0}^t : x^0 \mapsto x(t; t_0, x^0)$ which maps an initial condition $x^0 \in U$ from time $t_0$ to its image $x(t; t_0, x^0)$ at time $t$.

In other words, a trajectory of (1) starting from $x^0$ at time $t_0$ is given by $\phi_{t_0}^t(x^0)$. The path-averaged value $\bar{f}$ of a scalar quantity $f$ along a trajectory is then defined as follows.

**Definition 1.** Consider a measurable, time-dependent, scalar function $f : U \times \mathbb{R}^+ \rightarrow \mathbb{R}$. The path-average $\bar{f}$ of the function $f$ along a trajectory $\phi_{t_0}^t(x^0)$ over the time window $[t_0, t_0 + T] \subset I$ is
\begin{equation}
\bar{f}(x^0, t_0, T) = \frac{1}{T} \int_{t_0}^{t_0 + T} f(\phi_{t_0}^t(x^0), t) dt.
\end{equation}

For fixed initial time $t_0$ and integration time $T$, the function $\bar{f}(\cdot, t_0, T)$ is a scalar field defined over the phase space $U$. When no confusion may arise, we omit the dependence of the path-average $\bar{f}$ on $t_0$ and $T$, and simply write $\bar{f}(x^0)$.

In the following, we assume that the vector field $u$ is divergence-free: $\nabla \cdot u = 0$. This implies that the maps $\phi_{t_0}^t$ are area preserving. In the context of fluid dynamics, the divergence-free assumption puts us in the framework of incompressible fluid flow.
2.2 Hyperbolic LCSs

Hyperbolic Lagrangian coherent structures are the locally most repelling or attracting material lines. A material line is a one-parameter family of curves $\gamma(t) \subset U$ advected passively under the flow of (1) such that $\gamma(t_2) = \phi_{t_2}^{t_1}(\gamma(t_1))$, for all $t_1, t_2 \in I$. Hyperbolic LCSs enhance mixing due to their excessive repulsion or attraction of nearby initial conditions. This intuitive idea has been formulated in terms of a rigorous theory over the last few years [3, 9, 14–16, 23, 24]. Here, we summarize the relevant results from this theory.

For a given time interval $[t_0, t_0 + T]$, the Cauchy–Green strain tensor $C_{t_0}^{t_0+T}$ at a point $x^0 \in U$ is defined in terms of the deformation gradient $\nabla \phi_{t_0}^{t_0+T}$ as

$$C_{t_0}^{t_0+T}(x^0) = \left[\nabla \phi_{t_0}^{t_0+T}(x^0)\right]^\top \nabla \phi_{t_0}^{t_0+T}(x^0),$$

(3)

where the gradient $\nabla$ is taken with respect to the initial conditions $x^0$ and $\top$ denotes the matrix transposition. The Cauchy-Green strain tensor is symmetric and positive definite. Therefore, it has an orthonormal set of eigenvectors $\{\xi_1(x^0), \xi_2(x^0)\}$, corresponding to positive eigenvalues $0 < \lambda_1(x^0) \leq \lambda_2(x^0)$, satisfying

$$C_{t_0}^{t_0+T}(x^0)\xi_k(x^0) = \lambda_k(x^0)\xi_k(x^0), \quad \langle \xi_i(x^0), \xi_j(x^0) \rangle = \delta_{ij}, \quad i, j, k \in \{1, 2\},$$

where $\delta_{ij}$ is the Kronecker delta function and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product. For notational brevity, we omit the dependence of the eigenvalues and eigenvectors on the initial time $t_0$ and the integration time $T$.

Each eigenvector (respectively, eigenvalue) of the Cauchy–Green strain tensor defines a vector field (respectively, scalar field) on the phase space $U$. A material line $\gamma(t)$ whose time-$t_0$ position $\gamma(t_0)$ is everywhere tangent to the eigenvector field $\xi_1$ is referred to as a strainline [14]. Strainlines coincide with the locally most compressing direction in the flow.

The eigenvector field $\xi_2$, on the other hand, coincides with the locally most repelling (or stretching) direction, with the local repulsion rate being $\sqrt{\lambda_2}$ [3]. Motivated by this observation, the ridges of the finite-time Lyapunov exponent (FTLE) $\Lambda_{t_0}^{t_0+T}$ defined as

$$\Lambda_{t_0}^{t_0+T}(x^0) = \frac{1}{2|T|} \log \left[\lambda_2(x^0)\right],$$

(4)

are often used as an indicator of hyperbolic LCSs [25, 26]. Haller [3] showed, however, that FTLE ridges can lead to false positives and false negatives in
LCS detection and developed a first self-consistent LCS theory (see also Haller and Beron-Vera [9]).

Efficient and accurate computational methods have been developed to implement this theory [14, 15, 27]. Following Ref. [14], we compute a repelling LCS \( \mathcal{R}(t) \) over the time interval \([t_0, t_0 + T]\), with \( T > 0 \), as a material line such that its initial position \( \mathcal{R}(t_0) \) satisfies

(A) \( \mathcal{R}(t_0) \) is a strainline, i.e., \( \xi_1(x^0) \parallel \mathcal{R}(t_0), \forall x^0 \in \mathcal{R}(t_0) \)

(B) \( \lambda_1(x^0) \neq \lambda_2(x^0) > 1, \quad \forall x^0 \in \mathcal{R}(t_0) \)

(C) \( \langle \xi_2(x^0), \nabla^2 \lambda_2(x^0) \xi_2(x^0) \rangle \leq 0, \quad \forall x^0 \in \mathcal{R}(t_0) \)

(D) The average of \( \lambda_2 \) along \( \mathcal{R}(t_0) \) is locally maximal among all nearby strainlines.

Roughly speaking, condition (A) implies that a repelling LCS is oriented such that, at each point \( x^0 \), its normal is aligned with the locally most stretching (or repelling) direction. Condition (B) ensures that the LCS is away from degenerate points where the normal repulsion rate \( \sqrt{\lambda_2} \) is equal to one. Condition (C) ensures that the graph of \( \lambda_2 \) in the direction orthogonal to the LCS is concave, implying that there is a local maximum of \( \lambda_2 \) near each point \( x^0 \) on the LCS. Finally, condition (D) ensures that the LCS is on average more repelling than any nearby material line.

Attracting LCSs can be computed similarly with backward-time integration, i.e., with substituting \( T \) with \(-T\). This can be done since an attracting material line becomes repelling in backward-time. As pointed out in Ref. [15], however, one should be careful with the interpretation of the resulting attracting and repelling LCSs: the repelling LCSs computed over the time interval \([t_0, t_0 + T]\) belong to a finite-time dynamical system which is different from that of the attracting LCSs computed over the time interval \([t_0, t_0 - T]\).

### 2.3 Time evolution of hyperbolic LCSs

If the vector field \( u \) in (1) is divergence free, the length of a strainline shrinks as it is advected to the final time \( t_0 + T \). More precisely, we have the following identity for the final length of a strainline [15].

**Proposition 1.** Let \( S(t) \) denote a strainline corresponding to the time interval \([t_0, t_0 + T]\). Let \( \ell(t) \) denote the Euclidean length of \( S(t) \) and denote
the arc-length parametrization of the initial position of the strainline, \( S(t_0) \),
by \( r: [0, \ell(t_0)] \to U \). We have
\[
\ell(t_0 + T) = \int_0^{\ell(t_0)} \frac{1}{\lambda_2(r(s))} \, ds. \tag{5}
\]

Proof. The position of the strainline at the final time \( t_0 + T \) is given by
\( S(t_0 + T) = \phi_{t_0 + T}^t(S(t_0)) \). This curve is parametrized by
\( p: [0, \ell(t_0)] \to U \),
where \( p(s) = \phi_{t_0 + T}^r(r(s)) \) for all \( s \in [0, \ell(t_0)] \). Its length is given by
\[
\ell(t_0 + T) = \int_0^{\ell(t_0)} \| p'(s) \| \, ds
= \int_0^{\ell(t_0)} \| \nabla \phi_{t_0 + T}^r(r(s)) r'(s) \| \, ds,
\]
where the prime sign denotes differentiation with respect to the parameter \( s \).

Since \( r \) is the arc-length parametrization of \( S(t_0) \), the tangent vector \( r' \)
is a unit-length vector. Therefore, from the definition of strainlines we have
\( r'(s) = \xi_1(r(s)) \) for all \( s \in [0, \ell(t_0)] \). Hence,
\[
\ell(t_0 + T) = \int_0^{\ell(t_0)} \| \nabla \phi_{t_0 + T}^r(r(s)) \xi_1(r(s)) \| \, ds
= \int_0^{\ell(t_0)} \sqrt{\langle \xi_1(r(s)), C_{t_0 + T}^r(r(s)) \xi_1(r(s)) \rangle} \, ds
= \int_0^{\ell(t_0)} \sqrt{\lambda_1(r(s))} \, ds.
\]
This completes the proof since for incompressible flow \( \lambda_1 = 1/\lambda_2 \) [28].

Since, for incompressible flow, \( \lambda_2 \geq 1 \) throughout the phase space [28], we have \( \ell(t_0 + T) \leq \ell(t_0) \). That is the length of any strainline shrinks under advection to the final time \( t_0 + T \). The length-shrinkage is particularly dramatic for hyperbolic LCSs since they are a subset of strainlines along which the values of \( \lambda_2 \) are relatively large. For instance, \( \lambda_2 \) becomes as large as \( 1.6 \times 10^5 \) for the turbulent flow studied in Section 4. In fact, the length of a typical repelling LCS decreases exponentially in time [3], motivating the following assumption.

\[ \lambda_2 \geq 1 \]

[28]
Let $S(t)$ denote a hyperbolic LCS corresponding to the time interval $[t_0, t_0 + T]$ (with $T > 0$ for repelling and $T < 0$ for attracting LCSs). Denoting the length of the hyperbolic LCS at time $t$ by $\ell(t)$, we assume that there exists a constant $\lambda > 0$ such that

$$\ell(t) \leq \ell(t_0)e^{-\lambda |t-t_0|}, \quad \forall t \in [t_0, t_0 + T].$$

This assumption allows for oscillations in the length $\ell(t)$ of the LCS. It however implies that the length of a repelling LCS (respectively, attracting LCS) has an exponentially decaying upper envelope as advected forward (respectively, backward) in time.

3 Hyperbolic LCSs coincide with contours of path-averaged scalars

The shrinking length of a hyperbolic LCS leads to nearly uniform values of the path-averaged scalar along it. To see this, consider two points $x^0_1, x^0_2 \in R(t_0)$ on the time-$t_0$ image of a repelling LCS $R(t)$ (see Fig. 2). Although the points $x^0_1$ and $x^0_2$ may be far apart from one another, as time evolves and the LCS shrinks, their later positions $\phi_{t_0}^t(x^0_1)$ and $\phi_{t_0}^t(x^0_2)$ approach each other exponentially fast. As a result, the values $f(\phi_{t_0}^t(x^0_1), t)$ and $f(\phi_{t_0}^t(x^0_2), t)$ of the scalar $f$ converge. In other words, after some initial transient, the values of the integrand in (2) are virtually identical for the two trajectories. Therefore, for reasonably long integration times $T$, the path-averaged scalar $\bar{f}$ is almost uniform along a repelling LCS. In contrast, if the points $x^0_1$ and $x^0_2$ are not initially on an LCS, their paths typically diverge resulting in quite different values of $\bar{f}(x^0_1, t_0, T)$ and $\bar{f}(x^0_2, t_0, T)$.

In the following, we state and prove the above observations in rigorous terms. The proofs are elementary. We start with a general crude estimate.

Proposition 2. Assume $f : U \times \mathbb{R}^+ \to \mathbb{R}$ is Lipschitz continuous in the spatial argument with Lipschitz constant $K_f$. For two initial conditions $x^0_1, x^0_2 \in U$, the path-averaged scalar field $\bar{f}$ computed over the time interval $[t_0, t_0 + T]$ satisfies

$$|\bar{f}(x^0_1) - \bar{f}(x^0_2)| \leq \frac{K_f}{T} \int_{t_0}^{t_0+T} \|\phi_{t_0}^t(x^0_1) - \phi_{t_0}^t(x^0_2)\| dt.$$
Figure 2: An illustration of the fact that a repelling LCS $\mathcal{R}(t)$ shrinks in length as advected in forward-time. The same is true for attracting LCSs advected in backward-time. $\mathcal{R}(t_0)$ marks a segment of the LCS lying between points $x_1^0, x_2^0 \in U$ at the initial time $t_0$.

**Proof.** We have

$$
|\bar{f}(x_1^0) - \tilde{f}(x_2^0)| = \frac{1}{T} \left| \int_{t_0}^{t_0+T} \left[ f(\phi_{t_0}^t(x_1^0), t) - f(\phi_{t_0}^t(x_2^0), t) \right] dt \right|
$$

$$
\leq \frac{1}{T} \int_{t_0}^{t_0+T} \left| f(\phi_{t_0}^t(x_1^0), t) - f(\phi_{t_0}^t(x_2^0), t) \right| dt
$$

$$
\leq \frac{K_f}{T} \int_{t_0}^{t_0+T} \|\phi_{t_0}^t(x_1^0) - \phi_{t_0}^t(x_2^0)\| dt,
$$

(7)

where the first line follows from Definition 1; the second line follows from the monotonicity of integrals and the last line follows from the Lipschitz continuity of $f$. 

The upper bound in (6) is not tight. In turbulent fluid flow on unbounded domains, for instance, the Richardson’s law [29, 30] states that the relative dispersion $\|\phi_{t_0}^t(x_1^0) - \phi_{t_0}^t(x_2^0)\|$ grows on average proportionally to $(t-t_0)^{3/2}$. This leads to a blow up of the right hand side of (6) as $T \to \infty$. However, for scalar fields $f$ bounded by a constant $L_f$, the maximum deviation $|\bar{f}(x_1^0) - \tilde{f}(x_2^0)|$ is clearly bounded by

$$
|\bar{f}(x_1^0) - \tilde{f}(x_2^0)| \leq 2 \sup_{t \in [t_0, t_0+T]} \left( \sup_{x \in U} |f(x, t)| \right) \leq 2L_f,
$$

(8)

for any $x_1^0, x_2^0 \in U$. 

9
Restricting the initial conditions \( x_0^1 \) and \( x_0^2 \) to belong to a hyperbolic LCS, however, the loose upper bound \( (7) \) becomes finite for all times. More surprisingly, this upper bound converges to zero as \( T \to \infty \).

**Theorem 1.** Assume (H) and that \( f : U \times \mathbb{R} \to \mathbb{R} \) is Lipschitz continuous in the spatial argument with Lipschitz constant \( K_f \). Let \( x_0^1, x_0^2 \in \mathcal{R}(t_0) \subset U \) be two initial conditions lying on time-\( t_0 \) position of a repelling LCS \( \mathcal{R}(t) \), corresponding to the time interval \( [t_0, t_0 + T] \), with \( T > 0 \). Denote the initial length of the LCS segment lying between the points \( x_0^1 \) and \( x_0^2 \) by \( \ell_0 \).

There exists \( \lambda > 0 \) such that the path-averaged scalar field \( \bar{f} \) computed over the time interval \( [t_0, t_0 + T] \) satisfies

\[
|\bar{f}(x_0^1) - \bar{f}(x_0^2)| \leq \frac{K_f \ell_0}{\lambda T} \left( 1 - e^{-\lambda T} \right).
\]

(9)

**Proof.** Let \( \ell(t) \) denote the length of the repelling LCS \( \mathcal{R}(t) \) at time \( t \). We have

\[
\|\phi^{t_0}_{t_0}(x_0^1) - \phi^{t_0}_{t_0}(x_0^2)\| \leq \ell(t),
\]

with the equality satisfied only when the LCS is a straight line. This, together with inequality \( (7) \) and assumption (H), implies

\[
|\bar{f}(x_0^1) - \bar{f}(x_0^2)| \leq \frac{K_f \ell_0}{\lambda T} \int_{t_0}^{t_0+T} \ell_0 e^{-\lambda(t-t_0)} dt = \frac{K_f \ell_0}{\lambda T} \left( 1 - e^{-\lambda T} \right)
\]

As a result, as the integration time \( T \) increases, \( |\bar{f}(x_0^1) - \bar{f}(x_0^2)| \) converges to zero, i.e., the path-averaged scalar along the repelling LCS \( \mathcal{R}(t_0) \) converges to a uniform value.

Theorem 1 holds for an attracting LCS in backward time. More precisely, the values of the path-averaged scalar \( \bar{f}(x_0, t_0, -T) \) is almost uniform along the time-\( t_0 \) position of attracting LCSs corresponding to the time interval \( [t_0, t_0 - T] \). The proof is identical to the repelling case and therefore is omitted.

Before presenting the numerical results on the turbulent flow, we illustrate the implications of Theorem 1 on a simple analytic example.

**Example 1.** Consider the linear saddle

\[
\dot{x} = x/10, \quad \dot{y} = -y/10.
\]

(10)
Figure 3: Example 1: The repelling LCS segment $R(t_0)$ between $y = -1$ and $y = 1$ (red curve). A straight line $M(t_0)$ is shown for reference (blue curve). The straight line $M(t_0)$ is obtained by a $\pi/10$ clock-wise rotation of the repelling LCS segment. Some trajectories of system (10) are shown in gray color.

whose solution, with the initial condition $x^0 = (x^0, y^0)$ at the initial time $t_0 = 0$, is given by

$$x(t) = (x^0 e^{t/10}, y^0 e^{-t/10}).$$

For the scalar function, we choose $f(x, t) = \cos(\pi x) \cos(\pi y) \cos(t)$ where $x = (x, y)$. Therefore, the path-averaged scalar is by definition

$$\bar{f}(x^0) = \frac{1}{T} \int_0^T \cos(\pi x^0 e^{t/10}) \cos(\pi y^0 e^{-t/10}) \cos(t) dt,$$

for an initial condition $x^0$ where, for simplicity, we set $t_0 = 0$. The multiplier $1/10$ in (10) is chosen to reduce the oscillations of the function $\cos(\pi x^0 e^{t/10})$ as $t$ increases, rendering the accurate numerical approximation of the above integral feasible.

Here, the repelling LCS $R(t_0)$ is the stable manifold $x = 0$. We consider the segment of this repelling LCS satisfying $-1 \leq y \leq 1$. For comparison, we also take a straight line $M(t_0)$ obtained from $\pi/10$ clock-wise rotation of this LCS segment (cf. Fig. 3).
Figure 4: The path-averaged scalar along a typical material line (a) and along the repelling LCS (b). The averages are computed for four integration times $T = 10, 20, 40$ and $60$. As the integration time increases the path-averaged scalar converges to a constant along the LCS while having rapid variations along the typical material line.

Points $x_0 \in \mathcal{R}(t_0)$ are parametrized as $x_0(s) = (0, s), \ s \in [-1, 1]$. Similarly, points $x_0 \in \mathcal{M}(t_0)$ are parametrized as $x_0(s) = (s \cdot \sin(\pi/10), s \cdot \cos(\pi/10))$. Fig. 4 shows the variations of the path-averaged scalar $\bar{f}$ as a function of the parameter $s$ as it varies along the repelling LCS $\mathcal{R}(t_0)$ and the straight line $\mathcal{M}(t_0)$. We compute the path-averages for four integration times $T = 10, 20, 40$ and $60$.

As the integration time $T$ increases the variations along the straight line $\mathcal{M}(t_0)$ increase. This is expected since for longer integration times $T$, the spatial structure of $\bar{f}(x_0)$ becomes more complex, leading to rapid variations along an arbitrary curve. The repelling LCS $\mathcal{R}(t_0)$, however, defies this trend. By increasing the integration time $T$, the variations of $\bar{f}$ along the repelling LCS decrease. For longer integration times, the LCS becomes indistinguishable from a contour of $\bar{f}$.

Note that if the prefactor $1/T$ is omitted from Definition (2), all variations in Fig. 4 multiply by $T$. The variations of the path-averaged scalar along the repelling LCS would still be small compared to its variations along a typical material line.
Figure 5: (a) The initial distribution of the scalar function $f$, i.e., $f(x, 0)$. (b) The path-average of scalar $f$, i.e., $\bar{f}(x^0)$. (c) The finite-time Lyapunov exponent $\Lambda_{t_0}^{t_0+T}$ with $t_0 = 0$ and $T = 50$.

4 Two-dimensional turbulence

4.1 Numerical set-up

For our numerical experiment, we first solve the forced Navier–Stokes equations

\begin{align}
\partial_t u + u \cdot \nabla u &= - \nabla p + \nu \Delta u + F, \\
\nabla \cdot u &= 0,
\end{align}

using a pseudo-spectral method in space with $512 \times 512$ modes and a forth order Runge–Kutta time integrator. The forcing $F$ is band-limited with random phase applied to a few intermediate wave-numbers, $3 \leq k \leq 5$. The spatial domain is $U = [0, 2\pi] \times [0, 2\pi]$ with periodic boundary conditions and we solve the equations over the time interval $[0, T]$ with $T = 50$. The initial condition $u(x, 0)$ is a fully developed turbulent field obtained from a separate simulation. This yields a turbulent velocity field $u(x, t)$ as a function of position $x \in U$ and time $t \in [0, 50]$.

We compute fluid trajectories by solving (1) for $512 \times 512$ initial conditions $x^0$ uniformly distributed in space. This yields the maps $\phi_{t_0}^t : x^0 \mapsto \phi_{t_0}^t(x^0)$ with $t \in [0, T]$ where $t_0 = 0$.

For the scalar field $f$ to be averaged along these trajectories, we choose

\begin{equation}
f(x, t) = \sin(nx) \cos(my) \cos(2\pi kt/T),
\end{equation}
Figure 6: (a) Probability density function (PDF) of the angle $\theta$ between the contours of $\bar{f}$ and the strainlines. The dotted line (vertical) marks the mean of the angles. The dashed line (horizontal) marks the uniform distribution. (b) The probability density function of the maximum variation of $\bar{f}$ along material lines normalized by their initial length $\ell_0$ (cf. quantity (13)). The red curve (star symbols) shows the distribution for a total of 249 repelling LCSs. The blue curve (circles) shows the distribution for 249 arbitrary material lines.

where $\mathbf{x} = (x, y)$ and $m, n, k \in \mathbb{N}$. The following analysis was carried out for several values of $(n, m, k)$ and similar results where obtained. For brevity, we only present the results for $n = m = 4$ and $k = 1$.

Lagrangian diagnostics often choose the scalar field $f$ to be a functional of the velocity field $\mathbf{u}$ [10, 11, 21]. We choose scalar field (12), which is independent of the dynamics, to demonstrate that Theorem 1 is valid for arbitrary (but sufficiently smooth) scalars.

4.2 Numerical results

Fig. 5(a) shows the initial distribution of the scalar function, $f(\mathbf{x}, 0)$. The path-averaged scalar $\bar{f}$ inherits from the complexity of the trajectories and therefore shows a more convoluted spatial distribution (Fig. 5(b)). The finite-time Lyapunov exponent (FTLE) field (Fig. 5(c)) shows a striking resemblance to the path-averaged function $\bar{f}$. The ridges of the FTLE field are frequently associated with hyperbolic LCSs. But, as mentioned before, FTLE ridges return false positives and false negatives in LCS detection[3] and therefore are not a reliable measure of hyperbolicity. Therefore, we use
the method discussed in Section 2.2 to locate hyperbolic LCSs. Fig. 1 shows the initial positions of the repelling LCSs (black curves) corresponding to the time interval $[0, 50]$. The path-averaged scalar $\bar{f}$ is shown in the background.

Visually, repelling LCSs seem to coincide with contours of the path-averaged scalar. More strikingly, we find a strong alignment between the contours of the path-averaged scalar $\bar{f}$ and strainlines, regardless of whether they are an LCS or not. As a measure of their alignment, we examine the angle $\theta$ between their normal vectors $\nabla \bar{f}$ and $\xi_2$, i.e. $\theta = \angle (\nabla \bar{f}, \xi_2)$. This is an appropriate measure of alignment since contours of $\bar{f}$ are everywhere orthogonal to the gradient $\nabla \bar{f}$ and the strainlines are everywhere orthogonal to Cauchy–Green eigenvector $\xi_2$. Fig. 6(a) shows the probability density function (PDF) of the angle $\theta$ computed for all $512 \times 512$ grid points. The range of the angle is between 0 and $\pi/2$ since the eigenvectors do not possess a well-defined orientation: $\xi_2$ and $-\xi_2$ are both eigenvectors of the Cauchy–Green strain tensor.

The mean of the angles $\theta$ is approximately $\pi/7$ with 63% of the points having an angle less than this mean. At approximately 80% of the grid points, the normal vectors $\xi_2$ and $\nabla \bar{f}$ make an angle less than $\pi/4$. A similar alignment between Lyapunov vectors and the contours of passive scalars has been reported before [31]. While the alignment between strainlines and passive scalars is expected, their alignment with the contours of the path-averaged scalars is surprising. This strong alignment is only explained by the length-shrinking property of strainlines (Proposition 1) and Theorem 1.

Focusing on repelling LCSs as a subset of strainlines, we measure the maximum variation of the path-averaged scalar $\bar{f}$ between two points $x_1^0$ and $x_2^0$ on an LCS $R(t_0)$. To this end, we compute the quantity

$$\sup_{x_1^0, x_2^0 \in R(t_0)} \left| \frac{f(x_1^0) - f(x_2^0)}{\ell_0} \right|,$$

for each LCS $R(t_0)$. For a fair comparison between LCSs with various lengths, we normalize the difference by the initial length $\ell_0$ of the LCS $R(t_0)$. Quantity (13) assigns a single number to each LCS. If the LCS coincides exactly with a contour of the path-averaged scalar, this number is equal to zero. Furthermore, it follows from Theorem 1 that the quantity (13) is bounded from above by $(K_f/\lambda T)(1 - \exp(-\lambda T))$, for some $\lambda > 0$.

Fig. 6(b) shows the distribution of maximum variation (13) (red curve) for 249 repelling LCSs. For comparison, we also compute this quantity for 249 arbitrary material lines. We obtain these material lines from the repelling LCSs by a $-\pi$ shift in $x$ and a $\pi$ shift in $y$ (Note the periodic
boundary conditions). More precisely, for each point \((x, y) \in \mathcal{R}(t_0)\), we consider the shift \((x - \pi, y + \pi) \in \mathcal{M}(t_0)\). The collection of these points forms a material line \(\mathcal{M}(t_0)\). Repeating this for all LCSs, we get 249 material lines that differ from the LCSs, but are of the same length and shape. The distribution of the maximum variation corresponding to these material lines is also shown in Fig. 6(b) (blue curve). Comparing the support of the two distributions shows that the maximum variation of \(\bar{f}\) tends to be smaller for the repelling LCSs compared to the arbitrary material lines. Furthermore, the likelihood of the maximum variation along an LCS being small is significantly higher.

Better statistics are achieved by transverse section analysis used by Kel-ley et al. [21]. Here, we reiterate their method. At each point of an LCS, we take a transverse line-segment of length 0.2 centered on the LCS and locally orthogonal to it. Since the LCSs are strainlines, the line segments are in the direction of \(\xi_2\). This amounts to a total of 249000 straight line-segments centered on 1000 equidistant points along each one of the 249 LCSs (See Fig. 7(a) for an illustration.). The value of the scalar \(\bar{f}\) is interpolated on 401 equi-spaced points on each line segment. Fig. 7(b) shows the ensemble average of \(\bar{f}\).
average of $\overline{f}$ along these segments. To avoid cancellation of negative and positive values, we choose the direction of each line segment such that the value of $\overline{f}$ is locally increasing in that direction on the LCS.

This figure shows that the LCSs tend to coincide with the zero level sets of the path-averaged scalar $\overline{f}$. The zero of the ensemble average of $\overline{f}$ is located at $s = 0.0035$, where $s$ is the signed distance from the LCSs along the transverse line segment. Given the spacing of the underlying grid over which $\overline{f}$ is computed, i.e. $\sim 0.012$, the LCSs coincide with zero level sets of $\overline{f}$ to the discretization precision. We also note that these results are statistically significant since the amplitude of the ensemble average is approximately $0.045$ while the spatial mean of $\overline{f}$ is approximately $5 \times 10^{-5}$.

Theorem 1 predicts that the repelling LCSs coincide with the contours of $\overline{f}$. It does not, however, favor one contour value over the other. Yet, Fig. 7(b) shows that the LCSs tend to coincide with the zero level set of path-averaged scalar. Kelley et al. [21] observe the same correlation between LCSs and their scalar field, i.e., the scale-to-scale energy flux. Since the energy flux is a function of the fluid velocity $u$, they conclude that LCSs impede the transfer of energy to smaller scales in two-dimensional turbulence. This explanation does not apply here since the scalar field (12) evolves independently of the dynamics.

We propose that the alignment of LCS with zero level curves is explained by studying the length of trajectories. In a stationary frame of reference, trajectories starting on and near hyperbolic LCSs tend to be longer and explore more of the domain compared to trajectories that are far from any LCSs [1, 32–34]. Therefore, it is more likely for a ‘chaotic trajectory’ to sample all the values in the range of the scalar. As a result, the path-average of the scalar $f$ along a chaotic trajectory is more likely to be close (compared to non-chaotic trajectories) to the spatial mean of $f$.

Fig. 8 serves to validate this proposal. Fig. 8(a) shows two sets of sample trajectories. The black-colored trajectories have small path-averages satisfying $|\overline{f}(x_0)| < 10^{-5}$ while the path-averages for the magenta-colored trajectories are relatively larger, satisfying $|\overline{f}(x_0)| > 10^{-2}$. The black curves tend to have longer length and explore more of the domain while the magenta trajectories are shorter and more localized. Fig. 8(b) better demonstrates the correlation between the length of a trajectory and its corresponding path-average $\overline{f}$. Each point on this scatter plot corresponds to a single trajectory $\phi^t_0(x_0)$, comparing its length $\ell$ to its corresponding averaged scalar value $\overline{f}(x_0)$. For shortest trajectories $\overline{f}$ takes a wide range of values from $-0.4$ to $0.4$. However, as the length increases the points tend to concentrate more
Figure 8: (a) Sample trajectories with $|\mathcal{f}(x_0)| < 10^{-5}$ (black) and $|\mathcal{f}(x_0)| > 10^{-2}$ (magenta). The square symbols mark the initial condition for each trajectory. The domain is periodic in $x$ and $y$. (b) Scatter plot of the length $\ell$ of all $512^2$ computed trajectories versus their corresponding path-averaged scalar $\mathcal{f}$. Each point on the plot corresponds to a single trajectory.

and more near the zero values of $\mathcal{f}$. Eventually, all the trajectories longer than 12 units assume the path-averaged values $\bar{\mathcal{f}}$ between $-0.08$ and $0.07$.

5 Conclusion

We investigated the missing link between hyperbolic Lagrangian coherent structures (LCS), as drivers of mixing in unsteady dynamical systems, and diagnostics obtained as time averages of some scalar field along trajectories of the dynamical system. We showed that LCSs tend to align with the contours of path-averaged scalars.

The results are obtained from the fact that repelling (respectively, attracting) LCSs shrink in length under advection in forward (respectively, backward) time. Therefore, the points on an LCS converge to one another under advection, in contrast to typical behavior of chaotic trajectories that tend to separate over time. As a result, as the integration time $T$ increases, the variations of the path-averaged scalars along an LCS tend to zero (cf. Theorem 1).

The converse is, however, not correct: not every contour of a path-
averaged scalar coincides with an LCS. Therefore, averaged scalars cannot be immediately used for LCS detection and a filtering criterion is required to remove the undesired contours. Pérez-Muñuzuri and Huhn [11], for instance, use a cutoff threshold of 20% for their averaged scalars while Mendoza and Mancho [18] and Rypina et al. [12] consider the contours with sharp gradients.

We illustrated our results on a direct numerical simulation of two-dimensional turbulence. While Lagrangian diagnostics are often a function of the vector field \( \mathbf{u} \), the scalar field (12) used here is dynamically independent of the turbulent flow. Yet, LCSs align with its path-averaged contours (cf. Fig. 1).

A closer inspection showed that LCSs aligned mostly with the zero level curves of the path-averaged scalar. A similar alignment was observed by Kelley et al. [21] with the scalar being the scale-to-scale energy flux. They attribute this alignment to the LCSs separating dynamically distinct regions in fluid flows. The same argument does not apply to our simulation since the scalar field is independent of the dynamics. Instead, we show that relatively long lengths of trajectories starting on the LCSs leads to a more even sampling of the scalar throughout space. As a result, the value of the path-averaged scalar on an LCS tends to the spatial average of the scalar, which in this case is zero. We point out that this argument resembles that of Birkhoff’s ergodic theorem (See Ref. [35], Theorem 7.3.1). The theorem, however, does not immediately apply to our turbulent flow since it is far from ergodic and, moreover, the scalar field is time-dependent.

Finally, we remark that a result similar to Theorem 1 should hold in three-dimensions where hyperbolic LCSs are the locally most repelling or attracting hyper-surfaces [3]. Analogous to the two-dimensional case, the surface area of a repelling LCS (respectively, attracting LCS) shrinks in forward-time (respectively, backward-time) [16]. This leads to a nearly uniform sampling of scalars along such special hyper-surfaces. We will present a detailed analysis of this higher dimensional case elsewhere.

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