Probes of holographic thermalization in a simple model with momentum relaxation

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Abstract

From the viewpoint of AdS/CFT correspondence, we investigate the holographic thermalization process in a four dimensional Einstein-Maxwell-axions gravity theory, which is considered as a simple bulk theory dual to a boundary theory with momentum relaxation. We probe the thermalization process using the equal time two-point functions and the entanglement entropy with the circle profile. We analyze the effects of momentum relaxation on the process in details and results show that the momentum relaxation gives longer thermalization time, which means it suppresses the holographic thermalization process. This matches the properties of the quasi-normal frequencies for the bulk fluctuations which the frequency violates from zero mode more profoundly for stronger momentum relaxation. We claim that is reasonable because the decay of the bulk fluctuations holographically describes the approach to thermal equilibrium in the dual theory.

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I. INTRODUCTION

The AdS/CFT correspondence [1–3] provides deep insights into the mapping between the strongly coupled field theory and the dynamics of a classical gravitational theory in the bulk. Plenty of applications have been made as this correspondence has been treated as a powerful tool to deal with the strongly coupled systems. Among them, the holographic thermalization process, which constructs a proper model in the bulk gravity to investigate the thermalization process of quark-gluon plasma (QGP) [4], has attracted considerable attentions in the theoretical study. Two main reasons may contribute to the situation. On one hand, the perturbative quantum chromodynamics (QCD) breaks down during the far from equilibrium process of QGP formation. Such that the study of non-equilibrium dynamics in strongly coupled theories are involved in Relativistic Heavy Ion Collider (RHIC) [5, 6]. On the other hand, the holographic study of hot QCD matter with equilibrium and near-equilibrium aspects has been well addressed in [7] and further use of the powerful AdS/CFT correspondence to study the thermalization of strongly coupled plasma is a natural clue. Previous attempts were made to describe the thermalization process holographically dual to the formation of black hole via gravitational collapse in AdS space [8–13].

Later, a simpler model for holographic thermalization was proposed in [14, 15], which captures many critical features of the thermalization process. In the proposal, the thermalization process is dual to the collapse of a thin shell of matter described by an Vaidya-AdS metric leading to a thermal equilibrium configuration given by a Schwarzschild-AdS black hole. The time evolution of non-local thermalization probes of the boundary field theory are then dual to the related descriptions in terms of geometric quantities. The authors studied the equal time two-point correlation functions of local gauge invariant operators, expectation values of Wilson loop operators, and entanglement entropy which are dual to minimal lengths, areas, and volumes in AdS space, respectively. Especially, they found that the UV modes thermalize first while IR modes thermalize later which are different from that in perturbative approaches [16]. And the thermalization time scales typically as $\sim \ell$, where $\ell$ is the characteristic length of the probe. This proposal was soon extended to describe more real RHIC processes by including the effect of a non-vanishing chemical potential [17, 18]. In this case, the charged matters was involved and the thermal equilibrium configuration was given by a Reissner-Nordström(RN) AdS black hole. They found that the thermalization time for renormalized geodesic lengths and minimal area surfaces is longer for the larger charge. Considerable efforts has then been made in further investigations on this directs, see for example [19–49] and therein.

In this paper, we shall study the holographic thermalization via Einstein-Maxwell gravity coupled with two linear spacial-dependent scalar fields in the bulk. It was pointed out in [50] that the scalar fields in the bulk source a spatially dependent field theory with momentum relaxation and the linear coefficient of the scalar fields describes the strength of the momentum relaxation. This means that our thermalization process will involve in momentum relaxation, which is more closer to that may happen in heavy ion collisions as it has been holographically described via boost-invariant plasma and hydrodynamic features in fully inhomogeneous case [11, 51, 52].

Our aim is to study the effect of momentum relaxation during the thermalization process from a simple bulk gravity without translational invariance. We will mainly focus on the equal time two-point correlation functions of local gauge invariant operators and entanglement entropy which are dual to minimal lengths and volumes in AdS space, and study the
effect of momentum relaxation on the two observables. The effect of momentum relaxation on the time dependent optical conductivity\cite{53} and the equilibrium chiral magnetic effect \cite{54} in the thermalization process of this model has been investigated. It is noticed that the thermalization process in massive gravity has been studied in \cite{55}. The authors mainly studied the effect of dissipation of momentum dually introduced by the massive graviton in the bulk gravity on the equal time two-point correlation functions, and it was found that the dissipation of momentum deduced in massive gravity increases the holographic thermalization process. Even so, our study of momentum relaxation in the thermalization process is still deserved from two aspects. One is that both the shell collapsing into the Vaidya-AdS black brane but the thermal equilibrium configuration in our study are different from that in massive gravity. The other is that here the mechanism of dissipation of momentum is dual to breaking of translational invariance in the present bulk while it is dual to the breaking of diffeomorphism symmetry in massive gravity \cite{56}. Moreover, we will include non-vanishing chemical potential in the study.

The paper is organized as follows. In section II, we present the generalized Vaidya-AdS black brane in Einstein-Maxwell-axions theory. We analyze the effect of momentum relaxation on the holographic thermalization process via numerically computing the two observables, i.e., equal time two-point correlation functions and entanglement entropy in section III A and section III B, respectively. The last section is our conclusion and discussion.

II. VAIDYA ADS BLACK BRANES IN EINSTEIN-MAXWELL-AXIONS THEORY

We consider the AdS black branes in Einstein-Maxwell-axions gravity theory proposed in \cite{50}. The action of the four dimensional theory was given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R + \frac{6}{l^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \sum_{l=1}^{2} (\partial \psi_l)^2 \right),$$

(1)

By setting the scalar fields to linearly depend on the two dimensional spatial coordinates $x^a$, i.e., $\psi_l = \beta \delta_{l_a} x^a$, the action admits the charged black brane solution

$$ds^2 = -r^2 f(r) dt^2 + \frac{1}{r^2 f(r)} dr^2 + r^2 (dx_1^2 + dx_2^2),$$

$$f(r) = \frac{1}{l^2} - \frac{\beta^2}{2r^2} - \frac{m_0}{r^3} + \frac{q_0^2}{r^4},$$

$$A = A_t(r) dt, \quad A_t = \left( 1 - \frac{r_h}{r} \right) \frac{2q_0}{r_h},$$

(2)

where the horizon $r_h$ satisfies $f(r_h) = 0$; $l$ describes the radius of AdS spacetime, and for simplicity we will set $l = 1$. $m_0$ and $q_0$ are the mass and charge of the black brane, with the

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1 In general, the linear combination form of the scalar fields are $\psi_l = \beta_{l_a} x^a$. Then defining a constant $\beta^2 \equiv \frac{1}{2} (\sum_{a=1}^{2} \sum_{l=1}^{2} \beta_{l_a} \beta_{l_a})$ with the coefficients satisfying the condition $\sum_{l=1}^{2} \beta_{l_a} \beta_{l_b} = \beta^2 \delta_{ab}$, we will obtain the same black hole solution. Since there is rotational symmetry on the $x^a$ space, we can choose $\beta_{l_a} = \beta \delta_{l_a}$ without loss of generality.
relation given by
\[ 1 - \frac{\beta^2}{2r_h^2} - \frac{m_0}{r_h^3} + \frac{q_0^2}{r_h^4} = 0. \] (3)

It is worthwhile to mention that the scalar fields in the bulk source a spatially dependent boundary field theory with momentum relaxation, which is dual to a homogeneous and isotropic black brane (2). The linear coefficient \( \beta \) of the scalar fields is usually considered to describe the strength of the momentum relaxation in the dual boundary theory [50]. A general action with axions terms and the holography has been studied in [57]. We note that the extended thermodynamics of the black brane has also been studied in [58, 59]. The Hawking temperature of the black brane reads
\[ T = \frac{1}{4\pi} \left( \frac{d}{dr} \frac{r^2 f(r)}{dr} \right) \Big|_{r_h}. \] (4)

Such a temperature is treated as the temperature of the dual boundary field. The condition \( T = 0 \) then restricts the maximal value of charge parameter \( q_0 \) if we fix \( m_0 \) and \( r_h \). Besides, the chemical potential of the dual field on the boundary can be modeled by the non-zero electric field and is given by [17]
\[ \mu = \lim_{r \to \infty} A_t(r) = \frac{2q_0}{r_h}. \] (5)

With properly chosen coordinate transformation, the above black hole brane (2) can be represented as in the Eddington-Finkelstein coordinates,
\[ ds^2 = \frac{1}{z^2} \left[ -f(z)dv^2 - 2dvdz + dx^adx^a \right], \] (6)
\[ f(z) = 1 - \frac{1}{2} \beta^2 z^2 - m_0 z^3 + q_0^2 z^4, \] (7)
with
\[ dv = dt - \frac{1}{f(z)} dz \quad \text{and} \quad z = \frac{1}{r}. \] (8)

We note that the coordinates \( v \) and \( t \) coincide on the boundary.

In the framework of AdS/CFT duality, the rapid injection of energy followed by the thermalization process in the boundary theory corresponds to the collapse of a black brane or a falling thin shell of dust in the AdS spacetime. Thus, in order to holographically describe the thermalization process, one usually frees the mass and charge parameter as smooth functions of \( v \) as [15, 18]
\[ m(v) = \frac{m_0}{2} \left[ 1 + \tanh \left( \frac{v}{v_0} \right) \right], \] (9)
\[ q(v) = \frac{q_0}{2} \left[ 1 + \tanh \left( \frac{v}{v_0} \right) \right], \] (10)

Note that metric (6) is an ingoing metric in Vaidya-AdS spacetime, which means that the shell will “fall” from \( z = \infty \) to \( z = 0 \). Correspondingly, in AdS spacetime, the shell is outgoing till it reaches the equilibrium at \( r = \infty \).

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where \( v_0 \) represents the finite thickness of the falling charged dust shell and the relation between \( m_0 \) and \( q_0 \) is still given by (3). Then the related Vaidya AdS black brane is

\[
ds^2 = \frac{1}{z^2} \left[ -f(v, z) dv^2 - 2 dv dz + dx^a dx^a \right],
\]

\[(11)\]

\[
f(v, z) = 1 - \frac{1}{2} \beta^2 z^2 - m(v) z^3 + q(v)^2 z^4.
\]

\[(12)\]

\[
A_v = 2q(v)(1 - z).
\]

\[(13)\]

It is easy to check when \( v \rightarrow \infty \), the above formula denotes a Vaidya-AdS metric (6) while in the limit \( v \rightarrow -\infty \), it corresponds to a pure AdS spacetime. According to[17], the external sources of current and energy-momentum tensor for the above solution (11)-(13) are

\[
J_{(ext)}^z = 2 \frac{dq(v)}{dv},
\]

\[(14)\]

\[
T_{(ext)}^{vv} = z^2 \frac{dm(v)}{dv} - 2z^3 q(v) \frac{dq(v)}{dv}.
\]

\[(15)\]

It is noticed that here in our solution, we consider the simple case that only the mass and charge depend on the time but the momentum relaxation coefficient \( \beta \) does not vary with time, which is similar as that studied for five dimensional EMA theory[54]. More external sources were considered to construct the solution in Vaidya setup of this theory [53], where the authors probed the time-dependent optical conductivity without translation invariance.

III. HOLOGRAPHIC THERMALIZATION

For the purpose of exploring the dynamics and the scale dependence of the thermalization processes, the non-local observables are required to provide sufficient information. In this work, we will focus on two non-local observables, i.e., the equal time two-point function and entanglement entropy. Using the AdS/CFT correspondence, they can be calculated by the spacelike geodesic and the extremal volume in AdS space, respectively [15]. With the Vaidya-AdS black brane (9)-(13) of the bulk theory in hands, we are ready to study the effect of momentum relaxation on these processes.

A. Effect of momentum relaxation on two-point correlation functions

According to the study in [15], the on-shell equal time two-point correlation function for a scalar operator \( \mathcal{O} \) with conformal dimension \( \Delta \) can be holographic evaluated as

\[
\langle \mathcal{O}(t_0, x_i) \mathcal{O}(t_0, x'_i) \rangle \sim \exp(-\Delta \mathcal{L})
\]

\[(16)\]

\[3\] It is noticed that the quenches we consider here are homogeneous in spatial directions because the metric in the bulk is homogeneous. However, as we mentioned before that the boundary theory has momentum relaxation sourced by the spacial dependent scalar fields in the bulk.
where $L$ is length of a bulk geodesic connecting two points $(t_0, x_i)$ and $(t_0, x'_i)$ on the AdS boundary. Thus, to disclose the properties of the equal time two-point correlation function, one needs to minimize the length of $L$.

To proceed, we consider a space-like geodesic connecting the two boundary points: $(t, x_1) = (t_0, -\ell/2)$ and $(t', x'_1) = (t_0, \ell/2)$, with all other spatial directions identical at the two end points. $\ell$ is the boundary separation along $x_1 \equiv x$. Such a geodesic then can be parametrized by $v = v(x)$ and $z = z(x)$ and the boundary conditions satisfied by this geodesic are

\begin{align}
    z(\pm \ell/2) &= z_0, \quad v(\pm \ell/2) = t_0, \\
    z(0) &= z_*, \quad v(0) = v_*, \\
    z'(0) &= v'(0) = 0.
\end{align}

where $z_0$ is the UV radial cut-off near the boundary; $z_*$ and $v_*$ are two parameters characterizing the extremal case that will be stated soon. The length element of the geodesic is then given by

\begin{equation}
    L_{\text{AdS}} = \int_{-\ell/2}^{\ell/2} dx \sqrt{1 - f(v, z)(v')^2 - 2 v' z'},
\end{equation}

where the prime denotes the derivative with respect to $x$. In order to evaluate the correlation functions, we need to minimize the above length. It is obvious that the integral function in (20), which can be treated as ‘Lagrangian’, does not explicitly depend on $x$ which plays the role of time coordinate in classical mechanics. Thus, the corresponding Hamiltonian is conservative with regard to $x$ and the related conservation equation is

\begin{equation}
    1 - f(v, z)(v')^2 - 2 v' z' = \left(\frac{z_*}{z}\right)^2.
\end{equation}

Subsequently, the equations of motion from the ‘Lagrangian’ in (20) can be reduced as

\begin{align}
    z v'' + 2 z' v' - 1 + (v')^2 \left[f(v, z) - \frac{1}{2} z \frac{\partial f}{\partial z}\right] &= 0, \\
    z'' + f(v, z) v'' + z' v' \frac{\partial f}{\partial z} + \frac{1}{2} (v')^2 \frac{\partial f}{\partial v} &= 0.
\end{align}

The minimal length of geodesic can be numerically calculated by solving the above equations of motion with the boundary conditions (17) and (18). However, only the finite part of the geodesic length is physical and interested, so one usually considers the renormalized length of the geodesic [15]

\begin{equation}
    \delta L_{\text{AdS}} = L_{\text{AdS}} + 2 \ln(z_0),
\end{equation}

where $2 \ln z_0$ is the contribution of a pure AdS boundary to eliminate the divergent term.

Before presenting the numerical results, we consider the ratio of the chemical potential
FIG. 1: The relative renormalized geodesic length $\delta \tilde{L}_{AdS}$ as a function of boundary time $t_0$ with different $\beta$ for the boundary separation $\ell = 1, 2, 3, 4$. Clearly, for small $\beta$, the thermalization process is slightly effected by $\beta$.

and the temperature as a new varying parameter

$$\chi = \frac{1}{4\pi} \left( \frac{\mu}{T} \right) = \frac{2 q_0 z_h}{2 + m_0 z_h^3 - 2 q_0^2 z_h^4},$$

(25)

Note the chemical potential $\mu$ defined by (5) is dimensionless but the temperature has dimension $T \sim [\text{length}]^{-1}$. So if we want to keep $\chi$ dimensionless we have to redefine $\mu$ by a scale with length unit that depends on the particular compactification. However, such a scale will not significantly change the behavior of $\chi$ but only as a factor. Besides, if we fix the mass parameter $m_0$ and the horizon radius $z_h$ then we can easily obtain a relation between $q_0$ and $\beta$ by (3) as well as $T = 0$,

$$\beta^2 = 6 z_h^2 - 2 q_0^2 z_h^2.$$  

(26)

Hereafter, for the sake of simplicity, we will fix $m_0 = 1$ and $z_h = 1$. With such consideration,

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4 Since the boundary field theory is conformal, it has been suggested that the only relevant parameter one can vary is this dimensionless ratio $\chi$ constructed from the chemical potential and the temperature if we consider the thermalization process with a chemical potential [17, 30]. However, if we guarantee the existence of horizon and set $r_h = 1$, one can easily show that varying $\chi$ is equal to varying $\beta$, as we will do later.
the maximal values of $q_0$ and $\beta$ are given by

$$\beta^2 = 2 q_0^2, \quad q_0^2 \leq 3/2, \quad \beta^2 \leq 3.$$  \hfill (27)

Hence, $\chi$ is a monotonic function of $q_0$ or $\beta$ with range $[0, \infty]$. We therefore adopt $\beta$ as the varying parameter to proceed the numerical calculation.

In Fig. 1, we show the relative renormalized geodesic length $\tilde{\mathcal{L}}_{\text{AdS}} = (\delta \mathcal{L}_{\text{AdS}} - \delta \mathcal{L}_{\text{AdS-thermal}})/\ell$ as a function of boundary time $t_0$ for different boundary separation $\ell$ and different $\beta$, where $\delta \mathcal{L}_{\text{AdS-thermal}}$ denotes that the quantity is computed in static Vaidya-AdS spacetime. The larger $\beta$ corresponds to larger renormalized geodesic. For small $\beta$ (for example, $\beta < 0.1$), the length of geodesics are almost same. The figures also exhibit the tiny dependence of the thermalization time $\tau_{\text{real}}$ on $\beta$ when $\ell$ is small, where $\tau_{\text{real}}$ is defined as the time when the probes reach the thermal equilibrium value. For large $\ell$, the curves show a slightly complicated behavior for different $\beta$. The intersection of different curves near $\tau_{\text{real}}$ with large $\ell$ denotes the fact that different $\beta$ may lead to same renormalization geodesic, see the right panel of Fig. 2. Further details will be discussed together with the results of holographic entanglement entropy.

Another interesting thermalization time is the so-called thermalization critical time $\tau_{\text{crit}}$, at which the tip of the extremal line or surface grazes the middle of the shell at $v = 0$, defined as [32]

$$\tau_{\text{crit}} = \int_{z_0}^{z^*} \frac{1}{f(z)} \, dz,$$  \hfill (28)

where $f(z) = 1 - \frac{1}{2} \beta^2 z^2 - m_0 z^3 + q_0^2 z^4$. The left panel of Fig. 2 shows the relation between the relative critical thermalization time $\delta \tau_{\text{crit}}$ and $\ell$, where $\delta \tau_{\text{crit}}$ is re-scaled as

$$\delta \tau_{\text{crit}} = \frac{\tau_{\text{crit}} - \ell}{\ell}.$$  \hfill (29)

The numerical results show several interesting properties. First, $\tau_{\text{crit}}$ is also nearly independent of the coefficient $\beta$ when $\ell$ is small, and appears as a constant. In other words,
\( \tau_{\text{crit}} \) saturates the causality bound \( \tau_{\text{crit}} \sim \frac{\ell}{2} [32, 60] \). This feature can be understood easily, since when the boundary separation is small the metric could be seen as an asymptotically AdS, and the geodesic in the static black brane will only extend near the boundary. Second, the critical thermalization time has monotonic dependence on \( \beta \), i.e., \( \tau_{\text{crit}} \) increases as \( \beta \) increases, when \( \ell \) is large enough. This is similar with \( \tau_{\text{real}} \) from the right-bottom panel of Fig. 1, even though we need to technically improve our numerical precision of \( \tau_{\text{real}} \) to make a direct comparison.

### B. Effect of momentum relaxation on holographic entanglement entropy

We now consider the case of holographic entanglement entropy which is modeled by the extremal surfaces of a spherical region in the boundary. The prescription for computing entanglement entropy using the AdS/CFT correspondence has been initially proposed in [61, 62], in which it was addressed that for a system \( A \) in the boundary CFT which has a gravity dual, the information included in a subsystem \( B \) is evaluated by the entanglement entropy \( S_A \) as

\[
S_A = \frac{\text{Area}(\gamma_A)}{4G_N^{(d+2)}}. \tag{30}
\]

where \( \gamma_A \) is a \( d \)-dimensional minimal surface with boundary given by the \( (d-1) \)-dimensional manifold \( \partial \gamma_A = \partial A \), and \( G_N^{(d+2)} \) is the Newton constant of the general gravity in AdS\( _{d+2} \) theory.

Since we are working in a four dimensional Vaidya-AdS spacetime, the spherical region in the boundary is actually a disk, and the computation of the entanglement entropy is then identical to the Wilson loop computation. Thus, we will use the polar coordinates \((\rho, \phi)\) to rewrite the disk. The extremal surface then can be represented by \( z(\rho) \) and \( v(\rho) \) because of the azimuthal symmetry in the \( \phi \)-direction. Subsequently, the volume element of hyper-surfaces which measures the entanglement entropy in our model is given by

\[
\mathcal{V}_{\text{AdS}} \equiv \text{Area}(\gamma_A) = \int_{0}^{R} d\rho \frac{\rho}{z^2} \sqrt{1 - f(v, z) (v')^2 - 2 v' z'}, \tag{31}
\]

where \( ' \equiv \frac{d}{d\rho} \), and we have absorbed a factor \((2\pi)^{-1} \) to \( \mathcal{V}_{\text{AdS}} \). The equations of motion derived from (31) are

\[
z'' = \frac{1}{2\rho z} \left[ 4\rho f^2(v')^2 + f \left( 8\rho v' z' - 4\rho + z(v')^2(2z' - \rho \frac{\partial f}{\partial z}) \right) + z \left( 4v'(z')^2 - 2z'(1 + \rho v' \frac{\partial f}{\partial v}) - (v')^2(2z' - \rho \frac{\partial f}{\partial v}) \right) \right], \tag{32}
\]

\[
v'' = \frac{1}{2\rho z} \left[ 4\rho \left( 1 - (v')^2 f - 2v' z' \right) + zv' \left( 2(v')^2 f - 2 + v'(4z' + \rho \frac{\partial f}{\partial z}) \right) \right]. \tag{33}
\]

Different with the case of the two-point function, here the integral function or Lagrangian of (31) implicitly depends on \( \rho \), which means there is no conservation equation. Meanwhile, we cannot construct the solutions from the midpoint to impose the boundary conditions because \( \rho \) in the denominator will cause numerical issues at \( \rho = 0 \). Therefore we expand the
equations of motion at a point $\rho_p$ near $\rho = 0$ to quadratic order, and the results will fix the boundary conditions at this point as
\[ z(\rho_p) = z_* - \frac{f(v_* z_*)}{2z_*} \rho_p^2, \quad v(\rho_p) = v_* + \frac{\rho_p^2}{2z_*}, \] (34)
where $z_*, v_*$ are two free parameters, and $z'(\rho_p), v'(\rho_p)$ can also be set by this expansion. The other two boundary conditions read
\[ z(R) = z_0, \quad v(R) = t_0, \] (35)
where again, $z_0$ is the radial IR cut-off and $t_0$ is the boundary time.

Then we can solve the equations of motion (32) and (33) via the boundary conditions (34) and (35). During the calculation, we renormalize the AdS volume by subtracting the cut-off dependent part [15]
\[ \delta V_{AdS} = V_{AdS} - \frac{R}{z_0}. \] (36)

The results of $\delta V_{AdS} = (\delta V_{AdS} - \delta V_{AdS-thermal})/(\pi R^2)$ against $t_0$ are shown in Fig. 3. It is obvious that the features of entanglement entropy during the thermalization processes are very similar with those in geodesic case. The larger $\beta$ gives larger renormalized entanglement

FIG. 3: The relative renormalized volume element $\delta V_{AdS}$ as a function of boundary time $t_0$ with different $\beta$ for the boundary radius $R = 0.5, 1, 1.5, 2$. 
entropy while again for small $\beta$ they are almost same. Moreover, when $R$ is small, the thermalization time $\tau_{\text{real}}$ is almost the same for different $\beta$, while for large $R$ the time $\tau_{\text{real}}$ behaves as a multi-valued function of $\beta$ as we observed in the two-point function case. This means that different $\beta$ may lead to same renormalized entropy, see the right panel of Fig. 4.

The relative critical time $\delta \tau_{\text{crit}}$ as a function of the boundary radius $R$ has been presented in the left panel of Fig. 4. For small $\beta$ and $R$ the critical time is linear with $R$, but for large $\beta$ the linearity does not hold. In particular, the critical time $\tau_{\text{crit}}$ violates the causal bound, i.e., $\tau_{\text{crit}} > R$. It is also obvious when $R$ is large enough, the critical thermalization time increase monotonically as $\beta$ increases, which matches the observe obtained from two-point correlation function. This suppression effect of momentum relaxation on the thermalization process may be explained as follows. In holography, the approach to thermal equilibrium in the boundary field theory is described by the decay of the bulk fluctuations. And it was found in [63] that the shift of the quasi-normal frequencies from zero for the bulk fluctuations is larger if the system has bigger $\beta$. However, the deep physical understanding of this suppression from the field theory is still missing.

In the right panel of Fig. 4, we observe similar but weaker feature compared to the geodesic case that for fixed large $R$, different $\beta$ leads to same $\delta \mathcal{V}_{\text{AdS}}$ if $t_0$ is in a certain range. The thermalization process shows such time range is usually a short interval before it reaches the thermal equilibrium. For example, see the right-bottom panel of Fig. 3, the interval is around $1.6 < t_0 < 2.0$. With smaller boundary separation leads to narrower interval. It can be expected that for small enough $\ell(R)$ the renormalized $\delta \mathcal{L}_{\text{AdS}}(\delta \mathcal{V}_{\text{AdS}})$ will be a monotonic function of $\beta$. Taking care of the two observables, we find that the right panels of Fig. 2 and Fig. 4 illustrate another interesting property. The relative renormalized geodesic length $\delta \mathcal{L}_{\text{AdS}}(\delta \mathcal{V}_{\text{AdS}})$ becomes positive for fixed large boundary separation (radius, $\ell = 4$ or $R = 2$) and time when $\beta \gtrsim 1.47$ for both probes. However, this feature is not hold for small enough $\ell$ or $R$. For clarity, we zoom in the thermalization curves near the thermal equilibrium point, and plot the relation between $z_*$ and $t_0$. In details, from the left panel of Fig. 5, we see that for two-point function when $\ell = 4$ the thermalization process will cross the zero point and then rapidly return to thermal equilibrium, while for holographic entanglement entropy.

FIG. 4: (Left panel) The relative critical thermalization time $\delta \tau_{\text{crit}}$ as a function of boundary radius $R$ with different $\beta$. (Right panel) The relative renormalized volume $\delta \mathcal{V}_{\text{AdS}}$ as a function of linear coefficient $\beta$ with fixed boundary radius $R = 2$ and time $t_0 = 1.8$. 
FIG. 5: (Left panel) The relative renormalized quantities $\delta \mathcal{L}_{AdS}$ or $\delta \mathcal{Y}_{AdS}$ as a function of boundary separation scale with $\beta = 1.5$ for $\ell = 4$, $R = 2$ and $R = 2.5$. (Right panel) $z_*$ as a function of $t_0$ for $\beta = 1.5$ with boundary separation scale $\ell = 4$, $R = 1.5$ and $R = 2.5$. The embedded picture shows the case when $\ell = 2$. Note here the dashed green line means different $R$ because for $R = 1.5$ its behavior is close to zero when approaching the thermal equilibrium point.

The process is quite sluggish. These behaviors are similar with the evolution of holographic entanglement entropy and holographic complexity in the massive BTZ black hole, which also states the appearance of momentum relaxations in the boundary CFT theory [64]. In the right panel of Fig. 5, the rapid process of two-point function for large $\ell$ corresponds to a jump of the curve $z_*(t_0)$ (see the red line), while the relatively slow process of the holographic entanglement entropy for large $R$ results from a gradual change of $z_*(t_0)$ (see dot-dashed blue line). In contrast, we note that for small $\ell$ or $R$ the thermalization process of two observers are similar, see the green line and the inserted picture in the right panel of Fig. 5. We claim that these positive-features result from the effect of momentum relaxation. If $\beta = 0$ these features will disappear, for example, see figure 10 and 23 in [17].

We also note that when the boundary separation is large enough, the equilibrium value of $z_*$ is larger than the non-equilibrium value. Thus it will be interesting to present the relation of $z_*$ with $\beta$ or the boundary separation at fixed time for two probes, see Fig. 6 and Fig. 7. Clearly the corresponding curves of two probes behave similarly. So we can focus on the two-point function to discuss these behaviors.

The top two panels of Fig. 6 show the behavior of $z_*$ as a function of $\beta$ at three time points for two boundary separations. These time points are chosen as the initial time, one arbitrary non-equilibrium time and the equilibrium time. For large separation $\ell = 4$, we chose $t_0 = 0.01$, 1, 2.5 and for $\ell = 1$, $t_0 = 0.01$, 0.3, 0.8. The figures illustrate that the thermal equilibrium value of $z_*$ is always smaller than 1. While for small enough $\ell$, $z_* < 1$ is true for entire thermalization process. Besides, we can tell that for large enough $\ell$ the intersection of $z_*$ at different time solely happens around $\beta \approx 1.5$, which means the case that the equilibrium value of $z_*$ is larger than non-equilibrium $z_*$ can only be true at large $\beta$.

The bottom two panels of Fig. 6 present the behavior of $z_*$ as a function of $\ell$ for $\beta = 0.1$, 1.5 at $t_0 = 0.01$, 1, 2.5. For left panel, $\beta$ is small enough. We note that for the non-equilibrium
FIG. 6: (Top) The relation between $z_*$ and $\beta$ with $\ell = 1$, $t_0 = 0.01$, 0.3, 0.8 and $\ell = 4$, $t_0 = 0.01$, 1, 2.5, respectively. (Bottom) The relation between $z_*$ and $\ell$ with $\beta = 0.1$, 1.5 at fixed boundary time $t_0 = 0.01$, 1, 2.5, respectively.

For $t_0 = 1$ (the red solid line), the shell position appears at a tipping point $z_*(\ell_{tip})$. For $\ell < \ell_{tip}$ the spacetime is Vaidya-AdS type so the curve for non-equilibrium and equilibrium $z_*$ is same, see the red solid line and red dashed line. For $\ell > \ell_{tip}$ the equilibrium value $z_*$ is smallest, as expected. Meanwhile, for large enough $\ell$ satisfying $\ell > \ell_{tip}$ the curve of $z_*(\ell)$ is parallel to the $z_*$ at initial time (the green dashed line). For right panel where $\beta$ is large, the differences of $z_*$ curves at different time are weak. The comparison between two panels suggests the effect of $\beta$ on the thermalization process is significant.

IV. CONCLUSION AND DISCUSSION

In this paper, we investigated the holographic thermalization process in a four dimensional Einstein-Maxwell-axions gravity theory, which is dual to a boundary theory with momentum relaxation. We mainly calculated the equal time two-point correlation function of a scalar operator and entanglement entropy, which are modeled by the minimal lengths of geodesic and volumes of spherical region in AdS space. We focused on the effects of momentum

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5 One should be aware of that for different non-equilibrium time $t_0$ the tipping point is different.
FIG. 7: (Top) The relation between $z_*$ and $\beta$ with $R = 0.5, t_0 = 0.01, 0.2, 1$ and $R = 2, t_0 = 0.01, 1, 2.5$, respectively. (Bottom) The relation between $z_*$ and $R$ with $\beta = 0.1, 1.5$ at fixed boundary time $t_0 = 0.01, 1, 2.5$, respectively.

It was found that the momentum relaxation would suppress the holographic thermalization process from the behaviors of two probes. This result is different from that found in massive gravity [55]. It may be because the thermal equilibrium configurations in two models are different, or here the momentum relaxation is introduced through new matter fields so that new degree of freedom is involved. Or we may extend our studies in a generalized axion theory with higher order(see [57]) to consider a generic potential and investigate the thermalization process. Deep physical explanation deserves more efforts.

We argued that the suppression effect on the thermalization process can be explained by profound violation of the quasi-normal frequencies from zero mode for the bulk fluctuations studied in [63], since the decay of the bulk fluctuations holographically describes the approach to thermal equilibrium in the dual boundary field theory. We believe that the deep physical insight of this suppression in the viewpoint of the field theory is encouraged to be investigated.

Here we chose the circle geometry to evaluate the holographic entanglement entropy. It is straightforward to study it by the method of stripe geometry with the same strategy. Moreover, it was addressed in [65] that the entanglement entropy is usually not enough to describe the rich geometric structure because it grows in a very short time during the thermalization process of a strongly coupled system. Recently, another active studies in holographic frame-
work is the holographic complexity evaluated from gravity side via ‘complexity=volume’ conjecture [66] or ‘complexity=action’ conjecture [67, 68]. The study on the evolution of sub-region complexity may help us supplement the description of thermalization process. This proposal has been addressed for Einstein gravity in [69] and recently was generalized in [70–72]. Thus, it would be very interesting to study the effect of momentum relaxation on the evolution of complexity during the thermalization process in this model, which will be presented elsewhere.

Acknowledgments

We appreciate Matteo Baggioli and Shang-Yu Wu for helpful corresponding. This work is supported by the Natural Science Foundation of China under Grant No.11705161 and Natural Science Foundation of Jiangsu Province under Grant No.BK20170481.

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