Quantum repeater protocol in mixed single- and two-mode Tavis-Cummings models

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Abstract – In this paper we study the production of entanglement between two atoms which are far from each other. We consider a system including eight two-level atoms (1, 2, ···, 8) such that any atom with its adjacent atom is in an atomic Bell state, so that we have four separate pairs of maximally entangled states (i, i + 1) where i = 1, 3, 5, 7. Our purpose is to produce entanglement between the atomic pair (1, 8), while these two distant atoms have no interaction. By performing the interaction between adjacent non-entangled atomic pairs (2, 3) as well as (6, 7), each pair with a two-mode quantized field, the entanglement is produced between atoms (1, 4) and (5, 8), respectively. Finally, by applying an appropriate Bell state measurement (BSM) on atoms (4, 5) or performing an interaction between them with a single-mode field (quantum electrodynamic: QED method), the qubit pair (1, 8) becomes entangled and so the quantum repeater is successfully achieved. This swapped entanglement is then quantified via concurrence measure and the effects of coupling coefficients and detuning on the concurrence and success probability are numerically investigated. The maxima of concurrence and success probability and the corresponding time periods have been decreased by increasing the detuning in asymmetric condition in the BSM method. Also, the effects of detuning, initial interaction time and coupling coefficient on the produced entanglement by the QED method are considered. Increasing (decreasing) of the detuning (interaction time) has a destructive effect on the swapped entanglement in asymmetric condition.

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Introduction. – Extending the quantum phenomena to macroscopic distances is of considerable interest [1,2]. In particular, entangled quantum states can be transferred over long distances using quantum repeater protocols [2]. In fact, to prevail on the entanglement attenuation in the process of transferring the entanglement the quantum repeater is proposed. These transferred states are useful in quantum communication [3], quantum key distribution [4] and quantum cryptography [5]. In ref. [6] the authors have investigated the optimal probability of creating a maximally entangled state by quantum repeater with respect to classical communication. The hybrid quantum repeater is investigated in [7,8] using bright coherent light. A quantum repeater which uses photon pair sources in combination with memories has been proposed [9]. Also, it is shown that the local generation of entangled pairs of atomic excitations, in combination with two-photon detections permits the implementation of a quantum repeater protocol [10]. A quantum repeater model using quantum dot spins and photons with spatial entanglement in optical microcavities has been studied in [11]. An experimental realization of entanglement concentration and a quantum repeater has been reported in [12]. As two sticking points in the quantum repeater protocols one may refer to the entanglement swapping and quantum memories. The entanglement can be swapped in short parts of a long distance by performing the interaction described by Tavis-Cummings [13] or Jaynes-Cummings [14] models. After producing the entanglement, by Bell state measurement (BSM) [15–17] or performing interaction between separable parts [18] the entanglement is swapped to non-entangled segments (for instance, see [19–29]).

In this paper we consider a quantum repeater protocol to distribute entanglement between two distant atoms. To achieve the purpose, by inserting six atoms between the two far atoms we divide the distance into seven shorter parts (see fig. 1). Then, by performing the interaction between atoms (2, 3) and (6, 7) in two two-mode cavities, the entanglement is produced in the two pairs (1, 4) and (5, 8), separately. Finally, the atoms (1, 8) can be
entangled either by operating the BSM (performing inter-
action) between (on) atoms (4, 5) in a single-mode cavity. The degree of entanglement of the produced entangled 
states of atoms (1, 8) is evaluated by concurrence [30]. In 
the following, we have analyzed the influence of the atom-
field coupling coefficient and detuning on the concurrence 
and success probability of the proposed model. Also, the 
effect of the interaction time on the concurrence has been 
investigated.

This paper is organized as follows: In the next section 
we have introduced our quantum repeater protocol model 
and presented the detailed calculations. The related nu-
merical results are collected in the third section. Finally, 
the paper is ended by the conclusions in the fourth section.

Quantum repeater protocol. – As we demonstrated in 
the Introduction of the paper, in the model shown in 
fig. 1, our aim is the generation of entanglement between 
two separable distant atoms 1, 8. At first, we consider the 
interaction between atoms (2, 3), where the initial state 
for atoms (1–4) reads as $|\Psi\rangle_{1,2} \otimes |\Psi\rangle_{3,4}$ with

$$
|\Psi\rangle_{i,i+1} = \frac{1}{\sqrt{2}}(|eg\rangle - |ge\rangle)_{i,i+1}, \quad i = 1, 3.
$$

(1)

Two two-level atoms (2, 3) with excited (ground) state $|e\rangle$ ($|g\rangle$) interact in a two-mode cavity field by the Tavis-
Cummings Hamiltonian introduced by

$$
\hat{H} = \hat{H}_0 + \hat{H}_1,
$$

(2)

where the free and interacting parts are respectively de-
fined as ($\hbar = 1$) [31]

$$
\hat{H}_0 = \omega \hat{a}^\dagger \hat{a} + \omega' \hat{b}^\dagger \hat{b} + \sum_{i=2,3} \frac{\omega_i}{2} \hat{\sigma}_{iz},
$$

$$
\hat{H}_1 = \sum_{i=2,3} g_i (\hat{a} \hat{b}^{\dagger} + \hat{a}^\dagger \hat{b}),
$$

(3)

where in fact we have considered the two-photon transi-
tion process in (3). It should be mentioned that the 
two-photon process [32,33] is not far from experimental re-
alization (for a few efforts in this relation see refs. [34–36]). 
In the above relations, the creation (annihilation) oper-
ators of the two modes have been shown by $\hat{a}^\dagger$, $\hat{b}^\dagger$ ($\hat{a}$, $\hat{b}$) and $\hat{\sigma}_i^+ = |e\rangle_i \langle g|$ ($\hat{\sigma}_i^- = \hat{\sigma}_i^+ \dagger$). $\hat{\sigma}_{iz}$ are raising (lower-
ing) and population inversion operators of the $i$-th atom,
respectively. The $i$-th atom with frequency $\omega_i$ (in this 
case, $i = 2, 3$) interacts with a two-mode field with fre-
cuencies $\omega$, $\omega'$. The corresponding Hamiltonian in 
the interaction picture reads as

$$
\hat{H}_{int} = \sum_{i=2,3} g_i (\hat{a} \hat{b}^{\dagger} e^{i \Delta_i t} + \text{H.C.}),
$$

(4)

which may be obtained from $\hat{H}_{int} = e^{i \hat{H}_0 t} \hat{H}_1 e^{-i \hat{H}_0 t}$ by the 
Baker-Hausdorff formula, where the detuning is defined as 
$\Delta_i = \omega_i - (\omega + \omega')$. We assume that the cavity is initially 
in the vacuum state and also $\Delta_i \gg g_i$. So, following the 
path of refs. [37,38] the effective Hamiltonian is obtained 
as follows:

$$
\hat{H}_{eff} = g_{23} (\hat{\sigma}_2^+ \hat{\sigma}_3^- e^{i \Delta_{23} t} + \text{H.C.}) + \sum_{i=2,3} \lambda_i \hat{\sigma}_i^+ \hat{\sigma}_i^-,
$$

(5)

$$
\lambda_i = \frac{g_i^2}{\Delta_i}, \quad \frac{1}{\omega_{23}} = \frac{1}{2} \left( \frac{1}{\Delta_1} + \frac{1}{\Delta_3} \right), \quad g_{23} = \frac{g_2 g_3}{\omega_{23}}.
$$

(6)

Now, we calculate the time evolution operator with the help 
of effective Hamiltonian (5) and using the relation $\hat{U}(t) = H_{eff} \hat{U}(t)$ in the basis spanned by 
$\{|ee\rangle_{2,3}, |eg\rangle_{2,3}, |ge\rangle_{2,3}, |gg\rangle_{2,3}\}$.

see eq. (7) on the next page

where $j = 2, 6$ and

$$
d_{23} = \Delta_2 - \Delta_3 + (\lambda_2 - \lambda_3), \quad f = \sqrt{d_{23}^2 + 4g_{23}^2}.
$$

(8)

Then, using the initial state $|\Psi\rangle_{1,2} \otimes |\Psi\rangle_{3,4}$ (eq. (1)) 
and (7), the state of atoms (1, 2, 3, 4) after the inter-
action at time $t$ can be deduced as follows:

$$
|\Psi(t)\rangle_{1-4} = \hat{U}_{2,3}(t)|\Psi\rangle_{1,2} \otimes |\Psi\rangle_{3,4}
$$

$$
= \frac{1}{2} \left[ (U_{23}(t)|ee\rangle + \sum_{i=3} U_{33}(t)|eg\rangle - |eg\rangle - U_{21}(t)|ge\rangle + \sum_{i=2} U_{22}(t)|ge\rangle
$$

$$
+ \sum_{i=2} U_{32}(t)|gg\rangle \rangle_{1-4},
$$

(9)

where $U_{ij}(t)$ is the matrix element of (7) locating in the 
row $i$ and column $j$. If the atoms (2, 3) are in $|ge\rangle_{2,3}$ 
or $|eg\rangle_{2,3}$, the state of the atomic pair (1, 4) is achieved 
respectively as follows:

$$
|\Psi(t)\rangle_{1,4} = \frac{1}{\sqrt{|U_{23}(t)|^2 + |U_{33}(t)|^2}} (U_{33}(t)|eg\rangle
$$

$$
+ U_{32}(t)|ge\rangle)_{1,4},
$$

(10)

$$
|\Psi(t)\rangle_{1,4} = \frac{1}{\sqrt{|U_{23}(t)|^2 + |U_{22}(t)|^2}} (U_{23}(t)|eg\rangle
$$

$$
+ U_{22}(t)|ge\rangle)_{1,4}.
$$

(11)

Similarly, atoms (5, 6, 7, 8) evolve with the effective 
Hamiltonian (5) as shown in the above procedure, how-
ever, with the time evolution operator $\hat{U}_{6,7}(t)$ in (7), by
which the entangled states of atoms (5, 8) can be obtained as follows:

\[
|\Psi(t)\rangle_{5,8} = \frac{1}{\sqrt{|U_{33}(t)|^2 + |U_{32}(t)|^2}} (U_{33}(t)|eg\rangle + U_{32}(t)|ge\rangle)_{5,8},
\]

\[
|\Psi'(t)\rangle_{5,8} = \frac{1}{\sqrt{|U_{23}(t)|^2 + |U_{22}(t)|^2}} (U_{23}(t)|eg\rangle + U_{22}(t)|ge\rangle)_{5,8},
\]

depending on measuring \(|ge\rangle_{6,7}\) and \(|eg\rangle_{6,7}\), respectively. Consequently, we have four possible non-entangled states \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}, |\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}, |\Psi'(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\) and \(|\Psi'(t)\rangle_{1,4} \otimes |\Psi'(t)\rangle_{5,8}\). Considering all these states we are going to introduce our quantum repeater protocol and compare the obtained results.

The case \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\). In this stage, we operate the Bell state measurement performed with the state \(|B\rangle_{4,5} = 1/2(|ee\rangle + |gg\rangle)_{4,5}\) on state \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\). As a result, the state of atoms (1, 8) is converted to the Bell state,

\[
|B\rangle_{1,8} = \frac{1}{\sqrt{2}} (|ee\rangle + |gg\rangle)_{1,8},
\]

with the success probability

\[
S_{|B\rangle_{1,8}}(t) = \frac{|U_{33}(t)|^2 + |U_{32}(t)|^2}{(|U_{33}(t)|^2 + |U_{32}(t)|^2)^2}.
\]

Now, we perform a BSM with Bell state \(|B'\rangle_{4,5} = 1/\sqrt{2}(|eg\rangle + |ge\rangle)_{4,5}\) on state \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\), so the atoms (1, 8) are converted to the following entangled state:

\[
|\gamma\rangle_{1,8} = \frac{1}{\sqrt{|U_{33}(t)|^4 + |U_{32}(t)|^4}} (U_{33}(t)|eg\rangle + U_{32}(t)|ge\rangle)_{1,8},
\]

whose concurrence reads as

\[
C_{1}(t) = \frac{2(|U_{33}(t)|^2|U_{32}(t)|)}{|U_{33}(t)|^4 + |U_{32}(t)|^4}.
\]

Next, we consider the initial state \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\) and investigate the interaction between atoms (4, 5) in a single-mode cavity using the effective Hamiltonian

\[
\hat{H}'_{\text{eff}} = \lambda' \sum_{i=4,5} \hat{a}_i^\dagger \hat{a}_i \hat{a}_i^\dagger + \lambda' (\hat{a}_4^\dagger \hat{a}_5^\dagger + \hat{a}_4 \hat{a}_5),
\]

\[
\lambda' = \frac{g^2}{\delta}, \quad \delta = \omega_4 - \omega = \omega_5 - \omega.
\]

Notice that the above Hamiltonian has been readily obtained by using the path of refs. [37, 38] and the following Hamiltonian:

\[
\hat{H}' = \omega \hat{a}^\dagger \hat{a} + \sum_{i=4,5} \frac{\omega_i}{2} \hat{\sigma}_{ix} + g \hat{a} \hat{a}^\dagger.\]

By performing the interaction (18) between atoms (4, 5) for the initial state \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\), the following entangled state can be achieved for atoms (1, 4, 5, 8):

\[
|\phi\rangle_{1,4,5,8} = \frac{1}{\sqrt{|U_{32}(t)|^2 + |U_{33}(t)|^2}} \left( U_{32}(t)\frac{1}{2} (e^{2i\lambda' t} e^{-2i\lambda' \tau} - 1)|ee\rangle + \frac{1}{2} (e^{2i\lambda' t} e^{-2i\lambda' \tau} + 1)|ge\rangle + U_{32}^* (t)\frac{1}{2} (e^{2i\lambda' t} e^{-2i\lambda' \tau}) - 1)|gg\rangle + \frac{1}{2} (e^{2i\lambda' t} e^{-2i\lambda' \tau} + 1)|ge\rangle + U_{32}^* (t)U_{33} (t)|ge\rangle + e^{-2i\lambda' (\tau - t)} |ge\rangle \right)_{1,4,5,8}.
\]

By making a measurement on the state (21), one obtains \(|eg\rangle_{4,5}\) and \(|ge\rangle_{4,5}\), the result of which arrives us respectively at

\[
|\phi\rangle_{1,8}^1 = \frac{1}{\sqrt{|a_{1,1}(t, \tau)|^2 + |b_{1,1}(t, \tau)|^2}} (a_{1,1}(t, \tau)|eg\rangle + b_{1,1}(t, \tau)|ge\rangle)_{1,8},
\]

\[
a_{1,1}(t, \tau) = U_{33}^* (t)e^{2i\lambda' t} e^{-2i\lambda' \tau} - 1,
\]

\[
b_{1,1}(t, \tau) = U_{33}^* (t)e^{2i\lambda' t} e^{-2i\lambda' \tau} + 1,
\]

with the concurrence

\[
C_{1}^1(t, \tau) = \frac{2|a_{1,1}^* (t, \tau) b_{1,1}(t, \tau)|}{|a_{1,1}(t, \tau)|^2 + |b_{1,1}(t, \tau)|^2},
\]

and

\[
|\phi\rangle_{1,8}^1 = \frac{1}{\sqrt{|a_{1,1}^* (t, \tau)|^2 + |b_{1,1}^* (t, \tau)|^2}} (a_{1,1}^* (t, \tau)|eg\rangle + b_{1,1}^* (t, \tau)|ge\rangle)_{1,8},
\]

\[
a_{1,1}^* (t, \tau) = U_{33}^* (t)e^{2i\lambda' t} e^{-2i\lambda' \tau} + 1,
\]

\[
b_{1,1}^* (t, \tau) = U_{33}^* (t)e^{2i\lambda' t} e^{-2i\lambda' \tau} - 1,
\]
with the concurrence
\[ C_1^\prime(t, \tau) = \frac{2|\alpha_1^\prime(t, \tau)\beta_1^\prime(t, \tau)|}{|\alpha_1^\prime(t, \tau)|^2 + |\beta_1^\prime(t, \tau)|^2}. \] (25)

The case $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$. By the operation of BSM performed with the state $|B\prime\rangle_{4,5} = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)_{4,5}$ on state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$, the atoms (1, 8) are converted to the Bell state
\[ |B\prime\rangle_{1,8} = \frac{1}{\sqrt{2}}(|eg\rangle + e^{-2i\varphi}|ge\rangle)_{1,8}, \] (26)
where $e^{i\varphi} = \pm \frac{\cos(\frac{\pi}{4}) + i\frac{\sqrt{2}}{2}\sin(\frac{\pi}{4})}{\sqrt{\cos(\frac{\pi}{4}) + \frac{\sqrt{2}}{2}\sin(\frac{\pi}{4})}}$ with $|e^{i\varphi}| = 1$ and the success probability reads as
\[ S_{|B\prime\rangle_{1,8}}(t) = \frac{|U_{33}(t)|^2|U_{23}(t)|^2 + |U_{32}(t)|^2|U_{22}(t)|^2}{2(|U_{33}(t)|^2 + |U_{32}(t)|^2)(|U_{23}(t)|^2 + |U_{22}(t)|^2)^2}. \] (27)

Notice that $S_{|B\prime\rangle_{1,8}}(t) = S_{|B\rangle_{1,8}}(t)$. But by measuring the Bell state $|B\rangle_{4,5} = \frac{1}{\sqrt{2}}(|ee\rangle + |gg\rangle)_{4,5}$ on state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$, the atoms (1, 8) are converted to the entangled state
\[ |\gamma\rangle_{1,8}^2 = \frac{1}{\sqrt{|U_{22}(t)|^2 + |U_{23}(t)|^2 + |U_{24}(t)|^2}} \times (U_{22}(t)|U_{33}(t)|e^{-i\varphi} + U_{32}(t)|U_{33}(t)|g^i\varphi)_{1,8}, \] (28)
with the concurrence
\[ C_2(t) = \frac{2|U_{22}(t)|^2|U_{32}(t)|^2|U_{33}(t)|^2}{|U_{22}(t)|^2 + |U_{23}(t)|^2 + |U_{24}(t)|^2}. \] (29)

It can be seen that $C_2(t) = C_1(t)$.

Now, the entangled state for atoms (1, 4, 5, 8) can be achieved after performing the interaction according to the Hamiltonian (18) between atoms (4, 5) in a single-mode cavity with initial state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$ which results in
\[ |\phi\rangle_{1,4,5,8} = \frac{1}{\sqrt{|U_{32}(t)|^2 + |U_{33}(t)|^2 + |U_{23}(t)|^2}} \times \left\{ U_{33}(t)|U_{23}(t)|\left[\frac{1}{2}(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} - 1)|eege\rangle + \frac{1}{2}(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} + 1)|gege\rangle + U_{32}(t)|U_{22}(t)|\left[\frac{1}{2}(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} + 1)|geege\rangle + \frac{1}{2}(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} - 1)|gege\rangle + \frac{1}{2}(e^{2i\lambda'\tau}e^{-2i\lambda'\tau})U_{33}(t)|U_{22}(t)|gege\rangle \right]\right\}_{1,4,5,8}. \] (30)

By applying a measurement on the state (30), one obtains $|eg\rangle_{1,4,5}$ and $|ge\rangle_{4,5,8}$, where the results for atoms (1, 8) are respectively as
\[ |\phi\rangle_{1,8}^2 = \frac{1}{\sqrt{|a_2(t, \tau)|^2 + |b_2(t, \tau)|^2}}(a_2(t, \tau)|eg\rangle + b_2(t, \tau)|ge\rangle)_{1,8}, \] (31)
\[ a_2(t, \tau) = U_{23}(t)U_{33}(t)(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} - 1), \]
\[ b_2(t, \tau) = U_{23}(t)U_{33}(t)(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} + 1), \]
whose concurrence reads as
\[ C_2^\prime(t, \tau) = \frac{2|a_2^\prime(t, \tau)b_2^\prime(t, \tau)|}{|a_2^\prime(t, \tau)|^2 + |b_2^\prime(t, \tau)|^2}, \] (32)
and
\[ |\phi\rangle_{1,8}^2 = \frac{1}{\sqrt{a_2^\prime(t, \tau)^2 + b_2^\prime(t, \tau)^2}}(a_2^\prime(t, \tau)|eg\rangle + b_2^\prime(t, \tau)|ge\rangle)_{1,8}, \] (33)
\[ a_2^\prime(t, \tau) = U_{23}(t)U_{33}(t)(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} - 1), \]
\[ b_2^\prime(t, \tau) = U_{23}(t)U_{33}(t)(e^{2i\lambda'\tau}e^{-2i\lambda'\tau} + 1), \]
with the concurrence
\[ C_2^\prime(t, \tau) = \frac{2|a_2^\prime(t, \tau)b_2^\prime(t, \tau)|}{|a_2^\prime(t, \tau)|^2 + |b_2^\prime(t, \tau)|^2}. \] (34)

It is easy to check that $C_2^\prime(t, \tau) = C_2(t, \tau)$.

The case $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$. The above processes can be repeated for the initial state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$. With a BSM using the maximally entangled state $|B\prime\rangle_{4,5} = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)_{4,5}$ on state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$, the state of atoms (1, 8) is converted to the Bell state $|B\prime\rangle_{1,8}$ in eq. (26) with the success probability (27). Meanwhile, measuring the Bell state $|B\rangle_{4,5} = \frac{1}{\sqrt{2}}(|ee\rangle + |gg\rangle)_{4,5}$ on state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$, the entangled state
\[ |\gamma\rangle_{1,8}^3 = \frac{1}{\sqrt{|U_{32}(t)|^2 + |U_{23}(t)|^2 + |U_{24}(t)|^2}} \times (U_{32}(t)|U_{23}(t)|e^{-i\varphi} + U_{22}(t)|U_{33}(t)|g^i\varphi)_{1,8}, \] (35)

is obtained, where its concurrence is calculated as follows:
\[ C_3(t) = \frac{2|U_{32}(t)|^2|U_{33}(t)|^2|U_{23}(t)|^2}{|U_{32}(t)|^2 + |U_{23}(t)|^2 + |U_{24}(t)|^2}. \] (36)

Notice that $C_3(t) = C_2(t) = C_1(t)$.

On the other hand, the entangled state for atoms (1, 4, 5, 8) after performing the interaction according to the Hamiltonian (18) with initial state $|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}$ can
be obtained as
\[
|\phi''\rangle_{1,4,5,8} = \frac{1}{\sqrt{|U_{32}(t)|^2 + |U_{33}(t)|^2 + |U_{23}(t)|^2}} \times \left\{ U_{33}(t)U_{23}(t) \left[ \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} - 1)|eege\rangle + \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} + 1)|egeg\rangle \right] + U_{32}(t)U_{22}(t) \left[ \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} + 1)|gege\rangle \right] + \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} - 1)|ggee\rangle \right\}
\]
\[
+ U_{23}(t)|gege\rangle + e^{-2i\lambda'(\tau - t)}U_{22}(t)|gege\rangle \right\} |\phi''\rangle_{1,4,5,8}.
\]

If the output of the measurement on the state (37) reads as \(|eg\rangle_{1,4,5,8}\) and \(|ge\rangle_{4,5,8}\), then the entangled states for atoms (1, 8) are achieved as \(|\phi''\rangle_{1,8}\) and \(|\phi''\rangle_{1,8}\), where \(|\phi''\rangle_{1,8}\) and \(|\phi''\rangle_{1,8}\) are \(|\phi''\rangle_{1,8}\) in eqs. (31) and (33) with the concurrence \(C_3(t) = C'_3(t)\) and \(C_{10}(t) = C''_3(t)\) eqs. (32) and (34), respectively (notice that \(C_3(t) = C''_3(t)\) = \(C_3(t) = C''_3(t)\)).

The case \(|\Psi(t)\rangle_{1,4,5,8} \otimes |\Psi(t)\rangle_{5,8}\). For this case, after a BSM with \(|B'\rangle_{4,5} = \frac{1}{\sqrt{2}}(|eg\rangle + |ge\rangle)_{4,5}\) on state \(|\Psi(t)\rangle_{1,4,5,8} \otimes |\Psi(t)\rangle_{5,8}\), the entangled state for atoms (1, 8) is achieved as
\[
|\gamma\rangle_{1,8} = \frac{1}{\sqrt{|U_{23}(t)|^2 + |U_{22}(t)|^2}}(U_{23}(t)|eg\rangle + U_{22}(t)|ge\rangle)_{1,8}.
\]

with the concurrence
\[
C_4(t) = \frac{2|U_{23}(t)|^2|U_{22}(t)|}{|U_{23}(t)|^2 + |U_{22}(t)|^2}.
\]

Notice that \(C_4(t) = C_2(t) = C_1(t)\).

Also, using BSM performed with \(|B\rangle_{4,5} = \frac{1}{\sqrt{2}}(|ee\rangle + |gg\rangle)_{4,5}\) on state \(|\Psi(t)\rangle_{1,4,8} \otimes |\Psi(t)\rangle_{5,8}\), the Bell state (14) is obtained for atoms (1, 8) with the success probability
\[
S'_\{B\}_{1,8} = \frac{|U_{23}(t)|^2|U_{22}(t)|^2}{(|U_{23}(t)|^2 + |U_{22}(t)|^2)^2}.
\]

Notice that \(S'_\{B\}_{1,8} = S_{\{B\}_{1,8}} = S_{\{B\}_{1,8}}\).

Moreover, by performing the interaction according to the Hamiltonian (18) between atoms (4, 5) with initial state \(|\Psi(t)\rangle_{1,4} \otimes |\Psi(t)\rangle_{5,8}\), the following entangled state of atoms (1, 4, 5, 8) is obtained as follows:
\[
|\phi''\rangle_{1,4,5,8} = \frac{1}{\sqrt{|U_{32}(t)|^2 + |U_{22}(t)|^2}} \times \left\{ U_{32}(t) \left[ \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} - 1)|eege\rangle + \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} + 1)|egeg\rangle \right] + U_{22}(t) \left[ \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} + 1)|gege\rangle + \frac{1}{2}(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} - 1)|ggee\rangle \right] \right\}
\]
\[
+ U_{22}(t)|gege\rangle + e^{-2i\lambda'(\tau - t)}U_{22}(t)|gege\rangle \right\} |\phi''\rangle_{1,4,5,8}.
\]

Now, via measuring \(|eg\rangle_{1,5,8}\) and \(|ge\rangle_{4,5,8}\) on the obtained state in (41), the atoms (1, 8) are respectively converted to the states
\[
|\phi\rangle_{1,8} = \frac{1}{\sqrt{|a_4(t, \tau)|^2 + |b_4(t, \tau)|^2}}(a_4(t, \tau)|eg\rangle + b_4(t, \tau)|ge\rangle),
\]

\[
a_4(t, \tau) = U_{23}(t)(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} - 1),
\]

\[
b_4(t, \tau) = U_{22}(t)(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} + 1),
\]

with the concurrence
\[
C'_4(t, \tau) = \frac{2|a_4(t, \tau)|^2|b_4(t, \tau)|}{|a_4(t, \tau)|^2 + |b_4(t, \tau)|^2},
\]

and
\[
|\phi\rangle_{1,8} = \frac{1}{\sqrt{|a_4(t, \tau)|^2 + |b_4(t, \tau)|^2}}(a_4(t, \tau)|eg\rangle + b_4(t, \tau)|ge\rangle),
\]

\[
a'_4(t, \tau) = U_{23}(t)(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} + 1),
\]

\[
b'_4(t, \tau) = U_{22}(t)(e^{2i\lambda'\tau} - e^{-2i\lambda'\tau} - 1),
\]

with the concurrence
\[
C''_4(t, \tau) = \frac{2|a'_4(t, \tau)|^2|b'_4(t, \tau)|}{|a'_4(t, \tau)|^2 + |b'_4(t, \tau)|^2}.
\]

Notice that \(C'_4(t, \tau) = C''_4(t, \tau) = C'_4(t, \tau)\).

Results and discussion. In this section we present our numerical results and investigate on the degree of entanglement of the entangled states and success probabilities which have been calculated in the previous section. We have shown the time evolution of concurrence and success probability of entangled states for atoms (1, 8) produced by the BSM method, where we have considered the effects of the atom-field coupling coefficient and detuning on concurrence and success probability, in figs. 2 and 3, respectively. In fig. 2(a) concurrence has been reached to its maximum value in some intervals of time, in symmetric condition, however, in asymmetric condition the maximum of concurrence and its time period have been decreased. The success probability in fig. 2(b) has been reached to 0.25 in some intervals of time for symmetric condition and in asymmetric condition this value and its time period have been decreased.

In fig. 3 we have considered the effect of detuning on concurrence and success probability. In figs. 3(a) and (b) the maximum of concurrence and success probability have been considerably decreased by increasing the detuning.
Effect of the coupling coefficient on the time evolution of (a) concurrence $C_1(t) = C_2(t) = C_3(t) = C_4(t)$, (b) success probability $S_{[B_{1, s}, t]}(t) = S_{[B_{1, s}, t]}(t)$, for symmetric condition $g_2 = g_3$ (solid red line) and asymmetric condition $g_2 = g$, $g_3 = 3g$ (dashed black line) with $\Delta_2 = 2g$ and $\Delta_3 = 3g$ (BSM method).

Fig. 2: (Colour online) The effect of the coupling coefficient on the time evolution of (a) concurrence $C_1(t) = C_2(t) = C_3(t) = C_4(t)$, (b) success probability $S_{[B_{1, s}, t]}(t) = S_{[B_{1, s}, t]}(t)$, for symmetric condition $g_2 = g_3$ (solid red line) and asymmetric condition $g_2 = g$, $g_3 = 3g$ (dashed black line) with $\Delta_2 = 2g$ and $\Delta_3 = 3g$ (BSM method).

The effect of the coupling coefficient on the time evolution of concurrence: (a) $C'_1(t, \tau) = C''_4(t, \tau)$, (b) $C'_4(t, \tau) = C''_1(t, \tau)$, (c) $C'_2(t, \tau) = C''_2(t, \tau) = C''_3(t, \tau) = C''_4(t, \tau)$, for $\Delta_3 = 3g$ (solid red line) and $\Delta_3 = 10g$ (dashed black line) with $\delta = \Delta_2 = 2g$, $g_2 = g$, $g_3 = 5g$ and $gt = 10$ (QED method).

Fig. 4: (Colour online) The effect of the coupling coefficient on the time evolution of concurrence: (a) $C'_1(t, \tau) = C''_4(t, \tau)$, (b) $C'_4(t, \tau) = C''_1(t, \tau)$, (c) $C'_2(t, \tau) = C''_2(t, \tau) = C''_3(t, \tau) = C''_4(t, \tau)$, for $\Delta_3 = 3g$ (solid red line) and $\Delta_3 = 20g$ (dashed black line) with $\delta = \Delta_2 = 2g$, $g_2 = g$, $g_3 = 5g$ and $gt = 10$ (QED method).

The effect of detuning on the concurrence of entangled atoms (1, 8) produced by the QED method has been considered in fig. 4. In figs. 4(a) and (b) the death of entanglement has been shown to occur in large time intervals by increasing the detuning in asymmetric condition and the entangled state of atoms (1, 8) has been converted to the atomic Bell state for decreased detuning by many times. Varying detuning has no observable effect on concurrence in fig. 4(c); in addition, the death of entanglement has been shown to occur only for some instants of time.

We have considered the effect of the interaction time on the concurrence in fig. 5. The entanglement attenuation has been shown to occur in large time intervals for decreased interaction time in figs. 5(a) and (b). In fig. 5(c) we can see a similar evolution of concurrence for the two cases, but, maxima and minima of concurrence have been relocated.

The effects of symmetric and asymmetric conditions have been considered in fig. 6. The maxima of concurrence
have been shown to occur by many times for asymmetric condition in figs. 6(a) and (b). Also, the death of entanglement has been shown to occur for some finite time intervals and then it suddenly revives in symmetric condition. In fig. 6(c) it is shown that the evolution of concurrence is independent of the coupling coefficient.

**Summary and conclusions.** – In this paper we considered the quantum repeater protocol using mixed single- and two-mode Tavis-Cummings models. We investigated eight two-level atoms prepared in four entangled pairs (1, 2), (3, 4), (5, 6) and (7, 8). By performing the interaction between atoms (2, 3) and (6, 7) in two-mode cavities, the entangled atoms (1, 4) and (5, 8) were produced. Then, by using a BSM or performing the interaction between atoms (4, 5) in a single-mode cavity (QED method), the atoms (1, 8) were converted to an entangled state. The atomic states (1, 8) are converted to atomic Bell states by choosing suitable Bell states in the BSM method. The effect of the coupling coefficient and detuning have been investigated on the concurrence and success probability of entangled state of atoms (1, 8) produced by the BSM method. In asymmetric condition the maxima of concurrence and success probability and their time periods have been decreased by increasing the detuning in the BSM method. Also, we considered the effects of detuning, interaction time and coupling coefficient on entanglement produced by the QED method. We observed the destructive effects of increased detuning, *i.e.*, the maxima of concurrence occurred less times and the death of entanglement happened in large time intervals when the detuning was increased. The effect of decreasing the interaction time *gt* on the entanglement has been discussed, *i.e.*, the entanglement attenuation happened in large intervals of time by decreasing the interaction time. The death of entanglement in symmetric condition has been observed in large time intervals for atoms (1, 8) which are produced by the QED method.