Symmetrized Poly-Bernoulli Numbers and Combinatorics

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Abstract

Poly-Bernoulli numbers are one of the generalizations of the classical Bernoulli numbers. Since a negative indexed poly-Bernoulli number is an integer, it is an interesting problem to study this number from a combinatorial viewpoint. In this short article, we give a new combinatorial relation between symmetrized poly-Bernoulli numbers and Dumont-Foata polynomials.

1 Introduction

A poly-Bernoulli polynomial $B_{m}^{(\ell)}(x)$ of index $\ell \in \mathbb{Z}$ is defined by the generating series

$$e^{-xt}\frac{\text{Li}_{\ell}(1-e^{-t})}{1-e^{-t}} = \sum_{m=0}^{\infty} B_{m}^{(\ell)}(x) \frac{t^{m}}{m!},$$

where $\text{Li}_{\ell}(z)$ is the polylogarithm function given by

$$\text{Li}_{\ell}(z) = \sum_{m=1}^{\infty} \frac{z^{m}}{m^{\ell}} \quad (|z| < 1).$$
The polynomial $B_m^{(1)}(x)$ coincides with the classical Bernoulli polynomial $B_m(1 - x) = (-1)^m B_m(x)$ since $\text{Li}_1(z) = -\log(1 - z)$ holds. Following Kaneko [7], the special value $B_m^{(\ell)} := B_m^{(\ell)}(0)$ is called a poly-Bernoulli number of index $\ell$.

The aim of this study is giving a combinatorial perspective to the special values of $B_m^{(\ell)}(x)$ at integers $k \in \mathbb{Z}$. We assume that the index $\ell \leq 0$. In this case the values $B_m^{(\ell)}(k)$ are always integers. One of the first such investigations was Brewbaker’s study [4]. He noticed the coincidence of two numbers, the poly-Bernoulli number $B_m^{(\ell)}(0)$ and the number of 01 lonesum matrices of size $m \times |\ell|$. Recently, Bényi and Hajnal [3] also established more combinatorial relations in this direction.

In this article, we take a step in another direction similar to Kaneko, Sakurai, and Tsumura [9]. To describe this more precisely, let $G_n$ be the Genocchi number A110501 defined by $G_n = 2(2^{n+1} - 1)|B_{n+1}|$, where $B_m = B_m^{(1)}(1)$ is the classical Bernoulli number. In [9, Theorem 4.2] the authors showed

$$\sum_{\ell=0}^{n} (-1)^\ell B_n^{(-\ell-1)}(1) = (-1)^{n/2} G_n$$

for any $n \geq 0$. As they mentioned, this equation is an analogue of the result of Arakawa and Kaneko [2],

$$\sum_{\ell=0}^{n} (-1)^\ell B_n^{(-\ell)}(0) = \begin{cases} 1, & \text{if } n = 0; \\ 0, & \text{if } n > 0. \end{cases}$$

In addition, Sakurai asked in her master’s thesis whether we can generalize these equations for any positive integers $x = k$, and give some combinatorial meaning to them. Our main result provides an answer to these two questions in terms of the Dumont-Foata polynomial as follows.

**Theorem 1.** Let $G_n(x, y, z)$ be the $n$-th Dumont-Foata polynomial defined in (3), and $B_m^{(-\ell)}(k)$ the symmetrized poly-Bernoulli number defined in (5). Then we have

$$\sum_{\ell=0}^{n} (-1)^\ell B_n^{(-\ell)}(k) = k! \cdot (-1)^{n/2} G_n(1, 1, k)$$

for any non-negative integers $n, k \geq 0$. In particular, both sides equal zero for odd $n$.

This theorem recovers the equations (1) and (2) since $B_m^{(-\ell)}(0) = B_m^{(-\ell)}(0), B_m^{(-\ell)}(1) = B_m^{(-\ell-1)}(1)$, and $G_n(1, 1, 1) = G_n$ hold as we see later.

**Remark 2.** I feel that there are a lot of possibilities of establishing similar identities as this theorem. Recent work of Bényi and Hajnal [3, Section 6] pointed out that the sequence of diagonal sums

$$\sum_{\ell=0}^{n} B_n^{(-\ell-1)}(1) \ (n \geq 0)$$
appears as \textbf{A136127} in the On-Line Encyclopedia of Integer Sequences (OEIS) \cite{OEIS}. Furthermore, there are many types of generalizations of poly-Bernoulli numbers as referred in \cite{polyBernoulli}, and analogues such as poly-Euler numbers \cite{polyEuler}, and poly-cosecant numbers \cite{polyCosecant}.

2 Definitions

2.1 Dumont-Foata polynomials

We review the work of Dumont and Foata \cite{DumontFoata} here. For a positive even integer $n \in 2\mathbb{Z}$, we consider the surjective map $p : \{1, 2, \ldots, n\} \to \{2, 4, \ldots, n\}$ with $p(x) \geq x$ for each $x \in \{1, 2, \ldots, n\}$. This map is called a (surjective) \textit{pistol} of size $n$, and corresponds to the following diagram. Here we draw an example for $n = 6$.

![Diagram of pistol for n=6](image)

Figure 1: $p(1) = 2, p(2) = p(4) = 4, p(3) = p(5) = p(6) = 6$

Let $\mathcal{P}_n$ be the set of all pistols of size $n$. For each pistol $p \in \mathcal{P}_n$, we define three quantities called bulging, fixed, and maximal points. First, the number $x \in \{1, 2, \ldots, n\}$ is a \textit{bulging point} of $p \in \mathcal{P}_n$ if $p(y) < p(x)$ for any $0 < y < x$. We let $b(p)$ denote the number of bulging points of $p$. In the diagram, $b(p)$ corresponds to the number of steps of the minimal stair covering all check marks. For the above example $p$, the points $x = 1, 2, 3$ are bulging points, so that $b(p) = 3$.

![Minimal stair examples](image)

Figure 2: minimal stair (left), not minimal stair (right)

Next, the point $x \in \{1, 2, \ldots, n\}$ is called a \textit{fixed point} of $p \in \mathcal{P}_n$ if $p(x) = x$. Finally, the point $x \in \{1, 2, \ldots, n-1\}$ is a \textit{maximal point} of $p \in \mathcal{P}_n$ if $p(x) = n$. We let $f(p)$ and $m(p)$ denote the numbers of fixed points and maximal points of $p \in \mathcal{P}_n$, respectively. For the above example, we have $f(p) = 2$ and $m(p) = 2$. Dumont and Foata \cite[Théorème 1a, 2]{DumontFoata} established the following interesting theorem.
Theorem 3. [6] Let \( n \in 2\mathbb{Z}_{>0} \) be a positive even integer. The polynomial defined by
\[
G_n(x, y, z) := \sum_{p \in \mathcal{P}_n} x^{b(p)} y^{f(p)} z^{m(p)}
\] is a symmetric polynomial in three variables, and satisfies \( G_n(1, 1, 1) = G_n \).

In addition, \( G_0(x, y, z) = 1 \) and \( G_n(x, y, z) = 0 \) for a positive odd integer \( n \in \mathbb{Z} \). The polynomial \( G_n(x, y, z) \) is called the \( n \)-th Dumont-Foata polynomial. Furthermore, Dumont and Foata showed that the polynomial for \( n > 0 \) has the form
\[
G_n(x, y, z) = xyzF_n(x, y, z)
\]
and the polynomial \( F_n(x, y, z) \) satisfies the recurrence relation
\[
F_n(x, y, z) = (x+z)(y+z)F_{n-2}(x, y, z + 1) - z^2F_{n-2}(x, y, z)
\]
with initial values \( F_1(x, y, z) = 0 \) and \( F_2(x, y, z) = 1 \). This implies that the polynomial \( G_n(z) := G_n(1, 1, z) \) called the Gandhi polynomial satisfies
\[
G_{n+2}(z) = z(z+1)G_n(z+1) - z^2G_n(z)
\]
with \( G_0(z) = 1, G_1(z) = 0 \).

For instance, there exist three pistols of size 4. The pistols have \((b(p), f(p), m(p)) = (2, 2, 1), (2, 1, 2) \) and \((1, 2, 2)\), so that the Dumont-Foata polynomial is given by
\[
G_4(x, y, z) = x^2y^2z + x^2yz^2 + xy^2z^2 = xyz(xy + yz + zx).
\]
Indeed, \( G_4(1, 1, 1) = 3 \) coincides the 4-th Genocchi number given by \( G_4 = 2(2^6 - 1)|B_6| = 3 \).

\[\text{Figure 3: All elements of } \mathcal{P}_4\]

2.2 Symmetrized poly-Bernoulli numbers

Table 1. includes the first few values of poly-Bernoulli numbers \( \{B_m^{(-\ell)}(0)\} \) and \( \{B_m^{(-\ell)}(1)\} \) with \( m, \ell \geq 0 \).

\begin{tabular}{|c|c|c|c|c|}
\hline
\ell \backslash m & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 2 & 4 & 8 & 16 \\
2 & 1 & 4 & 14 & 46 & 146 \\
3 & 1 & 8 & 46 & 230 & 1066 \\
4 & 1 & 16 & 146 & 1066 & 6902 \\
\hline
\end{tabular}

\begin{tabular}{|c|c|c|c|c|}
\hline
\ell \backslash m & 0 & 1 & 2 & 3 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 1 & 3 & 7 & 15 & 31 \\
3 & 1 & 7 & 31 & 115 & 391 \\
4 & 1 & 15 & 115 & 675 & 3451 \\
\hline
\end{tabular}

Table 1: \( B_m^{(-\ell)}(0) \) \textcolor{blue}{A099594}, \( B_m^{(-\ell)}(1) \) \textcolor{blue}{A136126}
We can see the symmetric property of these numbers at a glance. On the other hand, for $k \geq 2$ it seems unlikely that such a simple symmetric property can be given. In order to reproduce the symmetric properties for any $k \geq 2$, Kaneko-Sakurai-Tsumura [9] considered combinations of $B_m^{(-\ell)}(k)$. To make this precise, let $m, \ell, k \geq 0$ be non-negative integers. We now define the symmetrized poly-Bernoulli number $B_m^{(-\ell)}(k)$ by

$$B_m^{(-\ell)}(k) = \sum_{j=0}^{k} \left[ \begin{array}{c} k \\ j \end{array} \right] B_m^{(-\ell-j)}(k),$$

(5)

where $\left[ \begin{array}{c} k \\ j \end{array} \right]$ is the unsigned Stirling number of the first kind A130534 defined in [1, Definition 2.5]. This number satisfies the symmetry property

$$B_m^{(-\ell)}(k) = B_\ell^{(-m)}(k)$$

for any $m, \ell, k \geq 0$. Note that

$$B_m^{(-\ell)}(0) = B_m^{(-\ell)}(0), \quad B_m^{(-\ell)}(1) = B_m^{(-\ell-1)}(1).$$

| $\ell \backslash m$ | 0 | 1 | 2 | 3 | 4 |
|------------------|---|---|---|---|---|
| 0                | 1 | -1| 1 | -1| 1 |
| 1                | 1 | 0 | 0 | 0 | 0 |
| 2                | 1 | 2 | 2 | 2 | 2 |
| 3                | 1 | 6 | 18| 42| 90|
| 4                | 1 | 14| 86| 374|1382|

Table 2: $B_m^{(-\ell)}(2)$ and $B_m^{(-\ell)}(2)$

Moreover, the authors showed the following explicit formula for $B_m^{(-\ell)}(k)$.

$$B_m^{(-\ell)}(k) = \sum_{j=0}^{\min(m, \ell)} j!(k+j)! \left\{ \begin{array}{c} m+1 \\ j+1 \end{array} \right\} \left\{ \begin{array}{c} \ell+1 \\ j+1 \end{array} \right\},$$

(6)

where $\left\{ \begin{array}{c} k \\ j \end{array} \right\}$ is the Stirling number of the second kind A008277 defined in [1, Definition 2.2].

We prove our main theorem using this formula in the next section.

3 Proof

To prove Theorem 1, it suffices to show that the function

$$\tilde{G}_n(k) := \frac{(-1)^{n/2}}{k!} \sum_{\ell=0}^{n} (-1)^\ell B_m^{(-\ell)}(k)$$

5
satisfies the recurrence relation (4) for any integer \( k \geq 0 \). First, we can easily see that \( \tilde{G}_0(k) = 1 \) and \( \tilde{G}_1(k) = 0 \), which are the initial cases. Moreover, for any odd integer \( n \), \( \tilde{G}_n(k) = 0 \) follows from the symmetric property of \( \mathcal{B}_m^{(-\ell)}(k) \). For an even integer \( n \geq 2 \), by the formula (6) we have

\[
(-1)^{n/2}k! \left( k(k+1)\tilde{G}_n(k+1) - k^2\tilde{G}_n(k) - \tilde{G}_{n+2}(k) \right)
\]

\[
= k \sum_{j=0}^{n/2} j!(k+j+1)! \sum_{\ell=j}^{n-j} (-1)^\ell \left\{ \begin{array}{c} n - \ell + 1 \\ j + 1 \end{array} \right\} \left\{ \begin{array}{c} \ell + 1 \\ j + 1 \end{array} \right\} \tag{7}
\]

\[
- k^2 \sum_{j=0}^{n/2} j!(k+j)! \sum_{\ell=j}^{n-j} (-1)^\ell \left\{ \begin{array}{c} n - \ell + 1 \\ j + 1 \end{array} \right\} \left\{ \begin{array}{c} \ell + 1 \\ j + 1 \end{array} \right\} \tag{8}
\]

\[
+ \sum_{j=0}^{n/2+1} j!(k+j)! \sum_{\ell=j}^{n+2-j} (-1)^\ell \left\{ \begin{array}{c} n - \ell + 3 \\ j + 1 \end{array} \right\} \left\{ \begin{array}{c} \ell + 1 \\ j + 1 \end{array} \right\}. \tag{9}
\]

Since \( \left\{ \begin{array}{c} k \\ 1 \end{array} \right\} = 1 \) holds for any \( k \geq 1 \), we can split the third line (9) according as \( j = 0 \) or not, which equals

\[
k! + \sum_{j=0}^{n/2} (j+1)! (k+j+1)! \sum_{\ell=j+1}^{n+1-j} (-1)^\ell \left\{ \begin{array}{c} n + 3 - \ell \\ j + 2 \end{array} \right\} \left\{ \begin{array}{c} \ell + 1 \\ j + 2 \end{array} \right\}. \tag{10}
\]

Let

\[
a_{n,j} := \sum_{\ell=j}^{n-j} (-1)^\ell \left\{ \begin{array}{c} n - \ell + 1 \\ j + 1 \end{array} \right\} \left\{ \begin{array}{c} \ell + 1 \\ j + 1 \end{array} \right\}. \tag{11}
\]

Then the total of (7), (8), and (9) equals

\[
k! + \sum_{j=0}^{n/2} j!(k+j)! \left( k(j+1)a_{n,j} + (j+1)(k+j+1)a_{n+2,j+1} \right). \tag{11}
\]

Once the sum is 0, the proof completes. By using the generating function given in [1, Proposition 2.6 (8)], we have

\[
\frac{t^j}{(1-t)(1-2t) \cdots (1-(j+1)t)} = \sum_{\ell \geq j} \left\{ \begin{array}{c} \ell + 1 \\ j + 1 \end{array} \right\} t^\ell = \sum_{\ell \leq n-j} \left\{ \begin{array}{c} n - \ell + 1 \\ j + 1 \end{array} \right\} t^{n-\ell}
\]

for any non-negative integers \( n, j \in \mathbb{Z}_{\geq 0} \). Multiplying these two expressions, we obtain

\[
\frac{s^j t^j}{(1-s)(1-t) \cdots (1-(j+1)s)(1-(j+1)t)} = \sum_{\ell \geq j} \sum_{k \leq n-j} \left\{ \begin{array}{c} \ell + 1 \\ j + 1 \end{array} \right\} \left\{ \begin{array}{c} n - k + 1 \\ j + 1 \end{array} \right\} s^\ell t^{n-k}.
\]
By specializing at $s = -x, t = x$,

$$
\frac{(-1)^j x^{2j}}{(1 - x^2) \cdots (1 - (j + 1)^2 x^2)} = \sum_{\ell \geq j} \sum_{k \leq n-j} (-1)^\ell \binom{\ell + 1}{j + 1} \binom{n - k + 1}{j + 1} x^{n + \ell - k}.
$$

(12)

Thus, we see that the number $a_{n,j}$ defined in (10) appears as the $n$-th coefficient of (12). By the expression of the left-hand side of (12), we easily see that $a_{n,j} = 0$ when $n$ is an odd integer or $2j > n$. Further, we get the initial values $a_{2j,j} = (-1)^j, a_{n,0} = 1$ for even $n$, and the recurrence relation

$$
a_{n+2,j} = (j + 1)^2 a_{n,j} - a_{n,j-1}.
$$

Applying this to the equation (11), we get

$$
k! + \sum_{j=0}^{n/2} j!(k + j)!(k + j + 1)(k + j + 2)(j + 2)! a_{n,j+1} - a_{n,j})
\]

$$
= k! + \sum_{j=0}^{n/2} (j + 2)(j + 2)! a_{n,j+1} - \sum_{j=0}^{n/2} (j + 1)(j + 1)! a_{n,j}.
$$

Since $a_{n,n/2+1} = 0$ and $a_{n,0} = 1$, this equals 0, which concludes the proof of Theorem 1.

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