Parameter estimation for Gaussian processes with application to the model with two independent fractional Brownian motions

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Abstract The purpose of the article is twofold. Firstly, we review some recent results on the maximum likelihood estimation in the regression model of the form $X_t = \theta G(t) + B_t$, where $B$ is a Gaussian process, $G(t)$ is a known function, and $\theta$ is an unknown drift parameter. The estimation techniques for the cases of discrete-time and continuous-time observations are presented. As examples, models with fractional Brownian motion, mixed fractional Brownian motion, and sub-fractional Brownian motion are considered. Secondly, we study in detail the model with two independent fractional Brownian motions and apply the general results mentioned above to this model.

Key words: discrete observations, continuous observations, maximum likelihood estimator, strong consistency, fractional Brownian motion, Fredholm integral equation of the first kind
1 Introduction

Gaussian processes with drift arise in many applied areas, in particular, in telecommunication and on financial markets. An observed process often can be decomposed as the sum of a useful signal and a random noise, where the last one mentioned is usually modeled by a centered Gaussian process, see, e.g., [19, Ch. VII].

The simplest example of such model is the process

\[ Y_t = \theta t + W_t, \]

where \( W \) is a Wiener process. In this case the MLE of the drift parameter \( \theta \) by observations of \( Y \) at points \( 0 = t_0 \leq t_1 \leq \cdots \leq t_N = T \) is given by

\[
\hat{\theta} = \frac{1}{t_N - t_0} \sum_{i=0}^{N-1} (Y_{t_{i+1}} - Y_{t_i}) = \frac{Y_T - Y_0}{T},
\]

and depends on the observations at two points, see e.g. [5]. Models of such type are widely used in finance. For example, Samuelson’s model [42] with constant drift parameter \( \mu \) and known volatility \( \sigma \) has the form

\[
\log S_t = \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t,
\]

and the MLE of \( \mu \) equals

\[
\hat{\mu} = \frac{\log S_T - \log S_0}{T} + \frac{\sigma^2}{2}.
\]

At the same time, the model with Wiener process is not suitable for many processes in natural sciences, computer networks, financial markets, etc., that have long- or short-term dependencies, i.e., the correlations of random noise in these processes decrease slowly with time (long-term dependence) or rapidly with time (short-term dependence). In particular, the models of financial markets demonstrate various kinds of memory (short or long). However, a Wiener process has independent increments, and, therefore, the random noise generated by it is “white”, i.e., uncorrelated. The most simple way to overcome this limitation is to use fractional Brownian motion. In some cases even more complicated models are needed. For example, the noise can be modeled by mixed fractional Brownian motion [9], or by the sum of two fractional Brownian motions [28]. Moreover, recently Gaussian processes with non-stationary increments have become popular such as sub-fractional [6], bifractional [14] and multifractional [2, 36, 40] Brownian motions.

In this paper we study rather general model where the noise is represented by a centered Gaussian process \( B = \{ B_t, t \geq 0 \} \) with known covariance function, \( B_0 = 0 \). We assume that all finite-dimensional distributions of the process \( \{ B_t, t > 0 \} \) are multivariate normal distributions with nonsingular covariance matrices. We observe the process \( X_t \) with a drift \( \theta G(t) \), that is,
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\[ X_t = \theta G(t) + B_t, \quad (1) \]

where \( G(t) = \int_{0}^{t} g(s) \, ds \), and \( g \in L_1[0,t] \) for any \( t > 0 \).

The paper is devoted to the estimation of the parameter \( \theta \) by observations of the process \( X \). We consider the MLEs for discrete and continuous schemes of observations. The results presented are based on the recent papers \([33, 32]\). Note that in \([32]\) the model (1) with \( G(t) = t \) was considered, and the driving process \( B \) was a process with stationary increments. Then in \([33]\) these results were extended to the case of non-linear drift and more general class of driving processes. In the present paper we apply the theoretical results mentioned above to the models with fractional Brownian motion, mixed fractional Brownian motion and sub-fractional Brownian motion.

Similar problems for the model with linear drift driven by fractional Brownian motion were studied in \([5, 16, 26, 35]\). The mixed Brownian — fractional Brownian model was treated in \([7]\). In \([4, 39]\) the nonparametric functional estimation of the drift of a Gaussian processes was considered (such estimators for fractional and subfractional Brownian motions were studied in \([13] \) and \([43]\) respectively).

In the present paper special attention is given to the model of the form

\[ X_t = \theta t + B_{H1}^t + B_{H2}^t \]

with two independent fractional Brownian motions \( B_{H1} \) and \( B_{H2} \). This model was first studied in \([28]\), where a strongly consistent estimator for the unknown drift parameter \( \theta \) was constructed for \( 1/2 < H_1 < H_2 < 1 \) and \( H_2 - H_1 > 1/4 \) by continuous-time observations of \( X \). Later, in \([34]\), the strong consistency of this estimator was proved for arbitrary \( 1/2 < H_1 < H_2 < 1 \). The details on this approach are given in Remark 2 below. However, the problem of drift parameter estimation by discrete observations in this model was still open. Applying our technique, we obtain the discrete-time estimator of \( \theta \) and prove its strong consistency for any \( H_1, H_2 \in (0,1) \). Moreover, we also construct the continuous-time estimator and prove the convergence of the discrete-time estimator to the continuous-time one in the case where \( H_1 \in (1/2,3/4] \) and \( H_2 \in (H_1,1) \).

It is worth mentioning that the drift parameter estimation is developed for more general models involving fBm. In particular, the fractional Ornstein–Uhlenbeck process is a popular and well-studied model with fBm. The MLE of the drift parameter for this process was constructed in \([20]\) and further investigated in \([3, 44, 47]\). Several non-standard estimators for the drift parameter of an ergodic fractional Ornstein–Uhlenbeck process were proposed in \([15]\) and studied in \([17]\). The corresponding non-ergodic case was treated in \([3, 10, 45]\). In the papers \([8, 11, 12, 13, 48, 49]\) drift parameter estimators were constructed via discrete observations. More general fractional diffusion models were studied in \([21, 27]\) for continuous-time estimators and in \([23, 29, 31]\) for the case of discrete observations. An estimator of the volatility parameter was constructed in \([25]\). For Hurst index estimators see, e.g., \([22]\) and references cited therein. Mixed diffusion model including fractional Brownian motion and Wiener process was investigated in \([21]\). We refer to the paper \([30]\) for a survey of the results on parameter estimation in fractional and mixed diffusion models and to the books \([24, 38]\) for a comprehensive study of this topic.
The paper is organized as follows. In Section 2 we construct the MLE by discrete-time observations and formulate the conditions for its strong consistency. In Section 3 we consider the estimator constructed by continuous-time observations and the relations between discrete-time and continuous-time estimators. In Section 4 these results are applied to various models mentioned above. In particular, the new approach to parameter estimation in the model with two independent fractional Brownian motions is presented in Subsection 4.5. Auxiliary results are proved in the appendices.

2 Construction of drift parameter estimator for discrete-time observations

Let the process $X$ be observed at the points $0 < t_1 < t_2 < \ldots < t_N$. Then the vector of increments
\[
\Delta X^{(N)} = (X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_N} - X_{t_{N-1}})^\top
\]
is a one-to-one function of the observations. We assume in this section that the inequality $G(t_k) \neq 0$ holds at least for one $k$.

Evidently, vector $\Delta X^{(N)}$ has Gaussian distribution $\mathcal{N}(\theta \Delta G^{(N)}, \Gamma^{(N)})$, where
\[
\Delta G^{(N)} = (G(t_1), G(t_2) - G(t_1), \ldots, G(t_N) - G(t_{N-1}))^\top.
\]

Let $\Gamma^{(N)}$ be the covariance matrix of the vector
\[
\Delta B^{(N)} = (B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_N} - B_{t_{N-1}})^\top.
\]

The density of the distribution of $\Delta X^{(N)}$ w.r.t. the Lebesgue measure is
\[
\text{pdf}_{\Delta X^{(N)}}(x) = \frac{(2\pi)^{-N/2}}{\det \Gamma^{(N)}} \exp \left\{ -\frac{1}{2} \begin{pmatrix} x - \theta \Delta G^{(N)} \end{pmatrix}^\top \left( \Gamma^{(N)} \right)^{-1} \begin{pmatrix} x - \theta \Delta G^{(N)} \end{pmatrix} \right\}.
\]

Then one can take the density of the distribution of the vector $\Delta X^{(N)}$ for a given $\theta$ w.r.t. the density for $\theta = 0$ as a likelihood function:
\[
L^{(N)}(\theta) = \exp \left\{ \theta (\Delta G^{(N)})^\top (\Gamma^{(N)})^{-1} \Delta X^{(N)} - \frac{\theta^2}{2} (\Delta G^{(N)})^\top (\Gamma^{(N)})^{-1} \Delta G^{(N)} \right\}.
\]

The corresponding MLE equals
\[
\hat{\theta}^{(N)} = \frac{(\Delta G^{(N)})^\top (\Gamma^{(N)})^{-1} \Delta X^{(N)}}{(\Delta G^{(N)})^\top (\Gamma^{(N)})^{-1} \Delta G^{(N)}}.
\]
Theorem 1 (Properties of the discrete-time MLE [33]). 1. The estimator \( \hat{\theta}^{(N)} \) is unbiased and normally distributed:

\[
\hat{\theta}^{(N)} - \theta \sim \mathcal{N}\left(0, \frac{1}{(\Delta G^{(N)})\Gamma^{(N)}\Gamma^{(N)}}\right).
\]

2. Assume that

\[
\text{var} B_t = \frac{\text{var}(B_t)}{G^2(t)} \to 0, \quad \text{as } t \to \infty. \tag{4}
\]

If \( t_N \to \infty \), as \( N \to \infty \), then the discrete-time MLE \( \hat{\theta}^{(N)} \) converges to \( \theta \) as \( N \to \infty \) almost surely and in \( L_2(\Omega) \).

3 Construction of drift parameter estimator for continuous-time observations

In this section we suppose that the process \( X_t \) is observed on the whole interval \([0, T]\). We investigate MLE for the parameter \( \theta \) based on these observations.

Let \( \langle f, g \rangle = \int_0^T f(t)g(t)dt \). Assume that the function \( G \) and the process \( B \) satisfy the following conditions.

(A) There exists a linear self-adjoint operator \( \Gamma = \Gamma_T : L_2[0, T] \to L_2[0, T] \) such that

\[
\text{cov}(X_s, X_t) = \mathbb{E} B_s B_t = \int_0^T \Gamma_T 1_{[0, t]}(u)\,du = \langle \Gamma_T 1_{[0, t]}, 1_{[0, t]} \rangle. \tag{5}
\]

(B) The drift function \( G \) is not identically zero, and in its representation \( G(t) = \int_0^T g(s)\,ds \) the function \( g \in L_2[0, T] \).

(C) There exists a function \( h_T \in L_2[0, T] \) such that \( g = \Gamma h_T \).

Note that under assumption (A) the covariance between integrals of deterministic functions \( f \in L_2[0, T] \) and \( g \in L_2[0, T] \) w. r. t. the process \( B \) equals

\[
\mathbb{E} \int_0^T f(s)\,dB_s \int_0^T g(t)\,dB_t = \langle \Gamma_T f, g \rangle.
\]

Theorem 2 (Likelihood function and continuous-time MLE [33]). Let \( T \) be fixed, assumptions (A)–(C) hold. Then one can choose

\[
L(\theta) = \exp \left\{ \theta \int_0^T h_T(s)\,dX_s - \frac{\theta^2}{2} \int_0^T g(s)h_T(s)\,ds \right\} \tag{6}
\]

as a likelihood function. The MLE equals

\[
\hat{\theta}_T = \frac{\int_0^T h_T(s)\,dX_s}{\int_0^T g(s)h_T(s)\,ds}. \tag{7}
\]
It is unbiased and normally distributed:

\[
\hat{\theta}_T - \theta \sim \mathcal{N} \left(0, \frac{1}{\int_0^T g(s)h_T(s)\,ds} \right).
\]

**Theorem 3 (Consistency of the continuous-time MLE [33])**. Assume that assumptions (A)–(C) hold for all \( T > 0 \). If, additionally,

\[
\liminf_{t \to \infty} \frac{\text{var} B_t}{G(t)^2} = 0,
\]

then the estimator \( \hat{\theta}_T \) converges to \( \theta \) as \( T \to \infty \) almost surely and in mean square.

**Theorem 4 (Relations between discrete and continuous MLEs [33])**. Let the assumptions of Theorem 2 hold. Construct the estimator \( \hat{\theta}^{(N)} \) from (3) by observations \( X_{T/\sqrt{N}}, k = 1, \ldots, N \). Then

1. the estimator \( \hat{\theta}^{(N)} \) converges to \( \hat{\theta}_T \) in mean square, as \( N \to \infty \),
2. the estimator \( \hat{\theta}^{(2n)} \) converges to \( \hat{\theta}_T \) almost surely, as \( n \to \infty \).

4 Application of estimators to models with various noises

**4.1 The model with fractional Brownian motion and linear drift**

**Definition 1.** The fractional Brownian motion \( B^H = \{B_t^H, t \geq 0\} \) with Hurst index \( H \in (0, 1) \) is a centered Gaussian process with \( B_0 = 0 \) and covariance function

\[
E B_t^H B_s^H = \frac{1}{2} \left( t^{2H} + s^{2H} - |t - s|^{2H} \right).
\]

Let \( H \in (0, 1) \) be fixed. Consider the model

\[
X_t = \theta t + B_t^H. \tag{9}
\]

where \( X \) is an observed stochastic process, \( B^H \) is an unobserved fractional Brownian motion with Hurst index \( H \), and \( \theta \) is a parameter of interest. Any finite slice of the stochastic process \( \{B_t^H, t > 0\} \) has a multivariate normal distribution with non-singular covariance matrix. Since \( \text{var} (B_t^H) = t^{2H} \), the random process \( B^H \) satisfies Theorem 1. Hence, we have the following result.

**Corollary 1.** Under condition \( t_N \to +\infty \) as \( N \to \infty \), the estimator \( \hat{\theta}^{(N)} \) in the model (9) is \( L_2 \)-consistent and strongly consistent.

**Remark 1.** Bertin et al. [5] considered the MLE in the model (9) in the discrete scheme of observations, where the trajectory of \( X \) was observed at the points \( t_k = \frac{k}{N^\alpha}, k = 1, 2, \ldots, N^\alpha, \alpha > 1 \). Hu et al. [16] investigated the MLE by discrete observations
at the points $t_k = kh$, $k = 1, 2, \ldots, N$. They considered even more general model of the form $X_t = \theta t + \sigma B^H_t$ with unknown $\sigma$. In both papers $L_2$-consistency and strongly consistency of the MLEs were proved. Note that in Corollary 1 both these schemes of observations are allowed, since the only condition $t_N \to \infty$ is required.

Now we consider the case of continuous-time observations and apply the results of Section 3 to the model (9). Let $H \in \left( \frac{1}{2}, 1 \right)$. Denote by $\Gamma_H$ the corresponding operator $\Gamma$ for the model (9). Then

$$
(\Gamma_H f)(t) = H(2H - 1) \int_0^T \frac{f(s)}{|t-s|^{2-2H}} ds.
$$

For the function

$$
h_T(s) = CH^{1/2-H}(T-s)^{1/2-H},
$$

$C_H = (H(2H-1)B(H+\frac{1}{2};\frac{3}{2}-H))^{-1}$, we have that

$$
\Gamma_H h_T = 1_{[0,T]},
$$

see [35]. The MLE is given by

$$
\hat{\theta}_T = \frac{T^{2H-2}}{B(\frac{3}{2}-H, \frac{3}{2}-H)} \int_0^T s^{1/2-H}(T-s)^{1/2-H} dX_s.
$$

This estimator was studied in [26, 35], see also [32, Example 3.11].

**Corollary 2.** Let $H \in \left( \frac{1}{2}, 1 \right)$. The conditions of Theorems 2, 3 and 4, are satisfied. The estimator $\hat{\theta}_T$ is $L_2$-consistent and strongly consistent. For fixed $T$, it can be approximated by discrete-sample estimator in mean-square sense.

### 4.2 Model with fractional Brownian motion and power drift

Now we generalize the model (9) for the case of the non-linear drift function $G(t) = t^{\alpha+1}$. Let $0 < H < 1$ and $\alpha > -1$. Consider the process

$$
X_t = \theta t^{\alpha+1} + B^H_t
$$

This is a particular case of model (1), with $g(t) = (\alpha + 1)t^\alpha$.

Now verify the conditions of the theorems. The condition (4) holds true if and only if $\alpha > H - 1$.

**Corollary 3.** If $\alpha > H - 1$, the model (12) satisfies the conditions of Theorem 1. The estimator $\hat{\theta}^{(N)}$ in the model (12) is $L_2$-consistent and strongly consistent (provided that $\lim_{N \to \infty} t_N = +\infty$).

The condition $(B)$: $g \in L_2[0,T]$ holds true if and only if $\alpha > -\frac{1}{2}$. The integral equation $\Gamma h = g$ is rewritten as

$$
\int_0^T \frac{f(s)}{|t-s|^{2-2H}} ds.
$$
\[
\int_0^T \frac{h(s)\,ds}{|t-s|^{2H-2}} = \frac{\alpha + 1}{(2H-1)\Gamma(\alpha)}.
\]

If \( \alpha > 2H - 2 \), then the solution is

\[
h(t) = \text{const} \cdot \left( \frac{T^\alpha}{r^{\alpha+1-2H}} - \alpha^{\alpha+1-2H} W \left( \frac{T^\alpha}{r}, \alpha, H - \frac{1}{2} \right) \right),
\]

where \( W \left( \frac{T^\alpha}{r}, \alpha, H - \frac{1}{2} \right) = \int_0^{T^\alpha/r} (v + 1)^{-\alpha-1} v^{\frac{1}{2} - H} \,dv \). The asymptotic behaviour of the function \( W \left( \frac{T^\alpha}{r}, \alpha, H - \frac{1}{2} \right) \) as \( t \to 0^+ \) is

\[
W \left( \frac{T^\alpha}{r}, \alpha, H - \frac{1}{2} \right) \sim \begin{cases} 
B \left( \frac{3}{2} - H, H - \frac{1}{2} - \alpha \right) & \text{if } \alpha < H - \frac{1}{2}, \\
\ln(T/t) & \text{if } \alpha = H - \frac{1}{2}, \\
\frac{\Gamma(\alpha+1)}{\alpha(\alpha+1-2H)} & \text{if } \alpha > H - \frac{1}{2}.
\end{cases}
\]

Therefore, the function \( h(t) \) defined in (13) is square integrable if \( \alpha + 1 - 2H - \max\left(0, \alpha - H + \frac{1}{2}\right) > -\frac{1}{2} \), which holds if \( \alpha > 2H - \frac{3}{2} \). Note that if \( \alpha > 2H - \frac{3}{2} \), then the following inequalities hold true: \( \alpha > 2H - 2 \) (whence \( h \) defined in (13) is indeed a solution to the integral equation \( \Gamma h = g \), \( \alpha > H - 1 \) (whence conditions (3) and so (8) are satisfied), and \( \alpha > \frac{1}{2} \) (whence condition (B) is satisfied).

**Corollary 4.** If \( \alpha > 2H - \frac{3}{2} \), the conditions of Theorems 2, 3 and 4 are satisfied. The estimator \( \hat{\theta}_T \) is \( L_2 \)-consistent and strongly consistent. For fixed \( T \), it can be approximated by discrete-sample estimator in mean-square sense.

### 4.3 The model with Brownian and fractional Brownian motion

Consider the following model:

\[
X_t = \theta t + W_t + B^H_t,
\]

where \( W \) is a standard Wiener process, \( B^H \) is a fractional Brownian motion with Hurst index \( H \), and random processes \( W \) and \( B^H \) are independent. The corresponding operator \( \Gamma \) is \( \Gamma = I + \Gamma_H \), where \( \Gamma_H \) is defined by (10). The operator \( \Gamma_H \) is self-adjoint and positive semi-definite. Hence, the operator \( \Gamma \) is invertible. Thus Assumption (C) holds true.

In other words, the problem is reduced to the solving of the following Fredholm integral equation of the second kind

\[
h_T(u) + H_2(2H_2 - 1) \int_0^T h_T(s)|s-u|^{2H_2-2} \,ds = 1, \quad u \in [0,T].
\]
This approach to the drift parameter estimation in the model with mixed fractional Brownian motion was first developed in [7].

Note also that the function \( h_T = \Gamma_T^{-1}|_{[0,T]} \) can be evaluated iteratively

\[
h_T = \sum_{k=0}^{\infty} \left( \frac{1}{2} \left\| \Gamma_T^H \right\| \left( 1 - \Gamma_T^H \right)^k \right) \frac{1}{k+1},
\]

(16)

4.4 Model with subfractional Brownian motion

Definition 2. The subfractional Brownian motion \( \tilde{B}^H = \{ \tilde{B}^H_t, t \geq 0 \} \) with Hurst parameter \( H \in (0,1) \) is a centered Gaussian random process with covariance function

\[
\text{cov}(\tilde{B}^H_s, \tilde{B}^H_t) = \frac{2 |s|^{2H} + 2 |s|^{2H} - |t-s|^{2H} - |t+s|^{2H}}{2}.
\]

(17)

We refer [6, 46] for properties of this process. Obviously, neither \( \tilde{B}^H \), nor its increments are stationary. If \( \{ B^H_t, t \in \mathbb{R} \} \) is a fractional Brownian motion, then the random process \( \frac{B^H + \tilde{B}^H}{\sqrt{2}} \) is a subfractional Brownian motion. Evidently, mixed derivative of the covariance function \( \tilde{K}^H \) equals

\[
\tilde{K}^H(s,t) := \frac{\partial^2 \text{cov}(\tilde{B}^H_s, \tilde{B}^H_t)}{\partial t \partial s} = H(2H-1) \left( |t-s|^{2H-2} - |t+s|^{2H-2} \right).
\]

(18)

If \( H \in \left( \frac{1}{2}, 1 \right) \), then the operator \( \Gamma = \tilde{\Gamma}^H \) that satisfies \( \tilde{K}^H \) for \( \tilde{B}^H \) equals

\[
\tilde{\Gamma}^H f(t) = \int_{0}^{T} K^H(s,t) f(s) \, ds.
\]

(19)

Consider the model \( (1) \) for \( G(t) = t \) and \( B = \tilde{B}^H \):

\[
X_t = \theta t + \tilde{B}^H_t.
\]

(20)

Let us construct the estimators \( \hat{\theta}^{(N)} \) and \( \hat{\theta}_T \) from [3] and [7] respectively and establish their properties. In particular, Proposition [1] allows to define finite-sample estimator \( \hat{\theta}^{(N)} \).

Proposition 1. The linear equation \( \tilde{\Gamma}^H f = 0 \) has only trivial solution in \( L_2[0,T] \). As a consequence, the finite slice \( \left( \tilde{B}^H_{t_1}, \ldots, \tilde{B}^H_{t_N} \right) \) with \( 0 < t_1 < \ldots < t_N \) has a multivariate normal distribution with nonsingular covariance matrix.

Since \( \text{var} \left( \tilde{B}^H_t \right) = (2 - 2^{2H-1}) t^{2H-2} \), the random process \( \tilde{B}^H \) satisfies Theorem [1]. Hence, we have the following result.
Corollary 5. Under condition \( t_N \to +\infty \) as \( N \to \infty \), the estimator \( \hat{\theta}^{(N)} \) in the model (20) is \( L^2 \)-consistent and strongly consistent.

In order to define the continuous-time MLE (7), we have to solve an integral equation. The following statement guarantees the existence of the solution.

Proposition 2. If \( \frac{1}{2} < H < \frac{3}{4} \), then the integral equation \( \tilde{\Gamma}_H h = 1_{[0,T]} \), that is
\[
\int_0^T K_H(s,t)h(s)\,ds = 1 \quad \text{for almost all } t \in (0,T)
\] (21)
has a unique solution \( h \in L^2[0,T] \).

Corollary 6. If \( \frac{1}{2} < H < \frac{3}{4} \), then the random process \( \tilde{B}^H \) satisfies Theorems 2, 3, and 4. As the result, \( L(\theta) \) defined in (6) is the likelihood function in the model (20), and \( \hat{\theta}_T \) defined in (7) is the MLE. The estimator is \( L^2 \)-consistent and strongly consistent.

For fixed \( T \), it can be approximated by discrete-sample estimator in mean-square sense.

4.5 The model with two independent fractional Brownian motions

Consider the following model:
\[
X_t = \theta t + B_{t_1}^{H_1} + B_{t_2}^{H_2}, \quad (22)
\]
where \( B_{t_1}^{H_1} \) and \( B_{t_2}^{H_2} \) are two independent fractional Brownian motion with Hurst indices \( H_1, H_2 \in (\frac{1}{2}, 1) \). Obviously, the condition (4) is satisfied:
\[
\frac{\text{var} \left( B_{t_1}^{H_1} + B_{t_2}^{H_2} \right)}{t^2} = \frac{t^{2H_1} + t^{2H_2}}{t^2} \to 0, \quad t \to \infty.
\]

Theorem 5. Under condition \( t_N \to +\infty \) as \( N \to \infty \), the estimator \( \hat{\theta}^{(N)} \) in the model (22) is \( L^2 \)-consistent and strongly consistent.

Evidently, the corresponding operator \( \Gamma \) for the model (22) equals \( \Gamma_{H_1} + \Gamma_{H_2} \), where \( \Gamma_{H_i} \) is defined by (10). Therefore, in order to verify the assumptions of Theorem 4 we need to show that there exists a function \( h_T \) such that \( (\Gamma_{H_1} + \Gamma_{H_2})h_T = 1_{[0,T]} \). This is equivalent to \( (I + \Gamma_{H_1}^{-1} \Gamma_{H_2})h_T = \Gamma_{H_1}^{-1} 1_{[0,T]} \), since the operator \( \Gamma_{H_1} \) is injective and its range contains \( 1_{[0,T]} \), see (11) and Theorem 7. Hence, it suffices to prove that the operator \( I + \Gamma_{H_1}^{-1} \Gamma_{H_2} \) is invertible. This is done in Theorem 8 in Appendix 8 for \( H_1 \in (1/2, 3/4), H_2 \in (H_1, 1) \). Thus, in this case the assumptions of Theorem 4 hold with
\[
h_T = (I + \Gamma_{H_1}^{-1} \Gamma_{H_2})^{-1} \Gamma_{H_1}^{-1} 1_{[0,T]}.
\]
Therefore, we have the following result for the estimator

\[ \hat{\theta}_T = \frac{\int_0^T h_T(s) \, dX_s}{\int_0^T h_T(s) \, ds}. \]

**Theorem 6.** If \( H_1 \in (1/2, 3/4) \) and \( H_2 \in (H_1, 1) \), then the random process \( B^{H_1} + B^{H_2} \) satisfies Theorems 2, 3, and 4. As the result, \( L(\theta) \) defined in (6) is the likelihood function in the model (22), and \( \hat{\theta}_T \) is the maximum likelihood estimator. The estimator is \( L_2 \)-consistent and strongly consistent. For fixed \( T \), it can be approximated by discrete-sample estimator in mean-square sense.

**Remark 2.** Another approach to the drift parameter estimation in the model with two fractional Brownian motions was proposed in [28] and developed in [34]. It is based on the solving of the following Fredholm integral equation of the second kind

\[ (2 - 2H_1) \tilde{h}_T(u) u^{1-2H_1} + \int_0^T h_T(s) k(s, u) \, ds = (2 - 2H_1) u^{1-2H_1}, \quad u \in (0, T], \quad (23) \]

where

\[ k(s, u) = \int_0^{s/2} \frac{\partial_s K_{H_1, H_2}(s, v) \partial_u K_{H_1, H_2}(u, v)}{v} \, dv, \]

\[ K_{H_1, H_2}(t, s) = c_{H_1} \beta_{H_2} s^{3/2-H_2} \int_s^t (t-u)^{1/2-H_1} u^{H_1-H_2} (u-s)^{H_2-3/2} \, du, \]

\[ c_{H_1} = \left( \frac{\Gamma(3 - 2H_1)}{2H_1 \Gamma(\frac{3}{2} - H_1)^3 \Gamma(\frac{1}{2} + H_1)} \right)^{1/2}, \quad \beta_{H_2} = \left( \frac{2H_2 (H_2 - \frac{1}{2}) \Gamma(3 - 2H_2)}{\Gamma(2 - 2H_2)} \right)^{1/2}. \]

Then for \( 1/2 < H_1 < H_2 < 1 \) the estimator is defined as

\[ \hat{\theta}(T) = \frac{N(T)}{\delta_{H_1} \langle N \rangle(T)}, \]

where \( \delta_{H_1} = c_{H_1} B \left( \frac{3}{2} - H_1, \frac{3}{2} - H_1 \right) \), \( N(t) \) is a square integrable Gaussian martingale,

\[ N(T) = \int_0^T \tilde{h}_T(t) \, dX(t), \]

\( \tilde{h}_T(t) \) is a unique solution to (23) and

\[ \langle N \rangle(T) = (2 - 2H_1) \int_0^T \tilde{h}_T(t) t^{1-2H_1} \, dt. \]

This estimator is also unbiased, normal and strongly consistent. The details of this method can be found also in [24, Sec. 5.5].
5 Integral equation with power kernel

**Theorem 7.** Let $0 < p < 1$ and $b > 0$.

1. If $y \in L_1[0, b]$ is a solution to integral equation

$$
\int_0^b \frac{y(s) \, ds}{|t-s|^p} = f(t) \quad \text{for almost all } t \in (0, b),
$$

(24)

then $y(x)$ satisfies

$$
y(x) = \frac{\Gamma(p) \cos \frac{\pi p}{2}}{\pi x^{(1-p)/2}} \hat{y}_{b-}^{(1-p)/2} \left( x^{1-p} \hat{y}_{0+}^{(1-p)/2} \left( \frac{f(x)}{x^{(1-p)/2}} \right) \right)
$$

(25)

almost everywhere on $[0, b]$, where $\hat{y}_{a+}^\alpha$ and $\hat{y}_{b-}^\alpha$ are the Riemann–Liouville fractional derivatives, that is

$$
\hat{y}_{a+}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left( \int_a^x f(t) \left( \frac{x-t}{x-a} \right)^\alpha \, dt \right),
$$

(27)

$$
\hat{y}_{b-}^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left( \int_x^b f(t) \left( \frac{x-t}{b-x} \right)^\alpha \, dt \right).
$$

2. If $y_1 \in L_1[0, b]$ and $y_2 \in L_1[0, b]$ are two solutions to integral equation (24), then $y_1(x) = y_2(x)$ almost everywhere on $[0, b]$.

3. If $y \in L_1[0, b]$ satisfies (25) almost everywhere on $[0, b]$ and the fractional derivatives are solutions to respective Abel integral equations, that is

$$
\frac{1}{\Gamma \left( \frac{1-p}{2} \right)} \int_0^t \frac{\hat{y}_{0+}^{(1-p)/2} \left( f(x) x^{p-1/2} \right)}{(t-x)^{p+1/2}} \, dx = \frac{f(t)}{t^{(1-p)/2}},
$$

(26)

for almost all $t \in (0, b)$ and

$$
\frac{1}{\Gamma \left( \frac{1-p}{2} \right)} \int_x^b \frac{\pi y(s) s^{(1-p)/2}}{\Gamma(p) \cos \frac{\pi p}{2}} \frac{ds}{(s-x)^{p+1/2}} = x^{1-p} \hat{y}_{0+}^{(1-p)/2} \left( \frac{f(x)}{x^{(1-p)/2}} \right),
$$

(27)

for almost all $x \in (0, b)$, then $y(x)$ is a solution to integral equation (24).

**Proof.** Firstly, transform the left-hand side of (24). By [35] Lemma 2.2(i), for $0 < s < t$

$$
\int_0^s \left( \frac{\pi}{(t-\tau)^{p+1/2}(s-\tau)^{p+1/2}} \tau^{1-p} \right) \, d\tau = \frac{B \left( p, \frac{1-p}{2} \right)}{s^{1-p/2} t^{1-p/2} (t-s)^p}.
$$

Hence, for $s > 0, t > 0, s \neq t$
\[
\int_0^{\min(s,t)} \frac{d \tau}{(t-\tau)^{(p+1)/2}(s-\tau)^{(p+1)/2}r^{1-p}} = \frac{B\left(p, \frac{1-p}{2}\right)}{s^{(1-p)/2}t^{(1-p)/2}r^{1-p}}
\]

Hence
\[
\int_0^b \frac{y(s)ds}{|t-s|^p} = \int_0^b \frac{d \tau}{B\left(p, \frac{1-p}{2}\right)} \int_0^{\min(s,t)} \frac{d \tau}{(t-\tau)^{(p+1)/2}(s-\tau)^{(p+1)/2}r^{1-p}} \frac{1}{B\left(p, \frac{1-p}{2}\right)} \int_0^t \left(\frac{1}{x^{1-p}} I_{0+}^{(1-p)/2}(x^{1-p}/2) y(s)\right) ds.
\]

Change the order of integration, noting that \(\{(s, \tau) : 0 < s < b, 0 < \tau < \min(s,t)\} = \{(s, \tau) : 0 < \tau < t, \tau < s < b\} \) for \(0 < t < b\):
\[
\int_0^b \frac{y(s)ds}{|t-s|^p} = \frac{1}{B\left(p, \frac{1-p}{2}\right)} \int_0^t \frac{1}{\tau^{(p+1)/2}} \int_\tau^b \frac{1}{x^{1-p}} I_{0+}^{(1-p)/2}(x^{1-p}/2) y(s) ds \int_\tau^b \frac{1}{x^{1-p}} I_{0-}^{(1-p)/2}(x^{1-p}/2) y(s) \frac{d \tau}{\tau^{1-p}}.
\]

The right-hand side of (28) can be rewritten with fractional integration:
\[
\int_0^b \frac{y(s)ds}{|x-s|^p} = \frac{\Gamma\left(\frac{1-p}{2}\right)^2}{B\left(p, \frac{1-p}{2}\right)} x^{(1-p)/2} I_{0+}^{(1-p)/2} \frac{1}{x^{1-p}} I_{0-}^{(1-p)/2}(x^{1-p}/2) y(x)
\]

for \(0 < x < b\), where \(I_{0+}\) and \(I_{0-}\) are fractional integrals
\[
I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty \frac{f(t)}{(x-t)^{(1-\alpha)}} dt
\]

and
\[
I_{0-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{(1-\alpha)}} dt
\]

The constant coefficient can be simplified:
\[
\frac{\Gamma\left(\frac{1-p}{2}\right)^2}{B\left(p, \frac{1-p}{2}\right)} = \frac{\Gamma\left(\frac{1-p}{2}\right) \Gamma\left(\frac{p+1}{2}\right)}{\Gamma(p)} = \frac{\pi}{\Gamma(p) \cos\left(\frac{\pi}{2} p\right)}.
\]

Thus integral equation (24) can be rewritten with use of fractional integrals:
\[
\frac{\pi}{\Gamma(p) \cos\left(\frac{\pi}{2} p\right)} x^{(1-p)/2} I_{0+}^{(1-p)/2} \frac{1}{x^{1-p}} I_{0-}^{(1-p)/2}(x^{1-p}/2) y(x) = f(x)
\]

for almost all \(x \in (0, b)\).

Whenever \(y \in L_1[0, b]\), the function \(x^{(1-p)/2} y(x)\) is obviously integrable on \([0, b]\). Now prove that the function \(x^{p-1} I_{0-}^{(1-p)/2}(x^{1-p}/2) y(x)\) is also integrable on \([0, b]\).
Indeed, 
\[
\left| \int \frac{I_{b-}^{(1-p)/2}(x^{(1-p)/2}y(x))}{x^{1-p}} \, dx \right| \leq \frac{1}{\Gamma \left( \frac{1-p}{2} \right)} \int_x^b \frac{t^{(1-p)/2}|y(t)|}{(t-x)^{(p+1)/2}} \, dt,
\]
\[
\int_0^b \left| \frac{I_{b-}^{(1-p)/2}(x^{(1-p)/2}y(x))}{x^{1-p}} \right| \, dx \leq \frac{1}{\Gamma \left( \frac{1-p}{2} \right)} \int_0^b \frac{1}{x^{1-p}} \int_x^b \frac{t^{(1-p)/2}|y(t)|}{(t-x)^{(p+1)/2}} \, dx \, dt
\]
\[
= \frac{1}{\Gamma \left( \frac{1-p}{2} \right)} \int_0^b \frac{y(t)}{x^{1-p}(t-x)^{(p+1)/2}} \, dt < \infty,
\]
and the integrability is proved.

Due to [41, Theorem 2.1], the Abel integral equation \( f(x) = I_{a-}^{\alpha} \phi(x), x \in (a,b) \), may have not more that one solution \( \phi(x) \) within \( L_1[a,b] \). If the equation has such a solution, then the solution \( \phi(x) \) is equal to \( \mathcal{G}_{a-}^{\alpha} f(x) \). Similarly, the Abel integral equation \( f(x) = I_{b-}^{\alpha} \phi(x) \) may have not more that one solution \( \phi(x) \in L_1[a,b] \), and if it exists, \( \phi = \mathcal{G}_{b-}^{\alpha} f \).

Therefore, if \( y \in L_1[0,b] \) is a solution to integral equation, then is also satisfies (29), so
\[
I_{b-}^{(1-p)/2} \left( \frac{\pi}{\Gamma(p) \cos \left( \frac{\pi p}{2} \right)} x^{(1-p)/2} \right) = \frac{f(x)}{x^{(1-p)/2}}, \quad (30)
\]
\[
\frac{1}{\Gamma(p) \cos \left( \frac{\pi p}{2} \right)} \frac{\pi x^{(1-p)/2}y(x)}{\Gamma(p)} = \mathcal{G}_{b-}^{(1-p)/2} \left( \frac{f(x)}{x^{(1-p)/2}} \right),
\]
\[
\frac{\pi x^{(1-p)/2}y(x)}{\Gamma(p) \cos \left( \frac{\pi p}{2} \right)} = \mathcal{G}_{b-}^{(1-p)/2} \left( \frac{f(x)}{x^{(1-p)/2}} \right)
\]
for almost all \( x \in (0,b) \). Thus \( y(x) \) satisfies (25). Statement 1 of Theorem 7 is proved, and statement 2 follows from statement 1.

From equations (26) and (27), which can be rewritten with fractional integration operator,
\[
I_{b-}^{(1-p)/2} \mathcal{G}_{b-}^{(1-p)/2} (f(x)x^{(1-p)/2}) = \frac{f(x)}{t^{(1-p)/2}},
\]
\[
I_{b-}^{(1-p)/2} \left( \frac{\pi y(x)x^{(1-p)/2}}{\Gamma(p) \cos \left( \frac{\pi p}{2} \right)} \right) = x^{1-p} \mathcal{G}_{b-}^{(1-p)/2} \left( \frac{f(x)}{x^{(1-p)/2}} \right)
\]
(30) follows, and (30) is equivalent to (24). Thus statement 3 of Theorem 7 holds true.
Remark 3. The integral equation (24) was solved explicitly in [26, Lemma 3] under the assumption $f \in C([0, b])$. Here we solve this equation in $L_1[0, b]$ and prove the uniqueness of a solution in this space. Note also that the formula for solution in the handbook [37, formula 3.1.30] is incorrect (it is derived from the incorrect formula 3.1.32 of the same book, where an operator of differentiation is missing; this error comes from the book [50]).

6 Boundedness and invertibility of operators

This appendix is devoted to the proof of the following result, which plays the key role in the proof of the strong consistency of the MLE for the model with two independent fractional Brownian motions.

Theorem 8. Let $H_1 \in (\frac{4}{7}, \frac{3}{4}]$, $H_2 \in (H_1, 1)$, and $\Gamma_H$ be the operator defined by (10). Then $\Gamma_H^{-1} \Gamma_{H_2} : L_2[0, T] \rightarrow L_2[0, T]$ is a compact linear operator defined on the entire space $L_2[0, T]$, and the operator $I + \Gamma_H^{-1} \Gamma_{H_2}$ is invertible.

The proof consists of several steps.

6.1 Convolution operator

If $\phi \in L_1[-T, T]$, then the following convolution operator

$$L f(x) = \int_0^T \phi(t - s) f(s) \, ds$$

is a linear continuous operator $L_2[0, T] \rightarrow L_2[0, T]$, and

$$\|L\| \leq \int_{-T}^T |\phi(t)| \, dt.$$  \hspace{1cm} (32)

Moreover, $L$ is a compact operator.

The adjoint operator of the operator (31) is

$$L^* f(x) = \int_0^T \phi(s - t) f(s) \, ds.$$  \hspace{1cm} (33)

If the function $\phi$ is even, then the linear operator $L$ is self-adjoint.

Let us consider the following convolution operators.

Definition 3. For $\alpha > 0$, the Riemann–Liouville operators of fractional integration are defined as
\[ I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(s) \frac{ds}{(t-s)^{1-\alpha}}, \]
\[ I_T^- f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T f(s) \frac{ds}{(t-s)^{1-\alpha}}. \]

The operators \( I_0^\alpha \) and \( I_T^- \) are mutually adjoint. Their norm can be bounded as follows
\[
\| I_T^- \| = \| I_0^\alpha \| \leq \frac{1}{\Gamma(\alpha)} \int_0^T \frac{ds}{s^{1-\alpha}} = \frac{T^{\alpha}}{\Gamma(\alpha+1)}. \quad (33)
\]

Let \( \frac{1}{2} < H < 1 \) and \( \Gamma_H \) be the operator defined by (10). Then
\[
\Gamma_H = H \Gamma(2H) (I_{0+}^{2H-1} + I_{T-}^{2H-1}). \quad (34)
\]

The linear operators \( I_0^\alpha, I_T^- \) for \( \alpha > 0 \), and \( \Gamma_H \) for \( \frac{1}{2} < H < 1 \) are injective.

### 6.2 Semigroup property of the operator of fractional integration

**Theorem 9.** For \( \alpha > 0 \) and \( \beta > 0 \) the following equalities hold
\[
I_0^\alpha I_0^\beta = I_0^{\alpha+\beta},
I_T^- I_T^- = I_T^{\alpha+\beta}.
\]

This theorem is a particular case of [41, Theorem 2.5].

**Proposition 3.** For \( 0 < \alpha \leq \frac{1}{2} \) and \( f \in L_2[0,T] \),
\[
\langle I_{0+}^\alpha f, I_{T-}^\alpha f \rangle \geq 0.
\]

Equality is achieved if and only if
- \( f = 0 \) almost everywhere on \([0,T]\) for \( 0 < \alpha \leq \frac{1}{2} \);
- \( \int_0^T f(t) dt = 0 \) for \( \alpha = \frac{1}{2} \).

**Proof.** Since the operators \( I_{0+}^\alpha \) and \( I_{T-}^\alpha \) are mutually adjoint, by semigroup property, we have that
\[
\langle I_{0+}^\alpha f, I_{T-}^\alpha f \rangle = \langle I_{0+}^\alpha I_{0+}^\alpha f, f \rangle = \langle I_{0+}^{2\alpha} f, f \rangle,
\]
\[
\langle I_{0+}^\alpha f, I_{T-}^\alpha f \rangle = \langle f, I_{T-}I_{T-}^\alpha f \rangle = \langle f, I_{T-}^{2\alpha} f \rangle.
\]

Adding these equalities, we obtain
\[
\langle I_{0+}^\alpha f, I_{T-}^\alpha f \rangle = \frac{1}{2} \langle I_{0+}^{2\alpha} f + I_{T-}^{2\alpha} f, f \rangle. \quad (35)
\]

If \( 0 < \alpha \leq \frac{1}{2} \), then
Parameter estimation for Gaussian processes

\[
\langle I^\alpha_H f, I^\alpha_H f \rangle = \frac{1}{2H\Gamma(2H)} \langle \Gamma_H f, f \rangle = \\
= \frac{1}{2H\Gamma(2H)} E\left( \int_0^T f(t) dB_t^H \right) \geq 0.
\]

where \( H = \alpha + \frac{1}{2}, \frac{1}{2} < H < 1, \) and \( B_t^H \) is a fractional Brownian motion.

Let us consider the case \( \alpha = \frac{1}{2} \). Since

\[
I^1_{0+} f(t) + I^1_{T-} f(t) = \int_0^T f(s) ds + \int_T^T f(s) ds = \int_0^T f(s) ds,
\]

\[
I^1_{0+} f + I^1_{T-} f = \int_0^T f(s) ds 1_{[0,T]},
\]

\[
\langle I^1_{0+} f + I^1_{T-} f, f \rangle = \int_0^T f(s) ds \langle 1_{[0,T]}, f \rangle = \left( \int_0^T f(s) ds \right)^2,
\]

we see from (35) that

\[
\langle I^{1/2}_{0+} f, I^{1/2}_{T-} f \rangle = \frac{1}{2} \left( \int_0^T f(s) ds \right)^2 \geq 0. \tag{36}
\]

Conditions for the equality \( \langle I^\alpha_H f, I^\alpha_H f \rangle = 0 \) can be easily found by analyzing the proof. Indeed, if \( 0 < \alpha < \frac{1}{2} \) and \( H = \alpha + \frac{1}{2} \), then \( \Gamma_H \) is a self-adjoint positive compact operator whose eigenvalues are all positive. Then \( 2H\Gamma(2H)\langle I^\alpha_H f, I^\alpha_H f \rangle = \langle \Gamma_H f, f \rangle = \| \Gamma_H^{1/2} f \|^2 \). In this case, the equality \( \langle I^\alpha_H f, I^\alpha_H f \rangle = 0 \) holds true if and only if \( f = 0 \) almost everywhere on \( [0,T] \). If \( \alpha = \frac{1}{2} \), then the condition for the equality follows from (36).

\[\square\]

**Proposition 4.** For \( 0 < \alpha \leq \frac{1}{2} \) and \( f \in L_2[0,T] \),

\[
\| I^\alpha_{0+} f \| + \| I^\alpha_{T-} f \| \leq \sqrt{2} \| I^\alpha_{0+} f + I^\alpha_{T-} f \|. \tag{37}
\]

Consequently, for \( \frac{1}{2} < H \leq \frac{3}{4} \)

\[
\| I^{2H-1}_{0+} f \| + \| I^{2H-1}_{T-} f \| \leq \frac{\sqrt{2}}{H\Gamma(2H)} \| \Gamma_H f \|.
\]

**Proof.** Taking into account Proposition\[3\] we get

\[
(\| I^\alpha_{0+} f \| + \| I^\alpha_{T-} f \|)^2 \leq 2 \| I^\alpha_{0+} f \|^2 + 2 \| I^\alpha_{T-} f \|^2 \\
\leq 2 \| I^\alpha_{0+} f \|^2 + 4 \langle I^\alpha_{0+} f, I^\alpha_{T-} f \rangle + 2 \| I^\alpha_{T-} f \|^2 \\
= 2 \| I^\alpha_{0+} f + I^\alpha_{T-} f \|^2,
\]

whence the inequality (37) follows.

The second statement is obtained by the representation (34). \[\square\]
6.3 Transposition of operators

Lemma 1. Let $A$ be a linear continuous operator on $L^2[0,T]$, and $B$ be an injective self-adjoint compact linear operator on $L^2[0,T]$. If the linear operator $A^*B^{-1}$ is bounded, that is $\|A^*B^{-1}\| = K < \infty$, then the linear operator $B^{-1}A$ is defined on the entire space $L^2[0,T]$, bounded, and $\|B^{-1}A\| = K$.

Proof. For the self-adjoint compact linear operator $B$ one can find an orthonormal eigenbasis $\{e_1, e_2, \ldots\}$ such that

$$B\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{k=1}^{\infty} \lambda_k x_k e_k \quad \text{for} \quad \sum_{k=1}^{\infty} x_k^2 < +\infty.$$ 

Then $\lim_{k \to \infty} \lambda_k = 0$ (by compactness), but for all $k$ the inequality $\lambda_k \neq 0$ holds (by injectivity).

The inverse operator is a self-adjoint linear operator defined by the equation

$$B^{-1}\left(\sum_{k=1}^{\infty} x_k e_k\right) = \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} e_k \quad \text{for} \quad \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^2} < +\infty.$$ 

The domain of the operator $B^{-1}$ is the subset

$$B(L^2[0,T]) = \left\{ \sum_{k=1}^{\infty} x_k e_k : \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^2} < \infty \right\}$$

of the Hilbert space $L^2[0,T]$.

Let us prove that the operator $B^{-1}A$ is defined on $L^2[0,T]$. Assume the opposite, i.e., $B^{-1}A$ is undefined at some point $f \in L^2[0,T]$. This means that $Af \not\in B(L^2[0,T])$.

Decompose $Af$ into a series by the eigenfunctions of the operator $B$:

$$Af = \sum_{k=1}^{\infty} x_k e_k. \quad (38)$$

Since $Af \not\in B(L^2[0,T])$, we see that

$$\sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^2} = +\infty,$$

and for

$$s_n = \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2}$$

it holds that $\lim_{n \to \infty} s_n = +\infty$ and $s_n \geq 0$ for all $n \in \mathbb{N}$. Therefore, there exists $N \in \mathbb{N}$ such that

$$s_N > K^2 \|f\|^2. \quad (39)$$

Put
\[ g = \sum_{k=1}^{N} \frac{x_k}{\lambda_k} e_k. \]

Then

\[ \|g\|^2 = \sum_{k=1}^{N} \frac{x_k^2}{\lambda_k^2} = s_N, \quad \|g\| = \sqrt{s_k}, \quad g \in B(L_2[0,T]), \]

\[ B^{-1}g = \sum_{k=1}^{N} \frac{x_k}{\lambda_k^2} e_k, \quad \langle A^*B^{-1}g, f \rangle = \langle B^{-1}g, Af \rangle = \sum_{k=1}^{N} \frac{x_k}{\lambda_k^2} x_k = s_N. \]

By the Cauchy–Schwarz inequality,

\[ \left| \langle A^*B^{-1}g, f \rangle \right| \leq \|A^*B^{-1}g\| \|f\| \leq \|A^*B^{-1}\| \|g\| \|f\| = K \sqrt{s_N} \|f\|. \]

Hence,

\[ s_N \leq K \sqrt{s_N} \|f\|. \quad (40) \]

The inequalities (39) and (40) contradict each other. Thus, the operator \( B^{-1}A \) is defined on the entire space \( L_2[0,T] \).

Now let us prove boundedness of the operator \( B^{-1}A \) and the inequality

\[ \|B^{-1}Af\| \geq K \|f\|. \quad (41) \]

We use the same decomposition of the vector \( Af \) into the eigenvectors of \( B \) as above, see (38). Then

\[ B^{-1}Af = \sum_{k=1}^{\infty} \frac{x_k}{\lambda_k} e_k, \quad \|B^{-1}Af\|^2 = \sum_{k=1}^{\infty} \frac{x_k^2}{\lambda_k^2}, \]

\[ \lim_{n \to \infty} s_n = \|B^{-1}Af\|^2 > K^2 \|f\|^2, \]

by (41). Therefore, there exists \( N \in \mathbb{N} \) such that the inequality (39) holds. Arguing as above, we get a contradiction. Hence, \( \|B^{-1}A\| \leq K \).

It remains to prove the opposite inequality \( \|B^{-1}A\| \geq K \). The operator \( A^*B^{-1} \) is defined on the set \( B(L_2[0,T]) \). For all \( f \in B(L_2[0,T]) \) from the domain of the operator \( A^*B^{-1} \), we have

\[ \|A^*B^{-1}f\|^2 = \langle B^{-1}A A^*B^{-1} f, f \rangle \leq \|B^{-1}AA^*B^{-1}f\| \|f\| \leq \|B^{-1}A\| \|A^*B^{-1}f\| \|f\|, \]

whence

\[ \|A^*B^{-1}f\| \leq \|B^{-1}A\| \|f\|. \]

Therefore \( K = \|A^*B^{-1}\| \leq \|B^{-1}A\|. \)

\[ \square \]
6.4 The proof of boundedness and compactness

Proposition 5. Let $\frac{1}{2} < H_1 < H_2 < 1$ and $H_1 \leq \frac{3}{4}$. Then $\Gamma_{H_1}^{-1}\Gamma_{H_2}$ is a compact linear operator defined on the entire space $L_2[0,T]$.

Proof. The operator $\Gamma_{H_1}$ is an injective self-adjoint compact operator $L_2[0,T] \to L_2[0,T]$. The inverse operator $\Gamma_{H_1}^{-1}$ is densely defined on $L_2[0,T]$. By Proposition 4 the operators $\Gamma_{0+}^{-1}\Gamma_{H_1}$ and $\Gamma_{T-}^{-1}\Gamma_{H_1}$ are bounded. Therefore, by Lemma 1 the operators $\Gamma_{H_1}^{-1}\Gamma_{0+}^{-1}$ and $\Gamma_{H_1}^{-1}\Gamma_{T-}^{-1}$ are also bounded and defined on the entire space $L_2[0,T]$. By (34) and the semigroup property (Theorem 9),

$$\Gamma_{H_1}^{-1}\Gamma_{H_2} = H \Gamma (2H) \left( \Gamma_{H_1}^{-1}\Gamma_{0+}^{2H_2-1} + \Gamma_{H_1}^{-1}\Gamma_{T-}^{-1} \right),$$

where $H$ is a positive definite self-adjoint and injective operator. Hence $\Gamma_{H_1}^{-1}\Gamma_{H_2}$ is a compact linear operator defined on the entire space $L_2[0,T]$.

6.5 The proof of invertibility

Now prove that $-1$ is not an eigenvalue of the linear operator $\Gamma_{H_1}^{-1}\Gamma_{H_2}$. Indeed, if $\Gamma_{H_1}^{-1}\Gamma_{H_2} f = -f$ for some function $f \in L_2[0,T]$, then $\Gamma_{H_2} f + \Gamma_{H_1} f = 0$. Since $\Gamma_{H_2}$ and $\Gamma_{H_1}$ are positive definite self-adjoint (and injective) operators, $\Gamma_{H_2} + \Gamma_{H_1}$ is also a positive definite self-adjoint and injective operator. Hence $f = 0$ almost everywhere on $[0,T]$.

Because $-1$ is not an eigenvalue of the compact linear operator $\Gamma_{H_1}^{-1}\Gamma_{H_2}$, $-1$ is a regular point, i.e., $-1 \notin \sigma(\Gamma_{H_1}^{-1}\Gamma_{H_2})$, and the linear operator $\Gamma_{H_1}^{-1}\Gamma_{H_2} + I$ is invertible.

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References

1. Belfadli, R., Es-Sebaiy, K., Ouknine, Y.: Parameter estimation for fractional Ornstein–Uhlenbeck processes: non-ergodic case. Frontiers in Science and Engineering 1(1), 1–16 (2011)
2. Benassi, A., Cohen, S., Istas, J.: Identifying the multifractional function of a Gaussian process. Stat. Probab. Lett. 39(4), 337–345 (1998)
3. Bercu, B., Coutin, L., Savy, N.: Sharp large deviations for the fractional Ornstein–Uhlenbeck process. Teor. Veroyatn. Primen. 55(4), 732–771 (2010)
4. Berger, J., Wolpert, R.: Estimating the mean function of a Gaussian process and the Stein effect. J. Multivariate Anal. 13(3), 401–424 (1983)
5. Bertin, K., Torres, S., Tudor, C.A.: Maximum-likelihood estimators and random walks in long memory models. Statistics 45(4), 361–374 (2011)
6. Bojdecki, T., Gorostiza, L.G., Talarczyk, A.: Sub-fractional Brownian motion and its relation to occupation times. Stat. Probab. Lett. 69(4), 405–419 (2004)
7. Cai, C., Chigansky, P., Kleptsyna, M.: Mixed Gaussian processes: A filtering approach. Ann. Probab. 44(4), 3032–3075 (2016)
8. Céncar, P., Es-Sebaiy, K.: Almost sure central limit theorems for random ratios and applications to LSE for fractional Ornstein–Uhlenbeck processes. Probab. Math. Statist. 35(2), 285–300 (2015)
9. Cheridito, P.: Mixed fractional Brownian motion. Bernoulli 7(6), 913–934 (2001)
10. El Machkouri, M., Es-Sebaiy, K., Ouknine, Y.: Least squares estimator for non-ergodic Ornstein–Uhlenbeck processes driven by Gaussian processes. J. Korean Statist. Soc. 45(3), 329–341 (2016)
11. Es-Sebaiy, K.: Berry-Esseen bounds for the least squares estimator for discretely observed fractional Ornstein–Uhlenbeck processes. Stat. Probab. Lett. 83(10), 2372–2385 (2013)
12. Es-sebaiy, K., Ndiaye, D.: On drift estimation for non-ergodic fractional Ornstein–Uhlenbeck process with discrete observations. Afr. Stat. 9(1), 615–625 (2014)
13. Hu, Y., Song, J.: Parameter estimation for fractional Ornstein–Uhlenbeck processes with discrete observations. In: Malliavin calculus and stochastic analysis. A Festschrift in honor of David Nualart, pp. 427–442. New York, NY: Springer (2013)
14. Ibragimov, I.A., Rozanov, Y.A.: Gaussian random processes, Applications of Mathematics, vol. 9. Springer-Verlag, New York-Berlin (1978)
15. Kleptsyna, M.L., Le Breton, A.: Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Statist. Inference Stoch. Process. 5, 229–248 (2002)
16. Kubilius, K., Mishura, Y.: On drift parameter estimation in models with fractional Brownian motion. Statistics 49(1), 35–62 (2015)
17. Kubilius, K., Mishura, Y.: The rate of convergence of Hurst index estimate for the stochastic differential equation. Stochastic Processes Appl. 122(11), 3718–3739 (2012)
18. Kubilius, K., Mishura, Y., Baltrusaitis, J.: Consistency of the drift parameter estimator for the discretized fractional Ornstein–Uhlenbeck process with Hurst index $H \in (0,1/3)$. Electron. J. Stat. 9(2), 1799–1825 (2015)
19. Kubilius, K., Mishura, Y., Baltrusaitis, J.: Parameter estimation in fractional diffusion models, Bocconi & Springer Series, vol. 8. Bocconi University Press, [Milan]; Springer, Cham (2017)
20. Kukush, A., Mishura, Y., Valkie, E.: Statistical inference with fractional Brownian motion. Stat. Inference Stoch. Process. 8(1), 71–93 (2005)
21. Le Breton, A.: Filtering and parameter estimation in a simple linear system driven by a fractional Brownian motion. Statist. Probab. Lett. 38(3), 263–274 (1998)
27. Mishura, Y.: Stochastic calculus for fractional Brownian motion and related processes, vol. 1929. Springer Science & Business Media (2008)
28. Mishura, Y.: Maximum likelihood drift estimation for the mixing of two fractional Brownian motions. In: Stochastic and Infinite Dimensional Analysis, pp. 263–280. Springer (2016)
29. Mishura, Y., Ralchenko, K.: On drift parameter estimation in models with fractional Brownian motion by discrete observations. Austrian Journal of Statistics 43(3), 218–228 (2014)
30. Mishura, Y., Ralchenko, K.: Drift parameter estimation in the models involving fractional brownian motion. In: V. Panov (ed.) Modern Problems of Stochastic Analysis and Statistics: Selected Contributions In Honor of Valentin Konakov, pp. 237–268. Springer International Publishing, Cham (2017)
31. Mishura, Y., Ralchenko, K., Seleznev, O., Shevchenko, G.: Asymptotic properties of drift parameter estimator based on discrete observations of stochastic differential equation driven by fractional Brownian motion. In: Modern stochastics and applications, Springer Optim. Appl., vol. 90, pp. 303–318. Springer, Cham (2014)
32. Mishura, Y., Ralchenko, K., Shklyar, S.: Maximum likelihood drift estimation for Gaussian process with stationary increments. Austrian J. Statist. 46(3-4), 67–78 (2017)
33. Mishura, Y., Ralchenko, K., Shklyar, S.: Maximum likelihood drift estimation for Gaussian process with stationary increments. Nonlinear Anal. Model. Control 23(1), 120–140 (2018)
34. Mishura, Y., Voronov, I.: Construction of maximum likelihood estimator in the mixed fractional–fractional Brownian motion model with double long-range dependence. Mod. Stoch. Theory Appl. 2(2), 147–164 (2015)
35. Norros, I., Valkeila, E., Virtamo, J.: An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions. Bernoulli 5(4), 571–587 (1999)
36. Peltier, R.F., Lévy Véhel, J.: Multifractional Brownian motion: definition and preliminary results. INRIA research report, vol. 2645 (1995)
37. Polyanin, A., Manzhirov, A.: Handbook of integral equations, second edn. Chapman & Hall/CRC, Boca Raton, FL (2008)
38. Prakasa Rao, B.L.S.: Statistical inference for fractional diffusion processes. Chichester: John Wiley & Sons (2010)
39. Privault, N., Réveillac, A.: Stein estimation for the drift of Gaussian processes using the Malliavin calculus. Ann. Statist. 36(5), 2531–2550 (2008)
40. Ralchenko, K.V., Shevchenko, G.M.: Paths properties of multifractal Brownian motion. Theory Probab. Math. Statist. 80, 119–130 (2010)
41. Samko, S., Kilbas, A., Marichev, O.: Fractional Integrals and Derivatives. Taylor & Francis (1993)
42. Samuelson, P.A.: Rational theory of warrant pricing. Industrial Management Review 6(2), 13–32 (1965)
43. Shen, G., Yan, L.: Estimators for the drift of subfractional Brownian motion. Comm. Statist. Theory Methods 43(8), 1601–1612 (2014)
44. Tanaka, K.: Distributions of the maximum likelihood and minimum contrast estimators associated with the fractional Ornstein-Uhlenbeck process. Stat. Inference Stoch. Process 16, 173–192 (2013)
45. Tanaka, K.: Maximum likelihood estimation for the non-ergodic fractional Ornstein–Uhlenbeck process. Stat. Inference Stoch. Process. 18(3), 315–332 (2015)
46. Tudor, C.: Some properties of the sub-fractional Brownian motion. Stochastics 79(5), 431–448 (2007)
47. Tudor, C.A., Viens, F.G.: Statistical aspects of the fractional stochastic calculus. The Annals of Statistics 35(3), 1183–1212 (2007)
48. Xiao, W., Zhang, W., Xu, W.: Parameter estimation for fractional Ornstein-Uhlenbeck processes at discrete observation. Applied Mathematical Modelling 35, 4196–4207 (2011)
49. Xiao, W.L., Zhang, W.G., Zhang, X.L.: Maximum-likelihood estimators in the mixed fractional Brownian motion. Statistics 45, 73–85 (2011)
50. Zabreyko, P.P., Koshelev, A.I., Krasnosel’skii, M.A., Mikhailin, S.G., Rakovshchik, L.S., Sht’ senko, V.Y.: Integral equations: A reference text. Noordhoff, Leyden (1975)