On non-relativistic $Q\bar{Q}$ potential via Wilson loop in Galilean space-time

Haryanto M. Siahaan

Theoretical and Computational Physics Group, Department of Physics,
Faculty of Information Technology and Sciences,
Parahyangan Catholic University, Bandung 40141 - INDONESIA

Abstract

We calculate the static Wilson loop from string/gauge correspondence to obtain the $Q\bar{Q}$ potential in non-relativistic quantum field theory, i.e. CFT with Galilean symmetry. We analyze the convexity conditions for $Q\bar{Q}$ potential in this theory, and obtain restrictions for the acceptable dynamical exponent $z$.

PACS numbers: 11.25.Tq

*Electronic address: haryanto.siahaan@gmail.com, haryanto.siahaan@home.unpar.ac.id
I. INTRODUCTION

It has been shown by Maldacena that large N superconformal gauge theories have a dual description in terms of string theory in AdS space [1]. This proposal was realized by Maldacena to compute the energy between quark ($Q$) and anti-quark ($\bar{Q}$) pairs [2]. His method was to calculate expectation values of an operator similar to the Wilson loop in the large N limit of field theories. Maldacena’s idea was improved later by Rey, Theisen, and Yee [3]. It turns Wilson loop into a physical gauge invariant property that can be read from the string picture. The $Q\bar{Q}$ energy in the large N superconformal $\mathcal{N} = 4$ Yang-Mills theory can be obtained from the Wilson loop of the corresponding string in AdS space. It is proposed that quark and anti-quark pairs live on the boundary, connected by a U-shaped string in the bulk. In the discussion on this spacetime, the energy has a non-confining Coulomb-like behavior, as expected for a conformal field theory. Later this approach was applied to many other spaces and models, as summarized in ref. [4].

Recently, gravity duals for a certain Galilean-invariant conformal field theory has attracted some attention in theoretical high energy physics community [5–9]. A special case when we take the dynamical exponent $z = 2$ of this theory (whose isometry is the Schrodinger group $Sch(d−1)$) is considered to be the basis in constructing duality between gravity and unitary Fermi gas. However, our interest in this paper is the theory with an arbitrary dynamical exponent $z$, i.e. Galilean invariant CFT. In this general scheme, one can discuss the non-relativistic version of the AdS/CFT dictionary, i.e. the operator-state correspondence between the particle on the boundary and the string in the bulk. Scaling transformation in this non-relativistic conformal symmetry can be written as [8–10]

$$x^i \rightarrow \lambda x^i, t \rightarrow \lambda^z t.$$  

The asymptotic metric in this case can be written as

$$ds^2 = \frac{R^2}{r^2} \left( -\frac{dt^2}{r^{2(z−1)}} + dtd\xi + (dx^i)^2 + dr^2 \right) + ds^2_{X_5},$$

where $R$ is the characteristic radius of space-time, $\xi$ is a compact light-like coordinate, $x^i$ for $i = 1,\ldots,d$ together with $t$ are the space-time coordinates on the boundary where (2) is mapped at $r = 0$, and finally $ds^2_{X_5}$ is the metric of a suitable internal manifold geometry which allows (2) to be a solution of the supergravity equations of motion. The extra dimension $\xi$ is usually associated with quantum numbers interpreted as the particle number. However, the relation between translation in $\xi$ and its interpretation as particle number operator is still an unclear topic [11, 12]. Thus we just set this time-like extra dimension $\xi$ to be constant.
The holographic Wilson loop in non-relativistic CFT had been studied by Kluson in ref. \[11\]. He assumed general time dependence of $\xi$ and also the moving $Q\bar{Q}$ pair cases in the context of non-relativistic quantum field theory. His study was devoted to the space-time with Galilean symmetry \[20\]. Nevertheless, he still does not include analysis of convexity conditions \[12\] and \[13\] yet. One needs to verify these conditions in $Q\bar{Q}$ potential discussions to make sure that the corresponding potential function $V(L)$ is a monotone non-decreasing and convex function of the separation $L$. The goal of this paper is to verify these conditions for $Q\bar{Q}$ potential, which is obtained by calculating the Wilson loop in the string picture in Galilean space-time. Furthermore, we would like to see the restrictions which may appear for acceptable dynamical exponent $z$.

This paper is organized as follows. In section 2, we will perform calculations to acquire the $Q\bar{Q}$ potential energy in Galilean space-time. In section 3, we will derive some conditions for acceptable $z$ due to convexity inequality. Finally in section 4, there is a summary of our findings.

II. $Q\bar{Q}$ POTENTIAL IN NON-RELATIVISTIC CFT WITH GALILEAN SYMMETRY

We will start with the Nambu-Goto action
\[
S = -\frac{1}{2\pi\alpha'} \int d\tau d\sigma \sqrt{-\det G_{MN} \partial_{\alpha} x^M \partial_{\beta} x^N} \quad (3)
\]
for metric (2) where $x^M = (t, r, \xi, x^i)$, $G_{MN}$ is space-time metric in (2), and impose suitable ansatzs in describing static strings, i.e. $t = x^0 = \tau$, $r = r(\sigma)$, $x = x(\sigma)$, and $\xi = \text{constant}$. Kluson in ref. \[11\] has considered a more general case for an extra time-like dimension $\xi$ as a $\tau$-dependent variable, but we can simply set $\xi$ to be constant (for example as discusussed in ref. \[10\]) since the $Q\bar{Q}$ potential would depend on their separation distance \[21\] only. The corresponding action can be written as
\[
S = -\frac{T}{2\pi\alpha'} \int d\sigma \sqrt{f^2(r) (r')^2 + (x')^2} \quad (4)
\]
for $f(r) = R^2 r^{-(z+1)}$ and we have used $(\cdot)' \equiv \partial_{\sigma} (\cdot)$. Variable $T$ in (4) is the loop period and can be written this way due to the time translation invariance of action (3) for metric (2). We have followed a standard prescription that has been used in some literature, for example refs. \[14\]-\[18\], in obtaining the action (4) as well as the corresponding $Q\bar{Q}$ potential as a function of $Q\bar{Q}$ pair’s distance. Though the metric (2.1) is not diagonal, but action (4) leads us to a problem of Wilson loop computation which can be started by finding a geodesic in the effective 2-dimensional geometry \[18\]
\[
(d_{s_{eff}})^2 = f^2(r) \left(dx^2 + dr^2\right). \quad (5)
\]
The equation of motion (geodesic line) from (4) is

$$\frac{dx}{dr} = \pm \frac{f(r_0)}{\sqrt{f^2(r) - f^2(r_0)}}.$$  (6)

$r_0$ is the maximum position of the U-shaped string with respect to the $r$-coordinate (bulk radius, see Fig. 1). From (6) one can obtain the separation distance of quark and anti-quark on the boundary, by integrating the geodesic with respect to $r$. Since the boundary is at $r = 0$, then the separation as the function of $r_0$ can be obtained by the following integration

$$L(r_0) = 2 \int_0^{r_0} \frac{f(r_0)}{\sqrt{f^2(r) - f^2(r_0)}} dr.$$  (7)

Related to the expression for the $Q\bar{Q}$ separation above, one may provide such an illustration as depicted in Fig. 1.

Inserting $f(r) = R^2 r^{-(z+1)}$ to (7) and using the beta function in our computation give the following exact result

$$L(r_0, z) = 2 \int_0^{r_0} \frac{r^{z+1}}{\sqrt{r_0^{2z+2} - r^{2z+2}}} = \frac{2r_0\sqrt{\pi}}{\Gamma\left(\frac{z+2}{2z+2}\right)} \frac{1}{\Gamma\left(\frac{1}{2z+2}\right)}.$$  (8)

Then we follow a general prescription in refs. 4, 15, 17, 18 to compute the energy between quark and anti-quark. We have a general form of total $Q\bar{Q}$ energy as

$$E(r_0) = \frac{1}{\pi\alpha'} \int_0^{r_0} \frac{f^2(r)}{\sqrt{f^2(r) - f^2(r_0)}} dr - 2m_Q.$$  (9)
where $m_Q$ is considered as the energy of non interacting quark \[14, 15, 17, 18\]. Thus the $Q\bar{Q}$ potential can be written as

$$V_{QQ} (r_0) = E (r_0) - 2m_Q$$

$$= \frac{1}{\pi \alpha'} \int_0^{r_0} \int_0^{r_0} f^2 (r) \sqrt{f^2 (r) - f^2 (r_0)} dr$$

which can also be computed by the use of beta function. The potential is

$$V_{QQ} (r_0, z) = 2R^2 r_0^{z+1} \int_0^{r_0} \frac{dr}{r^{z+1} \left( r_0^{2z + 2} - r^{2z + 2} \right)} = \frac{2R^2 \sqrt{\pi} \Gamma \left( \frac{-z}{2z + 2} \right)}{r_0^{2z + 2} \Gamma \left( \frac{1}{2z + 2} \right)}. \quad (11)$$

In the next section we will see the compatibility of the potential \[11\] with convexity conditions.

### III. CONVEXITY CONDITIONS AND STRING EMBEDDINGS

There are some conditions that should be satisfied by any potential which describes interaction between quark and anti-quark whose name 'convexity' conditions \[13, 18\]

$$\frac{dV}{dL} > 0 \quad (12)$$

and

$$\frac{d^2 V}{dL^2} \leq 0. \quad (13)$$

Condition \[12\] means quark and anti-quark are attractive everywhere, and \[13\] tells us that the potential is a monotone non-increasing function of their separation. These conditions can be verified as follows

$$\frac{dV_{QQ} (r_0, z)}{dL (r_0, z)} = \frac{dV_{QQ} (r_0, z)}{dr_0} \frac{dr_0}{dL (r_0, z)} = \frac{-z R^2}{r_0^{2z + 2} \Gamma \left( \frac{z + 2}{2z + 2} \right)} \Gamma \left( \frac{z + 2}{2z + 2} \right) \frac{1}{r_0^{2z + 2}} \Gamma \left( \frac{z + 2}{2z + 2} \right) > 0 \quad (14)$$

and

$$\frac{d^2 V_{QQ} (r_0, z)}{dL (r_0, z)^2} = \frac{d \left( \frac{dV_{QQ} (r_0, z)}{dL (r_0, z)} \right)}{dr_0} \frac{dr_0}{dL (r_0, z)}$$

$$= \frac{z R^2}{4 \sqrt{\pi} r_0^{2z + 2}} \left( \frac{1}{r_0^{2z + 2}} \right)^2 \left( \Gamma \left( \frac{z + 2}{2z + 2} \right) \right)^2 \leq 0. \quad (15)$$
The two last equations are inequalities for physically accepted \( z \) based on convexity conditions for the \( Q \bar{Q} \) pair.

In ref. [19], the authors present simple embeddings of duals for nonrelativistic critical points, where the dynamical critical exponent can take many values \( z \neq 2 \) [22]. They find that \( z = 1 \) and \( z \geq 3/2 \) as the possible dynamical critical exponents that allow string embeddings in gauge/gravity dual picture. From their paper [19], we could learn that our \( f(r) \) would depend on the coordinates of the internal manifold \( X_5 \) [23]. Hartnoll and Yoshida write the non-compact part of the metric which can accommodate a large number of values of \( z \) by the following ansatz [24]

\[
ds^2 = \frac{R^2}{r^2} \left( -\frac{dt^2}{h^2(X_5) r^{2(z-1)}} + dt d\xi + (dr)^2 \right)
\]

(16)

which modifies our previous \( f(r) \) from \( R^2 r^{-(z+1)} \) to \( R^2 r^{-(z+1)} h(X_5)^{-1} \). Nevertheless, the function \( h(X_5) \) would not appear in (8) and (11). Thus our findings on the restrictions for \( z \) can be applied to the work of Hartnoll and Yoshida in ref. [19]. One can verify that conditions (14) and (15) are fulfilled for \( z = 1 \), and also for \( z \geq 3/2 \). The negativity of \( \Gamma \left( \frac{-z}{z+2} \right) \) for \( z \geq 1 \) guarantees both (14) and (15) are satisfied.

IV. SUMMARY

We have calculated the potential between \( Q \) and \( \bar{Q} \) in the non-relativistic quantum field theory by using the Wilson loop analysis in the gauge/gravity correspondence in the Galilean bulk. Our findings are inequalities (14) and (15) for physically acceptable dynamical exponent \( z \) from convexity conditions. Yoshida and Hartnoll [19] have found families of \( z \) for string embeddings in Galilean space-time, i.e. \( z = 1 \) and \( z \geq 3/2 \), which agree with inequalities (14) and (15) above.

Acknowledgments

I would thank LPPM-UNPAR for supporting my research. I also thank to my colleagues in physics department of Parahyangan Catholic University for all their supports, and to Frank Landsman of PPB-UNPAR for correcting my manuscript.

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2 (1998) 231 [Int. J. Theor. Phys. 38 (1999) 1113] [arXiv:hep-th/9711200].

6
[2] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80, 4859 (1998) [arXiv:hep-th/9803002].

[3] S. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C 22 (2001) 379 [arXiv:hep-th/9803001].

[4] J. Sonnenschein, “What does the string/gauge correspondence teach us about Wilson loops?” [arXiv:hep-th/0003032]

[5] S. A. Hartnoll, “Lectures on holographic methods for condensed matter physics,” Class. quant. Grav. 26:224002 (2009) [arXiv:0903.3246 [hep-th]].

[6] D. T. Son and M. Wingate, “General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas,” Annals Phys. 321, 197 (2006) [arXiv:cond-mat/0509786].

[7] Y. Nishida and D. T. Son, “Nonrelativistic conformal field theories,” Phys. Rev. D 76, 086004 (2007) [arXiv:0706.3746 [hep-th]].

[8] D. T. Son, “Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry,” Phys. Rev. D 78, 046003 (2008) [arXiv:0804.3972 [hep-th]].

[9] K. Balasubramanian and J. McGreevy, “Gravity duals for non-relativistic CFTs,” Phys. Rev. Lett. 101, 061601 (2008) [arXiv:0804.4053 [hep-th]].

[10] A. Akhavan, M. Alishahiha, A. Davody and A. Vahedi, “Non-relativistic CFT and Semi-classical Strings,” JHEP 0903 (2009) 053 [arXiv:0811.3067 [hep-th]].

[11] J. Kluson, “Open String in Non-Relativistic Background,” Phys. Rev. D81, 106006 (2010) [arXiv:0912.4587 [hep-th]].

[12] K. Balasubramanian and J. McGreevy, “The particle number in Galilean holography,” JHEP 1101 (2011) 137 [arXiv:1007.2184 [hep-th]].

[13] C. Bachas, “Convexity Of The Quarkonium Potential,” Phys. Rev. D 33 (1986) 2723.

[14] C. Nunez, M. Piai, A. Rago, “Wilson Loops in string duals of Walking and Flavored Systems,” Phys. Rev. D 81, 086001 (2010) [arXiv:0909.0748 [hep-th]].

[15] H. Boschi-Filho and N. R. F. Braga, “Wilson Loops for a quark anti-quark pair in D3-brane space,” JHEP 03 (2005) 051 [arXiv:hep-th/0411135].

[16] E. C´aceres, M. Natsume and T. Okamura, “Screening length in plasma winds,” JHEP 0610 (2006) 011 [arXiv:hep-th/0607233].

[17] Y. Kinar, E. Schreiber, J. Sonnenschein, “Q̅ Q Potential from Strings in Curved Spacetime - Classical Results,” Nucl. Phys. B 566 (2000) 103-125 [arXiv:hep-th/9811192].

[18] R. E. Arias and G. A. Silva, “Wilson loops stability in the gauge/string correspondence,” JHEP 1001 (2010)023 [arXiv:0910.0662 [hep-th]].

[19] S. A. Hartnoll and K. Yoshida, “Families of IIB duals for nonrelativistic CFTs,” JHEP 0812 (2008)071 [arXiv:0810.0298 [hep-th]].

[20] From now on this will be abbreviated as Galilean space-time.

[21] A distance between Q and ¯Q in our 3 + 1 dimensional world, i.e. on the boundary of the Galilean bulk,
see Fig. 1.

[22] I thank Koushik Balasubramanian to give me know this work.

[23] I thank reviewer for pointing this out to me.

[24] We follow the form of metric by Balasubramanian and McGreevy [9]. $f(X_5)$ in ref. [19] is $h^2(X_5)$ in this paper.