GREEN’S CONJECTURE FOR CURVES ON RATIONAL SURFACES WITH AN ANTICANONICAL PENCIL

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ABSTRACT. Green’s conjecture is proved for smooth curves $C$ lying on a rational surface $S$ with an anticanonical pencil, under some mild hypotheses on the line bundle $L = \mathcal{O}_S(C)$. Constancy of Clifford dimension, Clifford index and gonality of curves in the linear system $|L|$ is also obtained.

1. INTRODUCTION

Green’s Conjecture concerning syzygies of canonical curves was first stated in [G] and proposes a generalization of Noether’s Theorem and the Enriques-Babbage Theorem in terms of Koszul cohomology, predicting that for a curve $C$

$$K_{p,2}(C, \omega_C) = 0 \quad \text{if and only if } p < \text{Cliff}(C).$$

Quite remarkably, this would imply that the Clifford index of $C$ can be read off the syzygies of its canonical embedding. The implication $K_{p,2}(C, \omega_C) \neq 0$ for $p \geq \text{Cliff}(C)$ was immediately achieved by Green and Lazarsfeld ([G, Appendix]) and the conjectural part reduces to the vanishing $K_{c-1,2}(C, \omega_C) = 0$ for $c = \text{Cliff}(C)$, or equivalently, $K_{g-c-1,1}(C, \omega_C) = 0$.

One naturally expects the gonality $k$ of $C$ to be also encoded in the vanishing of some Koszul cohomology groups. In fact, Green-Lazarsfeld’s Gonality Conjecture predicts that any line bundle $A$ on $C$ of sufficiently high degree satisfies

$$K_{p,1}(C, A) = 0 \quad \text{if and only if } p \geq h^0(C, A) - k.$$ 

Green ([G]) and Ehbauer ([E]) have shown that the statement holds true for any curve of gonality $k \leq 3$. As in the case of Green’s Conjecture, one implication is well-known (cf. [G, Appendix]); it was proved by Aprodu (cf. [A1]) that the conjecture is thus equivalent to the existence of a non-special globally generated line bundle $A$ such that $K_{h^0(C, A) - k,1}(C, A) = 0$.

Both Green’s Conjecture and Green-Lazarsfeld’s Gonality Conjecture are in their full generality still open. However, by specialization to curves on $K3$ surfaces, they were proved for a general curve in each gonality stratum of $M_g$ by Voisin and Aprodu (cf. [V1, V2, A2]). Combining this with an earlier result of Hirschowitz and Ramanan (cf. [HR]), the two conjectures follow for any curve of odd genus $g = 2k - 3$ and maximal gonality $k$.

In [A2], Aprodu provided a sufficient condition for a genus $g$ curve $C$ of gonality $k \leq (g+2)/2$ to satisfy both conjectures; this is known as the linear growth condition and is expressed in terms of the Brill-Noether theory of $C$ only:

$$\dim W^1_d(C) \leq d - k \quad \text{for } k \leq d \leq g - k + 2.$$ 

Aprodu and Farkas ([AF]) used the above characterization in order to establish Green’s Conjecture for smooth curves lying on arbitrary $K3$ surfaces. It is natural to ask whether a similar strategy can solve Green’s Conjecture for curves lying on anticanonical rational surfaces, since these share some common behaviour with $K3$ surfaces. The situation gets more complicated because such a surface $S$ is in general non-minimal and its canonical bundle is non-trivial; in particular, given a line bundle $L \in \text{Pic}(S)$, smooth curves in the linear system $|L|$ do not form
a family of curves with constant syzygies, as it happens instead in the case of K3 surfaces. Our main result is the following:

**Theorem 1.1.** Let $S$ be a smooth, projective, rational surface with an anticanonical pencil and let $L$ be a line bundle on $S$ such that $L \otimes \omega_S$ is nef and big. In the special case where $h^0(S, \omega_S^2) = \chi(S, \omega_S^2) = 2$, also assume that the Clifford index of a general curve in $|L|$ is not computed by the restriction of the anticanonical bundle $\omega_S^2$.

Then, any smooth, irreducible curve $C \in |L|$ satisfies Green’s Conjecture.

With no hypotheses on the line bundle $L$, we obtain Green’s Conjecture and Green-Lazarsfeld’s Gonality Conjecture for a general curve in $|L|_s$, where $|L|_s$ denotes the locus of smooth and irreducible curves in the linear system $|L|$ (cf. Proposition 5.2). For later use, we denote by $g(L) := 1 + (c_1(L)^2 + c_1(L) \cdot K_S)/2$ the genus of any curve in $|L|_s$.

Examples of surfaces as in Theorem 1.1 are given by all rational surfaces $S$ whose canonical divisor satisfies $K_S^2 > 0$, or equivalently, having Picard number $\rho(S) \leq 9$, such as Del Pezzo surfaces ($-K_S$ ample), generalized Del Pezzo surfaces ($-K_S$ is nef and big), some blow-ups of Hirzebruch surfaces. However, the class of surfaces that we are considering also includes surfaces $S$ with $K_S^2 \leq 0$, such as rational elliptic surfaces (i.e., smooth complete complex surfaces that can be obtained by blowing up $\mathbb{P}^2$ at 9 points, which are the base locus of a pencil of cubic curves with at least one smooth member).

We also obtain the following:

**Theorem 1.2.** Assume the same hypotheses as in Theorem 1.1 and let $g(L) \geq 4$. Then, all curves in $|L|_s$ have the same Clifford dimension $r$, the same Clifford index and the same gonality. Moreover, if the curves in $|L|_s$ are exceptional, then one of the following occurs:

(i) $r = 2$ and any curve in $|L|_s$ is the strict transform of a smooth, plane curve under a morphism $\phi : S \to \mathbb{P}^2$ which is the composition of finitely many blow-ups.

(ii) $r = 3$ and $S$ can be realized as the blow-up of a normal cubic surface $S' \subset \mathbb{P}^3$ at a finite number of points (possibly infinitely near); any curve in $|L|_s$ is the strict transform under the blow-up map of a smooth curve in $|−3K_{S'}|$.

This generalizes results of Pareschi (cf. [P1]) and Knutsen (cf. [K]) concerning the Brill-Noether theory of curves lying on a Del Pezzo surface $S$. In [K], the author proved that line bundles violating the constancy of the Clifford index only exist when $K_S^2 = 1$; they are described in terms of the coefficients of the generators of $\text{Pic}^0(S)$ in their presentation. In fact, one can show that such line bundles are exactly those satisfying $L \otimes \omega_S$ is nef and big and the restriction of the anticanonical bundle $\omega_S^2$ to a general curve in $|L|_s$ computes its Clifford index (cf. Remark 2).

The proofs of Theorem 1.1 and Theorem 1.2 rely on vector bundle techniques à la Lazarsfeld (cf. [La1]); in particular, we consider rank-2 bundles $E_{C,A}$, which are the analogue of the Lazarsfeld-Mukai bundles for K3 surfaces. The key fact proved in Section 3 is that, if $A$ is a complete, base point free pencil on a general curve $C \in |L|_s$, the dimension of $\text{ker} \mu_{0,A}$ is controlled by $H^2(S, E_{C,A} \otimes E_{C,A}^\vee)$; if this is nonzero, the bundle $E_{C,A}$ cannot be slope-stable with respect to any polarization $H$ on $S$.

By considering Harder-Narasimhan and Jordan-Hölder filtrations, in Section 4 we perform a parameter count for pairs $(C, A)$ such that $E_{C,A}$ is not $\mu_H$-stable; this gives an upper bound for the dimension of any irreducible component $W$ of $W_d^1(|L|)$ which dominates $|L|$ under the natural projection $\pi : W_d^1(|L|) \to |L|_s$. It turns out (cf. Proposition 5.1) that, if a general curve $C \in |L|_s$ is exceptional, the same holds true for all curves in $|L|_s$ and one is either in case (i) or (ii) of Theorem 1.2 in this context we recall that Green’s Conjecture for curves of Clifford dimension 2 and 3 was verified by Loose in [Lo]. If instead $C$ has Clifford dimension 1, our parameter count ensures that it satisfies the linear growth condition (3). In order to deduce Green’s
Then, for every smooth integral divisor\( \mathcal{Y} \), we make use of the hypotheses made on \( L \) and show that the Koszul group\( K_{g−c−1,1}(C,\omega_C) \) does not depend (up to isomorphism) on the choice of \( C \in |L|_s \), as soon as \( c \) equals the Clifford index of a general curve in \( |L|_s \). Semicontinuity will imply the constancy of the Clifford index and the gonality.

**Acknowledgements:** I am grateful to my advisor Gavril Farkas, who suggested me to investigate the topic.

### 2. Syzygies and Koszul Cohomology

If \( L \) is an ample line bundle on a complex projective variety \( X \), let \( S := \text{Sym}^*H^0(X, L) \) be the homogeneous coordinate ring of the projective space \( \mathbb{P}(H^0(X, L)^\vee) \) and set \( R(X) := \bigoplus_m H^0(X, L^m) \). Being a finitely generated \( S \)-module, \( R(X) \) admits a minimal graded free resolution

\[
0 \to E_k \to \cdots \to E_1 \to E_0 \to R(X) \to 0,
\]

where for \( k \geq 1 \) one can write \( E_k = \sum_{i \geq k} S(-i - 1)^{\beta_{k,i}} \). The syzygies of \( X \) of order \( k \) are by definition the graded components of the \( S \)-module \( E_k \). We say that the pair \((X, L)\) satisfies property \((N_p)\) if \( E_0 = S \) and \( E_k = S(-k - 1)^{\beta_{k,k}} \) for all \( 1 \leq k \leq p \). In other words, property \((N_0)\) is satisfied whenever \( \phi_L \) embeds \( X \) as a projectively normal variety, while property \((N_1)\) also requires that the ideal of \( X \) in \( \mathbb{P}(H^0(X, L)^\vee) \) is generated by quadrics; for \( p > 1 \), property \((N_p)\) means that the syzygies of \( X \) up to order \( p \) are linear.

The most effective tool in order to compute syzygies is Koszul cohomology, whose definition is the following. Let \( L \in \text{Pic}(X) \) and \( F \) be a coherent sheaf on \( X \). The Koszul cohomology group \( K_{p,q}(X, F, L) \) is defined as the cohomology at the middle-term of the complex

\[
\bigwedge^{p+1} H^0(L) \otimes H^0(F \otimes L^{q-1}) \to \bigwedge^p H^0(L) \otimes H^0(F \otimes L^q) \to \bigwedge^{p-1} H^0(L) \otimes H^0(F \otimes L^{q+1}).
\]

When \( F \simeq \mathcal{O}_X \), the Koszul cohomology group is conventionally denoted by \( K_{p,q}(X, L) \). It turns out (cf. [G]) that property \((N_p)\) for the pair \((X, L)\) is equivalent to the vanishing

\[
K_{i,q}(X, L) = 0 \quad \text{for all } i \leq p \text{ and } q \geq 2.
\]

In particular, Green’s Conjecture can be rephrased by asserting that \((C, \omega_C)\) satisfies property \((N_p)\) whenever \( p < \text{Cliff}(C) \).

In the sequel we will make use of the following results, which are due to Green. The first one is the Vanishing Theorem (cf. [G] Theorem (3.a.1)), stating that

\[
K_{p,q}(X, E, L) = 0 \quad \text{if } p \geq h^0(X, E \otimes L^q).
\]

The second one (cf. [G] Theorem (3.b.1)) relates the Koszul cohomology of \( X \) to that of a smooth hypersurface \( Y \subset X \) in the following way.

**Theorem 2.1.** Let \( X \) be a smooth irreducible projective variety and assume \( L, N \in \text{Pic}(X) \) satisfy

\[
H^0(X, N \otimes \mathcal{L}^q) = 0 \quad \forall q \geq 0.
\]

Then, for every smooth integral divisor \( Y \in |L| \), there exists a long exact sequence

\[
\cdots \to K_{p,q}(X, L^q, N) \to K_{p,q}(X, N) \to K_{p,q}(Y, N \otimes \mathcal{O}_Y) \to K_{p-1,q+1}(X, L^q, N) \to \cdots.
\]
3. Petri map via vector bundles

Let $S$ be a smooth rational surface with an anticanonical pencil and $C \subset S$ be a smooth, irreducible curve of genus $g$. We set $L := \mathcal{O}_S(C)$. If $A$ is a complete, base point free $g^r_d$ on $C$, as in the case of $K3$ surfaces, let $F_{C,A}$ be the vector bundle on $S$ defined by the sequence

$$0 \to F_{C,A} \to H^0(C, A) \otimes \mathcal{O}_S \xrightarrow{ev_A} A \to 0,$$

and set $E_{C,A} := F_{C,A}^\vee$. Since $N_{C|S} = \mathcal{O}_C(C)$, by dualizing the above sequence we get

$$(7) \quad 0 \to H^0(C, A)^\vee \otimes \mathcal{O}_S \to E_{C,A} \to \mathcal{O}_C(C) \otimes A^\vee \to 0.$$\hspace{1cm}

This trivially implies that:

- $\chi(S, E_{C,A} \otimes \omega_S) = h^0(S, E_{C,A} \otimes \omega_S) = g - d + r$,
- $\text{rk} E_{C,A} = r + 1, c_1(E_{C,A}) = L, c_2(E) = d$,
- $h^2(S, E_{C,A}) = 0, \chi(S, E_{C,A}) = g - d + r - c_1(L) \cdot K_S$.

Being a bundle of type $E_{C,A}$ is an open condition. Indeed, a vector bundle $E$ of rank $r + 1$ is of type $E_{C,A}$ whenever $h^1(S, E \otimes \omega_S) = h^2(S, E \otimes \omega_S) = 0$ and there exists $\Delta \in G(r + 1, H^0(S, E))$ such that the degeneracy locus of the evaluation map $ev_A : \Lambda \otimes \mathcal{O}_S \to E$ is a smooth connected curve.

Notice that the dimension of the space of global sections of $E_{C,A}$ depends not only on the type of the linear series $A$ but also on $A \otimes \omega_S$. In particular, one has

$$h^0(S, E_{C,A}) = r + 1 + h^0(C, \mathcal{O}_C(C) \otimes A^\vee),$$

$$h^1(S, E_{C,A}) = h^0(C, A \otimes \omega_S).$$

Moreover, if the line bundle $\mathcal{O}_C(C) \otimes A^\vee$ has sections, then $E_{C,A}$ is generated off its base points. In the case $r = 1$, we prove the following.

**Lemma 3.1.** Let $A$ be a complete, base point free $g^1_d$ on $C \subset S$. If either

- $h^0(S, \omega_C^\vee) > 2$, or
- $h^0(S, \omega_C^\vee) = 2$ and $A \not\simeq \omega_C^\vee \otimes \mathcal{O}_C$

holds, then $h^0(C, A \otimes \omega_S) = 0$.

**Proof.** Since $L \otimes \omega_S$ is effective, the short exact sequence

$$0 \to L^\vee \otimes \omega_S^\vee \to \omega_S^\vee \to \omega_S^\vee \otimes \mathcal{O}_C \to 0$$

implies $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) \geq h^0(S, \omega_C^\vee)$ and the statement follows trivially if $h^0(S, \omega_C^\vee) > 2$. Let $h^0(S, \omega_C^\vee) = 2$ and $h^0(C, A \otimes \omega_S) > 0$. Then necessarily $h^0(C, \omega_S^\vee \otimes \mathcal{O}_C) = 2$ and $A \otimes \omega_S$ is the fixed part of the linear system of sections of $A$. Since $A$ is base point free by hypothesis, then $A \simeq \omega_S^\vee \otimes \mathcal{O}_C$. \hfill $\Box$

Under the hypotheses of the above Lemma, the bundle $E_{C,A}$ is globally generated off a finite set and $\chi(S, E_{C,A}) = h^0(S, E_{C,A}) = g - d + 1 - c_1(L) \cdot K_S$. We remark that the vanishing of $h^1(S, E_{C,A})$ turns out to be crucial in most of the following arguments and this is why the assumptions on the anticanonical linear system of $S$ are needed.

The following proposition gives a necessary and sufficient condition for the injectivity of the Petri map $\mu_{0,A} : H^0(C, A) \otimes H^0(C, \omega_C \otimes A^\vee) \to H^0(C, \omega_C)$.

**Proposition 3.2.** If $C \subset |L|_s$ is general and either $h^0(S, \omega_C^\vee) > 2$ or $h^0(S, \omega_C^\vee) = 2$ and $A \not\simeq \omega_C^\vee \otimes \mathcal{O}_C$, then for any complete, base point free pencil $A$ on $C$ one has:

$$\ker \mu_{0,A} \simeq H^2(S, E_{C,A} \otimes E_{C,A}^\vee).$$

In particular, the vanishing of the one side implies the vanishing of the other.
Corollary 3.3 can be alternatively proved by arguing in the following way. W dominates Ext corollary 3.3. Let duality.

Indeed, as in [P2, Lemma 1], one finds a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \omega_S \otimes \mathcal{O}_C & \to & F_{C,A} \otimes \omega_C \otimes A^\vee & \to & \omega_C \otimes A^{-2} & \to & 0 \\
\| & & \| & & \| & & \| & & \\
0 & \to & \omega_S \otimes \mathcal{O}_C & \to & \Omega^1_{C} \otimes \omega_C & \to & \omega^2_C & \to & 0,
\end{array}
\]

where the homomorphism induced by s on global sections is \( \mu_{1,A} \) and the coboundary map \( H^0(C, \omega_C^2) \to H^1(C, \omega_S \otimes \mathcal{O}_C) \) equals (up to a scalar factor) \( \rho^0\).

If \( A \) has degree \( d \), look at the natural projection \( \pi : W^1_d(\vert L \vert) \to \vert L \vert_s \). First order deformation arguments (see, for instance, [ACG, p. 722]) imply that

\[
\mathrm{Im}(d\pi_{(C,A)}) \subset \mathrm{Ann}(\mathrm{Im}(\rho^0 \circ \mu_{1,A})).
\]

Therefore, by Sard’s Lemma, if \( C \in \vert L \vert_s \) is general, the short exact sequence (8) is exact on the global sections for any base point free \( A \in W^1_d(C \setminus W^1_d(C)) \), and \( \ker \mu_{0,A} \cong H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee) \).

By tensoring short exact sequence (7) with \( F_{C,A} \otimes \omega_S \), one finds that

\[
H^0(C, F_{C,A} \otimes \omega_C \otimes A^\vee) \cong H^0(S, E^\vee_{C,A} \otimes E_{C,A} \otimes \omega_S)
\]

because \( H^1(S, F_{C,A} \otimes \omega_S) \cong H^2 \Gamma(S, E^\vee_{C,A} \otimes E_{C,A} \otimes \omega_S) = 0 \) for \( i = 0, 1 \). The statement follows now by Serre duality.

Corollary 3.3. Let \( H \) be any polarization on \( S \) and \( W \) be an irreducible component of \( W^1_d(\vert L \vert) \) which dominates \( \vert L \vert \) and whose general points correspond to \( \mu_H \)-stable bundles; in the special case where \( h^0(S, \omega_S^\vee) = 2 \), assume that general points of \( W \) are not of the form \((C, \omega_S^\vee \otimes \mathcal{O}_C)\).

Then, \( \rho(g, 1, d) \geq 0 \) and \( W \) is reduced of dimension equal to \( \dim \vert L \vert + \rho(g, 1, d) \).

Proof. Let \((C, A)\) be a general point of \( W \). If \( E_{C,A} \) is stable, \( E_{C,A} \otimes \omega_S \) also is. The inequality \( \mu_H(E_{C,A}) > \mu_H(E_{C,A} \otimes \omega_S) \) implies that \( H^2(S, E^\vee_{C,A} \otimes E_{C,A} \otimes \omega_S) \cong \mathrm{Hom}(E_{C,A}, E_{C,A} \otimes \omega_S) = 0 \). As a consequence, \( W \) is smooth in \((C, A)\) of the expected dimension.

Remark 1. Corollary 3.3 can be alternatively proved by arguing in the following way. Let \( M := M_H^m(c) \) be the moduli space of \( \mu_H \)-stable vector bundles on \( S \) of total Chern class \( c = 2 + c_1(L) + d \omega \in H^{2r}(S, \mathbb{Z}) \), where \( \omega \) is the fundamental cocycle. Since every \([E] \in M \) satisfies \( \mathrm{Ext}^2(E, E) = 0 \), it turns out that \( M \) is smooth, irreducible projective variety of dimension \( 4d - c_1(L)^2 - 3 \) (cf. [CoMR, Remark 2.3]), as soon as it is non-empty. Let \( M^0 \) be the open subset of \( M \) parametrizing vector bundles \([E]\) of type \( E_{C,A} \); if this is non-empty, define \( G \) as the Grassmann bundle on \( M^0 \) with fiber over \([E]\) equal to \( G(2, H^0(S, E)) \). Look at the rational map \( h : G \to W^1_d(\vert L \vert) \) sending a general \((E, \Lambda) \in G\) to the pair \((C_\Lambda, A_\Lambda)\), where \( C_\Lambda \) is the degeneracy locus of the evaluation map \( ev_\Lambda : \Lambda \otimes \mathcal{O}_S \to E \) and \( \mathcal{O}_{C_\Lambda}(C_\Lambda) \otimes A^\vee \) is its cokernel. Since any
$[E] \in M^0$ is simple, one easily checks that $h$ is birational onto its image, that is denoted by $\mathcal{W}$. As a consequence, the dimension of $\mathcal{W}$ equals:

$$4d - c_1(L)^2 - 3 + 2(g - d - 1 - c_1(L) \cdot K_S) = 2d - 3 - c_1(L) \cdot K_S \leq \dim |L| + \rho(g, 1, d).$$

4. Parameter Count

By the analysis performed in the previous section, given a polarization $H$ on $S$, the linear growth condition for a general curve in $|L|_s$ can be verified by controlling the dimension of every dominating component $\mathcal{W} \subset W_j(|L|)$, whose general points are pairs $(C, A)$ such that $A \not\cong \omega_S^\vee \otimes \mathcal{O}_C$ and the bundle $E_{C,A}$ is not $\mu_H$-stable. Indeed, if $A \cong \omega_S^\vee \otimes \mathcal{O}_C$ for a general point of $\mathcal{W}$, then $\omega_S^\vee \otimes \mathcal{O}_C$ is an isolated point of $W^1_\delta(C')$ for every $C' \in |L|_s$.

Let $A$ be a complete, base point free $g^1_1$ on a curve $C \in |L|_s$ such that the bundle $E := E_{C,A}$ is not $\mu_H$-stable and $A \not\cong \omega_S^\vee \otimes \mathcal{O}_C$ if $h^0(S, \omega_S^\vee) = 2$. The maximal destabilizing sequence of $E$ has the form

$$(9) \quad 0 \to M \to E \to N \otimes I_\xi \to 0,$$

where $M, N \in \text{Pic}(S)$ satisfy

$$(10) \quad \mu_H(M) \geq \mu_H(E) \geq \mu_H(N),$$

with both or none of the inequalities being strict, and $I_\xi$ is the ideal sheaf of a 0-dimensional subscheme $\xi \subset S$ of length $l = d - c_1(N) \cdot c_1(M)$.

**Lemma 4.1.** In the above situation, assume that general curves in $|L|_s$ have Clifford index $c$. If $\mu_{0,A}$ is non-injective and $C$ is general in $|L|$, then the following inequality holds:

$$(11) \quad c_1(M) \cdot c_1(N) + c_1(N) \cdot K_S \geq c.$$

**Proof.** Being a quotient of $E := E_{C,A}$ off a finite set, $N$ is base component free and is non-trivial since $H^2(S, N \otimes \omega_S) = 0$. As a consequence, $h^0(S, N) \geq 2$. Proposition 3.2 implies that $\text{Hom}(E, E \otimes \omega_S) \neq 0$. Applying $\text{Hom}(E, -)$ to the short exact sequence (9) twisted with $\omega_S$, we obtain

$$0 \to \text{Hom}(E, M \otimes \omega_S) \to \text{Hom}(E, E \otimes \omega_S) \to \text{Hom}(E, N \otimes \omega_S \otimes I_\xi) \to \cdots.$$  

Apply now $\text{Hom}(-, N \otimes \omega_S \otimes I_\xi)$ (respectively $\text{Hom}(-, M \otimes \omega_S)$) to exact sequence (9), and find that $\text{Hom}(E, N \otimes \omega_S \otimes I_\xi) = 0$ (resp. $\text{Hom}(E, M \otimes \omega_S) \simeq \text{Hom}(N \otimes I_\xi, M \otimes \omega_S)$), hence $N^\vee \otimes M \otimes \omega_S$ is effective and $h^0(S, M \otimes \omega_S) \geq 2$. This ensures that $N \otimes \mathcal{O}_C$ contributes to the Clifford index of $C$ and

$$c \leq \text{Cliff}(N \otimes \mathcal{O}_C) = c_1(N) \cdot (c_1(N) + c_1(M)) - 2h^0(C, N \otimes \mathcal{O}_C) + 2 \leq c_1(N)^2 + c_1(N) \cdot c_1(M) - 2h^0(S, N) + 2 = c_1(N) \cdot c_1(M) + c_1(N) \cdot K_S.$$  

Now, upon fixing a nonnegative integer $l$ and a line bundle $N$ such that (10) is satisfied for $M := L \otimes N^\vee$, we want to estimate the number of moduli of pairs $(C, A)$ such that the bundle $E_{C,A}$ sits in a short exact sequence like (9). The following construction is analogous to the one performed in [LC, Section 4] in the case of K3 surfaces. Let $\mathcal{E}_{N,l}$ be the moduli stack of extensions of type (9), where $l(\xi) = l$. Having fixed $c \in H^2(S, \mathbb{Z})$, we denote by $\mathcal{M}(c)$ the moduli stack of coherent sheaves of total Chern class $c$. We consider the natural maps $p : \mathcal{E}_{N,l} \to \mathcal{M}(c(M)) \times \mathcal{M}(c(N \otimes I_\xi))$ and $q : \mathcal{E}_{N,L} \to \mathcal{M}(c(E))$, which send the $C$-point of $\mathcal{E}_{N,l}$ corresponding to extension (9) to the classes of $(M, N \otimes I_\xi)$ and $E$ respectively. Notice that, since $M, N$ lie in $\text{Pic}(S)$, the stack $\mathcal{M}(c(M))$ has a unique $C$-point endowed with a
1-dimensional space of automorphisms, while $\mathcal{M}(c(N \otimes I_\xi))$ is corepresented by the Hilbert scheme $S[^l_0]$ parametrizing 0-dimensional subschemes of $S$ of length $l$.

We denote by $\hat{Q}_{N,l}$ the closure of the image of $q$ and by $Q_{N,l}$ its open substack consisting of vector bundles of type $E_{C,A}$ for some $C \in |L|_s$ and $A \in W^1_1(C)$, with $d := l + c_1(M) \cdot c_1(N)$ and $A \not= \omega_S^N \otimes O_C$ if $h^0(S, \omega_S^N) = 2$. Let $\mathcal{G}_{N,l} \rightarrow Q_{N,l}$ be the Grassmann bundle whose fiber over $[E] \in Q_{N,l}(\mathbb{C})$ is $G(2, H^0(S, E))$. By construction, a $\mathbb{C}$-point of $\mathcal{G}_{N,l}$ corresponding to a pair $([E], \Lambda)$, with $\Lambda \in G(2, H^0(S, E))$, comes endowed with an automorphism group equal to $\text{Aut}(E)$. We define $\mathcal{W}_{N,l}$ to be the closure of the image of the rational map $\mathcal{G}_{N,l} \dashrightarrow W^1_1(|L|)_s$, which sends a general point $([E], \Lambda) \in \mathcal{G}_{N,l}(\mathbb{C})$ to the pair $(C_A, A_A)$ where the evaluation map $\text{ev}_{\Lambda} : \Lambda \otimes O_S \rightarrow E$ degenerates on $C_\Lambda$ and has $O_{C_\Lambda}(C_\Lambda) \otimes A^\Lambda_\Lambda$ as cokernel. The following proposition gives an upper bound for the dimension of $\mathcal{W}_{N,l}$.

**Proposition 4.2.** Assume that general curves in $|L|_s$ have Clifford index $c$. Then, every irreducible component $\mathcal{W}$ of $W^1_1(|L|)_s$ which dominates $|L|$ and is contained in $\mathcal{W}_{N,l}$ satisfies

$$\dim \mathcal{W} \leq \dim |L| + d - c - 2.$$  

**Proof.** The fiber of $p$ over a $\mathbb{C}$-point of $\mathcal{M}(c(M)) \times \mathcal{M}(c(N \otimes I_\xi))$ corresponding to $(M, N \otimes I_\xi)$ is the quotient stack

$$[\text{Ext}^1(N \otimes I_\xi, M)/\text{Hom}(N \otimes I_\xi, M)],$$

where the action of the Hom group on the Ext group is trivial. The fiber of $q$ over $[E] \in \hat{Q}_{N,l}(\mathbb{C})$ is the Quot-scheme $\text{Quot}_S(E, P)$, where $P$ is the Hilbert polynomial of $N \otimes I_\xi$. The condition $\mu_H(M) \geq \mu_H(N)$ implies that $\text{Ext}^2(N \otimes I_\xi, M) \simeq \text{Hom}(M, N \otimes O_S \otimes I_\xi)^\vee = 0$, hence the dimension of the fibers of $p$ is constant and equals

$$-\chi(S, N \otimes M^\vee \otimes O_S \otimes I_\xi) = -g + 2c_1(N) \cdot c_1(M) + c_1(M) \cdot K_S + l.$$

By looking at the tangent and obstruction spaces at any point, one shows that the Quot schemes constructing the fibers of $q$ are either all 0-dimensional or all smooth of dimension 1; indeed, $\text{Hom}(M, N \otimes I_\xi) = 0$ unless $M \simeq N$ and $l = 0$, in which case $\text{Ext}^1(M, N \otimes I_\xi) = H^1(S, O_S) = 0$. As a consequence, if nonempty, $Q_{N,l}$ has dimension at most $3l - 2 - g + 2c_1(N) \cdot c_1(M) + c_1(M) \cdot K_S$.

Since the map $h_{N,l}$ forgets the automorphisms, its fiber over a pair $(C, A) \in \mathcal{W}_{N,l}$ is the quotient stack

$$[U/\text{Aut}(E_{C,A})],$$

where $U$ is the open subscheme of $\mathbb{P}(\text{Hom}(E_{C,A}, O_{C}(C) \otimes A^\vee))$ whose points correspond to morphisms with kernel equal to $O_{S}[^2_0]$, and $\text{Aut}(E_{C,A})$ acts on $U$ by composition. Using the vanishing $h^i(S, E_{C,A} \otimes O_S) = 0$ for $i = 1, 2$, one checks that

$$\text{Hom}(E_{C,A}, O_C(C) \otimes A^\vee) \simeq H^0(S, E_{C,A} \otimes E_{C,A}^\vee),$$

and $U$ is isomorphic to $\mathbb{P}\text{Aut}(E_{C,A})$. Hence, the fibers of $h_{N,l}$ are stacks of dimension $-1$ and

$$\dim \mathcal{W}_{N,l} \leq 3l - 1 - g + 2c_1(N) \cdot c_1(M) + c_1(M) \cdot K_S + 2(g - d - 1 - c_1(L) \cdot K_S) = d + g - 3 - c_1(N) \cdot c_1(M) - c_1(N) \cdot K_S - c_1(L) \cdot K_S.$$

The conclusion follows now from the fact that $\dim |L| \geq g - 1 - c_1(L) \cdot K_S$, along with Lemma 4.1. 

5. PROOF OF THE MAIN RESULTS

We recall some facts about exceptional curves, that is, curves of Clifford dimension greater than 1. Coppens and Martens ([CM]) proved that, if $C$ is an exceptional curve of gonality $k$ and Clifford dimension $r$, then $\text{Cliff}(C) = k - 3$ and $C$ possesses a 1-dimensional family of $g^r_k$. Furthermore, if $r \leq 9$, there exists a unique line bundle computing $\text{Cliff}(C)$ (cf. [ELMS]).
this is conjecturally true for any $r$. Curves of Clifford dimension 2 are precisely the smooth plane curves of degree $\geq 5$, while the only curves of Clifford dimension 3 are the complete intersections of two cubic surfaces in $\mathbb{P}^3$ (cf. [Ma]). We will use these results in the proof of the following:

**Proposition 5.1.** Let $L$ be a line bundle on a smooth, rational surface $S$ with an anticanonical pencil. If $g(L) \geq 4$ and a general curve $C \in |L|_s$ is exceptional, then any other curve inside $|L|_s$ has the same Clifford dimension $r$ of $C$ and either case (i) or (ii) of Theorem 1.2 occurs.

**Proof.** Since any curve of odd genus and maximal gonality has Clifford dimension 1 (cf. [A3, Corollary 3.11]), we can assume that general curves in $|L|_s$ have gonality $k \leq (g+2)/2$ and are exceptional. There exists a component $W$ of $W^1_2(|L|)$ of dimension at least $\dim |L| + 1$ and, by Corollary 3.3, this is contained in $W_{N,l}$ for some $N$ and $l$. Notice that the line bundle $N$ is nef since it is globally generated off a finite set. Furthermore, it follows from the proof of Proposition 4.2 that $N$ and $M := L \otimes N^\vee$ satisfy equality in (1), that is,

$$k - 3 = c_1(M) \cdot c_1(N) + c_1(N) \cdot K_S = k - l + c_1(N) \cdot K_S;$$

in particular, $N \otimes O_C$ computes the Clifford index of a general $C \in |L|_s$ and $h^1(S, M^\vee) = 0$. Having at least a 2-dimensional space of sections, the line bundle $\omega_S^\vee \otimes O_C$ has degree $\geq k$, thus $-c_1(M) \cdot K_S \geq k - 3 + l$.

Assume $h^0(S, N \otimes \omega_S) \geq 2$; the restriction of $M$ to a general curve $C \in |L|_s$ contributes to its Clifford index and

$$k - 3 \leq \text{Cliff}(M \otimes O_C) = c_1(M) \cdot c_1(N) + c_1(M) \cdot K_S \leq 3 - 2l.$$ 

As $k \geq 2r$ (cf. [ELMS, Proposition 3.2]), we have $r \leq 3$; if $r = 3$, then $l = 0$, while $r = 2$ implies $l \leq 1$. Let $r = 2$; since $\chi(S, N) = h^0(S, N) = h^0(C, N \otimes O_C) = 3$ and $h^1(S, N \otimes \omega_S) = 0$ for $i = 1, 2$ (as one can check twisting (2) with $\omega_S$ and taking cohomology), then $c_1(N)^2 = l + 1$ and $h^0(S, N \otimes \omega_S) = l \leq 1$, contradicting our assumption. Hence, the inequality $h^0(S, N \otimes \omega_S) \geq 2$ implies $r = 3$ and $l = 0$.

Assume instead that $h^0(S, N \otimes \omega_S) \leq 1$; we get $c_1(N)^2 \leq 3 - l$ and $h^0(C, N \otimes O_C) = h^0(S, N) = \chi(S, N) \leq 4 - l$. Since $N \otimes O_C$ computes the Clifford index of $C$, then $r \leq 3$ holds in this case as well. Moreover, $l = 0$ when $r = 3$, and $l \leq 1$ if $r = 2$.

Let $r = 2$ and $l = 1$; then, $c_1(N)^2 = -c_1(N) \cdot K_S = 2$. By [Ha, Lemma 2.6, Theorem 2.11], $N$ is base point free and not composed with a pencil, hence it defines a generically finite map $\psi_S : S \to |L|_{\psi_S}$ splitting into the composition of a birational morphism $\phi : S \to S'$, which contracts the finitely many curves $E_1, \ldots, E_h$ having zero intersection with $c_1(N)$, and a ramified double cover $\pi : S' \to \mathbb{P}^2$. Let $N' := \pi^*(O_{\mathbb{P}^2}(1))$; since $N = \psi_S^*(N')$ and $\psi_S$ preserves both the intersection products and the dimensions of cohomology groups, we have $c_1(N')^2 = -c_1(N') \cdot K_{S'} = 2$ and

$$1 = h^0(S, N \otimes \omega_S) \geq h^0(S, N \otimes \omega_S(-E_1 - \cdots - E_h)) = h^0(S', N' \otimes \omega_{S'}).$$

We apply Theorem 3.3. in [Ha] and get $N' = \omega_{S'}^\vee$ and $K_{S'} = 2$ (cases (b), (c), (d) of the aforementioned theorem cannot occur since they would contradict $c_1(N')^2 = 2$). The line bundle $N \otimes O_C$ is very ample because it computes $\text{Cliff}(C)$ (cf. [ELMS, Lemma 1.1]); hence, $C$ is isomorphic to $C' = \psi(C)$ and $\omega_{S'}^\vee \otimes O_{C'}$ is also very ample. Proceeding as in the proof of [Pi] Lemma 2.6 (where the ampleness of $\omega_{S'}^\vee$ is not really used), one shows that $\phi(C') \in | - 2K_{S'}|$. This gives a contradiction because it implies $g(C') = g(C) = 3$.

Up to now, we have shown that $r \leq 3$ and $l = 0$, hence $-c_1(N) \cdot K_S = 3$ and $c_1(N)^2 > 0$. By [Ha, Proposition 3.2], the line bundle $N$ defines a morphism $\phi_N : S \to \mathbb{P}^n$ which is birational to its image and only contracts the finitely many curves which have zero intersection with $c_1(N)$. If $r = 2$, then $\phi_N$ is the blow-up of $\mathbb{P}^2$ at finitely many points (maybe infinitely near) and any
curve in $|L|_s$ is the strict transform of a smooth plane curve. For $r = 3$, the image of $\phi_N$ is a normal cubic surface $S' \subset \mathbb{P}^3$ and any curve in $|L|_s$ is the strict transform of a smooth curve in $|−3K_{S'}|$ hence has Clifford dimension 3.

The following result is now straightforward.

**Proposition 5.2.** Let $C$ be a smooth, irreducible curve lying on a rational surface $S$ with an anticanonical pencil. If $C$ is general in its linear system, then $C$ satisfies Green’s Conjecture; if moreover $C$ is not isomorphic to the complete intersection of two cubics in $\mathbb{P}^3$, then it satisfies Green-Lazarsfeld’s Gonality Conjecture as well.

**Proof.** We assume that $C$ has genus $g \geq 4$, Clifford dimension 1, Clifford index $c$ and gonality $k \leq (g + 2)/2$. Having fixed $k \leq d \leq g − k + 2$, Corollary 3.3 and Proposition 4.2 imply that every dominating component $W$ of $W_d^1(|L|)$ has dimension $\leq \dim |L| + d − k$, hence $C$ satisfies the linear growth condition. Green’s Conjecture for smooth plane curve and complete intersection of two cubics in $\mathbb{P}^3$ was established by Loose in [Lo], while Aprodu proved Green-Lazarsfeld’s Gonality Conjecture for curves of Clifford dimension 2 in [Al].

We proceed with the proof of Theorem 1.1.

**Proof of Theorem 1.1.** We can assume $g(L) \geq 4$. By Proposition 5.2, if $C \in |L|_s$ is general then $K_{g−c−1,1}(C, \omega_C)$ is regular and has geometric genus 0. We remark that this also implies that

$$H^0(C, \omega_C) \cong H^0(S, L \otimes \omega_S), \quad \forall C \in |L|_s.$$ 

Equality (6) for $q = 0$ is trivial since $|L|$ contains a smooth, irreducible curve. For $q \geq 2$, the line bundle $N^{q−1}$ is nef and big and the Kawamata-Viehweg Vanishing Theorem (cf. [La2, Theorem 4.3.1]) implies that

$$0 = H^1(S, N^{−(q−1)} \otimes \omega_S) \cong H^1(S, (L \otimes \omega_S)^{q−1} \otimes \omega_S) = H^1(S, N^q \otimes L^q).$$

By adjunction, for any curve $C \in |L|_s$, we obtain the following long exact sequence

$$\cdots \rightarrow K_{g−c−1,1}(S, L^q, L \otimes \omega_S) \rightarrow K_{g−c−1,1}(S, L \otimes \omega_S) \rightarrow K_{g−c−1,1}(C, \omega_C) \rightarrow K_{g−c−2,2}(S, L^q, L \otimes \omega_S) \rightarrow \cdots.$$

The group $K_{g−c−1,1}(S, L^q, L \otimes \omega_S)$ trivially vanishes since $H^0(S, \omega_S) = 0$. By the Vanishing Theorem applied to $K_{g−c−2,2}(S, L^q, L \otimes \omega_S)$, we can conclude that

$$H^0(C, \omega_C) \cong K_{g−c−1,1}(C, \omega_C),$$

provided that $g − c − 2 \geq h^0(S, L \otimes \omega_S^2)$. We can assume $h^0(S, L \otimes \omega_S^2) \geq 2$ and we are under the hypothesis that the anticanonical system contains a pencil. Hence, $\omega_S^2 \otimes \mathcal{O}_C$ contributes to the Clifford index and, if $C \in |L|_s$ is general, then:

$$c = \text{Cliff}(C) \leq \text{Cliff}(\omega_S^2 \otimes \mathcal{O}_C) = −c_1(L) \cdot K_S − 2h^0(C, \omega_S^2 \otimes \mathcal{O}_C) + 2 \leq −c_1(L) \cdot K_S − 2h^0(S, \omega_S^2) + 2.$$

Since $H^1(S, L \otimes \omega_S^2) \cong H^1(S, L^q \otimes \omega_S^q) = 0$, we have

$$h^0(S, L \otimes \omega_S^2) = χ(S, L \otimes \omega_S^2) = g + c_1(L) \cdot K_S + K_S^2 \leq g − c + 1 − h^0(S, \omega_S^2) = h^1(S, \omega_S^2).$$
Knutsen’s conditions because, if \( \Gamma \)
\( \text{base point free pencil on} \)
\( C \) when
\( C \) is a net on
\( \chi \)
\( C \) has degree 1; we denote by \( c \) their Clifford index.

By semicontinuity of the gonality, all curves in \( |L|_s \) have gonality
\( k \) and Clifford index \( k - 2 \); in particular, \( \omega^\vee \otimes \mathcal{O}_C \) computes \( \text{Cliff}(C) \). The easiest example where the gonality is not constant is provided by \( L = \omega^{-n}_S \) for \( n \geq 3 \).

Vice versa, if \( S \) has degree 1 and \( \text{Cliff}(\omega^\vee_S \otimes \mathcal{O}_C) = \text{Cliff}(C) \) for a general \( C \in |L|_s \), one recovers Knutsen’s conditions because, if \( \Gamma \) is a smooth rational curve with \( \Gamma^2 = 0 \), then \( \mathcal{O}_S(\Gamma) \) cuts out a base point free pencil on \( C \), and if \( c_1(L)^2 \geq 8 \) and \( E \) is a \((-1)\)-curve, then \( \mathcal{O}_C(-K_S + E) \) defines a net on \( C \) which contributes to its Clifford index. This shows that the extra hypothesis we make when \( \chi(S, \omega^\vee_S) = h^0(S, \omega^\vee_S) = 2 \) is unavoidable.

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