Square root of gerbe holonomy and invariants of
time-reversal-symmetric topological insulators

Krzysztof Gawędzki∗
Université de Lyon, ENS de Lyon,
Université Claude Bernard, CNRS
Laboratoire de Physique, F-69342 Lyon, France

ABSTRACT
The Feynman amplitudes with the two-dimensional Wess-Zumino action functional have a geometric interpretation as bundle gerbe holonomy. We present details of the construction of a distinguished square root of such holonomy and of a related 3d-index and briefly recall the application of those to the building of topological invariants for time-reversal-symmetric two- and three-dimensional crystals, both static and periodically forced.

1. INTRODUCTION
The central theme of the present paper is a specific refinement of the two-dimensional Wess-Zumino (WZ) field-theoretic action functional that finds its application in the construction of invariants of the time-reversal-symmetric topological insulators in two and three dimensions. The WZ functional has appeared in the context of field theory anomalies [30]. A two-dimensional WZ action related to the chiral anomaly was used in [32] as an important component in the construction of a particular conformal field theory, the so called Wess-Zumino-Witten sigma model. As opposed to local action functionals, the 2d WZ action $S_{WZ}(\phi)$ of a classical field $\phi$ was originally defined by a local functional of its three-dimensional extension. This resulted in $S_{WZ}(\phi)$ defined only modulo $2\pi$, leading, however, to the univalued Feynman amplitude $e^{iS_{WZ}(\phi)}$ (we set the Planck constant $\hbar$ to 1 here). The first attempts to write local formulae for $S_{WZ}(\phi)$ were based on cohomological approaches [1,11]. In particular, [11] realized that the Deligne cohomology [8] provided the proper tool for such problems, permitting to define the WZ action in more general situations. With the advent of the theory of bundle gerbes [21,24], this approach has gained a geometric interpretation: the Feynman amplitudes $e^{iS_{WZ}(\phi)}$ got the interpretation of the “bundle gerbe holonomy” [5,13].

The refinement of the WZ action that we shall discuss here will permit to fix a square root of the WZ amplitude for “equivariant” fields $\phi$ that intertwine an orientation preserving involution of the closed surface on which they are defined with an involution in the target space. Giving a unique value to the square root of the WZ amplitude of equivariant fields will require to define their WZ action modulo $4\pi$ rather than only modulo $2\pi$. In a somewhat implicit way, the square root of the WZ amplitude was used in [6,7] to define dynamical torsion invariants of periodically forced crystalline systems with time-reversal symmetry. In the case of such Floquet systems, the surface was the Brillouin 2-torus equipped with the involution reversing the sign of the quasimomentum and the target space was the unitary group with the involution corresponding to the time reversal. On the way, it was shown in those references that the Fu-Kane-Mele invariant [19,9] of the static time-reversal-symmetric topological insulators may be expressed as the square root of a WZ amplitude.

From the point of view of gerbes, in order to fix the square root of the bundle gerbe holonomy describing the WZ amplitude of equivariant fields, one needs an additional structure expressing the equivariance of the gerbe under the target-space involution. The situation bears some similarity to that where the WZ amplitudes are extended to fields defined on non-oriented surfaces, studied previously in [23,14,26]. There are, however, some important differences. The most notable of these is that, in general, the orientation preserving involutions of surfaces possess fixed points that require special treatment, unlike the orientation-reversing involutions of oriented covers of non-oriented surfaces. The aspects of the bundle gerbe theory needed to define the square root of gerbe holonomy were reviewed in much detail in the lecture notes [12].

* directeur de recherche émérite, email: kgawedzk@ens-lyon.fr
of the present author, together with their applications to two- and three-dimensions topological insulators and Floquet systems. What was omitted there, however, was the proofs that the presented formulae for the square root of the gerbe holonomy for a related 3d-index define those quantities in an unambiguous way. The main goal of the present paper is to fill that gap. We use here a different but equivalent presentation of the additional structure on the gerbe needed for the construction of the square root of gerbe holonomy. This permits to streamline the proofs.

The paper is organized as follows. In Sec. 2 we recall the definition of bundle gerbes and in Sec. 3 that of gerbe holonomy. Sec. 4 introduces the notion of an equivariant extension of a gerbe with respect to a Z2-action defined by an involution on the base space. The central Sec. 6 presents a local formula for the square root of gerbe holonomy and proves that it determines the latter in an unambiguous way under some topological conditions. In Sec. 7 we establish a non-local formula for the same quantity. Such a formula was employed in [2] as the definition of the square root of the WZ amplitude. Sec. 8 describes the equivariant extension of gerbes and the square root of gerbe holonomy in terms of local data. Sec. 9 presents a formula, involving the square root of gerbe holonomy, for the 3d-index and proves that the latter is unambiguously defined. The last two sections cover the subjects discussed in detail in [12] and are included for completeness. In Sec. 10 we briefly describe how the general scheme can be extended to the case of the basic gerbe on the unitary group with the involution given by the time reversal and, in Sec. 11 we summarize the application of such an extension to the construction of invariants of time-reversal-symmetric crystalline systems, both static and periodically driven. Appendix discusses the relation between the equivariant extension of gerbes used in the present paper and the equivariant structure on gerbes employed in [12], making explicit the relation between the constructions of both papers.

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2. BUNDLE GERBES

Bundle gerbes are examples of higher structures, 1-degree higher than line bundles. They were introduced by M. K. Murray [21] in 1996, see also [22], as geometric examples of more abstract gerbes of J. Giraud [16] and J.-L. Brylinski [4]. Below, we shall only consider bundle gerbes and line bundles equipped with hermitian structure and unitary connection without further mention. The aspect of bundle gerbes that we shall be interested in here is that they provide local formulae for topological Feynman amplitudes of the Wess-Zumino (WZ) type for two-dimensional classical fields, as already mentioned in Introduction.

Let us start by recalling some notations. We shall work in the category of smooth manifolds. If \( \pi : Y \to M \) then by \( Y^{[n]} \) we shall denote the subset of \( Y^n \) composed of \( (y_i)_{i=1}^n \) with all \( \pi(y_i) \) equal. For a sequence \((i_1, \ldots, i_m)\) with \( 1 \leq i_j \leq n \), \( p_{i_1 \ldots i_m} \) will denote the map from \( Y^{[n]} \) to \( Y^{[m]} \) such that \( p_{i_1 \ldots i_m}(y_1, \ldots, y_n) = (y_{i_1}, \ldots, y_{i_m}) \). If \( \pi \) is a submersion of manifolds then \( Y^{[n]} \) is a submanifold of \( Y^n \) and the maps \( p_{i_1 \ldots i_k} \) are smooth.

Definition 1 [21]. A bundle gerbe \( G \) (below, a “gerbe” for short) over \( M \) is a quadruple \((Y, B, \mathcal{L}, t)\), where \( \pi : Y \to M \) is a surjective submersion, \( B \) is a real 2-form on \( Y \) (called the curving), \( \mathcal{L} \) is a line bundle over \( Y^{[2]} \) with curvature 2-form \( F_\mathcal{L} = p_2^*B - p_1^*B \), and \( t \) is a line-bundle isomorphism over \( Y^{[3]} \)

\[
t : p_{12}^*\mathcal{L} \otimes p_{23}^*\mathcal{L} \to p_{13}^*\mathcal{L}, \tag{2.1}
\]

acting fiber-wise\(^1\) as \( \mathcal{L}_{y_1, y_2} \otimes \mathcal{L}_{y_2, y_3} \xrightarrow{t} \mathcal{L}_{y_1, y_3} \) for \((y_1, y_2, y_3) \in Y^{[3]}\), that defines an (associative) groupoid multiplication on \( \mathcal{L} \subset Y \).

The condition on the curving 2-form implies that \( p_1^*dB = p_2^*dB \) so that \( dB = \pi^*H \) for some closed 3-form \( H \) on \( M \) called the curvature of the gerbe \( G \). The isomorphism \( t \) provides a canonical trivialization of the

\(^1\) We denote by \( \mathcal{L}_{y_1, y_2} \) the fiber of \( \mathcal{L} \) over \((y_1, y_2) \in Y^{[2]}\).
line bundle $d^*\mathcal{L}$, where $d$ is the diagonal embedding of $Y$ into $Y^{[2]}$ and a canonical isomorphism of $\sigma^*\mathcal{L}$ with $\mathcal{L}^{-1}$, where $\sigma(y_1,y_2) = (y_2,y_1)$ and $\mathcal{L}^{-1}$ denotes the line bundle dual to $\mathcal{L}$. In particular, $\mathcal{L}_{y,y} \cong \mathbb{C}$ and $\mathcal{L}^{-1}_{y_1,y_2} \cong \mathcal{L}_{y_2,y_1}$ canonically.

3. BUNDLE GERBE HOLONOMY

Let $\Sigma$ be a closed oriented surface. If $\mathcal{G} = (Y,B,L,t)$ is a gerbe over $M$ and $\phi : \Sigma \to M$ then one may associate to $\phi$ a phase in $U(1)$ denoted $\text{Hol}_{\mathcal{G}}(\phi)$ and called the holonomy of $\mathcal{G}$ along $\phi$. We shall need an explicit representation of such a phase.

To this end, let us choose a triangulation of $\Sigma$ composed of triangles $c$ (with orientation inherited from $\Sigma$), edges $b$ and vertices $v$, see Fig. 1. We suppose that it is sufficiently fine so that one may choose maps $s_c : c \to Y$ and $s_b : b \to Y$ and elements $s_v \in Y$ such that

$$\pi \circ s_c = \phi|_c, \quad \pi \circ s_b = \phi|_b, \quad \pi(s_v) = \phi(v). \quad (3.1)$$

Then the holonomy of $\mathcal{G}$ along $\phi$ may be given by the expression

$$\text{Hol}_{\mathcal{G}}(\phi) = e^{\sum_c s_c^*B \otimes \text{hol}_L(s_c|_b, s_b)}, \quad (3.2)$$

where we use a slightly abusive notation in which $\text{hol}_L(\ell)$ stands for the parallel transport in the line bundle $\mathcal{L}$ along the curve $\ell$ in $Y^{[2]}$, a linear map from the fiber of $\mathcal{L}$ over the initial point of $\ell$ to the one over the final point. A priori, the expression on the right hand side of (3.2) is an element of the line

$$\bigotimes_{v\in b\subset c} \mathcal{L}^\pm_{s_v(v), s_b(v)}, \quad (3.3)$$

where the minus power (the dual line) is chosen if $v$ has a negative orientation, i.e. is the initial point of the edge $b$ with orientation inherited from $c$. The groupoid structure on $\mathcal{L}$ defined by the isomorphism $t$ of (2.1), however, makes the line (3.3) canonically isomorphic to $\mathbb{C}$. Indeed, for a fixed vertex $v_0$ as in Fig. 2

we have

$$\bigotimes_{v_0 \in b\subset c} \mathcal{L}^\pm_{s_c(v_0), s_b(v_0)} \cong \mathcal{L}_{s_{c_1}(v_0), s_{b_1}(v_0)} \otimes \mathcal{L}_{s_{c_2}(v_0), s_{b_2}(v_0)} \otimes \mathcal{L}_{s_{c_3}(v_0), s_{b_3}(v_0)} \otimes \mathcal{L}_{s_{c_4}(v_0), s_{b_4}(v_0)} \otimes \mathcal{L}_{s_{c_5}(v_0), s_{b_5}(v_0)}$$

FIG. 1: Triangulation of $\Sigma$ for gerbe holonomy calculation

FIG. 2: Triangulation of $\Sigma$ around vertex $v_0$
and a cyclic permutation of terms does not change the isomorphism with \( \mathbb{C} \) because of the associativity of the groupoid multiplication in \( \mathcal{L} \). Hence, the right hand side of (3.2) may be canonically viewed as a complex number that, in fact, is a phase in \( U(1) \).

**Proposition 1** [13]. The phase associated to the right hand side of (3.2) is independent of the choice of the maps \( s_a \) and \( s_b \) and of the triangulation of \( \Sigma \).

**Proof**. We give here a brief proof of Proposition 1 since below we shall need its refinements.

1. If we change the map \( s_{c_0} \) to \( s'_{c_0} \) for a triangle \( c_0 \) then

\[
e^{i \int_{c_0} s'_{c_0} B} = e^{i \int_{c_0} s_{c_0} B} e^{i \int_{c_0} (s'_{c_0} B - s_{c_0} B)} = e^{i \int_{c_0} s_{c_0} B} e^{i \int_{c_0} (s_{c_0} s'_{c_0})^* F_L} \cong e^{i \int_{c_0} s_{c_0} B} \otimes \text{hol}_{\mathcal{L}}(s_{c_0}, s'_{c_0} b)\),
\]

(3.5)

where the last tensor product that belongs to \( \otimes_{c \in b \subset \mathcal{C}} \mathcal{L}^\pm \) gives rise to the holonomy of \( \mathcal{L} \) along the loop \( (s_{c_0}, s'_{c_0}) \partial_{c_0} \) in \( Y^2 \). Now the isomorphisms \( t \) map \( \text{hol}_{\mathcal{L}}(s_{c_0} b, s'_{c_0} b) \otimes \text{hol}_{\mathcal{L}}(s'_{c_0} b, s_b) \) to \( \text{hol}_{\mathcal{L}}(s_{c_0} b, s_b) \) so that one may identify \( e^{i \int_{c_0} s'_{c_0} B} \otimes \text{hol}_{\mathcal{L}}(s_{c_0} b, s_b) \) with \( e^{i \int_{c_0} s_{c_0} B} \otimes \text{hol}_{\mathcal{L}}(s_{c_0} b, s_b) \) and this identification commutes with both the expressions with numbers in \( \mathbb{C} \) also based on applying isomorphisms \( t \), as the latter are associative.

2. Similarly, if we change the map \( s_{b_0} \) to \( s'_{b_0} \) then we may identify \( \otimes_{c \supset b_0} \text{hol}_{\mathcal{L}}(s_{c} b_{0}, s'_{b_0}) \) with \( \otimes_{c \supset b_0} \text{hol}_{\mathcal{L}}(s_{c} b_{0}, s_{b_0}) \) using \( t \) but \( \otimes_{c \supset b_0} \text{hol}_{\mathcal{L}}(s_{b_0}, s'_{b_0}) \) is canonically equal to 1 as the terms appear in dual pairs corresponding to two triangles bordering the edge \( b_0 \) that induce on it opposite orientations.

3. To show that the phase associated to the right hand side of (3.2) is independent of the triangulation, let us change the latter by subdividing one of the triangles \( \triangle \) as on the left hand side of Fig. 3 defining \( s_{c'} = s_{c} |_{c'} \), \( s_{b'} = s_{b} |_{b'} \), etc. Then the right hand side of (3.2) picks up additionally only trivial factors \( \text{hol}_{\mathcal{L}}(s_{c} b_{0}, s_{b}) \) etc. canonically identified with 1. Similarly, if we subdivide one of the edges \( b \) and the neighboring triangles \( c_1, c_2 \) as on the right hand side of Fig. 3 choosing \( s_{c_1'} = s_{c_1} |_{c_1'} \), \( s_{c_2'} = s_{c_1} |_{c_2'} \), \( s_{b'} = s_{b} |_{b'} \) and \( s_{b''} = s_{b} |_{b''} \), \( s_{b''} = s_{b} |_{b''} \), then the right hand side of (3.2) changes only by decomposing \( \text{hol}_{\mathcal{L}}(s_{c_1} b, s_b) \) as \( \text{hol}_{\mathcal{L}}(s_{c_1} b, s_b) \otimes \text{hol}_{\mathcal{L}}(s_{c_1} b, s_b |_{b'}) \) and by adding factors canonically equal to 1 and its numerical value remains unchanged. The above shows that the phases associated to the right hand side of (3.2) are equal for triangulations differing by two-dimensional Pachner moves [27] whose chains allow to relate any two triangulations of \( \Sigma \) to a common third one.

![FIG. 3: Two ways of generating a finer triangulation of \( \Sigma \)](image)

**Example 1.** If \( \phi : \Sigma \to M \) is a constant map then \( \text{Hol}_G(\phi) = 1 \). Indeed, one may choose in this case all \( s_c \) and \( s_b \) to be constant and taking the same value and all contributions to the right hand side of (3.2) become canonically equal to 1.

If \( \mathcal{D} : \Sigma \to \Sigma \) is a diffeomorphism that preserves or changes the orientation then, respectively,

\[
\text{Hol}_G(\phi) = \text{Hol}_G(\phi \circ \mathcal{D})^{\pm 1}.
\]

(3.6)

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2 Such proofs may be also done in the cohomological language using local data for gerbes that we discuss in Sec. 5; see [11, 13].
This follows by computing the right hand side with the triangulation obtained from that used for the left hand side by application of \( D^{-1} \) and with the maps \( s_c \circ D \) and \( s_b \circ D \).

**Proposition 2.** If \( \phi : \Sigma \rightarrow M \) has an extension \( \psi : \mathcal{T} \rightarrow M \) to an oriented compact 3-manifold \( \mathcal{T} \) with boundary \( \partial \mathcal{T} = \Sigma \) then for any gerbe \( \mathcal{G} \) with curvature \( H \)

\[
\text{Hol}_{\mathcal{G}}(\phi) = e^{i \int \psi^* H}.
\]  

**Proof.** Let us triangulate \( \mathcal{T} \) denoting by \( h, c, b, v \) the corresponding simplices assumed sufficiently small so that one may choose over them the lifts \( s_b, s_c \) and \( s_b \) of \( \psi \) to \( Y \). The tetrahedra \( h \) will be taken with the orientation induced from \( \mathcal{T} \). Then

\[
e^{i \int \psi^* H} = e^{i \sum_{c} \int_{c} s_b^* dB} = e^{i \sum_{c \subset h} \int_{c} s_b^* B} = e^{i \sum_{c \subset h} \int_{c} (s_b^* B - s_c^* B)} = e^{i \sum_{c \subset h} \int_{c} s_b^* B} = e^{i \sum_{c \subset h} \int_{c} (s_b \circ s_c|_e)^* F_L} = e^{i \sum_{c \subset h} \int_{c} s_b^* B} = e^{i \sum_{c \subset h} \int_{c} (s_b \circ s_c|_e)^* F_L} = e^{i \sum_{c \subset h} \int_{c} s_b^* B} \prod_{b \subset c \subset h} \text{hol}_L((s_c, s_b)|_b) \cong e^{i \sum_{c \subset h} \int_{c} s_b^* B} \bigotimes_{b \subset c \subset h} \text{hol}_L((s_c, s_b)|_b) = e^{i \sum_{c \subset h} \int_{c} s_b^* B} \bigotimes_{b \subset c \subset h} \text{hol}_L((s_c, s_b)|_b) = \text{Hol}_{\mathcal{G}}(\phi),
\]

where the last but one equality arises since in the preceding expression each term \( \text{hol}_L((s_c, s_b)|_b) \) appears twice with opposite orientations of \( b \) shared by two faces \( c \) of \( h \) and, similarly, each term \( \text{hol}_L((s_c, s_b)|_b) \) appears twice with opposite orientations of \( b \subset c \subset \partial \mathcal{T} = \Sigma \) corresponding to two \( h \) sharing the face \( c \).

**Remark.** The homotopic formula (3.7) coincides with Witten’s definition of the WZ Feynman amplitude \( e^{iS_{WZ}(\phi)} \) [21,22] which requires that \( \int_{\mathcal{T}} \psi^* H \) for an extension \( \psi \) of \( \phi \) be well defined modulo \( 2\pi \). This holds whenever \( H \) is a curvature of a gerbe. There may, however, be \( \phi \) with no extension \( \psi \) and then \( \text{Hol}_{\mathcal{G}}(\phi) \) cannot be defined this way. In such cases, which correspond to \( M \) with non-trivial 2nd homology, \( \text{Hol}_{\mathcal{G}}(\phi) \) depends also on the gerbe \( \mathcal{G} \) and not only on its curvature.

### 4. Equivariance of Gerbes Under an Involution

Equivariant bundle gerbes were studied by several authors, see [17,18,29,14,15,26,2,22]. We shall discuss here a simple version of such an equivariance under an involution \( \Theta : M \rightarrow M \) that induces a \( \mathbb{Z}_2 \)-action on \( M \). Let \( \mathcal{G} = (Y, B, \mathcal{L}, t) \) be a bundle gerbe over \( M \). A \( \mathbb{Z}_2 \)-equivariant extension of \( \mathcal{G} \) is a gerbe \( \tilde{\mathcal{G}} = (\tilde{Y}, \tilde{B}, \tilde{\mathcal{L}}, \tilde{t}) \) over \( M \) such that \( \tilde{Y} = Z_2 \times Y \) with the projection \( \tilde{Y} \ni (z, y) \rightarrow z \pi(y) \in M \) for \( z = \pm 1 \) and \( \tilde{B} = p^* B \) for \( p(z, y) = y \). We may decompose

\[
\tilde{Y}^{[n]} = \bigsqcup_{(z_1, \ldots, z_n)} \tilde{Y}^{[n]}_{(z_1, \ldots, z_n)} \quad \text{for} \quad \tilde{Y}^{[n]}_{(z_1, \ldots, z_n)} \subset \times_{m=1}^{n} \{ z_m \} \times Y.
\]

In particular, we may identify \( \tilde{Y}^{[n]}_{1,1} \) with \( Y^{[n]} \) by the restriction of the map \( p^{\times n} \). We demand that under this identification,

\[
\tilde{\mathcal{L}}|_{\tilde{Y}^{[2]}_{1,1}} = \mathcal{L}, \quad \tilde{t}|_{\tilde{Y}^{[3]}_{1,1}} = t.
\]

Finally, note that \( \mathbb{Z}_2 \) acts on \( \tilde{Y} \) by \( z(z', y) = (z'z, y) \) covering the \( \mathbb{Z}_2 \) action on \( M \) and this action lifts diagonally to \( \tilde{Y}^{[n]} \). We demand that the \( \mathbb{Z}_2 \)-action on \( \tilde{Y}^{[2]} \) lifts to a \( \mathbb{Z}_2 \)-action on \( \tilde{\mathcal{L}} \) by bundle isomorphisms that commute with \( \tilde{t} \) and we fix such a lift. It is easy to see by considering the curvature of the line bundle \( \tilde{\mathcal{L}} \) restricted to \( \tilde{Y}^{[2]}_{1,1} \) that the existence of the gerbe \( \tilde{\mathcal{G}} \) implies that the curvature form \( H \) of the gerbe

\[3\] We shall view \( \mathbb{Z}_2 \) as the multiplicative group composed of \( \pm 1 \).
must be preserved by the $\mathbb{Z}_2$-action on $M$. The notion of the $\mathbb{Z}_2$-equivariant extension of a gerbe $G$ is equivalent to the one of the $\mathbb{Z}_2$-equivariant structure on $\mathcal{G}$ as defined in [14] or [12], see Appendix.

In order to simplify the notations, we shall identify below $\tilde{Y}_{1,1}^{[2]}$ with the manifold

$$Z = \{(y, y') \in Y \times Y \mid \pi(y) = \Theta(\pi(y'))\} \quad (4.3)$$

and the line bundle $\tilde{E}$ restricted to $\tilde{Y}_{1,1}^{[2]}$ with a line bundle $\mathcal{K}$ on $Z$. The groupoid multiplication $\tilde{t}$ and the $\mathbb{Z}_2$ symmetry of $\tilde{E}$ induce the isomorphisms

$$L_{y,y'} \otimes \mathcal{K}_{y',y''} \cong \mathcal{K}_{y,y''}, \quad \mathcal{K}_{y',y''} \otimes L_{y',y'} \cong \mathcal{K}_{y',y'}, \quad \mathcal{K}_{y',y''} \cong \mathcal{K}_{y',y''}^{-1} \quad (4.4)$$

for $(y, y') \in Y^{[2]}$ and $(y', y'') \in Z$. We shall abundantly use below.

If $\Theta$ acts without fixed points, then the $\mathbb{Z}_2$-equivariant extension $\tilde{G}$ of $G$ induces a gerbe $\tilde{G} = (\tilde{Y}, \tilde{B}, \tilde{E}, \tilde{t})$ over the quotient manifold $\tilde{M} = M/\mathbb{Z}_2$, see [13, 14] for a discussion of gerbes on smooth discrete quotients. One takes $\tilde{Y} = Y$ but projected to $\tilde{M}$ rather than to $M$ and $\tilde{B} = B$. Then

$$\tilde{Y}^{[n]} = \bigcup_{(z_1, \ldots, z_n)} \tilde{Y}_{1;z_1,\ldots,z_n}^{[n]} \subset \tilde{Y}^{[n]} \quad (4.5)$$

and one sets $\tilde{E} = \tilde{E}|_{\tilde{Y}^{[2]}}$ and $\tilde{t} = \tilde{t}|_{\tilde{Y}^{[3]}}$, the latter after the composition with the $\mathbb{Z}_2$-symmetry of $\tilde{E}$. The $\mathbb{Z}_2$ equivariant extension $\tilde{G}$ will serve as the replacement for $\tilde{G}$ in the case when $\Theta$ has fixed points.

### 5. SURFACES WITH ORIENTATION-PRESERVING INVOLUTIONS

Suppose that the closed oriented surface $\Sigma$ is equipped with an orientation preserving involution $\vartheta$. We shall consider $\Sigma$ with the $\mathbb{Z}_2$-action induced by $\vartheta$. Let $\Sigma'$ denote the set of fixed points of $\vartheta$. If $\Sigma' = \emptyset$ then $\Sigma/\mathbb{Z}_2$ is again a closed oriented surface. We shall be interested here in the case when $\Sigma' \neq \emptyset$. The canonical example will be given by the torus $T^2 = \mathbb{R}^2/(2\pi \mathbb{Z}^2)$ viewed as a square $[-\pi, \pi]^2$ with the periodic identifications of the boundary points and the involution $\vartheta$ given by $k \mapsto -k$ with four fixed points, see Fig. 4. The most general examples with connected $\Sigma$ and $\Sigma' \neq \emptyset$ are provided by the doubly ramified covers between Riemann surfaces of genus $g$ and $h$, including the hyperelliptic cover with $h = 0$. The case of $T^2$ with the $k \mapsto -k$ involution corresponds to $g = 1$ and $h = 0$. The cardinality $|\Sigma'|$ of $\Sigma'$ satisfies the identity $|\Sigma'| = 2g + 2 - 4h$ following from the Riemann-Hurwitz formula. In particular, it is even. Around each fixed point, $\vartheta$ acts as $z \mapsto -z$ in an appropriate local complex coordinate.

If $\Sigma' \neq \emptyset$ then we shall view the quotient space $\Sigma = \Sigma/\mathbb{Z}_2$ as a $\mathbb{Z}_2$-orbifold rather than a smooth lower genus surface. As such, it possesses an orbifold triangulation with triangles $\tilde{c}$, edges $\tilde{b}$ and vertices $\tilde{v}$, the latter including the images of the fixed points of $\vartheta$. The preimages of the simplices of that triangulation form a triangulation of $\Sigma$ with triangles $c$, edges $b$ and vertices $v$. The latter include the fixed points of $\vartheta$ whereas the other simplices of the triangulation of $\Sigma$ form pairs whose elements are interchanged by $\vartheta$.

![FIG. 4: Periodized square with the fixed points of the involution $k \mapsto -k$](image-url)
6. SQUARE ROOT OF GERBE HOLONOMY

As we have seen, if the involution \( \Theta : M \to M \) acts without fixed points then the \( \mathbb{Z}_2 \)-equivariant extension \( \tilde{G} = (\tilde{Y}, \tilde{B}, \tilde{L}, \tilde{t}) \) of a gerbe \( G = (Y, B, L, t) \) over \( M \) induces a gerbe \( \tilde{G} \) over \( \tilde{M} = M/\mathbb{Z}_2 \). Any map \( \tilde{\phi} \) from a closed oriented surface \( \tilde{\Sigma} \) to the quotient manifold \( \tilde{M} \) may be viewed as a map \( \phi : \Sigma \to M \) from a double cover \( \Sigma \) of \( \tilde{\Sigma} \) to \( M \) that satisfies an equivariance condition

\[
\phi \circ \vartheta = \Theta \circ \phi
\]

for the orientation-preserving deck involution \( \vartheta \) of \( \Sigma \) interchanging the two preimages of the points of \( \tilde{\Sigma} \). One has the relation

\[
\left( \text{Hol}_{\tilde{G}}(\tilde{\phi}) \right)^2 = \text{Hol}_G(\phi).
\]  

(6.2)

The present section is devoted to a construction that provides an extension of such a relation to cases when the involutions \( \vartheta \) and \( \Theta \) have fixed points. We shall show that, under special conditions that will be specified below, a \( \mathbb{Z}_2 \)-equivariant extension \( \tilde{G} = (\tilde{Y}, \tilde{B}, \tilde{L}, \tilde{t}) \) of a gerbe \( G = (Y, B, L, t) \) over \( M \) permits to define a distinguished square root of the holonomy \( \text{Hol}_G(\phi) \) of maps \( \phi : \Sigma \to M \) satisfying the equivariance condition (6.1). We shall construct such a square root via a local formula, a refinement of the one for the gerbe holonomy described in Sec.3.

For every triangle \( \tilde{c} \) and every edge \( \tilde{b} \) of a sufficiently fine orbifold triangulation of \( \tilde{\Sigma} = \Sigma/\mathbb{Z}_2 \), we shall select their lifts \( c \) and \( b \) to \( \Sigma \) and then lifts \( s_c : c \to Y \) and \( s_b : b \to Y \) of \( \phi|_c \) and \( \phi|_b \), respectively. Triangles \( c \) will be considered with the orientation inherited from \( \Sigma \). If \( \tilde{b} \subset \tilde{c} \) then either \( b \subset c \) or \( \vartheta(b) \subset c \). An example for \( \Sigma = T^2 \) and \( \vartheta \) given by \( k \mapsto -k \) is presented in Fig.5.

**FIG. 5**: Lift of a triangulation of \( T^2/\mathbb{Z}_2 \) with colored selected triangles \( c \) and thick selected edges \( b \)

Consider now the expression

\[
\sum_{c} \int_{b \subset c} s_{c,b}^* \left( \otimes_{b \subset c} \text{hol}_L(s_c|_b, s_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c} \text{hol}_K(s_c \circ \vartheta|_b, s_b) \right),
\]

(6.3)

where here and below the sums and tensor products involve only the selected lifts of triangles \( \tilde{c} \) and edges \( \tilde{b} \). The parallel transports on the right hand side are well defined because if \( b \subset c \) then \( \pi \circ s_c|_b = \phi|_b = \pi \circ s_b \) and if \( \vartheta(b) \subset c \) then \( \pi \circ s_c \circ \vartheta|_b \circ \pi = \phi \circ \vartheta|_b = \Theta \circ \phi|_b = \Theta \circ \pi \circ s_b \). Note that

\[
\left( \otimes_{b \subset c} \text{hol}_L(s_c|_b, s_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c} \text{hol}_K(s_c \circ \vartheta|_b, s_b) \right) \in \left( \otimes_{v \in b \subset c} L_{s_c(v),s_b(v)}^{\pm 1} \right) \otimes \left( \otimes_{v \in \vartheta(b) \subset c} K_{s_c(v),s_b(\vartheta(v))}^{\pm 1} \right) = \otimes_{\tilde{v}} \left( \otimes_{e \in \tilde{b} \subset \tilde{c}} \right) \left( \otimes_{v \in \vartheta(b) \subset c} \right) P_{\vartheta} \]  

(6.4)

where on the right hand sides we regrouped together the tensor factors involving vertices \( v \in \Sigma \) projecting to a given vertex \( \tilde{v} \in \tilde{\Sigma} \).
Let us analyze the line $\mathcal{P}_\tilde{v}_0$ for a fixed vertex $\tilde{v}_0$. To this end, let us number the triangles $\tilde{c} \ni \tilde{v}_0$ counterclockwise as $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_k$ and the edges $b \ni \tilde{v}_0$ as $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_k$ oriented towards $\tilde{v}_0$, starting from the edge shared by $\tilde{c}_1$ and $\tilde{c}_2$, as in Fig.2 but now for triangles and edges of $\tilde{\Sigma}$. Let us denote by $v_i \ (v'_i)$ the vertex in $c_i \ (b_i)$ that projects to $\tilde{v}_0$. We shall define

$$\tilde{x}_{\tilde{c}}(\tilde{v}_0) = (z_i, s_c(v_i)) \in \tilde{Y}, \quad \tilde{x}_{\tilde{b}}(\tilde{v}_0) = (z'_i, s_b(v'_i)) \in \tilde{Y} \quad (6.5)$$

with the rules

$$z_i' = \begin{cases} 
    z_i & \text{if } b_i \subset c_i, \quad i = 1, \ldots, k, \\
    (0)z_i & \text{if } \vartheta(b_i) \subset c_i, \quad i = 1, \ldots, k,
\end{cases} \quad z_{i+1}' = \begin{cases} 
    z_{i}' & \text{if } b_i \subset c_{i+1}, \quad i = 1, \ldots, k - 1, \\
    (0)z_{i}' & \text{if } \vartheta(b_i) \subset c_{i+1}, \quad i = 1, \ldots, k - 1.
\end{cases} \quad (6.6)$$

This fixes all $z_i$ and $z_i'$ except for $z_1$ whose choice will not matter. The above choices of $z_i$ and $z_i'$ guarantee that

$$\tilde{L}_{\tilde{x}_{\tilde{c}_1}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \cong \begin{cases} 
    \mathcal{L}_{s_{c_1}(v_1), s_b(v_1)} & \text{if } b_i \subset c_i, \quad i = 1, \ldots, k, \\
    \mathcal{K}_{s_{c_1}(v_1), s_b(\vartheta(v_1))} & \text{if } \vartheta(b_i) \subset c_i, \quad i = 1, \ldots, k,
\end{cases} \quad (6.7)$$

$$\tilde{L}^{-1}_{\tilde{x}_{\tilde{c}_1}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \cong \begin{cases} 
    \mathcal{L}^{-1}_{s_{c_1}(v_1), s_b(v_1)} & \text{if } b_i \subset c_{i+1}, \quad i = 1, \ldots, k - 1, \\
    \mathcal{K}^{-1}_{s_{c_1}(v_1), s_b(\vartheta(v_1))} & \text{if } \vartheta(b_i) \subset c_{i+1}, \quad i = 1, \ldots, k - 1.
\end{cases} \quad (6.8)$$

There is still the instance $\tilde{v}_0 \subset \tilde{b}_k \subset \tilde{c}_1$ not covered by the previous formulae. There are two cases here. If $\tilde{v}_0$ is not the image of a fixed point of $\vartheta$ then $z_1 = z_1$ if $v_1 = v_1$ and $z_1 = (0)z_1$ if $v_1 = \vartheta(v_1)$. Similarly, $z_i' = z_1'$ if $v'_i = v_1$ and $z_i' = (0)z_1'$ if $v'_i = \vartheta(v_1)$. Taking $i = k$, we infer that

$$\tilde{L}^{-1}_{\tilde{x}_{\tilde{c}_1}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \cong \begin{cases} 
    \mathcal{L}^{-1}_{s_{c_1}(v_1), s_b(v_1)} & \text{if } b_k \subset c_1, \\
    \mathcal{K}^{-1}_{s_{c_1}(v_1), s_b(\vartheta(v_1))} & \text{if } \vartheta(b_k) \subset c_1
\end{cases} \quad (6.9)$$

in that case so that

$$\mathcal{P}_{\tilde{v}_0} \cong \tilde{L}_{\tilde{x}_{\tilde{c}_1}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \tilde{L}_{\tilde{x}_{\tilde{c}_2}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \tilde{L}_{\tilde{x}_{\tilde{c}_3}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \cdots \otimes \tilde{L}_{\tilde{x}_{\tilde{c}_k}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \tilde{L}_{\tilde{x}_{\tilde{b}_k}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \cong \mathbb{C}, \quad (6.10)$$

where the last canonical isomorphism is obtained as in (6.4) using the line-bundle isomorphisms $\tilde{t}$ of the gerbe $\tilde{\mathcal{G}}$. The isomorphism $\mathcal{P}_{\tilde{v}_0} \cong \mathbb{C}$ does not depend on the choice of the triangle $\tilde{c}_1$ nor on the choice of $z_1$ due to the associativity of the groupoid multiplication in $\tilde{L}$ and its commutation with the $\mathbb{Z}_2$-action. If, however, $\tilde{v}_0 = \vartheta(v_0)$ for $v_0 \in \Sigma'$ then $z'_k = z_1$ if $v'_k \subset c_1$ and $z'_k = (0)z_1$ if $b_k \subset c_1$ and we have

$$\tilde{L}^{-1}_{(-1)\tilde{x}_{\tilde{c}_1}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \cong \begin{cases} 
    \mathcal{L}^{-1}_{s_{c_1}(v_1), s_b(v_1)} & \text{if } b_k \subset c_1, \\
    \mathcal{K}^{-1}_{s_{c_1}(v_1), s_b(\vartheta(v_1))} & \text{if } \vartheta(b_k) \subset c_1
\end{cases} \quad (6.11)$$

so that

$$\mathcal{P}_{\tilde{v}_0} \cong \tilde{L}_{\tilde{x}_{\tilde{c}_1}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \tilde{L}_{\tilde{x}_{\tilde{c}_2}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \tilde{L}_{\tilde{x}_{\tilde{c}_3}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \cdots \otimes \tilde{L}_{\tilde{x}_{\tilde{c}_k}(\tilde{v}_0), \tilde{x}_{\tilde{b}_c}(\tilde{v}_0)} \otimes \tilde{L}_{\tilde{x}_{\tilde{b}_k}(\tilde{v}_0), (-1)\tilde{x}_{\tilde{c}_1}(\tilde{v}_0)} \cong \tilde{L}_{(1, s_{c_1}(v_0)), (-1, s_{c_1}(v_0))} = \mathcal{K}_{s_{c_1}(v_0), s_{c_1}(v_0)} \quad (6.12)$$

in this case. Can we canonically trivialize the latter lines? Let

$$M' = \{ x \in M \mid \Theta(x) = x \} \quad (6.13)$$

be the set of fixed points of $\Theta$ that, for simplicity, we shall assume to be a submanifold of $M$. Let $Y' = \pi^{-1}(M') \subset Y$. Note that the equivariance (6.11) implies that $\vartheta(\Sigma') \subset M'$ so that $s_{c_1}(v_0) \in Y'$.

Consider the map

$$Y' \ni y' \mapsto ((1, y'), (-1, y')) \in \tilde{Y}[2] \quad (6.14)$$

and the flat line bundle $\mathcal{N}' = r^* \tilde{L}$. Note that (6.12) may be rewritten as the relation

$$\mathcal{P}_{\tilde{v}_0} \cong \mathcal{N}'_{s_{c_1}(v_0)}. \quad (6.15)$$
What is easy to see is that the square of the line bundle $\mathcal{N}'$ possesses a natural trivialization
\[ \mathcal{N}'^2 \cong Y'[2] \times \mathbb{C} \] (6.16)
given on the fibers by the $\mathbb{Z}_2$-action on $\tilde{\mathcal{L}}$ and its groupoid multiplication $\tilde{t}$:
\[ \mathcal{N}'^2 = \tilde{\mathcal{L}}^2_{(1,y'),(-1,y')} \cong \tilde{\mathcal{L}}_{(1,y'),(1,y')} \cong \tilde{\mathcal{L}}_{(1,y'),(1,y')} \cong \mathbb{C}. \] (6.17)

Denote by $\pi'$ the restriction of the submersion $\pi$ and by $B'$ the restriction of the 2-form $B$ to $Y'$. The map $\pi': Y' \to M'$ is a surjective submersion and $\tilde{Y}'[2]$ may be identified with a submanifold of $Y'[2]$. It makes then sense to consider the line bundles $\mathcal{L}' = \mathcal{L}|_{\tilde{Y}'[2]}$ with the groupoid multiplication $t'$ induced from the one of $\mathcal{L}$. There is a natural isomorphism of line bundles over $Y'[2]$
\[ \mathcal{L}' \otimes p'^*_2 \mathcal{N}' \xrightarrow{\nu'} p'^*_1 \mathcal{N}' \otimes \mathcal{L}' \] (6.18)
given again by the groupoid multiplication $\tilde{t}$ and the $\mathbb{Z}_2$-action on $\tilde{\mathcal{L}}$. Indeed, fiber-wise,
\[ \mathcal{L}'_{y_1,y_2} \otimes \mathcal{N}'_{y_2} \cong \tilde{\mathcal{L}}_{(1,y_1'),(1,y_2')} \otimes \tilde{\mathcal{L}}_{(1,y_1'),(-1,y_2')} \cong \tilde{\mathcal{L}}_{(1,y_1'),(1,y_2')} \otimes \tilde{\mathcal{L}}_{(-1,y_1'),(-1,y_2')} \cong \mathcal{N}'_{y_1} \otimes \mathcal{L}'_{y_1,y_2} \] (6.19)
which commutes with the groupoid multiplication in $\mathcal{L}'$, i.e. such that for $(y_1',y_2',y_3') \in Y'[3]$ the isomorphism of lines
\[ \mathcal{L}'_{y_1,y_2} \otimes \mathcal{L}'_{y_2,y_3} \otimes \mathcal{N}'_{y_3} \xrightarrow{\nu' \otimes \mathbb{I}_d} \mathcal{L}'_{y_1,y_3} \otimes \mathcal{N}'_{y_3} \xrightarrow{\nu'} \mathcal{N}'_{y_1} \otimes \mathcal{L}'_{y_1,y_3} \] (6.20)
coincides with
\[ \mathcal{L}'_{y_1,y_2} \otimes \mathcal{L}'_{y_2,y_3} \otimes \mathcal{N}'_{y_3} \xrightarrow{\nu' \otimes \mathbb{I}_d} \mathcal{L}'_{y_1,y_2} \otimes \mathcal{N}'_{y_2} \otimes \mathcal{L}'_{y_2,y_3} \otimes \mathcal{N}'_{y_3} \xrightarrow{\nu' \otimes \mathbb{I}_d} \mathcal{N}'_{y_1} \otimes \mathcal{L}'_{y_1,y_3}. \] (6.21)

The isomorphism $\nu'$ allows to canonically identify the lines $\mathcal{N}'_{y'}$ for all $y'$ over the same point $x \in M'$ and the bundle $\mathcal{N}'$ with a pullback $\pi^*\mathcal{N}'$ of a flat bundle $\mathcal{N}'$ over $M'$. A straightforward check shows that the trivialization (6.16) commutes with $\nu'^2$ so that it defines a trivialization of the flat line bundle $\mathcal{N}'^2$ over $M'$. In general, that does not imply the trivializability of the flat line bundle $\mathcal{N}'$. If, however, $M'$ is simply connected then $\mathcal{N}'$ is trivializable (as any flat line bundle over a simply connected manifold) and we may choose its trivialization so that it squares to the trivialization of $\mathcal{N}'^2$ induced by (6.16). Besides, if $M'$ is also connected then such a trivialization of $\mathcal{N}'$ is defined up to a global sign. It induces a preferred trivialization of $\mathcal{N}'$ also defined modulo a global sign. Such a trivialization allows to identify the lines $\mathcal{P}_{\tilde{x}_0}$ of (6.12) with $\mathbb{C}$, again up to a global sign. Let us check that the above identification does not depend on the choice of the initial triangle $\tilde{c}_1 \ni \tilde{x}_0$. The choice of $\tilde{c}_2$ as the initial triangle gives
\[ \mathcal{P}_{\tilde{x}_0} \cong \tilde{\mathcal{L}}_{\tilde{x}_2}(\tilde{x}_0) \otimes \cdots \otimes \tilde{\mathcal{L}}_{\tilde{x}_k}(\tilde{x}_0) \] (6.22)
\[ \otimes \tilde{\mathcal{L}}_{\tilde{x}_k}(\tilde{x}_0),(-1)\tilde{x}_k(\tilde{x}_0) \otimes \tilde{\mathcal{L}}_{(-1)\tilde{x}_k}(\tilde{x}_0) \otimes \mathcal{L}_{(-1)\tilde{x}_k}(\tilde{x}_0) \] (6.23)
i.e.
\[ \mathcal{P}_{\tilde{x}_0} \cong \mathcal{N}'_{\tilde{x}_2}(\tilde{x}_0) \]

The isomorphisms (6.16) and (6.23) may be summarized as resulting from the ones
\[ \tilde{\mathcal{L}}_{\tilde{x}_1}(\tilde{x}_0),(-1)\tilde{x}_1(\tilde{x}_0) \otimes \mathcal{L}_{\tilde{x}_2}(\tilde{x}_0) \cong \tilde{\mathcal{L}}_{\tilde{x}_1}(\tilde{x}_0),(-1)\tilde{x}_1(\tilde{x}_0) \] (6.24)
\[ \tilde{\mathcal{L}}_{\tilde{x}_2}(\tilde{x}_0),(-1)\tilde{x}_1(\tilde{x}_0) \otimes \tilde{\mathcal{L}}_{\tilde{x}_2}(\tilde{x}_0) \cong \tilde{\mathcal{L}}_{\tilde{x}_2}(\tilde{x}_0),(-1)\tilde{x}_2(\tilde{x}_0) \] (6.25)
respectively. Tensoring the both sides of (6.24) with $\tilde{\mathcal{L}}_{(-1)\tilde{x}_1}(\tilde{x}_0),(-1)\tilde{x}_2(\tilde{x}_0)$ and the both sides of (6.25) with $\tilde{\mathcal{L}}_{\tilde{x}_1}(\tilde{x}_0),\tilde{x}_2(\tilde{x}_0)$, we make the left hands equal whereas the identification of the right hand sides agrees with
the interpretation of the line bundle $N'$ as the pullback of the line bundle $N$, see (6.18) and (6.19). We infer that (6.15) and (6.23) induce the same isomorphisms of lines

$$
P_{v_0} \cong N'_\vartheta(v_0)$$

(6.26)

which is then independent of the choice of the initial triangle. If $M'$ a 1-connected then, using the trivialization of $N'$ described above and defined modulo a global sign, we obtain an isomorphism $P_{v_0} \cong \mathbb{C}$ defined up to a sign that is the same for all fixed points $v_0$ of $\vartheta$. Since the number of such fixed points is even, this sign ambiguity disappears when we take the tensor product of such identifications over all $v_0$.

Summarizing the above discussion, we conclude that if $M'$ is a 1-connected submanifold of $M$ then the expression (6.3) may be identified with a phase\(^4\) in $U(1)$ that is independent of the sign in the choice of the trivialization of $N'$.

**Proposition 3.** The $U(1)$-phase associated to the expression (6.3) is independent of the choice of maps $s_c$ and $s_b$ lifting $\varphi|_c$ and $\varphi|_b$ to $Y$, of the lifts $c$ and $b$ of simplices $\bar{c}$ and $\bar{b}$ to $\Sigma$ and of the orbifold triangulation of $\Sigma$.

**Proof.** We shall proceed similarly as the the proof of Proposition 1.

1. If we change the map $s_{c_0}$ to $s'_{c_0}$ for the lift $c_0$ of a triangle $\bar{c}$ then

$$
e^i \int_{c_0} s'_{c_0}^* B = e^i \int_{c_0} s_{c_0}^* B e^i \int_{c_0} (s'_{c_0}^* B - s_{c_0}^* B) = e^i \int_{c_0} s_{c_0}^* B e^i \int_{c_0} (s_{c_0}^* s'_{c_0})^* F_{L}$$

$$= e^i \int_{c_0} s_{c_0}^* B \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_L(s_{c_0}|_b, s_{c_0}'|_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_K(s_{c_0}|_b|, s_{c_0}'|_b) \right)$$

$$= e^i \int_{c_0} s_{c_0}^* B \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_L(s_{c_0}|_b, s_{c_0}'|_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_K(s_{c_0}|_b|, s_{c_0}'|_b) \right).$$

Using the relations

$$
\left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_L(s_{c_0}|_b, s_{c_0}'|_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_K(s_{c_0}|_b|, s_{c_0}'|_b) \right)
$$

one shows that the expression (6.28) after the change is equivalent to the one before the change and the associativity of $\tilde{t}$ guarantees that both define the same phase if $M'$ is a 1-connected submanifold of $M$.

If we change the lift $c_0$ of a triangle $\bar{c}$ to $c_0' = \vartheta(c_0)$ and the map $s_{c_0}$ to $s'_{c_0}$ then

$$
e^i \int_{c_0'} s'_{c_0}^* B = e^i \int_{c_0} s_{c_0}^* B e^i \int_{c_0} (s'_{c_0}^* B - s_{c_0}^* B) = e^i \int_{c_0} s_{c_0}^* B e^i \int_{c_0} (s_{c_0}^* s'_{c_0}|_b)\right)^* F_{L}$$

$$= e^i \int_{c_0} s_{c_0}^* B \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_L(s_{c_0}|_b, s_{c_0}'|_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_K(s_{c_0}|_b|, s_{c_0}'|_b) \right)$$

$$= e^i \int_{c_0} s_{c_0}^* B \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_L(s_{c_0}|_b, s_{c_0}'|_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_K(s_{c_0}|_b|, s_{c_0}'|_b) \right).$$

Using the relations

$$
\left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_L(s_{c_0}|_b, s_{c_0}'|_b) \right) \otimes \left( \otimes_{\vartheta(b) \subset c_0} \text{hol}_K(s_{c_0}|_b|, s_{c_0}'|_b) \right)
$$

\(^4\) The parallel transports occurring in (6.3) and the isomorphisms $P_{v} \cong \mathbb{C}$ preserve the hermitian structures.
Proof of Lemma 1.

\[
\frac{1}{\vartheta(b)\subset c_0} \left( \text{hol}_K(\vartheta|_b, s_{c_0}'|_b) \otimes \text{hol}_L(s_{c_0}|_b, s_b) \right) \overset{\tilde{\vartheta}}{=} \frac{1}{\vartheta(b)\subset c_0} \text{hol}_K(\vartheta|_b, s_b) \quad (6.30)
\]

one shows that, again, the expression (6.3) after the change is equivalent to the one before the change and, as before, the associativity of \( \tilde{\vartheta} \) guarantees that both define the same phase.

2. Similarly, if we change the map \( s_{b_0} \) to \( s'_{b_0} \) for the lift \( \tilde{b}_0 \) of an edge \( \tilde{b}_0 \) then

\[
\otimes_{c \supset \vartheta(b_0)} \text{hol}_L(s_c|_b, s'_{b_0}) \overset{\tilde{\vartheta}}{=} \otimes_{c \supset \vartheta(b_0)} \left( \text{hol}_L(s_c|_b, s_b) \otimes \text{hol}_L(s_b, s'_{b_0}) \right),
\]

\[
\otimes_{c \supset \vartheta(b_0)} \text{hol}_K(s_c|_b, s'_{b_0}) \overset{\tilde{\vartheta}}{=} \otimes_{c \supset \vartheta(b_0)} \left( \text{hol}_K(s_c|_b, s_b) \otimes \text{hol}_L(s_b, s'_{b_0}) \right).
\]

But

\[
\left( \otimes_{c \supset \vartheta(b_0)} \text{hol}_L(s_{b_0}, s'_{b_0}) \right) \otimes \left( \otimes_{c \supset \vartheta(b_0)} \text{hol}_L(s_{b_0}, s'_{b_0}) \right) \overset{1}{=} (6.33)
\]

as the terms appear in dual pairs corresponding to two triangles \( \tilde{c} \) bordering the same edge \( \tilde{b}_0 \) that induce on it and on the corresponding edge \( b_0 \) opposite orientations.

If we change the lift \( b_0 \) of an edge \( \tilde{b}_0 \) to \( b_0' = \vartheta(b_0) \) and the map \( s_{b_0} \) to \( s_{b_0'} \) then

\[
\otimes_{c \supset b_0'} \text{hol}_L(s_c|_{b_0}, s'_{b_0}) = \otimes_{c \supset \vartheta(b_0)} \left( \text{hol}_K(s_c|_{b_0}, s_{b_0}) \otimes \text{hol}_K(s_{b_0}, s_{b_0'}) \right),
\]

\[
\otimes_{c \supset b_0} \text{hol}_K(s_c|_{b_0}, s'_{b_0}) = \otimes_{c \supset \vartheta(b_0)} \left( \text{hol}_K(s_c|_{b_0}, s_{b_0}) \otimes \text{hol}_K(s_{b_0}, s_{b_0'}) \right).
\]

However,

\[
\left( \otimes_{c \supset \vartheta(b_0)} \text{hol}_K(s_{b_0}, s_{b_0'}) \right) \otimes \left( \otimes_{c \supset \vartheta(b_0)} \text{hol}_K(s_{b_0}, s_{b_0'}) \right) \overset{1}{=} (6.36)
\]

as, again, the terms appear in dual pairs corresponding to two triangles \( \tilde{c} \) bordering the same edge \( \tilde{b}_0 \) and inducing on \( b_0 \) opposite orientations.

3. The independence of the phase associated to (6.3) on the orbifold triangulation of \( \Sigma \) is proven by using the two Pachner moves depicted on Fig. 3 for that triangulation and the corresponding subdivisions of the lifted triangles and edges. The argument that such moves lead to equivalent expressions (6.3) is then essentially the same as in the proof of Proposition 1.

We are now ready to define the square root of the gerbe holonomy of equivariant maps.

**Definition 2.** Suppose that \( M \) is a manifold with a \( \mathbb{Z}_2 \)-action induced by an involution \( \Theta \) with the fixed-point set \( M' \) that is a 1-connected submanifold of \( M \). Let \( \mathcal{G} \) be a \( \mathbb{Z}_2 \)-equivariant extension of a gerbe \( \mathcal{G} \) over \( M \) and \( \phi: \Sigma \to M \) be a map satisfying the equivariance condition \( \boxed{[b.1]} \) for an orientation-preserving involution \( \vartheta: \Sigma \to \Sigma \) with discrete fixed points. Then we set

\[
\sqrt{\text{hol}_\mathcal{G}(\phi)} = \left( \sum_{c \subset c_0} \int_c s^*_c B \right) \left( \otimes_{b \subset c} \text{hol}_L(s_c|_b, s_b) \right) \otimes \left( \otimes_{b \subset c} \text{hol}_K(s_c|_b, s_b) \right),
\]

where the right hand side is identified with a \( U(1) \)-phase the way described above.

It remains to show

**Lemma 1.**

\[
\left( \sqrt{\text{hol}_\mathcal{G}(\phi)} \right)^2 = \text{hol}_\mathcal{G}(\phi). \quad (6.38)
\]

**Proof of Lemma 1.** Recall that on the right hand side of (6.37), \( c \) and \( b \) run over the selected lifts of simplices \( \tilde{c} \) and \( \tilde{b} \) of the orbifold triangulation of \( \Sigma \). By Proposition 2, we may also use on the right hand side of (6.37) the opposite choices \( c' = \vartheta(c) \), \( b' = \vartheta(b) \) of such lifts. Now

\[
\left( \sum_{c \subset c_0} \int_c s^*_c B \right) \left( \otimes_{b \subset c} \text{hol}_L(s_c|_b, s_b) \right) \otimes \left( \otimes_{b \subset c} \text{hol}_K(s_c|_b, s_b) \right)
\]


\[ e^{i \sum f_c \ast^o_c B} \left( \bigotimes_{b \in C} \text{hol}_c(s_c|b, s_b) \right) \otimes \left( \bigotimes_{b' \in C} \text{hol}_{c'}(s_c|b', s_b) \right) \]

\[ \tilde{\tau} \cong e^{i \sum f_c \ast^o_c B} \left( \bigotimes_{b \in C} \text{hol}_c(s_c|b, s_b) \right) \otimes \left( \bigotimes_{b' \in C} \left( \text{hol}_c(s_c|b', s_b) \otimes \text{hol}_{c'}(s_c|b, s_b) \right) \right). \] (6.39)

Similarly

\[ e^{i \sum f_c \ast^o_c B} \left( \bigotimes_{b \in C} \text{hol}_c(s_c|b, s_b) \right) \otimes \left( \bigotimes_{b' \in C} \text{hol}_{c'}(s_c|b', s_b) \right) \]

\[ \tilde{\tau} \cong e^{i \sum f_c \ast^o_c B} \left( \bigotimes_{b \in C} \text{hol}_c(s_c|b, s_b) \right) \otimes \left( \bigotimes_{b' \in C} \left( \text{hol}_c(s_c|b, s_b) \otimes \text{hol}_{c'}(s_c|b', s_b) \right) \right). \] (6.40)

Using the relation

\[ \otimes_{b \in C} \left( \text{hol}_c(s_c|b, s_b) \otimes \text{hol}_{c'}(s_c|b, s_b) \right) \tilde{\tau} \cong 1 \] (6.41)

(involving also the $\mathbb{Z}_2$-symmetry of line bundle $\tilde{L}$), we infer that the tensor product of the two versions of the right hand side of (6.37) is equivalent to

\[ e^{i \sum (f_c \ast^o_c B + f_c \ast^o_c B)} \left( \bigotimes_{b \in C} \text{hol}_c(s_c|b, s_b) \right) \otimes \left( \bigotimes_{b' \in C} \text{hol}_{c'}(s_c|b', s_b) \right) \]

\[ \otimes_{b \in C} \left( \bigotimes_{b' \in C} \left( \text{hol}_c(s_c|b', s_b) \otimes \text{hol}_{c'}(s_c|b, s_b) \right) \right). \] (6.42)

which is a version of the right hand side of (6.32) for the triangulation of $\Sigma$ induced from that of $\tilde{\Sigma}$. A straightforward (although somewhat tedious) check shows that the above identifications commute with the ones associating $U(1)$-phases to the right hand sides of the two versions of (6.37) and to (3.2). This proves the identity (6.38).

\[ \square \]

**Example 2.** If $\phi : \Sigma \to M$ is constant with the value $m' \in M'$ then $\sqrt{\text{hol}_c(\phi)} = 1$. This is easily shown choosing all $s_c$ and $s_b$ involved in the expression on the right hand side of (6.37) constant and taking the same value $y' \in Y'$. Then the right hand side of (6.37) is equal to 1 as an element of the line

\[ \left( \bigotimes_{v \in b \in C} \mathcal{L}_{y', y'}^{\pm 1} \right) \otimes \left( \bigotimes_{v \in b \in C} \mathcal{K}_{y', y'}^{\pm 1} \right) \cong \mathbb{C}, \] (6.43)

see (6.4), if the last isomorphism results from the fact that each line is accompanied by its dual corresponding to the opposite end of $b$. But also for the individual lines one has the canonical isomorphisms $\mathcal{L}_{y', y'}^{\pm 1} \cong \mathbb{C}$ and $\mathcal{K}_{y', y'}^{\pm 1} = \mathcal{N}^{\pm 1}_{y', y'} \cong \mathbb{C}$ (in the last case up to a sign) and the latter isomorphisms agree with the ones resulting in the identifications $P_y \cong \mathbb{C}$ on which the interpretation of the right hand side of (6.37) as an $U(1)$-phase is based. Clearly the two identifications of the line (6.43) with $\mathbb{C}$ also agree proving the announced equality.

If $\mathcal{D} : \Sigma \to \Sigma$ is a diffeomorphism that commutes with $\vartheta$ and preserves or reverses the orientation then, respectively,

\[ \sqrt{\text{hol}_c(\phi)} = \sqrt{\text{hol}_c(\phi \circ \mathcal{D})}^{\pm 1}, \] (6.44)

as may be easily seen by computing the left and the right hand sides using, the triangulations, the lifts $c, b$ and the maps $s_c, s_b$ related by $\mathcal{D}$.

### 7. Homotopic Formula for the Square Root of Gerbe Holonomy

Let $\mathcal{T}$ be a compact oriented 3-manifold with boundary $\partial \mathcal{T} = \Sigma$ equipped with an orientation-preserving involution $\zeta$ reducing to $\vartheta$ on the boundary. We shall assume that at the fixed points of $\zeta$ its derivative
has one eigenvalue 1 and two eigenvalues $-1$. Then fixed-point set of $\zeta$ forms necessarily a $1d$ submanifold with boundary $T' \subset T$ such that $\partial T' = \Sigma'$. An example for $\Sigma = \mathbb{T}^2 = \mathbb{R}^2/(2\pi\mathbb{Z}^2)$ with the $k \mapsto -k$ involution would be $T = D \times S$, where $D$ is the unit disc and $S$ the unit circle in the complex plane, with the boundary identification induced by the map $(k_1, k_2) \mapsto (e^{ik_1}, e^{ik_2})$ and with $\zeta$ given by $(z, v) \mapsto (\bar{z}, \bar{v})$. In this case, $T' = [-1, 1] \times \{1\} \cup [-1, 1] \times \{-1\}$, see Fig. 6.

![Diagram](image)

**FIG. 6:** $T = D \times S$ with the fixed-point set $T'$ of the involution $\zeta : (z, v) \mapsto (\bar{z}, \bar{v})$

**Proposition 4.** Suppose that a map $\phi : \Sigma \to M$ satisfying \[ \text{(3.8)} \] has an extension $\psi : T \to M$ such that

$$
\psi \circ \zeta = \Theta \circ \psi .
$$

(7.1)

Assume that the fixed-point set $M'$ of $\Theta$ is a 1-connected submanifold of $M$. Then for any $\mathbb{Z}_2$-equivariant extension $\tilde{G}$ of a gerbe $G$ on $M$ with curvature $H$,

$$
\sqrt{\text{Hol}_G(\phi)} = e^{\frac{i}{2} \int \psi^* H} .
$$

(7.2)

In particular, given $\phi$, the right hand side does not depend on its equivariant extension $\psi$.

**Proof.** The involution $\zeta$ generates on $T$ a $\mathbb{Z}_2$-action and we shall view the quotient space $\tilde{T} = T/\mathbb{Z}_2$ as a $\mathbb{Z}_2$-orbifold. Let us fix a sufficiently fine orbifold triangulation of $\tilde{T}$ with simplices $\tilde{h}, \tilde{c}, \tilde{b}, \tilde{v}$. By definition, its restrictions to the 2-dimensional boundary $\Sigma = \Sigma/\mathbb{Z}_2$ and to the 1-dimensional fixed-point set $T'$ induce triangulations of the latter. The preimages of simplices of the orbifold triangulation give rise to a triangulation of $T$ with simplices permuted within pairs by $\zeta$, except for the ones in $T'$ that are left invariant. Let us fix for the simplices $\tilde{h}, \tilde{c}, \tilde{b}$ their lifts $h, c, b$ to $T$ and the maps $s_h : h \to Y$, $s_c : c \to Y$, $s_b : b \to Y$ such that

$$
\pi \circ s_h = \psi|_h , \quad \pi \circ s_c = \psi|_c , \quad \pi \circ s_b = \psi|_b .
$$

(7.3)

Similarly as in \[ \text{(6.8)} \], we have

$$
\frac{i}{2} \int \psi^* H = \frac{i}{2} \sum_h (f_h \psi^* H + f_{(h)} \psi^* H) = \frac{i}{2} \sum_h f_h \psi^* H = \frac{i}{2} \sum_{c \subset h} f_c s_c^* B + i \sum_{c \subset h} f_{(c)} \psi^* H .
$$

(7.4)
where $\otimes'$ means that $b \subset T'$ are omitted to avoid an overcount. Reshuffling the terms on the right hand side, we obtain

\[
\int_{\partial' T'} \phi^* H \approx e^{i \sum_{b \in c} F_b s_b B} \left( \otimes_{b \in c \subset h} \text{hol}_{\Sigma}(s_b, s_{b_1}) \right) \left( \otimes_{b \in c \subset \xi(b)} \text{hol}_{\Sigma}(s_b, s_{b_1}) \right) \left( \otimes_{b \in c \subset \xi(c)} \text{hol}_{\Sigma}(s_b, s_{b_1}) \right) = \text{hol}_{\Sigma}(s_b, s_{b_1}) \text{hol}_{\Sigma}(s_b, s_{b_1}) \text{hol}_{\Sigma}(s_b, s_{b_1}),
\]

(7.5)

where we used the fact that, in the term between $\equiv$ signs, in the first line for each pair $(b,c)$ such that $b \subset c \notin \Sigma$ and in the second line for $(b,c)$ such that $\xi(b) \subset c \notin \Sigma$ there are two of tetrahedra $h$ or $\xi(h)$ containing $c$ and inducing on it opposite orientations. Similarly in the third line for each $(b,h)$ such that $b \subset h$ and in the forth line for each $(b,h)$ such that $h \subset \xi(h)$ and $b \subset \xi(h)$ there are two of triangles $\xi(c)$ or $c$ in $h$ containing $b$ and inducing on it opposite orientations. We still have to analyze the last line of (7.5). To this end, let us consider a small $\xi$-invariant neighborhood $U \subset T'$ around a fixed edge $b \subset T'$ that is diffeomorphic to $I \times D$ where $I$ is an interval and $D$ is a unit disc in the complex plane, with $\xi$ acting by $(t, z) \mapsto (t, -z)$ so that $T' \cap U$ is represented by $I \times \{0\}$. We may assume that the lifts $h, c$ of tetrahedra $\tilde{h}$ and triangles $\tilde{c}$ that share the edge $\tilde{b}$ are chosen so that their intersections with the disc $(x) \times D$ transverse to $b$ are as on Fig. 7 (otherwise, we change those lifts). Then the contribution of $b$

(considered with the orientation of the interval $I$) to the last line of (7.5) is

\[
\text{hol}_{\Sigma}(s_b, s_{h_1}) \otimes \text{hol}_{\Sigma}(s_{h_2}, s_b) \otimes \text{hol}_{\Sigma}(s_b, s_{h_2}) \otimes \cdots \otimes \text{hol}_{\Sigma}(s_b, s_{h_1}) \otimes \text{hol}_{\Sigma}(s_{h_1}, s_b)
\]

\[
\equiv \text{hol}_{\Sigma}(s_b, s_{b_1}) \equiv \text{hol}_{\Sigma}(s_b, s_b)
\]

(7.6)

where we used the fact that $\psi(T') \subset M'$ due to the equivariance (7.4) so that $s_b$ takes values in $Y'$. Hence the last line in (7.5) builds to

\[
\text{hol}_{\Sigma}(\psi|_{T'}) \in \otimes_{v \in \partial' T'} N'_{\phi(v)}^{(1)} \cong \otimes_{v \in \Sigma'} N'^{-1}_{\phi(v)},
\]

(7.7)

where the last isomorphism uses the fact that $\partial T' = T' \cap \Sigma = \Sigma'$ and the natural isomorphism $N' \cong N'^{-1}$, see (6.2) and the remark under (6.2). In fact, the last line in (7.5) results in (7.7) for any choice of the lifts $h, c$. Recall from (6.4), (6.10) and (6.26) that the first line in (7.5) may be naturally interpreted as an element of the line $\otimes_{v \in \Sigma'} N'_{\phi(v)}$, which is consistent with the fact that the tensor product of both lines describes a $U(1)$-phase. Now, if $M'$ is a 1-connected submanifold of $M$ then $N' \cong \Sigma$ as a flat line bundle and both lines of (7.5) may be viewed as contributing $U(1)$-phases, the first one equal to $\sqrt{\text{hol}_{\Sigma}(\phi)}$ and the second one equal to 1. This proves the identity (7.2).
Remark. The right hand side of (7.2) could be taken as Witten-type definition of the Feynman amplitude $e^{iS_{WZ}(\phi)}$ for equivariant maps $\phi$. The net result of imposing the equivariance condition (7.1) on the extension $\psi$ of $\phi$ is to make $\int_{\mathcal{F}} \psi^* H$ well defined modulo $4\pi$ rather than $2\pi$.

8. LOCAL DATA

Let $L$ be a line bundle over $M$ (as always here, equipped with a hermitian structure and a unitary connection) and $(O_i)_{i \in J}$ a sufficiently fine open covering of $M$. It is well known that $L$ may be represented (up to an isomorphism) by local data $(a_i, h_{ij})$ with real connection 1-forms $a_i$ and $U(1)$-valued transition functions, defined on $O_i$ and $O_i \cap O_j \equiv O_{ij}$, respectively. Such local data are obtained from local normalized sections $\sigma_i : O_i \to L$ of $L$ by the formulae

$$i \nabla_L \sigma_i = a_i \sigma_i, \quad \sigma_{ij} = h_{ij} \sigma_i \sigma_j.$$

They satisfy the relations

$$a_{ij} - a_i = id h_{ij}, \quad h_{ij} h_{jk} = h_{ij}.$$

Remark. We demand that $|\sigma_{ij}(z)| = 1$ and $\sigma_{ij}(z) = \sigma_{ji}(\bar{z})^{-1}$ (i.e. is the dual element to $\sigma_{ij}(z)$). The local data $(B_i, A_{i1}, g_{i12})$ for the gerbe $G$ are then defined by the relations

$$B_i = s_i^* B, \quad i \nabla_L \sigma_{i1} = \sigma_{i1} A_{i1}, \quad t \circ (\sigma_{i1} \otimes \sigma_{i2}) = g_{i12} \sigma_{i1} \sigma_{i2}.$$

Similarly one may extract local data for a gerbe $G = (Y, B, \mathcal{L}, t)$ over $M$. Let $s_i : O_i \to Y$ be maps such that $\pi(s_i(x)) = x$ and let $\sigma_{i12} : O_{i12} \equiv O_i \cap O_j \to \mathcal{L}$ be such that $\sigma_{i12}(x) \in \mathcal{L}(s_i(x), s_j(x))$. The local data $(B_i, A_{i1}, g_{i12})$ for the gerbe $G$ are then defined by the relations

$$B_i = s_i^* B, \quad i \nabla_L \sigma_{i1} = \sigma_{i1} A_{i1}, \quad t \circ (\sigma_{i1} \otimes \sigma_{i2}) = g_{i12} \sigma_{i1} \sigma_{i2}.$$

The curvature $H$ of $G$ satisfies $H = dB_i$ on $O_i$. In terms of the local data, the expression for the gerbe holonomy takes the form

$$\text{Hol}_G(\phi) = \exp \left[ \sum_c \int_c \phi^* B + i \sum_{b \subset c} \int_b \phi^* A_{i_b} \right] \prod_{v \in \partial c} g_{i_c i_b} \phi(v)^{\pm 1},$$

where $c$, $b$ and $v$ are the triangles, edges and vertices of a sufficiently fine triangulation of $\Sigma$ and the indices $i_c$, $i_b$ and $i_v$ are chosen so that

$$\phi(c) \subset O_i, \quad \phi(b) \subset O_i, \quad \phi(v) \in O_i.$$

Let $\Theta$ be an involution of $M$. We shall assume that the covering $(O_i)_{i \in J}$ is invariant under the $\mathbb{Z}_2$-action induced by the involution $\Theta$, i.e. $zO_i = O_i$ for some $\mathbb{Z}_2$-action on the index set $J$. We shall write $-z, -x, -i$ for $(1)z, (1)x = \Theta(x)$ and $(1)i$. Let $\tilde{G} = (\tilde{Y}, \tilde{B}, \tilde{\mathcal{L}}, \tilde{t})$ be a $\mathbb{Z}_2$-equivariant extension of $G$. We shall repeat the previous construction of local data for $G$ defining maps $\tilde{s}_i^* : O_i \to \tilde{Y}$ such that $\tilde{s}_i^* = (z, s_i - 1)(z^{-1} x)$ (of course $z^{-1} = z$ in $\mathbb{Z}_2$) and $\tilde{\sigma}_{i12}^z : O_{i12} \to \tilde{\mathcal{L}}$ such that $\tilde{\sigma}_{i12}^z(x) \in \tilde{\mathcal{L}}(\tilde{s}_i^z(x), \tilde{s}_j^z(x))$. The local data $(\tilde{B}_i, \tilde{A}_{i1}, \tilde{g}_{i12})$ for the gerbe $G$ are defined by the identities similar to (8.3):

$$\tilde{B}_i = (\tilde{s}_i)^* \tilde{B}, \quad i \nabla_L \tilde{\sigma}_{i12} = \tilde{\sigma}_{i12} A_{i1}, \quad \tilde{t} \circ (\tilde{\sigma}_{i12} \otimes \tilde{\sigma}_{i2}) = \tilde{g}_{i12} \tilde{\sigma}_{i12} \tilde{\sigma}_{i2}.$$

where $c$, $b$ and $v$ are the triangles, edges and vertices of a sufficiently fine triangulation of $\Sigma$ and the indices $i_c$, $i_b$ and $i_v$ are chosen so that

$$\phi(c) \subset O_i, \quad \phi(b) \subset O_i, \quad \phi(v) \in O_i.$$
and they satisfy relations similar to (8.5):

\[ \tilde{B}_{1i}^{(2z)} - B_{1i}^{(2z)} = d\tilde{A}_{1i}^{(2z)}, \quad \tilde{A}_{1i}^{(2z)} - \tilde{A}_{1i}^{(z)} + \tilde{A}_{1i}^{(2z)} = i d\ln \tilde{g}_{1i1z}^{(2z)}, \quad \tilde{g}_{1i1z}^{(2z)}(\tilde{g}_{1i1z}^{(2z)})^{-1} = 1. \tag{8.9} \]

One has:

\[ \tilde{B}_{i}^{(1)} = B_{i}, \quad \tilde{A}_{1i}^{(1)} = A_{1i}, \quad \tilde{g}_{1i1z}^{(1)} = g_{1i1z}. \tag{8.10} \]

The local data \( (\tilde{B}_{i}, \tilde{A}_{1i}^{(1)}, \tilde{g}_{1i1z}^{(1)}) \) may be reduced to simpler ones that provide local data for a \( \mathbb{Z}_2 \)-equivariant structure on gerbe \( \mathcal{G} \) as defined in (4.12). To this end, we shall impose the identities

\[ \tilde{\sigma}_{(-i)} = (-1)\tilde{\sigma}_{(-i)} \circ \Theta, \tag{8.11} \]
\[ \tilde{\sigma}_{i} = \tilde{\sigma}_{i} \circ (\tilde{\sigma}_{i} \otimes \tilde{\sigma}_{i}) \quad \text{i.e.} \quad \tilde{\sigma}_{i} = \tilde{1}. \tag{8.12} \]

and shall define 1-forms \( \Pi_{i} \) on \( O_{i}, \quad U(1) \)-valued functions \( \chi_{1i} = \chi_{1i}^{-1} \) on \( O_{1i} \) and \( f_{i} \) on \( O_{i} \) by the relations

\[ \Pi_{i} = \tilde{A}_{i}^{(-1)}, \quad \chi_{1i} = \tilde{g}_{1i1z}^{(-1)}, \quad (-1)\tilde{\sigma}_{i} = f_{i} \tilde{\sigma}_{i}^{-1} \circ \Theta. \tag{8.13} \]

Let us notice that from the last of the relations (8.8) it follows that

\[ 1 = \tilde{g}_{1i1z}^{-1}(\tilde{g}_{1i1z}^{-1})^{-1} \tilde{g}_{1i1z}^{-1}(\tilde{g}_{1i1z}^{-1})^{-1} = \chi_{1i} \chi_{1i}^{-1} \tag{8.14} \]

i.e. \( \chi_{1i} = \chi_{1i}^{-1} \).

**Proposition 5.** We have the following identities:

\[ \theta^{*} B_{-i} = B_{i} + d\Pi_{i}, \tag{8.15} \]
\[ \theta^{*} A_{(-i)}(-i) = A_{1i} + \Pi_{i} - i d\ln \chi_{1i}, \tag{8.16} \]
\[ \theta^{*} g(-i)(-i) = g_{1i1z}^{-1} \chi_{1i}^{-1} \chi_{1i}^{-1} \tag{8.17} \]
\[ \theta^{*} \Pi_{i} = i d\ln f_{i}, \tag{8.18} \]
\[ \theta^{*} f_{i} = f_{i}. \tag{8.19} \]

**Proof.** From the first of the relations (8.8), we have:

\[ \tilde{B}_{i}^{(-1)} = \theta^{*} B_{-i}. \tag{8.21} \]

On the other hand, (8.9) and the first of the definitions (8.13) imply that

\[ \tilde{B}_{i}^{(-1)} - \tilde{B}_{i}^{(1)} = d\tilde{A}_{1i}^{(1)} = d\Pi_{i} \tag{8.22} \]

so that (8.15) follows.

Next, since the action of \(-1\) on \( \tilde{L} \) preserves the connection, we infer from (8.11) and the 2nd of the definitions (8.8) that

\[ \tilde{A}_{1i}^{(1)} = \theta^{*} A_{(-i)}(-i). \tag{8.23} \]

Now, the 2nd of the relations (8.9) gives:

\[ \tilde{A}_{1i}^{(-1)} - \tilde{A}_{1i}^{(-1)} = i d\ln \tilde{g}_{1i1z}^{(-1)}, \quad \text{i.e.} \quad \theta^{*} A_{(-i)}(-i) = \tilde{A}_{1i}^{(-1)} - \Pi_{i} - i d\ln \chi_{1i}. \tag{8.24} \]

Similarly, by (8.12),

\[ \tilde{A}_{1i}^{(1)} - \tilde{A}_{1i}^{(1)} = i d\ln \tilde{g}_{1i1z}^{(1)} = 0, \quad \text{i.e.} \quad \tilde{A}_{1i}^{(1)} = A_{1i} + \Pi_{i}. \tag{8.25} \]

Substituted to (8.21), this implies (8.16).
From the last of the relation (8.11), the commutation of the \( \mathbb{Z}_2 \) action with the groupoid multiplication \( \tilde{t} \) and the definition (8.8), it follows that
\[
\tilde{g}_{i_1 i_2 i_3}^{(-1)(-1)(-1)} = \Theta^* g_{(-i_1)(-i_2)(-i_3)}
\] (8.26)
and from the last of the relations (8.29) that
\[
\tilde{g}_{i_1 i_2 i_3}^{(-1)(-1)(-1)} = \tilde{g}_{i_1 i_2 i_3}^{(-1)(-1)(-1)} \frac{1}{1-i_1-i_2} \tilde{g}_{i_1 i_2 i_3}^{(-1)(-1)} = \frac{i}{1-i_1-i_2} \chi_{i_1 i_2} (\chi_{i_1 i_2})^{-1}
\] (8.27)
The latter four relations imply (8.17).

From the middle one of the definitions (8.8) and the last one of (8.13), it follows that
\[
-\Theta^* \tilde{A}_{(-i)(-i)}^{(-1)} = \Theta^* \tilde{A}_{(-i)(-i)}^{(-1)} = A_{i_1 i_2}^{(-1)} - i \ln f_{i_1}.
\] (8.28)
This gives (8.18).

In order to prove (8.19) and (8.20), let us additionally define \( U(1) \)-valued functions \( f_{i_1 i_2} \) on \( O_{i_1 i_2} \) by the relation
\[
(-1)\tilde{g}_{i_1 i_2}^{(-1)} = f_{i_1 i_2} \tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)} \circ \Theta
\] (8.29)
so that, in particular, \( f_{i} = f_{1} \). Applying \(-1\) to both sides of (8.29) and composing the result with the action of \( \Theta \), we obtain the relation
\[
\tilde{g}_{i_1 i_2}^{(-1)} \circ \Theta = \Theta^* f_{i_1 i_2} (-1)\tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)}.
\] (8.30)
The symmetry \( \tilde{\sigma}_{i_1 i_2}^{(-1)} = (\tilde{\sigma}_{i_1 i_2}^{(-1)})^{-1} \) that is preserved by the action of \(-1\) implies then that
\[
\tilde{\sigma}_{i_1 i_2}^{(-1)} \circ \Theta = (\Theta^* f_{i_1 i_2})^{-1} (-1)\tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)}
\] (8.31)
and, by comparison to (8.29), that
\[
\Theta^* f_{(-i_1)(-i_2)} = f_{i_1 i_2}.
\] (8.32)
In particular, setting \( i_1 = i_2 = i \), we obtain (8.20). Next, using the commutation of the action of \(-1\) with \( \tilde{t} \), we obtain the identity
\[
\tilde{t} \circ ((-1)\tilde{\sigma}_{i_1 i_2}^{(-1)} \otimes (-1)\tilde{\sigma}_{i_1 i_2}^{(-1)}) = \tilde{t} \circ (\tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)} \otimes f_{i_1 i_2} \tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)} \circ \Theta)
\] \[= f_{i_1 i_2} \Theta^* \tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)} \tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)} \circ \Theta
\] \[= (-1)\tilde{t} \circ (\tilde{\sigma}_{i_1 i_2} \otimes \tilde{\sigma}_{i_1 i_2}^{(-1)}) = \tilde{g}_{i_1 i_2} (-1)\tilde{\sigma}_{i_1 i_2}^{(-1)} = f_{i_1 i_2} \tilde{\sigma}_{(-i_1)(-i_2)}^{(-1)} \circ \Theta.
\] (8.33)
It follows that
\[
f_{i_1 i_2} = f_{i_1 i_2} \Theta^* \tilde{g}_{(-i_1)(-i_2)}^{(-1)} = f_{i_1 i_2} \Theta^* \chi_{(-i_1)(-i_2)}.
\] (8.34)
Combining this with (8.32), we infer that
\[
f_{i_1 i_2} \Theta^* \chi_{(-i_1)(-i_2)} = \Theta^* f_{-i_1} \chi_{i_1 i_2}.
\] (8.35)
that, together with (8.20), implies (8.19).

Conversely, we may recover the local data \((\tilde{B}_i, \tilde{A}_{i_1 i_2}, \tilde{g}_{i_1 i_2 i_3})\) of a \( \mathbb{Z}_2 \)-equivariant extension \( \tilde{G} \) of gerbe \( G \) from the local data \((B_i, A_{i_1 i_2}, g_{i_1 i_2 i_3})\) and \((\Pi_{i_1}, \chi_{i_1 i_2}, \tilde{t}_i)\) by setting
\[
\tilde{B}_i^1 = B_i,
\] (8.36)
\[
\tilde{A}_{i_1 i_2}^{1(-1)} = A_{i_1 i_2},
\] (8.37)
\[
\tilde{g}_{i_1 i_2 i_3}^{111} = g_{i_1 i_2 i_3},
\] \[
\tilde{g}_{i_1 i_2 i_3}^{11(-1)} = g_{i_1 i_2 i_3},
\] \[
\tilde{g}_{i_1 i_2 i_3}^{1(-1)(-1)} = g_{i_1 i_2 i_3} \chi_{i_1 i_2} (\chi_{i_1 i_2})^{-1}.
\]
\begin{equation}
\theta_{i_1 i_2 i_3}^{-1} g_{i_1 i_2 i_3} = \Theta^* g_{(i_1)}^{-1}(i_2)(i_3) \tag{8.38}
\end{equation}
and imposing the desired symmetry in the indices. The identities (8.9) follow then from the relations (8.15), (8.16) and (8.17).

We shall also need to find local data for the flat line bundle \( N' \) over the fixed-point manifold \( M' \) of \( \Theta \). Let us denote \( O_i \cap M' \equiv O'_i \). For \( x' \in O'_i \), \((s_i(x'), s_i(x')) \in \mathbb{Z}, \) and \( \tilde{\sigma}_{i(-i)}^{1(-1)}(x') \in \tilde{L}(1,s_i(x'),-1,s_i(x')) = N'_i(x') \). We have

\begin{equation}
\ln_{N'}(\tilde{\sigma}_{i(-i)}^{1(-1)}(x')) = \tilde{A}_{i(-i)}^{1(-1)}(x') \tilde{\sigma}_{i(-i)}^{1(-1)}(x') = (A_i(i_1) + \Pi_{-i})(x') \tilde{\sigma}_{i(-i)}^{1(-1)}(x'), \tag{8.39}
\end{equation}
and for \( x' \in O'_{i_1 i_2}, \)

\begin{equation}
\tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tilde{\sigma}_{i_2}^{1(-1)}(x') = \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x'). \tag{8.40}
\end{equation}

On the other hand,

\begin{equation}
\tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tilde{\sigma}_{i_2}^{1(-1)}(x') = \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x'). \tag{8.41}
\end{equation}

From (6.48), (6.49) and (8.31), we infer that

\begin{equation}
\nu'(\tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tilde{\sigma}_{i_2}^{1(-1)}(x')) = \frac{\tilde{g}_{i_1 i_2}^{1(-1)}(x')}{\tilde{g}_{i_2}^{1(-1)}(x')} \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tag{8.42}
\end{equation}

It follows that, as elements of the line bundle \( N' \) over \( M' \),

\begin{equation}
\tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') = \frac{g_{i_1 i_2}^{1(-1)}(x') \chi(i_1)(i_2)(x')}{g_{i_2}^{1(-1)}(x')} \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x'). \tag{8.43}
\end{equation}

Hence \((a'_i, h'_{i_1 i_2}), \) where

\begin{equation}
a'_i = (A_i(i_1) + \Pi_{-i}) |O'_i, \quad h'_{i_1 i_2} = \frac{g_{i_1 i_2}^{1(-1)}(i_1)(i_2)}{g_{i_2}^{1(-1)}(i_1)(i_2)} \chi(i_1)(i_2) \tilde{\sigma}_{i_1 i_2}^{1(-1)}(x') \tag{8.44}
\end{equation}

are local data of the flat line bundle \( N' \). They satisfy the relations

\begin{equation}
da'a' = 0, \quad a'_{i_1} - a'_{i_2} = i \ln h'_{i_1 i_2}, \quad h'_{i_1 i_2} h'_{i_2 i_3} h'_{i_1 i_3} = h'_{i_1 i_2} \tag{8.45}
\end{equation}

that are straightforward to check. Let us define

\begin{equation}
l'_i = (\chi_{i(-i)}(i_1) i_i) |O'_i \tag{8.46}
\end{equation}

Then

\begin{equation}
i \ln l'_i = (i \ln \chi_{i(-i)} + i \ln f_{i_i}) |O'_i = (A_i(i_1) - \Theta^* A_{i_1} + \Pi_{-i} - \Pi_i + \Theta^* \Pi_{-i} + \Pi_i) |O'_i = 2a'_i \tag{8.47}
\end{equation}

and

\begin{equation}
\ln(l'_{i_1 i_2})^{-1} l'_{i_1 i_2} = (\chi_{i_1 i_1}(i_1) - \chi_{i_1 i_2}(i_1)(i_2) - \chi_{i_1 i_1}(i_2) - \chi_{i_1 i_2}(i_1)(i_2)) |O'_{i_1 i_2} \tag{8.48}
\end{equation}
The family \((l'_i)\) provides local data for the trivialization of the flat line bundle \((N')^2\) considered in Sec.\[6\]. If \(M'\) is 1-connected then one may choose the square roots \(\sqrt{l'_i}\) so that
\[
i d \ln \sqrt{l'_i} = a'_i, \quad (\sqrt{l'_i})^{-1} \sqrt{l'_{i_1 i_2}} = k'_{i_1 i_2}.
\]
(8.49)
Such \((\sqrt{l'_i})\), defined modulo a global sign, provide local data for a trivialization of the flat line bundle \(N'\) used in Sec.\[6\].

We have now all elements at hand to present the local-data formula for the square of the gerbe holonomy defined in Sec.\[5\]. A straightforward but somewhat tedious verification that we omit here shows that
\[
\sqrt{\mathrm{Hol}_G(\phi)} = \exp \left[ i \sum_c \int \phi^* B_{i_c} + i \sum_{b \subset c} \int_b \phi^* A_{i_{eb}} + i \sum_{b \subset c} \int_b \phi^* A_{(i_{eb})} - i \sum_{b \subset \vartheta(c)} \int_b \phi^* \Pi_{-i_{eb}} \right] \prod_{v \in b \subset \vartheta(c) \subset \Sigma'} (g_{i_{ev}, i_{ev}}(\phi(v)))^{\mp 1} \prod_{v \in \vartheta(c) \subset \Sigma'} (f_{i_{ev}}(\phi(v)))^{\mp 1}
\]
\[
\times \prod_{v \in \vartheta(c) \subset \Sigma'} \left( (g_{i_{ev}, i_{ev}}(\phi(v)))^{\mp 1} \chi_{i_{ev}}(\phi(v)) \right)^{\mp 1} \left( \sqrt{l'_i}(\phi(v)) \right)^{\mp 1},
\]
(8.50)
where \(c\) and \(b\) run, as before, over the selected lifts of triangles \(\tilde{c}\) and \(\tilde{b}\) of a sufficiently fine orbifold triangulation of \(\Sigma\), whereas \(v\) runs over all vertices projecting to vertices \(\bar{v}\) with the exception of the product \(\prod_{v \in \vartheta(c)}\) where only one of two vertices \(v \notin \Sigma'\) projecting to each vertex \(\bar{v}\) is taken into consideration. The indices \(i_c, i_b,\) and \(i_v\) are chosen so that the relations \(\[5, 7\]\) hold and, that, additionally, \(i_{\vartheta(v)} = -i_v\) for \(v \notin \Sigma'\).

9. 3d-INDEX

Let, as above, \(M\) be a manifold equipped with an involution \(\Theta\) whose fixed-point set \(M'\) is a 1-connected submanifold of \(M\). Let \(G = (Y, B, L, t)\) be a gerbe on \(M\) with curvature \(H = \Theta^* H\) and let \(\tilde{G} = (\tilde{Y}, \tilde{B}, \tilde{L}, \tilde{t})\) be a \(\mathbb{Z}_2\)-equivariant extension of \(G\).

Let \(\mathcal{R}\) be a compact oriented 3d-manifold with an orientation-reversing involution \(\rho\) with the derivative equal to \(-I\) at the fixed points. The fixed-point set \(\mathcal{R}'\) of \(\rho\) must then be discrete. As the main example of such a 3d-manifold with involution, we shall consider the 3d-torus \(T^3 = \mathbb{R}^3/(2\pi \mathbb{Z}^3)\), viewed as the cube \([-\pi, \pi]^3\) with the periodic identifications, with \(\rho\) given by \(k \mapsto -k\).

Let \(\Phi : \mathcal{R} \rightarrow M\) be a map satisfying the equivariance condition
\[
\Phi \circ \rho = \Theta \circ \Phi.
\]
(9.1)
We shall again view \(\mathcal{R}/\mathbb{Z}_2 = \tilde{\mathcal{R}}\) as a \(\mathbb{Z}_2\) orbifold, choosing for it a sufficiently fine orbifold triangulation, with simplices \(\tilde{h}, \tilde{c}, \tilde{b}, \tilde{v}\), that lifts to a triangulation of \(\mathcal{R}\) with simplices \(h, c, b, v\), the ones of positive dimension interchanged in pairs by \(\rho\). We shall select one \(h\), one \(c\) and one \(b\) in each such pair, together with the maps \(s_h : h \rightarrow Y\), \(s_c : c \rightarrow Y\) and \(s_b : b \rightarrow Y\) such that
\[
\pi \circ s_h = \Phi|_h, \quad \pi \circ s_c = \Phi|_c, \quad \pi \circ s_b = \Phi|_b.
\]
(9.2)
Given such choices, we shall consider the expression
\[
K_\phi(\Phi) = e^{-\frac{i}{2} \sum_h s_h \Psi^*H + i \sum_{c \subset h} s_c^*B} \left( \otimes_{b \subset c \subset h} \mathrm{hol}_L(s_c|_b, s_b) \right) \otimes \left( \otimes_{b \subset c \subset h} \mathrm{hol}_K(s_c \circ \rho|_b, s_b) \right),
\]
(9.3)
where only the selected \(h, c, b\) are considered, the tetrahedra \(h\) are taken with the orientation of \(\mathcal{R}\), and \(c \subset h\), \(b \subset c \subset h\), and \(\rho(b) \subset c \subset h\) with the inherited orientations and, in the last case, \(b\) with the orientation related to that of \(\rho(b)\) by \(\rho\). The contributions to the right hand side of \(\[9.3\]\) from triangles \(c\) that are shared by two tetrahedra \(h\) cancel out as such \(c\) appear twice with opposite orientations. Denote
by \( \mathcal{F} \) the simplicial complex that is the union of all selected tetrahedra \( h \) and by \( \Sigma \) its boundary \( \partial \mathcal{F} \) which is the union of all triangles of the triangulation of \( \mathcal{R} \) that belong to only one of the selected \( h \). If \( c \subset \Sigma \) then also \( \rho(c) \subset \Sigma \) so that \( \Sigma \) is \( \rho \)-invariant and \( \rho \) preserves its orientation. The right hand side of (9.4) takes values in the line
\[
\left( \bigotimes_{v \in b, c \subset \Sigma} L_{s, v}^{+1} \right) \otimes \left( \bigotimes_{v \in b, c \subset \Sigma} K_{s, v}^{+1}(\vartheta(v)) \right).
\]
(9.4)

If the simplicial complex \( \mathcal{F} \) forms a submanifold with boundary of \( \mathcal{R} \) then \( \Sigma = \partial \mathcal{F} \) is a closed oriented surface and the line (9.4) is canonically isomorphic to \( \mathbb{C} \), as was shown in Sec.6. We may then identify
\[
K_{\mathcal{F}}(\Phi) = e^{-\frac{i}{\hbar} \int_{\mathcal{F}} \Phi^* H} \sqrt{\text{Hol}_{\mathcal{F}}(\Phi|_{\partial \mathcal{F}})} \in U(1).
\]
(9.5)

For simplicity, we shall limit ourselves to such situations of which an example is provided by \( \mathcal{R} = \mathbb{T}^3 \) where we may take as \( \mathcal{F} \) the subset obtained by restricting one of the components of \( k \in [-\pi, \pi]^3 \) to non-negative (or non-positive) values. Note that the right hand side of (9.3) squares to 1 by Lemma 1 of Sec.6 and Proposition 2 of Sec.6. Hence \( K_{\mathcal{F}}(\Phi) = \pm 1 \).

**Proposition 6.** The phase associated to the right hand side (9.3) is independent of the choice of the simplices \( h, c, b \), the maps \( s_c, s_b \) and the orbifold triangulation of \( \mathcal{R} \).

**Proof.** We shall proceed similarly as in the proof of Proposition 3 in Sec.6 showing that the changes of the selected simplices and/or of their maps to \( Y \) lead to expressions that are equivalent under the use of line-bundle isomorphisms \( \tilde{t} \) and the ones induced by the \( \mathbb{Z}_2 \) action, part of the structure of the gerbe \( \mathcal{G} \).

1. If the tetrahedron \( h_0 \) is changed to \( h'_0 = \rho(h_0) \) then we have:
\[
e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} = e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{h_0} \Phi^* H} = e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{h_0} s_{h_0} dB}
\]
\[
e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{\rho(c) \subset h_0} f_c s_{\rho(c)} B + \int_{c \subset h_0} f_c s_{c} B} = e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{\rho(c) \subset h_0} f_c s_{\rho(c)} B + \int_{c \subset h_0} f_c s_{c} B} \cong e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{\rho(c) \subset h_0} f_c s_{\rho(c)} B + \int_{c \subset h_0} f_c s_{c} B}
\]
\[
\times \left( \bigotimes_{b \subset c \subset h_0} \text{hol}_{\mathcal{L}}(s_c|b, s_{h_0}|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{L}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \right) \cong e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{\rho(c) \subset h_0} f_c s_{\rho(c)} B + \int_{c \subset h_0} f_c s_{c} B}
\]
\[
\times \left( \bigotimes_{b \subset c \subset h_0} \text{hol}_{\mathcal{L}}(s_c|b, s_{h_0}|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{L}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \right).
\]
(9.6)

But
\[
\sum_{c \subset h_0} f_c s_{c} B \cong \sum_{\rho(b) \subset \rho(c) \subset h_0} f_{\rho(b)} s_{\rho(c)} B \cong \sum_{b \subset c \subset h_0} f_{b} s_{b} B
\]
\[
e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{\rho(b) \subset \rho(c) \subset h_0} f_{\rho(b)} s_{\rho(c)} B} \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{L}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \cong e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H} e^{i \int_{b \subset c \subset h_0} f_{b} s_{b} B} \bigotimes_{b \subset c \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b)
\]
\[
\bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b)
\]
(9.7)

Hence
\[
e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H + \sum_{c \subset h_0} f_c s_{c} B} \bigotimes_{b \subset c \subset h_0} \text{hol}_{\mathcal{L}}(s_c|b, s_{h_0}|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b)
\]
\[
\cong e^{-\frac{i}{\hbar} \int_{h_0} \Phi^* H + \sum_{c \subset h_0} f_c s_{c} B} \bigotimes_{b \subset c \subset h_0} \text{hol}_{\mathcal{L}}(s_c|b, s_{h_0}|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_c \circ \rho|b, s_{h_0} \circ \rho|b)
\]
\[
\bigotimes_{b \subset c \subset h_0} \text{hol}_{\mathcal{L}}(s_b, s_{h_0}|b) \bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_b, s_{h_0} \circ \rho|b)
\]
\[
\bigotimes_{\rho(b) \subset \rho(c) \subset h_0} \text{hol}_{\mathcal{K}}(s_b, s_{h_0} \circ \rho|b)
\]
in applying the construction of
need completely fixed by Witten’s rule (3.7). In the application considered in the sequel, we shall, however, also

is 1-connected, there is no $\mathbb{Z}$

involution of $\Sigma$ and $\Theta$ is the involution on

$\theta$

reversal

nor simply connected, so that the construction of Sec. 6 does not work. On the other hand, although for

Different construction of such a gerbe are possible but they all lead to the same holonomy $\text{Hol}_G(\Phi)$ to the old one

is shown the same way as in the proof of Proposition 3 in Sec. 6.

FIG. 8: Three-dimensional Pachner moves

Remark. Proposition 6 implies, in particular, that the right hand side of (10.1) does not depend on the choice of the submanifold with boundary $\mathcal{F} \subset \mathcal{R}$ forming the closure of a fundamental domain for the involution $\rho$ of $\mathcal{R}$. Although the right hand side of (10.1) may be often defined using a homotopic non-local formula (7.2) for the square root of gerbe holonomy, the local approach based on gerbe theory is useful to establish such a result that, in application to topological insulators, is a powerful source of equalities between different forms of invariants [12, 20].

10. BASIC GERBE ON THE GROUP $U(N)$ AND THE TIME REVERSAL

We would like to apply the constructions of the previous sections to the case when $M$ is the unitary group $U(N)$ in $N$ dimensions and $G$ is the so called basic gerbe on $U(N)$ with curvature given by the closed bi-invariant 3-form

$$H = \frac{1}{12\pi} \text{tr}(u^{-1} du)^3.$$ (10.1)

Different construction of such a gerbe are possible but they all lead to the same holonomy $\text{Hol}_G(\phi)$ that is completely fixed by Witten’s rule (3.7). In the application considered in the sequel, we shall, however, also need $\sqrt{\text{Hol}_G(\phi)}$ for maps $\phi$ satisfying the equivariance condition (6.1), where $\theta$ is an orientation-preserving involution of $\Sigma$ and $\Theta$ is the involution on $U(N)$ generated by the adjoint action $u \mapsto \theta u \theta^{-1}$ of the time reversal $\theta : \mathbb{C}^N \to \mathbb{C}^N$. The transformation $\theta$ is an anti-unitary map that squares to $\pm I$. There is a problem in applying the construction of $\sqrt{\text{Hol}_G(\phi)}$ from Sec. 6 for such an involution $\Theta$. On the one hand, when $\theta^2 = I$ then there exists a $\mathbb{Z}_2$-equivariant extension $\tilde{G}$ of the basic gerbe $G$. However, in that case the fixed-point set $U(N)^\theta \subset U(N)$ of $\Theta$ is conjugate to the subgroup $O(N) \subset U(N)$ and is neither connected nor simply connected, so that the construction of Sec. 6 does not work. On the other hand, although for $\theta^2 = -I$ (requiring an even $N$) the fixed point set $U(N)^\theta$ is conjugate to the subgroup $Sp(N) \subset U(N)$ and is 1-connected, there is no $\mathbb{Z}_2$-equivariant extension $\tilde{G}$ of the basic gerbe $G$ in that case. This was discussed
in detail in Sec. I of [12] using the equivalent concept of a $\mathbb{Z}_2$-equivariant structure on the gerbe $\mathcal{G}$. In [12] a slight modification of the construction of $\mathcal{G}$ from [24] was used, see Sec. H of [12], but the conclusions are independent of the choice of the basic gerbe $\mathcal{G}$ on $U(N)$.

What partially saved the day in the case when $\theta^2 = -I$ was the passage to the double-cover group

$$\tilde{U}(N) = \{(u, \omega) \in U(N) \times U(1) \mid \omega^2 = \det(u)\},$$

(10.2)

with the lift $\tilde{\Theta}(u, \omega) = (\Theta(u), \omega^{-1})$ of the involution $\Theta$. The covering map from $\tilde{U}(N)$ to $U(N)$ just forgets $\omega$ so that the corresponding deck transformation of $\tilde{U}(N)$ is given by the multiplication by $(I, -1)$. Let $\mathcal{G}$ be the pullback to $\tilde{U}(N)$ by the covering map of the basic gerbe $\mathcal{G}$ on $U(N)$ (obtained by naturally pulling back all the elements of the structure of $\mathcal{G}$). If $\tilde{U}(N)$ is considered with the $\mathbb{Z}_2$-action induced by $\tilde{\Theta}$ then, as discussed in Sec. J of [12], there exists a $\mathbb{Z}_2$-equivariant extension $\hat{\mathcal{G}}$ of the gerbe $\mathcal{G}$. Besides, if the involution $\vartheta$ on $\Sigma$ corresponds to a hyperelliptic cover, see Sec. 5 then any map $\phi : \Sigma \to U(N)$ satisfying the equivariance condition (6.1) has det($\phi$) that winds even number of times along the cycles of $\Sigma$. It follows that $\phi$ lifts to a map $\hat{\phi} : \Sigma \to \tilde{U}(N)$. Besides, the lift $\hat{\phi}$ satisfies the equivariance condition relative to the involution $\hat{\Theta}$ and is unique up to a composition with the multiplication by $(I, -1)$ in $\tilde{U}(N)$. The fixed-point set $\tilde{U}(N)' \subset \tilde{U}(N)$ of $\hat{\Theta}$ is composed of two disjoint simply connected components, one isomorphic to $Sp(N)$ and the other obtained by the deck transformation of the first one. Although $\tilde{U}(N)'$ is not 1-connected and the trivialization of the flat line bundle $N'$ over $\tilde{U}(N)'$ used in the definition of the square root of gerbe holonomy in Sec. 6 has independent sign ambiguities on each connected component of $\tilde{U}(N)'$, it was shown in Sec. F of [12] that such ambiguities cancel for the maps $\hat{\phi}$ as above allowing an unambiguous definition of $\sqrt{\text{Hol}_\hat{G}(\hat{\phi})}$. Furthermore, the latter quantity is equal for the two possible choices of the lift $\hat{\phi}$. It was used in [12] as a definition of $\sqrt{\text{Hol}_\hat{G}(\hat{\phi})}$ for the maps $\phi : \Sigma \to U(N)$ satisfying the equivariance condition (6.1) if $\vartheta$ corresponds to a hyperelliptic cover and $\Theta$ is given by the adjoint action of the time reversal $\theta$ squaring to $-I$. In particular, this covers the case of the 2-torus $\mathbb{T}^2$ with the involution $\vartheta$ induced by $k \mapsto -k$, see Sec. 5 above. It follows by a simple extension of the results of [7] that any map $\phi : \mathbb{T}^2 \to U(N)$ equivariant with respect to the $k \mapsto -k$ and the time-reversal involutions, the latter with $\theta^2 = -I$ can be extended after a composition with an $SL(2, \mathbb{Z})$ diffeomorphism of $\mathbb{T}^2$ (in order to render the winding of det($\phi$) around the $k_1$-cycle trivial) to an equivariant map $\psi : \mathcal{T} = D \times S \to U(N)$, where $\mathcal{T}$ is taken with the involution $(z, v) \mapsto (\bar{z}, \bar{v})$, see the beginning of Sec. 7. Again, $\psi$ may be lifted to an equivariant map $\hat{\psi} : \mathcal{T} \to \tilde{U}(N)$. Then the application of (6.44) and of Proposition 4 shows that the equality (7.2) holds in that case.

A similar construction permits to define unambiguously the $3d$-index $K_\mathcal{G}(\Phi)$ of Sec. 9 for the maps $\Phi : \mathbb{T}^3 \to U(N)$ satisfying the equivariance condition (9.1) if the involution $\rho$ of the 3-torus $\mathbb{T}^3$ is induced by $k \mapsto -k$ and $\Theta$ is given by the adjoint action of the time reversal $\theta$ with $\theta^2 = -I$, see again Sec. F of [12]. One shows that $\Phi$ lifts to the map $\hat{\Phi} : \mathbb{T}^3 \to \tilde{U}(N)$ equivariant with respect to $\hat{\Theta}$ and that $K_\mathcal{G}(\hat{\Phi})$ is unambiguously defined and independent of the choice of the lift $\hat{\Phi}$.

11. APPLICATIONS TO THE TIME-REVERSAL INVARIANT TOPOLOGICAL INSULATORS

The 2nd part of lectures [12] discussed how the square root of the gerbe holonomy and the corresponding $3d$-index provide invariants of the topological time-reversal-symmetric insulators in two and three space dimensions. For completeness, we shall list those results here.

In the simplest case, the $d$-dimensional insulators are described by Hamiltonians on a crystalline lattice that, after the discrete Fourier-Bloch transformation, give rise to a (smooth) map

$$\mathbb{T}^d \ni k \mapsto h(k) = h(k) \in \text{End}(\mathbb{C}^N)$$

(11.1)

The gerbe $\hat{\mathcal{G}}$ introduced here should not be confused with the quotient gerbes discussed at the end of Sec 4 for which we used the same notation.
and all the hermitian matrices $h(k)$ have a spectral gap around the Fermi energy $\epsilon_F$. Denote by $p(k)$ the spectral projector on the eigenstates of $h(k)$ with energies $<\epsilon_F$ which then depends smoothly on $k \in \mathbb{T}^d$.

For the electronic time-reversal-symmetric insulators,

$$\theta h(k)\theta^{-1} = h(-k) \quad \text{and} \quad \theta p(k)\theta^{-1} = p(-k),$$

(11.2)

where $\theta: \mathbb{C}^N \to \mathbb{C}^N$ is an anti-unitary with $\theta^2 = -I$. Denote by $u_p(k)$ the unitary matrix $I - 2p(k)$. In two or three dimensions, the map $\mathbb{T}^d \ni k \mapsto u_p(k) \in U(N)$ is then equivariant, i.e. $\Theta(u_p(k)) = u_p(-k)$, where $\Theta(u) = \theta u \theta^{-1}$. This is still the case for the restriction of the three dimensional $u_p$ to any 2-torus $\mathbb{T}^2 \subset \mathbb{T}^3$ preserved by the $k \mapsto -k$ involution of $\mathbb{T}^3$.

Proposition 7 \[6, 7, 12\]. Let $\mathcal{G}$ be the basic gerbe on $U(N)$.

1. For $d = 2$, $\sqrt{\text{Hol}_\mathcal{G}(u_p)} = (-1)^{KM}$, where $KM \in \{0,1\}$ is the Fu-Kane-Mele invariant \[19, 9\] of the time-reversal-symmetric 2d topological insulators.

2. For $d = 3$, $K_{\mathcal{G}}(u_p) = (-1)^{KM^s}$ where $KM^s \in \{0,1\}$ is the “strong” Fu-Kane-Mele invariant \[10\] of the time-reversal-symmetric 3d topological insulators.

Remark. \quad The expression (10.3) for $K_{\mathcal{G}}(u_p)$ for $\mathcal{F} \subset \mathbb{T}^3$ bounded by the 2-tori $\mathbb{T}^2_0$ and $\mathbb{T}^2_1$, see Fig. 9, leads to the relation $KM^s = KM|_{\mathbb{T}^2_0} + KM|_{\mathbb{T}^2_1}$ known from \[10\], between the strong and the weak Fu-Kane-Mele invariants, the latter defined for the involution-preserved $\mathbb{T}^2 \subset \mathbb{T}^3$ by the relation $(-1)^{KM|_{\mathbb{T}^2}} = \sqrt{\text{Hol}_\mathcal{G}(u_p|_{\mathbb{T}^2})}$. Of course, the pair $(\mathbb{T}^2_0, \mathbb{T}^2_1)$ could be replaced by similar pairs orthogonal to other axes.

Further applications concern the so called Floquet systems described by lattice Hamiltonians periodically depending on time. After the discrete Fourier-Bloch transformation, such a Hamiltonian gives rise to a map

$$\mathbb{R} \times \mathbb{T}^d \ni (t,k) \mapsto h(t,k) = h(t,k)^\dagger = h(t + 2\pi, k) \in \text{End}(\mathbb{C}^N),$$

(11.3)

where, for convenience, we fixed the period of temporal driving to $2\pi$. The time evolution of the corresponding systems is described by the unitary matrices $u(t,k)$ such that

$$i\partial_t u(t,k) = h(t,k) u(t,k), \quad u(0,k) = I, \quad u(t + 2\pi,k) = u(t,k) u(2\pi,k).$$

(11.4)

The Floquet theory that deals with such systems is based on the diagonalization of the unitary matrices $u(2\pi,k)$ whose eigenvalues are written as $e^{-ie_n(k)}$, where $e_n(k)$ are called the (band) “quasienergies”. Let us suppose that $\epsilon \in [-2\pi,0]$ is such that $e^{-i\epsilon}$ is not in the spectrum of $u(2\pi,k)$ for all $k$ (i.e. $\epsilon$ is in the quasienergy gap). Then the “effective Hamiltonian”

$$h_\epsilon(k) \equiv h_\epsilon(u(2\pi,k)) = \frac{i}{2\pi} \ln_{-\epsilon}(u(2\pi,k)),$$

(11.5)

where, by definition, $\ln_{-\epsilon}(e^{i\varphi}) = i\varphi$ if $-\epsilon - 2\pi < \varphi < -\epsilon$, is well defined and depends smoothly on $k \in \mathbb{T}^d$.

It satisfies $u(2\pi,k) = e^{-2i\pi h_\epsilon(k)}$. For two gap quasienergies $-2\pi \leq \epsilon \leq \epsilon' < 0$,

$$h_{\epsilon'}(k) - h_{\epsilon}(k) = p_{\epsilon,\epsilon'}(u(2\pi,k)) \equiv p_{\epsilon,\epsilon'}(k),$$

(11.6)
where $p_{\epsilon,\epsilon'}(k)$ is the spectral projector of $u(2\pi, k)$ on quasienergies $\epsilon < \epsilon_n(k) < \epsilon'$. One may use the effective Hamiltonians $h_\epsilon(k)$ to introduce the periodized evolution operators

$$v_\epsilon(t, k) = u(t, k) e^{ith_\epsilon(k)} = v_\epsilon(t + 2\pi, k)$$

that define a map $v_\epsilon : \mathbb{T}^{d+1} \to U(N)$.

For the electronic time-reversal-symmetric Floquet systems,

$$\theta h(t, k) \theta^{-1} = h(-t, -k)$$

for an anti-unitary $\theta$ with $\theta^2 = -I$. It follows then that

$$\Theta(v_\epsilon(t, k)) \equiv \theta v_\epsilon(t, k) \theta^{-1} = v_\epsilon(-t, -k)$$

and

$$\theta p_{\epsilon,\epsilon'}(k) \theta^{-1} = p_{\epsilon,\epsilon'}(-k).$$

In particular, for $-2\pi \leq \epsilon \leq \epsilon' < 0$, one may consider the Fu-Kane-Mele invariants $KM_{\epsilon,\epsilon'}$ and $KM_{\epsilon,\epsilon'}^s$ of the quasienergy bands between $\epsilon$ and $\epsilon'$, defined in two and three dimensions, respectively, by the relations

$$(-1)^{KM_{\epsilon,\epsilon'}} = \sqrt{\text{Hol}_G(u_{p_{\epsilon,\epsilon'}})}, \quad (-1)^{KM_{\epsilon,\epsilon'}^s} = \tilde{G}_G(u_{p_{\epsilon,\epsilon'}^s}),$$

where, as before, $G$ is the basic gerbe on $U(N)$ and $u_{p_{\epsilon,\epsilon'}(k)} = I - 2p_{\epsilon,\epsilon'}(k)$.

In [8, 12] additional dynamical invariants $K_\epsilon$ and $K_\epsilon^s$ with values in $\{0, 1\}$ were introduced for the time-reversal-invariant Floquet systems in two and three dimensions, respectively, such that

$$(-1)^K = G_G(v_\epsilon), \quad (-1)^{K^s} = G_G(v^s_{\epsilon|_{\mathbb{R}/2\pi\mathbb{Z}} \times \mathbb{T}^2}).$$

In $3d$, one can also define weak dynamical invariants $K_{\epsilon|_{\mathbb{T}^2}}$ for 2-tori $\mathbb{T}^2 \subset \mathbb{T}^3$ preserved by the $k \mapsto -k$ involution of $\mathbb{T}^3$ setting

$$(-1)^{K_{\epsilon|_{\mathbb{T}^2}}} = G_G(v_\epsilon|_{\mathbb{R}/2\pi\mathbb{Z}} \times \mathbb{T}^2).$$

One has the following

**Proposition 8** [8, 12].

1. (Relation between strong and weak invariants).

$$K_\epsilon^s = K_{\epsilon|_{\mathbb{T}^2}} - K_{\epsilon|_{\mathbb{T}^2}} \mod 2.$$  

2. (Relation to the Fu-Kane-Mele invariants). For two gap quasienergies $-2\pi \leq \epsilon \leq \epsilon' < 0$,

$$K_{\epsilon'} - K_\epsilon = KM_{\epsilon,\epsilon'}, \quad K_{\epsilon'}^s - K_\epsilon^s = KM_{\epsilon,\epsilon'}^s.$$  

**Remark.** The invariants $K_\epsilon$ are the counterparts for time-reversal-symmetric gapped Floquet systems of the dynamical invariants for such systems without time-reversal symmetry introduced in [28], see also [25]. They are supposed to count modulo 2 the “Kramers pairs” (related by the time-reversal) of eigenstates of the evolution operator over one period on a half-lattice that have quasienergy $\epsilon$ and are localized near the lattice edge $[0]$.  

**APPENDIX**

We shall describe here the relation between a $\mathbb{Z}_2$-extension $\tilde{G} = (\tilde{Y}, \tilde{B}, \tilde{L}, \tilde{t})$ of the gerbe $G = (Y, B, L, t)$ and a $\mathbb{Z}_2$-equivariant structure on $G$ defined in [12] following [14]. The definition given in [12] presupposed a simplified situation when the involution $\Theta$ induces the $\mathbb{Z}_2$-action on the base space $M$, lifts to an involutive map $\Theta_Y$ of $Y$ (that induces involutions $\Theta_{Y}^{[p]}$ on $Y^{[p]}$). The $\mathbb{Z}_2$-equivariant structure on $G$ was specified in [12] by a line-bundle $N$ over $Y$ with curvature $\Theta_Y^* B - B$ and an isomorphism $\nu : L \otimes p_{\log N}^* N \to p^*_{\log N} \otimes (\Theta_{Y}^{[2]})^* L$ of line-bundles over $Y^{[2]}$ that commutes with the groupoid multiplication $t$ in $L$. Furthermore, the flat line bundle $N \otimes \Theta_{Y}^* N = Q$ was assumed to be equipped with a trivializing section $S$ such that $\Theta_Q \circ S = S \circ \Theta$ for the involutive isomorphism $\Theta_Q$ of the line bundle $Q$ defined by the permutation of the tensor factors
that lifts the involution $\Theta_Y$ on the base, see Sec. C and E of [12]. From those data, one may recover the line bundle $\bar{L}$ of gerbe $\bar{G}$ setting

$$\bar{L}(1,y_1),(1,y_2) = \mathcal{L}_{y_1,y_2} = \bar{L}(-1,y_1),(-1,y_2), \quad \bar{L}(1,y_1),(-1,y_3) = N_{y_1} \otimes \mathcal{L}_{\Theta_Y y_1,y_3} = \bar{L}(-1,y_1),(1,y_3)$$

(2.2)

for $\pi(y_1) = \pi(y_2) = \Theta(\pi(y_3))$, with the obvious $\mathbb{Z}_2$-symmetry. The groupoid multiplication $\bar{t}$ is then defined by the linear maps

$$\bar{L}(1,y_1),(1,y_2) \otimes \bar{L}(1,y_2),(1,y_3)^{\prime} = \mathcal{L}_{y_1,y_2} \otimes \mathcal{L}_{y_2,y_3'} \xrightarrow{\bar{t}} \mathcal{L}_{y_1,y_3'} = \bar{L}(1,y_1),(1,y_3'), \quad \bar{L}(1,y_1),(1,y_2) \otimes \bar{L}(1,y_2),(-1,y_3) = \mathcal{L}_{y_1,y_2} \otimes \mathcal{L}_{\Theta_Y y_2,y_3} \xrightarrow{\nu \otimes \text{id}} N_{y_1} \otimes \mathcal{L}_{\Theta_Y y_1,\Theta_Y y_3} \otimes \mathcal{L}_{\Theta_Y y_2,y_3} \xrightarrow{\text{id} \otimes \text{id}} \mathcal{L}_{\Theta_Y y_1,y_3} = \bar{L}(-1,y_1),(-1,y_3), \quad \bar{L}(1,y_1),(-1,y_3) \otimes \bar{L}(-1,y_3),(-1,y_3') = N_{y_1} \otimes \mathcal{L}_{\Theta_Y y_1,y_3} \otimes \mathcal{L}_{\Theta_Y y_3,y_3'} \xrightarrow{\text{id} \otimes \nu \otimes \text{id}} \mathcal{L}_{\Theta_Y y_1,y_3} = \bar{L}(1,y_1),(-1,y_3'), \quad \bar{L}(1,y_1),(-1,y_3) \otimes \bar{L}(-1,y_3),(-1,y_3') = N_{y_1} \otimes \mathcal{L}_{\Theta_Y y_1,y_3} \otimes \mathcal{L}_{\Theta_Y y_3,y_3'} \xrightarrow{\text{id} \otimes \nu \otimes \text{id}} \mathcal{L}_{\Theta_Y y_1,y_3} = \bar{L}(1,y_1),(-1,y_3'),$$

(2.3)

(2.4)

(2.5)

(2.6)

and the $\mathbb{Z}_2$-symmetry of $\bar{L}$.

Conversely, given the $\mathbb{Z}_2$-equivariant extension $\tilde{G}$ of gerbe $G$, we may obtain a $\mathbb{Z}_2$-equivariant structure on $G$ by setting $N_y = L(1,y)(-1,\Theta y)$ and defining the line-bundle isomorphism $\nu$ by the linear maps on the fibers

$$\nu : \mathcal{L}_{y_1,y_2} \otimes \mathcal{L}_{y_2,y_3'} \xrightarrow{\nu} \mathcal{L}_{y_1,y_3'} \otimes \mathcal{L}_{\Theta_Y y_1,\Theta_Y y_3} \otimes \mathcal{L}_{\Theta_Y y_2,y_3} \xrightarrow{\text{id} \otimes \nu \otimes \text{id}} \mathcal{L}_{\Theta_Y y_1,y_3} = \bar{L}(1,y_1),(-1,y_3').$$

(2.7)

The trivialization of the line bundle $Q = N \otimes \Theta_s N$ defined by its section $S$ is then given by

$$\mathcal{L}_{y_1,y_2} \otimes \mathcal{L}_{y_2,y_3} \otimes \mathcal{L}_{\Theta_Y y_2,y_3} \xrightarrow{\text{id} \otimes \nu \otimes \text{id}} \mathcal{L}_{\Theta_Y y_1,y_3} = \bar{L}(1,y_1),(-1,y_3').$$

(2.8)

If, as in Sec. [8] $(O_i)_{i \in I}$ is a sufficiently fine $\Theta$-invariant open covering of $M$ and the maps $s_i : O_i \rightarrow Y$ obey the relation $s_{i_1} \circ s_{i_2} = \Theta_Y \circ s_{i_1} \circ \Theta_Y$ then the collection $(\Pi_i, \chi_{i_1,i_2}, f_i)$ with the entries defined in Sec. [8] provides local data for the $\mathbb{Z}_2$-equivariant structure on the gerbe $G$. In particular, the maps $s_{i_1}^{-1}$ introduced there take values in the line bundle $N'$ and

$$i\nabla_{\mathcal{N}} s_{i_1}^{-1} = \Pi_i s_{i_1}^{-1} = \nu(\sigma_{i_1} \otimes \sigma_{i_2}^{-1}) = \chi_{i_1,i_2} \sigma_{i_1}^{-1} \otimes \sigma_{(i_1),(-i_2)} \circ \Theta), \quad \bar{\sigma}_{i_1}^{-1} \otimes (\bar{\sigma}_{(i_1),(-i_2)} \circ \Theta) = f_i S.$$

(2.9)

(2.10)

The definition of $\sqrt{\text{Hol}_G(\phi)}$ in Sec. C of [12], based on the use of $\mathbb{Z}_2$-equivariant structure on $G$, is equivalent to the one from Sec. [8] of the present paper for the lifts $c$ of triangles $c$ forming the domain $F$ and lifts $b$ of edges $\tilde{b}$ either shared by two triangles of $F$ or belonging to the curves $b$ such that $\partial F = \ell \cup \partial(\ell)$ and $\partial \ell \subset \Sigma'$. Indeed, for such a choice of $c$ and $b$ the expression (6) reduces to

$$\frac{i}{e} \int_{\partial F} s^* c \left( \otimes_{b \in \mathcal{F}} \text{hol}_{\mathcal{L}}(s_{c}|b, s_b) \right) \otimes \left( \otimes_{b \in \partial \mathcal{L}} \text{hol}_{\mathcal{L}}(s_b, s_c \circ \partial(b)) \right)$$

$$\cong \frac{i}{e} \int_{\partial \mathcal{F}} s^* c \left( \otimes_{b \in \mathcal{F}} \text{hol}_{\mathcal{L}}(s_{c}|b, s_b) \right) \otimes \left( \otimes_{b \in \partial \mathcal{L}} \text{hol}_{\mathcal{L}}(\Theta_Y \circ s_b, s_c \circ \partial(b)) \right)$$

$$\cong \frac{i}{e} \int_{\partial \mathcal{F}} s^* c \left( \otimes_{b \in \mathcal{F}} \text{hol}_{\mathcal{L}}(s_{c}|b, s_b) \right) \otimes \left( \otimes_{b \in \partial \mathcal{L}} \text{hol}_{\mathcal{L}}(s_{\partial(b)}, s_{c \circ \partial(b)}) \right)$$

$$= \frac{i}{e} \int_{\partial \mathcal{F}} s^* c \left( \otimes_{b \in \mathcal{F}} \text{hol}_{\mathcal{L}}(s_{c}|b, s_b) \right) \otimes \left( \otimes_{b \in \partial \mathcal{L}} \text{hol}_{\mathcal{L}}(s_b) \right).$$

(11)
where $c$ and $b$ run here over the triangles and edges of the triangulation of $\Sigma$ with specified restrictions that determine their orientations and we used the relation (A.2) to represent the line bundle $K$. It was the right hand side of (A.11) that was used in Sec.~F of [12] to define $\sqrt{\text{Hol}_G(\phi)}$, employing a trivializing section of the flat line bundle $N'$ over $M'$ whose definition in [12] agrees with the one given in Sec.~F here.

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[1] O. Alvarez: Topological quantization and cohomology. Commun. Math. Phys. 100 (1985), 279-309
[2] O. Ben-Bassat: Equivariant gerbes on complex tori. J. Geom. Phys. 64 (2013), 209-221
[3] F. Bonahon: Geometric structures on 3-manifolds. In: Handbook of Geometric Topology, eds. R. B. Sher and R. J. Daverman, Elsevier Amsterdam 2002, p. 114
[4] J.-L. Brylinski: Loop Spaces, Characteristic Classes and Geometric Quantization. Birkhauser, Boston 1993
[5] A. L. Carey, J. Mickelsson, M. Murray: Bundle gerbes applied to quantum field theory. Rev. Math. Phys. 12 (2000), 65-90
[6] D. Carpentier, P. Delplace, M. Fruchart and K. Gawdzki: Topological index for periodically driven time-reversal-invariant 2D systems. Phys. Rev. Lett. 114 (2015), 106806
[7] D. Carpentier, P. Delplace, M. Fruchart, K. Gawdzki and C. Tauber: Construction and properties of a topological index for periodically driven time-reversal-invariant 2D crystals. Nucl. Phys. B 896 (2015), 779-834
[8] P. Deligne: Théorie de Hodge : II. Publ. Math. de l'IHÉS 40 (1970), 557
[9] L. Fu and C. L. Kane: Time reversal polarization and a $Z_2$ adiabatic spin pump. Phys. Rev. B 74 (2006), 195312
[10] L. Fu, C. L. Kane and E. J. Mele: Topological insulators in three dimensions. Phys. Rev. Lett. 98 (2007), 106803
[11] K. Gawdzki: Topological actions in two-dimensional quantum field theory. In: Non-Perturbative Quantum Field Theory, eds. G. ’t Hooft, A. Jaffe, G. Mack, P. Mitter et R. Stora, Plenum Press, New York, London 1988, pp. 101-142
[12] K. Gawdzki: Bundle gerbes for topological insulators. Preprint arXiv:1512.01028 [math-ph], to appear in Proceedings of Advanced School on Topological Field Theory, Warsaw, December 7-9, 2015
[13] K. Gawdzki, N. Reis: WZW branes and gerbes. Rev. Math. Phys. 14 (2002), 1281-1334
[14] K. Gawdzki, R. R. Suszek and K. Waldorf: Bundle gerbes for orientifold sigma models. Adv. Theor. Math. Phys. 15 (2011), 621-688
[15] K. Gawdzki, R. R. Suszek and K. Waldorf: Global gauge anomalies in two-dimensional bosonic sigma models. Commun. Math. Phys. 302 (2011), 513-580
[16] J. Giraud: Cohomologie Non-Abélienne. Springer 1971
[17] K. Gomi: Equivariant smooth Deligne cohomology. Osaka J. Math. 42 (2005), 309-337
[18] K. Gomi: Relationship between equivariant gerbes and gerbes over the quotient space. Commun. Contemp. Math. 7 (2005), 207-226
[19] C. L. Kane and E. J. Mele: $Z_2$ topological order and the quantum spin Hall effect. Phys. Rev. Lett. 95 (2005) 146802
[20] D. Monaco and C. Tauber: Gauge-theoretic invariants for topological insulators: A bridge between Berry, Wess-Zumino, and Fu-Kane-Mele. Lett. Math. Phys., online first
[21] M. K. Murray: Bundle gerbes. J. London Math. Soc. 54 (1996), 403-416
[22] M. K. Murray, D. M. Roberts, D. Stevenson and R. F. Vozzo: Equivariant bundle gerbes. Preprint arXiv: 1506.07931 [math.DG]
[23] M. K. Murray, D. Stevenson: Bundle gerbes: stable isomorphism and local theory. J. Lond. Math. Soc. 62 (2000), 925-937
[24] M. K. Murray and D. Stevenson: The basic bundle gerbe on unitary groups. J. Geom. Phys. 58 (2008), 1571-1590
[25] F. Nathan, M. S. Rudner: Topological singularities and the general classification of Floquet-Bloch systems. New J. Phys. 17 (2015), 125014
[26] T. Nikolaus, C. Schweigert: Equivariance in higher geometry. Adv. Math. 226 (2011), 3367-3408
[27] U. Pachner: P.L. homeomorphic manifolds are equivalent by elementary shellings. European J. Combin. 12 (1991), 1291-145
[28] M. S. Rudner, N. H. Lindner, E. Berg and M. Levin: Anomalous edge states and the bulk-edge correspondence for periodically driven two-dimensional systems. Phys. Rev. X 3 (2013), 031005
[29] U. Schreiber, C. Schweigert and K. Waldorf: Unoriented WZW models and holonomy of bundle gerbes. Commun. Math. Phys. 274 (2007), 31-64
[30] J. Wess and B. Zumino: Consequences of anomalous Ward identities. Phys. Lett. B 37 (1971), 95-97
[31] E. Witten: Global aspects of current algebra. Nucl. Phys. B 223 (1983), 422-432
[32] E. Witten, Non-abelian bosonization in two dimensions. Commun. Math. Phys. 92 (1984), 455-472