A renormalization group approach to non-Hermitian topological quantum criticality

Boran Zhou
Department of Physics, Nanjing University, Nanjing 210093, China

Rui Wang and Baigeng Wang
Department of Physics, Nanjing University, Nanjing 210093, China and National Laboratory of Solid State Microstructures and Collaborative Innovation Center of Advanced Microstructures, Nanjing University, Nanjing 210093, China

Critical transition points between symmetry-broken phases are characterized as fixed points in the renormalization group (RG) theory. We show that, following the standard Wilsonian procedure that traces out the large momentum modes, this well known fact can break down in non-Hermitian systems. Based on non-Hermitian Su-Schrieffer-Hegger (SSH)-type models, we propose a real-space decimation scheme to study the criticality between the topological and trivial phase. We provide concrete examples and an analytic proof to show that the real-space scheme perfectly overcomes the insufficiency of the standard method, especially in the sense that it always preserves the system at criticality as fixed points under RG. The proposed method can also greatly simplify the search of critical points for complicated non-Hermitian models by ruling out the irrelevant operators. These results pave the way towards more advanced RG-based techniques for the interacting non-Hermitian quantum systems.

Introduction. – The theory of RG achieved enormous success on the critical phenomena and phase transitions accompanied by symmetry breaking. A key observation that evokes fundamental developments of RG is that, the correlation length \( \xi \) emergent from Gaussian approximation is divergent and responsible for the singularities in various of thermal dynamical functions at the criticality \([1]\). Thus, the critical behavior is dominated by fluctuations that are statistically self-similar within the length scale \( \xi \), ensuring that the critical point must be a fixed point under the dilation of short-distance fluctuations. This invariance of critical point is a fundamental requirement, which is also obeyed by transitions between phases characterized by distinct topological numbers \([2,3]\). Moreover, since the topological numbers, being integers robust against the continuous perturbation, cannot flow under dilation, RG trajectories cannot cross the critical point from one phase to another.

Recent years also witnessed developments on non-Hermitian topological systems. The non-Hermicity, which can originate either from the loss and gain \([4,7]\) or the decaying quasi-particles in open systems \([8,13]\), bring to forth a number of remarkable phenomena when it meets the nontrivial topology, including the breakdown of bulk-boundary correspondence \([14,28]\), the emergence of new topological invariants \([16,17,19,29,37]\), and the non-Hermitian skin effect \([38,44]\). These results further lead to the born of the non-Bloch topological band theory, the enriched topological classifications according to symmetries \([45,47]\), new universal behaviors in conformal field theory \([48]\), and more \([49,52]\). On the other hand, the quantum phase transitions and the critical phenomena are much less studied in non-Hermitian topological systems. The RG approach that properly incorporates the non-Hermicity is yet to be investigated carefully.

In this work, we propose a RG method applicable to non-Hermitian topological systems. For purpose of clarity, we firstly introduce the general idea and results as follows. Assuming that a general model has \( N \) tuning parameters forming a vector, \( \mathbf{g} = \{g_1, g_2, ..., g_N\} \), then, in the phase diagram, the boundary separating different phases can in general be a manifold embedded in...
which we term as the critical hyper surface (CHS) for brevity, as schematically exemplified by the $S^2$ embedded in $\mathbb{R}^3$ in Fig.1(a),(b). The correlation function, as discussed before, is divergent in the manifold, therefore the RG transformation must keep the CHS invariant; namely, any RG flows with initial points located in the CHS should always remain within it. To obtain the flows, the coarse-graining is a standard way that traces out the high-energy degrees of freedom. In contrary to the Hermitian case where the modes with $|E| \in [E_0/b,E_b]$ ($b > 1$) are eliminated (Fig.1(c)), there occurs two scaling parameters in non-Hermitian systems $b_i$ and $b_r$, corresponding to the decimation of the real and imaginary window respectively, as shown by Fig.1(d). Intuitively, one expects that, unlike the Hermitian model where the flows always remain in CHS as indicated by Fig.1(a), the ratio $b_i/b_r$ cannot be chosen arbitrarily in the non-Hermitian case, otherwise the CHS may not remain invariant as shown by Fig.1(b). This poses a fundamental question, i.e., the standard coarse-graining does not ensure the critical points to be fixed points with non-Hermiticity. To overcome this problem, we propose a block-decimation RG scheme. Using the non-Hermitian Su-Schrieffer-Heeger (SSH)-type models as examples, we show that, the proposed RG transformation successively generates the low-energy effective models describing the most relevant degrees of freedom with preserving the criticality as fixed points, ensuring the invariance of CHS, and greatly simplifying the calculation of CHS by ruling out the irrelevant operators. These results pave the way for developments of advanced analytical and numerical RG techniques on interacting non-Hermitian systems.

Coarse-graining of momentum space. — We demonstrate different RG schemes using the non-Hermitian SSH models with extended hoppings as typical examples. The general Hamiltonian we consider reads as,

$$H = \sum_{n,j,\mu} t_{j,\mu}(c_{n+j,\mu}^\dagger c_{n,\mu} + h.c.) + \frac{\gamma_{j,\mu\nu}}{2} c_{n+j,\mu} c_{n,\mu} c_{n+j,\nu} c_{n,\nu} - h.c.,$$

(1)

where $c_{n,\mu}$ is the annihilation operator at site $n$ with sublattice $\mu = A, B$. $t_{j,\mu\nu}$ represents for the first several nearest neighbor hoppings and $\gamma_{j,\mu\nu}$ indicates the non-Hermiticity. We consider in this work up to the next nearest neighbor hopping terms as shown by Fig.2, where the sublattice indices are rearranged giving rise to parameters, $t_1$ and $\gamma_i$ with $i = 1,\ldots,4$. Models with different $t_1$ and $\gamma_i$ are considered in the following in order to exemplify the generality of our methods. We firstly apply the momentum space RG to Eq. (1) for a simple case with only nonzero $t_1$, $t_2$, and $\gamma_1$. The Hermitian SSH model with $\gamma_1 = 0$ is well studied; for $t_1 \neq t_2$, the system acquires a mass gap at $k = \pi$ (mod by $2\pi$). Thus, one can obtain the long-distance continuum field theory near $k = \pi$, to which we perform the coarse-graining of fast modes, leading to the fixed point, $t_1 = t_2$.

The system at $t_1 = t_2$ is gapless, scaling-invariant critical state between the topological trivial and nontrivial phase with winding number 0 and 1, respectively. As expected, the standard method characterizes the critical state as fixed point in RG sense, as explicitly shown by the red line $t_1 = t_2$ in Fig.2, which is the RG-invariant CHS embedded in 2D plane $(t_1, t_2)$.

After turning on $\gamma_1$, the CHS for the non-Hermitian model can be calculated utilizing the notion of the generalized Brillion zone (GBZ) [16,19,29,31], leading to $|t_1^2 - \gamma_1^2| = |t_2^2 - \gamma_1^2|$. As shown in Fig.3, two branches of CHS embedded in space $(t_1, t_2, \gamma_1)$ occur, where the grey surface intersects with the “Hermitian plane” at $t_1 = t_2$ and with the blue surface at $t_1 = \gamma_1/2$. Then, we apply the same RG transformation as above to the long-distance continuum model derived near the blue branch of CHS. It turns out that the CHS does not remain invariant as long as $\gamma_1 \neq 0$, as shown in Fig.3, the RG trajectories, i.e., the dashed curves with arrows, generically pass through the CHS, crossing from the trivial to nontrivial phase. The example shows that the non-
Hermitian system at criticality is not respected by the standard momentum space RG.

**Block-decimation in real space.** – The issue of the above conventional method is twofold. First, the decimation of the large momentum modes does not necessarily trace out the high energy windows with the proper ratio $b_i/b_0$, as indicated by Fig.1(d), and second, it is known that the wave vector emergent from periodic boundary condition is not able to characterize intrinsic feature of non-Hermitian states. We now propose a general block-decimation method to overcome these issues.

Firstly, we rewrite Eq.(1) more compactly as, 
\[ H = \sum_{j=-N}^N \sum_{\mu,\nu} \tilde{t}_{j,\mu\nu} \bar{c}_n \hat{c}_{n+j,\mu\nu}, \]
where the hopping $\tilde{t}_{j,\mu\nu}$ has absorbed the non-Hermitian terms. Then, we formally reshape the 1D chain into a ladder composed of an upper and a lower chain, each of which consists of two sublattices in a unit cell, as shown by Fig.4(a). Meanwhile, the total Hamiltonian is separated into three parts $H = H_u + H_d + H_{ud}$, where $H_u$ and $H_d$ describes the upper and lower chain respectively, and $H_{ud}$ the interchain hopping. The partition function of the total system is then cast into $Z = \int D\bar{\psi}_u D\psi_u D\bar{\psi}_d D\psi_d e^{-\mathcal{S}_u + \mathcal{S}_d + \mathcal{S}_{ud}}$, where
\[ \mathcal{S}_u[c, \bar{c}] = -\sum_{\omega_n} \sum_{k \in BZ^*} \bar{\psi}_k, i\omega_n, a (i\omega_n - \bar{f}^0_k) \psi_k, i\omega_n, a, \]
with $a = u, d$. $\psi_{k, i\omega_n, a} = [c_{k, i\omega_n, a}, \bar{c}_{k, i\omega_n, a}]^T$ is the Grassmann field. $f^0_k$ is the kinetic term, a matrix in sublattice space whose elements are $f^0_{k,\mu\nu} = \sum_{l=-1}^{l_{\text{max}}} \tilde{t}_{2l,\mu\nu} e^{-(2l+1)k}$ with $l$ being the largest integer less than or equal to $N/2$. Since the reciprocal space is also modified under decimation, the sum of $k$ in Eq.(2) is restricted within $BZ^*$. The action for the interchain coupling can be written as,
\[ \mathcal{S}_{ud} = \sum_{i\omega_n} \sum_{k \in BZ^*} \sum_\mu \bar{\psi}_{k, i\omega_n, \mu} \tilde{f}^0_k \psi_{k, i\omega_n, \mu}, \]
where the components of the matrix $\tilde{f}^0_k$ reads as $\tilde{f}^0_{k,\mu\nu} = \sum_{l=-N}^{N} (l_{\text{max}} - l) t_{2l+1,\mu\nu} e^{-(2l+1)k}$. Since the action is bilinear in terms of Grassmann fields, we exactly integrate out the lower chain as indicated by Fig.4(a), generating an effective action for the renormalized upper chain as,
\[ \mathcal{S}'_u = \sum_{i\omega_n} \sum_{k \in BZ^*} \bar{\psi}_{k, i\omega_n, u} G^{-1}(i\omega_n, k) \psi_{k, i\omega_n, u}. \]

The retarded Green’s function $G(\omega, k) = \omega^+ - \bar{f}^0_k (\omega^+ - \bar{f}^0_k)^{-1} f^0_k$ with $\omega^+ = \omega + i0^+$ is obtained from the Matsubara Green’s function by analytic continuation. Since our main focus in this work is to investigate the critical points which take place in the low-energy window, it is convenient to take the low-frequency limit in $G(\omega, k)$, which well preserves the low-energy states as long as the lower chain remains gapped. Then, one can read off an effective Hamiltonian from Eq.(1) as,
\[ H_{\text{eff}} = \sum_{k \in BZ^*} \bar{\psi}_{k, u} \left( H_{k, u}^0 \right) \psi_{k, u}, \]
which, after rescaling back to the original BZ, serves as the starting point for the next RG step.

The block-decimation was firstly invented for classical and quantum spin models [67, 68]. Its application to the non-Hermitian topological systems in the fashion of functional representation has not been explored to date. Unlike the momentum coarse-graining, which inevitably introduces bias between the decision of the real and imaginary energy modes, the real-space approach treats them on equal footing because both are traced out in the same manner.

Now we apply the block-decimation RG to some specific models. The first example we consider has nonzero parameters, $t_i$ and $\gamma_i$ with $i = 1, 2, 3$, as shown by Fig.2. Following the above steps, we obtain $H_{\text{eff}}$ describing the renormalized upper chain, which shares the same form as the bare one but with renormalized parameters. The procedure generates a relation between the bare parameters $g = \{ t_1, t_2, t_3, \gamma_1, \gamma_2, \gamma_3 \}$ and the renormalized ones $g' = \{ t'_1, t'_2, t'_3, \gamma'_1, \gamma'_2, \gamma'_3 \}$, i.e., $g' = \mathcal{R}(g)$, which is a complicated algebraic mapping in the 6D parameter space.

An analytic solution of the mapping $\mathcal{R}$ can be obtained for $t_3 = \gamma_3 = 0$, which leads to $t'_1 = t_1$, $\gamma'_1 = \gamma_1$, $t'_2 = -2t_2 \gamma_2 + t_1 (4t_2^2 + \gamma_2^2)/(4t_2^2 - \gamma_2^2)$ and $\gamma'_2 = -8t_1 t_2 \gamma_2 + \gamma_1 (4t_2^2 - \gamma_2^2)$. Using the above relations, one then obtains $t_2^2 - \gamma_2^2/4 = t'_2 - \gamma'_2/4$, where the CHS in this case can be exactly found as a hypersurface in $\mathbb{R}^4$ satisfying $|t_2^2 - \gamma_2^2/4| = |t'_2 - \gamma'_2/4|$. Given a bare model located in the CHS, it then becomes obvious that $|t_2^2 - \gamma_2^2/4| = |t'_2 - \gamma'_2/4|$ is always satisfied after any steps of the RG mapping $\mathcal{R}$, clearly showing the invariance of the CHS. For more complicated case with $t_3 \neq 0$ and $\gamma_3 \neq 0$, we numerically evaluate the winding number $W$ [11, 10] for each RG step $l$, whose sudden change indicates the critical points as a function of $l$. Fig.4(b) shows the winding numbers as a function of $l$ with other parameters fixed. The sudden change of $W$ indicates the critical point being around $t_{2c} = 1.5$. The blue and red horizontal data lines in the inset of Fig.4(b) show the positions of $t_{2c}$ as a function of the RG step $l$, for $t_1 = 1.8$ and $t_1 = 1.0$, respectively, clearly displaying the invariance of the critical points under the proposed RG transformation.

As another example, we consider a slightly different model with $t_3 = 0$ and $t_4 \neq 0$ in Fig.2. In this case, longer-range hopping terms absent in the bare Hamiltonian emerge after RG transformation [60]. To testify whether the CHS is invariant, we calculate the energy spectrum $|E|$ for the renormalized model, $H_{\text{eff}}$, under open boundaries, where the critical points are indicated...
by the gapless nodes $|E| = 0$. On the other hand, the critical points of the bare model can be accurately obtained by examining the winding number $W$. As shown by Fig. 4(c), the critical points of the renormalized and bare model are found to match exactly with each other. Therefore, the block decimation still respects the CHS even if new hopping terms are generated under RG.

More generally, suppose a new set of hopping terms $t_a$ emerge that are absent in $g$ of the bare model. In this case, a CHS described by $S(g, t_a) = S(R(g)) = 0$, can in principle be found, but embedded in the higher dimensional space of renormalized parameters, $\mathbb{R}^D$, where $D = \text{dim}[g] + \text{dim}[t_a]$. The “intersection” between $S(g, t_a) = 0$ and the manifold $\mathbb{R}^\text{dim}[g]$ automatically gives rise to a submanifold of $S(g, t_a) = 0$, which is nothing but the CHS embedded in $\mathbb{R}^\text{dim}[g]$, i.e., $S_0(g) = 0$. Being a submanifold, $S_0(g) = 0$ is thereby equivalent to the “intersection” of the two CHSs, $S(g, t_a) = S_0(g) = 0$, therefore $S(R(g)) = S_0(g) = 0$, i.e., the CHS remains intact under the proposed block-decimation.

An analytic proof. – We now provide a rigorous proof for the following claim, i.e., the block-decimation preserves the CHS of any 1D non-Hermitian systems as long as the subsystem to be traced out, e.g., the lower chain in Fig. 2, remains gapped. To achieve so, we start from the general renormalized Hamiltonian $H_{\text{eff}}$ in Eq. (6) under the open boundaries. Then, we promote the Bloch wave-vector to $k = -i n \beta$ and modify BZ to the GBZ [19]. Since the CHS for the $H_{\text{eff}}$ at each RG step is determined by the gapless nodes in GBZ with $|E| = 0$, it is sufficient to prove that the gapless nodes remain unchanged under the block-decimation.

Suppose that $H_{\text{eff}}^{(l)}$ at the $l$-th RG step is obtained. Using Eq. (2) and Eq. (3), it can be written into a matrix as, $H_{\text{eff}}^{(l)} = \tau^0 f^{(0),0}_\beta + \tau^x f^{(0),t}_\beta$, where $\tau$ denotes the upper/lower chain space and $f^{(0),0}_\beta$, $f^{(0),t}_\beta$ are the matrices at the $l$-th step, similar to those of $E_{\text{eff}}$. Then for the $l + 1$-th step, the GBZ can be uniquely determined by the eigen-equation, $H_{\text{eff}}^{(l+1)} = E^{(l+1)}$, where $H_{\text{eff}}^{(l+1)} = f^{(l+1),0}_\beta - \beta f^{(l),0}_\beta$, as well as a condition $|\beta_1| \leq |\beta_2| \leq \ldots \leq 2M$ are the solutions of the eigen-equation satisfying $|\beta_1| \leq |\beta_2| \leq \ldots \leq 2M$. Here, $2M$ is the degree of the algebraic equation for $\beta$ [19]. Moreover, as long as there exists gapless nodes in the spectrum of $H_{\text{eff}}^{(l+1)}$, they are determined by following equation,

$$\text{det}[H_{\text{eff}}^{(l+1)}] = 0, \quad \text{and } \beta \in \text{GBZ}.$$ (6)

The first condition in Eq. (6) generates $\beta_l$ as function of the model parameters $g_i$ at step $l$, since both $f^{(l),0}_\beta$ and $f^{(l),t}_\beta$ are dependent on $g_i$. The second condition $\beta \in \text{GBZ}$ further requires that $|\beta(g_i)| = |\beta_{M+1}(g_i)|$, which uniquely produces the implicit expression of the CHS at the $l + 1$-th step. Similar to Eq. (6), the CHS at the $l$-th step is then described by

$$\text{det}[H_{\text{eff}}^{(l)}] = 0, \quad \text{and } \beta \in \text{GBZ}.$$ (7)

Last, we note that Eq. (6) and Eq. (7) are related to each other via a simple matrix product as,

$$H_{\text{eff}}^{(l)} \left( -\frac{1}{f^{(0),0}_\beta} \left( f^{(0),t}_\beta \right)^{-1} \right) = \left( H_{\text{eff}}^{(l+1)} \left( f^{(0),t}_\beta \right)^{-1} \right),$$ (8)

such that we obtain for $\beta \in \text{GBZ}$ that

$$\text{det}[H_{\text{eff}}^{(l)}] = 0 \iff \text{det}[H_{\text{eff}}^{(l+1)}] = 0,$$ (9)

as long as $f^{(0),0}_\beta$ is invertible with $\text{det}[(f^{(0),0}_\beta)^{-1}] \neq 0$. Noting that this essentially requires that subsystem to be integrated out remains gapful, in consistent with what we assumed for the low-frequency approximation above and is satisfied by a large class of models as exemplified above. The equivalence in Eq. (9) proves that the effective Hamiltonian at two successive RG step share the same equation of CHS, i.e., the claim raised above.

Conclusion and discussion. – To summarize, we investigate the criticality of non-Hermitian system based
on SSH-type models. We show that the standard RG with momentum coarse-graining is inappropriate to non-Hermitian phases because it cannot preserve the criticality as RG fixed point. Instead, we propose a block-decimation scheme based on functional representation and prove that it produces RG flows that always respect the CHSs of non-Hermitian phases. It is worthwhile to point out that, the method can be easily applied to complicated models with longer-range hopping whose CHS is difficult to obtain [66], and it turns out that the proposed RG method can rule out the irrelevant operators and greatly simplify the calculation of the CHS for the renormalized models [66]. These results are not specific to SSH-type models but have general applications to other low-dimensional non-Hermitian quantum systems. The success of the real-space block-decimation sheds light onto further developments of numerical algorithms and perturbative RG calculations for the realm of non-Hermitian physics.

This work was supported by the National Program on Key Research Project (No. 2016YFA0300501) and the Youth Program of National Natural Science Foundation of China (No. 11904225).

[1] Mehran Kardar, [Statistical Physics of Fields] (2007).
[2] Wei Chen, Markus Legner, Andreas Ruegg, and Manfred Sigrist, “Correlation length, universality classes, and scaling laws associated with topological phase transitions,” Phys. Rev. B 95, 075116 (2017).
[3] Wei Chen, “Scaling theory of topological phase transitions,” J. Phys. Condens. Matter 28, 055601 (2016).
[4] Bo Zhen, Chia Wei Hsu, Yuichi Igarashi, Ling Lu, Ido Kaminer, Adi Pick, Song-Liang Chua, John Joannopoulos, and Marin Soljačić, “Spawning rings of exceptional points out of dirac cones,” Nature 525, 354–358 (2015).
[5] Ramy El-Ganainy, Konstantinos Makris, Mercedeh Khajavikhan, Ziad Musslimani, Stefan Rotter, and Demetrios Christodoulides, “Non-hermitian physics and pt symmetry,” Nature Physics 14, 11–19 (2018).
[6] Tomoki Ozawa, Hannah M. Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael C. Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, and Iacopo Carusotto, “Topological photonics,” Rev. Mod. Phys. 91, 015006 (2019).
[7] Liang Feng, Ramy El-Ganainy, and Li Ge, “Non-hermitian photonics based on paritytime symmetry,” Nat. Photonics 11 (2017).
[8] Hengyun Zhou, Chao Peng, Yoseob Yoon, Chia Wei Hsu, Keith A. Nelson, Liang Fu, John D. Joannopoulos, Marin Soljačić, and Bo Zhen, “Observation of bulk fermi arc and polarization half charge from paired exceptional points,” Science 359, 1009–1012 (2018).
[9] Vladyslav Kozić and Liang Fu, “Non-hermitian topological theory of finite-lifetime quasiparticles,” Phys. Rev. B 98, 035141 (2018).
[10] Tsuneiya Yoshida, Robert Peters, and Norio Kawakami, “Non-hermitian perspective of the band structure in heavy-fermion systems,” Phys. Rev. B 98, 035141 (2018).
[11] Michal Papaj, Hiroki Ioseb, and Liang Fu, “Nodal arc of disordered dirac fermions and non-hermitian band theory,” Phys. Rev. B 99, 201107 (2019).
[12] Paul A. McClarty and Jeffrey G. Rau, “Non-hermitian topology of spontaneous magnon decay,” Phys. Rev. B 100, 100405 (2019).
[13] Huitao Shen and Liang Fu, “Quantum oscillation from in-gap states and a non-hermitian landau level problem,” Phys. Rev. Lett. 121, 026403 (2018).
[14] Huitao Shen and Liang Fu, “Quantum oscillation from in-gap states and a non-hermitian landau level problem,” Phys. Rev. Lett. 121, 026403 (2018).
[15] Tony E. Lee, “Anomalous edge state in a non-hermitian lattice,” Phys. Rev. Lett. 116, 133903 (2016).
[16] Shunyu Yao and Zhong Wang, “Edge states and topological invariants of non-hermitian systems,” Phys. Rev. Lett. 121, 086803 (2018).
[17] Daniel Leykam, Konstantin Y. Bliokh, Chunli Huang, Y. D. Chong, and Franco Nori, “Edge modes, degeneracies, and topological numbers in non-hermitian systems,” Phys. Rev. Lett. 118, 040401 (2017).
[18] Flore K. Kunst, Elisabet Edvardsson, Jan Carl Budich, and Emil J. Bergholtz, “Biorthogonal bulk-boundary correspondence in non-hermitian systems,” Phys. Rev. Lett. 121, 026408 (2018).
[19] Kazuki Yokomizo and Shuichi Murakami, “Non-bloch band theory of non-hermitian systems,” Phys. Rev. Lett. 123, 066404 (2019).
[20] Victor Martinez Alvarez, Jos Eduardo Barrios Vargas, Matias Berdakin, and Luis Foa Torres, “Topological states of non-hermitian systems,” Eur. Phys. J. Spec. Top. 227, 1295–1308 (2018).
[21] Heinrich-Gregor Zirnstein, Gil Refael, and Bernd Rosenow, “Bulk-boundary correspondence for non-hermitian hamiltonians via block-decimation,” Phys. Rev. B 99, 201103 (2019).
[22] Ye Xiong, “Why does bulk boundary correspondence fail in some non-hermitian topological models,” J. Phys. Commun. 2, 035043 (2018).
[23] V. M. Martinez Alvarez, J. E. Barrios Vargas, and L. E. F. Foa Torres, “Non-hermitian robust edge states in one dimension: Anomalous localization and eigenstate condensation at exceptional points,” Phys. Rev. B 97, 121401 (2018).
[24] Ching Hua Lee and Ronny Thomale, “Anatomy of skin modes and topology in non-hermitian systems,” Phys. Rev. B 99, 201103 (2019).
[25] L. Jin and Z. Song, “Bulk-boundary correspondence in a non-hermitian system in one dimension with chiral inversion symmetry,” Phys. Rev. B 99, 081103 (2019).
[26] Loïc Herviou, Jens H. Bardarson, and Nicolas Regnault, “Defining a bulk-edge correspondence for non-hermitian systems,” Phys. Rev. A 99, 052118 (2019).
[27] Simon Pocock, Paloma Huidobro, and Vincenzo Giannini, “Bulk-edge correspondence and long-range hopping in the topological plasmonic chain,” Nanophotonics 8, 1337–1347 (2019).
[28] Huaqi Wang, Jiawei Ruan, and Haijun Zhang, “Non-hermitian nodes and bulk-boundary correspondence at an anomalous bulk-boundary correspondence.”
[29] Shunyu Yao, Fei Song, and Zhong Wang, “Non-hermitian chern bands,” Phys. Rev. Lett. 121, 136802 (2018).

[30] Tian-Shu Deng and Wei Yi, “Non-bloch topological invariants in a non-hermitian domain wall system,” Phys. Rev. B 100, 035102 (2019).

[31] Tao Liu, Yu-Ran Zhang, Qing Ai, Zongping Gong, Kohei Kawabata, Masahito Ueda, and Franco Nori, “Second-order topological phases in non-hermitian systems,” Phys. Rev. Lett. 122, 076801 (2019).

[32] Simon Lieu, “Topological phases in the non-hermitian su-schrieffer-heeger model,” Phys. Rev. B 97, 045106 (2018).

[33] Zongping Gong, Yuto Ashida, Kohei Kawabata, Kazuaki Takasan, Shoo Higashikawa, and Masahito Ueda, “Topological phases of non-hermitian systems,” Phys. Rev. X 8, 031079 (2018).

[34] Ananya Ghatak and Tanmoy Das, “New topological invariants in non-hermitian systems,” J. Phys. Condens. Matter 31, 263001 (2019).

[35] Chuanhao Yin, Hui Jiang, Linhu Li, Rong Lü, and Shu Chen, “Geometrical meaning of winding number and its characterization of topological phases in one-dimensional chiral non-hermitian systems,” Phys. Rev. A 97, 052115 (2018).

[36] Keita Esaki, Masatoshi Sato, Kazuki Hasebe, and Mahito Kohmoto, “Edge states and topological phases in non-hermitian systems,” Phys. Rev. B 84, 205128 (2011).

[37] Fei Song, Shunyu Yao, and Zhong Wang, “Non-hermitian topological invariants in real space,” Phys. Rev. Lett. 123, 246801 (2019).

[38] Xiaoseng Yang, Yang Cao, and Yunjia Zhai, “Non-hermitian weyl semimetals: Non-hermitian skin effect and non-bloch bulk-boundary correspondence,” arXiv: 1911.02492.

[39] Motoko Ezawa, “Braiding of majorana-like corner states in electric circuits and its non-hermitian generalization,” Phys. Rev. B 100, 045407 (2019).

[40] Zi-Yong Ge, Yu-Ran Zhang, Tao Liu, Si-Wen Li, Heng Fan, and Franco Nori, “Topological band theory for non-hermitian systems from the dirac equation,” Phys. Rev. B 100, 054105 (2019).

[41] Hui Jiang, Li-Jun Lang, Chao Yang, Shi-Liang Zhu, and Motohiko Ezawa, “Braiding of majorana-like corner states in electric circuits and its non-hermitian generalization,” Phys. Rev. B 100, 041031 (2019).

[42] Simon Lieu, “Topological symmetry classes for non-hermitian models and connections to the bosonic bogoliubov-de gennes equation,” Phys. Rev. B 98, 115135 (2018).

[43] C. Yuce, “Majorana edge modes with gain and loss,” Phys. Rev. A 93, 062130 (2016).

[44] C. Yuce, “Majorana edge modes with gain and loss,” Phys. Rev. A 93, 062130 (2016).

[45] Marcel Klett, Holger Carusso, Dennis Dast, Jörg Main, and Günter Wunner, “Relation between PT-symmetry breaking and topologically nontrivial phases in the su-schrieffer-heeger and kitaev models,” Phys. Rev. A 95, 053626 (2017).

[46] Jinho J. Lee, Hermang active, and Ji-Yang Kim, “Majorana fermions in the bulk of a non-hermitian system,” Phys. Rev. Lett. 115, 040402 (2015).

[47] Paolo San-Jose, Jorge C. Yao, and Ramon Aguado, “Majorana bound states from exceptional points in non-topological superconductors,” Sci. Rep. 6, 21427 (2016).

[48] J. Avila, Fernando Peiranda, Elsa Prada, Pablo San-Jose, and Ramon Aguado, “Non-hermitian topology as a unifying framework for the andreev versus majorana
states controversy," Comm. Phys. 2, 133 (2019).

[64] Mário G. Silveirinha, “Topological theory of non-hermitian photonic systems,” Phys. Rev. B 99, 125155 (2019).

[65] Kazuki Yamamoto, Masaya Nakagawa, Kyosuke Adachi, Kazuaki Takasan, Masahito Ueda, and Norio Kawakami, “Theory of non-hermitian fermionic superfluidity with a complex-valued interaction,” Phys. Rev. Lett. 123, 123601 (2019).

[66] “See supplemental material for details of calculation.”.

[67] Leo P. Kadanoff, “The application of renormalization group techniques to quarks and strings,” Rev. Mod. Phys. 49, 267–296 (1977).

[68] Leo P. Kadanoff, Wolfgang Götze, David Hamblen, Robert Hecht, E. A. S. Lewis, V. V. Palciauskas, Martin Rayl, J. Swift, David Aspnes, and Joseph Kane, “Static phenomena near critical points: Theory and experiment,” Rev. Mod. Phys. 39, 395–431 (1967).