Introduction to light cone field theory and high energy scattering

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Abstract

In this set of four lectures, we provide an elementary introduction to light cone field theory and some of its applications in high energy scattering.
1 Introduction

In these lectures, we will attempt to provide a “hands on” introduction to some of the ideas and methods in light cone field theory and its application to high energy scattering. Light cone quantization as an approach to study the Hamilton dynamics of fields was first investigated by Dirac, who pointed out several of its elegant features in a landmark paper \cite{1}. It was first applied to high energy physics in the 60’s in the context of current algebra \cite{2}. Light cone field theory currently finds applications in most areas of high energy physics, from perturbative QCD to string theories.

The elegance and simplicity of the light cone approach results from the analogy of relativistic field theories quantized on the light cone to non–relativistic quantum mechanics. In fact, this correspondence runs deep and it was shown by Susskind that there is an exact isomorphism between the Galilean subgroup of the Poincaré group and the symmetry group of two dimensional quantum mechanics \cite{3}. Furthermore, as was first shown by Weinberg \cite{4}, the vacuum structure of field theories simplifies greatly in the infinite momentum limit. The combination of the non–relativistic kinematics of light cone field theories as well as their simple vacuum structure, has given rise to the belief that potential methods of quantum mechanics can be applied to field theories quantized on the light cone. This observation is at the heart of recent attempts to understand bound state problems in QCD in the light cone formalism \cite{5}. Indeed, beginning with the t’Hooft model \cite{6} for mesons in 1+1–dimensional large \( N_c \) QCD, which made use of the light cone formalism, there have been many attempts to study confinement and chiral symmetry breaking in this approach (see Ref. \cite{7} and references therein).

Light cone field theory also provides much of the intellectual support for the intuitive quark–parton picture of high energy scattering. Frequently, the phrases ‘the theory of strong interactions, QCD’ and the ‘quark–parton picture of strong interac-
tions’ are used interchangeably. However, it is only in light cone quantization (and light cone gauge) that the quark–parton structure of QCD is manifest and multi–parton Fock states can be constructed as eigenstates of the QCD Hamiltonian [8].

One can therefore construct Lorentz invariant light cone wavefunctions – a fact which has been particularly useful in the study of exclusive processes in QCD [9]. Further, in deeply inelastic scattering, the experimentally measured structure functions are simply related (in leading twist) to the light cone quark distribution functions. The partonic picture of light cone quantum field theory was demonstrated very clearly in the papers of Kogut and Soper [10] and of Bjorken, Kogut and Soper [11].

The goal of these lectures is to illustrate both of the above points, the attractive features of light cone field theory and its applications to high energy scattering, in the simplest possible fashion by working out concrete examples. In the first lecture, we begin by introducing the light cone notation and the two component formalism. We then define the light cone Fock states and their equal light cone time commutation relations. We conclude by discussing the structure of the Poincaré group and demonstrate the above mentioned isomorphism to two dimensional quantum mechanics. In the second lecture, we explicitly derive the light cone QCD Hamiltonian in the two component formalism making use of the light cone constraint equations. It is shown that the Hamiltonian can be expressed as the sum of non–interacting and “potential” terms. For simplicity, in lecture three, we specialize to the case of QED and use the form of the Hamiltonian derived in lecture 2 to illustrate the parton picture of high energy scattering. In particular, we study high energy scattering off an external potential in the eikonal approximation in QED. In the fourth and final lecture we show how Bjorken scaling can be derived in QCD using the light cone commutation relations and briefly discuss the relation of light cone distribution functions to structure functions.

There are several reviews that the reader may study to learn more about the subject. An introductory review which also includes a guide to the literature for
beginners is that by Harindranath [12]. Another introductory review which stresses recent advances is that by Burkardt [7]. The most recent and comprehensive review of the subject is by Brodsky, Pauli and Pinsky [13]. A part of our lectures relies heavily on the classic papers of Kogut and Soper [10] and Bjorken, Kogut and Soper [11]. The reader should keep in mind that a wide variety of conventions are in use in the literature. Some of these are discussed in the review of Brodsky, Pauli and Pinsky.

The lectures below were delivered at the Cape Town lecture school and for spacetime reasons are the “short” form of lectures delivered previously at the University of Jyväskylä international summer school. The topics that were omitted in the short version include light cone perturbation theory, the renormalization group and the operator product expansion, and small x physics. The longer version of these lectures will be published separately at a later date [14].
Lecture 1: Light cone quantization and the light cone algebra.

We begin by defining our convention and notations. Our metric here is the +2 metric $\hat{g}^{\mu\nu} = (-,+,+,+)$. Note: for my convenience (and unfortunately, your inconvenience) I may change notations in the latter lectures. But you will have fair warning! The gamma matrices in usual space–time co–ordinates are denoted by carets. In the chiral representation,

$$\hat{\gamma}^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} ; \quad \hat{\gamma}^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} ; \quad \hat{\gamma}^5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

and $\{\hat{\gamma}^\mu, \hat{\gamma}^\nu\} = -2\hat{g}^{\mu\nu}$. Above, $\sigma^i, i = 1,2,3$ are the usual $2\times2$ Pauli matrices and $I$ is the $2\times2$ identity matrix. In light cone co–ordinates, $\gamma^\pm = (\hat{\gamma}^0 \pm \hat{\gamma}^3)/\sqrt{2}$ and $\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}$, where $g^{++} = g^{--} = 0, g^{+-} = g^{-+} = -1$. Also, $g_{t_1,t_2} = 1$ with $t_1, t_2 = 1, 2$ denoting the two transverse co–ordinates. We define $x^\mu \equiv (x^0, x^1, x^2, x^3) = (t, \vec{x})$ and

$$x^\pm = \frac{(t + z)}{\sqrt{2}} ; \quad \partial_{\pm} \equiv \frac{\partial}{\partial x^\pm} = \frac{1}{\sqrt{2}}(\partial_t \pm \partial_z) ; \quad A^\pm = \frac{(A^0 \pm A^z)}{\sqrt{2}}. \quad (1)$$

Note for instance that in this convention $A_+ = A^-$ and $A_t = +A^i$. Also, $q^2 = -2q^-q^+ + q_t^2$. Hence, a “space–like” $q^2$ implying large space–like components would correspond to $q^2 > 0$.

We now define the projection operators

$$\alpha^\pm = \frac{\hat{\gamma}^0 \gamma^\pm}{\sqrt{2}} \equiv \frac{\gamma^\mp \gamma^\pm}{2}, \quad (2)$$

which project out the two component spinors $\psi^\pm = \alpha^\pm \psi$. 

$$\psi_+ = \begin{pmatrix} 0 \\ \psi_2 \\ \psi_3 \\ 0 \end{pmatrix} ; \quad \psi_- = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \\ \psi_4 \end{pmatrix}, \quad (3)$$
where $\psi_1, \cdots \psi_4$ are the four components of $\psi$. It follows from the above that $\psi_+ + \psi_- = \psi$.

Some relevant properties of the projection operators $\alpha^\pm$ are

$$
(\alpha^\pm)^2 = \alpha^\pm; \quad \alpha^\pm \alpha^\mp = 0; \quad \alpha^+ + \alpha^- = 1; \quad (\alpha^\pm)^\dagger = \alpha^\mp.
$$

(4)

We can use these to show that

$$
\alpha^\pm \psi_\mp = 0; \quad \alpha^\pm \psi_\pm = \psi_\pm; \quad \alpha^\pm \gamma^0 = \frac{1}{2} \gamma^0 \alpha^\pm; \quad \alpha^\pm \gamma_\perp = \gamma_\perp \alpha^\pm.
$$

(5)

We will make liberal use of these identities in deriving the light cone Hamiltonian in lecture 2.

A particular property of light cone quantization is that it is the two component spinor $\psi_+$ above that is the dynamical spinor in the light cone QCD Hamiltonian $P_{QCD}^-$. Interestingly, the same feature is observed for fermion fields which obey equal time commutation relations when they are boosted to the infinite momentum frame. The dynamical spinors $\psi_+$ are defined in terms of creation and annihilation operators as

$$
\psi_+ = \int_{k^+ > 0} \frac{d^3k}{2^{1/4}(2\pi)^3} \sum_{s = \pm \frac{1}{2}} \left[ e^{ik \cdot x} b_s(k; x^+) + e^{-ik \cdot x} d_{s^\dagger}(k; x^+) \right],
$$

(6)

where $b_s(k)$ is a quark destruction operator and destroys a quark with momentum $k$ while $d_{s^\dagger}(k)$ is an anti–quark creation operator and creates an anti–quark with momentum $k$. They obey the equal light cone time ($x^+$) anti–commutation relations

$$
\{b_s(k, x^+), b^\dagger_{s'}(\vec{k}', x^+)\} = \{d_s(\vec{k}, x^+), d^\dagger_{s'}(\vec{k}', x^+)\} = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \delta_{ss'}.
$$

(7)

The above definitions ensure that the fermionic contribution to the light cone QCD Hamiltonian can be written as the sum of kinetic and potential pieces, $P_{f,QCD}^- = P_{f,0}^- + V_{QCD}$, where the kinetic piece of the Hamiltonian is defined as

$$
P_{f,0}^- = \int \frac{d^3k}{(2\pi)^3} \sum_{s = \pm \frac{1}{2}} \frac{(k^2 + M^2)}{2k^+} \left( b_{s^\dagger}(k)b_s(k) + d_{s^\dagger}(k)d_s(k) \right).
$$

(8)
These points will become clearer when we explicitly derive the QCD light cone Hamiltonian in lecture 2.

The gauge field $A^\mu$ has two dynamical components $A^\mu_i(x)$ with $i = 1, 2$ in light cone gauge $A^+ = 0$. These are defined in terms of creation–annihilation operators as

$$A^\mu_i(x) = \int_{k^+ > 0} \frac{d^3 k}{\sqrt{2 |k^+| (2\pi)^3}} \sum_{\lambda = 1, 2} \delta_{\lambda i} \left[ e^{ik \cdot x} a^\lambda_i(k; x^+) + e^{-ik \cdot x} a^{\dagger \lambda}_i(k; x^+) \right],$$

where the $\lambda$'s here correspond to the two independent polarizations and $a^\lambda_i (a^{\dagger \lambda}_i)$ creates (destroys) a gluon with momentum $k$. They obey the commutation relations

$$[a^\lambda_i(k), a^{\dagger \lambda'}_{i'}(k')] = (2\pi)^3 \delta^{(3)}(k - k') \delta_{\lambda \lambda'} \delta_{ii'}.$$

In an analogous fashion to Eq. (8), the bosonic kinetic energy can be written (after normal ordering) as

$$\int \frac{d^3 k}{(2\pi)^3} \sum_{\lambda = 1, 2} \frac{(k^2 + M^2)}{2k^+} a^{\dagger \lambda}(k) a^\lambda(k).$$

We will now discuss the structure of the Poincaré group on the light cone. For the field $\hat{\phi}_r$, which here denotes vector or scalar bosons, we can define the stress–energy tensor

$$\hat{T}^{\lambda\nu} = \hat{\Pi}^\lambda \partial^\nu \hat{\phi}_r - \hat{g}^{\lambda\nu} \mathcal{L},$$

where $\mathcal{L}$ is the Lagrangean density, and $\hat{\Pi}^\lambda$ is the generalized momentum

$$\hat{\Pi}^\lambda = \frac{\delta \mathcal{L}}{\delta (\partial^\lambda \hat{\phi}_r)}.$$

Keep in mind that the carets denote quantities in the usual spacetime co–ordinates. Define now the following generalized quantity

$$\hat{\Sigma}_{\alpha\beta}^{\mu\nu} = \begin{cases} \frac{1}{4} [\hat{\gamma}^{\mu}, \hat{\gamma}^{\nu}]_{\alpha\beta} & \text{for spinors} \\ (\hat{\gamma}_\alpha^\mu \hat{\gamma}_\beta^\nu - \hat{\gamma}_\beta^\mu \hat{\gamma}_\alpha^\nu) & \text{for vectors} \end{cases}$$

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One can then define the boost–angular momentum stress tensor

\[ \hat{J}^{\lambda\mu\nu} = \hat{x}^{\mu} \hat{T}^{\lambda\nu} - \hat{x}^{\nu} \hat{T}^{\lambda\mu} + \hat{\Pi}^{\lambda}_{\rho} \hat{\Sigma}_{\rho\nu}^{\mu} \hat{\phi}_{s}. \] (15)

There are ten conserved currents

\[ \partial_{\lambda} \hat{T}^{\lambda\mu} = 0, \]
\[ \partial_{\lambda} \hat{J}^{\lambda\mu\nu} = 0, \] (16)
and correspondingly, ten conserved charges,

\[ \hat{P}^{\mu} = \int d\hat{x}^{1} d\hat{x}^{2} d\hat{x}^{3} \hat{T}^{0\mu}, \]
\[ \hat{M}^{\mu\nu} = \int d\hat{x}^{1} d\hat{x}^{2} d\hat{x}^{3} (\hat{x}^{\mu} \hat{T}^{0\nu} - \hat{x}^{\nu} \hat{T}^{0\mu} + \hat{\Pi}^{0}_{\rho} \hat{\Sigma}_{\rho\nu}^{\mu} \hat{\phi}_{s}). \] (17)

The four components of the energy–momentum vector \( \hat{P}^{\mu} \) and the six components of the boost–angular momentum \( \hat{M}^{\mu\nu} \) comprise the ten generators of the Poincaré group\(^1\). These generators satisfy the Poincaré algebra

\[ [\hat{P}^{\mu}, \hat{P}^{\nu}] = 0; [\hat{M}^{\mu\nu}, \hat{P}^{\rho}] = i \left( \hat{g}^{\nu\rho} \hat{P}^{\mu} - \hat{g}^{\mu\rho} \hat{P}^{\nu} \right), \]
\[ [\hat{M}^{\mu\nu}, \hat{M}^{\rho\sigma}] = i \left( \hat{g}^{\mu\sigma} \hat{M}^{\nu\rho} + \hat{g}^{\nu\rho} \hat{M}^{\mu\sigma} - \hat{g}^{\mu\rho} \hat{M}^{\nu\sigma} - \hat{g}^{\nu\sigma} \hat{M}^{\mu\rho} \right). \] (18)

The six components of the boost–angular momentum can be further split into the three generators of rotations \( \hat{M}^{ij} = \hat{e}^{ijk} \hat{J}^{k} \) (where \( \hat{J}^{k} \) is the angular momentum operator) and three generators of boosts \( \hat{K}^{i} = \hat{M}^{i0} \). In the language of Ref. \[1\], these are referred to as kinematic and dynamic operators, respectively, since the former is independent of the interaction while the latter isn’t.

Transforming the above to light cone co–ordinates, we obtain \( P^{\mu} \equiv (P^{+}, P^{1}, P^{2}, P^{-}) \),

\(^1\)The Poincaré group is a sub–group of the conformal group which contains 15 generators, the additional generators being 4 conformal transformations and 1 dilatation.
where \( P^\pm = (\hat{P}^0 \pm \hat{P}^3)/\sqrt{2} \), and

\[
M^{\mu\nu} = \begin{pmatrix}
0 & -S_1 & -S_2 & K_3 \\
S_1 & 0 & J_3 & B_1 \\
S_2 & -J_3 & 0 & B_2 \\
-K_3 & -B_1 & -B_2 & 0
\end{pmatrix},
\]

(19)

Above we used the following definitions

\[
B_1 = \frac{(K_1 + J_2)}{\sqrt{2}}; \quad B_2 = \frac{(K_2 - J_1)}{\sqrt{2}},
\]

\[
S_1 = \frac{(K_1 - J_2)}{\sqrt{2}}; \quad S_2 = \frac{(K_2 + J_1)}{\sqrt{2}}.
\]

The commutation relations among the \( M^{\mu\nu} \)'s and the \( P^\mu \)'s are of course the same as in Eq. [8]. The operators \( B_1 \) and \( B_2 \) are kinematic and boost the system in the x and y directions respectively. In addition, the operators \( J_3 \) and interestingly, \( K_3 \) are kinematic and rotate the system in the x–y plane and boost it in the longitudinal direction respectively.

An interesting observation by Susskind [3] related to the above is that the commutation relations among the seven generators \( P^\pm, \vec{P}_t, J_3, B_1 \) and \( B_2 \) are the same as the commutation relations among the symmetry operators of non–relativistic quantum mechanics in two dimensions. Indeed, one can formally make the correspondence,

- \( P^- \rightarrow \) Hamiltonian.
- \( \vec{P}_t \rightarrow \) Momenta.
- \( P^+ \rightarrow \) Mass.
- \( J_3 \rightarrow \) Angular Momentum.
- \( \vec{B}_t \rightarrow \) generators of Galilean boosts in x–y plane.
These seven generators obey the commutation relations

\[
[P^-, P_t] = [P^-, P^+] = [P_t, P^+] = [J_3, P^-] = [J_3, P^+] = [B_t, P^+] = 0. \tag{20}
\]

and

\[
[J_3, P^t] = i\epsilon_{tl} P^l, [J_3, B^l] = i\epsilon_{tl} B^l, [B^l, P^-] = -i P^l, [B^l, P^t] = -i\delta_{tl} P^+. \tag{21}
\]

Above \(\epsilon_{ij}\) is the Levi–Civita tensor in two dimensions. Since they are kinematic operators, they leave the planes of \(x^+ = \text{constant}\) invariant under their operations.

Susskind, Bardacki & Halpern \[15\] and Kogut & Soper have shown that the above mentioned isomorphism is responsible for the non–relativistic quantum–mechanical structure of quantum field theories on the light cone. The simplest illustration of this isomorphism is the fact that the free particle Hamiltonian takes the form

\[
H \equiv P^- = \frac{P_t^2 + M^2}{2P^+}.
\]

Recalling the form of the energy in two dimensional quantum mechanics, we obtain the isomorphisms above. For QED and QCD, the above form is modified by the addition of a potential term which we will discuss in detail in lecture 2. Finally, we should mention that the other kinematic operator, \(K_3\), the boost operator in the longitudinal direction, serves to rescale the other operators

\[
\exp(i\omega K_3)P^- \exp(-i\omega K_3) = \exp(\omega)P^-.
\]

\[
\exp(i\omega K_3)J_3 \exp(-i\omega K_3) = J_3.
\]

\[
\exp(i\omega K_3)\vec{S} \exp(-i\omega K_3) = \exp(-\omega)\vec{S}. \tag{22}
\]

This property of \(K_3\) will come in handy in lecture 3.
Lecture 2: The light cone QCD Hamiltonian.

In this lecture, we will derive an explicit form for the light cone QCD Hamiltonian making use of the light cone constraint relations. Consider first the fermionic part of the QCD action

\[ S_F = \int d^4x \bar{\psi} (P + M) \psi. \]

Above, \( P^\mu = -iD^\mu \equiv -i(\partial^\mu - igA^\mu). \) For convenience, we will not write the integral \( \int d^4x, \) in the following but it must be understood to be there. Then writing out the above action explicitly,

\[ S_F = \bar{\psi} \gamma^\mu \gamma^\nu P_{\mu\nu} \psi + \bar{\psi} \gamma^\nu P_{\mu} \psi + \bar{\psi} \gamma^\mu P_\mu \psi + \bar{\psi} M \psi. \]

Consider now the first term in the above:

\[ \bar{\psi} \gamma^\nu P_{\mu\nu} \psi = \bar{\psi} \gamma^\nu P_{\mu\nu} \psi \to \sqrt{2} \psi^\dagger P_\mu \psi. \tag{23} \]

To dissect the above, we first decomposed \( \gamma^0 = (\gamma^+ + \gamma^-)/\sqrt{2}, \) made use of \( (\gamma^-)^2 = 0, \) and the properties of the projector \( \alpha^+ \) in Eq. 4 to obtain the RHS. Similarly, it is recommended to the serious student that he or she show that

\[ \bar{\psi} \gamma^\mu \gamma^\nu P_{\mu\nu} \psi = \sqrt{2} \psi^\dagger P_\mu \psi, \]

\[ \bar{\psi} \gamma^\nu P_{\mu} \psi = \frac{1}{\sqrt{2}} \left( \psi^\dagger \gamma^\nu P_{\mu} + \psi^\dagger \gamma^\mu P_{\nu} \right), \]

\[ \bar{\psi} M \psi = M \left( \psi^\dagger \gamma^0 \psi - \psi^\dagger \gamma^0 \psi \right). \tag{24} \]

One then obtains

\[ S_F = \sqrt{2} \psi^\dagger P_\mu \psi + \sqrt{2} \psi^\dagger P_\mu \psi + \frac{1}{\sqrt{2}} \left( \psi^\dagger \gamma^\nu P_{\mu} + \psi^\dagger \gamma^\mu P_{\nu} \right) + M \left( \psi^\dagger \gamma^0 \psi - \psi^\dagger \gamma^0 \psi \right), \tag{25} \]

where we have written the fermionic piece of the action in terms of the two–spinors \( \psi_- \) and \( \psi_+ \) and their hermitean conjugates.
Following Eq. [14], the momenta conjugate to these two–spinor fields are

\[
\begin{align*}
\Pi_+ &= \frac{\delta L}{\delta (\partial_+ \psi_+)} = \left( \frac{\sqrt{2}}{i} \right) \psi_+^\dagger, \\
\Pi_- &= \frac{\delta L}{\delta (\partial_+ \psi_-)} = 0.
\end{align*}
\]

Since trivially \([\Pi_-, \psi_-] = 0\), the two–spinor \(\psi_-\) is, unlike \(\psi_+\), not an independent quantum field. We will now show that one may derive a constraint equation (i.e., independent of the light cone time \(x^+\)) for \(\psi_-\) in terms of the dynamical field \(\psi_+\). The light cone constraint relations can be obtained from the operator equations of motion. In this case, it is the Dirac equation \((P + M) = 0\), or

\[
(-i\partial_+ - gA_+)\gamma^- \psi + (-i\partial_+ - gA_+)\gamma^+ \psi + \sum_{j=1}^{2} (-i\partial_j - gA_j) \gamma^j \psi + M \psi = 0. \tag{27}
\]

Multiply the above through by \(\gamma^+\). Since \((\gamma^+)^2 = 0\), this projects out the \(x^+\)–light cone time–dependence in the above and we obtain (after liberally using our projection operator tricks from Eq. [14]) the equation

\[
\sqrt{2}P_- \psi_- = -\gamma^0 (P_t + M) \psi_+. \tag{28}
\]

In light cone gauge, \(A_- = -A^+ = 0\), hence \(P_- = (-i\partial_+ - gA_-) \rightarrow -i\partial_-.\) With this gauge condition therefore, one can easily invert the \(P_- = -P^+\) operator and one obtains the light cone constraint equation

\[
\psi_- = \frac{\gamma^0}{\sqrt{2}P^+} (P_t + M) \psi_+. \tag{29}
\]

Thus for light cone time \(x^+\), \(\psi_-\) is determined completely by \(\psi_+\) at that time. Only the two components of the spinor \(\psi\) corresponding to \(\psi_+\) are independent dynamical fields on the light cone.

We can now use the above obtained constraint equation to replace \(\psi_-\) in Eq. [25] for \(S_F\). For instance,

\[
\sqrt{2}\psi_+^\dagger P_- \psi_- = \sqrt{2} \left( \frac{\gamma^0}{\sqrt{2}P^+} (P_t + M) \psi_+ \right)^\dagger P_- \left( \frac{\gamma^0}{\sqrt{2}P^+} (P_t + M) \psi_+ \right) \rightarrow \frac{(-)}{\sqrt{2}} \psi_+^\dagger (M - P^t) \frac{1}{P^+} (P_t + M) \psi_+.
\]

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Now rescale the fields $\psi \rightarrow 2^{-1/4}\psi$. As an exercise, the reader should use the light cone constraint equation above, the properties of the projection operators $\alpha^{\pm}$ in Eq. 3 and those of the light cone gamma matrices to first substitute for $\psi$ everywhere and then demonstrate the following identities,

$$\frac{M}{\sqrt{2}} \left( \psi^\dagger \gamma^0 \psi_+ + \psi^{\dagger 0} \psi_+ \right) = \frac{M}{2} \left( \psi^\dagger \gamma^0 \psi_+ \right)$$

$$\frac{1}{2} \left( \psi^\dagger \gamma^0 (P_t + M) \psi_+ + \psi^{\dagger 0} (P_t + M) \psi_+ \right) = \psi^\dagger \left( M - P_0 \right) \frac{1}{P^+} (P_t + M) \psi_+ .$$

Putting these together with the other term above, our result for the fermionic action expressed solely in terms of the dynamical two-spinor $\psi_+$ is

$$S_F = -\psi^\dagger P^- \psi_+ + \frac{1}{2} \psi^\dagger \left( M - P_0 \right) \frac{1}{P^+} (P_t + M) \psi_+ . \quad (30)$$

We now turn to the bosonic contribution to the action,

$$S_B = \frac{1}{4} F^{a}_{\mu \nu} F^{\mu \nu, a} , \quad (31)$$

and following a procedure analogous to the fermionic case, shall write it in terms of $A_t$, the two transverse, dynamical components of the gauge field $A^{\mu}$. We have seen earlier that the choice of light cone gauge $A^- = -A^+ = 0$ greatly simplifies the light cone constraint relation for the fermions. In this gauge, the various components of the field strength tensor also simplify to

$$F^a_{\mu \nu} = -\partial_\nu A^a_\mu ,$$

$$F^a_{t+} = \partial_t A^a_+ - \partial_+ A^a_t + g f^{abc} A^b_+ A^c_t ,$$

$$F^a_{t-} = -\partial_- A^a_t \equiv -E^a_t .$$

In addition, there are of course the purely transverse pieces $F_{ij}$; with $i, j = 1, 2$. From the above it is evident that there is no (light cone) time derivative $\partial_+ A_+$ in

\[2\]This rescaling gets rid of the $\sqrt{2}$ factors in the action. This also explains the peculiar $2^{-1/4}$ normalization factor in Eq. 6 for the properly normalized $\psi_+$ field.
the action. The field $A_+$ therefore has no momentum conjugate and we may use the operator equations of motion to eliminate the field $A_+ = -A^-$. The equations of motion are the Yang–Mills equations of course. The light cone constraint equation is just Gauss’ law on the light cone since it must be valid at all times. This condition is then

\[(D_t F^+)^a + (D_- F^-)^a = -J^{+a} \implies -\partial^2 A^{-a} = J^{+a} + (D_t E_t)^a,\]

where $E_t^a$ are the two transverse components of the electric field and $D_t$ is the covariant derivative $\partial_t - ig A_t$. We can write our light cone constraint equation for $A^-$ compactly below as

\[A^{-a} = \frac{1}{(P^+)^2} (J^{+a} + (D_t E_t)^a).\]  

(32)

Returning to the action $S_B = \frac{1}{4} F^2 = \frac{1}{4} F_t^2 - F_+^a F_-^a - \frac{1}{2} F_-^a F_+^a$, substituting the expressions for the field strength components in terms of the gauge fields and performing an integration by parts, we obtain

\[S_B = \frac{1}{4} F_t^2 - A_+^a (D_t E_t)^a - \frac{1}{2} (\partial_- A_+^a)^2 - (\partial_- A_t^a) (\partial_+ A_t^a).\]  

(33)

Before we substitute for $A_+$ above, we will first write out the full action $S_{QCD} = S_F + S_B - J_{ext} \cdot A$:

\[S_{QCD} = -\psi_+^\dagger (-i\partial^- - gA^-) \psi_+ + \frac{1}{2} \psi_+^\dagger (M - P_t) \frac{1}{P^+} (P_t + M) \psi_+ + \frac{1}{4} F_t^2 - A_+^a (D_t E_t)^a - \frac{1}{2} (\partial_- A_+^a)^2 - (\partial_- A_t^a) (\partial_+ A_t^a) - J_{ext}^+ A_+.\]

Consider the first term above. We can write this as

\[-\psi_+^\dagger (-i\partial^- - gA^-) \psi_+ = -i\psi_+^\dagger \partial_+ \psi_+ - J_{dyn}^+ A_+,\]

where $J_{dyn}^+ = \psi_+^\dagger \lambda^a \psi_+$. (The $\lambda^a$ are the Gell–Mann SU(3) matrices.) We now substitute the above result in our expression for the action and after
a) defining \( J^+ = J_{\text{dyn}}^+ + J_{\text{ext}}^+ \),
b) performing an integration by parts,
c) making use of the constraint relation Eq. 32 to eliminate \( A_+ \),
we obtain finally,

\[
S_{\text{QCD}} = -i \psi_+^\dagger \partial_+ \psi_+ - (\partial_- A_t) (\partial_+ A_t) + \frac{1}{4} F_t^2 + \frac{1}{2} \psi_+^\dagger (M - P_t) \frac{1}{P^+} (P_t + M) \psi_+ \\
+ \frac{1}{2} (J^+ + D_t E_t) \frac{1}{(P^+)^2} (J^+ + D_t E_t) .
\] (34)

The final step before we obtain the Hamiltonian is to identify the momenta conjugate to the dynamical fields (now with the proper normalization!),

\[
\Pi_{\text{fermi}} = \frac{\delta S_{\text{QCD}}}{\delta \partial_+ \psi_+} = -i \psi_+^\dagger , \\
\Pi_{\text{bose}} = \frac{\delta S_{\text{QCD}}}{\delta \partial_+ A_t} = -\partial_- A_t \equiv -E_t .
\] (35)

Writing out the fields and their momentum conjugates in terms of the creation and annihilation operators introduced in Eqs. 6 and 9, and making use of their commutation relations, the reader may confirm that

\[
[\Pi_{\text{bose}}(x), A_t(x')] = \{\Pi_{\text{fermi}}(x), \psi_+(x')\} = i \delta^{(3)}(x - x') .
\] (36)

The Hamiltonian density in our convention is defined as

\[
H \equiv P_{\text{QCD}}^- = S_{\text{QCD}} - \Pi_{\text{fermi}} \partial_+ \psi_+ - \Pi_{\text{bose}} \partial_+ A_t ,
\]

We can therefore write our final expression for the Hamiltonian density as

\[
P_{\text{QCD}}^- = \frac{1}{4} F_t^2 + \frac{1}{2} (J^+ + D_t E_t) \frac{1}{(P^+)^2} (J^+ + D_t E_t) \\
+ \frac{1}{2} \psi_+^\dagger (M - P_t) \frac{1}{P^+} (P_t + M) \psi_+ .
\] (37)

We have therefore succeeded in obtaining the light cone Hamiltonian in QCD, expressed solely in terms of the two–spinor \( \psi_+ \) and \( A_t \), the two transverse components of the gauge field. The following observations can be made regarding the above
expression. Firstly, one can show straightforwardly that the light cone Hamiltonian can be written as

\[ P^-_{QCD} = P^+_0 + V_{QCD}, \]  

(38)

where \( P^+_0 \) (the sum of the RHS of Eqs. 8 and 11) is the piece of the Hamiltonian not containing any factors of the coupling \( g \) and \( V_{QCD} \) is the rest, which can also be written out in terms of creation–annihilation operators. Furthermore, the ground state of the non–interacting Hamiltonian \( P^+_0 \) is also, remarkably, the ground state of the full Hamiltonian. This is the meaning behind statements one may have heard that the light cone vacuum is ‘trivial’. Because the vacuum is trivial, one may simply construct any eigenstate of the full Hamiltonian in terms of a complete Fock eigen–basis corresponding to eigenstates of the non–interacting Hamiltonian. As we shall demonstrate in the next lecture with a specific example, this point forms the basis for the quark–parton model in quantum field theory.

Just as in non–relativistic quantum mechanics then, one can use light cone time ordered perturbation theory to construct these states. Unfortunately, there is no room to discuss time ordered perturbation theory here but it will be discussed in the “long” version of these lectures [14].

There is one point we have not mentioned thus far but it threatens the entire pretty picture above. This has to do with the terms \( 1/P^+ \) and \( 1/(P^+)^2 \) above. Recall that they were obtained by inverting the light cone constraint equations in light cone gauge. Clearly, that operation and these terms are not well defined for \( P^+ = 0 \). The simple vacuum is thus only deceptively so and all the complications are now hidden in the zero–mode. That this would be the case should have been clearer in retrospect. Defining the operator \( 1/P^+ \) requires knowing the boundary conditions of the fields at large distances and therefore, should be sensitive to confining and chiral symmetry breaking effects. Attempts to regulate the zero mode, a well know example of which is discretized light cone quantization [13], also re-
sult in a non-trivial vacuum. On the other hand, perturbative physics should not be terribly sensitive to how fields are regulated at large distances. Different ‘epsilon’ prescriptions corresponding to different boundary conditions at infinity give the same short distance physics [16]. The justification of the above approach is therefore the success of the parton model in describing physics at large transverse momenta in QCD. The program to describe non-perturbative physics in the same framework is very advanced and we refer the reader to Ref. [13] to read of the latest developments.
Lecture 3: High energy Eikonal scattering and the parton model in QED.

In the previous lectures we developed some of the basic formalism of light cone field theory. We will now apply this formalism to a specific example; high energy scattering of an electron from an external potential in QED. We will show how one recovers the standard Eikonal picture in this formalism. More importantly, our results clearly can be interpreted in terms of a parton model picture of high energy scattering. This lecture closely follows the excellent paper of Bjorken, Kogut and Soper [11] where this example and others are discussed. For convenience, we will also use their “-2” convention (for eg., $A_-$ = $A^+$ and $A_t = -A^t$).

The light cone Hamiltonian in QED is similar to the QCD Hamiltonian derived above in Eq. 37 and of course much simpler. To treat the problem of scattering off an external potential, we introduce an external potential $a_\mu$ using the gauge invariant minimal substitution $p_\mu \rightarrow p_\mu - g a_\mu$. The QED Hamiltonian including the external potential $a_\mu$ is then

$$P_{\text{scatt}}(x^+) = \int d^2x_t dx^- \left\{ e a_+ \psi_+^\dagger \psi_+ + \frac{1}{2} e^2 \psi_+^\dagger \psi_+ \left( \frac{1}{(p^+ + e a^+)} \right)^2 \psi_+^\dagger \psi_+ + \psi_+^\dagger (M - i \vec{\sigma} \cdot (\vec{p} - e \vec{A} - e \vec{a})) \left( \frac{1}{2(p^+ + e a^+)} \right) (M + i \vec{\sigma} \cdot (\vec{p} - e \vec{A} - e \vec{a})) \psi_+ + e \psi_+^\dagger \psi_+ \left( \frac{1}{p^+ + e a^+} \right) \vec{p} \cdot \vec{A} + \frac{1}{2} \sum_{i=1,2} A_i \vec{p}^2 A_i^t \right\}. \tag{39}$$

Note that one can define

$$\left[ \frac{1}{p^+ + e a^+} \psi_+ \right] (x) = \int d\xi \frac{1}{2i} \epsilon(x^- - \xi) \exp \left( -ie \int_{\xi}^{x^-} d\xi' a_+^t (x^+, x_t, \xi') \right) \psi_+ (x^+, x_t, \xi), \tag{40}$$

where $\epsilon$ is the sign function. This can be checked by multiplying through by $p^+ - ea^+$. \footnote{In QCD, the sole change is to replace the exponential on the RHS by a path ordered exponential.}

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Now write $P_{\text{scatt}}^- = P_{QED}^- + V$, where $P_{QED}$ is the usual time independent QED Hamiltonian with $a^\mu = 0$ and $V(x^+) = P_{\text{scatt}}^- - P_{QED}^-$. We wish to construct the scattering matrix $S_{fi}$ in the interaction picture. In the usual quantum mechanical treatment,

$$
\psi_I(x^+, x_t, x^-) = e^{iP_{QED}^- x^+} \psi_+(0, x_t, x^-) e^{-iP_{QED}^- x^+}.
$$

$$
\bar{A}_I(x^+, x_t, x^-) = e^{iP_{QED}^- x^+} \bar{A}(0, x_t, x^-) e^{-iP_{QED}^- x^+}.
$$

Then, the scattering matrix for the scattering of an electron off an external potential is given by

$$
S_{fi} = \langle f | T \left\{ \exp \left( -i \int dx^+ V(x^+) \right) \right\} | i \rangle,
$$

where ‘T’ denotes light cone time ordering and $| i \rangle$ and $| f \rangle$ are asymptotic states which are eigenstates of the QED Hamiltonian $P_{QED}$. They can thus be evaluated in Rayleigh–Schrödinger perturbation theory (see the discussion at the end of lecture 2).

We want to compute the scattering matrix in the high energy scattering limit $P_i, P_f \to \infty$. Consider the states $| I \rangle$ and $| F \rangle$, which may be states in the rest frame of the electron. They are related by boosts to the states $| i \rangle$ and $| f \rangle$ above. Then $| i \rangle = e^{-i\omega K_3} | I \rangle$ and $| f \rangle = e^{-i\omega K_3} | F \rangle$, where $K_3$ (defined previously in lecture 1) is the generator of boosts in the longitudinal direction. In QED, $K_3$ is the operator

$$
K_3 = \int d^2 x_t \int dx^- \int x^+ \left[ \frac{i}{2} \bar{\psi}^+_+ \partial_- \psi_+ + (\partial_- A_t) (\partial_+ A_t) \right]_{x^+ = 0},
$$

and $\omega$ is the rapidity corresponding to the boost. The scattering matrix element between the scattering states in the rest frame is then

$$
\langle F | e^{i\omega K_3} T \left\{ \exp \left( -i \int dx^+ V(x^+) \right) \right\} e^{-i\omega K_3} | I \rangle.
$$
Using the definition of path ordered exponentials, this relation can be written as
\[ <F | T \left\{ \exp \left( -i \int dx^+ e^{i\omega K_3} V(x^+) e^{-i\omega K_3} \right) \right\} | I > . \] (44)

A great advantage of the light cone formalism is that the fields transform simply under boosts. We have
\[ e^{i\omega K_3} \psi_I(x^+, t, x^-) e^{-i\omega K_3} = e^{\omega/2} \psi_I(e^{-\omega} x^+, t, e^{\omega} x^-). \]
\[ e^{i\omega K_3} A_I(x^+, t, x^-) e^{-i\omega K_3} = A_I(e^{-\omega} x^+, t, e^{\omega} x^-). \]

The above can be shown explicitly by computing the commutators \([K_3, \psi_I]\) and \([K_3, A_I]\) using the definition of \(K_3\) in Eq. [43]. The field \(a\) however commutes with \(K_3\) and therefore does not transform under boosts.

Consider now the argument of the exponential in Eq. [44]. We can show that all but one of the terms in \(V(x^+)\) are invariant under the boost operation. For example,
\[ e^{i\omega K_3} \psi_I^\dagger \psi_I \frac{1}{(P^+)^2} \psi_I^\dagger \psi_I e^{-i\omega K_3} \rightarrow \psi_I^\dagger \psi_I \frac{1}{(P^+)^2} \psi_I^\dagger \psi_I. \]

Above, we have used the fact that elements of the Lorentz group are simply rescaled by boosts, \(e^{i\omega K_3} P^+ e^{-i\omega K_3} = e^{\omega} P^+\), as well as Eq. [43]. The only term that does not remain invariant is
\[ e a_+ \psi_I^\dagger \psi_I \rightarrow e^{-\omega} e a_+ \psi_I^\dagger \psi_I. \]

Hence,
\[ e^{i\omega K_3} V(x^+) e^{-i\omega K_3} = \int d^2 x_t \; dx^- \; e^{i\omega} e a_+ (x^+, x_t, x^-) \psi_I^\dagger (e^{-\omega} x^+, x_t, e^{\omega} x^-) \psi_I (e^{\omega} x^+, x_t, e^{-\omega} x^-) + O(e^{-\omega}). \]

Now let \(x^- \rightarrow e^{\omega} x^-\) above. Then
\[ e^{i\omega K_3} V(x^+) e^{-i\omega K_3} = \int d^2 x_t \; dx^- \; e a_+ (x^+, x_t, e^{-\omega} x^-) \psi_I^\dagger (e^{-\omega} x^+, x_t, x^-) \psi_I (e^{\omega} x^+, x_t, x^-) + O(e^{-\omega}). \] (45)
Going to the infinite rapidity limit $\omega \to \infty$ corresponding to very high energy scattering, we note from the above that the operators are all evaluated at $x^+ = 0$ so the time ordering in $x^+$ is irrelevant in that limit. Then one can show in that limit (and this is a subtle point) that

$$S_{fi} = \langle F | \mathcal{P} | I \rangle + O(e^{-\omega}) \equiv \langle f | \mathcal{P} | i \rangle + O(e^{-\omega}).$$  \tag{46}

Thus we have expressed $S_{fi}$ again in terms of the states $|i\rangle, |f\rangle$, thereby demonstrating the Lorentz invariance of these states in the infinite momentum limit. Also, above

$$\mathcal{P} = \exp \left( -i \int d^2x_t \chi(x_t) \rho(x_t) \right),$$  \tag{47}

where

$$\chi(x_t) = e \int dx^+ a_\pm(x^+, x_t, 0),$$  \tag{48}

and

$$\rho(x_t) = \int dx^- \psi^\dagger_I(0, x_t, x^-) \psi_I(0, x_t, x^-).$$  \tag{49}

We have therefore recovered the well known eikonal scattering limit in QED.

We shall now show that the above derivation has a deep connection with the parton model. The asymptotic ‘in’ state of the electron, $|i\rangle$, is an eigenstate of the QED Hamiltonian $P_{QED}^-$. We can expand $|i\rangle$ in terms of the “bare” quanta associated with the fields $\psi_+(0, x_t, x^-)$ and $A_\mu(0, x_t, x^-)$ at $x^+ = 0$:

$$|i\rangle = \int d^2k_1 \int_{k^+_1 > 0} \frac{dk^+_1}{k^+_1} \sum_\lambda \left\{ g(k_t, k^+_1, \lambda) a^\dagger(k_t, k^+_1, \lambda) |0\rangle + \int d^2k_{t2} \frac{dk^+_2}{k^+_2} \sum_{s_1, s_2} h(k_{t1}, k_{t2}; s_1, s_2) b^\dagger(k_{t1}; s_1) d^\dagger(k_{t2}; s_2) |0\rangle + \cdots \right\} + \cdots$$  \tag{50}

$^4$These are eigenstates (in QED!) of $P_0^-$ in Eq. 68
The creation and annihilation operators introduced here are the same as those in lecture 1. The coefficient $h$ above can be interpreted simply as the amplitude for $|i\rangle$ to contain a bare electron with momenta $\vec{k}_1$ and spin $s_1$, and a bare positron with momenta $\vec{k}_2$ and spin $s_2$. It was shown first by Drell, Levy and Yan that the amplitude squared for an arbitrary number of parton eigenstates, integrated over phase space could be simply related to the structure functions $W_1, W_2$.\[17\]

We can also see this here if we similarly expand $|f\rangle$ in terms of the bare quanta. The scattering matrix $S_{fi}$ in Eq. 46 can then be evaluated if we move $P$ past the creation–annihilation operators till it acts on $|0\rangle$:

$$P b^\dagger d^\dagger a^\dagger \cdots a^\dagger |0\rangle = P b^\dagger P^{-1} \cdots P a^\dagger P^{-1} P |0\rangle . \quad (51)$$

Since it is evident that $P$ is invariant under translations in the $x^-$ direction, it commutes with the generator of $x^-$ translations–$P^+$. One can check that $P |0\rangle = |0\rangle$. This follows formally by expanding $P$ and requiring that the operators in $\rho(x_t)$ are normal ordered. How do the creation–annihilation operators transform with $P$?

Using the light cone commutation relations, \{\psi_+(x), \psi_+^\dagger(x')\} = \delta^{(3)}(x - x')$, we find

$$P \psi_+^\dagger(0, x_t, x^-) P^{-1} = \exp (-i\chi) \psi_+^\dagger(0, x_t, x^-). \quad (52)$$

Fourier transforming the above, and using Eq. 48, we obtain for the electron creation operator

$$P b^\dagger(k_t, k^+, s) P^{-1} = \int \frac{d^2k'_t}{(2\pi)^2} b^\dagger(k'_t, k^+, s) \tilde{P}(k'_t - k_t) , \quad (53)$$

where (with $q_t = k'_t - k_t$)

$$\tilde{P}(q_t) = \int d^2x_t e^{-i\vec{q}_t \cdot \vec{x}_t} e^{-i\chi(x_t)} . \quad (54)$$

Similarly for the positron creation operator

$$P d^\dagger(k_t, k^+, s) P^{-1} = \int \frac{d^2k'_t}{(2\pi)^2} d^\dagger(k'_t, k^+, s) \tilde{P}_c(k'_t - k_t) , \quad (55)$$
with

\[ \hat{P}_c(q_t) = \int d^2 x_t e^{-i\vec{q}_t \cdot \vec{x}_t} e^{+i\chi(x_t)}. \] (56)

Finally, \( \mathcal{P} a^\dagger \mathcal{P}^{-1} \), since all the operators in \( \mathcal{P} \) commute with \( a^\dagger \).

What we have learnt from the above is that when a high energy bare electron or bare positron interacts with a potential at \( x_t \), the net effect is to multiply its wavefunction by the eikonal phase \( e^{-i\chi} \) or \( e^{+i\chi} \) respectively. The following physical picture then emerges from our manipulations above.

- The scattering of high energy particles (denoted here by ‘\( |i> \)’, which is an eigenstate of the Hamiltonian \( P_{QED}^- \)) is not simple–i.e., it cannot be described by a simple overall phase.

- However, due to the “potential” structure of QED on the light cone, the physical particle states (\( |i> \)) can be expanded in a complete basis of multi–parton eigenstates (eigenstates of \( P_{0,QED}^- \)).

- The scattering of these partons is simple–they acquire an eikonal phase in the scattering.

- The mutual interactions of partons in the physical state \( |i> \) is complex, but as the rapidity \( \omega \to \infty \), these interactions are slowed down by time dilation. Recall that in Eq. 45, the only term that survives is the one that contains the coupling to the external field \( a \) and all the other terms which contain the interactions of the partons with each other are suppressed.

Chronologically, one can view the scattering as follows. Partons in the initial state interact strongly for \( -\infty < x^+ < 0 \) with the potential \( V_{QED} \). At \( x^+ = 0 \), each individual parton scatters simply off the external potential, acquiring an eikonal phase. For \( 0 < x^+ < \infty \), the partons then again interact among each other with the potential \( V_{QED} \). This picture of scattering is also known as the impulse
approximation. It explains the striking phenomenon of Bjorken scaling observed in deep inelastic scattering at very large momentum transfers.

Finally, for completeness, we will mention that the cross-section for electron scattering off an external potential is given by

\[
\frac{d\sigma}{dk^0} = \int_{k^0_1, \ldots, k^0_n > 0} \frac{d^2 k_{11} dk_{12}^+ \cdots d^2 k_{nn} dk_n^+}{(2\pi)^3 k_1^+} \frac{d^2 k_{11} dk_{12}^+ \cdots d^2 k_{nn} dk_n^+}{(2\pi)^3 k_n^+}(2\pi)\delta(k^+ - \sum_{i=1}^{n} k_i^+) \times |<f|\mathcal{T}|i>|^2, \tag{57}
\]

where the transition amplitude is defined as

\[
|<f|\mathcal{T}|i>| = <f|U(\infty, 0)[\mathcal{P}-1]U(0, -\infty)|i>. \tag{58}
\]

Above, \(U\) is the light cone analog of the usual unitary evolution operator in quantum mechanics.
Lecture 4: Bjorken scaling and light cone Fock distributions.

In this last lecture, we will discuss the “light cone” limit $x^2 \to 0$ of deep inelastic scattering, in QCD. For very large momentum transfers, in this limit, one observes the phenomenon known as Bjorken scaling. Unfortunately, we will not have room for a discussion of the renormalization group ideas which predict, in QCD, the experimentally observed logarithmic violations of Bjorken scaling. These will be presented in the “longer” version of these lectures at a later date [14].

In deep inelastic scattering of an incident lepton off a hadron or nucleus, the kinematic invariants are the square of the momentum carried by the “space–like” virtual photon $q^2 = -Q^2 < 0$, (note: we use the ‘-2’ convention here) and $x_{Bj} = \frac{Q^2}{2P \cdot q}$, where $P^\mu$ is the four–momentum of the target. The cross–section expressed in terms of these invariants is a product of the point particle Rutherford cross section times a form factor, the electromagnetic form factor of the hadron $F_2$. In general, $F_2 \equiv F_2(x_{Bj}, Q^2)$, but in QCD, as $Q^2 \to \infty$, $F_2(x_{Bj}, Q^2) \to F_2(x_{Bj})$. The scaling of the structure function as a function of $x_{Bj}$ is what is known as Bjorken scaling. In this lecture, we will derive Bjorken scaling using the free field commutation relations.

The cross section for the inclusive deep inelastic scattering process $l(k) + (h,A)(P) \to l(k') + X$, where $X$ denotes undetected final states, is a tensor product of the leptonic tensor $l_{\mu\nu}$ and the hadronic tensor $W_{\mu\nu}$. The hadronic tensor is defined as [19]

$$W_{\mu\nu}(q^2, P \cdot q) = \sum_n (2\pi)^4 \delta^{(4)}(q + P - p_n) < P|J_\mu(0)|n> < n|J_\nu(0)|P >$$

$$\to \int d^4x e^{iq \cdot x} < P|J_\mu(x)J_\nu(0)|P > .$$

(59)

The sum above is over all hadronic final states with momenta $p_n$. Since $q^0 + P^0$
and $p_n^0$ are $+$ve, we can write the above as
\[ W_{\mu\nu} = \int d^4 x \, e^{i q \cdot x} < P[J_\mu(x), J_\nu(0)]|P > . \] (60)

Since the commutator vanishes outside the forward light cone, we will write the above as
\[ W_{\mu\nu} = \int d x^- e^{i q^+ x^-} \int d x^+ e^{i q^- x^+} \int_{x_t^2 < 2 x^+ x^-} d^2 x_t < P[J_\mu(x), J_\nu(0)]|P > . \] (61)

Above, $J_\mu^a = \bar{\psi} \gamma_\mu \lambda^a \psi(x)$.

In the high energy limit $q^+ \to \infty$, $q^- = $ fixed, the largest contribution to $W_{\mu\nu}$ comes from the region of the integral with the smallest oscillations, or $x^+$ finite, $x^- \to 0$. Since causality demands that $x^2 = 2 x^+ x^- - x_t^2 < 2 x^+ x^-$, the largest contribution to $W_{\mu\nu}$ is from the region of the light cone $x^2 \to 0$. In other words, the structure function is dominated by the light cone singularities of the commutator of currents. The limit $q^+ \to \infty$ and $q^- = $ fixed, corresponds to the limit $\nu = P \cdot q/M \to \infty$, $Q^2 \to \infty$ and $x_B j = Q^2 \to 2 P q$ fixed.

Let us examine the commutator in Eq. (61) in the limit $x^2 \to 0$. Here, using the “free field” current commutation relation which is reasonable in the weak coupling limit,
\[ \{\bar{\psi}(x), \psi(-x)\} = \frac{1}{8\pi} \gamma_\mu \partial_\mu \epsilon(x^0) \delta(x^2) + O(M^2 x^2), \] (62)
we obtain
\[ [J_\mu(x), J_\nu(-x)] \approx [\bar{\psi}(x) \gamma_\mu \gamma_\alpha \gamma_\nu \psi(-x) - \bar{\psi}(-x) \gamma_\nu \gamma_\alpha \gamma_\mu \psi(x)] \frac{1}{8\pi} \partial^0 \epsilon(x^0) \delta(x^2). \] (63)

We now use the identity
\[ \gamma_\mu \gamma_\alpha \gamma_\nu = S_{\mu\nu\alpha\beta} \gamma^\beta + i \epsilon_{\mu\nu\alpha\beta} \gamma^\beta \gamma^5, \] (64)
where
\[ S_{\mu\nu\alpha\beta} = (g_{\mu\alpha} g_{\nu\beta} + g_{\nu\alpha} g_{\mu\beta} - g_{\mu\nu} g_{\alpha\beta}), \] (65)
and $\epsilon_{\mu\nu\alpha\beta}$ is the anti-symmetric Levi-Civita tensor in four dimensions.

Substituting this identity in the current commutator, we obtain

$$[J_\mu(x), J_\nu(-x)] \xrightarrow{x^2 \to 0} \left[ \psi(x) S_{\mu\nu\alpha\beta} \gamma^\beta \psi(-x) + i \epsilon_{\mu\nu\alpha\beta} \bar{\psi}(x) \gamma^\beta \gamma^5 \psi(-x) \right. \left. - \bar{\psi}(-x) S_{\nu\mu\alpha\beta} \gamma^\beta \psi(x) - i \epsilon_{\nu\mu\alpha\beta} \bar{\psi}(-x) \gamma^\beta \gamma^5 \psi(x) \right] \frac{1}{8\pi} \partial^\alpha \epsilon(x^0) \delta(x^2). \tag{66}$$

We now perform a Taylor expansion on $\psi$ and $\bar{\psi}$,

$$\bar{\psi}(x) \psi(-x) = \sum_n \frac{1}{n!} x^{\mu_1} \cdots x^{\mu_n} \bar{\psi}(0) \frac{\partial}{\partial \mu_1} \cdots \frac{\partial}{\partial \mu_n} \psi(0) \, \tag{67}$$

Putting this back into our expression for the commutator, we obtain

$$[J_\mu(x), J_\nu(-x)] = \sum_{n=1,3}^{\infty} \frac{1}{n!} x^{\mu_1} \cdots x^{\mu_n} \mathcal{O}^{(n+1)}_{\beta\mu_1,\cdots;\mu_n}(0) S_{\mu\nu\alpha\beta} \frac{1}{4\pi} \partial^\alpha \epsilon(x^0) \delta(x^2), \tag{68}$$

where

$$\mathcal{O}^{(n+1)}_{\beta\mu_1,\cdots;\mu_n}(0) = \bar{\psi}(0) \gamma^\beta \frac{\partial}{\partial \mu_1} \cdots \frac{\partial}{\partial \mu_n} \psi(0). \tag{69}$$

We may note the following points regarding the above result.

- Only the odd terms in the sum survive. The even terms cancel out.
- We have expanded the operators in the vicinity of the light cone in a series of local operators—each of which multiplies the same singular function.
- Only a particular combination of Lorentz indices appears. We are interested only in the parity conserving terms, which is why the terms multiplying the anti-symmetric tensor $\epsilon_{\mu\nu\alpha\beta}$ do not appear. In general however, there will be an additional piece proportional to $\epsilon_{\mu\nu\alpha\beta}$ which contributes to $W_{\mu\nu}$. The corresponding structure function often referred to as $F_3$ is measured by parity violating currents, as for example is the case in deep inelastic neutrino scattering.
• $O$ is a twist two operator. Twist is a term which refers to the ‘dimension’ - ‘spin’ of an operator. Our operator above has dimension $= 3/2 \times 2 + n$ and spin $= n + 1$. In general, the expansion of the operators on the light cone can be organized into an expansion over successively higher twists, called the operator product expansion (often known by its acronym OPE) the coefficients of higher twist operators being suppressed by powers of $x^2$. The dominant operators at short distances are those with the smallest twist. There are a finite number of twist two operators.

In general, the naive dimensions of the operators are modified by interactions and they acquire ‘anomalous dimensions’, which may be determined by a renormalization group analysis. We will not discuss the OPE any further, but refer the reader to some of the textbooks with excellent discussions of the topic [18, 19, 20].

We return from this digression to topic of immediate interest: the derivation of Bjorken scaling. Recall that we had

$$W_{\mu\nu} = \int d^4 y e^{i q \cdot y} < P| [J_\mu(y), J_\nu(-y)] | P > .$$

We now substitute Eq. 68 in the RHS of the above. The matrix element of the symmetric, traceless operator $O^{(n+1)}$ between the hadronic states, has the tensorial structure,

$$< P| O^{(n+1)}_{\beta\mu_1,\ldots,\mu_n}(0) | P > = A_{n+1} p_\beta p_{\mu_1} \cdots p_{\mu_n} + B_{n+1} \delta_{\mu_1\mu_2} p_\beta p_{\mu_3} \cdots p_{\mu_n} + \text{ less singular terms} .$$

(70)

The second term above gives an additional power of $x^2$ when contracted with the coefficients and is therefore suppressed. The leading contribution then is

$$W_{\mu\nu} = \int d^4 y e^{i q \cdot y} \sum_{n=1,3} \frac{(p \cdot y)^n}{n!} A_{n+1} S_{\mu\nu,\alpha\beta} \frac{1}{4\pi} p^\beta \partial^\alpha \epsilon(x^0) \delta(x^2) .$$

(71)

Define a function and its Fourier transform

$$\tilde{f}(p \cdot y) = \sum_{n=1,3} \frac{(p \cdot y)^n}{n!} A_{n+1} = \int \frac{dx}{2\pi} e^{i x y p} \frac{f(x)}{x} .$$

(72)
Substituting the above into $W_{\mu\nu}$ and using the identity
\[ \int d^4 y e^{iky} \delta(y^2) \epsilon(y^0) = (2\pi)^2 \epsilon(k^0) \delta(k^2), \] (73)
we obtain
\[ W_{\mu\nu} = \int \frac{dx}{2\pi} \frac{f(x)}{(p \cdot q)} p^\alpha (q + xP)^\alpha S_{\mu\nu\alpha\beta}(2\pi)^2 \epsilon(xP^0 + q^0) \frac{1}{4\pi} \delta((xP + q)^2). \] (74)
Using the definition of $S_{\mu\nu\alpha\beta}$ in Eq. 58 and performing the delta function integration which sets $x = x_{Bj} = -q^2/2P \cdot q$, we can write the above finally as
\[ W_{\mu\nu} = \frac{f(x)}{(p \cdot q)} \left( p^\mu - \frac{(p \cdot q)q_\mu}{q^2} \right) \left( p^\nu - \frac{(p \cdot q)q_\nu}{q^2} \right) - \frac{f(x)}{2x} \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right). \] (75)

The electromagnetic tensor $W_{\mu\nu}$ has the most general tensorial decomposition,
\[ W_{\mu\nu} = a_1 P_\mu P_\nu + a_2 P_\mu q_\nu + a_3 P_\nu q_\mu + a_4 q_\mu q_\nu + a_5 g_{\mu\nu}. \]
The symmetry properties require however that $a_2 = a_3$ and from current conservation $q^\mu W_{\mu\nu} = 0$, and similarly for $q^\nu W_{\mu\nu} = 0$, we obtain,
\[ W_{\mu\nu} = \frac{F_2}{(p \cdot q)} \left( p^\mu - \frac{(p \cdot q)q_\mu}{q^2} \right) \left( p^\nu - \frac{(p \cdot q)q_\nu}{q^2} \right) - F_1 \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right), \] (76)
where $F_1$ and $F_2$ are the structure functions. Comparing the above to our result Eq. 75, we observe that $F_2 = f(x_{Bj})$, which is the famous scaling phenomenon known as Bjorken scaling. Further, in this limit $F_1 = F_2/2x$–this result is known as the Callan–Gross relation.

We shall now show that the structure functions derived above can be simply related, in leading twist, to the light cone parton distributions and further show that $F_2$ thereby has the interpretation of being the probability that a quark has a fraction $x$ of the total hadron momentum $p^+$ on the light front.

Consider the forward Compton scattering amplitude for the virtual photon scattering of the hadron in deep inelastic scattering,
\[ T_{\mu\nu}(q^2, p \cdot q) = i \int d^4 z e^{iqz} < P | T(J_\mu(z)J_\nu(0)) | P > \equiv 2\text{Im} W_{\mu\nu}. \] (77)
This can be decomposed into longitudinal and transverse pieces

\[ T_{\mu
u} = \frac{P_{\mu} P_{\nu}}{M^2} t_\perp(x, q^2) - g_{\mu\nu} t_L(x, q^2), \]

just as for \( W_{\mu
u} \). Now, in the Bjorken limit, the Callan–Gross relation implies that the longitudinal piece above vanishes. To leading twist then, just as for the hadronic tensor, we can decompose the transverse component of the Compton amplitude as

\[ t^T_\perp = 2 \sum_{n=1}^{\infty} \int d^4 z e^{i q \cdot z} C^\beta_n(z^2) z^{\mu_1 \cdots \mu_n} < P | O_{\beta \mu_1 \cdots \mu_n} | P >, \]

where, making the analogy to Eq. 68, the coefficient functions \( C^\beta_n(z^2) \) are the same for all odd values of \( n \) and zero otherwise. Also, \( O \) is the operator defined in Eq. 69.\(^5\)

One can define

\[ \frac{2^n q^{\mu_1} \cdots q^{\mu_n}}{(-q^2)^{n+1}} \tilde{C}^\beta_n(q^2) = i \int d^4 z \exp(i q \cdot z) z^{\mu_1} \cdots z^{\mu_n} C^\beta_n(z^2). \]

Typically, the functions \( \tilde{C}^\beta_n(q^2) \) are different and are the coefficient functions in the operator product expansion. However, in the scaling limit, they are constants. Substituting the above identity into our expression for \( t^T_\perp \), we obtain

\[ \frac{p \cdot q}{M^2} t^T_\perp(x, q^2) = \frac{-2q^2}{p \cdot q} \sum_{n=1,3}^{\infty} \left( \frac{2q_\beta}{q^2} \right) \left( \frac{2q_{\mu_1}}{q^2} \right) \cdots \left( \frac{2q_{\mu_n}}{q^2} \right) < P | O^{\beta \mu_1 \cdots \mu_n} | P >. \]

Since \( O \) is traceless and symmetric, we can again use the tensorial decomposition in Eq. 70. Then, since \( x = -q^2 / 2p \cdot q \), we obtain

\[ \frac{p \cdot q}{M^2} t^T_\perp = 4x \sum_{n=1,3}^{\infty} \left( \frac{1}{x} \right)^{n+1} A_{n+1}. \]

We can determine \( A_{n+1} \) by setting all the Lorentz indices in Eq. 70 to +. Then,

\[ A_{n+1} = \left( \frac{1}{p^+} \right)^{n+1} < P | O^{++ \cdots +} | P >_C. \]

From the definition of the operator \( O \) in Eq. 69, the matrix element above is given, in light cone gauge \( A^+ = 0 \), by all two particle irreducible insertions of the vertex \( \bar{\psi}\gamma^+(h^+)\psi \) (see Ref. 21 and references therein).

\(^5\)In general, the partial derivatives in Eq. 69 should be replaced by covariant derivatives.
Let us now digress a little to discuss the light cone Fock space distribution. We will relate it subsequently to the structure functions above. Recall the decomposition we had in Eq. 6 of lecture 1 of the dynamical 2–spinor $\psi_+$. We can then define the light cone parton distribution function as

$$\frac{dN}{d^3k} = \frac{1}{(2\pi)^3} \sum_{\lambda} \left[ b^\dagger_\lambda b_\lambda + d^\dagger_\lambda d_\lambda \right].$$

(84)

Writing this in terms of $\psi_+$ and using the light cone identity

$$\text{Tr} \left[ \sum_{\lambda} \gamma^+ \bar{\psi}_\lambda(x) \psi(y) \right] = \sqrt{2} \text{Tr} \left[ \sum_{\lambda} \psi_{\lambda}(x) \psi_{\lambda}(y) \right],$$

(85)

we obtain

$$\frac{dN}{d^3k} = \frac{2}{(2\pi)^3} \int d^3x d^3y e^{-ik(x-y)} \text{Tr} \left[ \gamma^+ S(x,y) \right],$$

(86)

where $S(x,y) = -i < T(\psi(x) \bar{\psi}(y)) >$. The light cone distribution function integrated over all momenta is the function

$$H(\alpha) = \int \frac{d^4k}{(2\pi)^4} \delta(\alpha - \frac{k^+}{p^+}) \frac{1}{p^+} \text{Tr} \left[ \gamma^+ \tilde{S}(p,k) \right],$$

(87)

where $\tilde{S}(p,k)$ is the fermion Green’s function in momentum space. We will now show that the function $H(\alpha)$ is, in leading twist, the structure function $F_2$.

Returning now to Eq. 83, we find

$$A_{n+1} = \frac{1}{(p^+)^{n+1}} \int \frac{d^4k}{(2\pi)^4} (k^+)^n \text{Tr} \left[ \gamma^+ \tilde{S}(p,k) \right].$$

(88)

In terms of $H(\alpha)$ then,

$$A_{n+1} = \int_{-\infty}^{\infty} d\alpha \frac{\alpha^n H(\alpha)}{\alpha}. $$

(89)

From the analytic properties of the function $H(\alpha)$, specifically the anti–commutation properties of the operators $\bar{\psi} \gamma^+ \psi$ on the light cone [21], one may conclude that $H(\alpha) = 0$ for $|\alpha| > 1$. Substituting the expression for $A_{n+1}$ in the transverse Compton amplitude, we obtain,

$$\frac{p \cdot q}{M^2} t_{\perp}^{T=2} = 4 \int_{-1}^{1} d\alpha \sum_{n=1,3}^\infty \frac{(\alpha)}{x}^n H(\alpha).$$

(90)
Performing the sum over $n$ and analytically continuing $t_\perp$ to the physical region $x \to x - i\epsilon$, with $x$ real and $0 < x \leq 1$,

$$\frac{p \cdot q}{M^2} t_\perp = 2x \int_{-1}^{1} d\alpha H(\alpha) \left\{ \frac{1}{x - \alpha - i\epsilon} - \frac{1}{x + \alpha - i\epsilon} \right\}.$$ (91)

Taking the imaginary part of the amplitude to obtain the structure functions, we get

$$F_2(x) = x(H(x) - H(-x)).$$ (92)

From the definition of $H(x)$ in Eq. 87, it is the probability to find a quark with momentum $k^+ = xp^+$ in the target. The function $-H(-x)$ has the interpretation of finding an anti–quark with momentum $k^+ = xp^+$ in the target. We have therefore, with Eq. 92, obtained the usual parton model interpretation of structure functions. In general, for a large but finite $Q^2$, the above result can be slightly modified to read

$$F_2(x, Q^2) = \int_{0}^{Q^2} d^2k_t \frac{dN}{d^2k_t dx}.$$ (93)

This follows simply from putting an upper cut–off $Q^2$ on the $k_t$ integration in Eq. 87. Finally, we should mention that the multi–parton Fock distributions discussed in lecture 3 can be related by a similar analysis to the higher twist contributions to the forward Compton scattering amplitude \cite{21, 22}.

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