Global spectral representations of black hole spacetimes in the complex plane

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Binary black hole coalescence produces a finite burst of gravitational radiation which propagates towards quiescent infinity. These spacetimes are analytic about infinity and contain a dimensionless coupling constant $M/s$, where $M$ denotes the total mass-energy and $s$ an imaginary distance. This introduces globally convergent Fourier series on a complex radial coordinate, allowing spectral representation of black hole spacetimes in all three dimensions. We illustrate this representation theory on a Fourier-Legendre expansion of Boyer-Lindquist initial data and a scalar wave equation with signal recovery by Cauchy’s integral formula.

I. INTRODUCTION

Rapid progress in sensitivity in the broadband gravitational-wave observatories is creating new opportunities for searches for the bursts of gravitational-wave emissions produced by coalescing black holes. Detection strategies will benefit greatly from a priori understanding of their gravitational-wave emissions, which includes matched filtering techniques against a catalogue of signals precomputed by numerical relativity. This poses an interesting challenge of designing highly efficient and stable computational algorithms.

Numerical relativity is particularly challenging in calculating bursts and preceding chirps over many wave-periods in the presence of singularities associated with black hole spacetimes. Spectral methods provide an attractive approach. They are optimal in efficiency and accuracy, both in amplitude and phase, provided that the metric is smooth everywhere. Here, we focus on bursts of radiation produced by black hole coalescence which propagate towards quiescent infinity. By causality, these spacetimes preserve quiescence at infinity for all finite time.

The asymptotic structure of black hole spacetimes which are asymptotically flat and quiescent at infinity shares the same asymptotic properties as the Green’s function of Minkowski spacetime. Consequently, these spacetimes preserve quiescence at infinity and contain a dimensionless coupling constant $M/s$ which shows that $\text{spacetimes carry a dimensionless coupling constant } M/s$. This introduces globally convergent Fourier series on a complex radial coordinate, allowing spectral representation of black hole spacetimes in all three dimensions. We illustrate this representation theory on a Fourier-Legendre expansion of Boyer-Lindquist initial data and a scalar wave equation with signal recovery by Cauchy’s integral formula.

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FIG. 1: The Boyer-Lindquist scalar field $\Phi(z, \theta)$ possesses a Fourier-Legendre expansion in $w = 1/z, z = x + is$, (top two windows) and $\cos \theta$ (bottom two windows), here shown for a symmetric configuration consisting of two black holes of mass $M_1, M_2 = 1$ positioned at the north and south pole $p = \pm 1$ of a spherical coordinate system $(r, \theta, \phi)$. The Fourier coefficients show exponential decay by analyticity in $\text{Im}(z) \geq s > s^* = 0.9682$. This gives rise to spectral accuracy in the first and second coordinate derivatives (top right). The norm of the Legendre coefficients shows exponential decay when $s$ is larger than $p$ (left). This gives rise to spectral accuracy in the first and second derivatives with respect to $\theta$ (bottom right).

The application to black holes spacetimes in the complex plane can be considered in the line-element

$$ds^2 = -N^2 dt^2 + h_{ij} dx^i dx^j, \quad N(z) = \frac{1 - M/2z}{1 + M/2z}, \quad (9)$$

for a three-metric $h_{ij}$. Working in the complex plane, we are at liberty to choose a lapse function $N$ and vanishing shift functions in accord with the asymptotic Schwarzschild structure of spacetime at large distances. This applies without loss of generality to any binary black hole spacetime with total mass-energy $M$ as measured at infinity, defined by the residue

$$M = \text{Re} \left\{ \frac{i}{\pi} \int_{-\infty}^{\infty} \left( \frac{1}{h_{zz}^{1/4}} - 1 \right) dz \right\}. \quad (10)$$
For binary black hole coalescence, $M$ is the sum $M_1 + M_2$ of the masses of the two black hole, plus the energy in gravitational radiation and minus the binding energy in the system. Thus, $M$ is a constant. In contrast, $M$ is a non-increasing function of time on any finite computational domain with outgoing radiation boundary conditions. For large $z$, the line-element (9) satisfies

$$ds^2 = -N^2 dt^2 + \left(1 + \frac{M}{2z}\right)\left(dz^2 + d\Sigma^2\right) + \cdots$$

up to order $(M/z)^2$, where $d\Sigma^2 = z^2 d\theta^2 + z^2 \sin^2 \theta d\phi^2$ denotes the line-element of the coordinate sphere of radius $z$. Accordingly, the three-metric $h_{ij}$ is decomposed into diagonal and off-diagonal elements given by the matrix factorization

$$h_{ij} = \gamma \begin{pmatrix} A_1 & B_1 & B_2 \\ B_1 & A_2 & B_3 \\ B_2 & B_3 & A_3 \end{pmatrix} \gamma,$$  

(12)

where $\gamma$ is the diagonal matrix

$$\gamma_{11} = 1, \gamma_{22} = z, \gamma_{33} = z \sin \theta.$$  

(13)

In accord with the spectral representation of \ref{4} and the potential \ref{4}, the coefficients $(A_i, B_i)$ are expanded in spherical harmonics on $z = x + is$,

$$A_i = MS_{|m| \leq l} a_i^m(t) w^{l+1} Y_{ml}(\theta, \phi),$$

(14)

$$B_i = MS_{|m| \leq l} b_i^m(t) w^{l+1} Y_{ml}(\theta, \phi).$$

(15)

Together with the Fourier expansion on $w = M/z$, \ref{14} \ref{15} defines a spectral representation in all three coordinates. Fig. \ref{2} shows the results in case of the axisymmetric Boyer-Lindquist initial data, including the computational error behavior in the scalar Ricci tensor. Here, we use the general expansions \ref{5} to probe independently the dependencies on $N$ and $L$ (to go beyond $N = L + 1$ in any truncated series of \ref{14} \ref{15}).

Time-evolution of $h_{ij}$ on $z = x + is$ gives rise to burst of radiation which propagates towards quiescent infinity, where the observer's signal is defined on non-negative radii $z \geq 0$. This signal can be calculated by projection according to Cauchy's integral formula,

$$u(r, \theta, \phi, t) = \text{Re} \left\{ \frac{i}{\pi} \int_{-\infty+is}^{\infty+is} \frac{u(z, \theta, \phi, t)}{z - r} dz \right\},$$

(16)

where $u$ denotes a relevant metric component or tensor quantity. Time-evolution on $z = x + is$ is stable when constant translation $s$, since this preserves reality of the dispersion relation. This can be illustrated by the linear wave-equation for a scalar field $u(t, r)$ in spherical coordinates $(t, r, \theta, \phi)$ with time-symmetric initial data, given by

$$K : \begin{cases} u_{tt}(t, r) = r^{-2}[r^2 u_{rr}(t, r)], \\ u(0, r) = u_0(r), \quad u_t(0, r) = 0. \end{cases}$$

(17)

By analytic continuation, the equivalent initial value problem $K'$ with complex radial coordinate on $-\infty + is < z < \infty + is$ is

$$K' : \begin{cases} u_{tt}(t, z) = w^4 u_{ww} \quad (w = 1/z) \\ u(0, z) = u_0(z), \quad u_t(0, z) = 0. \end{cases}$$

(18)
FIG. 3: Signal recovery of a wave propagating to quiescent infinity is shown for a quantity $u(t, z)$ on $z = x + is$, here satisfying the scalar wave equation $u_{tt} = r^{-2}(r^2 u_{rr})$, with time-symmetric initial data at $t = 0$ $u(0, x) = e^{-2x^2} \cos(10x)$ and $u_t(0, x) = 0$. The simulations use the Fourier representation on the complex radius under $w = M/z$ and the wave-equation $u_{tt} = w^4 u_{ww}$ using $N = 1024$ points. Plotted is the quantity $ru$ on $r \geq 0$ for $t_0 = 0$ and $t_1 = 4$. The numerical results (dots) accurately track the exact results (continuous line, by the method of characteristics). The errors are essentially independent of $s$, i.e., 4.5e-4 for $s = 0.5$ and 4.7e-4 for $s = 1.0$, as implied by analyticity in $s > 0$.

(In general, the analytic continuation of the initial data is the extension of data on $(r, \theta, \phi)$ and $(r, \pi - \theta, \phi + \pi)$). We can implement $K'$ numerically using finite-differencing and leapfrog time-stepping. Fig. 3 shows a representative numerical result, which is verified to satisfy independence of $s$. The scalar wave-equation with time-symmetric initial data shows the appearance of both left and right movers, the first propagating towards $z = -\infty + is$ and the second propagating towards $z = \infty + is$. Only the right movers are projected onto the observer’s non-negative radii $z \geq 0$. The left movers are ultimately dissipated “unseen” in their propagation to $-\infty + is$, while maintaining a contribution to the total mass-energy $M$ in $\Omega$.

To conclude, the complex plane offers a unique opportunity for circumventing spacetime singularities and enabling spectral representations in all three dimensions, consisting of spherical harmonics in the angular coordinates and Fourier series in the complex radial coordinate. This approach creates the most efficient representation of the metric, promising a reduction in computational effort by orders of magnitude compared to standard finite differencing approaches. It will be of interest to take advantage of this approach in time-dependent calculations described by the fully nonlinear equations for general relativity in any of its hyperbolic forms (e.g., 7).

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