The 3 + 1 decomposition of conformal Yano–Killing tensors and ‘momentary’ charges for the spin-2 field

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Abstract

The ‘fully charged’ spin-2 field solution is presented. This is an analog of the Coulomb solution in electrodynamics and represents the ‘non-waving’ part of the spin-2 field theory. Basic facts and definitions of the spin-2 field and conformal Yano–Killing tensors are introduced. Application of those two objects provides a precise definition of quasi-local gravitational charge. Next, the 3 + 1 decomposition leads to the construction of the momentary gravitational charges on the initial surface, which is applicable for Schwarzschild-like spacetimes.

Keywords: Yano–Killing tensors, conformal transformations, Weyl tensor

1. Introduction

Purpose of this paper. The charges associated with the gravitational field play a significant role in the general theory of relativity. Our goal is to provide a definition of energy, momentum, and angular momentum of a gravitational field using analogies between linearized gravitation and classical Maxwell’s electrodynamics. Following [1] we reiterate how, using the geometrical objects characterizing a gravitational field (spin-2 field $W_{\alpha \beta \mu \nu}$ and conformal Yano–Killing (CYK) tensors), one can obtain the quasi-local charges in Minkowski spacetime.

Using a $3 + 1$ decomposition of the spacetime, we propose to define the momentary charges in terms of natural initial value tensors: the electric and the magnetic parts of the spin-2 field and the initial value data for the CYK tensor—the conformal Killing vector (CKV) field. This is applicable for the Schwarzschild spacetime, which possesses fewer CYK tensors than Minkowski (fewer ‘hidden’ symmetries). Moreover, this construction works for
any initial data that is conformally flat, particularly for any spherically symmetric three-
metric. Obviously, this leads to quasi-local quantities defined on a Cauchy surface.

**Remarks about the notation.** In this paper we assume that the metric $g$ has a positive
signature (-, +, +, +), and we are using units, where $c = G = 1$. Antisymmetrization of a
tensor is denoted with square brackets, and symmetrization with round brackets. A four-
dimensional covariant derivative (for a Levi–Civita connection) is denoted by $\nabla$ or with a
semicolon. A three-dimensional covariant derivative is, on the other hand, denoted by $\nabla_3$ or
with a vertical segment. Greek letters $\alpha, \beta, \ldots, \mu, \nu, \ldots$ are indices with values in the set
{0, 1, 2, 3}, whereas Latin letters $i, j, \ldots$ assume values in {1, 2, 3}.

### 1.1. Spin-2 field

We will begin with defining a spin-2 field, which can also be identified with a Weyl tensor in
linearized gravitation.

**Definition 1.** Tensor field $W_{\alpha\beta\mu\nu}$ is called a spin-2 field if and only if the following conditions
are fulfilled:

\[
\begin{aligned}
\text{algebraic:} & \\
W_{\alpha\beta\mu\nu} & = W_{\mu\alpha\beta\nu} = W_{[\alpha\beta]\mu\nu}, \\
W_{[\alpha\beta]\mu\nu} & = 0, \\
g_{\alpha\mu}W_{\rho\nu\sigma} & = 0,
\end{aligned}
\]

(1.2)

\[
\begin{aligned}
\text{differential:} & \\
V_{[\alpha}W_{\beta]\mu\nu} & = 0.
\end{aligned}
\]

(1.3)

$W_{\alpha\beta\mu\nu}$ is antisymmetric in the first and in the second pair of indices, so we can define two
dual tensors for the first and the second pair of indices, respectively:

\[
\begin{aligned}
W^*_{\rho\sigma\mu\nu} & := \frac{1}{2} \epsilon_{\rho\sigma\mu\nu} W_{\alpha\beta\mu\nu}, \\
W^*_{\rho\sigma\mu\nu} & := \frac{1}{2} W_{\rho\sigma\mu\nu} \epsilon^{\alpha\beta}.
\end{aligned}
\]

(1.4)

A contraction of a spin-2 field with a normed vector $n^\mu$ perpendicular to the foliation $\Sigma_t$
allows us to define two new tensors: the electric and the magnetic parts of the Weyl tensor.

The electric part:

\[
E_{\alpha\beta} := W_{\mu\alpha\beta\nu} n^\mu n^\nu, 
\]

(1.5)

and the magnetic part:

\[
H_{\alpha\beta} := W^*_{\mu\alpha\beta\nu} n^\mu n^\nu. 
\]

(1.6)

Both defined tensors are symmetric (because of the symmetry of the field $W$) and traceless.
Applying a contraction with $n^\mu$ to the electric or the magnetic part gives us zero (because of
the antisymmetry of $W$ in pairs of indices):

\[
E_{\mu\nu} n^\mu = H_{\mu\nu} n^\mu = 0. 
\]

(1.7)

The property (1.3) can be rewritten in an equivalent form [1]:

\[
\begin{aligned}
V_{[\alpha}W_{\beta]\mu\nu} & = 0 \iff V^{\alpha}W_{\alpha\beta\mu\nu} = 0 \iff \\
\iff V_{[\alpha}W^*_{\beta]\mu\nu} & = 0 \iff V^{\alpha}W^*_{\alpha\beta\mu\nu} = 0.
\end{aligned}
\]

(1.8)
1.2. Conformal Yano–Killing tensors

The previously introduced field \( W \) is a tensor field with four indices. If we would like to follow the analogy with the classical electrodynamics and the Maxwell tensor, we need to define a tensor field that has two indices. Additionally, we would be pleased if the integral of this new object didn’t depend on the choice of the two-dimensional integration surface. We will show that the contraction of a field \( W \) with a conformal Yano–Killing tensor has both mentioned properties.

**Definition 2.** The antisymmetric tensor \( Q_{\mu\nu} \) is a conformal Yano–Killing (CYK) tensor for a metric \( g_{\mu\nu} \) iff:

\[
Q_{\mu\nu}(Q, g) = 0,
\]

where \( Q_{\text{kin}}(Q, g) := Q_{\kappa\lambda,\sigma} + Q_{\sigma,\kappa\lambda} - \frac{2}{n-1}(g_{\kappa\lambda} Q^{\sigma,\nu} + g_{\sigma,\nu} Q_{\kappa\lambda})^\mu.\)

We are only considering a four-dimensional case \((n = 4)\); therefore:

\[
Q_{\text{kin}}(Q, g) := Q_{\kappa\lambda,\sigma} + Q_{\sigma,\kappa\lambda} - \frac{2}{3}(g_{\kappa\lambda} Q^{\sigma,\nu} + g_{\sigma,\nu} Q_{\kappa\lambda})^\mu.\]

2. Minkowski spacetime

**Introduction.** The flat Minkowski spacetime is the simplest possible background for linearized gravitation. The reasoning presented below will lead us to exact results, which are very important. It will also help us gain some insight that will be useful in considering less trivial metrics. According to the conditions imposed in the introduction, we will be looking for a natural object that can be integrated over two-dimensional surfaces and will fulfill Gauss’s Law.

**Quasi-local charges—definition.** Now we combine a spin-2 field and a CYK tensor to define gravitational charge in the Minkowski spacetime.

Let us assume that \( W_{\alpha\beta\mu\nu} \) is a spin-2 field and \( Q_{\mu\nu} \) is any antisymmetric tensor. The new tensor field \(^1\)

\[
F_{\mu\nu}(W, Q) := W_{\mu\nu\alpha\beta} Q^{\alpha\beta}
\]

obeys the following

**Lemma 1.** Divergence of a tensor \( F_{\mu\nu} \) takes the form:

\[
V_e F^{\mu\nu}(W, Q) = \frac{2}{3} W^{\mu\nu\alpha\beta} Q_{\alpha\beta}.\]

If \( Q_{\mu\nu} \) is a CYK tensor, then \( Q_{\alpha\beta\mu\nu} = 0 \), and lemma 1 gives \( V_e F^{\mu\nu}(W, Q) = 0 \).

Let \( V \) be a three-dimensional volume with a boundary \( \partial V \); therefore:

\[
\int_{\partial V} F^{\mu\nu}(W, Q) d\sigma_{\mu\nu} = \int_V V_e F^{\mu\nu}(W, Q) d\Sigma_{\mu} = 0.
\]

Using this last equality, we can claim that each CYK tensor \( Q_{\mu\nu} \) defines a charge connected with a spin-2 field, because the flux of a tensor \( F^{\mu\nu} \) through any two closed two-dimensional

\(^1\) Intentionally we are using the same symbol as for the Maxwell tensor.
surfaces $S_1$ and $S_2$ is equal if there exists a three-dimensional volume $V$ with a boundary $\partial V$ equal to the sum of the surfaces $S_1$ and $S_2$.

**CYK tensors in Minkowski spacetime.** The basis of the space of solutions for the equation $Q_{\alpha\beta\mu
u} = 0$ (i.e., the basis of CYK tensors) in Minkowski spacetime consists of the following 20 tensors (see [1]):

$$T_0 \wedge T_k, \quad T_0 \wedge D, \quad T_k \wedge D, \quad D \wedge \mathcal{L}_{0k} = \frac{1}{2} \eta(D, D) T_0 \wedge T_k,$$

where $D := x^\mu \partial_\mu$, $T_\mu := \partial_\mu$, $\mathcal{L}_{\mu
u} := x_\mu \partial_\nu - x_\nu \partial_\mu$. Each CYK tensor in Minkowski spacetime can be expressed as a linear combination (with constant coefficients) of these 20 tensors.

**Gravitational charges in a 3 + 1 decomposition.** We begin with the following simple observation.

**Lemma 2.** Each CYK tensor in Minkowski spacetime can be expressed in the following way:

$$Q = a(t) T_0 \wedge X + b(t) * (T_0 \wedge Y),$$

where $X, Y$ are (three-dimensional) conformal Killing fields; $a(t), b(t)$ are functions of time only.

We will prove this lemma by giving the proper decomposition of the basis tensors described by the formulae (2.3).

Let us introduce the basis of CKVs in a flat, three-dimensional space:

$$T_k := \frac{\partial}{\partial x^k}, \quad S := x^k \frac{\partial}{\partial x^k}, \quad \mathcal{R}_k := e_k \partial_j x_l \frac{\partial}{\partial x^j}, \quad \mathcal{K}_k := x_k S - \frac{1}{2} \nu^2 \frac{\partial}{\partial x^k}.$$  

(2.5)

The fields written above correspond² (respectively) to: translation (3), scaling (1), rotation (3), and proper conformal transformation (3). Now we are able to provide the decomposition of each tensor in basis (2.3) in a form given in lemma 2.

1° $T_0 \wedge T_k = T_0 \wedge T_k$
2° $T_k \wedge D = T_0 \wedge S$
3° $T_k \wedge D = -t (T_0 \wedge T_k) - * (T_0 \wedge \mathcal{R}_k)$
4° $D \wedge \mathcal{L}_{0k} = \frac{1}{2} \eta(D, D) T_0 \wedge T_k = -\frac{1}{2} t^2 T_0 \wedge T_k - t * (T_0 \wedge \mathcal{R}_k) + T_0 \wedge \mathcal{K}_k$
5° $* (T_0 \wedge T_k) = * (T_0 \wedge T_k)$
6° $* (T_0 \wedge D) = * (T_0 \wedge S)$
7° $* (T_k \wedge D) = -t * (T_0 \wedge T_k) + (T_0 \wedge \mathcal{R}_k)$

² The number of vectors in each class is represented by the number in the bracket.
We have obtained 20 tensors of CYK basis in a 3 + 1 decomposition. To calculate the charges, we have to contract each CYK tensor with a spin-2 field and then integrate the result over a two-dimensional surface. The integration is done for a fixed moment in time $t$ (formally we would have to write: for a fixed value of parameter $t$ which enumerates the leaves of the foliation $\Sigma_t$ in the 3 + 1 decomposition). The contraction of a spin-2 field with a CYK tensor (written in a way proposed in lemma 2) reduces to the contraction of the electric part with a conformal field (for tensors of the form $T_0 \wedge X$) or to the contraction of the magnetic part with a conformal field (for the tensors of the form $\ast(T_0 \wedge X)$).

We will write $E(X)$ and $H(X)$ to represent the charges obtained from a contraction of a conformal Killing field $X$ with the electric and the magnetic parts, respectively. $E_0(X)$ and $H_0(X)$ are the initial values of the charge; $E_t(X)$ and $H_t(X)$ are the charges in time $t$.

Four of the tensors mentioned in section 2 are not time dependent (i.e., tensors with numbers: 1, 2, 5, 6), which also means that the charges associated with those tensors are constant in time. Four other tensors include expressions multiplied by the first or second power of time. Now we can use the equation (2.2) and write equations of the evolution. Let’s contract the tensors with numbers 3, 4, 7, 8 from section 2 with the spin-2 field then rewrite the result in terms of the electric and the magnetic parts. Finally we use equation (2.2), writing that this contraction has to be zero, and now we can put all time-dependent components on one side of the equation. For the charges linear in time we have:

$$
\begin{align*}
&\left\{\begin{array}{l}
E_t(R_k) = t H(T_k) + E_0(R_k), \\
H_t(R_k) = -t E(T_k) + H_0(R_k),
\end{array}\right.
\end{align*}
$$

(2.14)

and for quadratic in time:

$$
\begin{align*}
&\left\{\begin{array}{l}
E_t(K_k) = \frac{1}{2} t^2 E(T_k) + t H(R_k) + E_0(K_k), \\
H_t(K_k) = \frac{1}{2} t^2 H(T_k) - t E(R_k) + H_0(K_k),
\end{array}\right.
\end{align*}
$$

(2.15)

We can classify the charges according to time dependence (the number in brackets enumerates the charges in each class):

- (8) charges constant in time: $E(S), H(S), E(T_k), H(T_k)$,
- (6) charges linear in time: $E(R_k), H(R_k)$,
- (6) charges quadratic in time: $E(K_k), H(K_k)$.

### 2.1. Partially charged solution—electric charge counterpart

In [1] we have proposed a ‘charged’ spin-2 field configuration which is a non-oscillating monodipole solution of equation (1.3) with singularity at $r = 0$, and it possesses a global ‘potential’ (linearized metric tensor). This is a spin-2 field analog of the Coulomb solution in electrodynamics with electric charge (only). Let us denote by $p, k, s$ the dipole functions on a two-dimensional sphere that correspond to constant three-vectors in Cartesian coordinates $(x^k)$. More precisely, vector $p^i$ corresponds to $p = (p^i x_i)/r$, and, analogously, $s = (s^i x_i)/r$, and
\( k = (k^i x_i)/r \). The `charged’ solution in spherical coordinates \( y^0 = t, y^A = (\theta, \varphi) \), \( y^3 = r \), \( A = 1, 2 \) takes the following form:

\[
W_{BCDA} = -\frac{3}{r^2} \epsilon_{BC} \left( \frac{s_A}{r} - \epsilon_A^D \mathbf{p}_D \right),
\]

(2.16)

\[
W_{AB03} = \frac{6}{r^4} \epsilon_{AB} \mathbf{s},
\]

(2.17)

\[
W_{3A00} = \frac{3}{r^2} \left( \frac{\epsilon_A^D \mathbf{s}_D}{r} + \mathbf{p} \right),
\]

(2.18)

\[
W_{3AB0} = \frac{3}{r^4} \epsilon_{AB} \mathbf{s},
\]

(2.19)

\[
W_{5030} = -\frac{2}{r^3} \left( m + \frac{3k}{r} \right),
\]

(2.20)

\[
W_{0A03} = \frac{3}{r^3} \mathbf{k}_A,
\]

(2.21)

\[
W_{4B3C} = \frac{2}{r^4} \left( m + \frac{3k}{r} \right)(\eta_{AC} \eta_{BD} - \eta_{AD} \eta_{BC}),
\]

(2.22)

\[
W_{3AB3} = -W_{0AB0} = \frac{\eta_{AB}}{r^3} \left( m + \frac{3k}{r} \right),
\]

(2.23)

\[
W_{BC3A} = -\frac{3}{r^3} \epsilon_{BC} \epsilon_A^D \mathbf{k}_D,
\]

(2.24)

where indices \( A, B, C, \ldots \) correspond to angular coordinates on \( S^2 \), and \( x^3 = r \) is the radial coordinate.

According to [1], the spin-2 field solution (2.16)–(2.24) results from the linearized metric:

\[
h_{00} = \frac{2m}{r} + \frac{2k}{r^2},
\]

(2.25)

\[
h_{0A} = -6p_A - \frac{2}{r} \epsilon_A^B \mathbf{s}_B,
\]

(2.26)

\[
h_{03} = \frac{6}{r} \mathbf{p},
\]

(2.27)

\[
h_{33} = \frac{2m}{r} + \frac{6k}{r^2},
\]

(2.28)

which in Cartesian coordinates \((x^i)\) takes the following form:

\[
h_{00} = \frac{2m}{r} + \frac{2k m a^m}{r^2},
\]

(2.29)

\[
h_{0k} = \frac{6p_k}{r} - \frac{2}{r^3} \epsilon_{klm} \epsilon^l \mathbf{x}^m,
\]

(2.30)
\[ h^{kl} = \frac{x^k x^l}{r^2} \left( \frac{2m}{r} + \frac{6k_m x^m}{r^3} \right) \]  

(2.31)

The above symmetric tensor \( h_{\mu\nu} \) is the solution of linearized Einstein equations with the energy-momentum tensor localized on a timelike curve \( r = 0 \) (as a distribution with the support in the center):

\[
T^{00} = m\delta - k^m\delta_{,m}, \quad T^{0k} = \rho^k \delta + \frac{1}{2} \epsilon^{kmn} s_{,m}, \quad T^{kl} = 0,
\]

(2.32)

where \( \delta \) denotes a three-dimensional Dirac delta ‘function’, and \( \epsilon^{kmn} \) is a skew-symmetric Levi–Civita tensor (\( \epsilon^{123} = 1 \)).

2.2. Fully charged solution—magnetic monopole counterpart

One can obtain a generalization of (2.16)–(2.24), corresponding in electrodynamics to a magnetic monopole, by introducing additional charges \( q = (q^k x_k)/r, \ w = (w^k x_k)/r, \ d = (d^k x_k)/r \). This is a ‘fully charged’ spin-2 field solution, which is an analog of an electromagnetic monopole—a Coulomb solution with electric and magnetic charges. The quantities \( q, w, \) and \( d \) are obstructions for the existence of global linearized metric \( h \).

The fully charged spin-2 field solution in spherical coordinates takes the following form [1]:

\[
W_{BC0A} = \epsilon_{BC} \left( \frac{3}{2r} q_A + \frac{3}{r^2} \epsilon_A D_p D_q - \frac{3}{r^3} s_A \right),
\]

(2.33)

\[
W_{AB03} = \epsilon_{AB} \left( \frac{3}{r^2} q_A + \frac{2b}{r^3} + \frac{6s}{r^4} \right),
\]

(2.34)

\[
W_{A30} = -\frac{3}{2r} \epsilon_A D_q + \frac{3}{r^3} p_A + \frac{3}{r^3} \epsilon_A D_s D_D,
\]

(2.35)

\[
W_{AB00} = \epsilon_{AB} \left( \frac{3}{2r^2} q_A + \frac{b}{r^3} + \frac{3s}{r^4} \right),
\]

(2.36)

\[
W_{0303} = \frac{3}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4},
\]

(2.37)

\[
W_{A003} = \frac{3}{2r} w_A - \frac{3}{r^3} k_A - \frac{3}{r^2} \epsilon_A d_C,
\]

(2.38)

\[
W_{ABCD} = \left( \frac{3}{2r} w + \frac{2m}{r^3} + \frac{6k}{r^4} \right) \epsilon_{AB} \epsilon_{CD},
\]

(2.39)

\[
W_{AAB3} = -W_{0AB0} = \eta_{AB} \left( \frac{3}{2r} w + \frac{m}{r^3} + \frac{3k}{r^4} \right),
\]

(2.40)

\[
W_{ABC} = \epsilon_{BC} \left( \frac{3}{2r} \epsilon_A D_w D_q - \frac{3}{r^2} \epsilon_A D_d D_D - \frac{3}{r^3} \epsilon_A D_k \right),
\]

(2.41)

In the appendix we give more information about the Cartesian form of the formulae (2.33)–(2.41). Charges \( q, w \) correspond to metric tensors, which are not vanishing at spatial infinity (\( h_{\mu\nu} = O(1) \)). One can show that \( b \) is contained in the metric.
\[ h_{\theta \phi} = 4b \cos \theta. \] (2.42)

Similarly, charge \( d \) with direction along the \( z \)-axis (\( d = d \cos \theta \)) corresponds to the singular metric

\[ h_{\theta \theta} = 2rd \sin \theta \cos \theta \] (2.43)

or

\[ h_{r \theta} = 2d \left( \sin^2 \theta \log \tan \frac{\theta}{2} - \cos \theta \right). \] (2.44)

Definition

\[ E_{ij} := W_{ijkl} \epsilon^{ij}. \] (2.45)

\[ H_{ij} := \frac{1}{2} W_{ijkl} \epsilon^{ij}. \] (2.46)

of the electro-magnetic part of a spin-2 field applied to the fully charged solution gives:

\[ E_{ij} = E \left( n_i \partial_i + y^A \partial_A, n_j \partial_j + y^B \partial_B \right) \]
\[ = -\eta_{ij} \left( \frac{3w}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \right) - \frac{3}{r^2} \eta^{kl} d_{ij} \left( \epsilon_{i'k} n_i + \epsilon_{i'k} n_j \right) \]
\[ + \frac{3}{2r^2} \left( n_i \omega_j + n_j \omega_i \right) - \frac{3}{r^3} \left( n_i k_j + n_j k_i \right) \]
\[ - n_i n_j \left( -\frac{3w}{2r^2} - \frac{3m}{r^3} - \frac{15k}{r^4} \right). \] (2.47)

where \( \eta_{ij} = \delta_{ij} \) is the Euclidean metric and \( n^k := x^k/r \), or, in equivalent form:

\[ E_{ij} = -m \left( \eta_{ij} - 3n_i n_j \right) - \frac{3}{r^2} \eta + 3n_i n_j \left( \epsilon_{i'k} n_i + \epsilon_{i'k} n_j \right) \]
\[ - k \frac{3}{r^4} \left( n_i \eta_{ij} + n_i \eta_{ij} + n_j \eta_{il} - 5n_i n_j n_i \right) \]
\[ - w \frac{3}{2r^2} \left( n_i \eta_i - n_i \eta_j - n_j \eta_i - n_i n_j \eta_i \right). \] (2.48)

and one more possibility in terms of derivatives of \( 1/r \):

\[ E_{ij} = m \left[ \left( \frac{1}{r} \right)_{ij} - \tilde{k} \cdot \tilde{V} \left( \frac{1}{r} \right)_{ij} - \left( \tilde{d} \times \tilde{V} \right)_{ij} + \left( \tilde{d} \times \tilde{V} \right)_{ij} \right] \]
\[ - w \frac{3}{2r^2} \left( n_i \eta_i - n_i \eta_j - n_j \eta_i - n_i n_j \eta_i \right). \] (2.49)

and, analogously,

\[ H_{ij} = H \left( n_i \partial_i + y^A \partial_A, n_j \partial_j + y^B \partial_B \right) \]
\[ = -\eta_{ij} \left( \frac{3q}{r^2} + \frac{2b}{r^3} + \frac{6s}{r^4} \right) + \frac{3}{r^2} \eta^{kl} p, \left( \epsilon_{i'k} n_i + \epsilon_{i'k} n_j \right) \]
\[ + \frac{3}{2r^2} \left( n_i q_j + n_j q_i \right) - \frac{3}{r^2} \left( n_i s_j + n_j s_i \right) + n_i n_j \left( \frac{3q}{2r^3} + \frac{3b}{r^3} + \frac{15s}{r^4} \right). \] (2.50)
\[
H_{ij} = -\frac{b}{r^3} \left( \eta_{ij} - 3n_in_j \right) + p^j \frac{3}{r^3} \left[ u^k \left( \epsilon_{ijk}n_i + \epsilon_{ijl}n_j \right) \right] \\
- s^i \frac{3}{r^4} \left( n_i\eta_{ij} + n_j\eta_{ij} - 5n_in_j \right) \\
- q^j \frac{3}{2r^2} \left( \eta_{ij}n_i - n_i\eta_{ij} - n_j\eta_{ij} - n_i n_j \right),
\]
(2.51)

and finally in terms of the derivatives of \(1/r\):

\[
H_{ij} = \left( \left( \frac{1}{r} - \hat{\sigma} \cdot \vec{\nu} \right) \vec{V}_i \vec{V}_j + \left( \vec{p} \times \vec{\nu} \right)_j \vec{V}_i + \left( \vec{p} \times \vec{\nu} \right)_i \vec{V}_j \right) \frac{1}{r} \\
- q^j \frac{3}{2r^2} \left( \eta_{ij}n_i - n_i\eta_{ij} - n_j\eta_{ij} - n_i n_j \right),
\]
(2.52)

Let us observe that by exchanging \(\vec{w}, \vec{k}, -\vec{d}, \vec{m}\) in the electric part \(E\), we get the magnetic part \(H\). \(E(\vec{w}, \vec{k}, -\vec{d}, \vec{m}) = H\) represents a spin-2 field version of electromagnetic symmetry between the electric and magnetic monopoles.

Finally, we can check the values of quasi-local charges for the ‘fully charged’ solution:

\[
E(S) = Q(E, S) = \int_{S(r)} E_{ij} S^{ij} n^dS = 8\pi m, \quad (2.53)
\]
\[
E(T_k) = Q(E, T_k) = \int_{S(r)} E_{ij} T^i_k n^dS = 8\pi w_k, \quad (2.54)
\]
\[
E(R_k) = Q(E, R_k) = \int_{S(r)} E_{ij} R^i_k n^dS = -8\pi d_k, \quad (2.55)
\]
\[
E(K_k) = Q(E, K_k) = \int_{S(r)} E_{ij} K^i_k n^dS = 8\pi k, \quad (2.56)
\]
\[
H(S) = Q(H, S) = \int_{S(r)} H_{ij} S^{ij} n^dS = 8\pi b, \quad (2.57)
\]
\[
H(T_k) = Q(H, T_k) = \int_{S(r)} H_{ij} T^i_k n^dS = 8\pi q_k, \quad (2.58)
\]
\[
H(R_k) = Q(H, R_k) = \int_{S(r)} H_{ij} R^i_k n^dS = 8\pi p_k, \quad (2.59)
\]
\[
H(K_k) = Q(H, K_k) = \int_{S(r)} H_{ij} K^i_k n^dS = 8\pi s_k. \quad (2.60)
\]

CYK tensors (2.6) and (2.7) correspond to four time-independent charges:

\[
\hat{Q}(E, T_k) = \hat{Q}(E, S) = \hat{Q}(H, T_k) = \hat{Q}(H, S) = 0. \quad (2.61)
\]

Moreover, from (2.14) and (2.15) we obtain time dependence of other quantities:

\[
\hat{Q}(E, R_k) = Q(H, T_k), \quad (2.62)
\]
\[
\hat{Q}(E, K_k) = Q(H, R_k), \quad (2.63)
\]
\[
\hat{Q}(H, R_k) = -Q(E, T_k), \quad (2.64)
\]
\[ \dot{Q}(H, K_k) = -Q(E, R_k). \] (2.65)

Finally, we have the following time evolution for the charges: eight quantities are constant, six of them are linear, and the other six are quadratic in time. More precisely, we have constant charges:

\[ m(t) = m(0), \quad w_i(t) = w_i(0), \quad b(t) = b(0), \quad q_i(t) = q_i(0), \] (2.66)

linear in time:

\[ p_i(t) = -tw_i(0) + p_i(0), \quad d_i(t) = -tq_i(0) + d_i(0), \] (2.67)

and quadratic in time:

\[ k_i(t) = -\frac{1}{2}t^2w_i(0) + tp_i(0) + k_i(0), \quad s_i(t) = -\frac{1}{2}t^2q_i(0) + td_i(0) + s_i(0), \] (2.68)

where \( m(0) \) denotes the initial value of \( m \) at \( t = 0 \) and similarly for the rest of the quantities.

**Traditional Poincaré charges.** Let us observe that traditional relations between angular momentum (or center of mass) and Killing vectors (e.g., ADM (Arnowitt–Deser–Misner) or the Komar formula) are substituted by conformal acceleration. More precisely, we have the following table:

| KV | Charges | CKV |
|----|---------|-----|
| \( T_0 \) | \( p_0 \leftrightarrow m \) | \( S \) | (1) | energy |
| \( T_e \) | \( p_e \leftrightarrow p \) | \( R_k \) | (3) | linear momentum |
| \( \mathcal{L}_{\mu} \) | \( j_{\mu} \leftrightarrow s \) | \( K_k \) | (3) | angular momentum |
| \( \mathcal{L}_{\psi} \) | \( j_{\psi} \leftrightarrow k \) | \( K^i \) | (3) | center of mass |

Other quantities: \( b \)—dual mass, \( d \)—dual momentum, \( w \)—linear acceleration, and \( q \)—angular acceleration are usually vanishing if we want to have global ‘potentials’ (linearized metric \( h \), like vector potential \( A \) for a magnetic monopole). However, some parameters in Einstein metrics can be interpreted as topological charges; e.g., dual mass appears in the Taub–NUT solution \([5, 6]\) and dual momentum appears in Demiański metrics \([1, 7, 8]\). In \([7]\), a large class of metric tensors is given (see also equation \((4.50)\) in \([6]\)). It would be interesting to investigate if some parameters in those spacetimes correspond to charges \( q \) and \( w \) in some asymptotic regime.

**3. Schwarzschild spacetime**

**Introduction.** For the Schwarzschild metric, the construction described above for the Minkowski spacetime cannot be repeated, because the equation defining CYK tensors \((Q_{\mu\nu} = 0)\) has only two solutions (for Minkowski we had 20). One of them corresponds to the mass; i.e., after calculating the integral we get the charge corresponding to the mass, which is equal to the parameter \( M \) appearing in a standard form of the Schwarzschild metric. The second solution can be classified as a dual mass, and in our case it vanishes. Therefore we are forced to apply a different construction. Our goal is to define charges which are ‘local’ in time.
Let’s begin by writing down the Schwarzschild metric in the parametrization \((t, r, \theta, \phi)\):
\[
g_{\mu\nu} \, dx^\mu \, dx^\nu = - \left( 1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \tag{3.1}
\]

Now let us introduce a new coordinate \(\bar{r}\) defined by the equality
\[
r = \bar{r} (1 + \frac{M}{2\bar{r}}). \tag{3.2}
\]
We observe that for a fixed value of \(t\), the metric (3.2) is conformally flat (i.e., it takes the form of a flat-space metric multiplied by a conformal factor \(1 + \frac{M}{2\bar{r}}\)). Using this fact, we conclude that we have a set of 10 conformal Killing fields that are identical to the ones that we had for a flat three-dimensional space, since both metrics are conformally equivalent. Moreover, using the formula
\[
\nabla_{\mu} K^\mu_{\nu} = 0\]
where \(K^\mu_{\nu}\) is the extrinsic curvature tensor, we can easily check that the extrinsic curvature tensor \(K_{\mu\nu}\) vanishes.

**Definition of momentary charges.** In the previous sections we have used the contraction of a spin-2 field with CYK tensors to define global charges; then we have shown (lemma 2) that CYK tensors can be expressed as contractions of electric (or magnetic) parts with conformal Killing fields. Following this lead, we will try to provide a definition of the momentary charges for the Schwarzschild metric as contractions of \(E\) and \(H\) with conformal fields (ignoring the fact that we do not have enough CYK tensors for this metric).

Let \(E_{\mu
u}\) be an electric part, \(H_{\mu
u}\) a magnetic part, and \(X^\mu\) a conformal Killing field; then
\[
\lambda_{\mu} = \lambda = E_{\mu} X^{\mu} = \frac{1}{2N} (N_{\mu\nu} + N_{\nu\mu} - \frac{\partial N_{\nu}}{\partial \bar{r}}), \tag{3.3}
\]
where we have used the fact that \(E_{\mu\nu}\) is a symmetric traceless tensor. Identical calculation can be repeated for the magnetic part \(H_{\mu\nu}\).

Now we have to provide an expression for a three-dimensional divergence of an electric and a magnetic part. Let’s use the following formula:
\[
\left( g^\mu_{\alpha\beta} n^\alpha n^\beta \right)_{\mu} = (W_{\alpha\beta\mu} n^\alpha n^\beta + g^\mu_{\alpha\beta} W_{\alpha\beta\mu} n^\alpha n^\beta)_{\mu}. \tag{3.4}
\]
The first component on the right-hand side is zero, because of one of the properties of the tensor \(W\) (i.e., the four-dimensional divergence of \(W\) is zero iff (1.3), from the definition of the spin-2 field). Now let’s work on the left-hand side:
\[
\left( g^\mu_{\alpha\beta} n^\alpha n^\beta \right)_{\mu} = \left( g^\mu_{n^\alpha n^\beta} W_{\alpha\beta\mu} n^\alpha n^\beta \right)_{\mu} = -\left( \frac{3}{g^\mu} E_{\mu\nu} \right)_{\nu}. \tag{3.5}
\]
In the first equality we have added \(n^\alpha n^\beta\), using the fact that \(W\) is antisymmetric; then we applied the definition of the electric part.
\[
\left( \frac{3}{g^\mu} E_{\mu\nu} \right)_{\nu} = \left( \sqrt{-g} E^i_{\nu} \right)_{\nu} \frac{1}{\sqrt{-g}} = \left( N_{\nu} \sqrt{-g} E^i_{\nu} \right)_{\nu} \frac{1}{\sqrt{-g}} = E^i_{\nu} + E^i_{\nu} (\log N)_{\nu}. \tag{3.6}
\]
We managed to transform the left-hand side of (3.4) to the form containing three-dimensional covariant divergence of the electric part. In equality (3.6), we used the formula
\[
(\sqrt{-g} E^i_{\nu})_{\nu} = (N_{\nu} \sqrt{-g} E^i_{\nu})_{\nu}, \tag{3.7}
\]
which is proven in the appendix to this paper. The right-hand side of (3.4) was calculated in a straightforward way for the metric (3.2). The details of this
calculation are presented in the appendix. Finally, we derived the following formula:

\[ E^i_{\mu} = 0, \quad (3.7) \]

and from an identical reasoning for the magnetic part we have

\[ H^i_{\mu} = 0. \quad (3.8) \]

Both three-dimensional covariant divergences are zero. This means that the momentary gravitational charges (defined as a contraction of an electric or a magnetic part with a conformal Killing field) fulfill Gauss’s Law without sources. We can conclude that the value of the charge would not depend on the choice of the integration surface if there exists an appropriate volume, as was true for the Minkowski spacetime. This means that if we want to calculate the total charge inside any closed surface, we don’t need to know the configuration of fields \( E_{ab} \) and \( H_{ab} \) inside this surface; we only need to know the values at the boundary. The results presented in this section (i.e., that the three-dimensional covariant divergence of an electric and a magnetic part is zero) could also be reproduced using the formulas described in paper [9]:

\[ \epsilon^{\mu ab} K_{\mu}^a H^b, \quad (3.9) \]

\[ \epsilon^{\mu ab} H_{\mu}^a E^b, \quad (3.10) \]

where \( \wedge \) is the operation defined below for two symmetric tensors:

\[ (A \wedge B)_a = e^a_{bc} A_b^d B_d^c. \quad (3.11) \]

Equations (3.9) and (3.10) imply that this construction of momentary charges can be applied to any metric fulfilling the following conditions:

- the spatial part of the metric is conformally flat for fixed time\(^3\),
- the extrinsic curvature tensor vanishes.

4. Conclusions

In this paper we reconsider the definition of quasi-local gravitational charges for the Minkowski spacetime in terms of conformal Yano–Killing tensors and a spin-2 field. The set of 20 charges, defined in that way, has properties very similar to the ones that are also valid for the electric charge; for example, our gravitational charges fulfill Gauss’s Law. We extend the definition of quasi-local gravitational charges. The lemma describing the decomposition of CYK tensors into an exterior product of a time translation vector field and a conformal Killing vector has led us to the idea of defining momentary charges (as a contraction of an electric and a magnetic part with the conformal field) for a wide range of metrics (e.g., when the spatial part is conformally flat for a fixed time and the extrinsic curvature vanishes), including the Schwarzschild spacetime. Moreover, we have proven by straightforward calculation that the charges, defined using the described procedure, fulfill Gauss’s Law and we have given the conditions that are satisfactory to repeat this construction for a well-defined class of metrics.

In the future we would like to apply this construction for the case of asymptotically flat initial data. It is well known that some spacetimes admit (exact) CYK tensors ([3–6]) but in general one should consider asymptotic CYK tensors [2], which correspond to the notion of

\(^3\) Because we need all ten CKVs.
strong asymptotic flatness. The existence of asymptotic conformal Killing vectors is less restrictive, and it should lead to the definition of a global momentary charge for different asymptotics at spatial infinity. In particular, angular momentum and center of mass correspond to conformal acceleration \( \mathbf{K} \).

The content of this paper is the answer to the following question:

*What is the analog of a Coulomb solution (electric and magnetic monopole) for the spin-2 field?* For the spin-1 field the solution is ‘monopole’; for the spin-2 field we also have the dipole part. In Maxwell theory we have only time-independent charges; for gravity we also get time-dependent quantities. The ‘wave part’ of the theory starts from dipoles \((l = 1)\) for electrodynamics and quadrupoles \((l = 2)\) for gravity, respectively. Hence the ‘charged part’ for the spin-1 field is represented by \(l = 0\), but for the spin-2 field we have \(l = 0\) and \(l = 1\). Finally, the analog of the electric–magnetic monopole in electrodynamics is given by the mono-dipole solution \((2.16)–(2.24)\) for the spin-2 field.

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**Appendix. Fully charged solution**

Cartesian components:

\[
W_{ij0} = \frac{3m}{r^3} n_i n_j + \frac{15k}{r^4} n_i n_j - \frac{3}{r^4} \left( k_j n_i + k_i n_j \right) - \frac{\eta_{ij}}{r^3} \left( m + \frac{3k}{r} \right).
\]

\[
W_{ijkl} = \frac{3}{r^3} \left( m + \frac{3k}{r} \right) \left( n_i n_l \eta_{jk} - n_i n_k \eta_{jl} + n_j n_l \eta_{ik} - n_j n_k \eta_{il} \right)
+ \frac{2}{r^3} \left( m + \frac{3k}{r} \right) \left( \eta_{ik} \eta_{jl} - \eta_{il} \eta_{jk} \right)
+ \frac{3}{r^3} e_{ijklm} n^m n^p k^h \left( e_{pi}^h n_j - e_{pj}^h n_i \right)
+ \frac{3}{r^3} e_{ijklm} n^m n^p k^h \left( e_{pk}^h n_i - e_{pl}^h n_k \right).
\]

\[
W_{ijkl} = \frac{3}{r^3} \left[ \left( p_j - n_{pj} \right) n_i n_k - \left( p_k - n_{pk} \right) n_i n_j \right]
+ \frac{3}{r^4} \left( n_i n_k n^m e_{mj}^l s_j - n_i n_j n^m e_{mk}^l s_j \right)
+ \frac{3}{r^4} n^m s \left( 3 e_{mjk} n_i + e_{mk} n_j + e_{mj} n_k \right)
- \frac{3}{r^4} e_{mjk} n^m s_i
+ \frac{3}{r^4} e_{mjk} n^m e_{hi}^l p_i.
\]
Contraction with CKV: $T_k = \frac{\partial}{\partial x^k}$, $S = x^k \frac{\partial}{\partial x^k}$, $R_k = \epsilon_k = \partial_k$, $K_k = x_k = S = \frac{1}{2} r^2 \frac{\partial}{\partial \omega}$ gives

\[
E_{ik} \ T^j_k = -\frac{\eta_{ij}}{2} \left( \frac{3m}{r^2} + \frac{2m}{r^3} + \frac{6k}{r^4} \right) - \frac{3}{r} n^i \partial_d \partial_j \left( \epsilon_{ij} n_i + \epsilon_{kj} n_j \right)
+ \frac{3}{2r^2} \left( n_i w_j + n_j w_i \right) - \frac{3}{r^4} \left( n_i k_j + n_j k_i \right)
- n_i n_j \left( -\frac{3m}{2r^2} - \frac{3m}{r^3} - \frac{15k}{r^4} \right).
\] (A.4)

\[
E^i j R^k _j = -\epsilon_k \ n_i \left( \frac{3m}{2r} + \frac{m}{r^2} + \frac{3k}{r^3} \right) - \frac{3}{r} n^i d_k - \epsilon_k m_j x_m \left( \frac{3}{r^4} n^i k_j - \frac{3}{2r^2} n^i w_j \right).
\] (A.5)

\[
E_{ij} K^k = \frac{m}{2r} \left( n_i n_k + \eta_{ik} \right) - n^6 \left( \frac{3}{2r} \right) n^m \left( \epsilon_{mi} n_k - \epsilon_{nk} n_i \right)
- k^2 \left( \frac{3}{2r^2} \right) \left( -n_i n_k n_j + n_k \eta_{ij} - n_i \eta_{ijk} - n_j \eta_{ik} \right)
- w^3 \left( \frac{3}{4} \right) \left( -n_i n_k n_j - n_k \eta_{ij} + n_j \eta_{ik} - n_i \eta_{ik} \right).
\] (A.6)

\[
E^i j S^l = n^l \left( \frac{2m}{r^2} + \frac{9k}{r^3} \right) - \frac{3}{r} n^i d_k - \epsilon_k d^i d^j + \frac{3}{2r} \left( n^i w^j + w^i \right) - \frac{3}{r^3} k^i.
\] (A.7)

\[
H_{ik} T^j_k = -\frac{\eta_{ij}}{2} \left( \frac{3a}{r^2} + \frac{2a}{r^3} + \frac{6s}{r^4} \right) + \frac{3}{r} n^i \partial_d \partial_j \left( \epsilon_{ij} n_i + \epsilon_{kj} n_j \right)
+ \frac{3}{2r^2} \left( n_i q_j + n_j q_i \right) - \frac{3}{r^4} \left( n_i s_j + n_j s_i \right)
+ n_i n_j \left( \frac{3a}{2r^2} + \frac{3a}{r^3} + \frac{15s}{r^4} \right).
\] (A.8)

\[
H^i j R^k _j = \epsilon_k ^i n_i \left( -\frac{3a}{2r} - \frac{b}{r^2} - \frac{3s}{r^3} \right) + \frac{3}{r} n^i p_d + \epsilon_k m_j x_m \left( -\frac{3}{r^4} n^i s_j + \frac{3}{2r^2} n^i q_j \right).
\] (A.9)

\[
H_{ij} K^k = \frac{b}{2r} \left( n_i n_k + \eta_{ik} \right) + \frac{\eta_{ij}}{2} \left( \frac{3a}{r^2} + \frac{2a}{r^3} + \frac{6s}{r^4} \right) + \frac{3}{r} n^i \partial_d \partial_j \left( \epsilon_{ij} n_i + \epsilon_{kj} n_j \right)
+ \frac{3}{2r^2} \left( n_i q_j + n_j q_i \right) - \frac{3}{r^4} \left( n_i s_j + n_j s_i \right)
+ n_i n_j \left( \frac{3a}{2r^2} + \frac{3a}{r^3} + \frac{15s}{r^4} \right).
\] (A.10)

\[
H^i j S^l = n^l \left( \frac{2b}{r^2} + \frac{9s}{r^3} \right) + \frac{3}{r} n^i p_d + \epsilon_k d^i d^j + \frac{3}{2r} \left( n^i q + q^i \right) - \frac{3}{r^3} s^j.
\] (A.11)

After contraction with normal, we get:

\[
E_{ij} T^i j n^i = \frac{2m}{r^3} n_k - \frac{3}{r^3} d^l n^m e_{ml} - \frac{3}{r^4} k + \frac{9}{r^4} k l n k + \frac{3}{2r^2} \left( w_k + w^i n_k \right).
\] (A.12)

\[
E_{ij} R^i j n^i = -\frac{3}{r^3} \left( d^k - d^l n^k \right) - \frac{3}{r^3} k l n_p e_{pk} + \frac{3}{2r} n^i p_e n^i p_k.
\] (A.13)
\[ E_{ij} \mathcal{K}^i n^j = \frac{m}{r} n_k + \frac{3}{2r} d' n^m \epsilon_{mkl} + \frac{3}{2r^2} k^l (n_i n_k + \eta_{il}) - \frac{3}{4} w' (n_{kl} - 3 n_k n_l), \] (A.14)

\[ E_{ij} S^i n^j = \frac{2m}{r^2} + \frac{6}{r^3} \xi_j n^j + \frac{3}{r} w_l n^l, \] (A.15)

\[ H_{ij} T^i n^j = \frac{2b}{r^3} n_k + \frac{3}{2r^3} p^l n^m \epsilon_{mkl} - \frac{3}{r^3} s_k + \frac{9}{2r^2} s^l n_k + \frac{3}{r^2} q_k + q' n_k, \] (A.16)

\[ H_{ij} R^i n^j = \frac{3}{r^3} (p^k - p' n_i n^k) - \frac{3}{r^3} s^l n_k \epsilon_{ik} + \frac{3}{2r} q^l n_k \epsilon_{ik}, \] (A.17)

\[ H_{ij} \mathcal{K}^i n^j = \frac{b}{r} n_k + \frac{3}{2r^3} p^l n^m \epsilon_{mkl} + \frac{3}{2r^2} s^l (n_i n_k + \eta_{il}) - \frac{3}{4} q^l (n_{kl} - 3 n_k n_l), \] (A.18)

\[ H_{ij} S^i n^j = \frac{2b}{r^2} + \frac{6}{r^3} \xi_j n^j + \frac{3}{r} q_l n^l. \] (A.19)

Integrating expressions (A.12)–(A.19) on \( S^2 \), we obtain (2.53)–(2.60).

**Proof of lemma 1**

\[ V_\nu F^{\mu
u} (W, Q) = V_\nu (W^{\mu\nu\alpha\beta} Q_{\alpha\beta}) = \left( V_\nu W^{\mu\nu\alpha\beta} \right) Q_{\alpha\beta} + W^{\mu\nu\alpha\beta} (V_\nu Q_{\alpha\beta}) \]

The first component is zero because of equation (1.8), for the second component:

\[ W^{\mu\nu\alpha\beta} Q_{\alpha\beta} = W^{\mu\nu\alpha\beta} \left[ Q_{\alpha\beta} + Q_{\alpha\beta} - \frac{2}{3} (g_{\alpha\beta} Q_{\nu,\nu} + g_{\alpha\beta} Q_{\nu,\nu}) \right] \]

\[ = W^{\mu\nu\alpha\beta} (Q_{\alpha\beta} + Q_{\alpha\beta}) = \left( W^{\mu\nu\alpha\beta} + W^{\mu\nu\alpha\beta} \right) Q_{\alpha\beta} \]

\[ = \left( W^{\mu\nu\alpha\beta} + \frac{1}{2} W^{\mu\nu\alpha\beta} - \frac{1}{2} W^{\mu\nu\alpha\beta} \right) Q_{\alpha\beta} \]

\[ = \left( W^{\mu\nu\alpha\beta} + \frac{1}{2} W^{\mu\nu\alpha\beta} + \frac{1}{2} W^{\mu\nu\alpha\beta} \right) Q_{\alpha\beta} \]

\[ = \frac{3}{2} W^{\mu\nu\alpha\beta} Q_{\alpha\beta}. \]

In the first line we used the fact that \( W \) is traceless; in the second we renamed the indices; in the third we observed that \( Q \) is antisymmetric; and in the fourth line we applied equality \( W^{\mu(\nu\alpha\beta)} = 0 \).

Finally we get:

\[ V_\nu F^{\mu\nu} (W, Q) = \frac{2}{3} W^{\mu\nu\alpha\beta} Q_{\alpha\beta}. \]

15
Divergence of a tensor density with two indices

Let $L^i_{\ j}$ be a tensor density, so
\[
L^i_{\ j} = \sqrt{g} \left( \frac{1}{\sqrt{g}} L^i_{\ j} \right) = \sqrt{g} \left( L^i_{\ j},_i + L^k_{\ j} \Gamma^i_{\ kj} - L^i_{\ k} \Gamma^k_{\ ji} \right)
\]
\[
= \sqrt{g} L^i_{\ j},_i + L^i_{\ j} \sqrt{g},_i - \sqrt{g} L^k_{\ j} \Gamma^k_{\ ji} = \left( L^i_{\ j} \right),_i - \sqrt{g} L^i_{\ j} \Gamma^k_{\ ji}.
\]

Let us assume that $L^k$ is a symmetric tensor and perform the contraction with a Christoffel symbol:
\[
L^i_{\ k} \Gamma^k_{\ ji} = \frac{1}{2} L^k \left( g_{ki,i} - g_{jk,i} + g_{ik,j} \right) = \frac{1}{2} L^k g_{ik,j}.
\]
Finally for a symmetric tensor $L^k$ we get:
\[
L^i_{\ j} = \left( L^i_{\ j} \right),_i - \frac{1}{2} \sqrt{g} L^k g_{ik,j} = \left( L^i_{\ j} \right),_i - \frac{1}{2} L^k g_{ik,j}.
\]

Now we can prove the formula (\ref{eq:3.6}). Let $\xi_{\ mu} = \sqrt{-g} W_{\ mu\alpha\sigma} n^\nu n^\sigma$. Therefore we can write:
\[
\xi_{\ k} = g^{\mu\nu} \xi_{\ k\nu} = \left( g^{\mu\nu} + n^\mu n^\nu \right) \xi_{\ k\nu} = g^{\mu\nu} \xi_{\ k\nu}.
\]

Next, using equation (\ref{eq:A.20}) we obtain:
\[
\xi_{\ k\mu} = \xi_{\ k\mu} = \left( N \sqrt{g} E^i_{\ j} \right),_j - \frac{1}{2} \sqrt{g} \xi_{\ k\nu} g_{ij,k} = \left( N \sqrt{g} E^i_{\ j} \right),_j - \frac{1}{2} \sqrt{g} \xi_{\ k\nu} g_{ij,k}.
\]

Right-hand side of \ref{eq:3.4} for the metric \ref{eq:3.2}
\[
g^{\mu\nu} W_{\ alpha\beta} \left( n^\mu n^\nu \right)_{,\mu} = g^{\mu\nu} W_{\ alpha\beta} \left( n^\mu n^\nu \right)_{,\mu} + g^{\mu\nu} W_{\ alpha\sigma} n^\nu n^\sigma \Gamma^\sigma_{\ mu}
\]
\[
+ g^{\mu\nu} W_{\ alpha\sigma} n^\nu n^\sigma \Gamma^\sigma_{\ mu}
\]
\[
= g^{11} W_{0\alpha\sigma} \left( \frac{1}{N^2} \right),_1 + g^{\mu\nu} W_{0\alpha\beta} \frac{1}{N^2} \Gamma^\nu_{\ 0\mu} + g^{\mu\nu} W_{0\alpha\sigma} \frac{1}{N^2} \Gamma^\sigma_{\ 0\mu}
\]
\[
= g^{11} W_{0\alpha\sigma} \left( \frac{1}{N^2} \right),_1 + \frac{1}{N^2} \Gamma^0_{\ 01} \Gamma^{11}_{\ 00} W_{0\alpha\sigma} + \frac{1}{N^2} \left( \Gamma^0_{\ 01} \Gamma^{11}_{\ 00} - g^{00} \Gamma^1_{\ 01} \right) W_{0\alpha\sigma}
\]
\[
= \frac{1}{N^2} W_{0\alpha\sigma} \left[ 2 \Gamma^0_{\ 01} \Gamma^{11}_{\ 00} - g^{00} \Gamma^1_{\ 01} \right] - 2 \left( g^{00} - \log(N) \right) \right) .
\]

In the calculation above, we used the fact that each Christoffel symbol of the form $\Gamma^\sigma_{\ mu}$, excluding $\Gamma^0_{\ 01}$ and $\Gamma^1_{\ 01}$, is zero.

Proof of formula \ref{eq:3.7}

To derive the final formula for the divergence of the electric part, we have to use the results from equations (\ref{eq:3.6}) and (\ref{eq:A.22}):
\[ E^i_{kli} = -\frac{1}{N^2} W_{\alpha \nu 0} \left[ 2 T^{\alpha 0}_{\ 0} g^{11} - g^{00} T^{1}_{\ 00} - 3 g^{11} \log (N) \right]. \] (A.23)

After substituting the values for the metric (3.2), we obtain
\[ E^i_{kli} = 0. \] (A.24)

**Summary of basic facts used in a 3 + 1 decomposition**

A four-dimensional metric can be expressed using the lapse \((N)\) and the shift \((N^m)\) in the following manner:

\[
\begin{bmatrix}
  g_{00} & g_{0k} \\
  g_{0k} & g_{kk}
\end{bmatrix} = \begin{bmatrix}
  (N_i N^i - N^2) & N_k \\
  N_i & g_{kk}
\end{bmatrix}.
\]

Inverse metric:

\[
\begin{bmatrix}
  g^{00} & g^{0k} \\
  g^{0k} & g^{kk}
\end{bmatrix} = \begin{bmatrix}
  \left(1/N^2\right) & \left(N^m/N^2\right) \\
  \left(N^k/N^2\right) & \left(g^{km} - N^k N^m/N^2\right)
\end{bmatrix}.
\]

Unit normal time-like vector:

\[ n_{\mu} = (-N, 0, 0, 0), \quad n^\mu = \left(1/N, -(N^m/N)\right). \]

Connection of a three-dimensional metric with a four-dimensional one:

\[ g^{\mu \nu} + n^\mu n^\nu = 3 g^{\mu \nu}, \quad g^{\mu \nu} + n^\mu n_\nu = 3 g_{\mu \nu}. \]

Volume element:

\[ \left(-g\right)^{1/2} dx^0 dx^1 dx^2 dx^3 = N^3 \ g^{1/2} dt dx^1 dx^2 dx^3. \]

**Non-zero Christoffel symbols of metric (3.2)**

\[ \Gamma^0_{1i} = -\frac{4M}{M^2 - 4\bar{r}^2}, \quad \Gamma^i_{00} = \frac{64M^{\bar{r}} (M - 2\bar{r})}{(M + 2\bar{r})^7}, \quad \Gamma^1_{1i} = -\frac{2M}{\bar{r} (M + 2\bar{r})} \]

\[ \Gamma^2_{22} = \frac{\bar{r} (M - 2\bar{r})}{M + 2\bar{r}}, \quad \Gamma^3_{33} = -\sin \theta \cos \theta, \quad \Gamma^3_{13} = -\frac{M - 2\bar{r}}{\bar{r} (M + 2\bar{r})}, \quad \Gamma^3_{23} = \cot \theta \]

**References**

[1] Jezierski J 1995 The relation between metric and spin-2 formulations of linearized Einstein theory Gen. Rel. Grav. 27 821–43
[2] Jezierski J 1997 Conformal Yano–Killing tensors and asymptotic CYK tensors for the Schwarzschild metric Class. Quantum Grav. 14 1679–88
[3] Jezierski J 2002 CYK tensors, Maxwell field and conserved quantities for spin-2 field Class. Quantum Grav. 19 4405–29
[4] Jezierski J and Łukasik M 2006 Conformal Yano–Killing tensor for the Kerr metric and conserved quantities Class. Quantum Grav. 23 2895–918
[5] Jezierski J and Łukasik M 2007 Conformal Yano–Killing tensors for the Taub NUT metric Class. Quantum Grav. 24 1331–40
[6] Jezierski J 2008 Asymptotic conformal Yano–Killing tensors for asymptotic anti de Sitter spacetimes and conserved quantities Acta Phys. Pol. B 39 75–114
[7] Griffiths J B and Podolsky J 2006 A new look at the Plebański–Demiański family of solutions Int. J. Mod. Phys. D 15 335–70
[8] David Kubiznak and Krtous Pavel 2007 Conformal Killing–Yano tensors for the Plebański–Demiański family of solutions Phys. Rev. D 76 084036
[9] Andersson L and Moncrief V 2004 Future complete vacuum spacetimes 50 Years of the Cauchy Problem in General Relativity ed P T Chruściel and H Friedrich (Basel: Birkhauser) pp 299–330