DARBOUX PARTNERS OF PSEUDOSCALAR
DIRAC POTENTIALS ASSOCIATED WITH
EXCEPTIONAL ORTHOGONAL POLYNOMIALS

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Abstract
We introduce a method for constructing Darboux (or supersymmetric) pairs of pseudoscalar and
scalar Dirac potentials that are associated with exceptional orthogonal polynomials.
Properties of the transformed potentials and regularity conditions are discussed. As an
application, we consider a pseudoscalar Dirac potential related to the Schrödinger model for
the rationally extended radial oscillator. The pseudoscalar partner potentials are constructed
under first- and second-order Darboux transformations.

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1 Introduction
In recent years, a considerable amount of research has been devoted to studying exceptional $X_l$
orthogonal polynomials that lead to rational extensions of quantum mechanical potentials. Ex-
ceptional orthogonal polynomials (EOPs) of codimension one were first introduced in [1] as the
polynomial eigenfunctions of a Sturm-Liouville problem. In contrast to the families of classical
orthogonal polynomials, which start at degree 0, the exceptional ones start at degree $m(m \geq 1)$
but still form complete sets with respect to some positive definite measure. The relationship
between exceptional orthogonal polynomials, the Darboux transformation and shape invariant
potentials was shown in [2] followed by [3]. Higher codimensional families were obtained in
[4]. Multi-indexed families associated to Darboux-Crum or iterated Darboux transformations
were studied in [5]. Along with the study of the mathematical properties of these exceptional
polynomial families [6], the latter has found applications in a number of physical problems,
e.g. quantum superintegrability [7], Dirac operators minimally or non minimally coupled to
external fields and Fokker Planck equations [8], entropy measures in quantum information the-
ory [9], solutions of Schrödinger equation with some conditionally exactly solvable potentials
[10], solutions for position dependent mass systems [12], discrete quantum mechanics [17], quan-
tum Hamilton-Jacobi formalism [11], $N$-fold Supersymmetry and its dynamical breaking in the
context of position dependent mass [13].
In this note our objective is to show, in principle, how to obtain via Darboux transformation, exactly solvable scalar and pseudoscalar potentials and their supersymmetric partners for \((1+1)\)-dimensional Dirac equation associated to exceptional orthogonal polynomials. It should be mentioned here that a Darboux transformation for Dirac equation has been constructed for both one and two dimensional stationary case [14] and the fully time-dependent equation in \((1+1)\) dimensions [15]. The motivation of the present work stems from the fact that with the experimental realization of graphene [16] and of the massless Dirac nature of its electron low-energy spectrum, a renovated interest in obtaining analytical solutions for the energy eigenvalues and eigenfunctions for electrons and holes of the massless or mass dependent on position Dirac equation in \((2+1)\) as well as in \((1+1)\) dimensions has started to emerge [18].

The remainder of this article is organized as follows. In section 2 we demonstrate how one-parameter classes of pseudoscalar and scalar Dirac potentials can be generated by starting out from a solvable Schrödinger equation. Section 3 is devoted to the Darboux transformation for the Dirac equation, where we will mainly be summarizing results from [14]. In section 4 we apply the Darboux formalism to a particular class of pseudoscalar Dirac potentials that arise from a Schrödinger equation for a rationally extended oscillator potential, while section 5 contains a final discussion concerning regularity of the transformed potentials and related issues.

### 2 Construction of Dirac potentials

We start out by considering the conventional stationary Schrödinger equation in atomic units

\[
\psi'' + (\epsilon - V_0) \psi = 0, \tag{1}
\]

where \(\epsilon\) is the real-valued energy and the continuous function \(V_0\) stands for the potential. Let us assume that \(\psi\) is a solution of the Schrödinger equation (1). For our subsequent considerations it is essential to rewrite the potential \(V_0\) in the following form

\[
V_0 = q_0^2 + q_0', \tag{2}
\]

introducing a differentiable function \(q_0\). For a given potential \(V_0\), the general solution to the Riccati equation (2) determines a one-parameter family of functions \(q_0\) that we will use to construct a pseudoscalar or scalar Dirac potential. Before we do so, let us elaborate on how to find the general solution of (2). It is well-known [19] that the latter solution can be constructed once a particular solution is known. We can come by such a particular solution after linearizing (2) via \(q_0 = \dot{q}_0/q_0\), leading to the result

\[
\ddot{q}_0 - V_0 \dot{q}_0 = 0. \tag{3}
\]

Comparison of this equation and (1) shows that \(\dot{q}_0 = \psi|_{\epsilon=0}\) is a solution of (3), such that

\[
q_{0,p} = \frac{d}{dx} \log (\dot{q}_0) = \frac{d}{dx} \log (\psi|_{\epsilon=0}), \tag{4}
\]

is a particular solution to our Riccati equation (2). Once we have found (4), we can determine the general solution \(q_0\) of (2) through the formula [19]

\[
q_0 = q_{0,p} + \frac{\exp \left( -2 \int q_{0,p} \, dt \right)}{c + \int \exp \left( -2 \int q_{0,p} \, ds \right) \, dt}, \tag{5}
\]

where \(c\) is an arbitrary constant. Expression (5) simplifies if we insert the form of \(q_{0,p}\) given in (4):

\[
q_0 = \left[ \frac{d}{dx} \log (\dot{q}_0) \right] + \left( c q_0^2 + q_0^2 \int \frac{1}{q_0^2} \, dt \right)^{-1}, \tag{6}
\]
recall that \( \hat{q}_0 = \psi_{\epsilon=0} \). As mentioned before, (6) is a one-parameter family that will now be implemented as the parametrizing function of a Dirac potential. To this end, let us distinguish the following two cases.

**Pseudoscalar case.** First, we use (6) and the solution \( \psi \) to the Schrödinger equation (1) for defining two functions \( \Psi_1 \) and \( \Psi_2 \) as follows

\[
\begin{align*}
\Psi_1 &= \psi \\
\Psi_2 &= \frac{1}{E + m} (q_0 \Psi_1 - \Psi_1'),
\end{align*}
\]

where \( m \) is a positive constant and \( |E| = \sqrt{\epsilon + m^2} \). The function \( \Psi = (\Psi_1, \Psi_2)^T \) is a solution of the Dirac equation

\[
i \sigma_2 \Psi' + (U_0 - E) \Psi = 0.
\]

Here, \( \sigma_2 \) denotes the usual Pauli Matrix and the potential is given by

\[
U_0 = m \sigma_3 + q_0 \sigma_1,
\]

for Pauli matrices \( \sigma_1 \) and \( \sigma_3 \). Expression (10) represents a pseudoscalar potential for the Dirac equation (9). In conclusion, we have converted a solution \( \psi \) of our initial Schrödinger equation to a solution of the Dirac equation (9) for the pseudoscalar potential (10).

**Scalar case.** We start out by defining a function \( S_0 \) as follows:

\[
S_0 = q_0 + m,
\]

where \( q_0 \) is given in (6). Next, we introduce two functions \( \Psi_1 \) and \( \Psi_2 \) by

\[
\begin{align*}
\Psi_1 &= \psi \\
\Psi_2 &= \frac{1}{E} \left[ (m + S_0) \Psi_1 - \Psi_1' \right],
\end{align*}
\]

for a positive constant \( m \) and \( |E| = \sqrt{\epsilon} \). The function \( \Psi = (\Psi_1, \Psi_2)^T \) is a solution of the Dirac equation

\[
i \sigma_2 \Psi' + (U_0 - E) \Psi = 0,
\]

where the potential has scalar form

\[
U_0 = (m + S_0) \sigma_1.
\]

Thus, relations (12), (13) establish a link between the solutions of the Schrödinger equation (1) and its Dirac counterpart (14) for the scalar potential (15). Recall that the function \( S_0 \) can be given in explicit form, once we combine (11) and (6).

### 3 Darboux transformation for the Dirac equation

In this section we will explain how the Darboux transformation can be applied to a Dirac equation for potentials of pseudoscalar or scalar form. We will summarize particular results from [14], the reader may refer to the latter reference for a more detailed discussion of the topic. We consider a Dirac equation in the form

\[
i \sigma_2 \Psi' + (V_0 - E) \Psi = 0,
\]
where $V_0$ takes either pseudoscalar (10) or scalar form (15). Note that in contrast to the previous section, the parametrizing functions $q_0$ and $S_0$ are allowed to remain undetermined for now. If we assume that $\Psi = (\Psi_1, \Psi_2)^T$ is a solution to our Dirac equation (16), then its first component $\Psi_1 = \psi$ solves a stationary Schrödinger equation of the form

$$\psi'' + (\epsilon - q_0^2 - q_0') \psi = 0. \quad (17)$$

If the potential $V_0$ in our initial Dirac equation is pseudoscalar, then in (17) we have $\epsilon = E^2 - m^2$. In case of a scalar potential $V_0$, the function $q_0$ is defined through (11), and $\epsilon = E^2$. We will now apply the Darboux transformation to equation (17). To this end, let $u_1, \ldots, u_N$ be $N$ auxiliary solutions of (17) for the respective, pairwise different energy values $\lambda_1, \ldots, \lambda_N$. Then,

$$\phi = \frac{W_{u_1, \ldots, u_N}}{W_{u_1, \ldots, u_N}}, \quad (18)$$

where $W$ stands for the Wronskian of the functions denoted in its index, is a solution of the Schrödinger equation

$$\phi'' + \left[ \epsilon - q_0^2 - q_0' + 2 \frac{d}{dx} \log (W_{u_1, \ldots, u_N}) \right] \phi = 0. \quad (19)$$

We must now find an interrelation between (19) and a Dirac equation of the form (16). To this end, we impose the condition

$$q_0^2 + q_0' - 2 \frac{d}{dx} \log (W_{u_1, \ldots, u_N}) = q_1^2 + q_1', \quad (20)$$

which represents a Riccati equation for the new function $q_1$. This function can be found in the same way as it was done for its counterpart $q_0$ in (2). Adopting the notation from (4), we find that a particular solution $q_{1,p}$ to (20) is given by

$$q_{1,p} = \frac{d}{dx} \log (\hat{q}_1) = \frac{d}{dx} \log (\phi_{|\epsilon=0}). \quad (21)$$

Recall that $\phi$ is known from (18). The general solution of (20) can be constructed by means of the formula

$$q_1 = \left[ \frac{d}{dx} \log (\hat{q}_1) \right] + \left( c \hat{q}_1^2 + \hat{q}_1^2 \int \frac{1}{\hat{q}_1^2} \, dt \right)^{-1}. \quad (22)$$

Now that $q_1$ has been determined, we are ready to state the transformed Dirac equation as well as its solutions. Starting out with the equation, it reads

$$i \sigma_2 \Phi' + (U_1 - E) \Phi = 0, \quad (23)$$

where the potential $U_1$ has either pseudoscalar or scalar form. In the first case, the potential is given by

$$U_1 = m \sigma_3 + q_1 \sigma_1, \quad (24)$$

the function $q_1$ being stated in (22), while the corresponding solution $\Phi = (\Phi_1, \Phi_2)^T$ of (23) takes the same shape as in (7), (8), that is,

$$\Phi_1 = \frac{W_{u_1, \ldots, u_N, \Psi_1}}{W_{u_1, \ldots, u_N}},$$

$$\Phi_2 = \frac{1}{E + m} (q_1 \Phi_1 - \Phi_1').$$
recall that $\phi$ is defined in (13). If the potential $U_1$ takes scalar form, it reads

$$U_1 = (m + S_1) \sigma_1,$$

introducing a new function $S_1 = q_1 + m$. The corresponding solution $\Phi = (\Phi_1, \Phi_2)^T$ of (23) is obtained from (12), (13) as

$$\Phi_1 = \frac{W_{u_1, \ldots, u_N} \psi_1}{W_{u_1, \ldots, u_N}}, \quad \Phi_2 = \frac{1}{E} [(m + S_1) \Phi_1 - \Phi_1'] .$$

In summary, we have established a connection between the initial Dirac equation (16) and its transformed counterpart (23) by means of the Darboux transformation (18).

4 Application: the extended radial oscillator model

We will now combine the results from sections 2, 3 and apply them to a Dirac model that can be generated by means of the Schrödinger equation for the rationally extended radial oscillator potential. The main reasons for choosing this particular model are its simplicity and the fact that the transformed potentials turn out to be non-singular. We will first discuss the case of a pseudoscalar Dirac potential under first- and second-order Darboux transformations.

Pseudoscalar case. Our starting point is the Schrödinger equation for the rationally extended radial oscillator potential. The corresponding boundary-value problem has the following form

$$\psi''(x) + \left[ \epsilon - \frac{1}{4} \omega^2 x^2 - \frac{l (l + 1)}{x^2} - \frac{4 \omega}{\omega x^2 + 2 l + 1} + \frac{8 \omega (2 l + 1)}{(\omega x^2 + 2 l + 1)^2} \right] \psi(x) = 0, \quad x > 0$$

$$\psi(0) = \lim_{x \to \infty} \psi(x) = 0, \quad (25, 26)$$

where $\omega > 0$, $l$ is a nonnegative integer, and the real-valued constant $\epsilon$ represents the spectral parameter. The problem (25), (26) has an infinite number of solutions ($\psi_n$) and a corresponding discrete spectrum ($\epsilon_n$), where $n$ is a positive integer or zero [2]

$$\epsilon_n = \omega \left( 2 n + l + \frac{3}{2} \right)$$

$$\psi_n(x) = \frac{x^{l+1}}{\omega x^2 + 2 l + 1} \exp \left( -\frac{1}{4} \omega x^2 \right) \mathcal{L}_{n+1}^{l+\frac{1}{2}} \left( \frac{1}{2} \omega x^2 \right).$$

Here, the symbol $\mathcal{L}$ stands for an exceptional Laguerre polynomial of $X_1$-type, defined as follows [2]

$$\mathcal{L}_n^k(x) = -(x + k + 1) L_n^k(x) + L_{n+1}^k(x),$$

where $L$ denotes a conventional associated Laguerre polynomial [20]. Let us now construct a pseudoscalar potential for the Dirac equation from the boundary-value problem (25), (26). To this end, we must set up and solve the Riccati equation (2), which reads in the present case

$$\frac{1}{4} \omega^2 x^2 + \frac{l (l + 1)}{x^2} + \frac{4 \omega}{\omega x^2 + 2 l + 1} - \frac{8 \omega (2 l + 1)}{(\omega x^2 + 2 l + 1)^2} = q_0^2 + q_0' ,$$

observe that we extracted the Schrödinger potential $V_0$ from equation (23). Since the general solution of equation (30) for $q_0'$ involves very long expressions, we will restrict ourselves to the
particular solution \( q_{0,p} \). According to (4), we can construct \( q_{0,p} \) by evaluating the solution (28) at the energy \( \epsilon = 0 \). Since the latter solution does not contain \( \epsilon \) itself, we must find the numerical value for \( n \) that corresponds to vanishing \( \epsilon \). Inspection of (27) gives this value as

\[
\frac{\omega}{2} x + \frac{l - 1}{x} + \frac{4 l + 1}{\omega x^3 + 2 l x + x} + \omega x \frac{d}{du} \log \left[ \mathcal{L}_{\frac{l}{2} + \frac{3}{4}} \left( \frac{1}{2} \omega x^2 \right) \right] \bigg|_{u = \omega x^2 / 2},
\]

We therefore obtain the following particular solution \( q_0 = q_{0,p} \) of our Riccati equation (30):

\[
q_0 = d \frac{dx}{\omega x^2 + 2 l + 1} \exp \left( -\frac{1}{4} \omega x^2 \right) \mathcal{L}_{-\frac{l}{2} + \frac{3}{4}} \left( \frac{1}{2} \omega x^2 \right)
\]

We observe that the function \( \mathcal{L} \) in this expression is not a polynomial anymore, because its lower index is not an integer. However, the definition (29) still holds, but the associated Laguerre polynomials turn into associated Laguerre functions [20]. Figure 1 shows a plot of the function \( q_0 \), which parametrizes a pseudoscalar Dirac potential \( U_0 \) of the form (10):

\[
U_0 = m \sigma_3 + \left\{ \frac{d}{dx} \log \left[ \frac{x^{l+1}}{\omega x^2 + 2 l + 1} \exp \left( -\frac{1}{4} \omega x^2 \right) \mathcal{L}_{-\frac{l}{2} + \frac{3}{4}} \left( \frac{1}{2} \omega x^2 \right) \right] \right\} \sigma_1.
\]

The corresponding boundary-value problem for the Dirac equation (9) reads

\[
i \sigma_2 \Psi' + (U_0 - E) \Psi = 0, \quad x > 0
\]

\[
\Psi(0) = \lim_{x \to \infty} \Psi(x) = (0, 0)^T.
\]

The solution \( \Psi = (\Psi_1, \Psi_2)^T \) of this problem can be obtained by combining the general formulas (7), (8) with the above solution (28):

\[
\Psi_1 = \frac{x^{l+1}}{\omega x^2 + 2 l + 1} \exp \left( -\frac{1}{4} \omega x^2 \right) \mathcal{L}_{n+1} \left( \frac{1}{2} \omega x^2 \right)
\]

\[
\Psi_2 = \frac{1}{m + \sqrt{m^2 + \omega \left( 2 n + l + \frac{3}{2} \right)}} \left( q_0 \Psi_1 - \Psi_1' \right),
\]

where we omit to substitute the explicit expressions (28) and (32) for the solution and the parametrizing function \( q_0 \), respectively. Before we continue, let us point out that the function
Figure 2: Normalized probability densities $|\Psi_1|^2 + |\Psi_2|^2$ for the solutions (35), (36) with the settings $m = \omega = l = 1$, $n = 0$ (solid curve), $n = 1$ (dashed curve), and $n = 2$ (dotted curve).

$\Psi_2$, as given in (36), in fact vanishes at positive infinity, because both the product $q_0 \Psi_1$ and the derivative $\Psi_1'$ become zero there. This is illustrated in figure 2, where normalized probability densities associated with the functions (35), (36) are displayed. Finally, the discrete spectral values $E_n$, $n$ a nonnegative integer, that are associated with the solutions given in (35), (36), are found from (27) as

$$|E_n| = \sqrt{m^2 + \omega \left(2n + l + \frac{3}{2}\right)}.$$  \hfill (37)

Now that the first of the two Dirac partners has been determined, the remaining task is to find its Darboux partner. To this end, let us return to the initial Schrödinger equation and apply the Darboux transformation to it. As a first step, we must find an auxiliary function $u_1$ that solves our initial equation (25) at an energy $\lambda \neq \epsilon$. We set $n = -1/2$ in (27), giving

$$\lambda = \omega \left(l + \frac{1}{2}\right).$$  \hfill (38)

We observe from (27) that this energy $\lambda$ lies below the ground state energy $\epsilon_0$ of the problem (25), (26). We are therefore guaranteed that the latter spectral problem and its Darboux partner will admit the same discrete spectrum, that is, no spectral value will be added or deleted as a result of the Darboux transformation. Next, we choose the auxiliary solution of (25) at energy (38) from the set (28) by incorporating $n = -1/2$:

$$u_1 = \frac{x^{l+1}}{\omega x^2 + 2l + 1} \exp \left(-\frac{1}{4} \omega x^2\right) \mathcal{L}_{l+\frac{1}{2}}^\frac{1}{2} \left(\frac{1}{2} \omega x^2\right).$$  \hfill (39)

We can now perform the first-order Darboux transformation after substituting (39) and (28) into formula (18) for $N = 1$. The Darboux transformation delivers the following function $\phi$:

$$\phi = -\frac{u'_1}{u_1} \psi_n + \psi'_n$$  \hfill (40)

$$= -\frac{x^{l+2} \exp \left(-\frac{1}{4} \omega x^2\right) \omega \mathcal{L}_{n+1}^{l+\frac{1}{2}} \left(\frac{1}{2} \omega x^2\right)}{(\omega x^2 + 2l + 1) \mathcal{L}_{l+\frac{1}{2}}^\frac{1}{2} \left(\frac{1}{2} \omega x^2\right)} \left[ \frac{d}{du} \mathcal{L}_{n+1}^{l+\frac{1}{2}} (u) \right]_{u=\omega x^2/2} +$$

$$+ \frac{x^{l+2} \exp \left(-\frac{1}{4} \omega x^2\right) \omega}{\omega x^2 + 2l + 1} \left[ \frac{d}{du} \mathcal{L}_{n+1}^{l+\frac{1}{2}} (u) \right]_{u=\omega x^2/2}.$$  \hfill (41)
Before we relate this function to the transformed Dirac equation, let us construct the corresponding pseudoscalar potential. To this end, we need to determine the parametrizing function \( q_1 \) according to (21) and (22). Due to the length of the expressions involved, we will restrict ourselves to the simplest case, choosing \( q_1 = q_{1,p} \) as in (21). The argument \( \phi_{|\epsilon=0} \) of the logarithm in the latter reference is obtained from (41) by setting \( n \) as in (31):

\[
\phi_{|\epsilon=0} = \phi_{|n=-\frac{1}{2} \omega x^2}. \tag{42}
\]

Taking into account (21), we have the following expression for the parametrizing function \( q_1 = q_{1,p} \) that will enter in the transformed Dirac equation:

\[
q_1 = \frac{d}{dx} \log \left\{ -x^{l+2} \exp \left( -\frac{1}{4} \omega x^2 \right) \frac{\omega}{L^{l+\frac{1}{2}} \left( \frac{\omega}{2} x^2 \right)} \right\} + \frac{x^{l+2} \exp \left( -\frac{1}{4} \omega x^2 \right) \omega}{\omega x^2 + 2 \omega + 1} \left[ \frac{d}{du} L^{l+\frac{1}{2}} (u) \right]_{u=\omega x^2/2}.
\tag{43}
\]

Figure (3) shows a plot of this function \( q_1 \), compared to its counterpart \( q_0 \). The corresponding pseudoscalar potential \( U_1 \) is then given by (24), where the function (43) is to be inserted for \( q_1 \). Next, let us set up the transformed boundary-value problem for the Dirac equation, which takes almost the same form as (33), (34)

\[
i \sigma_2 \Phi' + (U_1 - E) \Phi = 0, \quad x > 0
\]

\[
\Phi(0) = \lim_{x \to \infty} \Phi(x) = (0, 0)^T. \tag{45}
\]

The solution \( \Phi = (\Phi_1, \Phi_2)^T \) of this problem is constructed in the same way as for the initial counterpart. The formulas (7), (8) are combined with the solution (41) to give

\[
\Phi_1 = \phi
\]

\[
\Phi_2 = \frac{1}{m + \sqrt{m^2 + \omega \left( 2n + l + \frac{3}{2} \right)}} (q_1 \Phi_1 - \Phi_1'). \tag{47}
\]

As before, we do not display the explicit form of the latter solution, because it involves very long expressions. Instead, we show a plot in figure 4. The discrete spectrum admitted by the
Figure 4: Normalized probability densities $|\Phi_1|^2 + |\Phi_2|^2$ for the solutions (46), (47) with the settings $m = \omega = l = 1$, $n = 0$ (solid curve), $n = 1$ (dashed curve), and $n = 2$ (dotted curve).

The transformed problem (44), (45) is the same as in (37), due to the choice of $\lambda$ in (38). Before we conclude this example, let us comment once more on the latter energy $\lambda$. For the numerical value that we chose in our example, the initial and transformed functions $q_0$ and $q_1$ look very similar, as can be seen by inspection of figure 3. This changes once we modify the value of $\lambda$, approaching the ground state $\epsilon_0$ by employing values of $n$ that are negative and close to zero. We did not use such values in our calculations, because the resulting expressions would become lengthy due to very small numbers. Figure 5 shows three examples of functions $q_1$ for different values of $\lambda$. Let us now have a look at the effect that the deformation of the transformed function $q_1$ has on the solutions of the transformed boundary-value problem (44), (45). For the parameter values used in figure 5 we display three corresponding normalized probability densities, see figure 6. We observe that the closer $\lambda$ is to the ground state energy $\epsilon_0$, the more apparent becomes the deformation of the probability density. Before we conclude this paragraph, let us briefly comment on higher-order Darboux transformations for the present example, which are obtained when instead of (40) we employ (18) for $N > 1$. Since in this case the expressions for transformed potentials and solutions are in general very complicated, we do not show them here, but just indicate how they were obtained. Starting with the case $N = 2$, we need two auxiliary solutions $u_1$ and $u_2$ of (17) at different energies $\lambda_1, \lambda_2 \neq \epsilon$ in order to set up the Darboux transformation.

Figure 5: The parametrizing function $q_1$ for the parameters settings $\omega = l = 1$. Furthermore, we chose the number $n$ in (27) as $n = -1/2$ (solid curve), $n = -1/50$ (dashed curve), and $n = -10^{-4}$ (dotted curve).
Figure 6: Normalized probability densities $|\Phi_1|^2 + |\Phi_2|^2$ for the settings $m = \omega = l = n = 1$. The energy $\lambda$ of the auxiliary solution is taken from (27) for $n = -1/2$ (solid curve), $n = -1/50$ (dashed curve), and $n = -10^{-4}$ (dotted curve).

The latter solutions are taken from (28), such that instead of (40) we have

$$\phi = \frac{W_{u_1, u_2, \psi_n}}{W_{u_1, u_2}}. \quad (48)$$

This expression, evaluated at $\epsilon = 0$ as in (42), then determines the transformed pseudoscalar potential via its parametrizing function $q_1$ in (22). Figure 7 shows an example of such a function for particular settings of the parameters, compared to the counterpart (32). The solutions of the transformed Dirac equation are constructed as for the first-order case, that is, by means of the formulas (46), (47), where $\phi$ is given in (48). Higher-order transformations can be performed in a completely similar fashion by simply employing a higher value for $N$ in (18), selecting the auxiliary solutions and their corresponding pairwise different energies, and afterwards following the same procedure that was outlined for the second-order case.

**Scalar case and position-dependent masses.** The pseudoscalar Dirac potential in its general form (11) contains the scalar case, as can be seen by comparison with (15). In order to obtain a scalar potential from a pseudoscalar counterpart, we must first set $m = 0$ in (11) and redefine $q_0 = m + S_0$. As a consequence, we obtain the following two special cases:
(a) We can generate a massless Dirac equation for a scalar potential if we further put \( m = 0 \) in (15).

(b) A position-dependent mass can be incorporated by allowing the constant \( m \) in the scalar potential (15) to be spatially dependent, that is, \( m = m(x) \).

These two special cases can be constructed for the particular example discussed in the previous paragraph. However, since the results for the transformed potential and the associated solutions would be very similar, we skip the illustrations for the above mentioned cases in this section.

5 Discussion

As shown in the previous sections, the process of constructing Darboux partners of Dirac equations for pseudoscalar or scalar potentials that can be expressed through EOPs, is in principle straightforward. At this point we would like to discuss certain technical and conceptual difficulties that may arise, and that are not easily visible from our example in section 4 or from the preceding general considerations. The first issue to be addressed is related to the regularity of the parametrizing functions \( q_0 \) and \( q_1 \) that generate the pseudoscalar Dirac potential (or the corresponding scalar case). Starting out with \( q_0 \) in its particular form (4), we observe that the latter form does not have singularities if the solution \( \hat{q}_0 \) of the linearized equation (3) does not vanish. If instead of (4) we consider the general solution (6) for \( q_0 \), we can avoid singularities by additionally imposing that \( \int x \, 1/\hat{q}_0^2 \, dt \) is bounded from above or below. This choice implies that there is a value for the constant \( c \) in (6), such that the denominator in the second term does not vanish. In case the general solution of the linearized equation (3) is available, the solution formula (6) can be rewritten in a more transparent way

\[
q_0 = \left[ \frac{d}{dx} \log (\hat{q}_0) \right] + \frac{1}{c \, \hat{q}_0^2 + \hat{q}_0 \, \bar{q}_0}, \tag{49}
\]

where \( \hat{q}_0 \) and \( \bar{q}_0 \) form a fundamental system of (3). It is clear that the same argumentation, as well as a formula analogous to (49), can be applied to the transformed parametrizing function \( q_1 \), as it obeys equations (21) and (22). Let us now make one more comment on formula (49). Since the functions contained in the latter formula are solutions of a linear, second-order equation, they are usually special functions. As a consequence, the full expression (49) in general becomes very complicated, such that it cannot be used in practical applications. This is especially true when using rationally extended potentials that are expressed through EOPs, because these potentials generally take a very long and involved form. As a possible consequence, it is common that the integral in (22) cannot be evaluated. This is true for our example presented in section 4, where we had to restrict ourselves to the particular form (4) and (21) of the functions \( q_0 \) and \( q_1 \), respectively. The same issue arises when processing Darboux transformations of orders higher than one, as mentioned in section 4, where a comparably simple example was considered. Let us finally comment on the spectral properties between the initial Dirac problem and its Darboux partner. It is well-known that the Darboux transformation of order \( N \), when applied to a Schrödinger problem that admits a discrete spectrum, can generate or remove at most \( N \) spectral values in the transformed problem, depending on the choice of the auxiliary solutions (21). This property transfers to the Dirac systems, because both the initial and the transformed problem are related to an associated Schrödinger problem. As a consequence, the discrete spectra of the Dirac systems are modified accordingly.

6 Concluding remarks

We have presented a method to generate spectral problems for the Dirac equation involving pseudoscalar or scalar potentials that are expressed through EOPs. As can be seen from the
example we showed in section 4, a principal restriction of our approach lies the feasibility of the calculations, which can become very cumbersome due to the length of the expressions involved. Since at this point our method is restricted to potentials of pseudoscalar or equivalent type, our future research will be dedicated to overcoming the latter restriction, such as to introduce potentials related to EOPs in more general potentials and possibly in higher dimensions.

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