Spinning Wormholes in Scalar-Tensor Theory

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We consider spinning generalizations of the Ellis wormhole in scalar-tensor theory. Analogous to other compact objects these wormholes can carry a non-trivial scalarization. We determine the domain of existence of the scalarized wormholes and investigate the effect of the scalarization on their properties. Depending on the choice of the coupling function, they may possess multiple throats and equators in the Jordan frame, while possessing only a single throat in the Einstein frame.

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I. INTRODUCTION

Among the contenders of General Relativity (GR) scalar-tensor theories (STT) hold a prominent place (see e.g. [1–7] for reviews). When considering besides the gravitational tensor field the presence of an additional gravitational scalar field, the formulations of STT are usually restricted by a number of physical requirements. In particular, the STT should obey the well-known observational constraints.

STT predict a number of new phenomena, not present in GR. One such phenomenon is gravitational dipole radiation, which would be emitted, for instance, from inspiralling close binaries [11–13]. Another such phenomenon is the existence of compact solutions, which possess a finite gravitational scalar field, when the coupling exceeds a critical strength. Dubbed spontaneous scalarization, this effect was first observed for static neutron stars [12, 13] (see also [14–21]) and recently demonstrated for rapidly rotating neutron stars [22–25]. Spontaneous scalarization is also known to occur in boson stars [26–29] and hairy black holes [29]. Here we show, that scalarization also arises for wormholes.

Wormholes represent intriguing topologically non-trivial solutions, connecting either two asymptotically flat universes by a throat or connecting two distant regions within a single universe. Whereas the non-traversable Einstein-Rosen bridge [30] of GR represents a feature of the Schwarzschild spacetime, traversable wormholes in GR need exotic matter for their existence [31, 32]. The simplest such traversable wormholes based on phantom fields are the static Ellis wormholes [33–35]. We note that exotic matter in the form of phantom fields can be employed in cosmology to model the accelerated expansion of the Universe (see e.g. [36]).

Rotating generalizations of the Ellis wormholes have only been found recently. These include analytically constructed slowly rotating perturbative wormhole solutions [37, 38] as well as rapidly rotating non-perturbative solutions [39, 40] that were obtained numerically. The rotating wormhole metric presented by Teo [41], however, does not represent a solution of a specified set of Einstein-matter equations.

While wormholes represent hypothetical objects, they have been searched for observationally [42–44], and a number of their observational signatures have been addressed already, such as their gravitational lensing effects [45–52] including their Einstein rings [52], their shadows [53–55], or their accretion disks [56]. Also combined neutron star–wormhole systems (see e.g. [57–59] and references therein) and boson star–wormhole systems [60–62] have been considered.

Here we construct rotating wormhole solutions in STT, which are based on the presence of a phantom field. Thus they fundamentally differ from wormholes obtained previously in STT, which were pure (static) STT solutions without any further (exotic) matter fields present [63–73]. Our main insight here consists in the realization that once the wormhole solutions are known in GR, analytically or numerically, then in order to obtain scalarized wormhole solutions in the Einstein frame, only the two scalar field equations must be considered. In fact, a single solution for a wormhole metric in GR gives rise to a whole family of STT wormhole solutions in the Einstein frame, which differ only in their scalar fields.

In the transformation between the Einstein frame and the Jordan frame the nonminimal coupling $\mathcal{A}$ plays a major role. It represents the interaction of the gravitational scalar field with the matter fields, i.e., here with the phantom field. Clearly the wormhole solutions should possess a strong dependence on the choice of this coupling. To illustrate this dependence, we present wormhole solutions for 3 examples of the coupling $\mathcal{A}$. The first example corresponds...
to the one employed in the discovery of the scalarization of neutron stars [12, 13], where we obtain particularly interesting scalarized wormhole solutions for positive values of the coupling constant (when there is no scalarization in neutron stars). These wormholes may possess many equators and throats in the Jordan frame. The second example corresponds to Brans-Dicke theory [4], while the third example has been inspired by [74]. While in the first and third example the rotating wormhole solutions of GR [39, 40] are also solutions of the STT equations, this is not the case in the second example.

In section II we state the theoretical setting. We briefly recall the action for scalar-tensor theories, and discuss the transition from the Jordan to the Einstein frame. We present the ansatz, the boundary conditions and the equations of motion. We then determine the domain of existence for the scalar charges and address the mass, the angular momentum, and the quadrupole moment as well as the geometric properties and the null energy condition. The results for the scalarized wormholes for the 3 examples of the nonminimal coupling $\mathcal{A}$ are presented in section III, while section IV gives our conclusions.

II. THEORETICAL SETTING

A. Scalar-Tensor Theories

Let us consider STT with a single gravitational scalar field $\Phi$. The most general action giving rise to second order field equations then contains three functions of the gravitational scalar field $\Phi$, $F(\Phi)$, $Z(\Phi)$, and $W(\Phi)$, and reads in the (physical) Jordan frame

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-\tilde{g}} \left( F(\Phi)\tilde{R} - Z(\Phi)\tilde{g}^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - 2W(\Phi) \right) + S_m[\Psi_m; \tilde{g}_{\mu\nu}].$$

(1)

Here $G_*$ is the bare gravitational constant, $\tilde{g}_{\mu\nu}$ is the spacetime metric, and $\tilde{R}$ is the Ricci scalar curvature. The matter fields $\Psi_m$ are contained in the $S_m[\Psi_m; \tilde{g}_{\mu\nu}]$ part of the action, which depends on the space-time metric $\tilde{g}_{\mu\nu}$. It does not involve the gravitational scalar field $\Phi$ to satisfy the weak equivalence principle.

The functions $F(\Phi)$ and $Z(\Phi)$ should satisfy a set of physical restrictions. We require that $F(\Phi) > 0$, since gravitons should carry positive energy, while the requirement $2F(\Phi)Z(\Phi) + 3[dF(\Phi)/d\Phi]^2 \geq 0$ guarantees, that the kinetic energy of the gravitational scalar field is not negative. When the potential function $W(\Phi)$ is chosen to vanish, this would correspond to a massless scalar field without self-interaction.

The gravitational and matter field equations in the Jordan frame are obtained by varying the action with respect to the metric components, the gravitational scalar field and the matter fields. This procedure then leads to a rather intricate set of coupled field equations. Alternatively, one may consider a mathematically equivalent approach to STT, obtained by invoking the conformally related Einstein frame with the metric $g_{\mu\nu}$

$$g_{\mu\nu} = F(\Phi)\tilde{g}_{\mu\nu}.$$  

(2)

Substitution of $g_{\mu\nu}$ in the action Eq. (1) leads to the action in the Einstein frame, which reads (up to a boundary term)

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left( R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - 4V(\varphi) \right) + S_m[\Psi_m; A^2(\varphi)g_{\mu\nu}].$$

(3)

Here $R$ is the Ricci scalar curvature with respect to the Einstein metric $g_{\mu\nu}$. The new gravitational scalar field $\varphi$ is defined via

$$\left( \frac{d\varphi}{d\Phi} \right)^2 = \frac{3}{4} \left( \frac{d\ln(F(\Phi))}{d\Phi} \right)^2 + \frac{Z(\Phi)}{2F(\Phi)}.$$  

(4)

and the new functions $A(\varphi)$ and $V(\varphi)$ are given by

$$A(\varphi) = F^{-1/2}(\Phi), \quad 2V(\varphi) = W(\Phi)F^{-2}(\Phi).$$

(5)

As a consequence of the transformation, the gravitational scalar field appears in the matter action in the Einstein frame via the nonminimal coupling $A^2(\varphi)$.
B. Action and Ansatz

We now turn to the construction of wormhole solutions in STT, supported by a phantom field \( \Psi \) as the matter field. We further assume the potential \( V(\varphi) \) to vanish. Thus both scalar fields have only a kinetic term in the action, while they interact via the nonminimal coupling \( A(\varphi) \). In the Einstein frame the action then reads

\[
S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left( R - 2g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 8\pi G_* A^2 g^{\mu\nu} \partial_\mu \Psi \partial_\nu \Psi \right)
\]  

(6)

In the following we change to the scaled phantom field \( \psi = \sqrt{4\pi G_*} \Psi \). The Einstein equations can then be cast in the form

\[
R_{\mu\nu} = 2 \left( \partial_\mu \varphi \partial_\nu \varphi - A^2 \partial_\mu \psi \partial_\nu \psi \right)
\]

(7)

and the field equations for the scalar fields yield

\[
\partial_\mu \left( \sqrt{-g} g^{\mu\nu} A^2 \partial_\nu \psi \right) = 0 ,
\]

(8)

\[
\partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \varphi \right) = -\frac{1}{2} \frac{dA^2}{d\varphi} \sqrt{-g} \partial_\lambda \psi \partial_\lambda \psi .
\]

(9)

For stationary rotating spacetimes we employ the line element

\[
ds^2 = -e^f dt^2 + p^2 e^{-f} \left( \psi^2 (d\eta^2 + q d\theta^2) + q \sin^2 \theta (d\phi - \omega dt)^2 \right).
\]

(10)

Here the functions \( f, p, \nu \) and \( \omega \) depend only on the coordinates \( \eta \) and \( \theta \), and \( q = \eta^2 + \eta_0^2 \) is an auxiliary function. The radial coordinate \( \eta \) takes positive and negative values, covering the real line \(-\infty < \eta < \infty\). In the limits \( \eta \to \pm \infty \) two distinct asymptotically flat regions \( \Sigma_{\pm} \) are approached. The two scalar fields \( \varphi \) and \( \psi \) depend only on the coordinates \( \eta \) and \( \theta \), as well.

C. Equations of Motion and Boundary Conditions

By substituting the Ansatz (10) into the general set of equations of motion a system of non-linear partial differential equations (PDEs) is obtained. As noted before [39], the PDE for the metric function \( p \) decouples and is given by

\[
\partial_\eta^2 p + \frac{3\eta}{q} \partial_\eta p + \frac{2 \cos \theta}{q \sin \theta} \partial_\theta p + \frac{1}{q} \partial_\theta^2 p = 0 .
\]

(11)

Imposing the boundary conditions \( p(\eta \to \infty) = p(\eta \to -\infty) = 1 \) and \( \partial_\theta p(\theta = 0) = \partial_\theta p(\theta = \pi) = 0 \), a trivial solution of this equation is given by

\[
p = 1 .
\]

(12)

Inserting the resulting metric Ansatz into the phantom field equation (8) leads to

\[
\partial_\eta \left( q \sin \theta A^2 \partial_\eta \psi \right) + \partial_\theta \left( \sin \theta A^2 \partial_\theta \psi \right) = 0 .
\]

(13)

Assuming \( \partial_\theta \psi = 0 \), a first integral is obtained

\[
\partial_\eta \psi = \frac{Q_\psi A^{-2}}{q} .
\]

(14)

We will assume that in the asymptotic region \( \Sigma_+ \) the gravitational scalar field vanishes, \( \varphi(+\infty) = 0 \), and \( A(0) = 1 \). Then the integration constant \( Q_\psi \) can be identified with the phantom scalar charge. We will also assume that the phantom field vanishes in the region \( \Sigma_+ \), i.e. \( \psi(+\infty) = 0 \).

Addressing next the gravitational scalar field \( \varphi \), we now assume that \( \partial_\theta \varphi = 0 \), as well. Thus Eq. (10) reduces to

\[
\partial_\eta \left( q \partial_\eta \varphi \right) = -\frac{1}{2} \frac{dA^2}{d\varphi} \frac{1}{A^4} \frac{1}{q} Q_\psi^2 .
\]

(15)
Let us next turn to the Einstein equations. Inserting \( p = 1, \partial_\psi \psi = 0 \) and \( \partial_\varphi \varphi = 0 \) shows that the Einstein equations \( R_{\varphi \varphi} = 0, \ R_{\theta \theta} = 0, \ R_{\varphi \theta} = 0, \ R_\psi = 0 \) and \( R_{\psi \varphi} = 0 \) are independent of the scalar fields. These equations lead to three second order PDEs for the metric functions \( f, \omega \) and \( \nu \), and to a constraint,

\[
0 = \partial_\nu (q \sin \theta \partial_\eta f) + \partial_\eta (\sin \theta \partial_\nu f) - q \sin^3 \theta e^{-2f} \left( q(\partial_\psi \omega)^2 + (\partial_\theta \omega)^2 \right)
\]

(16)

\[
0 = \partial_\nu (q^2 \sin^3 \theta e^{-2f} \partial_\eta \omega) + \partial_\eta (q \sin^3 \theta e^{-2f} \partial_\nu \omega)
\]

(17)

\[
0 = -q \sin \theta \partial_\eta f \partial_\theta f + q \cos \theta \partial_\eta \nu - \eta \sin \theta \partial_\nu \nu + q \sin^3 \theta e^{-2f} \partial_\nu \omega \partial_\theta \omega
\]

(18)

\[
0 = \left[ -q \sin \theta \partial_\eta f \partial_\theta f + q \cos \theta \partial_\eta \nu - \eta \sin \theta \partial_\nu \nu + q \sin^3 \theta e^{-2f} \partial_\nu \omega \partial_\theta \omega \right]
\]

(19)

Let us now consider the boundary conditions, which should be imposed in the asymptotic regions \( \Sigma_{\pm} \) and on the axis \( \theta = 0, \pi \). In the asymptotic region \( \Sigma_{+} \), i.e., for \( \eta \to +\infty \), we require that the metric approaches the Minkowski spacetime

\[
f|_{\eta \to \infty} = 0, \ \omega|_{\eta \to \infty} = 0, \ \nu|_{\eta \to \infty} = 0.
\]

(20)

In asymptotic region \( \Sigma_{-} \), i.e., for \( \eta \to -\infty \) we allow for finite values of the functions \( f \) and \( \omega \),

\[
f|_{\eta \to -\infty} = \gamma, \ \omega|_{\eta \to -\infty} = \omega_{-\infty}, \ \nu|_{\eta \to -\infty} = 0.
\]

(21)

The parameter \( \gamma \) controls the symmetry of the wormhole solutions. We call the solutions symmetric, when \( \gamma = 0 \), and non-symmetric, when \( \gamma \neq 0 \). Therefore we refer to \( \gamma \) as the asymmetry parameter. The parameter \( \omega_{-\infty} \) controls the rotation of the spacetime. For static wormhole solutions \( \omega_{-\infty} = 0 \). Static wormholes are known in closed form,

\[
f = \frac{\gamma}{2} \left[ 1 - 2 \pi \arctan \left( \frac{\eta}{\eta_0} \right) \right], \ \omega = 0, \ \nu = 0.
\]

(22)

We note, that in order to obtain the Minkowski spacetime in the limit \( \eta \to -\infty \), a suitable coordinate transformation needs to be performed (see below in subsection II D 2).

The last set of boundary conditions concerns the symmetry axis. Here regularity requires

\[
\partial_\theta f|_{\theta = 0} = 0, \ \partial_\theta \omega|_{\theta = 0} = 0, \ \nu|_{\theta = 0} = 0,
\]

(23)

together with the analogous conditions for \( \theta = \pi \).

We have not yet addressed the remaining Einstein equation for \( R_{\varphi \varphi} \). Substituting the solutions for \( f, \omega \) and \( \nu \) in \( R_{\varphi \varphi} \) shows, that it satisfies

\[
R_{\varphi \varphi} = -2 D^2 q^2,
\]

(24)

where the constant \( D \) depends on the mass and the angular momentum of the spacetime. Explicitly we find

\[
D^2 = \frac{q}{4} \left[ g(\partial_\eta f)^2 - (\partial_\theta f)^2 \right] - \frac{q}{2} \left( \eta \partial_\eta \nu - \frac{\cos \theta}{\sin \theta} \partial_\theta \nu \right) - \frac{q^2}{4} \sin^2 \theta e^{-2f} \left[ g(\partial_\psi \omega)^2 - (\partial_\theta \omega)^2 \right] + \frac{\eta_0^2}{4}.
\]

(25)

In the pure Einstein case and for vanishing gravitational scalar field \( \varphi \), the constant \( D \) is simply given by the phantom scalar charge \( Q_\varphi \), as seen by substituting Eq. (4) into the right hand side (rhs) of the Einstein equation. Since the scalar fields do not enter the PDSs for \( f, \omega \) and \( \nu \), the rhs of Eq. (24) must retain this constant also in the presence of both scalar fields. Consequently,

\[
-\frac{D^2}{q^2} = \partial_\varphi \varphi = -A^{-2} Q_\varphi^2/q^2.
\]

(26)

This leads to the first order ODE for the gravitational scalar field \( \varphi \)

\[
\partial_\eta \varphi = \pm \frac{1}{q} \sqrt{A^{-2} Q_\varphi^2 - D^2}.
\]

(27)

We note that any solution of Eq. (24) is also a solution of the second order equation (15).

From Eq. (24) we can read off the scalar charge of the gravitational scalar field, \( Q_\varphi = \pm \sqrt{Q_\varphi^2 - D^2} \). Hence, for any wormhole spacetime the two scalar charges are related by

\[
Q_\varphi^2 - Q_\psi^2 = D^2,
\]

(28)

where the quantity \( D \) depends on the mass and angular momentum of the spacetime.
D. Mass and Angular Momentum

Let us next address the mass and the angular momentum of the wormholes, which should be obtained in the physical Jordan frame. In general, STT give rise to different types of mass, such as the gravitational mass $M_K$, the tensor mass $M_T$ or the Schwarzschild mass $M_S$ (see e.g. [75, 77]). The tensor mass simply corresponds to the ADM mass in the Einstein frame, $M_T = M_E$. It has appealing properties, such as being positive definite, or exhibiting a monotonic decrease in the emission of gravitational waves [75, 77]. In the following we consider the mass and angular momentum first in the asymptotic region $\Sigma_+$ and then in $\Sigma_-$. 

1. Asymptotic region $\Sigma_+$

Let us first recall the mass $M_{E+}$ and the angular momentum $J_+$ in the Einstein frame. They are encoded in the asymptotic behavior of the metric functions $f(\eta)$ and $\omega(\eta)$ in the asymptotic region $\Sigma_+$. Therefore we can read off $M_{E+}$ and $J_+$ in the Einstein frame directly,

$$f_{\eta \to +\infty} \sim \frac{2M_{E+}}{\eta}, \quad \omega_{\eta \to +\infty} \sim \frac{2J_+}{\eta^3}. \quad (29)$$

Let us now turn to the Jordan frame and find the relations between the mass and the angular momentum in the two frames. In order to express the masses $M_K$ and $M_S$ in the Jordan frame by the mass and the scalar charge in the Einstein frame we consider the asymptotic behaviour of the metric in the Jordan frame as $\eta \to \infty$,

$$-\tilde{g}_{tt} = 1 - \frac{2M_{K+}}{\eta} + O(\eta^{-2}) = A(\varphi)^2 \left( 1 - \frac{2M_{E+}}{\eta} \right) + O(\eta^{-2})$$

$$= 1 - \frac{2}{\eta} \left( M_{E+} + \frac{dA}{d\varphi} \big|_{\varphi=0} Q\varphi \right) + O(\eta^{-2}), \quad (30)$$

$$\tilde{g}_{\eta\eta} = 1 + \frac{2M_{S+}}{\eta} + O(\eta^{-2}) = A(\varphi)^2 \left( 1 + \frac{2M_{E+}}{\eta} \right) + O(\eta^{-2})$$

$$= 1 + \frac{2}{\eta} \left( M_{E+} - \frac{dA}{d\varphi} \big|_{\varphi=0} Q\varphi \right) + O(\eta^{-2}), \quad (31)$$

where we used the expansions

$$A(\varphi)^2 = 1 + 2 \frac{dA}{d\varphi} \big|_{\varphi=0} \varphi + O(\varphi^2) = 1 - 2 \frac{dA}{d\varphi} \big|_{\varphi=0} Q\varphi + O(\eta^{-2}) \quad (32)$$

with $\varphi = -Q_\varphi/\eta + O(\eta^{-2})$. Hence we find for the gravitational mass $M_{K+}$ and the Schwarzschild mass $M_{S+}$

$$M_{K+} = M_{E+} + \frac{d\ln A}{d\varphi} \big|_{\varphi=0} Q\varphi, \quad (33)$$

$$M_{S+} = M_{E+} - \frac{d\ln A}{d\varphi} \big|_{\varphi=0} Q\varphi, \quad (34)$$

respectively, where we used $\frac{dA}{d\varphi} \big|_{\varphi=0} = \frac{d\ln A}{d\varphi} \big|_{\varphi=0}$ (since $A(0) = 1$) for convenience. We read off the simple relations

$$M_{K+} + M_{S+} = 2M_{E+}, \quad M_{K+} - M_{S+} = 2 \frac{d\ln A}{d\varphi} \big|_{\varphi=0} Q\varphi. \quad (35)$$

We observe that the gravitational mass, the Schwarzschild mass and the tensor mass coincide in the Jordan frame (and also coincide with the ADM mass in the Einstein frame), provided $\frac{dA}{d\varphi} \big|_{\varphi=0} = 0$.

The angular momentum, as read off at plus infinity in the Einstein frame agrees with the angular momentum in the Jordan frame [22],

$$J_{E+} = J_{J+} = J_+, \quad \frac{g_{\phi\phi}}{g_{\phi\phi}} = \frac{\tilde{g}_{\phi\phi}}{\tilde{g}_{\phi\phi}} \rightarrow \frac{2J_+}{\eta^3}. \quad (36)$$
2. Asymptotic region $\Sigma_-$

In the next step we consider the mass and the angular momentum in the asymptotic region $\Sigma_-$, i.e. $\eta \to -\infty$. In this region the expansion in the Einstein frame reads

$$f \to \gamma + \frac{2M_\Sigma}{\eta} + \frac{2J_\Sigma}{\eta^2}.$$  (37)

Here, to identify the mass and the angular momentum $M_\Sigma$ and $J_\Sigma$ in the Einstein frame, a coordinate transformation has to be performed to obtain an asymptotically flat spacetime in this limit. This is achieved by the transformation

$$\tilde{t} = e^{\gamma/2}t, \quad \tilde{\eta} = e^{-\gamma/2}\eta, \quad \tilde{\phi} = \phi - \omega_{-\infty}t,$$  (38)

leading to $\bar{M}_\Sigma$ and $\bar{J}_\Sigma$ in terms of the quantities $M_\Sigma$ and $J_\Sigma$,

$$\bar{J}_\Sigma = J_\Sigma e^{-2\gamma}, \quad \bar{M}_\Sigma = M_\Sigma e^{-\gamma/2}.$$  (39)

In order to relate the mass and angular momentum in the Jordan frame to the corresponding quantities in the Einstein frame we have to take into account that the gravitational scalar field assumes a finite value for $\eta \to -\infty$. Let us define the asymptotic quantities $\varphi_- = \varphi(\eta \to -\infty)$ and $A_- = A(\varphi_-)$. The coordinate transformation which yields an asymptotically flat metric is now given by

$$\tilde{t} = e^{\gamma/2}A_-t, \quad \tilde{\eta} = e^{-\gamma/2}A_-\eta, \quad \tilde{\phi} = \phi - \omega_{-\infty}t.$$  (40)

In these coordinates

$$\tilde{g}_{tt} = \frac{A_-^2 e^{\gamma - \gamma}}{\eta \to -\infty} \left( 1 + \frac{1}{2} \frac{dA_-}{d\varphi_-} (\varphi_- \varphi_-) \right) \left( 1 + \frac{2M_-}{\eta} \right) + O(\eta^{-2})$$

$$= \left( 1 + \frac{2dA_- Q_{\varphi_-} e^{-\gamma/2}}{\tilde{\eta}} \right) \left( 1 + \frac{2M_- e^{-\gamma/2} A_-}{\tilde{\eta}} \right) + O(\eta^{-2})$$

$$= 1 + \frac{2}{\eta} A_- \left( M_- e^{-\gamma/2} + \left[ \frac{dlnA_-}{d\varphi_-} \right] Q_{\varphi_-} e^{-\gamma/2} \right) + O(\eta^{-2}),$$

leading to

$$M_{K_-} = A_- \left( M_- e^{-\gamma/2} + \left[ \frac{dlnA_-}{d\varphi_-} \right] Q_{\varphi_-} e^{-\gamma/2} \right) = A_- \left( \bar{M}_{\Sigma} + \left[ \frac{dlnA_-}{d\varphi_-} \right] \bar{Q}_{\varphi_-} \right),$$  (44)

where we defined $\bar{Q}_{\varphi_-} = e^{-\gamma/2} Q_{\varphi_-}$ with $Q_{\varphi_-} = -(\eta^2 \partial_\eta \varphi_-)_{-\infty}$. Similarly we find

$$M_{S_-} = A_- \left( M_- e^{-\gamma/2} - \left[ \frac{dlnA_-}{d\varphi_-} \right] Q_{\varphi_-} e^{-\gamma/2} \right) = A_- \left( \bar{M}_{\Sigma} - \left[ \frac{dlnA_-}{d\varphi_-} \right] \bar{Q}_{\varphi_-} \right).$$  (45)

We read off the simple relations

$$M_{K_-} + M_{S_-} = 2A_- \bar{M}_{\Sigma}, \quad M_{K_-} - M_{S_-} = 2A_- \left. \frac{dlnA_-}{d\varphi_-} \right|_{\varphi_-} \bar{Q}_{\varphi_-},$$  (46)

which are analogous to the relations Eq. (35) in the asymptotic region $\Sigma_+$. The angular momentum can be obtained from the asymptotic behaviour of $\bar{\omega}$,

$$\bar{\omega} = -\frac{\bar{g}_{t\varphi}}{\bar{g}_{\varphi\varphi}} = (\omega - \omega_{-\infty}) \frac{e^{-\gamma/2}}{A_-} \to \frac{2J_{\Sigma}}{\eta^2},$$  (47)

which yields

$$J_{\Sigma} = J_{E} e^{-2\gamma} A_-^2 = \bar{J}_{\Sigma} A_-^2.$$  (48)
Let us reintroduce the Newton constant $G$. In the asymptotic region $\eta \to -\infty$ the effective Newton constant is $G_{\text{eff}} = G_* / F(\Phi_{-\infty}) = G_* A_*^2$. This leads to the simple relation

$$G_{\text{eff}} J_{J-} = G_J J_{E-}.$$  

(49)

In [39, 40] several relations between the global charges in the Einstein frame have been found, such as

$$J_{E+} = e^{-2\gamma} J_{E-} = J_{E-}.$$  

(50)

In the Jordan frame these relations then become

$$G_{\text{eff}} J_{J-} = G J_{E-} = G J_{J+}.$$  

(51)

Note that the simple relation between masses and angular momentum in the Einstein frame [39, 40],

$$M_{E+} + M_{E-} = 2\omega_{-\infty} e^{-2\gamma} J_{E-} = 2\omega_{-\infty} J_{E+}.$$  

(52)

has no simple analog in the Jordan frame for the gravitational mass or for the Schwarzschild mass.

### E. Quadrupole Moment

Let us now turn the derivation of the quadrupole moment $Q$ in both frames in both asymptotic regions. For these calculations we employ the definition of the quadrupole moment as given in [80].

#### 1. Asymptotic region $\Sigma_+$

The quadrupole moment for rotating wormholes in the Einstein frame in the asymptotic region $\Sigma_+$ was derived in [40],

$$Q_{E+} = -f_3 + M_{E+} \eta_0^2 + \frac{M_{E+} (M_{E+}^2 - D^2)}{3},$$  

(53)

where $f_3$ is a coefficient appearing in the third order terms of the expansion of $f$ in $1/\eta$.

Let us now calculate the quadrupole moment in the Jordan frame, repeating the suitably modified steps in [40]. To this end we consider a time-like Killing vector field $K$ on the space-time manifold with metric $\tilde{g}$, and $\tilde{\lambda}$ is the squared norm of $K$. We define the metric $\tilde{h}$ on a 3-dimensional space by the projection

$$\tilde{h} = -\tilde{\lambda} \tilde{g} + \tilde{K} \otimes \tilde{K},$$  

(54)

where the function $\tilde{\lambda}$ and the 1-form $\tilde{K}$ in the Jordan frame are related to $\lambda$ and $K$ in the Einstein frame via

$$\tilde{\lambda} = A^2 \lambda, \quad \tilde{K} = A^2 K,$$  

(55)

respectively. Consequently, the projected metric in the Jordan frame is related to the projected metric in the Einstein frame by

$$\tilde{h} = A^4 h,$$  

(56)

with

$$h = \left(1 + \frac{r_0^2}{r^2}\right)^2 \left[e^\nu (d\rho^2 + dz^2) + \rho^2 d\varphi^2\right].$$  

(57)

where we have introduced quasi-isotropic coordinates, and $r = \sqrt{\rho^2 + z^2}$. Here we have taken into account that the function $\omega$ does not contribute to the quadrupole moment. We note that the function $A$ behaves like

$$A = 1 + \frac{a_1}{r} + \frac{a_2}{r^2} + \cdots$$  

(58)

in the asymptotic region $\Sigma_+$.

We recall that a 3-dimensional space $(M, h)$ is called asymptotically flat if it can be conformally mapped to a manifold $(\tilde{M}, \tilde{h})$ with the properties
(i) $\hat{\mathcal{M}} = \mathcal{M} \cup \Lambda$ where $\Lambda \in \hat{\mathcal{M}}$

(ii) $\Omega|_\Lambda = \hat{\nabla}_i \Omega|_\Lambda = 0$ and $\hat{\nabla}_i \hat{\nabla}_j \Omega|_\Lambda = 2 \hat{h}_{ij}|_\Lambda$, where $\hat{h}_{ij} = \Omega^2 h_{ij}$.

In [80] the complex multipole tensors are defined recursively,

\[
\hat{P}^{(0)} = \hat{\Phi},
\hat{P}^{(1)} = \partial_\phi \hat{\Phi},
\hat{P}^{(n+1)}_{i_1 \cdots i_{n+1}} = \mathcal{C} \left[ \hat{\nabla}_{i_1} \cdots \hat{\nabla}_{i_{n+1}} \hat{P}^{(n)}_{i_1 \cdots i_n} - \frac{1}{2} n (2n - 1) \hat{R}_{i_1 i_2} \hat{P}^{(n)}_{i_3 \cdots i_{n+1}} \right].
\]

(59)

Here $\mathcal{C}$ denotes the symmetric trace-free part, $\hat{R}_{ij}$ is the Ricci tensor, $\hat{\nabla}_i$ is the covariant derivative on $(\hat{\mathcal{M}}, \hat{h})$, and $\Phi = \Omega^{-1/2} \hat{\Phi}$, where $\hat{\Phi} = (\hat{\lambda} - 1/\hat{\lambda})/4$ is the complex mass potential. For $n = 1$ we find for the complex quadrupole

\[
\hat{P}^{(2)}_{ij} = \mathcal{C} \left[ \hat{\nabla}_j \hat{\nabla}_i \Phi - \frac{1}{2} \hat{R}_{ij} \Phi \right].
\]

(60)

Next we consider the coordinate transformation

\[
\rho' = \frac{\rho}{\rho^2 + z^2}, \quad z' = \frac{z}{\rho^2 + z^2},
\]

(61)

which leads to

\[
\hat{h} = \frac{1}{r^4} \mathcal{A}^4 \left( 1 + \frac{\rho'^2 + z'^2}{\rho'^2} \right)^2 \left[ e^{\nu} \left( d\rho'^2 + dz'^2 \right) + \rho'^2 d\phi^2 \right]
\]

(62)

with $r'^2 = \rho'^2 + z'^2$.

An obvious choice for the conformal factor $\Omega$ seems to be

\[
\Omega = r'^2 \mathcal{A}^{-2} \left( 1 + \frac{\rho'^2 + z'^2}{\rho'^2} \right)^{-1}.
\]

(63)

However, this choice would introduce non-analytic terms in the expressions and even a divergent quadrupole moment. It was noted in [81] and [82] that the asymptotic flatness condition does not uniquely determine the conformal factor. In fact the freedom of the choice for the conformal factor is related to the choice of the origin in $\hat{\mathcal{M}}$. The preferred choice is the centre of mass, where the dipole moment $\hat{P}^{(1)}_i (0)$ vanishes.

Therefore we consider the conformal factor of the form

\[
\Omega' = \Omega (1 + \sigma_1 r' + \sigma_2 r'^2 + \cdots)^2
\]

(64)

and determine the constants $\sigma_1, \sigma_2, \cdots$, such that $\hat{P}^{(1)}_i$ and $\hat{P}^{(1)}_z$ vanish at $r' = 0$. This yields

\[
\sigma_1 = - \frac{1}{2} \frac{a_1^2 - 2 M_{E+} a_1 + 2 a_2}{M_{E+} - a_1},
\]

(65)

with $\sigma_2$ arbitrary. This choice for the conformal factor also leads to a finite quadrupole moment. Using the expansions of the metric functions $f$ and $\nu$ we then find for the mass and the quadrupole moment in the Jordan frame

\[
\mu_{J+} = - \hat{P}^{(0)} = M_{E+} - a_1,
\]

(66)

\[
Q_{J+} = \frac{1}{2} \hat{P}^{(2)}_{zz} (0) = - f_3 + \frac{2}{3} M_{E+} \eta_0^2 + \frac{1}{3} M_{E+} c_2 - \frac{1}{3} a_1 c_2,
\]

(67)

respectively, with $c_2 = M_{E+}^2 + \eta_0^2 - D^2$. The coefficient $a_1$ may we written conveniently as

\[
a_1 = - \frac{dA}{d\phi} \bigg|_{\phi=0} Q_{\phi} = - \frac{d\ln A}{d\phi} \bigg|_{\phi=0} Q_{\phi},
\]

(68)

since $A(0) = 1$. We observe that the zeroth multipole moment agrees with the gravitational mass, $\mu_{J+} = M_{K+}$. 
Comparison with the quadrupole moment in the Einstein frame yields
\[ Q_{J+} = Q_{E+} + \frac{1}{3} \frac{d \ln A}{d \varphi} \bigg|_{\varphi=0} Q_{\varphi} c_2. \tag{69} \]

Consequently, for STT with \([dA/d\varphi]_{\varphi=0} = 0\) the masses and quadrupole moments in the asymptotic region \(\Sigma_+\) coincide in the Jordan and the Einstein frame. Finally, we give the quadrupole moment in the Jordan frame in terms of the gravitational mass and the charges
\[ Q_{J+} = -f_3 + \mu_{J+} \eta_0^2 + \frac{1}{3} \mu_{J+} \left( \mu_{J+}^2 - Q_{J+}^2 + Q_{J-}^2 \right) - \frac{\alpha_{\infty}}{3} \frac{Q_{\varphi}}{\psi} \left( 2 \mu_{J+}^3 + 2 \mu_{J+}^2 - \mu_{J+} \alpha_{\infty} Q_{\varphi} \right), \tag{70} \]
with \(\alpha_{\infty} = \frac{d \ln A}{d \varphi} \bigg|_{\varphi=0}\).

2. Asymptotic region \(\Sigma_-\)

Let us now turn to the asymptotic region \(\Sigma_-\). Here we first apply the coordinate transformation Eq. (40) to obtain the line element in the Jordan frame, such that it tends asymptotically to the Minkowski form,
\[ ds^2 = \tilde{A}^2 \left[ -e^f dl^2 + e^{-f} \left( e^{\nu} (d\eta^2 + q d\theta^2) + \bar{q} \sin^2 \theta (d\bar{\phi} - \omega d\bar{\phi})^2 \right) \right], \tag{71} \]
where we defined
\[ \tilde{A} = \frac{A}{A_-}, \quad \tilde{f} = f - \gamma, \quad \bar{q} = \eta^2 + \eta_0^2, \quad \tilde{\eta} = e^{-\gamma/2} A_- \eta, \quad \tilde{\eta}_0 = e^{-\gamma/2} A_- \eta_0, \quad \tilde{\omega} = e^{-\gamma/2} \frac{\bar{q}}{A_-} (\omega - \omega_{-\infty}). \tag{72} \]
Now we can proceed as before. Note however, since the limit \(\tilde{\eta} \to -\infty\) corresponds to the limit \(r \to 0\) in isotropic coordinates we do not introduce the coordinates \(\rho', z'\). In the region \(\Sigma_-\) the coordinates \(\tilde{\eta}\) and \(r\) are related by
\[ \tilde{\eta} = \frac{1}{r_0} \left( \frac{r}{r_0} - \frac{r_0}{r} \right) = -\frac{1}{r} \left( 1 - \frac{r^2}{r_0^2} \right), \tag{73} \]
and \(\tilde{\eta}_0 = 2/r_0\).

Using the expansions of the metric functions \(\tilde{f}\) and \(\nu\) in the limit \(\tilde{\eta} \to -\infty\) the outcome for the mass and the quadrupole moment in the Jordan frame is of the form
\[ \mu_{J-} = -\tilde{\mu}(0) = M_{E-} - \tilde{a}_1, \tag{74} \]
\[ Q_{J-} = \frac{1}{2} \tilde{\mu}(2)(0) = -\tilde{f}_3 + \frac{2}{3} M_{E-} \tilde{\eta}_0^2 + \frac{1}{3} M_{E-} \tilde{c}_2 - \frac{1}{3} \tilde{a}_1 \tilde{c}_2, \tag{75} \]
where the constants can be read off from the asymptotic expansion of the metric functions and the gravitational scalar field in terms of \(\tilde{\eta}^{-1}\),
\[ \tilde{a}_1 = e^{-\gamma/2} A_- a_1, \quad \tilde{f}_3 = e^{-3\gamma/2} A_- f_3, \quad \tilde{c}_2 = e^{-\gamma} A_- c_2, \tag{76} \]
where \(a_1, f_3\) and \(c_2\) are the coefficients in the expansion with respect to \(\eta^{-1}\). The coefficient \(a_1\) is related to the charge of the gravitational scalar field,
\[ a_1 = -\frac{d \ln A}{d \varphi} \bigg|_{\varphi=0} Q_{\varphi-}. \tag{77} \]
Consequently, \(\mu_{J-} = M_{K-}\) and
\[ Q_{J-} = e^{-3\gamma/2} A_- \left\{ -\tilde{f}_3 + \frac{2}{3} M_{E-} \tilde{\eta}_0^2 + \frac{1}{3} M_{E-} c_2 + \frac{1}{3} \frac{d \ln A}{d \varphi} \bigg|_{\varphi=0} Q_{\varphi-} c_2 \right\}. \tag{78} \]
In the Einstein frame the quadrupole moment reads
\[ Q_{E-} = e^{-3\gamma/2} \left\{ -f_3 + \frac{2}{3} M_{E-} \eta_0^2 + \frac{1}{3} M_{E-} c_2 \right\}. \tag{79} \]
Thus the quadrupole moments in the Jordan frame and the Einstein frame are related by
\[ Q_{J-} = A^3 \left\{ Q_{E-} + \frac{1}{3} \left. \frac{d \ln A}{d \varphi} \right|_{\varphi=\bar{\varphi}} \varphi - \bar{\varphi} c_2 \right\}, \tag{80} \]
with \( \bar{c}_2 = e^{-\gamma}c_2 \).

F. Geometric Properties

Let us next consider the geometric properties of the wormhole solutions in scalar-tensor theory. Clearly, the geometrical properties of the spacetime depend on the frame. We first address the equatorial (or circumferential) radius \( R_e \). Because of the rotation, the throat deforms and its circumference is largest in the equatorial plane. Therefore a study of \( R_e \) reveals the location of the throat. In the Jordan frame \( R_e \) is given by
\[ R_e = \sqrt{\eta^2 + \eta_0^2 \left[ A(\varphi) e^{-f/2} \right]} \theta=\pi/2 = \eta_0 \cos(x), \tag{81} \]
where we defined \( \eta = \eta_0 \tan(x) \). Consequently, the conditions that the throat is located at \( x_t \) are given by
\[ \frac{d}{dx_t} R_e(x_t) = 0, \quad \text{and} \quad \frac{d^2}{dx_t^2} R_e(x_t) > 0. \tag{82} \]

G. Violation of the Null Energy Condition

Let us finally address the violation of the Null Energy Condition (NEC) in both frames. In the Einstein frame, we consider the quantity
\[ \Xi = R_{\mu\nu} k^\mu k^\nu, \tag{83} \]
with null vector \[ k^\mu = (e^{-f/2}, e^{f/2-\nu/2}, 0, \omega e^{-f/2}) \].

Taking into account the Einstein equations and the phantom field equation we obtain
\[ \Xi = -2D^2 e^{f-\nu}/q^2. \tag{85} \]

Since \( \Xi \) is non-positive, the NEC is violated everywhere.

In the Jordan frame, we consider the Einstein equations
\[ \tilde{G}_{\mu\nu} = \tilde{T}_{\mu\nu}, \tag{86} \]
where \( \tilde{G}_{\mu\nu} \) is the Einstein tensor computed with the metric \( \tilde{g}_{\mu\nu} \), and \( \tilde{T}_{\mu\nu} \) is defined by the left hand side of Eq. (86). Analogously to Eq. (83) we define
\[ \tilde{\Xi} = \tilde{R}_{\mu\nu} \tilde{k}^\mu \tilde{k}^\nu = \tilde{T}_{\mu\nu} \tilde{k}^\mu \tilde{k}^\nu, \tag{87} \]
with null vector \( \tilde{k}^\mu = k^\mu \). This yields
\[ \tilde{\Xi} = -e^{f-\nu} \left( 2 \frac{D^2}{q^2} - \frac{\partial \nu}{\partial \eta} A \frac{\partial \eta A}{A} + 2 \frac{\partial \nu}{\partial \eta} A \left( 2 \left( \frac{\partial \nu}{\partial A} \right)^2 \right) \right). \tag{88} \]

In order to demonstrate the violation of the NEC we consider the densities
\[ \Xi' = \sqrt{-g} \Xi = -2D^2 \frac{\sin \theta}{q}, \tag{89} \]
\[ \tilde{\Xi}' = \sqrt{-g} \tilde{\Xi} = -4 \frac{\sin \theta}{q} \left( 2 \frac{D^2}{q^2} - q \frac{\partial \nu}{\partial \eta} A \frac{\partial \eta A}{A} + 2 q^2 \frac{\partial \nu}{\partial \eta} A \left( 2 \frac{\partial \nu}{\partial A} \right)^2 \right). \tag{90} \]

In the Einstein frame \( \Xi' \) does not involve the metric functions (except for the auxiliary function \( q \)). Thus the violation of the NEC is the same for all solutions with the same value of \( D \). In constrast, in the Jordan frame \( \tilde{\Xi}' \) depends on the metric function \( \nu \) and on the gravitational scalar field \( \varphi \).
III. WORMHOLES IN THREE EXAMPLES OF SCALAR TENSOR THEORIES

In the following we will consider three examples of STT, which we specify by their respective coupling function $A(\varphi)$:

$$A_1(\varphi) = e^{\beta\varphi^2/2} \quad (\text{STT-1}), \quad A_2(\varphi) = e^{\alpha\varphi} \quad (\text{STT-2}), \quad A_3(\varphi) = \cosh(\varphi/\sqrt{3}) \quad (\text{STT-3}) .$$  

(91)

The first example has been employed by Damour and Esposito-Farese, when demonstrating the scalarization of neutron stars [12, 13]. The second example represents Brans-Dicke theory [4], while the third example appears to be new and has been inspired by [74]. We note, that the rotating wormhole solutions of GR [39, 40] are also solutions of the STT equations in the first and the third example. However, this is not the case in the second example.

In all cases the boundary condition for the gravitational scalar field $\varphi$ in the asymptotic region $\Sigma_+$ is chosen such that it vanishes there, $\varphi(+\infty) = 0$. Therefore in all examples the coupling function tends to $A(0) = 1$ in $\Sigma_+$. In contrast, in the asymptotic region $\Sigma_-$ the gravitational scalar field $\varphi$ assumes a finite value $\varphi_-$, yielding

$$\frac{d\ln A_1}{d\varphi} \bigg|_{\varphi_-} = \beta \varphi_-, \quad \frac{d\ln A_2}{d\varphi} \bigg|_{\varphi_-} = \alpha, \quad \frac{d\ln A_3}{d\varphi} \bigg|_{\varphi_-} = \sqrt{3} \tanh \left( \frac{\varphi_-}{\sqrt{3}} \right) .$$  

(92)

A. Model STT-1

We first consider the model STT-1 employed by Damour and Esposito-Farese [12] with the non-minimal coupling

$$A = e^{\frac{\beta}{2}\varphi^2} .$$  

(93)

With this STT they discovered the phenomenon of spontaneous scalarization in neutron stars, when choosing the coupling parameter $\beta$ below a critical negative value [12]. On the other hand, observations on the binary pulsar PSR J1738+0333 impose a lower bound on $\beta$, $\beta > -4.5$ [83], leaving only a small interval of $\beta$ to achieve scalarization for a massless gravitational scalar field in neutron stars [22, 83], but being much less restrictive in the massive case [84, 85]. We note that for boson stars and hairy black holes the upper bound on $\beta$ is similar to the one for neutron stars [26–29].

In neutron stars and boson stars the phenomenon of spontaneous scalarization is restricted to negative values of the coupling $\beta$. Here we show, that scalarization of wormholes arises for arbitrary (finite) values of $\beta$. This includes, in particular, positive values of $\beta$. For positive $\beta$ the solutions develop an oscillating gravitational scalar field, which can give rise to wormholes with a multitude of throats and equators in the Jordan frame. In the following subsections we present the solutions for general $\beta$, discuss the domain of existence for negative $\beta$ and subsequently for positive $\beta$, and then analyze the physical properties of these solutions.

1. Solutions

In STT-1 Eq. (15) for the gravitational scalar field becomes

$$\partial_\eta (q\partial_\eta \varphi) = -\beta \frac{\varphi^2}{q} e^{-\beta \varphi^2} Q_\psi^2 .$$  

(94)

Clearly, $\varphi = 0$ is always a solution. Hence any wormhole solution of Einstein gravity is also a solution of STT-1, although with a trivial gravitational scalar field.

However, there are in addition solutions with a non-trivial gravitational scalar field, i.e., scalarized wormhole solutions, which can be obtained from Eq. (27), which now reads

$$\partial_\psi \varphi = \pm \frac{Q_\psi}{\eta_0} e^{\frac{\varphi^2}{2}} \sqrt{1 - a_2 e^{\varphi^2}} ,$$  

(95)

with $a_2 = D^2/Q_\psi^2$. These solutions are obtained numerically.
2. Domain of existence

We start our discussion of the domain of existence with the case of negative $\beta$. Here it is convenient to introduce the scaled functions and charges

\[ \hat{\varphi} = \sqrt{-\beta} \varphi, \quad \hat{Q}_\psi = \sqrt{-\beta} \frac{Q_\psi}{\eta_0}, \quad \hat{D} = \sqrt{-\beta} \frac{D}{\eta_0}. \]  

(96)

In terms of these quantities Eq. (95) reads

\[ \partial_x \hat{\varphi} = \pm \hat{Q}_\psi e^{\frac{1}{2} \hat{\varphi}^2} \sqrt{1 - a^2 e^{-\hat{\varphi}^2}}. \]  

(97)

We note from the ODE Eq. (97) that $\partial_x \hat{\varphi} \neq 0$ since $a^2 e^{-\hat{\varphi}^2} < 1$. Therefore $\hat{\varphi}(x)$ is a monotonic function. Consequently, we can rewrite this equation in integral form

\[ \int_0^{\hat{\varphi}(x)} e^{-\frac{1}{2} \hat{\varphi}^2} \left\{ 1 - a^2 e^{-\hat{\varphi}^2} \right\}^{-\frac{1}{2}} d\hat{\varphi}' = \pm \hat{Q}_\psi \left( x - \frac{\pi}{2} \right). \]  

(98)

To obtain a regular solution we demand that $\hat{\varphi}(x)$ is finite on the interval $-\infty \leq \eta \leq \infty$, corresponding to $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Thus in the limiting case, $\hat{\varphi}(x)$ approaches zero.

An analytical expression for the critical phantom scalar charge $\hat{Q}_\psi^{\text{cr}}$ is given in the Appendix.

The corresponding bounds for the gravitational scalar charge $Q_\psi^C = \sqrt{-\beta} Q_\psi^C$ can be found from Eq. (28), i.e., $(\hat{Q}_\psi^C)^2 = (\hat{Q}_\psi^{\text{cr}})^2 - D^2$. The bounds $Q_\psi^C$ and $\hat{Q}_\psi^{\text{cr}}$ are shown in Fig. 1(a) as functions of $\hat{D}$. We observe that with increasing $|\hat{D}|$ the phantom scalar charge $|\hat{Q}_\psi^{\text{cr}}|$ tends rapidly to $|\hat{D}|$, while the gravitational scalar charge $|Q_\psi^C|$ tends to zero.

For $\hat{D} = 0$, the scalar charges are equal. Here the bound is given by $|\hat{Q}_\psi^{\text{cr}}(0)| = |\hat{Q}_\psi^{\text{cr}}(\eta)| = 1/\sqrt{2\pi}$. However, the case $\hat{D} = 0$ is special, since the contributions of the gravitational scalar field and the phantom scalar field in the stress-energy tensor cancel exactly. Hence the spacetime metric corresponds to the metric of a Kerr black hole. In fact, the only acceptable $\hat{D} = 0$ solution carries no scalar fields at all, i.e., $Q_\psi^C = \hat{Q}_\psi^{\text{cr}} = 0$, since otherwise the scalar fields would diverge at the horizon.

In Fig. 1(b) we exhibit the domain of existence in the $\hat{Q}_\psi^C - \hat{Q}_\psi$ plane. For each value of $\hat{Q}_\psi^C$ we determine the bound $\hat{Q}_\psi^{\text{cr}}$ from Eq. (150) and show $\hat{Q}_\psi^{\text{cr}}$ as a function of $\hat{Q}_\psi$ in the interval $[0, \hat{Q}_\psi^{\text{cr}}]$. In the figure the values of $\hat{D}$ can be read off at the intersections of the lilac curves with the abscissa.

In fact, the abscissa corresponds to the family of non-scalarized wormhole solutions. Thus we see that for each value of $\hat{D}$ a branch of scalarized wormhole solutions emerges from the non-scalarized ones. For a given $\hat{D}$, all wormhole solutions possess the same mass and the same angular momentum. The branches end, when the corresponding critical value of $\hat{Q}_\psi$ is reached.

We exhibit in Fig. 2 examples of solutions with negative $\beta$. Here the scaled functions $\hat{\varphi}$ and $\hat{\psi}$ are shown versus the scaled coordinate $x/\pi$ for $\hat{D} = 0.5$ (a), $\hat{D} = 1$ (b) and $\hat{D} = 1.5$ (c), and several values of $\hat{Q}_\psi$ ranging from $\hat{D}$ up to a value close to the critical value $\hat{Q}_\psi^{\text{cr}}$, Eq. (99).

We note that the solution for the phantom field is the same in the Jordan frame and in the Einstein frame.

Let us next consider the case of positive $\beta$. Here we rewrite the first order ODE as

\[ \partial_x \varphi = \pm \frac{Q_\psi}{\eta_0} \sqrt{e^{-\beta \varphi^2} - a^2}. \]  

(100)

From the boundary condition $\varphi(\pi/2) = 0$ we find $\partial_x \varphi(\pi/2) = \sqrt{1 - a^2 Q_\psi/\eta_0} = Q_\psi/\eta_0$. Let us suppose $Q_\psi > 0$, then $\varphi$ decreases with decreasing $x$. If $\varphi$ assumes the value $-\varphi_{\text{ex}} = -\sqrt{-2 \ln(a)/\beta}$ at some point $x_{\text{ex}}$, then $\partial_x \varphi(x_{\text{ex}}) = 0$ corresponding to a maximum. To continue to $x \leq x_{\text{ex}}$ smoothly we have to choose the lower sign in Eq. (100). Thus $\varphi$ increases with further decreasing $x$, until it reaches the value $\varphi_{\text{ex}}$, which is a maximum, and so on, until the boundary $x = -\pi/2$ is reached. In this way we have established a bound for the solutions, $-\varphi_{\text{ex}} \leq \varphi(x) \leq \varphi_{\text{ex}}$. Consequently, solutions exist for all (finite) $Q_\psi$, $-1 < a < 1$ and $\beta > 0$. Note that also a bound for the derivative $\partial_x \varphi$ exists, $-|Q_\psi/\eta_0| \leq \partial_x \varphi \leq |Q_\psi/\eta_0|$. 

3. Physical properties

We now consider the mass, the angular momentum and the quadrupole moment in STT-1. As discussed in Section 2.5 the angular momentum is the same in both frames. In the asymptotic region $\Sigma_+$ the coupling function $A$ satisfies $d\ln A/d\varphi = 0$, and thus the gravitational mass and the Schwarzschild mass in the Jordan frame are the same as the ADM mass in the Einstein frame,

$$M_{K+} = M_{S+} = M_{E+}. \quad (101)$$
In the static case there is a simple relation between the mass in the Einstein frame and the scalar charges,

\[ M_{E+}^2 = Q_\psi^2 - Q_\varphi^2 - \eta_0^2 . \]  

(102)

Consequently,

\[ M_{K+}^2 = M_{S+}^2 = Q_\psi^2 - Q_\varphi^2 - \eta_0^2 \]  

(103)

for the scalarized static wormholes in STT. Since for the non-scalarized wormholes \( Q_\varphi = 0 \), the comparison of the masses of scalarized and non-scalarized wormholes with the same phantom scalar charge yields

\[ M_{\text{scal}}^2 - M_{\text{non-scal}}^2 = -Q_\varphi^2 . \]  

(104)

Hence for a fixed phantom scalar charge the static scalarized wormholes possess less mass than the non-scalarized wormholes.

The quadrupole moments are the same in the Jordan frame and in the Einstein frame. This equivalence arises again because \( d\ln A/d\varphi = 0 \) holds in the asymptotic region \( \Sigma_+ \).

In the asymptotic region \( \Sigma_+ \) the gravitational scalar field assumes a finite value \( \varphi_- \). Consequently, the masses in the Jordan frame and in the Einstein frame differ,

\[ M_{K-} + M_{S-} = 2e^{\beta \varphi^2} M_{E-} , \quad M_{K-} - M_{S-} = 2\beta \varphi_- \sqrt{1 - a^2 e^{\beta \varphi^2}} \bar{Q}_\psi . \]  

(105)

The angular momentum in the Jordan frame differs from the angular momentum in the Einstein frame by a factor,

\[ J_{J-} = J_{E-} e^{-\beta \varphi^2} . \]  

(106)

For the quadrupole moment we find from the general expression, Eq. (100),

\[ Q_{J-} = e^{\frac{3}{2} \beta \varphi^2} Q_{E-} + \frac{1}{3} \beta \varphi_- e^{\beta \varphi^2} \sqrt{1 - a^2 e^{\beta \varphi^2}} \bar{Q}_\psi \beta_2 . \]  

(107)

Let us now consider the geometric properties of the wormholes in the Jordan frame. We begin with the location and the equatorial radius of the throat. In the Jordan frame the circumferential radius is defined in Eq. (81), and the condition for the throat coordinate is given in Eq. (82). In STT-1 they read

\[ \bar{R}_e(x) = \frac{n_0}{\cos x} e^{-(-\beta \varphi^2 + f)/2} \bigg|_{\theta=\pi/2} , \quad \left( \tan x + \beta \varphi \partial_x \varphi - \frac{1}{2} \partial_x f \right)_{x,\theta=\pi/2} = 0 . \]  

(108)

Starting again with the case of negative \( \beta \), we note that the geometry of the throat of the wormholes in the Jordan frame is almost identical to the one in the Einstein frame for this value of \( \beta \). The smallness of the deviation is demonstrated for \( \beta = -0.1 \) in Fig. 3, where we show the circumferential radius of the throat in the equatorial plane as a function of the mass. Here we set \( Q_\psi = Q^e_\psi \) and choose a sequence of rotating wormhole solutions with fixed \( D = 1 \). Also shown is the throat radius of the non-scalarized wormhole solutions. The two curves represent the limits of maximal, respectively vanishing scalarization for fixed \( D = 1 \).

Let us now turn to the case of positive \( \beta \). In Fig. 4 we illustrate the equatorial radius of the throat in the Jordan frame \( \bar{R}_e \) (top) and the scalar fields (bottom) for two sets of static scalarized wormhole solutions with varying \( \beta \). Clearly, the oscillating behavior of the gravitational scalar field translates into an oscillating behavior or the equatorial radius of the throat in the Jordan frame. The larger the value of \( \beta \) is chosen, the more oscillations manifest in \( \bar{R}_e \).

Thus the presence of a single throat (and no equator) in the Einstein frame can lead to wormholes with multiple throats and equators in the Jordan frame. We show in Fig. 5 an isometric embedding of the equatorial plane of two wormhole solutions in the Jordan frame, highlighting the oscillating behavior or the equatorial radius.

**B. Model STT-2**

As our second example we choose for the non-minimal coupling function \( A = e^{\alpha \varphi} \). Thus the model corresponds to Brans-Dicke theory, for which there are rather stringent observational bounds for its parameter \( \alpha \), \( \alpha < 4 \cdot 10^{-3} \).
1. Solutions

From Eq. (15) we find in this case for the gravitational scalar field the second order equation

\[
\partial_q(\partial_q \varphi) = -\alpha e^{-2\alpha \varphi} \frac{Q^2}{q} \iff \partial^2_x \varphi = -\alpha e^{-2\alpha \varphi} \left( \frac{Q_\psi}{\eta_0} \right)^2
\]

(109)

Thus the rhs does not vanish for \( \varphi = 0 \). Hence the solutions of General Relativity are not solutions of STT-2.
To find wormhole solutions of STT-2 we consider the first order equation
\[ \partial_x \varphi = \frac{Q_\psi}{\eta_0} e^{-\alpha \varphi} \sqrt{1 - a^2 e^{2\alpha \varphi}}, \] (110)
which is consistent with Eq. (109).
This ODE has solutions in closed form
\[ \varphi(x) = \frac{1}{\alpha} \ln \left[ \frac{1}{\cos \sigma} \cos \left( \cos \sigma \hat{Q}_\psi \left\{ \frac{\pi}{2} - x \right\} + \sigma \right) \right] \] (111)
with \( \hat{Q}_\psi = \alpha Q_\psi / \eta_0 \) and \( \cos \sigma = a \), which satisfy the condition \( \varphi(x = \pi/2) = 0 \).
For the phantom scalar field we find the first order ODE
\[ \partial_x \psi = \frac{\hat{Q}_\psi}{\alpha \cos^2 \sigma} \frac{\cos^2 \sigma}{\cos \sigma \hat{Q}_\psi \left\{ \frac{\pi}{2} - x \right\} + \sigma}. \] (112)
Integration yields
\[ \psi(x) = -\frac{\cos \sigma}{\alpha} \left[ \tan \left( \cos \sigma \hat{Q}_\psi \left\{ \frac{\pi}{2} - x \right\} + \sigma \right) - \tan \sigma \right], \] (113)
where the integration constant has been chosen such that \( \psi(x = \pi/2) = 0 \).
Let us now consider the solutions in the Jordan frame. With \( F(\Phi) = A^{-2} = e^{-2\alpha \varphi} \) and \( d\ln F/d\Phi = -2\alpha d\varphi/d\Phi \) we find from Eq. (110)
\[ \left( \frac{d(e^{-\alpha \varphi})}{d\Phi} \right)^2 = \frac{1}{2} \frac{\alpha^2}{1 - 3\alpha^2} Z(\Phi). \] (114)
This implies \( Z(\Phi) > 0 \) if \( \alpha^2 < 3 \), and \( Z(\Phi) < 0 \) if \( \alpha^2 > 3 \), respectively. In the simple case of constant \( Z(\Phi) \), \( |Z(\Phi)| = Z_0^2 \), this yields
\[ \Phi_{\pm}(x) = \pm \frac{\sqrt{2(1 - 3\alpha^2)}}{\alpha Z_0} \left( e^{-\alpha \varphi(x)} - 1 \right) \] (115)
\[ = \Phi_{\pm 0} \left( \frac{\cos \sigma}{\cos \left( \cos \sigma \hat{Q}_\psi \left\{ \frac{\pi}{2} - x \right\} + \sigma \right)} - 1 \right), \] (116)
\[ F(\Phi) = \left( 1 + \frac{\Phi_{\pm}}{\Phi_{\pm 0}} \right)^2, \] (117)
with $\Phi_{\pm 0} = \pm \frac{\sqrt{2(1-\alpha^2)}}{\alpha \xi_0}$.

The scalar charge $Q_\Phi$ can be computed from

$$Q_\Phi = \eta_0 \partial_x \Phi|_{x=\pi/2} = -\alpha \Phi_{\pm 0} \eta_0 \partial_x \varphi|_{x=\pi/2} = -\alpha \Phi_{\pm 0} Q_\varphi.$$  

\[(118)\]

2. Domain of existence

The domain of existence is determined from $|\hat{Q}_\psi| \geq |\hat{Q}_\varphi|$ and from the condition $e^{\alpha \varphi} > 0$. For convenience we write the solution as

$$e^{\alpha \varphi} = \chi = \cos \left( y \alpha D/\eta_0 + \sigma \right) \cos \sigma,$$

\[(119)\]

where $y = \pi/2 - x$. We note that $\chi$ does not change when we add any integer times $\pi$ to $\sigma$. Therefore it is sufficient to restrict to $0 \leq \sigma \leq \pi$. However, we have to exclude $\sigma = \pi/2$. The domain of existence is determined by the condition $\chi(y) > 0$ for $0 \leq y \leq \pi$. First we note that this condition implies $|\alpha D/\eta_0| \leq 1$, otherwise $y \alpha D/\eta_0$ would cover an interval larger than $\pi$ and $\chi$ would vanish at some point, no matter what value of $\sigma$ is chosen.

Let us now turn to the limits of $\sigma$. We start with $0 \leq \sigma < \pi/2$, when $\cos \sigma > 0$. In this case the domain of existence is determined by

$$-\frac{\pi}{2} - \pi \alpha D/\eta_0 < \sigma < \frac{\pi}{2} - \alpha D \pi.$$  

\[(120)\]

For $y = 0$ this condition is already satisfied. For $y = \pi$ this yields

$$-\frac{\pi}{2} - \frac{\alpha D}{\eta_0} < \sigma < \frac{\pi}{2} - \alpha D \pi.$$  

\[(121)\]

Employing the condition $0 \leq \sigma < \pi/2$ we find

$$\max \left( -\frac{\pi}{2} - \frac{\alpha D}{\eta_0}, 0 \right) < \sigma < \min \left( \frac{\pi}{2} - \frac{\alpha D}{\eta_0}, \pi \right).$$  

\[(122)\]

Similarly we find for the case $\pi/2 < \sigma \leq \pi$

$$\max \left( \frac{\pi}{2} - \frac{\alpha D}{\eta_0}, \pi \right), \frac{\pi}{2} < \sigma < \min \left( \frac{3\pi}{2} - \frac{3 \alpha D}{\eta_0}, \pi \right).$$  

\[(123)\]

The domain of existence in the $\alpha D/\eta_0 - \sigma$ plane is shown in Fig. 6(a) by the blue areas. In order to find the domain of existence for the charges we express the scaled quantities $\hat{Q}_\psi$ and $\hat{Q}_\varphi$ in terms of $\alpha D/\eta_0$ and $\sigma$

$$\hat{Q}_\psi = \alpha \frac{Q_\psi}{\eta_0} = \pm \alpha \frac{D}{\eta_0 \cos \sigma}, \quad \hat{Q}_\varphi = \alpha \frac{Q_\varphi}{\eta_0} = \pm \alpha \frac{D}{\eta_0 \tan \sigma}.$$  

\[(124)\]

The domain of existence in the $\hat{Q}_\psi - \hat{Q}_\varphi$ plane is shown in Fig. 6(b).

For the static wormholes the charges and the mass are related by

$$Q_\psi^2 - Q_\varphi^2 = \eta_0^2 + M_E^2.$$  

\[(125)\]

This leads to an additional reduction of the domain of existence,

$$Q_\psi^2 - Q_\varphi^2 \geq \eta_0^2.$$  

\[(126)\]

3. Properties

Next we consider the mass, the angular momentum and the quadrupole moment in STT-2. As discussed in Section 2.5 the angular momentum is the same in both frames. In the asymptotic region $\Sigma_+$ the coupling function $A$ satisfies
\[ d\ln A/d\varphi = \alpha \neq 0, \text{ thus the gravitational mass and the Schwarzschild mass in the Jordan frame differ from the ADM mass in the Einstein frame,} \]

\[ M_{K+} - M_{S+} = 2\alpha Q_\psi. \tag{127} \]

In STT-2 the quadrupole moments differ in the Jordan frame and in the Einstein frame,

\[ Q_{J+} = Q_{E+} + \frac{1}{3}\alpha Q_\psi e_2. \tag{128} \]

In the asymptotic region \( \Sigma_- \) the gravitational scalar field assumes a finite value \( \varphi_- \). The conformal factor and the scalar charges in \( \Sigma_- \) can be expressed in terms of \( \hat{Q}_\psi \) and \( a \) as

\[ A_\pm^2 = \frac{\cos^2 (\cos \sigma \pi \hat{Q}_\psi + \sigma)}{\cos^2 \sigma}, \tag{129} \]

\[ Q_{\psi \pm} = Q_\psi \frac{\cos^2 \sigma}{\cos^2 (\cos \sigma \pi \hat{Q}_\psi + \sigma)}, \tag{130} \]

\[ Q_{\varphi \pm} = Q_\psi \cos \sigma \tan (\cos \sigma \pi \hat{Q}_\psi + \sigma). \tag{131} \]

This yields for the masses in the Jordan frame

\[ M_{K-} + M_{S-} = \frac{2}{\cos \sigma} \cos (\cos \sigma \pi \hat{Q}_\psi + \sigma) \hat{M}_{E-}, \tag{132} \]

\[ M_{K-} - M_{S-} = -2\alpha Q_\psi e^{-\frac{2}{3}} \sin (\cos \sigma \pi \hat{Q}_\psi + \sigma), \tag{133} \]

where \( \hat{M}_{E-} \) denotes the mass in the Einstein frame.

The angular momentum in the Jordan frame differs from the angular momentum in the Einstein frame by some factor,

\[ J_{J-} = J_{E-} \frac{1}{\cos^3 \sigma} \cos^2 (\cos \sigma \pi \hat{Q}_\psi + \sigma). \tag{134} \]

For the quadrupole moment we find from the general expression, Eq. (80),

\[ Q_{J-} = \frac{1}{\cos^3 \sigma} \cos^3 (\cos \sigma \pi \hat{Q}_\psi + \sigma) \left( Q_{E-} + \frac{\alpha}{3} Q_\psi e^{-\frac{2}{3}} e_2 \right). \tag{135} \]

Considering the geometric properties of the wormholes in the Jordan frame, the circumferential radius and the condition for the throat coordinate read

\[ \hat{R}_\theta(x) = \frac{\eta_0}{\cos x} e^{a x - f/2} \bigg|_{\theta=\pi/2}, \quad \left( \tan x + \alpha \partial_\varphi x - \frac{1}{2} \partial_x f \right)_{x,\theta=\pi/2} = 0. \tag{136} \]

Let us remark, that the static wormhole solutions of Brans-Dicke theory obtained without a phantom field do not exist in the Einstein frame \([63, 65]\).
C. Model STT-3

As our third example we consider the coupling function \( A(\varphi) = \cosh(\varphi/\sqrt{3}) \). We are not aware of any previous investigations with this coupling function.

1. Solutions

From Eq. (15) we find for the gravitational scalar field the equation

\[
\partial_q (q \partial_q \varphi) = - \frac{1}{\sqrt{3}} \frac{\sinh(\varphi/\sqrt{3})}{\cosh^3(\varphi/\sqrt{3})} Q^2 \eta \Leftrightarrow \partial^2_x \varphi = - \frac{1}{\sqrt{3}} \frac{\sinh(\varphi/\sqrt{3})}{\cosh^3(\varphi/\sqrt{3})} \left( \frac{Q}{\eta_0} \right)^2,
\]

which allows for the trivial solution \( \varphi = 0 \), present in General Relativity. In order to obtain non-trivial solutions for the scalar field we turn to the first order equation,

\[
\partial_x \varphi = \frac{Q}{\eta_0} \frac{\sqrt{1 - a^2 \cos^2(\varphi/\sqrt{3})}}{\cosh(\varphi/\sqrt{3})},
\]

which is consistent with Eq. (137). This ODE has solutions

\[
\varphi(x) = \sqrt{3} \text{arsinh} \left( \frac{\sqrt{1 - a^2} \sin(\frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \})}{a} \right),
\]

which obey the condition \( \varphi(\pi/2) = 0 \).

The ODE of the phantom field becomes

\[
\partial_x \psi = \frac{Q\psi}{\eta_0} \frac{1}{1 + \frac{1 - a^2}{a^2} \sin^2( \frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \})},
\]

and has the solution

\[
\psi(x) = \sqrt{3} \text{arctan} \left( \frac{\sqrt{\frac{1 - a^2}{a^2}} \tan(\frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \})}{a} \right),
\]

which satisfies the boundary condition \( \psi(\pi/2) = 0 \).

Let us now go to the Jordan frame. With \( F(\Phi) = A(\varphi)^{-2} \) and \( Z(\Phi) = 1 \) we find from Eq. (4)

\[
F(\Phi) = 1 - \frac{\Phi^2}{6},
\]

and

\[
\Phi(\varphi) = \pm \sqrt{6} \tanh \left( \frac{\varphi}{\sqrt{3}} \right).
\]

Substitution of the solution Eq. (139) in Eq. (143) gives the gravitational scalar field in the Jordan frame

\[
\Phi(x) = \sqrt{6} \frac{\sqrt{1 - a^2} \sin(\frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \})}{\sqrt{a^2 + (1 - a^2) \sin^2( \frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \})}}.
\]

For the scalar charge in the Jordan frame we find \( Q_\Phi = \sqrt{2} Q_\varphi = \sqrt{2} \sqrt{1 - a^2} Q_\psi \). Moreover, substitution in \( F(\Phi) = 1 - \Phi^2/6 \) yields

\[
F(x) = \frac{a^2}{a^2 \cos^2( \frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \}) + \sin^2( \frac{a}{\sqrt{3}} \eta_0 \{ x - \frac{\pi}{2} \})}.
\]

Thus we find that \( F(\Phi) \geq 0 \). We note that the solutions are regular on the interval \(-\pi/2 \leq x \leq \pi/2\) in the Jordan frame and in the Einstein frame as well. Thus there are no constraints on the scalar charges except for \( |Q_\psi| > |Q_\varphi| \).
Figure 7: Wormholes in STT-3: The metric components in the equatorial plane and the scalar fields in the Einstein frame and in the Jordan frame, $-g_{tt}$ (a), $g_{\eta\eta}$ (b), $g_{\phi\phi}$ (c), $\psi$ (d), $\varphi$ and $\Phi$ (e). In (f) the function $F$ is shown. (Parameters: $a = 1/4$, $\frac{a Q_1}{\sqrt{3} \eta_0} = 2$.)

2. Properties

Let us now consider the geometry of the solutions. The case $\frac{d}{dx} \tilde{R}_c(x_t) < 0$ corresponds to a (local) maximum of the circumference in the equatorial plane $\tilde{R}_e$, and since $\tilde{R}_c$ grows without bound in the asymptotic regions, there exist two minima of $\tilde{R}_c$ located to the left and to the right of the coordinate $x_t$, respectively. Such a spacetime possesses two throats with a belly (or an equator) in between them.

As an example we consider a solution which possesses a single throat in the Einstein frame but two throats and a belly in between in the Jordan frame. The non-scalarized solution corresponds to the static Ellis wormhole. The scalarized solution is given by Eqs. (139), (141) and (144) and is characterized by the parameters $a = 1/4$, $\frac{a Q_1}{\sqrt{3} \eta_0} = 2$, and $\gamma = -2\pi \sqrt{11}$, corresponding to $M_{E+} = -\sqrt{11} \eta_0$. Note, that in this example the scalar fields are anti-symmetric.
Consequently, the masses coincide in the Jordan frame and in the Einstein frame in both asymptotically flat regions. In Fig.7 we illustrate the \( tt \), \( \eta \eta \), and \( \phi \phi \) components of the metric (in the equatorial plane) in the Jordan frame and in the Einstein frame. We observe that in the Einstein frame the \( \phi \phi \)-component possesses only one minimum corresponding to a single throat. In the Jordan frame, however, the \( \phi \phi \)-component possesses two minima and one local maximum corresponding to two throats and one belly, respectively. Fig.7(d) shows the phantom field, which is the same in both frames. The scalar fields \( \varphi \) and \( \Phi \) are exhibited in Fig.7(e). Fig.7(f) shows the function \( F(\Phi(x)) \).

In Fig.8 we show an isometric embedding of the equatorial plane of this solution. Here the presence of a single throat in the Einstein frame (a) and two throats and the belly in the Jordan frame (b) are clearly visible.

In Fig.9(a) we demonstrate the violation of the NEC in the equatorial plane for static STT-3 solutions for fixed \( D/\eta_0 = 2 \) and several values of \( a \). The NEC is always violated both in the Einstein frame and in the Jordan frame. Clearly, the violation depends on the parameter \( a \) only in the Jordan frame. Seemingly the violation increases with decreasing \( a \). Note however, that for small values of \( a \) the NEC in the Jordan frame is not violated in some part of the region \( \Sigma_- \) extending to the asymptotic region. For comparison we exhibit in Fig.9(b) the violation of the NEC also for static STT-2 solutions (\( \alpha = 0.125 \)) with the same value of \( D/\eta_0 \) and several values of \( a \). Here we observe violation of the NEC in the full spatial domain in both frames. But in contrast to the static STT-3 solutions the violation of the NEC in the Jordan frame decreases with decreasing \( a \).
IV. CONCLUSION

In General Relativity the non-trivial topology of traversable wormholes can be achieved by means of phantom fields, allowing for static and rotating Ellis wormholes \[33\ 37\ 40\], when no further fields are present. In particular, the rotating Ellis wormholes possess many interesting properties, e.g., they satisfy a Smarr relation, they possess as limiting configuration an extremal Kerr black hole, they possess bound orbits, etc.

Here we have considered scalarized wormholes in STT. We have shown, that once the corresponding (non-scalarized) wormhole solutions are known in General Relativity, it is no longer necessary to solve the Einstein equations for the metric. But only the equations for the gravitational scalar field and the phantom scalar field need to be solved. Indeed, each solution in General Relativity, and thus each solution for the metric, is characterized by a constant \(D\), which relates the charges \(Q_\phi\) and \(Q_\psi\) of the gravitational scalar field and the phantom scalar field, respectively, in the Einstein frame, \(D^2 = Q_\psi^2 - Q_\phi^2\).

Regarding the Jordan frame as the physical frame, we have considered various mass definitions like the gravitational mass, the tensor mass and the Schwarzschild mass of the wormholes. Assuming that the gravitational scalar field tends to zero in one asymptotically flat region, the wormhole solutions then lead in general to a non-vanishing value of the gravitational scalar field in the other asymptotically flat region (where a coordinate transformation needs to be performed to make it approach Minkowski space). Therefore the global charges mass and angular momentum have been considered separately in each asymptotically flat region.

We have also derived the general expressions for the quadrupole moment in both asymptotic regions in the Einstein frame and in the Jordan frame, obtaining relations between the quadrupole moments in these two frames. Likewise we have considered the geometric properties of these wormholes including the location of their throat(s) or equator(s). The NEC is always violated in the Einstein frame, independent of the specific STT considered. In the Jordan frame the violation of the NEC depends on the coupling function and thus on the STT.

To give some concrete examples we have then chosen 3 specific STT, by specifying their coupling functions \(A\): STT-1 with \(A_1(\varphi) = e^{\beta \varphi^2/2} [12\ 13]\), STT-2 with \(A_2(\varphi) = e^{n \varphi/\sqrt{3}} [4]\, and STT-3 with \(A_3(\varphi) = \cosh(\varphi/\sqrt{3})\). Clearly, the known Ellis wormholes of General Relativity are also solutions of STT-1 and STT-3, but not of STT-2. For these STT we have solved the scalar field equations, studied the domain of existence of the solutions and investigated their physical properties. Interestingly, in STT-1 scalarization arises for any positive or negative value of the parameter \(\beta\) (recall the presence of a critical threshold value in the case of neutron stars and boson stars). Here the solutions for positive values of \(\beta\) may possess multiple throats and equators in the Jordan frame, while they possess a single throat in the Einstein frame. Also in STT-3 we obtained wormhole solutions, which possess a single throat in the Einstein frame, while they possess an equator and a double throat in the Jordan frame.

Let us end with a comment on the stability of the solutions. Static Ellis wormholes are known to be unstable \[56\ 58\]. Matos and Nunez \[59\] have argued that rotating wormholes might, however, be more stable and thus traversable. A mode analysis of rotating wormholes in 5 dimensions has lent support to this conjecture \[60\]. In 4 dimensions an analogous mode analysis still remains a challenge for GR wormholes, and even more so in the case of STT wormholes. Another road to be followed to obtain stable wormholes could be to employ other generalized theories of gravity, as, e.g., Einstein-Gauss-Bonnet-dilaton gravity \[91\ 92\].

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Appendix

Here we derive an analytical formula for the critical value of the scalar phantom charge \(\hat{Q}_\psi^{cr}\). We start from Eq. (100). In order to compute its rhs we note that \(0 < a^2 \leq 1\) and consequently \(0 < a^2 e^{-\varphi^2} \leq 1\). Excluding the equality, we can expand the square root,

\[
\left[1 - a^2 e^{-\varphi^2}\right]^{-\frac{1}{2}} = \sum_{n=0}^{\infty} c_n a^{2n} e^{-n \varphi^2}, \quad \text{with} \quad c_n = \frac{(2n-1)(2n-3) \cdots 1}{n! 2^n}.
\] (146)
Figure 10: Wormholes in STT-1 for negative values of $\beta$: The function $B(a)$ (a) and its inverse (b).

Substitution in Eq. (99) leads to

$$\hat{Q}^\psi_{cr} = \pm \sum_{n=0}^{\infty} c_n a^{2n+1} \frac{1}{\pi} \int_0^{\infty} e^{-\left(n+1/2\right)\hat{\phi}^2} \, d\hat{\phi} = \pm \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n+1/2}} a^{2n}$$

(147)

when evaluating the Gauss integral. Multiplication of both sides with $a = \hat{D}/\hat{Q}_{\psi}^{cr}$, with $\hat{D} = \sqrt{-\beta D/\eta_0}$ yields

$$\hat{D} = \pm \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{c_n}{\sqrt{n+1/2}} a^{2n+1} =: \pm B(a) = B(\pm a).$$

(148)

Denoting by $B^{-1}$ the inverse function of $B$, we find

$$\pm \frac{\hat{D}}{\hat{Q}_{\psi}^{cr}} = B^{-1}(\hat{D}) \iff \hat{Q}_{\psi}^{cr} = \pm \frac{\hat{D}}{B^{-1}(\hat{D})}.$$ 

(149)

Thus, once the function $B(a)$ and its inverse are computed (see Fig. 10), Eq. (149) gives the upper and the lower bound for the scalar charge $\hat{Q}_{\psi}$ for given $\hat{D}(M, J)$ (and $\eta_0$, $\beta$).

A more convenient form of Eq. (149) is

$$\hat{Q}_{\psi}^{cr} = \hat{D}/B^{-1}(\hat{D}) =: G(\hat{D}).$$

(150)

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