OPTIMAL CONTROL OF A LARGE DAM, TAKING INTO ACCOUNT THE WATER COSTS

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ABSTRACT. This paper studies large dam models where the difference between lower and upper levels, \( L \), is assumed to be large. Passage across the levels leads to damage, and the damage costs of crossing the lower or upper level are proportional to the large parameter \( L \). Input stream of water is described by compound Poisson process, and the water cost depends upon current level of water in the dam. The aim of the paper is to choose the parameters of output stream (specifically defined in the paper) minimizing the long-run expenses. The particular problem, where input stream is ordinary Poisson and water costs are not taken into account, has been studied in [Abramov, J. Appl. Prob., 44 (2007), 249-258]. The present paper addresses the question How does the structure of water costs affect the optimal solution? Under natural assumptions we prove an existence and uniqueness of a solution and study the case of linear structure of the costs.

1. INTRODUCTION

A large dam is defined by the parameters \( L^{\text{lower}} \) and \( L^{\text{upper}} \), which are, respectively, the lower and upper levels of the dam. If the current level is between these bounds, the dam is assumed to be in a normal state. The difference \( L = L^{\text{upper}} - L^{\text{lower}} \) is large, and this is the reason for calling the dam large. This property enables us to use asymptotic analysis as \( L \to \infty \) and solve different problems of optimal control, which by a direct way, that is without using an asymptotic analysis, become very hard.

Let \( L_t \) denote the water level at time \( t \). If \( L^{\text{lower}} < L_t \leq L^{\text{upper}} \), then the state of the dam is called normal. Passage across lower or upper level leads to damage. The costs per time unit of this damage are \( J_1 = j_1 L \) for the lower level and, respectively, \( J_2 = j_2 L \) for the upper level, where \( j_1 \) and \( j_2 \) are given real constants. The water inflow is described by a compound Poisson process. Namely, the probability generating function of input amount of water (which is assumed to be an integer-valued random variable) in an interval \( t \) is given by

\[
 f_i(z) = \exp \left\{ -\lambda t \left( 1 - \sum_{i=1}^{\infty} r_i z^i \right) \right\},
\]

where \( r_i \) is the probability that at a specified moment of Poisson arrival the amount of water will increase by \( i \) units. In practice this means that the arrival of water is registered at random instants \( t_1, t_2, \ldots \); the times between consecutive instants are mutually independent and exponentially distributed with parameter \( \lambda \), and quantities of water (number of water units) of input flow are specified as a quantity.
i with probability \( r_i \) \((r_1 + r_2 + \ldots = 1)\). Clearly, this assumption is more applicable to real world problems than the assumption of [2] where the inter-arrival times of water units are exponentially distributed with parameter \( \lambda \). For example, the assumption made in the present paper enables us to approach a continuous dam model, assuming that the water levels \( L_t \) take the discrete values \( \{j\Delta\} \), where \( j \) is a positive integer and step \( \Delta \) is a positive small real constant. In the paper, however, the water levels \( L_t \) are assumed to be integer-valued.

The outflow of water is state-dependent as follows. If the level of water is between \( L_{lower} \) and \( L_{upper} \), then an interval between departures of water units (inverse output flow) has the probability distribution function \( B_1(x) \). If the level of water exceeds \( L_{lower} \) exactly, then output of water is frozen, and it resumes again as soon as the level of water exceeds the level \( L_{lower} \). (The exact mathematical formulation of the problem taking into account some specific details is given below.)

Let \( c_{L_t} \) denote the cost of water at level \( L_t \). The sequence \( c_i \) is assumed to be positive and non-increasing. The problem of the present paper is to choose the parameter \( \int_0^\infty x\,dB_1(x) \) of the dam in the normal state minimizing the objective function

\[
J = p_1 J_1 + p_2 J_2 + \sum_{i=L_{lower}+1}^{L_{upper}} c_i q_i,
\]

where

\[
p_1 = \lim_{t\to\infty} \Pr\{L_t = L_{lower}\},
\]

\[
p_2 = \lim_{t\to\infty} \Pr\{L_t > L_{upper}\},
\]

\[
q_i = \lim_{t\to\infty} \Pr\{L_t = L_{lower} + i\}, \quad i = 1, 2, \ldots, L.
\]

Usually the level \( L_{lower} \) is identified with an empty queue (i.e. \( L_{lower} := 0 \) and \( L_{upper} := L \)), and the dam model is the following queueing system with service depending on queue-length. If immediately before a service beginning the queue-length exceeds the level \( L \), then the customer is served by the probability distribution function \( B_2(x) \). Otherwise, the service time distribution is \( B_1(x) \). The value \( p_1 \) is the stationary probability of an empty system, the value \( p_2 \) is the stationary probability that a customer is served by probability distribution \( B_2(x) \), and \( q_i, \quad i = 1, 2, \ldots, L \), are the stationary probabilities of the queue-length process, so

\[
p_1 + p_2 + \sum_{i=1}^L q_i = 1.
\]

(For the described queueing system, the right-hand side limits in relations (1.3)-(1.5) do exist.)

In our study, the parameter \( L \) increases unboundedly, and we deal with the series of queueing systems. The above parameters, such as \( p_1, \, p_2, \, J_1, \, J_1 \) as well as other parameters are functions of \( L \). The argument \( L \) will be often omitted in these functions.

Similarly to [2], it is assumed that the input parameter \( \lambda \), the probabilities \( r_1, \, r_2, \ldots \) and probability distribution function \( B_2(x) \) are given, while the appropriate probability function \( B_1(x) \) should be chosen from the specified parametric family of functions \( B_1(x, C) \). (Actually, we deal with the family of probability distribution
functions $B_1(x)$ depending on two parameters $\delta$ and $L$ in series, i.e. $B_1(x, \delta, L)$. Then the parametric family of distributions $B_1(x, C)$ is described in the limiting scheme as $\delta L \to C$, so the parameter $C$ belongs to the family of possible limits of $\delta L$ as $\delta \to 0$ and $L \to \infty$.

The outflow rate, should be chosen such that to minimize the objective function of (1.2) with respect to the parameter $C$, which results in choice of the corresponding probability distribution function $B_1(x, C)$ of that family.

A particular problem have been studied in [2]. A circle of problems associated with the results of [2] are discussed in a review paper [3].

The simplest model with Poisson input stream and the objective function having the form $J = p_1 J_1 + p_2 J_2$ (i.e. the water costs are not taken into account), has been studied in [2]. Denote $\rho_2 = \lambda \int_0^\infty x dB_2(x)$ and $\rho_1(C) = \lambda \int_0^\infty x dB_1(x, C)$. (The parameter $C$ is a unique solution of a specific minimization problem precisely formulated in [2].) In was shown in [2] that the solution to the control problem is unique and has one of the following three forms:

(i) in the case $j_1 = j_2 \frac{p_2}{p_1}$, the optimal solution is $\rho_1 = 1$;
(ii) in the case $j_1 > j_2 \frac{p_2}{1-p_2}$, the optimal solution has the form $\rho_1 = 1 + \delta$, where $\delta(L)$ is a small positive parameter, and $\delta(L)L \to C$ as $L \to \infty$;
(iii) in the case $j_1 < j_2 \frac{p_2}{1-p_2}$, the optimal strategy has the form $\rho_1 = 1 - \delta$, and $\delta(L)L \to C$ as $L \to \infty$.

It has been also shown in [2] that the solution to the control problem is insensitive to the type of probability distributions $B_1(x)$ and $B_2(x)$. Specifically, it is expressed via the first moment of $B_2(x)$ and the first two moments of $B_1(x)$.

The aforementioned cases (i), (ii) and (iii) fall into the category of heavy traffic analysis in queueing theory. There are many papers related to this subject. We mention the books of Chen and Yao [4] and Whitt [19], where a reader can find many other references. The aforementioned paper [2] as well as the present paper, however, are conceptually close to the well-known paper of Halfin and Whitt [6].

The heavy-traffic conditions in queueing systems with large number of identical servers arise naturally if we assume that a high-level, associated with the loss in that system, is reached with a given positive probability, while the traffic intensities converge to 1 from the below and the arrival rates and number of servers increase to infinity. In the case of the single-server state-dependent queueing systems of the present paper that model a large dam, we assume that the specified costs for reaching the lower and upper levels multiplied by the corresponding probabilities must converge to the given fixed values in limit, and the resulting functional containing these quantities must be minimized. This leads to the study of the family of systems under the heavy-traffic behaviour, in which the sequence of products $\delta L$ must converge to the optimal value $C$.

Compared to the earlier studies in [2], the solution of the problems in the present paper requires a much deepen and delicate analysis. The results of [2] are extended in two directions: (1) the arrival process is compound Poisson rather than Poisson, and (2) structure of water costs in dependence of the level of water in the dam is included.

The first extension leads to new techniques of stochastic analysis. The main challenge in [2] was to reduce the certain characteristics of the system during a busy period to the convolution type recurrence relation such as $Q_n = \sum_{i=0}^{n} Q_{n-i+1} f_i$.
(Q_0 \neq 0), where f_0 > 0, f_i \geq 0 for all i \geq 1, \sum_{i=0}^{\infty} f_i = 1 and then to use the known results on the asymptotic behaviour of Q_n as n \to \infty. In the case when arrivals are compound Poisson, the same characteristics of the system cannot be reduced to the aforementioned convolution type of recurrence relation. Instead, we obtain a more general scheme including as a part the aforementioned recurrence relation. In this case, asymptotic analysis of the required characteristics becomes very challenging. It is based on special stochastic domination methods, which will be explained in details later.

The second extension leads to new analytic techniques of asymptotic analysis. Asymptotic methods of [2] do not longer work, and one should use more delicate techniques instead. That is, instead of Takác's asymptotic theorems [17], p. 22-23, one should use special Tauberian theorems with remainder by Postnikov [9], Sect. 25. For different applications of the aforementioned Takác's asymptotic theorems and Tauberian theorems of Postnikov see [3].

Another challenging problem for the dam model in the present paper is the solution to the control problem, that is, the proof of a uniqueness of the optimal solution. In the case of the model in [2] the existence and uniqueness of a solution follows automatically from the explicit representations of the functionals obtained there. (The existence of a solution follows from the fact that in the case \( \rho_1 = 1 \) we get a bounded value of the functional, while in the cases \( \rho_1 < 1 \) and \( \rho_1 > 1 \) the functional is unbounded. Then the uniqueness of a solution reduces to elementary minimization problem for smooth convex functions.) In the case of the model in the present paper, the solution of the present problem with extended criteria (1.2) is related to the same class of solutions as in [2]. That is, it must be either \( \rho_1 = 1 \) or one of two limits of \( \rho_1 = 1 + \delta, \rho_1 = 1 - \delta \) for positive small vanishing \( \delta \) as \( L \) increases unboundedly, and \( L\delta \to C \). While the existence of a solution follows trivially as in [2], the proof of a uniqueness of the solution requires elegant techniques of the theory of analytic functions and majorization inequalities (see [7] and [8]).

Similarly to [2], we use the notation \( \rho_{1,l} = \lambda^l \int_0^\infty x^l dB_1(x), l = 2, 3 \). The existence of \( \rho_{1,l} \) (i.e. the moments of the third order of \( B_1(x) \)) will be specially assumed in the formulations of the statements corresponding to case studies.

It is assumed in the present paper that \( c_i \) is a non-increasing sequence. If the cost sequence \( c_i \) were an arbitrary bounded sequence, then a richer class of possible cases could be studied. However, in the case of arbitrary cost sequence, the solution need not be unique, and arbitrary costs \( c_i \), say increasing in \( i \), seem not to be useful and, therefore, are not considered here. A non-increasing sequence \( c_i \) depends on \( L \) in series. This means that as \( L \) changes (increasing to infinity) we have different non-increasing sequences (see example in Section 7). The initial value \( c_1 \) and final value \( c_L \) are taken fixed and strictly positive, and the limit of \( c_L \) as \( L \to \infty \) is assumed to be positive as well.

Realistic models arising in practice assume that the probability distribution function \( B_1(x) \) should also depend on \( i \), i.e have the representation \( B_{1,i}(x) \). The model of the present paper, where \( B_1(x) \) is the same for all \( i \), under appropriate additional information can approximate those more general models. Namely, one can suppose that the stationary service time distribution \( B_1(x) \) has the representation
\[ B_1(x) = \sum_{i=1}^{L} q_i B_{1,i}(x) \quad (q_i, \ i = 1, 2, \ldots, L \text{ are the state probabilities}), \] and the solution to the control problem for \( B_1(x) \) enables us to find then the approximate solutions to the control problem for \( B_{1,i}(x) \), \( i = 1, 2, \ldots, L \) by using the Bayes rule.

For example, the simplest model can be of the form

\[ B_1(x) = a B^*_1(x) + b B^{**}_1(x), \]

where \( a := \sum_{i=1}^{L^0} q_i \ (L^0 < L) \), and, respectively, \( b := \sum_{i=L^0+1}^{L} q_i \).

In the present paper we address the following questions.

- Uniqueness of an optimal solution and its structure.
- Interrelation between the parameters \( j_1, j_2, \rho_2, c_i \) (\( i = 1, 2, \ldots, L \)) when the optimal solution is \( \rho_1 = 1 \).

The uniqueness of an optimal solution is given by Theorem 6.4. In the case of the model considered in [2] the condition for \( \rho_1 = 1 \) is \( j_1 = j_2 \rho_2 \). Intuitive explanation of this result is based on the well-known property of the stream of losses during a busy period of \( M/GI/1/n \) queues, under the assumption that the expected interarrival and service times are equal (see Abramov [1], Righter [10] and Wolff [22]). In the case of the model in this paper the interrelation between the aforementioned and some additional parameters involves the inequality (see Section 6, Corollary 6.5). Exact results are obtained in the particular case of linearly decreasing costs as the level of water increases (for brevity, this case is called linear costs). In this case, a numerical solution of the problem is given.

The rest of the paper is organized as follows. In Section 2 the main ideas and methods of asymptotic analysis are given. In Section 2.1 we recall the basic methods related to state dependent queueing system with ordinary Poisson input that have been used in [2]. Then in Section 2.2 extensions of these methods for the model considered in this paper are given. Specifically, the methodology of constructing linear representations between mean characteristics given during a busy period is explained. In Section 3 the asymptotic behavior of the stationary probabilities is studied. In Section 3.1 known Tauberian theorems that are used in the asymptotic analysis in the paper are recalled. In Section 3.2 exact formulae for the stationary probabilities \( p_1 \) and \( p_2 \) are derived. On the basis of these formulae, in Sections 3.3 and 3.4 the asymptotic theorems for the stationary probabilities \( p_1 \) and \( p_2 \) have been established. Section 4 is devoted to asymptotic analysis of the stationary probabilities \( q_{L-i}, i = 1, 2, \ldots, L \). In Section 4.1 the explicit representation for the stationary probabilities \( q_i \) is derived. On the basis of this explicit representation and Tauberian theorems, in following Sections 5.1, 5.2 and 5.3 asymptotic theorems for these stationary probabilities are established in the cases \( \rho_1 = 1, \rho_1 = 1 + \delta \) and \( \rho_1 = 1 - \delta \) correspondingly, where positive \( \delta \) is assumed to vanish such that \( \delta L \to C \) as \( L \to \infty \). In Section 5 the objective function given in (1.2) is studied. In following Sections 5.1, 5.2 and 5.3 the asymptotic theorems for this objective function are established for the cases \( \rho_1 = 1, \rho_1 = 1 + \delta \) and \( \rho_1 = 1 - \delta \) correspondingly. In Section 6 the theorem on existence and uniqueness of a solution is proved. In Section 7 the case of linear costs is studied and relevant numerical results are provided.
2. Methodology of analysis

In this section we describe the methodology used in the present paper. This is a very important step because the earlier methods of [2] do not work for this extended model and hence need in substantial revision.

We start from the model where arrivals are Poisson, and then we explain how the methods should be developed for the model where an arrival process is compound Poisson.

2.1. State dependent queueing system with Poisson input and its characteristics. In this section we consider the simplest model in which arrival flow is Poisson with parameter \( \lambda \). Let \( T_L \) denote the length of a busy period of this system, and let \( T_L^{(1)}, T_L^{(2)} \) denote the cumulative times spent for service of customers arrived during that busy period with probability distribution functions \( B_1(x) \) and \( B_2(x) \) correspondingly. For \( k = 1, 2 \), the expectations of service times will be denoted by \( \mu_k = \int_0^\infty x dB_k(x) \), and the loads by \( \rho_k = \frac{\lambda}{\mu_k} \). Let \( \nu_L, \nu_L^{(1)} \) and \( \nu_L^{(2)} \) denote correspondingly the number of served customers during a busy period, and the numbers of those customers served by the probability distribution functions \( B_1(x) \) and \( B_2(x) \). The random variable \( T_L^{(1)} \) coincides in distribution with a busy period of the \( M/GI/1/L \) queueing system (\( L \) is the number of waiting places excluding the place for server). The elementary explanation of this fact is based on a property of level crossings and the property of the lack of memory of exponential distribution (e.g. [2]), so the analytic representation for \( ET_L^{(1)} \) is the same as this for the expected busy period of the \( M/GI/1/L \) queueing system. The recurrence relation for the Laplace-Stieltjes transform and consequently that for the expected busy period of the \( M/GI/1/L \) queueing system has been derived by Tomko [18].

So, for \( ET_L^{(1)} \) the following recurrence relation is satisfied:

\[
ET_L^{(1)} = \sum_{i=0}^{L} ET_{L-i+1}^{(1)} \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^i}{i!} dB_1(x),
\]

where \( ET_0^{(1)} = \frac{1}{\mu_1} \). (The random variable \( T_i^{(1)} \) is defined similarly to that of \( T_L^{(1)} \). The only difference is in the state parameter \( i \) that is given instead of \( L \).) Recurrence relation (2.1) is a particular form of the recurrence relation

\[
Q_n = \sum_{i=0}^{n} Q_{n-i} f_i,
\]

where \( Q_0 \neq 0, f_0 > 0, f_i \geq 0, i = 1, 2, \ldots \) and \( \sum_{i=0}^{\infty} f_i = 1 \) (see Takács [17]).

Using the obvious system of equations:

\[
ET_L = ET_L^{(1)} + ET_L^{(2)},
\]

\[
E\nu_L = E\nu_L^{(1)} + E\nu_L^{(2)},
\]

and Wald’s equations (see [2], p.384)

\[
ET_L^{(1)} = \frac{1}{\mu_1} E\nu_L^{(1)},
\]

\[
ET_L^{(2)} = \frac{1}{\mu_2} E\nu_L^{(2)},
\]
one can express the quantities $E_{T_L}$, $E_{\nu_L}$, $E_{T_L}^{(2)}$, $E_{\nu_L}^{(1)}$ and $E_{\nu_L}^{(2)}$ all via $E_{T_L}^{(1)}$ as the linear functions. Indeed, taking into account that the number of arrivals during a busy cycle coincides with the total number of customers served during a busy period we have

\begin{equation}
\lambda E_{T_L} + 1 = E_{\nu_L},
\end{equation}

which together with (2.3)-(2.6) yields the linear representations required.

For example,

\begin{equation}
E_{\nu_L}^{(2)} = \frac{1}{1 - \rho_2} - \frac{1}{\mu_1} \cdot \frac{1 - \rho_1}{1 - \rho_2} E_{T_L}^{(1)},
\end{equation}

and

\begin{equation}
E_{T_L}^{(2)} = \frac{\rho_2}{\lambda(1 - \rho_2)} - \frac{\rho_2}{\lambda} \cdot \frac{1 - \rho_1}{1 - \rho_2} E_{T_L}^{(1)}.
\end{equation}

As a result, the stationary probabilities $p_1$ and $p_2$ both are expressed via $E_{\nu_L}^{(1)}$ as follows:

\begin{align*}
p_1 &= \frac{1 - \rho_2}{1 + (\rho_1 - \rho_2)E_{\nu_L}^{(1)}}, \\
p_2 &= \frac{\rho_2 + \rho_2(\rho_1 - 1)E_{\nu_L}^{(1)}}{1 + (\rho_1 - \rho_2)E_{\nu_L}^{(1)}}.
\end{align*}

(see Section 2 of [2] for further details). It is interesting to note that the coefficients in linear representation all are insensitive to the probability distribution functions $B_1(x)$ and $B_2(x)$ and are only expressed via parameters such as $\mu_1$, $\mu_2$ and $\lambda$.

The asymptotic behaviour of $E_{T_L}^{(1)}$ as $L \to \infty$ that given by (2.1) is established on the basis of the known asymptotic behaviour of the sequence $Q_n$ as $n \to \infty$ that given by (2.2) (see [17], p.22, [9] as well as recent paper [3]). To make the paper self-contained, the necessary results about the asymptotic behaviour of $Q_n$ as $n \to \infty$ are given in Section 3.1.

2.2. State dependent queueing system with compound Poisson input and its characteristics. For $M^X/GI/1/L$ queues, certain characteristics associated with busy periods have been studied by Rosenlund [11]. Developing the results of Tomko [18], Rosenlund [11] has derived the recurrence relations for the joint Laplace-Stieltjes and $z$-transform of two-dimensional distributions of a generalized busy period and the number of customers served during that period. In turn, both of these approaches [18] and [11] are based on a well-known Takács’ method (see [15] or [16]).

For further analysis, [11] used matrix-analytic techniques and techniques of the theory of analytic functions. This type of analysis is very hard and seems cannot be easily adapted for the purposes of the present paper, where a more general model than that from a paper [14] is studied.

In this section we explain how the method of Section 2.1 can be extended, and how the characteristics of the system can be expressed via the similar convolution type recurrence relations.

Notice first, that the linear representations similar to those derived for the state dependent queueing system with ordinary Poisson input are satisfied for the present system as well. Indeed, equations (2.3)-(2.6) all hold in the case of the present
queueing system. The only difference is that instead of (2.7), the relation between $E_T$ and $E_{\nu}$ should be

$$\lambda E_{\nu} + E_{\xi} = E_{\nu}, \tag{2.10}$$

where $\xi$ denotes a batch size of an arrival. (The random variable $\xi$ has the distribution $\Pr\{\xi = i\} = r_i$.) This leads to a slight change of the linear representations mentioned in Section 2.1. The main difficulty, however, is that the recurrence relation for $E_{(1)}(T)$ (or the corresponding quantity $E_{(1)}(\nu)$) is no longer a convolution type recurrence relation as (2.2). So, we should use another type of analysis, which is explained below.

For this model, let $\tilde{T}_j, j = 1, 2, \ldots, L$, denote the time interval starting from the moment when there are $L - j + 1$ customers in the system until the moment when there remain $L - j$ customers for the first time since its beginning. Similarly to the notation used in Section 2.1 let us introduce the random variables $\tilde{T}_j^{(1)}, \tilde{T}_j^{(2)}, \tilde{\nu}_j, \tilde{\nu}_j^{(1)}, \tilde{\nu}_j^{(2)}, j = 1, 2, \ldots, L$, which have the same meaning as before. Specifically, when $j$ takes the value $L$, $\tilde{T}_L$ is the length a busy period starting from a single customer (1-busy period); $\tilde{\nu}_L$ is the number of customers that served during a 1-busy period, and so on.

With the aid of the aforementioned Takács’ method [15], [16], one can derive the recurrence relation similar to that of (2.1). Namely,

$$E_{(1)}(\tilde{T}) = \sum_{i=0}^{L} E_{(1)}(\tilde{T})_{L-i+1} \int_{0}^{\infty} \frac{1}{z^i} \frac{d^i f_x(z)}{dz^i} \bigg|_{z=0} dB_1(x), \tag{2.11}$$

where $E_{(1)}(\tilde{T})_0 = \frac{1}{\mu_1}$, and the generating function $f_x(z)$ is given by (1.1). So, the only difference between (2.1) and (2.11) is in their integrands, and in particular case $r_1 = 1, r_i = 0, i \geq 2$ we clearly arrive at the same expressions. The explicit results associated with recurrence relation (2.11) is given later in the paper. Apparently, the similar system of equations as (2.1) - (2.6) is satisfied for the characteristics of the state dependent queueing system $M^\infty/GI/1$. Namely,

$$E_{(1)}(\tilde{T}) = E_{(1)}(\tilde{T}) + E_{(2)}(\tilde{T}), \tag{2.12}$$

$$E_{(1)}(\tilde{\nu}) = E_{(1)}(\tilde{\nu}) + E_{(2)}(\tilde{\nu}), \tag{2.13}$$

$$E_{(1)}(\tilde{T}) = \frac{1}{\mu_1} E_{(1)}(\tilde{\nu}), \tag{2.14}$$

$$E_{(2)}(\tilde{T}) = \frac{1}{\mu_2} E_{(2)}(\tilde{\nu}). \tag{2.15}$$

Therefore, the same linear representations via $E_{(1)}(\tilde{T})$ hold for characteristics of these systems, where by $\rho_1$ and $\rho_2$ one now should mean the expected numbers of arrived customers per service time (not the expected number of arrivals) having the probability distribution function $B_1(x)$ and, respectively, $B_2(x)$.

Let us now consider the length of a busy period $T_L$ and associated random variables $T_L^{(1)}, T_L^{(2)}, \nu_L, \nu_L^{(1)}$ and $\nu_L^{(2)}$. Let $\varsigma_1$ denote a size of batch that starts a busy period. (An integer random variable $\varsigma_1$ has the distribution $\Pr\{\varsigma = i\} = r_i$.)
Then $T_L$ can be represented

\[(2.16) \quad T_L \overset{d}{=} \sum_{i=1}^{\varsigma_1 \wedge (L+1)} \bar{T}_{L-i+1} + \sum_{i=1}^{\varsigma_1 - (L+1)} \bar{T}_{0,i},\]

where 1-busy periods $\bar{T}_{L-i+1}, i = 1, 2, \ldots, L$ are mutually independent;

$\bar{T}_0$ denotes a special 1-busy period that starts from a service time having the probability distribution function $B_1(x)$ and all other service times are mutually independent and identically distributed random variables having the probability distribution $B_2(x)$, and the distributions of interarrival times and batch sizes are the same as in the original state dependent queueing system;

$\bar{T}_{0,i}, i = 1, 2, \ldots$, is a sequence of independent and identically distributed 1-busy periods of the $\text{M}^X/\text{G}/1$ queueing system, the service times of which all are independent and identically distributed random variables having the probability distribution function $B_2(x)$, and the distributions of interarrival times and batch sizes are the same as in the original state dependent queueing system;

$a \wedge b$ denotes $\min\{a, b\}$;

d denotes the equality in distribution;

in the case where $\varsigma_1 - (L + 1) \leq 0$, the empty sum in (2.16) is assumed to be zero.

In turn, the representation for $T_L^{(1)}$ is as follows:

\[(2.17) \quad T_L^{(1)} \overset{d}{=} \sum_{i=1}^{\varsigma_1 \wedge (L+1)} \bar{T}_{L-i+1}^{(1)},\]

where $\bar{T}_0^{(1)}$ denotes a single service time having the probability distribution function $B_1(x)$. Whereas $\bar{E}_L^{(1)}$ is determined by recurrence relation (2.11), which is a particular case of (2.2), a convolution type recurrence relation is no longer valid for $\bar{E}_L^{(1)}$.

For the following asymptotic analysis of $\bar{E}_L^{(1)}$ and other mean characteristics such as $\bar{E}_L^{(1)}, \bar{E}_L^{(2)}$ we will use the following techniques. We first study the mean characteristics $\bar{E}(T_L | \varsigma_1 \wedge L), \bar{E}(T_{L}^{(1)} | \varsigma_1 \wedge L), \bar{E}(T_{L}^{(2)} | \varsigma_1 \wedge L), \bar{E}(\nu_{L}^{(1)} | \varsigma_1 \wedge L) \text{ and } \bar{E}(\nu_{L}^{(2)} | \varsigma_1 \wedge L)$. Then, assuming that $L \to \infty$, we have $\lim_{L \to \infty} \Pr\{\varsigma_1 \wedge L = i\} = \Pr\{\varsigma_1 = i\}$, as well as $\lim_{L \to \infty} \bar{E}(T_L | \varsigma_1 \wedge L) = \lim_{L \to \infty} \bar{E}T_L$, and the similar limits hold for the other mean characteristics. The following asymptotic behaviour of the probabilities $p_1$ and $p_2$ as $L \to \infty$ is then established similarly to that in [2].

Let us show the justice of the linear representations that similar to those (2.8) and (2.9). Write $\bar{E}_L = a + b\bar{E}_L^{(1)}$, where $a$ and $b$ are specified constants. Then,
by the total probability formula,

\[
\mathbb{E}\{T_L|\varsigma \land L\} = \sum_{i=1}^{L} \Pr\{\varsigma \land L = i\} \sum_{j=1}^{i} E\check{T}_{L-j+1}
\]

(2.18)

\[
= \sum_{i=1}^{L} \Pr\{\varsigma \land L = i\} \sum_{j=1}^{i} (a + b\check{T}_{L-i+1}^{(1)})
\]

\[
= a \sum_{i=1}^{L} i\Pr\{\varsigma \land L = i\} + b \sum_{i=1}^{L} \Pr\{\varsigma \land L = i\} \sum_{j=1}^{i} \check{T}_{L-i+1}^{(1)}
\]

\[
= a\mathbb{E}\{\varsigma \land L\} + b\mathbb{E}\{T_{L}^{(1)}|\varsigma \land L\}.
\]

This representation is *quazi-linear* in the sense that only for \( J \geq L \) (but not for all \( J \geq 1 \))

\[
\mathbb{E}\{T_{J}|\varsigma \land L\} = a\mathbb{E}\{\varsigma \land L\} + b\mathbb{E}\{T_{J}^{(1)}|\varsigma \land L\}.
\]

Apparently, the similar quazi-linear representations are satisfied for the mean characteristics \( \mathbb{E}\{\nu_{L}|\varsigma \land L\} \), \( \mathbb{E}\{T_{L}^{(2)}|\varsigma \land L\} \), and \( \mathbb{E}\{\nu_{L}^{(2)}|\varsigma \land L\} \) all via \( \mathbb{E}\{T_{L}^{(1)}|\varsigma \land L\} \).

The exact values of the coefficients in these quazi-linear representations will be derived in the next sections.

3. Asymptotic theorems for the stationary probabilities \( p_1 \) and \( p_2 \)

In this section, the explicit expressions are derived for the stationary probabilities, and their asymptotic behavior is studied. These results will be used in our further findings of the optimal solution.

3.1. Preliminaries. In this section we recall the main properties of recurrence relation (2.2). The detailed theory of these recurrence relations can be found in Takács [17]. For the generating function \( Q(z) = \sum_{j=0}^{\infty} Q_j z^j \), \( |z| \leq 1 \) we have

\[
Q(z) = \frac{Q_0 F(z)}{F(z) - z},
\]

where \( F(z) = \sum_{j=0}^{\infty} f_j z^j \).

Asymptotic behavior of \( Q_n \) as \( n \to \infty \) has been studied by Takács [17] and Postnikov [9]. Recall the theorems that are needed in this paper.

Denote \( \gamma_m = \lim_{z \to 1} \frac{d^m F(z)}{dz^m} \).

**Lemma 3.1.** (Takács [17], p. 22-23). If \( \gamma_1 < 1 \) then

\[
\lim_{n \to \infty} Q_n = \frac{Q_0}{1 - \gamma_1},
\]

If \( \gamma_1 = 1 \) and \( \gamma_2 < \infty \), then

\[
\lim_{n \to \infty} \frac{Q_n}{n} = \frac{2Q_0}{\gamma_2}.
\]

If \( \gamma_1 > 1 \), then

\[
\lim_{n \to \infty} \left( Q_n - \frac{Q_0}{\delta^n [1 - F'(\delta)]} \right) = \frac{Q_0}{1 - \gamma_1},
\]

where \( \delta \) is the least in absolute value root of the functional equation \( z = F(z) \).
Lemma 3.2. (Postnikov [9], Sect. 25). Let \( \gamma_1 = 1, \gamma_2 < \infty \) and \( f_0 + f_1 < 1 \). Then, as \( n \to \infty \),
\[
Q_{n+1} - Q_n = \frac{2Q_0}{\gamma_2} + o(1).
\]

3.2. Exact formulae for \( p_1 \) and \( p_2 \). In this section we derive exact representations for \( p_1 \) and \( p_2 \) via \( \mathcal{E}_L^{(1)} \). We also obtain some preliminary asymptotic representations that easily follow from the explicit results. Those asymptotic representations will be used in the sequel.

We first start from the linear representations for \( \mathcal{E}_L^{(2)} \) in terms \( \mathcal{E}_L^{(1)} \). Namely, we have the following lemma.

Lemma 3.3. For \( \mathcal{E}_L^{(2)} \), \( L = 1, 2, \ldots \), we have the following representation
\[
\mathcal{E}_L^{(2)} = \frac{1}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} \mathcal{E}_L^{(1)},
\]
where \( \rho_1 = \frac{\lambda_1}{\mu_1} \) and \( \rho_2 = \frac{\lambda_1}{\mu_2} < 1 \), and \( \mathcal{E}_L^{(1)} \) is given by
\[
\mathcal{E}_L^{(1)} = \sum_{i=0}^{L} \mathcal{E}_{L-i+1} \int_{0}^{\infty} \frac{1}{i!} \left. \frac{d^if(z)}{dz^i} \right|_{z=0} dB_1(x),
\]
\( \mathcal{E}_0^{(1)} = 1 \).

Proof. Taking into account that the number of arrivals during 1-busy cycle (1-busy period plus idle period) coincides with the number of customers served during the same 1-busy period, according to Wald’s identity we have:
\[
\lambda \left( \mathcal{E}_{T_L} + \frac{1}{\lambda} \right) = \lambda \mathcal{E}_{T_L} + 1 = \mathcal{E}_L = \mathcal{E}_L^{(1)} + \mathcal{E}_L^{(2)}.
\]
This equality together with (2.12, 2.13) yields the desired statement of the lemma, where (3.6) in turn follows from (2.11) and Wald’s identity (2.14). \( \square \)

The next step is to derive representations for \( \mathcal{E} \{ \nu_L^{(1)} | \varsigma_1 \land L \} \) and \( \mathcal{E} \{ \nu_L^{(2)} | \varsigma_1 \land L \} \). We have the following lemma.

Lemma 3.4. For \( \mathcal{E} \{ \nu_L^{(2)} | \varsigma_1 \land L \} \) we have
\[
\mathcal{E} \{ \nu_L^{(2)} | \varsigma_1 \land L \} = \frac{\mathcal{E} \{ \varsigma_1 \land L \}}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} \mathcal{E} \{ \nu_L^{(1)} | \varsigma_1 \land L \},
\]
where
\[
\mathcal{E} \{ \nu_L^{(1)} | \varsigma_1 \land L \} = \sum_{i=1}^{L} \Pr \{ \varsigma_1 \land L = i \} \sum_{j=1}^{i} \mathcal{E}_{L-j+1}^{(1)},
\]
and \( \mathcal{E}_{L-j+1}^{(1)}, j = 1, 2, \ldots, L, \) are given by (3.6).

Proof. Following the same arguments as in (2.18), one can write
\[
\mathcal{E} \{ \nu_L^{(2)} | \varsigma_1 \land L \} = a \mathcal{E} \{ \varsigma_1 \land L \} + b \mathcal{E} \{ \nu_L^{(1)} | \varsigma_1 \land L \}
\]
for specified constants \( a \) and \( b \) for which the linear representation \( \mathcal{E}_L^{(2)} = a + b \mathcal{E}_L^{(1)} \) is satisfied. Hence, according to relation (3.5) of Lemma 3.3 \( a = \frac{1}{1 - \rho_2} \) and \( b = \frac{1 - \rho_1}{1 - \rho_2} \). The proof is completed. \( \square \)
The following lemma yields exact estimates for the difference $\mathbb{E}\nu_L^{(1)} - \mathbb{E}\{\nu_L^{(1)} \mid \varsigma_1 \wedge L\}$.

**Lemma 3.5.** We have the following estimate:

\begin{equation}
(3.8) \quad \mathbb{E}\nu_L^{(1)} - \mathbb{E}\{\nu_L^{(1)} \mid \varsigma_1 \wedge L\} = \Pr\{\varsigma_1 > L\},
\end{equation}

Proof. Similarly to (2.17) we have

\[ \nu_L^{(1)} \overset{d}{=} \sum_{i=1}^{\varsigma_1 \wedge (L+1)} \nu_{L-i+1}^{(1)}, \]

where $\nu_{L-i+1}^{(1)}$, $i = 1, 2, \ldots, L$ are mutually independent, and $\nu_0^{(1)} = 1$.

Hence,

\begin{equation}
(3.9) \quad \mathbb{E}\nu_L^{(1)} = \sum_{i=1}^{L+1} \Pr\{\varsigma_1 \wedge (L+1) = i\} \sum_{j=1}^{i} \mathbb{E}\nu_{L-j+1}^{(1)}.
\end{equation}

In turn, the representation for $\mathbb{E}\{\nu_L^{(1)} \mid \varsigma_1 \wedge L\}$ is

\begin{equation}
(3.10) \quad \mathbb{E}\{\nu_L^{(1)} \mid \varsigma_1 \wedge L\} = \sum_{i=1}^{L} \Pr\{\varsigma_1 \wedge L = i\} \sum_{j=1}^{i} \mathbb{E}\nu_{L-j+1}^{(1)}.
\end{equation}

Subtracting (3.10) from (3.9) we obtain:

\begin{align*}
\mathbb{E}\nu_L^{(1)} - \mathbb{E}\{\nu_L^{(1)} \mid \varsigma_1 \wedge L\} &= \Pr\{\varsigma_1 = L\} \sum_{j=1}^{L} \mathbb{E}\nu_j^{(1)} + \Pr\{\varsigma_1 > L\} \sum_{j=0}^{L} \mathbb{E}\nu_j^{(1)} \\
&\quad - \Pr\{\varsigma_1 \geq L\} \sum_{j=1}^{L} \mathbb{E}\nu_j^{(1)} \\
&= \Pr\{\varsigma_1 > L\}.
\end{align*}

Relation (3.8) is proved. \(\Box\)

From Lemma 3.5 we have the following important corollary.

**Corollary 3.6.** As $L \to \infty$,

\begin{equation}
(3.11) \quad \mathbb{E}\nu_L^{(1)} - \mathbb{E}\{\nu_L^{(1)} \mid \varsigma_1 \wedge L\} = o(1),
\end{equation}

and

\begin{equation}
(3.12) \quad \mathbb{E}\nu_L^{(2)} - \mathbb{E}\{\nu_L^{(2)} \mid \varsigma_1 \wedge L\} = o(1).
\end{equation}

Proof. Asymptotic relation (3.11) follows immediately from (3.8). In order to show (3.12) let us first derive the linear representation of $\mathbb{E}\nu_L^{(2)}$ via $\mathbb{E}\nu_L^{(1)}$. From relation (2.10) and equations (2.3)-(2.6) in Section 2.1, which also hold true in the case of the present queueing system with batch arrivals, we obtain:

\begin{equation}
(3.13) \quad \mathbb{E}\nu_L^{(2)} = \frac{\mathbb{E}\varsigma}{1 - \rho_2} - \frac{1 - \rho_1}{1 - \rho_2} \mathbb{E}\nu_L^{(1)}.
\end{equation}

As well, for $\mathbb{E}\{\nu_L^{(2)} \mid \varsigma_1 \wedge L\}$ from Lemma 3.4 we have representation (3.7). Hence, comparing the terms of (3.13) and (3.7) and taking into account (3.11) we easily arrive at asymptotic relation (3.12). Lemma 3.6 is proved. \(\Box\)
The following lemma presents the exact formulae for the stationary probabilities $p_1$ and $p_2$ via the term $E\nu_L^{(1)}$.

Lemma 3.7. We have:

$$p_1 = \frac{(1 - \rho_2) E\xi}{E\xi + (\rho_1 - \rho_2) E\nu_L^{(1)}},$$

and

$$p_2 = \frac{\rho_2 E\xi + (\rho_1 - 1) E\nu_L^{(1)}}{E\xi + (\rho_1 - \rho_2) E\nu_L^{(1)}},$$

Proof. Using renewal arguments (e.g. [12]) and relation (2.10), we have:

$$p_1 = \frac{1}{E T_L^{(1)} + E T_L^{(2)} + \frac{1}{\lambda}} = \frac{E\xi}{E\nu_L^{(1)} + E\nu_L^{(2)}},$$

and

$$p_2 = \frac{E T_L^{(2)}}{E T_L^{(1)} + E T_L^{(2)} + \frac{1}{\lambda}} = \frac{\rho_2 E\nu_L^{(2)}}{E\nu_L^{(1)} + E\nu_L^{(2)}},$$

Now, substituting (3.13) for the right sides of (3.16) and (3.17) we obtain relations (3.14) and (3.15) of this lemma. □

3.3. Asymptotic theorems for $p_1$ and $p_2$ under ‘usual assumptions’. By ‘usual assumption’ we only mean the standard cases as $\rho_1 = 1$, $\rho_1 < 1$ or $\rho_1 > 1$ for the asymptotic behaviour as $L \to \infty$. In the following sections the heavy load assumptions are assumed.

The main result of Section 3.2 is Lemma 3.7, where the stationary probabilities $p_1$ and $p_2$ are expressed explicitly via $E\nu_L^{(1)}$. The aim of this section is to obtain the analogue of asymptotic Theorem 3.1 of [2]. To this end, we will derive an asymptotic representation for $E \{\nu_L^{(1)} | \kappa_1 < L\}$ as $L \to \infty$.

Let us first study the asymptotic behavior of $E\nu_L^{(1)}$ as $L \to \infty$. For this purpose derive the representation for the generating function $\sum_{j=0}^{\infty} E\nu_L^{(1)} z^j$. Using representation (3.16), we have:

$$\sum_{j=0}^{\infty} E\nu_L^{(1)} z^j = \sum_{j=0}^{\infty} z^j \sum_{i=0}^{j} E\nu_L^{(1)} \int_0^{\infty} \frac{1}{i!} f_x(u) \left| \right. \bigg|_{u=0} dB_1(x)$$

$$= U(z) - z$$

where

$$U(z) = \int_0^{\infty} \exp \left\{ -\lambda x \left( 1 - \sum_{i=1}^{\infty} r_i z^i \right) \right\} dB_1(x)$$

$$= \hat{B}_1(\lambda - \lambda \hat{R}(z)).$$
(By \( \hat{B}_1(s) \) we denote the Laplace-Stieltjes transform of \( B_1(x) \) \((\Re(s) \geq 0)\), and \( \hat{R}(z) = \sum_{i=1}^{\infty} r_i z_i, \mid z \mid \leq 1 \). Hence, from (3.19) and (3.18) we obtain:

\[
(3.20) \quad \sum_{j=0}^{\infty} \mathbb{E}\hat{\nu}^{(1)}(j) z^j = \frac{\hat{B}_1(\lambda - \lambda \hat{R}(z))}{\hat{B}_1(\lambda - \lambda \hat{R}(z)) - z}.
\]

Notice, that the right-hand side of (3.18) and hence that of (3.20) has the same form as (3.1). Therefore, according to Lemmas 3.1 and 3.2, the asymptotic behaviour of \( \mathbb{E}\hat{\nu}^{(1)}_L \), as \( L \to \infty \), is given by the following statements.

**Lemma 3.8.** If \( \rho_1 < 1 \), then

\[
(3.21) \quad \lim_{L \to \infty} \mathbb{E}\hat{\nu}^{(1)}_L = \frac{1}{1 - \rho_1}.
\]

If \( \rho_1 = 1 \), and additionally \( \rho_{1,2} = \int_0^{\infty} (\lambda x)^2 dB_1(x) < \infty \) and \( E_2 < \infty \), then

\[
(3.22) \quad \mathbb{E}\hat{\nu}^{(1)}_L - \mathbb{E}\hat{\nu}^{(1)}_{L-1} = \frac{2E_2}{\rho_{1,2}(E_2)^3 + E_2 - E_2} + o(1).
\]

If \( \rho_1 > 1 \), then

\[
(3.23) \quad \lim_{L \to \infty} \left[ \mathbb{E}\hat{\nu}^{(1)}_L - \frac{1}{\varphi L[1 + \lambda \hat{B}_1'(\lambda - \lambda \hat{R}(\varphi))\hat{R}'(\varphi)]} \right] = \frac{1}{1 - \rho_1},
\]

where \( \varphi \) is the root of the functional equation \( z = \hat{B}_1(\lambda - \lambda \hat{R}(z)) \) that is least in absolute value.

**Proof.** Asymptotic relations (3.21) and (3.23) follow by application of those (3.2) and, respectively, (3.3) of Lemma 3.1.

In order to prove asymptotic relation (3.22) we should apply the Tauberian theorem of Postnikov (Lemma 3.2). Then asymptotic relation (3.22) is to follow from (3.3) if we prove that the Tauberian condition \( f_0 + f_1 < 1 \) of Lemma 3.2 is satisfied. In the case of the present model, we must prove that for some \( \lambda_0 > 0 \) the equality

\[
(3.24) \quad \int_0^{\infty} e^{-\lambda_0 x}(1 + \lambda_0 r_1 x)dB_1(x) = 1
\]
is not the case. Without loss of generality \( r_1 \) in (3.24) can be set to be equal to 1, since

\[
\int_0^{\infty} e^{-\lambda_0 x}(1 + \lambda_0 r_1 x)dB_1(x) \leq \int_0^{\infty} e^{-\lambda_0 x}(1 + \lambda_0 x)dB_1(x).
\]

Thus, we should prove the inequality

\[
\int_0^{\infty} e^{-\lambda x}(1 + \lambda x)dB_1(x) < 1.
\]

Indeed, \( \int_0^{\infty} e^{-\lambda x}(1 + \lambda x)dB_1(x) \) is an analytic function in \( \lambda \), and hence, according to the theorem on the maximum module of an analytic function, equality (3.24) where \( r_1 = 1 \) must hold for all \( \lambda_0 \geq 0 \). This means that (3.24) is valid if and only if

\[
\int_0^{\infty} e^{-\lambda_0 x}(\lambda_0 x)^{t} dB_1(x) = 0
\]
for all $i \geq 2$ and $\lambda_0 \geq 0$. Since $\int_0^\infty e^{-\lambda x}(-x)^i dB_1(x)$ is the $i$th derivative of the Laplace-Stieltjes transform $\hat{B}_1(\lambda)$, then in this case the Laplace-Stieltjes transform $\hat{B}_1(\lambda)$ must be a linear function in $\lambda$, i.e. $\hat{B}_1(\lambda) = d_0 + d_1 \lambda$, $d_0$ and $d_1$ are some constants. However, since $|\hat{B}_1(\lambda)| \leq 1$, we have $d_0 = 1$ and $d_1 = 0$. This is a trivial case where $B_1(x)$ is concentrated in point 0, and therefore it is not a probability distribution function having a positive mean. Thus (3.21) is not the case, and the aforementioned Tauberian conditions are satisfied.

Now, the final part of the proof of (3.22) reduces to an elementary algebraic calculations:

$$\gamma_2 := \frac{d^2}{dz^2} \hat{B}_1(\lambda - \lambda \hat{R}(z))|_{z=1} = E\zeta^2 - 1 + \rho_{1,2}(E\zeta)^2.$$ 

The lemma is proved. \hfill \Box

With the aid of Lemma 3.8 one can easily obtain the statements on asymptotic behavior of $E\nu_1^{(1)}$, $E\{\nu_1^{(1)}|\zeta \wedge L\}$ and, consequently, $p_1$ and $p_2$. The theorem below characterizes asymptotic behavior of the probabilities $p_1$ and $p_2$ as $L \to \infty$.

**Theorem 3.9.** If $\rho_1 < 1$, then

$$\lim_{L \to \infty} p_1(L) = 1 - \rho_1,$$

$$\lim_{L \to \infty} p_2(L) = 0.$$ 

If $\rho_1 = 1$, and additionally $\rho_{1,2} = \int_0^\infty (\lambda x)^2 dB_1(x) < \infty$ and $E\zeta^2 < \infty$, then

$$\lim_{L \to \infty} L p_1(L) = \frac{\rho_{1,2}(E\zeta)^3 + E\zeta^2 - E\zeta}{2E\zeta},$$

$$\lim_{L \to \infty} L p_2(L) = \frac{\rho_2}{1 - \rho_2} \frac{1 - \rho_{1,2}(E\zeta)^3 + E\zeta^2 - E\zeta}{2E\zeta}.$$ 

If $\rho_1 > 1$, then

$$\lim_{L \to \infty} \frac{p_1(L)}{\varphi^L} = \frac{(1 - \rho_2)[1 + \lambda \hat{B}_1'(\lambda - \lambda \hat{R}(\varphi)) \hat{R}'(\varphi)][1 - \varphi]E\zeta}{(\rho_1 - \rho_2)[1 - \hat{R}(\varphi)]},$$

$$\lim_{L \to \infty} p_2(L) = \frac{\rho_2(\rho_1 - 1)}{\rho_1 - \rho_2},$$

where $\varphi$ is defined in the formulation of Lemma 3.8.

**Proof.** Let us first find asymptotic representation for $E\nu_1^{(1)}|\zeta \wedge L)$ as $L \to \infty$. According to Lemma 3.8 and explicit representation (3.10) we obtain as follows.

If $\rho_1 < 1$, then

$$\lim_{L \to \infty} \frac{E\nu_1^{(1)}|\zeta \wedge L)}{\varphi^L} = \frac{1}{1 - \rho_1} \lim_{L \to \infty} \sum_{i=1}^L i \Pr\{\zeta \wedge L = i\}$$

$$= \frac{E\zeta}{1 - \rho_1}. $$

If $\rho_1 = 1$, and additionally $\rho_{1,2} = \int_0^\infty (\lambda x)^2 dB_1(x) < \infty$ and $E\zeta^2 < \infty$, then

$$\lim_{L \to \infty} \frac{p_1(L)}{\varphi^L} = \frac{(1 - \rho_2)[1 + \lambda \hat{B}_1'(\lambda - \lambda \hat{R}(\varphi)) \hat{R}'(\varphi)][1 - \varphi]E\zeta}{(\rho_1 - \rho_2)[1 - \hat{R}(\varphi)]},$$

$$\lim_{L \to \infty} p_2(L) = \frac{\rho_2(\rho_1 - 1)}{\rho_1 - \rho_2},$$

where $\varphi$ is defined in the formulation of Lemma 3.8.

If $\rho_1 > 1$, then

$$\lim_{L \to \infty} \frac{p_1(L)}{\varphi^L} = \frac{(1 - \rho_2)[1 + \lambda \hat{B}_1'(\lambda - \lambda \hat{R}(\varphi)) \hat{R}'(\varphi)][1 - \varphi]E\zeta}{(\rho_1 - \rho_2)[1 - \hat{R}(\varphi)]},$$

$$\lim_{L \to \infty} p_2(L) = \frac{\rho_2(\rho_1 - 1)}{\rho_1 - \rho_2},$$

where $\varphi$ is defined in the formulation of Lemma 3.8.
If \( \rho_1 = 1, \rho_{1,2} < \infty \) and \( \Xi \varsigma^2 < \infty \), then

\[
\lim_{L \to \infty} \frac{\mathbb{E}\{
u_L^{(1)} | \varsigma_1 \land L \}}{L} = \frac{2\Xi \varsigma}{\rho_{1,2}(\Xi \varsigma)^3 + \Xi \varsigma^2 - \Xi \varsigma} \lim_{L \to \infty} \sum_{i=1}^{L} \text{Pr}\{\varsigma_1 \land L = i\} = \frac{2(\Xi \varsigma)^2}{\rho_{1,2}(\Xi \varsigma)^3 + \Xi \varsigma^2 - \Xi \varsigma}.
\]

If \( \rho_1 > 1 \), then

\[
\lim_{L \to \infty} \frac{\mathbb{E}\{
u_L^{(1)} | \varsigma_1 \land L \}}{\varphi L} = \frac{1}{1 + \lambda \hat{B}_L'(\lambda - \lambda \hat{R}(\varphi))\hat{R}'(\varphi)} \times \lim_{L \to \infty} \sum_{i=1}^{L} \text{Pr}\{\varsigma_1 \land L = i\} \sum_{j=0}^{i-1} \varphi^j
\]

\[
= \frac{1}{[1 + \lambda \hat{B}_L'(\lambda - \lambda \hat{R}(\varphi))\hat{R}'(\varphi)](1 - \varphi)} \times \lim_{L \to \infty} \sum_{i=1}^{L} \text{Pr}\{\varsigma_1 \land L = i\}(1 - \varphi^i)
\]

\[
= \frac{1 - \hat{R}(\varphi)}{[1 + \lambda \hat{B}_L'(\lambda - \lambda \hat{R}(\varphi))\hat{R}'(\varphi)](1 - \varphi)}.
\]

Therefore, taking into account these limiting relations by virtue of \( (3.11) \) (Corollary 3.6) and explicit representations \( (3.14) \) and \( (3.15) \) (Lemma 3.7) for \( \rho_1 \) and \( \rho_2 \), we finally arrive at the statements of the theorem. The theorem is proved.

**3.4. Asymptotic theorems for \( \rho_1 \) and \( \rho_2 \) under special heavy load conditions.** In this section we establish asymptotic theorems for \( \rho_1 \) and \( \rho_2 \) under heavy load assumptions where (i) \( \rho_1 = 1 + \delta \) or (ii) \( \rho_1 = 1 - \delta \), and \( \delta \) is a vanishing positive parameter as \( L \to \infty \). The theorems presented in this section are analogues of the theorems of \( [2] \) given in Section 4 of that paper. The conditions are special, because these heavy load conditions include the change of the parameter \( \rho_1 \) as \( L \) increases to infinity and \( \delta \) vanishes, but the other load parameter \( \rho_2 \) remains unchanged.

In case (i) we have the following two theorems.

**Theorem 3.10.** Assume that \( \rho_1 = 1 + \delta, \delta > 0 \) and that \( L\delta \to C > 0 \) as \( \delta \to 0 \) and \( L \to \infty \). Assume that \( \rho_{1,2}(L) \) is a bounded sequence, assume that \( \Xi \varsigma^3 < \infty \) and that the limit \( \lim_{L \to \infty} \rho_{1,2}(L) = \tilde{\rho}_{1,2} \) exists. Then,

\[
p_1 = \frac{\exp \left( \frac{2\Xi \varsigma}{\tilde{\rho}_{1,2}(\Xi \varsigma)^3 + \Xi \varsigma^2 - \Xi \varsigma} \right) - 1}{\delta \exp \left( \frac{2\Xi \varsigma}{\rho_{1,2}(\Xi \varsigma)^3 + \Xi \varsigma^2 - \Xi \varsigma} \right) - 1} \left[ 1 + o(1) \right],
\]

\[
p_2 = \frac{\delta \rho_2 \exp \left( \frac{2\Xi \varsigma}{\rho_{1,2}(\Xi \varsigma)^3 + \Xi \varsigma^2 - \Xi \varsigma} \right)}{(1 - \rho_2) \left[ \exp \left( \frac{2\Xi \varsigma}{\rho_{1,2}(\Xi \varsigma)^3 + \Xi \varsigma^2 - \Xi \varsigma} \right) - 1 \right]} \left[ 1 + o(1) \right].
\]
Proof. Note first, that under assumptions of the theorem there is the following expansion for \( \varphi \):

\[
\varphi = 1 - \frac{2\delta \varepsilon}{\tilde{\rho}_{1,2}(\varepsilon)^3 + \varepsilon^2 - \varepsilon} + O(\delta^2).
\]

This expansion is similar to that given originally in the book of Subhankulov [13], p.362, and its proof is provided as follows. Write the equation \( \varphi = \hat{B}_1(\lambda - \hat{\lambda}\hat{R}(\varphi)) \) and expand the right-hand side by Taylor’s formula taking \( \varphi = 1 - z \), where \( z \) is small enough, when \( \delta \) is small. We obtain:

\[
1 - z = 1 - (1 + \delta)z + \frac{\tilde{\rho}_{1,2}(\varepsilon)^3 + (1 + \delta)(\varepsilon^2 - \varepsilon)}{2\varepsilon} z^2 + O(z^3).
\]

Disregarding the small term \( O(z^3) \) in (3.37) we arrive at the quadratic equation

\[
\delta z - \frac{\tilde{\rho}_{1,2}(\varepsilon)^3 + (1 + \delta)(\varepsilon^2 - \varepsilon)}{2\varepsilon} z^2 = 0.
\]

The positive solution of (3.38),

\[
z = \frac{2\delta \varepsilon}{\tilde{\rho}_{1,2}(\varepsilon)^3 + (1 + \delta)(\varepsilon^2 - \varepsilon)}
\]

leads to the expansion given by (3.36).

Let us now expand the right-hand side of (3.33) when \( \delta \) is small. For the term \( 1 + \hat{\lambda}\hat{B}_1'(\lambda - \hat{\lambda}\hat{R}(\varphi))\hat{R}'(\varphi) \) we have the expansion

\[
1 + \lambda\hat{B}_1'(\lambda - \hat{\lambda}\hat{R}(\varphi))\hat{R}'(\varphi) = \delta + O(\delta^2),
\]

Then, according to the l’Hospital rule

\[
\lim_{u \to 1} \frac{1 - \hat{R}(u)}{1 - u} = \varepsilon.
\]

Hence

\[
\frac{1 - \hat{R}(\varphi)}{1 - \varphi} = \varepsilon[1 + o(1)].
\]

Substituting (3.36), (3.39) and (3.40) into (3.33) we obtain the expansion

\[
\mathbb{E} \{ \nu_{1,2}^{(1)} | \xi \land L \} = \exp\left( \frac{2C\varepsilon}{\tilde{\rho}_{1,2}(\varepsilon)^3 + \varepsilon^2 - \varepsilon} \right) - 1 - \varepsilon[1 + o(1)].
\]

Hence, relations (3.34) and (3.35) of the theorem follow by virtue of (3.11) (Corollary 3.6) and explicit representations (3.14) and (3.15) (Lemma 3.7) for \( p_1 \) and \( p_2 \).

\[ \square \]

**Theorem 3.11.** Under the conditions of Theorem 3.10 assume that \( C = 0 \). Then,

\[
\lim_{L \to \infty} L p_1(L) = \frac{\tilde{\rho}_{1,2}(\varepsilon)^3 + \varepsilon^2 - \varepsilon}{2\varepsilon},
\]

\[
\lim_{L \to \infty} L p_2(L) = \frac{\rho_{1,2}(\varepsilon)^3 + \varepsilon^2 - \varepsilon}{2\varepsilon}.
\]

**Proof.** The statement of the theorem follows by expanding the main terms of asymptotic relations (3.31) and (3.35) for small \( C \).

\[ \square \]

In case (ii) we have the following two theorems.
Theorem 3.12. Assume that \( \rho_1 = 1 - \delta, \delta > 0 \) and that \( L\delta \to C > 0 \) as \( \delta \to 0 \) and \( L \to \infty \). Assume that \( \rho_{1,3}(L) \) is a bounded function, assume that \( \mathbb{E}\xi^3 < \infty \) and that the limit \( \lim_{L \to \infty} \rho_{1,2}(L) = \bar{\rho}_{1,2} \) exists. Then,

\[
\begin{align*}
(3.44) \quad p_1 &= \delta \exp \left( \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2C\xi} \right) \left[ 1 + o(1) \right], \\
(3.45) \quad p_2 &= \delta \rho_2 \exp \left( \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2C\xi} \right) \left[ 1 + o(1) \right].
\end{align*}
\]

Proof. The explicit representation for the generating function for \( \mathbb{E}^{\nu^{(1)}_L} \) is given by (3.20). Since the sequence \( \{\mathbb{E}^{\nu^{(1)}_L}\} \) is increasing, then the asymptotic behavior of \( \mathbb{E}^{\nu^{(1)}_L} \) as \( L \to \infty \) under the assumptions \( \rho_1 = 1 - \delta, L\delta \to C \) as \( L \to \infty \) can be found according to a Tauberian theorem of Hardy and Littlewood (see e.g. [9], [13], [14], [20], and [17], p.203). Namely, according to that theorem, the behaviour of \( \mathbb{E}^{\nu^{(1)}_L} \) as \( L \to \infty \) and \( \delta \to 0 \) such that \( \delta L \to C > 0 \) can be found from the asymptotic expansion of

\[
(3.46) \quad (1 - z) \frac{\hat{B}_1(\lambda - \lambda R(z))}{\hat{B}_1(\lambda - \lambda R(z))} - z
\]
as \( z \uparrow 1 \). Similarly to the evaluation given in the proof of Theorem 4.3 [2], we have:

\[
(3.47) \quad (1 - z) \frac{\hat{B}_1(\lambda - \lambda R(z))}{\hat{B}_1(\lambda - \lambda R(z))} - z = \frac{1 - z}{1 - z - \rho_1(1 - z) + \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2\xi} (1 - z)^2 + O((1 - z)^3)} = \frac{1}{\delta + \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2\xi} (1 - z) + O((1 - z)^2)}
\]

\[
= \frac{1}{\delta \left[ 1 + \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2\delta\xi} (1 - z) \right] + O((1 - z)^2)}
\]

\[
= \frac{1}{\delta \exp \left( \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2\delta\xi} (1 - z) \right) [1 + o(1)].
\]

Therefore, assuming that \( z = \frac{L - 1}{L} \to 1 \) as \( L \to \infty \), from (3.47) we arrive at the following estimate:

\[
(3.48) \quad \mathbb{E}^{\nu^{(1)}_L} = \frac{1}{\delta} \exp \left( - \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2C\xi} \right) [1 + o(1)].
\]

Comparing (3.23) with (3.33) and taking into account (3.40), which holds true in the case of this theorem as well, we obtain:

\[
(3.49) \quad \mathbb{E}\{\nu^{(1)}_L \mid \xi_1 \wedge L\} = \frac{\xi}{\delta} \exp \left( - \frac{\bar{\rho}_{1,2}(\xi) + \xi^2 - \xi}{2C\xi} \right) [1 + o(1)].
\]
Hence, relations (3.44) and (3.45) of the theorem follow by virtue of (3.11) (Corollary 3.6) and explicit representations (3.14) and (3.15) (Lemma 3.7) for \( p_1 \) and \( p_2 \).

\[ \square \]

**Theorem 3.13.** Under the conditions of Theorem 3.12 assume that \( C = 0 \). Then we have (3.42) and (3.43).

**Proof.** The proof of the theorem follows by expanding the main terms of the asymptotic relations (3.44) and (3.45) for small \( C \).

\[ \square \]

### 4. Asymptotic theorems for the stationary probabilities \( q_i \)

The aim of this section is asymptotic analysis of the stationary probabilities \( q_i \), \( i = 1, 2, \ldots, L \) as \( L \to \infty \). The challenge is to first obtain the explicit representation for \( q_i \) in terms of \( E^{(1)}_{\nu_i} \), and then to study the asymptotic behavior of \( q_i \) as \( L \to \infty \) on the basis of the known asymptotic results for \( E^{(1)}_{\nu_i} \) as \( L \to \infty \). The asymptotic results are obtained in the following three cases: \( \rho_1 = 1 \), \( \rho_1 = 1 + \delta \) and \( \rho_1 = 1 - \delta \), where \( \delta \) is a positive small value.

#### 4.1. Explicit representation for the stationary probabilities \( q_i \)

The aim of this section is to prove the following statement.

**Lemma 4.1.** For \( i = 1, 2, \ldots, L \) we have

\[
q_i = \rho_1 p_1 \left( E^{(1)}_{\nu_i} - E^{(1)}_{\nu_{i-1}} \right).
\]

**Proof.** Using renewal arguments (e.g. [12]), relation (2.10) and Wald’s identities:

\[
E^{(1)}_{T_i} = \frac{p_1}{\lambda} E^{(1)}_{\nu_i}, \quad i = 1, 2, \ldots, L,
\]

we have:

\[
q_i = \frac{E^{(1)}_{T_i} - E^{(1)}_{T_{i-1}}}{E^{(1)}_{T_L} + \frac{1}{\lambda}} = \rho_1 \frac{E^{(1)}_{\nu_i} - E^{(1)}_{\nu_{i-1}}}{E^{(1)}_{\nu_L}}, \quad i = 1, 2, \ldots, L.
\]

Taking into account that \( E^{(1)}_{\nu_L} = E^{(1)}_{\nu_L} + E^{(2)}_{\nu_L} \) and then applying the linear representation for \( E^{(2)}_{\nu_L} \) given by (3.13), from (4.2) we obtain:

\[
q_i = \frac{\rho_1(1 - \rho_2)}{E_{\xi} + (\rho_1 - \rho_2)E^{(1)}_{\nu_L}} \left( E^{(1)}_{\nu_i} - E^{(1)}_{\nu_{i-1}} \right), \quad i = 1, 2, \ldots, L.
\]

Hence, representation (4.1) follows from (3.14) (Lemma 3.7), and Lemma 4.1 is proved.

#### 4.2. Asymptotic analysis of the stationary probabilities \( q_i \): The case \( \rho_1 = 1 \)

Let us study asymptotic behavior of the stationary probabilities \( q_i \). We start from the following modified version of (3.22) (Lemma 3.8):

\[
E^{(1)}_{\nu_{L-j}} - E^{(1)}_{\nu_{L-j-1}} = \frac{2E_{\xi}}{\rho_{1,2}(E_{\xi})^3 + E_{\xi}^2 - E_{\xi}} + o(1),
\]
which is assumed to be satisfied under the conditions \( \rho_{1,2} < \infty \) and \( \mathbb{E}_\varsigma^2 < \infty \). Under the same conditions, similarly to (3.32) we obtain:

\[
\mathbb{E}\{q_{L-j}\} \mid \varsigma_1 \wedge L - \mathbb{E}\{q_{L-j-1}\} \mid \varsigma_1 \wedge L = \frac{2\mathbb{E}_\varsigma}{\rho_{1,2}(\mathbb{E}_\varsigma)^3 + \mathbb{E}_\varsigma^2 - \mathbb{E}_\varsigma} \times \sum_{i=1}^{L} i\Pr\{\varsigma_1 \wedge L = i\} + o(1)
\]

(4.4)

Hence, according to (3.11) (Corollary 3.6) and (4.4) we have the estimate

\[
\mathbb{E}q_{L-j} - \mathbb{E}q_{L-j-1} = \frac{2(\mathbb{E}_\varsigma)^2}{\rho_{1,2}(\mathbb{E}_\varsigma)^3 + \mathbb{E}_\varsigma^2 - \mathbb{E}_\varsigma} + o(1).
\]

Asymptotic relations (4.5), (3.27) together with explicit relation (4.1) of Lemma 4.2 leads to the following theorem.

**Theorem 4.2.** In the case \( \rho_1 = 1 \) under the additional conditions \( \rho_{1,2} < \infty \) and \( \mathbb{E}_\varsigma^2 < \infty \) for any \( j \geq 0 \) we have

\[
\lim_{L \to \infty} Lq_{L-j} = 1.
\]

Note, that the asymptotic relation given by (4.6) is not expressed via \( \mathbb{E}_\varsigma \) and, therefore, it is invariant and hence the same as for the queueing system with ordinary Poisson arrivals.

### 4.3. Asymptotic analysis of the stationary probabilities \( q_i \):

**The case** \( \rho_1 = 1 + \delta, \delta > 0 \). In the case \( \rho_1 = 1 + \delta, \delta > 0 \) the asymptotic behaviour of \( q_i \) is specified by the following theorem.

**Theorem 4.3.** Assume that \( \rho_1 = 1 + \delta, \delta > 0 \), and \( L\delta \to C > 0 \) as \( \delta \to 0 \) and \( L \to \infty \). Assume that \( \rho_{1,3}(L) \) is a bounded sequence, assume that \( \mathbb{E}_\varsigma^3 < \infty \) and there exists \( \tilde{\rho}_{1,2} = \lim_{L \to \infty} \rho_{1,2}(L) \). Then, for any \( j \geq 0 \)

\[
q_{L-j} = \frac{\exp\left(\frac{2C\mathbb{E}_\varsigma}{\tilde{\rho}_{1,2}(\mathbb{E}_\varsigma)^3 + \mathbb{E}_\varsigma^2 - \mathbb{E}_\varsigma}\right)}{\exp\left(\frac{2C\mathbb{E}_\varsigma}{\tilde{\rho}_{1,2}(\mathbb{E}_\varsigma)^3 + \mathbb{E}_\varsigma^2 - \mathbb{E}_\varsigma}\right) - 1} \times\left(1 - \frac{2\delta\mathbb{E}_\varsigma}{\tilde{\rho}_{1,2}(\mathbb{E}_\varsigma)^3 + \mathbb{E}_\varsigma^2 - \mathbb{E}_\varsigma}\right)^j + o(\delta).
\]

(4.7)

**Proof.** Expanding (4.5,23) for large \( L \), we have:

\[
\mathbb{E}q_{L-j}^{(1)} = \frac{\varphi^j}{\varphi L[1 + \lambda\hat{B}_1^j(\lambda - \lambda\hat{R}(\varphi))\hat{R}'(\varphi)\varphi^j]} + \frac{1}{1 - \rho_1} + o(1).
\]

In turn, from (4.8) for large \( L \) we obtain:

\[
\mathbb{E}q_{L-j}^{(1)} - \mathbb{E}q_{L-j-1}^{(1)} = \frac{(1 - \varphi)\varphi^j}{\varphi L[1 + \lambda\hat{B}_1^j(\lambda - \lambda\hat{R}(\varphi))\hat{R}'(\varphi)\varphi^j]} + o(1).
\]

(4.9)
From (4.9), similarly to (3.33), we further have:
\[
EE\{\nu_{L-j}^{(1)}|s_1 \land L\} - EE\{\nu_{L-j-1}^{(1)}|s_1 \land L\} = \frac{(1 - \hat{R}(\varphi))(1 - \varphi)\varphi^j}{[1 + \lambda\hat{B}_1'(\lambda - \lambda\hat{R}(\varphi))\hat{R}'(\varphi)][1 - \varphi]} + o(1),
\]
and, according to (3.11) (Corollary 3.6),
\[
(4.10) \quad \mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)} = \frac{(1 - \hat{R}(\varphi))(1 - \varphi)\varphi^j}{[1 + \lambda\hat{B}_1'(\lambda - \lambda\hat{R}(\varphi))\hat{R}'(\varphi)][1 - \varphi]} + o(1).
\]

Next, under the conditions of the theorem, asymptotic expansions (3.36), (3.39) and (3.40) hold. Taking into consideration these expansions we arrive at the following asymptotic relations for \( j = 0, 1, \ldots \):
\[
\mathbb{E}\nu_{L-j}^{(1)} - \mathbb{E}\nu_{L-j-1}^{(1)} = \exp\left(\frac{2\mathbb{E}\xi}{\rho_{1,2}((\mathbb{E}\xi)\delta + \mathbb{E}\xi^2 - \mathbb{E}\xi)}\right)
\times \left(1 - \frac{2\delta\mathbb{E}\xi}{\rho_{1,2}((\mathbb{E}\xi)\delta + \mathbb{E}\xi^2 - \mathbb{E}\xi)}\right)^j \frac{2\delta\mathbb{E}\xi}{\rho_{1,2}((\mathbb{E}\xi)\delta + \mathbb{E}\xi^2 - \mathbb{E}\xi)}[1 + o(1)].
\]

Now, taking into account asymptotic relation (3.34) of Theorem 3.10 and the explicit formula given by (4.11) (Lemma 4.1) we arrive at the statement of the theorem.

4.4. Asymptotic analysis of the stationary probabilities \( q_i \): The case \( \rho_1 = 1 - \delta, \delta > 0 \). In the case \( \rho_1 = 1 - \delta, \delta > 0 \), the study is more delicate and based on special analysis. The additional assumption of this case is that there is a unique root \( \tau > 1 \) of the equation
\[
(4.11) \quad z = \hat{B}_1(\lambda - \lambda\hat{R}(z)),
\]
and there exists the first derivative \( \hat{B}_1'(\lambda - \lambda\hat{R}(\tau)) \).

Under the assumption that \( \rho_1 < 1 \) the unique root of (4.11) is not necessarily unique. Such type of condition has been considered by Willmot [21] to obtain the asymptotic behavior for high queue-level probabilities in stationary \( M/GI/1 \) queues. Denote the stationary probabilities in the \( M/GI/1 \) queueing system by \( q_i[M/GI/1], i = 0, 1, \ldots \). It was shown in [21] that
\[
(4.12) \quad q_i[M/GI/1] = \frac{(1 - \rho_1)(1 - \tau)}{\tau[1 + \lambda\hat{B}_1'(\lambda - \lambda\tau)][1 + o(1)]} \text{ as } i \to \infty,
\]
where \( \hat{B}_1(s) \) denotes the Laplace-Stieltjes transform of the service time distribution in the \( M/G/1 \) queueing system, and \( \tau \) denotes a root of the equation \( z = \hat{B}_1(\lambda - \lambda z) \) greater than 1, which is assumed to be unique. On the other hand, according to the Pollaczek-Khintchine formula (e.g. Takács [16], p.242), \( q_i[M/GI/1] \) can be represented explicitly
\[
(4.13) \quad q_i[M/GI/1] = (1 - \rho_1) \left( \mathbb{E}\nu_i^{(1)} - \mathbb{E}\nu_{i-1}^{(1)} \right), i = 1, 2, \ldots,
\]
where the random variable \( \nu_i^{(1)} \) in this formula is associated with the number of served customers during a busy period of the state dependent \( M/GI/1 \) queueing.
system, where the value of the system parameter, where the service is changed, is $i$ (see Section 2.1). Representation (4.13) can be easily checked, since in this case

$$
\sum_{j=0}^{\infty} E\nu_j^{(1)} z^j = \frac{\hat{B}_1(\lambda - \lambda z)}{\hat{B}_1(\lambda - \lambda z) - z},
$$

and multiplication of the right-hand side of (4.14) by $(1 - \rho_1)(1 - z)$ leads to the well-known Pollaczek-Khintchine formula. Then, from (4.12) and (4.13) there is the asymptotic proportion for large $L$ and any $j \geq 0$:

$$
E\nu_{L-j}^{(1)} - E\nu_{L-j-1}^{(1)} \to \tau^j[1 + o(1)].
$$

In the case of batch arrivals the results are similar. One can prove that the same proportion as (4.15) holds in this case as well, where $\tau$ in the case of batch arrivals denotes a unique real root of the equation of (4.11), which is greater than 1. (Recall that our convention is an existence of a unique real solution of (4.11) greater than 1.) Indeed, the arguments of [21] are elementary extended for the queueing system with batch arrivals. The simplest way to extend these results straightforwardly is to consider the stationary queueing system with batch Poisson arrivals, in which the first batch in each busy period is equal to 1. Denote this system by $M^{1,X}/G/1.

For this specific system, similarly to (4.12) we obtain:

$$
q_i[M^{1,X}/GI/1] = \frac{(1 - \rho_1)(1 - \tau)}{\tau'[1 + \lambda \hat{B}_1'(\lambda - \lambda R)(\tau)]} [1 + o(1)] as i \to \infty,
$$

where $q_i[M^{1,X}/GI/1], i = 0, 1, \ldots$, denotes the stationary probabilities in this system. Then, taking into account (3.20), similarly to (4.13) one can write

$$
q_i[M^{1,X}/GI/1] = (1 - \rho_1) \left( E\nu_{i}^{(1)} - E\nu_{i-1}^{(1)} \right), i = 1, 2, \ldots
$$

From (4.16) and (4.17) we obtain

$$
\frac{E\nu_{L-j}^{(1)} - E\nu_{L-j-1}^{(1)}}{E\nu_{L}^{(1)} - E\nu_{L-1}^{(1)}} = \tau^j[1 + o(1)].
$$

From (4.18) and the results of Sections 3.2 and 3.3 (see (3.21), (3.31) and (3.11) we also have the estimate

$$
\frac{E\nu_{L-j}^{(1)} - E\nu_{L-j-1}^{(1)}}{E\nu_{L}^{(1)} - E\nu_{L-1}^{(1)}} = \tau^j[1 + o(1)],
$$

which coincides with (4.15).

Now we formulate and prove a theorem on asymptotic behavior of the stationary probabilities $q_i$ in the case $\rho_1 = 1 - \delta$, $\delta > 0$. The special assumption in this theorem is that the class of probability distributions $\{B_1(x)\}$ is defined according to the above convention. More precisely, in the case $\rho_1 = 1 - \delta$, $\delta > 0$, and vanishing $\delta$ as $L \to \infty$ this means that there exists $\epsilon_0 > 0$ (small enough) such that for all $0 \leq \epsilon \leq \epsilon_0$, the above family of probability distribution functions $\tilde{B}_{1,\epsilon}(x)$ (depending now on the parameter $\epsilon$) satisfies the following properties. Let $\tilde{B}_{1,\epsilon}(s)$
denote the Laplace-Stieltjes transform of $B_1(x)$. We assume that any $\tilde{B}_{1,\epsilon}(s)$ is an analytic function in a small neighborhood of zero, and

$$
(4.20) \quad \tilde{B}'_{1,\epsilon}(s) < \infty.
$$

Property (4.20) is required for the existence of the probabilities $q_i$. Relation (4.16) contains the term $\tilde{B}'_1(\lambda - \lambda \tilde{R}(\tau))$, and this term must be finite. In addition, the term $\tilde{R}(\tau) < \infty$ must be finite as well, that is, the additional to (4.20) associated assumption is that

$$
(4.21) \quad \tilde{R}'(1 + \epsilon) < \infty
$$

for any $\epsilon$ of the defined neighborhood. Choice of small parameter $\epsilon$ is continuously connected with that choice of the parameter $\delta$ (or $L$) in the theorem below.

**Theorem 4.4.** Assume that the class of probability distribution functions $\{B_1(x)\}$ and the probabilities $r_1, r_2, \ldots$ are defined according to the conventions made and, respectively, satisfy (4.20) and (4.21), $\rho_1 = 1 - \delta, \delta > 0$, and $L \delta \to C > 0$, as $\delta \to 0$ and $L \to \infty$. Assume that $\rho_{1,3} = \rho_{1,3}(L)$ is a bounded sequence, $E_3^3 < \infty$, and there exists $\tilde{p}_{1,2} = \lim_{L \to \infty} \rho_{1,2}(L)$. Then,

$$
q_{L-j} = \frac{1}{\exp \left( \frac{2C E_3}{\rho_{1,2}(E_3)^3 + E_3^2 - E_3} \right) - 1} \times \frac{2\delta E_3}{\rho_{1,2}(E_3)^3 + E_3^2 - E_3} \left( 1 + \frac{2\delta E_3}{\rho_{1,2}(E_3)^3 + E_3^2 - E_3} \right)^j [1 + o(1)]
$$

for any $j \geq 0$.

**Proof.** Under the assumptions of this theorem let us first derive the following asymptotic expansion:

$$
(4.23) \quad \tau = 1 + \frac{2\delta E_3}{\rho_{1,2}(E_3)^3 + E_3^2 - E_3} + O(\delta^2).
$$

Asymptotic expansion (4.23) is similar to that of (3.36), and its proof is also similar. Namely, taking into account that the equation $z = \tilde{B}_1(\lambda - \lambda \tilde{R}(\tilde{z}))$ has a unique solution in the set $(1, \infty)$, and this solution approaches 1 as $\delta$ vanishes. Therefore, by the Taylor expansion of this equation around the point $z = 1$, we have:

$$
(4.24) \quad 1 + z = 1 + (1 - \delta) z + \frac{\tilde{p}_{1,2}(E_3)^3 + (1 - \delta)(E_3^2 - E_3)}{2E_3} z^2 + O(z^3).
$$

Disregarding the term $O(z^3)$, from (4.24) we arrive at the quadratic equation

$$
\delta z - \tilde{p}_{1,2}(E_3)^3 + (1 - \delta)(E_3^2 - E_3) z^2 = 0,
$$

and obtain a positive solution

$$
z = \frac{2\delta E_3}{\tilde{p}_{1,2}(E_3)^3 + (1 + \delta)(E_3^2 - E_3)}.
$$

This proves (4.23).

Next, from (4.19), (4.23) and explicit formula (4.1) we obtain

$$
(4.25) \quad q_{L-j} = qL \left( 1 + \frac{2\delta E_3}{\tilde{p}_{1,2}(E_3)^3 + E_3^2 - E_3} \right)^j [1 + o(1)].
$$
Taking into consideration

\[
\sum_{j=0}^{L-1} \left( 1 + \frac{2\delta E \varsigma}{\rho_{1,2}(E \varsigma)^3 + E \varsigma^2 - E \varsigma} \right)^j
\]

from the normalization condition \( p_1 + p_2 + \sum_{i=1}^{L} q_i = 1 \) and the fact that both \( p_1 \) and \( p_2 \) have the order \( O(\delta) \), we obtain:

\[
q_L = \frac{2\delta E \varsigma}{\rho_{1,2}(E \varsigma)^3 + E \varsigma^2 - E \varsigma} \cdot \frac{1}{\exp \left( \frac{2CE \varsigma}{\rho_{1,2}(E \varsigma)^3 + E \varsigma^2 - E \varsigma} \right) - 1} [1 + o(1)].
\]

The desired statement of the theorem follows from (4.26). □

5. Objective function

5.1. The case \( \rho_1 = 1 \). In this section we prove the following result.

**Proposition 5.1.** In the case \( \rho_1 = 1 \), under the additional conditions \( \rho_{1,2} < \infty \) and \( E \varsigma^2 < \infty \) we have:

\[
\lim_{L \to \infty} J(L) = j_1 \frac{\rho_{1,2}(E \varsigma)^3 + E \varsigma^2 - E \varsigma}{2E \varsigma} + j_2 \frac{\rho_{1,2}(E \varsigma)^3 + E \varsigma^2 - E \varsigma}{2E \varsigma} + c^*,
\]

where

\[
c^* = \lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} c_i.
\]

**Proof.** The first two terms in the right-hand side of (5.1) follow from asymptotic relations (3.27) and (3.28) (Theorem 3.9). The last term \( c^* \) of the right-hand side of (5.1) follows from (4.6) (Theorem 4.2), since

\[
\lim_{L \to \infty} \sum_{i=1}^{L} q_i c_i = \lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} c_i = c^*.
\]

5.2. The case \( \rho = 1 + \delta, \delta > 0 \). In the case \( \rho = 1 + \delta, \delta > 0 \) we have the following statement.
Proposition 5.2. Under the assumptions of Theorem 4.3 denote the objective function $J$ by $J_{\text{upper}}$. We have the following representation:

$$
J_{\text{upper}} = C \left[ j_1 \frac{1}{\exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right) - 1} \right. \\
+ j_2 \frac{\rho_2 \exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right)}{(1 - \rho_2) \left( \exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right) - 1 \right)} \left. \right] \\
+ c_{\text{upper}},
$$

(5.2)

where

$$
c_{\text{upper}} = \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right) - 1 \\
\times \lim_{L \to \infty} \frac{1}{L} \tilde{C}_L \left( 1 - \frac{2CE_\xi}{(\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right),
$$

(5.3)

and $\tilde{C}_L(z) = \sum_{j=0}^{L-1} c_{L-j} z^j$ is a backward generating cost function.

Proof. The representation for the term

$$
C \left[ j_1 \frac{1}{\exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right) - 1} \right. \\
+ j_2 \frac{\rho_2 \exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right)}{(1 - \rho_2) \left( \exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right) - 1 \right)} \left. \right] 
$$

of the right-hand side of (5.2) follows from (4.34) and (4.35) (Theorem 3.10). This term is similar to that in (5.2) in [2]. The new term which takes into account the water costs is $c_{\text{upper}}$. Taking into account representation (4.7), for this term we obtain:

$$
c_{\text{upper}} = \lim_{L \to \infty} \sum_{j=0}^{L-1} q_{L-j} c_{L-j} \\
= \lim_{L \to \infty} \sum_{j=0}^{L-1} c_{L-j} \cdot \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \exp \left( \frac{2CE_\xi}{\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi} \right) - 1 \\
\times \left( \frac{2\delta LE_\xi}{(\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \frac{2\delta LE_\xi}{(\rho_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L},
$$
and, because of $\lim_{L \to \infty} \delta L = C$, representation (5.3) follows.

5.3. **The case $\rho = 1 - \delta$, $\delta > 0$.** In the case $\rho = 1 - \delta$, $\delta > 0$ we have the following statement.

**Proposition 5.3.** Under the assumptions of Theorem 4.4 denote the objective function $J$ by $J_{\text{lower}}$. We have the following representation

$$
J_{\text{lower}} = C \left[ j_1 \exp \left( \frac{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma}{2C\bar{E}_\Sigma} \right) + j_2 \frac{\rho_2}{1 - \rho_2} \left( \exp \left( \frac{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma}{2C\bar{E}_\Sigma} \right) - 1 \right) \right]
$$

(5.4)

where

$$
c_{\text{lower}} = \frac{2C\bar{E}_\Sigma}{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma} \cdot \frac{1}{\exp \left( \frac{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma}{2C\bar{E}_\Sigma} \right) - 1}
$$

(5.5)

and $\hat{C}_L(z) = \sum_{j=0}^{L-1} c_{L-j}z^j$ is a backward generating cost function.

**Proof.** The representation for the term

$$
C \left[ j_1 \exp \left( \frac{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma}{2C\bar{E}_\Sigma} \right) + j_2 \frac{\rho_2}{1 - \rho_2} \left( \exp \left( \frac{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma}{2C\bar{E}_\Sigma} \right) - 1 \right) \right]
$$

of the right-hand side of (5.4) follows from (3.44) and (3.45) (Theorem 3.12). This term is similar to that (5.3) in [2]. The new term, which takes into account the water costs, is $c_{\text{lower}}$. Taking into account representation (4.22), for this term we obtain:

$$
c_{\text{lower}} = \lim_{L \to \infty} \sum_{j=0}^{L-1} q_{L-j}c_{L-j}
$$

$$
\lim_{L \to \infty} \sum_{j=0}^{L-1} c_{L-j} \cdot \frac{1}{\exp \left( \frac{2C\bar{E}_\Sigma}{\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma} \right) - 1}
$$

$$
\times \left( 1 + \frac{2\delta L\bar{E}_\Sigma}{(\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma) L} \right)^j \frac{2\delta L\bar{E}_\Sigma}{(\tilde{\rho}_{1,2}(\bar{E}_\Sigma)^3 + \bar{E}_\Sigma^2 - \bar{E}_\Sigma) L},
$$

and, because of $\lim_{L \to \infty} \delta L = C$, representation (5.5) follows.

6. **A solution to the control problem and its properties.**

In this section we discuss the solution to the control problem and study its properties. The functionals $J_{\text{upper}}$ and $J_{\text{lower}}$ are correspondingly given by (5.2) and (5.4), and the last terms in these functionals are correspondingly given by (5.3) and (5.5). For our further analysis we need in other representations for these last terms.
Denote
\[
\psi(C) = \lim_{L \to \infty} \frac{\sum_{j=0}^{L-1} c_{L-j} \left( 1 - \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j}{\sum_{j=0}^{L-1} \left( 1 - \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j},
\]
and
\[
\eta(C) = \lim_{L \to \infty} \frac{\sum_{j=0}^{L-1} c_{L-j} \left( 1 + \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j}{\sum_{j=0}^{L-1} \left( 1 + \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j}.
\]
Since \( \{c_i\} \) is a bounded sequence, then the limits of (6.1) and (6.2) do exist.

The relations between \( c_{\text{upper}} \) and \( \psi(C) \) and, respectively, between \( c_{\text{lower}} \) and \( \eta(C) \) are given in the lemma below.

**Lemma 6.1.** We have:
\[
c_{\text{upper}} = \psi(C),
\]
and
\[
c_{\text{lower}} = \eta(C).
\]

**Proof.** From (6.1) and (6.2) we correspondingly have the representations:
\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left( 1 - \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j = \psi(C) \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j,
\]
and
\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left( 1 + \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j = \eta(C) \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 + \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j.
\]
The desired results follow by direct transformations of the corresponding right-hand sides of (6.5) and (6.6).

Indeed, for the right-hand side of (6.5) we obtain:
\[
\psi(C) \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j = \psi(C) \left[ 1 - \exp \left( -\frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right) \right] \frac{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L}{2CE_\xi}.
\]
On the other hand, from (5.3) we have:

\[
(6.8)
\]

\[
c^{\text{upper}} \left[ 1 - \exp \left( - \frac{2CE_\zeta}{\bar{p}_{1,2}(E_\zeta)^3 + E_\zeta^2 - E_\zeta} \right) \right] \frac{\bar{p}_{1,2}(E_\zeta)^3 + E_\zeta^2 - E_\zeta}{2CE_\zeta}
\]

\[
= \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left( 1 - \frac{2CE_\zeta}{(\bar{p}_{1,2}(E_\zeta)^3 + E_\zeta^2 - E_\zeta)L} \right)^j.
\]

Hence, from (6.5), (6.7) and (6.8) we obtain (6.1). The proof of (6.4) is completely analogous and uses the representations of (5.5) and (6.6). □

The next lemma establishes the main properties of functions \( \psi(C) \) and \( \eta(C) \).

**Lemma 6.2.** The function \( \psi(C) \) is a non-increasing function, and its maximum is \( \psi(0) = c^* \). The function \( \eta(C) \) is a non-decreasing function, and its minimum is \( \eta(0) = c^* \). (Recall that \( c^* = \lim_{L \to \infty} \frac{1}{L} \sum_{i=1}^{L} c_i \) is defined in Proposition 5.1)

**Proof.** Let us first prove that \( \psi(0) = c^* \) is a maximum of \( \psi(C) \). For this purpose we use the following well-known inequality (e.g. Hardy, Littlewood and Pólya [7] or Marschall and Olkin [8]). Let \{\( a_n \)\} and \{\( b_n \)\} be arbitrary sequences, one of them is increasing and another decreasing. Then for any finite sum we have

\[
(6.9)
\]

\[
\sum_{n=1}^{l} a_n b_n \leq \frac{1}{l} \sum_{n=1}^{l} a_n \sum_{n=1}^{l} b_n.
\]

Applying inequality (6.9) to the finite sums of the left-hand side of (6.5) and passing to the limit as \( L \to \infty \), we have

\[
(6.10)
\]

\[
\leq \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2CE_\zeta}{(\bar{p}_{1,2}(E_\zeta)^3 + E_\zeta^2 - E_\zeta)L} \right)^j.
\]

Then, comparing (6.5) with (6.10) enables us to conclude,

\[
\psi(0) = c^* \geq \psi(C),
\]

i.e. \( \psi(0) = c^* \) is the maximum value of \( \psi(C) \).

Prove now, that \( \psi(C) \) is a not increasing function, i.e. for any nonnegative \( C_1 \leq C \) we have \( \psi(C) \leq \psi(C_1) \).

To prove this note, that for small positive \( \delta_1 \) and \( \delta_2 \) we have \((1-\delta_1-\delta_2) = (1-\delta_1)(1-\delta_2) + O(\delta_1\delta_2)\). Using this idea, one can prove the monotonicity of \( \psi(C) \) by the
replacement

\[ 1 - \frac{2CE_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \]

(6.11)

\[ = \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right) \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right) \]

\[ + O \left( \frac{1}{L^2} \right), \quad C > C_1 \]

in the above asymptotic relations for large \( L \). Indeed, notice that

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \times \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j
\]

(6.12)

\[ = \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \]

\[ \times \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j. \]

Therefore, for any non-decreasing sequence \( a_j \)

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \times \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j
\]

(6.13)

\[ \leq \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \times \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j. \]

Indeed, assume for contrary that

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \times \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j
\]

(6.14)

\[ > \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C_1E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j \times \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1)E_\xi}{(\bar{\rho}_{1,2}(E_\xi)^3 + E_\xi^2 - E_\xi)L} \right)^j. \]
Then, applying inequality (6.9) to the right-hand side of (6.14), we obtain:

\[
\begin{align*}
\lim_{L \to \infty} & \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C_1 E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j \\
\times & \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1) E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j \\
\leq & \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2C_1 E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j \\
\times & \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1) E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j \\
= & \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2C E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j.
\end{align*}
\]

(6.15)

Since the left-hand side of (6.14) is

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j
\]

(see relation (6.11)), then comparison of the last obtained term in (6.15) with the left-hand side of (6.14) enables us to write:

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \left( 1 - \frac{2C E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j > \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} a_j \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} \left( 1 - \frac{2C E_\varsigma}{(\bar{\rho}_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma) L} \right)^j.
\]

The contradiction with basic inequality (6.9) proves (6.13).
Taking into account \((6.12)\) and \((6.13)\), the extended version of \((6.11)\) after an application of \((6.9)\) now looks

\[
\lim_{L \to \infty} \sum_{j=0}^{L-1} \frac{C_{L-j}}{L} \left( 1 - \frac{2CE\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\]

\[
= \lim_{L \to \infty} \sum_{j=0}^{L-1} \frac{C_{L-j}}{L} \left( 1 - \frac{2C_1E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\times \left( 1 - \frac{2(C - C_1)E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\]

\[
(6.16)
\]

\[
\leq \lim_{L \to \infty} \sum_{j=0}^{L-1} \frac{C_{L-j}}{L} \left( 1 - \frac{2C_1E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\times \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1)E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\]

\[
= \psi(C_1) \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2C_1E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\times \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1)E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j .
\]

On the other hand, the right-hand side of \((6.5)\) can be rewritten

\[
\psi(C) \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2CE\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\]

\[
= \psi(C) \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2C_1E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\times \left( 1 - \frac{2(C - C_1)E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\]

\[
(6.17)
\]

\[
= \psi(C) \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2C_1E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j
\times \lim_{L \to \infty} \sum_{j=0}^{L-1} \left( 1 - \frac{2(C - C_1)E\varsigma}{(\rho_{1,2}(E\varsigma)^3 + E\varsigma^2 - E\varsigma)\cdot L} \right)^j .
\]

The last equality in \((6.17)\) is the application of \((6.12)\). From \((6.10)\) and \((6.17)\) we finally obtain \(\psi(C_1) \leq \psi(C)\) for any positive \(C_1 \geq C\).

The first statement of Lemma \(6.2\) is proved. The proof of the second statement of this lemma is similar.

In the following we need in a stronger result that is given by Lemma \(6.2\). Namely, we will prove the following lemma.
Lemma 6.3. If the sequence \( \{c_i\} \) contains at least two distinct values, then the function \( \psi(C) \) is a strictly decreasing function, and the function \( \eta(C) \) is a strictly increasing function.

Proof. In order to prove this lemma it is sufficient to prove that if the sequence \( \{c_i\} \) is nontrivial, that is there are at least two distinct values of this sequence, then for any distinct real numbers \( C_1 \neq C_2 \) the values of functions are also distinct, that is, \( \psi(C_1) \neq \psi(C_2) \) and \( \eta(C_1) \neq \eta(C_2) \). Let us prove the first inequality: \( \psi(C_1) \neq \psi(C_2) \). Rewrite (6.1) as

\[
\psi(C) = \lim_{L \to \infty} \frac{1}{L} \sum_{j=0}^{L-1} c_{L-j} \left( 1 - \frac{2CE_\varsigma}{(\rho_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma)L} \right)^j.
\]

The limit of the denominator is equal to \( \exp \left( - \frac{2CE_\varsigma}{\rho_{1,2}(E_\varsigma)^3 + E_\varsigma^2 - E_\varsigma} \right) \). The limit of the numerator does exist and bounded, since the sequence \( \{c_i\} \) is assumed to be bounded. As well, according to the other representation following from Lemma 6.1 and relation (5.3), it is an analytic function in \( C \) taking a nontrivial set of values.

The analytic function \( \psi(C) \) is defined for all real \( C \geq 0 \) and it can be extended analytically for the whole complex plane. Namely, for real negative values \( C \) we arrive at the function \( \eta(C) = \psi(-C) \). According to the maximum principle for the module of an analytic function, if an analytic function takes the same values in two distinct points inside a domain, then the function must be the constant. If \( c_i = c^* \) for all \( i = 1, 2, \ldots \), then this is just the case where \( \psi(C) = c^* \) for all \( C \). If there exist \( i_0 \) and \( i_1 \) such that \( c_{i_0} \neq c_{i_1} \), then the function \( \psi(C) \) cannot be a constant, because the analytic function is uniquely defined by the coefficients in the series expansion. So, the inequality \( \psi(C_1) \neq \psi(C_2) \) for distinct values \( C_1 \) and \( C_2 \) follows. The proof of the second inequality \( \eta(C_1) \neq \eta(C_2) \) is similar. \( \square \)

We are ready now to formulate and prove a main theorem on optimal control of the dam model considered in the present paper.

Theorem 6.4. Under the assumption that the costs \( c_i \) are not increasing, and under additional mild conditions of Theorems 4.3 and 4.4, there is a unique solution to the control problem. The solution to the control problem is defined by choice of the parameter \( p_1 \) as follows.

Let \( \overline{C} \) be the minimum value of the functional \( J^{\text{upper}} \) defined in (5.2) and (5.3) and, respectively, let \( \underline{C} \) be the minimum value of the functional \( J^{\text{lower}} \) defined in (5.4) and (5.5). Then at least one of two parameters \( \overline{C} \) or \( \underline{C} \) must be equal to zero.

1. In the case \( \overline{C} = 0 \) and \( \underline{C} > 0 \), the solution to the control problem is achieved for \( p_1 = 1 - \delta \), where positive \( \delta \) vanishes such that \( \delta L \to \underline{C} \) as \( L \to \infty \).
2. In the case \( \overline{C} = 0 \) and \( \underline{C} > 0 \), the solution to the control problem is achieved for \( p_1 = 1 + \delta \), where positive \( \delta \) vanishes such that \( \delta L \to \underline{C} \) as \( L \to \infty \).
3. In the case where both \( \overline{C} = 0 \) and \( \underline{C} = 0 \), the solution to the control problem is \( p_1 = 1 \).

Proof. Note first, that under the assumptions made there is a unique solution to the control problem considered in this paper. Indeed, a solution contains two terms one of them corresponds to the expression for \( p_1J_1 + p_2J_2 \) in (1.2) and another
one corresponds to the term \( \sum_{i=L_{\text{lower}}+1}^{L_{\text{upper}}} c_i q_i \) in (1.2). The first term of a solution is related to the models where the water costs are not taken into account, while the additional second term is related to the extended problem, where the water costs are taken into account.

In the case where the water costs are not taken into account, the existence of a unique solution to the control problem for the particular system in [2] follows from the main result of that paper. The same result holds true for a more general model with compound Poisson input flow but without water costs included. The last is supported by Theorems 3.10 - 3.13, which are similar to those Theorems 4.1 - 4.4 of [2].

In the case of the dam model, where the water costs are taken into account, the minimum of \( J^* \) is achieved for \( p_1 J_1 + p_2 J_2 \) of the objective function in (1.2) coincides with the term

\[
\sum_{i=L_{\text{lower}}+1}^{L_{\text{upper}}} c_i q_i
\]

in (5.1). That is both the minimum of \( J^* \) and that of \( J^\text{lower} \) are the same, and they are equal to the right-hand side of (5.1). In this case the minimum of the objective function in (1.2) is achieved for \( p_1 = 1 \).

If the minimum of \( J^\text{lower} \) is achieved for \( C = \overline{C} > 0 \), then, since \( \psi(C) \) is strictly increasing, we have \( c^\text{lower} > c^* \), and hence the term \( p_1 J_1 + p_2 J_2 \) of the objective function in (1.2) satisfies the inequality:

\[
p_1 J_1 + p_2 J_2 < j_1 \frac{\bar{\rho}_{1,2}(E\xi)^3 + \xi^2 - \xi}{2E\xi} + j_2 \frac{\rho_2}{1 - \rho_2} \frac{\bar{\rho}_{1,2}(E\xi)^3 + \xi^2 - \xi}{2E\xi}.
\]

This implies that \( J^\text{lower} \) is less than the right-hand side of (6.1). As well, in this case \( c^\text{upper} < c^* \), and the minimum of \( J^\text{upper} \) must be achieved for \( C = \overline{C} = 0 \). In this case the minimum of the objective function in (1.2) is achieved for \( p_1 = 1 - \delta \), where positive \( \delta \) vanishes as \( L \to \infty \), and \( L\delta \to \overline{C} \).

In the opposite case, if the minimum of \( J^\text{upper} \) is achieved for \( C = \overline{C} > 0 \), then the arguments are similar to those above, and \( J^\text{upper} \) is not greater than the right-hand side of (5.1). The minimum of \( J^\text{lower} \) must be achieved for \( C = \overline{C} = 0 \). In this case the minimum of the objective function in (1.2) is achieved for \( p_1 = 1 + \delta \), where positive \( \delta \) vanishes as \( L \to \infty \), and \( L\delta \to \overline{C} \).

From Theorem 6.4 we have the following evident property of the optimal control.

**Corollary 6.5.** The solution to the control problem can be \( p_1 = 1 \) only in the case \( j_1 \leq j_2 \frac{E^2}{1 - \rho_2} \). Specifically, the equality is achieved only for \( c_i \equiv c, i = 1, 2, \ldots, L \), where \( c \) is an arbitrary positive constant.
Although Corollary 6.5 provides a result in the form of simple inequality, this result is not really useful, since it is an evident extension of the result of [2]. A more constructive result is obtained for the special case considered in the next section.

7. EXAMPLE OF LINEAR COSTS

In this section we study an example related to the case of linear costs.

Assume that \( c_1 \) and \( c_L < c_1 \) are given. Then the assumption that the costs are linear means, that

\[
(7.1) \quad c_i = c_1 - \frac{i - 1}{L - 1}(c_1 - c_L), \quad i = 1, 2, \ldots, L.
\]

It is assumed that as \( L \) is changed, the costs are recalculated as follows. The first and last values of the cost \( c_1 \) and \( c_L \) remains the same. Other costs in the intermediate points are recalculated according to (7.1).

Therefore, to avoid confusing with the appearance of the index \( L \) for the fixed (unchangeable) values of cost \( c_1 \) and \( c_L \), we use the other notation: \( c_1 = \xi \) and \( c_L = \zeta \). Then (7.1) has the form

\[
(7.2) \quad c_i = \xi - \frac{i - 1}{L - 1}(\xi - \zeta), \quad i = 1, 2, \ldots, L.
\]

In the following we shall also use the inverse form of (7.2). Namely,

\[
(7.3) \quad c_{L-i} = \xi + \frac{i}{L - 1}(\xi - \zeta), \quad i = 0, 1, \ldots, L - 1.
\]

Apparently,

\[
(7.4) \quad c^* = \frac{\xi + \zeta}{2}.
\]

For \( c^{\text{upper}} \) we have

\[
(7.5) \quad c^{\text{upper}} = \psi(C)
= \lim_{L \to \infty} \frac{\sum_{j=0}^{L-1} \left( \xi + \frac{j}{L - 1}(\xi - \zeta) \right) \left( 1 - \frac{2CE\xi}{(\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi)L} \right)^j}{\sum_{j=0}^{L-1} \left( 1 - \frac{2CE\xi}{(\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi)L} \right)^j}

= \frac{\xi + (\xi - \zeta)}{L - 1} \cdot \frac{1 - \frac{2CE\xi}{(\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi)L}}{\sum_{j=0}^{L-1} \left( 1 - \frac{2CE\xi}{(\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi)L} \right)^j}

= \frac{\xi + (\xi - \zeta)}{2CE\xi} \frac{\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi}{\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi} \times \frac{2CE\xi}{\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi} + \exp \left( \frac{2CE\xi}{\tilde{\rho}_{1,2}(E\xi)^3 + E\xi^2 - E\xi} \right) - 1

\]

For example, as \( C \) converges to zero in (7.5), then \( c^{\text{upper}} \) converges to \( \xi + \frac{1}{2}(\xi - \zeta) = c^* \). This is in agreement with the statement of Proposition 5.1.
In turn, for $c_{\text{lower}}$ we have
\[
c_{\text{lower}} = \eta(C)
= \lim_{L \to \infty} \frac{\sum_{j=0}^{L-1} \left( \frac{\xi}{L-1} \right) \left( 1 + \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)L} \right)^j}{\sum_{j=0}^{L-1} \left( 1 + \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)L} \right)^j}.
\]
(7.6)
\[
= \xi + (\tau - \xi) \lim_{L \to \infty} \frac{1}{L-1} \left( \frac{\sum_{j=0}^{L-1} \left( 1 + \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)L} \right)^j}{\sum_{j=0}^{L-1} \left( 1 + \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)L} \right)^j} \right)
\]
\[
= \xi + (\tau - \xi) \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)} \left( 1 - \exp \left( \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)} \right) \right) \left( \frac{2CE\xi}{(\rho_{1,2}(\xi)^3 + E\xi^2 - E\xi)} \right).
\]
Again, as $C$ converges to zero in (7.6), then $c_{\text{lower}}$ converges to $\xi + \frac{1}{2}(\tau - \xi) = \xi^*$. So, we arrive at the agreement with the statement of Proposition 5.1

We cannot give the explicit solution because the calculations are very routine and cumbersome. However, we provide a numerical result. For simplicity, the input flow in the numerical example is assumed to be ordinary Poisson, that is we set $E\xi = 1$ and $E\xi^2 = 1$ in our calculations.

Following Corollary 5.5, take first $j_1 = j_2 \frac{\rho_2}{1 - \rho_2}$. Clearly, that for these relation between parameters $j_1$ and $j_2$ the minimum of $J_{\text{lower}}$ must be achieved for $C = 0$, while the minimum of $J_{\text{upper}}$ must be achieved for a positive $C$. Now, keeping $j_1$ fixed assume that $j_2$ increases. Then, the problem is to find the value for parameter $j_2$ such that the value of $C$ corresponding to the minimization problem of $J_{\text{upper}}$ reaches the point 0.

In our example we take $j_1 = 1$, $\rho_2 = \frac{1}{2}$, $\xi = 1$, $\tau = 2$, $\rho_{1,2} = 1$. In the table below we outline some values $j_2$ and the corresponding value $C$ for optimal solution of functional $J_{\text{upper}}$. It is seen from the table that the optimal value is achieved in the case $j_2 \approx 1.34$. Therefore, in the present example $j_1 = 1$ and $j_2 \approx 1.34$ lead to the optimal solution $\rho_1 = 1$.

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| Parameter   | Argument of optimal value |
|------------|----------------------------|
| $j_2$      | $C$                        |
| 1.06       | 0.200                      |
| 1.08       | 0.182                      |
| 1.10       | 0.165                      |
| 1.12       | 0.149                      |
| 1.14       | 0.134                      |
| 1.16       | 0.120                      |
| 1.18       | 0.104                      |
| 1.20       | 0.090                      |
| 1.25       | 0.055                      |
| 1.30       | 0.022                      |
| 1.33       | 0.010                      |
| 1.34       | 0                           |

Table 1. The values of parameter $j_2$ and corresponding arguments of optimal value $C$
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