POLARIZING ANISOTROPIC HEISENBERG GROUPS

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ABSTRACT. We expand the class of polarizable Carnot groups by implementing a technique to polarize anisotropic Heisenberg groups.

1. BACKGROUND AND MOTIVATION

In [2], Balogh and Tyson establish the concept of polarizable Carnot groups. Polarizable Carnot groups are marked by the ability to create a system of polar coordinates that properly integrates with the sub-Riemannian environment. This produces consequences such as sharp constants for the Moser-Trudinger inequality, capacity formulas, and closed-form fundamental solutions to the p-Laplace equation. The only non-Euclidean examples are those in the class of groups of Heisenberg-type. (See Section 3 for further discussion.) We extend the class of polarizable groups to include anisotropic Heisenberg groups. After a brief discussion of general Carnot groups in Section 2, groups of Heisenberg-type in Section 3, and polarizable Carnot groups in Section 4, we present our technique in Section 5.

2. CARNOT GROUPS

We begin by denoting an arbitrary Carnot group in \( \mathbb{R}^N \) by \( G \) and its corresponding Lie Algebra by \( g \). Recall that \( g \) is nilpotent and stratified, resulting in the decomposition

\[
g = V_1 \oplus V_2 \oplus \cdots \oplus V_k
\]

for appropriate vector spaces that satisfy the Lie bracket relation \([V_1, V_j] = V_{1+j}\). We set \( \dim V_i = n_i \) and denote a basis for \( g \) by

\[
X_{11}, X_{12}, \ldots, X_{n_1}, X_{21}, X_{22}, \ldots, X_{kn_k}
\]

so that

\[
V_i = \text{span}\{X_{i1}, X_{i2}, \ldots, X_{in_i}\}.
\]

The Lie Algebra \( g \) is associated with the group \( G \) via the exponential map \( \exp : g \to G \). For \( X \in g \), we let \( \Theta_X : \mathbb{R} \to G \) be the unique integral curve of \( X \) with the following properties:

\[
\Theta'(t)|_{t=0} = X \quad \Theta(0) = 0.
\]

We define the diffeomorphism \( \exp : g \to G \) by \( \exp X = \Theta_X(1) \). Coordinates in \( G \) arise from the image of the exponential map. That is,

\[
\exp \left( \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij} X_{ij} \right) = (a_{11}, a_{12}, \ldots, a_{n_1}, a_{21}, a_{22}, \ldots, a_{kn_k}) \in G.
\]

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One can choose the basis so that \( \exp \) is the identity, that is, so that in the relation above, \( a_{ij} = x_{ij} \).

The product of exponentials obeys the Baker-Campbell-Hausdorff formula (see, for example, [7])

\[
(\exp X)(\exp Y) = \exp(X + Y + \frac{1}{2}[X, Y] + R(X, Y))
\]

where \( R(X, Y) \) are terms formed by iterated brackets of \( X \) and \( Y \) of order at least 3.

In particular, if for \( X, Y \in g \), we have \((a_{11}, a_{12}, \ldots, a_{1n_1}, a_{21}, a_{22}, \ldots, a_{kn_k}) = \exp X \) and \((b_{11}, b_{12}, \ldots, b_{1n_1}, b_{21}, b_{22}, \ldots, b_{kn_k}) = \exp Y \), then Equation (2.1) induces a (non-abelian) algebraic group law on \( G \). The identity element of \( G \) is denoted by 0 and called the origin.

Endowing \( g \) with an inner product \( \langle \cdot, \cdot \rangle \) and related norm \( \| \cdot \| \), induces a natural metric on \( G \), called the Carnot-Carathéodory distance, defined for the points \( p \) and \( q \) as follows:

\[
d_C(p, q) = \inf_{\Gamma} \int_0^1 \| \gamma'(t) \| dt
\]

where the set \( \Gamma \) is the set of all curves \( \gamma \) such that \( \gamma(0) = p, \gamma(1) = q \) and \( \gamma'(t) \in V_1 \). By Chow’s theorem (see, for example, [4]) any two points can be connected by such a curve, which means \( d_C(p, q) \) is an honest metric. We may define a Carnot-Carathéodory ball of radius \( r \) centered at a point \( p_0 \) by

\[
\mathcal{B}(p_0, r) = \{ p \in G : d_C(p, p_0) < r \}.
\]

2.1. Calculus. When the basis is orthonormal, a smooth function \( u : G \to \mathbb{R} \) has the horizontal derivative given by

\[
\nabla_0 u = (X_{11}u, X_{12}u, \ldots, X_{1n_1}u)
\]

and the symmetrized horizontal second derivative matrix, denoted by \((D^2u)^*\), with entries

\[
((D^2u)^*)_{ij} = \frac{1}{2}(X_{1i}X_{1j}u + X_{1j}X_{1i}u)
\]

for \( i, j = 1, 2, \ldots, n_1 \).

Remark 1. For notational purposes, we shall set \( X_i = X_{1i} \).

We recall that for any open set \( O \subset G \), the function \( f \) is in the horizontal Sobolev space \( W^{1,p}(O) \) if \( f \) and \( X_if \) are in \( L^p(O) \) for \( i = 1, 2, \ldots, n_1 \). Replacing \( L^p(O) \) by \( L^p_{\text{loc}}(O) \), the space \( W^{1,p}_{\text{loc}}(O) \) is defined similarly. The space \( W^{1,p}_0(O) \) is the closure in \( W^{1,p}(O) \) of smooth functions with compact support. For more complete details on calculus on Carnot groups, see [10], [11], [12], and [15].

Using the above derivatives, we define the horizontal \( p \)-Laplacian of a smooth function \( f \) for \( 1 < p < \infty \) by

\[
\Delta_p f = \text{div}((\| \nabla_0 f \|^{p-2}\nabla_0 f)) = \sum_{i=1}^{n_1} X_i((\| \nabla_0 f \|^{p-2}\nabla_0 f))
\]

\[
= \| \nabla_0 f \|^{p-2} \text{tr}((D^2f)^*) + (p-2)\| \nabla_0 f \|^{p-4}((D^2f)^*\nabla_0 f, \nabla_0 f).
\]
Formally taking the limit as \( p \) goes to infinity results in the infinite Laplacian which is defined by
\[
\Delta_{\infty} f = \sum_{i,j=1}^{n_1} X_i f X_j f X_i f = \langle (D^2 f)^\ast \nabla_0 f, \nabla_0 f \rangle.
\]

3. Groups of Heisenberg type.

Groups of Heisenberg type were first introduced in [13]. We begin by recalling that the center \( Z \) of a Lie Algebra is defined by
\[
Z = \{ z \in g : [v, z] = 0 \ \forall v \in g \}.
\]
A Heisenberg-type group is then defined as follows:

**Definition 1.** [6, Definition 18.1.1] A Heisenberg-type Lie Algebra is a finite-dimensional real Lie algebra \( g \) which can be endowed with an inner product \( \langle \cdot, \cdot \rangle \) such that \([Z^\perp, Z^\perp] = Z\) where \( Z \) is the center of \( g \) and moreover, for every fixed \( z \in Z \), the map \( J_z : Z^\perp \to Z^\perp \) defined by \( \langle J_z(v), w \rangle = \langle z, [v, w] \rangle \ \forall w \in Z^\perp \) is an orthogonal map whenever \( \langle z, z \rangle = 1 \).

A Heisenberg-type group (also called group of Heisenberg-type) is a Carnot group with a Heisenberg-type Lie Algebra.

**Remark 2.** By scaling, we have
\[
(J_z)^2 = -\|z\|^2 Id
\]
for all \( z \) in \( Z \).

Given a group \( G \) of Heisenberg-type with corresponding Lie Algebra \( g \), we have
\[
g = Z^\perp \oplus Z
\]
where \( Z \) is the center of \( g \). The exponential map then yields for all \( x \in G \),
\[
x = \exp(v(x) + z(x))
\]
for \( v(x) \in Z^\perp \) and \( z(x) \in Z \). We then define the following on \( G \):
\[
Q = \dim Z^\perp + 2 \cdot \dim Z
\]
\[
\rho(x) = (\|v(x)\|^4 + 16\|z(x)\|^2)^{\frac{1}{2}}
\]
\[
B_r = \{ x \in G : \rho(x) < r \}
\]
\[
|B_r|_p = \int_{B_r} \|\nabla_0 \rho\|^p
\]
and \( \omega_p = |B_1|_p \).

In [8], Capogna, Danielli, and Garofalo find a closed-form formulation for the fundamental solution to the \( p \)-Laplace equation in groups of Heisenberg-type via the following theorem.

**Theorem 3.1.** [8, Theorem 2.1] Fix \( 1 < p < \infty \). Define the constant \( C_p \) by
\[
C_p = \begin{cases} 
\frac{p-1}{p-Q}(Q\omega_p)^{-\frac{1}{p-1}} & \text{when } p \neq Q \\
(Q\omega_p)^{-\frac{1}{p-1}} & \text{when } p \neq Q 
\end{cases}
\]
and the function $\Gamma_p(x)$ by

$$
\Gamma_p(x) = \begin{cases} 
C_p \rho_p \frac{1}{x} & \text{when } p \neq Q \\
C_p \log \rho & \text{when } p \neq Q.
\end{cases}
$$

Then $\Gamma_p(x)$ is a fundamental solution to the equation

$$
\Delta_p u = 0
$$

with singularity at the identity element $e \in G$. A fundamental solution with singularity at any other point of $G$ is obtained by left-translation of $\Gamma_p$.

This result motivated the following corollary:

**Corollary 3.2.** [5] Let $G$ be a Heisenberg-type group. Then on $G \setminus \{e\}$, the function $\rho$ in the theorem above satisfies

$$
\Delta_\infty \rho = 0.
$$

### 4. Polarizable Carnot Groups

In [2], Balogh and Tyson produce a procedure for constructing polar coordinates in certain Carnot groups, called polarizable Carnot groups. Polarizable Carnot groups are defined via the following definition:

**Definition 2.** [2, Definition 2.12] We say that a Carnot group $G$ is polarizable if the homogeneous norm $N = u_2^{\frac{1}{n}}$ associated to Folland’s [see [9]] solution $u_2$ for the 2-Laplacian $\Delta_2$ satisfies $\Delta_\infty N = 0$ in $G \setminus \{0\}$.

It is shown in [2, Section 5] that groups of Heisenberg-type are polarizable. To date, these are the only examples of polarizable Carnot groups outside of $\mathbb{R}^n$. In particular, [2, Section 6] shows that “polarizable” is a fragile concept, unstable under small perturbations of the underlying Lie Algebra. The specific counterexample given is an anisotropic Heisenberg group in $\mathbb{R}^5$ generated by the vectors

$$
X = \frac{\partial}{\partial x} + y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - x \frac{\partial}{\partial t},
$$

$$
Z = \frac{\partial}{\partial z} + 2w \frac{\partial}{\partial t}, \quad \text{and} \quad W = \frac{\partial}{\partial w} - 2z \frac{\partial}{\partial t}.
$$

This counterexample can be generalized as in the following:

**Counterexample 4.1.** For $j = 1, 2, \ldots, n$, let $L_j \in \mathbb{R} \setminus \{0\}$. Consider the following vector fields in $\mathbb{R}^{2n+1}$:

$$
X_j = \begin{cases} 
\frac{\partial}{\partial x_j} - L_{j} x_{j+n} \frac{\partial}{\partial t} & \text{for } j = 1, 2, \ldots n \\
\frac{\partial}{\partial x_j} + L_{j-n} x_{j-n} \frac{\partial}{\partial t} & \text{for } j = n + 1, n + 2, \ldots 2n.
\end{cases}
$$

Resulting Lie brackets are given for $j < k$ by

$$
[X_j, X_k] = \begin{cases} 
2L_j \frac{\partial}{\partial t} & \text{for } k = n + j \\
0 & \text{for } k \neq n + j.
\end{cases}
$$
We then add the vector $T = \frac{\partial}{\partial t}$ to form a basis for $\mathbb{R}^{2n+1}$ and stratify by $\mathbb{R}^{2n+1} = V_1 \oplus V_2$ where

$$V_1 = \text{span}\{X_1, X_2, \ldots, X_{2n}\} \text{ and } V_2 = \text{span}\{T\}.$$  

We use this Lie Algebra and the exponential map of Section 2 to produce a step-two Carnot group. Standard calculations yield that the exponential map is the identity, that is,

$$\exp \left( \sum_{j=1}^{2n} x_j X_j + tT \right) = (x_1, x_2, \ldots, x_{2n}, t).$$

Consequently the algebraic group law is given by

$$(x_1, x_2, \ldots, x_{2n}, t) \ast (y_1, y_2, \ldots, y_{2n}, s) = (x_1 + y_1, x_2 + y_2, \ldots, x_{2n} + y_{2n}, t + s + 2 \sum_{j=1}^{n} L_j (x_j y_{j+n} - y_j x_{j+n})).$$

We have the following theorem.

**Theorem 4.2.** Let $n = 2$ in Counterexample 4.1 and let $L_2 = 2L_1$. Set

$$N = \frac{(B^2 + t^2)^{\frac{3}{2}}(A - B + \sqrt{B^2 + t^2})^{\frac{3}{2}}}{(B + \sqrt{B^2 + t^2})^{\frac{3}{2}}}$$

where

$$B = B(x_1, x_2, x_3, x_4) = |L_1| \left( \frac{1}{2} x_1^2 + x_2^2 + \frac{1}{2} x_3^2 + x_4^2 \right)$$

and

$$A = A(x_1, x_2, x_3, x_4) = |L_1| (x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

Then, for an appropriate constant $C$, we have $CN^{-1}$ is the fundamental solution to $\Delta_2 u = 0$ but $\Delta_\infty N \neq 0$.

**Proof.** The theorem was proved in [2, Section 6] for the case $L_1 = 1$ by using the Beals-Gaveau-Greiner [3] formula for the fundamental solution. In the general case, the computations are similar and omitted. $\square$

Definition 2 produces the following corollary.

**Corollary 4.3.** Counterexample 4.1 need not be polarizable.

Theorem 3.1 and the discussion after Definition 2 produce the following corollary.

**Corollary 4.4.** Counterexample 4.1 need not be a group of Heisenberg-type.

5. COUNTEREXAMPLE REVISITED

In this section, we will take a closer look at Counterexample 4.1 and show that while the theorem is true, we can produce a procedure to polarize these Carnot groups and, in effect, falsify the Corollary 4.3.

We begin by examining the underlying assumptions on the Beals-Gaveau-Greiner [3] formula Subsection 2.1. In particular, we assumed in this formula, and consequently Theorem 4.2, that the vector fields are orthonormal. We will alter this assumption by operating under the following main assumption:
Main Assumption. The vector fields in Counterexample 4.1 satisfy the following:

\[
\begin{align*}
\|X_j\|^2 &= \langle X_j, X_j \rangle = 2|L_j| \text{ for } j = 1, 2, \ldots, n \\
\|X_j\|^2 &= \langle X_j, X_j \rangle = 2|L_{j-n}| \text{ for } j = n+1, n+2, \ldots, 2n \\
\langle X_j, X_k \rangle &= 0 \text{ for } j \neq k \\
\langle X_j, T \rangle &= 0 \\
\text{and } \langle T, T \rangle &= 1.
\end{align*}
\]

In particular, the basis is orthogonal but no longer orthonormal.

Remark 3. We note that these assumptions do not change the center \(Z\) of the Lie Algebra, the exponential map or the algebraic group law. However, the metric space properties have been altered, as Equation (2.2) relies on this norm.

Under the Main Assumptions, we consider the map \(J_T : V_1 \to V_1\) from Definition 1. By construction of the Lie Algebra, we have

\[
\begin{align*}
\langle T, 2L_j T \rangle &= 2L_j \quad \text{when } j = 1, 2, \ldots, n \text{ and } k = j + n \\
\langle T, [X_j, X_k] \rangle &= -\langle T, 2L_j T \rangle = -2L_j \quad \text{when } j = n+1, n+2, \ldots, 2n \text{ and } k = j - n \\
0 &\quad \text{otherwise}.
\end{align*}
\]

We conclude that the matrix \(J\) of the map \(J_T\) is given by

\[
\begin{bmatrix}
0_{n \times n} & -I_{n \times n} \\
I_{n \times n} & 0_{n \times n}
\end{bmatrix}
\]

and thus by Definition 1, the resulting Carnot group is a group of Heisenberg-type.

5.1. Calculus using the Orthogonal Vector Field. Because we are not employing orthonormal vectors, we must modify the formula for divergence, and hence the formula for the \(\Delta_p\) Laplace operator. This is detailed in the following lemma, whose proof results from the formulas for gradient and divergence in curvilinear coordinates ([11 Chapter 2] or [14 Chapter 7]).

Lemma 5.1. Consider a vector field \(F = \sum_{i=1}^{2n} g_i X_i\), where the vector fields \(X_i\) satisfy our Main Assumption. The divergence formula produces

\[
\text{div } F = \sum_{i=1}^{n} \frac{1}{\sqrt{2|L_i|}} X_i g_i + \sum_{i=n+1}^{2n} \frac{1}{\sqrt{2|L_{i-n}|}} X_i g_i.
\]

Now, let \(f(x_1, x_2, \ldots, x_{2n}, t)\) be a smooth function and define the operator \(\mathcal{M}(f)\) by

\[
\mathcal{M}(f) = \left( \sum_{j=1}^{n} \frac{1}{2|L_j|} (X_j f)^2 + \sum_{j=n+1}^{2n} \frac{1}{2|L_{j-n}|} (X_j f)^2 \right)^{\frac{1}{2}}.
\]
Then, for $1 < p < \infty$, the $p$-Laplace operator is given by
\[
\Delta_p f = \sum_{i=1}^{n} \mathcal{M}(f)\rho_i X_i X_i f + \sum_{j=n+1}^{2n} \mathcal{M}(f)\rho_j X_j X_j f + (p - 2)\mathcal{M}(f)^{-4} \rho_i \rho_j X_i X_j f.
\]

We now may invoke [2, Section 3] to construct polar coordinates.

**Theorem 5.2.** Let
\[
\rho(x_1, x_2, \ldots, x_n, t) = \left( \left( \sum_{i=1}^{n} 2|L_i|^2 x_i^2 + \sum_{i=n+1}^{2n} 2|L_{i-n}|^2 x_i^2 \right)^{\frac{1}{2}} + 16t \right)^{\frac{1}{p}}.
\]

Then the function $\Gamma_p(x)$ defined in Equation (3.2) is the fundamental solution to the $p$-Laplace equation for $1 < p < \infty$ and on $G \setminus \{e\}$, we have $\Delta_\infty \rho = 0$.

We now may invoke [2, Section 3] to construct polar coordinates.

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