Chaos control in the fractional order logistic map via impulses

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Abstract In this paper, the chaos control in the discrete logistic map of fractional order is obtained with an impulsive control algorithm. The underlying discrete initial value problem of fractional order is considered in terms of Caputo delta fractional difference. Every $\Delta$ steps, the state variable is instantly modified with the same impulse value, chosen from a bifurcation diagram versus impulse. It is shown that the solution of the impulsive control is bounded. The numerical results are verified via time series, histograms and the 0-1 test. Several examples are considered.

Keywords Caputo delta fractional difference · Impulsive chaos control · Discrete logistic map of fractional order · Lyapunov exponent of discrete maps of fractional order · 0-1 test

1 Introduction

The models which involve abruptly change of variables are called impulsive equations.

The concept of impulsive control has a long history. Many impulsive control methods were successfully developed under the framework of optimal control. In mechanical systems, impulsive phenomena had been studied for different scenarios such as mechanical systems with impacts.

The theory of impulsive differential equations studies systems evolution, when some process is interrupted by abrupt changes (impulses) of state [1]. These systems are modeled by differential equations which describe the period of continuous variation of state and by conditions which describe the discontinuities of first kind of the solution or of its derivatives at the moments of impulses. Many real-world problems can experience abrupt external forces which can change completely their dynamics. For instance, an example of a real-world problem that can be represented by an impulsive differential equation is a medicine intake, where the
user must take regular doses of the medicine, which causes abrupt changes in the amount of medicine in their body, to control the disease or making it disappear (examples of impulsive systems can be found in, e.g., [2–5]).

Details concerning existence and uniqueness of solutions, dependence of solutions on initial values, variation of parameters, oscillation and stability can be found in [6,7].

There exist different kinds of impulses [1], for instance systems with impulses applied at fixed times (presented first in [8,9]) and systems with impulses applied at variable times [10,11] (see also [2,12]). Impulses applied at vary time are important due to their applicability in the real-world problems. For example, the billiard-type system can be modeled by differential systems with impulses which act on the first derivatives of the solutions. Thus, the positions of the colliding balls do not change at the moments of impact (impulse), but their velocities gain finite increments (the velocity will change according to the position of the ball) [1].

In the last years, results arising from impulsive effects have been adapted easily to the discrete case.

Difference equations or discrete dynamical systems are a diverse field which impacts almost every branch of pure and applied mathematics. Discrete systems with memory in population, economy price option and signal processing have been considered almost at the same time since the fractional differential models are used. To note that, often real systems may encounter abrupt changes at certain time moments and therefore cannot be considered continuously but discrete time (see, e.g., [13]).

On the other side, the control of chaos, or control of chaotic systems, is the boundary field between control theory and dynamical systems theory studying when and how it is possible to control systems exhibiting irregular, chaotic behavior (see, e.g., [14]).

For discrete equations of integer order, the impulsive control algorithm utilized in this paper has the following form

\[
x_{n+1} = \begin{cases} 
  f(x_n), & \text{if } \mod (n, \Delta) \neq 0 \\
  (1 + \gamma)x_{n+1}, & \text{if } \mod (n, \Delta) = 0
\end{cases}
\]

(1)

where \( f \in C(\mathbb{R}, \mathbb{R}) \) is some discrete map which depend on a real bifurcation parameter, \( \Delta \in \mathbb{N}^* \) and the impulse \( \gamma \in \mathbb{R} \) a relative small real number. One assumes that for some parameter ranges, the system evolves chaotic.

The algorithm perturbs \( x \) every \( \Delta \) steps with the quantity \( (1 + \gamma) \) and acts “instantaneously” in the sense it modifies \( x_{n+1} \) while it is calculated, the system dynamics being subject to abrupt changes \( (1 + \gamma)x_{n+1} \) (in nature, these can be shocks, harvesting, natural disasters, etc.). If, without impulses \( \gamma \), for some parameter value the system behaves chaotically, an adequate design of the chaos control (i.e., adequate values of impulses \( \gamma \) and time moments \( \Delta \)) may force the system to become stable evolving along some regular trajectory.

The impulsive moment \( \Delta \) is fixed in advance, while the impulse \( \gamma \) chosen from the bifurcation plot of \( x \) versus \( \gamma \) shows the periodic windows where \( \gamma \) generates stable periodic orbits.

The impulsive control (1) has important implications, for example, in ecology. Thus, the phenomenon of a population increasing in response to an increase in its per-capita mortality rate has to be taken into consideration to design strategies in fisheries and pest management. This paradoxical phenomenon is known as the hydra effect [15]. The algorithm can also be applied successfully in chemical systems [16].

Impulsive equations modeled with continuous or discontinuous differential equations of integer order or fractional order [17–20], or by discrete equations [21] have been developed in impulsive problems in physics, orbital transfer of satellite, population dynamics, dosage supply in pharmacokinetics, biotechnology, pharmacokinetics, ecosystems management, industrial robotics, synchronization in chaotic secure communication systems and so forth (see [7] for a deep background on impulsive differential equations and inclusions and references, or [6]).

The discrete fractional calculus has been an increased interest due to its importance in real-world problems. More generalized chaos does has been found in fractional discrete systems [22–25].

The stability of impulsive fractional difference equations is studied in [26] and the first study of the fractional standard map with memory, derived from a differential equation, can be found in [27] (see also [22,23,28–31]).

In this paper, one considers the chaos control of the discrete logistic map of fractional order. It is proved that the impulsive solution of the controlled logistic map of fractional order is bounded. The numerical results are
verified with the 0-1 test, time series, histograms and the Lyapunov exponent. Because of the memory history effect, the numerical determination of the Lyapunov exponent requires a special approach.

The paper is organized as follows: Sect. 2 deals with the discrete logistic map of fractional order, and the applicative Sect. 3 presents the numerical implementation of the chaos control algorithm (1) in the case of the fractional logistic map of fractional order. The Appendix presents briefly the 0-1 test. The Conclusion section ends the manuscript.

2 The discrete logistic map of fractional order

Let \( q \in (0, 1), N_{1-q} = \{1 - q, 2 - q, 3 - q, \ldots\}, 0 < q \leq 1 \) and \( f \in C(\mathbb{R}, \mathbb{R}) \) a discrete map.

The difference equations of fractional order (FO) studied in this paper are modeled by the following initial value problem

\[
\Delta^q_k x(k) = f(x(k - 1 + q)),
\]

\( k \in N_{1-q}, \]

\( x(0) = x_0, \tag{2} \)

where \( \Delta^q_k x(k) \) is the Caputo delta fractional difference \([32, 33] \).

Hereafter, the FO equations are considered with initial condition \( x(0) = x_0 \).

The equivalent discrete integral form of (2) is (see, e.g., [32, 33])

\[
x(n) = x(0) + \frac{1}{\Gamma(q)} \times \sum_{j=1-q}^{n-q} \frac{\Gamma(n-j)}{\Gamma(n-j-q)} f(x(j - 1 + q)),
\]

which with \( j \leftrightarrow j + q \) becomes

\[
x(n) = x(0) + \frac{1}{\Gamma(q)} \times \sum_{j=1}^{n} \frac{\Gamma(n-j+q)}{\Gamma(n-j+1)} f(x(j - 1)),
\]

\( n = 1, 2, \ldots \) \( \tag{3} \)

Consider (2) in the case of the discrete logistic map of FO [34]

\[
\Delta^q_k x(k) = f(x(k + q - 1)) := \mu x(k + q - 1)(1 - x(k + q - 1)),
\]

\( k \in N_{1-q}. \) \( \tag{4} \)

Then, the underlying discrete integral (3) becomes (see also [34])

\[
x(n) = x(0) + \frac{\mu}{\Gamma(q)} \times \sum_{j=1}^{n} \frac{\Gamma(n-j+q)}{\Gamma(n-j+1)} f(x(j - 1)(1 - x(j - 1)).
\]

\( \tag{5} \)

Because, due to the discrete memory effect, the Jacobian matrix necessary for Lyapunov exponent (LE) cannot be obtained directly. Therefore, in [35] is proposed the following natural way of linearization of (5) along the orbit \( x_n \)

\[
a(n) = a(0) + \frac{\mu}{\Gamma(q)} \times \sum_{j=1}^{n} \frac{\Gamma(n-j+q)}{\Gamma(n-j+1)} f(x(j - 1)(1 - 2x(j - 1)),
\]

\( a(0) = 1, \) \( \tag{6} \)

wherefrom, from (6) via (5), the finite-time local LE, \( \lambda, \) is obtained as follows

\[
\lambda(x_0) \simeq \frac{1}{n} \ln |a(n-1)|. \]

Due to the discrete memory effect (the present status depends on all previous information), one of the main impediments to implement (3) is the instability of

\[
R_q(n) := \sum_{j=1}^{n} \frac{\Gamma(n-j+q)}{\Gamma(n-j+1)}, \quad n = 1, 2, \ldots
\]

Thus, one has the following

**Proposition 1**

\[
\lim_{n \to \infty} \frac{1}{n^q} R_q(n) = \frac{1}{q} \tag{7}
\]

**Proof** By using Gautchi inequality [36, 37]

\[
\frac{1}{(x+1)^{1-q}} \leq \frac{\Gamma(x+q)}{\Gamma(x+1)} \leq \frac{1}{(x+\frac{q}{2})^{1-q}}
\]

for any \( x \geq 0, \) one derives

\[
R_q(n) \geq \sum_{j=1}^{n} \frac{1}{(n-j+1)^{1-q}} \geq \sum_{j=1}^{n} \frac{1}{j^{1-q}} \geq \int_1^{n+1} \frac{dx}{x^{1-q}} = \frac{1}{q} (n+1)^q - 1 \]

\[
\geq \frac{1}{q} (n^q - 1),
\]
and
\[
R_q(n) \leq \sum_{j=1}^{n} \frac{1}{(n-j+\frac{q}{2})^{1-q}} = \left(\frac{q}{2}\right)^{q-1} + \sum_{j=1}^{n-1} \frac{1}{(j+\frac{q}{2})^{1-q}} \\
\leq \left(\frac{q}{2}\right)^{q-1} + \int_0^{n-1} \frac{dx}{(x+\frac{q}{2})^{1-q}} \\
= \frac{1}{2} \left(\frac{q}{2}\right)^{q-1} + \frac{1}{q} \left(n-1+\frac{q}{2}\right)^q \\
\leq \frac{1}{q} (n^q + 1).
\]

Thus
\[
\left|R_q(n) - \frac{n^q}{q}\right| \leq \frac{1}{q}, \quad \forall n \geq 1.
\]

\[\square\]

Relation (7) means that the terms of \((R_q(n))\) and \((n^q)\) grow similarly. Therefore, small errors in each steps may lead to large final errors. In Fig. 1 is presented the evolution of \(R_q(n)\) for \(n \leq 1500\) and \(q = k0.1, k = 1, 2, \ldots, 10\).

**Remark 1** (i) Denote by \(\Phi(t, x_0)\), a solution of (2).

Because, due to the memory history of solutions, \(\Phi\) does not verify the relation \(\Phi(t) \circ \Phi(s) = \Phi(t+s)\), one cannot consider (2) as defining a dynamical system. On the other side, motivated by the numerical calculation of the solutions utilized in this paper, the definition of a dynamical system of integer order modeled by a numerical scheme [45, 37]

\[
x(n+1) = \left\{ \begin{array}{ll}
\frac{x_0 + \frac{1}{\Gamma(q)} \sum_{j=1}^{n+1} \frac{\Gamma(n-j+q)}{\Gamma(n-j+1)} f(x(j-1))}{(1+\gamma)} \\
(1+\gamma) \frac{x_0 + \frac{1}{\Gamma(q)} \sum_{j=1}^{n+1} \frac{\Gamma(n-j+q)}{\Gamma(n-j+1)} f(x(j-1))}{(1+\gamma)} \\
\end{array} \right.
\]

where \(f(x(n))\) is the logistic map.

The sequence \((x(n))\) is bounded, as proved by the following result

**Theorem 1** For (8) with \(\gamma + 1 > 0\), it holds

\[
x(n) \leq \max \left\{ x_0 + \frac{(n+1)^q + 1}{4q}\right\}, (1+\gamma)
\]
\[
\left\{ x_0 + \frac{(n+1)^q + 1}{4q} \mu \right\} \quad \forall n \in \mathbb{N}.
\]

**Proof** It is clear that \( f(x) \leq \frac{\mu}{4} \) for any \( x \in \mathbb{R} \). Then, we get
\[
x(n + 1) \leq x_0 + \frac{(n+1)^q + 1}{4q} \mu,
\]
\[
\text{mod } (n, \Delta) \neq 0,
\]
\[
x(n + 1) \leq (1 + \gamma) \left( x_0 + \frac{(n+1)^q + 1}{4q} \right),
\]
\[
\text{mod } (n, \Delta) = 0.
\]

To implement the impulsive algorithm (8) and visualize the effect of impulses \( \gamma \), one draws the bifurcation diagram of the impulsive system versus \( \gamma \in [-0.15, 0.15] \). The tools utilized to analyze the regularity obtained with the impulsive control are the LE, time series, the '0-1' test (Appendix) with \( p, q \), the mean square displacement \( M \) and the median \( K \) value, and also histograms. First transients have been removed. For this system, the errors of \( K \) are about \( 1e^{-3} \) for the '0' value, and \( 1e^{-4} \) for '1'.

Note that if \( \gamma > 0 \), the system suffers a positive variation of internal energy, and the received energy, because at the moment \( \Delta \), when the impulse is applied, the new value of \( x_{n+1}, (1 + \gamma)x_{n+1} \), is bigger than the previous value [18]. The received energy is used to maintain the orbit on a regular motion. Similarly, if \( \gamma < 0 \), the system is forced to dissipates energy to reach the necessary one along the regular orbit. However, these phenomena should be interpreted under the mentioned history memory effect.

The bifurcation diagram of the FO logistic map (4) versus \( \mu \in [1.8, 2.8] \) with \( q = 0.9 \) is presented in Fig. 2a, while in Fig. 2b is presented the bifurcation plot versus \( q \in [0.01, 1] \) with \( \mu = 2.5 \). All bifurcation diagrams in this paper are overplotted with LE exponent (red plot) and \( K \) median value (blue plot).

**Remark 2** i) As known, because of the memory effect, general differential equations of FO cannot have nonconstant periodic solutions (see, e.g., [38]). Similarly, this happens with discrete difference equations of FO [39]. Therefore, these orbits are called “numerically stable periodic” (NSP), in the sense that the trajectory, from numerical point of view, can be an extremely near periodic trajectory [40].

ii) Even the bifurcation scenario versus \( \mu \) (Fig. 2a) looks similar to the case of integer-order logistic map, the Feigenbaum parameter sequence is different. Also, over \( \mu > 2.8 \), depending on \( q \), orbits may diverge (gray zones in Figs. 5 and 6).

iii) As the detail in the bifurcation diagram of \( x \) versus \( q \) in Fig. 2b shows, the system does not present the typical periodic direct and reverse period doubling bifurcations (dotted red lines), as for the integer-order counterpart (compare also with [34, Figs. 4–7] and [46, Fig. 1]). Because at this point the LE is positive and not zero as in the case of the logistic map of integer order, but \( K \) is approximately 0 like for regular motion, it is difficult to characterize this kind of dynamics. Similarly, but
Fig. 3 NSP period-4 orbit obtained with the algorithm (8) applied every step ($\Delta = 1$), for $\gamma = -0.05$; a Bifurcation diagram versus $\gamma \in [-0.15, 0.15]$. Two values of $\gamma$ for which the system behaves regularly have been chosen (dotted lines): $\gamma = -0.05$ and $\gamma = 0.059$; b time series for $\gamma = -0.05$ indicates a NSP orbit of period-4; c histogram for $\gamma = -0.05$ with 4 bars which indicates a NSP orbit of period-4; d $q$ and $p$ plot for $\gamma = -0.05$; e mean square displacement $M$. (Color figure online)
A period-5 NSP orbit, obtained with the algorithm (8), with $\Delta = 1$, but for $\gamma$ chosen within other stable window: $\gamma = 0.059$ [see 3 (a)]. a Time series indicates a NSP orbit of period-5; b histogram with 5 bars which indicates a NSP orbit of period-5; c $q$ and $p$ plot; d mean square displacement $M$. (Color figure online)

less visible, it happens in the bifurcation diagram versus $\mu$ (Fig. 2a).

For the next numerical experiments $q = 0.9$ and $\mu = 2.5$ (Fig. 2a), when $\text{LE}=0.1744$ and $K = 0.9538$ fact which indicates a chaotic behavior.

Next, several values of moments $\Delta$ are considered.

1. $\Delta = 1$. In this case, impulses applied every step (Fig. 3). As can be seen, there are several NSP windows, corresponding to negative but also positive ranges of $\gamma$, from which each value $\gamma$ leads to NSP orbits. Note the points corresponding to $\gamma_1, \gamma_2$ and $\gamma_3$, where $\text{LE} > 0$ and $K \approx 0$ [see Remark 2 (iii)]. Two representative values are considered.

   - For $\gamma = -0.05$ (dotted line in Fig. 3a), with the impulsive control applied every step, one obtains a NSP orbit of period-4, as revealed by the time series (Fig. 3b) and histogram (Fig. 3c). The regularity of this NSP orbit is underlined by the circular plot of $p$ and $q$ (Fig. 3d) and the bounded mean square displacement $M$ (Fig. 3e). In this case, $\text{LE}=-0.0061$ and $K = -0.0055$;
   - For $\gamma = 0.059$ (dotted line in Fig. 3a), the impulsive control leads to a NSP of period-5 (see time series and histogram in Fig. 4a, b, respectively). The regularity is underlined by the shape of the plot of $p$ and $q$ (Fig. 4c) and also by $M$ (Fig. 4d). $\text{LE}=0.0471$ and $K = -0.0076$.

2. The case $\Delta = 3$ is presented in Figs. 5, where a NSP orbit of period-12 is obtained with $\gamma = -0.031$. The period can be remarked in the time series and histogram in Fig. 5a, b, respectively, while the graphs of $p$ and $q$ (Fig. 5c) and the bounded mean square displacement $M$ show that the orbit is regular. $\text{LE}=-0.0033$ and $K = -0.0056$. 
Fig. 5 NSP orbit of period-12 obtained with the impulsive chaos control (8), applied every three step ($\Delta = 3$) and $\gamma = -0.031$ (dotted line in the bifurcation diagram). 

(a) Bifurcation diagram versus $\gamma \in [-0.15, 0.15]$; 
(b) time series indicating a NSP orbit of period-12; 
(c) histogram with 12 bars which indicates a NSP orbit of period-12; 
(d) $q$ and $p$ plot; 
(e) mean square displacement $M$. (Color figure online)
3. $\Delta_{\text{max}}$ for which chaos still can be controlled, for $\gamma \in [-0.15, 0.15]$, is $\Delta_{\text{max}} = 15$. Thus, for $\gamma = -0.12$ within a narrow periodic window $D$ (Fig. 6a), one obtains the NSP orbit of period-15 (Fig. 6b).

4. For $\Delta$ larger than 16 and $\gamma \in [-0.15, 0.15]$, chaos cannot be controlled. This is illustrated by the bifurcation diagram for $\Delta = 17$, and the positive LE and values of $K$ close to 1 (Fig. 6b).

4 Conclusion

In this paper, the impulsive chaos control of the discrete logistic map of fractional order (8) has been investigated. The discrete logistic map of fractional order has been proposed by Wu and Baleanu in [34] in terms of
Caputo delta fractional difference. The impulsive control, previously used in integer order continuous and discrete systems, is obtained by perturbing periodically (every $\Delta$ steps) the state variable with a constant impulse: $x_{n+1} \leftarrow (1 + \gamma)x_{n+1}$, where $\gamma$ is a relatively small real number. If, for a chosen $\Delta$, the control algorithm is applied for a $\gamma$ value which generates in the bifurcation diagram versus $\gamma$ a chaotic behavior, regular motions can be obtained. Several numerical cases are considered.

It is proved that the impulsed orbits remain bounded. To verify the obtained results, time series, histograms and the ‘0-1’ test are utilized. Because of the discrete memory effect, the Lyapunov exponent is obtained by linearization of the discrete integral of the initial value problem of fractional order. Note that the numerical implementation of the discrete integral of the underlying initial value of FO requires numerical precaution. Otherwise only few terms of iterations can be calculated.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

The ‘0-1’ test

The ‘0-1’ test has its roots in [41] been developed in [42] (see also [43] or [44]). It is designed to distinguish chaotic behavior from regular behavior in deterministic systems. The input being a time series, the test is easy to implement and does not need the system equations. Consider a discrete or continuous-time dynamical system and a one-dimensional observable dataset,
constructed from a time series, $\phi(j)$, $j = 1, 2, \ldots, N$, with $N$ some positive integer. The ‘0-1’ test bases on a theorem, which states that a nonchaotic motion is bounded, while a chaotic dynamic behaves like a Brownian motion [41].

(1) First, for $c \in [0, 2\pi]$, one computes the translation variables $p$ and $q$ [42]

$$p(n) = \sum_{j=1}^{n} \phi(j) \cos(jc), \quad q(n) = \sum_{j=1}^{n} \phi(j) \sin(jc),$$

for $n = 1, 2, \ldots, N$. The choice of $c$ represents an important and sensible algorithm variable (see, for example, [43] where for $c$, the interval $[\pi/5, 4\pi/5]$ is proposed).

(2) To determine the growths of $p$ and $q$, the mean square displacement $M$ is determined:

$$M(n) = \lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} [p(j+n) - p(j)]^2 + [q(j+n) - q(j)]^2,$$

where $n \ll N$ (in practice, $n = N/10$ represents a good choice).

(3) Next, the asymptotic growth rate $K$ is defined as

$$K = \lim_{n \to \infty} \log M(n) / \log n.$$ 

If the underlying dynamics is regular (i.e., periodic or quasiperiodic), then $K \approx 0$; if the underlying dynamics is chaotic, then $K \approx 1$.

In Fig. 7, the case of the logistic map of integer order is presented. In Fig. 7a are presented the plots of $q$ and $p$ while in Fig. 7b the mean square displacement $M$ as a function of $n$. In Fig. 1, the regular orbit of the logistic map $x_{n+1} = \mu x_n(1-x_n)$ for $\mu = 3.55$ while Fig. 2 presents the chaotic orbit of the logistic map for $\mu = 4$.

References

1. Bonotto, E.M., Bortolan, M.C., Caraballo, T., Collegari, R.: A survey on impulsive dynamical systems. Electron. J. Qual. Theory Differ. Equ. 7, 1–27 (2016)
2. Bainov, D.D., Simeonov, P.S.: Systems with Impulsive Effect. Stability, Theory and Applications. Wiley, London (1989)
3. Bouchard, B., Dang, N.-M., Lehalle, C.-A.: Optimal control of trading algorithms: a general impulse control approach. SIAM J. Financ. Math. 2, 404–438 (2011)
4. Davis, M.H.A., Guo, X., Guoliang, W.: Impulse control of multidimensional jump diffusions. SIAM J. Control Optim. 48, 5276–5293 (2010)
5. Fere, J.A.: Existence and stability of multiple impulse solutions of a nerve equation. SIAM J. Appl. Math. 42(2), 235–246 (1982)
6. Yang, T.: Impulsive Control Theory. Springer, Berlin (2001)
7. Benchohra, M., Henderson, J., Ntouyas, S.: Impulsive Differential Equations and Inclusions. Hindawi Publishing Corporation, London (2006)
8. Rožko, V.F.: A certain class of almost periodic motions in systems with pulses. Differencialnye Uravnenija 8, 2012–2022 (1972). (in Russian)
9. Rožko, V.F.: Ljapunov stability in discontinuous dynamic systems. Differencialnye Uravnenija 11, 1005–1012 (1975). (in Russian)
10. Kaul, S.K.: On impulsive semidynamical systems. J. Math. Anal. Appl. 150(1), 120–128 (1990)
11. Kaul, S.K.: On impulsive semidynamical systems, II. Recursive properties. Nonlinear Anal. 16, 635–645 (1991)
12. Lakshmikantham, V., Bainov, D.D., Simeonov, P.S.: Theory of Impulsive Differential Equations. World Scientific, Singapore (1989)
13. Allen, L.J.S., Aulbach, B., Elaydi, S., Sacker, R.: Difference equations and discrete dynamical systems. In: Proceedings of the 9th International Conference, University of Southern California, Los Angeles, California, USA, 2–7 August 2004. October 2005, 336p. Linda J S Allen (Texas Tech University, USA), Bernd Aulbach (University of Augsburg, Germany), Saber Elaydi (Trinity University, USA) and Robert Sacker (University of Southern California, Los Angeles, USA). https://doi.org/10.1142/5957
14. Fradkov, A.L., Pogromsky, A.Y.: Introduction to Control of Oscillations and Chaos. World Scientific Series on Nonlinear Science Series A, vol. 35. World Scientific, Singapore (1998)
15. Liz, E.: How to control chaotic behaviour and population size with proportional feedback. Phys. Lett. A 374, 725–728 (2010)
16. Matías, M.A., Güémez, J.: Stabilization of chaos by proportional pulses in the system variables. Phys. Rev. Lett. 72(10), 1455–1458 (1994)
17. Danca, M.-F.: Chaos suppression via periodic change of variables in a class of discontinuous dynamical systems of fractional order. Nonlinear Dyn. 70(1), 815–823 (2012)
18. Danca, M.-F., Fećkan, M., Chen, G.: Impulsive stabilization of chaos in fractional-order systems. Nonlinear Dyn. 89(3), 1889–1903 (2017)
19. Danca, M.-F., Tang, W., Chen, G.: Suppressing chaos in a simplest autonomous memristor-based circuit of fractional order by periodic impulses. Chaos Soliton Fract. 84, 31–40 (2016)
20. Danca, M.-F., Garrappa, R.: Suppressing chaos in discontinuous systems of fractional order by active control. Appl. Math. Comput. 257, 89–102 (2015)
21. Danca, M., Fećkan, M., Pospíšil, M.: Difference equations with impulses. Opuscula Mathematica 39(1), 5–22 (2019)
22. Tarasov, V.E., Zaslavsky, G.M.: Fractional equations of kicked systems and discrete maps. J. Phys. A 41, 435101 (2008)
23. Edelman, M., Tarasov, V.: Fractional standard map. Phys. Lett. A 374(2), 279–285 (2009)
24. Xiao, H., Ma, Y., Li, C.: Chaotic vibration in fractional maps. J. Vib. Control 20, 964–72 (2014)
25. Wu, G.C., Baleanu, D., Zeng, S.D.: Discrete chaos in fractional sine and standard maps. Phys. Lett. A. 378, 484–487 (2014)
26. Wu, G.-C., Baleanu, D.: Stability analysis of impulsive fractional difference equations. Fract. Calc. Appl. Anal. 21(2), 354–375 (2018)
27. Mainardi, F.: Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models. World Scientific, Singapore (2010)
28. Xiao, H., Ma, Y., Li, C.: Chaotic vibration in fractional maps. J. Vib. Control 20(7), 964–972 (2014)
29. Golev, A., Hristova, S., Nenov, S.: Monotone-iterative method for solving antiperiodic nonlinear boundary value problems for generalized delay difference equations with maxima. Abstr. Appl. Anal. 2013, 571954 (2013). https://doi.org/10.1155/2013/571954
30. Ruszewski, A.: Practical and asymptotic stability of fractional discrete-time scalar systems described by a new model. Arch. Control Sci. 26(4), 441–452 (2016)
31. He, J.-W., Zhang, L., Zhou, Y., Ahmad, B.: Existence of solutions for fractional difference equations via topological degree methods. Adv. Diff. Equ. 2018, 153 (2018). https://doi.org/10.1186/s13662-018-1610-2
32. Goodrich, C., Peterson, A.C.: Discrete Fractional Calculus. Springer, Berlin (2015)
33. Fečkan, M., Pospíšil, M.: Note on fractional difference Gronwall inequalities. Electron. J. Qual. Theory Diff. Equ. Article number 44 (2014)
34. Wu, G.-C., Baleanu, D.: Discrete fractional logistic map and its chaos. Nonlinear Dyn. 75(1–2), 283–287 (2014)
35. Wu, G.-C., Baleanu, D.: Jacobian matrix algorithm for Lyapunov exponents of the discrete fractional maps. Commun. Nonlinear Sci. 22(1–3), 95–100 (2015)
36. Gautschi, W.: Some elementary inequalities relating to the gamma and incomplete gamma function. J. Math. Phys. Camb. 38(1–4), 77–81 (1959)
37. Kershaw, D.: Some extensions of W. Gautschi’s inequalities for the gamma function. Math. Comput. 41(164), 607–611 (1983)
38. Tavazoie, M., Haeri, M.: A proof for non existence of periodic solutions in time invariant fractional order systems. Automatica 45(8), 1886–1890 (2009)
39. Diblik, J., Fečkan, M., Pospíšil, M.: Nonexistence of periodic solutions and S-asymptotically periodic solutions in fractional difference equations. Appl. Math. Comput. 257, 230–240 (2015)
40. Danca, M.-F., Fečkan, M., Kuznetsov, N., Chen, G.: Complex dynamics, hidden attractors and continuous approximation of a fractional-order hyperchaotic PWC system. Nonlinear Dyn. 91(4), 2523–2540 (2018)
41. Nicol, M., Melbourne, I., Ashwin, P.: Euclidean extensions of dynamical systems. Nonlinearity 14(2), 275–300 (2001)
42. Gottwald, G., Melbourne, I.: A new test for chaos in deterministic systems. P. R. Soc. A Math. Phys. Sci. 460(2042), 603–611 (2004)
43. Gottwald, G., Melbourne, I.: On the implementation of the 0–1 test for chaos. SIAM J. Appl. Dyn. Syst. 8(1), 129–145 (2009)
44. Gopal, R., Venkatesan, A., Lakshmanan, M.: Applicability of 0–1 test for strange nonchaotic attractors. Chaos 23(2), 023123 (2013). https://doi.org/10.1063/1.4808254
45. Stuart, A., Humphries, A.: Dynamical Systems and Numerical Analysis. Cambridge University Press, Cambridge (1998)
46. Xin, B., Liu, L., Hou, G., Ma, Y.: Chaos synchronization of nonlinear fractional discrete dynamical systems via linear control. Entropy 19(7), 351 (2017). https://doi.org/10.3390/e19070351

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