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Abstract. Given a holomorphic vector bundle $E : EX \to X$ over a compact Kähler manifold, one introduces twisted GW-invariants of $X$ replacing virtual fundamental cycles of moduli spaces of stable maps $f : \Sigma \to X$ by their cap-product with a chosen multiplicative characteristic class of $H^0(\Sigma, f^*E) - H^1(\Sigma, f^*E)$. Using the formalism [17] of quantized quadratic hamiltonians, we express the descendent potential for the twisted theory in terms of that for $X$. The result (Theorem 1) is a consequence of Mumford’s Riemann – Grothendieck formula [31, 13] applied to the universal stable map.

When $E$ is concave, and the inverse $C^\times$-equivariant Euler class is chosen, the twisted theory yields GW-invariants of $EX$. The “non-linear Serre duality principle” [19, 20] expresses GW-invariants of $EX$ via those of the supermanifold $\Pi E^*X$, where the Euler class and $E^*$ replace the inverse Euler class and $E$. We derive from Theorem 1 the nonlinear Serre duality in a very general form (Corollary 2).

When the bundle $E$ is convex, and a submanifold $Y \subset X$ is defined by a global section, the genus 0 GW-invariants of $\Pi EX$ coincide with those of $Y$. We prove a “quantum Lefschetz hyperplane section principle” (Theorem 2) expressing genus 0 GW-invariants of a complete intersection $Y$ via those of $X$. This extends earlier results [4, 25, 7, 28, 15] and yields most of the known mirror formulas for toric complete intersections.

Introduction. The mirror formula of Candelas et al [8] for the virtual numbers $n_d$ of degree $d = 1, 2, 3, ...$ holomorphic spheres on a quintic 3-fold $Y \subset X = \mathbb{CP}^4$ can be stated [11] as the coincidence of the 2-dimensional cones over the following two curves in $H^{even}(Y; \mathbb{Q}) = \mathbb{Q}[P]/(P^5)$:

$$J_Y(\tau) := e^{P\tau} + \frac{P^2}{5} \sum_{d>0} n_d d^3 \sum_{k>0} \frac{e^{(P+kd)\tau}}{(P+kd)^2}$$

and

$$I_Y(t) = \sum_{d \geq 0} e^{(P+d)t} \frac{(5P + 1)(5P + 2)...(5P + 5d)}{(P + 1)^5(P + 2)^5...(P + d)^5}.$$ 

The new proof given in this paper shares with earlier work [4, 22, 20, 7, 28, 15] the formulation of sphere counting in a hypersurface $Y \subset X$ as a problem in Gromov – Witten theory of $X$.

Gromov – Witten invariants of a compact almost Kähler manifold $X$ are defined as intersection numbers in moduli spaces $X_{g,n,d}$ of stable pseudo-holomorphic maps $f : \Sigma \to X$. All results of this paper can be stated and hold true in this generality, but we prefer to stay on the firmer ground of algebraic geometry where the most applications belong.

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Given a holomorphic vector bundle $E$ over a complex projective manifolds $X$ and an invertible multiplicative characteristic class $c$ of complex vector bundles, we introduce twisted Gromov–Witten invariants as intersection indices in $X_{g,n,d}$ with the characteristic classes $c(E_{g,n,d})$ of the virtual bundles $E_{g,n,d} = "H^0(\Sigma, f^*E) \ominus H^1(\Sigma, f^*E)"$. The “quantum Riemann–Roch theorem” (Theorem 1) expresses the twisted Gromov–Witten invariants (of any genus) and their gravitational descendents via untwisted ones.

The totality of gravitational descendents in the genus 0 Gromov–Witten theory of $X$ can be encoded by a semi-infinite cone $L_X$ in the cohomology algebra of $X$ with coefficients in the field of Laurent series in $1/z$ (see Section 6). Another such cone corresponds to each twisted theory. Let $L_E$ be the cone corresponding to the total Chern class

$$c = \lambda^{\dim} + c_1 \lambda^{\dim-1} + ... + c_{\dim}.$$ 

Theorem 1 specialized to this case says that the cones $L_X$ and $L_E$ are related by a linear transformation. It is described in terms of the stationary phase asymptotics $a_{\rho}(z)$ of the oscillating integral

$$\frac{1}{\sqrt{2\pi z}} \int_0^\infty e^{-\frac{x^2+(\lambda+\rho)\ln x}{z}} dx$$

as multiplication in the cohomology algebra by $\prod_i a_{\rho_i}(z)$, where $\rho_i$ are the Chern roots of $E$.

Assuming $E$ to be a line bundle, we derive a “quantum hyperplane section theorem” (Theorem 2). It is more general than the earlier versions [4, 25, 28, 15] in the sense that the restrictions $t \in H^{\leq 2}(X; \mathbb{Q})$ on the space of parameters and $c_1(E) \leq c_1(X)$ on the Fano index are removed.

In the quintic case when $X = \mathbb{C}P^4$ and $\rho = 5P$, the cone $L_X$ is known to contain the curve

$$J_X(t) = \sum_{d \geq 0} \frac{e^{(P+zd)t/z}}{(P+z)^5...(P+zd)^5},$$

and Theorem 2 says that the cone $L_E$ contains the curve

$$I_E(t) = \sum_{d \geq 0} \frac{e^{(P+zd)t/z}(\lambda+5P+z)...(\lambda+5P+5dz)}{(P+z)^5...(P+zd)^5}.$$ 

One obtains the quintic mirror formula by passing to the limit $\lambda = 0$.

The idea of deriving mirror formulas by applying the Grothendieck–Riemann–Roch theorem to universal stable maps is not new. Apparently this was the initial plan of M. Kontsevich back in 1993. In 2000, we had a chance to discuss a similar proposal with R. Pandharipande. We would like to thank these authors as well as A. Barnard and A. Knutson for helpful conversations.

The second author is grateful to D. van Straten for the invitation to the workshop “Algebraic aspects of mirror symmetry” held at Kaiserslautern in June 2001. The discussions at the workshop and particularly the lectures on “Variations of semi-infinite Hodge structures” by S. Bannikov proved to be very useful in our work on this project.
1. Generating functions. Let $X$ be a compact complex projective manifold of complex dimension $D$. Denote by $X_{g,n,d}$ the moduli orbispace of genus $g$, $n$-pointed stable maps $\overline{M}_{g,n,d}$ to $X$ of degree $d$, where $d \in H_2(X;\mathbb{Z})$. The moduli space is compact and can be equipped with a (rational-coefficient) virtual fundamental cycle $[X_{g,n,d}]$ of complex dimension $n + (1-g)(D-3) + \int_c c_1(TX)$.

The total descendent potential of $X$ is a generating function for Gromov-Witten invariants. It is defined as

$$D_X := \exp \left( \sum \frac{\hbar^{g-1}}{g} \mathcal{F}_X^g \right),$$

where $\mathcal{F}_X^g$ is the genus $g$ descendent potential,

$$\mathcal{F}_X^g = \sum_{n,d} \frac{Q_d}{n!} \int_{[X_{g,n,d}]} (\infty \sum_{k=0}^{\infty} (\text{ev}_1^* t_k) \psi_k^1) \cdots (\infty \sum_{k=0}^{\infty} (\text{ev}_n^* t_k) \psi_k^n).$$

Here $\psi_i$ is the 1-st Chern class of the universal cotangent line bundle over $X_{g,n,d}$ corresponding to the $i$-th marked point, $\text{ev}_i^* t_k$ are the pull-backs by the evaluation map $\text{ev}_i : X_{g,n,d} \to X$ at the $i$-th marked point of the cohomology classes $t_0, t_1, \ldots \in H^*(X,\mathbb{Q})$, and $Q_d$ is the representative of $d$ in the semigroup ring of degrees of holomorphic curves in $X$.

Let $E : EX \to X$ be a holomorphic vector bundle. We regard it as an element of the Grothendieck group $K^0(X)$. Given $E$, one associates to it an element $E_{g,n,d}$ in the Grothendieck group $K^0(X_{g,n,d})$ of coherent orbisheaves as follows. Consider the universal stable map with the indices $(g,n,d)$

$$\begin{array}{ccc}
X_{g,n+1,d} & \xrightarrow{\text{ev}} & X \\
\downarrow \text{ft} & & \\
X_{g,n,d} & & \\
\end{array}$$

formed by the maps of forgetting and evaluation at the last marked point. We pull-back $E$ to the universal curve and then apply the $K$-theoretic push-forward to $X_{g,n,d}$. Namely, there exists a complex $0 \to E_{g,n,d}^0 \to E_{g,n,d}^1 \to 0$ of locally free orbisheaves on $X_{g,n,d}$ whose cohomology sheaves are respectively $R^0 \text{ft}_*(\ev^* E)$ and $R^1 \text{ft}_*(\ev^* E)$. Moreover, the difference

$$E_{g,n,d} := [E_{g,n,d}^0] - [E_{g,n,d}^1]$$

in the Grothendieck group of vector orbibundles does not depend on the choice of the complex. These facts are based on some standard general results about local complete intersection morphisms. We refer to Appendix 1 for a further discussion of these properties of the maps $\text{ft}$.

A rational invertible multiplicative characteristic class of complex vector bundles takes on the form

$$c(\cdot) = \exp \left\{ \sum_{k=0}^{\infty} s_k \text{ch}_k(\cdot) \right\}$$

where $\text{ch}_k$ are components of the Chern character, and $s = (s_0, s_1, s_2, \ldots)$ are arbitrary coefficients or indeterminates. Given such a class and a vector bundle
$E \in K^0(X)$, one introduces the $(c, E)$-twisted descendent potentials $D_{c, E}$ and $F_{c, E}$ by replacing the virtual fundamental cycles $[X_{g,n,d}]$ in (1, 2) with the cap-products $c(E_{g,n,d}) \cap [X_{g,n,d}]$. For example, the Poincaré intersection pairing arises in Gromov–Witten theory as the intersection index in $X_{0,3,0} = X$, and in the twisted theory therefore takes on the form

$$ (a, b)_{c(E)} := \int_{[X_{0,3,0}]} c(E_{0,3,0}) \ev_1^*(a) \ev_2^*(1) \ev_3^*(b) = \int_X c(E) \ a \ b. $$

The genus 0 potentials $F_{c, E}$ when reduced modulo $Q$ have zero 2-jet at $t = 0$ since $X_{0,n,0} = X \times \overline{M}_{0,n}$ and dim $\overline{M}_{0,n} = n - 3$. This shows that the twisted potentials $D_{c, E}$ are well-defined (despite of the occurrence of both $\hbar$ and $\hbar^{-1}$ in the exponent of (1)) as formal power series in $t/\hbar, Q/\hbar$ and $\hbar$.

We will often assume that all vector bundles carry the $S^1$-action defined as the fiberwise multiplication by the unitary scalars. In this case $\text{ch}_k$ are understood as $S^1$-equivariant characteristic classes and all GW-invariants take values in the coefficient ring of $S^1$-equivariant cohomology theory. We will always identify the ring $H^*(BS^1; \mathbb{Q})$ with $\mathbb{Q}[\lambda]$ where $\lambda$ is the 1-st Chern class of the Hopf bundle over $\mathbb{C}P^\infty$.

2. Quantization formalism. Theorem 1 below expresses $D_{c, E}$ via $D_X$ in terms of the formalism of quadratic hamiltonians and their quantization in the Fock space (see [17]).

Consider the cohomology space $H = H^*(X; \mathbb{Q})$ as a super-space equipped with the non-degenerate symmetric bilinear form defined by the Poincaré intersection pairing $\langle a, b \rangle = \int_X ab$. Let $\mathcal{H} = H[z, z^{-1}]$ denote the super-space of Laurent polynomials in one even indeterminate $z$ with coefficients in $H$. We equip $\mathcal{H}$ with the even symplectic form

$$ \Omega(f, g) = \frac{1}{2\pi i} \oint (f(-z), g(z)) dz = -(-1)^{\bar{f}}\Omega(g, f). $$

The polarization $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ defined by the Lagrangian subspaces $\mathcal{H}_+ = H[z]$, $\mathcal{H}_- = z^{-1}H[z^{-1}]$ identifies $(\mathcal{H}, \Omega)$ with the cotangent bundle $T^*\mathcal{H}_+$. Then the standard quantization convention associates to quadratic hamiltonians $G$ on $(\mathcal{H}, \Omega)$ differential operators $\hat{G}$ of order $\leq 2$ acting on functions on $\mathcal{H}_+$.

More precisely, let $\{q_\alpha\}$ be a $\mathbb{Z}_2$-graded coordinate system on $\mathcal{H}_+$ and $\{p_\alpha\}$ be the dual coordinate system on $\mathcal{H}_-$ so that the symplectic structure in these coordinates assumes the Darboux form

$$ \Omega(f, g) = \sum_\alpha [p_\alpha(f)q_\alpha(g) - (-1)^{\bar{p}_\alpha}\bar{q}_\alpha q_\alpha(f)p_\alpha(g)]. $$

For example, when $H$ is the standard one-dimensional Euclidean space then $f = \sum q_k z^k + \sum p_k (-z)^{-1-k}$ is such a coordinate system.

In a Darboux coordinate system the quantization convention reads

$$ (q_\alpha q_\beta) := \frac{q_\alpha q_\beta}{\hbar}, \quad (q_\alpha p_\beta) := q_\alpha \frac{\partial}{\partial q_\beta}, \quad (p_\alpha p_\beta) := \hbar \frac{\partial^2}{\partial q_\alpha \partial q_\beta}. $$
The quantization is only a projective representation of the Lie algebra of quadratic hamiltonians on $\mathcal{H}$ to the Lie algebra of differential operators. For quadratic hamiltonians $F$ and $G$ we have
\[
\{F, G\}^\ast = [\hat{F}, \hat{G}] + \mathcal{C}(F, G)
\]
where $\{\cdot, \cdot\}$ is the Poisson bracket, $[\cdot, \cdot]$ is the super-commutator, and $\mathcal{C}$ is the cocycle
\[
\begin{align*}
\mathcal{C}(p_\alpha p_\beta, q_\alpha q_\beta) &= (-1)^{\tilde{g}_\alpha \tilde{g}_\beta} & \text{if } \alpha \neq \beta, \\
\mathcal{C}(q_\alpha^2, q_\beta^2) &= 1 + (-1)^{\tilde{g}_\alpha \tilde{g}_\beta}, \\
\mathcal{C} &= 0 & \text{on any other pair of quadratic Darboux monomials.}
\end{align*}
\]
We associate the quadratic hamiltonian $(Tf, f)/2$ to an infinitesimal symplectic transformation $T$. If $A, B$ are self-adjoint operators on $H$, then the operators $f \mapsto (A/z)f$ and $f \mapsto (Bz)f$ in $\mathcal{H}$ are infinitesimal symplectic transformations, and
\[
\mathcal{C}(A/z, Bz) = \text{str}(AB)/2.
\]

The differential operators act on formal functions (with coefficients depending on $\hbar^{k+1}$) on the space $\mathcal{H}^+$ of vector-polynomials $q = q_0 + q_1 z + q_2 z^2 + \ldots$ with the coefficients $q_0, q_1, q_2 \ldots \in H$. We will often refer to such functions as elements of the Fock space. We will assume that the ground field $\mathbb{Q}$ of constants is extended to the Novikov ring $\mathbb{Q}[[Q]]$ (or to $\mathbb{Q}(\lambda)[[Q]]$ in the $S^1$-equivariant setting) and will denote the ground ring by $\Lambda$.

On the other hand, the potentials $F^q_X$ are naturally defined as formal functions on the space of vector-polynomials $t(\psi) = t_0 + t_1 \psi + t_2 \psi^2 + \ldots$ where $t_0, t_1, t_2, \ldots \in H$ are cohomology classes of $X$ with coefficients in $\Lambda$, and $\psi$ is an indeterminate to be substituted by successive $\psi_i$'s in the definition $F^i$. We identify $\psi$ with $z$, and the total descend potential $F^q$ — with an element of the Fock space by means of the dilaton shift
\[
q(z) = t(z) - z.
\]

The twisted descend potentials $\mathcal{D}_{c,E}$ can be similarly considered as elements of the Fock spaces corresponding to the super-space $H$ equipped with the twisted inner products $\langle \cdot, \cdot \rangle$. Alternatively, we can identify the inner product spaces $(H, (\cdot, \cdot)_{c(E)})$ with $(H, (\cdot, \cdot))$ by means of the maps $a \mapsto a \sqrt{c(E)}$. Using the corresponding identification of the Fock spaces we consider the twisted descend potentials $\mathcal{D}_{c,E}$ as elements of the original Fock space via the convention:
\[
q(z) = \sqrt{c(E)}(t(z) - z).
\]
We obtain therefore a family $\mathcal{D}_s := \mathcal{D}_{c,E}$ of elements of the Fock space depending on the parameters $s = (s_0, s_1, s_2, \ldots)$. It is easy to see that the potentials $\mathcal{D}_{c,E}$ are well-defined at least as formal functions of variables $t$ and parameters $s$. We should stress however that, due to the dilaton shift, $\mathcal{D}_s$ as an element of the Fock space is a formal function of $q$ near the shifted origin $q(z) = -\sqrt{c(E)}z$ (drifting therefore with $s$).

3. Quantum Riemann–Roch. The operator of multiplication by $\text{ch}_1(E)$ in the cohomology algebra $H$ of $X$ is self-adjoint with respect to the Poincare pairing. Consequently the operator $A$ of multiplication by $\text{ch}_1(E)z^{2m-1}$ in the algebra $H = H[z, z^{-1}]$ is anti-self-adjoint with respect to the symplectic structure $\Omega$. Thus this operator defines an infinitesimal symplectic transformation, and $(\text{ch}_1(E)z^{2m-1})^\ast$
denotes the operator on the Fock space defined by quantization of the corresponding quadratic hamiltonian Ω(Af, f)/2.

Let $B_{2m}$ denote Bernoulli numbers: $\frac{x}{1-e^x} = \frac{x}{2} + \sum_{m\geq 0} B_{2m} \frac{x^{2m}}{(2m)!}$.

**Theorem 1.**

$$\exp\left\{ \frac{-1}{24} \sum_{l>0} s_{l-1} \int_X c_{D-1}(T_X) \right\} \text{ (sdet } \sqrt{c(E)})^{-\frac{1}{24}} D_s =$$

$$\exp\left\{ \sum_{m>0} \sum_{l>0} s_{2m-1} B_{2m} \frac{1}{(2m)!} (\frac{c_i(E) z^{2m-1}}{z}) \right\} \exp\left\{ \sum_{l>0} s_{l-1} (\frac{c_i(E)}{z}) \right\} D_0.$$

**Remarks.** (1) The variable $s_0$ is present on the RHS of (3) only in the form $\exp(s_0 \rho/z)$ where $\rho = ch_1(E)$. For any $\rho \in H^2(X)$ the operator $(\rho/z)^+$ is in fact a divisor operator, that is the total descendent potential satisfies the following divisor equation:

$$D_0 = \sum \rho_i Q_i \frac{\partial}{\partial Q_i} D_0 - \frac{1}{24} \int_X \rho \ c_{D-1}(T_X) \ D_0.$$

Here $Q_i$ are generators in the Novikov ring corresponding to a choice of a basis in $H_2(X)$, and $\rho_i$ are coordinates of $\rho$ in the dual basis. For $\rho = ch_1(E)$ the $c_{D-1}$-term cancels with the $s_0$-term on the LHS of (3). Thus the action of the $s_0$-flow reduces to the change $Q^d \mapsto Q^d \exp(s_0 \int_X \rho)$ in the descendent potential $D_0$ combined with the multiplication by the factor $\exp\{s_0[(\dim E)/48]\}$ which comes from the super-determinant.

(2) When $E = \mathbb{C}$, we have $E_{g,n,d} = \mathbb{C} - \mathbb{E}_g$, where $\mathbb{E}_g$ is the Hodge bundle. The Hodge bundles are known to satisfy $ch_k(\mathbb{E}_g) = -ch_k(\mathbb{E}_g^*)$ (in fact we reprove this in the next section). In view of this, Theorem 1 in this case turns into Theorem 4.1 in [17] and is a reformulation in terms of the formalism explained in Section 3 of the results of Mumford [31] and Faber – Pandharipande [13]. The proof of Theorem 1 is based on a similar application of Mumford’s Grothendieck – Riemann – Roch argument to our somewhat more general situation. The argument, no doubt, was known to the authors of [13]. The main new observation here is that the combinatorics of the resulting formula which appears rather complicated at a first glance fits quite nicely with the formalism of quantized quadratic hamiltonians. A verification of this — somewhat tedious but straightforward — is presented in Appendix 1.

**4. The Euler class.** The $S^1$-equivariant Euler class of $E$ is written in terms of the (non-equivariant) Chern roots $\rho_i$ as

$$e(E) = \prod_i (\lambda + \rho_i).$$

Using the identity $(\lambda + x) = \exp(\ln \lambda - \sum_k (-x)^k / k \lambda^k)$ we can express it via the components of the non-equivariant Chern character:

$$e(E) = \exp\{ ch_0(E) \ln \lambda + \sum_{k>0} ch_k(E) (-1)^{k-1}(k-1)! \lambda^k \}.$$
Denote by $D_e$ the element $D_s$ of the Fock space corresponding to $s_0 = \ln \lambda$ and $s_k = (-1)^{k-1}(k-1)!/\lambda^k$ for $k > 0$. Substituting the values of $s_k$ into (9), replacing $\text{ch}_i(E)$ by $\sum \rho_i^j/!!$ and using the binomial formula

$$(1 + x)^{1-2m} = \sum_{l \geq 0} \frac{(-1)^l(2m-2+l)!}{(2m-2)! l!} x^l$$

we arrive at the following conclusion.

**Corollary 1.**

$$\prod_i \text{exp}\{-\frac{1}{24} \int_X [(\lambda + \rho_i) \ln(\lambda + \rho_i) - (\lambda + \rho_i)] c_{D-1}(T_X) \} \prod_i (\text{sdet} \sqrt{\lambda + \rho_i})^{-\frac{1}{2}} D_e =$$

$$\prod_i \text{exp}\{\sum_{n>0} \frac{B_{2m}}{2m(2m-1)} \left( \frac{z}{\lambda + \rho_i} \right)^{2m-1} \} \prod_i \text{exp}\{\frac{(\lambda + \rho_i) \ln(\lambda + \rho_i) - (\lambda + \rho_i)}{z} \} \cdot D_0.$$  

**Remark.** The $1/z$-term in this formula actually arises in the form

$$\rho \ln \lambda + \sum (-1)^{k-1} \frac{k+1}{k(k+1)} \frac{\rho^k}{\lambda^k} = \int_0^\rho \ln(\lambda + x) dx = [(\lambda + x) \ln(\lambda + x) - (\lambda + x)] |^\rho_0.$$  

It has positive cohomological degree and is small in this sense. The constant term $(\lambda \ln \lambda - \lambda)/z$ is thrown away on the following grounds. According to [3], $(1/z)$ is the string operator and annihilates the descendent potential $D_0$. Thus the operators $\exp((\lambda \ln \lambda - \lambda)/z)$ do not change $D_0$. The rest of the series in the exponent converges in the $1/\lambda$-adic topology.

5. **Quantum Serre.** Introduce the multiplicative characteristic class

$$c^*(\cdot) = \exp\{\sum (-1)^{k+1} s_k \text{ch}_k(\cdot)\}.$$  

Since $\text{ch}_i(E) = (-1)^i \text{ch}_i(E)$ we have

$$c^*(E^*) = \frac{1}{c(E)}.$$  

There is no obvious relationship between $c^*((E^*)_{g,n,d})$ and $c(E_{g,n,d})$, but nonetheless the twisted descendant potentials $D_s = D_{c,E}$ and $D_s^* := D_{c^*,E^*}$ are closely related.

**Corollary 2.** We have $D_s^* = (\text{sdet} c(E))^{-1/24} D_s$. More explicitly,

$$D_{c^*,E^*}(t^*) = (\text{sdet} c(E))^{-\frac{1}{24}} D_{c,E}(t),$$

where $t^*(z) = c(E)t(z) + (1 - c(E))z$.

**Proof.** Replacing $\text{ch}_i(E)$ with $(-1)^i \text{ch}_i(E)$, and $s_k$ with $(-1)^{k+1} s_k$ in (7) preserves all terms except the super-determinant.

**Corollary 3.** Consider the dual bundle $E^*$ equipped with the dual $S^1$-action, and the $S^1$-equivariant inverse Euler class $e^{-1}$. Put

$$t^*(z) = z + (-1)^{\dim E/2} e(E)(t(z) - z)$$

and introduce the change $\pm : Q^d \mapsto Q^d(-1)^l \text{ch}_1(E)$ in the Novikov ring. With this notation

$$D_{e^{-1},E^*}(t^*, Q) = \text{sdet}[(\lambda)^{-\frac{1}{24}} e(E)]^{-\frac{1}{24}} D_{e,E}(t, \pm Q).$$
Proof. We have \( e^{-1}(E^*) = \prod_{i} (-\lambda - \rho_i)^{-1} \). Since
\[
(-\lambda + x)^{-1} = \exp\{-\ln(-\lambda) + \sum_{k} \frac{x^k}{k!}\} 
\]
we find that \( e^{-1}(\cdot) = \exp \sum s_k^* \operatorname{ch}_k(\cdot) \) where \( s_k^* = (k - 1)!/\lambda^k \) for \( k > 0 \) and \( s_0 = -\ln(-\lambda) \). For \( k > 0 \), \( s_k^* = (-1)^{k+1} s_k \) as in the situation of Corollary 2. However, \( s_0^* = -s_0 - \pi \sqrt{-1} \). We compensate for the discrepancy \(-\pi \sqrt{-1}\) using the divisor equation \( \mathfrak{D} \) described in Remark 1 following Theorem 1.

6. Genus 0. The genus 0 descendent potential \( \mathcal{F}_X^0 \) can be recovered from the so-called “J-function” of finitely many variables due to a reconstruction theorem essentially due to Dubrovin [12] and going back to Dijkgraaf and Witten [11]. The J-function is a formal function of \( t \in H \) and \( 1/z \) with vector coefficients in \( H \) defined by
\[
\forall a \in H, \quad (J_X(t, z), a) := (z + t, a) + \sum_{d, n} \frac{Q^d}{n!} \int_{[X_0, n+1, d]} \prod_{i=1}^n \frac{\operatorname{ev}_{n+1}^* t a}{z - \psi_{n+1}}. 
\]
We need the following reformulation of the reconstruction theorem in terms of the geometry of the symplectic space \((\mathcal{H}, \Omega)\), where we take \( \mathcal{H} \) to be the completion \( H((-1)^*) \).

The genus-0 descendent potential \( \mathcal{F}_X^0 \) considered as a formal function of \( q \in \mathcal{H}_+ \) via the dilaton shift \( \mathfrak{D} \) generates (the germ of) a Lagrangian section \( \mathcal{L}_X \subset \mathcal{H} = T^* \mathcal{H}_+ \). In Darboux coordinates
\[
\mathcal{L}_X = \{(p, q) : p = d_q \mathcal{F}_X^0 \}.
\]

Proposition. \( \mathcal{L}_X \) is a homogeneous Lagrangian cone swept by a moving semi-infinite subspace depending on \( \dim H \) parameters. More precisely,
(i) the tangent space \( L_\mathfrak{t} \subset \mathcal{H} \) to \( \mathcal{L}_X \) at a point \( \mathfrak{t} \) satisfies \( \mathcal{L}_X \cap L_\mathfrak{t} = z L_\mathfrak{t} \);
(ii) \( J_X(t, -z) \in \mathcal{H} \) is the intersection of \( \mathcal{L}_X \) with \( (t - z) + \mathcal{H}_- \).

Remarks. (1) Part (i) implies that the tangent spaces \( L_\mathfrak{t} \) are Lagrangian subspaces invariant under multiplication by \( z \). They consequently belong to the loop group Grassmannian (of the “twisted” series \( A^{(2)} \)) or to its super-version.

(2) Part (i) of the Proposition means that the spaces \( L_\mathfrak{t} \) actually depend only on \( \dim H \) parameters and form a variation of semi-infinite Hodge structures in the sense of [4]. It also shows that the cone \( \mathcal{L}_X \) is determined by its generic (\( \dim H \))-parametric slice \( (J(t) \in \mathcal{H})_{t \in H} \). Indeed, if the first \( t \)-derivatives of \( J \) span \( L_{J(t)}/z L_{J(t)} \) over \( \Lambda \), then they span the tangent space \( L_{J(t)} \) over \( \Lambda[z] \) and the cone is the union of the isotropic spaces \( z L_{J(t)} \). Part (ii) identifies one such slice with the J-function.

Part (ii) of the Proposition follows immediately from the definitions of \( J_X \) and \( \mathcal{L}_X \).

Part (i) follows easily from Dubrovin’s reconstruction formula [12] in the axiomatic theory of Frobenius structures. Indeed, the main feature of the formula is that the 2nd differentials \( d^2_q \mathcal{F}_X^0 \) of the genus 0 descendent potential depend on the application point \( q(z) = t(z) + z \) only through some finite-dimensional function \( q \mapsto t(q) \in H \), and the levels of the function are (germs at \( q = -z \) of) linear subspaces of codimension \( \dim H \). In geometric terms this means that the tangent
spaces to \( \mathcal{L}_X \) (regarded as affine spaces in \( \mathcal{H} \)) remain constant along these subspaces. The tangent spaces actually pass through the origin since \( \mathcal{L}_X \) is a cone — this follows from the genus 0 dilaton equation (see \([12, 19]\)). The tangent spaces therefore form a family \( \{\mathcal{L}_t\} \) of Lagrangian spaces depending only on \( t \in H \) and intersecting the cone \( \mathcal{L}_X \) along subspaces \( I_t \) of codimension \( \dim H \). Invariance of \( \mathcal{L}_X \) with respect to the flow of the string vector field \( f \mapsto f/z \) implies \( z^{-1}I_t \subset L_t \).

Considering the Fredholm index of projections to \( H_t \) allows us to state a field. In Appendix 2 we give another, more direct proof applicable in \([17]\) which relates gravitational descendents with ancestors in \([17]\) which relates gravitational descendents with ancestors in \([17]\). The tangent spaces actually pass through the origin since \( \mathcal{L}_L \) is a cone — this follows from the genus 0 dilaton equation (see \([12, 19]\)). The tangent spaces therefore form a family \( \{\mathcal{L}_t\} \) of Lagrangian spaces depending only on \( t \in H \) and intersecting the cone \( \mathcal{L}_X \) along subspaces \( I_t \) of codimension \( \dim H \). Invariance of \( \mathcal{L}_X \) with respect to the flow of the string vector field \( f \mapsto f/z \) implies \( z^{-1}I_t \subset L_t \).

In the above argument, we assume that the ground ring \( A \) is (or has been extended to) a field. In Appendix 2 we give another, more direct proof applicable in Gromov–Witten theory and free of this defect. It is based on Theorem 5.1 stated in \([17]\) which relates gravitational descendents with ancestors.

In the quasi-classical limit \( h \to 0 \), quantized symplectic transformations \exp \( \hat{A} \) of Theorem 1 acting on the total potentials considered as elements \( D \) in the Fock space turn into the “unquantized” symplectic transformations acting by \( \mathcal{L}_s \mapsto (exp A)\mathcal{L}_s \) on the Lagrangian cones \( \mathcal{L}_s \) generated by the genus 0 potentials \( \mathcal{F}_{c(s),E}^0 \).

**Corollary 4.**

\[
\mathcal{L}_s = \exp \left\{ \sum_{m \geq 0} \sum_{0 \leq l \leq D} s_{2m-1+l}(B_{2m}/(2m)! \text{ch}_l(E)z^{2m-1}) \right\} \mathcal{L}_0.
\]

**7. Quantum Lefschetz.** In the case of genus 0 GW-theory twisted by the Euler class \( e(E) \), the corresponding Lagrangian cone \( \mathcal{L}_{s(e)} \) is obtained from \( \mathcal{L}_X \) by multiplication in \( \mathcal{H} \) defined by the product over the Chern roots \( \rho \) of the series

\[
b_{\rho}(z) = \exp \left\{ \frac{(\lambda + \rho) \ln(\lambda + \rho) - (\lambda + \rho)}{z} + \sum_{m > 0} \frac{B_{2m}}{2m(2m - 1)} \left( \frac{z}{\lambda + \rho} \right)^{2m - 1} \right\}
\]

The series \([14]\) is well-known \([14]\) in connection with the asymptotic expansion of the gamma function \( \Gamma((\lambda + \rho)/z) \). More precisely, \([14]\) coincides with the stationary phase asymptotics of the integral

\[
\int_0^\infty e^{\frac{\lambda + (\lambda + \rho) \ln(x)}{x} \ln x} dx
\]

near the critical point \( x = \lambda + \rho \) of the phase function.

Let us assume now that \( E \) is the direct sum of \( r \) line bundles with the 1-st Chern classes \( \rho_i \) — in what follows we will need the Chern roots to be integer — and consider the J-function \( J_X(t,z) = \sum_d J_d(t,z)Q^d \). Put \( \rho_i(d) = \int_d \rho_i \) and introduce the following hypergeometric modification of \( J_X \):

\[
I_E(t,z) = \sum_d J_d(t,z)Q^d \prod_{k=0}^r \prod_{k=\infty}^{\infty} \frac{(\lambda + \rho_i + kz)}{(\lambda + \rho_i + k)}. \]

**Theorem 2.** The hypergeometric modification \( I_E \) considered as a family \( t \mapsto I_E(t,-z) \) of vectors in the symplectic space \( \langle H, \Omega_{c(e)} \rangle \) corresponding to the twisted inner product \( \langle a, b \rangle_{c(e)} = \int_X e(E)ab \) on \( H \), is situated on the Lagrangian section \( \mathcal{L}_{c,E} \subset \mathcal{H} \) defined by the differential of the twisted genus 0 descendent potential \( \mathcal{F}_{c,E}^0 \).
The following comment is in order. The series $I_E$ does not necessarily belong to $H((z^{-1}))$ because of possible unbounded growth of the numbers $\rho_i$. However the coefficients at each particular monomial $Q^d$ do. Similarly, multiplication by the series \([11]\) moves the cone $L_X$ out of the space $H((z^{-1}))$. However modulo each particular power of $1/\lambda$ it does not (the invariance of the cone with respect to the string flow $\exp(\lambda \ln \lambda - \lambda)/z$ is once again essential here). In fact all our formulas make sense as operations with generating functions, i. e. give rise to legitimate operations with their coefficients, because of the presence of suitable auxiliary variables (such as $s_k$ in Corollary 3, $1/\lambda$ in \([11]\), $Q$ in \([12]\), etc.) More formally, this means that (i) the ground ring $\Lambda$ in $H = H^*(X, \Lambda)$ should be completed in a suitable “adic” topology, and (ii) in the role of the symplectic space $H$ this means that (i) the ground ring $\Lambda$ in formulas make sense as operations with generating functions, i. e. give rise to legitimate operations with their coefficients, because of the presence of suitable auxiliary variables (such as $s_k$ in Corollary 3, $1/\lambda$ in \([11]\), $Q$ in \([12]\), etc.) More formally, this means that (i) the ground ring $\Lambda$ in $H = H^*(X, \Lambda)$ should be completed in a suitable “adic” topology, and (ii) in the role of the symplectic space $H$, we should take the space $H\{z, z^{-1}\}$ of Laurent series $\sum_{k \in \mathbb{Z}} h_k z^k$ “convergent” in the sense that $\lim_{k \to \infty} h_k \to 0$ in the topology of $\Lambda$. In the following proof we will have to similarly replace $\Lambda[z]$ by $\Lambda\{z\}$, and the ring $\Lambda$ should be also extended by $\sqrt{\lambda}$.

8. Proof of Theorem 2. Due to the equivariance properties (see \([19]\), Section 6) of $J$-functions with respect to the string and divisor flows \([8]\) we have

$$J_X(t + \sum (\lambda + \rho_i) \ln x_i) = e^{\sum (\lambda + \rho_i) \ln x_i} \sum_d J_d(t, z) \prod \rho_i^{\rho_i(d)}.$$ 

Integrating by parts (as in the proof of the identity $\Gamma(x + 1) = x\Gamma(x)$) we find

$$\begin{align*}
(2\pi z)^{-\frac{1}{2}} & \int_0^\infty dx_1 \ldots \int_0^\infty dx_r \ e^{-\sum x_i/z} J_X(t + \sum (\lambda + \rho_i) \ln x_i) = \\
I_E(t, z) \sqrt{e(E)} \prod_i \frac{1}{\sqrt{2\pi z(\lambda + \rho_i)}} & \int_0^\infty e^{-x_i/(\lambda + \rho_i)} dx_i.
\end{align*}$$

We conclude that the asymptotic expansion of the integral \([13]\) coincides with $I_E(t, z) \sqrt{e(E)} \prod_i b_{\rho_i}(z)$.

The multiplication by $\sqrt{e(E)}$ identifies the Lagrangian cone $L_{e,E} \subset (H, \Omega_{e(E)})$ with its normalized incarnation $L_{e(e)} \subset (H, \Omega)$. Therefore Theorem 2 is equivalent to the inclusion $I_E(t, z) \sqrt{e(E)} \prod_i b_{\rho_i}(z) \in L_{e(e)}$ and, due to Corollary 4, — to $I_E(t, z) \sqrt{e(E)} \prod_i b_{\rho_i}(z) \in L_0 = L_X$. It remains to show therefore that the asymptotic expansion of the integral \([13]\) belongs to the cone determined by the $J$-function $J_X(t, z)$. In fact we will prove the following

**Lemma.** For each $t$, the asymptotic expansion of the integral \([13]\) differs from $\lambda^{\dim E/2} J_X(t^*, z)$ (at some other point $t^*(t)$) by a linear combination of the first $t$-derivatives of $J_X$ at $t^*$ with coefficients in $z\Lambda\{z\}$.

For this, we are going to use another property of the $J$-function $J_X$ well-known in quantum cohomology theory and in the theory of Frobenius structures (see for instance Section 6 in \([13]\) and \([14]\)). The first derivatives $\partial_{\alpha} J_X$ satisfy the system of linear PDEs

$$z \frac{\partial}{\partial t^\alpha} \frac{\partial}{\partial t^\beta} J_X(t, z) = \sum_\gamma A_{\alpha\beta}^\gamma(t) \frac{\partial}{\partial t^\gamma} J_X(t, z).$$

where we use a coordinate system $t = \sum t^\alpha \phi_\alpha$ on $H$. Indeed, following \([8]\) we can argue that the second derivatives are linear combination of the first derivatives over $z^{-1} \Lambda\{z\}$ since infinitesimal variations of the tangent spaces $L_t$ spanned by $\partial J_X/\partial t^\alpha$
are to stay inside $z^{-1}L_t$, and that on the other hand the second derivatives are contained in $\mathcal{H}_-$ since $J_X \in z + t + \mathcal{H}_-$. Further analysis reveals that $A^\alpha_{\beta}$ are structural constants of the quantum cohomology algebra $\phi_{\alpha} \cdot \phi_{\beta} = \sum A^\gamma_{\alpha\beta} \phi_{\gamma}$.

In particular, $z\partial_t J_X = J_X$ since $1 \cdot = \text{id}$ (we use here the notation $\partial_v$ for the directional derivative in the direction of $v \in H$ and take $v = 1$).

We can interpret (14) as the relations defining the D-module generated by $J_X$, i.e. obtained from it by application of all differential operators. Using Taylor’s formula $J_X(t + y\partial_v) = \exp(y\partial_v)J_X(t)$ we now view (13) as the asymptotic expansion of the oscillating integral taking values in this D-module:

$$
(2\pi z)^{-\frac{5}{2}} \int_0^\infty dx_1 \ldots \int_0^\infty dx_r \ e^{-\sum x_i + \sum \frac{m}{z} \ln x_i} J_X(t, z) \sim
$$

$$
\prod_i e^{\frac{(\lambda z \partial_{\rho_i}) \ln(\lambda z \partial_{\rho_i}) - (\lambda z \partial_{\rho_i})}{z}} J_X(t, z),
$$

The exact form of the series in not relevant this time. What matters is that the relations (14) in the D-module allow us to rewrite any high order derivation as a differential operator of first order and that composition of derivations coincides with the quantum cup-product modulo higher order terms in $z$:

$$
z\partial_{u_1} \ldots z\partial_{u_N} = z\partial_{v_1} \ldots z\partial_{v_N} + o(z),$$

where $o(z)$ stands for a linear combination of $z\partial_{\rho_i}$ with coefficients in $z\Lambda \{z\}$. Using this (and also the relation $\lambda J_X = z\partial_{\lambda} J_X$ mentioned earlier) we see that

$$
\prod_i e^{\frac{(\lambda z \partial_{\rho_i}) \ln(\lambda z \partial_{\rho_i}) - (\lambda z \partial_{\rho_i})}{z}} J_X(t, z) = \prod_i e^{(\lambda z \partial_{(\lambda z \partial_{\rho_i})}) \ln(\lambda z \partial_{\rho_i}) - (\lambda z \partial_{\rho_i})} J_X(t, z)
$$

$$
= e^{\frac{m}{z}} J_X(t^*, z),
$$

where $t^*(t) = t + [\sum (\lambda + \rho_i \cdot) \ln(\lambda + \rho_i \cdot) - (\lambda + \rho_i \cdot)] 1$. Processing next the factor $e^{\frac{m}{z} \ln(1 + z\partial_{\rho_i})}$, we take out $\sqrt{A}$. The remaining factor $e^{\frac{m}{z} \ln(1 + z\partial_{\rho_i})}$ together with the rest of the exponent in the asymptotic expansion (15) yields an expression of the type $e^{o(z)/z} J_X(t^*, z)$ too. We conclude that the expansion (15) assumes the form

$$
\lambda z \partial_{\rho} J_X(t^*, z) + \sum C_{\alpha}(t^*, z) z\partial_{\rho_{\alpha}} J_X(t^*, z),
$$

where the coefficients $C_{\alpha}(t^*, \cdot)$ are in $\Lambda \{z\}$ as required.

**Remark.** The proof of the Lemma actually shows that for any phase function $\Phi(v)$ of $v \in H$ the asymptotics of the oscillating integral $\int dv \ e^{\Phi(z\partial_v) / z} J_X(t, z)$ generates the same cone as $J_X$.

**Corollary 5.** Let $\mathcal{L}_{e,E} \subset (\mathcal{H}, \Omega_{e(E)})$ be the Lagrangian cone determined by $I_E(t, -z)$ and let $L_t$ be the tangent space to $\mathcal{L}_{e,E}$ at the point $I_E(t, -z)$. Then the intersection (unique due to some transversality property) of $zL_t$ with the affine subspace $-z + z\mathcal{H}_-$ coincides with the value $J_{e,E}(t, z) \in -z + \tau(t) + \mathcal{H}_-$ of the $J$-function corresponding to the $(e, E)$-modified GW-theory. In other words,

$$
J_{e,E}(t, z) = I_E(t, z) + \sum c_{\alpha}(t, z) \ z\partial_{\rho_{\alpha}} I_E(t, z), \quad \text{where} \ c_{\alpha}(t, \cdot) \in \Lambda \{z\},
$$

for $c_{\alpha}(t, \cdot) \in \Lambda \{z\}$. 

and $\tau(t)$ is determined by the asymptotics $z + \tau \pmod{\mathcal{H}_-}$ of the RHS.

Remarks. (1) The procedure of computing $J_{e,E}$ in terms of $I_E$ is reminiscent of the Birkhoff factorization $U(z, z^{-1}) = V(z^{-1})W(z)$ in the theory of loop groups. Moreover, the procedure applied to the first derivatives of $I_E$ instead of $I_E$ is an example of Birkhoff factorization. Indeed, the derivatives form a $\Lambda\{z\}$-basis $U(z, z^{-1})$ in $L_t$, and $W(z)$ is the transition matrix to another, canonical basis $V(z^{-1}) \in 1 + \mathcal{H}_-$ formed by the first derivatives of $J_{e,E}$.

(2) A by-product of Corollary 5 is a geometrical description of the “mirror map” $t \mapsto \tau$: the $J$-function obtained as the intersection $L_t \cap (-z + z\mathcal{H}_-)$ comes naturally parameterized by $t$ which may have little common with the projections $\tau - z$ of the intersection points along $\mathcal{H}_-$.

9. Mirror formulas. Let us assume now that the bundle $E$ (which is still the sum of line bundles with first Chern classes $\rho_i$) is convex, i.e. spanned fiberwise by global sections, and apply the above results to the genus 0 GW-theory for a complete intersection $j : Y \subset X$ defined by a global section. While the above proof of Theorem 2 fails miserably in the limit $\lambda = 0$, the definition of the series $J_{e,E}$ and $I_E$ and the relation between them described by Corollary 5 survive the non-equivariant specialization. Namely, at $\lambda = 0$ the $J$-function $J_{e,E}$ degenerates into

$$J_{X,Y}(t, z) = z + t + \sum_{d,n} \frac{Q^d}{d!} (ev_{n+1})_* \left[ e(E_{0,n+1,d}) \frac{e(E_{0,n+1,d})}{z - \psi_{n+1}} \wedge_{i=1}^n ev_i^* t \right],$$

where $(ev_{n+1})_*$ is the cohomological push-forward along the evaluation map $ev_{n+1} : X_{0,n+1,d} \to X$ and $e$ is the (non-equivariant!) Euler class. Here $E_{0,n+1,d} \subset E_{0,n+1,d}^\ell$ is the subbundle defined as the kernel of the evaluation map $E_{0,n+1,d} \to ev_{n+1}^* E$ of sections (from $H^0(\Sigma, f^* E)$) at the $n + 1$-st marked point.

The function $J_{X,Y}$ is related to the GW-invariants of $Y$ by

$$e(E)J_{X,Y}(j^* u, z) = H_2(Y) \to H_2(X) j_* J_Y(u, z)$$

since $[Y_{0,n+1,d}] = e(E_{0,n+1,d}) \cap [X_{0,n+1,d}]$ (see for instance [10]). The long subscript here is to remind us that the corresponding homomorphism between Novikov rings should be applied to the RHS.

On the other hand, the series $I_E$ in the limit $\lambda = 0$ specializes to

$$I_{X,Y}(t, z) = \sum_d J_d(t, z)Q^d \prod_{i=1}^{\rho_i(d)} (\rho_i + k z)$$

since $\rho_i(d) \geq 0$ for all degrees $d$ of holomorphic curves. Passing to the limit $\lambda = 0$ in Theorem 2 and Corollary 5 we obtain the following “mirror theorem”.

Corollary 6. The series $I_{X,Y}(t, -z)$ and $J_{X,Y}(\tau, -z)$ determine the same cone. In particular, the series $J_{X,Y}$ related to the $J$-function of $Y$ by (17) is recovered from $I_{X,Y}$ via the “Birkhoff factorization procedure” followed by the mirror map $t \mapsto \tau$ as described in Corollary 5.

Remark. Corollary 6 is more general than the (otherwise similar) quantum Lefschetz hyperplane section theorems by Bertram and Lee [4, 8] and Gathmann [13] for (i) it is applicable to arbitrary complete intersections $Y$ without the restriction $c_1(Y) \geq 0$ and (ii) it describes the $J$-functions not only over the small space of
parameters \( t \in H^{≤2}(X, \Lambda) \) but over the entire Frobenius manifold \( H^{*}(X, \Lambda) \). In fact the results of [15] allow one to deal with both generalizations and to compute recursively the corresponding GW-invariants one at a time. What has been missing so far is the part that Birkhoff factorization plays in the formulations.

Now restricting \( J_{X,Y} \) and \( I_{X,Y} \) to the small parameter space \( H^{≤2}(X, \Lambda) \) and assuming that \( c_1(E) \leq c_1(X) \) we can derive the quantum Lefschetz theorems of [4, 25, 26, 33]. A dimensional argument shows that the series \( I_{X,Y} \) on the small parameter space has the form

\[
I_{X,Y}(t, z) = z F(t) + \sum G^{i}(t) \phi_{i} + O(z^{-1}),
\]

where \( \{ \phi_{i} \} \) is a basis in \( H^{≤2}(X, \Lambda) \), \( G^{i} \) and \( F \) are scalar formal functions and \( F \) is invertible (we have \( F = 1 \) and \( G^{i} = t^{i} \) when the Fano index is not too small).

**Corollary 7.** When \( c_1(E) \leq c_1(X) \) the restriction of \( J_{X,Y} \) to the small parameter space \( \tau \in H^{2}(X, \Lambda) \) is given by

\[
J_{X,Y}(\tau, z) = \frac{I_{X,Y}(t, z)}{F(t)}, \quad \text{where} \quad \tau = \sum \frac{G^{i}(t)}{F(t)} \phi_{i},
\]

The J-function of \( X = \mathbb{C}P^{n-1} \) restricted to the small parameter plane \( t_{0} + t P \) (where \( P \) is the hyperplane class generating the algebra \( H^{*}(X, \Lambda) = \Lambda[P]/(P^n) \)) takes on the form

\[
J_X = z e^{(t_0 + P t) / z} \sum_{d \geq 0} \frac{Q^d e^{dt}}{\prod_{k=1}^{d} (P + k z)^n}.
\]

For a hypersurface \( Y \) of degree \( l \) in \( \mathbb{C}P^{n-1} \) we then have

\[
I_{X,Y} = z e^{(t_0 + P t) / z} \sum_{d \geq 0} \frac{Q^d e^{dt}}{\prod_{k=1}^{d} (P + k z)^n}.
\]

**Corollary 8.** On the small parameter space

(i) \( J_{X,Y}(t_{0}, t, z) = I_{X,Y}(t_{0}, t, z) \) when \( l < n - 1 \);

(ii) \( J_{X,Y}(t_{0}, t, z) = I_{X,Y}(t_{0}, t, z) \), \( t_{0} = t_{0} + l! Q e^{t} \), when \( l = n - 1 \);

(iii) \( J_{X,Y}(t_{0}, \tau, z) = I_{X,Y}(t_{0}, \tau, z)/F(t), \quad \tau = G(t)/F(t), \) when \( l = n \), and the series \( F \) and \( G \) are found from the expansion \( I_{X,Y} = \exp(t_{0}/z)|z F + GP + O(z^{-1})| \).

Projecting \( J_{X,Y} \) by \( j^{*} \) onto the cohomology algebra \( \Lambda[P]/(P^n) \subset H^{*}(Y, \Lambda) \) we recover the mirror theorem of [17], and in the case \( l = n = 5 \) — the quintic mirror formula of Candelas et al [4].

10. Further comments.

On quantum Riemann – Roch. The operators \( \text{ch}_{l}(E)z^{2m-1} \) commute. In the non-equivariant setting this property is preserved under quantization for the operators with \( m \geq 0 \) which occur in Theorem 1. This is due to the nilpotency of \( \text{ch}_{l}(E) \) with \( l > 0 \). Also, the summand with \( l = 1 \) on the LHS of (7) is the only one left in this case. Thus formula (7) simplifies in the non-equivariant case:

\[
D_{s} = \left( e^{\text{ch}_{l}(E), c_{l-1}(T_{X})} \right) \text{sdet} \sqrt{c(E)} \sum_{m \geq 0} \sum_{l \geq 0} \frac{p_{m}^{l} \text{ch}_{l}(E)z^{2m-1}}{l!} D_{0}.
\]

The formula defines a formal group homomorphism from the group of invertible multiplicative characteristic classes to invertible operators acting on elements of the Fock space. It would be interesting to find a quantum-mechanical interpretation of
the normalizing factor in this formula. Since the Fock space should consist of top-degree forms on \( H^+ \) rather than function, the super-determinant probably takes on the role of the Jacobian of our “bare hands” identification \( q \mapsto \sqrt{c(E)} q \). We do not have however a plausible “physical” interpretation for the other factor.

On the Lagrangian cones. In the case of genus 0 GW-theory of \( X = pt \) the cone \( L_{pt} \) is generated by the family of functions in one variable \( x \):

\[
F(x, q) := \frac{1}{2} \int_0^x Q^2(u) \, du, \quad \text{where} \quad Q(x) = \sum q_k \frac{x^k}{k!}.
\]

In particular, under analytic continuation the cone \( L_X \) acquires singularities studied in geometrical optics on manifolds with boundary (see for instance [1, 22, 32]) and called open swallowtails. It would be interesting to study singularities of \( L_X \) under analytic continuation and to understand significance of the relationship with geometrical optics.

According to some results and conjectures of [12] and [17], the Lagrangian cones \( L \) corresponding to semisimple Frobenius structures are linearly isomorphic to a closure of the Cartesian products of \( \dim H \) copies of \( L_{pt} \), and various models in genus 0 GW-theory differ only by the position of the product with respect to the polarization. The same is true for the cones \( L_s \) corresponding to the different twisted theories on the same \( X \): according to Corollary 4 they are obtained from each other by linear symplectic transformations.

The transformations form the multiplicative group \( \exp(\sum \tau_m x^{2m-1}) \) where \( \tau_m \) are even elements of the algebra \( H \). The action of this group on the semi-infinite Grassmannian resembles the abstract Grassmannian interpretation of the KdV hierarchy. It would be interesting to further this analogy.

On the mirror theory. When \( X = pt \), the function \( J_{pt} = \exp(t_0/z) \). When \( E = \mathbb{C}^n \) is the trivial bundle over the point, the integral (13) turns into

\[
\int_0^\infty \cdots \int_0^\infty e^{-\frac{x_1 + \cdots + x_n}{z}} (x_1 \ldots x_n)^{\frac{1}{2}} \, dx_1 \wedge \ldots \wedge dx_n.
\]

It would be interesting to find a “quantum symplectic reduction theorem” which would explain how this integral is related to the \( J \)-functions of toric manifolds \( X \) (see [18]) obtained by symplectic reduction from \( \mathbb{C}^n \). For example, when \( X = \mathbb{C}P^{n-1} = \mathbb{C}^n/S^1 \), components of the \( J \)-function (18) coincide with the complex oscillating integral

\[
J_X(t) = z \int_{\gamma \subset \{u_1 \ldots u_n = e^t\}} e^{u_1 + \cdots + u_n} \frac{d \ln u_1 \wedge \ldots \wedge d \ln u_n}{dt}
\]

over suitable cycles. For a degree \( l \leq n \) hypersurface \( Y \subset X \), this yields integral representations for \( I_{X,Y} \) and \( I_Y \). Indeed the \( I \)-function (19) is proportional to the convolution (13)

\[
\int_0^\infty dv \, e^{-v} J_X(t + l \ln v) = \int_{\{u_1 \ldots u_n = e^{v + t}\}} e^{u_1 + \cdots + u_n - v} \frac{dv \wedge d \ln u_1 \wedge \ldots \wedge d \ln u_n}{dt}.
\]
Using the change $u_i \mapsto u_i v$ for $i = 1, \ldots, l \leq n$, we transform it to the “mirror partner” of $Y$:

$$
\frac{1}{2\pi i} \int_{\{u_1, \ldots, u_n = e^t\}} \frac{e^{(u_{i+1} + \ldots + u_n)/z} d\ln u_1 \wedge \ldots \wedge d\ln u_n}{(1 - u_1 - \ldots - u_l) dt} = \\
\int_{\{u_1, \ldots, u_n = e^t; u_1 + \ldots + u_l = 1\}} \frac{e^{u_{i+1} + \ldots + u_n} d\ln u_1 \wedge \ldots \wedge d\ln u_n}{d(1 - u_1 - \ldots - u_l) \wedge dt}.
$$

Another question. According to physics literature \[35\], the mirror maps $t \mapsto \tau$ arise from the mysterious renormalization. According to \[3\] the mathematical content of some important examples of renormalization in quantum field theory is Birkhoff factorization in suitable infinite-dimensional groups. Are renormalization and Birkhoff factorization synonymous?

**On Serre duality.** In the genus 0 theory, when $E$ is convex and $E^*$ is concave, the sheaves $E_{0,n,d}$ and $-E^*_{g,n,d}$ are vector bundles with the fibers $H^0(\Sigma, f^* E)$ and $H^1(\Sigma, f^* E^*)$ respectively. Using the Euler class of $E^*_{g,n,d}$ one obtains Gromov–Witten invariants of the non-compact total space $E^*X$ of the bundle $E^*$. The invariants twisted with the Euler class of $E_{0,n,d}$ can be interpreted as genus 0 Gromov–Witten invariants of the “super-manifold” $(\Pi E)X$, i.e. the total space of the bundle $E$ with the parity of the fibers reversed.

The “non-linear Serre duality” phenomenon emerged in \[19, 20\] in the context of fixed point localization for genus 0 Gromov–Witten invariants of $(\Pi E)X$ and $E^*X$. The duality was formulated as identification (modulo minor adjustments such as $\lambda \mapsto -\lambda, Q \mapsto \pm Q$) of certain genus 0 potentials written in Dubrovin’s canonical coordinates of the semi-simple Frobenius structures associated with the two modified theories. According to \[14, 21\] the total descendant potential of a semi-simple Frobenius structure can be described in terms of genus 0 data presented in canonical coordinates. This implies a higher genus version of the quantum Serre duality principle whenever the fixed point localization technique \[21\] applies. Corollary 2 and its particular case described by Corollary 3 assert the principle in much greater generality and show that both the localization technique and the reference to semi-simplicity and canonical coordinates in this matter are redundant.

Theorem 2, Corollary 5 and the mirror formulas of Section 9 have Serre-dual partners. Replacing $e$ and $E$ with with $e^{-1}$ and $E^*$ (equipped with the dual $S^1$-action as in Section 5) we should change the inner product to $(a, \beta)_{e^{-1}(E^*)} = \int_X \prod(\lambda - \rho_i)^{-1}ab = \int_X (-1)^r e^{-1}(E)ab$ and $I_E$ to $I_{E^*}$:

$$
\sum_d Q^d J_d \prod_i \prod_{k=\infty}^0 \frac{(-\lambda - \rho_i + kz)}{\prod_{k=\infty}^0 \rho_i(d)} = \sum_d (\pm Q)^d J_d \prod_i \prod_{k=\infty}^0 \frac{\rho_i(d)^{-1} (\lambda + \rho_i + kz)}{\prod_{k=\infty}^0 (\lambda + \rho_i + kz)}.
$$

When the classes $\rho_i$ are positive, the bundle $E^*$ is concave in the sense that $H^0(\Sigma, f^* E^*) = 0$ for all compact curves $\Sigma$ of any genus. The GW-invariants twisted by $(e^{-1}, E^*)$ admit the non-equivariant specialization $\lambda = 0$ (in moduli spaces $X_{g,m,d}$ of positive degrees $d \neq 0$). The reader can check that the results of Section 9, appropriately adjusted to the case of $I_{E^*}$, reproduce genus 0 mirror results of \[20\].
Appendix 1. The proof of Theorem 1

An application of the Grothendieck – Riemann – Roch theorem to the bundle $ev^*_{n+1}(E)$ over the universal curve $\pi : X_{g,n+1,d} \to X_{g,n,d}$ yields the following equality $(r,l,a,b \geq 0)$:

$$(21) \quad \text{ch}_k(E_{g,n,d}) = \pi_* \left[ \sum_{l+r+l=k+1} \frac{B_r}{r!} \text{ch}_l(ev^*(E)) \cdot \left( \psi^r - \sum_{i=1}^n (\sigma_i)_* \psi_i^{r-1} + \frac{1}{2} \iota_* \sum_{a+b=r-2} (-1)^a \psi_a^* \psi_b^* \right) \right]$$

Here $\psi = \psi_{n+1}, \sigma_i : X_{g,n,d} \to X_{g,n+1,d}$ is the section of the universal family defined by the $i$-th marked point, $i$ is the embedding into $X_{g,n+1,d}$ of the stratum $X^{Sing}_{g,n,d}$ of virtual codimension 2 formed by nodes of the curves, and $\psi_+, \psi_-$ denote the 1-st Chern classes of the line orbibundles over $X^{Sing}_{g,n,d}$ formed by the cotangent lines to the two branches of the curves at the nodes.

In order to justify (21) let us first recall the Grothendieck – Riemann – Roch Theorem 

$$\text{ch}(p_*V) = p_*(\text{ch}(V) \Td(T_{Y/B})), \quad (22)$$

where $V$ is a vector bundle on $Y$ and $p : Y \to B$ is a local complete intersection morphism. The latter hypothesis means (see [3]) that for some (and hence for any) embedding $j : Y \subset M$ into a non-singular space, the embedding $j \times p : Y \subset M \times B$ has a normal bundle $N_Y$ (i.e., the normal sheaf is locally free). The difference $j^*T_M \cap N_Y$ then takes on the role of the virtual relative tangent sheaf $T_{Y/B}$ which in fact does not depend on the choice of $j$. Under the hypothesis on $p$ there exists a complex $0 \to A^0 \to \ldots \to A^N \to 0$ of locally free sheaves on $B$ with cohomological sheaves equal to $R^ip_*V$ (later we will explicitly describe such resolutions for $\pi_*E = E_{g,n,d}$). The $K$-theoretic push forward $p_*V$ is defined as an element $\sum (-1)^i[A^i]$ in the Grothendieck group $K^0(B)$ of vector bundles on $B$, and $\text{ch}(p_*V)$ denotes the topological Chern character $\sum (-1)^i \text{ch}(A^i) \in H^*(B; \mathbb{Q})$.

We will apply the theorem in the orbispace/orbibundle situation which reduces to the following. The moduli orbispace $X_{g,n,d}$ (together with the universal stable map) can be described (see for instance [3]) as the quotient $P/G$ of a space $P$ by a semisimple complex Lie group $G$ acting on $P$ algebraically with at most finite stabilizers. By definition, orbi-sheaves and orbi-bundles on $P/G$ are $G$-equivariant sheaves and bundles on $P$. Their $G$-equivariant characteristic classes are elements of $H^{*}_G(P, \mathbb{Q})$ which coincides with $H^*(P/G, \mathbb{Q})$ since the action of $G$ on $P$ is almost free. Moreover, the $G$-space $P$ comes together with the $G$-equivariant universal family $C \to P$ of stable maps $ev : C \to X$, and $ev^*(E)$ is a $G$-equivariant bundle on $C$. Equivariant sheaves and bundles induce ordinary sheaves and bundles over finite-dimensional approximations to the homotopy quotients $C_G \to P_G$. The equivariant characteristic classes are determined by the ordinary characteristic classes of such approximations. Technically speaking, the formula (22) should be applied to these approximations. The hypotheses needed in (22) are satisfied because (a) the projection $C \to P$ is a local complete intersection morphism and (b) the space $C$ admits an equivariant embedding into a non-singular space.

The statement (a) is a local property of the map and follows from the fact that the universal family of curves $C \to P$ is flat and therefore in a neighborhood of a
nodal point of one of the fibers can be induced from the semi-universal unfolding (23)
\[ \mathbb{C}^2 \to \mathbb{C} : (x, y) \mapsto xy \]
of the nodal singularity \( xy = 0 \).

The statement (b) follows from the construction \([3,4]\) (in terms of Hilbert schemes) of an equivariant embedding of \( C \to P \) into a larger flat family of curves \( \tilde{C} \to \tilde{P} \) with \( \tilde{C} \) and \( \tilde{P} \) non-singular. In particular, near the locus \( \tilde{C}^{\text{Sing}} \) of nodes the family of curves is transversally described by the local normal form (23). We will exploit this property soon.

Further derivation of (21) does not differ much from Mumford’s argument \([31]\).

The normal form (23) allows one to express the relative cotangent orbisheaf \( \mathcal{T}^* = \mathcal{T}^*_{X_{g,n+1,d}/X_{g,n,d}} \) via the universal cotangent line bundle \( \mathcal{L} \) at the last marked point. Put \( \mathcal{O} = \mathcal{O}_{X_{g,n+1,d}} \). The sheaf \( \mathcal{O}(L) \) consists of meromorphic differentials on the curves allowed poles of order \( \leq 1 \) at the marked points and identified near the nodes with sections of the relative dualizing sheaf (which in the notations (23) have the form \( a(x, y)d\lambda \wedge dy/d(xy) \)). The sheaf \( \mathcal{O}(L) \) contains \( \mathcal{T}^* \) as a subsheaf of differentials holomorphic at the marked points and of the form \( b(x, y)x + c(x, y)ydx \wedge dy/d(xy) \equiv bdy - cdx \mod \mathbb{C}[[x, y]]d(xy) \) near the node (22). The coordinate expression \( dx \wedge dy/d(xy) \) represents a well-defined locally constant section of the orbibundle \( \mathcal{C}_x := \Lambda^2(L_+ \oplus L_-) \otimes L_+^{r_1} \otimes L_-^{r_2} \) over the singular locus \( X_{g,n,d}^{\text{Sing}} \) where \( L_+ \) and \( L_- \) are cotangent lines at the nodes. Using this and the residue of meromorphic differentials at the marked points we find

\[ \mathcal{O}(L)/\mathcal{T}^* = \iota_* \mathcal{O}_{X_{g,n,d}^{\text{Sing}}}((\mathbb{C}_x) \oplus (\sigma_1)_* \mathcal{O}_{X_{g,n,d}} \oplus \ldots \oplus (\sigma_n)_* \mathcal{O}_{X_{g,n,d}}). \]

The class \( c_1(L) = \psi \) vanishes when restricted to the pairwise disjoint strata \( D_i = \sigma_i(X_{g,n,d}) \) and \( Z = \iota(X_{g,n,d}^{\text{Sing}}) \). This translates the multiplicative property of the dual Todd class \( \text{Td}^\vee(\cdot) \) to additivity of \( \text{Td}^\vee(\cdot) - 1 \):

(24)
\[ \text{Td}(\mathcal{T}) = \text{Td}^\vee(\mathcal{T}^*) = 1 + \left[ \text{Td}^\vee(\mathcal{O}(L)) - 1 \right] + \sum_i \frac{1}{\text{Td}^\vee(\mathcal{O}(D_i)) - 1} + \left[ \frac{1}{\text{Td}^\vee(\mathcal{O}(Z)) - 1} \right]. \]

The first two terms yield
\[ \text{Td}^\vee(\mathcal{O}(L)) = \exp_\psi \frac{1}{\psi - 1} = \sum_{r \geq 0} \frac{B_r}{r!} \psi^r. \]

Using \( \sigma_i^*(-D_i) = \psi_i \) and the exact sequence \( 0 \to \mathcal{O}(-D_i) \to \mathcal{O} \to \mathcal{O}_{D_i} \to 0 \) we find
\[ \frac{1}{\text{Td}^\vee(\mathcal{O}(D_i)) - 1} = \text{Td}(\mathcal{O}(-D_i)) - 1 = \sum_{r \geq 1} \frac{B_r}{r!} ((-D_i)^r) = -(\sigma_i)_* \sum_{r \geq 1} \frac{B_r}{r!} \psi_i^{-r-1}. \]

The codimension-2 summand in (24) is processed using the inclusion-exclusion formula for the bi-graded Poincaré polynomial of \( \mathbb{C}[x, y]/(xy) \):

\[ \frac{1 - uv}{(1 - u)(1 - v)} = \frac{1}{1 - u} + \frac{1}{1 - v} - 1. \]

Consider the enlarged nodal locus \( \tilde{Z} \subset \tilde{C}/G \). On a double cover of its neighborhood \( \tilde{Z} \) is the normal crossing of the divisors \( D_\pm \) with the conormal bundles \( L_\pm \). We see from the Koszul complex
\[ 0 \to \mathcal{O}(L_+ \oplus L_-) \to \mathcal{O}(L_+) \oplus \mathcal{O}(L_-) \to \mathcal{O} \to \mathcal{O}_Z \to 0 \]
that in the neighborhood of $Z \subset X_{g,n+1,d} = C/G$

\[
\frac{1}{\text{Td}^\vee(\mathcal{O}_Z)} - 1 = \frac{1 - e^{-D_+} - D_-}{D_+ + D_-} \cdot \frac{e^{-D_+} - 1}{1 - e^{-D_-}} - 1
\]

\[
= \frac{D_+ + D_-}{D_+ + D_-} \left( \frac{1}{1 - e^{-D_+}} + \frac{1}{1 - e^{-D_-}} - 1 - \frac{1}{D_+} - \frac{1}{D_-} \right)
\]

\[
= \frac{D_+ - D_-}{D_+ + D_-} \left( \frac{D_+}{D_+ - 1} + \frac{D_-}{D_- - 1} - \frac{1}{D_+} \right)
\]

\[
= \frac{1}{2} \left[ \sum_{r \geq 2} \frac{B_r (\psi_+^{r-1} + \psi_-^{r-1})}{r!} \right] = \frac{1}{2} \left[ \sum_{r \geq 2} \frac{B_r (\psi_+^{r-1} + \psi_-^{r-1})}{r!} \right] (-1)^a \psi_a \psi_b.
\]

We use here $\psi_\pm = -\nu^*(D_\pm)$, $B_0 = 1, B_1 = -1/2$, $B_r = 0$ for odd $r > 1$, and assume that the push-forward $\iota_*$ is taken with respect to the virtual fundamental class $[Z]$ described in a neighborhood of $Z$ as as the cap-product of $[X_{g,n+1,d}]$ with the Euler class $D_+D_-$ of the normal bundle of $Z$.

Combining the formulas for $\text{Td}(T)$ with the Grothendieck – Riemann – Roch theorem we arrive at (21).

The formula (21) is the main geometric ingredient in out proof of Theorem 1. We also need the following facts.

(i) The comparison formula $\psi_\pm - \pi^*(\psi_\pm) = D_\pm$.

(ii) The naturality of the virtual fundamental cycles $\pi^*[X_{g,n,d}] = [X_{g,n+1,d}]$ under the flat morphism $\pi$.

(iii) The composition rule for $X_{g,n,d}^{\text{Sing}}$. Namely, the singular locus coincides with the total range of the gluing maps

\[(25) \quad X_{g_+, n+, \bullet+, d+, \bullet} \times X_{X_{0,1},+\bullet, 0, 0} \times X_{X_{\bullet-}, n-, +d, \bullet} \to X_{g,n,d}^{\text{Sing}} \subset X_{g,n+1,d}
\]

(over all splittings $g = g_+ + g_-$, $n = n_+ + n_-$, $d = d_+ + d_-$) and

\[(26) \quad X_{g-1, n+, \bullet+, \bullet, 0, 0} \to X_{g,n,d}^{\text{Sing}} \subset X_{g,n+1,d}.
\]

The composition rule says that images of the virtual fundamental classes under the gluing maps add up to the virtual fundamental class $[Z]$ of the singular locus.

These properties (ii) and (iii) are part of the axioms in [27] proved in [2], and (i) is well known too — see for instance [34].

Next, we need similar results about $E_{g,n,d}$ as elements in the Grothendieck groups of coherent orbisheaves:

(iv) $\pi^* E_{g,n,d} = E_{g,n+1,d}$

(v) $\gamma^* \iota^* E_{g,n+1,d} = \pi^*_+ E_{g_+, n+, \bullet+, d_+} + \pi^*_- E_{g_-, n-, +d_-, \bullet} - \text{ev}_\Delta^* E
\]

(vi) $\gamma^* \iota^* E_{g,n+1,d} = E_{g-1, n+, \bullet, 0, d} - \text{ev}_\Delta^* E$

where $\gamma$ are the gluing maps (25) and (26) respectively, $\text{ev}_\Delta = \text{ev}_\bullet = \text{ev}_0$ is the evaluation at the point of gluing, and $\pi^*_\pm$ are projections to the factors.

The properties can be verified by representing the bundle $E$ on $X$ as the quotient $A/B$ of two concave bundles. For this, pick a positive line bundle $L$ and let the exact sequence $0 \to \text{Ker} \to H^0(X; E \otimes L^N) \otimes L^{-N} \to E \to 0$ take on the role of $0 \to B \to A \to E \to 0$. Then $H^0(\Sigma; f^*A)$ and $H^0(\Sigma; f^*B)$ vanish for sufficiently large $N$ and any non-constant $f : \Sigma \to X$ so that $0 \to H^0(\Sigma; f^*E) \to$
\( H^1(\Sigma; f^*B) \to H^1(\Sigma; f^*A) \to H^1(\Sigma; f^*E) \to 0 \) is exact. This construction applied to a universal stable map of degree \( d \neq 0 \) yields a locally free resolution \( 0 \to R^1\pi_* (ev^* B) \to R^1\pi_* (ev^* A) \to 0 \) for \( E_{g,n,d} = R^0\pi_* (ev^* E) \oplus R^1\pi_* (ev^* E) \) mentioned earlier and reduces the problem about the sheaves \( E_{g,n,d} \) with \( d \neq 0 \) to the case of \( H^1 \)-vector bundles \(-A_{g,n,d}-B_{g,n,d}. \) When \( d = 0, H^0 \) is non-zero but has constant rank too. In either case the formulas (iv), (v), (vi) are easy to check directly, for instance, using Serre duality (it identifies elements of \( H^1(\Sigma; f^*A)^* \) with holomorphic differentials on \( \Sigma - \text{nodes} \) with values in \( f^*A \) and allowed poles of order \( \leq 1 \) at the nodes provided that the sum of the two residues at each node equals zero).

Finally, we will need three integrals over low-genus moduli spaces. Let us introduce the following correlator notation: suppose that

\( a^i(\psi) = a_0^{(i)} + a_1^{(i)} \psi + \ldots \)

are polynomials in \( \psi \) with coefficients in \( H^*(X; \Lambda) \), and \( \beta \in H^*(X_{g,n,d}; \Lambda) \). Write

\( \langle \beta; a^1, \ldots a^n \rangle_{g,n,d} = \int_{[X_{g,n,d}]} \beta \wedge \left( \sum_{j \geq 0} \text{ev}_1^* (a_j^{(1)}) \psi_j^1 \right) \cdots \left( \sum_{j \geq 0} \text{ev}_n^* (a_j^{(n)}) \psi_j^n \right). \)

Also, set \( c_{g,n,d} = \exp \left( \sum s_k \text{ch}_k (E_{g,n,d}) \right) \in H^*(X_{g,n,d}; \Lambda). \) In this notation

\( \langle c_{0,3,0}; t, t, \text{ch}_{k+1}(E) \rangle_{0,3,0} = \int_X c(E) t_0^{k+2} \text{ch}_{k+1}(E), \)

\( \langle c_{1,1,0}; \text{ch}_k(E) \psi \rangle_{1,1,0} = \frac{1}{24} \int_X e(X) \text{ch}_k(E), \)

\( \langle c_{1,1,0}; \text{ch}_{k+1}(E) \rangle_{1,1,0} = \)

\( \frac{1}{24} \int_X e(X) \text{ch}_{k+1}(E) \left( \sum s_j \text{ch}_{j-1}(E) \right) - \frac{1}{24} \int_X c(E) cD_{-1}(E) \text{ch}_{k+1}(E). \)

The equality (vii) is obvious since \( [X_{0,3,0}] = [X] \), and (viii) and (ix) follow easily from the well-known facts: \( X_{1,1,0} = X \times M_{1,1}, [X_{1,1,0}] = e(T_X \otimes \mathcal{E}^{-1}) \cap [X \times M_{1,1}] \) (where \( \mathcal{E} \) is the Hodge line bundle over the Deligne – Mumford space \( \overline{M}_{1,1} \)), \( E_{1,1,0} = E \otimes (1 \otimes \mathcal{E}^{-1}), c_1(\mathcal{E}) = \psi \) and \( \int_{[M_{1,1}]} \psi = 1/24. \)

Using (P1) and the properties (i – ix) we now derive Theorem 1. At \( s = 0, \) Theorem 1 holds trivially, so it suffices to prove the infinitesimal version

\[
\frac{\partial}{\partial s_k} D_s = \left( \sum_{2m+r=k+1} B_{2m} \left( \frac{2m!}{(2m)!} \right) (\text{ch}_r(E) z^{2m-1}) \right) D_s
\]

\[
+ \left( \frac{1}{24} \int_X cD_{-1}(X) \wedge \text{ch}_{k+1}(E) \right) + \frac{1}{24} \int_X e(X) \wedge \text{ch}_k(E) \wedge \text{ch}_{k+1}(E) \wedge \left( \sum_l s_{l+1} \text{ch}_l(E) \right) D_s
\]
Here the first two exceptional terms come from the factors on the LHS of (28); in particular the second one is due to

\[
\left( \text{sdet} \sqrt{c(E)} \right) = \exp \left( \text{str ln} \sqrt{c(E)} \right) = \exp \left( \int_X e(X) \wedge \left( \frac{1}{2} \sum_j s_j ch_j(E) \right) \right)
\]

The third exceptional term is the cocycle value

\[
C(B_2) = \sum s_{i+1}ch_i(E)z, ch_{k+1}(E)/z = -\frac{1}{2} \text{str} \left( \frac{1}{2} \sum_j s_j \frac{ch_{k+1}(E)}{z} \right)
\]

which arises from commuting the derivative of the \( \frac{1}{z} \) terms (on the RHS in (28)) past the terms involving \( z \).

In the above correlator notation,

\[
D_s = \exp \left( \sum_{g,n,d} \frac{h^{g-1}Q^d}{n!} \langle c_{g,n,d}; t, \ldots, t \rangle_{g,n,d} \right)
\]

and so

\[
D_s^{-1} \frac{\partial}{\partial s_k} D_s = \sum_{g,n,d} \frac{Q^d h^{g-1}}{n!} \langle ch_k(E_{g,n,d}) \wedge c_{g,n,d}; t, \ldots, t \rangle_{g,n,d}
\]

\[
+ \sum_{g,n,d} \frac{Q^d h^{g-1}}{(n-1)!} \left( \langle c_{g,n,d}; t, \ldots, t, \frac{\partial t}{\partial s_k} \rangle_{g,n,d} \right).
\]

We apply our expression (21) for \( ch_k(E_{g,n,d}) \) and compare the result with (27) by extracting terms involving the same Bernoulli numbers.

We begin with \( B_0 = 1 \). Due to the comparison formula (i) we have

\[
\pi^* (t(\psi_i)) = t(\psi_1) - (\sigma_i)_+ t(\psi_i)/\psi_1 + \psi_1
\]

where \([ \cdot ]_+ \) means power series truncation. Together with the naturality (iv) of the class \( c_{g,n,d} \) under \( \pi^* \), this implies that

\[
\langle \pi_* [\text{ev}^* ch_{k+1}(E)]_+ c_{g,n,d}; t, \ldots, t \rangle_{g,n,d} = \langle c_{g,n+1,d}; t, \ldots, t, ch_{k+1}(E) \rangle_{g,n+1,d} - n \langle c_{g,n,d}; t, \ldots, t, ch_{k+1}(E) \left[ t(\psi) \right] \rangle_{g,n,d}. \]

Summing over \( g, n, d \) we find

\[
\sum_{g,n,d} \frac{h^{g-1}Q^d}{n!} \langle \pi_1 [B_0 \text{ev}^* ch_{k+1}(E)]_+ c_{g,n,d}; t, \ldots, t \rangle_{g,n,d} = \]

\[
- \sum_{g,n,d} \frac{h^{g-1}Q^d}{(n-1)!} \left( \langle c_{g,n,d}; t, \ldots, t, \frac{t(\psi)}{\psi} \rangle_{g,n,d} \right) + \frac{1}{2h} \langle c_{0,3,0}; t, ch_{k+1}(E) \rangle_{0,3,0} - \langle c_{1,1,0}; ch_{k+1}(E) \rangle_{1,1,0}.
\]

Here the exceptional terms arise from the fact that the moduli spaces \( X_{0,2,0} \) and \( X_{1,0,0} \) are empty and therefore \( X_{0,3,0} \) and \( X_{1,1,0} \) cannot be interpreted as universal curves.
The first two summands on the right actually add up to \( D_{s}^{-1}(ch_{k+1}/z)D_{s} \). Indeed, the corresponding quadratic hamiltonian has \( pq \)-, \( q^2 \)-, but no \( p^2 \)-terms. Quantization of the \( pq \)-terms yields a linear vector field defined by the operator \( q(z) \mapsto - [ch_{k+1}(E)q(z)/z]_{\star} \), while the \( q^2 \)-term is \(-(q_0,q_0)/2\) and matches the 2-nd summand in (29) due to (vii). Evaluating the third summand via (ix) we conclude that (29) coincides with (30) since the corresponding quadratic hamiltonian has \( ch_{pq} \) as the quantization of the \( ch_{pq} \)

\[
D_{s}^{-1} \left( \left( \frac{ch_{k+1}(E)}{z} \right) + \frac{1}{2h} \int_X c_{D-1}(X) \ ch_{k+1}(E) \right) D_{s}.
\]

Next, proceeding as above with \( B_1 = -1/2 \) and using \( \sigma'_{\star} \psi = 0 \) we find

\[
\sum_{g,n,d} \frac{h^{g-1}Q^d}{n!} \langle \pi_{\star} [B_1 ev^* ch_{k}(E) (\psi - D_1 - \ldots - D_n)] c_{g,n,d; t, \ldots, t} \rangle_{g,n,d}
\]

\[
= \frac{1}{2} \sum_{g,n,d} \frac{h^{g-1}Q^d}{(n-1)!} \langle c_{g,n,d; t, \ldots, t, ch_{k}(E)(t(\psi) - \psi)} \rangle_{g,n,d} + \frac{1}{2} \langle c_{1,1,0; ch_{k}(E)\psi} \rangle_{1,1,0}.
\]

In view of (viii) this coincides with

\[
- \sum_{g,n,d} \frac{h^{g-1}Q^d}{(n-1)!} \langle c_{g,n,d; t, \ldots, t, \partial t/\partial s_k} \rangle_{g,n,d} + \frac{1}{48} \int_X e(X) \ ch_{k}(E)
\]

since \( t(z) = c(E)^{-1/2} q(z) + z \) and hence \( \partial t(z)/\partial s_k = - ch_{k}(E)(t(z) - z)/2 \).

Finally, it remains to check the equality of the \( B_{2m} \)-terms with \( m > 0 \):

\[
\sum_{g,n,d} \frac{h^{g-1}Q^d}{n!} \langle \pi_{\star} [ev^* ch_{k+1-2m}(E) \Psi_{m}] c_{g,n,d; t, \ldots, t} \rangle_{g,n,d}
\]

\[
= D_{s}^{-1}(ch_{k+1-2m}(E)z^{2m-1})D_{s}
\]

where

\[
\Psi_{m} = \psi^{2m} - \sum_{i=1}^{n} (\sigma_i)_{\star} \psi^{2m-1} + \frac{1}{2} t_{\star} \left( \frac{\psi^{2m-1} + \psi^{2m-1}}{\psi_+ + \psi_-} \right).
\]

Processing the first two summands in \( \Psi_{m} \), as before yields

\[
- \sum_{g,n,d} \frac{h^{g-1}Q^d}{(n-1)!} \langle c_{g,n,d; t, \ldots, t, ev^* ch_{k+1-2m}(E) \psi^{2m-1} (t(\psi) - \psi)} \rangle_{g,n,d},
\]

which coincides with the derivative of \( \ln D_{s} \) along the linear vector field defined by the multiplication operator \( q(z) \mapsto - ch_{k+1-2m}(E)z^{2m-1}q(z) \). This vector field is the quantization of the \( pq \)-terms in the quadratic hamiltonian corresponding to \( ch_{k+1-2m}(E)z^{2m-1} \). For \( m > 0 \) the hamiltonian contains no \( q^2 \)-terms.

Let us identify the cohomology space \( H \) with its dual by means of the intersection pairing \( (\cdot, \cdot) \). We use the coordinate notation \( \sum_{\alpha\beta} \partial_\alpha ch^{\alpha\beta} \partial_\beta \) for the bi-derivation \( H \) corresponding to the self-adjoint operator of multiplication by \( ch_{k+1-2m}(E) \) on \( H^* = H \). Applying the composition rules (iii),(v),(vi) we can express contributions of the last summand in \( \Psi_{m} \) as

\[
D_{s}^{-1} \left[ \frac{h}{2} \sum_{a+b=2m-2} (-1)^a \sum_{\alpha\beta} \partial_{q^a} ch^{\alpha\beta} \partial_{q^b} \right] D_{s}.
\]
Appendix 2. Descendents and ancestors

The aim of the appendix is to justify part (i) of the Proposition in Section 6 describing properties of the genus 0 descendent potential $\mathcal{F}_X^0$ in terms of the geometry of the symplectic space $(\mathcal{H}, \Omega)$. In fact we intend to do more, namely — to derive the Proposition from a relationship between gravitational descendents of any genus and their counterparts from Deligne – Mumford space $\overline{M}_{g,m}$ — expressed in terms of the quantization formalism of Section 2. The theorem in question, which is a reformulation of a result by Kontsevich – Manin [27], has been announced [17]. We recall the formulation and furnish a proof below.

Consider the composition $X_{g,m+l,d} \to \overline{M}_{g,m}$ of the operations of forgetting the last $l$ marked points and contraction. Denote by $\psi_i$ the pull-backs from Deligne – Mumford space $\overline{M}_{g,m}$ of the 1-st Chern classes of universal cotangent lines. They differ from the descendent classes $\psi_i$ on $X_{g,m+l,d}$. Following [17], introduce the genus $g$ ancestor potentials

$$F^g_X := \sum_{d,m,l} \frac{Q^d}{m!l!} \int_{[X_{g,m+l,d}]} \left[ \prod_{i=1}^m \left( \sum_{k \geq 0} (\text{ev}_i^* \bar{\psi}_k^k) \bar{\psi}_i^k \right) \right] \tau, $$

which are formal functions of the ancestor variables $\bar{\psi} = \sum k \bar{\psi}_k^k, \bar{\psi}_k \in H$, and of the parameters $\tau \in H$. The total ancestor potential is defined as

$$A_\tau = \exp\left\{ \sum h^{g-1} F^g_X \right\}$$

and is identified via the dilaton shift $q(z) = \bar{t}(z) - z$ with an element in the Fock space depending on the parameter $\tau \in H$.

We will use the abbreviated correlator notation

$$\langle a_1(\psi, \bar{\psi}), \ldots, a_m(\psi, \bar{\psi}) \rangle_{g,m}(\tau) := \sum_{i,d} \frac{Q^d}{i!} \langle 1; a_1(\psi, \bar{\psi}), \ldots, a_m(\psi, \bar{\psi}), \tau, \ldots, \tau \rangle_{g,m+l,d}$$

for Taylor series in $\tau$ with coefficients possibly mixing descendent and ancestor classes.

Introduce the operator series $S_\tau(z^{-1}) = 1 + S_1 z^{-1} + S_2 z^{-2} + \ldots$ acting on the space $\mathcal{H} = H(\langle z^{-1} \rangle)$ and defined in terms of genus 0 descendents:

$$S_\tau(z^{-1})u, v := (u, v) + \left( \frac{u}{z - \psi}, v \right)_{0,2}(\tau).$$

The series $S_\tau$ depends on the parameter $\tau \in H$. According to [19, 20] it satisfies the identity $S^*_\tau(-z^{-1}) S_\tau(z^{-1}) = 1$ and consequently defines a symplectic transformation on $(\mathcal{H}, \Omega)$. By quantization $\hat{S}$ of symplectic transformations we mean $\exp_{\text{lin}} \hat{S}$.

The action of the operator $\hat{S}_\tau^{-1}$ on an element $\mathcal{G}$ of the Fock space is explicitly described by the formula:

$$(\hat{S}_\tau^{-1} \mathcal{G})(\mathbf{q}) = e^{(\mathbf{a}, \mathbf{q})_{0,2}(\tau)/2\hbar} \mathcal{G}([S, \mathbf{q}]_+),$$

In particular, the factor $c(E)$ due $\partial_k = \sqrt{c(E)} \partial_q$ cancels with $1/c'(\text{ev}_k^+ (E))$ from (v) and (vi). This matches up with the quantization of $p^2$-terms in the quadratic hamiltonian of $c_{k+1-2m}(E) \bar{z}^{2m-1}$.

Combining (30), (31), (32) with (28) and (27) we recover Theorem 1.
where \([S_\tau \mathbf{q}]_+\) is the power series truncation of \(S_\tau (z^{-1}) \mathbf{q}(z)\). The formula is easy to check by generalizing it to \(\exp(-\epsilon A)\) as in Proposition 5.3 in [17] and taking \(A = \ln S, \epsilon = 1\). The quadratic hamiltonian of \(A\) contains no \(p^2\)-terms (since \(S\) is a power series in \(1/z\)), and quantization of \(\exp(-\epsilon A)\) amounts to solving a 1-st order linear PDE by the method of characteristics. The \(pq\)-terms give rise to the linear change of variables \(\mathbf{q} \mapsto [\exp(\epsilon A) \mathbf{q}]_+\). The exponential factor can be verified — by differentiation in \(\epsilon\) — using the WDVV-like identity
\[
\langle \mathbf{q}(\psi), 1, \mathbf{q}(\psi) \rangle_{0,3}(\tau) = \sum_{\alpha\beta} \langle \mathbf{q}(\psi), 1, \phi_\alpha \rangle_{0,3}(\tau) g^{\alpha\beta} \langle \phi_\beta, 1, \mathbf{q}(\psi) \rangle_{0,3}(\tau)
\]
(where \((g^{\alpha\beta})\) is the inverse to the intersection matrix \(g_{\alpha\beta} = (\phi_\alpha, \phi_\beta)\)) together with the string equation. We leave some details here to the reader.

**Theorem.** \(D = e^{\tau_1(t)} \hat{S}_\tau^{-1} A_{\tau}\).

**Proof.** Let \(\mathcal{L}\) be a universal cotangent line bundle over \(X_{g,m+l,d}\), and \(\hat{L}\) be its counterpart pulled back from \(\overline{\mathcal{M}}_{g,m}\) and corresponding to the same index (let it be 1) of the marked point. Let \(\psi = c_1(\mathcal{L})\) and \(\tilde{\psi} = c_1(\hat{L})\). There exists a section of \(\text{Hom}(\mathcal{L}, \mathcal{L})\) regular outside some virtual divisor \(D\) consisting of stable maps with the following property: the 1-st marked point \(\mathbf{1}\) is situated on a component of the curve which is subject to contraction under the map \(X_{g,m+l,d} \to \overline{\mathcal{M}}_{g,m}\). It is easy to see that \(D\) is the total range of the gluing maps

\[
X_{0,1+l'+l''} \times \mathcal{X}_{X_{g,m-1+o+l',d'+d''}} X_{g,m+1,l,d} \to X_{g,m+l,d}
\]

over all splittings \(l' + l'' = l, d' + d'' = d\). The virtual normal bundle to \(D\) (outside self-intersections of \(D\)) is canonically identified with \(\text{Hom}(\mathcal{L}, \mathcal{L})\). This implies that the section vanishes on \(D\) with 1-st order, and hence that \(\psi - \tilde{\psi}\) is Poincaré-dual to the virtual divisor: \([X_{g,m+l,d}] \cap (\psi - \tilde{\psi}) = [D]\). Thus we have

\[
\langle u_{\psi^{a+1} \tilde{\psi}^b}, \ldots \rangle_{g,m}(\tau) = \langle u_{\psi^a \tilde{\psi}^{b+1}}, \ldots \rangle_{g,m}(\tau) + \sum_{\alpha\beta} \langle u_{\psi^a \tilde{\psi}^b}, \phi_\alpha \rangle_{0,2}(\tau) g^{\alpha\beta} \langle \phi_\beta \tilde{\psi}^{b+1}, \ldots \rangle_{g,m}(\tau),
\]

where dots mean the descendant/ancestor content of other marked points (to be the same in all three places). Applying this identity inductively to reduce descendents to ancestors we conclude that the descendant potentials \((t(\psi), \ldots, t(\psi))_{g,m}(\tau)\) are obtained from the corresponding ancestor potentials \((\tilde{t}(\psi), \ldots, \tilde{t}(\psi))_{g,m}(\tau)\) by the substitution \(t(z) = [S_\tau(z^{-1}) \tilde{t}(z)]_+\). This is essentially the result from [27].

Let us compare this conclusion with the statement of the theorem. Noting the presence of the similar change \(\mathbf{q} \mapsto [S_\tau \mathbf{q}]_+\) in the explicit description of the operator \(\hat{S}_\tau^{-1}\) we should also notice that \(\mathbf{q}\) and \(t\) are not the same: \(\mathbf{q}(z) = t(z) - z\). This gives rise to the discrepancy \([z - S_\tau z]_+\). Expanding

\[
[S_\tau]_\beta^\alpha = \delta_\beta^\alpha + z^{-1} \sum_\mu g^{\alpha\mu} \langle \phi_\mu, \phi_\beta \rangle_{0,2}(\tau) + o(z^{-1}),
\]

we find the discrepancy equal to \(-\tau\) because in components

\[
\sum_\mu g^{\alpha\mu} \langle \phi_\mu, 1 \rangle_{0,2}(\tau) = \sum_\mu g^{\alpha\mu} \langle 1; \phi_\mu, 1 \rangle_{0,3,0} = \sum_\mu g^{\alpha\mu} g_{\mu\beta} \tau^\beta = \tau^\alpha.
\]
Thus $\tilde{q} = S_\tau q$ is equivalent to

$$t = [S_\tau t]_+ - \tau = [S_\tau (t - \tau)]_+.$$  

By Taylor's formula, we have

$$F^g(t) = \sum_{m=0}^{\infty} \frac{1}{m!} \langle t(\psi), ..., t(\psi) \rangle_{g,m}(0) = \sum_{m=0}^{\infty} \frac{1}{m!} \langle t(\psi) - \tau, ..., t(\psi) - \tau \rangle_{g,m}(\tau).$$

We conclude that for $g > 1$ the descendent potentials $F^g$ (which do not depend on $\tau$) are obtained from the ancestor potentials $\bar{F}^g$ (which do depend on $\tau$) by the substitution $\bar{q}(z) = [S_\tau (z^{-1})q(\bar{z})]_+.$ In order to make the same true for $g = 0, 1$ we have to include the terms corresponding to the unstable indices $(g, m) = (0, 0), (0, 1), (0, 2)$ and $(1, 0)$ and hence missing from the ancestor potentials. The first three of them give rise to the factor $\exp(q, q)_{0,2}(\tau)/2\hbar.$ Indeed,

$$\frac{1}{2} \langle t(\psi) - \psi, t(\psi) - \psi \rangle_{0,2}(\tau) = \langle \psi, \psi \rangle_{0,0}(\tau) + \langle t(\psi) - \tau, t(\psi) - \tau \rangle_{0,1}(\tau) + \frac{1}{2} \langle t(\psi) - \tau, t(\psi) - \tau \rangle_{0,2}(\tau).$$

This can be easily derived from the dilaton equation $\langle \psi, ... \rangle_{g,n+1,d} = (2g - 2 + n)(...)$ applied with $g = 0.$ Finally, the missing summand $\langle \psi \rangle_{1,0}(\tau)$ coincides with $F^1(\tau),$ and the Theorem follows.

Passing to the quasi-classical limit $\hbar \to 0$ we obtain the following result.

**Corollary.** The Lagrangian sections $L$ and $\bar{L}_\tau$ which represent respectively the differentials of the genus 0 descendent potential $F^0$ and ancestor potentials $\bar{F}^0_\tau$ are related by the symplectic transformations: $L_\tau = S_\tau L.$

Finally, we derive part (i) of the Proposition.

When the ancestor variable $\bar{q}$ belongs to $zH_+$ (i. e. $\bar{t}_0 = 0$), the genus 0 ancestor potential $\bar{F}^0$ has identically zero 2-jet at $\bar{q}$. This follows from $\dim \bar{M}_{0,m+2} < m.$ Thus the cone $\bar{L}_\tau$: (a) contains the isotropic space $zH_+$ and (b) at any point $\bar{q} \in zH_+$ has the tangent space $\bar{L}_{\bar{q}} = H_+.$ Applying the symplectic transformation $S_\tau^{-1}$ we see that the tangent spaces $L_\tau$ to $L$ at $\tau = S_\tau^{-1}\bar{q}$ intersect $L$ along $zL_{\bar{q}}$ provided that $\bar{q} \in zH_+.$ The condition $S_\tau f \in zH_+$ on $f = (p, q) \in L$ is equivalent to the system of equations

$$\langle 1, \bar{q}(\psi), v \rangle_{0,2}(\tau) = 0 \text{ for all } v \in H.$$ 

In other words, $\tau$ must be a critical point of $\langle 1, \bar{q}(\psi) \rangle_{0,2}(\tau)$ considered as a function of $\tau \in H$ (depending on the parameter $q \in H_+.$ When $\bar{q}(z) = q_0 - z$, the function turns into $(q_0, \tau) - (\tau, \tau)/2$ and has the nondegenerate critical point $\tau = q_0.$ This guarantees existence of a unique critical point $\tau(q)$ in a formal neighborhood of $q = -z.$ The result follows.
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