Stability of generalized Cauchy equations

Roman Badora, Barbara Przebieracz and Peter Volkmann

Dedicated to Professor János Aczél on his 90th Birthday

Abstract. We investigate the stability of the functional equation
\[ f(xy) = g(x)h(y) + k(y) \]
on amenable semigroups. This equation is a common generalization of two Pexider equations stemming from Cauchy’s additive and multiplicative functional equations, and it is a simple case of the Levi-Civita equation.

Mathematics Subject Classification. Primary 39B82; Secondary 39B22.

Keywords. Cauchy equations, stability in the sense of Hyers–Ulam, Pexider equations, Levi-Civita functional equation.

1. Introduction

A common generalization of Pexider’s equations
\[ f(xy) = g(x) + k(y) \]
and
\[ f(xy) = g(x)h(y) \]
is
\[ f(xy) = g(x)h(y) + k(y). \] (1.1)
For real functions this equation has already been treated in J. Aczél’s fundamental monograph [1], where the composition \( xy \) means addition of real numbers (therefore we have \( f(x + y) = g(x)h(y) + k(y) \) on p. 120 of [1]). Then chapter 15 of the book [3] is devoted to equation (1.1). By referring to J. Aczél [2], it is solved under the assumptions that \( f, g, h, k : S \to \mathbb{F} \), where \( S \) is an abelian groupoid with neutral element and \( \mathbb{F} \) is a field. More precisely, when solving Exercise 1 on p. 250 of [3], it turns out that \( f, g, h, k \) satisfy (1.1) if and only if they have one of the following forms (where \( b, c, u, v \in \mathbb{F} \)):
1. \( f(x) = b, \) \( g \) arbitrary, \( h = 0, \) \( k(x) = b. \)
2. \( f(x) = b, \) \( g(x) = c, \) \( h \) arbitrary, \( k(x) = b - ch(x). \)
3. \( f(x) = v(a(x) + b), \) \( g(x) = a(x) + b - c, \) \( h(x) = v, \) \( k(x) = v(a(x) + c), \)
    where \( a: S \to \mathbb{F} \) solves the additive Cauchy equation
    \[ a(xy) = a(x) + a(y), \quad x, y \in S. \]
4. \( f(x) = v(ce(x) + b), \) \( g(x) = ce(x) + u, \) \( h(x) = ve(x), \) \( k(x) = v(b - ue(x)), \)
    where \( e: S \to \mathbb{F} \) solves the multiplicative Cauchy equation
    \[ e(xy) = e(x)e(y), \quad x, y \in S. \]

The just given four types of functions \( f, g, h, k: S \to \mathbb{F} \) are also solutions of \((1.1)\) in arbitrary (not necessarily commutative) groupoids \( S. \)

Equation \((1.1)\) is a special case of the Levi-Civita functional equation
\[ f(xy) = \sum_{i=1}^{n} g_i(x)h_i(y). \]

For information concerning the solutions of this equation see, for example, \([5–10]\) and \([12]\). Recently, solutions of \((1.1)\), on various non-abelian groups, have been examined in \([4]\).

In the present paper we investigate the stability in the sense of Hyers-Ulam of equation \((1.1)\) on amenable semigroups. Let us recall that a semigroup \( S \) is called right amenable if there exists a right invariant mean on the space \( \mathcal{B}(S) \) of all bounded complex-valued functions defined on \( S. \) By a right invariant mean we understand a linear functional \( M \) satisfying
\[ \inf_{s \in S} f(s) \leq M(f) \leq \sup_{s \in S} f(s), \quad \text{for all real-valued } f \in \mathcal{B}(S), \]
and
\[ M(f_x) = M(f), \quad f \in \mathcal{B}(S), \quad x \in S, \]
where \( f_x(s) := f(sx), s \in S. \) It is easily seen that the linear functional \( M \) has the properties
\[ \|M\| \leq 2 \quad (1.2) \]
and \( M(c) = c, \) for all complex numbers \( c. \) Moreover, for convenience, we will write \( M_s(f(s)) \) instead of \( M(f). \)

In paper \([11]\) the stability of the functional equation
\[ u(Lx) = \alpha(L)u(x) + \beta(L) \]
is under consideration, where \( X \) is a set, \( \mathcal{L} \) is an amenable group of self-mappings of \( X, \) \( \mathbb{K} \) is the field of real or complex numbers, \( u: X \to \mathbb{K}, \alpha, \beta: \mathcal{L} \to \mathbb{K}. \)

The main result of our paper is contained in the next section.
2. Main theorem

**Theorem 2.1.** Let $S$ be a right amenable semigroup with neutral element 1. Suppose $f, g, h, k : S \to \mathbb{C}$, $\varepsilon \geq 0$ and

$$|f(xy) - g(x)h(y) - k(y)| \leq \varepsilon, \quad x, y \in S. \tag{2.1}$$

Then there exist $F, G, H, K : S \to \mathbb{C}$ satisfying

$$F(xy) = G(x)H(y) + K(y), \quad x, y \in S, \tag{2.2}$$

so that the differences $f - F$, $g - G$, $h - H$ and $k - K$ are bounded.

We start with two lemmas.

**Lemma 2.2.** Suppose that $S$ is a semigroup, $\varphi, \psi, \xi : S \to \mathbb{C}$, $\delta \geq 0$, $\varphi$ is unbounded and

$$|\varphi(xy) - \varphi(x)\psi(y) - \xi(y)| \leq \delta, \quad x, y \in S. \tag{2.3}$$

Then

$$\psi(xy) = \psi(x)\psi(y), \quad x, y \in S.$$

**Proof.** Let $(z_n)_{n \in \mathbb{N}}$ be a sequence such that

$$0 \neq |\varphi(z_n)| \uparrow \infty.$$

Using (2.3) with $x = z_n$ and dividing the obtained inequality by $|\varphi(z_n)|$, side by side, we have

$$\left| \frac{\varphi(z_n y)}{\varphi(z_n)} - \psi(y) - \frac{\xi(y)}{\varphi(z_n)} \right| \leq \frac{\delta}{|\varphi(z_n)|}, \quad n \in \mathbb{N}, y \in S.$$

Letting $n \to \infty$ we obtain

$$\psi(y) = \lim_{n \to \infty} \frac{\varphi(z_n y)}{\varphi(z_n)}, \quad y \in S. \tag{2.4}$$

Using (2.3) with $z_n x$ instead of $x$ and dividing the obtained inequality by $|\varphi(z_n)|$, side by side, we have

$$\left| \frac{\varphi(z_n x y)}{\varphi(z_n)} - \frac{\varphi(z_n x)}{\varphi(z_n)} \psi(y) - \frac{\xi(y)}{\varphi(z_n)} \right| \leq \frac{\delta}{|\varphi(z_n)|}, \quad n \in \mathbb{N}, x, y \in S.$$

Passing with $n$ to infinity and taking (2.4) into account we infer that

$$|\psi(xy) - \psi(x)\psi(y) - 0| \leq 0, \quad x, y \in S,$$

which completes the proof. \qed
Lemma 2.3. Suppose that $S$ is a right amenable semigroup, $\varphi, \psi, \xi : S \to \mathbb{C}$ and $\delta \geq 0$. If
\begin{equation}
|\varphi(xy) - \varphi(x)\psi(y) - \xi(y)| \leq \delta, \quad x, y \in S,
\end{equation}
and
$$\psi(xy) = \psi(x)\psi(y), \quad x, y \in S,$$
then there exists a function $\eta : S \to \mathbb{C}$ such that
$$\eta(xy) = \eta(x)\psi(y) + \eta(y), \quad x, y \in S,$$
and
$$|\eta(x) - \xi(x)| \leq 2\delta, \quad x \in S.$$  

Proof. Let $M$ be a right invariant mean on $B(S)$. By (2.5) we infer that, for every $y \in S$, the mapping
$$S \ni x \mapsto \varphi(xy) - \varphi(x)\psi(y)$$
is bounded. Hence we can define $\eta : S \to \mathbb{C}$ by the formula
$$\eta(y) := M_z(\varphi(zy) - \varphi(z)\psi(y)), \quad y \in S.$$  

For $x, y \in S$ we have
\begin{align*}
\eta(xy) &= M_z(\varphi(zxy) - \varphi(z)\psi(xy)) \\
&= M_z(\varphi(zxy) - \varphi(zx)\psi(y) + \varphi(zx)\psi(y) - \varphi(z)\psi(xy)) \\
&= M_z(\varphi(zxy) - \varphi(zx)\psi(y) + (\varphi(zx) - \varphi(z)\psi(x))\psi(y)) \\
&= M_z(\varphi(zxy) - \varphi(zx)\psi(y)) + M_z(\varphi(zx) - \varphi(z)\psi(x))\psi(y) \\
&= \eta(y) + \eta(x)\psi(y).
\end{align*}

Moreover, since $\|M\| \leq 2$ (cf.(1.2)), for $y \in S$ we have
\begin{align*}
|\eta(y) - \xi(y)| &= |M_z(\varphi(zy) - \varphi(z)\psi(y)) - \xi(y)| \\
&= |M_z(\varphi(zy) - \varphi(z)\psi(y)) - M_z(\xi(y))| \\
&= |M_z(\varphi(zy) - \varphi(z)\psi(y) - \xi(y))| \\
&\leq 2 \sup_{z \in S} |\varphi(zy) - \varphi(z)\psi(y) - \xi(y)|,
\end{align*}
which jointly with (2.5) gives (2.6) and completes the proof. \qed

Proof of Theorem 2.1. From (2.1) we get
\begin{equation}
|f(x) - g(x)h(1) - k(1)| \leq \varepsilon, \quad x \in S.
\end{equation}

By (2.7), applied for $xy$, and (2.1) we obtain
\begin{equation}
|g(xy)h(1) + k(1) - g(x)h(y) - k(y)| \leq 2\varepsilon, \quad x, y \in S.
\end{equation}

Let us consider the following cases.
1. $h = 0$.

Then, by (2.8), $k$ is bounded, and by (2.7), so is $f$. Functions $F = 0$, $G = g$, $H = 0$ and $K = 0$ are as required.

2. $h(1) = 0$ and there is an $x_0 \in S$ with $h(x_0) \neq 0$.

By (2.8) with $y = x_0$ we see that $g$ is bounded, and therefore, by (2.7), $f$ is also bounded.

- If $g(x) = a \in \mathbb{C}$ is constant, then putting $F(x) = k(1)$, $G(x) = a$, $H = h$ and $K(x) = -ah(x) + k(1)$, we have

$$F(xy) = k(1) = G(x)H(y) + K(y), \quad x, y \in S,$$

and, by (2.8),

$$|K(x) - k(x)| \leq 2\varepsilon, \quad x \in S.$$

- If there are $x_1, x_2 \in S$ with $g(x_1) = a \neq b := g(x_2)$ then, by (2.8) applied for $x_1$ and $x_2$ in place of $x$, we have

$$|ah(y) + k(y) - k(1)| \leq 2\varepsilon, \quad y \in S,$$

and

$$|bh(y) + k(y) - k(1)| \leq 2\varepsilon, \quad y \in S,$$

respectively. We conclude that

$$|a - b||h(y)| \leq 4\varepsilon, \quad y \in S,$$

hence $h$ is bounded. Now, by (2.9), $k$ is also bounded. With $F = G = H = K = 0$ we get what is required.

3. $h(1) \neq 0$.

We put

$$h_1(x) := \frac{h(x)}{h(1)}, \quad k_1(x) := \frac{k(x) - k(1)}{h(1)}, \quad x \in S.$$

After dividing both sides of (2.8) by $|h(1)|$ we get

$$|g(xy) - g(x)h_1(y) - k_1(y)| \leq \frac{2\varepsilon}{|h(1)|} =: \delta, \quad x, y \in S. \quad (2.10)$$

3.1. $g$ and $k_1$ are bounded.

By (2.7) $f$ is also bounded, so it is enough to put $F = 0$, $G = 0$, $H = h$ and $K = 0$ to get the assertion of the Theorem.

3.2. $g$ is bounded and $k_1$ is not.

By (2.7) $f$ is also bounded ($M_1 := \sup_{x \in S}|f(x)|$), and by (2.10) we infer that $h_1$ is unbounded. Let $M_2 > 0$ be such that $|g(x)| \leq M_2$ for $x \in S$ and let $(y_n)_{n \in \mathbb{N}}$ be a sequence such that $0 \neq |h_1(y_n)| \xrightarrow{n \to \infty} \infty$.

By (2.10), for any $x_1, x_2 \in S$, we obtain

$$|g(x_1)h_1(y_n) + k_1(y_n)| \leq \delta + M_2, \quad n \in \mathbb{N},$$
and 
\[ |g(x_2)h_1(y_n) + k_1(y_n)| \leq \delta + M_2, \quad n \in \mathbb{N}, \]
so
\[ |g(x_1) - g(x_2)| |h_1(y_n)| \leq 2\delta + 2M_2, \quad n \in \mathbb{N}. \]
Thereby, \( g \) is constant \( (g(x) = a \) for \( x \in S \)). We put \( F = 0, G = g, H = h \) and \( K(x) = -ah(x), x \in S \). It is obvious that (2.2) is satisfied.

In order to finish the proof in this case it is enough to check that the difference \( k - K \) is bounded. By (2.1) we have
\[ |k(x) - K(x)| = |ah(x) + k(x)| \leq \varepsilon + M_1, \quad x \in S. \]

### 3.3. \( g \) is unbounded.
By Lemma 2.2 we infer that
\[ h_1(xy) = h_1(x)h_1(y), \quad x, y \in S. \quad (2.11) \]
On account of Lemma 2.3 there exists \( k_2: S \to \mathbb{C} \) such that
\[ k_2(xy) = k_2(x)h_1(y) + k_2(y), \quad x, y \in S, \quad (2.12) \]
and
\[ |k_2(x) - k_1(x)| \leq 2\delta, \quad x \in S. \]
We define
\[ F(x) = [k_2(x) + g(1)h_1(x)]h(1), \]
\[ G(x) = k_2(x) + g(1)h_1(x), \]
\[ H(x) = h_1(x)h(1) = h(x), \]
\[ K(x) = k_2(x)h(1), \quad x \in S. \]
It easy to check, using (2.11) and (2.12), that (2.2) is fulfilled. Moreover, we have
\[ |k(y) - K(y)| = |k(y) - k_2(y)h(1)| \]
\[ \leq |k(y) - k_1(y)h(1)| + |k_1(y)h(1) - k_2(y)h(1)| \]
\[ \leq |k(1)| + 2\delta |h(1)|, \quad y \in S, \]
\[ |g(x) - G(x)| = |g(x) - k_2(x) - g(1)h_1(x)| \]
\[ \leq |g(x) - g(1)h_1(x) - k_1(x)| + |k_1(x) - k_2(x)| \]
\[ \leq \delta + 2\delta, \quad x \in S, \]
\[ |f(x) - F(x)| = |f(x) - G(x)h(1)| \]
\[ \leq |f(x) - g(x)h(1) - k(1)| + |k(1)| + |g(x) - G(x)| |h(1)| \]
\[ \leq \varepsilon + |k(1)| + 3\delta |h(1)|, \quad x \in S. \]
Acknowledgements

The research of the second author was supported by Institute of Mathematics, University of Silesia, Katowice, Poland (Stability of selected functional equations program). We gratefully acknowledge the Referee’s suggestions to improve the style of the presentation.

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Peter Volkmann  
Institut für Analysis  
KIT  
76128 Karlsruhe  
Germany  

Received: February 23, 2014  
Revised: July 10, 2014