MEAN CURVATURE FLOW IN SUBMANIFOLDS

HIROSHI NAKAHARA

ABSTRACT. We obtain explicit solutions of the mean curvature flow in some submanifolds in the Euclidean space. We give particularly an explicit solution of the mean curvature flow of some hyperplane in the Lagrangian self-expander which is constructed in Joyce, Lee and Tsui [2] and find therein a minimal hypersurface.

1. Introduction

Mean curvature flow evolves submanifolds of a riemannian manifold in the direction of their mean curvature vector. It is the steepest descent flow for the area functional and is described by a parabolic system of partial differential equations for the immersed map of evolving submanifolds. Put \( M_0 \) to be a hypersurface in \( \mathbb{R}^{n+1} \) and \( \{ M_t \}_{t \in [0, \epsilon)} \) to be the solution of mean curvature flow. By the weak maximum principle of it [1], we can see that if the initial manifold \( M_0 \) is in an open ball \( B(0, r) \), where \( r > 0 \), then \( M_t \subset B(0, \sqrt{r^2 - 2nt}) \), for any \( t \in [0, \epsilon) \). Furthermore, other properties of the mean curvature flow in \( \mathbb{R}^N \) have been extensively studied. For example, Wang investigates the mean curvature flow of graphs in [5] and the author constructs explicit self-similar solutions and translating solitons for the mean curvature flow in \( C^n (= \mathbb{R}^{2n}) \) in [4]. In this paper, however, we consider the mean curvature flow in some submanifolds of \( \mathbb{R}^N \). We give explicit solutions of the mean curvature flow in some Lagrangian submanifolds \( L \) of \( \mathbb{C}^n \) and in the paraboloid of revolution in \( \mathbb{R}^3 \).

2. Results

In order to discuss the mean curvature flow in submanifolds, firstly, we consider the following well known Proposition.

**Proposition 2.1.** Let \( l, L \) be submanifolds in \( \mathbb{C}^n \). Suppose that \( l \) is a submanifold in \( L \). Put \( H \) to be the mean curvature vector of \( l \) in \( L \), and \( \bar{H} \) to be the mean curvature vector of \( l \) in \( \mathbb{C}^n \). Fix \( p \in l \). Then

\[
H(p) = \bar{H}(p) - \sum_j A_{L,\mathbb{C}^n}(e_j, e_j),
\]

where \( A_{L,\mathbb{C}^n} \) is the second fundamental form of \( L \) in \( \mathbb{C}^n \) and \( \{ e_j \}_j \) is an orthonormal basis of \( T_p l \). So we can see that

\[
H(p) = \pi_{T_p L}(\bar{H}(p)),
\]

2010 Mathematics Subject Classification. 53A07 Primary, 53A10, 53C42(secondary).
where $\pi_{T_pL}(\tilde{H}(p))$ is the orthogonal projection of $\tilde{H}(p)$ to $T_pL$.

In this paper, if a manifold $M$ is a submanifold in a riemannian manifold $N$, then we denote $A_{M,N}$ the second fundamental form of $M$ in $N$ and $\nabla^N, \nabla^M$ the Levi-Civita connections on $N$ and $M$ respectively. Hence $A_{M,N} \in C^\infty(M, (TN/TM) \otimes T^*M \otimes T^*M)$.

**Proof.** From the definitions of the mean curvature vector and the second fundamental form we have
\[
H(p) = \sum_j A_{l,l}(e_j, e_j) = \sum_j (\nabla^L e_j e_j - \nabla^l e_j e_j) = \sum_j (\nabla^c e_j e_j - A_{L,C}^c(e_j, e_j) - \nabla^l e_j e_j)
\]
\[
= \sum_j (A_{l,C}^c(e_j, e_j) - A_{L,C}^c(e_j, e_j)) = \bar{H}(p) - \sum_j A_{L,C}^c(e_j, e_j).
\]
This finishes the proof. \(\square\)

In the following Theorem 2.2 from a direct calculation, the submanifolds $L$ are Lagrangian submanifold.

**Theorem 2.2.** Let $I$ be an interval of $\mathbb{R}$ and $\omega : I \rightarrow \mathbb{C} \setminus \{0\}$ be a smooth function. Suppose that $\tilde{\omega}(s) \neq 0$, for any $s \in I$. Define submanifolds $l_s$, for $s \in I$, in $\mathbb{C}^n$ by
\[
l_s = \{(x_1\omega(s), \ldots, x_n\omega(s)); \sum_{j=1}^n x_j^2 = 1, x_1, \ldots, x_n \in \mathbb{R}\},
\]
and submanifold $L$ in $\mathbb{C}^n$ by
\[
L = \bigcup_{s \in I} l_s.
\]
(Clearly, $l_s \subset L \subset \mathbb{C}^n$.) Let $H_s$ be the mean curvature vector of $l_s$ in $L$. Then
\[
H_s(x_1\omega(s), \ldots, x_n\omega(s)) = -\frac{(n - 1)\text{Re}(\tilde{\omega}(s)\tilde{\omega}(s))}{|\omega(s)|^2|\tilde{\omega}(s)|^2} \cdot \frac{\partial}{\partial s}
\]
holds, where $\partial/\partial s = (x_1\omega(s), \ldots, x_n\omega(s)) \in T_{(x_1\omega(s), \ldots, x_n\omega(s))}L$. Thus, by the definition of the mean curvature flow, if we suppose that $f$ is a solution of the following ordinal differential equation
\[
\frac{df(t)}{dt} = -\frac{(n - 1)\text{Re}(\tilde{\omega}(f(t))\tilde{\omega}(f(t)))}{|\omega(f(t))|^2|\tilde{\omega}(f(t))|^2},
\]
then $\{l_{f(t)}\}_t$ is the mean curvature flow in $L$.

The following Lemma 2.3 is a lemma of Theorem 2.2.

**Lemma 2.3.** Let $\alpha \in \mathbb{C} \setminus \{0\}$ be a constant. Define a submanifold $S$ in $\mathbb{C}^n$ by
\[
S = \{\alpha(x_1, \ldots, x_n) \in \mathbb{C}^n; \sum_{j=1}^n x_j^2 = 1, x_1, \ldots, x_n \in \mathbb{R}\}.
\]
Fix $p \in S$. Then
\[
H(p) = -\frac{n - 1}{|\alpha|^2} p.
\]
where $H(p)$ is the mean curvature vector of $S$ at $p$.

Proof. Let $\{e_1, \ldots, e_{n-1}\}$ be an orthonormal basis of $T_pS$. Let $V_j$ be the plane which is generated by $e_j$ and $\overrightarrow{Op}$, where $O = (0, \ldots, 0) \in \mathbb{C}^n$. Since the intersection of $S$ and $V_j$ is a circle of radius $|\alpha|$ with center $O$, we can get curves $c_1, \ldots, c_{n-1} : \mathbb{R} \to S$ such that

$$c_j(0) = p, \quad \dot{c}_j(0) = e_j, \quad \ddot{c}_j(0) = -\frac{1}{|\alpha|^2}p,$$

for any $j$. We compute

$$H(p) = \sum_{j=1}^{n-1} A_{S,\mathbb{C}^n}(e_j, e_j) = \sum_{j=1}^{n-1} \left( \nabla_{e_j} e_j \right)^\perp = \sum_{j=1}^{n-1} (\ddot{c}_j(0))^\perp = \sum_{j=1}^{n-1} \left( -\frac{1}{|\alpha|^2}p \right)^\perp = -\frac{n-1}{|\alpha|^2}p,$$

where $\perp$ is the orthogonal projection to $T_p^\perp S$. This completes the proof. 

Now we prove Theorem 2.2.

Proof of Theorem 2.2. We denote by $\bar{H}_s$ the mean curvature vector of $l_s$ in $\mathbb{C}^2$. Fix $p = (x_1\omega(s), \ldots, x_n\omega(s)) \in l_s$. By Lemma 2.3,

$$\bar{H}_s(p) = -\frac{n-1}{|\omega(s)|^2}(p).$$

By Proposition 2.1, we have

$$H(p) = \pi_{p,l}(\bar{H}(p)) = -\frac{n-1}{|\omega(s)|^2} \cdot \pi_{p,l}(p).$$

From a direct calculation, we can see $\partial/\partial s \perp T_pl_s$. Hence we obtain

$$H(p) = -\frac{n-1}{|\omega(s)|^2} \cdot \frac{p \cdot \partial/\partial s}{\partial/\partial s \cdot \partial/\partial s} \cdot \frac{\partial}{\partial s} = -\frac{(n-1)\text{Re} (\bar{\omega}(s)\dot{\omega}(s))}{|\omega(s)|^2 |\dot{\omega}(s)|^2} \cdot \frac{\partial}{\partial s}.$$ 

This finishes the proof.

Remark 2.3.1. Let $a > 0$ and $\alpha \geq 0$ be constants. Define $r : \mathbb{R} \to \mathbb{R}$ by $r(s) = \sqrt{1/a + s^2}$ and $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(s) = \int_0^s \frac{|t|dt}{(1/a + t^2)\sqrt{(1 + at^2)^n e^{\alpha t^2} - 1}}.$$

In the situation of Theorem 2.2 if we put $I = \mathbb{R}$ and $\omega(s) = r(s)e^{i\phi(s)}$, then $L$ is the Lagrangian self-similar solution constructed by Theorem C in Joyce, Lee and Tsui [2].
Then we compute
\[
\frac{\text{Re} (\bar{\omega}(s) \dot{\omega}(s))}{|\omega(s)|^2 |\dot{\omega}(s)|^2} = \frac{\text{Re} \left( r(s)e^{-i\phi(s)}(\dot{r}(s)e^{i\phi(s)} + ir\dot{\phi}(s)e^{i\phi(s)}) \right)}{r(s)^2 \cdot |\dot{r}(s)e^{i\phi(s)} + ir\dot{\phi}(s)e^{i\phi(s)}|^2} \\
= \frac{r(s)^2 \cdot |\dot{r}(s) + ir\dot{\phi}(s)|^2}{r(s)\dot{r}(s)} \\
= \frac{r(s)^2 \dot{r}(s)^2 + r(s)^4 \dot{\phi}(s)^2}{s^2 + s^2/((1 + at^2)^n e^{as^2} - 1)} \\
= \frac{1}{s + s/((1 + at^2)^n e^{as^2} - 1)} \\
= \frac{1}{s(1 + as^2)^n e^{as^2}/((1 + as^2)^n e^{as^2} - 1)} \\
= \frac{(1 + as^2)^n e^{as^2} - 1}{s(1 + as^2)^n e^{as^2}} \\
= \frac{(1 + as^2)^n - e^{-as^2}}{s(1 + as^2)^n}.
\]

By the equation (1) and L’Hôpital’s rule, we obtain
\[
H_0(x_1 \omega(0), \ldots, x_n \omega(0)) = -(n - 1) \cdot \frac{\text{Re} (\bar{\omega}(0) \dot{\omega}(0))}{|\omega(0)|^2 |\dot{\omega}(0)|^2} \cdot \frac{\partial}{\partial s} \\
= -(n - 1) \cdot \lim_{s \to 0} \frac{(1 + as^2)^n - e^{-as^2}}{s(1 + as^2)^n} \cdot \frac{\partial}{\partial s} \\
= -(n - 1) \cdot \lim_{s \to 0} \frac{n(1 + as^2)^{n-1} \cdot 2as + 2as e^{-as^2}}{(1 + as^2)^n + s \cdot n(1 + as^2)^{n-1} \cdot 2as} \cdot \frac{\partial}{\partial s} \\
= 0.
\]

Therefore \( l_0 \) is minimal in \( L \). See Figure 1. Furthermore, if we fix \( \alpha = 0 \), then from Theorem 2.2 \( \{ l_{\sqrt{2(n-1)t+1}} \} \), \( 0 \leq t \leq 1/2(n-1) \), is a solution of the mean curvature flow in \( L \). So we obtain the following Corollary 2.4.

**Corollary 2.4.** Let \( a > 0 \) be a constant. Define \( r : \mathbb{R} \to \mathbb{R} \) by \( r(s) = \sqrt{1/a + s^2} \) and \( \phi : \mathbb{R} \to \mathbb{R} \) by
\[
\phi(s) = \int_0^s \frac{|t| dt}{(1/a + t^2)\sqrt{(1 + at^2)^n - 1}}.
\]

We define the submanifold \( L \) by
\[
L = \{ r(s)e^{i\phi(s)}(x_1, \ldots, x_n); \sum_{j=1}^n x_j^2 = 1, s, x_1, \ldots, x_n \in \mathbb{R} \}.
\]
(This submanifold \( L \) is one of Lawlor's examples of special Lagrangian submanifolds \[3\].) Then the map
\[
F_t : S^{n-1} \to L; \ (x_1, \ldots, x_n) \mapsto (1/a + 1 - 2(n - 1)t)e^{i\phi(\sqrt{-2(n-1)t+1})}(x_1, \ldots, x_n)
\]
is a solution of the mean curvature flow in \( L \).

\[\text{Figure 1. Remark 2.3.1}\]

Lastly we give an example of the mean curvature flow in the paraboloid of revolution. The proof of following Proposition 2.5 is left to the reader.

**Proposition 2.5.** Define \( F : \mathbb{R} \times (0, \infty) \to \mathbb{R}^3 \) by
\[
F(\theta, r) = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ r^2 \end{pmatrix}.
\]
(By a direct calculation, \( \partial F/\partial r \perp \partial F/\partial \theta \).) Put \( L = F(\mathbb{R} \times (0, \infty)) \) and
\[
l_r = \{ F(\theta, r); \theta \in \mathbb{R} \}, \ r \in (0, \infty).
\]
(Clearly, \( l_r \subset L \subset \mathbb{R}^3 \).) Write \( H_r \) the mean curvature vector of \( l_r \) in \( L \). Then
\[
H_r(\theta) = -\frac{1}{r(1+4r^2)} \cdot \frac{\partial F}{\partial r}(\theta, r)
\]
holds. Thus, by the definition of the mean curvature flow, if we suppose that \( x \) is a solution of the following ordinal differential equation
\[
\frac{dx(t)}{dt} = -\frac{1}{x(t)(1+4x(t)^2)},
\]
then \( \{ l_{x(t)} \}_t \) is the mean curvature flow in \( L \).

**Proof.** The proof is left to the reader. \( \square \)
REFERENCES

[1] K. Ecker, Regularity theory for mean curvature flow, Birkhäuser.
[2] D. Joyce, Y.-I. Lee and M.-P. Tsui, Self-similar solutions and translating solitons for Lagrangian mean curvature flow, J. Differential Geom. 84 (2010), 127–161.
[3] G. Lawlor, The angle criterion, Inventiones math. 95 (1989), 437–446.
[4] H. Nakahara, Some examples of self-similar solutions and translating solitons for Lagrangian mean curvature flow, Tohoku Mathematical Journal, Vol. 65, No. 3.
[5] M.-T. Wang, Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension, Invent. Math. 148 (2002) 3, 525–543.

DEPARTMENT OF MATHEMATICS
TOKYO INSTITUTE OF TECHNOLOGY
2-21-1 O-okayama, MEGURO, TOKYO
JAPAN
E-mail address: 12d00031@math.titech.ac.jp