Abstract

A class of exact general relativistic rotating thick disks surrounded by rotating dust is constructed by applying the “displace, cut, fill and reflect” method on the van Stockum class of metrics. Two particular solutions are considered in detail. We find that the disks have annular regions with negative energy density and heat flow is present in most of the radial extension of the disk. It is shown that thin disks are composed of exotic matter with an equation of state of cosmic strings or struts. We also comment the recent relativistic galactic model proposed by Cooperstock and Tieu and the presence of an additional disk of exotic matter in their model.

Keywords: Relativistic disks; galaxies; dark matter.

1 Introduction

Exact solutions of Einstein’s field equations describing stationary or static axially symmetric configurations of matter are of great astrophysical interest, since the natural shape of an isolated self-gravitating fluid is axially
symmetric. In particular, disk-like solutions can be used as models of galaxies and accretion disks. Solutions for static thin disks without radial pressure were first studied by Bonnor and Sackfield [1] and Morgan and Morgan [2], and with radial pressure by Morgan and Morgan [3]. Several classes of exact solutions of the Einstein field equations corresponding to static thin disks with or without radial pressure have been obtained by different authors [4]–[13]. Thin rotating disks that can be considered as a source of the Kerr metric were presented by Bičák and Ledvinka [14], while rotating disks with heat flow were studied by González and Letelier [15]. Also thin disks with radial tension [16], magnetic fields [17] and magnetic and electric fields [18] were considered. The nonlinear superposition of a disk and a black hole was first obtained by Lemos and Letelier [7]. Perfect fluid disks with halos [19] and charged perfect fluid disks [20] were studied by Vogt and Letelier. The stability of some General Relativistic thin disk models using a first order perturbation of the energy-momentum tensor was investigated by Ujevic and Letelier [21]. For a survey on self gravitating relativistic disks, see for instance [22].

Even though in a first approximation thin disks can be used as useful models, in more realistic models the thickness of the disk should be considered. The addition of a new dimension may change the dynamical properties of the disk source, e.g., its stability. Thick static relativistic disks in various coordinate systems were presented by González and Letelier [23] and Vogt and Letelier [24]. General Relativistic versions of some well known Newtonian three-dimensional galactic potential-density pairs were studied by Vogt and Letelier [25].

In the works cited above an inverse style method was used to solve the Einstein equations, i.e., the energy-momentum tensor is computed from the metric representing the disk. Another approach to generate disks is by solving the Einstein equations given a source (energy-momentum tensor). Essentially, they are obtained by solving a Riemann-Hilbert problem and are highly nontrivial. This has been used by a German group to generate several exact solutions of disks [26]–[33].

Recently Cooperstock and Tieu [34] devised a new approach to modelling galactic dynamics via General Relativity. A galaxy is idealised as an axially symmetric uniform rotating fluid without pressure (dust). The authors show that the density distribution of the galaxy is related to the rotation parameter in a nonlinear way, and then show that several measured rotation curves for galaxies are consistent with the mass distribution of matter in flattened disks, therefore excluding the need for a massive halo of dark matter. However, their model has been criticized by some authors [35, 36, 37].
particular, Korzyński [35] has argued that the model possesses an additional singular rotating thin disk at $z = 0$ and Vogt and Letelier [36] have shown that this singular disk is made of exotic matter either like cosmic strings or struts with negative energy density. Balasin and Grumiller [38], using a somewhat different mathematical formulation, have shown that it is possible to obtain flat rotation curves without unphysical sources at $z = 0$, but in turn distributional sources may arise on the $z$-axis. Motivated by Cooperstock and Tieu’s work, we investigate the properties of thick rotating disks immersed in rotating dust obtained by using the inverse style method (image method).

The work is divided as follows. In Sec. 2 we present the van Stockum [39] class of metrics that will be used to construct the disks and the formalism to calculate their physical variables. A class of rotating disks based on the Bonnor [40] solution is presented in Sec. 3. In Sec. 4 another class of rotating disks based on an extension of Bonnor’s solution is discussed. The thin disk limit and some comments about the Cooperstock and Tieu galactic model are presented in Sec. 5. Sec. 6 is devoted to the discussion of results.

2 The van Stockum class of metrics and construction of disks

The class of solutions found by van Stockum [39] describing stationary axially symmetric distributions of dust matter has a line element given by

$$ds^2 = (dt - N d\varphi)^2 - r^2 d\varphi^2 - e^\nu (dr^2 + dz^2),$$  \hspace{1cm} (1)

where $N$ and $\nu$ are functions of the quasi-cylindrical coordinates $r$ and $z$. For metric Eq. (1), Einstein’s field equations $G_{\mu\nu} = (8\pi G/c^4)T_{\mu\nu}$ can be cast as

$$N_{,rr} + N_{,zz} - \frac{N_r}{r} = 0,$$  \hspace{1cm} (2a)

$$\nu_{,z} = -\frac{N_x N_z}{r}, \hspace{1cm} \nu_{,r} = \frac{N_{,z}^2 - N_{,r}^2}{2r},$$  \hspace{1cm} (2b)

$$\rho = \frac{c^2}{8\pi G r^2 e^\nu} \left( N_{,r}^2 + N_{,z}^2 \right),$$  \hspace{1cm} (2c)

where $\rho$ denotes the energy density of dust. Eq. (2a) can be expressed as

$$\Phi_{,rr} + \Phi_{,zz} + \frac{\Phi_{,r}}{r} = 0,$$  \hspace{1cm} (3)
by the transformation $N = r\phi_r$.

In order to obtain a thick disk surrounded by dust, given a solution of Eqs. (2a)–(2c), we apply the transformation $z \rightarrow h(z) + b$, where $b$ is a positive constant and $h(z)$ an even function of $z$ such that $h(0) = 0$. The parameter $b$ controls how much matter is concentrated near the symmetry axis (larger values of $b$ generate smoother distributions of matter); whereas the function $h(z)$ dictates the matter distribution along the $z$ direction. This is equivalent to construct a disk by the “displace, cut, fill and reflect” method that is described at length in [23]. The nonzero components of the energy-momentum tensor for metric equation (1) are

\begin{align*}
T^t_t &= \frac{c^4}{16\pi Ge^\nu} \left[ \frac{3}{2r^2} (N^2_r + N^2_z) - \left( \nu_{,rr} + \nu_{,zz} \right) + \frac{N}{r^2} \left( N_{,rr} + N_{,zz} - \frac{N^2_r}{r} \right) \right], \\
T^t_\phi &= -\frac{c^4}{8\pi Ge^\nu} \left[ \frac{1}{2} \left( N_{,rr} + N_{,zz} - \frac{N^2_r}{r} \right) \left( 1 + \frac{N^2}{r^2} \right) + \frac{N}{r^2} (N^2_r + N^2_z) \right], \\
T^\phi_t &= \frac{c^4}{16\pi Gr^2e^\nu} \left( N_{,rr} + N_{,zz} - \frac{N^2}{r} \right), \\
T^\phi_\phi &= -\frac{c^4}{16\pi Ge^\nu} \left[ \frac{N}{r^2} \left( N_{,rr} + N_{,zz} - \frac{N^2_r}{r} \right) + \frac{1}{2r^2} (N^2_r + N^2_z) + \nu_{,rr} + \nu_{,zz} \right], \\
T^r_r &= -T^z_z = \frac{c^4}{32\pi Gr^2e^\nu} (N^2_z - N^2_r - 2r\nu_{,r}), \\
T^z_r &= -\frac{c^4}{16\pi Gr^2e^\nu} (N_{,r}N_z + r\nu_{,z}).
\end{align*}

Applying the transformation $z \rightarrow h(z) + b$ and using Eqs. (2a)–(2c) we obtain the following expressions for the components of the energy-momentum
tensor of the disk

\[ T_t^t = \rho_0 c^2 + \frac{c^4}{16\pi Ge^\nu} \left[ h'' \left( \frac{NN_h}{r^2} - \nu_h \right) + (1 - h'^2) \left( \nu_{,hh} - \frac{3N^2_{,h}}{2r^2} - \frac{NN_{,hh}}{r^2} \right) \right], \]

(5a)

\[ T_\varphi^t = -c^2 N\rho_0 - \frac{c^4}{8\pi Ge^\nu} \left[ h'' N_h \left( 1 + \frac{N^2}{r^2} \right) - (1 - h'^2) \times \right. \]

\[ \left. \left( \frac{NN_{,h}^2}{r^2} + \frac{N_{,hh}^2}{2} \left( 1 + \frac{N^2}{r^2} \right) \right) \right], \]

(5b)

\[ T_t^\varphi = \frac{c^4}{16\pi Gr^2 e^\nu} \left[ h'' N_h - (1 - h'^2) N_{,hh} \right], \]

(5c)

\[ T_\varphi^\varphi = -\frac{c^4}{16\pi Ge^\nu} \left[ h'' \left( \frac{NN_h}{r^2} + \nu_h \right) - (1 - h'^2) \left( \frac{NN_{,hh}}{r^2} + \frac{N^2_{,h}}{2r^2} + \nu_{,hh} \right) \right], \]

(5d)

\[ T_r^r = -T_z^z = \frac{c^4}{32\pi Gr^2 e^\nu} N_{,h}^2 (h'^2 - 1), \quad T_z^r = 0, \]

(5e)

where primes indicate differentiation with respect to \( z \) and \( \rho_0 \) is given by Eq. (2c) with \( N_{,z} \) replaced by \( N_{,h} \).

The physical variables of the disk are obtained by solving the eigenvalue problem \( T_b^a \xi^b = \lambda \xi^a \), and has the solutions

\[ \lambda_{\pm} = \frac{1}{2} (S \pm \sqrt{D}), \]

(6)

\[ S = T_t^t + T_\varphi^\varphi = \rho_0 c^2 + \frac{c^4}{8\pi Ge^\nu} \left[ -h'' \nu_h + (1 - h'^2) \left( \nu_{,hh} - \frac{N^2_{,h}}{2r^2} \right) \right], \]

(7)

\[ D = (T_t^t - T_\varphi^\varphi)^2 + 4T_\varphi^t T_\varphi^\varphi = \left[ \rho_0 c^2 - \frac{c^4 N_{,h}^2 (1 - h'^2)}{8\pi Gr^2 e^\nu} \right]^2 \]

\[ -\frac{c^8}{64\pi^2 G^2 r^2 e^{2\nu}} \left[ h'' N_h - N_{,hh}(1 - h'^2) \right]^2, \]

(8)

\[ \lambda_r = T_r^r, \quad \lambda_z = T_z^z = -T_r^r. \]

(9)

Defining an orthonormal tetrad \((V^a, W^a, X^a, Y^a)\) with \( V^a = N_1(1, \Gamma, 0, 0) \), \( W^a = N_2(\Delta, 1, 0, 0) \), \( X^a = e^{-\nu/2}(0, 0, 1, 0) \) and \( Y^a = e^{-\nu/2}(0, 0, 0, 1) \), where
\(N_1, N_2\) are normalization factors and
\[
\Gamma = \begin{cases} \frac{(\lambda_+ - T^t_t)}{T^t_t}, & D \geq 0 \\ \frac{(T^t_t - T^\varphi_\varphi)}{2T^t_t}, & D \leq 0 \end{cases},
\]
\[
\Delta = \begin{cases} \frac{(\lambda_- - T^\varphi_\varphi)}{T^\varphi_\varphi}, & D \geq 0 \\ 0, & D \leq 0 \end{cases},
\]
the energy-momentum tensor can be written in the canonical form
\[
T_{ab} = \rho V_a V_b + P_\varphi W_a W_b + \kappa(V_a W_b + W_a V_b) + P_r X_a X_b + P_z Y_a Y_b.
\]
The energy density \(\rho\), the azimuthal stress \(P_\varphi\), the heat flow function \(\kappa\), the radial pressure \(P_r\) and the tension \(P_z\) along \(z\) are, respectively,
\[
\rho = \begin{cases} \frac{\lambda_+}{c^2}, & D \geq 0 \\ \frac{S}{2(c^2)}, & D \leq 0 \end{cases},
\]
\[
P_\varphi = \begin{cases} -\lambda_- & D \geq 0 \\ -S/2, & D \leq 0 \end{cases},
\]
\[
\kappa = \begin{cases} 0 & D \geq 0 \\ \sqrt{D}/2, & D \leq 0 \end{cases},
\]
\[
P_r = -T^r_r, \quad P_z = T^r_r.
\]
For the functions \(h(z)\) we use a class of even polynomials derived in [24]
\[
h(z) = \begin{cases} -z + C, & z \leq -a, \\ A z^2 + B z^{2n+2}, & -a \leq z \leq a, \\ z + C, & z \geq a, \end{cases}
\]
with
\[
A = \frac{2n + 1 - ad}{4na}, \quad B = \frac{ad - 1}{4n(n + 1)a^{2n+1}}, \quad C = -\frac{a(2n + 1 + ad)}{4(n + 1)}.
\]
Here \(n = 1, 2, \ldots\); \(a\) is the half-thickness of the disk and \(d\) is the jump of the second derivative on \(z = \pm a\). We have \(|h'(z)| = 1\) and \(h'' = 0\) for \(|z| > a\), so Eqs. [5a]–[16] are reduced to van Stockum’s solution outside the disk. For \(|z| \leq a\) we always have \(|h'(z)| \leq 1\) and Eqs. [5c] and [16] show that we have radial pressures and tensions along \(z\). Also when \(d = 0\) \(h''(|z| = a) = 0\) and Eqs. [7]–[8] show that the energy density of the disk is continuously matched with the surrounding dust cloud on the vertical borders of the disk.
3 Thick disks from the Bonnor solution

Bonnor [40] found a simple solution of equations (2a)-(3) by starting with the following solution of Eq. (3)

\[
\Phi = \frac{2k}{\sqrt{r^2 + z^2}},
\]

where \(k\) is a constant. The other functions read

\[
N = -\frac{2kr^2}{(r^2 + z^2)^{3/2}}, \quad \nu = \frac{k^2r^2(r^2 - 8z^2)}{2(r^2 + z^2)^4},
\]

\[
\rho = \frac{c^2k^2(r^2 + 4z^2)}{2\pi Ge\nu(r^2 + z^2)^4};
\]

The energy density is singular at the origin. Now applying the transformation \(z \to h(z) + b\) on Eqs. (19a)-(19b) and using Eqs. (7)-(8), (5e) and (16), we obtain

\[
D = \frac{c^8k^4}{4\pi^2Ge^2e^2\nu[r^2 + (h + b)^2]^4}\left\{[r^2 - 2(h + b)^2]^2 + 9r^2(h + b)^2h'^2\right\}
\]

\[
-\frac{9c^8k^2r^2}{16\pi^2Ge^2e^2\nu[r^2 + (h + b)^2]^3}\left\{(b + h)[r^2 + (h + b)^2]h''
\right\}
\]

\[
-(1 - h'^2)[r^2 - 4(h + b)^2]\right\}^2,
\]

\[
S = \frac{c^4k^2[r^2 + 4(h + b)^2]}{2\pi Ge\nu[r^2 + (h + b)^2]^4} + \frac{3c^4k^2r^2}{4\pi Ge\nu[r^2 + (h + b)^2]^5}\left\{2h''(h + b)\times
\right\}
\]

\[
[r^2 - 2(h + b)^2][r^2 + (h + b)^2] - (1 - h'^2)[2r^4 - 27r^2(h + b)^2 + 31(h + b)^4]\right\},
\]

\[
P_r = -P_z = \frac{9c^4k^2r^2(h + b)^2(1 - h'^2)}{8\pi Ge\nu[r^2 + (h + b)^2]^5}.
\]

We nondimensionalise the quantities by defining \(\tilde{r} = r/a, \tilde{z} = z/a, \tilde{h} = h/a, \tilde{d} = a/d, \tilde{k} = k/a^2, \tilde{D} = (G^2a^4/c^8)D, \tilde{S} = (Ga^2/c^4)S, \tilde{\rho} = (Ga^2/c^4)\rho, \tilde{k} = (Ga^2/c^4)\kappa\) and \(\tilde{P}_t = (Ga^2/c^4)P_t\). Fig. (11) displays curves of \(\tilde{D} = 0\) as function of \(\tilde{r}\) and \(\tilde{z}\) for the polynomial \(\tilde{h}\) with \(n = 1\) and some values of \(\tilde{k}, \tilde{d}\) and \(\tilde{b}\). On the left of each curve the discriminant \(\tilde{D}\) is positive. We observe that increasing the value of the parameter \(\tilde{k}\) keeping the others fixed, the level curve \(\tilde{D} = 0\) is moved to the right. The same happens when \(\tilde{b}\) is decreased and the other parameters are kept fixed.

Fig. (2a)-(e) show, respectively, curves of the discriminant Eq. (20a), the energy density Eq. (13), the azimuthal stress Eq. (14), the heat flow
function \((15)\) and the radial pressure Eq. \((16)\). The parameters used are \(n = 1, \tilde{k} = 0.5, \tilde{d} = 0\) and \(\tilde{b} = 1\). Each curve is a cut taken on the planes \(\tilde{z} = 0\) (the curves with largest amplitudes), \(\tilde{z} = 0.2, 0.4, 0.6, 0.8, 1\). From Fig. 2(b) we have an annular region of the disk with negative energy density. For the azimuthal stress (Fig. 2(c)) there are two rings with pressure (positive stress), however near the vertical borders of the disk we only have tension. Most of the disk radial extension has heat flow (Fig. 2(d)) that is zero on \(\tilde{z} = 1\). From Fig. 2(e) we see that the radial pressure (as well as tension along \(\tilde{z}\)) is concentrated in the central regions of the disk.

\section{Thick disks from a “monopole” and “quadrupole” solution}

We consider another solution of Eq. \((3)\), namely, a sum of a monopole-like and a quadrupole-like term

\[
\Phi = \frac{k_1}{\sqrt{r^2 + z^2}} + \frac{k_2(r^2 - 2z^2)}{(r^2 + z^2)^{3/2}},
\]

\((21)\)
Figure 2: (a) The discriminant Eq. (20a), (b) the energy density Eq. (13), (c) the azimuthal stress Eq. (14), (d) the heat flow function (15) and (e) the radial pressure Eq. (16) as functions of $\tilde{r}$ for $\tilde{z} = 0, 0.2, 0.4, 0.6, 0.8, 1$ and parameters $n = 1, \tilde{k} = 0.5, \tilde{d} = 0$ and $\tilde{b} = 1$. In (a) the small plot shows the curves for $\tilde{z} = 0.8$ and $\tilde{z} = 1$. 
where $k_1$ and $k_2$ are constants (note that $k_1 = 2k$ when compared with Bonnor’s solution). Eqs. (2b)–(2c) and the relation $N = r \Phi, r$ give

$$N = -\frac{r^2}{(r^2 + z^2)^{7/2}} \left[ k_1 (r^2 + z^2)^2 + 3 k_2 (r^2 - 4 z^2) \right],$$  \hfill (22a)

$$\nu = \frac{r^2}{16 (r^2 + z^2)^8} \left[ 2k_1^2 (r^2 + z^2)^4 (r^2 - 8 z^2) + 24 k_1 k_2 (r^2 + z^2)^2 \times (r^4 - 18 r^2 z^2 + 16 z^4) + 9 k_2^2 (9 r^6 - 288 r^4 z^2 + 672 r^2 z^4 - 256 z^6) \right],$$  \hfill (22b)

$$\rho = \frac{c^2}{8 \pi Ge^\nu (r^2 + z^2)^8} \left[ k_1^2 (r^2 + z^2)^4 (r^2 + 4 z^2) + 6 k_1 k_2 (r^2 + z^2)^2 \times (3 r^4 + 12 r^2 z^2 - 16 z^4) + 9 k_2^2 (9 r^6 + 72 r^4 z^2 - 48 r^2 z^4 + 64 z^6) \right].$$  \hfill (22c)

Applying again the transformation $z \rightarrow h(z) + b$ on the above solutions, we obtain for the physical quantities of the disk exact but long expressions that are not very illuminating. In Fig. 3 we show how the Eq. $\tilde{D} = 0$ is changed with the parameter $\tilde{k}_2 = k_2 / a^4$ (all other quantities were adimensionalised as in Sec. 3), for $\tilde{h}$ with $n = 1, \tilde{d} = 0, \tilde{k}_1 = 1$ and $\tilde{b} = 1$. Values for $\tilde{k}_2$ somewhat greater than 0.053 give rise to a second level curve where $\tilde{D} = 0$. 

Fig. 4(a)–(e) display, respectively, curves of the discriminant $\tilde{D}$ the en-
Figure 4: (a) The discriminant $\tilde{D}$ (b) the energy density Eq. (13), (c) the azimuthal stress Eq. (14), (d) the heat flow function (15) and (e) the radial pressure Eq. (16) as functions of $\tilde{r}$ for $\tilde{z} = 0, 0.2, 0.4, 0.6, 0.8, 1$ and parameters $n = 1$, $\tilde{k}_1 = 1$, $\tilde{d} = 0$ and $\tilde{b} = 1$. In (a) the small plot shows the curves for $\tilde{z} = 0.8$ and $\tilde{z} = 1$. 
ergy density Eq. (13), the azimuthal stress Eq. (14), the heat flow function (15) and the radial pressure Eq. (16). The parameters used are \( n = 1, \tilde{k}_1 = 1, \tilde{k}_2 = 0.05, \tilde{d} = 0 \) and \( \tilde{b} = 1 \). As in Fig. 2 each curve is a cut taken on the planes \( \tilde{z} = 0 \) (the curves with largest amplitudes), \( \tilde{z} = 0.2, 0.4, 0.6, 0.8, 1 \). When compared with Fig. 2(a)–(e), the curves are shifted to the right. We also note the appearance of a new local maximum point in the energy density (Fig. 4(b)) and a new local minimum in the azimuthal stress (Fig. 4(c)). The curves for the heat flow function and radial pressure are not changed qualitatively.

5 Thin disks and the Cooperstock and Tieu galactic model

The thin disk limit is achieved by choosing the function \( h(z) = |z| \) in the transformation \( z \to h(z) + b \). Since \( h' = 2\theta(z) - 1 \) and \( h'' = 2\delta(z) \), where \( \theta(z) \) and \( \delta(z) \) are, respectively, the Heaviside function and the Dirac distribution, the components (5a)–(5e) of the energy-momentum tensor will have a distributional part with support on the plane \( z = 0 \):

\[
T_{ab} = Q_{ab} \delta(z).
\]

The nonzero components of \( Q_{ab} \) are

\[
Q_{tt} = \frac{c^4}{8\pi Ge^{\nu}} \left( \frac{NN_h}{r^2} - \nu_h \right), \quad Q_{\varphi\varphi} = -\frac{c^4}{8\pi Ge^{\nu}} \left( 1 + \frac{N^2}{r^2} \right), \quad (23a)
\]

\[
Q_{t\varphi} = \frac{c^4}{8\pi G r^2 e^{\nu}}, \quad Q_{\varphi t} = -\frac{c^4}{8\pi G r^2 e^{\nu}} \left( \frac{NN_h}{r^2} + \nu_h \right). \quad (23b)
\]

The calculation of the eigenvalues result in

\[
S = Q_{tt} + Q_{\varphi\varphi} = -\frac{c^4 \nu_h}{4\pi Ge^{\nu}}, \quad (24)
\]

\[
D = (Q_{tt} - Q_{\varphi\varphi})^2 + 4Q_{t\varphi}^2 Q_{\varphi t} = -\frac{c^8 N^2}{16\pi^2 G^2 r^4 e^{2\nu}} < 0. \quad (25)
\]

Because the discriminant is always negative, the surface energy density \( \sigma \) of the thin disk is given by \( \sigma = S/(2c^2) \) and the azimuthal stress by \( P_{\varphi} = -S/2 \). Thus the disk is composed of matter with an equation of state \( P_{\varphi} = -c^2 \sigma \).

If \( \sigma > 0 \) we have tensions and it may be interpreted as an equation of state of matter formed by concentric loops of cosmic strings \[41\]. If \( \sigma < 0 \) we have pressure and matter with negative energy density. Objects with this equation of state are known in the literature as struts and they appear to stabilize certain superpositions of static isolated bodies in General Relativity
(see, for instance, [42]). In either case the thin disks are made of exotic matter with unusual properties. Note that this result is general in the sense that it is independent of particular forms for the metric functions \(N\) and \(\nu\).

In the galactic model presented by Cooperstock and Tieu [34] the relevant field equations that describe the distribution of galactic matter are, to order \(G\), given by

\[
N_{,rr} + N_{,zz} - \frac{N_r}{r} = 0, \tag{26}
\]

\[
\rho = \frac{c^2}{8\pi G r^2} \left( N_r^2 + N_z^2 \right). \tag{27}
\]

The metric function \(N(r, z)\) is calculated from the measured tangential velocity \(V = cN/r = c\Phi, r\), and then the density distribution is determined by the nonlinear relation (27). A convenient set of basis functions that are a solution of equation (26) is given by

\[
N = -\sum C_n k_n e^{-k_n |z|} r J_1(k_n r), \tag{28}
\]

where \(J_1\) is the Bessel function of order 1 and the \(C_n\) are constants. The absolute value of \(z\) must be used to ensure the reflexive symmetry with respect to the plane \(z = 0\). However, this is equivalent to make the transformation \(z \rightarrow |z|\) and by the results stated above, introduces on the plane \(z = 0\) an additional rotating disk made of exotic matter. It is worth to note that in another context de Araújo and Wang [43] have considered a solution of equation (26) in the form \(e^{-|z|} r J_1(r)\) and also comment that this introduces an additional mass layer with an equation of state \(P_\varphi = -c^2 \sigma\) and with heat flow.

In principle the exact solution (28) may also be used to generate thick disks via the transformation \(z \rightarrow h(z)\) (with \(h = 0\)), however the calculation of the other metric function \(\nu\) using relations (2b) is far from trivial. But at least the zeros of the discriminant equation (8) can be determined straightforwardly, since they do not depend on the function \(\nu\). As an example we take \(N\) as

\[
N = -a \sum_{n=1}^{10} C_n k_n e^{-a k_n h(z)} r J_1(a k_n r), \tag{29}
\]

where the \(C_n k_n\) and \(k_n\) were fitted for the Milky Way as given in Table 1 of [34]. Fig. 5 shows some curves of \(\tilde{D} = 0\) for \(h\) with \(n = 1\) and \(n = 3\). The variables were adimensionalised as in Sec. 3 and the half-thickness of the Milky Way was set to \(a = 0.75\ Kpc\). On the left of the curves we have small
Figure 5: Curves of $\tilde{D} = 0$ as function of $\tilde{r}$ and $\tilde{z}$ for $h$ with $n = 1$ and $n = 3$. The curves were calculated with Eq. (29) fitted for the Milky Way (table 1 of [34]).

intervals of radii where $\tilde{D} > 0$. In the limit $a \to 0$ these intervals would shrink to zero, leaving a singular layer on $z = 0$ with exotic matter.

6 Discussion

Using the van Stockum class of metrics that describe spacetimes with stationary axially symmetric distributions of dust, and applying the “displace, cut, fill and reflect” method, we constructed thick rotating disks surrounded by rotating dust. In general the disks have equal radial pressures and tensions in the $z$ direction. The solution found by Bonnor [40] and a sum of a “monopole” and “quadrupole” solution were used to study particular models of rotating disks. They were found to have annular regions with negative energy density and azimuthal pressure. Heat flow is concentrated near the plane $z = 0$ and is present in most of the radial extension of the disk. These particular examples suggest that the “displace, cut, fill and reflect” method used on the van Stockum class of metrics will generate disks with some unrealistic physical properties and thus are hardly useful in galactic modelling. However, the application of the above mentioned method to solutions obtained using a different approach (see, for instance, [38]) may result in
more realistic and useful models.

In the thin limit we obtain disks with heat flow everywhere and composed of exotic matter with an equation of state $P_\phi = -c^2 \sigma$, which may be interpreted as an equation of state of cosmic strings or struts. We also comment the new galactic model proposed by Cooperstock and Tieu \cite{34} and the presence of an additional singular layer of exotic matter in their model.

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