Comments on the Gravitational Lensing Magnification

Takashi Hamana

Astronomical Institute, Tohoku University, Sendai 980-8578, Japan

(Received March 21, 2022)

We rederive a relation between gravitational lensing magnification relative to the standard Friedmann distance and one relative to the Dyer-Roeder distance by investigating the null geodesic deviation equation. We show that the relation comes from a natural consequence of the definition of the lensing magnification matrices and is not based on the averaging of the magnifications, which has conventionally been used to derive it. We therefore conclude that the relation is true for each individual ray bundle.

§1. Introduction

Since the discovery of the first multiply imaged quasars and the first observations of gravitational arcs and arclets, gravitational lensing has rapidly become one of the most promising tools for cosmology. In studies of gravitational lensing, the lensing magnification plays an important role, such as the magnification bias in the statistical study of multiply imaged quasars and the number count of distant galaxies in the field of cluster of galaxies.

Conventionally, there are two definitions of the lensing magnification, namely, the magnification relative to the smooth Friedmann distance and that relative to the Dyer-Roeder distance. The relation between these two was derived from the averaging of magnifications over an ensemble of sources based on the argument of flux conservation (see, e.g., Section 4 of Ref. 8). Therefore, it is not clear whether the relation is true for each individual ray bundle.

The main purpose of this paper is to show that this relation results from a natural consequence of the definition of the lensing magnification matrices. Accordingly, we find that the relation is true for each individual ray bundle.

Throughout this paper, we use units for which $c = H_0 = 1$ and the scale factor $a$ is normalized to be unity at the present epoch ($a_0 = 1$). The density parameter $\Omega_0$ and normalized cosmological constant $\lambda_0$ are defined in the usual manner.

§2. Basic equations

The propagation of a bundle of light rays in an inhomogeneous universe was investigated in Refs. 9 and 10 in detail. In this section, we simply describe only the aspects which are directly relevant to this paper.
2.1. Universe model

It is well known that the metric of a realistic model of our universe is well approximated by the cosmological Newtonian metric of the form\(^21\), \(^22\)

\[
d\hat{s}^2 = a^2(\eta) \left[ -(1 + 2\phi) d\eta^2 + (1 - 2\phi) \gamma_{ij} dx^i dx^j \right],
\]

\[
\gamma_{ij} dx^i dx^j = d\chi^2 + f^2(\chi) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right),
\]

\[
f(\chi) = \begin{cases} 
K^{-1/2} \sin(\sqrt{K} \chi) & \text{for } K > 0 \\
\chi & \text{for } K = 0 \\
(-K)^{-1/2} \sinh(\sqrt{-K} \chi) & \text{for } K < 0,
\end{cases}
\]

where \(\eta\) is a conformal time and \(\gamma_{ij}\) is the metric on the constant curvature 3-space with curvature \(K = \Omega_0 + \lambda_0 - 1\). The scale factor \(a\) and Newtonian potential \(\phi\) are determined by the following equations to the lowest order:

\[
\left( \frac{a'}{a} \right)^2 = \frac{\Omega_0}{a} - K + \lambda_0 a^2,
\]

\[
\Delta^{(3)} \phi = 4\pi G (\rho - \rho_b) a^2 = \frac{3}{2} \Omega_0 \delta.
\]

Here \(\dot{} \equiv d/d\eta\), \(\Delta^{(3)}\) is the Laplacian operator in the spatial section, \(\rho_b\) is a mean matter density, and \(\delta\) is the density contrast defined by \(\delta \equiv (\rho - \rho_b)/\rho_b\). We write the above metric as \(\hat{g}_{\mu\nu} = a^2 g_{\mu\nu}\). Since the light cone structure is invariant under the conformal transformation of the metric, in the following we work in the conformally related \(g\) world.

2.2. Propagation of a bundle of light rays

Let us consider an infinitesimal bundle of light rays intersecting at the observer. We denote a connecting vector which connects the fiducial light ray \(\gamma\) to one of its neighbors as \(\xi^\mu\). All gravitational focusing and shearing effects on the infinitesimal light ray bundle are described by the geodesic deviation equation,

\[
\frac{d^2 \xi^\mu}{d\lambda^2} = -R^{\mu\alpha\beta\gamma} \xi^\alpha k^\beta k^\gamma,
\]

where \(k^\alpha = dx^\alpha/d\lambda\), and \(\lambda\) is the affine parameter along the fiducial light ray \(\gamma\). We introduce a dyad basis \(\epsilon^A_{\mu} (A, B, C, ... = 1, 2)\) in the two-dimensional screen orthogonal to \(k^\mu\) and parallel-propagated along \(\gamma\). The screen components of the connection vector are given by

\[
Y^A = \epsilon^A_{\mu} \xi^\mu.
\]

From the geodesic deviation equation (6), one can immediately find that \(Y^A\) satisfies the Jacobi differential equation

\[
\frac{d^2 Y_A}{d\lambda^2} = T_{AB} Y^B,
\]
here $\mathcal{T}_{AB}$ is the so-called optically tidal matrix. From the metric (1), up to first order in $\phi$, this matrix is given by

$$\mathcal{T} = -K\mathcal{I} - \begin{pmatrix} \mathcal{R} + \text{Re}[\mathcal{F}] & \text{Im}[\mathcal{F}] \\ \text{Im}[\mathcal{F}] & \mathcal{R} - \text{Re}[\mathcal{F}] \end{pmatrix}, \quad (9)$$

where $\mathcal{I}$ is a $2 \times 2$ identity matrix, and

$$\mathcal{R} = \Delta^{(3)} \phi, \quad (10)$$

$$\mathcal{F} = \phi_{11} - \phi_{22} + 2i\phi_{12}, \quad (11)$$

represent the Ricci and Weyl focusing induced by the density inhomogeneities, respectively. By the linearity of (8), the solution of $Y^A$ is written in terms of its initial value $dY^A/d\lambda|_{\lambda=0} = \vartheta^A$ and the $\lambda$-dependent linear transformation matrix $\mathcal{D}_{AB}$ as

$$Y^A(\lambda) = \mathcal{D}^A_B(\lambda)\vartheta^B. \quad (12)$$

Substituting the last equation into the Jacobi differential equation (8), we obtain

$$\frac{d^2\mathcal{D}_{AB}}{d\lambda^2} = \mathcal{T}_{AC}\mathcal{D}_{CB}. \quad (13)$$

This is our principal equation.

§3. Lensing magnifications

Now, we derive an evolution equation of the lensing magnification matrix relative to the smooth Friedmann distance from (13). First, we write (9) as $\mathcal{T} = \mathcal{T}^{(0)} + \delta\mathcal{T}$, with $\mathcal{T}^{(0)} = -K\mathcal{I}$, and $\delta\mathcal{T}$ is the second term in (9). In the homogeneous case, $\delta\mathcal{T}$ is vanishing, and the solution of $\mathcal{D}$ is $\mathcal{D}_{AB}(\lambda) = D_f(\lambda)\delta_{AB} = f(\lambda)\delta_{AB}$, where $D_f$ is, of course, the standard angular diameter distance in the background Friedmann universe.

It is natural to define the lensing magnification matrix relative to the corresponding Friedmann universe as

$$\mathcal{M}_{AB}(\lambda) \equiv \frac{\mathcal{D}_{AB}(\lambda)}{D_f(\lambda)}. \quad (14)$$

Differentiating $\mathcal{M}_{AB}$ twice with respect to $\lambda$ and using (13), one finds

$$\frac{d^2\mathcal{M}_{AB}}{d\lambda^2} = -\frac{2}{D_f} \frac{dD_f}{d\lambda} \frac{d\mathcal{M}_{AB}}{d\lambda} + \delta\mathcal{T}_{AC}\mathcal{M}_{CB}. \quad (15)$$

With the initial conditions $\mathcal{M}(\lambda)|_{\lambda=0} = \mathcal{I}$ and $d\mathcal{M} (\lambda)/d\lambda|_{\lambda=0} = \mathcal{O}$, the last equation can be written in the integral form

$$\mathcal{M}_{AB}(\lambda) = \delta_{AB} + \int_0^\lambda d\lambda' \frac{D_f(\lambda - \lambda')}{D_f(\lambda)} \delta\mathcal{T}_{AC}(\lambda')\mathcal{M}_{CB}(\lambda'). \quad (16)$$
This is the general form of the evolution equation of the lensing magnification matrix relative to the Friedmann distance in the multiple gravitational lensing theory.\textsuperscript{4}

Next, we derive an evolution equation of the lensing magnification matrix relative to the Dyer-Roeder distance in the same manner as above. First, we rederive the Dyer-Roeder distance from (13) under the following assumptions:\textsuperscript{3, 4}

(I) The intergalactic space where the light rays propagate has a uniform matter density $\tilde{\alpha}\rho_b$, where $0 \leq \tilde{\alpha} \leq 1$.

(II) The shear of the bundle of light rays can be ignored.

(III) The relation between the affine parameter and the redshift is still given by that in the homogeneous background universe.

From assumptions (I) and (II), the optically tidal matrix $T$ becomes

$$T = T^{\tilde{\alpha}(0)} = \left[ -K + \frac{(1 - \tilde{\alpha}) \Omega_0}{2a} \right] I,$$

and the Jacobi differential equation reduces to the scalar form

$$\frac{d^2 D}{d\lambda^2} = T^{\tilde{\alpha}(0)} D.$$  \hfill (18)

It was shown in Ref.\textsuperscript{3} that, by using the assumption (III), the last equation can be shown to be equivalent to the usual Dyer-Roeder differential equation. Therefore the solution of (18) is, of course, the Dyer-Roeder distance, and we denote it as $D_{\tilde{\alpha}}$.

We also assume that the matter density in the universe can be decomposed into a uniform part and a clumpy part as

$$\rho = \rho_{un} + \rho_{cl} = \tilde{\alpha}\rho_b + \rho_{cl},$$ \hfill (19)

with $\langle \rho_{cl} \rangle = (1 - \tilde{\alpha})\rho_b$. By using the above definitions, the optically tidal matrix can be rewritten as

$$T = T^{\tilde{\alpha}(0)} I + \delta T^{cl},$$ \hfill (20)

where $\delta T^{cl}$ has the same form as the second term of the right-hand side of (18), but the Newtonian potential $\phi$ is replace by $\phi^{cl}$, which is determined by the following Poisson equation:

$$\Delta^{(3)}\phi^{cl} = 4\pi G\rho_{cl}a^2.$$ \hfill (21)

It is again natural to define the lensing magnification matrix relative to the Dyer-Roeder distance as $\mathcal{M}^{\tilde{\alpha}} \equiv \mathcal{D} / D_{\tilde{\alpha}}$. Performing the same procedure as in the case of (14) and below, we obtain

$$\mathcal{M}^{\tilde{\alpha}}_{AB}(\lambda) = \delta_{AB} + \int_0^\lambda d\lambda' \frac{D_{\tilde{\alpha}}(\lambda - \lambda')D_{\tilde{\alpha}}(\lambda')}{D_{\tilde{\alpha}}(\lambda)} \delta T_{AC}^{cl}(\lambda') \mathcal{M}^{\tilde{\alpha}}_{CB}(\lambda').$$ \hfill (22)

The last integral equation of the lensing magnification matrix relative to the Dyer-Roeder distance has the same form as that of one relative to the Friedmann distance, (16), but the distances and gravitational potential are replaced by $D_{\tilde{\alpha}}$ and $\phi^{cl}$. 
As a consequence of the definitions of the lensing magnification matrices relative to the background Friedmann and Dyer-Roeder distances, the relation between these two is obviously

\[ \mathcal{M} = \frac{D_\alpha}{D_f} \mathcal{M}^{\tilde{\alpha}}. \] (23)

The image magnification of a point-like source is given by the inverse of the determinant of the magnification matrix. We denote the magnification relative to the standard Friedmann distance as \( \mu_f \) and to the Dyer-Roeder distance as \( \mu_{\tilde{\alpha}} \); i.e., \( \mu_f = |\det \mathcal{M}|^{-1} \) and \( \mu_{\tilde{\alpha}} = |\det \mathcal{M}^{\tilde{\alpha}}|^{-1} \), respectively. From the definitions and the relation (23), they are related by

\[ \mu_f = \left( \frac{D_f}{D_{\tilde{\alpha}}} \right)^2 \mu_{\tilde{\alpha}}. \] (24)

It is important to note that the relation (24) itself is well known, and has been used in gravitational lensing theory. However, the relation was, conventionally, derived by the averaging of magnifications over an ensemble of sources based on an argument of flux conservation.\(^8\) Strictly speaking, the conventional relation is written as

\[ \langle \mu_f \rangle = \left( \frac{D_f}{D_{\tilde{\alpha}}} \right)^2 \langle \mu_{\tilde{\alpha}} \rangle, \] (25)

where \( \langle \rangle \) represents an ensemble average over sources at the same redshift. It is, therefore, not clear whether the relation is true for each individual ray bundle. It should be emphasized that, as we have shown above, the relation comes from the natural consequence of the definitions of the lensing magnification matrices. Accordingly, the relation is true for each ray bundle.

\section{Summary}

In the present paper, we have shown that the integral equations of the lensing magnification matrix are obtained from the null geodesic deviation equation with the natural definitions for the magnification matrices. The integral equations \( (\mathbb{1}) \) and \( (\mathbb{2}) \) may be regarded as the general form for the evolution equation of the lensing magnification matrix in the multiple gravitational thin lensing theory.\(^9\) Therefore our definitions of the lensing magnification matrices evidently give the general form for that in the gravitational thin lensing theory. As a natural consequence of the definitions of the lensing magnification matrices, the relation between the magnification relative to the Friedmann distance and that relative to the Dyer-Roeder distance \( (\mathbb{2}) \) is obtained. It should be noted that the averaging of magnifications over an ensemble of sources was not performed to derive the relation \( (\mathbb{2}) \). Accordingly, we found that the relation is true for each individual ray bundle as well as the ensemble average of magnifications of sources at the same redshift.
Acknowledgements

The author would like to thank Professor T. Futamase for valuable discussions and carefully reading the manuscript. He would also like to thank Professor P. Schneider, Dr. M. Hattori, Dr. P. Premadi and M. Takada for fruitful discussions.

[1] D. Walsh, R. F. Carswell and R. J. Weymann, Nature 279 (1979), 381.
[2] R. Lynds and V. Petrossian, Bulletin of the American Astronomical Society 18 (1986), 1014.
[3] G. Socail, B. Fort, Y. Mellier and J.-P. Picat, Astron. Astrophys. 172 (1987), L14.
[4] B. Fort, J.-L. Prieur, G. Mathez, Y. Mellier and G. Socail, Astron. Astrophys. 200 (1988), L17.
[5] M. Fukugita and E. L. Turner, Mon. Not. R. Astron. Soc. 253 (1991), 99.
[6] T. Hamana, K. Futamase, T. Futamase and M. Kasai, Mon. Not. R. Astron. Soc. 287 (1997), 344.
[7] B. Fort, Y. Mellier and M. Dantel-Fort, Astron. Astrophys. 321 (1997), 335.
[8] P. Schneider, J. Ehlers and E. E. Falco, Gravitational Lenses (Springer-Verlag, Newyork, 1992).
[9] M. Sasaki, Prog. Theor. Phys. 90 (1993), 753.
[10] S. Seitz, P. Schneider and J. Ehlers, Class. Quant. Gravi. 11 (1994), 2345.
[11] T. Futamase, Phys. Rev. Lett. 61 (1988), 2175.
[12] T. Futamase, Mon. Not. R. Astron. Soc. 237 (1989), 187.
[13] C. C. Dyer and R. C. Roeder, Astrophys. J. 174 (1972), L115.
[14] C. C. Dyer and R. C. Roeder, Astrophys. J. 180 (1973), L31.
[15] J. Ehlers and P. Schneider, Astron. Astrophys. 168 (1986), 57.