Response functions in multicomponent Luttinger liquids

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Abstract. We derive an analytic expression for the zero temperature Fourier transform of the density–density correlation function of a multicomponent Luttinger liquid with different velocities. By employing a Schwinger identity and a generalized Feynman identity exact integral expressions are derived, and approximate analytical forms are given for frequencies close to each component singularity. We find power-law singularities and compute the corresponding exponents. Numerical results are shown for $N = 3$ components and implications for experiments on cold atoms are discussed.

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1. Introduction

It is well known that as a result of spin–charge separation, interacting one-dimensional spin-$1/2$ fermions with a repulsive interaction at incommensurate filling form a two-component Luttinger liquid (LL) [1]–[3]. A multicomponent LL with more than two components can be obtained in fermionic systems with repulsive interactions and larger internal degeneracies [4]–[6]. Possible condensed matter realizations of such systems are provided by multichannel quantum wires [7, 8], two-leg ladders [9], carbon nanotubes [10]–[12] and biased bilayer graphene [13]. The interaction of acoustic phonons with spin-$1/2$ fermions in one dimension can also give rise to a three-component LL [14, 15]. In Mott insulating materials, a multicomponent LL can be formed in spin–orbital chains [16]–[24] and spin tubes [25] under the effect of an applied magnetic field [26, 27] and in spin-1 chains with biquadratic interactions [28]–[30]. More recently, atom trapping technology has permitted the realization of Bose–Fermi mixtures [31]–[34], as well as degenerate gases with internal degrees of freedom [35, 36]. In the latter case, it has been suggested theoretically that these systems could realize Sp(2N) [37] or SU(N) spin systems [37]–[43] in low dimensions. In parallel, techniques for trapping atoms in one dimension have been developed [44]–[51]. In Bose–Fermi mixtures trapped in one
dimension, a multicomponent LL behavior is also expected [52]–[66]. The real space
correlation function of a multicomponent Luttinger liquid can be readily obtained [14,
59, 67]. However, the majority of experimental observables are actually Fourier
transforms (FTs) of these correlation functions [68]–[70] and thus their FTs must be derived. Because
of the branch cut structure of the correlation functions in an LL, this is a non-trivial task.
In the single-component case, the calculation can be carried out in closed form [71]–[73].
For two-component systems, only the exponents of the power-law singularities can be
predicted [74]–[76]. Recently a closed form of the $2k_F$ component of the density–density
response function in a two-component LL (i.e. the spin-1/2 case with different charge and
spin velocities) at zero temperature was obtained [77] in terms of Appell hypergeometric
functions [78, 79]. Such an expression permits the description of the crossovers between
the different power-law singularities of the response functions. In this paper, we derive
an exact expression for the FT of the density–density correlation function in the general
case of a multicomponent LL with different velocities for the modes. We show that in this
general case, the FTs of the Matsubara correlation functions are expressed in terms of the
Srivastava–Daoust generalized hypergeometric functions. We give the full analytical
continuation of the correlation functions to real frequencies, recovering the leading power-
law singularities and describing the various crossovers between them.

The paper is organized as follows. In section 2 we give the Hamiltonian of a general
multicomponent system and derive the general expression for the Matsubara correlation
functions at zero temperature. In sections 2.1 and 2.2 we derive two exact integral
representations of the Fourier transforms of the Matsubara correlation functions by means
of a Schwinger identity and a Feynman identity, respectively. The integral representation
obtained from the Schwinger identity is used to predict the exponents of the power-law
singularities. In section 2.3, starting from the integral representation derived from the
Feynman identity, we obtain the analytic continuation of the Matsubara correlator in
various cases and give the asymptotic expression close to the singular points. Finally, we
give some conclusions in section 3.

2. Model

In this paper we wish to consider the case of a general multicomponent system. The
continuum Hamiltonian is

$$
H = \sum_{a=1}^{N} \int dx \left[ -\psi_a^\dagger \frac{\hbar^2}{2M_a} \frac{\partial^2}{\partial x^2} \psi_a \right] + \sum_{1 \leq a < b \leq N} \int dx \, dx' \, V_{ab}(x - x') \rho_a(x) \rho_a(x'),
$$

where $\psi_a$ annihilates a particle of type $a$, $\rho_a(x) = \psi_a^\dagger(x) \psi_a$ is the corresponding particle
density, and $M_a$ is the corresponding mass. The interaction of particles of type $a$
with particles of type $b$ is $V_{ab}$. The particles may be either bosons or fermions. The Hamiltonian (1) can be bosonized [80] and its expression reads

$$
H = \sum_{a,b} \int \frac{dx}{2\pi} \left[ (\pi \Pi_a) M_{ab} (\pi \Pi_b) + (\partial_x \phi_a) N_{ab} (\partial_x \phi_b) \right],
$$

where $[\phi_a(x), \Pi_b(x')] = i \delta_{ab} \delta(x - x')$, and the real symmetric matrices $M$ and $N$ can be obtained from the variation of the ground state energy with change of boundary conditions.
and particle numbers respectively [80, 81]. The Hamiltonian (2) can be diagonalized, and the equal time correlations can then be obtained [80]. We wish to calculate the Matsubara time dependent correlation functions. In the case of density correlations, since the density is expressed as [82]

$$
\rho_a(x) = -\frac{1}{\pi} \partial_x \phi_a + \sum_{m=-\infty}^{+\infty} A_m \cos[2m(\phi_a(x) - \pi \rho^0_a x)],
$$

we will need to calculate correlation functions of the form $$\langle T_x e^{i m \phi_a(x,\tau)} e^{-im\phi_a(0,0)} \rangle$$, where $$T_x$$ is the Matsubara time ordered operator. If we have bosonic particles, as the bosonized form of the annihilation operator is

$$
\psi_a(x) = e^{i \theta_a(x)} \left[ \sum_m B_m e^{i 2m \phi_a(x) - \pi \rho^0_a x} \right],
$$

where $$\nabla \theta_a = \pi \Pi_a$$, the leading term in the single-particle Green’s function is proportional to $$\langle T_x e^{i \theta_a(x,\tau)} e^{-i \theta_a(x,\tau)} \rangle$$. Because of the duality transformation [80] $$M \leftrightarrow N$$ and $$\theta_a \leftrightarrow \phi_a$$, it is sufficient to calculate the correlation functions of the form $$\langle T_x e^{i \sum_n \alpha_n \phi_a(x,\tau)} e^{-i \sum_n \alpha_n \phi_a(0,0)} \rangle$$. We have

$$
\langle T_x e^{i \sum_n \alpha_n \phi_a(x,\tau)} e^{-i \sum_n \alpha_n \phi_a(0,0)} \rangle = \exp \left[ -\sum_{n<m} \alpha_n \alpha_m G_{nm}(x, \tau) \right],
$$

with [80]

$$
G_{nm}(x, \tau) = \pi \sum_{\omega_n} \int \frac{dq}{2\pi} e^{-|q|^a (1 - e^{i(qx - \omega_n \tau)})} [\omega_n^2 + (MN)q^2]^{-1} M_{nm},
$$

where $$a$$ is a cutoff and $$\omega_n$$ are the Matsubara frequencies. Let us introduce the projection operator $$P$$ that projects on the eigenspace of eigenvalue $$u^2_{\alpha}$$ of the matrix $$MN$$ to rewrite the matrix $$(\omega_n^2 + (MN)q^2)^{-1}$$ as

$$
(\omega_n^2 + (MN)q^2)^{-1} = \sum_n \frac{P_n}{\omega_n^2 + u^2_n q^2}.
$$

In the zero temperature limit, we have

$$
\int \frac{d\omega}{2\pi} (\omega^2 + (MN)q^2)^{-1} (1 - e^{i(qx - \omega \tau)}) e^{-|q|^a} = \sum_n \frac{P_n}{2u_{\alpha} \lambda |q|} (1 - e^{i(qx - u_n |\tau|)}) e^{-|q|^a},
$$

so that

$$
G(x, \tau) = \frac{1}{4} \sum_n \frac{P_n}{u_n} M \ln \left( \frac{x^2 + (u_n |\tau| + \alpha)^2}{\alpha^2} \right),
$$

and thus at zero temperature

$$
\langle T_x e^{i \sum_n \alpha_n \phi_a(x,\tau)} e^{-i \sum_n \alpha_n \phi_a(0,0)} \rangle = \prod_n \left( \frac{\alpha^2}{x^2 + (u_n |\tau| + \alpha)^2} \right)^{\eta_n},
$$

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with
\[ \eta_n = \sum_{a,b} \alpha_a \frac{(P_n M)_{ab}}{u_n} \alpha_b. \]  
(11)

In the case of the \( \theta \) correlation function, one obtains a formula analogous to (10), with the exponents \( \eta_n \) replaced by \( \bar{\eta}_n \), where
\[ \bar{\eta}_n = \sum_{a,b} \alpha_a \frac{(Q_n N)_{ab}}{u_n} \alpha_b, \]  
(12)

\( Q_n \) being the projector on the eigenstate of \( NM \) having the eigenvalue \( u_n^2 \). We note that for \( \tau = 0 \), using equations (10) and (11), we recover the expression of the exponents of equal time correlations from [80]. These formulas are particularly useful when implementing numerical methods like DMRG and exact diagonalization as one could use the following results to predict the response function from ground state energy computations. Knowing the Matsubara time ordered Green’s function, equation (10), we wish to obtain the corresponding Matsubara response function
\[ \chi_M(q, i\omega_n) = \int dx \, d\tau \, e^{i(q x - \omega_n \tau)} \langle T_\tau e^{i \sum_n \alpha_n \phi_n(x, \tau)} e^{-i \sum_n \alpha_n \phi_n(0, 0)} \rangle, \]  
(13)

and the retarded response function \( \chi(q, \omega) = \chi_M(q, i\omega_n \rightarrow \omega + i0) \). In the following, we use two complementary approaches. The first one is based on the Schwinger identity [83] in section 2.1, and will allow us to predict the singularities of the retarded response function. The second one is based on the Feynman identity [83] and will allow us to make a connection with the results for the two-component case [77]. We will assume that we can take the limit \( \alpha \rightarrow 0 \) in the integrals, i.e. \( \eta = \sum \eta_n < 1 \).

Let us recall that for the density–density correlation function the Fourier transform is related to the scattering cross section \( \sigma \) of light at a frequency \( \omega \) and angle \( \Omega \) incident on a sample, by the relation
\[ \frac{d^2 \sigma}{d\omega d\Omega} \propto S(q, \omega) = \text{Sgn}(\omega) \text{Im} \chi(q, \omega), \]  
(14)

where \( S(q, \omega) \) is the dynamic structure factor. This quantity is accessible, e.g., by means of inelastic neutron/light scattering when spin/density fluctuations are induced in the system and their subsequent relaxation is measured.

2.1. Schwinger identity

The Schwinger identity is [83]
\[ \frac{1}{(x^2 + (u \tau)^2)^\alpha} = \int_0^\infty \frac{d\lambda}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda(x^2 + (u \tau)^2)}. \]  
(15)

In the multicomponent case, it allows us to rewrite the expression (10) as
\[ \prod_n \left( \frac{\alpha^2}{x^2 + (u_n |\tau| + \alpha)^2} \right)^{\eta_n} = \alpha^{2n} \int \prod_n d\lambda_n \lambda_n^{\eta_n-1} \frac{\exp \left[ -x^2 \left( \sum_n \lambda_n \right) - \tau^2 \left( \sum_n \lambda_n u_n^2 \right) \right]}{\Gamma(\eta_n)} \]  
(16)
The Fourier transformation in (13) reduces to a Gaussian integral, and we find
\[
\chi^M(q, i\omega_n) = \pi \alpha^{2\eta} \int \frac{d\lambda_n \lambda^{-1}_n}{\Gamma(\eta_n)} e^{-\frac{1}{4}[q^2/(\sum_n \lambda_n) + (\omega_n^2/(\sum_n \lambda_n u_n^2))]} \sqrt{\sum_n \lambda_n/(\sum_n \lambda_n u_n^2)}. \tag{17}
\]

With the change of variables
\[
\lambda_j = \lambda_1 \mu_{j-1} \quad (j = 2, \ldots, N), \tag{18}
\]
we rewrite (17) as
\[
\chi^M(q, i\omega_n) = \pi \alpha^{2\eta} \int \frac{d\lambda_1 \lambda_1^{-1}}{\Gamma(\eta_1)} \prod_{j=1}^{N-1} \frac{d\mu_j \mu_j^{-1}}{\Gamma(\eta_j+1)} e^{-\frac{1}{4}[q^2/(1+\sum_{j=1}^{N} \mu_j) + \omega_n^2/(\sum_{j=1}^{N} \mu_j u_j^2)]} \sqrt{(1 + \sum_{j=1}^{N-1} \mu_j)(u_1^2 + \sum_{j=1}^{N-1} \mu_j u_j^2)}. \tag{19}
\]

With the change of variable \(\lambda_1 = \mu^{-1}\), the integration over \(\lambda_1\) can be carried out in closed form. We find, provided that \(\eta < 1\),
\[
\chi^M(q, i\omega_n) = \pi \alpha^{2\eta} \frac{\Gamma(1 - \sum_j \eta_j)}{\prod_j \Gamma(\eta_j)} \int \prod_{j=1}^{N-1} d\mu_j \mu_j^{-1} e^{-\frac{1}{4}[q^2/(1+\sum_{j=1}^{N} \mu_j) + \omega_n^2/(\sum_{j=1}^{N} \mu_j u_j^2)]} \sqrt{(1 + \sum_{j=1}^{N-1} \mu_j)(u_1^2 + \sum_{j=1}^{N-1} \mu_j u_j^2)}. \tag{20}
\]

We can perform the analytic continuation on (17) by substituting \(i\omega_n \rightarrow \omega + i0\). When \(\omega \simeq u_1 q\), we have to consider the integral
\[
\int \prod_{j=1}^{N-1} d\mu_j \mu_j^{-1} \left[u_1^2 q^2 - \omega^2 + \sum_{j=1}^{N-1} \mu_j(u_{j+1}^2 q^2 - \omega^2) \right]^{-1}. \tag{21}
\]

With the change of variables \(\mu_j = (u_1^2 q^2 - \omega^2) \xi_j\), we finally find that for \(|\omega| \rightarrow u_1 q\)
\[
\chi(q, \omega) \sim |(u_1 q)^2 - \omega^2|^{2\eta - \eta_1}, \tag{22}
\]
provided \(2\eta - \eta_1 - 1 < 0\). In the general case, we expect to find \(\chi(q, \omega \simeq u_j q) \sim |(u_j q)^2 - \omega^2|^{2\eta - \eta_j - 1}\) when \(2\eta - \eta_j - 1 < 0\). We note that \(\eta\) is the exponent of the equal time correlation functions [80]. The exponents of the singularities satisfy a ‘sum rule’
\[
\sum_{n=1}^{N} (2\eta - \eta_n - 1) = (2N - 1)\eta - N. \tag{23}
\]
2.2. Feynman identity

Using the identity from [83] (appendix B), we can rewrite the correlation function (10) as a multiple integral,

\[
\prod_{j=1}^{N} \frac{1}{(x^2 + u_j^2 \tau^2)^{\eta_j}} = \frac{\Gamma(\sum_{j=1}^{N} \eta_j)}{\prod_{j=1}^{N} \Gamma(\eta_j)} \int \prod_{j=1}^{N} dw_j \, w_j^{\eta_j-1} \delta \left(1 - \sum_{j=1}^{N} w_j\right) \left(x^2 + \sum_{j=1}^{N} w_j u_j^2 \tau^2\right)^{-(\sum_{j=1}^{N} \eta_j)}.
\]

The Matsubara response function (13) is then found from the integral (11.4.16) of [84],

\[
\int dx \, d\tau \, e^{i(qx - \omega_n \tau)} \left(x^2 + \sum_{j=1}^{N} w_j u_j^2 \tau^2\right)^{-\eta} = \frac{2^{2(1-\eta)} \pi^{(1-\eta)} \Gamma(1-\eta)}{\Gamma(\eta) \sqrt{\sum_{j=1}^{N} w_j u_j^2}} \left(q^2 + \sum_{j=1}^{N} \omega_n^2 - \omega_n^2 \right)^{\eta-1},
\]

where \(1/4 < \eta < 1\) as

\[
\chi^M(q, i\omega_n) = \frac{\Gamma(1-\eta)}{\prod_{j=1}^{N} \Gamma(\eta_j)} \int \prod_{j=1}^{N} dw_j \, w_j^{\eta_j-1} \delta \left(1 - \sum_{j=1}^{N} w_j\right) \times \frac{2^{2(1-\eta)} \pi^{2\eta} \pi}{(\sum_{j=1}^{N} w_j u_j^2)^{\eta-1/2}} \left(\sum_{j=1}^{N} w_j u_j^2\right)^{\eta-1}.
\]

For the case of \(\eta = 1/2\), the integral (26) can be expressed as a Lauricella function \(F_D\), which actually reduces to a simple product. The result is simply

\[
\chi^M(q, i\omega_n) = 2\pi \alpha \prod_{j=1}^{N} \left(\omega_n^2 + u_j^2 q^2\right)^{-\eta_j}.
\]

This result generalizes the one obtained for the two-component case in [77].

When \(\eta > 1/2\), we can use the Feynman identity again [83] to write

\[
\prod_{j=1}^{N} \frac{1}{((\omega_n^2/q^2) + \sum_j u_j^2 w_j)^{1-\eta}(\sum_j u_j^2 w_j)^{\eta-1/2}} = \frac{\Gamma(1/2)}{\Gamma(\eta - 1/2) \Gamma(1-\eta)} \times \int_0^1 ds \, s^{\eta-1/2} \left(s^{-\eta}(1-s)^{\eta-3/2} \left(s^{\omega_n^2/q^2} + \sum_j u_j^2 w_j\right)^{-1/2}\right),
\]

and find

\[
\chi^M(q, i\omega_n) = \pi 2^{2(1-\eta)} \alpha^2 \Gamma(1/2) / \Gamma(\eta - 1/2) \Gamma(1-\eta) \int_0^1 ds \, s^{\eta-1/2} \left(\sum_j u_j^2 w_j\right)^{-1/2} \times F^{(N-1)}_{D\omega_n^2/q^2} \left(1/2; \eta_1, \ldots, \eta_{N-1}; \eta; u_N^2\right) \left(s^{-\eta}(1-s)^{\eta-3/2} \left(s^{-\omega_n^2/q^2} + u_N^2\right)^{-1/2}\right),
\]

where \(F^{(N-1)}_{D\omega_n^2/q^2}\) is a Lauricella hypergeometric function of \(N-1\) variables. The integral (29) can be expressed in closed form [85] using a Srivastava–Daoust generalized hypergeometric
series \([86]\) of \(N\) variables as
\[
\chi^M(q, i\omega_n) = F(\eta)(q\alpha)^{(n-1)} F_{1;1;1;1}^{1;1;1;1}
\]
\[
\times \left[ \frac{1}{\eta_1, \ldots, \eta_{N-1}, 1 - \eta, 1 - u_1^2, \ldots, 1 - u_{N-1}^2, - \frac{\alpha}{u_N^2}, - \frac{\alpha}{u_N^2}} \right], \quad (30)
\]
where \(F(\eta) = \pi 2^{(1-\eta)\alpha} \Gamma(1 - \eta)/\Gamma(\eta) u_N\). When \(N = 2\), the Srivastava–Daoust hypergeometric series reduces to an Appell \(F_2\) hypergeometric function of two variables. Using the identity (16.16.3) from \([79]\), this function is seen to reduce to the \(F_2\) hypergeometric series \([86]\) of \(\eta_2\) functions.

Each term in the sum is then expressible with Srivastava–Daoust hypergeometric functions. Therefore, we can apply the Feynman identity to each term in the sum. We thus find
\[
\chi(q, i\omega_n) = \pi \alpha^{2n} q^{2n-2} 2^{2(1-\eta)} \Gamma(3/2) \Gamma(\eta + 1)
\]
\[
\times \left[ \frac{1}{\eta_1, \ldots, \eta_{N-1}, 1 - \eta, 1 - u_1^2, \ldots, 1 - u_{N-1}^2, s^{\alpha} u_1^2 F_D^{(N-1)}} \right]
\]
\[
\times \left( \frac{3}{2}; \{\eta_j + \delta_j\} \right) \quad \text{for } 1 \leq j \leq N-1; \eta + 1: \frac{u_1^2 - u_1^2}{s^{\alpha} u_1^2 F_D^{(N-1)}} + \frac{u_2^2}{s^{\alpha} u_1^2 F_D^{(N-1)}} + \ldots + \frac{u_{N-1}^2}{s^{\alpha} u_1^2 F_D^{(N-1)}} \right). \quad (34)
\]

Each term in the sum is then expressible with Srivastava–Daoust hypergeometric functions.
2.3. Analytic continuation of the Matsubara correlator

To obtain the retarded response function, we have to find the analytic continuation $i\omega_n \rightarrow \omega + i\epsilon$ of (32). We will first discuss the special case of $\eta = 1/2$, where the continuation is straightforward, leading to a simple picture of the behavior of the retarded response function. Then, we will turn to the more complicated case of $\eta > 1/2$, for which the calculations are more involved. We will see, however, that the simple picture of the $\eta = 1/2$ case is preserved.

2.3.1. The case of $\eta = 1/2$. In the case of $\eta = 1/2$, the analytic continuation is easily obtained from equation (27). We have for $u_j q < \omega < u_{j+1} q$

$$\chi(q, \omega + i0) = 2\pi\alpha e^{i\pi\sum_{j=1}^{N} \eta_j} \prod_{j=1}^{N} |\omega^2 - (u_j q)^2|^{-\eta_j},$$

so that

$$\text{Im}\chi(q, \omega \rightarrow u_j q + 0) \sim 2\pi\alpha \sin \left[\pi \sum_{l=1}^{j} \eta_l \right] |\omega^2 - (u_j q)^2|^{-\eta_j} \prod_{l \neq j} |(u_j^2 - u_l^2)q^2|^{-\eta_l},$$

and

$$\text{Im}\chi(q, \omega \rightarrow u_{j+1} q - 0) \sim 2\pi\alpha \sin \left[\pi \sum_{l=1}^{j} \eta_l \right] |\omega^2 - (u_{j+1} q)^2|^{-\eta_{j+1}} \prod_{l \neq j+1} |(u_{j+1}^2 - u_l^2)q^2|^{-\eta_l},$$

showing that the spectral function has singularities with exponent $-\eta_j$ whenever $\omega \sim u_j q$. This result is agreement with the result of section 2.1 for $\eta = 1/2$. The spectral function has a threshold for $\omega < u_1 q$ as there are no excitations of the system having a lower energy. Moreover, we note that for $j > 1$

$$\frac{\text{Im}\chi(q, \omega \rightarrow u_j q + 0)}{\text{Im}\chi(q, \omega \rightarrow u_j q - 0)} = \frac{\sin[\pi \sum_{l=1}^{j} \eta_l]}{\sin[\pi \sum_{l=1}^{j-1} \eta_l]}$$

which implies that the peaks of the spectral functions are asymmetric around $\omega = u_j q$. The imaginary part of the response function is represented in figure 1 for the case $\eta = 1/2$.

2.3.2. The case of $\eta > 1/2$. In the general case, we need to find the analytic continuation of equation (32) for $\eta > 1/2$ or equation (34) for $\eta < 1/2$. Since the method is similar in the two cases, we will concentrate on the case of $\eta > 1/2$.

Formally, under the analytic continuation, the variables in equation (32) become

$$\frac{u_j^2 - u_{j+1}^2}{u_N^2} \left(1 - \frac{\omega_n^2 t}{u_N^2 + u_N^2 q^2} \right) \rightarrow \left(1 - \frac{u_j^2}{u_N^2} \right) \left(1 + \frac{(\omega + i\epsilon)^2}{(u_N q)^2 - (\omega + i\epsilon)^2} \right).$$

The Lauricella function $F_D$ has cuts every time the real part of one of the variables is larger than one. In the case of $\omega > u_N q$, the real parts of all the variables, according to
The imaginary part of the response function $\chi(q, \omega)$ for the case $\eta_1 = 0.25$, $\eta_2 = 0.15$ and $\eta_3 = 0.1$. The velocities are $u_2 = 2u_1$ and $u_3 = 3u_1$. The unit of frequency $\omega$ is $u_1q$. The unit of $\text{Im}\chi(q, \omega)$ is $2\pi a/(u_1q)$. Power-law divergences are obtained for $\omega = u_1, 2u_1, 3u_1q$.

(39), will remain less than one, and the analytic continuation is straightforward. We can then derive an equivalent for the Lauricella function in the limit of $\omega \to u_Nq + 0$. We find

$$F_D^{(N-1)}\left(\frac{1}{2}; \eta_1, \ldots, \eta_{N-1}; \eta; \frac{u_N^2 - u_1^2}{u_N^2} \right) \times \left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right), \ldots, \frac{u_N^2 - u_{N-1}^2}{u_N^2} \left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right)$$

$$\sim \frac{\Gamma(\eta)\Gamma(\eta N + 1/2 - \eta)}{\Gamma(1/2)\Gamma(\eta N)} \left(\frac{\omega^2 - (u_Nq)^2}{\omega^2 t}\right)^{\eta - \eta N} \prod_{j=1}^{N-1} \left(1 - \frac{u_j^2}{u_N^2}\right)^{-\eta_j},$$

yielding

$$\text{Im}\chi(q, \omega) = \pi 2^{1-\eta} \alpha^2 \Gamma(\eta N + 1/2 - \eta) \sin(\pi \eta) \left(\frac{\omega^2 - (u_Nq)^2}{(\omega^2 - u_N^2 q^2)^{1/2}}\right)^{\eta - \eta N} \prod_{j=1}^{N-1} \left(1 - \frac{u_j^2}{u_N^2}\right)^{-\eta_j} \times e^{i\pi(\eta - \eta N) \text{sgn}\omega} \prod_{j=1}^{N-1} \left(1 - \frac{u_j^2}{u_N^2}\right)^{-\eta_j},$$

In equation (41), the exponent predicted in section 2.1 is recovered. For $\omega \to u_Nq - 0$, it is also possible to find an asymptotic estimation of the Lauricella function in the form

$$F_D^{(N-1)}\left(\frac{1}{2}; \eta_1, \ldots, \eta_{N-1}; \eta; \frac{u_N^2 - u_1^2}{u_N^2} \right) \times \left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right), \ldots, \frac{u_N^2 - u_{N-1}^2}{u_N^2} \left(1 - \frac{\omega^2 t}{\omega^2 - u_N^2 q^2}\right)$$

$$\sim \frac{\Gamma(\eta)\Gamma(\eta N + 1/2 - \eta)}{\Gamma(1/2)\Gamma(\eta N)} \left(\frac{u_Nq^2 - \omega^2}{\omega^2 t}\right)^{\eta - \eta N} \times e^{i\pi(\eta - \eta N) \text{sgn}\omega} \prod_{j=1}^{N-1} \left(1 - \frac{u_j^2}{u_N^2}\right)^{-\eta_j},$$

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\[ \text{Im}(q, \omega \to u_N q - 0) \sim \frac{\pi^{2-\epsilon} \alpha^2 \Gamma(\eta_{N+1} - 2\eta) \sin[\pi(\eta - \eta_N)]}{\Gamma(\eta_N) u_N} \times \frac{((u_N q)^2 - \omega^2)^{2\eta - \eta_{N-1}}}{(\omega^2)^{\eta_{N-1}} (u_N^2 / \alpha^2)^{\eta_{N-1}}} \prod_{j=1}^{N-1} \left( 1 - \frac{u_j^2}{u_N^2} \right)^{-\eta_j}, \]

i.e. the same power-law divergence as in equation (41) but with a different prefactor. The ratio of the two expressions is \( \sin[\pi(\eta - \eta_N)] / \sin(\pi \eta) \), as previously noted in the special case of \( \eta = 1/2 \). For \( \epsilon \to 0 \), the imaginary part of (39) is positive for \( \omega < u_N q \) and the real part of (39) is equal to one for \( t = t_j = (u_N q / \omega)^2 - 1 / (u_N / u_j)^2 - 1 \). We have \( t_1 < t_2 < \cdots < t_{N-1} \).

In particular, one finds that when \( \omega < u_1 q, t_1 > 1 \), so that no analytic continuation of the Lauricella function under the integral sign in (32) is needed. The response function remains purely real in this case, and the spectral function vanishes. In the case \( u_j q < \omega < u_{j+1} q \), we find that \( t_1 < \cdots < t_j < 1 < t_{j+1} < \cdots < t_{N-1} \). As a result, the integral has to be split into a sum of integrals over the intervals \([0, t_1], [t_l, t_{l+1}]\) with \( l \leq j \leq 1 \) and \([t_j, 1]\). For each interval, the analytic continuation of the Lauricella function, equation (B.3), must be used in order to express the full integral. To give a concrete example of the procedure, we will at first focus on the case \( N = 3 \). We need to consider the following integral:

\[
I(q, i\omega_n) = \int_0^1 dt \, t^{-\eta} (1 - t)^{\eta - 3/2} F_1 \left( \frac{1}{2}; \eta_1, \eta_2; \eta; \frac{u_3^2 - u_1^2}{u_3^2} \left( 1 - \frac{\omega_n^2 t}{\omega_n^2 + u_3^2 q^2} \right) \right), \]

and its analytic continuation \( I(q, i\omega_n \to \omega + i\epsilon) \). For \( \omega < u_1 q \), we have

\[
I(q, \omega) = \int_0^1 dt \, t^{-\eta} (1 - t)^{\eta - 3/2} F_1 \left( \frac{1}{2}; \eta_1, \eta_2; \eta; \frac{u_3^2 - u_1^2}{u_3^2} \left( 1 + \frac{\omega^2 t}{u_3^2 q^2 - \omega^2} \right) \right). \]

Let us now consider the case of \( u_1 q < \omega < u_2 q \). First, we split the integral with the rule given above and consider the intervals \( t \in [0, ((u_3 q)^2 / \omega^2 - 1) / (u_3^2 / u_1^2 - 1)] \) and \( t \in [((u_3 q)^2 / \omega^2 - 1) / (u_3^2 / u_1^2 - 1), 1] \). In the first interval, the analytic continuation is straightforward. In the second interval, we must use (A.4).

To calculate the integrals, it is convenient to perform the following change of variables in the first and second interval respectively:

\[
t = \frac{u_3 q / \omega - 1}{u_3 / u_1 - 1} s_1, \quad t = \frac{u_3 q / \omega - 1}{u_3 / u_1 - 1} + \left( 1 - \frac{u_3 q / \omega - 1}{u_3 / u_1 - 1} \right) s_2.
\]

In the integration over \( s_1 \), we do not need an analytic continuation of the Appell function. In the \( s_2 \) integration, one of the two variables is larger than 1 and we find the analytic continuation using equation (A.4).

Since we are interested in calculating the imaginary part of (44), only the term proportional to \( e^{i\pi \eta_1} \) gives a contribution and we can perform the integral explicitly when...
Im \chi = \begin{array}{c}
\frac{\pi}{\Gamma(1/2)\Gamma(1/2 + \eta - \eta_3)} \frac{(u_2^2/u_3^2)^{\eta-1/2-n_3}}{((u_3^2 - u_1^2)/u_3^2)^{\eta_3-1}} ((u_2^2 - u_4^2)/u_3^2)^{\eta_2} \\
\times \left( \frac{\omega^2 - 1}{(q^2 - \omega^2/u_3^2)} \right)^{2n_3-1-n_1} \left( \frac{u_2^3}{u_1^3} - \frac{1}{u_2^3} \right)^{1/2} \left( \frac{u_2^3/q^2}{\omega^2 - 1} \right)^{1/2} \\
\times \int_0^1 \frac{ds}{ds} s^{\eta_3-1/2}(1-s)^{\eta_3-3/2} \left( \frac{(\omega^2 - 1)}{(q^2 - \omega^2/u_3^2)} \right)^{\eta_3} \left( 1 + \left( \frac{(\omega^2 - u_2^2q^2)}{(u_3^2q^2 - \omega^2)} \right)s \right)^{\eta_3-1} \\
\times F_1 \left( 1 - \eta_1; 1/2, \eta_2; \eta_3 - 1 - \eta_1; \frac{\omega^2 - u_2^2q^2}{u_3^2q^2 - \omega^2} s; \frac{u_2^3}{u_1^3} \frac{u_2^3}{u_3^2q^2} - \frac{1}{u_2^3} \right). \tag{46}
\end{array}

When \omega \to u_1q, the integral is behaving as \((\omega^2 - u_1^2q^2)^{2\eta_3-1-n_1}, in agreement with (22). For \omega \to u_2q, we need to consider the behavior of the Appell function as one of its arguments is going to unity, while the other is negative. Using the results from appendix C, we find that when \eta_1 < 1/2, we can use equation (C.8) to approximate equation (46) as

\begin{align*}
\mathfrak{Im}(q, \omega) \simeq & \int_0^1 ds s^{n_3-3/2} \left( \frac{u_2^3}{u_1^3} - \frac{1}{u_2^3} \left( u_2^3q^2 - \omega^2 \right) - s \right)^{\eta_3-1/2} \\
& \times \left( \frac{u_2^3}{u_1^3} - \frac{1}{u_2^3} \left( u_3^3q^2 - \omega^2 \right) \right)^{2\eta_3-1} \\
& \simeq \left( \frac{u_2^3}{u_1^3} - \frac{1}{u_2^3} \left( u_3^3q^2 - \omega^2 \right) \right) \left( \frac{u_2^3}{u_1^3} - \frac{1}{u_2^3} \left( u_3^3q^2 - \omega^2 \right) \right)^{2\eta_3-1}. \tag{47}
\end{align*}

Again, this result is in agreement with the power-law divergence expected from (22). The expression (47) can be calculated numerically, and the result is plotted in figure 2 for a choice of parameters \(u_1 = 0.25, u_2 = 0.75, u_3 = 1, \eta_1 = 0.3, \eta_2 = 0.1, \eta = 0.6\).

We now turn to the case \(u_2q < \omega < u_3q\). In this case, we have to split the integral in equation (44) into three integrations. The first one, on \([0, (u_3q/\omega)^2 - 1/(u_3/u_1)^2 - 1], does not contribute to the imaginary part. The second one, on the interval \([(u_2q/\omega)^2 - 1/(u_3/u_1)^2 - 1, (u_3q/\omega)^2 - 1/(u_3/u_2)^2 - 1], requires an analytic continuation of the \(F_1\)
function using (A.4) and gives a contribution to the imaginary part

\[
\frac{\pi \Gamma(\eta)(1 - (u_1/u_3)^2)^{1/2}}{\Gamma(1/2)\Gamma(\eta_1)\Gamma(1/2 + \eta - \eta_1)} \left( \frac{u_3^2}{u_1^2} \right)^{2-\eta} \left( \frac{u_2^2 - u_1^2}{u_3^2 - u_2^2} \right)^{\eta - \eta_1 + 1/2} \\
\times \left( \frac{u_3^2 - u_1^2}{u_3^2 - u_2^2} \right)^{\eta_1} \left( \frac{(u_3 q)^2 - \omega^2}{\omega^2 - (u_1 q)^2} \right)^{\eta - 1} \left( 1 - \frac{u_1^2 q^2}{\omega^2} \right)^{-1/2} \\
\times \int_0^1 ds \, s^{\eta - 1/2} \left( 1 + \frac{u_2^2 - u_1^2}{u_3^2 - u_2^2} \right)^{1-\eta} \left( 1 + \frac{u_2^2 u_2 - u_1^2}{u_3^2 u_3 - u_2^2} \right)^{-\eta} \\
\times \left( 1 - \frac{u_2^2}{u_3^2} \frac{(u_3 q)^2 - \omega^2}{(u_1 q)^2} \right)^{\eta - 3/2} \\
\times F_1 \left( 1 - \eta_1; \frac{1}{2}, \eta_2; \eta + 1 - \eta_1; \frac{u_2^2 - u_1^2}{u_3^2 - u_2^2} s, s \right). \quad (48)
\]

The last integration, on \([u_3 q/\omega)^2 - 1/(u_3/u_2)^2 - 1, 1]\], requires an analytic continuation of \(F_1\) using (A.7), and contributes two terms. The first one is

\[
\frac{\pi \Gamma(\eta)\Gamma(1 - \eta_2)|\omega|((u_3^2)^{2\eta - m - 1}(u_2^2)^{-\eta}(u_3^2)^{m + 1/2 - \eta}}{\Gamma(1/2)\Gamma(\eta_1)\Gamma(\eta - 1/2)\Gamma(1 - \eta_1 - \eta_2)} \\
\times \left( \frac{u_3^2 - u_1^2}{u_3^2 - u_2^2} \right)^{1-\eta} \left( \frac{\omega^2 - (u_2 q)^2}{(u_3 q)^2 - \omega^2} \right)^{m - 1/2} \\
\times \int_0^1 ds \, (1 - s)^{\eta - 3/2} \left( 1 + \frac{u_3^2 \omega^2 - (u_2 q)^2}{u_3^2 (u_3 q)^2 - \omega^2} \right)^{1-\eta} \\
\times \left( 1 + \frac{\omega^2 - (u_2 q)^2}{(u_3 q)^2 - \omega^2} \right)^{-1/2} \left( 1 + \frac{u_3^2 \omega^2 - (u_2 q)^2}{u_3^2 (u_3 q)^2 - \omega^2} \right)^{\eta - 1/2} \\
\times F_1 \left( 1 - \eta_1; \frac{1}{2}, \frac{3}{2} - \eta; \frac{3}{2} - \eta_1 - \eta_2; \frac{u_3^2 - u_2^2}{u_3^2 - u_2^2} s, s \right) \\
\times \left( 1 + s((u_3^2 - u_1^2)(\omega^2 - (u_2 q)^2)) / [[(u_3^2 - u_1^2)(\omega^2 - (u_2 q)^2)] \right) \quad (49)
\]

and for \(\eta - \eta_2 - 1/2 < 0\) behaves as \((\omega^2 - (u_2 q)^2)^{2\eta_2 - m - 1}\) in agreement with (22). The second one is

\[
\frac{\Gamma(\eta)\Gamma(1 - \eta_2) \sin[\pi (\eta + \eta_2)]}{\Gamma(1/2)\Gamma(1/2 + \eta - \eta_2)} \left( \frac{u_3^2 - u_2^2}{u_3^2 - u_1^2} \right)^{\eta_1} \left( \frac{\omega^2 - u_3^2}{u_3^2 (u_3 q)^2 - \omega^2} \right)^{\eta} \\
\times \left( \frac{u_3^2 \omega^2 - (u_2 q)^2}{\omega^2} \right)^{\eta - 1/2} \left( \frac{\omega^2 - (u_2 q)^2}{(u_3 q)^2 - \omega^2} \right)^{\eta_2 - 1/2} \\
\times \int_0^1 s^{\eta - m - 1/2} (1 - s)^{-\eta - 3/2} \left( 1 + \frac{u_3^2 \omega^2 - (u_2 q)^2}{u_3^2 (u_3 q)^2 - \omega^2} \right)^{\eta} \\
\times \left( 1 + \frac{\omega^2 - (u_2 q)^2}{(u_3 q)^2 - \omega^2} \right)^{-1/2} \left( 1 + \frac{u_3^2 \omega^2 - (u_2 q)^2}{u_3^2 (u_3 q)^2 - \omega^2} \right)^{\eta - 1/2} \\
\times F_1 \left( 1 - \eta_2; \frac{1}{2}, \eta; \frac{1}{2} - \eta_2; \frac{u_3^2 - u_2^2}{u_3^2 - u_2^2} s, s \right) \\
\times \left( \frac{\omega^2 - (u_2 q)^2}{(u_3 q)^2 - \omega^2} s, - \frac{u_3^2 - u_2^2}{u_3^2 - u_2^2} (u_3 q)^2 - \omega^2 s) \right). \quad (50)
\]
The latter term contributes a divergence \((\omega^2 - (u_2q)^2)^{2\eta - \eta_2 - 1}\) as \(\omega \to u_2q + 0\) in agreement with (22).

To summarize, the qualitative behavior of the spectral function is the same as in the special case of \(\eta = 1/2\). The spectral function has a threshold at \(\omega = u_1q\), and has power-law singularities for \(\omega = u_jq\) with an exponent given by equation (22).

The case of a general \(N\) is treated in appendix D. It is found that the leading singularity for \(\omega \to u_jq\) is again a power-law divergence, with exponent given by (22).

3. Conclusion

We derived analytical expressions for the zero temperature Fourier transform of the density–density correlation function and the bosonic Green’s function of a multicomponent Luttinger liquid with different velocities. By using both a Schwinger identity and a generalized Feynman identity, we derived exact integral representations while an approximate analytical form was given for frequencies close to the characteristic frequencies of the different collective modes of the system. We derived in detail the analytic continuation for generic \(N\) and discussed, as an example, the case \(N = 3\). Power-law singularities are found every time the frequency is equal to the characteristic frequency of a collective mode \((\omega_j(q) \sim u_jq)\), with the same exponent \(2\eta - \eta_j - 1\), but a different weight when approaching the singularity from the left or from the right. The power-law singularity replaces the expected delta function for noninteracting particles. Moreover, if the characteristic exponent at the singularity becomes negative, as in the case of systems with attractive interactions when considering density–density correlations or in systems with repulsion when considering bosonic Green’s functions, a cusp is expected to replace the power-law divergence. All these results are valid in the ground state. For nonzero temperature \(T\), the power-law divergence at \(\omega = u_jq\) is replaced by a maximum diverging as \(T^{2\eta - \eta_j - 1}\) as \(T \to 0\). In the vicinity of the maximum, from simple scaling, we expect \(\Im \chi(q, \omega) \sim (k_B T)^{2\eta - \eta_j - 1} f_{\text{temp}}[|\omega - u_jq|/k_B T],\) with \(f_{\text{temp}}(x \gg 1) \sim x^{2\eta - \eta_j - 1}\). If we now turn to a zero temperature system of finite length \(L\), we expect the power-law divergence at \(\omega = u_jq\) to be replaced by a maximum diverging as \((1/L)^{2\eta - \eta_j - 1}\), and in the vicinity of the maximum \(\Im \chi(q, \omega) \sim (1/L)^{2\eta - \eta_j - 1} f_{\text{len}}[L(\omega/u_j - q)],\) with \(f_{\text{len}}(x \gg 1) \sim x^{2\eta - \eta_j - 1}\). If the calculation of both functions \(f_{\text{len}}\) and \(f_{\text{temp}}\) remains an open problem, the scaling arguments suggest that the power-law behavior of \(\Im \chi(q, \omega)\) is observable for finite temperature and finite size provided \(|\omega - u_jq| \gg k_B T, u_j/L\). Such behavior could be probed by Bragg or time of flight spectroscopy in the case of atomic gases of mixed species or by the inelastic neutron scattering technique in one quantum magnets with orbital and spin modes. An obvious extension of our results is the calculation of fermion (or anyon) spectral functions in multicomponent Luttinger liquids. In the case of a two-component liquid, the fermion spectral functions are expressible in terms of Appell hypergeometric functions [81]. For the case of three or more components, our results hint that the fermion spectral functions should be expressible as Srivastava–Daoust hypergeometric functions or a suitable generalization.

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Appendix A. Analytic continuation of the Appell $F_1$ function

In the calculation of the response function, an analytic continuation of the Lauricella hypergeometric function $F_D^{(N-1)}$ is necessary. In this appendix, we present the analytic continuation of the Appell $F_1$ hypergeometric function that corresponds to the particular case of $N = 3$. The analysis of this case is a stepping stone for the case of general $N$.

In order to find the analytic continuation, we start from the integral representation of the Appell hypergeometric function [78, 79]

$$F_1(a; b_1, b_2; c; z_1, z_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tz_1)^{b_1}(1-tz_2)^{b_2}}. \quad (A.1)$$

When $|z_1|, |z_2| < 1$, expanding in the series (A.1) and integrating w.r.t. $t$ gives back the series expansion. We wish to use (A.1) to express $\lim_{x_1,x_2 \to a_+} F_1(a; b_1, b_2; c; x_1 + i\epsilon_1, x_2 + i\epsilon_2)$ for $x_1$ and $x_2$ real for the cases $x_2 < 1 < x_1$ and $1 < x_2 < x_1$.

A.1. The case $x_2 < 1 < x_1$

In order to calculate the integral (A.1), we need to take into account the sole branch cut of $(1 - t(x_1 + i\epsilon_1))^{b_1}$ for $t > 1/x_1$. Because of this cut, we have for $t > x_1$, $(1 - t(x_1 + i\epsilon_1))^{b_1} = e^{-i\pi b_1 (tx_1 - 1)^{b_1}}$. Therefore, we split the integral in (A.1) into two integrations on $[0, 1/x_1]$ and $[1/x_1, 1]$. After a change of variable $t = s/x_1$, we find for the $[0, x_1]$ integral

$$\int_0^{1/x_1} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx_1)^{b_1}(1-tx_2)^{b_2}} = \frac{1}{x_1^a} \frac{\Gamma(a)\Gamma(1-b_1)}{\Gamma(1+a-b_1)} \times F_1(a; a + 1 - c, b_2; a + 1 - b_1; 1/x_1; x_2/x_1), \quad (A.2)$$

a purely real expression. Then, for the $[1/x_1, 1]$ integration, we find

$$\int_{1/x_1}^1 dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t(x_1 + i0_+))^b_1(1-tx_2)^{b_2}} = e^{i\pi b_1} \frac{x_1^{c-b_2-1}(x_1 - x_2)^{b_2}}{\Gamma(1 + c - a - b_1)} \times F_1 \left(1 - b_1; 1 - a; b_2; c - a + 1 - b_1; 1 - x_1, \frac{x_2(x_1 - 1)}{x_1 - x_2} \right), \quad (A.3)$$

where we have used the linear change of variables $t = 1/x_1 + s(1 - 1/x_1)$. Our result for the analytic continuation is then

$$F_1(a; b_1, b_2; c; x_1 + i0_+, x_2) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(1+a-b_1)} \frac{1}{x_1^a} \times F_1(a; a + 1 - c, b_2; a + 1 - b_1; 1/x_1; x_2/x_1)$$

$$+ e^{i\pi b_1} \frac{\Gamma(c)\Gamma(1-b_1)}{\Gamma(1 + c - a - b_1)} \frac{(x_1 - 1)^{c-a-b_1}}{x_1^{c-b_2-1}(x_1 - x_2)^{b_2}} \times F_1 \left(1 - b_1; 1 - a; b_2; c - a + 1 - b_1; 1 - x_1, \frac{x_2(x_1 - 1)}{x_1 - x_2} \right). \quad (A.4)$$

A.2. The case $1 < x_2 < x_1$

In this case, we need to consider both the branch cut of $(1 - t(x_1 + i0_+))^{b_1}$ and that of $(1 - t(x_2 + i0_+))^{b_2}$. Thus, we are led to split the integral (A.1) into three integrations over...
\[ [0, 1/x_1], \ [1/x_1, 1/x_2] \text{ and } [1/x_2, 1]. \] The first of these integrals is still given by (A.2). The second integral is given by
\[
\int_{1/x_1}^{1/x_2} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t(x_1+i0_+))^{b_1}(1-t(x_2+i0_+))^{b_2}} = e^{i\pi b_1} \frac{\Gamma(1-b_1)\Gamma(1-b_2) (x_1-x_2)^{1-b_1-b_2}(x_1-1)^{c-a-1}}{\Gamma(2-b_1-b_2)} x_1^{c-b_2-1}x_2^{b_1-1} \times F_1 \left(1-b_1; 1-a, 1+a-c; 2-b_1-b_2; 1-x_1, x_1-x_2 \bigg/ x_2(x_1-1) \right),
\] (A.5) where we have used the change of variable \( t = 1/x_1+s(1/x_2-1/x_1) \) and taken into account the branch cut of \((1-t(x_1+i0_+))^{b_1}\). For the third integral, we have
\[
\int_{1/x_2}^{1} dt \frac{t^{a-1}(1-t)^{c-a-1}}{(1-t(x_1+i0_+))^{b_1}(1-t(x_2+i0_+))^{b_2}} = e^{i\pi(b_1+b_2)} \frac{\Gamma(1-b_1)\Gamma(1-b_2) (x_2-1)^{c-a-b_2}}{\Gamma(1+c-a-b_2)} x_2^{a-b_1}(x_1-x_2)^{b_1} \times F_1 \left(1-b_2; 1-a, b_1; 1+c-a-b_2; 1-x_2, x_1(1-x_2) \bigg/ x_1-x_2 \right),
\] (A.6) where we have taken both branch cuts into account, and we have used the change of variables \( t = 1/x_2 + (1-1/x_2)s \). The final result is
\[
F_1(a, b_1, b_2; c; x_1+i0_+, x_2+i0_+) = \frac{\Gamma(c)\Gamma(1-b_1)}{\Gamma(c-a)\Gamma(1+a-b_1)} \frac{1}{x_1^a} = F_1 \left(a; 1+a-c, b_2; 1+a-b_1; 1-x_1 \bigg/ x_1 \right) + e^{i\pi b_1} \frac{\Gamma(c)\Gamma(1-b_1)\Gamma(1-b_2)}{\Gamma(a)\Gamma(c-a)\Gamma(2-b_1-b_2)} (x_1-x_2)^{1-b_1-b_2}(x_1-1)^{c-a-1} \times F_1 \left(1-b_1; 1-a, 1+a-c; 2-b_1-b_2; 1-x_1 \bigg/ x_2(x_1-1) \right) + e^{i\pi(b_1+b_2)} \frac{\Gamma(c)\Gamma(1-b_2)}{\Gamma(a+c-a-b_2)} x_2^{a-b_1}(x_1-x_2)^{b_1} \times F_1 \left(1-b_2; 1-a, b_1; 1+c-a-b_2; 1-x_2, x_1(1-x_2) \bigg/ x_1-x_2 \right).
\] (A.7)

For the case \( N = 3 \), expressions (A.4) and (A.7) must be injected into the integral (32) after analytic continuation to yield the response function.

**Appendix B. Analytic continuation of the Lauricella \( F_D \) function**

The Lauricella \( F_D \) function has the integral representation
\[
F_D(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 dt \frac{t^{a-1}(1-t)^{c-a-1}}{\prod_{j=1}^n(1-z_jt)^{b_j}}.
\] (B.1)
If we want to calculate
\[
\lim_{\epsilon_i \to 0} F_D(a; b_1, \ldots, b_n; x_1 + i \epsilon_1, \ldots, x_n + i \epsilon_n)
\] (B.2)
for \(x_n < x_{n-1} < \cdots < x_2 < x_1\), we have to consider the cuts of the functions \((1 - t x_j + i 0_-)^{-b_j}\). We are thus led to consider \(n\) cases separately, i.e. \(x_n < \cdots < x_{j+1} < 1 < x_j < \cdots < x_1\) with \(j = 1, \ldots, n - 1\) and \(1 < x_n < \cdots < x_1\).

In the case of \(x_n < \cdots < x_{j+1} < 1 < x_j < \cdots < x_1\), the integral (B.1) has to be split into \(j + 1\) integrations over the intervals \([0, 1/x_1], [1/x_1, 1/x_2], \ldots, [1/x_j, 1]\). In analogy to the case of the Appell \(F_1\) function, each integral gives a contribution proportional to a Lauricella \(F_D\) function. The resulting expression is
\[
F_D^{(N)}(a; b_1, \ldots, b_N; x_1 + i 0, \ldots, x_N + i 0) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(c - a)} \left[ \frac{\Gamma(a) \Gamma(1 - b_1)}{\Gamma(1 + a - b_1)} x_1^a \right] F_D^{(N)}(a; 1 + a - c, b_2, \ldots, b_N; 1 + a - b_1; \frac{x_2}{x_1}, \frac{x_3}{x_1}, \ldots, \frac{x_N}{x_1})
\]
\[
+ \sum_{m=1}^{j-1} \frac{\Gamma(1 - b_m) \Gamma(1 - b_{m+1})}{\Gamma(2 - b_m - b_{m+1})} (x_m - x_{m+1})^{1 - b_m - b_{m+1}} \left(1 - x_m\right)^{c - a - 1} \prod_{l \neq m, m+1} \left| \frac{x_l - x_m}{x_m} \right|^{-b_l} e^{i \pi \sum_{i=1}^{m} b_i} F_D^{(N)}(1 - b_m; b_1, \ldots, b_{m-1}, 1 - a, 1 + a - c, b_{m+2}, \ldots, b_N; 2 - b_m - b_{m+1}; \frac{1 - x_m/x_{m+1}}{1 - x_m/x_1}, \ldots, \frac{1 - x_m/x_{m-1}}{1 - x_m/x_{m+1}}, \frac{1 - x_m}{1 - x_m/x_{m+1}}, \frac{1 - x_m/x_{m+1}}{1 - x_m/x_{m+2}}, \ldots, \frac{1 - x_m/x_N}{1 - x_m/x_{m+1}}) \]
\[
+ \frac{\Gamma(1 - b_j) \Gamma(c - a)}{\Gamma(1 + c - a - b_j)} \frac{1}{x_j^{c-1}} \prod_{l \neq j} \left| \frac{x_l - x_j}{x_j} \right|^{-b_l} e^{i \pi \sum_{i=1}^{j} b_i} F_D^{(N)}(1 - b_j; b_1, \ldots, b_{j-1}, 1 - a, b_{j+1}, \ldots, b_N; 1 + c - a - b_j; \frac{1 - x_j}{1 - x_j/x_1}, \ldots, \frac{1 - x_j}{1 - x_j/x_{j-1}}, \frac{1 - x_j}{1 - x_j/x_{j+1}}, \ldots, \frac{1 - x_j}{1 - x_j/x_N}).
\] (B.3)
By reducing to \(N = 2\) it can be checked that the results of section A are recovered.

In the case \(1 < x_1 < \cdots < x_n\), we have to split the integral (B.1) into \(n + 1\) integrations over the intervals \([0, 1/x_1], [1/x_1, 1/x_2], \ldots, [1/x_n, 1]\). Each integration contributes a term proportional to a Lauricella \(F_D\) function.

**Appendix C. Asymptotic expansion of the Appell \(F_1\) function**

We wish to obtain an asymptotic expansion of the Appell function \(F_1(a; b_1, b_2; c; x_1, x_2)\) in the case of \(x_2 < 0\) and \(x_1 \to 1_-\). First, we need to obtain an expression of the Appell function in the form of a convergent series for all \(x_2 < 0\). We consider the series expansion

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for the Appell hypergeometric function $F_1$,

$$F_1(a; b_1, b_2; c; x_1, x_2) = \sum_{n_1, n_2} \frac{(a)_{n_1+n_2} (b_1)_{n_1} (b_2)_{n_2}}{(c)_{n_1+n_2}} \frac{x_1^{n_1} x_2^{n_2}}{n_1! n_2!} = \sum_{n_1} \frac{(a)_{n_1} (b_1)^{n_1} x_1^{n_1}}{(c)_{n_1}} \frac{2F_1(a + n_1, b_2; c + n_1; x_2)}{n_1!} \tag{C.1}$$

where we have used the notation [84]

$$(a)_n = \frac{\Gamma(n + a)}{\Gamma(a)}. \tag{C.3}$$

Using the second line of (C.1), we can define the function $F_1$ for all $x_2 \not\in [1, +\infty[$. For $x_2 < 0$, we can use equation (15.3.4) from [84] to rewrite

$$F_1(a; b_1, b_2; c; x_1, x_2) = (1 - x_2)^{-b_2} \sum_{n_1} \frac{(a)_{n_1} (b_1)^{n_1}}{(c)_{n_1}} \frac{x_1^{n_1}}{n_1!} \frac{2F_1(b_2, c - a; c + n_1; x_2/x_2 - 1)}{2F_1(a + n_1, b_2; c + n_1; x_2)}, \tag{C.4}$$

where we have used the convergence of the Gauss hypergeometric series to obtain the last two lines. With this convergent series, we can analyze its behavior as $x_1 \to 1_-$. First, when $c + m - a - b_1 > 0$, we have from equation (15.1.20) in [84]

$$\lim_{x \to 1^-} 2F_1(a, b_1; c + m; x_1) = \frac{\Gamma(c + m)\Gamma(c + m - a - b_1)}{\Gamma(c + m - a)\Gamma(c + m - b_1)}, \tag{C.6}$$

When $c + m - a - b_1 < 0$, using the first equation of (15.3.3) in [84], we find that as $x \to 1_-$,

$$2F_1(a, b_1; c + m; x_1) \sim (1 - x_1)^{-c - a - b_1 + m} \frac{\Gamma(c + m)\Gamma(a + b_1 - c - m)}{\Gamma(a)\Gamma(b_1)}. \tag{C.7}$$

Thus, when $c - a - b_1 < 0$, we have that

$$F_1(a; b_1, b_2; c; x_1, x_2) \sim (1 - x_2)^{-b_2} \frac{\Gamma(c)\Gamma(a + b_1 - c)}{\Gamma(a)\Gamma(b_1)} (1 - x_1)^{c - a - b_1}, \tag{C.8}$$

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while for $c - a - b_1 > 0$

$$
\lim_{x_1 \to 1} F_1(a; b_1; b_2; c; x_1, x_2) = (1 - x_2)^{-b_2} \frac{\Gamma(c) \Gamma(c + a - b_1)}{\Gamma(c - a) \Gamma(c - b_1)} \times \frac{\Gamma(c)}{\Gamma(c + a)} \times \frac{\Gamma(c)}{\Gamma(c + b_1)} \\
\times \sum_{n=0}^\infty \frac{\Gamma(n + a)}{\Gamma(n + c)} \frac{\Gamma(n + b_1)}{\Gamma(n + c - b_1)} \frac{\Gamma(n + a)}{\Gamma(n + c)} \times \frac{\Gamma(c)}{\Gamma(c + a)} \times \frac{\Gamma(c)}{\Gamma(c + b_1)}
$$

Let us now return to the case of a general $N$. We have to consider the integral

$$
I^{(N-1)}(q, \omega) = \int_0^1 dt t^{-\eta} (1 - t)^{-3/2} F_D^{(N-1)} \left( \frac{1}{2}; \{\eta_j\}_{1 \leq j \leq N-1}; \frac{t}{u_N^2 q^2/(\omega + i0)^2 - 1} \right)_{1 \leq j \leq N-1}
$$

(C.9)

Appendix D. Analytic continuation of the Matsubara correlator for generic $\eta$ and $N$ components

With the help of equation (B.3) we can write for $t_i < t < t_{i+1}$

$$
F_D^{(N-1)} \left( \frac{1}{2}; \{\eta_j\}_{1 \leq j \leq N-1}; \frac{t}{u_N^2 q^2/(\omega + i0)^2 - 1} \right)_{1 \leq j \leq N-1}
$$

$$
= \sum_{m=0}^{l-1} \varphi_m(t) + \psi_l(t),
$$

(D.1)

where

$$
\varphi_0(t) = \frac{\Gamma(\eta) \Gamma(1 - \eta_1)}{\Gamma(\eta - 1/2) \Gamma(3/2 - \eta_1)} \left[ (1 - u_1^2/u_N^2) (1 + t/(u_N^2 q^2/(\omega + i0)^2 - 1)) \right]^{1/2}
$$

$$
\times F_D^{(N-1)} \left( \frac{1}{2}; \{\eta_j\}_{1 \leq j \leq N-1}; \frac{t}{u_N^2 q^2/(\omega + i0)^2 - 1} \right)_{1 \leq j \leq N-1}
$$

$$
= \frac{1}{(1 - u_1^2/u_N^2) (1 + t/(u_N^2 q^2/(\omega^2 - 1)))} \left\{ \frac{u_N^2 - u_1^2}{u_N^2 - u_1^2} \right\}_{2 \leq j \leq N-1}.
$$

(D.2)

For $m \geq 1$

$$
\varphi_m(t) = \frac{\Gamma(\eta) \Gamma(1 - \eta_m) \Gamma(1 - \eta_{m+1})}{\Gamma(1/2) \Gamma(\eta - 1/2) \Gamma(2 - \eta_m - \eta_{m+1})} \left( \frac{u_{m+1}^2 - u_m^2}{u_N^2 - u_m^2} \right)^{1-\eta_m}
$$

$$
\times \left\{ \frac{u_{m+1}^2 - u_m^2}{u_N^2 - u_m^2} \right\}^{-\eta_{m+1}} \prod_{k \neq m, m+1} \left[ \frac{u_k^2 - u_m^2}{u_N^2 - u_m^2} \right]^{-\eta_k} e^{i \pi \sum \eta_k}
$$

$$
\times [1 - (1 - u_1^2/u_N^2) (1 + t/(u_N^2 q^2/(\omega^2 - 1)))]^{\eta-3/2}
$$

$$
\times F_D^{(N-1)} \left( 1 - \eta_m, \{\eta_k\}_{1 \leq k \leq N-1}; \frac{1}{2}, \frac{1}{2} - \eta; \{\eta_k\}_{m+2 \leq k \leq N-1}; 2 - \eta_m - \eta_{m+1} \right);
$$

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\[
\psi(l) = \Gamma(1, \eta_l) / \Gamma(\eta_l + 1/2) \prod_{k \neq l} \left| u_2^l - u_2^k \right|^{-\eta_k} e^{i \pi \sum_k \eta_k} \times \left[ \left( 1 - \frac{u_2^l}{u_2^k} \right) \left( 1 + \frac{t}{u_2^l q^2 / \omega^2 - 1} \right) \right]^{1/2} F_N^{(N-1)} \left( 1 - \eta, \{ \eta_k \}_{1 \leq k \leq l-1, \frac{1}{2}, \{ \eta_k \}_{l+1 \leq k \leq N-1; \frac{1}{2} + \eta - \eta_l} \left( \frac{u_2^l - u_2^k}{u_2^l} \right) \times \left( 1 - \frac{t}{u_2^l q^2 / \omega^2 - 1} \right) \right) \right]_{1 \leq k \leq N-1}.
\]

so that for \( u_l q < \omega < u_{l+1} q \)

\[
I^{(N-1)}(q, \omega) = \sum_{m=0}^{j-1} \int_{t_1}^{t_1} dt \ t^{-\eta} (1-t)^{\eta-3/2} \left( \sum_{m=0}^{j-1} \varphi_m(t) + \psi(t) \right) + \int_{t_1}^{t_j} dt \ t^{-\eta} (1-t)^{\eta-3/2} \left( \sum_{m=0}^{j-1} \varphi_m(t) + \psi(t) \right) + \int_{0}^{t_1} dt \ t^{-\eta} (1-t)^{\eta-3/2} F_N^{(N-1)} \left( \frac{1}{2}; \{ \eta_j \}_{1 \leq j \leq N-1; \eta_l}; \left( \frac{u_2^l - u_2^k}{u_2^l} \right) \times \left( 1 + \frac{t}{u_2^l q^2 / \omega^2 - 1} \right) \right) \right]_{1 \leq j \leq N-1}.
\]

We can rearrange that sum into

\[
I^{(N-1)}(q, \omega) = \sum_{m=0}^{j-1} \int_{t_1}^{t_1} dt \ t^{-\eta} (1-t)^{\eta-3/2} \varphi_m(t) + \sum_{l=1}^{j-1} \int_{t_1}^{t_l} dt \ t^{-\eta} (1-t)^{\eta-3/2} \psi(t) + \int_{t_1}^{t_j} dt \ t^{-\eta} (1-t)^{\eta-3/2} \psi(t) + \int_{0}^{t_1} dt \ t^{-\eta} (1-t)^{\eta-3/2} F_N^{(N-1)} \left( \frac{1}{2}; \{ \eta_j \}_{1 \leq j \leq N-1; \eta_l}; \left( \frac{u_2^l - u_2^k}{u_2^l} \right) \times \left( 1 + \frac{t}{u_2^l q^2 / \omega^2 - 1} \right) \right) \right]_{1 \leq j \leq N-1}.
\]
Therefore, we have to calculate $2j$ integrals. When in the last integral we do substitute the expression of (D.4), a factor $(\omega^2 - (u_jq)^2)^{2j+1-1}$ appears with an integral that is regular in the limit $\omega \to u_jq$, so the asymptotic behavior previously predicted is recovered. The first class of integrals in (D.6) can be, instead, manipulated by using the definition (D.3) and performing the following change of variables:

$$t = \frac{((u_N q)^2/\omega^2 - 1)}{(u_N)^2/\omega_{m+1}^2 - 1} + \left(1 - \frac{((u_N q)^2/\omega^2 - 1)}{(u_N)^2/\omega_{m+1}^2 - 1}\right) s.$$  \hspace{1cm} (D.7)

The integrals turn out to be regular for $\omega > u_jq$ and for $\omega \to u_jq + 0$ they behave as $(\omega^2 - (u_jq)^2)^{j-1/2}$, giving only a subdominant contribution.

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