Isolated Minkowski vacua, and stability analysis for an extended brane in the rugby ball

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We study a recently proposed model, where a codimension one brane is wrapped around the axis of symmetry of an internal two dimensional space compactified by a flux. This construction is free from the problems which plague delta-like, codimension two branes, where only tension can be present. In contrast, arbitrary fields can be localized on this extended brane, and their gravitational interaction is standard 4d gravity at large distance. In the first part of this note, we study the de Sitter (dS) vacua of the model. The landscape of these vacua is characterized by discrete points labeled by two integer numbers, related to the flux responsible for the compactification and to the current of a brane field. A Minkowski external space emerges only for a special ratio between these two integers, and it is therefore (topologically) isolated from the nearby dS solutions. In the second part, we show that the Minkowski vacua are stable under the most generic axially-symmetric perturbations (we argue that this is sufficient to ensure the overall stability).

I. INTRODUCTION

Models with two extra dimensions have been the object of many studies. Among them, there are the first examples of supersymmetric compactifications to four dimensional Minkowski space [1], where a stable spherical internal space is achieved through the flux of a gauge field and a cosmological constant. More recently, six dimensional models have been reconsidered in the brane-world scenario [2], since two is the minimum number of flat extra dimensions to have a fundamental scale of order TeV and a compactification of (sub)millimeter range. The background solution of [1] can be easily extended to a 6d braneworld. If one imposes that the internal and the external coordinates are factorized, then the presence of a brane modifies the internal geometry by creating a deficit angle [3], in the same way as a point mass modifies a two dimensional space [4]. In this way, the spherical internal space of [1] is modified to a (so called) rugby ball, with conical singularities at the two poles generated by the codimension two brane and its $Z_2$ image. It is manifest from the solution that the only quantity related to the brane tension is the deficit angle in the bulk, so that one may hope that this construction can provide a self tuning mechanism responsible for the vanishing of the 4d cosmological constant. Unfortunately, this is not the case for a number of reasons [5]. Nonetheless, the construction of [3] remains of great interest, since it is an extremely simple but complete solution of unwarped extra dimensions, which can be employed in a number of applications of physical relevance [6].

There is, however, a further problem that these applications face, related to the localization of matter and gauge fields on the codimension two branes. In field theoretical studies, matter and gauge fields on the brane are usually treated as test fields, which do not contribute to the background geometry (namely, they are neglected in the Einstein equations of the system). However, a more complete approach, with gravity also taken into account, shows that only tension can be present on the brane, while fields with a different equation of state necessary lead to worse than conical singularities at the brane location [5]. Contrary to what happens for conical singularities (where the scalar curvature is a delta function supported at the tip of the cone) such singularities are not integrable, so that the codimension two brane cannot be consistently treated even in a distributional sense. This appears as a specific example of the problem of defining sources of codimension two and higher in general relativity [8] (see also [9] for a recent discussion focused on exact codimension 2 solutions).

A way out of this problem was proposed in [10], through the addition of Gauss-Bonnet terms in the bulk, and in [11], where the codimension two brane emerges at the intersection between two codimension one branes. A possibly more conservative approach is to replace the delta-like strict codimension two brane with an extended codimension one defect [12] (extended defects on a codimension two bulk were also discussed in [13]). In this model, a region close to the singularity is replaced by a spherical cap, and a codimension one defect is placed at the junction between the two spaces (the same is done for both singularities, so to preserve the $Z_2$ symmetry of the system; moreover, the defect wraps around the main axis of symmetry of the system; the overall configuration can be seen in fig. 1 of [12]). Fields with arbitrary equation of state can be localized on the defect, and their zero modes (that is, the axially-symmetric ones) can be identified with the 4d fields of the observable sector. It was shown in [12] that the gravitational interaction between brane fields is described by Einstein 4d gravity at large distance. Therefore, this construction appears as a phenomenologically viable and complete theory of two extra dimensions, without the need of invoking higher order gravity terms, or a more complicated system of branes.

A natural extension of this construction is the study
of more general background solutions. For instance, ref. [14] embedded the codimension one brane in a Minkowski compactification characterized by a warped internal space. Alternatively, one can study cases of cosmological relevance, where the brane is embedded in a time dependent background. We perform the first step in this direction by studying and classifying the de Sitter (dS) vacua of this model. More precisely, we look for solutions characterized by a dS external geometry, and a static internal space. More general solutions can presumably be obtained analytically for small deviations from the dS or Minkowski ones; in general, the values of the bulk cosmological constants and of the brane tension determine both the dS expansion rate, and the bulk parameters, such as the compactification radius and the deficit angle; if more general sources are present, we expect a nearly static internal space, and an approximate Friedmann-Robertson-Walker cosmology, as long as their energy densities are smaller than the brane and bulk cosmological constants (this is typical of extra dimensional models with a stabilized internal space). At higher energies, we expect a strong evolution of the entire space, which can be presumably studied only through numerical computations [15].

We find two main results: first, the dS rate must be smaller than a given value, related to the inverse of the compactification radius. This is in agreement with the findings of [14], where it was observed that any compactification mechanism can stabilize the internal space only up to some given expansion rate (by causality reasons, one cannot expect compactification when the horizon size becomes parametrically smaller than the size of the internal space). A second, more surprising, result, is that the Minkowski solutions, obtained for a special relation between the bulk cosmological constants and the brane tension, are separated from the dS ones by topology. Indeed, the possible vacua of the model are labeled by two integer numbers. The first integer $N$ is related to the winding number of bulk fields charged under the gauge symmetry responsible for the compactification of the space (if present, these fields impose a quantization condition analogous to the one taking place for the Dirac monopole). This can also be seen as quantization of the flux of the gauge field in the compact space. The second integer $n$ is related to the current of a brane field; this current is necessary for matching the discontinuity of the gauge field across the two sides of the brane, and it controls the position of the brane in the internal space. Ref. [12] studied only the Minkowski solutions for the system, showing that they occur for $N/n = -2$. We find that a nonvanishing dS rate is instead achieved for greater ratios. In the Conclusions, we comment on the implications that this can have for the cosmological constant problem.

Clearly, for the construction of [12] to be of any interest, one should show that it is stable. Only the zero modes of the perturbations of this geometry were studied in [12]. In the second part of this work, we study the massive modes, to ensure that the system has no tachyonic instability (we do so only for the Minkowski vacua; from what we already mentioned, and from the stability study for the spherical compactification without branes [17], we expect that the system becomes unstable at high $H_i$).

More precisely, we concentrate only on axially symmetric perturbations (around the axis of symmetry of the background). The reason is mostly technical, since, as we will see, already this system of modes is quite involved. However, we have a second (although non rigorous) justification for this restricted study. The construction is obtained by cutting in an axially symmetric way the rugby ball and the spherical compactifications, and by joining them across the brane. Before cutting them, both these configurations are stable [1], so we expect that an instability - if any - will be related to their interface. For instance, the rugby ball may prevail over the spherical cap, so that the brane in between them would shrink towards the pole. Alternatively, the spherical cap may be favored, and the string would then extend towards the equator. Such instability would show up as a tachyonic axially symmetric mode; recently, the stability of 6D chiral gauged supergravity, including the unwarped “rugby-ball” solution, was studied in [18]. Also that analysis is restricted to axially symmetric perturbations; it is argued that the study of more general modes is unnecessary, since any angular dependence would contribute positively to the corresponding Kaluza-Klein mass.

We decompose the perturbations into scalar, vector, and tensor modes (where the names refer to how they transform with respect to transformations along the non-compact coordinates). These three sectors are decoupled at the linearized level, and they can be studied separately. The equations for the zero modes studied in [12] could be solved analytically. Unfortunately, for massive modes, only the tensor mode can be obtained analytically, while the bulk equations for the vector and scalar modes have to be solved numerically. 1 We do so with a shooting method. No tachyonic solution emerged from the (rather extensive) numerical computation, so that we can conclude

The paper is organized as follows. In Section 2 we review the model introduced in [12]. Section 3 is devoted to the study of the dS backgrounds. In Section 4 we show that the system is stable under the most general set of axially symmetric perturbations. The concluding Section 5 contains some remarks on the phenomenological

1 We are aware of two studies which are close to the present one, where the equations in the bulk could be decoupled, and then studied analytically. For a spherical bulk, the modes can be decomposed on spherical harmonics, and then decoupled due to orthogonality in the bulk [19]. This cannot be done for the “composite” bulk solution that we are investigating. Second, in the study of [18] the modes could be decoupled by using the equation for the dilaton present in the supergravity action. This mode, and the corresponding equation, is absent in our case (for other studies of linearized gravity in 6d contexts, see [18]).
II. MINKOWSKI COMPACTIFICATION

We summarize the construction of \[12\]. The action of the model is

$$S = S_o + S_i + S_b$$

$$S_{o,i} = \int d^6x \sqrt{-g} \left[ \frac{M^4}{2} R - \Lambda_{o,i} - \frac{1}{4} F^2 \right]$$

$$S_b = -\int d^5x \sqrt{-\gamma} \left[ \lambda_s + \frac{\nu^2}{2} \left( \partial \sigma - eA \right)^2 \right]$$

where \(S_b\) is the action of the brane, while \(S_{o,i}\) is the action in two bulk regions separated by the brane. The line element in the two regions is

$$\begin{align*}
\text{"out"} & : ds^2_6 = \eta_{\mu\nu} dx^\mu dx^\nu + R^2 d\theta^2 + R^2 \beta^2 \cos^2 \theta d\phi^2 \\
\text{"in"} & : ds^2_6 = \eta_{\mu\nu} dx^\mu dx^\nu + R^2 \beta^2 d\theta^2 + R^2 \beta^2 \cos^2 \theta d\phi^2
\end{align*}$$

where here and in the following \(x\) denotes the noncompact directions (notice the choice of the Minkowski metric; in the next Section we will instead consider a dS external space). The brane is at the background position \(\bar{\theta}\). The region \(0 < \theta < \bar{\theta}\), denoted as the “out” bulk, is a portion of the so call bulk compactification, characterized by the deficit angle \(1 - \beta\). The region \(\bar{\theta} < \theta < \pi/2\), denoted as “in” bulk, is a spherical cap, with the pole at \(\theta = \pi/2\). A \(Z_2\) symmetry extends this geometry to the region \(-\pi/2 < \theta < 0\). In addition, the background is axially symmetric (around the axis connecting the poles at \(\theta = \pm \pi/2\)).

The internal compactification is achieved through the gauge field configuration

$$F_{\theta\phi} = \partial_\theta A_\phi = M^2 R \beta \cos \theta$$

and the cosmological constants

$$\sqrt{2\Lambda_i} = \frac{M^2}{R \beta}, \quad \sqrt{2\Lambda_o} = \frac{M^2}{R}$$

If we suppress the external dimensions, the brane can be viewed as a string wrapped around this axis of symmetry. The brane field \(\sigma\) acts as a Goldstone boson (\(\nu\) is most easily interpreted as the vacuum expectation value of a field which breaks the \(U(1)\) symmetry on the brane). It generates a current which is necessary to provide the discontinuity of the magnetic field between the two bulk regions. From its own equation of motion, \(\sigma = n \phi\), where (due to the periodicity of the \(\phi\) coordinate) \(n\) is an integer. The brane position is then found to be \(12\)

$$\bar{\theta} = \arctan \frac{1 - \beta}{R \beta q^2}, \quad q \equiv e v$$

while the deficit angle in the out bulk is

$$1 - \beta = \frac{T}{2 M^4 \pi \sin \bar{\theta}}$$

where \(T\) is the four dimensional (i.e., after an integration along \(\phi\)) energy density of the brane. This result, in the limit \(\bar{\theta} \to \pi/2\) (when the brane shrinks to the north pole), reproduces the known relation between the tension of a codimension two brane, and the deficit angle generated by it.

To conclude the description of the background, we note that, if some field, with charge \(e\) under the \(U(1)\) symmetry is present, the deficit angle must satisfy a quantization condition

$$\beta = \frac{N}{2 e M^2 R}, \quad N \text{ integer}$$

Such quantization is also known as flux quantization, since it can be recast in the form

$$\Phi_B = \int d\theta d\phi F_{\theta\phi} = \frac{2 \pi}{e} N$$

where the integral is performed over the entire internal space.

From the Einstein equations, one then finds \(12\) that this integer \(N\) must be related to the winding \(n\) of the brane field \(\sigma\) by \(N/n = -2\). As we show in the next Section, this relation is actually due to the assumption of Minkowski noncompact space. Different ratios between these two integers result in a dS external geometry.

III. DE-SITTER COMPACTIFICATION

We now generalize the Minkowski solution described above to the case of a dS noncompact space, characterized by the expansion rate \(H\) (namely, we replace \(\eta_{\mu\nu}\) by the dS metric in the line elements \(12\)). The bulk compactification is achieved for

$$\Lambda_0 = \frac{M^4}{2 R^2} (1 + 9 H^2 R^2)$$

$$F_{\theta\phi} = M^2 R \beta \sqrt{1 - 3 H^2 R^2} \cos \theta$$

in the out bulk, and

$$\Lambda_i = \frac{M^4}{2 R^2 \beta} (1 + 9 \beta^2 H^2 R^2)$$

$$F_{\theta\phi} = M^2 R \beta \sqrt{1 - 3 H^2 R^2 \beta^2} \cos \theta$$

in the in bulk. We observe that the expansion rate cannot exceed the value of \(1/\sqrt{3} R\). This is not surprising in the light of the findings of \(10\), where it was shown that, for any dS compactification, the expansion of the external coordinates has the generic effect of destabilizing the internal space.

We still look for an axially symmetric solutions, so that \(A_\theta = 0\), while \(A_\phi\) depends only on \(\theta\). Moreover, \(A_\phi =
0 at the poles (from regularity), and $A_φ$ is continuous across the two branes (so that the brane action is well defined). This determines $A_φ$ in the bulk; the solution cannot be provided on a unique chart. In presence of a bulk field charged under this $U(1)$ symmetry, a consistent solution is possible only when the quantization condition

$$2eM^2Rβ \left\{ (1 - 3H^2R^2)^{1/2} \sin θ + (1 - 3H^2β^2R^2)^{1/2} (1 - \sin θ) \right\} = N$$

(11)

with $N$ integer, holds (this computation can be performed exactly as in the Minkowski case; see [12] for details). Also in this case, the quantization condition can be recast in the form (9).

Let us now discuss the brane equations. As for the Minkowski case, $σ = n φ$, where $n$ is an integer. By construction, the transverse metric components ($g_{µν}$ and $g_{ϕϕ}$) are already continuous across the brane. We are then left with three nontrivial brane equations (second Israel conditions, plus Ampere law, relating the discontinuity of $F_ϕ$ in the bulk to the current on the brane). With some algebra, they can be recast in the form

$$\frac{1 - β}{Rβ} = q^2 \tan θ F^2$$

(12)

$$1 + 2n = \frac{\sin θ(\sqrt{1 - 3H^2R^2} - F)}{\sqrt{1 - 3H^2R^2β^2(1 - \sin θ)} + \sqrt{1 - 3H^2β^2R^2 β}}$$

(13)

$$1 - \frac{β}{Rβ} \tan θ = \frac{2λs}{M^2}$$

(14)

where $q = e v$, and we have defined, for shortness,

$$F = \frac{1 - β}{(1 - 3H^2β^2R^2)^{1/2} (1 - 3H^2R^2)^{1/2} β}$$

(15)

To solve the above system of equations, we eliminate $\bar{θ}$ from (12) and (14), and we combine the resulting equation with the two bulk expressions for $Λ_0$ and $Λ_i$. We obtain

$$1 - β^2 = \frac{16λq^2(Λ_i - Λ_0)}{3(Λ_i - Λ_0)^2 + 2λq^2(5Λ_i - 3Λ_0) + 3λ^2q^4}$$

$$R^2 = \frac{16λ^2}{3(Λ_i - Λ_0)^2 - 2λq^2(3Λ_i - 5Λ_0) + 3λ^2q^4}$$

$$H^2 = -\frac{(Λ_i - Λ_0)^2 + 2λq^2(Λ_i + Λ_0) - λ^2q^4}{48λq^2}$$

(16)

where we have rescaled

$$\frac{2Λ_0}{M^2} → Λ_0 , \quad \frac{2Λ_i}{M^4} → Λ_i , \quad \frac{2λs}{M^2} → λ$$

(17)

These solutions are valid only for $λ ≤ (Λ_i - Λ_0)/q^2$ (this is because they are obtained by squaring some of the above equations). The maximal allowed value leads to $H = 1/(\sqrt{3}R)$, which, as we saw from eq. (12), is the highest possible value that the system can have for the dS expansion rate.

We see that the Minkowski compactification requires the tuning $\sqrt{λ} = (\sqrt{Λ_i} - \sqrt{Λ_0})/q$. For a small deviation

$$\sqrt{λ} = \frac{\sqrt{Λ_i} - \sqrt{Λ_0}}{q} + δ$$

(18)

eqs. (10) give

$$β^2 = \frac{Λ_0}{Λ_i} - 3\sqrt{Λ_0} (\sqrt{Λ_i} + \sqrt{Λ_0}) q δ + O (δ^2)$$

$$R^2 = \frac{1}{Λ_0} + \frac{3\sqrt{Λ_0} q δ + O (δ^2)}{2Λ_0^{3/2} (\sqrt{Λ_i} - \sqrt{Λ_0})}$$

$$H^2 = \frac{\sqrt{Λ_0} \sqrt{Λ_i} q}{6 (\sqrt{Λ_i} - \sqrt{Λ_0})} δ - \frac{q^2 (Λ_i + \sqrt{Λ_i} \sqrt{Λ_0} + Λ_0)}{12 (\sqrt{Λ_i} - \sqrt{Λ_0})^2} δ^2 + O (δ^3)$$

(19)

From either of (12) and (14), the brane position then satisfies

$$\tan θ = \frac{\sqrt{Λ_i} - \sqrt{Λ_0}}{q^2} + \frac{5}{4q} δ + O (δ^2)$$

(20)

The possible values for the above parameters are constrained by the two integer values $N$ and $n$. Eqs. (11) and (13) give

$$N = 2eM^2/\sqrt{Λ_i} + O (H^2)$$

$$1 + \frac{2n}{N} = -\frac{3(\sqrt{Λ_i} - \sqrt{Λ_0})^2}{2Λ_0 √Λ_i (\sqrt{Λ_i} - √Λ_0) + q^4} H^2 + O (H^4)$$

(21)

where the expansion in $δ$ has been replaced by an expansion in $H^2$ through the last of (19).

From the first quantization condition, we see that we cannot vary $λ$ alone without also varying the bulk cosmological constants. However, the Minkowski compactification ($δ = 0$) can be still deformed continuously into a dS one. However, once also the second condition is taken into account, the Minkowski solution appears to be detached from the dS ones. Indeed, the choice $2n/N = −1$ is only compatible with $H = 0$, with the only exception of the trivial case of $Λ_i = Λ_0$ (in which case, $β = 1$, and the brane is actually absent). 

We can gain further insight by estimating the parameters entering in eq. (21). Neglecting the subleading $H^2$ terms, and for non hierarchical values of the deficit angle (that is, $β$ and $1 - β$ of order one), both $\sqrt{Λ_i}$ and $√Λ_0$, as well as their difference, are of order $1/R^2$. In addition,

2 This conclusion actually holds for arbitrary values of $H$, and not just at small $δ$, as can be seen by studying eq. (13) for $2n/N = −1$. 


$q^2 R$ does not exceed one (as can be seen in eq. (20) - this value controls the ratio between the radius of the brane and that of the internal space). Therefore, we find

\[ 1 + \frac{2n}{N} = O \left( R^2 H^2 \right) \tag{22} \]

From the same reasoning, we also see that the expansion at small $\delta$ is actually an expansion for $RH \ll 1$.

In the concluding Section we comment on the implication of these findings for the cosmological constant problem.

IV. STABILITY OF THE MINKOWSKI COMPACTIFICATION

The goal of this Section is to obtain the massive perturbations of the system, to verify whether the background solution described in Section 2 has tachyonic instabilities. For the reasons mentioned in the Introduction, we focus on axially symmetric perturbations. 3 The most general perturbations of the geometry of this type are

\[ ds_6^2 = (1 + 2\Phi) \, dt^2 + 2Ad dl \hat{\phi} + (1 + 2C) \cos^2 \theta d\hat{\phi}^2 + 2 (T_{\mu} + \partial_{\mu}T) \, d\theta dx^\mu + 2 (V_{\mu} + \partial_{\mu}V) \, d\hat{\phi} dx^\mu + \left\{ \eta_{\mu\nu} (1 + 2\Psi) + 2E_{(\mu,\nu)} + h_{\mu\nu} \right\} dx^\mu dx^\nu \tag{23} \]

where we have defined the “dimensionful angular coordinates”

\[ dl \equiv \left\{ \begin{array}{ll} R\delta d\theta \, \text{“in”}, & d\hat{\phi} \equiv R\delta d\phi \end{array} \right. \tag{24} \]

and $E_{(\mu,\nu)} = \partial_\nu E_\mu + \partial_\mu E_\nu$. The vector modes $E_\mu$, $T_\mu$, $V_\mu$ are transverse, and the tensor mode $h_{\mu\nu}$ is transverse and traceless. The remaining modes are scalar (the denomination refers to how these modes transform under 4d coordinate transformations). Simultaneously, one needs to consider the perturbations of the gauge field,

\[ \delta A_\phi = a_\phi, \quad \delta A_i = a_i, \quad \delta A_\mu = \partial_\mu a + \hat{a}_\mu \tag{25} \]

where $\hat{a}_\mu$ is a transverse vector mode. Following the discussion of [12], we fix part of gauge freedom by setting $E_\mu = T = V = a_i = a' = 0$ (here and in the following, prime denotes derivative with respect to the rescaled coordinate $l$). We further impose

\[ \theta_{\text{brane}} = \bar{\theta}, \quad \Phi(\bar{\theta}) = 0 \tag{26} \]

that is, we require that the brane lies at the unperturbed background position, and that $g_{ll} = 1$ there (this choice includes the Gaussian normal coordinate choice at the brane location, which is the most convenient one to interpret the gravitational effects measured by brane observers). These choices do not fix the gauge completely (see [12] for details); however, we can still have a general (and unambiguous) study if we perform our computations in terms of the combination of modes which are invariant under the residual gauge freedom.

The tensor mode $h_{\mu\nu}$, and the vector modes $T_\mu$, $\hat{a}_\mu$, and $V_\mu$, are already invariant. For the scalar sector, the invariant combinations are instead

\[ \hat{\Phi} = \Phi + E' \]

\[ \hat{C} = C - \theta' \tan \theta E' \]

\[ \hat{a}_\phi = a_\phi + M^2 \theta' \cos \theta E' \]

\[ \Psi \tag{27} \]

Tensor, vector and scalar modes are decoupled at the linearized level, so we can study the three sectors separately. We do so in the next three Subsections. The relevant equations were obtained in [12], where the zero modes of the system were then studied. The derivation of these equations is not repeated here.

A. Tensor Modes

The axially symmetric tensor perturbation can be decomposed as

\[ h_{\mu\nu} (x, \theta) = \sum_n h_n (\theta) C_{\mu\nu\, n} (x) \tag{28} \]

where $C_{\mu\nu\, n}$ are 4d Kaluza Klein (KK) tensor modes, and $h_n$ their wavefunctions in the bulk. Our goal is to find the allowed perturbations, and their 4d masses $m_n$. The bulk equation

\[ \partial^2 h_{\mu\nu} + h_{\mu\nu}' - \theta' \tan \theta h_{\mu\nu}' = 0 \tag{29} \]

(\text{where $\partial^2$ denotes the d’Alambertian operator in 4d}) can be separated in

\[ \frac{d^2 h_n}{d\theta^2} - \tan \theta \frac{dh_n}{d\theta} + \mu_n^2 h_n = 0 \]

\[ \partial^2 C_{\mu\nu\, n} = m_n^2 C_{\mu\nu\, n} \tag{30} \]

where we have moved back to the $\theta$ coordinate, and where the parameter $\mu_n$ is different in the two bulk regions:

\[ \mu_n^2 \equiv \left\{ \begin{array}{ll} m_n^2 R^2 \beta^2 & \text{“in”} \nonumber \\ m_n^2 R^2 & \text{“out”} \end{array} \right. \tag{31} \]

From now on, we suppress the index $n$ for brevity, understanding that we are studying one KK mode at a time.

The bulk equations must be supplemented by a set of boundary and parity conditions. First, we require regularity at the two poles, imposing that the first derivative of $h$ vanishes there (we can impose this condition

3. Although involved, it is not hard to extend this analysis to general modes. Since the background is axially symmetric, the general dependence of the modes on the angular coordinate can only be of the form $\exp (in \phi)$, with $n$ integer.
either on the derivative with respect to \( \theta \) or \( l \), since the two variables are simply related by a constant rescaling). Second, parity considerations impose that the modes are even across the equator, \( h(-\theta) = h(\theta) \). Finally, we must satisfy the junction conditions across the brane, 

\[
\begin{align*}
  h_{in}(\theta) &= h_{out}(\theta) \\
  h_{in}^{\prime}(\theta) &= h_{out}^{\prime}(\theta) \\
  or \quad \frac{\partial h_{in}}{\partial \theta}(\theta) &= \beta \frac{\partial h_{out}}{\partial \theta}(\theta) \quad (32)
\end{align*}
\]

The bulk equations are solved by the Legendre functions \( P_\nu(x) \) and \( Q_\nu(x) \), where \( x = \sin \theta \) and \( \nu = -1/2 + \sqrt{1/4 + \mu^2} \) (we denote by \( \nu_i \) and \( \nu_o \) the values of this parameter in the in and out bulk, respectively). The bulk solution which is regular at the poles, even across the equator, and satisfies the first of (32) is 

\[
\begin{align*}
  h_{in} &= P_{\nu_i}(|x|) \\
  h_{out} &= A \left[ \cos \left( \frac{\pi \nu_i}{2} \right) P_{\nu_o}(x) - \frac{2}{\pi} \sin \left( \frac{\pi \nu_o}{2} \right) Q_{\nu_o}(x) \right] \\
  (33)
\end{align*}
\]

where 

\[
A = \frac{P_{\nu_i}(|\bar{x}|)}{\cos \left( \frac{\pi \nu_i}{2} \right) P_{\nu_o}(\bar{x}) - \frac{2}{\pi} \sin \left( \frac{\pi \nu_o}{2} \right) Q_{\nu_o}(\bar{x})} \quad , \quad \bar{x} = \sin \bar{\theta} \quad (34)
\]

The only undetermined parameter is the mass square of the mode, which enters in the two parameters \( \nu_{i,o} \). It can be found by imposing the only remaining condition to be satisfied, namely the second of (32). Specifically, for any fixed values of \( \beta \) and \( \bar{\theta} \), we (numerically) look for the roots of 

\[
f(m^2) = \frac{\partial}{\partial x} \left[ h_{in} - \beta h_{out} \right]_x \quad (35)
\]

As an example, Fig. (11) shows the behavior of \( f(m^2) \) for the specific choices of \( \beta = 0.9 \) and \( \bar{\theta} = 85^0 \). As can be seen in the figure, there are no tachyonic modes in the spectrum (this is the case also for more negative values of \( m^2 \) than those plotted). We verified that no tachyonic modes appear also for several other values of the brane position (those reported in figure (2)) and the deficit angle (We varied \( \beta = 0.2, ..., 0.9 \) in steps of 0.1 ). We also notice the presence of the zero mode already studied in (12).

Fig. (2) shows instead the mass spectrum for different brane positions, and for the specific choice \( \beta = 0.9 \). The points at \( \bar{\theta} = 90^0 \) have been obtained for a codimension two brane located at the pole. We see that the mass spectrum for the codimension-1 model converges continuously to the one of codimension-2 (we will see that this is not the case in the scalar sector). This can be proven analytically, from the study of the equations which determine the allowed modes. In the limit of \( \bar{\theta} \to \pi/2 \), the in part of the bulk shrinks to zero, so that all the bulk geometry is described by the rugby ball, as it is the case for the codimension two case; so, the bulk equations converge to that of codimension two. Moreover, the boundary/parity conditions that we have discussed above reduce to 

\[
\frac{dh}{d\theta} \bigg|_{\pi/2} = 0 \quad , \quad h(-\theta) = h(\theta) \quad , \quad as \ \bar{\theta} \to \pi/2 \quad (36)
\]

which coincide with those of the codimension two case.

In the codimension two case, the spectrum does not depend on \( \beta \) (this is strictly true for axially symmetric perturbations); this also emerges from our numerical results (not shown here): the dependence of the spectra on the deficit angle is very weak for any finite \( \bar{\theta} \), and it disappears as the codimension one brane is shrunk to the pole.

**B. Vector Modes**

The linearized Einstein and Maxwell equations for the vector modes have been derived in (12). The one for \( T_{\mu\nu} \)
simply reads
\[ \partial^2 T_\mu = 0 \] (37)
which immediately indicates that this perturbation has not massive modes. We decompose the two remaining modes as we did for the tensor case,
\[ \tilde{a}_\mu = \sum_n a_{\mu,n} (x) \ a_n (\theta) \ , \ \tilde{V}_\mu = \sum_n w_{\mu,n} (x) \ W_n (\theta) / M^2 \] (38)
Omitting the index \( n \) for brevity reasons, the bulk wave-functions satisfy the following system of equations
\[ \frac{d^2 a}{d\theta^2} + \tan \theta \frac{da}{d\theta} + \mu^2 a - \frac{1}{\cos \theta} \frac{dW}{d\theta} = 0 \]
\[ \frac{d^2 W}{d\theta^2} + \tan \theta \frac{dW}{d\theta} + \mu^2 W + 2 \cos \theta \frac{da}{d\theta} = 0 \] (39)
where \( \mu \) is defined as in eq. (31).

Due to the parity choice of the background, the mode \( a \) must be odd across the equator, while the mode \( W \) must be even,
\[ a (-\theta) = -a (\theta) \ , \ W (-\theta) = W (\theta) \] (40)
In addition, there are regularity conditions at the poles,
\[ \left. \frac{da}{d\theta} \right|_{\pm \pi/2} = \left. \frac{dW}{d\theta} \right|_{\pm \pi/2} = 0 \] (41)
and junction conditions across the brane.
\[ a_{in} (\bar{\theta}) = a_{out} (\bar{\theta}) \ , \ W_{in} (\bar{\theta}) = W_{in} (\bar{\theta}) \]
\[ \left. \frac{da}{d\theta} \right|_{\bar{\theta},in} = \beta \left. \frac{da}{d\theta} \right|_{\bar{\theta},out} \]
\[ \left. \frac{dW}{d\theta} + 2a \cos \bar{\theta} \right|_{\bar{\theta},in} = \beta \left. \frac{dW}{d\theta} + 2a \cos \bar{\theta} \right|_{\bar{\theta},out} \] (42)

For massless modes, the bulk equations (39) form a system of two coupled first order differential equations in terms of \( da/d\theta \) and \( dW/d\theta \). This system can be solved analytically. However, for nonvanishing mass, these equations must be integrated numerically.

Therefore, to find the spectrum of vector modes, we resort to a shooting method, which is appropriate for boundary value problems. Each mode is specified by its mass, and by a series of parameters which determine the initial conditions at one of the poles. For definiteness, we start from the south pole. As we discuss below, we actually need only one such parameter, which we denote by \( C \). We start from some guessed values for \( m^2 \) and \( C \), and we then solve the bulk equations (39) (when we cross a brane, we impose the conditions (42)). If the resulting solution turns out to be regular, and to have the correct parity assignment across the equator, then we have managed to identify one physical mode of the system.

In practice, the bulk solutions that we obtain numerically never satisfy these properties, signaling that the initial guess for the parameters \( m^2 \) and \( C \) was wrong. We can define the two “distances”,
\[ d_1 \equiv \frac{a(-\bar{\theta}) + a(\bar{\theta})}{a(-\bar{\theta}) - a(\bar{\theta})} \ , \ d_2 \equiv \frac{W(-\bar{\theta}) - W(\bar{\theta})}{W(-\bar{\theta}) + W(\bar{\theta})} \] (43)
which indicate how far the solution is from being \( Z_2 \) symmetric. We then proceed in two steps: (i) we densely scan the parameter space \( \{m^2, C\} \) within some given range; the values leading to the smallest distances are regarded as our best guesses; (ii) we use a Newton’s method to find the zeros of these distances, starting from the best guesses. Provided the initial conditions are dense enough, Newton’s method converges to all the physical solutions of the system, having values of \( \{m^2, C\} \) not too far from the probed range of values. Indeed, the two-dimensional nature of the initial parameter space, and the fact that the bulk geometry is regular everywhere, make the numerical problem a relatively simple one. It is easy to verify (for instance, by increasing the density of the initial scan) that all the solutions are reached with this method.

The main numerical difficulty occurs at the south pole, where the coordinate system used is singular. To overcome this, we actually solve (by Taylor expansion) the bulk equations analytically in a neighborhood of the south pole. As for the tensor sector, there is one overall normalization which cannot be determined by these linearized equations. We fix this by imposing \( a (-\pi/2) = 1 \).

We then find
\[ a (-\pi/2 + \epsilon) = 1 + C \epsilon^2 + O (\epsilon^4) \]
\[ W (-\pi/2 + \epsilon) = (2C + \frac{\mu^2}{2}) \epsilon^2 + O (\epsilon^4) \] (44)
The solutions of a system of second order equations, are usually specified by the values of the functions and their first derivatives at a given point. In the present case, due to the coordinate singularity at the poles, we also need to specify one of the second derivatives (we also note that the linear terms in the expansions vanish, due to the regularity conditions (41)). We started our numerical

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5 More accurately, the vanishing of \( d_1 \) and \( d_2 \) is a necessary, but not sufficient, condition for a mode to be a physical perturbation of the system; indeed, in some cases those conditions were (accidentally) satisfied, although the modes did not have the correct parity all throughout the bulk. The easiest way to solve this problem (in an automated way) is to check the parity condition also at other bulk positions. Specifically, we discharged all those solutions which satisfied \( d_1 = d_2 = 0 \), but which had the wrong parity at the equator (physical solutions satisfy \( W = da/d\theta = 0 \) at \( \theta = 0 \)). In all the cases we attempted, this was enough to eliminate all the spurious solutions (we always verified, by direct inspection, that all the modes which passed this second check had the correct parity all throughout the bulk).

6 This choice does not include the possibility of \( a = 0 \) at the pole. In this case, the numerical computation only leads to the zero mode characterized by constant \( a \) and \( W \) in the entire bulk.
evolutions with $\epsilon = 10^{-6}$, with the values of the wave functions and their derivatives obtained from (44).

We performed the analysis for several values of $\beta$ (namely $\beta = 0.3, 0.6, 0.9$) and $\theta (\bar{\theta} = 60, 65, 70, 75, 80, 85$). The initial region scanned was at $-100 \leq m^2 \leq 100$ and $-20 \leq \bar{C} \leq 20$ (the density was progressively increased until the final roots did not change. The final density we used was such that the $\mu^2$ values are varied in steps of 0.02 in every iteration). In our opinion, this range is sufficiently large to probe the stability of the model against tachyonic modes. Indeed, the first KK modes are expected to have a mass of the order of the inverse compactification radius, corresponding to $\mu^2$ values that are the order one. Also the parameter $\bar{C}$ is naturally expected to be of order one (since it comes from a Taylor expansion). Moreover, Newton’s method can converge to solutions outside this range of starting values for $m^2$ and $\bar{C}$ (this is indeed what happened in some cases). In all the cases studied, the two distances $d_1$ and $d_2$ strongly increased at negative $m^2$ (they were typically several orders of magnitudes greater than for positive $m^2$), indicating that no tachyons are present in the model; to have a further check, we started Newton’s method also from some of the guessed values with negative $m^2$ (despite the corresponding distances $d_1$ and $d_2$ were always very high); the method never converged to any tachyonic mode.

In figure 3 we show the obtained spectrum for several brane positions and for the specific choice $\beta = 0.9$. We note the presence of one massless mode. As for the tensor case, we also show the results for the strict codimension two case ($\bar{\theta} = 90^0$). We observe that, also for the vector sector, the limit of shrinking the extended brane to a codimension two defect is continuous (this can also be proven analytically from the study of the linearized equations, in the same way as we did for the tensor modes). Moreover, also for the vector sector, we observed a weak dependence (not shown here) of the spectrum on the value of the deficit angle $\beta$ (the reason is the same as for the tensor sector).

C. Scalar Modes

There are four scalar gauge invariant combinations, satisfying the set of linearized equations derived in [12]. Two bulk constraint equations (containing at most first order derivatives) can be used to express $\bar{C}$ and $\bar{\phi}$ in terms of the other two modes $\hat{\Phi}$ and $\Psi$. This leaves us with the two bulk equations

$$\frac{d^2\hat{\Phi}}{d\theta^2} + \tan \theta \left( \frac{d\Psi}{d\theta} - \frac{d\hat{\Phi}}{d\theta} \right) + \left( 3\mu^2 - 2 \right) \hat{\Phi} - \frac{\mu^2}{2} \Psi = 0$$

$$\frac{d^2\Psi}{d\theta^2} + \tan \theta \frac{d\Psi}{d\theta} + \frac{\mu^2}{2} \Psi + \frac{\mu^2}{2} \hat{\Phi} = 0$$

(45)

where $\mu$ is related to the physical mass as in (31).

The parity assignment of the background imposes that both modes are even. Moreover, regularity at the poles requires

$$\left. \frac{d\hat{\Phi}}{d\theta} \right|_{\pm \pi/2} = \left. \frac{d\hat{\Psi}}{d\theta} \right|_{\pm \pi/2} = \left. \frac{d\Psi}{d\theta} \right|_{\pm \pi/2} = \left. \frac{d\hat{\Phi}}{d\theta} \right|_{\pm \pi/2} = 0$$

(46)

Once we insert these conditions in one of the constraint bulk equations (which is legitimate, since the involved quantities are continuous as we approach the poles), we find

$$\left. (\hat{\Phi} + \Psi) \right|_{\pm \pi/2} = 0$$

(47)

The junction conditions at the brane location were also expressed in [12] in terms of all 4 gauge invariant scalar combinations, plus the the quantity $E'$. This quantity can be identified with the (scalar) perturbation of the brane position. In an arbitrary gauge, the brane is at the perturbed position $\theta = \bar{\theta} + \zeta (x^\nu)$. This quantity changes when we performed a change of coordinates involving the bulk coordinate $l$. The combination which does not change under such change of coordinate is $\zeta = \zeta - E'$, which can be then interpreted as the gauge invariant perturbation of the brane position. Not surprisingly, this is the quantity which enters in the junction conditions, when they are written in terms of the gauge invariant perturbations (27). We further restricted the gauge freedom by choosing a system of coordinates where the brane remains at the background position, that is $\zeta = 0$. In this case, $\zeta = -E'$, which is the quantity entering in the junction conditions.

We can combine the junctions conditions given in [12] to eliminate $E'$. This requires the use of the bulk equations (which is however legitimate, since the junction conditions relate bulk quantities at the two sides of the
brane). After some algebra, we find
\[
[f]_J = 0
\]
\[
[(1 + \sin^2 \theta) \hat{\Phi} + \sin \theta \cos \hat{\theta}(\Psi' - \hat{\Phi}')]_J = 0
\]
\[
\left( \theta' \left( 4\Psi' - \mu^2 \cos \theta \sin \hat{\theta}(\Psi - \hat{\Phi}) \right) \right)_J = 0
\]
\[
\left( -5\Psi' - \tan \hat{\theta} \hat{\Phi} + \hat{\Phi}' \right) \theta' - \frac{1 + 1/\beta}{\cos \theta \sin \hat{\theta}} \hat{\Phi} = 0
\]
(48)

where \([f]_J \equiv f_{\text{out}} - f_{\text{in}}\) denotes the difference of the quantity \(f\) between the two sides of the brane.

Also for the scalar sector, the bulk equations must be solved numerically. We therefore perform a numerical analysis analogous to the one done for the vector modes. We first solve the bulk equations analytically in a neighborhood of the south pole. Fixing the overall normalization by setting \(\Psi = 1\) at the south pole \(^7\), and taking into account the regularity conditions mentioned above, we find
\[
\Psi (-\pi/2 + \epsilon) = 1 + C \epsilon^2 + O (\epsilon^4)
\]
\[
\hat{\Phi} (-\pi/2 + \epsilon) = -1 + \frac{\mu^2 + 4C}{4} \epsilon^2 + O (\epsilon^4)
\]
(49)

Also in this case, the mode is uniquely determined by the two parameters \(m^2\) and \(C\). The numerical investigation then proceeds in the same way as for vectors. Fig. 4 shows the lightest masses in the spectrum, for the specific choice of \(\beta = 0.9\) and for different brane positions (the small “oscillatory” behavior of the eigenmasses visible in the figure is probably due to numerical errors, and it gives a measure of the precision of the computation). As for the other two sectors, the computation does not show any tachyonic modes (the modes exhibit a very bad parity for all negative values of \(m^2\) we have attempted). However, there are two interesting differences between the scalar and the other two sectors.

The first difference is the absence of scalar zero modes (as can be also verified by solving the equations analytically, which is possible for vanishing mass). This indicates that all the moduli of the model have been lifted (by the fluxes and tensions in the system), and that the compactification is stable (this is the case also for a codimension 2 brane at the pole \(^20\)). The present stability analysis is done in absence of (matter or gauge) fields localized on the brane. Ref. \(^12\) studied the gravitational interaction between brane sources; it was found that two zero modes are then excited, and contribute to reproduce Einstein 4d gravity at large distances. A similar situation is also encountered in 5d models, for instance the Randall-Sundrum model with a single brane \(^21\). The background solution of \(^21\) has no scalar perturbations; however, when (matter) fields are localized on the brane, a scalar zero mode - often denoted as brane bending \(^22\) - is excited, and gives a relevant contribution to the gravitational interactions between the brane fields.

The second peculiarity of the scalar sector is that the limit \(\theta \to \pi/2\) is discontinuous. To see this, we solve the linearized equations when the brane is close to the pole, at the position \(\theta = -\pi/2 + \epsilon\). The analytical solution then accurately describes the modes in the in bulk immediately before the brane. We can then expand the junction conditions for small \(\epsilon\), and obtain the values of the modes in the out part of the bulk immediately after the brane. They are
\[
\Psi_o = 1 + O (\epsilon^2)
\]
\[
\frac{d \Psi_o}{d\theta} = \frac{8C + 3(\beta - 1)\beta m^2}{4\beta} \epsilon + O (\epsilon^3)
\]
\[
\hat{\Phi}_o = -2 + \beta + O (\epsilon^2)
\]
\[
\frac{d \hat{\Phi}_o}{d\theta} = \frac{2(1 - \beta)}{\epsilon} + O (\epsilon)
\]
(50)

The limit \(\epsilon \to 0\) (i.e. \(\theta \to \pi/2\)) would be continuous if, these values converged to those which must be imposed for a codimension two brane at the pole. The latter values are \(\Psi + \hat{\Phi} = d\Psi/d\theta = d\hat{\Phi}/d\theta = 0\), which clearly shows that the limit is not continuous (the only exception is the trivial case of a vanishing deficit angle, \(\beta = 1\), when both cases collapse to a spherical compactification with an empty brane at the pole).

This discontinuity, however, does not lead to any appreciable discontinuity on the lowest eigenmasses, as can be observed from fig. 4 (the values for \(\theta = \pi/2\) refers to the codimension two brane). For small \(\epsilon\), the eigenfunctions, although starting from a different value on the outside bulk, quickly approach the ones of the codimension two case, leading to nearly identical eigenmasses (within the accuracy of the numerical computation).

We actually observe from the last of (49) that the limit \(\theta \to \pi/2\) actually leads to a divergent derivative of \(\hat{\Phi}\) on

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\(^7\) This choice does not include the possibility of \(\Psi = 0\) at the pole (such modes could in principle exist, since they could have a nonvanishing second derivative at the pole). We performed a separate numerical investigation for this case, which however did not show the existence of any such mode.

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FIG. 4: Smallest masses for the scalar modes, for various brane positions and for the specific choice \(\beta = 0.9\).
the outside bulk. This results in divergent terms in the linearized Einstein tensor. An analogous result is also encountered for the scalar modes excited by brane fields. It was found in [12] that one scalar mode diverges when the codimension one brane is shrunk to the pole. This singular limit in the scalar sector is what precludes the localization of matter fields on a strict codimension two brane.

V. CONCLUSIONS

We studied a brane-world model in a six dimensional space-time, in which two of the dimensions are compactified by a flux. The model is characterized by a codimension one brane, with one dimension extending inside the compact space. This construction avoids the singularities which plague codimension two and higher defects, where only tension can be localized. Indeed, it was shown in [12] that fields with arbitrary tension can be localized on this defect, and that their gravitational interaction is described by Einstein 4d gravity at large distance.

We do not regard this construction as a regularized codimension two brane, in the sense that one does not recover a well-behaved solution as the extended defect is shrunk to a zero size in the bulk. More accurately, the linearized computation of gravity in this model breaks down (in the scalar sector) as the size of the brane decreases. While the only safe claim that one can make is that gravity becomes strong in this limit, it is hard to expect a regular nonperturbative limit, in the light of the fact that the strict codimension two case is itself badly singular. The positive aspect of this statement is that this construction leads to distinct and potentially testable predictions for short-range gravitational and electroweak interactions (corrections to standard gravity would show up at distances comparable to the compactification radius, while electroweak effects would appear at energies close to the inverse size of the brane in the internal space).

We expect that other inequivalent regular constructions are possible, but that they would lead to different short range observables.

To make clear predictions, one needs to obtain the massive spectrum of perturbations of the model. While only the massless modes were studied in [12], we performed this computation in the second part of this note. Even more importantly, the computation is mandatory to verify that no tachyonic mode is present, so that the construction is stable. As also argued in [18], the study of axially symmetric perturbations should be enough for this check (since the background is itself axially symmetric, modes with a nontrivial angular dependence are expected to have a higher mass). For this reason, we focused our investigation on these modes.

Due to technical difficulties that we have discussed in the paper, we were not able to decouple the system of bulk equations for the vector and scalar modes, so that we had to resort to a numerical investigation. We did so with a shooting method (slightly modified, to cope with the coordinate singularities at the poles). Clearly, numerical methods can only guarantee the stability within the range probed. However, we conducted a rather extensive search. While Kaluza-Klein masses are naturally expected to be of the order of the inverse compactification radius, we densely investigated a parameter space in the interval $-100/R^2 \leq m^2 \leq 100/R^2$, and for several bulk parameters (brane position and deficit angle). In no case we found evidence for tachyonic solutions. Since this is a relatively easy numerical problem (the bulk is regular, and there are no strong hierarchies present), we believe that the present analysis ensures the stability of the construction.

While the above considerations are valid for a Minkowski external space, in the first part of this analysis we studied the dS solutions of this model. We found that the space of vacua is characterized by discrete points labeled by two integer numbers, related to the quantized values of the flux in the bulk ($N$) and of the current of a brane field ($n$), which, in turns, controls the position of the brane in the internal space. The Minkowski compactification requires $N = -2n$. If this is the case, we can actually have different Minkowski compactifications, provided the cosmological constant on the brane ($\Lambda$) and on the two sides of the bulk ($\Lambda_i,0$) satisfy

$$\sqrt{\Lambda} = \frac{\sqrt{\Lambda_1} - \sqrt{\Lambda_0}}{q}, \quad \Lambda_i = \frac{2e^2 M^8}{N^2}$$

As long as these relations are satisfied, a change in the brane and bulk tensions leads to a different internal space, but only to Minkowski external geometry. Since the ratio of the two integers $N$ and $n$ cannot be varied continuously, it is tempting to ask whether this can be of some relevance for the cosmological constant problem.

Discrete vacua typically arise in presence of fluxes, and several studies have already attempted to use this as a solution of the cosmological constant problem. The original mechanism of [22] is realized with a 4-form in four dimensions. This form can acquire only quantized values in units of a charge $q$, and its energy density behaves as a cosmological constant. The value of the form can change through membrane nucleation; this is however a very slow process, and it could be possible that the present universe is trapped in a metastable state, where the vacuum energy and of the 4-form add up to the observed value of the cosmological constant $\Lambda_{\text{tot}}$. This mechanism allows for several possible values of $\Lambda_{\text{tot}}$, and, provided these values are densely packed together, one may hope to reproduce the observed expansion rate for some value of the quantized flux (even if this is not the case initially, one should simply wait until the flux tunnels to a value compatible with observations). This, however, requires

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8 We actually employed a Newton method which could converge to solutions also outside this interval, if present.
very small values of $q$, which - besides being unnatural - do not lead to the known cosmology (they would lead to a reheating temperature much smaller than the one needed for primordial nucleosynthesis, see [24] for a discussion). String theory can offer a big improvement in this respect. There, one requires an internal space which can be stabilized by fluxes [25], and which can have a complex topology, with several non-contractible cycles. There are various ways of wrapping fluxes around such internal space, leading to several quantized fluxes, and to a multi-dimensional set of discrete vacua. Due to the high dimensionality, it is natural to expect vacua with a value of $\Lambda_{\text{tot}}$ compatible with the observed one, even if the distance between different vacua (namely, the value of the charge $q$) is large [24]. This is one of the possible realizations of the landscape of string theory [26].

In the present context, inserting the measured value of $H$ in eq. (22), we find

$$1 + \frac{2n}{N} \sim 10^{-60} \left( \frac{R}{0.1 \text{ nm}} \right)^2$$

(52)

In the mechanism of [23], the present value of $H$ is achieved at the price of an unnaturally small charge, and very large flux. Our realization does not put a significant constraint of the charge. However, the value (52) requires a very tiny “mismatch” of the relation $2n/N = -1$, which can be achieved only when the two winding numbers are themselves $O(10^{60})$. Although the corresponding request (very large flux) is usually not listed as a drawback of [23], it is hard to regard such high windings as natural.

We can think of some possible improvement. The necessity of large windings is probably due to the extreme simplicity of the model. As we mentioned, the problem of localizing fields in general relativity is present for any defect of codimension higher than one, and not just for codimension two. Assuming that such a construction must be done also for more realistic (and richer) models, we can expect that the presence of more fluxes can allow for a solution without too large winding, in a similar way as the string realization improves over the one of [24]. Alternatively, we may be satisfied in providing at least a partial solution to the cosmological constant problem. Rather than requiring large windings, it is probably more natural to assume that we are locked in one of the Minkowski vacua, with $N = -2n$. This relation (which by itself does not appear unnatural) could possibly explain why the “big” cosmological constant vanishes. The coincidence problem could instead be solved by some additional field, which would then play the role of quintessence. This would still be an improvement with respect to the usual models of quintessence, where the absence of a “big” cosmological constant is typically left unexplained.

To improve over these considerations requires a better understanding of the background solutions of the model. For instance, it may be possible to have $N = -2n$, even if the cosmological constants do not satisfy the relations (51). This may be compatible with a more general solution than the restricted ansatz (Minkowski or dS external geometry, times a static internal space) assumed here. Moreover, in order to study quintessence in this context, one needs to include sources with a different equation of state than vacuum. It is usually hard, if not impossible, to obtain analytical solutions with a time evolving internal space. However, such questions can be possibly addressed analytically at low energies, or numerically along the lines of [19].

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