Asymmetric and Moving-Frame Approaches to
the 2D and 3D Boussinesq Equations

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Abstract

Boussinesq systems of nonlinear partial differential equations are fundamental equations in geophysical fluid dynamics. In this paper, we use asymmetric ideas and moving frames to solve the two-dimensional Boussinesq equations with partial viscosity terms studied by Chae (Adv. Math. 203 (2006), 497-513) and the three-dimensional stratified rotating Boussinesq equations studied by Hsia, Ma and Wang (J. Math. Phys. 48 (2007), no. 6, 06560). We obtain new families of explicit exact solutions with multiple parameter functions. Many of them are the periodic, quasi-periodic, aperiodic solutions that may have practical significance. By Fourier expansion and some of our solutions, one can obtain discontinuous solutions. In addition, Lie point symmetries are used to simplify our arguments.

1 Introduction

Both the atmospheric and oceanic flows are influenced by the rotation of the earth. In fact, the fast rotation and small aspect ratio are two main characteristics of the large scale atmospheric and oceanic flows. The small aspect ratio characteristic leads to the primitive equations, and the fast rotation leads to the quasi-geostrophic equations (cf. [2], [6], [7], [9]). A main objective in climate dynamics and in geophysical fluid dynamics is to understand and predict the periodic, quasi-periodic, aperiodic, and fully turbulent characteristics of the large scale atmospheric and oceanic flows (e.g., cf. [4], [5]).
The Boussinesq system for the incompressible fluid follows in $\mathbb{R}^2$ is

$$ u_t + uu_x + vu_y - \nu \Delta u = -p_x, \quad v_t + uv_x + vv_y - \nu \Delta v - \theta = -p_y, $$

\hspace{1cm} (1.1)

$$ \theta_t + u\theta_x + v\theta_y - \kappa \Delta \theta = 0, \quad u_x + v_y = 0, $$

\hspace{1cm} (1.2)

where $(u, v)$ is the velocity vector field, $p$ is the scalar pressure, $\theta$ is the scalar temperature, $\nu \geq 0$ is the viscosity and $\kappa \geq 0$ is the thermal diffusivity. The above system is a simple model in atmospheric sciences (e.g., cf. [8]). Chae [1] proved the global regularity, and Hou and Li [3] obtained the well-posedness of the above system.

Another slightly simplified version of the system of primitive equations is the three-dimensional stratified rotating Boussinesq system (e.g., cf. [7], [9]):

$$ u_t + uu_x + vu_y + wu_z - \frac{1}{R_0}v = \sigma(\Delta u - p_x), $$

\hspace{1cm} (1.3)

$$ v_t + uv_x + vv_y + wv_z + \frac{1}{R_0}u = \sigma(\Delta v - p_y), $$

\hspace{1cm} (1.4)

$$ w_t + uw_x + vw_y +ww_z - \sigma RT = \sigma(\Delta w - p_z), $$

\hspace{1cm} (1.5)

$$ T_t + uT_x + vT_y + wT_z = \Delta T + w, $$

\hspace{1cm} (1.6)

$$ u_x + v_y + w_z = 0, $$

\hspace{1cm} (1.7)

where $(u, v, w)$ is the velocity vector field, $T$ is the temperature function, $p$ is the pressure function, $\sigma$ is the Prandtl number, $R$ is the thermal Rayleigh number and $R_0$ is the Rossby number. Moreover, the vector $(1/R_0)(-v, u, 0)$ represents the Coriolis force and the term $w$ in (1.6) is derived using stratification. So the above equations are the extensions of Navier-Stokes equations by adding the Coriolis force and the stratified temperature equation. Due to the Coriolis force, the two-dimensional system (1.1) and (1.2) is not a special case of the above three-dimensional system. Hsia, Ma and Wang [4] studied the bifurcation and periodic solutions of the above system (1.3)-(1.7).

In [10], we used the stable range of nonlinear term to solve the equation of nonstationary transonic gas flow. Moreover, we [11] solved the three-dimensional Navier-Stokes equations by asymmetric techniques and moving frames. Based on the algebraic characteristics of the above equations, we use in this paper asymmetric ideas and moving frames to solve the above two Boussinesq systems of partial differential equations. New families of explicit exact solutions with multiple parameter functions are obtained. Many of them are the periodic, quasi-periodic, aperiodic solutions that may have practical significance. Using Fourier expansion and some of our solutions, one can obtain discontinuous solutions. The symmetry transformations for these equations are used to simplify our arguments.
For convenience, we always assume that all the involved partial derivatives of related functions always exist and we can change orders of taking partial derivatives. The parameter functions are so chosen that the involved expressions make sense. We also use prime ' to denote the derivative of any one-variable function.

Observe that the two-dimensional Boussinesq system (1.1) and (1.2) is invariant under the action of the following symmetry transformation:

\[
T(u) = a^{-1} \epsilon_1 u(a^2(t + b), a\epsilon_1(x + \alpha), a\epsilon_2(y + \beta)) - \alpha',
\]

\[
T(v) = a^{-1} \epsilon_2 v(a^2(t + b), a(x + \alpha), a(y + \beta)) - \beta', \tag{1.8}
\]

\[
T(p) = a^{-2} p(a^2(t + b), a\epsilon_1(x + \alpha), a\epsilon_2(y + \beta)) + \alpha'' x + \beta'' y + \gamma,
\]

\[
T(\theta) = a^{-3} \epsilon_2 \theta(a^2(t + b), a\epsilon_1(x + \alpha), a\epsilon_2(y + \beta)), \tag{1.10}
\]

where \(a, b \in \mathbb{R}\) with \(a \neq 0\), \(\epsilon_1, \epsilon_2 \in \{1, -1\}\) and \(\alpha, \beta, \gamma\) are arbitrary functions of \(t\). The above transformation transforms a solution of the equation (1.1) and (1.2) into another solution with additional three parameter functions.

Denote \(\vec{x} = (x, y)\). The three-dimensional stratified rotating Boussinesq system is invariant under the following transformations:

\[
T_1[(u, v, w)] = ((u(t + b, \vec{x}A, \epsilon z), v(t + b, \vec{x}A), \epsilon wz)A, \epsilon w), \tag{1.12}
\]

\[
T_1(p) = p(t + b, \vec{x}A, \epsilon z), \quad T_1(T) = T(t + b, \vec{x}A, \epsilon z); \tag{1.13}
\]

\[
T_2(u) = u(t, x + \alpha, y + \beta, z + \gamma) - \alpha', \quad T_2(v) = v(t, x + \alpha, y + \beta, z + \gamma) - \beta', \tag{1.14}
\]

\[
T_2(w) = w(t, x + \alpha, y + \beta, z + \gamma) - \gamma', \quad T_2(T) = T(t, x + \alpha, y + \beta, z + \gamma) - \gamma,
\]

\[
T_2(p) = p(t, x + \alpha, y + \beta, z + \gamma) + \sigma^{-1}(\alpha'' x + \beta'' y + \gamma'' z) - R\gamma z + \mu, \tag{1.16}
\]

where \(\epsilon = \pm 1\), \(b \in \mathbb{R}\), \(A \in O(2, \mathbb{R})\), and \(\alpha, \beta, \gamma, \mu\) are arbitrary functions of \(t\). The above transformations transform a solution of the equation (1.3)-(1.7) into another solution. In particular, applying the transformation \(T_2\) to any solution in this paper yields another solution with extra four parameter functions.

To simplify problems, we always solve the Boussinesq systems modulo the above corresponding symmetry transformations, which is an idea that geometers and topologists often use.

The paper is organized as follows. In Section 2, we solve the two-dimensional Boussinesq equations (1.1)-(1.2) and obtain four families of explicit exact solutions. In Section 3, we present an approach with \(u, v, w, T\) linear in \(x, y\) to the equations (1.3)-(1.7), and obtain two families of explicit exact solutions. Assuming \(u_z = v_z = w_{zz} = T_{zz} = 0\) in
Section 4, we find another two families of explicit exact solutions of the equations (1.3)-(1.7). In Section 5, we obtain a family of explicit exact solutions of (1.3)-(1.7) that are independent of \( x \). The status can be changed by applying the transformation in (1.12) and (1.13) to them.

2 Solutions of the 2D Boussinesq Equations

In this section, we solve the two-dimensional Boussinesq equations (1.1)-(1.2) by an asymmetric method and by a moving frame.

According to the second equation in (1.2), we take the potential form:

\[
\begin{align*}
\xi_y &= \xi_x, \\
-\xi_x \xi_y &= \nu \Delta \xi_x + \theta = p_y,
\end{align*}
\]

for some functions \( \xi \) in \( t, x, y \). Then the two-dimensional Boussinesq equations become

\[
\begin{align*}
\xi_{yt} + \xi_x \xi_{xy} - \xi_x \xi_{yy} - \nu \Delta \xi_y &= -p_x, \\
\xi_{xt} + \xi_y \xi_{xx} - \xi_x \xi_{xy} - \nu \Delta \xi_x + \theta &= p_y, \\
\theta_t + \xi_y \theta_x - \xi_x \theta_y - \kappa \Delta \theta &= 0.
\end{align*}
\]

By our assumption \( p_{xy} = p_{yx} \), the compatible condition of the equations in (2.2) is

\[
(\Delta \xi)_t + \xi_y (\Delta \xi)_x - \xi_x (\Delta \xi)_y - \nu \Delta^2 \xi + \theta_x = 0.
\]

Now we first solve the system (2.3) and (2.4).

Our asymmetric approach is to assume

\[
\theta = \varepsilon(t, y), \quad \xi = \phi(t, y) + x\psi(t, y)
\]

for some functions \( \varepsilon, \phi \) and \( \psi \) in \( t, y \). Then (2.3) becomes

\[
\varepsilon_t - \psi \varepsilon_y - \kappa \varepsilon_{yy} = 0.
\]

Moreover, (2.4) becomes

\[
\phi_{yxt} + x\psi_{yxt} + (\phi_y + x\psi_y)\psi_{yy} - \psi(\phi_{yyy} + x\psi_{yyy}) - \nu(\phi_{yyyy} + x\psi_{yyyy}) = 0,
\]

equivalently,

\[
\phi_{yxt} + \phi_y \psi_{yy} - \psi \phi_{yyy} - \nu \phi_{yyyy} = 0,
\]

\[
\psi_{yxt} + \psi_y \psi_{yy} - \psi \psi_{yyy} - \nu \psi_{yyyy} = 0.
\]

The above two equations are equivalent to:

\[
\phi_{yt} + \phi_y \psi_y - \psi \phi_{yy} - \nu \phi_{yyy} = \alpha_1,
\]

\( 4 \)
\[
\psi_{yt} + \psi_y^2 - \psi \psi_{yy} - \nu \psi_{yyy} = \alpha_2
\] 
(2.11)
for some functions \(\alpha_1\) and \(\alpha_2\) of \(t\) to be determined.

Let \(c\) be a fixed real constant and let \(\gamma\) be a fixed function of \(t\). We define
\[
\zeta_1(s) = \frac{e^{\gamma s} - ce^{-\gamma s}}{2}, \quad \eta_1 = \frac{e^{\gamma s} + ce^{-\gamma s}}{2},
\] 
(2.12)
\[
\zeta_0(s) = \sin \gamma s, \quad \eta_0(s) = \cos \gamma s.
\] 
(2.13)
Then
\[
\eta_r^2(s) + (-1)^r \zeta_r^2(s) = e^r
\] 
(2.14)
and
\[
\partial_s(\zeta_r(s)) = \gamma \eta_r(s), \quad \partial_s(\eta_r(s)) = -(-1)^r \gamma \zeta_r(s),
\] 
(2.15)
where we treat \(0^0 = 1\) when \(c = r = 0\). First we assume
\[
\psi = \beta_1 y + \beta_2 \zeta_r(y)
\] 
(2.16)
for some functions \(\beta_1\) and \(\beta_2\) of \(t\), where \(r = 0, 1\). Then (2.11) becomes
\[
\beta_1' + c^r \beta_2^2 \gamma^2 + \beta_1^2 + [(\beta_2 \gamma)' + (-1)^r \nu \beta_2 \gamma^3 + 2 \beta_1 \beta_2 \gamma] \eta_r(y)
\]
\[
+ (-1)^r \beta_2 \gamma (\beta_1 \gamma - \gamma') y \zeta_r(y) = \alpha_2,
\] 
(2.17)
which is implied by the following equations:
\[
\beta_1' + c^r \beta_2^2 \gamma^2 + \beta_1^2 = \alpha_2, \quad \beta_1 \gamma - \gamma' = 0,
\] 
(2.18)
\[
(\beta_2 \gamma)' + (-1)^r \nu \beta_2 \gamma^3 + 2 \beta_1 \beta_2 \gamma = 0.
\] 
(2.19)
For convenience, we assume \(\gamma = \sqrt{\alpha'}\) for some function \(\alpha\) of \(t\). Thus we have
\[
\beta_1 = \frac{\gamma'}{\gamma} = \frac{\alpha''}{2 \alpha'}, \quad \beta_2 = \frac{b_1 e^{-(-1)^r \nu \alpha}}{\sqrt{(\alpha')^3}}, \quad b_1 \in \mathbb{R}.
\] 
(2.20)
To solve (2.10), we assume
\[
\phi = \beta_3 \eta_r(y)
\] 
(2.21)
for some function \(\beta_3\), modulo the transformation in (1.8)-(1.11). Now (2.10) becomes
\[
[(-1)^r ((\beta_3 \gamma)' + \beta_1 \beta_3 \gamma) + \nu \beta_3 \gamma^3] \zeta_r(y) = -\alpha_1,
\] 
(2.22)
which is implied by
\[
(-1)^r ((\beta_3 \gamma)' + \beta_1 \beta_3 \gamma) + \nu \beta_3 \gamma^3 = 0.
\] 
(2.23)
Thus
\[
\beta_3 = \frac{b_2 e^{-(-1)^r \nu \alpha}}{\alpha'},
\] 
(2.24)
where \( b_2 \) is a real constant.

In order to solve (2.6), we assume

\[
\varepsilon = b e^{\gamma_1 \eta_r(y)},
\]

(2.25)

where \( b \) is a real constant and \( \gamma_1 \) is a function of \( t \). Then (2.6) is implied by

\[
\gamma'_1 \eta_r(y) + (-1)^r \beta_2 \gamma_1 \zeta_r^2(y) + \kappa \gamma^2 \gamma_1 ((-1)^r \eta_r(y) - \gamma_1 \zeta_r^2(y)) = 0,
\]

(2.26)

which is implied by

\[
\gamma'_1 + (-1)^r \kappa \gamma^2 \gamma_1 = 0, \quad (-1)^r \beta_2 - \kappa \gamma_1 = 0.
\]

(2.27)

Then the first equation implies

\[
\gamma_1 = b_3 e^{(-1)^r \kappa \alpha}
\]

(2.28)

for some constant \( b_3 \). By the second equations in (2.20) and (2.27), we have:

\[
(-1)^r \frac{b_1 e^{(-1)^r \nu \alpha}}{\sqrt{(\alpha')^3}} = b_3 \kappa \sqrt{\alpha'} e^{(-1)^r \kappa \alpha}.
\]

(2.29)

For convenience, we take

\[
b_1 = (-1)^r b_0^2 \kappa b_3, \quad b_0 \in \mathbb{R}.
\]

(2.30)

Then (2.29) is implied by

\[
\alpha' e^{(-1)^r (\nu - \kappa) \alpha/2} = b_0.
\]

(2.31)

If \( \nu = \kappa \), then we have \( \alpha = b_0 t + c_0 \). Modulo the transformation in (1.8)-(1.11), we take \( b_0 = 1 \) and \( c_0 = 0 \), that is, \( \alpha = t \). When \( \nu \neq \kappa \), we similarly take \( b_0 = 1 \) and

\[
\alpha = \frac{2(-1)^r}{\nu - \kappa} \ln[(-1)^r (\nu - \kappa) t/2 + c_0], \quad c_0 \in \mathbb{R}.
\]

(2.32)

Suppose \( \nu = \kappa \). Then \( \gamma = 1 \) and

\[
\phi = b_2 e^{(-1)^r \nu t \eta_r(y)}, \quad \psi = (-1)^r b_3 \nu e^{(-1)^r \nu t \zeta_r(y)}.
\]

(2.33)

Moreover,

\[
\theta = b \exp(b_3 e^{(-1)^r \nu t \eta_r(y)}),
\]

(2.34)

\[
\xi = b_2 e^{(-1)^r \nu t \eta_r(y)} + (-1)^r b_3 \nu e^{(-1)^r \nu t x \zeta_r(y)}
\]

(2.35)

by (2.5). According to (2.1),

\[
u = -\xi_x = (-1)^r b_3 \nu e^{(-1)^r \nu t \zeta_r(y)}.
\]

(2.37)
Note
\[ u_t + uu_x + vu_y - \nu \Delta u = b_3^2 \nu^2 e^{-(-1)\nu t} x, \quad (2.38) \]
\[ v_t + uv_x + vv_y - \nu \Delta v - \theta = vv_y - b \exp(b_3 e^{-(-1)\nu t} \eta_t(y)). \quad (2.39) \]

By (1.1), we have
\[ p = b \int \exp(b_3 e^{-(-1)\nu t} \eta_t(y)) dy - \frac{1}{2} b_3^2 \nu^2 e^{-(-1)\nu t} \left(c^r x^2 + \zeta^2 (y)\right) \quad (2.40) \]
modulo the transformation in (1.8)-(1.11).

Consider the case \( \nu \neq \kappa \). Then
\[ \gamma = \sqrt{\alpha'} = \frac{1}{\sqrt{(-1)^r(\nu - \kappa) t/2 + c_0}} \quad (2.41) \]
by (2.32). Moreover,
\[ \phi = b_2 [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)+1} \eta_t(y) \quad (2.42) \]
by (2.21) and (2.24). Furthermore,
\[ \psi = \frac{(-1)^r(\kappa - \nu)y}{4[(-1)^r(\nu - \kappa)t/2 + c_0]} + (-1)^r b_3 \kappa [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)+3/2} \zeta_t(y) \quad (2.43) \]
by (2.16), (2.20) and (2.30). According to (2.25), (2.28) and (2.32),
\[ \theta = b e^{b_3 [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)} \eta_t(y)}. \quad (2.44) \]

Similarly, we have
\[ u_t + uu_x + vu_y - \nu \Delta u = b_3^2 c^r \kappa^2 [(-1)^r(\nu - \kappa)t/2 + c_0]^{4\nu/(\kappa-\nu)+2} x \]
\[ + \frac{3(\nu - \kappa)^2 x}{16[(-1)^r(\nu - \kappa)t/2 + c_0]^2}, \quad (2.45) \]
\[ v_t + uv_x + vv_y - \nu \Delta - \theta = -\psi_t + \psi \psi_y + \nu \psi_yy - \theta \]
\[ = -b e^{b_3 [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)} \eta_t(y)} + \frac{3}{4} b_3 \kappa [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)+1} \zeta_t(y) \]
\[ + \frac{3(\nu - \kappa)^2 y}{16[(-1)^r(\nu - \kappa)t/2 + c_0]^2} + \frac{b_3^2}{2} \kappa^2 [(-1)^r(\nu - \kappa)t/2 + c_0]^{4\nu/(\kappa-\nu)+3} \partial_y \zeta^2_t(y). \quad (2.46) \]

According (1.1), we have
\[ p = b \int e^{b_3 [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)} \eta_t(y)} dy - \frac{b_3^2}{2} c^r \kappa^2 [(-1)^r(\nu - \kappa)t/2 + c_0]^{4\nu/(\kappa-\nu)+2} x^2 \]
\[ - \frac{3(\nu - \kappa)^2 (x^2 + y^2)}{32[(-1)^r(\nu - \kappa)t/2 + c_0]^2} - \frac{b_3^2}{2} \kappa^2 [(-1)^r(\nu - \kappa)t/2 + c_0]^{4\nu/(\kappa-\nu)+3} \zeta^2_t(y) \]
\[ + \frac{3}{4} [(-1)^r b_3 \kappa (\nu - \nu)] [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa-\nu)+1} \eta_t(y) \quad (2.47) \]
modulo the transformation in (1.8)-(1.11).

**Theorem 2.1.** Let \( b, b_2, b_3, c, c_0 \in \mathbb{R} \) and let \( r = 0, 1 \). If \( \nu = \kappa \), we have the solution (2.34), (2.36), (2.37) and (2.40) of the two-dimensional Boussinesq equations (1.1)-(1.2), where \( \zeta_r(y) \) and \( \eta_r(y) \) are defined in (2.12)-(2.13) with \( \gamma = 1 \). When \( \nu \neq \kappa \), we have the following solutions of the two-dimensional Boussinesq equations (1.1)-(1.2):

\[
\begin{align*}
\nu &= \frac{(-1)^r(\kappa - \nu)x}{4[(1)^r(\nu - \kappa)t/2 + c_0]} \times (-1)^r b_3 \kappa [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa - \nu) + 1} y \eta_r(y) \\
&\quad - (-1)^r b_2 [(-1)^r(\nu - \kappa)t/2 + c_0]^{2\nu/(\kappa - \nu) + 1/2} \zeta_r(y), \\
\psi &= 6\nu y^{-1}
\end{align*}
\]

is another solution of (2.11). In order to solve (2.10), we assume

\[
\phi = \sum_{i=1}^{\infty} \gamma_i y^i
\]

modulo the transformation in (1.8)-(1.11), where \( \gamma_i \) are functions of \( t \) to be determined. Now (2.10) becomes

\[
-6\nu \gamma_1 y^{-2} - 18\nu \gamma_2 y^{-1} + \sum_{i=1}^{\infty} [i\gamma'_i - \nu(i + 2)(i + 3)(i + 4)\gamma_{i+2}] y^{-1} = \alpha_1,
\]

equivalently,

\[
\begin{align*}
\gamma_1 &= \gamma_2 = 0, & \alpha_1 &= -60\nu \gamma_3, \\
i\gamma'_i - \nu(i + 2)(i + 3)(i + 4)\gamma_{i+2} &= 0, & i &> 1.
\end{align*}
\]

Thus

\[
\begin{align*}
\gamma_{2i+2} &= \frac{2i\gamma'_i}{\nu(2i + 2)(2i + 3)(2i + 4)} = 0, & i &\geq 1, \\
\gamma_{2i+3} &= \frac{(2i + 1)\gamma'_{2i+1}}{\nu(2i + 3)(2i + 4)(2i + 5)} = \frac{360\gamma^{(i)}_3}{\nu^i(2i + 2)(2i + 5)!}, & i &\geq 1.
\end{align*}
\]

Hence

\[
\phi = 360 \sum_{i=0}^{\infty} \frac{\alpha^{(i)} y^{2i+3}}{\nu^i(2i + 3)(2i + 5)!},
\]

where \( \alpha \) is an arbitrary function of \( t \) such that the series converges, say, a polynomial in \( t \).
To solve (2.6), we also assume

\[ \varepsilon = \sum_{i=0}^{\infty} \beta_i y^i, \quad (2.58) \]

where \( \beta_i \) are functions of \( t \). Then (2.6) becomes

\[ 6\nu \beta_1 y^{-1} + \sum_{i=0}^{\infty} [\beta_i y^i + (i + 2)(6\nu - (i + 1)\kappa)\beta_{i+2}] y^i = 0, \quad (2.58) \]

that is, \( \beta_1 = 0 \) and

\[ \beta_i - (i + 2)(6\nu + (i + 1)\kappa)\beta_{i+2} = 0, \quad i \geq 0. \quad (2.59) \]

Hence

\[ \theta = \beta + \sum_{i=1}^{\infty} \frac{\beta^{(i)} y^{2i}}{2^i i! \prod_{r=1}^{i} (6\nu + (2r - 1)\kappa)}, \quad (2.60) \]

where \( \beta \) is an arbitrary function of \( t \) such that the series converges, say, a polynomial in \( t \). In this case,

\[ u_t + uu_x + vu_y - \nu \Delta u = -60\nu \alpha, \quad (2.61) \]

\[ v_t + uw_x + vv_y - \nu \Delta - \theta = -36\nu^2 y^{-3} - \beta - \sum_{i=1}^{\infty} \frac{\beta^{(i)} y^{2i}}{2^i i! \prod_{r=1}^{i} (6\nu + (2r - 1)\kappa)}. \quad (2.62) \]

According (1.1), we have

\[ p = 60\nu \alpha x - 18\nu^2 y^{-2} + \beta y + \sum_{i=1}^{\infty} \frac{\beta^{(i)} y^{2i+1}}{2^i i! (2i + 1) \prod_{r=1}^{i} (6\nu + (2r - 1)\kappa)} \quad (2.63) \]

modulo the transformation in (1.8)-(1.11).

**Theorem 2.2.** We have the following solutions of the two-dimensional Boussinesq equations (1.1)-(1.2):

\[ u = 360 \sum_{i=0}^{\infty} \frac{\alpha^{(i)} y^{2i+2}}{\nu^i (2i + 5)!} - 6\nu xy^{-2}, \quad v = -6\nu y^{-1}, \quad (2.64) \]

\( \theta \) is given in (2.60) and \( p \) is given in (2.63), where \( \alpha \) and \( \beta \) are arbitrary functions of \( t \) such that the related series converge, say, polynomials in \( t \).

Let \( \gamma \) be a function of \( t \). Denote the moving frame

\[ \tilde{\omega} = x \cos \gamma + y \sin \gamma, \quad \tilde{\sigma} = y \cos \gamma - x \sin \gamma. \quad (2.65) \]

Then

\[ \partial_t (\tilde{\omega}) = \gamma' \tilde{\sigma}, \quad \partial_t (\tilde{\sigma}) = -\gamma' \tilde{\omega}. \quad (2.66) \]
Moreover,
\[
\partial_{\tilde{\omega}} = \cos \gamma \partial_x + \sin \gamma \partial_y, \quad \partial_{\bar{\omega}} = -\sin \gamma \partial_x + \cos \gamma \partial_y.
\] (2.67)

In particular,
\[
\Delta = \partial_x^2 + \partial_y^2 = \partial_{\tilde{\omega}}^2 + \partial_{\bar{\omega}}^2.
\] (2.68)

We assume
\[
\xi = \phi(t, \tilde{\omega}) - \frac{\gamma'}{2}(x^2 + y^2), \quad \theta = \psi(t, \tilde{\omega}),
\] (2.69)
where \(\phi\) and \(\psi\) are functions in \(t, \tilde{\omega}\). Then (2.3) becomes
\[
\psi_t - \kappa \psi_{\tilde{\omega}\bar{\omega}} = 0
\] (2.70)
and (2.4) becomes
\[
-2\gamma'' + \phi_{t\tilde{\omega}} - \nu \phi_{\tilde{\omega}\bar{\omega}} + \psi_{\tilde{\omega}} \cos \gamma = 0.
\] (2.71)

Modulo the transformation in (1.8)-(1.11), the above equation is equivalent to
\[
-2\gamma'' \tilde{\omega} + \phi_{t\tilde{\omega}} - \nu \phi_{\tilde{\omega}\bar{\omega}} + \psi \cos \gamma = 0.
\] (2.72)

Assume \(\nu = \kappa\). We take the following solution of (2.70):
\[
\psi = \sum_{i=1}^{m} a_i d_i e^{\alpha_i^2 k t \cos 2b_i + a_i \tilde{\omega} \cos b_i} \sin(a_i^2 k t \sin 2b_i + a_i \tilde{\omega} \sin b_i + b_i + c_i)
\] (2.73)
where \(a_i, b_i, c_i, d_i\) are real numbers. Moreover, (2.72) is equivalent to solving the following equation:
\[
2\nu \gamma' - \gamma'' \tilde{\omega}^2 + \phi_t - \nu \phi_{\tilde{\omega}\bar{\omega}} + \sum_{i=1}^{m} d_i e^{\alpha_i^2 k t \cos 2b_i + a_i \tilde{\omega} \cos b_i} \times \sin(a_i^2 k t \sin 2b_i + a_i \tilde{\omega} \sin b_i + c_i) \right) \cos \gamma = 0
\] (2.74)
by (2.1). Thus we have the following solution of (2.74):
\[
\phi = -\left[\sum_{i=1}^{m} d_i e^{\alpha_i^2 k t \cos 2b_i + a_i \tilde{\omega} \cos b_i} \sin(a_i^2 k t \sin 2b_i + a_i \tilde{\omega} \sin b_i + c_i) \right] \int \cos \gamma \ dt + \gamma' \tilde{\omega}^2 + \sum_{s=1}^{n} \hat{d}_s e^{\hat{\alpha}_s^2 k t \cos 2b_s + \hat{a}_s \tilde{\omega} \cos b_s} \sin(\hat{\alpha}_s^2 k t \sin 2b_s + \hat{a}_s \tilde{\omega} \sin \hat{b}_s + \hat{c}_s),
\] (2.75)
where \(\hat{a}_s, \hat{b}_s, \hat{c}_s, \hat{d}_s\) are real numbers.

Suppose \(\nu \neq \kappa\). To make (2.72) solvable, we choose the following solution of (2.70):
\[
\psi = \sum_{i=1}^{m} a_i d_i e^{\alpha_i^2 k t + a_i \tilde{\omega}}.
\] (2.76)
Now (2.72) is equivalent to solving the following equation:
\[\nu \gamma' - \gamma'' \tilde{\omega}^2 + \phi_t - \nu \phi_{\tilde{\omega} \tilde{\omega}} + \sum_{i=1}^{m} d_i e^{a_i^2 \kappa t + a_i \tilde{\omega}} \cos \gamma = 0 \tag{2.77}\]
by (2.1). We obtain the following solution of (2.77):
\[
\phi = \gamma' \tilde{\omega}^2 + \sum_{s=1}^{n} \hat{d}_s e^{\hat{a}_s^2 \kappa t \cos 2b_s + \hat{a}_s \tilde{\omega} \cos \hat{b}_s} \sin (\hat{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \tilde{\omega} \sin \hat{b}_s + \hat{c}_s) \\
- \sum_{i=1}^{m} d_i e^{a_i^2 \kappa t + a_i \tilde{\omega}} \int e^{a_i^2 (\kappa - \nu) t} \cos \gamma \, dt. \tag{2.78}\]

Note
\[u = \phi_x \sin \gamma - \gamma' y, \quad v = \gamma' x - \phi_x \cos \gamma. \tag{2.79}\]

By (2.72),
\[
u u_t + uu_x + vv_y - \nu \Delta u \\
= (\phi_x \kappa t - \nu \phi_{\tilde{\omega} \tilde{\omega}}) \sin \gamma + 2\gamma' \phi_x \cos \gamma - \gamma'' x - \gamma' y \\
= (2\gamma'' \tilde{\omega} - \psi \cos \gamma) \sin \gamma + 2\gamma' \phi_x \cos \gamma - \gamma'' x - \gamma' y, \\
= \gamma''(x \sin 2\gamma - y \cos 2\gamma) + (2\gamma' \phi_x - \psi \sin \gamma) \cos \gamma - \gamma'' x, \tag{2.80}\]
\[
u v_t + uv_x + vv_y - \nu \Delta v - \theta \\
= (\nu \phi_{\tilde{\omega} \tilde{\omega}} - \kappa t \phi_{\tilde{\omega} \tilde{\omega}}) \cos \gamma + 2\gamma' \phi_x \sin \gamma - \gamma'' y + \gamma'' x - \psi \\
= (\psi \cos \gamma - 2\gamma'' \tilde{\omega}) \cos \gamma + 2\gamma' \phi_x \sin \gamma - \gamma'' y + \gamma'' x - \psi \\
= -\gamma''(x \cos 2\gamma + y \sin 2\gamma) + (2\gamma' \phi_x - \psi \sin \gamma) \sin \gamma - \gamma'' y. \tag{2.81}\]

According to (1.1),
\[p = \frac{\gamma'^2 - \gamma'' \sin 2\gamma}{2} x^2 + \frac{\gamma'^2 + \gamma'' \sin 2\gamma}{2} y^2 + \gamma' x y \cos 2\gamma + \int \psi \tilde{\omega} \sin \gamma - 2\gamma' \phi \tag{2.82}\]
modulo the transformation in (1.8)-(1.11).

**Theorem 2.3.** Let \(\gamma\) be any function of \(t\) and denote \(\tilde{\omega} = x \cos \gamma + y \sin \gamma\). Take
\[\{a_i, b_i, c_i, d_i, \hat{a}_s, \hat{b}_s, \hat{c}_s, \hat{d}_s \mid i = 1, \ldots, m; s = 1, \ldots, n\} \subset \mathbb{R}. \tag{2.83}\]

If \(\nu = \kappa\), we have the following solutions of the two-dimensional Boussinesq equations (1.1)-(1.2):
\[
u u_t + uu_x + vv_y - \nu \Delta u \\
= -\gamma' y + \sin \gamma \{2\gamma' \tilde{\omega} + \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\hat{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \tilde{\omega} \cos \hat{b}_s} \sin (\hat{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \tilde{\omega} \sin \hat{b}_s + \hat{c}_s) \\
- \sum_{i=1}^{m} a_i d_i e^{a_i^2 \kappa t \cos 2b_i + a_i \tilde{\omega} \cos b_i} \sin (a_i^2 \kappa t \sin 2b_i + a_i \tilde{\omega} \sin b_i + c_i) \int \cos \gamma \, dt\}, \tag{2.84}\]

\[v = \gamma' x - \phi_x \cos \gamma. \tag{2.79}\]
\[ v = \gamma' x - \cos \gamma \{ 2 \gamma' \bar{\omega} + \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\tilde{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \bar{\omega} \cos \hat{b}_s} \sin (\tilde{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \bar{\omega} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s) \] 

\[ - \frac{m}{i=1} a_id_i e^{\tilde{a}_i^2 \kappa t \cos 2b_i + a_i \bar{\omega} \cos b_i} \sin (\tilde{a}_i^2 \kappa t \sin 2b_i + a_i \bar{\omega} \sin b_i + b_i + c_i) \int \cos \gamma \, dt \}, \quad (2.85) \]

\[ \theta = \psi \text{ in } (2.73), \text{ and} \]

\[ p = (\sin \gamma + 2\gamma' \int \cos \gamma) \left[ \sum_{i=1}^{m} d_i e^{\tilde{a}_i^2 \kappa t \cos 2b_i + a_i \bar{\omega} \cos b_i} \sin (\tilde{a}_i^2 \kappa t \sin 2b_i + a_i \bar{\omega} \sin b_i + b_i + c_i) \right] 

+ \frac{\gamma^2 - \gamma''}{2} \sin 2\gamma \frac{y^2}{x^2} + \frac{\gamma^2 + \gamma''}{2} \sin 2\gamma x^2 y + \gamma'' xy \cos 2\gamma - 2\gamma'^2 \bar{\omega}^2 

- 2\gamma' \sum_{s=1}^{n} \tilde{a}_s e^{\tilde{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \bar{\omega} \cos \hat{b}_s} \sin (\tilde{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \bar{\omega} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s). \quad (2.86) \]

When \( \nu \neq \kappa \), we have the following solutions of the two-dimensional Boussinesq equations (1.1)-(1.2):

\[ u = \left\{ \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\tilde{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \bar{\omega} \cos \hat{b}_s} \sin (\tilde{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \bar{\omega} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s) \right. 

+ 2\gamma' \bar{\omega} - \sum_{i=1}^{m} a_id_i e^{\tilde{a}_i^2 \kappa t + a_i \bar{\omega}} \int e^{\tilde{a}_i^2 (\kappa - \nu) t \cos \gamma \, dt} \sin \gamma - \gamma' y, \quad (2.87) \]

\[ v = \left\{ \sum_{s=1}^{n} \hat{a}_s \hat{d}_s e^{\tilde{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \bar{\omega} \cos \hat{b}_s} \sin (\tilde{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \bar{\omega} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s) \right. 

+ 2\gamma' \bar{\omega} - \sum_{i=1}^{m} a_id_i e^{\tilde{a}_i^2 \kappa t + a_i \bar{\omega}} \int e^{\tilde{a}_i^2 (\kappa - \nu) t \cos \gamma \, dt} \cos \gamma + \gamma' x, \quad (2.88) \]

\[ \theta = \psi \text{ in } (2.76), \text{ and} \]

\[ p = \frac{\gamma^2 - \gamma''}{2} \sin 2\gamma \frac{x^2}{y^2} + \frac{\gamma^2 + \gamma''}{2} \sin 2\gamma \frac{y^2}{x^2} + \gamma'' xy \cos 2\gamma - 2\gamma'^2 \bar{\omega}^2 

- 2\gamma' \sum_{s=1}^{n} \tilde{a}_s e^{\tilde{a}_s^2 \kappa t \cos 2\hat{b}_s + \hat{a}_s \bar{\omega} \cos \hat{b}_s} \sin (\tilde{a}_s^2 \kappa t \sin 2\hat{b}_s + \hat{a}_s \bar{\omega} \sin \hat{b}_s + \hat{b}_s + \hat{c}_s) 

+ \sum_{i=1}^{m} d_i e^{\tilde{a}_i^2 \kappa t + a_i \bar{\omega}} \int e^{\tilde{a}_i^2 (\kappa - \nu) t \cos \gamma \, dt}. \quad (2.89) \]

**Remark 2.4.** By Fourier expansion, we can use the above solution to obtain the one depending on two piecewise continuous functions of \( \bar{\omega} \).

### 3 Asymmetric Approach I to the 3D Equations

Starting from this section, we use asymmetric approaches developed in [11] to solve the stratified rotating Boussinesq equations (1.3)-(1.7).
For convenience of computation, we denote
\[
\Phi_1 = u_t + uu_x + vu_y + wu_z - \frac{1}{R_0} v - \sigma(u_{xx} + u_{yy} + u_{zz}),
\]
(3.1)
\[
\Phi_2 = v_t + uv_x + vv_y + wv_z + \frac{1}{R_0} u - \sigma(v_{xx} + v_{yy} + v_{zz}),
\]
(3.2)
\[
\Phi_3 = w_t + uw_x + vw_y + ww_z - \sigma RT - \sigma(w_{xx} + w_{yy} + w_{zz}).
\]
(3.3)
Then the equations (1.3)-(1.5) become
\[
\Phi_1 + \sigma p_x = 0, \quad \Phi_2 + \sigma p_y = 0, \quad \Phi_3 + \sigma p_z = 0.
\]
(3.4)
Our strategy is to solve the following compatibility conditions:
\[
\partial_y(\Phi_1) = \partial_x(\Phi_2), \quad \partial_z(\Phi_1) = \partial_x(\Phi_3), \quad \partial_z(\Phi_2) = \partial_y(\Phi_3).
\]
(3.5)
First we assume
\[
u = \tau(t, z)x + \psi(t, z)y + \epsilon(t, z), \quad \tau = \sigma(t, z)x + \psi(t, z)y + \epsilon(t, z),
\]
(3.6)
\[
w = -\phi(t, z) - \psi(t, z), \quad T = \vartheta(t, z) + z,
\]
(3.7)
where \(\phi, \vartheta, \psi, \mu, \tau, \) and \(\epsilon\) are functions of \(t, z\) to be determined. Then
\[
\Phi_1 = \phi_{tz} + \phi_t + \phi_z(\phi_x + \psi_y + \mu) + (\sigma - 1/R_0)(\tau x + \psi z + \epsilon)
\]
\[-(\phi + \psi)(\phi_{zz} x + \psi_{zz} y + \mu_{zz}) - \sigma(\phi_{zzzz} x + \psi_{zzzz} y + \mu_{zzzz})
\]
\[= [\phi_{tz} + \phi^2_t + \tau(\psi - 1/R_0) - \phi_{zz}(\phi + \psi) - \sigma\phi_{zzzz}]x
\]
\[+ [\psi_t + \phi \psi_z(\psi - 1/R_0) - \psi_z(\phi + \psi) - \sigma\psi_{zzzz}]y
\]
\[+ \mu_t + \mu \phi_z + (\sigma - 1/R_0) \epsilon - \mu_z(\phi + \psi) - \sigma\mu_{zzz},
\]
(3.8)
\[
\Phi_2 = \tau x + \psi_{tz} y + \epsilon_t + \psi_z(\tau x + \psi z + \epsilon) + (\tau + 1/R_0)(\phi_x + \psi_y + \mu)
\]
\[-(\phi + \psi)(\tau z x + \psi_{zz} y + \epsilon_z) - \sigma(\tau_{zz} x + \psi_{zzzz} y + \epsilon_{zz})
\]
\[= [\psi_{tz} + \psi^2_t + \psi_t(\tau + 1/R_0) - (\phi + \psi)\psi_{zz} - \sigma\psi_{zzzz}]y
\]
\[+ [\tau_t + \psi \psi_z + (\tau + 1/R_0) \phi_z - (\phi + \psi)\tau_z - \sigma\tau_{zz}]x
\]
\[+ \epsilon_t + \epsilon \psi_z + (\tau + 1/R_0) \mu - (\phi + \psi) \epsilon_z - \sigma\epsilon_{zzz},
\]
(3.9)
\[
\Phi_3 = -\phi_t - \psi_t + (\phi + \psi)(\phi_z + \psi_z) - \sigma R(\vartheta + z) + \sigma(\phi_{zz} + \psi_{zz}).
\]
(3.10)
Thus (3.5) is equivalent to the following system of partial differential equations:
\[
\phi_{tz} + \phi^2_t + \tau(\psi - 1/R_0) - \phi_{zz}(\phi + \psi) - \sigma\phi_{zzzz} = \alpha_1,
\]
(3.11)
Solving (3.21) and (3.24) for γ gives

\[ \kappa_t + \kappa \phi_z + \psi_z (\kappa - 1/R_0) - \kappa_z (\phi + \psi) - \sigma \kappa_{zz} = \alpha, \]

(3.12)

\[ \mu_t + \mu \phi_z + (\kappa - 1/R_0) \varepsilon - \mu_z (\phi + \psi) - \sigma \mu_{zz} = \alpha_2, \]

(3.13)

\[ \psi_{tz} + \psi_z^2 + \kappa (\tau + 1/R_0) - (\phi + \psi) \psi_{zz} - \sigma \psi_{zzz} = \beta_1, \]

(3.14)

\[ \tau_t + \tau \psi_z + (\tau + 1/R_0) \phi_z - (\phi + \psi) \tau_z - \sigma \tau_{zz} = \alpha, \]

(3.15)

\[ \varepsilon_t + \varepsilon \psi_z + (\tau + 1/R_0) \mu - (\phi + \psi) \varepsilon_z - \sigma \varepsilon_{zz} = \beta_2 \]

(3.16)

for some \( \alpha, \alpha_1, \alpha_2, \beta_1, \beta_2 \) are functions of \( t \).

Let \( 0 \neq b \) and \( c \) be fixed real constants. Recall the notions in (2.12) and (2.13) with \( \gamma = b \). We assume

\[ \phi = b^{-1} \gamma_1 \zeta_r (z), \quad \psi = b^{-1} (\gamma_2 \zeta_r (z) + \gamma_3 \eta_r (z)), \]

(3.17)

\[ \zeta = \gamma_4 (\gamma_2 \eta_r (z) - (-1)^r \gamma_3 \zeta_r (z)), \quad \tau = \gamma_5 \gamma_1 \eta_r (z), \quad \gamma_4 \gamma_5 = 1, \]

(3.18)

where \( \gamma_i \) are functions of \( t \) to be determined. Moreover, (3.11) becomes

\[ (\gamma_1' + (-1)^r b^2 \sigma \gamma_1 - \gamma_1 \gamma_5/R_0) \eta_r (z) + (\gamma_1 + \gamma_2) \gamma_1 c^r = \alpha_1, \]

(3.19)

which is implied by

\[ \alpha_1 = (\gamma_1 + \gamma_2) \gamma_1 c^r, \]

(3.20)

\[ \gamma_1' + (-1)^r b^2 \sigma \gamma_1 - \gamma_1 \gamma_5/R_0 = 0. \]

(3.21)

On the other hand, (3.15) becomes

\[ [(\gamma_1 \gamma_5)'+ \gamma_1/R_0 + (-1)^r b^2 \sigma \gamma_1 \gamma_5] \eta_r + \gamma_1 \gamma_5 (\gamma_1 + \gamma_2) c^r = \alpha, \]

(3.22)

which gives

\[ \alpha = \gamma_1 \gamma_5 (\gamma_1 + \gamma_2) c^r, \]

(3.23)

\[ (\gamma_1 \gamma_5)' + (-1)^r b^2 \sigma \gamma_1 \gamma_5 + \gamma_1/R_0 = 0. \]

(3.24)

Solving (3.21) and (3.24) for \( \gamma_1 \) and \( \gamma_1 \gamma_5 \), we get

\[ \gamma_1 = b_1 e^{-(1)^r b^2 \sigma t} \sin \frac{t}{R_0}, \quad \gamma_1 \gamma_5 = b_1 e^{-(1)^r b^2 \sigma t} \cos \frac{t}{R_0}, \]

(3.25)

where \( b_1 \) is a real constant. In particular, we take

\[ \gamma_5 = \cot \frac{t}{R_0}. \]

(3.26)

Observe that (3.12) becomes

\[ [(\gamma_2 \gamma_4)' + (-1)^r b^2 \sigma \gamma_2 \gamma_4 - \gamma_2/R_0] \eta_r (z) + \gamma_4 (\gamma_1 \gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2) c^r \]

\[ -(-1)^r [(\gamma_3 \gamma_4)' + (-1)^r b^2 \sigma \gamma_2 \gamma_4 - \gamma_3/R_0] \zeta_r (z) = \alpha \]

(3.27)
and (3.14) becomes
\[\begin{align*}
[\gamma'_2 &+ (-1)^r b^2 \sigma \gamma_2 + \gamma_2 \gamma_4/R_0] \eta_r(z) + (\gamma_1 \gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2) c^r \\
-(-1)^r [\gamma'_3 + (-1)^r b^2 \sigma \gamma_3 + \gamma_3 \gamma_4/R_0] \zeta_r(z) &= \beta_1,
\end{align*}\]
(3.28)
equivalently,
\[\begin{align*}
\alpha &= \gamma_4 (\gamma_1 \gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2) c^r, \\
\beta_1 &= (\gamma_1 \gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2) c^r, \\
(\gamma_2 \gamma_4)' + (-1)^r b^2 \sigma \gamma_2 \gamma_4 - \gamma_2/R_0 &= 0, \\
\gamma'_2 + (-1)^r b^2 \sigma \gamma_2 + \gamma_2 \gamma_4/R_0 &= 0, \\
(\gamma_3 \gamma_4)' + (-1)^r b^2 \sigma \gamma_2 \gamma_4 - \gamma_3/R_0 &= 0, \\
\gamma'_3 + (-1)^r b^2 \sigma \gamma_3 + \gamma_3 \gamma_4/R_0 &= 0.
\end{align*}\]
(3.29)-(3.34)
Solving (3.31)-(3.34) under the assumption $\gamma_4 \gamma_5 = 1$, we obtain
\[\begin{align*}
\gamma_2 \gamma_4 &= b_2 e^{(-1)^r b^2 \sigma t} \sin t R_0, \\
\gamma_2 &= b_2 e^{(-1)^r b^2 \sigma t} \cos t R_0, \\
\gamma_3 \gamma_4 &= b_3 e^{(-1)^r b^2 \sigma t} \sin t R_0, \\
\gamma_3 &= b_3 e^{(-1)^r b^2 \sigma t} \cos t R_0.
\end{align*}\]
(3.35)-(3.36)
In particular, we have:
\[\gamma_4 = \tan \frac{t}{R_0}.\]
(3.37)
According to (3.23) and (3.29),
\[\gamma_1 \gamma_5 (\gamma_1 + \gamma_2) c^r = \gamma_4 (\gamma_1 \gamma_2 + \gamma_2^2 + (-1)^r \gamma_3^2) c^r,\]
(3.38)
equivalently
\[-2b_1 b_2 \cos \frac{2t}{R_0} + (b_2^2 - b_1^2 + (-1)^r b_3^2) \sin \frac{2t}{R_0} = 0.\]
(3.39)
Thus
\[b_1 b_2 = 0, \quad b_2^2 - b_1^2 + (-1)^r b_3^2 = 0.\]
(3.40)
So
\[r = 0, \quad b_2 = 0, \quad b_1 = b_3\]
(3.41)
or
\[r = 1, \quad b_1 = 0, \quad b_2 = b_3.\]
(3.42)
Assume $r = 0$ and $b_1 \neq 0$. Then
\[\phi = b^{-1} b_1 e^{-b^2 \sigma t} \sin bz \sin \frac{t}{R_0}, \quad \psi = b^{-1} b_1 e^{-b^2 \sigma t} \cos bz \cos \frac{t}{R_0}.\]
(3.43)
\[
\varsigma = -b_1 e^{-b_2 \sigma t} \sin bz \sin \frac{t}{R_0}, \quad \tau = b_1 e^{-b_2 \sigma t} \cos bz \cos \frac{t}{R_0}. \tag{3.44}
\]

Moreover, we take \(\mu = \varepsilon = \vartheta = 0\). Then

\[
\Phi_1 = \gamma_1^2 (x + \gamma_3 y) = b_1^2 e^{-2b_2 \sigma t} \sin \frac{t}{R_0} \left( x \sin \frac{t}{R_0} + y \cos \frac{t}{R_0} \right) \tag{3.45}
\]

by (3.8), (3.11)-(3.12), (3.20) and (3.23). Similarly

\[
\Phi_2 = b_1^2 e^{-2b_2 \sigma t} \cos \frac{t}{R_0} \left( x \sin \frac{t}{R_0} + y \cos \frac{t}{R_0} \right). \tag{3.46}
\]

According to (3.10)

\[
\Phi_3 = \left[ b^{-1} R_0^{-1} b_1 e^{-b_2 \sigma t} - b^{-1} b_1^2 e^{-2b_2 \sigma t} \cos \left( bz - \frac{t}{R_0} \right) \right] \sin \left( bz - \frac{t}{R_0} \right) - R\sigma z. \tag{3.47}
\]

By (3.4), we have

\[
p = \frac{Rz^2}{2} + \frac{b_1 e^{-b_2 \sigma t}}{b^2 \sigma R_0} \cos \left( bz - \frac{t}{R_0} \right) - \frac{b_1^2 e^{-2b_2 \sigma t}}{2 \sigma b^2} \cos^2 \left( bz - \frac{t}{R_0} \right) - \frac{b_1^2 e^{-2b_2 \sigma t}}{2 \sigma} \left( y^2 \cos^2 \frac{t}{R_0} + x^2 \sin^2 \frac{t}{R_0} + xy \sin \frac{2t}{R_0} \right). \tag{3.48}
\]

modulo the transformation in (1.14)-(1.16).

Suppose \(r = 1\) and \(b_2 \neq 0\). Then

\[
\phi = \tau = \mu = \varepsilon = \vartheta = 0, \quad \psi = b^{-1} b_2 e^{b_2 b + b^2 \sigma t} \cos \frac{t}{R_0}, \quad \varsigma = b_2 e^{b_2 b + b^2 \sigma t} \sin \frac{t}{R_0}. \tag{3.49}
\]

Moreover,

\[
\Phi_1 = \Phi_2 = 0, \quad \Phi_3 = b^{-1} b_2 R_0^{-1} e^{b_2 b + b^2 \sigma t} \sin \frac{t}{R_0} + b^{-1} b_2^2 e^{2(b_2 b + b^2 \sigma t)} \cos^2 \frac{t}{R_0} - R\sigma z. \tag{3.50}
\]

According to (3.4),

\[
p = \frac{Rz^2}{2} - \frac{b_2 e^{b_2 b + b^2 \sigma t}}{b^2 \sigma R_0} \sin \frac{t}{R_0} - \frac{b_2^2 e^{2(b_2 b + b^2 \sigma t)}}{2b^2 \sigma} \cos^2 \frac{t}{R_0} \tag{3.51}
\]

modulo the transformation (1.14)-(1.16).

**Theorem 3.1.** Let \(b, b_1, b_2 \in \mathbb{R}\) with \(b \neq 0\). We have the following solutions of the three-dimensional stratified rotating Boussinesq equations (1.3)-(1.7): (1)

\[
u = b_1 e^{-b_2 \sigma t} (x \cos bz - y \sin bz) \sin \frac{t}{R_0}, \quad v = b_1 e^{-b_2 \sigma t} (x \cos bz - y \sin bz) \cos \frac{t}{R_0}, \tag{3.52}
\]

\[
w = -b^{-1} b_1 e^{-b_2 \sigma t} \cos \left( bz - \frac{t}{R_0} \right), \quad T = z \tag{3.53}
\]

and \(p\) is given in (3.48); (2)

\[
u = b_2 e^{b_2 b + b^2 \sigma t} y \sin \frac{t}{R_0}, \quad v = b_2 e^{b_2 b + b^2 \sigma t} y \cos \frac{t}{R_0}, \tag{3.54}
\]

and
\[ w = -b^{-1}b_2e^{b_z+b^2\sigma t}\cos \frac{t}{R_0}, \quad T = z \]  

and \( p \) is given in (3.51).

Next we assume \( \phi = \zeta = \psi = \tau = 0 \). Then

\[ \mu_t - \frac{1}{R_0} \varepsilon - \sigma \mu_{zz} = \alpha_2, \quad \varepsilon_t + \frac{1}{R_0} \nu - \sigma \varepsilon_{zz} = \beta_2, \quad \partial_t - \partial_{zz} = 0. \]  

Solving them, we get:

**Theorem 3.2.** Let \( a_i, b_i, c_i, d_i, \hat{a}_r, \hat{b}_r, \hat{c}_r, \hat{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s \) be real numbers. We have the following solutions of the three-dimensional stratified rotating Boussinesq equations (1.3)-(1.7):

\[ u = \cos \frac{t}{R_0} \sum_{i=1}^{m} d_1 e^{a_1^2\sigma t} \cos 2b_i + a_i z \cos b_i \sin(a_i^2 \sigma t \sin 2b_i + a_i z \sin b_i + c_i) \]

\[ + \sin \frac{t}{R_0} \sum_{r=1}^{n} \tilde{d}_r e^{a_r^2 \sigma t} \cos 2\tilde{b}_r + a_r z \cos \tilde{b}_r \sin(\tilde{a}_r^2 \sigma t \sin 2\tilde{b}_r + \tilde{a}_r z \sin \tilde{b}_r + \tilde{c}_r), \]  

\[ v = -\sin \frac{t}{R_0} \sum_{i=1}^{m} d_1 e^{a_1^2\sigma t} \cos 2b_i + a_i z \cos b_i \sin(a_i^2 \sigma t \sin 2b_i + a_i z \sin b_i + c_i) \]

\[ + \cos \frac{t}{R_0} \sum_{r=1}^{n} \tilde{d}_r e^{a_r^2 \sigma t} \cos 2\tilde{b}_r + a_r z \cos \tilde{b}_r \sin(\tilde{a}_r^2 \sigma t \sin 2\tilde{b}_r + \tilde{a}_r z \sin \tilde{b}_r + \tilde{c}_r), \]  

\[ w = 0, \quad T = z + \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{a_s^2 \sigma t} \cos 2\tilde{b}_s + \tilde{a}_s z \cos \tilde{b}_s \sin(a_s^2 \sigma t \sin 2b_s + \tilde{a}_s z \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s), \]

\[ p = \frac{Rz^2}{2} + R \sum_{s=1}^{m} \tilde{d}_s e^{a_s^2 \sigma t} \cos 2\tilde{b}_s + \tilde{a}_s z \cos \tilde{b}_s \sin(a_s^2 \sigma t \sin 2b_s + \tilde{a}_s z \sin \tilde{b}_s + \tilde{c}_s). \]

**Remark 3.3.** By Fourier expansion, we can use the above solution to obtain the one depending on three arbitrary piecewise continuous functions of \( z \).

### 4 Asymmetric Approach II to the 3D Equations

In this section, we solve the stratified rotating Boussinesq equations (1.4)-(1.7) under the assumption

\[ u_z = v_z = w_{zz} = T_{zz} = 0. \]  

Let \( \gamma \) be a function of \( t \) and we use the moving frame \( \tilde{\omega} \) in (2.65). Assume

\[ u = f(t, \tilde{\omega}) \sin \gamma - \gamma' y, \quad v = -f(t, \tilde{\omega}) \cos \gamma + \gamma' x, \]
According to (4.3), we assume
\[ w = \phi(t, \varpi), \quad T = \psi(t, \varpi) + z, \] (4.3)
for some functions \( f, \phi \) and \( \psi \) in \( t \) and \( \varpi \). Using (2.66)-(2.68), we get
\[ \Phi_1 = -(\gamma'^2 + \gamma'/R_0)x - \gamma'' y + f_t \sin \gamma + (2\gamma' + 1/R_0)f \cos \gamma - \sigma f_\varpi \sin \gamma, \] (4.4)
\[ \Phi_2 = -(\gamma'^2 + \gamma'/R_0)y + \gamma'' x - f_t \cos \gamma + (2\gamma' + 1/R_0)f \sin \gamma + \sigma f_\varpi \cos \gamma, \] (4.5)
\[ \Phi_3 = \phi_t - \sigma \phi_\varpi - \sigma R(\psi + z). \] (4.6)

By (3.5), we have
\[ -2\gamma'' + f_\varpi - \sigma f_\varpi = 0, \] (4.7)
\[ \phi_t - \sigma \phi_\varpi - \sigma R\psi = 0. \] (4.8)

Moreover, (1.6) becomes
\[ \psi_t - \psi_\varpi = 0. \] (4.9)

Solving (4.7), we have:
\[ f = 2\gamma' \varpi + \sum_{i=1}^{m} a_i d_i \sin(a_i \varpi \sin b_i + a_i \varpi \sin b_i + b_i + c_i), \] (4.10)
where \( a_i, b_i, c_i, d_i \) are arbitrary real numbers. Moreover, (4.8) and (4.9) yield
\[ \phi = \sum_{r=1}^{n} \hat{a}_r e^{\hat{a}_r t \cos 2b_r + \hat{a}_r \varpi \cos b_r} \sin(\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{c}_r) + \sigma R t \psi, \] (4.11)
\[ \psi = \sum_{s=1}^{k} \tilde{a}_s e^{\tilde{a}_s t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s) \] (4.12)
if \( \sigma = 1 \), and
\[ \phi = \sum_{r=1}^{n} \hat{a}_r e^{\hat{a}_r t \cos 2b_r + \hat{a}_r \varpi \cos b_r} \sin(\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{c}_r) + \frac{\sigma R}{1 - \sigma} \sum_{s=1}^{k} \tilde{a}_s e^{\tilde{a}_s t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s), \] (4.13)
\[ \psi = \sum_{s=1}^{k} \tilde{a}_s^2 \tilde{a}_s e^{\tilde{a}_s t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin(\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s + \tilde{b}_s), \] (4.14)
when \( \sigma \neq 1 \), where \( \hat{a}_r, \hat{b}_r, \hat{c}_r, \tilde{d}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s \) are arbitrary real numbers.

Now
\[ \Phi_1 = (\gamma'' \sin 2\gamma - \gamma'^2 - \gamma'/R_0)x - \gamma'' y \cos 2\gamma + (2\gamma' + 1/R_0)f \cos \gamma, \] (4.15)
\[ \Phi_2 = - (\gamma'' \sin 2\gamma + \gamma'^2 + \gamma' \gamma R_0) y - \gamma'' x \cos 2\gamma (2\gamma' + 1/R_0) f \sin \gamma \]  \tag{4.16} \\

and \( \Phi_3 = -\sigma R z \). According to (3.4), we have

\[
p = \frac{-2\gamma' + 1/R_0}{\sigma} [\sum_{i=1}^{m} d_i e^{a_i'^2 \kappa t} \cos 2b_i + a_i \bar{\omega} \sin b_i + c_i] + \frac{R z}{2} + \frac{(\gamma'^2 + \gamma' R_0)(x^2 + y^2) + \gamma'' (y^2 - x^2) \sin 2\gamma}{2\sigma} + \frac{\gamma'' y \cos 2\gamma}{\sigma} \tag{4.17}
\]

modulo the transformation in (1.14)-(1.16).

**Theorem 4.1.** Let \( a_i, b_i, c_i, d_i, \bar{a}_r, \bar{b}_r, \bar{c}_r, \bar{d}_r, \bar{a}_s, \bar{b}_s, \bar{c}_s, \bar{d}_s \) be real numbers and let \( \gamma \) be any function of \( t \). Denote \( \bar{\omega} = x \cos \gamma + y \sin \gamma \). We have the following solutions of the three-dimensional stratified rotating Boussinesq equations (1.3)-(1.7):

\[
u = \sum_{i=1}^{m} a_i d_i e^{a_i'^2 \kappa t} \cos 2b_i + a_i \bar{\omega} \sin b_i + c_i 
+ 2\gamma' \bar{\omega} \sin \gamma - \gamma' y, \tag{4.18}
\]

\[
w = \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r'^2 \sigma t} \cos 2\hat{b}_r + \hat{a}_r \bar{\omega} \sin \hat{b}_r + \hat{c}_r 
+ \sigma R t \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s'^2 t} \cos 2\tilde{b}_s + \tilde{a}_s \bar{\omega} \sin \tilde{b}_s + \tilde{c}_s, \tag{4.20}
\]

\[
T = z + \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s'^2 t} \cos 2\tilde{b}_s + \tilde{a}_s \bar{\omega} \sin \tilde{b}_s + \tilde{c}_s \tag{4.21}
\]

if \( \sigma = 1 \), and

\[
w = \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r'^2 \sigma t} \cos 2\hat{b}_r + \hat{a}_r \bar{\omega} \sin \hat{b}_r + \hat{c}_r 
+ \sigma R \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s'^2 t} \cos 2\tilde{b}_s + \tilde{a}_s \bar{\omega} \sin \tilde{b}_s + \tilde{c}_s, \tag{4.22}
\]

\[
T = z + \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s'^2 t} \cos 2\tilde{b}_s + \tilde{a}_s \bar{\omega} \sin \tilde{b}_s + \tilde{c}_s \tag{4.23}
\]

when \( \sigma \neq 1 \).
Remark 4.2. By Fourier expansion, we can use the above solution to obtain the one depending on three arbitrary piecewise continuous functions of $\tilde{\omega}$.

Next we let $\alpha$ be any fixed function of $t$ and set

$$\tilde{\omega} = \alpha(x^2 + y^2).$$  \hfill (4.24)

We assume

$$u = y\phi(t, \tilde{\omega}) - \frac{\alpha'}{2\alpha}x, \quad v = -x\phi(t, \tilde{\omega}) - \frac{\alpha'}{2\alpha}y,$$  \hfill (4.25)

$$w = \psi(t, \tilde{\omega}) + \frac{\alpha'}{\alpha}z, \quad T = \vartheta(t, \tilde{\omega}) + z$$  \hfill (4.26)

where $\phi, \psi$ and $\vartheta$ are functions in $t, \tilde{\omega}$. Note

$$\Phi_1 = -\frac{\alpha'^2 + 2\alpha\alpha''}{4\alpha^2}x + \frac{\alpha'}{2R_0\alpha}y + y\phi_t + \left(\frac{x}{R_0} - \frac{\alpha'}{\alpha}y\right)\phi - x\phi^2 - 4\sigma\alpha y(\tilde{\omega}\phi)_{\tilde{\omega}\tilde{\omega}},$$  \hfill (4.27)

$$\Phi_2 = -\frac{\alpha'^2 + 2\alpha\alpha''}{4\alpha^2}y - \frac{\alpha'}{2R_0\alpha}x - x\phi_t + \left(\frac{y}{R_0} + \frac{\alpha'}{\alpha}x\right)\phi - y\phi^2 + 4\sigma\alpha x(\tilde{\omega}\phi)_{\tilde{\omega}\tilde{\omega}}.$$  \hfill (4.28)

According to the first equation in (3.5),

$$\left[\tilde{\omega}\left(\phi_t - \frac{\alpha'}{\alpha}\phi - 4\sigma\alpha(\tilde{\omega}\phi)_{\tilde{\omega}\tilde{\omega}}\right)\right]_\tilde{\omega} + \frac{\alpha'}{2R_0\alpha} = 0,$$  \hfill (4.29)

equivalently,

$$(\tilde{\omega}\phi)_t - \frac{\alpha'}{\alpha}\tilde{\omega}\phi - 4\sigma\alpha\tilde{\omega}(\tilde{\omega}\phi)_{\tilde{\omega}\tilde{\omega}} + \frac{\alpha'\tilde{\omega}}{2R_0\alpha} = \alpha\beta'$$  \hfill (4.30)

for some function $\beta$ of $t$. Write

$$\hat{\phi} = \frac{\tilde{\omega}\phi}{\alpha} + \frac{\tilde{\omega}}{2R_0\alpha} - \beta.$$  \hfill (4.31)

Then (4.30) becomes

$$\hat{\phi}_t - 4\sigma\alpha\tilde{\omega}\hat{\phi}_{\tilde{\omega}\tilde{\omega}} = 0.$$  \hfill (4.32)

Suppose

$$\hat{\phi} = \sum_{i=1}^{\infty} \gamma_i\tilde{\omega}^i,$$  \hfill (4.33)

where $\gamma_i$ are functions of $t$ to be determined. Equation (4.32) yields

$$(\gamma_i)_t = 4i(i + 1)\sigma\alpha\gamma_{i+1}.$$  \hfill (4.34)

Hence

$$\gamma_{i+1} = \frac{(\alpha^{-1}\partial_t)^i(\gamma)}{i!(i + 1)!(4\sigma)^i}.$$  \hfill (4.35)

for some function $\gamma$ of $t$. Thus

$$\hat{\phi} = \sum_{i=0}^{\infty} \frac{(\alpha^{-1}\partial_t)^i(\gamma)\tilde{\omega}^{i+1}}{i!(i + 1)!(4\sigma)^i}.$$  \hfill (4.36)
By \((4.31)\), we get
\[
\phi = \frac{\alpha \beta}{\omega} - \frac{1}{2R_0} + \alpha \sum_{i=0}^{\infty} \frac{(\alpha^{-1} \partial_t)^i(\gamma)\omega^i}{i!(i+1)!(4\sigma)^i}.
\] (4.37)

Note
\[
\Phi_3 = \psi_t + \frac{\alpha'}{\alpha} \psi - 4\sigma(\omega \psi_\omega)_\omega - \sigma R (\psi + z).
\] (4.38)

By the last two equations in (3.5),
\[
\psi_t + \frac{\alpha'}{\alpha} \psi - 4\sigma(\omega \psi_\omega)_\omega - \sigma R\psi = 0 \tag{4.39}
\]
modulo the transformation in (1.14)-(1.16). On the other hand, (1.6) becomes
\[
\vartheta_t - 4(\omega \vartheta_\omega)_\omega = 0.
\] (4.40)

Hence
\[
\vartheta = \sum_{i=0}^{\infty} \frac{\theta_1^{(i)} \omega^{i+1}}{4i((i+1)!)^2}
\] (4.41)
modulo the transformation in (1.14)-(1.16), where \(\theta_1\) is an arbitrary function of \(t\). Substituting (4.41) into (4.39), we obtain
\[
\psi = \alpha^{-1} \theta_2 \omega + \alpha^{-1} \sum_{i=1}^{\infty} \frac{\theta_2^{(i)} + R \sum_{r=0}^{i-1} \sigma^{i-r} (\alpha \theta_1^{(i-s-1)}(\gamma))(s)}{(4\sigma)^i((i+1)!)^2} \omega^{i+1},
\] (4.42)
where \(\theta_2\) is another arbitrary function of \(t\).

Now
\[
\Phi_1 = -\frac{\alpha'^2 + 2\alpha \alpha''}{4\sigma^2} x + \frac{\alpha \beta' y}{\omega} + \frac{x}{R_0} \phi - x \phi^2,
\] (4.43)
\[
\Phi_2 = -\frac{\alpha'^2 + 2\alpha \alpha''}{4\sigma^2} y - \frac{\alpha \beta' x}{\omega} + \frac{y}{R_0} \phi - y \phi^2
\] (4.44)
by (4.27) and (4.28), and
\[
\Phi_3 = (\alpha^{-1} \alpha' - \sigma R) z
\] (4.45)
by (4.38). According to (3.4), we obtain
\[
p = \left(\frac{\alpha^2 + 2\alpha \alpha''}{4\sigma \alpha^2} + \frac{3}{8\sigma R_0^2}\right)(x^2 + y^2) + \frac{\beta'}{\sigma} \arctan \frac{y}{x} + \frac{(R_0 \alpha \gamma - 1) \beta}{\sigma R_0} \ln \alpha(x^2 + y^2)
\]
\[
- \frac{\sigma^{-1} \beta^2}{2(x^2 + y^2)} + \frac{\sigma R - \alpha^{-1} \alpha' R}{2\sigma} z^2 - \frac{1}{\sigma R_0} \sum_{i=0}^{\infty} \frac{(\alpha_i \partial_t)^i(\gamma)\alpha^{-1+i}(x^2 + y^2)^{i+1}}{(i+1)!^2(4\sigma)^{i+1}}
\]
\[
+ \frac{\alpha}{2\sigma} \sum_{i,j=0}^{\infty} \frac{(\alpha_i \partial_t)^i(\gamma)(\alpha_j \partial_t)^j(\gamma)(\alpha(x^2 + y^2))^{i+j+1}}{i!(i+1)!(j+1)!(i+j+1)(4\sigma)^{i+j}}
\]
\[
+ \frac{\alpha \beta}{2\sigma} \sum_{i=1}^{\infty} \frac{(\alpha_i \partial_t)^i(\gamma)(\alpha(x^2 + y^2))^i}{i!(i+1)!(4\sigma)^i}
\] (4.46)
modulo the transformation in (1.14)-(1.16). By (4.25), (4.26), (4.37), (4.41) and (4.42), we have:

**Theorem 4.3** Let $\alpha, \beta, \gamma, \theta_1, \theta_2$ be any function of $t$ such that the following involved series converge. We have the following solutions of the three-dimensional stratified rotating Boussinesq equations (1.3)-(1.7):

$$u = \frac{\beta y}{x^2 + y^2} - \frac{y}{2R_0} - \frac{\alpha'}{2\alpha} x + \alpha y \sum_{i=0}^{\infty} \frac{(\alpha^{-1} \partial_t)^i (\gamma) \alpha^i (x^2 + y^2)^i}{i!(i + 1)! (4\sigma)^i},$$

(4.47)

$$v = \frac{x}{2R_0} - \frac{\alpha'}{2\alpha} y - \beta x + \alpha x \sum_{i=0}^{\infty} \frac{(\alpha^{-1} \partial_t)^i (\gamma) \alpha^i (x^2 + y^2)^i}{i!(i + 1)! (4\sigma)^i},$$

(4.48)

$$w = \theta_2 (x^2 + y^2) + \frac{\alpha'}{\alpha} z + \frac{1}{\alpha} \sum_{i=1}^{\infty} \theta_2^{(i)} + R \frac{1}{(4\sigma)^i ((i + 1)!)^2} \sum_{r=0}^{i-1} \alpha^{i-r} \phi_1^{(i-s-1)} (x^2 + y^2)^{i+1},$$

(4.49)

$$T = z + \sum_{i=0}^{\infty} \frac{\theta_1^{(i)} \alpha^{i+1} (x^2 + y^2)^{i+1}}{4^i ((i + 1)!)^2},$$

(4.50)

and $p$ is given in (4.46).

### 5 Asymmetric Approach III to the 3D Equations

In this section, we solve (1.3)-(1.7) with $v_x = w_x = T_x = 0$.

Let $c$ be a real constant. Set

$$\varpi = y \cos c + z \sin c.$$  

(5.1)

Suppose

$$u = f(t, \varpi), \quad v = \phi(t, \varpi) \sin c,$$

(5.2)

$$w = -\phi(t, \varpi) \cos c, \quad T = \psi(t, \varpi) + z,$$

(5.3)

where $f$, $\phi$ and $\psi$ are functions in $t$ and $\varpi$. Then

$$\Phi_1 = f_t - \sigma f_{\varpi \varpi} - \frac{\sin c}{R_0} \phi,$$

(5.4)

$$\Phi_2 = (\phi_t - \sigma \phi_{\varpi \varpi}) \sin c + \frac{1}{R_0} f,$$

(5.5)

$$\Phi_3 = (\sigma \phi_{\varpi \varpi} - \phi_t) \cos c - \sigma R (\psi + z).$$

(5.6)

By (3.5),

$$f_{\varpi t} - \sigma f_{\varpi \varpi \varpi} - \frac{\sin c}{R_0} \phi_{\varpi} = 0,$$

(5.7)

$$(\phi_t - \sigma \phi_{\varpi \varpi})_\varpi + \frac{\sin c}{R_0} f_\varpi + \sigma R \psi_\varpi \cos c = 0.$$  

(5.8)
Modulo (1.14)-(1.16), we have

\[ f_t - \sigma f_{\varpi\varpi} - \frac{\sin c}{R_0} \phi = 0, \quad (5.9) \]

\[ \phi_t - \sigma \phi_{\varpi\varpi} + \frac{\sin c}{R_0} f + \sigma R \psi \cos c = 0. \quad (5.10) \]

Denote

\[
\begin{pmatrix}
\dot{f} \\
\dot{\phi}
\end{pmatrix} =
\begin{pmatrix}
\cos \frac{t \sin c}{R_0} & -\sin \frac{t \sin c}{R_0} \\
\sin \frac{t \sin c}{R_0} & \cos \frac{t \sin c}{R_0}
\end{pmatrix}
\begin{pmatrix}
f \\
\phi
\end{pmatrix}.
\quad (5.11)
\]

Then (5.9) and (5.10) become

\[ \dot{f}_t - \sigma \dot{f}_{\varpi\varpi} - \sigma R \psi \cos c \sin \frac{t \sin c}{R_0} = 0, \quad (5.12) \]

\[ \dot{\phi}_t - \sigma \dot{\phi}_{\varpi\varpi} + \sigma R \psi \cos c \cos \frac{t \sin c}{R_0} = 0. \quad (5.13) \]

On the other hand, (1.6) becomes

\[ \psi_t - \psi_{\varpi\varpi} = 0. \quad (5.14) \]

Assume \( \sigma = 1 \). We have the following solution:

\[ \psi = \sum_{i=1}^{m} a_i d_i e^{\alpha_i^2 t \cos 2b_i + a_i \varpi \cos b_i} \sin (a_i^2 t \sin 2b_i + a_i \varpi \sin b_i + b_i + c_i), \quad (5.15) \]

\[
\dot{f} = -RR_0 \cot c \cos \frac{t \sin c}{R_0} \sum_{i=1}^{m} a_i d_i e^{\alpha_i^2 t \cos 2b_i + a_i \varpi \cos b_i} \sin (a_i^2 t \sin 2b_i + a_i \varpi \sin b_i + b_i + c_i) \\
+ \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\alpha_r^2 t \cos 2b_r + \hat{a}_r \varpi \cos b_r} \sin (\alpha_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r), \quad (5.16) \]

\[
\dot{\phi} = -RR_0 \cot c \sin \frac{t \sin c}{R_0} \sum_{i=1}^{m} a_i d_i e^{\alpha_i^2 t \cos 2b_i + a_i \varpi \cos b_i} \sin (a_i^2 t \sin 2b_i + a_i \varpi \sin b_i + b_i + c_i) \\
+ \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\alpha_s^2 t \cos 2b_s + \tilde{a}_s \varpi \cos b_s} \sin (\alpha_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s), \quad (5.17) \]

where \( a_i, b_i, c_i, \hat{a}_r, \hat{b}_r, \hat{c}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s \) are arbitrary real numbers. According to (5.11),

\[
f = -RR_0 \cot c \cos \frac{2t \sin c}{R_0} \sum_{i=1}^{m} a_i d_i e^{\alpha_i^2 t \cos 2b_i + a_i \varpi \cos b_i} \sin (a_i^2 t \sin 2b_i + a_i \varpi \sin b_i + b_i + c_i) \\
+ \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\alpha_r^2 t \cos 2b_r + \hat{a}_r \varpi \cos b_r} \sin (\alpha_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r) \\
+ \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\alpha_s^2 t \cos 2b_s + \tilde{a}_s \varpi \cos b_s} \sin (\alpha_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s), \quad (5.18) \]
\[
\phi = -\sin \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 t \cos 2b_r + \hat{a}_r \varpi \cos \hat{b}_r} \sin (\hat{a}_r^2 t \sin 2b_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r)
+ \cos \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2b_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin (\tilde{a}_s^2 t \sin 2b_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s). \tag{5.19}
\]

Suppose \( \sigma \neq 1 \). We take the following solution of (5.11)-(5.14):
\[
\psi = \sum_{i=1}^{m} a_i d_i e^{a_i^2 t + a_i \varpi}, \tag{5.20}
\]
\[
\hat{f} = \sigma R \sum_{i=1}^{m} a_i d_i e^{a_i^2 t + a_i \varpi} \cos c \left[ a_i^2 (1 - \sigma) \sin \frac{t \sin c}{R_0} - R_0^{-1} \sin c \cos \frac{t \sin c}{R_0} \right] \frac{a_i^4 (1 - \sigma)^2 + R_0^{-2} \sin^2 c}{a_i^4 (1 - \sigma)^2 + R_0^{-2} \sin^2 c}
+ \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 t \cos 2b_r + \hat{a}_r \varpi \cos \hat{b}_r} \sin (\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r), \tag{5.21}
\]
\[
\hat{\phi} = \sigma R \sum_{i=1}^{m} a_i d_i e^{a_i^2 t + a_i \varpi} \cos c \left[ a_i^2 (1 - \sigma) \cos \frac{t \sin c}{R_0} - R_0^{-1} \sin c \cos \frac{t \sin c}{R_0} \right] \frac{a_i^4 (1 - \sigma)^2 + R_0^{-2} \sin^2 c}{a_i^4 (1 - \sigma)^2 + R_0^{-2} \sin^2 c}
+ \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin (\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s), \tag{5.22}
\]

where \( a_i, b_i, c_i, \hat{a}_r, \hat{b}_r, \hat{c}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, \tilde{d}_s \) are arbitrary real numbers. According to (5.11),
\[
f = \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 t \cos 2b_r + \hat{a}_r \varpi \cos \hat{b}_r} \sin (\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r)
+ \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin (\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s)
- \sigma R \sum_{i=1}^{m} \frac{a_i d_i e^{a_i^2 t + a_i \varpi} \sin 2c}{2R_0 (a_i^4 (1 - \sigma)^2 + R_0^{-2} \sin^2 c) \sin 2c}, \tag{5.23}
\]
\[
\phi = -\sin \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{a}_r \hat{d}_r e^{\hat{a}_r^2 t \cos 2b_r + \hat{a}_r \varpi \cos \hat{b}_r} \sin (\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{b}_r + \hat{c}_r)
+ \cos \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{a}_s \tilde{d}_s e^{\tilde{a}_s^2 t \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s} \sin (\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{b}_s + \tilde{c}_s)
- \sigma R \sum_{i=1}^{m} \frac{a_i^3 d_i (\sigma - 1) e^{a_i^2 t + a_i \varpi} \cos c}{a_i^4 (1 - \sigma)^2 + R_0^{-2} \sin^2 c}, \tag{5.24}
\]

By (5.4)-(5.6), (5.9) and (5.10), \( \Phi_1 = 0 \),
\[
\Phi_2 = \left( \frac{\cos c}{R_0} f - \sigma R \psi \sin c \right) \cos c, \tag{5.25}
\]
\[ \Phi_3 = \left( \frac{\cos c}{R_0} f - \sigma R \psi \sin c \right) \sin c - \sigma Rz. \]  

(5.26)

According to (3.4),

\[ p = \frac{R \cos^2 c}{\sin c} \cos \frac{2t \sin c}{R_0} \sum_{i=1}^{m} d_i e^{a_i^2 t} \cos 2b_i + a_i \varpi \sin b_i \sin (a_i^2 t \sin 2b_i + a_i \varpi \sin b_i + c_i) \]

\[ - \frac{\cos c}{R_0} \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r^2 t} \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r \sin (\hat{a}_r^2 t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{c}_r) \]

\[ - \frac{\cos c}{R_0} \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s^2 t} \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s \sin (\tilde{a}_s^2 t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s) \]

\[ + R \sin c \sum_{i=1}^{m} d_i e^{a_i^2 t} \cos 2b_i + a_i \varpi \sin b_i + c_i \]

\[ + \frac{R}{2} \varpi, \]  

(5.27)

modulo if \( \sigma = 1 \), and

\[ p = - \frac{\cos c}{\sigma R_0} \cos \frac{t \sin c}{R_0} \sum_{r=1}^{n} \hat{d}_r e^{\hat{a}_r^2 t} \cos 2\hat{b}_r + \hat{a}_r \varpi \cos \hat{b}_r \sin (\hat{a}_r^2 \sigma t \sin 2\hat{b}_r + \hat{a}_r \varpi \sin \hat{b}_r + \hat{c}_r) \]

\[ - \frac{\cos c}{\sigma R_0} \sin \frac{t \sin c}{R_0} \sum_{s=1}^{k} \tilde{d}_s e^{\tilde{a}_s^2 t} \cos 2\tilde{b}_s + \tilde{a}_s \varpi \cos \tilde{b}_s \sin (\tilde{a}_s^2 \sigma t \sin 2\tilde{b}_s + \tilde{a}_s \varpi \sin \tilde{b}_s + \tilde{c}_s) \]

\[ + \frac{R}{2} R \cos \frac{2c}{2 R_0} (a_i^4 (1 - \sigma)^2 + R_0^2 \sin^2 c) + R \sin c \sum_{i=1}^{m} d_i e^{a_i^2 t + a_i \varpi} + \frac{R}{2} \varpi, \]  

(5.28)

modulo the transformation in (1.14)-(1.16).

In summary, we have:

**Theorem 5.1.** Let \( a_i, b_i, c_i, \hat{a}_r, \hat{b}_r, \hat{c}_r, \tilde{a}_s, \tilde{b}_s, \tilde{c}_s, d_i, c \) be arbitrary real numbers. Denote \( \varpi = y \cos x + z \sin c \). We have the following solutions of the three-dimensional stratified rotating Boussinesq equations (1.3)-(1.7):

\[ u = f, \quad v = \phi \sin c, \quad w = -\phi \cos c, \quad T = \psi + z, \]  

(5.29)

where (1) \( f \) is given in (5.18), \( \phi \) is given in (5.19), \( \psi \) is given in (5.15) and \( p \) is given in (5.27) if \( \sigma = 1 \); (2) \( f \) is given in (5.23), \( \phi \) is given in (5.24), \( \psi \) is given in (5.20) and \( p \) is given in (5.28) when \( \sigma \neq 1 \).

**Remark 5.2.** By Fourier expansion, we can use the above solution to obtain the one depending on three arbitrary piecewise continuous functions of \( \varpi \). Applying the transformation \( \mathcal{T}_1 \) in (1.12)-(1.13) to the above solution, we get a solution involving all the variables \( t, x, y, z \).
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