RATIONAL CURVES IN FANO HYPERSURFACES AND TROPICAL CURVES

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Abstract. Using ideas from the theory of tropical curves and degeneration, we prove that any Fano hypersurface (and more generally Fano complete intersections) is swept by at most quadratic rational curves.

1. Introduction

We begin with recalling a famous theorem of S. Mori [4].

Theorem 1. Let $X$ be a nonsingular projective variety with $-K_X$ ample. Then for any point $x \in X$, there is a rational curve through $x$.

Although more than thirty years have passed, it seems that essentially there is no other proof of this theorem other than Mori’s original one, which uses reduction to positive characteristic. In this paper, we attempt to prove this for Fano hypersurfaces in the realm of characteristic zero. Namely, we prove the following. We assume $n \geq 3$.

Theorem 2. Let $X$ be an irreducible hypersurface of degree $d \leq n$ in $\mathbb{P}^n$. If $d = n$, then $X$ is swept by quadratic rational curves. If $d < n$, then $X$ is swept by lines.

This is already known ([2], Exercises 4.4 and 4.10) but we give a simple proof using tropical geometry, which may be of independent interest. Our tools are ideas from the theory of tropical curves [3, 5, 7] and a calculation in the degenerate setting [6]. Using degeneration, it suffices to consider rational curves in a projective space, and we further reduce the problem to a combinatorial one of finding a rational tropical curve with appropriate properties.

In the next section, we recall definitions and facts about tropical curves. The proof of the theorem is given in Section 3.

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2. Tropical curves

Here we quickly recall some definitions and facts about tropical curves. See [3, 5, 7] for more details.

Let $\Gamma$ be a weighted, connected finite graph. Its sets of vertices and edges are denoted $\Gamma^{[0]}$, $\Gamma^{[1]}$, and $w_{\Gamma} : \Gamma^{[1]} \rightarrow \mathbb{N} \setminus \{0\}$ is the weight function. An edge $E \in \Gamma^{[1]}$ has adjacent vertices $\partial E = \{V_1, V_2\}$. Let $\Gamma^{[0]}_{\infty} \subset \Gamma^{[0]}$ be the set of one-valent vertices. We write $\Gamma = \Gamma \setminus \Gamma^{[0]}_{\infty}$. Noncompact edges of $\Gamma$ are called unbounded edges. Let $\Gamma^{[1]}_{\infty}$ be the set of unbounded edges.

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unbounded edges. Let $\Gamma^0, \Gamma^1, w_\Gamma$ be the sets of vertices and edges of $\Gamma$ and the weight function of $\Gamma$ (induced from $\mathbf{w}_\Gamma$ in an obvious way), respectively. Let $N$ be a free abelian group of rank $n \geq 2$ and $N_\mathbb{R} = N \otimes \mathbb{Z} \mathbb{R}$.

**Definition 3** ([3, Definition 2.2]). A parametrized tropical curve in $N_\mathbb{R}$ is a proper map $h : \Gamma \to N_\mathbb{R}$ satisfying the following conditions.

(i) For every edge, $E \subset \Gamma$ the restriction $h|_E$ is an embedding with the image $h(E)$ contained in an affine line with rational slope, or $h(E)$ is a point.

(ii) For every vertex $V \in \Gamma^0$, $h(V) \in N_\mathbb{Q}$ and the following balancing condition holds. Let $E_1, \ldots, E_m \in \Gamma^1$ be the edges adjacent to $V$ and let $u_i \in N$ be the primitive integral vector emanating from $h(V)$ in the direction of $h(E_i)$. Then

\[ \sum_{j=1}^m w(E_j)u_j = 0. \]

An isomorphism of parametrized tropical curves $h : \Gamma \to N_\mathbb{R}$ and $h' : \Gamma' \to N_\mathbb{R}$ is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights such that $h = h' \circ \Phi$.

**Definition 4.** A tropical curve is an isomorphism class of parametrized tropical curves. A tropical curve is trivalent if $\Gamma$ is a trivalent graph. The genus of a tropical curve is the first Betti number of $\Gamma$. The set of flags of $\Gamma$ is

$F_\Gamma = \{(V, E) \mid \partial E \}$.

We often call a tropical curve of genus zero as a rational tropical curve.

**Definition 5.** We call a tropical curve $(\Gamma, h)$ immersive if $h$ is an immersion and if $V \in \Gamma^0$, then $h^{-1}(h(V)) = \{V\}$.

In this paper, all the tropical curves we consider are trivalent and immersive. By (i) of Definition 3 we have a map $u : FT \to N$ sending a flag $(V, E)$ to the primitive integral vector emanating from $h(V)$ in the direction of $h(E)$.

**Definition 6.** The (unmarked) combinatorial type of a tropical curve $(\Gamma, h)$ is the graph $\Gamma$ together with the map $u : FT \to N$. We write this by the pair $(\Gamma, u)$.

**Definition 7.** The degree of a type $(\Gamma, u)$ is a function $\Delta : N \setminus \{0\} \to \mathbb{N}$ with finite support defined by

$\Delta(\Gamma, u)(v) := \#\{(V, E) \in FT \mid E \in \Gamma^1_{\infty}, w(E)u_{(V,E)} = v\}$

Let $e = |\Delta| = \sum_{v \in N \setminus \{0\}} \Delta(v)$. This is the same as the number of unbounded edges of $\Gamma$ (not necessarily of $h(\Gamma)$).

2.1. Toric varieties associated to tropical curves and pre-log curves in them.

**Definition 8.** A toric variety $X$ defined by a fan $\Sigma$ is called associated to a tropical curve $(\Gamma, h)$ if the set of the rays of $\Sigma$ contains the set of the rays spanned by the vectors in $N$ which are contained in the support of the degree map $\Delta : N \setminus \{0\} \to \mathbb{N}$ of $(\Gamma, h)$.

Given a tropical curve $(\Gamma, h)$ in $N_\mathbb{R}$, we can construct a polyhedral decomposition $\mathcal{P}$ of $N_\mathbb{R}$ such that $h(\Gamma)$ is contained in the 1-skeleton of $\mathcal{P}$ ([4, Proposition 3.9]). Given such $\mathcal{P}$, we construct a degenerating family $X \to \mathbb{C}$ of a toric variety $X$ associated to $(\Gamma, h)$.
Section 3]). We call such a family a degeneration of \( X \) defined respecting \((\Gamma, h)\). Let \( X_0 \) be the central fiber. It is a union \( X_0 = \bigcup_{v \in \mathcal{P}[0]} X_v \) of toric varieties intersecting along toric strata. Here \( \mathcal{P}[0] \) is the set of the vertices of \( \mathcal{P} \).

**Definition 9** ([7, Definition 4.1]). Let \( X \) be a toric variety. A holomorphic curve \( C \subset X \) is torically transverse if it is disjoint from all toric strata of codimension greater than one. A stable map \( \phi : C \to X \) is torically transverse if \( \phi^{-1}(\text{int}X) \subset C \) is dense and \( \phi(C) \subset X \) is a torically transverse curve. Here \( \text{int}X \) is the complement of the union of toric divisors.

**Definition 10.** Let \( C_0 \) be a prestable curve. A pre-log curve on \( X_0 \) is a stable map \( \varphi_0 : C_0 \to X_0 \) with the following properties.

1. For any \( v \), the restriction \( C \times_{X_0} X_v \to X_v \) is a torically transverse stable map.
2. Let \( P \in C_0 \) be a point which maps to the singular locus of \( X_0 \). Then \( C \) has a node at \( P \), and \( \varphi_0 \) maps the two branches \( (C'_0, P), (C''_0, P) \) of \( C_0 \) at \( P \) to different irreducible components \( X_{v'}, X_{v''} \subset X_0 \). Moreover, if \( w' \) is the intersection index of the restriction \( (C'_0, P) \to (X_{v'}, D') \) with the toric divisor \( D' \subset X_{v'} \), and \( w'' \) accordingly for \( (C''_0, P) \to (X_{v''}, D'') \), then \( w' = w'' \).

Suppose we are given a torically transverse rational curve \( \varphi : \mathbb{P}^1 \to X \) in a toric variety. Then there is some rational tropical curve \( (\Gamma, h) \) with the following properties.

- The toric variety \( X \) is associated to \((\Gamma, h)\).
- Let \( \mathcal{X} \to \mathbb{C} \) be a degeneration of \( X \) respecting \((\Gamma, h)\) and \( X_0 = \bigcup_{v \in \mathcal{P}[0]} X_v \) the central fiber. Then there is a family of prestable curves \( \mathcal{C} \to \mathbb{C} \) whose generic fiber is \( \mathbb{P}^1 \), and a family of stable maps over \( \mathbb{C} \)
  \[ \Phi : \mathcal{C} \to \mathcal{X}, \]
  such that the restriction to \( 1 \in \mathbb{C} \) is \( \varphi \).
- The restriction to \( 0 \in \mathbb{C} \), \( \varphi_0 : C_0 \to X_0 \) is maximally degenerate (see below).
- The tropical curve \((\Gamma, h)\) is the dual intersection graph of this maximally degenerate rational curve.

Here, a pre-log curve \( \varphi_0 : C_0 \to X_0 \) is maximally degenerate if for any \( v \in \mathcal{P}[0] \), the projection \( \pi_v : C_0 \times_{X_0} X_v \to X_v \) satisfies the following properties:

- Let \( D_v \) be the union of toric divisors of \( D_v \). When \( \dim X \geq 3 \), then \( \pi_v^{-1}(D_v) \) is at most three points, and the image of \( \pi_v \) is contained in the closure of the orbit of a one or two dimensional subtorus of the torus acting on \( X \) (note that this torus also acts on each component of \( X_0 \)).
- When \( \dim X = 2 \), then the case where the image of \( \pi_v \) is the union of the closures of transversally intersecting orbits of one dimensional subtori is also allowed.

Conversely, given a rational tropical curve, we can construct a maximally degenerate rational curve in \( X_0 \), and we can lift it to a smooth rational curve in a generic fiber of \( \mathcal{X} \to \mathbb{C} \) (when \( \dim X = 2 \), a nodal rational curve). See [7] for more information about these results and definitions. See also Remark [14] below.

### 3. RATIONAL CURVES IN FANO HYPERSURFACES

Here we give a proof of Theorem [2] It suffices to prove the claim for a hypersurface of degree \( d \) defined by a generic polynomial \( f \). Consider the degeneration
\[ z_0 z_1 \cdots z_{d-1} + tf = 0, \]
where $z_i$ are homogeneous coordinates of $\mathbb{P}^n$. The central fiber $X_0$ is a union of $d \mathbb{P}^{n-1}$s, intersecting along toric divisors. Due to the assumption that the degree $d$ is less than $n + 1$, each component of $X_0$ has a divisor which is not contained in other components (we call it a free toric divisor). We mainly argue the case where each component of $X_0$ has just one free toric divisor (i.e., $d = n$), since the other cases are easier.

Singular locus $S$ of the total space $\mathfrak{X}$ of the degeneration is given by the equations

$$z_i = z_j = f = t = 0, \ i \neq j.$$ 

Let

$$X_0 = \cup_{i=1}^d \mathbb{P}^{n-1}_i$$

be the decomposition to irreducible components. In $\mathbb{P}^{n-1}_i$, consider a rational curve of degree $d$. If it is smooth, then it has $dn + n - 4$ dimensional moduli. As we argued in [5], a necessary condition for a curve $\varphi_0: C \to \mathbb{P}^{n-1}_i$ to be liftable to a general fiber of $\mathfrak{X} \to \mathbb{C}$ is that any intersection of $\varphi_0(C)$ with the toric divisors of $\mathbb{P}^{n-1}_i$ is contained in $S$. This condition gives at most $d(n - 1)$ dimensional condition. We call them incidence condition. Here the factor of $n - 1$ is the maximal number of non-free toric divisors.

Thus, the expected dimension of smooth rational curves of degree $d$ satisfying the incidence condition is

$$dn + n - 4 - d(n - 1) = n + d - 4.$$ 

To sweep the hypersurface, we need at least $n - 2$ dimensional family. So we consider rational curves of degree two. It suffices to prove that the obstruction cohomology class of a general member of such a family vanishes.

Recall that a general embedded rational curve in $\mathbb{P}^{n-1}$ can be described using a tropical curve ([7]). We use this description to calculate the obstruction cohomology class, as we did in [5, 6].

By perturbing the incidence condition if necessary, we can assume the rational curve is generic, so that it corresponds to a trivalent, embedded tropical curve.

**Remark 11.** By the statement that ”rational curves in $\mathbb{P}^{n-1}$ are described by tropical curves”, we mean the following:

- Fix a general rational curve in $\mathbb{P}^{n-1}$ and a general rational tropical curve in $\mathbb{R}^{n-1}$ of the same degree (we do not impose any relation between these objects).
- Then, we can take
  - a neighborhood $U$ of the rational curve in the moduli space of rational curves of the given degree in $\mathbb{P}^{n-1}$, and
  - a neighborhood $V$ of the rational tropical curve in the moduli space of rational tropical curves of the given degree in $\mathbb{R}^{n-1}$ ($V$ can be taken so that it is diffeomorphic to an open subset of $\mathbb{R}^N$ for some $N$),
- so that $U$ can naturally be considered as a complexification of $V$.

In particular, given a tropical curve, we cannot tell what the rational curve precisely corresponding to it is. In other words, there is no canonical correspondence between holomorphic and tropical curves (we need to specify some artificial incidence conditions to obtain
Given a trivalent rational tropical curve of degree $d$, we can construct a degeneration $P \rightarrow C$ of $\mathbb{P}^{n-1}$ and also construct a $(dn + n - 4)$-dimensional family $F$ of maximally degenerate curves in the central fiber $P_0$.

- We can lift the curves in $F$, so that we obtain a $(dn + n - 4)$-dimensional family of smooth rational curves in a generic fiber of $P$.
- By suitably choosing $f$, some member of this $(dn + n - 4)$-dimensional family satisfies the incidence condition imposed by $S$.

See [7] for the construction of the family of rational curves from tropical curves. The last claim can be seen by a simple dimension count.

A usual tropical curve, as defined in Definition 4, is a proper map $\varphi : \Gamma \rightarrow \mathbb{R}^{n-1}$ from an abstract graph to the affine space. In particular, it has unbounded edges, which correspond to the intersections with the toric divisors. However, in our situation, there are two types of intersections with the toric divisors: one free toric divisor and $(n - 1)$ non-free toric divisors. So we add two types of one-valent vertices to unbounded edges to distinguish them, see Figure 1.

Figure 1. Two types of unbounded edges with one-valent vertex attached. The left figure represents the unbounded edge corresponding to the intersection with a non-free toric divisor, and the right figure represents the unbounded edge corresponding to the intersection with a free toric divisor.

Now we describe the normal sheaf of a nodal rational curve corresponding to a tropical curve. We take a degeneration of $\mathbb{P}_i^{n-1}$ so that the rational curve becomes maximally degenerate (see [7]), and study the normal sheaf on each component. The singular locus $S$ also degenerates, and we write its degeneration by the same letter $S$ (see [6], Subsection 3.2.3). A similar calculation was done in [5], the difference here is the existence of the non-free toric divisors.

A component of a maximally degenerate curve corresponds to a trivalent vertex of the corresponding tropical curve. In particular, each component is contained in the closure of the orbit of a two dimensional subtorus of the torus acting on $\mathbb{P}_i^{n-1}$. By perturbing the incidence condition again, we can assume the following:

(*) At each intersection of the maximally degenerate rational curve and $S$, this orbit closure of the two dimensional torus is transversal to $S$.

There are three types of edges:
- An edge corresponding to a node.
- An edge corresponding to an intersection with a free toric divisor.
- An edge corresponding to an intersection with a non-free toric divisor.
If we do not take into account any special conditions (in other words, if each of the three edges is the one corresponding to an intersection with a free toric divisor), the normal sheaf of a torically transverse rational curve in an $n - 1(\geq 2)$ dimensional toric variety which is contained in the orbit closure of a two dimensional subtorus is given by
\[ \mathcal{O}(1) \oplus \mathcal{O}^{\oplus n-3}. \]
Here the component $\mathcal{O}(1)$ is the normal sheaf as a map to the orbit closure, and the component $\mathcal{O}^{\oplus n-3}$ is the part transverse to the orbit closure. By Serre duality, the first cohomology of it is dual to
\[ H^0(\mathbb{P}^1, (\mathcal{O}(-1) \oplus \mathcal{O}^{\oplus n-3}) \otimes \omega_{\mathbb{P}^1}), \]
where $\omega_{\mathbb{P}^1}$ is the canonical sheaf. This gives the (dual of) obstruction class, and we will calculate it when there are above extra conditions.

The two from the three types of edges affect this calculation as follows:

- A node changes $\omega_{\mathbb{P}^1}$ to $\omega_{\mathbb{P}^1}(1)$ (Serre duality for nodal curves).
- An edge corresponding to an intersection with a non-free toric divisor changes $\mathcal{O}(1)$ component to $\mathcal{O}$, by the calculation in [5] and the above assumption (*).

Our purpose was to show that there is a family of rational curves satisfying the incidence condition $S$, such that the obstruction cohomology class of a general member of it vanishes. In view of Remark [11], it suffices to find a tropical curve whose obstruction cohomology class (more precisely, the obstruction cohomology class of the nodal rational curve associated to the tropical curve) calculated according to the above rule vanishes.

Such a tropical curve is given in the following way. Let
\[ e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \ldots, \quad e_{n-1} = (0, \ldots, 0, 1) \]
be the standard basis of $\mathbb{R}^{n-1}$. A rational tropical curve of degree $n - 1$ in $\mathbb{R}^{n-1}$ has $(n-1)$ unbounded edges in each directions
\[ -e_1, \quad -e_2, \ldots, \quad -e_{n-1}, \quad d_n = e_1 + e_2 + \cdots + e_{n-1}, \]
where we take the direction of an unbounded edge to be the one emanating from the unique adjacent vertex. We assume $-e_1$ is the direction corresponding to the (unique) free face.

Then consider the following rational quadratic tropical curve. Here and hereafter, all the edges have weight one.

![Figure 2](image_url)

Figure 2.

Now we prove the following, which completes the proof of Theorem [2].
Lemma 12. The maximally degenerate curve corresponding to the tropical curve of Figure 2 has vanishing obstruction.

Proof. The tropical curve has four types of trivalent vertices:

According to the rule described above, the normal sheaf of the component corresponding to the leftmost vertex is given by

$$\mathcal{O} \oplus \mathcal{O}^{n-3}.$$  

By Serre duality for nodal curves, the obstruction cohomology of this component is isomorphic to the dual of the following:

$$H^0(\mathbb{P}^1, (\mathcal{O} \oplus \mathcal{O}^{n-3}) \otimes \omega_{\mathbb{P}^1}(1)) = H^0(\mathbb{P}^1, \mathcal{O}(-1) \oplus \mathcal{O}^{n-3}(-1)) = 0.$$  

Similarly, the normal sheaves are given by the following:

$$\mathcal{O}(1) \oplus \mathcal{O}^{n-3},$$  
$$\mathcal{O} \oplus \mathcal{O}^{n-3},$$  
$$\mathcal{O}(-1) \oplus \mathcal{O}^{n-3}.$$  

Their corresponding (dual of the) obstruction classes are given as follows:

$$H^0(\mathbb{P}^1, (\mathcal{O} \oplus \mathcal{O}^{n-3}) \otimes \omega_{\mathbb{P}^1}(2)) = H^0(\mathbb{P}^1, \mathcal{O}(-1) \oplus \mathcal{O}^{n-3}),$$  
$$H^0(\mathbb{P}^1, (\mathcal{O} \oplus \mathcal{O}^{n-3}) \otimes \omega_{\mathbb{P}^1}(2)) = H^0(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}^{n-3}),$$  
$$H^0(\mathbb{P}^1, (\mathcal{O}(1) \oplus \mathcal{O}^{n-3}) \otimes \omega_{\mathbb{P}^1}(1)) = H^0(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}^{n-3}(-1)).$$  

In each of these three cases, the cohomology does not vanish, but one sees that if one knows that a section representing the cohomology class is zero at one point, then that section is itself zero.

Now let us look at the tropical curve of Figure 2. There is a vertex of the leftmost type, and the obstruction class restricted to this component is zero. Then by above observation, one sees that the obstruction of the whole maximally degenerate curve corresponding to the tropical curve is zero.

To see that the family of these curves sweeps the hypersurface, one has to check that the family is not contained in a proper subvariety. However, it is easy to see that in the tropical curve of Figure 2, with the normal sheaf given above, one of the two vertices intersecting the free face has $n-2$ dimensional freedom to move, so sweeps an open subset of the free face (once the place of this vertex is chosen, the place of the other vertex is determined uniquely by incidence conditions). This shows that the family of curves cannot contained in a proper subvariety. □
Remark 13. By the same proof as above and the calculation at the beginning of this section, it is easy to see that a Fano hypersurface of degree \( d \leq n - 2 \) in \( \mathbb{P}^n \) is swept by lines, which may be easily proved by induction taking various hyperplane sections.

Remark 14. As in [1], the argument for the degeneration of hypersurfaces is extended to complete intersection without any essential change. Thus, a Fano complete intersection

\[ X = \cap_{i=1}^{k} V_i \subset \mathbb{P}^{k+n-1}, \]

where \( V_i \) is a hypersurface of degree \( n_i > 1 \), with

\[ \sum_{i=1}^{k} n_i = d \leq k + n - 1 \]

is swept by quadratic rational curves (when \( d = k + n - 1 \)), or by lines (when \( d < k + n - 1 \)).

Remark 15. Combining with the results in [5] and moderate tropical intersection theory, we can study more general higher genus curves in Fano (or more general varieties which have toric degeneration) varieties using tropical technique.

References

[1] Katz, S., Lines on complete intersection threefolds with \( K = 0 \). Math. Z. 191 (1986), no. 2, 293-296.
[2] Kollár, J., Rational curves on algebraic varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge A Series of Modern Surveys in Mathematics, Vol. 32.
[3] Mikhalkin, G., Enumerative tropical algebraic geometry in \( \mathbb{R}^2 \). J. Amer. Math. Soc. 18 (2005), no. 2, 313–377.
[4] Mori, S., Projective manifolds with ample tangent bundles. Ann. Math. 110 (3): 593-606.
[5] Nishinou, T., Correspondence theorems for tropical curves. Preprint.
[6] Nishinou, T., Counting curves via degeneration. Preprint.
[7] Nishinou, T. and Siebert, B., Toric degenerations of toric varieties and tropical curves. Duke Math. J. 135 (2006), no. 1, 1–51.

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