Coulomb blockade of tunneling between disordered conductors

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Coulomb blockade of tunneling [1] has been investigated extensively in the last decade. Most of these studies were focussed on ultrasmall tunnel junctions with small capacitance $C$. Then, the relevant energy scale is the charging energy $E_C = e^2/2C$. In these effectively zero–dimensional junctions the charge of a tunneling electron is distributed over the junction area fast compared to the time scale $h/eV$ associated with the applied voltage $V$, and Coulomb blockade effects are observable for single junctions only because of the slow charge relaxation via the environmental impedance $Z(\omega)$ [2].

The situation changes for tunnel junctions with large lateral extensions where the time scale of propagation of the electromagnetic field over the system size becomes relevant. Then screening and relaxation processes within the electrodes need to be considered explicitly. Particularly, in junctions with disordered low–dimensional electrodes, the field propagates diffusively, and a suppression of the tunneling conductance arises from the reduction of the electron density of states by electron–electron interactions [3,4].

The pioneering papers on the diffusive anomalies in disordered conductors [5,6] are based on perturbation theory in the Coulomb interaction and fail at very low energy scales. As is well–known from the conventional theory of Coulomb blockade in effectively zero–dimensional junctions [3,4] a non–perturbative approach is needed for a complete treatment of the Zero–Bias Anomaly (ZBA) of the conductance. Furthermore, apart from the interaction within the electrodes, the Coulomb interaction between the electrodes needs to be taken into account. Only in limiting cases, e.g., for a strongly disordered electrode screened by a bulky electrode, can the mutual interaction be incorporated into a screened Coulomb potential.

Here, we present a microscopic approach treating all low energy excitations of the Coulomb field non–perturbatively and obtain explicit results for the current–voltage relation of zero–, one–, and two–dimensional junctions at finite temperature. Specifically, we consider a tunnel junction consisting of two disordered metallic electrodes separated by a thin insulating barrier with a chemical potential difference $\mu_1 - \mu_2 = eV$. The tunneling current $I$ can be calculated from the tunneling Hamiltonian

$$H_T = \sum_{k,q} T_{kq} c_{kq}^\dagger c_{kq} + \text{h.c.,} \quad (1)$$

where $k$ and $q$ label the momenta of electrons in the right and left electrode, respectively [3]. To leading order in $H_T$ the current reads

$$I(V) = -2e \text{Im}[X^{\text{ret}}(eV)]. \quad (2)$$

Here $X^{\text{ret}}(\omega)$ is the Fourier transform of

$$X^{\text{ret}}(t) = -i \theta(t) \sum_{k,q} T_{kq} T_{k'q'}^* \langle [c_{kq}^\dagger(t)c_{kq}(t), c_{k'q'}^\dagger c_{k'q'}] \rangle \quad (3)$$

where we put $\hbar = 1$, and the time evolution of the Fermi operators arises from the Hamiltonian in the absence of tunneling, but in the presence of disorder and interactions.

To evaluate the expectation value with four Fermi operators in Eq. (3), we employ the Keldysh $\sigma$–model technique [3,4] leading to the result in Eq. (1) below. We start from the partition function

$$Z = \mathcal{N} \int D\tilde{\psi}_1 D\psi_1 D\tilde{\psi}_2 D\psi_2 D\phi \ e^{iS[\tilde{\psi}_1, \psi_1, \tilde{\psi}_2, \psi_2, \phi]}, \quad (4)$$

where $\mathcal{N}$ is a normalization constant. The action is given by

$$S[\tilde{\psi}_j, \psi_j, \phi] = \sum_j \int_C dt \int dV_j \tilde{\psi}_j(r,t) \left[ i \frac{\partial}{\partial t} + \frac{1}{2m} \Delta \right]$$

$$- W_j(r) + e \phi(r,t) \right] \psi_j(r,t)$$

$$+ \frac{1}{8\pi} \int_C dt \int dr \left[ \nabla \phi(r,t) \right]^2, \quad (5)$$

where $C$ is the Keldysh contour, $W_j(r)$ are Gaussian distributed disorder potentials, $\phi$ is the electric potential, and $e$ is the electric charge. While the fermionic terms are integrated over the volume of the electrodes only, the last term is integrated over entire space, which leads to electrostatic coupling between the electrodes. For simplicity we disregard polarization effects in the tunnel barrier, that may easily be incorporated in the capacitance $C_0$ introduced below.
Assuming that the impurity potentials in the two electrodes are statistically independent, the Gaussian integrations over the realizations of $W_j(r)$ lead to quartic interactions of the fermion fields within each electrode. Within the $\sigma$–model approach these interactions can be decoupled by introducing Hubbard–Stratonovich matrix fields $\hat{Q}_j\text{[3]}. Subsequently, the fermionic fields can be integrated out in the usual way yielding a representation of the impurity averaged partition function as a functional integral over the electric potential $\phi$ and the matrix fields $\hat{Q}_j$. Since for given field $\phi$ all of these transformations can be done independently for each electrode, the resulting action $S[\phi, \hat{Q}_1, \hat{Q}_2]$ can immediately be inferred from earlier work [3].

To proceed we follow Kamenev and Andreev [7] and evaluate the integrals over the matrix fields $\hat{Q}_j$ using the saddle point approximation. The saddle point solution can be obtained analytically for spatially uniform fields $\phi(t)$ only. Provided $\phi$ is essentially constant on scales of the size of the mean free path $l$, an effective action was derived in [3] which incorporates dynamic screening in the random phase approximation (RPA) and the diffuson vertex correction within each electrode.

One is left with a Gaussian integral over the Coulomb field with a purely electromagnetic action which determines the field propagators. Since these depend on the geometry of the electrodes, we first restrict ourselves to the case of thin films. Then, for low energy excitations with wave vectors $q \ll a^{-1}$, where $a$ is the thickness of the films, we have effectively a two–dimensional problem, where the Coulomb field depends on two spacial coordinates in each electrode (see also the discussion below). Although only the large distance behavior of the Coulomb interaction matters, we have to keep the barrier thickness $\Delta$ finite since dipolar interactions arise [3].

With the sources restricted to two–dimensional films at $z_j = \pm \Delta/2$, the bare Coulomb interactions reads

$$U_{ij}^{(0)}(x - x', y - y') = \int dz dz' \frac{e^{2\delta(z - z_i)\delta(z' - z_j)}}{|r - r'|}.$$  

The resulting equation for the matrix $U^{rest}(q, \omega)$ of the Fourier transformed field propagator then takes the form of the diffusive RPA

$$U^{rest}(q, \omega) = \left[ \left( U^{(0)}(q, \omega) \right)^{-1} + P(q, \omega) \right]^{-1},$$  

where

$$U^{(0)}(q, \omega) = \begin{pmatrix} u(q) & v(q) \\ v(q) & u(q) \end{pmatrix},$$  

($U^{(0)}$) is the Coulomb interaction matrix.

Here $u(q)$ and $v(q)$ are the two–dimensional bare Coulomb interactions inside and between the electrodes, respectively. Neglecting polarization effects in the barrier, we obtain $u(q) = 2\pi e^2/q$ and $v(q) = 2\pi e^2 e^{-q\Delta}/q$.

The polarization function

$$P(q, \omega) = \begin{pmatrix} \nu_1 & D_{1}\nu_{1}^2 \\ 0 & \nu_2 D_{2}\nu_{2}^2 \end{pmatrix}$$  

contains the bare electron densities $\nu_j$ and the electron diffusion constants $D_j$ of the two electrodes.

Supplementing the partition function $\hat{Z}$ with source fields that couple to the fermion operators of each electrode, it may be employed as a generating functional to evaluate impurity averaged correlation functions like

$$X^{>}(t) = -i \sum_{k,q} T_k q e^{\frac{1}{2} c_k^\dagger(t)c_q(t) c_q c_k}. $$  

The pre–exponential factors resulting from functional derivatives with respect to the sources are replaced by their saddle point approximation. In terms of an average tunnel matrix element $T$, we find

$$X^{>}(t) = -i \nu_1 \nu_2 |T|^2 \int \frac{d\epsilon}{2\pi} \int \frac{d\epsilon'}{2\pi} e^{\frac{i}{2} (\epsilon - \epsilon') t}$$  

$$\times n(\epsilon)[1 - n(\epsilon')] e^{\frac{i}{2}(\epsilon - \epsilon')t},$$  

where the interaction effects are incorporated in

$$J(t) = 2 \int_0^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{1}{1 - e^{-\beta \omega}} \text{Im } Y(\omega).$$  

Here $\beta$ is the inverse temperature and

$$Y(\omega) = \sum_{i,j} \sum_q (2\delta_{ij} - 1) \frac{U_{ij}(q, \omega)}{(D_j q^2 - i\omega)(D_j q^2 - i\omega)}. $$  

The factors $(D_j q^2 - i\omega)^{-1}$ come from the vertex correction terms in the action and are seen to obey the classical diffusion equation. When the two–dimensional result for $Y(\omega)$ is inserted into Eq. (11), the $\omega$–integral needs to be cut off at frequencies of order $|\omega| \approx 1/\tau_0 \approx a^2/D$ where the crossover from two– to three–dimensional behavior occurs.

Now, combining Eqs. (3) and (11), the expression for the current is found to read

$$I(V) = \frac{G_0}{e} \int_{-\infty}^{\infty} d\epsilon \text{P}(\epsilon|V - \epsilon) \frac{1 - e^{-\beta \epsilon}}{1 - e^{-\beta \epsilon}},$$  

where $G_0 = 4\pi \nu_1 \nu_2 |T|^2$ is the bare tunneling conductance, and where we have introduced the spectral density

$$P(\epsilon) = \frac{1}{\pi} \text{Re } \int_0^{\infty} dt e^{i\epsilon t} e^{J(t)}. $$  

The expression (13) constitutes the central result of this work. It is formally identical to the conventional expression for the current–voltage relation of ultrasmall tunnel junctions (zero–dimensional case) [3], and determines the non–perturbative effective of Coulomb interactions on the $I-V$–curve of spatially extended disordered tunnel junctions. The very same form can also be derived in the case of effectively one–dimensional wires (see below).
To make contact with previous work, we first consider the (somewhat unrealistic) case where Coulomb interactions between the electrodes are disregarded, i.e., $U_{12} = U_{21} = 0$. Then $J(t) = J_1(t) + J_2(t)$ where $J_j(t)$ contains the contribution from $U_{jj}$ only. Replacing $J(t)$ by $J_j(t)$ in Eq. (14), we may define spectral densities $P_j(\epsilon)$ for each electrode. The current can then be written in the familiar form

$$I = 4\pi e|T|^2 \int d\epsilon \nu_j(\epsilon)\nu_j(\epsilon - eV) [n(\epsilon) - n(\epsilon - eV)]$$  \hspace{1cm} (15)

with the densities of states

$$\nu_j(\epsilon) = V_j \int_{-\infty}^{\infty} d\epsilon' \frac{1 + e^{-\beta\epsilon}}{1 + e^{-\beta\epsilon}} P_j(\epsilon - \epsilon').$$  \hspace{1cm} (16)

This relation gives a non-perturbative result for the effective tunneling density of states of an electrode in presence of Coulomb interactions expressed in terms of the spectral density $P_j(\epsilon)$ familiar from Coulomb blockade theory. When $P_j(\epsilon)$ is replaced by its perturbative approximation, the seminal result by Altshuler, Aronov, and Lee [1] is recovered.

In particular, for a two-dimensional film at zero temperature the density of states [14] reads

$$\nu(\epsilon) = \nu \exp\left[ -\frac{1}{4\pi g} \log |\tau_0| \log \left( \frac{|\epsilon|}{(D\kappa^2)^2\tau_0} \right) \right],$$  \hspace{1cm} (17)

where we have suppressed the index $j$. Further, $g = 2\pi eD$ is the conductance in units of $e^2/2\pi$ and $\kappa = 2\pi e^2\nu$ is the inverse screening length in two dimensions. The non-perturbative result [14] has been obtained previously by Kamenev and Andreev [3].

We now return to the full problem and determine the non-perturbative $I-V$ curve for two-dimensional interacting electrodes at zero temperature. Then, the four terms in Eq. (12) cannot be split into contributions of each electrode, and the spectral function $P(\epsilon)$ characterizes the coupled system. In the parameter range relevant for tunneling experiments $u(q) - v(q) \approx 2\pi e^2\Delta$, and we obtain from Eq. (12)

$$Y(\omega) = \frac{e^2}{C_0} \sum_q \sum_{j=1}^2 \frac{\lambda_j}{(D_j^* q^2 - i\omega)(D_j q^2 - i\omega)}.$$  \hspace{1cm} (18)

Here we have introduced the field diffusion constant $D^* = (2\delta_1 D_1 + \delta_2 D_2 + D_1^2)/(\delta_1 + \delta_2)$, with $\delta_j = D_j \kappa \Delta$ and the numerical factors $\lambda_j = (2\delta_i + D_j - D_i)/(2\delta_i + \delta_j)$ ($i \neq j$). While $D$ and $\kappa$ are properties of the electrodes, $\Delta$ may also be expressed in terms of the capacitance per unit area $C_0 = 1/4\pi \Delta$ of the junction. With the result (18) we readily obtain from Eq. (13) for the zero temperature differential conductance $G(V) = \partial I/\partial V$ at voltages $V \ll V_0$

$$\frac{G(V)}{G_0} = \frac{e^{-2\gamma/g}}{\Gamma(1 + 2/g)} \left( \frac{V}{V_0} \right)^{2/g}.$$  \hspace{1cm} (19)

where $\gamma = 0.577\ldots$ is Euler’s constant, $g = \frac{8\pi^2}{\pi} C_0 D^* \left( \sum_j \lambda_j \ln(1/\xi_j)/(1 - \xi_j) \right)^{-1}$ with $\xi_j = D_j/D^*$ is a dimensionless parameter, and $V_0 = 2\pi/e\tau_0 \approx 2\pi D/ea^2$.

When $g \gg 1$ we recover the perturbative result

$$G(V) = G_0[1 + 2/g \ln(V/V_0)].$$  \hspace{1cm} (20)

Hence in the non-perturbative approach the logarithmic corrections of perturbation theory are exponentiated to a power-law dependence on $V$. In the limiting case where one of the electrodes is bulky, i.e., $\kappa_1 \gg \kappa_2$, this result can be rewritten by means of Eq. (20) in terms of an effective tunneling density of states of the other electrode

$$\nu(\epsilon) = \nu_0 \frac{e^{-2\gamma/g}}{\Gamma(1 + 2/g)} \left( \frac{\epsilon}{eV_0} \right)^{2/g}.$$  \hspace{1cm} (21)

This gives a non-perturbative generalization of the result obtained by Altshuler, Aronov, and Zyzulin [2].

We now turn to one-dimensional contacts. Then the result [15] for $Y(\omega)$ remains valid provided the bare interactions $u(q)$ and $v(q)$ in Eq. (12) are replaced by $u(q) = 2e^2\ln(1/qa)$ and $v(q) = 2e^2\ln(1/q\Delta)$. Further the momentum sum is one-dimensional, and $C_0 = [4\ln(\Delta/a)]^{-1}$ becomes the capacitance per unit length. The zero temperature differential conductance now takes the form

$$\frac{G(V)}{G_0} = 1 - \text{erf} \left( \sqrt{\frac{V}{V_0}} \right)$$  \hspace{1cm} (22)

with $V_0 = \frac{e^3}{8\pi D^* C_0^*} \left[ \sum_j \lambda_j (1 + 1^{1/2}) \right]^2$. This result is formally equivalent to the conductance of an ultrasmall junction biased via a RC transmission line [14] and is in quantitative agreement with a recent experimental study of long tunnel junctions [1].

If one of the electrodes is bulky, we again may use Eq. (10) to obtain an effective tunneling density of states of a one-dimensional electrode

$$\nu(\epsilon) = \nu_0 \left( 1 - \text{erf} \left( \sqrt{\frac{eV_0}{\epsilon}} \right) \right).$$  \hspace{1cm} (23)

While for large energies we recover the perturbative result $\nu(\epsilon) = \nu_0(1 - 2\sqrt{eV_0/\pi\epsilon})$, the non-perturbative result (23) does not diverge at low energies but approaches an exponential suppression of the densities of states near the Fermi surface $\nu(\epsilon) = \nu_0\sqrt{\pi eV_0} \exp(-eV_0/\epsilon)$.

For the realistic case of a system with finite size the diffusive spreading of the transferred charge reaches the boundaries of the electrode for long times, and then the charge relaxes via the external circuit characterized by an impedance $Z_{\text{ext}}(\omega)$. This modifies the ZBA at very low voltages. For simplicity we assume that the external charge relaxation, which is the only process relevant for ultrasmall junctions [3], is slow and couples only to the
q = 0 component of the field [2]. Then Eq. (18) is modified to read

\[ Y(\omega) = \frac{e^2}{\mathcal{C}_0} \sum_{q \neq 0} \sum_{j=1}^{2} \frac{\lambda_j}{(D_j^* q^2 - i\omega)(D_j q^2 - i\omega)} + i \frac{e^2}{\omega} Z_{\text{ext}}^{-1}(\omega) - i\omega C. \]  

(24)

For one-dimensional electrodes of length L the discrete q values are of the form \((2\pi/L)n, n \text{ integer}\), and \(C = C_0 L\) is the total capacitance of the contact, while for two-dimensional quadratic electrodes \(C = C_0 L^2\).

For three-dimensional electrodes of finite size there are several energy scales in \(Y(\omega)\): The charging energy \(E_C = e^2/2C\) and for each spatial dimension \(\ell\) of the electrodes a Thouless energy \(E_{\text{Th}} = D/\ell^2\) as well as a field Thouless energy \(E^* = D^*/\ell^2\). Here \(\ell\) stands for the length, width \(w\), and thickness \(a\), respectively. The effective dimension of the electrodes is determined by \(E^*\). Whenever \(eV\) exceeds one of the scales \(E^*_\ell\) \((\ell = L, w, a)\) the effective dimension increases by one.

As an example we consider a symmetric one-dimensional contact of length \(L\) with an ohmic external impedance \(Z_{\text{ext}} = R\). Then, from Eq. (24) we obtain

\[ \text{Im} Y(\omega) = \text{Im} Y_{\infty}(\omega) + \frac{\omega}{1 + \alpha^2(\omega R_K C)^2} \]  

(25)

where \(\text{Im} Y_{\infty}(\omega) = e^2/[2\sqrt{2D^2C_0}(1 + \xi^{1/2})\omega^{3/2}]\) describes an infinitely long contact, and the function \(f(z) = (g(z) - \xi^{1/2}g(z/\xi))/(1 - \xi^{1/2})\) with \(g(z) = \text{Im}[-\sqrt{2i}\cot(\sqrt{2}\pi/2)]\) incorporates all size effects. The last term in Eq. (25) describes the environmental impedance characterized by the dimensionless parameter \(\alpha = R/R_K\) with \(R_K = h/e^2\).

In Fig. 1 we present results for typical values of metallic quasi-one-dimensional tunnel junctions \([1]\) \((\xi = 4 \times 10^{-4}, E^*/eV_0 = 100, E_C/eV_0 = 25)\). The full line corresponds to a finite length junction shunted by a rather large environmental impedance \((\alpha = 1)\) to display the crossover between zero-dimensional and one-dimensional behavior clearly. The dotted line gives the conventional environmental Coulomb blockade of a zero-dimensional junction which determines the behavior for voltages below \(E_C/e\) (see inset). On the other hand, for large voltages above \(E^*/e\) the result \([22]\) for a long one-dimensional junction (dashed line) is approached. The chain dotted line results from a perturbative calculation of the diffusive anomaly \([3]\).

In summary, we have studied tunneling in large junctions with diffusive electron motion. Treating the Coulomb interaction non-perturbatively, the \(I-V\)-curve was written in a form familiar from Coulomb blockade theory for ultrasmall junctions. Interaction effects are again incorporated in a spectral function \(P(\epsilon)\). However, while for ultrasmall junctions \(P(\epsilon)\) depends only on the effective impedance \([Z_{\text{ext}}^{-1}(\omega) - i\omega C]^{-1}\), for large junctions this quantity is replaced by \(-i\omega Y(\omega)/e^2\) which incorporates also the diffusive anomaly of disordered metals.

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**FIG. 1.** Differential conductance \(G(V)\) of a quasi one-dimensional tunnel junction of finite length with an external impedance at \(T = 0\) (solid line), see text for details. This is compared with the results of a zero-dimensional (dotted line) and a long one-dimensional junction (dashed line). The chain dotted line corresponds to the perturbative result.

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