Loop Variables and Gauge Invariant Interactions of Massive Modes in String Theory

B. Sathiapalan

Physics Department
Penn State University
120 Ridge View Drive
Dunmore, PA 18512

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Abstract

The loop variable approach used earlier to obtain free equations of motion for the massive modes of the open string, is generalized to include interaction terms. These terms, which are polynomial, involve only modes of strictly lower mass. Considerations based on operator product expansions suggest that these equations are particular truncations of the full string equations. The method involves broadening the loop to a band of finite thickness that describes all the different interacting strings. Interestingly, in terms of these variables, the theory appears non-interacting.
1 Introduction

A physically compelling symmetry principle would greatly advance our understanding of string theory. In the first quantized formalism, reparametrization invariance and Weyl invariance have been the guiding principles. This is particularly transparent in the Polyakov approach \[1\]. Via the BRST device this has been carried over into second quantized string field theory \[2, 3, 4, 5, 6\]. A space-time interpretation for these symmetries would thus be very useful.

In \[20, 21\], following the renormalization group approach \[7, 8, 9, 10\], the gauge invariant free equations and transformation laws of string fields were written down. The basic ingredient is a ‘loop variable’, which is essentially a collection of all the vertex operators of first quantized string theory. In this approach the gauge transformations have a simple form. In \[20\] there was also some speculation about a possible space-time interpretation.

In this paper we would like to report on some progress in the study of (gauge invariant) interactions in this ‘loop variable’ approach. We will deviate a little from the renormalization group approach that has been used so far \[11-34\]. Starting from the free equations of motion and the free gauge transformation laws of \[20, 21\], we will make an educated “guess” as to the form of the interacting equations and transformation laws. The result will be an interacting equation of motion, the form of which is such that the invariance of these equations under the modified gauge transformation follows naturally from the invariance of the free equations. This is reminiscent of the usual field theory trick of obtaining interactions by covariantising derivatives. However, in detail, our approach is quite different. In fact, the interacting equations look like the free equations of a string that has been broadened into a ‘band’. This is an intriguing result and one can speculate that in some approximation an interacting string behaves like a free membrane.

We should, however, emphasize that we do not obtain the full set of interactions of string theory. The equation for any particular mode contains only terms involving modes of strictly lower mass. In effect, we are picking the ‘naive product’-term \[2\] in the infinite series of terms contained in the operator

\[1\] We will be dealing, in any case, with strings that carry no Yang-Mills or U(1) quantum numbers.

\[2\] By ‘naive product’, we mean the term that involves no contractions. This will be made precise in the last section.
product expansion (OPE) of two (or more) vertex operators. Thus each equation is a truncation of the full equation, where higher mass fields have been set to zero and in addition higher order terms involving fields of the same mass are also set to zero. Only terms that have the same oscillator number \( \sum_n n a_n^\dagger a_n \) are present in any equation. This truncation is consistent, in the sense that the equations are gauge invariant, with the same number of gauge parameters as the free theory. Hopefully, it should be possible to generalize the construction to include the full set of interactions.

This paper is organized as follows: Section II describes how interacting equations and transformation laws are obtained from the free ones. Section III makes contact with string theory by describing how the field content is restricted to agree with that of string theory and how massive equations are obtained from massless ones, as in the loop variable approach. Section IV contains some conclusions and also explains the connection with the OPE of vertex operators and the nature of the truncation that is being performed on the full equation.
2 Interactions and Loop Variables

In this section we describe how one obtains interacting equations starting from free ones. We refer the reader to [20, 21] for a description of how one obtains the free equations of motion.

The loop variable is defined by

\[ e^{i \int_c \alpha(t)k(t)\partial_x X(z+t)dt + ik_0X} \]

where

\[ Y_n = \frac{\partial^n X}{(n-1)!} + \alpha_1 \frac{\partial^{n+1} X}{n!} + ... \alpha_m \frac{\partial^{n+m} X}{(n+m-1)!} + ... \]

and

\[ \alpha(t) = \alpha_0 + \frac{\alpha_1}{t} + \frac{\alpha_2}{t^2} + ... \]

The space-time fields are obtained from a string field \( \Psi[k_n] \) as follows: ([\( dk_n \) \( \equiv dk_1dk_2...dk_n... \) and \( \Psi[k_n] \equiv \Psi(k_0,k_1,k_2,...k_n...) \)])

\[ \int [dk_n] \Psi[k_n] \equiv \phi(k_0) \] (2.4)

is the tachyon,

\[ \int [dk_n] k_1^\mu \Psi[k_n] \equiv A^\mu(k_0) \] (2.5)

is the vector,

\[ \int [dk_n] k_2^\mu \Psi[k_n] \equiv S^\mu(k_0) \] (2.6)

and

\[ \int [dk_n] k_1^\mu k_1^\nu \Psi[k_n] \equiv S^{\mu\nu}(k_0) \] (2.7)

are the (auxiliary) spin-1 and spin-2 fields.

In order to define an interacting theory, we introduce an additional coordinate ‘\( \sigma \), (0 ≤ \( \sigma \) ≤ 1), to parametrize different space-time fields as follows: We make \( k(t) \) a function of \( \sigma \) as well, i.e. \( k(t, \sigma) \). Thus \( k_i \) in eq. (2.2) becomes \( k_i(\sigma) \). Thus \( \Psi \) also becomes an implicit function of \( \sigma \): \( \Psi[k_n(\sigma)] \). We retain eqns. (2.4), (2.3) and (2.6) and modify (2.7) to

\[ \int [dk_n] k_1^\mu k_1^\nu \Psi[k_n] = \delta(\sigma_1 - \sigma_2)S^{\mu\nu} + A^\mu(k_0)A^\nu(q_0) \] (2.8)
We have replaced $dk_n$ by $\mathcal{D}k_n$ to denote a functional integral. Eqn. (2.8) has the following interpretation: When $\sigma_1$ coincides with $\sigma_2$ we get (a higher excitation of) the same string, but otherwise we get two different strings at the same space-time point - which is an interaction. $k_0(\sigma)$ stands for the momentum of a particular string. Thus, in (2.8), we have called $k_0(\sigma_1)$ and $k_0(\sigma_2)$, $k_0$ and $q_0$ respectively.

One can thus imagine that $\Psi[k_n(\sigma)]$ defines a field theory, where the fields are $k_i(\sigma)$. (2.8) then describes the two point correlation function of $k_1(\sigma)$:

$$< k_1^\mu(\sigma_1) k_1^\nu(\sigma_2) >= \delta(\sigma_1 - \sigma_2) S^{\mu\nu} + < k_1^\mu(\sigma_1) >= k_1^\nu(\sigma_2) >$$  \hspace{0.5cm} (2.9)

where, by (2.5),

$$< k_1^\mu(\sigma_1) >= A_\mu(k_0)$$  \hspace{0.5cm} (2.10)

$$< k_1^\mu(\sigma_2) >= A_\mu(q_0)$$  \hspace{0.5cm} (2.11)

One can generalize this to three point correlation functions:

Thus

$$\int[dk_n] k_1^\mu(\sigma_1) k_1^\nu(\sigma_2) k_1^\rho(\sigma_3) = \delta(\sigma_1 - \sigma_2) \delta(\sigma_2 - \sigma_3) \phi_{111} + [\delta(\sigma_1 - \sigma_2) S^{\mu\nu} A^\rho + \delta(\sigma_1 - \sigma_3) S^{\mu\rho} A^\nu + \delta(\sigma_2 - \sigma_3) S^{\nu\rho} A^\mu] + A^\mu A^\nu A^\rho$$  \hspace{0.5cm} (2.12)

If we integrate over $\sigma_1, \sigma_2, \sigma_3$, RHS of (2.12) becomes

$$\phi_{111} + S^{\mu\nu} A^\rho + S^{\mu\rho} A^\nu + S^{\nu\rho} A^\mu + A^\mu A^\nu A^\rho$$  \hspace{0.5cm} (2.13)

(2.12) describes a higher string excitation along with a quadratic and a cubic interaction. The generalization to N-point correlation functions is obvious. One can also define correlators involving $k_0(\sigma)$. Thus

$$\int[\mathcal{D}k_n(\sigma)] k_0^\mu(\sigma_1) \Psi[k_n] = k_0^\mu \phi(k_0)$$  \hspace{0.5cm} (2.14)

$$\int[\mathcal{D}k_n(\sigma)] k_0^\mu(\sigma_1) k_1^\nu(\sigma_2) \Psi[k_n] = \delta(\sigma_1 - \sigma_2) k_0^\mu A^\nu(k_0) + k_0^\mu \phi(k_0) A^\nu(q_0)$$   \hspace{0.5cm} (2.15)

which can be rewritten as $e^{-\phi} \partial^\mu e^\phi A^\nu$.

$$\int[\mathcal{D}k_n(\sigma)] k_0^\mu(\sigma_1) k_1^\nu(\sigma_2) k_1^\rho(\sigma_3) \Psi[k_n]$$

$$= \delta(\sigma_1 - \sigma_2) \delta(\sigma_2 - \sigma_3) k_0^\mu S^{\nu\rho} + \delta(\sigma_2 - \sigma_3) k_0^\mu \phi(k_0) S^{\nu\rho} + \delta(\sigma_1 - \sigma_2) k_0^\mu A^\nu(k_0) A^\rho(p_0) + \delta(\sigma_1 - \sigma_2) \delta(\sigma_2 - \sigma_3) k_0^\mu A^\nu(k_0) A^\rho(q_0)$$
\[\delta(\sigma_1 - \sigma_3)p_0^\mu A^\nu(p_0)A^\rho(q_0) + k_0^\mu \phi(k_0)A^\nu(q_0)A^\rho(p_0) \quad (2.16)\]

If we integrate over \(\sigma_1, \sigma_2, \sigma_3\) eq. \((2.16)\) can be summarized as
\[e^{-\phi} \partial_\mu e^\phi (S^{\mu \rho} + A^\nu A^\rho) \quad (2.17)\]

For convenience, henceforth we will set \(\phi = 0\).

Using the above, the prescription for obtaining interacting equations is very simple: Take the free equation and replace \(k_i\) by \(\int d\sigma k_i(\sigma)\) and use eqns \((2.8),(2.12),(2.14)-(2.16)\). In terms of fields all we are doing is replacing \(S^{\mu \nu}\) by \(S^{\mu \nu} + A^\mu A^\nu\) and \(\phi_{\mu \nu \rho}^{111}\) by eqn. \((2.13)\). To obtain the \(\phi\) dependence, simply replace the derivative \(\partial_\mu\) by \(e^{-\phi} \partial_\mu e^\phi\).

Now let us apply this prescription to obtain interacting equations for \(S^\mu\) and \(S^{\mu \nu}\). The free equation for \(S^\mu\) is \((20)\):
\[k_0^2 k_2^\mu - k_0.k_1 k_1^\mu + k_0^\mu k_1.k_1 - k_0^\mu k_0.k_2 = 0. \quad (2.18)\]

This becomes the interacting equation:
\[
\int d\sigma_1 d\sigma_2 d\sigma_3 \{k_0^\mu(\sigma_1)k_{0 \nu}(\sigma_2)k_2^\nu(\sigma_3) - k_{0 \nu}(\sigma_1)k_1^\mu(\sigma_2)k_1^\nu(\sigma_3) + k_0^\mu(\sigma_1)k_1^\nu(\sigma_2)k_1^\mu(\sigma_3) - k_0^\mu(\sigma_1)k_0^\nu(\sigma_2)k_2^\nu(\sigma_3)\} = 0. \quad (2.19)\]

which becomes, in terms of space-time fields:
\[- \partial^2 S^\mu - i \partial_\nu (S_{\mu \nu}^\rho + A_\mu A^\nu) + i \partial_\nu (S_{\nu \rho}^\nu + A.A) + \partial_\rho \partial_\nu S^\nu = 0. \quad (2.20)\]

Similarly the free equation for \(S^{\mu \nu}\) is
\[- k_0^2 k_2^\mu k_2^\nu + k_1.k_0(k_1^\mu k_2^\nu + k_2^\mu k_0^\nu) - k_1.k_1 k_0^\mu k_0^\nu = 0. \quad (2.21)\]

and it becomes the interacting equation
\[\partial^2 (S^{\mu \nu} + A^\mu A^\nu) + \partial_\mu \partial_\nu S^{\mu \rho} + \partial_\rho \partial_\nu(A^\mu A^\rho) - \partial_\mu \partial_\nu(S_\rho^\rho + A_\rho A_\rho) = 0. \quad (2.22)\]

Equations \((2.21)\) and \((2.22)\) describe only cubic interactions of the type \(S.A.A\). Equations for higher spin fields such as \(\phi_{\mu \nu \rho}^{111}\) will describe cubic as well as quartic interactions of the type \(\phi.S.A\) and \(\phi.A.A.A\). However, as we shall see in the last section interactions of this type are only a small subset of the interactions actually present in string theory.
Eqns. (2.20) and (2.22) describe massless modes. To make contact with string theory one has to introduce masses by a process of dimensional reduction and impose constraints on the \( k_i \) \(^{[20, 21]}\). We will describe this in the next section.

Let us turn, now, to the gauge transformation law. For the free theory, gauge transformations are succinctly described by the equation

\[
k(t) \rightarrow k(t) \lambda(t)
\]  

(2.23)

In terms of modes,

\[
k_n \rightarrow k_n + \lambda_1 k_{n-1} + \lambda_2 k_{n-2} + \ldots + \lambda_n k_0
\]  

(2.24)

This is easily seen to leave (2.18) and (2.21) invariant. To translate this into a law for spacetime fields we follow the prescription described in \[^{[20, 21]}\] viz., do as in in equations (2.4)-(2.7), but with \( \Psi \) taken to be a function of the gauge parameter \( \lambda(t) \) also. Thus,

\[
\int [dk_n][d\lambda_n]\lambda_1 \Psi[k_n, \lambda_n] \equiv \Lambda_1
\]  

(2.25)

and

\[
\int [dk_n][d\lambda_n]\lambda_1 k_1^\mu \Psi[k_n, \lambda_n] \equiv \Lambda_2^\mu
\]  

(2.26)

\[
\int [dk_n][d\lambda_n]\lambda_2[k_n, \lambda_n] \equiv \Lambda_2
\]  

(2.27)

are some of the gauge parameters that enter the gauge transformation laws of the space-time fields \( A^\mu, S^\mu, S^{\mu\nu} \). Using (2.24) and (2.25) - (2.27) one finds:

\[
A_\mu \rightarrow A_\mu + \partial_\mu \Lambda_1
\]  

(2.28)

\[
S_\mu \rightarrow S_\mu + \partial_\mu \Lambda_2 + \Lambda_2^\mu
\]  

(2.29)

\[
S_{\mu\nu} \rightarrow S_{\mu\nu} + \partial_\mu \Lambda_2^\nu + \partial_\nu \Lambda_2^\mu
\]  

(2.30)

Now we have to generalize (2.23) and (2.24) such that it is a symmetry of the interacting equations. This is achieved by the following gauge transformation:

\[
\int d\sigma k(t, \sigma) \rightarrow \int d\sigma k(t, \sigma) \int d\sigma_1 \lambda(t, \sigma_1)
\]  

(2.31)

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The interacting equations were obtained from the free ones by the prescription \( k(t) \rightarrow \int d\sigma k(t, \sigma) \). Thus the invariance of the free equations under (2.23) guarantees that the interacting equations are invariant under (2.31). Of course, one has to choose the gauge transformation laws for a space-time field so that it is consistent with the already fixed gauge transformation laws of the lower mass fields. This is always possible since one can use (2.31) to define the gauge transformation law of that field. We illustrate this below.

We set \( \phi = 0 \) for convenience. Consider the gauge transformations of the various \( k_i \)'s and their products. We start with \( k_1 \):

\[
\delta[\int d\sigma k_1^\mu(\sigma)] = \delta A^\mu
\]

\[
= \int d\sigma_1 k_0^\mu(\sigma_1) \int d\sigma_2 \lambda_1(\sigma_2) = \partial_\mu \lambda_1
\]

which is unchanged from the free case. \( \Box \)

\[
\delta[\int d\sigma k_2^\mu(\sigma)] = \delta S^\mu
\]

\[
= \int d\sigma_1 d\sigma_2 [k_1^\mu(\sigma_1) \lambda_1(\sigma_2) + k_0^\mu(\sigma_1) \lambda_2(\sigma_2)]
\]

\[
= \int d\sigma_1 d\sigma_2 [\delta(\sigma_1 - \sigma_2) \Lambda_2^\mu + A^\mu \Lambda_1 + \delta(\sigma_1 - \sigma_2) \partial_\mu \Lambda_2]
\]

Thus

\[
\delta S^\mu = \Lambda_2^\mu + \partial_\mu \Lambda_2 + A^\mu \Lambda_1
\]

(2.33)

And,

\[
\delta[\int d\sigma_1 d\sigma_2 k_1^\mu(\sigma_1) k_1^\nu(\sigma_2)] = \delta[S^{\mu\nu} + A^\mu A^\nu]
\]

\[
= \int d\sigma_1 d\sigma_2 d\sigma_3 k_0^{(\mu}(\sigma_1) k_1^{\nu)}(\sigma_2) \lambda(\sigma_3)
\]

\[
= \int d\sigma_1 d\sigma_2 d\sigma_3 [\delta(\sigma_1 - \sigma_2) \delta(\sigma_2 - \sigma_3) \partial(\mu \Lambda_2^\nu) + \delta(\sigma_1 - \sigma_2) \partial(\mu A^\nu) \Lambda_1 +
\]

\[
\delta(\sigma_1 - \sigma_3) \partial(\mu \Lambda_1) A^\nu]
\]

\[
= \partial(\mu \Lambda_2^\nu) + A^{(\mu \partial(\nu)} \Lambda_1 + \partial(\mu A^\nu) \Lambda_1
\]

(2.34)

\( ^3 \)If \( \phi \neq 0 \), we we would get \( e^{-\phi} \partial_\mu e^\phi \Lambda_1 \).
If we use (2.33) and (2.34) as the gauge transformation laws, we are guaranteed that the equations will be invariant. Using (2.32) and (2.34) we find that
\[
\delta S^{\mu\nu} = \partial^{(\mu} \Lambda_{2}^{\nu)} + \partial^{(\mu} A^{\nu)} \Lambda_{1}
\] (2.35)
Thus (2.32), (2.33) and (2.35) are the gauge transformation laws that leave the (non linear) equations invariant.

The substitutions that take one from the free equations and free gauge transformations to the interacting equations and gauge transformations can be summarized by the following rules:
\[
S^{\mu\nu} \rightarrow S^{\mu\nu} + A^{\mu} A^{\nu}, \\
S^{\mu} \rightarrow S^{\mu}; \quad A^{\mu} \rightarrow A^{\mu}, \\
\Lambda_{2}^{\mu} \rightarrow \Lambda_{2}^{\mu} + \Lambda_{1} A^{\mu}
\] (2.36)
It can easily be checked that the gauge transformation laws (2.33) and (2.34) are obtained from (2.29) and (2.30) by these substitutions. It is very easy to generalize (2.32)-(2.35) to the third level:
\[
\int [dk_{n}]k_{1}^{\mu} k_{1}^{\nu} k_{1}^{\rho} \Psi[k_{n}] \equiv \phi_{111}^{\mu\nu\rho} \quad (2.37)
\]
\[
\int [dk_{n}]k_{2}^{(\mu} k_{1}^{\nu)} \Psi[k_{n}] \equiv \phi_{12}^{\mu\nu} \quad (2.38)
\]
\[
\int [dk_{n}]k_{2}^{[\mu} k_{1}^{\nu]} \Psi[k_{n}] \equiv \phi_{12}^{\mu\nu} \quad (2.39)
\]
\[
\int [dk_{n}]k_{3}^{\mu} \Psi[k_{n}] \equiv \phi_{3}^{\mu} \quad (2.40)
\]
The interacting version of (2.37) was already given in eq.(2.13). Similar calculations can easily be done for the rest of the fields. One can then use (2.31) to obtain the gauge transformation law for expression (2.13), whence, using (2.32) and (2.35), one deduces the gauge transformation law for \(\phi_{111}^{\mu\nu\rho}\). Since there is nothing conceptually new we do not give the results here. Instead we will work out the massive case in the next section.

In this section we have described a simple method of obtaining, starting from the free equations for (massless) higher spin fields (described in [20, 21]), interacting equations and the corresponding gauge transformation laws. In the next section we make contact with string theory following the procedure of [20, 21].
3 Massive Equations

In [20, 21] a two step procedure was described for obtaining gauge invariant massive equations starting from the massless ones, and then for reducing the number of fields to agree with that of the usual BRST formalism for string theory [3]. We will apply this procedure to the interacting theory described in Sec II and show how one obtains equations for massive modes (that interact with other massive and massless modes), with the field content at each mass level being exactly that of string theory. Furthermore, these equations have all the gauge invariances that the free equations have. It is therefore very likely that these equations are a truncated version of the full tree level equations of string theory. Consideration of OPE of vertex operators (Sec IV) supports this view.

To obtain massive equations from massless ones, we perform a dimensional reduction. We split the generalized momentum $k^\mu(t); \mu = 1, ..., D$ into $(k^\mu(t), q(t)); \mu = 1, ..., D - 1$. We then set $q_0 = mass$. Thus, for the tachyon $q_0^2 = -2$, for the massless vector $q_0^2 = 0$, for the first mass level $q_0^2 = 2$ and so on. We will also set $\int [dk_n][dq_n] q_1 \Psi[k_n, q_n] = 0$. The necessity for this was demonstrated in [3]. Thus at the massless level we just have a vector, $k_1^\mu$. At the next level there are more fields.

3.1 Spin-2

There are two fields $k_1^\mu k_1^\nu$ and $k_2^\mu$. Equation (2.18) splits into two equations ($q_0^2 = 2$).

\[
(k_0^2 + q_0^2) k_2^\mu - (k_0, k_1 + q_0, q_1) k_1^\mu + k_0^\mu (k_1, k_1 + q_1 q_1) - k_0^\mu (k_0, k_2 + q_0 q_2) = 0. \tag{3.1}
\]

\[
(k_0^2 + q_0^2) q_2 - (k_0, k_1 + q_0, q_1) q_1 + q_0 (k_1, k_1 + q_1 q_1) - q_0 (k_0, k_2 + q_0 q_2) = 0. \tag{3.2}
\]

Equation (2.21) can also be split up into three equations involving $k_1^\mu k_1^\nu, k_1^\mu q_1$ and $q_1 q_1$. Furthermore we will make the identifications

\[
\int [dk_n][dq_n] q_1 k_1^\mu \Psi[k_n, q_n] = \int [dk_n][dq_n] k_2^\mu q_0 \Psi[k_n] = S^\mu q_0
\]

and

\[
\int [dk_n][dq_n] q_1 q_1 \Psi[k_n] = \int [dk_n][dq_n] q_2 q_0 \Psi[k_n] = \eta q_0 \tag{3.3}
\]
In all the equations we can use (3.3) to eliminate $q_1$. The equations for $q_1 k_1^\mu$ and $q_1 q_1$ then reduce to (3.1) and (3.2). The equation for $k_1^\mu k_1^\nu$ becomes:

$$-(k_0^2 + q_0^2)k_1^\mu k_1^\nu + k_1 k_0 (k_1^\mu k_0^\nu + k_0^\mu k_1^\nu) + q_0^2 (k_2^\mu k_0^\nu + k_2^\nu k_0^\mu) - (k_1 k_1 + q_2 q_0)k_0^\mu k_0^\nu = 0.$$  

(3.4)

Thus we end up with three fields, $S^{\mu \nu}$, $S^\mu$, and $\eta$ - which is exactly the field content (including auxiliary fields) at the second mass level in string theory [3].

We also have to eliminate $q_1$ from the gauge transformations. Thus in

$$q_2 \to q_2 + \lambda q_0 + \lambda_1 q_1$$

(3.5)

we will set

$$q_1 \lambda_1 = \lambda q_0$$

(3.6)

so that

$$q_2 \to q_2 + 2 \lambda q_0$$

(3.7)

One can also check, then, that

$$q_1 q_1 \to q_1 q_1 + 2 \lambda q_1 q_0 = q_1 q_1 + 2 \lambda q_0^2$$

(3.8)

Comparing (3.7) and (3.8) we see that (3.3) is consistent.

Now that we have the correct field content and gauge transformations for a free massive spin-2, we can go ahead and introduce interactions in the manner described in Sec II. Note that first we have to get rid of all $q_1$-dependence in the free equations by replacing $q_1 k_1^\mu$ and $q_1 q_1$ using (3.3), and only after that can we introduce interactions. The net result is that $S^{\mu \nu}$ is replaced by $S^{\mu \nu} + A^\mu A^\nu$, just as in Sec II. The gauge transformations are also the same as before.

For $\eta$ we have:

$$\delta [\int d\sigma q_2(\sigma)] = \delta \eta = 2 \int d\sigma_1 \lambda_2(\sigma_1) \int d\sigma_2 q_0(\sigma_2)$$

$$\Rightarrow \delta \eta = 2 q_0 \Lambda_2$$

(3.9)

Equation (3.4) becomes, in the interacting case ($q_0^2 = 2$):

$$- \partial^2 S^\mu - i \partial_\nu (S^{\nu \mu} + A^\nu A^\mu) + i \partial^\mu (S^\nu_{\nu} + A^\nu A^\nu) - i \partial^\mu \partial_\nu S^\nu = 0.$$  

(3.10)
(3.2) becomes:

\[- \partial^2 \eta - 2i \partial_\nu S^\nu q_0 + q_0 (S^\nu + A^\nu A_\nu) = 0. \quad (3.11)\]

and (3.4) becomes:

\[
(- \partial^2 + q_0^2) S^{\mu \nu} - [\partial_\rho \partial^{\nu} (S_{\mu \rho} + A^\mu A_\rho) + (\mu \leftrightarrow \nu)] + 
q_0^2 \partial^{(\nu} S^{\mu)}
+ \partial^\mu \partial^{\nu} (S_\rho + A_\rho + \eta q_0) = 0.
\quad (3.12)
\]

Thus we have a set of equations for a massive spin-2 system interacting with a massless vector \(A^\mu\). Note that (3.10) and (3.11) are unchanged from the massless case, while (3.12) is changed. The gauge transformations are given by (2.32), (2.33), (2.35) and (3.9). We reproduce them here for convenience. \((q_0^2 = 2)\)

\[
\begin{align*}
\delta A^\mu &= \partial^\mu \Lambda_1 \\
\delta S^{\mu} &= \partial^\mu \Lambda_2 + A^\mu \Lambda_1 + \Lambda_2^\mu \\
\delta S^{\mu \nu} &= \partial^{(\mu} \Lambda_2^{\nu)} + \Lambda_1 \partial^{(\mu} A^{\nu)} \\
\delta \eta &= 2q_0 \Lambda_2
\end{align*}
\quad (3.13)
\]

3.2 Spin-3

We now turn to the spin-3 case. We only give an outline. The equations for the free theory are given in [20]. The procedure for obtaining the interacting theory was described in Sec II. But before we apply this prescription, we have to first perform the dimensional reduction and impose constraints similar to (3.3) in order to obtain the correct field content. In order to do this we have to eliminate (terms involving) \(q_1\) from the equations. The various fields are:

\[
k_1^\mu k_1^\nu k_1^\rho, q_1 k_1^\mu k_1^\nu, k_2^\mu k_1^\nu, q_1 k_1^\mu, q_2 k_1^\mu, q_3 k_1^\mu, q_4, q_5, q_6, q_7, q_8, q_9, q_{10}, q_{11}, q_{12}, and q_3
\quad (3.14)
\]

In making identifications between fields, it is crucial that they do not involve derivatives, since this could introduce higher derivatives in the kinetic terms. We start with the fields that have two Lorentz indices, \(q_1 k_1^\mu k_1^\nu\) and \(k_2^\mu k_1^\nu\). Let us compare their transformation laws:

\[
\delta[2q_1 k_1^\mu k_1^\nu] = 2\lambda_1 q_0 k_1^\mu k_1^\nu + 2\lambda_1 q_1 k_0 (\mu k_1^\nu)
\quad (3.15)
\]
\[\delta[q_0 k_2^\mu k_1^\nu] = 2\lambda_1 q_0 k_1^\mu k_0^\nu + q_0 \lambda_2 k_0^\mu k_1^\nu + q_0 \lambda_1 k_0^\mu k_2^\nu \]  
(3.16)

If we want the terms involving derivatives, in (3.15) and (3.16), to be equal, we must require that

\[2\lambda_1 q_0 k_1^\mu = q_0 \lambda_2 k_1^\mu + q_0 \lambda_1 k_2^\mu \]  
(3.17)

In that case we can set

\[q_0 k_1^\mu = 2 q_1 k_1^\mu k_1^\nu \]  
(3.18)

We can thus eliminate \(q_1\) from the two index (symmetric) fields and reduced their number to one. In (3.17), we can observe a pattern: \(q_1\) attaches itself, first to \(\lambda_1\) converting it to \(\lambda_2\), and then to \(k_1\) converting it to \(k_2\). The same pattern is evident in (3.18). We will observe this in other cases also.

Let us turn to the fields with one Lorentz index, of which there are four: \(q_2 k_1^\mu\), \(q_1 k_2^\mu\), \(q_1 k_1^\mu\), and \(k_3^\mu\). Applying the identifications (3.17), we find that:

\[\delta[q_2 k_1^\mu] = \frac{1}{2} \lambda_1 k_2^\mu q_0 + \frac{3}{2} \lambda_2 k_1^\mu q_0 + \lambda_1 q_2 k_0 \]  
(3.19)

\[\delta[q_1 k_2^\mu] = \frac{3}{2} \lambda_1 k_2^\mu q_0 + \frac{1}{2} \lambda_2 k_1^\mu q_0 + \lambda_2 q_1 k_0^\mu \]  
(3.20)

\[\delta[q_1 k_1^\mu] = \lambda_1 k_1^\mu q_0^2 + \lambda_2 k_1^\mu q_0^2 + \lambda_1 q_1 q_0 k_0 \]  
(3.21)

\[\delta[k_3 q_0^2] = \lambda_1 k_2^\mu q_0^2 + \lambda_2 k_1^\mu q_0^2 + \lambda_3 q_0^2 k_0 \]  
(3.22)

We can easily see that

\[q_1 q_1 k_1^\mu = 1/2(q_2 k_1^\mu + q_1 k_2^\mu) \]  
(3.23)

is a consistent identification, provided

\[q_1 q_1 \lambda_1 = 1/2(\lambda_1 q_2 + \lambda_2 q_1) \]  
(3.24)

Note that (3.23) and (3.24) are also consistent with the pattern described above. In any case, we can impose both (3.23) and (3.24) consistently. If we further impose

\[\lambda_3 q_0^2 = \lambda_1 q_1 q_1 \]  
(3.25)

we can also make the identification

\[q_1 q_1 k_1^\mu = k_3^\mu q_0^2 \]  
(3.26)
As a result of all these identifications, we have two independent vectors that we can choose to be \( q_2 k_1^\mu \) and \( k_3^\mu \), and also two independent vector gauge parameters, \( \lambda_1 k_2^\mu \) and \( \lambda_2 k_1^\mu \). These are thus sufficient to gauge the vectors away.

We now turn to the scalars: \( q_1 q_1, q_1 q_2, q_3 \). By comparing their gauge transformations one finds that the identification

\[
q_1 q_1 = q_2 q_0 = q_3 q_0^2
\]

is consistent. The only scalar gauge parameter is \( \lambda_3 \).

One can now rewrite all the free equations without any \( q_1 \)'s. At this point one can apply the prescription of the last section to obtain interacting equations. Similarly, one can also deduce the non-linear transformation laws. We will write down the gauge transformation laws and one of the equations by way of illustration.

The gauge transformation laws are:

\[
\delta \phi_{111}^{\mu \nu \rho} = \left( \partial_\mu \epsilon_\nu + \Lambda_2 \partial_\mu A_\nu + \Lambda_2 \partial_\mu A_\nu + \Lambda_1 \partial_\mu S_\nu \rho \right) + (\mu \leftrightarrow \nu) + (\mu \leftrightarrow \rho).
\]

\[
\delta (k_1^{(2)} k_2^{1\nu}) \equiv \delta \phi_{12}^{(12)} \mu \nu = \{ \partial_\mu \epsilon_\nu + \partial_\mu \epsilon_2 \nu + \epsilon_{\mu \nu} \}
+ \Lambda_2 A_\nu + \Lambda_2 \partial_\mu A_\nu + \Lambda_1 \partial_\mu S_\nu \}
+ \{ \mu \leftrightarrow \nu \}
\]

\[
\delta \phi_{3\mu} = \partial_\mu \epsilon + \epsilon_\mu + \epsilon_2 \mu + \Lambda_2 A_\mu + \Lambda_1 S_\mu.
\]

\[
\delta (q_2 k_1^\mu) \equiv \delta \phi_{12}^{(12)} = \partial_\mu \epsilon_2 + \frac{1}{2} \epsilon_1 \mu + \frac{3}{2} \epsilon_2 \mu + \frac{1}{2} \Lambda S_\mu - \frac{1}{2} \Lambda_2 A_\mu + \Lambda \partial_\mu \eta.
\]

\[
\delta \eta_3 = \epsilon
\]

(3.28)

Where

\[
\lambda_1 q_2 \sim \epsilon_2; \lambda_3 \sim \epsilon; \lambda_1 k_2^\mu \sim \epsilon_2^\mu; \lambda_2 k_1^\mu \sim \epsilon_1^\nu; \lambda_1 k_1^\mu k_1^\nu \sim \epsilon^{\mu \nu}.
\]

(3.29)

Finally, there is a tracelessness condition that the gauge parameters of the free theory satisfy \[20, 21\]:

\[
\lambda_1 k_1 k_1 + \lambda_1 q_1 q_1 (= \lambda_3 q_0^2) = 0,
\]

or

\[
\epsilon_\nu + \epsilon q_0^2 = 0.
\]

(3.30)
which gets modified to

$$
\epsilon^\nu + 2\Lambda_2^\nu A_\nu + \Lambda A^\nu A_\nu + \epsilon q_0^2 = 0. \quad (3.31)
$$

in the interacting theory.

The equation for \( k_3^\mu \) in the free theory is [20] \((q_0^2 = 4)\):

\[
(k_0^2 + q_0^2)k_3^\mu - k_0^\mu (q_3q_0 + k_3k_0) - k_1^\mu (q_2q_0 + k_2k_0) - \\
k_2^\mu (q_1q_0 + k_1k_0) + k_1^\mu (k_1k_1 + q_1q_1) + 2(k_0^\mu (k_2k_1 + q_2q_1)) = 0. \quad (3.32)
\]

First \( q_1k_2^\mu \) is replaced by \((2k_3^\mu q_0 - q_2k_1^\mu)\), \( q_1k_1^\mu \) by \( k_2^\mu q_0^2 \) and \( q_2q_1 \) by \( q_3q_0 \). Then, following the rule for introducing interactions, we replace \( k_i \) by \( \int d\sigma k_i(\sigma) \) and \( q_i \) by \( \int d\sigma q_i(\sigma) \) in (3.32):

We have the following relations:

\[
\int d\sigma_1 d\sigma_2 q_2(\sigma_1)k_1^\mu(\sigma_2) = \phi_{12}^\mu(\sigma) + \eta A^\mu,\\n\int d\sigma_1 d\sigma_2 k_1^{(\mu}(\sigma_1)k_2^{\nu)}(\sigma_2) = \phi_{(12)}^{\mu\nu} + A^{(\mu} S^{\nu)},
\]

and

\[
\int d\sigma_1 d\sigma_2 d\sigma_3 k_1^\mu(\sigma_1)k_1^\nu(\sigma_2)k_1^\rho(\sigma_3) = \phi_{111}^{\mu\nu\rho} + S^{\mu\nu} A^\rho + S^{\mu\rho} A^\nu + S^{\nu\rho} A^\mu + A^{(\mu} A^{\nu} A^\rho.
\]

Inserting (3.33) in (3.32) we get an interacting equation that is invariant under the gauge transformation (3.28). Similar calculations can be performed for all the other equations given in [20]. Thus we can obtain equations for a massive spin-3 field interacting with massive spin-2 and massless spin-1 fields.
4 Conclusions

In this paper we have shown how the loop variable approach of \cite{20, 21} that was used there to obtain free equations of motion, can be generalized to include interactions. The interactions are obtained by introducing an additional parameter $\sigma$, that broadens the loop to a band. Each loop in the band stands for a separate string. If $\sigma_1 = \sigma_2$, then we have the same string, and otherwise two different strings, but at the same space-time point, i.e. an interaction. Thus in this approach a free string breaks up into a number of lighter strings in a well defined way to give interactions. The net result is that one obtains a non-linear equation describing the propagation of a massive mode and interacting with all the (strictly) lighter modes. Furthermore gauge invariance of this equation, which is essential for consistency of the theory, follows naturally from the gauge invariance of the free equation. This is reminiscent of what happens in Yang-Mills theories where interactions are introduced by gauging global symmetries.

The equations do not contain all the non-linear terms that one expects from string theory. This is obvious since we only have interactions with modes that are lighter. For e.g. $S^{\mu\nu}$ interacts with $A_\mu$ only, but not with $S^{\mu\nu}$ itself - there are no self interactions. Thus it corresponds to a certain well defined truncation. To understand the nature of this truncation, we turn to the OPE of two vertex operators:

\begin{equation}
: \partial_z X^\mu (z) e^{ik.X(z)} : \partial_w X^\nu (w) e^{iq.X(w)} :
\end{equation}

\begin{equation}
= (z-w)^{k.q}: \partial_z X^\mu (z) \partial_w X^\nu (w) e^{ik.X(z)+iq.X(w)} :
\end{equation}

\begin{equation}
+ \{ (\frac{iq_\mu^\nu}{z-w} \partial_w X^\nu (w) - \frac{ik^\mu_0}{z-w} \partial_z X^\mu (z) + \frac{(\delta^{\mu\nu} + q_\mu^\nu q_\nu^\mu)}{(z-w)^2}) e^{ik.X(z)+iq.X(w)} \} : (4.1)
\end{equation}

The first term in the RHS of (4.1) can be Taylor expanded in powers of $(w-z)$ and we get:

\begin{equation}
: \partial_z X^\mu (z) \partial_w X^\nu (w) e^{ik.X(z)+ip.X(w)} :
\end{equation}

\begin{equation}
= : \partial_z X^\mu (z)(\partial_z X^\nu (z)+(w-z)\partial^2 X^\nu (z)+...) e^{ik.X(z)+p[X(z)+(w-z)\partial_x X+...]} : (4.2)
\end{equation}

\begin{equation}
= : \partial_z X^\mu \partial_x X^\nu (z) e^{i(k+p)X(z)} : + O(z-w) \quad (4.3)
\end{equation}

The presence of a cubic interaction term $S^{\mu\nu}A_\mu A_\nu$ can be inferred from the first term of (4.3). The next term would correspond to $\phi_{12}^{\mu\nu}A_\mu A_\nu$. However,
such a term is not present in the equation of motion of $\phi_{12}^{\mu\nu}$ derived by the procedure of Sec II. Similarly, the second term in (4.1) contains, among other things, a Yang-Mills type term $\partial_\mu A_\nu [A^\mu, A^\nu]$ (which would vanish in the U(1) case). This is also absent from the equations of motion obtained by the procedure of this paper.

Thus it is clear that we have retained only one term from the OPE - this is the first term in (4.3). $S_{\mu\nu}$ has the same value ($N=2$) of the oscillator number, $N = \sum_n a_n^\dagger a_n$, as the two states involved in the product, $A^\mu$ and $A^\nu$. This is true of all the equations we have written down, i.e. all the terms of an equation have the same oscillator number. In the case of the spin-3 field $\phi_{111}^{\mu\nu\rho}$ also, which has $N = 3$, one can check that each term in the equation of motion has $N = 3$. From this observation we can conclude that the equation of motion that one obtains by the procedure described in this paper is a consistent truncation of the full equations of string theory - a truncation, where oscillator number is conserved. The consistency reflects itself in the fact that the equations have the full gauge invariance necessary for consistent propagation of massive higher spin particles.

What is perhaps most intriguing is the form of the equations and the gauge transformations. The interacting equations look exactly like the free equations of a loop variable that has been broadened to a band. This is also true of the gauge transformations. One can speculate that this band represents a membrane and that, in some approximation, an interacting string looks like a free membrane. This is somewhat analogous to large-N Yang-Mills theory being a string theory.

Finally, it should be possible to generalize this construction to include, in the equations, interaction terms that do not have the same oscillator number, and thereby obtain the full string equation. We hope to return to this issue.
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