Abstract
We consider the Calderón type inverse problem of recovering an isotropic quasilinear conductivity from the Dirichlet-to-Neumann map when the conductivity depends on the solution and its gradient. We show that the conductivity can be recovered on an open subset of small gradients, hence extending a partial result of Muñoz and Uhlmann to all real analytic conductivities. We also recover non-analytic conductivities with additional growth assumptions along large gradients. Moreover, the results hold for non-homogeneous conductivities if the non-homogeneous part is assumed known.

Keywords: Calderon problem, elliptic, nonlinear, quasilinear, Dirichlet-to-Neumann map, conductivity, linearization

1. Introduction
1.1. Discussion
The classical Calderón problem considers an isotropic linear elliptic equation and its Dirichlet-to-Neumann (DN) map

$$\nabla \cdot (a(x) \nabla u) = 0,$$

$$\Gamma_a : u|_{\partial \Omega} \mapsto a(x) \frac{\partial u}{\partial \nu}|_{\partial \Omega}.$$  (1)

The inverse problem is to determine $a(x)$ from $\Gamma_a$. See [U] for a survey of this problem and its extensions. The matrix extension $a(x) = a_{ij}(x)$, or the anisotropic Calderón problem, is still unsolved in general for dimension three or higher.

Recent work has featured the quasilinear generalization $a = a(x, u, \nabla u)$. The case $a = a(x, u)$ was considered in [S1], using a linearization technique similar to that originally
introduced in [11]. This was extended in [SU] to the anisotropic case \( a = a_j(x, u) \) (matrix). See [HS] for analysis in the case \( a = a_j(x, \nabla u) \) in dimension two, and [KN] for the recovery of a Taylor polynomial of \( a = a_j(x, \nabla u) \). The linearization technique was also applied to the semilinear equations \( \Delta u + a(x, u) = 0 \) and \( \Delta u + a(x, \nabla u) = 0 \) by [IS] and [S2], respectively. More recently, a multilinearization technique was developed for \( \Delta u + a(x, u) = 0 \) in [FO, KU, LLLS], where \( a \) is analytic in \( u \). The equation \( \Delta u + f(x)|\nabla u|^2 + V(x, u) = 0 \) with \( V \) analytic was determined in the partial data case in [KU1] using multilinearization. In [I2], the equation \( \Delta u + c(u, \nabla u) = 0 \) was considered using singular solutions of the linearized equation which concentrate at the boundary. The gradient structure was used for the quasi-linear equation \( \nabla \cdot (a(u)\nabla u) + c(x)u = 0 \) with lower order term in [EPS].

In the work [MU] by Muñoz and Uhlmann, attention was brought to the quasilinear case \( a = a(u, \nabla u) \), and results were obtained assuming conditions on the holomorphic extension of \( a \). These conditions ensure that linearization is possible around complex-valued affine solutions. The question is raised whether we can allow for arbitrary \( a(s, p) \) real analytic, \( a(s, p) \) smooth but non-analytic, and non-homogeneous \( \partial a(x, s, p)/\partial x \neq 0 \).

The purpose of this work is to extend the results of [MU] to several new classes of quasilinear conductivities:

(a) \( a(u, \nabla u) \) real analytic, without additional hypotheses, theorems 1 and 2.
(b) \( a(u, \nabla u) \) smooth, assuming additional conditions (7) and (8), theorem 3.
(c) Non-homogeneous cases \( a = a(x, u, \nabla u) \) with known \( x \) part, corollary 1.

An apparent \textit{a priori} difficulty of the inverse problem for \( a = a(u, \nabla u) \) (and truly one for general \( a = a(x, u, \nabla u) \)) is that the linearized inverse problem is anisotropic. Not only is the anisotropic problem not understood in dimension three or higher, but the matrix \( a_j(x) \) can only be recovered up to pushforward by a boundary-preserving diffeomorphism; see [SU] and [S2] for similar difficulties. In the \( p \)-Laplace case \( a = A(x)|\nabla u|^{p-2}\nabla u \), the ‘nonlinear part’ of the quasilinearity is known, unlike for equation (2), and boundary determination was possible in [SX] using explicit Wolff-type oscillatory solutions, with full recovery using a monotonicity identity [GKS]. If instead the anisotropic and nonlinear part is small, then one may linearize near the zero solution as in [KN] and recover the \( x \)-dependent Taylor coefficients using linear theory. This is similar to the idea of the multilinearization approach [FO, KU, KU1, LLLS] for which dependence on \( u \) or \( \nabla u \) is analytic. Constant solutions were used for linearizing the conductivity equation in [SU], and complex-valued affine solutions were used for linearization in [MU]. However, it is not clear what non-constant, explicit solutions quasilinear equation (2) has in full generality. If \( a(s, p) \) is smooth but not real analytic, then complex-valued affine solutions no longer make sense.

Our approach is to use linear boundary determination theory, which is well understood even in the anisotropic case. We first linearize the equation around any solution with a prescribed boundary jet, then recover the tangential part of the linearized conductivity matrix (3) using boundary determination theory. Next, the isotropic structure of the linearized conductivity matrix yields a solvable algebraic system for the conductivity evaluated at the jet. According to the comparison principle, we can use logarithmic barriers to prescribe boundary jets on a small open subset, which will complete the proof of theorem 1. For theorem 3, decay condition (8) ensures that exponential barriers exist, hence solutions with any prescribed boundary jet in \( \mathbb{R} \times \mathbb{R}^n \).

This work is organized as follows. In section 2.1, we show that the linearized BVP is well-posed and has a well-defined DN map. In section 2.2, we use the comparison principle to...
construct solutions with prescribed boundary jets. In section 2.3, we geometrically reformulate the linearized problem, which allows us to invoke known boundary determination theory in section 2.4. In section 3, we complete the proofs of theorems 1 and 3.

1.2. Statement of results

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a smooth bounded domain, and $u(x) \in C^\infty(\overline{\Omega})$ be the smooth solution of the quasilinear boundary value problem

$$
\nabla \cdot (a(u, \nabla u) \nabla u) = 0, \quad x \in \Omega,
$$

$$
u u|_{\partial \Omega} = f, \tag{2}
$$

where $a(s, p) \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ satisfies standard conditions to make this BVP wellposed, see [MU] and [GT]. We assume summation over repeated indices.

Ellipticity:

$$
0 < \lambda(s, p)|\xi|^2 \leq a_{ij}(s, p)\xi_i\xi_j, \quad (s, p, \xi) \in \mathbb{R} \times \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}),
$$

$$
a_{ij} = a\delta_{ij} + \frac{1}{2}(a_{pi}p_j + a_{pj}p_i). \tag{3}
$$

Coercivity:

$$
\lambda(s, p) \geq \lambda_0(|s|) > 0,
$$

$$
a(s, p) \geq 1. \tag{4}
$$

Growth:

$$
|p||\nabla a| + |a| \leq \mu_0(|s|),
$$

$$
(1 + |p|)a_j \leq \mu_0(|s|)|p|. \tag{5}
$$

The bounds $\lambda$, $\mu_0$, $\lambda_0$ and domain $\Omega$ are known quantities. Positive functions $\lambda_0$ and $\mu_0$ are non-increasing and non-decreasing in $|s|$, respectively, and we may assume they are continuous. We also assume known bounds on the $C^k(K)$ seminorms of $a$ for all $k \geq 0$ and all compact subsets $K$ of the domain of $a$.

Now, the DN map is defined by

$$
f \mapsto \Gamma_a(f) = a(u, \nabla u)\frac{\partial u}{\partial \nu} \, d\Sigma, \tag{6}
$$

where $d\Sigma$ is the Euclidean area form on $\partial \Omega$, and $\frac{\partial u}{\partial \nu} = \nu \cdot \nabla u$ is the Euclidean normal derivative. Under no additional assumptions, we can recover $a$ from $\Gamma_a$ along sufficiently small gradient values.

**Theorem 1.** If $\Gamma_a = \Gamma_{\tilde{a}}$, then $a(s, p) = \tilde{a}(s, p)$ on an open set:

$$
\{(s, p) : |p| < \Pi(s)\},
$$

where $\Pi : \mathbb{R} \to (0, \infty)$ is a positive continuous function which depends on the known quantities.
We next recover two classes of \(a(s,p)\) along large gradients. The first follows immediately from theorem 1.

**Theorem 2.** Let \(a(s,p)\) and \(\tilde{a}(s,p)\) be real-analytic. If \(\Gamma_a = \Gamma_{\tilde{a}}\), then \(a(s,p) \equiv \tilde{a}(s,p)\).

Theorem 2 extends the result of [MU], which assumed conditions on the holomorphic extension of real analytic \(a(s,p)\) in order to solve a complex-valued BVP.

**Theorem 3.** Let \(a\) and \(\tilde{a}\) satisfy two additional conditions:

(a) Uniform elliptic lower bound:

\[
a_{ij}(s,p), \quad \tilde{a}_{ij}(s,p) \geq \lambda_0 > 0, \quad (s, p) \in \mathbb{R} \times \mathbb{R}^n, \tag{7}
\]

where the inequality is in the quadratic form sense, and \(\lambda_0\) is constant.

(b) Decay along large gradients:

\[
|a_i(s,p)|, \quad |\tilde{a}_i(s,p)| \leq \frac{\varepsilon_0}{|p|}, \quad (s, p) \in \mathbb{R} \times \mathbb{R}^n, \tag{8}
\]

for some sufficiently small \(\varepsilon_0 > 0\) depending on \(\Omega\) and \(\lambda_0\).

If \(\Gamma_a = \Gamma_{\tilde{a}}\), then \(a(s,p) \equiv \tilde{a}(s,p)\).

This seems to be the first example of global uniqueness of a non-analytic conductivity depending on \((u, \nabla u)\). Decay condition (8) is known from minimal surface type equations \(a_n(s,p) = f(s) / \sqrt{1 + |p|^2}\), see [MU, p 5].

Finally, as shown in remarks 2–4, the above results hold for non-homogeneous conductivities \(a = a(x,s,p)\) as well, provided the ‘\(x\)-dependent part’ is either known (i.e. is the same for both \(a\) and \(\tilde{a}\)) or symmetric along a direction, say \((0, \ldots, 0, 1) \in \mathbb{R}^n\). We assume the known quantities in (3)–(5) are unchanged and do not depend on \(x\), for simplicity.

**Corollary 1.** If \(n \geq 3\), suppose also \(\Omega\) is convex and \(a\) and \(\tilde{a}\) are of the form

\[
a(x,s,p) = F(x, s, p, b(x^1, \ldots, x^{n-1}, s, p)),
\]

\[
\tilde{a}(x,s,p) = F(x, s, p, \tilde{b}(x^1, \ldots, x^{n-1}, s, p)), \tag{9}
\]

where \(t \mapsto F(x, s, p, t)\) is injective for each fixed \((x, s, p)\). (If \(n = 2\), remove convexity of \(\Omega\) and assume \(b\) and \(\tilde{b}\) are functions of \((s, p)\) only).

Then the statements of theorems 1–3 are true with replacements of \((s,p)\) by \((x,s,p)\), with \(x \in \Omega\).

The fully general case where \(b = b(x,s,p)\) is unclear to the author and may require new methods.

2. Preliminaries

2.1. The linearized problem

For arbitrary solutions to the quasilinear equation (2), the linearized Dirichlet problem centered at that solution may not be solvable. Under a small gradient assumption or the decay assumption (8), we will show the uniqueness of weak solutions to the linearized equation, and standard techniques will again apply after invoking higher regularity. Note that the non-homogeneous case \(a = a(x, u, \nabla u)\) applies here without change as well.
We denote by \( u = u[f] \) the solution of the BVP (2). If \( f, h \in C^\infty(\partial\Omega) \), then formally, we let
\[
v(x) := \frac{d}{dt}\bigg|_{t=0} u[f + th](x) = \lim_{t\to 0} \frac{u[f + th](x) - u[f](x)}{t}
\]
solve the linearized Dirichlet problem
\[
\partial_t(au + ap_vu_i + av_i) = 0,
\]
\[v|_{\partial\Omega} = h.
\]
Here, \( u = \partial u/\partial \chi \) and \( v_j = \partial v/\partial \chi_j \). The linearized DN map:
\[
\Gamma_a[u](h) := \frac{d}{dt}\bigg|_{t=0} \Gamma_a(f + th) = \left( a_i\frac{\partial u}{\partial \nu} + a_{pj}v_j\frac{\partial u}{\partial \nu} + \frac{\partial v}{\partial \nu} \right) dS.
\]

For brevity, we say that the linearization of (2) exists at \( u \) if the limit (10) is well defined in \( C^1(\overline{\Omega}) \) and the Dirichlet problem (11) for \( v \) is uniquely solvable in \( W^{1,2}(\Omega) \). If \( \Gamma_a = \Gamma_\theta \) and the linearization exists at \( u \) and \( \bar{u} \), it then follows from \( C^1(\overline{\Omega}) \) convergence that \( \Gamma_a[u] = \Gamma_a[\bar{u}] \).

To show that the linearization of (2) exists, we first specialize to solutions \( u \) with small gradients, under no additional assumptions on \( a(s, p) \), proved below.

**Proposition 1.** Given \( s \in \mathbb{R} \), there exists \( \Pi_1 > 0 \) sufficiently small depending on known quantities and \( s \) such that if \( f \in C^\infty(\partial\Omega) \) satisfies
\[
\|f - s\|_{C^\infty(\partial\Omega)} \leq \Pi_1,
\]
then the linearization of (2) exists at \( u = u[f] \). Here, \( \alpha \in (0, 1) \) is as in [MU, theorem 2.10] and depends on the known quantities and \( s \).

To linearize near constant solutions, we need to verify the solution operator is Lipschitz continuous at constants.

**Lemma 1.** Given \( k \in \mathbb{R} \) and \( h \in C^{2,\alpha}(\partial\Omega) \), there exists \( \alpha \in (0, 1) \) sufficiently small such that the following estimate holds:
\[
\|u[k + h] - k\|_{C^{1,\alpha}(\overline{\Omega})} \leq C(k, \|h\|_{C^1(\overline{\Omega})})\|h\|_{C^{2,\alpha}(\partial\Omega)},
\]

**Proof.** Letting \( f = k + h \) and \( u = k + w \) in (2), we obtain the BVP
\[
\nabla \cdot (a(k + w, \nabla w)\nabla w) = 0,
\]
\[w|_{\partial\Omega} = h.
\]

The structural conditions (3)–(5) are still satisfied, with lower and upper bounds now also depending on a parameter \( k \). It was shown in [MU] that (2) has an a priori \( C^{1,\alpha}(\overline{\Omega}) \) estimate in terms of \( \|f\|_{C^{1,\alpha}(\overline{\Omega})} \) for some \( \alpha \in (0, 1) \) depending on \( a \) and \( \|f\|_{C^{2,\alpha}(\partial\Omega)} \), which means each coefficient in the nondivergence form of (14),
\[
a_{ij}(k + w, \nabla w)w_{ij} + (a_i(k + w, \nabla w)w_i) = 0,
\]
(15)
has a $C^\alpha(\Omega)$ estimate also in terms of $\|f\|_{C^2(\partial\Omega)}$. Therefore, the linear Schauder $C^{2,\alpha}$ estimate [GT, theorem 6.6] yields

$$\|u\|_{C^{2,\alpha}(\overline{\Omega})} \leq C(a, \Omega, \|f\|_{C^2(\partial\Omega)})\|f\|_{C^{2,\alpha}(\partial\Omega)}. \quad (16)$$

In the case of (14), the same estimate is therefore true, with $C(a, \Omega, \|f\|)$ replaced by $C(k, a, \Omega, \|h\|) \equiv C(k, \|h\|)$, and $\alpha$ depending also on $k$ and $\|h\|_{C^2}$. Substitution of $w = u - k$ into the above completes the proof.

We also need the continuity of the solution operator near any small solution, which we record for completeness. Given $M > 0$ and the sufficiently small $\alpha = \alpha(M)$ of (16), we give the subset $B_M = \{ f \in C^{2,\alpha}(\partial\Omega) : \|f\|_{C^2(\partial\Omega)} < M \}$ the $C^{2,\alpha}(\partial\Omega)$ norm.

**Lemma 2.** For each $M > 0$ and $\beta \in (0, \alpha(M))$, the operator $B_M \to C^{2,\beta}(\overline{\Omega})$ given by $f \mapsto u[f]$ is continuous.

**Proof.** If not, then there exists a sequence $f_k \in C^{2,\alpha}(\partial\Omega)$ with $\|f_k\|_{C^2} \leq M$ converging to $f \in C^{2,\alpha}(\partial\Omega)$ but with $\|u[f_k] - u[f]\|_{C^2(\Omega)} \geq 1$, say. By (16), the solutions $u[f_k]$ are uniformly bounded in $C^{2,\alpha}(\overline{\Omega})$, so the Arzela–Ascoli theorem yields a subsequence, also named $u[f_k]$, which converges in $C^{2,\beta}(\Omega)$ to some $v$. It follows that $\|v - u[f]\|_{C^{2,\beta}(\Omega)} \geq 1$. However, since each $u[f_k]$ solves (2) with boundary data $f_k$, sending $k \to \infty$ in (2) shows that $v$ solves (2) as well with boundary data $f$. By the comparison principle [[GT], theorem 10.7] for quasilinear divergence form operators, it follows that $C^{2,\beta}$ solutions are unique, hence $v = u[f]$, a contradiction.

**Proof of proposition 1.** Defining $t$-dependent quantities,

$$u(x; t) := u[f + th](x), \quad v(x; t) := \frac{u(x; t) - u(x; 0)}{t}, \quad a(x; t) := a(u(x; t)); \quad u(; \sigma; t) = u(0) + \sigma(u(t) - u(0)), \quad (17)$$

with $a[u] := a(u, \nabla u)$, we take a ‘discrete derivative’:

$$a(t)u(t; t) - a(0)u(t; 0) = ta(t)v(t; t) + (a(t) - a(t; 0))u(t; 0) = tv(t; t) + ta(t)v(t; t)u(t; 0) + ta_{pf}(t)v(t; t)u(t; 0), \quad (18)$$

where, by the fundamental theorem of calculus,

$$a(t) := \int_0^t a_u[u(\sigma; t)]d\sigma, \quad a_{pf}(t) := \int_0^t a_u[u(\sigma; t)]d\sigma. \quad (19)$$

We see that $v(t)$ then solves the Dirichlet problem

$$\partial_t \left( a(t)v(t)u(t; 0) + a_{pf}(t)v(t)u(t; 0) + a(t)v(t) \right) = 0, \quad v(t)|_{\partial\Omega} = h, \quad (20)$$
with associated Cauchy data (DN map)

\[ \left( a_j(t)\psi(t;\partial u(0);0) + a_p(t)\psi(\partial u(0);0) + a(t;\partial u(0);t) \right) \, dS. \]  

(21)

**Step 1:**

We verify the Dirichlet problem (20) for \( \psi(t) \) has a unique solution in \( W^{1,2}_{0}(\Omega) \), which is uniformly bounded in \( t \). Let \( w \in W^{1,2}_{0}(\Omega) \) solve

\[ \partial_t \left( a_j(t)\psi w_j(0) + a_p(t)\psi(\mu_j(0) + a(t;\psi w_j) = 0. \]  

(22)

Multiplying (22) by \( w \) and integrating by parts, we obtain

\[ \int_{\Omega} a_{ij}(t)w_i w_j \, dx = -\int_{\Omega} a_j(t)u(0)\psi w_i \, dx, \]  

(23)

where we defined

\[ a_{ij}(t) := a(t)\delta_{ij} + \frac{1}{2} \int_0^1 u_i(0)a_p[u(\sigma; t)] + u_j(0)a_p[u(\sigma; t)] \, d\sigma. \]  

(24)

For \( |t| \leq 1 \), estimate (16) shows \( u(t) \) is uniformly bounded in \( C^{2,\alpha}_{(\Omega)} \) with respect to \( t \), for \( \alpha \) sufficiently small. Moreover, lemma 2 shows that \( t \mapsto u(t) \) is continuous in \( C^{2,\beta}_{(\Omega)} \) if \( \beta \in (0, \alpha) \), so by the smoothness of \( a \), we can write \( a_j(t) = a_j(0) + r_j(t) \), where remainder \( r_j(t) \) vanishes in \( C^{1,\beta}_{(\Omega)} \) as \( t \to 0 \). By (4), we can therefore suppose \( t \) is sufficiently small so that

\[ a_{ij}(t)\xi^i \xi^j \geq \frac{1}{2} \lambda_0(|u(0)|)|\xi|^2, \quad \xi \in \mathbb{R}^n. \]  

(25)

Similarly, we write \( a_j(t) = a_j(0) + r_2(t) \), and recalling (5), we suppose \( t \) is sufficiently small so that

\[ |a_j(t)| \leq 2\mu_0(|u(0)|). \]  

(26)

We then apply the Cauchy–Schwarz inequality to (23) using (25) and (26):

\[ \int_{\Omega} \lambda_0(|u(0)|)|Du|^2 \, dx \leq 16 \int_{\Omega} \frac{\lambda_0^2}{\lambda_0}(|u(0)|)^2 |Du(0)|^2 w^2 \, dx \]  

(27)

Since \( Du = D(u - s) \), the estimate (13) for \( k = s \) and \( h = f - s \) implies the gradient is small:

\[ \|Du(0)\|_{L^\infty} \leq C(s) \|f - s\|_{C^{2,\alpha}} \leq C(s)\Pi_1. \]  

(28)

Here \( \Pi_1 \) is to be fixed. If \( \Pi_1 \leq 1 \), then \( \|f\|_{C^{2,\alpha}} \leq |s| + C(s) \), and \( C^{2,\alpha} \) estimate (16) yields

\[ \|u(0)\|_{L^\infty} \leq C(s), \quad \lambda_0(|u(0)|) \geq \lambda_0(C(s)), \quad \mu_0(|u(0)|) \leq \mu_0(C(s)), \]  

(29)
for another $C(s)$; we recall the monotonicity of $\lambda_0$, $\mu_0$. Substituting these and the Poincaré inequality for $w \in W^{1,2}_0(\Omega)$ into (27) yields
\[
\int_{\Omega} |Dw|^2 \, dx \leq \frac{\mu_0^2}{\lambda_0^2} C(s) \cdot C(s, \Omega) \cdot \Pi_s^2 \int_{\Omega} |Dw|^2 \, dx.
\] (30)
Choosing $\Pi_s$ small depending on $s, \Omega$ confirms that $w = 0$. The Fredholm alternative [[GT], theorem 5.3] shows that Dirichlet problem (20) has a unique solution $\Pi_s$; theorem 9.13] that estimate $[\text{GT}, \text{theorem } 9.13]$ yields the $W^{1,2}$ estimate
\[
\|v(t)\|_{W^{1,2}(\Omega)} \leq C((20), \Omega)\|h\|_{W^{1,2}(\Omega)},
\] (31)
where $C((20), \Omega)$ indicates (uniform) dependence on the coefficients of the differential operator in (20). For small $t$, the coefficients and their derivatives have controlled $t$ dependence in the $C^3(\overline{\Omega})$ norm, so we conclude that $v(t)$ has a uniform-in-$t$ $W^{1,2}$ bound. Moreover, since $f, h \in C^\infty$, standard Schauder bootstrapping yields higher regularity $C^{k,\alpha}(\overline{\Omega}) \rightarrow C^{k,\alpha}(\overline{\Omega})$ estimates for $u(t)$ as in (16), so the $t$ dependence of the coefficients in (20) is uniform in any $C^k$ norm. Therefore, the higher regularity $W^{k+2,2}(\Omega)$ estimate [[GT], theorem 8.13] combined with Sobolev embeddings yield uniform-in-$t$ estimates for $v(t)$ in any $C^k$ norm.

**Step 2:**
We now show $v(t)$ converges to $v$ in $C^4(\overline{\Omega})$, where $v$ is the unique solution of Dirichlet problem (11). Note that the unique solvability for $v$ is similar to that for $v(t)$ in step 1, so we omit the proof. In (20), as before, we split
\[
a_s(t) = a_s(t; 0) + r_s(t), \quad a_{p_j}(t) = a_{p_j}(t; 0) + r_{p_j}(t), \quad a(t) = a(t; 0) + r_a(t),
\] (32)
where each $r_s(t)$ vanishes in $C^4(\overline{\Omega})$ as $t \rightarrow 0$. Thus, omitting $t = 0$ arguments, $v(t)$ solves the inhomogeneous equation
\[
\partial_t (a_s v(t) u_i + a_{p_j} v(t) u_i) + a v(t) = o(1)_{C^4(\overline{\Omega})},
\] (33)
where $o(1)_{C^4(\overline{\Omega})}$ indicates a quantity vanishing as $t \rightarrow 0$ in $C^4(\overline{\Omega})$, and its uniformity in $t$ with respect to terms including $v$ or $v_i$ follows from step 1. We then put $w(t) = v(t) - v$ and see that $w(t)$ uniquely solves the Dirichlet problem
\[
\partial_t (a_s w(t) u_i + a_{p_j} w(t) u_i) + a w(t) = o(1)_{C^4(\overline{\Omega})},
\] (34)
\[
w(t)|_{\partial \Omega} = 0.
\]
The right-hand side clearly vanishes in $L^p$ for any $p > 1$, so it follows from the strong $W^{2,p}$ estimate [[GT], theorem 9.13] that $\|w(t)\|_{W^{2,p}(\Omega)} \rightarrow 0$ as $t \rightarrow 0$. By Sobolev embedding, we deduce that $v(t) \rightarrow v$ in $C^4(\Omega)$, as desired. \hfill $\blacksquare$

We next linearize around solutions $u$ with large gradients if we assume that $o(s, p)$ satisfies additional conditions.
Proposition 2. Let \( a \) satisfy uniform lower bound (7) and decay condition (8). Then for any solution \( u \) to (2) with smooth boundary data \( f \), the linearization of (2) exists at \( u \).

Proof. The proof of step 2 in proposition 1 is unchanged, and we repeat step 1 until (25). To proceed further, we invoke additional conditions (7) and (8). We suppose \( t \) is sufficiently small so that
\[
|a(\xi, t)Du(\xi, 0)| \leq \varepsilon_0 + o(1)c_{1, \beta}\|Du(\xi, 0)\|_{L^\infty} \leq 2\varepsilon_0.
\]
Applying the Cauchy–Schwartz inequality and the Poincaré inequality, we obtain
\[
\int_\Omega |Du|^2 \, dx \leq \frac{16\varepsilon_0^2}{\lambda_0} \cdot C(\Omega)\int_\Omega |Du|^2 \, dx.
\]
Choosing \( \varepsilon_0 \) small depending on \( \lambda_0, \Omega \) yields the desired conclusion. The rest of the proof proceeds identically.  

2.2. Solutions with prescribed boundary jets

We first introduce a coordinate normalization, with \( x^i, i = 1, \ldots, n \) the usual Euclidean coordinates in this section.

Definition 1. We say that \( \Omega \) sits above the origin, if \( \{x^n = 0\} \) is a supporting hyperplane for \( \Omega \) with \( 0 \in \partial \Omega \), and \( \Omega \subset \{x^n > 0\} \).

The class of PDEs (2) with conditions (3) and (5) considered is invariant under Euclidean isometries. Let \( I = R \circ T \) be a Euclidean isometry (rotation \( R \) plus translation \( T \)) such that \( I^{-1}(\Omega) \) sits above the origin. The resulting PDE for \( I^*u(x) := u(Ix) \) is also of class (2):
\[
\nabla \cdot [R_*a(I^*u, \nabla I^*u)\nabla I^*u] = 0, \quad x \in I^{-1}(\Omega),
\]
where \( R_*a(s, p) = a(s, R^{-1}p) \). It is clear that \( R_*a \) satisfies any of conditions (3), (5), or (8) if and only if \( a \) does.

We now construct solutions with prescribed boundary jet, provided the gradient at the boundary point is sufficiently small compared to the solution. No additional assumptions on \( a \) are needed. Recall \( \Pi_1(s) \) as in proposition 1.

Proposition 3. Let \( \Pi : \mathbb{R} \to (0, \infty) \) be the positive continuous function defined in (55). Then for any \( x_0 \in \partial \Omega \) and \( (s, p) \in \mathbb{R} \times \mathbb{R}^n \) such that \( |p| < \Pi(s) \), there exists boundary data \( f \in C^\infty(\partial \Omega) \) for (2) such that \( u(x_0) = s \) and \( \nabla u(x_0) = p \), with \( \|f - s\|_{C^\infty} \leq \Pi_1(s) \).

Proof. Step 1. We construct sub/super-solutions.

First suppose \( \Omega \) sits above the origin and \( x_0 = 0 \). Given \( (s, p) \in \mathbb{R} \times \mathbb{R}^n \) with \( p = p' + p_ne_n \) and \( p' \perp e_n \), we define
\[
u^{x,p}(x) := s + p' \cdot x - A p_n \log(1 - x^n/A),
\]
where $A$ is the sufficiently large constant

$$A := 2 \, \text{diam } \Omega \geq 2 \max_{x \in \Omega} x^n. \tag{39}$$

We have $u^{x,p}(0) = s$ and $Du^{x,p}(0) = p$. The derivatives:

$$Du^{x,p}(x) = p' + \frac{p_n}{A - x^n} e_n, \quad D^2u^{x,p}(x) = \frac{p_n}{(A - x^n)^2} e_n \otimes e_n. \tag{40}$$

The expanded form of equation (2) is

$$a_{ij}u_{ij} + a_i \|\nabla u\|^2 = 0. \tag{41}$$

For $\pm = \text{sign } p_n$, we thus obtain

$$\pm (a_{ij}u_{ij} + a_i |\nabla u|^2) \geq \frac{\lambda_0(|u^{x,p}|)}{(A - x^n)^2} \frac{|p_n|}{\mu_0(|u^{x,p}|)} |p_n| - \left( \frac{A^2}{4} |p'|^2 + p_n^2 \right), \tag{42}$$

where

$$U' := s + \frac{A}{2} |p'| + A |p_n| \log 2$$

satisfies $|u^{x,p}(x)| \leq U'$ on $\Omega$.

Let us now restrict $p$ to satisfy

$$\max(|p'|, |p_n|) \leq 1, \quad U' \leq U := s + 2A. \tag{43}$$

Then we obtain

$$\pm (a_{ij}u_{ij}^p + a_i |Du^{x,p}|^2) \geq \frac{\mu_0(U)}{(A - x^n)^2} \left[ \lambda_0(U) |p_n| - \left( \frac{A^2}{4} |p'|^2 + p_n^2 \right) \right]. \tag{44}$$

Choosing now $p$ to be any vector defined in the paraboloid

$$\left( \frac{2\lambda_0(U)}{A^2 \mu_0(U)} \right)^{-1} |p'|^2 \leq |p_n| \leq \frac{\lambda_0(U)}{2 \mu_0(U)}, \quad \max(|p'|, |p_n|) \leq 1, \tag{45}$$

we find that $u^{x,p}$ is a one-sided solution:

$$\pm (a_{ij}u_{ij}^p + a_i |Du^{x,p}|^2) \geq 0. \tag{46}$$

To remove the ‘corners’ of the paraboloid (45), we denote by $U = U'$ the constant in (43), and denote the various constants by

$$C_1^1 := \min\left( \frac{\lambda_0(U')}{2 \mu_0(U')}, 1 \right), \quad C_2^1 := \min\left( \frac{2 \lambda_0(U')}{A^2 \mu_0(U')}, 1 \right),$$
so that after shrinking the bounds, the paraboloid becomes
\[
\frac{1}{C_s^2} |p'|^2 \leq |p_n| \leq C_s^1.
\]  
(47)

To satisfy the smallness condition of proposition 1, we shrink the bounds again. Defining
\[
B_s^1 := \min(C_s^1, \frac{\Pi_1(s)}{N \text{diam} \Omega}), \quad B_s^2 := C_s^2,
\]
for some sufficiently large $N$, it is easy to verify that $u^{s,p}$ satisfies the desired estimate
\[
\|u^{s,p} - s\|_{C^{2,\alpha}(\partial \Omega)} \leq \Pi_1(s)
\]
provided $p = (p', p_n)$ lies within the paraboloid
\[
\frac{1}{B_s^2} |p'|^2 \leq |p_n| \leq B_s^1.
\]  
(48)

We next construct solutions with boundary jet inside this paraboloid, and more generally in the smallest rectangle containing it.

**Step 2.** A comparison principle argument.

Given $s$, we let $p \in \mathbb{R}^n$ be any vector in the vertical boundary of the paraboloid (48):
\[
|p'|^2 < B_s^1 B_s^2, \quad p_n = B_s^1.
\]  
(49)

Since $p_n > 0$, it follows that $u^{s,p}$ is a subsolution of (2). Choosing boundary data $f \in C^\infty(\partial \Omega)$ of the form
\[
f^{s,p}(x) := u^{s,p}(x), \quad x \in \partial \Omega,
\]
the comparison principle [[GT], theorem 10.7] shows that
\[
u[f^{s,p}](x) \geq u^{s,p}(x), \quad x \in \Omega,
\]  
(50)

with equality at $x = 0$, so the inner normal derivatives satisfy the same inequality:
\[
\frac{\partial u[f^{s,p}]}{\partial x^p}(0) \geq \frac{\partial u^{s,p}}{\partial x^p}(0) = p_n.
\]  
(52)

If we instead choose
\[
p_n = -B_s^1,
\]
then the inequalities reverse because $u^{s,p}$ becomes a supersolution. This means that the set of normal derivatives with tangential jet $(s, p')$ and small boundary data,
\[
\mathcal{N}_{s,p'} := \left\{ \frac{\partial u[g]}{\partial x^p}(0) | g \in C^\infty(\partial \Omega), g(0) = s, \nabla g(0) = p', \text{ and } \|g - s\|_{C^{2,\alpha}(\partial \Omega)} \leq \Pi_1(s) \right\},
\]
\]  
(53)
contains numbers at least as large as \( B^1 \) and at least as small as \(-B^1\). Since the set of \( g \) with boundary jet \((g, \nabla g)(0) = (s, p')\) is convex, it is connected. So, of course, is the set of such \( g \) also satisfying the smallness estimate

\[
\|g - s\|_{C^2(\Omega \setminus \partial \Omega)} \leq \Pi_1(s).
\]

By the smoothness of solutions to (2), we deduce that \( N^{s,p'} \) is connected, and it therefore contains the interval \([-B^1, B^1]\). Varying \( p' \) according to (49), we conclude that for each \( s \) and any vector \( p \) in the rectangle

\[
|p'|^2 \leq B^1_s B^2_s, \quad |p_n| \leq B^1_s,
\]

there exists a solution with boundary jet \((u(0), \nabla u(0)) = (s, p)\) with boundary data \( g \) satisfying (54). Defining \( \Pi \) by the radius of the largest sphere inside the rectangle,

\[
\Pi(s) = \min(\sqrt{B^1_s B^2_s}, B^1_s).
\]

This completes the proof of the proposition, in the case that \( \Omega \) sits above the origin.

**Step 3. Rotational invariance.**

Now suppose \( \Omega \subset \mathbb{R}^n \) is any smooth domain, and choose \( s, p \) such that \( |p| < \Pi(s) \), and \( x_0 \in \partial \Omega \). Letting \( T(x) = x - x_0 \) translate \( x_0 \) to the origin, we choose a rotation \( R \in O(n) \) such that the inner normal vector is mapped to \( e_n \), i.e., \( R(-\nu|_{x_0}) = e_n \). Then for Euclidean isometry \( I = R \circ T \), we see that \( I^{-1}(\Omega) \) sits above the origin. Moreover, inequalities (3)–(5) and the constant \( A \) in (39) are preserved under the isometry. Since \( |R| = \Pi(s) \), we can find a solution \( I' u \) of the rotated equation (37) such that \( I' u(0) = s \) and \( \nabla I' u(0) = p \). In original variables, we have \( u(x_0) = s \) and \( \nabla u(x_0) = p \), which completes the proof.

If we also assume the decay condition (8), then we can choose \( \Pi = +\infty \). That is, we can find a solution with any prescribed boundary jet. We can also change conditions (7) and (8), in this proof, to

\[
|a_i(s, p)| \leq \frac{\lambda(s, p)}{|p|},
\]

where \( \lambda(s, p) \) is in (3), and \( C > 0 \) is arbitrary. This condition makes (2) behave similar to a linear equation which admits exponential solutions.

**Proposition 4.** Let \( s \) satisfy (56), or alternatively (7) and (8). For any \((s, p) \in \mathbb{R} \times \mathbb{R}^n \) and \( x_0 \in \partial \Omega \), there exists boundary data \( f \in C^\infty(\partial \Omega) \) for (2) such that \( u(x_0) = s \) and \( \nabla u(x_0) = p \).

**Proof.** As in step 3 in the proof of proposition 3, by the invariance of the problem under Euclidean isometries, it suffices to assume that \( \Omega \) sits above the origin.

Fix \((s, p) \in \mathbb{R} \times \mathbb{R}^n \) with \( p_n \neq 0 \) and \( p' := p - p_n e_n \), and for \( h > 0 \) small depending on the constant \( C \) in (56) and \( p \), we introduce the function

\[
u^{s,p}(x) := s + p' \cdot x + h p_n (\alpha^x / h - 1).
\]

If \( p_n > 0 \) (resp. \( p_n < 0 \)) then we claim that \( \nu^{s,p} \) is a subsolution (supersolution) of (2), with jet

\[
\nu^{s,p}(0) = s, \quad \nabla \nu^{s,p}(0) = p.
\]
We have

\[ |\nabla u_s'|^2 = |p'|^2 + p_n^2 e^{2e'/h}, \quad u_s' = \frac{p_n}{h} e^{e'/h} \delta_m \delta_m', \tag{59} \]

so ellipticity (3) yields, for \( \pm 1 = \text{sign} p_n \),

\[ \pm (a_n |\nabla u_s'|^2 + a_{ij} u_{ij}') \geq \pm a_n |\nabla u_s'|^2 + \lambda \frac{|p_n|}{h} e^{e'/h}. \tag{60} \]

Applying decay condition (56), we obtain

\[ \pm (a_n |\nabla u_s'|^2 + a_{ij} u_{ij}') \geq \lambda e^{e'/h} \left( \frac{|p_n|}{h} - C \sqrt{|p'|^2 e^{-2e'/h} + p_n^2} \right). \tag{61} \]

Choosing \( h \) such that \( \frac{|p_n|}{h} \geq C \sqrt{|p'|^2 + p_n^2} \) verifies the claim.

We next observe that by continuity, the set of possible normal derivatives with tangential jet \((s, p')\)

\[ \left\{ \frac{\partial u[g]}{\partial x^n}(0) \mid g \in C^\infty(\partial \Omega) \text{ and } g(0) = s, \nabla' g(0) = p' \right\} \tag{62} \]

is connected and nonempty, so to show it is \( \mathbb{R} \), we need only show it is unbounded from above and below. If \( p_n \gg 1 \) is large, we let \( f = u_s'|_{\partial \Omega} \) and \( u \) solve (2) with boundary value \( f \). Then \( u_{s'} \) is a subsolution, and the comparison principle [[GT], theorem 10.7] says

\[ u(x) \geq u_{s'}(x), \quad x \in \Omega, \]

with equality at \( x = 0 \), so

\[ \frac{\partial u}{\partial x^n}(0) \geq \frac{\partial u_{s'}}{\partial x^n}(0) = p_n. \]

Repeating the argument with \( p_n \ll -1 \) and using supersolution \( u_{s'} \) yield the desired conclusion. \( \Box \)

**Remark 1.** The results of this section hold true unchanged for non-homogeneous conductivities \( a = a(x, s, p) \), provided the inequalities (3) and (5) remain independent of \( x \) and we include in (5) the structural condition (1.7) from [MU], namely

\[ |\nabla_x a(x, s, p)| \leq \mu_0(|s|)|p|. \]

Under this assumption, proposition 3 goes through, since estimate (42) is only changed by a factor of 2. To capture the global jet prescription proposition 4, we could instead use the condition

\[ |\nabla_x a(x, s, p)| \leq C \lambda(s, p), \]

for some \( C > 0 \), or just \( \leq C \) if \( \lambda(s, p) \geq \lambda_0 \) is uniformly elliptic. In these cases, estimate (61) goes through. At the final step of rotational invariance, we instead define the transformed conductivity in (37) by \( I_* a(x, s, p) = a(Ix, s, R^{-1} p) \).
2.3. Geometric reformulation of the linearized problem

We next geometrically reformulate the linearized equation (11) and its DN map (12), see e.g. [LU] for the same with the conductivity equation via the Laplace operator. We assume throughout this section that the linearization of (2) exists at a given solution $u$, which we addressed in section 2.1.

Let us first rewrite the equation. We can decompose the principal part of (11), namely

$$\partial_i (a_p v_i u_i + a v_i),$$

into its symmetric and anti-symmetric parts and rewrite the linearized differential operator as:

$$L_a[u] v := \partial_i (a_s v_i u_i + a v_i) = \partial_i (a_i j v_j + (a_s u v + A_i v_j)) = \partial_i (a_i j v_j + b' v),$$

where linearized conductivity $a_{ij}$, given in (3), is the symmetric part, and

$$b' := a_s u_i - \partial_j A_{ij}, \quad A_{ij} := \frac{1}{2} (a_p u_i - a_p u_j)$$

accounts for the anti-symmetric part; the last equality in (63) follows from anti-symmetry. We can also rewrite the linearized DN map (12):

$$\Gamma_a[u](h) = (a_i j v_i v_j + A_i j v_j + a_s u v) dS.$$  

(65)

Henceforth in this section only, we distinguish Euclidean coordinates from ordinary subscripts (which will denote local coordinates) unless otherwise stated.

We next rewrite this equation in terms of geometric quantities. Let $x'$ denote local coordinates on $\Omega$, and denote $x'_e$ to be global Euclidean coordinates. We define a $(2,0)$ tensor field $G \in T^{2,0}_0(\Omega)$ by its values in Euclidean coordinates, using the symmetric part $a_{ij}$ in (3) of the linearized operator. If $n \geq 3$,

$$G^i_j := G(dx'_e, dx'_j) = \sqrt{\text{det } a_{k\ell}} a_{ij}, \quad G = G^i_{j\ell} \partial_i \partial_{j\ell} = G^i_j \partial_i \partial_{j,e}.$$  

(66)

If $n = 2$, we first define a conductivity function $\sigma \in C^\infty(\Omega)$ by

$$\sigma := \text{det } a_{k\ell}.$$  

(67)

We can then define $G$ as follows:

$$G^i_j := G(dx'_e, dx'_j) = (\det a_{k\ell})^{-1/2} a_{ij}, \quad G = G^{ij} \partial_i \partial_j.$$  

(68)

In each case, the relationship can be written as

$$n \geq 3: \quad \frac{1}{\sqrt{\det G^i_j}} G^i_j = a_{ij},$$  

$$n = 2: \quad \sigma G^i_j = a_{ij}, \quad \det G^i_j = 1.$$  

(69)
We next define a Riemannian metric $g \in \mathcal{T}_2^1(\Omega)$ by the inverse of $G$ in Euclidean coordinates, which is well defined by ellipticity (3)

$$g_{\epsilon,ij} := g \left( \frac{\partial}{\partial x_{\epsilon}^i}, \frac{\partial}{\partial x_{\epsilon}^j} \right) = (G^{-1})_{ij}, \quad g = g_{ij} \, dx^i \, dx^j = g_{ij,\epsilon} \, dx^i_{\epsilon} \, dx^j_{\epsilon}. \quad (70)$$

We will instead denote $\partial_a = \partial / \partial x_a$, along with $g^{ij} = (g_{\epsilon,ij})^{-1}$ and $\sqrt{g} = \sqrt{\det g_{\epsilon,ij}}$. The divergence of the linearized conductivity takes two forms in various dimensions:

$$n \geq 3: \quad \partial_a (a_{ij} \partial_a v) = \sqrt{g} \Delta g v,$$

$$n = 2: \quad \partial_a (a_{ij} \partial_a v) = \text{div}(\sigma \nabla g v) = \sigma \Delta g v + \langle \nabla g \sigma, \nabla g v \rangle_g, \quad (71)$$

where, in local coordinates, the Laplace operator, gradient, and divergence are given by

$$\Delta g v = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} \, g^{ij} \partial_a v), \quad \nabla g v = g^{ij} \partial_a v \partial_a, \quad \text{div } B = \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} B^a). \quad (72)$$

To account for the lower order terms in (63), we define the magnetic Schrödinger operator with potential:

$$\Delta_{g,A,q} v := \frac{1}{\sqrt{g}} (\partial_i + A_i A_q) \sqrt{g} \, g^{ij} (\partial_j + A_j) v + q v$$

$$= \Delta g v + 2 \langle A^\#_g, \nabla g v \rangle + \left( \text{div } A^\#_g + |A^\#_g|^2 + q \right) v, \quad (73)$$

where $A = A_q dx^q, q$ are to be chosen, $A^\#_g = g^{ij} A_i \partial_j$, and $|A^\#_g|^2 = g^{ij} A_i A_j$. Recalling (71), we obtain the following relations:

$$n \geq 3: \quad \mathcal{L}_a [u] v = \sqrt{g} \Delta_{g,A,q} v,$$

$$n = 2: \quad \mathcal{L}_a [u] v = \sigma \Delta_{g,A,q} v, \quad (74)$$

provided the following relations are true:

$$n \geq 3: \quad b' = 2 \sqrt{g} \, g^{ij} A_{ij}, \quad \partial_a b' = \sqrt{g} (\text{div } A^\#_g + |A^\#_g|^2 + q),$$

$$n = 2: \quad b' + g^{ij} \partial_a \sigma = 2 \sigma g^{ij} A_{ij}, \quad \partial_a b' = \sigma (\text{div } A^\#_g + |A^\#_g|^2 + q). \quad (75)$$

This system can be solved for $A$ and $q$ in terms of $b$, $\sigma$, and $g$. This completes the reformulation of the linearized PDE (11) via (74).

We next reformulate the linearized DN map (65), and start with the anti-symmetric part of the linearized operator (64). We define a $(1,1)$ tensor $\alpha \in \mathcal{T}_2^1(\Omega)$ in terms of $g$:

$$\alpha_{\epsilon,i} := \alpha(dx_{\epsilon}^i, \partial_{\epsilon}^i) = \frac{1}{\sqrt{g}} g_{\epsilon,\delta} A_{\delta,\epsilon}, \quad \alpha = \alpha'_{ij} \partial_i \, dx^j, \quad (76)$$
Here, $\sqrt{g_e} = \sqrt{\det g_{e,ij}}$, and $A_{ik}$ is as in (64). The tensor $\alpha$ is anti-symmetric in the inner product induced by $g$:

$$\langle \alpha \cdot V, W \rangle_g = - \langle V, \alpha \cdot W \rangle_g, \quad V, W \in T\Omega,$$

(77)

since in Euclidean coordinates, with $V = V^i \partial_{e,i}, W = W^i \partial_{e,i}$, we have

$$\langle \alpha \cdot V, W \rangle_g = g_{e,\ell} \left( \frac{1}{2\sqrt{g_e}} g_{e,\ell,\beta} (a_{\beta e} u_i - a_{\beta i} u_e) V^i \right) W^\ell_g =: (g_{e,\ell} T^{\ell k}_{g,\beta,\ell}) V^\ell_g W^\ell_g,$$

(78)

where $T^{\ell k}$ is anti-symmetric in its indices, so the claim follows.

The Riemannian volume form $dV_g$ has the Euclidean representation

$$dV_g = \sqrt{g_e} \, dx^1_e \wedge \cdots \wedge dx^n_e = \sqrt{g_e} \, dx_e.$$

(79)

The induced area form $dS_g$ on $\partial\Omega$ is

$$dS_g = \nu_g \cdot dV_g|_{\partial\Omega},$$

(80)

where $\nu_g$ is the unit normal to $\partial\Omega$. Letting $f$ be a defining function for $\partial\Omega$, or a smooth function $\mathbb{R}^n \to \mathbb{R}$ with $\partial\Omega = f^{-1}(0)$ and $df|_{\partial\Omega} \neq 0$, the unit normals for Euclidean and $g$ metrics can be written in Euclidean coordinates as

$$\nu = (|df|_g)^{-1} \partial_{e,f} \partial_{e,i}, \quad \nu_g = (G(df, df))^{-1/2} G^\ell_{e,i} \partial_{e,f} \partial_{e,i},$$

(81)

As sections of the product bundle $\partial(T\Omega) \otimes \Lambda^0_{n-1}(\partial\Omega)$, the following relationship between the normal vectors holds:

$$n \geq 3: \quad (a_i \nu_{e,i}) \otimes dS = \nu_g \otimes dS_g,$$

$$n = 2: \quad (a_i \nu_{e,i}) \otimes dS = \sigma \nu_g \otimes dS_g.$$

(82)

In particular, the main part of the linearized DN map (65) has the form

$$n \geq 3: \quad (a_i \nu_{e,i} + A_{ij} \nu_{e,j}) dS = \langle \nabla_g v + \alpha \cdot \nabla_g v, \nu_g \rangle_g \, dS_g,$$

$$n = 2: \quad (a_i \nu_{e,i} + A_{ij} \nu_{e,j}) dS = \langle \sigma \nabla_g v + \alpha \cdot \nabla_g v, \nu_g \rangle_g \, dS_g,$$

(83)

where $v \in C^\infty(\overline{\Omega})$, $a_{ij}$, $\sigma$ are in (69), $A_{ij}$ is in (64), and $\alpha$ is in (76).

For the final term in (65), we define a vector field $\beta = \beta^i \partial_i$ so that the following equation holds on $\partial\Omega$:

$$n \geq 3: \quad (a_i u_e \nu_{i}) dS = \langle A^\# + \beta, \nu_g \rangle \, dS_g,$$

$$n = 2: \quad (a_i u_e \nu_{i}) dS = \langle \sigma A^\# + \beta, \nu_g \rangle \, dS_g.$$

(84)
We now recall the magnetic DN map, $C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$:

$$\Lambda_{g,A,q}(h) : h \mapsto \langle \nabla_g v[h], A^# v[h], \nu_g \rangle,$$

where: $\Delta_{g,A,q} v = 0,$ 
$$v|_{\partial\Omega} = h.$$  \hfill (85)

Then we have the following geometric reformulation. The linearized PDE (63) corresponds to a magnetic Schrödinger operator with potential, and the linearized DN map (65) corresponds to an oblique magnetic normal derivative with lower order term.

**Proposition 5.** Let $u$ be a solution of (2) and $f \in C^\infty(\partial\Omega)$ such that the linearization of (2) exists at $u$. Let also $h \in C^\infty(\partial\Omega)$ and $v$ solve the linearized problem (11).

Then for $n \geq 3$:

$$\mathcal{L}_u[u] v = \sqrt{g} \Delta_{g,A,q} v,$$
$$\Gamma_{\delta} [u](h) = \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g.$$ \hfill (86)

For $n = 2$:

$$\mathcal{L}_u[u] v = \sigma \Delta_{g,A,q} v,$$
$$\Gamma_{\delta} [u](h) = \sigma \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g.$$ \hfill (87)

Here, metric $g$ is given by (69) and (70), and $\sigma$ by (67) for $n = 2$. Magnetic potential $A$ and potential $q$ are determined by (75), anti-symmetric tensor $\alpha$ is given by (76), and vector field $\beta$ is defined according to (84). The map $\Lambda$ is defined by (85), and $\Gamma_{\delta}[u]$ is defined by (65).

In particular, if $\Gamma_{\delta} = \Gamma_{\delta^*}$ then for $n \geq 3$:

$$\Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g$$
$$= \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g.$$ \hfill (88)

For $n = 2$:

$$\sigma \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g$$
$$= \sigma \Lambda_{g,A,q}(h) dS_g + \langle \alpha \cdot \nabla_g h + \beta h, \nu_g \rangle_g dS_g.$$ \hfill (89)

**Proof.** The only claim remaining to prove is the identity

$$\langle \alpha \cdot \nabla_g v[h], \nu_g \rangle_g dS_g = \langle \alpha \cdot \nabla_g h, \nu_g \rangle_g dS_g,$$ \hfill (90)

or that this term depends only on the tangential projection of $\nabla_g v[h]$. This follows from the anti-symmetry of $\alpha$, since the normal contribution is

$$\langle \nabla_g v[h], \nu_g \rangle \langle \alpha \cdot \nu_g, \nu_g \rangle = 0,$$

which follows from (77).
2.4. Boundary determination of the linearized conductivity

In this section, we determine the tangential part of the linearized conductivity $a_{ij}$ at the boundary $\partial \Omega$. The main technical tool we need is the calculation of the symbol of the magnetic DN map (85) in boundary normal coordinates, [DKSU, lemma 8.6]. See [LU] for the construction in the case of the ordinary Laplace operator.

Letting $x^1, \ldots, x^{n-1}$ be local coordinates on a neighborhood $\Sigma \subset \partial \Omega$, we denote boundary normal coordinates in a neighborhood of $\Sigma$ in $\Omega$ by $(x^1, \ldots, x^n)$, with $x^n$ the $g$-distance function from $\partial \Omega$. The metric and its dual in boundary normal coordinates are

$$ g = (dx^n)^2 + \sum_{i,j < n} g_{ij} \, dx^i \, dx^j, \quad G = \partial_n \otimes \partial_n + \sum_{i,j < n} g_{ij} \partial_i \otimes \partial_j. $$

(91)

Note that

$$ \sum_{i,j < n} g_{ij} \, dx^i \, dx^j = g|_{T^* \partial \Omega}, \quad \sum_{i,j < n} g_{ij} \partial_i \otimes \partial_j = G|_{T^* \partial \Omega}, $$

(92)

if $x \in \partial \Omega$. For two metrics $g, \tilde{g}$, we use the same local coordinates $x^1, \ldots, x^{n-1}$ for $\Sigma$ when defining boundary normal coordinates.

We also need the principal symbol of pseudo-differential operators on a manifold using left quantization, see e.g. [H, definition 18.1.20]. The differential operator $C^\infty(\partial \Omega) \to C^\infty(\partial \Omega)$ defined by

$$ h \mapsto \langle \alpha \cdot \nabla g h + \beta h, \nu_g \rangle_g = \sum_{k < n} (-\alpha \cdot \nu_g)^k h_k + \langle \beta, \nu_g \rangle h, $$

clearly has principal symbol

$$ i\langle \alpha \cdot \xi^#, \nu_g \rangle_g = \sum_{k < n} (-\alpha \cdot \nu_g)^k (i \xi)_h, \quad (x, \xi) \in T^*_x (\partial \Omega). $$

(93)

To compute the final term in the DN maps (86) and (87), we now apply [DKSU, lemma 8.6], see also [LU], where the symbol of the magnetic DN map (85) is computed in boundary normal coordinates in $n \geq 3$ dimensions. The proof there goes through unchanged for the case of $n = 2$, although the subsequent boundary determination problem is different due to conformal invariance; see the remark in [LU].

**Lemma 3 (From lemma 8.6 in [DKSU]).** Let $(\Omega, g)$ be a compact Riemannian manifold with boundary with dimension $n \geq 2$. Then $\Lambda_{g,A,q}$ is a pseudo-differential operator of order $1$ on $\partial \Omega$, and its principal symbol is

$$ \langle \alpha \cdot \xi^#, \nu_g \rangle_g = -\sum_{k < n} g^{ij}(x) \xi_i \nu_j, \quad (x, \xi) \in T^*_x (\partial \Omega). $$

(94)

We thus obtain the principal symbol of $\Gamma_a[n]$, thought of as a form-valued pseudo-differential operator:

$$ n \geq 3: \quad -\langle \alpha \cdot \xi^#, \nu_g \rangle_g dS_g, $$

$$ n = 2: \quad -\sigma \langle \alpha \cdot \xi^#, \nu_g \rangle_g dS_g. $$

(95)

In dimension $n \geq 3$, we can recover the tangential projection of the metric on $\partial \Omega$. In dimension $n = 2$, we can recover the ‘effective conductivity’ $\sigma$ on the boundary. In both cases, we can recover the normal elements of the anti-symmetric tensor.
Lemma 4. If $\Gamma_\alpha = \Gamma_\tilde{\alpha}$, then $G|_{T^*\partial\Omega} = \tilde{G}|_{T^*\partial\Omega}$ if $n \geq 3$, where $G$ is given in terms of $a$ in (69). If $n = 2$, then $\sigma = \tilde{\sigma}$ on $\partial\Omega$, where we defined $\sigma$ by (67). In both cases, we have

$$(\alpha \cdot \nu_g) dS_g = (\tilde{\alpha} \cdot \nu_{\tilde{g}}) dS_{\tilde{g}}$$

on $\partial\Omega$.

Proof. By proposition 5, equalities (86) and (87) hold, so the principal symbols on $T^*\partial\Omega$ of the associated pseudo-differential operators on $\partial\Omega$ must coincide. By (95) and (88)–(89), we obtain

$$n \geq 3: (-|\xi|_g + i(\alpha \cdot \xi^#, \nu_g)) dS_g = (-|\tilde{\xi}|_{\tilde{g}} + i(\tilde{\alpha} \cdot \tilde{\xi}^#, \nu_{\tilde{g}})) dS_{\tilde{g}},$$

$$n = 2: (-\sigma|\xi|_g + i(\alpha \cdot \xi^#, \nu_g)) dS_g = (-\tilde{\sigma}|\xi|_{\tilde{g}} + i(\tilde{\alpha} \cdot \tilde{\xi}^#, \nu_{\tilde{g}})) dS_{\tilde{g}},$$

(97)

where we denote $\tilde{\xi}^# = \tilde{g}^{ij} \xi_j \partial_i$. In boundary normal coordinates, we have

$$dS_g = \partial_n \cdot dV_g|_{v=0} = \sqrt{g} \, dx^1 \wedge \cdots \wedge dx^{n-1} = \sqrt{g}|_{T^*\partial\Omega} \, dx^1 \wedge \cdots \wedge dx^{n-1},$$

so equating the real parts in (97) yields

$$n \geq 3: (\det g)^{i/j} = (\det \tilde{g})^{i/j}, \quad i, j < n,$$

$$n = 2: \quad \sigma = \tilde{\sigma}.$$  

(98)

The imaginary parts in (97) yield (96) in both cases. Note that the dimension $n = 2$ equality in (98) follows from $|\xi|_g dS_g = dx^1$.

Taking the determinant of both sides of (98) shows that $(\det g)^{n-2} = (\det \tilde{g})^{n-2}$, hence $g^{ij} = \tilde{g}^{ij}$ for $i, j < n$. The $n \geq 3$ case thus recovers $G|_{T^*\partial\Omega}$. □

We thus obtain the main result of this section, a boundary determination result for the quasilinear PDE (2) in Euclidean coordinates.

Proposition 6. Let $u$ be a solution of (2) and $f \in C^\infty(\partial\Omega)$ such that the linearization of (2) exists at $u$. Similarly with $\tilde{u}$ and $\tilde{a}$ replacing $a$.

If $n \geq 3$ and $\Gamma_\alpha = \Gamma_{\tilde{\alpha}}$, then the following holds on $\partial\Omega$:

$$[aI + \frac{1}{2}(\nabla_\mu \otimes \nabla f + \nabla f \otimes \nabla_\mu a)]|_{T^*\partial\Omega} = [\tilde{a}I + \frac{1}{2}(\nabla_{\tilde{\mu}} \otimes \nabla f + \nabla f \otimes \nabla_{\tilde{\mu}} \tilde{a})]|_{T^*\partial\Omega}.$$  

(99)

If $n \geq 2$ and $\Gamma_\alpha = \Gamma_{\tilde{\alpha}}$, then we have on $\partial\Omega$,

$$(\nabla_\mu a \otimes \nabla u - \nabla u \otimes \nabla_\mu a) \cdot \nu = (\nabla_{\tilde{\mu}} \tilde{a} \otimes \nabla \tilde{u} - \nabla \tilde{u} \otimes \nabla_{\tilde{\mu}} \tilde{a}) \cdot \nu.$$  

(100)
If $n = 2$ and $\Gamma_a = \Gamma_{\tilde{a}}$, then we also have
\[
det a_{ij} = det \tilde{a}_{ij} \tag{101}
\]
on $\partial \Omega$, where $a_{ij}$ is in (3).

**Proof.** If $\Gamma_a = \Gamma_{\tilde{a}}$, then lemma 4 applies. Since for $n \geq 3$ we have $G|_{T^*\partial\Omega} = \tilde{G}|_{T^*\partial\Omega}$, the characterization (69) of $a_{ij}$ implies (99); indeed, the restriction of $\nabla u$ to $T^*\partial\Omega$ is precisely $\nabla f$ by (2). Recalling the definition (67) of $\sigma$ shows that $\det a_{ij} = \det \tilde{a}_{ij}$ on $\partial\Omega$.

For (96), we recall from (83) that
\[
-(\alpha \cdot \nu_g) dS_g = (A_{ij} \nu_j) dS, \tag{102}
\]
and (96) yields
\[
A_{ij} \nu_j = \tilde{A}_{ij} \nu_j, \tag{103}
\]
which is precisely (100), recalling the definition (64) of $A_{ij}$. $\square$

3. Proofs of theorems

Suppose first that $\Omega$ sits above the origin, see definition 1. Evaluating equalities (99)–(101) at $x = 0$ and identifying $T_0\partial\Omega$ with $\mathbb{R}^{n-1} \times \{0\}$, we obtain the following equalities at $x = 0$:

If $n \geq 3$:
\[
a \delta_{ij} + \frac{1}{2}(a_{ij} f_i + a_{ji} f_j) = \tilde{a} \delta_{ij} + \frac{1}{2}(\tilde{a}_{ij} f_i + \tilde{a}_{ji} f_j), \quad i, j < n. \tag{104}
\]

If $n \geq 2$:
\[
a_{ij} u_n - a_{ji} f_i = \tilde{a}_{ij} \tilde{u}_n - \tilde{a}_{ji} f_i, \quad i < n. \tag{105}
\]

If $n = 2$:
\[
(a + a_{p1} f_1)(a + a_{p2} u_2) - \frac{1}{4}(a_{p1} u_2 + a_{p2} f_1)^2 = (\tilde{a} + \tilde{a}_{p1} f_1)(\tilde{a} + \tilde{a}_{p2} u_2) \tag{106}
\]
\[
- \frac{1}{4}(\tilde{a}_{p1} \tilde{u}_2 + \tilde{a}_{p2} f_1)^2.
\]

**Idea of proving theorem 1.** Our approach below is to evaluate the above equalities at solutions with prescribed boundary jet. After exploiting the algebraic structure of these matrix equalities, we eventually deduce $a = \tilde{a}$ at this boundary jet. By varying the jet, we obtain equality along a subspace of $\mathbb{R} \times \mathbb{R}^n$. By using Euclidean isometry invariance to rotate this subspace, we conclude the full result. We consider the dimension cases $n = 2$ and $n = 3$ separately below.

**Proof of theorem 2.** This follows from theorem 1 since $a - \tilde{a}$ is an analytic function which vanishes on an open set.

**Idea of proving theorem 3.** This is proved exactly as in theorem 1 below. The only difference is we can prescribe an arbitrarily large boundary jet, by proposition 4, which ensures global uniqueness.
Idea of proving corollary 1. This is identical to proving theorem 1, except that the homogeneity assumption $\partial u(x, s, p)/\partial x = 0$ is used in the last step of Euclidean invariance. In remarks 2 and 3 for dimension $n = 3$ and remark 4 for dimension $n = 2$, we show how this can be handled to obtain corollary 1.

The case $n > 3$

Fixing $s \in \mathbb{R}$, we choose $p \in \mathbb{R}^{n-1} \times \{0\}$ such that $0 < |p| < \Pi(s)$. Using proposition 3, we can find $f \in C^\infty(\partial \Omega)$ such that the solution $u = u[f]$ to (2) has the following properties:

$$f(0) = s, \quad \nabla f(0) = p, \quad \frac{\partial u}{\partial x^n}(0) = 0, \quad \|f - s\|_{C^{2}(\partial \Omega)} \leq \Pi(s). \quad (107)$$

By proposition 1, the linearization thus exists at $u = u[f]$, and we can invoke proposition 6.

We now use this same $f$ as boundary data to create solution $\tilde{u}[f]$ to (2) with $\tilde{a}$ in place of $a$. Since $\Gamma_a = \Gamma_{\tilde{a}}$, we infer at $x = 0$,

$$\tilde{a} \frac{\partial \tilde{u}}{\partial x^n}(0) = a \frac{\partial u}{\partial x^n}(0) = 0, \quad (108)$$

and hence $\tilde{a}_n(0) = 0$ as well. Substituting this into (104) yields

$$a\delta_{ij} + \frac{1}{2}(a_jp_i + a_ip_j)_{(s,p)} = \tilde{a}\delta_{ij} + \frac{1}{2}(\tilde{a}_j p_i + \tilde{a}_i p_j)_{(s,p)}, \quad i, j < n. \quad (109)$$

Let us put $A = a - \tilde{a}$:

$$A\delta_{ij} + \frac{1}{2}(A_j p_i + A_i p_j)_{(s,p)} = 0, \quad i, j < n. \quad (110)$$

To find the spectrum, we first note that $A$ is the eigenvalue in directions orthogonal to $D_{p,A}$ and $p$. Afterwards, we expand an eigenvector in this basis and diagonalize the resulting $2 \times 2$ matrix. The spectrum of the matrix on the left-hand side of (110), listed in increasing order, is as follows

$$A + \frac{1}{2}p \cdot \nabla_x A - \frac{1}{2}|p|\nabla_x A, \quad A, \ldots, A, \quad A + \frac{1}{2}p \cdot \nabla_x A + \frac{1}{2}|p|\nabla_x A, \quad (111)$$

where $\nabla_x A = (A_{p_1}, \ldots, A_{p_{n-1}})$, and middle eigenvalue $A$ is listed $n - 3$ times. From (110), these eigenvalues are all zero. Subtracting the largest and smallest eigenvalues yields

$$|\nabla_x A(s, p)| = 0, \quad s \in \mathbb{R}, \quad p \in \mathbb{R}^{n-1} \times \{0\}, \quad |p| < \Pi(s) \quad (112)$$

so the smallest eigenvalue yields the result on a hyperplane, even if $n = 3$,

$$A(s, p) = 0, \quad s \in \mathbb{R}, \quad p \in \mathbb{R}^{n-1} \times \{0\}, \quad |p| < \Pi(s). \quad (113)$$

We can also conclude this equality from the vanishing of the middle eigenvalue in (111). Now choose any Euclidean isometry $I = R \circ T$ such that $I^{-1}(\Omega)$ sits above the origin. Then the above argument applies to the rotated PDE (37), and we obtain

$$R \cdot A(s, p) = 0, \quad s \in \mathbb{R}, \quad p \in \mathbb{R}^{n-1} \times \{0\}, \quad |p| < \Pi(s). \quad (114)$$
The set of such rotations $R$ is all of $O(n)$. Recalling that
\[ R_s A(s, p) = A(s, R^{-1} p) \]
and varying $R \in O(n)$, we conclude in general that
\[ A(s, p) = 0, \quad s \in \mathbb{R}, \quad p \in \mathbb{R}^n, \quad |p| < \Pi(s). \tag{115} \]
This completes the proof of theorem 1 if $n \geq 3$.

**Remark 2.** Theorem 1 still holds if $n \geq 3$ and we replace $a(s, p)$ and $\tilde{a}(s, p)$ by
\[ F(x, s, p, b(s, p)) \quad \text{and} \quad F(x, s, p, \tilde{b}(s, p)), \]
where $t \mapsto F(x, s, p, t)$ is injective for each fixed $(x, s, p)$. To see this, we merely observe that (113) and the injectivity of $F$ imply
\[ b(s, p) = \tilde{b}(s, p), \quad s \in \mathbb{R}, \quad p \in \mathbb{R}^{n-1} \times \{0\}, \quad |p| < \Pi(s). \]
The Euclidean invariance argument completes the proof.

**Remark 3.** Another non-homogeneous extension. Let $a$ be non-homogeneous with a symmetry in one direction:
\[ a = a(x^1, \ldots, x^{n-1}, s, p). \]
If $\Omega$ is convex, then theorem 1 still holds. Indeed, instead choosing a general vector $p \in \mathbb{R}^n$ with $|p| < \Pi(s)$, we simply repeat the proof starting at (110) and deduce
\[ A(0, 0, \ldots, 0, s, p) = 0 \]
directly. Since $\Omega$ is convex, we can place $\Omega$ above a hyperplane at any point $x \in \partial \Omega$, so a Euclidean invariance argument means there holds
\[ A(\pi_{\mathbb{R}^{n-1} \times \{0\}} (x), s, p) = 0 \]
for any $x \in \partial \Omega$. Since $\partial A / \partial x^a = 0$, this completes the proof. It is not clear whether this works for $n = 2$ or $\Omega$ non-convex.

**The case $n = 2$**

Fixing $s \in \mathbb{R}$, we choose $p \in \mathbb{R}$ such that $0 < |p| < \Pi(s)$. Using proposition 3, we can find $f \in C^\infty(\partial \Omega)$ such that the solution $u = u[f]$ to (2) has the following properties:
\[ f(0) = s, \quad \frac{\partial f}{\partial x^a}(0) = p, \quad \frac{\partial u}{\partial x^a}(0) = 0, \quad \|f - s\|_{C^\infty(\partial \Omega)} \leq \Pi_1(s). \tag{116} \]
We now use this same $f$ as boundary data to create solution $\tilde{u}[f]$ to (2) with $\tilde{a}$ in place of $a$. Since $\Gamma_a = \Gamma_{\tilde{a}}$, we infer at $x = 0$,
\[ \tilde{a} \frac{\partial \tilde{u}}{\partial x^a}(0) = a \frac{\partial u}{\partial x^a}(0) = 0, \tag{117} \]
and hence \( \tilde{u}_2(0) = 0 \) as well. Substituting into (105) yields
\[
p a_{p_2}(s, p, 0) = \tilde{p} \tilde{a}_{p_2}(s, p, 0).
\] (118)
Substituting this and \( u_2 = \tilde{u}_2 = 0 \) into (106) yields
\[
(a + pa_{p_1})a|_{(s,p,0)} = (\tilde{a} + \tilde{p} \tilde{a}_{p_1})\tilde{a}|_{(s,p,0)}.
\] (119)
Putting
\[
A(s, p) := a(s, p, 0)^2 - \tilde{a}(s, p, 0)^2,
\] (120)
we deduce
\[
A + \frac{1}{2} p A_p = 0,
\] (121)
and hence
\[
A(s, p) = c(s)|p|^{-2}.
\] (122)
Since \( A(s, p) \) is smooth, we conclude \( c = 0 \):
\[
A(s, p) = 0, \quad s \in \mathbb{R}, \quad |p| < \Pi(s).
\] (123)
As in the proof for \( n \geq 3 \), since \( O(2) \cdot e_1 = S^1 \), where \( e_1 = (1, 0) \) and \( O(2) \) is the group of orthogonal transformations on \( \mathbb{R}^2 \), invoking Euclidean isometries completes the proof.

**Remark 4.** As in remark 2, theorem 1 still holds for \( n = 2 \) if we replace \( a \) and \( \tilde{a} \) by
\[
F(x, s, p, b(s, p)) \quad \text{and} \quad F(x, s, p, \tilde{b}(s, p)),
\]
where \( t \mapsto F(x, s, p, t) \) is injective for each fixed \((x, s, p)\). Indeed, from (123), we obtain
\[
F(0, s, p, 0, b(s, p, 0))^2 = F(0, s, p, 0, \tilde{b}(s, p, 0))^2, \quad s, p \in \mathbb{R}, \quad |p| < \Pi(s).
\]
By (4), we have \( F \geq 1 \), so the map \( t \mapsto F(x, s, p, t)^2 \) is also injective for fixed \((x, s, p)\), and we obtain
\[
b(s, p, 0) = \tilde{b}(s, p, 0), \quad s, p \in \mathbb{R}, \quad |p| < \Pi(s).
\]
As before, the Euclidean invariance argument completes the proof.

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References

[DKSU] Ferreira D D S, Kenig C E, Salo M and Uhlmann G 2009 Limiting Carleman weights and anisotropic inverse problems Invent. Math. 178 119–71

[EPS] Egger H, Pietschmann J-F and Schlottbom M 2014 Simultaneous identification of diffusion and absorption coefficients in a quasilinear elliptic problem Inverse Problems 30 035009

[FO] Feizmohammadi A and Oksanen L 2019 An inverse problem for a semi-linear elliptic equation in Riemannian geometries (arXiv:1904.00608)

[GKS] Guo C Y, Kar M and Salo M 2016 Inverse problems for p-Laplace type equations under monotonicity assumptions (arXiv:1602.02591)

[GT] Gilbarg D and Trudinger N 2001 Elliptic Partial Differential Equations of Second Order, Reprint of the 1998 Edition (Berlin: Springer)

[H] Hörmander L 2007 The Analysis of Linear Partial Differential Operators III: Pseudo-differential Operators (Berlin: Springer)

[HS] Hervas D and Sun Z 2002 An inverse boundary value problem for quasilinear elliptic equations Commun. PDE 27 2449–90

[I1] Isakov V 1993 On uniqueness in inverse problems for semilinear parabolic equations Arch. Ration. Mech. Anal. 124 1–12

[I2] Isakov V 2001 Uniqueness of recovery of some quasilinear partial differential equations Commun. PDE 26 1947–73

[IS] Isakov V and Sylvester J 1994 Global uniqueness for a semilinear elliptic inverse problem Commun. Pure Appl. Math. 47 1403–10

[KN] Kang H and Nakamura G 2002 Identification of nonlinearity in a conductivity equation via the Dirichlet-to-Neumann map Inverse Problems 18 1079

[KU] Krupchyk K and Uhlmann G 2019 A remark on partial data inverse problems for semilinear elliptic equations (arXiv:1905.01561)

[KU1] Krupchyk K and Uhlmann G 2019 Partial data inverse problems for semilinear elliptic equations with gradient nonlinearities (arXiv:1909.08122)

[LLLS] Lassas M, Liiimatainen T, Lin Y H and Salo M 2019 Partial data inverse problems and simultaneous recovery of boundary and coefficients for semilinear elliptic equations (arXiv:1905.02764)

[LU] Lee J M and Uhlmann G 1989 Determining anisotropic real-analytic conductivities by boundary measurements Commun. Pure Appl. Math. 42 1097–112

[MU] Munoz C and Uhlmann G 2018 The Calderon problem for quasilinear elliptic equations (arXiv:1806.09586)

[S1] Sun Z 1996 On a quasilinear inverse boundary value problem Math. Z 221 293–305

[S2] Sun Z 2004 Inverse boundary value problems for a class of semilinear elliptic equations Adv. Appl. Math. 32 791–800

[SU] Sun Z and Uhlmann G 1997 Inverse problems in quasilinear anisotropic media Am. J. Math. 119 771–97

[SX] Salo M and Zhong X 2012 An inverse problem for the 2pε-Laplacian: boundary determination SIAM J. Math. Anal. 44 2474–95

[U] Uhlmann G 2009 Electrical impedance tomography and Calderón’s problem Inverse Problems 25 123011