Deformation Quantization:
Twenty Years After\textsuperscript{1}

Daniel Sternheimer

Laboratoire Gevrey de Mathématique Physique, CNRS ESA 5029
Département de Mathématiques, Université de Bourgogne
BP 400, F-21011 Dijon Cedex France.
e-mail: dastern@u-bourgogne.fr

Abstract. We first review the historical developments, both in physics and in mathematics, that preceded (and in some sense provided the background of) deformation quantization. Then we describe the birth of the latter theory and its evolution in the past twenty years, insisting on the main conceptual developments and keeping here as much as possible on the physical side. For the physical part the accent is put on its relations to, and relevance for, “conventional” physics. For the mathematical part we concentrate on the questions of existence and equivalence, including most recent developments for general Poisson manifolds; we touch also noncommutative geometry and index theorems, and relations with group theory, including quantum groups. An extensive (though very incomplete) bibliography is appended and includes background mathematical literature.

I. BACKGROUND

In this Section we briefly present the fertile ground which was needed in order for deformation quantization to develop, even if from an abstract point of view one could have imagined it on the basis of Hamiltonian classical mechanics. Indeed there are two sides to “deformation quantization”. The philosophy underlying the rôle of deformations in physics has been consistently put forward by Flato since more than 30 years and was eventually expressed by him in [66] (see also [58,67]). In short, the passage from one level of physical theory to another, more refined, can be understood (and might even have been predicted) using what mathematicians call deformation theory. For instance one passes from Newtonian physics to special relativity by deforming the invariance group (the Galilei group \(\text{SO}(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4\)) to the Poincaré group \(\text{SO}(3, 1) \cdot \mathbb{R}^4\) with deformation parameter \(c^{-1}\), where \(c\) is the velocity of light. There are many other examples among which quantization is perhaps the most seminal.

\textsuperscript{1} This review is dedicated to the memory of our good friend Ryszard Rączka, in whose honor the meeting in Łódź “Particles, Fields and Gravitation”, was held in April 1998. The e-print is infinitesimally updated from the August 1998 version to be published in the Proceedings entitled\textit{ Particles, Fields and Gravitation}, edited by Jakub Rembieliński for AIP Press.
As a matter of fact it seems that the idea that quantum mechanics is some kind of deformed classical mechanics has been, almost from the beginning of quantum theory, “in the back of the mind of many physicists” (after we came out with the preprint of [17], a scientist even demanded that we quote him for that!). This is attested by the notion of classical limit and even more by that of semi-classical approximation, a good presentation of which can be found in [157]. But the idea remained hidden “in the back of the minds” for a long time, in particular due to the apparently insurmountable “quantum jump” in the nature of observables – and probably also because the mathematical notion of deformation and the relevant cohomologies were not available. A long maturation was needed which eventually gave birth to full-fledged deformation quantization about 20 years ago [17].

A word of caution may be needed here. It is possible to intellectually imagine new physical theories by deforming existing ones. Even if the mathematical concept associated with an existing theory is mathematically rigid, it may be possible to find a wider context in which nontrivial deformations exist. For instance the Poincaré group may be deformed to the simple (and therefore rigid in the category of Lie groups) anti De Sitter group SO(3,2) very popular recently – though it had been studied extensively by us 15-20 years ago [3], resulting in particular in a formulation of QED with photons dynamically composed of two singletons [70] in AdS universe. As is now well-known, there exist deformations of the Hopf algebras associated with a simple Lie group (these “quantum groups” [55], which are in fact an example of deformation quantization [25], have been extensively studied and applied to physics). Nevertheless such intellectual constructs, even if they are beautiful mathematical theories, need to be somehow confronted with physical reality in order to be taken seriously in physics. So some physical intuition is still needed when using deformation theory in physics.

I.1 Weyl Quantization and Related Developments

We assume the reader somewhat familiar with (classical and) quantum mechanics. In an “impressionist” fashion we only mention the names of Planck, Einstein, de Broglie, Heisenberg, Schrödinger and finally Hermann Weyl. What we call here “deformation quantization” is related to Weyl’s quantization procedure. In the latter [161], starting with a classical observable $u(p,q)$, some function on phase space $\mathbb{R}^2\ell$ (with $p, q \in \mathbb{R}^\ell$), one associates an operator (the corresponding quantum observable) $\Omega(u)$ in the Hilbert space $L^2(\mathbb{R}^\ell)$ by the following general recipe:

$$u \mapsto \Omega_w(u) = \int_{\mathbb{R}}^{2\ell} \hat{u}(\xi, \eta) \exp(i(P.\xi + Q.\eta)/\hbar) w(\xi, \eta) \, d\xi d\eta$$

where $\hat{u}$ is the inverse Fourier transform of $u$, $P_\alpha$ and $Q_\alpha$ are operators satisfying the canonical commutation relations $[P_\alpha, Q_\beta] = i\hbar \delta_{\alpha\beta}$ ($\alpha, \beta = 1, ..., \ell$), $w$ is a weight function and the integral is taken in the weak operator topology. What is now called normal ordering corresponds to choosing the weight $w(\xi, \eta) = \exp(-\frac{1}{4}(\xi^2 \pm \eta^2))$, 

Where does this document come from? Why was it written? What is the primary purpose?
DEFORMATION QUANTIZATION

standard ordering (the case of the usual pseudodifferential operators in mathematics) to \( w(\xi, \eta) = \exp(-\frac{i}{2} \xi \eta) \) and the original Weyl (symmetric) ordering to \( w = 1 \). An inverse formula was found shortly afterwards by Eugene Wigner [162] and maps an operator into what mathematicians call its symbol by a kind of trace formula. For example \( \Omega_1 \) defines an isomorphism of Hilbert spaces between \( L^2(\mathbb{R}^{2\ell}) \) and Hilbert-Schmidt operators on \( L^2(\mathbb{R}^{2\ell}) \) with inverse given by

\[
\Omega_1(u) = (2\pi\hbar)^{-\ell} \text{Tr}[(\xi.P + \eta.Q)/(i\hbar)]
\]

(2)

and if \( \Omega_1(u) \) is of trace class one has \( \text{Tr}(\Omega_1(u)) = (2\pi\hbar)^{-\ell} \int u \omega^\ell \) where \( \omega^\ell \) is the (symplectic) volume \( dx \) on \( \mathbb{R}^{2\ell} \). Numerous developments followed in the direction of phase-space methods, many of which can be found described in [2]. Of particular interest to us here is the question of finding an interpretation to the classical function \( u \), symbol of the quantum operator \( \Omega_1(u) \); this was the problem posed (around 15 years after [162]) by Blackett to his student Moyal. The (somewhat naïve) idea to interpret it as a probability density had of course to be rejected (because \( u \) has no reason to be positive) but, looking for a direct expression for the symbol of a quantum commutator, Moyal found [126] what is now called the Moyal bracket:

\[
M(u, v) = \nu^{-1} \sinh(\nu P)(u, v) = P(u, v) + \sum_{r=1}^{\infty} \nu^{2r} P^{2r+1}(u, v)
\]

(3)

where \( 2\nu = i\hbar \), \( P^r(u, v) = \Lambda^{i_1j_1} \ldots \Lambda^{i_rj_r}(\partial_{i_1\ldots i_r} u)(\partial_{j_1\ldots j_r} v) \) is the \( r \)th power \( (r \geq 1) \) of the Poisson bracket bidifferential operator \( P^r \), \( i_k, j_k = 1, \ldots, 2\ell, k = 1, \ldots, r \) and \( (\Lambda^{i_kj_k}) = \begin{pmatrix} 0 & -I \end{pmatrix} \). To fix ideas we may assume here \( u, v \in \mathcal{C}^\infty(\mathbb{R}^{2\ell}) \) and the sum taken as a formal series (the definition and convergence for various families of functions \( u \) and \( v \) was also studied, including in [17]). A similar formula for the symbol of a product \( \Omega_1(u)\Omega_1(v) \) had been found a little earlier [95] and can now be written more clearly as a (Moyal) star product:

\[
u \ast_M v = \exp(\nu P)(u, v) = uv + \text{Tr}(P^r(u, v)).
\]

(4)

Several integral formulas for the star product have been introduced and the Wigner image of various families of operators (including bounded operators on \( L^2(\mathbb{R}^{\ell}) \)) were studied, mostly after deformation quantization was developed (see e.g. [44,123,107]). An adaptation to Weyl ordering of the mathematical notion of pseudodifferential operators (ordered, like differential operators, “first \( q \), then \( p \)” was done in [96] – and the converse in [102]. Starting from field theory, where normal (Wick) ordering is essential (the rôle of \( q \) and \( p \) above is played by \( q \pm ip \)), Berezin [18,19] developed in the mid-seventies an extensive study of what he called “quantization”, based on the correspondence principle and Wick symbols. It is essentially based on Kähler manifolds and related to pseudodifferential operators in the complex domain [32]. However in his theory (which we noticed rather late), as in the studies of various orderings [2], the important concepts of deformation and autonomous formulation of quantum mechanics in general phase space are absent.
I.2 Classical Mechanics on General Phase Space and its Quantization

Initially classical mechanics, in Lagrangean or Hamiltonian form, assumed implicitly a “flat” phase space $\mathbb{R}^{2\ell}$, or at least considered only an open connected set thereof. Eventually more general configurations were needed and so the mathematical notion of manifold, on which mechanics imposed some structure, was needed. This has lead in particular to using the notions of symplectic and later of Poisson manifolds, which have been introduced also for purely mathematical reasons. One of these reasons has to do with families of infinite-dimensional Lie algebras, which date back to works by Élie Cartan at the beginning of this century and regained a lot of popularity (including in physics) in the past 30 years.

A typical example can be found with Dirac constraints [50]: second class Dirac constraints restrict phase space from some $\mathbb{R}^{2\ell}$ to a symplectic manifold $W$ imbedded in it (with induced symplectic form), while first class constraints further restrict to a Poisson manifold with symplectic foliation (see e.g. [72]). Some of the references where one can find detailed information on the symplectic approach to classical (Hamilton) mechanics are [118,1,98] and (which includes the derivation of symplectic manifolds from Lagrangean mechanics) [149]. The question of quantization on such manifolds was certainly treated by many authors (including in [50]) but did not go beyond giving some (often useful) recipes and hoping for the best.

A first systematic attempt started around 1970 with what was called soon afterwards geometric quantization [113], a by-product of Lie group representations theory where it gave significant results [10,109]. It turns out that it is geometric all right, but its scope as far as quantization is concerned has been rather limited since few classical observables could be quantized, except in situations which amount essentially to the Weyl case considered above. In a nutshell one considers phase-spaces $W$ which are coadjoint orbits of some Lie groups (the Weyl case corresponds to the Heisenberg group with the canonical commutation relations $\hbar$ as Lie algebra); there one defines a “prequantization” on the Hilbert space $L^2(W)$ and tries to halve the number of degrees of freedom by using polarizations (often complex ones, which is not an innocent operation as far as physics is concerned) to get a Lagrangean submanifold $L$ of dimension half that of $W$ and quantized observables as operators in $L^2(L)$; “Moyal quantization” on a symplectic groupoid $\mathbb{R}^{2\pi} \times \mathbb{R}^{2\pi}$ was obtained therefrom in [92]. A recent exposition can be found in [163].

I.3 Pseudodifferential Operators and Index Theorems

One may argue that physicists had invented the theory of distributions (with Dirac’s $\delta$) and symbols of pseudodifferential operators (with standard ordering) much before mathematicians developed the corresponding theories. These may also be considered as belonging to the large family of examples of a fruitful interaction between physics and mathematics, even if in the latter case (symbols) it seems
DEFORMATION QUANTIZATION

that the two developments were largely independent at the beginning, and in fact converged only with the advent of deformation quantization.

In this connection one should not forget that there is a significant difference in attitude to Science (with notable exceptions): in physics a very good idea may be enough to earn you a Nobel prize (with a little bit of luck, enough PR and provided you live long enough to see it well recognized); in mathematics one usually needs to have, young enough, several good ideas and prove that they are really good (often with hard work, because “problems worthy of attack prove their worth by hitting back”) in order to be seriously considered for the Fields medal. This rather ancient difference may explain Goethe’s sentence (much before Nobel and Fields): “Mathematicians are like Frenchmen: They translate everything into their own language and henceforth it is something completely different”.

In the fifties [38] the notion of Fourier integral operators was introduced, generalizing and making precise the sometimes heuristic calculus of “differential operators of noninteger order”. Soon it evolved into what are now called pseudodifferential operators, defined on general manifolds [101] and as indispensable to theories of partial differential equations as distributions (with which they are strongly mixed). But what gained to this tool fame and respectability among all mathematicians was the proof, in 1963 (and following years for various generalizations) of the so-called \textit{index theorem} for elliptic (pseudo)differential operators on manifolds by Atiyah, Singer, Bott, Patodi and others [9,136,39,89]. The (analytical) index of a linear map between two vector spaces is defined (when both terms are finite) as the dimension of its kernel minus the codimension of its image. Elliptic partial differential operators $d$ on compact manifolds $X$ have an index $i(d)$, equal (and this is the original theorem) to a “topological index” which depends only on topological invariants associated with the manifold (the Todd class \(\tau(X)\) of the complexified cotangent bundle of $X$ and the fundamental class $[X]$) and on a cohomological invariant chd (a Chern character) associated with the symbol of the principal part of the operator $d$. To give the flavor of the result we write a precise formula (see e.g. Atiyah’s lecture in [39]), valid for compact manifolds (without or with boundary):

$$i(d) = \langle (\text{chd}) \tau(X), [X] \rangle.$$  \hspace{1cm} (5)

The existence of such a formula had also been conjectured by Gel’fand. The proof is very elaborate and one cannot avoid doing it also for pseudodifferential operators. Topological arguments and factorization (which imposes consideration of continuous symbols – this was my share in [39]) permit eventually to reduce the proof of the equality to the cases of the Dirac operator on even dimensional manifolds and one particular (convolution) operator on the circle. There have been numerous developments in a wide array of mathematical domains provoked by this seminal result. The formula itself has been very much generalized, including to “algebraic” index theorems where the algebra of pseudodifferential operators is replaced by more abstract algebras (this is an major ingredient in noncommutative geometry [41] and is strongly related to star products [42,131]). Throughout the theory a capital rôle
is played by the symbol $\sigma(d)$ (the classical function associated with the standard-ordered operator $d$). Note that the principal part of a differential operator is independent of the ordering, but eventually the whole symbol was used. In the proof of the reduction one needs an expression for the symbol of a product of operators, given by an integral formula analogous to a star product. So mathematicians had been using star products (albeit corresponding to a different ordering and without formal series development in some parameter like $\hbar$) before they were systematically defined. This permitted eventually to give original proofs of existence of star products on quite general manifolds [31,97] by adapting techniques and results developed [32] in the theory of pseudodifferential operators.

I.4 Cohomologies and Deformation Theory

In an often ignored section of a paper, I.E. Segal [146] and (independently) a little later Wigner and Inon"u [104,143] have introduced in the early fifties a kind of inverse [117] to the mathematical notion of deformation of Lie groups and algebras, notion which was precisely defined only in 1964 by Murray Gerstenhaber [86]. That inverse was called contraction and typical examples (mentioned at the beginning of this Section) are the passage from De Sitter to Poincaré groups (by taking the limit of zero curvature in space-time) or from Poincaré to Galilei (by taking $c^{-1} \to 0$). Intuitively speaking a contraction is performed by neglecting in symmetries, at some level of physical reality, a constant (like $c^{-1}$) which has negligible impact at this level but significant effects at a more “refined” level. Note that this may be realized mathematically in varying generality (e.g. [143] is more general than [104] but both have for inverse a Gerstenhaber deformation). The notion of deformation of algebras, which may be seen as an outcome of the notion (introduced a few years before) of deformations of complex analytic structures [110], gives rise to a better defined mathematical theory which, for completeness, we shall briefly present in the following two subsections. All this is by now well-known, even to physicists, and we shall keep details to a minimum, referring the reader interested in more details to papers and textbooks cited here and references quoted therein.

It also turns out that recently we had to introduce deformations which are even more general than those introduced by Gerstenhaber (see [53,139,128] and [67] in these Proceedings) in order e.g. to quantize Nambu mechanics. They are still inverse to some contraction procedure, applied (like deformation quantization, where the algebra is that of classical observables and the parameter Planck’s constant) to algebras which are not geometrical symmetries. So one should keep an open mathematical mind, let physics be a guide and develop if needed completely new mathematical tools. This is true physical mathematics, in contradistinction with standard mathematical physics where one mainly applies existing tools or with theoretical physics where mathematical rigor is too often left aside and a good physical intuition (which Dirac certainly had e.g. when he worked with his $\delta$ “function”) is then required as a guide.
I.4.1 Hochschild and Chevalley-Eilenberg cohomologies

Let first \( A \) be an associative algebra (over some commutative ring \( K \)) and for simplicity we consider it as a module over itself with the adjoint action (algebra multiplication); the generalization to cohomology valued in a general module is straightforward. A \( p \)-cochain is a \( p \)-linear map \( C \) from \( A^p \) into (the module) \( A \) and its \( \delta \)-coboundary \( \delta C \) is given by

\[
\delta C(u_0, \ldots, u_p) = u_0 C(u_1, \ldots, u_p) - C(u_0 u_1, u_2, \ldots, u_p) + \cdots \\
+ (-1)^p C(u_0, u_1, \ldots, u_{p-1} u_p) + (-1)^{p+1} C(u_0, \ldots, u_{p-1}) u_p.
\]

(6)

One checks that we have here what is called a complex, i.e. \( \delta^2 = 0 \). We say that a \( p \)-cochain \( C \) is a \( p \)-cocycle if \( \delta C = 0 \). We denote by \( Z^p(A, A) \) the space of \( p \)-cocycles and by \( B^p(A, A) \) the space of those \( p \)-cocycles which are coboundaries (of a \( (p-1) \)-cochain). The \( p \)th Hochschild cohomology space (of \( A \) valued in \( A \)) is defined as \( H^p(A, A) = Z^p(A, A)/B^p(A, A) \). Cyclic cohomology is defined using a bicomplex which includes the Hochschild complex and we shall briefly present it at the end of this review in the example of interest for us here.

For Lie algebras (with bracket \{·,·\}) one has a similar definition, due to Chevalley and Eilenberg [40]. The \( p \)-cochains are here skew-symmetric, i.e. linear maps \( B^p: \wedge^p A \to A \), and the Chevalley coboundary operator \( \partial \) is defined on a \( p \)-cochain \( B \) by (where \( \hat{u}_j \) means that \( u_j \) has to be omitted):

\[
\partial C(u_0, \ldots, u_p) = \sum_{j=0}^{p} (-1)^j \{u_j, C(u_0, \ldots, \hat{u}_j, \ldots, u_p)\} \\
+ \sum_{i<j} (-1)^{i+j} C(\{u_i, u_j\}, u_0, \ldots, \hat{u}_i, \ldots, \hat{u}_j, \ldots, u_p).
\]

(7)

Again one has a complex (\( \partial^2 = 0 \)), cocycles and coboundaries spaces \( Z^p \) and \( B^p \) (resp.) and by quotient the Chevalley cohomology spaces \( H^p(A, A) \), or in short \( H^p(A) \); the collection of all cohomology spaces is often denoted \( H^* \).

I.4.2 Gerstenhaber theory of deformations of algebras

Let \( A \) be an algebra. By this we mean an associative, Lie or Hopf algebra, or a bialgebra. Whenever needed we assume it is also a topological algebra, i.e. endowed with a locally convex topology for which all needed algebraic laws are continuous. For simplicity we may think that the base (commutative) ring \( K \) is the field of complex numbers \( \mathbb{C} \) or that of the real numbers \( \mathbb{R} \). Extending it to the ring \( K[[\nu]] \) of formal series in some parameter \( \nu \) gives the module \( \hat{A} = A[[\nu]] \), on which we can consider the preceding various algebraic (and topological) structures.

I.4.2.1 Deformations and cohomologies. A concise formulation of a Gerstenhaber deformation of an algebra (which we shall call in short a DrG-deformation whenever a confusion may arise with more general deformations) is [86,87,25]:
Definition 1 A deformation of such an algebra $A$ is a $K[[\nu]]$-algebra $\hat{A}$ such that $\hat{A}/\nu \hat{A} \approx A$. Two deformations $\hat{A}$ and $\hat{A}'$ are said equivalent if they are isomorphic over $K[[\nu]]$ and $\hat{A}$ is said trivial if it is isomorphic to the original algebra $A$ considered by base field extension as a $K[[\nu]]$-algebra.

Whenever we consider a topology on $A$, $\hat{A}$ is supposed to be topologically free. For associative (resp. Lie algebra) Definition 1 tells us that there exists a new product $\ast$ (resp. bracket $[\cdot, \cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{\cdot, \cdot\}$) this means that, for $u, v \in A$ (we can extend this to $A[[\nu]]$ by $K[[\nu]]$-linearity) we have:

$$u \ast v = uv + \sum_{r=1}^{\infty} \nu^r C_r(u, v)$$

$$[u, v] = \{u, v\} + \sum_{r=1}^{\infty} \nu^r B_r(u, v)$$

where the $C_r$ are Hochschild 2-cochains and the $B_r$ (skew-symmetric) Chevalley 2-cochains, such that for $u, v, w \in A$ we have $(u \ast v) \ast w = u \ast (v \ast w)$ and $S[[u, v], w] = 0$, where $S$ denotes summation over cyclic permutations. At each level $r$ we therefore need to fulfill the equations ($j, k \geq 1$):

$$D_r(u, v, w) \equiv \sum_{j+k=r} (C_j(C_k(u, v), w) - C_j(u, C_k(v, w))) = bC_r(u, v, w)$$

$$E_r(u, v, w) \equiv \sum_{j+k=r} S B_j(B_k(u, v), w) = \partial B_r(u, v, w)$$

where $b$ and $\partial$ denote (respectively) the Hochschild and Chevalley coboundary operator. In particular we see that for $r = 1$ the driver $C_1$ (resp. $B_1$) must be a 2-cocycle. Furthermore, assuming one has shown that (10) or (11) are satisfied up to some order $r = t$, a simple calculation shows that the left-hand sides for $r = t + 1$ are then 3-cocycles, depending only on the cochains $C_k$ (resp. $B_k$) of order $k \leq t$. If we want to extend the deformation up to order $r = t + 1$ (i.e. to find the required 2-cochains $C_{t+1}$ or $B_{t+1}$), this cocycle has to be a coboundary (the coboundary of the required cochain): The obstructions to extend a deformation from one step to the next lie in the 3-cohomology. In particular (and this was Vey’s trick) if one can manage to pass always through the null class in the 3-cohomology, a cocycle can be the driver of a full-fledged (formal) deformation.

For a (topological) bialgebra (an associative algebra $A$ where we have in addition a coproduct $\Delta : A \rightarrow A \otimes A$ and the obvious compatibility relations), denoting by $\otimes_\nu$ the tensor product of $K[[\nu]]$-modules, we can identify $\hat{A} \hat{\otimes}_\nu \hat{A}$ with $(A \otimes A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some operator topology (e.g. projective for Fréchet nuclear topology), we similarly have a deformed coproduct $\Delta = \Delta + \sum_{r=1}^{\infty} \nu^r D_r$, $D_r \in \mathcal{L}(A, A \otimes A)$ and in this context appropriate
DEFORMATION QUANTIZATION

Cohomologies can be introduced. Here we shall not elaborate on these, nor on the additional requirements for Hopf algebras, referring for more details to original papers and books; there is a huge literature on the subject, among which we may mention [55, 88, 23, 25, 26, 148, 151] and references quoted therein.

I.4.2.2 Equivalence means that there is an isomorphism $T_\nu = I + \sum_{r=1}^{\infty} \nu^r T_r$, $T_r \in \mathcal{L}(A, A)$ so that $T_\nu (u \ast' v) = (T_\nu u \ast T_\nu v)$ in the associative case, denoting by $\ast'$ (resp. $\ast$) the deformed laws in $\tilde{A}$ (resp. $\tilde{A}'$); and similarly in the Lie case. In particular we see (for $r = 1$) that a deformation is trivial at order 1 if it starts with a 2-cocycle which is a 2-coboundary. More generally, exactly as above, we can show [17] that if two deformations are equivalent up to some order $t$, the condition to extend the equivalence one step further is that a 2-cocycle (defined using the $T_k$, $k \leq t$) is the coboundary of the required $T_{t+1}$ and therefore the obstructions to equivalence lie in the 2-cohomology. In particular, if that space is null, all deformations are trivial.

I.4.2.3 Unit. An important property is that a deformation of an associative algebra with unit (what is called a unital algebra) is again unital, and equivalent to a deformation with the same unit. This follows from a more general result of Gerstenhaber (for deformations leaving unchanged a subalgebra) and a proof can be found in [87].

I.4.2.4. In the case of (topological) bialgebras or Hopf algebras, equivalence of deformations has to be understood as an isomorphism of (topological) $\mathbb{K}[[\nu]]$-algebras, the isomorphism starting with the identity for the degree 0 in $\nu$. A deformation is again said trivial if it is equivalent to that obtained by base field extension. For Hopf algebras the deformed algebras may be taken (by equivalence) to have the same unit and counit, but in general not the same antipode.

I.4.3 Examples of special interest: the differentiable cases

Consider the algebra $N = C^\infty(X)$ of functions on a differentiable manifold $X$. When we look at it as an associative algebra acting on itself by pointwise multiplication, we can define the corresponding Hochschild cohomologies. Now let $\Lambda$ be a skew-symmetric contravariant two-tensor (possibly degenerate) defined on $X$, satisfying $[\Lambda, \Lambda] = 0$ in the sense of the supersymmetric Schouten-Nijenhuis bracket [132, 145] (a definition of which, both intrinsic and in terms of local coordinates, can be found in [17, 71]). Then the inner product $P(u, v) = i(\Lambda)(du \wedge dv)$ of $\Lambda$ with the 2-form $du \wedge dv$, $u, v \in N$, defines a Poisson bracket $P$: it is obviously skew-symmetric, satisfies the Jacobi identity because $[\Lambda, \Lambda] = 0$ and the Leibniz rule $P(uw, v) = P(u, w)v + uP(v, w)$. It is a bidifferential 2-cocycle for the (general or differentiable) Hochschild cohomology of $N$, skew-symmetric of order $(1, 1)$, therefore [17] nontrivial and thus defines an infinitesimal deformation of the pointwise product on $N$. We say that $X$, equipped with such a $P$, is a Poisson manifold [17, 120].

When $\Lambda$ is everywhere nondegenerate ($X$ is then necessarily of even dimension $2\ell$), its inverse $\omega$ is a closed everywhere nondegenerate 2-form ($d\omega = 0$ is then
equivalent to \([\Lambda, \Lambda] = 0\) and we say that \(X\) is symplectic; \(\omega^\ell\) is a volume element on \(X\). Then one can in a consistent manner work with differentiable cocycles \([17, 71]\) and the differentiable Hochschild \(p\)-cohomology space \(H^p(N)\) is \([103, 155]\) that of all skew-symmetric contravariant \(p\)-tensor fields, and therefore is infinite-dimensional. Thus, except when \(X\) is of dimension 2 (because then necessarily \(H^3(N) = 0\)), the obstructions belong to an infinite-dimensional space where they may be difficult to trace. On the other hand, when \(2\ell = 2\), any 2-cocycle can be the driver of a deformation of the associative algebra \(N\): “anything goes” in this case; some examples for \(\mathbb{R}^2\) can be found in \([155]\).

Now endow \(N\) with a Poisson bracket: we get a Lie algebra and can look at its Chevalley cohomology spaces. Note that \(P\) is bidifferential of order \((1, 1)\) so it is important to check whether the Gerstenhaber theory is consistent when restricted to differentiable cochains (both of arbitrary order and of order at most 1), especially since the general (Gelfand-Fuks) cohomology is very complicated \([85]\) (but in fact the pathology arises only when non-continuous cochains are allowed). This is a nontrivial question and we gave it a positive answer \([71, 17]\); in brief, if a coboundary is differentiable, it is the coboundary of a differentiable cochain.

Again, since \(P\) is of order \((1,1)\), we first studied the 1-differentiable cohomologies. When the cochains are restricted to be of order \((1, 1)\) with no constant term (then they annihilate constant functions, which we write “n.c.” for “null on constants”) it was found \([119]\) that the Chevalley-Eilenberg cohomology \(H^*_\text{diff,n.c.}(N)\) of the Lie algebra \(N\) (acting on itself with the adjoint representation) is exactly the de Rham cohomology \(H^*(X)\). Thus \(\dim H^p_\text{diff,n.c.}(N) = b_p(X)\), the \(p\)th Betti number of the manifold \(X\). Without the n.c. condition one gets a slightly more complicated formula \([119]\); in particular if \(X\) is symplectic with an exact 2-form \(\omega = d\alpha\), one has here \(H^p_\text{diff}(N) = H^p(X) \oplus H^{p-1}(X)\).

All this allowed us “three musketeers” to study in 1974 what we called in \([71]\) 1-differentiable deformations of the Poisson bracket Lie algebra \(N\) and to give some applications \([72]\). In particular we noticed that the “pure” order \((1, 1)\) deformations correspond to a deformation of the 2-tensor \(\Lambda\); allowing constant terms and taking the deformed bracket in Hamilton equations instead of the original Poisson bracket gave (at this classical level) a kind of friction term.

Shortly afterwards, triggered by our works, a “fourth musketeer” J. Vey \([155]\) noticed that in fact \(\dim H^2_\text{diff,n.c.}(N) = 1 + \dim H^2_\text{diff,n.c.}(N)\) and that \(H^3_\text{diff,n.c.}(N)\) is also finite-dimensional, which allowed him to study differentiable deformations. Incidentally, in the \(\mathbb{R}^{2\ell}\) case, Vey rediscovered (independently, because he ignored it) the Moyal bracket. The latter was then rather “exotic” and few authors (except for a number of physicists, like \([2]\) or J. Plebański who described it in Polish lecture notes \([140]\) he gave us later) paid any attention to it. In Mathematical Reviews this bracket, for which \([126]\) is nowadays often quoted, is not even mentioned in the review! We then came back to the problem \([73]\), this time with differentiable deformations, and deformation quantization was conceived.
II. DEFORMATION QUANTIZATION AND ITS RAMIFICATIONS

II.1 The Birth of Deformation Quantization

Though we had mentioned the main features in 1976 and 1977 in short papers [72,16], meetings [78] and a long preprint, it is only with the publication of the latter [17], our first major (and often quoted) contribution in this new domain, that what eventually became known as deformation quantization [159] took off. Incidentally, as for the two other parts of our “deformation trilogy” (see e.g. [69,70] and [74,75,78]) which deal (resp.) with singleton physics and with nonlinear evolution equations, true recognition was slow to come (it took about twenty years). Let me stress once more (and this will become evident with the section on physical applications), that the important and most original conceptual aspect is that quantization is here an autonomous theory based on a deformation of the composition law of classical observables, not on a radical change in the nature of the observables. In addition to this important conceptual advantage, our approach is more general (simple examples can be given); it can be shown to coincide with the conventional (operatorial) approach in known applications (see below) whenever a (possibly generalized) Weyl mapping can be defined; it also paves the way to better conventional quantizations in field theory (e.g. on the infinite-dimensional symplectic manifold of initial conditions for nonlinear evolution equations) via a kind of cohomological renormalization.

II.1.1 Differentiable deformations and star products

Let $X$ be a differentiable manifold (of finite, or possibly infinite, dimension; to be precise, in the former case, we assume it is locally of finite dimension, paracompact and Hausdorff, and by differentiable we mean infinitely differentiable; the base field may be $\mathbb{R}$ or $\mathbb{C}$). We assume given on $X$ a Poisson structure ($\text{a Poisson bracket } P$).

**Definition 2** A star product is a deformation of the associative algebra of functions $N = C^\infty(X)$ of the form $\ast = \sum_{n=0}^{\infty} \nu^n C_n$ where for $u, v \in N$, $C_0(u,v) = uv$, $C_1(u,v) - C_1(v,u) = 2P(u,v)$ and the $C_n$ are bidifferential operators (locally of finite order). We say a star product is strongly closed if $\int_X (u \ast v - v \ast u) dx = 0$ where $dx$ is a volume element on $X$.

**Remark 1.** a. The parameter $\nu$ of the deformation is in physical applications taken to be $\nu = \frac{i}{\hbar}$.

b. Using equivalence one may take $C_1 = P$. The latter is the case of Moyal, but other orderings like standard or normal do not verify this condition (only the skew-symmetric part of $C_1$ is $P$). Again by equivalence, in view of Gerstenhaber’s result mentioned in I.4.2.3, we may take cochains $C_r$ which are without constant term (what we called n.c. or null on constants). In fact, in the original paper [17], we considered only this case and we also concentrated on “Vey products” [121] for
which the cochains $C_r$ have the same parity as $r$ and have $P^r$ for principal symbol in any Darboux chart, with $X$ symplectic; when $X$ is symplectic of dimension $2\ell$ with symplectic form $\omega$, the (Liouville) volume element is $dx = \omega^\ell$.

c. It is also possible to consider star products for which the cochains $C_n$ are allowed to be slightly more general. Allowing them to be \textit{local} ($C_n(u, v) = 0$ on any open set where $u$ or $v$ vanish) gives nothing new [36]. (Note that this is not the same as requiring the whole associative product to be local; in fact [142] the latter condition is very restrictive and, like true pseudodifferential operators, a star product is a nonlocal operation). In some cases (e.g. for star representations of Lie groups) it may be practical to consider pseudodifferential cochains. As far as the cohomologies are concerned, it has been recently shown [94,138] that as long as one requires at least continuity for the cochains, the theory is the same as in the differentiable case. Incidentally this indicates that one has to go beyond continuous cochains to get the pathological features of the general Gelfand-Fuks cohomology of infinite Lie algebras.

Also, due to formulas like (2) and the relation with Lie algebras (see II.4.1), it is sometimes convenient [131,63,30] to take $\mathbb{K}[\nu^{-1}, \nu]$ (Laurent series in $\nu$, polynomial in $\nu^{-1}$ and formal series in $\nu$) for the ring on which the deformation is defined. Again, this will not change the theory.

d. By taking the corresponding commutator $[u, v]_\nu = (2\nu)^{-1}(u * v - v * u)$, since the skew-symmetric part of $C_1$ is $P$, we get a deformation of the Poisson bracket Lie algebra $(N, P)$. This is a crucial point because (at least in the symplectic case) we know the needed Chevalley cohomologies and (in contradistinction with the Hochschild cohomologies) they are small [155,48]. The interplay between both structures gives existence and classification; in addition it will explain why (in the symplectic case) the classification of star products is based on the 1-differentiable cohomologies, hence ultimately on the de Rham cohomology of the manifold.

\section*{II.1.2 Invariance and covariance}

Since the beginning [17] we realized that there is a big difference between Poisson brackets and their deformations, from the point of view of geometric invariance. Indeed, while a Poisson bracket $P$ is (by definition) invariant under all symplectomorphisms, i.e. transformations of the manifold $X$ which preserve the symplectic form $\omega$ (generated by the flows $x_u = i(\Lambda)(du)$ defined by Hamiltonians $u \in N$), already on $\mathbb{R}^{2\ell}$ one sees easily that its powers $P^r$, hence also the Moyal bracket (3), are invariant only under flows generated by Hamiltonians $u$ which are polynomials of maximal order 2, forming the “affine” symplectic Lie algebra $\mathfrak{sp}(\mathbb{R}^{2\ell}) \cdot h_\ell$. For other orderings the invariance is even smaller (only $h_\ell$ remains). For general Vey products the first terms of a star product are [17,121] $C_2 = P^2_\ell + bH$ and $C_3 = S^3_\ell + T + 3\partial H$. Here $H$ is a differential operator of maximal order 2, $T$ a 2-tensor corresponding to a closed 2-form, $\partial$ the Chevalley coboundary operator. $P^2_\ell$ is given (in canonical coordinates) by an expression similar to $P^2$ in which usual derivatives are replaced by
covariant derivatives with respect to a given symplectic connection $\Gamma$ (a torsionless connection with totally skew-symmetric components when all indices are lowered using $\Lambda$). $S^3_\Gamma$ is a very special cochain given by an expression similar to $P^3$ in which the derivatives are replaced by the relevant components of the Lie derivative of $\Gamma$ in the direction of the vector field associated to the function $(u \text{ or } v)$. Fedosov’s algorithmic construction [61] shows that the symplectic connection $\Gamma$ plays a role at all orders. Therefore the invariance group of a star product is a subgroup of the finite-dimensional group of symplectomorphisms preserving a connection. Its Lie algebra is $g_0 = \{a \in N; [a, u]_\nu = P(a, u) \forall u \in N\}$, preferred observables Hamiltonians for which the classical and quantum evolutions coincide. We are thus led to look for a weaker notion and shall call a star product covariant under a Lie algebra $g$ of functions if $[a, b]_\nu = P(a, b) \forall a, b \in g$. It can be shown [8] that $\ast$ is $g$-covariant iff there exists a representation $\tau$ of the Lie group $G$ whose Lie algebra is $g$ into $\text{Aut}(N[[\nu]]; \ast)$ such that $\tau_g u = (I + \sum_{r=1}^{\infty} \nu^r \tau^r_g ) (g. u)$ where $g \in G, u \in N$, $G$ acts on $N$ by the natural action induced by the vector fields associated with $g$, $(g \cdot u)(x) = u(g^{-1}x)$, and where the $\tau^r_g$ are differential operators on $W$. Invariance of course means that the geometric action preserves the star product: $g \cdot u \ast g \cdot v = g \cdot (u \ast v)$. This is the basis for the theory of star representations which we shall briefly present below (II.4.1).

II.2 Existence and Classification of Deformation Quantizations

II.2.1 Symplectic finite-dimensional manifolds; reduction

As early as 1975, Vey [155] had shown that on a symplectic manifold $X$ with $b_3(X) = 0$, there exists a globally defined deformation of the Poisson bracket $P$. He did this by a careful study allowing him to show (by induction) that at each order of deformation (each order of $\hbar$), he can manage to pass via the zero class of the finite-dimensional obstructions space $H^3_{\text{diff,n.c.}}(N)$. This was later easily extended to star products [130] and in essence tells us that we can in this case “glue” Moyal products defined on local charts (equivalence will take care of intersection of two charts and the vanishing of $b_3(X)$ permits to do it in a way compatible with multiple intersections).

The restriction $b_3(X) = 0$ seemed purely technical and indeed already in [17], with the important case of the hydrogen atom $(X = T^*(S^3))$, we showed that it is not essential. The latter case was generalized by Gutt to $X = T^*(M)$ where $M$ is a Lie group [93] (see also regular star representations in [35]) or more generally a parallelizable manifold. Shortly afterwards De Wilde and Lecomte were able to find a proof of existence, first for a general cotangent bundle, then for exact symplectic manifolds and finally for a general symplectic manifold. The latter required at first a very abstract proof which the authors eventually made more “palatable” [47] along
the lines used by the Japanese group (gluing Moyal on Darboux charts), and the question of invariant star products was also studied [49].

What is behind the scene for cotangent bundles $T^\ast(M)$, is that there one has globally defined “momentum coordinates” which permits to work with globally defined differentiations on $M$ and polynomials in them (differential operators). From there a natural step forward is to localize everything on a general symplectic manifold $X$ and to work with a bundle $\mathcal{W}(X)$ of Weyl algebras; a Weyl algebra $\mathcal{W}_\ell$ is the enveloping algebra of the Heisenberg canonical commutation relations Lie algebra $h_\ell$, possibly completed to formal series, and the product there is the $\text{Sp}(2\ell)$-invariant Moyal product. The “miracle” is that this bundle has a flat connection and a global section; therefore locally defined (Moyal) star products on polynomials (in any given Darboux chart) can be glued together to a globally defined star product. (A Darboux chart on a symplectic manifold is a chart with local coordinates the $q$’s and $p$’s of physicists, i.e. where $\omega = \sum_{j=1}^{2\ell} dq_j \wedge dp_j$). This line of conduct was taken by Fedosov since 1985 (in an obscure paper [60] which was made detailed and precise later [61,63]), using a “germ” approach (infinitesimal neighborhoods), symplectic connections and an algorithmic construction of the flat connection on the Weyl bundle, canonically constructed starting from a given symplectic connection on $X$. Independently a Japanese group [134] obtained also a general proof of existence, gluing together Moyal products defined on Darboux charts by “projecting” from the Weyl bundle, and the method could also be easily adapted to give the existence [134] of closed star products [42]. For more details on these (especially Fedosov’s) constructions we refer to the original papers and to the reviews in [20,99,159]. The relation between the “Russian” approach and the “Belgian” one was made [46] by a famous Belgian mathematician (with a Russian wife), translating both into his own language of “gerbes”. Eventually the Fedosov construction was shown (cf. e.g. [131]) to be “generic” in the sense that any differentiable star product is equivalent to a Fedosov star product.

Now an important tool in symplectic mechanics is that of reduction [124] caused by the action of an invariance group $G$ and subsequent reduction of the algebra of observables. In fact, already in [17], a reduction of this general type was used in connection with the hydrogen atom. Fedosov has recently showed [64] that the classical reduction theory can be “quantized” in the same conditions, i.e. that reduction commutes with $G$-invariant deformation quantization (at least for $G$ compact); note however that, as shown in a simple example [158], reductions may give nonequivalent star products.

Remark 2: connection with 1-differentiable deformations. We have indicated that equivalence classes of star products are in one-to-one correspondence with formal series in the deformation parameter $\nu$ with coefficients in $H^2(X)$, i.e. series of the form $[\Lambda] + \nu[H^2(X)][[\nu]]$, where $\Lambda$ is the (nondegenerate) Poisson 2-tensor given on $X$. Since $H^2(X)$ classifies 1-differentiable deformations of the Lie algebra $(N,P)$, one expects that any Fedosov deformation can be obtained by a sequence of successive 1-differentiable deformations of the initial Poisson bracket, starting with any given
one, at least for $X$ symplectic. This is indeed true. Intuitively, one starts – if needed by equivalence – with a star product n.c. at order 1, $(u, v) \mapsto uv + \nu P(u, v) + O(\hbar^2)$. Associativity is satisfied mod $\nu^2$. Then one takes a 1-differentiable infinitesimal deformation of $P$ of the form $P + \nu P_1 + \cdots$, corresponding in fact to an infinitesimal deformation $\omega + \nu \omega_1 + O(\nu^2)$. Fedosov tells us that we can find $P_1'$ and higher order terms so that the new product $(u, v) \mapsto uv + \nu (P(u, v) + \nu P_1(u, v)) + \nu^2 P_1'(u, v) + \cdots$ is associative to order 2 in $\nu$. The classes of the 2-tensors associated with $P_1$ give all possible choices to order 2. One does the same at the next step with $(P + \nu P_1 + \nu^2 P_2)$ and $P_2'$, and so on. Indeed, in [24], where at every order in $\nu$ the effect of adding a de Rham 2-cocycle was traced in the star product, Bonneau gave a detailed proof of this fact. So by “plugging in” 1-differentiable deformations of Poisson brackets one can cover all possible equivalence classes, starting from any given Fedosov star product.

In a more abstract form a similar conclusion is a consequence of results developed in the nice review [99]. The mathematically oriented reader will find there a detailed presentation (in a Čech cohomology context) of the equivalence question and related problems.

II.2.2 Poisson finite-dimensional manifolds

As we explained earlier, physics (e.g. with first class Dirac constraints) requires sometimes manifolds which have a Poisson bracket $P$, but a degenerate one and are therefore not symplectic. These are called Poisson manifolds [17] and they are foliated with symplectic leaves. A typical example is the dual of a finite-dimensional Lie algebra, foliated by coadjoint orbits (see e.g. [109]). As this example shows (even in the case of the Heisenberg Lie algebra $\mathfrak{h}_\ell$, where one must not forget the trivial orbit), Poisson manifolds are in general not regular; a regular Poisson manifold is foliated with symplectic leaves of constant even dimension (before introducing Poisson manifolds we had considered [71] “canonical manifolds”, regular Poisson manifolds where the leaves have codimension 1). In the regular case the theory we just explained extends in a straightforward manner. Some non conclusive attempts were made in the general case, most notably in [135] following more “traditional” lines and [156] in the direction indicated by Kontsevich’s formality conjecture [111].

The solution to this difficult problem was recently given by Maxim Kontsevich [112] and this is in a way “the cherry on the cake” of deformation quantization (and contributed to getting him the Fields medal in 1998). It involves very elaborate constructions, both conceptually and computationally and makes an essential use of ideas coming from string theory. We shall not attempt here to describe it in detail but give the flavor of the development. The reader interested in more details should refer to [112] and follow carefully subsequent developments. The mathematically oriented reader may be interested in the “Bourbaki-style” presentation (by a French mathematician [133], see quotation above...) of the context and results.

Since, as we noted above, a Poisson bracket $P$ is a nontrivial 2-cocycle for the
Hochschild cohomology of the algebra $N = C^\infty(X)$, a natural question is to decide whether this infinitesimal deformation can be extended to a star product on $N$. Kontsevich answers positively, and more:

**Theorem 1** Let $X$ be a differentiable manifold and $N = C^\infty(X)$. There is a natural isomorphism between equivalence classes of deformations of the null Poisson structure on $X$ and equivalence classes of differentiable deformations of the associative algebra $N$; in particular, any Poisson bracket $P$ on $X$ comes from a canonically defined (modulo equivalence) star product.

With this concise formulation of the result (which gives a positive answer to Kontsevich’s formality conjecture) we see that, in this more general context, a main result from the symplectic case is still valid: classes of star products correspond to classes of deformations of the Poisson structure. A deformation of the null Poisson structure is a formal series $\Lambda(\hbar) = \sum_{n=1}^{\infty} \Lambda_n \hbar^n$ having vanishing Schouten-Nijenhuis bracket with itself: $[\Lambda(\hbar), \Lambda(\hbar)] = 0$. Any given Poisson structure $\Lambda_0$ can be identified with the series $\Lambda_0 \hbar$.

**Remark 3.** From Theorem 1 it is natural to conjecture that the relation with 1-differentiable deformations of the Poisson bracket mentioned at the end of (II.2.1) extends to general Poisson manifolds, but full proofs of that and a study of the relation with some 2-cohomology on the manifold have not yet been given. Furthermore (and this is one of the developments to come) a comparison of the proof with e.g. that of Fedosov in the symplectic case is not done either. Finally, in his Berlin August 25, 1998 “Fields medal” lecture, Motivic Galois group and deformation quantization, Kontsevich stressed that the isomorphism described in Theorem 1 should be taken as one of a family of isomorphisms and indicates that the motivic Galois group should act on the moduli space of non-commutative algebras.

The bulk of the proof is the “affine case”, essentially when $X$ is some $\mathbb{R}^\ell$ (or an open set in it), the result being formulated in such a way that “gluing” charts, though still nontrivial, is not too difficult to perform for an experienced mathematician. Doing so Kontsevich gives an interesting explicit universal formula for the star product on such an $X$ where graphs (and Stokes formula) play a crucial rôle. The formula looks like

$$u \ast v = \sum_{n=0}^{\infty} \sum_{\Gamma \in G_n} w_\Gamma B_\Gamma,\Lambda(u, v)$$

where $\Lambda$ is an arbitrary Poisson structure on an open domain in $\mathbb{R}^\ell$ and $G_n$ a set of labeled oriented graphs $\Gamma$. The latter are pairs $(V_\Gamma, E_\Gamma)$ such that $E_\Gamma \subset V_\Gamma \times V_\Gamma$ with $n + 2$ vertices $V_\Gamma$ and $2n$ edges $E_\Gamma$ satisfying some additional conditions (see [112], section 2). $G_n$ has $(n(n + 1))^n$ elements (1 for $n = 0$). $B_\Gamma$ is an explicitly defined bidifferential operator (of total order $2n$, as in (4)) and $w_\Gamma$ a weight defined by an absolutely convergent integral (of the exterior product of the differentials of $2n$ harmonic angular variables associated with $\Gamma$, taken over the space of configurations of $n$ numbered pairwise distinct points on the Lobatchevsky upper half-plane).
II.2.3 Remarks on the infinite-dimensional case

Poisson structures are known on infinite-dimensional manifolds since a long time and there is an extensive literature on this subject, which alone would require a book. A typical structure, for our purpose, is a symplectic structure such as that defined by Segal [147] (see also [114]) on the space of solutions of a classical field equation like \( \Box \Phi = F(\Phi) \), where \( \Box \) is the d’Alembertian. Now if one considers scalar valued functionals \( \Psi \) over such a space of solutions, i.e. over the phase space of initial conditions \( \varphi(x) = \Phi(x, 0) \) and \( \pi(x) = \frac{\partial}{\partial t} \Phi(x, 0) \), one can consider a Poisson bracket defined by

\[
P(\Psi_1, \Psi_2) = \int (\frac{\delta \Psi_1}{\delta \varphi} \frac{\delta \Psi_2}{\delta \pi} - \frac{\delta \Psi_1}{\delta \pi} \frac{\delta \Psi_2}{\delta \varphi}) \, dx
\]

where \( \delta \) denotes the functional derivative. The problem is that while it is possible to give a precise mathematical meaning to (13), the formal extension to powers of \( P \), needed to define the Moyal bracket, is highly divergent, already for \( P^2 \).

The same difficulty is met if one takes e.g. a space \( N \) of differentiable functions on a Hilbert space with orthonormal basis \( \{ p_k, q_k; k = 1, \ldots, \infty \} \) and a Poisson bracket

\[
P(u, v) = \sum_{k=1}^{\infty} (\frac{\partial u}{\partial p_k} \frac{\partial v}{\partial q_k} - \frac{\partial u}{\partial q_k} \frac{\partial v}{\partial p_k}).
\]

This is not so surprising for physicists who know from experience that the correct approach to field theory is via normal ordering, and that there are infinitely many inequivalent representations of the canonical commutation relations (as opposed to the von Neumann uniqueness in the finite-dimensional case, for projective representations). Integral formulas, related to Feynman path integrals, can also be used with some success. The participants at the \( \acute{L}ódź \) meeting may be interested to learn that an analogue of the pseudodifferential calculus in the infinite-dimensional case, and especially the (“Wigner”) notion of symbols of operators, has been developed already in 1978 by Paul Krée and Ryszard Rączka [115].

We shall come back to this question in (II.3.2) with more specific examples and give indications showing that with proper care the deformation quantization approach can help making better mathematical sense of field theory calculations done by theoretical physicists.

II.2.4 Generalized deformations, \( n \)-gebras and related structures

One of the mathematical reasons we started with the study of deformations of Poisson brackets is related to the fact that it is the only one, among classical infinite-dimensional algebras, which is not rigid, even at the level of 1-differentiable deformations. In particular unimodular structures (defined by a determinant) are rigid. It turns out that in connection with Nambu mechanics, where the Poisson bracket is replaced by an \( n \)-bracket, say a functional determinant, one meets structures of this type and it is not a big surprise that a specific quantization (not of Heisenberg type) was difficult to find.
Roughly speaking, a generalized deformation of a $\mathbb{K}$-algebra $A$ (associative, Lie or other) is a $\mathbb{K}$-algebra $A_\nu$ having $A$ for limit as the deformation parameter $\nu \to 0$. Among the “other” algebras are of course the bialgebras to which we shall come back in (II.4.2) when dealing with quantum groups (incidentally, since ‘al’ means ‘the’ in Arabic, applied to a set containing only one element, the French denomination ‘bigèbre’, imposed by Cartier, is far better).

Here we shall be concerned mostly with the so-called $n$-gebras, algebras $A$ endowed with a composition law $A^n \to A$ satisfying some conditions including skew-symmetry. Structures of this kind were introduced by Nambu [129] in connection with his “generalized mechanics” and (in a paper published in an obscure journal) by Filippov [65]. Serious interest in them developed only from 1992, when Takhtajan [152] and independently Flato and Frønsdal (unpublished) discovered that Nambu $n$-brackets satisfy a generalization of Jacobi identity, called the Fundamental Identity (FI); surprisingly enough, this identity had not been discovered before.

The resurgence of operads which occurred at the same time [90,122] and are related to $n$-gebras [91], as well as the new notion of strong homotopy Lie algebras introduced then by Stasheff and is also related to deformation theory [150] add to the interest in these structures.

Recently, there have been several works dealing with various generalizations of Poisson structures by extending the binary bracket to an $n$-bracket. The main point for these generalizations is to look for the corresponding identity which would play the rôle of Jacobi identity for the usual Poisson bracket. Indeed, in view of generalizations, the Jacobi identity for a Lie 2-bracket can be presented in a number of ways [84] among which two have been recently extensively studied. The most straightforward way is to require that the sum over the symmetric group $\mathfrak{S}_3$ of the composed brackets $[[\cdot, \cdot], \cdot]$ is zero. When extended to $n$-brackets, leads to the notion of generalized Poisson structures studied e.g. in [11]; the corresponding identity is obtained by complete skew-symmetrization of the $2n - 1$ composed brackets when $n$ is even; this is equivalent to require that the Schouten bracket of the $n$-tensor defining the $n$-bracket with itself vanishes.

A physically more appealing way is to say that the adjoint map $b \mapsto [a, b]$ is a Lie algebra derivation. Indeed this means that the bracket of conserved quantities is again a conserved quantity. The two formulations coincide only for $n = 2$ and for $n \geq 3$ the latter is a stronger requirement. This Fundamental Identity of Nambu Mechanics can be written:

$$[x_1, \ldots, x_{n-1}, [y_1, \ldots, y_n]] - \sum_{1 \leq i \leq n} [y_1, \ldots, y_{i-1}, y_i+1, \ldots, y_n, [x_1, \ldots, x_{i-1}, y_i, x_i, \ldots, x_{n-1}]] = 0. \quad (14)$$

Nambu brackets (like Poisson brackets and commutators such as the Moyal bracket, for $n = 2$) are $n$-brackets required to satisfy, with respect to the usual algebra multiplication and in addition to skew-symmetry $\{x_1, \ldots, x_n\} = \epsilon(\sigma)\{x_{\sigma_1}, \ldots, x_{\sigma_n}\}$ $\forall \sigma \in \mathfrak{S}_n$ and the FI, a Leibniz rule:
\[ \{x_0 x_1, x_2, \ldots, x_n\} = x_0 \{x_1, x_2, \ldots, x_n\} + \{x_0, x_2, \ldots, x_n\} x_1. \]  

(15)

The related cohomologies are not yet completely known, though a major step in this direction was done in [84] where one can also find a very interesting and detailed study of all intermediate possibilities between the two generalizations described here (generalized Poisson and Nambu). In (II.3.3) we shall nevertheless indicate, specializing to the case \( A = N \), how one can quantize Nambu brackets using generalized deformations based on the factorization of polynomials and methods of second quantization. One of the steps there (and this produces a non-DrG-deformation) is an operation, the effect of which is that in products the deformation parameter \( \hbar \) behaves as if it was nilpotent (e.g. multiplied by a Dirac \( \gamma \) matrix).

This last fact has very recently induced Pinczon [139] and Nadaud [128] to generalize Gerstenhaber theory to the case of a deformation parameter \( \sigma \) which does not commute with the algebra. A similar theory can be done in this case, with appropriate cohomologies. While that theory does not reproduce the above mentioned Nambu quantization, it gives new and interesting results. For instance [139], while the Weyl algebra \( W_1 \) (generated by the Heisenberg Lie algebra \( h_1 \)) is known [56] to be DrG-rigid, it can be nontrivially deformed in such a supersymmetric deformation theory to the supersymmetry enveloping algebra \( \mathcal{U}(\mathfrak{osp}(1, 2)) \); or [128], on the polynomial algebra \( \mathbb{C}[x, y] \) in 2 variables, Moyal-like products of a new type were discovered. This is another example of a motivated study which goes beyond a generally accepted framework.

II.3 Physical Applications

In this subsection and the following, I shall present a few of the numerous developments which have made use of deformation quantization and/or are strongly related to it. The presentation made, and therefore the bibliography, is by no means exhaustive – more than a whole volume would be needed for that – and the absence of reference to any specific work does not (in general) reflect a lack of appreciation; I did not even quote all of my publications in the domain. The aim of these two last subsections (in fact, of all this review) is mainly to give the flavor of the many facets of deformation quantization and quite naturally the presentation will be somewhat biased towards the works of our group. Nevertheless the interested reader should be able to complete whatever is missing by a kind of “hyper-referencing”, looking at references of references a few times.

II.3.1 Quantum mechanics

Let us start with a phase space \( X \), a symplectic (or Poisson) manifold and \( N \) an algebra of classical observables (functions, possibly including distributions if proper care is taken for the product). We shall call star quantization a star product on \( N \) invariant (or sometimes only covariant) under some Lie algebra \( \mathfrak{g}_0 \) of “preferred
observables”. Invariance of the star product ensures that the classical and quantum evolutions of observables under a Hamiltonian $H \in \mathfrak{g}_0$ will coincide [17]. The typical example is the Moyal product on $W = \mathbb{R}^{2\ell}$.

II.3.1.1 Spectrality. Physicists want to get numbers matching experimental results, e.g. for energy levels of a system. That is usually achieved by describing the spectrum of a given Hamiltonian $\hat{H}$ supposed to be a self-adjoint operator so as to get a real spectrum and so that the evolution operator (the exponential of $i\hbar \hat{H}$) is unitary (thus preserves probability). A similar spectral theory can be done here, in an autonomous manner. The most efficient way to achieve it is to consider [17] the star exponential (corresponding to the evolution operator)

$$\text{Exp}(Ht) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{t}{i\hbar} \right)^n (H^*)^n$$

where $(H^*)^n$ means the $n$th star power of the Hamiltonian $H \in N$ (or $N[[\nu]]$). Then one writes its Fourier-Stieltjes transform $d\mu$ (in the distribution sense) as $\text{Exp}(Ht) = \int e^{\lambda t/i\hbar} d\mu(\lambda)$ and defines the spectrum of $(H/\hbar)$ as the support $S$ of $d\mu$ (incidentally this is the definition given by L. Schwartz for the spectrum of a distribution, out of motivations coming from Fourier analysis). In the particular case when $H$ has discrete spectrum, the integral can be written as a sum (see the top equation in (18) below for a typical example): the distribution $d\mu$ is a sum of “delta functions” supported at the points of $S$ multiplied by the symbols of the corresponding eigenprojectors.

In different orderings with various weight functions $w$ in (1) one gets in general different operators for the same classical observable $H$, thus different spectra. For $X = \mathbb{R}^{2\ell}$ all orderings are mathematically equivalent (to Moyal under the Fourier transform $T_w$ of the weight function $w$). This means that every observable $H$ will have the same spectrum under Moyal ordering as $T_w H$ under the equivalent ordering. But this does not imply physical equivalence, i.e. the fact that $H$ will have the same spectrum under both orderings. In fact the opposite is true [34]: if two equivalent star products are isospectral (give the same spectrum for a large family of observables and all $\hbar$), they are identical.

It is worth mentioning that our definition of spectrum permits to define a spectrum even for symbols of non-spectrable operators, such as the derivative on a half-line which has different deficiency indices; this corresponds to an infinite potential barrier (see also [154] for detailed studies of similar questions). That is one of the many advantages of our autonomous approach to quantization.

II.3.1.2 Applications. In quantum mechanics it is preferable to work (for $X = \mathbb{R}^{2\ell}$) with the star product that has maximal symmetry, i.e. $\mathfrak{sp}(\mathbb{R}^{2\ell}) \cdot \mathfrak{h}_\ell$ as algebra of preferred observables: the Moyal product. One indeed finds [17] that the star exponential of these observables (polynomials of order $\leq 2$) is proportional to the usual exponential. More precisely, if $H = \alpha p^2 + \beta pq + \gamma q^2 \in \mathfrak{sl}(2)$ with $p, q \in \mathbb{R}^\ell$, $\alpha, \beta, \gamma \in \mathbb{R}$, setting $d = \alpha \gamma - \beta^2$ and $\delta = |d|^{1/2}$ one gets (the sums and integrals
appearing in the various expressions of the star exponential being convergent as distributions, both in phase-space variables and in $t$ or $\lambda$)

$$\text{Exp}(Ht) = \begin{cases} 
(cos \delta t)^{-1} \exp((H/\hbar \delta) \tan(\delta t)) & \text{for } d > 0 \\
\exp(Ht/\hbar) & \text{for } d = 0 \\
(cosh \delta t)^{-1} \exp((H/\hbar \delta) \tanh(\delta t)) & \text{for } d < 0
\end{cases} \quad (17)$$

hence the Fourier decompositions

$$\text{Exp}(Ht) = \begin{cases} 
\sum_{n=0}^{\infty} \Pi_n^{(\ell)} e^{(n+\frac{\delta}{2})t} & \text{for } d > 0 \\
\int_{-\infty}^{\infty} e^{\lambda t/\hbar} \Pi(\lambda, H) d\lambda & \text{for } d < 0
\end{cases} \quad (18)$$

We thus get the discrete spectrum $(n + \frac{\ell}{2})\hbar$ of the harmonic oscillator and the continuous spectrum $\mathbb{R}$ for the dilation generator $pq$. The eigenprojectors $\Pi_n^{(\ell)}$ and $\Pi(\lambda, H)$ are given [17] by known special functions on phase-space (generalized Laguerre and hypergeometric, multiplied by some exponential). Formulas (17) and (18) can, by analytic continuation, be given a sense outside singularities and even (as distributions) for singular values of $t$.

Other examples can be brought to this case by functional manipulations [17]. For instance the Casimir element $C$ of $\mathfrak{so}(\ell)$ representing angular momentum, which can be written $C = p^2 q^2 - (pq)^2 - \ell(\ell - 1)\hbar^2$, has $n(n + (\ell - 2))\hbar^2$ for spectrum. For the hydrogen atom, with Hamiltonian $H = \frac{1}{2} p^2 - |q|^{-1}$, the Moyal product on $\mathbb{R}^{2\ell+2}$ ($\ell = 3$ in the physical case) induces a star product on $X = T^*S^\ell$; the energy levels, solutions of $(H - E) \phi = 0$, are found from (18) and the preceding calculations for angular momentum to be (as they should, with $\ell = 3$) $E = \frac{1}{2}(n + 1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum.

We thus have recovered, in a completely autonomous manner entirely within deformation quantization, the results of “conventional” quantum mechanics in those typical examples (and many more can be treated similarly). It is worth noting that the term $\frac{\ell}{2}$ in the harmonic oscillator spectrum, obvious source of divergences in the infinite-dimensional case, disappears if the normal star product is used instead of Moyal – which is one of the reasons it is preferred in field theory.

**II.3.1.3 Remark on convergence.** We have always considered star products as formal series and looked for convergence only in specific examples, and then generally in the sense of distributions. The same applies to star exponentials, as long as each coefficient in the formal series is well defined. In the case of the harmonic oscillator or more generally the preferred observables $H$ in Weyl ordering, this study was facilitated by the fact that the powers $(H*)^n$ are polynomials in $H$. Moreover, in the case of star exponentials, a notion of convergence stronger than as distributions would require considerations analogous to the problem of analytic vectors in Lie groups representations [76] and pose problems also when looking at their Fourier decomposition. Nevertheless some authors (see e.g. [141] and references therein) insist in making a stronger parallel with operator algebras and look for domains (in
II.3.2 Path integrals, field theory and statistical mechanics

II.3.2.1 Path integrals are intimately connected to star exponentials. In fact, in quantum mechanics the path integral of the action is nothing but the partial Fourier transform of the star exponential (16) with respect to the momentum variables, for $X = \mathbb{R}^{2\ell}$ as phase space with the Moyal star product [137]. For normal ordering the path integral is essentially the star exponential [51] and we shall come back to it in (II.3.2.2).

For compact groups the star exponential $E$ (defined in a similar manner, see below (II.4.1)) can be expressed in terms of unitary characters using a global coherent state formalism [33] based on the Berezin dequantization of compact group representation theory used in [4,125] (it gives star products somewhat similar to normal ordering); the star exponential of any Hamiltonian on $G/T$ (where $T$ is a maximal torus in the compact group $G$) is then equal to the path integral for this Hamiltonian.

II.3.2.2 Field theory. The deformation quantization of a given classical field theory consists in the giving a proper definition for a star product on the infinite-dimensional manifold of initial data for the classical field equation (see II.2.3) and constructing with it, as rigorously as possible, whatever physical expressions are needed. As in other approaches to field theory, here also one faces serious divergence difficulties as soon as one is considering interacting fields theory, and even at the free field level if one wants a mathematically rigorous theory. But the philosophy in dealing with the divergences is significantly different and one is in position to take advantage of the cohomological features of deformation theory to perform what can be called cohomological renormalization.

Starting with some star product $*$ (e.g. an infinite-dimensional version of a Moyal-type product or, better, a star product similar to the normal star product (19)) on the manifold of initial data, one would interpret various divergences appearing in
DEFORMATION QUANTIZATION

the theory in terms of coboundaries (or cocycles) for the relevant Hochschild coho-

mology. Suppose that we are suspecting that a term in a cochain of the product \(*\) is

responsible for the appearance of divergences. Applying the procedure described in

(I.4.2.2), we can try to eliminate it, or at least get a lesser divergence, by subtracting

at the relevant order a coboundary; we would then get a better theory with a new

star product, equivalent to the original one. Furthermore, since in this case we can

expect to have at each order an infinity of non equivalent star products, we can try

to subtract a cocycle and then pass to a non equivalent star product whose lower

order cochains are identical to those of the original one. We would then make an

analysis of the divergences up to order \(h^r\), identify a divergent cocycle, remove it,

and continue the procedure (at the same or hopefully a higher order). Along the

way one should preserve the usual properties of a quantum field theory (Poincaré

covariance, locality, etc.) and the construction of adapted star products should be

done accordingly. The complete implementation of this program should lead to a

cohomological approach to renormalization theory.

A very good test for this approach would be to start from classical electrodynam-

ics, where (among others) the existence of global solutions and a study of infrared

divergencies were recently rigorously performed [77], and go towards mathematically

rigorous QED. Physicists will think that spending so much effort in trying to give

complete mathematical sense to recipes that work so well is a waste of time,

but I am sure that the mathematical tools needed will prove very efficient. Would

De Gaulle have been a mathematician he could have said about this scheme “vaste

programme” (supposedly his answer to a minister who wanted to get rid of all stupid

bureaucrats); but had he been a scientist he would probably have been a physicist

and share the attitude of too many physicists towards mathematics: “l’intendance

suivra” (“Supply Corps will follow”, needed logistics will be provided).

In the case of free fields, one can write down an explicit expression for a star pro-

duct corresponding to normal ordering. Consider a (classical) free massive scalar field

\(\Phi\) with initial data \((\phi, \pi)\) in the Schwartz space \(S\). The initial data \((\phi, \pi)\) can

advantageously be replaced by their Fourier modes \((\bar{a}, a)\) which after quantization become

the usual creation and annihilation operators, respectively. The normal star product

\(*_N\) is formally equivalent to the Moyal product and an integral representation for

\(*_N\) is given by:

\[
(F *_N G)(\bar{a}, a) = \int_{S' \times S'} d\mu(\xi, \xi) F(\bar{a} + \bar{\xi}, a + \xi) G(\bar{a} + \xi, a),
\]

(19)

where \(\mu\) is a Gaussian measure on \(S' \times S'\) defined by the characteristic function

\[\exp\left(-\frac{1}{\hbar} \int dk \bar{a}(k)a(k)\right)\) and \(F, G\) are holomorphic functions with semi-regular kernels.

Likewise, Fermionic fields can be cast in that framework by considering functions

valued in some Grassmann algebra and super-Poisson brackets (for the deformation

quantization of the latter see e.g. [27]).

For the normal product (19) one can formally consider interacting fields. It turns

out that the star exponential of the Hamiltonian is, up to a multiplicative well-
defined function, equal to Feynman’s path integral. For free fields, we have a mathematical meaningful equality between the star exponential and the path integrals as both of them are defined by a Gaussian measure, and hence well-defined. In the interacting fields case, giving a rigorous meaning to either of them would give a meaning to the other.

The interested reader will find in [51] calculations performing some steps in the above direction, for free scalar fields and the Klein-Gordon equation, and an example of cancellation of some infinities in \( \lambda \phi^4 \)-theory via a \( \lambda \)-dependent star product equivalent to a normal star product. Finally we mention here for more completeness (though neither is directly related to what precedes) the symbolic calculus of [115] and the Fedosov-like approach to self-dual Yang-Mills and gravity of [83].

II.3.2.3 Statistical mechanics. In view of our philosophy on deformations, a natural question to ask is their stability: Can deformations be further deformed, or does “the buck stops there”? As we indicated at the beginning of this review and shall exemplify with quantum groups, the answer to that question may depend on the context. Here is another example.

If one looks for deformations of the Poisson bracket Lie algebra \((N, P)\) one finds (assuming mild technical assumptions on parity of cochains in [13] which, in view of the classification of star products, are not required) that a further deformation of the Moyal bracket, with another deformation parameter \( \rho \), is again a Moyal bracket for a \( \rho \)-deformed Poisson structure; in particular, for \( X = \mathbb{R}^2 \), quantum mechanics viewed as a deformation is unique and stable.

Now, for the associative algebra \( N \), the only local associative composition law is [142] of the form \((u, v) \mapsto u f v\) for some \( f \in N \). If we take \( f = f_\beta \in N[[\beta]] \) we get a \( 0 \)-differentiable deformation (with parameter \( \beta \)) of the usual product, which for convenience we shall call here a Rubio product. We were thus lead [14] to look, starting from a \( * \) product, for a new composition law

\[
(u, v) \mapsto u \tilde{*}_{\nu, \beta} v = u *_{\nu} f_{\nu, \beta} *_{\nu} v \quad \text{with} \quad f_{\nu, \beta} = \sum_{r=0}^{\infty} \nu^{2r} f_{2r, \beta} \in N[[N[[\nu^2]], \beta]] \tag{20}
\]

where \( f_{0, \beta} \equiv f_\beta \neq 0 \) and \( f_0 = 1 \). The transformation \( u \mapsto T_{\nu, \beta} u = f_{\nu, \beta} *_{\nu} u \) intertwines \( *_{\nu} \) and \( \tilde{*}_{\nu, \beta} \) but it is not an equivalence of star products because \( \tilde{*}_{\nu, \beta} \) is not a star product: it is a \((\nu, \beta)\)-deformation of the usual product (or a \( \nu \)-deformation of the Rubio product) with at first order in \( \nu \) the driver given by \( P_{\beta}(u, v) = f_{\beta} P(u, v) + u P(f_{\beta}, v) - P(f_{\beta}, u)v \), a conformal Poisson bracket associated with a conformal symplectic structure given by the 2-tensor \( \Lambda_{\beta} = f_{\beta} \Lambda \) and the vector \( E_{\beta} = [\Lambda, f_{\beta}] \).

In view of applications we suppose given a star product, denoted \( * \), on some algebra \( \mathcal{A} \) of observables (possibly defined on some infinite-dimensional phase-space) and take for \( f_{\nu, \beta} \) the exponential \( g_{\beta} \equiv \exp_{\nu}(c_{\beta} H) = 1 + \sum_{n=1}^{\infty} \frac{(c_{\beta})^n}{n!} (H*)^n \) with \( c = -\frac{1}{2} \) (we omit \( \nu \) from now on and write \( * \) for \( \tilde{*}_{\nu, \beta} \)). The star exponential \( \text{Exp}(Ht) \) defines an automorphism \( u \mapsto \alpha_t(u) = \text{Exp}(-Ht) * u * \text{Exp}(Ht) \). A KMS state \( \sigma \) on \( \mathcal{A} \) is a state (linear functional) satisfying, \( \forall a, b \in \mathcal{A} \), the Kubo–Martin–Schwinger
condition $\sigma(\alpha_t(a) * b) = \sigma(b * \alpha_{t+i\hbar\beta}(a))$. Then the (quantum) KMS condition can be written \[14\], with $[a, b]_\beta = (i/\hbar)(a \tilde{\omega} b - b \tilde{\omega} a)$, simply $\sigma(g - \beta \cdot [a, b]_\beta) = 0$: up to a conformal factor, a KMS state is like a trace with respect to this new product. The (static) classical KMS condition is the limit for $\hbar = 0$ of the quantum one. So we can recover known features of statistical mechanics by introducing a new deformation parameter $\beta = (kT)^{-1}$ and the related conformal symplectic structure. This procedure commutes with usual deformation quantization. Finally let us mention that recently several people [29,160] have considered the question of KMS states and related modular automorphisms from a more conventional point of view in deformation quantization.

II.3.3 Nambu mechanics and its quantization

We mention this aspect here mainly for the sake of completeness, as an example of generalized deformation. A somewhat detailed recent review can be found in [68] (see also [67]), so we shall just briefly indicate a few highlights.

Nambu [129] started with a kind of “Hamilton equations” on $\mathbb{R}^3$, of the form $\frac{dr}{dt} = \nabla g(r) \wedge \nabla h(r)$, where $x, y, z$ are the dynamical variables and $g, h$ are two functions of $r$. Liouville theorem follows directly from the identity $\nabla \cdot (\nabla g(r) \wedge \nabla h(r)) = 0$, which tells us that the velocity field in the above equation is divergenceless. From this we derive the evolution of a function $f$ on $\mathbb{R}^3$:

$$\frac{df}{dt} = \frac{\partial(f, g, h)}{\partial(x, y, z)},$$

(21)
where the right-hand side is the Jacobian of the mapping $\mathbb{R}^3 \to \mathbb{R}^3$ given by $(x, y, z) \mapsto (f, g, h)$. In this “baby model for integrable systems”, Euler equations for the angular momentum of a rigid body are obtained when the dynamical variables are taken to be the components of the angular momentum vector $L = (L_x, L_y, L_z)$, $g$ is the total kinetic energy $L_x^2 + L_y^2 + L_z^2$ and $h$ the square of the angular momentum $L_x^2 + L_y^2 + L_z^2$. Other examples can be given, in particular Nahm’s equations for static $\mathfrak{su}(2)$ monopoles, $\dot{x}_i = x_j x_k$ ($i, j, k = 1, 2, 3$) in $\mathfrak{su}(2)^* \sim \mathbb{R}^3$, with $h = x_1^2 - x_2^2$, $g = x_1^2 - x_3^2$, etc. Here the principle of least action, which states that the classical trajectory $C_1$ is an extremal of the action functional $A(C_1) = \int_{C_1} (pdq - Hdt)$, is replaced by a similar one [152] with a 2-dimensional cycle $C_2$ and “action functional” $A(C_2) = \int_{C_2} (xdy \wedge dz - hdg \wedge dt)$ (which bears some flavor of strings and some similitude with the cyclic cocycles of Connes [41]).

Expression (21) was easily generalized to $n$ functions $f_i$, $i = 1, \ldots, n$. One introduces an $n$-tuple of functions on $\mathbb{R}^n$ with composition law given by their Jacobian, linear canonical transformations $\text{SL}(n, \mathbb{R})$ and a corresponding $(n - 1)$-form which is the analogue of the Poincaré-Cartan integral invariant. The Jacobian has to be interpreted as a generalized Poisson bracket: It is skew-symmetric
with respect to the \( f_i \)'s, satisfies the FI which is an analogue of the Jacobi identity (but was discovered much after [129]) and a derivation of the algebra of smooth functions on \( \mathbb{R}^n \) (i.e., the Leibniz rule is verified in each argument, e.g. \( \{f_1 f_2, f_3, \ldots, f_{n+1}\} = f_1 \{f_2, \ldots, f_{n+1}\} + \{f_1, f_3, \ldots, f_{n+1}\} f_2 \), etc.). Hence there is a complete analogy with the Poisson bracket formulation of Hamilton equations, including the important fact that the components of the \((n-1)\)-tuple of “Hamiltonians” \((f_2, \ldots, f_n)\) are constants of motion.

Shortly afterwards it was shown [15,127] that Nambu mechanics could be seen as a coming from constrained Hamiltonian mechanics; e.g. for \( \mathbb{R}^3 \) one starts with \( \mathbb{R}^6 \) and an identically vanishing Hamiltonian, takes a pair of second-class constraints to reduce it to some \( \mathbb{R}^4 \) and one more first-class Dirac constraint, together with time rescaling, will give the reduction. This “chilled” the domain for almost 20 years – and gives a physical explanation to the fact that Nambu could not go beyond Heisenberg quantization.

In order to quantize the Nambu bracket, a natural idea is to replace, in the definition of the Jacobian, the pointwise product of functions by a deformed product. For this to make sense, the deformed product should be Abelian, so we are lead to consider commutative DrG-deformations of an associative and commutative product. Looking first at polynomials (this restriction can be removed [138]) we are lead to the commutative part of Hochschild cohomology called Harrison cohomology, which is trivial [12,87]. Dealing with polynomials, a natural idea is to factorize them and take symmetrized star products of the factors. More precisely we introduce an operation \( \alpha \) which maps a product of factors into a symmetrized tensor product (in a kind of Fock space) and an evaluation map \( T \) which replaces tensor product by star product. Associativity will be satisfied if \( \alpha \) annihilates the deformation parameter \( \hbar \) (there are still \( \hbar \)-dependent terms in a product due to the last action of \( T \)); intuitively one can think of a deformation parameter which is \( \hbar \) times a Dirac \( \gamma \) matrix. This fact brought us to generalized deformations, but even this was not enough. Dealing with distributivity of the product with respect to addition and with derivatives posed difficult problems. In the end we took for observables Taylor developments of elements of the algebra of the semi-group generated by irreducible polynomials (“polynomials over polynomials”, inspired by second quantization techniques) and were then able to perform a meaningful quantization of these Nambu-Poisson brackets (cf. [53] for more details and [52] for subsequent development).

II.4 Related Mathematical Developments

II.4.1 Star representation theory of Lie groups

Let \( G \) be a Lie group (connected and simply connected), acting by symplectomorphisms on a symplectic manifold \( X \) (e.g. coadjoint orbits in the dual of the Lie algebra \( \mathfrak{g} \) of \( G \)). The elements \( x, y \in \mathfrak{g} \) will be supposed realized by functions \( u_x, u_y \) in \( N \) so that their Lie bracket \( [x, y]_\mathfrak{g} \) is realized by \( P(u_x, u_y) \). Now take a \( G \)-covariant
star-product *, that is
\[ P(u_x, u_y) = [u_x, u_y] = (u * v - v * u) / 2\nu, \]
which shows that the map \( g \ni x \mapsto (2\nu)^{-1}u_x \in N \) is a Lie algebra morphism. The appearance of \( \nu^{-1} \) here and in the trace (see (I.1)) cannot be avoided and explains why we have often to take into account both \( \nu \) and \( \nu^{-1} \). We can now define the star exponential

\[ E(e^x) = \text{Exp}(x) = \sum_{n=0}^{\infty} (n!)^{-1} (u_x/2\nu)^n \]

where \( x \in g \), \( e^x \in G \) and the power \(*n\) denotes the \( n \)th star-power of the corresponding function. By the Campbell-Hausdorff formula one can extend \( E \) to a group homomorphism \( E : G \to (N[[\nu, \nu^{-1}]], *) \) where, in the formal series, \( \nu \) and \( \nu^{-1} \) are treated as independent parameters for the time being. Alternatively, the values of \( E \) can be taken in the algebra \( (P[[\nu^{-1}]], *) \), where \( P \) is the algebra generated by \( g \) with the *-product (a representation of the enveloping algebra).

We call star representation [17,82] of \( G \) a distribution \( E \) (valued in \( \text{Im}E \)) on \( X \) defined by \( D \ni f \mapsto E(f) = \int_G f(g)E(g^{-1})dg \) where \( D \) is some space of test-functions on \( G \). The corresponding character \( \chi \) is the (scalar-valued) distribution defined by \( D \ni f \mapsto \chi(f) = \int_X E(f)d\mu \), \( d\mu \) being a quasi-invariant measure on \( X \).

The character is one of the tools which permit a comparison with usual representation theory. For semi-simple groups it is singular at the origin in irreducible representations, which may require caution in computing the star exponential (22). In the case of the harmonic oscillator that difficulty was masked by the fact that the corresponding representation of \( \mathfrak{sl}(2) \) generated by \( (p^2, q^2, pq) \) is integrable to a double covering of \( \text{SL}(2, \mathbb{R}) \) and decomposes into a sum \( D(\frac{1}{4}) \oplus D(\frac{3}{4}) \): the singularities at the origin cancel each other for the two components.

This theory is now very developed, and parallels in many ways the usual (operatorial) representation theory. It is not possible here to give a detailed account of all of them, but among notable results one may quote:

i) An exhaustive treatment of nilpotent or solvable exponential [5] and even general solvable Lie groups [6]. The coadjoint orbits are there symplectomorphic to \( \mathbb{R}^{2\ell} \) and one can lift the Moyal product to the orbits in a way that is adapted to the Plancherel formula. Polarizations are not required, and “star-polarizations” can always be introduced to compare with usual theory. Wavelets [45], important in signal analysis, are manifestations of star products on the (2-dimensional solvable) affine group of \( \mathbb{R} \) or on a similar 3-dimensional solvable group [21].

ii) For semi-simple Lie groups an array of results is already available, including [4,125] a complete treatment of the holomorphic discrete series (this includes the case of compact Lie groups) using a kind of Berezin dequantization, and scattered results for specific examples. Similar techniques have also been used [37,108] to find invariant star products on Kähler and Hermitian symmetric spaces (convergent for an appropriate dense subalgebra). Note however, as shown by recent developments of unitary representations theory (see e.g. [144]), that for semi-simple groups the coadjoint orbits alone are no more sufficient for the unitary dual and one needs far more elaborate constructions.
iii) For semi-direct products, and in particular the Poincaré and Euclidean groups, an autonomous theory has also been developed (see e.g. [7]).

Comparison with the usual results of “operatorial” theory of Lie group representations can be performed in several ways, in particular by constructing an invariant Weyl transform generalizing (1), finding “star-polarizations” that always exist, in contradistinction with the geometric quantization approach (where at best one can find complex polarizations), study of spectra (of elements in the center of the enveloping algebra and of compact generators) in the sense of (II.3.1), comparison of characters, etc. Note also in this context that the pseudodifferential analysis and (non autonomous) connection with quantization developed extensively by Unterberger, first in the case of $\mathbb{R}^{2\ell}$, has been recently extended to the above invariant context [154]. But our main insistence is that the theory of star representations is an autonomous one that can be formulated completely within this framework, based on coadjoint orbits (and some additional ingredients when required).

II.4.2 Quantum groups

Around 1980 Kulish and Reshetikhin [116], for purposes related to inverse scattering and 2-dimensional models, discovered a strange modification of the $\mathfrak{sl}(2)$ Lie algebra, where the commutation relation of the two nilpotent generators is a sine in the semi-simple generator instead of being a multiple of it – this in fact requires some completion of the enveloping algebra $\mathcal{U}(\mathfrak{g})$. The theory was developed in the first half of the 80’s by the Leningrad school of L. Faddeev [59], systematized by V. Drinfeld who developed the Hopf algebraic context and coined the extremely effective (though somewhat misleading) term of quantum group [55] and from the enveloping algebra point of view by Jimbo [105]. Shortly afterwards, Woronowicz [164] realized these models in the context of the noncommutative geometry of Alain Connes [41] by matrix pseudogroups, with coefficients (satisfying some relations) in $C^*$ algebras. A typical example of such Hopf algebras is a Poisson Lie group, a Lie group $G$ with compatible Poisson structure i.e. a Poisson bracket $P$ on $\hat{N} = C^\infty(G)$, considered as a bialgebra with coproduct defined by $\Delta u(g, g') = u(gg')$, $g, g' \in G$, satisfying $\Delta P(u, v) = P(\Delta u, \Delta v)$, $u, v \in \hat{N}$.

Now the topological dual of $N$ is the space $N'$ of distributions with compact support on $G$; it includes $G$ (Dirac’s $\delta$s at the points of $G$) and a completion of $\mathcal{U}(\mathfrak{g})$ (differential operators). Taking an adequate subspace $N_0'$ of $N'$ (generated by the coefficients of suitably chosen representations, e.g. the “well-behaved” vectors of Harish Chandra) will give a dual $N_0' \supset N'$. All these are reflexive (the bidual coincides with the original space; the algebraic dual of a Hopf algebra is in general not a Hopf algebra). This is the basis of the theory of topological Hopf algebras developed recently, first for $G$ compact [26,25] and then for $G$ semi-simple and in general [22]. In the compact or semi-simple case the quantum group is obtained by giving a star product on $N$ or $N_0$ and keeping unchanged the coproduct (what is called a preferred deformation) or equivalently by deforming the coproduct in
the dual (and keeping the product unchanged). Associativity of the star product corresponds to the Yang-Baxter equation, and the Faddeev-Reshetikhin-Takhtajan and Jimbo models of quantum groups can be seen in this way. Also, all Poisson-Lie groups can be quantized [57,22], though not necessarily with preferred deformations. We have therefore shown that quantum groups are in fact a special case of star products. For more details see e.g. the original papers and [79,80].

**II.4.3 Noncommutative geometry and index theorems**

Noncommutative geometry arose by a kind of “distillation” from the works of Connes on $C^*$-algebras and the use in that connection of methods and results of algebraic geometry. It involves in particular cyclic cohomology which was introduced by A. Connes in connection with trace formulas for operators (cyclic homology was introduced independently by Tsygan [153]). In particular cyclic cocycles are higher analogues of traces (see [54] for a generalization of the notion of trace). Thus they facilitate (by setting it algebraically) the computation of the index, which can obviously be viewed as the trace of some operator, and permit to generalize the index theorem, producing algebraic index formulas [41] of which the Atiyah-Singer formula (5) is a special case. As a matter of fact, Fedosov worked first in problems related to the index theorem and this brought him naturally to star product algebras of functions and to the index question in that context [62] as a fruitful alternative to algebras of pseudodifferential operators. Recently Nest and Tsygan [131] gave a nice proof of general algebraic index theorems in the framework of deformation quantization; doing so they show the existence of a “formal trace” (for $X$ symplectic of dimension $2\ell$) given by $\text{Tr}_N = \frac{1}{\ell!} \int_X (u_\omega^\ell + \nu \tau_1(u) + \nu^2 \tau_2(u) + \cdots)$ where the $\tau_k$ are local expressions in $u$. That trace satisfies $\text{Tr}_N (u * v - v * u) = 0$; thus the integrand will give an equivalence, over $\mathbb{K}[\nu^{-1}, \nu]]$, between any given n.c. star product and a strongly closed one.

Cyclic cohomology is based on a bicomplex containing a Hochschild complex with coboundary operator $b$ of degree 1 and another one with operation $B$ of degree -1 anticommuting with $b$. For a precise definition and properties, see [41]. The concept does not require to make reference to operator algebras; formulated abstractly, it applies even better to star products algebras provided the star products considered are closed (see [42], where an explanation of cyclic cohomology in this context can be found). Indeed, if $*$ is closed (see Def. 2) and a trace $\tau$ is defined on $u = \sum_{r=0}^{\infty} \nu^r u_r \in N[[\nu]]$ by $\tau(u) = \int u_\nu \omega^\ell$, we can consider the quasi-homomorphism (that measures the noncommutativity of the $*$-algebra and is also a Hochschild 2-cocycle) $\theta(u_1, u_2) = u_1 * u_2 - u_1 u_2$; then $\varphi_{2k}(u_0, \ldots, u_{2k}) = \tau(u_0 * \theta(u_1, u_2) * \ldots * \theta(u_{2k-1}, u_{2k}))$ defines the components of a cyclic cocycle $\varphi$ in the $(b, B)$ bicomplex on $N$ that is called the character of the closed star product. In particular $\varphi_{2\ell}(u_0, \ldots, u_{2\ell}) = \int u_0 du_1 \wedge \ldots \wedge du_{2\ell}$. The composition of symbols of pseudodifferential operators is [42] a closed star product, the character of which coincides with that defined by the trace on these operators.
A natural extension of the associative algebra context of noncommutative geometry is to Hopf algebras (in the line of [25]) and this indeed permitted now Connes and Moscovici [43] to compute the index of transversally elliptic operators on foliations, a longstanding problem (which among many other tools required hypoelliptic pseudodifferential operators). Another extension, motivated by physics, is to supersymmetric data, and this has been the subject of recent studies by Fröhlich and coworkers [81], first in the context of usual differential geometry and now in that of noncommutative geometry. There are many more developments in this framework, including quantized space, but we shall not develop these further.

Acknowledgements. I want to thank Giuseppe (= Joseph) Dito and Moshé Flato for very useful comments, and Piotr Rączka and the organizers of PFG98 in Łódź (especially Jakub Rembieliński) for excellent hospitality in Poland.

REFERENCES

1. Abraham R. and Marsden J.E., *Foundations of mechanics*, Benjamin/Cummings Publ. Co., Advanced Book Program, Reading, Mass. (Second edition, 1978).
2. Agarwal G.S. and Wolf E. “Calculus for functions of noncommuting operators and general phase-space methods in quantum mechanics I, II, III”, Phys. Rev. D2 (1970), 2161-2186, 2187-2205, 2206-2225.
3. Angelopoulos E., Flato M., Fronsdal, C. and Sternheimer D. “Massless particles, conformal group and De Sitter universe”, Phys. Rev. D23 (1981), 1278-1289.
4. Arnal D., Cahen M. and Gutt S. “Representations of compact Lie groups and quantization by deformation”, Bull. Acad. Royale Belg. 74 (1988), 123-141; “Star exponential and holomorphic discrete series”, Bull. Soc. Math. Belg. 41 (1989), 207-227.
5. Arnal D. and Cortet J.C. “Nilpotent Fourier transform and applications”, Lett. Math. Phys. 9 (1985), 25-34; “Star-products in the method of orbits for nilpotent Lie groups”, J. Geom. Phys. 2 (1985), 83-116; “Représentations star des groupes exponentiels”, J. Funct. Anal. 92 (1990), 103-135.
6. Arnal D., Cortet J.C. and Ludwig J. “Moyal product and representations of solvable Lie groups”, J. Funct. Anal. 133 (1995), 402-424.
7. Arnal D., Cortet J.C. and Molin P. “Star-produit et représentation de masse nulle du groupe de Poincaré”, C.R. Acad. Sci. Paris Sér. A 293, 309-312 (1981).
8. Arnal D., Cortet J.C., Molin P. and Pinczon G. “Covariance and geometrical invariance in star-quantization”, J. Math. Phys. 24 (1983), 276-283.
9. Atiyah M.F. and Singer I.M. “The index of elliptic operators on compact manifolds”, Bull. Amer. Math. Soc. 69 (1963) 422-433; “The index of elliptic operators I,III and IV,V”, Ann. of Math. 87 (1968) 484-530, 546-604 and 93 (1971), 119-138, 139-149; Atiyah M., Bott R. and Patodi V.K. “On the heat equation and the index theorem”, Invent. Math. 19 (1973), 279-330 and 28 (1975), 277-280.
10. Auslander L. and Kostant B. “Polarization and unitary representations of solvable groups”, Inv. Math. 14 (1971), 255-354.
11. de Azcárraga J.A., Perelomov A.M. and Pérez Bueno J.C. “New generalized Poisson structures”, J. Phys. A 29 (1996), L151–L157.
12. Barr M. “Harrison Homology, Hochschild Homology, and Triples”, J. Alg. 8 (1968), 314-323.
13. Basart H. and Lichnerowicz A. “Déformations d’un star-produit sur une variété symplectique”. C. R. Acad. Sci. Paris 293 I (1981), 347-350; “Conformal Symplectic Geometry, Deformations, Rigidity and Geometrical (KMS) Conditions”, Lett. Math. Phys. 10 (1985), 167-177.
14. Basart H., Flato M., Lichnerowicz A. and Sternheimer D. “Deformation theory applied to quantization and statistical mechanics”, Lett. Math. Phys. 8 (1984), 483-494; “Mécanique statistique et déformations”, C.R. Acad. Sci. Paris Sér. I 299 (1984), 405-410.
15. Bayen F. and Flato M. “Remarks concerning Nambu’s generalized mechanics”, Phys. Rev. D11 (1975), 3049-3053.
16. Bayen F., Flato M., Fronsdal C., Lichnerowicz A. and Sternheimer D. “Quantum mechanics as a deformation of classical mechanics”. Lett. Math. Phys. 1 (1977), 521-530.
17. Bayen F., Flato M., Fronsdal C., Lichnerowicz A. and Sternheimer D. “Deformation theory and quantization I, II”, Ann. Phys. (NY) (1978) 111, 61-110, 111-151.
18. Berezin F. A. “General concept of quantization”, Comm. Math. Phys. 40 (1975), 153-174; “Quantization”, Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116-1175; “Quantization in complex symmetric spaces”, Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), 363-402, 472. [English translations: Math. USSR-Izv. 38 1109-1165 (1975) and 39, 341–379 (1976)].
19. Berezin F. A. and Šubin M. A. “Symbols of operators and quantization” in: Hilbert space operators and operator algebras (Proc. Internat. Conf., Tihany, 1970), Colloq. Math. Soc. Janos Bolyai 5, 21-52. North-Holland, Amsterdam (1972).
20. Bertelson M., Cahen M. and Gutt S. “Equivalence of star products”, Classical and Quantum Gravity 14 (1997), A93-A107.
21. Bertrand J. and Bertrand P. “Symbolic calculus on the time-frequency half-plane”, J. Math. Phys. 39 (1998), 4071-4090.
22. Bidegain F. and Pinczon G. “A star product approach to noncompact quantum groups”, Lett. Math. Phys. 33 (1995), 231-240; “Quantization of Poisson-Lie Groups and Applications”, Comm. Math. Phys. 179 (1996), 295-332.
23. Bonneau P. “Cohomology and associated deformations for not necessarily coassociative bialgebras”, Lett. Math. Phys. 26 (1992), 277-280.
24. Bonneau P. “Fedosov star-products and 1-differentiable deformations”, Lett. Math. Phys. (in press; expanded version in math.QA/9809032).
25. Bonneau P., Flato M., Gerstenhaber M. and Pinczon G. “The hidden group structure of quantum groups: strong duality, rigidity and preferred deformations”, Comm. Math. Phys. 161 (1994), 125-156.
26. Bonneau P., Flato M. and Pinczon G. “A natural and rigid model of quantum groups”. Lett. Math. Phys. 25 (1992), 75-84.
27. Bordemann M. “On the Deformation Quantization of super-Poisson Brackets”, q-alg/9605038.
28. Bordemann M., Neumaier N. and Waldmann S., “Homogeneous Fedosov Star Products on Cotangent Bundles” I, II, q-alg/9711016, q-alg/9707030.
29. Bordemann M., Roemer H. and Waldmann S., “A Remark on Formal KMS States in Deformation Quantization”, math/9801139.
30. Bordemann M. and Waldmann S. “Formal GNS Construction and States in Deformation Quantization”, q-alg/9607019, Comm. Math. Phys. 195 (1998), 549-583.
31. Boutet de Monvel L. “Star products on conic Poisson manifolds of constant rank”, Mat. Fiz. Anal. Geom. 2 (1995), 143-151; “Star produit associé à un crochet de Poisson de rang constant”, in: Partial differential equations and functional analysis, 111-119, Progr. Nonlinear Diff. Eqs. Appl., 22, Birkhäuser Boston, (1996).
32. Boutet de Monvel L. and Guillemin V. The spectral theory of Toeplitz operators. Annals of Mathematics Studies 99, Princeton University Press (1981).
33. Cadavid C. and Nakashima M. “The star-exponential and path integrals on compact groups”, Lett. Math. Phys. 23 (1991), 111-115.
34. Cahen M., Flato M., Gutt S. and D. Sternheimer “Do different deformations lead to the same spectrum?”, J. Geom. Phys. 2 (1985), 35-48.
35. Cahen M. and Gutt S. “Regular *-representations of Lie algebras”, Lett. Math. Phys. 6 (1982), 395-404.
36. Cahen M., Gutt S. and De Wilde M. “Local cohomology of the algebra of $C^\infty$ functions on a connected manifold”, Lett. Math. Phys. 4 (1980), 157-167.
37. Cahen M., Gutt S. and Rawnsley J. “Quantization of Kähler manifolds IV, Lett. Math. Phys. 34 (1995), 159-168.
38. Calderon A.P. and Zygmund A. “Singular integral operators and differential equations”, Amer. J. Math. 77 (1957), 901-921.
39. Cartan H. and Schwartz L. Séminaire Henri Cartan 1963/64, “Théorème d’Atiyah-Singer sur l’indice d’un opérateur différentiel elliptique”, fasc. 1 & 2, Secrétariat mathématique, Paris (1965).
40. Chevalley C. and Eilenberg S. “Cohomology theory of Lie groups and algebras”, Trans. Amer. Math. Soc. 63 (1948), 85-124.
41. Connes A. Noncommutative Geometry, Academic Press, San Diego (1994).
42. Connes A., Flato M. and Sternheimer D. “Closed star-products and cyclic cohomology”, Lett. Math. Phys. 24 (1992), 1-12.
43. Connes A. and Moscovici H. “Hopf algebras, cyclic cohomology and the transverse index theorem”, preprint math.DG/9806109 (1998).
44. Daubechies I. “On the distributions corresponding to bounded operators in the Weyl quantization”, Comm. Math. Phys. 75 (1980), 229-238; Daubechies I. and Grossmann A. “An integral transform related to quantization”, J. Math. Phys. 21 (1980), 2080-2090.
45. Daubechies I. “Wavelets and other phase space localization methods”, in: Proc. Int. Congress of Mathematicians, (Zürich, 1994), 56-74, Birkhäuser, Basel (1995).
46. Deligne P. “Déformations de l’Algèbre des Fonctions d’une Variété Symplectique: Comparaison entre Fedosov et De Wilde, Lecomte”, Selecta Math. N.S. 1 (1995), 667-697.

32
47. De Wilde M. and Lecomte P.B.A. “Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds”, Lett. Math. Phys. 7 (1983), 487-496; “Star-products on cotangent bundles”, Lett. Math. Phys. 7 (1983), 235-241 and 8 (1984), 79; “Existence of star-products on exact symplectic manifolds”, Ann. Inst. Fourier (Grenoble) 35 (1985), 117-143; “Existence of star-products revisited”, dedicated to the memory of Professor Gottfried Köthe, Note Mat. 10 (1990), suppl. 1, 205-216 (1992); “Formal deformations of the Poisson Lie algebra of a symplectic manifold and star-products. Existence, equivalence, derivations” in Deformation Theory of Algebras and Structures and Applications (M. Hazewinkel and M. Gerstenhaber Eds.), NATO ASI Ser. C 247, 897-960, Kluwer Acad. Publ., Dordrecht (1988).

48. De Wilde M., Lecomte P.B.A. and Gutt S. “A propos des deuxième et troisième espaces de cohomologie de l’algèbre de Lie de Poisson d’une variété symplectique”, Ann. Inst. H. Poincaré Sect. A (N.S.) 40 (1984), 77-83.

49. De Wilde M., Lecomte P.B.A. and Mélotte D. “Invariant star-products on symplectic manifolds”, J. Geom. Phys. 2 (1985), 121-129; “Invariant cohomology of the Poisson Lie algebra of a symplectic manifold”, Comment. Math. Univ. Carolin. 26 (1985), 337-352.

50. Dirac P.A.M. Lectures on Quantum Mechanics, Belfer Graduate School of Sciences Monograph Series No. 2 (Yeshiva University, New York, 1964). “Generalized Hamiltonian dynamics”, Canad. J. Math. 2 (1950), 129-148.

51. Dito J. “Star-product approach to quantum field theory: the free scalar field”, Lett. Math. Phys. 20 (1990), 125-134; “Star-products and nonstandard quantization for K-G equation”, J. Math. Phys. 33 (1992), 791-801; “An example of cancellation of infinities in star-quantization of fields”, Lett. Math. Phys. 27 (1993), 73-80.

52. Dito G. and Flato M. “Generalized Abelian Deformations: Application to Nambu Mechanics”, Lett. Math. Phys. 39 (1997), 107-125.

53. Dito G., Flato M., Sternheimer D. and Takhtajan L. “Deformation Quantization and Nambu Mechanics”, Comm. Math. Phys. 183 (1997), 1-22.

54. Dixmier J. “Existence de traces non normales”, C.R. Acad. Sci. Paris Sér. A 262 (1966), 1107-1108.

55. Drinfeld V.G. “Quantum Groups”, in: Proc. ICM86, Berkeley, 1, 101-110, Amer. Math. Soc., Providence (1987); “Quasi-Hopf algebras”, Leningrad Math. J. 1 (1990), 1419-1457; “On almost co-commutative Hopf algebras”, ibid., 321-431.

56. Du Cloux F. “Extensions entre représentations unitaires irréductibles des groupes de Lie nilpotents”, Astérisque 125 (Soc. Math. Fr. 1985), 129-211.

57. Etingof P. and Kazhdan D. “Quantization of Lie Bialgebras” I. Selecta Math. (N.S.) 2 (1996), 1-41 (q-alg/9506005); II. q-alg/9701038, III. q-alg/9610030 (ibid.); IV. math.QA/9801043; V. math.QA/9808121; “Quantization of Poisson algebraic groups and Poisson homogeneous spaces” q-alg/9510020, in Quantum Symmetries, Les Houches Summer School Proceedings 64, North Holland (1998).

58. Faddeev L. D. “On the relation between mathematics and physics”. in: Integrable systems (Tianjin, 1987), 3–9, Nankai Lectures Math. Phys., World Sci. Pub. (1990).

59. Faddeev L.D., Reshetikhin N.Yu. and Takhtajan L.A. “Quantization of Lie groups and Lie algebras”, Leningrad Math. J. 1 (1990), 193-225.
60. Fedosov B.V. “Formal quantization” in *Some topics of modern mathematics and their applications to problems of mathematical physics*, 129-139 (Moscow, 1985); “Quantization and index”, Dokl. Akad. Nauk SSSR 291, 82-86 (1986).

61. Fedosov B.V. “A simple geometrical construction of deformation quantization”, J. Diff. Geom. 40 (1994), 213-238.

62. Fedosov B.V. “Analytical formulas for the index of elliptic operators”, Trans. Moscow Math. Soc. 30 (1974), 159-240; “An index theorem in the algebra of quantum observables, Dokl. Akad. Nauk SSSR 305 (1989), 835-838; “The Index Theorem for Deformation Quantization”, in: *Boundary Value Problems, Schrödinger Operators, Deformation Quantization*, 206-318, Math. Top. 8, Akademie Verlag, Berlin (1995); “A trace density in deformation quantization”, *ibid.*, 319-333.

63. Fedosov B.V. *Deformation Quantization and Index Theory*, Mathematical Topics 9, Akademie Verlag, Berlin (1996).

64. Fedosov B.V. “Non-abelian reduction in deformation quantization”, Lett. Math. Phys. 43 (1998), 137-154; “Reduction and eigenstates in deformation quantization”, in: *Pseudo-differential calculus and mathematical physics*, 277-297, Math. Top. 5, Akademie Verlag, Berlin (1994).

65. Filippov V.T. “$n$-Lie algebras”, Siberian Math. J. 26 (1985) 875–879.

66. Flato, M. “Deformation view of physical theories”, Czechoslovak J. Phys. B32 (1982), 472-475.

67. Flato, M. “Two Disjoint Aspects of the Deformation Programme: Quantizing Nambu Mechanics; Singleton Physics”, these Proceedings.

68. Flato M., Dito G. and Sternheimer D. “Nambu mechanics, $n$-ary operations and their quantization” in *Deformation theory and symplectic geometry, Proceedings of Ascona meeting, June 1996* (D. Sternheimer, J. Rawnsley and S. Gutt, Eds.), Math. Physics Studies 20, 43-66, Kluwer Acad. Publ., Dordrecht (1997).

69. Flato M. and Fronsdal C. “One massless particle equals two Dirac singletons”, Lett. Math. Phys. 2 (1978), 421-426.

70. Flato M. and Fronsdal C. “Composite Electrodynamics”. J. Geom. Phys. 5 (1988), 37-61; “Interacting Singletons”, Lett. Math. Phys. 44 (1998), 249-259.

71. Flato M., Lichnerowicz A. and Sternheimer D. “Déformations 1-différentiables d’algèbres de Lie attachées à une variété symplectique ou de contact”, C.R. Acad. Sci. Paris Sér. A 279 (1974), 877-881 and Compositio Mathematica, 31 (1975), 47-82; “Algèbres de Lie attachées à une variété canonique”, J. Math. Pures Appl. 54 (1975), 445-480.

72. Flato M., Lichnerowicz A. and Sternheimer D. “Deformations of Poisson brackets, Dirac brackets and applications”. J. Math. Phys. 17 (1976), 1754-1762.

73. Flato M., Lichnerowicz A. and Sternheimer D. “Crochets de Moyal-Vey et quantification”, C.R. Acad. Sci. Paris Sér. A 283 (1976), 19-24.

74. Flato M., Pinczon G. and Simon J. “Non-linear representations of Lie groups”, Ann. Sci. Éc. Norm. Sup. (4) 10 (1977), 405-418.

75. Flato M. and Simon J., “Non-linear wave equations and covariance”, Lett. Math. Phys. 2 (1977), 115-160.

76. Flato M., Simon J., Snellman H. and Sternheimer D. “Simple facts about analytic vectors and integrability”, Ann. Sci. Éc. Norm. Sup. (4) 5 (1972), 432-434.
DEFORMATION QUANTIZATION

77. Flato M., Simon J.C.H. and Taflin E. The Maxwell-Dirac equations: the Cauchy problem, asymptotic completeness and the infrared problem, Memoirs of the American Mathematical Society 127 (number 606, May 1997).

78. Flato M. and Sternheimer D. “Deformations of Poisson brackets, separate and joint analyticity in group representations, nonlinear group representations and physical applications.” in: Lectures at the Advanced Summer Institute on Harmonic Analysis (Liège 1977), Mathematical Physics and Applied Mathematics 5, 385-448. D. Reidel, Dordrecht (1980).

79. Flato M. and Sternheimer D. “Closedness of star products and cohomologies”, in: Lie Theory and Geometry: In Honor of B. Kostant (J.L. Brylinski et al., eds.), 241-259. Progress in Mathematics, Birkhäuser, Boston (1994).

80. Flato M. and Sternheimer D. “Topological Quantum Groups, star products and their relations”, St. Petersburg Mathematical Journal (Algebra and Analysis) 6 (1994), 242-251.

81. Fröhlich J., Grandjean O. and Recknagel A. “Supersymmetric quantum theory and differential geometry”, Comm. Math. Phys. 193 (1998), 527-594; “Supersymmetric quantum theory and non-commutative geometry”, math-ph/9807006 (1998).

82. Fronsdal C. “Some ideas about quantization”, Rep. Math. Phys. 15 (1978), 111-145.

83. García-Compeán H., Plebański J. and Przanowski M. “Geometry Associated with Self-dual Yang-Mills and the Chiral Model Approaches to Self-dual Gravity”, Acta Phys.Polon. B29 (1998), 549-571.

84. Gautheron Ph. “Some remarks concerning Nambu mechanics”, Lett. Math. Phys. 37 (1996), 103-116; “Simple facts concerning Nambu algebras”, Comm. Math. Phys. 195 (1998), 417-434.

85. Gel’fand I.M., Kalinin D.I. and Fuks D.B. “The cohomology of the Lie algebra of Hamiltonian formal vector fields”, Funkcional. Anal. i Priložen. 6 (1972), 25-29.

86. Gerstenhaber M. “On the deformation of rings and algebras”, Ann. Math. 79 (1964), 59-103; and (IV), ibid. 99 (1974), 257-276.

87. Gerstenhaber M. and Schack S.D. “Algebraic cohomology and deformation theory”, in Deformation Theory of Algebras and Structures and Applications (M. Hazewinkel and M. Gerstenhaber Eds.), NATO ASI Ser. C 247, 11-264, Kluwer Acad. Publ., Dordrecht (1988).

88. Gerstenhaber M. and Schack S.D. “Bialgebra cohomology, deformations and quantum groups”, Proc. Nat. Acad. Sci. USA 87 (1990), 478-481; “Algebras, bialgebras, quantum groups and algebraic deformations”, in Deformation Theory and Quantum Groups with Applications to Mathematical Physics (M. Gerstenhaber and J. Stasheff, eds.), Contemporary Mathematics 134, 51-92, American Mathematical Society, Providence (1992).

89. Gilkey P. Invariance theory, the heat equation and the Atiyah-Singer index theorem, Second edition, Studies in Advanced Mathematics, CRC Press, Boca Raton FL (1995).

90. Ginzburg G. and Kapranov M. “Koszul duality for operads”, Duke Math. J. 76 (1994), 203-272.

91. Gnedenbaye A.V. “Les algèbres k-aires et leurs opérades”, C. R. Acad. Sci. Paris Sér.I 321 (1995), 147-152.
92. Gracia-Bondía J.M. and Várilly J.C. “From geometric quantization to Moyal quantization”, J. Math. Phys. 36 (1995), 2691-2701.
93. Gutt S. “An explicit *-product on the cotangent bundle to a Lie group”, Lett. Math. Phys. 7 (1983), 249-258.
94. Gutt S. “On some second Hochschild cohomology spaces for algebras of functions on a manifold”, Lett. Math. Phys. 37 (1997), 157-162.
95. Groenewold A. “On the principles of elementary quantum mechanics”, Physica 12 (1946), 405-460.
96. Grossman A., Loupias G. and Stein E.M. “An algebra of pseudodifferential operators and quantum mechanics in phase space”, Ann. Inst. Fourier Grenoble 18 (1968), 343-368.
97. Guillemin V. “Star products on compact pre-quantizable symplectic manifolds”. Lett. Math. Phys. 35 (1995), 85-89.
98. Guillemin V. and Sternberg S. Symplectic techniques in physics. Cambridge University Press (1984).
99. Gutt S. and Rawnsley J. “Equivalence of star products on a symplectic manifold: an introduction to Deligne’s Čech cohomology classes”, preprint (June 1998).
100. Hansen F. “The Moyal product and spectral theory for a class of infinite-dimensional matrices”, Publ. RIMS, Kyoto Univ., 26 (1990), 885-933; “Quantum Mechanics in Phase Space”, Reports On Math. Phys. 19 (1984), 361-381.
101. Hörmander L. “Pseudo-differential operators and non-elliptic boundary problems”, Ann. Math. 83 (1966), 129-209. “Fourier Integral Operators I”, Acta Math. 127 (1971), 79-183.
102. Hörmander L. “The Weyl calculus of pseudodifferential operators”, Comm. Pure Appl. Math. 32 (1979), 360-444; “Symbolic calculus and differential equations” in 18th Scandinavian Congress of Mathematicians (Aarhus, 1980), 56-81, Progr. Math. 11, Birkhäuser Boston (1981).
103. Hochschild G., Kostant B. and Rosenberg A. “Differential forms on regular affine algebras”, Trans. Am. Math. Soc. 102 (1962), 383-406.
104. Inonü E. and Wigner E.P. “On the contraction of groups and their representations”, Proc. Nat. Acad. Sci. U. S. A. 39 (1953), 510-524.
105. Jimbo M. “A q-difference algebra of U(g) and the Yang-Baxter equation”, Lett. Math. Phys. 10 (1985), 63-69.
106. Jurzak J.P. Unbounded noncommutative integration, Mathematical Physics Studies 7, D. Reidel Publ. Co., Dordrecht (1985).
107. Kammerer J.B. “Analysis of the Moyal product in a flat space”, J. Math. Phys. 27 (1986), 529-535.
108. Karabegov, A. V. “Cohomological classification of deformation quantizations with separation of variables”, Lett. Math. Phys. 43 (1998), 347-357; “Berezin’s quantization on flag manifolds and spherical modules”, Trans. Amer. Math. Soc. 350 (1998), 1467-1479.
109. Kirillov A.A. Elements of the theory of representations, Springer, Berlin (1976).
110. Kodaira K. and Spencer D.C. “On deformations of complex analytic structures” I, II, Ann. of Math. 67 (1958), 328-466; III “Stability theorems for complex structures” ibid. 71 (1960), 43-76.
111. Kontsevich M. “Formality conjecture”, in: Deformation Theory and Symplectic Geometry, 139-156, Math. Phys. Stud. 20, Kluwer Acad. Publ., Dordrecht (1997).
112. Kontsevich M. “Deformation quantization of Poisson manifolds, I” q-alg/9709040 (and private communications); “Deformation quantization”, Arbeitstagung (1997).
113. Kostant B. Quantization and unitary representations, in: Lecture Notes in Math. 170, 87-208, Springer Verlag, Berlin (1970); “On the definition of quantization”, in: Géométrie symplectique et physique mathématique (Colloq. Int. CNRS, No 237), 187-210, Éds. CNRS, Paris (1975); “Graded manifolds, graded Lie theory, and pre-quantization” in Differential geometrical methods in mathematical physics, 177-306, Lecture Notes in Math. 570, Springer, Berlin (1977).
114. Kostant B. “Symplectic spinors”, Symposia Mathematica 14 (1974), 139-152.
115. Krée P. and Rączka R. “Kernels and symbols of operators in quantum field theory”, Ann. Inst. H. Poincaré Sect A (N.S.) 28 (1978), 41-73.
116. Kulish P.P. and Reshetikhin N.Yu. “Quantum linear problem for the sine-Gordon equation and higher representations”, Zap. Nauch. Sem. LOMI 101 (1981), 101-110 (English translation in Jour. Sov. Math. 23 (1983), 24-35).
117. Lévy-Nahas M. “Deformations and contractions of Lie algebras”, J. Math. Phys. 8 (1967), 1211-1222. “Déformations du groupe de Poincaré”, in: L’extension du groupe de Poincaré aux symétries internes des particules élémentaires (Colloque Int. CNRS N° 159), 25-45, Éds CNRS, Paris (1968).
118. Libermann P. and Marle C.-M. Symplectic geometry and analytical mechanics, Mathematics and its Applications 35. D. Reidel Publ. Co., Dordrecht (1987).
119. Lichnerowicz A. “Cohomologie 1-différentiable des algèbres de Lie attachées à une variété symplectique ou de contact”, J. Math. Pures Appl. 53 (1974), 459-483.
120. Lichnerowicz A. “Les variétés de Poisson et leurs algèbres de Lie associées”, J. Diff. Geom. 12 (1977), 253-300.
121. Lichnerowicz A. “Déformations d’algèbres associées à une variété symplectique (les *ν-produits)”, Ann. Inst. Fourier, Grenoble, 32 (1982), 157-209.
122. Loday J.-L. “La renaissance des opérades”, Séminaire Bourbaki, Exposé 792 (Novembre 1994).
123. Maillard J.M. “On the twisted convolution product and the Weyl transform of tempered distributions”, J. Geom. Phys. 3 (1986), 231-261.
124. Marsden, J. E. and Weinstein, A. “Reduction of symplectic manifolds with symmetry”, Rep. Math. Phys. 5 (1974), 121-130.
125. Moreno C. “Invariant star products and representations of compact semi-simple Lie groups”, Lett. Math. Phys. 12 (1986), 217-229.
126. Moyal J.E. “Quantum mechanics as a statistical theory”, Proc. Cambridge Phil. Soc. 45 (1949), 99-124.
127. Mukunda, N. and Sudarshan, E.C.G. “Relation between Nambu and Hamiltonian mechanics”, Phys. Rev. D 13 (1976), 2846-2850.
128. Nadaud F. “Generalized deformations, Koszul resolutions, Moyal Products”, Reviews Math. Phys. 10 (5) (1998), 685-704.
129. Nambu Y. “Generalized Hamilton dynamics”, Phys. Rev D7 (1973), 2405-2412.
130. Neroslavsky O.M. and Vlasov A.T. “Sur les déformations de l’algèbre des fonctions d’une variété symplectique”, C.R. Acad. Sc. Paris Sér. I 292 (1981), 71-76.

131. Nest R. and Tsygan B. “Algebraic Index Theorem”, Comm. Math. Phys. 172 (1995), 223-262; “Algebraic Index Theorem For Families”, Advances in Mathematics 113 (1995), 151-205; “Formal deformations of symplectic manifolds with boundary”, J. Reine Angew. Math. 481 (1996), 27-54.

132. Nijenhuis A. “Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II ”, Indag. Math. 17 (1955), 390-397, 398-403.

133. Oesterlé J. “Quantification formelle des variétés de Poisson [d’après Maxim Kontsevich]”, Séminaire Bourbaki, exposé 843 (mars 1998).

134. Omori H., Maeda Y. and Yoshioka A. “Weyl manifolds and deformation quantization”, Adv. in Math. 85 (1991), 225-255; “Existence of a closed star product”, Lett. Math. Phys. 26 (1992), 285-294.

135. Omori H., Maeda Y. and Yoshioka A. “Deformation Quantizations of Poisson Algebras”, Contemporary Mathematics Amer. Math. Soc. 179 (1994), 213-240 and Proc. Japan Acad. 68 (1992), 97-101; “Deformation quantizations of the Poisson algebra of Laurent polynomials”, Lett. Math. Phys. (in press).

136. Palais R.S. Seminar on the Atiyah-Singer index theorem, Annals of Mathematics Studies No. 57, Princeton University Press (1965).

137. Pankaj Sharan “Star-product representation of path integrals”, Phys. Rev. D 20 (1979), 414-418.

138. Pinczon G. “On the equivalence between continuous and differential deformation theories”, Lett. Math. Phys. 39 (1997), 143-156.

139. Pinczon G. “Non commutative deformation theory”, Lett. Math. Phys. 41 (1997), 101-117.

140. Plebański J. “Naviasy Poissona i komutatory”, Torun preprint Nr 69 (1969).

141. Rieffel M. “Questions on quantization”, quant-ph/9712009, to be published in Proceedings of the International Conference on Operator Algebras and Operator Theory (Shanghai, July 1997).

142. Rubio R. “Algèbres associatives locales sur l’espace des sections d’un fibré en droites”, C.R. Acad. Sci. Paris Sér. I 299 (1984), 699-701.

143. Saletan E.J. “Contraction of Lie groups”, J. Math. Phys. 2 (1961), 1-21 and 742.

144. Schmid W., “Character formulas and localization of integrals”, in Deformation theory and symplectic geometry, Proceedings of Ascona meeting, June 1996 (D. Sternheimer, J. Rawnsley and S. Gutt, Eds.), Math. Physics Studies 20, 259-270, Kluwer Acad. Publ., Dordrecht (1997).

145. Schouten J.A. “On the differential operators of first order in tensor calculus”, Convegno Internazionale di Geometria Differenziale, Italia, 1953, 1-7, Edizioni Cremonese, Roma (1954).

146. Segal I.E. “A class of operator algebras which are determined by groups”, Duke Math. J. 18 (1951), 221-265.

147. Segal I.E. “Symplectic structures and the quantization problem for wave equations”, Symposia Mathematica 14 (1974), 79-117.

148. Shnider S. and Sternberg S. Quantum Groups, Graduate Texts in Mathematical Physics vol. II, International Press (Boston and Hong-Kong, 1993). [With an extended reference list of 1264 items!].
DEFORMATION QUANTIZATION

149. Souriau, J.-M. Structure of dynamical systems. A symplectic view of physics. Progress in Mathematics 149. Birkhäuser Boston (1997); Edited translation from the French Structure des systèmes dynamiques, Dunod, Paris (1970); “Des particules aux ondes: quantification géométrique” in: Huygens’ principle 1690–1990: theory and applications (The Hague and Scheveningen, 1990), 299–341, Stud. Math. Phys. 3, North-Holland, Amsterdam (1992); “La structure symplectique de la mécanique décrite par Lagrange en 1811”, Math. Sci. Humaines 94 (1986), 45-54.

150. Stasheff J. “Deformation theory and the Batalin-Vilkovisky master equation”, in Deformation theory and symplectic geometry (Ascona, 1996), 271-284, Math. Phys. Stud. 20, Kluwer Acad. Publ., Dordrecht (1997).

151. Takhtajan L.A. Introduction to quantum groups in: Springer Lecture Notes in Physics, 370, 3-28 (1990); Lectures on quantum groups in: Nankai Lectures on Math. Phys. (M. Ge and B. Zhao eds.) 69-197, World Scientific, Singapore (1990).

152. Takhtajan L.A. “On foundation of the generalized Nambu mechanics”, Comm. Math. Phys. 160 (1994), 295-315.

153. Tsygan B. “Homology of matrix Lie algebras over rings and Hochschild homology”, Uspekhi Math. Nauk 38 (1983), 217-218.

154. Unterberger A. and Upmeier H. Pseudodifferential analysis on symmetric cones, Studies in Advanced Mathematics, CRC Press, Boca Raton FL (1996); “The Berezin transform and invariant differential operators”, Comm. Math. Phys. 164 (1994), 563-597; Unterberger A. and J. “Quantification et analyse pseudo-différentielle”, Ann. Sci. Éc. Norm. Sup. (4) 21 (1988), 133-158.

155. Vey J. “Déformation du crochet de Poisson sur une variété symplectique”, Comment. Math. Helv. 50 (1975), 421-454.

156. Voronov A. “Quantizing Poisson manifolds”, q-alg/9701017.

157. Voros A. Développements semi-classiques, Thèse (Orsay et Saclay), mai 1977. “Semi-classical approximations”, Ann. Inst. Henri Poincaré 24 (1976), 31-90.

158. Waldman S. “A Remark on Non-equivalent Star Products via Reduction for $\mathbb{C}P^n$” math.QA/9802078.

159. Weinstein A. “Deformation quantization”, Séminaire Bourbaki, exposé 789 (juin 1994), Astérisque 227, 389-409.

160. Weinstein A. “The modular automorphism group of a Poisson manifold”, J. Geom. Phys. 23 (1997), 379-394.

161. Weyl, H. The theory of groups and quantum mechanics, Dover, New-York (1931), translated from Gruppentheorie und Quantenmechanik, Hirzel Verlag, Leipzig (1928); “Quantenmechanik und Gruppentheorie”, Z. Physik 46 (1927), 1-46.

162. Wigner, E.P. “Quantum corrections for thermodynamic equilibrium”, Phys. Rev. 40 (1932), 749-759.

163. Woodhouse N. M. J. Geometric quantization. Oxford Science Publications, Oxford Mathematical Monographs. Oxford University Press, New York (2nd edition, 1992).

164. Woronowicz S.L. “Compact matrix pseudogroups”, Comm. Math. Phys. 111 (1987), 613-665; Comm. Math. Phys. 160 (1989), 125-170; “Quantum E(2) group and its Pontryagin dual”, Lett. Math. Phys. 23 (1991), 251-263.

PACS (1998): 02.10.Tq, 02.10.Vr, 03.65.-w, 02.40.Vh, 11.10.-z

MSC (1991): 81S30, 81S10, 81T70, 46M20, 58B30, 58G12, 58F06, 17B37, 19K56, 22E45.

39