On the solutions of the $Z_n$-Belavin model with arbitrary number of sites

Kun Hao$^{a,b}$, Fakai Wen$^{a,c}$, Junpeng Cao$^{c,d}$, Guang-Liang Li$^e$, Wen-Li Yang$^{a,b,f}$, Kangjie Shi$^{a,b}$ and Yupeng Wang$^{c,d}$

$^a$Institute of Modern Physics, Northwest University, Xian 710069, China
$^b$Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xian 710069, China
$^c$Beijing National Laboratory for Condensed Matter Physics, Institute of Physics, Chinese Academy of Sciences, Beijing 100190, China
$^d$Collaborative Innovation Center of Quantum Matter, Beijing, China
$^e$Department of Applied Physics, Xian Jiaotong University, Xian 710049, China
$^f$Beijing Center for Mathematics and Information Interdisciplinary Sciences, Beijing, 100048, China

Abstract

The periodic $Z_n$-Belavin model on a lattice with an arbitrary number of sites $N$ is studied via the off-diagonal Bethe Ansatz method (ODBA). The eigenvalues of the corresponding transfer matrix are given in terms of a unified inhomogeneous $T-Q$ relation. In the special case of $N = nl$ with $l$ being also a positive integer, the resulting $T-Q$ relation recovers the homogeneous one previously obtained via algebraic Bethe Ansatz.

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$^1$Corresponding author: wlyang@nwu.edu.cn
$^2$Corresponding author: yupeng@iphy.ac.cn
1 Introduction

Our understanding to phase transitions and critical phenomena has been greatly enhanced by the study on lattice integrable models [1]. Such exact results provide valuable insights into the key theoretical development of universality classes in areas ranging from modern condensed physics [2, 3] to string and super-symmetric Yang-Mills theories [4, 5, 6]. Among solvable models [1, 7, 8], elliptic ones stand out as a particularly important class due to the fact that most others can be reduced to from them by taking trigonometric or rational limits. The $\mathbb{Z}_n$-Belavin model [9] is a typical elliptic quantum integrable model, with the celebrated XYZ spin chain as the special case of $n = 2$.

The first exact solution of the $\mathbb{Z}_2$-model with periodic boundary condition was given by Baxter [10], where the fundamental equation (the Yang-Baxter equation [1, 11]) was emphasized and the $T-Q$ method was proposed. Takhtadzhan and Faddeev [12] resolved the model with the algebraic Bethe Ansatz method [7, 13]. By employing the intertwiners vectors [14] which constitute the face-vertex correspondence between the $\mathbb{Z}_n$-Belavin model and the associated face model, Hou et al [15] generalized Takhatadzhan and Faddeev’s approach to the $\mathbb{Z}_n$-Belavin model with a generic $n$. In their approach, local gauge transformation played a central role to obtain local vacuum states (reference states) with which the algebraic Bethe Ansatz analysis can be performed. However, such reference states are so far only available for some very particular number of lattice sites, namely, $N = nl$ with $l$ being a positive integer, but not for the other $N$. This leads to the fact that the conventional Bethe Ansatz methods have been quite hard to apply to the latter case for many years. In fact, the lack of a reference state is a common feature of the integrable models without $U(1)$ symmetry and had been a very important and difficult issue in the field of quantum integrable models.

Recently, a systematic method, i.e., the off-diagonal Bethe Ansatz (ODBA) [16, 17] was proposed to solve the eigenvalue problem of integrable models without $U(1)$-symmetry. The closed XYZ spin chain (or the $\mathbb{Z}_2$-model) with arbitrary number of sites [18] and several other long-standing models [16, 19, 20, 21] have since been solved. In this paper, we adopt ODBA to solve the eigenvalue problem of the periodic $\mathbb{Z}_n$-Belavin model with a generic positive integer $n \geq 2$ and an arbitrary lattice number $N$.

The paper is organized as follows. Section 2 serves as an introduction of our notations and some basic ingredients. The commuting transfer matrix associated with the periodic
\(Z_n\)-Belavin model is constructed to show the integrability of the model. In section 3, based on some intrinsic properties of the \(Z_n\)-Belavin’s \(R\)-matrix, we construct the fused transfer matrices by anti-symmetric fusion procedure and derive some operator identities and the quasi-periodicities of these matrices. Taking the \(Z_3\) model as a concrete example, we express the eigenvalues of the transfer matrix in terms of a nested inhomogeneous \(T - Q\) relation and the associated Bethe Ansatz equations (BAEs) in Section 4. Generalization to \(Z_n\) case is presented in Section 5. We summarize our results and give some discussions in Section 6. A slightly detailed description about the \(Z_4\) case, which might be crucial to understand the procedure for \(n \geq 4\), is given in Appendix A. In addition, we discuss the ODBA solution of \(Z_n\)-Belavin model with twisted boundary condition in Appendix B.

2 \(Z_n\)-Belavin model with periodic boundary condition

Let us fix a positive integer \(n \geq 2\), a complex number \(\tau\) such that \(\text{Im}(\tau) > 0\) and a generic complex number \(w\). For convenience, let us introduce the elliptic functions

\[
\theta \left[ \begin{array}{c} a \\ b \end{array} \right] (u, \tau) = \sum_{m=-\infty}^{\infty} \exp \left\{ \sqrt{-1} \pi \left[ (m+a)^2 \tau + 2(m+a)(u+b) \right] \right\}, \quad (2.1)
\]

\[
\theta^{(j)}(u) = \theta \left[ \begin{array}{c} \frac{1}{2} - \frac{j}{n} \\ \frac{1}{2} \end{array} \right] (u, n\tau), \quad \sigma(u) = \theta \left[ \begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (u, \tau), \quad (2.2)
\]

\[
\zeta(u) = \frac{\partial}{\partial u} \{ \ln \sigma(u) \}.
\]

Among them the \(\sigma\)-function\(^1\) satisfies the following identity:

\[
\sigma(u + x)\sigma(u - x)\sigma(v + y)\sigma(v - y) - \sigma(u + y)\sigma(u - y)\sigma(v + x)\sigma(v - x) = \sigma(u + v)\sigma(u - v)\sigma(x + y)\sigma(x - y).
\]

Let \(V\) denote an \(n\)-dimensional linear space with an orthonormal basis \(\{|i\}|i = 1, \cdots, n\}\}, and \(g, h\) be two \(n \times n\) matrices with the elements

\[
h_{ij} = \delta_{i+1,j}, \quad g_{ij} = \omega_n^i \delta_{i,j}, \quad \text{with} \quad \omega_n = e^{\frac{2\pi i}{n}}, \quad i, j \in \mathbb{Z}_n.
\]

\(^1\)Our \(\sigma\)-function is the \(\vartheta\)-function \(\vartheta_1(u)\) \[22\]. It has the following relation with the \textit{Weierstrassian} \(\sigma\)-function denoted by \(\sigma_w(u)\): \(\sigma_w(u) \propto e^{nw^2} \sigma(u), \quad \eta_1 = \pi^2(\frac{1}{6} - 4 \sum_{n=1}^{\infty} \frac{w^{2n}}{4n^2})\) and \(q = e^{\sqrt{-1} \tau}\).
namely,

\[
g = \begin{pmatrix}
1 & \omega_n & \omega_n^2 & \cdots & \omega_n^{n-1} \\
\omega_n & 1 & \omega_n & \cdots & \omega_n^{n-2} \\
\omega_n^2 & \omega_n & 1 & \cdots & \omega_n^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\omega_n^{n-1} & \omega_n^{n-2} & \omega_n^{n-3} & \cdots & 1
\end{pmatrix},
\]

\[
h = \begin{pmatrix}
1 & \vdots & \vdots & \vdots & \vdots \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & 1 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & 1
\end{pmatrix}.
\]

(2.3)

It is easy to verify that the matrices satisfy the relation

\[
gh = \omega_n^{-1} hg.
\]

(2.4)

Associated with any \(\alpha = (\alpha_1, \alpha_2), \alpha_1, \alpha_2 \in \mathbb{Z}_n\), one can introduce an \(n \times n\) matrix \(I_\alpha\) defined by

\[
I_\alpha = I_{(\alpha_1, \alpha_2)} = g^{\alpha_2} h^{\alpha_1},
\]

(2.5)

and an elliptic function \(\sigma_\alpha(u)\) given by

\[
\sigma_\alpha(u) = \theta \left[ \frac{\frac{1}{2} + \frac{\alpha_1}{n}}{\frac{1}{2} + \frac{\alpha_2}{n}} \right] (u, \tau), \quad \text{and} \quad \sigma_{(0,0)}(u) = \sigma(u).
\]

The \(\mathbb{Z}_n\)-Belavin R-matrix \(R(u) \in \text{End}(V \otimes V)\) is given by \([9, 23, 14]\)

\[
R(u) = \sum_{\alpha \in \mathbb{Z}_n^2} \frac{\sigma_\alpha(u + \frac{w}{n})}{n\sigma_\alpha(w/n)} I_\alpha \otimes I_{-1}\alpha,
\]

(2.6)

which satisfies the quantum Yang-Baxter equation (QYBE)

\[
R_{12}(u_1 - u_2) R_{13}(u_1 - u_3) R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3) R_{13}(u_1 - u_3) R_{12}(u_1 - u_2),
\]

(2.7)

and the properties \([23]\).

Initial condition :

\[
R_{12}(0) = P_{1,2},
\]

(2.8)

Unitarity :

\[
R_{12}(u) R_{21}(-u) = \frac{\sigma(w + u) \sigma(w - u)}{\sigma(w) \sigma(w)} \times \text{id},
\]

(2.9)

Crossing-unitarity :

\[
R_{21}^t(-u - nw) R_{12}^t(u) = \frac{\sigma(u) \sigma(-u - nw)}{\sigma(w) \sigma(w)} \times \text{id},
\]

(2.10)

\(Z_n\)-symmetry :

\[
g_1 g_2 R_{12}(u) g_1^{-1} g_2^{-1} = R_{12}(u), \quad h_1 h_2 R_{12}(u) h_1^{-1} h_2^{-1} = R_{12}(u),
\]

(2.11)

Fusion conditions :

\[
R_{12}(-w) = P_{1,2}^{-1} S_{12}^{(-)}, \quad R_{12}(w) = S_{12}^{(+)} P_{1,2}^{-1}.
\]

(2.12)
Here $R_{21}(u) = P_{1,2}R_{12}(u)P_{1,2}$ with $P_{1,2}$ being the usual permutation operator, $P^{(\pm)}_{1,2} = \frac{1}{2}(1 \pm P_{1,2})$ is anti-symmetric (symmetric) project operator in the tensor product space $V \otimes V$, $S^{(\pm)}_{12}$ are some non-degenerate matrices $\in \text{End}(V \otimes V)$ and $t_i$ denotes the transposition in the $i$-th space. Here and below we adopt the standard notation: for any matrix $A \in \text{End}(V)$, $A_j$ is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as $A$ on the $j$-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the $i$-th and $j$-th ones.

As usual, the corresponding “row-to-row” monodromy matrix $T(u)$ \textsuperscript{7}, an $n \times n$ matrix with operator-valued elements acting on $(V) \otimes^N$ reads

$$ T_0(u) = R_{0N}(u - \theta_N)R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1). \quad (2.13) $$

Here $\{\theta_i| i = 1, \cdots, N\}$ are arbitrary free complex parameters which are usually called the inhomogeneous parameters. With the help of the QYBE (2.7), one can show that $T(u)$ satisfies the Yang-Baxter algebra relation

$$ R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \quad (2.14) $$

Let us introduce the transfer matrix $t(u)$

$$ t(u) = tr_0(T_0(u)) = tr(T(u)). \quad (2.15) $$

The $Z_n$-Belavin model \textsuperscript{9} with periodic boundary condition is a quantum spin chain described by the Hamiltonian

$$ H = \frac{\partial}{\partial u} \{ \ln t(u) \}_{u=0,\{\theta_i\}=0} - \text{Constant} = \sum_{i=1}^{N} H_{i,i+1}, \quad (2.16) $$

where the local Hamiltonian $H_{i,i+1}$ is

$$ H_{i,i+1} = \frac{\partial}{\partial u} \{ P_{i,i+1}R_{i,i+1}(u) \}_{u=0}, \quad (2.17) $$

with the periodic boundary condition, namely,

$$ H_{N,N+1} = H_{N,1}. \quad (2.18) $$

The commutativity of the transfer matrices

$$ [t(u), t(v)] = 0, $$

follows as a consequence of (2.14). This ensures the integrability of the inhomogeneous $Z_n$-Belavin model with periodic boundary.
3 Relations of the eigenvalues

Following the method developed in [19] (see also Chapter 7 of [17]), we apply the fusion techniques [24, 25, 26, 27] to study the $\mathbb{Z}_n$-Belavin model. Besides the fundamental transfer matrix $t(u)$ some other fused transfer matrices $\{t_j(u)\} \ (j = 1, \cdots, n)$ (see below (3.8)), which commute with each other and include the original one as $t_1(u) = t(u)$, are constructed through an anti-symmetric fusion procedure with the help of the fusion condition (2.12) of the $R$-matrix.

3.1 Operator product identities

The quasi-periodicity of the $\sigma$-function

\[
\sigma(u + 1) = -\sigma(u), \quad \sigma(u + \tau) = -e^{-2i\pi(u+\frac{\tau}{2})}\sigma(u),
\]

indicates that the $R$-matrix $R(u)$ given by (2.6) possesses the quasi-periodic properties

\[
R_{12}(u + 1) = -g_1^{-1} R_{12}(u) g_1 = -g_2 R_{12}(u) g_2^{-1},
\]

\[
R_{12}(u + \tau) = -e^{-2i\pi(u+\frac{\tau}{2})} h_1^{-1} R_{12}(u) h_1 = -e^{-2i\pi(u+\frac{\tau}{2})} h_2 R_{12}(u) h_2^{-1},
\]

which lead to the quasi-periodicity of the transfer matrix $t(u)$ given by (2.15)

\[
t(u + 1) = (-1)^N t(u),
\]

\[
t(u + \tau) = (-1)^N e^{-2i\pi(N(u+\frac{\tau}{2})-\sum_{i=1}^N \theta_i)} t(u).
\]

Let us introduce the usual (or non-deformed) anti-symmetric projectors $\{P_{1,\cdots,m}^{(-)} | m = 2, \cdots, n\}$ in a tensor space of $V$ defined by the induction relations

\[
P_{1,\cdots,m+1}^{(-)} = \frac{1}{m+1} \left(1 - \sum_{j=2}^{m+1} P_{1,j}^{(-)}\right) P_{2,\cdots,m+1}^{(-)}, \quad m = 2, \cdots, n-1.
\]

Iterating the above relation yields alternative definition of the projectors

\[
P_{1,\cdots,m}^{(-)} = \frac{1}{m!} \sum_{\kappa \in S_m} (-1)^{\text{sign}(\kappa)} P_\kappa, \quad m = 2, \cdots, n,
\]

where $S_m$ is the permutation group of $m$ indices, $P_\kappa$ is a permutation in the group, and $\text{sign}(\kappa)$ is 0 for an even permutation $\kappa$ and 1 for an odd permutation. With the above anti-symmetric projectors, we can construct the fused monodromy matrices

\[
T_{1,\cdots,m}(u) = P_{1,\cdots,m}^{(-)} T_1(u) \cdots T(u - (m-1)w) P_{1,\cdots,m}^{(-)}, \quad m = 2, \cdots, n.
\]
The corresponding fused transfer matrices \( \{ t_j(u) \mid j = 1, \cdots, n \} \) (including the original one as \( t_1(u) = t(u) \)) are then given by

\[
t_m(u) = tr_{1,\cdots,m} \{ T_{(1,\cdots,m)}(u) \}, \quad m = 2, \cdots, n.
\] (3.8)

The last fused transfer matrix \( t_n(u) \) is the so-called quantum determinant \([28]\) which plays the role of the generating functional of the centers of the associated quantum algebras \([29]\). For generic values of \( \{ \theta_j \} \), \( t_n(u) \) is proportional to the identity operator, namely,

\[
t_n(u) = \text{Det}_q(T(u)) \times \text{id}, \quad \text{Det}_q(T(u)) = \prod_{l=1}^N \frac{\sigma(u - \theta_l + w)}{\sigma(w)} \prod_{k=1}^{n-1} \frac{\sigma(u - \theta_l - kw)}{\sigma(w)}. \] (3.9)

The QYBE \((2.7)\), the fusion condition \((2.12)\) and the relations \((2.14)\) imply that these fused matrices commute with each other,

\[
[t_i(u), t_j(v)] = 0, \quad i, j = 1, \cdots, n.
\] (3.10)

Using the method (see Chapter 7.2 of \([17]\)), we can show the relations

\[
T_1(\theta_j) T_{(2,\cdots,m)}(\theta_j - w) = P_{1,\cdots,m}^{(-)} T_1(\theta_j) T_{(2,\cdots,m)}(\theta_j - w), \quad m = 2, \cdots, n; \quad j = 1, \cdots, N,
\]

which immediately lead to the following recursive relations

\[
t(\theta_j) t_m(\theta_j - w) = t_{m+1}(\theta_j), \quad m = 1, \cdots, n-1; \quad j = 1, \cdots, N.
\] (3.11)

Moreover, the fusion condition \((2.12)\) and the fact that \( P_{12}^{(-)} P_{12}^{(+)} = 0 = P_{12}^{(+)} P_{12}^{(-)} \) enable us to derive some zeros for the fused matrices,

\[
t_m(\theta_j + kw) = 0, \quad j = 1, \cdots, N; \quad k = 1, \cdots, m-1; \quad m = 2, \cdots, n.
\] (3.12)

Similarly as deriving the relations \((3.4)-(3.5)\), we have the fused transfer matrices enjoy the following periodicity,

\[
t_m(u + 1) = (-1)^m N t_m(u), \quad m = 1, \cdots, n,
\] (3.13)

\[
t_m(u + \tau) = (-1)^m N e^{-2i\pi m N u + \frac{\pi}{4} + \frac{\pi}{2} - \frac{m-1}{2} w - m \sum_{l=1}^N \theta_l} t_m(u), \quad m = 1, \cdots, n.
\] (3.14)

Let us evaluate the transfer matrix of the closed chain at some special points. The initial condition of the \( R \)-matrix \((2.6)\) implies that

\[
t(\theta_j) = R_{j,j-1}(\theta_j - \theta_{j-1}) \cdots R_{j,1}(\theta_j - \theta_1) R_{j,N}(\theta_j - \theta_N) \cdots R_{j,j+1}(\theta_j - \theta_{j+1}).
\]

The unitarity relation \((2.9)\) allows us to derive the following identity:

\[
\prod_{l=1}^N t(\theta_l) = \prod_{l=1}^N a(\theta_l) \times \text{id}, \quad a(u) = \prod_{l=1}^N \frac{\sigma(u - \theta_l + w)}{\sigma(w)}, \quad d(u) = a(u - w).
\] (3.15)
3.2 Functional relations of eigenvalues

The commutativity (3.10) of the transfer matrices \( \{t_m(u)|m = 1, \cdots, n\} \) with different spectral parameters implies that they have common eigenstates. Let \( |\Psi\rangle \) be a common eigenstate of \( \{t_m(u)\} \), which does not depend upon \( u \), with the eigenvalues \( \Lambda_m(u) \) (we shall take the convention: \( \Lambda(u) = \Lambda_1(u) \)),

\[
t_m(u)|\Psi\rangle = \Lambda_m(u)|\Psi\rangle, \quad m = 1, \cdots n.
\]

The properties (3.9), (3.11) and (3.12) of the transfer matrices \( \{t_m(u)|m = 1, \cdots, n\} \) imply that the corresponding eigenvalues \( \{\Lambda_m(u)|m = 1, \cdots, n\} \) satisfy the functional relations

\[
\Lambda(\theta_j) \Lambda_m(\theta_j - w) = \Lambda_{m+1}(\theta_j), \quad m = 1, \cdots, n - 1; \quad j = 1, \cdots, N; \quad (3.16)
\]

\[
\Lambda_m(\theta_j + kw) = 0, \quad j = 1, \cdots, N; \quad k = 1, \cdots, m - 1; \quad m = 2, \cdots, n - 1; \quad (3.17)
\]

\[
\Lambda_n(u) = \text{Det}_q(T(u)) = a(u) \prod_{k=1}^{n-1} d(u - kw), \quad (3.18)
\]

where the functions \( a(u) \) and \( d(u) \) are given by (3.15). From the definitions (2.6), (2.15) and (3.8) of the \( R \)-matrix \( R(u) \) and the associated transfer matrices \( \{t_m(u)|m = 1, \cdots, n\} \), we have that \( \Lambda_m(u) \), as a function of \( u \), is an elliptical polynomial of degree \( mN, m = 1, \cdots, n-1 \). (3.19)

The periodicity (3.13)-(3.14) of these transfer matrices imply that the eigenvalues \( \{\Lambda_m(u)\} \) are some elliptic polynomials of the fixed degrees (3.19) with the periodicity

\[
\Lambda_m(u + 1) = (-1)^{mN} \Lambda_m(u), \quad m = 1, \cdots, n - 1; \quad (3.20)
\]

\[
\Lambda_m(u + \tau) = (-1)^{mN} e^{-2i\pi\{mN(u + \frac{\tau}{n} + \frac{w}{2}) - m \sum_{i=1}^{N} \theta_i\}} \Lambda_m(u), \quad m = 1, \cdots, n - 1. \quad (3.21)
\]

Moreover the product identity (3.15) of the transfer matrix \( t(u) \) leads to the relation

\[
\prod_{i=1}^{N} \Lambda(\theta_i) = \prod_{i=1}^{N} a(\theta_i), \quad (3.22)
\]

which serves as the selection rule [18] for the eigenvalues of the transfer matrix from the solutions of (3.16)-(3.21)

The relations (3.16)-(3.22) allow us to determine the eigenvalues \( \{\Lambda_m(u)\} \) of the transfer matrices \( \{t_m(u)\} \) completely.
4 ODBA solution of the $Z_3$ case

Similarly as that [18] for the eight-vertex model (the $Z_2$-case), we demonstrate that (3.16)-(3.22) enable us to express the eigenvalues $\{\Lambda_m(u)\}$ of the transfer matrices $\{t_m(u)\}$ simultaneously in terms of some inhomogeneous $T - Q$ relations [17].

For the $Z_3$-Belavin model, the corresponding (3.16)-(3.22) read

\begin{align*}
\Lambda(\theta_j)\Lambda_m(\theta_j - w) &= \Lambda_{m+1}(\theta_j), \quad m = 1, 2, \quad j = 1, \cdots, N, \quad (4.1) \\
\Lambda_3(u) &= a(u) d(u - w) d(u - 2w), \quad (4.2) \\
\Lambda_2(\theta_j + w) &= 0, \quad j = 1, \cdots, N. \quad (4.3) \\
\Lambda_m(u + 1) &= (-1)^{mN} \Lambda_m(u), \quad m = 1, 2, \quad (4.4) \\
\Lambda_m(u + \tau) &= (-1)^{mN} e^{-2i\pi(mN u + \frac{1}{8} + \frac{1}{8} - \frac{m+1}{2} w - m \sum_{l=1}^{N} \theta_l)} \Lambda_m(u), \quad m = 1, 2. \quad (4.5) \\
\prod_{l=1}^{N} \Lambda(\theta_l) &= \prod_{l=1}^{N} a(\theta_l). \quad (4.6)
\end{align*}

Keeping the fact that $\Lambda_m(u)$ is an elliptical polynomial of degree $mN$ in mind, we can express $\Lambda(u)$ and $\Lambda_2(u)$ in terms of the inhomogeneous $T - Q$ relation [18] as follows. Let us introduce some $Q$-functions

\begin{equation}
Q^{(i)}(u) = \prod_{j=1}^{N} \sigma(u - \lambda_j^{(i)}) \sigma(w), \quad i = 1, \cdots, 4, \quad (4.7)
\end{equation}

parameterized by $4N$ parameters $\{\lambda_j^{(i)}|j = 1, \cdots, N; i = 1, \cdots, 4\}$ (the so-called Bethe roots) determined later by the associated BAEs (see below (4.15)-(4.23)). Associated with the above $Q$-functions, we introduce 5 functions $\{Z_i(u)|i = 1, 2, 3\}$ and $\{X_i(u)|i = 1, 2\}$ as

\begin{align*}
Z_1(u) &= a(u) e^{2i\pi l_1 u + \phi_1} \frac{Q^{(1)}(u - w)}{Q^{(2)}(u)}, \quad (4.8) \\
Z_2(u) &= d(u) e^{2i\pi l_2 u + \phi_2} \frac{Q^{(2)}(u + w) Q^{(3)}(u - w)}{Q^{(1)}(u) Q^{(4)}(u)}, \quad (4.9) \\
Z_3(u) &= d(u) e^{-2i\pi (l_1 + l_2) u + (l_1 + l_2 w) - \phi_1 - \phi_2} \frac{Q^{(4)}(u + w)}{Q^{(3)}(u)}, \quad (4.10) \\
X_1(u) &= c_1 a(u) d(u) e^{2i\pi l_3 u} \frac{Q^{(3)}(u - w)}{Q^{(1)}(u) Q^{(2)}(u)}, \quad (4.11) \\
X_2(u) &= c_2 a(u) d(u) e^{2i\pi l_4 u} \frac{Q^{(2)}(u + w)}{Q^{(3)}(u) Q^{(4)}(u)}, \quad (4.12)
\end{align*}
where \( \{ t_i \mid i = 1, \cdots, 4 \} \) are 4 integers, \( \{ \phi_i, c_i \mid i = 1, 2 \} \) are 4 complex numbers. Then we can introduce the inhomogeneous \( T - Q \) relations,

\[
\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + X_1(u) + X_2(u), \tag{4.13}
\]

\[
\Lambda_2(u) = Z_1(u)Z_2(u - w) + Z_1(u)Z_3(u - w) + Z_2(u)Z_3(u - w) + X_1(u)Z_3(u - w) + Z_1(u)X_2(u - w). \tag{4.14}
\]

In order that the above parameterizations of \( \Lambda(u) \) and \( \Lambda_2(u) \) become a solution to \( (4.1) - (4.6) \), the \( 4(N + 1) \) parameters \( \{ \lambda_j^{(i)} \mid j = 1, \cdots, N; i = 1, \cdots, 4 \} \) and \( \{ \phi_i, c_i \mid i = 1, 2 \} \) have to satisfy the associated BAEs

\[
e^{2i\pi l_1 \lambda_j^{(1)} + \phi_2} Q^{(2)}(\lambda_j^{(1)} + w)Q^{(2)}(\lambda_j^{(1)}) + c_1 e^{2i\pi l_3 \lambda_j^{(1)}} a(\lambda_j^{(1)})Q^{(4)}(\lambda_j^{(1)}) = 0, \quad j = 1, \cdots, N; \tag{4.15}
\]

\[
e^{2i\pi l_1 \lambda_j^{(2)} + \phi_1} Q^{(1)}(\lambda_j^{(2)} - w)Q^{(1)}(\lambda_j^{(2)}) + c_1 e^{2i\pi l_3 \lambda_j^{(2)}} d(\lambda_j^{(2)})Q^{(3)}(\lambda_j^{(2)} - w) = 0, \quad j = 1, \cdots, N; \tag{4.16}
\]

\[
e^{-2i\pi (l_1 + l_2) \lambda_j^{(3)} + (2l_1 + l_2)w - \phi_1 - \phi_2} Q^{(4)}(\lambda_j^{(3)} + w)Q^{(4)}(\lambda_j^{(3)}) + c_2 e^{2i\pi l_4 \lambda_j^{(3)}} a(\lambda_j^{(3)})Q^{(2)}(\lambda_j^{(3)} + w) = 0, \quad j = 1, \cdots, N; \tag{4.17}
\]

\[
e^{2i\pi l_2 \lambda_j^{(4)} + \phi_2} Q^{(3)}(\lambda_j^{(4)} - w)Q^{(3)}(\lambda_j^{(4)}) + c_2 e^{2i\pi l_4 \lambda_j^{(4)}} a(\lambda_j^{(4)})Q^{(1)}(\lambda_j^{(4)}) = 0, \quad j = 1, \cdots, N; \tag{4.18}
\]

\[
-\Theta^{(1)} + \Theta^{(2)} - \frac{N}{3} w = m_1 + l_1 \tau, \tag{4.19}
\]

\[
-\Theta^{(2)} - \Theta^{(3)} + \Theta^{(1)} + \Theta^{(4)} - \frac{N}{3} w = m_2 + l_2 \tau, \tag{4.20}
\]

\[
-\Theta - \Theta^{(3)} + \Theta^{(1)} + \Theta^{(2)} - \frac{N}{3} w = m_3 + l_3 \tau, \tag{4.21}
\]

\[
-\Theta - \Theta^{(2)} + \Theta^{(3)} + \Theta^{(4)} + \frac{5N}{3} w = m_4 + l_4 \tau, \tag{4.22}
\]

\[
\prod_{j=1}^{N} \frac{Q^{(1)}(\theta_j - w)}{Q^{(2)}(\theta_j)} = e^{-2i\pi l_1 \Theta - N\phi_1}, \tag{4.23}
\]

where \( \{ m_i \mid i = 1, \cdots, 4 \} \) are 4 integers and

\[
\Theta = \sum_{l=1}^{N} \theta_l, \quad \Theta^{(i)} = \sum_{l=1}^{N} \lambda_l^{(i)}, \quad i = 1, \cdots, 4. \tag{4.24}
\]

We have checked that the functions \( \Lambda(u) \) and \( \Lambda_2(u) \) given by the inhomogeneous \( T - Q \) relations \( (4.13) - (4.14) \) are solutions to \( (4.1) - (4.6) \) provided that the \( 4(N + 1) \) parameters
\{\lambda^{(i)}_{j} \mid j = 1, \cdots, N; i = 1, \cdots, 4\} \) and \(\phi_1, \phi_2, c_1\) and \(c_2\) satisfy the associated BAEs \((4.15)-(4.23)\) for arbitrary fixed integers \(\{l_i, m_i \mid i = 1, \cdots, 4\}\). Therefore the corresponding \(\Lambda(u)\) becomes an eigenvalue of the transfer matrix \(t(u)\) given by \((2.15)\). In the homogeneous limit: \(\{\theta_j \to 0\}\), the resulting \(T - Q\) relation \((4.13)\) and the associated BAEs \((4.15)-(4.23)\) give rise to the eigenvalue and BAEs of the corresponding homogeneous spin chain (i.e., the \(Z_3\)-Belavin model with periodic boundary condition described by the Hamiltonian \((2.16)\) for the case of \(n = 3\)).

Some remarks are in order. The integers \(\{l_i, m_i \mid i = 1, \cdots, 4\}\) appeared in the BAEs \((4.19)-(4.22)\) are due to the quasi-periodicity \((3.2)-(3.3)\) of the \(R\)-matrix in terms of the spectral parameter \(u\). Any choice of these integers may give rise to the complete set of eigenvalues \(\Lambda(u)\). Numerical solutions of the BAEs \((4.15)-(4.23)\) with random choice of \(w\) and \(\tau\) for some small size imply that the solution \((4.13)\) indeed gives the complete solutions of the model. Here we present the numerical solutions of the BAEs for the \(N = 2\) case in Table 1. The eigenvalue calculated from \((4.13)\) is the same as that from the exact diagonalization of the Hamiltonian \((2.16)\) with periodic boundary condition \((2.18)\). Moreover, for a generic \(w\) and an arbitrary site number \(N\), the eigenvalue \(\Lambda(u)\) should be given by an inhomogeneous \(T - Q\) relation such as \((4.13)\) with non-vanishing terms related to \(X_i(u)\). However, when \(N\) is some particular number (i.e., \(N = 3l\) for a positive integer \(l\)) or the crossing parameter \(w\) takes some particular values (i.e., see below \((4.33)-(4.34)\) ), the relation \((4.13)\) is reduced to a homogeneous one \([1]\), which corresponds to the \(c_1 = c_2 = 0\) solutions of \((4.15)-(4.23)\).
Table 1: Solutions of BAEs (4.15)-(4.23) for the $Z_3$ case, $N = 2$, $\{\theta_j\} = 0$, $w = -0.5$, $\tau = i$ and the parameters $m_1 = 1$, $m_2 = m_3 = m_4 = l_1 = l_2 = l_3 = l_4 = 0$. The symbol $m$ indicates the number of the eigenenergy $E$.

| $\lambda_1^{(1)}$ | $\lambda_2^{(1)}$ | $\lambda_1^{(2)}$ | $\lambda_2^{(2)}$ | $\lambda_1^{(3)}$ |
|-------------------|-------------------|-------------------|-------------------|-------------------|
| $-1.5000 - 0.2862i$ | $1.5000 + 0.2862i$ | $-0.1667 + 0.1501i$ | $0.8333 - 0.1501i$ | $-0.3053 - 1.0000i$ |
| $-1.5000 + 0.2862i$ | $1.5000 - 0.2862i$ | $-0.1667 - 0.1501i$ | $0.8333 + 0.1501i$ | $-0.3053 + 1.0000i$ |
| $0.2550 + 0.0000i$ | $-0.2550 - 0.0000i$ | $-0.1667 + 0.1501i$ | $0.8333 - 0.1501i$ | $0.7205 - 0.5000i$ |
| $-0.2795 + 0.5000i$ | $0.2795 + 0.5000i$ | $0.5667 + 0.5000i$ | $0.1034 + 0.5000i$ | $0.2550 + 1.0000i$ |
| $-0.2795 - 0.5000i$ | $0.2795 + 0.5000i$ | $0.5667 - 0.5000i$ | $0.1034 - 0.5000i$ | $0.2550 - 1.0000i$ |
| $0.7601 + 0.5000i$ | $-0.7601 + 0.5000i$ | $0.5667 - 0.5000i$ | $0.1034 + 0.5000i$ | $1.7517 - 0.0000i$ |
| $0.3053 + 0.0000i$ | $-0.3053 + 0.0000i$ | $-0.3053 + 0.0000i$ | $0.5000 - 0.0000i$ | $-0.5000 + 0.2862i$ |
| $0.3053 - 0.0000i$ | $-0.3053 - 0.0000i$ | $-0.3053 + 0.0000i$ | $0.5000 - 0.0000i$ | $0.5000 - 0.2862i$ |

| $\lambda_2^{(3)}$ | $\lambda_1^{(4)}$ | $\lambda_2^{(4)}$ | $\phi_1$ | $\phi_2$ |
|-------------------|-------------------|-------------------|----------|----------|
| $1.3053 - 1.0000i$ | $0.3333 + 0.0000i$ | $0.3333 - 0.0000i$ | $1.0000 + 0.0000i$ | $-0.7466 + 6.2832i$ |
| $1.3053 + 1.0000i$ | $0.3333 + 0.0000i$ | $0.3333 - 0.0000i$ | $1.0000 - 0.0000i$ | $-0.7466 - 6.2832i$ |
| $0.2795 + 0.5000i$ | $0.2795 + 0.5000i$ | $0.2795 - 0.5000i$ | $0.0054 + 3.1416i$ | $0.3020 - 6.2239i$ |
| $0.7450 - 1.0000i$ | $0.6667 - 0.8499i$ | $0.6667 + 0.8499i$ | $0.0374 + 0.3110i$ | $-1.9185 + 5.7091i$ |
| $0.7450 + 1.0000i$ | $0.6667 - 0.8499i$ | $0.6667 + 0.8499i$ | $0.0374 - 0.3110i$ | $-1.9185 - 5.7091i$ |
| $-0.7517 + 0.0000i$ | $-0.6667 + 0.0000i$ | $2.0000 - 0.0000i$ | $0.0055 + 15.3832i$ | $0.0035 + 12.7554i$ |
| $1.5000 + 0.2862i$ | $1.6667 + 0.1501i$ | $-0.3333 + 0.1501i$ | $0.4211 - 3.1416i$ | $-1.1676 + 3.1416i$ |
| $1.5000 - 0.2862i$ | $1.6667 - 0.1501i$ | $-0.3333 - 0.1501i$ | $0.4211 + 3.1416i$ | $-1.1676 - 3.1416i$ |
| $0.5000 + 0.2862i$ | $1.6667 + 0.1501i$ | $-0.3333 - 0.1501i$ | $-5.8621 - 0.6954i$ | $5.1156 + 0.6954i$ |

| $c_1$ | $c_2$ | $E$ | $m$ |
|-------|-------|-----|-----|
| $-0.0069 - 0.0057i$ | $-138.0911 + 115.2344i$ | $-5.619523$ | $1$ |
| $-0.0069 + 0.0057i$ | $-138.0911 - 115.2344i$ | $-5.619523$ | $1$ |
| $0.0243 - 0.1299i$ | $-2.7217 + 0.8747i$ | $-5.619523$ | $1$ |
| $-0.0053 - 0.0007i$ | $-346.9166 + 5524.2128i$ | $0.157726$ | $2$ |
| $-0.0053 + 0.0007i$ | $-346.9166 - 5524.2128i$ | $0.157726$ | $2$ |
| $-2.8680 + 0.1801i$ | $-0.0054 + 0.0427i$ | $0.157726$ | $2$ |
| $-0.1592 + 0.0000i$ | $3.1476 + 0.0000i$ | $5.461797$ | $3$ |
| $-0.1592 + 0.0000i$ | $3.1476 + 0.0000i$ | $5.461797$ | $3$ |
4.1 Generic $w$ and $\tau$ case

It follows from (4.15)-(4.18) that for the solution with $c_1 = c_2 = 0$, the parameters $\{\lambda_j^{(i)}\}$ have to form the pairs:

$$\begin{cases} 
\lambda_j^{(1)} = \lambda_k^{(2)}, & \text{or} \quad \lambda_j^{(1)} = \lambda_k^{(2)} - w, \\
\lambda_j^{(3)} = \lambda_k^{(4)}, & \text{or} \quad \lambda_j^{(3)} = \lambda_k^{(4)} - w.
\end{cases} \quad (4.25)$$

Without losing generality, let us suppose that

$$\lambda_j^{(1)} = \lambda_j^{(2)} \overset{\text{Redef}}{=} \bar{\lambda}_j^{(1)}, \quad j = 1, \ldots, \bar{M}_1,$$

$$\lambda_{M_1+j}^{(1)} = \lambda_{M_1+j}^{(2)} - w \overset{\text{Redef}}{=} \bar{\lambda}_{M_1+j}^{(1)}, \quad j = 1, \ldots, N - \bar{M}_1,$$

$$\lambda_j^{(3)} = \lambda_j^{(4)} \overset{\text{Redef}}{=} \bar{\lambda}_j^{(2)}, \quad j = 1, \ldots, \bar{M}_2,$$

$$\lambda_{M_2+j}^{(3)} = \lambda_{M_2+j}^{(4)} - w \overset{\text{Redef}}{=} \bar{\lambda}_{M_2+j}^{(2)}, \quad j = 1, \ldots, N - \bar{M}_2,$$

where $\bar{M}_1$ and $\bar{M}_2$ are two non-negative integers. The corresponding $T - Q$ relation (4.13) is reduced to

$$\Lambda(u) = a(u)e^{2\pi i l_1 u + \phi_1} \frac{\tilde{Q}^{(1)}(u - w)}{Q^{(1)}(u)} + d(u)e^{2\pi i l_2 u + \phi_2} \frac{\tilde{Q}^{(1)}(u + w)\tilde{Q}^{(2)}(u - w)}{Q^{(1)}(u)Q^{(2)}(u)}$$

$$+ d(u)e^{-2\pi i (l_1 + l_2)u + (2l_1 + l_2)w - \phi_1 - \phi_2} \frac{\tilde{Q}^{(2)}(u + w)}{Q^{(2)}(u)},$$

where the reduced $Q$-functions are given by

$$\tilde{Q}^{(i)}(u) = \prod_{j=1}^{\bar{M}_i} \frac{\sigma(u - \bar{\lambda}_j^{(i)})}{\sigma(w)}, \quad i = 1, 2, \quad (4.27)$$

provided that the two non-negative integers $\bar{M}_1$ and $\bar{M}_2$ satisfy the relations

$$\begin{cases} 
(\frac{2}{3}N - \bar{M}_1)w = m_1 + l_1\tau \\
(\bar{M}_1 - \bar{M}_2 - \frac{N}{3})w = m_2 + l_2\tau
\end{cases} \quad (4.28)$$

where $m_1$, $m_2$, $l_1$ and $l_2$ are some integers.

- For the case of $N = 3l$ with a positive integer $l$. The only solution to (4.28) is

$$m_1 = m_2 = l_1 = l_2 = 0, \quad \text{and} \quad \bar{M}_1 = 2l, \bar{M}_2 = l.$$
The resulting $T - Q$ relation becomes

$$
\Lambda(u) = a(u)e^{\phi_1} \frac{\bar{Q}^{(1)}(u - w)}{Q^{(1)}(u)} + d(u)e^{\phi_2} \frac{\bar{Q}^{(1)}(u + w)\bar{Q}^{(2)}(u - w)}{Q^{(1)}(u)Q^{(2)}(u)}
$$

$$
+d(u)e^{-\phi_1 - \phi_2} \frac{\bar{Q}^{(2)}(u + w)}{Q^{(2)}(u)}, \quad (4.29)
$$

the $3l$ parameters $\{\bar{\lambda}^{(1)}_j | j = 1, \cdots, 2l\}$ and $\{\bar{\lambda}^{(2)}_j | j = 1, \cdots, l\}$ satisfy the associated BAEs and the selection rule

$$
a(\bar{\lambda}^{(1)}_j) \bar{Q}^{(2)}(\bar{\lambda}^{(1)}_j) e^{\phi_1 - \phi_2} = -\frac{\bar{Q}^{(1)}(\bar{\lambda}^{(1)}_j + w)}{\bar{Q}^{(1)}(\bar{\lambda}^{(1)}_j - w)}, \quad j = 1, \cdots, 2l, \quad (4.30)
$$

$$
e^{\phi_1 + 2\phi_2} \frac{\bar{Q}^{(1)}(\bar{\lambda}^{(2)}_j + w)}{\bar{Q}^{(1)}(\bar{\lambda}^{(2)}_j)} = -\frac{\bar{Q}^{(2)}(\bar{\lambda}^{(2)}_j + w)}{\bar{Q}^{(2)}(\bar{\lambda}^{(2)}_j - w)}, \quad j = 1, \cdots, l, \quad (4.31)
$$

$$
\prod_{j=1}^N \frac{Q^{(1)}(\theta_j - w)}{Q^{(1)}(\theta_j)} = e^{-N\phi_1}. \quad (4.32)
$$

For this case (i.e., $N = 3l$), the algebraic Bethe Ansatz can also be applied to and our results recover those obtained in [15, 30].

- $N \neq 3l$ case. Since $\tau$ and $w$ are generic complex numbers, generally $[4.28]$ can not be satisfied in this case and the eigenvalue $\Lambda(u)$ should be given by an inhomogeneous $T - Q$ relation.

### 4.2 Degenerate $w$ case

For some degenerate values of $w$, $c_1 = c_2 = 0$ solutions indeed exist for an arbitrary site number $N$. In this case, the parameters $w$ and $\tau$ are no longer independent but related with the constraint condition:

$$
w = \frac{3m_1 + 3l_1 \tau}{2N - 3\bar{M}_1}, \quad \text{for} \quad m_1, l_1 \in \mathbb{Z}; \quad \bar{M}_1 \in \mathbb{Z}^+, \quad (4.33)
$$

and there exists an integer $n_1$ such that

$$
\bar{M}_2 = (n_1 + 1)\bar{M}_1 - \frac{2n_1 + 1}{3} N \in \mathbb{Z}^+. \quad (4.34)
$$

In this case the relation $[4.28]$ is fulfilled by

$$
\begin{cases}
(\frac{2}{3}N - \bar{M}_1)w = m_1 + l_1 \tau \\
(\bar{M}_1 - \bar{M}_2 - \frac{N}{3})w = n_1(m_1 + l_1 \tau)
\end{cases}
$$
The resulting $T - Q$ relation becomes
\[
\Lambda(u) = a(u) e^{2\pi l_1 u + \phi_1} \frac{\bar{Q}^{(1)}(u - w)}{Q^{(1)}(u)} + d(u) e^{2\pi n_1 l_1 u + \phi_2} \frac{\bar{Q}^{(1)}(u + w)\bar{Q}^{(2)}(u - w)}{Q^{(1)}(u)Q^{(2)}(u)}
\]
\[+ d(u) e^{-2\pi (n_1 + 1) l_1 u + (n_1 + 2) l_1 w - \phi_1 - \phi_2} \frac{\bar{Q}^{(2)}(u + w)}{Q^{(2)}(u)}.
\] (4.35)

The resulting BAEs and selection rule read
\[
e^{2\pi(1-n_1)\lambda_j^{(1)}} \frac{a(\bar{\lambda}_j^{(1)}) \bar{Q}^{(2)}(\bar{\lambda}_j^{(1)}) e^{\phi_1 - \phi_2}}{d(\bar{\lambda}_j^{(1)}) \bar{Q}^{(2)}(\bar{\lambda}_j^{(1)} - w)} = -\frac{\bar{Q}^{(1)}(\bar{\lambda}_j^{(1)} + w)}{\bar{Q}^{(1)}(\bar{\lambda}_j^{(1)} - w)}, \quad j = 1, \cdots, M_1, \] (4.36)
\[
e^{2\pi(2n_1 + 1)\lambda_j^{(2)} + (n_1 + 2) l_1 w + \phi_1 + 2\phi_2} \frac{\bar{Q}^{(1)}(\bar{\lambda}_j^{(2)} + w)}{\bar{Q}^{(1)}(\bar{\lambda}_j^{(2)})} = -\frac{\bar{Q}^{(2)}(\bar{\lambda}_j^{(2)} + w)}{\bar{Q}^{(2)}(\bar{\lambda}_j^{(2)} - w)}, \quad j = 1, \cdots, M_2, \] (4.37)
\[
\prod_{j=1}^N \frac{Q^{(1)}(\theta_j - w)}{Q^{(1)}(\theta_j)} = e^{-N\phi_1 - 2\pi l_1 \sum_{j=1}^N \theta_j}.
\] (4.38)

5 Results for the $Z_n$ case

In Sections 3, we have obtained the very operator product identities (3.11)-(3.15) for the fused transfer matrices $\{t_j(u)|j = 1, \cdots, n\}$. These identities lead to that the corresponding eigenvalues $\{\Lambda_j(u)|j = 1, \cdots, n\}$ of the transfer matrices satisfy the associated relations (3.16)-(3.22). Similarly as those for the $Z_3$ case, the relations allow us to determine the eigenvalues of the transfer matrix of the $Z_n$-Belavin model completely.

Let us introduce some functions $\{Q^i|i = 1, \cdots, 2n - 2\}$, $\{Z_i|i = 1, \cdots, n\}$ and $\{X_i|i =
1, \cdots, n - 1} \text{ as follows:}

\[ Q^{(i)}(u) = \prod_{j=1}^{N_i} \frac{\sigma(u - \lambda_{j}^{(i)})}{\sigma(w)}, \quad i = 1, \ldots, 2n - 2, \tag{5.1} \]

\[ Z_1(u) = e^{2i\pi l_1 u + \phi_1 a(u)} \frac{Q^{(1)}(u - w)}{Q^{(2)}(u)}, \]

\[ Z_2(u) = e^{2i\pi l_2 u + \phi_2 d(u)} \frac{Q^{(2)}(u + w)Q^{(3)}(u - w)}{Q^{(1)}(u)Q^{(4)}(u)}, \]

\[ \vdots \]

\[ Z_{n-1}(u) = e^{2i\pi l_{n-1} u + \phi_{n-1} d(u)} \frac{Q^{(2n-4)}(u + w)Q^{(2n-3)}(u - w)}{Q^{(2n-5)}(u)Q^{(2n-2)}(u)}, \]

\[ Z_n(u) = e^{-2i\pi \sum_{k=1}^{n-1} l_k(u + (n-k)w) - \sum_{j=1}^{n-1} \phi_j d(u)} \frac{Q^{(2n-2)}(u + w)}{Q^{(2n-3)}(u)}, \tag{5.2} \]

and

\[ X_1(u) = c_1 e^{2i\pi l_n u a(u)} d(u) \frac{Q^{(3)}(u - w)}{Q^{(1)}(u)Q^{(2)}(u)}, \]

\[ X_2(u) = c_2 e^{2i\pi l_{n+1} u a(u)} d(u) \frac{Q^{(2)}(u + w)Q^{(5)}(u - w)}{Q^{(3)}(u)Q^{(4)}(u)}, \]

\[ \vdots \]

\[ X_{j}(u) = c_{j} e^{2i\pi l_{n+j-1} u a(u)} d(u) \frac{Q^{(2j-2)}(u + w)Q^{(2j+1)}(u - w)}{Q^{(2j-1)}(u)Q^{(2j)}(u)}, \]

\[ \vdots \]

\[ X_{n-1}(u) = c_{n-1} e^{2i\pi l_{2n-2} u a(u)} d(u) \frac{Q^{(2n-4)}(u + w)}{Q^{(2n-3)}(u)Q^{(2n-2)}(u)} \tag{5.3} \]

where the 2n - 2 positive integers \( \{N_i| i = 1, \cdots, 2n - 2\} \) are given by \( \{5.3\} = \{5.1\} \), \( \{l_i| i = 1, \cdots, 2n - 2\} \) are arbitrary integers, \( \{\phi_i, c_i| i = 1, \cdots, n - 1\} \) are 2n - 2 complex numbers.

It is remarked that for an even \( n \), an extra factor function \( f_{\frac{n}{2}}(u) \) should be added to the function \( X_{\frac{n}{2}}(u) \), namely,

\[ X_{\frac{n}{2}}(u) = c_{\frac{n}{2}} e^{2i\pi l_{\frac{3n}{2}-1} u} d(u) \frac{Q^{(n-2)}(u + w)Q^{(n+1)}(u - w)}{Q^{(n-1)}(u)Q^{(n)}(u)} \times f_{\frac{n}{2}}(u), \tag{5.4} \]
which ensures that all the numbers \( \{N_i \mid i = 1, \ldots, 2n - 2\} \) are positive integers. The explicit expression of the function \( f_\Phi(u) \) is given by (5.11) (or (5.13)) below.

We are now in position to construct the associated inhomogeneous \( T-Q \) relations similar to those given by (4.13)-(4.14). Let us introduce the functions \( \{Y_l(u) \mid l = 1, \ldots, 2n - 1\} \),

\[
\begin{cases}
Y_{2j-1}(u) = Z_j(u), & j = 1, \cdots, n, \\
Y_{2j}(u) = X_j(u), & j = 1, \cdots, n - 1.
\end{cases}
\] (5.5)

We further take the notation:

\[
Y_j^{(l)}(u) = Y_j(u - lw), \quad l = 1, \cdots, n, \quad j = 1, \cdots, 2n - 1.
\] (5.6)

Then the eigenvalue \( \Lambda_m(u) \mid m = 1, \cdots n - 1 \) which satisfy the relations (3.16)-(3.22) can be given in terms of the inhomogeneous \( T-Q \) relations

\[
\Lambda_m(u) = \sum_{1 \leq i_1 < i_2 < \cdots < i_m \leq 2n-1} Y_{i_1}(u)Y_{i_2}^{(1)}(u) \cdots Y_{i_m}^{(m-1)}(u), \quad m = 1, \cdots, n - 1.
\] (5.7)

The sum \( \sum' \) in the above expression is over the constrained increasing sequences \( 1 \leq i_1 < i_2 < \cdots < i_m \leq 2n - 1 \) such that when any \( i_k = 2j \) (i.e., \( Y_{i_k}^{(k-1)}(u) = Y_{2j}^{(k-1)}(u) = X_j^{(k-1)}(u) \)), \( i_{k-1} \leq 2j - 3 \) and \( i_{k+1} \geq 2j + 3 \). Namely, when \( Y_{i_k}(u) = Y_{2j} = X_j(u) \), the previous element \( Y_{i_{k-1}}(u) \) can not be chosen as \( Y_{2j-1} = Z_j(u) \) or \( Y_{2j-2} = X_{j-1}(u) \), while the next element \( Y_{i_{k+1}}(u) \) can not be chosen as \( Y_{2j+1} = Z_{j+1}(u) \) or \( Y_{2j+2} = X_{j+1}(u) \) (e.g., once a \( X_j \) element was chosen, its nearest neighbors (namely, \( X_{j-1}(u) \), \( Z_j(u) \), \( Z_{j+1}(u) \) and \( X_{j+1}(u) \) in the diagram (5.8) can not be chosen any more.).

\[
Z_1 \quad Z_2 \quad Z_3 \quad Z_4 \quad \cdots \quad Z_n
\]

\[
X_1 \quad X_2 \quad X_3 \quad \cdots \quad X_{n-1}
\] (5.8)

The \( 2n - 2 \) positive integers \( \{N_i \mid i = 1, \cdots, 2n - 2\} \) and the function \( f_\Phi(u) \) in (5.4) are given as follows:

- For the case of odd \( n \), we have

\[
N_{2i-1} = N_{2i} = N_{2(n-i)-1} = N_{2(n-i)} = \frac{i(n - i)}{2}N, \quad i = 1, \cdots, \frac{n-1}{2};
\] (5.9)

and there is no function \( X_\Phi(u) \);
• For the case of even \( n \) and even \( N \), we have

\[
N_{2i-1} = N_{2i} = N_{2(n-i)-1} = N_{2(n-i)} = \frac{i(n-i)}{2}N, \quad i = 1, \cdots, \frac{n}{2};
\]

and

\[
f_{2i}^{q}(u) = 1; (5.11)
\]

• For the case of even \( n \) and odd \( N \), we have

\[
N_{2i-1} = N_{2i} = N_{2(n-i)-1} = N_{2(n-i)} = \frac{i(n-i)}{2}N + \frac{i}{2}, \quad i = 1, \cdots, \frac{n}{2};
\]

the function \( f_{2i}^{q}(u) \) is given by

\[
f_{2i}^{q}(u) = \sigma(u).
\]

Moreover, the vanishing condition of the residues of \( \Lambda_m(u) \) at the points \( \lambda_{j}^{(i)} \) gives rise to the
Further, the periodicities (3.21) of the eigenvalues as well as the selection rule (3.22) give

\[ e^{2\pi l_2 \lambda_j^{(1)}} + \phi_2 \frac{Q^{(2)}(\lambda_j^{(1)} + w)}{Q^{(4)}(\lambda_j^{(1)})} + c_1 e^{2\pi l_n \lambda_j^{(1)}} a(\lambda_j^{(1)}) \frac{Q^{(2)}(\lambda_j^{(1)})}{Q^{(4)}(\lambda_j^{(1)})} = 0, \quad j = 1, \cdots, N_1, \]  

\[ e^{2\pi l_1 \lambda_j^{(2)}} + \phi_1 Q^{(1)}(\lambda_j^{(2)} - w) + c_1 e^{2\pi l_n \lambda_j^{(2)}} d(\lambda_j^{(2)}) \frac{Q^{(3)}(\lambda_j^{(2)} - w)}{Q^{(1)}(\lambda_j^{(2)})} = 0, \quad j = 1, \cdots, N_2, \]  

\[ \vdots \]

\[ e^{2\pi (i+1) \lambda_j^{(2i-1)}} + \phi_i+1 \frac{Q^{(2i)}(\lambda_j^{(2i-1)} + w)}{Q^{(2i+2)}(\lambda_j^{(2i-1)})} + c_i e^{2\pi l_n + i - 1 \lambda_j^{(2i-1)}} a(\lambda_j^{(2i-1)}) \frac{Q^{(2i)}(\lambda_j^{(2i-1)} + w)}{Q^{(2i+2)}(\lambda_j^{(2i-1)})} = 0, \]

\[ i = 2, \cdots, n - 2; \quad j = 1, \cdots, N_{2i-1}, \]  

\[ e^{2\pi l_1 \lambda_j^{(2i)}} + \phi_i Q^{(2i-1)}(\lambda_j^{(2i)} - w) + c_i e^{2\pi l_n + i - 1 \lambda_j^{(2i)}} a(\lambda_j^{(2i)}) \frac{Q^{(2i+1)}(\lambda_j^{(2i)} - w)}{Q^{(2i-1)}(\lambda_j^{(2i)})} = 0, \]

\[ i = 2, \cdots, n - 2; \quad j = 1, \cdots, N_{2i}, \]  

\[ \vdots \]

\[ e^{-2\pi \sum_{k=1}^{n-1} l_k (\lambda_j^{(2n-3)} + (n-k)w) - \sum_{l=1}^{n-1} \phi_l Q^{(2n-2)}(\lambda_j^{(2n-3)} + w)} + c_{n-1} e^{2\pi l_{2n-1} \lambda_j^{(2n-3)}} a(\lambda_j^{(2n-3)}) \frac{Q^{(2n-4)}(\lambda_j^{(2n-3)} + w)}{Q^{(2n-2)}(\lambda_j^{(2n-3)})} = 0, \quad j = 1, \cdots, N_{2n-3}, \]  

\[ e^{2\pi l_{n-1} \lambda_j^{(2n-2)}} + \phi_{n-1} \frac{Q^{(2n-3)}(\lambda_j^{(2n-2)} - w)}{Q^{(2n-5)}(\lambda_j^{(2n-2)})} + c_{n-1} e^{2\pi l_{2n-2} \lambda_j^{(2n-2)}} a(\lambda_j^{(2n-2)}) \frac{Q^{(2n-5)}(\lambda_j^{(2n-2)})}{Q^{(2n-3)}(\lambda_j^{(2n-2)})} = 0, \]

\[ j = 1, \cdots, N_{2n-2}. \]  

Further, the periodicities (3.21) of the eigenvalues as well as the selection rule (3.22) give
rise to the associated BAEs:

\[
\Theta^{(2)} - \Theta^{(1)} = (N_1 - N + \frac{N}{n})w + m_1 + l_1 \tau,
\]

\[
\Theta^{(1)} - \Theta^{(2)} - \Theta^{(3)} + \Theta^{(4)} = (N_3 - N_2 + \frac{N}{n})w + m_2 + l_2 \tau,
\]

\[
\vdots
\]

\[
-\Theta^{(2i-2)} - \Theta^{(2i-1)} + \Theta^{(2i-3)} + \Theta^{(2i)} = (N_{2i-1} - N_{2i-2} + \frac{N}{n})w + m_i + l_i \tau,
\]

\[
\vdots
\]

\[
-\Theta^{(2n-4)} - \Theta^{(2n-3)} + \Theta^{(2n-5)} + \Theta^{(2n-2)} = (N_{2n-3} - N_{2n-4} + \frac{N}{n})w + m_{n-1} + l_{n-1} \tau,
\]

\[
-\Theta + \Theta^{(1)} + \Theta^{(2)} - \Theta^{(3)} + \frac{(n-1)Nw}{n} - N_3w = l_n \tau + m_n,
\]

\[
\vdots
\]

\[
-\Theta - \Theta^{(2j-2)} + \Theta^{(2j-1)} + \Theta^{(2j)} - \Theta^{(2j+1)} + (N_{2j-2} - N_{2j+1} + \frac{(n-1)N}{n})w
\]

\[
= l_{n+j-1} \tau + m_{n+j-1},
\]

\[
\vdots
\]

\[
-\Theta - \Theta^{(2n-4)} + \Theta^{(2n-3)} + \Theta^{(2n-2)} + (N_{2n-4} + \frac{(n-1)N}{n})w = l_{2n-2} \tau + m_{2n-2},
\]

\[
\prod_{j=1}^{N} \frac{Q^{(1)}(\theta_j - w)}{Q^{(2)}(\theta_j)} = e^{-2i\pi l_1 \Theta - N\phi_1},
\]

where \(\{m_i|i = 1, \cdots, 2n - 2\}\) are arbitrary integers and

\[
\Theta = \sum_{l=1}^{N}\theta_l, \quad \Theta^{(i)} = \sum_{l=1}^{N_i}\lambda_l^{(i)}, \quad i = 1, \cdots, 2n - 2.
\]

We have checked that for a generic \(w\) and \(\tau\) but the number of sites \(N = nl\) with \(l\) being a positive integer, the inhomogeneous \(T - Q\) relations (5.7) can be reduced to homogeneous ones which were previously obtained by the algebraic Bethe ansatz \([15, 30]\). Moreover, it is also found that when the crossing parameter \(w\) takes some discrete values (like (4.33) for the \(n = 3\) case) the resulting \(T - Q\) relations can also become the homogeneous ones.
6 Conclusions

The periodic $Z_n$-Belavin model with an arbitrary site number $N$ and generic coupling constants $w$ and $\tau$ described by the Hamiltonian (2.16) and (2.18) is studied via the off-diagonal Bethe Ansatz method. The eigenvalues $\{\Lambda_i(u)|i = 1, \ldots, n-1\}$ of the corresponding transfer matrix and fused ones $\{t_i(u)|i = 1, \ldots, n-1\}$ given by (3.8) are derived in terms of the inhomogeneous $T-Q$ relations (5.7). In the special case of $N = nl$ with a positive integer $l$, the resulting $T-Q$ relation is reduced to a homogeneous one (such as (4.29)), which recovers the result obtained by the algebraic Bethe Ansatz method [15]. On the other hand, if the crossing parameter $w$ take some special values (such as (4.33) for the $n = 3$ case), the resulting $T-Q$ relation also becomes a homogeneous one (such as (4.35) for the $n = 3$ case).

We remark that the $Z_n$-symmetry (2.11) of the $R$-matrix $R(u)$ ensures that the $Z_n$-Belavin model with the twisted boundary condition given by

$$H_{N,N+1} = G_1 H_{N,1} G_1^{-1}, \quad G = I_\alpha = I_{(\alpha_1, \alpha_2)}, \quad \alpha_i \in \mathbb{Z}_n, \quad (6.1)$$

is also integrable. The corresponding transfer matrix $t^{(\alpha)}(u)$ can be constructed by [31, 32]

$$t^{(\alpha)}(u) = tr_0 (G_0 T_0(u)), \quad G = I_\alpha = I_{(\alpha_1, \alpha_2)}, \quad \alpha_i \in \mathbb{Z}_n. \quad (6.2)$$

The Hamiltonian can be derived the same way as the periodic one (c.f., (2.16)). Using the similar method developed in previous sections, we can construct the corresponding ODBA solution, which is given in Appendix B.

The eigenvalues of the transfer matrix for the $Z_n$-Belavin model with periodic (or twisted) boundary condition obtained in this paper might help one to construct the corresponding eigenstates, thus further giving rise to studying correlation functions [7] of the model. For this purpose, some particular basis such as the separation of variable (SoV) [33] basis [34, 35] or its higher-rank generalization [36] will play an important role.

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Appendix A: $T-Q$ relations for the $Z_4$ case

In this Appendix, we take the $Z_4$ case as an example to show the procedure for constructing the inhomogeneous $T-Q$ relations $[5,7]$. The functions $[5,1]-[5,3]$ now read

\[ Q^{(i)}(u) = \prod_{j=1}^{N_i} \frac{\sigma(u - \lambda^{(i)}_j)}{\sigma(w)}, \quad i = 1, \ldots, 6, \quad (A.1) \]

\[ Z_1(u) = e^{2i\pi l_1 u + \phi_1} a(u) \frac{Q^{(1)}(u - w)}{Q^{(2)}(u)}, \]
\[ Z_2(u) = e^{2i\pi l_2 u + \phi_2} d(u) \frac{Q^{(2)}(u + w)Q^{(3)}(u - w)}{Q^{(1)}(u)Q^{(4)}(u)}, \]
\[ Z_3(u) = e^{2i\pi l_3 u + \phi_3} d(u) \frac{Q^{(4)}(u + w)Q^{(5)}(u - w)}{Q^{(3)}(u)Q^{(6)}(u)}, \]
\[ Z_4(u) = e^{-2i\pi \sum_{k=1}^{4} l_k (u + (4-k)w) - \sum_{j=1}^{3} \phi_j} d(u) \frac{Q^{(6)}(u + w)}{Q^{(5)}(u)}, \quad (A.2) \]

and

\[ X_1(u) = c_1 e^{2i\pi l_1 u} a(u) d(u) \frac{Q^{(3)}(u - w)}{Q^{(1)}(u)Q^{(2)}(u)}, \]
\[ X_2(u) = c_2 e^{2i\pi l_2 u} a(u) d(u) \frac{Q^{(2)}(u + w)Q^{(5)}(u - w)f_2(u)}{Q^{(3)}(u)Q^{(4)}(u)}, \]
\[ X_3(u) = c_3 e^{2i\pi l_3 u} a(u) d(u) \frac{Q^{(4)}(u + w)}{Q^{(5)}(u)Q^{(6)}(u)}. \quad (A.3) \]
The inhomogeneous $T - Q$ relations (5.7) become

$$
\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + Z_4(u) + X_1(u) + X_2(u) + X_3(u),
$$

(A.4)

$$
\Lambda_2(u) = Z_1(u)Z_2(u - w) + Z_1(u)Z_3(u - w) + Z_1(u)Z_4(u - w) + Z_2(u)Z_3(u - w)
+ Z_2(u)Z_4(u - w) + Z_3(u)Z_4(u - w) + X_1(u)(Z_3(u - w) + Z_4(u - w))
+ (Z_1(u) + Z_2(u))X_3(u - w) + X_1(u)X_3(u - w) + Z_1(u)X_2(u - w)
+ X_2(u)Z_4(u - w),
$$

(A.5)

$$
\Lambda_3(u) = Z_1(u)Z_2(u - w)Z_3(u - 2w) + Z_1(u)Z_2(u - w)Z_4(u - 2w)
+ Z_1(u)Z_3(u - w)Z_4(u - 2w) + Z_2(u)Z_3(u - w)Z_4(u - 2w)
+ Z_1(u)Z_2(u - w)X_3(u - 2w) + Z_1(u)X_2(u - w)Z_4(u - 2w)
+ X_1(u)Z_3(u - w)Z_4(u - 2w),
$$

(A.6)

$$
\Lambda_4(u) = Z_1(u)Z_2(u - w)Z_3(u - 2w)Z_4(u - 3w).
$$

(A.7)

The positive integers \( \{N_i|i = 1, \cdots, 6\} \) and the function \( f_2(u) \) are given as follows:

- When \( N \) is even, we have

$$
N_1 = N_2 = N_5 = N_6 = \frac{3}{2}N, \quad N_3 = N_4 = 2N,
$$

(A.8)

and the function \( f_2(u) \) is

$$
f_2(u) = 1.
$$

(A.9)

- When \( N \) is odd, we have

$$
N_1 = N_2 = N_5 = N_6 = \frac{3N + 1}{2}, \quad N_3 = N_4 = 2N + 1,
$$

(A.10)

and the functions \( f_2(u) \) is

$$
f_2(u) = \sigma(u).
$$

(A.11)
The associated BAEs (5.14)-(5.27) become

\[
    e^{2\pi i \ell_1 \lambda_j^{(1)} + \phi_1} \frac{Q^{(2)}(\lambda_j^{(1)} + w)}{Q^{(4)}(\lambda_j^{(1)})} + c_1 e^{2\pi i \ell_4 \lambda_j^{(1)}} \frac{a(\lambda_j^{(1)})}{Q^{(2)}(\lambda_j^{(1)})} = 0, \quad j = 1, \ldots, N_1, \quad (A.12)
\]

\[
    e^{2\pi i \ell_1 \lambda_j^{(2)} + \phi_1} Q^{(1)}(\lambda_j^{(2)} - w) + c_1 e^{2\pi i \ell_4 \lambda_j^{(2)}} d(\lambda_j^{(2)}) \frac{Q^{(3)}(\lambda_j^{(2)} - w)}{Q^{(1)}(\lambda_j^{(2)})} = 0, \quad j = 1, \ldots, N_2, \quad (A.13)
\]

\[
    e^{2\pi i \ell_3 \lambda_j^{(3)} + \phi_3} \frac{Q^{(4)}(\lambda_j^{(3)} + w)}{Q^{(6)}(\lambda_j^{(3)})} + c_2 e^{2\pi i \ell_5 \lambda_j^{(3)}} a(\lambda_j^{(3)}) \frac{Q^{(2)}(\lambda_j^{(3)} + w) f_2(\lambda_j^{(3)})}{Q^{(4)}(\lambda_j^{(3)})} = 0, \quad j = 1, \ldots, N_3, \quad (A.14)
\]

\[
    e^{2\pi i \ell_2 \lambda_j^{(4)} + \phi_2} \frac{Q^{(3)}(\lambda_j^{(4)} - w)}{Q^{(1)}(\lambda_j^{(4)})} + c_2 e^{2\pi i \ell_5 \lambda_j^{(4)}} a(\lambda_j^{(4)}) \frac{Q^{(5)}(\lambda_j^{(4)} - w) f_2(\lambda_j^{(4)})}{Q^{(3)}(\lambda_j^{(4)})} = 0, \quad j = 1, \ldots, N_4, \quad (A.15)
\]

\[
    e^{-2\pi \sum_{k=1}^3 \ell_k \lambda_j^{(5)} - \sum_{t=1}^3 (4-t) \ell_t w - \sum_{t=1}^3 \phi t Q^{(6)}(\lambda_j^{(5)} + w)} + c_3 e^{2\pi i \ell_6 \lambda_j^{(5)}} a(\lambda_j^{(5)}) \frac{Q^{(4)}(\lambda_j^{(5)} + w)}{Q^{(6)}(\lambda_j^{(5)})} = 0, \quad j = 1, \ldots, N_5, \quad (A.16)
\]

\[
    e^{2\pi i \ell_3 \lambda_j^{(6)} + \phi_3} \frac{Q^{(5)}(\lambda_j^{(6)} - w)}{Q^{(3)}(\lambda_j^{(6)})} + c_3 e^{2\pi i \ell_6 \lambda_j^{(6)}} a(\lambda_j^{(6)}) \frac{Q^{(5)}(\lambda_j^{(6)})}{Q^{(5)}(\lambda_j^{(6)})} = 0, \quad j = 1, \ldots, N_6, \quad (A.17)
\]

\[
    \Theta^{(2)} - \Theta^{(1)} = (N_1 - N + \frac{N}{4}) w + m_1 + l_1 \tau, \quad (A.18)
\]

\[
    \Theta^{(1)} - \Theta^{(2)} - \Theta^{(3)} + \Theta^{(4)} = (N_3 - N_2 + \frac{N}{4}) w + m_2 + l_2 \tau, \quad (A.19)
\]

\[
    \Theta^{(3)} - \Theta^{(4)} - \Theta^{(5)} + \Theta^{(6)} = (N_5 - N_4 + \frac{N}{4}) w + m_3 + l_3 \tau, \quad (A.20)
\]

\[
    -\Theta + \Theta^{(1)} + \Theta^{(2)} - \Theta^{(3)} - N_3 w + \frac{3Nw}{4} = m_4 + l_4 \tau, \quad (A.21)
\]

\[
    -\Theta - \Theta^{(2)} + \Theta^{(3)} + \Theta^{(4)} - \Theta^{(5)} + N_2 w - N_5 w + \frac{3Nw}{4} = m_5 + l_5 \tau, \quad (A.22)
\]

\[
    -\Theta - \Theta^{(4)} + \Theta^{(5)} + \Theta^{(6)} + N_4 w + \frac{3Nw}{4} = m_6 + l_6 \tau, \quad (A.23)
\]

\[
    \prod_{j=1}^N \frac{Q^{(1)}(\theta_j - w)}{Q^{(2)}(\theta_j)} = e^{-2\pi i \ell_1 \Theta - N\phi_1}. \quad (A.24)
\]
The main purpose of this Appendix is to show the new features occurred in $Z_4$ case, which are crucial to understand the structure of the inhomogeneous $T-Q$ relations (5.7) for general $Z_n$ case.

**Appendix B: $Z_3$-Belavin model with twisted boundary condition**

The Yang-Baxter algebra relation (2.14) and the $Z_n$ symmetry (2.11) properties of $Z_n$-Belavin R-matrix lead to the fact that the transfer matrix $t^{(\alpha)}(u)$ given by (6.2) with different spectral parameters are mutually commuting $[t^{(\alpha)}(u), t^{(\alpha)}(v)] = 0$. This ensures the integrability of the inhomogeneous $Z_n$-Belavin model with twisted boundary condition.

Without loss of generality, we take the $Z_3$-model with the twisted boundary matrix $G = h$ as an example to construct the solution. The corresponding transfer matrix then reads
\[
t^{(1,0)}(u) = tr_0 (h_0 T_0(u)).
\]

The invariant relation and operator identities of this transfer matrix $t^{(\alpha)}(u)$ can be derived in the same way as in dealing with the $su(n)$ spin torus [36]. The properties of this transfer matrix imply that the corresponding eigenvalues $\{\Lambda_m(u) | m = 1, \cdots, 3\}$ satisfy the following functional relations
\[
\Lambda(\theta_j)\Lambda_m(\theta_j - w) = \Lambda_{m+1}(\theta_j), \quad m = 1, 2, \quad j = 1, \cdots, N, \quad (B.1)
\]
\[
\Lambda_3(u) = \text{Det}_q\{h\} \text{Det}_q\{T(u)\} = a(u)d(u - w)d(u - 2w), \quad (B.2)
\]
\[
\Lambda_2(\theta_j + w) = 0, \quad j = 1, \cdots, N. \quad (B.3)
\]
The periodicity of the $Z_n$-Belavin R-matrix and commuting relation (2.4) of operators $g, h$ give rise to that the eigenvalues are some elliptic polynomials of the fixed degrees $mN$ with the periodicity
\[
\Lambda_m(u + 1) = (-1)^{mN} e^{\frac{4m\pi}{3}} \Lambda_m(u), \quad m = 1, 2, \quad (B.4)
\]
\[
\Lambda_m(u + \tau) = (-1)^{mN} e^{-2i\pi\left\{mn(u + \frac{w}{3} + \frac{r}{2} - \frac{m-1}{2}w) - m\sum_{i=1}^{N} \theta_i\right\}} \Lambda_m(u), \quad m = 1, 2. \quad (B.5)
\]
Moreover, the unitarity relation (2.9) and $h^3 = \text{id}$ allow us to derive the following identity
\[
\left\{ \prod_{l=1}^{N} \Lambda(\theta_l) \right\}^3 = \left\{ \prod_{l=1}^{N} a(\theta_l) \right\}^3. \quad (B.6)
\]
Similar as the periodic case, the relations (B.1)-(B.6) allow us to determine the eigenvalues \( \{\Lambda_m(u)\} \) of the corresponding transfer matrices completely. We can thus express \( \Lambda(u) \) and \( \Lambda_2(u) \) in terms of an inhomogeneous \( T - Q \) relation as follows. Let us introduce some \( Q \)-functions

\[
Q^{(i)}(u) = \prod_{j=1}^{N} \frac{\sigma(u - \lambda_j^{(i)})}{\sigma(u)}, \quad i = 1, \cdots, 4,
\]

parameterized by \( 4N \) parameters \( \{\lambda_j^{(i)}|j = 1, \cdots, N; i = 1, \cdots, 4\} \) determined later by the associated BAES (see below (B.14)-(B.22)). Associated with the above \( Q \)-functions, we introduce 5 functions \( \{Z_i(u)|i = 1, 2, 3\} \) and \( \{X_i(u)|i = 1, 2\} \) as

\[
Z_1(u) = a(u)e^{2\pi i (l_1 + \frac{1}{4})u + \phi_1} \frac{Q^{(1)}(u - w)}{Q^{(2)}(u)},
\]

\[
Z_2(u) = \omega_3 d(u)e^{2\pi i (l_2 + \frac{1}{4})u + \phi_2} \frac{Q^{(2)}(u + w)Q^{(3)}(u - w)}{Q^{(1)}(u)Q^{(4)}(u)},
\]

\[
Z_3(u) = \omega_3 d(u)e^{-2\pi i \{l_1 + l_2 + \frac{1}{4}\}u + (2l_1 + l_2)w} - \phi_1 - \phi_2 \frac{Q^{(4)}(u + w)}{Q^{(3)}(u)},
\]

\[
X_1(u) = c_1 a(u)d(u)e^{2\pi i (l_3 + \frac{1}{4})u} \frac{Q^{(3)}(u - w)}{Q^{(1)}(u)Q^{(2)}(u)},
\]

\[
X_2(u) = c_2 a(u)d(u)e^{2\pi i (l_4 + \frac{1}{4})u} \frac{Q^{(2)}(u + w)}{Q^{(3)}(u)Q^{(4)}(u)},
\]

where \( \omega_3 = e^{\frac{2\pi}{5}}, \{l_i|i = 1, \cdots, 4\} \) are 4 integers, \( \{\phi_1, c_i|i = 1, 2\} \) are 4 complex numbers. Then we can introduce the inhomogeneous \( T - Q \) relations,

\[
\Lambda(u) = Z_1(u) + Z_2(u) + Z_3(u) + X_1(u) + X_2(u),
\]

\[
\Lambda_2(u) = Z_1(u)Z_2(u - w) + Z_1(u)Z_3(u - w) + Z_2(u)Z_3(u - w)
\]

\[
+X_1(u)Z_3(u - w) + Z_1(u)X_2(u - w).
\]

In order that the above parameterizations of \( \Lambda(u) \) and \( \Lambda_2(u) \) become a solution to (B.1)-(B.6), the \( 4(N + 1) \) parameters \( \{\lambda_j^{(i)}|j = 1, \cdots, N; i = 1, \cdots, 4\} \) and \( \{\phi_i, c_i|i = 1, 2\} \) have
to satisfy the associated BAEs

\[
\begin{align*}
\omega_3 e^{2i\pi(l_2 + \frac{1}{3})\lambda_j^{(1)} + \phi_2} Q^{(2)}(\lambda_j^{(1)} + \tau) Q^{(2)}(\lambda_j^{(1)}) + c_1 e^{2i\pi(l_3 + \frac{2}{3})\lambda_j^{(1)}} a(\lambda_j^{(1)}) Q^{(4)}(\lambda_j^{(1)}) &= 0, \\
& j = 1, \cdots, N, \\
\end{align*}
\]

(B.14)

\[
\begin{align*}
e^{2i\pi(l_1 + \frac{1}{4})\lambda_j^{(2)} + \phi_1} Q^{(1)}(\lambda_j^{(2)} - \tau) Q^{(1)}(\lambda_j^{(2)}) + c_1 e^{2i\pi(l_3 + \frac{2}{3})\lambda_j^{(2)}} d(\lambda_j^{(2)}) Q^{(3)}(\lambda_j^{(2)} - \tau) &= 0, \\
& j = 1, \cdots, N,
\end{align*}
\]

(B.15)

\[
\begin{align*}
\omega_3 e^{-2i\pi((l_1 + l_2 + \frac{1}{3})\lambda_j^{(3)} + (2l_1 + l_2 + 2)\lambda_j^{(4)}) - \phi_1 - \phi_2} Q^{(4)}(\lambda_j^{(3)} + \tau) Q^{(4)}(\lambda_j^{(3)}) \\
&+ c_2 e^{2i\pi(l_4 + \frac{1}{4})\lambda_j^{(3)}} a(\lambda_j^{(3)}) Q^{(2)}(\lambda_j^{(3)} + \tau) = 0, \\
& j = 1, \cdots, N,
\end{align*}
\]

(B.16)

\[
\begin{align*}
\omega_3 e^{2i\pi(l_2 + \frac{1}{3})\lambda_j^{(4)} + \phi_2} Q^{(3)}(\lambda_j^{(4)} - \tau) Q^{(3)}(\lambda_j^{(4)}) + c_2 e^{2i\pi(l_4 + \frac{2}{3})\lambda_j^{(4)}} a(\lambda_j^{(4)}) Q^{(1)}(\lambda_j^{(4)}) &= 0, \\
& j = 1, \cdots, N,
\end{align*}
\]

(B.17)

\[
\begin{align*}
-\Theta^{(1)} + \Theta^{(2)} - \frac{N}{3} w &= m_1 + (l_1 + \frac{2}{3})\tau, \\
-\Theta^{(2)} - \Theta^{(3)} + \Theta^{(1)} + \Theta^{(4)} - \frac{N}{3} w &= m_2 + (l_2 + \frac{2}{3})\tau, \\
-\Theta - \Theta^{(3)} + \Theta^{(1)} + \Theta^{(2)} - \frac{N}{3} w &= m_3 + (l_3 + \frac{2}{3})\tau, \\
-\Theta - \Theta^{(2)} + \Theta^{(3)} + \Theta^{(4)} + \frac{5N}{3} w &= m_4 + (l_4 + \frac{2}{3})\tau,
\end{align*}
\]

(B.18) - (B.21)

\[
\prod_{j=1}^N \frac{Q^{(1)}(\theta_j - \tau) \theta_j^3}{Q^{(2)}(\theta_j)^3} = e^{-6i\pi(l_1 + \frac{1}{3})\Theta - 3N\phi_1}.
\]

(B.22)

where \(\{m_i|i = 1, \cdots, 4\}\) are 4 integers and

\[
\Theta = \sum_{l=1}^N \theta_l, \quad \Theta^{(i)} = \sum_{l=1}^N \lambda_l^{(i)}, \quad i = 1, \cdots, 4.
\]

Numerical solutions of the BAEs (B.14)-(B.22) for small size with random choice of \(w\) and \(\tau\) imply that the Bethe ansatz solution (B.12) indeed give the complete solutions of the model. Here we present the numerical solutions of the BAEs for the \(N = 2\) case in Table 2. The eigenvalue calculated from (B.12) is the same as that from the exact diagonalization of the Hamiltonian (2.16) with the twisted boundary condition (6.1) associated with \(G = h\).
Table 2: Solutions of BAEs (B.14)-(B.22) for the $Z_3$-Belavin model with the twisted boundary condition, $N = 2$, $\{ \theta_j \} = 0$, $w = -0.5$, $\tau = i$ and the parameters $m_1 = 1$, $m_2 = m_3 = m_4 = l_1 = l_2 = l_3 = l_4 = 0$. The symbol $m$ indicates the number of the eigenenergy $E$.

| $\lambda_1^{(1)}$ | $\lambda_2^{(1)}$ | $\lambda_1^{(2)}$ | $\lambda_2^{(2)}$ | $\lambda_1^{(3)}$ |
|------------------|------------------|------------------|------------------|------------------|
| $-0.2987 + 0.0905i$ | $0.2987 - 0.0905i$ | $0.8147 + 0.0484i$ | $-0.1481 + 0.6183i$ | $0.3220 + 0.1800i$ |
| $0.7013 + 0.0905i$ | $-0.7013 - 0.0905i$ | $1.8147 + 0.0484i$ | $-1.1481 + 0.6183i$ | $-0.3220 - 0.1800i$ |
| $0.3498 + 0.1256i$ | $-0.3498 - 0.1256i$ | $-0.1853 + 1.0484i$ | $0.8519 - 0.3817i$ | $0.3436 - 0.1110i$ |
| $-0.3220 - 0.1800i$ | $0.3220 + 0.1800i$ | $0.1667 + 0.6667i$ | $0.5000 - 0.0000i$ | $0.7013 - 0.0905i$ |
| $0.6780 - 0.1800i$ | $-0.6780 + 0.1800i$ | $1.1667 - 0.3333i$ | $-0.5000 + 1.0000i$ | $-0.7013 - 0.0905i$ |
| $0.6780 + 0.1800i$ | $-0.6780 - 0.1800i$ | $1.1667 - 0.3333i$ | $-0.5000 + 1.0000i$ | $-0.7013 - 0.0905i$ |
| $1.2648 + 0.0940i$ | $-1.2648 - 0.0940i$ | $-0.5215 - 0.3974i$ | $1.1882 + 1.0641i$ | $1.7274 + 0.1920i$ |
| $-0.3436 + 0.1110i$ | $0.3436 - 0.1110i$ | $-1.8118 + 0.0641i$ | $2.4785 + 0.6026i$ | $1.3498 + 0.1256i$ |
| $2.6564 + 0.1110i$ | $-2.6564 - 0.1110i$ | $-0.8118 + 0.0641i$ | $1.4785 + 0.6026i$ | $-0.3498 - 0.1256i$ |

| $\lambda_2^{(3)}$ | $\lambda_1^{(4)}$ | $\lambda_2^{(5)}$ | $\phi_1$ | $\phi_2$ |
|------------------|------------------|------------------|--------|--------|
| $0.6780 - 0.1800i$ | $1.0000 + 0.0000i$ | $0.3333 + 1.3333i$ | $1.4622 - 1.0718i$ | $4.5236 - 3.7306i$ |
| $1.3220 + 0.1800i$ | $1.0000 + 1.0000i$ | $0.3333 + 0.3333i$ | $1.4622 + 9.4002i$ | $2.4292 - 16.2969i$ |
| $0.6564 + 0.1110i$ | $0.0215 + 1.3974i$ | $1.3118 - 0.0641i$ | $4.3225 + 9.5401i$ | $2.5056 - 12.6652i$ |
| $1.7013 + 0.0905i$ | $0.6853 + 0.9516i$ | $0.6481 + 0.3817i$ | $1.7970 - 6.8967i$ | $1.7596 + 3.7306i$ |
| $1.7013 + 0.0905i$ | $0.6853 + 0.9516i$ | $0.6481 + 0.3817i$ | $3.8914 + 18.2360i$ | $-0.3348 - 10.9302i$ |
| $-0.7274 - 0.1920i$ | $-1.0000 + 1.0000i$ | $2.3333 + 0.3333i$ | $4.2878 + 4.8973i$ | $-0.6765 + 4.5508i$ |
| $-0.3498 - 0.1256i$ | $1.6481 + 0.3817i$ | $-0.3147 + 0.9516i$ | $1.8336 - 2.8543i$ | $1.8807 - 8.7834i$ |
| $1.3498 + 0.1256i$ | $2.6481 + 0.3817i$ | $-1.3147 + 0.9516i$ | $1.8336 + 18.0896i$ | $1.8807 - 19.2553i$ |

| $c_1$ | $c_2$ | $E$ | $m$ |
|-------|-------|-----|-----|
| $3.4493 - 2.9903i$ | $34.0526 + 77.9548i$ | $-5.0705 - 0.1943i$ | 1 |
| $-4.3143 - 1.4921i$ | $4.1934 + 9.5997i$ | $-5.0705 - 0.1943i$ | 1 |
| $-102.6655 - 26.2513i$ | $2.4711 - 8.2348i$ | $-5.0705 - 0.1943i$ | 1 |
| $3.1926 - 4.5605i$ | $-6.1152 + 0.6177i$ | $0.8364 + 1.3278i$ | 2 |
| $25.9258 - 37.0339i$ | $0.4424 + 0.6141i$ | $0.8364 + 1.3278i$ | 2 |
| $19.1094 + 40.9693i$ | $-0.7531 + 0.0761i$ | $0.8364 + 1.3278i$ | 2 |
| $24.2590 - 49.1261i$ | $-0.6505 - 0.0544i$ | $4.2341 - 1.1336i$ | 3 |
| $-4.3389 - 0.6734i$ | $-8.8842 + 2.4665i$ | $4.2341 - 1.1336i$ | 3 |
| $2.7526 - 3.4208i$ | $2.3061 - 8.9272i$ | $4.2341 - 1.1336i$ | 3 |
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