Abstract. We study the transverse geometric behavior of 2-dimensional foliations in 3-manifolds. We show that an $R$-covered transversely orientable foliation with Gromov hyperbolic leaves in a closed 3-manifold admits a regulating, transverse pseudo-Anosov flow (in the appropriate sense) in each atoroidal piece of the manifold. The flow is a blow up of a one prong pseudo-Anosov flow. In addition we show that there is a regulating flow for the whole foliation. We also determine how deck transformations act on the universal circle of the foliation.

Keywords: Foliations, transverse flows, atoroidal pieces, pseudo-Anosov flows, universal circle, periodic orbits, group actions on the circle.

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1. Introduction

This article studies the transverse geometric behavior of 2-dimensional foliations in 3-manifolds. We will assume that the foliation is transversely orientable so there is a transverse flow. Any such flow can be used to understand how the geometry of leaves varies transversely — at least locally. By change in geometry in this article we mean the following: consider a geodesic arc in a leaf of the foliation and use the chosen transverse flow to push this arc to an arc in a nearby leaf. Does the length increase or decrease and by how much? Notice that there are other important ways to consider changes in transverse geometry: for example consider how the spacing between distinct leaves varies, which leads to holonomy invariant transverse measures.

The obvious first example to analyze is when the foliation is a fibration, where we consider the case of closed 3-manifolds. The foliation is encoded by the monodromy, which is a homeomorphism of a closed surface. By the Nielsen-Thurston theory [Th4, Bl-Ca], the monodromy is either homotopically periodic, reducible or pseudo-Anosov. Reducible means that there is a simple closed curve and a power preserves this curve. Pseudo-Anosov means that the homeomorphism preserves a pair of singular, transverse one dimensional foliations, whose leaves are either contracted by the map (stable) or expanded (unstable). The singularities are $p$-prong type with $p \geq 3$. The pseudo-Anosov option very strongly describes how the geometry of leaves is changing by some appropriate transverse suspension flow, describing the directions of contraction and expansion. This geometric information was crucial to geometrize such manifolds [Th1, Th2].

In this article we study more general foliations. One general problem is the use of a transverse flow. In the case of fibrations any transverse flow induces homeomorphisms between leaves, so we can compare how it affects the geometry. Whenever there is non trivial holonomy of closed curves [Ca-Co], the transverse
flow cannot even take a closed curve to a closed curve. This leads to the first
adjustment: the transverse change of geometry is best understood in the universal
cover and for Reebless foliations: then leaves in the universal cover are simply
connected [No], so any compact set in a leaf can be pushed by the transverse
flow to nearby leaves. But in general this cannot be accomplished for entire
leaves. In fact one necessary condition for the transverse flow in the universal
cover to be a homeomorphism between arbitrary leaves is that the foliation is
what is called $\mathbb{R}$-covered [Fe1]: the leaf space of the foliation in the universal
cover is homeomorphic to the reals $\mathbb{R}$. In this article we will prove results about
$\mathbb{R}$-covered foliations.

Fibrations are $\mathbb{R}$-covered. But even in this situation the pseudo-Anosov case
is best understood by looking at the action on the ideal boundary as follows. Re-
strict to the case that the fiber is negatively curved which is the generic case, so
one can assume the fiber is a hyperbolic surface. The universal cover is the hyper-
bolic plane, compactified with an ideal circle [Be-Pe]. Any lift of the monodromy
to the universal cover induces a homeomorphism of this ideal circle. Thurston
[Th4, Bl-Ca], following ideas of Nielsen, did a very thorough study of this ac-
tion on a circle, yielding (in the non periodic, irreducible case) invariant geodesic
laminations on the surface, which blow down to the singular foliations associated
with the pseudo-Anosov monodromy.

This program of seeing all ideal circles of leaves in the universal circle as one sin-
gle object can be carried out to a certain extent for any foliation with hyperbolic
leaves: this is the theory of the universal circle of foliations [Th5, Th6, Ca-Du,
Cal2], which introduces a powerful way to collate all circles at infinity into a single
circle, called the universal circle of the foliation. This has some powerful conse-
quences for the geometry of the foliation and the manifold [Ca-Du, Cal2]. The
general expectation is that either geometry does not change very much transver-
-sally — yielding a Seifert fibered space structure; or there is some region with
unbounded distortion yielding to some pseudo-Anosov behavior in at least part
of the manifold.

This strategy has been carried out very successfully when the foliation is $\mathbb{R}$-
covered (again the case of Gromov hyperbolic leaves is the generic case) [Fe1,
Cal1], and $M$ is atoroidal. The atoroidal case is the most common one as the
manifold is then atoroidal and hence hyperbolic by Perelman's results. In this case
it was proved in [Fe1, Cal1] that (when the foliation is transversely orientable),
there is a pseudo-Anosov flow transverse to the foliation and regulating. Regulat-
ing means that in the universal cover every lifted flow line intersects every leaf of
the foliation. One of the important consequences is that this provided a proof of
the weak hyperbolization conjecture: either there is a $\mathbb{Z}^2$ subgroup or the funda-
mental group of the manifold is Gromov hyperbolic [Gr]. This result on Gromov
hyperbolicity was of course superseded by the full proof of the geometrization
conjecture by Perelman.

What was left unanswered in [Fe1, Cal1] is the question of what happens in
the intermediate case: that is when $M$ is not Seifert fibered or atoroidal. In
particular the JSJ decomposition of $M$ is not trivial. The purpose of this article
is to analyze the transverse geometry of $\mathbb{R}$-covered foliations in this intermediate
situation. Our main result is the following:

**Theorem 1.1.** *(Main theorem)* Let $\mathcal{F}$ be a two dimensional foliation in $M^3$
closed so that $\mathcal{F}$ is transversely oriented, $\mathbb{R}$-covered, and has Gromov hyperbolic leaves.
Let $P$ be an atoroidal piece of the JSJ decomposition of the manifold. Then there
is a flow $\Phi$ in $P$ which is a blow up of a one prong pseudo-Anosov flow, so that $\Phi$
is transverse to $\mathcal{F}$ restricted to $P$ and it is regulating for $\mathcal{F}$ restricted to $P$. The union of the regular periodic orbits of $\Phi$ is dense in $P$.

Once this result is proved it is not very hard to obtain the following consequences. The first involves the action of $\pi_1(M)$ in the universal circle.

**Corollary 1.2.** Let $\mathcal{F}$ as in the Main theorem and $\gamma$ in $\pi_1(M)$ associated with a periodic orbit of $\Phi$ in the interior of $P$. Then up to a finite iterate the action of $\gamma$ on the universal circle of $\mathcal{F}$ has finitely many fixed points which are alternatively attracting and repelling. If the orbit is regular there are exactly 4 fixed points up to iterate.

In section 2 we provide the necessary background on the universal circle of foliations.

We also prove the following, which extends well known results for $\mathbb{R}$-covered foliations in atoroidal manifolds [Fe1, Cal1]:

**Corollary 1.3.** Suppose that $\mathcal{F}$ is a transversely oriented foliation which is $\mathbb{R}$-covered. Then there is a flow transverse to $\mathcal{F}$ which is regulating for $\mathcal{F}$.

The Main Theorem also helps to understand in general the action of $\pi_1(M)$ on the universal circle of the foliation $\mathcal{F}$.

Besides their intrinsic interest, the results of this article, particularly the Main Theorem has some uses in other situations. For example in partially hyperbolic dynamics in dimension 3 the properties of two dimensional foliations are essential due to the results of Burago-Ivanov [Bu-Iv] who produced some “branching foliations” associated with the dynamics. In many cases these foliations are $\mathbb{R}$-covered and the change in transverse geometry can give important information. In particular in [FP2] we use the results of this article on transverse pseudo-Anosov flows on atoroidal pieces and group actions on the universal cover to prove that some partially hyperbolic diffeomorphisms in dimension 3 are not dynamically coherent, amongst other results in [FP2].

1.1. **Ideas of the proof of the main theorem.** Let $\tilde{\mathcal{F}}$ be the foliation lifted to the universal cover $\tilde{M}$. As indicated above the main idea is to use the universal circle of the foliation. In the case of an $\mathbb{R}$-covered foliation the universal circle is canonically homeomorphic to the circle at infinity of any leaf of $\tilde{\mathcal{F}}$. This is described more carefully in the next section. To understand the change of geometry one uses geometric shapes in the leaves of $\tilde{\mathcal{F}}$ determined by ideal points in the leaves. Any pair of geodesics in the hyperbolic plane are isometric and so are any ideal triangles. To see distortion one has to look at ideal quadrilaterals in leaves of $\tilde{\mathcal{F}}$ determined by four ideal points in these leaves. Now change the leaves and move the ideal points according to the universal circle identifications. The ideal quadrilaterals change and may become thinner in one direction or in the opposite direction. In [Fe1] we used this distortion quadrilaterals to produce a pair two dimensional immersed laminations transverse to $\mathcal{F}$ and intersecting leaves of $\mathcal{F}$ in geodesics. These laminations capture some of the distortion in the transverse direction. The atoroidal property was then used heavily to show that these immersed laminations fill $M$ and they are in fact embedded, and eventually produce the transverse pseudo-Anosov flow.

We continue this analysis in the toroidal case. It is expected that the immersed laminations do not fill $M$. The constructions and proofs in [Fe1] are to a certain extent specific to the atoroidal case. In particular Theorem 5.1 of [Fe1] produces what is called a leafwise geodesic embedded lamination. The passage from an
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immersed to an embedded lamination is fundamental in the understanding of the problem. The process is done by a convex hull procedure. In the toroidal case the geodesic lamination obtained by this convex hull process theoretically can well be a cutting torus in the JSJ decomposition manifold. We definitely do not want such a lamination (a torus), as it would not describe the transversal change of geometry. At this point the proofs in the atoroidal and non atoroidal case diverge. In the toroidal case we study in more detail the actual immersed laminations which are the limits of the distortion quadrilaterals, we show that components of these laminations are contained in the interior of the atoroidal pieces (appropriately adjusted). In particular they cannot be any of the tori of the JSJ decomposition. This is the hardest fact to prove and it uses a lot the definition of the distortion quadrilaterals. We first show that the geodesics produced by the limiting process cannot cross the tori of the JSJ decomposition. Then we show that the geodesics cannot be contained in the tori either. The second property is harder to obtain and involves manipulating the foliation we start with. After this is done, We will show that the a priori immersed laminations are in fact embedded. We will also show that the two laminations we obtain are not the same, they are transverse to each other and they fill an atoroidal piece \( P \). Then there is a blow down process to produce a pseudo-Anosov like flow — but there may be one prongs. So we generalize the notion of a pseudo-Anosov flow to a one prong pseudo-Anosov flow. Finally, an appropriate blow up of this flow produces a flow in \( P \) which is transverse to \( F \) in \( P \) and regulating.

Once this is done the construction of the regulating flow for the whole foliation is not so complicated.

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2. \( \mathbb{R} \)-covered foliations with Gromov hyperbolic leaves

Here we explain the basics about these foliations and review the information we need about them. The details are in [Fe1]. Let \( F \) be an \( \mathbb{R} \)-covered foliation with Gromov hyperbolic leaves. By Candel’s theorem [Cand] there is a metric in \( M \) making every leaf of \( F \) into a hyperbolic surface (notice that \( \mathbb{R} \)-covered is not necessary for Candel’s theorem). Let \( \tilde{F} \) be the lifted foliation to \( \tilde{M} \). Each leaf \( F \) of \( \tilde{F} \) with its induced Riemannian metric is isometric to the hyperbolic plane and is compactified with an ideal circle \( S^1 \). First we introduce the ideal annulus \( A \). As a set \( A \) is the union of \( S^1(E) \) where \( E \) are the leaves of \( \tilde{F} \). The topology is as follows: consider a transversal \( \tau \) to \( \tilde{F} \). For each point \( x \) in \( \tau \) with \( x \) in \( E \) leaf of \( \tilde{F} \), consider the unit tangent bundle of \( E \) at \( x \) which is a circle. Each unit vector determines a geodesic ray in \( E \) starting at \( x \) with that direction. This determines an ideal point in \( E \), hence a point in \( S^1(E) \). The map between directions and \( S^1(E) \) is a bijection for each \( E \). As \( x \) varies in \( \tau \) this provides a bijection between the unit tangent bundle of \( \tilde{F} \) restricted to \( \tau \) and the union of the circles at infinity of the leaves intersecting \( \tau \). The union of the unit tangent bundles to \( \tilde{F} \) along \( \tau \) has a natural topology coming from the geometry of \( \tilde{M} \). We put a topology in \( A \) induced by these local bijections. In [Fe1] we proved that this topology is well defined, and deck transformations act by homeomorphisms on \( A \). Notice that this topology in \( A \) clearly induces the natural topology in each \( S^1(E) \). One important property is the following: suppose that \( \alpha, \beta \) are continuous curves in \( A \) transverse to the foliation by circles at infinity of leaves. Suppose that for each \( E \) in \( \tilde{F} \) the intersection of \( \alpha, \beta \) with
$S^1(E)$, denoted respectively by $\alpha_E, \beta_E$, are distinct points in $S^1(E)$. As $E$ varies let $\ell_E$ be the geodesic in $E$ with ideal points $\alpha_E, \beta_E$. Then the geodesics $\ell_E$ vary continuously in $\tilde{M}$ with $E$.

We now describe the universal circle $\mathcal{U}$ of $\mathcal{F}$. There are two possibilities:

- **The uniform case**—Here for any two leaves $E, F$ of $\tilde{F}$, the Hausdorff distance between them (as subsets of $\tilde{M}$) is finite [Th5]. The bound obviously depends on the pair of leaves. For any pair of leaves $E, F$ there is a map $\tau : E \to F$ which is a quasi-isometry. This quasi-isometry is coarsely well defined and induces a homeomorphisms $\tau_\infty$ between $S^1(E)$ and $S^1(F)$. The map $\tau_\infty$ is as follows: given $p$ in $S^1(E)$ there is a unique $q$ in $S^1(F)$ so that if $r$ is a geodesic ray in $E$ with ideal point $p$ and $r'$ is a geodesic ray in $F$ with ideal point $q$ then $p, p'$ are a finite Hausdorff distance from each other in $\tilde{M}$. The homeomorphisms between ideal circles satisfy a cocycle property and they are obviously equivariant under the action of deck transformations.

- **The non uniform case**—In particular there are no compact leaves. In this case $\mathcal{F}$ has a unique minimal sublamination $\mathcal{F}'$. The complementary regions of $\mathcal{F}'$ are $I$-bundles over non compact surfaces. One can then collapse the complementary regions to produce a new $\mathbb{R}$-covered foliation which is minimal. All the results proved for this induced minimal foliation pull back to $\mathcal{F}$. So when necessary we assume in this case that $\mathcal{F}$ is minimal. Under this condition and not uniform then for any leaves $E, F$ of $\tilde{F}$ there is a dense set of directions in $E$ (and in $F$ too) so that if $r$ is a ray in $E$ with one of these directions, then $r$ is asymptotic to $F$. In fact it is asymptotic to a geodesic ray in $F$. This gives a way to identify a dense set of points in $E$ with a dense set of points in $F$. This extends to a unique homeomorphism $\tau_\infty$ between $S^1(E)$ and $S^1(F)$. Again these homeomorphisms satisfy a cocycle and are equivariant under deck transformations.

The universal circle $\mathcal{U}$ of $\mathcal{F}$ is the quotient of $\mathcal{A}$ by these identifications: that is, $x$ in $S^1(E)$ is identified with $y$ in $S^1(F)$ if $\tau_\infty(x) = y$, where $\tau_\infty$ is the map described above.

In both the uniform and non-uniform cases the universal circle induces a vertical foliation in $\mathcal{A}$: two points in $\mathcal{A}$ are in the same leaf of the vertical foliation if they represent the same point of the universal circle. This foliation is by continuous curves in $\mathcal{A}$.

We now introduce the ideal quadrilaterals and parallelepipeds in $\tilde{M}$. Let $a, b, c, d$ be 4 distinct points in $\mathcal{U}$, which are circularly ordered. Let $J$ be a compact interval in the leaf space of $\tilde{F}$. For each $F$ leaf of $\tilde{F}$ in $J$ let $Q_F$ be the ideal quadrilateral in $F$ with ideal points $a_F, b_F, c_F, d_F$ which are the representatives of $a, b, c, d$ in $S^1(F)$. By the properties of the universal circle, the quadrilaterals $Q_F$ vary continuously with $F$. Let $\mathcal{P}$ be the union of $Q_F$ over all $F$ in $J$.

In [Fe1] we proved the following: if $M$ is not Seifert then there is a sequence of parallelepipeds $\mathcal{P}_i$ with tops $Z_i$ and bottoms $X_i$ so that $Z_i$ is very thin in one direction and $X_i$ is very thin in the opposite direction. The thinness of a quadrilateral is the minimum distance between opposites sides of the quadrilateral. An ideal quadrilateral is regular if both such minimum distances are equal. See Figure 1 for a depiction of such a parallelepiped.

In [Fe1] we construct these so that thinness of $Z_i$ converges to 0 in one direction while thinness of $X_i$ converges to 0 in the other direction. These parallelepipeds
are measuring the distortion of the geometry transversally to $\tilde{F}$. Pick a height $F$ where $Q_F$ is a regular quadrilateral. Going up to the top $Z_i$ makes the quadrilateral very thin — in other words stretching the leaf in the direction of the side of the quadrilateral which are very near and contracting the other direction. Going down the opposite happens.

Since the thinness of $Z_i$ converges to 0 they are getting closer and closer to geodesics. Project to $M$ and consider such a limit geodesic $\ell_0$. Lift to a geodesic $\ell$ in a leaf $F$ of $\tilde{F}$. Now saturate $\ell$ by the universal circle, that is take all geodesics in $E$ leaf of $\tilde{F}$ so that ideal points correspond to the same points in the universal circle $\mathcal{U}$ as the ideal points of $\ell$. The union of all of these is a closed set in $\tilde{M}$. The projection is the a priori immersed lamination $\mathcal{L}^s$, with lift $\tilde{\mathcal{L}}^s$. Each leaf $L$ of $\mathcal{L}^s$ intersects the leaves of $\tilde{F}$ in geodesics. We call $\mathcal{L}^s$ a leafwise geodesic lamination.

In the same way considering limits of the bottoms of the parallelepipeds produces the immersed lamination $\mathcal{L}^b$.

We state these results formally for future reference: Given a geodesic $\ell$ in a leaf $F$ of $\tilde{F}$ the saturation of $\ell$ is the union over all $E$ leaves of $\tilde{F}$ of the geodesic $\ell_E$ in $E$, so that the ideal points of $\ell_E$ in $S^1(E)$ and the ideal points of $\ell$ in $S^1(F)$ are the same points under the universal circle identification. We also call this the saturation of $\ell$ by the universal circle. In [Fe1] it is proved that this is a properly embedded plane in $\tilde{M}$. We call it a wall.

**Proposition 2.1.** Let $\mathcal{P}_i$ be a sequence of parallelepipeds in $\tilde{M}$ with tops $Z_i$ very thin in one direction and bottoms $X_i$ very thin in the other direction. Let $\mathcal{L}^s$ be the leafwise geodesic lamination obtained as follows: consider all limits of deck translates of $Z_i$. These form a collection of geodesics in leaves of $\tilde{F}$. Saturate

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{A parallelepiped $\mathcal{P}_i$: this is a 3-dimensional set in $\tilde{M}$ made up of ideal quadrilaterals in an interval of leaves of $\tilde{F}$. The top quadrilateral is $Z_i$ and the bottom one is $X_i$. The curves $a, b, c, d$ in the figure are not made up of points in $\tilde{M}$, rather they are curves of ideal points of leaves of $\tilde{F}$. Each of these curves denotes ideal points of different leaves, but corresponding to the same point in the universal circle. On the top the ideal quadrilateral $Z_i$ is thin in one direction: the geodesic sides $e_1, e_2$ are very close to each other in the respective leaf of $\tilde{F}$. In the bottom, the ideal quadrilateral $X_i$ is thin in the opposite direction: the geodesics $e_3, e_4$ are now close to each other in the respective leaf of $\tilde{F}$.}
\end{figure}
these geodesics by the universal circle. \( \mathcal{L}^s \) is the projection of this collection of saturated walls to \( M \). Let \( \mathcal{L}^u \) be the immersed geodesic lamination obtained by doing the same procedure with the bottoms \( X_i \).

Notice that since different walls may intersect, we keep track of the leaves of \( \mathcal{L}^s \) and not just the set.

2.1. JSJ decomposition and \( \mathbb{R} \)-covered foliations. Here we review some results from from [FP1]. The JSJ decomposition of an irreducible manifold splits it into Seifert and atoroidal pieces [He, Ja, Ja-Sh]. Since we are considering non orientable manifolds we allow Klein bottles amongst the cutting surfaces. Let \( M \) be a 3-manifold with a Reebless, \( \mathbb{R} \)-covered foliation with hyperbolic leaves.

Let \( T \) be a torus or Klein bottle in the JSJ decomposition and \( \tilde{T} \) be a lift to \( \tilde{M} \). Then \( \tilde{T} \) with its path metric is quasi-isometrically embedded in \( \tilde{M} \). This follows from [Ka-Le, Theorem 1.1], see also [Ng, Section 3.1]. Then one can isotope \( T \) so that \( \tilde{T} \) intersects each leaf \( F \) of \( \tilde{\mathcal{F}} \) in a single component which is a quasigeodesic in \( F \). Then one can pull tight these quasigeodesics, and assume that \( T \) satisfies that \( \tilde{T} \) intersects leaves of \( \tilde{\mathcal{F}} \) in geodesics. We always assume this is the case for any \( T \) a cutting surface of the JSJ decomposition. We say that \( T \) is in good position.

In addition we have the following very important fact (Proposition 4.4 of [FP1]): for any \( F \) leaf of \( \tilde{\mathcal{F}} \) let \( \ell_F = \tilde{T} \cap F \), a geodesic in \( F \) with ideal points \( a_F, b_F \) in \( S^1(F) \). Then the set of \( b_F \) as \( F \) varies in \( \tilde{\mathcal{F}} \) is a leaf of the vertical foliation in \( A \). In other words all \( b_F \in S^1(F) \) correspond to the same point in the universal circle \( U \) of \( \mathcal{F} \).

3. Properties of the immersed leafwise geodesic laminations

In this section we prove the main ingredients to produce the pseudo-Anosov flow in an atoroidal piece \( P \).

**Proposition 3.1.** Let \( \mathcal{L}^s, \mathcal{L}^u \) be the immersed leafwise geodesic laminations transverse to \( \mathcal{F} \), as in Proposition 2.1. Then no leaf of \( \mathcal{L}^s \) transversely intersects a torus or Klein bottle of the JSJ decomposition.

**Proof.** Suppose that some \( L \) in \( \tilde{\mathcal{L}}^s \) transversely intersects a lift \( \tilde{T} \) for \( T \) one of the tori or Klein bottles of the JSJ decomposition. If necessary lift to a double cover and we can assume that \( M \) is orientable. Hence we can assume that \( T \) is a torus.

Recall that both \( \tilde{T} \) and \( L \) intersect leaves of \( \tilde{\mathcal{F}} \) in geodesics so that endpoints are constant under the universal circle identification. Therefore this transverse intersection of \( \tilde{T} \) and \( L \) is seen in every leaf of \( \tilde{\mathcal{F}} \). In particular up to deck transformations there are parallelepipeds \( \mathcal{P}_i \) so that the top quadrilaterals, call them \( Z_i \), converge to \( \ell = F \cap L \) for some \( F \) in \( \tilde{\mathcal{F}} \), see construction of \( \mathcal{L}^s \) and section 2. The quadrilaterals \( Z_i \) intersect \( L \cap F \) in segments whose length converge to zero.

We can assume that all the tops of the parallelepipeds are in \( F \). Then \( Z_i \cap L = \mu_i \) are segments converging to \( L \cap F \cap T = p_1 \). Let \( \eta = F \cap \tilde{T} \).

The bottoms of the parallelepipeds \( \mathcal{P}_i \), call them \( X_i \) are quadrilaterals that are very thin in the other direction. They still intersect \( \tilde{T} \) in a geodesic arc. This is because the parallelepoid intersects leaves of \( \tilde{\mathcal{F}} \) in ideal quadrilaterals with ideal points constant when identified with the universal circle. Now this geodesic arc is not very short, as was the case for the top quadrilaterals, but rather very long. Call these geodesic arcs \( \zeta_i \). These project in \( M \) into \( T \). Since the foliation induced by \( \mathcal{F} \) on \( T \) is minimal, by adjusting the bottom slightly, we can assume
that \( \pi(\zeta_i) \) contains \( \pi(\mu_i) \) and distance of \( \pi(p) \) from both endpoints of \( \pi(\zeta_i) \) along \( \pi(\zeta_i) \) goes to infinity with \( i \).

Let \( \gamma_i \) be deck transformations of \( \tilde{M} \) which are also deck transformations of \( \tilde{T} \) which take \( \zeta_i \) to segments in \( \eta \) containing \( \mu_i \) and so that distance along \( \eta \) from \( p \) to endpoints of \( \gamma_i(\zeta_i) \) goes to infinity. This is the crucial property: we are using that \( T \) is compact so we can always bring (by deck transformations) long segments of the lifted foliation \( \tilde{\mathcal{F}} \cap \tilde{T} \) to intersect a compact set.

We now analyze the action of \( \gamma_i \) on the universal circle \( \mathcal{U} \). We use the identification of \( \mathcal{U} \) with \( S^1(F) \). Here \( \tilde{T} \cap F \) is the geodesic \( \eta \) in \( F \). Let \( I \) be a complementary interval in \( S^1(F) \) of the ideal points of \( \eta \). We can parametrize \( I \) as follows: for each \( q \) in \( I \) there is a unique \( q' \) in \( \eta \) so that the geodesic ray in \( F \) from \( q' \) with ideal point \( q \) is perpendicular to \( \eta \). In this way \( I \) is parametrized by \( \eta \). The action of \( \pi_1(T) \) on \( \mathcal{U} \) preserves the points of \( \mathcal{U} \) corresponding to the ideal points of \( \eta \) in \( F \). In other words, when expressing this action in terms of \( S^1(F) \), it follows that \( \pi_1(T) \) preserves \( I \). We analyze the action of \( \pi_1(T) \) on \( I \). Since \( I \) is canonically identified with \( \eta \), this induces an action of \( \pi_1(T) \) on \( \eta \). Let this action be denoted by \( \rho \).

Let the endpoints of \( \mu_i \) be \( x_i, x_i' \) and the endpoints of \( \gamma_i(\zeta_i) \) be \( y_i, y_i' \). By renaming we assume that \( y_i', x_i', x_i, y_i \) are always linearly ordered in \( \eta \). Therefore we have the following property:

**Property 1** – We have points \( x_i \) in \( \eta \) converging to \( p \) which are taken by \( \rho(\gamma_i) \) to \( y_i \) which escape in \( \eta \).

We use the following result. This result almost surely has more hypothesis than what is needed to get a global fixed point of a \( \mathbb{Z}^2 \) action on \( \mathbb{R} \), but it suffices for our needs:

**Lemma 3.2.** Let \( G \cong \mathbb{Z}^2 \) acting on \( \mathbb{R} \cong \eta \) so that there are points \( x_i \) in \( \mathbb{R} \) in a compact set of \( \mathbb{R} \), and let \( g_i \) in \( G \) with \( g_i(x_i) \to \infty \) and \( g_i \) has a fixed point \( < x_i \). For each \( i \) let \( z_i = \lim_{n \to -\infty} g_i^n(x_i) \). Suppose that \( d(z_i, x_i) \) converges to 0 as \( i \to \infty \) and \( z_i \) converges to \( z_0 \) in \( \mathbb{R} \). Then \( z_0 \) is a global fixed point of \( G \).

**Proof.** Since \( G \cong \mathbb{Z}^2 \), the action is by orientation preserving homeomorphisms.

Let \( \beta \) in \( G \) and suppose that \( \beta \) does not fix \( z_0 \). Up to taking an inverse we assume that \( z_0 < \beta(z_0) \). Now \( x_i, z_i \) converge to \( z_0 \) as \( i \to \infty \). Fix \( i \) big enough so that \( \beta(z_i) > x_i \), but \( \beta(z_i) < g_i(x_i) \). Then

\[
g_i \beta(z_i) = \beta g_i(z_i) = \beta(z_i) > x_i.
\]

In other words \( x_i < \beta(z_i) < g_i(x_i) \), and \( \beta(z_i) \) fixed by \( g_i \). This is a contradiction and finishes the proof of the lemma.

**Conclusion of the proof of Proposition 3.1.** Let \( g_i = \rho(\gamma_i) \) acting on \( \mathbb{R} \cong \eta \). By Property 1, we have \( x_i \) in \( \mathbb{R} \) with \( g_i(x_i) = y_i \) and \( y_i \) converging to \( \infty \). In addition \( g_i \) has a fixed point \( z_i \) in \( [x_i', x_i] \) and \( [x_i', x_i] \) converges to \( p \). So \( g_i, z_i, x_i \) satisfy the other properties of Lemma 3.2 with \( z_0 = p \). The lemma shows that \( \pi_1(T^2) \) has a global fixed point.

Consider the set \( A \) of \( \tilde{M} \) which in each leaf \( E \) of \( \tilde{\mathcal{F}} \) is the geodesic ray in \( E \) satisfying

- The starting point of \( r \) is in \( \tilde{T} \cap E \) and \( r \) is perpendicular in \( E \) to \( \tilde{T} \cap E \).
- The ideal point of \( r \) in \( S^1(E) \) and the global fixed point of the action of \( \pi_1(T^2) \) on \( I \subset S^1(F) \) correspond to the same point in the universal circle \( \mathcal{U} \).
Since \( \widetilde{T} \) is a properly embedded plane, the union of the geodesic rays \( r \) as above forms an embedding of a closed half plane in \( \widetilde{M} \).

We claim that \( H = \pi_1(T^2) \) leaves \( A \) invariant. Any \( \gamma \in \pi_1(T^2) \) leaves \( \widetilde{T} \) invariant. When acting on the universal circle \( U \) then \( \pi_1(T^2) \) fixes the point corresponding to the global fixed point of \( \pi_1(T^2) \) acting on \( I \). Let \( E \) in \( \widetilde{F} \) and \( r = E \cap \widetilde{T} \). Then \( \gamma(r) \) is contained in a leaf \( D \) of \( \widetilde{F} \), and since \( \gamma \) is an isometry then \( \gamma(r) \) is a geodesic ray in \( D \), \( \gamma(r) \) starts in \( \widetilde{T} \) and \( \gamma(r) \) is perpendicular to \( \widetilde{T} \cap D \) in \( D \). Finally \( \gamma(r) \) has ideal point in \( S^1(D) \) which is the same point as the global fixed point of \( \pi_1(T^2) \) acting on \( I \) under the universal circle identification. It follows that \( \gamma(r) = D \cap \widetilde{T} \), hence \( \gamma \) preserves \( A \).

In particular \( \pi_1(T^2) \) leaves invariant the infinite curve \( \partial A \). This is impossible since \( \pi_1(T^2) \) has to act freely and properly discontinuously on \( \partial A \).

This finishes the proof of Proposition 3.1. \( \square \)

We now prove a further property:

**Proposition 3.3.** Suppose that \( P \) is an atoroidal piece of \( M \). Then there is an immersed leafwise geodesic lamination \( \mathcal{L}^s \) (as in Proposition 2.1) in \( P \) and no leaf of \( \mathcal{L}^s \) isotopic to a component of \( \partial P \). Similarly for \( \mathcal{L}^u \).

**Proof.** We do the proof for \( \mathcal{L}^s \). In order to do this we will use the following doubling trick. Cut \( M \) along the components of the boundary of \( P \) and double \( P \) along the boundary. Let this be the manifold \( N \), which we think of \( P \cup P' \) where \( P' \) is a copy of \( P \). Now do the whole analysis for \( N \). As in the proof of Proposition 3.1 we initially lift to a double cover if necessary and assume that \( N \) is orientable.

The foliation \( N \) has an \( \mathbb{R} \)-covered foliation \( \mathcal{F}' \) which is the double of \( \mathcal{F}|_P \). This foliation \( \mathcal{F}' \) has hyperbolic leaves. Let \( \mathcal{V} \) be the universal circle of \( \mathcal{F}' \). If the action of \( \pi_1(N) \) on \( \mathcal{V} \) is uniformly quasisymmetric, then \( N \) is Seifert fibered. In particular \( P \) is Seifert fibered, contradiction to the hypothesis on \( P \). It follows that the action of \( \pi_1(N) \) on \( \mathcal{V} \) is not uniformly quasisymmetric. Hence there are parallelepipeds \( P_i \) producing laminations in \( N \) given by Proposition 2.1, and still denoted by \( \mathcal{L}^s, \mathcal{L}^u \). We will prove that they induce laminations in \( M \) as required.

By the previous proposition we know that no leaf \( \mathcal{L}^s \) intersects a boundary component of \( P \) transversely. If we prove that no boundary component of \( P \) is a leaf of \( \mathcal{L}^s \) then we get a sublamination of \( \mathcal{L}^s \) contained in either \( P \) or \( P' \). If contained in \( P' \) then the mirror image is contained in \( P \).

Suppose that a component \( T \) of \( \partial P \) is contained in \( \mathcal{L}^s \). We use the setup of the proof of the previous proposition. Suppose that some leaf of \( \mathcal{L}^s \) is a torus \( T \) of the JSJ decomposition of \( N \). Let \( \widetilde{T} \) a lift. As in the proof of the previous proposition there are parallelepipeds denoted by \( P_i' \) so the tops are in a fixed leaf \( F \) of \( \widetilde{F} \) and converge to \( \eta = F \cap \widetilde{T} \). Let the ideal points of \( \eta \) in \( S^1(F) \) be \( x_1, x_2 \). We denote the tops by \( Z_i' \). We will adjust the tops and produce a new set of parallelepipeds \( P_i \), still satisfying the thin conditions as before.

Let \( a_i', b_i', c_i', d_i' \) be the ideal points of \( Z_i' \) in \( S^1(F) \), so that the geodesics \( (a_i', b_i') \) and \( (c_i', d_i') \) are very close in \( F \) to \( \eta \). Up to renaming the points, assume that \( (a_i', d_i') \) are very close in \( F \cup S^1(F) \) to \( x_1 \) and \( (b_i', c_i') \) are very close to \( x_2 \). Now we do the symmetrization of \( Z_i' \) with respect to \( \eta \). Recall that \( N \) is the double of \( P \), so there is a reflection in the universal circle \( \mathcal{V} \) of \( \mathcal{F}' \) with respect to the ideal points of \( \eta \) (seen as points in \( \mathcal{V} \)). Denote this reflection map by \( \xi : \mathcal{V} \to \mathcal{V} \). Replace \( Z_i' \) by an ideal quadrilateral \( Z_i \) in \( F \) with ideal points \( a_i, b_i, c_i, d_i \) as follows. Consider the pair \( b_i', c_i' \). Both are very close to \( x_2 \) and are distinct from
each other, so at least one is distinct from \( x_2 \). If \( \xi(b'_i) = c'_i \), we choose \( b_i = b'_i \)
and \( c_i = c'_i \). Otherwise one of \( b'_i \) or \( c'_i \) is farther from \( x_2 \) — use the reflection \( \xi \) to
compare them if on opposite sides of \( x_2 \). Suppose the farthest point is \( b'_i \). Then
let \( b_i = b'_i \) and choose \( c_i = \xi(b_i) \). Do the same for the pair \( a'_i, d'_i \). The resulting
quadrilateral with ideal points \((a_i, b_i, c_i, d_i)\) is denoted by \( X_i \). It has ideal points
still very close to \( x_1, x_2 \) respectively, so it is very thin in the same direction that
\( X'_i \) is. In particular by construction the thinness in this direction goes to 0 as
\( i \to \infty \). But \( X_i \) is less thin in this direction than \( X'_i \) since we may have pushed
a pair of endpoints slightly farther away from \( x_1, x_2 \) respectively.

Let now \( P_i \) be parallelepiped intersecting the same set of leaves of \( \tilde{F} \) that
\( P'_i \) intersects, but the top is now \( Z_i \) instead of \( Z'_i \). Let \( X_i \) denote the bottoms
of \( P_i \). The tops are very thin in the direction very close to \( \eta \). Since we made
the tops slightly thinner in the opposite direction — they are still very thick in
that direction, but slightly less thick, then the bottoms \( X_i \) still have thinness
converging to 0 in the opposite direction. We explain a bit more: when moving
the ideal quadrilaterals from \( X_i \) across leaves of \( \tilde{F}' \) using the universal circle to
move the ideal points, the following happens: the top quadrilateral \( Z'_i \) moves to
\( X'_i \) which is very thin in the opposite direction. Since \( Z_i \) is slightly thinner than
\( Z'_i \) in the opposite direction then \( Z_i \) moves to even thinner quadrilaterals \( X_i \) in
the opposite direction.

We use this sequence of parallelepipeds \( P_i \). Notice that by construction of \( N \)
as the double of \( P \) the universal circle \( V \) is symmetric with respect to \( \eta \). In other
words this implies that for any leaf \( E \) of \( \tilde{F} \) intersecting \( P_i \) then the quadrilateral
\( Q'_E = P_i \cap E \) in \( E \) is symmetric with respect to \( \tilde{T} \cap E \). In particular the bottom
\( X_i \) of \( P_i \) is also symmetric with respect to the intersection of \( \tilde{T} \) with that leaf.
In addition any deck translate of \( P_i \) under an element of \( \pi_1(T) \) is still symmetric
with respect to \( \tilde{T} \).

Orient the one dimensional foliation \( F' \cap T \).

![Figure 2. An example of action on the double \( N = P \cup P' \).](image)

We will use the setup of Proposition 3.1. Now \( I \) is an open interval in the
universal circle \( V \) of \( F' \). Recall that \( X_i \) are the bottoms of the parallelepipeds
\( P_i \). Consider \( X_i \cap \tilde{T} \) which is a compact segment denoted by \( \zeta_i \) of the foliation
induced by \( F' \) in \( T \). Let \( v_i \) be the positive endpoint of \( \zeta_i \) with respect to the
orientation of $\mathcal{F}' \cap T$. Up to subsequence assume that $\pi(v_i)$ converges. Up to a slight modification of the bottoms $X_i$, we can assume that $\pi(v_i)$ are always in the same local leaf of the foliation in $T$. So there are $g_i$ in $\pi_1(T)$ with $g_i(v_i)$ in a fixed leaf $F$ of $\tilde{\mathcal{F}}$ which we can assume is $F$, and suppose that $g_i(v_i)$ converging to a point $v_0$. In other words $g_i(v_i)$ are all in the curve $\eta = F \cap T$. If the lengths of $g_i(\zeta_i)$ do not converge to infinity, then the induced action of $g_i$ on $\eta \cong I$ has big intervals which are contracted to a bounded subcompact interval. This brings us to a setup that Lemma 3.2 produced a global fixed point of the action of $\pi_1(T)$ on $I$. As seen in the proof of Proposition 3.1 this leads to a contradiction.

We conclude that the lengths of $g_i(\zeta_i)$ converge to infinity. This means that if the other endpoint of $g_i(\zeta_i)$ is denoted by $q_i$ then $q_i$ escapes in $\eta$. But the bottoms $X_i$ are symmetric with respect to $\tilde{T}$ and the action of $\pi_1(T)$ on $\mathcal{V}$ is symmetric with respect to $\tilde{T}$. Let the ideal points of $X_i$ be $a_i^1, b_i^1, c_i^1, d_i^1$ in $S^1(E_i)$, corresponding to $a_i, b_i, c_i, d_i$ respectively in $S^1(F)$. Here $E_i$ are the leaves of $\tilde{\mathcal{F}}'$ containing the bottoms $X_i$ of the parallelepipeds $\mathcal{P}_i$. By the above $g_i(a_i^1)$ converges to a point $t_0$ in the interior of $I$. Also $g_i(b_i^1)$ converges to an ideal point $x_2$ of $\eta$ in $S^1(F)$. By the symmetry property, $g_i(c_i^1)$ converges to $t_1$ and $g_i(d_i^1)$ converges to the mirror image of $t_0$ on the other side of $\eta$. See figure 2.

This means that the sides $(g_i(d_i^1), g_i(a_i^1))$ and $(g_i(c_i^1), g_i(b_i^1))$ of $g_i(X_i)$ are not getting close to each other in $F$. Hence $(d_i^1, a_i^1)$ and $(c_i^1, b_i^1)$ are not getting closer to each other in $E_i$ as well. This is a contradiction, by construction of the parallelepipeds $\mathcal{P}_i$.

This proves that $T$ cannot be a leaf of $\mathcal{L}^s$. In fact we proved the following: for any choice of distortion parallelepipeds $\mathcal{P}_i$ in $\tilde{N}$ so that the tops $Z_i$ converge to a geodesic $\ell$ in any leaf $F$ of $\tilde{\mathcal{F}}'$ then $\ell$ cannot cross any lift of torus in $N$, nor be contained in any such lift.

Dealing with a subtle point

We obtained an immersed leafwise geodesic lamination $\mathcal{L}^s$ in $N$ which is contained in $P$ and it is obtained by taking limits of distortion parallelepipeds $\mathcal{P}_i$. In fact $\mathcal{L}^s$ is contained in the interior of $P$ and so induces an immersed leafwise geodesic lamination in $M$. The subtle point is that the distortion parallelepipeds $\mathcal{P}_i$ are contained in $\tilde{N}$, but do not necessarily generate distortion parallelepipeds in $\tilde{M}$ which will generate $\mathcal{L}^s$ in $M$. We will adjust our construction of the distortion parallelepipeds.

For notational reasons we will rename our parallelepipeds $\mathcal{P}'_i$. We will adjust the $\mathcal{P}'_i$ to obtain new parallelepipeds (to be denoted $\mathcal{P}_i$) with the properties we need. What we want is that the distortion parallelepipeds can be chosen contained in a fixed lift $\tilde{P}$ of $P$ to $\tilde{N}$.

First we prove a preliminary fact. Consider the leaf $F$ of $\tilde{\mathcal{F}}'$ as in the beginning of the proof of the proposition. The intersection $F \cap \tilde{P}$ is a hyperbolic surface with geodesic boundary. The tops $Z'_i$ of the parallelepipeds $\mathcal{P}'_i$ converge to a geodesic $\ell$ in $F$. We proved that $\ell$ is contained in the interior of $F \cap \tilde{P}$.

We claim that $\ell$ is not asymptotic to a geodesic $g$ in $F$ which is the intersection of $\tilde{T}$ with $F$ for some JSJ surface $T$ of $N$ (notice that here we are not taking a double cover of $N$ to make it orientable). Since we are taking all $\pi_1(N)$ translates and closures of the limits this means that some geodesic $E \cap \tilde{T}$ in $\tilde{T}$ ($E$ leaf of $\tilde{\mathcal{F}}'$) is contained in all the limits. This we proved above that it is impossible, proving the claim. In addition any ideal point $p$ of $\ell$ in $F$ is accumulated in $F \cup S^1(F)$ by geodesics $g_i$ which are intersections of lifts of JSJ surfaces with $F$. Otherwise
a half plane in \( F \) does not intersecting such a lift, and hence taking limits a full leaf does not intersect such a lift contradiction.

Now we adjust the parallelepipeds \( P_i' \). Let \( \ell \) have ideal points \( x_1, x_2 \). Let the tops of \( P_i' \) be \( Z_i' \) with ideal points \( a_i, b_i, c_i, d_i \) so that \( a_i, d_i \) are very close to \( x_1 \) and \( b_i, c_i \) very close to \( x_2 \). We will enlarge the ideal quadrilateral \( Z_i' \) still keeping it very thin in the direction close to the geodesic \( \ell \). Let \( I_i \) be the interval in \( S^1(F) \) with ideal points \( a_i, d_i \) and very close to \( x_1 \). Recall that there are sequences of endpoints of \( \tilde{T}' \cap F \) for \( \tilde{T}' \) lifts of JSJ surfaces converging to \( x_1 \). Hence for \( i \) big we can choose \( a_i \) arbitrarily close to \( a_i', \) \( a_i \) not in the interior of \( I_i \) and \( a_i \) an ideal point of \( \tilde{T}' \cap F \) for some lift \( \tilde{T}' \) of a JSJ surface. Do the same for \( d_i', b_i', c_i' \), producing \( a_i, b_i, c_i, d_i \). These are distinct and define an ideal quadrilateral \( Z_i \).

Let \( P_i \) the parallelepipeds intersecting the same leaves of \( \tilde{F} \) that \( P_i' \) does but defined by the ideal points \( a_i, b_i, c_i, d_i \). Now we prove properties of \( P_i \). The geodesics \( (a_i, d_i) \) are very close to \( (a_i', d_i') \) and similarly \( (b_i, c_i) \) are very close to \( (b_i', c_i') \). So \( Z_i \) is very thin in the same direction that \( Z_i' \) is. Let \( X_i \) be the bottoms of the parallelepipeds \( P_i \). The \( Z_i \) are very thin in the \( \ell \) direction, but less thin than \( Z_i' \) in this direction. This implies that the \( Z_i \) are a little bit thinner than the \( Z_i' \) in the other direction. This implies that the \( X_i \) are even thinner than the \( X_i' \) in the opposite direction.

We conclude that the \( P_i \) have thinness in the top \( (Z_i) \) in one direction converging to zero, and in the bottom \( (X_i) \) thinness in the other direction converging to zero. Hence the \( P_i \) are distortion parallelepipeds which yield laminations \( \mathcal{L}^s \) and \( \mathcal{L}^u \).

We finally prove the crucial property we want. By choice of the points \( a_i, b_i, c_i \) and \( d_i \) they are ideal points of the hyperbolic surface \( F \cap \tilde{P} \). In particular the geodesics \( (a_i, b_i), (b_i, c_i), (c_i, d_i) \) and \( (d_i, a_i) \) are contained in \( F \cap \tilde{P} \). Hence the tops \( Z_i \) are entirely contained in \( \tilde{P} \). Since the quadrilaterals in \( P_i \) are obtained by following the universal circle and so is \( \tilde{P} \), it follows that \( P_i \cap E \) is contained in \( \tilde{P} \) for any leaf \( E \) of \( \tilde{F} \) that it intersects. In particular \( P_i \) is entirely contained in \( \tilde{P} \). This is the fact we wanted to prove.

Since \( P_i \) is entirely contained in \( \tilde{P} \) then we can think of them also as contained in \( \tilde{M} \). The quadrilaterals in \( P_i \) have the same leafwise geometry whether seen in \( \tilde{N} \) or in \( \tilde{M} \). It follows that \( P_i \) are distortion quadrilaterals in \( \tilde{M} \), and of course they generate the immersed leafwise geodesic laminations \( \mathcal{L}^s \) and \( \mathcal{L}^u \) in \( \tilde{M} \). At this point we can completely forget \( \tilde{N} \) and consider all objects \( P_i, \mathcal{L}^s, \mathcal{L}^u \) in \( \tilde{M} \) or in \( M \).

This finishes the proof of the proposition.

By the above proposition there is an immersed lamination \( \mathcal{L}^s \) contained in the interior of \( P \) and an immersed lamination \( \mathcal{L}^u \) contained in the interior of \( P \).

**Remark 3.4.** This proof used the auxiliary manifold \( N \) a doubling of \( P \). A priori some other construction using distortion parallelepipeds in \( M \) could yield a lamination so that a component of \( \partial P \) is a leaf. We strongly believe this is not possible, but we are not able to prove this at this point.

**Proposition 3.5.** The laminations \( \mathcal{L}^s, \mathcal{L}^u \) constructed in Proposition 3.3 are embedded.

**Proof.** We will extensively use the analysis of [Fe1]. In pages 458 and 459 of [Fe1] we showed that there are 3 options for the immersed leafwise geodesic laminations \( \mathcal{L}^s, \mathcal{L}^u \):
• **Option A.** No leaf of \( \mathcal{L}^s \) transversely intersects another leaf of \( \mathcal{L}^s \). In other words \( \mathcal{L}^s \) is an embedded leafwise geodesic lamination. There is an analogous statement for \( \mathcal{L}^u \).

• **Option B.** No leaf of \( \mathcal{L}^s \) intersects transversely a leaf of \( \mathcal{L}^u \).

• **Option C.** There is a leaf of \( \mathcal{L}^s \) transversely intersecting a leaf of \( \mathcal{L}^u \).

In [Fe1] pages 459-464 it is proved that Option C implies Option A for both \( \mathcal{L}^s \) and \( \mathcal{L}^u \). This is explicitly stated in Lemma 5.6 of [Fe1].

Fix a lift \( \tilde{P} \) of \( P \).

Suppose that one of the laminations is not embedded, without loss of generality suppose that \( \mathcal{L}^s \) self intersects transversely. We will prove that option C holds in this case. This will produce a contradiction since Option C implies Option A for both \( \mathcal{L}^s \) and \( \mathcal{L}^u \). This will prove that both \( \mathcal{L}^s \) and \( \mathcal{L}^u \) are embedded.

Suppose then that Option C does not hold. By hypothesis there are leaves \( L, L' \) of \( \tilde{\mathcal{L}}^s \) contained in \( \tilde{P} \) which intersect transversely. We consider all leaves \( L' \) of \( \tilde{\mathcal{L}}^s \) in \( \tilde{P} \) so that there is a sequence \( L_0 = L, L_1, \ldots, L_k = L' \) of leaves of \( \tilde{\mathcal{L}}^s \cap \tilde{P} \) with \( L_i \) intersecting \( L_{i-1} \) transversely for all \( 1 \leq i \leq k \). This forms a subset \( \mathcal{B} \) of leaves of \( \tilde{\mathcal{L}}^s \) in \( \tilde{P} \).

We consider the convex hull of the set of leaves in \( \mathcal{B} \). envelope of the these set of leaves. The boundary of the convex hull is made up of geodesics. When one varies the leaf in \( \tilde{\mathcal{F}} \) each boundary component of the convex hull varies according to the universal circle of \( \mathcal{F} \). In addition there are no transverse self intersections when projecting to \( M \). This is because it is the convex hull of these chains of consecutively intersecting leaves. In addition the projection of the boundary of the convex hull is made up of compact surfaces, for the same reason, see Thurston [Th6] or [Fe1]. Hence these compact surfaces in \( M \) are either tori or Klein bottles. They are contained in \( P \), hence have to be homotopic to boundary components of \( P \). But since these surfaces intersect leaves of \( \mathcal{F} \) in geodesics it follows that these surfaces are all components of \( \partial P \). But no leaf of \( \mathcal{L}^u \) in \( P \) can intersect any leaf of \( \mathcal{L}^s \) in \( \pi(\mathcal{B}) \). It follows that some component of the projection of the boundary of the convex hull separates \( \pi(\mathcal{B}) \) from some leaves of \( \mathcal{L}^u \) in \( P \). This contradicts that this projection is a component of \( \partial P \).

The contradiction implies that both \( \mathcal{L}^s \) and \( \mathcal{L}^u \) are embedded and finishes the proof of the proposition. \( \square \)

Fix a piece \( P \). Fix a lift \( \tilde{P} \) to \( \tilde{M} \). Let \( \mathcal{L}^s_m \), be a minimal sublamination of \( \mathcal{L}^s \), and similarly define \( \mathcal{L}^u_m \) both leafwise geodesic laminations in the interior of \( P \). We now know that they are embedded. We will obtain properties of \( \mathcal{L}^s_m, \mathcal{L}^u_m \). In fact most of the properties are proved in [Fe1]. A *crown* is a hyperbolic surface which is a half open annulus: its completion has one boundary component which is a closed geodesic. There are finitely many boundary components. The other boundary components are infinite geodesics, which are consecutively asymptotic – see [Bl-Ca].

**Lemma 3.6.** \( \mathcal{L}^s_m, \mathcal{L}^u_m \) are not compact leaves. \( \mathcal{L}^s_m, \mathcal{L}^u_m \) are distinct and intersect transversely. The complementary regions of \( \mathcal{L}^s_m \) (or \( \mathcal{L}^u_m \)) in \( P \) are either \( S^1 \) bundles over open finite sided ideal polygons (generating open solid tori or solid Klein bottles) or \( S^1 \) bundles over crowns, generating sets homeomorphic to torus \( \times [0,1) \) or Klein bottle \( \times [0,1) \).}

**Proof.** We do the proof for \( \mathcal{L}^s_m \). Suppose that \( \mathcal{L}^s_m \) is a compact leaf \( B \). Since it has a one dimensional foliation, then it is either a torus or a Klein bottle. Since \( P \) is atoroidal, then \( B \) is isotopic to a component of \( \partial P \). Lift \( B \) to a cover \( \tilde{B} \)
boundedly isotopic to a boundary component $S$ of $\tilde{P}$. For each $F$ leaf of $\tilde{F}$ then $B \cap F$ and $S \cap F$ are geodesics which are boundedly isotopic. Hence they are the same geodesic. In other words $B$ is a component of $\partial P$. We proved in Proposition 3.3 that this is impossible. This finishes the proof of the first assertion.

Now consider complementary regions. First of all $\mathcal{L}_s^m$ is not a foliation in $P$, since it does not intersect $\partial P$. For each complementary region $V$ consider the boundary $S$ of the set in $V$ which is $\epsilon$ near $\mathcal{L}_m^s$. In [Fe1, Lemma 6.3] it is proved that each component of $S$ is either a torus or a Klein bottle. If this complementary region is not peripheral, then the torus or Klein bottle is compressible and bounds a solid torus or solid Klein bottle. Proposition 6.1 of [Fe1] further shows that $V$ is a $S^1$ bundle over a finite sided ideal polygon. If the region is peripheral then $S$ is isotopic to a component $S'$ of the boundary. $S'$ is either a torus or Klein bottle and $V$ is homeomorphic to $S' \times [0, 1]$. The other boundary components are in annular or Möbius band leaves of $\mathcal{L}_m^s$. As in Proposition 6.1 of [Fe1] there are finitely many of them, they are asymptotic, leading to $V$ being an $S^1$ bundle over a crown surface. This finishes the proof of the lemma.

\begin{lemma}
$\mathcal{L}_s^m, \mathcal{L}_n^m$ are distinct and intersect transversely. The interior complementary regions of $A = \mathcal{L}_s^m \cup \mathcal{L}_n^m$ in $P$ are either finite sided polygons (with compact completion) times $\mathbb{R}$ or an $S^1$ bundle over a finite side polygon (with compact completion). The peripheral complementary regions of $A$ have completion torus or Klein bottle times $[0, 1]$. The non peripheral boundary is made up of compact annuli or Möbius bands contained in leaves of $\mathcal{L}_s^m$ or $\mathcal{L}_n^m$.
\end{lemma}

\begin{proof}
The proof that they are distinct is exactly as in [Fe1]. Let $L'$ be a boundary leaf of (say) $\mathcal{L}_s^m$. Then it is an annulus or Möbius band with $\pi_1(L')$ generated by $\gamma$. Let $L$ be the lift of $L'$ to $\tilde{M}$ with $\gamma(L) = L$. This is analyzed in detail in Lemma 6.6 of [Fe1]. There the following is proved: Let $\theta(\gamma)$ be the action of $\gamma$ on the universal circle $U$ of $F$, and suppose that $\gamma$ is monotone increasing on the leaf space of $\tilde{F}$. Let $I$ be the open interval in $U$ determined by the ideal points of $L \cap F$ for some fixed leaf $F$ of $\tilde{F}$ – under the identification $S^1(F) \cong U$.

Satisfying the following: there are leaves $E_n$ in $\mathcal{L}_m^s$ converging to $L$ and with ideal points of $E_n \cap F$ in $I$ – again under the identification $S^1(F) \cong U$. Recall that $L$ is isolated on one side, but not on the other. This is because $\mathcal{L}_s^m$ is minimal, but not a compact leaf. Then Proposition 6.6 of [Fe1] proves that the action of $\theta(\gamma)$ in $I$ is a contraction with a single fixed point. In the case of the lamination $\mathcal{L}_m^s$ the same proposition shows that $\theta(\gamma)$ acts as an expansion in $I$. These are contradictory and show that $\mathcal{L}_n^s$ and $\mathcal{L}_n^m$ are not the same lamination.

Since both $\mathcal{L}_m^s, \mathcal{L}_n^s$ are minimal it follows that they do not share any leaf. By the properties of the complementary regions of $\mathcal{L}_m^s$ and $\mathcal{L}_m^n$ (separately) it now follows that they have to intersect transversely. The components of the intersection of complementary regions of $A$ with any leaf of $F$ have to have compact completion. The description of the interior complementary regions of $A$ is done in Proposition 6.11 of [Fe1]. The description of the peripheral components follows from the description of the peripheral complementary components of $\mathcal{L}_s^m$ and $\mathcal{L}_n^m$ in $P$ separately, done in the previous Lemma.
\end{proof}

4. One Prong Pseudo-Anosov Flows and Blow Ups in Atoroidal Pieces

We generalize the notion of pseudo-Anosov flows to include one prongs:

\begin{definition}
(One prong pseudo-Anosov flows) A flow $\phi$ in a closed 3 manifold $Q^3$ is a one prong topological pseudo-Anosov flow if there are no point orbits of...
\( \varphi \) and orbits of \( \varphi \) are contained in a pair of (possibly singular) two dimensional foliation \( \mathcal{E}^s, \mathcal{E}^u \) weak stable and weak unstable of \( \varphi \), satisfying:

- All flow lines in a leaf of \( \mathcal{E}^s \) are forward asymptotic. In the backwards direction the orbits diverge from each other in the intrinsic metric of the two dimensional leaves. Similarly for \( \mathcal{E}^u \) with the reversed direction.
- The (topological) singularities of \( \mathcal{E}^s, \mathcal{E}^u \) are all of \( p \)-prong type where \( p \) is a positive integer which can be equal to one. The singular locus is a finite union of periodic orbits of \( \varphi \). The singular locus of \( \mathcal{E}^s \) is the same as the singular locus of \( \mathcal{E}^u \).
- The foliations \( \mathcal{E}^s, \mathcal{E}^u \) are (topologically) transverse to each other and intersect exactly along the flow lines of \( \varphi \).

**Theorem 4.2.** Let \( \mathcal{F} \) be a transversely oriented \( \mathbb{R} \)-covered foliation with hyperbolic leaves in a 3-manifold \( M \). Suppose that there is an atoroidal piece \( P \) in the JSJ decomposition of \( M \). Then there is a one prong pseudo-Anosov flow in a closed manifold \( P \), obtained from \( P \) by collapsing each boundary component of \( P \) to a circle.

**Proof.** A part of this is done carefully in section 7 of [Fe1], which itself just follows the constructions of Mosher [Mos1, Mos2]. There is a problem with the collapsing near each component of the boundary of \( P \) which we will explain how to adjust. Let \( A = \mathcal{L}_m^s \cup \mathcal{L}_m^u \). For each leaf \( F \) of \( \mathcal{F} \) restricted to \( P \) we collapse every closure of a component of

\[
F - (\mathcal{L}_m^s \cup \mathcal{L}_m^u)
\]

not intersecting the boundary of \( P \) to a point. The laminations \( \mathcal{L}_m^s \) and \( \mathcal{L}_m^u \) collapse to two dimensional foliations \( \mathcal{E}^s, \mathcal{E}^u \) in the collapsed set. Most of these closures of complementry regions in \( F \) are compact quadrilaterals. Finitey many of these closures which are also in the interior of \( P \) are finite sided polygons with compact closure having \( 2p \) sides. Here \( p \geq 3 \). The \( p \) boundary leaves of \( \mathcal{L}_m^s \) associated with this complementary region collapse to a \( p \)-prong singularity of \( \mathcal{E}^s \). Proposition 6.11 of [Fe1] states that for every such complementary region of \( \mathcal{L}_m^s \) there is also a complementary region of \( \mathcal{L}_m^u \) which intersects the leaf \( F \) in a \( p \)-sided ideal polygon. So the same complementary region of \( A \) also generates a \( p \)-prong singularity of \( \mathcal{E}^u \).

But there is a problem with the peripheral complementary components: the problem in peripheral components is that the leaves of \( \mathcal{F} \) may not intersect them in sets with compact completion. For the interior components, since \( \mathcal{L}_m^s, \mathcal{L}_m^u \) fill \( P \) then they are either solid tori or solid Klein bottles, and a leaf \( F \) intersects such a component locally in a finite sided ideal polygon with compact closure in \( P \). That is, if you look at a local leaf \( F \) intersecting the boundary, and go around the boundary, the intersection with \( F \) closes up and bounds a disk in \( F \) in the complementary component. But for a peripheral complementary component \( W \) look at how a leaf \( F \) intersects the boundary of \( W \) contained in \( \mathcal{L}_m^s \cup \mathcal{L}_m^u \): when you go around it may not close up. In fact if you continue going around maybe it will be dense in this boundary component of \( W \) and the collapsing of points in \( F \) will be an awful topological space.

In order to deal with this we do the following: Let \( \alpha \) be a closed curve which is an intersection of a leaf \( L \) of \( \mathcal{L}_m^s \) intersecting the boundary of \( W \) with a leaf of \( \mathcal{L}_m^u \) intersecting the boundary of \( W \). Consider a local annulus \( C \) in \( L \) with one boundary in \( \alpha \) and entering \( P - W \). Cut \( P - W \) along this annulus \( C \). Do this for all peripheral complementary components. Now do the collapsing along closures of intersections of \( F \) leaf of \( \mathcal{F} \) with the complement of \( A = \mathcal{L}_m^s \cup \mathcal{L}_m^u \).
The laminations $\mathcal{L}_m^s, \mathcal{L}_m^u$ project to foliations in the collapsed set. The curve $\alpha$ is still a closed curve in the collapsed space. The intersection of the laminations $\mathcal{L}_m^s$ and $\mathcal{L}_m^u$ is a one dimensional foliation in this object. It is orientable because the intersection of $\mathcal{L}_m^s$ and $\mathcal{L}_m^u$ was transverse to the foliation $\mathcal{F}$ before collapsing. Therefore the collapse of this intersection induces a flow. We orient the flow going in the negative direction transverse to the foliation. Finally glue the two sides of the opened up annulus $C$ so that flow lines glue to flow lines. The quotient is a closed manifold $P_s$ with two induced foliations $\mathcal{E}^s, \mathcal{E}^u$.

As proved in Proposition 7.2 of [Fe1] orbits in the same leaf of $\mathcal{E}^s$ are forward asymptotic and in a leaf of $\mathcal{E}^u$ they are backwards asymptotic. Hence this flow is a one prong pseudo-Anosov flow. The reason for the possible one prongs is because of the peripheral components of $P$: the boundary of a component $W$ as above in the interior of $P$ may be made up of a single annulus or Möbius band in $\mathcal{L}_m^u$ and a single annulus or Möbius band in $\mathcal{L}_m^s$. Then the collapsing will fold over the two sides of the annulus or Möbius band leaves producing a one pronged leaf for each of $\mathcal{E}^s$ and $\mathcal{E}^u$ in the quotient. This situation is in fact extremely common: for example consider the suspension case where $\mathcal{F}$ is a fibre over the circle and the monodromy has a pseudo-Anosov component. The pseudo-Anosov map associated with this monodromy may have one prongs when collapsing a boundary component to a point. This is exactly the same that happens here.

This finishes the proof of the theorem.

4.1. Blow ups of one pronged pseudo-Anosov flows. We now do blow ups of one pronged pseudo-Anosov flows following Fried’s method [Fr]. Consider a periodic orbit $\alpha$ of a one pronged pseudo-Anosov flow in a manifold $Q$. The return map of flow lines in a local a cross section satisfies the following: in a prong of the weak stable leaf of $\alpha$ is conjugated to $x \mapsto 4x$. In a weak unstable leaf it is conjugated to $x \mapsto 2x$. This obviously is only topological conjugation. Fried [Fr] blew up the orbit $\alpha$ to its projective tangent bundle. The derivative of the return map above induces a flow in the blown up set. This extends the flow in $Q - \alpha$ to the blown up set with boundary.

Now return to the situation of Theorem 4.2. Given the construction of the laminations $\mathcal{L}_m^s, \mathcal{L}_m^u$ and the collapsed one prong pseudo-Anosov flow in $P^s$ the next result follows immediately.

**Corollary 4.3.** Under the hypothesis of Theorem 4.2 the blow up of the one pronged pseudo-Anosov flow in $P_s$ produces a flow $\Phi$ in $P$ which is transverse to $\mathcal{F}$ restricted to $P$ and which is regulating for $\mathcal{F}$ restricted to $P$.

As remarked above the singular foliations $\mathcal{E}^s, \mathcal{E}^u$ in $P_s$ may have one prong singularities. The blown up objects in $P$ are denoted by $\mathcal{E}^s_b, \mathcal{E}^u_b$. They are foliations in the interior of $P$ but are not foliations in the boundary. The are only points in the boundary that are parts of leaves are those that come from the blow up of the leaves in $P_s$ that originated from collapsing the boundary of $P$. There only finitely many of these in $\mathcal{E}^s_b$ and in $\mathcal{E}^u_b$. For example if $\mathcal{E}^s$ has a one prong in $\gamma$, then the blow up of $\gamma$ will be a torus intersecting leaves of $\mathcal{E}^s_b$ in a single component. This is illustrated in figure 3. The figure shows the 3 steps in the process: Figure (a) depicts the laminations $\mathcal{L}_m^s, \mathcal{L}_m^u$. These do not fill $P$, a local cross section intersects each in a Cantor set. Figure (b) shows the singular foliations $\mathcal{E}^s, \mathcal{E}^u$ in $P_s$. Figure (c) shows the blow up of figure (b). The blow up of the foliations $\mathcal{E}^s, \mathcal{E}^u$ is a foliation in the interior of $P$, but not on the boundary.
Figure 3. Figures of the objects in the construction. a) Depicts leaves of the laminations $L^s_m, L^u_m$. The leaves $A, C$ are extremal, there is a path from boundary of $P$ to these leaves not intersecting any other leaf of the appropriate lamination. b) The corresponding figure in $P^\ast$. The leaf $A$ of the lamination collapses to a one prong leaf (this is the situation of a one prong), denoted by $A'$ in this figure. The leaf $C$ collapses to $C'$. $A', C'$ intersect locally in the one prong depicted. c) The blow up. The singularity blows up to a torus. $A'$ blows up to the leaf $A''$ intersecting the boundary in a circle, similarly $C'$ blows to $C''$.

4.2. Density of regular periodic orbits of $\Phi$ in $P$. We denote by $\Phi_\ast$ the one prong pseudo-Anosov flow in $P_\ast$. The closed orbits of $\Phi_\ast$ which are obtained by collapsing components of $\partial P$ are called the boundary collapsed orbits. These are exactly the orbits that may be one prong orbits of the one prong pseudo-Anosov flow $\Phi_\ast$.

**Proposition 4.4.** The flow $\Phi_\ast$ is transitive, that is, the set of periodic orbits of $\Phi_\ast$ is dense. The same is true of the blow up flow $\Phi$ in $P$. In fact, the set of regular periodic orbits of $\Phi$ in $P$ is dense in $P$.

**Proof.** If necessary lift to a double cover so that $P^\ast$ (and hence $P$ is orientable).

We will use a result about pseudo-Anosov flows. However the flow $P_\ast$ may have one prongs and in general many results do not hold for one prong pseudo-Anosov flows. To get around that we will do Fried’s surgery. More specifically do Fried’s surgery on the boundary collapsed orbits to obtain $p$ prong orbits with $p > 1$. Fried’s surgery [Fr] has two steps: blow up the orbit using the action on the tangent bundle, then blow down using a new meridian. The blow up is the procedure to go from the one prong pseudo-Anosov flow $\Phi_\ast$ in $P_\ast$ to the blown up one prong pseudo-Anosov flow $\Phi$ in $P$. The blow down is chosen by a new choice of a meridian, call it $m$. If the intersection number of $m$ with the collection of the blow ups of the prongs of $\Phi_\ast$ is $p$, then the resulting orbit is a $p$-prong orbit. One can always do this so that $p \geq 2$, resulting in a pseudo-Anosov flow $\Phi_2$ in a
closed manifold $P_2$. The manifold $P_2$ is obtained by Dehn surgery on $P_s$ on the curves associated with the boundary collapsed orbits of $\Phi_s$.

What is important is that the flows $\Phi_s$ in $P_s$ and $\Phi_2$ in $P_2$ are what is called *almost equivalent* [De-Sh]: $P_s$ minus the boundary collapsed orbits is the same as $P_2$ minus the union of the Dehn surgery orbits; and the flows restricted to these sets have the same orbits. Hence if the set of periodic orbits of $\Phi_2$ in $P_2$ is dense in $P_2$ the same is true for the flow $\Phi_s$ in $P_s$.

We prove the result for $\Phi_2$ in $P_2$. Suppose that $\Phi_2$ is not transitive. Then Mosher [Mos1] proved that there is an incompressible torus $T$ transverse to the flow $\Phi_2$ and which separates basic sets of $\Phi_2$. This transverse torus does not intersect periodic orbits of $\Phi_2$. The blow up produces a torus $T'$ in $P$ transverse to the blown up one prong pseudo-Anosov flow $\Phi$ in $P$. Since $P$ is atoroidal then $T'$ is isotopic to a boundary component of $P$. Projecting to $P_2$, it follows that the torus $T$ that bounds a solid torus $B$ in $P_2$ containing a periodic orbit $\alpha$ associated with the blow down of the corresponding boundary component of $P$. This orbit is obtained by a Dehn surgery of a possible one prong orbit of $\Phi_s$ in $P_s$.

For simplicity assume the flow is outgoing from $B$ along $T_2$. Then the weak stable leaf of $\alpha$ cannot intersect $T_2$ — since $T^2$ is outgoing from $B$, and so this stable leaf is entirely contained in $B$. A lift of $B$ to the universal cover $\tilde{P}_2$ of $P_2$ is a solid tube with bounded cross section. The lift $E$ of the weak stable leaf of $\alpha$ is a $p$-prong leaf which is properly embedded in $\tilde{P}_2$ [Mos1, Ga-Oe]. This uses that $\Phi_2$ is a pseudo-Anosov flow. This is not true in general if there are one prongs. This is why we did the surgery in $P_s$ to obtain $P_2$ and a pseudo-Anosov flow $\Phi_2$ in $P_2$. But $E$ is contained in a solid torus with compact cross sections, so this is impossible.

We conclude that this is impossible. Hence the flow $\Phi_2$ is transitive and so is the flow $\Phi_s$ in $P_s$. In particular $\Phi$ is also transitive in $P$. Since there are finitely many possibly singular orbits of $\Phi$ in $P$, then the union of the regular periodic orbits is dense in $P$.

This finishes the proof of the proposition.

\begin{remark}
After this proposition one may ask the following: there is a lot of freedom in the initial collapsing map from $P$ to $P_s$ (collapsing the laminations $L^m, L^u$ to the singular foliations $E^s, E^u$ respectively). Topologically the type of the singular foliations is determined by the new meridian, which determines the topological type of the collapsing. Hence why consider one prong pseudo-Anosov flows in $P_s$ instead of always choosing a collapsing that yields a true pseudo-Anosov flow? This is a valid question. Here is one important reason to consider one prong pseudo-Anosov flows: Suppose that in each component $D$ of $\partial P$ the foliation $\mathcal{F}$ induces a foliation by circles in $D$. For example this happens if the foliation in $\mathcal{F}$ is a foliation by compact surfaces, that is $\mathcal{F}$ is a fibration over the circle in $P$. Then the preferred collapsing is the one that collapses each circle of $\mathcal{F}|_D$ to a point. This yields a foliation in $P_s$ which is transverse to the induced flow $\Phi_s$, so $\Phi_s$ is a suspension flow. But clearly the suspension flow $\Phi_s$ may have one prongs. In this way the theory generalizes the theory of pseudo-Anosov homeomorphisms of compact surfaces with boundary: it is well known that pseudo-Anosov homeomorphisms with one prongs in the boundary are extremely common and cannot be disregarded. For example $S^2 \times S^1$ has a one prong pseudo-Anosov flow which is a suspension of a one prong pseudo-Anosov homeomorphism of $S^2$. Obviously $S^2$ does not admit pseudo-Anosov homeomorphisms, but admits one prong pseudo-Anosov homeomorphisms, for example there is one with exactly 4 singularities, all one prongs. The blow up
\end{remark}
of the suspension is a sphere minus 4 disks times $S^1$. There are also many other situations where the foliation restricted to each boundary component is by circles, but the foliation in $P$ may not even have compact leaves. Whenever this happens, the natural collapsing produces a foliation in $P^*$ transverse to the flow $\Phi_*$. 

4.3. **Regulating flows transverse to $\mathbb{R}$-covered foliations.**

**Theorem 4.6.** Suppose that $F$ is a Reebless, $\mathbb{R}$-covered, transversely oriented foliation. Then $F$ admits a transverse regulating flow. There is such a flow which is essentially flowing along Seifert fibers in the Seifert pieces and is the blow up one prong pseudo-Anosov flow in the atoroidal pieces.

**Proof.** We first deal with the case that $F$ admits a holonomy invariant transverse measure. See Definition.9.2.14 of [Ca-Co] for a definition of such a measure. If $F$ has a compact leaf (hence clearly generating a holonomy invariant transverse measure supported on it), then we proved in [Fe1] that the compact leaf is a fiber. In addition there is a suspension flow transverse to the foliation which is regulating for $F$, see [Fe1]. Suppose that $F$ does not have compact leaves. Then $F$ has a unique minimal set and one can blow down complementary regions, to produce a minimal foliation, see [Fe1]. Since the foliation is minimal the holonomy invariant measure has full support. By a result of Tischler [Tic] (see also Theorem 9.4.2 of [Ca-Co]), it follows that $M$ fibers over the circle. In addition the foliation can be approximated arbitrarily well by a fibration over the circle. Suspension flows to the fibrations are also regulating for the foliation $F$, see [Th3]. This finishes the proof in this case.

From now on assume that there is no holonomy invariant transverse measure. By Candel’s theorem [Cand] there is a metric in $M$ making all leaves hyperbolic, and we assume this metric.

If the JSJ decomposition of $M$ is trivial then $M$ is either Seifert fibered or atoroidal. Consider the case that $M$ is Seifert. By Brittenham’s theorem [Br] the foliation $F$ has a sublamination which is either vertical or horizontal. If it is vertical then $F$ cannot be $\mathbb{R}$-covered. It follows that the sublamination is horizontal and then so is $F$. Hence we can assume that the Seifert fibration is transverse to the foliation. Hence it is orientable, so its lift to $\tilde{M}$ shows that the flow generated by the Seifert fibration is regulating. If $M$ is atoroidal the result is proved by the construction in [Fe1].

Assume from now on that the JSJ decomposition of $M$ is not trivial. Using the results of section 2 we assume that all decomposing tori and Klein bottles of the JSJ decomposition are in good position with respect with the foliation $F$. Let $P$ be a piece of the JSJ decomposition. If $P$ is Seifert the Seifert fibration provides a transverse flow regulating for $F|_P$. If $P$ is atoroidal Corollary 4.3 provides a blow up of a one prong pseudo-Anosov flow in $P$ which is transverse to $F|_P$ and regulating.

Now all one has to do is to match the flows in between the pieces. Suppose that $T$ is a torus or Klein bottle of the JSJ decomposition. On either side of $T$ there are pieces $P_1, P_2$ of the JSJ decomposition with flows transverse to the foliations $F|_{P_1}$ and $F|_{P_2}$ respectively. Hence there are two flows in $T$ which are regulating for $F_T$.

We do an isotopy between these flows in $T$ through flows transverse to $F|_T$ and regulating for this foliation. We do the case where $T$ is a torus, which is the more complicated case. Choose a basis for the homology of $T$. All the foliations in $T$: either $F|_T$ or the two foliations by flow lines do not have Reeb components. Hence all the leaves in any of these foliations have the same slope. The slope
of $\mathcal{F}|_T$ defines a half line of slopes positively transverse to $\mathcal{F}|_T$. Both the flows induced in $T$ are in this half space of positive slopes. Then one can isotope one to the other keeping it in this half space of positive slopes and keeping it regulating for $\mathcal{F}|_T$. Then enlarge $T$ to $T \times [0,1]$ and interpolate the flows from one side to the other.

This constructs a flow transverse to $\mathcal{F}$ and regulating for $\mathcal{F}$. This finishes the proof of the Theorem. □

5. Action of the fundamental group on the universal circle

Let $\mathcal{F}$ be a transversely orientable, Reebless, $\mathbb{R}$-covered foliation in a closed 3-manifold $M$ so that its leaves are hyperbolic. We obtain some information about the action of deck transformations on the universal circle $U$ of $\mathcal{F}$.

This builds up on several prior works: for example on actions of lifts of homeomorphisms of closed surfaces on the circle at infinity (see an account in the appendix of [BFFP]). The case of atoroidal manifolds, has already been worked out previously, see [Fe2].

We first work out one specific example which is the direct consequence of the results of this article. First assume that $M$ has a non trivial JSJ decomposition. Let $\varphi$ be a regulating flow as construted in Theorem 4.6. We can assume that $\varphi$ preserves the tori and Klein bottles of the JSJ decomposition. If there is an atoroidal piece $P$ we may assume that $\varphi$ restricted to $P$ is $\Phi$: a blow up of a one prong pseudo-Anosov flow.

Let $\tilde{\varphi}$ the lift to $\tilde{M}$. Given any two leaves $E, F$ of $\tilde{\mathcal{F}}$ define $\tau_{E,F}: E \to F$, $\tau_{E,F}(x) = \tilde{\varphi}_E(x) \cap F$ Since $\varphi$ is regulating for $\mathcal{F}$ the maps $\tau_{E,F}$ are always homeomorphisms. They clearly satisfy a cocycle condition. Let $\tilde{T}$ be a lift to $\tilde{M}$ of a torus or Klein bottle in the JSJ decomposition. Clearly $\tau_{E,F}(\tilde{T} \cap E) = \tilde{T} \cap F$.

If $M$ is Seifert or atoidal there is a transverse regulating flow, so this defines a map $\tau_{E,F}$ for any $E, F$.

Lemma 5.1. For any $E, F$, the homeomorphism $\tau_{E,F}$ extends to a homeomorphism, $\zeta_{E,F}: E \cup S^1(E) \to F \cup S^1(F)$. In addition for any $x$ in $S^1(E)$, $x$ and $\zeta_{E,F}(x)$ define the same point in the universal circle $U$ of $\mathcal{F}$.

Proof. Consider first the case that $M$ is Seifert. Then we choose the transverse flow $\Phi$ so that it is an isometry between the leaves in the universal cover, so it extends as a homeomorphism between compactifications. The foliation is uniform and the map $\tau_{E,F}$ sends a point to a point boundedly near it, so a geodesic ray to a geodesic ray boundedly near it. Hence the ideal points in $E, F$ project to the same point in the universal circle. If $M$ is atoroidal, then [Fe1, Cal1] proved that there is a transverse pseudo-Anosov $\Phi$ to $F$. This is constructed using the universal circle, with the distortion quadrilaterals. In particular a leaf $L$ of $\mathcal{L}^a_m, \mathcal{L}^a_n$ satisfies that the ideal points of $L \cap E$ as $E$ varies in $\tilde{\mathcal{F}}$ define the same point in $U$. This implies the result in this case.

Finally suppose that $M$ has a non trivial JSJ decomposition, so there is at least one torus or Klein bottle $T$ JSJ cutting surface. Here we need more information from [FP1]. Let $\tilde{T}$ be a lift of $T$ to $\tilde{M}$. Proposition 4.4 of [FP1] shows that $\tilde{T} \cap E$ as $E$ varies in $\tilde{\mathcal{F}}$ defines a constant pair of points in $U$. Given $F$ in $\tilde{\mathcal{F}}$ let $\mathcal{G}_F$ be the lamination in $\tilde{F}$ obtained by intersecting all lifts $\tilde{T}$ of JSJ tori or Klein bottles $T$ with $F$. It is a lamination by geodesics. Lemma 4.8 of [FP1] states
given \( F \) in \( \widetilde{F} \) then the set of ideal points of \( \mathcal{G}_F \) is dense in \( S^1(F) \) and for any non degenerate interval \( J \) of \( S^1(F) \) there are leaves of \( \mathcal{G}_F \) with both ideal points in \( J \).

Now fix \( E,F \) leaves of \( \widetilde{F} \). We know that for any \( \widetilde{T} \) lift of a JSJ torus or Klein bottle then \( \tau_{E,F}(\widetilde{T} \cap E) = \widetilde{T} \cap F \). No two leaves of \( \mathcal{G}_E \) share an ideal point. In addition the circular order in \( S^1(E) \) induced by the ideal points of leaves of \( \mathcal{G}_E \) is preserved by \( \tau_{E,F} \): the circular order induced in \( S^1(F) \) by \( \tau_{E,F}(\widetilde{T}) \) as \( \widetilde{T} \) varies over the lifts is the same, when one identifies \( S^1(E) \) with \( S^1(F) \) using the universal circle.

We know that for any ideal point \( q \) in \( S^1(E) \) either it is an ideal point of some leaf of \( \mathcal{G}_E \) or is accumulated by ideal points of leaves of \( \mathcal{G}_E \) (so that both endpoints converge to \( q \)). These facts imply that \( \tau_{E,F} \) extends to a homeomorphism from \( E \cup S^1(E) \) to \( F \cup S^1(F) \). Since the homeomorphism satisfies that \( q, \tau_{E,F}(q) \) project to the same point in \( \mathcal{U} \) for any \( q \) ideal point of leaf of \( \mathcal{G}_E \), it follows that this is true for all \( q \) in \( S^1(E) \). This proves the lemma.

We fix a transverse, regulating flow \( \varphi \) as above. Let \( \gamma \) in \( \pi_1(M) \) be a deck transformation. For any \( E \) leaf of \( \widetilde{F} \) define

\[
h_E = \gamma \circ \tau_{E,\gamma^{-1}(E)}
\]

This is a homeomorphism from \( E \) to itself. By Lemma 5.1 this induces a homeomorphism \( h_\infty \) from \( S^1(E) \) to itself. Recall that \( \tau_{E,\gamma^{-1}(E)} \) induces the identity map in the universal circle level. It follows that under the identification of \( \mathcal{U} \) with \( S^1(E) \) then \( h_\infty \) is the representation of the action of \( \gamma \) on the universal circle \( \mathcal{U} \).

Suppose that there is at least one atoroidal piece \( P \). Recall the “singular foliations” \( \mathcal{E}_b^s, \mathcal{E}_b^u \) in \( P \) (they are singular foliations in the interior of \( P \)). Consider the lift of these to a lifted \( \widetilde{P} \) of \( P \) to \( \widetilde{M} \). They induce foliations in \( \widetilde{P} \) so that intersected with any leaf \( E \) of \( \widetilde{F} \) they are foliations by quasigeodesics. Some are \( p \)-prong leaves, each prong is a quasigeodesic. The transverse flow \( \widetilde{\varphi} \) expands length along the unstable leaves and contracts along stable leaves.

Let \( \gamma \) be an arbitrary element of \( \pi_1(M) \) and \( \rho(\gamma) \) the induced action on the universal circle \( \mathcal{U} \) of \( \mathcal{F} \).

**Proposition 5.2.** Suppose that \( M \) has an atoroidal piece \( P \). Let \( \gamma \) be a deck transformation associated with an interior periodic orbit of \( P \). Then up to a finite power \( \rho(\gamma) \) has finitely many fixed points in \( \mathcal{U} \), alternating between attracting and repelling. In case \( \gamma \) is associated with a regular orbit, then \( \rho(\gamma) \) (up to power) has exactly 4 fixed points.

Finally if \( E \) is in \( \widetilde{F} \) and \( q \) is an ideal point in \( S^1(E) \) associated with a fixed point of \( \rho(\gamma) \), the following happens: there is a neighborhood basis of \( q \) in \( S^1(E) \) defined by geodesics \( \ell_i \) in \( E \) so that for any \( x \) in \( \ell_i \) and \( y \) in \( \gamma \circ \tau_{E,\gamma^{-1}(E)}(\ell_i) \), then \( d_E(x,y) \to \infty \) if \( i \to \infty \).

**Proof.** Fix a leaf \( E \) of \( \widetilde{F} \), we do the analysis in \( E \). Let \( \alpha \) be the periodic orbit associated with \( \gamma \) and \( \widetilde{\alpha} \) the lift of \( \alpha \) fixed by \( \gamma \). Up to a power \( \gamma \) fixes the stable and unstable prongs of \( \widetilde{\alpha} \). Let \( z = \widetilde{\alpha} \cap E \). Let \( \widetilde{P} \) be the lift of \( P \) containing \( \widetilde{\alpha} \). Then the intersections of the stable and unstable prongs of \( \widetilde{\alpha} \) with \( E \) are quasigeodesic rays in \( E \) entirely contained in \( \widetilde{P} \). These prongs are contained in leaves of \( \mathcal{E}_b^s, \mathcal{E}_b^u \) respectively.

Fix one unstable prong \( r \), that is, contained in \( \mathcal{E}_b^u(\widetilde{\alpha}) \cap E \). Fix a regular stable leaf \( \zeta \) (that is a leaf of \( \mathcal{E}_b^s \cap E \)) intersecting \( r \). Recall that \( h_E = \gamma \circ \tau_{E,\gamma^{-1}(E)} \). Consider \( h_E^r(\zeta) \). The flow \( \varphi \) preserves \( \mathcal{E}_b^s, \mathcal{E}_b^u \), contracts length exponentially along
the stables and expands along the unstables. Hence \( h_E \) preserves the foliations by stables and unstables in \( \tilde{P} \cap E \). Then \( h_E^i(\zeta) \) converges to the stable leaf of \( z \) when \( i \to -\infty \) (if \( z \) is a \( p \)-prong it converges to properly embedded real line in this leaf). Let this limit be \( \zeta' \). In addition \( h_E^i(r) \) escapes in \( E \) when \( i \to \infty \). Notice that for any \( i \), \( h_E^i(\zeta) \) is entirely contained in \( \tilde{P} \). In addition \( h_E^i(\zeta) \) are uniform quasigeodesics, independent of \( i \). As \( i \to \infty \) they escape in \( E \) and are nested so they converge to a single ideal point, which is the ideal point of \( r \).

Let \( a_1, a_2 \) be the ideal points in \( S^1(E) \) of \( \zeta' \) and \( b_1 \) the ideal point of \( r \). Let \( I \) be the interval in \( S^1(E) \) with endpoints \( a_1, a_2 \) and containing \( b_1 \). The above shows that \( h_\infty \) fixes \( a_1, a_2, b_1 \) and acts as a contraction on the interior of \( I \) with single fixed point \( b_1 \). This proves the first assertion of the proposition.

We now consider the last statement from the proposition. Consider the geodesics which are obtained by pulling \( h_E^i(\zeta) \) tight. These are leaves of \( \mathcal{L}_m^s \cap E \). These form a neighborhood basis of the ideal point \( b_1 \) of \( r \) in \( E \cup S^1(E) \). These geodesics are also a uniform bounded distance in \( E \) from \( h_E^i(\zeta) \) (for the same \( i \)) and so are their images under \( h_E \).

The angle between leaves of \( \mathcal{L}_m^s \cap E \) and leaves of \( \mathcal{L}_m^u \cap E \) is uniformly bounded below. Let \( \ell_i \) be the geodesic obtained by pulling \( h_E^i(\zeta) \) tight. Let \( r_g \) be the geodesic obtained by pulling \( r \) tight. The action of \( h_E \) on \( r \) expands length exponentially. This follows because \( \varphi \) expands length exponentially. The geodesics \( \ell_i \) are a bounded distance from \( h_E^i(\zeta) \) and so is \( r_g \) from \( g \). It follows that the distance from \( \ell_i \cap r_g \) to \( \ell_{i+1} \cap r_g \) converges to infinity as \( i \to \infty \). The angle condition implies that for any \( x \) in \( \ell_i \) and \( y \) in \( \ell_{i+1} \) then \( d_E(x, y) \to \infty \) as \( i \to \infty \). Since the Hausdorff distance between \( \ell_{i+1} \) and \( h_E(\ell_i) \) is uniformly bounded, we obtain the bound desired.

If the ideal point is an ideal point of a stable prong, then we use inverses instead in the above argument.

This finishes the proof of the proposition. \( \square \)

**Remark 5.3.** The last condition in the proposition is what is called **super attracting** in [FP2]. The super attracting definition is particularly useful in the case that \( \mathcal{F} \) is uniform. Since we do not use that here we do not define it formally, and we refer the interested reader to [FP2]. The specific result of this proposition is used in [FP2] to help analyze partially hyperbolic diffeomorphisms homotopic to the identity in 3-manifolds and to prove that some of them are not dynamically coherent.

### 5.1. Some general properties of action on the universal circle \( \mathcal{U} \)

We are now ready to describe more general information about the action \( \rho(\gamma) \) on the universal circle \( \mathcal{U} \). There are too many cases to enumerate in the statement of a single result. Instead we little by little describe each individual case. The foliation \( \mathcal{F} \) satisfies the properties announced in the beginning of this section.

1) Suppose that \( M \) is Seifert. As explained above we can assume that the Seifert fibration is transverse to \( \mathcal{F} \), and we can put a metric so that flowing along Seifert fibers is a local isometry between leaves of \( \mathcal{F} \). Any deck transformation preserves the Seifert fibration so induces an isometry on the quotient of \( \tilde{M} \) by the lift of the Seifert fibration. This quotient \( R \) is isometric to the hyperbolic plane and the ideal circle of this plane is canonically identified to the universal circle \( \mathcal{U} \). Let \( \gamma \) be a deck transformation, so it induces an isometry of \( R \).

- If the isometry is elliptic then \( \gamma \) is associated with a fiber of the Seifert fibration. Then a finite power of \( \gamma \) is the identity on \( R \). If \( \gamma \) preserves orientation
then the action of $\rho(\gamma)$ on $\mathcal{U}$ is either free or fixes every point. If $\gamma$ reverses orientation on $R$ then $\rho(\gamma)$ has two fixed points on $\mathcal{U}$.

- If the isometry induced on $R$ is hyperbolic there are exactly two fixed points of $\rho(\gamma)$ on $\mathcal{U}$.

- The isometry on $R$ cannot be parabolic, because $M$ quotient the Seifert fibration is a closed orbifold surface.

2) Suppose that $M$ is atoroidal. Then the results of $[Fe1]$ imply that there is a pseudo-Anosov flow $\Phi$ transverse to $\mathcal{F}$ and regulating for $\mathcal{F}$. The action of elements of $\pi_1(M)$ on $\mathcal{U}$ was determined by Proposition 5.3 of $[Fe2]$. If $\gamma$ fixes 3 or more points of $\mathcal{U}$ then $\gamma$ is associated with a periodic orbit of $\Phi$ and $\rho(\gamma)$ has a finite even number of fixed points on $\mathcal{U}$, which are alternatively attracting and repelling. If $\gamma$ is not associated with a periodic orbit of $\Phi$, then $\rho(\gamma)$ has exactly two fixed points on $\mathcal{U}$, one attracting, one repelling. Finally it could be that $\gamma$ is associated with an orbit of $\Phi$, but permutes the local prongs. In this case $\rho(\gamma)$ acts freely on $\mathcal{U}$, but a power of $\rho(\gamma)$ fixes at least 4 point on $\mathcal{U}$.

3) Finally suppose that the JSJ decomposition of $M$ is non trivial. As in section 2 we put the JSJ tori and Klein bottles in good position with respect to the foliation $\mathcal{F}$. Let $T$ be the JSJ tree of $M$: the vertices are lifts of pieces of the JSJ decomposition of $M$, an edge is a lift of a torus or Klein bottle of the JSJ decomposition which connects two lifts of pieces of the JSJ decomposition. This tree has a more or less canonical embedding into any leaf $F$ of $\mathcal{F}$ (modulo moving vertices in complementary regions, and isotoping edges) preserving the ordering — see details in section 4 of $[FP1]$. The universal circle is canonically homeomorphic to a quotient of the set of ends of this tree, see $[FP1]$. There are many possibilities.

3.A) If $\gamma$ acts freely on the tree $T$, then $\gamma$ does not fix any lift of a piece. Then $\rho(\gamma)$ has exactly two fixed points on $\mathcal{U}$.

3.B) Suppose that $\gamma$ fixes a lift $\tilde{P}$ of a Seifert piece $P$. As in case 1) above $\gamma$ induces an isometry of the quotient of $\tilde{P}$ by the lift of the Seifert fibration, which is an isometry of a surface embedded in the hyperbolic plane, but with infinitely many geodesic boundaries. As in case 1), $\gamma$ could be elliptic, with $\rho(\gamma)$ either not fixing any point on $\mathcal{U}$ or $\rho(\gamma)$ fixing at least a Cantor set of points in $\mathcal{U}$, or exactly 2 points on $\mathcal{U}$ (if $\rho(\gamma)$ reverses orientation). See similar analysis in the Appendix of $[BFFP]$. If the action of $\gamma$ on the quotient surface is hyperbolic then $\rho(\gamma)$ fixes exactly 2 points on $\mathcal{U}$.

3.C) Finally if $P$ is atoroidal, look at the action of $\gamma$ on the leaf spaces of the lifts of the blow ups of $\mathcal{E}^s, \mathcal{E}^u$ as constructed in section 2. If these actions on the leaf spaces are free, then $\gamma$ acts as a translation on these leaf spaces and $\rho(\gamma)$ fixes exactly two points on $\mathcal{U}$, one attracting, one repelling. If $\gamma$ fixes some leaf of the blow up of $\mathcal{E}^s$ then it is associated with a periodic orbit of the blown up one prong pseudo-Anosov flow in $P$. If the orbit is in the interior of $P$ (not peripheral), then a power of $\rho(\gamma)$ fixes finitely many ($\geq 4$) points in $\mathcal{U}$, which are alternatively attracting and repelling. If the associated orbit is peripheral, then $\rho(\gamma)$ fixes infinitely many points on $\mathcal{U}$. This is Proposition 5.2.

References

[BFFP] T. Barthelme, S. Fenley, S. Frankel, R. Potrie, Dynamical incoherence for a large class of partially hyperbolic diffeomorphisms, arxiv.math.2002.10315, to appear in Erg. Th. Dyn. Sys.
