Euler-Poincaré functions

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Introduction

Euler-Poincaré functions and pseudo-coefficients are important tools in harmonic analysis since they help to single out a given representation in direct integrals or sums. Given a real reductive group $G$ that admits a compact Cartan subgroup, these functions can be attached to a given finite dimensional representation $\tau$ of a maximal compact subgroup $K$. In [10] a construction was given in the case when $\tau$ extends to the bigger group $G$. In this paper we give a new construction for an arbitrary representation $\tau$. Instead, we put a condition on the group $G$, namely, that it acts orientation preservingly on the symmetric space $G/K$. This condition is satisfied if $G$ is connected or if it respects a complex structure on $G/K$. For connected $G$ the existence of Euler-Poincaré functions is known [4], but this is not always sufficient for applications since Levi components are not in general connected, even if
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the ambient group is. After having constructed Euler-Poincaré functions we compute the orbital integrals of semisimple elements.

1 Notations

We denote Lie groups by upper case roman letters $G, H, K, \ldots$ and the corresponding real Lie algebras by lower case German letters with index 0, that is: $g_0, h_0, t_0, \ldots$. The complexified Lie algebras will be denoted by $g, h, t, \ldots$, so, for example: $g = g_0 \otimes \mathbb{C}$.

Let $G$ be a real reductive group \cite{13} and fix a maximal compact subgroup $K$ of $G$. Fix a Cartan involution $\theta$ with fixed point set the maximal compact subgroup $K$ and let $t_0$ be the Lie algebra of $K$. The group $K$ acts on $g_0$ via the adjoint representation and there is a $K$-stable decomposition $g_0 = k_0 \oplus p_0$, where $p_0$ is the eigenspace of (the differential of) $\theta$ to the eigenvalue $-1$. Write $g = t \oplus p$ for the complexification. This is called the Cartan decomposition.

Fix a symmetric bilinear form $B : g_0 \times g_0 \to \mathbb{R}$ such that

- $B$ is invariant, that is $B(\text{Ad}(g)X, \text{Ad}(g)Y) = B(X, Y)$ for all $g \in G$ and all $X, Y \in g_0$ and

- $B$ is negative definite on $t_0$ and positive definite on its orthocomplement $p_0 = t_0^\perp \subset g_0$.

When $G$ is semisimple we can choose $B$ to be the Killing form of $g_0$.

Let $U(g)$ be the universal enveloping algebra of $g$, then $B$ gives rise to a Casimir element $C \in U(g)$. For an irreducible admissible representation $(\pi, V_\pi)$ the operator $C$ will act on $V_\pi$ by a scalar $\pi(C) \in \mathbb{C}$.

Let $X$ denote the quotient manifold $G/K$. The tangent space at $eK$ identifies with $p_0$ and the form $B$ gives a $K$-invariant positive definite inner product on this space. Translating this by elements of $G$ defines a $G$-invariant Riemannian metric on $X$. This makes $X$ the most general globally symmetric space of the noncompact type \cite{8}. 
Let $\hat{G}$ denote the unitary dual of $G$, i.e., $\hat{G}$ is the set of isomorphism classes of irreducible unitary representations of $G$.

The form $\langle X, Y \rangle = -B(X, \theta(Y))$ is positive definite on $\mathfrak{g}_0$ and therefore induces a positive definite left invariant top differential form $\omega_L$ on any closed subgroup $L$ of $G$. If $L$ is compact we set

$$v(L) = \int_L \omega_L.$$ 

Let $H = AB$ be a $\theta$-stable Cartan subgroup where $A$ is the connected split component of $H$ and $B$ is compact. The double use of the letter $B$ here will not cause any confusion. Then $B \subset K$. Let $\Phi$ denote the root system of $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the complexified Lie algebras of $G$ and $H$. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the root space decomposition. Let $x \to x^c$ denote the complex conjugation on $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0 = \text{Lie}(G)$. A root $\alpha$ is called imaginary if $\alpha^c = -\alpha$ and it is called real if $\alpha^c = \alpha$. Every root space $\mathfrak{g}_\alpha$ is one dimensional and has a generator $X_\alpha$ satisfying:

$$[X_\alpha, X_{-\alpha}] = Y_\alpha \quad \text{with} \quad \alpha(.) = B(Y_\alpha, .)$$

$$B(X_\alpha, X_{-\alpha}) = 1$$

and $X_\alpha^c = X_{\alpha^c}$ if $\alpha$ is non-imaginary and $X_\alpha^c = \pm X_{-\alpha}$ if $\alpha$ is imaginary. An imaginary root $\alpha$ is called compact if $X_\alpha^c = -X_{-\alpha}$ and noncompact otherwise. Let $\Phi_0$ be the set of noncompact imaginary roots and choose a set $\Phi^+$ of positive roots such that for $\alpha \in \Phi^+$ nonimaginary we have that $\alpha^c \in \Phi^+$. Let $W = W(G, H)$ be the Weyl group of $(G, H)$, that is

$$W = \frac{\text{normalizer}(H)}{\text{centralizer}(H)}.$$ 

Let $\text{rk}_\mathbb{R}(G)$ be the dimension of a maximal $\mathbb{R}$-split torus in $G$ and let $\nu = \dim G/K - \text{rk}_\mathbb{R}(G)$. We define the Harish-Chandra constant of $G$ by

$$c_G = (-1)^{\Phi^+_0} |(2\pi)^{\Phi^+}| 2^{\nu/2} \frac{v(T)}{v(K)} |W|.$$ 

2 Normalization of Haar measures

Although the results will not depend on normalizations we will need to normalize Haar measures for the computations along the way. First for any
compact subgroup $C \subset G$ we normalize its Haar measure so that it has total mass one, i.e., $\text{vol}(C) = 1$. Next let $H \subset G$ be a reductive subgroup, and let $\theta_H$ be a Cartan involution on $H$ with fixed point set $K_H$. The same way as for $G$ itself the form $B$ restricted to the Lie algebra of $H$ induces a Riemannian metric on the manifold $X_H = H/K_H$. Let $dx$ denote the volume element of that metric. We get a Haar measure on $H$ by defining
\[ \int_H f(h)dh = \int_{X_H} \int_{K_H} f(xk)dkdx \]
for any continuous function of compact support $f$ on $H$.

Let $P \subset G$ be a parabolic subgroup of $G$ (see [13] 2.2). Let $P = MAN$ be the Langlands decomposition of $P$. Then $M$ and $A$ are reductive, so there Haar measures can be normalized as above. Since $G = PK = MANK$ there is a unique Haar measure $dn$ on the unipotent radical $N$ such that for any constant function $f$ of compact support on $G$ it holds:
\[ \int_G f(x)dx = \int_M \int_A \int_N \int_K f(mank)dkdndadm. \]

Note that these normalizations coincide for Levi subgroups with the ones met by Harish-Chandra in ([13] sect. 7).

## 3 Invariant distributions

In this section we shall throughout assume that $G$ is a real reductive group of inner type [13]. A distribution $T$ on $G$, i.e., a continuous linear functional $T : C^\infty_c(G) \to \mathbb{C}$ is called invariant if for any $f \in C^\infty_c(G)$ and any $y \in G$ it holds: $T(f^y) = T(f)$, where $f^y(x) = f(yxy^{-1})$. Examples are:

- orbital integrals: $f \mapsto \mathcal{O}_g(f) = \int_{G_g\backslash G} f(x^{-1}gx)dx$ and
- traces: $f \mapsto \text{tr } \pi(f)$ for $\pi \in \hat{G}$.

These two examples can each be expressed in terms of the other. Firstly, Harish-Chandra proved that for any $\pi \in \hat{G}$ there exists a conjugation invariant locally integrable function $\Theta_\pi$ on $G$ such that for any $f \in C^\infty_c(G)$
\[ \text{tr } \pi(f) = \int_G f(x)\Theta_\pi(x)dx. \]
Recall the *Weyl integration formula* which says that for any integrable function $\varphi$ on $G$ we have
\[
\int_G \varphi(x) dx = \sum_{j=1}^r \frac{1}{|W(G, H_j)|} \int_{H_j} \int_{G/H_j} \varphi(xhx^{-1}) |\det(1 - h|\mathfrak{g}/\mathfrak{h}_j)| dx dh,
\]
where $H_1, \ldots, H_r$ is a maximal set of nonconjugate Cartan subgroups in $G$ and for each Cartan subgroup $H$ we let $W(G, H)$ denote its Weyl group, i.e., the quotient of the normalizer of $H$ in $G$ by its centralizer.

An element $x$ of $G$ is called *regular* if its centralizer is a Cartan subgroup. The set of regular elements $G^{reg}$ is open and dense in $G$ therefore the integral above can be taken over $G^{reg}$ only. Letting $H^{reg}_j := H_j \cap G^{reg}$ we get

**Proposition 3.1** Let $N$ be a natural number bigger than $\frac{\dim G}{2}$, then for any $f \in L^1_2(G)$ and any $\pi \in \hat{G}$ we have
\[
\text{tr} \pi(f) = \sum_{j=1}^r \frac{1}{|W(G, H_j)|} \int_{H^{reg}_j} \mathcal{O}_h(f) \Theta_\pi(h) |\det(1 - h|\mathfrak{g}/\mathfrak{h}_j)| dh.
\]
That is, we have expressed the trace distribution in terms of orbital integrals.

In the other direction it is also possible to express semisimple orbital integrals in terms of traces.

At first let $H$ be a $\theta$-stable Cartan subgroup of $G$. Let $\mathfrak{h}$ be its complex Lie algebra and let $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ be the set of roots. Let $x \rightarrow x^c$ denote the complex conjugation on $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0 = \text{Lie}_\mathbb{R}(G)$. Choose an ordering $\Phi^+ \subset \Phi$ and let $\Phi^+_I$ be the set of positive imaginary roots. To any root $\alpha \in \Phi$ let
\[
H \rightarrow \mathbb{C}^\times
h \mapsto h^\alpha
\]
be its character, that is, for $X \in \mathfrak{g}_\alpha$ the root space to $\alpha$ and any $h \in H$ we have $\text{Ad}(h)X = h^\alpha X$. Now put
\[
'\Delta_I(h) = \prod_{\alpha \in \Phi^+_I} (1 - h^{-\alpha}).
\]
Let $H = AT$ where $A$ is the connected split component and $T$ is compact. An element $at \in AT = H$ is called split regular if the centralizer of $a$ in $G$ equals the centralizer of $A$ in $G$. The split regular elements form a dense open subset containing the regular elements of $H$. Choose a parabolic $P$ with split component $A$, so $P$ has Langlands decomposition $P = MAN$. For $at \in AT = H$ let

$$\triangle_+(at) = \left| \det((1 - \text{Ad}((at)^{-1}))|_{g/a \oplus m}) \right|^\frac{1}{2}$$

$$= \left| \det((1 - \text{Ad}((at)^{-1}))|_n) \right| \rho_P$$

$$= \left| \prod_{\alpha \in \Phi^+ - \Phi^+_f} (1 - (at)^{-\alpha}) \right| \rho_P,$$

where $\rho_P$ is the half of the sum of the roots in $\Phi(P, A)$, i.e., $a^{2\rho_P} = \det(a|_n)$. We will also write $h^{\rho_P}$ instead of $a^{\rho_P}$.

For any $h \in H^{reg} = H \cap G^{reg}$ let

$$'F^H_f(h) = 'F_f(h) = '\triangle_f(h)\triangle_+(h) \int_{G/A} f(xhx^{-1})dx.$$

It then follows directly from the definitions that for $h \in H^{reg}$ it holds

$$O_h(f) = \frac{'F_f(h)}{h^{\rho_P} \det(1 - h^{-1}||\mathfrak{g}/\mathfrak{h})^+||},$$

where $(\mathfrak{g}/\mathfrak{h})^+$ is the sum of the root spaces attached to positive roots. There is an extension of this identity to nonregular elements as follows: For $h \in H$ let $G_h$ denote its centralizer in $G$. Let $\Phi^+(\mathfrak{g}_h, \mathfrak{h})$ be the set of positive roots of $(\mathfrak{g}_h, \mathfrak{h})$. Let

$$\varpi_h = \prod_{\alpha \in \Phi^+(\mathfrak{g}_h, \mathfrak{h})} Y_\alpha,$$

then $\varpi_h$ defines a left invariant differential operator on $G$.

**Lemma 3.2** For any $f \in L^1_{2N}(G)$ and any $h \in H$ we have

$$O_h(f) = \frac{\varpi'_h F_f(h)}{c_h h^{\rho_P} \det(1 - h^{-1}||\mathfrak{g}/\mathfrak{h})^+||},$$

Our aim is to express orbital integrals in terms of traces of representations. By the above lemma it is enough to express \( 'F_f(h) \) in terms of traces of \( f \) when \( h \in H^{reg} \). For this let \( H_1 = A_1 T_1 \) be another \( \theta \)-stable Cartan subgroup of \( G \) and let \( P_1 = M_1 A_1 N_1 \) be a parabolic with split component \( A_1 \). Let \( K_1 = K \cap M_1 \). Since \( G \) is connected the compact group \( T_1 \) is an abelian torus and its unitary dual \( \hat{T}_1 \) is a lattice. The Weyl group \( W = W(M_1, T_1) \) acts on \( \hat{T}_1 \) and \( \hat{t}_1 \in \hat{T}_1 \) is called regular if its stabilizer \( W(\hat{t}_1) \) in \( W \) is trivial. The regular set \( \hat{T}_1^{reg} \) modulo the action of \( W(\hat{t}_1) \) parameterizes the discrete series representations of \( M_1 \) (see [9]).

For \( \hat{t}_1 \in \hat{T}_1 \) Harish-Chandra [7] defined a distribution \( \Theta_{\hat{t}_1} \) on \( G \) which happens to be the trace of the discrete series representation \( \pi_{\hat{t}_1} \) attached to \( \hat{t}_1 \) when \( \hat{t}_1 \) is regular. When \( \hat{t}_1 \) is not regular the distribution \( \Theta_{\hat{t}_1} \) can be expressed as a linear combination of traces as follows. Choose an ordering of the roots of \((M_1, T_1)\) and let \( \Omega \) be the product of all positive roots. For any \( w \in W \) we have \( w \Omega = \epsilon(w) \Omega \) for a homomorphism \( \epsilon : W \to \{ \pm 1 \} \). For nonregular \( \hat{t}_1 \in \hat{T}_1 \) we get \( \Theta_{\hat{t}_1} = \frac{1}{|W(\hat{t}_1)|} \sum_{w \in W(\hat{t}_1)} \epsilon(w) \Theta'_{w, \hat{t}_1} \), where \( \Theta'_{w, \hat{t}_1} \) is the character of an irreducible representation \( \pi_{w, \hat{t}_1} \) called a limit of discrete series representation.

Let \( \nu : a \mapsto a^\nu \) be a unitary character of \( A_1 \) then \( \hat{h}_1 = (\nu, \hat{t}_1) \) is a character of \( H_1 = A_1 T_1 \). Let \( \Theta_{\hat{h}_1} \) be the character of the representation \( \pi_{\hat{h}_1} \) induced parabolically from \((\nu, \pi_{\hat{t}_1})\). Harish-Chandra has proven

**Theorem 3.3** Let \( H_1, \ldots, H_r \) be maximal a set of nonconjugate \( \theta \)-stable Cartan subgroups. Let \( H = H_j \) for some \( j \). Then for each \( j \) there exists a continuous function \( \Phi_{H|H_j} \) on \( H^{reg} \times H_j \) such that for \( h \in H^{reg} \) it holds

\[
'F_f^H(h) = \sum_{j=1}^r \int_{H_j} \Phi_{H|H_j}(h, \hat{h}_j) \text{ tr} \pi_{\hat{h}_j}(f) \, d\hat{h}_j.
\]

Further \( \Phi_{H|H_j} = 0 \) unless there is \( g \in G \) such that \( g A g^{-1} \subset A_1 \). Finally for
H_j = H the function can be given explicitly as

\[ \Phi_{H/H}(h, \hat{h}) = \frac{1}{|W(G, H)|} \sum_{w \in W(G, H)} \epsilon(w|T) \langle w\hat{h}, h \rangle \]

\[ = \frac{1}{|W(G, H)|} '\triangle(h) \Theta_{\hat{h}}(h), \]

where '\triangle = \triangle_+ \triangle_I.

Proof: \[\square\].

4 Existence of Euler-Poincaré functions

Let G be a real reductive group that admits a compact Cartan. Fix a maximal compact subgroup K of G and a Cartan T of G which lies inside K. The group G is called orientation preserving if G acts by orientation preserving diffeomorphisms on the manifold \( X = G/K \). For example, the group \( G = SL_2(\mathbb{R}) \) is orientation preserving but the group \( PGL_2(\mathbb{R}) \) is not. Recall the Cartan decomposition \( g_0 = k_0 \oplus p_0 \). Note that G is orientation preserving if and only if its maximal compact subgroup K preserves orientations on \( p_0 \).

Lemma 4.1 The following holds:

- Any connected group is orientation preserving.
- If \( X \) carries the structure of a complex manifold which is left stable by \( G \), then \( G \) is orientation preserving.

Proof: The first is clear. The second follows from the fact that biholomorphic maps are orientation preserving.

Q.E.D.

Let \( t \) be the complexified Lie algebra of the Cartan subgroup T. We choose an ordering of the roots \( \Phi(g, t) \) of the pair \((g, t)\) \[\square\]. This choice induces a decomposition \( p = p_- \oplus p_+ \), where \( p_\pm \) is the sum of the positive/negative root spaces which lie in \( p \). As usual denote by \( \rho \) the half sum of the positive roots.
The chosen ordering induces an ordering of the compact roots $\Phi(\mathfrak{g}, t)$ which form a subset of the set of all roots $\Phi(\mathfrak{g}, t)$. Let $\rho_K$ denote the half sum of the positive compact roots. Recall that a function $f$ on $G$ is called $K$-central if $f(k x k^{-1}) = f(x)$ for all $x \in G$, $k \in K$. For any $K$-representation $(\rho, V)$ let $V^K$ denote the space of $K$-fixed vectors and let $(\hat{\tau}, V_{\hat{\tau}})$ denote the dual representation. The restriction from $G$ to $K$ gives a ring homomorphism of the representation rings:

$$res_K^G : \text{Rep}(G) \rightarrow \text{Rep}(K).$$

**Theorem 4.2 (Euler-Poincaré functions)** Let $(\tau, V_{\tau})$ a finite dimensional representation of $K$. If either $G$ is orientation preserving or $\tau$ lies in the image of $res_K^G$, then there is a compactly supported smooth $K$-central function $f_{\tau}$ on $G$ such that for every irreducible unitary representation $(\pi, V_{\pi})$ of $G$ we have

$$\text{tr} \ \pi(f_{\tau}) = \sum_{p=0}^{\dim(p)} (-1)^p \dim(V_{\pi} \otimes \wedge^p(V_{\tau})^K).$$

We call $f_{\tau}$ an Euler-Poincaré function for $\tau$.

If, moreover, $K$ leaves invariant the decomposition $p = p_+ \oplus p_-$ then there is a compactly supported smooth $K$-central function $g_{\tau}$ on $G$ such that for every irreducible unitary representation $(\pi, V_{\pi})$ we have

$$\text{tr} \ \pi(g_{\tau}) = \sum_{p=0}^{\dim(p_-)} (-1)^p \dim(V_{\pi} \otimes \wedge^p p_- \otimes V_{\hat{\tau}})^K.$$  

Remark: If the representation $\tau$ lies in the image of $res_K^G$ or the group $G$ is connected then the theorem is well known, [4], [10]. We only included it for the sake of completeness. We will therefore concentrate on the proof in the case when $G$ is orientation preserving.

**Proof:** We will concentrate on the case when $G$ is orientation preserving, suffice to say that the other case can be treated similarly. Without loss of generality assume that $\tau$ is irreducible. Suppose given a function $f$
which satisfies the claims of the theorem except that it is not necessarily $K$-central, then the function

$$x \mapsto \int_K f(kxk^{-1})dk$$

will satisfy all claims of the theorem. Thus one only needs to construct a function having the claimed traces.

If $G$ is orientation preserving the adjoint action gives a homomorphism $K \to \text{SO}(p)$. If this homomorphism happens to lift to the double cover $\text{Spin}(p)$ we let $\tilde{G} = G$ and $\tilde{K} = K$. In the other case we apply the

**Lemma 4.3** If the homomorphism $K \to \text{SO}(p)$ does not factor over $\text{Spin}(p)$ then there is a double covering $\tilde{G} \to G$ such that with $\tilde{K}$ denoting the inverse image of $K$ the induced homomorphism $\tilde{K} \to \text{SO}(p)$ factors over $\text{Spin}(p) \to \text{SO}(p)$. Moreover the kernel of the map $\tilde{G} \to G$ lies in the center of $\tilde{G}$

**Proof:** At first $\tilde{K}$ is given by the pullback diagram:

$$\begin{array}{ccc}
\tilde{K} & \to & \text{Spin}(p) \\
\downarrow & & \downarrow \alpha \\
K & \to & \text{SO}(p) \\
\text{Ad} & & \\
\end{array}$$

that is, $\tilde{K}$ is given as the set of all $(k, g) \in K \times \text{Spin}(p)$ such that $\text{Ad}(k) = \alpha(g)$. Then $\tilde{K}$ is a double cover of $K$.

Next we use the fact that $K$ is a retract of $G$ to show that the covering $\tilde{K} \to K$ lifts to $G$. Explicitly let $P = \exp(p_0)$ then the map $K \times P \to G, (k, p) \mapsto kp$ is a diffeomorphism [13]. Let $g \mapsto (\underline{k}(g), \underline{p}(g))$ be its inverse map. We let $\tilde{G} = \tilde{K} \times P$ then the covering $\tilde{K} \to \tilde{K}$ defines a double covering $\beta : \tilde{G} \to G$. We have to install a group structure on $\tilde{G}$ which makes $\beta$ a homomorphism and reduces to the known one on $\tilde{K}$. Now let $k, k' \in K$ and $p, p' \in P$ then by

$$k'p'kp = k'k^{-1}p'kp$$

it follows that there are unique maps $a_K : P \times P \to K$ and $a_P : P \times P \to P$ such that

$$\underline{k}(k'p'kp) = k'ka_K(k^{-1}p'k, p)$$

$$\underline{p}(k'p'kp) = a_P(k^{-1}p'k, p).$$
Since $P$ is simply connected the map $a_K$ lifts to a map $\tilde{a}_K : P \times P \rightarrow \tilde{K}$. Since $P$ is connected there is exactly one such lifting with $\tilde{a}_K(1, 1) = 1$. Now the map

$$ (\tilde{K} \times P) \times (\tilde{K} \times P) \rightarrow \tilde{K} \times P $$

$$(k', p'), (k, p) \mapsto (kk'\tilde{a}_K(k^{-1}p'k, p), a_P(k^{-1}p'k, p))$$

defines a multiplication on $\tilde{G} = \tilde{K} \times P$ with the desired properties.

Finally $\ker(\beta)$ will automatically be central because it is a normal subgroup of order two. Q.E.D.

Let $S$ be the spin representation of $Spin(p)$ (see [11], p.36). It splits as a direct sum of two distinct irreducible representations

$$ S = S^+ \oplus S^- $$

We will not go into the theory of the spin representation, we only need the following properties:

- The virtual representation

$$ (S^+ - S^-) \otimes (S^+ - S^-) $$

is isomorphic to the adjoint representation on $\wedge^{even} p - \wedge^{odd} p$ (see [11], p. 36).

- If $K$ leaves invariant the spaces $p_-$ and $p_+$, as is the case when $X$ carries a holomorphic structure fixed by $G$, then there is a one dimensional representation $\epsilon$ of $\tilde{K}$ such that

$$ (S^+ - S^-) \otimes \epsilon \cong \wedge^{even} p_- - \wedge^{odd} p_- $$

The proof of this latter property will be given in section 5.

**Theorem 4.4** (Pseudo-coefficients) Assume that $G$ is orientation preserving. Then for any finite dimensional representation $(\tau, V_\tau)$ of $\tilde{K}$ there is a compactly supported smooth function $h_\tau$ on $\tilde{G}$ such that for every irreducible unitary representation $(\pi, V_\pi)$ of $\tilde{G}$ it holds:

$$ \text{tr } \pi(h_\tau) = \dim(V_\pi \otimes S^+ \otimes V_\tau)^\tilde{K} - \dim(V_\pi \otimes S^- \otimes V_\tau)^\tilde{K}. $$
The functions given in this theorem are also known as pseudo-coefficients [10]. This result generalizes the one in [10] in several ways. First, the group $G$ needn’t be connected and secondly the representation $\tau$ needn’t be spinorial. The proof of this theorem relies on the following lemma.

**Lemma 4.5** Let $(\pi, V_\pi)$ be an irreducible unitary representation of $\tilde{G}$ and assume
\[
\dim(V_\pi \otimes S^+ \otimes V_\pi)^\tilde{K} - \dim(V_\pi \otimes S^- \otimes V_\pi)^\tilde{K} \neq 0,
\]
then the Casimir eigenvalue satisfies $\pi(C) = \tilde{\tau}(C_\tilde{K}) - B(\rho) + B(\rho_K)$.

**Proof:** Let the $\tilde{K}$-invariant operator
\[
d_\pm : V_\pi \otimes S^\pm \rightarrow V_\pi \otimes S^\pm
\]
be defined by
\[
d_\pm : v \otimes s \mapsto \sum_i \pi(X_i)v \otimes c(X_i)s,
\]
where $(X_i)$ is an orthonormal base of $\mathfrak{p}$. The formula of Parthasarathy, [1], p. 55 now says
\[
d_-d_+ = d_+d_- = \pi \otimes s^\pm(C_\tilde{K}) - \pi(C) \otimes 1 - B(\rho) + B(\rho_K).
\]
Our assumption leads to $ker(d_+d_-) \cap \pi \otimes S(\tau) \neq 0$, and therefore $0 = \tau(C_\tilde{K}) - \pi(C) - B(\rho) + B(\rho_K)$. Q.E.D.

For the proof of Theorem 4.4 let $(\tau, V_\tau)$ a finite dimensional irreducible unitary representation of $\tilde{K}$ and write $E_\tau$ for the $\tilde{G}$-homogeneous vector bundle over $X = \tilde{G}/\tilde{K}$ defined by $\tau$. The space of smooth sections $\Gamma^\infty(E_\tau)$ may be written as $\Gamma^\infty(E_\tau) = (C^\infty(\tilde{G}) \otimes V_\tau)^\tilde{K}$, where $\tilde{K}$ acts on $C^\infty(\tilde{G})$ by right translations. The Casimir operator $C$ of $\tilde{G}$ acts on this space and defines a second order differential operator $C_\tau$ on $E_\tau$. On the space of $L^2$-sections $L^2(X, E_\tau) = (L^2(\tilde{G}) \otimes V_\tau)^\tilde{K}$ this operator is formally selfadjoint with domain, say, the compactly supported smooth functions and extends to a selfadjoint operator. Consider a Schwartz function $f$ on $\mathbb{R}$ such that the Fourier transform $\hat{f}$ has compact support. By general results on hyperbolic equations ([12], chap IV) it follows that the smoothing operator $f(C_\tau)$, defined by the spectral theorem has finite propagation speed. Since $f(C_\tau)$ is $\tilde{G}$-equivariant.
it follows that $f(C_\tau)$ can be represented as a convolution operator $\varphi \mapsto \varphi \ast \tilde{f}_\tau$, for some $\tilde{f}_\tau \in (C^\infty_c(\hat{G}) \otimes \text{End}(V_\tau))^\hat{K} \times \hat{K}$ and with $\tilde{f}_\tau$ denoting the pointwise trace of $f_\tau$ we have for $\pi \in \hat{\hat{G}}$: $\text{tr} \left( \pi(f_\tau) \right) = f(\pi(C)) \dim(V_\pi \otimes V_\tau)^\hat{K}$, where $\pi(C)$ denotes the Casimir eigenvalue on $\pi$. This construction extends to virtual representations by linearity.

Choose $f$ such that $f(\bar{C}_K - B(\rho) + B(\rho_K)) = 1$. Such an $f$ clearly exists. Let $\gamma$ be the virtual representation of $\tilde{K}$ on the space

$$V_\gamma = (S^+ - S^-) \otimes V_\tau,$$

then set $h_\tau = \tilde{f}_\gamma$. Theorem 4.4 follows. Q.E.D.

To get the first part of Theorem 4.2 from Theorem 4.4 one replaces $\tau$ in the proposition by the virtual representation on $(S^+ - S^-) \otimes V_\tau$. Since $(S^+ - S^-) \otimes (S^+ - S^-)$ is as $\tilde{K}$ module isomorphic to $\wedge^p$ we get the desired function, say $j$ on the group $\hat{G}$. Now if $\hat{G} \neq G$ let $z$ be the nontrivial element in the kernel of the isogeny $\hat{G} \rightarrow G$, then the function

$$f(x) = \frac{1}{2}(j(x) + j(zx))$$

factors over $G$ and satisfies the claim.

To get the second part of the theorem one proceeds similarly replacing $\tau$ by $\epsilon \otimes \tau$. Q.E.D.

5 Clifford algebras and Spin groups

This section is solely given to provide a proof of the properties of the spin representation used in the last section. We will therefore not strive for the utmost generality but plainly state things in the form needed. For more details the reader is referred to [11].

Let $V$ be a finite dimensional complex vector space and let $q : V \rightarrow \mathbb{C}$ be a non-degenerate quadratic form. We use the same letter for the symmetric bilinear form:

$$q(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y)).$$
Let $SO(q) \subset GL(V)$ be the special orthogonal group of $q$. The Clifford algebra $Cl(q)$ will be the quotient of the tensorial algebra

$$TV = \mathbb{C} \oplus V \oplus (V \otimes V) \oplus \ldots$$

by the two-sided ideal generated by all elements of the form $v \otimes v + q(v)$, where $v \in V$.

This ideal is not homogeneous with respect to the natural $\mathbb{Z}$-grading of $TV$, but it is homogeneous with respect to the induced $\mathbb{Z}/2\mathbb{Z}$-grading given by the even and odd degrees. Hence the latter is inherited by $Cl(q)$:

$$Cl(q) = Cl^0(q) \oplus Cl^1(q).$$

For any $v \in V$ we have in $Cl(q)$ that $v^2 = -q(v)$ and therefore $v$ is invertible in $Cl(q)$ if $q(v) \neq 0$. Let $Cl(q)^\times$ be the group of invertible elements in $Cl(q)$. The algebra $Cl(q)$ has the following universal property: For any linear map $\varphi : V \to A$ to a $\mathbb{C}$-algebra $A$ such that $\varphi(v)^2 = -q(v)$ for all $v \in V$ there is a unique algebra homomorphism $Cl(v) \to A$ extending $\varphi$.

Let $Pin(q)$ be the subgroup of the group $Cl(q)^\times$ generated by all elements $v$ of $V$ with $q(v) = \pm 1$. Let the complex spin group be defined by

$$Spin(q) = Pin(q) \cap Cl^0(q),$$

i.e., the subgroup of $Pin(q)$ of those elements which are representable by an even number of factors of the form $v$ or $v^{-1}$ with $v \in V$. Then $Spin(q)$ acts on $V$ by $x.v = xv x^{-1}$ and this gives a double covering: $Spin(q) \to SO(q)$.

Assume the dimension of $V$ is even and let

$$V = V^+ \oplus V^-$$

be a polarization, that is $q(V^+) = q(V^-) = 0$. Over $\mathbb{C}$ polarizations always exist for even dimensional spaces. By the nondegeneracy of $q$ it follows that to any $v \in V^+$ there is a unique $\hat{v} \in V^-$ such that $q(v, \hat{v}) = -1$. Further, let $V^{-,w}$ be the space of all $w \in V^-$ such that $q(v, w) = 0$, then

$$V^- = \mathbb{C}\hat{v} \oplus V^{-,w}.$$

Let

$$S = \wedge^* V^- = \mathbb{C} \oplus V^- \oplus \wedge^2 V^- \oplus \ldots \oplus \wedge^{top} V^-,$$

then we define an action of $Cl(q)$ on $S$ in the following way:
• for $v \in V^-$ and $s \in S$ let
  $$v.s = v \wedge s,$$
• for $v \in V^+$ and $s \in \wedge V^-$ let
  $$v.s = 0,$$
• and for $v \in V^+$ and $s \in S$ of the form $s = \hat{v} \wedge s'$ with $s' \in \wedge V^-$ let
  $$v.s = s'.$$
By the universal property of $Cl(V)$ this extends to an action of $Cl(q)$. The module $S$ is called the \textit{spin module}. The induced action of $Spin(q)$ leaves invariant the subspaces
$$S^+ = \wedge^{even} V^-, \quad S^- = \wedge^{odd} V^-,$$
the representation of $Spin(q)$ on these spaces are called the \textit{half spin representations}. Let $SO(q)^+$ the subgroup of all elements in $SO(q)$ that leave stable the decomposition $V = V^+ \oplus V^-$. This is a connected reductive group isomorphic to $GL(V^+)$, since, let $g \in GL(V^+)$ and define $\hat{g} \in GL(V^-)$ to be the inverse of the transpose of $g$ by the pairing induced by $q$ then the map $Gl(v) \rightarrow SO(q)^+$ given by $g \mapsto (g, \hat{g})$ is an isomorphism. In other words, choosing a basis on $V^+$ and a the dual basis on $V^-$ we get that $q$ is given in that basis by $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$. Then $SO(q)^+$ is the image of the embedding
$$GL(V^-) \hookrightarrow SO(q)$$
$$A \mapsto \left( \begin{array}{cc} A & 0 \\ 0 & t_A^{-1} \end{array} \right).$$
Let $Spin(q)^+$ be the inverse image of $SO(q)^+$ in $Spin(q)$. Then the covering $Spin(q)^+ \rightarrow SO(q)^+ \cong GL(V^-)$ is the “square root of the determinant”, i.e., it is isomorphic to the covering $GL(V^-) \rightarrow GL(V^-)$ given by the pullback diagram of linear algebraic groups:
$$\begin{array}{ccc}
GL(V^-) & \rightarrow & GL(1) \\
\downarrow & & \downarrow x \mapsto x^2 \\
GL(V^-) & \rightarrow & GL(1).
\end{array}$$
As a set, $\tilde{GL}(V^-)$ is given as the set of all pairs $(g, z) \in GL(V^-) \times GL(1)$ such that $\det(g) = z^2$ and the maps to $GL(V^-)$ and $GL(1)$ are the respective projections.

**Lemma 5.1** There is a one dimensional representation $\epsilon$ of $Spin(q)^+$ such that

$$S^\pm \otimes \epsilon \cong \wedge^\pm V$$

as $Spin(q)^+$-modules, where $\wedge^\pm$ means the even or odd powers respectively.

**Proof:** Since $Spin(q)^+$ is a connected reductive group over $\mathbb{C}$ we can apply highest weight theory. If the weights of the representation of $Spin(q)^+$ on $V$ are given by $\pm \mu_1, \ldots, \pm \mu_m$, then the weights of the half spin representations are given by

$$\frac{1}{2}(\pm \mu_1 \pm \cdots \pm \mu_m)$$

with an even number of minus signs in the one and an odd number in the other case. Let $\epsilon = \frac{1}{2}(\mu_1 + \cdots + \mu_m)$ then $\epsilon$ is a weight for $Spin(q)^+$ and $2\epsilon$ is the weight of, say, the one dimensional representation on $\wedge^{top} V^+$. By Weyl's dimension formula this means that $2\epsilon$ is invariant under the Weyl group and therefore $\epsilon$ is. Again by Weyl's dimension formula it follows that the representation with highest weight $\epsilon$ is one dimensional. Now it follows that $S^+ \otimes \epsilon$ has the same weights as the representation on $\wedge^+ V$, hence must be isomorphic to the latter. The case of the minus sign is analogous.

Q.E.D.

### 6 Orbital integrals

It now will be shown that $\text{tr} \pi(f_\tau)$ vanishes for a principal series representation $\pi$. To this end let $P = MAN$ be a nontrivial parabolic subgroup with $A \subset \exp(p_0)$. Let $(\xi, V_\xi)$ denote an irreducible unitary representation of $M$ and $e^\nu$ a quasicharacter of $A$. Let $\pi_{\xi, \nu} := \text{Ind}^G_P(\xi \otimes e^\nu \otimes 1)$.

**Lemma 6.1** We have $\text{tr} \pi_{\xi, \nu}(f_\tau) = 0$. 
Proof: By Frobenius reciprocity we have for any irreducible unitary representation $\gamma$ of $K$:

$$\text{Hom}_K(\gamma, \pi_{\xi,\nu}|_K) \cong \text{Hom}_{K_M}(\gamma|_{K_M}, \xi),$$

where $K_M := K \cap M$. This implies that $\text{tr} \pi_{\xi,\nu}(f \tau)$ does not depend on $\nu$. On the other hand $\text{tr} \pi_{\xi,\nu}(f \tau) \neq 0$ for some $\nu$ would imply $\pi_{\xi,\nu}(C) = \hat{\tau}(C_K) - B(\rho) + B(\rho_K)$ which only can hold for $\nu$ in a set of measure zero.

Q.E.D.

Recall that an element $g$ of $G$ is called elliptic if it lies in a compact Cartan subgroup. Since the following relies on results of Harish-Chandra which were proven under the assumption that $G$ is of inner type, we will from now on assume this.

**Proposition 6.2** Assume that $G$ is of inner type. Let $g$ be a semisimple element of the group $G$. If $g$ is not elliptic, then the orbital integral $O_g(f \tau)$ vanishes. If $g$ is elliptic we may assume $g \in T$, where $T$ is a Cartan in $K$ and then we have

$$O_g(f \tau) = \text{tr} \tau(g) \ c_g^{-1}|W(t, g)| \prod_{\alpha \in \Phi^+_g} (\rho_g, \alpha),$$

where $c_g$ is Harish-Chandra’s constant, it does only depend on the centralizer $G_g$ of $g$. Its value is given in [4].

Proof: The vanishing of $O_g(f \tau)$ for nonelliptic semisimple $g$ is immediate by the lemma above and Theorem [3.3]. So consider $g \in T \cap G'$, where $G'$ denotes the set of regular elements. Note that for regular $g$ the claim is $O_g(f \tau) = \text{tr} \tau(g)$. Assume the claim proven for regular elements, then the general result follows by standard considerations as in [1], p.32 ff. where however different Haar-measure normalizations are used that produce a factor $[G_g : G_g^0]$, therefore these standard considerations are now explained. Fix
$g \in T$ not necessarily regular. Let $y \in T^0$ be such that $gy$ is regular. Then

$$\text{tr} \tau(gy) = \int_{T \setminus G} f_\tau(x^{-1}gyx) \, dx$$

$$= \int_{T^0 \setminus G} f_\tau(x^{-1}gyx) \, dx$$

$$= \int_{G_g \setminus G} \int_{T^0 \setminus G_g} f_\tau(x^{-1}z^{-1}gyzx) \, dz \, dx$$

$$= \int_{G_g \setminus G} \sum_{\eta : G_g / G_g^0} \frac{1}{[G_g : G_g^0]} \int_{T^0 \setminus G_g^0} f_\tau(x^{-1}z^{-1}gyz\eta x) \, dz \, dx.$$ 

The factor $\frac{1}{[G_g : G_g^0]}$ comes in by the Haar-measure normalizations. On $G_g^0$ consider the function

$$h(y) = f(x^{-1}\eta^{-1}ggyx).$$

Now apply Harish-Chandra’s operator $\omega_{G_g}$ to $h$ then for the connected group $G_g^0$ it holds

$$h(1) = \lim_{y \to 1} c_{G_g^0} \omega_{G_g} (O_{G_g}^0(h)).$$

When $y$ tends to 1 the $\eta$-conjugation drops out and the claim follows.

So in order to prove the proposition one only has to consider the regular orbital integrals. Next the proof will be reduced to the case when the compact Cartan $T$ meets all connected components of $G$. For this let $G^+ = TG^0$ and assume the claim proven for $G^+$. Let $x \in G$ then $xTx^{-1}$ again is a compact Cartan subgroup. Since $G^0$ acts transitively on all compact Cartan subalgebras it follows that $G^0$ acts transitively on the set of all compact Cartan subgroups of $G$. It follows that there is a $y \in G^0$ such that $xTx^{-1} = yTy^{-1} \subset TG^0 = G^+$, which implies that $G^+$ is normal in $G$.

Let $\tau^+ = \tau|_{G^+ \cap K}$ and $f_{\tau^+}$ the corresponding Euler-Poincaré function on $G^+$.

**Lemma 6.3** $f_{\tau^+} = f_\tau|_{G^+}$

Since the Euler-Poincaré function is not uniquely determined the claim reads that the right hand side is a EP-function for $G^+$.

**Proof:** Let $\tau^+ = \tau|_{K^+}$, where $K^+ = TK^0 = K \cap G^+$. Let $\varphi^+ \in (C^\infty_c(G^+) \otimes V_\tau)^{K^+}$, which may be viewed as a function $\varphi^+ : G^+ \to V_\tau$ with $\varphi^+(xk) = \cdots$
\(Euler-Poincaré\) Functions

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\(\tau(k^{-1})\varphi^+(x)\) for \(x \in G^+, k \in K^+\). Extend \(\varphi^+\) to \(\varphi : G \to V_\tau\) by \(\varphi(xk) = \tau(k^{-1})\varphi^+(x)\) for \(x \in G^+, k \in K\). This defines an element of \((C_c^\infty(G) \otimes V_\tau)^K\) with \(\varphi|_{G^+} = \varphi^+\). Since \(C_\tau\) is a differential operator it follows \(f(C_\tau)\varphi|_{G^+} = f(C_\tau)\varphi^+\), so

\[
(\varphi \ast \tilde{f_\tau})|_{G^+} = \varphi^+ \ast \tilde{f_\tau}.
\]

Considering the normalizations of Haar measures gives the lemma. Q.E.D.

For \(g \in T\) we compute

\[
\mathcal{O}_g(f_\tau) = \int_{T \setminus G} f_\tau(x^{-1}gx)dx = \sum_{yG/G^+} \frac{1}{[G : G^+]} \int_{T \setminus G^+} f_\tau(x^{-1}y^{-1}gyx)dx,
\]

where the factor \(\frac{1}{[G : G^+]}\) stems from normalization of Haar measures and we have used the fact that \(G^+\) is normal. The latter equals

\[
\frac{1}{[G : G^+] \sum_{yG/G^+} \mathcal{O}^G_{y^{-1}xy}(f_\tau) = \frac{1}{[G : G^+] \sum_{yG/G^+} \mathcal{O}^G_{y^{-1}xy}(f_\tau^+)}.
\]

Assuming the proposition proven for \(G^+\), this is

\[
\frac{1}{[G : G^+] \sum_{yG/G^+} \text{tr} \tau(y^{-1}gy) = \text{tr} \tau(g).
\]

From now on one thus may assume that the compact Cartan \(T\) meets all connected components of \(G\). Let \((\pi, V_\pi) \in \hat{G}\). Harish-Chandra has shown that for any \(\varphi \in C_c^\infty(G)\) the operator \(\pi(\varphi)\) is of trace class and there is a locally integrable conjugation invariant function \(\Theta_{\pi}\) on \(G\), smooth on the regular set such that

\[
\text{tr} \pi(\varphi) = \int_{G^+} \varphi(x)\Theta_{\pi}(x)dx.
\]

For any \(\psi \in C^\infty(K)\) let \(\pi|_K(\psi) = \int_K \psi(k)\pi(k)dk\).
Lemma 6.4 Assume $T$ meets all components of $G$. For any $\psi \in C^\infty(K)$ the operator $\pi|_K(\psi)$ is of trace class and for $\psi$ supported in the regular set $K' = K \cap G'$ we have

$$\text{tr} \pi|_K(\psi) = \int_K \psi(k) \Theta_\pi(k) dk.$$  

(For $G$ connected this assertion is in [4] p.16.)

Proof: Let $V_\pi = \bigoplus_i V_\pi(i)$ be the decomposition of $V_\pi$ into $K$-types. This is stable under $\pi|_K(\psi)$. Harish-Chandra has proven $[\pi|_K : \tau] \leq \dim \tau$ for any $\tau \in \hat{K}$. Let $\psi = \sum_j \psi_j$ be the decomposition of $\psi$ into $K$-bitypes. Since $\psi$ is smooth the sequence $\| \psi_j \|_1$ is rapidly decreasing for any enumeration of the $K$-bitypes. Here $\| \psi \|_1$ is the $L^1$-norm on $K$. It follows that the sum $\sum_i \text{tr}(\pi|_K(\psi)|V_\pi(i))$ converges absolutely, hence $\pi|_K(\psi)$ is of trace class.

Now let $S = \exp(p_0)$ then $S$ is a smooth set of representatives of $G/K$. Let $G.K = \cup_{g \in G} gKg^{-1} = \cup_{s \in S} sKs^{-1}$, then, since $G$ has a compact Cartan, the set $G.K$ has non-empty interior. Applying the Weyl integration formula to $G$ and backwards to $K$ gives the existence of a smooth measure $\mu$ on $S$ and a function $D$ with $D(k) > 0$ on the regular set such that

$$\int_{G.K} \varphi(x) dx = \int_S \int_K \varphi(sks^{-1}) D(k) dk d\mu(s)$$

for $\varphi \in L^1(G.K)$. Now suppose $\varphi \in C^\infty_c(G)$ with support in the regular set. Then

$$\text{tr} \pi(\varphi) = \int_{G.K} \varphi(x) \Theta_\pi(x) dx$$

$$= \int_S \int_K \varphi(sks^{-1}) D(k) \Theta_\pi(k) d\mu(s)$$

$$= \int_K \int_S \varphi(s) d\mu(s) D(k) \Theta_\pi(k) dk,$$
where we have written \( \varphi^s(k) = \varphi(sks^{-1}) \). On the other hand

\[
\text{tr } \pi(\varphi) = \text{tr } \int_{G.K} \varphi(x)\pi(x)dx = \text{tr } \int_S \int_K \varphi(sks^{-1})D(k)\pi(sks^{-1})dkd\mu(s) = \int_S \text{tr } \pi|_K(\varphi^sD(\pi(s)^{-1})d\mu(s) = \int_S \text{tr } \pi|_K(\varphi^sD)d\mu(s) = \text{tr } \pi|_K(\int_S \varphi^s d\mu(s)D).
\]

This implies the claim for all functions \( \psi \in C_c^\infty(K) \) which are of the form

\[
\psi(k) = \int_S \varphi(sks^{-1})d\mu(s)D(k)
\]

for some \( \varphi \in C_c^\infty(G) \) with support in the regular set. Consider the map

\[
F: S \times K' \to G.K'
\quad (s,k) \mapsto sks^{-1}
\]

Then the differential of \( F \) is an isomorphism at any point and by the inverse function theorem \( F \) locally is a diffeomorphism. So let \( U \subset S \) and \( W \subset K' \) be open sets such that \( F|_{U \times W} \) is a diffeomorphism. Then let \( \alpha \in C_c^\infty(U) \) and \( \beta \in C_c^\infty(W) \), then define

\[
\phi(sks^{-1}) = \alpha(s)\beta(k) \quad \text{if } s \in U, k \in W
\]

and \( \varphi(g) = 0 \) if \( g \) is not in \( F(U \times W) \). We can choose the function \( \alpha \) such that \( \int_S \alpha(s)d\mu(s) = 1 \). Then

\[
\int_S \varphi(sks^{-1})d\mu(s)D(k) = \beta(k)D(k).
\]

Since \( \beta \) was arbitrary and \( D(k) > 0 \) on \( K' \) the lemma follows. \( \text{Q.E.D.} \)

Let \( W \) denote the virtual \( K \)-representation on \( \Lambda^{even}p \otimes V_\tau - \Lambda^{odd}p \otimes V_\tau \) and write \( \chi_W \) for its character.
Lemma 6.5 Assume $T$ meets all components of $G$, then for any $\pi \in \hat{G}$ the function $\Theta_{\pi \chi_W}$ on $K' = K \cap G'$ equals a finite integer linear combination of $K$-characters.

**Proof:** It suffices to show the assertion for $\tau = 1$. Let $\varphi$ be the homomorphism $K \rightarrow O(p)$ induced by the adjoint representation, where the orthogonal group is formed with respect to the Killing form. We claim that $\varphi(K) \subset SO(p)$, the subgroup of elements of determinant one. Since we assume $K = K^0T$ it suffices to show $\varphi(T) \subset SO(p)$. For this let $t \in T$. Since $t$ centralizes $t$ it fixes the decomposition $p = \oplus_{\alpha} p_{\alpha}$ into one dimensional root spaces. So $t$ acts by a scalar, say $c$ on $p_{\alpha}$ and by $d$ on $p_{-\alpha}$. There is $X \in p_{\alpha}$ and $Y \in p_{-\alpha}$ such that $B(X,Y) = 1$. By the invariance of the Killing form $B$ we get

$$1 = B(X,Y) = B(\text{Ad}(t)X, \text{Ad}(t)Y) = cdB(X,Y) = cd.$$

So on each pair of root spaces $\text{Ad}(t)$ has determinant one hence also on $p$.

Replacing $G$ by a double cover if necessary, which doesn’t effect the claim of the lemma, we may assume that $\varphi$ lifts to the spin group $\text{Spin}(p)$. Let $p = p^+ \oplus p^-$ be the decomposition according to an ordering of $\phi(t,g)$. This decomposition is a polarization of the quadratic space $p$ and hence the spin group acts on $S^+ = \wedge^{even} p^+$ and $S^- = \wedge^{odd} p^+$ in a way that the virtual module $(S^+ - S^-) \otimes (S^+ - S^-)$ becomes isomorphic to $W$. For $K$ connected the claim now follows from [1] (4.5). An inspection shows however that the proof of (4.5) in [1], which is located in the appendix (A.12), already applies when we only assume that the homomorphism $\varphi$ factors over the spin group.

Q.E.D.

We continue the proof of the proposition. Let $\hat{T}$ denote the set of all unitary characters of $T$. Any regular element $\hat{t} \in \hat{T}$ gives rise to a discrete series representation $(\omega, V_\omega)$ of $G$. Let $\Theta_{\hat{t}} = \Theta_{\omega}$ be its character which, due to Harish-Chandra, is known to be a function on $G$. Harish-Chandra’s construction gives a bijection between the set of discrete series representations of $G$ and the set of $W(G,T) = W(K,T)$-orbits of regular characters of $T$.

Let $\Phi^+$ denote the set of positive roots of $(g,t)$ and let $\Phi^+_{c}, \Phi_{n}^+$ denote the subsets of compact and noncompact positive roots. For each root $\alpha$ let $t \mapsto t^\alpha$
denote the corresponding character on $T$. Define

$$'\Delta_c(t) = \prod_{\alpha \in \Phi_c^+} (1 - t^{-\alpha})$$

$$'\Delta_n = \prod_{\alpha \in \Phi_n^+} (1 - t^{-\alpha})$$

and $'\Delta = '\Delta_c '\Delta_n$. If $\hat{t} \in \hat{T}$ is singular, Harish-Chandra has also constructed an invariant distribution $\Theta_{\hat{t}}$ which is a virtual character on $G$. For $\hat{t}$ singular let $W(\hat{t}) \subset W(\mathfrak{g}, t)$ be the isotropy group. One has $\Theta_{\hat{t}} = \sum_{w \in W(\hat{t})} \epsilon(w) \Theta_{w, \hat{t}}'$ with $\Theta_{w, \hat{t}}'$ the character of an induced representation acting on some Hilbert space $V_{w, \hat{t}}$ and $\epsilon(w) \in \{\pm 1\}$. Let $E_2(G)$ denote the set of discrete series representations of $G$ and $E_2'(G)$ the set of $W(G, T)$-orbits of singular characters.

By Theorem 3.3 the proposition will follow from the

Lemma 6.6 For $t \in T$ regular we have

$$\text{tr} \tau(t) = \frac{1}{|W(G, T)|} \sum_{i \in \hat{T}} \Theta_i(f_\tau) \Theta_i(t).$$

Proof: Let $\gamma$ denote the virtual $K$-representation on $(\wedge^{\text{even}} p - \wedge^{\text{odd}} p) \otimes V_\tau$. Harish-Chandra has shown (7 Theorem 12) that for any $\hat{t} \in \hat{T}$ there is an irreducible unitary representation $\pi^0_\hat{t}$ such that $\Theta_{\hat{t}}$ coincides up to sign with the character of $\pi^0_\hat{t}$ on the set of elliptic elements of $G$ and $\pi^0_\hat{t} = \pi^0_{\hat{t}'}$ if and only if there is a $w \in W(G, T) = W(K, T)$ such that $\hat{t}' = w \hat{t}$.

Further (7, Theorem 14) Harish-Chandra has shown that the family

$$\left( \frac{'\Delta(t) \Theta_i(t)}{\sqrt{|W(G, T)|}} \right)_{i \in \hat{T}/W(G, T)}$$

forms an orthonormal basis of $L^2(T)$. Here we identify $\hat{T}/W(G, T)$ to a set of representatives in $\hat{T}$ to make $\Theta_i$ well defined.
Consider the function $g(t) = \frac{\text{tr} \gamma(t) \, ' \Delta(t)}{\text{tr} \gamma(t) \, ' \Delta(t)} = \text{tr} \tau(t) \, ' \Delta(t)$. Its coefficients with respect to the above orthonormal basis are

$$
\left\langle g, \frac{\text{tr} \gamma(t) \, ' \Delta(t)}{\sqrt{|W(G,T)|}} \right\rangle = \frac{1}{\sqrt{|W(G,T)|}} \int_T \text{tr} \gamma(t) \, ' \Delta(t) \, |\Theta_i(t)| dt
$$

$$
= \sqrt{|W(G,T)|} \int_K \text{tr} \gamma(k) \Theta_i(k) dk
$$

where we have used the Weyl integration formula for the group $K$ and the fact that $W(G,T) = W(K,T)$. Next by Lemma 6.5 this equals

$$
\sqrt{|W(G,T)|} \dim((\wedge^\text{even} p - \wedge^\text{odd} p) \otimes \tilde{\gamma} \otimes \pi_0^i)^K = \sqrt{|W(G,T)|} \Theta_i(f_\tau).
$$

Hence

$$
g(t) = \text{tr} \tau(t) \, ' \Delta(t)
$$

$$
= \sum_{i \in \tilde{T}/W(G,T)} \Theta_i(f_\tau) \, ' \Delta(t) \Theta_i(t)
$$

$$
= \frac{1}{|W(G,T)|} \sum_{i \in \tilde{T}} \Theta_i(f_\tau) \, ' \Delta(t) \Theta_i(t).
$$

The lemma and the proposition are proven. Q.E.D.

**Corollary 6.7** If $\bar{g} \in \tilde{G}$ is semisimple and not elliptic then $O_{\bar{g}}(g_\tau) = 0$. If $\bar{g}$ is elliptic regular then

$$
O_{\bar{g}}(g_\tau) = \frac{\text{tr} \tau(\bar{g})}{\text{tr} (\bar{g}| S^+ - S^-)}.
$$

**Proof:** Same as for the last proposition with $g_\tau$ replacing $f_\tau$. Q.E.D.

**Proposition 6.8** Assume that $\tau$ extends to a representation of the group $G$ on the same space. For the function $f_\tau$ we have for any $\pi \in \hat{G}$:

$$
\text{tr} \, \pi(f_\tau) = \sum_{p=0}^{\dim \pi} (-1)^p \dim \text{Ext}^p_{(\mathfrak{g},K)}(V_\tau, V_\pi),
$$

i.e., $f_\tau$ gives the Euler-Poincaré numbers of the $(\mathfrak{g}, K)$-modules $(V_\tau, V_\pi)$, this justifies the name Euler-Poincaré function.
**Proof:** By definition it is clear that

\[
\text{tr} \, \pi(f_\tau) = \sum_{p=0}^{\dim} (-1)^p \dim H^p(g, K, V_\tau \otimes V_\pi).
\]

The claim now follows from [2], p. 52. Q.E.D.

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