Leakage of the Josephson flux qubit

Daniel Domínguez and Ezequiel N. Pozzo
Centro Atomico Bariloche, 8400 San Carlos de Bariloche, Argentina
E-mail: domingd@cab.cnea.gov.ar

Abstract. We study the dynamics of the Josephson flux qubit, which consists on a SQUID
loop with three Josephson junctions operated at or near a magnetic flux of half quantum. We
perform simulations of the time dependent Schrödinger equation when the system is started in
the ground state. The leakage from the two lowest energy levels of the qubit computational
subspace to higher energy levels is calculated, when pulses in the magnetic field are applied. We
obtain, as expected, that constant field pulses have a higher leakage than rf field pulses at the
resonant Rabi frequency. In the case of the constant field pulse we study the average leakage
as a function of the pulse intensity, $f_p$. For low $f_p$ we obtain that the leakage is quadratic in $f_p$
and we compare with a perturbative calculation.

1. Introduction
In recent years, different devices of mesoscopic Josephson junctions have been studied
experimentally as candidates to be used for the design of qubits for quantum computation.
[1, 2, 3, 4, 5, 6] A large effort is devoted to succeed in the coherent manipulation of their quantum
states in a controllable way. [3, 4, 5] Real qubit devices, however, are not perfect isolated two-
level systems. First, coupling to the external environment induces relaxation and dephasing
[6]. Second, additional higher energy levels always exist in solid state devices. Therefore
leakage effects, i.e., transitions from the allowed qubit states to higher excited states of the
system can occur during quantum computation operations.[7] Leakage to the higher energy
levels produced through the interaction with the environment has been studied in some cases.
[8, 9] Even neglecting the interaction with the external environment, intrinsic leakage can occur
due to direct transitions outside the computational subspace during the application of pulses for
computational operations.[10]

Here, we will study the Josephson flux qubit (JFQ), which consists of a SQUID loop with
three Josephson junctions operated at or near a magnetic flux of half quantum.[2, 5, 6] We
will evaluate quantitatively the amount of intrinsic leakage when rf and constant pulses in the
magnetic field are applied.

2. Model
The JFQ consists on three Josephson junctions in a superconducting ring [2] that encloses
a magnetic flux $\Phi = f \Phi_0$, with $\Phi_0 = h/2e$. Two of the junctions have the same coupling
energy $E_J$ and capacitance $C$, while the third junction has couplings $\alpha E_J$ and $\alpha C$, respectively
($0.5 < \alpha < 1$). Typically the circuit inductance can be neglected and the phase difference of the
third junction is: \( \varphi_3 = 2\pi f + \varphi_1 - \varphi_2 \), leading to the Hamiltonian [2]

\[
\mathcal{H} = \frac{1}{2} \vec{P}^T \mathbf{M}^{-1} \vec{P} + E_J V(\vec{\varphi})
\]

(1)

where \( \vec{\varphi} = (\varphi_1, \varphi_2) \), \( \vec{P} = -i\hbar \nabla_\varphi = -i\hbar \left( \frac{\partial}{\partial \varphi_1}, \frac{\partial}{\partial \varphi_2} \right) \), and

\[
\mathbf{M} = \left( \frac{\Phi_0}{2\pi} \right)^2 C \begin{pmatrix} 1 + \alpha + \gamma & -\alpha \\ -\alpha & 1 + \alpha + \gamma \end{pmatrix} = \frac{\hbar^2}{\eta^2 E_J} \mathbf{m}.
\]

Here \( \eta = \sqrt{8E_C/E_J} \) with \( E_C = e^2/2C \), we include in \( \mathbf{M} \) the on-site gate capacitance \( C_g = \gamma C \), and

\[
V(\vec{\varphi}) = 2 + \alpha - \cos \varphi_1 - \cos \varphi_2 - \alpha \cos(2\pi f + \varphi_1 - \varphi_2)
\]

(2)

The time-dependent Schrödinger equation is

\[
i \frac{\partial \Psi(\vec{\varphi})}{\partial t} = \left[ -\frac{\eta^2}{2} \nabla_\varphi^T \mathbf{m}^{-1} \nabla_\varphi + V(\vec{\varphi}) \right] \Psi(\vec{\varphi})
\]

(3)

where we normalized time by \( t_J = h/E_J \), energy by \( E_J \) and momentum by \( h/\eta \). For quantum computation implementations [2, 5, 6] the JFQ is operated at magnetic fields near the half-flux quantum \( f = 1/2 + \delta f \). In this case the two lowest energy eigenstates are symmetric and antisymmetric superpositions of two states corresponding to macroscopic persistent currents of opposite sign. These two eigenstates are energetically separated from the others (for small \( \delta f \)) and therefore the system has been used as a qubit [2, 5, 6]. Recently, we have reported how the dynamics at the higher energy levels can show quantum signatures of classical chaos.[11] Moreover, the existence of higher levels can lead to leakage effects in the qubit operation, as we will study here.

We calculate the quantum dynamics of the JFQ integrating numerically Eq. (3) with a second order split-operator algorithm [12], with a discretization grid of \( \Delta \varphi = 2\pi/128 \) and \( \Delta t = 0.1t_J \). We use \( 2\pi \)-periodic boundary conditions on \( \vec{\varphi} = (\varphi_1, \varphi_2) \). Eigenstates \( |\Phi_i\rangle \) and eigenenergies \( E_i \) are also calculated by numerical diagonalization of the \( \mathcal{H} \). We consider the case of \( \eta = 0.48 \) (i.e., \( E_J/E_C = 35 \)), \( \gamma = 0 \), and \( \alpha = 0.8 \), corresponding to the experimental values of [5].

We start the JFQ at the ground state \( |\Phi_0\rangle \) of the qubit operation point at \( f_0 = 0.5 \), and we apply different types of pulses in the magnetic field, \( f(t) = f_0 + \delta f(t) \) during a time interval \( \tau \). After the pulse is applied, the wave function has evolved to \( |\Psi(\tau)\rangle \) and the qubit returns to \( f = f_0 = 0.5 \). We calculate the population \( P_1 \) at the end of the pulse of the different eigenstates \( |\Phi_i\rangle \) at \( f = f_0 \), \( P_1(\tau) = |\langle \Psi(\tau) | \Phi_i \rangle|^2 \). If the JFQ device were an ideal qubit, all the population should be subspace expanded by the two lowest eigenstates, \( |\Phi_0\rangle \), \( |\Phi_1\rangle \), and the leakage outside of the quantum computational subspace is given by \( \mathcal{L}(\tau) = 1 - P_0 - P_1 = \sum_{i=2}^{\infty} P_i \).

3. Results

We first consider the case of an rf field pulse, \( f(t) = f_0 + f_{\text{rf}} \sin(\omega_r t), \) for \( 0 < t < \tau \), with the resonant frequency \( \omega_r = E_1 - E_0 \). In [5], pulses of intensity in the range \( 10^{-4} \lesssim f_{\text{rf}} \lesssim 10^{-3} \) were used. Here we calculate the leakage for \( f_{\text{rf}} = 0.001 \). In Fig.1 we show \( \mathcal{L}(\tau) \) as a function of the pulse length \( \tau \). We also show in the inset of Fig.1 the population of the ground state \( P_0(\tau) \) and the first excited state \( P_1(\tau) \). We see that \( \mathcal{L}(\tau) \) oscillates with \( \tau \), as expected. The average value of the leakage is very small, \( \mathcal{L} \sim 10^{-7} \), showing that under a purely resonant pulse the JFQ behaves very close to an ideal qubit (in the absence of noise to the external environment).

Next, we consider a less favorable case, a non-resonant pulse, given by a square wave, constant perturbation: \( f(t) = f_0 + f_p \), for \( 0 < t < \tau \). In Fig.2 we show the leakage \( \mathcal{L}(\tau) \) for \( f_p = 0.001 \).
Figure 1. Leakage $L$ as a function of the pulse length $\tau$, for an rf field pulse of strength $f_{\text{rf}} = 0.001$ at the resonant Rabi frequency $\omega_\text{r} = E_1 - E_0$. Inset Fig.1: populations of the ground state, $P_0(\tau)$ (thick line), and the first excited state, $P_1(\tau)$ (thin line). Time is normalized by $t_J = \hbar/E_J$.

Figure 2. Leakage $L$ as a function of the pulse length $\tau$, for a constant field pulse of strength $f_p = 0.001$. Inset Fig.1: populations of the ground state, $P_0(\tau)$ (thick line), and the first excited state, $P_1(\tau)$ (thin line). Time is normalized by $t_J = \hbar/E_J$. as a function of the pulse of length $\tau$. We also show in the inset of Fig.2 the population of the ground state $P_0(\tau)$ and the first excited state $P_1(\tau)$. Here we find that the leakage is two orders of magnitude larger than in the previous case, $L \sim 10^{-5}$. Still, this is a very small amount of leakage.

A question we want to address is for which values of $f_p$ the amount leakage can become important. Therefore, we calculate the dependence of the time averaged leakage $\bar{L}$ as a function of the pulse intensity $f_p$. We show this result in Fig.3. For very small pulse intensities, the average leakage can be estimated by perturbation theory, if we write $H = H(f_0) + V(\delta f(t))$, obtaining (when the initial state is the ground state $|\Phi_0\rangle$):

$$\bar{L} = \sum_{n=2}^{\infty} \frac{2|V_{n0}|^2}{(E_n - E_0)^2}$$
where, for small $f_p$, 

$$V_{n0} = \langle \Phi_n | V(\delta f(t)) | \Phi_0 \rangle \approx 2\pi f_p \alpha \langle \Phi_n | \sin(\pi + \phi_1 - \phi_2) | \Phi_0 \rangle;$$

which predicts a dependence $\mathcal{L} \propto f_p^2$.

We compare in Fig.3 the calculated results with the perturbative approximation (summing up to the first 10 levels). We find that for $f_p \lesssim 0.001$ the perturbative approximation is adequate. For higher values of the pulse strength the average leakage grows faster with $f_p$. In particular we find that for $f_p \sim 0.03$ the amount of leakage starts to become important (i.e., above 1%).

In conclusion, we find for the JFQ that for pulse strengths $f_p < 0.01$ the intrinsic leakage is negligible small, even in the unfavorable case of a constant field pulse. This result, however, neglects the effect of external noise due to the environment, which could significantly increase the expected amount of leakage; this interesting problem will be the subject of future studies.

Acknowledgments
We acknowledge support from ANPCyT (PICT-13829 and PICT-13511), Conicet (PIP-5596) and CNEA.

References
[1] Y. Nakamura, Y. A. Paskin, J. S. Tsai, Nature 398, 786 (1999).
[2] J. E. Mooij et al., Science 285, 1036 (1999); T.P.Orlando et al., Phys. Rev. B 60, 15398 (1999).
[3] D. Vion et al., Science 296, 886 (2002).
[4] J. M. Martinis et al., Phys. Rev. Lett. 89, 117901 (2002).
[5] I. Chiorescu et al., Science 299, 1869 (2003).
[6] F. Yoshihara et al., Phys. Rev. Lett 97, 167001 (2006); K. Kakuyanagi et al., Phys. Rev. Lett. 98, 047004 (2007).
[7] R. Fazio et al., Phys. Rev. Lett. 83, 5385 (1999).
[8] G. Burkard, R. H. Koch, and D. P. DiVincenzo, Phys. Rev. B 69, 064503 (2004).
[9] F. Meier and D. Loss, Phys. Rev. B 71, 094519 (2005); T. Hakioglu and K. Savran, Phys. Rev. B 71, 115115 (2005).
[10] X. Hu and S. Das Sarma, Phys. Rev. A 66, 012312 (2002); F. Troiani et al., Phys. Rev. Lett. 94, 207208 (2005).
[11] E. N. Pozzo and D. Dominguez, Phys. Rev. Lett. 98, 057006 (2007); E. N. Pozzo, D. Dominguez, and M. J. Sánchez, Phys. Rev. B 77, 024518 (2008).
[12] M. D. Feit et al., J. Comp. Phys. 47, 412 (1982).