ON THE FILTERED SYMPLECTIC HOMOLOGY OF PREQUANTIZATION BUNDLES

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Abstract. We study Reeb dynamics on prequantization circle bundles and the filtered (equivariant) symplectic homology of prequantization line bundles, aka negative line bundles, with symplectically aspherical base. We define (equivariant) symplectic capacities, obtain an upper bound on their growth, prove uniform instability of the filtered symplectic homology and touch upon the question of stable displacement. We also introduce a new algebraic structure on the positive (equivariant) symplectic homology capturing the free homotopy class of a closed Reeb orbit – the linking number filtration – and use it to give a new proof of the non-degenerate case of the contact Conley conjecture (i.e., the existence of infinitely many simple closed Reeb orbits), not relying on contact homology.

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1. Introduction

In this paper we study dynamics of Reeb flows on prequantization $S^1$-bundles and the filtered (equivariant) symplectic homology of the associated prequantization line bundles $E$, aka negative line bundles, over symplectically aspherical manifolds. The new features in this case, as compared to the symplectic homology of exact fillings, come from the difference between the Hamiltonian action (i.e., the symplectic area), giving rise to the action filtration of the homology, and the contact action. We define equivariant symplectic capacities, obtain an upper bound on their growth, prove uniform instability of the filtered symplectic homology, and touch upon the question of stable displacement in $E$. We also introduce a new algebraic structure on the positive (equivariant) symplectic homology capturing the free homotopy class of a closed Reeb orbit – the linking number filtration. We then use this filtration to give a new proof of the non-degenerate case of the contact Conley conjecture, not relying on contact homology.

Prequantization $S^1$-bundles $M$ form an interesting class of examples to study Reeb dynamics, and, in particular, the question of multiplicity of closed Reeb orbits with applications, for instance, to closed magnetic geodesics and geodesics on CROSS's; see, e.g., [GGM17]. The range of possible dynamics behavior in this case is similar to that for Hamiltonian diffeomorphisms and more limited than for all contact structures where it should be, more adequately, compared with the class of symplectomorphisms; see [GG15]. Furthermore, in many instances, various flavors of the Floer-type homology groups associated to $M$ can be calculated explicitly providing convenient basic tools to study the multiplicity questions.

Most of the Floer theoretic constructions counting closed Reeb orbits require a strong symplectic filling $W$ of $M$ and, in general, the properties of the resulting groups depend on the choice of $W$ unless $W$ is exact and $c_1(TW) = 0$. (The exceptions are the cylindrical contact homology and the contact homology linearized by an augmentation, but here we are only concerned with the symplectic homology.) The most natural filling $W$ of a prequantization $S^1$-bundle $M$ is that by the disk bundle or, to be more precise, by the region bounded by $(M, \alpha)$, where $\alpha$ is the contact form, in the line bundle $E$.

However, the filling $W$ is also quite awkward to work with. The main reason is that $W$ is never exact, although it is aspherical when the base $B$ is aspherical. As a consequence, the Hamiltonian and contact actions of closed Reeb orbits differ and the Hamiltonian action, giving rise to the action filtration on the homology, is not necessarily non-negative; see Proposition 4.3. The second difficulty comes from that the natural map $\pi_1(M) \to \pi_1(W)$ fails, in general, to be one-to-one. This is the case, for instance, when $B$ is symplectically aspherical: the fiber, which is not contractible in $M$, becomes contractible in $W$. This fact has important conceptual consequences. For instance, the proof of the contact Conley conjecture for prequantization bundles with aspherical base (i.e., the existence of infinitely many simple closed Reeb orbits) from [GGM15, GGM17], which relies on the free homotopy class grading of the cylindrical contact homology, fails to directly translate to the symplectic homology framework.

One of the goals of this paper is to systematically study the filtered symplectic homology of $W$, equivariant and ordinary. Note that total Floer theoretic "invariants" such as the equivariant and/or positive symplectic homology of $E$ has been calculated explicitly; see [GGM17, Oa08, Ri14]. Moreover, in many cases the total
symplectic homology of $E$ vanishes and this fact alone is sufficient for many applications. The results of the paper comprise a part of the second author’s Ph.D. thesis, [Sh].

The paper is organized as follows. In Section 2, we set our conventions and notation and very briefly recall the constructions of various flavors of symplectic homology. The only non-standard point here is the definition of the negative/positive symplectic homology. Namely, since the filling is not required to be exact, this homology cannot be defined as the subcomplex generated by the orbits with negative action. Instead, following [BO17], it is defined essentially as the homology of the subcomplex generated by the constant one-periodic orbits of an admissible Hamiltonian. In Section 3, we investigate the consequences of vanishing of the total symplectic homology. We introduce a class of (equivariant) symplectic capacities, prove upper bounds on their growth, show that vanishing is equivalent to a seemingly stronger condition of uniform instability, and revisit the relation between vanishing of the symplectic homology and displacement. In Section 4, we specialize these results to prequantization line bundles with symplectically aspherical base and also briefly touch upon the question of stable displacement in prequantization bundles. A new algebraic structure on the positive (equivariant) symplectic homology of such bundles – the linking number filtration – is introduced in Section 5, where we also calculate the associated graded homology groups. This filtration is given by the linking number of a closed Reeb orbit with the base $B$. It is then used in Section 6 to reprove the non-degenerate case of the contact Conley conjecture circumventing the foundational difficulties inherent in the construction of the contact homology.

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2. Preliminaries

In this section we set our conventions and notation used throughout the paper and briefly discuss various flavors of contact homology.

2.1. Conventions. The conventions and notation adopted in this paper are similar to those used in [GG16], resulting ultimately, through cancellations of several negative signs, in the same grading and the action filtration as in [BO13, BO17].

Throughout the paper, we usually assume, unless specified otherwise that the underlying symplectic manifold $(W^{2n}, \omega)$ is symplectically aspherical, i.e., $[\omega]|_{\pi_2(W)} = 0 = c_1(TW)|_{\pi_2(W)}$ and compact with contact type boundary $M = \partial W$. The symplectic completion of $W$ is

$$\widehat{W} = W \cup_M (M \times [1, \infty]),$$

equipped with the symplectic form $\omega$ extended to the cylindrical part as $d(r\alpha)$, where $\alpha$ is a contact primitive of $\omega$ on $M$ and $r$ is the coordinate on $[1, \infty)$.

We denote by $P(\alpha)$ or $P(W)$ the set of contractible in $W$ periodic orbits $x$ of the Reeb flow on $(M^{2n-1}, \alpha)$. Since the form $\omega$ is not required to be exact there are two different ways to define an action of $x$. One is the contact action $A_{\alpha}(x)$
given by the integral of $\alpha$ over $x$, i.e., the period of $x$ as a closed orbit of the Reeb flow. The second one is the \textit{symplectic or Hamiltonian action} $A_\omega(x)$. This is the symplectic area bounded by $x$, i.e., the integral of $\omega$ over a disk bounded by $x$ in $W$. By Stokes’ formula, $A_\alpha(x) = A_\omega(x)$ when $x$ is contractible in $M$, but in general $A_\alpha(x) \neq A_\omega(x)$. We denote the resulting contact and symplectic action spectra by $S(\alpha)$ and $S_\omega(W)$, respectively.

The circle $S^1 = \mathbb{R}/\mathbb{Z}$ plays several different roles throughout the paper. We denote it by $G$ when we want to emphasize the role of the group structure on $S^1$.

The Hamiltonians $H$ on $\tilde{W}$ are always required to be one-periodic in time, i.e., $H: S^1 \times \tilde{W} \rightarrow \mathbb{R}$. In fact, most of the time the Hamiltonians we are actually interested in are \textit{autonomous}, i.e., independent of time. The (time-dependent) Hamiltonian vector field $X_H$ of $H$ is given by the Hamilton equation $i_{X_H} \omega = -dH$. For instance, on the cylindrical part $M \times [1, \infty)$ with $\omega = d(r\alpha)$ the Hamiltonian vector field of $H = r$ is the Reeb vector field of $\alpha$.

We focus on contractible one-periodic orbits of $H$. Such orbits can be identified with the critical points of the \textit{action functional} $A_H: \Lambda \rightarrow \mathbb{R}$ on the space $\Lambda$ of contractible loops $x$ in $\tilde{W}$ given by

$$A_H(x) = A_\omega(x) - \int_{S^1} H(t, x(t)) \, dt. \quad (2.1)$$

We will always require $H$ to have the form $H = kr + c$, where $k \notin S_\omega(\alpha)$, outside a compact set. Under this condition the Floer homology of $H$ is defined; see, e.g., [Vi99]. Note, however, that the homology depends on $k$.

The \textit{action spectrum} of $H$, i.e., the collection of action values for all contractible one-periodic orbits of $H$, will be denoted by $S(H)$. When $H$ is autonomous, a one-periodic orbit $y$ is said to be a \textit{reparametrization} of $x$ if $y(t) = x(t + \theta)$ for some $\theta \in G = S^1$. Two one-periodic orbits are said to be \textit{geometrically distinct} if one of them is not a reparametrization of the other. We denote by $P(H)$ the collection of all geometrically distinct contractible one-periodic orbits of $H$.

With our sign conventions in the definitions of $X_H$ and $A_H$, the Hamiltonian actions on $x$ converge to $S_\omega(x)$ in the construction of the symplectic homology. In particular, $S(H) \rightarrow S_\omega(W) \cup \{0\}$.

We normalize the \textit{Conley–Zehnder index}, denoted throughout the paper by $\mu$, by requiring the flow for $t \in [0, 1]$ of a small positive definite quadratic Hamiltonian $Q$ on $\mathbb{R}^{2n}$ to have index $n$. More generally, when $Q$ is small and non-degenerate, the flow has index equal to $(\text{sgn} \, Q)/2$, where $\text{sgn} \, Q$ is the signature of $Q$. In other words, the Conley–Zehnder index of a non-degenerate critical point $x$ of a $C^2$-small autonomous Hamiltonian $H$ on $W^{2n}$ is equal to $n - \mu_M$, where $\mu_M = \mu_M(H)$ is the Morse index of $H$ at $x$. The \textit{mean index} of a periodic orbit $x$ will be denoted by $\hat{\mu}(x)$; see, e.g., [Lo, SZ] for the definitions and also [GG16] for additional references and a detailed discussion.

We denote by $HF(H)$ the Floer homology of $H$ (when it is defined) and by $HF^G(H)$ the filtered Floer homology, where $I \subset \mathbb{R}$ is an interval, possibly infinite, with end points not in $S(H)$. Likewise, the (filtered) $G = S^1$-equivariant Floer homology is denoted by $HF^G(H)$ and $HF^{G, I}(H)$; see, e.g., [BO13, GG16] and references therein. Note that for $I = \mathbb{R}$ the filtered homology groups turn into total Floer homology $HF(H)$ and $HF^G(H)$. These homology groups are graded by the
Conley–Zehnder index, but the grading is suppressed in the notation when it is not essential.

Our choice of signs in (2.1) effects the signs in the Floer equation. Recall that the Floer equation is the $L^2$-anti-gradient flow equation for $A_H$ on $\Lambda$ with respect to a metric $\langle \cdot, \cdot \rangle$ on $\hat{W}$ compatible with $\omega$:

$$\partial_s u = -\nabla_{L^2} A_H(u),$$

where $u: \mathbb{R} \times S^1 \to V$ and $s$ is the coordinate on $\mathbb{R}$. Explicitly, this equation has the form

$$\partial_s u + J\partial_t u = \nabla H,$$

where $t$ is the coordinate on $S^1$. Here the almost complex structure $J$ is defined by the condition $\langle \cdot, \cdot \rangle = \omega(J \cdot, \cdot)$ making $J$ act on the first argument in $\omega$ rather than the second one, which is more common, to ensure that the left hand side of the Floer equation is still the Cauchy–Riemann operator $\bar{\partial}_J$. (Thus $J = -J_0$, where $J_0$ is defined by acting on the second argument in $\omega$, and $X_H = -J \nabla H$.) Note, however, that now the right hand side of (2.2) is $\nabla H$ with positive sign.

The differential in the Floer or Morse complex is defined by counting the downward Floer or Morse trajectories. As a consequence, a monotone increasing homotopy of Hamiltonians induces a continuation map preserving the action filtration in homology. (Clearly, $H \geq K$ on $S^1 \times \hat{W}$ if and only if $A_H \leq A_K$ on $\Lambda$.) Thus we have natural maps $HF^I(K) \to HF^I(H)$ and similar maps in the equivariant setting.

### 2.2. Different flavors of symplectic homology.

Our main goal in this section is to recall the definition of several kinds of symplectic homology groups associated with a compact symplectic manifold $W$ with contact type boundary $M$. Our treatment of the subject is intentionally brief, for the most part the material is standard or nearly standard, and we refer the reader to numerous other sources for a more detailed discussion; see, e.g., [BO09a, BO09b, BO13, BO17, CO, GG16, Se, Vi99] and references therein. Throughout the paper, all homology groups are taken with rational coefficients unless specifically stated otherwise. This choice of the coefficient field is essential, although suppressed in the notation, and some of the results are simply not true when, say, the coefficient field has finite characteristic.

Recall that a Hamiltonian $H: S^1 \times \hat{W} \to \mathbb{R}$ is said to be admissible if $H = \kappa r + c$, where $\kappa \notin S_\infty(\alpha)$, outside a compact set and $H \leq 0$ on $W$. The former condition guarantees that the (equivariant) Floer homology of $H$ is defined. The (equivariant) filtered symplectic homology groups of $W$ are, by definition, the limits

$$SH^I(W) = \lim_{\overset{\longrightarrow}{H}} HF^I(H)$$

and

$$SH^{G,I}(W) = \lim_{\overset{\longrightarrow}{H}} HF^{G,I}(H)$$

taken over all admissible Hamiltonians $H$. It is sufficient to take the direct limit over a cofinal sequence. For instance, we can require that $H = \text{const} < 0$ on $W$ and that $H$ depends only on $r$ on the cylindrical part. Just as the Floer homology, the symplectic homology groups are graded by the Conley–Zehnder index. The total (equivariant) symplectic homology groups $SH(W)$ (respectively, $SH^G(W)$) are obtained by setting $I = \mathbb{R}$. 
Note that the choice of the contact primitive $\alpha$ on $M$ implicitly enters the definition of the homology. However, one can show that the filtered homology is independent of $\alpha$; see [Vi99].

The (equivariant) homology comprises roughly speaking two parts: one – the “negative” homology – coming from the constant orbits of $H$ and the other – the positive” part – generated by the non-constant orbits. Let us describe the construction in the non-equivariant setting. The equivariant case can be dealt with in a similar fashion.

Assume first that $\alpha$ on $M$ is non-degenerate. This can always be achieved by replacing $M$ by its $C^\infty$-small perturbation. Let $H$ be a non-positive $C^2$-small Morse function on $W$ and a monotone increasing, convex function $h$ of $r$ on the cylindrical part such that $h'' > 0$ in the region containing one-periodic orbits of $H$. Clearly, $H$ is a Morse–Bott non-degenerate Hamiltonian and the Hamiltonians meeting the above conditions form a cofinal family. Let $CF^-(H)$ be the subspace in the Floer complex, or, to be more precise, the Morse–Bott Floer complex $CF(H)$ of $H$ generated by the critical points of $H$; see, e.g., [BO09a]. The key observation now is that $CF^-(H)$ is actually a subcomplex of $CF(H)$; see [BO17, Rmk. 2]. This is absolutely not obvious and the reason is that, as shown in [BO09b, p. 654] (see also [CO, Lemma 2.3]), a Floer trajectory asymptotic at $+\infty$ to a periodic orbit on a level $r = r_0$ cannot stay entirely in the union of $W$ with the domain $r \leq r_0$. Therefore, by the standard maximum principle such a trajectory cannot be asymptotic to a critical point of $H$ in $W$ at $-\infty$. Hence $CF^-(H)$ is closed under the Floer differential. Now we can interpret $CF^+(H) = CF(H)/CF^-(H)$, as a Morse–Bott type Floer complex arising from the non-trivial one-periodic orbits of $H$. We denote the resulting negative/positive Floer homology by $HF^\pm(H)$.

Passing to the limit over $H$, we obtain the negative/positive symplectic homology groups $SH^\pm(W)$, which fit into the long exact sequence

$$\ldots \to SH^-(W) \to SH(W) \to SH^+(W) \to \ldots$$

(2.3)

When $\alpha$ is degenerate, we approximate it by non-degenerate forms $\alpha'$ (or, equivalently, approximate $W$ by small perturbations $W'$ in $\hat{W}$ with non-degenerate, in the obvious sense, characteristic foliation), and pass to the limit as $\alpha' \to \alpha$. It is easy to see that the resulting negative/positive homology is well defined and we still have the long exact sequence (2.3). We will give a slightly different description of the positive homology in Section 5.1.

Note that this construction works essentially under no restrictions on $W$ as long as the Floer homology of $H$ is defined; e.g., $W$ can be weakly monotone. Moreover, the Bourgeois-Oancea “maximum principle” argument also gives a filtration of the Floer homology of $H$ by the “level of $r$” with $HF^-(H)$ lying the lowest level. (Alternatively, one can use the approach from [McLR, Appendix D].)

The homology $HF^\pm(H)$ inherits the action filtration in the obvious way and thus we have the groups $HF^\pm,I(H)$, where the end points of $I$ are required to be outside $S_0(H)$. As a consequence, we obtain the filtered groups $SH^\pm,I(W)$ with the end points of $I$ outside $S_0(W) \cup \{0\}$. Occasionally, we will use the notation $SH^\pm,(-\infty,0](W)$ (respectively, $SH^{(-\infty,0]}(W)$) for the inverse limit of the groups $SH^\pm,(-\infty,\epsilon](W)$ (respectively, $SH^{(-\infty,\epsilon]}(W)$) as $\epsilon \searrow 0$. 

The long exact sequence (2.3) still holds for the filtered groups $\text{SH}^{\pm, I}(H)$ and $\text{SH}^I(W)$. Since the constant orbits of $H$ have non-positive action, $\text{SH}^{-, I}(W) = 0$ if $I \subset (0, \infty)$ and there is a natural map

$$\text{SH}^-(W) \to \text{SH}^{[-\infty, 0]}(W).$$

When the form $\omega$ is exact, this map is an isomorphism. Although we do not have an explicit example, there seems to be no reason to expect this map to be either one-to-one or onto when $\omega$ is just aspherical; cf. Proposition 4.3 and [Oa08]. (Hence the terms “negative/positive” symplectic homology is somewhat misleading.)

These constructions extend to the equivariant setting in the standard way.

A word is also due on the functoriality and invariance of symplectic homology. When $W$ is exact the subject is quite standard and treated in detail and greater generality in numerous papers starting with [Vi99]; see, e.g., [CO, Gut, Se] and references therein. However, without some form of the exactness condition, the questions of functoriality and invariance of the homology are less understood. In fact, already when $W$ is aspherical, functoriality becomes a rather delicate question and we opt here for the minimalist approach to it which however is sufficient for our purposes.

Let $X$ be a Liouville vector field on $\hat{W} \setminus Z$, where $Z \subset W$ is a compact set, which agrees with $r\partial_r$ on the cylindrical part of $\hat{W}$. (For instance, we can take as $Z$ a closed codimension-two submanifold such that $[Z] \in H_{2n-2}(W)$ is Poincaré dual to the relative cohomology class in $H^2(W, M)$ of the pair $(\omega, \alpha_0)$.) Let $M_\tau$, $\tau \in [0, 1]$, be a family of closed hypersurfaces smoothly depending on $\tau$, enclosing $Z$ and transverse to $X$. We denote by $W_\tau \supset Z$ the domain in $\hat{W}$ bounded by $M_\tau$. Assume furthermore that $M_0 = M$, and hence $W_0 = W$. Then, as is easy to see, the total (equivariant, positive, negative, etc.) symplectic homology of $W_\tau$ is independent of $\tau$; see [Vi99]. For the filtered homology, this is no longer true. However, when $W_1 \supset W$, there is a natural map from the filtered homology of $W_1$ to the filtered homology of $W$, a very particular case of the “Viterbo transfer” morphism, which is essentially given by a monotone homotopy of the Hamiltonians. On the level of total homology, this map is an isomorphism.

The relations between the equivariant and non-equivariant symplectic homology are similar to the relations between the ordinary equivariant and non-equivariant homology.

On the one hand, we have the Gysin exact sequence for the positive/negative and filtered symplectic homology (see [BO09b, BO13]):

$$\ldots \to \text{SH}_\ast^I(W) \to \text{SH}_\ast^I(G)(W) \xrightarrow{D} \text{SH}_\ast^{I-2}(W) \to \text{SH}_\ast^{I-1}(W) \to \ldots$$

where $\ast = \{\pm, I\}$ (in all combinations including $\ast = I$) and we refer to $D$ as the shift operator; see also [GG16]. As a consequence, vanishing of $\text{SH}^{I, G}(W)$ implies vanishing of $\text{SH}^I(W)$.

On the other hand, there is a Leray–Serre type spectral sequence starting with $E^2 = \text{SH}_\ast^I(W) \otimes H_\ast(\mathbb{CP}^\infty)$, where $\ast = \pm$ or nothing, and converging to $\text{SH}_\ast^{I, G}(W)$; see [Hu, BO13, BO17, Se, Vi99]. As a consequence, vanishing of $\text{SH}_\ast^I(W)$ (with $\ast = \pm$ or nothing) implies and is, by the Gysin sequence, equivalent to vanishing of $\text{SH}_\ast^{I, G}(W)$. 
3. Vanishing of symplectic homology and its consequences

In this section we analyze general quantitative and qualitative consequences of vanishing of the symplectic homology, focusing mainly on symplectically aspherical manifolds.

3.1. Homology calculations and equivariant capacities. The condition that $\text{SH}(W) = 0$ readily lends itself for an explicit calculation of the (equivariant) positive symplectic homology, which then can be used to define several variants of the homological symplectic capacities. We start with a calculation of the negative and positive equivariant symplectic homology of $W$.

**Proposition 3.1.** Assume that $W^{2n}$ is symplectically aspherical and $\text{SH}(W) = 0$. Then we have the following natural isomorphisms:

(i) $\text{SH}^-(W) = H_*(W, \partial W)[-n]$ and $\text{SH}^{-G}(W) = H_*(W, \partial W) \otimes H_*(\mathbb{CP}^\infty)[-n]$;

(ii) $\text{SH}^+(W) = H_*(W, \partial W)[-n + 1]$ and $\text{SH}^{+G}(W) = H_*(W, \partial W) \otimes H_*(\mathbb{CP}^\infty)[-n + 1]$;

(iii) combined with the identification (3.1), the Gysin sequence shift map

$$
\text{SH}^+_{r+2}(W) \xrightarrow{D} \text{SH}^+_{r}(W)
$$

is the identity on the first factor and the map $H_{q+2}(\mathbb{CP}^\infty) \to H_q(\mathbb{CP}^\infty)$ on the second, given by the pairing with a suitably chosen generator of $H^2(\mathbb{CP}^\infty)$. In particular, $D$ is an isomorphism when $r \geq n + 1$.

**Proof.** Assertion (i) is an immediate consequence of the definitions and the condition that $W$ is symplectically aspherical; see, e.g., [BO13, Vi99]. With our grading conventions (which ultimately result in the same grading as in [BO13]), we have

$$
\text{SH}^-(W) = H_{n+r}(W, \partial W), \\
\text{SH}^{-G}(W) = \bigoplus_{p+q=r} H_{p+r}(W, \partial W) \otimes H_q(\mathbb{CP}^\infty),
$$

where all homology groups are taken with coefficients in $\mathbb{Q}$. In particular, as $W$ is oriented,

$$
\text{SH}_n^-(W) = \mathbb{Q}, \\
\text{SH}_q^-(W) = 0 \text{ if } q \geq n + 1
$$

Combining the assumption $\text{SH}(W) = 0$ with the long exact sequence

$$
\text{SH}(W) \xrightarrow{[-1]} \text{SH}^+(W) \xrightarrow{\text{SH}^-(W)} \text{SH}^+(W)
$$

we see that $\text{SH}^+_{n+1}(W) = \text{SH}_n^-(W)$. Hence, we have

$$
\text{SH}^+_{n+1}(W) = \text{SH}_n^-(W) = \mathbb{Q}, \\
\text{SH}_q^+(W) = 0 \text{ if } q \geq n + 2.
$$

(3.2)
This proves the second assertion.

Next consider the Gysin sequence

\[
\begin{array}{ccc}
SH^+_G(W) & \xrightarrow{D} & SH^+_{G-2}(W) \\
& \searrow & \nwarrow [+1] \\
& & SH^+_G(W)
\end{array}
\]

where \(D\) is the shift operator. Then

\[
SH^+_q(G)(W) \cong SH^+_q(G)(W) \quad \text{if} \quad q \geq n + 1.
\]

Recall also from Section 2.2 that \(SH(W) = 0\) if and only if \(SH_G(W) = 0\). From the long exact sequence, we see that

\[
SH^+_r(G)(W) = SH^-_r(G)(W) = \bigoplus_{p+q=r} H_{p+n}(W, \partial W) \otimes H_q(CP_\infty).
\]

This isomorphism commutes with \(D\) and, on the right, \(D\) is given by the pairing

\[
H_q(CP_\infty) \to H_{q-2}(CP_\infty) \quad \text{with a generator of} \quad H^2(CP_\infty).
\]

This proves assertion (iii) and completes the proof of the theorem. \(\square\)

With this calculation in mind, we are in the position to define (equivariant) homological symplectic capacities, aka spectral invariants or action selectors, depending on the perspective. The construction follows the standard path which goes back to [EH, HZ, Sc, Vi92]. (See also [GG16, GH] for a recent detailed treatment in the case where \(W\) is a ball.)

To a non-zero class \(\beta \in SH^+(W)\), we associate the “capacity”

\[
c(\beta, W) = \inf\{a \in \mathbb{R} \mid \beta \in \text{im}(i_a)\} \in \mathbb{R},
\]

where the map \(i_a : SH^+(-\infty, a)(W) \to SH^+(W)\) is induced by the inclusion of the complexes. (When \(\beta = 0\), we have, by definition, \(c(\beta, W) = -\infty\).) This capacity can be viewed as a function of \(\beta\) or \(W\). In the latter case, \(c(\beta, W)\) has all expected features of a symplectic capacity as long as \(W\) varies within a suitably chosen class of manifolds with naturally isomorphic homology groups \(SH^+(W)\). (We omit a detailed and formal discussion of the general capacity properties of \(c(\beta, W)\) and other capacities introduced below, for they are not essential for our purposes.) For \(\beta \in SH^+_G(W)\), the equivariant capacity \(c^G_G(\beta, W)\) is defined in a similar fashion.

By assertion (ii) of Proposition 3.1, every class \(\zeta \in H_*(W, \partial W)\) gives rise to class \(\zeta^+ \in SH^+(W)\) and we set \(c_\zeta(W) = c(\zeta^+, W)\). The capacity arising from the unit \(\zeta = [W, \partial W] = 0\) is of particular interest and we denote it by \(c(W)\). Likewise, by (3.1), we can associate to \(\zeta\) a sequence of classes \(c_{\zeta}^G = \zeta^+ \otimes \sigma_k \in SH^+_G(W)\), \(k = 0, 1, 2, \ldots\), where \(\sigma_k\) is a generator in \(H_{2k}(CP_\infty)\) and \(D(c_{\zeta}^G) = c_{\zeta}^G\), and we set

\[
c_{\zeta,k}(W) := c(c_{\zeta}^G, W).
\]

When \(\zeta = [W, \partial W]\), we will simply write \(c_k^G := c_{\zeta,k}\). The operator \(D\) does not increase the action filtration (see, e.g., [BO13, GG16]), and hence

\[
c_{\zeta,0}(W) \leq c_{\zeta,1}(W) \leq c_{\zeta,2}(W) \leq \ldots. \quad (3.3)
\]
Lemma 3.2. The capacities are non-negative:
\[ c_\zeta(W) \geq 0 \text{ and } c^{G}_{\zeta,k}(W) \geq 0 \] (3.4)

and
\[ c^{G}_{\zeta,0}(W) \leq c_\zeta(W). \] (3.5)

These inequalities are well known when \( W \) is exact. (Moreover, then all capacities are strictly positive.) However, when \( W \) is only assumed to be symplectically aspherical, non-trivial closed Reeb orbits on \( \partial W \) can possibly have negative Hamiltonian action (i.e., symplectic area), and (3.4) is not entirely obvious.

Proof. To prove (3.4) for, say, \( c_\zeta(W) \), consider an admissible Hamiltonian \( H \), which is non-degenerate and bounded from below by \( -\delta < 0 \) on \( W \). It is clear that the action selector corresponding to \( \zeta^+ \) for \( H \) is also bounded from below by \( -\delta \). Indeed, after a small non-degenerate perturbation of \( H \) outside \( W \), the value of the selector is attained on an orbit which is connected by a Floer trajectory to a critical point of \( H \) in \( W \). Passing to the limit, we see that \( c_\zeta(W) \geq 0 \). For the capacities \( c^{G}_{\zeta,k} \) the argument is similar.

The proof of (3.5) is identical to the argument in the case where \( W \) is exact. Namely, reasoning as in the proof of Proposition 3.1, it is easy to show that the natural map
\[ H_*(W,\partial W) \to SH^-(W) \xrightarrow{\cong} SH^+(W) \to SH^+,G(W), \]
where we suppressed in the notation the grading shift by the second arrow isomorphism, sends \( \zeta \) to \( c^G_0 = \zeta^+ \otimes \sigma_0 \). With this in mind, (3.5) follows from the commutative diagram

\[
\begin{array}{ccc}
SH^+,(-\infty,a) (W) & \longrightarrow & SH^+ (W) \\
\downarrow & & \downarrow \\
SH^+,(-\infty,a),S^1 (W) & \longrightarrow & SH^+,S^1 (W)
\end{array}
\]

Remark 3.3. We expect that the strict inequalities also hold in (3.4). However, proving this would require a more subtle argument. One could use, for instance, a continuation or “transfer” map between \( W \) and a slightly shrunk domain \( W' \) to show that this map decreases the action by a certain amount and reasoning as in the proof of Theorem 3.8. In Section 4, we will show that the strict inequalities hold for prequantization bundles by a rather simple and different argument.

Remark 3.4. Proposition 3.1 readily extends to the setting where \( W \) is monotone (or negative monotone, provided that the Floer homology is defined) once a Novikov ring is incorporated into the isomorphisms.
3.2. Uniform instability of the symplectic homology. Let us now turn to quantitative consequences of vanishing of the symplectic homology.

We say that the filtered symplectic homology of $W$ is \textit{uniformly unstable} if the natural “quotient-inclusion” map

$$\text{SH}^I(W) \to \text{SH}^{I+c}(W) \quad (3.6)$$

is zero for every interval $I$ (possibly infinite) and some constant $c \geq 0$ independent of $I$. One way to interpret this definition, inspired by the results in [Su], is that every element of the filtered homology is “noise” on the $c$-scale or, equivalently, that all bars in the barcode associated with this homology have length no longer than $c$. (See, e.g., [PS, UZ] for a discussion of barcodes and persistence modules in the context of symplectic topology.)

The requirement that the homology is uniformly unstable is seemingly stronger than that the total homology vanishes: setting $I = \mathbb{R}$ we conclude that $\text{SH}(W) = 0$. However, as was pointed out to us by Kei Irie, [Ir], the two conditions are equivalent for Liouville domains. In other words, somewhat surprisingly, vanishing of the total homology is equivalent to the uniform instability of the filtered homology. The next proposition is a minor generalization of this observation.

\textbf{Proposition 3.5.} Assume that $\omega|_{\pi_2(W)} = 0$. The following two conditions are equivalent:

(i) $\text{SH}(W) = 0$ and

(ii) there exists a constant $c_0 > 0$ such that for any $c > c_0$ and any interval $I \subset \mathbb{R}$ the map (3.6) is zero.

Moreover, the smallest constant $c_0$ with this property is exactly the capacity $c(W)$.

\textbf{Proof.} As has been pointed out above, to prove the implication (ii) $\Rightarrow$ (i), it is enough to set $I = \mathbb{R}$ in (ii). Indeed, then (3.6) is simultaneously zero and the identity map, which is only possible when $\text{SH}(W) = 0$.

Let us prove the converse. Assume that $\text{SH}(W) = 0$ and consider the natural map $\psi: \text{SH}^-(W) \to \text{SH}^{(-\infty,c)}(W)$. By definition, $c(W) = \inf\{c \mid \psi(\zeta) = 0\} < \infty$ where we took $\zeta$ to be the image of the fundamental class $[W, \partial W]$ in $\text{SH}^-(W)$; see Proposition 3.1. Our goal is to show that the map (3.6) vanishes for any $c > c(W)$ and any interval $I$.

\textbf{Step 1.} For $a \notin S_\omega(W)$, consider an interval $I = (-\infty, a)$. We have the following commutative diagram where the horizontal maps are given by the pair-of-pants product:

$$\begin{array}{ccc}
\text{SH}^{(-\infty,a)}(W) \otimes \text{SH}^{(-\infty,0)}(W) & \longrightarrow & \text{SH}^{(-\infty,a)}(W) \\
\text{id} \otimes \psi & & \phi \\
\text{SH}^{(-\infty,a)}(W) \otimes \text{SH}^{(-\infty,c)}(W) & \longrightarrow & \text{SH}^{(-\infty,a+c)}(W)
\end{array}$$

Recall that $\zeta \in \text{SH}^-(W)$ is a unit with respect to this product. (We refer the reader to [AS] for the definition of the pair-of-pants product applicable in this case and also to, e.g., [Ri13].) Thus, for any $\sigma \in \text{SH}^{(-\infty,a)}(W)$,
Hence the map $\phi$ vanishes.

**Step 2.** For $a, b \notin S_\omega(W)$, consider an interval $I = (a, b)$. We have the following commutative diagram:

$$
\begin{array}{cccc}
\rightarrow & SH_k^{(-\infty, a)}(W) & \rightarrow & SH_k^{(-\infty, b)}(W) \\
\downarrow \phi_1 & \downarrow \phi_2 & \downarrow \psi & \\
\rightarrow & SH_k^{(-\infty, a+c)}(W) & \rightarrow & SH_k^{(-\infty, b+c)}(W)
\end{array}
$$

By Step 1, the maps $\phi_1, \phi_2$ are zero maps. Hence the map $\psi$ vanishes.

**Step 3.** For $a \notin S_\omega(W)$ consider an interval $I = (a, \infty)$. At Step 2, we obtained the zero map $\psi: SH^{(a, b)}(W) \to SH^{(a+c, b+c)}(W)$. By taking $b$ to $\infty$, we see the map $\psi: SH^{(a, \infty)}(W) \to SH^{(a+c, \infty)}(W)$ vanishes. \hfill $\square$

**Remark 3.6.** It is worth pointing out that Proposition 3.5 and Theorem 3.8 below do not have a counterpart in the equivariant setting. Indeed, when $W$ is the standard symplectic ball $B^{2n}$ the maps

$$
SH_{G, \infty}(a, b)(W) \to SH_{G, \infty}^{(a+c, b+c)}(W)
$$

are non-zero for any $a$ and $c \geq 0$ while $SH_{G}(W) = 0$.

### 3.3. Growth of symplectic capacities

Another consequence of vanishing of the symplectic homology is an upper bound on the growth of the equivariant symplectic capacities.

**Proposition 3.7.** Assume that $\omega|_{\pi_2(W)} = 0$. Then, for every $\zeta \in H_d(W, \partial W)$ and $k$ such that $2k \geq 2n - d$, we have

$$
0 \leq c_{\zeta,k+1}(W) - c_{\zeta,k}(W) \leq c(W).
$$

**Proof.** The first inequality is simply the assertion that the sequence $c_{\zeta,k}(W)$ is (non-strictly) monotone increasing (see (3.3)) and, as has been pointed out in Section 3.1, this is a consequence of the fact that the operator $D$ does not increase the action filtration (see, e.g., [BO13, GG16]).

Let us show that $c_{\zeta,k+1}(W) - c_{\zeta,k}(W) \leq c(W)$. By Proposition 3.1,

$$
SH_{G, k+1}^{(b)}(W) \cong \bigoplus_{r=0}^{n} H_{2n-2r}(W; \partial W),
$$

$$
SH_{G, k}^{(b)}(W) \cong \bigoplus_{r=1}^{n} H_{2n-2r+1}(W; \partial W).
$$

For $k \geq 1$ and $c > c(W)$, we have the following commutative diagram:
Consider a class $\zeta \in H_d(W, \partial W)$. Since $2k \geq 2n - d$, the class $\zeta^G$ lies in $\text{SH}^+(W)$ for some $r \geq 1$. From (3.2), we see that $\text{SH}^+(W) = 0$ and $\text{SH}^+(W) = 0$ for all $r \geq 1$. Hence the map $D_2$ is an isomorphism. Let $\xi$ be the preimage of $\zeta^G$ under $D_2$. Assume that there exists a class $\xi' \in \text{SH}^{(e, b), G}(W)$ which is sent to $\zeta^G$ by the map $i_r$. Then we see that $(i_r \circ j_r)(\xi') = 0$. By commutativity of the diagram, $(D_1 \circ j_{r+2})(\xi) = 0$. Hence, there exists a class $\xi' \in \text{SH}^{(e, b), G}(W)$ such that $\pi_*(\xi') = j_{r+2}(\xi)$. Again, by commutativity of the diagram, $(f \circ j_{r+2})(\xi) = 0$. Hence, $c_{e, k+1}(W) \leq b + e$.

### 3.4. Vanishing and displacement

A geometrical counterpart of the condition that $\text{SH}(W) = 0$ is the requirement that $W$ is (stably) displaceable in $\widehat{W}$. In this section, we will revisit and generalize the well-known fact that $\text{SH}(W) = 0$ for displaceable Liouville domains $W$. In particular, we extend this result to monotone or negative monotone symplectic manifolds.

**Theorem 3.8.** Assume that $W$ is positive or negative monotone and that $W$ is displaceable in $\widehat{W}$ with displacement energy $e(W)$. Then, for any $c > e(W)$ and any interval $I \subset \mathbb{R}$, the quotient-inclusion map (3.6) is zero. Thus the filtered symplectic homology is uniformly unstable and, in particular, $\text{SH}(W) = 0$.

This theorem generalizes the result that $\text{SH}(W) = 0$ for Liouville domains displaceable in $\widehat{W}$ proved in [CF, CFO] via vanishing of the Rabinowitz Floer homology. (See also [Vi99] for the first results in this direction.) The proof of Theorem
3.8 when \( \omega|_{\pi_2(W)} = 0 \) is implicitly contained in [Su]. Thus the main new point here is that this condition can be relaxed as that \( W \) is allowed to be positive or negative monotone. Note also that when \( W \) is symplectically aspherical one can obtain the uniform instability as a consequence of Proposition 3.5 and of vanishing of the homology although with a possibly different lower bound on \( c \), which turns out to be better. Namely, combining this proposition with Theorem 3.8, and also using Proposition 3.7, we have the following:

**Corollary 3.9.** Assume that \( W \) is symplectically aspherical and displaceable in \( \hat{W} \) with displacement energy \( e(W) \). Then

\[
e(W) \leq c(W)
\]

and thus, for \( \zeta \in H_d(W, \partial W) \) and \( 2k \geq 2n - d \),

\[
0 \leq e_{\xi,k}^G(W) - e_{\xi,k}^G(W) \leq e(W).
\]

Furthermore, recall that \( W \) is called *stably displaceable* when \( W \times S^1 \) is displaceable in \( \hat{W} \times T^*S^1 \) and the displacement energy of \( W \times S^1 \) is then referred to as the stable displacement energy \( e_{st}(W) \) of \( W \). Combining the Künneth formula from [Oa06] with Theorem 3.8, we obtain the following well-known result.

**Corollary 3.10.** Assume that \( W \) is a Liouville domain and stably displaceable in \( \hat{W} \) with stable displacement energy \( e_{st}(W) \). Then for any \( c > e_{st}(W) \) and any interval \( I \subset \mathbb{R} \) the map (3.6) is zero. In particular, \( SH(W) = 0 \).

**Proof of Theorem 3.8.** First, assume that \( W \) is aspherical and \( I = [a, b] \) for \( a, b \notin S(\alpha) \). Consider a cofinal sequence of admissible Hamiltonians \( \{ H_i : S^1 \times \hat{W} \to \mathbb{R} \} \) satisfying the following conditions:

(i) \( H_i \) is \( C^2 \)-small on \( W \);

(ii) \( H_i = h_i(r) \) on the cylindrical part \( \partial W \times [1, \infty) \), where \( h_i \) is an increasing function such that, for some \( r_i > 1 \), \( h_i'' \geq 0 \) on \([1, r_i] \) and \( h_i(r) = k_i r + l_i \) with \( k_i \notin S(\alpha) \) for \( r \in [r_i, \infty) \);

(iii) \( k_i \to \infty \) and \( r_i \to 1 \) as \( i \to \infty \).

Define a sequence of Hamiltonians \( \{ F_i : S^1 \times \hat{W} \to \mathbb{R} \} \) by

(i) \( F_i \) is on \( C^2 \)-small on \( W \);

(ii) \( F_i = f_i(r) \) on the cylindrical part \( \partial W \times [1, \infty) \), where for \( \epsilon > 0 \)

\[
f_i(r) = \begin{cases} h_i(r) & \text{if } r \in [1, r_i] \\ k^-_i(r-r_i) & \text{if } r \in [r_i, r_i^- - \epsilon] \\ c_i & \text{if } r \in [r_i^-, r_i^+] \\ k^+_i(r-r_i) & \text{if } r \in [r_i^+, \infty) 
\end{cases}
\]

for \( k^+_i = k_i, k^-_i \notin S(\alpha) \) and \( k^-_i > k^+_i \);

(iii) \( f_i''(r) \leq 0 \) if \( r \in [r_i^- - \epsilon, r_i^-] \) and \( f_i''(r) \geq 0 \) if \( r \in [r_i^+, r_i^+ + \epsilon] \);

(iv) \( \min F_i \to 0 \) as \( i \to \infty \).

The graphs of \( F_i \) and \( H_i \) are shown in Fig. 1.

Then the sequence \( \{ F_i \} \) is cofinal and \( F_i \geq H_i \). Thus we have

\[
SH^i(W) = \limHF^i(F_i),
\]

where the limit is taken over the sequence \( \{ F_i \} \).
Let us show that the map $HF^I(F) \to HF^{I+\epsilon}(F)$ is zero for a Hamiltonian $F \in \{F_i\}_{i \geq N}$ and sufficiently large $N$. By the assumption that $W$ is displaceable in $\hat{W}$, there exists a Hamiltonian $K : S^1 \times \hat{W} \to \mathbb{R}$ such that $\phi^1_K(W) \cap W = \emptyset$ and $\epsilon(W) < \|K\| < \epsilon$, where $\phi^1_K$ is the Hamiltonian flow of $K$. Consider the positive and negative parts of Hofer’s norm of $K$:

$$
\|K\|_+ = \int_{S^1} \max_{x \in \hat{W}} K(t, x) \, dt
$$

and

$$
\|K\|_- = \int_{S^1} - \min_{x \in \hat{W}} K(t, x) \, dt.
$$

Then $\|K\| = \|K\|_+ + \|K\|_-$. Choose a constant $s \gg 0$ meeting the following conditions:

$$
\inf S(H) + s > b + \|K\|_+,
$$

$$
\inf S(K) + s > b + \|K\|_+.
$$

Select constants $c$ and $r_\pm$ such that

$c > s$;

$\text{supp} \ K \subset W \cup \partial W \times [1, r^+]$,

$\phi^1_K$ displaces $W \cup \partial W \times [1, r^-]$.

Define a Hamiltonian $K \# F$ by

$$
K \# F(t, x) = K(t, x) + F \left( t, \left( \phi^1_K \right)^{-1}(x) \right).
$$

Then $\phi^1_{K \# F} = \phi^1_K \circ \phi^1_F$ which is homotopic to the catenation of $\phi^1_F$ with $\phi^1_K$. 

**Figure 1.** Functions $F_i$ and $H_i$
Let $\mathcal{P}(K)$ be the collection of one-periodic orbits of $K$ which are contractible in $W$. The collections $\mathcal{P}(F)$ and $\mathcal{P}(K#F)$ are defined similarly. Then there is a one-to-one correspondence between $\mathcal{P}(H)$ and $\mathcal{P}(F)$ for the orbits lying on a level $r \in [r^-, r^+ + \epsilon]$. Denote by $\mathcal{P}(F, r^+)$ the collection of such orbits in $\mathcal{P}(F)$. It is not hard to see that $\mathcal{P}(K#F)$ consists of the orbits in $\mathcal{P}(K)$ and the orbits in $\mathcal{P}(F, r^+)$. Indeed, all of the orbits in $\mathcal{P}(F)$ on $r \leq r^-$ are displaced by $\phi^1_K$. Thus, the orbits near $r = r^+$ survive the displacement by $\phi^1_K$.

Evaluating the action functional for $x \in \mathcal{P}(K)$ and $y \in \mathcal{P}(F, r^+)$, we have

\[ A_{K#F}(x) = -\int_{\bar{x}} \hat{\omega} + \int_{S^1} K#F(t, x(t)) \, dt \]

\[ = A_K(x) + \int_{S^1} F \left( t, \left( \phi^t_K \right)^{-1} (x(t)) \right) \, dt \]

\[ = A_K(x) + c \]

\[ \geq b + \|K\|_+ , \]

where $\bar{x}$ is a capping of $x$.

Let $z \in \mathcal{P}(H)$ be the orbit corresponding to $y \in \mathcal{P}(F, r^+)$. Then

\[ A_{K#F}(y) = -\int_{\bar{y}} \hat{\omega} + \int_{S^1} K#F(t, y(t)) \, dt \]

\[ = -\int_{\bar{z}} \hat{\omega} + (r^+ - 1) \int_z \alpha + \int_{S^1} H(t, z(t)) \, dt + c \]

\[ = A_H(z) + (r^+ - 1) \int_z \alpha + c \]

\[ \geq b + \|K\|_+ , \]

where $\bar{y}$ and $\bar{z}$ are cappings of $y$ and $z$, respectively.

Let $H_s$ be a linear homotopy from $F$ to $K#F$. For $x \in \mathcal{P}(F)$ and $y \in \mathcal{P}(K#F)$, consider the moduli space

\[ \mathcal{M}(x, y, H_s, J_s) = \{ u \in C^\infty(S^1 \times \mathbb{R}, \hat{W}) | \lim_{s \to -\infty} u(t, s) = x, \lim_{s \to +\infty} u(t, s) = y, \partial_s u + J_s(\partial_t u - X_{H_s}(u)) = 0 \} . \]

If the moduli space is not empty,

\[ A_{K#F}(y) \leq A_F(x) + \int_{S^1} \int_{\mathbb{R}} \frac{\partial H_s}{\partial s} (u) \, ds \, dt \]

\[ \leq A_F(x) + \int_{S^1} \max_{x \in \hat{W}} (K#F - F) \, dt \]

\[ = A_F(x) + \|K\|_+ . \]

Similarly, consider a linear homotopy from $K#F$ to $F$. Then for $x \in \mathcal{P}(F)$ and $y \in \mathcal{P}(K#F)$, we have

\[ A_F(x) \leq A_{K#F}(y) + \|K\|_- . \]

For $e > \|K\|$, we have the following commutative diagram:
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\[ \text{HF}^I(F) \xrightarrow{(1)} \text{HF}^{I+\tau}(F) \]
\[ \text{HF}^{I+\|K\|+}(K\#F) \xrightarrow{(2)} \text{HF}^{I+\|K\|+\|K\|+}(F) \]

Since \( \mathcal{A}_{K\#F}(x) \geq b + \|K\|_+ \) for every \( x \in \mathcal{P}(K\#F) \), the map (2) vanishes. The map (1) vanishes as well. By taking direct limit of the map (1) over the cofinal sequence \( \{F_i\} \), we see that the map (3.6) vanishes.

Next let us show that the map (3.6) also vanishes when \( W \) is monotone, i.e., \( [\omega]_{\pi_2(W)} = \lambda c_1(TW)_{\pi_2(W)} \) for some nonzero constant \( \lambda \). Let \( H \) be an admissible Hamiltonian. Consider the set
\[ S_q(H) = \{ A_H(\bar{x}) \mid \Delta_H(\bar{x}) \in [q, q + 2n] \} \]
where \( \dim W = 2n \) and \( \Delta_H(\bar{x}) \) is the mean index of \( \bar{x} \), and the set \( S_q(K) \) defined similarly for the Hamiltonian \( K \). Clearly, these sets are compact. Now, choose a constant \( s \gg 0 \) meeting the following conditions:
\[ \inf S_q(H) + s > b + \|K\|_+; \]
\[ \inf S_q(K) + s > b + \|K\|_+. \]
By the same argument as the case where \( W \) is aspherical, we conclude that the map (3.6) is zero. \( \square \)

4. PREQUANTIZATION BUNDLES

4.1. Generalities. Let \((B^{2m}, \sigma)\) be a symplectically aspherical manifold such that \( \sigma \) is integral. Then there exists a principal \( S^1 \)-bundle \( \pi: M \to B \) and an \( S^1 \)-invariant one-form \( \alpha_0 \) on it (a connection form with curvature \( \sigma \)) such that \( \pi^*\sigma = d\alpha_0 \). This is a prequantization \( S^1 \)-bundle. The form \( \alpha_0 \) is automatically contact and the contact structure \( \ker \alpha_0 \) is a connection on \( \pi \). The Reeb flow of \( \alpha_0 \) is (up to a factor) the \( S^1 \)-action on \( M \). The factor is the integral of \( \alpha_0 \) over the fiber and its value depends on conventions and essentially boils down to the definition of an integral form and an integral de Rham class; see, e.g., [GGK, App. A]. Here we assume that this integral is \( \pi \), i.e., \( \sigma \) is integral if and only if \( [\sigma] \in H^2(B; \pi\mathbb{Z}) \).
(We use the same notation \( \pi \) for the number 3.14... and for the projection of a principle \( S^1 \)-bundle to the base; what is what should be clear from the context.)
The associated line bundle \( \pi: E \to B \) is a symplectic manifold with symplectic form
\[ \omega = \frac{1}{2}(\pi^*\sigma + d(r^2\alpha_0)) \]
where \( r: E \to [0, \infty) \) is the fiberwise distance to the zero section. We call \((E, \omega)\) a prequantization line bundle. (In the context of algebraic geometry prequantization line bundles are often referred to as negative line bundles.) Note that away from the zero section we have
\[ \omega = \frac{1}{2}d((1 + r^2)\alpha_0). \]
Let \( \alpha = f\alpha_0 \) be a contact form on \( M \) supporting \( \ker \alpha_0 \). Without loss of generality we may assume that \( f > 1/2 \). Then the fiberwise star shaped hypersurface given by the condition \((1 + r^2)/2 = f\) and denoted by \( M_f \) or \( M_\alpha \) has contact type,
and the restriction of the primitive $(1 + r^2)\alpha_0/2$ to $M_f$ is exactly $\alpha$. The domain bounded by $M_f$ in $E$, which is sometimes denoted by $W_f$ or $W_\alpha$ in what follows, is a strong symplectic filling of $M_f$ diffeomorphic to the associated disk bundle. Clearly, we can identify $E$ with the completion $\tilde{E}$.

Denote by $\tilde{\pi}_1(M)$ the collection of free homotopy classes of loops in $M$ or equivalently the set of conjugacy classes in $\pi_1(M)$. Furthermore, let $f$ be the free homotopy class of the fiber in $M$ or in $E \setminus B$ and $\tilde{f}^k = \{fk \mid k \in \mathbb{Z}\}$.

Since $\sigma$ is aspherical, the homotopy long exact sequence of the $S^1$-bundle $M \to B$ splits and we have

$$1 \to \pi_1(S^1) \to \pi_1(M) \to \pi_1(B) \to 1.$$  

It is not hard to see that $\tilde{f}^k$ is the image of $\mathbb{Z} \cong \pi_1(S^1)$ in $\tilde{\pi}_1(M) = \tilde{\pi}_1(E \setminus B)$; [GGM15, Lemma 4.1]. Furthermore, this is exactly the set of free homotopy classes of loops $x$ with contractible projections to $B$, i.e., of loops contractible in $E$. The one-to-one correspondence $\tilde{f}^k \to \mathbb{Z}$ is given by the linking number $L_B(x)$ of $x$ with $B$. This is simply the intersection index of a generic disk bounded by $x$ with $B$.

In what follows, all loops and periodic orbits are assumed to be contractible in $E$ unless stated otherwise.

Recall that to every such loop $x$ in $M_\alpha$, we can associate two actions: the symplectic action $A_\omega(x)$ obtained by integrating $\omega$ over a disk bounded by $x$ in $E$ or $W$ and the contact action $A_\alpha(x)$ which is the integral of $\alpha$ over $x$. These two actions are in general different.

Example 4.1. Let $M$ be the $S^1$-bundle $r = \epsilon$ in $E$. Then the closed Reeb orbits $x$ in $M$ are the iterated fibers. (In particular, every Reeb orbit is closed.) As a straightforward calculation shows, we have $A_\omega(x) = \pi k \epsilon^2/2$ and $A_\alpha(x) = \pi k (1 + \epsilon^2)/2$ for $x \in \tilde{f}^k$.

Lemma 4.2. Let $x$ be a loop in $M_\alpha$ in the free homotopy class $\tilde{f}^k$, $k \in \mathbb{Z}$, i.e., $L_B(x) = k$. Then

$$A_\omega(x) = A_\alpha(x) - \frac{\pi}{2} k.$$  

Proof. It is clear that the difference $A_\omega(x) - A_\alpha(x)$ is a purely topological invariant completely determined by the free homotopy class of $x$ in $E \setminus B$. By Example 4.1, we see that the difference is equal to $-\pi k/2$. \hfill $\square$

It follows from Lemma 4.2 that the Hamiltonian action is bounded from below by $-\pi k/2$ and the question if it can really be negative was raised in, e.g., [Oa08].

Our next proposition gives an affirmative answer to this question.

Proposition 4.3. For every $k$ there exists a contact form $\alpha = f\alpha_0$, where $f > 1/2$, with a closed Reeb orbit $x$ in the class $\tilde{f}^k$ such that $A_\omega(x)$ is arbitrarily close to $-\pi k/2$.

Proof. Recall that every free homotopy class can be realized by an embedded smooth oriented loop which is tangent to the contact structure; see, e.g., [EM] and references therein. Let $y$ be such a loop in the class $\tilde{f}^k$. By moving $y$ slightly in the normal direction to $y$ in $\xi = \ker \alpha_0$, i.e., in the direction of $J\dot{x}$ where $J$ is an almost complex structure on $\xi$ compatible with $d\alpha_0$, we obtain a transverse embedded loop $x$. The loop $x$ is nearly tangent to $\xi$ and we can ensure that

$$0 < \int_x \alpha_0 < \epsilon$$
for an arbitrarily small \( \epsilon > 0 \).

It is a standard (and easy to prove) fact that there exists a contact form \( \beta \) on \( M \) supporting \( \xi \) such that \( x \), up to a parametrization, a closed Reeb orbit of \( \beta \). Let \( g = \alpha_0/\beta \), i.e., \( g \) is defined by \( \alpha_0 = g\beta \). By scaling \( \beta \) if necessary, we can ensure that \( g \leq 1 \). In other words, \( 1/g \geq 1 \). It is not hard to see that \( g^2x \) extends to a function \( h \) on \( M \) so that \( h/g > 1/2 \) and the derivative of \( h \) in the normal direction to \( x \) is zero, i.e., \( \ker dh \supset \xi \) at all points of \( x \).

The latter condition guarantees that \( x \) is still a closed Reeb orbit of \( \alpha := h\beta = f\alpha_0 \), where \( f = h/g \). By construction, \( f > 1/2 \). Furthermore, we have \( \alpha|_x = \alpha_0|_x \), and hence

\[ 0 < \int_x \alpha = \int_x \alpha_0 < \epsilon. \]

Thus, by \( (4.1) \),

\[ -\pi k/2 < A_\omega(x) \leq \epsilon - \pi k/2 \]

and \( A_\omega(x) \) can be made arbitrarily close to \(-\pi k/2\).

\[ \square \]

### 4.2. Applications of homology vanishing to prequantization bundles

When \( B \) is aspherical, \( \text{SH}(W) = 0 \) for \( W = W_f \) by the Künneth formula from \([\text{Oa08}]\), and the results from Section 3 directly apply to \( W \).

For instance, combining the Thom isomorphism \( H_*(B) = H_*(W, \partial W)[-2] \) with Proposition 3.1, we obtain

**Corollary 4.4.** Assume that \( B^{2m} \) is symplectically aspherical and \( W = W_f \). Then we have natural isomorphisms

(i) \( \text{SH}^-(W) = H_*(B)[-m + 1] \) and \( \text{SH}^{-,G}(W) = H_*(B) \otimes H_*(\mathbb{C}P^\infty)[-m + 1] \);

(ii) \( \text{SH}^+(W) = H_*(B)[-m + 2] \) and

\[ \text{SH}^{+,G}(W) = H_*(B) \otimes H_*(\mathbb{C}P^\infty)[-m + 2]; \quad (4.2) \]

(iii) combined with the identification \( (4.2) \), the Gysin sequence shift map

\[ \text{SH}^{+,G}_{r+2}(W) \xrightarrow{D} \text{SH}^{+,G}_r(W) \]

is the identity on the first factor and the map \( H_{q+2}(\mathbb{C}P^\infty) \rightarrow H_q(\mathbb{C}P^\infty) \), given by the pairing with a suitably chosen generator of \( H^2(\mathbb{C}P^\infty) \), on the second. In particular, \( D \) is an isomorphism when \( q \geq m + 2 \).

**Remark 4.5.** Recall that even when \( B \) is not aspherical but simply meets the standard conditions sufficient to have the (equivariant) symplectic homology of \( W_f \) defined (e.g., that \( E \) is weakly monotone), this homology is independent of \( f \); see \([\text{Vi99}]\) and also \([\text{Ri14}]\) and Section 2.2.

Likewise, and Lemma 3.2 and Proposition 3.7 yield

**Corollary 4.6.** Assume that \( B \) is symplectically aspherical. Then, for every \( \zeta \in \text{H}_d(B^{2m}) \) and \( W = W_f \), we have

\[ 0 \leq c_{\zeta,0}^G(W) \leq c_{\zeta,1}^G(W) \leq c_{\zeta,2}^G(W) \leq \ldots \text{ and } c_{\zeta,0}^G(W) \leq c_{\zeta}(W), \quad (4.3) \]

and, when \( 2k \geq 2m - d \),

\[ 0 \leq c_{\zeta,k+1}^G(W) - c_{\zeta,k}^G(W) \leq c(W). \]

Moreover, \( c_{\zeta,0}^G(W) > 0 \), and hence all capacities are strictly positive.
The new point here, when compared to the general results, is the last assertion that the capacities are strictly positive. To see this, note first that these capacities are monotone (with respect to inclusion) on the domains $W_f$. Thus it suffices to show that $c^G_{\xi_0}(U) > 0$ for a small tubular neighborhood $U$ of $B$ in $E$ bounded by the $S^1$-bundle $r = \epsilon$. It is not hard to see that in this case $c^G_{\xi_0}(U) = A_\omega(x)$ for a closed Reeb orbit $x$ on $M$; see Section 5.2. Hence, by Example 4.1, $c^G_{\xi_0}(U) \geq \pi \epsilon^2 / 2 > 0$.

4.3. Stable displacement. The zero section of a prequantization bundle $E$ is never topologically displaceable since its intersection product with itself is Poincaré dual, up to a non-zero factor, to $[\sigma] \neq 0$. As a consequence, no compact subset containing the zero section is topologically displaceable either. However, the situation changes dramatically when one considers stable displaceability. (Recall that $K \subset W$ is stably displaceable if $K \times S^1$ is displaceable in $W \times T^*S^1$). The following observation essentially goes back to [Gü].

**Proposition 4.7.** The zero section is stably displaceable in $E$.

**Proof.** The zero section $B$ is a symplectic submanifold of $E$. Thus $B' := B \times S^1$ is nowhere coisotropic in $E \times T^*S^1$, i.e., at no point the tangent space to $B'$ is coisotropic. Furthermore, $B'$ is smoothly infinitesimally displaceable: there exists a non-vanishing vector field along $B'$ which is nowhere tangent to $B'$. Now the proposition follows from [Gü, Thm. 1.1]. (When $\dim B = 2$, one can also use the results from [LS, Po95]).

**Remark 4.8.** This argument shows that every closed symplectic submanifold $B$ of any symplectic manifold is stably displaceable.

As a consequence of Proposition 4.7, a sufficiently small tubular neighborhood of $B$ is also stably displaceable in $E$. However, in contrast with the case of Liouville manifolds, this is not enough to conclude that arbitrary large tubular neighborhoods, and thus all compact subsets of $E$, are also stably displaceable and in fact they need not be.

**Example 4.9.** Let $E$ be the tautological bundle over $B = \mathbb{C}P^1$, i.e., a blow up of $\mathbb{C}^2$. Then $E$ contains a monotone torus $L$ which is the restriction of the $S^1$-bundle (for a suitable radius) to the equator. It is known that $HF(L, L) \neq 0$; [Sm, Sect. 4.4]. Then, by the Künneth formula, $HF(L', L') \neq 0$ where $L' = L \times S^1$ in $E \times T^*S^1$. See [RS, Ve] for generalizations of this example and its connections with non-vanishing of symplectic homology.

**Remark 4.10.** Note that Proposition 4.7 holds for any base $B$, but gives no information about $SH(W)$. The reason is that the Künneth formula does not directly apply in this case even when $B$ is aspherical; see [Oa06]. The boundary of a tubular neighborhood $U$ of $B \times S^1$ in $E \times T^*S^1$ does not have contact type. Moreover, the symplectic form is not even exact near the boundary. As a consequence, the symplectic homology of $U$ is not defined. One can still introduce an ad hoc variant of such a homology group to have the Künneth formula and then reason along the lines of the proof of Theorem 3.8 to show that this homology, and hence $SH(W)$, vanishes. However, this argument is not much simpler than the proof in [Oa08]. (We refer the reader to [Ri14, RS, Ve] for a variety of vanishing/non-vanishing results for $SH(W)$ expressed in terms of the conditions on the base $B$.)
On the other hand, the proposition does imply, via the Künneth formula, vanishing of the Rabinowitz Floer homology for low energy levels in $E$, i.e., for $r$ close to zero, proved originally in [AK]. Note in this connection that, as was pointed out to us by Alex Oancea, the Rabinowitz Floer homology might depend on the energy level in this case.

Proposition 4.7 has some standard consequences along the lines of the almost existence theorem, the Weinstein conjecture and lower bounds on the growth of periodic orbits, which all are proved via variants of the displacement energy–capacity inequalities and are accessible by several methods requiring somewhat different assumptions on $(B, \sigma)$; see, e.g., [Po98] and also [Gi05]. Here, dealing with the almost existence, we adopt the setting from [Sc] which is immediately applicable.

The condition which $E$ must satisfy then is that it is stably strongly semi-positive in the sense of [Sc], which is the case if and only if $N_B \geq n+1$ or $(B^{2n}, \sigma)$ is positive monotone, i.e., $c_1(TB) = \lambda [\omega]$ on $\pi_2(B)$ where $\lambda \geq 0$ and in addition we require that $\langle [\omega], \pi_2(B) \rangle = 0$ whenever $\langle c_1(TB), \pi_2(B) \rangle = 0$. (Here $N_B$ is the minimal Chern number of $B$. Note that $N_E = N_B - 1$.)

**Corollary 4.11** (Almost Existence in $E$ near $B$; [Lu]). Assume that $B$ is as above. Let $H$ be a smooth, proper, autonomous Hamiltonian on $E$ and let $I$ be a (possibly empty) interval such that \{ $H = c$ \} is contained in a sufficiently small neighborhood of $B$ in $E$. Then, for almost all $c \in I$ in the sense of measure theory, the level \{ $H = c$ \} carries a periodic orbit of $H$.

This is an immediate consequence of the displacement energy–capacity inequality from [Sc]. (Note that in this case the level \{ $H = c$ \} automatically bounds a domain in a small tubular neighborhood of $B$, and hence the Hofer–Zehnder capacity of this domain, as a function of $c$ with finite values, is defined; see [MS] for a different approach.) The corollary is not the most general result of this kind. It is a particular case of the main theorem from [Lu]. However, our proof is simpler than the argument *ibid* and, in fact, Remark 4.8 can be used to simplify some parts of that argument. Corollary 4.11 implies the Weinstein conjecture for contact type hypersurfaces in $E$ near $B$. There are, of course, many other instances where the Weinstein conjecture is known to hold for hypersurfaces in $E$. For example, although to the best of knowledge it is still unknown if it holds in general for prequantization bundles, it does hold under suitable additional conditions. For instance, this is the case when $\sigma$ is aspherical, [Oa08], or more generally if $\pi^* [\sigma]$ is nilpotent in the quantum cohomology of $E$; [Ri14].

The second application is along the lines of the Conley conjecture (see [GG15] or [Vi92, Prop. 4.13]) and concerns the number or the growth of simple periodic orbits of compactly supported Hamiltonian diffeomorphisms. For the sake of simplicity, we assume that $\sigma$ is aspherical although this condition can be relaxed.

Let $H: S^1 \times E \to \mathbb{R}$ be a compactly supported Hamiltonian.

**Corollary 4.12.** Assume that $\sigma$ is symplectically aspherical and $\text{supp} \ H$ is contained in a sufficiently small neighborhood of $B \times S^1$ in $E \times T^* S^1$. Then $\varphi_H$ has infinitely many simple contractible periodic orbits with non-zero action provided that $\varphi_H \neq \text{id}$. Moreover, when $H \geq 0$ the number of such orbits of period up to $k$ and with positive action grows at least linearly with $k$ unless, of course, $H = 0$.

Here the first assertion follows readily from [FS, Thm. 2] and the second assertion is a consequence of [Gü, Thm. 1.2].
5. Linking number filtration

In this section we construct and utilize a new filtration on the positive (equivariant) symplectic homology of a prequantization disk bundle $W \to B$. This filtration is, roughly speaking, given by the linking number of a closed Reeb orbit and the zero section. It “commutes” with the Hamiltonian action filtration and plays essentially the same role as the grading by the free homotopy class in the contact homology of the corresponding $S^1$-bundle. Although the linking number filtration can be defined in a more general setting, it is of particular interest to us when the base $B$ is symplectically aspherical. This is the assumption we will make henceforth. We will then use the linking number filtration to reprove the non-degenerate case of the contact Conley conjecture, originally established in [GGM15, GGM17], without relying on the machinery of contact homology.

5.1. Definition of the linking number filtration. Throughout this section we keep the notation and convention from Section 4. In particular, $E$ and $M$ are the prequantization line and, respectively, $S^1$-bundles over a symplectically aspherical manifold $(B,\sigma)$, and $f > 1/2$ is a function on $M$. The domain $W = W_f$ is bounded by the fiberwise star shaped hypersurface $(1 + r^2)/2 = f$. This hypersurface $M_f$ has contact type and the restriction of the primitive $(1 + r^2)\alpha_0/2$ (on $E \setminus B$) to $\partial W$ is $\alpha = f\alpha_0$. Furthermore, recall that all loops and periodic orbits we consider are assumed to be contractible in $E$ unless stated otherwise.

Assume first that $\alpha$ is non-degenerate and let $H$ be a time-dependent admissible Hamiltonian on $E = \tilde{W}$. We require $H$ to be constant on a neighborhood $U$ of $B$ and such that all one-periodic orbits $x$ of $H$ outside $U$ are small perturbations of closed Reeb orbits. We will call these orbits non-constant. For a generic choice of such a Hamiltonian $H$ all non-constant orbits are non-degenerate.

Fix an almost complex structure $J$ on $E$ compatible with $\omega$, which we require to be independent of time near $B$ and outside a large compact set, and such that $B$ is an almost complex submanifold of $E$. Consider solutions $u : \mathbb{R} \times S^1 \to E$ of the Floer equation for $(H, J)$ asymptotic as $s \to \pm \infty$ to non-constant orbits. By the results from [FHS], the regularity conditions are satisfied for a generic pair $(H, J)$ meeting the above requirements and, moreover, for a generic Hamiltonian $H$ as above when $J$ is fixed. With this in mind, we have the complex $\text{CF}^+(H)$ generated by non-constant one-periodic orbits of $H$, contractible in $E$, and equipped with the standard Floer differential. Clearly, the homology of this complex is $HF^+(H)$.

The complex $\text{CF}^+(H)$ carries a natural filtration by the linking number with $B$. Indeed, since $H$ is constant near $B$, every solution $u$ of the Floer equation is a holomorphic curve near $B$, and the intersection index of $u$ with $B$ is non-negative since $B$ is an almost complex submanifold of $E$. When $u$ is a solution connecting $x$ to $y$, the difference $L_B(x) - L_B(y)$ is exactly this intersection number. Thus

$$L_B(x) \geq L_B(y)$$

and the Floer differential does not increase $L_B$. In other words, for every $k \in \mathbb{Z}$, the subspace $\text{CF}^+(H, f^\leq k)$ generated by the orbits $x$ with $L_B(x) \leq k$ is a subcomplex and we obtain an increasing filtration of the complex $\text{CF}^+(H)$. Set

$$\text{CF}^+(H, f^k) := \text{CF}^+(H, f^\leq k)/\text{CF}^+(H, f^\leq k-1).$$
We denote the homology of the resulting complexes by $HF^+ (H, f^k)$ and, respectively, $HF^+ (H, f^k)$.

Passing to the direct limit over $H$, we obtain the homology groups $SH^+ (W, f^k)$ and, respectively, $SH^+ (W, f^k)$, which fit into a long exact sequence

$$\cdots \to SH^+ (W, f^{k-1}) \to SH^+ (W, f^k) \to SH^+ (W, f^k) \to \cdots.$$ 

Furthermore, the complexes $CF^+ (H, f^k)$ and $CF^+ (H, f^k)$ inherit the filtration by the Hamiltonian action from the complex $CF(H)$ and this filtration descends to the homology groups.

The construction extends to the equivariant setting in a straightforward way and we only briefly outline it. Following [BO13, BO17], consider a parametrized action on the homology groups.

It is convenient to assume that $\hat{H}$ is independent of $(t, \zeta)$ and admissible in the standard sense. We will sometimes write $\hat{H}_t(z, \zeta)$ for $\hat{H}(t, z, \zeta)$. Let $\Lambda$ be the space of contractible loops $S^1 \to E$. The Hamiltonian $\hat{H}$ gives rise to the action functional

$$A_{\hat{H}} : \Lambda \times S^{2m+1} \to \mathbb{R}$$

which is a parametrized version of the standard action functional. The critical points of $A_{\hat{H}}$ are the pairs $(x, \zeta)$ satisfying the condition:

the loop $x$ is a one-periodic orbit of $H_\zeta$ and $\int_{S^1} \nabla_\zeta \hat{H}_\zeta(t, x(t)) \, dt = 0$.

It is convenient to assume that $\hat{H}$ is a small perturbation of an ordinary non-degenerate Hamiltonian $H$. Then $x$ in such a pair $(x, \zeta)$ is small perturbation of a one-periodic orbit of $H$.

Due to $S^1$-invariance of $\hat{H}$, the pairs $(x, \zeta)$ come in families $S$, called critical families, even when $\hat{H}$ is non-degenerate. The complex $CF^G (\hat{H})$ is generated by the families $S$. As in the non-equivariant case, one has a subcomplex $CF^{-,G} (\hat{H})$ generated by the critical families $S$ where $x$ is a constant orbit and the quotient complex $CF^{+,G} (\hat{H})$. The essential point is that again the Hamiltonian $\hat{H}$ can be taken constant on $U$ on every slice $E \times (t, \zeta)$. Then the quotient complex $CF^{+,G} (\hat{H})$ is generated by the critical families $S$ such that $x$ is a non-constant one-periodic orbit of $H$.

The parametrized Floer equation has the form

$$\partial_s u + J \partial_t u = \nabla E \hat{H},$$

$$\frac{d\lambda}{ds} = \int_{S^1} \nabla_\zeta \hat{H}(u(t, s), t, \lambda(s)) \, dt,$$

where $\lambda : \mathbb{R} \to S^{2m+1}$ and $u : S^1 \times \mathbb{R} \to E$. Thus, when $\hat{H}$ is constant (e.g., near $B$), $u$ is a holomorphic curve. It follows that (5.1) holds when $(u, \lambda)$ connects a critical family containing $x$ to a critical family containing $y$. As a consequence, the complex $CF^{+,G} (\hat{H})$ is again filtered by the linking number. Passing to the limit as $m \to \infty$ and then over $\hat{H}$ we obtain the linking number filtration on $SH^{+,G} (W)$.

We denote the resulting homology groups by $SH^{+,G} (W, f^k)$ and $SH^{+,G} (W, f^k)$. As in the non-equivariant case, these groups fit into the long exact sequence

$$\cdots \to SH^{+,G} (W, f^{k-1}) \to SH^{+,G} (W, f^k) \to SH^{+,G} (W, f^k) \to \cdots$$
and also inherit the action filtration. By Lemma 4.2, this filtration is essentially given by the contact action up to a constant depending on $k$. Moreover, as is easy to see, the shift map $D$ from the Gysin sequence respects the linking number filtration.

It is clear that the linking number filtration is preserved by the continuation maps because the continuation Hamiltonians can also be taken constant on $U$. As a consequence, the resulting groups are independent of the original contact form $\alpha$ or, equivalently, the domain $W$.

**Remark 5.1.** The construction of the linking number filtration can be generalized in several ways. Here we only mention one of them. Let $W$ be a compact symplectic manifold with contact type boundary $M$ and let $\Sigma \subset W$ be a closed codimension-two symplectic submanifold such that its intersection number with any sphere $A \in \pi_1(W)$ is zero. Note that for $\Sigma = B \subset E$ this intersection number is, up to a factor, $-\langle [\omega], A \rangle$, and hence the condition is satisfied automatically when $\sigma$ is aspherical. Then for every loop $x$ in $M$ contractible in $W$ the linking number $L_{\Sigma}(x)$ is well defined and we have a linking number filtration on the (equivariant) positive symplectic homology. This filtration has the same general properties as in the particular case discussed in this section.

5.2. **Calculation of the homology groups.** The calculation of the linking number filtration groups $\mathcal{SH}^+(W, f^k)$ and $\mathcal{SH}^{+,G}(W, f^k)$ can be easily carried out by using the standard Morse–Bott type arguments in Floer homology.

**Proposition 5.2.** Let, as above, $W$ be the prequantization disk bundle over a symplectically aspherical manifold $(B^{2m}, \sigma)$. Then

(i) $\mathcal{SH}^+(W, f^k) = 0$ and $\mathcal{SH}^{+,G}(W, f^k) = 0$ for $k \leq 0$,

(ii) $\mathcal{SH}^+(W, f^k) = H_\ast(M)[2k - m]$ and

(iii) $\mathcal{SH}^{+,G}(W, f^k) = H_\ast(B)[2k - m]$ for $k \in \mathbb{N}$,

where all homology groups are taken with rational coefficients.

In particular,

$$\mathcal{SH}^{+,G}_{2k+m}(W, f^k) = \mathbb{Q} \text{ for } k \in \mathbb{N}$$

and, as expected, $\mathcal{SH}^{+,G}(W, f^k)$ is isomorphic to the contact homology groups of $(M, \xi)$ for the free homotopy class $f^k$, cf. [BO17].

**Proof.** Consider an admissible Hamiltonian of the form $H = h(r^2)$, where $h$ is monotone increasing, convex function equal to zero on $[0, 1 - \epsilon]$ and to $ar^2 + b$ on $[1 + \epsilon, \infty)$, where $\epsilon > 0$ is sufficiently small for a fixed $a$. The non-trivial one-periodic orbits of $H$ occur on Morse–Bott non-degenerate levels $r = r_1, \ldots, r_l$, where $l = \lfloor a/\pi \rfloor$, when the form $\alpha_0$ is normalized to have integral $\pi$ over the fiber. The linking number of the orbits on the level $r = r_k$ with $B$ is exactly $k$. Now the proposition follows by the standard Morse–Bott argument in Floer homology (see, e.g., [BO09a, Poz] and also [GG16]) together with an index calculation as in, e.g., [GG04].

Comparing Case (iii) of Proposition 5.2 and Case (ii) of Proposition 4.4, we see that

$$\mathcal{SH}^{+,G}(W) = \bigoplus_{k \in \mathbb{N}} \mathcal{SH}^{+,G}(W, f^k),$$

(5.3)
although the isomorphism is not canonical in contrast with \((4.2)\). Note that there is no similar isomorphism in the non-equivariant case: \(\text{SH}^+(W) \neq \bigoplus_{k \in \mathbb{N}} \text{SH}^+(W, f^k)\) and, in fact, the sum on the right is much bigger than \(\text{SH}^+(W)\).

\textbf{Remark 5.3.} Although this is not immediately obvious, one can expect the natural maps \(\text{SH}^{+, \mathcal{G}}(W, f^k) \to \text{SH}^{+, \mathcal{G}}(W)\) to be monomorphisms, resulting in a filtration of \(\text{SH}^{+, \mathcal{G}}(W)\) by the groups \(\text{SH}^{+, \mathcal{G}}(W, f^k)\). Then the right hand side of \((5.3)\) would be the graded space associated with this filtration. On the other hand, the decomposition \((4.2)\) gives rise to a similarly looking filtration \(\bigoplus_{q \leq k} \text{H}_q(B) \otimes \text{H}_2(\mathbb{CP}^\infty)\). However, these two filtrations are different. Indeed, the shift operator \(D\) is strictly decreasing with respect to the filtration coming from \((4.2)\) and thus the induced operator on the graded space is zero. On the other hand, under the identification \(\text{SH}^+(W, f^k) \cong \text{H}_*(B)\) from Case (ii) of Proposition 5.2, the operator \(D\) is given by pairing with \([\sigma] \in \text{H}^2(B)\) (see \cite[Prop. 2.22]{GG16}) and this pairing is non-trivial. To put this somewhat informally, the decompositions \((4.2)\) and \((5.3)\) do not match term-wise.

\textbf{Remark 5.4} (Lusternik–Schnirelmann inequalities). We can also use the linking number filtration to extend the Lusternik–Schnirelmann inequalities established in \cite[Thm. 3.4]{GG16} for exact fillings to prequantization bundles. Namely, assume that all closed Reeb orbits on \(M_\alpha\) are isolated. Then, for any \(\beta \in \text{SH}^{+, \mathcal{G}}(W)\), we have the \textit{strict} inequality

\[
c(\beta, W_\alpha) > c(D(\beta), W_\alpha),
\]

where the right hand side is by definition \(-\infty\) when \(D(\beta) = 0\). In particular, when the orbits are isolated,

\[
0 < c_{\zeta, 0}^G(W) < c_{\zeta, 1}^G(W) < c_{\zeta, 2}^G(W) < \ldots
\]

in \((4.3)\) for every \(\zeta \in \text{H}_*(B)\). For the sake of brevity we only outline the proof of \((5.4)\). Consider a “sufficiently large” admissible autonomous Hamiltonian \(H\) constant on \(U\). Then, by \cite[Thm. 2.12]{GG16}, the strict Lusternik–Schnirelmann inequality holds for \(H\). As a consequence, there exist two one-periodic orbits \(x\) and \(y\) of \(H\) contractible in \(E\), the carriers for the corresponding action selectors for \(\beta\) and \(D(\beta)\) in \(\text{HF}^{+, \mathcal{G}}(H)\), such that \(A_H(x) > A_H(y)\) and \(x\) and \(y\) are connected by a solution \(u\) of the Floer equation. As in the proof of \cite[Thm. 3.4]{GG16}, we need to show that this inequality remains strict as we pass to the limit. When \(x\) and \(y\) are in the same free homotopy class (i.e., \(L_B(x) = L_B(y)\)), that proof goes through word-for-word. When, \(L_B(x) > L_B(y)\), the Floer trajectory \(u\) has to cross \(U\), where it is a holomorphic curve, passing through a point of \(B\). By the standard monotonicity argument, \(A_H(x) - A_H(y) = E(u) > \epsilon > 0\), where \(E(u)\) is the energy of \(u\) and \(\epsilon\) is independent of \(H\).

6. Contact Conley conjecture

\textbf{6.1. Local symplectic homology.} In this section we recall the definitions of the (equivariant) local symplectic homology and of symplectically degenerate maxima (SDM) for Reeb flows—the ingredients essential for the statement and the proof of the non-degenerate case of the contact Conley conjecture.

Let \(x\) be an isolated closed Reeb orbit of period \(T\), not necessarily simple, for a contact form \(\alpha\) on \(M^{2m+1}\). The Reeb vector field coincides with the Hamiltonian
vector field of the Hamiltonian $r$ on $M \times (1 - \epsilon, 1 + \epsilon)$ equipped with the symplectic form $d(r\alpha)$. Consider now the Hamiltonian $H = T \cdot h(r)$, where $h'(1) = 1$ and $h''(1) > 0$ is small. On the level $r = 1$, this flow is simply a reparametrization of the Reeb flow and the orbit $x$ corresponds to an isolated one-periodic orbit $\tilde{x}$ of $H$. By definition, the equivariant local symplectic homology $SH^G(x)$ of $x$ is the local $G = S^1$-equivariant Floer homology $HF^G(\tilde{x})$ of $\tilde{x}$; see [GG16, Sect. 2.3]. It is easy to see that $SH^G(x) := HF^G(\tilde{x})$ is independent of the choice of the function $h$. Note also that this construction is purely local: it only depends on the germ of $\alpha$ along $x$.

In what follows, we will use the notation $x$ for both orbits $\tilde{x}$ and $x$.

These local homology groups do not carry an absolute grading. To fix such a grading by the Conley–Zehnder index, it is enough to pick a symplectic trivialization of the contact structure $\xi$ natural ways to do this. For instance, one can start with a trivialization of the $T$-grading by the Conley–Zehnder index, it is enough to pick a symplectic trivialization $\xi$.

Example 6.1. Assume that $x$ is non-degenerate. Then $SH^G(x) = \mathbb{Q}$, concentrated in degree $\mu(x)$, when $x$ is good; and $SH^G(x) = 0$ when $x$ is bad; see [GG16, Sect. 2.3] and, in particular, Examples 2.18 and 2.19 therein.

Furthermore, it is worth keeping in mind that $\tilde{x}$ is necessarily degenerate even when $x$ is non-degenerate. Indeed, the linearized flow along $\tilde{x}$ has 1 as an eigenvalue and its algebraic multiplicity is at least 2.

As a consequence, $SH^G(x)$ is supported in the interval of length $2m$ centered at the mean index $\hat{\mu}(x)$ of $x$, i.e., only for the degrees in this range the homology can be non-zero. (If $\tilde{x}$ were non-degenerate the length of the interval would be $2m + 2$.)

In other words, using self-explanatory notation, we have

$$\supp SH^G(x) \subset [\hat{\mu}(x) - m, \hat{\mu}(x) + m];$$

see [GG16, Prop. 2.20]. Moreover,

$$\supp SH^G(x) \subset (\hat{\mu}(x) - m, \hat{\mu}(x) + m)$$

when $x$ is weakly non-degenerate, i.e., at least one of its Floquet multipliers is different from 1.

Remark 6.2. Conjecturally, when $x$ is the $k$th iteration of a simple orbit,

$$SH^G(x) \cong HC(x) \cong HF(\varphi)^{\mathbb{Z}_k},$$

where $HC(x)$ is the local contact homology of $x$ introduced in [HM] (see also [GH2M]), $\varphi$ is the return map of $x$, and $HF(\varphi)^{\mathbb{Z}_k}$ is the $\mathbb{Z}_k$-invariant part in the local Floer homology of $\varphi$ with respect to the natural $\mathbb{Z}_k$-action. When $x$ is simple, i.e., $k = 1$, this has been proved. Indeed, in this case, $HC(x)$ is rigorously defined and the first isomorphism is a local version of the main result in [BO17]. The second
isomorphism is established in [HM] and can also be thought of as a local variant of the isomorphism between the Floer and contact homology from [EKP]. When $k \geq 1$, there are foundational problems with the construction of $HC(x)$ common to many versions of the contact homology (see, however, [Ne]) and proving directly that the first and the last term in (6.3) are isomorphic might be a simpler approach. We will return to this question elsewhere.

When $M$ is compact, the groups $SH^G(x)$, where $x$ ranges over all closed Reeb or orbits of $\alpha$ (not necessarily simple), are the building blocks for $SH^+,G(W)$ where $W$ is a symplectically aspherical filling of $M$. For instance, vanishing of the local homology groups for all $x$ in a certain degree $d(x)$ implies vanishing of the global (i.e., total) homology in a fixed degree $d$. However, there might be a shift of degrees, i.e., $d \neq d(x)$, which depends on the choice of trivializations along the orbits $x$. This shift is obviously zero when $x$ is contractible in $W$ and the trivialization of $T_xW$ comes from a capping of $x$.

The lemma readily follows from the observation that under the above conditions $SH^+,G(H,f^k) = 0$ for a suitable cofinal family of admissible Hamiltonians $H$. (More generally, there is a spectral sequence starting with $\bigoplus_x SH^G(x)$ and converging to $SH^+,G(W)$, which also implies the lemma. We do not need this fact and we omit its proof for the sake of brevity, for it is quite standard; see, e.g., [GGM15] where such a spectral sequence is constructed for the contact homology and [GG17, Sect. 2.2.2] for a discussion of this spectral sequence for the Floer homology.)

Next recall that an iteration $k$ of $x$ is called admissible when none of the Floquet multipliers of $x$, different from 1, is a root of unity of degree $k$. For instance, every $k$ is admissible when no Floquet multiplier is a root of unity or, as the opposite extreme, when $x$ is totally degenerate, i.e., all Floquet multipliers are equal to 1. Furthermore, every sufficiently large prime $k$ (depending on $x$) is admissible.

For our purposes it is convenient to adopt the following definition. Namely, $x$ is a symplectically degenerate maximum (SDM) if there exists a sequence of admissible iterations $k_i \to \infty$ such that

$$SH^G_{q,m}(x^{k_i}) \neq 0 \text{ for } q = \hat{\mu}(x^{k_i}) + m = k_i\hat{\mu}(x) + m. \quad (6.4)$$

This condition is obviously independent of the choice of a trivialization along $x$. It follows from (6.2) that then $x^{k_i}$, and hence $x$, must be totally degenerate. Thus the definition can be rephrased as that $x$ is totally degenerate and (6.4) holds for some sequence $k_i \to \infty$.

Remark 6.4. Continuing the discussion in Remark 6.2 note there are several, hypothetically equivalent, ways to define a closed SDM Reeb orbit. The definition above is a contact analog of the original definition of a Hamiltonian SDM from [Gi10] and it lends itself conveniently to the proof of the non-degenerate case of the contact Conley conjecture. Alternatively, a contact SDM was defined in [GH2M] as a closed isolated Reeb orbit $x$ with $HC_{\hat{\mu}(x)+m}(x) \neq 0$. By (6.3), its symplectic homology
analogue would be that $SH^G_{\hat{\mu}(x)+m}(x) \neq 0$. For a simple orbit this is equivalent to that the fixed point of $\varphi$ is an SDM. Furthermore, one can show that the $\mathbb{Z}_k$-action on the homology is trivial for totally degenerate orbits and thus $HF(\varphi)_{\mathbb{Z}_k} = HF(\varphi)$. Hence, the equivalence of the two definitions would then follow from the identification of the first and the last term in (6.3) combined with the persistence of the local Floer homology; [GG10].

6.2. Conley conjecture. Now we are in the position to give a symplectic homology proof of the non-degenerate case of the contact Conley conjecture, a contact analogue of the main result from [SZ].

**Theorem 6.5** (Contact Conley Conjecture). Let $M \to B$ be a prequantization bundle and let $\alpha$ be a contact form on $M$ supporting the standard (co-oriented) contact structure $\xi$ on $M$. Assume that

(i) $B$ is symplectically aspherical and
(ii) $\pi_1(B)$ is torsion free.

Then the Reeb flow of $\alpha$ has infinitely many simple closed Reeb orbits with contractible projections to $B$, provided that none of the orbits in the free homotopy class $f$ of the fiber is an SDM. Assume in addition that the Reeb flow of $\alpha$ has finitely many closed Reeb orbits in the class $f$. Then for every sufficiently large prime $k$ the Reeb flow of $\alpha$ has a simple closed orbit in the class $f_k$.

Before proving this theorem let us compare it with other results on the contact Conley conjecture. Theorem 6.5 was proved in [GGM15, GGM17] without the assumption that none of the orbits is an SDM. However, that argument relied on the machinery of linearized contact homology which is yet to be made completely rigorous. (See, however, [Ne] where some of the foundational issues have been resolved in dimension three.) The key difficulty in translating the proof from those two papers into the symplectic homology framework in the non-degenerate case was purely conceptual: the grading by the free homotopy classes $f$ of the fiber is an SDM. Assume in addition that the Reeb flow of $\alpha$ has finitely many closed Reeb orbits in the class $f$. Then for every sufficiently large prime $k$ the Reeb flow of $\alpha$ has a simple closed orbit in the class $f_k$.

There are also some minor discrepancies between the conditions of Theorem 6.5 and its counterpart in [GGM15, GGM17]. Namely, there the base $B$ is assumed to be aspherical, i.e., $\pi_r(B) = 0$ for $r \geq 2$, but as is pointed out in [GGM15, Sect. 2.2], this condition is only used to make sure that $\sigma$ is aspherical and $\pi_1(B)$ is torsion free. Then, the class $c_1(\xi)$ is required *ibid* to be atoroidal. This is a minor technical restriction imposed only for the sake of simplicity and it does not arise in the symplectic homology setting because the fiber is contractible in $W$.

**Proof of Theorem 6.5.** The argument closely follows the reasoning from [GGM15], which in turn is based on the proof in [SZ]. We need the following simple, purely algebraic fact, proved in [GGM15, Lemma 4.2], which only uses the conditions that $\sigma$ is aspherical and that $\pi_1(B)$ is torsion free.
Lemma 6.6. Under the conditions of the theorem, for every \( k \in \mathbb{N} \) the only solutions \( h \in \tilde{\pi}_1(P) \) and \( l \geq 0 \) of the equation \( h^l = f^k \) are \( h = f^r \), for some \( r \in \mathbb{N} \), and \( l = k/r \). (In particular, \( f \) is primitive.)

Next, without loss of generality we may assume that there are only finitely many closed Reeb orbits in the class \( f \), for otherwise there is nothing to prove. We denote these orbits by \( x_1, \ldots, x_r \) and set \( \Delta_j = \hat{\mu}(x_j) \), where we equipped \( T_xW \) with a trivialization coming from a capping of \( x_i \) in \( W \). Let \( k \) be a large prime. Then, unless there is a simple closed Reeb orbit in the class \( f^k \), every closed Reeb orbit in this class has the form \( x_j^k \) by Lemma 6.6.

We will show that in this case \( \text{SH}^G_{m+2k}(W, f^k) = 0 \) when \( k \) is large, which contradicts Proposition 5.2 and more specifically (5.2). By Lemma 6.3, it is enough to show that

\[
\text{SH}^G_{m+2k}(x_j^k) = 0. \tag{6.5}
\]

Pick a prime \( k \) so large that \( k|\Delta_j - 2| > 2m \) for all \( x_j \) with \( \Delta_j \neq 2 \). Then, since \( \hat{\mu}(x_j^k) = k\Delta_j \), we have

\[
\text{supp} \text{SH}^G(x_j^k) \subset [k\Delta_j - m, k\Delta_j + m]
\]

by (6.1), and hence \( m + 2k \) is not in the support. Thus (6.5) holds in this case. On the other hand, when \( \Delta_j = 2 \), (6.5) holds when \( k \) is sufficiently large, for otherwise (6.4) would be satisfied for some sequence of primes \( k_i \to \infty \) and \( x_j \) would be an SDM. This completes the proof of the theorem. \qed

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