Private and Secure Distributed Matrix Multiplication Schemes for Replicated or MDS-Coded Servers

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Abstract—In this paper, we study the problem of private and secure distributed matrix multiplication (PSDMM), where a user having a private matrix \( A \) and \( N \) non-colluding servers sharing a library of \( L \) \((L > 1)\) matrices \( B^{(0)}, B^{(1)}, \ldots, B^{(L-1)} \), for which the user wishes to compute \( AB^{(\theta)} \) for some \( \theta \in [0, L) \) without revealing any information of the matrix \( A \) to the servers, and keeping the index \( \theta \) private to the servers. Previous work is limited to the case that the shared library \( \text{i.e., the matrices} \) \( B^{(0)}, B^{(1)}, \ldots, B^{(L-1)} \) is stored across the servers in a replicated form and schemes are very scarce in the literature, there is still much room for improvement. In this paper, we propose two PSDMM schemes, where one is limited to the case that the shared library is stored across the servers in a replicated form but has a better performance than state-of-the-art schemes in that it can achieve a smaller recovery threshold and download cost. The other one focuses on the case that the shared library is stored across the servers in an MDS-coded form, which requires less storage in the servers. The second PSDMM code does not subsume the first one even if the underlying MDS code is degraded to a repetition code as they are totally two different schemes.

Index Terms—Distributed computation, privacy, secure, distributed matrix multiplication.

I. INTRODUCTION

MATRIX multiplication is one of the key operations in many science and engineering applications, such as machine learning and cloud computing. Carrying out the computation on distributed servers are desirable for improving efficiency and reducing the user’s computation load, as the user can divide the computation at hand into several sub-tasks to be carried out by the helper servers. As a downside, when scaling out computations across many distributed servers, the computation latency can be affected by orders of magnitude due to the presence of stragglers, see e.g., [2], [3]. Fortunately, recent works have shown that coding techniques can reduce the computation latency [4]–[9].

As the computations are scaling out across many distributed servers, besides stragglers, security is also a concern as the servers might be curious about the matrix contents. To this end, consider the problem where the user has two matrices \( A \) and \( B \) and wishes to compute their product with the assistance of \( N \) distributed servers, which are honest but curious in that any \( X \) of them may collude to deduce information about either \( A \) or \( B \) [10], [11]. This raises the problem of SDMM, which has recently received a lot of attention from an information-theoretic perspective [10]–[19].

In another variant of this problem, user privacy should also be taken into account. Consider the scenario where the user has a private matrix \( A \) and there are \( N \) non-colluding servers sharing a library of \( L \) matrices \( B^{(0)}, B^{(1)}, \ldots, B^{(L-1)} \), for which the user wishes to compute \( AB^{(\theta)} \) for some \( \theta \in [0, L) \) without revealing any information of the matrix \( A \) to the servers and keeping the index \( \theta \) of the desired matrix \( B^{(\theta)} \) hidden from the servers. This problem is termed PSDMM, which is also quite relevant to the problem of private information retrieval (PIR) [20]–[35].

Generally, four performance metrics are of particular interest for PSDMM schemes and more generally for any matrix multiplication scheme:

- The upload cost: the amount of data transmitted from the user to the servers to assign the sub-tasks;
- Server storage cost: the amount of data stored in each server;
- The download cost: the amount of data to be downloaded from the servers;
- The recovery threshold \( R_c \): the number of servers that need to complete their tasks before the user can recover the desired matrix product(s).

The goal is to design PSDMM schemes that can minimize one or several of the above metrics.

Up to now, the study of PSDMM is still very scarce in the literature. In [29], a PSDMM code based on PIR was proposed, which can provide a flexible tradeoff between the upload cost and download cost by adjusting the partitioning sizes of the matrices, but requires a high sub-packetization degree and cannot mitigate stragglers. In [36], PSDMM codes based on the polynomial codes in [5] were presented. In [16], Aliasgari et al. proposed a new PSDMM code based on the entangled polynomial codes in [9], which generalizes the one in [36] and allows for a flexible tradeoff between the upload cost and the download cost (or equivalently, recovery threshold). Very recently, Yu et al. [17] studied the problem of PSDMM based on Lagrange coded computing with bilinear complexity [37].

The detailed performance of the above codes together with the new codes proposed in this work will be illustrated in the following sections.

In this paper, we propose two PSDMM codes \( C_{\text{PSDMM}} \) and \( C_{\text{PSDMM}}' \) with the shared library being stored across the servers in a replicated form and in an MDS-coded form, respectively. The proposed codes have the following advantages:
The new code $C_{\text{PSDMM}}$ outperforms the ones in [16] and [36], in that it has a smaller recovery threshold as well as download cost under the same upload cost, and it provides a more flexible tradeoff between the upload and download costs than the code in [36]. The new code $C_{\text{PSDMM}}$ also has a smaller recovery threshold and download cost compared to the codes in [17] and [29], for some parameter regions. In addition, it can tolerate stragglers and does not require sub-packetization, hence being superior to the code in [29].

The new PSDMM code $C'_{\text{PSDMM}}$ with the shared library being stored across the servers in an MDS-coded form, can greatly reduce the storage overhead in the servers. To the best of our knowledge, this is the first time to present a new efficient PSDMM code from replicated servers, followed by detailed comparisons with previous work. Section IV presents a PSDMM code from replicated servers, with each server stores $L_{\text{sr}}$ elements from $\mathbf{F}$, and returns the result $Y_i$ to the user, where $i \in [0, N)$.

Query, communication and computation phase (MDS-coded servers): The user sends a query $q_i^{(\theta)}$ (which contains $A_i$) to server $i$, who then performs some calculation and returns the result $Y_i$ to the user, where $i \in [0, N)$.

Decoding phase: From the results returned by the fastest $R_c$ servers, the user can then decode the desired matrix multiplication $AB^{(\theta)}$.

We say that an encoding scheme for PSDMM is private if the index $\theta$ is kept secret from any individual server, i.e.,

$$I(\theta; q_i^{(\theta)}, \tilde{A}_i, B_i) = 0,$$

for any $i \in [0, N)$ and $\theta \in [0, L)$, where $B_i$ denotes the data stored in server $i$, e.g., $B_i$ stands for $B^{(0)}, B^{(1)}, \ldots, B^{(L-1)}$ for PSDMM schemes for replicated servers. The scheme is secure if no information is leaked about the matrix $A$ to any server, i.e.,

$$I(q_i^{(\theta)}, \tilde{A}_i; A) = 0,$$

for any $i \in [0, N)$, where $I(X;Y)$ denotes the mutual information between $X$ and $Y$.

The upload cost is defined as the number of elements in $\mathbf{F}$ transmitted from the user to the servers, i.e.,

$$\sum_{i=0}^{N-1} |\tilde{A}_i| + |q_i^{(\theta)}|$$

for replicated servers and

$$\sum_{i=0}^{N-1} |q_i^{(\theta)}|$$

for MDS-coded servers, while the download cost is defined as the number of elements in $\mathbf{F}$ that the user downloaded from the fastest $R_c$ servers $j_0, \ldots, j_{R_c-1}$, maximized over $\{j_0, \ldots, j_{R_c-1}\} \subseteq [0, N)$, i.e.,

$$\max_{\{j_0, \ldots, j_{R_c-1}\} \subseteq [0, N)} \sum_{i=j_0}^{j_{R_c-1}} |Y_i|,$$

where $|Y|$ denotes the size of the matrix $Y$ counted as $\mathbf{F}$ symbols.

B. Main results

This subsection includes the main results derived in this paper. The proofs are carried out in the later sections.

**Theorem 1.** There exists an explicit PSDMM code $C_{\text{PSDMM}}$ from replicated servers, with the upload cost $N \frac{tr}{mp}$, download cost $R_c \frac{tr}{mn}$, recovery threshold

$$R_c = pmn + pm + n,$$

and each server stores $L_{\text{sr}}$ elements from $\mathbf{F}$.

**Theorem 2.** There exists an explicit PSDMM code $C'_{\text{PSDMM}}$ from MDS-coded servers, with each server stores $L \frac{tr}{mp}$ elements from $\mathbf{F}$, the upload cost $LN \frac{tr}{mp}$, download cost $R_c \frac{tr}{mn}$, and recovery threshold

$$R_c = pmn + pm + n - 1$$

if the underlying finite field is sufficiently large.
III. PRIVATE AND SECURE DISTRIBUTED MATRIX MULTIPLICATION FROM REPLICATED SERVERS

In this section, we propose a new PSDMM code from replicated servers. The construction is similar to the one in [16], but the encoding phase of the new PSDMM scheme is more efficient. This leads to a smaller recovery threshold and download cost. Before presenting the general construction, we first give a motivating example.

A. A motivating example

Assume that the user possesses a matrix $A \in \mathbb{F}^{t \times s}$ and there are $L = 2$ matrices $B^{(0)}, B^{(1)} \in \mathbb{F}^{s \times r}$ stored across the $N$ servers in a replicated form, i.e., each server holds the matrices $B^{(0)}, B^{(1)}$, where $t, s, r$ are even. Suppose the user wishes to compute $AB^{(0)}$. Partition the matrices $A$ and $B^{(i)}$ into block matrices

$$A = \begin{bmatrix} A_{0,0} & A_{0,1} \\ A_{1,0} & A_{1,1} \end{bmatrix}, \quad B^{(i)} = \begin{bmatrix} B_{0,0}^{(i)} & B_{0,1}^{(i)} \\ B_{1,0}^{(i)} & B_{1,1}^{(i)} \end{bmatrix},$$

where $A_{k,j} \in \mathbb{F}^{t \times \tilde{t}}$ and $B_{j,k}^{(i)} \in \mathbb{F}^{\tilde{t} \times \tilde{t}}$. Then

$$AB^{(0)} = \begin{bmatrix} C_{0,0} & C_{0,1} \\ C_{1,0} & C_{1,1} \end{bmatrix}$$

where

$$C_{i,j}^{(0)} = A_{i,0}B_{0,j}^{(0)} + A_{i,1}B_{1,j}^{(0)}, \quad i, j \in \{0, 1\}.$$ (1)

Let $Z$ be a random matrix over $\mathbb{F}^{\tilde{t} \times \tilde{t}}$. Then, define a polynomial

$$f(x) = A_{0,0}x^{\alpha_{0,0}} + A_{0,1}x^{\alpha_{0,1}} + A_{1,0}x^{\alpha_{1,0}} + A_{1,1}x^{\alpha_{1,1}} + Zx^{\gamma}$$ (2)

where $\alpha_{k,j}, \gamma$ are some integers to be specified later.

Let $a_{1}$ and $a_{0,0}, \ldots, a_{0,N-1}$ be $N + 1$ pairwise distinct elements from $\mathbb{F}$. For every $i \in \{0, N\}$, the user first evaluates $f(x)$ at $a_{0,i}$, then sends $f(a_{0,i})$ and the query

$$q_{i}^{(0)} = (a_{0,i}, a_{1})$$ (3)
to server $i$. Upon receiving the query $q_{i}^{(0)}$, server $i$ encodes the library into a matrix as $g(a_{0,i})$ where $g(x)$ is defined as

$$g(x) = \sum_{j=0}^{1} \sum_{k=0}^{s} B_{j,k}^{(0)}x^{\beta_{j,k}} + \sum_{j=0}^{1} \sum_{k=0}^{s} B_{j,k}^{(0)}a_{1}^{k}.$$ (4)

After encoding the library, server $i$ computes $f(a_{0,i})g(a_{0,i})$ and then returns the result to the user.

Let $h(x) = f(x)g(x)$, the expression is shown in (5) in the next page, then the results returned from the servers are exactly the evaluations of $h(x)$ at some evaluation points.

The user wishes to obtain the data in (1) (related to the useful terms in $h(x)$) from any $R_c$ out of the $N$ evaluations of $h(x)$, which can be fulfilled if the following conditions hold.

(i) For $k \in \{0, 2\}$ and $k' \in \{0, 2\}$,

$$\alpha_{k,0} + \beta_{0,k'} = \alpha_{k,1} + \beta_{1,k'}.$$ (ii) For $U = \{\alpha_{k,0} + \beta_{0,k'} | 0 \leq k < 2, 0 \leq k' < 2\}$ and $I = \{\alpha_{k,j} + \beta_{j,k'} | 0 \leq k, k' < 2, 0 \leq j \neq j' < 2\}$

$$\cup \{\gamma + \beta_{j,k'} | 0 \leq j, k' < 2\} \cup \{\alpha_{k,j} | 0 \leq k, j < 2\} \cup \{\gamma\},$$

$|U| = 4$ and $U \cap I = \emptyset$.

(iii) $R_c = \deg(h(x)) + 1$.

The task can be finished because

- (i) guarantees that each $C_{k,k'}$ appears in $h(x)$,
- (ii) guarantees that $C_{k,k'}^{(0)}$, $k, k' = 0, 1$ are coefficients of different terms of $h(x)$ with different degrees, which are different from the degrees of the interference terms of $h(x)$, i.e., each $C_{k,k'}^{(i)}$ is the coefficient of a unique term in $h(x)$,
- (iii) guarantees the decodability from Lagrange interpolation [38].

By (2) and (ii), we can get $\gamma \neq \alpha_{k,j}$ for $k, j = 0, 1$; thus one easily obtains $I(\{f(a_{i}); A\} = 0$ for any $i \in \{0, N\}$ and $a_{i} \in \mathbb{F}$ as $f(a_{i})$ is masked by the random matrix $Z$. And, $I(q_{i}^{(0)}; A) = 0$ as $q_{i}^{(0)}$ and $A$ are independent, then security is fulfilled. While the privacy condition is met by the definition of the query vector in (3) for the desired index, the detailed proof is similar to [36].

We provide a concrete exponent assignment for this example in Table I. From the given assignment, we see that $\deg(h(x)) = 13$ and thus $R_c = 14$.

| $+$ | $\beta_{0,0} = 4$ | $\beta_{0,1} = 9$ | $\beta_{1,0} = 3$ | $\beta_{1,1} = 8$ |
|-----|-------------------|-------------------|-------------------|-------------------|
| $a_{0,0} = 1$ | $5$ | $\ominus$ | $10$ | $\vartriangle$ |
| $a_{0,1} = 2$ | $7$ | $\heartsuit$ | $12$ | $\heartsuit$ |
| $a_{1,0} = 3$ | $\spadesuit$ | $10$ | $\ominus$ | $9$ |
| $a_{1,1} = 4$ | $8$ | $\ominus$ | $13$ | $\spadesuit$ |

| $\gamma = 0$ | $4$ | $\ominus$ | $9$ | $\ominus$ |

| $\ominus$ | $8$ | $\heartsuit$ | $0$ | $\ominus$ |

TABLE I

AN ASSIGNMENT FOR EXPOENTS OF $f(x)$ IN \(2\) AND $g(x)$ IN \(3\), WHERE $\blacklozenge, \oslash, \blackspadesuit$ ARE USED TO HIGHLIGHT THE EXPONENTS OF THE USEFUL TERMS OF $h(x)$ IN (5), I.E., ALL THE ELEMENTS IN $U$.

B. General construction

In the following, we propose a general construction for PSDMM from replicated servers. Partition the matrices $A$ and $B^{(i)} (i \in \{0, L\})$ into block matrices as

$$A = [A_{k,j}]_{k \in \{0,m\}, j \in \{0,p\}}, \quad B^{(i)} = [B_{j,k'}^{(i)}]_{j \in \{0,p\}, k' \in \{0,n\}},$$

where $A_{k,j} \in \mathbb{F}^{\tilde{t} \times \tilde{t}}$ and $B_{j,k'}^{(i)} \in \mathbb{F}^{\tilde{t} \times \tilde{t}}$.

Then

$$AB^{(i)} = [C_{k,k'}^{(i)}]_{k \in \{0,m\}, k' \in \{0,n\}},$$

where

$$C_{k,k'}^{(i)} = \sum_{j=0}^{p-1} A_{k,j}B_{j,k'}^{(i)}, \quad k \in \{0, m\}, \quad k' \in \{0, n\}.$$ (6)

Let $Z$ be a random matrix over $\mathbb{F}^{\tilde{t} \times \tilde{t}}$, and define a polynomial

$$f(x) = \sum_{k=0}^{m-1} \sum_{j=0}^{p-1} A_{k,j}x^{\alpha_{k,j}} + Zx^{\gamma},$$ (6)
Let $h(x) = f(x)g(x)$, i.e.,
\[
h(x) = \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} A_{k,j} B_{j,k} x^{\alpha_{k,j}} + \sum_{j=0}^{n-1} B_{j,k} x^{\gamma_{j,k} + \beta_{j,k}} + \sum_{j=0}^{n-1} x^{\alpha_{k,j}} x^{\gamma_{j,k} + \beta_{j,k}}.
\]
then the results returned from the servers are exactly the evaluations of $h(x)$ at some evaluation points, thus we can derive the following result.

**Proposition 1.** For PSDMM schemes for replicated servers, the multiplication of $A$ and $B^{(\theta)}$ can be securely computed with the upload cost $N \frac{n \times \theta}{m \times \theta}$, download cost $R_c \frac{(\gamma_{j,k} + \beta_{j,k})}{m \times \theta}$ with $R_c$ being the recovery threshold, if the following conditions hold. 

(i) For $k \in [0, m)$ and $k' \in [0, n)$, 
\[
\alpha_{k,0} + \beta_{0,k'} = \cdots = \alpha_{k,p-1} + \beta_{p-1,k'}.
\]

(ii) The set $U = \{\alpha_{k,0} + \beta_{0,k'} | 0 \leq k < m, 0 \leq k' < n\}$ containing all the exponents of the useful terms of $h(x)$ is disjoint with the set 
\[
I = \{\alpha_{k,j} + \beta_{j,k'} | 0 \leq k < m, 0 \leq j' \neq j < p, 0 \leq k' < n\} \cup \{\gamma + \beta_{j,k'} | 0 \leq j' < p, 0 \leq k' < n\} \cup \{\gamma\}
\]
containing all the exponents of the interference terms of $h(x)$, and $|U| = mn$.

(iii) $R_c = \deg(h(x)) + 1 = \max(U \cup I) + 1$.

**Proof.** By (ii), one can deduce that $\alpha_{0,0}, \ldots, \alpha_{m-1,p-1}, \gamma$ are pairwise distinct, then $I(f(\alpha_{0,i})A) = 0$ for any $i \in [0, N)$ and $a_i \in F$, together with the fact that $q_i^{(\theta)}$ and $A$ are independent, we conclude that security is guaranteed. The privacy condition follows from three conditions: 1) the definition of the query in Eq. (7) for the desired index $\theta \in [0, L]$, 2) the matrices $B^{(\theta)}, B^{(\theta)}$, \ldots, $B^{(\theta)}$ are independent of $\theta$, and 3) the matrix $f(\alpha_{0,i})$ is random to each server as it is masked by the random matrix $Z$ according to (5). The detailed and rigorous proof is similar to [30], thus we omit it here. By (i), each $C_k,k'$ appears in $h(x)$, while (ii) guarantees that each
$C_{k,k'}$ is the coefficient of a unique term in $h(x)$, and finally (iii) guarantees the decodability.

It is obvious that the upload cost is $\sum_{i=0}^{N-1} |f(a_i)| = N \frac{t_s}{m_p}$ and the download cost is $R \frac{t_r}{m_n}$. This finishes the proof.

By Proposition 2, there may exist other assignment methods of the exponents of $f(x)$ and $g(x)$ (or $g_i(x)$).

Proposition 2. Conditions (i)-(iii) of Proposition 7 can be satisfied if $R = pmn + pm + n$ and

$\alpha_{k,j} = j + kp + 1, \beta_{j,k'} = pm - j + k'(pm + 1)$

for $k \in [0,m), j \in [0,p)$, and $k' \in [0,n)$, and $\gamma = 0$.

Before the proof of the above proposition, below we provide some intuitions and insights about the assignment method:

- We wish the degrees of the terms of $f(x)$ and $g(x)$ (or $g_i(x)$) are as small as possible to possibly lower the degree of $h(x)$. W.O.L.G., for $f(x)$, we first set the degree of the term related to the random matrix $Z$ to be 0, i.e., $\alpha = 0$, and then set the degrees of the rest $pm$ terms of $f(x)$ from 1 to $pm$ for simplicity, i.e., $\alpha_{k,j} = j + kp + 1$.

- Given $\alpha_{k,j} = j + kp + 1$, Proposition 1(i) implies that $\beta_{i,k'} = \beta_{0,k'} - i$ for $i \in [1,p)$, i.e., $\beta_{0,k'}$ is the largest integer among $\beta_{0,k'}, \beta_{1,k'}, \ldots, \beta_{p-1,k'}$. Thus it is suffice to fix the values of $\beta_{0,0}, \beta_{0,1}, \ldots, \beta_{0,n-1}$. Note that Proposition 1(ii) requires $U \cap I = \emptyset$, which implies that $\alpha_{0,0} + \beta_{0,0} \not\in \{\alpha_{k,j} | 0 \leq k < m, 0 \leq j < p\}$, it can be fulfilled if

$\alpha_{0,0} + \beta_{0,0} > \max\{\alpha_{k,j} | 0 \leq k < m, 0 \leq j < p\} = pm,

i.e., \beta_{0,0} \geq pm$. W.O.L.G., we set $\beta_{0,0} = pm$.

- $\beta_{0,k'}, k' \in [1,n)$ can be determined similarly according to Proposition 1(i) and (ii).

Proof. From Proposition 2 we have

$\alpha_{k,j} + \beta_{j,k'} \equiv (k' + 1)(pm + 1) + kp \tag{9}$

for $j \in [0,p), k \in [0,m)$ and $k' \in [0,n)$, thus Proposition 1(i) holds, and then we easily have

$|U| = |\{(k'+1)(pm+1)+kp | 0 \leq k < m, 0 \leq k' < n\}| = mn.

Now let us prove $U \cap I = \emptyset$.

$1)$ When $j, j' \in [0,p)$ with $j \neq j'$, i.e., $0 < |j - j'| \leq p - 1$, it is easy to see that the element

$\alpha_{k,j} + \beta_{j',k'} = (k' + 1)(pm + 1) + kp + (j - j')$

in $I$ is not in the set $U$ for all $k \in [0,m), k' \in [0,n)$, and $j, j' \in [0,p)$, with $j \neq j'$;

$2)$ By Proposition 2 it is also easy to see

$\gamma + \beta_{j,k'} = \beta_{j,k'} = (k' + 1)(pm + 1) - j - 1 \not\in U;

3)$ Since $\min U = pm + 1$, $\alpha_{k,j} \leq pm$, and $\gamma = 0$, thus $\not\in U$ and $\alpha_{k,j} \not\in U$ for all $k \in [0,m)$ and $j \in [0,p)$.

Combining 1)-3), we thus proved $U \cap I = \emptyset$, i.e., Proposition 1(ii) holds. It is easy to check that

$max(U \cap I) = \alpha_{m-1,p-1} + \beta_{0,n-1} = pmn + pm + n - 1$

thus Proposition 1(iii) holds.

By Propositions 1 and 2 we can derive the result in Theorem 1.

Remark 1. There may exist other assignment methods of the exponents of $f(x)$ and $g(x)$ that are better than those in Proposition 2. Finding the optimal assignment of the exponents of $f(x)$ and $g(x)$ is an interesting problem.

C. Comparison

Next, we provide comparisons of some key parameters of the proposed PSDMM code $\text{C}_{\text{PSDMM}}$ and some previous codes. Table II illustrates the first comparison. To this end, note that Chang–Tandon code [29] cannot mitigate stragglers and the number of servers can be an arbitrary integer larger than $m$ but should be prefixed. Table III gives a (fairer) comparison of the normalized parameters of the proposed PSDMM code and Chang–Tandon code, where the normalized upload cost is the ratio of upload cost to $|A|$ and the normalized download cost is the ratio of the download cost to $|AB(1)|$. Figures 1 and 2 visualize the comparison in Table III by assuming $N = R_c$ in the new PSDMM code so that it is comparable with the Chang–Tandon code [29].

Remark 2. Although it is well known that $R(p, m, n) < pmn$, the value of $R(p, m, n)$ is not yet known in general, even for some small parameters. Finding out the value of $R(p, m, n)$ is challenge in general [9]. In the case that the value of $R(p, m, n)$ is not yet known, the constructions in [7] are then built on known upper bound constructions, which are scattered.
TABLE III
A COMPARISON OF THE NORMALIZED UPLOAD COST AND NORMALIZED DOWNLOAD COST BETWEEN $C_{PSDMM}$ AND CHANG–TANDON CODE.

|                  | Normalized upload cost | Normalized download cost | Recovery threshold $R_c$ | References |
|------------------|------------------------|--------------------------|--------------------------|------------|
| Chang–Tandon code | $R_c/m$                | $m + 1 + \frac{m+1}{m} + \cdots + \frac{(m+1)}{L-1}$ | $\geq m + 1$ | [29]       |
| The new code $C_{PSDMM}$ | $N/mp$           | $R_c/mn$                 | $pmn + pm + n$ | Thm. [1]   |

As the value of $R(p,m,n)$ in Table II is unknown in general by Remark 2, we show in Table IV that the code $C_{SDMM}$ outperforms the SDMM code in [17] in terms of the recovery threshold for some small parameters.

TABLE IV
A COMPARISON OF THE RECOVERY THRESHOLDS BETWEEN THE PSDMM CODE BASED ON BILINEAR COMPLEXITY IN [17] AND THE NEW CODE $C_{PSDMM}$ FOR SOME SMALL $p,m,n$, WHERE $R^* (p,m,n)$ Denotes the Best Upper Bound of $R(p,m,n)$ and the Values in the Last Column Are Rounded to 4 Decimal Places

| $p$ | $m$ | $n$ | $R^* (p,m,n)$ | $R_c$ [17] | $R_c^{PSDMM}$ | $R_c^{PSDMM} - R_c^{PSDMM}$ |
|-----|-----|-----|---------------|------------|--------------|---------------------------|
| 2   | 2   | 2   | 14            | 14         | 6.67%        |                           |
| 2   | 3   | 2   | 15            | 15         | 12.93%       |                           |
| 3   | 4   | 2   | 16            | 16         | 17.02%       |                           |
| 4   | 4   | 3   | 17            | 17         | 15.15%       |                           |
| 5   | 5   | 4   | 18            | 18         | 22.11%       |                           |
| 6   | 6   | 4   | 19            | 19         | 19.63%       |                           |
| 7   | 7   | 5   | 20            | 20         | 20.36%       |                           |
| 8   | 8   | 5   | 21            | 21         | 14.99%       |                           |
| 9   | 9   | 5   | 22            | 22         | 21.33%       |                           |

From Tables II and III, the numerical comparison in Table IV and Figures 1 and 2, we can see that the new PSDMM code $C_{PSDMM}$ has some advantages over the previous constructions:

- The new PSDMM code $C_{PSDMM}$ is more general than Kim–Lee code in [36]. Namely, it can provide a more flexible tradeoff between the upload cost and the download cost. In particular, when $p = 1$, the recovery threshold of the new code is one smaller than that of Kim–Lee code in [36] under the same upload cost.
- The recovery threshold and the download cost of the new PSDMM code $C_{PSDMM}$ are smaller than those of Alia's code [17] and the PSGPD code [3] in [16] under the same upload cost and the SDMM code in [17] for some small parameters, more specially, when $R(p,m,n) \geq (pmn + pm + n - 1)/2$, see Table IV for some small parameter regions.
- The download cost of Chang–Tandon code in [29] depends on $L$ and is a mononotonically increasing function w.r.t. $L$ (see the trends in Figures 1 and 2). Besides, Chang–Tandon PSDMM code cannot mitigate stragglers and requires a large sub-packetization degree. In contrast, the new PSDMM code $C_{PSDMM}$ does not have such defects. Furthermore, Figures 1 and 2 show that the new PSDMM code has a smaller normalized download cost for some given normalized upload cost compared with Chang–Tandon’s PSDMM code.

Remark 3. Although there are non-Pareto-optimal points in Figures 1 and 2 for the new PSDMM code $C_{PSDMM}$, they are still of interest as the chosen computational complexity (submatrix dimension for the private matrix $A$) dictates the normalized upload cost. A larger submatrix dimension for the private matrix $A$ results in a smaller normalized upload cost.

In [14], there was a small mistake in the recovery threshold of the PSGPD code. A quick correction could be adding $st - s + 2$ to the exponents of each term in the polynomial $F_{P^k_i}(z)$ in Eq. (43). In Table II, we give the correct value of the recovery threshold.
so we may not be able to choose a small normalized upload cost in resource-constrained situations.

IV. PRIVATE AND SECURE DISTRIBUTED MATRIX MULTIPLICATION FROM MDS-CODED SERVERS

In this section, we propose a new PSDMM code $C_{PSDMM}$ from MDS-coded servers, i.e., the matrices $B^{(0)}, B^{(1)}, \ldots, B^{(L-1)}$ are stored across the servers in an MDS-coded form by an $N$ servers, where $t,s,r$ are even. Partition the matrices $A$ and $B^{(i)}$ into block matrices the same as that in Section III-A.

Let $a_{0}, \ldots, a_{N-1}$ be $N$ pairwise distinct elements from $F$, and suppose server $i$ stores two coded pieces of $B^{(0)}, B^{(1)}$ as $g_{0}(a_{i}), g_{1}(a_{i})$ for $i \in [0, N)$, where

$$g_{i}(x) = B^{(i)}_{0,0}x^{\beta_{0,0}} + B^{(i)}_{0,1}x^{\beta_{1,0}} + B^{(i)}_{1,0}x^{\beta_{0,1}} + B^{(i)}_{1,1}x^{\beta_{1,1}},$$

for $i = 0, 1$, with $\beta_{0,0}, \beta_{1,0}, \beta_{0,1}, \beta_{1,1}$ being some integers such that the matrices $B^{(0)}$ and $B^{(1)}$ can be decoded from any 4 out of the $N$ servers, that is

$$\det \begin{bmatrix}
\alpha_{0,0} & \alpha_{0,1} & \alpha_{0,2} & \ldots & \alpha_{0,N-1} \\
\alpha_{1,0} & \alpha_{1,1} & \alpha_{1,2} & \ldots & \alpha_{1,N-1} \\
\alpha_{2,0} & \alpha_{2,1} & \alpha_{2,2} & \ldots & \alpha_{2,N-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{N-1,0} & \alpha_{N-1,1} & \alpha_{N-1,2} & \ldots & \alpha_{N-1,N-1}
\end{bmatrix} \neq 0$$

for any $\{i_{0}, i_{1}, i_{2}, i_{3}\} \subset [0, N)$ and $|\{i_{0}, i_{1}, i_{2}, i_{3}\}| = 4$.

Suppose that the user wishes to obtain $AB^{(0)}$. Let $Z_{0}$ be a random matrix over $F^{2 \times 2}$. Then, define a polynomial

$$f(x) = A_{0,0}x^{\alpha_{0,0}} + A_{0,1}x^{\alpha_{0,1}} + A_{1,0}x^{\alpha_{1,0}} + A_{1,1}x^{\alpha_{1,1}} + Z_{0}x^{\gamma},$$

where $\alpha_{k,j}, \gamma$ are some integers to be specified later.

Let $S$ be a random matrix over $F^{2 \times 2}$. For every $i \in [0, N)$, the user first evaluates $f(x)$ at $a_{i}, i \in [0, N)$, and then sends the query

$$q_{i}^{(0)} = (f(a_{i}), S)$$

to server $i$. Upon receiving the query $q_{i}^{(0)}$, server $i$ computes $f(a_{i})g_{0}(a_{i}) + Sg_{1}(a_{i})$ and then returns the result to the user.

Let $h(x) = f(x)g_{0}(x) + Sg_{1}(x)$, the expression is shown in (14) in the next page, then the result returned from server $i$ is exactly $h(a_{i})$, where $i \in [0, N)$.

The user wishes to obtain the data in $h(x)$ (related to the useful terms in $h(x)$) from any $R_{c}$ out of the $N$ evaluations of $h(x)$, which can be fulfilled if the following conditions hold.

(i) For $k \in [0, 2)$ and $k' \in [0, 2)$,

$$\alpha_{k,0} + \beta_{0,k'} = \alpha_{k,1} + \beta_{1,k'}.$$

(ii) For $U = \{\alpha_{k,0} + \beta_{0,k'} | 0 \leq k < 2, 0 \leq k' < 2 \}$ and

$$I = \{\alpha_{k,j} + \beta_{j',k'} | 0 \leq k < k', 0 \leq j < j' < 2 \} \cup \{\gamma + \beta_{j',k'} | 0 \leq j, k' < 2 \} \cup \{\beta_{j,k} | 0 \leq j, k < 2 \},$$

$|U| = 4$ and $U \cap I = \emptyset$.

(iii) $R_{c} = \deg(h(x)) + 1$.

The task can be finished because of the similar reason as in Section III-A. By (ii), we can get $\gamma \neq \alpha_{k,j}$ for $k, j = 0, 1$, thus one easily obtains $I(S; A) = 0$ as $S$ is random and $I(f(a_{i}); A) = 0$ for any $i \in [0, N)$ and $a_{i} \in F$ according to [12], then security is fulfilled. While the privacy condition is met by the definition of the query vector in [13] for the desired index, the detailed proof is similar to [36].

Fig. 2. Comparison of the normalized upload cost and normalized download cost between the new PSDMM code $C_{PSDMM}$ and the Chang–Tandon code under the parameters $N = R_{c} = 11$ (resp. $N = R_{c} = 17$).
$$h(x) = (A_{0,0}B_{0,0}^{(0)}x^{\beta_{0,0}+\beta_{0,0}} + A_{0,1}B_{0,1}^{(0)}x^{\beta_{0,1}+\beta_{1,0}}) + (A_{0,0}B_{0,0}^{(0)}x^{\beta_{0,0}+\beta_{0,1}} + A_{0,1}B_{1,1}^{(0)}x^{\alpha_{0,1}+\beta_{1,1}})$$

useful terms

$$+ (A_{1,0}B_{0,0}^{(0)}x^{\alpha_{0,1}+\beta_{0,0}} + A_{1,1}B_{1,0}^{(0)}x^{\alpha_{1,1}+\beta_{1,0}}) + (A_{1,0}B_{0,1}^{(0)}x^{\alpha_{0,1}+\beta_{0,1}} + A_{1,1}B_{1,1}^{(0)}x^{\alpha_{1,1}+\beta_{1,1}}) + \sum_{j'=0}^{1} \sum_{k'=0}^{1} Z_{0}B_{j',k'}^{(0)}x^{\alpha_{j',k'}+\beta_{j',k'}}$$

useful terms

$$+ \sum_{k=0}^{1} \sum_{k'=0}^{1} (A_{k,0}B_{1,k}^{(0)}x^{\alpha_{k,0}+\beta_{1,k'}} + A_{k,1}B_{1,k}^{(0)}x^{\alpha_{k,1}+\beta_{1,k'}} + S \left( B_{0,0}^{(1)}x^{\beta_{0,0}} + B_{1,0}^{(1)}x^{\beta_{0,1}} + B_{0,1}^{(1)}x^{\beta_{1,0}} + B_{1,1}^{(1)}x^{\beta_{1,1}} \right) , \right)$$

(14) interference terms

We provide a concrete exponent assignment for this example in Table V. From the given assignment, we see that \(\deg(h(x)) = 11\) and thus \(R_{c} = 12\). The upload cost is \(N\frac{L}{2}\) and the download cost is \(R_{c}\frac{L}{2}\), while each server only needs to store \(\frac{L}{2}\) elements from \(F\).

| \(\alpha_{0,0}\) | \(\beta_{0,0}\) | \(\beta_{0,1}\) | \(\beta_{1,0}\) | \(\beta_{1,1}\) |
|---|---|---|---|---|
| 4 | 0 | 1 | 2 | 3 |
| 2 | 0 | 2 | 3 | 4 |
| 8 | 0 | 6 | 7 | 9 |
| 6 | 0 | 1 | 2 | 3 |

B. General construction

In the following, we propose a general construction for PSDMM from MDS-coded servers according to an \([N, pn]\) MDS code. Partition the matrices \(A\) and \(B^{(i)}\) (\(i \in [0, L]\)) into block matrices the same as in Section II-B.

Let \(a_{0}, ..., a_{N-1}\) be \(N\) pairwise distinct elements from \(F\), then server \(i\) stores the \(L\) matrices

\[ g_{0}(a_{i}), g_{1}(a_{i}), ..., g_{L-1}(a_{i}) \]

for \(i \in [0, N]\), where

\[ g_{i}(x) = \sum_{j=0}^{p-1} \sum_{k=0}^{n-1} B_{j,k}^{(i)}x^{\beta_{j,k}} \]

with

\[ \det \begin{bmatrix} a_{0,0}^{(i)} & a_{0,1}^{(i)} & \cdots & a_{0,p-1}^{(i)} \\ a_{1,0}^{(i)} & a_{1,1}^{(i)} & \cdots & a_{1,p-1}^{(i)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{p-1,0}^{(i)} & a_{p-1,1}^{(i)} & \cdots & a_{p-1,p-1}^{(i)} \end{bmatrix} \neq 0 \]

for any \(\{i_{0,1}, ..., i_{p-1,1}\} \subseteq [0, N]\) and \(|\{i_{0,1}, ..., i_{p-1,1}\}| = pn\).

Suppose that the user wishes to compute \(AB^{(i)}\) for some \(\theta \in [0, L]\). Let \(Z_{0}\) be a random matrix over \(F^{\frac{L}{2} \times \frac{L}{2}}\), and define a polynomial

\[ f(x) = \sum_{k=0}^{m-1} \sum_{j=0}^{p-1} A_{k,j}x^{\alpha_{k,j}} + Z_{0}x^{\gamma}, \]

where \(\alpha_{k,j}, \gamma\) are some integers to be specified later. Additionally, let \(S_{0}, S_{1}, ..., S_{L-2}\) be \(L-1\) random matrices over \(F^{\frac{L}{2} \times \frac{L}{2}}\).

For every \(i \in [0, N]\), the user first evaluates \(f(x)\) at \(a_{i}\) and then sends the query

\[ q_{i}^{(\theta)} = (S_{0}, \ldots, S_{\theta-1}, f(a_{i}), S_{\theta}, \ldots, S_{L-2}) \]

(16)

upon receiving the query, server \(i\) computes

\[ \sum_{j=0}^{L-1} q_{i}^{(\theta)}(j)g_{j}(a_{i}) = f(a_{i})g_{0}(a_{i}) + \sum_{j=0, j \neq \theta}^{L-1} S_{j}g_{j}(a_{i}) \]

and then sends the result back to the user.

Let \(h(x) = f(x)g_{\theta}(x) + \sum_{j=0, j \neq \theta}^{L-1} S_{j}g_{j}(x)\), i.e.,

\[ h(x) = \sum_{k=0}^{m-1} \sum_{j=0}^{p-1} \sum_{k=0}^{n-1} A_{k,j}B_{j,k}^{(\theta)}x^{\alpha_{k,j}+\beta_{j,k'}} + \sum_{j=0}^{\theta} \sum_{k=0}^{n-1} Z_{k}B_{j,k}^{(\theta)}x^{\gamma+\beta_{j,k'}} \]

then the result returned from server \(i\) is exactly \(h(a_{i})\), thus we can derive the following result.

**Proposition 3.** For PSDMM schemes for MDS-coded servers, the multiplication of \(A\) and \(B^{(i)}\) can be securely computed with the upload cost \(LN \frac{\ell}{m_{2}}\), download cost \(R_{c} \frac{\ell}{m_{2}}\) with \(R_{c}\) being the recovery threshold, and each server stores \(L \frac{m_{1}}{pn}\) elements from \(F\) if the following conditions hold.

(i) For \(k \in [0, m]\) and \(k' \in [0, n]\),

\[ \alpha_{k,0} + \beta_{0,k'} = \cdots = \alpha_{k,p-1} + \beta_{p-1,k'}. \]

(ii) \(U = \{\alpha_{k,0} + \beta_{0,k'} | 0 \leq k \leq m, 0 \leq k' \leq n \} \) is disjoint with

\[ I = \{\alpha_{k,j} + \beta_{j,k'} | 0 \leq k \leq m, 0 \leq j < \theta \leq p, 0 \leq k' \leq n \} \cup \{\gamma + \beta_{j,k'} | 0 \leq j < \theta < p, 0 \leq k' \leq n \} \cup \{\beta_{j,k} | 0 \leq j < p, 0 \leq k < n \} \]

and \(|U| = mn\).

(iii) \(R_{c} = \deg(h(x)) + 1 = \max(U \cup I) + 1\).
Proof. By (ii), one can deduce that $\alpha_{0,0}, \ldots, \alpha_{m-1, p-1}, \gamma$ are pairwise distinct, then
\[ I(\{a_i\}; A) = 0 \]
for any $i \in [0, N)$ and $a_i \in F$ as $f(a_i)$ is masked by the random matrix $Z_p$. Since $S_0, \ldots, S_{L-2}$ are random matrices, we further have
\[ I(f(a_i), S_0, \ldots, S_{L-2}; A) = 0 \]
i.e., security is guaranteed. The privacy condition follows from the definition of the query in Eq. (16) for the desired index $\theta \in [0, L)$, which is random to each server. The detailed proof is similar to [56], thus we omit it here. By (i), each $C_{k,k'}$ appears in $h(x)$, while (ii) guarantees that each $C_{k,k'}$ is the coefficient of a unique term in $h(x)$, and finally (iii) guarantees the decodability.

It is obvious that the upload cost is $\sum_{i=0}^{N-1} |q_i(\theta)| = NL \frac{\alpha}{mp}$ and the download cost is $R_c \frac{\alpha}{mn}$. This finishes the proof.

Remark 4. In the above proposition, we can also broadcast the query vector component by component to all the servers at once, except for the coordinate $f(a_i)$, which has to be individually sent to each server. Depending on the relation of the size of $L$ versus $N$, this may imply significant saving in the upload cost. In the case of broadcasting, the upload cost would be $(N + L - 1) \frac{\alpha}{mp}$.

In the following, we provide an assignment method for the exponents of $f(x)$ and $g(x)$.

**Proposition 4.** Conditions (i)–(iii) of Proposition 3 can be satisfied if $R_c = pmn + pn - 1$,
\[ \alpha_{k,j} = (k + 1)pn - jn, \beta_{j,k'} = jn + k' \]
for $k \in [0, m), j \in [0, p), k' \in [0, n)$, and $\gamma = 0$.

Some intuitions and insights about the assignment method:
- If $\beta_{0,0}, \ldots, \beta_{0,n-1}, \ldots, \beta_{p-1,0}, \ldots, \beta_{p-1,n-1}$ are in succession, then the matrix in (15) is a Vandermonde matrix and thus (15) is satisfied. Thus we set $\beta_{j,k'} = jn + k'$ for $j \in [0, p)$, and $k' \in [0, n)$.
- We set $\gamma = 0$, because we will have
  \[ \{\gamma + \beta_{j,k'}|0 \leq j' < p, 0 \leq k' < n\} \]
  \[ = \{\beta_{j,k}|0 \leq j < p, 0 \leq k < n\}, \]
  thus the cardinality of the set $I$ can be reduced by its definition in Proposition 3.(ii).

- Given $\beta_{j,k'} = jn + k'$, Proposition 3.(i) implies that $\alpha_{k,j} = \alpha_{k,0} - jn$ for $j \in [1, p)$. Since
  \[ \min\{\beta_{j,k}|0 \leq j < p, 0 \leq k < n\} = pn - 1 \]
  and
  \[ U \cap \{\beta_{j,k}|0 \leq j < p, 0 \leq k < n\} = \emptyset, \]
  which can be fulfilled if $\min U \geq pn$. W.O.L.G., we set $\alpha_{0,0} + \beta_{0,0} = pm$, then $\alpha_{0,j} = pm - jn$ for $j \in [1, p)$.
- $\alpha_{k,0}, k \in [1, n)$ can be determined similarly according to Proposition 3.(i) and (ii).

Proof. With the above intuitions and insights, it is straightforward and easy to check conditions (i)–(iii) of Proposition 3; therefore, we omit the proof here.

Remark 5. Note that if the polynomial $h(x)$ contains $N_{dt}$ distinct terms with $N_{dt} \leq \deg(h(x))$, then the coefficients of the polynomial $h(x)$ can be possibly retrieved if one gets $N_{dt}$ points on the curve $y = h(x)$ and the corresponding $N_{dt}$ equations are linear independent, which can be fulfilled if the scheme is built over a sufficiently large finite field. That is, in order to minimize the recovery threshold, it is also feasible to minimize the number $N_{dt}$ of distinct terms of $h(x)$ instead of its degree if the scheme is over a sufficiently large finite field, then the recovery threshold is $R_c = N_{dt}$. This is indeed the case as in the SDMM problem in [10], [11].

In this paper, we aim to build schemes over a small finite field, i.e., characterized the recovery threshold by $\deg(h(x))$ other than $N_{dt}$. Nevertheless, we still provide an alternative assignment method for the case that the underlying finite field is sufficiently large. In this case, Proposition 3.(iii) can be replaced by
(iii’) $R_c = N_{dt}$ if the underlying finite field is sufficiently large.

In the meanwhile, $\beta_{0,0}, \ldots, \beta_{0,n-1}, \ldots, \beta_{p-1,0}, \ldots, \beta_{p-1,n-1}$ do not need to be in succession as in Proposition 4 as (15) can be easily satisfied over a sufficiently large finite field. In this case, we can provide a more efficient exponent assignment in terms of the recovery threshold. Before presenting the general assignment, Table VII gives an example of the assignment following the example in Section IV-A which leads to a smaller recovery threshold, i.e., 11.

**Proposition 5.** Conditions (i), (ii) of Proposition 3 and (iii’) in the above can be satisfied if $R_c = pmn + n + p - 1$,
\[ \alpha_{k,j} = j + kp + 1, \beta_{j,k'} = p - 1 - j + k'(pm + 1) \]
for $k \in [0, m), j \in [0, p), k' \in [0, n)$, $\gamma = 0$, and the underlying finite field is sufficiently large.

| Storage per server | (Broadcast) upload cost | Download cost | Recovery threshold $R_c$ | References |
|--------------------|-------------------------|---------------|--------------------------|------------|
| $C_{psdmm}$        | $L \frac{\alpha}{m}$   | $N \frac{\alpha}{mp}$ | $R_c \frac{\alpha}{mn}$ | $pmn + pm + n$ | Theorem 4 |
| $C_{psdmm}$        | $L \frac{\alpha}{mp}$  | $(N + L - 1) \frac{\alpha}{mp}$ | $R_c \frac{\alpha}{mn}$ | $pmn + pn - 1$ | Theorem 2 |
By Propositions 3, 4 and 5 we immediately have the result in Theorem 2.

C. Comparison and Difference between $C^I_{PSDMM}$ and $C^I_{PSDMM}$ in Section III

In this section, we give a comparison of some key parameters between the proposed PSDMM code $C^I_{PSDMM}$ from MDS-coded servers and $C^I_{PSDMM}$ from replicated servers in Section III see Table VII, and illustrate the difference between the two proposed PSDMM codes.

From Table VII, we see that compared to $C^I_{PSDMM}$ from replicated servers in Section III the proposed PSDMM code $C^I_{PSDMM}$ from MDS-coded servers requires much less storage per server, i.e., only a fraction of $1/p$, as that of $C^I_{PSDMM}$, but at the cost of increasing the upload cost.

Note that although MDS codes subsume repetition codes, it does not mean that $C^I_{PSDMM}$ subsume $C^I_{PSDMM}$, as they are totally two different schemes. The difference between the two schemes lies in the following aspects:

- **Upload phase:** In $C^I_{PSDMM}$, the user sends an encoded piece of $A$ and $L$ field elements (i.e., $q^{(0)}_i$ in (7)) to each server, whereas in $C^I_{PSDMM}$ from MDS-coded servers, the user sends an encoded piece of $A$ together with other $L - 1$ random matrices (i.e., $q^{(0)}_i$ in (16)) to each server.

- **Matrix partitioning:** In $C^I_{PSDMM}$ from replicated servers, the partition of the matrices $B^{(0)}, B^{(0)}, \ldots, B^{(L-1)}$ is carried out by each server before it encodes the library after receiving the query, whereas in $C^I_{PSDMM}$ from MDS-coded servers, the partition of the matrices $B^{(0)}, B^{(0)}, \ldots, B^{(L-1)}$ is predetermined as these matrices are stored across the servers in an MDS-coded form in advance.

- **Polynomial evaluation:** In $C^I_{PSDMM}$ from replicated servers, each server should first encode the library by evaluating $L$ encoding polynomials $g_i(x)$ ($t \in [0, L)$) in (8) at $L$ evaluation points, respectively, and then carry out the matrix multiplication between a $\frac{1}{m} \times \frac{1}{m}$ matrix and a $\frac{1}{p} \times \frac{1}{p}$ matrix. Whereas in $C^I_{PSDMM}$ from MDS-coded servers, each server needs to perform $L$ times the matrix multiplication between a $\frac{1}{m} \times \frac{1}{m}$ matrix and a $\frac{1}{p} \times \frac{1}{m}$ matrix, but does not need to evaluate polynomials.

V. Conclusion

We considered the problem of PSDMM and proposed two new coding schemes, i.e., $C^I_{PSDMM}$ from replicated servers and $C^I_{PSDMM}$ from MDS-coded servers. The proposed codes have a better performance than state-of-the-art schemes in that $C^I_{PSDMM}$ can achieve a smaller recovery threshold and download cost as well as providing a more flexible tradeoff between the upload and download costs, whereas $C^I_{PSDMM}$ can significantly save the storage in the servers. Characterizing the optimal trade-off between the upload and download costs as well as various theoretical bounds for PSDMM provide interesting open problems for further study.

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