Fault-Tolerant Coherent $H^\infty$ Control for Linear Quantum Systems

Yanan Liu, Senior Member, IEEE, Daoyi Dong, Senior Member, IEEE, Ian R. Petersen, Fellow, IEEE, Qing Gao, Senior Member, IEEE, Steven X. Ding, Shota Yokoyama, and Hidehiro Yonezawa

Abstract—Robustness and reliability are two key requirements for developing practical quantum control systems. The purpose of this article is to design a coherent feedback controller for a class of linear quantum systems suffering from Markovian jumping faults so that the closed-loop quantum system has both fault tolerance and $H^\infty$ disturbance attenuation performance. This article first extends the physical realization conditions from the time-invariant case to the time-varying case for linear stochastic quantum systems. By relating the fault-tolerant $H^\infty$ control problem to the dissipation properties and the solutions of Riccati differential equations, an $H^\infty$ controller for the quantum system is then designed by solving a set of linear matrix inequalities. In particular, an algorithm is employed to introduce additional quantum inputs and to construct the corresponding input matrices to ensure the physical realizability of the quantum controller. Also, we propose a real application of the developed fault-tolerant control strategy to quantum optical systems. A linear quantum system example from quantum optics, where the amplitude of the pumping field randomly jumps among different values due to the fault processes, can be modeled as a linear Markovian jumping system. It is demonstrated that a quantum $H^\infty$ controller can be designed and implemented using some basic optical components to achieve the desired control goal for this class of systems.

Index Terms—Coherent feedback control, fault-tolerant quantum control, $H^\infty$ control, linear quantum systems, quantum controller.

I. INTRODUCTION

Developing robust and reliable quantum control systems is a fundamental task with practical significance in implementing various quantum technologies [1]–[9]. In practice, quantum systems may suffer from various kinds of faults. This will degrade the system performance and may even make the quantum system unstable [10]. Generally, a fault in a dynamical system is a deviation of the system structure or the system parameters from the nominal situation [11]. For example, the laser output can be seen as a fault signal, which results in fluctuations of the pumping field in an optical parametric oscillator (OPO). Thus, a time-varying system parameter will be introduced in the system dynamics. To improve the performance of a quantum system, it is necessary to develop fault-tolerant control theory for quantum systems. However, many unique features of quantum systems, such as measurement reduction and noncommutative observables [12], make fault-tolerant control strategies for classical systems difficult to be extended to their quantum counterparts [10], [11], [13]. It is one of the main goals of this article to develop a fault-tolerant feedback control approach for a class of linear quantum systems with fault signals. Many examples of physical systems can be modeled as linear quantum systems, such as optical cavities, nondegenerate parametric amplifiers, degenerate parametric amplifiers, and large atomic ensembles [14]. The modeling and control theory development of linear quantum systems has been widely studied [15]–[17]. Feedback control, including measurement-based feedback control and coherent feedback control, plays an important role.
in quantum control theory since it has the capacity to suppress uncertainties and noises, thus having good robustness [18], [19]. In the measurement-based feedback control scheme, a measurement device is used to extract the plant information, which is then fed back to the original system through the control channel to achieve desired closed-loop behavior [20]–[23]. However, quantum measurement is different from the classical one, in the sense that it inevitably causes quantum state collapse and introduces additional stochastic noises [24]–[26]. In addition, the time to process the measurement outcomes and calculate the control signal cannot be ignored in general, which causes a time-delay problem in measurement-based feedback control [27].

Coherent feedback control, where the controller itself is also a quantum system and has no such disadvantages, is used in our approach [17], [28]–[30]. From both the control theory and the physical aspects, coherent feedback control has been widely applied in many quantum technologies. Realistic coherent control schemes have shown significantly better performance than optimal measurement-based schemes [31]. Coherent feedback control theory has been used to enhance the squeezing level of an OPO in quantum optical experiments [32]. A simple squeezer, which has better robustness to the gain fluctuations, has been proposed in [33] based on the idea of coherent feedback control. Furthermore, the proposed coherent feedback controller for linear quantum systems derived from physics first principles has been experimentally implemented on a quantum optical platform [16].

Unlike the cases in classical control systems, where we usually assume that all the designed controllers are physically realizable [28], the controller designed using mathematical models for a quantum plant may not correspond to a real physical quantum system. Hence, the physical realization in designing a coherent feedback controller needs to be considered. James et al. [28] have deduced necessary and sufficient conditions of physical realizability for a class of quantum systems described by quantum stochastic differential equations (QSDEs). They also pointed out that if the designed quantum controller does not satisfy the realizability conditions, one can introduce additional quantum inputs (also called additional quantum noises in the literature; see, e.g., [28] and [34]) and adjust the corresponding input matrices to make the controller physically realizable. For a class of linear quantum systems described by complex transfer function matrices, a construction algorithm has been proposed to physically synthesize them in [35]. Nurdin et al. [34], [36], [37] proposed several ways to synthesize a quantum system described by time-invariant linear differential equations and discussed how to implement the quantum systems using some basic optical components. These synthesis schemes are of significance in designing coherent feedback control strategies even though the experimental setup is often complex and challenging.

On the other hand, \( H^\infty \) control is a well-known robust control method that has been used in both classical and quantum systems [28], [38]–[40]. A quantum version of the standard dissipation properties has been proposed in [28], where the \( H^\infty \) control problem for quantum systems was formulated using two Riccati equations. By solving these Riccati equations, a controller is obtained and can be implemented as either a fully classical system, a purely quantum system, or a mixture of both quantum and classical elements. While James et al. [28] only considered the cases of time-invariant quantum systems, in practical applications, time-varying linear quantum systems are often encountered. The main difference between this article and [28] is that we consider time-varying linear quantum systems, as a result of the fault processes that the linear quantum system suffers from. A dynamic game approach to designing a classical \( H^\infty \) controller for a class of time-varying linear quantum systems has been proposed in [41], by recognizing the equivalence between a quantum system and a corresponding auxiliary classical stochastic system. A linear–quadratic Gaussian optimal controller has been designed in [42] to optimize the squeezing level achieved in one of the quadratures of the fundamental optical field for the time-varying quantum systems.

This article aims to solve the time-varying \( H^\infty \) coherent feedback control problem for a linear quantum system suffering from a fault signal. The strictly bounded real lemma of the time-varying quantum systems is first presented, by which the \( H^\infty \) control problem is formulated in several Riccati differential equations and a group of linear matrix inequalities (LMIs) [43]. The fault under consideration is modeled as a Markovian chain on a probability space [44], [45]. The physical realization conditions for time-varying quantum systems are then presented, and an algorithm is given to construct a physically meaningful quantum controller. In many practical applications, quantum optical systems have shown powerful potential for developing future quantum technologies [46]–[49]. In this article, we use a squeezer that has been widely used in quantum optics [49] to test the effectiveness of our control approach. A purely quantum \( H^\infty \) controller consisting of basic optical components is designed to ensure that the system has desired robust performance even when suffering from faults. Although the preliminary results in this article have been presented in the conference paper [43], this article presents a more comprehensive quantum fault-tolerant control theory than [43]. Also, this article includes physical realization analysis of time-varying quantum systems and an application of the proposed theory to quantum optics that were not covered in [43].

The main findings of this article are summarized as follows.
1) The bounded real properties of time-varying quantum systems are illustrated and used to design an \( H^\infty \) quantum controller.
2) Physical realizability conditions for time-varying linear quantum systems are investigated, through which the designed controller can be implemented as a quantum system.
3) The proposed quantum fault-tolerant \( H^\infty \) control method is applied to quantum optical systems. In particular, an OPO is recognized to be a time-varying linear quantum system, where the time-varying system parameter is caused by a fault signal in the pumping field.
4) A quantum controller is implemented by using several basic optical components to achieve fault-tolerant coherent \( H^\infty \) control for a class of quantum systems.
The rest of this article is organized as follows. Section II presents the system model and the problem formulation. In Section III, a main theorem is obtained to illustrate the equivalence between the strictly bounded real lemma, $H^\infty$ control problem, and relevant Riccati differential equations. Section IV presents a controller design in terms of LMIs. In Section V, we provide an application of fault-tolerant quantum control for a class of quantum systems in quantum optics. A squeezer where the pumping field suffers from a fault signal is considered. A purely quantum controller is implemented by some basic optical components. Finally, Section VI concludes this article.

Notation: $A$ represents an operator in Hilbert space and is a matrix with proper dimension. $A^T$ represents the transpose of $A$; $A^\dagger$ represents the adjoint of a Hilbert space operator; $A^{-1}$ is the inverse of $A$; $X^\#$ represents the operation of taking adjoint of each element of $X$, where $X$ is a matrix/array of operators, and $X^\dagger = (X^\#)^\dagger$; and $\|A\|_\infty$ represents the $H^\infty$ norm of the operator $A$. $\text{Tr}(A)$ is the trace of $A$; $\Im(A)$ represents the imaginary part of $A$; $i$ means imaginary unit, i.e., $i = \sqrt{-1}$; $I$ is the identity matrix with proper dimension; $\hbar$ is the reduced Planck constant; and $\mathbb{S}$ represents the state space.

II. SYSTEM MODEL AND PROBLEM FORMULATION

The dynamics of linear quantum systems can usually be described by time-invariant linear differential equations, for example, many systems in quantum optical experiments [32], [33]. In particular, an optical cavity composed by three mirrors may be described by the time-invariant linear differential equations in [28]. However, when the system suffers from a fault process, the equations may no longer be time invariant. In this article, we consider the following time-varying linear quantum system:

\[
\begin{align*}
\frac{dx(t)}{dt} &= A(t)x(t)dt + Bd\omega(t), \quad x(0) = x_0 \\
\frac{dy(t)}{dt} &= Cx(t)dt + Dd\omega(t)
\end{align*}
\]

where $A(t) \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n_\omega}$, $C \in \mathbb{R}^{n_\omega \times n}$, $D \in \mathbb{R}^{n_\omega \times n}$, and $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ is a vector of self-adjoint possibly noncommutative system variables. Here, $A(t)$ may be taken as a generalization to time-varying linear quantum systems from time-invariant linear quantum systems with constant matrix $A$. It shows the time-varying nature of the systems. For example, a time-varying Hamiltonian in the quantum harmonic oscillators will result in a time-varying $A(t)$. The initial system variables satisfy the commutation relation (CR) [28]

\[
[x_i(0), x_j(0)] = 2i\Theta_{jk}, \quad j, k = 1, \ldots, n.
\]

Here, the commutator is defined as $[A, B] = AB - BA$. $\Theta_{jk}$ is defined to be of one of the following form [28]:

1. canonical if $\Theta = \text{diag}(J, J)$;
2. degenerate canonical if $\Theta = \text{diag}(0_{n' \times n'}, J, \ldots, J)$, where $0 < n' \leq n$.

$J$ is the real skew-symmetric matrix $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, which represents the disturbance input and is assumed to have the form

\[
d\omega(t) = \beta_\omega(t)dt + d\tilde{\omega}(t)
\]

where $\beta_\omega(t)$ is a self-adjoint process and $d\tilde{\omega}(t)$ is the noise part.

The quantum noise satisfies Itô table $d\tilde{\omega}(t)d\tilde{\omega}(t)^T = F_\omega dt$ with a nonnegative matrix $F_\omega$ [50]. We write

\[
S_\omega = \frac{1}{2}(F_\omega + F_\omega^T)
\]

and

\[
T_\omega = \frac{1}{2}(F_\omega - F_\omega^T)
\]

where $T_\omega$ satisfies the following equation:

\[
[d\tilde{\omega}(t), d\tilde{\omega}(t)^T] = d\tilde{\omega}(t)d\tilde{\omega}(t)^T - (d\tilde{\omega}(t)d\tilde{\omega}(t)^T)^T = 2T_\omega dt.
\]

The fault process that we consider for linear quantum systems in this article is introduced in the time-varying system matrix $A(t)$. We can write $A(t) = A(F(t))$, where $F(t)$ represents the fault signal. Many linear quantum systems can be described by time-invariant linear differential equations, for example the empty cavity considered in [28] and the OPOs in quantum optical experiments [49], [51]. When a fault process is introduced in an OPO, this will result in a time-varying system matrix $A(t)$ in the linear differential equations. Controller design for this time-varying linear quantum system is more challenging. More details of the fault processes will be explained later in the application of the theory to quantum optical systems in Section V.

Theoretically, a possible quantum system that is described by the time-varying linear differential equations (TLDEs) (1) can be related to an open quantum harmonic oscillator with a time-varying Hamiltonian. In the following, we also show how to obtain the corresponding TLDEs (1) from an open quantum harmonic oscillator.

We refer to a quantum system with a quadratic Hamiltonian $H = \frac{1}{2} X^T R(t) X$ and subject to weak interaction with $N_\omega$ external environment fields though the coupling operator $L = \Lambda X$ as an open quantum harmonic oscillator. Here, $R(t)$ is a real and symmetric matrix of dimension $n \times n$ and $\Lambda$ is a complex-valued coupling matrix of dimension $2N_\omega \times n$. For a given vector of self-adjoint operators $X = [X_i]_{i=1,\ldots,n}$, its Lindblad generator is given as

\[
\mathcal{L}[X] = \frac{i}{\hbar}[H, X] + \frac{1}{2} (L^\dagger [X, L] + [L^\dagger, X]).
\]

Substituting the quadratic Hamiltonian and the coupling operator into (6), we obtain

\[
\mathcal{L}[X] = \Theta(R(t) + \hbar\Im(\Lambda^\dagger \Lambda))X
\]

where $2\hbar\Im(\Lambda^\dagger \Lambda) = \Lambda^\dagger \Lambda - \Lambda^T \Lambda^*$. Let $A_t = [(A_t)_i]_{i=1,\ldots,N_\omega}$ be a vector of annihilation operators of the environmental fields. From the Langevin equation, we obtain [52]

\[
\begin{align*}
\frac{dX_t}{dt} &= \mathcal{L}[X_t]dt + [X_t, L]dA_t^\dagger - [X_t, L^\dagger]dA_t \\
&= \Theta(R(t) + \hbar\Im(\Lambda^\dagger \Lambda))X_t dt + i\hbar\Theta(\Lambda^T dA_t^\dagger - \Lambda dA_t).
\end{align*}
\]

The dynamical equations of the quadratures for the output fields are given as

\[
\begin{align*}
\frac{dY_t}{dt} &= (L_t + L_t^\dagger)dt + dA_t + dA_t^\dagger \\
\frac{dY_t}{dt} &= -i(L_t - L_t^\dagger)dt - i(dA_t - dA_t^\dagger)
\end{align*}
\]
and \( Y_n = [(Y_t)_{i=1,\ldots,N_y}] \) is a vector of the amplitude quadratures of the output fields and \( Y'_n = [(Y'_t)_{i=1,\ldots,N_y}] \) are the phase quadratures of the output fields.

Let \( x(t) = X_t, \omega(t) = [(A_{i,0})_{i=1,\ldots,N_y} - i(A_{i,0})_{i=1,\ldots,N_y}], \) and \( y(t) = [Y_t] \). We obtain the linear differential equations (1), where the matrices \( A, B, C, \) and \( D \) are given by

\[
A = 2\Theta(R(F(t)) + \mathcal{S}(A^1 A^2)) \quad (9)
\]

\[
B = 2\Theta[-\Lambda^1 \Lambda^T] \quad (10)
\]

\[
C = P_N^T \left[ \sum_{N_y} 0_{N_y \times N_x} \right] \left[ \begin{array}{c} \Lambda + \Lambda^# \\ \Lambda^T + \Lambda^T \end{array} \right] \quad (11)
\]

\[
D = P_N^T \left[ \sum_{N_y} 0_{N_y \times N_x} \right] P_N \quad (12)
\]

where \( P_N \) is the permutation matrix satisfying

\[
P_N a = \left[ a_1 \quad a_3 \quad \cdots \quad a_{2m-1} \quad a_2 \quad a_4 \quad \cdots \quad a_{2m} \right]^T
\]

for an arbitrary vector \( a = \left[ a_1 \quad a_2 \quad \cdots \quad a_{2m} \right]^T \). Also,

\[
\sum_{N_y} \left[ I_{N_y \times N_y} \quad 0_{N_y \times (N_x - N_y)} \right].
\]

The main goals of this article are to analyze the strictly bounded real lemma of the time-varying quantum systems described by (1) and to design a coherent quantum feedback controller to achieve closed-loop \( H^\infty \) performance. One can conclude that if the matrices of a quantum system described by TLDEs (1) can be written in the form of (9)–(11), this quantum system will be physically realizable. Sufficient and necessary conditions for the physical realizability of the system (1) will be deduced later in Section IV-C.

### III. Bounded Real Properties

In this section, we consider the bounded real properties for the time-varying quantum system (1). The strictly bounded real lemma states a relation between a storage function and supply functions in terms of system energy \([53]\). Details of bounded real properties can be found in \([54]\). Relevant discussions of bounded real properties in the context of time-invariant linear quantum systems have been given in \([28]\). Here, we make a further extension to the time-varying cases. In addition, we formulate the strict bounded real lemma, which will be used in the controller synthesis later.

We consider a quantum system described as follows:

\[
dx(t) = A(t)x(t)dt + \left[ B \quad G \right] \left[ d\omega(t)^T \quad dv(t)^T \right]^T
\]

\[
dz(t) = Cz(t)dt + \left[ D \quad H \right] \left[ d\omega(t)^T \quad dv(t)^T \right]^T.
\]

(13)

Here, \( d\omega(t) = \beta(t)dt + d\bar{\omega} \) represents the disturbance input with the quantum noise \( d\bar{\omega} \), and \( dv \) represents other inputs including unexpected quantum noises in quantum systems.

We first define a storage function \( V(x(t)) = x(t)^T P(t)x(t) \), where \( P(t) \) is a time-varying positive-definite symmetric matrix, and then define the following operator valued quadratic function:

\[
\gamma(x, \beta) = \left[ x^T \beta^T \right] R \left[ \begin{array}{c} x \\ \beta \end{array} \right]
\]

as the supply function, where \( R \) is a constant real symmetric matrix.

**Definition 1 (see [28]):** The quantum system (13) is said to be bounded real with disturbance attenuation \( g \) if there exists a positive time-varying storage function \( V(x(t)) = x(t)^T P(t)x(t) \), a constant \( \lambda > 0 \), and a supply rate \( \gamma(x, \beta) \) such that the following inequality holds:

\[
\langle V(x(t)) + \int_0^t \gamma(x(s), \beta(s))ds \rangle \leq \langle V(x(0)) \rangle + \lambda t \quad \forall t > 0.
\]

(14)

Here, \( \langle V(x(t)) \rangle \) represents the expectation of the operator \( V(x(t)) \), and \( \beta(t) = C x(t) + D \beta(t) \). The supply rate function has the form

\[
\gamma(x, \beta) = \beta^T \beta - g^2 \beta^T \beta
\]

\[
= \left[ x^T \beta^T \right] \left[ C^T C \quad C^T D \quad D^T C \quad D^T D - g^2 \right] \left[ \begin{array}{c} x \\ \beta \end{array} \right].
\]

(15)

Also, we say that the system (13) is strictly bounded real with disturbance attenuation \( g \) if there exists a constant \( \epsilon > 0 \) such that inequality (14) holds for the supply function with \( R + \epsilon I \).

With this definition, the following theorem states the relationship between the bounded real properties and the Riccati differential equations, as well as the \( H^\infty \) control problem, which will be used to design a coherent controller.

**Theorem 2:** For the system (13), the following four statements are equivalent.

1. The system (13) is strictly bounded real with disturbance attenuation \( g \).
2. There exists a positive-definite matrix \( \tilde{P}(t) \) such that

\[
\dot{\tilde{P}}(t) + A(t)^T \tilde{P}(t) + \tilde{P}(t) A(t) + C^T C
\]

\[
+ (C^T D + \tilde{P}(t) B)(B^2 - D^T D)^{-1} (D^T C + B^T \tilde{P}(t)) < 0.
\]

3. The Riccati differential equation

\[
\dot{\tilde{P}}(t) + A(t)^T \tilde{P}(t) + \tilde{P}(t) A(t) + C^T C
\]

\[
+ (C^T D + \tilde{P}(t) B)(B^2 - D^T D)^{-1} (D^T C + B^T \tilde{P}(t))
\]

\[
= 0
\]

has a stabilizing solution \( \tilde{P}(t) \geq 0 \).

4. The homogeneous system \( \dot{z}(t) = A(t) x(t) \) is exponentially stable, and the operator mapping \( \omega \) to \( z \) satisfies

\[
\| T_\omega \| \leq g.
\]

**Proof:** Since the equivalence between statements 2)–4) has been proved in \([55]\), we here only prove the equivalence between 1) and 2).
For a given storage function \( V(x(t)) = x(t)^T P(t) x(t) \), we calculate
\[
d(V(x(t))) = \langle dx^T(t) \cdot [P(t)x(t)] + x^T(t) \cdot d[P(t)x(t)] \rangle
\]
\[
= \langle x^T(t) \left( A^T(t)P(t) + P(t)A(t) + \hat{P}(t) \right) x(t) + \beta_\omega^T(t)B^T P(t)x(t) + x^T(t)P(t)B\beta_\omega(t) + \lambda_0 \rangle \ dt
\]
where \( \hat{P}(t) = \frac{dP(t)}{dt} \) and
\[
\lambda_0(t) = \text{Tr} \left\{ \begin{bmatrix} B^T \\ G^T \end{bmatrix} P(t) \begin{bmatrix} B \\ G \end{bmatrix} F \right\}
\]
and \( F \) is defined as
\[
F dt = \begin{bmatrix} d\omega \\ d\nu \\ d\omega^T \\ d\nu^T \end{bmatrix}.
\]

Suppose \( \rho \) is an initial Gaussian state, and let \( E_0 \) denote the expectation with respect to a random state \( \phi \). We have \( \{V(x(t))\} = \langle \rho, E_0[V(x(t))] \rangle \) [56]. Here, \( \langle \rho, \cdot \rangle \) represents the expectation with respect to the Gaussian state \( \rho \). If the system (13) is bounded real with disturbance attenuation \( g \), then we have
\[
\langle \rho, \int_0^t E_0 \left[ x^T(s) \left( A^T(t)P(t) + P(t)A(t) + \hat{P}(t) \right) x(s) + \beta_\omega^T(s)B^T P(t)x(s) + x^T(s)P(t)B\beta_\omega(s) + \lambda_0(s) + \gamma(x(s), \beta_\omega(s)) \right] ds \rangle \leq \lambda t
\]
where \( \gamma(x(s), \beta_\omega(s)) \) is defined as (15).

Let
\[
f(s) = E_0 \left[ x^T(s) \left( A^T(t)P(t) + P(t)A(t) + \hat{P}(t) \right) x(s) + \beta_\omega^T(s)B^T P(t)x(s) + x^T(s)P(t)B\beta_\omega(s) + \lambda_0(s) + \gamma(x(s), \beta_\omega(s)) \right].
\]
Then, we have \( \int_0^t f(s) ds = F(t) - F(0) \), where \( F(t) \) is defined as \( F(t) = f(t) \). Lagrange’s mean value theorem implies that there exists a time \( t' \in (0, t) \) such that \( F(t) - F(0) = f(t') t \). Combined with (17) we then obtain
\[
\langle \rho, f(t') t \rangle \leq \lambda t
\]
which means
\[
\langle \rho, f(t') \rangle \leq \lambda.
\]
For all times \( t \), there exists a time \( t' \) such that (19) holds. Hence, we can rewrite \( t' = t \).

Consider a real matrix \( X \) and corresponding operator valued quadratic form \( x^T X x \) for the system (13). From [28, Lemma A.1], we know that the following statements are equivalent.

i) There exists a constant \( \lambda \geq 0 \) such that \( \langle \rho, x^T X x \rangle \leq \lambda \) for all Gaussian states \( \rho \).

ii) The matrix \( X \) is negative semidefinite.

Hence, we obtain
\[
\begin{bmatrix}
\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + CT C & CT D + P(t)B \\
B^T P(t) + D^T C & D^T D - g^2 I
\end{bmatrix} \leq 0.
\]

Furthermore, the system is strictly bounded real with disturbance attenuation \( g \) if and only if there exists a real positive-definite symmetric matrix such that the following matrix inequality is satisfied:
\[
\begin{bmatrix}
\dot{P}(t) + A^T(t)P(t) + P(t)A(t) + CT C & CT D + P(t)B \\
B^T P(t) + D^T C & D^T D - g^2 I
\end{bmatrix} < 0
\]
which means (2).

From statement 2) to statement 1):

Based on Schur’s complement, statement 2) implies (20). Then, we can obtain (19) from (20). From Lagrange’s mean value theorem, we can always find a time range \( t \in (0, t_f) \) (for each \( t \), we can find the corresponding \( t_f \) such that (18) holds with \( \langle \rho, f(t)/t_f \rangle \leq \lambda t_f \). By choosing the storage function as \( V(x) = x^T P(t_f) x \) and by setting \( \lambda_0(t) = B^T P(t_f) B \), one obtains
\[
\langle V(x(t_f)) \rangle - \langle V(x(0)) \rangle + \int_0^{t_f} \langle \gamma(x(s), \beta_\omega(s)) \rangle ds \leq \lambda_0 t_f.
\]
From Definition 1, one has that the system (13) is strictly bounded real with disturbance attenuation \( g \).

The proof is then complete. \( \square \)

If we let \( A(t) \) be a time-invariant matrix \( A \), Theorem 2 will be the same as [28, Corollary 4.5].

IV. COHERENT \( H^\infty \) CONTROL DESIGN

Quantum optical systems usually are sensitive to external disturbance. In this article, we consider a linear quantum system suffering from abrupt variation in its parameters, structure, or system dynamics such that the system dynamics may randomly transit between a finite number of different modes, named faulty modes. It is then appropriate to model the fault process on a probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \) by a continuous-time Markov chain \( \{F(t)\}_{t \geq 0} \) [44], which results in a Markovian jump linear quantum system. To be specific, \( F(t) \) values within a finite set \( S = \{e_1, e_2, \ldots, e_N\} \) for an integer \( N \). The transition rate matrix is priorly known as \( \Pi = (\pi_{jk}) \in \mathbb{R}^{N \times N} \), with \( \pi_{jj} = -\sum_{j \neq k} \pi_{jk} \), and \( \pi_{jk} \geq 0, j \neq k \). Here, the transition rate matrix describes the instantaneous rate at which a continuous time Markov chain transitions between states. In this section, we develop a coherent \( H^\infty \) control design for a class of linear quantum systems whose Hamiltonian is dependent on the fault process \( F(t) \).
A. Closed-Loop Systems

The system with a disturbance input and a control input is described as

\[ dx(t) = A(F(t))x(t)dt + B_1dw(t) + B_2du(t) \]
\[ dz(t) = C_1x(t)dt + D_1du(t) \]
\[ dy(t) = C_2x(t)dt + D_2dw(t) \] (23)

\( A(F(t)) \) takes finite values in \( \{A_1, A_2, \ldots, A_N\} \), \( A_i = A(e_i) \) since the fault process \( F(t) \) has been assumed to be a Markov chain, which has values within a finite set \( \mathbb{S} = \{e_1, e_2, \ldots, e_N\} \).

Assume that the controller is described by the following dynamical equations:

\[ d\xi(t) = A(t)\xi(t)dt + B(t)d\eta(t) + \mathcal{E}(t)d\nu_K(t) \]
\[ du(t) = C(t)\xi(t)dt + D(t)d\nu_K(t) \] (24)

where \( \xi(t) = \begin{bmatrix} \xi_1(t) & \xi_2(t) & \cdots & \xi_n(t) \end{bmatrix}^T \) is a vector of self-adjoint controller variables. The input \( \nu_K \) is introduced to ensure the physical realizable conditions, which is assumed to be a vector of noncommutative Wiener processes satisfying the Itô table with canonical Hermitian Itô matrix \( F_{\nu_K} \).

To coincide with a Markovian jump linear plant, the controller is also assumed to jump between different modes with \( \langle A_1, B_1, C_1 \rangle, \ldots, \langle A_N, B_N, C_N \rangle \).

We obtain the closed-loop systems by identifying \( \beta_n(\eta) = C_n(t)\xi(t) + \mathcal{D}(t)d\nu_K(t) \) as

\[ d\eta(t) = \begin{bmatrix} A_1 & B_2C_1 \\ B_1C_2 & A_1 \end{bmatrix} \eta(t)dt + \begin{bmatrix} B_1 \\ B_1D_2 \end{bmatrix} d\nu(t) \]
\[ + \begin{bmatrix} B_2D_1 \\ E_i \end{bmatrix} d\nu_K(t) \]
\[ dz(t) = \begin{bmatrix} C_1 & D_1C_i \end{bmatrix} \eta(t)dt + D_1D_i d\nu_K(t). \] (25)

The control objective here is to design a controller (24) such that the closed-loop system (25) is strictly bounded real with a given disturbance attenuation \( g \), that is, there exists a positive-definite matrix \( P(t) \) such that

\[ \langle \eta^T(t)P(t)\eta(t) \rangle + \int_0^t \left\{ \beta_2^T(s)\beta_2(s) - g^2\beta_\omega^T(s)\beta_\omega(s) \right\} ds \]
\[ + \epsilon\eta^T(s)\eta(s) + \epsilon\beta_\omega^T(s)\beta_\omega(s) \leq \langle \eta^T(0)P_0\eta(0) \rangle + \lambda t \quad \forall t > 0. \] (26)

B. \( H^\infty \) Controller Design

Substituting \( d\omega(t) = \beta_\omega(t)dt + d\hat{\omega}(t) \) and \( du(t) = \beta_\omega(t)dt + d\hat{u}(t) \) into (25), we have

\[ d\eta(t) = \tilde{A}_i\eta(t)dt + \tilde{B}_i\beta_\omega(t)dt + \tilde{B}_i d\hat{\omega}(t) + \tilde{B}_2i d\nu_K(t) \]
\[ dz(t) = \tilde{C}_i\eta(t)dt + \tilde{D}_i d\nu_K(t). \] (27)

Here

\[ \tilde{A}_i = \begin{bmatrix} A_i & B_2C_1 \\ B_1C_2 & A_i \end{bmatrix}, \quad \tilde{B}_i = \begin{bmatrix} B_1 \\ B_1D_2 \end{bmatrix}, \quad \tilde{C}_i = \begin{bmatrix} C_1 & D_1C_i \end{bmatrix}, \quad \tilde{D}_i = D_1D_i. \]

For the quantum system, we define \( \langle V(\eta(t)) \rangle = \langle \eta^T(t)P(t)\eta(t) \rangle \) with a positive-definite matrix \( P(t) \) and \( \gamma(\eta(t), \beta_\omega(t)) = \beta_\omega^T(t)\beta_\omega(t) - g^2\beta_\omega^T(t)\beta_\omega(t) + \epsilon\eta^T(t)\eta(t) + \epsilon\beta_\omega^T(t)\beta_\omega(t) \).

Substituting (28) into (26), the control objective can be written as

\[ \langle \eta^T(t)P(t)\eta(t) \rangle + \int_0^t \left\{ \tilde{C}_i^T \tilde{C}_i + \epsilon I \right\} \eta(s)ds \]
\[ - \int_0^t \langle (g^2 - \epsilon)\beta_\omega^T(s)\beta_\omega(s) \rangle ds \leq \langle \eta^T(0)P_0\eta(0) \rangle + \lambda t. \] (29)

We consider a classical system composed of a plant

\[ dx_c(t) = A(F(t))x_c(t)dt + B_1dw_c(t) + B_2du_c(t) \]
\[ dz_c(t) = C_1x_c(t)dt + D_1du_c(t) \]
\[ dy_c(t) = C_2x_c(t)dt + D_2dw_c(t) \] (30)

and a controller

\[ d\xi_c(t) = A(t)\xi_c(t)dt + B(t)d\nu_c(t) + \mathcal{E}(t)d\nu_c(t) \]
\[ du_c(t) = C(t)\xi_c(t)dt + D(t)d\nu_c(t). \] (31)

Here, \( \nu_c(t) = \beta_\omega(t)dt + S_{\nu_c}^T d\hat{\omega}(t), \quad d\nu_c(t) = S_{\nu_c}^T d\nu(t) \), and \( x_c(0) \) is a Gaussian random vector with mean \( \hat{x}_c(0) \) and covariance matrix \( Y_{c0}, S_{\nu_c}^T \) and \( S_{\nu_c} \) are defined as in (4).

It is possible to prove that if the classical system with a controller in the form of (31) is strictly bounded real with disturbance attenuation \( g \), the quantum system with the same control parameters in controller (24) is also strictly bounded real with \( g \). The detailed proof is in Appendix B. The following proposition on \( H^\infty \) control design for classical systems is cited here, and it will be applied to the quantum case in this article.

Proposition 3 (see [57]): If there exists \( P = (P_1, \ldots, P_N) \) and \( P_i > 0 \) satisfies

\[ A_i^T P_i + P_i A_i + \sum_{j=1}^N \pi_{ij} P_j + g^{-2} P_i B_1 B_1^T P_i + C_i^T C_i < 0 \] (32)

for \( i = 1, \ldots, N \), then \( \|T_{\nu\omega}\|_\infty < g \).

Here, the norm \( \|T_{\nu\omega}\|_\infty \) is the \( H^\infty \)-norm for the system transfer function \( T_{\nu\omega} \) from disturbance input \( \omega(t) \) to the error.
output $z(t)$. Now, we have the following conclusion for quantum systems under consideration.

**Theorem 4:** If there exists a controller of the form (24) such that the closed-loop system (25) is strictly bounded real with disturbance attenuation $g$, then the LMIs (33), (34) shown at the bottom of this page, have feasible solutions $X_i, Y_i$ and $L_i, F_i$. Here, for $i = 1, \ldots, N$, we define

$$S_i(Y) = -\text{diag}(Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_N)$$

and

$$R_i(Y) = \left[\pi_1 Y_1 \cdots \pi_{i-1} Y_i \pi_{i+1} \cdots \pi_N Y_i\right].$$

In this case, the controller is given by

$$C_i = F_i Y_i^{-1},$$

$$B_i = \left(Y_i^{-1} - X_i\right)^{-1} L_i,$$

$$A_i = \left(Y_i^{-1} - X_i\right)^{-1} M_i Y_i^{-1}$$

where

$$M_i = -A_i^T X_i A_i Y_i - X_i B_i F_i - L_i C_i Y_i - C_i^T (C_i Y_i + D_{12} F_i) - g^2 (X_i B_i + L_i D_{21}) B_i^T - \sum_{j=1}^{N} \pi_{ij} Y_j^{-1} Y_i.$$ 

Similarly, if the LMIs (33) and (34) have feasible solutions and the controller is defined as in (35)–(37), then the closed-loop system (25) is strictly bounded real with the disturbance attenuation $g$.

**Proof:** According to Theorem 2, the system (25) is strictly bounded real with disturbance attenuation $g$ if and only if the transfer function $T_{\omega z}$ from $\omega$ to $z$ satisfies $\|T_{\omega z}\|_\infty < g$. Since we have illustrated the equivalence of the $H_\infty$ performance of the quantum system (25) and the classical system composed of (30) and (31), Proposition 3 can be applied to the closed-loop system (25) directly by replacing the parameters in (32) as $A_1, B_{1j}, B_{2j}$, and $C_1$. Through some matrix calculations, we can then prove Theorem 4. The detailed proof can be completed using the result in [57].

**Remark 5:** Theorem 4 converts the $H_\infty$ controller design problem to the solutions of LMIs (33) and (34). However, it is not easy to obtain existence conditions for the solutions to these two LMIs or their analytical solutions. We will only consider numerical solutions for a specific quantum system in Section V.

### C. Physical Realization of the Controller

Unlike the classical systems in which we assume that a differential equation always corresponds to some real plant, the linear differential equation in (23) does not necessarily represent a meaningful quantum system. Sufficient and necessary conditions for physical realization of the following time-invariant linear system have been proposed in [28]:

$$\begin{aligned}
    dx(t) &= Ax(t)dt + Bd\omega(t), \quad x(0) = x_0 \\
    dy(t) &= Cx(t)dt + Dd\omega(t)
\end{aligned}$$

where the first condition is to preserve the CR during the whole evolution time. In this article, we need to consider the physical realization of time-varying linear quantum systems.

The following proposition shows sufficient and necessary conditions for preserving CR of systems (1) with only the parameter $A(t)$ being time varying.

**Proposition 6:** The time-invariant system (38) preserves CR, if and only if the system (1) preserves CR with time evolving.

**Proof:** According to [28, Th. 2.1] (see Appendix A for details), if the system (38) preserves CR, one has

$$iA(t) + i\Theta A^T(t) + BT \beta B^T = 0$$

where $\Theta$ is any $N \times N$ complex matrix and $A(t)$ is a time-varying parameter.

Hence, the time-varying parameter $A(t)$ does not affect the CR of quantum systems, which proves the proposition.

Based on this proposition, we find that the sufficient and necessary conditions of physical realizability can be directly extended to the following proposition from [28, Theorem 3.4].

**Proposition 7:** The system (1) is physically realizable if and only if

$$iA(t) + i\Theta A^T(t) + BT \beta B^T = 0$$

where $\Theta$ is any $N \times N$ complex matrix and $A(t)$ is a time-varying parameter.
And the matrix $D$ satisfies (12).

**Proof:** If the system is physically realizable, then (10)–(12) hold. The dynamics of the unitary operator $U_t$ are calculated as

$$dU_t = \left(-iHdt - \frac{1}{2}Ldt + \begin{bmatrix} -L^T & \Gamma dt(t) \end{bmatrix} U_t. \right)$$

(44)

Since $U_t$ is unitary for each $t \geq 0$, we have $d(x(t)x(t)^T - x(t)x(t)^T) = 0$ due to the preservation of the CR. We can then obtain (39) for the time-invariant linear differential equation (38). From Proposition 6, we obtain (42). Then, using (10) and (11), we obtain (43). The detailed calculations can be found in [28]. Hence, we have proved that (12), (42), and (43) are necessary for realizability.

Then, we suppose (12), (42), and (43) hold to prove that there exists a symmetric matrix $R$ and a coupling matrix $\Lambda$ such that the following two equations:

$$\sum_{j=1}^{m} \kappa_j \rho_j = \sum_{j=1}^{m} \kappa_j \rho_j = 0$$

(49)

since the CR is the annihilation operator of the fundamental field, while the second term represents the decay rate for the jth mirror. Also, $\rho_{\in j}$ represents the field of the jth mirror. The output of each mirror is described as

$$d\rho_{\text{out},j}(t) = \sqrt{\kappa_j} \rho_{\text{in},j}(t) dt - d\rho_{\text{out},j}(t).$$

(46)

**A. Basic Optical Components**

1) Cavities: The optical cavity is a widely used component in quantum optics. It is composed of two mirrors in the simplest case as well as a group of mirrors for more general cases. The following linear differential equation describes the dynamics of an optical field in an empty cavity with $m$ mirrors:

$$\dot{\rho}(t) = -\frac{\kappa}{2} \rho(t) dt + \sum_{j=1}^{m} \sqrt{\kappa_j} d\rho_{\text{in},j}(t).$$

(48)

Here, $\dot{\rho}(t)$ is the annihilation operator inside the cavity, $\kappa = \sum_{j=1}^{m} \kappa_j$ represents the total decay rate, while $\kappa_j$ is the decay rate for the jth mirror. Also, $\rho_{\text{in},j}$ represents the input field of the jth mirror. The output of each mirror is described as

$$d\rho_{\text{out},j}(t) = \sqrt{\kappa_j} \rho_{\text{in},j}(t) dt - d\rho_{\text{out},j}(t).$$

(49)



2) Second-Order Nonlinear Effect: Nonlinear crystals are materials causing nonlinear effects and have been widely used in many quantum optical experiments. They have the ability to couple a fundamental field (oscillating at $\nu$) and a second harmonic field (oscillating at $2\nu$). Many materials that have a nonlinear effect have been used in different optical experiments, for example, potassium niobate (KNBs) and potassium titanyl phosphate (KTP, KTiOPO4). Materials with a $\chi^{(2)}$ nonlinearity are mainly used for parametric nonlinear frequency conversion (e.g., OPOs), which is essential in the generation of squeezed states. The Hamiltonian of a crystal with $\chi^{(2)}$ nonlinearity is given by [59]

$$H = i\hbar \chi^{(2)} (\hat{b} \hat{a}^2 - (\hat{a} \hat{b})^2).$$

(50)

where $\hat{a}$ is the annihilation operator of the fundamental field, while $\hat{b}$ is that of the harmonic field. We may understand this Hamiltonian from the upconversion and downconversion aspect, where the first term can be explained as the annihilation of two photons at the fundamental frequency and the creation of one at the harmonic frequency, while the second term represents the reverse process. Hence, the Heisenberg equations of these two fields are written as

$$\dot{\hat{a}} = -\chi^{(2)} \hat{b} \hat{a}^2$$

$$\dot{\hat{b}} = \chi^{(2)} \hat{a}^2$$

(51)

where $\chi^{(2)}$ is the second-order nonlinear coefficient of the crystal.

3) Dynamical Squeezers: A squeezer is one of the most fundamental components in continuous-variable quantum information processing. The main function of a squeezer is to generate a squeezed state [33]. A popular system that works as a squeezer is an OPO, which is composed of several mirrors and a nonlinear crystal. Then, the nonlinear interaction of the crystal can be enhanced by the optical cavity. By combining the

V. APPLICATIONS OF FAULT-TOLERANT CONTROL TO QUANTUM OPTICAL SYSTEMS

In this section, we consider possible applications of fault-tolerant coherent $H^\infty$ control of linear quantum systems in quantum optics. Here, we consider that a quantum plant suffers from a fault signal, and a coherent feedback controller is designed to make the quantum system fault tolerant and robust against external disturbance inputs. We first give a brief introduction to some necessary optical components that will be used to set up the quantum plant as well as the controller.
equations of motion from the crystal and an empty cavity, we obtain the dynamical equation of the internal cavity mode as
\[
d\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t)dt - \chi^{(2)}\hat{a}^\dagger(t)\hat{b}(t)dt + \sum_{j=1}^{m}\sqrt{\kappa_j}d\hat{A}_{in,j}(t)
\]
while the output equations remain as in (49).

We assume that the harmonic field (or what we call the pumping field) is an intense field, which can be undepleted by its interaction with the nonlinear crystal and the fundamental field of interest [59]. Under this assumption, we can replace the operator \(\hat{b}\) with a complex number \(\beta\), which means we may ignore the dynamics of the pumping field, leaving the dynamics of fundamental mode as
\[
d\hat{a}(t) = -\frac{\kappa}{2}\hat{a}(t)dt - \chi\hat{a}^\dagger(t)dt + \sum_{j=1}^{m}\sqrt{\kappa_j}d\hat{A}_{in,j}(t)
\]
where \(\chi = \chi^{(2)}\beta\). Here, we only consider the case \(\chi \in \mathbb{R}\). The parameter \(\beta = |\beta|e^{i\phi}\) is composed of the real amplitude \(|\beta|\) and the phase \(\phi\) of the pump field. We assume that only the amplitude of the pump field suffers from the fault signal, while its phase remains unchanged as \(\phi = 2k\pi, k = 1, 2, \ldots\). Under this assumption, \(\chi\) becomes real. This parameter is pumping dependent, which means we can change its value by changing the pumping field. In this article, we define this class of OPO as a dynamical squeezer, which has been widely used in quantum optical experiments to generate the squeezing states.

4) Static Squeezers: For some practical applications, we may not be interested in the internal dynamics of a squeezer and only focus on the transformation matrix between its input and output fields. To obtain this relation, the evolution time inside the cavity may be assumed to be very short. This relation has been obtained in [60] by assuming the evolution time of the fundamental mode is extremely short. To directly understand the static squeezer from an experimental point of view, we first transform the operators of cavity mode and input and output fields to the frequency domain by using the Fourier transform
\[
-i\omega\hat{a}(\omega) = -\frac{\kappa}{2}\hat{a}(\omega) - \chi\hat{a}^\dagger(-\omega) - i\omega\sum_{j=1}^{m}\sqrt{\kappa_j}\hat{A}_{in,j}(\omega)
\]
\[
-i\omega\hat{a}^\dagger(-\omega) = -\frac{\kappa}{2}\hat{a}^\dagger(-\omega) - \chi\hat{a}(\omega) - i\omega\sum_{j=1}^{m}\sqrt{\kappa_j}\hat{A}_{in,j}^\dagger(-\omega)
\]
\[
(-i\omega)\hat{A}_{out,j}(\omega) = \sqrt{\kappa_j}\hat{a}(\omega) - (-i\omega)\hat{A}_{in,j}(\omega).
\]
By solving the two equations in (54), we obtain
\[
\hat{a}(\omega) = \frac{-i\omega}{(\frac{\kappa}{2} - i\omega)^2 - |\chi|^2}\sum_{j=1}^{m}\left\{\left(\frac{\kappa}{2} - i\omega\sqrt{\kappa_j}\right)\hat{A}_{in,j}(\omega) - \chi\sqrt{\kappa_j}\hat{A}_{in,j}^\dagger(-\omega)\right\}.
\]

Substituting (56) into (55), we obtain the transfer function for the squeezer as
\[
\hat{A}_{out,j}(\omega) = \frac{1}{(\frac{\kappa}{2} - i\omega)^2 - |\chi|^2}\left\{\sum_{j=1}^{k}\left[\left(\frac{\kappa}{2} + i\omega\right)\kappa_j\hat{A}_{in,j}(\omega) - \chi\kappa_j\hat{A}_{in,j}^\dagger(-\omega)\right]\right. - \left[\frac{\kappa^2}{2} - |\chi|^2\right]\hat{A}_{in,j}(\omega)\right\}.
\]

The assumption that the evolution time is extremely short means that the squeezer has a broad squeezing spectrum. Squeezers with broad bandwidth have been realized in many experimental setups such as a monolithic cavity [51] and a single-pass waveguide [61]. For this kind of squeezer, if we only focus on a range of frequencies, the noise power can be taken as a constant, which means the relation between input and output of the squeezer remains unchanged with frequency. Without loss of generality, we consider the case with \(\omega \ll \kappa\). We can then obtain a simplification of (57)
\[
\hat{A}_{out,j} = \frac{1}{(\frac{\kappa}{2} - i\omega)^2 - |\chi|^2}\left\{\sum_{j=1}^{m}\left[\frac{\kappa}{2}\kappa_j\hat{A}_{in,j} - \chi\kappa_j\hat{A}_{in,j}^\dagger\right]\right. - \left[\frac{\kappa^2}{2} - |\chi|^2\right]\hat{A}_{in,j}\right\}.
\]

A squeezer with the relationship between input and output as in (58) is called static squeezer, which has been proposed in [60]. From the perspective of quantum control theory, we usually analyze a system in the time domain. In this case, the static squeezer can be seen as a squeezer at a stable state, where the dynamics between input and output have been assumed to be static. The relation has been deduced in [60], where the time of light going through the squeezer has been assumed to be short enough. From the experimental point of view, the dynamics are usually analyzed in the frequency domain by using the Fourier transformation. In this case, the static squeezer is an approximation of a dynamical squeezer with a broad squeezing spectrum, and the output of the fundamental field at any frequency within the interested frequency range remains constant. This assumption results in (58) for the relation between input and output.

B. Fault-Tolerant Control Design for Quantum Optical Systems

In this section, we consider a linear quantum system arising in quantum optics. When this system suffers from a fault signal, an \(H^\infty\) coherent feedback controller can be designed to deal with the fault process as well as the disturbance input. The system has been designed to generate squeezed light in [49]. The system is a dynamical squeezer composed of three mirrors, and its simplified diagram is shown as in Fig. 1.
The system may be described by
\[
\begin{align*}
 dx(t) &= A(t)x(t)dt + B_1d\omega(t) + B_2du(t) \\
 dz(t) &= C_1x(t)dt + D_1du(t) \\
 dy(t) &= C_2x(t)dt + D_2d\omega(t) \tag{61}
\end{align*}
\]
where
\[
A(t) = \begin{bmatrix}
-\frac{\kappa}{2} - \chi(t) & 0 \\
0 & \chi(t) - \frac{\kappa}{2}
\end{bmatrix}
\]
\[
B_1 = \sqrt{\kappa_1}, \quad B_2 = \sqrt{\kappa_2}
\]
\[
C_1 = \sqrt{\kappa_2}, \quad D_1 = -I
\]
\[
C_2 = \sqrt{\kappa_1}, \quad D_2 = -I.
\]

Here, \(I\) is the \(2 \times 2\) identity matrix, and we write \(\chi(t) = \chi = \chi^{(2)}\beta(t)\) as a time-varying parameter, where \(\beta(t)\) represents the fault signal due to the unstable voltage of the laser generator.

Since the pump laser is treated in a classical way, if the macroscopic laser device is subject to an undesired fault signal, a time-varying Hamiltonian is introduced. In some practical applications, we may assume that the amplification of the laser is not changing with time continuously and only jumps among several values. This makes it reasonable to model the fault process as a Markovian chain. Therefore, the whole system is a Markovian jump linear quantum system. To deal with this fault process, as well as the disturbance input that the dynamical squeezer itself suffers from, a coherent feedback controller is designed and connected to the plant directly without any measurement. After applying a controller to the plant, the whole closed system is shown in Fig. 2.

For the experimental system in [49], the round-trip length is 45 mm and the corresponding optical path length is 53 mm (including the length of crystal). One of the mirrors with a partial-reflection coating has power transmissivity \(T_1 = 14.6\%\) at 860 nm. The other two mirrors have power transmissivity \(T_2 = 0.02\%\) and \(T_3 = 0\), respectively. We then calculate the decay rates by using \(\kappa_i = \frac{T_i}{\tau}\), with \(\tau\) being the cavity round trip time as \(\kappa_1 = 0.8264, \kappa_2 = 0.0011\), and \(\kappa = 0.8275\). We here only consider the case that the system is acting as an amplifier. In this case \(\chi \leq \kappa\). In the numerical example, we take three different values \(\chi \in \{0.1\kappa, 0.2\kappa, 0.3\kappa\}\), which results in three modes for the system with

\[
A_1 = \begin{bmatrix}
-0.4551 & 0 \\
0 & -0.3724
\end{bmatrix}
\]
\[
A_2 = \begin{bmatrix}
-0.4965 & 0 \\
0 & -0.3310
\end{bmatrix}
\]
\[
A_3 = \begin{bmatrix}
-0.5379 & 0 \\
0 & -0.2896
\end{bmatrix}. \tag{62}
\]
We first consider the case where the transition rate matrix is known as
\[
\begin{bmatrix}
-0.02 & 0.01 & 0.01 \\
0.01 & -0.01 & 0 \\
0.01 & 0 & -0.01
\end{bmatrix}.
\]

By solving the LMIs in (33) and (34), we obtain the controller as
\[
\begin{align*}
A_1 &= \begin{bmatrix}
-1.7535 & 0 \\
0 & -2.1226
\end{bmatrix}, & B_1 &= \begin{bmatrix}
1.2524 & 0 \\
0 & 1.8944
\end{bmatrix} \\
C_1 &= \begin{bmatrix}
-0.0331 & 0 \\
0 & -0.0331
\end{bmatrix} \\
A_2 &= \begin{bmatrix}
-1.5796 & 0 \\
0 & -2.2738
\end{bmatrix}, & B_2 &= \begin{bmatrix}
0.9713 & 0 \\
0 & 2.2099
\end{bmatrix} \\
C_2 &= \begin{bmatrix}
-0.0331 & 0 \\
0 & -0.0331
\end{bmatrix} \\
A_3 &= \begin{bmatrix}
-1.3992 & 0 \\
0 & -2.4340
\end{bmatrix}, & B_3 &= \begin{bmatrix}
0.7024 & 0 \\
0 & 2.5600
\end{bmatrix} \\
C_3 &= \begin{bmatrix}
-0.0331 & 0 \\
0 & -0.0331
\end{bmatrix}.
\end{align*}
\]

(63)

Here, \(\{A_i, B_i, C_i\}\) are the parameters of the \(i\)th mode of the controller.

By using Proposition 7, it can be checked that the controller for each \(i = 1, 2, 3\) is not physically realizable without additional quantum inputs. Hence, we need to construct the matrices \(\mathcal{E}_{11}\) and \(\mathcal{E}_{12}\) in terms of the additional quantum inputs such that the controller \(\{A_i, B_i, C_i\}\) satisfies the physical realizability conditions, where \(\mathcal{B}_i = [B_{1i}, \mathcal{E}_{11}, \mathcal{E}_{12}]\). According to the algorithm in [58], we can calculate the input matrices as follows:
\[
\begin{align*}
\mathcal{E}_{11} &= \begin{bmatrix}
0.0331 & 0 \\
0 & 0.0331
\end{bmatrix}, & \mathcal{E}_{12} &= \begin{bmatrix}
1.2258 & 0 \\
0 & 1.2258
\end{bmatrix} \\
\mathcal{E}_{21} &= \begin{bmatrix}
0.0331 & 0 \\
0 & 0.0331
\end{bmatrix}, & \mathcal{E}_{22} &= \begin{bmatrix}
1.3057 & 0 \\
0 & 1.3057
\end{bmatrix} \\
\mathcal{E}_{31} &= \begin{bmatrix}
0.0331 & 0 \\
0 & 0.0331
\end{bmatrix}, & \mathcal{E}_{32} &= \begin{bmatrix}
1.4202 & 0 \\
0 & 1.4202
\end{bmatrix}.
\end{align*}
\]

(66)

Here, these control parameters imply that the controller at each mode \(i\) has three inputs, \(y, \nu_1,\) and \(\nu_2,\) and the corresponding input matrices are \(B_i, \mathcal{E}_{11},\) and \(\mathcal{E}_{12},\) respectively.

We further present an implementation of the controller, which should switch between different modes according to the plant. The structure diagram of the controller is shown in Fig. 3, where the left cavity with pump field \(\epsilon_1\) is the static squeezer with the total decay rate \(\kappa' = \kappa_1',\) while the right one is the general dynamical squeezer with the total decay rate as \(\kappa = \kappa_1 + \kappa_2 + \kappa_3.\) According to the relation between input and output of the static squeezer in (58), we first can obtain the input–output equation for the left static squeezer in Fig. 3:
\[
\begin{align*}
\begin{bmatrix}
y' + y_1' \\
y' - y_1'
\end{bmatrix} = & \begin{bmatrix}
-1 + \frac{\kappa' - \chi \kappa'}{(\frac{2}{3})^2 - \chi^2} & 0 \\
0 & -1 + \frac{\kappa' + \chi \kappa'}{(\frac{2}{3})^2 - \chi^2}
\end{bmatrix} \begin{bmatrix}
y + y_1 \\
y - y_1
\end{bmatrix}.
\end{align*}
\]

(67)

With (67), we can write its dynamical and output equations for the system in Fig. 3 as follows:
\[
\begin{align*}
d\begin{bmatrix}
\hat{a}(t) + \hat{a}^\dagger(t) \\
\hat{a}(t) - \hat{a}^\dagger(t)
\end{bmatrix}/dt & = \begin{bmatrix}
-\frac{\kappa}{2} + \chi & 0 \\
0 & -\frac{\kappa}{2} - \chi
\end{bmatrix} \begin{bmatrix}
\hat{a}(t) + \hat{a}^\dagger(t) \\
\hat{a}(t) - \hat{a}^\dagger(t)
\end{bmatrix} \\
+ \sqrt{\kappa_1} & \begin{bmatrix}
-1 + \frac{\kappa' - \chi \kappa'}{(\frac{2}{3})^2 - \chi^2} & 0 \\
0 & -1 + \frac{\kappa' + \chi \kappa'}{(\frac{2}{3})^2 - \chi^2}
\end{bmatrix} \begin{bmatrix}
y(t) + y_1(t) \\
y(t) - y_1(t)
\end{bmatrix} \\
+ \sqrt{\kappa_2} & d\begin{bmatrix}
\nu_1(t) + \nu_1^\dagger(t) \\
\nu_1(t) - \nu_1^\dagger(t)
\end{bmatrix} + \sqrt{\kappa_3} & d\begin{bmatrix}
\nu_2(t) + \nu_2^\dagger(t) \\
\nu_2(t) - \nu_2^\dagger(t)
\end{bmatrix}.
\end{align*}
\]

(68)
Fig. 3. Diagram for the controller composed of a static squeezer pumped by $\chi'$, an OPO pumped by $\chi$, and a phase shifter $\pi$.

Here, $\hat{a}(t)$ represents the annihilation operator of the fundamental field inside the dynamical OPO; $\kappa' = \kappa'_3$ and $\chi'$ are the total decay rate and the pumping dependent parameter of the static squeezer, respectively; while for the dynamical squeezer with pump field $\epsilon_2$, $\kappa = \kappa_1 + \kappa_2 + \kappa_3$, and $\chi$ is the pumping-dependent parameter. We should note that $\kappa_1, \kappa_2$, and $\kappa_3$ need to be changed for different modes of the controller. This can be realized by tunable mirrors. Usually, it can be achieved by making the mirror as a cavity itself and manipulating the decay rate by controlling the separation of the cavity [59]. Also, we can change the parameters $\chi$ and $\chi'$ by changing the pump field.

Comparing (63)–(66) and (68), we obtain

$$A_i = \begin{bmatrix} -\frac{\kappa}{2} - \chi & 0 \\ 0 & -\frac{\kappa}{2} + \chi \end{bmatrix}$$

$$B_i = \sqrt{\kappa_1} \begin{bmatrix} -1 + \frac{\epsilon_2 \kappa - \chi \kappa'}{(\epsilon_2^2 - \chi^2)} & 0 \\ 0 & -1 + \frac{\epsilon_2 \kappa' + \chi \kappa'}{(\epsilon_2^2 - \chi^2)} \end{bmatrix}$$

$$\mathcal{E}_{i1} = \sqrt{\kappa'_2}$$

$$\mathcal{E}_{i2} = \sqrt{\kappa'_3}.$$  \hspace{1cm} (69)

According to the parameters for different modes, we can calculate the corresponding parameters as follows:

**Mode 1**

$$\kappa = 3.8761, \chi = -0.1846$$

$$\kappa_1 = 2.3724, \kappa_2 = 0.0011, \kappa_3 = 1.5026$$

$$\kappa'_1 = 10$$

$$\chi' = 0.6237$$

**Mode 2**

$$\kappa = 3.8534, \chi = -0.3471$$

$$\kappa_1 = 2.1475, \kappa_2 = 0.0011, \kappa_3 = 1.7046$$

$$\kappa'_1 = 10$$

$$\chi' = 1.1953$$

**Mode 3**

$$\kappa = 3.8332, \chi = -0.5174$$

$$\kappa_1 = 1.7981, \kappa_2 = 0.0011, \kappa_3 = 2.0340$$

$$\kappa'_1 = 10$$

$$\chi' = 1.7650.$$

Here, we have used $\chi$ and $\kappa$ to represent the pumping coefficient and decay rate for the dynamical squeezer, and $\chi'$ and $\kappa'$ to represent the parameters for the static squeezer. In the static squeezer, we have assumed that only one mirror has a decay rate and the other two are ideal and fully reflective. The parameters of the squeezer also show that the total decay rate remains unchanged for the three modes of the controller and only the pumping field needs to be adjusted.

**Remark 9:** It should be noted that the controller designed above is mode dependent, which means the controller should switch with the modes of the plant. Mode-dependent controllers are also often designed in classical systems since the mode-independent one usually does not have satisfactory performance. In classical cases, the modes of the plant are often assumed to be known to the controller, and we have also made this assumption in quantum cases here. For the dynamical squeezer in Fig. 1, it is possible to know the modes of the plant by measuring the output of the pump field, which is shown as $B_{out}$. Since the pump field is an intense laser and treated in a classical way, measuring the output does not destroy any quantum information. Furthermore, since we have assumed that the parameter $\chi$ in system equation (53) is real, we only consider fluctuations in the amplitude of the pump field. Therefore, we can measure changes in the amplitude of $B_{out}$ without a reference laser.

**VI. CONCLUSION**

In this article, we have extended the sufficient and necessary conditions of physical realization to the time-varying linear quantum systems described by QSDEs, which plays an important role in designing a coherent feedback controller. Then, the $H^\infty$ control problem has been considered and related to Riccati differential equations. This makes it possible to design an $H^\infty$ controller by solving a group of LMIs. The physical realizability of the time-varying controller is then ensured by introducing additional quantum inputs and constructing the corresponding input matrices. For a dynamic squeezer used in quantum optical experiments, the fault signal of the pump laser has been recognized, and this results in a Markovian jump linear quantum system. A coherent feedback controller has been designed to bound the effect of disturbance input on the output even when the plant suffers from a fault signal. The physical realizability of the controller has been ensured theoretically, and it has also been implemented by some basic optical components, including squeezers, beamsplitters, and phase shifters. This article has assumed that the system has precisely known transit rate matrix and the designed controller is mode dependent. In some practical applications, we may only know estimated or partial values of the transit rate matrix. This leads to the control problem for
Markovian lumped systems with partly known or uncertain rate matrices, which needs to be further considered. Moreover, existence conditions for solutions to the LMIs in this article may be further considered to make the proposed control strategy more practical.

**APPENDIX A**

**Theorem 2.1** [28]: For the system (38), \( x_i(t), x_j(0) = 2i\Theta_{ij} \) implies \( x_i(t), x_j(t) = 2i\Theta_{ij} \) for all \( t \geq 0 \) if and only if \( iA + i\Theta A^T + i\Theta B^T = 0 \).

**APPENDIX B**

For the closed-loop quantum system (27), define \( Q(t) = \frac{1}{2} \langle \eta(t)\eta^T(t) + (\eta(t)\eta^T(t))^T \rangle \), where \( (\eta(t)\eta^T(t))^T = T \). Then \( Q(t) \) has the following properties:

\[
\frac{d}{dt} Q(t) = \tilde{A}_i Q(t) + Q(t) \tilde{A}_i^T + \eta_i \tilde{B}_i \eta_i^T dt + \tilde{B}_i \eta_i^T (\tilde{B}_i \eta_i^T) dt + \tilde{B}_i \eta_i^T (\tilde{A}_i \eta_0) dt + \tilde{B}_i \eta_i^T (\tilde{B}_i \eta_0) dt + \tilde{A}_i \eta_0 (\tilde{A}_i \eta_0) dt + \tilde{A}_i \eta_0 (\tilde{B}_i \eta_0) dt.
\]

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Yanan Liu received the master's degree in engineering from the Department of Automation, University of Science and Technology of China, Hefei, China, in 2017, and the Ph.D. degree in engineering from the University of New South Wales, Canberra, ACT, Australia, in 2021. She is currently a Postdoctoral Scholar with the Okinawa Institute of Science and Technology, Okinawa, Japan. Her current research interests include measurement-based quantum feedback control and coherent feedback control.

Daoyi Dong (Senior Member, IEEE) received the B.E. degree in automatic control and the Ph.D. degree in engineering from the University of Science and Technology of China, Hefei, China, in 2001 and 2006, respectively. He is currently a Senior Associate Professor at the University of New South Wales, Canberra, ACT, Australia. He was with the Institute of Systems Science, Chinese Academy of Sciences, Beijing, China, and with the Institute of Cyber-Systems and Control, Zhejiang University, Hangzhou, China. He has visiting positions with Princeton University, Princeton, NJ, USA, RIKEN, Wako-Shi, Japan, University of Duisburg-Essen, Duisburg, Germany, and the University of Hong Kong, Hong Kong. His research interests include quantum control and machine learning.

Dr. Dong was awarded an ACA Temasek Young Educator Award by the Asian Control Association. He is a recipient of an International Collaboration Award, a Humboldt Research Fellowship from the Alexander von Humboldt Foundation of Germany, and an Australian Post-Doctoral Fellowship from the Australian Research Council. He is an Associate Editor for IEEE TRANSACTIONS ON NEURAL NETWORKS AND LEARNING SYSTEMS.

Ian R. Petersen (Fellow, IEEE) was born in Victoria, Australia. He received the Ph.D. degree in electrical engineering from the University of Rochester, Rochester, NY, USA, in 1984.

From 1983 to 1985, he was a Postdoctoral Fellow with the Australian National University, Canberra, ACT, Australia. In 1985, he joined the University of New South Wales Canberra, Canberra. In 1987, he moved to The Australian National University, where he is currently a Professor with the Research School of Engineering.

He was the Australian Research Council Executive Director for Mathematics, Information and Communications in 2002 and 2003. He was an Acting Deputy Vice-Chancellor Research for the University of New South Wales in 2004 and 2005. His main research interests include robust control theory, quantum control theory, and stochastic control theory.

Dr. Petersen was an Associate Editor for IEEE TRANSACTIONS ON AUTOMATIC CONTROL, SYSTEMS AND CONTROL LETTERS, Automatica, and SIAM Journal on Control and Optimization. He is an Editor for Automatica, in the area of optimization in systems and control. He held an Australian Research Council Professorial Fellowship from 2005 to 2007, an Australian Research Council Federation Fellowship from 2007 to 2012, and an Australian Research Council Laureate Fellowship from 2012 to 2017. He was elected IFAC Council Member for the 2014–2017 Triennium. He was also elected a member of the IEEE Control Systems Society Board of Governors for the periods 2011–2013 and 2015–2017. He is the Vice-president for Technical Activity for the Asian Control Association and was General Chair of the 2012 Australia Control Conference. He was General Chair of the 2015 IEEE Multi-Conference on Systems and Control. He is a Fellow of the International Federation of Automatic Control and the Australian Academy of Science.

Qing Gao (Senior Member, IEEE) received the B.Eng. and Ph.D. degrees in mechanical and electrical engineering from the University of Science and Technology of China, Hefei, China, in 2002 and 2013, respectively, and the second Ph.D. degree in mechatronics engineering from the City University of Hong Kong, Hong Kong, in 2014.

From 2014 to 2016, he was a Postdoctoral Research Associate with the School of Engineering and Information Technology, University of New South Wales Canberra at the Australian Defence Force Academy, Canberra, ACT, Australia. Since 2018, he has been a Full Professor with the School of Automation Science and Electrical Engineering, Beihang University, Beijing, China. His research interests include intelligent systems and control theory.

Dr. Gao is the recipient of the Alexander von Humboldt Fellowship of Germany and the 21st Guan Zhao-Zhi Award at the 34th Chinese Control Conference.

Steven X. Ding received the Ph.D. degree in electrical engineering from the Gerhard-Mercator University of Duisburg, Duisburg, Germany, in 1992.

From 1992 to 1994, he was a R&D Engineer with Rheinmetall GmbH, Munich, Germany. From 1995 to 2001, he was a Professor of Control Engineering and the Head of the Institute of Automatic Control and Complex Systems, University of Duisburg-Essen, Duisburg, Germany. His research interests include model-based and data-driven fault diagnosis, control, and fault-tolerant systems as well as their applications in industry with a focus on automotive systems, chemical processes, and renewable energy systems.

Shota Yokoyama received the B.E., M.E., and Ph.D. degrees in engineering from the Department of Applied Physics, University of Tokyo, Tokyo, Japan, in 2010, 2012, and 2015, respectively.

From 2015 to 2017, he was a Visiting Fellow with the School of Engineering and Information Technology, University of New South Wales, Canberra, ACT, Australia, where he has been a Research Associate since 2017. His current research interests include experimental quantum optics, quantum information, and quantum computation.

Dr. Yokoyama was a Postdoctoral Fellow for Research Abroad at the Japan Society for the Promotion of Science from 2015 to 2017.

Hidehiro Yonezawa received the B.E., M.E., and Ph.D. degrees in engineering from the Department of Applied Physics, University of Tokyo, Tokyo, Japan, in 2002, 2004, and 2007, respectively.

He was a Research Associate from 2007 to 2009 and a Project Assistant Professor from 2009 to 2013 with the University of Tokyo. He is currently a Senior Lecturer with the University of New South Wales, Canberra, ACT, Australia. Since 2015, he has been a Co-Program Manager with the Centre for Quantum Computation and Communication Technology, Australian Research Council, Canberra. His research interests include experimental quantum optics, quantum information, and quantum control.