\textbf{m-almost everywhere convergence of intuitionistic fuzzy observables induced by Borel measurable function}

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Abstract: In paper [4] we studied the upper and the lower limits of sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state $m$ for a definition the notion of almost everywhere convergence. We compared two concepts of $m$-almost everywhere convergence. The aim of this paper is to show the connection between almost everywhere convergence in classical probability space induced by Kolmogorov construction and $m$-almost everywhere convergence in intuitionistic fuzzy space. We studied the sequence of intuitionistic fuzzy observables induced by Borel measurable function.

Keywords: Intuitionistic fuzzy event, Intuitionistic fuzzy observable, Intuitionistic fuzzy state, Joint intuitionistic fuzzy observable, Product, Upper limit, Lower limit, $m$-almost everywhere convergence, Function of several intuitionistic fuzzy observables, Borel measurable function, Kolmogorov construction.

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1 Introduction

In [1–3] K. T. Atanassov introduced the notion of intuitionistic fuzzy sets. Then in [7] B. Riečan defined the intuitionistic fuzzy state on the family of intuitionistic fuzzy events

$$\mathcal{F} = \{(\mu_A, \nu_A) : \mu_A + \nu_A \leq 1\Omega\},$$
where $\mu_A, \nu_A$ are $\mathcal{S}$-measurable functions, $\mu_A, \nu_A : \Omega \to [0, 1]$, as a mapping $\mathfrak{m}$ from the family $\mathcal{F}$ to the set $R$ by the formula

$$\mathfrak{m}(\mu_A, \nu_A) = (1 - \alpha) \int_{\Omega} \mu_A dP + \alpha \left(1 - \int_{\Omega} \nu_A dP\right),$$

where $\mu : \mathcal{S} \to [0, 1]$ is a probability measure and $\alpha \in [0, 1]$.

In paper [4] we defined the upper and the lower limits for sequence of intuitionistic fuzzy observables. We used an intuitionistic fuzzy state $\mathfrak{m}$ for a definition the notion of almost everywhere convergence. We compared two concepts of $\mathfrak{m}$-almost everywhere convergence.

In this paper we study the $\mathfrak{m}$-almost everywhere convergence of sequence of intuitionistic fuzzy observables $g_n(x_1, \ldots, x_n) : \mathcal{B}(R) \to \mathcal{F}$ given by

$$g_n(x_1, \ldots, x_n) = h_n \circ g_n^{-1},$$

where $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ is the joint intuitionistic fuzzy observable of intuitionistic fuzzy observables $x_1, \ldots, x_n$ and $g : R^n \to R$ is a Borel measurable function. We show the connection between $\mathfrak{m}$-almost everywhere convergence of this sequence of intuitionistic fuzzy observables and $P$-almost everywhere convergence of random variables in classical probability space induced by Kolmogorov construction.

**Remark.** Note that in a whole text we use a notation “IF” in short as the phrase “intuitionistic fuzzy.”

2 **IF-events, IF-states and IF-observables**

First we start with definitions of basic notions.

**Definition 2.1.** Let $\Omega$ be a nonempty set. An IF-set $A$ on $\Omega$ is a pair $(\mu_A, \nu_A)$ of mappings $\mu_A, \nu_A : \Omega \to [0, 1]$ such that $\mu_A + \nu_A \leq 1_{\Omega}$.

**Definition 2.2.** Start with a measurable space $(\Omega, \mathcal{S})$. Hence $\mathcal{S}$ is a $\sigma$-algebra of subsets of $\Omega$. An IF-event is called an IF-set $A = (\mu_A, \nu_A)$ such that $\mu_A, \nu_A : \Omega \to [0, 1]$ are $\mathcal{S}$-measurable.

The family of all IF-events on $(\Omega, \mathcal{S})$ will be denoted by $\mathcal{F}$, $\mu_A : \Omega \to [0, 1]$ will be called the membership function, $\nu_A : \Omega \to [0, 1]$ be called the non-membership function.

If $A = (\mu_A, \nu_A) \in \mathcal{F}$, $B = (\mu_B, \nu_B) \in \mathcal{F}$, then we define the Łukasiewicz binary operations $\oplus, \odot$ on $\mathcal{F}$ by

$$A \oplus B = ((\mu_A + \mu_B) \land 1_{\Omega}, (\nu_A + \nu_B - 1_{\Omega}) \lor 0_{\Omega}),$$

$$A \odot B = ((\mu_A + \mu_B - 1_{\Omega}) \lor 0_{\Omega}, (\nu_A + \nu_B) \land 1_{\Omega})$$

and the partial ordering is then given by

$$A \leq B \iff \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$
In paper we use max-min connectives defined by

\[ A \lor B = (\mu_A \lor \mu_B, \nu_A \land \nu_B), \]
\[ A \land B = (\mu_A \land \mu_B, \nu_A \lor \nu_B) \]

and the de Morgan rules

\[ (a \lor b)^* = a^* \land b^*, \]
\[ (a \land b)^* = a^* \lor b^*, \]

where \( a^* = 1 - a \).

**Example 2.3.** Fuzzy set \( f : \Omega \rightarrow [0, 1] \) can be regarded as IF-set, if we put

\[ A = (f, 1 - f, \Omega - f). \]

If \( f = \chi_A \), then the corresponding IF-set has the form

\[ A = (\chi_A, 1 - \chi_A) = (\chi_A, \chi_A'). \]

In this case \( A \oplus B \) corresponds to the union of sets, \( A \odot B \) to the intersection of sets and \( \leq \) to the set inclusion.

In the IF-probability theory \([7, 9]\) instead of the notion of probability, we use the notion of state.

**Definition 2.4.** Let \( \mathcal{F} \) be the family of all IF-events in \( \Omega \). A mapping \( m : \mathcal{F} \rightarrow [0, 1] \) is called an IF-state, if the following conditions are satisfied:

(i) \( m((1_{\Omega}, 0_\Omega)) = 1 \), \( m((0_\Omega, 1_\Omega)) = 0 \);

(ii) if \( A \odot B = (0_\Omega, 1_\Omega) \) and \( A, B \in \mathcal{F} \), then \( m(A \oplus B) = m(A) + m(B) \);

(iii) if \( A_n \nearrow A \) (i.e. \( \mu_{A_n} \nearrow \mu_A \), \( \nu_{A_n} \searrow \nu_A \)), then \( m(A_n) \nearrow m(A) \).

Probably the most useful result in the IF-state theory is the following representation theorem \([7]\):

**Theorem 2.5.** To each IF-state \( m : \mathcal{F} \rightarrow [0, 1] \) there exists exactly one probability measure \( P : \mathcal{S} \rightarrow [0, 1] \) and exactly one \( \alpha \in [0, 1] \) such that

\[ m(A) = (1 - \alpha) \int_{\Omega} \mu_A dP + \alpha \left( 1 - \int_{\Omega} \nu_A dP \right) \]

for each \( A = (\mu_A, \nu_A) \in \mathcal{F} \).

The third basic notion in the probability theory is the notion of an observable. Let \( \mathcal{J} \) be the family of all intervals in \( R \) of the form

\[ [a, b) = \{ x \in R : a \leq x < b \}. \]

Then the \( \sigma \)-algebra \( \sigma(\mathcal{J}) \) is denoted \( \mathcal{B}(R) \) and it is called the \( \sigma \)-algebra of Borel sets, its elements are called Borel sets.
Definition 2.6. By an IF-observable on \( \mathcal{F} \) we understand each mapping \( x : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F} \) satisfying the following conditions:

(i) \( x(\mathbb{R}) = (1_\Omega, 0_\Omega), x(\emptyset) = (0_\Omega, 1_\Omega) \):

(ii) if \( A \cap B = \emptyset \), then \( x(A) \odot x(B) = (0_\Omega, 1_\Omega) \) and \( x(A \cup B) = x(A) \oplus x(B) \);

(iii) if \( A_n \nearrow A \), then \( x(A_n) \nearrow x(A) \).

If we denote \( x(A) = (x^\flat(A), 1_\Omega - x^\sharp(A)) \) for each \( A \in \mathcal{B}(\mathbb{R}) \), then \( x^\flat, x^\sharp : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{T} \) are observables, where \( \mathcal{T} = \{ f : \Omega \rightarrow [0, 1]; f \text{ is } \mathcal{S} \text{-measurable} \} \).

Remark 2.7. Sometimes we need to work with \( n \)-dimensional IF-observable \( x : \mathcal{B}(\mathbb{R}^n) \rightarrow \mathcal{F} \) defined as a mapping with the following conditions:

(i) \( x(\mathbb{R}^n) = (1_\Omega, 0_\Omega), x(\emptyset) = (0_\Omega, 1_\Omega) \):

(ii) if \( A \cap B = \emptyset, A, B \in \mathcal{B}(\mathbb{R}^n) \), then \( x(A) \odot x(B) = (0_\Omega, 1_\Omega) \) and \( x(A \cup B) = x(A) \oplus x(B) \);

(iii) if \( A_n \nearrow A \), then \( x(A_n) \nearrow x(A) \) for each \( A, A_n \in \mathcal{B}(\mathbb{R}^n) \).

If \( n = 1 \) we simply say that \( x \) is an IF-observable.

Similarly to the classical case, the following theorem can be proved ([9]).

Theorem 2.8. Let \( x : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{F} \) be an IF-observable, \( m : \mathcal{F} \rightarrow [0, 1] \) be an IF-state. Define the mapping \( m_x : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1] \) by the formula

\[
m_x(C) = m(x(C)).
\]

Then \( m_x : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1] \) is a probability measure.

3 Product operation, joint IF-observable and function of several IF-observables

In [5] we introduced the notion of product operation on the family of IF-events \( \mathcal{F} \) and showed an example of this operation.

Definition 3.1. We say that a binary operation \( \cdot \) on \( \mathcal{F} \) is product if it satisfying the following conditions:

(i) \( (1_\Omega, 0_\Omega) \cdot (a_1, a_2) = (a_1, a_2) \) for each \( (a_1, a_2) \in \mathcal{F} \);

(ii) the operation \( \cdot \) is commutative and associative;
Definition 3.3. \textbf{IF-observables} $x, y$ \textit{(1, Theorem 1)}

The joint IF-observable of IF-observables $x, y$ is a mapping $h : B(R) \rightarrow F$ satisfying the following conditions:

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A, B \in B(R^2)$ and $A \cap B = \emptyset$, then

$$h(A \cup B) = h(A) \oplus h(B) \text{ and } h(A) \odot h(B) = (0_\Omega, 1_\Omega);$$

(iii) if $A, A_1, \ldots \in B(R^2)$ and $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ for each $C, D \in B(R)$.

Theorem 3.2 ([5, Theorem 1]). \textit{The operation} defined by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$$

\textit{for each} $(x_1, y_1), (x_2, y_2) \in F$ is product operation on $F$.

In [8] B. Riečan defined the notion of a joint IF-observable and proved its existence.

Definition 3.3. \textit{Let} $x, y : B(R) \rightarrow F$ \textit{be two IF-observables. The joint IF-observable of the IF-observables} $x, y$ \textit{is a mapping} $h : B(R^2) \rightarrow F$ \textit{satisfying the following conditions:}

(i) $h(R^2) = (1_\Omega, 0_\Omega)$, $h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) if $A, B \in B(R^2)$ and $A \cap B = \emptyset$, then

$$h(A \cup B) = h(A) \oplus h(B) \text{ and } h(A) \odot h(B) = (0_\Omega, 1_\Omega);$$

(iii) if $A, A_1, \ldots \in B(R^2)$ and $A_n \nearrow A$, then $h(A_n) \nearrow h(A)$;

(iv) $h(C \times D) = x(C) \cdot y(D)$ for each $C, D \in B(R)$.

Theorem 3.4 ([8, Theorem 3.3]). \textit{For each two IF-observables} $x, y : B(R) \rightarrow F$ \textit{there exists their joint IF-observable.}

Remark 3.5. \textit{The joint IF-observable of IF-observables} $x, y$ \textit{from Definition 3.3 is a two-dimensional IF-observable.}

If we have several IF-observables and a Borel measurable function, we can define the IF-observable, which is the function of several IF-observables. Regarding this we provide the following definition.

Definition 3.6. \textit{Let} $x_1, \ldots, x_n : B(R) \rightarrow F$ \textit{be IF-observables,} $h_n$ \textit{their joint IF-observable and} $g_n : R^n \rightarrow R$ \textit{a Borel measurable function. Then we define the IF-observable} $g_n(x_1, \ldots, x_n) : B(R) \rightarrow F$ \textit{by the formula}

$$g_n(x_1, \ldots, x_n)(A) = h_n(g_n^{-1}(A)).$$
4 Lower and upper limits, m-almost everywhere convergence

In [4] we defined the notions of lower and upper limits for a sequence of IF-observables and showed the connection between two kinds of m-almost everywhere convergence.

Definition 4.1. We shall say that a sequence \((x_n)_n\) of IF-observables has \(\limsup\) if there exists an IF-observable \(\overline{x} : \mathcal{B}(R) \to \mathcal{F}\) such that
\[
\overline{x}((-\infty, t)) = \bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, t - \frac{1}{p}\right)\right)
\]
for every \(t \in R\). We write \(\overline{x} = \limsup_{n \to \infty} x_n\).

Note that if another IF-observable \(y\) satisfies the above condition, then \(m \circ y = m \circ \overline{x}\).

Definition 4.2. A sequence \((x_n)_n\) of IF-observables has \(\liminf\) if there exists an IF-observable \(\underline{x}\) such that
\[
\underline{x}((-\infty, t)) = \bigwedge_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigvee_{n=k}^{\infty} x_n \left(\left(-\infty, t - \frac{1}{p}\right)\right)
\]
for all \(t \in R\). Notation: \(\underline{x} = \liminf_{n \to \infty} x_n\).

Theorem 4.3 ([4, Theorem 3.3]). The IF-observables \(\overline{x}, \underline{x}\) from Definition 4.1 and Definition 4.2 can be expressed in the following form
\[
\overline{x}(A) = \left(\overline{x}^\flat(A), 1_\Omega - \overline{x}^\sharp(A)\right),
\]
\[
\underline{x}(A) = \left(\underline{x}^\flat(A), 1_\Omega - \underline{x}^\sharp(A)\right),
\]
for each \(A \in \mathcal{B}(R)\). Here \(\overline{x}^\flat, \overline{x}^\sharp\) are upper and lower limits of sequence \((x_n^\flat)_n\) of observables in tribe \(T\) and \(\underline{x}^\flat, \underline{x}^\sharp\) are upper and lower limits of sequence \((x_n^\sharp)_n\) of observables in tribe \(T\) (see [6]).

Proposition 4.1 ([4, Proposition 3.1]). If a sequence of IF-observables \((x_n)_n\) has \(\overline{x} = \limsup_{n \to \infty} x_n\) and \(\underline{x} = \liminf_{n \to \infty} x_n\), then
\[
\overline{x}((-\infty, t)) \leq \underline{x}((-\infty, t)),
\]
for every \(t \in R\).

Proposition 4.2 ([4, Proposition 4.1]). A sequence \((x_n)_n\) of an IF-observables converges m-almost everywhere to 0 if and only if
\[
\mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}\left(\bigvee_{p=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n \left(\left(-\infty, t - \frac{1}{p}\right)\right)\right) = \mathbf{m}(0_F((-\infty, t))),
\]
for every \(t \in R\).
In accordance to Proposition 4.2 we can extend the notion of $m$-almost everywhere convergence in the following way.

**Definition 4.4.** A sequence $(x_n)_n$ of an IF-observables converges $m$-almost everywhere to an IF-observable $x$, if

$$m\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(\begin{array}{c} -\infty, t - \frac{1}{p} \end{array}\right)\right)\right) = m\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} x_n\left(\left(\begin{array}{c} -\infty, t - \frac{1}{p} \end{array}\right)\right)\right) = m(x((-\infty, t)))$$

for every $t \in \mathbb{R}$.

### 5 $P$-almost everywhere convergence and $m$-almost everywhere convergence

The main result of this section is given in Theorem 5.1. The main step is presented in the following proposition.

Recall, that the corresponding probability space is $(\mathbb{R}^N, \sigma(C), P)$, where $C$ is the family of all sets of the form

$$\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\},$$

and $P$ is the probability measure determined by the equality

$$P(\{(t_i)_{i=1}^{\infty} : t_1 \in A_1, \ldots, t_n \in A_n\}) = m(x_1(A_1) \cdot \ldots \cdot x_n(A_n)).$$

The corresponding projections $\xi_n : \mathbb{R}^N \to \mathbb{R}$ are defined by the equality

$$\xi_n((t_i)_{i=1}^{\infty}) = t_n.$$

**Proposition 5.1.** Let $(x_n)_n$ be a sequence of IF-observables, $(\xi_n)_n$ the sequence of corresponding projections, $g_n : \mathbb{R}^n \to \mathbb{R}$ be a Borel measurable functions ($n = 1, 2, \ldots$). Then

$$P\left(\{u \in \mathbb{R}^N : \limsup_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\} \right) \leq m\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\left(\left(\begin{array}{c} -\infty, t - \frac{1}{p} \end{array}\right)\right)\right),$$

$$P\left(\{u \in \mathbb{R}^N : \liminf_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\} \right) \geq m\left(\bigvee_{p=1}^{\infty} \bigvee_{k=1}^{\infty} \bigwedge_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\left(\left(\begin{array}{c} -\infty, t - \frac{1}{p} \end{array}\right)\right)\right).$$
Proof. We have
\[ P\left(\{u \in R^N : \limsup_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\}\right) = \]
\[ = P\left(\bigcup_{p=1}^{\infty} \bigcap_{k=1}^{\infty} \bigcap_{n=1}^{\infty} \left\{ u \in R^N : g_n(u_1, \ldots, u_n) < t - \frac{1}{p} \right\} \right) = \]
\[ = \lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} P\left( \bigcap_{n=k}^{k+i} \left( \bigcap_{n=k}^{k+i} g_n^{-1}\left(\left[ -\infty, t - \frac{1}{p} \right) \right) \right) \right) = \]
\[ = \lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} P\left( \bigcap_{n=k}^{k+i} \left( \bigcap_{n=k}^{k+i} g_n^{-1}\left(\left[ -\infty, t - \frac{1}{p} \right) \right) \right) \right) \leq \]
\[ \leq \lim_{p \to \infty} \lim_{k \to \infty} \lim_{i \to \infty} m\left( \bigcup_{n=k}^{k+i} g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t - \frac{1}{p} \right) \right) \right) = \]
\[ = m\left( \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t - \frac{1}{p} \right) \right) \right). \]

The second inequality can be proved similarly. \( \square \)

Theorem 5.1. Let \((x_n)_n\) be a sequence of IF-observables, \((\xi_n)_n\) be the sequence of corresponding projections, \((g_n)_n\) be a sequence of Borel measurable functions \(g_n : R^n \to R\). If the sequence \((g_n(\xi_1, \ldots, \xi_n))_n\) converges \(P\)-almost everywhere, then the sequence \((g_n(x_1, \ldots, x_n))_n\) converges \(m\)-almost everywhere and
\[ m\left( \lim_{n \to \infty} \sup g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t \right) \right) \right) = m\left( \lim_{n \to \infty} \inf g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t \right) \right) \right) \]
for each \(t \in R\). Moreover
\[ P\left(\{u \in R^N : \limsup_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\}\right) = m\left( \lim_{n \to \infty} \sup g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t \right) \right) \right) \]
for each \(t \in R\).

Proof. Let the sequence \((g_n(\xi_1, \ldots, \xi_n))_n\) converges \(P\)-almost everywhere, then
\[ P\left(\{u \in R^N : \limsup_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\}\right) = \]
\[ = P\left(\{u \in R^N : \liminf_{n \to \infty} g_n(\xi_1(u), \ldots, \xi_n(u)) < t\}\right). \]
(1)

Put
\[ \varphi(t) = \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t - \frac{1}{p} \right) \right), \]
\[ \psi(t) = \bigcup_{p=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcup_{n=1}^{\infty} g_n(x_1, \ldots, x_n)\left(\left[ -\infty, t - \frac{1}{p} \right) \right), \]
and

$$
\eta_n(u) = g_n(\xi_1(u), \ldots, \xi_n(u)).
$$

Since \( \varphi(t) \leq \psi(t) \), then

$$
m(\varphi(t)) \leq m(\psi(t)).
$$

By Proposition 5.1 and (1) we obtain

$$
m(\psi(t)) \leq P\left( \{ u \in \mathbb{R}^N : \lim_{n \to \infty} \eta_n(u) < t \} \right) = m(\varphi(t)).
$$

Hence

$$
m(\varphi(t)) = m(\psi(t)),
$$

and moreover

$$
P\left( \{ u \in \mathbb{R}^N : \lim_{n \to \infty} \sup_{n} \eta_n(u) < t \} \right) = m(\varphi(t)) = m(\psi(t)).
$$

Denote the common value by

$$
F(t) = m\left( \lim_{n \to \infty} \bigcap_{n=1}^{\infty} \varphi(-n) \right) = m(\varphi(t)) = m(\psi(t)).
$$

Since \( \limsup_{n \to \infty} \eta_n \) is a random variable, then \( F : \mathbb{R} \to [0,1] \) is a distribution function. Evidently

$$
m\left( \bigcup_{n=1}^{\infty} \varphi(n) \right) = \lim_{n \to \infty} m(\varphi(n)) = \lim_{n \to \infty} F(n) = 1,
$$

$$
m\left( \bigcap_{n=1}^{\infty} \varphi(-n) \right) = \lim_{n \to \infty} m(\varphi(-n)) = \lim_{n \to \infty} F(-n) = 0.
$$

Since \( m \) is faithful, we obtain

$$
\bigcup_{n=1}^{\infty} \varphi(n) = (1, 0], \quad \bigcap_{n=1}^{\infty} \varphi(-n) = (0, 1]. \quad (2)
$$

Let \( t_n \nearrow t \). Evidently \( \varphi(t_n) \leq \varphi(t) \), hence

$$
\bigcup_{n=1}^{\infty} \varphi(n) \leq \varphi(t).
$$

On the other hand to each \( p \in N \) there exist \( j, q \in N \) such that \( (\infty, t - \frac{1}{p}) \subset (\infty, t_j - \frac{1}{q}) \), hence

$$
g_n(x_1, \ldots, x_n)\left( (\infty, t - \frac{1}{p}) \right) \leq g_n(x_1, \ldots, x_n)\left( (\infty, t_j - \frac{1}{q}) \right)
$$

and therefore

$$
\bigcap_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\left( (\infty, t - \frac{1}{p}) \right) \leq \bigcap_{n=k}^{\infty} g_n(x_1, \ldots, x_n)\left( (\infty, t_j - \frac{1}{q}) \right) \leq \varphi(t_j) \leq \bigcup_{j=1}^{\infty} \varphi(t_j).
$$
Since the relation holds for every $k$ and $p$, we obtain
\[
\varphi(t) = \bigvee_{p=1}^{\infty} \bigwedge_{n=k}^{\infty} g_n(x_1, \ldots, x_n) \left( \left( -\infty, t - \frac{1}{p} \right) \right) \leq \bigvee_{j=1}^{\infty} \varphi(t_j).
\]
Hence
\[
\varphi(t) = \bigvee_{j=1}^{\infty} \varphi(t_j). \tag{3}
\]
Let us summarize: $\varphi$ is non-decreasing, $\bigvee_{n=1}^{\infty} \varphi(n) = (1_\Omega, 0_\Omega)$, $\bigwedge_{n=1}^{\infty} \varphi(-n) = (0_\Omega, 1_\Omega)$, $t_n \nearrow t$ implies $\varphi(t_n) \nearrow \varphi(t)$.

Put $y_n = g_n(x_1, \ldots, x_n)$. Using max-min connectives $\vee$, $\wedge$, the De Morgan rules and equality $y_n(A) = (y_n^\uparrow(A), 1_\Omega - y_n^\downarrow(A))$ we obtain
\[
\bigvee_{p=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n(A) = \left( \bigvee_{p=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\uparrow(A), 1_\Omega - \bigvee_{p=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\downarrow(A) \right)
\]
for each $A \in \mathcal{B}(R)$, where $y_n^\uparrow$, $y_n^\downarrow : \mathcal{B}(R) \to \mathcal{T}$ are observables. Put
\[
\varphi^\uparrow(t) = \bigvee_{p=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\uparrow \left( \left( -\infty, t - \frac{1}{p} \right) \right),
\]
\[
\varphi^\downarrow(t) = \bigvee_{p=1}^{\infty} \bigwedge_{n=k}^{\infty} y_n^\downarrow \left( \left( -\infty, t - \frac{1}{p} \right) \right).
\]
Then
\[
\varphi(t) = (\varphi^\uparrow(t), 1_\Omega - \varphi^\downarrow(t)).
\]
Since $\varphi$ is non-decreasing, therefore $\varphi^\uparrow$, $\varphi^\downarrow$ are non-decreasing. More by (2) we have
\[
(1_\Omega, 0_\Omega) = \bigvee_{n=1}^{\infty} \varphi(n) = \bigvee_{n=1}^{\infty} (\varphi^\uparrow(n), 1_\Omega - \varphi^\downarrow(n)) = \left( \bigvee_{n=1}^{\infty} \varphi^\uparrow(n), 1_\Omega - \bigvee_{n=1}^{\infty} \varphi^\downarrow(n) \right),
\]
\[
(0_\Omega, 1_\Omega) = \bigwedge_{n=1}^{\infty} \varphi(-n) = \bigwedge_{n=1}^{\infty} (\varphi^\uparrow(-n), 1_\Omega - \varphi^\downarrow(-n)) = \left( \bigwedge_{n=1}^{\infty} \varphi^\uparrow(-n), 1_\Omega - \bigwedge_{n=1}^{\infty} \varphi^\downarrow(-n) \right).
\]
Hence
\[
\bigvee_{n=1}^{\infty} \varphi^\uparrow(n) = 1_\Omega, \quad \bigwedge_{n=1}^{\infty} \varphi^\downarrow(-n) = 0_\Omega, \tag{4}
\]
\[
\bigvee_{n=1}^{\infty} \varphi^\uparrow(n) = 1_\Omega, \quad \bigwedge_{n=1}^{\infty} \varphi^\downarrow(-n) = 0_\Omega. \tag{5}
\]
Since $t_n \nearrow t$ implies
\[
(\varphi^\uparrow(t_n), 1_\Omega - \varphi^\downarrow(t_n)) = \varphi(t_n) \nearrow \varphi(t) = (\varphi^\uparrow(t), 1_\Omega - \varphi^\downarrow(t)),
\]
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then
\[ \varphi^\flat(t_n) \nearrow \varphi^\flat(t), \quad 1_\Omega - \varphi^\flat(t_n) \searrow 1_\Omega - \varphi^\flat(t) \iff \varphi^\flat(t_n) \nearrow \varphi^\flat(t). \] (6)

For fixed \( \omega \in \Omega \) and arbitrary \( t \in \mathbb{R} \) put
\[ F^\flat_\omega(t) = \varphi^\flat(t)(\omega), \quad F^\sharp_\omega(t) = \varphi^\sharp(t)(\omega). \]

Evidently \( F^\flat_\omega, F^\sharp_\omega : \mathbb{R} \to [0, 1] \) are the non-decreasing functions and by (4), (6) and by (5), (7) we obtain that \( F^\flat_\omega, F^\sharp_\omega \) are the distribution functions. Denote by \( \lambda^\flat_\omega, \lambda^\sharp_\omega \) the corresponding Stieltjes probability measures and define \( \overline{\gamma}^\flat, \overline{\gamma}^\sharp : \mathcal{B}(\mathbb{R}) \to \mathcal{T} \) by the equalities
\[ \overline{\gamma}^\flat(A)(\omega) = \lambda^\flat_\omega(A), \quad \overline{\gamma}^\sharp(A)(\omega) = \lambda^\sharp_\omega(A). \]

Then \( \overline{\gamma}^\flat, \overline{\gamma}^\sharp \) are the observables and
\[ \overline{\gamma}^\flat((\infty, t])(\omega) = \lambda^\flat_\omega((\infty, t]) = F^\flat_\omega(t) = \varphi^\flat(t)(\omega), \]
\[ \overline{\gamma}^\sharp((\infty, t])(\omega) = \lambda^\sharp_\omega((\infty, t]) = F^\sharp_\omega(t) = \varphi^\sharp(t)(\omega), \] (8) (9)

for each \( \omega \in \Omega \). Using Theorem 4.3 and (8), (9) we have that there exists IF-observable \( \overline{y} = \lim \sup_{n \to \infty} y_n \) given by
\[ \overline{y}((\infty, t]) = \left( \overline{\gamma}^\flat((\infty, t]), 1_\Omega - \overline{\gamma}^\sharp((\infty, t]) \right) = (\varphi^\flat(t), 1_\Omega - \varphi^\sharp(t)) = \varphi(t). \]

The existence of IF-observable \( \underline{y} = \lim \inf_{n \to \infty} y_n = \psi \) can be proved similarly. Since \( \mathbf{m}(\varphi(t)) = \mathbf{m}(\overline{y}(t)) \), then
\[ \mathbf{m}\left( \overline{\gamma}((\infty, t]) \right) = \mathbf{m} \left( \overline{y}((\infty, t]) \right) \]
for each \( t \in \mathbb{R} \) and by Definition 4.4 we have that \( (y_n)_n = (g_n(x_1, \ldots, x_n))_n \) converges \( \mathbf{m} \)-almost everywhere. Moreover
\[ P\left( \{ u \in \mathbb{R}^N : \lim \sup_{n \to \infty} \eta_n(u) < t \} \right) = \mathbf{m} \left( \overline{\gamma}((\infty, t]) \right) \]
for each \( t \in \mathbb{R} \).

\( \square \)

6 Conclusion

The Theorem 5.1 is important for the proof of the Individual ergodic theorem in intuitionistic fuzzy case, where we work with the sequence of several IF-observables induced by the Borel function.
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