Gravity, Stability and Energy Conservation on the Randall-Sundrum Brane-World

Misao Sasaki\(^1,4\), Tetsuya Shiromizu\(^2,4,5\) and Kei-ichi Maeda\(^3,6\)

\(^1\)Department of Earth and Space Science, Graduate School of Science, Osaka University, Toyonaka 560-0043, Japan

\(^2\)DAMTP, University of Cambridge
Silver Street, Cambridge CB3 9EW, United Kingdom

\(^3\)Isaac Newton Institute, University of Cambridge,
20 Clarkson Road, Cambridge CB3 0EH, United Kingdom

\(^4\)Department of Physics, The University of Tokyo, Tokyo 113-0033, Japan

\(^5\)Research Centre for the Early Universe(RESCEU),
The University of Tokyo, Tokyo 113-0033, Japan

\(^6\)Department of Physics, Waseda University, Shinjuku, Tokyo 169-8555, Japan

We carefully investigate the gravitational perturbation of the Randall-Sundrum (RS) single brane-world solution [hep-th/9906064], based on a covariant curvature tensor formalism recently developed by us. Using this curvature formalism, it is known that the 'electric' part of the 5-dimensional Weyl tensor, denoted by \(E_{\mu\nu}\), gives the leading order correction to the conventional Einstein equations on the brane. We consider the general solution of the perturbation equations for the 5-dimensional Weyl tensor caused by the matter fluctuations on the brane. By analyzing its asymptotic behaviour in the direction of the 5th dimension, we find the curvature invariant diverges as we approach the Cauchy horizon. However, in the limit of asymptotic future in the vicinity of the Cauchy horizon, the curvature invariant falls off fast enough to render the divergence harmless to the brane-world. We also obtain the asymptotic behavior of \(E_{\mu\nu}\) on the brane at spatial infinity, assuming the matter perturbation is localized. We find it falls off sufficiently fast and will not affect the conserved quantities at spatial infinity. This indicates strongly that the usual conservation law, such as the ADM energy conservation, holds on the brane as far as asymptotically flat spacetimes are concerned.

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I. INTRODUCTION

A recent discovery by Randall and Sundrum of the exact solution that describes Minkowski branes in the 5-dimensional anti-de Sitter space [1] has attracted much attention. In particular, in their second paper [2], they showed that a single Minkowski brane solution is possible if the brane has positive tension, and opened up the possibility of dimensional reduction without compactifying extra dimensions. Subsequently, a various aspects of, and variants of the RS solution have been discussed by many authors. Among them are work on non-linear plane-waves [3], on black strings and black cigars [4,5], on the AdS/CFT correspondence [6,7], on cosmological solutions [8–16], on the stability of the RS solution [17] and on the quantum creation of the brane world [18](see also [19,20]).

Meanwhile, we formulated a covariant set of equations that describes both the 5-dimensional gravity and the 4-dimensional gravity on the brane [21]. With \(Z_2\)-symmetry, which is expected to hold from an \(M\)-theoretic point of view [22,23], we found the negative tension brane-world would not be allowed since it would be a world of anti-gravity.
Then Garriga and Tanaka showed that the negative tension brane is a world of a negative Brans-Dicke parameter at the linear perturbation order [17].

On the other hand, we found a positive tension brane has the correct sign of gravity, and the equations reduce to the conventional Einstein equations in the low energy limit, provided that the extra term due to the ‘electric’ part of the 5-dimensional Weyl tensor, $E_{\mu\nu}$, is negligible [21]. Thus it is urgently important to clarify the effect of $E_{\mu\nu}$ to the brane-world. Part of this program was first done by Randall and Sundrum themselves [2] and more rigorously by Garriga and Tanaka [17], and they showed that the effect is small at large distances from the source. However, this conclusion was obtained only for static perturbations. Furthermore, since they adopted the metric perturbation formalism, the asymptotic behavior of the curvature perturbations in 5 dimensions as the Cauchy horizon is approached was not straightforward to see.

In this paper, we carefully investigate the first order perturbation of the RS brane-world in terms of the 5-dimensional Weyl tensor. We focus on the single brane model. We first briefly review our curvature tensor formalism. Then we derive the evolution equation for $E_{\mu\nu}$, and give an expression for the retarded Green function with appropriate boundary condition on the brane. We recover the result obtained in [2,17] on the brane for static sources. For general spacetime dependent sources, we evaluate the asymptotic behavior of the Weyl tensor. We find the Weyl curvature diverges away. This infinite redshift effect was suggested by Chamblin and Gibbons [3]. Then we discuss the energy conservation on the brane. Local energy-momentum conservation is guaranteed by the 4-dimensional covariance. We find globally conserved energy exists as well, just as in the conventional Einstein gravity, for asymptotically flat spacetimes.

II. THE EFFECTIVE EINSTEIN EQUATIONS ON THE BRANE

In this section, we briefly review the effective gravitational equations on the brane and the equations for the 5-dimensional Weyl tensor which were derived in [21].

We consider a 5-dimensional spacetime with negative vacuum energy but otherwise vacuum,

$$G_{\mu\nu} = \kappa_5^2 T_{\mu\nu}; \quad T_{\mu\nu} = \Lambda g_{\mu\nu},$$

and a brane in this spacetime as a fixed point of the $Z_2$-symmetry. $\kappa_5$ is the 5-dimensional gravitational coupling constant ($\kappa_5^2 = 8\pi G_5$). We assume the form of the metric,

$$ds^2 = (\eta_{\mu\nu} + q_{\mu\nu}) dx^\mu dx^\nu = d\chi^2 + q_{\mu\nu} dx^\mu dx^\nu,$$

where $n_\mu dx^\mu = d\chi$ is unit normal to the $\chi =$constant hypersurfaces, one of which corresponds to the brane, and $q_{\mu\nu}$ is the induced metric on the $\chi =$constant hypersurfaces. Then, thanks to the $Z_2$-symmetry, the effective 4-dimensional gravitational equations on the brane take a form that resembles the conventional Einstein equations:

$$(4)G_{\mu\nu} = -\Lambda_4 q_{\mu\nu} + 8\pi G_N \tau_{\mu\nu} + \kappa_5^4 \pi_{\mu\nu} - E_{\mu\nu},$$

where

$$\Lambda_4 = \frac{1}{2} \kappa_5^2 \left( \Lambda + \frac{1}{6} \kappa_5^2 \lambda^2 \right),$$

$$G_N = \frac{\kappa_5^2 \lambda}{48\pi},$$

$$\pi_{\mu\nu} = -\frac{1}{4} \tau_{\mu\alpha} \tau^\alpha_{\nu} + \frac{1}{12} \tau_{\mu\nu} + \frac{1}{8} q_{\mu\nu} \tau_{\alpha\beta} \tau^{\alpha\beta} - \frac{1}{24} q_{\mu\nu} \tau^2,$$

$$E_{\mu\nu} = C_{\mu\nu\alpha\beta} n^\alpha n^\beta$$

and $C_{\mu\nu\alpha\beta}$ is the 5-dimensional Weyl tensor. $\lambda$ is the vacuum energy on the brane and gives the brane tension. $\tau_{\mu\nu}$ is the energy-momentum tensor of the matter on the brane. It should be noted that $E_{\mu\nu}$ here is not the one exactly on the brane (which is proportional to the delta function), but the one that is evaluated by taking the limiting value to the brane. But for simplicity, we call it $E_{\mu\nu}$ on the brane in the rest of the paper. It can be evaluated from either side of the brane due to $Z_2$-symmetry.

As $G_N$, which corresponds to the 4-dimensional Newton constant, has the same sign as $\lambda$, we assume a positive tension brane to obtain the conventional gravity on the brane. In the case of a two brane model with positive and
negative tensions, it has been argued that it is possible to recover the normal gravity if the distance between the two branes is fine-tuned \(^{[15]}\) and if some mechanism to stabilize the distance is introduced \(^{[13]}\). In this paper, however, we focus on a single brane model and assume \(\lambda > 0\).

The difference of Eq. (3) from the Einstein gravity is the presence of \(\pi_{\mu \nu}\) and \(E_{\mu \nu}\) on the right-hand side. With \(\kappa_5\) and \(\Lambda\) of a very high energy scale, it is easy to see that \(\pi_{\mu \nu}\) can be safely neglected in the low energy limit. On the other hand, since \(E_{\mu \nu}\) comes from the 5-dimensional Weyl tensor, there is no a priori reason to expect that it is small even in the low energy limit. In fact, in the two-brane model, it is this part that contributes dominantly to recover Einstein gravity on the negative tension brane when the distance between the two branes is fine-tuned \(^{[17]}\). We note that because of the contracted Bianchi identities, we have

\[
\bar{D}_\mu E_{\mu \nu} = \kappa_4^2 D^\mu \pi_{\mu \nu},
\]

where \(D_\mu\) is the covariant differentiation with respect to \(q_{\mu \nu}\). Hence, in the low energy limit when we can neglect \(\pi_{\mu \nu}, E_{\mu \nu}\) is transverse-traceless with respect to \(q_{\mu \nu}\).

The effective gravitational equations on the brane (3) are not closed but one must solve the gravitational field in the bulk at the same time. The 5-dimensional equations in the bulk we have to solve are

\[
\mathcal{L}_n E_{\alpha \beta} = D^\mu B_{\mu (\alpha \beta)} + \frac{1}{6} \kappa_5^2 \Lambda (K_{\alpha \beta} - q_{\alpha \beta} K) + K_{\mu \nu \alpha \beta} R_{\mu \nu \alpha \beta} + 3K_{\mu (\alpha \beta) \mu} - KE_{\alpha \beta} + (K_{(\mu \nu} K_{\beta \nu)} - K_{\alpha \beta} K_{\mu \nu}) K_{\mu \nu},
\]

(9)

\[
\mathcal{L}_n B_{\mu \nu \alpha \beta} = -2D_{[\mu} E_{\nu\alpha \beta]} + K_{\alpha \beta} B_{\mu \nu \sigma} - 2B_{(\alpha \beta |\mu \nu \sigma]},
\]

(10)

\[
\mathcal{L}_n (4) R_{\mu \nu \alpha \beta} = -2(4) R_{\mu \nu \sigma [\alpha K_{\beta \sigma]} - 2D_{[\mu} B_{|\beta \alpha \beta |\nu]},
\]

(11)

where \(K_{\mu \nu} = (1/2) \mathcal{L}_n q_{\mu \nu}\) and \(B_{\mu \nu \alpha \beta} = q_{\mu \nu} q_{\alpha \beta} C_{\rho \sigma \alpha \beta \kappa \lambda}.\) These equations are derived from the 5-dimensional Bianchi identities \(^{[21]}\).

The equation for \(E_{\mu \nu}\) has an alternative form that will be more convenient for later use;

\[
\mathcal{L}_n E_{\alpha \beta} = D^\mu B_{\mu (\alpha \beta)} + K^{\mu \nu \alpha \beta} C_{\mu \nu \alpha \beta} + 4K^\mu_{(\alpha} E_{\beta) \mu} - \frac{3}{2} K E_{\alpha \beta} - \frac{1}{2} q_{\alpha \beta} K_{\mu \nu} E_{\mu \nu} + 2 \tilde{K}_{\mu \nu} \tilde{K}_{\alpha \beta} - \frac{7}{6} \tilde{K}_{\mu \nu} \tilde{K}_{\alpha \beta},
\]

(12)

where \(\tilde{K}_{\alpha \beta}\) is the traceless part of \(K_{\alpha \beta},\)

\[
\tilde{K}_{\alpha \beta} = K_{\alpha \beta} - \frac{1}{4} q_{\alpha \beta} K_{\mu \nu}.
\]

Equations (3) (or (12)), (10) and (11) are to be solved under the boundary condition at the brane,

\[
D^\mu E_{\mu \nu \mid \text{brane}} = \kappa_4^2 D^\mu \pi_{\mu \nu},
\]

(14)

\[
B_{\mu \nu \alpha \beta \mid \text{brane}} = 2D_{[\mu} K_{\nu \alpha \beta]} = -\kappa_5^2 D_{[\mu} (\tau_{\nu |\alpha - \frac{1}{3} q_{\nu |\alpha}}),
\]

(15)

where we used the expression of \(K_{\mu \nu}\) which is obtained by the Israel junction condition and \(Z_2\)-symmetry,

\[
K_{\mu \nu \mid \text{brane}} = -\frac{1}{6} \kappa_5^2 \lambda_{\mu \nu} - \frac{1}{2} \kappa_5^2 \left(\tau_{\mu \nu} - \frac{1}{3} q_{\mu \nu} \tau\right).
\]

(16)

III. THE FIRST ORDER PERTURBATION AROUND THE RS-BRANE WORLD

In this section, we derive the equations for the first order perturbation around the RS solution \(^{[2]}\). As we have noted in the previous section, we focus on the case of a single brane with positive tension.
A. The perturbation equations

The background bulk spacetime is taken to be the anti-de Sitter spacetime whose metric is

\[ ds^2 = d\chi^2 + e^{-2\chi/\ell} \eta_{ij} dx^i dx^j = \ell^2 \left( d\tau^2 + \frac{1}{\ell^2} \eta_{ij} dx^i dx^j \right), \]  

(17)

where \( \ell = \sqrt{-6/\Lambda}, \) \( z := e^{\chi/\ell}, \) \( i = 0, 1, 2, 3 \) and \( \eta_{ij} \) is the metric of the Minkowski spacetime, \( \eta_{ij} dx^i dx^j = -dt^2 + dx^2. \) \( \chi = \infty \) or \( z = \infty \) is the Cauchy horizon. The RS solution is obtained by putting a brane on a surface \( z = z_* \) and gluing two identical copies of the region \( z \geq z_* \) of the AdS \( \text{AdS}^5 \). Since the coordinate \( z \) is scale-free, we may set \( z_* = 1 \) without loss of generality. This solution is obtained from Eq. (8) by putting \( \Lambda_4 = 0 \) and \( \tau_{\mu\nu} = E_{\mu\nu} = 0. \)

We now consider the first order perturbation of the RS solution. We assume \( \tau_{\mu\nu} \) is a small quantity of \( \epsilon \) and solve the perturbation linear in \( \epsilon \). Hence, we have

\[ (4) G_{\mu\nu} = 8\pi G_N \tau_{\mu\nu} - E_{\mu\nu} + O(\epsilon^2) \]  

(18)

on the brane.

In the first order of \( \tau_{\mu\nu} \), the extrinsic curvature \( K_{\mu\nu} \) can be written as

\[ K_{\alpha\beta} = -\frac{1}{\ell} \eta_{\alpha\beta} + k_{\alpha\beta}, \]  

(19)

where \( k_{\alpha\beta} \) is \( O(\epsilon) \). The Lie derivative of \( k_{\mu\nu} \) with respect to \( n^\nu \) is just \( E_{\mu\nu} \) apart from the signature,

\[ \mathcal{L}_n k_{\alpha\beta} = k_{\alpha}{}^\mu k_{\beta\mu} - E_{\alpha\beta} = -E_{\alpha\beta} + O(\epsilon^2). \]  

(20)

Using Eq. (19), Eq. (12) simplifies to

\[ \mathcal{L}_n E_{\alpha\beta} = D^\mu B_{\mu(\alpha\beta)} + \frac{2}{\ell} E_{\alpha\beta} + O(\epsilon^2). \]  

(21)

On the other hand, Eq. (10) reduces to

\[ \mathcal{L}_n B_{\mu\nu\alpha} = -2D_{[\mu} E_{\nu]\alpha} + O(\epsilon^2). \]  

(22)

Combining these two equations, we obtain the wave equation for \( E_{\mu\nu} \) in 5 dimensions,

\[ \left[ \mathcal{L}_n \left( \mathcal{L}_n + K_0 \right) + D^2 + \frac{4}{\ell^2} \right] E_{\alpha\beta} = \left[ \Box_5 + \frac{4}{\ell^2} \right] E_{\alpha\beta} = 0, \]  

(23)

where \( \Box_5 \) is the 5-dimensional d’Alembertian. In the coordinates \((z, x^\mu)\), this becomes

\[ \left[ \partial_z^2 - \frac{3}{z} \partial_z + \frac{1}{z^2} \partial_4^2 + \frac{4}{z^2} \right] E_{\alpha\beta} = 0, \]  

(24)

where \( \Box_4 \) is the 4-dimensional d’Alembertian. Substituting Eq. (15) to Eq. (23), we obtain the boundary condition of \( E_{\mu\nu} \) on the brane,

\[ \partial_z \left( \frac{E_{\mu\nu}}{z^2} \right)_{z=1} = -\frac{1}{2} \kappa_5^2 \ell \left[ D^2 \tau_{\mu\nu} + \frac{1}{3} D_{\mu} D_{\nu} \tau - \frac{1}{3} q_{\mu\nu} D^2 \tau \right]. \]  

(25)

In addition to the above, since we have \( D^\mu E_{\mu\nu} = O(\epsilon^2) \) on the brane, and \( \mathcal{L}_n D^\mu E_{\mu\nu} = O(\epsilon^2) \), \( E_{\mu\nu} \) is transverse-traceless with respect to the 4-metric \( q_{\mu\nu} \) everywhere in the bulk.

Once \( E_{\mu\nu} \) is solved, we can integrate Eqs. (14) and (11) to obtain \( B_{\alpha\beta\mu} \) and \( (4) C_{\alpha\beta\mu\nu} \). To the first order, these are expressed in the coordinates \((z, x^\mu)\) as

\[ \partial_z B_{\mu\nu\alpha} = -\frac{2\ell}{z} \partial_\mu E_{\nu}\alpha \]  

(26)

\[ \partial_z (z^2 (4) C_{\nu\mu\alpha\beta}) = \ell z \left[ \partial_\nu B_{\rho(\mu}\alpha} \eta_{\beta\rho} - \partial_\rho B_{\nu(\alpha} \eta_{\beta\rho]} \right] - 2\ell z \partial_\nu B_{(\alpha\beta]\nu} \]  

(27)

The above equations will be used when we evaluate the asymptotic behaviour of the Weyl curvature near the Cauchy horizon. To discuss the behaviour near the Cauchy horizon, we consider an invariant quantity defined by the 5-dimensional Weyl tensor,

\[ C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = (4) C_{\mu\nu\alpha\beta} (4) C^{\mu\nu\alpha\beta} + 6 E_{\mu\nu} E^{\mu\nu} + 4 B_{\mu\nu\alpha} B^{\mu\nu\alpha} + O(\epsilon^3). \]  

(28)
B. The retarded Green function

The general solution of \( E_{\mu\nu} \) can be expressed in terms of the Green function satisfying

\[
(\Box + 4/\ell^2) G(z, x) = -\delta^5(z, x). \tag{29}
\]

As usual, we assume there is no incoming waves from past Cauchy horizon. Hence we consider the retarded Green function. In addition to this causal boundary condition, we have one more boundary condition at the brane. Because of the boundary condition for \( E_{\mu\nu} \), Eq. (24), we must impose the corresponding condition on the Green function,

\[
\partial_z \left( \frac{G}{z^2} \right)_{z=1} = 0. \tag{30}
\]

With the Green function satisfying this condition, \( E_{\mu\nu} \) is given by

\[
E_{\mu\nu}(z, x) = \int_{z' = 1} d^4x' (-g(x'))^{1/2} g^{zz}(x') \left[ \partial_z G(z, x; z', x') E_{\mu\nu}(x') - G(z, x; z', x') \partial_z E_{\mu\nu}(x') \right]
\]

\[
= - \int_{z' = 1} d^4x' (-g(x'))^{1/2} g^{zz}(x') G(z, x; z', x') \partial_z \left( \frac{E_{\mu\nu}(x')}{z'^2} \right)
\]

\[
= \frac{\kappa^2}{2} \int_{z' = 1} d^4x' G(z, x; z', x') \left( \Box_4 \gamma_{\mu\nu} + \frac{1}{3} \eta_{\mu\nu} \Box_4 \tau - \frac{1}{3} \eta_{\mu\nu} \Box_4 \tau(x') \right). \tag{31}
\]

For bounded sources, which we are mainly interested in, the last line of the above equation can be re-expressed as

\[
E_{\mu\nu}(z, x) = \frac{\kappa^2}{2} \left[ \delta_\mu^\beta \delta_\nu^\alpha \Box_4 + \frac{1}{3} \eta^{\alpha\beta} (\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box_4) \right] \int_{z' = 1} d^4x' G(z, x; z', x') \gamma_\alpha_\beta(x'). \tag{32}
\]

Now let us construct the retarded Green function. The general form of a mode function satisfying Eq. (23) is

\[
u_{m,p}(z, x) = N_m z^2 (J_0(mz) + b_m N_0(mz)) \frac{e^{-i\omega_{pm} t + i\vec{p} \cdot \vec{x}}}{(2\pi)^{3/2} \sqrt{2\omega_{pm}}}, \tag{33}\]

where \( \omega_{pm} = \sqrt{\vec{p}^2 + m^2/\ell^2} \) and \( N_m \) is a normalization constant. The boundary condition (23) requires that the mode functions should satisfy (30), which determines \( b_m \) as

\[
\begin{align*}
\frac{J_1}{N_1} & = - \frac{J_1}{N_1}(m).
\end{align*} \tag{34}\]

Then \( N_m \) is fixed by requiring that the mode functions should be properly normalized with respect to the Klein-Gordon norm. This gives

\[
N_m = \sqrt{\frac{m^{1/2}}{\sqrt{1 + b_m^2}}}. \tag{35}\]

Using the mode functions obtained above, the retarded Green function is constructed as

\[
G_R(z, x; z', x')|_{z' = 1} = -\frac{2z^2}{\pi \ell} \int_0^\infty dm \frac{N_1(m) J_0(mz) - J_1(m) N_0(mz)}{N_1(m)^2 + J_1(m)^2} g_m(x, x') \tag{36}\]

where \( g_m(x, x') \) is the 4-dimensional Green function given by

\[
\begin{align*}
g_m(x, x') & = \int d\omega d^3p \frac{e^{-i\omega(t-t') + i\vec{p} \cdot \vec{x} - \vec{x}'}}{(2\pi)^4 (m/\ell)^2 + \vec{p}^2 - (\omega + i\epsilon)^2}.
\end{align*} \tag{37}\]
C. Static case

First let us consider the static case. In this case, the Green function is given by the integration of the retarded Green function over the time. Then the 4-dimensional part gives rise to the factor $e^{-mr/\ell}$. Hence the integration with respect to $m$ is dominated by the contribution of small $m$, and we obtain

$$G(z, x; 1, x') = \int_{-\infty}^{\infty} dt \frac{G_R(z, t, x; 1, 0, x') \simeq \frac{\ell z^2}{4\pi \sqrt{r^2 + (\ell z)^2}}}{r^3},$$

(38)

where $r = |x - x'|$.

On the brane the far field approximation gives

$$G(1, x; 1, x') \simeq \frac{\ell}{4\pi r^3}.$$  

(39)

From Eq. (32), we obtain the leading order of $E_{\mu\nu}$ as

$$E_{00} \simeq \frac{4G_N \ell^2}{\pi} \int d^3x \left( \frac{3}{2} P \right),$$

$$E_{ij} \simeq \frac{1}{3} \delta_{ij} E_{00}, \quad E_{0i} \simeq 0,$$

(40)

where we assumed the perfect fluid form, $\tau_{\mu\nu} = p u_{\mu} u_{\nu} + P(q_{\mu\nu} + u_{\mu} u_{\nu})$, for the energy momentum tensor. The result is consistent with those obtained in Refs. [2,17].

For $z \gg 1$, that is, near the Cauchy horizon, the static Green function is approximately given by

$$G(z, x; 1, x') \simeq \frac{1}{4\pi \ell^2 z} \left[ 1 - \frac{3}{2} \left( \frac{r}{\ell z} \right)^2 \right].$$

(41)

Hence we obtain the asymptotic behaviour as

$$E_{\mu\nu} = O(\partial^2 G) = O(z^{-3}), \quad \partial_\mu E_{\alpha\beta} = O(z^{-5}).$$

(42)

With the help of Eqs. (26) and (27), the above gives

$$B_{\mu\nu\alpha} = O(z^{-5}), \quad (4) C_{\mu\nu\alpha\beta} = O(z^{-7}).$$

(43)

Therefore we have

$$E_{\mu\nu} E^{\mu\nu} = O(z^{-2}), \quad B_{\mu\nu\alpha} B^{\mu\nu\alpha} = O(z^{-4}), \quad (4) C_{\mu\nu\alpha\beta}(4) C^{\mu\nu\alpha\beta} = O(z^{-6}),$$

(44)

from which we obtain the estimate,

$$C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = O(z^{-2}).$$

(45)

This means that the perturbation remains regular at the Cauchy horizon for the static case.

D. General case

We now turn to the general case of time-dependent sources. We first note that the 4-dimensional Green function, $g_m(x, x')$, can be evaluated exactly as

$$g_m(x, x') = \frac{1}{2\pi} \theta(t - t') \frac{\partial}{\partial s^2} \left[ \theta(s^2) J_0 \left( m \frac{|s|}{\ell} \right) \right]$$

(46)

where $s^2 = (t - t')^2 - r^2$ and $r = |x - x'|$. Inserting this expression to Eq. (38), we find the part that contains the derivative of $\theta(s^2)$ (hence is proportional to $\delta(s^2)$) will not contribute. This is because this part of $g_m$ becomes independent of $m$, hence the integration with respect to $m$ gives $\delta(z - 1)$, but $z = 1$ is outside the domain of definition of $E_{\mu\nu}$ (recall that $E_{\mu\nu}$ on the brane is actually defined by taking the limit to the brane). Thus we have
\[
G_R(z, x; z', x')|_{z' = 1} = -\frac{z^2}{\pi^2 \ell^2} \theta(t-t') \theta(s^2) \frac{\partial}{\partial s^2} \int_0^\infty dm \frac{N_1(m)J_0(mz) - J_1(m)N_0(mz)}{N_1(m^2) + J_1(m^2)} J_0 \left( \frac{m|s|}{\ell} \right).
\] (47)

To evaluate the asymptotic behavior of \( G_R \), we may approximate the above integral by assuming \( m \gg 1 \). Physically this is because only the high frequency modes (in the 5-dimensional sense) can reach the null infinity, which corresponds to the Cauchy horizon in the present case. Since \( z > 1 \) we can then use the asymptotic form of the Bessel functions except the function \( J_0(m|s|) \); we keep it as it is since we do not want to restrict the range of \( s^2 \).

Thus, we obtain
\[
G_R(z, x; z', x')|_{z' = 1} \simeq -\frac{\theta(t-t') \theta(s^2 z^{3/2})}{\pi^2 \ell^2} \frac{\partial}{\partial s^2} \int_0^\infty dm \cos[m(z-1)] J_0 \left( \frac{m|s|}{\ell} \right)
\]
\[
= \frac{\theta(t-t') \theta(s^2) \theta(s^2 - \ell^2(z-1)^2)}{2\pi^2 s^2 - \ell^2(z-1)^2} z^{3/2}.
\] (48)

On the brane \( (z = 1 + 0) \), the Green function becomes
\[
G_R(1, x; 1, x') \simeq \frac{\theta(s^2) \theta(t-t')}{2\pi^2 s^3},
\] (49)

which is in accordance with a naive expectation. This implies the behaviour of \( E_{\mu\nu} \) in the far-field region as
\[
E_{\mu\nu} \sim O(\partial^2 G_R) = O(s^{-5}).
\] (50)

This should be compared with the energy momentum tensor of a radiative field, \( \tau_{\mu\nu} \sim s^{-2} \) at null infinity. Thus we conclude that \( E_{\mu\nu} \) cannot carry away the energy momentum from a system to infinity. We will come back to this point in the next section.

To investigate the asymptotic behavior near the Cauchy horizon, it is more convenient to work in the null coordinates, \( u = s - \ell(z-1) \) and \( v = s + \ell(z+1) \). Then with the help of Eqs. (20) and (24), we find
\[
E_{\mu\nu} = O(\partial^2 s G_R) = O\left( \frac{(v-u)^{3/2}}{(vu)^{5/2}} \right),
\]
\[
B_{\mu\nu\alpha} = O(\partial\alpha E_{\mu\nu}) = O(\partial^3 s G_R) = O\left( \frac{(v+u)(v-u)^{3/2}}{(vu)^{7/2}} \right),
\]
\[
(4) C_{\mu\nu\alpha\beta} = O(\partial\alpha B_{\mu\nu\beta}) = O(\partial^4 s G_R) = O\left( \frac{(v-u)^{3/2}}{(vu)^{7/2}} \right).
\] (51)

We take the limit \( v \to \infty \) along \( u = \text{constant} \) to approach the Cauchy horizon. Then \( E_{\mu\nu} = O(v^{-1}) \), \( B_{\mu\nu\alpha} = O(v^{-1}) \) and \( (4) C_{\mu\nu\alpha\beta} = O(v^{-2}) \). Therefore, the curvature invariant \((5)\) diverges in this limit as
\[
C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = O(v^4).
\] (52)

This means that the Cauchy horizon is unstable to the perturbation. The bad behaviour of the Weyl tensor, however, does not necessarily imply an instability of the brane-world. It will not affect the brane if the divergence disappears in the limit of distant future; \( u = \alpha v \to \infty (\alpha = \text{const.} < 1) \). This is the infinite redshift effect. In the present case, if we take this limit, we find \( C_{\mu\nu\alpha\beta} C^{\mu\nu\alpha\beta} = O(u^{-3}) \). Hence the brane-world is unaffected by the instability of the Cauchy horizon.

**IV. THE ENERGY CONSERVATION ON THE BRANE**

In the conventional Einstein gravity, the ADM energy is conserved if the energy-momentum tensor \( \tau_{\mu\nu} \) decays faster than \( \sim r^{-3} \) towards spatial infinity. For the matter source of compact support, this condition is trivially satisfied.

In the present case, we have \( E_{\mu\nu} \), which is a part of the 5-dimensional Weyl curvature, in addition to the matter energy-momentum tensor. However, as we have seen in the previous section, although the 4-geometry on the brane is affected by \( E_{\mu\nu} \), the effect seems to be quite subtle: First, \( E_{\mu\nu} \) itself is locally conserved at the linear order, independent of \( \tau_{\mu\nu} \). Second, both the static and dynamic perturbations fall off sufficiently rapidly at large distances from the source. So we expect that the conservation of the total energy holds also in the brane-world, provided the brane geometry is asymptotically flat.
To re-confirm this expectation, in this section we present a more detailed discussion of the energy conservation. A brief review of the asymptotic structure at spatial infinity is given in Appendix A. The following argument is based on a recently developed formalism of the conformal infinity \cite{25,26}, which is much easier to deal with than the old formalism \cite{24}.

First we express the asymptotic behavior of the 4-dimensional Ricci tensor as

\[ L_{\mu\nu} = (4)R_{\mu\nu} - \frac{1}{6}g_{\mu\nu}(4)R \simeq L_{\mu\nu}^{(0)} + L_{\mu\nu}^{(1)}r^{-1} + L_{\mu\nu}^{(2)}r^{-2} + L_{\mu\nu}^{(3)}r^{-3} + O\left(\frac{1}{r^4}\right). \]

The total energy of a system is naturally defined by the electric part of the 4-dimensional Weyl tensor, \((4)E_{\mu\nu} = (4)C_{\mu\nu\alpha\beta}r^\alpha r^\beta\), because it is the one that describes the tidal force. \(r^\alpha\) is the unit vector of the radial direction. By a conformal transformation, one can expand the spatial infinity to introduce the structure of a unit 3-dimensional timelike hyperboloid. We denote the metric and the covariant derivative of it by \(g_{\mu\nu}\) and \(\nabla_\mu\), respectively. Then, assuming \(L_{\mu\nu}^{(0)} = L_{\mu\nu}^{(1)} = L_{\mu\nu}^{(2)} = 0\), the leading order behavior of \((4)E_{\mu\nu}\) for \(r \to \infty\) can be expressed as

\[ (4)E_{\mu\nu} \simeq \frac{1}{r}\left[ \frac{1}{2}(\mathcal{D}_\mu \mathcal{D}_\nu + p_{\mu\nu})F^{(1)} - \frac{1}{2}(L_{\mu\nu}^{(3)} + p_{\mu\nu}L_{\alpha\beta}^{(3)}r^\alpha r^\beta) \right] + O\left(\frac{1}{r^2}\right), \]

where \(F^{(1)}\) is a function on the unit 3-dimensional timelike hyperboloid. In the present case, we have contributions from not only \(\tau_{\mu\nu}\) but also \(E_{\mu\nu}\) to the Ricci tensor. From the analysis in the previous section at the spatial infinity, \(E_{\mu\nu} = O(r^{-5})\). Hence, if the matter source is localized, we have \(L_{\mu\nu}^{(n)} = 0\) for \(n = 0, 1, 2, 3\). Thus,

\[ (4)E_{\mu\nu} \simeq \frac{1}{2r}(\mathcal{D}_\mu \mathcal{D}_\nu + p_{\mu\nu})F^{(1)} + O\left(\frac{1}{r^2}\right), \]

Since \((4)E_{\mu\nu}\) is traceless, \((1)F\) satisfies

\[ (\mathcal{D}^2 + 3)F^{(1)} = 0. \]

Using this and introducing \((4)\tilde{E}_{\mu\nu} = r^{(4)}E_{\mu\nu}\), we can show

\[ \mathcal{D}^{\mu}(4)\tilde{E}_{\mu\nu} = 0. \]

Now we define the ADM energy by

\[ E_{\text{ADM}} := -\frac{1}{8\pi} \int_C dS^{\mu\nu} \epsilon_{\mu\nu\alpha\beta} \tilde{E}^{\alpha\beta} t_\beta. \]

\textbf{V. SUMMARY AND DISCUSSION}

We have carefully investigated the asymptotic behavior of the linear perturbation around the single brane solution of the RS brane-world scenario \cite{3}. It is known that the effect of the 5-dimensional curvature appears in the form of \(E_{\mu\nu}\), which is the ‘electric’ part of the 5-dimensional Weyl tensor, on the right-hand side of the effective Einstein equations on the brane \cite{23}. To deal with this part of the Weyl tensor directly, we have employed the curvature tensor perturbation formalism developed in \cite{24} instead of the metric perturbation formalism. For static perturbations, we have found that \(E_{\mu\nu}\) on the brane falls off as \(r^{-5}\) from the source and the Weyl curvature in the bulk behaves regularly near the Cauchy horizon \((z = \infty)\) as \(C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = O(z^{-2})\), in accordance with the previous result obtained in the metric perturbation formalism \cite{24,25}.

For generic perturbations, we have found the perturbation diverges as \(C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = O(u^4)\) near the Cauchy horizon \((v = \infty, u = \text{const.})\). This implies instability of the Cauchy horizon. However, we have found that this instability does not affect the brane-world because \(C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta} = O(u^{-3})\) for \(u = \alpha v \to \infty\) \((\alpha < 1)\) due to the infinite redshift effect. This supports the conjecture by Chamblin and Gibbons \cite{3} that the Cauchy horizon instability would not affect the brane-world.
We have also found that $E_{\mu\nu}$ decays rapidly as $s^{-5}$ on the brane for generic spacetime-dependent perturbations, where $s^2 = t^2 - r^2$. This is to be contrasted with the case of a radiative field (like electromagnetic radiation) that behaves like $s^{-2}$ (To avoid the confusion, a radiation field behaves like $\sim r^{-4}$ at spatial infinity). This implies $E_{\mu\nu}$ does not carry energy away from the system to infinity. Together with the fact that $E_{\mu\nu} = O(r^{-5})$ for static perturbations, we have concluded that the ADM energy is well-defined also in the brane-world and it is conserved.

One remaining issue is the positivity of the ADM energy. In the linear perturbation, the right-hand side of the effective Einstein equations has the additional term, $-E_{\mu\nu}$. If $-E_{\mu\nu}u^\mu u^\nu > 0$, where $u^\mu$ is an arbitrary timelike vector, the brane will be stable in the semi-classical level. In this connection, recent work on cosmological solutions that $-E_{00}$ is proportional to the mass of the 5-dimensional Schwarzshild-AdS spacetime [10–12] implies it is positive. On the contrary, our result of the perturbation analysis given in Eq. (40) has the opposite sign. One might suspect that our result is incorrect. However, there is a good reason to believe that our result is indeed correct. Remember that the ADM energy is guaranteed or not within the present perturbation analysis. Some global non-linear analysis will be necessary to resolve this issue.

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**APPENDIX A: ASYMPTOTIC FLATNESS AT SPATIAL INFINITY**

We briefly describe the definition of asymptotic flatness and some useful results obtained in 4-dimensional asymptotically flat spacetimes (see Refs. [27][28] for the details and exact descriptions). In this appendix the notation basically follows Ref. [27], which is slightly different from the main text of this paper. Below the suffix $a, b, ...$ denotes the abstract index $\mathcal{E}$.

**Definition:** A spacetime $(M, q_{ab})$ will be said to be asymptotically flat at spacelike infinity $\mathcal{I}$ if there exists a smooth function $\Omega$ satisfying the following features (i), (ii) and the energy momentum tensor satisfies the fall off condition (iii):

(i) $\Omega|_{\mathcal{I}} = 0$ and $d\Omega|_{\mathcal{I}} \neq 0$.

(ii) The following quantities have smooth limits on $\mathcal{I}$.

\[
\begin{align*}
    r^a &= \Omega^{-4} g^{ab}D_b \Omega, \\
    \hat{p}_{ab} &= \Omega^2 (q_{ab} - F^{-1} \Omega^{-4} D_a \Omega D_b \Omega) = \Omega^2 p_{ab},
\end{align*}
\]

where $F = \mathcal{L}_r \Omega$.

(iii) $T_{\mu\nu} := T_{ab} (e_{\mu}^a)(e_{\nu}^b) = O(\Omega^4)$ near $\mathcal{I}$, where $\{(e_{\mu}^a)\}_{\mu = 0, 1, 2, 3}$ is a tetrad of the metric $q_{ab}$.

In this formalism, $\Omega \sim r^{-1}$. For example, the extrinsic curvature of the $\Omega = \text{const.}$ hypersurface is written as

\[
\kappa_{ab} := \frac{1}{2} \mathcal{L}_r p_{ab} = \Omega^{-1} F^{1/2} \hat{p}_{ab} - \frac{1}{2} F^{-1/2} \mathcal{L}_r \hat{p}_{ab},
\]

where $\hat{r}^a := r^a/\sqrt{\hat{p}_{ab} \hat{p}^{ab}}$. From the 4-dimensional Einstein equations under the condition (i), (ii) and (iii), we find
\[ F \doteq 1 \quad \text{and} \quad (3) \hat{R}_{ab} \doteq 2\hat{\rho}_{ab}, \quad (A4) \]

where the hatted equality, \( \doteq \), denotes the evaluation on \( I \). This gives the 3-dimensional Riemann tensor as

\[ (3) \hat{R}_{abcd} \doteq 2\hat{\rho}_{a[c} \hat{\rho}_{d]b}, \quad (A5) \]

which means that the 3-dimensional \( \Omega = 0 \) surface is locally a unit 3-dimensional timelike hyperboloid.

A part of the gravitational field is described by the electric part of the 4-dimensional Weyl tensor,

\[ (4) E_{ab} = \kappa_{ac} \kappa_{b}^c - \mathcal{L}_r \kappa_{ab} + \mathcal{D}_{(a} \kappa_{b)} - a_a a_b - \frac{1}{2} (p^c a_b p_d^b + p_{ab} \mathcal{P}^d_r \mathcal{P}^c_r) L_{cd}, \quad (A6) \]

where \( a^a = \hat{\rho}^b D_b \hat{\rho}^a \) and \( L_{ab} = (4) R_{ab} - (1/6) q_{ab}(4) R \). Around \( \Omega = 0 \), \( (4) E_{ab} \) is expanded as \( (4) E_{ab} = \sum_{\ell=0} F^{(\ell)}(4) \Omega^\ell \).

\[ (4) E^{(1)}_{ab} = \frac{1}{2} (\hat{D}_a \hat{D}_b + \hat{\rho}_{ab}) F^{(1)} - \frac{1}{2} (F^{(3)}_{ab} + p_{ab} F^{(3)}_{cd} \hat{\rho}^c \hat{\rho}^d), \quad (A7) \]

where \( F = 1 + \sum_{\ell=1} F^{(\ell)}(4) \Omega^\ell \).

From a rather lengthy argument, we find there exists an asymptotic conformal killing vector, \( \hat{\xi}^a \), such that \( \hat{\xi}^a \doteq \hat{\rho}^a \alpha \) and \( \mathcal{L}_\xi \hat{\rho}_{ab} \doteq 2\alpha \hat{\rho}_{ab} \) or \( \hat{D}_a \hat{D}_b \alpha \doteq 0 \). The conformal Killing vector induces the time-translation symmetry at spatial infinity. The asymptotic conformal Killing vector \( t^\mu \) appeared in Sec. IV is just \( \hat{D}^\mu \alpha \).

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