Dark Matter Relics and the Expansion Rate in Scalar-Tensor Theories

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ABSTRACT: We study the impact of a modified expansion rate on the dark matter relic abundance in a class of scalar-tensor theories. The scalar-tensor theories we consider are motivated from string theory constructions, which have conformal as well as disformally coupled matter to the scalar. We investigate the effects of such a conformal coupling to the dark matter relic abundance for a wide range of initial conditions, masses and cross-sections. We find that exploiting all possible initial conditions, the annihilation cross-section required to satisfy the dark matter content can differ from the thermal average cross-section in the standard case. We also study the expansion rate in the disformal case and find that physically relevant solutions require a nontrivial relation between the conformal and disformal functions. We study the effects of the disformal coupling in an explicit example where the disformal function is quadratic.

KEYWORDS: Dark energy theory, dark matter theory, annihilation cross-section, scalar-tensor theories, string theory and cosmology

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1 Introduction

The most recent cosmological observations support the so called $\Lambda$CDM model of cosmology. These observations provide overwhelming evidence for the existence of non-baryonic (cold) dark matter (DM), constituting about 27% of the Universe’s energy density budget, while another $\sim 68\%$ is believed to be in the form of dark energy, which can simply be a cosmological constant, $\Lambda$. The other $\sim 5\%$ being formed by baryonic matter, described by the standard model (SM) of particles.

A popular framework to understand the origin of DM is the thermal relic scenario. In this scenario, at very early times when the universe was at a very high temperature, thermal equilibrium was obtained and the number density of DM particles $\chi$ was roughly equal to the number density of photons. During equilibrium the dark matter number density decayed exponentially as $n_\chi^{eq} \sim e^{-m_\chi/T}$ for a non-relativistic DM candidate, where $m_\chi$ is the mass of the DM particle $\chi$. As the universe cooled down as it expanded, DM
interactions became less frequent and eventually, the DM interaction rate dropped below the expansion rate \((\Gamma_\chi < H)\). At this point the density number froze-out and the universe was left with a “reluc” of DM particles. Therefore, the dependence of the number density at the time of freeze-out is crucial to determine the DM relic abundance. The longer the DM particles remain in equilibrium, the lower its density will be at freeze-out and vice-versa. In the standard \(\Lambda CDM\) scenario, particle freeze-out happens during the radiation era and DM species with weak scale interaction cross-section freeze-out with an abundance that matches the present observed value. The weakness of the interactions is reflected in the predicted thermally-averaged annihilation cross section, \(\langle \sigma v \rangle\), which is around \(3.0 \times 10^{-26} \text{cm}^3\text{s}^{-1}\). Despite such a small value, the Fermi-LAT and Planck experiments have been exploring upper bounds on \(\langle \sigma v \rangle\) (see [1, 2]). From observations, it appears that the annihilation cross-section can be smaller than the thermal average value for lower dark matter masses \((\leq 100 \text{ GeV})\), whereas an annihilation cross-section larger than the thermal average value can still be allowed for larger DM mass.

If, from future measurements, \(\langle \sigma v \rangle \neq 3.0 \times 10^{-26} \text{cm}^3\text{s}^{-1}\) is established, what can we say about the origin of the dark matter? Can it still be the thermal dark matter or do we need non-thermal origin of dark matter? In the case of non-thermal origin, the DM can arise from the decay of a heavy particle, e.g., moduli, and can satisfy the DM content with any value of \(\langle \sigma v \rangle\). The primary motivation of this paper is to find out whether the DM content can still have a thermal origin with larger or smaller \(\langle \sigma v \rangle\) by utilising non-standard cosmology.

The phenomenological \(\Lambda CDM\) model complemented with the inflationary paradigm to provide the seeds of large scale structure, is very successful in describing our current universe. However, the physics describing the universe’s evolution from the end of inflation (reheating) to just before big-bang nucleosynthesis (BBN) \((t \lesssim 200 \text{ s})\) remains relatively unconstrained. During this period the universe may have gone through a “non-standard” period of expansion, and still be compatible with BBN. If such modification happened during DM decoupling, the DM freeze-out may be modified with measurable consequences for the relic DM abundances.

Departures from the standard cosmology between reheating and BBN will mainly be a consequence of a modified expansion rate \((\dot{H})\), due to a modification of General Relativity (GR). Such modifications are well motivated by attempts to embed the \(\Lambda CDM\) and inflation models into a fundamental theory of gravity and particle physics, such as theories with extra-dimensions, supergravity and string theory. Indeed, our main motivation in this paper is to develop further tools that may allow us to connect such fundamental theories with observations. In this paper our approach will be mostly phenomenological, but we have in mind a scenario that can be derived in the context of a fundamental theory of gravity, such as string theory. Furthermore, we will be concerned with modifications to the standard picture due to the presence and interactions with scalar fields only.

Modifications to the relic abundances were first discussed by Catena et al. [3] in the context of conformally coupled scalar-tensor theories (ST), such as generalisations of the Brans-Dicke theory. Further studies on conformally coupled ST models have been performed in the last years in [4–7] (see also [8–14]) with the most recent work of [15] where it
is shown that the boundary conditions used in [3] cannot have \( \langle \sigma v \rangle \neq 3.0 \times 10^{-26} \text{cm}^3 \text{s}^{-1} \). Indeed ST theories where dark matter and dark energy are correlated constitute an attractive way to address the dark matter and dark energy problems via an attraction mechanism towards standard general relativity (GR) [3].

In the context of ST theories, the most general physically consistent relation between two metrics in the presence of a scalar field, is given by\(^1\) [16]:

\[
\tilde{g}_{\mu\nu} = C(\phi) g_{\mu\nu} + D(\phi) \partial\mu \phi \partial\nu \phi .
\]

The first term in (1.1) is the well-known conformal transformation which characterises the Brans-Dicke class of scalar-tensor theories explored in [3–7, 15]. However, in reference [15], it is shown that there is no change in the thermal cross-section arising from the conformally modified metric compared to the standard cosmology after satisfying all the constraints based on the boundary conditions chosen in [3, 4]. The second term is the disformal contribution, which is generic in extensions of general relativity. In particular, it arises naturally in D-brane models, as discussed in [17] in a model of coupled dark matter and dark energy. In this paper we revisit the expansion rate modification and impact on the DM relic abundances for the conformal case, providing new interesting results with new boundary conditions to show that \( \langle \sigma v \rangle \neq 3.0 \times 10^{-26} \text{cm}^3 \text{s}^{-1} \), while satisfying the DM content. We further discuss the general modifications to the expansion rate and Boltzmann equation, due to the disformal coupling and present an explicit non-trivial example for the case in which the conformal term in (1.1) is a monomial.

The paper is organised as follows. We start in section 2 introducing the scalar-tensor theory conformally and disformally coupled to matter. Then, we examine the formulation of this theory in the Einstein and Jordan frames, comment on their physical interpretation and derive the equations that describe the cosmological evolution of the Universe. Subsequently, after discussing the expansion rate modifications caused by the presence of the conformal and disformally coupled scalar field in section 3, we investigate in detail its impact on the dark matter relic abundance by exploring a concrete pure conformal example as well as an example where we also turn on the disformal contribution. Finally, in section 4 we conclude.

2 The scalar-tensor theory set-up

We are interested in scalar-tensor theories coupled to matter both conformally and disformally [16]. Our motivation comes from theories with extra dimensions and in particular string theory compactifications, where several additional scalar fields appear, from closed and open string theory sectors of the theory [17]. Our approach in this paper nonetheless, will be phenomenological and therefore our equations will be simplified. However, we present the more general set-up, which can accommodate a realisation from concrete string

\(^1\)In the more general case, \( C \) and \( D \) can be functions of \( X = \frac{1}{2} (\partial \phi)^2 \) as well. However, we will not consider this case in the present paper.
theory compactifications in appendix B and C. The action we want to consider is given by:

\[ S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} \left( R - \frac{1}{2}(\partial \phi)^2 + V(\phi) \right) - \int d^4x \sqrt{-g} \mathcal{L}_M(\tilde{g}_{\mu\nu}). \]  

Here the disformally coupled metric is given by

\[ \tilde{g}_{\mu\nu} = C(\phi)g_{\mu\nu} + D(\phi)\partial_{\mu}\phi\partial_{\nu}\phi, \]  

and the inverse by:

\[ \tilde{g}^{\mu\nu} = \frac{1}{C} \left[ g^{\mu\nu} - \frac{D}{C + D(\partial\phi)^2} \partial_{\mu}\phi\partial_{\nu}\phi \right]. \]  

Moreover, \( \kappa^2 = M_P^{-2} = 8\pi G \), but keep in mind that that \( G \) is not in general equal to Newton’s constant as measured by e.g. local experiments. Further, \( C(\phi), D(\phi) \) are functions of \( \phi \), which can be identified as a conformal and disformal couplings of the scalar to the metric, respectively (note that the conformal coupling is dimensionless, whereas the disformal one has units of mass\(^{-4}\)).

The action in (2.1) is written in the Einstein frame\(^2\), which is identified in the literature of scalar-tensor theories (including conformal and disformal couplings) with the frame respect to which the scalar field, gravity is coupled. We follow this use and refer to “Jordan” or “disformal frame” to identify the frame in which dark matter is coupled only to the metric \( \tilde{g}_{\mu\nu} \), rather than to the metric \( g_{\mu\nu} \) and a scalar field \( \phi \).

The equations of motion obtained from (2.1) are:

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa^2 \left( T^\phi_{\mu\nu} + T_{\mu\nu} \right), \]  

where, in the frame relative to \( g_{\mu\nu} \), the energy-momentum tensors are defined as

\[ T^\phi_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_\phi}{\delta g^{\mu\nu}}, \quad T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta (-\sqrt{-g} \mathcal{L}_M)}{\delta g^{\mu\nu}}, \]  

and we model the energy-momentum tensor for matter and both dark components as perfect fluids, that is:

\[ T^i_{\mu\nu} = P_i g_{\mu\nu} + (\rho_i + P_i) u_\mu u_\nu \]  

where \( \rho_i, P_i \) are the energy density and pressure for each fluid \( i \) with equation of state \( P_i/\rho_i = \omega_i \). For the scalar field, the energy-momentum tensor takes the form:

\[ T^\phi_{\mu\nu} = -g_{\mu\nu} \left[ \frac{1}{2}(\partial \phi)^2 + V \right] + \partial_\mu \phi \partial_\nu \phi, \]  

\(^2\)In string theory, the Einstein frame refers to the frame in which the dilaton and graviton degrees of freedom are decoupled, while the string (or Jordan) frame is that in which they are not. Further, the dilaton field as well as all other moduli (scalar) fields not relevant for the cosmological discussion are stabilised, massive, and are therefore decoupled from the low energy effective theory. In the literature of scalar-tensor theories however, the Einstein and Jordan frames are identified with respect to the (usually single) scalar field to which gravity is coupled, but such scalar has no particular physical nor geometrical interpretation.
and one can define the energy density and pressure of the scalar field as:
\[ \rho_\phi = -\frac{1}{2} (\partial \phi)^2 + V, \quad P_\phi = -\frac{1}{2} (\partial \phi)^2 - V. \] (2.8)

Finally the equation of motion for the scalar field dark energy becomes:
\[ -\nabla_\mu \nabla^\mu \phi + V' - \frac{T_{\mu\nu}^\phi}{2} \left[ \frac{C'}{C} g_{\mu\nu} + \frac{D'}{C} \partial_\mu \phi \partial_\nu \phi \right] + \nabla_\mu \left[ \frac{D}{C} T_{\mu\nu} \partial_\nu \phi \right] = 0. \] (2.9)

Due to the nontrivial coupling, the individual conservation equations for the two fluids are modified. However, the conservation equation for the full system is preserved, and given in the usual way by
\[ \nabla_\mu \left( T_{\mu\nu} + T_{\mu\nu}^\phi \right) = 0. \] (2.10)

Thus using (2.7) and the equation of motion for the scalar field we can write
\[ \nabla_\mu T_{\mu\nu}^\phi = Q \partial^\nu \phi = -\nabla_\mu T_{\mu\nu}, \] (2.11)

where
\[ Q \equiv \nabla_\mu \left[ \frac{D}{C} T_{\mu\lambda} \partial_\lambda \phi \right] - \frac{T_{\mu\nu}^\phi}{2} \left[ \frac{C'}{C} g_{\mu\nu} + \frac{D'}{C} \partial_\mu \phi \partial_\nu \phi \right]. \] (2.12)

In the Jordan, or disformal frame, as defined above, matter is conserved,
\[ \tilde{\nabla}_\mu \tilde{T}_{\mu\nu} = 0, \] (2.13)

where \( \tilde{\nabla}_\mu \) is the covariant derivative computed with respect to the disformal metric (2.2) with the Christoffel symbols given by
\[ \tilde{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \frac{C'}{C} \delta^\mu_{(\alpha} \partial_{\beta)} \phi - \gamma^2 \frac{C'}{2C} \partial^\mu \phi g_{\alpha\beta} + \frac{D}{C} \gamma^2 \partial^\mu \phi \left[ \nabla_\alpha \nabla_\beta \phi + \left( \frac{D'}{2D} - \frac{C'}{C} \right) \partial_\alpha \phi \partial_\beta \phi \right], \] (2.14)

and we have introduced the “Lorentz factor” \( \gamma \) defined as
\[ \gamma = \frac{1}{\sqrt{1 + \frac{D}{C} (\partial \phi)^2}}. \] (2.15)

In this frame, the energy-momentum tensor is defined as
\[ \tilde{T}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \] (2.16)

and the disformal energy-momentum tensor can be written as:
\[ \tilde{T}_{\mu\nu} = (\tilde{\rho} + \tilde{P}) \tilde{u}^\mu \tilde{u}^\nu + \tilde{P} \tilde{g}_{\mu\nu}, \] (2.17)

where \( \tilde{u}^\mu = C^{-1/2} \gamma w^\mu \). Using (2.16), we obtain a relation between the energy momentum tensor in both frames as:
\[ \tilde{T}_{\mu\nu} = C^{-3} \gamma T_{\mu\nu}. \] (2.18)

Further using (2.17) we arrive at a relation among the energy densities and pressures in both frames, given by
\[ \tilde{\rho} = C^{-2} \gamma^{-1} \rho, \quad \tilde{P} = C^{-2} \gamma P, \] (2.19)

and therefore the equations of state in both frames are related by \( \tilde{\omega} = \omega \gamma^2 \). Note that in the pure conformal case, \( D = 0, \gamma = 1 \) and therefore \( \tilde{\omega} = \omega \).
2.1 Cosmological equations

Consider an homogeneous and isotropic FRW metric \( g_{\mu\nu} \),
\[
    ds^2 = -dt^2 + a(t)^2 dx_i dx^i, \tag{2.20}
\]
where \( a(t) \) is the scale factor. In this background, the Einstein and Klein-Gordon equations become, respectively
\[
    H^2 = \frac{\kappa^2}{3} \left[ \rho_\phi + \rho \right], \tag{2.21}
\]
\[
    \dot{H} + H^2 = -\frac{\kappa^2}{6} \left[ \rho_\phi + 3P_\phi + \rho + 3P \right], \tag{2.22}
\]
\[
    \ddot{\phi} + 3H\dot{\phi} + V_{,\phi} + Q_0 = 0. \tag{2.23}
\]
where, \( H = \frac{\dot{a}}{a} \), dots are derivatives with respect to \( t \) and we have denoted \( V_{,\phi} \equiv \frac{dV}{d\phi} \). Also the Lorentz factor becomes
\[
    \gamma = (1 - D\phi^2/C)^{-1/2}.
\]
The continuity equations for the scalar field and matter are given by
\[
    \dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = -Q_0 \dot{\phi}, \tag{2.24}
\]
\[
    \dot{\rho} + 3H(\rho + P) = Q_0 \dot{\phi}, \tag{2.25}
\]
where \( Q_0 \) is given by
\[
    Q_0 = \rho \left[ \frac{D}{C} \ddot{\phi} + \frac{D}{C} \dot{\phi} \left( 3H + \frac{\dot{\phi}}{\rho} \right) + \left( \frac{D}{2C} - \frac{D C}{C} \right) \dot{\phi}^2 + \frac{C\phi}{2C}(1 - 3\omega) \right]. \tag{2.26}
\]

Using (2.25) we can rewrite this in a more compact and useful form as
\[
    Q_0 = \rho \left( \frac{\dot{\gamma}}{\phi \gamma} + \frac{C\phi}{2C}(1 - 3\omega \gamma^2) - 3H \omega \frac{(\gamma^2 - 1)}{\phi} \right). \tag{2.27}
\]

Plugging this into the (non-)conservation equation for dark matter (2.25), gives:
\[
    \dot{\rho} + 3H(\rho + P \gamma^2) = \rho \left[ \frac{\dot{\gamma}}{\gamma} + \frac{C\phi}{2C} (1 - 3\omega \gamma^2) \right]. \tag{2.28}
\]

Using the relations for the physical proper time and the scale factors in the two frames, given by
\[
    \tilde{a} = C^{1/2} a, \quad d\tilde{\tau} = C^{1/2} \gamma^{-1} d\tau, \tag{2.29}
\]
we can define the disformal-frame Hubble parameter \( \tilde{H} \equiv \frac{d\ln \tilde{a}}{d\tilde{\tau}} \), as
\[
    \tilde{H} = \frac{\gamma}{C^{1/2}} \left[ H + \frac{C\phi}{2C} \dot{\phi} \right], \tag{2.30}
\]
so that (2.13) takes the standard form in terms of \( \tilde{H} \):
\[
    \frac{d\tilde{\rho}}{d\tilde{\tau}} + 3\tilde{H}(\tilde{\rho} + \tilde{P}) = 0. \tag{2.31}
\]

Equations (2.30) and (2.31) give the background evolution equations for the modified expansion rate and matter’s density evolution.
2.2 Master equations

In order to solve the cosmological equations, it is convenient to replace time derivatives with derivatives with respect to the number of e-folds \( N \), defined as \( N = \ln \frac{a}{a_0} \) and define \( \lambda = \frac{V}{\rho} (= \frac{C}{\rho}) \). With these definitions, we can rewrite the Friedmann equation (2.21) and \( Q_0 \) as:

\[
H^2 = \frac{k^2 \rho}{3} \frac{(1 + \lambda)}{\left(1 - \frac{k^2 \rho^2}{6}\right)},
\]

\[
Q_0 = \frac{\gamma^2 H^2}{2} \left[ \frac{2D}{C} \phi'' - \frac{2D}{C} \phi' \left(3\omega + \frac{k^2 \rho^2}{2} + \frac{3(1 + \omega) B}{2(1 + \lambda)}\right) + \left(\frac{D}{C}\phi^2 + \frac{C \phi}{H^2 C} (\gamma^{-2} - 3\omega)\right) \right],
\]

where here we denote \( \phi' = d/dN \). Note also that (2.32) implies that \( \kappa \phi' \leq \pm \sqrt{6} \).

Using these equations and further defining a dimensionless scalar field \( \varphi = \kappa \phi \), we can rewrite (2.22) and (2.23) as:

\[
H' = -H \left[ \frac{3B}{2(1 + \lambda)} (1 + \omega) + \frac{\varphi'^2}{2} \right],
\]

\[
\varphi'' \left[ 1 + 3H^2 \gamma^2 B D \frac{D}{C} \right] + 3 \varphi' \left[ 1 + \omega \frac{3H^2 \gamma^2 B D}{\kappa^2 (1 + \lambda) C} \right] + \frac{H'}{H} \varphi' \left[ 1 + \frac{3H^2 \gamma^2 B D}{\kappa^2 (1 + \lambda) C} \right]
\]

\[
+ \frac{3B}{1 + \lambda} \alpha(\varphi) (1 - 3\omega \gamma^2) + \frac{3B\lambda}{V} \frac{V}{V} + \frac{3H^2 \gamma^2 B D}{\kappa^2 (1 + \lambda) C} \left[ (\delta(\varphi) - \alpha(\varphi)) \varphi'^2 \right] = 0,
\]

where we defined:

\[
B \equiv 1 - \frac{\varphi'^2}{6},
\]

\[
\gamma^{-2} = 1 - \frac{H^2 D}{\kappa^2 C} \varphi'^2,
\]

\[
\alpha(\varphi) = \frac{d \ln C^{1/2}}{d \varphi},
\]

\[
\delta(\varphi) = \frac{d \ln D^{1/2}}{d \varphi}.
\]

One can solve the system of coupled equations above for \( H \) and \( \varphi \) as functions of \( N \). However, in some cases it is simpler to use (2.34) into (2.35) and solve the following disformal master equation:

\[
\frac{2(1 + \lambda)}{3B} \varphi'' + (2\lambda + 1 - \omega) \varphi' + 2\lambda \frac{d \ln V}{d \varphi} + 2(1 - 3\omega \gamma^2) \alpha(\varphi)
\]

\[
+ \frac{2\gamma^2 (1 + \lambda) \frac{D \rho}{C}}{3B} \left[ \varphi'' - 3 \varphi' \left[ \omega + \frac{\varphi'^2}{6} + \frac{(1 + \omega) B}{2(1 + \lambda)} \right] + \frac{C}{2D} \left( \frac{D}{C} \phi^2 \right) \right] = 0,
\]
with $\gamma$ given by:

$$\gamma^{-2} = 1 - \frac{(1 + \lambda) D \rho}{3B C \phi^2}. \quad (2.41)$$

From (2.40) we see that the conformal case is recovered for $D = 0$, when the second line vanishes. Moreover, the disformal piece appears always together with derivatives of the scalar field, as expected and also nontrivially coupled to the energy density. This complicates considerably the analysis of the disformal case, as we will see below.

### 2.3 Modified expansion rate

The effect of the expansion rate during the early time evolution due to the presence of a scalar field can be extracted from the Hubble parameter evolution in the disformal frame defined as:

$$\tilde{H} = d(\log \tilde{a})/d\tilde{\tau},$$

which can be written using (2.29) as:

$$\tilde{H} = \frac{H_\gamma C^{1/2}}{\left(1 + \alpha(\varphi)\varphi'\right)}, \quad (2.42)$$

where remember that $\gamma$ depends on $H$ (or $\rho$) as seen from (2.37), while in the pure conformal case $D = 0$ and $\gamma = 1$. Note that in principle, the factor $(1 + \alpha(\varphi)\varphi')$ can be positive or negative, indicating an expansion or contraction modified rate. We stick to positive definite values for this factor and therefore only modified expansion rates, though in principle, one could have a brief contraction period during the early universe evolution, before the onset of BBN\(^3\). Moreover, notice that while $\tilde{H}$ can grow during the cosmological evolution, the null energy condition (NEC) is not violated. This is because the Einstein frame expansion rate $H$ is dictated by the energy density $\rho$ and pressure $p$, which obey the NEC and therefore $\dot{H} < 0$ during the whole evolution, as it should (see for example [19]).

We further want to relate the modified expansion rate to the expected expansion rate in general relativity (GR), that is:

$$H_{GR}^2 = \frac{\kappa^2}{3} \tilde{\rho}. \quad (2.43)$$

We can do this be using the Friedmann equation (2.32) and the relation between the energy densities (2.19) to write

$$\gamma^{-1} H^2 = \frac{\kappa^2}{\kappa_{GR}^2} C^2 \frac{(1 + \lambda)}{B} H_{GR}^2. \quad (2.44)$$

Using the definition of $\gamma$ (see (2.37)) into this equation, one finds a cubic equation for $H^2$ in terms of all the other parameters. The real positive solution to that equation can then be replaced into (2.42) to find the modified expansion rate $\tilde{H}$, which will thus be a complicated function of $H_{GR}$ as we now see. The cubic equation for $H$ takes the form:

$$d_1 H^6 - H^4 + d_2^2 = 0, \quad (2.45)$$

\(^3\)See [18] for a review on scenarios with a possible contraction phase in the early universe.
The solutions to (2.45) can be written as
\[
H^2 = \frac{1}{3d_1} \left( 1 + \left( \frac{2}{\Delta} \right)^{1/3} + \left( \frac{2}{\Delta} \right)^{1/3} \right)
\]
with \(\Delta = 2 - 27d_1^2d_2^2 + d_1d_2\sqrt{27(27d_1^2d_2^2 - 4)}\). The other two solutions can be obtained by replacing
\[
\left( \frac{2}{\Delta} \right)^{1/3} \rightarrow e^{2\pi i/3} \left( \frac{2}{\Delta} \right)^{1/3}
\]
and
\[
\left( \frac{2}{\Delta} \right)^{1/3} \rightarrow e^{4\pi i/3} \left( \frac{2}{\Delta} \right)^{1/3}.
\]
We are interested in real positive solutions for \(H^2\). One possibility to get this is to have the imaginary part of \((\Delta/2)^{1/3}\) vanish by requiring that \(\Delta > 0\), which is impossible. Therefore, the way to obtain real solutions for \(H\) is to have the imaginary parts of \((\Delta/2)^{1/3}\) and \((\Delta/2)^{-1/3}\) cancel each other, leaving a real positive solution.

For this, we need that \(27d_1^2d_2^2 \leq 4\), which implies the following relation between the conformal and disformal functions:
\[
\frac{3\sqrt{3} D \varphi^2 (1 + \lambda) H_{GR}^2}{B \kappa_{GR}^2} \leq 2.
\]
Under this condition, we can rewrite \(\Delta\) as:
\[
\Delta = 2 - 27d_1^2d_2^2 + id_1d_2\sqrt{27(4 - 27d_1^2d_2^2)},
\]
which allows us to define a complex number \(Z \equiv \Delta/2\) and it is easy to check that \(\bar{Z} = 2/\Delta\) and thus \(|Z|^2 = 1\). Denoting further \(Z_i\) with \(i = 1, 2, 3\) denoting the three solutions to \(H\) as explained above, the solutions for \(H\), (2.47) takes the simple form:
\[
H^2_i = \frac{1}{3d_1} \left[ 1 + Z_i^{1/3} + \bar{Z}_i^{1/3} \right],
\]
and remember that we are interested only in the real positive solution. We can now plug in (2.49), as well as the expression for \(\gamma\) in terms of \(H\) into the Jordan frame expansion rate (2.42), can be written as:
\[
\tilde{H}^2 = \frac{\kappa_{GR}^2}{\kappa_{GR}^2} \frac{\gamma^3 C(1 + \lambda)(1 + \alpha(\varphi)\varphi')^2}{B} H_{GR}^2, \tag{2.50}
\]
where there is a non-trivial dependence of \(H_{GR}\) encoded in
\[
\gamma_i = \frac{1}{3d_1d_2} \left[ 1 + Z_i^{1/3} + \bar{Z}_i^{1/3} \right]. \tag{2.51}
\]
In the conformal case, $D = 0$, $\gamma = 1$ and therefore (2.50) is simply
\[
\tilde{H}^2 = \frac{\kappa^2}{\kappa_{GR}^2} \frac{C(1 + \lambda)(1 + \alpha(\varphi)\varphi')^2}{B} H_{GR}^2. \tag{2.52}
\]
From this relation we define a speed-up parameter $\xi$, which will be useful below to measure the departures from the $GR$ expansion rate result:
\[
\xi \equiv \frac{\tilde{H}}{H_{GR}}. \tag{2.53}
\]

3 Modifications of the dark matter relic abundances

In this section we discuss the modifications to the DM relic abundance’s predictions due to modifications of the expansion rate before the onset of nucleosynthesis caused by the presence of a scalar field conformal and disformally coupled to matter. We start by re-visiting the conformal case, discussed originally in [3]\(^4\). We first solve (numerically) the master equation for the scalar field (2.40) in order to compute the modified expansion rate $\tilde{H}$ and compare it with the standard expansion rate, $H_{GR}$. We then use this to compute the modifications to the dark matter relic abundances by solving the Boltzmann equation using the modified expansion rate. We start revisiting by the conformal case by exploring a wide range of initial conditions, masses and cross-sections. We then look at an explicit disformal example.

Before solving the master equation (2.40), we would like to write it in terms of Jordan frame quantities $\tilde{\omega} = \omega \gamma^2$, $\tilde{\rho} = C^{-2} \gamma^{-1} \rho$. Moreover, the number of e-folds $N$ can be expressed in terms of Jordan frame quantities as follows. In this frame, the entropy is conserved and is given by $\tilde{S} = \tilde{a} \tilde{s}$, where $\tilde{s} = \frac{2}{45} \pi g_s(\tilde{T}) \tilde{T}^3$. So, the conservation of entropy and (2.29) show that $N$ is a function of temperature and the scalar field as:
\[
N \equiv \ln \frac{a}{a_0} = \ln \left[ \frac{\tilde{T}_0}{T} \left( \frac{g_s(\tilde{T}_0)}{g_s(T)} \right)^{1/3} \right] + \ln \left[ \frac{C_0}{C} \right]^{1/2}. \tag{3.1}
\]
Therefore, we can introduce the parameter, $\tilde{N}$, defined as
\[
\tilde{N} \equiv \ln \left[ \frac{\tilde{T}_0}{T} \left( \frac{g_s(\tilde{T}_0)}{g_s(T)} \right)^{1/3} \right]. \tag{3.2}
\]
and transform to derivatives w.r.t. $\tilde{N}$ ($\tilde{N}$ (assuming well behaved functions)):
\[
\varphi' = \frac{1}{1 - \alpha(\varphi) \frac{d\varphi}{d\tilde{N}}} \frac{d\varphi}{d\tilde{N}}, \quad \varphi'' = \frac{1}{(1 - \alpha(\varphi) \frac{d\varphi}{d\tilde{N}})^3} \left( \frac{d^2\varphi}{d\tilde{N}^2} + \frac{d\alpha}{d\varphi} \left( \frac{d\varphi}{d\tilde{N}} \right)^3 \right). \tag{3.3}
\]
In a slight abuse of notation and to keep expressions neat, in what follows we denote derivatives w.r.t. $\tilde{N}$ with a prime $'$.\(^4\)

\(^4\)Modifications to the Boltzmann equation due to a conformal coupling in (non-critical) string theory have been discussed in [8].
3.1 Conformal case

We start with the pure conformal case. That is, we take \( D(\phi) = 0 \) in (2.40) and therefore \( \gamma = 1 \) (and \( \tilde{\omega} = \omega \)). Moreover, during the radiation and matter dominated eras, of interest for us, the potential energy of the scalar field is subdominant and therefore, we take \( \lambda \sim 0 \). Therefore the master equation (2.40) simplifies to:

\[
\frac{2}{3(1 - \varphi'^2/6)} \varphi'' + (1 - \tilde{\omega}) \varphi' + 2(1 - 3 \tilde{\omega}) \alpha(\varphi) = 0, \tag{3.4}
\]

which in terms of derivatives wrt \( \tilde{N} \) takes the form:

\[
\frac{1}{3B [1 - \alpha(\varphi)\varphi']} \left( \varphi'' + \frac{d\alpha}{d\varphi} (\varphi')^3 \right) + \frac{(1 - \tilde{\omega})}{[1 - \alpha(\varphi)\varphi']} \varphi' + (1 - 3 \tilde{\omega}) \alpha(\varphi) = 0, \tag{3.5}
\]

where \( B = 1 - \frac{(\varphi')^2}{6(1 - \alpha(\varphi)\varphi')} \). Using the relation between \( \tilde{H} \) and \( H_{GR} \) defined in (2.52), we can write the speed-up parameter as

\[
\xi = \frac{\tilde{H}}{H_{GR}} = \frac{C^{1/2}(\varphi)}{C^{1/2}(\varphi_0)} \frac{1}{(1 - \alpha(\varphi)\varphi') \sqrt{B \sqrt{1 + \alpha^2(\varphi_0)}}} \tag{3.6}
\]

where we have used the relation between the bare gravitational constant and that measured by local experiments for conformally coupled theories [20]:

\[
\kappa^2_{GR} = \kappa^2 C(\varphi_0)[1 + \alpha^2(\varphi_0)], \tag{3.7}
\]

where \( \varphi_0 \) is the value of the scalar field at present time.

3.1.1 Expansion rate modification

The scalar equation in the conformal case (3.4), as function of \( N \) (for \( \lambda = 0 \)) contains a term which can be interpreted as an effective potential, dictated by \( V_{eff} = \ln C^{1/2} \). For a strictly radiation dominated era, \( \tilde{\omega} = 1/3 \), the effective potential term vanishes and we are left with an equation that can be solved analytically [21], giving \( \varphi' \propto e^{-N} \). That is, any initial velocity will rapidly go to zero (remember that from the Friedmann equation (2.32), \( \varphi' \) is constrained to be \( \varphi' \lesssim \pm \sqrt{6} \)). Therefore we explore the effects of having a non-zero initial velocity in our analysis below (see also Appendix A for further examples). Since the scalar field is expressed in Planck units, we focus on order one or smaller field variations \( \Delta \varphi \). One can check, using the analytic solution to (3.4) deep in the radiation era, that for initial velocities \( \varphi' \ll \pm \sqrt{6} \), the total field displacement is of order \( \Delta \varphi \sim \varphi' \) [21]. However, given that we don’t know much about the theory before BBN, we explore different initial values for \( (\varphi_0, \varphi'_0) \) and study their consequences. In particular we explore initial values \( \varphi_0 \) and \( \varphi'_0 \in (-1.0, 1.0) \).

We now concentrate on an explicit conformal factor. We use the same conformal factor as that studied in [3], which is given by:

\[
C(\varphi) = (1 + b e^{-\beta \varphi})^2 \tag{3.8}
\]
with the values $b = 0.1$, $\beta = 8$, which have been shown to satisfy the constraints imposed by tests of gravity, for the parameters $\alpha$, $\beta$, $\xi$. As we will see, the requirement of reaching the GR expansion rate value by the time of the onset of BBN, drives these parameters to very small values, which are thus consistent with the constraints from gravity for their values today.

As we discussed above, in the equation of motion for $\varphi$, with $\tilde{\omega} \neq 1/3$, the conformal factor acts as an effective potential on which the scalar field moves, damped by the Hubble friction (see (2.23)). Since any initial velocity $\varphi'$ goes rapidly to zero deep in the radiation era, in the subsequent evolution of $\varphi$, the term $(\varphi')^2$ in the master equation will be negligible. In this regime, the equation is that of a particle moving in an effective potential with a damping term. Therefore, one can understand the evolution of $\varphi$ from the form of the effective potential ($V_{eff} = \ln C^{1/2}$) and the initial conditions chosen. For the conformal function we are considering (3.8), one sees that there is a set of initial conditions that will give rise to an interesting behaviour in $\varphi$, and therefore an interesting modified expansion rate in the Jordan frame $\tilde{H}$ (2.52), as we now explain.

In general, both the initial position and velocity of the scalar field can take any value, positive and negative. In the runaway effective potential dictated by $\ln C^{1/2}$ for the conformal factor (3.8) we consider here, we have the following possibilities. i) The scalar field starting somewhere up in the runaway effective potential with zero initial velocity. In this case the scalar field will roll-down the potential, eventually stopping due to Hubble friction, at some constant value of $\varphi$, which depends on the its initial value. So long as $(1 + \alpha(\varphi) \varphi')$ stays positive (see (2.42)), $C$ will evolve rapidly towards 1. More generally, the initial velocity can be different from zero. If the initial velocity is positive, the behaviour will be similar to the previous case. The field will roll-down the effective potential towards its final terminal value.

ii) A more interesting possibility arises when one allows for negative initial velocities. In this case, the field will start rolling-up the effective potential towards smaller values of the field, eventually turning back down and moving towards its terminal value. It is easy to see that an interesting effect happens when the field starts at an initial positive value. Given a sufficient initial negative velocity the field will move towards negative values until its velocity becomes zero and then positive again, as it rolls back down the effective potential. This change in sign for the scalar evolution will produce a pick in the conformal function that will give rise to a non-trivial modification of the Jordan’s frame expansion rate $\tilde{H}$, as we are looking for. As mentioned before, we are interested in (sub-)Planckian initial values $(\varphi_0, \varphi'_0)$, such that $\tilde{H} > 0$. With these requirements, one can see that given an initial negative velocity, there is a suitable initial value of the scalar field such that the behaviour just described holds and the expansion rate $\tilde{H}$ has an interesting evolution before the onset of BBN. At late times the conformal function goes to one and the GR expansion is recovered. We show this behaviour explicitly in Figures 1 and 2 where we plot the numerical solution for the evolution of $\varphi$ and $C(\varphi)$ as functions of the temperature. In these plots we find $\varphi = \varphi(\tilde{T})$ by first solving (3.5) numerically with initial conditions.
\( (\varphi_0, \varphi'_0) = (0.2, -0.99) \) and then use (3.2) to express \( \varphi(\tilde{N}) \) as function of \( \tilde{T} \). As we can see, the conformal factor starts growing towards a maximum value as \( \varphi \) moves to negative values, to rapidly drop down towards its GR value at \( C \to 1 \) as \( \varphi \) moves down the effective potential towards positive values. This non-trivial effect will give rise to the possibility of re-annihilation, as we discuss below.

**Figure 1:** Typical evolution of the scalar field as temperature decreases. The initial values are \( (\varphi, d\varphi/d\tilde{N}) = (0.2, -0.994) \).

**Figure 2:** Behaviour of the conformal factor, \( C(\varphi) \) as a function of the temperature for the same initial values as in Fig. 1.

Based on the discussion above, we have solved the master equation (3.5), to find the the scalar field as a function of \( \tilde{N} \) for various initial conditions, where we see the interesting behaviour explained above. The resulting modified expansion rate and its comparison with the standard case is shown in Figure 3 for the same initial conditions as in Figures

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\(^5\)In appendix A we show further examples of the thermal evolution of the scalar field.
1 and 2. In our numerical exploration, we choose initial conditions for which the notch in the expansion rate (see Fig. 3) occurs closer to the BBN time. This has interesting consequences for the dark matter annihilation, as we discuss below.

Figure 3: Comparing the Hubble expansion rate $\tilde{H}$ in the Jordan Frame with the standard Hubble expansion rate $H_{GR}$. The presence of the scalar field enhances and decreases the expansion rate during the radiation dominated era. This plot corresponds to initial conditions given by $(\varphi_0, \varphi'_0) = (0.2, -0.994)$.

When solving the master equation (3.5), we have taken into account an important effect that occurs during the radiation dominated era. Deep in this epoch, the equation of state is given by $\tilde{\omega} = 1/3$. When a particle species in the cosmic soup becomes non-relativistic, $\tilde{\omega}$ differs slightly from 1/3. When the temperature of the universe drops below the rest mass of each of the particle types, there are non-zero contributions to $1 - 3\tilde{\omega}$. This activates the effective potential, which can be seen in the last term of (3.5), and displaces, or “kicks” the field along $V_{eff}$.

To examine this effect in more detail, we start by writing $1 - 3\tilde{\omega}$ during the early stages of the universe as in [3] and [15]

$$1 - 3\tilde{\omega} = \frac{\tilde{\rho} - 3\tilde{p}}{\tilde{\rho}} = \sum_A \frac{\tilde{\rho}_A - 3\tilde{p}_A}{\tilde{\rho}} + \frac{\tilde{\rho}_m}{\tilde{\rho}},$$

(3.9)

where the sum runs over all particles in thermal equilibrium during the radiation dominated era and $\tilde{\rho}_m$ is the contribution from the non-relativistic decoupled and pressureless matter. The summation over all the particle is responsible for the kicking effect discussed above. Then, a kick function is defined as

$$\Sigma(\tilde{T}) \equiv \sum_A \frac{\tilde{\rho}_A - 3\tilde{p}_A}{\tilde{\rho}},$$

(3.10)

In appendix A we show more examples of modified expansion rate using different initial conditions.

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6In appendix A we show more examples of modified expansion rate using different initial conditions.
where the energy density $\tilde{\rho}_A$ and pressure $\tilde{p}_A$ of each type $A$ of particle are given by

$$
\tilde{\rho}_A(\tilde{T}) = \frac{g_A}{2\pi^2} \int_{m_A}^{\infty} \left( E^2 - m_A^2 \right)^{1/2} \frac{E^2}{\exp(E/\tilde{T}) \pm 1} dE \tag{3.11}
$$

$$
\tilde{p}_A(\tilde{T}) = \frac{g_A}{6\pi^2} \int_{m_A}^{\infty} \left( E^2 - m_A^2 \right)^{3/2} \frac{E^2}{\exp(E/\tilde{T}) \pm 1} dE \tag{3.12}
$$

with $g_A$ being the number of internal degrees of freedom of species of type $A$ and the plus (minus) sign in the integral corresponds to fermions (bosons). To compute (3.10), we use the Standard Model particle spectrum. In particular, we take into account the top quark, the Higgs boson, Z boson, W bosons, bottom quark, tau lepton, charm quark, charged pions, neutral pion, muon lepton and the electron\textsuperscript{7}. As we show in Figure 10, $\Sigma$ is mostly zero, except when the kicks happen.

During the radiation dominated era $\tilde{\rho} \simeq \pi^2 g_{\text{eff}}(\tilde{T}) \tilde{T}^4/30$, where $\tilde{T}$ is the Jordan frame temperature and $g_{\text{eff}}$ is the total number of relativistic degrees of freedom. Also, during this stage $\tilde{\rho}_m$ is negligible and as we have shown $\Sigma$ is slightly different than zero. Therefore, we can compute the equation of state from (3.9) as $\tilde{\omega} = (1 - \Sigma(\tilde{T}))/3$.

In Figure 4 we show the evolution of $\tilde{\omega}$ between 10 TeV and 10 eV. This figure shows four troughs, which are the “kicks” mentioned above. Each kick corresponds to the transition of one or more particles to the non-relativistic regime. For example, the trough at around 0.5 MeV is due to the electron, while the one at around 100 GeV is due to the heavy particles ($t, H, Z$ and $W$).

Towards the end of the radiation era, approaching the transition to the matter dominated era, eq. (3.9) takes the approximate form:

$$
1 - 3 \tilde{\omega} \simeq \frac{\tilde{\rho}_m}{\tilde{\rho}_m + \tilde{\rho}_r} \simeq \frac{1}{1 + \tilde{T}/\tilde{T}_{eq}}, \tag{3.13}
$$

where $\tilde{T}_{eq} \sim O(10^{-9})$ GeV is the temperature at matter-radiation equality, that is, $\tilde{\rho}_m(\tilde{T}_{eq}) = \tilde{\rho}_r(\tilde{T}_{eq})$. We can now combine (3.10) and (3.13) to compute the thermal evolution of (3.9) in the radiation dominated and matter dominated eras and use it in the master equation.

### 3.1.2 Parameter Constraints

In scalar-tensor theories of gravity, there are some constraints on the parameters that need to be taken into account. Deviations from GR can be parametrised in terms of the post-Newtonian parameters $\gamma_{PN}$ and $\beta_{PN}$, which are given in terms of $\alpha(\varphi_0)$ defined in (2.38) and its derivative $\alpha'_0 = d\alpha/d\varphi |_{\varphi_0}$ as [22, 23]:

$$
\gamma_{PN} - 1 = -\frac{2\alpha_0^2}{1 + \alpha_0^2}, \quad \beta_{PN} - 1 = \frac{1}{2} \frac{\alpha'_0 \alpha_0^2}{(1 + \alpha_0^2)^2}, \tag{3.14}
$$

Solar system tests of gravity, including the perihelion shift of Mercury, Lunar Laser Ranging experiments, and the measurements of the Shapiro time delay by the Cassini spacecraft\textsuperscript{7} We present more details of the kick function in appendix A.

\textsuperscript{7}
\[ \dot{\omega} = \frac{2}{3} \dot{H} \omega - \langle \sigma v \rangle \left( n_x^2 - (n_{eq}^x)^2 \right), \]  

where, as we have discussed above, the relevant expansion rate is the Jordan frame one, which can give interesting effects due to the presence of the scalar field. Further \( n_{eq}^x \) is the equilibrium number density. We can rewrite this equation in terms of \( x = m_x / \tilde{T} \)

\[ \frac{dY}{dx} = -\frac{s(\langle \sigma v \rangle)}{xH} \left( Y^2 - Y_{eq}^2 \right). \]  

where \( Y = \frac{n_x}{s} \), \( s = \frac{2\pi^2}{45} g_4(\tilde{T}) \tilde{T}^3 \). Numerical solutions to the Boltzmann equation (3.16) with the modified expansion rate \( \dot{H} \) were found for dark matter particles with masses ranging from 5 GeV to 1000 GeV. For instance, we show solutions in figures 5 and 6 for two different masses. As we can see from (3.16), the annihilation cross-section influences the evolution of the abundance \( Y \). The current value of \( Y \) determines the present dark matter content of the universe. This can be seen clearly by recalling the current value of the energy density parameter \( \Omega_0 = \frac{\rho_m}{\rho_{c,0}} = mY_{0} s_{0}, \) where \( \rho_{c,0} \) and \( s_{0} \) are the well-known current values of the critical energy density and the entropy density of the universe, respectively. So, for each
single mass, the thermally-averaged annihilation cross section, \( \langle \sigma v \rangle \), was chosen such as the current DM content of the universe is 27 %, so \( \Omega_0 = 0.27 \).

In Figure 7 we show the annihilation cross-section, \( \langle \sigma v \rangle_{\text{Conformal}} \), found for all masses and compare it to the annihilation cross sections for the standard cosmology model, \( \langle \sigma v \rangle_{\text{Standard}} \). As it is shown, for large masses \( \langle \sigma v \rangle_{\text{Conformal}} \) is larger than \( \langle \sigma v \rangle_{\text{Standard}} \), up to a factor of four. As the mass decreases \( \langle \sigma v \rangle_{\text{Conformal}} \) decreases up to the point where is smaller than \( \langle \sigma v \rangle_{\text{Standard}} \). Then, for masses smaller than 100 GeV the figure shows that \( \langle \sigma v \rangle_{\text{Conformal}} \approx \langle \sigma v \rangle_{\text{Standard}} \). Thus, we have found that the annihilation cross-sections can be larger or smaller than the thermal average cross-section. Just to give an example of larger and smaller cross-section, the following table compares the numerical values of \( \langle \sigma v \rangle_{\text{Conformal}} \) and \( \langle \sigma v \rangle_{\text{Standard}} \) for two dark matter masses, 1000 GeV and 130 GeV.

| Mass (GeV) | \( \langle \sigma v \rangle_{\text{Conformal}} \times 10^{-26} \text{cm}^3/\text{s} \) | \( \langle \sigma v \rangle_{\text{Standard}} \times 10^{-26} \text{cm}^3/\text{s} \) |
|------------|-----------------|-----------------|
| 1000       | 9.57            | 2.23            |
| 130        | 1.68            | 2.04            |

Figures 5 and 6 show the evolution of the abundance \( \tilde{Y}(x) \) for DM particles with masses 130 GeV and 1000 GeV, respectively. These figures also include the abundance \( Y_{GR}(x) \) calculated in the standard cosmology model and the equilibrium abundance \( Y_{Eq}(x) \).

**Figure 5**: Evolution of the abundance as temperature changes for a DM particle of mass 130 GeV.

The temperature evolution of the abundance for a 130 GeV mass is not noticeable affected by the presence of the scalar field \( \phi \). In this case, \( \tilde{Y} \) and \( Y_{GR} \) are almost indistinguishable from one another. On the other hand, the scalar field \( \phi \) has a prominent effect on the temperature evolution of the abundance for a 1000 GeV DM particle. First of all, the freeze-out happens earlier than expected due to the enhancement of the expansion rate, \( \tilde{H} \). Then, an unusual effect appears. As the temperature decreases, \( \tilde{H} \) becomes...
smaller than the interaction rate\textsuperscript{8} $\tilde{\Gamma}$ and a short period of annihilation starts again called “re-annihilation”. The re-annihilation process reduces the abundance of dark matter until a second and final freeze-out happens. After this final freeze-out the abundance remains constant.

\textbf{Figure 6}: Abundance for a mass of 1000 GeV.

The re-annihilation phase can be described better by discussing the relation between the expansion rate $\tilde{H}$ and the interaction rate $\tilde{\Gamma}$. The first freeze-out happens when $\tilde{\Gamma}$ becomes smaller than $\tilde{H}$ which can be seen in figure 8 to happen around a temperature of 50 GeV for a 1000 GeV particle. Then, near to 7 GeV $\tilde{H}$ drops below $\tilde{\Gamma}$, and so the re-annihilation process starts and goes on until the second freeze-out occurs. Around 2 GeV $\tilde{H}$ becomes much larger than $\tilde{\Gamma}$ and so the abundance becomes almost constant. Our

\textsuperscript{8}The interaction rate is defined as $\tilde{\Gamma} \equiv \langle \sigma v \rangle_{\text{Conformal}} \delta \tilde{Y}$. 
Figure 8: Expansion rate (as in figure 3) and interaction rate as function of temperature. The interaction rate, \( \tilde{\Gamma} \), is given by \( \langle \sigma v \rangle_{\text{Conformal}} \tilde{s} \tilde{Y} \). We use \( \tilde{Y} \) from figures 5 and 6 and the values of \( \langle \sigma v \rangle_{\text{Conformal}} \) presented previously for 130 GeV and 1000 GeV masses.

analysis shows that, as found in [3], re-annihilation occurs for this particular choice of conformal factor. However, we found that when fully integrating the master equation, the re-annihilation occurs only for very large masses of the dark matter particles (in [3] it was found for \( m = 50 \text{GeV} \)). On the other hand, in [15], no re-annihilation was found\(^9\), which was probably due to the initial conditions used and the values of the DM masses explored.

3.2 Disformal case

We now discuss briefly the effect of the disformal factor in the metric (2.2) to the expansion rate of the universe, \( \tilde{H} \), and compare it to the conformal modification to \( \tilde{H} \)\(^10\). Hence, we explore \( D(\phi) \neq 0 \) for the same conformal factor studied before, that is, \( C(\varphi) = (1 + b e^{-\beta \varphi})^2 \) for \( b = 0.1, \beta = 8 \). To investigate these modifications, we first need to look at the the scalar field evolution with temperature.

In the pure conformal case studied above, we found the thermal evolution of the scalar field by solving the master equation (3.5) numerically, which is (2.40) for \( D(\phi) = 0 \). However, to study the effects of the disformal factor on the scalar field, it is more convenient to solve the system of two coupled equations (2.34) and (2.35). Using these equations we find solutions for the dimensionless scalar field \( \varphi \), and for the expansion rate in the Einstein frame \( H \).

Notice that solving the system of coupled equations or solving the master equation to find the thermal evolution of the scalar field are equivalent methods (as we have explicitly checked), because (2.40) it is nothing but a combination (2.34) and (2.35). However, while

\(^9\)Although [15] used a different conformal factor to [3], we expect the re-annihilation effect to be present also in that case.

\(^10\)We leave a detailed exploration for a future publication.
in the pure conformal case the master equation can be made independent of \(H\) (or \(\rho\)), this is not the case for the more general disformal case, as we can see in eq. (2.40).

In the same way as for the conformal case, we are interested mainly in the radiation and matter eras and therefore we can neglect the potential energy of the scalar field. Thus, we consider \(V \sim 0\) and \(\lambda = 0\). Also, while solving the coupled equations we have to express \(\omega\) in the Jordan frame by using \(\hat{\omega} = \omega \gamma^2\) and transform all derivatives w.r.t. \(N\) to derivatives w.r.t. \(\hat{N}\) by using (3.3).

With this information, we solve the system of coupled equations numerically to find the dimensionless scalar field \(\phi\) and the Hubble parameter \(H\), as functions of the number of e-folds \(\hat{N}\) (and the temperature). We choose the same initial conditions for the scalar field and its derivative as in the conformal case and to obtain the initial condition for \(H\), we use (2.49).

Once we have the solutions for \(\phi\) and \(H\) as functions of temperature, we can go back to (2.42) and (2.37) to obtain the expansion rate for the disformal model. As an example, in Figure 9 we show the effects of a disformal factor given by \(D(\varphi) = D_0 \varphi^2\) with \(D_0 = -4.9 \times 10^{-14}\). In this plot, we illustrate the effect of the disformal contribution on the expansion rate (\(\hat{H}_{\text{Disformal}}\)) and compare it to the modified expansion rate for the conformal case (\(\hat{H}_{\text{Conformal}}\)) and the standard case (\(H_{\text{GR}}\)). We use the same initial conditions as in Figures 1 and 2 for the scalar field and its derivative.

Also, it is important to mention that for the case shown the parameter constraints described in section 3.1.2 are satisfied. In particular we find \(\alpha_0^2 \approx 2 \times 10^{-5}\), \(\alpha'_0 > 0\) and \(\xi \approx 1.02\).

![Figure 9](image)

**Figure 9:** Comparing the modified expansion rate of the universe in the disformal and conformal scenarios for the same initial conditions as in Fig. 3.

From our example, with \(C\) and \(D\) as indicated above, we can clearly see the differences from the disformally modified expansion rate \(\hat{H}_{\text{Disformal}}\) compared to the conformally modified and standard case, \(H_{\text{GR}}\). The evolution of \(\hat{H}_{\text{Disformal}}\) is similar to that of \(\hat{H}_{\text{Conformal}}\), having an (two in our example) enhancement and a decrement compared to the standard expansion rate \(H_{\text{GR}}\). Moreover, the main differences with respect to the conformal...
modification are the position of the notch and its shape. The notch is moved to higher temperatures and it becomes a little bit sharper.

These differences between the expansion rates can be understood from (2.35). First, in this equation we see that the factor $\frac{3H^2\gamma^2BD}{\kappa(1+\lambda)C}$ in the coefficients of $\phi''$, $\phi'$ and $\phi^2$ vanishes when $D = 0$. For the disformal example shown in Figure 9, this factor is a very small correction to the equation, which is reflected in the slight shape modification of $\tilde{H}_{\text{Disformal}}$ compared to $\tilde{H}_{\text{Conformal}}$. Second, the term proportional to $\delta(\varphi)$ plays a more important role, being responsible for the shifting of the notch.

As was discussed previously in section 3.1.3, a modification to the expansion rate of the universe prior to BBN has tremendous consequences on the abundance, $\tilde{Y}$, of dark matter particles. Also, this modification implies that the thermally-averaged annihilation cross section, $\langle \sigma v \rangle$, for dark matter particles differs significantly from the one predicted by the standard cosmological model, which is approximately $3.0 \times 10^{-26}$ cm$^3$/s (see Figure 7). We have seen than the enhancement of $\tilde{H}$ allows bigger values of $\langle \sigma v \rangle$ for particles with masses within certain range, and also, the lower value of $\tilde{H}$ implies smaller $\langle \sigma v \rangle$ for particles with masses within a small interval. Thus, the location and shape of the notch determines for which masses the annihilation cross section is smaller. In the disformal scenario, the notch has been moved to higher temperatures, which allows particles with higher masses to have smaller and larger annihilation cross sections for the observed DM content. We leave the detailed study of the modification of $\tilde{H}$ for more general conformal and disformal factors for a separate forthcoming publication.

4 Conclusions

Scalar-tensor theories of gravity are a useful method to explore departures of the expansion rate of the universe from the standard cosmological model in the early universe. The expansion rate of the universe had a strong influence on the evolution of the dark matter abundance during the early stages of the universe’s evolution, specially prior to BBN. Modifications to the expansion rate during that time would be reflected in the calculation of the dark matter relic abundance and so can be used as a probe to the predictions of scalar-tensor theories. In this paper, we explored the role played by the scalar field in the modification of the expansion rate of the universe on scalar-tensor theories coupled both conformally and disformally to matter.

For the conformal case, we explored a conformal factor of the type $C = (1 + b e^{-\beta \varphi})^2$. We made no approximations and solved numerically the master equation for the scalar (3.5) for a suitable range of initial conditions. Using this result we then computed the expansion rate modification $\tilde{H}$ under the presence of the scalar field during the radiation dominated era prior to BBN. When comparing the expansion rate, $\tilde{H}$, to the standard expansion rate, $H_{\text{GR}}$, we found that the speed-up factor, $\xi = \frac{\tilde{H}}{H_{\text{GR}}}$, increases up to 200 and then become of order 1 prior to BBN (see Fig. 3). Previously, in reference [15], it was shown that there is no change in the expansion rate arising from the modification due to $C$ compared to the standard cosmology, after satisfying all the constraints based on the boundary conditions chosen in [3, 4].
This enhancement on the expansion rate has important consequences on the evolution of the abundance of dark matter particles. So, we also investigated the effect on the abundance of dark matter particles. We observed that for dark matter particles of large mass \((m \sim 10^3 \text{GeV in our example, compared to the } m \sim 50 \text{GeV reported in [3] for which we found no re-annihilation})\), the particles undergo a second annihilation process and then freeze-out once and for all in (see Fig. 6). Moreover, we found that for large masses the annihilation cross-section has to be up to four times larger than that of standard cosmology models in order to satisfy the dark matter content of the universe of 27%. On the other hand, for small masses this re-annihilation process is not present, but we found that for masses around 130 GeV, the annihilation cross-section can be smaller than the annihilation cross-section for the standard cosmological model (see Fig. 7).

We also started to investigate the effects on the early evolution of the expansion rate of a disformal factor in the metric (2.2). We noticed that in order to have a consistent solution, i.e. a real positive \(\tilde{H}\), the conformal and disformal factors need to satisfy a very specific relation, (2.48). We studied the effect of a disformal factor by turning on a small disformal contribution to the conformal case, given by \(D(\varphi) = -4.9 \times 10^{-14} \varphi^2\). To find the modified expansion rate \(\tilde{H}\) during the radiation dominated era prior to BBN, we solved numerically the disformal system of coupled equations (2.35) and (2.34).

We found that when the disformal function is turned on, \(\tilde{H}\) has a very similar profile as for pure conformal case with an enhancement and a notch compare to the standard expansion rate. However, the position of the notch changes and there may be a second enhancement of \(H\) compared to \(H_{GR}\), depending on the initial conditions (see also Appendix A). The disformal factor, is moving the notch to higher temperatures which means that we can have larger and smaller annihilation cross-section for any mass of the DM candidate for the observed DM content.

We have shown the analysis of the disformal case in a concrete example. We plan to study the disformal effects in more detail in order to assess their general features in a forthcoming publication. Moreover, as we mentioned at the beginning of the paper, we are aiming at models that can be embedded in a more fundamental theory of gravity such as string theory. We plan to analyse these cases in a future publication.

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A Numerical Implementation

In this section we describe in detail our numerical method to study the master equation and expansion rate evolution and present additional examples for different initial conditions.

As we explained in section 3.1.1 the result shown in Figure 4 for \(\tilde{\omega}(\tilde{T})\) was obtained using the so-called “kick” function, defined in (3.10). After using \(\tilde{\rho} \simeq \pi^2 g_{\text{eff}}(\tilde{T})\tilde{T}^4/30\),
Particle | Mass (GeV) | $g_A$
---|---|---
$t$ | 173.2 | 12
$H$ | 125.1 | 1
$Z$ | 91.19 | 3
$W^\pm$ | 80.39 | 6
$b$ | 4.18 | 12
$\tau$ | 1.78 | 4
$c$ | 1.27 | 12
$\pi^0$ | 0.140 | 12
$\pi^\pm$ | 0.135 | 2
$\mu$ | 0.106 | 4
$e$ | 0.000511 | 4

| Table 1 | Spectrum of particles used to calculate the kick function (A.1). For each particle, we show its mass and number of internal degrees of freedom, $g_A$. |

(3.11) and (3.12) $\Sigma$ becomes

$$
\Sigma(\tilde{T}) = \sum_A \frac{15}{\pi^2} \frac{g_A}{g_{\text{eff}}(\tilde{T})} y_A^2 \int_{y_A}^{\infty} dx \frac{\sqrt{x^2 - y_A^2}}{e^{x \pm 1}},
$$

(A.1)

where $g_{\text{eff}}(\tilde{T})$ is the total number of relativistic degrees of freedom\(^\text{11}\), $g_A$ is the number of internal degrees of freedom of particles of type $A$, $y_A = m_A/\tilde{T}$ and the plus (minus) sign in the integral is for fermions (bosons). The particle spectrum used to compute $\Sigma$ is shown in Table 1 and in Figure we plot the resulting kick function during the radiation dominated era. Then, we compute the equation of state from (3.9) as $\tilde{\omega} = (1 - \Sigma(\tilde{T}))/3$, which is shown in Figure 4.

\textbf{A.1 Conformal case solutions}

Using the numerical solution for $\tilde{\omega}$, we plugged it into the master equation, and solve it numerically for the interesting initial conditions as described in section 3.1.1 for the conformal case. To look for suitable solutions, we fixed the initial value of the velocity and then look for an initial value of the scalar such that $(1 + \alpha(\varphi)\varphi')$ stayed positive, thus giving a positive modified expansion rate. As we explained in that section, we were aiming at solutions where the scalar field passed from positive to negative to positive values again. In Figure 11, we show the thermal evolution of the scalar field in the pure conformal scenario for different initial velocities $\varphi'$. We let the initial value of $\varphi$, fixed at $\varphi_i = 0.2$ and solve the master equation (3.5) for the different initial velocities and initial temperature to be 1000 GeV. This plot shows the behaviour of the scalar field as described in section 3.1.1. In Figure 12, we show the resulting modified expansion rates for the different initial conditions considered in Fig. 11.

\(^{11}\)To calculate $g_{\text{eff}}$ we follow the numerical procedure described in Appendix A of [27].
Figure 10: Thermal evolution of the kick function during the radiation dominated era. Outside the interval of temperatures shown, $\Sigma$ vanishes.

Figure 11: Scalar field as function of temperature in the pure conformal scenario for various initial conditions. We use $\varphi_i = 0.2$ and every curve shown corresponds to a different value of $\varphi'_i$.

A.2 Disformal case solutions

As discussed in the main text, when both conformal and disformal functions are non-zero, we solve the system of coupled equations (2.35) and (2.34) numerically with Mathematica. We explored different boundary conditions for the scalar field and its first derivative. With the solutions of these equations we used (2.42) to find the modified expansion rate. In Figure 13 we show the modified expansion rates for a disformal factor given by $D =$
Figure 12: Modified expansion rate as function of temperature in the conformal scenario for different boundary conditions. We use $\varphi_i = 0.2$ and every curve shown corresponds to a different value of $\varphi'_i$.

$D_0\varphi^2$ with $D_0 = -4.9 \times 10^{-14}$ and the conformal factor being the same as before, that is $C = (1 + b e^{-\beta \varphi})^2$ with $b = 0.1$, $\beta = 8$. We have added one additional initial condition with respect to the conformal case, $\varphi'_\text{initial} = -1$. It is interesting to notice that for this initial condition, the pure conformal case does not give a solution satisfying the necessary constraints explained in the main text. In this sense, we see that the disformal contribution is important in order to find solutions otherwise excluded.

B General Disformal Set-Up

The general scalar-tensor action coupled to matter, which can include a realisation in string theory compactifications is given by:

$$S = S_{EH} + S_\phi + S_m,$$

(B.1)

where:

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R,$$

(B.2)

$$S_\phi = -\int d^4x \sqrt{-g} \left[ \frac{b}{2} (\partial \phi)^2 + M^4 C_1^2(\phi) \sqrt{1 + \frac{D_1(\phi)}{C_1(\phi)} (\partial \phi)^2 + V(\phi)} \right],$$

(B.3)

$$S_m = -\int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{DM}(\tilde{g}_{\mu\nu}),$$

(B.4)

and the disformally coupled metric is given by

$$\tilde{g}_{\mu\nu} = C_2(\phi) g_{\mu\nu} + D_2(\phi) \partial_\mu \phi \partial_\nu \phi.$$

(B.5)
Figure 13: Modified expansion rate as function of temperature in the disformal scenario for various initial conditions. We use $\varphi_i = 0.2$ and every curve shown corresponds to a different value of $\varphi'_i$.  

$b$ is a constant equal to 1 or 0, depending on the model one wants to consider; $C_i(\phi), D_i(\phi)$ are functions of $\phi$, which can be identified as conformal and disformal couplings of the scalar to the metric, respectively. Finally, we have introduced the mass scale $M$ to keep units right (remember that the conformal coupling is dimensionless, whereas the disformal has units of $Mass^{-4}$.)

The connection of the general action (B.1) to the different models in the literature can be obtained as follows: the case $C_1 = D_2$, $D_1 = D_2$, $b = 0$ arises when considering a D-brane moving along an extra dimension. This case was studied in [17] as a model of a coupled dark matter dark energy sector scenario, where scaling solutions arise naturally. Note that in this case, the kinetic term for the scalar field, identified with dark energy for example, is automatically non-canonical and dictated by the DBI action (see [17]). On the other hand, phenomenological models considering a disformal coupling between matter and a scalar field, usually consider a canonical kinetic term, and therefore, in that case, $C_1 = D_1 = 0$ and $b = 1$ ($C_1$ can be taken to be non-zero and will be part of the scalar potential). Furthermore, the widely studied case of a conformal coupling is obtained for $b = 1, C_1 = D_1 = D_2 = 0$ or, as in the case of a D-brane for example\(^{12}\), simply considering small velocities with $b = 0, C_1 = C_2$ and $D_1 = D_2$, and normalising canonically the scalar field (see Appendix C).

Finally, let us clarify further our nomenclature on frames. The action in (B.1) is written in the Einstein frame, which in string theory, is usually related to the frame in which the

\(^{12}\)For the system corresponding to a D-brane moving in a typically warped compactification in string theory, the functions $C(\phi)$ and $D(\phi)$ are identified with powers of the so-called warp factor, usually denoted as $h(\phi)$. In this approach, the longitudinal and transverse fluctuations of the D-brane are identified with the dark matter and dark energy fluids respectively [17].
dilaton and the graviton degrees of freedom are decoupled. From this point of view, the dilaton field as well as all other moduli fields not relevant for the cosmological discussion are considered as stabilised, massive, and are therefore decoupled from the low energy effective theory. In the literature of scalar-tensor theories however (including conformal and disformal couplings), the Einstein and Jordan frames are identified with respect to the (usually single) scalar field to which gravity is coupled. In this paper, we follow this and call “Jordan” or “disformal frame” the frame in which dark matter is coupled only to the metric $\tilde{g}_{\mu\nu}$, rather than to the metric $g_{\mu\nu}$ and a scalar field $\phi$.

The equations of motion obtained from (B.1) are (2.4):

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 \left( T^\phi_{\mu\nu} + T^D_{\mu\nu} \right) ,$$  \hspace{1cm} (B.6)

where in the frame relative to $g_{\mu\nu}$ the energy momentum tensors are defined in (2.5) and (2.6). The energy-momentum tensor for the scalar field in the general case is modified from (2.7) to:

$$T^\phi_{\mu\nu} = - g_{\mu\nu} \left[ M^4 C_1^2 \gamma_1^{-1} + \frac{b}{2} (\partial \phi)^2 + V \right] + (M^4 C_1 D_1 \gamma_1 + b) \partial_\mu \phi \partial_\nu \phi$$  \hspace{1cm} (B.7)

where now the energy density and pressure are given by:

$$\rho_\phi = - \frac{b}{2} (\partial \phi)^2 + M^4 C_1^2 \gamma_1 + V , \quad P_\phi = - \frac{b}{2} (\partial \phi)^2 - M^4 C_1^2 \gamma_1^{-1} - V ,$$  \hspace{1cm} (B.8)

and the “Lorentz factor” $\gamma_1$ introduced above is defined by

$$\gamma_1 \equiv \left( 1 + \frac{D_1}{C_1} (\partial \phi)^2 \right)^{-1/2} .$$  \hspace{1cm} (B.9)

We can rewrite (B.8) in a more succinct way, by defining $V \equiv V + C_1^2 M^4$

$$\rho_\phi = - \left[ \frac{b}{2} + \frac{M^4 C_1 D_1 \gamma_1}{\gamma + 1} \right] (\partial \phi)^2 + V , \quad P_\phi = - \left[ \frac{b}{2} + \frac{M^4 C_1 D_1 \gamma_1^{-1}}{\gamma + 1} \right] (\partial \phi)^2 - V .$$  \hspace{1cm} (B.10)

The equation of motion for the scalar field becomes (compare with (2.9))

$$- \nabla_\mu \left[ (M^4 D_1 C_1 \gamma_1 + b) \partial^\mu \phi \right] + \frac{\gamma_1^{-1} M^4 C_1^2}{2} \left[ \frac{D_1'}{D_1} + 3 \frac{C_1'}{C_1} \right] + \frac{\gamma_1 M^4 C_1^2}{2} \left[ \frac{C_1'}{C_1} - \frac{D_1'}{D_1} \right] + V''$$

$$- \frac{T^\mu\nu}{2} \left[ \frac{C_2'}{C_2} g_{\mu\nu} + \frac{D_2'}{C_2} \partial_\mu \phi \partial_\nu \phi \right] + \nabla_\mu \left[ \frac{D_2}{C_2} T^\mu\nu \partial_\nu \phi \right] = 0 .$$  \hspace{1cm} (B.11)

Finally, the energy-momentum conservation equation gives rise to (2.11), where $Q$ now is given in terms of $C_2, D_2$:

$$Q \equiv \nabla_\mu \left[ \frac{D_2}{C_2} T^\mu\lambda \partial_\lambda \phi \right] - \frac{T^\mu\nu}{2} \left[ \frac{C_2'}{C_2} g_{\mu\nu} + \frac{D_2'}{C_2} \partial_\mu \phi \partial_\nu \phi \right] .$$  \hspace{1cm} (B.12)
B.1 General cosmological equations

The equations of motion for the general system in an FRW background become:

\[ H^2 = \frac{\kappa^2}{3} [\rho_\phi + \rho], \]  
\[ (B.13) \]

\[ \dot{H} + H^2 = -\frac{\kappa^2}{6} [\rho_\phi + 3P_\phi + \rho + 3P], \]  
\[ (B.14) \]

\[ \ddot{\phi} \left[ 1 + \frac{b}{M^4 C'_1 D_1 \gamma^3_1} \right] + 3H\dot{\phi} \gamma^{-2}_1 \left[ \frac{b}{M^4 C'_1 D_1 \gamma_1} + 1 \right] \]
\[ + \frac{C_1}{2D_1} \left( \gamma^{-2}_1 - \frac{5C'_1}{C_1} D_1' - \frac{D_1}{D_1} - \frac{C'_1}{C_1} - 4\gamma^{-3}_1 \frac{C'}{C} \right) + \frac{1}{M^4 C'_1 D_1 \gamma^3_1} (V' + Q_0) = 0, \]  
\[ (B.15) \]

where, \( H = \frac{\dot{a}}{a} \), dots are derivatives with respect to \( t \), \( ' = d/d\phi \) and

\[ \gamma_1 = (1 - D_1 \dot{\phi}^2/C_1)^{-1/2}. \]

We also have the continuity equations for the scalar field and matter given by

\[ \dot{\rho}_\phi + 3H(\rho_\phi + P_\phi) = -Q_0 \dot{\phi}, \]  
\[ (B.16) \]

\[ \dot{\rho} + 3H(\rho + P) = Q_0 \dot{\phi}, \]  
\[ (B.17) \]

where \( Q_0 \) is given by

\[ Q_0 = \rho \left( \frac{D_2}{C_2} \frac{\ddot{\phi}}{\phi} + \frac{D_2}{C_2} \frac{\dot{\phi}}{\rho} \left( 3H + \frac{\dot{\rho}}{\rho} \right) + \left( \frac{D'_2}{2C_2} - \frac{D'_2 C'_2}{C_2 C'_2} \right) \phi^2 + \frac{C'_2}{2C_2} (1 - 3\omega) \right). \]  
\[ (B.18) \]

Using (B.17) we can rewrite this in a more compact and useful form as

\[ Q_0 = \rho \left( \frac{\ddot{\gamma}_2}{\phi \gamma_2} + \frac{C'_2}{2C_2} (1 - 3\omega \gamma^2_2) - 3H \omega \left( \frac{\gamma_2 - 1}{\phi} \right) \right), \]  
\[ (B.19) \]

where

\[ \gamma_2 = (1 - D_2 \dot{\phi}^2/C_2)^{-1/2}. \]

Plugging this into the (non-)conservation equation for dark matter (B.17), gives:

\[ \dot{\rho} + 3H(\rho + P \gamma^2_2) = \rho \left( \frac{\ddot{\gamma}_2}{\gamma_2} + \frac{C'_2}{2C_2} \phi (1 - 3\omega \gamma^2_2) \right). \]  
\[ (B.20) \]

The energy densities and pressures in the Einstein and Jordan frames are now related similarly to (2.19), replacing \( \gamma \to \gamma_2 \):

\[ \tilde{\rho} = C_2^{-2\gamma_2^{-1}} \rho, \quad \tilde{P} = C_2^{-2\gamma_2} P, \]  
\[ (B.21) \]

and therefore the equation of states in both frames are related by \( \tilde{\omega} = \omega \gamma^2_2 \). Similarly the physical proper time and the scale factors in the two frames are related via \( \gamma_2 \):  

\[ \tilde{a} = C_2^{1/2} a, \quad d\tilde{\tau} = C_2^{1/2} \gamma^{-1}_2 d\tau. \]  
\[ (B.22) \]
Defining the disformal frame Hubble parameter $\dot{H} \equiv \frac{d\ln \hat{a}}{d\tau}$, gives:

$$\dot{H} = \frac{\gamma_2}{C_2^{1/2}} \left[ H + \frac{C_2'}{2C_2} \dot{\phi} \right]. \quad (B.23)$$

To solve the equations of motion one now can proceed as in section 2.2 to write the equations in terms of derivatives w.r.t. the number of e-folds $N$ and consider different cases by choosing appropriately the parameters $b, C_i, D_i$. We leave the analysis of these for a future publication.

C  The conformal case in D-brane scenarios

In this section we show how to recover the pure conformal case from the D-brane picture, that is, $b = 0, C_1 = C_2, D_1 = D_2$. We start by expanding the square root in the scalar part of the action (B.1). Doing this we get

$$S_\phi = -\int d^4x \sqrt{-g} \left[ M^4 C_1^2 \left( 1 + \frac{D_1}{2C_1} (\partial \phi)^2 + \ldots \right) + V(\phi) \right]$$

$$= -\int d^4x \sqrt{-g} \left[ M^4 C_1 D_1 \left( \partial \phi \right)^2 + M^4 C_1^2 (\phi) + V(\phi) + \ldots \right]$$

$$= -\int d^4x \sqrt{-g} \left[ \frac{M^4 C_1 D_1}{2} (\partial \phi)^2 + V(\phi) + \ldots \right], \quad (C.1)$$

On the other hand, the matter Lagrangean takes the form

$$S_{DM} = -\int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{DM}(\tilde{g}_{\mu\nu})$$

$$= -\int d^4x \sqrt{-\tilde{g}} C_1^2(\phi) \left( 1 + \frac{D_1}{2C_1} (\partial \phi)^2 + \ldots \right) \mathcal{L}_{DM}(\tilde{g}_{\mu\nu})$$

$$= -\int d^4x \sqrt{-\tilde{g}} C_1^2(\phi) \mathcal{L}_{DM}(\tilde{g}_{\mu\nu}) + \ldots = -\int d^4x \sqrt{-\tilde{g}} \mathcal{L}_{DM}(\tilde{g}_{\mu\nu}) + \ldots \quad (C.2)$$

where now $\tilde{g}_{\mu\nu} = C_1(\phi)g_{\mu\nu}$ (and we have used that $\det \tilde{g}_{\mu\nu} = C_1^4(1 + D_1/C_1(\partial \phi)^2))$.

Finally, to compare the D-brane case with the pure conformal case, we need to canonically normalise $\phi$. Calling $\varphi$ the canonically normalised field, this is obtained from $\phi$ as

$$\varphi = \int M^2 \sqrt{D_1 C_1} d\phi. \quad (C.3)$$

It is clear that when $D_1 = 1/(M^4 C_1)$, $\varphi = \phi$ and therefore the action for the scalar field (C.1) is already in the required form. We can now take the limit $\gamma \to 1$ into the equations of motion (B.13)-(B.15) and make the identification $D_1 = 1/(M^4 C_1)$ to recover the conformal case equations of motion. Note that in this limit $Q_0 \to \rho C_1'/2C_1$, and is independent of $D_1$ (see (B.19) with $C_1 = C_2, D_1 = D_2$).
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