Calculating hadron properties from dynamical hadronization in the Functional Renormalisation Group

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Abstract. We report on a functional renormalisation group approach to bound state properties similar to the Dyson-Schwinger–Bethe-Salpeter approach. The current approach is set-up for the two-flavour quark-meson model for an illustration of the basic properties. This allows us to access the pion and sigma poles and decay properties. First numerical results are presented and evaluated. We also discuss the next steps.

1. Introduction

Hadron properties such as masses or decay widths are, at least in principle, determined by the underlying dynamics of quarks and gluons. Due to their strong interactions at small scales and the non-perturbative nature of bound states, non-perturbative methods as well as real-time computations are required for calculating such observables. Functional methods, mainly the Dyson-Schwinger and Bethe-Salpeter equations (DSE-BSE) approach, have been successfully applied to access hadron resonances, for a recent review on the successes and open problems within this approach see [1].

In the present work we report on first steps towards a similar programme within the functional renormalisation group (FRG) approach. This programme is based on two cornerstones: the dynamical hadronisation procedure, [2, 3, 4], as formulated for QCD in [5, 6, 7], and the recent development of real-time approaches to the FRG in the context of Minkowski space-time, see e.g. [8, 9, 10, 11]. Dynamical hadronisation allows to reparameterise resonant hadronic channels in multi-quark interactions in terms of composite operators, the effective hadronic degrees of freedom. For an access to bound state properties one has to evaluate the correlation functions of these composite operators for real-time frequencies. In such an approach the pole masses of the hadronic resonances are encoded in the analytic properties of the propagators of the effective hadronic degrees of freedom. Decay widths are encoded in the vertex functions that are directly related to Bethe-Salpeter wave functions. In summary this provides a uniform approach only based on the effective action of the theory. More details on this approach will be presented elsewhere.
2. The Functional Renormalisation Group
The FRG approach in its modern form is based on the effective action \( \Gamma_k[\Phi] \) of the theory, where the superfield \( \Phi \) includes all fields in the theory at hand. In the present work we consider the two-flavour quark-meson model with \( \Phi = (q, \bar{q}, \phi) \) with \( \phi = (\sigma, \vec{\pi}) \). After introducing an infrared cutoff \( k \) the scale-dependent effective action \( \Gamma_k \) satisfies the Wetterich equation [12],

\[
\partial_t \Gamma_k = \frac{1}{2} \text{Tr} \left( \frac{1}{\Gamma_k^{(2)}} + R_k \right),
\]

where \( k \) is the RG scale, \( \partial_t = k \partial_k \), and \( \Gamma_k^{(n)} \) is the \( n \)th field derivative of the effective action. The trace sums over species of fields, internal indices and four-momenta, and \( R_k(p^2) \) is a regulator function that suppresses the propagation of the infrared modes of all fields \( \Phi \). The present work is based on and uses the same notation as the QCD-works [5, 6, 7], for more details and literature we refer to these works. Finally, we use analytic regulators that have the full Euclidean \( O(4) \) space-time symmetry, and hence also preserves \( O(1,3) \) Minkowski space-time symmetry. In our opinion keeping the full space-time symmetry in the regularisation is key to an access to the higher resonances. Also, it allows for a fully numerical access to correlation functions at real-time also at finite temperature and density, see [10, 11].

3. The Quark-Meson model with dynamical hadronisation
Two-flavour QCD with dynamical hadronisation for the scalar–pseudo-scalar multiplet has been studied in [5, 6, 7] in the Landau gauge. There it has been shown, that the decoupling of gluons at very low energies leads to a low-energy effective theory in terms of quarks \( q, \bar{q} \) and \( \phi = (\sigma, \vec{\pi}) \) below some cutoff scale \( k_{\text{EFT}} \) that serves as an ultraviolet (UV) cutoff for the effective theory. Accordingly, the two-flavour quark-meson model serves as an ideal testing ground for the techniques and mechanisms at work that will be applied to QCD later.

3.1. QCD-induced quark-meson model
Let us briefly describe how the quark-meson model emerges as a low energy effective theory from QCD with dynamical hadronisation: the QCD-flows in [5, 6, 7] are initiated at a high initial cutoff scale \( k \approx 10^2 \text{ GeV} \). There, perturbation theory works and the effective action at this initial scale is well approximated by the classical action. When flowing down to non-perturbative scales \( k \approx 1 \text{ GeV} \) multi-quark interactions to all order are generated by (1). In particular scalar-pseudo-scalar four-quark interactions are generated. The \( \eta \) and \( \vec{a} \) fields have masses that are far larger than the UV cutoff scale \( k_{\text{EFT}} \) of the low-energy effective theory, and hence do not contribute to the dynamics below this scale. Note however, that they are present in the effective action and cannot be neglected in the evaluation of physical processes. In the present structural work we simply drop them. Concentrating on the local part of the remaining scalar-pseudoscalar interactions we are led to

\[
\Gamma_k^{(4-\text{int})} = -\lambda_k \int_x \left( \frac{(q\bar{q})^2}{4} + (q\bar{q}\gamma_5\tau^a q)^2 \right),
\]

where the \( \tau^a = \sigma^a/2 \) with the Pauli matrices \( \sigma^a \) and \( \int_x = \int d^4x \). The scale-dependent coupling \( \lambda_k \) changes via quark-gluon diagrams as well as pure quark or gluon diagrams. In the vicinity of the chiral symmetry breaking scale the four-fermi interaction \( \lambda_k \) grows large and even diverges in the chiral limit. It is convenient to describe such resonant channels in terms of effective degrees of freedom. To that end we rewrite (2) as

\[
\Gamma_k^{(4-\text{int})} = \int_x \left[ \frac{1}{2} (\sigma^2 + \vec{\pi}^2) (\rho^2 + m_k^2) + h_k \bar{q} \left( \frac{\sigma}{2} + i\gamma_5\tau^a \pi^a \right) q \right] \text{EoM}(\sigma, \vec{\pi})
\]
by means of a scale-dependent Hubbard-Stratonovich (HS) transformation: with the identification \( \lambda_k = h_k^2/(2m^2) \) the right hand side of (3) reduces to (2) on the equations of motion for \( \sigma, \vec{\pi}. \) This procedure introduces effective mesonic \( \sigma \) and \( \vec{\pi} \) degrees of freedom.

3.2. Dynamical hadronisation

If the above procedure is performed at one cutoff scale \( k \), the four-fermi interaction (2) is regenerated in QCD in the next RG-step via quark-gluon diagrams as well as pure quark or gluon diagrams. Even if switching off the remaining QCD interactions the hadronised interaction (3) itself regenerates the four fermi-interaction. A suitable scale-dependent HS transformation will reshuffle this remnant four-fermi interaction again into the Yukawa interaction which in turn induces a four-fermi interaction in the next RG step. This successive interplay of generating the four-quark interaction, or more generally multi-quark interactions, and then nullifying them, is called Dynamical Hadronization \([2, 3, 4]\). Here we have described it for the local (momentum-independent) part of a four-fermi interaction, but it can be applied to a full momentum-dependent channel, see \([5, 7]\) for the application to full QCD. By construction the hadronised theory still is full QCD, no information is lost and no double-counting problem arises as typically in the case of a fixed-scale HS transformation. For a more detailed discussion see \([5, 7]\).

While simply being an identity transformation, dynamical hadronisation provides a significant practical advantage: A resonant channel of the four-quark interaction is rewritten in the form of Yukawa terms, and therefore one channel of the four-point correlation function is represented by a meson exchange, which facilitates calculations. The scale-adaptive nature of the dynamical hadronisation implies scale-dependent mesonic fields, \( \Phi = (\sigma, \vec{\pi}) \rightarrow \Phi_k = (\sigma_k, \vec{\pi}_k) \) leading to

\[
\partial_t \Gamma_k \rightarrow \partial_t \Gamma_k + \int \frac{\delta \Gamma_k}{\delta \Phi_k} \partial_t \Phi_k, \tag{4}
\]

on the left hand side of the flow equation (1). Even in the case discussed above, where we remove the local four-quark interaction, \( \lambda_k \rightarrow 0 \) we still have a reparameterisation freedom of the mesonic fields due to \( \lambda_k = h_k^2/(2m^2) \): neither \( h_k \) nor \( m_\pi \) is fixed. This is an important example as it entails that \( m_\pi \) is not necessarily the physical mass of the pion. Here we choose

\[
\partial_t \sigma_k = \frac{1}{2} \int \partial_t A_k \bar{q}q, \quad \partial_t \pi^a_k = \int \partial_t A_k \bar{q}i\gamma_5 \tau^a q, \tag{5}
\]

such that the requirement \( \partial_t \lambda_k = \text{flow}(4) - h_k \partial_t A_t = 0 \) (as suggested in refs. \([2, 3, 4, 5, 6, 7]\)) determines the quantity \( A_k \) in every RG step. Here \( \text{flow}(4) \) stands for the diagrams that are derived from the right hand side of (1) for the flow of the local part of the scalar-pseudoscalar part of the four-quark interaction.

3.3. Current Approximation to the quark-meson model

Above we have outlined how the quark-meson model emerges as a low-energy effective theory from full QCD flows with dynamical hadronisation. Apart from the quark-meson interactions discussed in the previous section further interactions as well as the kinetic terms are generated. At the UV scale of the effective field theory, \( k_{\text{EFT}} \) we can read off the UV action of the quark-meson model as the result of QCD flows from the initial scale \( k \approx 10^2 \text{ GeV} \) to \( k_{\text{EFT}} \).

Here we concentrate on the extraction of real-time quantities and take the effective field theory point of view: we fix the UV couplings of the model at \( k_{\text{EFT}} \) by adjusting observables at \( k = 0 \). It is left to discuss the approximation used in the present work. Finally we want to use and extend the approximation to the quark-meson sector used in the QCD flows \([5, 7]\), for more details see the discussion there. Here we shall use a much reduced version of this
approximation which still far surpasses those used before in quark-meson model computations in terms of momentum and frequency dependence. This approximation will allow us to study the qualitative feature of the real-time properties of the quark-meson sector of QCD. Moreover, in terms of momentum and frequency dependence. This approximation will allow us to study

\[ \Gamma_k = \int_x \left[ Z_{k,\psi} \bar{q} \gamma_\mu q + \frac{1}{2} \left( Z_{k,\sigma} (\partial \sigma)^2 + Z_{k,\pi} (\partial \pi)^2 \right) + h_k \bar{q} \left( \frac{\sigma}{2} + i \gamma_\mu \tau^a \pi^a \right) q + V_k[\rho] \right], \tag{6} \]

with \( \rho = 1/(\sigma^2 + \pi^2) \) and four-momentum dependent wave function renormalisations \( Z_{k,\psi} = Z_{k,\phi}(p) \) as well as a momentum-dependent Yukawa coupling \( h_k = h_k(p^2/2 + q^2/2) \) with quark and anti-quark momentum \( p \) and \( q \). The full effective potential \( V_k(\rho) \) is expanded about a fixed value \( \rho_0 \) of the unrenormalised field which leads to rapid convergence, see [13],

\[ V_k[\rho] = \sum_{n=0}^{\infty} \frac{V^{(n)}_k(\rho_0)}{n!} (\rho - \rho_0)^n. \tag{7} \]

\( \rho_0 \) is chosen such that it is close to the physical expansion point \( \rho_{\text{phys}} \propto f_\pi^2 \). Note that the latter is not at the minimum of \( V_k \) in the case of explicit chiral symmetry breaking, for a discussion see e.g. [13]. At the minimum the first and second expansion coefficients of \( V_k \) define the curvature masses of the mesons, \( m_{\pi,\text{cur}}^2 = V_k^{(1)}(\rho_{\text{phys}}) \) and \( m_{\sigma,\text{cur}}^2 = V_k^{(1)}(\rho_{\text{phys}}) + \sqrt{2\rho_{\text{phys}} V_k^{(2)}(\rho_{\text{phys}})} \).

In principle, one may consider all orders of the potential. However, it has been checked that a polynomial approximation about a fixed expansion point \( \rho_0 \) up to the order \( \rho^6 \) provides a robust and quantitatively accurate representation of the potential, [5, 6, 7, 13].

4. Observables and analytical continuation

At vanishing cutoff scale \( k = 0 \) we are now in the position to determine observables such as the pole masses. To that end we need to perform an analytical continuation of the Euclidean results. In future work this is done by a direct computation along the lines of [10, 11].

However, here we will exploit two different and less demanding methods of analytical continuation leaving the direct calculation for future work. The first one is using a Padé approximant and determining the respective poles, e.g. [14], while the second method is a modified Padé approach based on the Schlessinger point method, e.g. [15].

The following results are obtained by tuning to certain infrared values. In particular we require the curvature masses \( m_{k=0} \) to be \( m_{\pi,\pi} = 138 \text{ MeV} \), \( m_{\pi,\sigma} = 536 \text{ MeV} \), and \( m_{\psi} = 297 \text{ MeV} \). Furthermore, \( f_\pi = 93 \text{ MeV} \) is used as input. Starting the RG flow from the UV scale to be chosen to be 0.95 GeV we obtained the results displayed in figs. 1 and 2.

Using the analytical continuation methods described above, one obtains pole masses that can be compared with the curvature masses, both for \( k \to 0 \). Hereby the uncertainties for the Padé pole masses are derived from the convergence pattern of different orders of the approximant. The results agree very well with the curvature mass for the pion, in accordance with [14], but not with the same precision for the sigma meson. Further results will be published elsewhere.

| Particle       | Curvature Mass (Input) | Pole Mass Padé   | Pole Mass Schlessinger |
|----------------|------------------------|------------------|-----------------------|
| Pion           | 138.22                 | 136.2 ± 0.1      | 136.4 ± 1.0           |
| Sigma meson    | 536.37                 | 472 ± 7          | 479 ± 10              |

Table 1. Pole masses vs. curvature masses, all in MeV.
5. Summary and Outlook

In this preliminary study we have demonstrated that the FRG is a functional method suitable for calculating observables related to bound states. Within the Quark-Meson model we determined the propagators of the sigma and the pion fields, and from them the respective pole masses via Padé approximants. In a next step we will compute real time correlation functions with the methods devised in [10, 11] as well as improving the approximation towards full QCD along the lines of [5, 6, 7].

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