Abstract
Given a connected reductive algebraic group $G$ defined over a finite field $\mathbb{F}_q$ together with a representation $\rho^\flat : G^\flat \to \mathrm{GL}_N$ of the dual group of $G$ (in the sense of Deligne-Lusztig), Braverman and Kazhdan [4] defined an exotic Fourier operator on the space of complex valued functions on $G(\mathbb{F}_q)$. Under some assumption on $\rho^\flat$, they gave a conjectural formula for the Fourier kernel which they prove when $G = \mathrm{GL}_n$ for some $n$. In these notes we give a simple proof of their conjecture for any $G$ without any assumption on $\rho^\flat$.
1 Introduction

Let $G$ be a connected reductive algebraic groups over $\overline{\mathbb{F}}_q$ with geometric Frobenius $F : G \to G$. Let $G^b$ be a connected reductive algebraic group over $\overline{\mathbb{F}}_q$ with geometric Frobenius $F^b$ such that $(G, F)$ and $(G^b, F^b)$ are in duality (in the sense of Deligne-Lusztig [7]). Lusztig defined a partition of the set $\hat{G}^F$ of all irreducible $\overline{\mathbb{Q}}_\ell$-characters of $G^F$ (with $\ell$ a prime not dividing $q$) indexed by the set of $F^b$-stable semisimple conjugacy classes of $G^b$. We call the parts of this partition the Lusztig series and we denote by $\text{LS}(G)$ the set of Lusztig series of $(G, F)$.

Let $(G', F)$ be a standard pair, i.e. $G'$ is an $F$-stable Levi subgroup of some parabolic subgroup of $\text{GL}_n$ where $F : \text{GL}_n \to \text{GL}_n$ is the standard Frobenius. We assume given a morphism $\rho^b : G^b \to G'^b = G'$ which commutes with Frobenius. It induces a map between semisimple conjugacy classes fixed by Frobenius. Using Lusztig’s parametrization we thus get a map

$$t_\rho : \text{LS}(G) \to \text{LS}(G').$$

Notice that in the case where $\rho^b$ is normal, i.e. its image is a normal subgroup of $G'$, there exists a morphism $\rho : G' \to G$ in duality with $\rho^b$ (see Proposition 3.5.2) and in this case the map $t_\rho$ is given by the pull-back functor along the map $\rho^F : G'^F \to G^F$ (see Proposition 3.5.3). Although the morphism $\rho$ does not exists in general, we keep it in the notation $t_\rho$.

Exotic Fourier operators

Consider the space $C(G'^F)$ of all functions $G'^F \to \overline{\mathbb{Q}}_\ell$. Fix a non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell$ and consider the standard Fourier kernel
The standard Fourier operator $F^{G'} : C(G^F) \to C(G^F)$ is then defined by

$$F^{G'}(f)(x) = \sum_{y \in G^F} \phi^{G'}(xy)f(y).$$

We also consider the standard gamma function $\gamma^{G'} : \hat{G}^F \to \overline{\mathbb{Q}}_\ell$ defined by the formula

$$\gamma^{G'}(\chi) = \sum_{g \in G^F} \phi^{G'}(g)\chi(g) \chi(1).$$

For any irreducible character $\chi$ of $G^F$ we then have

$$F^{G'}(\chi) = \gamma^{G'}(\chi)\chi^\vee$$

where $\chi^\vee$ denotes the dual character of $\chi$, and one can recover the Fourier kernel from the gamma function by the formula

$$\phi^{G'} = \sum_{\chi \in \hat{G}^F} \chi(1)\gamma^{G'}(\chi)\chi^\vee. \quad (1.1)$$

The function $\gamma^{G'}$ is admissible, i.e. it is constant on Lusztig series (see below Theorem 5.1.3), and so induces a function on $LS(G')$ which we still denote by $\gamma^{G'}$ as no confusion should arise. Using the morphism $\rho$, we can thus transfer $\gamma^{G'}$ into an admissible function on $\hat{G}^F$ by putting

$$\gamma^G_{\rho} := c_{G,G'}\gamma^{G'} \circ t_p,$$

where $c_{G,G'} = \epsilon_G \epsilon_{G'} q^{v_G - v_{G'}}$ with $\epsilon_G = (-1)^{\ell_G - \mathrm{rank}(G)}$ and $v_G$ the dimension of the unipotent radical of a Borel subgroup of $G$.

Define then the corresponding Fourier kernel $\phi^G_{\rho} : G^F \to \overline{\mathbb{Q}}_\ell$ by the right hand side of the formula $(1.1)$ with $\gamma^{G'}$ (resp. $G'$) replaced by $\gamma^G_{\rho}$ (resp. $G$) and denote by $F^G_{\rho} : C(G^F) \to C(G^F)$ the corresponding exotic Fourier operator with kernel $\phi^G_{\rho}$, i.e.

$$F^G_{\rho}(f)(x) = \sum_{y \in G_F} \phi^G_{\rho}(xy)f(y)$$

for all $f \in C(G^F)$.

These Fourier transforms $F^G_{\rho}$ were first studied by Braverman and Kazhdan in [4]. They conjectured an explicit formula for the kernel $\phi^G_{\rho}$.
The aim of these notes is to provide a simple proof of their conjecture. Moreover we do not make any assumption on \( \rho \) unlike in [4].

**Description of \( \phi^G_{\rho} \) in terms of Deligne-Lusztig theory**

We will consider the quotient stack \([G/G]\) for the conjugation action and we will identify freely the space \( C([G/G]^F) \) of functions on \([G/G]^F\) with the subspace of \( C(G^F) \) of \( G^F \)-invariant functions (this is possible because \( G \) is connected).

Fix a maximally split \( F \)-stable maximal torus \( T \) of \( G \) and denote by \( N \) the normalizer of \( T \) in \( G \).

Recall that the \( G^F \)-conjugacy classes of \( F \)-stable maximal tori of \( G \) are parametrized by the set \( H^1(F, N) \) of \( F \)-conjugacy classes. For \( \overline{w} \in H^1(F, N) \) and \( w \in N \) a representative of \( \overline{w} \), we denote by \( T_w \) an \( F \)-stable maximal torus of \( G \) in the \( G^F \)-class of \( F \)-stable maximal tori corresponding to \( \overline{w} \). Then the Frobenius \( F \) on \( T_w \) corresponds to the Frobenius \( F \circ w \) on \( T \).

A way to put all together the \( F \)-stable maximal tori of \( G \) is to consider the quotient stack \([T/N] \). Indeed

\[
[T/N]^F = \bigsqcup_{\overline{w} \in H^1(F, N)} [T^{F\circ w}/N^{F\circ w}]
\]

\[
\cong \bigsqcup_{\overline{w} \in H^1(F, N)} [T_w^F/N_w^F]
\]

where \( N_w = N_G(T_w) \).

For any \( F \)-stable maximal torus \( H \) of \( G \), put

\[
\gamma^H_{\rho} := c_{H,G} \gamma^G \circ t_{\rho_H} : H^F \to \overline{Q}_\ell
\]

where \( \rho_H^b : H^b \to G' \) is the composition \( H^b \leftrightarrow G^b \to G' \) and denote by \( \phi^H_{\rho} : H^F \to \overline{Q}_\ell \) the associated kernel.

By Lemma [5.2.1] we have the following explicit description : let \( T' \) be an \( F \)-stable maximal torus of \( G' \) that contains the image of \( H^b \) and denote by \( \rho_H : T' \to H \) the morphism in duality with \( H^b \to T' \) (see Remark [3.5.1]). Then

\[
\phi^H_{\rho} = c_{H,T'} \rho_H(\psi \circ \text{Tr}|_{T'}).
\]

The above collection of kernels \( \phi^H_{\rho} \) provides thus an explicit kernel

\[
\phi^H_{[T/N]} \in C([T/N]^F),
\]

and so a Fourier transform
\[
F_{[T/N]}^\rho : C([T/N]^F) \to C([T/N]^F).
\]

We denote by \(F_{[G/G]}^\rho : C([G/G]^F) \to C([G/G]^F)\) the restriction of \(F_{[G]}^\rho\) to invariant functions.

In §2.2 we define an injective \(\mathbb{Q}_\ell\)-linear map \(I_{[T/N]}^G : C([T/N]^F) \hookrightarrow C([G/G]^F)\) in terms of Deligne-Lusztig induction. If \(G\) is of type \(A\) with connected center, then it is an isomorphism.

We have the following theorem (see Theorem 5.2.3).

**Theorem 1.0.1.** The following diagram commutes

\[
\begin{array}{ccc}
C([T/N]^F) & \overset{F_{[T/N]}^\rho}{\longrightarrow} & C([G/G]^F) \\
\downarrow I_{[T/N]}^G & & \downarrow F_{[G]}^\rho \\
C([T/N]^F) & \overset{I_{[T/N]}^G}{\longrightarrow} & C([G/G]^F).
\end{array}
\]

In particular

\[
I_{[T/N]}^G(\phi_{[T/N]}^\rho) = \phi_{[G]}^\rho. \tag{1.3}
\]

We prove the commutativity of the diagram (4.5) using the spectral definition of \(F_{[T/N]}^\rho\) and \(F_{[G/G]}^\rho\) (i.e. the definition in terms of gamma functions). Since the function \(\phi_{[T/N]}^\rho\) has an explicit construction, Formula (4.6) gives an explicit realization of \(\phi_{[G]}^\rho\).

The fact that Formula (4.6) is a consequence of the commutativity of the diagram is explained in Remark 4.2.7(2).

**Geometric realization of \(\phi_{[G]}^\rho\)**

For simplicity we assume that \(G' = \text{GL}_n\) and that \(\rho^b\) restricts to a morphism \(\rho^b : T^b \to T'\) where \(T'\) is the maximal torus of \(G'\) of diagonal matrices. Then let \(\rho : T' \to T\) be a morphism in duality with \(\rho^b\) (see above Proposition 3.5.2). If we put \(W := N/T\), the morphism \(\rho\) is naturally \(W\)-equivariant.

Consider the Artin-Schreier sheaf \(L_{\rho}\) on \(\overline{\mathbb{F}}_q\) and put \(\Phi^{T'} = \text{Tr}'(L_{\rho})\). We then consider the complex

\[
\Phi_T := \rho^! \Phi^{T'} [\dim T']
\]

in the “derived category” \(\mathcal{D}_\ell^b(T)\) of constructible \(\ell\)-adic sheaves on \(T\). This complex is naturally \(W\)-equivariant, comes with a natural Weil structure \(F^* \Phi_T \simeq \Phi_T^F\), and the two are compatible (see §6.1). Moreover \(\Phi_T^F\) is a perverse sheaf (not necessarily irreducible) and so we get a perverse sheaf \(\Phi_{[T/N]}^\rho\) equipped with a Weil structure \(\varphi_{[T/N]}^\rho : F^* \Phi_{[T/N]}^\rho \simeq \Phi_{[T/N]}^F\).

For a stack \(\mathfrak{X}\) equipped with a geometric Frobenius, denotes by \(\mathcal{M}(\mathfrak{X}; F)\) the category of perverse sheaves on \(\mathfrak{X}\) equipped with a Weil structure.
Put

$$(\Phi^G_\rho, \varphi^G_\rho) := I^G_{[T/N]}(\Phi^{[T/N]}_\rho, \varphi^{[T/N]}_\rho)$$

where $I^G_{[T/N]} : \mathcal{M}([T/N]; F) \to \mathcal{M}([G/G]; F)$ is a geometric realization of $I^G_{[T/N]}$ (see §6.2).

**Theorem 1.0.2.** We have

$$X_{\Phi^G_\rho, \varphi^G_\rho} = 0$$

where $X_{\Phi^G_\rho, \varphi^G_\rho}$ is the characteristic function of $(\Phi^G_\rho, \varphi^G_\rho)$.

The complex $I^G_{[T/N]}(\Phi^{[T/N]}_\rho)$ is in fact the $W$-invariant part of the induced complex $\text{Ind}^G_T(\Phi^T_\rho)$, where $\text{Ind}^G_T$ is the parabolic induction from $T$ to $G$, and so the formula (1.4) is the one conjectured by Braverman-Kazhdan [4] under some restriction on $\rho^\flat$ (see Remark 6.3.4 for more details). They proved their conjecture when $G$ is a general linear group [4]. In the same paper, under the same assumption on $\rho^\flat$, they also conjectured that the geometric Fourier transform

$$F^G_{[G/G]} : \mathcal{D}^\Lambda([G/G]) \to \mathcal{D}^\Lambda([G/G])$$

defined from the kernel $\Phi^\Lambda_\rho$ commutes with $\text{Ind}^G_T$, and observed that this geometric conjecture implies Formula (1.4). This geometric conjecture reduces to the following one.

**Conjecture 1.0.3.** Let $B = TU$ be a Borel subgroup of $G$. The direct image with proper support of $\Phi^G_\rho$ along the map $G \to G/U$ is supported on $T = B/U \hookrightarrow G/U$.

Conjecture 1.0.3 was proved by Cheng and Ngô [6] when $G$ is a general linear group. Shortly after the first version of this paper was posted, Chen T.-H. [5] proved it for an arbitrary connected reductive group (under some restriction on the characteristic).

Conjecture 1.0.3 interprets commutativity of Fourier with induction in terms of kernels while our approach to prove (1.4) is to interpret commutativity of Fourier with induction in terms of gamma functions which are more flexible (see equivalence of assertions (1) and (3) in Proposition 4.2.2).

Let us finally remark that we have a natural geometric version $F^G_{[T/N]} : \mathcal{M}([T/N]) \to \mathcal{M}([T/N])$ of $\Phi^G_{[T/N]}$ and that when $G$ is of type $A$ with connected center, then $I^G_{[T/N]}$ is an equivalence $\mathcal{M}([T/N]) \simeq \mathcal{M}([G/G])$ (see [9]). The desired geometric Fourier transform $F^G_{[G/G]} : \mathcal{M}([G/G]) \to \mathcal{M}([G/G])$ can then be defined so that the geometric version of the diagram (4.5) commutes.
2 Reminders and notations

Our base field is a finite field $\mathbb{F}_q$ with $q$ elements and we choose an algebraic closure $\overline{\mathbb{F}}_q$. We fix a prime $\ell$ which is different from the characteristic of $k$ and we fix an algebraic closure $\overline{\mathbb{Q}}_\ell$ of the field of $\ell$-adic numbers. We choose an involution $\overline{\mathbb{Q}}_\ell \to \overline{\mathbb{Q}}_\ell$, $z \mapsto \overline{z}$ such that $\overline{z} = z^{-1}$ if $z$ is root of unity.

2.1 Reductive groups

If $T$ is a torus defined over $\mathbb{F}_q$ with Frobenius $F$, we have the multiplicative norm map $\text{Nr}_{F^r/F} : T(\mathbb{F}_{q^r}) \to T(\mathbb{F}_q)$, $x \mapsto xF(x) \cdots F^{r-1}(x)$.

Let $G$ be a connected reductive algebraic group over $\mathbb{F}_q$ with a geometric Frobenius $F : G \to G$ associated with some $\mathbb{F}_q$-structure on $G$ and denote by $L = L_G : G \to G$, $g \mapsto g^{-1}F(g)$ the Lang map. For a maximal torus $T$ of $G$ we denote by $X(T)$ the character group and by $Y(T)$ the co-character group. We denote by $Z_G$ the center of $G$ and for any $x \in G$ we denote by $C_G(x)$ the centralizer of $x$ in $G$.

Relative Weyl groups

For a Levi subgroup $L$ of (some parabolic subgroup of) $G$ we put

$$W_G(L) := N_G(L)/L,$$

where $N_G(L)$ denotes the normalizer of $L$ in $G$.

Example 2.1.1. A Levi factor $L$ of a parabolic subgroup of $GL_n$ is $GL_n$-conjugate to $L_\omega = (GL_{a_1})^{n_1} \times \cdots \times (GL_{a_r})^{n_r}$ for some positive integers $a_1, \ldots, a_r$ and $n_1 > \cdots > n_r$ such that $\sum_{i=1}^r a_in_i = n$. Note that the symmetric group in $m$ letters $S_m$ acts on each $(GL_x)^m$ as $w \cdot (g_1, \ldots, g_m) = (g_{w^{-1}(1)}, \ldots, g_{w^{-1}(m)})$. Therefore we have an action of $S_{a_1} \times \cdots \times S_{a_r}$ on $L_\omega$ and

$$N_{GL_n}(L_\omega) \simeq L_\omega \rtimes (S_{a_1} \times \cdots \times S_{a_r}),$$

and so $W_{GL_n}(L_\omega) \simeq S_{a_1} \times \cdots \times S_{a_r}$.

Recall that if we fix an $F$-stable Levi factor $L$ of some parabolic subgroup of $G$, then the $G^F$-conjugacy classes of $F$-stable Levi factors (of some parabolic subgroup of $G$) that are conjugate under $G$ to $L$ are parametrized by the set $H^1(F, N_G(L)) = H^1(F, W_G(L))$ of $F$-conjugacy classes of $W_G(L)$. For $w \in N_G(L)$ (or $w \in W_G(L)$), we then denote by $L_w$ a representative of the $G^F$-conjugacy class of $F$-stable Levi factors corresponding to the class of $w$ in $H^1(F, N)$. The pair $(L_w, F)$ is then isomorphic to the pair $(L, F \circ w)$ where $F \circ w$ is the Frobenius $t \mapsto F(w^{-1}tw)$ on $L$. More precisely, we have $L_w = g^{-1}Lg$ with $g \in G$ such that $gF(g^{-1}) = w$. 
We say that an $F$-stable maximal torus of $G$ is **maximally split** if it is contained in some $F$-stable Borel subgroup of $G$.

**Duality**

Let $T$ be a maximally split $F$-stable maximal torus of $G$ with corresponding Weyl group $W$ and let $B$ be an $F$-stable Borel subgroup of $G$ containing $T$. Consider another connected reductive group $G^b$ endowed with a Frobenius $F^b : G^b \to G^b$. Let $T^b$ be a maximally split $F^b$-stable maximal torus of $G^b$, $B^b$ an $F^b$-stable Borel subgroup containing $T^b$ and let $W^b$ the Weyl group of $G^b$ with respect to $T^b$. If there exists an isomorphism $\varphi : X(T) \to Y(T^b)$ which takes simple roots (with respect to $B$) to simple coroots (with respect to $B^b$) and which is compatible with the action of the Galois group $\text{Gal}(\overline{F}_q/F_q)$, then we say that $(G^b, F^b)$ and $(G, F)$ are dual groups [7, Definition 5.21]. In particular, for two tori $S$ and $S^b$ with Frobenius $F$ and $F^b$, the pairs $(S, F)$ and $(S^b, F^b)$ are dual if there exists an isomorphism $X(S) \cong Y(S^b)$ compatible with Galois group actions.

The classification of connected reductive groups in terms of root data ensures the existence of a dual group $(G^b, F^b)$ for $(G, F)$ (unique up to isomorphism). From now $(G^b, F^b)$ will denote a group in duality with $(G, F)$.

As the functor $X$ is contravariant and the functor $Y$ covariant, we have a canonical anti-isomorphism $W \to W^b, w \mapsto w^b$ such that for all $\chi \in X(T)$ and $w \in W$, we have

$$\varphi(w \cdot \chi) = w^b \cdot \varphi(\chi),$$

where $w \cdot \chi := \chi \circ w$.

It also satisfies

$$F(w)^b = (F^b)^{-1}(w^b),$$

for all $w \in W$.

The map $w \mapsto w^b$ defines a bijection $H^1(F, W) \to H^1(F^b, W^b)$ and so a bijection between the set of $G^F$-conjugacy classes of $F$-stable maximal tori of $G$ and the set of $F^b$-stable maximal tori of $G^b$.

Let $L$ be an $F$-stable Levi factor of some parabolic subgroup of $G$ and $S$ be an $F$-stable maximal torus of $L$ maximally split. There exists an $F^b$-stable Levi factor $L^b$ of some parabolic subgroup of $G^b$ together with an $F^b$-stable maximal torus $S^b$ of $L^b$ maximally split such that $(L, S, F)$ is in duality with $(L^b, S^b, F^b)$. More precisely, if $S$ is of the form $T_w$ for some $w \in W$ then $S^b$ is of the form $T^b_{w^b}$.
2.2 Lusztig induction

Let $L$ be an $F$-stable Levi subgroup of some parabolic subgroup $P$ of $G$ (which may not be $F$-stable) and denote by $U_P$ the unipotent radical of $P$. The variety $L^{-1}(U_P)$ is equipped with an action of $G^F$ by left multiplication and with an action of $L^F$ by right multiplication. These actions induce actions of $G^F$ and $L^F$ on the compactly supported $\ell$-adic cohomology groups $H^i_c(L^{-1}(U_P), \mathbb{Q}_\ell)$.

For any character $\pi$ of $L^F$, the $\mathbb{Q}_\ell$-vector space $M^i_{L\subset P}(\pi) := H^i_c(L^{-1}(U_P), \mathbb{Q}_\ell) \otimes_{\mathbb{Q}[L^F]} V_\pi$ is thus a $G^F$-module and we denote by $R^G_{L\subset P}(\pi)$ the virtual character of $G^F$ defined by

$$R^G_{L\subset P}(\pi)(g) := \sum_i (-1)^i \text{Tr} \left( g, M^i_{L\subset P}(\pi) \right),$$

for all $g \in G$.

Explicitly [8, Proposition 11.2] we have

$$R^G_{L\subset P}(\pi)(g) = \frac{1}{|L^F|} \sum_{l \in L^F} \text{Tr} \left( (g, l), H^*_c(L^{-1}(U_P), \mathbb{Q}_\ell) \right) \pi(l^{-1}),$$

(2.1)

for all $g \in G^F$, where $H^*_c(L^{-1}(U_P), \mathbb{Q}_\ell) := \sum_i (-1)^i H^i_c(L^{-1}(U_P), \mathbb{Q}_\ell)$.

Recall that the operator $R^G_{L\subset P}$ does not depend on the choice of the parabolic subgroup having $L$ as a Levi factor (see [ ]) and so from now we will denote simply by $R^G_L$ this operator.

We have the following basic properties [8, Propositions 9.1.7, 9.1.8].

**Proposition 2.2.1.** (i) If $L \subset M$ is an inclusion of Levi subgroups, then $R^G_L = R^G_M \circ R^M_L$.

(ii) If $\pi$ is a character of $L^F$, then $R^G_L(\pi^\vee) = R^G_L(\pi)^\vee$.

We have the following result [7, Corollary 7.7].

**Theorem 2.2.2** (Deligne-Lusztig). Any irreducible character of $G^F$ appears in some virtual character $R^G_T(\theta)$ for some $F$-stable maximal torus $T$ of $G$ and some linear character $\theta : T^F \to \mathbb{Q}_\ell^\times$.

2.3 Lusztig series

We have the following proposition [7, (5.21.5)].

**Proposition 2.3.1.** (i) There exists a bijective correspondence between the set of $G^F$-conjugacy classes of pairs $(T, \theta)$, where $T$ is an $F$-stable maximal torus of $G$ and $\theta \in T^F$, and the set
of $G^{bF^b}$-conjugacy classes of pairs $(T^b, s)$, where $T^b$ is an $F^b$-stable maximal torus of $G^b$ and $s \in T^{bF^b}$.

(ii) If $(T, F)$ and $(T^b, F^b)$ are in duality, then $\hat{T}^F \cong T^{bF^b}$.

The correspondence (i) and the isomorphism (ii) of the proposition depends on the choice of the isomorphism $\mathbb{F}_q^\times \cong (\mathbb{Q}_p/\mathbb{Z})_{F^b}$ and the embedding $\mathbb{F}_q^\times \hookrightarrow \mathbb{Q}_p^\times$. Indeed, to construct the correspondence between characters and $F$-stable points of tori from an isomorphism $X(S) \cong Y(S^b)$, we relate characters of $S^F$ with $X(S)$ and $F^b$-stable points of $S^b$ with $Y(S^b)$ as follows. The choice of an isomorphism $\mathbb{F}_q^\times \cong (\mathbb{Q}_p/\mathbb{Z})_{F^b}$ defines a surjective group homomorphism $Y(S) \to S^F$, $y \mapsto \text{Nr}_{F^b/F}(y(\zeta))$ where $n$ is such that $S$ is split over $\mathbb{F}_{q^n}$ and $\zeta$ is the $(q^n - 1)$-th root of unity corresponding to $1/(q^n - 1) \in (\mathbb{Q}_p/\mathbb{Z})_{F^b}$. The choice of the embedding $\mathbb{F}_q^\times \hookrightarrow \mathbb{Q}_p^\times$ defines a surjective group homomorphism $X(S) \to \hat{S}^F$ by restricting a character $S \to \mathbb{F}_q^\times$ to $S^F$.

If $H$ is an $F$-stable maximal torus of $G$ and $\eta \in \hat{H}^F$, we call $(H, \eta)$ a Deligne-Lusztig pair of $(G, F)$ (DL pair of $G$ for short). We say that two DL pairs $(T, \theta)$ and $(T', \theta')$ of $G$ are geometrically conjugate if there exists some positive integer $n$, some $g \in G^{F^n}$ such that

$$gTg^{-1} = T' \quad \text{and} \quad \theta \circ \text{Nr}_{F^n/F}(t) = \theta' \circ \text{Nr}_{F^n/F}(gtg^{-1}),$$

for all $t \in T^{F^n}$.

We have the following proposition [7, Proposition 5.22].

**Proposition 2.3.2.** Geometric conjugacy classes of DL pairs $(T, \theta)$ are in one-to-one correspondence with $F^b$-stable conjugacy classes of semi-simple elements of $G^b$.

When the $G^F$-conjugacy class of the DL pair $(T, \theta)$ corresponds to the $G^{bF^b}$-conjugacy class of a pair $(T^b, s)$ (see Proposition[2.3.1](i)) we will write sometimes $R^G_{T^b,s}$ instead of $R^G_T(\theta)$.

**Theorem 2.3.3** (Deligne-Lusztig). $R^G_{T^b,s}$ and $R^G_{T'^b,s'}$ have no commun irreducible constituent unless $s$ and $s'$ are $G^b$-conjugate.

A geometric Lusztig series (or Lusztig series for short) of $(G, F)$ associated to the geometric conjugacy class of some DL pair $(T, \theta)$ of $G$ is the set of all irreducible characters of $G^F$ which appear non-trivially in some $R^G_{T^b,s}(\theta')$ where $(T', \theta')$ is geometrically conjugate to $(T, \theta)$. Thanks to Theorem[2.2.2](i), any irreducible character of $G^F$ belongs to a Lusztig series and thanks to Theorem[2.3.3] the Lusztig series are disjoint and so form a partition of $G^F$ which is parametrized by the $F^b$-stable semisimple conjugacy classes of $G^b$.

Let $(T, \theta)$ be a DL pair of $G$ and $s \in G^{bF^b}$ be a corresponding semisimple element. We denote either by $\mathcal{E}_G(T, \theta)$ or $\mathcal{E}_G(s)$ the Lusztig series associated with the geometric conjugacy class of $(T, \theta)$. For $\pi \in G^F$ we will also denote by $\mathcal{E}_G(\pi)$ the Lusztig series which contains $\pi$.

We denote by $\text{LS}(G)$ the set of Lusztig series of $(G, F)$. 

3 Preliminaries

3.1 Gamma functions on finite groups

Fix a finite group $G$ and denote by $\widehat{G}$ the set of irreducible $\mathbb{Q}_\ell$-characters of $G$ and by $C(G)$ the $\mathbb{Q}_\ell$-vector space of all functions $G \to \mathbb{Q}_\ell$.

The action of $G$ on itself by right and left multiplication makes $C(G)$ into a $G \times G$-module by

$$((g, h) \cdot f)(x) = f(g^{-1}xh)$$

for all $f \in C(G)$, $g, h, x \in G$. It decomposes into irreducible as

$$C(G) = \bigoplus_{\pi \in \widehat{G}} C_\pi(G),$$

where $C_\pi(G)$ is the subspace generated by $\{(g, 1) \cdot \pi | g \in G\}$. In fact $C_\pi(G) \cong V_\pi \otimes V_\pi^\vee$ where $V_\pi$ is an irreducible representation with character $\pi$ and $V_\pi^\vee$ is the dual representation.

Denote by $C(G)^\iota$ the $G \times G$-module with underlying space $C(G)$ and with $G \times G$-action given

$$(g, h) * f = (h, g) \cdot f.$$  

If $F^G : C(G) \to C(G)^\iota$ is a morphism of $G \times G$-modules, we have a function $\gamma^G : \widehat{G} \to \mathbb{Q}_\ell$ (which we call a gamma function) such that for every $\pi \in \widehat{G}$ we have

$$F^G(\pi) = \gamma^G(\pi) \pi^\vee. \quad (3.1)$$

The function $\gamma^G$ determines completely $F^G$.

**Proposition 3.1.1.** Let $F^G : C(G) \to C(G)^\iota$ be a $\mathbb{Q}_\ell$-linear map. The following assertions are equivalent.

1) $F^G : C(G) \to C(G)^\iota$ is a morphism of $G \times G$-modules.

2) The function $F^G$ is given by a kernel, i.e. there exists a central function $\phi^G \in C(G)$ such that

$$F^G(f)(g) = \sum_{h \in G^F} \phi^G(gh)f(h), \quad (3.2)$$

for all $f \in C(G^F)$ and $g \in G^F$.

**Proof.** If (1) holds, the function $\phi^G$ in (2) is given by the formula

$$\phi^G(g) = \sum_{\pi \in \widehat{G}} \gamma^G(\pi)\pi(1)\overline{\pi(g)}, \quad (3.3)$$

for all $g \in G$. 

$\square$
Remark 3.1.2. From (3.1) and (3.2) we see that

\[ \gamma^G(\pi) = \pi(1)^{-1} \sum_{g \in G} \phi^G(g) \pi(g^{-1}), \]

(3.4)

for all \( \pi \in \widehat{G} \).

From the above discussion we see that it is equivalent to give oneself:

- a morphism \( F^G : C(G) \to C(G)^t \) of \( G \times G \)-modules,
- a function \( \gamma^G : \widehat{G} \to \overline{Q}_\ell \) (gamma function),
- a central function \( \phi^G : G \to \overline{Q}_\ell \) (kernel).

### 3.2 Quotient stacks over finite fields

Assume that \( Z \) is a finite set on which a finite group \( H \) acts on the right. By notation abuse and when the context is clear, we denote the map \( Z \to Z, z \mapsto z \cdot h \) by \( h \). Denote by \([Z/H]\) the category of \( G \)-equivariant maps \( G \to Z \) where \( G \) acts on itself by right translation and denote by \([Z/H]\) the set of isomorphism classes of \([Z/H]\). A function \([Z/H] \to \overline{Q}_\ell \) is a function on objects which is constant on isomorphism classes, i.e. it is a \( \overline{Q}_\ell \)-valued function on \([Z/H]\). We will often identify the \( \overline{Q}_\ell \)-space \( C([Z/H]) \) of functions \([Z/H] \to \overline{Q}_\ell \) with the space \( C(Z)^H \) of \( H \)-invariant functions on \( Z \). We consider on \( C([Z/H]) \) the inner form

\[
(f, h)_{[Z/H]} := \sum_{[x] \in [Z/H]} \frac{1}{|Aut([x])|} f([x]) h([x])
\]

\[
= \frac{1}{|H|} \sum_{x \in Z} f(x) \overline{h(x)}
\]

for all \( f, h \in C([Z/H]) \).

If \( X \) is a \( \overline{F}_q \)-scheme on which a \( \overline{F}_q \)-algebraic group \( G \) acts on the right, we denote by \([X/G]\) the quotient stack. We assume that \( X, G \) and the action of \( G \) on \( X \) are defined over \( F_q \) and we denote by \( F \) the associated geometric Frobenius on \( X, G \) and \([X/G]\). Recall that \([X/G]\) is defined as the functor

\[
\text{Sch}/\overline{F}_q \longrightarrow \text{Groupoids}
\]

sending a \( \overline{F}_q \)-scheme \( S \) to the groupoid of diagrams

\[
\begin{array}{ccc}
E & \longrightarrow & X \\
\downarrow & & \downarrow \\
S & & 
\end{array}
\]
where the vertical arrow is a $G$-torsor over $S$ and the horizontal arrow is a $G$-equivariant morphism.

Note that the $G$-torsors over $\mathbb{F}_q$ are parametrized by the set $H^1(F, G) = H^1(F, G/G^o)$ of $F$-conjugacy classes on $G$, and so the set of $\mathbb{F}_q$-points of $[X/G]$ decomposes as the disjoint union

$$[X/G](\mathbb{F}_q) = [X/G]^F \cong \bigcup_{\overline{h} \in H^1(F, G)} [X^{F_{\overline{h}}}/G^{F_{\overline{h}}}]$$

(3.5)

where $h \in G$ denotes a representative of $\overline{h}$ and $G^{F_{\overline{h}}} = \{ g \in G \mid F(h^{-1}gh) = g \}$.

Notice that if $G$ is connected, then $H^1(F, G)$ is trivial and so $[X/G](\mathbb{F}_q) = [X/F]^G$.

**Example 3.2.1.** Let $T$ be the torus of diagonal matrices in $SL_2$ and let $W$ be the Weyl group of $SL_2$ with respect to $T$. We have $[T/W] = [\mathbb{G}_m/\mu_2]$ where $\mu_2$ acts on $\mathbb{G}_m$ as $x \mapsto x^{-1}$. There are two torsors on $\text{Spec}(\mathbb{F}_q)$ which are $\text{Spec}(\mathbb{F}_q^2) \to \text{Spec}(\mathbb{F}_q)$ (with action of $\mu_2$ on $\text{Spec}(\mathbb{F}_q^2)$ given by the Frobenius) and the trivial one $\text{Spec}(\mathbb{F}_q \times \mathbb{F}_q) \to \text{Spec}(\mathbb{F}_q)$ (where $\mu_2$ acts on $\text{Spec}(\mathbb{F}_q \times \mathbb{F}_q)$ by exchanging the coordinates). Then

$$[\mathbb{G}_m/\mu_2](\mathbb{F}_q) = [\mathbb{F}_q^x/\mu_2] \sqcup \{ x \in \mathbb{F}_q^x \mid x^q = x^{-1}/\mu_2 \}.$$

From the decomposition (3.5) we have

$$C([X/G]^F) = \bigoplus_{\overline{h} \in H^1(F, G)} C([X^{F_{\overline{h}}}/G^{F_{\overline{h}}}]).$$

For $f \in C([X/G]^F)$ and $\overline{h} \in H^1(F, G)$ denote by $f_{\overline{h}}$ the coordinate of $f$ in $C([X^{F_{\overline{h}}}/G^{F_{\overline{h}}}]$. We consider on $C([X/G]^F)$ the inner product

$$(a, b)_{[X/G]^F} : = \sum_{[x] \in [X/G]^F} \frac{1}{|\text{Aut}([x])|} a(x)b(x)$$

$$= \sum_{\overline{h} \in H^1(F, G)} (a_{\overline{h}}, b_{\overline{h}})_{[X^{F_{\overline{h}}}/G^{F_{\overline{h}}}]}. $$

For a function $f \in C([X/G]^F)$ and $n \in \mathbb{Z}$, we let $f(n)$ be the function which takes the values $q^{-n}f(x)$ at any $x \in [X/G]^F$.

### 3.3 Lusztig induction on $[T/N]$

Let $T$ be an $F$-stable maximal torus of $G$ with normalizer $N = N_G(T)$ and Weyl group $W = N/T$. 

The quotient stack $[T/N]$

By §3.2 the set of $\mathbb{F}_q$-points of $[T/N]$ decomposes as the disjoint union

$$[T/N](\mathbb{F}_q) = [T/N]^F = \bigsqcup_{\pi \in H^1(F,N)} [T^{F_{\pi}}/N^{F_{\pi}}].$$

(3.6)

We thus have

$$C([T/N]^F) = \bigoplus_{\pi \in H^1(F,N)} C([T^{F_{\pi}}/N^{F_{\pi}}]).$$

Since the action of $F \circ w$ on $T$ corresponds to the action of $F$ on $T_w$, we have

$$C([T/N]^F) = \bigoplus_{\pi \in H^1(F,N)} C([T^F_w/N^F_w])$$

(3.7)

where $N_w = N_G(T_w)$. The inner product on $C([T/N]^F)$ is then given by

\[
(a, b)_{[T/N]^F} = \sum_{[x] \in [T/N]^F} \frac{1}{|\text{Aut}([x])|} a(x)\overline{b(x)}
\]

\[
= \sum_{\pi \in H^1(F,N)} \frac{1}{|W^F_w|} (a_w, b_w)_{[T^F_w/T_w]^F}.
\]

for any two functions $a, b \in C([T/N]^F)$, where $W_w = N_w/T_w$.

Remark 3.3.1. Notice that

$$C([T/W]^F) = C([T/N]^F)$$

but it is more natural to work with $N$ instead of $W$ (see §9). Working with $W$ instead of $N$ modifies the above inner form by a factor $|T^F_w|$.

Lusztig induction and restriction

Let $L$ be an $F$-stable Levi subgroup of some parabolic subgroup $P$ of $G$ (which may not be $F$-stable). In the following we identify $C([G/G]^F)$ with $C(G^F)^G^F$.

The Lusztig induction $\pi \mapsto R^G_{L,C,P}(\pi)$ defined on characters extends linearly to a $\overline{\mathbb{Q}}_L$-linear map $R^G_L : C([L/L]^F) \to C([G/G]^F)$. Explicitly

$$R^G_{L,C,P}(f)(g) = \frac{1}{|L|} \sum_{l \in L^F} \text{Tr} \left( (g, l), H^*_c(L^{-1}(U_P), \overline{\mathbb{Q}}_L) \right) f(l^{-1}),$$

(3.8)
for all $g \in G^F$ and $f \in C([L/L]^F)$.

We define the Lusztig restriction $^*R_L^G : C([G/G]^F) \to C([L/L]^F)$ by the formula \[8, Proposition 11.2\] 

$$^*R_L^G(f)(l) = \frac{1}{|GF|} \sum_{g \in GF} \text{Tr} \left( (g, l), H'_c(\mathcal{L}^{-1}(U_p), \overline{Q}_l) \right) f(g^{-1}),$$

for all $l \in L^F$ and $f \in C([G/G]^F)$.

Proposition 3.3.2. \[8, Definition 9.1.3, Proposition 9.1.6\] The two operators $^*R_L^G$ and $R_L^G$ are adjoint to each other with respect to $(, )_{[G/G]^F}$ and $(, )_{[L/L]^F}$.

Remark 3.3.3. If $f \in C([G/G]^F)$, then $^*R_L^G(f)$ is $N_G(L)^F$-invariant.

Define the induction \[3.3.4\] 

$$I_{[T/N]}^G : C([T/N]^F) \to C([G/G]^F), \quad f \mapsto \sum_{w \in H^1(F, N)} \frac{1}{|W^F_w|} R_{T^w_u}^G(f_w).$$

Notice that 

$$I_{[T/N]}^G(f) = \frac{1}{|W|} \sum_{w \in W} R_{T^w_u}^G(f_w).$$

The restriction is defined by 

$$^*I_{[T/N]}^G : C([G/G]^F) \to C([T/N]^F), \quad h \mapsto \sum_{\pi \in H^1(F, N)} ^*R_{T^w_u}^G(h).$$

From the above proposition we have the following result.

Lemma 3.3.4. For any $f \in C([T/N]^F)$ and $h \in C([G/G]^F)$, we have 

$$\left( I_{[T/N]}^G(h), f \right)_{[T/N]^F} = \left( h, I_{[T/N]}^G(f) \right)_{[G/G]^F}.$$

Say that a function in $C([G/G]^F)$ is uniform if it is a linear combination of Deligne-Lusztig characters $R_{T^w_u}(\theta)$ for various $w \in W$ and linear characters $\theta$ of $T^F_w$. We denote by $C([G/G]^F)_{unif}$ the $\overline{Q}_l$-subspace of $C([G/G]^F)$ of uniform functions.

Theorem 3.3.5. The map $I_{[T/N]}^G$ induces an isomorphism 

$$C([T/N]^F) \simeq C([G/G]^F)_{unif}$$

with inverse given by the restriction of $^*I_{[T/N]}^G$ to uniform functions.
Proof. This is a consequence of Mackey’s formula for Deligne-Lusztig induction.

Remark 3.3.6. If $G$ is of type $A$ with connected center, then $C([G/G]^F)_{uni} = C([G/G]^F)$ and so in this case we have

$$C([T/N]^F) \simeq C([G/G]^F)$$

### 3.4 Lusztig induction and Lusztig series

In this section we prove that Lusztig induction from a Levi induces a map between Lusztig series. More precisely we have the following result.

**Proposition 3.4.1.** Let $L$ be an $F$-stable Levi factor of some parabolic subgroup of $G$ and let $\pi \in \mathcal{E}_L(T, \theta) = \mathcal{E}_L(s)$. Then any irreducible constituent of $\mathcal{R}_L^G(\pi)$ belongs to $\mathcal{E}_G(T, \theta) = \mathcal{E}_G(s)$. Therefore the functor $\mathcal{R}_L^G$ induces a map $\mathcal{V}_L^G : \text{LS}(L) \to \text{LS}(G)$.

**Proof.** Let $P = LU_p$ and $B = TU$ be a parabolic subgroup and Borel subgroup of $G$ such that $B \subset P$ and consider the Borel subgroup $B_L = B \cap L = TV$ of $L$. Assume that $\pi$ is an irreducible constituent of $\mathcal{R}_L^G(\theta)$. Then $\pi$ is an irreducible constituent of some $H^i_c(L^{-1}(V), \overline{Q}_\ell) \otimes_{\mathbb{Q}_\ell}(T^F) \theta$. On the other hand, for all non-negative integer $k$, we have (transitivity of Lusztig induction)

$$H^k_c(L^{-1}(U), \overline{Q}_\ell) \otimes_{\mathbb{Q}_\ell}(T^F) \theta \simeq \bigoplus_{i+j=k} \left( H^i_c(L^{-1}(U), \overline{Q}_\ell) \otimes_{\mathbb{Q}_\ell}(T^F) H^j_c(L^{-1}(V), \overline{Q}_\ell) \right) \otimes_{\mathbb{Q}_\ell}(T^F) \theta.$$

Therefore any irreducible constituent of $\mathcal{R}_L^G(\pi)$ appears in some $H^k_c(L^{-1}(U), \overline{Q}_\ell) \otimes_{\mathbb{Q}_\ell}(T^F) \theta$ for some $k$. Note that a priori $\alpha$ could also appear in other cohomology groups and we could have cancellation in $\mathcal{R}_L^G(\theta)$. The proposition 13.3 of [8] says that $\alpha$ appears at least in some $\mathcal{R}_L^G(\theta')$ with $(T', \theta')$ in the geometric conjugacy class of $(T, \theta)$. □

### 3.5 Morphisms in duality and Lusztig series

A morphism $f : T' \to T$ of tori induces a morphism $Y(T') \to Y(T)$ between the co-character groups and so a map $X(T'^b) \to X(T^b)$ between the character groups of the dual tori. Since the contravariant functor $X$ is fully faithful, we get a morphism $f^b : T^b \to T'^b$. If $f$ commutes with Frobenius then so does $f^b$.

**Remark 3.5.1.** In the case where $T'$ is the maximal torus $T_N$ of diagonal matrices of $\text{GL}_N$, the morphism $f^b$ is constructed as follows. The morphism $f : T_N \to T$ is of the form $(t_1, \ldots, t_N) \mapsto \alpha_1(t_1) \cdots \alpha_N(t_N)$ for some cocharacters $\alpha_1, \ldots, \alpha_N$ of $T$. Regarding now the $\alpha_i$’s as characters of $T^b$ via the isomorphism $X(T^b) \simeq Y(T)$, we obtain $f^b : T^b \to T'^b = T_N, t \mapsto (\alpha_1(t), \ldots, \alpha_N(t))$. 

More generally, consider a morphism \( f : H' \to H \) of connected reductive algebraic groups defined over \( \mathbb{F}_q \), which is normal (i.e. the image of \( H' \) is a normal subgroup of \( H \)).

**Proposition 3.5.2.** There exists a normal morphism \( f^b : H^b \to H'^b \) defined over \( \mathbb{F}_q \) which extends any morphism \( T^b \to T'^b \) obtained by duality from the restriction \( T' \to T \) of \( f \) to maximal tori.

**Proof.** To see this, we are reduced to the case where \( f \) is a surjective morphism or the inclusion of a closed connected normal subgroup. First of all recall that any connected reductive group \( G \) is the almost-direct product of the connected component of its center and a finite number of quasi-simple groups \( G_1, \ldots, G_r \), i.e. the product map \( \mathbb{Z}_G \times G_1 \times \cdots \times G_r \to G \) is an isogeny (that is a surjective homomorphism with finite kernel). Therefore if \( N \) is a closed connected normal subgroup of \( H \), then there exists a closed connected normal subgroup \( S \) of \( H \) such that the \( H \) is the almost-direct product of \( S \) and \( N \). The isogeny \( S \times N \to H \) induces an isogeny between the root data and so an isogeny between the dual root data. By the isogeny theorem (see for instance [13, Theorem 23.9]) we thus get an isogeny \( H^b \to (S \times N)^0 \simeq S^b \times N^b \). Composing this isogeny with the projection \( S^b \times N^b \to N^b \) we get the required morphism \( H^b \to N^b \).

We now assume that \( f \) is surjective and denote by \( S \) the kernel of \( f \). As \( f \) factorizes through the isogeny \( H'/S^o \to H'/S \), we may assume that \( S \) is connected. By the above discussion, the inclusion \( i : S \hookrightarrow H' \) induces a surjective morphism \( \hat{i} : H'^b \to S^b \). Also if \( T' \) denotes a maximal torus of \( H' \), \( T_S := T' \cap S \) and \( T := f(T') \), then we have an exact sequence of tori \( 1 \to T_S \to T' \to T \to 1 \) and so an exact sequence \( 1 \to T^b \to T'^b \to (T_S)^b \to 1 \). Therefore \( \ker(\hat{i}) \cap T'^b \simeq T^b \) is connected from which we deduce that \( K^b := \ker(\hat{i}) \) is also connected. The map \( T^b \to T'^b \) induces an isomorphism between the root data of \( K^b \) and \( K^b \), and so extends to an isomorphism \( H^b \simeq K^b \) by the isogeny theorem.

Let \( f^b : H^b \to H'^b \) be a morphism in duality with \( f \). We denote by \( f^F : H'^F \to H^F \) the induced group homomorphism on rational points.

**Proposition 3.5.3.** The pull back functor \( (f^F)^* : \text{Rep}(H^F) \to \text{Rep}(H'^F) \), \( \alpha \mapsto \alpha \circ f^F \) between categories of finite dimensional representations (over \( \overline{\mathbb{Q}}_l \)) induces a map between the sets of Lusztig series. More precisely, if \( \alpha \in \mathcal{E}_H(s) \), with \( s \in H'^{F^b} \), then any irreducible constituent of \( \alpha \circ f^F \) belongs to \( \mathcal{E}_{H'}(f^b(s)) \).

**Proof.** The statement is clear if both \( H' \) and \( H \) are direct products of a torus by a quasi-simple group. Since both \( H' \) and \( H \) are such direct products up to central isogeny we are reduced to prove the proposition for \( f : H' \to H \) a central isogeny. Let \( T' \) be an \( F \)-stable maximal torus of a Borel subgroup \( B' = T'U' \) of \( H' \) and let \( B = TU \) be the image of \( B' \) by \( f \) with \( f(T) = T' \). By base change, the map \( f \) induces a finite surjective map

\[
\bar{f} : \bigsqcup_{z \in \ker(f)} \mathcal{L}^{-1}_{H'}(zU') \to \mathcal{L}^{-1}_H(U).
\]
Via $f$, the groups $H^F$ and $T'^F$ act on $L_H^{-1}(U)$ and $\hat{f}$ is invariant under these actions. We thus get for all $i$ an inclusion $H_i'(L_H^{-1}(U), \overline{\mathbb{Q}}_l) \hookrightarrow \bigoplus_{z \in \text{Ker}(f)} H_i'(L_H^{-1}(zU'), \overline{\mathbb{Q}}_l)$ of $H^F \times T'^F$-modules. Expressing any $z \in \text{Ker}(f)$ in the form $t'F(t'^{-1})$ for some $t' \in T'$ yields an $G^F \times T'^F$-equivariant isomorphism $L_H^{-1}(zU') \cong L_H^{-1}(U')$. Therefore if $\theta \in T^F$ and $\rho$ is a representation of $H^F$ appearing in $H_i'(L_H^{-1}(U), \overline{\mathbb{Q}}_l)$, the $H^F$-submodule of $H_i'(L_H^{-1}(U), \overline{\mathbb{Q}}_l)$ on which $T^F$ acts by $\theta$, then $(f^F)^*(\rho)$ appears in $H_i'(L_H^{-1}(U'), \overline{\mathbb{Q}}_l)(f^F)_*(\theta)$. \hfill \Box

### 3.6 Functoriality

Let $G$ and $G'$ be two connected reductive algebraic groups defined over $\mathbb{F}_q$ and, by notation abuse, use the letter $F$ for the corresponding geometric Frobenius on $G$ and $G'$. Let $\rho^b : G^b \to G'^b$ be an algebraic morphism which commutes with Frobenius $F^b$. The functoriality principle predicts a map $t_\rho$ from certain “packets” of irreducible representations of $G^F$ to “packets” of irreducible representations of $G'^F$.

The packets we consider are the Lusztig series and

$$t_\rho : \text{LS}(G) \to \text{LS}(G'), \quad \mathcal{E}_G(s) \mapsto \mathcal{E}_{G'}(\rho^b(s)).$$

**Remark 3.6.1.** In the following two cases, the map $t_\rho$ is given by a functor:

1. If $L^b$ is a Levi factor of some parabolic subgroup of $G^b$ and if $\rho^b : L^b \hookrightarrow G^b$ is the inclusion, then $t_\rho = t_{L^b}^G$.
2. If $\rho^b : G^b \to G'^b$ is normal. Then the map $t_\rho$ is given by $(\rho^F)^*$ where $\rho : G' \to G$ is dual to $\rho^b$.

**Remark 3.6.2.** We can also define $t_\rho$ from a morphism $\rho^b : N_{G^b}(T^b) \to G'^b$ as a morphism $N_{G^b}(T^b) \to G'^b$ defines a map between the sets of $F^b$-stable semisimple orbits of $G^b$ and $G'^b$.

Indeed let $s$ and $\sigma$ be two semisimple elements of $G^b$ that are $G^b$-conjugate. Then $s$ and $\sigma$ have $G^b$-conjugates $\overline{s}$ and $\overline{\sigma}$ respectively in $T^b$. The elements $\overline{s}$ and $\overline{\sigma}$ are $N_{G^b}(T^b)$-conjugate (indeed if $g\overline{s}g^{-1} = \overline{\sigma}$ then both $T^b$ and $gT^bg^{-1}$ are maximal tori of $C_{G^b}(T^b)$ and so are conjugate by an element of $C_{G^b}(T^b)$). Therefore the images of $\overline{s}$ and $\overline{\sigma}$ by $\rho^b$ are $G'^b$-conjugate. We thus have a well-defined map between semisimple orbits.

Let $T^b$ be an $F$-stable maximal torus of $G$ and $T'^b$ be an $F'^b$-stable maximal torus of $G'^b$ containing $\rho^b(T^b)$. Let $L'^b$ be the $F'^b$-stable Levi subgroup $C_{G'^b}(\rho^b(T^b))$ of $G'^b$, it contains $T'^b$. The morphism $\rho^b : T^b \to L'^b$ being normal, we get a dual morphism

$$\rho : L' \to T.$$
By propositions 3.4.1 and 3.5.3 we then have the following commutative diagram

\[
\begin{array}{ccc}
LS(T) & \xrightarrow{(\rho^F)_*} & LS(L') \\
\downarrow^{\rho_T} & & \downarrow^{\rho_{L'_T}} \\
LS(G) & \xrightarrow{t_\rho} & LS(G')
\end{array}
\] (3.9)

These commutative diagrams, where \( T^b \) runs over \( F^b \)-maximal tori of \( G^b \), characterize completely \( t_\rho \). In particular this shows that \( t_\rho \) does not depend on the choice of the isomorphism \( \overline{F}_q^\times \simeq (\mathbb{Q}_l/\mathbb{Z})_{p'} \) and the embedding \( \overline{F}_q^\times \hookrightarrow \overline{Q}_l^\times \).

4 Gamma functions on finite reductive groups

In this section \( G \) is an arbitrary connected reductive group equipped with a geometric Frobenius \( F : G \rightarrow G \). We will only be interested in gamma functions which are constant on Lusztig series. We call them admissible. A central function is then called admissible if the corresponding gamma function is admissible. The admissible central functions on \( G^F \) is the subspace generated by the functions of the form

\[ \sum_{\pi \in \mathcal{E}} \pi(1) \pi, \]

with \( \mathcal{E} \) a Lusztig series of \((G, F)\).

4.1 Gamma functions and normal morphisms

Let \( G' \) be another connected reductive group with Frobenius \( F \) and \( f : G' \rightarrow G \) be a normal morphism which commutes with Frobenius. Recall (see Proposition 3.5.3) that the pullback functor \( \pi \mapsto \pi \circ f^F \) induces a map \( t_f \) from the set of Lusztig series of \((G, F)\) to the set of Lusztig series of \((G', F)\).

We have the following lemma.

**Lemma 4.1.1.** Let \( \gamma^{G'} : \widehat{G'}^F \rightarrow \overline{Q}_l^\times \) be admissible. Let \( \gamma^G : \widehat{G}^F \rightarrow \overline{Q}_l^\times \) with corresponding central function \( \phi^G \). Then the following assertions are equivalent:

1. \( \gamma^G = \gamma^{G'} \circ t_f \),
2. \( \phi^G = (f^F)_!(\phi^{G'}) \).

If one of these two conditions is satisfied then \( \gamma^G \) is also admissible.

**Proof.** Let us assume (2). For \( \pi \in \widehat{G}^F \), we have
\[ \gamma^G(\pi) = \pi(1)^{-1}|G|^F \left( \phi^G, \pi^\vee \right)_{[G/G]^F} \]
\[ = \pi(1)^{-1}|G|^F \left( f_i(\phi^G), \pi^\vee \right)_{[G/G]^F} \]
\[ = \pi(1)^{-1}|G|^F \left( \phi^G, f^*\pi^\vee \right)_{[G'/G']^F} \]
\[ = \pi(1)^{-1}|G|^F \left( \phi^G, \sum_{\chi} a_{\chi} \chi^\vee \right)_{[G'/G']^F} \]

where the sum is over the irreducible characters of \( G' \).

We thus have
\[ \gamma^G(\pi) = \pi(1)^{-1} \sum_{\chi} a_{\chi} \chi(1) \gamma^{G'}(\chi). \]

Since the \( \chi \) such that \( a_{\chi} \neq 0 \) belongs to the Lusztig series \( t_f(\mathcal{E}_G(\pi)) \) and since \( \gamma^{G'} \) is constant on Lusztig series we deduce (1).

\[ \square \]

### 4.2 Gamma functions and Lusztig induction

We denote by \( v_G \) the dimension of the unipotent radical of a Borel subgroup of \( G \). If \( H \) is another connected reductive group defined over \( \mathbb{F}_q \), we put

\[ c_{H,G} := q^{v_H - v_G} \epsilon_H \epsilon_G, \]

We will use the following relations:

\[ c_{H,G}^{-1} = c_{G,H}, \quad c_{G_1,G_2} \cdot c_{G_2,G_3} = c_{G_1,G_3}. \]

Notice that \( v_L - v_G = -\dim U_P \) for any \( F \)-stable Levi factor of a parabolic \( P \) of \( G \).

Let \( L \) be an \( F \)-stable Levi factor of some parabolic subgroup \( P \) of \( G \). Recall (see Proposition \[3.4.1\]) that the Lusztig induction functor \( R^G_L : C([L/L]^F) \to C([G/G]^F) \) induces a map \( t^G_L \) from the set of Lusztig series of \( (L,F) \) to the set of Lusztig series of \( (G,F) \).

**Lemma 4.2.1.** Let \( \gamma^G : \widehat{G}^F \to \overline{\mathbb{Q}}_\ell \) be admissible and \( \gamma^L : \widehat{L}^F \to \overline{\mathbb{Q}}_\ell \) with corresponding central function \( \phi^L \). The following assertions are equivalent.

(i) \( \gamma^L = c_{L,G} (\gamma^G \circ t^G_L). \)

(ii) \( \phi^L = * R^G_L(\phi^G). \)

If one of the two conditions hold then \( \gamma^L \) is also admissible.
Proof. The assertion (ii) is equivalent to:

\[ \left( \phi^L, \pi^\vee \right)_{[L/L]^F} = \left( \pi R_L^G(\phi^G), \pi^\vee \right)_{[L/L]^F} = \left( \phi^G, R_L^G(\pi^\vee) \right)_{[G/G]^F}, \]

for all $\pi \in \widehat{L^F}$.

For $\pi \in \widehat{L^F}$, write

\[ R_L^G(\pi) = \sum_{\alpha \in \hat{G}} n_\alpha \alpha, \]

with $n_\alpha \in \mathbb{Z}$. Then

\[ |L^F| \left( \phi^G, R_L^G(\pi^\vee) \right)_{[G/G]^F} = |L^F| \sum_\alpha n_\alpha \left( \phi^G, \alpha^\vee \right)_{[G/G]^F} \]

\[ = \frac{\gamma^G(\alpha) |L^F|}{|G^F|} R_L^G(\pi^\vee)(1) \]

\[ = \frac{\gamma^G(t_L^G(E_L(\pi))) |L^F|}{|G^F|} R_L^G(\pi^\vee)(1). \]

The second equality holds for any $\alpha$ such that $n_\alpha \neq 0$ (because $\gamma^G$ is constant on Lusztig series by assumption).

But [8, Proposition 12.17]

\[ R_L^G(\pi^\vee)(1) = \epsilon_G e_L |G^F/L^F|_{p^h} \pi(1). \]

Hence if (ii) holds then

\[ \gamma^L(\pi) = \gamma^G(t_L^G(E_L(\pi))) \epsilon_G e_L q^{-\dim U_p} \]

hence (i).

\[ \square \]

Assume given, for any $F$-stable Levi factor $L$ of a parabolic subgroup of $G$, a function $\gamma^L : \overline{L^F} \to \overline{Q_{\ell}}$, with corresponding central function $\phi^L$. Denote by $\mathbf{F}^L : C(L^F) \to C(L^F)^{\mathfrak{t}}$ the corresponding isomorphism of $L^F \times L^F$-modules and by $\mathbf{F}^{[L/L]} : C([L/L]^F) \to C([L/L]^F)$ the restriction of $\mathbf{F}^L$ to the subspace of central functions.

Let $\mathcal{T}^G$ be the collection of gamma functions $\gamma^T$ where $T$ describes the set of $F$-stable maximal tori of $G$. We say that $\mathcal{T}^G$ is admissible if for any two geometrically conjugate DL pairs $(T, \theta)$ and $(T', \theta')$ of $G$ we have

\[ \gamma^T(\theta) = c_{T,T'} \gamma^{T'}(\theta'). \]
**Proposition 4.2.2.** The following assertions are then equivalent:

1. $\mathcal{T}^G$ is admissible and for any inclusion $M \subset L$ of $F$-stable Levi factors we have

\[
F^{[L/L]} \circ R_L^L = c_{L,M} R_M^L \circ F^{[M/M]}.
\]  

(4.1)

2. $\mathcal{T}^G$ is admissible and for any $F$-stable Levi factor $L$ (of some parabolic subgroup of $G$) and any $F$-stable maximal torus $T$ of $L$, we have

\[
\phi^L = \frac{1}{|W_L(T)|} \sum_{w \in W_L(T)} R_{T_w}^L (\phi^{T_w}).
\]  

(4.2)

3. The function $\gamma^G$ is admissible and for any inclusion of $F$-stable Levi factors $M \subset L$ we have

\[
\gamma^M = c_{M,L} \gamma^L \circ 1^L_M.
\]  

(4.3)

4. The function $\phi^G$ is admissible and for any inclusion of $F$-stable Levi factors $M \subset L$, we have

\[
\ast R_M^L (\phi^L) = \phi^M.
\]  

**Remark 4.2.3.** Notice that if the equivalent conditions of the above proposition are satisfied then the gamma functions $\gamma^L$ are admissible.

**Proof.** The equivalence of (3) with (4) follows from Lemma 4.2.1. Let us now prove that (1) implies (2). Formula (4.2) follows from the two formulas

\[
\phi^L = F^L(1_1), \quad 1^{[L/L]}_1 = \frac{1}{|W_L(T)|} \sum_{w \in W_L(T)} c_{T_w,L} R_{T_w}^L (1^T_w),
\]  

(4.3)

where $1^{[L/L]}_1$ and $1^T_w$ are the characteristic functions of 1.

Let us see that (2) implies (3). We may assume without loss of generalities that $M$ is a maximal torus $T$ of $L$. Let $\theta \in \overline{T^F}$ and let $\pi \in \mathcal{E}_L(T, \theta)$. Assume first that $\pi \in \overline{L^F}$ is an irreducible constituent of $R_L^T(\theta)$. By Formula (3.4)

\[
\gamma^L(\pi) = \frac{|L^F|}{\pi(1)} (\phi^L, \pi)_{[L/L]^F}.
\]

\[
(\phi^L, \pi)_{[L/L]^F} = \frac{1}{|W_L(T)|} \sum_{w \in W_L(T)} \left( R_{T_w}^L (\phi^{T_w}), \pi \right)_{[L/L]^F}
\]

\[
= \frac{1}{|W_L(T)|} \sum_{w \in W_L(T)} \sum_{\alpha \in \hat{T}^F} (\phi^{T_w}, \alpha)_{T_w^\alpha} \left( R_{T_w}^L (\alpha), \pi \right)_{[L/L]^F}.
\]
Since $T^G$ is admissible we see that if $(R_{T_w}^L (\alpha), \pi) \neq 0$, then

$$(\phi^T_w, \alpha)_{(T_w/T_w)} = |T_w^F|^{-1} \gamma^T_T (\alpha) = |T_w^F|^{-1} c_{T_w, T} \gamma^T (\theta).$$

Hence

$$(\phi^L, \pi)_{(L/L)^F} = \gamma^T_T (\theta) \left( \frac{1}{|W_L(T)|} \sum_{w \in W_L(T)} |T_w^F|^{-1} c_{T_w, T} \sum_{\alpha \in T^L_w} (R_{T_w}^L (\alpha), \pi)_{(L/L)^F} \right)$$

$$= \gamma^T_T (\theta) \left( \frac{1}{|W_L(T)|} \sum_{w \in W_L(T)} c_{T_w, T} R_{T_w}^L (1, 1)_{(L/L)^F} \right).$$

Using the second formula in (4.3) we get

$$(\phi^L, \pi)_{(L/L)^F} = c_{L, T} \gamma^T_T (1, 1)_{(L/L)^F}$$

$$= c_{L, T} \gamma^T_T (\theta) \pi(1)_{(L/F)}. $$

If $\pi$ does not appear in $R_{T_w}^L (\theta)$ it will appear in some $R_{T_w'}^L (\theta')$ in the geometric $L$-conjugacy class of $(T, \theta)$. Using the above calculation with $(T', \theta')$ instead of $(T, \theta)$ together with the admissibility of $T^G$, we get the required formula.

It remains to see that (3) implies (1). Let $\alpha$ be an irreducible character of $M^F$ and let

$$R_{M}^L (\alpha) = \sum_{\pi \in L^F} n_\pi \pi$$

be the decomposition into irreducible characters, then

$$F^L \left( R_{M}^L (\alpha) \right) = \sum_{\pi} \gamma^L (\pi) n_\pi \pi^\vee.$$

Let $(T, \theta)$ be a DL pair such that $\alpha$ is an irreducible constituent of $R_{T}^M (\theta)$. Then any irreducible constituent of $R_{M}^L (\alpha)$ lives in $E_G(T, \theta)$ by Proposition 3.4.1 and so

$$F^L \left( R_{M}^L (\alpha) \right) = c_{L, T} \gamma^T_T (\theta) \sum_{\pi} n_\pi \pi^\vee$$

Since $R_{L}^G (\alpha^\vee) = R_{L}^G (\alpha)^\vee$ we have

$$F^L \left( R_{M}^L (\alpha) \right) = c_{L, T} \gamma^T_T (\theta) R_{M}^L (\alpha^\vee)$$

On the other hand, $F^M (\alpha) = \gamma^M (\alpha) \alpha^\vee$ and
\[ \gamma^M(\alpha) = c_{M,T} \gamma^T(\theta). \]

Hence
\[
R_M^L \left( F^M(\alpha) \right) = c_{M,T} \gamma^T(\theta) R_M^L(\alpha^\gamma).
\]

Remark 4.2.4. If \( \gamma^G \) is admissible and if we put
\[ \gamma^L := c_{L,G} \gamma^G \circ t^G_L \]
for any \( F \)-stable Levi subgroup \( L \), then by transitivity of the \( t^G_L \), the family \( \{\gamma^L\}_L \) satisfies the assertion (3) of the above proposition.

Fix now an \( F \)-stable maximal torus \( T \) of \( G \) with normalizer \( N \). If the equivalent conditions of Proposition 4.2.2 are satisfied then for any \( w \in H^1(F, N) \) we have
\[ \phi^T_w = * R^G_{T_w}(\phi^G) \]
and so by Remark 3.3.3 \( \phi^T_w \) is \( N_w^F \)-invariant. Therefore, the collection of the \( \phi^T_w \) where \( w \) runs over \( H^1(F, N) \) defines a functions
\[ \phi^{[T/N]} = \sum_{w \in H^1(F, N)} \phi^T_w \]
in \( C([T/N]^F) \).

We thus have a well-defined operator \( F^{[T/N]} : C([T/N]^F) \to C([T/N]^F) \)
\[ F^{[T/N]}(f) = \text{pr}_2 ! (\text{pr}_1^* f) \otimes \overline{m} \phi^{[T/N]} \]
for all \( f \in C([T/N]^F) \), where
\[
\begin{array}{ccc}
[T/N] & \xrightarrow{\pi} & [T/N] \\
\downarrow & & \downarrow \text{pr}_1 \\
[T/N] & \xrightarrow{\text{pr}_1} & [(T \times T)/N] & \xrightarrow{\text{pr}_2} & [T/N]
\end{array}
\]
where \( N \) acts diagonally on \( T \times T \), and \( \overline{m} \) is the quotient of the multiplication \( m : T \times T \to T \).

Remark 4.2.5. We have
\[ F^{[T/N]} = \bigoplus_{w \in H^1(F, N)} F^{T_w}, \quad (4.4) \]
where \( F^{T_w} : C([T_w^F/N_w^F]) \to C([T_w^F/N_w^F]) \) is defined from the kernel \( \phi^T_w \).
Denote by \( \epsilon : C([T/N]^F) \to C([T/N]^F) \) the map \((f_w)_{w \in H^1(F,N)} \mapsto (\epsilon G \epsilon T_w f_w)_{w \in H^1(F,N)}\).

Recall that

\[
\epsilon G \epsilon T_w = (-1)^{\ell(w)}
\]

where \( \ell(w) \) is the length of the image of \( w \in N \) in \( W = N/T \).

**Proposition 4.2.6.** If the equivalent conditions of Proposition 4.2.2 are satisfied then the diagram

\[
\begin{array}{ccc}
C([T/N]^F) & \xrightarrow{I^G_{[T/N] \circ (v_G)}} & C([G/G]^F) \\
F^G [T/N] & \downarrow & F^G [G/G] \\
C([T/N]^F) & \xrightarrow{I^G_{[T/N]}} & C([G/G]^F),
\end{array}
\]

(4.5)

commutes, and

\[
I^G_{[T/N]}(\phi^{[T/N]}) = \phi^G.
\]

(4.6)

**Proof.** The diagram commutes thanks to the commutation formula (4.1) and the decomposition (4.4).

The equality (4.6) follows from Formula (4.2). \(\square\)

**Remark 4.2.7.** Formula (4.6) is a consequence of the commutativity of Diagram (4.5). Indeed

\[
F^G [G/G](1_1^{[G/G]}) = \phi^G, \quad F^G [T/N](1_1^{[T/N]}) = \phi^{[T/N]}
\]

where \( 1_1^{[G/G]} \) denote the characteristic function of the neutral element 1 and \( 1_1^{[T/N]} := \sum_{w \in H^1(F,N)} 1_{T_w} \).

The formula follows thus from the commutativity of Diagram (4.5) and the formula (see Formula (4.3))

\[
(I^G_{[T/N]} \circ \epsilon)(v_G)(1_1^{[T/N]}) = \frac{1}{|W|} \sum_{w \in W} \epsilon_G \epsilon T_w q^{-v_G} R^G_{T_w}(1_1^{T_w}) = 1_1^{[G/G]}.
\]
5 Exotic Fourier operator

5.1 Standard Fourier operators

Let $G'$ be a connected reductive group with geometric Frobenius $F : G' \to G'$. We assume that the pair $(G', F)$ is standard, namely that it is isomorphic to a pair of the form $((GL_{n_1})^{m_1} \times \cdots \times (GL_{n_r})^{m_r}, F)$, with $n_1 > n_2 > \cdots > n_r$ and where $F$ is the composition of the standard Frobenius with an element $\sigma$ of $S_{m_1} \times \cdots \times S_{m_r}$. In other words $(G', F)$ is isomorphic to a rational Levi factor of some parabolic subgroup of $GL_n$, with $n = \sum n_i m_i$, and where the $\mathbb{F}_q$-structure on $GL_n$ is standard. In particular, if $L'$ is an $F$-stable Levi factor of some parabolic subgroup of $G'$, then the pair $(L', F)$ is also standard.

For each $i = 1, \ldots, r$, let $\lambda_i = (\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,s_i})$ be the partition of $m_i$ given by the decomposition of the $i$-th coordinate of $\sigma$ into disjoint cycles. We have

$$G'^F \simeq \prod_{i=1}^r \prod_{j=1}^{s_i} GL_{m_i}(\mathbb{F}_q^{\lambda_{i,j}}).$$

We denote by $F$ the corresponding Frobenius on $g' := (gl_{n_1})^{m_1} \times \cdots \times (gl_{n_r})^{m_r}$, then

$$g'^F \simeq \prod_{i=1}^r \prod_{j=1}^{s_i} gl_{n_i}(\mathbb{F}_q^{\lambda_{i,j}}).$$

We consider on any algebra of the form $gl_{s_1} \times \cdots \times gl_{s_r}$ the trace form $\text{Tr}(x) = \sum_{i=1}^r \text{Tr}(x_i)$. It is compatible with restriction to Levi subalgebras and commutes with Frobenius. The algebra $g'$ is therefore equipped with a trace form $\text{Tr}$ which commutes with Frobenius and which is compatible with restriction to Levi subalgebras.

We fix a non-trivial additive character $\psi : \mathbb{F}_q \to \overline{\mathbb{Q}}_l$. We define the standard Fourier transform $F_{g'} : C(g'^F) \to C(g'^F)$ by the formula

$$F_{g'}(f)(x) = \sum_{y \in g'^F} \psi(\text{Tr}(xy)) f(y),$$

for all $f \in C(g'^F)$ and $x \in g'^F$.

Given an $F$-stable Levi subgroup $L'$ of $G'$ with corresponding Lie algebra $l'$, one can extend Lusztig induction $R^G_{L'} : C([L'/L']^F) \to C([G'/G']^F)$ using the embedding $j : [G'/G'] \to [g'/G']$ to an induction $R_{l'}^g : C([l'/L']^F) \to C([g'/G']^F)$ (see [11, Chapter 3]) such that the Lusztig inductions commute with $j_!$ and $j^*$, i.e.

$$R_{l'}^g \circ j_! = j_! \circ R^G_{L'}, \quad j^* \circ R_{l'}^g = R^G_{L'} \circ j^*.$$

We have the following commutation formula between Lusztig induction and Fourier transforms [11, Corollary 6.2.17].
Theorem 5.1.1. 
\[ F_{[g'/G']} \circ R_{v'} = c_{G',L'} R_{v'} \circ F_{[l'/L']} , \]
where \( F_{v'} \) is the standard Fourier transform, i.e. given by the kernel \( \psi \circ \text{Tr} \) on \( v'^F \).

Remark 5.1.2. As \((G', F)\) is a standard, the only proper Levi subgroups of \( G' \) which support a cuspidal pair are the maximal tori, and so by [11 Corollary 6.2.6], the proof of Theorem 5.1.1 reduces to the case where the Levi \( L' \) is a maximal torus of \( G' \). Fix an \( F \)-stable maximal torus \( T' \) of \( G' \) with normalizer \( N' \). For \( w \in N' \) denotes by \( t'_w \) the Lie algebra of \( T'_w \).

As the stack \([T'/N']\) takes care of all \( F \)-stable maximal tori of \( G' \) (up to rational conjugacy), the above theorem is equivalent to the commutativity of the following diagram

\[
\begin{array}{ccc}
C([t'/N']^F) \xrightarrow{I_{[t'/N']}(f)} C([g'/G']^F) \\
\downarrow \quad \downarrow \quad \downarrow \\
C([t'/N']^F) \xrightarrow{F_{[t'/N']}} C([g'/G']^F),
\end{array}
\]

where
\[
I_{[t'/N']}(f) = \sum_{w \in H^1(F, N')} \frac{1}{|W_w^F|} R_{t'_w}(f_w)
\]
for \( f = (f_w)_{w \in H^1(F, N')} \in C([t'/N']^F) \), and
\[
F_{[t'/N']} = \bigoplus_{w \in H^1(F, N')} F_{t'_w},
\]
with \( F_{t'_w} : C([t'_w^F/N'_w^F]) \to C([t'_w^F/N'_w^F]) \) defined from the kernel \( \psi \circ \text{Tr} \) on \( t'_w^F \).

We define the standard Fourier operator \( F^{G'} : C(G'^F) \to C(G'^F) \) as
\[
F^{G'} := j^* \circ F_{v'} \circ j,* ,
\]
where \( j : G' \hookrightarrow g' \) is the inclusion. The associated kernel \( \delta^{G'} \) is the function \( \psi \circ \text{Tr}|_{G'^F} \) on \( G'^F \).

The following result is a consequence of Theorem 5.1.1.

Corollary 5.1.3. For any \( F \)-stable Levi factor \( L' \) of some parabolic subgroup of \( G' \), we have
\[
F^{[G'/G']} \circ R_{L'} = c_{G',L'} R_{L'} \circ F^{[L'/L']}. 
\]

Let \( \gamma^{L'} \) be the gamma function associated to \( F^{L'} \). Since the center of \( G' \) is connected, the above commutation formula implies that \( F^{G'} \) is admissible and so we have the following result.
Corollary 5.1.4. The family \( \{ \gamma^L \}_{L} \) satisfies the equivalent conditions of Proposition 4.2.2. In particular, the functions \( \gamma^L \) are admissible and

\[
\ast R^G_{L'}(\psi \circ \text{Tr}|_{G'}) = \ast R^G_{L'}(\phi^G) = \phi^L = \psi \circ \text{Tr}|_{L'}.
\] (5.2)

Moreover the gamma function \( \gamma^L \) never vanishes as for any DL pair \( (T', \theta') \) of \( G' \), the value \( \gamma^T(\theta') \) is a Gauss sum and therefore is non-zero.

Proposition 5.1.5. Let \( S \) be a torus and assume given a morphism \( \rho : G' \to S \) defined over \( \mathbb{F}_q \). Then if \( T' \) is an \( F \)-stable maximal tori of \( G' \) we have

\[
(\rho^F)_* (\psi \circ \text{Tr}|_{G'}) = c_{G',T'} (\rho^F_T)_* (\psi \circ \text{Tr}|_{T'}),
\]

where \( \rho_{T'} : T' \to S \) is the restriction of \( \rho \) to \( T' \).

Proof. Let us give here a simple proof using gamma functions and (5.2). For a more direct approach see Appendix B.

The morphism \( \rho \) being normal is in duality with a morphism

\[
\rho^h : S^h \to G'^h = G'.
\]

By Remark 3.6.1 we thus have the following commutative triangle

\[
\begin{array}{ccc}
\text{LS}(S) & \xrightarrow{\rho^h} & \text{LS}(T') \\
\downarrow{\rho^*} & & \downarrow{\rho^*} \\
\text{LS}(G') & \xrightarrow{\rho^h} & \text{LS}(G')
\end{array}
\]

Therefore

\[
\gamma^S_{\rho} := \gamma^G \circ (\rho^F)_* = \gamma^G \circ t^G_{T'} \circ (\rho^F_{T'})^*,
\]

By (5.2) and Lemma 4.2.1 we have \( \gamma^{T'} = c_{T',G'} \gamma^G \circ t^G_{T'} \). Therefore

\[
\gamma^S_{\rho} = c_{G',T'} \gamma^{T'} \circ (\rho^F_{T'})^*
\]

and so we conclude by Lemma 4.1.1 as the kernel corresponding to the gamma function \( \gamma^S_{\rho} \) is \( (\rho^F)_* \phi^G \) and the kernel corresponding to the gamma function \( \gamma^{T'} \circ (\rho^F_{T'})^* \) is \( (\rho^F_{T'})_* \phi^{T'} \).
5.2 Spectral definition of exotic Fourier operators

Fix a standard pair \((G', F)\) as in §5.1 and let \(\rho^b : G^b \to G'^b = G'\) be defined over \(\mathbb{F}_q\).

Consider the gamma function on \(G^F = \hat{G}F\gamma_{G^F}\) where \(\gamma_{G'}\) is as in §5.1 and denote by \(G^F_{\rho} : C(G^F) \to C(G^F)^t\) the corresponding operator that we call exotic Fourier operator. They were first considered by Bravermann and Kazhdan [4].

For any \(F\)-stable maximal torus \(T\) of \(G\) put \(\gamma^T_{\rho} := c_{T,G}\gamma^G \circ t_G\).

The kernel \(\phi^T_{\rho}\) corresponding to \(\gamma^T_{\rho}\) can be explicitly computed as follows.

**Lemma 5.2.1.** If \(T'\) is an \(F\)-stable maximal torus of \(G'\) that contains the image \(\rho^b(T^b)\) then

\[
\phi^T_{\rho} = c_{T,T'} (\rho^F_{T'})(\phi^{T'})
\]

where \(\rho_{T'} : T' \to T\) is dual to the restriction \(T^b \to T'\) of \(\rho^b\).

**Proof.** The proof is completely similar to that of Proposition 5.1.5. We have the commutative diagram

\[
\begin{array}{ccc}
\text{LS}(G) & \xrightarrow{\rho^F_{T'}} & \text{LS}(G') \\
\downarrow{t_G} & & \downarrow{t^F_{G'}} \\
\text{LS}(T) & \xrightarrow{\rho^F_{T'}} & \text{LS}(T')
\end{array}
\]

from which we deduce

\[
\gamma^T_{\rho} = c_{T,G'} \gamma^{G'} \circ t_G \circ t^G_L = c_{T,G'} \gamma^{G'} \circ t^G_{T'} \circ \rho^*_T = c_{T,T'} \gamma^{T'} \circ \rho^*_T.
\]

The last equality follows from Lemma 4.2. as \(R^G_{\phi^G} = \phi^{T'}\) by Formula 5.2.1.

The lemma follows thus from Lemma 4.1.1.

\(\Box\)
Remark 5.2.2. The above lemma implies that if \( T'' \) is another \( F \)-stable maximal torus of \( G' \) containing \( \rho^b(T^b) \) then

\[
\varepsilon_{T'}(\rho_{T'}^F); (\psi \circ \Tr|_{T'}) = \varepsilon_{T''}(\rho_{T''}^F); (\psi \circ \Tr|_{T''}).
\]

This can be also deduced from Proposition 5.1.5. To see that we consider the Levi subgroup \( L' := C_{G'}(\rho^b(T^b)) \) of \( G' \). The restriction \( T^b \to L' \) of \( \rho^b \) is then normal and we apply the proposition to the dual morphism \( \rho_{L'} : L' \to T \). Notice that

\[
\phi^T_\rho = c_{T,L'}(\rho_{L'}^F); (\psi \circ \Tr|_{L'})
\]

We now fix a maximally split \( F \)-stable maximal torus \( T \) of \( G \) with normalizer \( N \).

From the definition of \( \gamma^T_\rho \) and Remark 4.2.4, the conditions of Proposition 4.2.2 are satisfied and so by Proposition 4.2.6 we have the following result.

**Theorem 5.2.3.** The diagram

\[
\begin{align*}
C([T/N]^F) & \xrightarrow{(\ell_G^F)^{[T/N]}(\circ \psi_G)} C([G/G]^F) \\
\ell_G^{[T/N]} & \downarrow \quad \downarrow \ell_G^{[G/G]} \\
C([T/N]^F) & \xrightarrow{\ell_G^{[T/N]}} C([G/G]^F),
\end{align*}
\]

(5.3)

commutes, and

\[
\ell_G^{[T/N]}(\phi^{[T/N]}_\rho) = \phi^G_\rho.
\]

(5.4)

Remark 5.2.4. Notice that in the above diagram, \( \ell_G^{[G/G]} \) is defined spectrally (using gamma functions) while the kernel of \( \ell_G^{[T/N]} \) can be explicitly defined in geometrical terms by Remark 5.2.1. The equality (5.4) gives then an explicit formula for \( \phi^G_\rho \).

6 Geometric realizations

6.1 Preliminaries

For an \( \mathbb{F}_q \)-algebraic stack \( \mathfrak{x} \) of finite type we denote by \( \mathcal{D}_\mathfrak{x}^b(\mathfrak{x}) \) the bounded “derived category” of complexes of \( \mathcal{O}_\mathfrak{x} \)-sheaves on \( \mathfrak{x} \) with constructible cohomology.

Assume that \( H \) is a finite group scheme acting on the right on \( \mathfrak{x} \). An \( H \)-equivariant complex is a pair \((K, \theta)\) with \( K \in \mathcal{D}_\mathfrak{x}^b(\mathfrak{x}) \) and \( \theta = (\theta_h)_{h \in H} \) a collection of isomorphisms.
$\theta_h : h^*(K) \simeq K$

satisfying the two conditions:

$\theta_1 = \text{Id}$,

$\theta_{uv} = \theta_u \circ u^*(\theta_v)$ for all $u, v \in H$.

We denote by $D^b_c(\mathfrak{X}; H)$ the category whose objects are the $H$-equivariant complexes and the morphisms $(K, \theta) \to (K', \theta')$ is $\text{Hom}(K, K')^H$ (see [9] §2.3] for more details).

If $H$ acts trivially on $\mathfrak{X}$, then an $H$-equivariant complex is a pair $(K, \theta)$ with $\theta : H \to \text{Aut}(K)$ a group homomorphism, in which case we say that $H$ acts on $K$.

By [10], the category $D^b_c(\mathfrak{X})$ is Krull-Remak-Schmidt, namely each object decomposes into a finite direct sum of indecomposable objects. Recall that in a Krull-Remak-Schmidt category, all idempotent of endomorphism rings splits.

In particular if $H$ is a finite group acting on $K \in D^b_c(\mathfrak{X})$ then we have an isomorphism

$$K \simeq \bigoplus_{\chi \in \hat{H}} K_{\chi},$$

where $K_{\chi}$ is the kernel of the idempotent $1 - e_{\chi} \in \text{End}(K)$ with

$$e_{\chi} := \frac{\chi(1)}{|H|} \sum_{h \in H} \chi(h) \theta_h.$$

We will denote by $K^H$ the $W$-invariant part $K_1$ of $K$.

If $\pi : \mathfrak{X} \to \mathfrak{Y}$ is an $H$-torsor (see [9] §2.8] for the definition), then for any $K \in D^b_c(\mathfrak{Y})$, there exists a canonical $H$-equivariant structure on $\pi^*(K)$ (induced by the trivial action of $H$ on $K$) and in this way $\pi^*$ realizes an equivalence of categories between $D^b_c(\mathfrak{Y})$ and the category of $H$-equivariant complexes on $\mathfrak{X}$ [9, Lemma 2.3.4]. The inverse functor $D^b_c(\mathfrak{X}; H) \to D^b_c(\mathfrak{Y})$ is $(K, \theta) \mapsto \pi_!(K, \theta)^H$.

Assume now that $\mathfrak{X}$ is defined over $\mathbb{F}_q$ and denote by $F : \mathfrak{X} \to \mathfrak{X}$ the corresponding geometric Frobenius. An $F$-equivariant complex is a pair $(K, \varphi)$ with $K \in D^b_c(\mathfrak{X})$ and $\varphi : F^*K \simeq K$. If $(K, \varphi)$ and $(K', \varphi')$ are two $F$-equivariant complexes, the Frobenius $F$ acts on $\text{Hom}(K, K')$ as $f \mapsto \varphi' \circ F^*(f) \circ \varphi^{-1}$. We let $D^b_c(\mathfrak{X}; F)$ be the category whose objects are $F$-equivariant complexes and the set morphisms $(K, \varphi) \to (K', \varphi')$ is $\text{Hom}(K, K')^F$. We denote by $X : D^b_c(\mathfrak{X}; F) \to C(\mathfrak{X}^F)$ the map which sends $(K, \varphi) \in D^b_c(\mathfrak{X}; F)$ to its characteristic function $X_{K, \varphi} : x \mapsto \sum_i (-1)^i \text{Tr}(\varphi^i_x, H^i_x K)$.

**Theorem 6.1.1.** If $f : \mathfrak{X} \to \mathfrak{Y}$ commutes with Frobenius endomorphisms, the following diagrams commute
\[ D^b_c(\mathfrak{X}; F) \xrightarrow{f} D^b_c(\mathfrak{Y}; F) \]
\[ \mathcal{X} \downarrow \quad \mathcal{X} \]
\[ C(\mathfrak{X}^F) \xrightarrow{(f^F)_*} C(\mathfrak{Y}^F) \]
\[ D^b_c(\mathfrak{Y}; F) \xrightarrow{f^*} D^b_c(\mathfrak{X}; F) \]
\[ \mathcal{X} \downarrow \quad \mathcal{X} \]
\[ C(\mathfrak{Y}^F) \xrightarrow{(f^F)^*} C(\mathfrak{X}^F) \]

where \( f^F : \mathfrak{X}^F \to \mathfrak{Y}^F \).

Assume now that \( H \) is also defined over \( \mathbb{F}_q \) and that the right action of \( H \) on \( \mathfrak{X} \) commutes with Frobenius. We denote by \( \mathcal{D}^b_c(\mathfrak{X}; H, F) \) the category whose objects are triples \((K, \theta, \varphi)\) with \( K \in \mathcal{D}^b_c(\mathfrak{X}) \), \( \theta \) an \( H \)-equivariant structure on \( K \) and \( \varphi : F^* K \simeq K \) such that the following diagram commutes for all \( h \in H \):

\[ h^* F^*(K) \xrightarrow{F^*(\theta_{F(h)})} F^* K \]
\[ h^*(\varphi) \downarrow \quad \varphi \]
\[ h^* K \xrightarrow{\theta_h} K \]

(6.1)

We see this compatibility condition as a cocycle condition in \( H \rtimes \langle F^{-1} \rangle \) which we let act on \( \mathfrak{X} \) on the right by

\[ [x] \cdot (hF^{-1}) = F([x] \cdot h) \]

for \([x] \in \mathfrak{X} \) and \( h \in H \).

Indeed we have \((hF^{-1})^* K = h^* F^*(K)\) and if we put

1. \( \theta_{F^{-1}} := \varphi \),
2. \( \theta_{hF^{-1}} := \theta_h \circ h^*(\theta_{F^{-1}}) : (hF^{-1})^* K \to K \) for \( h \in H \),

then, since \( hF^{-1} = h \cdot F^{-1} = F^{-1} \cdot F(h) \) in \( H \rtimes \langle F^{-1} \rangle \), we want to have

\[ \theta_{hF^{-1}} = \theta_{F^{-1}} \circ F^*(\theta_{F(h)}) \]

i.e. we want the commutativity of the diagram (6.1).

A morphism \((K, \theta, \varphi) \to (K', \theta', \varphi') \) in \( \mathcal{D}^b_c(\mathfrak{X}; H, F) \) is a morphism \( f : K \to K' \) compatible with the \( H \)-equivariant and \( F \)-equivariant structures and such that the following diagram commutes for all \( h \in H \):

\[ (hF^{-1})^* K \xrightarrow{(hF^{-1})^*(f)} (hF^{-1})^* K' \]
\[ \theta_{hF^{-1}} \downarrow \quad \theta'_{hF^{-1}} \]
\[ K \xrightarrow{f} K' \]
Note that if \( f : \mathfrak{x} \to \mathfrak{y} \) is an \( H \)-equivariant morphism which commutes with Frobenius, then \( f^* \) and \( f_* \) induces functors \( f_i : \mathcal{D}_c^b(\mathfrak{x}; H, F) \to \mathcal{D}_c^b(\mathfrak{y}; H, F) \) and \( f^* : \mathcal{D}_c^b(\mathfrak{y}; H, F) \to \mathcal{D}_c^b(\mathfrak{x}; H, F) \).

**Proposition 6.1.2.** Assume that \( \pi : \mathfrak{x} \to \mathfrak{y} \) is an \( H \)-torsor which commutes with Frobenius. Then \( \pi^* \) realizes an equivalence of categories \( \mathcal{D}_c^b(\mathfrak{y}; F) \to \mathcal{D}_c^b(\mathfrak{x}; H, F) \), where the \( H \)-equivariant structure on \( \pi^*(K) \), for \( K \in \mathcal{D}_c^b(\mathfrak{y}) \), is the one induced by the trivial action of \( H \) on \( K \).

**Proof.** Let us construct the inverse functor.

Let \( (K, \theta, \varphi) \in \mathcal{D}_c^b(\mathfrak{x}; H, F) \). From (6.1), we have the commutative diagram

\[
\begin{array}{ccc}
(F^* \pi_1 K)_{\chi \circ F^{-1}} & \xrightarrow{1-f_{\chi \circ F^{-1}}} & F^* \pi_1 K \\
\downarrow & & \downarrow \\
(\pi_1 K)_{\chi} & \xrightarrow{1-e_{\chi}} & \pi_1 K
\end{array}
\]

for any \( \chi \in \widetilde{H} \) where

\[
e_{\chi} = \frac{\chi(1)}{|H|} \sum_{h \in H} \chi(h)(\pi_1 \theta)_h, \quad f_{\chi} = \frac{\chi(1)}{|H|} \sum_{h \in H} \chi(h)(F^* \pi_1 \theta)_h.
\]

On the other hand

\[
F^* \left( (\pi_1 K)_{\chi} \right) \simeq (F^* \pi_1 K)_{\chi}
\]

as by definition \( (F^* \pi_1 \theta)_h = F^*((\pi_1 \theta)_h) \). Therefore when \( \chi = \chi \circ F^{-1} \), the Weil structure \( \pi_1^* \varphi \) restricts to a Weil structure \( (\pi_1^* \varphi)_c : F^* \left( (\pi_1 K)_{\chi} \right) \simeq (\pi_1 K)_{\chi} \).

The functor \( (K, \theta, \varphi) \mapsto ((\pi_1 K)^H, (\pi_1^* \varphi)^H) \) is the desired inverse functor. \( \square \)

Finally note that, for any \( h \in H \), we have a functor

\[
o_h : \mathcal{D}_c^b(\mathfrak{x}; H, F) \to \mathcal{D}_c^b(\mathfrak{x}; F \circ h), \quad (K, \theta, \varphi) \mapsto (K, \theta \circ h^*(\varphi))
\]

where \( F \circ h : \mathfrak{x} \to \mathfrak{x}, [x] \mapsto F([x] \cdot h) \).

**Remark 6.1.3.** We assume that \( H \) acts on the right on an \( \overline{\mathbb{F}}_q \)-scheme \( X \) and that this action is defined over \( \overline{\mathbb{F}}_q \). Let \( (K, \theta, \varphi) \in \mathcal{D}_c^b(X; H, F) \) and let \( (\overline{K}, \overline{\varphi}) \) be the corresponding object in \( \mathcal{D}_c^b([X/H]; F) \). Then, for \( \overline{h} \in H^1(F, H) \), the \( \overline{h} \)-coordinate of \( X_{\overline{K}, \overline{\varphi}} \) in the decomposition

\[
C([X/H]^F) = \bigoplus_{\overline{h} \in H^1(F, H)} C(X_{\overline{K}, \overline{\varphi}})^{H_{\overline{h}}}
\]

is the characteristic function of \( o_h(K, \theta, \varphi) \).
We will regard the constant sheaf $\mathcal{Q}_\ell$ on $X$ as an object of $\mathcal{D}_c^b(X)$ concentrated in degree 0 and if $X$ is irreducible we denote by $\text{IC}_X$ the intersection cohomology complex on $X$ (i.e. the intermediate extension of the smooth $\ell$-adic sheaf $\mathcal{Q}_\ell$ on some smooth non-empty open substack of $X$).

We denote respectively by $M(\mathcal{X})$ (resp. $M(\mathcal{X}; F)$, $M(\mathcal{X}; H, F)$) the subcategory of $\mathcal{D}_c^b(X)$ (resp. $\mathcal{D}_c^b(\mathcal{X}; F)$, $\mathcal{D}_c^b(\mathcal{X}; H, F)$) of perverse sheaves on $\mathcal{X}$ for the auto-dual perversity.

### 6.2 Geometric induction and Deligne-Lusztig induction

Let $T$ be an $F$-stable maximal torus of $G$ with normalizer $N$ and Weyl group $W = N/T$. Let $B$ be a Borel subgroup containing $T$.

Put

$$\text{car}_T := T//W$$

and consider the quotient stacks $[T/T]$, $[B/B]$ and $[G/G]$ with respect to the conjugation action.

Denote by $B(T)$ the classifying stack of $T$-torsors. Since $T$ acts trivially on itself, we have

$$[T/T] \cong T \times B(T).$$

From [9, §2.8, §7.2] we can define a pair of adjoint functors $(^*I^G_T, I^G_T)$

$$\text{M}([T/N]) \leftrightarrow \text{M}([G/G]).$$

Let us explain their construction (in [9] we work with $\text{M}([T/W])$ instead of $\text{M}([T/N])$, the two being equivalent).

Consider the commutative diagram

$$\begin{array}{c}
{T/T} \\ s \downarrow \downarrow q' \\
T \times_{\text{car}_T} [G/G] \\ \downarrow (q',p) \\
{G/G}
\end{array} \quad \begin{array}{c}
{B/B} \\
\downarrow p
\end{array}$$

where $s : [T/T] \to T$ is the projection.

By [BY], Lusztig functor $\text{Ind}^G_{[T/T]} : \mathcal{D}_c^b([T/T]) \to \mathcal{D}_c^b([G/G])$, $K \mapsto p_*q'^*(K)$ preserves perverse sheaves and we denote again by $\text{Ind}^G_{[T/T]}$ the induced functor on perverse sheaves. Moreover the functor $s'[\dim T](\dim T) : \text{M}(T) \to \text{M}([T/T])$ is an equivalence of categories with inverse functor $p^0H^0 \circ (s,[-\dim T])(-\dim T)$.

From diagram (6.2), the functor $\text{Ind}^G_{[T/T]}$ decomposes as

$$\text{Ind}^G_{[T/T]} = \text{Ind}^G_T \circ p^0H^0 \circ (s,[-\dim T])(-\dim T).$$
Consider the cartesian diagram

\[
\begin{array}{ccc}
[T/T] & \xrightarrow{s} & T \\
\downarrow{\pi_{[T/T]}} & \swarrow{\pi_T} & \downarrow{\pi_T} \\
[T/N] & \xrightarrow{\tau} & [T/W]
\end{array}
\]

We consider the functor \( \text{Ind}_T^G : \mathcal{M}(T) \to \mathcal{M}([G/G]), K \mapsto p_* q^!(K)[\dim T](\dim T) \).

\[
\text{Ind}_T^G = \text{Ind}_{[T/T]}^G \circ \pi_{[T/T]}^!.
\]  \tag{6.3}

and so we end up with a factorization

\[
\text{Ind}_{[T/T]}^G = \text{Ind}_{[T/N]}^G \circ \pi_{[T/T]}^!.
\]  \tag{6.4}

where

\[
\text{Ind}_{[T/N]}^G := \text{Ind}_{[T/W]}^G \circ p_* h^0 \circ (\pi^*_T[-\dim T](-\dim T)).
\]

**Remark 6.2.1.** While the factorization (6.3) holds if we replace categories of perverse sheaves by derived categories, we can not expect to have (6.4) with derived categories instead of perverse sheaves because the kernel \((q, p)_{!}\bar{Q}_L^t\) is not \(W\)-equivariant on \([T/T] \times_{\text{car}} [G/G]\) (see [9, Proposition 2.8.2]).

The construction of the left adjoint \( \ast \text{Ind}_{[T/N]}^G \) of \( \text{Ind}_{[T/N]}^G \) is also clear from [9, §2.8].

**Theorem 6.2.2.** (1) The co-unit

\[
\ast \text{Ind}_{[T/N]}^G \circ \text{Ind}_{[T/N]}^G \to 1
\]

is an isomorphism.

(2) For any \( K \in \mathcal{M}([T/T]) \) equipped with a \( W \)-equivariant structure, the perverse sheaf \( \text{Ind}_{[T/T]}^G(K) \) is endowed with an action of \( W \) and

\[
\text{Ind}_{[T/T]}^G(\overline{K}) = \text{Ind}_{[T/T]}^G(K)^W
\]

where \( \overline{K} \) satisfies \( \pi^*_T(\overline{K}) = K \).

(3) The functors \( \text{Ind}_{[T/N]}^G \) and \( \ast \text{Ind}_{[T/N]}^G \) induces functors between categories of \( F \)-equivariant perverse sheaves and the following diagrams commute
Proof. The first assertion is \cite[Theorem 7.2.1]{9}, the second one follows from \cite[Remark 2.4.3, Proposition 2.8.2]{9}. It is clear from the construction that $I_{T/N}^G$ and $I_{T/N}^G$ preserve $F$-equivariance. Lusztig \cite{12} proved that for any smooth $\ell$-adic sheaf $K$ equipped with a Weil structure $\varphi$, we have

$$X_{\text{Ind}_{T/N}^G(K,\varphi)} = R_T^G(X_{K,\varphi}).$$

(6.5)

Moreover for any $(K, \varphi) \in D^b_{\text{fl}}([T/T]; F)$ we have

$$X_{\text{Ind}_{T/N}^G(K,\varphi)}(x) = \sum_{t \in T^F} X_{K,\varphi}(t) N(t, x)$$

where $N(\cdot, \cdot) : [T/T]^F \times_{\text{cary}} [G/G]^F \to \overline{\mathbb{Q}_\ell}$ is the characteristic function of $(q, p)\overline{\mathbb{Q}_\ell}$.

Therefore, as the characteristic functions of $F$-equivariant smooth $\ell$-adic sheaves generate the space of all functions on $T^F$, the above formula (6.5) remains true for any $(K, \varphi) \in D^b_{\text{fl}}([T/T]; F)$.

We deduce that for $K \in M([T/T]; W, F)$,

$$X_{\text{Ind}_{T/N}^G(K)}^w = I_{T/N}^G(X_K)$$

where $K$ is the descent of $K$ to $M([T/N]; F)$. The assertion (3) follows thus from (2).

\hfill \Box

### 6.3 Braverman-Kazhdan conjecture

Let $T$ be a maximally split $F$-stable maximal torus of $G$ with normalizer $N$ and Weyl group $W = N/T$, $(G', F)$ a standard pair (see §5.1) and $\rho^b : G^b \to G'^b = G'$ a morphism that commutes with Frobenius. We let $L'$ be the centralizer of $\rho^b(T^b)$ in $G'$.

To simplify the presentation we will assume that $G' = \text{GL}_n$, that $L'$ is of the form $(\text{GL}_{n_1})^{a_1} \times \cdots \times (\text{GL}_{n_\ell})^{a_\ell}$ with $n_1 > \cdots > n_\ell$ (similar results will hold for arbitrary standard pairs $(G', F)$) and that $T'$ the maximal torus $T'_n$ of $\text{GL}_n$ of diagonal matrices. We will use the following identifications (see §2.1)

$$N_{G'}(L') = L' \rtimes (S_{a_1} \times \cdots \times S_{a_\ell}), \quad W_{G'}(L') = S_{a_1} \times \cdots \times S_{a_\ell}.$$
The action of $S_{a_1} \times \cdots \times S_{a_r}$ on $L'$ preserves $T'$ and defines a natural embedding of $S_{a_1} \times \cdots \times S_{a_r}$ in $S_n = W_G(T')$. The map $\rho^b$ induces thus a group homomorphism $W \to S_{a_1} \times \cdots \times S_{a_r} \hookrightarrow S_n$, $w \mapsto w'$ and so an action of $W$ on $T'$.

The morphism $\rho : T' \to T$ (obtained by duality from $\rho^b : T^b \to T'$) is $W$-equivariant.

Fix a non-trivial additive character $\psi$ of $\mathbb{F}_q$ and let $L_\psi$ the Artin-Schreier sheaf on the affine line over $\mathbb{F}_q$ equipped with its natural Weil structure $\varphi_\psi : F^* L_\psi \simeq L_\psi$ such that

$$X_{L_\psi, \varphi_\psi}^{(i)} = \psi \circ \text{Tr}_{\mathbb{F}_q / \mathbb{F}_q},$$

for all positive integer $i$.

The trace $\text{Tr}_{|T'} : T' \to \overline{\mathbb{F}_q}$ being $W$-invariant and the $F$-equivariant perverse sheaf $\Phi^{T'} := (\text{Tr}_{|T'})^*(L_\psi, \varphi_\psi)[\dim T']$ on $T'$ gives a natural object $(\Phi^{T'}, \theta^{T'}, \varphi^{T'})$ of $\mathcal{D}^b_{\text{c}}(T'; W, F)$.

**Proposition 6.3.1.** The complex $\rho^* \Phi^{T'}$ is a perverse sheaf.

**Proof.** As $\rho(T')$ is closed in $T$, we may assume without loss of generality that $\rho$ is surjective, i.e. that $\rho$ is the quotient map

$$T' \to T'/S = T$$

where $S := \text{Ker}(\rho)$. Denote by $j : T' = T_n \hookrightarrow \mathbb{A}^n$ the natural open embedding. The morphism $j$ is affine and quasi-finite and so $j^!$ preserves perverse sheaves.

Let $m : \mathbb{A}^n \times \mathbb{A}^n \to \mathbb{A}^n$ be the multiplication (coordinate by coordinate) and consider the standard geometric Fourier transform

$$\mathcal{F}^{\mathbb{A}^n} : M(\mathbb{A}^n) \to M(\mathbb{A}^n), \quad K \mapsto \text{pr}_2^* \left( \text{pr}_1^* (K) \otimes m^* (\text{Tr}_* (L_\psi)) \right) [\dim T'].$$

It preserves equivariance by $S$ (which acts by translation on $\mathbb{A}^n$) and so does its restriction

$$\mathcal{F}^{T'} := j^* \circ \mathcal{F}^{\mathbb{A}^n} \circ j_1^* : M(T') \to M(T').$$

The operator $\mathcal{F}^{T'} : M(T') \to M(T')$ descends to an operator $\mathcal{F}^T : M(T) \to M(T)$ such that the diagram

$$\begin{array}{ccc}
M(T') & \xrightarrow{\mathcal{F}^{T'}} & M(T') \\
\rho^! & \uparrow & \rho^! \\
M(T) & \xrightarrow{\mathcal{F}^T} & M(T)
\end{array}$$

commutes. Therefore

$$\mathcal{F}^T(\mathcal{Q}_{\ell, 1}) = \rho_! \Phi^{T'}$$
where \( \mathbb{Q}_{\ell,1} \) is the constant sheaf supported at \( 1 \in T \), is a perverse sheaf.

We then get an object \((\Phi^{[T/N]}, \varphi^{[T/N]}) \in \mathcal{M}([T/N]; F) \approx \mathcal{M}([T/W]; F)\) by descending the object \((\Phi^T, \theta^T, \varphi^T)\) of \( \mathcal{M}(T; W, F) \) obtained from

\[
\rho_!(\Phi^T', \theta^T', \varphi^T') \in \mathcal{M}(T; W, F)
\]

by multiplying its \( W \)-equivariant structure by the linear character \( \alpha : w \mapsto (-1)^{\ell(w)+\ell(w')} \) of \( W \) where \( \ell(w) \) (resp. \( \ell(w') \)) denotes the length of \( w \) in \( W \) (resp. the length of \( w' \) in \( S_n \)).

Lemma 6.3.2. We have

\[
X_{\Phi^{[T/N]}, \varphi^{[T/N]}} = \epsilon_G \Phi^{[T/N]}.
\]

where \( \Phi^{[T/N]} \) is the function defined in \( \S 5.2 \).

Proof. By Remark 6.1.3 we need to prove that \( X_o w (\Phi^T, \theta^T, \varphi^T) \in C(T^{\rho F}) \) corresponds to \( \epsilon_G \Phi^T \in C(T^F) \) under the identification \( C(T^F) \cong C(T^{\rho F}) \).

We have the commutative diagram

\[
\begin{array}{ccc}
\mathcal{D}^b_c(T'; W, F) & \xrightarrow{\rho_i} & \mathcal{D}^b_c(T; W, F) \\
\downarrow o_v & & \downarrow o_w \\
\mathcal{D}^b_c(T'; F \circ w) & \xrightarrow{\rho_i} & \mathcal{D}^b_c(T; F \circ w) \\
\downarrow x & & \downarrow x \\
C(T'^{\rho F}) & \cong & C(T^{\rho F}) \\
\cong & & C(T^F) \\
C(T^{\rho F}_{w'}) & \xrightarrow{\rho^F_{w'}} & C(T^F_{w'})
\end{array}
\]

from which we deduce that

\[
X_o w (\Phi^T, \theta^T, \varphi^T) = \alpha(w) \rho^F_{w'} \left( X_{o v (\Phi^T, \theta^T, \varphi^T)} \right)
\]

\[
= (-1)^{\dim T'} \alpha(w) \rho^F_{w'} (\Phi^T_{w'})
\]

\[
= (-1)^{\dim T'} \alpha(w) c^{T'}_{T'_w, T_w} \Phi^T_{w'}
\]

\[
= \epsilon_G \Phi^T_{w'}.
\]
Theorem 6.3.3.

\[ X_{\Phi^G, \varphi^G} = \varepsilon_{G, \rho}. \tag{6.6} \]

Proof. Follows from Theorem 6.2.2(3) and Formula (5.4).

Remark 6.3.4. Let \( \sigma : T \to \text{GL}_1 \) be a character. We say that a cocharacter \( \lambda : \text{GL}_1 \to T \) is \( \sigma \)-positive (see [6]) if \( \sigma \circ \lambda : \text{GL}_1 \to \text{GL}_1 \) is of the form \( t \mapsto t^m \) for some positive integer \( m \). The morphism \( \rho : T \to T' = T_n \) is given by \( n \) characters \( \lambda_1, \ldots, \lambda_n \) which can be regarded as cocharacters of \( T \) via \( X(T_n^\sigma) \simeq Y(T) \). Assume that the cocharacters \( \lambda_1, \ldots, \lambda_n \) are \( \sigma \)-positive. Braverman and Kazhdan [4, Theorem 4.2(3)] proved that the complex \( E \) is an irreducible perverse sheaf on the image of \( \rho : T' \to T \). Later on, Cheng and Ngô proved that this complex is actually a smooth \( \ell \)-adic sheaf on the image of \( \rho \) [6, Proposition 2.1]. Under the \( \sigma \)-positivity assumption and assuming that \( \rho \) is surjective, we thus regard the complex \( \text{Ind}_{T}^{G}(\rho \cdot \Phi') \) as the intermediate extension of some semisimple smooth \( \ell \)-adic sheaf on the open subset of semisimple regular elements of \( G \) on which the Weyl group \( W \) acts. Braverman and Kazhdan [4] conjectured that the characteristic function of \( \text{Ind}_{T}^{G}(\rho \cdot \Phi')^{W} \) coincides with the kernel \( \phi^G_{\rho} \). In light of Theorem 6.2.2(2) we see that Theorem 6.3.3 proves a more general statement than their conjecture as we do not make any assumption of \( \rho \).

7 Appendix A

The commutativity of Diagram (5.1) follows from a particular case of a result in [11]. Here we give a more elegant proof.

Assume that \( G \) is an arbitrary connected reductive group with a geometric Frobenius \( F : G \to G \). Its Lie algebra \( \mathfrak{g} \) is then equipped with a natural geometric Frobenius \( F : \mathfrak{g} \to \mathfrak{g} \) and the adjoint action \( \text{Ad} \) of \( G \) on \( \mathfrak{g} \) commutes with Frobenius. Let \( T \) be a maximally split \( F \)-stable maximal torus with Lie algebra \( \mathfrak{t} \) and denote by \( N \) the normalizer of \( T \) in \( G \) and put \( W := N/T \). We denote by \( \mathfrak{c} \mathfrak{a} \mathfrak{r}_t \) the affine scheme \( t//W \).

Define the geometric induction

\[ I_{[t/N]}^g : \mathcal{M}([t/N]; F) \to \mathcal{M}([\mathfrak{g}/G]; F) \]
as in §6.2 with Lie algebras instead of groups.

If $f : \mathfrak{x} \to \mathfrak{y}$ is a $W$-torsor, we denote by $\epsilon : D^b_c(\mathfrak{y}) \to D^b_c(\mathfrak{y})$ the functor that maps an $W$-equivariant complexes $(K, \theta)$ on $\mathfrak{x}$ to the $W$-equivariant complex $(K, \epsilon \theta)$ where $\epsilon \theta$ is the $W$-equivariant structure $\theta$ twisted by the sign character $\epsilon$ of $W$.

Define the geometric induction

$$I^g_{[t/N],\epsilon} : \mathcal{M}([t/N]; F) \to \mathcal{M}([g/G]; F)$$

where we use the kernel $\epsilon(\text{IC}_{[t/W] \times \text{cart}, [g/G]})$ instead of $\text{IC}_{[t/W] \times \text{cart}, [g/G]}$.

Assume given a $G$-invariant non-degenerate bilinear form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \overline{\mathbb{F}}_q$ defined over $\mathbb{F}_q$ (invariant for the diagonal action of $G$ on $\mathfrak{g} \times \mathfrak{g}$). The existence of such bilinear form requires some restriction on the characteristic (see [11, §2.5]).

Consider the geometric Fourier transform $\mathcal{F}^{[g/G]} : \mathcal{M}([g/G]; F) \to \mathcal{M}([g/G]; F)$ defined by

$$K \mapsto \text{pr}_2 \left( \text{pr}_1(K) \otimes \langle \cdot, \cdot \rangle^{\mathcal{L}_\varphi} \right) [\dim \mathfrak{g}]$$

where $\text{pr}_1, \text{pr}_2 : [((\mathfrak{g} \times \mathfrak{g})/G] \to [g/G]$ are the two projections ($G$ acting diagonally on $\mathfrak{g} \times \mathfrak{g}$).

Since the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{t} \times \mathfrak{t}$ remains non-degenerate and $N$-invariant, we also have a geometric Fourier transform $\mathcal{F}^{[t/N]} : \mathcal{M}([t/N]; F) \to \mathcal{M}([t/N]; F)$.

The aim of this section is to prove the following theorem.

**Theorem 7.0.1.** The following diagram commutes

$$\begin{array}{ccc}
\mathcal{M}([t/N]; F) & \xrightarrow{I^g_{[t/N],\epsilon}(\nu_{G})} & \mathcal{M}([g/G]; F) \\
\mathcal{F}^{[t/N]} \downarrow & & \downarrow \mathcal{F}^{[g/G]} \\
\mathcal{M}([t/N]; F) & \xrightarrow{I^g_{[t/N]}} & \mathcal{M}([g/G]; F)
\end{array}$$

where $\nu_{G}$ is the dimension of the unipotent radical of a Borel subgroup of $G$.

Fix an $F$-stable Borel subgroup $B$ containing $T$ with unipotent radical $U$. Let $\mathfrak{b}$ and $\mathfrak{u}$ be the Lie algebras of $B$ and $U$.

Consider the variety

$$Y := \mathfrak{g} \times \mathfrak{t} \times (G/B)$$

and the closed subschemes

$$X := \{(x, t, gB) \in Y \mid \text{Ad}(g^{-1})(x) \in t + \mathfrak{u}\}, \quad X' := \{(x', t', gB) \in Y \mid g^{-1}x'g \in -t' + \mathfrak{u}\}.$$
Then the vector bundle $X' \to G/B$ is the orthogonal of $X \to G/B$ in $Y \to G/B$ with respect to the form $\langle \cdot , \cdot \rangle_g + \langle \cdot , \cdot \rangle_t$ on $g \times t$.

Denote by $\mathcal{F}^Y : D^b_c(Y) \to D^b_c(Y)$ the geometric Fourier transform relative to $G/B$ with kernel $\langle \cdot , \cdot \rangle_Y(L_\phi)$ where

$$\langle \cdot , \cdot \rangle_Y : (g \times t) \times (g \times t) \times G/B \xrightarrow{pr_{12}} (g \times t) \times (g \times t) \xrightarrow{\langle \cdot , \cdot \rangle_g + \langle \cdot , \cdot \rangle_t} \mathbb{F}_q.$$ 

Namely if $pr_1, pr_2$ denote the two projections $(g \times t) \times (g \times t) \times G/B \to g \times t$, then

$$\mathcal{F}^Y(K) = pr_2^!(pr_1^!(K) \otimes \langle \cdot , \cdot \rangle_Y(L_\phi)) [\dim g \times t].$$

The following result is straightforward.

**Lemma 7.0.2.**

$$\mathcal{F}^Y(\overline{Q}_{\ell,X}) \simeq \overline{Q}_{\ell,X'}(-\dim b).$$

Moreover if $p : Y \to g \times t$ denotes the projection, then

$$p_1 \circ \mathcal{F}^Y = \mathcal{F}^{g \times t} \circ p_1$$

and so

$$\mathcal{F}^{g \times t}(p_1 \overline{Q}_{\ell,X}) \simeq p_1 \overline{Q}_{\ell,X'}(-\dim b).$$

Put $S := g \times \text{car} t$ and let $S' \subset g \times t$ be the subscheme of pairs $(x', t')$ such that the semisimple part of $x'$ is $\text{Ad}(G)$-conjugate to $-t'$.

The projections $\pi : X \to S$ and $\pi' : X' \to S'$ being small resolutions of singularities we have

$$p_1 \overline{Q}_{\ell,X} = \pi_1 \overline{Q}_{\ell,X} = \text{IC}_S, \quad p_1 \overline{Q}_{\ell,X'} = \pi'_1 \overline{Q}_{\ell,X'} = \text{IC}_{S'}.$$ 

Therefore

$$\mathcal{F}^{g \times t}(\text{IC}_S) \simeq \text{IC}_{S'}(-\dim b). \quad (7.1)$$

As the chosen Borel $B$ is $F$-stable, this isomorphism is compatible with $F$-equivariant structures. However, this isomorphism depends on the choice of $B$ and so apriori does not preserves the obvious $W$-equivariant structures on both sides (where the action of $W$ on $g \times t$ is given by the action of $W$ on the second factor).

**Proposition 7.0.3.** The isomorphism (7.1) induces an isomorphism

$$\mathcal{F}^{g \times t}(\epsilon(\text{IC}_S)) \simeq \text{IC}_{S'}(-\dim b). \quad (7.2)$$

in $\mathcal{M}(S'; W, F)$. 
To prove the proposition we need to see that the obvious $W$-equivariant structure on $\text{IC}_{S'}$ and the one induced by that of $\mathcal{F}^g(\text{IC}_S)$ through the isomorphism (7.1) differs by the sign character of $W$.

Notice that since $\text{IC}_{S'}$ is (up to a shift) a simple perverse sheaf, any two $W$-equivariant structures on $\text{IC}_{S'}$ differs by a linear character of $W$. We compute this linear character on the global compactly supported cohomology.

**Lemma 7.0.4.** The action of $W$ on the top cohomology of $R\Gamma_c(S', \text{IC}_{S'})$ is trivial.

**Proof.** Notice that the top cohomology of $R\Gamma_c(S', \text{IC}_{S'})$ is the top compactly supported cohomology of $X'$ and it is certainly well-know that the Springer action of $W$ on $H^{\text{top}}_c(X', \overline{Q}_\ell)$ is trivial. As we could not locate a reference in the literature, we give a proof involving $S'$ as an essential ingredient.

We have

\[
\mathcal{H}^{\text{top}} R\Gamma_c(S', \text{IC}_{S'}) = H^{2d}_c(S', \text{IC}_{S'})
\]

where $d = \dim S' = \dim g$.

Denote by $S'_{\text{rss}}$ the open subset of $S'$ of semisimple regular elements and by $j$ the inclusion $S'_{\text{rss}} \hookrightarrow S'$. The map

\[
H^{2d}_c(S'_{\text{rss}}, \overline{Q}_\ell) \rightarrow H^{2d}_c(S', \text{IC}_{S'})
\]

induced by the morphism $j_! j^* \text{IC}_{S'} \rightarrow \text{IC}_{S'}$ is naturally $W$-equivariant.

The action of $W$ on $H^{2d}_c(S'_{\text{rss}}, \overline{Q}_\ell)$ is induced by the action of $W$ on the set of irreducible components of $S'_{\text{rss}}$ of maximal dimension. As $S'_{\text{rss}}$ is irreducible, this action is thus trivial.

We are thus reduced to prove that the morphism (7.3) is an isomorphism. To see that we use the small map $\pi' : X' \rightarrow S'$. We have the following commutative diagram

\[
\begin{array}{ccc}
H^{2d}_c(S'_{\text{rss}}, \overline{Q}_\ell) & \xrightarrow{=} & H^{2d}_c(S', \text{IC}_{S'}) \\
\downarrow \cong & & \downarrow \cong \\
H^{2d}_c(X'_{\text{rss}}, \overline{Q}_\ell) & \rightarrow & H^{2d}_c(X', \overline{Q}_\ell)
\end{array}
\]

The bottom arrow is an isomorphism as

\[
dim (X' \setminus X'_{\text{rss}}) < d.
\]

\[\square\]

A simple calculation shows that

\[
R\Gamma_c(g \times t, \mathcal{F}^{g \times t}(\text{IC}_S)) = \text{IC}_{S,0}[\dim g \times t](\dim g \times t)
\]
where $IC_{S,0}$ denotes the pullback of $IC_S$ at $0 ∈ S$.

Proposition 7.0.3 follows thus from the following lemma.

**Lemma 7.0.5.** The group $W$ acts on $H^{\text{top}}(IC_{S,0}[-\dim g × t]) = H^{2\dim(G/B)}IC_{S,0}$ by the sign character.

**Proof.** If we notice that $H^{2\dim G/B}(IC_{S,0}) = H^{2\dim G/B}(G/B, \overline{Q}_ℓ)$ then the above lemma is well-known as the Springer action of $W$ on the top cohomology of $G/B$ is known to be the sign character. This can be indeed computed using the result of Borho-MacPherson [π] saying that the Springer action of $W$ on $H^i(G/B, \overline{Q}_ℓ)$ coincides with the so-called classical action of $W$ on $H^i(G/B, \overline{Q}_ℓ) = H^i(G/T; \overline{Q}_ℓ)$. □

**Proof of Theorem 7.0.1.** Since $M([t/N]) ≃ M([t/W])$, it is sufficient to prove the analogous statement with $[t/W]$ instead of $[t/N]$. Consider the quotient stacks

$$S := [S/G], \quad S′ := [S′/G]$$

for the action of $G$ on $g$. Then $S$ and $S′$ are left with an action of $W$ on the other factor.

The isomorphism (7.2) descends to an isomorphism

$$\mathcal{F}^{-[g/G] × [t/W]}(e(I_{[S/W]}(ν_G))) ≃ IC_{[S/W]}(−\dim t).$$

Since we have an isomorphism of functors (involutivity of Fourier)

$$\mathcal{F}^{[t/W]} ◦ \mathcal{F}^{[t/W]} ≃ a^∗ (\dim t)$$

where $a : [t/W] → [t/W]$ is induced by the map $t → t$, $t ↔ −t$, we deduce that

$$(\mathcal{F}^{-[g/G]} × 1) (e(I_{[S/W]}(ν_G))) = (1 × \mathcal{F}^{[t/W]})(IC_{[S/W]}), \quad (7.4)$$

where $\mathcal{F}^{-[g/G]} × 1 : M([g/G] × [t/W]) → M([g/G] × [t/W])$ is obtained by doing Fourier transform on the first factor and nothing on the second, i.e. it is defined from the cohomological correspondence

$$[g/G] × [t/W] \xrightarrow{pr_1 × \text{Id}_{[g/G]}} ([g × g]/ G] × [t/W] \xrightarrow{pr_2 × \text{Id}_{[g/G]}} [g/G] × [t/W]$$

with kernel $⟨, ν_gL_0⟩ \otimes \overline{Q}_ℓ{[t/W]}$, and where $1 × \mathcal{F}^{[t/W]} : M([g/G] × [t/W]) → M([g/G] × [t/W])$ is defined by doing Fourier transform on the second factor and nothing on the first one.

A simple calculation shows that the composition of functors $\mathcal{F}^{-[g/G]} ◦ I_{[t/W],e}^g (ν_G) : M([t/W]) → M([g/G])$ is given by the cohomological correspondence

$$[t/W] \longleftarrow [t/W] × [g/G] \longrightarrow [g/G] \quad (7.5)$$
whose kernel is the left hand side of (7.4), while the composition of functors \( I_{[1/W]} \circ \mathcal{F}_{[1/W]} \) is given by the correspondence (7.5) with kernel the right hand side of (7.4).

\[ \square \]

### 8 Appendix B

For a subvariety of \( gl_n \), we denote by \( \text{Tr}_Z \) and \( \text{det}_Z \) the restriction of the trace and the determinant to \( Z \). Denote by \( B_n = T_n U_n \) the Borel subgroup of \( GL_n \) of upper triangular matrices, by \( N_n \) the normalizer of \( T_n \) in \( GL_n \) and \( W_n = N_n / T_n \).

**Theorem 8.0.1.** We have

\[
(\text{det}_{GL_n})_! \left( \text{Tr}_{GL_n}^* L_\psi \right) \left[ \dim GL_n \right] (\dim U_n) \simeq (\text{det}_{T_n})_! \left( \text{Tr}_{T_n}^* L_\psi \right) \left[ \dim T_n \right].
\]

**Proof.** Let \( X_n \) be the complementary of \( B_n \) in \( GL_n \). Then by the Bruhat decomposition

\[
X_n = \bigsqcup_{w \in W_n \setminus \{1\}} B_n \dot{w} U_{n,w},
\]

where \( \dot{w} \) denotes a representative of \( w \) in \( N_n \) and \( U_{n,w} := U_n \cap \dot{w}^{-1} U_n \dot{w} \).

We have a distinguished triangle

\[
(\text{det}_{X_n})_! \left( \text{Tr}_{X_n}^* L_\psi \right) \xrightarrow{(\text{det}_{GL_n})_! \left( \text{Tr}_{GL_n}^* L_\psi \right) \left[ \dim GL_n \right] (\dim U_n)} (\text{det}_{T_n})_! \left( \text{Tr}_{T_n}^* L_\psi \right) \left[ \dim T_n \right] \xrightarrow{(\text{det}_{B_n})_! \left( \text{Tr}_{B_n}^* L_\psi \right)}
\]

Since \( \text{det}_{B_n} \) factorizes through \( \text{det}_{T_n} \) via the projection \( B_n \to T_n \) and since \( \text{Tr}_{B_n}^* L_\psi \) is the pullback of \( \text{Tr}_{T_n}^* L_\psi \) along that projection, we have

\[
(\text{det}_{B_n})_! \left( \text{Tr}_{B_n}^* L_\psi \right) \simeq (\text{det}_{T_n})_! \left( \text{Tr}_{T_n}^* L_\psi \right) \left[ -2 \dim U_n \right] (-\dim U_n).
\]

It remains to see that \( (\text{det}_{X_n})_! \left( \text{Tr}_{X_n}^* L_\psi \right) = 0 \). For \( w \neq 1 \), put

\[
X_{n,w} := B_n \dot{w} U_{n,w}.
\]

It is enough to show that

\[
(\text{det}_{X_{n,w}})_! \left( \text{Tr}_{X_{n,w}}^* L_\psi \right) = 0,
\]

for all \( w \neq 1 \). The morphism \( \text{det}_{X_{n,w}} \) factorizes through the projection \( p_w : X_{n,w} \to T_n \dot{w} U_{n,w} \) which is a \( U_n \)-torsor. It is thus enough to prove that

\[
(p_w)_! \left( \text{Tr}_{X_{n,w}}^* L_\psi \right) = 0.
\]
Let \( x \in T_n \mathfrak{w} U_{w,n} \) and \( p_{w,x} : U_n x \to \{ x \} \). We need to see that

\[
(p_{w,x})_! \left( \text{Tr}_{U_n x}^* \mathcal{L}_\psi \right) = 0. \tag{8.1}
\]

Writing \( U_n = 1 + u_n \), we see that

\[
\text{Tr}_{U_n x}^* \mathcal{L}_\psi \simeq \text{Tr}_x^* \mathcal{L}_\psi \boxtimes \text{Tr}_{u_n x}^* \mathcal{L}_\psi.
\]

Consider

\[
f : u_n x \simeq u_n \to \bigoplus_{i < j} \mathbb{A}^1, \quad \{ u_{ij} \}_{i < j} \mapsto \sum_{i < j} u_{ij} x_{ji}.
\]

Then \( \text{Tr}_{u_n x}^* \mathcal{L}_\psi = f^* \left( \bigotimes_{i < j} \mathcal{L}_\psi \right) = \bigotimes_{i < j} \mathcal{L}_{\psi,x_{ji}} \) where \( \mathcal{L}_{\psi,x_{ji}} \) is the pullback of \( \mathcal{L}_\psi \) along the map \( \mathbb{A}^1 \to \mathbb{A}^1, u \mapsto ux_{ji} \).

As \( x \not\in B_n \), for some \( i < j \) we have \( x_{ji} \neq 0 \) and so \( \mathcal{L}_{\psi,x_{ji}} \neq \overline{\mathbb{Q}}_\ell \). Therefore, the proper pushforward of \( \mathcal{L}_{\psi,x_{ji}} \) on a point is zero. From Kunneth formula we deduce (8.1). \( \square \)

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