A NOTE ON LOCAL BEHAVIOR OF EIGENFUNCTIONS OF THE
SCHRÖDINGER OPERATOR

IHYEOK SEO

Abstract. We show that a real eigenfunction of the Schrödinger operator changes
sign near some point in $\mathbb{R}^n$ under a suitable assumption on the potential.

1. Introduction

The time evolution of a non-relativistic quantum particle is described by the wave
function $\Psi(t, x)$ which is governed by the Schrödinger equation

$$i\partial_t \Psi(t, x) = H \Psi(t, x),$$

where the Hamiltonian $H = -\Delta + V(x)$ is called the Schrödinger operator. Here, $\Delta$
is the Laplace operator and $V$ is a potential.

The fundamental approach to find a solution of the above equation is by separation
of variables. In fact, considering the ansatz $\Psi(t, x) = f(t) \psi(x)$, the solution can be
written as $\Psi(t, x) = f(0) e^{-iEt} \psi(x) = e^{-iEt} \Psi(0, x)$, where $E$ is an eigenvalue with the
analogous eigenfunction $\psi$ which is a solution of the following eigenvalue equation
for the Schrödinger operator:

$$(\Delta + V(x)) \psi(x) = E \psi(x). \quad (1.1)$$

From the physical point of view, $E \in \mathbb{R}$ is the energy level of the particle.

In this note we are interested in local behavior of $\psi$ near some point in $\mathbb{R}^n$. By
using Brownian motion ideas, it was shown in [5] that for a certain class of potentials $V$, if $\psi(x_0) = 0$ for $x_0 \in \mathbb{R}^n$ and $\psi$ is real, then either

(a) $\psi$ is identically zero near $x_0$ or
(b) $\psi$ has both positive and negative signs arbitrarily close to $x_0$.

As remarked in [5], this asserts that the nodal set $\{x : \psi(x) = 0\}$ must have (Hausdorff)
dimension at least $n - 1$. Also, in many cases, the first cannot occur if $\psi \neq 0$, and so one can assert that the eigenfunction $\psi$ changes sign near $x_0$ in that case.

Here we will consider potentials $V$ given by

$$\|V\| := \sup_Q \left( \int_Q |V(x)| dx \right)^{-1} \int_Q \int_Q \frac{|V(x) V(y)|}{|x - y|^{n-2}} dx dy < \infty, \quad (1.2)$$

where the sup is taken over all dyadic cubes $Q$ in $\mathbb{R}^n$, $n \geq 3$. Our goal is to prove the
following theorem.

2010 Mathematics Subject Classification. Primary: 34L10; Secondary: 35J10.
Key words and phrases. Eigenfunctions, Schrödinger operator.

arXiv:1402.5701v1 [math.AP] 24 Feb 2014.
Theorem 1.1. Let $n \geq 3$. Assume that $\psi \in H^1(\mathbb{R}^n)$ is real and is an eigenfunction of \eqref{1.1} with $E \in \mathbb{C}$. If the potential $V$ satisfies \eqref{1.2} and $\psi$ has a zero of infinite order at $x_0 \in \mathbb{R}^n$, then either (a) holds or (b) holds.

Let us now give more details about the assumptions in the theorem. First, $H^1(\mathbb{R}^n)$ denotes the Sobolev space of functions whose derivatives up to order 1 belong to $L^2(\mathbb{R}^n)$, and by an eigenfunction of \eqref{1.1} we mean a weak solution such that for every $\phi \in H^1_0(\mathbb{R}^n)$

\[
\int_{\mathbb{R}^n} \nabla \psi \cdot \nabla \phi + (V(x) - E)\psi \phi \, dx = 0.
\]

\[\text{\eqref{1.3}}\]

Next, we say that $\psi$ has a zero of infinite order at $x_0 \in \mathbb{R}^n$ if for all $m > 0$

\[
\int_{B(x_0, \varepsilon)} \psi(x) \, dx = O(\varepsilon^m) \quad \text{as} \quad \varepsilon \to 0,
\]

\[\text{\eqref{1.4}}\]

where $B(x_0, \varepsilon)$ is the ball centered at $x_0$ with radius $\varepsilon$. For smooth $\psi$, \eqref{1.4} holds if and only if $D^\alpha \psi(x_0) = 0$ for every order $|\alpha|$.

Finally, the condition \eqref{1.2} is closely related to the global Kato and Rollnik potential classes, denoted by $\mathcal{K}$ and $\mathcal{R}$, respectively, which are defined by

\[V \in \mathcal{K} \iff \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|x - y|^{n-2}} \, dy < \infty\]

and

\[V \in \mathcal{R} \iff \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(x)V(y)|}{|x - y|^2} \, dx \, dy < \infty.
\]

These are fundamental classes of potentials in spectral and scattering theory. Indeed, it is not difficult to see that the Kato and Rollnik potentials satisfy the condition \eqref{1.2}. It should be also noted that there are potentials satisfying \eqref{1.2} which are not in $\mathcal{K}$. For example, $V(x) = 1/|x|^2$. More generally, potentials in the Fefferman-Phong class $\mathcal{F}^p$, which is defined by

\[V \in \mathcal{F}^p \iff \sup_{x,r} r^{2-n/p} \left( \int_{B(x,r)} |V(y)|^p \, dy \right)^{1/p} < \infty\]

for $1 < p \leq n/2$, satisfy \eqref{1.2} (see, for example, \cite{8}). Note that $L^{n/2} = \mathcal{F}^{n/2}$ and $1/|x|^2 \in L^{n/2, \infty} \subset \mathcal{F}^p$ if $p \neq n/2$. We also point out that the above theorem can be found in \cite{2} for $V \in L^{n/2, \infty}$.

2. Preliminaries

The key ingredient in the proof of the theorem is the following lemma, due to Kerman and Sawyer \cite{7} (see Theorem 2.3 there and also Lemma 2.1 in \cite{1}), which characterizes weighted $L^2$ inequalities for the fractional integral

\[I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n.
\]
Lemma 2.1. Let \( n \geq 3 \). Assume that \( w \) be a nonnegative measurable function on \( \mathbb{R}^n \). Then there exists a constant \( C_w \) depending on \( w \) such that the following inequality
\[
\| I_{\alpha/2} f \|_{L^2(w)} \leq C_w \| f \|_{L^2}(2.1)
\]
holds for all measurable functions \( f \) on \( \mathbb{R}^n \) if and only if
\[
\| w \|_\alpha := \sup_Q \left( \int_Q w(x) dx \right)^{-1/2} \int_Q \int_Q \frac{w(x)w(y)}{|x-y|^{n-\alpha}} dxdy < \infty. \tag{2.2}
\]
Furthermore, the constant \( C_w \) may be taken to be a constant multiple of \( \| w \|_1^{1/2} \).

To prove Theorem 1.1 in the next section, we will use the above lemma with \( \alpha = 2 \). (Recall that the condition (1.2) corresponds to the case \( \alpha = 2 \) in (2.2).) Also, the following simple lemma is needed for handling the energy constant \( E \).

Lemma 2.2. Let \( \chi_{B(x_0, r)} \) be the characteristic function of the ball \( B(x_0, r) \subset \mathbb{R}^n \). Then there exists \( r_0 > 0 \) such that \( w = \chi_{B(x_0, r)} \) satisfies (2.2) uniformly for all \( r < r_0 \).

Proof. First, note that
\[
\sup_Q \left( \int_Q w(x) dx \right)^{-1} \int_Q \int_Q \frac{w(x)w(y)}{|x-y|^{n-\alpha}} dxdy \leq \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \frac{w(y)}{|x-y|^{n-\alpha}} dy.
\]
Hence, it suffices to show that
\[
\lim_{r \to 0} \sup_{x \in \mathbb{R}^n} \int_{|y-x_0| < r} \frac{1}{|x-y|^{n-\alpha}} dy = 0.
\]
But, this is an easy consequence of the following computation:
\[
\sup_{x \in \mathbb{R}^n} \int_{|y-x_0| < r} \frac{1}{|x-y|^{n-\alpha}} dy \leq \sup_{|x-x_0| < 2r} \int_{|y-x_0| < r} |x-y|^{-(n-\alpha)} dy + \sup_{|x-x_0| \geq 2r} \int_{|y-x_0| < r} r^{-(n-\alpha)} dy \leq \sup_{x \in \mathbb{R}^n} \int_{|y-x| < 4r} |x-y|^{-(n-\alpha)} dy + Cr^\alpha \leq Cr^\alpha.
\]
□

Finally, we recall the following lemma (see, for example, [4]) concerning the doubling property.

Lemma 2.3. Let \( f \in L^1_{loc} \) be a function in a ball \( B(x_0, r_0) \subset \mathbb{R}^n \). Assume that the doubling property
\[
\int_{B(x_0, 2r)} f \, dx \leq C \int_{B(x_0, r)} f \, dx \tag{2.3}
\]
holds for all \( r \) with \( 2r < r_0 \). If \( f \geq 0 \) and \( f \) has a zero of infinite order at \( x_0 \), then \( f \) must vanish identically in \( B(x_0, r_0) \).
3. Proof of Theorem 1.1

Suppose that \((b)\) does not hold. Without loss of generality, we may then assume \(\psi \geq 0\) near \(x_0\) (since the other case \(\psi \leq 0\) follows clearly from the same argument). With this assumption, we will show that \(\psi\) must vanish identically in a sufficiently small neighborhood of \(x_0\), and thereby we prove the theorem. For simplicity of notation we shall also assume \(x_0 = 0\), and we will use the letter \(C\) to denote positive constants possibly different at each occurrence.

Now, let \(\eta\) be a smooth function supported in \(B(0, 2\delta)\) such that \(0 \leq \eta \leq 1\), \(\eta = 1\) on \(B(0, \delta)\) and \(|\nabla \eta| \leq C\delta^{-1}\). Here, \(\delta > 0\) is less than a fixed \(\delta_0\) chosen later. Putting \(\phi = \eta^2/(\psi + \varepsilon)\) with \(\varepsilon > 0\) in the integral in (1.3), we see that

\[
\int 2\eta \nabla \psi \cdot \nabla \eta dx - \int \frac{\eta^2}{\psi + \varepsilon} \nabla \psi \cdot \nabla \eta dx + \int (V(x) - E) \frac{\psi \eta^2}{\psi + \varepsilon} dx = 0.
\]

(Here it is an elementary matter to check \(\phi \in H_0^1\).) By setting \(\tilde{\psi} = \ln(\psi + \varepsilon)\), it follows now that

\[
\int 2\eta \nabla \tilde{\psi} \cdot \nabla \eta dx - \int \eta^2 \nabla \tilde{\psi} \cdot \nabla \tilde{\psi} dx + \int (V(x) - E) \frac{\psi \eta^2}{\psi + \varepsilon} dx = 0. \tag{3.1}
\]

Using the simple algebraic inequality

\[2ab \leq \left(\frac{a^2}{4} + 4b^2\right), \quad a, b \geq 0,\]

we bound the first integral in (3.1) as follows:

\[
\left| \int 2\eta \nabla \tilde{\psi} \cdot \nabla \eta dx \right| \leq \int 2|\eta \nabla \tilde{\psi}||\nabla \eta| dx \leq \int \frac{1}{4} \eta^2 |\nabla \tilde{\psi}|^2 dx + \int 4|\nabla \eta|^2 dx.
\]

Then by combining this and (3.1), it is not difficult to see that

\[
\int \eta^2 |\nabla \tilde{\psi}|^2 dx \leq \frac{16}{3} \int |\nabla \eta|^2 dx + \frac{4}{3} \int |V(x) - E| \eta^2 dx. \tag{3.2}
\]

Now, using Lemmas 2.1 and 2.2 in the previous section, the second term in the right-hand side of (3.2) is bounded as follows:

\[
\int |V(x) - E| \eta^2 dx \leq \int |V| \eta^2 dx + |E| \int \chi_{B(0, 2\delta)} \eta^2 dx \leq C\|V\| \int |\nabla \eta|^2 dx + C|E| \|\chi_{B(0, 2\delta)}\| \int |\nabla \eta|^2 dx \leq C(\|V\| + 1) \int |\nabla \eta|^2 dx
\]

if \(2\delta < \delta_0\) for a sufficiently small \(\delta_0\). Indeed, note that when \(\alpha = 2\) in Lemma 2.1 the inequality (2.1) is equivalent to

\[
\int |g|^2 w dx \leq C\|w\| \int |\nabla g|^2 dx, \quad g \in H^1.
\]
Then this and Lemma 2.2 give the above bound. Consequently, returning to (3.2) and recalling $\eta = 1$ on $B(0, \delta)$, we get
\[
\int_{B(0, \delta)} |\nabla \tilde{\psi}|^2 dx \leq \int \eta^2 |\nabla \tilde{\psi}|^2 dx \leq C \int |\nabla \eta|^2 dx \\
\leq C \int_{B(0, 2\delta)} \delta^{-2} dx \\
\leq C \delta^{-2n}.
\] (3.3)

At this point, one can apply the Poincaré inequality and the lemma of John and Nirenberg [6], as in [2], in order to conclude that for a small $\rho > 0$
\[
\left( \frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} e^{\rho \tilde{\psi}} dx \right) \left( \frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} e^{-\rho \tilde{\psi}} dx \right) < C.
\] (3.4)

In fact, by the Poincaré inequality and (3.3),
\[
\int_{B(0, \delta)} |\tilde{\psi} - \tilde{\psi}_B|^2 dx \leq C\delta^2 \int_{B(0, \delta)} |\nabla \tilde{\psi}|^2 dx \leq C\delta^n,
\]
where
\[
\tilde{\psi}_B = \frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} \tilde{\psi} dx.
\]

Now, by Hölder’s inequality
\[
\int_{B(0, \delta)} |\tilde{\psi} - \tilde{\psi}_B| dx \leq C \delta^n,
\]
and so $\tilde{\psi}$ belongs to the BMO space (in $B(0, \delta_0)$). Thus, by the lemma of John and Nirenberg [6], there exists a small $\rho > 0$ so that (3.3) holds. Since $\psi = \ln(\psi + \varepsilon)$, by Fatou’s lemma, (3.4) leads to
\[
\left( \frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} \psi^\rho dx \right) \left( \frac{1}{|B(0, \delta)|} \int_{B(0, \delta)} \psi^{-\rho} dx \right) < C.
\]

It is a well known fact\(^1\) (see [3]) that this implies the doubling property for $\psi^\rho$. Then, by Lemma 2.3, $\psi^\rho$ must vanish identically near $x_0 = 0$, and so $\psi \equiv 0$ near $x_0 = 0$.

**References**

[1] J. A. Barcelo, J. M. Bennett, A. Ruiz and M. C. Vilela, *Local smoothing for Kato potentials in three dimensions*, Math. Nachr. 282 (2009), 1391-1405.
[2] F. Chiarenza and N. Garofalo, *Unique continuation for nonnegative solutions of Schrödinger operators*, Preprint Series No. 122, University of Minnesota, 1984.
[3] R. R. Coifman and C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Math. 51 (1974), 241-250.
[4] M. Giaquinta, *Multiple integrals in the calculus of variations and nonlinear elliptic systems*, Annals of Math. Studies, no. 105, Princeton Univ. Press (1983).

\(^1\) The doubling property is satisfied for functions in the $A_2$ Muckenhoupt class.
[5] M. Hoffman-Ostenhof, T. Hoffman-Ostenhof and B. Simon, Brownian motion and a consequence of Harnack’s inequality: nodes of quantum wave functions, Proc. Amer. Math. Soc. 80 (1980), 301-305.

[6] F. John and L. Nirenberg, On functions of bounded mean oscillation, Comm. Pure Appl. Math. 14 (1961), 415-426.

[7] R. Kerman and E. Sawyer, The trace inequality and eigenvalue estimates for Schrödinger operators, Ann. Inst. Fourier (Grenoble) 36 (1986), 207-228.

[8] I. Seo, On minimal support properties of solutions of Schrödinger equations, J. Math. Anal. Appl. 414 (2014), 21-28.

School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Republic of Korea
E-mail address: ihseo@kias.re.kr