Mathematical Go: Revisited

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Abstract

In this paper we will look at the book Mathematical Go by Elwyn Berlekamp and David Wolfe [3], and demonstrate that the results are different when we consider Go in the world of scoring play games as presented by Fraser Stewart [4]. We also demonstrate how the scoring play theory can be applied to other Go scenarios such as Ko’s.

1 Introduction

The ancient Chinese game of Go has been of great interest to mathematicians for years. Along with African sowing games, it is one of the oldest combinatorial games in existence. The name of the game in Chinese is pronounced Weiqi, Igo in Japanese and Paduk in Korean, but all three names use the same characters. It literally translates as wrapping, or surrounding chess, the first character meaning to wrap or surround, the second meaning chess.

What is interesting is that while the theory of combinatorial games was constructed by John Conway after observing Elwyn Berlekamp playing Go, and noticing that the game would often break up into smaller games that are played side by side, but independently, it wasn’t until Fraser Stewart wrote his PhD thesis [5], that there was any notion of a general theory for scoring play combinatorial games.

The reason for this is unclear, but the theory of normal play combinatorial games, that is games where the last player to move is the winner was quickly established, and we also gained the Conway number system [2], [1]. Elwyn Berlekamp and David Wolfe then attempted to apply this theory to Go in their book Mathematical Go [3].
Their method appeared to be successful, and many mathematicians since have used it to try and advance our understanding of Go from a mathematical point of view. In this paper I intend to show that using scoring play theory as an alternative will yield much more substantial results, as well as increasing our understanding of the game of Go.

The basic rules of the game are as follows;

1. There are two players, Left plays with Black stones, and Right plays with White stones.

2. The game is played on a $n \times m$ grid, where $n$ and $m$ are the number of horizontal and vertical lines respectively.

3. Players take turns to place stones on the intersections of the lines.

4. Players can capture their opponents stones or intersections of the board by placing stones on all adjacent intersections.

5. Players get one point for every stone and intersection that is captured.

6. The game ends either when all areas of the board have been captured, a player concedes or a player makes two consecutive passes.

7. The player who gets the most points wins.

The game has other rules and variations depending upon if you are using Japanese rules, Chinese rules, American rules, Ancient rules and so on. For example under Japanese rules you may not place a stone in an area where it would be automatically captured, but in Chinese rules these moves are permitted. However in this paper will only be looking at the game under Japanese rules.

There is also a special rule called a “ko”. All variations of the game use the rule that you may place a stone in an area where it would normally be automatically captured if you are capturing one of the surrounding stones. This rule can lead to situations involving loops or kos as they are called in Japanese. I discuss this particular rule further in section 5.
1.1 Scoring Play Theory

Intuitively a scoring play combinatorial game is one that has the following three properties;

1. The rules of the game clearly define what points are and how players either gain or lose them.

2. When the game ends the player with the most points wins.

3. For any two games $G$ and $H$, a points in $G$ are equal to $a$ points in $H$, where $a \in \mathbb{R}$.

4. At any stage in a game $G$ if Left has $L$ points and Right has $R$ points, then the score of $G$ is $L - R$, where $L, R \in \mathbb{R}$.

Mathematically the definition is given as follows [4];

Definition 1. A scoring play game $G = \{G_L|G^S|G_R\}$, where $G_L$ and $G_R$ are sets of games and $G^S \in \mathbb{R}$, the base case for the recursion is any game $G$ where $G_L = G_R = \emptyset$.

$G_L = \{\text{All games that Left can move to from } G\}$

$G_R = \{\text{All games that Right can move to from } G\}$,

and for all $G$ there is an $S = (P, Q)$ where $P$ and $Q$ are the number of points that Left and Right have on $G$ respectively. Then $G^S = P - Q$, and for all $g^L \in G^L$, $g^R \in G^R$, there is a $p^L, p^R \in \mathbb{R}$ such that $g^{LS} = G^S + p^L$ and $g^{RS} = G^S + p^R$.

$G^S_L$ and $G^S_R$ are called the final scores of $G$ and are the largest scores that Left and Right can achieve when $G$ ends, moving first respectively, if both players play their optimal strategy on $G$.

For scoring play the disjunctive sum needs to be defined a little differently, because in scoring games when we combine them together we have to sum the games and the scores separately. For this reason we will be using two symbols $+_{\ell}$ and $+$. The $\ell$ in the subscript stands for “long rule”, this comes from [1], and means that the game ends when a player cannot move on any component on his turn.
Definition 2. The disjunctive sum is defined as follows:

\[ G + \ell H = \{ G^{L} + \ell H, G^{L} H + \ell H | G^{S} + H^{S} | G^{R} + \ell H, G^{R} H + \ell H \}, \]

where \( G^{S} + H^{S} \) is the normal addition of two real numbers.

The outcome classes also need to be redefined to take into account the fact that a game can end with a tied score. So we have the following two definitions.

Definition 3.

\( L_{>} = \{ G | G^{SL} > 0 \}, L_{<} = \{ G | G^{SL} < 0 \}, L_{=} = \{ G | G^{SL} = 0 \}. \)
\( R_{>} = \{ G | G^{SR} > 0 \}, R_{<} = \{ G | G^{SR} < 0 \}, R_{=} = \{ G | G^{SR} = 0 \}. \)
\( L_{\geq} = L_{>} \cup L_{=}, L_{\leq} = L_{<} \cup L_{=} \).
\( R_{\geq} = R_{>} \cup R_{=}, L_{\leq} = R_{<} \cup R_{=} \).

Definition 4. The outcome classes of scoring games are defined as follows:

- \( L = (L_{>} \cap R_{>) \cup (L_{>} \cap R_{=}) \cup (L_{=} \cap R_{>}) \)
- \( R = (L_{<} \cap R_{<}) \cup (L_{<} \cap R_{=}) \cup (L_{=} \cap R_{<}) \)
- \( N = L_{>} \cap R_{<} \)
- \( P = L_{<} \cap R_{>} \)
- \( T = L_{=} \cap R_{=} \)

We will also be using two conventions throughout this paper. The first is that the initial score of a game will be 0 unless stated otherwise. The second is that for a game \( G \) if \( G^{L} = G^{R} = 0 \), then we will write \( G \) as \( G^{S} \) rather than \( \{ G^{S} \} \). For example the game \( G = \{ \{l|0.\}|1|\{l|2.\} \} \), will be written as \( \{0|1|2\} \). The game \( \{l|n.\} \), will be written as \( n \), and so on. This is simply for convenience and ease of reading.

Definition 5. We define the following:

- \( -G = \{ G^{R} | -G^{S} | -G^{L} \} \).
- For any two games \( G \) and \( H \), \( G = H \) if \( G + \ell X \) has the same outcome as \( H + \ell X \) for all games \( X \).
For any two games $G$ and $H$, $G \geq H$ if $H + \ell X \in O$ implies $G + \ell X \in O$, where $O = L \geq, R \geq, L >$ or $R >$, for all games $X$.

For any two games $G$ and $H$, $G \leq H$ if $H + \ell X \in O$ implies $G + \ell X \in O$, where $O = L \leq, R \leq, L <$ or $R <$, for all games $X$.

$G \cong H$ means $G$ and $H$ have identical game trees.

$G \approx H$ means $G$ and $H$ have the same outcome.

Theorem 6. For any game $G$, if $G \not\cong 0$ then $G \neq 0$.

The proof of this can be found in [4].

2 Mathematical Go and Scoring Play Comparison

In this section I will be comparing scoring play theory with the theory from Mathematical Go. There are three main areas, and differences that I want to highlight. Temperature theory when applied to Go, the interpretation of zero, and dominated and reversible options.

2.1 Temperature Theory

In normal play combinatorial game theory a game is deemed to be “hot” if moving improves your position. That is, under normal play, the value of a game is the number of moves that a player has left, so moving on a hot game increases this number. For example the game $G = \{0|1\} = \frac{1}{2}$ is what we call a “cold” game, since moving is bad for both players. The value of $G$ is $\frac{1}{2}$ but if Left moves he moves to a game that has value 0, and if Right moves he moves to a game with value 1, so both both players will be worse off than they were.

In a “hot” game the opposite is true. For example consider the game $K = \{10| -10\}$. This means that by moving both players will move to a game where they have a 10 move advantage, which is clearly very good in games where you want to be the last player to move. But what about the game $L = \{20| -20\}$, given the choice between moving on $K$ or moving on $L$, both players would pick $L$, since they gain a 20 move advantage.
The problem is that these games are not comparable, since neither of them has a numerical value. So temperature theory tells us just how “hot” a game is, and when playing a hot game you move in the components with the highest temperature first until everything is numbers then simply add up the game values and determine the winner.

So how do we determine the temperature of a game. It’s relatively simple, we subtract a number \( x \) from the Left options and add the number \( x \) to the Right options until we get a game that is either a number or infinitesimally close to a number. \( x \) is the least such number required to do that.

Formally the definition is as follows;

**Definition 7.** \( G = \{G^L|G^R\} \) cooled by the non-negative number \( t \), denoted by \( G_t \), is defined by;

\[
G_t = \{G_t^L - t|G_t^R + t\},
\]

unless for some \( \tau < t \), \( \{G^L_\tau - \tau|G^R_\tau - \tau\} \) is infinitesimally close to a number \( x \), in which case;

\[
G_t = x
\]

So consider the games \( K \) and \( L \), with their game values from the book Mathematical Go. According to normal play temperature theory it is always undesirable to move to a cold game, and always desirable to move to a hot game. The game \( K \) has a “hot” value, and the game \( L \) has a cold value. In other words for any game \( X \), moving on \( K \) will always improve your position, but moving on \( L \) will not.

\[
K = \begin{array}{c|c|c|c|}
\bullet & \bullet & \bullet & \\
\hline
\bullet & \bullet & \bullet & \\
\hline
\bullet & \bullet & \bullet & \\
\hline
\end{array} = \{*|1\}
\]

\[
L = \begin{array}{c|c|c|c|}
\bullet & \bullet & \bullet & \\
\hline
\bullet & \bullet & \bullet & \\
\hline
\bullet & \bullet & \bullet & \\
\hline
\end{array} = \{**\} = 0
\]
The basis for this is that if Right moves on $K$ he will gain a point, but he gains no points moving on $L$, in other words moves that gain you points are *always* better than moves that would not gain you any points.

The reason why this is flawed, is because while scoring play games are partially ordered, there is very little comparibility between scoring play games \[4\]. In other words moves that gain you points in Go are not necessarily good moves. It may be the case that those moves are in fact good moves, *when you play Go*. Scoring play theory can allow us to understand why gaining points is good, if it is always good, but in general this is *not* true.

### 2.2 Zero

The next thing I need to talk about is the idea of zero. In normal play combinatorial game theory $0 = \{ \cdot | \cdot \}$ and is equivalent to all $P$ positions. By using this approach on Go it is possible to dismiss a lot of positions because they are equivalent to zero and therefore make no difference to the overall game.

A good example of this is the game $G$ in figure 1. Using the theory from Mathematical Go this position can be dismissed because it makes no difference to the overall game since it does not affect who moves last.

As an alternative using scoring play theory *if* this game really makes no difference then we can understand *why* it makes no difference. As we know from theorem 6 the above game is not identical to zero and therefore not equal to zero. This means that there may be Go positions where this game *can* affect the outcome of an overall game.

By using scoring play theory one could perhaps show that no such Go position can exist, and by doing so we can really understand how positions where neither player can gain any points affect the game, if they do at all. This can lead to a much greater understanding of the game.

\[ G = \begin{array}{c}
\bullet \bullet \\
\circ \\
\bullet \bullet \\
\end{array} \]

Figure 1: A game with normal play value 0

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2.3 Dominated and Reversible Options

Another difference between scoring play theory and Mathematical Go theory are dominated and reversible options. First I will look at dominated options. The authors were able to dismiss many positions because under normal play theory they are dominated options. However as an alternative using scoring play theory the same positions are not dominated, and as before if indeed these are bad moves when you play Go, scoring play theory can help us understand why.

To demonstrate this consider the game $K$ shown in figure 2. The Left options of $K$ are $G$ and $H$ and are shown in figures 3 and 4. According to the book Mathematical Go $G = \ast$ and $H = 1$, therefore $H \geq G$. This means that in the book Mathematical Go they do not consider the option $G$ from the game shown in figure 2.

Using scoring play theory however we can see that $H \not\geq G$. The game $G = \{0|0|0\}$ and $H = \{0|1|\}$. So consider the game $X = \{1|−2|\}$, Left wins $G +_\ell X$ moving second, but loses moving second on $H +_\ell X$, which means that $H \not\geq G$. In other words we have to consider the option $G$ from $K$.

![Figure 2: The Go Position K](image)

![Figure 3: G = {0|0|0}](image)
As with section 2.2, this difference is important because while it is most likely the case that the game $H$ is always better than $G$ when you play Go, scoring play theory can help us understand why.

Again the key is understand the true nature and structure of the game of Go, and scoring play theory provides us with the necessary tools to gain that understanding.

A very similar argument can be made for reversible options, to show that options the authors claimed were reversible are in fact not reversible. In fact in [5] it was conjectured that there are no reversible options in scoring play games. If this was proven to be true, it would mean that no Go positions are reversible, which would be a major result, and go a long way to further our understanding of Go.

3 Ko

Kos are an area that is very complicated, but extremely important when studying Go because they are a major part of the game. I’d like to demonstrate how scoring play theory can be used to study these positions as well.

A Ko is a position which repeats itself after two consecutive moves by the same player, i.e. Left plays, then Right plays, then Left plays again and we have returned to the original position. An example is the game shown in figure 5.

Interestingly in Chinese the word for ko is either qie or hukou, the first meaning a disaster, the second meaning a tiger’s mouth. The reason why a
position of this nature is so undesirable is because if white takes the black stone, black can immediately retake the white stone, and this could repeat, potentially, forever.

However under Go rules if white captures the black stone, then black may not recapture the white stone until his next turn. This rule is an attempt to prevent a potential cycle that may never end.

So how can scoring play theory be used to analyse such a position? With scoring play theory the best thing to do is not to draw in “loops” on the game tree, but rather in the form shown in figure 6, which corresponds to the game given in figure 5. The reason for this is that looping can change the score through repeated cycles so it makes more sense to think of “loopy” games as a game tree of infinite depth.

With this approach using scoring play theory it is a matter of determining for any given position when the best time to move to 0 is, or as they say in Go, “fill the ko”. Using our braces and slashes notation this game is written as \{0|1\{0|1|\ldots\}.

So I have demonstrated how scoring play theory can be applied to Go, and of course there are many other possible ways one can use this theory to analyse the game of Go, and increase our understanding of the game and the subtle strategies involved in a rigorous mathematical way.
4 Conclusion

I have looked at several different aspects of Go, and shown that there are big differences between scoring play theory and the theory presented in Mathematical Go. I hope I have convinced the reader that scoring play theory can go a long way to helping us genuinely understand mathematically how to play Go, and that applying normal play theory to Go does not really work in a deeper sense.

References

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