HOMOMORPHISMS OF COMMUTATOR SUBGROUPS OF
BRAID GROUPS WITH SMALL NUMBER OF STRINGS

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Abstract. For any \( n \), we describe all endomorphisms of the braid group \( B_n \) and of its commutator subgroup \( B'_n \), as well as all homomorphisms \( B'_n \to B_n \). These results are new only for small \( n \) because endomorphisms of \( B_n \) are already described by Castel for \( n \geq 6 \), and homomorphisms \( B'_n \to B_n \) and endomorphisms of \( B'_n \) are already described by Kordek and Margalit for \( n \geq 7 \). We use very different approaches for \( n = 4 \) and for \( n \geq 5 \).

Introduction

Let \( B_n \) be the braid group with \( n \) strings. It is generated by \( \sigma_1, \ldots, \sigma_{n-1} \) (called standard or Artin generators) subject to the relations

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1; \quad \sigma_i \sigma_j = \sigma_j \sigma_i \sigma_j \text{ for } |i - j| = 1.
\]

Let \( B'_n \) be the commutator subgroup of \( B_n \).

In this paper we describe all endomorphisms of \( B_n \) and \( B'_n \) and homomorphisms \( B'_n \to B_n \) for any \( n \). These results are new only for small \( n \) because endomorphisms of \( B_n \) are described by Castel in [4] for \( n \geq 6 \), and homomorphisms \( B'_n \to B_n \) and endomorphisms of \( B'_n \) are described by Kordek and Margalit in [11] for \( n \geq 7 \).

The automorphisms of \( B_n \) and \( B'_n \) have been already known for any \( n \): Dyer and Grossman [5] proved that the only non-trivial element of \( \text{Out}(B_n) \) corresponds to the automorphism \( \Lambda \) defined by \( \sigma_i \mapsto \sigma_i^{-1} \) for any \( i = 1, \ldots, n - 1 \), and in [17] we proved that the restriction map \( \text{Aut}(B_n) \to \text{Aut}(B'_n) \) is an isomorphism for \( n \geq 4 \) (\( B'_3 \) is a free group of rank 2, thus its automorphisms are known as well: see e.g. [15]).

The problem to study homomorphisms between braid groups and, especially, between their commutator subgroups was posed by Vladimir Lin [12–14] because he found its applications to the problem of superpositions of algebraic functions (the initial motivation for Hilbert's 13th problem), see [13] and references therein.

Let us formulate the main results. We start with those about homomorphisms of \( B'_n \) to \( B_n \) and to itself.

Theorem 1.1. (proven for \( n \geq 7 \) in [11]). Let \( n \geq 5 \). Then every non-trivial homomorphism \( B'_n \to B_n \) extends to an automorphism of \( B_n \).

We proof this theorem in §2. Since \( B''_n = B'_n \) and \( \text{Aut}(B_n) = \text{Aut}(B'_n) \) for \( n \geq 5 \), the following two corollaries are, in fact, equivalent versions of Theorem 1.1.

Corollary 1.2. If \( n \geq 5 \), then any non-trivial endomorphism of \( B'_n \) is bijective.
Corollary 1.3. *If* $n \geq 5$, *then any non-trivial homomorphism* $B'_n \to B_n$ *is an automorphism of* $B'_n$ *composed with the inclusion map.*

Let $R$ be the homomorphism

$$R : B_4 \to B_3, \quad \sigma_1, \sigma_3 \mapsto \sigma_1, \quad \sigma_2 \mapsto \sigma_2.$$  \hspace{1cm} (1)

(we denote it by $R$ because, if we interpret $B_n$ as $\pi_1(X_n)$ where $X_n$ is the space of monic squarefree polynomials of degree $n$, then $R$ is induced by the mapping which takes a degree 4 polynomial to its cubic resolvent).

For a group $G$, we denote its commutator subgroup, center, and abelianization by $G'$, $Z(G)$, and $G^{ab}$ respectively. We also denote the inner automorphism $y \mapsto xyx^{-1}$ by $\bar{x}$, the commutator $xyx^{-1}y^{-1}$ by $[x, y]$, and the centralizer of an element $x$ (resp. of a subgroup $H$) in $G$ by $Z(x; G)$ (resp. by $Z(H; G)$).

Given two group homomorphisms $f : G_1 \to G_2$ and $\tau : G_1^{ab} \to Z(\text{im} f; G_2)$, we define the *transvection of $f$* by $\tau$ as the homomorphism $f_{[\tau]} : G_1 \to G_2$ given by $x \mapsto f(x)\tau(x)$ where $x$ is the image of $x$ in $G_1^{ab}$. To simplify notation, we will not distinguish between $\tau$ and its composition with the canonical projection $G_1 \to G_1^{ab}$. So, we shall often speak of a transvection by $\tau : G_1 \to Z(\text{im} f; G_2)$.

We say that two homomorphisms $f, g : G_1 \to G_2$ are equivalent if there exists $h \in \text{Aut}(G_2)$ such that $f = hg$. If, moreover, $h \in \text{Inn}(G_2)$, we say that $f$ and $g$ are conjugate.

Theorem 1.4. *Any homomorphism* $\varphi : B'_4 \to B_4$ *either is equivalent to a transvection of the inclusion map, or* $\varphi = fR$ *for a homomorphism* $f : B'_3 \to B_4$ (*since* $B'_3$ *is free [9], it has plenty of homomorphisms to any group).*

We prove this theorem in §3.

Corollary 1.5. *Any endomorphism of* $B'_4$ *is either an automorphism or a composition of* $R$ *with a homomorphism* $B'_3 \to B'_4$.

As we already mentioned, $B'_3$ is free, thus its homomorphisms are evident. Now let us describe endomorphisms of $B_n$. We say that a homomorphism is cyclic if its image is a cyclic group (probably, infinite cyclic).

Theorem 1.6. (proven for $n \geq 6$ in [4]). *If* $n \geq 5$, *then any non-cyclic endomorphism of* $B_n$ *is a transvection of an automorphism.*

For $n \geq 7$, this result is derived in [11] from Theorem 1.1. The same proof works without any change for any $n \geq 5$.

Theorem 1.7. *Any endomorphism of* $B_4$ *is either a transvection of an automorphism, or it is of the form* $fR$ *for some* $f : B_3 \to B_4$ (*see Proposition 1.9 for a general form of such* $f$).

This theorem also can be derived from Theorem 1.4 in the same way as it is done in [11] for $n \geq 7$.

Let $\Delta = \Delta_n = \prod_{i=1}^{n-1} \prod_{j=1}^{n-i} \sigma_j$ (the Garside’s half-twist), $\delta = \delta_n = \sigma_{n-1} \ldots \sigma_2 \sigma_1$, and $\gamma = \gamma_n = \sigma_1 \delta_n$. One has $\delta^n = \gamma^{n-1} = \Delta^2$, and it is known that $Z(B_n)$ is generated by $\Delta^2$, and each periodic braid (i.e. a root of a central element) is conjugate to $\delta^k$ or $\gamma^k$ for some $k \in \mathbb{Z}$.

It is well-known that $B_3$ admits a presentation $\langle \Delta, \delta \mid \Delta^2 = \delta^3 \rangle$. By combining this fact with basic properties of canonical reduction systems, it is easy to prove the following descriptions of homomorphisms from $B_3$ to $B_n$ for $n = 3$ or 4.
Proposition 1.8. Any non-cyclic endomorphism of $B_3$ is equivalent to a transvection by $\tau$ of a homomorphism of the form $\Delta \mapsto \Delta$, $\delta \mapsto X\delta X^{-1}$ for some $X \in B_3$ and $\tau : B_3^{ab} \to Z(B_3) = \langle \Delta^2 \rangle$.

Proposition 1.9. For any non-cyclic homomorphism $\varphi : B_3 \to B_4$, one of the following two possibilities holds:

(a) $\varphi$ is equivalent to a transvection by $\tau$ of a homomorphism of the form $\Delta_3 \mapsto \Delta_4$, $\delta_3 \mapsto X\gamma_4 X^{-1}$ for some $X \in B_4$ and $\tau : B_3^{ab} \to Z(B_4) = \langle \Delta_4^2 \rangle$;

(b) $\varphi$ is equivalent to $(\psi \tau)$ where $\psi$ is a non-cyclic endomorphism of $B_3$, $\iota : B_3 \to B_4$ is the standard embedding, and $\tau$ is a homomorphism $B_3^{ab} \to Z(B_4) = \langle \Delta_4^2 \rangle$.

Remark 1.10. Since $B_n^{ab} \cong Z(B_n) \cong \mathbb{Z}$, the transvection in Theorem 1.6 (and in the non-degenerate case in Theorem 1.7) is uniquely determined by a single integer number. In contrast, $(B'_1)^{ab} \cong \mathbb{Z}^2$, thus the transvection in Theorem 1.4 depends on two integers (here $\operatorname{im} \varphi = B'_1$, hence $Z(\operatorname{im} \varphi; B_4) = Z(B_4) \cong \mathbb{Z}$). Notice also that two transvections are involved in the case (b) of Proposition 1.9, thus the general form of $\varphi$ in this case is

$$\Delta_3 \mapsto f(\mu(\Delta_3)^{6k+1}\Delta_4^{6l}), \quad \delta_3 \mapsto f(\mu(X\delta_3 X^{-1}\Delta_3^{4k})\Delta_4^{4l})$$

with $k, l \in \mathbb{Z}$, $X \in B_3$, $f \in \operatorname{Aut}(B_4)$.

2. The case $n \geq 5$

In this section we prove Theorem 1.1 which describes homomorphisms $B'_n \to B_n$ for $n \geq 5$. The proof is very similar to the proof of the case $n \geq 5$ of the main theorem of [17] which describes $\operatorname{Aut}(B_n)$. As we already mentioned, Theorem 1.1 for $n \geq 7$ is proven by Kordek and Margalit in [11]. Some elements of their proof are valid for $n \geq 5$ (see Proposition 2.4 below) which allowed us to omit a big part of our original proof based on [17].

Let $S_n$ be the symmetric group. Let $e : B_n \to \mathbb{Z}$ and $\mu : B_n \to S_n$ be the homomorphisms defined on the generators by $e(\sigma_i) = 1$ and $\mu(\sigma_i) = (i, i + 1)$ for $i = 1, \ldots, n - 1$. So, $e(X)$ is the exponent sum (signed word length) of $X$. Let $P_n = \ker \mu$ be the pure braid group. Following [12], we denote $P_n \cap B'_n$ by $J_n$, and $\mu|B'_n$, by $\mu'$, thus $J_n = \ker \mu'$.

For a pure braid $X$, we denote the linking number between the $i$-th and the $j$-th strings of $X$ by $\operatorname{lk}_{ij}(X)$. It can be defined as $\frac{1}{2}e(X_{ij})$ where $X_{ij}$ is the 2-braid obtained from $X$ by removal of all strings except the $i$-th and the $j$-th ones. For $1 \leq i < j \leq n$, we set $\sigma_{ij} = (\sigma_{j-1} \ldots \sigma_{i+1})\sigma_i(\sigma_{j-1} \ldots \sigma_{i+1})^{-1}$ (here $\sigma_{i+1} = \sigma_i$). Then $P_n$ is generated by $\{\sigma_{ij}^2 \ 1 \leq i < j \leq n\}$ (see [1]) and we denote the image of $\sigma_{ij}^2$ in $P_n^{ab}$ by $A_{ij}$. We use the additive notation for $P_n^{ab}$ and $J_n^{ab}$.

Lemma 2.1. ([17, Lemma 2.3]). $P_n^{ab}$ (for any $n$) is free abelian group with basis $(A_{ij})_{1 \leq i < j \leq n}$, and the natural projection $P_n \to P_n^{ab}$ is given by $X \mapsto \sum_{i < j} \operatorname{lk}_{ij}(X)A_{ij}$.

If $n \geq 5$, then the homomorphism $J_n^{ab} \to P_n^{ab}$ induced by the inclusion map defines an isomorphism of $J_n^{ab}$ with $\{\sum x_{ij} A_{ij} | \sum x_{ij} = 0\}$ (notice that this statement is wrong for $n = 3$ or 4; see [17, Proposition 2.4]).

From now on, till the end of this section, we assume that $n \geq 5$ and $\varphi : B'_n \to B_n$ is a non-cyclic homomorphism. Since any group homomorphism $G_1 \to G_2$ maps $G_1'$
to $G'_2$, we have $\varphi(B''_n) \subset B'_n$. By [9] (see also [17, Remark 2.2]), we have $B''_n = B'_n$, thus
\[
\varphi(B'_n) \subset B'_n.
\]
Then [12, Theorem D] implies that
\[
\varphi(J_n) \subset J_n.
\]
Thus we may consider the endomorphism $\varphi_*$ of $J_{ab}^n$ induced by $\varphi|_{J_n}$. We shall not distinguish between $J_{ab}^n$ and its isomorphic image in $P_{ab}^n$ (see Lemma 2.1).

Following [12], we set
\[
c_i = \sigma_i^{-1} \sigma_i \quad (i = 3, \ldots, n - 1) \quad \text{and} \quad c = c_3.
\]

**Lemma 2.2.** Suppose that $\mu \varphi = \mu'$ and $\varphi(c) = c$. Then $\varphi_* = \text{id}$.

**Proof.** The exact sequence $1 \to J_n \to B'_n \to A_n \to 1$ defines an action of $A_n$ on $J_{ab}^n$ by conjugation. Let $V$ be a complex vector space with base $e_1, \ldots, e_n$ endowed with the natural action of $S_n$ induced by the action on the base. We identify $P_{ab}^n$ with its image in the symmetric square $\text{Sym}^2 V$ under the homomorphism $A_{ij} \to e_i e_j$. Then, by Lemma 2.1, we may identify $J_{ab}^n$ with $\{ \sum x_{ij} e_i e_j \mid x_{ij} \in \mathbb{Z}, \sum x_{ij} = 0 \}$. These identifications are compatible with the action of $A_n$. Thus $W := J_{ab}^n \otimes \mathbb{C}$ is a $\mathbb{C}A_n$-submodule of $\text{Sym}^2 V$.

For an element $v$ of a $\mathbb{C}S_n$-module, let $\langle v \rangle_{\mathbb{C}S_n}$ be the $\mathbb{C}S_n$-submodule generated by $v$. It is shown in the proof of [17, Lemma 3.1], that $W = W_2 \oplus W_3$ where
\[
W_2 = \langle (e_1 - e_2)(e_3 + \cdots + e_n) \rangle_{\mathbb{C}S_n}, \quad W_3 = \langle (e_1 - e_2)(e_3 - e_4) \rangle_{\mathbb{C}S_n},
\]
and that $W_2$ and $W_3$ are irreducible $\mathbb{C}S_n$-modules isomorphic to the Specht modules corresponding to the partitions $(n - 1, 1)$ and $(n - 2, 2)$ respectively. Since the Young diagrams of these partitions are not symmetric, $W_2$ and $W_3$ are also irreducible as $\mathbb{C}A_n$-modules.

The condition $\mu \varphi = \mu'$ implies that $\varphi_*$ is $A_n$-equivariant. Hence, by Schur’s lemma, $\varphi_* = a \text{id}_{W_2} + b \text{id}_{W_3}$. We have the identity
\[
(n - 2)(e_1 - e_2)e_3 = (e_1 - e_2)(e_3 + \cdots + e_n) + \sum_{i \geq 4}(e_1 - e_2)(e_3 - e_i)
\]
whence, denoting $e_5 + \cdots + e_n$ by $e$,
\[
(n - 2)\varphi_*((e_1 - e_3)e_2) = (e_1 - e_3)(a(e_2 + e_4 + e) + b((n - 3)e_2 - e_4 - e)),
\]
\[
(n - 2)\varphi_*((e_2 - e_4)e_3) = (e_2 - e_4)(a(e_1 + e_3 + e) + b((n - 3)e_3 - e_1 - e)).
\]
The condition $\varphi(c) = c$ implies the $\varphi$-invariance of $c^2 \in J_n$. Since the image of $c^{-2}$ in $J_{ab}^n$ is $A_{12} - A_{34}$, we obtain that $e_1 e_2 - e_3 e_4$ is $\varphi_*$-invariant. Hence
\[
(n - 2)(e_1 e_2 - e_3 e_4) = (n - 2)\varphi_*((e_1 e_2 - e_3 e_4)
\]
\[
= (n - 2)\varphi_*((e_1 - e_3)e_2 + (e_2 - e_4)e_3)
\]
\[
= (2a + (n - 4)b)(e_1 e_2 - e_3 e_4) + (a - b)(e_1 + e_2 - e_3 - e_4).e
\]
Since \{\(e_i e_j\}_{i < j}\} is a base of $\text{Sym}^2 V$, it follows that $2a + (n - 4)b = n - 2$ and $a - b = 0$ whence $a = b = 1$.  \(\square\)
Lemma 2.3. Let \( \varphi_1 \) and \( \varphi_2 \) be equivalent homomorphisms \( B'_n \to B_n \). Then \( \mu \varphi_1 \) and \( \mu \varphi_2 \) are conjugate.

Proof. This fact immediately follows from Dyer - Grossman’s [5] classification of automorphisms of \( B_n \) (see the beginning of the introduction) because \( \mu \Lambda = \mu \).

Proposition 2.4. (Kordek and Margalit [11, §3, Proof of Thm. 1.1, Cases 1–3 and Step 1 of Case 4]). There exists \( f \in \text{Aut}(B_n) \) such that \( f \varphi(c_i) = c_i \) for each odd \( i \) such that \( 1 \leq i < n \) (recall that we assume \( n \geq 5 \)).

This proposition implies, in particular, that \( \mu \varphi \) is non-trivial, hence by Lin’s result [12, Theorem C] \( \mu \varphi \) is conjugate either to \( \mu' \) or to \( \nu \mu' \) (when \( n = 6 \)) where \( \nu \) is the restriction to \( A_6 \) of the automorphism of \( S_6 \) given by \( (12) \mapsto (12)(34)(56), (123456) \mapsto (123)(45) \) (it represents the only nontrivial element of \( \text{Out}(S_6) \)).

Lemma 2.5. If \( n = 6 \), then \( \mu \varphi \) is not conjugate to \( \nu \mu' \).

Proof. Let \( H \) be the subgroup generated by \( c_3 \) and \( c_5 \). By Lemma 2.3 and Proposition 2.4 we may assume that \( \varphi|_H = \text{id} \). Then we have

\[
\mu'(H) = \mu \varphi(H) = \{\text{id}, (12)(34), (12)(56), (34)(56)\}.
\]

In particular, no element of \( \{1, \ldots, 6\} \) is fixed by all elements of \( \mu \varphi(H) \). A straightforward computation shows that

\[
\nu \mu'(H) = \{\text{id}, (12)(34), (13)(24), (14)(23)\},
\]

thus 5 and 6 are fixed by all elements of \( \nu \mu'(H) \). Hence these subgroups are not conjugate in \( S_6 \).

Lemma 2.6. There exists \( f \in \text{Aut}(B_n) \) such that \( f \varphi(c) = c \) and \( \mu \varphi = \mu' \).

Proof. By Proposition 2.4 we may assume that

\[
\varphi(c) = c.
\]

Then \( \mu \varphi \) is non-trivial, hence, by [12, Thm. C] combined with Lemma 2.5, it is conjugate to \( \mu' \), i.e. there exists \( \pi \in S_n \) such that \( \tilde{\pi} \mu \varphi = \mu' \), i.e. \( \pi \mu(\varphi(x)) = \mu(x) \pi \) for each \( x \in B'_n \). For \( x = c \) this implies by (3) that \( \pi \) commutes with \( (12)(34) \), hence \( \pi = \pi_1 \pi_2 \) where \( \pi_1 \in V_4 \) (the group in the right hand size of (2)) and \( \pi_2(i) = i \) for \( i \in \{1, 2, 3, 4\} \). Let \( \tilde{V}_4 = \{1, c, \Delta_4, c \Delta_4\} \). This is not a subgroup but we have \( \mu(\tilde{V}_4) = V_4 \). We can choose \( y_1 \in \tilde{V}_4 \) and \( y_2 \in <\sigma_5, \ldots, \sigma_{n-1}> \) so that \( \mu(y_j) = \pi_j \) for \( j = 1, 2 \). Let \( y = y_1 y_2 \). Then we have \( \tilde{y}(c) = c^{\pm 1} \) and \( \mu \tilde{y} \varphi = \tilde{\pi} \mu \varphi = \mu' \). Thus, for \( f = A^k \tilde{y}, k \in \{0, 1\} \), we have \( f \varphi(c) = c \) and \( \mu f \varphi = \mu' \).

Due to Lemma 2.6, from now on we assume that \( \mu \varphi = \mu' \) and \( \varphi(c) = c \). Then, by Lemma 2.2, we have \( \varphi_* = \text{id} \), hence (see Lemma 2.1)

\[
l_k(x) = l_k(\varphi(x)) \quad \text{for any } x \in J_n \text{ and } 1 \leq i < j \leq n.
\]

Starting at this point, the proof of [17, Thm. 1.1] given in [17, §5], can be repeated almost word-by-word in our setting. The only exception is the proof of [17, Lemma 5.8] (which is Lemma 2.11 below) where the invariance of the isomorphism type of centralizers of certain elements is used as well as Dyer–Grossman result [5].
However, as pointed out in [17, Remark 5.15] (there is a misprint there: \( n \geq 6 \) should be replaced by \( n \geq 5 \)), there is another, even simpler, proof of Lemma 2.11 based on Lemma 2.7 (see below). This proof was not included in [17] by the following reason. At that time we new only Garside-theoretic proof of Lemma 2.7 while the rest of the proof of the main theorem for \( n \geq 6 \) used only Nielsen-Thurston theory and results of [12]. So we wanted to make the proofs (at least for \( n \geq 6 \)) better accessible for readers who are not familiar with the Garside theory. Now we learned from [11] that when we wrote that paper, Lemma 2.7 had been already known for a rather long time [2, Lemma 4.9] and the proof in [2] is based on Nielsen-Thurston theory.

In the rest of this section, for the reader’s convenience we re-expose Section 5.1 of [17] (Sections 5.2–5.3 can be left without any change). In this re-exposition we give another proof of [17, Lemma 5.8] and omit the lemmas which are no longer needed due to Proposition 2.4.

We shall consider \( B_n \) as a mapping class group of \( n \)-punctured disk \( \mathbb{D} \). We assume that \( \mathbb{D} \) is a round disk in \( \mathbb{C} \) and the set of the punctures is \( \{1, 2, \ldots, n\} \). Given an embedded segment \( I \) in \( \mathbb{D} \) with endpoints at two punctures, we denote with \( \sigma_I \) the positive half-twist along the boundary of a small neighborhood of \( I \). The set of all such braids is the conjugacy class of \( \sigma_1 \) in \( B_n \). The arguments in the rest of this section are based on Nielsen-Thurston theory. The main tool are the canonical reduction systems. One can use [3], [6] or [10] as a general introduction to the subject. In [17] we gave all precise definitions and statements needed there (using the language and notation inspired mostly by [8]).

**Lemma 2.7.** ([2, Lemma 4.9], [17, Lemma A.2]). Let \( x, y \in B_n \) be such that \( x y x = y x y \) and each of \( x \) and \( y \) is conjugate to \( \sigma_1 \). Then there exists \( u \in B_n \) such that \( \tilde{u}(x) = \sigma_1 \) and \( \tilde{u}(y) = \sigma_2 \).

Let \( \text{sh}_2 : B_{n-2} \to B_n \) be the homomorphism \( \text{sh}_2(\sigma_i) = \sigma_{i+2} \). We set

\[
\tau = \sigma_1^{(n-2)(n-3)} \text{sh}_2(\Delta_{n-2}^{-2}).
\]

We have \( \tau \in J_n \) (in the notation of [17], \( \tau = \psi_{2,n-2}(1; \sigma_1^{(n-2)(n-3)}, \Delta^{-2}) \)). Recall that we assume \( \varphi(c) = c \), \( \mu \varphi = \mu' \), and hence (4) holds.

**Lemma 2.8.** Let \( I \) and \( J \) be two disjoint embedded segments with endpoints at punctures. Then \( \varphi(\sigma_I^{-1} \sigma_J) = \sigma_J^{-1} \sigma_I \), where \( I_1 \) and \( J_1 \) are disjoint embedded segments such that \( \partial I_1 = \partial I \) and \( \partial J_1 = \partial J \).

**Proof.** The braid \( \sigma_I^{-1} \sigma_J \) is conjugate to \( c \), hence so is its image (because \( \varphi(c) = c \)). Therefore \( \varphi(\sigma_I^{-1} \sigma_J) = \sigma_J^{-1} \sigma_I \) for some disjoint \( I_1 \) and \( J_1 \). The matching of the boundaries follows from (4) applied to \( \sigma_I^{-2} \sigma_J^2 \). \( \square \)

**Lemma 2.9.** (cf. [17, Lemmas 5.1 and 5.3]). Let \( C_1 \) be a component of the canonical reduction system of \( \varphi(\tau) \). Then \( C_1 \) cannot separate the punctures 1 and 2, and it cannot separate the punctures \( i \) and \( j \) for \( 3 \leq i < j \leq n \).

**Proof.** Let \( u = \sigma_i^{-1} \sigma_{ij}, \ 3 \leq i < j \leq n \). By Lemma 2.8, \( \varphi(u) = \sigma_I^{-1} \sigma_J \) with \( \partial I = \{1, 2\} \) and \( \partial J = \{i, j\} \). Since \( \varphi(u) \) commutes with \( \varphi(\tau) \), the result follows. \( \square \)
Lemma 2.10. (cf. [17, Lemma 5.7]). \( \varphi(\tau) \) is conjugate in \( P_n \) to \( \tau \).

Proof. \( \varphi(\tau) \) cannot be pseudo-Anosov because it commutes with \( \varphi(c) \) which is \( c \) by our assumption, hence it is reducible.

If \( \varphi(\tau) \) were periodic, then it would be a power of \( \Delta^2 \) because it is a pure braid. This contradicts (4), hence \( \varphi(\tau) \) is reducible non-periodic.

Let \( C \) be the canonical reduction system for \( \varphi(\tau) \). By Lemma 2.9, one of the following three cases occurs.

Case 1. \( C \) is connected, the punctures 1 and 2 are inside \( C \), all the other punctures are outside \( C \). Then the restriction of \( \varphi(\tau) \) (viewed as a diffeomorphism of \( \mathbb{D} \)) to the exterior of \( C \) cannot be pseudo-Anosov because \( \varphi(\tau) \) commutes with \( \varphi(c) = c \), hence it preserves a circle which separates 3 and 4 from 5, \ldots, \( n \). Hence \( \varphi(\tau) \) is periodic which contradicts (4). Thus this case is impossible.

Case 2. \( C \) is connected, the punctures 1 and 2 are outside \( C \), all the other punctures are inside \( C \). This case is also impossible and the proof is almost the same as in Case 1. To show that \( \varphi(\tau) \) cannot be pseudo-Anosov, we note that it preserves a curve which encircles only 1 and 2.

Case 3. \( C \) has two components: \( C_1 \) and \( C_2 \) which encircle \( \{1, 2\} \) and \( \{3, \ldots, n\} \) respectively. Let \( \alpha \) be the interior braid of \( C_2 \) (that is \( \varphi(\tau) \) with the strings 1 and 2 removed). It cannot be pseudo-Anosov by the same reasons as in Case 1: because \( \varphi(\tau) \) preserves a circle separating 3 and 4 from 5, \ldots, \( n \). Hence \( \alpha \) is periodic. Using (4), we conclude that \( \varphi(\tau) \) is a conjugate of \( \tau \). Since the elements of \( Z(\tau; B_n) \) realize any permutation of \( \{1, 2\} \) and of \( \{3, \ldots, n\} \), the conjugating element can be chosen in \( P_n \). \( \Box \)

Lemma 2.11. (cf. [17, Lemma 5.8]). There exists \( u \in P_n \) such that \( \varphi(c_i) = \bar{u}(c_i) \) for each \( i = 3, \ldots, n - 1 \).

Proof. Due to Lemma 2.10, without loss of generality we may assume that \( \varphi(\tau) = \tau \) and \( \tau(C) = C \) where \( C \) is the canonical reduction system for \( \tau \) consisting of two round circles \( C_1 \) and \( C_2 \) which encircle \( \{1, 2\} \) and \( \{3, \ldots, n\} \) respectively. Since the conjugating element in Lemma 2.10 is chosen in \( P_n \), we may assume that (4) still holds.

By Lemma 2.8, for each \( i = 3, \ldots, n - 1 \), we have \( \varphi(c_i) = \sigma_{I_i}^{-1} \sigma_{J_i}^{-1} \) with \( \partial I_i = \{1, 2\} \) and \( \partial J_i = \{i, i + 1\} \). Since \( \tau \) commutes with each \( c_i \), the segments \( I_i \) and \( J_i \) can be chosen disjoint from the circles \( C_1 \) and \( C_2 \). Hence \( \sigma_{I_i} = \sigma_1 \) for each \( i \), and all the segments \( J_i \) are inside \( C_2 \).

Therefore the braids \( \sigma_{J_3}, \ldots, \sigma_{J_{n-1}} \) satisfy the same braid relations as \( \sigma_3, \ldots, \sigma_{n-1} \). Hence, by Lemma 2.7 combined with [17, Lemma 5.13], \( J_3 \cup \cdots \cup J_{n-1} \) is an embedded segment. Hence it can be transformed to the straight line segment \( [3, n] \) by a diffeomorphism identical on the exterior of \( C_2 \). Hence for the braid \( u \) represented by this diffeomorphism we have \( \bar{u}(c_i) = c_i \), \( i \geq 3 \). The condition \( \partial J_i = \{i, i + 1\} \) implies that \( u \in P_n \). \( \Box \)

The rest of the proof of Theorem 1.1 repeats word-by-word [17, §§5.2–5.3].

Remark 2.12. Besides Nielsen-Thurston theory, in the case \( n = 5 \), the arguments in [17, §§5.3] use an auxiliary result [17, Lemma A.1] for which the only proof we know is based on a slight modification of the main theorem of [16] which is proven there using the Garside theory.
3. The case $n = 4$

We shall use the same notation as in [17, §6]. The groups $B_3'$ and $B_4'$ were computed in [9], namely $B_3'$ is freely generated by $u = \sigma_2\sigma_1^{-1}$ and $t = \sigma_1^{-1}\sigma_2$, and $B_4' = K_4 \rtimes B_3'$ where $K_4 = \ker R$ (see (1)). The group $K_4$ is freely generated by $c = \sigma_3\sigma_1^{-1}$ and $w = \sigma_2\sigma_2^{-1}$. The action of $B_3'$ on $K_4$ by conjugation is given by

$$ucu^{-1} = w, \quad uwu^{-1} = w^2c^{-1}w, \quad tct^{-1} = cw, \quad twt^{-1} = cw^2.$$  \hfill (5)

The action of $\sigma_1$ and $\sigma_2$ on $K_4$ is given by

$$\sigma_1c\sigma_1^{-1} = c, \quad \sigma_1w\sigma_1^{-1} = c^{-1}w, \quad \sigma_2c\sigma_2^{-1} = w, \quad \sigma_2w\sigma_2^{-1} = wc^{-1}w.$$  \hfill (6)

So, we also have $B_4 = K_4 \rtimes B_3$.

Besides the elements $c, w, u, t$ of $B_4'$, we consider also $d = \Delta \sigma_1^{-3}\sigma_3^{-3}$ and $g = R(d) = \Delta_3^2\sigma_1^{-6}$.

One has (see Figure 1)

$$d = [c^{-1}t, u^{-1}], \quad g = [t, u^{-1}].$$  \hfill (7)

We denote the subgroup generated by $c$ and $d$ by $H$ and the subgroup generated by $c$ and $g$ by $G$.

![Figure 1](image-url)  \hfill Figure 1. the identity $d = [c^{-1}t, u^{-1}]$.

Let $\varphi : B_4' \to B_4$ be a homomorphism such that $K_4 \not\subset \ker \varphi$.

**Lemma 3.1.** The restriction of $\varphi$ to $H$ is injective, $\varphi(H) \subset B_4'$, and $\varphi(G) \subset B_4'$.

**Proof.** We have $H = \langle c \rangle \rtimes \langle d \rangle$ and $d$ acts on $c$ by $dcd^{-1} = c^{-1}$. Hence any non-trivial normal subgroup of $H$ contains a power of $c$. Thus, if $\varphi|_H$ were not injective, $\ker \varphi$ would contain a power of $c$ and hence $c$ itself because the target group $B_4$ does not have elements of finite order. Then we also have $w \in \ker \varphi$ because $w = ucu^{-1}$.

This contradicts the assumption $K_4 = \langle c, w \rangle \not\subset \ker \varphi$, thus $\varphi|_H$ is injective.

We have $dcd^{-1} = c^{-1}$, hence the image of $\varphi(c)$ under the abelianization $c : B_4 \to \mathbb{Z}$ is zero, i.e., $\varphi(c) \in B_4'$. By (7) we also have $\varphi(d) \in B_4'$ and $\varphi(g) \in B_4'$, thus $\varphi(H) \subset B_4'$ and $\varphi(G) \subset B_4'$. \hfill \Box
Lemma 3.2. \( \varphi(c) \) and \( \varphi(g) \) do not commute.

Proof. Suppose that \( \varphi(c) \) and \( \varphi(g) \) commute. Then \( \varphi(c) = \varphi(gcg^{-1}) \). Hence (see Figure 2) \( \varphi(c) = \varphi(w^{-1}c^{-1}w) \), i.e., \( \varphi \) factors through the quotient of \( B'_4 \) by the relation \( wc = c^{-1}w \). Let us denote this quotient group by \( \hat{B}'_4 \).

The relation \( wc = c^{-1}w \) allows us to put any word \( \prod_j c^{k_j} w^{l_j} \) with \( l_j = \pm 1 \) into the normal form \( c^{k_1-k_2+k_3-...-l_1+l_2+l_3+...} \) in \( \hat{B}'_4 \). Due to (5), the conjugation by \( t \) of the word \( w^{-1}cwc \) (which is equal to 1 in \( \hat{B}'_4 \)) yields

\[
1 = t(w^{-1}cwc)t^{-1} = (w^{-2}c^{-1})(cw)(cw^2)(cw) = w^{-1}cw^2cw = c^{-2}w^2
\]

(here in the last step we put the word into the above normal form). Conjugating once more by \( t \) and putting the result into the normal form, we get

\[
1 = t(c^{-2}w^2)t^{-1} = (w^{-1}c^{-1})(w^{-1}c^{-1})(cw^2)(cw) = w^{-1}c^{-1}wcw^2 = c^2w^2.
\]

Thus \( c^{-2}w^2 = c^2w^2 = 1 \), i.e., \( c^4 = 1 \) in \( \hat{B}'_4 \), hence \( \varphi(c^4) = 1 \) which contradicts Lemma 3.1. \( \square \)

As in [17], we denote the stabilizer of 1 under the natural action of \( B_3 \) on \( \{1, 2, 3\} \) by \( B_{1,2} \). It is well-known (and easy to prove by Reidemeister-Schreier method) that \( B_{1,2} \) is isomorphic to the Artin group of type \( B_2 \), that is \( \langle x, y \mid xyxy = yxxy \rangle \). The Artin generators \( x \) and \( y \) of the latter group correspond to \( \sigma_1^2 \) and \( \sigma_2 \).

Lemma 3.3. (cf. [17, Lemma 6.2]) We have \( G = Z(d^2c^6; B'_4) \) and this group is generated by \( g \) and \( c \) subject to the defining relation \( gcgc = cgcg \).

Proof. The centralizer of \( d^2c^6 \) in \( B_4 \) is the stabilizer of its canonical reduction system which is shown in Figure 4, and (see [8, Thm. 5.10]) it is the image of the injective homomorphism \( B_{1,2} \times \mathbb{Z} \to B_4 \), \( (X, n) \mapsto Y\sigma_1^n \), where the 4-braid \( Y \) is obtained from the 3-braid \( X \) by doubling the first strand. It follows that \( Z(d^2c^6; B'_4) \) is the isomorphic image of \( B_{1,2} \) under the homomorphism \( \psi : B_{1,2} \to B'_4 \) defined on the generators by \( \psi(\sigma_1^2) = g \), \( \psi(\sigma_2) = c \) (see Figure 3), thus \( Z(d^2c^6; B'_4) = G \). As we have pointed out above, \( B_{1,2} \) is the Artin group of type \( B_2 \), hence so is \( G \) and \( gcgc = cgcg \) is its defining relation. \( \square \)

Figure 2. The identity \( gcg^{-1} = w^{-1}c^{-1}w \).

Figure 3. The images of the generators under \( \psi : B_{1,2} \to B'_4 \).
Lemma 3.4. \( \varphi(d^2c^6) \) is conjugate in \( B_4 \) to \( d^{2k}, d^{2k}c^{6k} \), or \( h^k \) for some integer \( k \neq 0 \), where \( h = \Delta_3^{-2}\sigma_3\sigma_2\sigma_1^2\sigma_2\sigma_3 \).

Proof. Let \( x = d^2c^6 \). By Lemma 3.3, \( G = Z(x; B'_4) \), hence \( \varphi(G) \subset Z(\varphi(x); B_4) \).

By Lemma 3.1 we also have \( \varphi(G) \subset B'_4 \), hence \( \varphi(G) \subset Z(\varphi(x); B'_4) \). Then it follows from Lemma 3.2 that \( Z(\varphi(x); B'_4) \) is non-commutative. The isomorphism classes of the centralizers (in \( B'_4 \)) of all elements of \( B'_4 \) are computed in [17, Table 6.1]. We see in this table that \( Z(\varphi(x); B'_4) \) is non-commutative only in the required cases (see the corresponding canonical reduction systems in Figure 4) unless \( \varphi(x) = 1 \). However the latter case is impossible by Lemma 3.1. \( \square \)

![Figure 4. Canonical reduc. systems for \( d^m, c^m, (d^2c^6)^m, h^m, m \neq 0 \).](image)

Lemma 3.5. There exists an automorphism of \( B_4 \) which takes \( \varphi(c) \) and \( \varphi(d) \) to \( c^k \) and \( d^k \) respectively for an odd positive integer \( k \).

Proof. Let \( x = d^2c^6 \) and \( y = d^2c^{-6} \). Since \( y = dx^{-1} \), the images of \( x \) and \( y \) are conjugate and both of them belong to one of the conjugacy classes indicated in Lemma 3.4. The canonical reduction systems for \( d^{2k}, d^{2k}c^{6k} \), and \( h^k \) for \( k \neq 0 \) are shown in Figure 4. Since \( x \) and \( y \) commute, the canonical reduction systems of their images can be chosen disjoint from each other. Hence, up to composing \( \varphi \) with an inner automorphism of \( B_4 \), \( (\varphi(x), \varphi(y)) \) is either \( (h^{k_1}, h^{k_2}) \) or \( (d^{2k_1}c^{l_1}, d^{2k_2}c^{l_2}) \) where \( l_j \in \{0, \pm 6k_j\} \), \( j = 1, 2 \). Since \( x \) and \( y \) are conjugate, by comparing the linking numbers between different pairs of strings, we deduce that \( k_1 = k_2 \) and (in the second case) \( l_1 = \pm l_2 \). Moreover, \( \varphi(x) \neq \varphi(y) \) by Lemma 3.1. Hence, up to exchange of \( x \) and \( y \) (which is realizable by composing \( \varphi \) with \( \tilde{d} \)), we have \( \varphi(x) = d^{2k}c^{6k} \) and \( \varphi(y) = d^{2k}c^{-6k} \) whence, using that \( xy^{-1} = c^{12} \), we obtain \( \varphi(c^{12}) = \varphi(xy^{-1}) = c^{12k} \). Since the canonical reduction systems of any braid and its non-zero power coincide (see, e.g., [7, Lemmas 2.1–2.3]), we obtain \( \varphi(c) = c^k \) and \( \varphi(d) = d^k \). By composing \( \varphi \) with \( \Lambda \) if necessary, we can arrive to \( k > 0 \). The relation \( d^kc^kd^{-k} = c^{-k} \) combined with Lemma 3.1 implies that \( k \) is odd. \( \square \)

Lemma 3.6. \( \varphi(K_4) \subset K_4 \).

Proof. Lemma 3.5 implies that \( c^k \) is mapped to \( \varphi(c) \) by an automorphism of \( B_4 \). Since \( K_4 \) is a characteristic subgroup of \( B'_4 \) (see [17, Lemma 6.5]) and \( B'_4 \) is a characteristic subgroup of \( B_4 \), we deduce that \( \varphi(c) \in K_4 \). The same arguments can be applied to any other homomorphism of \( B'_4 \) to \( B_4 \) whose kernel does not contain \( K_4 \), in particular, they can be applied to \( \varphi\tilde{u} \) whence \( \varphi\tilde{u}(c) \in K_4 \). Since \( \varphi(w) = \varphi\tilde{u}(c) \), we conclude that \( \varphi(K_4) = \langle \varphi(c), \varphi(w) \rangle \subset K_4 \). \( \square \)

Let \( F = G \cap K_4 \).
Lemma 3.7. (a) The group $F$ is freely generated by $c$ and $c_1 = w^{-1}c^{-1}w$.

(b). Let $a_1, \ldots, a_{m-1}$ and $b_1, \ldots, b_m$ be non-zero integers, and let $a_0$ and $a_m$ be any integers. Then $c^{a_0}w^{b_1}c^{a_1} \ldots w^{b_m}c^{a_m}$ is in $F$ if and only if $m$ is even and $b_j = (-1)^j$ for each $j = 1, \ldots, m$.

Proof. The relation on $g$ and $c$ in Lemma 3.3 is equivalent to
\[ g^{-1}cg = cg^{-1}. \] (8)
Recall that $G = \langle c, g \rangle$. We have $R(c) = 1$ and, by (7), $g = R(d) \in B'_2$ whence $R(g) = g$. Hence $R(G)$ is generated by $g$. By definition, $F = \ker(R|_G)$, hence $F$ is the normal closure of $c$ in $G$, i.e., $F$ is generated by the elements $\tilde{g}^k(c)$, $k \in \mathbb{Z}$. We have $\tilde{g}(c) = c_1$ (see Figure 2) and
\[ \tilde{g}(c_1) = \tilde{g}^2(c) = g c^{-1}(cgcg^{-1})g^{-1} = g c^{-1}(g^{-1}cg)c^{-1} = c_1^{-1}c_1 \]
whence by induction we obtain $\tilde{g}^k(c) \in \langle c, c_1 \rangle$ for all positive $k$. Similarly,
\[ \tilde{g}^{-1}(c) = (g^{-1}cg)c^{-1} = c_1c^{-1} \]
and $\tilde{g}^{-1}(c_1) = c$ whence $\tilde{g}^k(c) \in \langle c, c_1 \rangle$ for all negative $k$. Thus $F = \langle c, c_1 \rangle$.

To check that $c$ and $c_1$ is a free base of $F$ (which completes the proof of (a)), it is enough to observe that if, in a reduced word in $x, y$, we replace each $x^k$ with $c^k$ and each $y^k$ with $w^{-1}c^{-k}w$, then we obtain a reduced word in $c$ and $w$. The statement (b) also easily follows from this observation. □

Lemma 3.8. If $x \in F$ and $x = [w^{-1}, A]$ with $A \in K_4$, then $x = [w^{-1}, c^k], k \in \mathbb{Z}$.

Proof. Let $A = w^{b_1}c^{a_1} \ldots w^{b_m}c^{a_m}w^{b_{m+1}}, m \geq 0$, where $a_1, \ldots, a_m$ and $b_2, \ldots, b_m$ are non-zero while $b_1$ and $b_{m+1}$ may or may not be zero. If $m = 0$, then $[w^{-1}, A] = 1 = [w^{-1}, c^0]$ and we are done. If $m = 1$, then $[w^{-1}, A] = w^{b_1-1}c^{a_1}w^{-1}c^{-a_1}w^{-b_1}$ where, by Lemma 3.7(b), we must have $b_1 = 0$, hence $[w^{-1}, A] = [w^{-1}, c^{a_1}]$ as required. Suppose that $m \geq 2$. Then
\[ [w^{-1}, A] = w^{b_1-1}c^{a_1} \ldots w^{b_m}c^{a_m}w^{-a_m}w^{-b_m} \ldots c^{-a_1}w^{-b_1} \]
and this is a reduced word in $c, w$. Hence, by Lemma 3.7(b), the sequence of the exponents of $w$ in this word (starting form $b_1 - 1$ when $b_1 \neq 1$ or from $b_2$ when $b_1 = 1$) should be $(-1, 1, -1, 1, \ldots, -1, 1)$. Such a sequence cannot contain $(\ldots, b_m, 1, -b_m, \ldots)$. A contradiction. □

Lemma 3.9. If $\varphi(d^2) = d^2$ and $\varphi(c) = c$, then $w^{-1}\varphi(w) \in F$.

Proof. For any $k \in \mathbb{Z}$ we have
\[ \sigma_3^k w = \sigma_3^k (\sigma_2 \sigma_3)(\sigma_1^{-1} \sigma_2^{-1}) = (\sigma_2 \sigma_3)^k (\sigma_1^{-1} \sigma_2^{-1}) = (\sigma_2 \sigma_3)^k \sigma_1^{-1} \sigma_2^{-1} = w \sigma_1^{-1} \sigma_2^{-1}, \]
hence $\sigma_3^k \sigma_1^{-1} \sigma_2^{-1} = w = \sigma_3^{-k} \sigma_1^{-1} \sigma_2^{-1}$ and we obtain
\[ d^2 w d^{-2} = \Delta^2 \sigma_1^{-6} (\sigma_3^{-6} w \sigma_1)^6 \sigma_3^{-6} \Delta^{-2} = \sigma_1^{-6} (\sigma_3^{-6} w \sigma_1)^6 \sigma_3^{-6} = c^6 w c^6. \] (9)
Set $x = w^{-1}\varphi(w)$, i.e., $\varphi(w) = wx$. The relation (9) combined with our hypothesis on $c$ and $d^2$ implies
\[ c^6 w x c^6 = \varphi(c^6 w c^6) = \varphi(d^2(w)) = d^2(w x) = d^2(w) d^2(x) = c^6 w c^6 d^2 x d^{-2} \]
whence $x(c^6 d^2) = (c^6 d^2)x$, i.e., $x \in Z(d^2 c^6)$. On the other hand, $\varphi(w) \in K_4$ by Lemma 3.6, hence $x = w^{-1}\varphi(w) \in K_4$. By Lemma 3.3 we have $Z(d^2; B'_4) = G$, thus $x \in Z(d^2 c^6) \cap K_4 = G \cap K_4 = F$. □
Lemma 3.10. There exists $f \in \text{Aut}(B_4)$ and a homomorphism $\tau : B_4' \rightarrow Z(B_4)$ such that $f \varphi(c) = c$, $f \varphi(d^2) = d^2$, and $Rf \varphi = R\text{id}_{[\tau]}$.

Proof. By Lemma 3.5 we may assume that $\varphi(c) = c^k$ and $\varphi(d) = d^k$ for an odd positive $k$. For $x \in K_4$, we denote its image in $K_4^{ab}$ by $\tilde{x}$ and we use the additive notation for $K_4^{ab}$. Consider the homomorphism $\pi : B_4 \rightarrow \text{Aut}(K_4^{ab}) = \text{GL}(2, \mathbb{Z})$, where $\pi(x)$ is defined as the automorphism of $K_4^{ab}$ induced by $\tilde{x}$; here we identify $\text{Aut}(K_4^{ab})$ with $\text{GL}(2, \mathbb{Z})$ by choosing $\tilde{c}$ and $\tilde{w}$ as a base of $K_4^{ab}$. By Lemma 3.6, $\varphi(w) \in K_4$, hence we may write $\varphi(w) = p\tilde{c} + q\tilde{w}$ with $p, q \in \mathbb{Z}$. Then, for any $x \in B_4$, we have

$$\pi \varphi(x) \cdot P = P \pi(x) \quad \text{where} \quad P = \begin{pmatrix} k & p \\ 0 & q \end{pmatrix}. \quad (10)$$

($P$ is the matrix of the endomorphism of $K_4^{ab}$ induced by $\varphi|_4$). By (9) we have

$$\pi(d^2) = \begin{pmatrix} 1 & 12 \\ 0 & 1 \end{pmatrix} \quad \text{hence} \quad \pi(d^{2k}) \cdot P = P \pi(d^2) = \begin{pmatrix} 0 & 12k(q - 1) \\ 0 & 0 \end{pmatrix}. \quad (11)$$

Since $\varphi(d^2) = d^{2k}$, we obtain from (10) combined with (11) that $q = 1$, i.e., $\varphi(w) = p\tilde{c} + \tilde{w}$. By (5) we have $\varphi(u)c^k\varphi(u)^{-1} = \varphi(ucu^{-1}) = \varphi(w)$, hence

$$k \varphi(u)c\varphi(u)^{-1} = \varphi(w) = p\tilde{c} + \tilde{w}.$$

Therefore $k = 1$ because $p\tilde{c} + \tilde{w}$ cannot be a multiple of another element of $K_4^{ab}$. Notice that $\tilde{\sigma}_1(c) = c$, $\tilde{\sigma}_1(d^2) = d^2$, and $\tilde{\sigma}_1(w) = c^{-1}w$ (see (6)). Hence, for $f = \tilde{\sigma}_1^p$, we have

$$f \varphi(c) = c, \quad f \varphi(d^2) = d^2, \quad f \varphi(w) = \tilde{w}. \quad (12)$$

It remains to show that $Rf \varphi = R\text{id}_{[\tau]}$ for some $\tau : B_4' \rightarrow Z(B_4)$. Let $x \in B_4'$. Since $B_4' = K_4 \times B_3'$ and $B_4 = K_4 \times B_3$, we may write $x = x_1 a_1$ and $f \varphi(x) = x_2 a_2$ with $x_1 \in R(x) \in B_3'$, $x_2 = Rf \varphi(x) \in B_3$, and $a_1, a_2 \in K_4$. The equation (10) for $f \varphi$ (and hence with the identity matrix for $P$ because (12) means that $f \varphi|_4$ induces the identity mapping of $K_4^{ab}$) reads $\pi f \varphi(x) = \pi(x)$, that is $\pi(x_2 a_2) = \pi(x_1 a_1)$. Since $a_1, a_2 \in K_4 \subset \ker \pi$, this implies that

$$\pi(x_1) = \pi(x_2). \quad (13)$$

Let $S_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $S_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. It is well-known that the mapping $\sigma_1 \mapsto S_1$, $\sigma_2 \mapsto S_2$ defines an isomorphism between $B_3/\langle \Delta_3^4 \rangle$ and $\text{SL}(2, \mathbb{Z})$. From (6) we see that $\pi(\sigma_1) = S_1$ and $\pi(\sigma_1^{-1}\sigma_2\sigma_1) = S_2$. Hence $\ker(\pi|_{B_3}) = \langle \Delta_3^4 \rangle = R(Z(B_4))$. Therefore (13) implies that $x_2 = x_1 R(\tau(x))$ for some element $\tau(x)$ of $Z(B_4)$. It is easy to check that $\tau$ is a group homomorphism, thus, recalling that $x_1 = R(x)$ and $x_2 = Rf \varphi(x)$, we get $Rf \varphi(x) = x_2 = x_1 R(\tau(x)) = R(x\tau(x)) = R \text{id}_{[\tau]}(x)$. □

Lemma 3.11. If $\varphi|_4 = \text{id}$ and $R \varphi = R\text{id}_{[\tau]}$ for some homomorphism $\tau : B_4' \rightarrow Z(B_4)$, then $\varphi = \text{id}_{[\tau]}$.

Proof. Since $B_4' = K_4 \times B_3'$ and $K_4 \subset \ker \pi$, it is enough to show that $\varphi|_{B_3'} = \text{id}_{[\tau]}$. So, let $x \in B_3'$. The condition $R \varphi = R\text{id}_{[\tau]}$ means that $\varphi(x) = x a \tau(x)$ with $a \in K_4$. 


Let $b$ be any element of $K_4$. Then $xbx^{-1} \in K_4$, hence $\varphi(xbx^{-1}) = xbx^{-1}$ (because $\varphi|_{K_4} = \text{id}$). Since $\varphi(x) = x\tau(x)$, $\varphi(b) = b$, and $\tau(x)$ is central, it follows that

$$xbx^{-1} = \varphi(xbx^{-1}) = \varphi(x)b\varphi(x)^{-1} = x\tau(x)b\tau(x)^{-1}a^{-1}x^{-1} = xaba^{-1}x^{-1}$$

whence $aba^{-1} = b$. This is true for any $b \in K_4$, thus $a \in Z(K_4)$. Since $K_4$ is free, we deduce that $a = 1$, hence $\varphi(x) = x\tau(x) = \text{id}_{[\tau]}(x)$.

**Proof of Theorem 1.4.** Recall that we assume in this section that $\varphi$ is a homomorphism $B'_4 \rightarrow B_4$ such that $K_4 \not\subset \ker \varphi$.

By Lemma 3.10 we may assume that $\varphi(c) = c$, $\varphi(d^2) = d^2$, and $R\varphi = R\text{id}_{[\tau]}$ for some $\tau : B'_4 \rightarrow Z(B_4)$, in particular, $R\varphi(u) = R(u\tau(u))$. The latter condition means that $\varphi(u) = u\tau(u)$ with $a \in K_4$. Then, by (5), we have

$$\varphi(w) = \varphi(ucu^{-1}) = uaca^{-1}u^{-1} = \tilde{u}(c[c^{-1}, a]), = w[w^{-1}, \tilde{u}(a)],$$

thus $w^{-1}\varphi(w) = [w^{-1}, A]$ for $A = \tilde{u}(a) \in K_4$. By Lemma 3.9 we have also $w^{-1}\varphi(w) \in F$. Then Lemma 3.8 implies that $w^{-1}\varphi(w) = [w^{-1}, c^k]$ for some integer $k$, that is $\varphi(w) = c^kw^{-k}$. Hence, $(\tilde{c}^{-k}\varphi)|_{K_4} = \text{id}$. Since $c \in \ker R$, we have $R\tilde{c}^{-k} = R$ whence $R\tilde{c}^{-k}\varphi = R\varphi = R\text{id}_{[\tau]}$. This fact combined with $(\tilde{c}^{-k}\varphi)|_{K_4} = \text{id}$ and Lemma 3.11 implies that $\tilde{c}^{-k}\varphi = \text{id}_{[\tau]}$, i.e., $\varphi$ is equivalent to $\text{id}_{[\tau]}$. □

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