On the Scaling Limit of the 1D Hubbard Model at Half Filling

Ezer Melzer

School of Physics and Astronomy
Beverly and Raymond Sackler Faculty of Exact Sciences
Tel-Aviv University
Tel-Aviv 69978, ISRAEL

email: melzer@ccsg.tau.ac.il

Abstract

The dispersion relations and $S$-matrix of the one-dimensional Hubbard model at half filling are considered in a certain scaling limit. (In the process we derive a useful small-coupling expansion of the exact lattice dispersion relations.) The resulting scattering theory is consistently identified as that of the $SU(2)$ chiral-invariant Thirring (or Gross-Neveu) field theory, containing both massive and massless sectors.
1. Introduction

The Hubbard model [1] describes electrons on a lattice with on-site interaction only, in addition to a standard nearest-neighbor hopping term. In two dimensions the model has received much attention lately in connection with high-$T_c$ superconductivity. Some of its properties are believed [2] to be similar to those exhibited by the one-dimensional model, which is exactly solvable by means of the Bethe Ansatz technique [3]. In this paper we discuss certain aspects of the scaling limit of the one-dimensional model, which are relevant for its large-distance asymptotic behavior.

The Hamiltonian of the linear Hubbard model is given by

$$H = -\frac{1}{2} \sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} (c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) - 2U \sum_{j=1}^{L} (n_{j,\uparrow} - \frac{1}{2})(n_{j,\downarrow} - \frac{1}{2}), \quad (1.1)$$

where $c_{j,\sigma}$ are canonical fermionic annihilation operators, $j$ labels the sites of a periodic chain of length $L$ (which is taken to be even), $\sigma$ labels the two spin degrees of freedom, $U$ is a real coupling constant, and $n_{j,\sigma} = c_{j,\sigma}^\dagger c_{j,\sigma}$ is the number operator for spin $\sigma$ on site $j$. (The overall normalization of $H$ chosen in (1.1) will be convenient later on.) Since $H$ commutes with the total number operator $\sum_{j=1}^{L} \sum_{\sigma=\uparrow,\downarrow} n_{j,\sigma}$, it can be diagonalized separately in eigenspaces of fixed number of “electrons” $N$.

The model has an $SO(4) = SU(2) \times SU(2)/\mathbb{Z}_2$ symmetry, the two $SU(2)$’s pertaining to spin ($s$) and charge ($c$). The spectrum is built out of four fundamental excitations (alias quasiparticles), forming a “spinon-antispinon” $SU(2)_s$-doublet and a “holon-antiholon” $SU(2)_c$-doublet. However, this separation of spin and charge seen in the quantum numbers of the quasiparticles does not mean that the theory decouples into a tensor product of two $SU(2)$-symmetric models. For instance, there is a selection rule which allows only representations with integer total $SU(2)_s$ and $SU(2)_c$ spin in the spectrum (implementing the $\mathbb{Z}_2$ quotient which reduces the symmetry from the naive $SU(2) \times SU(2)$ down to $SO(4)$).

The situation at half-filling $N = L$ is of special interest. In this case one of the two quasiparticle doublets develops a mass gap while the other remains massless. The fact that the mass gap vanishes as the coupling tends to zero opens up the possibility for the

---

1 For details on the following features of the model see [1] and references therein.
2 In particular, this restriction implies that there are no single-particle states in the spectrum and thus the fundamental quasiparticles are “confined”.

1
existence of a scaling limit in which both massive and massless excitations survive in the spectrum of the resulting field theory. Our aim is to explore this possibility, which we will do at the level of the dispersion relations of the quasiparticles and their scattering amplitudes. The $S$-matrix theory obtained this way is then identified as that of the $SU(2)$ chiral-invariant Thirring (or Gross-Neveu) model, whose lagrangian is given by [5-7]

$$
\mathcal{L} = i \bar{\psi} \partial_\tau \psi + g \left[ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma^5 \psi)^2 \right]
= i \bar{\psi} \partial_\tau \psi - \frac{1}{2} g \sum_{\alpha=1}^{3} (\bar{\psi} \gamma^\mu \sigma^\alpha \psi)^2 ,
$$

where $\psi$ is a doublet of Dirac spinors, $\gamma^\mu$ are Dirac matrices in 1+1 dimensions, and $\sigma^\alpha$ are the Pauli matrices (the equality between the two lines of (1.2) can be established with the aid of identities listed in the appendix of [5]).

The field theory (1.2) and its $S$-matrix have been discussed in [8-16]. It is known, using bosonization, that the theory essentially decouples into a massless and a massive sector. (This statement holds modulo certain orbifolding, cf. [15][17], which is reflected for instance by “kinky” restrictions [18] on the multiparticle spectrum.) The massless sector is described by the level one $SU(2)$ WZW conformal field theory. The massive sector, on the other hand, can be viewed as a marginally relevant $SU(2)$-preserving (integrable) perturbation of another copy of the same conformal field theory, where mass is generated dynamically through “dimensional transmutation” (for a construction of this sector of the theory from a scaling limit of the XXZ spin chain cf. [19]).

The emergence of the $SU(2)$ chiral-invariant Thirring field theory from the scaling limit of the half-filled Hubbard model, as described in the sequel, is not surprising. It was already noted on the basis of renormalization group and symmetry arguments in [13], where the continuum (low-energy) limit of (1.1) was considered. Nevertheless, we think that our complementary analysis is worthwhile. Related work can be found in [20].

The rest of the paper is organized as follows. In sect. 2 we define the scaling limit and derive the scaled dispersion relations. In the process the familiar Hubbard model dispersion relations (2.1)–(2.2) are rewritten in the form (2.3)–(2.4), which is most useful for analyzing the rather singular zero-coupling limit; we find this apparently new form, whose derivation is presented in the appendix, interesting by itself. The scaled $S$-matrix obtained in sect. 3 is discussed in the final section.

---

3 Somewhat confusingly, in the literature the $S$-matrix of the massive sector alone is occasionally referred to as that of the full theory (1.2).
2. The scaling limit

We restrict attention to the attractive regime $U > 0$, where the spin excitations (spinons) are massive while the charge excitations (holons) are massless. The repulsive regime is dual to the attractive one in the sense that the properties of the two excitations are interchanged \[4\][21]. The spin-wave dispersion relation is given in the parametric form (see \[4\] and references therein)

$$p_s(k) = k - \int_0^\infty \frac{d\omega}{\omega} \frac{J_0(\omega) \sin(\omega \sin k)}{\cosh(\omega U)} e^{-\omega U}$$

$$\epsilon_s(k) = U - \cos k + \int_0^\infty \frac{d\omega}{\omega} \frac{J_1(\omega) \cos(\omega \sin k)}{\cosh(\omega U)} e^{-\omega U}$$

(2.1)

where the $J_\nu(\omega)$ are Bessel functions. The charge-wave dispersion relation reads

$$p_c(\lambda) = -\int_0^\infty \frac{d\omega}{\omega} \frac{J_0(\omega) \sin(\omega \lambda)}{\cosh(\omega U)} \ , \quad \epsilon_c(\lambda) = \int_0^\infty \frac{d\omega}{\omega} \frac{J_1(\omega) \cos(\omega \lambda)}{\cosh(\omega U)}$$

(2.2)

where, for nonzero $U$, $\lambda \in (-\infty, \infty)$. (In the free case $U=0$ one has $|\lambda| \leq 1$, and, using formulas 6.693(1-2) of \[22\], eq. (2.2) reduces to $p_c(\lambda)|_{U=0} = - \arcsin \lambda$ and $\epsilon_c(\lambda)|_{U=0} = \sqrt{1 - \lambda^2}$, so that $\epsilon_c(p)|_{U=0} = \cos p$.)

In order to take the scaling limit we need a more convenient form for the dispersion relations. Let $U_n \equiv (n + \frac{1}{2}) \frac{\pi}{U}$. Then, as shown in the appendix, eq. (2.1) can be expanded as

$$p_s(k) = \frac{2}{U} \sum_{n=0}^\infty \frac{1}{U_n} K_0(U_n) \sinh(U_n \sin k)$$

$$\epsilon_s(k) = \frac{2}{U} \sum_{n=0}^\infty \frac{1}{U_n} K_1(U_n) \cosh(U_n \sin k)$$

(2.3)

for $k \in (-\frac{\pi}{2}, \frac{\pi}{2})$, while for $|\lambda| \geq 1$ the following expansion of eq. (2.2) is valid:

$$p_c(\lambda) = \text{sgn} \lambda \left( -\frac{\pi}{2} + \frac{\pi}{U} \sum_{n=0}^\infty \frac{(-1)^n}{U_n} I_0(U_n) e^{-U_n |\lambda|} \right)$$

$$\epsilon_c(\lambda) = \frac{\pi}{U} \sum_{n=0}^\infty \frac{(-1)^n}{U_n} I_1(U_n) e^{-U_n |\lambda|}$$

(2.4)

Here $K_\nu(z)$ and $I_\nu(z)$ are the modified Bessel functions.

\[4\] Our conventions differ from those in \[1\] by an overall factor of $\frac{1}{2}$ in the hamiltonian \[1\] and a change in the sign of $U$. 
From eq. (2.3) we read off the spinon mass gap (cf. [3][23])

$$\Delta(U) = \epsilon_s(0) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{K_1((2n+1) \frac{\pi}{2U})}{2n+1},$$

(2.5)

which vanishes for small $U$ according to [24]

$$\Delta(U) \sim \frac{4\sqrt{U}}{\pi} e^{-\frac{\pi}{2U}} \text{ as } U \to 0^+.$$  

(2.6)

(For large $U$, on the other hand, the leading behavior is $\Delta(U \gg 1) \sim \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{U^{-1}}{2n+1} = \frac{U}{4}$; in between $\Delta(U)$ increases monotonically for all $U > 0$.) The vanishing of the mass gap as $U \to 0$ is a necessary condition for the existence in this limit of a nontrivial field theory where the spinon survives as a massive particle of finite mass $M$. Introducing a dimensionful lattice spacing $a$ (so that $H$ of (1.1) is replaced by $H/a$), we define the corresponding scaling limit by

$$a, U \to 0^+ \text{ with } M \equiv \lim_{a, U \to 0} \frac{\Delta(U)}{a} = \lim_{a, U \to 0} \frac{4\sqrt{U}}{\pi a} e^{-\frac{\pi}{2U}} \text{ fixed.}$$

(2.7)

To obtain a whole massive dispersion curve in the scaling limit, the lattice rapidity variable $k$ has to be rescaled according to

$$k \to 0 \text{ such that } \theta = \frac{\pi \sin k}{2U} \text{ is finite.}$$

(2.8)

As a result, it follows that only the $n=0$ term in (2.3) survives in the limit (2.7), leading to the scaled momentum and energy

$$P_s = \lim_{a, U \to 0} \frac{p_s(k)}{a} = M \sinh \theta, \quad E_s = \lim_{a, U \to 0} \frac{\epsilon_s(k)}{a} = M \cosh \theta.$$  

(2.9)

This is a relativistic massive dispersion relation $E_s(P) = \sqrt{M^2 + P^2}$ in 1+1 dimensions, parameterized in terms of the customary rapidity variable $\theta$ in the continuum. Note that the overall normalization of the hamiltonian (1.1) is such that the “speed of light” is 1.

Turning to the holon dispersion relation, we see from (2.4) that the energy $\epsilon_c(\lambda)$ vanishes as $\lambda \to \pm \infty$ (for fixed $U$), the leading behavior being $\epsilon_c(|\lambda| \gg 1) \sim 2I_1(\frac{\pi}{2U})e^{-\pi|\lambda|/2U}$. The momentum $p_c(\lambda)$ approaches $\mp \frac{\pi}{2}$ in this limit, like $p_c(|\lambda| \gg 1) \sim \text{sgn}(\lambda)(-\frac{\pi}{2} + 2I_0(\frac{\pi}{2U})e^{-\pi|\lambda|/2U})$. This implies that the dispersion curve linearizes around these points:

$$\epsilon_c(p \sim \mp \frac{\pi}{2}) \sim v_c(U)(\frac{\pi}{2} - |p|), \quad v_c(U) = \frac{I_1(\frac{\pi}{2U})}{I_0(\frac{\pi}{2U})}.$$  

(2.10)
The speed of the charge wave \( v_c(U) \) was first given in [24] (cf. also [25]). We note that \( v_c(U \to 0^+) = 1 \), already suggesting that in the scaling limit the holons become massless particles, traveling with the same “speed of light” as obtained from the massive dispersion relation of the spinons.

To obtain the full massless dispersion relation in the scaling limit (2.7), we first note that as \( U \to 0 \) the holon dispersion relation (2.4) becomes linear for all \( |\lambda| \geq 1 \), and not just as \( |\lambda| \to \infty \). In fact, we may define the rescaled rapidity variable \( \beta \) by letting

\[
\lambda \to \pm 2 \quad \text{such that} \quad \beta = \pm \frac{\pi(2 - |\lambda|)}{2U} \quad \text{is finite},
\]

when the limit (2.7) is taken. Here and below the upper and lower sign choices apply to right- and left-moving excitations, respectively, which have to be treated separately. Now using the asymptotics of the functions \( I_\nu(z) \), one finds from (2.4) and (2.11) that the scaled dispersion relation takes the form

\[
P_c = \lim \frac{1}{a} \left( p_c(\lambda) \pm \frac{\pi}{2} \right) = \pm \frac{M}{2} e^{\pm \beta}, \quad E_c = \lim \frac{\epsilon_c(\lambda)}{a} = \frac{M}{2} e^{\pm \beta}, \quad (2.12)
\]

in the scaling limit (2.7).

Eq. (2.12) provides the standard parameterization [26-28] of the dispersion relation \( E_c(P) = |P| \) of a massless particle in a (1+1)-dimensional quantum field theory, with \( \beta \in (-\infty, \infty) \) (for both right- and left-movers) and \( M \) being some mass scale. This mass scale is arbitrary and irrelevant if the theory is conformal, since no observables depend on it and a change in it can be absorbed by a redefinition of \( \beta \). We find it satisfying that in our case, where the theory does have a massive sector, this mass scale turns out to be exactly equal to the mass \( M \) of the massive excitation (2.9), when one uses the most natural definition (2.11) of \( \beta \). (Of course one can define \( \beta \) as in (2.11) but with \( \lambda \to \pm(2 + \text{const} \cdot U) \), which would have shifted \( \beta \) and hence rescaled \( M \) of (2.12) by a finite factor; the choice \( \text{const}=0 \) is what we call the most natural one.)

3. The S-matrix in the scaling limit

In the previous section we found that the spectrum of the model in the scaling limit (2.7) consists of two doublets (labeled by \( s \) and \( c \)) of particles, one massive and the other massless. We can now use the known S-matrix computed in [4] for the spin chain to obtain...
the $S$-matrix of the continuum theory, simply by reexpressing the amplitudes of \cite{1} in terms of the rescaled variables (2.8), (2.11).

Due to the $SU(2)_s \times SU(2)_c$ symmetry of the model, the two-particle $S$-matrix is block-diagonal with four $4 \times 4$ blocks $S_{xy}$ ($x, y \in \{s, c\}$), corresponding to the scattering sectors $s-s$, $s-c$, $c-s$, and $c-c$. (In each block the rows and columns correspond to incoming and outgoing particles; they are labeled by a pair of $SU(2)$ quantum numbers, namely spin up ‘$+$’ or down ‘$-$’.) In the spin chain, the amplitudes in each sector are functions of a single variable $\mu$ which is defined as

$$\mu = \frac{\left| \sin k_1 - \sin k_2 \right|}{2U} \quad \text{in } s-s$$

$$= \frac{\left| \sin k - \lambda \right|}{2U} \quad \text{in } s-c \quad \text{or} \quad c-s$$

$$= \frac{\left| \lambda_1 - \lambda_2 \right|}{2U} \quad \text{in } c-c,$$

where the index $j=1,2$ refers to the two scattering particles. In terms of the rescaled rapidities (2.8) and (2.11), where $U \to 0^+$, $k_j \to 0$, and $\lambda_j \to \pm \frac{\pi}{2}$, this becomes

$$\mu \to \frac{|\theta_1 - \theta_2|}{\pi} \quad \text{in } s-s$$

$$\to \infty \quad \text{in } s-c$$

$$\to \begin{cases} -\frac{\left| \beta_1 - \beta_2 \right|}{\infty} & \text{for } R-R \text{ or } L-L \\ \infty & \text{for } R-L \text{ or } L-R \end{cases} \quad \text{in } c-c,$$

where $R$ and $L$ indicate right- and left-movers, respectively, in the massless sector.

Using the results of \cite{1} we now find the following $S$-matrix amplitudes in the scaling limit:

$$S_{ss}(\theta_1, \theta_2) = S_0(\theta) \left( \frac{\theta}{\theta - i\pi I - \frac{i\pi}{\theta - i\pi} \Pi} \right),$$

$$S_{sc}(\theta, \beta) = \lim_{\mu \to \infty} \left( -i \frac{1 + i e^{\pi \mu}}{1 - i e^{\pi \mu}} I \right) = iI,$$

$$S^{(RR)}_{cc}(\beta_1, \beta_2) = S^{(LL)}_{cc}(\beta_1, \beta_2) = -S_{ss}(\beta_1, \beta_2)$$

$$S^{(RL)}_{cc}(\beta_1, \beta_2) = \lim_{\mu \to \infty} S_0(-\mu)I = iI,$$

where $\theta = \theta_1 - \theta_2$, $I$ and $\Pi$ are the identity and permutation matrices (i.e. $I_{ab}^a' b' = \delta_a^a' \delta_b^b'$ and $\Pi_{ab}^a' b' = \delta_a^b' \delta_b^a$ with $a, b, a', b' \in \{+, -\}$), and

$$S_0(\theta) = \frac{\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right) \Gamma\left(1 + i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} + i\frac{\theta}{2\pi}\right) \Gamma\left(1 - i\frac{\theta}{2\pi}\right)} = \exp \left\{ i \int_0^\infty \frac{d\omega}{\omega} J_0(0) \sin(\omega \frac{2 \theta}{\pi}) e^{-\omega u} \right\}.$$
(Of course $J_0(0) = 1$ on the most rhs of (3.6); it is inserted there, as well as the arbitrary variable $u > 0$ which can simply be rescaled away, in order to exhibit an amusing relation between the phase shift associated with $S_0(\theta)$ and $p_s(k)$ of (2.1) at $k$ such that $\frac{\pi \sin k}{2u} = \theta$, cf. (2.8).)

4. Discussion

Eq. (3.3) is identified as the $S$-matrix of the massive sector of the $SU(2)$-Thirring model [9-10,12-14]. It is equal to the limit $g \to \left(-\frac{\pi}{2}\right)^+$ of the $S$-matrix of the ordinary massive Thirring model [29][30] (which, up to a sign [17], is also that of the sine-Gordon model in the limit $\beta \to \sqrt{8\pi}$). Eq. (3.5), on the other hand, has been recently proposed [27] (cf. also [31]) to describe the massless scattering theory associated with the level one $SU(2)$ WZW conformal field theory. Finally, the fact that $S_{sc}(\theta,\beta)$ turned out to be rapidity-independent indicates that the massive and massless sectors essentially decouple in the scaling limit. These observations are all in concert with the identification of the full scaled model as the $SU(2)$ chiral-invariant Thirring field theory.

Furthermore, as mentioned in the introduction, this field theory is not quite a true tensor product of two sub-theories, due to nontrivial “gluing” of sectors in their spectra. In the spirit of [17], it appears that the phase factor $i$ in (3.4) signals this effect at the $S$-matrix level. It would be interesting to investigate what consequences this factor may have on the correlation functions of the theory.\footnote{Recall the work of [32], where the spin correlation functions in the Ising field theory [33] were reconstructed using the form factor bootstrap program [14]. In this model, the nontrivial sign of the $S$-matrix $S(\theta) = -1$ leads to correlators which are expressed in terms of solutions to Painlev\`e equations [33], rather than simple Bessel functions which arise in fermion correlators in the trivial theory of a free massive Majorana fermion whose $S$-matrix is simply $S(\theta) = 1$. Cf. also [34].} However, for doing that a better understanding of scattering theories involving massless particles is needed, as well as an extension of the form factor bootstrap program to their framework.

Acknowledgements. I would like to thank F. Eßler for discussions. This work was supported in part by the US-Israel Binational Science Foundation.
Appendix A. Analysis of the dispersion relations

We start with the massless case which is relatively simple. To derive eq. (2.4) from (2.2) we first expand the Bessel functions \( J_\nu(\omega) \) in powers of \( \omega \) and integrate the resulting series term by term, using formulas 4.111(3,4,7) of [22]. This gives

\[
p_c(\lambda) = -\text{sgn } \lambda \left\{ -\frac{\pi}{2} + 2 \arctan e^x \right. \\
+ \sum_{k=1}^{\infty} \frac{1}{2^{2k}(k!)^2} \left( \frac{\pi}{2U} \right)^{2k} \left( \frac{d}{dx} \right)^{2k-1} \frac{1}{\cosh x} \right|_{x=\pi|\lambda|/2U} \quad (A.1)
\]

\[
\epsilon_c(\lambda) = \sum_{k=0}^{\infty} \frac{1}{2^{2k+1}k!(k+1)!} \left( \frac{\pi}{2U} \right)^{2k+1} \left( \frac{d}{dx} \right)^{2k} \frac{1}{\cosh x} \right|_{x=\pi|\lambda|/2U} 
\]

(Recall that throughout the paper \( U > 0 \).) Now for \( x > 0 \) expand \( 1/\cosh x = 2 \sum_{n=0}^{\infty} (-1)^n e^{-(2n+1)x} \) and \( 2 \arctan e^x = \pi - 2 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} e^{-(2n+1)x} \). Interchanging summations over \( n \) and \( k \) and using the power series expansion of \( I_\nu(z) \), we obtain eq. (2.4).

Since \( I_\nu(z) \sim (2\pi z)^{-1/2}e^z \) for large \( |z| \), we see that the expansions (2.4) absolutely converge for \( |\lambda| > 1 \), and in fact they are convergent also for \( |\lambda|=1 \). We note that a complementary small-(\( \lambda/U \)) expansion can be obtained from (A.1) by expanding

\[
1/\cosh x = \sum_{n=0}^{\infty} \frac{E_n}{n!} x^n, \quad \text{where } E_n \text{ are Euler's numbers. This way we arrive at}
\]

\[
p_c(\lambda) = -\sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\pi \lambda}{2U} \right)^n \sum_{k=0}^{\infty} \frac{E_{n+2k-1}}{2^{2k}(k!)^2} \left( \frac{\pi}{2U} \right)^{2k} \\
\epsilon_c(\lambda) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\pi \lambda}{2U} \right)^n \sum_{k=0}^{\infty} \frac{E_{n+2k}}{2^{2k+1}k!(k+1)!} \left( \frac{\pi}{2U} \right)^{2k+1} . \quad (A.2)
\]

This expansion converges for \( |\lambda| \leq U \).

In preparation to the analysis of the massive dispersion relation (2.1) we need several definitions and lemmas.

**Definition**: For \( m = 0, 1, 2, \ldots \) and integer \( \ell \), define the constants \( A^{(m)}_\ell \) and \( B^{(m)}_\ell \) as the coefficients appearing in the expansions

\[
\left( \frac{d}{d\alpha} \right)^{2m} \frac{1}{\sqrt{\alpha^2 + 1}} = \sum_{\ell \in \mathbb{Z}} (-1)^{m+\ell} (4m - 1 - 2\ell)!! \ A^{(m)}_\ell (\alpha^2 + 1)^{-(2m+\frac{3}{2}-\ell)} \\
\left( \frac{d}{d\alpha} \right)^{2(m+1)} \left( \sqrt{\alpha^2 + 1} - \alpha \right) = \sum_{\ell \in \mathbb{Z}} (-1)^{m+\ell} (4m + 1 - 2\ell)!! \ B^{(m)}_\ell (\alpha^2 + 1)^{-(2m+\frac{3}{2}-\ell)} , \quad (A.3)
\]
where, as usual, the double-factorial stands for \( n!! = 1 \cdot 3 \cdot \ldots \cdot n \) when \( n \) is a positive odd integer and \((-1)!! = 1\).

The coefficients \( A^{(m)}_{\ell}, B^{(m)}_{\ell} \) satisfy the following recursion relations (for \( m = 1, 2, \ldots \)) and initial conditions:

\[
A^{(m)}_{\ell} = A^{(m-1)}_{\ell} + (4m - 2\ell)A^{(m-1)}_{\ell-1} \quad \text{ , } \quad B^{(m)}_{\ell} = B^{(m-1)}_{\ell} + (4m + 2 - 2\ell)B^{(m-1)}_{\ell-1}
\]

\[
A^{(0)}_{\ell} = B^{(0)}_{\ell} = \delta_{\ell, 0}
\]

(\text{A.4})

from which it is easy to see that \( A^{(m)}_{\ell} = B^{(m)}_{\ell} = 0 \) for \( \ell \not\in \{0, 1, \ldots m\} \), so that the sums in (\text{A.3}) are in fact finite.

\textbf{Lemma 1:} For \( n = 0, 1, 2, \ldots \)

\[
\sum_{\ell=0}^{m} (-1)^{\ell}(4m - 1 - 2\ell)!! A^{(m)}_{\ell} \frac{2^{n}(2m + n + \frac{1}{2} - \ell)}{\Gamma(2m + \frac{1}{2} - \ell)} = \frac{[(2m + 2n - 1)!!]^2}{(2n-1)!!}
\]

\[
\sum_{\ell=0}^{m} (-1)^{\ell}(4m - 1 - 2\ell)!! B^{(m)}_{\ell} \frac{2^{n}(2m + n + \frac{3}{2} - \ell)}{\Gamma(2m + \frac{3}{2} - \ell)} = \frac{(2m + 2n - 1)!!(2m + 2n + 1)!!}{(2n-1)!!}
\]

(\text{A.5})

\textbf{Proof:} Equate powers of \( \alpha^2 \) on both sides of (\text{A.3}) after expanding them using the binomial expansion \( (\alpha^2 + 1)^s = \sum_{n=0}^{\infty} \binom{s}{n} \alpha^{2n}, \) for \( |\alpha| < 1 \).

\textbf{Corollary:} For \( k \in (-\frac{\pi}{2}, \frac{\pi}{2}) \)

\[
k = \sum_{m=0}^{\infty} \frac{\sin^{2m+1} k}{(2m+1)!} \sum_{\ell=0}^{m} (-1)^{\ell}(4m - 1 - 2\ell)!! A^{(m)}_{\ell}
\]

\[
\cos k = 1 - \sum_{m=1}^{\infty} \frac{\sin^{2m} k}{(2m)!} \sum_{\ell=0}^{m-1} (-1)^{\ell}(4m - 3 - 2\ell)!! B^{(m-1)}_{\ell}
\]

(\text{A.6})

\textbf{Proof:} For the first line use \( k = \arcsin(\sin k) = \sum_{m=0}^{\infty} \frac{[(2m-2)!!]^2}{(2m+1)!} \sin^{2m+1} k \) and then replace the numerator of the coefficient here by the lhs of the first line of (\text{A.5}), with \( n=0 \).

For the second line expand \( \cos k = \sqrt{1 - \sin^2 k} = 1 - \sum_{m=1}^{\infty} \frac{(2m-3)!!(2m-1)!!}{(2m)!} \sin^{2m} k \) and then use the second line of (\text{A.3}) with \( n=0 \) and \( m \) replaced by \( (m - 1) \).

\textbf{Lemma 2:} For \( m = 0, 1, 2, \ldots \), let

\[
S_m(z) = \sum_{\ell=0}^{m} (-1)^{\ell} A^{(m)}_{\ell} z^{2m-\ell} K_{2m-\ell}(z) \quad \text{ , } \quad T_m(z) = \sum_{\ell=0}^{m} (-1)^{\ell} B^{(m)}_{\ell} z^{2m+1-\ell} K_{2m+1-\ell}(z)
\]

(\text{A.7)}
where the $K_\nu(z)$ are modified Bessel functions. Then

$$ S_m(z) = z^{2m}K_0(z) \quad , \quad T_m(z) = z^{2m+1}K_1(z) \quad . \quad (A.8) $$

**Proof:** Using (A.4) and the recursion relation 8.486(10) of [22] for the $K_\nu(z)$, one obtains $S_{m+1}(z) = z^2S_m(z)$ and $T_{m+1}(z) = z^2T_m(z)$ with $S_0(z) = K_0(z)$ and $T_0(z) = zK_1(z)$, from which (A.8) immediately follows.

Equipped with the above results, consider (2.1). Expanding the $\sin(\omega \sin k)$ and $\cos(\omega \sin k)$ in power series in $\omega \sin k$ and integrating term by term using 6.621(4) of [22] we find

$$ p_s(k) = k - 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \sin^{2m+1}k \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{d}{d\alpha} \right)^{2m} \frac{1}{\sqrt{\alpha^2 + 1}} \bigg|_{\alpha=2nU} $$

$$ \epsilon_s(k) = U - \cos k + 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \sin^{2m}k \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{d}{d\alpha} \right)^{2m} \left( \sqrt{\alpha^2 + 1} - \alpha \right) \bigg|_{\alpha=2nU} . \quad (A.9) $$

Now using (A.3) and (A.6) yields, after reordering the summations over $\ell$ and $n$,

$$ p_s(k) = \sum_{m=0}^{\infty} \frac{\sin^{2m+1}k}{(2m+1)!} \sum_{\ell=0}^{m} (-1)^{\ell}(4m - 1 - 2\ell)!! \ A^{(m)}_{\ell} \times \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^{n} (\alpha^2 + 1 - \alpha) \bigg|_{\alpha=2nU} \right) $$

$$ \epsilon_s(k) = U - 1 + 2 \sum_{n=1}^{\infty} (-1)^{n-1} (\sqrt{\alpha^2 + 1} - \alpha) \bigg|_{\alpha=2nU} $$

$$ + \sum_{m=1}^{\infty} \frac{\sin^{2m}k}{(2m)!} \sum_{\ell=0}^{m-1} (-1)^{\ell}(4m - 3 - 2\ell)!! \ B^{(m-1)}_{\ell} \times \left( 1 + 2 \sum_{n=1}^{\infty} (-1)^{n} (\alpha^2 + 1 - \alpha) \bigg|_{\alpha=2nU} \right) . \quad (A.10) $$

The sums over $n$ encountered in the last step have been analyzed in detail by Fisher and Barber in [23], where they are referred to as “remnant functions” (they are related to the so-called Epstein-Hurwitz zeta function). Using the notation introduced in eq. (2.25)
of [23], we have
\[
\sum_{n=1}^{\infty} (-1)^n (\alpha^2 + 1)^{\sigma-1} \bigg|_{\alpha=n/y} = y^{2(1-\sigma)} \frac{\Gamma(\sigma)}{\Gamma(-1/2)} R_{\sigma,0}^{-}(y^2) \quad (\sigma = -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots)
\]
\[
\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{\alpha^2 + 1}} \bigg|_{\alpha=n/y} = -\frac{y}{2} \left[ R_{1/2,0}^{-}(y^2) + 2 \ln 2 \right]
\]
\[
\sum_{n=1}^{\infty} (-1)^n \left( \sqrt{\alpha^2 + 1} - \alpha \right) \bigg|_{\alpha=n/y} = \frac{1}{4y} \left[ R_{3/2,0}^{-}(y^2) + 2y^2 \ln 2 \right] .
\]

(A.11)

The crucial step now is to use eqs. (2.26), (6.20), and (6.21) of [23], which leads to

\[
p_s(k) = \sum_{m=0}^{\infty} \frac{\sin^{2m+1} k}{(2m+1)!} \sum_{\ell=0}^{m} (-1)^\ell (4m - 1 - 2\ell)!! A^{(m)}_{\ell} \times \frac{2U^{-1}}{(4m - 1 - 2\ell)!!} \sum_{n=0}^{\infty} U_n^{2m-\ell} K_{2m-\ell}(U_n)
\]

\[
= \frac{2}{U} \sum_{m=0}^{\infty} \frac{\sin^{2m+1} k}{(2m+1)!} \sum_{n=0}^{\infty} S_m(U_n) , \quad (A.12)
\]

\[
\epsilon_s(k) = \frac{2}{U} \sum_{n=0}^{\infty} \frac{K_1(U_n)}{U_n} + \sum_{m=1}^{\infty} \frac{\sin^{2m} k}{(2m)!} \sum_{\ell=0}^{m-1} (-1)^\ell (4m - 3 - 2\ell)!! B^{(m-1)}_{\ell} \times \frac{2U^{-1}}{(4m - 3 - 2\ell)!!} \sum_{n=0}^{\infty} U_n^{2m-1-\ell} K_{2m-1-\ell}(U_n)
\]

\[
= \frac{2}{U} \sum_{n=0}^{\infty} \frac{K_1(U_n)}{U_n} + \frac{2}{U} \sum_{m=1}^{\infty} \frac{\sin^{2m} k}{(2m)!} \sum_{n=0}^{\infty} T_{m-1}(U_n) ,
\]

where \( U_n \equiv (n + \frac{1}{2}) \pi \), and in the second lines in each formula we employ the definitions (A.7). Now invoking (A.8) and reordering the summations over \( n \) and \( m \), we finally obtain (2.3).
References

[1] M.C. Gutzwiller, Phys. Rev. Lett. 10 (1963) 159; J. Hubbard, Proc. Roy. Soc. (London), Ser. A, 276 (1963) 238.
[2] P.W. Anderson, Science 235 (1987) 1196.
[3] E.H. Lieb and F.Y. Wu, Phys. Rev. Lett. 20 (1968) 1445.
[4] F.H.L. Essler and V.E. Korepin, Phys. Rev. Lett. 72 (1994) 908, and Stony Brook preprint ITP-SB-93-45, cond-mat/9310056, to appear in Nucl. Phys. B.
[5] P.K. Mitter and P.H. Weisz, Phys. Rev. D8 (1973) 4410.
[6] D. Gross and A. Neveu, Phys. Rev. D10 (1974) 3235.
[7] R. Dashen and Y. Frishman, Phys. Rev. D11 (1975) 2781.
[8] T. Banks, D. Horn and H. Neuberger, Nucl. Phys. B108 (1976) 119.
[9] B. Berg, M. Karowski, P. Weisz and V. Kurak, Nucl. Phys. B134 (1978) 125.
[10] B. Berg and P. Weisz, Nucl. Phys. B146 (1978) 205.
[11] A.A. Belavin, Phys. Lett. B87 (1979) 117.
[12] N. Andrei and J.H. Lowenstein, Phys. Rev. Lett. 43 (1979) 1698, and Phys. Lett. B91 (1980) 401.
[13] C. Destri and J.H. Lowenstein, Nucl. Phys. B205 (1982) 369.
[14] F.A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory (World Scientific, Singapore, 1992)
[15] I. Affleck, talk given at the Nato Advanced Study Institute on Physics, Geometry and Topology, Banff, August 1989.
[16] S. Lukyanov, Rutgers preprint RU-93-30, hep-th/9307196.
[17] T.R. Klassen and E. Melzer, Int. J. Mod. Phys. A8 (1993) 4131.
[18] T.R. Klassen and E. Melzer, Nucl. Phys. B382 (1992) 441.
[19] B.M. McCoy and T.T. Wu, Phys. Lett. B87 (1979) 50.
[20] N. Andrei, summer course on Low-dimensional Quantum Field Theories for Condensed Matter Physicists, Trieste 1992, unpublished.
[21] F. Woynarovich, J. Phys. C16 (1983) 5293 and 6593.
[22] I.S. Gradshteyn and I.M. Rizhik, Table of Integrals, Series, and Products (Academic Press, Orlando, 1980).
[23] M.E. Fisher and M.N. Barber, Arch. Rational Mech. Annals 47 (1972) 205.
[24] A.A. Ovchinnikov, Sov. Phys. JETP 30 (1970) 1160.
[25] M. Takahashi, Progr. Theor. Phys. 43 (1970) 1619.
[26] Al.B. Zamolodchikov, Nucl. Phys. B358 (1991) 619.
[27] A.B. Zamolodchikov and Al.B. Zamolodchikov, Nucl. Phys. B379 (1992) 602.
[28] P. Fendley and H. Saleur, preprint USC-93-022, hep-th/9310058.
[29] A.B. Zamolodchikov and Al.B. Zamolodchikov, Ann. Phys. 120 (1979) 253.
[30] P.H. Weisz, Nucl. Phys. B122 (1977) 1.
[31] L.D. Faddeev and L. Takhtajan, Phys. Lett. A85 (1981) 375, and J. Sov. Math. 24 (1984) 241.
[32] B. Berg, M. Karowski and P. Weisz, Phys. Rev. D19 (1979) 2477.
[33] T.T. Wu, B.M. McCoy, C.A. Tracy and E. Barouch, Phys. Rev. B13 (1976) 316; B.M. McCoy, C.A. Tracy and T.T. Wu, Phys. Rev. Lett. 38 (1977) 793.
[34] M.Yu. Lashkevich, preprint LANDAU-94-TMP-4, hep-th/9406118.