Analysis of the Wilsonian Effective Potentials in Dynamical Chiral Symmetry Breaking

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Abstract

The non-perturbative renormalization group equation for the Wilsonian effective potential is given in a certain simple approximation scheme in order to study chiral symmetry breaking phenomena dynamically induced by strong gauge interactions. The evolving effective potential is found to be non-analytic in infrared, which indicates spontaneous generation of the fermion mass. It is also shown that the renormalization group equation gives the identical effective fermion mass with that obtained by solving the Schwinger-Dyson equation in the (improved) ladder approximation. Moreover introduction of the collective field corresponding to the fermion composite into the theory space is found to offer an efficient method to evaluate the order parameters; the dynamical mass and the chiral condensate. The relation between the renormalization group equation incorporating the collective field and the Schwinger-Dyson equation is also clarified.
1. Introduction

It has been a very important subject to understand the non-perturbative dynamics of the quantum field theories, especially the strongly coupled gauge theories like QCD. One of the most outstanding features of the non-perturbative dynamics of QCD is spontaneous breakdown of the chiral symmetry. So far, the approach by means of the Schwinger-Dyson equations (SDE) has been extensively studied [1, 2, 3] and been applied not only to the strongly coupled QED and QCD [4, 5], but also to the models beyond the standard model [6, 7]. In this approach the so-called (improved) ladder approximation [4] has been commonly applied to solve the SDEs. It should be noted that the SDEs offer us surprisingly good results on the chiral symmetry breaking phenomena in QCD in spite of the fact that the approximation scheme is fairly crude [5]. However, it has been also known that this SD approach suffers from some problems: difficulties in improvement of the approximation [8], large gauge dependence in the ladder approximation [9], and so on. Therefore it would be desirable to develop other non-perturbative methods so as to overcome such difficulties, for example, in order to perform gauge independent study of the chiral order parameters in QCD dynamics.

In the previous papers [10, 11], we proposed the Wilson renormalization group equation (RGE) [12] for the system of a massless fermion coupled by gauge interaction, with which we can study the critical behavior of the chiral symmetry in a remarkably simple manner. In this RG approach it is enough to calculate the beta functions for the multi-fermi couplings in order to evaluate the critical couplings, phase boundary and the anomalous dimensions of the fermion composites near the critical points. Specially the RG flows of the four-fermi coupling play an essential role to show this critical behavior. The procedure is quite straightforward in contrast to the calculations using the SDEs. It should be noted that the set of beta functions obtained in a certain approximation framework turns out to be free from gauge dependence [10]. Moreover, if we approximate these RGEs further by restricting the radiative corrections to a certain limited type, which we called the “ladder-like” corrections in [11], then it gives the same critical dynamics as obtained by solving the ladder SDE. Therefore the above mentioned gauge independent analysis indeed offer us an improvement of the ladder SDEs. Thus, it may be said that this Wilson RG approach is more powerful than the SD approach as long as the critical dynamics is concerned.

However the beta functions for the multi-fermi couplings do not tell us the order parameters in spontaneous chiral symmetry breaking. Therefore it is not known immediately in which phase the chiral symmetry is broken, even after the critical behavior is clarified by the RGEs. Actually the chiral symmetry breaking itself might look somehow puzzling in the framework of the Wilson RG. Because the Wilsonian effective action keeps chiral invariant manifestly, and therefore the fermion mass seems to have no chance to appear in it. On the other hand, the order parameters such as the dynamical mass of quarks, the quark condensation, the $\pi$-on decay constant, etc. are the relevant physical quantities to be evaluated in QCD-like gauge theories, which are of our main concern. Thus it is an interesting and important problem to study not only the critical behavior but also the chiral order parameters in this RG framework.

In this paper we discuss how the order parameters may be evaluated in the Wilson RG framework. For the sake of simplicity we adopt the simple approximation to take
only the “ladder like” corrections into account and propose a non-perturbative RGE in terms of the Wilsonian effective potential. Then the dynamical fermion mass is found to appear through non-analyticity of the Wilsonian effective potential in infrared instead of an explicit mass term. Also it will be shown that the RGE gives the same effective mass as the ladder SDE does in the entire region of the coupling space. Through this analysis the structure of the ladder SDE is also clarified in the point of view of the non-perturbative RG. Thus any improvement of this approximation in the RG framework exceeds the analysis done by the ladder SDE so far. However the purpose of this paper is to discuss the method to evaluate the order parameters in the Wilson RG framework and to make clear the relation to the SDE, not to improve results for the order parameters obtained so far. Study of the order parameters beyond the level of the SD approach, specially the gauge independent analysis, is currently under way.

This paper is organized as follows. Section 2 is devoted to a brief review of the SDE in the so-called improved ladder approximation, which will be discussed in relation to the RGE in the later sections. In section 3 the non-perturbative RGE for the Wilsonian effective potential is proposed. The spontaneous chiral symmetry breaking will be examined by solving the partial differential equation for the effective potential. It will be seen that in the chiral symmetry broken phase the non-analyticity comes out in the infrared effective potential. This singular behavior is found to indicate the dynamical generation of the fermion mass. The relation between the RGE and the SDE is clarified by an analytical method in section 4. Through this observation the improved ladder approximation used in the SD approach will be understood from the point of view of Wilson RG. However the non-analytic behavior of the Wilsonian effective potential makes it rather difficult to evaluate the order parameters. In section 5 we are going to consider the practical method to calculate the order parameters in this RG framework. In order to circumvent this problem we will propose a method incorporating the “collective field” corresponding to the composite operator of fermions in the theory space. This method will be found to enable us to perform the calculation very effectively compared with the former RGEs. Lastly the results of the effective mass of quarks and the chiral condensate in massless QCD will be presented using the RGEs in comparison with the results obtained in the ladder SD approach. In section 6 we summarize our conclusions and also discuss the further issues along the lines considered in this paper.

2. The ladder Schwinger-Dyson equation

First let us consider the massless QED extended so as to include the chiral invariant 4-fermi interactions; namely the gauged Namubu-Jona-Lasinio model [2, 3], which is given by the bare action

\[ S_0 = \int d^4x \bar{\psi}(i\partial - gA)\psi - \frac{1}{4}F_{\mu\nu}^2 + \frac{G}{2\Lambda_0^2} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right]. \]  

(1)

This should be regarded as the action of the effective theory at scale \( \Lambda_0 \), which is the overall cut-off of this model. The 2-point function of the fermion \( S(p) \), which is of our present interest in order to see chiral symmetry breaking, satisfies the SDE written in terms of the full photon propagator \( D_{\mu\nu}(p) \), the full vertex function \( g\Gamma_{\mu}(p,q) \) and \( S(p) \). However the SDE for the vertex cannot be given a closed form using only \( S, D_{\mu\nu} \) and
Therefore it is inevitable to approximate the SDE in any practical analyses.

In the ladder approximation, which has been mostly used, the photon propagator \( D_{\mu\nu} \) as well as the vertex function \( g \Gamma_\mu \) are simply replaced by the tree level ones;

\[
g \Gamma_\mu(p,q) = g \gamma_\mu, \\
iD_{\mu\nu}(p) = \frac{1}{p^2} \left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right),
\]

where the propagator is given in the Landau gauge. Actually the results obtained in this approximation have been known to depend on the gauge choice considerably [9]. However the SDE in the ladder approximation is supposed to be the best in the Landau gauge, since the Ward identity \( Z_1 = Z_2 \) is maintained only in this gauge. Indeed \( Z_1 = 1 \) by neglecting corrections to the vertex and also \( Z_2 = 1 \) as is seen from the form of the 2-point function in the Landau gauge,

\[
iS^{-1}(p) = \not{p} - \Sigma(p),
\]

where \( \Sigma(p) \) is the fermion self-energy i.e. the mass function.

The SDE for the mass function \( \Sigma \) is found to be given by the integral equation (in Euclidean)

\[
\Sigma(k) = \int_{|p|<\Lambda_0} \frac{d^4p}{(2\pi)^4} \left[ \frac{G}{\Lambda_0^2} + \frac{3g^2}{4(k-p)^2} \right] \text{tr} \left( \frac{1}{i\not{p} + \Sigma(p)} \right),
\]

which was first studied by Bardeen et. al. [2]. It is seen from this equation that the mass function may be evaluated as sum of bubble diagrams of the fermion loops which contain the “ladder” corrections of photon exchange. The spontaneous chiral symmetry breaking is indicated by appearance of a non-trivial solution of the eq.(4). The structure of the critical line in the space of the bare couplings \( (G,\alpha \equiv g^2/4\pi^2) \) has been clarified [3] and the symmetry broken phase was found to be given by

\[
\begin{cases}
G > \pi^2 \left(1 + \sqrt{1 - \alpha/\alpha_{cr}}\right)^2 & \text{for } \alpha < \alpha_{cr}, \\
\alpha > \alpha_{cr} & \text{for } G < \pi^2,
\end{cases}
\]

where \( \alpha_{cr} \) denotes the critical gauge coupling and is found to be \( \alpha_{cr} = \pi/3 \) [1].

The ladder approximation mentioned above, however, is so crude that the radiative corrections to the gauge coupling are totally ignored. Therefore we cannot apply the SDE (4) directly to the gauge theories such as QCD in order to obtain any physical results. Miransky and Higashijima [4] have invented independently the scheme to incorporate the correction to the gauge coupling to the SDE effectively, which is called the improved ladder approximation. This approximation is to simply replace the vertex \( g \gamma_\mu \) by

\[
g \Gamma_\mu = \bar{g} \left(\max(p^2, k^2)\right) \gamma_\mu T^a,
\]

where \( p \) and \( k \) are the Euclidean momentum carried by the fermions attached to the vertex, \( \bar{g}(p^2) \) is the running gauge coupling constant renormalized at the scale of \( p \). Also \( T^a \) denotes the color group representation of the fermion. The choice of the momentum
dependence in (6) is just for the practical reason to make analytical study easier. Thus the SDE in the improved ladder approximation is written after the angle integration as

$$\Sigma(k) = \frac{1}{2\pi^2} \int_0^{\Lambda_0} dp \, p^3 \left[ G + \left( \frac{\lambda(k)}{k^2} \theta(k - p) + \frac{\lambda(p)}{p^2} \theta(p - k) \right) \right] \frac{\Sigma(p)}{p^2 + \Sigma^2(p)}.$$  (7)

Here we introduced the running gauge coupling defined by

$$\lambda(p) = \frac{3}{4} C_2(R) g^2(p),$$  (8)

where $C_2(R)$ is the second Casimir of the color representation $R$. We may use the running gauge coupling evaluated by perturbation in the SDE, provided an infrared cutoff of running is performed to $\lambda(p)$ in order to avoid the divergent pole. Fortunately it has been found that the physical quantities are almost independent of this infrared cutoff [5].

Indeed the scheme in the improved ladder approximation makes it possible to study QCD-like gauge theories. Besides it has been known that the SDE (and also the Bethe-Salpeter equations) in this scheme leads to the order parameters for dynamical chiral symmetry breaking in QCD in a rather good accuracy [5]. However this method to use the running gauge coupling deviates from the scheme of SDE, and therefore, it cannot be regarded as a systematic improvement of the approximation. Hence we cannot help but stopping more or less in this level of analyses. However it will be seen later that this manipulation is understood as a quite natural improvement of the approximation in the framework of non-perturbative renormalization group rather than in this SD approach.

3. The Wilsonian effective potential and the RG equations

There have been known several formulations of the non-perturbative RG or the so-called exact renormalization group equations [12, 13, 14]. In this paper we discuss the dynamical chiral symmetry breaking phenomena only in the framework of the Wegner-Houghton RGE [13] for our present purpose. This RGE gives the variation of the so-called Wilsonian effective action defined by sharp cutoff under the infinitesimal shift of the cutoff scale. As is seen later this sharp cutoff scheme makes it easier to clarify the relation between the non-perturbative RGE’s and the SDE’s, though the physical consequences should not depend on the choice of cutoff scheme of course.

Before discussing the RGE for the system considered in the previous section, let us mention briefly the general formulation of the Wegner-Houghton RGE. If we devide the freedom of the quantum field $\phi(p)$ into the higher momentum modes with $|p| > \Lambda$ and the lower momentum modes with $|p| < \Lambda$ by introducing the cutoff scale $\Lambda$ in the Euclidean formalism, then the Wilsonian effective action at this scale, $S_{\text{eff}}[\phi; \Lambda]$, may be defined by integrating out the higher frequency modes in the partition function. Namely

$$Z = \int \prod_{|p| < \Lambda_0} d\phi(p) \, e^{-S_0[\phi; \Lambda_0]} = \int \prod_{|p| < \Lambda} d\phi(p) \, e^{-S_{\text{eff}}[\phi; \Lambda]},$$  (9)

where $S_0$ denotes the bare action with the bare cutoff $\Lambda_0$. This effective action contains the general operators consistent with the original symmetries of the bare action, for example the chiral symmetry of our concern. Interestingly enough the infinitesimal variation of
the effective action with respect to the cutoff \( \Lambda \) may be evaluated exactly. After taking account of the scale transformation of dimensionful variables the Wegner-Houghton RGE is found to be

\[
\frac{\partial S_{\text{eff}}}{\partial t} = d S_{\text{eff}} - \int \frac{d^d p}{(2\pi)^d} \frac{\partial}{\partial \phi_p'} \left( \frac{2 - d - \eta^i}{2} - p^\mu \frac{\partial'}{\partial p^\mu} \right) \delta S_{\text{eff}} \delta \phi_{p'}^i - \frac{1}{2} \int \frac{d^d p}{(2\pi)^d} \delta(\vert p \vert - 1) \left\{ \delta S_{\text{eff}} \left\{ \frac{\delta^2 S_{\text{eff}}}{\delta \phi_p^i \delta \phi_{-p}^j} \right\}^{-1} \delta S_{\text{eff}} - \text{str} \ln \left( \frac{\delta^2 S_{\text{eff}}}{\delta \phi_p^i \delta \phi_{-p}^j} \right) \right\},
\]

where \( t = \ln(\Lambda_0/\Lambda) \) is introduced as the scale parameter. The 1st line of the RGE represents nothing but the scaling of the effective action. \( \eta^i/2 \) denotes the anomalous dimension of the field \( \phi^i \). While the 2nd line comes from the radiative corrections which correspond to the tree and the one loop Feynman diagrams including only the propagators with the momentum of the scale \( \Lambda \). However this RGE is a rather formal object, though it has been derived exactly. It is inevitable to simplify this RGE by applying some approximations in practical analyses.

Now we shall consider the non-perturbative RGE for the system given by the bare action (1) in the approximation scheme, which will be found to correspond to the ladder approximation applied to the SDE. In the process to derive the RGE, the scheme of this approximation will be defined. First we adopt the so-called local potential approximation (LPA) \([13, 15]\), in which the radiative corrections to any operators containing derivatives are ignored. Therefore solely the potential part of \( S_{\text{eff}} \) may be evolved with respect to the cutoff scale. It should be noted that the wave function renormalizations, therefore the anomalous dimensions also, are ignored in this scheme. First let us also ignore the corrections to the operators including the gauge field. In the Wilson RG framework the gauge invariance is necessarily lost by introducing cutoff, though the chiral invariance is maintained. This problem makes it rather complicated to deal with gauge theories generally. Recently the formulations of the Wilson RG utilizing the Slavnov-Taylor identities have been developed and been examined in the application to the Yang-Milles theories \([16]\). However in the present approximation we avoid this problem just by ignoring such corrections. One the other hand we do not get the radiative corrections to the gauge interaction at the price of this approximation, which is in the same level as of the ladder SDE. Treatment of the running gauge coupling will be discussed later.

As the result of the approximation discussed so far, the form of the Wilsonian effective action at scale \( \Lambda \) is restricted to

\[
S_{\text{eff}}[\psi, \bar{\psi}, A_\mu; \Lambda] = \int d^4 x \, \bar{\psi}(\partial + gA)\psi + V(\psi, \bar{\psi}; \Lambda) + \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2,
\]

where \( V \) is the general potential invariant under the chiral symmetry. The last term in (11) is the gauge fixing term and we also adopt the Landau gauge hereafter. The potential \( V \) may be written down as a polynomial composed of the following parity and chiral invariant operators, which are mutually independent;

\[
O_1 = (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma_5 \psi)^2 = -\frac{1}{2} \left\{ (\bar{\psi} \gamma_\mu \psi)^2 - (\bar{\psi} \gamma_5 \gamma_\mu \psi)^2 \right\},
\]
\[
O_2 = \left( \bar{\psi} \gamma_\mu \psi \right)^2 + \left( \bar{\psi} \gamma_5 \gamma_\mu \psi \right)^2,
\]
\[
O_3 = \left\{ \left( \bar{\psi} \gamma_\mu \psi \right) \left( \bar{\psi} \gamma_5 \gamma_\mu \psi \right) \right\}^2.
\] (12)

Now the infinitesimal variation of the effective potential \( V(\psi, \bar{\psi}; \Lambda) \) under the shift of the cutoff \( \Lambda \rightarrow \Lambda - \delta \Lambda \) is given by

\[
\delta V(\psi, \bar{\psi}; \Lambda) = \int_{\Lambda - \delta \Lambda < |p| < \Lambda} \frac{d^4p}{(2\pi)^4} \text{str} \ln M
\]
\[
= \int_{\Lambda - \delta \Lambda < |p| < \Lambda} \frac{d^4p}{(2\pi)^4} \left[ \text{tr} \ln M_{BB} - \text{tr} \ln \left( M_{FF} - M_{BB} M_{BB}^{-1} M_{BF} \right) \right], \quad (13)
\]
where \( M \) is the matrix given by

\[
M = \begin{pmatrix}
M_{BB} & M_{BF} \\
M_{FB} & M_{FF}
\end{pmatrix} = \begin{pmatrix}
\frac{D_{\mu\nu}^{-1}}{g^2} & g^2 V_{\gamma_\mu} & -g^2 \gamma_\mu \psi \\
-\gamma_\nu & \delta^2 V_{\delta_\psi \delta_\psi} & -i \gamma_\mu \psi \\
g^2 \gamma_\nu & \delta^2 V_{\delta_\psi \delta_\psi} & \delta^2 V_{\delta_\psi \delta_\psi}
\end{pmatrix}.
\] (14)

Here \( D_{\mu\nu} = M_{BB}^{-1} \) denotes the photon propagator in the Landau gauge. Note that the combination of \( M_{BB} M_{BB}^{-1} M_{BF} \) in eq.(13) represents the effective 4-fermi interactions induced by one photon exchange.

In the previous paper \[\] we examined the critical dynamics of chiral symmetry breaking in this level of approximation. However it was found that this approximation is still better than the ladder approximation in the SD approach as far as the critical dynamics is concerned. In other words this correction contains the contributions from the “non-ladder” diagrams not only from the “ladder” diagrams. Here we shall simply extract the “ladder-like” corrections only by imposing further restriction to the effective potential. We consider the effective potential \( V \) composed of only the operator \( O_1 \) given in (12), and pick up only the parts propotional to 1 or \( \gamma_5 \) in the spinor structure from the vertices appearing in \( N_{FF} \equiv M_{FF} - M_{BB} M_{BB}^{-1} M_{BF} \). After performing the fierz transformation the resultant matrix is simply reduced to be

\[
N_{FF} = \begin{pmatrix}
0 & -i \gamma_\mu \psi \\
-i \gamma_\mu \psi & \left( \sigma + i \gamma_5 \pi \right) \left( V_\rho - \frac{3g^2}{4\pi^2} \right)
\end{pmatrix},
\] (15)

where the notations \( \sigma = \bar{\psi} \gamma_5 \psi, \pi = \bar{\psi} i \gamma_5 \psi \) and \( \rho = (\sigma^2 + \pi^2)/2 \) are introduced. \[\] Thus we obtain the non-perturbative RGE in the “ladder” approximation as

\[
\frac{\partial V}{\partial \tau} = 4V - 6\rho V_\rho - \frac{1}{4\pi^2} \ln \left[ 1 + 2\rho (V_\rho - 3\pi \alpha) \right], \quad (16)
\]

\[\text{This matrix (15) in LPA is evaluated simply by setting the lower momentum modes to zero modes, however there arises a subtle problem related with their Grassmann nature. The effective potential evaluated by the exact zero modes must terminate at a certain finite order, unless we consider the large \( N_f \) limit. However such zero mode parts do not represent the bulk structure of the general Green functions of fermions. Here we suppose that the correction to the effective potential is evaluated in small momentum limit with keeping \( \delta \Lambda \) finite rather than by the exact zero modes, and that the effective potential \( V \) is given by an infinite order of polynomial in terms of \( \sigma = \bar{\psi}(-p)\psi(p) \) and \( \pi = \bar{\psi}(-p)i\gamma_5\psi(p) \). The sharp cutoff limit is taken by \( \delta \Lambda > \mid p \mid \rightarrow 0. \]
after taking account of the canonical scaling. Indeed it will be shown that analysis of this RGE results in the identical effective mass, which is defined shortly, to the solution of the ladder SDE (4).

Before discussing the spontaneous chiral symmetry breaking, let us briefly mention the critical surface derived from this RGE. If we expand the effective potential $V(\rho)$ into a polynomial,

$$V(\rho; t) = -G(t)\rho + \frac{1}{2}G_8(t)\rho^2 + \frac{1}{3!}G_{12}(t)\rho^3 + \cdots,$$  \hspace{1cm} (17)

and substitute this into the RGE, then the beta functions for the couplings are easily obtained. For the 4-fermi coupling $G$ we find

$$\frac{dG}{dt} = -2G + \frac{1}{2\pi^2} \left( G + \frac{3}{\pi} \alpha \right)^2.$$  \hspace{1cm} (18)

Here we note that the higher order couplings are not involved in the evolution of the 4-fermi coupling due to 1-loop nature of the non-perturbative RGE. This RGE (18) shows the fixed points (line) at

$$G^*(\alpha) = \pi^2 \left( 1 \pm \sqrt{1 - \alpha/\alpha_{cr}} \right)^2,$$  \hspace{1cm} (19)

where $+$ ($-$) sign is for the UV (IR) fixed points respectively. Thus the phase boundary, which is shown in Fig.1, is found to just coincide to the result of the ladder SDE given in eq.(5). The anomalous dimension of the four-fermi operator is immediately obtained from this RGE \[11\], which is also shown to agree with that derived from the SDE (4) \[3\]. However it is not obvious whether the chiral symmetry is truly broken spontaneously in the upper region of the phase boundary in Fig.1, since the order parameters of chiral symmetry are not obtained in such RG analyses.

Let us consider the effective mass of the fermion, which is the direct result obtained by solving the SDE. In the RG framework the Wilsonian effective action keeps it’s chiral invariance under evolution. Therefore the effective mass cannot appear as the mass term of this action. On the other hand the effective mass $m_{\text{eff}}$ enters into the loop correction may be read from the effective interaction given in eq.(15) as

$$m_{\text{eff}} = \lim_{\Lambda \to 0} \left. \frac{\partial V}{\partial \rho} \right|_{\sigma = \pi = 0} = \lim_{\Lambda \to 0} \left. \frac{\partial V}{\partial \sigma} \right|_{\sigma = \pi = 0}.$$  \hspace{1cm} (20)

If we are allowed to expand the effective potential into a power series as in (17), namely, if the potential is a regular function of $\rho$ at the origin, then this effective mass cannot help but vanishing. On the other hand it is found that each coupling actually keeps growing very rapidly and eventually diverges in the broken phase. This is nothing but infrared singularity caused by massless particles. Note that each beta function is given through one loop diagrams with the massless fermion even in the chirally broken phase.

The above observation would imply that the expansion by the series of operators becomes invalid in the infrared. Hence we examined to numerically solve the partial differential equations by using the mesh method. In practice all the couplings diverge at a certain finite scale. This divergence occurring at a finite scale, rather than in the infrared limit, is supposed to be caused by the naive approximation applied here.
differential equation given by eq.(16). In Fig.2 evolution of the effective potential to infrared is presented in the case of bare couplings $G = 0, \alpha = 1.5\alpha_{\text{cr}}$. Actually it is seen that the effective potential turns out to be non-analytic at the origin ($\sigma = \pi = 0$). Therefore the effective mass given by eq.(20) is not well-defined. If we introduce the small bare mass of fermion, then the dynamical mass for the massless theory may be defined by the vanishing limit and is found to get a finite value. Anyway the dynamical generation of fermion mass appears through the non-analytic behavior of the Wilsonian effective potential in infrared. Actual evaluation of the effective mass and also the chiral condensate will be discussed in section 5.

Needless to say the approximation scheme for the non-perturbative RGE considered here is too crude to apply for the analysis of QCD-like gauge theories. It is necessary to improve the approximation so as to incorporate the running effect of the gauge coupling at least. As the lowest order approximation to the RGE, the beta function for the gauge coupling may well be estimated by the perturbative one. Namely we solve the RGE given by eq.(16) coupled with the perturbative RGE for $\alpha$. This may be regarded somehow as a similar idea to the improved ladder approximation performed for the SDEs. However it is a clear advantage for the Wilson RG approach to allow such improvement within the same framework of RG in contrast to the SD approach.

Also we need to take the color group representation into consideration in the case of QCD-like gauge theories. Suppose the fermions form an $N$ dimensional representation. Then the 1-loop correction given by (13) is multiplied by the factor $N$. On the other hand we have to take care of the fierz transformation for the color indices as well as the spinor indices to derive the effective interaction as given in eq.(15). If we note the following
Fig. 2: The evolution of the Wilsonian effective potential by the RGE (16) in the broken phase ($G = 0$, $\alpha = 1.5 \alpha_{cr}$). The effective potential becomes non-analytic at the origin.

\[ \sigma \]

\[ \text{Wilsonian effective potential } V(\sigma) \]

\[ t = 1.0 \]

\[ t = 1.5 \]

\[ t > 2.5 \]

\[ V(\sigma) \]

\[ \text{effective potential} \]

\[ \text{becomes non-analytic at the origin.} \]

\[ \begin{align*}
\sum_a \bar{\psi}_1 \gamma_\mu T^a \psi_2 \bar{\psi}_3 \gamma_\nu T^a \psi_4 &= -\frac{C_2(R)}{4N} \bar{\psi}_1 \psi_4 \bar{\psi}_3 \gamma_\nu \gamma_\mu \psi_2 + \cdots, \\
\end{align*} \]

(21)

then we may obtain the RGE similarly. The factor $N$ is found to be absorbed by the redefinition, $V \to NV$ and $\rho \to N^2 \rho$. Consequently the RGE for the QCD-like gauge theories is given by

\[ \frac{\partial V}{\partial t} = 4V - 6\rho V_\rho - \frac{1}{4\pi^2} \ln \left[ 1 + 2\rho(V_\rho - 3\pi C_2(R)\alpha)^2 \right], \]

(22)

where the gauge coupling $\alpha(t)$ is now subject to the perturbative RGE. This RGE will be discussed in the comparison with the SDE in the improved ladder approximation in the next section.

4. The relation between the RGE and the SDE

In this section it is shown analytically that the RGE’s, eq.(16) and eq.(22) proposed in the previous section offer us the same effective masses as the SDE’s in the (improved) ladder approximation do. Our strategy to this end is as follows. First we give an analytical expression for the Wilsonian effective potential, which is expected to be the solution of the RGE. We define also the mass function giving the effective mass introduced by eq.(20). The essential point in deriving these analytical expressions is to introduce the bilocal
auxiliary field. Then it is shown that the mass function satisfies the (improved) ladder SDE as the stationary condition with respect to this auxiliary field. On the other hand we also confirm that the effective potential given in this method indeed satisfies the RGE discussed before.

The Wilsonian effective potential is given by integration of the higher momentum modes. In the following discussions we use \( \psi, \bar{\psi} \) and \( A_\mu \) for the lower momentum modes, which are treated as zero modes in evaluation of the effective potential. The higher momentum modes are expressed by \( \chi_p, \bar{\chi}_p \) and \( a_\mu p \) respectively with index representing their momentum. For the time being also let us abbreviate the four fermi interaction \((\bar{\psi}_i \gamma_5 \psi_j)^2\) just for simplicity. In the approximation scheme considered in the previous section, we may give the following representation for the effective potential after integrating out \( a_\mu \);

\[
e^{-\Omega V(\sigma;\Lambda)} = e^{\Omega \frac{G}{2} \sigma^2} \int \mathcal{D} \chi \mathcal{D} \bar{\chi} e^{-S^x[\chi, \bar{\chi}; \sigma]} \tag{23}
\]

where \( \sigma \) denotes the product of the lower (zero) momentum fermions \( \bar{\psi} \psi \) and \( \Omega \) denotes the space-time volume. \( S^x \) is the action for the higher momentum fermions with the effective 4-fermi interaction induced by the photon exchange, which is given by

\[
S^x = \int'_{\bar{\chi}} - p \left[ i \bar{\chi} - \left( G + \frac{3g^2}{4p^2} \right) \right] \chi_p - \frac{1}{2} \int'_{\bar{\chi}} - p' \int'_{\bar{\chi}} - k \left( G + \frac{3g^2}{4(k-p)^2} \right) \chi_p \chi_k \chi_{k'} \chi_{p'}, \tag{24}
\]

where \( k' = k - p + p' \) by momentum conservation. Here we introduced the simplified notation for the momentum integration;

\[
\int'_{p} \equiv \int_{p < |p| < A_0} \frac{d^4p}{(2\pi)^4}. \tag{25}
\]

Now the effective 4-fermi interaction may be eliminated by adding the following term with a bilocal auxiliary field \( S(p, p') \) \([17]\);

\[
\delta S^{x,S} = \frac{1}{2} \int'_{\bar{\chi}} - p \int'_{\bar{\chi}} - p' \int'_{\bar{\chi}} - k \left( G + \frac{3g^2}{4(k-p)^2} \right) \left[ S(p, p') - \bar{\chi}_{-p'} \chi_p \right] \left[ S(k', k) - \bar{\chi}_{-k'} \chi_k \right]. \tag{26}
\]

Then we are able to integrate out the fermionic freedoms with higher momentum, \( \chi \) and \( \bar{\chi} \). After performing the integration we obtain the Wilsonian effective potential expressed by the path integration over the bilocal auxiliary field \( S \);

\[
e^{-\Omega V(\sigma;\Lambda)} = e^{\Omega \frac{G}{2} \sigma^2} \int \mathcal{D} S e^{-S^S[S; \sigma]} \tag{27}
\]

Here the effective action of the bilocal field is given by

\[
S^S = \frac{1}{2} \int'_{p} \int'_{p'} \int'_{k} \left( G + \frac{3g^2}{4(k-p)^2} \right) \left[ S(p, p') S(k', k) \right.
- \left. \int'_{p} \text{tr} \ln \left[ i \bar{\chi} - \left( G + \frac{3g^2}{4p^2} \right) \sigma + \int'_{k} \left( G + \frac{3g^2}{4(k-p)^2} \right) S(k, k) \right]. \tag{28}
\]

\[10\]
The ladder approximation considered in the analyses of the SDE corresponds to the mean field approximation in the bilocal auxiliary field method. Namely the effective potential in our approximation may be obtained by applying the saddle point method to the path integral given in eq.(27). The stationary condition for the saddle point determines the expectation value of the bilocal field, or the composite operator of the fermions with higher momentum. Then the bilocal auxiliary field \(S\) is now reduced to the VEV depending on only the relative momentum, 
\[ S(p, p') \rightarrow S(p) = \langle \bar{\chi}_{-p} \chi_p \rangle, \]
because of the translationary invariance. Consequently we can obtain the following analytical expression of the Wilsonian effective potential, which is expected to satisfy the RGE proposed in the preceding section;
\[ V(\sigma; \Lambda) = -\frac{G}{2} \sigma^2 - \int_p' \text{tr} \ln \left[ i\not{\phi} + \Sigma(p; \Lambda) \right] + \frac{1}{2} \int_p' \int_k' \left( G + \frac{3g^2}{4(k - p)^2} \right) S(p) S(k), \]
where \(\Sigma\) denotes the effective mass function for the higher momentum modes, which is defined as
\[ \Sigma(p; \Lambda) = -\left( G + \frac{3g^2}{4p^2} \right) \sigma - \int_k' \left( G + \frac{3g^2}{4(k - p)^2} \right) S(k). \]

Note that this mass function depends on the lower momentum modes \(\sigma\) as well as the IR cutoff \(\Lambda\). On the other hand the stationary condition with respect to \(S(p)\) is found to be
\[ \Sigma(p; \Lambda) = -\left( G + \frac{3g^2}{4p^2} \right) \sigma - \int_k' \left( G + \frac{3g^2}{4(k - p)^2} \right) \text{tr} \left( \frac{1}{ik} + \Sigma(k; \Lambda) \right). \]

The mass function \(\Sigma\) and also the auxiliary \(S\) are formally given in terms of \(\sigma\) by solving eq.(31) and eq.(32). Therefore the Wilsonian effective potential \(V\) may be regarded as a function of \(\sigma\) and \(\Lambda\). This set of the equations will be found to interplay between the ladder SDE (4) and the non-perturbative RGE in our approximation scheme (16).

First let us examine the effective mass defined by eq.(20). By taking account of the stationary condition eq.(32), we may deduce from the eq.(30) the following relation,
\[ \frac{\partial V}{\partial \sigma} = -G\sigma + \int_k' \left( G + \frac{3g^2}{4k^2} \right) \text{tr} \left( \frac{1}{ik} + \Sigma(k; \Lambda) \right) = -G\sigma + \frac{1}{2\pi^2} \int_\Lambda^{\Lambda_0} dk' k'^3 \left( G + \frac{3g^2}{4k'^2} \right) \frac{\Sigma(k; \Lambda) \Sigma(k; \Lambda)}{k'^2 + \Sigma(k; \Lambda)}, \]

Therefore the effective mass is found to be given in terms of \(\Sigma\) defined above by
\[ m_{\text{eff}} \equiv \lim_{\Lambda \to 0} \frac{\partial V}{\partial \sigma}(\sigma; \Lambda) \bigg|_{\sigma=0} = \lim_{p \to 0} \lim_{\Lambda \to 0} \Sigma(p; \Lambda) \big|_{\sigma=0}. \]

On the other hand it is readily seen by setting \(\sigma = 0\) that the mass function \(\Sigma(p; \Lambda)\) indeed satisfies the ladder SDE given by eq.(4) in the IR limit of \(\Lambda = 0\). Thus it is verified that
the effective mass obtained by solving the ladder SDE is also derived from the Wilsonian effective potential given by (30).

Next we shall confirm that the effective potential also satisfies the RGE given by eq.(16). Again by using the stationary condition (32), we may obtain

\[ \Lambda \frac{\partial V}{\partial \Lambda} = \Lambda \int \frac{d^4p}{(2\pi)^4} \delta(|p| - \Lambda) \text{tr} \ln[i\phi + \Sigma(p; \Lambda)]. \]  

(35)

Here if we set $|p| = \Lambda$ in eq.(32), then it is written down after performing the angle integration of the momentum as

\[ \Sigma(|p| = \Lambda; \Lambda) = - \left( G + \frac{3g^2}{4\Lambda^2} \right) \sigma - \frac{1}{2\pi^2} \int_{\Lambda}^{\Lambda_0} dk \; k^3 \left( G + \frac{3g^2}{4k^2} \right) \frac{\Sigma(k; \Lambda)}{k^2 + \Sigma(k; \Lambda)^2}. \]  

(36)

Therefore the variation of the effective potential $V(\sigma; \Lambda)$ given by eq.(35) may be represented in terms of the potential itself by using eq.(33) and is found to be

\[ \Lambda \frac{\partial V}{\partial \Lambda} = \frac{1}{2\pi^2} \Lambda^4 \ln \left[ \Lambda^2 + \left( V_\sigma - \frac{3g^2}{4\Lambda^2} \sigma \right)^2 \right]. \]  

(37)

Thus it is seen that the effective potential derived through the above discussion indeed satisfies the RGE given in (16) after considering the canonical scaling transformation.

The SDE in the improved ladder approximation also can be related to the RGE by the similar arguments given so far. In this case it is enough to replace the gauge coupling appearing in the analytical expression of the Wilsonian effective potential to the running one; $g^2 \rightarrow C_2(R)\bar{g}^2(\text{max}(p^2, k^2))$. Then it is now obvious that the effective mass function $\Sigma$ at $\sigma = 0$ satisfies the improved ladder SDE (7) in the IR limit. On the other hand the variation of the effective potential is found to be given by

\[ \Lambda \frac{\partial V}{\partial \Lambda} = \frac{1}{2\pi^2} \Lambda^4 \ln \left[ \Lambda^2 + \left( V_\sigma - \frac{3C_2(R)\bar{g}^2(\Lambda^2)}{4\Lambda^2} \sigma \right)^2 \right]. \]  

(38)

Here it should be noted that this variation may be given in terms of the potential owing to the special choice of the momentum dependence of the running gauge coupling in (6). Since the argument of the running gauge coupling appearing in eq.(38) is simply given by the cutoff $\Lambda$, we may realize that the effective potential is the solution of the coupled equations of the non-perturbative RGE given by eq.(22) and the RGE for the gauge coupling. Thus it has been shown that the improved ladder approximation, which looks rather artificial in the SD approach, turns out to be the naive improvement by evaluating the gauge beta function perturbatively in the non-perturbative renormalization group framework.

In this section we discussed the relation between the non-perturbative RGE and the (improved) ladder SDE via the Wilsonian effective potential given by using the bilocal auxiliary field. However we should note that the above argument is rather formal. As a matter of fact the ladder SDE (4) is known to have many solutions corresponding to the unstable states. Therefore the solution of the RGE for the effective potential might not be
uniquely determined at the singular point in the broken phase. We have not cleared this point yet. It has been also known that the unstable solutions of the SDE are related to the appearance of the tachyon pole for the bilocal auxiliary field [17]. More careful analyses with taking the bound states into consideration would be necessary to understand the unstable solutions in the SDE’s in the RG framework.

5. The RG equations with the collective field

In section 3 we discussed the dynamical mass of the fermion appears through non-analytic behavior of the effective potential in infrared. Therefore in order to see the spontaneous chiral symmetry breaking, it seems to be necessary to deal with infinitely many operators, or to solve the partial differential equation in practice, since otherwise such non-analyticity does not come out. However it turns out to be rather difficult to calculate the effective mass in a good accuracy with this manner. Hence we might suppose that the Wilson RG approach is unsuitable for evaluation of the order parameters, while it is indeed quite efficient to analyze the critical dynamics.

On the other hand the non-analytic behavior of the effective potential was found to be related with the infrared singularity caused by the massless fermions. However the fermions must acquire their mass dynamically in the broken phase and, therefore, the system should be free from the infrared singularity. Then how can we treat the mass to be generated in the RG framework with keeping it’s chiral symmetry? Actually it will be seen that this problem is resolved by introducing proper new operators into the theory space.

Our method is to extend the theory space by introducing the collective coordinate corresponding the order parameter \( \langle \bar{\psi}\psi \rangle \). Concretely we simply introduce the collective field, which denotes \( \phi \), as an auxiliary field in the original path integration. In this section we discuss how the order parameters may be evaluated from the non-perturbative RGE defined in the extended space in the relation with the SDE’s. It will be shown that the order parameters are calculated quite precisely by dealing with only several operators, that is, by truncating the power series of the effective potential by a few terms. Thus this method will be seen to be much more effective than the naive Wilson RG in the present case.

In the followings we shall consider the QED with the 4-fermi interaction by abbreviating \((\bar{\psi}i\gamma_5\psi)^2\) part for simplicity. Then we may rewrite the partition function as

\[
Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu \mathcal{D}\phi e^{-S_0[\psi,\bar{\psi},A_\mu,\phi]},
\]

where the bare action \( S_0 \) is given by

\[
S_0 = \int d^4x \bar{\psi} (\not{\partial} + gA)\psi - \frac{G}{2}(\bar{\psi}\psi)^2 + \frac{1}{2}(\phi - y\bar{\psi}\psi)^2 + \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2,
\]

which is invariant under the discrete chiral transformation;

\[
\psi \rightarrow \gamma_5\psi, \quad \bar{\psi} \rightarrow -\bar{\psi}\gamma_5, \quad \phi \rightarrow -\phi.
\]

Here if we choose the Yukawa coupling \( y \) so as to eliminate the four-fermi interaction, this action may be regarded as the gauged Yukawa system imposed the so-called compositeness
condition, which was first examined by using the one loop RGE by Bardeen, Hill and Lindner \[18\]. However it should be stressed that the collective coordinates are introduced not for the purpose of making the perturbative treatment possible by eliminating the 4-fermi interactions, but to avoid the infrared singularity by reflecting the dynamically generated fermion mass faithfully to the RGE. Note that we may treat the system equally even in the case that the four-fermi interaction is absent in the bare action. It is also noted that the resultant order parameters should not depend on the choice of the Yukawa coupling introduced in the bare action.

First we shall derive the RGE for this system in the same approximation scheme considered in section 3. Then the Wilsonian effective action at the general scale of $\Lambda$ will be given by

$$S_{\text{eff}} = \int d^4x \bar{\psi}(\partial + gA)\psi + U(\phi, \sigma; \Lambda) + \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{2\alpha}(\partial_\mu A_\mu)^2,$$  \hspace{1cm} \text{(42)}$$

where the effective potential $U$ is kept invariant under the chiral transformation (41). It is noted that the collective field $\phi$ does not appear as a propagating mode through the evolution due to the local potential approximation. Actually it is found that the contributions of these field to the radiative corrections are totally ignored in the approximation applied here. Therefore the RGE is derived with just same argument done in section 3 and is found to be

$$\frac{\partial U}{\partial t} = 4U - 3\sigma U_\sigma - \phi U_\phi - \frac{1}{4\pi^2} \ln \left[1 + \left(U_\sigma - 3\pi\alpha\sigma\right)^2\right].$$  \hspace{1cm} \text{(43)}$$

Thus the structure of the RGE incorporating the collective field has been reduced to be of the same form as the RGE considered so far due to this simple approximation. Nevertheless this formulation will be found to have a great advantage compared to the former one.

First let us explain the way to evaluate the order parameters with this RGE and the numerical results. If we expand the effective potential $U(\phi, \sigma; t)$ into a polynomial with respect to $\sigma$ as

$$U(\phi, \sigma; t) = U^{(0)}(\phi; t) + U^{(1)}(\phi; t)\sigma + \frac{1}{2}U^{(2)}(\phi; t)\sigma^2 + \cdots,$$  \hspace{1cm} \text{(44)}$$

then eq.(43) is reduced to the RGE’s for the functions $U^{(i)}(\phi; t)$. In the chiral symmetry broke phase the potential $U^{(0)}(\phi)$ will be found to show it’s non-trivial minimum at a certain scale under evolution, and the collective field acquires the vacuum expectation value $\langle \phi \rangle$. Shortly it will be shown that $\langle \phi \rangle$ is related with the chiral condensate by

$$\langle \bar{\psi}\psi \rangle = \frac{1}{y}\langle \phi \rangle,$$  \hspace{1cm} \text{(45)}$$

\hspace{1cm}^3\text{In this sense our formulation is a sort of the “environmentally friendly renormalization group”, which has been proposed by O’Connor and Stephens \[19\].}

\hspace{1cm}^4\text{For the NJL model this scheme is nothing but the large N leading approximation.}

\hspace{1cm}^5\text{The Wilsonian effective potential is shown to coincide in the infrared limit with the conventional effective potential defined by the Legendre transformation, which must be convex \[20\]. However the RG given eq.(43) leads us to evolution into the concave one due to the naive approximation adopted here. This behavior is related with the divergence of the RG flows at a finite scale discussed in section 3.}
which is independent of the Yukawa coupling $y$. The point of this method is to solve the RGE at the absolute minimum of this potential $U^{(0)}(\phi)$. There the fermion propagates with the effective mass appearing through the linear term with respect to $\sigma$ as

$$m_{\text{eff}}(t) = U^{(1)}(\langle \phi \rangle; t).$$

Thus we may avoid the problem of the infrared singularity. The RG flows are subject to the canonical scaling at the scale lower than the effective mass. Indeed the effective mass $m_{\text{eff}} = m_{\text{eff}}(t \to \infty)$ obtained in this method will be shown to coincide with the results in the ladder SD approach.

In the practical analysis we may well truncate the series given by eq.(44) at a certain order. The results are found to converge quite rapidly with increasing the number of the couplings. In Fig.3 the contour of the effective fermion mass obtained by this method is presented in the space of the bare couplings $(G, \alpha)$ compared with the corresponding results by numerical analysis of the ladder SDE. It is clearly seen that they coincide with each other perfectly as is expected from the analytical study done in the previous section. The proof of this coincidence based on the RGE with the collective field (43) will be also given shortly. We show also the effective mass generated in the pure gauge theories compared with the SD results in Fig.4. Notice that the effective mass of the theory in the very vicinity to the criticality is evaluated enough precisely. These observations may convince us that the RG method is quite efficient in analyses of the composite order parameters as well as the critical behavior in the dynamical chiral symmetry breaking.

In the primitive approximation scheme applied here, which reproduces the results obtained by solving the (improved) ladder SDE, any radiative corrections by the collective field has been ignored. However the collective field introduced just as an auxiliary in the bare action becomes the propagating degree of freedom as an bound state with evolution. Therefore it will be significant to improve the approximation so that the collective field is incorporated as the dynamical variable. For this purpose the analysis in the scheme beyond the local potential approximation will be necessary.

Now we shall show along the argument done in the previous section that the order parameters, i.e. the effective mass and the chiral condensate, are really independent of the Yukawa coupling and also accord with the SD results. For the time being $\phi$ and $\sigma$ denote the lower momentum modes, which are treated as zero modes. Since the collective field does not contribute to the radiative correction at all, we may immediately written down the Wilsonian effective potential $U(\phi, \sigma; \Lambda)$ by introducing the bilocal auxiliary field;

$$U(\phi, \sigma; \Lambda) = -\frac{G - y^2}{2} \sigma^2 - y\phi\sigma + \frac{1}{2} \phi^2 - \int_p \tr \ln [ip + \Sigma(p; \Lambda)]$$

$$+ \frac{1}{2} \int_p \int_k \left( G - y^2 + \frac{3g^2}{4(k - p)^2} \right) S(p)S(k),$$

(47)

6 There found two absolute minima correspondingly to the discrete chiral symmetry. If the small bare mass is introduced, the absolute minima is uniquely determined.

7 The non-perturbative RGE’s for the Higgs-Yukawa theories have been examined for study of the triviality mass bound in the standard model [21].
Fig. 3: Contour of the effective mass in the $(G, \alpha)$ plane. The results obtained by solving the non-perturbative RGE coincide with the ladder SD results.

Fig. 4: The effective mass in strongly coupled gauge theories; $(G = 0)$.

where the effective mass function $\Sigma$ denotes

$$
\Sigma(p; \Lambda) = - \left( G - y^2 + \frac{3g^2}{4p^2} \right) \sigma - y \phi - \int_k' \left( G - y^2 + \frac{3g^2}{4(k-p)^2} \right) S(k).
$$

Also the stationary condition is found to be given by

$$
\Sigma(p; \Lambda) = - \left( G - y^2 + \frac{3g^2}{4p^2} \right) \sigma - y \phi - \int_k' \left( G - y^2 + \frac{3g^2}{4(k-p)^2} \right) \text{tr} \left( \frac{1}{ik' + \Sigma(k; \Lambda)} \right).
$$

We note that the mass function depends on not only $\sigma$ but also $\phi$ in turn. It is straightforward to see that the effective potential $U(\phi, \sigma; \Lambda)$ defined by these equations satisfies the non-perturbative RGE (43) indeed. However the relation to the ladder SDE is not obvious due to the presence of the collective field $\phi$.

By solving the RGE (43) we may obtain the effective potential $U(\phi, \sigma; \Lambda)$ as the infrared limit. However the effective mass is defined through the effective potential in terms of the fermion, $V(\sigma; \Lambda)$ as is given by eq.(20). In order to derive this effective potential from $U(\phi, \sigma)$ we have to integrate out the collective zero mode $\phi$;

$$
e^{-\Omega V(\sigma)} = \int d\phi e^{-\Omega U(\phi, \sigma)}.
$$

However we may well evaluate this integral by the saddle point method at the absolute minimum of the potential $U$ in the infrared (infinite volume) limit. Then $\phi$ is given by a
function in terms of $\sigma$ after solving the stationary condition of $U$;

$$\frac{\partial U}{\partial \phi} = -y\sigma + \phi + y \int \frac{d^4p}{(2\pi)^4} \left( \frac{1}{i\not{p} + \Sigma(p)} \right) \equiv 0. \quad (51)$$

Let us represent the formal solution for this equation by $\phi^*(\sigma)$. By using this the effective potential in terms of the fermions is given by $V(\sigma) = U(\phi^*(\sigma), \sigma)$. On the other hand the minimum of the potential $U^{(0)}$, which was denoted by $\langle \phi \rangle$ above is nothing but

$$\langle \phi \rangle = \phi^*(\sigma = 0) = -y \int \frac{d^4p}{(2\pi)^4} \left( \frac{1}{i\not{p} + \Sigma(p)} \right) \bigg|_{\sigma=0}. \quad (52)$$

First we shall consider the effective mass function $\Sigma(p)$ which should be subject to the ladder SDE in the infrared limit. Note that the collective coordinate $\phi$ in the stationary condition (49) is now given by $\phi^*(\sigma)$. Therefore by setting $\sigma = 0$, we may obtain the effective mass function as

$$\Sigma(p) = -y\langle \phi \rangle + \int \frac{d^4k}{(2\pi)^4} \left( G - y^2 + \frac{3g^2}{4(k-p)^2} \right) \left( \frac{1}{i\not{k} + \Sigma(k)} \right) \bigg|_{\sigma=0}. \quad (53)$$

After substituting the expression for $\langle \phi \rangle$ given by eq.(52) into this equation, it is readily seen that the effective mass indeed satisfies the ladder SDE and, therefore, is independent of the free parameter $y$.

Now it is seen that the order parameters defined by eq.(45) and eq.(46) in the context of the RG are just the same quantities given by the ladder SDE, and, therefore, are independent of $y$. Coincidence of the effective mass is shown by the relation

$$\frac{dV(\sigma)}{d\sigma} \bigg|_{\sigma=0} = \lim_{t \to \infty} \frac{\partial U(\phi, \sigma; t)}{\partial \sigma} \bigg|_{\phi=\langle \phi \rangle, \sigma=0} = \lim_{t \to \infty} U^{(1)}(\langle \phi \rangle; t) = m_{\text{eff}}. \quad (54)$$

We note also that this effective mass is the momentum independent part of the mass function given by eq.(53); $m_{\text{eff}} = \Sigma(0)$. From eq.(52) we see that the chiral condensate is given by $\langle \bar{\psi}\psi \rangle = -\langle \phi \rangle / y$. Since the mass function has been shown to be independent of $y$, the chiral condensate is found to be so. Thus the non-perturbative RGE incorporating the collective field (43) also has been shown to offer the order parameters in the ladder approximation.

This method is found to be efficient equally in the analysis of the dynamical chiral symmetry breaking in QCD-like gauge theories. The order parameters may be evaluated by solving the RGE given by eq.(43) with the running gauge coupling imposed a proper IR cutoff. The bare cutoff scale $\Lambda_0$ should be large enough compared to the dynamical scale of the theory. Now it would be expected that this method using the non-perturbative RGE should reproduce the identical results with those obtained by using the improved ladder SDE, as long as we adopt the same IR cutoff to the running gauge coupling. In Fig.5 the results of the dynamical mass of quarks $m_{\text{eff}} = \Sigma(0)$ as well as the chiral condensate $\langle \bar{\psi}\psi \rangle$ obtained by the both method are shown in the case of SU(3) QCD with three triplet massless quarks. From this figure we may realize the efficiency of this RG method, with the collective filed, since the results converge quite rapidly against truncation of the
potential given by (44). In the practical calculation, the scale of $\Lambda_{\text{QCD}}$ is set to $490\text{MeV}$, which has been known to offer the $\pi$ decay constant of the experimental value $f_\pi=94\text{MeV}$ [5]. The chiral condensates are renormalized at $1\text{GeV}$.

6. Discussions

In this paper we have proposed non-perturbative RGE’s by which we can analyze the order parameters for the chiral symmetry breaking dynamically induced by strong gauge interactions. The RGE was derived in the simple approximation scheme, which exactly accords with the ladder approximation mostly used in the SD approach. Moreover the so-called improved ladder approximation performed in the SD framework has been understood as the very natural and simple extension of the RGE. The spontaneous symmetry breaking is found to appear through the non-analyticity of the infrared effective potential in the naive Wilson RG picture. We have proposed also the scheme of the non-perturbative RG extended so as to incorporate the collective coordinate corresponding to the composite order parameter in order to avoid the infrared singularity. This extended RGE is quite
efficient in the practical evaluation of the order parameters, which was demonstrated by the calculation in the case of the massless QCD.

Throughout of this paper we have been discussing in a quite simple approximation scheme in order to make clear relation with the SD approach. Now it has been found that any improvement of this approximation enable us to go beyond the analyses by the (improved) ladder approximation. As far as the critical dynamics is concerned, it has been achieved so as to incorporate the non-ladder diagrams in the previous works [10, 11]. We should stress that the gauge dependence, which has been a serious problem in the SD approach [3], remarkably disappears by taking account of the non-ladder diagrams [10]. Therefore it seems to be important to extend the analyses for the order parameters presented in this paper to the better approximation scheme including the non-ladder corrections.

Another interesting problem in the non-perturbative RG study would be the bound states and the composite particles. In this paper we have introduced the collective field $\phi$ representing the fermion composite operator. It is naturally expected that this field is closely related to the composite particle. Recently the appearance of the composite particle has been vigorously studied in the non-perturbative RG framework [22]. It would be extremely fascinating problem to describe the transition of the effective degree of freedom in QCD, quarks to hadrons, in the Wilson RG picture.

References

[1] T. Maskawa and H. Nakajima, Prog. Theor. Phys. 52 (1974) 1326. R. Fukuda and T. Kugo, Nucl. Phys. B117 (1976) 250. V. A. Miransky, Nuovo Cim. 90A (1985) 149.

[2] W. A. Bardeen, C. N. Leung and S. T. Love, Phys. Rev. Lett. 56 (1986) 1230. C. N. Leung, S. T. Love and W. A. Bardeen, Nucl. Phys. B273 (1986) 649.

[3] K.-I. Kondo, H. Mino and K. Yamawaki, Phys. Rev. D39 (1989) 2430. K. Yamawaki, in Proc. Johns Hopkins Workshop on Current Problems in Particle Theory 12, Baltimore, 1988, eds. G. Domokos and S. Kovesi-Domokos (World Scientific, Singapore, 1988). T. Appelquist, M. Soldate, T. Takeuchi and L.C.R. Wijewardhana, ibid.

[4] V. A. Miransky, Sov. J. Nucl. Phys. 38 (1984) 280. K. Higashijima, Phys. Rev. D29 (1984) 1228.

[5] K-I. Aoki, T. Kugo and M. G. Mitchard, Phys. Lett. B266 (1991) 467. K.-I. Aoki, M. Bando, T. Kugo, M. G. Mitchard and H. Nakatani, Prog. Theor. Phys. 84 (1990) 683.

[6] K. Yamawaki, M. Bando and K. Matumoto, Phys. Rev. Lett. 56 (1986) 1335. T. Akiba and T. Yanagida, Phys. Lett. B169 (1986) 432. B. Holdom, Phys. Rev. D24 (1981) 1441.

[7] V. A. Miransky, M. Tanabashi and K. Yamawaki, Phys. Lett. B221 (1989) 177. W. A. Bardeen, C. T. Hill and M. Lindner, Phys. Rev. D41 (1990) 1647. B. Holdom,
Phys. Rev. D54 (1996) 1068. K. Yamawaki, [hep-ph/9603293], in the proceedings of
14th Symposium on Theoretical Physics: Dynamical Symmetry Breaking and Effective
Field Theory, Cheju, Korea, 21-26 Jul 1995.

[8] K. Kondo and H. Nakatani, Nucl. Phys. B351 (1991) 236; Prog. Theor. Phys. 88
(1992) 7373. K. Kondo, Int. J. Mod. Phys. A7 (1992) 7239.

[9] K-I. Aoki, M. Bando, T. Kugo, K. Hasebe and H. Nakatani, Prog. Theor. Phys. 81
(1989) 866.

[10] K-I. Aoki, K. Morikawa, W. Souma, J.-I. Sumi and H. Terao, Prog. Theor. Phys. 97
(1997) 479.

[11] K-I. Aoki, K. Morikawa, J.-I. Sumi, H. Terao and M. Tomoyose, KANAZAWA-99-11,
KUCP-0139.

[12] K. G. Wilson and I. G. Kogut, Phys. Rep. 12 (1974) 75.

[13] F. Wegner and A. Houghton, Phys. Rev. A8 (1973) 401.

[14] J. Polchinski, Nucl. Phys. B231 (1984) 269. G. Keller, C. Kopper and M. Salmhofer,
Helv. Phys. Acta 65 (1992) 32. C. Wetterich, Phys. Lett. B301 (1993) 90.
M. Bonini, M. D’Attanasio, G. Marchesini, Nucl.Phys. B409 (1993) 441. T. R. Mor-
ris, Int. J. Mod. Phys, A9 (1994) 2411.

[15] A. Hazenfratz and P. Hazenfratz, Nucl. Phys. B270 (1986) 269. T. R. Morris,
Phys. Lett, B334 (1994) 355.

[16] C. Becchi, On the construction of renormalized quantum field theory using renor-
malization group techniques, in: Elementary Particles, Field Theory and Statistical
Mechanics, eds. M. Bonini, G. Marchesini, and E. Onofri, Parma University, 1993.
U. Ellwanger, Phys. Lett. B335 (1994) 364. U. Ellwanger, M. Hirsch, and A. Weber,
Z. Phys. C69 (1996) 687. M. Bonini, M. D’Attanasio, and G. Marchesini, Nucl. Phys.
B418 (1994) 81; B421 (1994) 429; B437 (1995) 163; Phys. Lett. B346 (1995) 87.
M. D’Attanasio, and T. R. Morris, Phys. Lett. B378 (1996) 213. F. Freire, and
C. Wetterich, Phys. Lett. B380 (1996) 337.

[17] H. Kleinert, Phys. Lett. 62B (1976) 429. E. Schrauner, Phys. Rev. D16 (1977) 1877.
T. Kugo, Phys. Lett. 76B (1978) 625. T. Morozumi and H. So, Prog. Theor. Phys.
77 (1987) 1434.

[18] W. A. Bardeen, C. T. Hill and M. Lindner, Phys. Rev. D41 (1990) 1647.

[19] D. O’Connor and C.R. Stephens, Phys. Rev. Lett. 72 (1994) 506; Int. J. Mod. Phys.
A9 (1994) 2805.

[20] R. Fukuda, Prog. Theor. Phys. 56 (1976) 258. L. O’Raifeartaigh, A. Wipf and
H. Yoneyama, Nucl. Phys. B271 (1986) 653. N. Tetradis and C. Wetterich,
Nucl. Phys. B383 (1992) 197.
[21] M. Maggiore, Z. Phys. C41 (1989) 687. T.E. Clark, B. Haeri and S.T. Love, Nucl. Phys. B402 (1993) 628.

[22] D. U. Jungnickel and C. Wetterich, Phys. Lett. B389 (1996) 600; Eur. Phys. J. C2 (1998) 557. B. Bergerhoff and C. Wetterich, Phys. Rev. D57 (1998) 1591.