FROM C*ALGEBRA EXTENSIONS TO CQMS, $SU_q(2)$, PODLES SPHERE AND OTHER EXAMPLES

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Abstract. We construct compact quantum metric spaces (CQMS) starting with some C*-algebra extension with a positive splitting. As special cases we discuss the case of Toeplitz algebra, quantum SU(2) and Podles sphere.

2000 Mathematics Subject Classification. 46L87(primary), 58B34(secondary).

Key words and phrases. Quantum SU(2), Podles sphere, Compact Quantum Metric Space.
1. Introduction

In noncommutative geometry, the natural way to specify a metric is by a “Lipschitz seminorm”. This idea was first suggested by Connes ([2]), and developed further in [3]. Connes pointed out ([2], [3]) that from a Lipschitz seminorm one obtains in a simple way an ordinary metric on the state space of a $C^*$-algebra. A natural question in this context is when does this metric topology coincides with the weak* topology. In his search for an answer to this question, Rieffel ([5], [6], [7]) has identified a larger class of spaces, namely order unit spaces on which one can answer these questions. He has introduced the concept of Compact Quantum Metric Spaces (CQMS) as a generalization of compact metric spaces, and used ([7]) this new concept for rigorous study of convergence questions of algebras much in the spirit of Gromov-Hausdorff convergence. One natural question in this regard is are there plenty of CQMS floating around? Rieffel ([5], [6]) has given some general principles for the construction of CQMS. In [1] we exploited one of his principles to construct CQMS. In fact Rieffel has shown ([8]) that indeed there are many examples. But in concrete $C^*$-algebras one would like to have more explicit description of these structures. Our objective here is construction of CQMS out of quantum SU(2) and Podles spheres. To achieve that we make a slightly general construction and produce CQMS starting from $C^*$-algebra extensions. Organization of the paper is as follows. In the next section we recall the basics of CQMS. In section 3 the basic construction has been described. In the final section we employ the principle developed in section 3 to special cases.

2. Compact Quantum Metric Space: Preliminaries

We recall some of the definitions from [7].

**Definition 2.1.** An order unit space is a real partially ordered vector space $A$ with a distinguished element $e$, the order unit satisfying

(i) (Order Unit property) For each $a \in A$ there is an $r \in \mathbb{R}$ such that $a \leq re$.

(ii) (The Archimidean property) If $a \in A$ and if $a \leq re$ for all $r \in \mathbb{R}$ with $r \geq 0$, then $a \leq 0$. 

Remark 2.2. The following prescription defines a norm on an order unit space.

\[ \|a\| = \inf \{ r \in \mathbb{R} \mid -re \leq a \leq re \} \]

Definition 2.3. By a state of an order unit space \((A, e)\) we mean a \(\mu \in A'\), the dual of \((A, \| \cdot \|)\) such that \(\mu(e) = 1 = \|\mu\|'.\) Here \(\| \cdot \|'\) stands for the dual norm on \(A'\). Collection of states on \((A, e)\) is denoted by \(S(A)\).

Remark 2.4. States are automatically positive.

Example 2.5. Motivating example of the above concept is the real subspace of selfadjoint elements in a C*-algebra with the order structure inherited from the C*-algebra.

Definition 2.6. Let \((A, e)\) be an order unit space. By a Lip norm on \(A\) we mean a seminorm \(L,\) on \(A\) such that

(i) For \(a \in A,\) we have \(L(a) = 0\) iff \(a \in \mathbb{R}e;\)
(ii) The topology on \(S(A)\) coming from the metric

\[ \rho_L(\mu, \nu) = \sup \{ |\mu(a) - \nu(a)| : L(a) \leq 1 \} \]

is the \(w^*\) topology.

Definition 2.7. A compact quantum metric space is a pair \((A, L)\) consisting of an order unit space \(A\) and a Lip norm \(L\) defined on it.

The following theorem of Rieffel will be of crucial importance.

Theorem 2.8 (Theorem 4.5 of [7]). Let \(L\) be a seminorm on the order unit space \(A\) such that \(L(a) = 0\) iff \(a \in \mathbb{R}e.\) Then \(\rho_L\) gives \(S(A)\) the \(w^*\)-topology exactly if

(i) \((A, L)\) has finite radius, i.e., \(\rho_L(\mu, \nu) \leq C\) for all \(\mu, \nu \in S(A)\) for some constant \(C,\) and
(ii) \(B_1 = \{ a | L(a) \leq 1, \text{ and } \|a\| \leq 1 \}\) is totally bounded in \(A\) for \(\| \cdot \|.)\)

3. Extensions to CQMS

In this section we describe the general principle of construction of CQMS from certain C*-algebra extensions. Let \(A\) be a unital C*-algebra. Fix a faithful representation \(A \subseteq B(H)\). Suppose we have a dense order unit space \(\text{Lip}(A) \subseteq A_{s,a},\) where \(A_{s,a}\) denotes
the real partially ordered subset of selfadjoint elements in $\mathcal{A}$. Let $L$ be a Lip norm on $\text{Lip}(\mathcal{A})$ such that $((\text{Lip}(\mathcal{A}), I), L)$ is a CQMS. Let $\nu$ be a state on $\mathcal{A}$, then define $\tilde{\mathcal{A}}_\nu$ to be the collection of $((a_{ij})) \in \mathcal{K}(l^2(\mathbb{N})) \otimes \mathcal{A}$ such that (i) $a_{ij} \in \text{Lip}(\mathcal{A})$, (ii) $a_{ij} = a_{ji}$, (iii) $\sup_{i,j \geq 1}(i + j)^k(L(a_{ij}) + |\nu(a_{ij})|) < \infty \forall k$. Clearly $\mathcal{A}_\nu := \tilde{\mathcal{A}}_\nu \oplus \mathbb{R}I$, where $I$ is the identity on $\mathcal{B}(l^2(\mathbb{N}) \otimes \mathcal{H})$ is an order unit space. Define $L_k : \mathcal{A}_\nu \to \mathbb{R}_+$ by $L_k(I) = 0$,

$$L_k((a_{ij})) = \sup_{i,j \geq 1}(i + j)^k(L(a_{ij}) + |\nu(a_{ij})|).$$

**Lemma 3.1.** Let $d = \text{diameter of } ((\text{Lip}(\mathcal{A}), I), L)$. Then for a “Lipschitz function” $a \in \text{Lip}(\mathcal{A})$ one has $||a|| \leq (L(a) + |\nu(a)|)(1 + d)$.

**Proof:** Let $\mu$ be an arbitrary state on $\mathcal{A}$. Then using $\sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\} \leq d$ we get,

$$|\mu(a)| \leq |\mu(a) - \nu(a)| + |\nu(a)|$$

$$\leq L(a) d + |\nu(a)|$$

$$\leq (L(a) + |\nu(a)|)(1 + d).$$

\hfill \square

**Lemma 3.2.** There exists a constant $C > 0$ such that for $((a_{ij})) \in \tilde{\mathcal{A}}_\nu$,

$$||(a_{ij})|| \leq C L_2((a_{ij})).$$

**Proof:** Let $\{e_i\}_{i \geq 1}$ be the canonical orthonormal basis for $l^2(\mathbb{N})$. Let $\sum \lambda_i e_i \otimes u_i$ and $\sum \mu_i e_i \otimes v_i$ be two generic elements in $l^2(\mathbb{N}) \otimes \mathcal{H}$. Here $u_i, v_i \in \mathcal{H}$ are unit vectors. Then clearly $||\sum \lambda_i e_i \otimes u_i||^2 = \sum |\lambda_i|^2, ||\sum \mu_i e_i \otimes u_i||^2 = \sum |\mu_i|^2$. Now observe that

$$|\langle \sum \lambda_i e_i \otimes u_i, ((a_{ij})) \sum \mu_j e_j \otimes v_j \rangle| \leq \sum |\lambda_i||\mu_j||\langle u_i, a_{ij} v_j \rangle|$$

$$\leq \sum |\lambda_i||\mu_j|(L(a_{ij}) + |\nu(a_{ij})|)(1 + d)$$

$$\leq (1 + d)\sum |\lambda_i||\mu_j| L_2((a_{ij})) )_{ij}$$

$$\leq L_2((a_{ij}))(1 + d) \sum_{n=1}^\infty \frac{1}{n^2} \sqrt{\sum |\lambda_i|^2} \sqrt{\sum |\mu_i|^2}.$$
Lemma 3.3. Let $B_1 = \{ a \in A_\nu | L_k(a) \leq 1, \|a\| \leq 1 \}$. Then $B_1$ is totally bounded in norm for $k > 2$.

Proof: Let $\epsilon > 0$ be given. Choose $N$ such that $(\frac{1}{N})^{k-2} < \epsilon$. For $G = ((g_{ij})) \in A_\nu$, let $P_N(G) \in K(l^2(\mathbb{N})) \otimes A$ be the element given by

$$P_N(G)_{ij} = \begin{cases} g_{ij} & \text{for } i, j \leq N, \\ 0 & \text{otherwise}. \end{cases}$$

Now observe that

$$L_k(G - P_N(G)) = \sup_{i \geq N, j \geq N} (i + j)^k (L(g_{ij}) + |\nu(g_{ij})|)$$

$$\geq N^{k-2} \sup_{i \geq N, j \geq N} (i + j)^2 (L(g_{ij}) + |\nu(g_{ij})|)$$

$$= N^{k-2} L_2(G - P_N(G)).$$

Note that for $G \in B_1$, $L_k(G - P_N(G)) \leq 1$, therefore

$$\|G - P_N(G)\| \leq C L_2(G - P_N(G))$$

$$\leq CN^{-(k-2)} L_k(G - P_N(G)) < C\epsilon.$$

Here the constant $C$ is the one obtained in the previous lemma. Note $C$ does not depend on $N$. By theorem 2.8 there exists $N \times N$ matrices $((a^{(r)}_{ij})) \in M_N(A)$, for $r = 1, \ldots, l$ such that for any $N \times N$ matrix $((a_{ij})) \in B_1$, there exists $r$ satisfying $\|((a_{ij})) - ((a^{(r)}_{ij}))\| < \epsilon$. Now for $G \in B_1$, get $((a^{(r)}_{ij}))$ such that $\|P_N(G) - ((a^{(r)}_{ij}))\| < \epsilon$. Then,

$$\|G - ((a^{(r)}_{ij}))\| \leq \|G - P_N(G)\| + \epsilon \leq (1 + C)\epsilon.$$

This completes the proof.

Theorem 3.4. $((A_\nu, I), L_k)$ is a compact quantum metric space for $k > 2$.

Proof: In view of theorem 2.8 and the previous lemma we only have to show that $(A_\nu, L_k)$ has finite radius. Let $\mu_1, \mu_2 \in S(A_\nu), a \in A_\nu$ with $L_k(a) \leq 1$. By lemma 3.2 $\|a\| \leq C$, because $L_2(a) \leq L_k(a)$. Hence $|\mu_1(a) - \mu_2(a)| \leq 2C$, that is $\text{diam}(A_\nu, L_k) \leq 2C$. 

$\square$
Proposition 3.5. Let 

\[ 0 \to A_0 \xrightarrow{i} A_1 \xrightarrow{\pi} A_2 \to 0 \]

be a short exact sequence of $C^*$-algebras, with $A_1, A_2$ unital and a positive linear splitting $\sigma : A_2 \to A_1$. Let $\phi : A'_1 \to A'_0 \oplus A'_2, \psi : A'_0 \oplus A'_2 \to A'_1$ be the bounded linear maps given by

\[
\phi(\mu) = (\mu_1, \mu_2), \mu_1 = \mu|_{i(A_0)}, \mu_2 = \mu \circ \sigma \\
\psi(\mu_1, \mu_2) = \mu, \mu(a) = \mu_2(\pi(a)) + \mu_1(a - \sigma \circ \pi(a))
\]

Then $\mu_1, \mu_2$ are inverse to each other.

Proof: Let $\phi(\mu) = (\mu_1, \mu_2), \psi(\mu_1, \mu_2) = \mu'$. Then

\[
\mu'(a) = \mu_2(\pi(a)) + \mu_1(a - \sigma \circ \pi(a)) \\
= \mu(\sigma \circ \pi(a)) + \mu(a - \sigma \circ \pi(a)) \\
= \mu(a).
\]

Therefore $\psi \circ \phi = Id_{A'_1}$. Similarly one can show that the other composition is also identity. 

Let $\mathcal{A}, Lip(\mathcal{A}), L$ be as above. Suppose we have a short exact sequence of $C^*$-algebras

\[ 0 \to \mathcal{K} \otimes \mathcal{A} \xrightarrow{i} \tilde{A}_1 \xrightarrow{\pi} \tilde{A}_2 \to 0 \]

with $\tilde{A}_1, \tilde{A}_2$ unital and a positive unital linear splitting $\sigma : \tilde{A}_2 \to \tilde{A}_1$. Let $(\mathcal{A}_2, L_2)$ be a compact quantum metric space with $\mathcal{A}_2$ a dense subspace of selfadjoint elements of $\tilde{A}_2$. Define $\mathcal{A}_1 = i(\tilde{A}_0) \oplus \sigma(\mathcal{A}_2)$. Then we have

Theorem 3.6. In the above set up $L_1 : \mathcal{A}_1 \to \mathbb{R}_+$, given by

\[ L_1(a) = L_2(\pi(a)) + L_k(a - \sigma \circ \pi(a)) \]

is a Lip norm for $k > 2$.

Proof: We break the proof in several steps.

Step (i) $L_1(a) = 0$ iff $a \in \mathbb{R} Id_{\mathcal{A}_1}$: If part is obvious for the only if part note $L_1(a) = 0$ gives $\pi(a) = \lambda Id_{\mathcal{A}_2}$ for some $\lambda \in \mathbb{R}$ and $L_0(a - \lambda Id_{\mathcal{A}_1}) = 0$. Hence $a = \lambda Id_{\mathcal{A}_1}$.

Step (ii) $(\mathcal{A}_1, L_1)$ has finite radius: Let $\mu, \lambda \in S(\mathcal{A}_1)$ and $(\mu_1, \mu_2) = \phi(\mu), (\lambda_1, \lambda_2) = \phi(\lambda)$,
where $\phi$ is as in proposition 3.5. Then from the norm estimate of $\phi$ obtained in proposition 3.5 we get $\|\mu_i\|, \|\lambda_i\| \leq (1 + \|\sigma\|)$, for $i = 1, 2$ and positivity of $\sigma$ implies $\|\mu_2\| = \|\lambda_2\| = 1$.

Let $x \in A_1$ with $L(x) \leq 1$, then

$$|\mu(x) - \lambda(x)| = |\mu_2(\pi(x)) + \mu_1(x - \sigma \circ \pi(x)) - \lambda_2(\pi(x)) - \lambda_1(x - \sigma \circ \pi(x))|$$

$$\leq |\mu_2(\pi(x)) - \lambda_2(\pi(x))| + |\mu_1(x - \sigma \circ \pi(x)) - \lambda_1(x - \sigma \circ \pi(x))|$$

$$\leq \text{diam}(A_2, L_2) + 2(1 + \|\sigma\|)C$$

where $C$ is the constant obtained in lemma 3.2. This proves $(A_1, L_1)$ has finite radius.

Step (iii) In view of theorem 2.8 it suffices to show that $B_1 = \{a \in A_1 : \|a\| \leq 1, L_1(a) \leq 1\}$ is totally bounded. Since $(A_\nu, L_k)$ and $(A_2, L_2)$ are compact quantum metric spaces it follows that if we have a sequence $a_n \in B_1$, then there exists a subsequence $a_{n_k}$ such that both $\pi(a_{n_k})$ and $a_{n_k} - \sigma \circ \pi(a_{n_k})$ converges in norm. Hence $a_{n_k}$ converges in norm implying the totally boundedness. $\square$

### 4. Examples

**Example 4.1.** Let $\Omega$ be a strongly pseudoconvex domain in $\mathbb{C}^n$. Let $H^2(\partial \Omega)$ be the closure in $L^2(\partial \Omega)$ of boundary values of holomorphic functions that can be continuously extended to $\tilde{\Omega}$. For $f \in C(\partial \Omega)$ let $T_f$ be the associated Toeplitz operator, that is the compression of the multiplication operator $M_f$ on $L^2(\partial \Omega)$ on $H^2(\partial \Omega)$. Let $\mathfrak{T}(\partial \Omega)$ be the associated Toeplitz extension, that is the $C^*$-algebra generated by the operators $T_f$ along with the compacts. Then we have a short exact sequence of $C^*$-algebras

$$0 \rightarrow \mathcal{K}(H^2(\partial \Omega)) \overset{i}{\rightarrow} \mathfrak{T}(\partial \Omega) \overset{\pi}{\rightarrow} C(\partial \Omega) \rightarrow 0$$

Since this sequence admits a positive unital splitting by the previous theorem we get CQMS structure on $\mathfrak{T}(\partial \Omega)$.

**Example 4.2.** The $C^*$-algebra of continuous functions on the quantum $SU(2)$, to be denoted by $C(SU_q(2))$, is the universal $C^*$-algebra generated by two elements $\alpha$ and $\beta$ satisfying the
following relations:

\[ \alpha^* \alpha + \beta^* \beta = I, \quad \alpha \alpha^* + q^2 \beta \beta^* = I, \]
\[ \alpha \beta - q \beta \alpha = 0, \quad \alpha \beta^* - q \beta^* \alpha = 0, \]
\[ \beta^* \beta = \beta \beta^*. \]

The \( C^* \)-algebra \( C(SU_q(2)) \) can be described more concretely as follows. Let \( \{e_i\}_{i \geq 0} \) and \( \{e_i\}_{i \in \mathbb{Z}} \) be the canonical orthonormal bases for \( L_2(\mathbb{N}_0) \) and \( L_2(\mathbb{Z}) \) respectively. We denote by the same symbol \( N \) the operator \( e_k \mapsto ke_k, \quad k \geq 0 \), on \( L_2(\mathbb{N}_0) \) and \( e_k \mapsto ke_k, \quad k \in \mathbb{Z} \), on \( L_2(\mathbb{Z}) \). Similarly, denote by the same symbol \( \ell \) the operator \( e_k \mapsto e_{k-1}, \quad k \geq 1, \quad e_0 \mapsto 0 \) on \( L_2(\mathbb{N}_0) \) and the operator \( e_k \mapsto e_{k-1}, \quad k \in \mathbb{Z} \) on \( L_2(\mathbb{Z}) \). Now take \( \mathcal{H} \) to be the Hilbert space \( L_2(\mathbb{N}_0) \otimes L_2(\mathbb{Z}) \), and define \( \pi \) to be the following representation of \( C(SU_q(2)) \) on \( \mathcal{H} \):

\[ \pi(\alpha) = \ell \sqrt{I - q^2N} \otimes I, \quad \pi(\beta) = qN \otimes \ell. \]

Then \( \pi \) is a faithful representation of \( C(SU_q(2)) \), so that one can identify \( C(SU_q(2)) \) with the \( C^* \)-subalgebra of \( B(\mathcal{H}) \) generated by \( \pi(\alpha) \) and \( \pi(\beta) \). Image of \( \pi \) contains \( K \otimes C(\mathbb{T}) \) as an ideal with \( C(\mathbb{T}) \) as the quotient algebra, that is we have a useful short exact sequence

\[
0 \longrightarrow K \otimes C(\mathbb{T}) \overset{i}{\longrightarrow} A \overset{\sigma}{\longrightarrow} C(\mathbb{T}) \longrightarrow 0.
\]

The homomorphism \( \sigma \) is explicitly given by \( \sigma(\alpha) = \ell, \sigma(\beta) = 0 \). It is easy to see that the above short exact sequence admits a positive splitting taking \( z^n \in C(\mathbb{T}) \) to \( \ell^n \otimes I \), for all \( n \geq 0 \). Hence we get a compact quantum metric space structure on \( C(SU_q(2)) \).

**Example 4.3.** Quantum sphere was introduced by Podles in [4]. This is the universal \( C^* \)-algebra denoted by \( C(S^2_{qc}) \), generated by two elements \( A \) and \( B \) subject to the following relations:

\[
A^* = A, \quad B^* B = A - A^2 + cI, \quad BA = q^2 A B, \quad BB^* = q^2 A - q^4 + cI.
\]

Here the deformation parameters \( q, c \) satisfy \( |q| < 1, c > 0 \). For later purpose we also note down two irreducible representations such that the representation given by the direct sum of
these two is faithful. Let \( \mathcal{H}_+ = l^2(N_0), \mathcal{H}_- = \mathcal{H}_+ \). Define \( \pi_+(A), \pi_-(B) : \mathcal{H}_+ \to \mathcal{H}_+ \) by

\[
\pi_+(A)(e_n) = \lambda_+ q^n e_n \quad \text{where} \quad \lambda_+ = \frac{1}{2} \pm \left( c + \frac{1}{4} \right)^{1/2} \\
\pi_-(B)(e_n) = c_- (n)^{1/2} e_{n-1} \quad \text{where} \quad c_- (n) = \lambda_- q^{2n} - (\lambda_- q^{2n})^2 + c, \text{ and } e_{-1} = 0.
\]

Since \( \pi = \pi_+ \oplus \pi_- \) is a faithful representation we have ([9]),

(i) \( C(S_{qc}^2) \cong C^*(\mathfrak{S}) \oplus_\sigma C^*(\mathfrak{S}) := \{ (x, y) : x, y \in C^*(\mathfrak{S}), \sigma(x) = \sigma(y) \} \) where \( C^*(\mathfrak{S}) \) is the Toeplitz algebra and \( \sigma : C^*(\mathfrak{S}) \to C(\mathbb{T}) \) is the symbol homomorphism.

(ii) We have a short exact sequence

\[
0 \longrightarrow \mathcal{K} \overset{i}{\longrightarrow} C(S_{qc}^2) \overset{\alpha}{\longrightarrow} C^*(\mathfrak{S}) \longrightarrow 0
\]

As in the earlier case this short exact sequence is also split exact. Here a positive splitting is given by \( \ell \in C^*(\mathfrak{S}) \mapsto (\ell, \ell) \). Now to apply the basic theorem note that by the earlier example on Toeplitz extensions we already have a Lip norm on a dense subspace of \( C^*(\mathfrak{S}) \).

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