The infinite rate symbiotic branching model: from discrete to continuous space

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Abstract

The symbiotic branching model describes a spatial population consisting of two types that are allowed to migrate in space and branch locally only if both types are present. We continue our investigation of the large scale behaviour of the system started in [BHO16], where we showed that the continuum system converges after diffusive rescaling. Inspired by a scaling property of the continuum model, a series of earlier works initiated by Klenke and Mytnik [KM12a, KM12b] studied the model on a discrete space, but with infinite branching rate. In this paper, we bridge the gap between the two models by showing that by diffusively rescaling this discrete space infinite rate model, we obtain the continuum model from [BHO16]. As an application of this convergence result, we show that if we start the infinite rate system from complementary Heaviside initial conditions, the initial ordering of types is preserved in the limit and that the interface between the types consists of a single point.

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1 Introduction

1.1 The symbiotic branching model and its interface

In [EF04] Etheridge and Fleischmann introduce a spatial population model that describes the evolution of two interacting types. On the level of a particle approximation, the dynamics follows locally a branching process, where each type branches with a rate proportional to the frequency of the other type. Additionally, types are allowed to migrate to neighbouring colonies. In the continuum space and large population limit, the rescaled numbers of the respective types \( u_t^{[\gamma]}(x) \) and \( v_t^{[\gamma]}(x) \) are given by the nonnegative solutions of the system of stochastic partial differential equations

\[
\begin{align*}
\frac{\partial}{\partial t} u_t^{[\gamma]}(x) &= \frac{\Lambda}{2} u_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x)v_t^{[\gamma]}(x)} \dot{W}_t^{(1)}(x), \\
\frac{\partial}{\partial t} v_t^{[\gamma]}(x) &= \frac{\Lambda}{2} v_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x)v_t^{[\gamma]}(x)} \dot{W}_t^{(2)}(x),
\end{align*}
\]

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with suitable nonnegative initial conditions $u_0(x) \geq 0, v_0(x) \geq 0, x \in \mathbb{R}$. Here, $\gamma > 0$ is the branching rate and $(\hat{W}^{(1)}, \hat{W}^{(2)})$ is a pair of correlated standard Gaussian white noises on $\mathbb{R} \times \mathbb{R}$ with correlation governed by a parameter $\rho \in [-1, 1]$. Existence (for $\rho \in [-1, 1]$) and uniqueness (for $\rho \in [-1, 1]$) was proved in [EF04] for a large class of initial conditions.

The model generalizes several well-known examples of spatial population dynamics. Indeed, for $\rho = -1$ and $u_0 = 1 - v_0$, the system reduces to the continuous-space stepping stone model analysed in [TR95], while for $\rho = 0$, the system is known as the mutually catalytic model due to Dawson and Perkins [DP98]. Finally, for $\rho = 1$ and the extra assumption $u_0 = v_0$, the model is an instance of the parabolic Anderson model, see for example [Mue91].

One of the central questions is how the local dynamics, where one type will eventually dominate over the other, interacts with the migration to shape the global picture. A particularly interesting situation is when initially both types are spatially separated and one would like to know how one type ‘invades’ the other, in other words we would like to understand the interface between the two types. Mathematically, this corresponds to ‘complementary Heaviside initial conditions’, i.e.

$$u_0(x) = 1_{\mathbb{R}^-}(x) \quad \text{and} \quad v_0(x) = 1_{\mathbb{R}^+}(x), \quad x \in \mathbb{R}.$$

**Definition 1.1.** The interface at time $t$ of a solution $(u_t^{[\gamma]}, v_t^{[\gamma]})_{t \geq 0}$ of the symbiotic branching model $cSBM(\varrho, \gamma)_{u_0, v_0}$ with $\rho \in [-1, 1], \gamma > 0$ is defined as

$$\text{If}_{c_t} = \{x \in \mathbb{R} : u_t^{[\gamma]}(x)v_t^{[\gamma]}(x) > 0\},$$

where $\text{cl}(A)$ denotes the closure of the set $A$ in $\mathbb{R}$.

The first question that arises is whether this interface is non-trivial. Indeed, in [EF04] it is shown that the interface is a compact set and moreover that the width of the interface grows at most linearly in $t$. This result is strengthened in [BDE11] Thm. 2.11 for all $\rho$ close to $-1$ by showing that the width is at most of order $\sqrt{t \log(t)}$.

Especially the latter bound on the width seems to suggest diffusive behaviour for the interface. This conjecture is supported by the following scaling property, see [EF04] Lemma 8:

If we rescale time and space diffusively, i.e. if given $K > 0$ we set

$$(u_t^{(K)}(x), v_t^{(K)}(x)) := (u_{K^2t}^{[\gamma]}(Kx), v_{K^2t}^{[\gamma]}(Kx)) \quad \text{for} \quad x \in \mathbb{R}, t \geq 0,$$

then this defines a solution to $cSBM(\varrho, K\gamma)_{u_0^{(K)}, v_0^{(K)}}$ with correspondingly transformed initial states $(u_0^{(K)}, v_0^{(K)})$.

Provided that the initial conditions are invariant under diffusive rescaling, a diffusive rescaling of the system is equivalent (in law) to a rescaling of just the branching rate. Since the complementary Heaviside initial conditions are invariant, we will in the following always consider the limit $\gamma \to \infty$. This scaling then includes the diffusive rescaling, while also giving us the flexibility to consider more general initial conditions.

For the continuous space model this programme has been carried out in [BHO16]. We define the measure-valued processes

$$\mu_t^{[\gamma]}(dx) := u_t^{[\gamma]}(x) \, dx, \quad \nu_t^{[\gamma]}(dx) := v_t^{[\gamma]}(x) \, dx$$  \quad (2)
obtained by taking the solutions of cSBM(ϱ, γ)u₀,v₀ as densities, where the initial conditions remain fixed. The following result was proved in [BHO16 Thm. 1.10]. Here and in the following, if S = ℝ or S = ℤᵈ with d ∈ ℕ we denote by Mtem(S) the space of tempered measures on S, and by Mrap(S) the space of rapidly decreasing measures. Similarly, B⁺ₙₖ(S) (resp. B⁺ₙₖ(S)) denotes the space of nonnegative, tempered (resp. rapidly decreasing) measurable functions on S. We collect all the relevant formal definitions in Appendix A.1.

**Theorem 1.2 (BHO16).** Let ϱ ∈ (−1, 0). Suppose the initial conditions satisfy (u₀, v₀) ∈ B⁺ₙₖ(ℝ)² resp. (u₀, v₀) ∈ B⁺ₙₖ(ℝ)², and for each γ > 0 we let (uᵣ⁺, vᵣ⁺)₀≥₀ be the solution to cSBM(ϱ, γ)u₀,v₀. Then as γ → ∞, the measure-valued process (µ⁺ₖ, ν⁺ₖ)₀≥₀ defined by [2] converges in law in D[₀,∞)(Mtem(ℝ)²) resp. in D[₀,∞)(Mrap(ℝ)²) equipped with the Meyer-Zheng ‘pseudo-path’ topology to a measure-valued process (µ̆, ν̆)₀≥₀ satisfying the following separation-of-types condition: For any x ∈ ℝ, t ∈ (0, ∞) we have

$$\mathbb{E}_{µ₀,v₀}[Sₜµ(x)Sₜν(x)] → 0, \quad as \: ε → 0, \quad (3)$$

where (Sₜ)ₜ≥₀ denotes the heat semigroup.

**Remark 1.3.** (a) We call the limit (µₜ, νₜ)ₜ≥₀ the continuous-space infinite rate symbiotic branching model cSBM(ϱ, ∞)u₀,v₀.

(b) We recall the definition of the Meyer-Zheng ‘pseudo-path’ topology in Appendix A.3. This topology is strictly weaker than the standard Skorokhod topology on D[₀,∞). Under the more restrictive condition that (u₀, v₀) = (1ₜ₋, 1ₜ₊) and ϱ ∈ (−1, −1/√2), we can also show tightness in the stronger Skorokhod topology, so that in particular (µ⁺ₖ, ν⁺ₖ) converges in C[₀,∞)(Mtem(ℝ)²) as γ → ∞, cf. Theorem 1.5 in [BHO16]. Also, we show that in this case, the limiting measures µₜ, νₜ are absolutely continuous with respect to Lebesgue measure and if we denote the densities also by µₜ and νₜ, we can derive the more intuitive separation-of-types condition:

$$µₜ(⋅)νₜ(⋅) = 0 \quad \mathbb{P} \otimes \ell\text{-a.s.} \quad (4)$$

For ϱ = −1 and complementary Heaviside initial conditions, the analogue of Theorem 1.2 was already proved in Tribe [Tri95] for the continuum stepping stone model, as one of the steps of understanding the diffusively rescaled interface. Under these assumptions it was shown that the process (µ⁺ₖ, ν⁺ₖ)₀≥₀ converges weakly for γ → ∞ to

$$(1_{x≤Bₜ}) dx, \: 1_{x≥Bₜ} \: dx)₀≥₀, \quad (5)$$

for (Bₜ)₀≥₀ a standard Brownian motion. Unfortunately, our previous work does not give such a truly explicit characterization of the infinite rate system for ϱ > −1. However, we do have a characterization in terms of a martingale problem (which we will recall below). This allows us to show that the limit is not of the form [5], see Remark 1.14 in [BHO16], even if we allow the position to be a general diffusion rather than a Brownian motion. In fact, even the case ϱ = −1 with general initial conditions is not covered by [Tri95]. However, this case is taken up in the work [HOV16], where we show in particular that for complementary initial conditions which do not necessarily sum up to one, the interface of the infinite rate limit moves like a Brownian motion with drift.
For $\varrho > -1$, in order to take a first step towards a more explicit characterization of the limit in Theorem 1.2, our aim in this paper is to make the connection to related results on the discrete lattice $\mathbb{Z}$. We first recall that for any $d \in \mathbb{N}$, the discrete-space finite rate symbiotic branching model on $\mathbb{Z}^d$ is given by the nonnegative solutions $((u_t(x), v_t(x)), x \in \mathbb{Z}^d, t \geq 0)$ of
\[
\begin{cases}
    du_t(x) = \frac{\Delta}{2} u_t(x) dt + \sqrt{\gamma u_t(x)} v_t(x) dW^{(1)}_t(x), \\
v_t(x) = \frac{\Delta}{2} v_t(x) dt + \sqrt{\gamma u_t(x)} v_t(x) dW^{(2)}_t(x),
\end{cases}
\]
with suitable nonnegative initial conditions $u_0(x) \geq 0, v_0(x) \geq 0, x \in \mathbb{Z}^d$. Here, $\gamma > 0$ is the branching rate, $\Delta$ is the discrete Laplace operator, defined for any $f : \mathbb{Z}^d \to \mathbb{R}$ as
\[
\Delta f(x) := \sum_{y : |y-x|=1} (f(y) - f(x)), \quad x \in \mathbb{Z}^d,
\]
and the pair $(W^1(x), W^2(x))$ is a $\varrho$-correlated two-dimensional Brownian motion which is independent for each $x \in \mathbb{Z}^d$.

Prior to our work, but also inspired by the scaling property for the continuous model, Klenke and Mytnik consider this discrete space model, where the branching rate is sent to infinity. Indeed, in a series of papers [KM10, KM12a, KM12b] show that a non-trivial limiting process exists for $\gamma \to \infty$ (on the lattice) and study its long-term properties. Moreover, Klenke and Oeler [KO10] give a Trotter type approximation. Their results concentrate on the case $\varrho = 0$, i.e. the mutually catalytic model, however analogous results have been derived by Döring and Mytnik for the case $\varrho \in (-1, 1)$ in [DM13, DM12]. In analogy with (4), the limiting process satisfies the separation-of-types property, i.e. at each site only one type is present almost surely. We will refer to the limit as the discrete-space infinite rate symbiotic branching model, abbreviated as $dSBM(\varrho, \infty)$.

What makes the results on the lattice especially interesting for our purpose of identifying the continuous infinite rate model is the fact that there is a very explicit description of the limit $dSBM(\varrho, \infty)$ in terms of an infinite system of jump-type stochastic differential equations (SDEs).

As noted in [EF04], the continuous finite rate symbiotic branching model $cSBM(\varrho, \gamma)$ can be obtained as a diffusive time/space rescaling of the discrete model $dSBM(\varrho, \gamma)$. Therefore, it seems natural to expect that by rescaling the discrete system with infinite branching rate diffusively we obtain the infinite rate continuous space system of Theorem 1.2. In other words, we expect that the following diagram (Figure 1) commutes.

Indeed, this will be our first main result in this paper. In future work, we will attempt to exploit this commutativity to give a more explicit description of the limiting object in Theorem 1.2 by rescaling the jump-type SDEs of [KM12a]. As the second main result in this paper, we can deduce from the scaling limit that the continuous model preserves the initial ordering of types in the limit and also that the interface consists of a single point.

## 2 Main results

In order to state our main result, we first recall the martingale problem that characterizes the limit in Theorem 1.2. This martingale problem is very much related to the martingale
Throughout, we use the notation defined in Appendix A.1. We can formulate the martingale problem in both discrete and continuous space simultaneously. Therefore, let $S$ be either $\mathbb{Z}^d$ or $\mathbb{R}$. We recall the self-duality function employed in [EF04]: Let $\varrho \in (-1, 1)$ and if either $(\mu, \nu, \phi, \psi) \in M_{\text{tem}}(S)^2 \times B_{\text{rap}}(S)^2$ or $(\mu, \nu, \phi, \psi) \in M_{\text{rap}}(S)^2 \times B_{\text{tem}}(S)^2$, define
\[
\langle (\mu, \nu, \phi, \psi) \rangle_\varrho := -\sqrt{1 - \varrho} \langle \mu + \nu, \phi + \psi \rangle_S + i \sqrt{1 + \varrho} \langle \mu - \nu, \phi - \psi \rangle_S,
\]
where $\langle \mu, \phi \rangle_S$ denotes the integral $\int_S \phi(x) \mu(dx)$, for $\mu$ a measure and $\phi$ a measurable function. Then, we define the self-duality function $F$ as
\[
F(\mu, \nu, \phi, \psi) := \exp(\langle \mu, \nu, \phi, \psi \rangle_\varrho).
\]

With this notation, we define a martingale problem, which in the continuous setting was called $\text{MP}'$ in [BHO15].

**Definition 2.1** (Martingale Problem $(\text{MP}_F(S))_{\mu_0,\nu_0}$). Fix $\varrho \in (-1, 1)$ and (possibly random) initial conditions $(\mu_0, \nu_0) \in M_{\text{tem}}(S)^2$ (resp. $M_{\text{rap}}(S)^2$). A càdlàg $M_{\text{tem}}(S)^2$-valued (resp. $M_{\text{rap}}(S)^2$-valued) stochastic process $(\mu_t, \nu_t)_{t \geq 0}$ is called a solution to the martingale problem $(\text{MP}_F(S))_{\mu_0,\nu_0}$ if the following holds: There exists an increasing càdlàg $M_{\text{tem}}(S)$-valued (resp. $M_{\text{rap}}(S)$-valued) process $(\Lambda_t)_{t \geq 0}$ with $\Lambda_0 = 0$ and
\[
\mathbb{E}_{\mu_0,\nu_0} [\Lambda_t(dx)] \in M_{\text{tem}}(S) \quad \text{(resp. } \mathbb{E}_{\mu_0,\nu_0} [\Lambda_t(dx)] \in M_{\text{rap}}(S)\text{)}
\]
for all $t > 0$, such that for all test functions $\phi, \psi \in (C_{\text{tem}}^{(2)}(S))^+$ (resp. $\phi, \psi \in (C_{\text{rap}}^{(2)}(S))^+$) the process
\[
F(\mu_t, \nu_t, \phi, \psi) - F(\mu_0, \nu_0, \phi, \psi)
- \frac{1}{2} \int_0^t F(\mu_s, \nu_s, \phi, \psi) \langle (\mu_s, \nu_s, \Delta \phi, \Delta \psi) \rangle_\varrho ds
- 4(1 - \varrho^2) \int_{[0,t] \times S} F(\mu_s, \nu_s, \phi, \psi) \phi(x) \psi(x) \Lambda(ds, dx)
\]
is a martingale, where $\Delta$ denotes the continuum Laplace operator if $S = \mathbb{R}$ and the discrete Laplace operator if $S = \mathbb{Z}^d$. 

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**Figure 1:** A commuting diagram.
In (11) we have interpreted the right-continuous and increasing process \( t \mapsto \Lambda_t(dx) \) as a (locally finite) measure \( \Lambda(ds, dx) \) on \( \mathbb{R}^+ \times S \), via
\[
\Lambda([0, t] \times B) := \Lambda_t(B).
\]

In order to characterize cSBM(\( \varrho, \infty \)), it does not suffice to require that the martingale problem \((\mathcal{MP}_F(\mathbb{R}))^\varrho_{u_0, v_0}\) is satisfied, since it holds for cSBM(\( \varrho, \gamma \)) for arbitrary \( \gamma < \infty \), see e.g. Proposition A.5 in [BHO16]. However, we do get uniqueness if we require additionally that the separation-of-types condition \( (3) \) is satisfied, as we recall from [BHO16, Thm. 1.10] (where the martingale problem \( \mathcal{MP}_F(\mathbb{R}) \) was denoted by \( \mathcal{MP}' \)).

We note that in the discrete context, our martingale problem \((\mathcal{MP}_F(\mathbb{Z}^d))^\varrho\) is not exactly the same as the martingale problem in [KM12a, Thm. 1.1]. Indeed, the main difference is the appearance of the measure \( \lambda \), which, in some sense that can be made precise, characterizes the correlations. The reason why we need this extra term in the continuous case can be understood if we recall that the martingale problem \( \mathcal{MP}_F \) is tailored to an application of a self-duality (introduced in this context by Mytnik [Myt98]), which characterizes the finite-dimensional distributions. In the discrete context it suffices to consider test functions \( \phi, \psi \) that satisfy \( \phi(x)\psi(x) = 0 \) for all \( x \in \mathbb{Z}^d \), see Corollary 2.4 in [KM10]. However, the same arguments do not carry over to the continuous space, where we need arbitrary test functions \( \phi, \psi \in \mathcal{C}^{(2)}(\mathbb{R}) \) (resp. \( \phi, \psi \in \mathcal{C}^{(2)}(\mathbb{R}) \) ).

But we note that obviously any solution of our martingale problem \( \mathcal{MP}_F(\mathbb{Z}^d) \) (together with separation-of-types) satisfies the martingale problem of Theorem 1.1 in [KM12a] (respectively Theorem 4.4 in [DM12] for general \( \varrho \)). So as a first preliminary result, we show that the converse is also true and that there is a unique solution to the discrete analogue of the martingale problem in [BHO16]. Moreover, we allow for more general initial conditions.

We combine the existence and uniqueness result for both the discrete and the continuous case in the following theorem, where for a measure \( \nu \) on \( \mathbb{Z}^d \) we write \( \nu(k) \) instead of \( \nu(\{k\}) \).

**Theorem 2.2.** Assume that \( \varrho \in (-1, 0) \). Consider \( S \in \{\mathbb{Z}^d, \mathbb{R}\} \).

a) For all initial conditions \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(S)^2 \) (resp. \((u_0, v_0) \in \mathcal{M}_{\text{rap}}(S)^2 \) ), there exists a unique solution \((\mu_t, \nu_t)_{t \geq 0} \) to the martingale problem \((\mathcal{MP}_F(S))^\varrho_{u_0, v_0} \) satisfying the separation-of-types property in the sense that

- if \( S = \mathbb{Z}^d \), then for all \( t > 0 \) and \( k \in \mathbb{Z}^d \) we have
  \[
  \mu_t(k)\nu_t(k) = 0 \quad \mathbb{P}_{\mu_0, \nu_0}-a.s.;
  \]
  \[
  \text{(12)}
  \]
- if \( S = \mathbb{R} \), then for all \( t > 0 \) and \( x \in \mathbb{R} \) we have
  \[
  S_{t+\varepsilon}\mu_0(x)S_{t+\varepsilon}\nu_0(x) \geq \mathbb{E}_{\mu_0, \nu_0}[S_{\varepsilon}\mu_t(x)S_{\varepsilon}\nu_t(x)] \xrightarrow{\varepsilon \to 0} 0.
  \]
  \[
  \text{(13)}
  \]
Moreover, the solution is a strong Markov process.

b) Let \((u_0, v_0) \in \mathcal{B}_{\text{tem}}^+(S)^2 \) (resp. \((u_0, v_0) \in \mathcal{B}_{\text{rap}}^+(S)^2 \) ). For each \( \gamma > 0 \) denote by \((u^{(\gamma)}_t, v^{(\gamma)}_t)_{t \geq 0} \) the solution to SBM(\( \varrho, \gamma \))_{u_0, v_0} , considered as measure-valued processes. Then, as \( \gamma \uparrow \infty \), the processes \((u^{(\gamma)}_t, v^{(\gamma)}_t)_{t \geq 0} \) converge in law in \( D_{[0, \infty)}(\mathcal{M}_{\text{tem}}(S)^2) \) (resp. in \( D_{[0, \infty)}(\mathcal{M}_{\text{rap}}(S)^2) \) ) equipped with the Meyer-Zheng “pseudo-path” topology to the unique solution of the martingale problem \((\mathcal{MP}_F(S))^\varrho_{u_0, v_0} \) satisfying the separation-of-types condition.

Moreover, the solution is a strong Markov process.
We call the unique solution to the martingale problem \((\mathbf{MP}_F(S))^\circ\) satisfying [12] resp. [13] the infinite rate symbiotic branching process and denote it by \(d\text{SBM}(\varrho, \infty)\) if \(S = \mathbb{Z}^d\) and by \(c\text{SBM}(\varrho, \infty)\) if \(S = \mathbb{R}\).

**Remark 2.3.** As noted above, for the discrete case our martingale problem is more restrictive than the version of [KM12a, DM12], since we require the martingale property to hold for a larger class of test functions. Thus, for \(S = \mathbb{Z}^d\) our theorem generalizes their results in two ways: We show that their solution also satisfies our stronger martingale problem. Further, we allow for more general initial conditions since we do not require the types to be separated initially, while [KM12a, DM12] assume that \(\mu_0(k)\nu_0(k) = 0\) for all \(k \in \mathbb{Z}^d\).

Under this condition, by uniqueness our solution coincides of course with the infinite rate process constructed in [KM12a] and [DM12].

Nevertheless, the work in [KM12a] goes substantially beyond what we claim here in the sense that they are also able to show that the solution of \(d\text{SBM}(\varrho, \infty)\) can be characterized as a solution to a jump-type SDE, see [KM12a, Thm 1.3] for \(\varrho = 0\) and [DM12, Prop. 4.14] for \(\varrho \neq 0\). Moreover, [KM12a] considers more general operators than the discrete Laplacian. Also, they define solutions as taking values in a Liggett-Spitzer space (characterized by a suitable test function \(\beta : \mathbb{Z}^d \to \mathbb{R}^+\), whereas we follow [DP98] in using tempered measures as state space. By choosing \(\beta\) in a suitable way, one can show that for initial conditions that satisfy [12] our solution agrees with theirs.

Now we can finally state the main result of our paper, which says that for \(\varrho \in (-1,0)\) the (one-dimensional) discrete-space infinite rate model \(d\text{SBM}(\varrho, \infty)\) converges under diffusive rescaling (in the Meyer-Zheng sense) to the continuous-space model \(c\text{SBM}(\varrho, \infty)\) introduced in [BHO16]. More precisely, given initial conditions \((\mu_0, \nu_0)\) for \(c\text{SBM}(\varrho, \infty)\) we define for each \(n \in \mathbb{N}\) initial conditions \((u_0^{(n)}, v_0^{(n)})\) for \(d\text{SBM}(\varrho, \infty)\) by

\[
\begin{align*}
\mu_0^{(n)}(k) &:= n \mu_0 \left( \frac{k}{n}, \frac{k+1}{n} \right) \quad \text{and} \quad v_0^{(n)}(k) := n \nu_0 \left( \frac{k}{n}, \frac{k+1}{n} \right), \quad k \in \mathbb{Z}. \tag{14}
\end{align*}
\]

It is easy to see that for \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)), we have \((u_0^{(n)}, v_0^{(n)}) \in \mathcal{M}_{\text{tem}}(\mathbb{Z})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{Z})^2\)). Denote by \((u_t^{(n)}, v_t^{(n)})_{t \geq 0}\) the solution to \(d\text{SBM}(\varrho, \infty)\) by \((u_0^{(n)}, v_0^{(n)})\).

We define a sequence \((\mu_t^{(n)}, \nu_t^{(n)})_{t \geq 0}\) of approximating processes for \(c\text{SBM}(\varrho, \infty)\) by diffusive rescaling, as follows: For any Borel subset \(B \subseteq \mathbb{R}\), let

\[
\begin{align*}
\mu_t^{(n)}(B) &:= \frac{1}{n} \sum_{k \in \mathbb{Z}} u_t^{(n)}(k) \mathbbm{1}_B(k/n) \quad \text{and} \quad \nu_t^{(n)}(B) := \frac{1}{n} \sum_{k \in \mathbb{Z}} v_t^{(n)}(k) \mathbbm{1}_B(k/n), \quad t \geq 0. \tag{15}
\end{align*}
\]

Observe that for each \(n \in \mathbb{N}\), the measures \(\mu_t^{(n)}\) are concentrated on the scaled lattice \(\frac{1}{n}\mathbb{Z}\), with \(\frac{1}{n}u_t^{(n)}(n \cdot)\) as density w.r.t. counting measure, and analogously for \(\nu_t^{(n)}\). Considered as discrete measures on \(\mathbb{R}\), it is easy to see that indeed \((\mu_t^{(n)}, \nu_t^{(n)})_{t \geq 0}\) takes values in \(D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^2)\) (resp. \(D_{[0,\infty)}(\mathcal{M}_{\text{rap}}(\mathbb{R})^2)\)). Also note that from [14], it is clear that

\[
(\mu_0^{(n)}, \nu_0^{(n)}) \to (\mu_0, \nu_0) \tag{16}
\]

in \(\mathcal{M}_{\text{tem}}(\mathbb{R})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)) as \(n \to \infty\).

**Theorem 2.4.** Let \(\varrho \in (-1,0)\) and consider initial conditions \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)). For each \(n \in \mathbb{N}\) denote by \((u_t^{(n)}, v_t^{(n)})_{t \geq 0}\) the solution to \(d\text{SBM}(\varrho, \infty)\) from
Theorem 2.2 for $S = \mathbb{Z}$, with initial conditions $(u_0^{(n)}, v_0^{(n)})$ defined by (14). Then as $n \to \infty$, the sequence of processes $(\mu_t^{(n)}, \nu_t^{(n)})_{t \geq 0}$ from (15) converges weakly in $D_{(0, \infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^2)$ (resp. $D_{(0, \infty)}(\mathcal{M}_{\text{rap}}(\mathbb{R})^2)$) equipped with the Meyer-Zheng 'pseudo-path' topology to the unique solution $(\mu_t, \nu_t)_{t \geq 0}$ of cSBM($\varrho, \infty)_{\mu_0, \nu_0}$ from Theorem 2.2 for $S = \mathbb{R}$.

Remark 2.5. For the convergence result of Theorem 2.4 it is of course not essential that the initial conditions for the rescaled discrete model are given exactly as in (14). For example, (14) may be replaced by

$$u_0^{(n)}(k) := n \mu_0 \left( \left( \frac{k}{n}, \frac{k+1}{n} \right] \right) \quad \text{and} \quad v_0^{(n)}(k) := n \nu_0 \left( \left( \frac{k}{n}, \frac{k+1}{n} \right] \right), \quad k \in \mathbb{Z}. \tag{17}$$

What we really need is that $(u_0^{(n)}, v_0^{(n)})$ be defined in such a way that (16) holds.

In order to state our next result, whose proof is an application of the convergence in Theorem 2.4, we need some more notation and definitions: For a Radon measure $\mu$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we denote by $\text{supp}(\mu)$ its measure-theoretic support, i.e.

$$\text{supp}(\mu) := \{ x \in \mathbb{R} : \mu(B(x)) > 0 \text{ for all } \varepsilon > 0 \}. \tag{18}$$

Further, let

$$L(\mu) := \inf \text{supp}(\mu) \in \mathbb{R}, \quad R(\mu) := \sup \text{supp}(\mu) \in \mathbb{R}$$

denote the leftmost resp. rightmost point in the support of $\mu$. Note that $\mu = 0$ if and only if $\text{supp}(\mu) = \emptyset$, which is equivalent to $L(\mu) = \infty$, $R(\mu) = -\infty$. The measure $\mu$ is called strictly positive if its support is the whole real line, or equivalently if it is non-zero on every non-empty open set.

Theorem 2.6. Suppose $\varrho \in (-1, 0)$, and assume initial conditions $(\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2$ or $\mathcal{M}_{\text{rap}}(\mathbb{R})^2$ which are mutually singular and such that $R(\mu_0) \leq L(\nu_0)$. Assume further that $\mu_0 + \nu_0$ is not the zero measure. Let $(\mu_t, \nu_t)_{t \geq 0}$ denote the solution to cSBM($\varrho, \infty)_{\mu_0, \nu_0}$. Then the following holds:

a) The process $(\mu_t, \nu_t)_{t \geq 0}$ preserves the initial ordering of types in the sense that, almost surely,

$$R(\mu_t) \leq L(\nu_t) \quad \text{for all } t \geq 0. \tag{18}$$

b) For all fixed $t > 0$, almost surely, the measures $\mu_t$ and $\nu_t$ are mutually singular and have a single-point interface in the sense that

$$R(\mu_t) = L(\nu_t),$$

and the sum $\mu_t + \nu_t$ is strictly positive.

Remark 2.7. Of course, Theorem 2.6 holds in particular for complementary Heaviside initial conditions $(\mu_0, \nu_0) = (\mathbb{I}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$ as considered in [BHO16]. Its proof proceeds by first showing the analogous result for the discrete-space model and then using the convergence result of Theorem 2.4 see Section 3 below. Note that the property (18) holds pathwise on a set of probability one, while the second part of the theorem is restricted to fixed times and does not ensure existence of an ‘interface process’ $(I_t)_{t \geq 0}$ such that almost surely we have $I_t := R(\mu_t) = L(\nu_t)$ for all $t > 0$. However, the restriction to fixed times applies also to the
symmetric (continuous-time) random walk if $S$, i.e. the semigroup of standard Brownian motion if $R$ be either $\pm$ close' to $S$ formulates these results, it is convenient to introduce the following notation: Again let $(\tilde{S},\mu)$ can show that $SBM(\varrho,\mu)$ is a generalization of [DM13, Thm. 1.2], who consider the discrete model. In fact, we $\tau$ where $\varrho$ is chosen sufficiently small.

For the proof of our convergence result Thm. 2.4, we need that in both discrete and continuous space, $SBM(\varrho,\mu)$ has $(2+\varepsilon)$-th moments if $\varrho < 0$ and $\varepsilon$ is chosen sufficiently small. This is a generalization of [DM13, Thm. 1.2], who consider the discrete model. In fact, we can show that $SBM(\varrho,\mu)$ has finite $p$-th moments for any $p > 2$ such that $\varrho$ is 'sufficiently close' to $-1$ w.r.t. $p$. Moreover, second moments can be calculated explicitly. In order to formulate these results, it is convenient to introduce the following notation: Again let $S$ be either $R$ or $Z^d$ for some $d \in N$, and let $(S_t)_{t \geq 0}$ denote the usual heat semigroup on $S$, i.e. the semigroup of standard Brownian motion if $S = R$ and the semigroup of simple symmetric (continuous-time) random walk if $S = Z^d$. Further, we write $(S_t^{(2)})_{t \geq 0}$ for the corresponding two- resp. $2d$-dimensional semigroup on $S^2$. Finally, we define a semigroup $(\tilde{S}_t)_{t \geq 0}$ of the respective process killed upon hitting the diagonal in $S^2$, i.e.

$$\tilde{S}_t f(x,y) := E_{x,y} \left[ f(X_t^{(1)},X_t^{(2)}) \mathbf{1}_{\{t < \tau^{1,2}\}} \right], \quad f : S^2 \to R, \ (x,y) \in S^2,$$

where $\tau^{1,2} := \inf\{t > 0 : X_t^{(1)} = X_t^{(2)}\}$ denotes the first hitting time of the diagonal. Here $(X^{(1)},X^{(2)})$ denotes a simple symmetric (continuous-time) $2d$-dimensional random walk if $S = Z^d$ and a two-dimensional standard Brownian motion if $S = R$. We remark that $\tilde{S}_t$ is symmetric and in the continuous case has a transition density $\tilde{\pi}_t$ w.r.t. Lebesgue measure which can be expressed in terms of the usual heat kernel, see (78) in the appendix. Thus we can also let $\tilde{S}_t$ act on tempered or rapidly decreasing measures $m$ on $R^2$ via

$$\tilde{S}_t m(x,y) := \int_{R^2} \tilde{\pi}_t(x,y; a,b) m(d(a,b)), \quad (x,y) \in R^2,$$

and from (78) it is easy to see that the function $\tilde{S}_t m$ is continuous and vanishes on the diagonal in $R^2$.

**Proposition 2.8 (Moments of $SBM(\varrho,\infty)$).** Assume $\varrho \in (-1,0)$. Consider initial conditions $(u_0, v_0) \in B_{tem}(S)^2$ (resp. $(u_0, v_0) \in B_{rap}(S)^2$), and let $(\mu_t, \nu_t)_{t \geq 0}$ denote the solution of $SBM(\varrho,\infty)_{u_0,v_0}$.

a) We have the explicit second moment formulas

$$E_{u_0,v_0} [\langle \mu_t, \phi \rangle S \langle \nu_t, \psi \rangle S] = \langle \phi \otimes \psi, \tilde{S}_t(u_0 \otimes v_0) \rangle_{S^2},$$

(20)
For all $t > 0$ and test functions $\phi, \psi \in \mathcal{C}^\infty_\lambda(S)$ (resp. $\mathcal{C}^\infty_{-\lambda}(S)$).

b) Let $p > 2$ such that $\rho + \cos(\pi/p) < 0$. Then we have finiteness of $p$-th moments

$$
\sup_{r \in [0,T]} \mathbb{E}_{u_0,v_0} [\langle \mu_r, \phi \rangle_{S}^{|P|}] < \infty, \quad \sup_{r \in [0,T]} \mathbb{E}_{u_0,v_0} [\langle \nu_r, \phi \rangle_{S}^{|P|}] < \infty
$$

for all $T > 0$ and test functions $\phi, \psi \in \mathcal{C}^\infty_\lambda(S)$ (resp. $\mathcal{C}^\infty_{-\lambda}(S)$).

Remark 2.9. a) In the discrete-space case $S = \mathbb{Z}^d$, and for integrable initial conditions with disjoint support, the mixed second moment formula in (20) is already known, see [DM13, Thm. 1.2]. Moreover, they state estimate (22) only for the total masses, i.e. $\phi \equiv 1$, and for $p = 2 + \varepsilon$. The key observation for the proof of Proposition 2.8 (also due to [DM13]) is that if $p > 2$ and $\rho + \cos(\pi/p) < 0$, then $p$-th moments of the finite rate processes $\text{SBM}(\rho, \gamma)$ are bounded uniformly in $\gamma > 0$, see Corollary 3.8 below.

b) Generalizing (20)-(21), an explicit expression for the moments of $\text{SBM}(\rho, \gamma)$ is in fact available for all integer values $p = n > 2$ such that $\rho + \cos(\pi/n) < 0$, which is however much more involved than for second moments. See [HOV16] where this is proved by establishing a new moment duality for $\text{SBM}(\rho, \infty)$.

Remark 2.10. The reader will have noticed that in all our results, we have omitted the boundary case $\rho = -1$. The reason is that in both discrete and continuous space, different techniques are required, since we can no longer use the self-duality with respect to the function $F$ from (9) to show uniqueness. Instead, if $\rho = -1$ and the initial conditions satisfy $u_0 + v_0 = 1$, then one can use the self-duality with a system of coalescing Brownian motions, see [Tri95]. However, if $u_0 + v_0 \neq 1$ a new approach is needed (as remarked in [DM13]). This challenge is taken up in [HOV16] where we construct $\text{SBM}(-1, \infty)$ for general initial conditions, using a new moment duality instead of the self-duality to establish uniqueness. Using these techniques, one can show that all results in this section, in particular the convergence in Theorem 2.4 continue to hold for $\rho = -1$ as well.

The remaining paper is structured as follows: In Section 3 we prove Theorem 2.2 and Proposition 2.8. Then, in Section 4 we show our main result Theorem 2.4. Finally, as an application of the convergence result of Theorem 2.4, in Section 5 we prove Proposition 2.6.

Notation: We have collected some of the standard facts and notations about measure-valued processes in Appendix A.1. In Appendix A.2 we recall some standard results for the (killed) heat semigroup and in Appendix A.3 we recall the Meyer-Zheng “pseudo-path” topology. Throughout this paper, we will denote by $c, C$ generic constants whose value may change from line to line. If the dependence on parameters is essential we will indicate this correspondingly.
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3 Existence, uniqueness and properties of SBM(ϱ, ∞)

In this section, we sketch the proof of Theorem 2.2, i.e. existence and uniqueness (subject to the separation-of-types property) of the solution to the martingale problem (MPF(S))_{t \geq 0, r_0} for S \in \{\mathbb{Z}^d, \mathbb{R}\}, \rho \in (-1, 0) and general initial conditions (not necessarily mutually singular). The solution is given as the \( \gamma \uparrow \infty \)-limit in the Meyer-Zheng topology of the finite rate processes SBM(ρ, γ). The continuous space version of Theorem 2.2 (apart from the strong Markov property) was already proved in [BHO10], and the discrete space case is very similar. In order to make this note self-contained and for the convenience of the reader, we restate the main steps of the proof in Section 3.1. This reminder also offers some guidance for Section 4 below, where we use a similar general strategy for the proof of the convergence of the discrete to the continuous model. In Section 3.2, we prove some additional properties of SBM(ρ, ∞) (in particular the moment results in Proposition 2.8), which will be used in Section 4 below, and the continuous-space versions of which were not contained in our earlier paper [BHO10]. Therefore, in Section 3.2 we provide some somewhat more detailed proofs.

We begin with some preliminaries on the finite rate model. Let S \in \{\mathbb{Z}^d, \mathbb{R}\}. For initial conditions \((u_0, v_0) \in B^{+}\text{tem}(S)^2\) resp. \(B^{+}\text{rap}(S)^2\), we denote by \((u^{[\gamma]}_t, v^{[\gamma]}_t)_{t \geq 0}\) the solution to SBM(ρ, γ) in discrete or continuous space with these initial conditions and finite branching rate γ \in (0, ∞). Considering the solutions as measure-valued processes, we have \((u^{[\gamma]}_t, v^{[\gamma]}_t)_{t \geq 0} \in C_{(0,\infty)}(\mathcal{M}_{\text{rap}}(S))^2\) resp. \(C_{(0,\infty)}(\mathcal{M}_{\text{tem}}(S))^2\). Further, we define a continuous \(\mathcal{M}_{\text{rap}}(S)\)-resp. \(\mathcal{M}_{\text{tem}}(S)\)-valued increasing process \((L^{[\gamma]}_t)_{t \geq 0}\) by

\[
L^{[\gamma]}_t(dx) := \gamma \int_0^t u^{[\gamma]}_s(x)v^{[\gamma]}_s(x) ds dx, \quad t \geq 0,
\]

where \(dx\) denotes counting measure if \(S = \mathbb{Z}^d\) and Lebesgue measure if \(S = \mathbb{R}\). We will also consider \(L^{[\gamma]}\) as a measure \(L^{[\gamma]}(ds, dx)\) on \(\mathbb{R}^+ \times S\) via \(L^{[\gamma]}([0,t] \times B) := L^{[\gamma]}_t(B)\). Recalling that \((S_t)_{t \geq 0}\) denotes the heat semigroup and \(\Delta\) the Laplacian, by the martingale problem formulation of the SPDE (1) and the system of SDEs (6) we have that for all test functions \(\phi, \psi \in \bigcup_{\lambda > 0} C^{(2)}_{-\lambda}(S)\) (resp. \(\bigcup_{\lambda > 0} C^{(2)}_{\lambda}(S)\))

\[
M^{[\gamma]}_t(\phi) := \langle u^{[\gamma]}_0, \phi \rangle_S - \langle u^{[\gamma]}_0, \phi \rangle_S - \frac{1}{2} \int_0^t \langle u^{[\gamma]}_s, \Delta \phi \rangle_S ds,
\]

\[
N^{[\gamma]}_t(\psi) := \langle v^{[\gamma]}_0, \psi \rangle_S - \langle v^{[\gamma]}_0, \psi \rangle_S - \frac{1}{2} \int_0^t \langle v^{[\gamma]}_s, \Delta \psi \rangle_S ds
\]

are square-integrable martingales with quadratic (co-)variation

\[
[M^{[\gamma]}(\phi), M^{[\gamma]}(\phi)]_t = [N^{[\gamma]}(\phi), N^{[\gamma]}(\phi)]_t = \langle L^{[\gamma]}_t, \phi^2 \rangle_S,
\]

\[
[M^{[\gamma]}(\phi), N^{[\gamma]}(\psi)]_t = \rho \langle L^{[\gamma]}_t, \phi \psi \rangle_S
\]

(24)
for all $t > 0$, with $L^\gamma_t$ from (23). Also, we have the Green function representation for SBM($\varrho, \gamma$)$_{u_0,v_0}$, see e.g. [DP98 Thm. 2.2(b)(ii)] or [BHO16 Cor. A.4]: For every $T > 0$ and $\phi, \psi \in \bigcup_{\lambda > 0} C^{-\lambda}(S)$ (resp. $\bigcup_{\lambda > 0} C_\lambda(S)$) we know that

\[
M^\gamma_t(\phi) := \langle u^\gamma_0, S_{T-t} \phi \rangle_S - \langle u_0, S_T \phi \rangle_S, \quad t \in [0,T],
\]
\[
N^\gamma_t(\phi) := \langle v^\gamma_0, S_{T-t} \phi \rangle_S - \langle v_0, S_T \phi \rangle_S, \quad t \in [0,T]
\]

are martingales on $[0,T]$ with quadratic (co-)variation

\[
[M^\gamma_t(\phi), M^\gamma_t(\phi)]_t = [N^\gamma_t(\phi), N^\gamma_t(\phi)]_t = \int_{[0,t] \times S} (S_{T-r} \phi(x))^2 L^\gamma(dx, dx), \quad t \in [0,T]
\]

(26)

\[
[M^\gamma_t(\phi), N^\gamma_t(\psi)]_t = \int_{[0,t] \times S} S_{T-r} \phi(x) S_{T-r} \psi(x) L^\gamma(dx, dx).
\]

(27)

### 3.1 Proof of Theorem 2.2

As in [BHO16], the first step in the proof of Theorem 2.2 is to show tightness of the family of finite rate models SBM($\varrho, \gamma$), $\gamma \in (0, \infty)$.

**Proposition 3.1.** Suppose $\varrho \in [-1, 0)$ and $(u_0, v_0) \in \mathcal{B}^\dagger_{rap}(S)^2$ (resp. $\mathcal{B}^\dagger_{tem}(S)^2$). Then the family of processes $\{(u^\gamma_t, v^\gamma_t, L^\gamma_t) : \gamma > 0\}$ is tight with respect to the Meyer-Zheng topology on $D_{[0,\infty]}(\mathcal{M}^\dagger_{rap}(S)^3)$ (resp. $D_{[0,\infty]}(\mathcal{M}^\dagger_{tem}(S)^3)$).

The key step in the proof of the Meyer-Zheng tightness is the following lemma (corresponding to [BHO16 Lemma 3.1]) which relies crucially on the colored particle moment duality for finite rate symbiotic branching, see [EF04 Prop. 9] for the discrete case and [EF04 Prop. 12] for the continuous case. The estimate shows that (27) is bounded in expectation, uniformly in $\gamma > 0$. Recall that $(S^\dagger_t)_{t \geq 0}$ and $(\tilde{S}_t)_{t \geq 0}$ denote the heat semigroup on $S^2$ and the killed semigroup defined in [19], respectively.

**Lemma 3.2.** Suppose $\varrho \in [-1, 0)$ and $(u_0, v_0) \in \mathcal{B}^\dagger_{rap}(S)^2$ (resp. $\mathcal{B}^\dagger_{tem}(S)^2$). Then for all $t > 0$ and test functions $\phi, \psi \in \bigcup_{\lambda > 0} \mathcal{B}^\dagger_{\lambda}(S)$ (resp. $\bigcup_{\lambda > 0} \mathcal{B}^\dagger_{\lambda}(S)$) we have monotone convergence as $\gamma \uparrow \infty$

\[
\mathbb{E}_{u_0,v_0} \int_{[0,t] \times S} S_{t-r} \phi(x) S_{t-r} \psi(x) L^\gamma(dx, dx) \leq \frac{1}{|\varrho|} \left< \phi \otimes \psi, (S^\dagger_t - \tilde{S}_t)(u_0 \otimes v_0) \right> S^2 < \infty.
\]  

(28)

For a proof in the continuous case $S = \mathbb{R}$, see [BHO16 Lemma 3.1]. The proof for the discrete case $S = \mathbb{Z}^d$ is virtually identical, replacing the Brownian motions by simple symmetric random walks and the corresponding local times. Note that in view of the definition of the semigroup $(\tilde{S}_t)_{t \geq 0}$, the limit in (28) coincides indeed with (35) in [BHO16].

We now give a brief sketch of the proof of Proposition 3.1. In a different but similar setting, in Section 4 we will carry out the full details.
Sketch of the proof of Proposition 3.1. First use the Green function representation \((26)\) combined with the lower bound from \((74)\) (for \(n = 1\)), the Burkholder-Davis-Gundy inequality and the upper bound \((28)\) to derive uniform moment estimates

\[
\sup_{\gamma > 0} \mathbb{E}_{u_0,v_0} \left[ \sup_{0 \leq t \leq T} \langle u_1^{[\gamma]}, \phi \rangle_S^2 \right] < \infty, \quad \sup_{\gamma > 0} \mathbb{E}_{u_0,v_0} \left[ \sup_{0 \leq t \leq T} \langle v_1^{[\gamma]}, \phi \rangle_S^2 \right] < \infty, \quad (29)
\]

\[
\sup_{\gamma > 0} \mathbb{E}_{u_0,v_0} \left[ \sup_{0 \leq t \leq T} \langle L_1^{[\gamma]}, \phi \rangle_S \right] < \infty. \quad (30)
\]

Compare also the proof of Lemma 3.2 below for the strategy how to derive these estimates from Lemma 3.2. As in [BHO16, Prop. 3.3], these estimates in turn imply the compact containment condition for the family of processes \(\{(u_1^{[\gamma]}, v_1^{[\gamma]}, L_1^{[\gamma]})_{t \geq 0} : \gamma > 0\}\). Tightness in the Meyer-Zheng topology is then proved similarly to Proposition 4.3 below, using the martingale problem formulation of SBM\((\varrho, \gamma)\) together with the bounds \((29)-(30)\).

Next, one has to check that limit points of the family \(\{(u_1^{[\gamma]}, v_1^{[\gamma]}, L_1^{[\gamma]})_{t \geq 0} : \gamma > 0\}\) solve the martingale problem \((\text{MP}_F(S))_{u_0,v_0}^\varrho\) and satisfy the separation-of-types property for positive times. The following corresponds to [BHO16 Prop. 4.3]:

**Proposition 3.3.** Let \(\varrho \in [-1,0)\) and \((u_0, v_0) \in (\mathcal{B}^{+}_{\text{rap}}(S))^2\) (resp. \((\mathcal{B}^{+}_{\text{tem}}(S))^2\)). Suppose that \((u_t, v_t, L_t)_{t \geq 0} \in D_{[0,\infty)}(\mathcal{M}^{\text{rap}}(S)^2)\) (resp. \(D_{[0,\infty)}(\mathcal{M}^{\text{tem}}(S)^2)\)) is any limit point with respect to the Meyer-Zheng topology of the family \(\{(u_1^{[\gamma]}, v_1^{[\gamma]}, L_1^{[\gamma]})_{t \geq 0} : \gamma > 0\}\). Then for all test functions \(\phi, \psi \in C^2_{\text{tem}}(S)^+\) (resp. \(C^2_{\text{rap}}(S)^+\)), the process

\[
\tilde{M}_t(\phi, \psi) := F(u_t, v_t, \phi, \psi) - F(u_0, v_0, \phi, \psi)
- \frac{1}{2} \int_0^t F(u_s, v_s, \phi, \psi) \langle \langle u_s, v_s, \Delta \phi, \Delta \psi \rangle_\varrho \rangle ds
- 4(1 - \varrho^2) \int_{[0,t] \times S} F(u_s, v_s, \phi, \psi) \phi(x) \psi(x) L(ds, dx)
\]

is a martingale, and the process \((L_t)_{t \geq 0}\) satisfies the requirements of Definition 2.1. In particular, \((u_t, v_t)_{t \geq 0}\) solves the martingale problem \((\text{MP}_F(S))_{u_0,v_0}^\varrho\).

As in the proof of [BHO16 Prop. 4.3], this follows from the fact \((31)\) holds for the finite rate model SBM\((\varrho, \gamma)\), by taking the limit \(\gamma \to \infty\). Compare also the proof of Proposition 4.3 below.

The next lemma gives the crucial bound on mixed second moments of limit points, from which the separation-of-types property can be derived as in [BHO16].

**Lemma 3.4** (Moment bounds). Let \(\varrho \in [-1,0)\) and \((u_0, v_0) \in (\mathcal{B}^{+}_{\text{rap}}(S))^2\) (resp. \((\mathcal{B}^{+}_{\text{tem}}(S))^2\)). Suppose that \((u_t, v_t)_{t \geq 0} \in D_{[0,\infty)}(\mathcal{M}^{\text{rap}}(S)^2)\) (resp. \(D_{[0,\infty)}(\mathcal{M}^{\text{tem}}(S)^2)\)) is any limit point with respect to the Meyer-Zheng topology of the family \(\{(u_1^{[\gamma]}, v_1^{[\gamma]}, L_1^{[\gamma]})_{t \geq 0} : \gamma > 0\}\). Then we have for all \(\phi, \psi \in \bigcup_{\lambda > 0} C^+_{\lambda}(S)\) (resp. \(\bigcup_{\lambda > 0} C^+_{\lambda}(S)\)) that

\[
\mathbb{E}_{u_0,v_0} \left[ \langle u_t, \phi \rangle_S \langle v_t, \psi \rangle_S \right] \leq \left\langle \phi \otimes \psi, \tilde{S}_t(u_0 \otimes v_0) \right\rangle_{S^2}. \quad (32)
\]
The mixed second moment estimate (32) is again a consequence of the colored particle moment duality and is proved exactly as in [BHO16, Lemma 4.4], see in particular inequality (51) there. Of course, by Proposition 2.8a), in fact equality holds in (32), which however we can prove only in Subsection 3.2 below.

**Corollary 3.5 (Separation of Types).** Under the assumptions of Lemma 3.4, we have for all $t > 0$ the separation-of-types property (12) if $S = \mathbb{Z}^d$ resp. (13) if $S = \mathbb{R}$.

**Proof.** For the continuous case $S = \mathbb{R}$, see [BHO16, Lemma 4.4] and also the proof of Cor. 4.6 below. For the discrete case $S = \mathbb{Z}^d$ and $k \in S$, simply choose $\phi := \psi := 1_{\{k\}}$ in (32).

By the above results, it is straightforward to show uniqueness in our martingale problem (under the separation-of-types condition), which as in [BHO16] follows from self-duality. Note that up to now, all results on tightness and properties of limit points included also the case $\varrho = -1$. However, in the next proposition we have to exclude this case since for $\varrho = -1$ the self-duality is no longer sufficient to deduce uniqueness.

**Proposition 3.6 (Uniqueness).** Fix $\varrho \in (-1,0)$ and (possibly random) initial conditions $(u_0, v_0) \in \mathcal{M}_{\text{tem}}(S)^2$ or $\mathcal{M}_{\text{rap}}(S)^2$. Then there is at most one solution $(u_t, v_t)_{t \geq 0}$ to the martingale problem $(\mathcal{M}_F(S))^\varrho_{u_0, v_0}$ satisfying the separation-of-types property (12) if $S = \mathbb{Z}^d$ resp. (13) if $S = \mathbb{R}$.

**Proof.** For the case $S = \mathbb{R}$, this is proved in [BHO16, Prop. 5.2], and the case $S = \mathbb{Z}^d$ follows along the same lines. As in [BHO16, Prop. 5.1], one shows first that solutions to $(\mathcal{M}_F(\mathbb{Z}^d))^\varrho$ satisfying the separation-of-types condition are self-dual w.r.t. the function $F$ from (9). For any solution $(u_t, v_t)_{t \geq 0}$ of the martingale problem $(\mathcal{M}_F(\mathbb{Z}^d))^\varrho_{u_0, v_0}$ with initial conditions $(u_0, v_0) \in \mathcal{M}_{\text{tem}}(\mathbb{Z}^d)^2$ and any solution $(\tilde{u}_t, \tilde{v}_t)_{t \geq 0}$ of $(\mathcal{M}_F(\mathbb{Z}^d))^\varrho_{\tilde{u}_0, \tilde{v}_0}$ with $(\tilde{u}_0, \tilde{v}_0) \in \mathcal{M}_{\text{rap}}(\mathbb{Z}^d)^2$, we have

$$E_{u_0, v_0}[F(u_t, v_t, \tilde{u}_t, \tilde{v}_t)] = E_{\tilde{u}_0, \tilde{v}_0}[F(u_0, v_0, u_t, v_t)], \quad t \geq 0.$$ (33)

In fact, for the discrete case the proof simplifies considerably: Since $\mathcal{M}_{\text{tem}}(\mathbb{Z}^d) = B_{\text{tem}}^+(\mathbb{Z}^d)$ and the discrete Laplace operator can be applied directly to the solution $(u_t, v_t)$, we do not need to perform a spatial smoothing via the heat kernel $S_\varepsilon$ as in the proof of [BHO16, Prop. 5.1]. See also the proof of [KMI2a, Prop. 4.7] for the slightly different martingale problem employed in that paper, or the proof of [DP98, Thm. 2.4(b)] for the discrete finite rate model. With the self-duality at hand, uniqueness follows by standard arguments, see e.g. [KMI2a, proof of Prop. 4.1] or [DP98, proof of Thm. 2.4(a)].

We are now ready to finish the proof of Theorem 2.2. For $(u_0, v_0) \in (B_{\text{tem}}^+(S))^2$ or $(B_{\text{rap}}^+(S))^2$, combining Prop. 3.1, Prop. 3.3, Cor. 3.5 and Prop. 3.6 yields convergence of the finite-rate models $\text{SBM}(\varrho, \gamma)_{u_0, v_0}$ to $\text{SBM}(\varrho, \infty)_{u_0, v_0}$ as $\gamma \uparrow \infty$, and in particular also existence and uniqueness (subject to separation-of-types) of solutions to the martingale problem $(\mathcal{M}_F(S))^\varrho_{u_0, v_0}$. Thus Theorem 2.2(b) is fully proved.

For part a) and $S = \mathbb{R}$, it remains to show existence of a solution to $(\mathcal{M}_F(\mathbb{R}))^\varrho_{u_0, v_0}$ satisfying the separation-of-types condition (13) if the initial conditions are from $\mathcal{M}_{\text{tem}}(\mathbb{R})^2.$
rather than from $B_{\text{tem}}^+(\mathbb{R})^2$. This can be done by approximating $(u_0, v_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2$ by absolutely continuous initial conditions $(S_{\epsilon}u_0, S_{\epsilon}v_0) \in B_{\text{tem}}^+(\mathbb{R})^2$ and then showing tightness, properties of limit points and convergence as $\epsilon \downarrow 0$ by the same strategy as above. Alternatively, one can also do the $\gamma \uparrow \infty$- and $\varepsilon \downarrow 0$-limits ‘at the same time’ and repeat all arguments in this subsection with variable initial conditions $(u_0^{[\gamma]}, v_0^{[\gamma]}) := (S_{\gamma^{-1}}u_0, S_{\gamma^{-1}}v_0)$, $\gamma > 0$, instead of fixed ones. Finally, the existence of a solution for initial conditions from $\mathcal{M}_{\text{tem}}(\mathbb{R})^2$ follows also from the proof of our convergence result Theorem 2.4 see Section 4 below.

Finally, in order to show the strong Markov property we argue as in the proof of [DFM+03, Lemma 5.4]: We know that for each $(u_0, v_0) \in \mathcal{M}_{\text{tem}}(\mathcal{S})^2$ resp. $\mathcal{M}_{\text{rap}}(\mathcal{S})^2$, there is a unique solution $(u_t, v_t)_{t \geq 0}$ to $(\mathbf{M}_F(\mathcal{S}))_{u_0, v_0}^t$. For $t > 0$, let $P_t((u_0, v_0); \cdot)$ denote the law of $(u_t, v_t)$ on $\mathcal{M}_{\text{tem}}(\mathcal{S})^2$ resp. on $\mathcal{M}_{\text{rap}}(\mathcal{S})^2$ under the corresponding probability measure $\mathbb{P}_{u_0, v_0}$. Using the self-duality [33] for $\mathcal{S} = \mathbb{Z}^d$ resp. the approximate self-duality [BHO16, eq. (56)] for $\mathcal{S} = \mathbb{R}$ and a monotone class argument, it is easy to see that $(u_0, v_0) \mapsto P_t((u_0, v_0); \cdot)$ is Borel measurable; consequently $P_t$ is a transition kernel. Now the strong Markov property of $(u_t, v_t)_{t \geq 0}$ follows along the same lines as in the proof of [DFM+03, Lemma 5.4].

### 3.2 Further properties of the limit

In this subsection, we prove some additional properties of the infinite rate model $\text{SBM}(\rho, \infty)$, including the identities for second moments from Proposition 2.8.

We start by proving a version of Lemma 3.2 in [DM13] that bounds $p$-th moments of $\text{SBM}(\rho, \gamma)$ uniformly in $\gamma$, provided $\rho$ is ‘sufficiently close’ to $-1$. This result is stated in [DM13] for $p = 2 + \varepsilon$ (in which case it holds for all $\rho < 0$, and which is all that we will need in the present paper), but using the ‘critical curve’ of [BDE11] the proof works in fact for other values of $p$ as well. Also in contrast to [DM13], we do not restrict to integrable initial conditions and do not consider the total mass, but test against suitable test functions.

In the following, for a $\rho$-correlated planar Brownian motion $(W^{(1)}, W^{(2)})$ starting at $(x, y) \in (\mathbb{R}^+)^2$, we denote by $\tau$ the first hitting time of $(W^{(1)}, W^{(2)})$ at the first quadrant $(\mathbb{R}^+)^2$, and by $\mathbb{E}_{x,y}[\cdot]$ the corresponding expectation.

**Lemma 3.7.** Let $\rho \in [-1, 1)$ and $p \geq 1$ such that $\rho + \cos(\pi/p) < 0$. Then there exists a constant $C = C(p)$ only depending on $p$ such that the following holds: For all $(u_0, v_0) \in B_{\text{tem}}^+(\mathcal{S})^2$ (resp. $B_{\text{rap}}^+(\mathcal{S})^2$), $\phi \in \bigcup_{\lambda > 0} C^+(\mathcal{S})$ (resp. $\bigcup_{\lambda > 0} C^+(\mathcal{S})$) and $T > 0$ we have

$$
\sup_{\gamma > 0} \mathbb{E}_{u_0, v_0} \left[ \sup_{t \in [0, T]} \left\langle u_t^{[\gamma]}, S_{T-t} \phi \right\rangle_{\mathcal{S}}^p \right]
\leq C \left( \mathbb{E}_{(u_0, S_T \phi)_{\mathcal{S}}, (v_0, S_T \phi)_{\mathcal{S}}} \left[ \tau^{p/2} \right] + \langle u_0, S_T \phi \rangle_{\mathcal{S}}^p \right) < \infty,
$$

and analogously for $v^{[\gamma]}$.

**Proof.** Fix $T > 0$. By the Green function representation of the finite rate model (see

---

*In [HOV16] however, we use the result also for values $n > 2$.**
Corollary 3.8. Let\footnote{\ref{26}--\ref{27}}, we know that
\[
M_t^{[\gamma,T]}(\phi) := \langle u_t^{[\gamma]}, S_{T-t}\phi\rangle_S - \langle u_0, S_T\phi\rangle_S,
\]
\[
N_t^{[\gamma,T]}(\phi) := \langle v_t^{[\gamma]}, S_{T-t}\phi\rangle_S - \langle v_0, S_T\phi\rangle_S,
\]
t \in [0,T], are continuous square-integrable zero-mean martingales with covariation structure given by
\[
[M^{[\gamma,T]}(\phi), M^{[\gamma,T]}(\phi)]_t = [N^{[\gamma,T]}(\phi), N^{[\gamma,T]}(\phi)]_t = \gamma \int_0^t \langle (S_{T-r}\phi)^2, u_r^{[\gamma]}v_r^{[\gamma]} \rangle_S \, dr,
\]
\[
[M^{[\gamma,T]}(\phi), N^{[\gamma,T]}(\phi)]_t = \varrho \int_0^t \langle (S_{T-r}\phi)^2, u_r^{[\gamma]}v_r^{[\gamma]} \rangle_S \, dr.
\]
By a version of the Dubins-Schwarz Theorem\footnote{see e.g. \cite[Lemma 4.2, Rem. 4.3]{BDE11}}, we can represent \( (M_t^{[\gamma,T]}(\phi), N_t^{[\gamma,T]}(\phi))_{t \in [0,T]} \) (possibly on an enlargement of the underlying probability space) as a time-changed \( \varrho \)-correlated planar Brownian motion \((W^{(1)}, W^{(2)})\), i.e.
\[
(M_t^{[\gamma,T]}(\phi), N_t^{[\gamma,T]}(\phi)) = (W_{A_t}^{(1)}, W_{A_t}^{(2)}), \quad t \in [0,T],
\]
where \( A_t := \gamma \int_0^t \langle (S_{T-s}\phi)^2, u_s^{[\gamma]}v_s^{[\gamma]} \rangle_S \, ds \). Now let \( \tau_Q := \inf \{t > 0 : (W_t^{(1)}, W_t^{(2)}) \notin Q \} \) denote the first hitting time of \((W^{(1)}, W^{(2)})\) of the boundary of the quadrant
\[
Q := \{(x,y) \in \mathbb{R}^2 : x \geq -\langle u_0, S_T\phi\rangle_S \text{ and } y \geq -\langle v_0, S_T\phi\rangle_S \}.
\]
Since \( u^{[\gamma]} \) and \( \phi \) are nonnegative, we have \((M_t^{[\gamma,T]}(\phi), N_t^{[\gamma,T]}(\phi)) \in Q \) and thus \( A_t \leq \tau_Q \) for all \( t \in [0,T] \). Then for each \( p \geq 1 \) we obtain by \ref{34} and the Burkholder-Davis-Gundy inequality that
\[
E_{u_0,v_0} \left[ \sup_{t \in [0,T]} \langle u_t^{[\gamma]}, S_{T-t}\phi\rangle_S^p \right] \leq C_p \left( E_{u_0,v_0} \left[ \sup_{t \in [0,T]} |M_t^{[\gamma,T]}(\phi)|^p \right] + \langle u_0, S_T\phi\rangle_S^p \right) \leq C_p \left( E_{0,0} \left[ \tau_Q^{p/2} \right] + \langle u_0, S_T\phi\rangle_S^p \right) = C_p \left( E_{u_0,v_0,S_T\phi_S} \langle u_0, S_T\phi\rangle_S \left[ \tau_Q^{p/2} \right] + \langle u_0, S_T\phi\rangle_S^p \right),
\]
and the last expectation is finite iff \( \varrho + \cos(\frac{x}{p}) < 0 \), see \cite[Thm. 5.1]{BDE11}. Since the constant \( C_p \) depends only on \( p \), the proof is complete.\qed

By combining the previous lemma with the lower bounds in Lemma\footnote{A.1} for continuous space (see \ref{72}) resp. Lemma\footnote{A.2} for discrete space\footnote{choose \( \eta = 1 \) in \ref{74}}), the following corollary is immediate, where we recall the notation \( \phi_\lambda(x) := e^{-\lambda|x|} \) for \( x \in S, \lambda \in \mathbb{R} \).

**Corollary 3.8.** Let \( \varrho \in [-1,1) \) and \( p \geq 1 \) such that \( \varrho + \cos(\pi/p) < 0 \). Then for each \( T > 0 \) and \( \lambda > 0 \) (resp. \( \lambda < 0 \)) there exists a constant \( C(p,\lambda,T) \) such that for all \( (u_0, v_0) \in B_{\text{tem}}(S)^2 \) (resp. \( B_{\text{rap}}(S)^2 \)) we have
\[
\sup_{T > 0} E_{u_0,v_0} \left[ \sup_{t \in [0,T]} \langle u_t^{[\gamma]}, \phi_\lambda \rangle_S^p \right] \leq C(p,\lambda,T) \left( E_{(u_0,S_T\phi_S),(v_0,S_T\phi_S)} \left[ \tau_Q^{p/2} \right] + \langle u_0, S_T\phi\rangle_S^p \right) < \infty,
\]
and analogously for \( v^{[\gamma]} \).
Proposition 3.9. Let \( \varrho < 0 \) and \( p > 2 \), the above uniform moment bounds allow us to prove Proposition 2.8. They also allow us to extend the martingale representation (24)-(25) and the Green function representation (26)-(27) to the infinite rate limit. Before turning to the proof of Prop. 2.8, we collect these and some additional properties of the limit (which will be of importance in Section 4 below) in the following proposition:

Proposition 3.9. Let \( \varrho \in (-1, 0) \) and \((u_0, v_0) \in \mathcal{B}_{\text{rap}}^+(S)^2\) (respectively \(\mathcal{B}_{\text{tem}}^+(S)^2\)). Then as \( \gamma \uparrow \infty \), the processes \((u_t, v_t, L_t)_{t \geq 0}\) converge with respect to the Meyer-Zheng topology to a process \((u_t, v_t, L_t)_{t \geq 0} \in D_{[0, \infty)}(|\mathcal{M}_{\text{rap}}(S)^2|)\) (respectively \(D_{[0, \infty)}(|\mathcal{M}_{\text{tem}}(S)^2|)\)). The limit has the following properties in addition to those stated in Theorem 2.2 and Proposition 2.8:

a) For all \( \phi, \psi \in \bigcup_{\lambda > 0} C^{(2)}_\lambda(S) \) (resp. \( \bigcup_{\lambda > 0} C^{(2)}_\lambda(S) \)) we have that

\[
M_t(\phi) := \langle u_t, \phi \rangle_S - \langle u_0, \phi \rangle_S - \frac{1}{2} \int_0^t \langle u_s, \Delta \phi \rangle_S \, ds,
\]

\[
N_t(\psi) := \langle v_t, \psi \rangle_S - \langle v_0, \psi \rangle_S - \frac{1}{2} \int_0^t \langle v_s, \Delta \psi \rangle_S \, ds
\]

are square-integrable martingales with quadratic (co-)variation

\[
[M(\phi), M(\phi)]_t = [N(\phi), N(\phi)]_t = \langle L_t, \phi^2 \rangle_S,
\]

\[
[M(\phi), N(\psi)]_t = \varrho \langle L_t, \phi \psi \rangle_S
\]

for all \( t > 0 \).

b) [Green function representation]

For all \( T > 0 \) and \( \phi, \psi \in \bigcup_{\lambda > 0} C^{(2)}_{-\lambda}(S) \) (resp. \( \bigcup_{\lambda > 0} C^2(S) \)) we have that

\[
\langle u_t, S_T \phi \rangle_S = \langle u_0, S_T \phi \rangle_S + M_T^\phi(\phi), \quad \langle v_t, S_T \psi \rangle_S = \langle v_0, S_T \psi \rangle_S + N_T^\psi(\psi)
\]

\( (38) \)

for \( t \in [0, T] \), where \((M_T^\phi(\phi))_{t \in [0, T]}\) and \((N_T^\psi(\psi))_{t \in [0, T]}\) are square-integrable martingales with quadratic (co-)variation

\[
[M_T^\phi(\phi), M_T^\phi(\phi)]_t = [N_T^\phi(\phi), N_T^\phi(\phi)]_t = \int_{[0,t] \times S} (S_{T-r} \phi(x))^2 L(dr, dx),
\]

\[
[M_T^\phi(\phi), N_T^\phi(\psi)]_t = \varrho \int_{[0,t] \times S} S_{T-r} \phi(x) S_{T-r} \psi(x) L(dr, dx)
\]

\( (39) \)

for \( t \in [0, T] \). In particular, we have the uniform second moment bound

\[
\mathbb{E}_{u_0, v_0} \left[ \sup_{t \in [0,T]} |M_T^\phi(\phi)|^2 \right] \leq \frac{4}{\varrho} \left\langle S_T^{(2)}(\phi) \left( S_T^{(2)}(\phi) - \phi \right) \right\rangle_{S^2},
\]

\( (40) \)

and analogously for \( N_T^\psi(\psi) \).

c) The quadratic variation of the (complex-valued) martingale \((\tilde{M}_t(\phi, \psi))_{t \geq 0}\) in (31) is given by

\[
\left[ M(\phi, \psi), \overline{M}(\phi, \psi) \right]_t = 4(1 - \varrho^2) \int_{[0,t] \times S} |F(u_s, v_s, \phi, \psi)|^2 (\phi(x)^2 + \psi(x)^2) L(ds, dx)
\]

\( (41) \)

for all \( t > 0 \).
d) We have the following first moment formula for the process \((L_t)_{t \geq 0}\): For all \(t > 0\) and \(\phi, \psi \in \bigcup_{\lambda > 0} C^+_{\lambda}(S)\) (resp. \(\bigcup_{\lambda > 0} C^\times_{\lambda}(S)\)),

\[
\mathbb{E}_{u_0, v_0} \left[ \int_{[0,t] \times S} S_{t-s} \phi(x) S_{t-s} \psi(x) L(ds, dx) \right] = \frac{1}{|\partial|} \left\langle \phi \otimes \psi, (S_t^{(2)} - \bar{S}_t)(u_0 \otimes v_0) \right\rangle_{S^2}.
\]

(42)

**Proof.** We give the proof for \((u_0, v_0) \in B^+_{\text{tem}}(S)^2\). By Prop. 3.1, we know that the family \(\{(u^\gamma_t, v^\gamma_t, L^\gamma_t)_{t \geq 0} : \gamma > 0\}\) is tight with respect to the Meyer-Zheng topology on \(D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(S)^3)\). Suppose that \((u_t, v_t, L_t)_{t \geq 0} \in D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(S)^3)\) is any limit point.

First of all, we note from the proof of Lemma 3.7 that for \(\varepsilon = \varepsilon(\varrho) > 0\) sufficiently close to zero we have

\[
\sup_{\gamma > 0} \mathbb{E}_{u_0, v_0} \left[ \left( \int_{[0,T] \times S} (S_{T-s} \phi(x))^2 L^{\gamma}(ds, dx) \right)^{(2+\varepsilon)/2} \right] < \infty
\]

for each \(T > 0\) and \(\phi \in \bigcup_{\lambda > 0} C_\lambda(S)\), see in particular (35) (with \(p = 2 + \varepsilon\) chosen such that \(\varrho + \cos(\frac{\pi}{2+\varepsilon}) < 0\)). Using the lower bound \(\mathbb{E}_{u_0, v_0} \left[ \left( \int_{[0,T] \times S} (S_{T-s} \phi(x))^2 L^{\gamma}(ds, dx) \right)^{(2+\varepsilon)/2} \right] \leq C(\phi, \lambda, T) \sup_{\gamma > 0} \mathbb{E}_{u_0, v_0} \left[ \left( \int_{[0,T] \times S} (S_{T-s} \phi(x))^2 L^{\gamma}(ds, dx) \right)^{(2+\varepsilon)/2} \right] \) for all \(T > 0\), \(\phi \in \bigcup_{\lambda > 0} C_\lambda(S)\) and a suitable \(\lambda = \lambda(\varrho) > 0\).

Now let \(M_t^{\gamma}(\phi), N_t^{\gamma}(\phi)\) be the martingales corresponding to (36) for finite \(\gamma > 0\), as defined in (24). Then (along a subsequence which we do not distinguish in notation) \((M_t^{\gamma}(\phi), N_t^{\gamma}(\phi))_{t \geq 0}\) converges to \((M_t(\phi), N_t(\phi))_{t \geq 0}\) w.r.t. Meyer-Zheng, and for each \(t > 0\) we have by (25) and (30) that

\[
\sup_{\gamma > 0} \mathbb{E}_{u_0, v_0} \left[ |M_t^{\gamma}(\phi)|^2 \right] = \sup_{\gamma > 0} \mathbb{E}_{u_0, v_0} \left[ \left( L_t^{\gamma}, \phi^2 \right)_S \right] < \infty.
\]

Applying [MZ84] Thm. 11 we get that the limit \((M(\phi), N(\phi))\) is again a martingale. But in fact, by the Burkholder-Davis-Gundy inequality and (43), we know even that for \(\varepsilon = \varepsilon(\varrho) > 0\) small enough we have

\[
\sup_{\gamma > 0} \mathbb{E}_{u_0, v_0} \left[ |M_t^{\gamma}(\phi)|^{2+\varepsilon} \right] < \infty
\]

for all \(t > 0\). Thus \(M_t^{\gamma}(\phi)\) converges to \(M_t(\phi)\) in \(L^2\). Consequently, we get for the quadratic variation of the martingales that \([M_t^{\gamma}(\phi), M_t^{\gamma}(\phi)]_t = (L_t^{\gamma}, \phi^2)_S\) converges to \([M(\phi), M(\phi)]_t\) in \(L^1\), and we conclude that \([M_t(\phi), M_t(\phi)]_t = (L_t, \phi^2)_S\). The proof for the covariation is the same. This is a).

Note that this implies in particular that the limit point \((L_t)_{t \geq 0}\) of the family \(\{(L_t^{\gamma})_{t \geq 0} : \gamma > 0\}\) is unique, since it is characterized by the covariation structure \((37)\) of the martingales in \(36\) which in turn are uniquely determined by \((u_t, v_t)_{t \geq 0}\), for which we have uniqueness.
by Prop. 4. Therefore we have now also proved that $L^{[\cdot]}_t$ converges in $D_{(0,\infty)}(\mathcal{M}_{tem}(\mathbb{R}))$ w.r.t. the Meyer-Zheng topology, as in the statement of the proposition.

The proof of the Green function representation in b) is very similar to a), using the corresponding property of the finite rate model and uniform boundedness of $(2+\varepsilon)$-th moments of the martingales $(\mathcal{M}^γ_\tau(\phi),\mathcal{N}^γ_\tau(\phi))$ from [26]-[27], which again follows from (43). The uniform second moment bound $\varrho(\phi)$ is a direct consequence of (38)-(39) and the Burkholder-Davis-Gundy inequality.

Similarly, for c) we use that the analogue of formula (41) holds for the finite rate model, as a straightforward calculation using Itô’s formula shows. Then again, estimate (43) allows us to extend the formula to the limit.

Finally, for d) we note that (43) allows us to pass to the limit in formula (28), yielding the first moment expression (42) for $L$.

Now the moment properties in Proposition 2.8 can be proved easily:

**Proof of Proposition 2.8.** Let $\varrho \in (-1,0)$. Then we can choose $\varepsilon = \varepsilon(\varrho) > 0$ sufficiently small such that $\varrho + \cos(\frac{\pi}{1+\varepsilon}) < 0$. Applying Corollary 3.8 with $p = 2+\varepsilon$, the families $\{\langle u_t^{[\cdot]},\phi \rangle_S : \gamma > 0 \}$ and $\{\langle u_t^{[\cdot]},\phi \rangle_S : \gamma > 0 \}$ are uniformly integrable, hence (by Hölder’s inequality) also $\{\langle u_t^{[\cdot]},\phi \rangle_S \langle v_t^{[\cdot]},\psi \rangle_S : \gamma > 0 \}$. Together with convergence of the finite rate processes to the infinite rate process $(u_t,v_t)_{t\geq 0}$, we get

$$
\mathbb{E}_{u_0,v_0} \left[ \langle u_t,\phi \rangle_S \langle v_t,\psi \rangle_S \right] = \lim_{\gamma \uparrow \infty} \mathbb{E}_{u_0,v_0} \left[ \langle u_t^{[\cdot]},\phi \rangle_S \langle v_t^{[\cdot]},\psi \rangle_S \right] = \langle \phi \otimes \psi, \bar{S}_t(u_0 \otimes v_0) \rangle_S^2,
$$

where the last equality follows directly by taking the limit in the finite rate moment duality from [EF04] and the definition of the semigroup $(\bar{S}_t)_{t \geq 0}$. This proves the mixed second moment formula (20). The second moment formulae (21) can be derived in a similar way from the finite rate moment duality. Alternatively, it follows also directly from Prop. 3.9 b) and d), by putting $t = T$ in the Green function representation (38)-(39) and using the covariance structure of the martingales together with the first moment formula (42) for $L$.

Finally, the bound (22) on $p$-th moments follows by an application of Fatou’s lemma upon letting $\gamma \uparrow \infty$ in Corollary 3.8. Thus we have now fully proved Proposition 2.8. \qed

### 4 Convergence of the discrete to the continuous model

In this section, we prove Theorem 2.4. Recall that given initial conditions $(\mu_0,\nu_0) \in \mathcal{M}_{tem}(\mathbb{R})^2$ (resp. $\mathcal{M}_{rap}(\mathbb{R})^2$) for cSBM($\varrho,\infty$), we define $(\mu_t^{(n)},\nu_t^{(n)})_{t \geq 0}$ by (14)-(15), and our goal is to show that $(\mu_t^{(n)},\nu_t^{(n)})_{t \geq 0} \xrightarrow{d} (\mu_t,\nu_t)_{t \geq 0}$, $n \to \infty$, as measure-valued processes, where $(\mu_t,\nu_t)_{t \geq 0}$ denotes the (unique) solution to cSBM($\varrho,\infty$) from Theorem 2.2 with $S = \mathbb{R}$. The general strategy is familiar and similar to Section 3. First we prove tightness, then we show that limit points solve the martingale problem (MP$_F(\mathbb{R})$)$_{\mu_0,\nu_0}$ from Definition 2.1 and the separation-of-types property 1.3.

The proof will consist of a series of lemmas and propositions. We begin with some preliminaries. Note that if $\phi \in \bigcup_{\lambda > 0} \mathcal{B}_\lambda(\mathbb{R})$ (resp. $\bigcup_{\lambda > 0} \mathcal{B}_{-\lambda}(\mathbb{R})$) is a test function and for each
We start with a lemma showing that second moments of \((\mu_t^{(n)}),\nu_t^{(n)})\) and first moments of \(\Lambda_t^{(n)}\) are bounded uniformly in \(n \in \mathbb{N}\).

**Lemma 4.1.** Suppose \(\varrho \in (-1,0)\) and \((\mu_0,\nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)). Then we have for all \(T > 0\) and \(\phi \in \bigcup_{\lambda > 0} \ell^*_\lambda(\mathbb{R})\) (resp. \(\bigcup_{\lambda > 0} C^*_\lambda(\mathbb{R})\)) that

\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \left[ \sup_{t \in [0,T]} \langle \phi, \mu_t^{(n)} \rangle^2 \right] < \infty, \quad \sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \left[ \sup_{t \in [0,T]} \langle \phi, \nu_t^{(n)} \rangle^2 \right] < \infty, \quad (47)
\]

and

\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \left[ \sup_{t \in [0,T]} \langle \phi, \Lambda_t^{(n)} \rangle \right] < \infty. \quad (48)
\]
Proof. We give the proof for \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\). Fix \(T > 0\) and assume w.l.o.g. that \(\phi = \phi_\lambda\) for some \(\lambda > 0\). By Lemma A.2 there is a constant \(C = C(\lambda, T)\) independent of \(n\) such that

\[
\phi_\lambda(k) \leq C d S_{n^2 T - s}(\phi_\lambda(z)(k)) \quad (49)
\]

for all \(s \in [0, n^2 T]\), \(k \in \mathbb{Z}\) and \(n \in \mathbb{N}\). Using this together with the Green function representation for the discrete model (see Proposition 3.9b), with \([0, n^2 T]\) in place of \([0, T]\), we get

\[
\langle \mu_t^{(n)}, \phi_\lambda \rangle = \langle u_t^{(n)}, \phi_\lambda \rangle_{\mathbb{Z}} \leq C \langle u_t^{(n)}, d S_{n^2 T - t} \phi_\lambda \rangle_{\mathbb{Z}} = C \langle u_0^{(n)}, d S_{n^2 T} \phi_\lambda \rangle_{\mathbb{Z}} + M_{n^2 T}^{\phi_\lambda} \quad (50)
\]

for all \(t \in [0, T]\), where the second moment of the martingale term is bounded by

\[
\mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left( \sup_{t \in [0, T]} \left| M_{n^2 T}^{\phi_\lambda} \right|^2 \right) \leq \frac{4}{\|\phi_\lambda\|} \left\langle \phi_\lambda^{(n)} \otimes \phi_\lambda^{(n)}, \left( d S_{n^2 T} - d \tilde{S}_{n^2 T}(u_0^{(n)} \otimes v_0^{(n)}) \right)_{\mathbb{Z}^2} \right\rangle \quad (51)
\]

for all \(n \in \mathbb{N}\) (recall estimate (40)). Combining (50)-(51) with Lemma A.4 in the appendix, the first inequality in (17) follows easily, and the proof of the second one is analogous.

For the increasing process \(\Lambda^{(n)}\), we observe that

\[
\mathbb{E}_{\mu_0^{(n)}, \nu_0^{(n)}} \left( \sup_{t \in [0, T]} \langle \phi_\lambda, \Lambda_t^{(n)} \rangle_{\mathbb{R}} \right) = \frac{1}{n^2} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ \langle \phi_\lambda(z), \Lambda_t^{(n)} \rangle_{\mathbb{Z}} \right] = \frac{1}{n^2} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ \int_{[0, n^2 T] \times \mathbb{Z}} \phi_\lambda/2(z)(k) L^{(n)}(ds, dk) \right] \leq \frac{C}{n^2} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ \int_{[0, n^2 T] \times \mathbb{Z}} \left( d S_{n^2 T - s} \phi_\lambda/2(z)(k) \right)^2 L^{(n)}(ds, dk) \right] = C \left\langle \phi_\lambda^{(n)} \otimes \phi_\lambda^{(n)}, \left( d S_{n^2 T} - d \tilde{S}_{n^2 T}(u_0^{(n)} \otimes v_0^{(n)}) \right)_{\mathbb{Z}^2} \right\rangle,
\]

where we used again estimate (49) (with \(\lambda/2\)) for the inequality and formula (42) for the last equality. Now we see as before in (51) that the RHS of the previous display is bounded uniformly in \(n \in \mathbb{N}\).

□

**Corollary 4.2** (Compact Containment). Suppose \(\varrho \in (-1, 0)\) and \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)). Then the compact containment condition holds for the family of processes \(\{(\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t \geq 0 : n \in \mathbb{N}}\}\), i.e., for every \(\varepsilon > 0\) and \(T > 0\) there exists a compact subset \(K = K_{\varepsilon, T} \subseteq \mathcal{M}_{\text{tem}}(\mathbb{R})\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)) such that

\[
\inf_{n \in \mathbb{N}} P\{\mu_t^{(n)} \in K_{\varepsilon, T} \text{ for all } t \in [0, T]\} \geq 1 - \varepsilon,
\]

and similarly for \(\nu_t^{(n)}\) and \(\Lambda_t^{(n)}\).
Proof. Given the uniform moment bounds from Lemma 4.1, the proof is virtually identical to that of [BHO16, Corollary 3.3].

Proposition 4.3. Suppose \( \varrho \in (-1,0) \) and \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2 \) (resp. \( \mathcal{M}_{\text{rap}}(\mathbb{R})^2 \)). Then the family of processes \( \{ (\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)}) \}_{t \geq 0} : n \in \mathbb{N} \} \) is tight with respect to the Meyer-Zheng topology on \( D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R}))^3 \) (resp. \( D_{[0,\infty)}(\mathcal{M}_{\text{rap}}(\mathbb{R}))^3 \)).

Proof. Suppose \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2 \). We will apply [Kur91, Cor. 1.4] that we recall in Appendix A.3 see after (80) below. This criterion requires us to check the Meyer-Zheng tightness condition (80) for the coordinate processes plus a compact containment condition. Corollary 4.2 takes care of the latter condition so that we only have to check tightness of the coordinate processes.

Let \( \phi \in \mathcal{C}^{(2)}_{\text{rap}}(\mathbb{R})^+ \) and fix \( T > 0 \). By Prop. 3.9a), we know that

\[
\langle \mu_t^{(n)}, \phi \rangle_R = \langle u_n^{(n)}, \phi \rangle_Z + \frac{1}{2} \int_0^{n^2 t} \left( u_s^{(n)} \right)^2 ds + M_{n^2 t}(\phi^{(n)}), \tag{52}
\]

where \( (M_{n^2 t}(\phi^{(n)}))_{t \geq 0} \) is a martingale with second moments bounded uniformly in \( t \in [0,T] \) by

\[
\mathbb{E} u_n^{(n)}, v_0 \left( \sup_{t \in [0,T]} \| M_{n^2 t}(\phi^{(n)}) \|^2 \right) \leq 4 \mathbb{E} u_n^{(n)}, v_0 \left( \langle L_{n^2 t}^{(n)}, (\phi^{(n)})^2 \rangle_Z \right),
\]

where we used the Burkholder-Davis-Gundy inequality. Choosing a suitable \( \lambda > 0 \) and using the lower bound from Lemma A.2, we see that there is a constant \( C = C(\phi, \lambda, T) \) such that the previous display is bounded by

\[
C \mathbb{E} u_n^{(n)}, v_0 \left( \int \left( dS_{n^2 T-S} \phi_{\lambda}^{(n)}(k) \right)^2 L^{(n)}(ds, dk) \right) \leq C \frac{\langle \phi_{\lambda}, \phi_{\lambda}^{(n)} \rangle + \langle dS_{n^2 T-S} \phi_{\lambda}^{(n)} \rangle + \langle u_0^{(n)} \rangle + \langle v_0^{(n)} \rangle}{\langle \phi_{\lambda}, \phi_{\lambda}^{(n)} \rangle + \langle u_0^{(n)} \rangle + \langle v_0^{(n)} \rangle},
\]

where we have also used (42). But the last display is bounded uniformly in \( n \in \mathbb{N} \) by Lemma A.4 hence we get

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E} u_n^{(n)}, v_0 \left( \| M_{n^2 t}(\phi^{(n)}) \|^2 \right) < \infty
\]

for all \( T > 0 \). This implies immediately the Meyer-Zheng tightness condition (80) for the sequence of martingales in (52).

In view of (52), it remains to show tightness of the term

\[
X_t^{(n)} := \int_0^{n^2 t} \langle u_{s}^{(n)} \rangle Z ds = n^2 \int_0^{t} \langle u_{n^2 s}^{(n)} \rangle Z ds, \quad t \in [0,T].
\]

But Lemma 4.4 implies that this term is tight in the stronger Skorokhod topology, as follows: Since \( \phi \in \mathcal{C}^{(2)}_{\text{rap}}(\mathbb{R})^+ \), there is a suitable \( \lambda > 0 \) and some constant \( C = C(\phi) \)
such that \( |d\Delta(\phi^{(n)})(k)| \leq \frac{C}{n^r}\phi_\lambda(\frac{k}{n}) = \frac{C}{n^r}\phi^{(n)}(k) \) for all \( k \in \mathbb{Z}, n \in \mathbb{N} \). Thus we get for \( 0 \leq s < t \leq T \) that

\[
\mathbb{E}_{u_0^{(n)},v_0^{(n)}}[|X_t^{(n)} - X_s^{(n)}|^2] \leq C(\phi) \mathbb{E}_{u_0^{(n)},v_0^{(n)}}\left(\left(\int_s^t \langle u_{n^2r}, \phi^{(n)} \rangle_Z \, dr\right)^2\right) \\
\leq C(t-s) \mathbb{E}_{u_0^{(n)},v_0^{(n)}}\left[\langle u_{n^2r}, \phi^{(n)} \rangle_Z \right] \\
= C(t-s) \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}}\left[\langle \mu_r^{(n)}, \phi_\lambda \rangle_R^2 \right] dr, \\
\tag{53}
\]

where we have used Jensen’s inequality in the second step. By Lemma 4.1 the integrand in the above display is bounded uniformly in \( n \in \mathbb{N} \) and \( r \in [0,T] \), whence we get

\[
\mathbb{E}_{u_0^{(n)},v_0^{(n)}}[|X_t^{(n)} - X_s^{(n)}|^2] \leq C'(t-s)^2, \quad 0 \leq s < t \leq T, 
\]

confirming Kolmogorov’s tightness criterion for the Laplace term.

This shows that the sequence of coordinate processes \( \langle \mu_t^{(n)}, \phi \rangle_R \), \( n \in \mathbb{N} \), is tight w.r.t. the Meyer-Zheng topology. The same argument works for \( \langle \nu_t^{(n)}, \phi \rangle_R \). For the increasing process \( t \mapsto \langle \Lambda_t^{(n)}, \phi \rangle_R \), condition (80) reduces to

\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}}[\langle \Lambda_T^{(n)}, \phi \rangle_R] < \infty 
\]

which is also ensured by Lemma 4.1.

Since \( C^{(2)}(\mathbb{R})^+ \) separates the points of \( \mathcal{M}_{\text{tem}}(\mathbb{R}) \) and the compact containment condition holds by Corollary 4.2 an application of [Kur91 Cor. 1.4] concludes the proof. \( \square \)

### 4.2 Properties of Limit Points

In this subsection, we check that limit points \( (\mu_t, \nu_t, \Lambda_t)_{t \geq 0} \) of the sequence \( (\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t \geq 0} \) satisfy the martingale problem \((\mathbb{MP}_F(\mathbb{R}))^\varrho_{\nu_0,\lambda_0}(\mathbb{R})^2\) and the separation-of-types property (13). By uniqueness in Theorem 2.2 this implies that the rescaled discrete processes converge indeed to the unique solution of \( \text{cSBM}(\varrho, \infty)_{\nu_0,\lambda_0} \).

**Proposition 4.4.** Let \( \varrho \in (-1,0) \) and \( (\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2 \) (resp. \( \mathcal{M}_{\text{rap}}(\mathbb{R})^2 \)). Suppose \( (\mu_t, \nu_t, \Lambda_t)_{t \geq 0} \in D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^3) \) (resp. \( D_{[0,\infty)}(\mathcal{M}_{\text{rap}}(\mathbb{R})^3) \)) is any limit point with respect to the Meyer-Zheng topology of the sequence \( (\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t \geq 0}, \ n \in \mathbb{N} \). Then \( (\mu_t, \nu_t)_{t \geq 0} \) solves the martingale problem \((\mathbb{MP}_F(\mathbb{R}))^\varrho_{\nu_0,\lambda_0}(\mathbb{R})^2\), with the process \( (\Lambda_t)_{t \geq 0} \) satisfying the requirements of Definition 2.1.

**Proof.** First of all, the limit point \( (\Lambda_t)_{t \geq 0} \) of the sequence \( (\Lambda_t^{(n)})_{t \geq 0} \) has the properties required in Definition 2.1. It is clear that \( (\Lambda_t)_{t \geq 0} \) is increasing with \( \Lambda_0 = 0 \), and condition (10) follows from the first moment estimate (48) together with an application of Fatou’s
lemma. It remains to check that for all test functions \( \phi, \psi \in C_\text{rap}^2(\mathbb{R})^+ \), the process

\[
M_t(\phi, \psi) := F(\mu_t, \nu_t, \phi, \psi) - F(\mu_0, \nu_0, \phi, \psi)
\]

\[
- \frac{1}{2} \int_0^t F(\mu_s, \nu_s, \phi, \psi) \langle (\mu_s, \nu_s, \Delta \phi, \Delta \psi) \rangle \, ds
\]

\[
- 4(1 - \rho^2) \int_{[0,t] \times \mathbb{R}} F(\mu_s, \nu_s, \phi, \psi) \phi(x) \psi(x) \Lambda(ds, dx), \quad t \geq 0
\]

is a martingale.

Since \((u^{(n)}, v^{(n)})\) solves the discrete martingale problem \((\text{MP}_E)_{u_0^{(n)}, v_0^{(n)}}\), we know that

\[
\bar{M}_{n^2t}(\phi^{(n)}, \psi^{(n)}) := F(u^{(n)}_{n^2t}, v^{(n)}_{n^2t}, \phi^{(n)}, \psi^{(n)}) - F(u^{(n)}_0, v^{(n)}_0, \phi^{(n)}, \psi^{(n)})
\]

\[
- \frac{1}{2} \int_0^{n^2t} F(u^s_n, v^s_n, \phi^{(n)}, \psi^{(n)}) \langle (u^s_n, v^s_n, d\Delta(\phi^{(n)}), d\Delta(\psi^{(n)})) \rangle \, ds
\]

\[
- 4(1 - \rho^2) \int_{[0,n^2t] \times \mathbb{R}} F(u^s_n, v^s_n, \phi^{(n)}, \psi^{(n)}) \phi^{(n)}(k) \psi^{(n)}(k) L^{(n)}(ds, dk)
\]

is a martingale for each \( n \in \mathbb{N} \). Choose a sequence \( n_k \uparrow \infty \) such that \((\mu_t^{(n_k)}, \nu_t^{(n_k)}, \Lambda_t^{(n_k)})_{t \geq 0}\) converges to \((\mu_t, \nu_t, \Lambda_t)_{t \geq 0}\) w.r.t. the Meyer-Zheng topology on \( D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^2) \). In view of (45) and (46) (and also using the usual approximation of the continuum Laplace operator by its rescaled discrete counterpart), we get that \((\bar{M}_{n^2t}(\phi^{(n)}, \psi^{(n)}))_{t \geq 0}\) converges to \((M_t(\phi, \psi))_{t \geq 0}\) w.r.t. Meyer-Zheng on \( D_{[0,\infty)}(\mathbb{R}) \) as \( k \to \infty \). Now fixing \( T > 0 \), we know by Burkholder-Davis-Gundy and (41) in Prop. 3.9c) that

\[
\mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ \sup_{t \in [0,T]} |\bar{M}_{n^2t}(\phi^{(n)}, \psi^{(n)})|^2 \right] \leq C T \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ \langle L^{(n)}_{n^2T}, (\phi^{(n)})^2 + (\psi^{(n)})^2 \rangle \right],
\]

where we have also used that \(|F(\cdot)| \leq 1\). Now we argue as in the proof of Proposition 4.3. Choosing a suitable \( \lambda > 0 \) and combining the lower bound from Lemma A.2 with formula (42), we see that there is a constant \( C' = C'(\phi, \psi, \lambda, T) \) such that the previous display is bounded by

\[
C' \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ \int_{[0,n^2T] \times \mathbb{Z}} \left( dS_{n^2T-s} \phi^{(n)}(k) \right)^2 L^{(n)}(ds, dk) \right]
\]

\[
= \frac{C'}{|\varrho|} \left( \phi^{(n)} \otimes \phi^{(n)}, (dS_{n^2T} - d\tilde{S}_{n^2T})(u_0^{(n)} \otimes v_0^{(n)}) \right)_Z^2
\]

\[
\leq \frac{C'}{|\varrho|} \left( \phi^{(n)}, dS_{n^2T} u_0^{(n)} \right)_Z \left( \phi^{(n)}, dS_{n^2T} v_0^{(n)} \right)_Z.
\]

Again by Lemma A.4, the last display is bounded uniformly in \( n \in \mathbb{N} \), hence we get

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [0,T]} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[ |\bar{M}_{n^2t}(\phi^{(n)}, \psi^{(n)})|^2 \right] < \infty
\]

for all \( T > 0 \). Applying [MZ84, Thm. 11], we infer that the Meyer-Zheng limit \((M_t(\phi, \psi))_{t \geq 0}\) is again a martingale, which completes our argument.
We now turn to the separation-of-types property. As in [BHO16 Lemma 4.4] (see in particular inequality (51) there), it can be derived from the following bound on mixed second moments:

**Lemma 4.5 (Mixed second moment bound).** Let \( \rho \in (-1, 0) \) and \((\mu_0, \nu_0) \in M_{\text{em}}(\mathbb{R})^2 \) (resp. \( M_{\text{rap}}(\mathbb{R})^2 \)). Suppose \((\mu_t, \nu_t)_{t \geq 0} \in D_{0, \infty}(M_{\text{em}}(\mathbb{R})^2) \) (resp. \( D_{0, \infty}(M_{\text{rap}}(\mathbb{R})^2) \)) is any limit point with respect to the Meyer-Zheng topology of the sequence \((\mu_t^{(n)}, \nu_t^{(n)})_{t \geq 0}, n \in \mathbb{N} \), from \((13)\). Then with \( \tilde{S}_t \) as defined in \((19)\), we have

\[
E_{\mu_0, \nu_0} \langle \mu_t, \phi \rangle_R \langle \nu_t, \psi \rangle_R \leq \langle \phi \otimes \psi, \tilde{S}_t(\mu_0 \otimes \nu_0) \rangle_{R^2}
\]

for all \( t \geq 0 \) and \( \phi, \psi \in \bigcup_{\lambda > 0} C^+_{\lambda}(\mathbb{R}) \) (resp. \( \bigcup_{\lambda > 0} C^-_{\lambda}(\mathbb{R}) \)).

**Proof.** By [MZ84 Thm. 5] (see also [Kur91 Thm. 1.1(b)]) we can find a sequence \( n_k \uparrow \infty \) and a set \( I \subseteq (0, \infty) \) of full Lebesgue measure such that the finite dimensional distributions of \((\mu_t^{(n_k)}, \nu_t^{(n_k)})_{t \in I} \) converge weakly to those of \((\mu_t, \nu_t)_{t \in I} \) as \( k \rightarrow \infty \). Fix \( t \in I \). Then for all test functions \( \phi, \psi \) as above, we can assume that almost surely

\[
\langle \mu_t^{(n_k)}, \phi \rangle_R \langle \nu_t^{(n_k)}, \psi \rangle_R \xrightarrow{k \rightarrow \infty} \langle \mu_t, \phi \rangle_R \langle \nu_t, \psi \rangle_R
\]

in \( \mathbb{R} \). Using Fatou’s lemma, we get

\[
E_{\mu_0, \nu_0} \left[ \langle \mu_t, \phi \rangle_R \langle \nu_t, \psi \rangle_R \right] \leq \liminf_{k \rightarrow \infty} E_{\mu_0^{(n_k)}, \nu_0^{(n_k)}} \left[ \langle \mu_t^{(n_k)}, \phi \rangle_R \langle \nu_t^{(n_k)}, \psi \rangle_R \right].
\]

But for all \( n \in \mathbb{N} \) we have by the mixed second moment formula \((20)\) (for \( S = \mathbb{R} \)) that

\[
E_{\mu_0^{(n)}, \nu_0^{(n)}} \left[ \langle \mu_t^{(n)}, \phi \rangle_R \langle \nu_t^{(n)}, \psi \rangle_R \right] = E_{u_0^{(n)}, v_0^{(n)}} \left[ \langle u_t^{(n)}, \phi^{(n)} \rangle_Z \langle v_t^{(n)}, \psi^{(n)} \rangle_Z \right]
\]

\[
= \langle \phi^{(n)} \otimes \psi^{(n)}, d\tilde{S}_{um}^Z(u_0^{(n)} \otimes v_0^{(n)}) \rangle_{Z^2}.
\]

As the usual discrete heat semigroup converges to its continuous counterpart under diffusive rescaling, the same holds for the killed semigroup \( \langle t \tilde{S}_t \rangle_{t \geq 0} \), see e.g. Lemma A.4 for details. Thus the RHS of the above display converges to the corresponding continuous quantity, namely to the RHS of \((55)\). This shows the estimate \((55)\) for all \( t \in I \). Using the fact that \( I \) has full Lebesgue measure together with right-continuity of the paths of \((\mu_t, \nu_t)_{t \geq 0} \) and Fatou’s lemma, we get the same estimate for all \( t > 0 \).

**Corollary 4.6 (Separation-of-types).** Under the assumptions of Lemma 4.5, the separation-of-types property \((13)\) holds for each \( t > 0 \).

**Proof.** Having shown the upper bound \((55)\) for the mixed second moment, the proof of the separation-of-types property is basically the same as that of [BHO16 Lemma 4.4]: For each \( t > 0, x \in \mathbb{R} \) and \( \varepsilon > 0 \) fixed, letting \( \phi(\cdot) := \psi(\cdot) := p_\varepsilon(x - \cdot) \) in \((55)\) gives

\[
E_{\mu_0, \nu_0} \left[ S_\varepsilon \mu_t(x) S_\varepsilon \nu_t(x) \right]
\]

\[
\leq \iint dydz \: p_\varepsilon(x - y)p_\varepsilon(x - z) \tilde{S}_t(\mu_0 \otimes \nu_0)(y, z)
\]

\[
\leq S_{1 + \varepsilon} \mu_0(x) S_{1 + \varepsilon} \nu_0(x).
\]

Since \((y, z) \mapsto \tilde{S}_t(\mu_0 \otimes \nu_0)(y, z)\) is continuous and vanishes on the diagonal, by taking \( \varepsilon \downarrow 0 \) in the first inequality in \((58)\) we get \( E_{\mu_0, \nu_0} \left[ S_\varepsilon \mu_t(x) S_\varepsilon \nu_t(x) \right] \rightarrow 0 \), which proves our claim.
Note that by combining Prop. 4.3 with Prop. 4.4 and Cor. 4.6 we have now fully proved the convergence result in Theorem 2.4.

5 Preservation of order and the single-point interface

In this section, we prove Theorem 2.6. For \( \varrho \in (-1, 0) \), consider the solution \((\mu_t, \nu_t)_{t \geq 0}\) to \( \text{cSBM}(\varrho, \infty)_{\mu_0, \nu_0} \) with initial conditions \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2 \) or \( \mathcal{M}_{\text{rap}}(\mathbb{R})^2 \) which are mutually singular and such that the types are strictly ordered, i.e. \( R(\mu_0) \leq L(\nu_0) \). The proof involves several parts: First we show that the initial ordering of types is preserved:

\[
\mathbb{P}_{\mu_0, \nu_0}[R(\mu_t) \leq L(\nu_t) \text{ for all } t > 0] = 1.
\]  

(59)

Then, we will show that the solution has a single-point interface, i.e.

\[
\mathbb{P}_{\mu_0, \nu_0}[R(\mu_t) = L(\nu_t)] = 1
\]  

(60)

for all \( t > 0 \). Both properties (in an analogous sense) are first shown for the discrete model and then extended to the continuous model by an application of Theorem 2.4. The central observation in Lemma 5.1 is here that the mapping \( \mu \mapsto R(\mu) \), respectively \( \mu \mapsto L(\mu) \), which assigns to each measure the rightmost, respectively leftmost, point in the support is lower (respectively upper) semicontinuous. Thus, (59) follows by arguing that the discrete approximation satisfies the same property combined with Theorem 2.4. In order to prove (60), we combine (59) with the observation that for fixed \( t > 0 \), the measure \( \mu_t + \nu_t \) is strictly positive almost surely, i.e. \( \text{supp}(\mu_t + \nu_t) = \mathbb{R} \), see Cor. 5.5 below. Finally, we derive the mutual singularity of the measures \( \mu_t \) and \( \nu_t \) for each \( t > 0 \) from the single-point interface and the separation-of-types property (13).

Lemma 5.1. The mapping \( R(\cdot) : \mathcal{M}_{\text{tem}}(\mathbb{R}) \rightarrow \overline{\mathbb{R}}, \mu \mapsto R(\mu) := \sup \text{supp}(\mu) \) is lower semicontinuous, and the mapping \( L(\cdot) : \mathcal{M}_{\text{tem}}(\mathbb{R}) \rightarrow \overline{\mathbb{R}}, \mu \mapsto L(\mu) := \inf \text{supp}(\mu) \) is upper semicontinuous. The same holds if \( \mathcal{M}_{\text{tem}}(\mathbb{R}) \) is replaced by \( \mathcal{M}_{\text{rap}}(\mathbb{R}) \).

Proof. We will prove lower semicontinuity of \( R(\cdot) \), since the proof for upper semicontinuity of \( L(\cdot) \) is completely analogous.

Let \( (\mu^{(n)})_{n \in \mathbb{N}} \) be a sequence in \( \mathcal{M}_{\text{tem}}(\mathbb{R}) \) with \( \mu^{(n)} \rightarrow \mu \in \mathcal{M}_{\text{tem}} \) as \( n \rightarrow \infty \). For the purposes of the proof, we write

\[
R_n := R(\mu^{(n)}) = \sup \text{supp}(\mu^{(n)}), \quad \tilde{R} := \liminf_{n \rightarrow \infty} R_n \in \overline{\mathbb{R}},
\]

and we have to show that

\[
R(\mu) \leq \tilde{R}.
\]

We distinguish the three possible cases \( \tilde{R} = \infty \), \( \tilde{R} = -\infty \) and \( \tilde{R} \in \mathbb{R} \), where in the first case the assertion is trivially true. Suppose that \( \tilde{R} = -\infty \), then we can find a subsequence such
that $R_{n_k} \to -\infty$. Take a test function $\phi \in C_c^+$. Then $(\mu^{(n_k)}, \phi) \to 0$ since $\mu^{(n_k)}$ is supported on $(-\infty, R_{n_k}]$ and $\phi$ is compactly supported. Since on the other hand $(\mu^{(n_k)}, \phi) \to (\mu, \phi)$, we conclude that $(\mu, \phi) = 0$. Since $\phi \in C_c^+$ was arbitrary, this implies $\mu = 0$, i.e. $R(\mu) = -\infty = R$.

It remains to consider the case $\tilde{R} \in \mathbb{R}$. We show that $\text{supp}(\mu) \subseteq (-\infty, \tilde{R}]$, whence we get $R(\mu) \leq \tilde{R}$ and our assertion is proved. Assume that $\text{supp}(\mu) \cap (\tilde{R}, \infty) \neq \emptyset$. Then by the definition of the support, we can find a function $\phi \in C_c^+$ with $\text{supp}(\phi) \subseteq (\tilde{R}, \infty)$ such that $(\mu, \phi) > 0$. On the other hand, let $(R_{n_k})_{k \in \mathbb{N}}$ be a subsequence such that $R_{n_k} \to \tilde{R}$ as $k \to \infty$. Then we have $(\mu, \phi) = \lim_{k \to \infty}(\mu^{(n_k)}, \phi) \to 0$ since $\mu^{(n_k)}$ is supported on $(-\infty, R_{n_k}]$ and $(-\infty, R_{n_k}] \cap \text{supp}(\phi) = \emptyset$ for sufficiently large $k$, giving a contradiction. This completes our proof.

\begin{corollary}
Let $(\mu^{(n)}, \nu^{(n)})_{n \in \mathbb{N}}$ be a convergent sequence in $\mathcal{M}_{\text{tem}}(\mathbb{R})^2$ or $\mathcal{M}_{\text{rap}}(\mathbb{R})^2$ with limit $(\mu, \nu)$. Further, assume that for all $n \in \mathbb{N}$ we have

$$R(\mu^{(n)}) \leq L(\nu^{(n)}).$$

Then we have also

$$R(\mu) \leq \liminf_{n \to \infty} R(\mu^{(n)}) \leq \limsup_{n \to \infty} L(\nu^{(n)}) \leq L(\nu).$$

\end{corollary}

\begin{lemma}
Suppose $(\mu^{(n)}_t, \nu^{(n)}_t)_{t \geq 0}, n \in \mathbb{N}$, is a sequence of processes in $D_{[0, \infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^2)$ (resp. $D_{[0, \infty)}(\mathcal{M}_{\text{rap}}(\mathbb{R})^2)$) such that $(\mu^{(n)}_t, \nu^{(n)}_t)_{t \geq 0} \to (\mu_t, \nu_t)_{t \geq 0}$ weakly in $D_{[0, \infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^2)$ (resp. $D_{[0, \infty)}(\mathcal{M}_{\text{rap}}(\mathbb{R})^2)$) w.r.t. the Meyer-Zheng topology. Suppose further that

$$\mathbb{P}[R(\mu^{(n)}_t) \leq L(\nu^{(n)}_t) \text{ for all } t > 0, n \in \mathbb{N}] = 1.$$

Then we have also

$$\mathbb{P}[R(\mu_t) \leq L(\nu_t) \text{ for all } t > 0] = 1. \quad (61)$$

\end{lemma}

\begin{proof}
Using [MZ84, Thm. 5] (see also [Kur91, Thm. 1.1(b)]), there exists a set $I \subseteq (0, \infty)$ of full Lebesgue measure such that by passing to a subsequence (which we do not distinguish in notation) we may assume that the finite dimensional distributions of $(\mu^{(n)}_t, \nu^{(n)}_t)_{t \in I}$ converge weakly to those of $(\mu_t, \nu_t)_{t \in I}$ as $n \to \infty$.

Fix $t \in I$. By Skorokhod’s representation theorem we may - and will - assume that $(\mu^{(n)}_t, \nu^{(n)}_t) \to (\mu_t, \nu_t)$ in $\mathcal{M}_{\text{tem}}(\mathbb{R})^2$ almost surely. Applying Corollary 5.2 pathwise on a set of full probability, we conclude that

$$\mathbb{P}[R(\mu_t) \leq L(\nu_t)] = 1, \quad t \in I. \quad (62)$$

Now choose a dense countable subset $J \subseteq I$ and intersect countably many sets of full probability to obtain

$$\mathbb{P}[R(\mu_t) \leq L(\nu_t) \text{ for all } t \in J] = 1. \quad (63)$$

For arbitrary $t > 0$, since $J$ is dense we can choose a sequence $t_n \downarrow t$ with $t_n \in J$. Now use the fact that (63) holds for each $t_n$ plus right-continuity of the paths of $(\mu_t, \nu_t)_{t \geq 0}$ and again apply Corollary 5.2 (with $(\mu^{(n)}, \nu^{(n)}) := (\mu_{t_n}, \nu_{t_n})$) pathwise on a set of full probability to see that (61) holds.

\end{proof}
We return to the discrete-space infinite rate model dSBM(\(\varrho, \infty\))\(u_0, v_0\), with initial conditions \((u_0, v_0) \in \mathcal{M}_{\text{tem}}(\mathbb{Z})^2\), and now discuss strict positivity of the solution \((u_t, v_t)\) for fixed positive times \(t > 0\). Let \(Q^x_{x,y}\) denote the exit measure of a planar \(\varrho\)-correlated Brownian motion \((B^{(1)}, B^{(2)})\) from the first quadrant \((\mathbb{R}^+)^2\), started at \((x, y) \in (\mathbb{R}^+)^2\). That is, defining \(\tau := \inf \{ t > 0 : B^{(1)}_t B^{(2)}_t = 0 \}\), let

\[
Q^x_{x,y} (\cdot) := \mathbb{P}_x [ (B^{(1)}_\tau, B^{(2)}_\tau) \in \cdot].
\]

Then \(Q^x_{x,y}\) is concentrated on the boundary of the first quadrant, i.e. on \((\mathbb{R}^+)^2 \setminus (0, 0)^2\), and has no atom at zero iff \((x, y) \neq (0, 0)\). Moreover, the mapping \((x, y) \mapsto Q^x_{x,y}\) is continuous.

For an exact description of \(Q^x_{x,y}\) for initial conditions that are already separated (i.e. \(u_0(k) v_0(k) = 0\) for all \(k \in \mathbb{Z}\)) the exit measure \(Q^x_{x,y}\) determines the distribution of the solution for fixed time \(t > 0\), as follows: We have

\[
\mathbb{E}_{u_0,v_0} \left[ Q^x_{(u_t, \phi)_Z, (v_t, \psi)_Z} (\cdot) \right] = Q^x_{(S_{t u_0}, \phi)_Z, (S_{t v_0}, \psi)_Z} (\cdot) \quad (64)
\]

for all suitable test functions \(\phi, \psi\). This was derived in [KO10] Thm. 2] for the case \(\varrho = 0\) as a consequence of the Trotter approximation, which can also be generalized to \(\varrho \in (-1, 1)\) (see e.g. [DM12], Sec. 4.3, pp. 34ff.). Choosing \(\varphi := \psi := 1\{k\}\) and using that obviously \(Q^x_{x,y} (\cdot) = \delta_{(x,y)} (\cdot)\) if \(xy = 0\) shows that the distribution of \((u_t(k), v_t(k))\) at fixed space-time points is given by

\[
\mathbb{P}_{u_0,v_0} [(u_t(k), v_t(k)) \in \cdot] = Q^x_{(S_{t u_0(k)}, S_{t v_0(k)})} (\cdot), \quad k \in \mathbb{Z}, t > 0. \quad (65)
\]

From this it follows immediately that \(u_t(k) + v_t(k) > 0\) almost surely if \(u_0\) and \(v_0\) are not both identically zero, since \( (S_{t u_0(k)}, S_{t v_0(k)}) \neq (0, 0)\).

It is straightforward to generalize (64)-(65) to arbitrary (not necessarily separated) initial conditions \((u_0, v_0) \in \mathcal{M}_{\text{tem}}(\mathbb{Z})^2\) as employed in our framework: In fact, since the separation of types holds at positive times, by the Markov property applied at time \(s \in (0, t)\) we get

\[
\mathbb{E}_{u_0,v_0} \left[ Q^x_{(u_t, \phi)_Z, (v_t, \psi)_Z} (\cdot) \right] = \mathbb{E}_{u_0,v_0} \left[ \mathbb{E}_{u_s,v_s} \left[ Q^x_{(u_{t-s}, \phi)_Z, (v_{t-s}, \psi)_Z} (\cdot) \right] \right] \quad (66)
\]

Letting \(s \downarrow 0\) and using the right-continuity of the paths of \((u_t, v_t)\) we obtain [64], and (65) follows as before.

As a first simple application of our convergence result Theorem 2.4, we can now easily extend (64) to continuous space:

**Lemma 5.4.** Let \(\varrho \in (-1, 0)\). Assume that \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\) (resp. \(\mathcal{M}_{\text{rap}}(\mathbb{R})^2\)). Then we have for the solution of cSBM(\(\varrho, \infty\))\(\mu_0, \nu_0\) that

\[
\mathbb{E}_{\mu_0,\nu_0} \left[ Q^x_{(\mu_t, \phi)_\mathbb{R}, (\nu_t, \psi)_\mathbb{R}} (\cdot) \right] = Q^x_{(S_{t \mu_0}, \phi)_\mathbb{R}, (S_{t \nu_0}, \psi)_\mathbb{R}} (\cdot) \quad (67)
\]

for all \(t \geq 0\) and test functions \(\phi, \psi \in \bigcup_{\lambda > 0} C^+_{\lambda}(\mathbb{R})\) (resp. \(\bigcup_{\lambda > 0} C^-_{\lambda}(\mathbb{R})\)).
Proof. We define a sequence of approximating discrete processes as in (14)-(15). By (45) and (64), we have for all $n \in \mathbb{N}$

\[
\mathbb{E}_{\rho_{\mu_0}^{(n)}} \left[ Q^\rho_{\mu_t^{(n)}} \phi \right] \mathbb{E}_{\mu_0^{(n)}} \left[ \langle \mu_t^{(n)}, \psi \rangle \right] = \mathbb{E}_{\mu_0^{(n)}} \left[ Q^\rho_{\mu_t^{(n)}} \phi \right] \mathbb{E}_{\mu_0^{(n)}} \left[ \langle \mu_t^{(n)}, \psi \rangle \right]
\]

Applying Theorem 2.4, Lemma A.4 and using continuity of $(x, y) \mapsto Q^\rho_{x, y}$, we can take the limit on both sides to obtain (67).

Corollary 5.5. Let $\rho \in (-1, 0)$ and $(\mu_0, \nu_0) \in \mathcal{M}_{tem}(\mathbb{R})^2$ or $\mathcal{M}_{rap}(\mathbb{R})^2$. Assume further that $\mu_0 + \nu_0$ is not the zero measure. Then we have for all $t > 0$ that

\[
P_{\mu_0, \nu_0} [\text{supp}(\mu_t + \nu_t) = \mathbb{R}] = 1.
\]

Proof. Fix $t > 0$ and consider a test function $\phi \in C^+_z$, $\phi \neq 0$. Then since $\langle \mu_t + \nu_t, \phi \rangle = 0$ implies $Q^\rho_{(\mu_t, \phi), \langle \nu_t, \phi \rangle} (0, 0) = 1$, we have by Lemma 5.4 that

\[
P_{\mu_0, \nu_0} [(\mu_t + \nu_t, \phi) = 0] = \mathbb{E}_{\mu_0, \nu_0} \left[ \mathbb{I}_{(\mu_t + \nu_t, \phi) = 0} Q^\rho_{(\mu_t, \phi), \langle \nu_t, \phi \rangle} (0, 0) \right]
\]

\[
\leq \mathbb{E}_{\mu_0, \nu_0} \left[ Q^\rho_{(\mu_t, \phi), \langle \nu_t, \phi \rangle} (0, 0) \right] = Q^\rho_{(\mu_0, \phi), \langle \nu_0, \phi \rangle} (0, 0) = 0.
\]

Here we used that $Q^\rho_{x, y}$ has an atom at zero iff $(x, y) = (0, 0)$. Since $\phi$ was arbitrary, this implies in particular that for all intervals $[a, b]$ with rational endpoints $a, b \in \mathbb{Q}$ we have $P_{\mu_0, \nu_0} [(\mu_t + \nu_t)[a, b] = 1$. Intersecting the countably many sets of probability one, we see that $\text{supp}(\mu_t + \nu_t) = \mathbb{R}$ almost surely.

Before turning to the proof of Theorem 2.6 we now sketch a proof of an analogous version for the discrete-space case. Again let $(u_t, v_t)_{t \geq 0}$ denote the solution to $dSBM(\rho, \infty)_{u_0, v_0}$, with $(u_0, v_0) \in \mathcal{M}_{tem}(\mathbb{Z})^2$. We consider $(u_t, v_t)$ as elements of $\mathcal{M}_{tem}(\mathbb{R})^2$ which are concentrated on the lattice $\mathbb{Z} \subseteq \mathbb{R}$. The support is given by $\text{supp}(u_t) = \{ k \in \mathbb{Z} : u_t(k) > 0 \}$, and $R(u_t)$, $L(v_t)$ are defined as before.

Proposition 5.6. Let $\rho \in (-1, 1)$ and consider initial conditions $(u_0, v_0) \in \mathcal{M}_{tem}(\mathbb{Z})^2$ or $\mathcal{M}_{rap}(\mathbb{Z})^2$ such that $R(u_0) < L(v_0)$. Then almost surely the initial ordering of types is preserved for all times, i.e.

\[
P_{u_0, v_0} [R(u_t) < L(v_t) \text{ for all } t > 0] = 1.
\]

Assume in addition that $u_0 + v_0$ is not identically zero. Then we have the discrete analogue of the ‘single-point interface’ property for all fixed times, in the sense that

\[
P_{u_0, v_0} [R(u_t) = L(v_t) - 1] = 1, \quad t > 0.
\]

Proof. We only sketch the argument: Recall the approximation procedure used in [KM12a] in order to construct the discrete model $dSBM(\rho, \infty)_{u_0, v_0}$ for $\rho = 0$ (see [DM12] for the extension to $\rho \neq 0$). Inspection of the definition in [KM12a], p. 15, eqs. (2.8)-(2.9), shows that the initial ordering of types is preserved by the dynamics of the approximating...
processes, thus (68) holds for the latter. By [KM12a Thm. 3.1], the approximating processes converge to \((u_t, v_t)_{t > 0}\) w.r.t. the Skorokhod topology. Applying Lemma 5.3 we infer that the limit satisfies \(P_{\mu_0, \nu_0}[R(u_t) \leq L(v_t)] \text{ for all } t > 0 = 1\). However, the inequality is in fact strict because we know already by the usual separation-of-types property (12) of \(dSBM(\varnothing, \infty)_{u_0, v_0}\) that almost surely \(u_t(k)v_t(k) = 0\) for all \(k \in \mathbb{Z}, t > 0\).

In addition, if \(u_0 + v_0\) does not vanish, then for all \(t > 0\), almost surely \(u_t + v_t\) is strictly positive by (65). But \(R(u_t) < L(v_t)\) together with \(u_t + v_t > 0\) implies \(R(u_t) = L(v_t) - 1\).

**Proof of Theorem 2.6.** Consider \(\varrho \in (-1, 0)\) and initial conditions \((\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2\) which are mutually singular and such that \(R(\mu_0) \leq L(\nu_0)\).

If \(R(\mu_0) < L(\nu_0)\), then for the initial conditions (14) for the discrete process we also have \(R(u_0^{(n)}) < L(v_0^{(n)})\), provided \(n \in \mathbb{N}\) is large enough. Now consider the case \(R(\mu_0) = L(\nu_0) = I_0\), where we can assume without loss of generality that \(I_0 = 0\). If \(\mu_0(\{0\}) = 0\), then (14) implies \(R(u_0^{(n)}) = -1 < 0 = L(v_0^{(n)})\). If on the other hand \(\mu_0(\{0\}) > 0\), we must have \(\nu_0(\{0\}) = 0\) since by assumption the measures \(\mu_0\) and \(\nu_0\) are mutually singular and thus cannot both have an atom at \(I_0 = 0\). Modifying the definition of the discrete initial conditions according to (17), we then again have \(R(u_0^{(n)}) = -1 < 0 = L(v_0^{(n)})\). Thus in both cases Proposition 5.6 is applicable, and using the definition (15) of the approximating processes we get that almost surely

\[ R(\mu_t^{(n)}) \leq L(\nu_t^{(n)}) \]

for all \(t > 0\) and \(n \in \mathbb{N}\). Another application of Lemma 5.3 in combination with Theorem 2.4 concludes the proof of the ‘preservation of order of types’-property (59).

For the rest of the proof, we fix \(t > 0\). As for the discrete model, the ‘single-point interface’ property (60) is now a simple consequence of (59) together with strict positivity from Corollary 5.5. We have to show that \(\mu_t\) and \(\nu_t\) are mutually singular almost surely. Since \(I_t := R(\mu_t) = L(\nu_t)\) a.s., it is clear by the definition of the measure-theoretic support that \(\mu_t(I_t, \infty) = 0\) and \(\nu_t(-\infty, I_t) = 0\) a.s. It only remains to rule out the possibility that both \(\mu_t\) and \(\nu_t\) have an atom at \(I_t\) with positive probability. Choose any strictly positive test function \(\phi \in \bigcup_{\lambda > 0} C^1_{\lambda}(\mathbb{R}), \phi(\cdot) > 0\). Then we have

\[
\int_{\mathbb{R}} \phi(x) \mathbb{E}_{\mu_0, \nu_0} \left[ S_t \mu_t(x) S_t \nu_t(x) \right] dx \\
= \int_{\mathbb{R}} \phi(x) \mathbb{E}_{\mu_0, \nu_0} \left[ \int_{\mathbb{R}} p_\epsilon(x - y) \mu_t(dy) \int_{\mathbb{R}} p_\epsilon(x - z) \nu_t(dz) \right] dx \\
\geq \int_{\mathbb{R}} \phi(x) \mathbb{E}_{\mu_0, \nu_0} \left[ p_\epsilon(x - I_t) \mu_t(\{I_t\}) p_\epsilon(x - I_t) \nu_t(\{I_t\}) \right] dx \\
= \mathbb{E}_{\mu_0, \nu_0} \left[ \mu_t(\{I_t\}) \nu_t(\{I_t\}) \int_{\mathbb{R}} \phi(x) p_\epsilon(x - I_t)^2 dx \right] \\
= \frac{1}{2\sqrt{\pi\epsilon}} \mathbb{E}_{\mu_0, \nu_0} \left[ \mu_t(\{I_t\}) \nu_t(\{I_t\}) \int_{\mathbb{R}} \phi(x/\sqrt{\epsilon}) p_\epsilon(x - \sqrt{\epsilon}I_t) dx \right].
\]

In the above display, the LHS goes to zero by the separation-of-types property (13), while the integral inside the expectation on the RHS converges to \(\phi(I_t) > 0\) a.s. This leads to a contradiction unless we have \(\mu_t(\{I_t\}) \nu_t(\{I_t\}) = 0\) a.s. Hence \(\mu_t\) and \(\nu_t\) cannot both have an atom at \(I_t\), and our proof is complete.

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A Appendix

A.1 Notation and spaces of functions and measures

In this appendix we have collected our notation and we recall some well-known facts concerning the spaces of functions and measures employed throughout the paper. Most of the material in this subsection can also be found e.g. in [DEF+02b], [DFM+03] or [EF04]. We can develop the notation for both the discrete and the continuous setting simultaneously, so throughout we let $S = \mathbb{Z}^d$ for some $d \in \mathbb{N}$ or $S = \mathbb{R}$.

For $\lambda \in \mathbb{R}$, let $\phi_\lambda(x) := e^{-\lambda |x|}$, $x \in S$, and for $f : S \to \mathbb{R}$ define $|f|_\lambda := \|f/\phi_\lambda\|_\infty$, where $\|\cdot\|_\infty$ is the supremum norm. We denote by $B_\lambda(S)$ the space of all measurable functions $f : S \to \mathbb{R}$ such that $|f|_\lambda < \infty$ and so that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x| \to \infty$. Next, we introduce the spaces of rapidly decreasing and tempered measurable functions, respectively, as $B_{\text{rap}}(S) := \bigcap_{\lambda > 0} B_\lambda(S)$ and $B_{\text{tem}} := \bigcap_{\lambda > 0} B_{-\lambda}(S)$. (69)

For $S = \mathbb{R}$, we write $C_{\lambda}(\mathbb{R}), C_{\text{rap}}(\mathbb{R}), C_{\text{tem}}(\mathbb{R})$ for the subspaces of continuous functions in $B_\lambda(\mathbb{R}), B_{\text{rap}}(\mathbb{R}), B_{\text{tem}}(\mathbb{R})$ respectively. Moreover, if we additionally assume that all partial derivatives up to order $k \in \mathbb{N}$ exist and belong to $C_{\lambda}(\mathbb{R}), C_{\text{rap}}(\mathbb{R}), C_{\text{tem}}(\mathbb{R})$, we write $C_{\lambda}^{(k)}(\mathbb{R}), C_{\text{rap}}^{(k)}(\mathbb{R}), C_{\text{tem}}^{(k)}(\mathbb{R})$. In order to formulate results for the continuous and discrete case simultaneously, it is convenient to employ all of these notations also for $S = \mathbb{Z}^d$, where of course (by convention) the spaces just introduced coincide with $B_\lambda(\mathbb{Z}^d), B_{\text{rap}}(\mathbb{Z}^d), B_{\text{tem}}(\mathbb{Z}^d)$ respectively. Finally, we will also use the space $C_{\infty}^{\text{c}}(\mathbb{R})$ of infinitely differentiable functions with compact support.

For each $\lambda \in \mathbb{R}$, the linear space $C_{\lambda}(S)$ endowed with the norm $|\cdot|_{\lambda}$ is a separable Banach space, and the spaces $C_{\text{rap}}(S), C_{\text{tem}}(S)$ can be topologized by a suitable metric to turn them into Polish spaces, for the details see e.g. Appendix A.1 in [BHO16].

If $\mathcal{F}$ is any of the above spaces of functions, we denote by $\mathcal{F}^+$ the subset of nonnegative elements of $\mathcal{F}$.

Let $\mathcal{M}(S)$ denote the space of (nonnegative) Radon measures on $S$. For $\mu \in \mathcal{M}(S)$ and a measurable function $f : S \to \mathbb{R}$, we will denote the integral of $f$ with respect to the measure $\mu$ (if it exists) by any of the following notations

$$\langle \mu, f \rangle, \quad \int_S \mu(dx) f(x), \quad \int_S f(x) \mu(dx).$$

In the case of the Lebesgue measure $\ell$ on $\mathbb{R}$, we will simply write $dx$ in place of $\ell(dx)$. If $\mu \in \mathcal{M}(\mathbb{R})$ is absolutely continuous w.r.t. $\ell$, we will identify $\mu$ with its density, writing $\mu(dx) = \mu(x) \ell(dx)$. 31
Similarly, for $\mu \in \mathcal{M}(\mathbb{Z})$, we will often write $\mu(k) := \mu(\{k\})$.

For $\lambda \in \mathbb{R}$, define

$$\mathcal{M}_\lambda(S) := \{ \mu \in \mathcal{M}(S) : \langle \mu, \phi_\lambda \rangle < \infty \}$$

and introduce the spaces

$$\mathcal{M}_{\text{tem}}(S) := \bigcap_{\lambda > 0} \mathcal{M}_\lambda(S), \quad \mathcal{M}_{\text{rap}}(S) := \bigcap_{\lambda > 0} \mathcal{M}_{-\lambda}(S)$$

of tempered and rapidly decreasing measures on $S$, respectively. Again by defining suitable metrics it can be seen that these spaces are Polish. Moreover, $\mu_n \to \mu$ in $\mathcal{M}_{\text{tem}}(S)$ iff $\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle$ for all $\varphi \in \bigcup_{\lambda > 0} C_\lambda(S)$. We write $\mathcal{M}_f(S)$ for the space of finite measures on $S$ endowed with the topology of weak convergence. With this notation we have $\mathcal{M}_{\text{rap}}(S) \subseteq \mathcal{M}_f(S)$. To topologize the space $\mathcal{M}_{\text{rap}}(S)$ we say that $\mu_n \to \mu$ in $\mathcal{M}_{\text{rap}}(S)$ iff $\mu_n \to \mu$ in $\mathcal{M}_f(S)$ (w.r.t. the weak topology) and $\sup_{n \in \mathbb{N}} \langle \mu_n, \phi_\lambda \rangle < \infty$ for all $\lambda < 0$ (see [DFM+03, p. 140]).

Finally, we remark that $B_{\text{tem}}^+(\mathbb{R})$ and thus also $C_{\text{tem}}^+(\mathbb{R})$ may be viewed as subspaces of $\mathcal{M}_{\text{tem}}(\mathbb{R})$: Indeed, we can take a function $u \in B_{\text{tem}}^+(\mathbb{R})$ as a density w.r.t. Lebesgue measure and thus identify it with the tempered measure $u(x) \, dx$. Note, however, that the topology of $\mathcal{M}_{\text{tem}}(\mathbb{R})$ restricted to $C_{\text{tem}}^+(\mathbb{R})$ is weaker than the topology on $C_{\text{tem}}^+(\mathbb{R})$ defined in Appendix A.1 in [BHO16]. The same relationship holds between $C_{\text{rap}}^+(\mathbb{R})$ and $\mathcal{M}_{\text{rap}}(\mathbb{R})$. In particular, we have continuous embeddings $C_{\text{tem}}^+(\mathbb{R}) \hookrightarrow \mathcal{M}_{\text{tem}}(\mathbb{R})$ and $C_{\text{rap}}^+(\mathbb{R}) \hookrightarrow \mathcal{M}_{\text{rap}}(\mathbb{R})$.

For $S = \mathbb{Z}^d$, we can identify any measure in $\mathcal{M}_{\text{tem}}(\mathbb{Z}^d)$ resp. $\mathcal{M}_{\text{rap}}(\mathbb{Z}^d)$ with its density w.r.t. counting measure, and moreover this density will be in $B_{\text{tem}}^+(\mathbb{Z}^d)$ resp. $B_{\text{rap}}^+(\mathbb{Z}^d)$, the reason being that a summable sequence is necessarily bounded. Thus we have equality of the sets $\mathcal{M}_{\text{tem}}(\mathbb{Z}^d) = B_{\text{tem}}^+(\mathbb{Z}^d)$ and $\mathcal{M}_{\text{rap}}(\mathbb{Z}^d) = B_{\text{rap}}^+(\mathbb{Z}^d)$. Nevertheless, as in the continuous case, the topology on $\mathcal{M}_{\text{tem}}(\mathbb{Z}^d)$ resp. $\mathcal{M}_{\text{rap}}(\mathbb{Z}^d)$ introduced above is strictly weaker than the topology on $B_{\text{tem}}^+(\mathbb{Z}^d)$ resp. $B_{\text{rap}}^+(\mathbb{Z}^d)$ defined in [BHO16, Appendix A.1].

### A.2 Semigroup estimates

Let $(p_t)_{t \geq 0}$ denote the heat kernel in $\mathbb{R}$ corresponding to $\frac{1}{2} \Delta$,

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp \left\{ -\frac{|x|^2}{2t} \right\}, \quad t > 0, \ x \in \mathbb{R}, \quad (70)$$

and write $(S_t)_{t \geq 0}$ for the associated heat semigroup. Similarly, let $(dS_t)_{t \geq 0}$ denote the semigroup corresponding to a continuous-time simple random walk $(X_t)_{t \geq 0}$ with generator $\frac{1}{2} d\Delta$, the discrete Laplace operator as defined in [7], and discrete heat kernel $d p_t$.

For $\mu \in \mathcal{M}(\mathbb{R})$ and $x \in \mathbb{R}$, let $S_t \mu(x) := \int_{\mathbb{R}} p_t(x - y) \mu(dy)$ and similarly for $dS$. The following estimates are well known and can be proved as in Appendix A of [DFM+03] (see also [Shi94, Lemma 6.2 (ii)]):

**Lemma A.1.** Fix $\lambda \in \mathbb{R}$ and $T > 0$.

1. For all $\varphi \in B_{\text{tem}}^+(\mathbb{R})$, we have

$$\sup_{t \in [0, T]} S_t \varphi(x) \leq C(\lambda, T) |\varphi|_{\lambda \phi_\lambda(x)}, \quad x \in \mathbb{R}, \quad (71)$$
Moreover, there is a constant $C'(\lambda, T) > 0$ such that
\begin{equation}
\inf_{t \in [0,T]} S_t \phi_\lambda(x) \geq C'(\lambda, T) \phi_\lambda(x), \quad x \in \mathbb{R}.
\end{equation}

b) Let $0 < \varepsilon < T$. Then for all $\mu \in \mathcal{M}_\lambda(\mathbb{R})$ we have
\begin{equation}
\sup_{t \in [\varepsilon, T]} S_t \mu(x) \leq C(\lambda, T, \varepsilon) \langle \mu, \phi_\lambda \rangle \phi_{-\lambda}(x), \quad x \in \mathbb{R}.
\end{equation}

Therefore, the heat semigroup preserves the space $\mathcal{B}_\lambda(\mathbb{R})$ and maps $\mathcal{M}_\lambda(\mathbb{R})$ into $\mathcal{B}_\lambda(\mathbb{R})$.

We have analogous estimates for the discrete space semigroup:

**Lemma A.2.** Let $\lambda \in \mathbb{R}$ and $T > 0$. Then there are constants $c(\lambda, T), C(\lambda, T) > 0$ such that for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and all $t \in [0, T]$ we have
\begin{equation}
c(\lambda, T) \phi_\lambda(\frac{k}{n}) \leq dS_{nt}(\phi_\lambda(\frac{k}{n}))(k) \leq C(\lambda, T) \phi_\lambda(\frac{k}{n}).
\end{equation}

**Proof.** The upper bound is just a reformulation of [DEF*02b, Corollary A3(n)]. For the lower bound, let $X_t$ be a continuous time random walk with generator $\frac{1}{2} \Delta$ started in 0. If we define $X_t^{(n)} := X_{nt}/n$, then we know from Donsker’s theorem that $(X_t^{(n)})_{t \geq 0}$ converges in distribution to a standard Brownian motion $B$. Fix $T > 0$, by Skorokhod’s representation theorem, we can choose a common probability space $\mathbb{P}$ (with expectation $\mathbb{E}$) such that $\sup_{t \in [0, T]} |X_t^{(n)} - B_t| \to 0$ almost surely.

For $\lambda \geq 0$, we can estimate using the triangle inequality
\begin{equation*}
dS_{nt}(\phi_\lambda(\cdot/n))(k) = \mathbb{E}[e^{-\lambda|X_t^{(n)}+k|}] \geq e^{-\lambda|k/n|} \mathbb{E}[e^{-\lambda|X_t^{(n)}|}].
\end{equation*}
So it remains to show that the expectation on the LHS is bounded from below uniformly in $t \in [0, T]$ and $n$. Now, choose $n_0$ large enough such that for all $n \geq n_0$
\begin{equation*}
\mathbb{P}\left\{ \sup_{t \in [0, T]} |X_t^{(n)} - B_t| \geq \frac{1}{2} \right\} \leq \frac{1}{2} \mathbb{E}[e^{-\lambda|B_T|}].
\end{equation*}
Using the above estimate and the fact that $t \mapsto \mathbb{E}[e^{-\lambda|B_t|}]$ is decreasing, we thus obtain for $n \geq n_0$
\begin{equation*}
\mathbb{E}[e^{-\lambda|X_t^{(n)}|}] \geq e^{-\frac{1}{2} \lambda} \mathbb{E}[e^{-\lambda|B_t|} \mathbb{1}_{\{\sup_{t \in [0, T]} |X_t^{(n)} - B_t| \leq \frac{1}{2}\}}] \geq e^{-\frac{1}{2} \lambda} \left( \mathbb{E}[e^{-\lambda|B_T|}] - \mathbb{P}\left\{ \sup_{t \in [0, T]} |X_t^{(n)} - B_t| \geq \frac{1}{2} \right\} \right) \geq \frac{1}{2} e^{-\frac{1}{2} \lambda} \mathbb{E}[e^{-\lambda|B_T|}].
\end{equation*}
This proves the claim for $\lambda \geq 0$, since for any $n \leq n_0$ we can use the trivial estimate
\begin{equation*}
\mathbb{E}[e^{-\lambda|X_s^{(n)}|}] \geq \mathbb{P}\{X_s^{(n)} = 0 \text{ for all } s \in [0, t]\} \geq \mathbb{P}\{X_s^{(n)} = 0 \text{ for all } s \in [0, T]\}.
\end{equation*}
so that we choose the constant $c(\lambda, T)$ as claimed.

Finally, if $\lambda < 0$, we can use that
\begin{equation*}
dS_{nt}(\phi_\lambda(\cdot/n))(k) = \mathbb{E}[e^{-\lambda|X_{nt}+k|/n}] \geq e^{-\lambda|k/n|} \mathbb{E}[e^{\lambda|X_t^{(n)}|}],
\end{equation*}
and the latter expectation can be bounded uniformly in $n$ and $t \in [0, T]$ as in case $\lambda \geq 0$.
\[\square\]
We also need the following local central limit theorem, which is just a reformulation of Lemma 8 and Lemma 59 in [DEF+02a].

**Lemma A.3.** Let $p_t$ resp. $p_t^{(2)}$ denote the usual one- resp. two-dimensional heat kernel, and $d_t$ resp. $d_t^{(2)}$ its discrete counterpart. Then we have for all $t > 0$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{Z}} \left| n d_{n^2 t}(x) - p_t(x/n) \right| = 0,$$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{Z}^2} \left| n^2 d_{n^2 t}^{(2)}(x) - p_t^{(2)}(x/n) \right| = 0. \quad (75)$$

For the next lemma, we recall that $(\bar{S}_t)_{t \geq 0}$ (resp. $(d_{\bar{S}_t})_{t \geq 0}$) denotes the semigroup of two-dimensional standard Brownian motion (resp. simple symmetric random walk) killed upon hitting the diagonal in $\mathbb{R}^2$ (resp. $\mathbb{Z}^2$), as defined in [19].

**Lemma A.4 (Convergence of semigroups).** Suppose $(\mu_0, \nu_0) \in \mathcal{M}_{\text{tem}}(\mathbb{R})^2$ (resp. $\mathcal{M}_{\text{rap}}(\mathbb{R})^2$) and $\phi, \psi \in \bigcup_{\lambda > 0} C^\lambda_\mu(\mathbb{R})$ (resp. $\bigcup_{\lambda > 0} C^\lambda_{\mathcal{Z}}(\mathbb{R})$). For each $n \in \mathbb{N}$, define $(u_0^{(n)}, v_0^{(n)})$ by [44] and $(\phi^{(n)}, \psi^{(n)})$ by [44]. Then we have the convergence

$$\left\langle \phi^{(n)}, d S_{n^2 t} u_0^{(n)} \right\rangle_{\mathbb{Z}} \overset{n \to \infty}{\longrightarrow} \left\langle \phi, S_t \mu_0 \right\rangle_{\mathbb{R}} \quad (76)$$

and

$$\left\langle \phi^{(n)} \otimes \psi^{(n)}, d S_{n^2 t} (v_0^{(n)} \otimes v_0^{(n)}) \right\rangle_{\mathbb{Z}^2} \overset{n \to \infty}{\longrightarrow} \left\langle \phi \otimes \psi, \bar{S}_{t} (\mu_0 \otimes \nu_0) \right\rangle_{\mathbb{R}^2} \quad (77)$$

for all $t > 0$.

Given the local central limit theorem in Lemma A.3, (76)-(77) follow by standard arguments, e.g. along the lines of the proof of [DEF+02b, Lemma 50]. For the killed semigroup, one uses that the transition density $\tilde{p}_t$ of $\bar{S}_t$ is given by

$$\tilde{p}_t(x, y; a, b) = \begin{cases} 
1_{\{a < b\}} \left( p_t^{(2)}(x - a, y - b) - p_t^{(2)}(x - b, y - a) \right) & \text{if } x < y \\
1_{\{a > b\}} \left( p_t^{(2)}(x - a, y - b) - p_t^{(2)}(x - b, y - a) \right) & \text{if } x > y
\end{cases}$$

$$= \left( 1_{\{x < y, a < b\}} + 1_{\{x > y, a > b\}} \right) \left( p_t^{(2)}(x - a, y - b) - p_t^{(2)}(x - b, y - a) \right). \quad (78)$$

where $p_t^{(2)}$ denotes the usual two-dimensional heat kernel. The corresponding discrete-space transition density reads

$$d_{\tilde{p}_t}(k, \ell; a, b) = \left( 1_{\{k < \ell, a < b\}} + 1_{\{k > \ell, a > b\}} \right) \left( d_t^{(2)}(k - a, \ell - b) - d_t^{(2)}(k - b, \ell - a) \right). \quad (79)$$

In particular, by the above form of the densities (and the symmetry of the usual heat kernel) it is immediately seen that these semigroups are symmetric.

### A.3 The topology on path space

Suppose $E$ is a Polish space and $I \subseteq \mathbb{R}$, then we denote by $D_I(E)$ resp. $C_I(E)$ the space of càdlàg resp. continuous $E$-valued paths $t \mapsto f_t$, $t \in I$. In this paper, we will always have $I = [0, \infty)$ or $I = (0, \infty)$ and $E \in \{(C_{\text{tem}})^m, (C_{\text{rap}})^m, \mathcal{M}_{\text{tem}}(\mathcal{S})^m, \mathcal{M}_{\text{rap}}(\mathcal{S})^m \}$ for $\mathcal{S}$ either $\mathbb{Z}$.
or $\mathbb{R}$ and some power $m \in \mathbb{N}$. The space $D_I(E)$ is then also Polish if we endow it with the usual Skorokhod ($J_1$)-topology.

In this paper, we will also make use of the weaker Meyer-Zheng ‘pseudo-path’ topology on $D_{[0,\infty)}(E)$. This topology was introduced in [MZ84] and can be formalized as follows: let\( \lambda(dt) := \exp(-t) \, dt \) and let $w(t), t \in [0, \infty)$ be an $E$-valued Borel function. Then, a ‘pseudo-path’ corresponding to $w$ is defined to be the probability law $\psi_w$ on $[0, \infty) \times E$ given as the image measure of $\lambda$ under the mapping $t \mapsto (t, w(t))$. Note that with this definition two functions which are equal Lebesgue-a.e. give rise to the same pseudo-path. Moreover, $w \mapsto \psi_w$ is one-to-one on the space of càdlàg paths $D_{[0,\infty)}(E)$, and thus we obtain an embedding of $D_{[0,\infty)}(E)$ into the space of probability measures on $[0, \infty) \times E$. The induced topology on $D_{[0,\infty)}(E)$ is then called the pseudo-path topology. Note that convergence in this topology is equivalent to convergence in Lebesgue measure (see [MZ84, Lemma 1]).

We will need the following sufficient condition for relative compactness of a sequence of stochastic processes on $D_{[0,\infty)}(E)$ equipped with this topology, due to [MZ84, Thm. 4] in the case $E = \mathbb{R}$. If $(X_t^{(n)})_{t \geq 0}, n \in \mathbb{N}$ is a sequence of càdlàg real-valued stochastic processes, with $(X_t^{(n)})_{t \geq 0}$ adapted to a filtration $(\mathcal{F}_t^{(n)})_{t \geq 0}$, then Meyer and Zheng require that
\[
\sup_{n \in \mathbb{N}} \left( V_T(X_t^{(n)}) + \sup_{t \leq T} \mathbb{E}[|X_t^{(n)}|]\right) < \infty
\]
for all $T > 0$, where $V_T(X_t^{(n)})$ denotes the conditional variation of $X_t^{(n)}$ up to time $T$, defined as
\[
V_T(X_t^{(n)}) := \sup \mathbb{E}\left[ \sum_{i} \mathbb{E}[|X_{t_{i+1}}^{(n)} - X_{t_i}^{(n)}| | \mathcal{F}_{t_i}^{(n)}]\right],
\]
and the sup is taken over all partitions of the interval $[0, T]$. For our purposes we need the version for processes taking values in general separable metric spaces $E$ stated in [Kur91]. In fact, according [Kur91, Cor. 1.4] we only have to check condition (80) for the coordinate processes and in addition a compact containment condition to deduce tightness of our measure-valued processes in the pseudopath topology (which again is equivalent to the topology of convergence in Lebesgue measure).

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