CRYPTANALYSIS OF ANSHEL-ANSHEL-GOLDFELD-LEMIEUX KEY AGREEMENT PROTOCOL

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Abstract. The Anshel-Anshel-Goldfeld-Lemieux (abbreviated AAGL) key agreement protocol [2] is proposed to be used on low-cost platforms which constrain the use of computational resources. The core of the protocol is the concept of an Algebraic Eraser TM (abbreviated AE) which is claimed to be a suitable primitive for use within lightweight cryptography. The AE primitive is based on a new and ingenious idea of using an action of a semidirect product on a (semi)group to obscure involved algebraic structures. The underlying motivation for AAGL protocol is the need to secure networks which deploy Radio Frequency Identification (RFID) tags used for identification, authentication, tracing and point-of-sale applications.

In this paper we revisit the computational problem on which AE relies and heuristically analyze its hardness. We show that for proposed parameter values it is impossible to instantiate the secure protocol. To be more precise, in 100% of randomly generated instances of the protocol we were able to find a secret conjugator $z$ generated by TTP algorithm (part of AAGL protocol).

1. THE COLORED BURAU KEY AGREEMENT PROTOCOL

A general mathematical framework of AAGL protocol is quite complicated. In this paper we try to omit unnecessary details and simplify the notation of [2] as much as possible. We refer an interested reader to [2, Sections 2 and 3] for a complete description. Here we start out by giving a particular implementation of the primitive called the Colored Burau Key Agreement Protocol (CBKAP).

1.1. A platform group. Fix an integer $n \geq 7$ and a prime $p$. Let $t = (t_1, \ldots, t_n)$ be a tuple of formal variables. Define matrices

$$x_1(t) = \begin{pmatrix} -t_1 & 1 \\ \vdots & \ddots \\ & & 1 \end{pmatrix}$$

and for $i = 2, \ldots, n-1$

$$x_i(t) = \begin{pmatrix} 1 \\ \vdots \\ t_i & -t_i & 1 \\ & & \ddots \end{pmatrix}$$

which is the identity matrix except for the $i$th row where it has successive entries $t_i, -t_i, 1$ with $-t_i$ on the diagonal. We look at the matrices $x_1(t), \ldots, x_{n-1}(t)$ as elements of the group $GL(n, \mathbb{F}_p(t))$ of $n \times n$ matrices with entries as Laurent polynomials over the finite field $\mathbb{F}_p$. The symmetric group on $n$ symbols $S_n$ acts on $GL(n, \mathbb{F}_p(t))$ by permuting the variables $t_1, \ldots, t_n$. We denote the result of the action of $s \in S_n$ on $x \in GL(n, \mathbb{F}_p(t))$ by $^sx$.

The semidirect product $GL(n, \mathbb{F}_p(t)) \rtimes S_n$ of the groups $GL(n, \mathbb{F}_p(t))$ and $S_n$ relative to the defined action of $S_n$ on matrices $GL(n, \mathbb{F}_p(t))$ is a set of pairs

$$\{(m, s) \mid m \in GL(n, \mathbb{F}_p(t)), \ s \in S_n\}$$
with multiplication given by
\[(m_1, s_1) \cdot (m_2, s_2) := (m_1 \cdot s_1^i m_2, s_1 \cdot s_2).\]
Denote by \(s_i = (i, i+1) \in S_n\) the transposition which interchanges \(i\) and \(i+1\) and by \(g_i\) the element of the semidirect product \(GL(n, \mathbb{F}_p(t)) \rtimes S_n\)
\[g_i = (x_i(t), s_i).\]
A subgroup
\[G = \langle g_1, \ldots, g_{n-1} \rangle\]
of \(GL(n, \mathbb{F}_p(t)) \rtimes S_n\) is called the colored Burau group. The group \(G\) is a platform group for AAGL key agreement protocol.

Recall that the group \(B_n\) of \(n\)-strand braids has the classical Artin’s presentation:
\[B_n = \left\{ \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \text{ if } |i - j| = 1, \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i - j| > 1 \right\}.
\]
A word over the group alphabet \(\{\sigma_1, \ldots, \sigma_{n-1}\}\) is called a braid word. Any \(n\)-strand braid can be represented by a braid word. The length of a shortest braid word representing an element \(g \in B_n\) is called the geodesic length of \(g\) relative to the Artin’s set of generators and is denoted by \(|g|\). The function \(|\cdot| : B_n \to \mathbb{N}\) is called the geodesic length function on \(B_n\).

**Lemma 1.1.** The elements \(g_i = (x_i(t), s_i)\), for \(i = 1, 2, \ldots, n - 1\), satisfy the braid relations and hence determine a representation of the braid group \(B_n\), i.e., the mapping \(\varphi : B_n \to G\) defines a group epimorphism.

**Proof.** Straightforward check. \(\square\)

### 1.2. Action of the platform group on \(GL(n, \mathbb{F}_p)\)

Fix elements \(\tau_1, \ldots, \tau_n \in \mathbb{F}_p\) and define a homomorphism \(\pi\) which maps \(GL(n, \mathbb{F}_p(t))\) into \(GL(n, \mathbb{F}_p)\) by assigning the value \(\tau_i\) to the variable \(t_i\), i.e., by evaluating a matrix at \(\tau_1, \ldots, \tau_n\). We call \(\pi\) the evaluation function.

**Assumption on \(\tau_1, \ldots, \tau_n\).** We assume that \(\pi\) defines a correct group homomorphism.

Relative to the chosen tuple \(\tau_1, \ldots, \tau_n \in \mathbb{F}_p\) and the corresponding function \(\pi\) one can define an action of \(GL(n, \mathbb{F}_p(t)) \rtimes S_n\) on \(GL(n, \mathbb{F}_p) \times S_n\) by putting
\[(m_1, s_1) \star (m_2, s_2) = (m_1 \cdot \pi(x_1 m_2, s_1 s_2))\]
where \(\star\) denotes the action. Indeed, it is not difficult to check that \(\star\) is an action and satisfies the property
\[((m_1, s_1) \star (m_2, s_2)) \star (m_3, s_3) = (m_1, s_1) \star ((m_2, s_2) \cdot (m_3, s_3)).\]
We say that \((m_1, t_1)\) and \((m_2, t_2)\) \(*\)-commute if the equality
\[(\pi(m_1), s_1) \star (m_2, s_2) = (\pi(m_2), s_2) \star (m_1, s_1)\]
holds. The next lemma is obvious.

**Lemma 1.2.** Let \(w = \prod_{k=1}^m (x_i(t), s_{i_k})\) and \(v = \prod_{p=1}^l (x_j(t), s_{j_p})\) be such that \(|i_k - j_p| > 1\) for every \(k = 1, \ldots, m\) and \(p = 1, \ldots, l\). Then the elements \(w\) and \(v\) \(*\)-commute.

### 1.3. The protocol

Before the parties perform actual transmissions the following data is being prepared by the Third Trusted Party (TTP).

- A matrix \(m_0 \in GL(n, \mathbb{F}_p)\) which has an irreducible characteristic polynomial over \(\mathbb{F}_p\). The choice of \(m_0\) is not relevant for the purposes of this paper, we refer the reader to [2] for more information on how \(m_0\) can be generated randomly.
- \(*\)-commuting subgroups \(A = \langle w_1, \ldots, w_r \rangle\) and \(B = \langle u_1, \ldots, u_r \rangle\) of the group \(G\). We want to point out that the elements \(w_i\) and \(u_j\) are given to us as products of generators of \(G\) and there inverses, i.e., as formal words in group alphabet \(\{g_1, \ldots, g_{n-1}\}\). We prefer this form because it allows us to avoid time consuming matrix multiplication in \(GL(n, \mathbb{F}_p(t))\).
Both, the matrix $m_0$ and subgroups $A$ and $B$, can be chosen only once. Now, the public and private keys are chosen as follows:

**Alice’s Private Key:** is a pair which consists of a matrix of the form

$$n_a = l_1 m_0^{\alpha_1} + l_2 m_0^{\alpha_2} + \ldots + l_r m_0^{\alpha_r} \in GL(n, \mathbb{F}_p)$$

(where $l_1, \ldots, l_r \in \mathbb{F}_p$ and $r, \alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$) and a random sequence $w_{i_1}^{\epsilon_1}, \ldots, w_{i_m}^{\epsilon_m}$ of generators of $A$ and their inverses.

**Alice’s Public Key:** is an element

$$A_{\text{public}} = (n_a, id) \ast w_{i_1}^{\epsilon_1} \ast \ldots \ast w_{i_m}^{\epsilon_m} \in GL(n, \mathbb{F}_p) \times S_n.$$  

Recall that each $w_{i_k}$ is given as a formal product of the generators of $G$. To perform the $\ast$-operation efficiently one should not directly compute $w_{i_k}$, but consequently apply the factors of $w_{i_k}$ to the argument.

**Bob’s Private Key:** is a pair which consists of a matrix of the form

$$n_b = l'_1 m_0^{\beta_1} + l'_2 m_0^{\beta_2} + \ldots + l'_r m_0^{\beta_r} \in GL(n, \mathbb{F}_p)$$

(where $l'_1, \ldots, l'_r \in \mathbb{F}_p$ and $r', \beta_1, \ldots, \beta_r \in \mathbb{Z}^+$) and a random sequence $v_{j_1}^{\delta_1}, \ldots, v_{j_l}^{\delta_l}$ of generators of $B$ and their inverses.

**Bob’s Public Key:** is a pair

$$B_{\text{public}} = (n_b, id) \ast v_{j_1}^{\delta_1} \ast \ldots \ast v_{j_l}^{\delta_l} \in GL(n, \mathbb{F}_p) \times S_n.$$  

Again, each $v_{j_k}$ is given as a formal product of the generators of $G$. To perform the $\ast$-operation efficiently one should not directly compute $v_{j_k}$, but consequently apply the factors of $v_{j_k}$ to the argument.

The **shared key:** is an element of $GL(n, \mathbb{F}_p) \times S_n$ obtained by Alice in the form

$$[(n_a, id) \cdot B_{\text{public}}] \ast w_{i_1}^{\epsilon_1} \ast \ldots \ast w_{i_m}^{\epsilon_m}$$

and by Bob in the form

$$[(n_b, id) \cdot A_{\text{public}}] \ast v_{j_1}^{\delta_1} \ast \ldots \ast v_{j_l}^{\delta_l}$$

It requires a little work to prove that the obtained elements are indeed equal in $GL(n, \mathbb{F}_p)$. We omit the proof.

1.4. **TTP algorithm.** The cornerstone part of the proposed key-exchange is the choice of $\ast$-commuting subgroups of the group $G$. The basic idea is to use Lemma 1.1 and choose commuting subgroups $A$ and $B$ in $B_n$ and then pull them into $G$ using the epimorphism $\varphi$. The resulting subgroups $\varphi(A)$ and $\varphi(B)$ of $G$ commute. Moreover, for any choice of $\pi$ the subgroups $\varphi(A)$ and $\varphi(B)$ $\ast$-commute.

Before we present the algorithm we need to give some details about the braid group $B_n$. The group $B_n$ has a cyclic center generated by an element $\Delta^2$ where $\Delta$ is an element called the half twist and can be expressed in the generators of $B_n$ as follows:

$$\Delta = (\sigma_1 \ldots \sigma_{n-1}) \cdot (\sigma_1 \ldots \sigma_{n-2}) \ldots \cdot (\sigma_1).$$

Any element $g \in B_n$ can be uniquely represented in a form

$$\Delta^p \xi_1 \ldots \xi_p$$

satisfying certain conditions and called the left Garside normal form.

Now, since $\Delta^2$ is a central element it follows that element $u, w$ commute in $B_n$ if and only $u \Delta^{2p}$ and $w \Delta^{2r}$ do (for any choice of $p, r \in \mathbb{Z}$). Hence, we may always assume that the normal forms of the generators $\{w_1, \ldots, w_n\}$ and $\{v_1, \ldots, v_r\}$ have the power of $\Delta$ equal to 0 or $-1$. When we say that we reduce a braid modulo $\Delta^2$ we mean changing the $\Delta$-power of its normal form to $-1$ or 0 depending on parity.

The algorithm below (originally proposed in [2]) generates two $\ast$-commuting subgroups.

**Algorithm 1.3. (TTP algorithm)**

1. Choose two secret subsets $BL = \{b_1, \ldots, b_a\}$, $BR = \{b_{r1}, \ldots, b_{r\beta}\}$ of the set of generators of $B_n$, where $|l_i - r_j| \geq 2$ for all $1 \leq i \leq l$ and $1 \leq j \leq r$.
2. Choose a secret element $z \in B_n$. 


the scheme the authors propose two slightly different sets of parameters:

Parameter set # 1.
- Let \( n = 14 \), \( p = 13 \), and \( r = 3 \).
- Choose the conjugator \( z \) randomly of length 17.
- Choose the words \( w_i \) and \( v_j \) randomly of length approximately 10.
- The number \( \gamma \) of the words \( w_i \) and \( v_j \) is 27.

• Parameter set # 2.

We want to point out that TTP algorithm produces generators of two commuting subgroups in \( B_n \). Alice and Bob need to compute their images in \( GL(n, \mathbb{F}_p(t)) \) to obtain \( \ast \)-commuting subgroups.

1.5. Security assumptions. It was noticed in [2] that if the conjugator \( z \) generated randomly by TTP algorithm is known then there exists an efficient linear attack on the scheme which is able to recover the shared key of the parties. The problem of recovering the exact \( z \) seems like a very difficult mathematical problem because it reduces to solving the system of equations

$$
\begin{align*}
\frac{w_1'}{w_z} &= \Delta^2 p_1 zw_1 z^{-1} \\
&\vdots \\
\frac{w_\gamma'}{w_z} &= \Delta^2 p_\gamma zw_\gamma z^{-1} \\
\frac{v_1'}{v_z} &= \Delta^2 r_1 zv_1 z^{-1} \\
&\vdots \\
\frac{v_\gamma'}{v_z} &= \Delta^2 r_\gamma zv_\gamma z^{-1}
\end{align*}
$$

(1)

which has too many unknowns, only left hand sides (i.e., elements \( w_1', \ldots, w_\gamma', v_1', \ldots, v_\gamma' \)) are known. Hence, it might be difficult to find the original \( z \).

Now observe that the AAGL key exchange protocol uses only the output of TTP algorithm, namely the tuples \( \{v_1', \ldots, v_\gamma'\} \) and \( \{w_1', \ldots, w_\gamma'\} \) since all internal values in TTP algorithm are not available to the parties. In other words it is irrelevant for the protocol how two particular commuting generating sets were constructed. This observation leads us to the following problem

For tuples \( \{v_1', \ldots, v_\gamma'\} \) and \( \{w_1', \ldots, w_\gamma'\} \) find any \( z' \) and any numbers \( p_1, \ldots, p_\gamma, r_1, \ldots, r_\gamma \in \mathbb{Z} \) such that the words \( \{\Delta^2 p_1 z^{-1} v_1' z', \ldots, \Delta^2 p_\gamma z^{-1} v_\gamma' z'\} \) and \( \{\Delta^2 r_1 z^{-1} w_1' z', \ldots, \Delta^2 r_\gamma z^{-1} w_\gamma' z'\} \) can be expressed as words over two disjoint commuting subsets of generators of \( B_n \).

This is a new problem for computational group theory. Let us refer to it as simultaneous conjugacy separation search problem (abbreviated SCSSP). We want to emphasize that SCSSP has little in common with the simultaneous conjugacy search problem often referenced in the papers on the braid group cryptanalysis. The main difference is that in the conjugacy search problem both conjugate elements are available and the goal is to recover the secret conjugator. And in case of SCSSP only the left side of the equation is known. It is not clear if one of the problems can be reduced to the other.

It follows from the observation above that any solution \( z' \) to a problem stated above plays a role of a conjugator \( z \) and can be used in a linear attack outlined in [2]. The main goal of this paper is to present an algorithm which for proposed parameter values solves SCSSP. Experimental results convince us that our attack is a serious threat for AAGL as the success rate is 100%. Furthermore, a slight modification of the algorithm produces the exact \( z \) generated by TTP in 40% of randomly generated instances.

1.6. Proposed parameter values. To provide 80 bits of security against the exhaustive search for \( z \) for the scheme the authors propose two slightly different sets of parameters:

• Parameter set # 1.
  - Let \( n = 14 \), \( p = 13 \), and \( r = 3 \).
  - Choose the conjugator \( z \) randomly of length 17.
  - Choose the words \( w_i \) and \( v_j \) randomly of length approximately 10.
  - The number \( \gamma \) of the words \( w_i \) and \( v_j \) is 27.

• Parameter set # 2.
2. TTP ATTACK

In this section we describe a heuristic attack which finds a solution to a given instance of SCSSP. The main ingredient in our attack is a length function on the group $B_n$. As it is explained in [6] there are no known efficiently computable and “sharp” length functions for braid groups. Therefore, for our attack we adopt the method of approximation of the geodesic length function originally proposed in [6]. In all our algorithms by $|·|$ we denote approximation of the geodesic length function.

We present results of experiments which show that a fast heuristic procedure based on the length-based reduction is extremely successful for the suggested parameters. In fact, every instance of TTP algorithm generated in our experiments has been broken.

2.1. Generation. The original paper [2] lacks any details on how to randomly generate the secret element $z$ and the words $\{w_1, \ldots, w_\gamma\}$, $\{v_1, \ldots, v_\gamma\}$ in TTP algorithm. Hence, in all our experiments:

- The word $z$ is taken uniformly randomly as a word of a particular length from the ambient free group $F(\sigma_1, \ldots, \sigma_{n-1})$.
- The words $w_1, \ldots, w_\gamma$ and $v_1, \ldots, v_\gamma$ are taken uniformly randomly as words of particular lengths from the ambient free groups $F(BL)$ and $F(BR)$.

Also, the authors suggest to take the sets $BL$ and $BR$ randomly on step (1) of TTP algorithm. Observe that in general this might result in a choice of $BL$ such that for some $1 \leq i < j < k \leq n - 1$

$$\sigma_i, \sigma_k \in BL, \text{ but } \sigma_j \in BR.$$ 

We think that this situation is not desirable as it excludes the use of at least two braid generators in the words $w_i$ and $v_j$. We think that the choice of the following sets

$$BL = \{\sigma_1, \ldots, \sigma_l\} \text{ and } BR = \{\sigma_{l+2}, \ldots, \sigma_{n-1}\}$$

(where $n$ is an even number and $l = (n - 2)/2$) is optimal as it excludes only $\sigma_{l+1}$ which maximizes the size of a space for the words $w_1, \ldots, w_\gamma$ and $v_1, \ldots, v_\gamma$.

2.2. Recovering $\Delta$-powers. The first stage in our attack is recovering $\Delta$ powers in the system [1], i.e., computing numbers $p_1, \ldots, p_l$ and $r_1, \ldots, r_\gamma$. The main tool in our computations below is the triangular inequality for the Cayley graph of the braid group $B_n$. Observe that the following inequalities hold.

(Parameter set #1) For each $i = 1, \ldots, \gamma$

$$|z^{-1}u_i z| \leq 2|z| + |u_i| = 44 \text{ and } |z^{-1}w_j z| \leq 2|z| + |w_j| = 44$$

and

$$|\Delta^{2p}| = pn(n - 1) = 182p.$$

Hence, $|\Delta^{2p}z^{-1}u_i z|, |\Delta^{2p}z^{-1}w_j z| \in [182p - 44, 182p + 44]$ and

$$|\Delta^{2p}z^{-1}u_i z| - |\Delta^{2(p-1)}z^{-1}u_i z| \geq 182 - 2 \cdot 44 = 94$$

$$|\Delta^{2p}z^{-1}w_j z| - |\Delta^{2(p-1)}z^{-1}w_j z| \geq 182 - 2 \cdot 44 = 94.$$ 

(Parameter set #2) For each $i = 1, \ldots, \gamma$

$$|z^{-1}u_i z| \leq 2|z| + |u_i| = 46 \text{ and } |z^{-1}w_j z| \leq 2|z| + |w_j| = 46$$

and

$$|\Delta^{2p}| = pn(n - 1) = 132p.$$

Hence $|\Delta^{2p}z^{-1}u_i z|, |\Delta^{2p}z^{-1}w_j z| \in [132p - 46, 132p + 46]$ and

$$|\Delta^{2p}z^{-1}u_i z| - |\Delta^{2(p-1)}z^{-1}u_i z| \geq 132 - 2 \cdot 46 = 40.$$

- Let $n = 12$, $p = 13$, and $r = 3$.
- Choose the conjugator $z$ randomly of length 18.
- Choose the words $w_i$ and $v_j$ randomly of length approximately 10.
- The number $\gamma$ of the words $w_i$ and $v_j$ is 27.
\[ |\Delta^{2p} z^{-1} w_j z| - |\Delta^{2(p-1)} z^{-1} w_j z| \geq 132 - 2 \cdot 46 = 40. \]

This observation implies that for both parameter sets the sequences \( \{ |\Delta^{2p} z^{-1} u_i z| \}_{p=0}^{\infty} \) and \( \{ |\Delta^{2p} z^{-1} w_j z| \}_{p=0}^{\infty} \) are strictly increasing. Thus, to recover the original power of \( \Delta \) one can repeatedly multiply \( u'_i \) (and \( w'_j \)) on the left by \( \Delta^2 \) until the length cannot be reduced anymore (see Algorithm 2.1). Moreover, since the difference between two elements differing by \( \Delta^2 \) is at least 40 even crude approximations of the length function must work.

**Algorithm 2.1** (\( \Delta \)-power recovery).  
**INPUT:** An element \( w \in B_n \).  
**OUTPUT:** An element \( u \) minimal in the left coset \( \langle \Delta^{-2} \rangle w \).

**COMPUTATIONS:**

A. Set \( u = w \).  
B. If \( |u| > |\Delta^{-2} u| \) then set \( u = \Delta^{-2} u \) and goto B.  
C. If \( |u| > |\Delta^2 u| \) then set \( u = \Delta^2 u \) and goto B.  
D. Otherwise output \( u \).

Clearly Algorithm 2.1 always terminates. The time complexity of the algorithm depends on the complexity of the procedure which approximates the geodesic length. The procedure is heuristic and its worst case complexity is not known. Experimental results in [5] suggest that the length approximation can be efficiently computed for most braid words and we estimate the expected complexity of the procedure as \( O(n) \). Under this assumption, it is easy to see that the power-recovery algorithm can be executed in at most \( O(|w| + n^2) |w|/n^2 = O(|w|^2/n^2 + |w|) \) steps as the algorithm performs up to \( |w|/n^2 \) iterations and on each iteration for a word \( u \) of length up to \( |w| \) the length of a word \( \Delta^2 u \) is estimated.

### 2.3. Recovering conjugator.

The second part of the attack computes a secret conjugator. At this point we assume that all \( \Delta \)-powers from the system (1) are successfully found and we have a system of equations of the form

\[
\begin{align*}
\exists \ z_1, z_2, \ldots, z_r & \ s.t. \\
& \begin{cases}
  u''_1 = z w_1 z^{-1} \\
  \cdots \\
  u''_r = z w_r z^{-1}
\end{cases} \\
\text{or} & \begin{cases}
  z^{-1} w'_1 z = w_1 \\
  \cdots \\
  z^{-1} w'_r z = w_r
\end{cases}
\end{align*}
\]

where only elements \( u''_i = \Delta^{-2p} u'_i \) and \( w''_j = \Delta^{-2r} w'_j \) are known. Let us call two sets of braids separated if they can be expressed as words over disjoint commuting sets of generators of \( B_n \). As mentioned in Section 1.5 to break the protocol it is sufficient to find any conjugator \( z' \) which conjugates two tuples of elements \( (u''_1, \ldots, u''_r) \) and \( (w''_1, \ldots, w''_r) \) into two separated tuples of elements \( (u_1, \ldots, u_r) \) and \( (w_1, \ldots, w_r) \). This is the main goal of our attack.

Let \( \tilde{u} = (u_1, \ldots, u_m) \) be a tuple of elements in \( B_n \) and \( x \) is an element of \( B_n \). Denote by \( |\tilde{u}| \) the total length of elements in \( \tilde{u} \), i.e., put

\[ |\tilde{u}| = \sum_{i=1}^{m} |u_i|. \]

Denote by \( \tilde{u}^x \) a tuple obtained from \( \tilde{u} \) by conjugation of each its element by \( x \). It is intuitively clear that conjugation of a tuple of braids by a random element \( x \) almost always increases the length of the tuple. In other words, for a random element \( x \) the inequality

\[ |\tilde{u}^x| > |\tilde{u}| \]

is almost always true. We do not have a proof of this fact, but numerous experiments convince us that it is true. Moreover, conjugation by longer elements almost always results in longer tuples.

The idea that conjugation consequently increases the length of tuples is not new. It was used in papers [4], [3] for different length functions with different success. But the most successful is a recent attack [6].
which uses approximation of the geodesic length. In this paper we use the idea of separating two tuples of braid. To find $z'$ we repeatedly conjugate the tuple $(u_1', \ldots, u_n', w_1', \ldots, w_n')$ by generators of $B_n$ and their inverses and if for some generator $\sigma_k^\pm 1$ the decrease of the total length of the tuple is observed then it is reasonable to guess that $\sigma_k^\pm 1$ is involved in $z'$.

Algorithm 2.2 (Recovering conjugator - I).

**INPUT:** Tuples $\tilde{a} = \{a_1, \ldots, a_\gamma\}$ and $\tilde{b} = \{b_1, \ldots, b_\gamma\}$.

**OUTPUT:** An element $z'$ separating tuples $\tilde{a}$ and $\tilde{b}$.

**INITIALIZE:** Set $z' = 1$.

**COMPUTES:**

A. For each $i = 1, \ldots, n - 1$ and $\varepsilon = \pm 1$ conjugate tuples $\tilde{a}$ and $\tilde{b}$ by a generator $\sigma_i^\varepsilon$ and compute

$$\delta_{i,\varepsilon} = |\tilde{a}\sigma_i^\varepsilon| + |\tilde{b}\sigma_i^\varepsilon| - (|\tilde{a}| + |\tilde{b}|).$$

B. If for some $\sigma_i^\varepsilon$ the sets $\tilde{a}\sigma_i^\varepsilon$ and $\tilde{b}\sigma_i^\varepsilon$ are separated then output $z' = \sigma_i^\varepsilon z'$.

C. Otherwise, if all $\delta_{i,\varepsilon}$ are positive (i.e., conjugation by $\sigma_i^\varepsilon$ cannot further decrease the total length) then output FAILURE.

D. Otherwise choose $i$ and $\varepsilon$ for which $\delta_{i,\varepsilon}$ is minimal. Set $z' = \sigma_i^\varepsilon z'$, $\tilde{a} = \tilde{a}\sigma_i^\varepsilon$, and $\tilde{b} = \tilde{b}\sigma_i^\varepsilon$. Goto step A.

The described attack is similar to the one described in [6]. Recall that the main problem in [6] was the existence of so-called peaks (see [6, Definition 2.5]). This phenomenon is a consequence of difficult structure of finitely generated subgroups of braid groups. In this paper, we do not have this problem as $z$ is chosen in the whole group $B_n$.

Note that Algorithm 2.2 is a greedy descend procedure. It may fail due to the fact that there exists a small fraction of words for which the inequality (3) does not hold. It is also prone to the length approximation errors. One can significantly reduce the failure rate of a descent procedure by introducing a backtracking algorithm which allows exploration of more than one search paths. Algorithm 2.3 gives an implementation of the attack with backtracking.

Algorithm 2.3 (Recovering conjugator with Backtracking).

**INPUT:** Tuples $\tilde{a} = \{a_1, \ldots, a_\gamma\}$ and $\tilde{b} = \{b_1, \ldots, b_\gamma\}$.

**OUTPUT:** An element $z'$ separating tuples $\tilde{a}$ and $\tilde{b}$.

**INITIALIZE:** Set $S = \{(\tilde{a}, \tilde{b}, 1)\}$.

**COMPUTES:**

A. If $S = \emptyset$ then output FAILURE.

B. Choose $(\bar{x}, \bar{y}, c) \in S$ such that $|\bar{x}| + |\bar{y}|$ is the minimal.

C. For each $i = 1, \ldots, n - 1$ and $\varepsilon = \pm 1$ conjugate tuples $\bar{x}$ and $\bar{y}$ by a generator $\sigma_i^\varepsilon$ and compute

$$\delta_{i,\varepsilon} = |\bar{x}\sigma_i^\varepsilon| + |\bar{y}\sigma_i^\varepsilon| - (|\bar{x}| + |\bar{y}|).$$

D. If for some $\sigma_i^\varepsilon$ the sets $\bar{x}\sigma_i^\varepsilon$ and $\bar{y}\sigma_i^\varepsilon$ are separated then output $z' = \sigma_i^\varepsilon c$.

E. Otherwise, for each $i = 1, \ldots, n - 1$ and $\varepsilon = \pm 1$ add the tuple $(\bar{x}\sigma_i^\varepsilon, \bar{y}\sigma_i^\varepsilon, \sigma_i^{\varepsilon} c)$ to the set $S$. Goto step A.

We must mention here that, although there is a possibility that Algorithm 2.3 outputs FAILURE or does not terminate on some inputs, this situation has never occurred in our experiments.

Finally, we present another modification of Algorithm 2.2.

Algorithm 2.4 (Recovering conjugator - II).

**INPUT:** Tuples $\tilde{a} = \{a_1, \ldots, a_\gamma\}$ and $\tilde{b} = \{b_1, \ldots, b_\gamma\}$.

**OUTPUT:** An element $z'$ separating tuples $\tilde{a}$ and $\tilde{b}$.

**INITIALIZE:** Set $z' = 1$.

**COMPUTES:**


A. For each \( i = 1, \ldots, n - 1 \) and \( \varepsilon = \pm 1 \) conjugate tuples \( \bar{a} \) and \( \bar{b} \) by a generator \( \sigma_i^\varepsilon \) and compute

\[
\delta_{i,\varepsilon} = |\bar{a}^{\sigma_i^\varepsilon}| + |\bar{b}^{\sigma_i^\varepsilon}| - (|\bar{a}| + |\bar{b}|).
\]

B. If all \( \delta_{i,\varepsilon} \) are positive (i.e., conjugation by \( \sigma_i^\varepsilon \) cannot further decrease the total length) and the sets \( \bar{a} \) and \( \bar{b} \) are separated then output \( z' \).

C. If all \( \delta_{i,\varepsilon} \) are positive (i.e., conjugation by \( \sigma_i^\varepsilon \) cannot further decrease the total length), but the sets \( \bar{a} \) and \( \bar{b} \) are not separated then output FAILURE.

D. Otherwise choose \( i \) and \( \varepsilon \) for which \( \delta_{i,\varepsilon} \) is minimal. Set \( z' = \sigma_i^\varepsilon z' \), \( \bar{a} = \bar{a}^{\sigma_i^\varepsilon} \), and \( \bar{b} = \bar{b}^{\sigma_i^\varepsilon} \). Goto step A.

Algorithms 2.2 and 2.3 are almost the same except that they have different termination conditions. Algorithm 2.2 stops as soon as the tuples are separated, while Algorithm 2.3 tries to minimize the total length of the tuple and when the minimal value is reached it checks if the current tuples are separated.

The complexity of step A in Algorithms 2.2 and 2.4 is \( O(\gamma n (|\pi_i| + |\bar{b}_i|)) \). The maximal number of iterations can be bounded by the total length of the input \( |\pi_i| + |\bar{b}_i| \). A very crude upper bound on the complexity of the two algorithms is \( O(\gamma n (|\pi_i| + |\bar{b}_i|)^2) \).

The complexity of Algorithm 2.3 is harder to estimate. Potentially, the backtracking mechanism may cause the algorithm to explore exponentially many potential solutions. However, our experiments show that a very few backtracking steps are required to find a solution.

2.4. Results of experiments. The attack was tested on different sets of instances of the protocol. In particular we generated the sets \( BL \) and \( BR \) randomly and used fixed sets \( BL = \{\sigma_1, \ldots, \sigma_l\} \) and \( BR = \{\sigma_{l+2}, \ldots, \sigma_{n-1}\} \). We used the proposed values of the parameters (see Section 1.6). In addition the attack was tested on instances generated with the increased length of the secret conjugator \( z \).

In all the experiments Algorithm 2.3 had 100% success of producing a separating conjugator \( z' \). The average time of a run of the algorithm was 4.5 seconds when executed on a Dual Core Opteron 2.2 GHz machine with 4GB of ram. The algorithm without backtracking had slightly smaller but still respectable success rate of 90%. It is very interesting to notice that Algorithm 2.3 actually recovered the original secret conjugator \( z \) in about 40% of the cases. That is the reason we mention this algorithm in the paper.

Experiments with instances of TTP protocol generated using \( |z| = 50 \) (which is almost three times greater than the suggested value) again showed 100% success rate. However, we need to point out that the attack may fail when the length of \( z \) is large relative to the length of \( \Delta^2 \). For instance when in the second parameter set the length of \( z \) is increased to 100, the algorithm recovering \( \Delta \)-powers sometimes output wrong values. Nevertheless, the success rate of Algorithm 2.3 is still about 90% in this case. We think it is possible to modify our algorithms to work with increased parameter values. But the biggest concern here is that the protocol with increased parameter values might be not suitable for purposes of lightweight cryptography.

References

[1] I. Anshel, M. Anshel, D. Goldfeld, An algebraic method for public-key cryptography, Math. Res. Lett. 6 (1999), 287–291.
[2] I. Anshel, M. Anshel, D. Goldfeld, S. Lemieux, "Key Agreement, The Algebraic EraserTM, and Lightweight Cryptography", In "Algebraic Methods in Cryptography", Contemporary Math. 418 (2006), 1–17.
[3] D. Garber, S. Kaplan, M. Teicher, B. Tsaban, U. Vishne "Length-based conjugacy search in the Braid group", available at [http://arxiv.org/abs/math.GR/0209267](http://arxiv.org/abs/math.GR/0209267)
[4] J. Hughes, A. Tannenbaum, "Length-based attacks for certain group based encryption rewriting systems". In: Workshop SEC2020 Sécurité de la Communication sur Intenet, September 2002, Tunis, Tunisia.
[5] A. G. Myasnikov, V. Shpilrain, A. Ushakov. A practical attack on some braid group based cryptographic protocols. In CRYPTO 2005, Lecture Notes Comp. Sc. 3621 (2005), 86-96.
[6] A. D. Myasnikov, A. Ushakov, Length Based Attack and Braid Groups: Cryptanalysis of Anshel-Anshel-Goldfeld Key Exchange Protocol. T. Okamoto and X. Wang (Eds.): PKC 2007, LNCS 4450, 2007, 768.
