WEAK CONVERGENCE RESULTS FOR INHOMOGENEOUS ROTATING FLUID EQUATIONS

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Abstract. We consider the equations governing incompressible, viscous fluids in three space dimensions, rotating around an inhomogeneous vector \( B(x) \): this is a generalization of the usual rotating fluid model (where \( B \) is constant). We prove the weak convergence of Leray–type solutions towards a vector field which satisfies the usual 2D Navier–Stokes equation in the regions of space where \( B \) is constant, with Dirichlet boundary conditions, and a heat–type equation elsewhere. The method of proof uses weak compactness arguments.

Résumé. On considère les équations modélisant des fluides incompressibles et visqueux en trois dimensions d’espace, en rotation rapide autour d’un vecteur non homogène \( B(x) \): on généralise ainsi le modèle habituel des fluides tournants (où \( B \) est constant). On montre la convergence des solutions de Leray vers un champ de vecteurs qui vérifie les équations habituelles de Navier–Stokes 2D dans les régions de l’espace où \( B \) est constant, avec des conditions aux limites de Dirichlet, et une équation de type chaleur ailleurs. La méthode de démonstration repose sur des arguments de compacité faible.

1. Introduction

The aim of this article is to study the asymptotics of solutions of rotating fluid equations, in the case when the rotation vector is non homogeneous. We consider a domain \( \Omega = \Omega_b \times \Omega_3 \), where \( \Omega_b \) denotes either the whole space \( \mathbb{R}^2 \) or any periodic domain of \( \mathbb{R}^2 \), and similarly \( \Omega_3 \) denotes \( \mathbb{R} \) or \( \mathbb{T} \). We are interested in the following system:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \frac{1}{\varepsilon} u \wedge B + \nabla p &= 0 \quad \text{on } \mathbb{R}^+ \times \Omega, \\
\nabla \cdot u &= 0 \quad \text{on } \mathbb{R}^+ \times \Omega, \\
u |_{t=0} &= u^0 \quad \text{on } \Omega \end{align*}
\]

where \( B = be_3 \) is the rotation vector, and \( b \) is a smooth vector field defined in \( \Omega_b \). We shall suppose throughout this paper that \( b \) does not vanish, and is equal to a positive constant \( b_0 \) perturbated by a smooth \( C^\infty \) function; more assumptions on \( b \) will be made as we go along. Before stating the result we shall prove here, let us recall some well-known facts in the constant case \( (b = 1) \). The rotating fluid equations, with \( b \) constant and homogeneous, modelize the movement of the atmosphere or the oceans at mid-latitudes (see for instance [9] or [16]).

fluid is supposed to be incompressible (which corresponds to the hydrostatic approximation), and its viscosity is \( \nu > 0 \). The vector field \( u \) is the velocity and the scalar \( p \) is the pressure, both are unknown. The parameter \( \varepsilon \) is the Rossby number, and its inverse stands for the speed of rotation of the Earth. Taking the limit \( \varepsilon \to 0 \) means that the scale of motion of the fluid is much smaller than that of the Earth. Note that one can also see \( B \) as a magnetic field, in which case it makes sense to understand what happens when \( b \) is not homogeneous; that also holds if one wants to study the movement of the atmosphere in other regions than mid-latitudes.

In the constant case, those equations have been studied by a number of authors. We refer for instance to the works of A. Babin, A. Mahalov and B. Nicolaenko \([2] - [1]\), I. Gallagher \([7]\), E. Grenier \([10]\) for the periodic case, and J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier \([5]\) for the whole space case as well as \([6]\) for the case of horizontal plates with Dirichlet boundary conditions (for such boundary conditions we refer also to the work of E. Grenier and N. Masmoudi \([11]\) as well as N. Masmoudi \([15]\)). The results in those papers concern both weak and strong solutions; in this article we shall only be concerned with Leray-type weak solutions \([12]\): we will see in Section 2 below that their existence is an easy adaptation of the proof of Leray’s existence theorem \([12]\). In the constant case, it is known that weak solutions converge towards the solution of the two-dimensional Navier-Stokes equations. Such a result in the whole space case is due to Strichartz-type estimates (which are obtained by writing the solution of the linearized problem in Fourier space), whereas in the periodic case it follows from the study of the (discrete) spectrum of the rotating fluid operator (following methods introduced by S. Schochet in \([17]\)).

The problem here if we want to follow those methods is that it does not seem a good idea to take the Fourier transform of the operator

\[
Lu \overset{\text{def}}{=} P(u \wedge B), \quad \nabla \cdot u = 0,
\]

where \( P \) denotes the Leray projection onto divergence free vector fields, when \( B \) is not homogeneous; moreover the study of the spectrum of \( L \) is not an easy matter. So our strategy to study this problem is first to try and recover the well-known results of the constant case without using any information on the spectrum of \( L \) (other than the determination of its kernel), and without using the Fourier transform. This will be achieved in Section 3. Then the study of the variable case will be an adaptation of the constant case, in Section 4.

Before stating the results we shall prove in this paper, let us comment on the difficulties compared with the constant case: as stated above, it is easy to construct a bounded family of weak solutions to our problem, whether \( b \) is constant or not. Hence one can construct a weak limit point \( \pi \), and the question we want to address is to find the equation satisfied by \( \pi \). Of course the problem consists in taking the limit in the non linear part of the equation. As noted above, we do not wish to study the spectrum of the operator \( L \) since that seems to be a difficult issue. So we cannot apply the usual, constant \( b \) methods, as to our knowledge they all involve spectral properties of \( L \). The idea therefore is to turn to what is known as “weak compactness methods”, in the spirit of Lions and Masmoudi \([13] - [14]\) (for the incompressible limit). We shall recall briefly below what those methods are, and then we shall state the main results of this paper.
1.1. **Weak compactness methods.** Let us explain what weak compactness methods are all about. The idea is as follows: as usual the trouble to find the limit of the equation comes from the bilinear terms. They can be separated into three categories:

- products involving only elements of the kernel of the penalization $L$, which can be shown to be compact (see Corollary 2.5);
- products of elements of the kernel against elements of $(\text{Ker} L)\perp$, for which one can take the limit since elements of $(\text{Ker} L)\perp$ converge weakly to zero (see Corollary 2.5);
- products involving only elements of $(\text{Ker} L)\perp$, which are the problem.

The idea now is to prove that in the last situation, the limit is in fact zero for algebraic reasons: in previous works on rotating fluids, that result was proved essentially by writing the product of two elements of $(\text{Ker} L)\perp$ by projection onto eigenvectors of $L$. In the periodic case, a “miracle” in the formulation yielded the result (see [2]-[4] or [7]), whereas in the whole space case, Strichartz estimates did the job (and the convergence was strong), see [5]. In this paper we will show that the result has in fact not much to do with spectral properties of $L$, but is due to simple algebraic properties. Let us recall the result in the case of the incompressible limit, where such properties were first used (see [13]).

**Proposition 1.1.** [13] Let $(\rho_\varepsilon), (u_\varepsilon), (\theta_\varepsilon)$ be bounded families of $L^2([0,T], H^1(\Omega))$ such that

\[\rho_\varepsilon \to \rho, \quad u_\varepsilon \to u, \quad \theta_\varepsilon \to \theta \text{ as } \varepsilon \to 0.\]

Assume that

\[
\varepsilon \partial_t \rho_\varepsilon + \nabla \cdot u_\varepsilon = 0,
\]

\[
\varepsilon \partial_t u_\varepsilon + \nabla (\rho_\varepsilon + \theta_\varepsilon) = s_\varepsilon,
\]

\[
\varepsilon \partial_t \theta_\varepsilon + \frac{2}{3} \nabla \cdot u_\varepsilon = s'_\varepsilon,
\]

where $s_\varepsilon, s'_\varepsilon \to 0$ in $L^1([0,T], H^{-s}(\Omega))$ for some $s > 0$. Then

\[PV\cdot (u_\varepsilon \otimes u_\varepsilon) \to P\nabla \cdot (u \otimes u) \text{ and } \nabla \cdot (u_\varepsilon \theta_\varepsilon) \to \nabla \cdot (u \theta)\]

in the sense of distributions.

**Proof.** This result has to be compared with the so-called “compensated compactness” theorems, in the sense that the convergences of some quadratic quantities in $\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon$ are established under the assumption that some combinations of the derivatives of these functions converge strongly in time to 0. The proof consists in checking that the acoustic oscillations do not bring any contribution to the limiting terms. We introduce the following decompositions:

\[u_\varepsilon = Pu_\varepsilon + \nabla \psi_\varepsilon, \quad \text{and} \quad \theta_\varepsilon = \frac{3\theta_\varepsilon - 2\rho_\varepsilon}{5} + \pi_\varepsilon,\]

so that

\[Pu_\varepsilon \text{ and } \frac{3\theta_\varepsilon - 2\rho_\varepsilon}{5} \text{ are bounded in } W^{1,1}([0,T], H^{-s}(\Omega)), \]

\[\varepsilon \partial_t \nabla \psi_\varepsilon + \nabla \pi_\varepsilon = (I - P)s_\varepsilon \to 0, \quad \varepsilon \partial_t \pi_\varepsilon + \frac{2}{3} \Delta \psi_\varepsilon = \frac{2}{3} s'_\varepsilon \to 0 \text{ in } L^1([0,T], H^{-s}(\Omega)).\]

We shall note in the following $S_\varepsilon \overset{\text{def}}{=} (I - P)s_\varepsilon$ and $S'_\varepsilon \overset{\text{def}}{=} \frac{2}{5} s'_\varepsilon$. The incompressibility and Boussinesq relations

\[\nabla \cdot u = 0, \quad \nabla (\rho + \theta) = 0\]
allow to identify the limits
\[ P u_\varepsilon \to u, \quad \frac{3\theta_\varepsilon - 2\rho_\varepsilon}{5} \to \theta \text{ in } L^2([0,T] \times \Omega), \]
\[ \nabla \psi_\varepsilon \to 0, \quad \pi_\varepsilon \to 0 \text{ in } w - L^2([0,T] \times \Omega), \]
from which we deduce that, in the sense of distributions
\[ u_\varepsilon \otimes u_\varepsilon - \nabla \psi_\varepsilon \otimes \nabla \psi_\varepsilon \to u \otimes u, \]
\[ \theta_\varepsilon u_\varepsilon - \pi_\varepsilon \nabla \psi_\varepsilon \to \theta u. \]
The key argument is therefore the following formal computation (which can be made rigorous by introducing regularizations with respect to the space variable \( x \))
\[ P \nabla \cdot (\nabla \psi_\varepsilon \otimes \nabla \psi_\varepsilon) = \frac{1}{2} P \nabla |\nabla \psi_\varepsilon|^2 + P (\Delta \psi_\varepsilon \nabla \psi_\varepsilon) \]
\[ = \frac{3}{2} P (\varepsilon \pi_\varepsilon \nabla \psi_\varepsilon) - \pi_\varepsilon \nabla \pi_\varepsilon + \pi_\varepsilon S_\varepsilon + S' \varepsilon \nabla \psi_\varepsilon) \]
\[ = \frac{3}{2} P (\varepsilon \pi_\varepsilon \nabla \psi_\varepsilon) + \pi_\varepsilon S_\varepsilon + S' \varepsilon \nabla \psi_\varepsilon), \]
\[ \nabla \cdot (\pi_\varepsilon \nabla \psi_\varepsilon) = \pi_\varepsilon \Delta \psi_\varepsilon + \nabla \psi_\varepsilon \cdot \nabla \pi_\varepsilon \]
\[ = \frac{3}{2} \pi_\varepsilon (S'_\varepsilon - \varepsilon \partial_t \pi_\varepsilon) + \nabla \psi_\varepsilon \cdot (S_\varepsilon - \varepsilon \partial_t \nabla \psi_\varepsilon) \]
\[ = \frac{3}{2} \pi_\varepsilon S'_\varepsilon + \nabla \psi_\varepsilon \cdot S_\varepsilon - \frac{3}{4} \varepsilon \partial_t |\pi_\varepsilon|^2 - \frac{3}{2} \pi_\varepsilon |\nabla \psi_\varepsilon|^2, \]
which shows that the contribution of the acoustic oscillations is negligible. \( \square \)

Inspired by the previous computation, we shall in this article try to use a similar method in the case of rotating fluids: we refer to the proofs of Propositions 3.4 and 4.4 for precise computations.

1.2. Main results. Since we consider incompressible flows, we introduce the following subspaces of \( L^2(\Omega) \) and \( H^1(\Omega) \)
\[ H = \{ u \in L^2(\Omega) \mid \nabla \cdot u = 0 \}, \quad V = \{ u \in H^1(\Omega) \mid \nabla \cdot u = 0 \}. \]
We will also use the following notation for the inhomogeneous Sobolev spaces
\[ H^s(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid (Id - \Delta)^{s/2}u \in L^2(\Omega) \}. \]
Similarly homogeneous Sobolev spaces will be defined by
\[ \dot{H}^s(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid (-\Delta)^{s/2}u \in L^2(\Omega) \}. \]
It will appear clearly in the following that the horizontal variables play a special role in this problem. Consequently we shall use the following notation: if \( x \) is a point in \( \Omega \), then we shall note its cartesian coordinates by \((x_1, x_2, x_3)\), and the horizontal part of \( x \) will be denoted \( x_h \defeq (x_1, x_2) \in \Omega_h \). Similarly we will denote the horizontal part of any vector field \( f \) by \( f_h \), the horizontal gradient by \( \nabla_h \defeq (\partial_1, \partial_2) \) and its orthogonal by \( \nabla_h^\perp = (\partial_2, -\partial_1) \), and the horizontal divergence and Laplacian respectively by \( \text{div}_h f \defeq \partial_1 f_1 + \partial_2 f_2 = \nabla_h \cdot f_h \) and \( \Delta_h \defeq \partial_1^2 + \partial_2^2 \).
Finally as usual, $C$ will denote a constant which can change from line to line, and $\nabla p$ will denote the gradient of a function which can also change from line to line.

Before stating the main theorems of this paper, let us give some additional definitions. We will note by

$$ S = \{ x \in \Omega / \nabla b(x) = 0 \} \quad \text{and} \quad O = \{ x \in \Omega / \nabla b(x) \neq 0 \}. $$

Finally $S$ will be the interior of the singular set $S$. We will assume in the sequel that

$$(H0) \quad \text{The set } \Omega \setminus (S \cup O) \text{ is of Lebesgue measure } 0;$$

$$(H1) \quad S \text{ is a smooth domain,}$$

$$(H2) \quad \text{On each connected component } O_j \text{ of } O, \text{ there is a smooth function } \sigma_j$$

such that $(b, \sigma_j, x_3)$ is a global smooth coordinate system

and $O_j = \{ x_h \in \mathbb{R}^2 \mid (b(x_h), \sigma_j(x_h)) \in B_j \times \Sigma_j \} \times \Omega_3$.

Now we are ready to state the main theorems of this paper. The first result, rather standard,

$$(\text{Theorem 1}) \quad \text{As we will see in Section 3, the}$$

interest of this result lies in its proof, which contrary to the references above, does not depend

on the boundary conditions (which can be the whole space or periodic, in each direction).

Theorem 2. Suppose that $B = be_3$ where $b$ is constant and homogeneous. Let $u^0$ be any vector field in $H$. Then for all $\varepsilon > 0$, Equation (1.1) has at least one weak solution $u_\varepsilon \in L^\infty(\mathbb{R}^+, H) \cap L^2(\mathbb{R}^+, H^1)$. Moreover, for all $t > 0$, the following energy estimate holds:

$$ \|u_\varepsilon(t)\|_{L^2}^2 + 2\nu \int_0^t \|\nabla u_\varepsilon(s)\|_{L^2}^2 ds \leq \|u^0\|_{L^2}^2. $$

Now the aim of the paper is to describe the limit of $u_\varepsilon$ as $\varepsilon$ goes to zero. We will first concentrate on the constant case.

Theorem 3. Suppose that $B = be_3$ where $b = b(x_h)$ is a nonnegative smooth function, say a $C^\infty_0(\Omega_h)$ perturbation of a constant, and where assumptions $(H0)$ to $(H2)$ are satisfied. Let $u^0$ be any vector field in $H$, and let $u_\varepsilon$ be any weak solution of (1.1) in the sense of Theorem 1. Then $u_\varepsilon$ converges weakly in $L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)$ to a limit $\overline{u}$ which if $\Omega_3 = \mathbb{R}$ is zero, and if $\Omega_3 = T$ is the solution of the two dimensional Navier–Stokes equations

$$(\text{NS2D}) \quad \partial_t \overline{u} - \nu \Delta \overline{u} + \overline{u} \cdot \nabla \overline{u} = (-\nabla_h p, 0), \quad \text{div}_h \overline{u}_h = 0, \quad \overline{u}_{|t=0} = \int_T u^0(x_h, x_3) \, dx_3.$$

Remark 1.2. This theorem is by no means a novelty, it is even rather less precise than other such results one can find in the literature ([2, 3, 4, 5, 6, 7]). As we will see in Section 3 the interest of this result lies in its proof, which contrary to the references above, does not depend on the boundary conditions (which can be the whole space or periodic, in each direction).

Now let us state the new result of this paper, concerning the case when $b$ is not homogeneous.

Theorem 3. Suppose that $B = be_3$ where $b = b(x_h)$ is a nonnegative smooth function, say a $C^\infty_0(\Omega_h)$ perturbation of a constant, and where assumptions $(H0)$ to $(H2)$ are satisfied. Let $u^0$ be any vector field in $H$, and let $u_\varepsilon$ be any weak solution of (1.1) in the sense of Theorem 1. Then $u_\varepsilon$ converges weakly in $L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)$ to a limit $\overline{u}$ which if $\Omega_3 = \mathbb{R}$ is zero, and if $\Omega_3 = T$ is defined as follows: the third component $\overline{u}^3$ satisfies the transport equation

$$ \partial_t \overline{u}^3 - \nu \Delta_h \overline{u}^3 + \overline{u} \cdot \nabla_h \overline{u}^3 = 0, \quad \partial_3 \overline{u}^3 = 0, \quad \overline{u}_{|t=0} = \int_T u^0,3(x_h, x_3) \, dx_3 \quad \text{in} \quad \mathbb{R}^+ \times (O \cup S),$$

while the horizontal component $\overline{u}_h$ depends on the region of space considered:
• in $\mathcal{S}$, $\overline{u}_h$ satisfies the two dimensional Navier–Stokes equations ($\text{NS2D}$) with Dirichlet boundary conditions.
• in $\mathcal{O}$, $\overline{u}_h$ satisfies the following heat equation:

$$\partial_t \overline{u}_h - \nu \Pi \Delta_h \overline{u}_h = 0,$$

where $\Pi$ is the $L^2$ orthogonal projection onto the kernel of $L$ (which can be extended to $\mathcal{D}'(\mathcal{O} \cup \mathcal{S})$) which satisfies in particular

$$-(\Pi \Delta_h \overline{u}_h | \overline{u}_h)_{L^2(\mathcal{O})} = \|\nabla_h \overline{u}_h\|_{L^2(\mathcal{O})}^2.$$ 

Remark 1.3. In the regions where $b$ is homogeneous, we recover at the limit the 2D Navier–Stokes equations as usual. The Dirichlet boundary conditions appear quite naturally, considering that on the other side of the boundary one finds that $\overline{u}_h$ is proportional to $\nabla_\perp \times b$ which vanishes on the boundary of $\mathcal{S}$. More surprising is certainly the result in the region where $b$ is not homogeneous. This can be understood as some sort of turbulent behaviour, where all scales are mixed due to the variation of $b$. Technically the result is due to the fact that the kernel of $L$ is very small as soon as $b$ is not a constant, which induces a lot of rigidity in the limit equation.

The structure of the paper is as follows. In the next section, we present the operator $L$ and study its main properties (proof of Theorem 1, kernel, projections onto Ker$L$). The following section is devoted to the proof of Theorem 2. Although the result is not new, we present an alternative proof which holds regardless of the domain (with no boundary). This serves as a warm–up to the final section, in which the general variable case is presented, with the proof of Theorem 3.

Remark 1.4. One can wonder about what remains of those results under more general assumptions on the rotation vector $B$. First let us consider the case when $B = b(t, x_h)e_3$ depends also on time. The results of Section 2 are identical, and if the sets $\mathcal{S}$ and $\mathcal{O}$ are independent of time one recovers the same type of theorem as in the constant case. In particular the equation on $\overline{u}$ in $\mathcal{O}$ is derived in an identical way to Section 4.2. If the sets $\mathcal{S}$ and $\mathcal{O}$ do depend on time, then one has to be a little bit more careful and this issue will not be treated here. A more physical problem is the case when the direction of $B$ is not fixed, in other words when $B$ is a three component vector, depending on all three variables. Then geometrical problems appear, simply to determine the kernel of $L$; this will be dealt with in a forecoming paper.

2. Study of the singular perturbation

2.1. Energy estimate. In this section we shall prove Theorem 1 stated in the introduction.

Proof. The structure of the equation (1.1) governing the rotating fluids is very similar to the one of the usual Navier-Stokes equation, since the singular perturbation is just a linear skew-symmetric operator. Therefore weak solutions “à la Leray” can be constructed by the approximation scheme of Friedrichs: approximate solutions are obtained by a standard truncation $J_n$ of high frequencies. In order to obtain uniform bounds on these approximate
solutions, we have just to check that the energy inequality is still satisfied. Computing formally the $L^2$ scalar product of (1.1) by $u$ leads to
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = -\int \left( \frac{1}{2} (u \cdot \nabla)|u|^2 - \nu u \cdot \Delta u + \frac{1}{\varepsilon} u \cdot B + u \cdot \nabla p \right) dx.
\]
Integrating by parts (without boundary) and using the incompressibility constraint, we get
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = -\nu \|\nabla u\|_{L^2}^2,
\]
which holds for any smooth solution of (1.1).

The energy inequality for weak solutions is obtained by taking limits in the approximation scheme. □

In particular, the energy estimate provides uniform bounds in $L^\infty(\mathbb{R}^+, H) \cap L^2(\mathbb{R}^+, \dot{H}^1)$ on any family $(u_\varepsilon)_{\varepsilon > 0}$ of weak solutions of (1.1) provided that the initial data $u^0$ belongs to $H$.

**Corollary 2.1.** Let $u^0$ be any vector field in $H$. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1). Then there exists $\overline{u} \in L^\infty(\mathbb{R}^+, H) \cap L^2(\mathbb{R}^+, \dot{H}^1)$, such that, up to extraction of a subsequence,
\[
u u \to \overline{u} \ 	ext{in } w-L^2_{loc}(\mathbb{R}^+ \times \Omega)\ 	ext{as } \varepsilon \to 0.
\]

2.2. **Characterization of the kernel.** We are interested in describing the asymptotic behaviour of $(u_\varepsilon)$, i.e. in characterizing its limit points. Of course, the equations satisfied by such a limit point $\overline{u}$ depend strongly on the structure of the singular perturbation
\[
(2.1) \quad L : u \in H \mapsto P(u \wedge B) \in H
\]
where $P$ denotes the Leray projection from $L^2(\Omega)$ onto its subspace $H$ of divergence-free vector fields. In particular, we will prove that $\overline{u}$ belongs to the kernel $\text{Ker}(L)$ of $L$, which is characterized in the following proposition.

**Proposition 2.2.** Define the linear operator $L$ by (2.1). Then $u \in H$ belongs to $\text{Ker}(L)$ if and only if there exist $\nabla_h \varphi \in L^2(\Omega_h)$ and $\alpha \in L^2(\Omega_h)$ with
\[
\nabla_h b \cdot \nabla_h \varphi = 0,
\]
such that
\[
u u = \nabla_h \varphi + \alpha e_3.
\]

**Proof.** We have
\[
u P(u \wedge B) = 0.
\]
Then, in the sense of distributions,
\[
\text{rot } (u \wedge B) = 0,
\]
which can be rewritten
\[
(\nabla \cdot B)u + (B \cdot \nabla)u - (u \cdot \nabla)B - (\nabla \cdot u)B = 0.
\]
As $\nabla \cdot B = \nabla \cdot u = 0$ and $B = b e_3$, we get
\[
(2.2) \quad b \partial_3 u - (u \cdot \nabla)be_3 = 0.
\]
In particular, $\partial_3 u_1 = \partial_3 u_2 = 0$ from which we deduce that
\begin{equation}
(2.3) \quad u_1, u_2 \in L^2(\Omega_h).
\end{equation}
Note that in the case where $\Omega_3 = \mathbb{R}$, the invariance with respect to $x_3$ and the fact that $u \in L^2(\Omega)$ imply that $u_1 = u_2 = 0$ (and therefore $u_3 = 0$ by the divergence free condition).

Differentiating the incompressibility constraint with respect to $x_3$ leads then to
\[ \partial_{33}^2 u_3 = -\partial_{13}^2 u_1 - \partial_{23}^2 u_2 = 0 \]
in the sense of distributions. The function $\partial_3 u_3$ depends only on $x_1$ and $x_2$, and satisfies $\int \partial_3 u_3 dx_3 = 0$. So $\partial_3 u_3 = 0$ and
\begin{equation}
(2.4) \quad u_3 \in L^2(\Omega_h), \quad \partial_1 u_1 + \partial_2 u_2 = 0.
\end{equation}
Combining (2.3) and (2.4) provides the existence of $\nabla_h \nabla \varphi \in L^2(\Omega_h)$ such that
\[ u_1 = \partial_2 \varphi, \quad u_2 = -\partial_1 \varphi. \]
Replacing in (2.2) leads to
\[ \nabla_h \nabla \varphi \cdot \nabla_h b = b \partial_3 u_3 = 0, \]
which concludes the proof. □

Before applying this result to the characterization of the weak limit $\overline{u}$, let us just specify it in two important cases. If $\nabla \varphi = 0$ almost everywhere, $u \in H$ belongs to $\text{Ker}(L)$ if and only if
\[ u = \nabla_h \nabla \varphi + \alpha e_3, \]
for some $\nabla_h \nabla \varphi \in L^2(\Omega_h)$ and $\alpha \in L^2(\Omega_h)$. If $\nabla \varphi \neq 0$ almost everywhere (in other words, if $\Omega \setminus \mathcal{O}$ is of Lebesgue measure zero), then the condition arising on $u$ is much more restrictive : $u \in H$ belongs to $\text{Ker}(L)$ if and only if on each connected component of $\mathcal{O}$,
\[ u = F(b) \nabla_h \nabla \varphi + \alpha e_3, \]
for some square integrable function $F(b)$ and some $\alpha \in L^2(\Omega_h)$.

From this characterization of $\text{Ker}(L)$, we deduce some constraints on the weak limit $\overline{u}$.

**Corollary 2.3.** Let $u^0$ be any vector field in $H$. Denote by $(u_\varepsilon)_{\varepsilon > 0}$ a family of weak solutions of (1.1), and by $\overline{u}$ any of its limit points. Then, there exist $\varphi \in L^\infty(\mathbb{R}^+, \dot{H}^1(\Omega_h)) \cap L^2(\mathbb{R}^+, H^2(\Omega_h))$ and $\alpha \in L^\infty(\mathbb{R}^+, L^2(\Omega_h)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\Omega_h))$ with
\[ \nabla_h b \cdot \nabla_h \varphi = 0, \]
such that
\[ \overline{u} = \nabla_h \nabla \varphi + \alpha e_3. \]

**Proof.** Let $\chi \in \mathcal{D}(\mathbb{R}^+ \times \Omega)$ be any divergence-free test function. Multiplying (1.1) by $\varepsilon \chi$ and integrating with respect to all variables leads to
\[ \iint u_\varepsilon (\varepsilon \partial_t \chi + \varepsilon u_\varepsilon \cdot \nabla \chi + \varepsilon \nu \Delta \chi + \chi \wedge B) dx dt = 0. \]
Because of the bounds coming from the energy estimate, we can take limits in the previous identity as \( \varepsilon \to 0 \) to get
\[
\int \mathbf{\pi} \wedge B \cdot \chi \, dx \, dt = 0.
\]
This means that there exists some \( p \) such that
\[
\mathbf{\pi} \wedge B = \nabla p.
\]
As \( u_\varepsilon \) satisfies the incompressibility relation for all \( \varepsilon > 0 \),
\[
\nabla \cdot u_\varepsilon = 0.
\]
Then \( \mathbf{\pi}(t) \in \text{Ker}(L) \) for almost all \( t \in \mathbb{R}^+ \), and we conclude by Proposition 2.2 that there exist \( \varphi \in L^\infty(\mathbb{R}^+, \dot{H}^1(\Omega_h)) \cap L^2(\mathbb{R}^+, \dot{H}^2(\Omega_h)) \) and \( \alpha \in L^\infty(\mathbb{R}^+, \dot{H}^1(\Omega_h)) \cap L^2(\mathbb{R}^+, \dot{H}^2(\Omega_h)) \) with
\[
\nabla_h b \cdot \nabla_h \varphi = 0,
\]
such that
\[
u = \nabla_h \varphi + \alpha e_3.
\]

2.3. Decomposition by projection on the kernel. To go further in the description of the asymptotic behaviour (i.e. in the characterization of \( \mathbf{\pi} \)), we have to isolate the fast oscillations generated by the singular perturbation \( L \), which produce “big” terms in (1.1), but converge weakly to 0.

Therefore we introduce the following decomposition
\[
u_\varepsilon = \mathbf{\pi}_\varepsilon + w_\varepsilon,
\]
where \( \mathbf{\pi}_\varepsilon = \Pi u_\varepsilon \) is the \( L^2 \) orthogonal projection of \( u_\varepsilon \) on \( \text{Ker}(L) \) and \( w_\varepsilon = \Pi_\perp u_\varepsilon \) is the projection of \( u_\varepsilon \) on \( \text{Ker}(L)\perp \).

We have seen in the previous paragraph that the characterization of the kernel \( \text{Ker}(L) \) is strongly linked to the geometry of the vector field \( b \). In order to obtain further regularity properties on \( \Pi \) and \( \Pi_\perp \), we then need a precise description of the singular set
\[
S = \{ x \in \Omega / \nabla b(x) = 0 \},
\]
which justifies Assumptions (H0) to (H2) given in the introduction.

Before stating the main properties of \( \Pi \) and \( \Pi_\perp \), let us give the following definitions: by Assumption (H0) it is enough to describe the limiting function \( \mathbf{\pi} \) on \( \mathcal{O} \cup S \). So it is natural to define the following function spaces: for all \( s \geq 0 \), \( H^s(\mathcal{O} \cup S) \) is the closure of \( C^\infty_c(\mathcal{O} \cup S) \) for the \( H^s \) norm, and we will note, for \( s \geq 0 \), \( H^{-s}(\mathcal{O} \cup S) \) the dual space of \( H^s(\mathcal{O} \cup S) \).

It will be useful in the following to note that
\[
\forall s \geq 0, \quad H^s(\mathcal{O} \cup S) \subset H^s(\Omega) \quad \text{and} \quad \forall s \leq 0, \quad H^s(\Omega) \subset H^s(\mathcal{O} \cup S).
\]

Proposition 2.4. Define the linear operator \( L \) by (2.1). Denote by \( \Pi \) the orthogonal projection of \( \mathcal{H} \) onto \( \text{Ker}(L) \) and by \( \Pi_\perp = \text{Id} - \Pi \) the orthogonal projection of \( \mathcal{H} \) onto \( \text{Ker}(L)\perp \). The operators \( \Pi \) and \( \Pi_\perp \) so defined have natural extensions to tempered distributions on \( \mathcal{O} \cup S \), and for all \( s \in \mathbb{R} \), there exists some \( C_s > 0 \) such that for any function \( u \in H^s(\mathcal{O} \cup S) \),
\[
\| \Pi u \|_{H^s(\mathcal{O} \cup S)} \leq C_s \| u \|_{H^s(\mathcal{O} \cup S)} \quad \text{and} \quad \| \Pi_\perp u \|_{H^s(\mathcal{O} \cup S)} \leq C_s \| u \|_{H^s(\mathcal{O} \cup S)}.
\]
Proof. By Proposition 2.2, for all $u \in H$

$$\Pi u = \nabla_h^\perp \varphi + \alpha e_3,$$

for some $\nabla_h \varphi \in L^2(\Omega_h)$ and $\alpha \in L^2(\Omega_h)$ with

$$\nabla_h b \cdot \nabla_h^\perp \varphi = 0.$$  

By definition $\Pi \perp u = u - \Pi u$ is orthogonal to any element of $\text{Ker}(L)$. So for all $\beta \in L^2(\Omega_h)$,

$$\int (u - \Pi u) \cdot \beta e_3 \, dx = 0,$$

which implies that

$$\alpha = \frac{1}{|\Omega_3|} \int u_3 dx_3. \quad (2.6)$$

In order to determine $\varphi$, we consider separately the domains $O$ and $S$.

On $S$, $\varphi$ is defined as

$$\nabla_h^\perp \varphi = \left( \frac{1}{|\Omega_3|} \int_{\Omega_3} u_h dx_3 \right),$$

and the smoothness properties stated in Proposition 2.4 are obvious.

On $O$, we use Assumption (H2) which implies that on each connected component $O_j$ of $O$, $u$ can be written $u(x) = f_j(b, \sigma_j, x_3)$ where $f_j$ has the same smoothness as $u$ since the change of coordinate is in $C^\infty(\Omega)$. Then clearly the projection $\Pi$ is simply defined by

$$\Pi u \overset{\text{def}}{=} \frac{1}{|\Omega_3| |\Sigma_j|} \int_{\Sigma_j} \int_{\Omega_3} f_j(b, \sigma_j, x_3) d\sigma_j dx_3,$$

and the result follows. \qed

Corollary 2.5. Let $u^0$ be any vector field in $H$. Denote by $(u_\varepsilon)_{\varepsilon > 0}$ a family of weak solutions of $(1.1)$, and by $\overline{u}$ any of its limit points. Consider a subsequence of $(u_\varepsilon)$ (abusively denoted $(u_\varepsilon)$) such that

$$u_\varepsilon \rightharpoonup \overline{u} \text{ in } w-L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega) \text{ as } \varepsilon \to 0.$$  

Then, if $\overline{u}_\varepsilon = \Pi u_\varepsilon$ is the projection of $u_\varepsilon$ on $\text{Ker}(L)$,

$$\overline{u}_\varepsilon \to \overline{u} \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^+ \times (O \cup S)) \text{ as } \varepsilon \to 0.$$  

Proof. By Proposition 2.4, the projection $\Pi$ is continuous in $L^2$. Then, by the energy estimate, $\overline{u}_\varepsilon$ is uniformly bounded in $L^2_{\text{loc}}(\mathbb{R}^+, \mathcal{K})$ for some compact subset $\mathcal{K}$ of $L^2(\Omega)$, which provides regularity with respect to space variables. The second step consists in getting regularity with respect to time. Apply $\Pi$ to the convection equation in $(1.1)$:

$$(2.7) \quad \partial_t \overline{u}_\varepsilon + \Pi (u_\varepsilon \cdot \nabla u_\varepsilon - \nu \Delta u_\varepsilon) = 0.$$  

As $u_\varepsilon$ is divergence-free,

$$u_\varepsilon \cdot \nabla u_\varepsilon = \nabla \cdot (u_\varepsilon \otimes u_\varepsilon) \text{ is uniformly bounded in } L^\infty(\mathbb{R}^+, W^{-1,1}(\Omega)).$$  

From the energy estimate, we also deduce that

$$\Delta u_\varepsilon \text{ is uniformly bounded in } L^2(\mathbb{R}^+, H^{-1}(\Omega)).$$

from which we infer that
\[ u_\varepsilon \cdot \nabla u_\varepsilon - \nu \Delta u_\varepsilon \] is uniformly bounded in \( L^2_{\text{loc}}(\mathbb{R}_+^+, H^{-5/2}_{\text{loc}}(\Omega)) \).

Combining this with (2.5), we get by Proposition 2.4
\[ \partial_t u_\varepsilon = -\Pi(u_\varepsilon \cdot \nabla u_\varepsilon - \nu \Delta u_\varepsilon) \] is uniformly bounded in \( L^2_{\text{loc}}(\mathbb{R}_+^+, H^{-5/2}_{\text{loc}}(\mathcal{O} \cup \mathcal{S})) \),
which provides the expected regularity in \( t \).

Aubin’s lemma [1] then gives the following interpolation result
\[ (\overline{u}_\varepsilon) \] is strongly compact in \( L^2_{\text{loc}}(\mathbb{R}_+ \times (\mathcal{O} \cup \mathcal{S})) \).

By Proposition 2.4, \( \Pi \) belongs to \( C(\mathcal{L}^2(\Omega), \mathcal{L}^2(\Omega)) \) and therefore to \( C(\varepsilon^{-\mathcal{L}^2(\Omega)}, \mathcal{L}^2(\Omega)) \),
from which we deduce that
\[ \overline{u}_\varepsilon = \Pi u_\varepsilon \rightharpoonup \Pi \overline{u} = \overline{u}. \]
Combining both results shows that \( u_\varepsilon \) converges strongly to \( u \) in \( L^2_{\text{loc}}(\mathbb{R}_+ \times (\mathcal{O} \cup \mathcal{S})) \). □

2.4. Remarks concerning the regularity.

2.4.1. Comparison with the gyrokinetic approximation. As mentioned in the introduction, the study of the asymptotics for an inhomogeneous penalization is a natural question in the magnetohydrodynamic framework, when \( \mathcal{B} \) represents the magnetic field. Such a study has been performed for the gyrokinetic approximation [3], that is for a kinetic model perturbed by a singular magnetic constraint:

- in the case where \( \mathcal{B} = b(x) e_3 \), the singular limit is exactly the same as in the constant case: the fast rotation has an averaging effect in the plane orthogonal to the magnetic lines;
- in the case where \( \mathcal{B} \) has constant modulus but variable direction, extra drift terms are obtained due to the curvature of the field.

A simplified version of this result can be written as follows.

**Theorem 4.** [3] Let \( f^0 \) be a function of \( L^\infty(\Omega \times \mathbb{R}^3) \), and \((f_\varepsilon)\) be a family of solutions of
\[ \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon} v \wedge B \cdot \nabla_v f_\varepsilon = 0, \quad t \in \mathbb{R}_+^*, (x,v) \in \Omega \times \mathbb{R}^3, \]
with initial condition
\[ f_\varepsilon(t = 0) = f^0. \]
Then the family \((f_\varepsilon)\) is relatively compact in \( w \ast L^\infty(\mathbb{R}_+^* \times \Omega \times \mathbb{R}^3) \), as well as the family \((g_\varepsilon)\) defined by
\[ g_\varepsilon(t, x, w) = f_\varepsilon(t, x, R(x, -\frac{t}{\varepsilon})w) \]
where \( R(x, \theta) \) denotes the rotation of angle \( \theta \) around the oriented axis of direction \( \mathcal{B}(x) \).
Moreover,
- if \( \mathcal{B} = b e_3 \) with \( b \in C^1(\Omega_h, \mathbb{R}_+^3) \), any limit point of \((g_\varepsilon)\) satisfies
\[ \partial_t g + \nu_3 \partial_{x_3} g = 0; \]
• if $B \in C^1(\Omega)$ with $\nabla_x \cdot B = 0$ and $|B| \equiv 1$, any of its limit points satisfies
\[
\partial_t g + (w \cdot B)B \cdot \nabla_x g = \frac{1}{2} w \wedge (3(w.B)(B \wedge \nabla B) - B \wedge \nabla wB - \nabla B \wedge wB) \cdot \nabla_w g
\]
with the notation $\nabla_V \Phi \overset{\text{def}}{=} V \cdot \nabla \Phi$.

The result obtained in this paper is very different because of the incompressibility constraint, which imposes a lot of rigidity to the system. In particular, the kernel of the penalization is much smaller and the limiting system has less degrees of freedom.

2.4.2. A remark in the inviscid case. The weak compactness method used here allows to study the singular limit without regularity with respect to the time variable. However it uses crucially the strong compactness in $x$ given by the energy estimate (1.2). Implicitly we have actually considered the penalization
\[
L_\varepsilon : u \in V \mapsto P(u \wedge B) - \varepsilon \Delta u \in H^{-1}(\Omega).
\]
That rules out the possibility to manage an analogous study for inviscid rotating fluids, the first obstacle being to prove the existence of solutions for
\[
\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla)u_\varepsilon + \frac{1}{\varepsilon} u_\varepsilon \wedge B + \nabla p = 0, \quad \nabla \cdot u_\varepsilon = 0
\]
on a uniform time interval $[0, T]$. Indeed the operator $\exp(tL/\varepsilon)$ is not even uniformly bounded on $H^s(\Omega)$ for $s \geq \frac{1}{2}$.

**Lemma 2.6.** Define the linear operator $L$ by (2.1). Then, the group $(\exp(tL))_{t \in \mathbb{R}}$ generated by $L$ is not uniformly bounded on $H^s(\Omega)$ for $s \geq \frac{1}{2}$.

**Proof.** The proof of that result is simply due to the fact that by definition of $\Pi$ seen above, the trace of $\Pi u$ is not defined on $\partial S$ even if $u \in V$. So $\Pi$ is not continuous on $H^s(\Omega)$ for $s \geq \frac{1}{2}$. Then the formula
\[
\Pi = \lim_{T \to +\infty} \frac{1}{T} \int_0^T e^{tL} dt
\]
implies that $e^{tL}$ cannot be uniformly bounded in $H^s(\Omega)$ for $s \geq \frac{1}{2}$ which proves the lemma. □

3. The case of a constant vector field $B$: the 2D Navier-Stokes limit

In the previous section, we have obtained a constraint equation on the limiting velocity field, which expresses that $\overline{u}$ belongs to the kernel of the singular perturbation $\mathcal{L}$. This comes from the fact that the component $u_\varepsilon = \Pi \perp u_\varepsilon$ has fast oscillations with respect to time, and consequently converges weakly to $0$. In the case where $\Omega = \mathbb{R}^3$, this characterizes completely the weak limit $\overline{u} = 0$.

Then it remains to get an evolution equation for $\overline{u}$ in the case where $\Omega = \mathbb{T}$. A natural idea consists in projecting the evolution equation (1.1) for $u_\varepsilon$ on the kernel of $\mathcal{L}$, and to study its limit as $\varepsilon \to 0$. The difficulty is to take limits in the nonlinear terms: as Corollary 2.5
provides strong compactness on the non-oscillating component \( u_\varepsilon \), the problem comes actually to prove that the oscillating terms \( w_\varepsilon \) do not produce any constructive interference.

In order to have a good understanding of this phenomena and of its mathematical formulation, we propose to consider first the case where the vector field \( B \) is constant and homogeneous. The convergence result established here is not so precise as the ones given in [2]-[4], [7] or [10], since it does not describe the oscillating component and consequently does not provide any strong convergence. Nevertheless the proof is less technical (in particular it does not require any knowledge on the spectral structure of \( L \)), which allows to consider more general cases in the sequel.

3.1. **Projection on the kernel.** In order to obtain the evolution of the limiting velocity field \( \overline{u} \), the idea is to use the strong convergence \( u_\varepsilon \to \overline{u} \) in \( L^2_{\text{loc}}(\mathbb{R}^+ \times (\mathcal{O} \cup S)) \), rather than the weak convergence \( u_\varepsilon \rightharpoonup \overline{u} \) in \( W^1_2 L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega) \).

Having this idea in view, we first determine the evolution equation for \( u_\varepsilon \).

**Proposition 3.1.** Let \( u_0 \) be any vector field in \( H \). For all \( \varepsilon > 0 \), denote by \( u_\varepsilon \) a weak solution of (1.1), and by \( u_\varepsilon = \Pi u_\varepsilon \) its projection on \( \text{Ker}(L) \). Then,

\[
\partial_t \overline{u}_\varepsilon - \Pi(u_\varepsilon \wedge \text{rot } u_\varepsilon) - \nu \Delta u_\varepsilon = 0.
\]

**Proof.** Identity (3.1) is essentially a variant of (2.7). Indeed, in the case of a constant \( B \), the projection \( \Pi \) commutes with any partial derivative, in particular with the Laplacian \( \Delta \):

\[
\Pi(\nu \Delta u_\varepsilon) = \nu \Delta \Pi u_\varepsilon = \nu \Delta \overline{u}_\varepsilon.
\]

Then the key argument is the following identity:

\[
u \Delta u_\varepsilon = \nu \Delta \Pi u_\varepsilon = \nu \Delta \overline{u}_\varepsilon.
\]

Replacing in (2.7) leads to the expected result. \( \square \)

3.2. **Brief description of the oscillations.**

**Proposition 3.2.** Let \( u_0 \) be any vector field in \( H \). For all \( \varepsilon > 0 \), denote by \( u_\varepsilon \) a weak solution of (1.1), and by \( w_\varepsilon = \Pi_1 u_\varepsilon \) its projection on \( \text{Ker}(L)_\perp \). Then there exists \( W_\varepsilon \in L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega) \) such that

\[
\partial_3 W_\varepsilon = \nu \partial_3 W_\varepsilon.
\]

Moreover,

\[
\varepsilon \partial_t (\partial_3 W_\varepsilon - \Delta W_\varepsilon) + \partial_3 W_\varepsilon \wedge B = r_\varepsilon,
\]

\[
\varepsilon \partial_t (\partial_2 W_\varepsilon,1 - \partial_1 W_\varepsilon,1) + b(\partial_1 W_\varepsilon,1 + \partial_2 W_\varepsilon,2) = s_\varepsilon,
\]

where \( r_\varepsilon, s_\varepsilon \) converge to 0 in \( L^2_{\text{loc}}(\mathbb{R}^+, H_{\text{loc}}^{5/2}(\mathcal{O} \cup S)) \) as \( \varepsilon \to 0 \).
Proof. As we have supposed in this section that \( S = \Omega \), the projection of \( u_\varepsilon \) on \( \text{Ker}(L) \) satisfies, due to Proposition 2.4
\[
\Pi u_\varepsilon = \frac{1}{|\Omega_3|} \int u_\varepsilon dx_3.
\]
Then
\[
\int w_\varepsilon dx_3 = \int (u_\varepsilon - \Pi u_\varepsilon) dx_3 = 0,
\]
and there exists \( W_\varepsilon \) such that
\[
w_\varepsilon = \partial_3 W_\varepsilon.
\]
Moreover, as \( \Omega_3 = T \), we can always choose \( W_\varepsilon \) so that
\[
\int W_\varepsilon dx_3 = 0.
\]
It remains then to determine the equations for \( W_\varepsilon \). Equation (1.1) implies
\[
\varepsilon \partial_t w_\varepsilon + w_\varepsilon \wedge \mathbf{b} c_3 + \nabla p = \varepsilon \nu \Pi_\perp \Delta u_\varepsilon - \varepsilon \Pi_\perp (u_\varepsilon \cdot \nabla u_\varepsilon),
\]
which can be rewritten in terms of \( W_\varepsilon \)
\[
\varepsilon \partial_t \partial_3 W_\varepsilon + \partial_3 W_\varepsilon \wedge \mathbf{b} c_3 + \nabla p = S_\varepsilon,
\]
with
\[
S_\varepsilon = \varepsilon (\nu \Pi_\perp \Delta u_\varepsilon - \Pi_\perp (u_\varepsilon \cdot \nabla u_\varepsilon)).
\]
From the energy bound, we deduce that the right hand side in (3.4) is of order \( \varepsilon \) in the space \( L^2_{\text{loc}}(\mathbb{R}^+, H^{-3/2}_{\text{loc}}(\mathcal{O} \cup S)) \). Indeed, using the continuity properties of \( \Pi_\perp \) as well as (2.5) we get
\[
\| \Pi_\perp \Delta u_\varepsilon \|_{L^2([0,T],H^{-1}(\mathcal{O} \cup S))} \leq \| \Delta u_\varepsilon \|_{L^2([0,T],H^{-1}(\Omega))} \leq \| u_\varepsilon \|_{L^2([0,T],H^1(\Omega))}
\]
and
\[
\| \Pi_\perp (u_\varepsilon \cdot \nabla u_\varepsilon) \|_{L^2([0,T],H^{-3/2}(K))} \leq \| u_\varepsilon \cdot \nabla u_\varepsilon \|_{L^2([0,T],H^{-3/2}(K))} \leq C \| u_\varepsilon \|_{L^\infty(\mathbb{R}^+, L^2(\Omega))} \| \nabla u_\varepsilon \|_{L^2([0,T],L^2(\Omega))}
\]
for all compact subsets \( K \) of \( \mathcal{O} \cup S \). Moreover, the right-hand side in (3.4) belongs to the image of \( \Pi_\perp \) and can therefore be written as a partial derivative with respect to \( x_3 \),
\[
S_\varepsilon = \partial_3 R_\varepsilon \quad \text{with} \quad \int R_\varepsilon \, dx_3 = 0.
\]
The equation on the third component provides then
\[
\varepsilon \partial_t \partial_3 W_{\varepsilon,3} + \partial_3 p = \partial_3 R_\varepsilon
\]
where \( R_\varepsilon \) converges to 0 in \( L^2_{\text{loc}}(\mathbb{R}^+, H^{-3/2}_{\text{loc}}(\mathcal{O} \cup S)) \) as \( \varepsilon \to 0 \). Integrating with respect to \( x_3 \) provides, since \( \int p \, dx_3 = 0 \),
\[
\varepsilon \partial_t W_{\varepsilon,3} + p = R_\varepsilon.
\]
Replacing in (3.4) leads to
\[
\varepsilon \partial_t \partial_3 W_\varepsilon + \partial_3 W_\varepsilon \wedge \mathbf{b} c_3 + \nabla (R_\varepsilon - \varepsilon \partial_t W_{\varepsilon,3}) = S_\varepsilon
\]
which is the first identity in Proposition 3.2.
In order to establish the second identity, we compute the rotational of \( \text{(3.4)} \) and write its last component
\[
\varepsilon \partial_t (\partial_3 W_{\varepsilon,1} - \partial_1 W_{\varepsilon,2}) + \partial_3 (\partial_1 W_{\varepsilon,1} + \partial_2 W_{\varepsilon,2}) = (\partial_2 S_{\varepsilon,1} - \partial_1 S_{\varepsilon,2}).
\]
Moreover, the right hand side in \( \text{(3.4)} \) belongs to the image of \( \Pi_\perp \) (recall that in this section \( \Pi_\perp \) commutes with all derivatives) and can therefore be written as a partial derivative with respect to \( x_3 \). Then integrating with respect to \( x_3 \) leads to the expected equality, where the right-hand side converges to 0 in \( L^2_{\text{loc}}(\mathbb{R}^+, H^{-5/2}_{\text{loc}}(\mathcal{O} \cup \mathcal{S})) \) as \( \varepsilon \) goes to 0. \( \square \)

### 3.3. Study of the coupling.

The algebraic structure of the propagator \( \text{(3.3)} \) implies that the oscillating terms cannot interact and produce some contribution in the limiting equation governing \( \mathbf{u} \). Indeed the nonlinear term \( w_\varepsilon \wedge \text{rot} w_\varepsilon \) can be rewritten as the sum of a total derivative with respect to \( x_3 \) and a total derivative with respect to \( t/\varepsilon \), modulo a remainder which converges formally to 0. Then, in order to prove a rigorous convergence result, the first step is to introduce a regularization of the equations \( \text{(3.3)} \) and to get a control of the source terms in some strong norm.

**Lemma 3.3.** Let \( u^0 \) be any vector field in \( H \). For all \( \varepsilon > 0 \), denote by \( u_\varepsilon \) a weak solution of \( \text{(1.1)} \) in \( L^\infty(\mathbb{R}^+, H) \cap L^2(\mathbb{R}^+, H^1) \), and by \( w_\varepsilon = \Pi_\perp u_\varepsilon \) its projection onto \( \text{Ker}(L)^\perp \). Then, for all \( \delta > 0 \), there exists \( W_\varepsilon^\delta \in L^2_{\text{loc}}(\mathbb{R}^+, \cap_s H^s(\Omega)) \) such that
\[
\varepsilon \partial_t (\partial_3 W_{\varepsilon}^\delta - \nabla W_{\varepsilon,3}^\delta) + \partial_3 (\partial_1 W_{\varepsilon,1}^\delta + \partial_2 W_{\varepsilon,2}^\delta) = r_\varepsilon^\delta,
\]
\[
\varepsilon \partial_t (\partial_2 W_{\varepsilon,1}^\delta - \partial_1 W_{\varepsilon,2}^\delta) + b(\partial_1 W_{\varepsilon,1}^\delta + \partial_2 W_{\varepsilon,2}^\delta) = s_\varepsilon^\delta,
\]
where for all \( \delta \), \( r_\varepsilon^\delta \) and \( s_\varepsilon^\delta \) converge to 0 in \( L^2_{\text{loc}}(\mathbb{R}^+, L^2_{\text{loc}}(\Omega)) \) as \( \varepsilon \to 0 \).

**Proof.** We introduce the following regularization: let \( \kappa \in C^\infty_c(\mathbb{R}^3, \mathbb{R}^+) \) such that \( \kappa(x) = 0 \) if \( |x| \geq 1 \) and \( \int \kappa dx = 1 \), we define
\[
\kappa_\delta : x \mapsto \frac{1}{\delta^3} \kappa \left( \frac{x}{\delta} \right),
\]
and
\[
w_\varepsilon^\delta = w_\varepsilon * \kappa_\delta, \quad W_\varepsilon^\delta = W_\varepsilon * \kappa_\delta,
\]
so that \( w_\varepsilon^\delta = \partial_3 W_{\varepsilon}^\delta \).

By the energy estimate, for all \( T > 0 \),
\[
u_\varepsilon \text{ is uniformly bounded in } L^2([0,T], H^1(\Omega)).
\]
By Proposition 2.4, we infer that
\[
w_\varepsilon = \Pi_\perp u_\varepsilon \text{ is uniformly bounded in } L^2([0,T], \mathcal{K})
\]
for some compact \( \mathcal{K} \) of \( L^2(\Omega) \). The result \( \text{(3.5)} \) then follows from the following fact:
\[
w_\varepsilon^\delta(t,x) - w_\varepsilon(t,x) = \int (w_\varepsilon(t,x-y) - w_\varepsilon(t,x)) \kappa_\delta(y) dy
\]
hence there is some continuity modulus \( \omega \) such that

\[
\forall \varepsilon, \quad |w^\delta_\varepsilon(t, x) - w_\varepsilon(t, x)| \leq \int \omega(y) \kappa_\delta(y) dy,
\]

and the result follows.

Regularizing (3.3) leads to

\[
\varepsilon \partial_t (\partial_3 W^\delta_\varepsilon - \nabla W^\delta_\varepsilon \wedge B) = r_\varepsilon \ast \kappa_\delta,
\]

\[
\varepsilon \partial_t (\partial_2 W^\delta_\varepsilon,1 - \partial_1 W^\delta_\varepsilon,2) + b(\partial_1 W^\delta_\varepsilon,1 + \partial_2 W^\delta_\varepsilon,2) = s_\varepsilon \ast \kappa_\delta,
\]

because \( b \) is homogeneous. Then, for all \( T > 0 \) and all compact subsets \( K \) of \( O \cup S \) and for \( \delta \) small enough,

\[
\|r^\delta_\varepsilon\|_{L^2([0,T], L^2(K))} = \|r_\varepsilon \ast \kappa_\delta\|_{L^2([0,T], L^2(K))} \leq C \|\kappa_\delta\|_{W^{5/2,1}(\mathbb{R}^3)} \|r_\varepsilon\|_{L^2([0,T], H^{-5/2}(K'))},
\]

where \( K' \) is a compact subset of \( O \cup S \) such that

\[
\{x \in \Omega / d(x, K) \leq \delta\} \subset K'.
\]

And, in the same way,

\[
\|r^\delta_\varepsilon\|_{L^2([0,T], L^2(K))} \leq C \delta^{5/2} \|r_\varepsilon\|_{L^2([0,T], H^{-5/2}(K'))}.
\]

For a fixed \( \delta \), Proposition 3.2 gives the expected convergences. \( \square \)

Equipped with this preliminary result, we are now able to study the coupling between the oscillating terms and to prove the following proposition.

**Proposition 3.4.** Let \( u^0 \) be any vector field in \( H \). For all \( \varepsilon > 0 \), denote by \( u_\varepsilon \) a weak solution of (1.1), and by \( w_\varepsilon = \Pi_\perp u_\varepsilon \) its projection onto \( \text{Ker}(L)^\perp \). Then,

\[
P \int w_\varepsilon \wedge \text{rot} w_\varepsilon dx_3 \to 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \Omega) \quad \text{as} \quad \varepsilon \to 0,
\]

where \( P \) denotes the Leray projection.

**Proof.** We start by proving that

\[
P \left( \int w_\varepsilon \wedge \text{rot} w_\varepsilon dx_3 - \int w^\delta_\varepsilon \wedge \text{rot} w^\delta_\varepsilon dx_3 \right) \to 0 \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^+ \times \Omega) \quad \text{as} \quad \delta \to 0 \quad \text{uniformly in} \quad \varepsilon.
\]

From the identity (3.2) and the relations \( \nabla \cdot w_\varepsilon = \nabla \cdot w^\delta_\varepsilon = 0 \), we deduce that

\[
P \left( w_\varepsilon \wedge \text{rot} w_\varepsilon - w^\delta_\varepsilon \wedge \text{rot} w^\delta_\varepsilon \right) = \mathcal{P} \nabla \left( \frac{|w_\varepsilon|^2 - |w^\delta_\varepsilon|^2}{2} \right) - \mathcal{P} \nabla \cdot (w_\varepsilon \otimes w_\varepsilon - w^\delta_\varepsilon \otimes w^\delta_\varepsilon)
\]

\[
= -\mathcal{P} \nabla \cdot ((w_\varepsilon - w^\delta_\varepsilon) \otimes w_\varepsilon) - \mathcal{P} \nabla \cdot (w^\delta_\varepsilon \otimes (w_\varepsilon - w^\delta_\varepsilon))
\]

By Lemma 3.3, we get (3.7).
It remains to prove that, for any fixed $\delta > 0$,  
\begin{equation}
\int w_\varepsilon^\delta \wedge \text{rot } w_\varepsilon^\delta \, dx_3 \to 0 \quad \text{as } \varepsilon \to 0.
\end{equation}

Define $\rho_\varepsilon^\delta = \text{rot } W_\varepsilon^\delta$:
\[
\begin{pmatrix}
\rho_{\varepsilon,1}^\delta \\
\rho_{\varepsilon,2}^\delta \\
\rho_{\varepsilon,3}^\delta
\end{pmatrix} = \begin{pmatrix}
\partial_3 W_{\varepsilon,3}^\delta - \partial_2 W_{\varepsilon,2}^\delta \\
-\partial_1 W_{\varepsilon,3}^\delta + \partial_3 W_{\varepsilon,1}^\delta \\
\partial_1 W_{\varepsilon,2}^\delta - \partial_2 W_{\varepsilon,1}^\delta
\end{pmatrix},
\]
which is uniformly bounded with respect to $\varepsilon$ in $L^2([0,T], \cap_8 H^s(\Omega))$ by Lemma 3.3. We have
\[
w_\varepsilon^\delta \wedge \text{rot } w_\varepsilon^\delta = \begin{pmatrix}
\partial_3 W_{\varepsilon,1}^\delta \\
\partial_3 W_{\varepsilon,2}^\delta \\
\partial_3 W_{\varepsilon,3}^\delta
\end{pmatrix} \wedge \begin{pmatrix}
\partial_{3,1}^\delta \\
\partial_{3,2}^\delta \\
\partial_{3,3}^\delta
\end{pmatrix}.
\]
From 3.6 and the divergence-free relation $\partial_3 W_{\varepsilon,3}^\delta = -\partial_1 W_{\varepsilon,1}^\delta - \partial_2 W_{\varepsilon,2}^\delta$, we deduce that the previous term can be rewritten
\[
\begin{pmatrix}
\varepsilon b^{-1} \partial_{3,1}^\delta (\partial_3 W_{\varepsilon,2}^\delta - \partial_2 W_{\varepsilon,3}^\delta) - b^{-1} r_{\varepsilon,2}^\delta \\
\varepsilon b^{-1} \partial_{3,2}^\delta (-\partial_1 W_{\varepsilon,3}^\delta + \partial_3 W_{\varepsilon,1}^\delta) + b^{-1} r_{\varepsilon,1}^\delta \\
\varepsilon b^{-1} \partial_{3,3}^\delta (\partial_2 W_{\varepsilon,1}^\delta - \partial_1 W_{\varepsilon,2}^\delta) - b^{-1} s_{\varepsilon}^\delta
\end{pmatrix} \wedge \begin{pmatrix}
\partial_{3,1}^\delta \\
\partial_{3,2}^\delta \\
\partial_{3,3}^\delta
\end{pmatrix},
\]
or equivalently
\[
\begin{pmatrix}
-\varepsilon b^{-1} \partial_{3,1}^\delta (\partial_2 r_{\varepsilon,2}^\delta + b^{-1} r_{\varepsilon,2}^\delta) \\
-\varepsilon b^{-1} \partial_{3,2}^\delta (\partial_1 r_{\varepsilon,1}^\delta + b^{-1} r_{\varepsilon,1}^\delta) \\
-\varepsilon b^{-1} \partial_{3,3}^\delta (\partial_1 s_{\varepsilon}^\delta - b^{-1} s_{\varepsilon}^\delta)
\end{pmatrix} \wedge \begin{pmatrix}
\partial_{3,1}^\delta \\
\partial_{3,2}^\delta \\
\partial_{3,3}^\delta
\end{pmatrix}.
\]
Integrating by parts with respect to $x_3$ leads then to
\[
b^{-1} \left( -\varepsilon \partial_{3,1}^\delta (\partial_2 r_{\varepsilon,2}^\delta + \partial_3 r_{\varepsilon,3}^\delta) + \partial_3 (r_{\varepsilon,2}^\delta \partial_{3,1}^\delta + s_{\varepsilon}^\delta \partial_{3,1}^\delta) \\
-\varepsilon \partial_{3,2}^\delta (\partial_1 r_{\varepsilon,1}^\delta + \partial_3 r_{\varepsilon,3}^\delta) + \partial_3 (r_{\varepsilon,2}^\delta \partial_{3,1}^\delta - s_{\varepsilon}^\delta \partial_{3,1}^\delta) \\
-\varepsilon \partial_{3,3}^\delta (\partial_1 s_{\varepsilon}^\delta + \partial_3 s_{\varepsilon}^\delta) + \partial_3 (r_{\varepsilon,2}^\delta \partial_{3,1}^\delta - r_{\varepsilon,1}^\delta \partial_{3,1}^\delta)
\right),
\]
from which we deduce that, for any fixed $\delta > 0$,
\[
P \int w_\varepsilon^\delta \wedge \text{rot } w_\varepsilon^\delta \, dx_3 \to 0
\]
in the sense of distributions, as $\varepsilon \to 0$.

Combining (3.7) and (3.8) gives the expected convergence. \hfill $\Box$

3.4. Passage to the limit. In order to determine the limiting velocity field $\overline{u}$, we have now to take limits in (3.1) which can be rewritten
\[
\partial_{3} \overline{u}_\varepsilon - \Pi(\overline{u}_\varepsilon \wedge \text{rot } \overline{u}_\varepsilon + w_\varepsilon \wedge \text{rot } \overline{u}_\varepsilon + \overline{u}_\varepsilon \wedge \text{rot } w_\varepsilon + w_\varepsilon \wedge \text{rot } w_\varepsilon) - \nu \Delta \overline{u}_\varepsilon = 0,
\]
using the decomposition
\[
u_\varepsilon = \overline{u}_\varepsilon + w_\varepsilon,
\]
where we recall that
\[
\overline{u}_\varepsilon \to \overline{u} \text{ strongly in } L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega),
\]
\[
w_\varepsilon \to 0 \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega).
\]
Then standard arguments using (3.2) show that
\[ \partial_t \overline{u} - \Pi(\overline{u} \wedge \text{rot} \overline{u}) - \nu \Delta \overline{u} = \lim_{\varepsilon \to 0} \Pi(w_\varepsilon \wedge \text{rot} w_\varepsilon), \]
and by Proposition 3.4 we get
\[ \partial_t \overline{u} - \Pi(\overline{u} \wedge \text{rot} \overline{u}) - \nu \Delta \overline{u} = 0. \]
In the case where \( B \) is constant, the projection \( \Pi \) reduces to a simple averaging with respect to \( x_3 \), and we recover the usual convergence result towards the 2D1/2 Navier-Stokes equation.

4. The case of a variable vector field \( B \):
a turbulent behaviour

In this section we shall prove Theorem 3 stated in the introduction, concerning the general case when the rotation vector \( B = b(x_3)e_3 \) is inhomogeneous. We suppose that Assumptions \( (H0) \) to \( (H2) \) are satisfied. If \( \Omega_3 = \mathbb{R} \), then \( \overline{u} = 0 \) simply because it is in \( L^2(\Omega) \) but only depends on the horizontal variables. So from now on we can suppose that \( \Omega_3 = \mathbb{T} \).

The strategy of proof is quite similar to the constant case, so we shall often be referring to the results of the previous section. The first remark to be made is that if \( b \) is constant in some positive measured region of \( \Omega_h \), then in that region the results of the previous section should apply and one should recover at the limit the usual two-dimensional behaviour.

Moreover, the results of Section 2 hold for any \( B \), so in particular any weak limit point \( \overline{u} \) of a sequence of weak solutions \( u_\varepsilon \) to (1.1) is in the kernel of \( L \) according to Corollary 2.3. That means in particular that the third component does not see the difference between \( \mathcal{O} \) and \( \mathcal{S} \) since the elements of the kernel of \( L \) have the same third component whether \( b \) is homogeneous or not. So in the following, we shall restrict the study of the limit system to the horizontal components only. As in the previous section, the proof of Theorem 3 consists in finding the equation satisfied by \( \overline{u} \) (at least its horizontal part \( \overline{u}_h \)), by taking the limit of the equation satisfied by the horizontal part of \( u_\varepsilon = \Pi u_\varepsilon \). The first result we shall establish is that in the general, variable \( b \) case, there is no coupling between oscillating vector fields yielding extra terms in the averaged equation. This will be a generalization of Proposition 3.4 to the variable case, and the analysis will follow closely the proof of Proposition 3.4. Then we shall write the averaged equation on \( \mathcal{O} \). Finally we shall concentrate on the \( \mathcal{S} \) case and show the limit \( \overline{u}_h \) satisfies a two-dimensional Navier–Stokes equation with homogeneous boundary conditions on the boundary of \( \mathcal{S} \).

4.1. The averaged equation. Let \( u_\varepsilon \) be a family of weak solutions to (1.1), and define \( \overline{u}_\varepsilon = \Pi u_\varepsilon \). Recalling that the elements of \( \text{Ker}(L) \) are divergence free, we have as in the constant case
\[ \partial_t \overline{u}_\varepsilon - \nu \Pi \Delta u_\varepsilon - \Pi(u_\varepsilon \wedge \text{rot} u_\varepsilon) = 0. \]
Of course the projector \( \Pi \) no longer commutes with (horizontal) derivatives. However as \( \Pi \) belongs to \( C(w - H^s(\mathcal{O} \cup \mathcal{S}), w - H^s(\mathcal{O} \cup \mathcal{S})) \) for \( s \leq 0 \), we clearly have as \( \varepsilon \) goes to zero,
\[ \Pi \Delta u_\varepsilon \to \Pi \Delta \overline{u}. \]
Let us now take the limit in the quadratic term.
Lemma 4.1. Let $u^0 \in L^2(\Omega)$ be a divergence-free vector field. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1), and by $\overline{u}_\varepsilon = \Pi u_\varepsilon$ (resp. $w_\varepsilon = \Pi_1 u_\varepsilon$) its projection onto $\text{Ker}(L)$ (resp. $\text{Ker}(L)^\perp$). Then the following results hold in $\mathcal{D}'(\mathbb{R}^+ \times (\mathcal{O} \cup \mathcal{S}))$, as $\varepsilon$ goes to zero:

\[(4.1) \quad \Pi(\overline{u}_\varepsilon \cdot \nabla \overline{u}_\varepsilon) \to \Pi(\overline{\nu} \cdot \nabla \overline{\nu}),\]

\[(4.2) \quad \Pi(w_\varepsilon \cdot \nabla \overline{u}_\varepsilon + \overline{u}_\varepsilon \cdot \nabla w_\varepsilon) \to 0,\]

\[(4.3) \quad \Pi(w_\varepsilon \cdot \nabla w_\varepsilon) \to 0.\]

Proof. The results (4.1) and (4.2) are simply due to the compactness of $\overline{u}_\varepsilon$ as well as the fact that $w_\varepsilon$ goes weakly to zero, results given in Corollary 2.5. We also use the continuity properties of $\Pi$ stated in Proposition 2.4. The more difficult result to prove is of course (4.3). The method will follow the proof of Proposition 3.4 and will be achieved in two steps. First we show that one can smooth out the equation satisfied by $w_\varepsilon$, and then we perform some algebra on the bilinear term in the equation, as in the constant case. We shall therefore continuously be referring to the methods of Section 3.

Let us start by proving the following result, analogous to Proposition 3.2.

Proposition 4.2. Let $u^0 \in L^2(\Omega)$ be a divergence-free vector field. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1), and by $w_\varepsilon = \Pi_1 u_\varepsilon$ its projection onto $\text{Ker}(L)^\perp$. Then there exists $W_{\varepsilon,3} \in L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega)$ such that $w_{\varepsilon,3} = \partial_3 W_{\varepsilon,3}$. Moreover,

\[(4.4) \quad \varepsilon \partial_t (w_{\varepsilon,h} - \nabla_h W_{\varepsilon,3}) + w_{\varepsilon,h} \wedge B = \tilde{r}_\varepsilon,\]

\[(4.5) \quad \varepsilon \partial_t (\partial_2 w_{\varepsilon,1} - \partial_1 w_{\varepsilon,2}) + \text{div}_h(b w_{\varepsilon,h}) = \tilde{s}_\varepsilon,\]

where $\tilde{r}_\varepsilon, \tilde{s}_\varepsilon$ converge to 0 in $L^2_{\text{loc}}(\mathbb{R}^+, H^5_{\text{loc}}(\mathcal{O} \cup \mathcal{S}))$ as $\varepsilon \to 0$.

Proof. We shall omit the proof of that result here, as it is identical to the proof of Proposition 3.2. We just have to notice that the third component $w_{\varepsilon,3}$ is of vertical mean zero, so can as in the constant case be replaced by $\partial_3 W_{\varepsilon,3}$. The other components, contrary to the constant case, cannot be transformed in that way, so remain as they are. The rest of the proof is identical to the constant case.

Now as in the constant case, let us smooth out Equation (4.4).

Lemma 4.3. Let $u^0 \in L^2(\Omega)$ be a divergence-free vector field. For all $\varepsilon > 0$, denote by $u_\varepsilon$ a weak solution of (1.1), and by $w_\varepsilon = \Pi_1 u_\varepsilon$ its projection on $\text{Ker}(L)^\perp$. Then, for all $\delta > 0$, there exists $w^\delta_{\varepsilon,h} \in L^2_{\text{loc}}(\mathbb{R}^+ \cap H^s(\Omega))$ and $W^\delta_{\varepsilon,3} \in L^2_{\text{loc}}(\mathbb{R}^+ \cap H^s(\Omega))$ such that

\[(4.6) \quad w_{\varepsilon,h} - w^\delta_{\varepsilon,h} \to 0 \text{ in } L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega) \text{ as } \delta \to 0 \text{ uniformly in } \varepsilon > 0,\]

\[(4.7) \quad w_{\varepsilon,3} - \partial_3 W^\delta_{\varepsilon,3} \to 0 \text{ in } L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega) \text{ as } \delta \to 0 \text{ uniformly in } \varepsilon > 0,\]

\[(4.8) \quad \delta \|\nabla_h w^\delta_{\varepsilon,h}\|_{L^2(\Omega)} \to 0 \text{ as } \delta \to 0 \text{ uniformly in } \varepsilon > 0.\]
and

\[ \varepsilon \partial_t (w^\delta_{\varepsilon, h} - \nabla_h W^\delta_{\varepsilon, 3}) + w^\delta_{\varepsilon, h} \wedge B = \tilde{r}^\delta_{\varepsilon} \]
\[ \varepsilon \partial_t (\partial_2 w^\delta_{\varepsilon, 1} - \partial_1 w^\delta_{\varepsilon, 2}) + \partial_1 (w^\delta_{\varepsilon, 1} b) + \partial_2 (w^\delta_{\varepsilon, 2} b) = \tilde{s}^\delta_{\varepsilon}, \]

where \( \tilde{r}^\delta_{\varepsilon} \) and \( \tilde{s}^\delta_{\varepsilon} \) converge to 0 in \( L^2_{\text{loc}}(\mathbb{R}^+, L^2_{\text{loc}}(O \cup S)) \) as \( \varepsilon \) and \( \delta \) go to 0. More precisely for any subset \( K \) of \( O \cup S \) there is a constant \( C \) (independent of \( \varepsilon \) and \( \delta \)) and a constant \( c_\delta \) depending only on \( \delta \) such that

\[ ||\tilde{r}^\delta_{\varepsilon} + \tilde{s}^\delta_{\varepsilon}||_{L^2(K)} \leq c_\delta o(1) + C\delta. \]

**Proof.** We shall not write all the details of the proof here, since it is very similar to the constant case (Lemma 3.3); let us simply point out where the fact that \( b \) is not constant appears — note that in \( (1.40) \), the part \( c_\delta o(1) \) is precisely due to the terms of the constant case, and we will see here that the fact that \( b \) is no longer constant yields terms which are estimated by \( C\delta \). In the approximation of the equation, the only difference with the constant case is that of course \( (w_\varepsilon \wedge B) * \kappa_\delta \) is not equal to \( (w_\varepsilon \wedge B) \wedge B \). Moreover of course \( (1.7) \) is obvious in the constant case, since \( \Pi_\perp \) commutes with partial derivatives. So we need to deal with those two problems due to the fact that \( b \) is not constant.

First of all, the difference between \( (w_\varepsilon \wedge B) * \kappa_\delta \) and \( (w_\varepsilon \wedge \kappa_\delta) \wedge B \) is small when \( \delta \) goes to zero, due to the following computation: we have

\[ (w_\varepsilon \wedge B) * \kappa_\delta(t, x) - (w_\varepsilon \wedge \kappa_\delta) \wedge B(t, x) = \int w_\varepsilon(t, x - y) \kappa_\delta(y) \wedge (B(x - y) - B(x)) \, dy \]

hence

\[ (w_\varepsilon \wedge B) * \kappa_\delta(t, x) - (w_\varepsilon \wedge \kappa_\delta) \wedge B(t, x) = \int_0^1 \int y \cdot \nabla B(x - \sigma y) \wedge w_\varepsilon(x - y) \kappa_\delta(y) \, dy \\ \times d\sigma. \]

To conclude we need to take the \( L^2 \) norm in \( x \) of that quantity, and Young’s inequality yields

\[ ||(w_\varepsilon \wedge B) * \kappa_\delta(t, \cdot) - (w_\varepsilon \wedge \kappa_\delta) \wedge B(t, \cdot)||_{L^2(\Omega)} \leq C_B \||| \cdot \kappa_\delta||_{L^1(\Omega)} ||w_\varepsilon(t)||_{L^2(\Omega)} \leq C\delta ||u_0||_{L^2} \]

uniformly in time. The result follows for the first equation in \( (1.8) \). The second one is of the same type, since

\[ \partial_1 (w_\varepsilon, 1 b) + \partial_2 (w_\varepsilon, 2 b) = \text{div}_h w_\varepsilon, h + w_\varepsilon, h \cdot \nabla_h b. \]

The term \( (w_\varepsilon, h \cdot \nabla_h b) * \kappa_\delta \) is approximated by \( (w_\varepsilon, h * \kappa_\delta) \cdot \nabla_h b \) exactly as above; to replace the term \( (\text{div}_h w_\varepsilon, h) * \kappa_\delta \) by \( (\text{div}_h w_\varepsilon, h * \kappa_\delta) \) we write the same type of computation, with

\[ (\text{div}_h w_\varepsilon, h) * \kappa_\delta(t, x) - b(\text{div}_h w_\varepsilon, h * \kappa_\delta)(t, x) = - \int \partial_3 w_\varepsilon, 3(t, x - y) \kappa_\delta(y)(b(x - y) - b(x)) \, dy \]

hence, since \( \Pi_\perp \) commutes with \( \partial_3 \),

\[ ||(\text{div}_h w_\varepsilon, h) * \kappa_\delta - b(\text{div}_h w_\varepsilon, h * \kappa_\delta)||_{L^2(\mathbb{R}^+ \times \Omega)} \leq C_B \||| \partial_3 w_\varepsilon, 3||_{L^2(\mathbb{R}^+ \times \Omega)} \delta \leq C_B \delta ||u_0||_{L^2}. \]

That ends the proof of \( (1.8) \).

Now to end the proof of the proposition, we still need to check \( (1.7) \). The idea is to use the following estimate, due to the fact that \( w_\varepsilon \) is bounded in \( L^2(K) \) for some compact subspace of \( L^2(\Omega) \): there is a continuity modulus \( \omega \) such that

\[ \forall y \in \Omega, \ ||w_\varepsilon(t, \cdot + y) - w_\varepsilon(t, \cdot)||_{L^2(\Omega)} \leq \omega(y), \quad \text{uniformly in} \ t \ and \ \varepsilon. \]
Now since \( w_{\varepsilon,h}^\delta = \kappa_\delta * w_{\varepsilon,h} \), we have
\[
\nabla_h w_{\varepsilon,h}^\delta(x) = \int_{\Omega} \frac{1}{\delta^d} (\nabla_h \kappa)(\frac{y}{\delta}) w_{\varepsilon,h}(x - y) \, dy.
\]

Since \( \int_{\Omega} \nabla_h \kappa(y) \, dy = 0 \), it follows that
\[
\nabla_h w_{\varepsilon,h}^\delta(x) = \int_{\Omega} \frac{1}{\delta^d} (\nabla_h \kappa)(\frac{y}{\delta}) (w_{\varepsilon,h}(x - y) - w_{\varepsilon,h}(x)) \, dy.
\]

Then by (4.10) we find that
\[
\| \nabla_h w_{\varepsilon,h}^\delta \|_{L^2(\Omega)} \leq C \eta(\delta) \| \nabla_h \kappa \|_{L^1(\Omega)}
\]
where \( \eta(\delta) \) goes to zero as \( \delta \) goes to zero. The result is proved.

\[\square\]

Now we are ready to prove the following result.

**Proposition 4.4.** Let \( u_0^0 \in L^2(\Omega) \) be a divergence-free vector field. For all \( \varepsilon > 0 \), denote by \( u_{\varepsilon} \) a weak solution of (1.1), and by \( w_{\varepsilon} = \Pi \perp u_{\varepsilon} \) its projection onto \( \text{Ker}(L)^\perp \). Then
\[
\Pi (w_{\varepsilon} \wedge \text{rot} w_{\varepsilon}) \rightarrow 0 \text{ in } \mathcal{D}'(\mathbb{R}^+ \times (\mathcal{O} \cup \mathcal{S})) \text{ as } \varepsilon \rightarrow 0.
\]

**Proof.** Since the result we are looking for is a weak limit, we can restrict our attention to the set \( \mathcal{O} \) where \( \nabla b \) does not vanish, as in \( \mathcal{S} \) the result is due to Proposition 3.4. Moreover, to prove the result one can restrict our attention to \( \int w_{\varepsilon} \wedge \text{rot} w_{\varepsilon} \, dx_3 \) and it is enough to prove that it is proportional to \( \nabla_h b \) (up to a small remainder term): taking the scalar product with a function in \( \text{Ker}(L) \) will then yield automatically zero by definition of \( \text{Ker}(L) \).

In order to simplify the computations, we shall directly prove the result replacing \( w_{\varepsilon,h} \) by \( w_{\varepsilon,h}^\delta \) and \( w_{\varepsilon,3} \) by \( \partial_3 W_{\varepsilon,3}^\delta \). The difference in the two computations is indeed small when \( \delta \) is small, uniformly in \( \varepsilon \), exactly as in the proof of (3.7) in the constant case. So writing \( \partial_3 W_{\varepsilon,3}^\delta = w_{\varepsilon,3}^\delta \), and dropping the index \( \delta \) to simplify, we can perform the following algebraic computations, which will prove the result.

Let us start by recalling that, due to Proposition 4.2, we have
\[
(4.11) \quad w_{\varepsilon,h} = -\frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,h} + r_{\varepsilon}),
\]
where similarly to Section 3 we have noted
\[
\rho_{\varepsilon,h} \overset{\text{def}}{=} \nabla_h \perp W_{\varepsilon,3} - w_{\varepsilon,h}^\perp.
\]
We shall also define
\[
\rho_{\varepsilon,3} \overset{\text{def}}{=} \partial_1 w_{\varepsilon,2} - \partial_2 w_{\varepsilon,1}.
\]

In (4.11) and in the following, the function \( r_{\varepsilon} \) denotes a remainder term, arbitrarily small in the space \( L^2_{\text{loc}}(\mathbb{R}^+ \times (\mathcal{O} \cup \mathcal{S})) \). It follows that
\[
w_{\varepsilon} \wedge \text{rot} w_{\varepsilon} = \begin{pmatrix}
-\frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,1} + r_{\varepsilon}) \\
-\frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,2} + r_{\varepsilon}) \\
\partial_3 W_{\varepsilon,3}
\end{pmatrix} \wedge \begin{pmatrix}
\partial_3 \rho_{\varepsilon,1} \\
\partial_3 \rho_{\varepsilon,2} \\
\rho_{\varepsilon,3}
\end{pmatrix},
\]
Recalling (4.11) we get
\[ w_\varepsilon \wedge \text{rot } w_\varepsilon = \begin{pmatrix} -\frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,2} + r_\varepsilon)\rho_{\varepsilon,3} - \partial_3 W_{\varepsilon,3} \partial_3 \rho_{\varepsilon,2} \\ \frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,1} + r_\varepsilon)\rho_{\varepsilon,3} + \partial_3 W_{\varepsilon,3} \partial_3 \rho_{\varepsilon,1} \\ \frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,2} + r_\varepsilon)\partial_3 \rho_{\varepsilon,1} - \frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,1} + r_\varepsilon)\partial_3 \rho_{\varepsilon,2} \end{pmatrix}. \]

Since the vertical component can be treated exactly as in the constant case, we shall now restrict our attention to the first two components. So calling \( \alpha_h \) the horizontal components of that vector field, we have after an integration by parts and using the divergence free condition \( \partial_3 W_{\varepsilon,3} = -\text{div}_h w_{\varepsilon,h} \):
\[ \int \alpha_h \, dx_3 = -\frac{1}{b} \int (\varepsilon \partial_t \rho_{\varepsilon,h} + r_\varepsilon)\rho_{\varepsilon,3} \, dx_3 - \int \rho_{\varepsilon,h} \text{div}_h w_{\varepsilon,h} \, dx_3. \]

Now we recall (calling once again generically \( r_\varepsilon \) the small remainder terms) that
\[ -\varepsilon \partial_t \rho_{\varepsilon,3} + \text{div}_h (bw_{\varepsilon,h}) = r_\varepsilon \]
so
\[ \text{div}_h w_{\varepsilon,h} = \frac{1}{b}(\varepsilon \partial_t \rho_{\varepsilon,3} - w_{\varepsilon,h} \cdot \nabla_h b + r_\varepsilon). \]

It follows that
\[
\int \alpha_h \, dx_3 = -\frac{1}{b} \int \varepsilon \partial_t \rho_{\varepsilon,h} \rho_{\varepsilon,3} \, dx_3 - \frac{1}{b} \int \rho_{\varepsilon,h} \varepsilon \partial_t \rho_{\varepsilon,3} \, dx_3 + \frac{1}{b} \int \rho_{\varepsilon,h} w_{\varepsilon,h} \cdot \nabla_h b \, dx_3 + \int \tilde{r}_\varepsilon \, dx_3 \\
= -\frac{1}{b} \int \varepsilon \partial_t (\rho_{\varepsilon,h} \rho_{\varepsilon,3}) \, dx_3 + \frac{1}{b} \int \rho_{\varepsilon,h} w_{\varepsilon,h} \cdot \nabla_h b \, dx_3 + \int \tilde{r}_\varepsilon \, dx_3,
\]
where now \( \tilde{r}_\varepsilon \) denotes generically the product of \( r_\varepsilon \) by a component of \( \rho_\varepsilon \). But \( \rho_{\varepsilon,3} \) is a combination of derivatives of \( w_\varepsilon \) whereas \( \rho_{\varepsilon,h} \) is a combination of components of \( w_\varepsilon \). A product of the type \( \rho_{\varepsilon,h} r_\varepsilon \) clearly goes to zero in \( \mathcal{D}'(\mathbb{R}^+ \times (O \cup S)) \). For the term \( r_\varepsilon \rho_{\varepsilon,3} \), one uses result (4.11) and (4.12) to infer that for any subset \( K \) of \( O \cup S \),
\[
\| r_\varepsilon \rho_{\varepsilon,3} \|_{L^1(K)} \leq \| r_\varepsilon \|_{L^2(K)} \| \rho_{\varepsilon,3} \|_{L^2(K)} \leq (c_\delta a_\varepsilon(1) + C\delta) \| \nabla_h w_{\varepsilon,h} \|_{L^2(K)} \rightarrow 0,
\]
as \( \varepsilon \) followed by \( \delta \) go to zero. So from now on \( \tilde{r}_\varepsilon \) will denote generically a term going to zero in \( \mathcal{D}'(\mathbb{R}^+ \times (O \cup S)) \).

Recalling (4.11) we get
\[ \int \alpha_h \, dx_3 = -\frac{1}{b} \int \varepsilon \partial_t (\rho_{\varepsilon,h} \rho_{\varepsilon,3}) \, dx_3 - \frac{1}{b^2} \int \rho_{\varepsilon,h} \varepsilon \partial_t \rho_{\varepsilon,3} \cdot \nabla_h b \, dx_3 + \int \tilde{r}_\varepsilon \, dx_3. \]
In particular we have
\[
\int \alpha_1 \, dx_3 = -\frac{1}{b} \int \varepsilon \partial_t (\rho_{\varepsilon,2} \rho_{\varepsilon,3}) \, dx_3 - \frac{1}{b^2} \int \rho_{\varepsilon,2} \varepsilon \partial_t \rho_{\varepsilon,1} \, dx_3 - \frac{1}{2b^2} \int \varepsilon \partial_t (\rho_{\varepsilon,2}^2) \, dx_3 + \int \tilde{r}_\varepsilon \, dx_3.
\]
Similarly
\[
\int \alpha_2 \, dx_3 = \frac{1}{b} \int \varepsilon \partial_t (\rho_{\varepsilon,1} \rho_{\varepsilon,3}) \, dx_3 + \frac{1}{b^2} \int \rho_{\varepsilon,1} \varepsilon \partial_t \rho_{\varepsilon,2} \, dx_3 + \frac{1}{2b^2} \int \varepsilon \partial_t (\rho_{\varepsilon,1}^2) \, dx_3 + \int \tilde{r}_\varepsilon \, dx_3.
\]
But one can also write
\[
\int \alpha_2 \, dx_3 = \frac{1}{b} \int \varepsilon \partial_t (\rho_{\varepsilon,1} \rho_{\varepsilon,3} + \frac{1}{2b} \rho_{\varepsilon,1}^2 \partial_1 b + \frac{1}{b} \rho_{\varepsilon,1} \rho_{\varepsilon,2} \partial_2 b) \, dx_3 - \frac{1}{b^2} \int \rho_{\varepsilon,2} \varepsilon \partial_t \rho_{\varepsilon,1} \, dx_3 + \int \tilde{r}_\varepsilon \, dx_3.
\]
It follows that up to full derivatives of the type \( \varepsilon \partial_t, \int \alpha_h\, dx_3 \) is equal to
\[
(4.12) \quad \left( -\frac{1}{b^2} \int \rho_{\varepsilon, 2} \varepsilon \partial_t \rho_{\varepsilon, 1}\, dx_3 \right) \nabla_h b + \int \tilde{r}_\varepsilon\, dx_3.
\]
Now the proof is almost finished: we recall that we want to take the projector of the term \( \int w_\varepsilon \wedge \text{rot} w_\varepsilon\, dx_3 \) onto the kernel of \( L \) restricted to the set \( \mathcal{O} \). Recalling that \( \text{Ker} L \) is made of vector fields of the type \( \nabla_\perp h \varphi \) with \( \nabla_h b \cdot \nabla_\perp h \varphi = 0 \), we have obviously
\[
\Pi \left( -\frac{1}{b^2} \int \rho_{\varepsilon, 2} \varepsilon \partial_t \rho_{\varepsilon, 1}\, dx_3 \right) \nabla_h b = 0.
\]
The result is therefore proved for the horizontal component of \( \int w_\varepsilon \wedge \text{rot} w_\varepsilon\, dx_3 \). The third component is identical to the constant case, so the proposition is proved.

This ends the proof of (4.3), hence of Lemma 4.1.

In the following we shall denote by \((L)\) the limiting system:
\[
(L) \quad \begin{cases} 
\partial_t \overline{u} - \nu \Delta_h \overline{u} + \Pi (\overline{u} \cdot \nabla_h \overline{u}) = 0 & \text{in } \mathcal{D}'(\mathbb{R}^+ \times (\mathcal{O} \cup \mathcal{S})), \\
\overline{u}|_{t=0} = \Pi u^0.
\end{cases}
\]
The existence of solutions to this system is easy to prove, as it is of the same form as a 2D Navier–Stokes equation: the only point which might be a problem is that \( \Pi \) does not in general commute with the Friedrich frequency truncation \( J_n \), recalled in Section 2; however approximating the system by for instance
\[
\partial_t J_n \overline{u}_n - \nu J_n \Pi J_n \Delta_h J_n \overline{u}_n + J_n \Pi (J_n \overline{u}_n \cdot \nabla_h J_n \overline{u}_n) = 0, \quad J_n \Pi J_n \overline{u}_n = 0
\]
will do the job. In any case in the next two sections we shall give precise formulations for the solution of \((L)\): in \( \mathcal{S} \, \overline{u}_n \) is the unique solution of \((\text{NS2D})\), and outside \( \mathcal{S} \) it is the solution of a heat equation. The uniqueness of \( \overline{u} \) implies in particular that the convergence holds for the whole sequence \( u_\varepsilon \) and not only for a subsequence.

As noted earlier in this section, the third component of \( \Pi u \) for any vector field \( u \) is the same whether \( b \) is constant or not and is simply the vertical average of \( u_3 \). It follows that the third component of this equation is simply
\[
\partial_t \overline{u}_3 - \nu \Delta_h \overline{u}_3 + \overline{u}_h \cdot \nabla_h \overline{u}_3 = 0 \quad \text{in } \mathbb{R}^+ \times (\mathcal{O} \cup \mathcal{S}), \quad \overline{u}_3|_{t=0} = \int u_3|_{t=0}\, dx_3.
\]
Now all the work consists in determining \( \overline{u}_h \). We shall consider separately the vector field on \( \mathcal{O} \) and on \( \mathcal{S} \), which is the object of the two following sections; so in those sections, our attention will be restricted to the horizontal component \( \overline{u}_h \).

4.2. The averaged equation on \( \mathcal{O} \). We shall prove the following result, which yields the part of Theorem 3 which lies in \( \mathcal{O} \).

**Proposition 4.5.** Let \( \overline{u} \) be a vector field satisfying \((L)\) with
\[
\overline{u} \in L^\infty(\mathbb{R}^+, H) \cap L^2(\mathbb{R}^+, \dot{H}^1) \quad \text{and} \quad \overline{u} \in \text{Ker}(L).
\]
Then the vector field \( \overline{\mathbf{u}}_h \) satisfies the following heat equation:
\[
\partial_t \overline{\mathbf{u}}_h - \nu \Delta_h \overline{\mathbf{u}}_h = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathcal{O}
\]
\[
\overline{\mathbf{u}}_{h|t=0} = \Pi u_{h,0}.
\]

Proof. The function \( \overline{\mathbf{u}}_h \) satisfies
\[
\partial_t \overline{\mathbf{u}}_h - \nu \Delta_h \overline{\mathbf{u}}_h + \Pi (\overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h) = 0,
\]
and since \( \overline{\mathbf{u}} \) is in Ker(\( L \)), we have
\[
- (\Pi \Delta_h \overline{\mathbf{u}}_h|\mathcal{O})_{L^2(\mathcal{O})} = \Pi (\overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h|\mathcal{O})_{L^2(\mathcal{O})} = \| \nabla_h \overline{\mathbf{u}}_h \|_{L^2(\mathcal{O})}^2.
\]
We note that \( \overline{\mathbf{u}}_h \) is equal to zero on the boundary of \( \mathcal{O} \), since it is a multiple of \( \nabla \perp b \).
So to prove the proposition, the only point we need to check is that
\[
(\forall \Phi \in \text{Ker}(L) \cap \dot{H}^s(\mathcal{O}), \ s > 1, \quad (\overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h|\Phi_h)_{L^2(\mathcal{O})} = 0.
\]
By definition of Ker(\( L \)), we have
\[
\Phi_h \cdot \nabla_h b = \overline{\mathbf{u}}_h \cdot \nabla_h b = 0,
\]
from which we infer that
\[
\Phi_h \wedge \overline{\mathbf{u}}_h = 0.
\]
Now as in (3.2) we can write
\[
\Pi (\overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h) = - \Pi (\overline{\mathbf{u}}_h \wedge \text{rot} \overline{\mathbf{u}}_h)
\]
hence
\[
(\overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h|\Phi_h)_{L^2(\mathcal{O})} = (\overline{\mathbf{u}}_h \wedge \Phi_h|\text{rot} \overline{\mathbf{u}}_h)_{L^2(\mathcal{O})}.
\]
Finally Identity (4.14) yields (4.13), and Proposition 4.5 is proved. \( \square \)

4.3. The 2D Navier-Stokes limit on \( S \). In this section we shall analyse the equation satisfied by the limit on \( S \), that is to say in the regions where \( b \) is a constant.

Proposition 4.6. Let \( \overline{\mathbf{u}} \) be a vector field satisfying \((L)\) with
\[
\overline{\mathbf{u}} \in L^\infty(\mathbb{R}^+, \mathbb{H}) \cap L^2(\mathbb{R}^+, \dot{H}^1) \quad \text{and} \quad \overline{\mathbf{u}} \in \text{Ker}(L).
\]
Then the vector field \( \overline{\mathbf{u}}_h \) satisfies the two–dimensional Navier–Stokes equations in \( S \), with homogeneous Dirichlet boundary conditions:
\[
\partial_t \overline{\mathbf{u}}_h - \nu \Delta_h \overline{\mathbf{u}}_h + \overline{\mathbf{u}}_h \cdot \nabla_h \overline{\mathbf{u}}_h = -\nabla_h p \quad \text{in} \quad \mathbb{R}^+ \times S
\]
\[
\overline{\mathbf{u}}_{h|t=0} = \int u_{h,0} \, dx_3, \quad \overline{\mathbf{u}}_{h|\partial S} = 0.
\]

Proof. First let us recall why the equation on \( \overline{\mathbf{u}}_h \) is the two–dimensional Navier–Stokes equation: we simply consider the weak formulation of the original rotating fluid equations and take its limit, by integrating against a test function \( \Phi \), divergence free and compactly supported in \( \mathbb{R}^+ \times S \). The weak formulation is as follows:
\[
(4.15) \int_{\mathbb{R}^+} \int_{\Omega} \left( -u_\varepsilon \cdot \partial_t \Phi + \nu \nabla u_\varepsilon \cdot \nabla \Phi - u_\varepsilon \otimes u_\varepsilon \cdot \nabla \Phi \right) \, dx \, dt = \int_{\Omega} u_0 \Phi_{|t=0} \, dx.
\]
Then taking the limit as \( \varepsilon \) goes to zero in (4.15) yields, due to Lemma 4.1,
\[
\int \left( -\mathbf{\Pi} \cdot \partial_t \Phi + \nu \nabla \mathbf{\Pi} \cdot \nabla \Phi - \mathbf{\Pi} \otimes \mathbf{\Pi} \cdot \nabla \Phi \right) \, dx = \int_{\Omega} u^0 \Phi_{|t=0} \, dx
\]
where we have noticed that on \( \mathcal{S} \), we have
\[
\Pi(\mathbf{\Pi} \cdot \nabla \mathbf{\Pi}) = \nabla \cdot (\mathbf{\Pi} \otimes \mathbf{\Pi}).
\]
Now recalling that \( \mathbf{\Pi} \) only depends on the horizontal variable, we deduce the expected equation on \( \mathcal{F} \), up to the boundary terms. To get the boundary terms, we simply recall that the limit \( \mathbf{\Pi} \) is in \( L^2_{\text{loc}}(\mathbb{R}^+, H^1(\Omega_h)) \) hence cannot have a jump on the boundary of \( \mathcal{S} \). Then we notice that \( \partial \mathcal{S} \subset \partial \mathcal{O} \), simply because if \( x \in \partial \mathcal{S} \), then \( \nabla b(x) = 0 \). So the result follows directly: the boundary condition is a homogeneous Dirichlet boundary condition.

Theorem 3 is proved.

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