Small – essential sub-modules relative to an arbitrary sub-module

M. S. ABBAS

Department of Mathematics, College of Science, Mustansiriya University, Baghdad, Iraq

mhdsabass@gmail.com

Abstract. Let M be an R-module and T a submodule of M. The notion of small-essential and T-essential sub module are well known. In this paper, we introduce small-essential submodule of M relative to T, which is a generalization of both ones. Several properties and characterizations have been introduced. We invest this concept to consider two socles of modules and study some of their properties.

1. Introduction

Throughout this work, all rings are associative with nonzero identity and all modules are unitary left R-module. We use the notation "⊆" and "≤" to denote the inclusion and submodule. The radical of module M denoted by J(M), is accordingly the intersection of all maximal submodules of M or is equal to the sum of small submodule of M. The concept dual to that of the radical is the socle. The socle of the module M, denoted by Soc(M), is the sum of all minimal submodules of M. It is equal to the intersection of all essential submodules of M. Small and essential are cornerstone of the radical and socle. A submodule N of an R-module M is essential, denoted by N ⪰ M if for every nonzero submodule K of M, N ∩ K is nonzero. Dually, a submodule N of M is small(denoted by N ≪ M) if for each submodule L of M , N + L = M implies that L = M. For more details about essential and small submodules see [1]. Recently in [2], the authors generalized the concept of essential submodules, by considering the class of small submodule in place of that of all submodules. A submodule N is called small-essential denoted by N ⪰ M. N ∩ K = 0, then K = 0 for all small submodules K of M. Also they introduced the small socle of modules, denoted by Soc_M(M) as the sum of all small and minimal submodules of M and proved that it is equal to the intersection of all small-essential submodules of M. On the other hand, the authors in [3] generalized the notion of essential submodules by replacing an arbitrary submodule T (say), instead of 0. Let T be an arbitrary submodule of an R-module M. A submodule N of M is called T-essential if for each submodule X of M, N ∩ X ⊆ T, implies that X ⊆ T. The notion of essential and T-essential are coincide if T = 0. Also they introduced the socle of an R-module M relative to T, denoted by Soc_T(M) as the sum of all T-minimal submodules of M and proved that it is equal to the intersection of all submodules L of M with T ⊆ L and L ⊆ T. These notions mentioned above motivate us to introduce a new generalization of T-essential submodules, as well as, of small-essential submodules. Let T be an arbitrary submodule of an R-module M. We say that a submodule N of M is a T-small-essential of M provided that N T X ⊆ T implies that X ⊆ T for all small submodules X of M, every T-essential submodule is T-small-essential, and every small-essential submodule of M is 0-small-essential.
In section one, we investigate the basic properties of T-small-essential submodules also we give some of their characterizations. For a submodule T of an R-module M, we investigate the relationship between small-essential submodules of M relative to T and small-essential left ideals of R relative to (T : m), for each m ∈ M \ T. After that we introduce T-small complement of a submodule in M. And consider conditions under which, the T-small complement of a submodule in M is the largest submodule of M with certain property.

In section two we use the concepts of section one to introduce two socles of an R-module M, SOC(TS)(M) and SOC’(TS)(M), one contains SOC(T)(M) and the other contained in SOC(T)(M). Finally, as an application of T-small-essential submodules we introduce T(TS)-finitely cogenerated modules to clarify when SOC(TS)(M) being T-small-essential submodule of M. Also we introduce the notion of T_S-uniform modules and use it to characterize small uniserial modules.

2. T-small-essential submodules.

We start with the following notion

Definition (1.1). Let R be a ring and T a proper submodule of an R-module M. The submodule K of M is called T-small-essential (simply T-essential ) if K ⊈ T and for each small submodule L of M, if K ∩ L ⊈ T, then L ⊈ T. In this case K is denoted by K ⊵ M. And M is called Ts-essential extension of K.

Remarks and example (1.2)
(a). It is clear that a submodule K is small-essential if and only if K is 0S-essential.
(b). It is clear that every T-essential submodule is Ts-essential. The converse is true in hollow modules.
(c). Recall that an R-module M is SF-module if zero is the only small submodule of M. Then every nonzero submodule of SF-module ( in particular, semisimple, Z as Z-module ) is small essential w.r.t. any proper submodule. in particular, for each m, n ∈ Z, mZ ⊵ (Z).
(d). In the Z-module Z_p^∞, each proper submodule is small. Let H_n= \langle \frac{1}{p^n} + Z \rangle and H_m = \langle \frac{1}{p^m} + Z \rangle. Then m ≤ n if and only if H_n ⊵ (H_m)_S Z_p^∞

In the following proposition, we give the relation between small-essential submodule and Ts-essential submodules.

Proposition (1.3) Let M be an R-module with submodules T ⊆ K and T is small. Then K ⊵ T M if and only if K/T ⊵ S M/T

Proof. Assume that K ⊵ T M. For each small submodule W/T of M, suppose that K/T ∩ W/T = 0. Then K ∩ W ⊈ T. By ( [1], 5-17 ) we have W is small in M and hence W ⊈ T. Thus K/T ⊵ S M/T. Conversely. Assume K ∩ L ⊈ T for some small submodule L of M. Then 0 = (K∩L)/T = K/T ∩ (L+T)/T. Hence K/T = 0, and so K ⊈ T
**Proposition (1.4)** Let $M$ be an $R$-module with submodules $K$ and $T$. Then $(K + T)/T \subseteq M/T$ implies that $K \subseteq M$. The converse is true in case $T$ is small.

**Proof.** $K \nsubseteq T$, since $(K + T)/T$ is nonzero. For each small submodule $W$ of $M$, assume that $K \cap W \subseteq T$. Since $W/T$ is small submodule of $M/T$ and $(K+T)/T \cap W/T = ((K \cap W)/T + T)/T = 0$. By hypothesis, we have $W \subseteq T$. Let $W/T$ be small submodule of $M/T$ with $(K + T)/T \cap W/T = 0$. Then $K \cap W \subseteq (K + T) \cap W \subseteq T$ where $W$ is a small in $M$, ([1], 5-17). Thus $W \subseteq T$ and hence $W/T = 0$.

The smallness property of the submodule $T$ cannot be dropped in the converse part of proposition (1.4) as in the following example, consider the $Z$-module $Z/24$. The submodule $3Z_{24}$ is not small in $Z_{24}$ and $8Z_{24} \notin (3Z_{24})S Z_{24}$, but $(8Z_{24} + 3Z_{24})/3Z_{24}$ is not small-essential in $Z_{24}/3Z_{24}$.

**Theorem (1.5).** Let $T$ and $K$ be submodules of an $R$-module $M$. Then the following statements are equivalent

1. $K \subseteq M$
2. For each $m \in M/T$ with $Rm$ is small in $M$, there exists $r \in R$ such that $rm \in K/T$.

**Proof.** (1) $\rightarrow$ (2). Let $m \in M/T$ with $Rm$ is small in $M$. Since $K \subseteq M$, then $K \cap Rm \subseteq T$. So there is $r \in R$ such that $rm \in K/T$. (2) $\rightarrow$ (1). By hypothesis $K \nsubseteq T$. For some small submodule $W$ of $M$, assume that $K \cap W \subseteq T$ and $W \nsubseteq T$. Then there is $w \in W/T$. By (2) there is an element $r \in R$ such that $rw \in K/T$. But on the other hand $rw \in K \cap W \subseteq T$ which is a contradiction. Thus $K \subseteq M$.

**Corollary (1.6) ([2], 2-4).** Let $M$ be an $R$-module with nonzero submodule $K$. Then $K \subseteq M$ if and only if for each $x (\neq 0) \in M$ with $Rx \ll M$, there exists an element $r$ in $R$ such that $rx (\neq 0) \in K$.

**Corollary (1.7).** Let $M$ be an $R$-module with submodule $K$ and $T$. Then $K \subseteq M$ if and only if for each $x \in J(M) \setminus T$, there exists $r \in R$ such that $rx \in K/T$.

**Proposition (1.8).** Let $M$ be an $R$-module and $K$ a submodule of $M$. If $K \subseteq M$, then $(K : m) \subseteq R$ for each $m$ in $M$.

**Proof.** For each small left ideal $A$ of $R$ and $m \in M$, assume that $(K : m) \cap A = 0$. Define the $R$-homomorphism $\alpha_m : R \rightarrow M$ by $\alpha_m (r) = rm$ for all $r \in R$. Then $Am$ is a small submodule of $M$. If $x \in K \cap Am$, then $x = am \in K$ for some $a \in A$. This implies that $a \in (K : m) \cap A$ and hence $a = 0$, so $K \cap Am = 0$, small essentiality of $K$ in $M$ implies that $Am = 0$ and hence $A \subseteq (K : m)$, so $A = 0$. This shows that $(K : m) \subseteq R$.

The converse of (1.8) may not be true generally, $8Z_{24}$ is not small-essential in the $Z$-module $Z_{24}$, but for each $\bar{m} \in Z_{24}$, $(8Z_{24} : \bar{m})$ is small-essential, because, $0$ is the only small submodule in $Z$.

Now consider the following
**Theorem (1.9)** Let \( K \) and \( T \) be submodules of an \( R \)-module \( M \). Then \( K \supseteq_{TS} M \) if and only if \((K : m) \supseteq_{(T : m)S} R \) for each \( m \in M \setminus T \)

**proof.** Assume that \( K \supseteq_{TS} M \) and \( m \in M \setminus T \). By theorem (1.5), there exists an element \( r \in R \) such that \( rm \in K \setminus T \). This is equivalent to saying that \((K : m) \not\subseteq (T : m)\). For each small left ideal \( A \) of \( R \), assume \((K : m) \cap A \not\subseteq (T : m)\). It is easy to see that \( K \cap Am \subseteq T \), but \( K \supseteq_{TS} M \), then \( Am \subseteq T \) and hence \( A \not\subseteq (T : m) \). Conversely, for each small submodule \( W \) of \( M \) with \( K \cap W \subseteq T \). If \( W \not\subseteq T \), then there is an element \( w \in W \setminus T \). Apply theorem (1.5) on \((K : w)\), there exists an element \( s \in R \) such that \( sw \notin K \setminus T \) which is a contradiction, because, \( sw \in K \cap W \subseteq T \). Thus \( K \supseteq_{TS} M \).

**Corollary (1.10).** Let \( M \) be an \( R \)-module with submodule \( K \). Then \( K \supseteq_{S} M \) if and only if \((K : m) \supseteq_{LR(m)S} R \) for each nonzero element \( m \) in \( M \).

**Proposition (1.11).** Let \( \{ N_i \} \) and \( \{ K_i \} \), \( i \in I \) be families of submodules of an \( R \)-module \( M \) and \( T \) a submodule of \( M \). If \( N_i \supseteq_{TS} M_i \) for each \( i \in I \), then \((\bigoplus_{i \in I} N_i) \supseteq_{(\bigoplus_{i \in I} T)S} (\bigoplus_{i \in I} M_i) \).

**Proof.** Assume \((m_i)_{i \in I} \in (\bigoplus_{i \in I} M_i) \setminus (\bigoplus_{i \in I} T)\). Since \( N_i \supseteq_{TS} M_i \) for each \( i \in I \), then by Theorem (1.5), there exists an element \( r_i \in R \) \( r_m \in N_i \setminus T_i \), so there exists an element \( r \in R \) such that \((r_i m_i) \in (\bigoplus_{i \in I} N_i) \setminus (\bigoplus_{i \in I} T)\). Again Theorem (1.5) completes the proof.

The proof of the following proposition is immediate by Theorem (1.5)

**Proposition (1.12).** Let \( M \) be an \( R \)-module with submodules \( K, W, \) and \( T \). Then

1. If \( K, W \supseteq_{TS} M \), then \( K \cap W \supseteq_{TS} M \).
2. Let \( K \subseteq W \). Then \( K \supseteq_{TS} W \) and \( W \supseteq_{TS} M \).

**Theorem (1.13).** Let \( M \) be an \( R \)-module and \( T \subseteq K_i \subseteq M_i \) \( i = 1, 2 \) submodules of \( M \) with \( M_i \cap M_2 = T_i \cap T_2 \). Then the following are equivalent

1. \( K \supseteq_{TS} M, i = 1, 2 \)
2. \((K_1 + K_2) \supseteq_{(T_1 + T_2)S} (M_1 + M_2) \)

**Proof.**

(1) \( \rightarrow \) (2). Let \( x + y \in M_1 + M_2 \setminus T_1 + T_2 \) where \( x \in M_1 \) and \( y \in M_2 \). Then either \( x \notin T_1 \) or \( y \notin T_2 \). Assume \( x \in M_1 \setminus T_1 \), then by Theorem (1.5), there is an element \( r \in R \) such that \( rx \in K_1 \setminus T_1 \). If \( ry \in K_2 \), then \( r(x+y) \in K_1 + K_2 \setminus T_1 + T_2 \) and the proof is complete. If \( ry \notin K_2 \subseteq M_2 \setminus T_2 \), then there is an element \( s \in R \) such that \( sry \in K_2 \setminus T_2 \). Hence \( sr(x+y) \in K_1 + K_2 \setminus T_1 + T_2 \).

(2) \( \rightarrow \) (1) Let \( W_1 \) be a small submodule of \( M_1 \) with \( K_1 \cap W_1 \subseteq T_1 \). We claim that \((K_1 + K_2) \cap W_1 \subseteq T_1 + T_2 \). If \( z = x + y \in (K_1 + K_2) \cap W_1 \) where \( x \in K_1 \), \( y \in K_2 \), then \( x - z = -y \in M_1 \cap M_2 = T_1 \cap T_2 \). So \( y \in T_1 \subseteq K_1 \). Thus \( z = x + y \in K_1 \cap W_1 \subseteq T_1 \) and hence \( x - z \in T_1 \) which implies \( x \in T_1 \). Thus \( z = x + y \in T_1 + T_2 \). By (2), we have \( W_1 \subseteq T_1 + T_2 \) and then \( W_1 \subseteq T_1 \). By the same argument \( K_2 \supseteq_{TS} M_2 \).

Recall that an \( R \)-homomorphism is small in case its kernel is small in its domain.
Theorem (1.14) Let $M$ and $N$ be $R$-modules, $T$ a submodule of $N$ and $\alpha : M \to N$ an $R$-homomorphism such that $\alpha(M) \not\subseteq T$. Then

(1) if $\alpha(M) \subseteq_{TS} N$, then for each small $R$-homomorphism $h$, $\ker(h) \cap \alpha(M) \subseteq T$ implies that $\ker(h) \subseteq T$.

(2) If for every $R$-homomorphism $h$, $\ker(h) \cap \alpha(M) \subseteq T$ implies that $\ker(h) \subseteq T$, then $\alpha(M) \subseteq_{TS} N$.

Proof. (1). It is clear by the definitions. (2). Let $W$ be a small submodule of $N$ with $\alpha(M) \cap W \subseteq T$. Consider the mapping $h : (M) + W \to M/\alpha^{-1}(T)$ by $h(\alpha(m) + w) = m + \alpha^{-1}(T)$ for all $m \in M$ and $w \in W$. If $(m) + w = 0$, then $(m) = -w \in (M) + W \subseteq T$ and hence $m \in \alpha^{-1}(T)$. It is clear that $h$ is an $R$-homomorphism with $\ker(h) \cap \alpha(M) \subseteq T$. By the hypothesis $\ker(h) \subseteq T$. But $W \subseteq \ker(h)$. Thus $(M) \subseteq_{TS} N$.

Corollary (1.15). Let $M$ and $N$ be $R$-module and $\alpha : M \to N$ be a nonzero $R$-homomorphism. Then

(1) if $\alpha(M) \subseteq_{S} N$, then for each small $R$-homomorphism $h$ with $\ker(h) \cap \alpha(M) = 0$ is monomorphism

(2) if for every $R$-homomorphism $h$ with $\ker(h) \cap \alpha(M) = 0$ is $R$-monomorphism, then $\alpha(M) \subseteq_{S} N$

Proposition (1.16). Let $M$, $N$ be $R$-modules. $W$, $T$ submodules of $N$ and $\alpha : M \to N$ an $R$-homomorphism. If $W \subseteq_{TS} N$, then $\alpha^{-1}(W) \subseteq_{\alpha^{-1}(T)S} M$.

Proof. Let $L$ be a small submodule of $M$ with $\alpha^{-1}(W) \cap L \subseteq \alpha^{-1}(T)$. Then $W \cap (L) \subseteq T$ and hence $(L) \subseteq T$, so $L \subseteq \alpha^{-1}(T)$.

Corollary (1.17). Let $\alpha : M \to N$ be an $R$-homomorphism, and $W$ a submodule of $N$. If $W \subseteq_{S} N$, then $\alpha^{-1}(W) \subseteq_{(\ker(\alpha))S} M$.

Definition (1.18). Let $N$ be a submodule of an $R$-module $M$. A submodule $N'$ of $M$ is called small-complement (s-complement) of $N$ in $M$ if $N'$ is small in $M$ and maximal with the property $N \cap N' = 0$. The submodule $N'$ is denoted by $N^{SC}$.

More generally, we introduce the following.

Definition (1.19). Let $K$ and $T$ ($\neq M$) be submodules of an $R$-module $M$. A submodule $K'$ of $M$ is called $Ts$-complement of $K$ in $M$, if $K'$ is small in $M$, and maximal with the property $K \cap K' \subseteq T$.

In the $Z$-module $Z_{24}$, $6Z_{24}$ is $(2Z_{24})s$-complement of $8Z_{24}$. In fact, Zorn’s lemma shows the existence of $Ts$-complement (s-complement) of every submodule infinitely generated module as in the following proposition which gives more details.
Proposition (1.20). Let $M$ be a finitely generated $R$-module with submodules $T ( \neq M ) \subseteq W$. Then there exists small submodule $W' ( \not\subseteq T )$ of $M$ such that $W + W' \subseteq_{TS} M$ and $( W + W' ) / T = W / T \oplus ( W + T ) / T$.

Proof. Consider the following family of submodules of $M \mathcal{C} = \{ N \ll M \mid N \cap W \subseteq T \}$. Clearly $\mathcal{C}$ is nonempty, we ordered $\mathcal{C}$ by inclusion. Let $\mathcal{C}' = \{ N_{\alpha} \mid \alpha \in \Lambda \}$ be a chain in $\mathcal{C}$. Set $C = \bigcup \mathcal{C}$. Since $M$ is finitely generated, then $C$ is small in $M$ and $N \cap C \subseteq T$. By Zorn's lemma, $\mathcal{C}$ has a maximal element $W'$ (say) which is Ts-complement of $W$ in $M$. For each small submodule $L$ of $M$, assume that $( W + W' ) \cap L \subseteq T$. Let $w = w' + l \in W \cap ( W' + L )$ for some $w' \in W'$ and $l \in L$. Then $w - w' = l \in ( W + W' ) \cap L \subseteq T \subseteq W$, so $w' = w - l \in W' \cap W' \subseteq T$ and hence $w' \in T$. Thus $W \cap ( W' + L ) \subseteq T$. By maximality of $W'$, $L \subseteq W'$ and hence $L \subseteq T$. This shows that $W + W' \subseteq M$. For the other part, it is enough to show that $W / T \cap ( W' + T ) / T = 0$. Suppose $w \in W$ and $w' \in W'$ with $w + T = w' + T$. Then $w - w' \in T$ and hence $w' \in W \cap W' \subseteq T$. Thus $w' + T = T$.

Corollary (1.21). Let $M$ be a finitely generated $R$-module. Then for each submodule $N$ of $M$, there exists a small-complement $N^{SC}$ of $N$ in $M$ such that $N \oplus N^{SC} \subseteq_{TS} M$.

Proposition (1.22). Let $N$ and $K$ be submodules of an $R$-module $M$ with $N$ is small and $T = N \cap K$. Then $N$ is Ts-complement of $K$ in $M$ if and only if $( N + K ) / N \subseteq_{S} M / N$.

Proof. Assume that $( N + K ) / N \subseteq_{S} M / N$. For each small submodule $W$ of $M$ with $N \subseteq W$ and $W \cap K \subseteq T$, let $d + N \in W / N \cap ( K + N ) / N$, then $d + N = k + N$ for some $k \in K$, this implies that $k \in W \cap K \subseteq T = N \cap K \subseteq N$ and hence $W / N \cap ( K + N ) / N = 0$. Small-essential of $( N + K ) / N$ in $M / N$ gives that $N = W$. Thus $W$ is Ts-complement of $K$ in $M$. Conversely, for each small submodule $W / N$ in $M / N$ such that $W / N \cap ( K + N ) / N = 0$. If $x \in W \cap K$, then $x + N \in W / N \cap ( K + N ) / N$ and hence $x + N = N$. therefore $W \cap K \subseteq N \cap K = T$. By assumption $W = N$ and this shows that $( K + N ) / N \subseteq_{S} M / N$.

By Proposition (1.22) and Proposition (1.4) we have the following

Proposition (1.23). Let $T$ and $K$ be submodules of an $R$-module $M$ with $T$ is small in $M$, Then $T \cap K$ is Ts-complement of $K$ in $M$, implies that $K \subseteq_{TS} M$.

Lemma (1.24). Let $T$ and $K$ be submodules of finitely generated $R$-module $M$. If $K \subseteq_{TS} M$, then $K^{SC} \subseteq T$. In additional, if $K \cap T = 0$ and $T$-small, then $K^{SC} = T$.

Proof. By the definition of small-complement, $K \cap K^{SC} \subseteq T$. Since $K \subseteq_{TS} M$, then $K^{SC} \subseteq T$. If $K \cap T = 0$ and $T$ small, then by maximality of $K^{SC}$, we have $T \subseteq K^{SC}$.

The following proposition shows when a small-complement of a submodule in finitely generated module, is the largest small submodule which has trivial intersection with the submodule.
Proposition (1.25). Let $K$ be a submodule of a finitely generated $R$-module $M$. Then the following are equivalent

1. $K \subseteq (K^{SC})_S M$
2. For each small submodule $N$ of $M$ with $K \cap N = 0$, $N \subseteq K^{SC}$.
3. For each $x \in M \setminus K^{SC}$, there exists $r \in R$ such that $rx \neq 0 \in K$.

Proof. (1) $\rightarrow$ (2), and (3) $\rightarrow$ (1), clearly by definition and Theorem (1.5) respectively. (2 $\rightarrow$ (1). Let $N$ be a small submodule of $M$ with $K \cap N = 0$. By (2) we have $N \subseteq K^{SC}$. By theorem (1.5) for each $x \in M \setminus K^{SC}$, there exists $r \in R$ such that $rx \neq 0$.

3. T-small Socles of a module

Let $M$ be an $R$-module with submodules $W$ and $T$. Recall that the submodule $W$ is T-minimal if $W \notin T$ and there exist no submodules of $M$ properly contain in $W + T$ and contain $T$ properly. This is equivalent to saying that $(W + T)/T$ is a simple $R$-module.

We introduce the following

Definition (2.1). Let $K$ and $T$ be submodules of an $R$-module $M$. The submodule $K$ is called $T$-minimal if $K \subseteq T$ and there are no small submodules $W$ of $M$ with $T \subseteq W \subseteq K$ + $T$, that is $(K + T)/T$ is nonzero $R$-module and $0$ is only small submodule of $(K + T)/T$, that is $(K + T)/T$ is nonzero SF-module.

Let $T$ be a submodule of an $R$-module $M$. The socle of $M$ with respect to $T$, denoted by $\text{Soc}_T(M)$, defined in [3] as $\text{Soc}_T(M) = \sum \{ K | K \text{ is T-minimal in } M \}$. It is proved in [3], that $\text{Soc}_T(M) = S_T(M)$ where $S_T(M) = \cap \{ L | T \subseteq L \text{ and } L \not\subseteq T \}$.

Definition (2.2). Let $T$ be a submodule of an $R$-module $M$. The small socle of $M$ relative to $T$, denoted by $\text{Soc}_{TS}(M)$ defined by $\sum \{ K \ll M | K \text{ is T-minimal } \}$

It is clear that $\text{Soc}_{TS}(M) \subseteq \text{Soc}_T(M)$ and $\text{Soc}_T(M) = \text{Soc}_{TS}(M)$ in case $K + T \ll M$.

Proposition (2.3). Let $T$ be a submodule of a finitely generated $R$-module $M$, and $S_{TS}(M) = \cap \{ L \ll M | T \subseteq L \text{ and } L \not\subseteq TS \}$ . Then every submodule $K/T$ of $S_{TS}(M)/T$ where $K$ is small in $M$ is direct summand.

Proof. Let $K/T$ be a submodule of $S_{TS}(M)$ with $K$ small in $M$. By proposition (1.19), there exists a small submodule $K'$ of $M$ such that $K + K' \not\subseteq TS$. Then $K + K'$ is small in $M$ and hence $K/T \subseteq S_{TS}(M)/T \subseteq (K + K')/T = K/T \oplus (K' + T)/T$. Then $S_{TS}(M)/T = S_{TS}(M)/T \cap (K/T \oplus (K' + T)/T)$.

Theorem (2.4). Let $T$ be a submodule of a finitely generated $R$-module $M$. Then $\text{Soc}_{TS}(M) = S_{TS}(M)$.

Proof. Let $W$ be a small T-minimal submodule of $M$ and $L$ a small submodule of $M$ with $T \subseteq L$ and $L \not\subseteq TS$. We show that $W \subseteq L$ and hence $\text{Soc}_{TS}(M) \subseteq S_{TS}(M)$. Since $((W \cap L) + T$
Theorem (2.8). Let $M$ be a finitely generated $R$-module with submodule $T$. Then
$$S_{T}(M) = \bigcap \{ L \ll M \mid T \subseteq L \subseteq M \}.$$ 

Proof. Let $W$ be a $T$-minimal submodule of $M$ and $L \ll M$ with $T \subseteq L \subseteq M$. We show that $W \subseteq L$ and hence $S_{T}(M) \subseteq S'_{T}(M)$. If $(W \cap L) + T = W + T$, then $W \subseteq L$ and since $L \subseteq T$ minimal, then either $(W \cap L) + T = T$ or $(W \cap L) + T = W + T$. If $(W \cap L) + T = T$, then $W \subseteq T$ and hence $W \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$ and $L \subseteq T$ minimal, then either $(W \cap L) + T = T$ or $(W \cap L) + T = W + T$. If $(W \cap L) + T = W + T$, then by modular law $(W + T) \cap L = W + T$ and hence $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$. For the converse inclusion, let $L$ be a submodule of $M$ with $T \subseteq L$. Then $L \subseteq T$ and since $L \subseteq M$, then $W \subseteq T$ which is a contradiction to the $T$-minimality of $W$. Thus $(W \cap L) + T = W + T$, and by modular law $(W + T) \cap L = W + T$ and hence $W + T \subseteq L$. Therefore $W \subseteq L$.
contradiction. Thus \( L/T \cap V/T = 0 \) and hence \( L/T = (L_0 + T)/T \). This shows that \( L/T = \sum_{\alpha \in \Lambda} (M_\alpha + T)/T \). But \((M_\alpha + T)/T \) is minimal and small and hence \( M_\alpha \) is \( T_\text{S-minimal} \) for each \( \alpha \in \Lambda \). Thus \( s_{T_\text{S}}(M) \subseteq \text{soc}_{T_\text{S}}(M) \).

**Example (2.9).** All submodules of the \( \mathbb{Z} \)-module \( Z_{24} \) have the following properties.

| \( N/M \) | \( N/\mathbb{Z} \) | \( N/\mathbb{Z}_{24} \) | \( N/\mathbb{Z}_{2} \) | \( N/\mathbb{Z}_{12} \) | \( N/\mathbb{Z}_{4} \) | \( N/\mathbb{Z}_{6} \) | \( N/\mathbb{Z}_{8} \) | \( N/\mathbb{Z}_{12} \) | \( N/\mathbb{Z}_{16} \) | \( N/\mathbb{Z}_{24} \) | \( \text{soc}_{Z_{24}}(N) \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( Z_{24} \) | \( x \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 2Z_{24} \) | \( x \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 3Z_{24} \) | \( x \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 4Z_{24} \) | \( x \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 6Z_{24} \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 8Z_{24} \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 12Z_{24} \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |
| \( 0 \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) | \( \checkmark \) |

Then we have the following according to the above table

- \( \text{soc}_{Z_{24}}(Z_{24}) = Z_{24} \), \( \text{soc}_{Z_{24}}^I(Z_{24}) = Z_{24} \), \( \text{soc}_{Z_{24}}^{I_1}(Z_{24}) = Z_{24} \).
- \( \text{soc}_{Z_{24}}(Z_{24}) = Z_{24} \), \( \text{soc}_{Z_{24}}^I(Z_{24}) = Z_{24} \), \( \text{soc}_{Z_{24}}^{I_1}(Z_{24}) = Z_{24} \).
- \( \text{soc}_{Z_{24}}(Z_{24}) = Z_{24} \), \( \text{soc}_{Z_{24}}^I(Z_{24}) = Z_{24} \), \( \text{soc}_{Z_{24}}^{I_1}(Z_{24}) = Z_{24} \).
- \( \text{soc}_{Z_{24}}(Z_{24}) = 2Z_{24} \), \( \text{soc}_{Z_{24}}(Z_{24}) = 2Z_{24} \), \( \text{soc}_{Z_{24}}(Z_{24}) = 2Z_{24} \).
- \( \text{soc}_{Z_{24}}(Z_{24}) = 2Z_{24} \), \( \text{soc}_{Z_{24}}(Z_{24}) = 2Z_{24} \), \( \text{soc}_{Z_{24}}(Z_{24}) = 2Z_{24} \).
- \( \text{soc}_{Z_{24}}(Z_{24}) = 0Z_{24} \), \( \text{soc}_{Z_{24}}(Z_{24}) = 0Z_{24} \), \( \text{soc}_{Z_{24}}(Z_{24}) = 0Z_{24} \).

**Definition (2.10).** Let \( M \) be an \( R \)-module and \( T \) be a submodule of \( M \). Then \( M \) is called \( T(T_\text{S}) \)-finitely cogenerated, if for every family \( \{ N_\alpha \mid \alpha \in \Lambda \} \) of (small) submodules \( N_\alpha \) of \( M \) with \( \bigcap_{\alpha \in \Lambda} N_\alpha \subseteq T \), there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( \bigcap_{\alpha \in \Lambda_0} N_\alpha \subseteq T \).

It is clear that each \( T \)-finitely cogenerated module is \( T_\text{S} \)-finitely cogenerated. Every submodule of \( T(T_\text{S}) \)-finitely generated is \( T(T_\text{S}) \)-finitely cogenerated. An \( R \)-module \( M \) is \( T \)-finitely cogenerated if and only if \( M/T \) is finitely cogenerated.

We call an \( R \)-module \( M \) is \( s \)-finitely cogenerated in case \( M \) is \( Os \)-finitely cogenerated. Finally if \( M \) is \( T_\text{S} \)-finitely cogenerated, then \( M/T \) is \( s \)-finitely cogenerated.

**Theorem (2.11).** Let \( M \) be an \( R \)-module with a submodule \( T \). Then
(1) If soc_{T_S}(M) is T-finitely cogenerated and soc_{T_S}(M) \subseteq_{T_S} M, then M is Ts-finitely cogenerated.

(2) In additional, if M is finitely generated and Ts-finitely cogenerated, then soc_{T_S}(M) \subseteq_{T_S} M and soc_{T_S}(M) is Ts-finitely cogenerated.

**Proof.** (1) Let \( \{N_{\alpha}| \alpha \in \Lambda \} \) be a family of small submodules \( N_{\alpha} \) of M with \( \prod_{\alpha \in \Lambda} N_{\alpha} \subseteq T \). Then \( (\prod_{\alpha \in \Lambda} N_{\alpha}) \cap soc_{T_S}(M) \subseteq T \). Since \( soc_{T_S}(M) \) is T-finitely cogenerated, then \( \prod_{\alpha \in \Lambda} N_{\alpha} \cap soc_{T_S}(M) \subseteq T \) for some finite subset \( \Lambda_0 \) of \( \Lambda \), but \( soc_{T_S}(M) \subseteq_{T_S} M \), then \( \prod_{\alpha \in \Lambda} N_{\alpha} \subseteq T \).

(2) Let K be a submodule of M with soc_{T_S}(M) \cap K \subseteq T. By theorem(2.4), \( \cap \{ L \subseteq M \cap L \subseteq_{T_S} M \} \cap K \subseteq T \). Since M is Ts-finitely cogenerated, then \( \cap_{\alpha \in \Lambda_0} L_{\alpha} \subseteq_{T_S} M \). By proposition(1.2), \( \cap_{\alpha \in \Lambda_0} L_{\alpha} \subseteq_{T_S} M \). Hence K \subseteq T and hence soc_{T_S}(M) \subseteq_{T_S} M. □

**Corollary(2.12).** Let M be an R-module. If soc_{s}(M) is finitely cogenerated and soc_{s}(M) \subseteq_{s} M, then M is s-finitely cogenerated. Moreover, if M is finitely generated and s-finitely cogenerated, then soc_{s}(M) \subseteq_{s} M and soc_{s}(M) is s-finitely cogenerated.

The following corollary in fact is exactly theorem(2.11) but in different words

**Corollary(2.13):** Let M be an R-module and T a submodule of M. Then

(1) If soc_{T_S}(M)/T is finitely generated and soc_{T_S}(M) \subseteq_{T_S} M, then M/T is s-finitely cogenerated.

(2) In additional, if M is finitely generated and M/T is s-finitely cogenerated, then soc_{T_S}(M) \subseteq_{T_S} M and soc_{T_S}(M) is s-finitely cogenerated.

Let T be a proper submodule of an R-module M. Recall that M is T-uniform, if for each submodule N of M with N \not\subseteq T, we have N \not\subseteq T M [3]

We introduce the following:

**Definition(2.14).** Let M be an R-module with proper submodule T. Then M is called T_{s}-uniform if N \subseteq_{T_S} M for each small submodule N of M with N \not\subseteq T. We call a nonzero R-module M is s-uniform, in case M is O_{s}-uniform.

The following proposition gives a characterization of T_{s}-uniform module.

**Proposition(2.15).** Let M be an R-module with proper submodule T. Then M is T_{s}-uniform if and only if for each small submodules N_{1}, N_{2} of M, N_{1} \cap N_{2} \subseteq T implies That either N_{1} \subseteq T or N_{2} \subseteq T

**Proof.** Let N_{1}, N_{2} be small submodules of M with N_{1} \cap N_{2} \subseteq T and N_{1} \not\subseteq T, since N_{1} \subseteq_{T_S} M then N_{2} \subseteq T. Conversely, let K, L be two small submodules of M with K \not\subseteq T such that k \cap L \subseteq T. Then the hypothesis implies that L \subseteq T and hence K \not\subseteq_{T_S} M.

**Corollary(2.16).** A nonzero R-module M is s-uniform if and only if every nonzero small submodules of M have nonzero intersection.
Every submodule of $T$-uniform module which contains $T$ property is $T$-uniform, if $M/T$ is s-uniform, then $M$ is $T$-uniform.

$Z_{24}$ as $Z$-module is $T$-uniform for every proper submodule $T$ of $Z_{24}$, but $Z_{24}$ is not $(3Z_{24})$-uniform. This example motivate us to consider some class of uniserial modules

We call an $R$-module $M$ is small uniserial if for any two small submodules $N_1$ and $N_2$, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$

**Proposition (2.17).** An $R$-module $M$ is small uniserial if and only if $M$ is $T$-uniform for each proper submodules $T$ of $M$.

**Proof.** For a proper submodule $T$ of $M$, assume $N_1$, $N_2$ are small submodules of $M$ with $N_1 \cap N_2 \subseteq T$. Since $M$ is small uniserial, then either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$ and hence either $N_1 \subseteq T$ or $N_2 \subseteq T$. Conversely, let $N_1$, $N_2$ be small submodules of $M$ with $N_2 \not\subseteq N_1$. Hence $N_2 \not\subseteq N_1 \cap N_2$. By the hypothesis, $N_2 \not\subseteq (N_1 \cap N_2)$ and hence $N_1 \subseteq N_2$. Thus $N_1 \subseteq N_1 \cap N_2$ and hence $N_1 \subseteq N_2$.

**Definition (2.18).** Let $M$ be an $R$-module and $T$ a proper submodule of $M$. A submodule $N$ of $M$ is called $T$($T$)-complemented, if there exists a (small) submodule $K$($\not\subseteq T$) of $M$ such that $N\cap K \subseteq T$. In particular, a submodule $N$ of a nonzero $R$-module $M$ is called (small) complemented, if there exists a nonzero (small) submodule $K$ of $M$ such that $N\cap K = 0$.

It's clear that $T$-complemented submodules are $T$-complemented. In the $z$-modules $Z_{24}$, $4Z_{24}$, $6Z_{24}$, $8Z_{24}$, $12Z_{24}$ and $0Z_{24}$ are $(4Z_{24})$-complemented, but $6Z_{24}$ is $(12Z_{24})$-complemented which is not $(12Z_{24})$-complemented. Note that, the concepts of $T$-complemented and $T$-essential submodules are completely different, $4Z_{24}$ is $(4Z_{24})$-complemented, but $4Z_{24}$ is not $(4Z_{24})$-essential in $Z_{24}$ and $2Z_{24}$ is $(4Z_{24})$-essential, but it is not $(14Z_{24})$-complemented in $Z_{24}$.

**Definitions (2.19).** Let $M$ be an $R$-module with proper submodule $T$. Then $M$ is called $T(\alpha)$-directly finite, if $M$ is not isomorphic to any proper $T(\alpha)$-complemented submodules of itself. In particular, a nonzero $R$-module $M$ is called $O$(small)-directly finite, if $M$ is not isomorphic to any proper $O$(small)-complemented submodules of itself.

Clearly, $T$-directly finite submodules are $T$-$\alpha$-directly finite. Any finite module is $T(\alpha)$-directly finite for any proper submodule $T$.

**Definition (2.20).** Let $M$ be an $R$-module with submodule $T$. Then $M$ is called weakly $T$-$\alpha$-essential co-Hopfian, denoted by $WTECO$-$H$ ($WTSECO$-$H$), if $\alpha(X) \not\subseteq T(\leq T)$ for every injective endomorphism $\alpha$ of $M$.

It's clear that every $WTECO$-$H$ module is $WTSECO$-$H$. Observe that a module $M$ is $WOECO$-$H$ if and only if $M$ is weakly co-Hopfian [4]. A module $M$ is $WOSECO$-$H$ if and only if $M$ is small weakly co-Hopfian [2].
Theorem (2.21). For an \( R \)-module \( M \) with a proper submodule \( T \), the following statements are equivalent

1. \( M \) is WTSECO-H (WTSECO-H)
2. \( M \) is \( (T) \)-directly finite and the image of every injective endomorphism of \( M \) is either \( (T) \)-essential or a proper \( (T) \)-complemented.

Proof. We shall prove the case WTSECO-H and the other by a similar way.

(1) \( \Rightarrow \) (2). Assume that \( M \) is not \( (T) \)-directly finite, then there is a proper \( (T) \)-complemented submodule \( N \) of \( M \) and isomorphism \( \alpha: M \rightarrow N \). So there exists a small submodule \( K(\not\subseteq T) \) of \( M \) such that \( N \cap K \subseteq T \). Now \( \alpha \) is an injective endomorphism submodule \( N \) of \( M \), but by (1) \( N = \alpha(M) \not\subseteq (T) \), which is a contradiction.

(2) \( \Rightarrow \) (1). Let \( \alpha \) be an injective endomorphism of \( M \) with \( \alpha(M) \not\subseteq (T) \). Then by (2) \( \alpha(M) \) is proper \( (T) \)-complemented submodule of \( M \) and \( \alpha(M) \cong M \), which contradicts the \( (T) \)-directly finite property of \( M \), thus \( M \) is WTSECO-H.

Corollary (2.22). The following statements are equivalent for a nonzero \( R \)-module \( M \)

1. \( M \) is weakly co-Hopfian (small weakly co-Hopfian)
2. \( M \) is \( O \)-directly finite and the image of every injective endomorphism of \( M \) is either \( O \)-essential or proper \( O \)-complemented.

The following theorem gives more characterization of WTSECO-H modules

Theorem (2.23). Let \( M \) be an \( R \)-module with submodule \( T \). The following statements are equivalent

1. \( M \) is WTSECO-H
2. For each submodule \( N \) of \( M \), if there is an \( R \)-monomorphism \( \alpha: M \oplus N \rightarrow M \), then \( N \subseteq \alpha^{-1}(T) \)
3. For each submodule \( N \) of \( M \), if there is T-essential \( R \)-monomorphism \( \alpha: M \oplus N \rightarrow M \), then \( N \subseteq \alpha^{-1}(T) \)
4. There exists a fully invariant T-essential submodule which is WTSECO-H.
5. For each injective endomorphism \( \alpha \) of \( M \), if \( N \subseteq (T) \), then \( \alpha(N) \subseteq (T) \).
6. For each submodule \( N(\not\subseteq T) \) of \( M \) all injective endomorphism \( \alpha \) of \( M \), \( \alpha^{-1}(N) \not\subseteq \alpha^{-1}(T) \).

Proof: We shall prove (1) \( \Rightarrow \) (2), (3) \( \Rightarrow \) (1), (4) \( \Rightarrow \) (1), (1) \( \Leftrightarrow \) (6), and (1) \( \Leftrightarrow \) (5), (2) \( \Rightarrow \) (3), and (1) \( \Rightarrow \) (4) are trivial.

(1) \( \Rightarrow \) (2) Suppose that \( \alpha: M \oplus N \rightarrow M \) is an \( R \)-monomorphism where \( N \) is a submodule of \( M \). Then \( \alpha \circ j \) is an injective endomorphism of \( M \) where \( j: M \rightarrow M \oplus N \) is the canonical injection. Then by (1), \( \alpha \circ j \) has \( T \)-essential image in \( M \), but \( \alpha(M \oplus O) = \alpha \circ j(M) \) and \( \alpha(M \oplus O) \cap \alpha(O \oplus N) \subseteq T \), hence \( \alpha(N) = \alpha(O \oplus N) \subseteq T \) and thus \( N \subseteq \alpha^{-1}(T) \).

(2) \( \Rightarrow \) (3) it is trivial.

(3) \( \Rightarrow \) (1) Assume \( \alpha \) is an injective endomorphism of \( M \) with \( \alpha(M) \not\subseteq (T) \). By ([3], lemma 3.1), there is a submodule \( K(\not\subseteq T) \) of \( M \) such that \( \alpha(M) \cap K \subseteq (T) \). Define \( \theta: M \oplus K \rightarrow M \) by \( \theta(m,k) = \alpha(m)+k \) for all \( m \in M \) and \( k \in K \). Then \( \theta \) is \( T \)-essential \( R \)-monomorphism and \( \theta(K) \not\subseteq (T) \) which contradicts (3).
(4)⇒(1) Suppose $K$ is a fully invariant T-essential submodule of $M$ which is WTECO-H. Let $\alpha$ be an injective endomorphism of $M$. Then $\alpha|_K$ is an injective endomorphism of $K$, hence $\alpha(K) \leq_T K$. But $K \leq_T M$, then $\alpha(K) \leq_T M$ and hence $\alpha(M) \leq_T M$.

(1)⇒(6) Let $N(\not\subseteq T)$ be a submodule of $M$ and $\alpha$ an injective endomorphism of $M$. Then by (1), $\alpha(M) \cap N \not\subseteq T$. If $n \in \alpha(M) \cap N$ with $n \not\in T$, then $n = \alpha(m)$ for some $m \in M$ and $m \not\in \alpha^{-1}(T)$. But $m \in \alpha^{-1}(N)$. Thus $\alpha^{-1}(N) \not\subseteq \alpha^{-1}(T)$.

(6)⇒(1) If there exists an injective endomorphism $\alpha$ of $M$ and $K$ a submodule of $M$ with $\alpha(M) \cap N \subseteq T$ for some $N \not\subseteq T$, then $\alpha^{-1}(N) = M \cap \alpha^{-1}(N) = \alpha^{-1}(\alpha(M) \cap N) \not\subseteq \alpha^{-1}(T)$ which contradicts (6). Thus $M$ is WTECO-H.

(1)⇒(5) Let $\alpha$ be an injective endomorphism of $M$ and $K$ a submodule of $M$ with $\alpha(N) \cap K \subseteq T$. Then $N \cap \alpha^{-1}(K) \subseteq \alpha^{-1}(T)$, and hence $\alpha^{-1}(K) \subseteq \alpha^{-1}(T)$, so $K \cap \alpha(M) \subseteq T$ By (1), $K \subseteq T$.

(1)⇒(5) Let $\alpha$ be an injective endomorphism of $M$. Since $M \not\subseteq_F M$ for any submodule $F$ of $M$, in particular $M \not\subseteq_{\alpha^{-1}(T)} M$. By (5) we have $\alpha(M) \not\subseteq_T M$.

**Corollary (2.24):** The following are equivalent statements for an $R$-module $M$

1. $M$ is weakly co-Hopfian
2. For each submodule $N$ of $M$, if there is a monomorphism $\alpha: M \oplus N \rightarrow M$, then $N = 0$
3. For each submodule $N$ of $M$, if there is an essential monomorphism $\alpha: M \oplus N \rightarrow M$, then $N = 0$
4. There is a fully invariant essential submodule which is weakly co-Hopfian.
5. For every injective endomorphism $\alpha$ of $M$, if $N \subseteq M$, then $\alpha(N) \subseteq M$
6. For each nonzero submodule $N$ of $M$ and injective endomorphism $\alpha$ of $M$, $\alpha^{-1}(N)$ is nonzero

**Corollary (2.25).** Let $M$ be an $R$-module with a submodule $T$ and $\alpha$ is an injective endomorphism of $M$. Then

1. $N \not\subseteq_{\alpha^{-1}(T)} M$ if and only if $\alpha(N) \not\subseteq_T M$, $N \subseteq_T M$ if and only if $\alpha^{-1}(N) \not\subseteq_{\alpha^{-1}(T)} M$.
2. $soc_T(M) = \cap \alpha^{-1}(N) = \cap \alpha(N)$ where the first(second) intersection runs over all T-essential (T-essential) submodules of $M$.

**Theorem (2.26).** Let $M$ be an $R$-module with a submodule $T$. Consider the following conditions:

1. $M$ is WTSECO-H
2. For each small submodule $N$ of $M$, if there is an $R$-monomorphism $\alpha: M \oplus N \rightarrow M$, then $N \subseteq \alpha^{-1}(T)$
3. For each small submodule $N$ of $M$, if there is a $T_S$-essential $R$-monomorphism $\alpha: M \oplus N \rightarrow M$, then $N \subseteq \alpha^{-1}(T)$.
4. There exists a fully invariant $T_S$-essential submodule which is WTSECO-H.

Then (1)⇒(2)⇒(3), (1)⇐(4), and (3)⇒(1) in case $M$ is finitely generated.

**Proof:** (2)⇒(1) Let $\alpha$ be an injective endomorphism of $M$ and $\alpha(M) \not\subseteq_{T_S} M$. Since $M$ is finitely generated, then by proposition (1.19) there exists a small submodule $K(\not\subseteq T)$ of $M$ such that $\alpha(M) + K \not\subseteq_{T_S} M$. Define $\theta: M \oplus K \rightarrow M$ by $\theta(m,k) = \alpha(m) + k$ for all $m \in M$ and $k \in K$. Then $\theta$ is $T_S$-essential monomorphism and $\theta(K) = K \not\subseteq T$ and hence $K \not\subseteq \theta^{-1}(T)$ which contradicts (3).
(1) \implies (4) It is clear, since the submodule M itself.

(4) \implies (1) Let K be a fully invariant T_\alpha-essential submodule of M which is WTSECO-H and \alpha be an injective endomorphism of M. Then \alpha|_K is an injective endomorphism of K, hence \alpha(K) \leq_{TS} K. But K \leq_{TS} M, then \alpha(K) \leq_{TS} M and hence \alpha(M) \leq_{TS} M.

**Corollary (2.27):** The following statements are equivalent for a nonzero finitely generated R-module M.

1. M is small weakly co-Hopfian
2. For each small submodule N of M, if there is an R-monomorphism \text{M} \oplus K \hookrightarrow \text{M}, then N = 0.
3. For each small submodule N of M, if there is a small-essential R-monomorphism \text{M} \oplus K \hookrightarrow \text{M}, then N = 0.
4. There exists of fully invariant small-essential submodule which is small weakly co-Hopfian.

### 4. References

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