Multigame Effect in Finite Populations Induces Strategy Linkage Between Two Games

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Abstract
Evolutionary game dynamics with two 2-strategy games in a finite population has been investigated in this study. Traditionally, frequency-dependent evolutionary dynamics are modeled by deterministic replicator dynamics under the assumption that the population size is infinite. However, in reality, population sizes are finite. Recently, stochastic processes in finite populations have been introduced into evolutionary games in order to study finite size effects in evolutionary game dynamics. However, most of these studies focus on populations playing only single games. In this study, we investigate a finite population with two games and show that a finite population playing two games tends to evolve toward a specific direction to form particular linkages between the strategies of the two games.

Keywords: Evolutionary game theory; Finite population; Stochastic dynamics; Linkage disequilibrium;

1. Introduction

Evolutionary game theory is a fundamental mathematical framework that enables the investigation of evolution in biological, social, and economic systems, and has been successfully applied to the study of the Darwinian process of natural selection (Lewontin, 1961; Maynard Smith, 1972; Maynard Smith and Price, 1973, Maynard Smith, 1974, 1982; Taylor and Jonker, 1978; Sugden, 1986; Hofbauer and Sigmund, 1998; Nowak, 2006; Nowak and Sigmund, 2004). The Darwinian process is an inherently frequency-dependent process. The fitness of an individual is not only linked to environmental conditions but also tightly coupled with the frequencies of its competitors. Replicator dynamics, introduced by Taylor and Jonker (Taylor and Jonker, 1978), is a system of deterministic differential equations, which model the frequency-dependent selection. It is the most popular model for the evolution of the frequencies of strategies in a population. However, this model intrinsically assumes that population sizes are infinite and it fails to consider stochastic effects.

Recently, various frequency-dependent stochastic processes in finite populations have been introduced into evolutionary games in order to study the finite size effect in evolutionary game dynamics (Nowak et al., 2004; Taylor et al., 2004; Fudenberg et al., 2006; Szabó and Tóke, 1998; Traulsen et al., 2006; Szabó and Hauert, 2002). One such stochastic process is the frequency-dependent Moran process (Nowak et al., 2004). This process is a stochastic birth-death process and comprises two procedures: (1) birth, in which a player is chosen as a parent to reproduce with a probability proportional to its fitness, and its offspring has the same strategy as the parent, and (2) death, in which the offspring replaces a randomly chosen individual. Thus, the population size $N$ is strictly constant in both these procedures. Another process is the local update process (Traulsen et al., 2005). In the local update process, one individual is chosen randomly, who compares his/her payoff to that of another randomly chosen individual, and the probability of the former switching to the latter's strategy is based on difference between their payoffs. Repeating the process $N$-times is regarded as the unit time. These models have enabled many analyses of evolutionary processes and have provided considerable insight into the stochastic effects in evolutionary game dynamics (Nowak et al., 2004; Taylor et al., 2004; Fudenberg et al., 2006; Traulsen et al., 2005; Ficici and Pollack, 2007; Ohtsuki et al., 2007). However, most of the previous studies have only focused on a single game with a maximum of two to three strategies.

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most systems that are of interest to us, we can see that players are playing many games simultaneously, such as biological games in ecosystems and social games in human societies. Such a situation where players play several games simultaneously is termed multigame (Hashimoto, 2006). With an infinite population, the multigame effect has been investigated; when the numbers of strategies of the games are more than two, the fate of the frequencies of the strategies in a single game may change dramatically with or without another game in general (Chamberland and Cressman, 2000; Hashimoto, 2006). Even if one of the games has Evolutionary Stable Strategy (ESS) (Maynard Smith, 1982; Hofbauer and Sigmund, 1998), the ESS point may be destabilized (Hashimoto, 2006). However, when both games have two strategies, their fates coincide with the fates of single games (Cressman et al., 2000). However, in a finite population, the manner in which multigame influence the dynamics remains unclear. In this article, we apply this motivation to the simplest case.

2. Evolutionary game dynamics with two 2-strategy games

In this section, we investigate evolutionary game dynamics with two 2-strategy games in a finite population. Let us consider a population playing two games, game-α and game-β, simultaneously. The reward matrices of the games are given by

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},
\]

respectively. The players can be divided into 2 × 2 groups, with group \((i, j)\) comprising players who play strategy-\(i\) for game-α and strategy-\(j\) for game-β. We assume that a player’s payoff for game-α and that for game-β additively influence his/her payoff. A player of strategy-\((i, j)\) playing against a player of strategy-\((k, l)\) will be rewarded \(a_{ik} + b_{jl}\). This player obtains \(\alpha_{ik}\) through game-α and \(\beta_{jl}\) through game-β. Let \(x_{ij}\) denote the frequency of players playing strategy-\((i, j)\) \((\sum x_{ij} = 1)\). Furthermore, the frequency of players playing strategy-\(i\) in game-α is denoted by \(y_i\) and that of strategy-\(j\) in game-β by \(z_j\):

\[
y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_{11} + x_{12} \\ x_{21} + x_{22} \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x_{11} + x_{21} \\ x_{12} + x_{22} \end{pmatrix}.
\]

If every individual interacts with a representative sample of the population, the expected payoff for an \((i, j)\)-strategy player is determined by \(f_{ij} = \sum_{kl} a_{ik} x_{kl} + b_{jl} x_{kl}\) and the average payoff of the population is \(\bar{f} = \sum x_{ij} f_{ij}\). Using \(y\) and \(z\), these equations can be rewritten as follows:

\[
f_{ij} = (Ay)_i + (Bz)_j, \quad \bar{f} = 'Ay + 'Bz.
\]

In this article, we assume that a coexistence equilibrium point exists in each game. At a coexistence equilibrium point, all strategies obtain the same payoff. Let \(p = ' (p, 1 - p)\) and \(q = ' (q, 1 - q)\) denote the coexistence equilibrium points in game-α and game-β, respectively. \(p\) and \(q\) are determined by \(p = \frac{a_{11} - a_{12}}{a_{11} - a_{21} + a_{22}}\) and \(q = \frac{b_{11} - b_{12}}{b_{11} - b_{21} + b_{22}}\), respectively. The existence of the coexistence equilibrium points yields \((a_{11} - a_{21})(a_{22} - a_{12}) > 0\) and \((b_{22} - b_{12})(b_{11} - b_{21}) > 0\). Note that in a multigame situation, system states that satisfy the coexistence equilibria in both the games are not a single point but rather points on a line determined by \(y = p\) and \(z = q\). Here, this line is termed \(L\) (see Fig. 3). Furthermore, we assume that the equilibrium points are stable in both the games and their stability are sufficiently strong. The stability of the equilibrium points ensures

\[
a_{11} - a_{21} < 0, \quad a_{22} - a_{12} < 0, \\
b_{11} - b_{21} < 0, \quad b_{22} - b_{12} < 0.
\]

This means that the two games are Hawk and Dove games which were initially introduced by J. Maynard Smith (Maynard Smith, 1982).

2
2.1. Infinite-population model

In an infinite population, the replicator equation that corresponds to this multigame situation is given by

\[ \dot{x}_{ij} = x_{ij} \left( f_{ij} - \bar{f} \right) = x_{ij} \left( (Ay)_i + (Bz)_j - \gamma Ay - \gamma Bz \right). \]

(1)

Behaviors in this system are rather simple. This differential equation leads

\[ \frac{d}{dt} \frac{x_{11}x_{22}}{x_{12}x_{21}} = \frac{x_{11}x_{22}}{x_{12}x_{21}} \left( f_{11} + f_{22} - f_{12} - f_{21} \right) = 0. \]

Thus, \( \frac{x_{11}x_{22}}{x_{12}x_{21}} \) is constant in time evolution and this implies that \( \frac{x_{11}x_{22}}{x_{12}x_{21}} = \text{(const)} \) forms an invariant manifold. Some invariant manifolds are shown in Fig. 1. Additionally, in Fig. 2, four orbits that follow Eq. (1) are plotted as examples. An orbit starting at time \( t = 0 \) from a point \( \left( \frac{x_{11}(0)}{x_{21}(0)} \frac{x_{12}(0)}{x_{22}(0)} \right) \) will move on an invariant manifold determined by \( \frac{x_{11}x_{22}}{x_{12}x_{21}} \) and will eventually converge to the intersection point of the invariant manifold and \( L \). Every point on \( L \) is a fixed point and is stable to the transverse direction of \( L \). Furthermore, because every point on \( L \) is a fixed point, the stability to the direction of \( L \) is neutral.

![Figure 1](image)

Figure 1: Since \( \sum x_{ij} = 1 \) and \( x_{ij} \geq 0 \) hold, a system state can be represented by a point in a tetrahedron. Five examples of invariant manifolds in the replicator equation are plotted. \( \frac{x_{11}x_{22}}{x_{12}x_{21}} \) is (a) \( \frac{1}{4} \), (b) \( \frac{1}{3} \), (c) 1, (d) 5, and (e) 25. The vertexes represent the system states that all players have the same strategy. \( e_1 \) represents one of the four vertexes: \( (x_{11}, x_{12}, x_{21}, x_{22}) = (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \) and \( (0, 0, 0, 1) \), respectively.

2.2. Finite-population model

In a finite population, stochastic effects by the demographic noise constantly perturb the system state. Selection pressures bring the system state to \( L \), however, stochastic effects prevent the system state from staying on \( L \). In this study, we are interested in determining the motion of the system state in \( L \)'s direction. From a naive intuition, it may seem to be a simple random walk because the stability in this direction is neutral in a system with an infinite population. However, we find that finite size effect breaks this neutrality and a flow emerges along \( L \). Consequently, the system state tends to evolve toward a specific direction of \( L \).

As mentioned above, several processes are proposed for game dynamics in finite populations. In this article, we concentrate on the local updating process proposed by Traulsen et al. (Traulsen et al., 2005). In the local update process, a player \( b \) is chosen randomly and his/her payoff is compared to that of another randomly chosen individual \( a \). The probability of player-\( b \) switching his/her strategy to the player-\( a \)'s strategy is given by

\[ \gamma(f_{a}, f_{b}) = \frac{1}{2} + \frac{\omega}{2}(f_{a} - f_{b}). \]
where $f_a$ and $f_b$ are the payoffs of player-a and player-b, respectively. Furthermore, $\omega (\geq 0)$ measures the strength of the selection pressure. Selection is weak when $\omega \ll 1$, the process is dominated by random updating and payoff differences have a negligible effect on the process. For larger $\omega$, the selection intensity increases. However, an upper limit is imposed on $\omega$ by the following requirement: $0 \leq \gamma \leq 1$. The probability of selecting an $(i, j)$-strategy player as player-a is $x_{ij}$ and the probability of selecting a $(k, l)$-strategy player as player-b is $x_{kl}$. Therefore, the probability, $P(i, j)_{(k, l)}$, that the number of $(i, j)$-strategy players increases by one and that of $(k, l)$-strategy players decreases by one in a single process is given by

$$
P(i, j)_{(k, l)} = x_{ij} x_{kl} \gamma (f_{ij}, f_{kl}) = x_{ij} x_{kl} \left\{ \frac{1}{2} + \frac{\omega}{2} (f_{ij} - f_{kl}) \right\}.
$$

The probability that the system state does not change is $1 - \sum_{(i, j) \neq (k, l)} P(i, j)_{(k, l)}$.

It is noteworthy that in the limit of $N \to \infty$, this process represents a replicator equation. The expected change of $x_{ij}$ in a single process is as follows:

$$
\langle \Delta x_{ij} \rangle = \frac{1}{N} \sum_{(k, l)} P(i, j)_{(k, l)} - P(i, j)_{(i, j)} = \frac{\omega}{N} x_{ij} (f_{ij} - \bar{f}).
$$

Because the process is iterated $N$ times in a unit time, a single process takes time $\Delta t = \frac{1}{N}$. In the limit of $N \to \infty$, the replicator equation is derived as

$$
\dot{x}_{ij} = \lim_{N \to \infty} \frac{\Delta x_{ij}}{\Delta t} = \omega x_{ij} (f_{ij} - \bar{f}).
$$

Therefore, the stochastic process indeed implements a replicator equation in a finite population with stochasticity of demographic noise.

In the process, the system state ultimately reaches one of the four homogeneous states (i.e., the states that all players have the same strategy) after long transient, and these four states correspond to the four vertexes of the tetrahedron. However, the time taken to reach such a homogeneous state is extremely long, especially in a large population, because of the stability of coexistence equilibrium points (Antal and Scheuring, 2006). In contrast, extinction of a single strategy resulting from a random walk along $L$ occurs in a short time-scale. Similar to the infinite-population case, first, the system state is immediately brought to the vicinity of $L$ from its initial point by the selection pressure roughly along the manifold that the initial state is located. Subsequently, the system state fluctuates around $L$ because of the stochastic effects. Eventually, one of the strategies become extinct, implying that the system state has reached one of the boundaries of $L$. Strictly speaking, this is inaccurate. In fact, the system state reaches a point on a boundary plane.
when a single strategy is extinct and this location is not necessarily a boundary of \( L \) but a point close to it. Following this, the system state fluctuates on the boundary plane around the intersection point of \( L \) and the boundary plane. In Fig. 3, an evolutionary path from an initial state to a state in which a single strategy is eliminated is plotted. Finally, in the limit of time evolution, two strategies will be eliminated, that is, the population becomes homogeneous. This is the absorbing state of the system and it takes an enormously long time because of the stability of the coexistence equilibrium. Therefore, the boundaries of \( L \) can be considered as absorbing states for a short time-scale evolution.

As mentioned above, we are not interested in determining how the homogeneous states are reached but rather how the boundaries of \( L \) are reached. This rather literary illustration of the system behavior raises following questions: Is the movement of the system state along \( L \) just a simple random walk? If not, which absorbing state has a higher likelihood of being observed? This article is aimed to answer these questions.

2.3. Variable transformation

To investigate the behavior of the system we here adopt a variable transformation \[
\begin{pmatrix}
    x_{11} \\
    x_{12} \\
    x_{21} \\
    x_{22}
\end{pmatrix}
\mapsto
\begin{pmatrix}
    r \\
    u \\
    v
\end{pmatrix}:
\]
\[
\begin{pmatrix}
    x_{11} \\
    x_{12} \\
    x_{21} \\
    x_{22}
\end{pmatrix}
= \begin{pmatrix}
    pq & p(1-q) \\
    (1-p)q & (1-p)(1-q)
\end{pmatrix}
+ \begin{pmatrix}
    r/2 & 1 & -1 \\
    -1 & 1 & -1
\end{pmatrix}
+ \begin{pmatrix}
    u/2 & 1 & -1 \\
    -1 & 1 & -1
\end{pmatrix}
+ \begin{pmatrix}
    v/2 & 1 & -1 \\
    -1 & 1 & -1
\end{pmatrix}.
\] (3)

Note that the origin of \((r, u, v)\)-coordinate depends on \( p \) and \( q \). Since Eq. (3) leads \( y = p + u \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( z = q + v \begin{pmatrix} 1 \\ -1 \end{pmatrix} \), \( u \) and \( v \) represents the deviations from equilibria in game-\( \alpha \) and game-\( \beta \), respectively. \( L \) is represented by \( u = v = 0 \). Therefore, \( r \)-axis is identical to \( L \). A sample trajectory in \((r, u, v)\)-coordinate is plotted in Fig. 4. A Markov process on \((r, u, v)\)-space is led from the transition probabilities (Eq. (2)) and this variable transformation. For an example, with a probability of \( P_{(1,2)}^{(1,1)} \), a \((1, 2)\)-strategy player is replaced by \((1, 1)\)-strategy player, then \( r \) and \( v \) increase by \( 1/N \) and \( u \) remains unchanged. Another example, a \((2, 2)\)-strategy player is replaced by \((1, 1)\)-strategy player with a probability of \( P_{(2,1)}^{(1,1)} \), then \( u \) and \( v \) increase by \( 1/N \) and \( r \) remains unchanged.

2.4. Langevin equations

To analyze the dynamics of a large but finite population, it is effective to approximate the dynamics using Langevin equations (Helbing, 1996; Traulsen et al., 2005, 2006, 2012). The objective of this study is to determine the time evolution of \( r \); the first step to this is to derive the Langevin equations for \((r, u, v)\) by approximating the Markov process, assuming that the population size \( N \) is sufficiently large.

Figure 3: A trajectory in a stochastic system with a finite population is plotted. The system state is brought to \( L \) due to the selection pressure in the same way as the infinite-population model. However, in contrast to the case of an infinite population, stochastic effects prevent the system state from staying on \( L \), and it fluctuates around \( L \). Eventually, the system state reaches one of the boundaries of \( L \).
Figure 4: A sample trajectory in \((r, u, v)\)-coordinate. \(r\)-axis is identical to the line \(L\) indicated in Fig. 3. The trajectory makes its way toward the \(r\)-axis and subsequently fluctuates around it. Eventually, it reaches one of the boundaries of \(L\).

The expected changes of \(r\), \(u\) and \(v\) in a single process are respectively given as

\[
\langle \Delta r \rangle = \frac{1}{N} g_r(r, u, v) = \frac{1}{N} \sum_{k,l} p^{(j)}_{(k,l)} - p^{(j)}_{(l,k)},
\]

\[
\langle \Delta u \rangle = \frac{1}{N} g_u(r, u, v) = \frac{1}{N} \sum_{k,l} p^{(k)}_{(1,l)} - p^{(k)}_{(l,1)},
\]

\[
\langle \Delta v \rangle = \frac{1}{N} g_v(r, u, v) = \frac{1}{N} \sum_{k,l} p^{(k)}_{(l,2)} - p^{(k)}_{(2,l)}.
\]

These are used for the drift terms in the Langevin equations. Based on a simple calculation using Eq. (2) and the variable transformation Eq. (3), \(g_r\), \(g_u\), and \(g_v\) are given by

\[
g_r(r, u, v) = \omega \left[ \Phi u \left\{ -p(1-p)(1-2p) - ru + \frac{r}{2}(1-2p) - \frac{u}{2}(1-2p) + \frac{v}{2} \right\} + \Psi v \left\{ -q(1-q)(1-2q) - rv + \frac{r}{2}(1-2q) - \frac{v}{2}(1-2q) + \frac{u}{2} \right\} \right]
\]

\[
g_u(r, u, v) = \omega \left[ \Phi u(p + u)(1-p - u) + \Psi v \left\{ \frac{r}{2} - uv + \frac{u}{2}(1-2q) + \frac{v}{2} \right\} \right]
\]

\[
g_v(r, u, v) = \omega \left[ \Psi v(q + v)(1-q - v) + \Phi u \left\{ \frac{r}{2} - uv + \frac{u}{2}(1-2q) + \frac{v}{2} \right\} \right]
\]

respectively, where \(\Phi = a_{11} + a_{22} - a_{12} - a_{21}\), \(\Psi = b_{11} + b_{22} - b_{12} - b_{21}\). Note that the stability of the equilibrium points in the two games yields \(\Phi < 0, \Psi < 0\). Moreover, \(\langle (\Delta r)^2 \rangle, \langle (\Delta u)^2 \rangle, \langle (\Delta v)^2 \rangle, \langle \Delta r \Delta u \rangle, \langle \Delta r \Delta v \rangle, \langle \Delta u \Delta v \rangle\) are also
respectively given as

\[
\langle (\Delta r)^2 \rangle = \frac{1}{N^2} h_{rr}(r, u, v) = \frac{1}{N^2} \sum_{k,l} P_{(k,l)}^{(1)} + P_{(k,l)}^{(2)},
\]

\[
\langle (\Delta u)^2 \rangle = \frac{1}{N^2} h_{uu}(r, u, v) = \frac{1}{N^2} \sum_{k,l} P_{(k,l)}^{(1)} + P_{(k,l)}^{(2)},
\]

\[
\langle (\Delta v)^2 \rangle = \frac{1}{N^2} h_{vv}(r, u, v) = \frac{1}{N^2} \sum_{k,l} P_{(k,l)}^{(1)} + P_{(k,l)}^{(2)},
\]

\[
\langle \Delta r \Delta u \rangle = \frac{1}{N^2} h_{ru}(r, u, v) = \frac{1}{N^2} \sum_{k,l} P_{(k,l)}^{(1)} - P_{(k,l)}^{(2)},
\]

\[
\langle \Delta r \Delta v \rangle = \frac{1}{N^2} h_{rv}(r, u, v) = \frac{1}{N^2} \sum_{k,l} P_{(k,l)}^{(1)} - P_{(k,l)}^{(2)},
\]

\[
\langle \Delta u \Delta v \rangle = \frac{1}{N^2} h_{uv}(r, u, v) = \frac{1}{N^2} \sum_{k,l} P_{(k,l)}^{(1)} - P_{(k,l)}^{(2)}.
\]

These are used for the diffusion terms in the Langevin equations. Remember that the process is repeated \(N\) times in a unit time. Assuming that \(N\) is sufficiently large, the central limit theorem yields the Langevin equations as follows:

\[
\dot{r} = g_r(r, u, v) + \frac{1}{\sqrt{N}} \zeta_r(t), \tag{7}
\]

\[
\dot{u} = g_u(r, u, v) + \frac{1}{\sqrt{N}} \zeta_u(t), \tag{8}
\]

\[
\dot{v} = g_v(r, u, v) + \frac{1}{\sqrt{N}} \zeta_v(t). \tag{9}
\]

Here, \(\zeta_r(t), \zeta_u(t)\) and \(\zeta_v(t)\) denote Gaussian white noise with

\[
\langle \zeta_r(t) \rangle = \langle \zeta_u(t) \rangle = \langle \zeta_v(t) \rangle = 0,
\]

\[
\langle \zeta_r(t) \zeta_r(t') \rangle = \langle \zeta_u(t) \zeta_u(t') \rangle = \langle \zeta_v(t) \zeta_v(t') \rangle = \langle \zeta_a(t) \zeta_a(t') \rangle = \langle \zeta_v(t) \zeta_v(t') \rangle = \langle \zeta_v(t) \zeta_v(t') \rangle = h_{rr} - g_r^2 \delta(t - t'),
\]

\[
\langle \zeta_u(t) \zeta_u(t') \rangle = \langle \zeta_v(t) \zeta_v(t') \rangle = h_{uv} - g_u g_v \delta(t - t'),
\]

\[
\langle \zeta_r(t) \zeta_v(t') \rangle = \langle \zeta_u(t) \zeta_v(t') \rangle = h_{rv} - g_r g_v \delta(t - t'),
\]

\[
\langle \zeta_u(t) \zeta_v(t') \rangle = \langle \zeta_v(t) \zeta_v(t') \rangle = h_{uv} - g_u g_v \delta(t - t').
\]

These noises are demographic stochasticity and should be interpreted in Ito’s sense. Assuming that the system size \(N\) is sufficiently large leads that the deviations of \(u\) and \(v\) are sufficiently small. Under this assumption, we disregard the effects of the deviations of \(u\) and \(v\) on the noise terms. For an example, \(\langle \zeta_r(t) \zeta_a(t') \rangle = h_{ra}(r, u, v) - g_r^2(r, u, v)\) can be approximated by \(h_{rr}(r, 0, 0) - g_r^2(r, 0, 0)\). The other correlations of the noise terms can be approximated in the same
manner. A simple calculation yields the following equations:

\[ h_{rr} - g_r^2 \approx h_{rr}(r, 0, 0) - g_r^2(r, 0, 0) \]
\[ = [pq + (1 - p)(1 - q) + r] (p(1 - q) + (1 - p)q - r) \]  \hspace{1cm} (10)

\[ h_{uv} - g_u^2 \approx h_{uv}(r, 0, 0) - g_u^2(r, 0, 0) = p(1 - p) \]  \hspace{1cm} (11)

\[ h_{vv} - g_v^2 \approx h_{vv}(r, 0, 0) - g_v^2(r, 0, 0) = q(1 - q) \]  \hspace{1cm} (12)

\[ h_{uu} - g_u g_v \approx h_{uu}(r, 0, 0) - g_u g_v(r, 0, 0) = \frac{r}{2} \]  \hspace{1cm} (13)

\[ h_{vu} - g_v g_u \approx h_{vu}(r, 0, 0) - g_v g_u(r, 0, 0) = -(1 - 2p)q(1 - q) \]  \hspace{1cm} (14)

\[ h_{rv} - g_r g_v \approx h_{rv}(r, 0, 0) - g_r g_v(r, 0, 0) = -(1 - 2q)p(1 - p) \]  \hspace{1cm} (15)

Thus, the correlations of noise \( \zeta_u \) and \( \zeta_v \) are approximated by

\[ \langle \zeta_u(t) \zeta_u(t') \rangle = p(1 - p)\delta(t - t'), \]
\[ \langle \zeta_v(t) \zeta_v(t') \rangle = q(1 - q)\delta(t - t'), \]
\[ \langle \zeta_u(t) \zeta_v(t') \rangle = \frac{r}{2}\delta(t - t'), \]

respectively. This approximation and linearizing Eqs. (8) and (9) yield

\[ \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J \begin{pmatrix} u \\ v \end{pmatrix} + \frac{1}{\sqrt{N}} \langle \zeta_u(t) \rangle \begin{pmatrix} \zeta_u(t) \\ \zeta_v(t) \end{pmatrix} \]  \hspace{1cm} (16)

where \( J \) is the Jacobi matrix of Eqs. (8) and (9) at \((u, v) = (0, 0)\):

\[ J = \begin{pmatrix} \frac{\partial \Phi}{\partial u} & \frac{\partial \Phi}{\partial v} \\ \frac{\partial \Psi}{\partial u} & \frac{\partial \Psi}{\partial v} \end{pmatrix} = \omega \begin{pmatrix} \Phi (1 - p) & \Psi \frac{1}{2} \\ \Psi \frac{1}{2} & \Psi q(1 - q) \end{pmatrix}. \]

Because \( N \) is sufficiently large, we can assume that the motion of \( r \) is rather slow and the time taken to reach one of the boundaries of \( L \) is sufficiently long. Thus, the asymptotic solution of the linear Langevin equation Eq. (16) is given as

\[ \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \frac{1}{\sqrt{N}} \int_{-\infty}^{\infty} e^{t(s - t)} \begin{pmatrix} \zeta_u(s) \\ \zeta_v(s) \end{pmatrix} \mathrm{d}s. \]  \hspace{1cm} (17)

Based on a simple algebra, it is shown that \( u \) and \( v \) obey the two-dimensional normal distribution with zero mean:

\[ \frac{1}{2\pi \sqrt{|F|}} \exp \left[ -\frac{1}{2}(u, v)F^{-1}(u, v) \right] \]

where \( F \) is the covariance matrix of \( u \) and \( v \):

\[ F = \begin{pmatrix} u^2 & uv \\ uv & v^2 \end{pmatrix} = -\frac{1}{2N\omega} \begin{pmatrix} \Phi & 0 \\ 0 & \Psi \end{pmatrix}. \]  \hspace{1cm} (18)

For the next step, we approximate \( g_r(r, u, v) \) by \( \langle g_r(r, u, v) \rangle_{u,v} \) in order to exclude \( u \) and \( v \) from the Langevin equation of \( r \) (Eq. (7)). By using Eq. (13), we obtain

\[ \langle g_r(r, u, v) \rangle_{u,v} = \frac{1}{N} \left\{ \frac{r}{2} + \frac{1}{2}(1 - 2p)(1 - 2q) \right\}. \]

Finally, Eq. (7) can be approximated by

\[ \langle \dot{r} \rangle_{u,v} = \frac{1}{N} \left\{ \frac{r}{2} + \frac{1}{2}(1 - 2p)(1 - 2q) \right\} + \frac{1}{\sqrt{N}} \zeta(t). \]  \hspace{1cm} (19)
where $\zeta(t)$ denotes Gaussian white noise with

$$\langle \zeta(t) \rangle = 0,$$

$$\langle \zeta(t) \zeta(t') \rangle = \{pq + (1-p)(1-q) + r\} \{p(1-q) + (1-p)q - r\} \delta(t - t').$$

In contrast to the case of an infinite population, Eq. (19) indicates that the neutrality on $L$ vanishes and the motion of $r$ is not a simple random walk. The drift term of Eq. (19) is positive (negative) when $r$ is larger (smaller) than $r^* = -\frac{2}{\sqrt{r}}$. Thus, the equilibrium point $r^*$ is unstable in Eq. (19) (see Fig. 5). The order of the drift term is $O(1/N)$, whereas that of the diffusion term is $O(1/\sqrt{N})$. Therefore, the flow by the drift term is obscured by the diffusion when the population size is large. However, if the population size is not large, distinct trends of behaviors can be observed as demonstrated by the numerical simulations in Section 3.

![Figure 5: A schematic view of the drift term of Eq. (19).](image)

Note that from Eq. (4), it is obvious that $\langle \Delta r \rangle$ is always zero when $(u, v) = (0, 0)$ (i.e., the system state is precisely on $L$) regardless of the value of $r$. Therefore, when the system state is precisely on $L$, the expected motion of $r$ is neutral. However, constantly occurring perturbation by finite size effect causes $(u^2)$ and $(v^2)$ to be non-zero, making flow presented in Fig. 5 in the expected motion of $r$. In this manner, emergence of such flow is a somewhat indirect effect of finiteness of the system size.

2.5. Strategy linkage between the two games

On $L$, $r$ measures the linkage between the strategies of the two games. $x_{ij} = y_i z_j$ holds if $r = 0$, that is, there is no linkage between the strategies of the two games. When $r$ is positively larger, a player playing strategy-1 (strategy-2) in game-$\alpha$ also plays strategy-1 (strategy-2) in game-$\beta$ with a higher probability and vice versa. In such a case, the strategies of the two games are positively linked in the population. On the other hand, when $r$ is negatively larger, a player playing strategy-1 (strategy-2) in game-$\alpha$ plays strategy-2 (strategy-1) in game-$\beta$ with higher probability and vice versa; in such a case, the strategies of the two games are negatively linked. Because the labels of the strategies can be assigned arbitrarily, we can assume that $p, q \leq \frac{1}{2}$ without the loss of generality. This assumption means that in each game, the minor strategy at the equilibrium point is labeled strategy-1 and the major one is labeled strategy-2.

Here, we show that the system state tends to have positive linkage under this condition. We can also assume that $p \leq q$. Even if $p > q$, game-$\alpha$ and game-$\beta$ just need to be relabeled to satisfy this assumption. These assumptions yield the following equations: $pq \leq p(1-q) \leq (1-p)q \leq (1-p)(1-q)$. Therefore, non-negativity of $x_{ij}$ indicates that the boundaries of $L$ are $\left( \begin{array}{cc} 0 & p \\ q & 1-p-q \end{array} \right)$ and $\left( \begin{array}{cc} p & 0 \\ q-p & 1-q \end{array} \right)$. Thus, the range of $r$ is given by

$$r^L \leq r \leq r^H, \quad \left( r^L = -2pq, \quad r^H = 2p(1-q) \right).$$

Since $p, q \leq \frac{1}{2}$, $r^*$ is always non-positive and the midpoint of $L$ is always non-negative (i.e., $r^* \leq 0 \leq \left( r^L + r^H \right)/2 = p(1-2q)$). Thus, the position of $r^*$ on $L$ can typically be depicted as Fig. 6(a). Furthermore, if $p + q \leq \frac{1}{2}$ is satisfied, $r^*$ is lesser than $r^L$ and in such a case, the drift term in Eq. (19) is always set to be positive regardless of the value of $r$, as presented in Fig. 6(b). Such an obvious asymmetry of flow introduces a particular trend in the system. If the initial state is chosen randomly, it is clear that the system state goes to the higher boundary of $L$ with a higher probability than the lower boundary. The population tends to have minor-minor and major-major linkages between the strategies of the two games.

Additionally, let us consider the rare mutation of strategies. Mutation restores extinct strategies and makes the system ergodic. Therefore, the system state can be observed to making round trips along $L$ repeatedly. Obviously,
3. Numerical simulations

Here, we demonstrate the tendency of the motion of \( r \) with several numerical simulations.

3.1. Round trips along \( L \)

To confirm the discussion in the last of the previous section, let us introduce mutation into the system. In each iteration of the process, a randomly chosen individual replaces his/her strategy with another strategy with a certain small probability \( \mu \). Since mutation makes the system ergodic, the system state shows round trips along \( L \) repeatedly. Therefore, we can observe the tendency of the motion. In Fig. 6, the time series of \( r \) takes more time to fluctuate around the lower and higher boundaries, we will confirm this bias with a numerical simulation in Section 3.

Figure 6: Schematic views of the drift term in Eq. (19). Since \( p, q \leq \frac{1}{2} \), \( r^* \leq 0 \leq \frac{r^L + r^H}{2} \) is always satisfied. (a) When \( p + q \geq \frac{1}{2} \), \( r^* \) is between \( r^L \) and \( r^0 \). (b) Otherwise, \( r^* \) is under \( r^H \).

states around \( \left( \begin{array}{c} p \\ q-p \\ 1-q \end{array} \right) \) are observed more often than \( \left( \begin{array}{c} 0 \\ q \\ 1-p-q \end{array} \right) \). This indicates that the population with positive linkage is observed more often than that with negative linkage. Although it is difficult to analytically derive the frequencies of the periods to stay around the lower and higher boundaries, we will confirm this bias with a numerical simulation in Section 3.

3.2. Direction of flow at \( (r, u, v) = (0, 0, 0) \)

Second, we calculated the expected values of \( r \) at \( t = \frac{1}{8} \) starting from \( (r, u, v) = (0, 0, 0) \) at \( t = 0 \) for a set of various \( (p, q) \). Eq. (19) indicates that if both of \( p \) and \( q \) are larger or smaller than \( \frac{1}{2} \), \( \langle r(t) \rangle \) is positive, otherwise, \( \langle r(t) \rangle \) is negative. A probability distribution, the initial distribution of which is concentrated on \( (r, u, v) = (0, 0, 0) \), is updated \( N \times \frac{1}{8} \) times with transition probabilities given by Eq. (5); subsequently, \( \langle r(t = 1/8) \rangle \) is evaluated with the distribution at \( t = 1/8 \) and plotted in Fig. 8. As expected from Eq. (19), Fig. 8 shows that when \( p \) and \( q \) are both larger or smaller than \( \frac{1}{2} \), \( \langle r \rangle \) is positive; otherwise, \( \langle r \rangle \) is negative.

To justify the framework of our analysis, let us compare the values obtained by the numerical simulations with analytically approximated values. Since the period from \( t = 0 \) to \( t = \frac{1}{8} \) is too short to use Eq. (19) as it is without any modifications, we here derive the approximated value of \( \langle r(t = 1/8) \rangle \) with the condition \( (r(0), u(0), v(0)) = (0, 0, 0) \). Since this initial state is a fixed point, diffusional effect dominates the dynamics when \( t \) is small. Thus, the mean values of \( u^2(t) \), \( v^2(t) \), and \( u(t)v(t) \) for \( t \ll 1 \) are simply obtained from Eqs. (11)-(13):

\[
\begin{bmatrix}
  u^2(t) & u(t)v(t) \\
  u(t)v(t) & v^2(t)
\end{bmatrix} = \frac{1}{N} \begin{bmatrix}
  p(1-p) & 0 \\
  0 & q(1-q)
\end{bmatrix} t + O(t^2).
\]

(20)
Additionally, the mean values of \(r(t)u(t)\) and \(r(t)v(t)\) for \(t \ll 1\) are also obtained from Eqs. (14) and (15):

\[
\begin{align*}
\left( \frac{r(t)u(t)}{r(t)v(t)} \right) &= -\frac{1}{N} \left( p(1-p)(1-2q) \right) t + O(t^2).
\end{align*}
\]

From Eqs. (20), (21), and (4), \(i\) for \(t \ll 1\) can be approximated by

\[
i = -\frac{\omega i}{N} \left[ \Phi p(1-p) + \Psi q(1-q) \right] (1-2p)(1-2q) + O(t^2) + \frac{1}{\sqrt{N}} \xi(t).
\]

\(\langle r(t) \rangle\) can be approximated by integrating Eq. (22) as follows:

\[
\langle r(t) \rangle = -\frac{\omega i^2}{2N} \left[ \Phi p(1-p) + \Psi q(1-q) \right] (1-2p)(1-2q) + O(t^2).
\]

Fig. 8 shows the approximation values calculated by Eq. (23). This figure clearly indicates that the values obtained by the numerical simulations are approximated effectively by Eq. (23).

4. Conclusion

In this article, we investigate a finite population with two games and show that a finite population playing two games tends to evolve toward a specific direction to form certain linkages between the strategies of the two games. We found that although the two games are not related, a population tends to form a linkage between the minor (major) strategies of the two games.

From the population genetics perspective, this means that two loci, which determine an individual’s traits that independently contribute to its fitness, may have a stronger tendency to form a particular linkage disequilibrium in smaller populations. In future studies, more complicated situations, such as games that have three or more strategies, populations that play three or more games, and diploid cases, could be investigated.

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Figure 7: Time series of $r$ in the system with mutation are plotted. The payoff matrices of the games are set $A = \begin{pmatrix} 0 & 0.2 \\ 0.8 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0.3 \\ 0.7 & 0 \end{pmatrix}$. Thus, $p = 0.2$ and $q = 0.3$. Moreover, the selection intensity is $\omega = 0.5$ and mutation rate is $\mu = 5 \times 10^{-5}$. The number of individuals is 500, 1,000 and 2,000 in (a), (b), and (c), respectively. The initial states are all set at $(x_1, x_2, y_1, y_2) = (pq, (1-p)(1-q), 0)$, $pq, (1-p)(1-q), 0)$. The dashed lines represent $r^H(= 0.28)$ and $r^L(= -0.12)$. The upper one is $r^H$ and the other is $r^L$. 

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Figure 8: The number of players is $N = 32^2 = 1024$. The expected values of $r$ at $t = \frac{1}{8}$, which start from $(r, u, v) = (0, 0, 0)$ at $t = 0$, are plotted for various $(p, q)$. $(p, q)$ is set as $(p, q) = \left( \frac{P}{32}, \frac{Q}{32} \right)$, $(P = 1, \ldots, 31, \ Q = 1, \ldots, 31)$. The payoff matrices are given by $A = \begin{pmatrix} 0 & p \\ 1-p & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & q \\ 1-q & 0 \end{pmatrix}$ here. The initial state $\begin{pmatrix} N_1 \ N_2 \\ N_3 \ N_4 \end{pmatrix}$ ($N_{ij}$ is the number of $(i, j)$-strategy players) corresponding to $(r, u, v) = (0, 0, 0)$ is $\begin{pmatrix} P^3 \ (32-P)^3 \ P(32-P)^2 \ (32-P)^2 \end{pmatrix}$. The colored surface represents the results of the numerical simulation and the red lines are plotted by Eq. (23). Numerical simulation has been conducted as follows: initially, a probability distribution on the system states that concentrated on $\begin{pmatrix} P^3 \ (32-P)^3 \ P(32-P)^2 \ (32-P)^2 \end{pmatrix}$ is prepared and updated 128 ($= N \times \frac{1}{8}$) times with the transition probabilities given in Eq. (2); subsequently, $\langle r(t = 1/8) \rangle$ is evaluated with the distribution at $t = \frac{1}{8}$. 