Parallel-propagated frame along null geodesics in higher-dimensional black hole spacetimes

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In [arXiv:0803.3259] the equations describing the parallel transport of orthonormal frames along timelike (spacelike) geodesics in a spacetime admitting a non-degenerate principal conformal Killing–Yano 2-form \( h \) were solved. The construction employed is based on studying the Darboux subspaces of the 2-form \( F \) obtained as a projection of \( h \) along the geodesic trajectory. In this paper we demonstrate that, although slightly modified, a similar construction is possible also in the case of null geodesics. In particular, we explicitly construct the parallel-transported frames along null geodesics in \( D = 4, 5, 6 \) Kerr-NUT-(A)dS spacetimes. We further discuss the parallel transport along principal null directions in these spacetimes. Such directions coincide with the eigenvectors of the principal conformal Killing–Yano tensor. Finally, we show how to obtain a parallel-transported frame along null geodesics in the background of the 4D Plebański–Demiański metric which admits only a conformal generalization of the Killing–Yano tensor.

I. INTRODUCTION

Solving the parallel transport equations along null geodesics in a four-dimensional spacetime is a well known problem with many physical applications. For example, in a geometric optics approximation linearly polarized photons and gravitational waves propagate along null geodesics while the corresponding polarization vectors are parallel-propagated along null geodesics. In particular, such a problem is relevant to the study of the Darboux subspaces of the 2-form \( F \) obtained as a projection of the Killing–Yano 2-form \( h \) along the geodesic trajectory. In this paper we demonstrate that, although slightly modified, a similar construction is possible also in the case of null geodesics. In particular, we explicitly construct the parallel-transported frames along null geodesics in \( D = 4, 5, 6 \) Kerr-NUT-(A)dS spacetimes. We further discuss the parallel transport along principal null directions in these spacetimes. Such directions coincide with the eigenvectors of the principal conformal Killing–Yano tensor. Finally, we show how to obtain a parallel-transported frame along null geodesics in the background of the 4D Plebański–Demiański metric which admits only a conformal generalization of the Killing–Yano tensor.

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Recently, models of gravity with extra dimensions attracted a lot of attention. In order to re-derive many of the important results of the four-dimensional optics in a curved spacetime in the higher-dimensional case it is necessary to consider first a problem of parallel transport along null geodesics. This is the main purpose of this paper.

We focus our attention on a special class of higher-dimensional spacetimes which admit a so-called principal conformal Killing–Yano (PCKY) tensor \( h \) (see also [14, 15, 16] for reviews). The most general metric element admitting such a tensor, the canonical (off-shell) metric, was studied in [17, 18]. Similar to 4D, when the vacuum Einstein equations with the cosmological constant are imposed the canonical metric becomes the Kerr-NUT-(A)dS spacetime [19].

Recently, a parallel-propagated frame along timelike (spacelike) geodesics in the canonical spacetime was constructed [20]. As it often happens, a limit when the velocity of the particle motion tends to the speed of light is singular, so that, the problem of parallel transport along null geodesics requires a special treatment. We deal with this problem in the present paper. The obtained results generalize the 4D results [21, 22, 23].

Consider a spacetime with a PCKY tensor \( h \), that is, a closed rank-2 non-degenerate conformal Killing–Yano tensor [24, 25]. In such a spacetime the geodesic motion is completely integrable [26, 27, 28]. Let us concentrate on a generic null ray and denote its velocity vector \( l \). In our construction, starting with \( l \) we first generate two additional parallel-propagated vectors, one of which, say \( n \), is ‘external’ to the null plane of \( l \) and can be made null. This means that the tangent space \( T \) at each point of \( l \) splits into a 2-dimensional
parallel-propagated subspace $U$ spread by \{\(l, n\)\} and a \((D-2)\)-dimensional parallel-propagated subspace $V$ orthogonal to $U$: $T = U \oplus V$.

The construction of the parallel-propagated frame is now similar to the timelike case. We consider the Darboux problem for the 2-form $F$ obtained as a projection of the PCKY tensor $h$ to a subspace $V$. Such a 2-form is automatically parallel-transported. In particular, each of the Darboux subspaces of $F$ is independently parallel-transported. The zeroth value Darboux subspace of dimensions, respectively. Operations \(\flat\) and \(\sharp\) correspond to ‘lowering’, ‘raising’ of indices of vectors, forms, respectively. \(\delta\) denotes the co-derivative. For a $p$-form $\alpha_p$, one has $\delta \alpha_p = \epsilon \cdot d \star \alpha_p$, where $d$ denotes the exterior derivative, $\star$ denotes the Hodge star operator, and $\epsilon = (-1)^{p(D-p)+p-1}$. The 'hook' operator $\dual$ denotes ‘contraction’. The scalar product of two vectors $a$ and $b$ is denoted by $a \cdot b = a\dual b$.

**Definition.** A PCKY tensor $h$ is a closed non-degenerate conformal Killing–Yano 2-form, $h = \frac{1}{2} h_{ab} dx^a \wedge dx^b$. It obeys
\[
\nabla_X h = X^b \wedge \xi^b, \tag{3}
\]
where $X$ is an arbitrary vector field.

The condition of non-degeneracy means that in a generic point of the manifold the skew symmetric matrix $h_{ab}$ has the (matrix) rank $2n$ and that the eigenvalues of $h$ are functionally independent in some spacetime domain (see [18] for more details). This means, that we exclude the possibility that $h$ possesses constant eigenvalues, and in particular, that it is covariantly constant; $\xi \neq 0$. (For the discussion of cases when such degeneracies are admitted see [29, 30].)

The equation (3) implies
\[
dh = 0, \quad \xi^b = -\frac{1}{D-1} \delta h. \tag{4}
\]

It can be shown ([18]), that for any spacetime admitting the PCKY tensor $h$, $\xi$ is a (primary) Killing vector. In tensor notations the definition (3) reads
\[
\nabla_c h_{ab} = 2g_{c[a} \xi_{b]} \quad \xi_b = \frac{1}{D-1} \nabla_d h^d_b. \tag{5}
\]

Let $\gamma$ be a null geodesic and $l^a = dx^a/d\tau$ be a tangent vector to it; $\tau$ denotes the affine parameter. We denote the covariant derivative of a tensor $T$ along $\gamma$ by
\[
\dot{T} = \nabla_\gamma T = l^a \nabla_a T. \tag{6}
\]
In particular, $\dot{l} = 0$. 

**II. BASIC NOTIONS AND NOTATIONS**

In what follows we use the notations of [15, 20]. We consider a $D$-dimensional spacetime $M^D$, equipped with the metric
\[
g = g_{ab} dx^a dx^b. \tag{1}
\]
To treat both cases of even and odd dimensions simultaneously we denote
\[
D = 2n + \epsilon, \tag{2}
\]
where $\epsilon = 0$ and $\epsilon = 1$ for even and odd number of dimensions, respectively. Operations $\dual$, $\flat$ correspond to ‘lowering’, ‘raising’ of indices of vectors, forms, respectively. \(\delta\) denotes the co-derivative. For a $p$-form $\alpha_p$, one has $\delta \alpha_p = \epsilon \cdot d \star \alpha_p$, where $d$ denotes the exterior derivative, $\star$ denotes the Hodge star operator, and $\epsilon = (-1)^{p(D-p)+p-1}$. The ‘hook’ operator $\dual$ denotes ‘contraction’. The scalar product of two vectors $a$ and $b$ is denoted by $a \cdot b = a\dual b$.

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\]
In particular, $\dot{l} = 0$. 

III. GENERIC CONSTRUCTION OF A PARALLEL-PROPAGATED FRAME

In this section we outline the general construction of parallel-transported frames along generic geodesics.

A. Construction of parallel-transported vectors \( m \) and \( n \)

Starting with \( l \) one can easily construct two additional parallel-propagated vectors. The construction is based on the following result: Let \( h \) be a PCKY tensor and \( u \) be a parallel-transported vector along a null geodesic \( l \), obeying \( u \cdot l = 0 \). Then the vector

\[
 w = (u \cdot h)^2 + \beta(u)l
\]

is parallel-transported along \( l \), provided that

\[
 \hat{\beta}(u) = u \cdot \xi .
\]

Here \( \xi \) is the primary Killing vector \( \xi^1 \).

To prove this statement we use Eq. (3) and the property of the hook operator. We find

\[
\dot{w} = (u \cdot h)^2 + \beta(u)l = [u \cdot (P \wedge \xi^a)]^2 + \beta(u)l
\]

\[
= \xi(u \cdot l) - l(u \cdot \xi) + \beta(u)l = l[\hat{\beta}(u) - u \cdot \xi] .
\]

Obviously, \( u = l \) obeys the requirements and we may construct \( [\beta = \beta(l)] \)

\[
m = \frac{1}{\sqrt{-\kappa_1}} [(l \cdot h)^2 + \beta l] .
\]

Here, \( \kappa_1 = Q_{ab} h^a h^b \) is a constant of geodesic motion corresponding to the conformal Killing tensor \( Q_{ab} = h_{ac} h^c_b \). Since \( \xi \) is a primary Killing vector, \( \beta = l \cdot \xi \) is also a constant and \( \beta \) can be immediately integrated to get

\[
\beta = (l \cdot \xi) \tau .
\]

We also find

\[
m \cdot l = 0 , \quad m \cdot m = 1 .
\]

So, \( m \) given by (11) and (10) is the normalized spacelike vector which is orthogonal to \( l \) and parallel-propagated along \( \gamma \).

Taking \( u = m \) in (7) we may construct another parallel-propagated vector

\[
n = \tilde{n} + \frac{1}{2}(\tilde{n} \cdot \tilde{n}) l ,
\]

where

\[
\tilde{n} = \frac{1}{\sqrt{-\kappa_1}} [(m \cdot h)^2 + \beta(m)l] , \quad \beta(m) = m \cdot \xi .
\]

We find

\[
n \cdot l = -1 , \quad n \cdot m = 0 , \quad n \cdot n = 0 .
\]

So, \( n \) is a null parallel-transported vector, external to \( l \) and orthogonal to \( m \). Moreover, we can easily show that it is independent of \( \beta(m) \). Indeed, using (13) we find that

\[
n = \frac{1}{\sqrt{-\kappa_1}} (m \cdot h)^2 + Cl , \quad C = \frac{1}{2\kappa_1} (Q_{ab}^{(2)} l^a - \kappa_1 \beta^2) , \quad Q_{ab}^{(2)} = Q_{ac} Q_b^c .
\]

One might wonder whether it is not possible to generate more parallel-propagated vectors in this way. Unfortunately, the fact that \( n \) is external, \( n \cdot l = -1 \), shows that \( n \) does not obey the requirements of the lemma anymore and we have to proceed differently.\(^1\)

B. Projection formalism: operator \( F \)

Let us consider the following 2-form \( F \):

\[
F_{ab} = P_a^c P_b^d h_{cd} ,
\]

where

\[
P_{ab} = g_{ab} + 2l(a n_b)
\]

is the projector to a \((D-2)\)-dimensional space \( V \), orthogonal to a 2-dimensional space \( U \) spanned by \( \{ l, n \} \). We have \( P_{ab} h^b = 0 \), \( P_{ab} h^b = 0 \).

Since vectors \( l \) and \( n \) are parallel-transported, so is \( P_{ab} \); \( P_{ab} = 0 \). Therefore we find

\[
\bar{F}_{ab} = P_a^c P_b^d h_{cd} = 2 P_a^c P_b^d [\xi, d] = 0 ,
\]

where we have used Eq. (5). So, the 2-form \( F \) is parallel-transported. This implies that the eigenvalues of \( F \) as well as its Darboux subspaces are independently parallel-transported (see [20] for more details).

The problem of finding remaining parallel-transported vectors along null geodesic \( \gamma \) is now

\(^1\) In fact, when \( n \) is used as a ‘seed’ in (11) one obtains a new linear independent vector. This vector may be used as a seed again and so on—to produce the whole tower of linear independent vectors. These vectors are, however, not parallel-transported.
quite analogous to the problem of parallel transport along timelike (spacelike) geodesics. By solving the eigenvalue problem for the operator $F^2$ one finds the eigenvectors spanning each of the Darboux subspaces of $F$. In each Darboux subspace, the parallel-transported vectors are obtained by a $\tau$-dependent orthogonal transformation of these eigenvectors.

The structure of the Darboux subspaces of $F$ depends crucially on the chosen geodesic $\gamma$. With increasing number of dimensions increases the number of degenerate cases (corresponding to special geodesics) which require their own special treatment. In what follows we concentrate on generic geodesics for which the discussion significantly simplifies.

C. Darboux subspace $V_0$

We denote the Darboux subspace corresponding to the zeroth eigenvalue of $F$ by $V_0$. For a generic geodesic $\gamma$, $V_0$ is 3-dimensional (4-dimensional) in the odd (even) number of spacetime dimensions. It is spanned by the parallel-transported vectors $\{l, n, m\}$ and, in the even-dimensional case, $z$ given by (20) below.

The fact that $l$ and $n$ are zero-value eigenvectors of $F$ is trivial. To prove that also $m$ belongs to $V_0$ we first notice that, $m$ being orthogonal to $\{l, n\}$, is unaffected by projector $P$, that is $P^b_m m^a = m^b$. Moreover, using (15) we realize that $s^a = h^a_b m^b$ is a linear combination of $l$ and $n$, $s = \alpha l + \beta n$, which when projected again by $P$ gives zero. So we have

$$F_{ad} m^d = P^b_{ad} h_{bc} P^c_d m^d = P^b_{ad} h_{bc} m^c = P^b_{ad} s^b = 0.$$  \hfill (19)

In an even number of spacetime dimensions we consider an additional vector $z$ given by

$$z = (l \perp f)^\perp,$$  \hfill (20)

where we have denoted

$$f = \ast h^\perp (n-1) = \ast (h \perp \ldots \perp h).$$  \hfill (21)

Tensor $f$ is a Killing–Yano 2-form (see, e.g., [31]) and therefore vector $z$ is automatically parallel-transported. Forms $f$ and $h$ are related as

$$h^a_c f^c_b = \delta^a_b.$$  \hfill (22)

It then follows that $h^a_b z^b \propto l^a$. Using this relation one can easily show that $z$ is spacelike parallel-transported vector, orthogonal to $\{l, n, m\}$,

$$z \cdot l = 0, \quad z \cdot m = 0, \quad z \cdot n = 0.$$  \hfill (23)

It can be normalized, so that, $z \cdot z = 1$. Moreover, it is the last zero-value eigenvector of $F$ spanning $V_0$. Indeed, we get

$$F^a_{\ b} z^b = P^a_d h^d_c P^c_{\ b} z^b = P^a_d h^d_c z^c \propto P^a_d 0 = 0.$$  \hfill (24)

To conclude, in the even-dimensional case, $V_0$ is spanned by explicitly constructed parallel-transported vectors $\{l, n, m, z\}$.

D. Parallel-transported vectors in remaining Darboux subspaces

We restrict ourselves by considering generic geodesics for which all the remaining Darboux subspaces of $F$ are two-dimensional. Denote by $V_1$ the Darboux subspace of $F$ corresponding to the non-zero eigenvalue $-\lambda^2_1$ of the operator $F^2$,

$$F^2 v = -\lambda^2_1 v, \quad v \in V_1,$$  \hfill (25)

and by $\{n_1, \tilde{n}_1\}$ the orthonormal basis in $V_1$. This basis is related to the parallel-propagated orthonormal basis $\{p_1, \tilde{p}_1\}$ spanning $V_1$ by a 2D rotation

$$p_1 = \cos \gamma_1 n_1 - \sin \gamma_1 \tilde{n}_1,$$  \hfill (26)

$$\tilde{p}_1 = \sin \gamma_1 n_1 + \cos \gamma_1 \tilde{n}_1,$$  \hfill (27)

$$\gamma_1 (\tau = 0) = 0.$$  \hfill (28)

The construction of parallel-propagated vectors in a different Darboux subspace is exactly analogous, independent of the other constructions.

IV. Kerr-NUT-(A)ds spacetimes and their properties

A. Canonical metric element and Kerr-NUT-(A)dS spacetimes

The most general canonical metric element admitting the PCKY tensor reads [18]

$$g = \sum_{\mu=1}^{n-1} (\omega^\perp \omega^\perp + \omega^\perp \omega^\perp) + \omega^\perp \omega^\perp - \omega^\perp \omega^\perp + \epsilon \omega^\perp \omega^\perp,$$  \hfill (29)

$$\omega^\perp = \sum_{\mu=1}^{n-1} (\omega^\perp) \omega^\perp + \omega^\perp \omega^\perp + \epsilon \omega^\perp \omega^\perp.$$  \hfill (30)
where the basis 1-forms are \((\mu = 1, \ldots, n - 1)\)

\[
\omega^{\hat{n}} = \frac{d\nu}{\sqrt{Q_\nu}}, \quad \omega^\mu = \frac{dx^\mu}{\sqrt{Q_\nu}},
\]

\[
\bar{\omega}^{\hat{n}} = \sqrt{Q_\nu} \sum_{j=0}^{n-1} A^{(j)}_\mu d\psi^j, \quad \bar{\omega}^\mu = \sqrt{Q_\nu} \sum_{j=0}^{n-1} A^{(j)}_\mu d\psi^j,
\]

\[
\omega^\nu = \sqrt{Q_\nu} \sum_{j=0}^n A^{(j)}_\nu d\psi_j, \quad Q_\nu = \frac{-c}{A^{(n)}}, \quad (28)
\]

We enumerate the basis \(\{\omega\}\) so that \(\bar{\omega}^\mu\) is (the only one) timelike 1-form. Here,

\[
A^{(j)}_\mu = \sum_{\nu_1 < \cdots < \nu_j} x_{\nu_1}^2 \cdots x_{\nu_j}^2, \quad A^{(j)}_\nu = \sum_{\nu_1 < \cdots < \nu_j} x_{\nu_1}^2 \cdots x_{\nu_j}^2,
\]

\[
Q_\nu = \frac{X_\mu}{U_\mu}, \quad U_\mu = \prod_{\nu=1}^n (x_{\nu}^2 - x_{\nu_j}^2), \quad x_n = -r^2, \quad (29)
\]

and \(X_\mu, X_\nu\) are arbitrary functions of \(x_{\mu}, r\), respectively. Time is denoted by \(\psi_0\), azimuthal coordinates by \(\psi_j, j = 1, \ldots, m = D - n - 1\), \(r\) is the Boyer-Lindquist type radial coordinate, and \(x_{\mu}, \mu = 1, \ldots, n - 1\), stand for latitude coordinates.

The inverse metric reads

\[
g^{-1} = \sum_{\mu=1}^{n-1} (\epsilon_\mu e_\mu + \tilde{\epsilon}_\mu \tilde{e}_\mu + e_\nu \tilde{e}_\nu + \epsilon_{\nu} e_\nu + \epsilon_\nu \tilde{e}_\nu), \quad (30)
\]

where

\[
e_\nu = \sqrt{Q_\nu} \partial_{\psi}, \quad e_\mu = \sqrt{Q_\nu} \partial_{x_{\mu}},
\]

\[
\tilde{e}_\nu = \frac{1}{\sqrt{X_\mu U_\mu}} \sum_{j=0}^m (x_{\nu}^2)^{n-1-j} \partial_{\psi_j},
\]

\[
\tilde{e}_\mu = \frac{1}{\sqrt{Q_\nu U_\mu}} \sum_{j=0}^m (x_{\nu}^2)^{n-1-j} \partial_{\psi_j},
\]

\[
e_\nu = \frac{-c}{\sqrt{-cA^{(n)}}}. \quad (31)
\]

The PCKY tensor for the canonical metric reads \(31, 12\)

\[
h = \sum_{\mu=1}^{n-1} x_{\mu} \omega^\mu \wedge \bar{\omega}^{\hat{n}} - r \omega^{\hat{n}} \wedge \bar{\omega}^\mu. \quad (32)
\]

Eqs. \(27\) and \(32\) mean that the basis \(\{\omega\}\) is an ‘orthogonal Darboux basis’ of \(h\). We call such a basis a canonical one. The canonical basis is fixed uniquely by the PCKY tensor up to 2D rotations in each of the (KY) 2-planes \(\omega^\mu \wedge \bar{\omega}^{\hat{n}}\), \(\omega^{\hat{n}} \wedge \bar{\omega}^\mu\). The basis \(\{\omega\}\) is a special canonical basis for which many of the Ricci coefficients of rotation vanish \(32, 18\). We call it a principal canonical basis.

The PCKY tensor \(h\) generates the whole towers of explicit and hidden symmetries \(13\). Namely, it generates all the isometries \(\partial_{\psi_0}\), and in particular, the primary Killing vector \(\xi, 14\),

\[
\xi = -\frac{1}{D - 1} (\delta h)^{\hat{\mu}} = \partial_{\psi_0}. \quad (33)
\]

It also generates the set of the second-rank irreducible Killing tensors \((j = 1, \ldots, m)\)

\[
K^{(j)} = \sum_{\mu=1}^{n-1} A^{(j)}_\mu (\omega^\mu \omega^{\hat{n}} + \bar{\omega}^{\hat{n}} \bar{\omega}^\mu)
\]

\[
+ A^{(j)}_n (\omega^\nu \omega^{\hat{n}} - \bar{\omega}^{\hat{n}} \bar{\omega}^\nu) + \epsilon A^{(j)} \omega^\nu \bar{\omega}^\nu.
\]

These objects are responsible for complete integrability of geodesic motion in the canonical space-time.

When the vacuum Einstein equations with the cosmological constants are imposed,

\[
R_{ab} = (-1)^n (D - 1)c_n g_{ab}, \quad (35)
\]

metric functions \(X_\mu(x_{\nu})\) and \(X_\nu(r)\) take the following specific form \(32\):

\[
X_\nu = -\sum_{k=1}^n c_k (r^2)^k - 2Mr_1^{1-\epsilon} + \frac{\epsilon c}{r^2},
\]

\[
X_\mu = \sum_{k=1}^n c_k x_{\mu}^{2k} - 2b_{\mu} x_1^{1-\epsilon} + \frac{\epsilon c}{x_{\mu}}, \quad (36)
\]

and the canonical element becomes the general Kerr-NUT-(A)dS space-time derived by Chen, Liu, and Pope \(19\). The parameter \(c_n\) is proportional to the cosmological constant and the remaining constants \(c_k, c > 0\), and \(b_{\mu}\) are related to rotation parameters, mass, and NUT parameters.

B. Geodesics

As we mentioned above, the geodesic motion in the canonical background \(27 - 29\) is completely integrable \(26, 27, 28\). In particular, the null geodesic velocity takes the following form \(12, 13\):

\[
\tilde{l}^\nu = \sum_{\mu=1}^n (l_\mu \omega^\mu + \bar{l}_\mu \bar{\omega}^\mu) + \epsilon l_\nu \omega^\nu, \quad (37)
\]

where

\[
l_\nu = \frac{\sigma_n}{(X_n U_\nu)^1/2} (W_n^2 - X_n V_n)^{1/2},
\]

\[
l_\mu = \frac{\sigma_n}{(X_\mu U_\nu)^1/2} (X_\mu V_\mu - W_\mu^2)^{1/2},
\]

\[
\bar{l}_\nu = \frac{W_n}{(X_n U_\nu)^1/2}, \quad \bar{l}_\mu = \frac{1}{\sqrt{Q_\nu U_\mu}} W_\mu, \quad (38)
\]

\[
l_\nu = \frac{\psi_0}{\sqrt{-cA^{(n)}}}.
\]
Here, the constants $\sigma_\mu = \pm 1$ ($\mu = 1, \ldots, n$) are independent of one another and we have defined

$$V_n = -\sum_{j=1}^{n} r^{2(n-1-j)}k_j, \quad \psi_j = \sum_{j=1}^{n} (-x_\mu^2)^{n-1-j}k_j, \quad W_n = \sum_{j=0}^{n} r^{2(n-1-j)}\psi_j, \quad W_\mu = \sum_{j=0}^{n} (-x_\mu^2)^{n-1-j}\psi_j.$$  

The quantities $\psi_j$ and $\kappa_j$ are conserved and connected with the Killing vectors and the Killing tensors, respectively. We also have

$$\kappa_n = -\frac{\psi_n^2}{c}. \quad (40)$$

The coordinate components of the velocity are

$$\dot{r} = \frac{\sigma_n}{U_n} \left( W_n^2 - X_n V_n \right)^{1/2},$$
$$\dot{x}_\mu = \frac{\sigma_\mu}{|U_\mu|} \left( X_\mu V_\mu - W_\mu^2 \right)^{1/2},$$
$$\dot{\psi}_k = \frac{\sum_{\mu=1}^{n-1} (-x_\mu^2)^{n-1-k}}{U_\mu X_\mu} W_n - \frac{\sigma_n}{U_n} X_n V_n - \frac{\sigma_\mu}{U_\mu} X_\mu V_\mu - W_\mu.$$  

(41)

One can symbolically integrate equations for $\psi_k$. Let $f$ be an arbitrary function of $r$ and $x_\mu$’s, obeying

$$\dot{f} = \frac{f_n(r)}{U_n} + \sum_{\nu=1}^{n-1} f_\nu(x_\nu). \quad (42)$$

Then $f$ can be written as (see Appendix C in [20])

$$f = \int \frac{\sigma_n f_n dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\nu=1}^{n-1} \int \frac{\sigma_\nu \text{sign}(U_\nu) f_\nu dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}}.$$  

(43)

In particular, using the following identities (see, e.g., [33]):

$$1 - \frac{1}{A(n)} = -\frac{1}{r^2 U_n} + \frac{1}{U_n} \sum_{\nu=1}^{n-1} \frac{1}{x_\nu^2 U_\nu}, \quad 1 = \frac{r^{2(n-1)}}{U_n} + \frac{1}{U_n} \sum_{\mu=1}^{n-1} \frac{(-x_\mu^2)^{n-1}}{U_\mu}, \quad (44)$$

we find that

$$\psi_k = \int \frac{\sigma_n f_n^{(k)} dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\mu=1}^{n-1} \int \frac{\sigma_\mu \text{sign}(U_\mu) f_\mu^{(k)} dx_\mu}{\sqrt{X_\mu V_\mu - W_\mu^2}},$$
$$f_n^{(k)} = \frac{W_n}{X_n} r^{2(n-1-k)} + \frac{\psi_n}{c^2} \delta_{kn},$$
$$f_\mu^{(k)} = \frac{W_\mu}{X_\mu} (-x_\mu^2)^{n-1-k} - \frac{\psi_\mu}{c^2} \delta_{kn}. \quad (45)$$

Similarly, we have

$$\tau = \int \frac{\sigma_n r^{2(n-1)} dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\nu=1}^{n-1} \int \frac{\sigma_\nu \text{sign}(U_\nu) (-x_\nu^2)^{n-1} dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}}. \quad (46)$$

V. PARALLEL TRANSPORT IN KERR-NUT-(A)DS SPACETIMES

A. Parallel-propagated frame

We shall construct the parallel-propagated frame for a geodesic motion in four steps. First, to simplify the calculations, we use the freedom of local 2D rotations in the KY 2-planes of $h$ to introduce the velocity adapted canonical basis in which $n$ components of the velocity vanish. Next, we generate parallel-transported vectors $m, n, \nu$, and possibly $\omega$. In the third step, by studying the eigenvalue problem for the operator $\mathcal{F}^2$, we find the orthonormal 1-forms $\lbrace \xi^1, \xi^2 \rbrace$ spanning each of the 2D Darboux subspaces $V_c$. Finally, in each $V_i$, we rotate these 1-forms by an (affine-parameter)-dependent rotation to obtain the (dual) parallel-transported frame.$^2$

1. Velocity adapted canonical basis

In order to construct the velocity adapted canonical basis, we perform the boost transformation in the $\lbrace \tilde{\omega}^\alpha, \tilde{\omega}^\beta \rbrace$ 2-plane and the rotation transformations in each of the $\lbrace \tilde{\omega}^\alpha, \tilde{\omega}^\beta \rbrace$ 2-planes

$$\tilde{\omega}^\alpha = \cosh \alpha_n \tilde{\omega}^\alpha + \sinh \alpha_n \tilde{\omega}^\beta, \quad \tilde{\omega}^\beta = \sinh \alpha_n \tilde{\omega}^\alpha + \cosh \alpha_n \tilde{\omega}^\beta,$$
$$\omega^\alpha = \cos \alpha_n \tilde{\omega}^\alpha + \sin \alpha_n \tilde{\omega}^\beta, \quad \omega^\beta = -\sin \alpha_n \tilde{\omega}^\alpha + \cos \alpha_n \tilde{\omega}^\beta.$$  

(47)

Here, we choose

$$\cosh \alpha_n = \frac{\ell_n}{k_n}, \quad \sinh \alpha_n = \frac{\ell_n}{k_n},$$
$$\cos \alpha_n = \frac{\ell_n}{k_n}, \quad \sin \alpha_n = \frac{\ell_n}{k_n}. \quad (48)$$

$^2$ In our setup it is somewhat more natural to work with 1-forms. One could, of course, similarly construct the parallel-transported frame of vectors.
and
\[ k_n = -\sqrt{r_n^2 - l_n^2} = -\sqrt{V_n}, \]
\[ k_\mu = \sqrt{r_\mu^2 + l_\mu^2} = \sqrt{V_\mu}. \] (49)

Such a transformation preserves the form of the metric as well as the form of the PCKY tensor.
\[ g = \sum_{\mu=1}^{n-1}(o^\mu o^{\bar{\mu}} + \tilde{o}^\mu \tilde{o}^{\bar{\mu}}) + o^\mu \tilde{o}^{\bar{n}} - \tilde{o}^\mu \tilde{o}^{\bar{n}} + \varepsilon o^\mu \tilde{o}^\nu, \]
\[ h = \sum_{\mu=1}^{n-1} x_\mu o^{\bar{\mu}} \wedge \tilde{o}^{\bar{n}} - r o^{\bar{\mu}} \wedge \tilde{o}^{\bar{n}}. \] (50)

Hence, the basis \{o\} is still canonical. Moreover, one obtains the following form of the velocity:
\[ v = \sum_{\mu=1}^{n} \tilde{k}_\mu \tilde{o}^{\bar{\mu}} + \varepsilon l_{\bar{k}} \tilde{o}^{\bar{\nu}}. \] (51)

This form simplifies considerably the subsequent calculations, especially the task of solving the eigenvalue problem for \( F^2 \). We remark, that in the adapted basis \{o\} the components of the velocity depend on constants \( \kappa_j \) only; the constants \( \Psi_j \) and \( \sigma_j \) are absorbed in the definition of the new frame.

2. Parallel-transported vectors in \( V_0 \)

The eigenspace \( V_0 \) is spread by \( l \), the vectors \( m \) and \( n \) given by (40) and (12), and, in an even number of spacetime dimensions, by \( z \) (20). Let us express these vectors in the velocity adapted basis (17). The vector \( l \) is given by (51) and (49). Using (51) and (49), we find
\[ m^b = \frac{1}{\sqrt{-\kappa_1}} \left[ \sum_{\mu=1}^{n-1} (\tilde{k}_\mu \beta \tilde{o}^{\bar{\mu}} - \tilde{k}_\mu x_\mu o^{\bar{\mu}}) \right. \]
\[ + \tilde{k}_\mu \beta \tilde{o}^{\bar{\nu}} - \tilde{k}_\mu r o^{\bar{\nu}} + \varepsilon \beta l_{\bar{k}} o^{\bar{\nu}}, \] (52)
\[ n^b = \sum_{\mu=1}^{n-1} \left[ \tilde{k}_\mu (C + \frac{x_\mu^2}{\kappa_1}) \tilde{o}^{\bar{\mu}} + \tilde{k}_\mu \beta x_\mu o^{\bar{\mu}} \right] \]
\[ + \tilde{k}_\mu (C - \frac{x_\mu^2}{\kappa_1}) \tilde{o}^{\bar{\nu}} + \tilde{k}_\mu \beta r_\mu o^{\bar{\nu}} + \varepsilon C l_{\bar{k}} o^{\bar{\nu}}, \] (53)
where
\[ C = \frac{1}{2\kappa_1} (-r^2 \tilde{k}_n^2 + \sum_{\mu=1}^{n-1} x_\mu^2 \tilde{k}_\mu^2 - \kappa_1 \beta^2). \] (54)

Moreover, using Eq. (53) we find
\[ \beta = l \cdot \xi = l \cdot \partial_{\psi_0} = \Psi_0, \]
and so
\[ \beta = \Psi_0 r. \] (55)

Using Eq. (10) one can express this angle as a function of \( r \) and \( x_\mu \)'s.

In an even number of spacetime dimensions we have an additional vector \( z \). We find
\[ f \propto x_1 \ldots x_{n-1} o^{\bar{\mu}} \wedge \tilde{o}^{\bar{n}} + \sum_{\mu=1}^{n-1} x_1 \ldots \hat{x}_\mu \ldots x_{n-1} o^{\bar{\mu}} \wedge \tilde{o}^{\bar{n}}. \] (56)

Here, the symbol \( \propto \) means equality up to a constant factor, and \( \hat{x}_\mu \) denotes that in the sum over \( \mu, x_\mu \) is replaced by \( r \). In consequence, we have the following expression for the (normalized) vector \( z \):
\[ z^b = \frac{1}{\sqrt{-\kappa_{n-1}}} \left( x_1 \ldots x_{n-1} \tilde{k}_n o^{\bar{n}} \right. \]
\[ - \sum_{\mu=1}^{n-1} x_1 \ldots \hat{x}_\mu \ldots x_{n-1} \tilde{k}_\mu o^{\bar{\mu}} \). \] (57)

Using the transformation inverse to (17),
\[ \tilde{\omega}^{\bar{\mu}} = \cosh \alpha_n \tilde{o}^{\bar{\mu}} - \sinh \alpha_n o^{\bar{\mu}}, \]
\[ \omega^{\bar{\nu}} = -\sinh \alpha_n \tilde{o}^{\bar{\nu}} + \cosh \alpha_n o^{\bar{\nu}}, \]
\[ \tilde{\omega}^{\bar{\mu}} = \cos \alpha_n \tilde{o}^{\bar{\mu}} - \sin \alpha_n o^{\bar{\mu}}, \]
\[ \omega^{\bar{\nu}} = \sin \alpha_n \tilde{o}^{\bar{\nu}} + \cos \alpha_n o^{\bar{\nu}}, \]
\[ \omega^{\bar{\nu}} = o^{\bar{\nu}}, \]
one can easily obtain the above parallel-transported vectors in the principal basis \{\omega\}.

3. Darboux subspaces \( V_i \)

Using (51) and (53) one can write down \( F \) in the velocity adapted basis. The general expression is quite involved and therefore we do not state it here. What is important is that one can show that \( F \) is independent of \( \beta \). Let us denote \{\xi\} the (dual) Darboux basis of \( F \). Then, for a generic geodesic, we have
\[ F = \sum_{i=1}^{n-2+\varepsilon} \lambda_i \xi^{\bar{i}} \wedge \xi^{\bar{j}}. \] (59)

Here, \{\xi^{\bar{i}}, \xi^{\bar{j}}\} are orthonormal vectors spanning the Darboux subspace \( V_i \), \( \lambda_i > 0 \) are all different and correspond to the eigenvalues of \( F^2 \).
\( F^2 v_i = -\lambda_i^2 v_i, \ v_i \in V_i. \) Obtaining the general form of \( \{ \xi, \zeta \} \) is the biggest obstacle in writing down a general formula for the parallel-propagated basis in an arbitrary number of dimensions. Concrete examples are in the next section.

4. Parallel-transported basis

In order to construct parallel-transported vectors in each of the Darboux subspaces \( V_i \) we perform the rotation

\[
\pi^i = \cos \gamma_i \xi^i - \sin \gamma_i \zeta^i, \\
\hat{\pi}^i = \sin \gamma_i \xi^i + \cos \gamma_i \zeta^i,
\]

with the initial conditions \( \gamma_i(\tau = 0) = 0 \).

When \( \hat{\gamma}_i \) given by the last equation can be written in the form

\[
\hat{\gamma}_i = M_i \left( (r^2 + \lambda_i^2) \prod_{\mu=1}^{n-1} (x_{\mu}^2 - \lambda_i^2) \right)^{1/2},
\]

where \( M_i \) is some constant, we can use the identity

\[
\left( r^2 + \lambda_i^2 \right) \prod_{\mu=1}^{n-1} \left( x_{\mu}^2 - \lambda_i^2 \right)^{-1} = \frac{1}{r^2 + \lambda_i^2} U_n - \sum_{\mu=1}^{n-1} \frac{1}{(x_{\mu}^2 - \lambda_i^2) U_{\mu}},
\]

the Carter’s class of solutions [34, 33]—describing among others a 4D rotating charged black hole in the cosmological background (see also Appendix C).

The metric reads

\[
g = -\bar{\omega}^2 \omega^2 + \omega^2 \omega^2 + \bar{\omega}^1 \omega^1 + \omega^1 \omega^1,
\]

where

\[
\bar{\omega}^2 = \sqrt{\frac{X_2}{U_2}} (d\psi_0 + x_1^2 d\psi_1), \ \omega^2 = \sqrt{\frac{U_2}{X_2}} dr,
\]

\[
\bar{\omega}^1 = \sqrt{\frac{X_1}{U_1}} (d\psi_0 - r^2 d\psi_1), \ \omega^1 = \sqrt{\frac{U_1}{X_1}} dx_1,
\]

and \( U_2 = -U_1 = x_1^2 + r^2 \). The PCKY tensor is

\[
h = x_1^2 \omega^1 \wedge \bar{\omega}^1 - r^2 \omega^2 \wedge \bar{\omega}^2.
\]

The components of the velocity are

\[
\begin{align*}
\bar{l}_2 &= \frac{W_2}{\sqrt{X_2 U_2}}, \ l_2 = \frac{\sigma_2}{\sqrt{X_2 U_2}} \sqrt{W_2^2 - X_2 V_2}, \\
\bar{l}_1 &= \frac{W_1}{\sqrt{X_1 U_1}}, \ l_1 = \frac{\sigma_1}{\sqrt{X_1 U_1}} \sqrt{X_1 V_1 - W_1^2},
\end{align*}
\]

where

\[
W_2 = r^2 \Psi_0 + \Psi_1, \ V_2 = -\kappa_1 > 0, \\
W_1 = -x_1^2 \Psi_0 + \Psi_1, \ V_1 = \kappa_1.
\]

In the velocity adapted frame \( \{ o \} \), the parallel-transported frame reads

\[
\begin{align*}
\nu^o &= \bar{k}_1 (-\delta^0 + \bar{\sigma}^1), \ \hat{k}_1 = -\bar{k}_2 = \sqrt{\frac{V_2}{U_2}} = \sqrt{-\kappa_1 U_2}, \\
\nu^o &= \bar{k}_1 \sqrt{\frac{\bar{k}_1}{-\kappa_1}} (-\beta \delta^2 + r \bar{\sigma}^2 + \bar{\beta} \bar{\sigma}^1 - x_1 o^1), \\
\beta &= \Psi_0 \tau, \ \text{or, in terms of } r \text{ and } x_1,
\end{align*}
\]

where \( \beta = \Psi_0 \tau \), or, in terms of \( r \) and \( x_1 \),

\[
\beta = \frac{\sigma_2 \Psi_0 r^2 dr}{\sqrt{W_2^2 - X_2 V_2}} + \frac{\sigma_1 \Psi_0 x_1 dx_1}{\sqrt{X_1 V_1 - W_1^2}}.
\]

VI. SPECIAL CASES

A. Parallel transport in 4D

The parallel transport along generic geodesics in the 4-dimensional canonical spacetime, derived earlier in [22, 23], is the point of view of the above described theory trivial. We write it only for completeness and because it encapsulates the important sub-case of parallel transport in the 5D canonical spacetime.

B. Parallel transport in 5D

Next, we consider the 5D canonical spacetime. The metric reads

\[
g = -\bar{\omega}^2 \omega^2 + \omega^2 \omega^2 + \bar{\omega}^1 \omega^1 + \omega^1 \omega^1 + \omega^0 \omega^0,
\]
The 2-form \( \omega^2 = \sqrt{X_2 / U_2} (d\psi_0 + x_1^2 d\psi_1) \), \( \omega^3 = \sqrt{U_2 / X_2} dr \),
\( \omega^1 = \sqrt{X_1 / U_1} (d\psi_0 - r^2 d\psi_1) \), \( \omega^4 = \sqrt{U_1 / X_1} dx_1 \),
\[ \omega^2 = \frac{\sqrt{c}}{rx_1} \left[ d\psi_0 + (x_1^2 - r^2) d\psi_1 - x_1^3 r^2 d\psi_2 \right], \]
and \( U_2 = -U_1 = x_1^2 + r^2 \). The PCKY tensor is

\[ \text{h} = x_1 \omega^1 \wedge \omega^4 - r \omega^2 \wedge \omega^3. \]

The components of the velocity are

\[
\begin{align*}
\dot{i}_2 &= \frac{W_2}{\sqrt{X_2} U_2}, \quad l_2 = \frac{\sigma_2}{\sqrt{X_2} U_2} \sqrt{W_2^2 - X_2 V_2}, \\
\dot{i}_1 &= \frac{-W_1}{\sqrt{X_1} U_1}, \quad l_1 = \frac{\sigma_1}{\sqrt{X_1} U_1} \sqrt{X_1 V_1 - W_1^2}, \\
\end{align*}
\]

where

\[
\begin{align*}
W_1 &= -x_1^2 \Psi_0 + \Psi_1 - \frac{\Psi_2}{x_1}, \quad V_1 = \kappa_1 + \frac{\Psi_2}{cx_1^2}, \\
W_2 &= r^2 \Psi_0 + \Psi_1 + \frac{\Psi_2}{r^2}, \quad V_2 = -\kappa_1 + \frac{\Psi_2}{cr^2}.
\end{align*}
\]

In the velocity adapted frame \( \{ o \} \), (17), we have

\[
\begin{align*}
t^i &= \tilde{k}_2 \hat{o}^3 + \tilde{k}_1 \hat{o}^1 + l_o o^i, \\
m^i &= \frac{1}{\sqrt{-\kappa_1}} (x_1^2 \tilde{k}_2 \hat{o}^2 - r^2 \tilde{k}_2 \hat{o}^3 + \beta \tilde{k}_1 \hat{o}^1 - x_1 \tilde{k}_1 o^1 + \beta x_1 o^i), \\
n^i &= \tilde{k}_2 \left( C - \frac{r^2}{\kappa_1} \right) \hat{o}^2 + \tilde{k}_1 \frac{\beta \kappa_1}{r^2} \hat{o}^3 + \tilde{k}_1 \left( C + \frac{r^2}{\kappa_1} \right) \hat{o}^1 \\
&\quad + \tilde{k}_1 \frac{\beta \kappa_1}{r^2} o^1 + C l_o o^i,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{k}_2 &= -\frac{\sqrt{2}}{\sqrt{U_2}}, \quad \tilde{k}_1 = \sqrt{\frac{\sqrt{U_1}}{U_1}}, \\
C &= \frac{1}{2\kappa_1} \left( -r^4 \tilde{k}_2^2 + x_1^4 \tilde{k}_2^2 - \kappa_1 \beta^2 \right), \\
\beta &= \int \frac{\sigma_2 \Psi_0 x_1^2 dx_1}{\sqrt{W_2^2 - X_2 V_2}} + \int \frac{\sigma_1 \Psi_0 x_1^2 dx_1}{\sqrt{X_1 V_1 - W_1^2}}.
\end{align*}
\]

The 2-form \( F \) reads

\[ F = \lambda \xi \wedge \zeta, \quad \lambda = \frac{\Psi_2}{\sqrt{-c \kappa_1}}. \]

where

\[
\begin{align*}
\xi &= \sqrt{(r^2 + \lambda^2)(x_1^2 - \lambda^2)} \left( -\frac{x_1 F_1}{\sqrt{U_2}} \hat{o}^2 + \frac{r F_2}{\sqrt{U_2}} \hat{o}^3 - \hat{o}^1 + o^i \right), \\
\zeta &= \sqrt{r^2 + \lambda^2} \left( -\frac{\lambda^2}{r^2 + \lambda^2} \frac{1}{r^2} F_1 F_2 \hat{o}^2 + \hat{o}^3 \right),
\end{align*}
\]

Here we have introduced

\[ F_r = \frac{x_1 l_t}{\sqrt{2}}, \quad F_{x_1} = -\frac{r l_t}{\sqrt{-V_1}}, \]

which are functions of \( r, x_1 \), respectively. Using (60) we find

\[ \gamma = \frac{2}{(x_1^2 - \lambda^2)(r^2 + \lambda^2)}, \quad M = \frac{\lambda^2 \Psi_2 \Psi_0 - \Psi_2 \Psi_1 - c \kappa_1}{\sqrt{-c \kappa_1}}, \]

which is of the form (61). Therefore, the parallel-propagated forms \( \{ \pi, \hat{\pi} \} \) are given by (60), where

\[ \gamma = \int \frac{\sigma_2 \gamma dr}{\sqrt{W_2^2 - X_2 V_2}} + \int \frac{\sigma_1 \gamma_{x_1} dx_1}{\sqrt{X_1 V_1 - W_1^2}}, \]

\[ \gamma r = \frac{M}{r^2 + \lambda^2}, \quad \gamma_{x_1} = \frac{M}{x_1^2 - \lambda^2}. \]

C. Parallel transport in 6D

Finally we consider the 6D canonical spacetime. The metric reads

\[ g = -\tilde{\omega}_2 \tilde{\omega}_3 + \omega^3 \omega^1 + \sum_{\mu=1}^{2} \left( \omega^2 \omega^\mu + \tilde{\omega}_2 \tilde{\omega}^\mu \right), \]

\[ \omega^3 = \sqrt{U_3 \frac{X_3}{U_3}} dr, \quad \omega^1 = \sqrt{U_1 \frac{X_1}{U_1}} dx_1, \]

\[ \tilde{\omega}_2 = \sqrt{X_2 \frac{U_2}{X_2}} d\psi_0, \quad \tilde{\omega}_3 = \sqrt{X_1 \frac{U_1}{X_1}} d\psi_1 + x_2^2 x_3^2 d\psi_2, \]

\[ \tilde{\omega}_1 = \sqrt{X_1 \frac{U_1}{X_1}} d\psi_0 + A_x d\psi_1 - x_3^2 r^2 d\psi_2, \]

\[ \tilde{\omega}_1 = \sqrt{X_1 \frac{U_1}{X_1}} d\psi_0 + A_x d\psi_1 - x_2^2 r^2 d\psi_2. \]

Here

\[ A_r = x_2^2 + x_3^2, \quad A_{x_1} = x_2^2 - r^2, \quad A_{x_2} = x_1^2 - r^2, \]

\[ U_3 = (x_1^2 + r^2)(x_1^2 + r^2), \quad U_2 = -x_2^2 r^2 (x_1^2 - x_2^2), \]

\[ U_1 = (x_1^2 + r^2)(x_1^2 - x_2^2). \]

The PCKY tensor is

\[ \text{h} = x_1 \omega^1 \wedge \tilde{\omega}_3 + x_2 \omega^2 \wedge \tilde{\omega}_3 - r \omega^3 \wedge \tilde{\omega}_3. \]

The components of the velocity are

\[
\begin{align*}
\tilde{l}_3 &= \frac{W_3}{\sqrt{X_3 U_3}}, \quad l_3 = \frac{\sigma_3}{\sqrt{X_3 U_3}} \sqrt{W_3^2 - X_3 V_3}, \\
\tilde{l}_2 &= -\frac{W_2}{\sqrt{X_2 U_2}}, \quad l_2 = \frac{\sigma_2}{\sqrt{X_2 U_2}} \sqrt{X_2 V_2 - W_2^2}, \\
\tilde{l}_1 &= \frac{W_1}{\sqrt{X_1 U_1}}, \quad l_1 = \frac{\sigma_1}{\sqrt{X_1 U_1}} \sqrt{X_1 V_1 - W_1^2},
\end{align*}
\]
The parallel-transported vectors spanning \( V_0 \) are
\[
\begin{align*}
\mathbf{l}^\flat &= \frac{1}{\sqrt{-\kappa_1}} \left( \beta \hat{k}_3 \hat{o}^3 + \hat{k}_2 \hat{o}^2 + \hat{k}_1 \hat{o}^1 \right), \\
\mathbf{m}^\flat &= \frac{1}{\sqrt{-\kappa_1}} \left( \beta \hat{k}_3 \hat{o}^3 + \hat{k}_2 \hat{o}^2 - x_2 \hat{k}_2 \hat{o}^1 + 3 \hat{k}_1 \hat{o}^1 \right), \\
\mathbf{n}^\flat &= \beta \hat{k}_3 \left( C - \frac{r^2}{\kappa_1} \right) \hat{o}^3 + \beta \hat{k}_3 \beta \hat{k}_2 \hat{o}^2 + \beta \hat{k}_2 \hat{o}^2 + \beta \hat{k}_1 \hat{o}^1, \\
\mathbf{z}^\flat &= \frac{1}{\sqrt{-\kappa_2}} \left( x_1 x_2 \hat{k}_3 \hat{o}^3 - x_1 \hat{k}_2 \hat{o}^2 - x_2 \hat{k}_1 \hat{o}^1 \right),
\end{align*}
\]
where
\[
\begin{align*}
\hat{k}_3 &= \frac{V_3}{U_3}, \quad \hat{k}_2 = \sqrt{\frac{V_2}{U_2}}, \quad \hat{k}_1 = \frac{1}{\sqrt{V_1}}, \\
C &= \frac{1}{2 \kappa_1} \left( -\frac{r^2}{\kappa_3} k_3^3 + x_2 k_2^2 + x_2 k_1^2 - \frac{1}{\kappa_1} \beta^2 \right), \\
\beta &= \int \frac{\Psi_0 \sigma r^3 dr}{\sqrt{W_3^2 - X_3 V_3}} - \int \frac{\Psi_0 \sigma x_2^2 dx_2}{\sqrt{X_2 V_2 - W_2}} + \int \frac{\Psi_0 \sigma x_2 dx_1}{\sqrt{X_1 V_1 - W_1}}.
\end{align*}
\]

The 2-form \( F \) reads
\[
F = \lambda \varsigma \wedge \xi, \quad \lambda = \frac{\sqrt{\kappa_2}}{\sqrt{\kappa_1}},
\]
where
\[
\begin{align*}
\varsigma &= \sqrt{\frac{(r^2 + \lambda^2)(\lambda^2 - x_2^2)}{U_1}} \left( F_1 \hat{o}^3 + F_2 \hat{o}^2 + \hat{o}^1 \right), \\
\xi &= \frac{1}{\lambda} \sqrt{\frac{(r^2 + \lambda^2)(\lambda^2 - x_2^2)}{U_1}} (r F_1 \hat{o}^3 + x_2 F_2 \hat{o}^2 + x_1 \hat{o}^1).
\end{align*}
\]
Here we have introduced
\[
F_1 = \frac{U_1 k_1 k_3}{\kappa_1 (r^2 + \lambda^2)}, \quad F_2 = \frac{U_1 k_2 k_1}{\kappa_1 (\lambda^2 - x_2^2)}.
\]
Using Eq. (90) we find
\[
\gamma = -\lambda \left( \Psi_2 - \lambda^2 \Psi_1 + \lambda^4 \Psi_0 \right) \left( x_1^2 - \lambda^2 \right) (x_2^2 - \lambda^2) (r^2 + \lambda^2).
\]

This means that also in 6D the angle \( \gamma \) can be symbolically integrated—it is given by (93)—and the parallel-transported forms \( \{ \pi, \tilde{\pi} \} \), (94), explicitly constructed.
coordinate components of $l_\pm$, which lead to the equations for geodesics
\begin{align}
\dot{r} &= \pm 1, \quad x_\mu = 0, \quad \dot{\psi}_j = \frac{\psi^{2(n-1-j)}}{X_n},
\end{align}
where $\mu = 1, \ldots, n-1$ and $j = 0, \ldots, m$. These can be integrated to get
\begin{align}
r &= \pm r, \quad \dot{\psi}_j = \pm \int_0^{\pm r} \frac{r^{2(n-1-j)}}{X_n} dr + \psi^{(0)}_j.
\end{align}

**B. Parallel transport**

Now, we turn to the task of parallel transport along the principal null direction $l_+$.\(^3\) The parallel-propagated frame can be obtained by a sequence of Lorentz transformations of the canonical basis $\{l, n, e_\mu, e_\mu, e_\xi\}$,
\begin{align}
l &= \frac{1}{\sqrt{Q_n}} l, \\
n &= \frac{1}{\sqrt{Q_n}} n + \sum_{\mu=1}^{n-1} \sqrt{2} \sqrt{Q_n} e_\mu + \varepsilon \sqrt{2} \sqrt{Q_n} e_\xi \\
&+ \left( \sum_{\mu=1}^{n-1} Q_{\mu} + \varepsilon Q_\xi \right) \frac{1}{\sqrt{Q_n}} l, \\
e_\mu &= \frac{x_\mu}{\sqrt{x_\mu^2 + r^2}} e_\mu - \frac{r}{\sqrt{x_\mu^2 + r^2}} \left( \tilde{e}_\mu + \sqrt{2} \sqrt{Q_n} l \right), \\
\tilde{e}_\mu &= \frac{r}{\sqrt{x_\mu^2 + r^2}} e_\mu + \frac{x_\mu}{\sqrt{x_\mu^2 + r^2}} \left( \tilde{e}_\mu + \sqrt{2} \sqrt{Q_n} l \right), \\
e_\xi &= e_\xi + \sqrt{2} \sqrt{Q_n} l.
\end{align}

Similar to 4D, it may be convenient to find a parallel-propagated complex null frame. This is done in Appendix B.

\footnote{The parallel transport along $l_-$ is analogous.}

**VIII. CONCLUSIONS**

In this paper we have studied the equations describing the parallel-transport along null geodesics in spacetimes which possess a (non-degenerate) principal conformal Killing-Yano (PCKY) tensor. When the vacuum Einstein equations with the cosmological constant are imposed, this class of metrics coincides with the Kerr-NUT-(A)dS spacetimes, describing the higher-dimensional rotating black holes with NUT parameters, in an asymptotically flat or (A)dS background. In particular, a solution of this problem gives effective tools for studying the polarization of light beams in backgrounds of considered black hole metrics.

A tangent vector to the null ray, which is evidently parallel-propagated, determines a $(D-1)$-dimensional null plane to which it is orthogonal. Our main observation is, that using the PCKY tensor one can obtain a parallel-propagated along the null geodesic vector which does not belong to this null plane. We used these two parallel-propagated vectors to construct a projection operator on a $(D-2)$-dimensional subspace. By using the eigenvectors of the PCKY tensor projected to this subspace we found two-dimensional Darboux planes invariant under the parallel-transport, and by proper rotations in these planes we constructed the required parallel-propagated basis. Though the idea of this construction is rather simple, concrete calculations in higher-dimensional spacetimes are quite involved. We performed them concretely in spacetimes with $D \leq 6$. In each of these cases the final first order ordinary differential equations specifying rotations in the 2D Darboux planes were solved by the separation of variables. We expect that this remains true for any number of dimensions.

The class of the Kerr-NUT-(A)dS spacetimes we considered in this paper belongs to the algebraic type D. These spacetimes possess two special congruences of null geodesics called the principal null directions. Tangent vectors to these geodesics are ‘eigenvectors’ of the Weyl tensor. At the same time they are null eigenvectors of the PCKY tensor. This property implies that for this subclass of null geodesics the problem of the parallel-transport becomes degenerate and requires special consideration. We have studied this degenerate case and directly solved the corresponding equations of parallel transport. This result might be useful for studying the peeling-off property in the Kerr-NUT-(A)dS spacetimes.
APPENDIX A: GEOMETRIC OPTICS IN HIGHER-DIMENSIONAL SPACETIMES

It is well known that if the wave length of massless field radiation is much smaller than a characteristic scale on which the gravitational field changes one can use the geometric optics approximation. In this approximation a normal to a surface of constant phase is a null vector which is tangent to a null geodesic describing a motion of a massless quantum. We collect here useful relations of the geometric optics in a higher dimensional curved spacetime. To make the presentation concrete we discuss the electromagnetic field propagation. We closely follow the nice presentation of the MTW book [1], which requires only tiny changes connected with the number of dimensions $D$ which is now not four but arbitrary. Maxwell equations in a spacetime with the metric $g_{ab}$, $(a, b = 0, \ldots, D - 1)$ in the Lorentz gauge have the form

$$
\nabla^b \nabla_b A^a - R_{a b}^a A^b = 0, \quad (A1)
$$

$$
\nabla_a A^a = 0. \quad (A2)
$$

We write the potential $A_a$ in the form

$$
A_a = \Re \left\{ [A + O(\epsilon)]_a e^{iS/\epsilon} \right\}. \quad (A3)
$$

Here $\epsilon$ is a small parameter.

Substituting (A3) into (A2) and keeping the term of the leading order $\epsilon^{-1}$ one obtains

$$
l^a A_a = 0, \quad l_a = \nabla_a S. \quad (A4)
$$

Similarly, substituting (A3) into (A1) and keeping the terms of order $\epsilon^{-2}$ and $\epsilon^{-1}$ one gets

$$
l^a l^a = 0, \quad (A5)
$$

$$
l^b \nabla_b A_a = -\frac{1}{2} A_a \nabla_b l^a. \quad (A6)
$$

Since $\nabla_b l_a = \nabla_b \nabla_a S = \nabla_a \nabla_b S = \nabla_a l_b$ the equation (A5) implies that

$$
l^b \nabla_b l^a = \nabla_l l^a = 0. \quad (A7)
$$

Hence integral lines of $l^a$

$$
\frac{dx^a}{d\tau} = l^a \quad (A8)
$$

are null geodesics and $\tau$ is an affine parameter.

We call $l$-plane a $(D - 1)$-dimensional null plane formed by the vectors $v$ orthogonal to $l$, $v \cdot l = 0$. Relation (A3) shows that the vector $\mathcal{A}$ lies in the $l$-plane. Consider a gauge transformation of the potential $A_a \to A_a + \nabla_a \alpha$, where $\alpha = \Re [\epsilon \exp(iS/\epsilon)]$. This transformation generates the following map $\mathcal{A} \to \mathcal{A} + \gamma l_a$. This means that the vector $\mathcal{A}$ is determined up to the transformation

$$
\mathcal{A} \to \mathcal{A} + \gamma l. \quad (A9)
$$

Thus for a non-trivial electromagnetic field the vector $\mathcal{A}$ is spacelike. Let us write $\mathcal{A} = \mathcal{A} e$, where $e \cdot e = 1$. We call $\mathcal{A}$ the amplitude and $e$ the polarization vector.

Since $\epsilon^2 \nabla_l e = 0$, the equation (A9) implies

$$
\nabla_l e = 0, \quad (A10)
$$

$$
\nabla_l A + \frac{1}{2} \mathcal{A} \nabla_l b = 0. \quad (A11)
$$

The first equation, (A10), shows that the vector of polarization $e$ is parallel-transported along the null geodesic, while the second equations implies

$$
\nabla_a (\mathcal{A}^2 l^a) = 0. \quad (A12)
$$

This conserved current gives the conservation law for the ‘number of photon’

$$
N = \int d\Sigma_a \mathcal{A}^2 l^a, \quad (A13)
$$

where $d\Sigma$ is a volume element of a $(D - 1)$-dimensional spacelike Cauchy surface.

Denote by $e_i$, $i = 1, \ldots, D - 2$ a set of $(D - 2)$ parallel-propagated mutually orthogonal unit vectors. An arbitrary vector of the linear polarization $e$ can be decomposed in this basis as follows

$$
e = \sum_{i=1}^{D-2} b_i e_i, \quad (A14)
$$

where $b_i$ are constant coefficients.

APPENDIX B: PARALLEL TRANSPORT ALONG PRINCIPAL NULL DIRECTIONS

In this appendix we present the details of the construction of a parallel-transported frame along the principal null directions. Besides the basis $\{l, n, e_\mu, \tilde{e}_\mu, e_x\}$, it is useful to consider also the complex null Darboux basis $\{l, n, m_\mu, \tilde{m}_\mu, e_x\}$,

$$
m_\mu = \frac{1}{\sqrt{2}} (\tilde{e}_\mu + i e_\mu), \quad \tilde{m}_\mu = \frac{1}{\sqrt{2}} (\tilde{e}_\mu - i e_\mu), \quad (B1)
$$

$\mu = 1, \ldots, n - 1$, with the only non-vanishing scalar products

$$
l \cdot n = -1, \quad m_\mu \cdot \tilde{m}_\mu = 1, \quad (B2)
$$
in which the PCKY tensor \( h \), takes the form

\[
    h = r \mathbf{l}^\sigma \wedge \mathbf{n}^\nu + i \sum_\mu x_\mu \mathbf{m}_\mu^b \wedge \mathbf{\bar{m}}_\mu^b.
\]  

(B3)

The covariant derivatives of the Darboux basis along the principal null direction \( \mathbf{l}_+ \propto \mathbf{l} \) are

\[
    \nabla_{\mathbf{l}_+} \mathbf{l} = \frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{l} + \varepsilon \sqrt{2} \frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{e}_\tilde{\varepsilon},
\]

\[
    \nabla_{\mathbf{l}_+} \mathbf{n} = -\frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{n} + \varepsilon \sqrt{2} \frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{e}_\tilde{\varepsilon} + \frac{\sqrt{2}}{Q_n} \left( \frac{x_\mu}{x_\mu^2 + r^2} \mathbf{e}_\tilde{\mu} + \frac{r}{x_\mu^2 + r^2} \mathbf{\bar{e}}_\tilde{\mu} \right),
\]

\[
    \nabla_{\mathbf{l}_+} \mathbf{\bar{e}}_\tilde{\mu} = \frac{x_\mu}{x_\mu^2 + r^2} \mathbf{\bar{e}}_\tilde{\mu} + \frac{\sqrt{2}}{Q_n} \frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{l},
\]

\[
    \nabla_{\mathbf{l}_+} \mathbf{\bar{e}}_\tilde{\varepsilon} = \frac{\sqrt{2} \mathbf{Q_e}}{Q_n} \mathbf{l}.
\]  

(B4)

For the null basis these are equivalent to

\[
    \nabla_{\mathbf{l}_+} \mathbf{m}_\mu = \frac{ix_\mu}{x_\mu^2 + r^2} \mathbf{m}_\mu + \frac{i}{x_\mu + ir} \sqrt{Q_n} \mathbf{l},
\]

\[
    \nabla_{\mathbf{l}_+} \mathbf{\bar{m}}_\mu = \frac{-ix_\mu}{x_\mu^2 + r^2} \mathbf{\bar{m}}_\mu + \frac{i}{x_\mu - ir} \sqrt{Q_n} \mathbf{l},
\]

\[
    \nabla_{\mathbf{l}_+} \mathbf{n} = -\frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{n} + \varepsilon \sqrt{2} \frac{\sqrt{Q_n}}{\sqrt{Q_n}} \mathbf{e}_\tilde{\varepsilon} + \sum_{\mu=1}^{n-1} \frac{\sqrt{Q_n}}{\sqrt{Q_n}} \left( \frac{-i}{x_\mu - ir} \mathbf{m}_\mu + \frac{i}{x_\mu + ir} \mathbf{\bar{m}}_\mu \right),
\]  

(B5)

with the equations for \( l \) and \( e_\varepsilon \) unchanged.

Now, we change these Darboux bases to the parallel-transported ones by a sequence of the local Lorentz transformations. (Such transformations preserve the orthogonality and the normalization of the frame.) Guided by Eq. (33), our first transformation is the boost in the \( \{ l, n \} \) plane

\[
    ^n l = \frac{l}{\sqrt{Q_n}} \mathbf{l}, \quad ^n n = \sqrt{Q_n} n.
\]

\[
    ^n e_\tilde{\mu} = e_\tilde{\mu}, \quad ^n e_\tilde{\tilde{\varepsilon}} = e_\tilde{\tilde{\varepsilon}}, \quad ^n e_\varepsilon = e_\varepsilon, \quad ^n \mathbf{m}_\mu = \mathbf{m}_\mu, \quad ^n \mathbf{\bar{m}}_\mu = \mathbf{\bar{m}}_\mu.
\]  

(B6)

The transformed vectors are orthogonal and have

\[
    \nabla_{\mathbf{l}_+} ^n \mathbf{n} = 0,
\]

\[
    \nabla_{\mathbf{l}_+} ^n \mathbf{\bar{e}}_\tilde{\mu} = \frac{x_\mu}{x_\mu^2 + r^2} ^n \mathbf{\bar{e}}_\tilde{\mu} + \frac{\sqrt{2} x_\mu \sqrt{Q_n}}{x_\mu^2 + r^2} ^n \mathbf{a}_\tilde{\varepsilon},
\]

\[
    \nabla_{\mathbf{l}_+} ^n \mathbf{\bar{e}}_\tilde{\varepsilon} = \frac{\sqrt{2} \mathbf{Q_e}}{r} ^n \mathbf{a}_\varepsilon,
\]

or, for the null frame,

\[
    \nabla_{\mathbf{l}_+} ^n \mathbf{m}_\mu = \frac{ix_\mu}{x_\mu^2 + r^2} ^n \mathbf{m}_\mu + \frac{i}{x_\mu + ir} \sqrt{Q_n} \mathbf{l},
\]

\[
    \nabla_{\mathbf{l}_+} ^n \mathbf{\bar{m}}_\mu = \frac{-ix_\mu}{x_\mu^2 + r^2} ^n \mathbf{\bar{m}}_\mu + \frac{i}{x_\mu - ir} \sqrt{Q_n} \mathbf{l},
\]

\[
    \nabla_{\mathbf{l}_+} ^n \mathbf{n} = \sum_{\mu=1}^{n-1} \sqrt{Q_n} \left( \frac{-i}{x_\mu - ir} ^n \mathbf{m}_\mu + \frac{i}{x_\mu + ir} ^n \mathbf{\bar{m}}_\mu \right) + \varepsilon \sqrt{2} \sqrt{Q_n} ^n \mathbf{a}_\tilde{\varepsilon}.
\]  

(B7)

Next transformation is a multi-null rotation, leaving \( ^n \mathbf{l} \) fixed. Actually, this transformation can be decomposed into a sequence of commuting null rotations each of which combines only vectors \( \mathbf{l}, \mathbf{n} \) and vectors \( \mathbf{m}_\mu, \mathbf{\bar{m}}_\mu \) from one KY 2-plane. Namely, for each \( \mu = 1, \ldots, n-1 \) we perform the null rotation characterized by the parameter \( \sqrt{Q_n} \),

\[
    ^{\mathbf{n}} \mathbf{l} = ^n \mathbf{l}, \quad ^{\mathbf{n}} \mathbf{m}_\mu = ^n \mathbf{m}_\mu + \sqrt{Q_n} ^n \mathbf{a}_\varepsilon, \quad ^{\mathbf{n}} \mathbf{\bar{m}}_\mu = ^n \mathbf{\bar{m}}_\mu + \sqrt{Q_n} ^n \mathbf{a}_\varepsilon, \quad ^{\mathbf{n}} e_\tilde{\mu} = ^n e_\tilde{\mu}, \quad ^{\mathbf{n}} e_\tilde{\tilde{\varepsilon}} = ^n e_\tilde{\tilde{\varepsilon}} + \sqrt{2} \sqrt{Q_n} ^n \mathbf{a}_\varepsilon.
\]  

(B8)

in odd dimension accompanied by

\[
    ^{\mathbf{n}} e_\varepsilon = ^n e_\varepsilon + \sqrt{2} \sqrt{Q_n} ^n \mathbf{a}_\varepsilon.
\]  

(B9)

The transformed vectors are orthogonal and have
to be completed by properly transformed vector $^{\nu}n$

$$^{\nu}n = ^{\nu}n + \sum_{\mu=1}^{n-1} \sqrt{Q_\mu} (^{\nu}m_\mu + ^{\nu}m_{\bar{\mu}}) + \varepsilon \sqrt{2} \sqrt{Q} e_\bar{i} + \left( \sum_{\mu=1}^{n-1} Q_\mu + \varepsilon Q_\bar{\mu} \right)^{\nu}l$$

$$= ^{\nu}n + \sum_{\mu=1}^{n-1} \sqrt{2} \sqrt{Q_\mu} ^{\nu}e_\bar{\mu} + \varepsilon \sqrt{2} \sqrt{Q} e_\bar{i} + \left( \sum_{\mu=1}^{n-1} Q_\mu + \varepsilon Q_\bar{\mu} \right)^{\nu}l .$$

(B11)

Let us remark here, that since the parameters of the null rotations are real, each of them actually mix only three directions $^{\nu}L, ^{\nu}n, ^{\nu}e_\mu$; the direction $^{\nu}e_\bar{\mu}$ remains fixed. The covariant derivatives of the null rotated frame are simple

$$\nabla_{l_+} ^{\nu}l = 0 , \quad \nabla_{l_+} ^{\nu}n = 0 , \quad \nabla_{l_+} ^{\nu}e_i = 0 ,$$

$$\nabla_{l_+} ^{\nu}e_\mu = \frac{x_\mu}{x_\mu^2 + r^2} ^{\nu}e_\bar{\mu} , \quad \nabla_{l_+} ^{\nu}e_\bar{\mu} = - \frac{x_\mu}{x_\mu^2 + r^2} ^{\nu}e_\bar{\mu} ,$$

$$\nabla_{l_+} ^{\nu}m_\bar{\mu} = - \frac{ix_\mu}{x_\mu^2 + r^2} ^{\nu}m_\bar{\mu} , \quad \nabla_{l_+} ^{\nu}m_\mu = \frac{ix_\mu}{x_\mu^2 + r^2} ^{\nu}m_\mu .$$

(B12)

Finally, we perform the spatial rotation in each spatial KY 2-plane, by the angle $\varphi_\mu = \arctan \frac{1}{x_\mu}$,

$$^{1}l = ^{\nu}l , \quad ^{1}n = ^{\nu}n , \quad ^{1}e_i = ^{\nu}e_i ,$$

$$^{1}e_\bar{\mu} = \frac{x_\mu}{\sqrt{x_\mu^2 + r^2}} ^{\nu}e_\bar{\mu} , \quad ^{1}e_\bar{\mu} = - \frac{r}{\sqrt{x_\mu^2 + r^2}} ^{\nu}e_\bar{\mu} ,$$

$$^{1}m_\bar{\mu} = \frac{ix_\mu}{\sqrt{x_\mu^2 + r^2}} ^{\nu}m_\bar{\mu} , \quad ^{1}m_\mu = \frac{ix_\mu}{\sqrt{x_\mu^2 + r^2}} ^{\nu}m_\mu .$$

(B13)

The resulting frame $\{ ^{1}l, ^{1}n, ^{1}e_\mu, ^{1}e_\bar{\mu}, ^{1}m_\mu, ^{1}m_\bar{\mu} \}$ or, alternatively, $\{ ^{1}l, ^{1}n, ^{1}m_\mu, ^{1}m_\bar{\mu}, ^{1}e_i \}$ is parallel-transported

$$\nabla_{l_+} ^{1}l = 0 , \quad \nabla_{l_+} ^{1}n = 0 , \quad \nabla_{l_+} ^{1}e_i = 0 ,$$

$$\nabla_{l_+} ^{1}e_\mu = 0 , \quad \nabla_{l_+} ^{1}e_\bar{\mu} = 0 ,$$

$$\nabla_{l_+} ^{1}m_\mu = 0 , \quad \nabla_{l_+} ^{1}m_\bar{\mu} = 0 .$$

(B14)

Combining all the transformations together we arrive at the result (98), or, for the complex null frame,

$$^{1}l = \frac{1}{\sqrt{Q_\bar{\mu}}} l ,$$

$$^{1}n = \sqrt{Q_\bar{\mu}} n + \left( \sum_{\mu=1}^{n-1} Q_\mu + \varepsilon Q_\bar{\mu} \right) \frac{1}{\sqrt{Q_\bar{\mu}}} l$$

$$+ \sum_{\mu=1}^{n-1} \sqrt{Q_\mu} (m_\bar{\mu} + m_\mu) + \varepsilon \sqrt{2} \sqrt{Q} e_\bar{i}$$

$$^{1}m_\bar{\mu} = \sqrt{x_\mu - ir} \left( m_\mu + \sqrt{Q_\mu} Q_\bar{\mu} l \right)$$

$$^{1}m_\mu = \sqrt{x_\mu + ir} \left( m_\bar{\mu} + \sqrt{Q_\bar{\mu}} Q_\mu l \right)$$

$$^{1}e_\bar{i} = e_\bar{i} + \sqrt{2} \sqrt{Q} \sqrt{Q_\bar{\mu}} l .$$

(B15)

APPENDIX C: PARALLEL TRANSPORT IN THE PLEBANSKI-DEMIAŃSKI FAMILY OF SOLUTIONS

In the main text we have demonstrated how to construct a parallel-propagated frame along null geodesics in spacetimes admitting the PCKY tensor. In this appendix we show how to modify this construction for an important family of 4D spacetimes described by the Plebański-Demiański metric. Such a family generally admits only a non-closed generalization of the PCKY tensor—the (non-degenerate) conformal Killing–Yano (CKY) tensor. Our construction generalizes the results presented in [22, 23].

1. Plebański-Demiański metric

The Plebański-Demiański metric [39] describes a large class of four-dimensional type D spacetimes. The concrete form of physical metrics is obtained by a due limiting procedure (see, e.g., [40] for a recent review). In this way one can obtain, for example, the metric of an accelerated rotating charged black hole in the cosmological background.

The whole Plebański-Demiański class of solutions possesses the CKY tensor [41]. Such a tensor is responsible for complete integrability of a null geodesic motion. When the acceleration parameter is removed the corresponding subclass of solutions obtained earlier by Carter [34, 35] allows the PCKY tensor [42] and the solution of parallel transport was already described in Section VI.A (see also [20] for the timelike case). In order to see
the impact of the presence of a non-trivial acceleration on parallel transport, we write the Plebański-Demiański metric in the notations of Section VI.A. So we have
\begin{equation}
g = -\Omega^2 \omega^2 + \omega^2 \omega^2 + \bar{\omega}^1 \omega^1 + \omega^1 \omega^1, \tag{C1}\end{equation}
where
\begin{align}
\bar{\omega}^2 &= \sqrt{\frac{X_2}{U_2}} (d\psi_0 + x_2^2 d\psi_1), \quad \omega^2 = \sqrt{\frac{U_2}{X_2}} dr, \\
\bar{\omega}^1 &= \sqrt{\frac{X_1}{U_1}} (d\psi_0 - r^2 d\psi_1), \quad \omega^1 = \sqrt{\frac{U_1}{X_1}} dx_1. \tag{C2}
\end{align}
Here, \( U_2 = -U_1 = x_1^2 + r^2 \) and \( \Omega = (1 - x_1 r)^{-1} \). For \( X_1 = X_1(x_1) \) and \( X_2 = X_2(r) \) we refer to the metric as the off-shell metric. Such a metric possesses two (Hodge dual) non-degenerate CKY 2-forms \[ h = \Omega \left( -r \omega^2 \wedge \bar{\omega}^2 + x_1 \omega^1 \wedge \bar{\omega}^1 \right), \tag{C3} \]
\[ k = -\Omega \left( x_1 \omega^2 \wedge \omega^2 + r \omega^1 \wedge \omega^1 \right), \tag{C4} \]
which are connected with the background isometries as
\begin{align}
\xi_{(h)} &= -\frac{1}{3} \delta h = \partial_{\psi_0}, \\
\xi_{(k)} &= -\frac{1}{3} \delta k = \partial_{\psi_1}. \tag{C5}
\end{align}
These isometries together with the hidden symmetry of the CKY tensor make the null geodesic motion in the off-shell background \[ \text{(C1), (C2)} \] completely integrable. The tetrad components of the velocity are
\begin{align}
\dot{l}_2 &= \frac{W_2}{\Omega \sqrt{X_2 U_2}}, \quad l_2 = \frac{\sigma_2}{\Omega \sqrt{X_2 U_2}} \sqrt{W_2^2 - V_2 X_2}, \\
\dot{l}_1 &= -\frac{W_1}{\Omega \sqrt{X_1 U_1}}, \quad l_1 = \frac{\sigma_1}{\Omega \sqrt{X_1 U_2}} \sqrt{V_1 X_1 - W_2^2}, \tag{C6}
\end{align}
where
\begin{align}
W_2 &= r^2 \psi_0 + \psi_1, \quad V_2 = -\kappa_1 > 0, \\
W_1 &= -x_1^2 \psi_0 + \psi_1, \quad V_1 = \kappa_1. \tag{C7}
\end{align}
The constant \( \kappa_1 \) corresponds to the conformal Killing tensor \( Q_{ab} = k_{ab} k_c^c \), whereas the constants \( \psi_0 \) and \( \psi_1 \) are associated with the Killing vectors \( \partial_{\psi_0} \) and \( \partial_{\psi_1} \), respectively.\(^4\)

For the special choice of metric functions
\begin{equation}
X_1 = -k - 2nx_1 + e^2 x_1^2 - 2mx_1^3 + (k + e^2 + g^2 + \Lambda/3)x_1^4, \\
X_2 = k + e^2 + g^2 - 2mr + e^2 - 2nr^3 - (k + \Lambda/3)r^4, \tag{C8}
\end{equation}
and the vector potential
\begin{equation}
A = -\frac{1}{\Omega} \left( \frac{e r}{\sqrt{U_2 X_2}} \omega^2 + \frac{g x_1}{\sqrt{U_1 X_1}} \bar{\omega}^1 \right), \tag{C9}
\end{equation}
the Plebański-Demiański metric obeys the Einstein-Maxwell equations with \( e \) and \( g \) the electric and magnetic charges and the cosmological constant \( \Lambda \).

2. Construction of parallel-propagated frame

Let us now construct a parallel-propagated frame in the off-shell background \[ \text{(C1), (C2)} \]. We start by observing that having a CKY 2-form \( \omega \), that is a 2-form obeying the CKY equations
\begin{equation}
\nabla_X \omega = \frac{1}{3} X \omega = 3 \omega \wedge \xi^X \tag{C7}
\end{equation}
and a null geodesic velocity vector \( l \), one can construct the following parallel-transported vector:
\begin{equation}
\dot{w} = (L \omega)^\sharp + \beta l, \quad \dot{\beta} = \beta \cdot \xi. \tag{C10}
\end{equation}
Indeed, using the defining property \[ \text{(C10)} \] we obtain
\begin{align}
\dot{w} &= (L \omega)^\sharp + \beta l \\
&= \frac{1}{3} L (L \omega) + [L (L \xi \wedge \xi)]^\sharp + \beta l \tag{C12}
\end{align}
Here we have used the obvious fact that the first term in the second line is zero and then proceeded in the same way as in Section III.A.

In particular, this means that for the off-shell Plebański-Demiański metric we may construct the following two parallel-transported vectors:
\begin{align}
\dot{m} &= \frac{1}{\sqrt{-\kappa_1}} [L (L \omega)^\sharp + \beta \omega], \tag{C13} \\
\dot{z} &= \frac{1}{\sqrt{-\kappa_1}} [L (L \omega)^\sharp + \beta \omega]. \tag{C14}
\end{align}
Moreover, using \[ \text{(C5)} \] we find\(^5\)
\begin{align}
\beta_h &= \psi_0 \tau, \quad \beta_k = \psi_1 \tau. \tag{C15}
\end{align}

\(^4\) One can formally recover the ‘non-accelerating’ class of solutions \[ \text{see } \text{[24], [28]} \] by taking the conformal factor \( \Omega = 1 \).

\(^5\) Let us remark here, that contrary to Section IV.B one cannot for the Plebański-Demiański metric ‘separate’ the affine parameter \( \tau \) as in \[ \text{[20]} \]. Formally, this is due to the presence of a nontrivial (non-separable) conformal factor \( \Omega \).
The last parallel-transported vector \( \mathbf{n} \) is simply determined by the normalization conditions

\[
\mathbf{n} \cdot \mathbf{l} = -1, \quad \mathbf{n} \cdot \mathbf{m} = 0, \quad \mathbf{n} \cdot \mathbf{z} = 0, \quad \mathbf{n} \cdot \mathbf{n} = 0.
\]

(C16)

Let us explicitly write down the form of the (of-shell) parallel-transported frame in the velocity adapted basis \( \{\mathbf{o}\} \). We have

\[
\mathbf{l} = \mathbf{\hat{l}}_1 (\beta^2 \mathbf{\hat{z}}^2 + \mathbf{\hat{o}}^2),
\]

\[
\mathbf{\hat{k}}_1 = \frac{1}{\Omega \sqrt{U_2}}
\]

\[
= \mathbf{\hat{m}}^b = \frac{\mathbf{\hat{k}}_1}{\mathbf{\hat{k}}_1} \left[ \frac{\Omega^2 U_2 + \beta^2 + \beta^2_k}{2} \mathbf{\hat{o}}^2 - \Omega (\beta r + \beta_k x_1) \mathbf{\hat{z}}^2 \right. \left. + \frac{\Omega^2 U_2 - \beta^2 - \beta^2_k}{2} \mathbf{\hat{z}}^2 \right. \left. + \Omega (\beta_k x_1 - \beta r) \mathbf{\hat{z}}^2 \right],
\]

\[
\mathbf{n}^b = \frac{\mathbf{\hat{k}}_1}{\mathbf{\hat{k}}_1} \left[ \frac{\Omega^2 U_2 + \beta^2 + \beta^2_k}{2} \mathbf{\hat{o}}^2 - \Omega (\beta r + \beta_k x_1) \mathbf{\hat{z}}^2 \right. \left. + \frac{\Omega^2 U_2 - \beta^2 - \beta^2_k}{2} \mathbf{\hat{z}}^2 \right. \left. + \Omega (\beta_k x_1 - \beta r) \mathbf{\hat{z}}^2 \right],
\]

\[
\mathbf{\hat{z}}^b = \frac{\mathbf{\hat{k}}_1}{\mathbf{\hat{k}}_1} \left[ \frac{\Omega^2 U_2 + \beta^2 + \beta^2_k}{2} \mathbf{\hat{o}}^2 - \Omega (\beta r + \beta_k x_1) \mathbf{\hat{z}}^2 \right. \left. + \frac{\Omega^2 U_2 - \beta^2 - \beta^2_k}{2} \mathbf{\hat{z}}^2 \right. \left. + \Omega (\beta_k x_1 - \beta r) \mathbf{\hat{z}}^2 \right].
\]

(C17)

It is easy to see, that one can formally recover the ‘non-accelerating’ limit of the PCKY tensor, Eq. [59], by setting \( \Omega = 1 \) (\( \beta_k = 0 \)).

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