Critical phase in non-conserving zero-range processes and equilibrium networks

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Zero-range processes, in which particles hop between sites on a lattice, are closely related to equilibrium networks, in which rewiring of links take place. Both systems exhibit a condensation transition for appropriate choices of the dynamical rules. The transition results in a macroscopically occupied site for zero-range processes and a macroscopically connected node for networks. Criticality, characterized by a scale-free distribution, is obtained only at the transition point. This is in contrast with the widespread scale-free real-life networks. Here we propose a generalization of these models whereby criticality is obtained throughout an entire phase, and the scale-free distribution does not depend on any fine-tuned parameter.

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Many driven, non-equilibrium models, reach a critical or scale invariant steady state only when their dynamical parameters are fine-tuned to reach a phase transition point. Examples include a wide range of systems such as jamming in traffic [1], coalescence in granular gases [2], gelation in networks [3] and wealth condensation in macroeconomics [4]. In all these processes one has a condensation phase transition which we shall discuss in detail below. In other non-equilibrium models, for example in driven lattice gases and sandpile models [2, 4], it has been argued that scale invariance and power-law distributions are generic, or at least one may have scale-free distributions across wide regions of the parameter space rather than just at critical points. This phenomenon has been termed self organized criticality.

In recent years considerable attention has been given to the study of real-life networks. Networks, defined as collections of nodes connected by links, are found in many fields of study, ranging from molecular biology to social communities and the Internet [5, 6]. With each node one associates a degree k which is the number of links connected to it. In general links may be directed or they may carry a weight, however for our purposes we do not consider such features. It has been observed that very often real-life networks are characterized by a degree distribution P(k) which decays algebraically for large k [5, 6]. These networks, termed scale-free networks, are indeed critical, suggesting the existence of a mechanism which drives them to this state. Subsequently, dynamical processes for growing networks have been proposed which result in a critical distribution for a wide range of the dynamical parameters. In these processes nodes and links are continually added to the network with some predetermined rates [5, 6, 7, 11]. On the other hand “equilibrium” networks [8, 12], whose dynamics constitutes rewiring processes with a fixed number of nodes, exhibit a critical distribution only at a critical point. This transition corresponds to condensation (also referred to as gelation) where a single node captures a finite fraction of the links. Note that although termed equilibrium networks, their dynamics does not always obey detailed balance, thus their steady state may not always be a thermal equilibrium one.

Instructive insight has been gained into the condensation transition through the analysis of simple interacting particle systems [12, 13, 14]. These systems form fundamental models which may be mapped onto particular applications. For example, the zero-range process (ZRP) [10] is a particularly simple and exactly soluble model in which each site μ of a lattice may contain an integer number of particles nμ and particles hop to a neighboring site with rate u(nμ). This model is closely related to the equilibrium networks discussed above which undergo rewiring dynamics [12]; in the following we will exploit this relationship.

Condensation in the ZRP will occur, for example, when u(n) decays to some finite, large-n asymptotic value β, as u(n) ∼ β(1+b/n) with b > 2. The transition is simply understood by considering p(n) the steady-state probability that a site contains n particles. For a subcritical density of particles one finds that p(n) decays exponentially with decay length dependent on the conserved particle density ρ = N/L where N is the number of particles and L is the number of sites. This p(n) describes the low density, fluid phase. As the density is increased the decay length increases until at the critical density, ρc, it diverges and one has a power-law distribution p(n) ∼ n−β, thus a critical fluid. Above ρc, in addition to the power law, a piece of p(n) emerges centered about n = L(ρ−ρc); this piece represents the condensate [15]. Thus the supercritical phase corresponds to a critical fluid coexisting with a condensate. Only at criticality does one have a pure power-law distribution.

In the present work we investigate how a critical phase may emerge in processes such as ZRPs and network models. We shall see that the introduction of non-conservation in an appropriate fashion, modifies the condensate phase into a scale-free phase by effectively suppressing the condensate and leaving the critical fluid. In this generalization of the ZRP particles are created at all sites at constant rate but are removed at a rate that depends non-linearly on the occupation number. Equiv-
alently in the network context links are created and destroyed. We begin by elucidating the mechanism for the generation of a critical phase within a generalized ZRP; later we will define an equilibrium network where this mechanism is also manifested.

Consider a lattice of \( L \) sites upon which reside a number of particles. With rate \( u(n) \) (probability per unit time) which depends on the occupation \( n_\mu \) of site \( \mu \), a particle is transferred from site \( \mu \) to another site. We consider a fully connected geometry, where the destination site is chosen randomly from the other \( L - 1 \) sites. In addition to the hopping dynamics, particles are added to site \( \mu \) with a constant rate \( c \), or removed with a rate \( a(n_\mu) \), which increases with the site occupation \( n_\mu \). Thus, our choice for the dynamical rates is given by

\[
\begin{align*}
n, m \to n - 1, m + 1 \, \text{ with rate } & \quad u(n) = (1 + \frac{b}{n}) \theta(n) \\
n \to n + 1 \, \text{ with rate } & \quad c = \left( \frac{1}{2} \right)^n \\
n \to n - 1 \, \text{ with rate } & \quad a(n) = \left( \frac{n}{2} \right)^k ,
\end{align*}
\]

where \( b, s \) and \( k \) are positive parameters, and \( \theta(n) \) is the usual Heaviside step function. The dynamical rates are conveniently implemented by using a random-sequential updating scheme, whereby at each time step a site \( \mu \) is chosen at random, and a hop, annihilation or creation event may occur with relative probabilities given by the rates \( \{ u, c, a \} \). With \( b \leq 2 \) the system is always found in a sub-critical phase, where the particle number distribution is exponential. We therefore restrict our discussion hereafter to the case \( b > 2 \).

In Fig. 1 we compare the particle number distribution \( p(n) \) of our model with that of the ZRP with conserving dynamics. It is clearly seen that for this choice of parameters, the creation/annihilation dynamics can selectively destroy any condensate and sustain the power-law distribution seen in the network context links. In what follows we analyze the model showing that this feature holds for an entire region in the parameter space.

The steady state of the model is fully described by the probability distribution \( P(n_1, n_2, \ldots, n_L) \) over all possible configurations. In contrast to the conserving ZRP \( \text{ (1)} \), the steady-state distribution of the model \( \text{ (1)} \) does not factorize generally. However, we make the mean field approximation that the steady state distribution does factorize, i.e. \( P(n_1, n_2, \ldots, n_L) \to \prod_{i=1}^{L} p(n_i) \). Due to the fully-connected geometry we expect this approximation to become exact in the limit \( L \to \infty \).

Using this approximation, the steady-state master equation is given by

\[
0 = [u(n+1) + a(n+1)]p(n+1) - [\lambda + c]p(n) - \{u(n) + a(n)\} p(n) - [\lambda + c]p(n-1) \theta(n).
\]

Here the current \( \lambda \) is given by

\[
\lambda = \sum_{n=1}^{\infty} u(n)p(n) . \quad (3)
\]

From (2) it follows that

\[
p(n) = \frac{(\lambda + c)^n}{\Pi_{m=1}^{n} [a(m) + u(m)]} p(0) . \quad (4)
\]

Note that this is not a closed solution as \( \lambda \) depends on \( p(n) \). The values of \( \lambda \) and \( p(0) \) should be set such that both the normalization condition \( 1 = \sum_{n=0}^{\infty} p(n) \) and the creation/annihilation balance condition

\[
c = \sum_{n=1}^{\infty} a(n)p(n) \quad (5)
\]

are obeyed. We now identify the three phases of the model by determining the asymptotic, large \( L \) behaviors of \( p(n) \) and \( \lambda \) which satisfy (3) and (5). The emergent phase diagram is summarised in Fig. 2. Deferring details to a later publication, we find the following results:

**Low-density phase, \( s > k \).** — Rewriting (5) as

\[
L^{k-s} = \sum_{n} n^s p(n) \quad (\text{as } L \to \infty) \quad \text{implies } p(n) \quad \text{is a rapidly decreasing function of } n, \quad \text{and the steady state density } \rho \quad \text{is } \ll 1.
\]

Thus, \( p(1) \approx \rho \sim L^{k-s} \). In the thermodynamic limit the density goes to zero.

For \( s < k \) the sum in (5) is controlled by the behavior of \( p(n) \) at large \( n \). We find the following regimes:

**High-density phase, \( s < k/(k+1) \).** — Here

\[
p(n) \sim n^{-b} \exp \left[ \frac{n}{L^s} - \frac{n^{k+1}}{(k+1)L^k} \right] , \quad (6)
\]

thus \( p(n) \) is strongly peaked at \( n \sim L^{1-s/k} \). This is the high density phase, where all sites are highly occupied. Note that the mean number of particles in the system, \( N \sim L^{2-s/k} \) is super-extensive.

**Critical phase, \( k > s > k/(k+1) \).** — In this phase the system relaxes to the critical density, and \( p(n) \) takes an algebraic form. However, for large but finite systems two
sub-phases are observed, distinguished by the finite-size corrections to the dominant power law.

For \( k > s > k b / (k+1) \)

\[
p(n) \sim n^{-b} \exp \left[ -\frac{n}{L^{x}} \right], 
\]

where \( x = (k - s) / (k - b + 1) \). This cuts off the power law at \( n \sim L^{x} \). We refer to this as critical sub-phase (a).

For \( kb / (k+1) > s > k / (k+1) \)

\[
p(n) \sim n^{-b} \exp \left[ nd \left( \frac{\ln L}{L} \right)^{1/x} - \frac{n^{k+1}}{(k+1)L^k} \right],
\]

where \( d = b - s(k+1)/k \). Here, on top of the algebraic part, \( p(n) \) is weakly peaked at \( n \sim L^{k/(k+1)} (\ln L)^{1/(k+1)} \). This peak will diminish as \( L \to \infty \). We refer to this as critical sub-phase (b).

In Fig. 3 we present typical data obtained from numerical simulations in the different phases and compare with theoretical curves of \( p(n) \). We found that in all phases, starting from random initial configurations of various densities, the system relaxes towards its expected steady state value. However, for the low-density phase the time scales for full relaxation were prohibitive and we do not present steady state data for this phase. In Fig. 3 we also provide data for a one-dimensional (1d) variant of the model, where sites are arranged in a 1d array, and particles are allowed to hop only to the right neighbor of the departure site. For the 1d geometry the mean-field approximation is not expected to be exact even in the limit \( L \to \infty \). Nevertheless, we find numerically that the three phases discussed above exist also in the 1d model.

We now apply the approach discussed above to equilibrium networks. To make the analogy with the ZRP, one identifies a site of the ZRP and its occupation number with a node in the network and its degree, respectively. We define a network model which incorporates both rewiring and creation/annihilation dynamics, and show how a proper choice of rates leads to the existence of a critical phase, much like that of the non-conserving ZRP, within which networks are scale free. Due to the introduction of annihilation of links, there is no simple mapping from the ZRP to the network model since this would require keeping track of pairs of linked particles in the ZRP. However, the two systems are closely related and as we shall see share the same phase diagram.

We consider a network of \( L \) nodes which are linked together by an integer number, \( N/2 \), of undirected links (\( N \) is the number of particles in the corresponding ZRP). With rate \( u(n_\mu) = 1 + b/n_\mu \), one of the \( n_\mu \) links is disconnected from node \( \mu \) and is rewired to another randomly chosen node. This does not change the number of links in the network. With rate \( a(n_\mu) = (n_\mu/L)^k \) one of the links connected to node \( \mu \) is removed from the network. In addition, a new link is created between node \( \mu \) and other randomly chosen node at a constant rate \( c = 1/L^s \). Again the dynamics is conveniently implemented by choosing a node \( \mu \) randomly at each time step, and changing the wiring with probabilities constructed from the relevant rates. The mean-field master equation for the network model differs slightly from that of the ZRP (2), and is given by

\[
\begin{align*}
0 &= [a(n + 1) + u(n + 1) + \Lambda(n + 1)]p(n + 1) \\
&\quad - [a(n) + u(n) + \Lambda(n)]p(n)\theta(n) \\
&\quad - [\lambda + 2c]p(n) + [\lambda + 2c]p(n - 1)\theta(n) .
\end{align*}
\]

Here \( \lambda = \sum u(n)p(n) \) as before, and

\[
\Lambda(n) = \frac{n}{N} \sum_{\ell=1}^{\infty} a(\ell)p(\ell) = \frac{n}{N} .
\]

The steady state solution to (10) is given by

\[
p(n) = \frac{(\lambda + 2c)^n}{\prod_{m=1}^{n} [\Lambda(m) + a(m) + u(m)]}p(0) .
\]
The main difference between this result and that of the ZRP [4] lies in $\Lambda(n)$ and its dependence on the total number of links in the system. This complicates slightly the analysis, however all phases persist including the critical phase. Thus the phase diagram in Fig. 2 also describes the network model: the low-density and high-density phases characterize extremely sparse and extremely dense networks; the critical fluid corresponds to extremely dense networks; the critical fluid corresponds to extremely sparse and extreme.

Further interesting observations are that the weak peak of $p(m)$ in critical sub-phase (b) may correspond to a number of highly connected nodes but is distinct from a condensate phase. This would correspond to a large number of hubs in the network. Also, we note that in the network model the suppression of events which link nodes to themselves has a considerable effect on the results [10], by introducing a new cut-off into the probability distribution which can become the dominant scale. A full analysis will be published elsewhere.

Our main interest in the systems studied lies in the emergence of a critical phase which we have shown exists for annihilation and creation indices $k$, $s$ in the range $k > s > k/(k+1)$. We conclude by comparing the critical phases we have identified to the critical points of the corresponding conserving ZRP and network models. In the latter, the average particle/link density $\rho$ is an external parameter. A power-law distribution of the occupation number/degree is only obtained at $\rho = \rho_c$. In contrast, in the non-conserving models we have studied, the steady-state density is set by the dynamics to be $\rho_c$ and a power-law distribution is obtained throughout the critical phase. In models exhibiting self-organized criticality [17], a critical phase is typically obtained only when the driving rate of the system vanishes with the system size [18], in order to ensure relaxation between stimuli. In comparison, in the present work the creation rate $c$ in e.g. [4] which could be thought of as a driving rate for the system, vanishes in the large $L$ limit whereas $u$ the hopping/rewiring rate does not vanish. Thus, there is a separation of timescales in the dynamical processes. On the other hand, there are no avalanches or underlying absorbing states which are features usually associated with self organized criticality.

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