Supersymmetry breakdown for an extended version of the Nicolai supersymmetric fermion lattice model

Hajime Moriya

Faculty of Mechanical Engineering, Institute of Science and Engineering,
Kanazawa University, Kakuma, Kanazawa 920-1192, Japan

(Dated: May 1, 2018)

Abstract

Sannomiya-Katsura-Nakayama have recently studied an extension of the Nicolai supersymmetric fermion lattice model which is named “the extended Nicolai model”. The extended Nicolai model is parameterized by an adjustable constant \( g \in \mathbb{R} \) in its defining supercharge, and satisfies \( N = 2 \) supersymmetry. We show that for any non-zero \( g \) the extended Nicolai model breaks supersymmetry dynamically, and the energy density of any homogeneous ground state for the model is strictly positive.
I. PURPOSE

In [1] Sannomiya-Katsura-Nakayama investigated supersymmetry breakdown for an extended version of the Nicolai supersymmetric fermion lattice model [2]. As this model satisfies the algebraic relation of $\mathcal{N} = 2$ supersymmetry, and it is reduced the original Nicolai model when its adjustable parameter $g \in \mathbb{R}$ is equal to 0, it is called the extended Nicolai model.

Supersymmetry breakdown for the extended Nicolai model has been shown for any non-zero $g$ on finite systems [1]. In the infinite-volume limit, however, Sannomiya-Katsura-Nakayama verified supersymmetry breakdown of the extended Nicolai model only when $g > g_0 := 4/\pi$. This restriction upon the parameter $g$ seems to be merely due to technical nature and its physics meaning is unclear. The purpose of this note is to remove this restriction upon $g$ in the case of the infinite-volume limit. We show that for any $g \neq 0$ the extended Nicolai model defined on $\mathbb{Z}$ breaks supersymmetry dynamically. Furthermore, we prove that for any $g \neq 0$ the energy density of any (homogeneous) ground state for the extended Nicolai model is strictly positive.

In [1] it is noted that even if supersymmetry is broken for any finite subsystem, there may be restoration of supersymmetry in the infinite-volume limit as exemplified in [3]. We clarify that such restoration can not happen for the extended Nicolai model by formulating the model as supersymmetric $C^*$-dynamics in our framework [4]. Our proof based on $C^*$-algebraic methods is rather model-independent. It makes essential use of a crucial finding by Sannomiya-Katsura-Nakayama (Eq.(15) of [1]) that will be reformulated in terms of superderivations of an infinite-volume $C^*$-system.

II. THE EXTENDED NICOLAI SUPERSYMMETRIC FERMION LATTICE MODEL

We consider spinless fermions over an infinitely extended lattice $\mathbb{Z}$. Let $c_i$ and $c_i^*$ denote the annihilation operator and the creation operator of a spinless fermion at $i \in \mathbb{Z}$, respectively. Those obey the canonical anticommutation relations (CARs): For $i, j \in \mathbb{Z}$

\[
\{c_i^*, c_j\} = \delta_{i,j} 1, \\
\{c_i^*, c_j^*\} = \{c_i, c_j\} = 0.
\]

The fermion number operator on each site $i \in \mathbb{Z}$ is given by $n_i := c_i^* c_i$.
For any \( g \in \mathbb{R} \) we take the following infinite sum of local fermion operators:

\[
Q(g) := \sum_{k \in \mathbb{Z}} (gc_{2k-1} + c_{2k-1}^* c_{2k} c_{2k+1}).
\]  

(2)

It is interpreted as perturbation of the supercharge of the original Nicolai model \( Q(0) \) by another supercharge \( \sum_{k \in \mathbb{Z}} c_{2k-1} \) multiplied by \( g \). Note that the perturbed term \( \sum_{k \in \mathbb{Z}} c_{2k-1} \) itself generates a trivial model. By some formal computation using (1) we see that \( Q(g) \) is nilpotent:

\[
0 = Q(g)^2 = Q(g)^*^2.
\]  

(3)

Let a supersymmetric Hamiltonian be given as

\[
H(g) := \{Q(g), Q(g)^*\}.
\]  

(4)

For any \( g \in \mathbb{R} \), the model has \( \mathcal{N} = 2 \) supersymmetry by definition. As noted above, if \( g = 0 \), it corresponds to the supersymmetric fermion lattice model defined by Nicolai in [2].

We note that in the infinite-volume system either \( Q(g) \) or \( Q(g)^* \), or both cannot exist as a well-defined linear operator if the supersymmetry associated with them breaks dynamically. In fact we will show that this is the case unless \( g = 0 \). Nevertheless, its supersymmetric dynamics always makes sense in the infinitely extended system as we will see later.

We shall consider the model under periodic boundary conditions as in [1]. Let \( M, N \in 2\mathbb{N} \). Define

\[
\tilde{Q}(g)_{[-M+1,N]} := \sum_{k=-M/2+1}^{N/2} (gc_{2k-1} + c_{2k-1}^* c_{2k} c_{2k+1}),
\]  

(5)

where \( N + 1 \) is identified with \(-M + 1\). We see that

\[
0 = \tilde{Q}(g)_{[-M+1,N]}^2 = \tilde{Q}(g)_{[-M+1,N]}^*^2.
\]  

(6)

Then we define the corresponding local supersymmetric Hamiltonian on the same region \([-M+1,N]\) as

\[
\tilde{H}(g)_{[-M+1,N]} := \{\tilde{Q}(g)_{[-M+1,N]}, \tilde{Q}(g)_{[-M+1,N]}^*\}.
\]  

(7)

Also we may consider free boundary conditions upon supercharges. Let

\[
\hat{Q}(g)_{[-M+1,N+1]} := \sum_{k=-M/2+1}^{N/2} (gc_{2k-1} + c_{2k-1}^* c_{2k} c_{2k+1}) + gc_{N+1}.
\]  

(8)
We see that

\[ 0 = \hat{Q}(g)^2_{[-M+1,N+1]} = \hat{Q}(g)^*_{[-M+1,N+1]} \cdot \quad (9) \]

We give a local supersymmetric Hamiltonian upon the same region \([-M + 1, N + 1]\) by the following supersymmetric form:

\[ \hat{H}(g)_{[-M+1,N+1]} := \{ \hat{Q}(g)_{[-M+1,N+1]}, \; \hat{Q}(g)_{[-M+1,N+1]}^* \}. \quad (10) \]

Note that we have imposed the free boundary condition upon local supercharges rather than local Hamiltonians. Thus \(\hat{H}(g)_{[-M+1,N+1]}\) differs from the usual free-boundary Hamiltonian upon \([-M + 1, N + 1]\) that has more terms near the edges although both of them are localized in \([-M + 1, N + 1]\) and give rise to the same time evolution as we let \([-M + 1, N + 1]\) to \(\mathbb{Z}\). (See the next section.)

III. MATHEMATICALLY RIGOROUS FORMULATION

In [4] a general framework of supersymmetric fermion lattice models is given. By using this framework we shall reformulate the extended Nicolai model introduced in the preceding section as supersymmetric C*-dynamics [5].

For each finite subset \(I \in \mathbb{Z}\), let \(\mathcal{A}(I)\) denote the finite-dimensional algebra generated by \(\{c_i, c_i^* ; i \in I\}\). For \(I \subset J \in \mathbb{Z}\), \(\mathcal{A}(I)\) is imbedded into \(\mathcal{A}(J)\). We define

\[ \mathcal{A}_o := \bigcup_{I \in \mathbb{Z}} \mathcal{A}(I), \quad (11) \]

where all finite subsets \(I\) of \(\mathbb{Z}\) are taken. The norm completion of the *-algebra \(\mathcal{A}_o\) (with the operator norm) yields a C*-algebra \(\mathcal{A}\) which is known as the CAR algebra. The dense subalgebra \(\mathcal{A}_o\) is usually called the local algebra.

Let \(\sigma\) denote the shift-translation automorphism group on \(\mathcal{A}\): For each \(k \in \mathbb{Z}\)

\[ \sigma_k(c_i) = c_{i+k}, \quad \sigma_k(c_i^*) = c_{i+k}^*, \quad \forall i \in \mathbb{Z}. \quad (12) \]

Let \(\gamma\) denote the grading automorphism on the C*-algebra \(\mathcal{A}\) determined by

\[ \gamma(c_i) = -c_i, \quad \gamma(c_i^*) = -c_i^*, \quad \forall i \in \mathbb{Z}. \quad (13) \]
The total system $\mathcal{A}$ is decomposed into the even part and the odd part:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-, \quad \mathcal{A}_+ = \{ A \in \mathcal{A} | \gamma(A) = A \}, \quad \mathcal{A}_- = \{ A \in \mathcal{A} | \gamma(A) = -A \}. \quad (14)$$

For each $I \subseteq \mathbb{Z}$

$$\mathcal{A}(I) = \mathcal{A}(I)_+ \oplus \mathcal{A}(I)_-, \quad \text{where} \quad \mathcal{A}(I)_+ := \mathcal{A}(I) \cap \mathcal{A}_+, \quad \mathcal{A}(I)_- := \mathcal{A}(I) \cap \mathcal{A}_-. \quad (15)$$

The graded commutator is defined as

$$[F_+, G]_\gamma = [F_+, G] \quad \text{for} \quad F_+ \in \mathcal{A}_+, \quad G \in \mathcal{A},$$

$$[F_-, G_-]_\gamma = \{ F_-, G_- \} \quad \text{for} \quad F_- \in \mathcal{A}_-, \quad G_- \in \mathcal{A}_-. \quad (16)$$

From the canonical anticommutation relations (1) the graded-locality follows:

$$[A, B]_\gamma = 0 \quad \text{for all} \quad A \in \mathcal{A}(I) \quad \text{and} \quad B \in \mathcal{A}(J) \quad \text{if} \quad I \cap J = \emptyset, \quad I, J \subseteq \mathbb{Z}. \quad (17)$$

The notation $Q(g)$ of Eq.(2) is a merely formal expression of the supercharge. Nevertheless it gives a well-defined infinitesimal fermionic transformation. Define a superderivation (a linear map that satisfies the graded Leibniz rule) from $\mathcal{A}_0 \to \mathcal{A}_0$ by

$$\delta_g(A) := [Q(g), A]_\gamma \quad \text{for every} \quad A \in \mathcal{A}_0. \quad (18)$$

Similarly the conjugate superderivation is given by

$$\delta^*_g(A) := [Q(g)^*, A]_\gamma \quad \text{for every} \quad A \in \mathcal{A}_0. \quad (19)$$

Let us explain the formula (18) in some depth. For each fixed local element $A \in \mathcal{A}_0$, only finite terms in the summation formula of $Q(g)$ are involved in $[Q(g), A]_\gamma$, because there is a least $I \subseteq \mathbb{Z}$ (with respect to the inclusion) such that $A \in \mathcal{A}(I)$, and by the graded-locality (17) only local fermion terms of (2) that have non-trivial intersection with $I$ may contribute to $[Q(g), A]_\gamma$. The other infinite number of terms give zero. Therefore there exist $M_0, N_0 \in 2\mathbb{N}$ such that $[-M_0 + 1, N_0] \supset I$ and the identity

$$\delta_g(A) = [\tilde{Q}(g)[-M+1,N], A]_\gamma$$

holds for all $M(\in 2\mathbb{N}) \geq M_0$ and $N(\in 2\mathbb{N}) \geq N_0$, where the periodic-boundary condition as in (5) is used. For example, if $I$ is a finite interval of the type $[-S + 1, -S + 2, \cdots , T - 1, T]$
with \((S, T \in 2\mathbb{N})\), then it is enough to take \(M_0 = S + 2, N_0 = T + 2\). An analogous identity by free-boundary supercharges \([8]\) is possible. For each \(A \in \mathcal{A}_o\) we have the asymptotic formula

\[
\delta_g(A) = \lim_{N \to \infty} \left[ \tilde{Q}(g)_{[-N+1,N]}, A \right]_\gamma,
\]

and similarly

\[
\delta_g(A) = \lim_{N \to \infty} \left[ \tilde{Q}(g)_{[-N+1,N+1]}, A \right]_\gamma.
\]

The nilpotent condition \([3]\) is expressed by the superderivation \(\delta_g\) as

\[
\delta_g \circ \delta_g = \delta^*_g \circ \delta^*_g = 0.
\]

Define the derivation generated by the Hamiltonian \(H(g)\).

\[
d_g(A) := [H(g), A] \text{ for every } A \in \mathcal{A}_o.
\]

This is the infinitesimal time-generator of the model. We can immediately verify the following supersymmetric relation:

\[
d_g(A) = \delta^*_g \circ \delta_g(A) + \delta_g \circ \delta^*_g(A) \text{ for every } A \in \mathcal{A}_o.
\]

It has been known that short-range interactions of fermion lattice systems give Hamiltonian dynamics in the infinite-volume limit \([6]\). Somewhat heuristically, we have for any \(A \in \mathcal{A}\) and \(t \in \mathbb{R}\)

\[
\alpha_g(t)(A) := \lim_{N \to \infty} \exp \left( it\tilde{H}(g)_{[-N+1,N]} \right) A \exp \left( -it\tilde{H}(g)_{[-N+1,N]} \right) \in \mathcal{A},
\]

where special local Hamiltonians given in \([7]\) are used for concreteness. However, we may take any boundary condition upon local Hamiltonians on finite subsystems.

We can construct supersymmetric dynamics in the infinitely extended system corresponding to the extended Nicolai model as in the following theorem. Note that it holds irrespective of broken-unbroken supersymmetry.

**Theorem III.1.** For each \(g \in \mathbb{R}\) the superderivation \(\delta_g\) generates a supersymmetric dynamics in \(\mathcal{A}\). Precisely, there exists a strongly continuous one parameter group of \(*\)-automorphisms \(\alpha_g(t)\) \((t \in \mathbb{R})\) on \(\mathcal{A}\) whose pre-generator is given by the derivation \(d_g \equiv \delta^*_g \circ \delta_g + \delta_g \circ \delta^*_g\) on the local algebra \(\mathcal{A}_o\).

**Proof.** From our work \([4]\) the statement follows immediately. \(\square\)
We need to fix the crucial terminology “supersymmetry(SUSY) breakdown” for the present paper. In physics literature, SUSY breakdown is usually identified with strict positivity of SUSY Hamiltonian, see e.g. \cite{7}. However, one should be caution when dealing with models on non-compact space. (In fact, we see a relevant remark by Witten \cite{3}.) As shown in Theorem \ref{thm:supersymmetry}, superderivations are building blocks for supersymmetric dynamics. So let us introduce the following definition which is based on invariance under superderivations. We consider that it is a straightforward expression of the physics concept of symmetry and symmetry breakdown.

**Definition III.2.** Suppose that a superderivation generates a supersymmetric dynamics as in Theorem \ref{thm:supersymmetry}. If a state of $\mathcal{A}$ is invariant under the superderivation defined on the local system $\mathcal{A}_0$, then it is called a supersymmetric state. If there exists no supersymmetric state of $\mathcal{A}$, then SUSY is spontaneously broken.

Sannomiya-Katsura-Nakayama employed a different definition \cite{1}: SUSY is spontaneously broken if the energy density of ground states is strictly positive.

This alternative definition based on the energy density seems not satisfactory in some respects. First, it is only limited to homogeneous ground states. There may exist non-periodic ground states that do not have a well-defined energy density as we have observed such states for the original Nicolai model \cite{8} \cite{9}. It is not obvious how the status of SUSY for homogeneous states implies that for non-homogeneous states in the infinite-volume limit. Second, its full justification has not yet been done even for the particular model (i.e. the extended Nicolai model) in \cite{1}. Actually, we shall discuss the second point in the next section.

**IV. SUPERSYMMETRY BREAKDOWN**

**A. Supersymmetry breakdown in the infinite-volume system**

The first theorem is a direct consequence of a crucial property of the extended Nicolai model \cite{1} which is stated below in (26) and (27). We only need to show that it is still valid in the infinite-volume limit.

**Theorem IV.1.** For any $g \neq 0$, the extended Nicolai supersymmetric fermion lattice model breaks SUSY spontaneously.
Proof. As given in Eq.(15) of [1], for each \( k \in \mathbb{Z} \) let

\[
O_k := c_{2k-1}^* \left( 1 - \frac{1}{g} (c_{2k}^* c_{2k+1} + c_{2k-3}^* c_{2k-2}) + \frac{2}{g^2} c_{2k-3}^* c_{2k-2}^* c_{2k-1} c_{2k+1} \right) .
\] (26)

We shall show that for all \( k \in \mathbb{Z} \)

\[
\delta_g(O_k) = g.
\] (27)

As the model is \( \sigma_2 \)-invariant, it is enough to show the statement for a specific \( k \in \mathbb{Z} \). So let us consider

\[
O_2 = c_3^* \left( 1 - \frac{1}{g} (c_4^* c_5 + c_1 c_2^*) + \frac{2}{g^2} c_1 c_2^* c_4^* c_5 \right) \in \mathcal{A}([1, 2, 3, 4, 5, 6])_-. 
\]

Then for the identity (20) to be valid it is enough to take \( M_0 = 0 - 2 = -2 \) and \( N_0 = 6 + 2 = 8 \). We compute

\[
\delta_g(O_2) = [\tilde{Q}(g|_{-1,8}), O_2]_\gamma \\
= \left[ \sum_{k=0}^{4} (gc_{2k-1} + c_{2k-1}^* c_{2k} c_{2k+1}) , O_2 \right]_\gamma \quad (9 = -1) \\
= [g (c_{-1} + c_1 + c_3 + c_5 + c_7) + (c_{-1} c_0^* c_1 + c_1 c_2^* c_3 + c_3 c_4^* c_5 + c_5 c_6^* c_7 + c_7 c_8^* c_{-1}) , O_2]_\gamma \\
= [g (c_1 + c_3 + c_5) + (c_{-1} c_0^* c_1 + c_1 c_2^* c_3 + c_3 c_4^* c_5 + c_5 c_6^* c_7) , O_2]_\gamma , 
\]

where the identification \( 9 = -1 \) is made and the graded-locality (17) is noted. Similarly, we can verify that

\[
\delta_g(O_2) = [\tilde{Q}(g|_{-1,7}), O_2]_\gamma \\
= g [(c_{-1} + c_1 + c_3 + c_5 + c_7) + (c_{-1} c_0^* c_1 + c_1 c_2^* c_3 + c_3 c_4^* c_5 + c_5 c_6^* c_7) , O_2]_\gamma \\
= g [(c_1 + c_3 + c_5) + (c_{-1} c_0^* c_1 + c_1 c_2^* c_3 + c_3 c_4^* c_5 + c_5 c_6^* c_7) , O_2]_\gamma . 
\]
By direct computation using the canonical anticommutation relations (11) we have

\[\delta_g(O_2)\]
\[= \left[ g (c_1 + c_3 + c_5) + (c_{-1} c_0^* c_1 + c_1 c_2 c_3 + c_3 c_4 c_5 + c_5 c_6 c_7), \ c_3^* \left( 1 - \frac{1}{g} (c_4^* c_5 + c_1 c_2^*) + \frac{2}{g^2} c_1 c_2^* c_4 c_5 \right) \right]_{\gamma}\]
\[= \left\{ g c_3 + c_1 c_2^* c_3 + c_3 c_4^* c_5, \ c_3^* \right\} \left( 1 - \frac{1}{g} (c_4^* c_5 + c_1 c_2^*) + \frac{2}{g^2} c_1 c_2^* c_4 c_5 \right) - c_3^* \cdot 0\]
\[= g - \left( c_4^* c_5 + c_1 c_2^* \right) + \frac{2}{g} c_1 c_2^* c_4 c_5 + \left( c_1 c_2^* + c_4 c_5 \right) - 2 \times \frac{1}{g} c_1 c_2^* c_4 c_5\]
\[= g,
\]
where we have noted
\[\delta_g \left( 1 - \frac{1}{g} (c_4^* c_5 + c_1 c_2^*) + \frac{2}{g^2} c_1 c_2^* c_4 c_5 \right)\]
\[= \left[ g (c_1 + c_3 + c_5) + (c_{-1} c_0^* c_1 + c_1 c_2 c_3 + c_3 c_4 c_5 + c_5 c_6 c_7), \ c_3^* \left( 1 - \frac{1}{g} (c_4^* c_5 + c_1 c_2^*) + \frac{2}{g^2} c_1 c_2^* c_4 c_5 \right) \right]_{\gamma}\]
\[= 0.
\]
Analogously we obtain (27) for any \(k \in \mathbb{Z}\).

Now take any (not necessarily homogeneous) state \(\omega\) of \(\mathcal{A}\). Then for any \(k \in \mathbb{Z}\)

\[\omega (\delta_g(O_k)) = \omega (g1) = g.\]  \hspace{1cm} (28)

Thus if \(g \neq 0\), then \(\omega\) is not invariant under \(\delta_g\). As \(\omega\) is arbitrary, there exists no invariant state under \(\delta_g\) and hence SUSY is spontaneously broken for any \(g \neq 0\).

**B. Positivity of energy density due to supersymmetry breakdown**

We shall discuss the energy density for homogeneous ground states. Let us fix relevant notation. A state \(\omega\) on \(\mathcal{A}\) is called translation invariant if \(\omega(A) = \omega(\sigma_1(A))\) for all \(A \in \mathcal{A}\). A state \(\omega\) on \(\mathcal{A}\) is called homogeneous (with periodicity 2) if \(\omega(A) = \omega(\sigma_2(A))\) for all \(A \in \mathcal{A}\).

For any homogeneous state \(\omega\) we can define the energy density for the extended Nicolai model by the expectation value of the local Hamiltonians per site in the infinite-volume
limit. As we can choose any boundary condition upon local Hamiltonians as noted in [10], we have
\[
e(g)(\omega) := \lim_{N \to \infty} \frac{1}{2N} \omega \left( \tilde{H}(g)_{[-N+1, N]} \right) = \lim_{N \to \infty} \frac{1}{2N} \omega \left( \hat{H}(g)_{[-N+1, N+1]} \right). \tag{29}
\]
Since all \(\tilde{H}(g)_{[-N+1, N]}\) (as well as \(\hat{H}(g)_{[-N+1, N]}\)) are positive operators by definition, we have
\[
e(g)(\omega) \geq 0. \tag{30}
\]
A general definition of ground states for \(C^\ast\)-systems is given in terms of the infinitesimal time evolution, see [6]. Now it is \(d_g\) given in (25). For homogeneous states, this general characterization of ground states is known to be equivalent to the minimum energy-density condition [11]. The extended Nicolai model on \(\mathbb{Z}\) is a homogeneous model of 2-periodicity. It has been known that for any translation invariant model there is at least one translation invariant (pure or non-pure) ground state. Hence there exists at least one homogeneous ground state \(\varphi\) of the periodicity 2 for the extended Nicolai model. The second theorem is as follows.

**Theorem IV.2.** Let \(\varphi\) be any homogeneous ground state for the extended Nicolai model. The energy density \(e(g)(\varphi)\) is strictly positive if the parameter \(g\) of the model is not 0.

**Proof.** Some idea of the proof which we will present below is owing to Buchholz [12] and Buchholz-Ojima [13]. We will use the standard formulation of GNS representations, see [6]. By \((\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)\) we denote the GNS representation associated to the state \(\varphi\) of \(\mathcal{A}\). Precisely, \(\pi_\varphi\) is a homomorphism from \(\mathcal{A}\) into \(\mathfrak{B}(\mathcal{H}_\varphi)\) (the set of all bounded linear operators on the Hilbert space \(\mathcal{H}_\varphi\)), and \(\Omega_\varphi \in \mathcal{H}_\varphi\) is a cyclic vector such that \(\varphi(A) = \langle \Omega_\varphi, \pi_\varphi(A)\Omega_\varphi \rangle\) for all \(A \in \mathcal{A}\).

We consider finite averages of local operators \(\{O_k\}\) defined in (26) under shift-translations: For \(n \in \mathbb{N}\) let
\[
o(n) := \frac{1}{n} \sum_{k=1}^{n} O_k \in \mathcal{A}([-1, 2n + 1])_-. \tag{31}
\]
We shall study the asymptotic behavior of \(\varphi(\delta_g(o(n)))\) as \(n \to \infty\). By (27) we have
\[
\varphi(\delta_g(o(n))) = \frac{1}{n} \sum_{k=1}^{n} \varphi(\delta_g(O_k)) = \frac{1}{n} \sum_{k=1}^{n} g = g. \tag{32}
\]
We can rewrite $\delta_g(o(n)) \in \mathcal{A}_o$ in terms of finite supercharges which are located in a slightly larger region including the support region of $o(n)$. As in (20) by using local supercharges (5) under periodic boundary conditions we have

$$\delta_g(o(n)) = \left[ \tilde{Q}(g)[-3,2(n+2)], o(n) \right]_\gamma. \quad (33)$$

Similarly, we may use free-boundary supercharges (8) as nothing will change due to the choice of boundary conditions. By using the GNS representation $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$ we have

$$\varphi(\delta_g(o(n))) = \left\langle \Omega_\varphi, \pi_\varphi \left( \left[ \tilde{Q}(g)[-3,2(n+2)], o(n) \right]_\gamma \right) \Omega_\varphi \right\rangle = \left\langle \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right), \pi_\varphi(o(n)) \right\rangle \Omega_\varphi + \left\langle \pi_\varphi(o(n))^* \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right) \Omega_\varphi \right\rangle. \quad (35)$$

As $\Omega_\varphi$ is a normalized vector, by using the triangle inequality and Cauchy-Schwarz inequality this yields the following estimate

$$|\varphi(\delta_g(o(n)))| \leq \left\| \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right)^* \Omega_\varphi \right\| \cdot \left\| \pi_\varphi(o(n)) \Omega_\varphi \right\| + \left\| \pi_\varphi(o(n))^* \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right) \Omega_\varphi \right\|. \quad (36)$$

By applying Lemma [IV.3] and Lemma [IV.4] which will be shown later into the above estimate (36) we obtain

$$\lim_{n \to \infty} |\varphi(\delta_g(o(n)))| = 0. \quad (37)$$

This contradicts with (32) when $g \neq 0$. Thus when $g \neq 0$ the assumption of Lemma [IV.4] does not hold, and accordingly $e(g)(\varphi)$ should be non-zero. \hfill \square

Now we will show two lemmas used in the above theorem. We recall the Landau notation: O is so called “big-O”, and o is so called “little-o”. The following mathematical statement claims non-existence of averaged fermion operators (fermion observables at infinity) in the infinite-volume limit. As the estimate is also essential, we shall recapture its derivation from the original work [14].
Lemma IV.3.

\[ \|o(n)\| \sim O\left(\frac{1}{\sqrt{n}}\right) \text{ as } n \to \infty. \] (38)

In particular, \( \lim_{n \to \infty} o(n) = 0 \) in norm.

Proof. For any \( F \in \mathcal{A} \) the inequality \( \|F\|^2 = \|F^*F\| \leq \|F^*F + FF^*\| \) holds. By using this obvious inequality we obtain

\[ \|o(n)\|^2 \leq \frac{1}{n^2} \sum_{k=1}^{n} \sum_{k'=1}^{n} \|\{O_k^*, O_{k'}\}\|. \] (39)

Each term is estimated from the above by some constant:

\[ \|\{O_k^*, O_{k'}\}\| \leq \|O_k^*O_{k'}\| + \|O_{k'}O_k^*\| \leq 2\|O_k^*\| \cdot \|O_{k'}\| = 2\|O_1\|^2 \equiv C^2/5 \] (40)

By the graded-locality (17) and the definition of \( O_k \) given in (26) we have

\[ \{O_k^*, O_{k'}\} = 0 \quad \text{if } |k - k'| > 2. \] (41)

Thus for each fixed \( k \in \{1, 2, \ldots, n\} \) there are at most five \( k' \in \{1, 2, \ldots, n\} \) such that \( \{O_k^*, O_{k'}\} \) does not vanish. By applying (40) (41) to (39) we obtain

\[ \|o(n)\|^2 \leq \frac{1}{n^2} \sum_{k=1}^{n} 5 \times C^2/5 = \frac{C^2}{n}. \] (42)

it is equivalent to \( \|o(n)\| \leq \frac{C}{\sqrt{n}} \) giving (38). \( \square \)

Lemma IV.4. If the energy density \( e(g)(\varphi) \) is equal to 0, then

\[ \left\| \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)]^* \right) \Omega_\varphi \right\| \sim o\left(\sqrt{n}\right), \left\| \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right) \Omega_\varphi \right\| \sim o\left(\sqrt{n}\right) \text{ as } n \to \infty. \] (43)

Proof. By the assumption \( e(g)(\varphi) = 0 \) we have

\[ 0 = e(g)(\varphi) = \lim_{n \to \infty} \frac{1}{2n + 8} \varphi \left( \tilde{H}(g)[-3,2(n+2)] \right) = \lim_{n \to \infty} \frac{1}{2n} \varphi \left( \tilde{H}(g)[-3,2(n+2)] \right). \] (44)

By (17)

\[ \varphi \left( \tilde{H}(g)[-3,2(n+2)] \right) = \varphi \left( \left\{ \tilde{Q}(g)[-3,2(n+2)], \tilde{Q}(g)^*[-3,2(n+2)] \right\} \right) \]

\[ = \left\{ \left\langle \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right) \Omega_\varphi, \pi_\varphi \left( \tilde{Q}(g)^*[-3,2(n+2)] \right) \Omega_\varphi \right\rangle \right\} \]

\[ + \left\{ \left\langle \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right) \Omega_\varphi, \pi_\varphi \left( \tilde{Q}(g)[3,2(n+2)] \right) \Omega_\varphi \right\rangle \right\} \]

\[ = \left\| \pi_\varphi \left( \tilde{Q}(g)^*[-3,2(n+2)] \right) \Omega_\varphi \right\|^2 + \left\| \pi_\varphi \left( \tilde{Q}(g)[-3,2(n+2)] \right) \Omega_\varphi \right\|^2. \] (45)
By this together with (44) we have
\[ \| \pi_\varphi \left( \tilde{Q}(g)_{[-3,2(n+2)]}^* \right) \Omega_\varphi \|^2 \sim o (2n) , \quad \| \pi_\varphi \left( \tilde{Q}(g)_{[-3,2(n+2)]} \right) \Omega_\varphi \|^2 \sim o (2n) . \] (46)

From these we obtain (43).

ACKNOWLEDGMENTS

I thank Hosho Katsura, Yu Nakayama and Noriaki Sannomiya for correspondence. I thank members of Kanazawa University for discussion.

[1] N. Sannomiya, H. Katsura, and Y. Nakayama, Phys. Rev. D. 94, 045014 (2016).
[2] H. Nicolai, J. Phys. A: Math. Gen. 9, 1497 (1976).
[3] E. Witten, Nucl. Phys. B 202, 253 (1982).
[4] H. Moriya, Ann. Inst. Henri. Poincaré 17, 2199 (2016).
[5] Usually one first considers finite systems and then takes their infinite volume limit. We have a different viewpoint here. First we are given an infinitely extended system upon \( \mathbb{Z} \) that represents the total system. The total system includes subsystems as its subalgebras. See [6] for C*-algebraic treatment of quantum statistical mechanics.
[6] O. Bratteli and D. W. Robinson, Operator algebras and quantum statistical mechanics 1 and 2 (Springer Verlag, 1987, 1997).
[7] S. Weinberg, The quantum theory of fields III (Cambridge University Press, 2000).
[8] H. Moriya, (2016), arXiv:1610.09142.
[9] H. Katsura, H. Moriya, and Y. Nakayama, (2017), arXiv:1710.04385.
[10] B. Simon, The Statistical Mechanics of Lattice Gases (Princeton University Press, 1993).
[11] O. Bratteli, A. Kishimoto, and D. Robinson, Comm. Math. Phys. 64, 41 (1978).
[12] D. Buchholz, Lect. Notes in Phys. 539, 211 (2000).
[13] D. Buchholz and I. Ojima, Nucl. Phys. B 498, 228 (1997).
[14] O. E. Lanford III and D. W. Robinson, J. Math. Phys. 9, 1120 (1968).