Process of the slope components of $\alpha$-regression quantile

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Abstract

We consider the linear regression model along with the process of its $\alpha$-regression quantile, $0 < \alpha < 1$. We are interested mainly in the slope components of $\alpha$-regression quantile and in their dependence on the choice of $\alpha$. While they are invariant to the location, and only the intercept part of the $\alpha$-regression quantile estimates the quantile $F^{-1}(\alpha)$ of the model errors, their dispersion depends on $\alpha$ and is infinitely increasing as $\alpha \to 0, 1$, in the same rate as for the ordinary quantiles.

We study the process of $R$-estimators of the slope parameters over $\alpha \in [0, 1]$, generated by the Hájek rank scores. We show that this process, standardized by $f(F^{-1}(\alpha))$ under exponentially tailed $F$, converges to the vector of independent Brownian bridges. The same course is true for the process of the slope components of $\alpha$-regression quantile.

AMS 2000 subject classifications. Primary 62J05, 62G32, 62G35.
Key words and phrases: R-estimator; slope parameters; Brownian bridge; Hájek’s rank scores.

1 Introduction

We start with the linear regression model

$$Y_{ni} = \beta_0 + x_{ni}^\top \beta + e_{ni}, \quad i = 1, \ldots, n \tag{1.1}$$

with observations $Y_{n1}, \ldots, Y_{nn}$, independent errors $e_{n1}, \ldots, e_{nn}$, identically distributed according to an unknown distribution function $F$; with the vector of covariates $x_{ni} = (x_{i1}, \ldots, x_{in})^\top$, $i = 1, \ldots, n$, unknown parameter $\beta = (\beta_1, \ldots, \beta_p)^\top$ of interest, and nuisance intercept $\beta_0$. The regression $\alpha$-quantiles, $0 \leq \alpha \leq 1$, introduced in [15], are an important tool mainly in economics, where the quantile regression became a technical term. Remind that the regression $\alpha$-quantile of model (1.1) is defined as the solution of the minimization

$$(\hat{\beta}_0(\alpha), \hat{\beta}(\alpha)) = \arg \min \{ \alpha \sum_{i=1}^{n} (Y_i - b_0 - x_i^\top \beta)^+ + (1 - \alpha) \sum_{i=1}^{n} (Y_i - b_0 - x_i^\top \beta)^- \}, \quad b_0 \in \mathbb{R}_1, \quad \beta \in \mathbb{R}_p, \quad 0 < \alpha < 1. \tag{1.2}$$

*The research was supported by the Grant GAČR 18-01137S
The population counterpart of (1.2) is
\[
\{ \hat{\beta}_0(\alpha) = \beta_0 + F^{-1}(\alpha), \hat{\beta}(\alpha) = (\beta_1, \ldots, \beta_p)^\top, \ 0 < \alpha < 1 \}.
\]
Hence, only the intercept part of the \( \alpha \)-regression quantile reflects the quantile \( F^{-1}(\alpha) \) of the probability distribution \( F \), while \( \hat{\beta}(\alpha) \) only reflects the slopes. If \( \alpha \) runs over the interval \((0, 1)\), we get the regression quantile process with step-functions trajectories with number of breakpoints increasing with the number \( n \) of observations. There is a rich literature devoted to the concepts connected with regression quantile, its processes and applications. As an excellent review we recommend Koenker’s book [16]. The choice of \( \alpha \) is an important decision, namely when \( Y_i \) reflects the loss, when we should consider the balance between the underestimation and overestimation of our risk. For the applications is important the shape of the limiting processes over \( 0 < \alpha < 1 \) and the shape of various functionals of the regression quantile, characterizing the economic properties.

Alternatively to the regression quantile, we can follow the intercept and slope components separately. The so called two-step regression quantile, proposed in [13], first estimates the slope components \( \beta \) with the aid of rank estimator \( \hat{\beta}_{nR} \) and then estimates the intercept as the \( \alpha \)-quantile of the residuals \( Y_i - \hat{\beta}_{nR} x^\top \), \( i = 1, \ldots, n \). The two-step regression quantile process is asymptotically equivalent to the ordinary regression quantile process, only its number of breakpoints differs, being exactly \( n \). The empirical processes corresponding to the regression quantiles and their inversions are numerically illustrated in [14].

The R-estimate of slopes \( \beta \) is generally defined as the minimizer \( \hat{\beta}_{nR} \) of the Jaeckel [6] measure of the rank dispersion
\[
D_n(b) = \sum_{i=1}^{n} (Y_{ni} - x_{ni}^\top b) \left( a_n(R_{ni}(Y_i - x_{ni}^\top b), - \bar{a}_n) \right)
\]

where \( R_{ni}(Y_{ni} - x_{ni}^\top b) \) is the rank of the residual \( Y_{ni} - x_{ni}^\top b \) among \( Y_{n1} - x_{n1}^\top b, \ldots, Y_{nn} - x_{nn}^\top b \), \( \bar{Y}_n = \frac{1}{n} \sum_{i=1}^{n} Y_{ni}, \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_{ni} \), \( a_n(i) \) are the scores and \( \bar{a}_n = \frac{1}{n} \sum_{i=1}^{n} a_n(i) \). Notice that \( \hat{\beta}_{nR} \) in invariant to the shift in location, hence it is independent of \( \beta_0 \).

The scores \( a_n(i), i = 1, \ldots, n \) are typically generated by a function \( \varphi(u) : (0, 1) \mapsto \mathbb{R}_1 \), nondecreasing and square integrable on \((0,1)\), such that
\[
\lim_{n \to \infty} \int_0^1 \left( a_n(1 + [nu]) - \varphi(u) \right)^2 du = 0.
\]
For instance, \( a_n(i) = \varphi \left( \frac{i}{n+1} \right) \), \( i = 1, \ldots, n \).

Particularly, we shall consider the following family of score functions \( \{ \varphi_\alpha(u), \ 0 \leq \alpha \leq 1, \ 0 \leq u \leq 1 \} \) :

\[
\varphi_\alpha(u) = \begin{cases} 
0 & 0 \leq u \leq \alpha \leq 1 \\
1 & 1 \geq u > \alpha \geq 0.
\end{cases}
\]

As \( n \to \infty \), the function \( \varphi_\alpha(u) \) generates the following scores :
\[
a_n(i, \alpha) = \begin{cases} 
0 & 0 \leq i \leq \alpha \\
i - \alpha & \alpha < i \leq \alpha + 1 \\
1 & i > \alpha + 1
\end{cases}
\]
\[ i = 1, \ldots, n. \] Notice that \( a_n(i, \alpha) \) is continuous in \( \alpha \in (0, 1) \). The scores \( a_n(i, \alpha), \ i = 1, \ldots, n \) are known as Hájek’s rank scores (see Hájek [3] and Hájek and Šidák [4]).

If \( R_{n1}, \ldots, R_{nn} \) are the ranks of random variables \( Z_1, \ldots, Z_n \), then the vector \( (a(R_{n1}, \alpha), \ldots, a(R_{nn}, \alpha)) \) is a solution of the linear programming

\[
\sum_{i=1}^{n} Z_i a_n(R_{ni}, \alpha) = \max
\]

under \( \sum_{i=1}^{n} a_n(R_{ni}, \alpha) = n(1 - \alpha) \)

\[ 0 \leq a_n(R_{ni}, \alpha) \leq 1, \ i = 1, \ldots, n \]

(cf. also [12]). In this case the Jaeckel criterion (13) asymptotically simplifies, as \( n \to \infty \), to

\[
D_{n\alpha}(b) = \sum_{i=1}^{n} \left[ (Y_{ni} - \bar{Y}_n) - (x_{ni} - \bar{x}_n)^\top b \right] I[R_{ni}(Y_i - x_{ni}^\top b) \geq n\alpha] + (R_{ni}(Y_i - x_{ni}^\top b) - n\alpha)I[n\alpha \leq R_{ni}(Y_i - x_{ni}^\top b) \leq n\alpha + 1] \approx \sum_{i=1}^{n} \left[ (Y_{ni} - \bar{Y}_n) - (x_{ni} - \bar{x}_n)^\top b \right] I[R_{ni}(Y_i - x_{ni}^\top b) \geq n\alpha].
\]

Jaeckel proved that \( D_{n\alpha}(b) \) is continuous, convex and piecewise linear function of \( b \in \mathbb{R}_p \), thus differentiable with gradient

\[
\frac{\partial D_{n\alpha}(b)}{\partial b} \big|_{b_0} = -\sum_{i=1}^{n} (x_{ni} - \bar{x}_n) I[R_{ni}(Y_i - x_{ni}^\top b_0) \geq n\alpha]
\]

at any point \( b_0 \in \mathbb{R}_p \) of differentiability. Notice that the gradients of the Jaeckel measure are just the Hájek scores. Using the uniform asymptotic linearity of the Hájek scores, (see Theorem 2.1), we can approximate the Jaeckel measure by a quadratic function.

Our subject of interest is to investigate the possible convergence of the process of R-estimators \( \{\hat{\beta}_{n\alpha}, \ 0 < \alpha < 1\} \), generated by the Hájek scores (14). Hájek and Šidák [4] proved the weak convergence of the process of Hájek’s scores to the Brownian bridge, under the i.i.d. observations as well as under contiguous (Pitman) alternatives. The intercept component of the \( \alpha \)-regression quantile reflects the population quantile \( F^{-1}(\alpha) \), but it is not the case of the slope components. However, the dispersion of the process of slopes expands for \( \alpha \to 0, 1 \) and its variance copies the variance of the \( \alpha \)-quantile, i.e. \( \alpha(1-\alpha)f^{-2}(F^{-1}(\alpha)) \).

Under conditions on the tails of distribution of model errors, such as imposed in [2], we can prove the weak convergence of the process \( \{f(F^{-1}(\alpha))(\hat{\beta}_{n\alpha} - \beta)\} \) to the vector of independent Brownian bridges over the compact subsets of \([0, 1]\).

### 2 Process of R-estimates of slopes and its asymptotics

Let \( \hat{\beta}_{n\alpha} \) be the R-estimator of \( \beta \), based on the Hájek rank scores, i.e. the minimizer of (17). Following the steps of [2], we shall first study the order of \( \hat{\beta}_{n\alpha} \) over \( (\alpha_{n\alpha}^*, 1 - \alpha_{n\alpha}^*) \) and show that the process of R-estimators converges to the vector of independent Brownian bridges.
for some $\alpha^*_n \downarrow 0$ as $n \to \infty$. This, in turn, will lead to the convergence over $\alpha \in (\alpha_0, 1 - \alpha_0)$ with any $0 < \alpha_0 < 1/2$ fixed.

Consider the process of the Hájek rank scores

$$A_n(n^{-1/2}b) = \left\{ A_{n\alpha}(n^{-1/2}b) = n^{-1/2} \sum_{i=1}^{n} (x_{ni} - \bar{x}_n) a_{n\alpha}(R_{ni}(Y_{i} - n^{-1/2}x_{ni}^\top b), \alpha) : 0 \leq \alpha \leq 1 \right\}$$

(2.1)

for $b \in \mathbb{R}^p$. The R-estimator $\beta_{n\alpha}$ is the minimizer of $D_{n\alpha}(n^{-1/2}b)$ and $A_{n\alpha}(n^{-1/2}b)$ is its gradient, due to (1.7) and (1.8). The results in [2] and [9] imply that the process (2.1) is uniformly asymptotically linear in $b$, what enables to approximate $D_{n\alpha}(n^{-1/2}b)$ by a quadratic function and then to approximate $\beta_{n\alpha}$ by its minimizer.

In order to realize these approximations, we impose the following conditions on the distribution of the model errors and on the triangular array of covariates $x_{n1}, \ldots, x_{nn}$. These conditions are only sufficient and apparently can be weakened.

(F1) The density $f(x) = F'(x)$ is absolutely continuous and bounded with bounded derivative $f'$ for $A < x < B$, where $-\infty \leq A = \sup\{x : F(x) = 0\}$ and $+\infty \geq B = \inf\{x : F(x) = 1\}$.

(F2) The density $f(x) = F'(x)$ is monotonically decreasing as $x \downarrow A$ or $x \uparrow B$ and $f'(x)/f(x) \leq c|x|$ for $x \geq K(0), c > 0$.

(F3) $|F^{-1}(\alpha)| \leq c(\alpha(1 - \alpha))^{-a}$ and similarly, $1/f(F^{-1}(\alpha)) \leq c(\alpha(1 - \alpha))^{-a-1}$ for $0 < \alpha \leq \alpha_0$ and $1 - \alpha_0 \leq \alpha < 1$ where $0 < a < \frac{1}{4} - \varepsilon, \varepsilon > 0, 0 < \alpha_0 \leq 1/2$.

(X1) The matrix

$$Q_n = \sum_{i=1}^{n} (x_{ni} - \bar{x}_n)(x_{ni} - \bar{x}_n)^\top, \quad \bar{x}_n = \frac{1}{n} \sum_{i=1}^{n} x_{ni}$$

has the rank $p$ and $n^{-1}Q_n \to C$ as $n \to \infty$, where $C$ is a positively definite $p \times p$ matrix. Moreover, we assume

$$\lim_{n \to \infty} \max_{1 \leq i \leq n} (x_{ni} - \bar{x}_n)^\top Q_n^{-1}(x_{ni} - \bar{x}_n) = 0 \quad (\text{Noether condition}).$$

(2.2)

(X2) $n^{-1} \sum_{i=1}^{n} \|x_{ni}\|^4 = O(1)$ as $n \to \infty$, and

$$\max_{1 \leq i \leq n} \|x_{ni}\| = O\left(n^{(2(b-a) - \delta)/(1+\delta)}\right) \text{ as } n \to \infty \text{ for some } b > 0, \delta > 0 \text{ such that } 0 < b - a < \frac{\delta}{2}.$$

As a consequence of Section V.3.5 in [4], we get the following weak convergence in the Prokhorov topology under $b = 0$

$$\left\{ n^{1/2}Q_n^{-1/2}A_{n\alpha}(0) : 0 \leq \alpha \leq 1 \right\} \overset{D}{\to} W^*_p$$

(2.3)

as $n \to \infty$, where $W^*_p$ is the vector of $p$ independent Brownian bridges (see [4] and [1]). Furthermore, under a sequence of contiguous alternatives, when $Y_{ni} = Y_{ni}^0 + n^{-1/2}x_{ni}^\top b$, $i = 1, \ldots, n$ with $Y_{ni}^0$ independent having distribution function $F$, there also applies the following convergence to the vector of $p$ independent Brownian bridges

$$\left\{ n^{1/2}Q_n^{-1/2}A_n(\alpha, n^{-1/2}b) - n^{-1/2}Q_n^{1/2}bf(F^{-1}(\alpha)) : 0 \leq \alpha \leq 1 \right\} \overset{D}{\to} W^*_p$$

(2.4)
as $n \to \infty$ (see [4], Theorem VI.3.2). The first result is the uniform asymptotic linearity of $A_n(\alpha, n^{-1/2}b)$ in $b$, proven in [9].

Denote

$$
\sigma_\alpha = \frac{(\alpha(1-\alpha))^{1/2}}{f(F^{-1}(\alpha))}, \quad 0 < \alpha < 1 \quad \text{and} \quad \alpha^*_n = \frac{1}{n^{1+4b}} \quad \text{with} \quad b \quad \text{from} \quad (X2).
$$

(2.5)

**Theorem 2.1** Assume that $F$ and $X_n$ satisfy (F1)–(F3) and (X1)–(X2). Then

$$
\sup \left\{ \left| \frac{1}{2} A_n(\alpha, n^{-1/2} \sigma_\alpha b) - A_n(\alpha, 0) + n^{-1} Q_n b \right| : \|b\| \leq K, \quad \alpha^*_n \leq \alpha \leq 1 - \alpha^*_n \right\} \overset{p}{\to} 0
$$

and

$$
\sup \left\{ \left| A_n(\alpha, n^{-1/2} b) - A_n(\alpha, 0) + f(F^{-1}(\alpha))n^{-1} Q_n b \right| : \|b\| \leq K, \quad 0 \leq \alpha \leq 1 \right\} \overset{p}{\to} 0
$$

as $n \to \infty$, for any fixed $K$, $0 < K < \infty$.

**Proof.** The theorem is proven in [9].

The following theorem gives the asymptotic behavior of the R-estimator of slope parameter over the interval $[\alpha^*_n, 1 - \alpha^*_n]$.

**Theorem 2.2** Under the conditions of Theorem 2.1, as $n \to \infty$,

$$
\sup \left\{ n^{1/2} \sigma_\alpha^{-1} \left\| \hat{\beta}_{na} - \beta \right\| : \alpha^*_n \leq \alpha \leq 1 - \alpha^*_n \right\} = O_p(1)
$$

(2.8)

and

$$
\sup \left\{ n^{1/2} \sigma_\alpha^{-1} \left\| \hat{\beta}_{na} - \beta - (\alpha(1-\alpha))^{-1/2} n Q_n^{-1} A_{na}(0) \right\| : \alpha^*_n \leq \alpha \leq 1 - \alpha^*_n \right\} = o_p(1).
$$

Moreover, the process

$$
\left\{ f(F^{-1}(\alpha)) Q_n^{1/2} (\hat{\beta}_{na} - \beta) : \alpha^*_n \leq \alpha \leq 1 - \alpha^*_n \right\}
$$

(2.9)

converges to the vector of independent Brownian bridges.

**Proof.** The theorem is proven in Section 3.

As a consequence, we conclude that the process $\left\{ f(F^{-1}(\alpha)) Q_n^{1/2} (\hat{\beta}_{na} - \beta) \right\}$ converges to the vector of Brownian bridges over the interval $[\alpha_0, 1 - \alpha_0]$ for any fixed $0 < \alpha_0 < 1/2$, i.e. over the compact subsets of $(0,1)$.

**Corollary 2.1** Under the conditions of Theorem 2.2, the process

$$
\left\{ f(F^{-1}(\alpha)) Q_n^{1/2} (\hat{\beta}_{na} - \beta) : 0 < \alpha < 1 \right\}
$$

(2.10)

converges to the vector of independent Brownian bridges in $\mathcal{D}(0,1)^p$. The convergence over $0 < \alpha < 1$ is in the sense that the process converges over the interval $[\alpha_0, 1 - \alpha_0]$ for any fixed $0 < \alpha_0 < 1/2$, i.e. converges over the compact subsets of $(0,1)$.
3 Proofs

Proof of Theorem 2.2

Notice that
\[ t_{n\alpha} = n^{1/2} \sigma^{-1}_\alpha (\tilde{\beta}_{n\alpha} - \beta) \] (3.1)
minimizes \([D_{n\alpha} (n^{-1/2} \sigma a b) - D_{n\alpha} (0)]\). Theorem 2.1 leads to the following quadratic approximation of \(D_{n\alpha}(b)\):
\[
\sup \left\{ \left( (a(1 - \alpha))^{-1/2} \sigma^{-1}_\alpha [D_{n\alpha} (n^{-1/2} \sigma a b) - D_n (0)] + b^\top A_{n\alpha} (0) \right) - \frac{1}{2} n^{-1} b^\top Q_n b \right\} : \left\langle b \right\rangle \leq K, \alpha_n^* \leq \alpha \leq 1 - \alpha_n^* \}
\] (3.2)
This further implies that as \(n \to \infty\)
\[
\min_{\left\langle b \right\rangle \leq K} \left[ (a(1 - \alpha))^{-1/2} \sigma^{-1}_\alpha [D_{n\alpha} (n^{-1/2} \sigma a b) - D_{n\alpha} (0)] \right] \] (3.3)
\[
= \min_{\left\langle b \right\rangle \leq K} \left[ \frac{1}{2} n^{-1} b^\top Q_n b - (a(1 - \alpha))^{-1/2} b^\top A_{n\alpha} (0) \right] + o_p(1)
\]
uniformly for \(\alpha_n^* \leq \alpha \leq 1 - \alpha_n^*\), for any \(K, 0 < K < \infty\).
Moreover,
\[
\min_{b \in \mathbb{R}^p} \left[ \frac{1}{2} n^{-1} b^\top Q_n b - (a(1 - \alpha))^{-1/2} b^\top A_{n\alpha} (0) \right] = -\frac{1}{2} (a(1 - \alpha))^{-1} A_{n\alpha}^\top (0) n Q_n^{-1} A_{n\alpha} (0)
\] (3.4)
and
\[
\arg \min_{b \in \mathbb{R}^p} \left[ \frac{1}{2} n^{-1} b^\top Q_n b - (a(1 - \alpha))^{-1/2} b^\top A_{n\alpha} (0) \right] = (a(1 - \alpha))^{-1/2} n Q_n^{-1} A_{n\alpha} (0)
\] (3.5)
Notice that \(\|u_{n\alpha}\| = O_p(1)\) uniformly in \(\alpha_n^* \leq \alpha \leq 1 - \alpha_n^*\) by (3.1). Inserting \(b = u_{n\alpha}\) in (3.2), we obtain
\[
\sup \left\{ \left( (a(1 - \alpha))^{-1/2} \sigma^{-1}_\alpha [D_{n\alpha} (n^{-1/2} \sigma a u_{n\alpha}) - D_n (0)] + \frac{1}{2} (a(1 - \alpha))^{-1/2} A_{n\alpha}^\top (0) n Q_n^{-1} A_{n\alpha} (0) \right) : \alpha_n^* \leq \alpha \leq 1 - \alpha_n^* \}\] (3.6)
Hence, using the convexity of \(D_n\), we apply the approach of Pollard in [18] and conclude
\[
\sup \left\{ \left\| t_{n\alpha} - u_{n\alpha} \right\| : \alpha_n^* \leq \alpha \leq 1 - \alpha_n^* \right\} = o_p(1).
\] (3.7)
The convergence of (2.10) to the vector of Brownian bridges follows from (2.3).

If \(1 - \alpha \geq 1 - \alpha_n^*\), then \(R_{ni} (Y_i - x_i^\top b) \geq n (1 - \alpha)\) only for the maximal residual \(Y_i - x_i^\top b\). Hence the estimator \(\tilde{\beta}_{n(1 - \alpha)}\) minimizes the maximal residual over \(b \in \mathbb{R}^p\). More precisely,
\[
\tilde{\beta}_{n(1 - \alpha)} = \arg \min_{b \in \mathbb{R}^p} \left\{ \left[ Y_{ni} - \bar{Y}_n - (x_{ni} - \bar{x}_n)^\top b \right]_{n \times p} \right\}. \] (3.8)
Denote as $D_n$ the antirank of the maximal residual. Then

$$[Y_{nD_n} - \bar{Y}_n - (x_{nD_n} - \bar{x}_n)^\top \tilde{\beta}_{n(1-\alpha)}] \leq [Y_{nD_n} - \bar{Y}_n - (x_{nD_n} - \bar{x}_n)^\top b],$$

hence

$$(x_{nD_n} - \bar{x}_n)^\top \tilde{\beta}_{n(1-\alpha)} \geq (x_{nD_n} - \bar{x}_n)^\top b$$

for any $b \in \mathbb{R}_p$, including 0 and any other estimator of $\beta$. Moreover, notice that $\tilde{\beta}_{n(1-\alpha)}$ is constant for $0 < \alpha \leq \alpha^*_n$, i.e.

$$\tilde{\beta}_{n(1-\alpha)} = \tilde{\beta}_{n(1-\alpha^*_n)} \text{ for } 0 < \alpha \leq \alpha^*_n.$$  

Analogously, $\tilde{\beta}_{n00} = \tilde{\beta}_{n00^*}$ for $0 < \alpha_0 \leq \alpha^*_n$, hence the convergence holds over $[\alpha_0, 1 - \alpha_0]$ for any $0 < \alpha_0, 1/2$ and thus for the compact subintervals of $(0, 1)$.

\[\blacksquare\]

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