THE COHOMOLOGY OF THE AFFINE DELIGNE-LUSZTIG
VARIETIES IN THE AFFINE FLAG MANIFOLD OF $GL_2$

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1. Introduction

Let $k$ be a field with $q$ elements, and let $ar{k}$ be an algebraic closure. Let further $F = k((t))$, $L = ar{k}((t))$ and $\sigma$ the Frobenius morphism of $L/F$. Thus $\sigma(\sum a_n t^n) = \sum \sigma(a_n) t^n$. Denote the valuation on $L$ by $v_L$ and the ring of integers in $F$ by $\sigma_F = k[[t]]$. Let $G$ be a split connected reductive group over $k$, and let $T$ be a split maximal torus in $G$. The aim of this work is to compute the induced representations of $G(L)/I$ of $G$ is defined by

$$X_w(b) = \{ xI \in G(L)/I : x^{-1} b \sigma(x) \in I w I \}.$$

It is a locally closed subset of $G(L)/I$, and one provides it with the reduced induced sub-Ind-scheme structure. In fact, it is a scheme locally of finite type, defined over $k$. Up to isomorphism, it depends only on the $\sigma$-conjugacy class of $b$: if $b' = g^{-1} b \sigma(g)$, then $x \mapsto g^{-1} x$ is an isomorphism $X_w(b) \rightarrow X_w(b')$. The subgroup $J_b = \{ g \in G(L) : g^{-1} b \sigma(g) = b \}$ of $G(L)$ acts by left multiplication on $X_w(b)$. If $b, c \in G(L)$ are $\sigma$-conjugate, then the groups $J_b$ and $J_c$ are conjugate. The action of $J_b$ on $X_w(b)$ induces an action of $J_b$ on the $l$-adic cohomology of $X_w(b)$ with compact supports.

The aim of this work is to compute the induced representations of $J_b$ on these cohomology groups for $G = GL_2$. Consider the diagonal torus $T \subset GL_2$, the standard Iwahori subgroup $I \subset GL_2(L)$ and the natural embedding $SL_2 \subset GL_2$. Let $W_a$ denote the affine Weyl group of $SL_2(L)$ corresponding to the torus $T^{SL_2} = T \cap SL_2$. The extended affine Weyl group $\tilde{W}$ of $GL_2(L)$ is given by a split extension $\tilde{W} = W_a \times \mathbb{Z}$. We choose a splitting by sending $v \in \mathbb{Z}$ to $\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}^v$.

Now fix a $b \in GL_2(L)$ and $w \in W_a$. Since the valuation of the determinant of all matrices in $I$ is zero, $X_{(w, v)}(b) \neq \emptyset$ implies $v = v_L(\det(b))$. We write $X_w(b) := X_{(w, v_L(\det(b)))}(b)$. As already mentioned, $X_w(b)$ depends up to isomorphism only on the $\sigma$-conjugacy class of $b$. Further if $c$ is in the center of $GL_2(L)$, then we have $X_w(b) = X_w(cb)$ for all $w \in W_a$ (Lemma 2.18). All in all, we have only three essentially different cases for $b$ in which we determine $X_w(b)$ explicitly together with the action of $J_b$. The three cases are presented in Table I. In each of these cases, for all $w \in W_a$ such that $X_w(b) \neq \emptyset$ (except for some special values of $w$, see below), we have the $J_b$-equivariant isomorphisms:
where $S$ is a smooth connected variety whose dimension depends on $\ell(w)$ and $b$, and $K^{(m)}_b$ is a subgroup of $J_b$, where $m \in \{0, 1\}$ depends on $w$. The group $J_b$ acts by permuting the connected components and $K^{(m)}_b$ is the stabilizer of one of them. All these objects for the three different cases are collected in Table 1.

We set $b_1 := \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}$. For $m \in \{0, 1\}$ we write $g_m := \begin{pmatrix} 1 & 0 \\ 0 & t^m \end{pmatrix}$. Further, let $D$ denote the quaternion division algebra over $F$, containing the unramified extension of $F$ of degree two, and $U_D$ the unit subgroup of its valuation ring.

| $b$ | $X_w(b) \neq \emptyset$ if | $J_b$ | $K^{(m)}_b$ | $S$ |
|-----|--------------------------|-------|-------------|-----|
| 1   | $\ell(w) > 0$ odd       | $GL_2(F)$ | $g_mGL_2(\mathfrak{o}_F)g_m^{-1}$ | $\mathbb{A}_{\ell(w)-1} \times (\mathbb{P}^1 - \mathbb{P}^1(k))$ |
| $(1 \ 0 \ 0 \ \alpha)$, $\alpha > 0$ | $\ell(w) - \alpha > 0$ odd | $T(F)$ | $T(\mathfrak{o}_F)$ | $\mathbb{A}_{\ell(w)-\alpha-1} \times (\mathbb{P}^1 - \{0, \infty\})$ |
| $b_1$ | $\ell(w)$ even         | $D^\times$ | $U_D$       | $\mathbb{A}_{\ell(w)-\frac{1}{2}}$ |

The special cases excluded in this table are:

(i) $b = 1$, $w = 1$;

(ii) $b = \begin{pmatrix} t^\alpha & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha > 0$, $w$ such that $\ell(w) = \alpha$.

In all these special cases the variety $X_w(b)$ is a disjoint union of points. The precise statements are Propositions 4.12, 4.13, and 4.15.

Now we turn to the cohomology. Therefore, we make a base change to $\bar{k}$ and denote $X_w(b) \times_{\text{Spec } k} \text{Spec } \bar{k}$ by $\overline{X}_w(b)$. The schemes $\overline{X}_w(b)$ are locally of finite type, but not of finite type. In our case the $\overline{X}_w(b)$ are disjoint unions of schemes of finite type. Therefore, we take the cohomology with compact supports which commutes with colimits. Thus the cohomology groups of $\overline{X}_w(b)$ are direct sums of the cohomology groups of the connected components. The groups $J_b$ and $\Gamma := \text{Gal}(\bar{k}/k)$ act on $H^*_c(\overline{X}_w(b), \mathbb{Q})$ in a natural way. The group $J_b$ is in every case locally profinite and the representations $H^*_c(\overline{X}_w(b), \mathbb{Q})$ are smooth. If we consider in the following a representation of a locally profinite group, then we mean a smooth representation.

Consider first the case $b = 1$ and let $1 \neq w \in W_a$ such that $X_w(1) \neq \emptyset$. Put $G := J_1 = GL_2(F)$ and $K := K^{(0)}_1 = GL_2(\mathfrak{o}_F)$. Assume for simplicity that $m \in \{0, 1\}$ corresponding to this $w$ equals 0 (compare (1.1)). Consider the representation

$$\overline{\text{St}} = \inf_{GL_2(k)}^{K} \text{St}_{GL_2(k)}$$
where \( \text{St}_{GL_2(k)} \) denotes the Steinberg representation of \( GL_2(k) \) and we inflate with respect to the natural projection \( K = GL_2(\mathfrak{o}_F) \to GL_2(k) \). We prove (Proposition 5.4) that there are the following isomorphisms of \( G \times \Gamma \)-modules:

\[
H^r_{\Gamma}(\overline{X}_w(1), \overline{Q}_l) \cong \begin{cases} 
\text{c} - \text{Ind}_K^G \text{St} \left( \frac{\ell(w)-1}{2} \right) & \text{if } r = \ell(w), \\
\text{c} - \text{Ind}_K^G \text{St} \left( \frac{\ell(w)-3}{2} \right) & \text{if } r = \ell(w) + 1, \\
0 & \text{else.}
\end{cases}
\]

The idea of the proof is to reduce the cohomology of \( X_w(1) \) to the cohomology of the Drinfeld upper halfplane \( \Omega^2_k = \mathbb{P}^1 - \mathbb{P}^1(k) \) and the action of \( K_1^{(m)} \) to the action of the finite group \( GL_2(k) \) on \( \Omega^2_k \).

To investigate these representations, we consider the \( \Gamma \)-modules \( \text{Hom}_G(H^r_{\Gamma}(\overline{X}_w(1), \overline{Q}_l), \pi) \) where \( \pi \) ranges over smooth irreducible representations of \( G \). As shown in [BH], there are two types of such representations: the cuspidal and the noncuspidal ones. First of all, there are no nonzero morphisms into cuspidal representations. The noncuspidal (irreducible) representations are of one of the following types: \( \text{Ind}_B^G \chi \), where \( \chi \) ranges over all character of the diagonal torus (with some nonrelevant exceptions) and \( B \) is the Borel subgroup of upper triangular matrices; \( \phi_G = \phi \circ \text{det} \) and \( \phi \cdot \text{St}_G \) where \( \phi \) ranges over all characters of \( F^\times \). A character of \( T(F) \) resp. of \( F^\times \) is said to be \( \text{unramified} \), if it is trivial on \( T(\mathfrak{o}_F) \) resp. on \( \mathfrak{o}_F^\times \).

**Theorem 1.1.** Let \( 1 \neq w \in W_o \) such that \( X_w(1) \neq \emptyset \).

(i) Let \( \chi \) be a character of \( T(F) \). Then

\[
\text{Hom}_G(H^r_{\Gamma}(\overline{X}_w(1), \overline{Q}_l), \text{Ind}_B^G \chi) = \begin{cases} 
\overline{Q}_l(\frac{3-\ell(w)}{2}) & \text{if } \chi \text{ unramified,} \\
0 & \text{else.}
\end{cases}
\]

(ii) Let \( \phi \) be a character of \( F^\times \). Then

\[
\text{Hom}_G(H^r_{\Gamma}(\overline{X}_w(1), \overline{Q}_l), \phi_G) = \begin{cases} 
\overline{Q}_l(\frac{3-\ell(w)}{2}) & \text{if } \phi \text{ unramified,} \\
0 & \text{else}
\end{cases}
\]

(iii) Let \( \phi \) be a character of \( F^\times \). Then \( \text{Hom}_G(H^r_{\Gamma}(\overline{X}_w(1), \overline{Q}_l), \phi \cdot \text{St}_G) = 0 \).

(iv) Let \( \pi \) be a cuspidal representation of \( G \). Then \( \text{Hom}_G(H^r_{\Gamma}(\overline{X}_w(1), \overline{Q}_l), \pi) = 0 \).

**Theorem 1.2.** Let \( 1 \neq w \in W_o \) such that \( X_w(1) \neq \emptyset \).

(i) Let \( \chi \) be a character of \( T(F) \). Then

\[
\text{Hom}_G(H^r(\overline{X}_w(1), \overline{Q}_l), \text{Ind}_B^G \chi) = \begin{cases} 
\overline{Q}_l(\frac{1-\ell(w)}{2}) & \text{if } \chi \text{ unramified,} \\
0 & \text{else.}
\end{cases}
\]

(ii) Let \( \phi \) be a character of \( F^\times \). Then \( \text{Hom}_G(H^r(\overline{X}_w(1), \overline{Q}_l), \phi_G) = 0 \).

(iii) Let \( \phi \) be a character of \( F^\times \). Then

\[
\text{Hom}_G(H^r(\overline{X}_w(1), \overline{Q}_l), \phi \cdot \text{St}_G) = \begin{cases} 
\overline{Q}_l(\frac{1-\ell(w)}{2}) & \text{if } \phi \text{ unramified,} \\
0 & \text{else}
\end{cases}
\]
(iv) Let \( \pi \) be a cuspidal representation of \( G \). Then \( \text{Hom}_G(H_c^{\ell(w)}(X_w(1), \mathbb{Q}_l), \pi) = 0 \).

The essential ingredient in the proof of these Theorems (as well as the corresponding Theorems for \( b \neq 1 \)) is the Frobenius reciprocity ( [BH] 2.4-5) which one has to apply several times.

Let now \( b = \begin{pmatrix} 1 & 0 \\ 0 & \ell^\alpha \end{pmatrix} \) with \( \alpha > 0 \) and \( w \in W_a \) with \( \ell(w) > \alpha \) such that \( X_w(b) \neq \emptyset \). We have: \( J_b = T(F) = T(\mathfrak{a}_F) \times \mathbb{Z}^2 \). The groups \( H_c^r(X_w(b), \mathbb{Q}_l) \) are zero for \( r \neq \ell(w) - \alpha, \ell(w) - \alpha + 1 \). For \( r = \ell(w) - \alpha, \ell(w) - \alpha + 1 \), we have an isomorphism of \( T(F) \)-representations:

\[
H_c^r(X_w(b), \mathbb{Q}_l) \cong c - \text{Ind}_{T(\mathfrak{a}_F)}^{T(F)} 1_{\mathbb{Q}_l}. 
\]

In particular, \( T(\mathfrak{a}_F) \) acts trivially on \( H_c^r(X_w(b), \mathbb{Q}_l) \), and this representation is inflated from the regular representation of \( T(F)/T(\mathfrak{a}_F) = \mathbb{Z}^2 \) (Theorem 5.13). Thus the only information encoded in these representations is the action of \( J_b \) on the set of the connected components of \( X_w(b) \) by permutation. A similar situation occurs also in the next case. An analogous result is proven for the groups \( SL_2, SL_3 \) in [Zb].

Let now \( b = b_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \), and let \( w \in W_a \) be such that \( X_w(b_1) \neq \emptyset \). Then \( J_{b_1} = D^\times \) is the multiplicative group of the quaternion division algebra \( D \) over \( F \), and

\[
H_c^r(X_w(b_1), \mathbb{Q}_l) \cong \begin{cases} 
\text{Ind}_{U_D}^{D^\times} 1_{\mathbb{Q}_l}(-\ell(w)/2) & \text{if } r = \ell(w), \\
0 & \text{else},
\end{cases}
\]

are isomorphisms of \( D^\times \times \Gamma \)-modules.

For any character \( \chi \) of \( F^\times \) let \( \chi_D = \chi \circ \det \) be a character of \( D^\times \). These are all one-dimensional representations of \( D^\times \) ([BH] 53.5). Let further \( R\mathbb{Z} \) denote the regular representation of \( D^\times /U_D = \mathbb{Z} \).

**Theorem 1.3.** Let \( w \in W_a \) such that \( X_w(b_1) \neq \emptyset \). Then

\[
H_c^{\ell(w)}(X_w(b_1), \mathbb{Q}_l) \cong \text{inf}_{\mathbb{Z}}^{D^\times} R\mathbb{Z},
\]

as \( D^\times \)-representations.

(i) Let \( \chi \) be a character of \( F^\times \). Then

\[
\text{Hom}_{D^\times}(H_c^{\ell(w)}(X_w(b_1), \mathbb{Q}_l), \chi_D) = \begin{cases} 
\mathbb{Q}_l(-\ell(w)/2) & \text{if } \chi \text{ unramified,} \\
0 & \text{else.}
\end{cases}
\]

(ii) Let \( \pi \) be an irreducible representation of \( D^\times \) of dimension \( \geq 2 \). Then

\[
\text{Hom}_{D^\times}(H_c^{\ell(w)}(X_w(b_1), \mathbb{Q}_l), \pi) = 0.
\]

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2. AFFINE DELIGNE-LUSZTIG VARIETIES INSIDE THE AFFINE FLAG MANIFOLD

2.1. Notations.
Let \( k \) be a field with \( q \) elements, and let \( \bar{k} \) be an algebraic closure of \( k \). Let \( \sigma \) denote the Frobenius morphism of \( \bar{k}/k \). We set \( F = k((t)) \) and \( L = \bar{k}((t)) \). We extend \( \sigma \) to the Frobenius morphism of \( L/F \). Thus \( \sigma(\sum a_n t^n) = \sum \sigma(a_n) t^n \). We write \( o = \bar{k}[[t]] \) and \( o_F = k[[t]] \) for the valuation rings of \( L \) and \( F \), and \( p \) and \( p_F \) for their maximal ideals. Furthermore, we denote the valuation on \( L \) by \( v_L \).

2.2. The Bruhat-Tits building.
In this subsection, we recall the Bruhat-Tits buildings of the groups \( SL_2(L) \) and \( SL_2(F) \), and prove some facts about them which we will need later on. A detailed discussion can be found in [Br], chapters 4 and 5.

Let \( \sim \) be the equivalence relation on the set of all \( o \)-lattices in \( L^2 \) given by \( L \sim L' \) if and only if there is a scalar \( c \in L \) with \( cL = L' \).

**Definition 2.1.** The Bruhat-Tits building \( B_\infty \) of \( SL_2(L) \) is the one-dimensional simplicial complex such that

(i) its 0-dimensional simplices (vertices) are equivalence classes under \( \sim \) of \( o \)-lattices \( L \subset L^2 \);

(ii) two vertices are connected by a 1-dimensional simplex (alcove) if and only if there are representatives \( L_0, L_1 \) of them, such that \( tL_1 \subset L_0 \subset L_1 \).

Then \( B_\infty \) is a tree (a graph in which every two vertices are connected by exactly one path) with infinitely many vertices and infinitely many alcoves containing a fixed vertex. One can accomplish the same construction with \( F \) and \( o_F \) instead of \( L \) and \( o_L \). We denote the simplicial complex arising from this construction by \( B_1 \). It is the Bruhat-Tits building of \( SL_2(F) \). It is again a tree with infinitely many vertices. For a fixed vertex there are exactly \( q + 1 \) alcoves containing it. We see \( B_1 \) as a subset of \( B_\infty \), by sending an \( o_F \)-module \( L \subset F^2 \) to \( L \otimes_{o_F} o \subset L^2 \). Further \( \sigma \) acts on \( B_\infty \), and \( B_1 \) are exactly the fixed points.

**Definition 2.2.** Let \( L \) be an \( o \)-lattice in \( L^2 \), and let \( m \in \mathbb{Z} \) be such that \( \wedge^2 L = t^m o \). The type of the vertex represented by \( L \) is

\[
\begin{cases}
0 & \text{if } m \text{ is even}, \\
1 & \text{if } m \text{ is odd}.
\end{cases}
\]

Since scalar multiplication changes this integer by some even number, the Definition is independent of the choice of the representing lattice.

**Definition 2.3.**

(i) A gallery in \( B_\infty \) is a sequence \( (C_0, C_1, ..., C_n) \) of alcoves such that \( C_i \) and \( C_{i+1} \) are adjacent (i.e. have a common vertex) for every \( 0 \leq i \leq n - 1 \). A gallery is non-stuttering if \( C_i \neq C_{i+1} \) for all \( i \).

(ii) The length of the gallery \( \Gamma = (C_0, C_1, ..., C_n) \) is defined as \( \ell(\Gamma) = n \).
(iii) If \( \Gamma = (C_0, C_1, ..., C_n) \) and \( \Gamma' = (C'_0, C'_1, ..., C'_m) \) are two galleries such that \( C_n, C'_0 \) are adjacent, then the composite gallery is
\[
(\Gamma, \Gamma') = (C_0, C_1, ..., C_n, C'_0, C'_1, ..., C'_m).
\]

(iv) The inverse gallery of \( \Gamma = (C_0, C_1, ..., C_n) \) is the gallery
\[
\Gamma^{-1} = (C_n, C_{n-1}, ..., C_0).
\]

If \( \Gamma, \Gamma' \) are galleries, then the length of the composed gallery \( (\Gamma, \Gamma') \) is given by
\[
\ell(\Gamma, \Gamma') = \ell(\Gamma) + \ell(\Gamma') + 1.
\]

For every two alcoves \( C, D \) in \( \mathcal{B}_\infty \) resp. \( \mathcal{B}_1 \) there is a gallery of minimal length containing them both and having \( C \) as the first alcove. Since \( \mathcal{B}_\infty \) resp. \( \mathcal{B}_1 \) is a tree, this gallery is unique.

**Definition 2.4.** A gallery \( \Gamma = (C_0, C_1, ..., C_n) \) is called minimal if it is the minimal gallery connecting \( C_0 \) and \( C_n \).

**Definition 2.5.** Let \( \Gamma = (C_0, C_1, ..., C_n) \) be a gallery in \( \mathcal{B}_\infty \). A first vertex of \( \Gamma \) is a vertex of \( C_0 \) which satisfies the following condition:
\[
(*) \text{ It is not a vertex of } C_1 \text{ if } n > 0 \text{ and } C_0 \neq C_1.
\]

A last vertex of \( \Gamma \) is a first vertex of the inverse gallery \( \Gamma^{-1} \).

This means the following: if \( \ell(\Gamma) = 0 \) or \( \ell(\Gamma) > 0 \) and \( C_0 = C_1 \), then every vertex of its first alcove is a first vertex of \( \Gamma \). If \( \ell(\Gamma) > 0 \) and \( C_0 \neq C_1 \), then \( \Gamma \) has a unique first vertex: it is the vertex of \( C_0 \) which is not a vertex of \( C_1 \). In particular, every gallery has at least one first resp. last vertex. If a gallery has more than one alcove, and is minimal, then the first resp. last vertex is unique. We say that a gallery contains a vertex if it contains an alcove which has this vertex as one of its faces. Since \( \mathcal{B}_\infty \) is a tree, there is a unique gallery with minimal length, containing a vertex \( P \) and an alcove \( C \) of \( \mathcal{B}_\infty \), and having \( P \) as a first vertex. Analogously, there is a unique gallery of minimal length, containing two distinct vertices \( P, Q \) of \( \mathcal{B}_\infty \), and having \( P \) as a first vertex. Such galleries are minimal in the sense of Definition 2.4.

If \( P, Q \) are some simplices in \( \mathcal{B}_\infty \), then we say that a gallery \( \Gamma \) is stretched from \( P \) to \( Q \) if \( \Gamma \) is minimal and has \( P \) resp. \( Q \) as a first resp. last vertex or alcove.

**Definition 2.6.**

(i) The distance between two alcoves \( C, D \) in \( \mathcal{B}_\infty \) is the length of the gallery stretched from \( C \) to \( D \).

(ii) The distance between a vertex \( P \) and an alcove \( D \) in \( \mathcal{B}_\infty \) is the length of the gallery stretched from \( P \) to \( D \).

We will need the following characterisation of minimal galleries:

**Lemma 2.7.**

(i) A gallery \( \Gamma = (C_0, C_1, ..., C_n) \) in \( \mathcal{B}_\infty \) resp. in \( \mathcal{B}_1 \) is minimal if the following conditions hold:
(a) $\Gamma$ contains no alcove twice.
(b) For every alcove $C_i$ with $0 < i < n$ the following holds: if $P$, $Q$ are the two vertices of $C_i$, and $P$ is a vertex of $C_{i-1}$, then $Q$ is a vertex of $C_{i+1}$.
(ii) Let $\Gamma = (C_0, C_1, ..., C_n)$, $\Gamma' = (C'_0, C'_1, ..., C'_m)$ be two minimal galleries such that $C_n$ and $C'_0$ are adjacent. Then the composite gallery $(\Gamma, \Gamma')$ is minimal if the following conditions hold:
(c) $\Gamma$, $\Gamma'$ have no common alcoves.
(d) A last vertex of $\Gamma$ is a first vertex of $\Gamma'$.

Proof. The proofs for $\mathfrak{B}_\infty$ and $\mathfrak{B}_1$ are the same, thus we restrict ourselves to the case of $\mathfrak{B}_\infty$. At first we prove (i). We proceed by induction. The cases $\ell(\Gamma) = 0$ and $\ell(\Gamma) = 1$ are trivial. Assume now that (a) and (b) imply minimality for every gallery of length $\leq n$ with $n \geq 1$. Let $\Gamma = (C_0, C_1, ..., C_{n+1})$ be a gallery satisfying the conditions (a) and (b). Then the gallery $\Gamma_1 = (C_0, C_1, ..., C_n)$ satisfies these conditions, too and is minimal by the induction hypothesis. Now, removing the alcove $C_n$ (without vertices) from $\mathfrak{B}_\infty$, divides $\mathfrak{B}_\infty$ in two connected components, both of which are trees. By (a), the alcoves $C_{n-1}$, $C_n$, $C_{n+1}$ are pairwise distinct and by (b), the alcoves $C_{n-1}$, $C_{n+1}$ are not adjacent (otherwise $C_{n-1}$, $C_n$, $C_{n+1}$ would give a non-trivial cycle) and thus $C_{n-1}$, $C_{n+1}$ lie in different connected components, described above. By minimality of $\Gamma_1$, the gallery $(C_0, C_1, ..., C_{n-1})$ does not contain $C_n$. Thus $C_0$ lies in the same connected component as $C_{n-1}$. Thus every gallery connecting $C_0$ with $C_{n+1}$ contains $C_n$. This holds for the unique gallery $\Gamma_{min}$ stretched from $C_0$ to $C_{n+1}$, and thus it will have the form $\Gamma_{min} = (C_0, D_1, ..., D_r, C_n, D_{r+1}, ..., D_s, C_{n+1})$. Hence $(C_0, D_1, ..., D_r, C_n)$ and $(C_n, D_{r+1}, ..., D_s, C_{n+1})$ are minimal. But $\Gamma_1 = (C_0, C_1, ..., C_n)$ is minimal and $C_n$, $C_{n+1}$ are adjacent, thus $\Gamma_{min} = (C_0, C_1, ..., C_n, C_{n+1}) = \Gamma$. Hence $\Gamma$ is minimal.

To prove (ii) we notice first that from (c) and minimality of $\Gamma$, $\Gamma'$ the condition (a) of part (i) for the composed gallery $(\Gamma, \Gamma')$ follows. The condition (b) is also clear for all alcoves of $(\Gamma, \Gamma')$ except for $C_n$ and $C'_0$. If $\ell(\Gamma) = 0$, then the condition (b) for $C_n$ is empty. Assume that $\ell(\Gamma) > 0$. Then, since $\Gamma$ is minimal, its last vertex is unique: it is the vertex $P$ of $C_n$ which is not a vertex of $C_{n-1}$. Now by (d), $P$ is a common vertex of $C_n$ and $C'_0$. This is exactly the condition (b) of (i) for the alcove $C_n$. The verification of the condition (b) for $C'_0$ can be done similarly (one can also invert all involved galleries and use above considerations again). 

Let

$$I = \begin{pmatrix} o^\times & o \\ p & o^\times \end{pmatrix}$$

be the standard Iwahori subgroup of $GL_2(L)$, and

$$I^{SL_2} = I \cap SL_2(L)$$

the standard Iwahori subgroup of $SL_2(L)$.

The groups $SL_2(L)$ and $GL_2(L)$ act transitively on the set of the alcoves of $\mathfrak{B}_\infty$. Further $SL_2(L)$ acts transitively on all vertices with the same type $m$ in $\mathfrak{B}_\infty$ (Lemma 4.10), and
$GL_2(L)$ acts transitively on the set of all vertices. One has an obvious base vertex of type 0 represented by $0 \oplus 0$ with stabilizer $SL_2(0)$, resp. $GL_2(0)$.

**Definition 2.8.** Let $C^0_M$ denote the base alcove, represented by $0 \oplus t \subseteq 0 \oplus 0$.

The stabilizer of $C^0_M$ under the action of $SL_2(L)$ is $I^{SL_2}$. Let $T$ be the diagonal torus of $GL_2$, and $T^{SL_2} = T \cap SL_2$ the diagonal torus of $SL_2$. Let further $N_{SL_2}(T^{SL_2})$ denote the normalizer of $T^{SL_2}$ in $SL_2$. Then we define:

**Definition 2.9.**

(i) The standard apartment $A_M$ is the minimal full subcomplex of $\mathcal{B}_\infty$ whose alcoves lie in the $N_{SL_2}(T^{SL_2})(L)$-orbit of $C^0_M$.

(ii) The affine Weyl group of $SL_2(L)$ is the group

$$W_a = N_{SL_2}(T^{SL_2})(L)/(T^{SL_2})(0).$$

One has $T^{SL_2}(0) = N_{SL_2}(T^{SL_2})(L) \cap I^{SL_2}$. Thus $W_a$ acts simply transitively on the set of the alcoves in the standard apartment. By the choice of the alcove $C^0_M$, we identify $W_a$ with $A_M$. Furthermore we number the alcoves in $A_M$ by integers and call the $i$-th alcove $C^i_M$ such that $C^i_M$ is represented by $0 \oplus t^{i+1} \subseteq 0 \oplus t^i 0$. Then $\begin{pmatrix} t^{-i} & 0 \\ 0 & t^i \end{pmatrix}$ corresponds under the above identification to $C^{-2i}_M$ and $\begin{pmatrix} 0 & t^{-i} \\ -t^i & 0 \end{pmatrix}$ corresponds to $C^2_{M-1}$ for $i \in \mathbb{Z}$.

The group $W_a$ is a Coxeter group on two generators of order two with no further relations. Let $\ell(w)$ denote the length of the element $w \in W_a$. If $wC^0_M = C^i_M$, then $\ell(w) = |i|$.

We have a Bruhat decomposition of $SL_2(L)$:

$$SL_2(L) = \bigcup_{w \in W_a} I^{SL_2}wI^{SL_2}.$$ 

**Definition 2.10.** The relative position map on the set of the alcoves is

$$\text{inv} : SL_2(L)/I^{SL_2} \times SL_2(L)/I^{SL_2} \to W_a.$$

It maps the cosets $xI^{SL_2}$, $yI^{SL_2}$ to the unique element $w$ of the affine Weyl group such that $x^{-1}y \in I^{SL_2}wI^{SL_2}$.

Via the above identification of $W_a$ with $A_M$, we see the relative position of two alcoves as an alcove in the standard apartment. Let $D, D'$ be two alcoves in $\mathcal{B}_\infty$ and let $\Gamma$ be the gallery, stretched from $D$ to $D'$, with length $i$. Then

$$\text{inv}(D, D') = \begin{cases} C^i_M & \text{if a first vertex of } \Gamma \text{ has type 0}, \\ C^{-i}_M & \text{if a first vertex of } \Gamma \text{ has type 1} \end{cases}.$$

**Definition 2.11.** Let $N_{GL_2}(T)$ be the normalizer of $T$ in $GL_2$. The extended affine Weyl group is

$$\tilde{W} = N_{GL_2}(T)(L)/T(0).$$
Then a Bruhat-decomposition of $GL_2(L)$ is given by:

$$GL_2(L) = \bigcup_{\tilde{w} \in \tilde{W}} I\tilde{w}I.$$  

The affine and the extended affine Weyl groups are closely related: the inclusion $N_{SL_2}(T^{SL_2})(L) \hookrightarrow N_{GL_2}(T)(L)$ induces a short exact sequence:

$$1 \to W_a \to \tilde{W} \to \mathbb{Z} \to 0,$$

where the last map is induced by $v_L \circ \det$. This sequence is split by the map sending $m \in \mathbb{Z}$ to \( \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \), and we have $\tilde{W} = W_a \rtimes \mathbb{Z}$. Thus an element of $\tilde{W}$ is given by a pair $(w, m)$, where $w \in W_a$ and $m \in \mathbb{Z}$.

### 2.3. The affine flag manifold.

In this subsection we define the affine flag manifolds of $SL_2$ and $GL_2$ and show that they are Ind-schemes. A similar situation in characteristic 0 with the affine Grassmanian instead of the affine flag manifold is discussed in ([BL] section 2).

If $R$ is a $k$-algebra, then a $R[[t]]$-submodule of $R((t))^2$ is a $R[[t]]$-submodule $\mathfrak{L}$ such that there exists a positive integer $N$ with $t^N R[[t]]^2 \subseteq \mathfrak{L} \subseteq t^{-N} R[[t]]^2$ and $t^{-N} R[[t]]^2 / \mathfrak{L}$ is a locally on $\text{Spec}(R)$ free module of finite rank.

**Definition 2.12.** The affine flag manifold $X^{SL_2}$ of $SL_2$ is the functor, whose $R$-valued points are

$$X^{SL_2}(R) = \begin{cases} \mathfrak{L}_0, \mathfrak{L}_1 \in R[[t]] \text{-lattices}, \wedge^2 \mathfrak{L}_0 = R[[t]], \\ t\mathfrak{L}_1 \subseteq \mathfrak{L}_0 \subseteq \mathfrak{L}_1, \mathfrak{L}_0/t\mathfrak{L}_1 \text{ is loc. on } \text{Spec}(R) \text{ free of rank 1} \end{cases}.$$

Then $X^{SL_2}$ is an Ind-scheme: set

$$X^{SL_2,n}(R) = \{ (\mathfrak{L}_0 \subseteq \mathfrak{L}_1) \in X^{SL_2}(R) : t^n R[[t]]^2 \subseteq \mathfrak{L}_0 \subseteq \mathfrak{L}_1 \subseteq t^{-n} R[[t]]^2 \}.$$

Then $X^{SL_2,n}$ is isomorphic to a closed subscheme of a partial flag manifold of $V = t^{-n} k[[t]]^2 / t^n k[[t]]^2$: let $\mathfrak{F}l_{2n,2n+1} (V)$ be the partial flag manifold of $V$ such that for every $k$-algebra $R$ the $R$-valued points are

$$\mathfrak{F}l_{2n,2n+1}(V)(R) = \begin{cases} L_0 \subseteq L_1 \subseteq R \otimes V : L_i, R \otimes V/L_i \text{ are loc. on } \text{Spec}(R) \text{ free } R\text{-modules} \\ \text{for } i = 0, 1; \text{rk}(L_0) = 2n, \text{rk}(L_1) = 2n + 1 \end{cases}.$$ 

Let $\mathfrak{F}l_{2n,2n+1}^i (V)$ be the closed subscheme of $\mathfrak{F}l_{2n,2n+1}(V)$ parametrizing the chains of $t$-stable subspaces of $V$ of dimensions $2n, 2n + 1$. There is a closed embedding:

$$X^{SL_2,n} \longrightarrow \mathfrak{F}l_{2n,2n+1}^i (V),$$

$$(\mathfrak{L}_0 \subseteq \mathfrak{L}_1) \mapsto (\mathfrak{L}_0 \cap t^i k[[t]]^2 \subseteq \mathfrak{L}_1 / t^n k[[t]]^2),$$

which is an isomorphism on $K$-valued points for every field $K$ containing $k$ (compare [BL] 2.4. There is also remarked that, as Genestier pointed out the scheme on the right hand side is in general not reduced, even in the easiest case). Thus we have a filtration of $X^{SL_2}$
by closed subschemes of finite type over $k$:

$$X^{SL_2,1} \subsetneq X^{SL_2,2} \subsetneq \ldots \subsetneq X^{SL_2}.$$ 

The group $SL_2(L)$ acts transitively by linear transformations on $X^{SL_2}(\bar{k})$ and the stabilizer of the $\bar{k}$-valued point $\mathfrak{o} \oplus t\mathfrak{o} \subsetneq \mathfrak{o} \oplus \mathfrak{o}$ is $I^{SL_2}$, thus

$$X^{SL_2}(\bar{k}) = SL_2(L)/I^{SL_2}.$$ 

We identify the set of $\bar{k}$-valued points of $X^{SL_2}$ with the set of alcoves in $\mathfrak{B}_\infty$. Under this identification the $k$-valued points $X^{SL_2}(k) = SL_2(F)/(I^{SL_2} \cap SL_2(F))$ correspond to the alcoves in $\mathfrak{B}_1$.

The Ind-scheme $X^{SL_2}$ is reduced ([PR] 6.1), and connected ([PR] 5.1).

**Definition 2.13.** The affine flag manifold $X$ of $GL_2$ is the functor whose $R$-valued points are

$$X(R) = \left\{ (\mathfrak{L}_0 \subsetneq \mathfrak{L}_1) \subseteq R(t)^2 : \mathfrak{L}_0, \mathfrak{L}_1 \text{ are } R[[t]]\text{-lattices, } t\mathfrak{L}_1 \subsetneq \mathfrak{L}_0 \subsetneq \mathfrak{L}_1, \mathfrak{L}_0/t\mathfrak{L}_1 \text{ is loc. on } \text{Spec}(R) \text{ free of rank 1} \right\}.$$ 

Then $X$ is again an Ind-scheme, and we have: $X(\bar{k}) = GL_2(L)/I$. Set further

$$X^{(v)}(R) = \left\{ (\mathfrak{L}_0 \subsetneq \mathfrak{L}_1) \in X(R) : \text{Spec}(K) \rightarrow \text{Spec}(R), \text{ with } K \text{ field: } \wedge^2(\mathfrak{L}_0 \otimes K) = t^vK[[t]] \right\}.$$ 

Then

$$X = \bigsqcup_v X^{(v)}.$$ 

The Ind-scheme $X$ is not reduced ([PR] 6.5) in contrast to $X^{SL_2}$. We have:

$$(X^{(v)})_{\text{red}}(R) = \{(\mathfrak{L}_0 \subsetneq \mathfrak{L}_1) \in X(R) : \wedge^2 \mathfrak{L}_0 = t^vR[[t]]\}.$$ 

If $r_v \in GL_2(L)$ satisfies $v_L(\det(r_v)) = v$, then the left multiplication by $r_v$ gives an isomorphism $X^{SL_2} \stackrel{\sim}{\rightarrow} (X^{(v)})_{\text{red}}$.

We identify $X^{SL_2}$ with $(X^{(0)})_{\text{red}}$ by choosing $r_0 = 1$:

$$X^{SL_2} \stackrel{\sim}{\rightarrow} (X^{(0)})_{\text{red}} \hookrightarrow X_{\text{red}}.$$ 

Let $H \subset GL_2(L)$ be the subgroup of all matrices with the valuation of the determinant equal 0. On the $\bar{k}$-valued points the above inclusion is given by

$$X^{SL_2}(\bar{k}) = SL_2(L)/I^{SL_2} = H/I \hookrightarrow GL_2(L)/I.$$ 

Thus, on the left hand side stands the set of all alcoves in $\mathfrak{B}_\infty$. The group $H$ acts on them by linear transformations, and this action corresponds on the right hand side to left multiplication on the set of the cosets. Further, the action of $H$ on the vertices of $\mathfrak{B}_\infty$ is type-preserving.

By the above discussion, $X_{\text{red}}$ is isomorphic to a disjoint union of $\mathbb{Z}$ copies of $X^{SL_2}$. 
Lemma 2.14. Let $R = \{r_v; v \in \mathbb{Z}\}$ be a subset of $GL_2(L)$ with $v_L(\det(r_v)) = v$ for every $v \in \mathbb{Z}$. Then there is an isomorphism of Ind-schemes:

$$\alpha_R : \coprod_v X^{SL_2} \to X_{\text{red}}$$

such that $\alpha_R(x_v I^{SL_2}) = r_v x_v I$ for every $v \in \mathbb{Z}$.

Since $X^{SL_2}$ is connected we have: $\pi_0(X_{\text{red}}) = \mathbb{Z}$ and the connected component of the coset $I$ is $X^{(0)}$.

2.4. Affine Deligne-Lusztig varieties in the affine flag manifold of $GL_2(L)$.

Definition 2.15. Let $\tilde{w} \in \tilde{W}$ and $b \in GL_2(L)$. The affine Deligne-Lusztig variety for $GL_2(L)$ is the locally closed subset of $X$ given by

$$X_{\tilde{w}}(b) = \{xI \in GL_2(L)/I : x^{-1}b\sigma(x) \in I\tilde{w}I\},$$

provided with the reduced sub-Ind-scheme structure.

That $X_{\tilde{w}}(b)$ is indeed locally closed will follow from Proposition 2.23 and the results of the next two sections. For any $b \in GL_2(L)$ the $\sigma$-stabilizer of $b$ in $GL_2(L)$ is given by

$$J_b = \{g \in GL_2(L) : g^{-1}b\sigma(g) = b\}.$$

Remark 2.16. The group $J_b$ is the group of $F$-valued points of the functor $\tilde{J}_b$ which associates to an $F$-algebra $R$ the group

$$\tilde{J}_b(R) = \{g \in GL_2(R \otimes_F L) : g^{-1}b\sigma(g) = b\}.$$

This functor is representable by a connected reductive group $\tilde{J}_b$ over $F$, which is an inner form of a certain Levi subgroup $M_b$ of $GL_2$ attached to $b$ (compare [RZ] (1.12) for the Witt ring case).

The group $J_b$ acts by left multiplication on $X_{\tilde{w}}(b)$. Now we will give the three examples, which will be relevant for us. In the rest of the paper we will use the following notation:

$$b_1 := \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

Examples.

(i) For $b = 1$ we have $J_1 = GL_2(F)$.

(ii) For $b = \begin{pmatrix} 1 & 0 \\ 0 & t^\alpha \end{pmatrix}$ with $\alpha > 0$ we have $J_b = T(F)$.

(iii) Let $b = b_1$. Then $J_{b_1}$ is the multiplicative group of the quaternion algebra over $F$.

Let $F \subsetneq E \subsetneq L$ be the unramified extension of $F$ of degree 2. Then

$$J_{b_1} = \left\{ \begin{pmatrix} a & \sigma(c) \\ t c & \sigma(a) \end{pmatrix} : a, c \in E, a\sigma(a) - tc\sigma(c) \neq 0 \right\}.$$

We can also write:

$$J_{b_1} \cong E[\pi]^*,$$
where $D = E[\pi]$ is the (non-commutative) quaternion algebra over $F$ defined by the relations $\pi^2 = t$ and $a\pi = \pi\sigma(a)$ for all $a \in E$. The isomorphism is given by sending $a \in E$ to \( \begin{pmatrix} a & 0 \\ 0 & \sigma(a) \end{pmatrix} \) and $\pi$ to $b_1$.

The determinant of matrices in $I$ always has the valuation 0 and

$$v_L(\det(x^{-1}b\sigma(x))) = v_L(\det(b)).$$

From this follows

**Lemma 2.17.** If $X_{(w,m)}(b)$ is non-empty, then $v_L(\det(b)) = m$.

**Convention.** For transparency, we omit $m$ from the notation and write $X_w(b)$ instead of $X_{(w,v_L(\det(b)))}(b)$ for every $w \in W_a$ and $b \in GL_2(L)$. This notation depends on the chosen splitting $\tilde{W} = W_a \rtimes \mathbb{Z}$.

**Lemma 2.18.**

(i) The varieties $X_w(b)$ and $X_w(g^{-1}b\sigma(g))$ are isomorphic.

(ii) Let $c = \begin{pmatrix} t^m & 0 \\ 0 & t^m \end{pmatrix} \in N_{GL_2}(T)(L)$. Then $X_w(b)$ and $X_w(cb)$ are equal as subvarieties of $X_{\text{red}}$.

**Proof.** If $g \in GL_2(L)$, then $x \mapsto g^{-1}x$ is an isomorphism $X_w(b) \to X_w(g^{-1}b\sigma(g))$. This proves (i). To prove (ii), we notice that $c$ is central and the image of $c$ in $\tilde{W}$ is the pair $(1, 2m)$. Let $v = v_L(\det(b))$. Now we have: $xI \in X_w(b) \Leftrightarrow x^{-1}b\sigma(x) \in I(w, v)I \Leftrightarrow x^{-1}cb\sigma(x) \in I(1, 2m)(w, v)I = I(w, v + 2m)I$. But $v + 2m = v_L(\det(cb))$, and thus $xI \in X_w(b)$ is equivalent to $xI \in X_w(cb)$. □

Now we show the connection between $X_w(b)$ and affine Deligne-Lusztig varieties in the affine flag manifold of $SL_2(L)$. We use the following

**Definition 2.19.** Let $w \in W_a$, $b \in GL_2(L)$ and $\bar{w} = (w, v_L(\det(b))) \in \tilde{W}$. The affine Deligne-Lusztig variety for $SL_2(L)$ is the locally closed subset of $X^{SL_2}$ defined by

$$X_w^{SL_2}(b) = \{ xI^{SL_2} \in SL_2(L)/I^{SL_2} : x^{-1}b\sigma(x) \in I\bar{w}I \},$$

provided with its reduced subscheme structure.

That $X_w^{SL_2}(b)$ is indeed locally closed will follow from the results of the next two sections. Now, we want to understand **Definition 2.19** better. We see $X_w^{SL_2}(b)(k)$ as a set of alcoves in $\mathcal{B}_\infty$.

**Lemma 2.20.** Consider the situation as in the Definition 2.19. An alcove $D$ lies in $X_w^{SL_2}(b)(k)$ if and only if $\inv(D, b\sigma D) = w$.

**Proof.** For better readability we assume that $\det(b) = t^m$ for some $m \in \mathbb{Z}$. This is in fact the only interesting case for the future. The proof without this assumption is very similar. We fix a representative of $w$ in $N_{SL_2}(T^{SL_2})(L) \subseteq N_{GL_2}(T)(L)$ and denote it again by $w$. 
Let $D = xC^0_M$ with $x \in SL_2(L)$. Since $b_1C^0_M = C^0_M$ we have: $b\sigma D = b\sigma(x)b^{-m}_1C^0_M$ and $b\sigma(x)b^{-m}_1 \in SL_2(L)$. Since $b_1I_{SL_2}b^{-1}_1 = I_{SL_2}$, we have:

$$\text{inv}(D, b\sigma D) = w \iff x^{-1}b\sigma(x)b^{-m}_1 \in I_{SL_2}w_{I_{SL_2}} \iff x^{-1}b\sigma(x) \in I_{SL_2}wb^{-m}_1I_{SL_2} \Rightarrow \Rightarrow x^{-1}b\sigma(x) \in I_{w}b^{-m}_1I = IwI \iff D \in X_{w}^{SL_2}(b)$. \\

Now we have to prove the converse of the third inclusion. Since $b_1$ normalizes $I$ and $I_{SL_2}$, it is enough to show that $IwI \cap SL_2(L) \subseteq I_{SL_2}w_{I_{SL_2}}$. If $iwj \in IwI \cap SL_2(L)$, then $\det(i) = 1$ and hence $\det(w) = 1$ and $i, j \in I$. Let $r = \begin{pmatrix} \det(i) & 0 \\ 0 & 1 \end{pmatrix}$. Now, $w$ normalizes $T(\mathfrak{o})$, and thus $rw = wr'$ for some $r' \in T(\mathfrak{o})$. Hence $iwj = ir^{-1}wr'j$ where $\det(ir^{-1}) = \det(w) = 1$ and hence $\det(r'j) = 1$. Moreover, since $r, r' \in T(\mathfrak{o})$, we have $ir^{-1}, r'j \in I$. Thus $ir^{-1}, r'j \in I_{SL_2}$ and $iwj = ir^{-1}wr'j \in I_{SL_2}w_{I_{SL_2}}$. □

Thus if $b \in SL_2(L)$, then $X_{w}^{SL_2}(b)$ is the usual affine Deligne-Lusztig variety inside the flag manifold of $SL_2$, attached to $b$ and $w$. If $b, b' \in GL_2(L)$ are $\sigma$-conjugate under $SL_2(L)$ and $w \in W$, then the varieties $X_{w}^{SL_2}(b)$ and $X_{w}^{SL_2}(b')$ are isomorphic as subvarieties of $X_{SL_2}$. Indeed if $g \in SL_2(L)$, then $x \mapsto g^{-1}x$ is an isomorphism $X_{w}^{SL_2}(b) \to X_{w}^{SL_2}(g^{-1}b\sigma(g))$.

The group $J_b^{SL_2} = J_b \cap SL_2(L)$ acts on $X_{w}^{SL_2}(b)$ by left multiplication. To express $X_{w}(b)$ in terms of $X_{w}^{SL_2}(b)$, we need the following

**Lemma 2.21.** Let $b \in GL_2(L)$. The restriction of $v_L \circ \det : GL_2(L) \to \mathbb{Z}$ to $J_b$ is surjective.

**Proof.** If $b, c \in GL_2(L)$ are $\sigma$-conjugate, then the groups $J_b$ and $J_c$ are conjugate in $GL_2(L)$. Since conjugation does not change the determinant, it is enough to prove the statement for some representatives of the $\sigma$-conjugacy classes of $GL_2(L)$. Those are given by

\[
\begin{cases} 
\begin{pmatrix} t^\alpha & 0 \\ 0 & t^\beta \end{pmatrix} : \alpha \leq \beta \\
\begin{pmatrix} 0 & t^{\alpha-1} \\ t^\alpha & 0 \end{pmatrix} : \alpha \text{ odd} 
\end{cases}
\]

which follows from ( [RR] 1.10).

The group $J_b$ stays unchanged if we multiply $b$ by some central element of $GL_2(L)$. Therefore, we have only two cases, in which we can prove the statement explicitly.

**First case:** $b = \begin{pmatrix} 1 & 0 \\ 0 & t^\alpha \end{pmatrix}$ and $\alpha \geq 0$. Then for all $v \in \mathbb{Z}$ we have: $\begin{pmatrix} 1 & 0 \\ 0 & t^v \end{pmatrix} \in J_b$ and $v_L(\det \begin{pmatrix} 1 & 0 \\ 0 & t^v \end{pmatrix}) = v$.

**Second case:** $b = b_1$. In this case we have $b_1^v \in J_b$, and $v_L(\det(b_1^v)) = v$ for all $v \in \mathbb{Z}$. □

We proved the surjectivity of the homomorphism $v_L \circ \det : J_b \to \mathbb{Z}$ for any $b \in GL_2(L)$. Let us now introduce its kernel.

**Definition 2.22.** For $b \in GL_2(L)$ set

$$H_b = \text{Ker}(v_L \circ \det : J_b \to \mathbb{Z}).$$
Now $H_b \subseteq H$ where $H \subseteq GL_2(L)$ is the subgroup of all matrices with valuation of the determinant equal zero. $H_b$ is exactly the stabilizer of $X_w(b) \cap (X^{(0)})_{\text{red}}$ under the action of $J_b$ on $X_w(b)$. Recall that in the last subsection we identified $X^{SL_2}$ with $(X^{(0)})_{\text{red}}$. Via this identification $H_b$ acts on $X^{SL_2}_w(b) = X_w(b) \cap (X^{(0)})_{\text{red}}$, and this action is the restriction of the action of $H$ on $X^{SL_2}$.

**Proposition 2.23.** We have

$$X_w(b) \cong \bigsqcup_{J_b/H_b} X^{SL_2}_w(b)$$

as $k$-varieties, and the $J_b$-action on the set of these components is given by left multiplication on the index set.

**Proof.** The scheme structure on $X_w(b)$ is the reduced one, thus the inclusion $X_w(b) \subset X$ factorizes through $X_{\text{red}} \to X$. By Lemma 2.21 we can choose $R = (r_v) \subseteq J_b$ with $r_0 = 1$ and $v_L(\det(r_v)) = v$ for all $v \in \mathbb{Z}$, and $\alpha_R$ from Lemma 2.14 restricts to the isomorphism:

$$X_w(b) \cong \bigsqcup_{v \in \mathbb{Z}} X^{SL_2}_w(b).$$

Now the action of $J_b$ permutes these components and $H_b$ is exactly the stabilizer of the component corresponding to $r_0 = 1$. Again, by Lemma 2.21, $J_b$ acts transitively on the set of these components. \hfill $\square$

In particular, $X_w(b)$ is non-empty if and only if $X^{SL_2}_w(b)$ is. We clearly have $J_b^{SL_2} \subseteq H_b$. By Lemma 2.18(i), to determine $X_w(b)$ for all $b \in GL_2(L)$ and all $w \in W_a$, it is enough to do so for all $b$ lying in a fixed set of representatives of the $\sigma$-conjugacy classes of $GL_2(L)$ (compare (2.2)). By Lemma 2.18(ii) it is enough to consider the following three cases:

(i) $b = 1$;
(ii) $b = \begin{pmatrix} 1 & 0 \\ 0 & t^\alpha \end{pmatrix}$ and $\alpha > 0$;
(iii) $b = b_1$.

By Proposition 2.23, $X_w(b)$ is the disjoint union of $\mathbb{Z}$ copies of $X^{SL_2}_w(b)$. In Lemma 2.20 we showed that $X^{SL_2}_w(b)(\bar{k})$ is the set of all alcoves in $\mathcal{B}_\infty$ with $\text{inv}(D, b\sigma D) = w$. In the following, we will determine $X^{SL_2}_w(b)(\bar{k})$.

### 3. The sets $X^{SL_2}_w(b)(\bar{k})$

#### 3.1. The vertex of departure.

**Lemma 3.1.** Let $\mathcal{C}$ be a full connected subcomplex of $\mathcal{B}_\infty$. Let $D$ be an alcove in $\mathcal{B}_\infty$, which is not contained in $\mathcal{C}$. Then there is a unique gallery $\Gamma_{D,\mathcal{C}}$ with minimal length in $\mathcal{B}_\infty$, containing a vertex $P_D$ in $\mathcal{C}$, whose first alcove is $D$. This vertex $P_D$ is uniquely determined by $D$.

**Proof.** The existence of such a gallery follows from the connectedness of $\mathcal{B}_\infty$. A gallery with given properties clearly contains exactly one vertex lying in $\mathcal{C}$. Such a gallery is uniquely determined by this vertex since $\mathcal{B}_\infty$ is a tree.
To prove the uniqueness of \( \Gamma_{D, \mathcal{E}} \), assume \( \Gamma'_{D, \mathcal{E}} \) is an other gallery with such properties stretched from \( D \) to a vertex \( P'_D \) in \( \mathcal{E} \). By the connectedness of \( \mathcal{E} \), there would be the minimal gallery \( \Gamma_{P_{PP'}} \) stretched from \( P_D \) to \( P'_D \), lying in \( \mathcal{E} \). Consider the composite gallery \((\Gamma_{D, \mathcal{E}}, \Gamma_{PP'})\). It is minimal by Lemma 3.4. In fact, \( \Gamma_{D, \mathcal{E}} \) and \( \Gamma_{PP'} \) are minimal, \( \Gamma_{D, \mathcal{E}} \) has no alcoves in \( \mathcal{E} \) and all alcoves of \( \Gamma_{PP'} \) lie in \( \mathcal{E} \); thus they have no common alcoves and \( P_D \) is a last vertex of \( \Gamma_{D, \mathcal{E}} \) and a first vertex of \( \Gamma_{PP'} \).

Thus there are two minimal galleries, \( \Gamma'_{D, \mathcal{E}} \) and \((\Gamma_{D, \mathcal{E}}, \Gamma_{PP'})\) stretched from \( D \) to \( P'_D \). Since \( \mathcal{B}_\infty \) is a tree, these galleries coincide. Since \( \Gamma'_{D, \mathcal{E}} \) contains exactly one vertex in \( \mathcal{E} \), this implies \( P_D = P'_D \). Therefore \( \Gamma_{D, \mathcal{E}} = \Gamma'_{D, \mathcal{E}} \).

**Definition 3.2.** Let \( \mathcal{E} \) be a full connected subcomplex of \( \mathcal{B}_\infty \). Let \( D \) be an alcove in \( \mathcal{B}_\infty \), which is not contained in \( \mathcal{E} \). Let \( \Gamma_{D, \mathcal{E}} \) and \( P_D \) be as in Lemma 3.1. We call \( P_D \) the vertex of departure for \( D \) from \( \mathcal{E} \). Set further

\[
d_{\mathcal{E}}(D) = 1 + \ell(\Gamma_{D, \mathcal{E}}).
\]

Now we introduce some notations which we will need in the following.

**Notation 3.3.** Let \( \mathcal{E} \) be a full connected subcomplex of \( \mathcal{B}_\infty \).

\( i \) For \( m \in \{0, 1\} \) let \( \mathcal{E}^{(m)} \) denote the set of all vertices in \( \mathcal{E} \) with type \( m \).

\( ii \) For a vertex \( P \) in \( \mathcal{E} \) and \( n > 0 \), set

\[
D_{\mathcal{E}}^{(n)}(P) = \{ D : D \text{ is an alcove in } \mathcal{B}_\infty \text{ with } P \text{ as vertex of departure from } \mathcal{E} \text{ and } d_{\mathcal{E}}(D) = n \}.
\]

**Lemma 3.4.** Let \( \mathcal{E} \) be a full connected subcomplex of \( \mathcal{B}_\infty \) and let \( D \) be an alcove in \( \mathcal{B}_\infty \), not lying in \( \mathcal{E} \). Let further \( \gamma \) be an automorphism of \( \mathcal{B}_\infty \) (as a simplicial complex), and assume that \( \gamma \) stabilizes the subcomplex \( \mathcal{E} \). Then \( \Gamma_{\gamma D, \mathcal{E}} = \gamma \Gamma_{D, \mathcal{E}} \).

**Proof.** As an automorphism of a simplicial complex \( \gamma \) inherits adjacency, and thus takes galleries to galleries. Since \( \gamma \) is invertible, it also inherits minimality of galleries. Thus \( \gamma \Gamma_{D, \mathcal{E}} \) is minimal. The first alcove of \( \gamma \Gamma_{D, \mathcal{E}} \) is \( \gamma D \). Further, \( \gamma \) stabilizes the set of vertices lying in \( \mathcal{E} \) and thus also the set of vertices not lying in \( \mathcal{E} \). Hence \( \gamma \Gamma_{D, \mathcal{E}} \) contains exactly one vertex in \( \mathcal{E} \). This vertex is the image of a last vertex of \( \Gamma_{D, \mathcal{E}} \), and thus itself a last vertex of \( \gamma \Gamma_{D, \mathcal{E}} \). Thus \( \gamma \Gamma_{D, \mathcal{E}} \) has all properties of Lemma 3.1 which uniquely characterize \( \Gamma_{\gamma D, \mathcal{E}} \). The Lemma follows.

### 3.2. The first case: the sets \( X_{wL^2(1)}^{\mathbb{Q}}(\mathbb{k}) \)

Let \( D \) be an alcove in \( \mathcal{B}_\infty \) which is not contained in \( \mathcal{B}_1 \). Apply Lemma 3.1 to the full connected subcomplex \( \mathcal{E} = \mathcal{B}_1 \) and \( D \). Thus there is a unique minimal gallery \( \Gamma_{D, \mathcal{B}_1} \) stretched from \( D \) to a uniquely determined vertex \( P_D \) in \( \mathcal{B}_1 \) which is the vertex of departure for \( D \) from \( \mathcal{B}_1 \).

**Lemma 3.5.** We have \( \Gamma_{\sigma D, \mathcal{B}_1} = \sigma \Gamma_{D, \mathcal{B}_1} \).

**Proof.** Follows from Lemma 3.4 applied to the automorphism given by \( \sigma \).

Now we construct the minimal gallery stretched from \( D \) to \( \sigma D \). Observe that the vertices of departure for \( D \) and for \( \sigma D \) from \( \mathcal{B}_1 \) are equal (the one is the image under \( \sigma \) of the other
and both are in $\mathcal{B}_1$, thus stable under $\sigma$). Thus we get a gallery connecting $D$ and $\sigma D$, which consists of two parts: the first part is $\Gamma_{D,\mathcal{B}_1}$ and the second part is the gallery stretched from $P_D$ to $\sigma D$ i.e. $\Gamma_{\sigma D,\mathcal{B}_1}^{-1} = \Gamma_{D,\mathcal{B}_1}^{-1}$. We denote the composite gallery by

$$\Gamma_D := (\Gamma_{D,\mathcal{B}_1}, \sigma \Gamma_{D,\mathcal{B}_1}^{-1}).$$

**Lemma 3.6.** The gallery $\Gamma_D$ is minimal and $\ell(\Gamma_D) = 2d_{\mathcal{B}_1}(D) - 1$.

**Proof.** $\Gamma_{D,\mathcal{B}_1}$ and $\sigma \Gamma_{D,\mathcal{B}_1}^{-1}$ are minimal. $P_D$ is a last vertex of $\Gamma_{D,\mathcal{B}_1}$ and a first vertex of $\sigma \Gamma_{D,\mathcal{B}_1}^{-1}$. Thus by Lemma 2.7 we have to prove that they have no common alcoves. If an alcove $C$ would lie in $\Gamma_{D,\mathcal{B}_1}$ and in $\sigma \Gamma_{D,\mathcal{B}_1}^{-1}$, then $\sigma C$ would too, and $\Gamma_{D,\mathcal{B}_1}$ would contain $C$ and $\sigma C$. Further $d_{\mathcal{B}_1}(C) = d_{\mathcal{B}_1}(\sigma C)$, since $\sigma_{\mathcal{B}_1,\mathcal{B}_1} = \sigma_{\mathcal{B}_1,\mathcal{B}_1}$ by Lemma 3.5. But all alcoves of $\Gamma_{D,\mathcal{B}_1}$ have different distances to $\mathcal{B}_1$ and hence $C = \sigma C$. This is equivalent to $C$ lying in $\mathcal{B}_1$ which leads to a contradiction, since $\Gamma_{D,\mathcal{B}_1}$ has no alcoves lying in $\mathcal{B}_1$.

The length of $\Gamma_D$ is:

$$\ell(\Gamma_D) = \ell(\Gamma_{D,\mathcal{B}_1}) + \ell(\sigma \Gamma_{D,\mathcal{B}_1}^{-1}) + 1 = (d_{\mathcal{B}_1}(D) - 1) + (d_{\mathcal{B}_1}(D) - 1) + 1 = 2d_{\mathcal{B}_1}(D) - 1.$$

The gallery $\Gamma_D$ is minimal and $\ell(\Gamma_D) > 0$, thus $\Gamma_D$ has a unique first vertex. If its type is $m \in \{0, 1\}$, then

$$\text{inv}(D, \sigma D) = C_M^{(1)m(2d_{\mathcal{B}_1}(D)-1)}.$$ 

The set $X_{w}^{SL_2(1)}(\bar{k})$ is non-empty exactly for $w = 1$ and $w \in W_a$ with odd length (Proposition 3.7 below or [Re] Proposition 2.1.2). The following picture illustrates this via the identification of $W_a$ and $A_M$ (the fat alcoves are those for which $X_{w}^{SL_2(1)}(\bar{k})$ is non-empty).

$$X_{w}^{SL_2(1)}(\bar{k}) = \prod_{P \in \mathcal{B}_1} D_{\mathcal{B}_1}(P).$$

**Proposition 3.7.** The set $X_{w}^{SL_2(1)}(\bar{k})$ is non-empty if and only if $w = 1$ or $\ell(w)$ is odd.

Let now $w \in W_a$ such that $X_{w}^{SL_2(1)}(\bar{k}) \neq \emptyset$.

(i) The set $X_{1}^{SL_2(1)}(\bar{k})$ is the set of all alcoves in $\mathcal{B}_1$.

(ii) If $\ell(w) = 2i - 1, i > 0$ (i.e. $wC_0 = C_M^{2i-1}$ or $wC_0 = C_M^{-2i+1}$), let

$$m = \begin{cases} 0 & \text{if } i \text{ is odd and } wC_0 = C_M^{-2i+1} \text{ or } i \text{ is even and } wC_0 = C_M^{2i-1}, \\ 1 & \text{if } i \text{ is odd and } wC_0 = C_M^{2i-1} \text{ or } i \text{ is even and } wC_0 = C_M^{-2i+1}. \end{cases}$$

Then

$$X_{w}^{SL_2(1)}(\bar{k}) = \prod_{P \in \mathcal{B}_1} D_{\mathcal{B}_1}(P).$$

**Lemma 3.8.** If $X_{w}^{SL_2(1)}(\bar{k}) \neq \emptyset$, then $w = 1$ or $\ell(w)$ is odd.

**Proof.** In fact, let $w \in W_a$ and $D \in X_{w}^{SL_2(1)}(\bar{k})$ be any alcove. If $D \in \mathcal{B}_1$, then $\text{inv}(D, \sigma D) = C_0$ and thus $w = 1$. Otherwise, the length of the minimal gallery $\Gamma_D$ constructed above is odd, and thus $\ell(w)$ is odd.

**Proof of Proposition.** (i) follows from the fact that $\mathcal{B}_1 = \mathcal{B}_\infty^{<\sigma>}$. To prove (ii) we consider the case $i = \frac{\ell(w)+1}{2}$ odd and $wC_0 = C_M^{2i-1}$. Thus $m = 1$. If an alcove $D$ lies in $X_{w}^{SL_2(1)}(\bar{k})$,
then $D$ is not contained in $\mathfrak{B}_1$ and the length of the gallery $\Gamma_D$ defined above must be equal to the length of the gallery stretched from $C_M^0$ to $C_M^{2i-1}$: $2d_{\mathfrak{B}_1}(D) - 1 = 2i - 1$, so we must have $d_{\mathfrak{B}_1}(D) = i$. The type of the first vertex of $\Gamma_D$ must coincide with the type of the first vertex of the gallery stretched from $C_M^0$ to $C_M^{2i-1}$, i.e. $0$, so the type of the vertex of departure for $D$ from $\mathfrak{B}_1$ must be $1$ (by parity of $i$). So $X_{w_1}^{SL_2}(1)(\bar{k}) \subseteq \bigcup_{P \in \mathfrak{B}_1} D_{\mathfrak{B}_1}(P)$.

If conversely, $D \in D_{\mathfrak{B}_1}(P)$ with $P \in \mathfrak{B}_1^{(1)}$, then the relative position of $D$ and $\sigma D$ will be $C_M^{2i-1}$ (this is clear by construction of $\Gamma_D$). Hence $X_{w_1}^{SL_2}(1)(\bar{k}) = \bigcup_{P \in \mathfrak{B}_1^{(1)}} D_{\mathfrak{B}_1}(P)$ and thus is non-empty. The sets $D_{\mathfrak{B}_1}(P)$ are disjoint for different $P \in \mathfrak{B}_1^{(1)}$, by uniqueness of the vertex of departure. The other three cases ($i$ is odd, $wC_M^0 = C_M^{-2i+1}$ and the two cases where $i$ is even) have similar proofs. □

3.3. The second case: the sets $X_{w_1}^{SL_2}(b)(\bar{k})$ for diagonal $b \neq 1$.

Let now $b = \left( \begin{array}{cc} 1 & 0 \\ 0 & \ell^\alpha \end{array} \right)$ with $\alpha > 0$. First of all, $b$ acts on $A_M$ by translation by $\alpha$ alcoves to the right: $b$ sends the alcove $C_M^i$ to the alcove $C_M^{i+\alpha}$. The distance from the main apartment will play the analogous role, which in the previous case was played by the distance to $\mathfrak{B}_1$. Let $D$ be an alcove in $\mathfrak{B}_\infty$ which is not contained in $A_M$. Apply Lemma 3.3 to the full connected subcomplex $\mathcal{C} = A_M$ of $\mathfrak{B}_\infty$ and $D$. Thus there is a unique minimal gallery $\Gamma_{D,A_M}$ stretched from $D$ to a uniquely determined vertex $P_D$ in $A_M$, which is the vertex of departure for $D$ from $A_M$. Then $P_D$ is a last vertex of $\Gamma_{D,A_M}$.

Lemma 3.9. We have: $\Gamma_{b\sigma D,A_M} = b\sigma \Gamma_{D,A_M}$.

Proof. Follows from Lemma 3.4 applied to the automorphism given by $b\sigma$. □

Like in the previous case, we want to construct a gallery stretched from $D$ to $b\sigma D$. Let $\Gamma_{D,\ell}$ be the gallery stretched from $P_D$ to $b\sigma P_D$. Thus $\Gamma_{D,\ell}$ has $P_D$ as a first vertex and $b\sigma P_D$ as a last vertex. All alcoves of $\Gamma_{D,\ell}$ lie in $A_M$. The length of $\Gamma_{D,\ell}$ is $\alpha - 1$. From Lemma 3.9 follows: $\Gamma_{b\sigma D,A_M}^{-1} = b\sigma \Gamma_{D,A_M}^{-1}$. This gallery has $b\sigma P_D$ as a first vertex and $b\sigma D$ as the last alcove. We set

$$\Gamma_D := (\Gamma_{D,A_M}, \Gamma_{D,\ell}, b\sigma \Gamma_{D,A_M}^{-1}).$$

It has $D$ as the first and $b\sigma D$ as the last alcove.

Lemma 3.10. The gallery $\Gamma_D$ is minimal and $\ell(\Gamma_D) = 2d_{A_M}(D) + \alpha - 1$.

Proof. The galleries $\Gamma_{D,A_M}, \Gamma_{D,\ell}$ and $b\sigma \Gamma_{D,A_M}^{-1}$ are minimal. We use Lemma 2.1(ii). At first, we prove that $\Gamma_{D,A_M}, \Gamma_{D,\ell}$ and $b\sigma \Gamma_{D,A_M}^{-1}$ pairwise have no common alcoves. In fact, $\Gamma_{D,A_M}$ and $b\sigma \Gamma_{D,A_M}^{-1}$ have no alcoves lying in $A_M$. Hence $\Gamma_{D,A_M}$ and $b\sigma \Gamma_{D,A_M}^{-1}$ have no common alcoves with $\Gamma_{D,\ell}$ (whose alcoves all lie in $A_M$). All alcoves in $\Gamma_{D,A_M}$ have the same vertex $P_D$ of departure from $A_M$. The vertex $b\sigma P_D$ is the vertex of departure from $A_M$ for every alcove in $\Gamma_{b\sigma D,A_M}^{-1} = b\sigma \Gamma_{D,A_M}^{-1}$. But $P_D \neq b\sigma P_D$. By the uniqueness of the vertex of departure, $\Gamma_{D,A_M}$ and $b\sigma \Gamma_{D,A_M}^{-1}$ contain no common alcoves.

Now the condition (d) of Lemma 2.1(ii) for $\Gamma_{D,A_M}$, $\Gamma_{D,\ell}$ is clear. Thus the composite gallery $(\Gamma_{D,A_M}, \Gamma_{D,\ell})$ is minimal.
Now we have to verify the condition (d) of Lemma 2.7(ii) for \((\Gamma_{D,A_M}, \Gamma_{D,tr})\) and \(b\sigma \Gamma_{D,A_M}^{-1}\). The vertex \(b\sigma P_D\) is a first vertex of \(b\sigma \Gamma_{D,A_M}^{-1}\). Further, \(b\sigma P_D\) is a last vertex of \(\Gamma_{D,tr}\). The only vertex contained in \(\Gamma_{D,A_M}\) and in \(\Gamma_{D,tr}\) is \(P_D\) (it is the unique vertex of \(\Gamma_{D,A_M}\) contained in \(A_M\)). But \(P_D \neq b\sigma P_D\), and thus \(b\sigma P_D\) is also a last vertex of the composite gallery \((\Gamma_{D,A_M}, \Gamma_{D,tr})\). Thus by Lemma 2.7(ii) the composite gallery \(\Gamma_D = (\Gamma_{D,A_M}, \Gamma_{D,tr}, b\sigma \Gamma_{D,A_M}^{-1})\) is minimal.

The length of \(\Gamma_D\) is:
\[
\ell(\Gamma_D) = \ell(\Gamma_{D,A_M}) + \ell(\Gamma_{D,tr}) + \ell(b\sigma \Gamma_{D,A_M}^{-1}) + 2
\]
\[
= 2\ell(\Gamma_{D,A_M}) + \ell(\Gamma_{D,tr}) + 2 = 2(d_{A_M}(D) - 1) + (\alpha - 1) + 2
\]
\[
= 2d_{A_M}(D) + \alpha - 1. \quad \square
\]

The gallery \(\Gamma_D\) is minimal and \(\ell(\Gamma_D) > 0\), hence it has a unique first vertex. If its type is \(m \in \{0, 1\}\), then
\[
\text{inv}(D, b\sigma D) = C_M^{(-1)m(\alpha + 2d_{A_M}(D) - 1)}.
\]

Now, \(X_w^{SL_2}(b)(\tilde{k})\) is non-empty if and only if \(w\) has length \(\alpha\) or \(\alpha + 2i - 1\) for some \(i > 0\) (see Proposition 3.11 below or [Re] 2.1.4 and 2.2). The following picture illustrates this in the case \(\alpha = 4\) (the fat alcoves are those for which \(X_w^{SL_2}(b)(\tilde{k})\) is non-empty).

---

**Proposition 3.11.** Let \(b = \begin{pmatrix} 1 & 0 \\ 0 & t^{\alpha} \end{pmatrix}\) with \(\alpha > 0\), and \(w \in W_a\). Then \(X_w^{SL_2}(b)(\tilde{k})\) is non-empty if and only if \(\ell(w) = \alpha\) or \(\ell(w) = \alpha + 2i - 1\) with \(i > 0\).

Let now \(w \in W_a\) such that \(X_w^{SL_2}(b)(\tilde{k}) \neq \emptyset\).

(i) If \(wC_M^0 = C_M^{\alpha}\), then \(X_w^{SL_2}(b)(\tilde{k}) = \prod_{j \in \mathbb{Z}} \{C_M^{2j}\}\).

(ii) If \(wC_M^0 = C_M^{-\alpha}\), then \(X_w^{SL_2}(b)(\tilde{k}) = \prod_{j \in \mathbb{Z}} \{C_M^{2j+1}\}\).

(iii) If \(\ell(w) = \alpha + (2i - 1)\) for \(i > 0\) (i.e. \(wC_M^0 = C_M^{\alpha+2(i-1)}\) or \(wC_M^0 = C_M^{-\alpha-(2i-1)}\)), let

\[
m = \begin{cases} 
0 & \text{if } i \text{ is odd and } wC_M^0 = C_M^{-\alpha-(2i-1)} \text{ or } i \text{ is even and } wC_M^0 = C_M^{\alpha+2(i-1)}, \\
1 & \text{if } i \text{ is even and } wC_M^0 = C_M^{-\alpha-(2i-1)} \text{ or } i \text{ is odd and } wC_M^0 = C_M^{\alpha+2(i-1)}.
\end{cases}
\]

Then
\[
X_w^{SL_2}(b)(\tilde{k}) = \prod_{P \in D_{A_M}^i(P)}
\]

**Lemma 3.12.** If \(X_w^{SL_2}(b)(\tilde{k}) \neq \emptyset\), then \(\ell(w) = \alpha\) or \(\ell(w) = \alpha + 2i - 1\) with \(i > 0\).

**Proof.** In fact, let \(w \in W_a\) and \(D \in X_w^{SL_2}(b)(\tilde{k})\) be any alcove. If \(D = C_M^{2j}\) resp. \(D = C_M^{2j+1}\) lies in \(A_M\), then \(\text{inv}(D, b\sigma D) = C_M^{\alpha}\) resp. \(C_M^{-\alpha}\) and \(\ell(w) = \alpha\). Otherwise, the length of the minimal gallery \(\Gamma_D\) constructed above is \(\alpha + 2i - 1\) for some \(i > 0\). Thus \(\ell(w) = \alpha + 2i - 1\). \(\square\)
Proof of Proposition. To prove (i) and (ii), we notice that if \( \ell(w) = \alpha \), then \( X_w^{SL2}(b)(\tilde{k}) \) is contained in \( A_M \): in fact if \( D \) is not in \( A_M \), then the length of the gallery \( \Gamma_D \) constructed above is \( \alpha + 2i - 1 \) for some \( i \in \mathbb{Z} \), which differs by an odd integer from \( \alpha \). Now \( b \) acts on the alcoves in \( A_M \) by shifting by \( \alpha \) alcoves to the right. Thus if \( wC^0_M = C^\alpha_M \), then \( X_w^{SL2}(b)(\tilde{k}) = \{ C^2_M \colon j \in \mathbb{Z} \} \). If \( wC^0_M = C^{-\alpha}_M \), then \( X_w^{SL2}(b)(\tilde{k}) = \{ C^{2j-1}_M \colon j \in \mathbb{Z} \} \).

Now we prove (iii) for \( i > 0 \) odd and \( wC^0_M = C^{\alpha+2i-1}_M \) (the other cases can be proven similarly). In this case \( m = 1 \). If \( D \) lies in the set \( X_w^{SL2}(b)(\tilde{k}) \), then \( D \) does not lie in \( A_M \), by the above considerations, and the length of the gallery \( \Gamma_D \) constructed above must be equal to \( \ell(w) = \alpha + 2i - 1 \). Therefore, we must have: \( \alpha + 2d_A(D) = \alpha + 2i \), which implies \( d_A(D) = i \). The first vertex of \( \Gamma_D \) must have the same type as the first vertex of the gallery stretched from \( C^0_M \) to \( C^\alpha_M \), i.e. 0. So, by parity of \( i \), the vertex of departure for \( D \) must have type 1. Thus, \( \bigcup_{P \in A^{(1)}_M} D_A \Gamma(P) \supseteq X_w^{SL2}(b)(\tilde{k}) \).

Conversely, if \( D \in D_A(P) \) for some \( P \in A^{(1)}_M \), then the corresponding gallery \( \Gamma_D \) has the length \( \alpha + (2i - 1) \) and the type of its first vertex is 0, thus it can be folded into the gallery stretched from \( C^0_M \) to \( C^\alpha_M \). Therefore, \( X_w^{SL2}(b)(\tilde{k}) = \bigcup_{P \in A^{(1)}_M} D_A \Gamma(P) \). This union is disjoint, since the vertex of departure is uniquely determined. Thus (iii) follows. Now, the first part of the Proposition follows from Lemma 3.12 and (i)-(iii).

3.4. The third case: the sets \( X_w^{SL2}(b)(\tilde{k}) \).

Recall that \( b_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} \). The action of \( b_1 \) on the main apartment is given by reflection about the midpoint of \( C^0_M \), thus it sends \( C^\alpha_M \) to \( C^{-\alpha}_M \). The vertex of \( C^0_M \) represented by \( \sigma \oplus 0 \) goes to the vertex represented by \( \sigma \oplus t0 \), and conversely.

In this case, \( C^0_M \) plays the same role as \( \mathfrak{B}_1 \) and \( A_M \) in the previous cases. Let \( D \neq C^0_M \) be an alcove in \( \mathfrak{B}_\infty \). By Lemma 3.11 applied to the full connected subcomplex \( \overline{C^0_M} \) of \( \mathfrak{B}_\infty \) containing only the alcove \( C^0_M \) and its vertices, there is a unique minimal gallery \( \Gamma_D^{\overline{C^0_M}} \) in \( \mathfrak{B}_\infty \) stretched from \( D \) to its vertex \( P_D \) of departure from \( \overline{C^0_M} \).

Lemma 3.13. We have: \( b_1\sigma \Gamma_{D,\overline{C^0_M}} = \Gamma_{b_1\sigma D,\overline{C^0_M}} \).

Proof. Follows from Lemma 3.14 applied to the automorphism given by \( b_1\sigma \).

Like above, we construct the minimal gallery \( \Gamma_D \) stretched from \( D \) to \( b_1\sigma D \). Let \( \Gamma_D^{\overline{C^0_M}} \) be the gallery consisting of the single alcove \( C^0_M \). Now, \( P_D \) is a last vertex of \( \Gamma_D^{\overline{C^0_M}} \) and a first vertex of \( \Gamma_D^{tr} \). Further, \( b_1\sigma P_D \) is a first vertex of \( \Gamma_D^{\overline{C^0_M}} = \Gamma^{\overline{C^0_M}}_{b_1\sigma D,\overline{C^0_M}} \) and a last vertex of \( \Gamma_D^{tr} \). Hence

\[
\Gamma_D := (\Gamma_D^{\overline{C^0_M}}, \Gamma_D^{tr}, b_1\sigma \Gamma^{\overline{C^0_M}}_{D,\overline{C^0_M}})
\]

is a gallery. Its first alcove is \( D \) and its last alcove is \( b_1\sigma D \).

Lemma 3.14. The gallery \( \Gamma_D \) is minimal and \( \ell(\Gamma_D) = 2d_{\overline{C^0_M}}(D) \).

Proof. The galleries \( \Gamma_D^{tr}, \Gamma_D^{\overline{C^0_M}} \) and \( b_1\sigma \Gamma^{\overline{C^0_M}}_{D,\overline{C^0_M}} \) are minimal. We use Lemma 2.7(ii). At first, we prove that \( \Gamma_D^{\overline{C^0_M}}, \Gamma_D^{tr} \) and \( b_1\sigma \Gamma^{\overline{C^0_M}}_{D,\overline{C^0_M}} \) have pairwise no common alcoves. \( \Gamma_D^{tr} \)
clearly has no common alcoves with $\Gamma_{D,C_M^0}$ and $b_1\sigma \Gamma_{D,C_M}^{-1}$. Further, every alcove of $\Gamma_{D,C_M^0}$ has $P_D$ as its vertex of departure from $C_M^0$ and every alcove of $b_1\sigma \Gamma_{D,C_M}^{-1} = \Gamma_{D,C_M}^{-1}$ has $b_1\sigma P_D \neq P_D$ as its vertex of departure from $C_M^0$. The vertex of departure from $C_M^0$ is uniquely determined for every alcove, thus $\Gamma_{D,C_M^0}$ and $b_1\sigma \Gamma_{D,C_M}^{-1}$ have no common alcoves.

The condition (d) of Lemma 2.7(ii) for $\Gamma_{D,C_M^0}$ is non-empty. Thus the composite gallery $(\Gamma_{D,C_M^0}, \Gamma_{D,tr})$ is minimal.

Now we have to verify the condition (d) of Lemma 2.7(ii) for $(\Gamma_{D,C_M^0}, \Gamma_{D,tr})$ and $b_1\sigma \Gamma_{D,C_M}^{-1}$.
The vertex $b_1\sigma P_D$ is a first vertex of $b_1\sigma \Gamma_{D,C_M}^{-1}$. Further $b_1\sigma P_D$ is a last vertex of $\Gamma_{D,tr}$. The only vertex contained in $\Gamma_{D,C_M^0}$ and in $\Gamma_{D,tr}$ is $P_D$. But $P_D \neq b_1\sigma P_D$. Thus $b_1\sigma P_D$ will also be a last vertex of the composite gallery $(\Gamma_{D,C_M^0}, \Gamma_{D,tr})$. Thus by Lemma 2.7(ii) the composite gallery $\Gamma_D = (\Gamma_{D,C_M^0}, \Gamma_{D,tr}, b_1\sigma \Gamma_{D,C_M}^{-1})$ is minimal.

The length of $\Gamma_D$ is:

$$\ell(\Gamma_D) = \ell(\Gamma_{D,C_M^0}) + \ell(\Gamma_{D,tr}) + \ell(b_1\sigma \Gamma_{D,C_M}^{-1}) + 2 = 2\ell(\Gamma_{D,C_M^0}) + 0 + 2 = 2d_{C_M^0}(D). \quad \Box$$

The gallery $\Gamma_D$ is minimal and $\ell(\Gamma_D) > 0$, thus it has a unique first vertex. If its type is $m \in \{0, 1\}$, then

$$\text{inv}(D, b_1\sigma D) = C_M^0 (-1)^m 2d_{C_M^0}(D).$$

Now, $X_w^{S_L^2}(b_1)(\bar{k})$ is non-empty if and only if $w$ has even length (Proposition 3.15 below or [Re] 2.2). The following picture illustrates this (the fat alcoves are those for which $X_w^{S_L^2}(b_1)(\bar{k})$ is non-empty).

\[ \begin{array}{cccccccccccc}
\mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 & \mathcal{B}_w^0 \\
\end{array} \]

For $m \in \{0, 1\}$, let $P_m$ denote the vertex of $\mathcal{B}_w$ represented by $\sigma \oplus t^m \sigma$.

**Proposition 3.15.** Let $w \in W_a$. Then $X_w^{S_L^2}(b_1)(\bar{k})$ is non-empty if and only if $\ell(w)$ is even.

Let now $w \in W_a$ such that $X_w^{S_L^2}(b_1)(\bar{k}) \neq \emptyset$.

(i) We have: $X_1^{S_L^2}(b_1)(\bar{k}) = \{C_M^0\}$.

(ii) If $\ell(w) = 2i$ for $i > 0$ (i.e. $wC_M^0 = C_M^{2i}$ or $wC_M^0 = C_M^{-2i}$), let

$$m = \begin{cases} 
0 & \text{if } i \text{ is odd and } wC_M^0 = C_M^{2i} \text{ or if } i \text{ is even and } wC_M^0 = C_M^{-2i}, \\
1 & \text{if } i \text{ is odd and } wC_M^0 = C_M^{2i} \text{ or if } i \text{ is even and } wC_M^0 = C_M^{-2i}.
\end{cases}$$

Then

$$X_w^{S_L^2}(b_1)(\bar{k}) = D_{C_M^0}(P_m).$$

**Lemma 3.16.** If $X_w^{S_L^2}(b_1)(\bar{k}) \neq \emptyset$, then $\ell(w)$ is even.

**Proof.** In fact, let $w \in W_a$ and $D \in X_w^{S_L^2}(b_1)(\bar{k})$ be any alcove. If $D = C_M^0$, then $w = 1$ has the length 0. Otherwise, the length of the minimal gallery $\Gamma_D$ constructed above is even and thus $\ell(w) = \ell(\Gamma_D)$ is even. \qed
Proof of Proposition. For (i) observe that for any alcove \( D \neq C^0_M \), the gallery \( \Gamma_D \) never has the length 0, thus \( X^{SL_2}_1(\bar{b}_1)(\bar{k}) \subseteq \{ C^0_M \} \). The inverse implication is clear.

Now we prove (ii) for \( i > 0 \) odd and \( wC^0_M = C^2_M \) (the other cases can be proven similarly). In this case \( m = 1 \). If \( D \) lies in the set \( X^{SL_2}_w(\bar{b}_1)(\bar{k}) \), then we have: \( 2d_{\bar{C}^0_M}(D) = \ell(D) = \ell(w) = 2i \), which implies \( d_{\bar{C}^0_M}(D) = i \). The first vertex of \( \Gamma_D \) must have the same type as the first vertex of the gallery stretched from \( C^0_M \) to \( C^2_M \), i.e. 0. So by parity of \( i \), the vertex of departure for \( D \) from \( \bar{C}^0_M \) must have type 1. Thus \( D_{\bar{C}^0_M}(P_1) \supseteq X^{SL_2}_w(\bar{b}_1)(\bar{k}) \).

Conversely, if \( D \in D_{\bar{C}^0_M}(P_1) \), then the corresponding gallery \( \Gamma_D \) has length \( 2i \) and the type of its first vertex is 0. Thus \( \Gamma_D \) can be folded into the gallery stretched from \( C^0_M \) to \( C^2_M \). So \( X^{SL_2}_w(\bar{b}_1)(\bar{k}) = D_{\bar{C}^0_M}(P_1) \). The first part of the Proposition follows from Lemma 3.16 and (i), (ii).

\[ \square \]

4. The variety structure on \( X_w(b) \)

In this section we choose the functorial point of view on a scheme, and work mainly with the set of its \( \bar{k} \)-valued points. In particular, a locally closed reduced sub-Ind-scheme of the affine flag manifold is uniquely determined by the set of its \( \bar{k} \)-valued points. In the last section we have determined the set-theoretical structure of \( X^{SL_2}_w(b)(\bar{k}) \). Now we determine the scheme-structure on \( X_w(b) \) and on \( X^{SL_2}_w(b) \). From Proposition [2.23] we have

\[ X_w(b) \neq \emptyset \iff X^{SL_2}_w(b) \neq \emptyset \iff X^{SL_2}_w(b)(\bar{k}) \neq \emptyset. \]

In all cases \( X^{SL_2}_w(b)(\bar{k}) \) were (disjoint unions of) sets of very similar types. At first we prove a general fact which shows that all these sets are locally closed subsets of \( X^{SL_2}(\bar{k}) \).

4.1. The crucial result.

**Lemma 4.1.** Let \( l \geq 0 \) and \( P \neq Q \) two vertices in \( \mathcal{B}_\infty \), represented by lattices \( \mathcal{L}_1, \mathcal{L}_2 \), respectively. Assume that \( \mathcal{L}_1 \supseteq \mathcal{L}_2 \) and \( \dim \mathcal{L}_1/\mathcal{L}_2 = l+1 \). Then the following are equivalent:

(i) The length of the minimal gallery in \( \mathcal{B}_\infty \) stretched from \( P \) to \( Q \) is \( l \).

(ii) The \( \bar{k}[t] \)-module \( \mathcal{L}_1/\mathcal{L}_2 \) is cyclic.

The same holds for \( \mathcal{B}_1 \) instead of \( \mathcal{B}_\infty \).

**Proof.** Choose by the elementary divisor theorem a \( \bar{k}[t] \)-basis \( \{ v_1, v_2 \} \) of \( \mathcal{L}_1 \) such that \( \{ t^{a_1}v_1, t^{a_2}v_2 \} \) is a \( \bar{k}[t] \)-basis of \( \mathcal{L}_2 \). Then \( 0 \leq a_1, a_2 \leq l+1 \) and \( a_1 + a_2 = l + 1 \). We can assume \( a_1 \leq a_2 \). The \( \bar{k}[t] \)-module \( \mathcal{L}_1/\mathcal{L}_2 \) is cyclic if and only if \( a_1 = 0 \). Consider the following lattice chain:

\[ \langle t^{a_1}v_1, t^{a_1+1}v_2 \rangle_{\bar{k}[t]} \supseteq \langle t^{a_1}v_1, t^{a_1+1}v_2 \rangle_{\bar{k}[t]} \supseteq \ldots \supseteq \langle t^{a_1}v_1, t^{a_2}v_2 \rangle_{\bar{k}[t]}, \]

which represents a minimal gallery between the vertex \( P \), represented by \( \langle t^{a_1}v_1, t^{a_1}v_2 \rangle_{\bar{k}[t]} \) and \( Q \), represented by \( \langle t^{a_1}v_1, t^{a_2}v_2 \rangle_{\bar{k}[t]} \). It is minimal by Lemma [2.7(i)] (verification of the two conditions is straightforward). The length of this gallery is \( a_2 - a_1 - 1 \). This number is equal \( l \) if and only if \( a_1 = 0 \), or equivalently if and only if \( \mathcal{L}_1/\mathcal{L}_2 \) is cyclic. The proof for \( \mathcal{B}_1 \) instead of \( \mathcal{B}_\infty \) is the same. \( \square \)
It is a well known fact that the alcoves around a vertex $P$ in $\mathfrak{B}_\infty$ form a closed subset of $X^{SL_2}(\bar{k})$. There is a unique reduced closed subscheme of $X^{SL_2}$ such that its $\bar{k}$-valued points are exactly the alcoves around $P$. This subscheme is defined over $k$ and isomorphic to $\mathbb{P}_k^1$ as a $k$-scheme.

**Definition 4.2.** Let $P$ be a vertex in $\mathfrak{B}_1$. Let $V$ be a subset of $\mathbb{P}_k^1(k)$. Thus $V$ corresponds to a finite set of alcoves in $\mathfrak{B}_1$, having distance 0 to $P$. Let $l \geq 0$. We set

$$\mathfrak{F}_{P,V,l}(\bar{k}) = \begin{cases} D : & D \text{ is an alcove in } \mathfrak{B}_\infty \text{ which has distance } l \text{ to } P, \\ & \text{and the minimal gallery stretched from } P \text{ to } D \text{ does not pass through } V \end{cases},$$

and let $\mathfrak{F}_{P,V,l}(k)$ be the intersection of $\mathfrak{F}_{P,V,l}(\bar{k})$ with the set of alcoves lying in $\mathfrak{B}_1$.

**Proposition 4.3.** Let $P, V, l$ be as in Definition 4.2. Then $\mathfrak{F}_{P,V,l}(\bar{k})$ is a locally closed subset of $X^{SL_2}(\bar{k})$. Denote by $\mathfrak{F}_{\bar{k}}$ the corresponding induced reduced sub-Ind-scheme of $X^{SL_2}_{\bar{k}}$. The Ind-scheme $\mathfrak{F}_{\bar{k}}$ is a scheme, it is defined over $k$ by a scheme $\mathfrak{F}$, and there is an isomorphism of $k$-schemes:

$$\mathfrak{F} \cong A_k^l \times (\mathbb{P}^1_k - V).$$

**Proof.** By homogeneity under the action of $GL_2(F)$, we can assume that $P$ is represented by $L_0 = \mathfrak{o}_F \oplus \mathfrak{o}_F$. Let us first prove the Proposition for $V$ consisting of only one element. In this case we can assume that this element is $C_0$ and its vertices are $P$ and $Q$, where $Q$ is represented by $L_1 = \mathfrak{o}_F \oplus t\mathfrak{o}_F$.

Before going on with the proof, we make the following

**Remark 4.4.** For $V$ and $P$ as described above, the set $\mathfrak{F}_{P,V,l}(\bar{k})$ is nothing other than the open Schubert cell for the element $v \in W_{0_0}$ with $vC_0^0 = C_0^{-(l+1)}$. There is another proof that the Schubert cell for $v \in W_a$ is isomorphic to the affine space of dimension $\ell(v)$. This proof works similarly in the affine case, as in the finite case (which is given in [Bo], 14.12, the first theorem). Nevertheless, we will give another proof below and then compare the two proofs.

**The sketch of the proof.** Consider $N = L_0/l+1L_0$ as a $k$-vector space carrying a nilpotent action of $t$. Let $\mathfrak{flag}_{l+1,l+2}(N) \subseteq \mathfrak{flag}_{l+1,l+2}(N)$ be the closed subscheme of the partial flag manifold of $N$, parametrizing the chains of $t$-stable subspaces of $N$ of dimensions $l+1, l+2$. Let also

$$S := \mathfrak{flag}_{l+1,l+2}(N)_{\text{red}}.$$

In particular $S$ has the same $k$-valued points as $\mathfrak{flag}_{l+1,l+2}(N)$. We have the following two closed immersions (the vertical one is given by the inverse of (2.1)):

$$\begin{array}{c}
X^{SL_2} \\
S \longrightarrow \mathfrak{flag}_{l+1,l+2}(N)
\end{array}$$

In the proof we consider the following commutative diagram of inclusions:
where $Y$ will later be defined as some open subscheme of $\mathfrak{Flag}_{l+1,l+2}(N)$ which is isomorphic to the affine space over $k$ of appropriate dimension. We will show that the two diagonal arrows from $\mathfrak{Flag}_{P,V,l}(k)$ exist and that there is a unique closed reduced subscheme of $Y$, defined by vanishing of linear polynomials whose $k$-rational points are $\mathfrak{Flag}_{P,V,l}(k)$.

**First step.** We prove that $\mathfrak{Flag}_{P,V,l}(k) \to X^{SL_2}(k)$ factorizes through $S(k) \to X^{SL_2}(k)$.

Observe that every alcove with distance $l$ to $P$ has a unique representing lattice chain $\mathcal{L}' \supseteq \mathcal{L} \supseteq t\mathcal{L}'$ such that $\mathcal{L}, \mathcal{L}' \subseteq \mathcal{L}_0$, the vertex represented by $\mathcal{L}$ has greater distance to $P$, and $\dim_k \mathcal{L}_0/\mathcal{L} = l + 1$ ($\mathcal{L}$ is uniquely determined for reasons of dimension and then the uniqueness of $\mathcal{L}'$ follows from the uniqueness of minimal galleries). After these choices $\mathcal{L}_0/\mathcal{L}$ is a cyclic $k[t]$-module by Lemma 4.4. We have $\mathcal{L}, \mathcal{L}' \supseteq t^{l+1}\mathcal{L}_0$.

The projections $\mathcal{L} \to L := \mathcal{L}/(t^{l+1}\mathcal{L}_0)$, $\mathcal{L}' \to L' := \mathcal{L}'/(t^{l+1}\mathcal{L}_0)$ give the embedding $\mathfrak{Flag}_{P,V,l}(k) \subseteq S(k)$ (which is injective, since the maps $\mathfrak{Flag}_{P,V,l}(k) \to X^{SL_2}(k), S(k) \to X^{SL_2}$ are). Note that $N/\mathcal{L} \cong \mathcal{L}_0/\mathcal{L}$ and $N/L' \cong \mathcal{L}_0/\mathcal{L}'$ are $k[t]$-cyclic too. Thus:

\[(4.1) \quad \mathfrak{Flag}_{P,V,l}(k) \subseteq \{(L \subseteq L') \in S(k): N/L \text{ and } N/L' \text{ are cyclic as } k[t]-\text{modules}\},\]

**Second step.** In this step we reformulate the condition that the minimal gallery stretched from the vertex, represented by $\mathcal{L}$, to $P$ does not pass through $Q$.

Let $\{e_1, e_2\}$ be the image of the canonical $\mathfrak{so}_F$-basis of $\mathcal{L}_0$ under the projection $\mathcal{L}_0 \to N$. Then $\{e_1, te_1, ..., t^l e_1, e_2, te_2, ..., t^l e_2\}$ is a $k$-basis of $N$. Let $N_1 = \mathcal{L}_1/t^{l+1}\mathcal{L}_0$. Then $N_1$ is a $(2l + 1)$-dimensional subspace of $N$ and $\{e_1, te_1, ..., t^l e_1, te_2, ..., t^l e_2\}$ is its basis.

**Lemma 4.5.** Let $\mathcal{L}' \supseteq \mathcal{L} \supseteq t\mathcal{L}'$ be a representative of an alcove $D$, having distance $l$ to $P$, chosen as in step one. Let $L := \mathcal{L}/t^{l+1}\mathcal{L}_0$. The following are equivalent:

(i) The minimal gallery from $P$ to the vertex represented by $\mathcal{L}$ does not pass through $Q$.

(ii) $L \not\subseteq N_1$.

**Proof of Lemma.** We have $t^{-1}\mathcal{L}_1 \supseteq \mathcal{L}_0 \supseteq \mathcal{L}$ and $\dim_k((t^{-1}\mathcal{L}_1)/\mathcal{L}_0) = 1$, $\dim_k(\mathcal{L}_0/\mathcal{L}) = l + 1$. Hence: $\dim_k((t^{-1}\mathcal{L}_1)/\mathcal{L}) = l + 2$. Further we have: $\mathcal{L}_1 \supseteq t\mathcal{L}$. By the elementary divisor theorem, choose a $k[[t]]$-basis $\{v_1, v_2\}$ of $\mathcal{L}_1$ such that $\{t^{a_1}v_1, t^{a_2}v_2\}$ is a $k[[t]]$-basis of $t\mathcal{L}$.

Now, (i) is equivalent to the statement that the minimal gallery from $Q$ to $D$ has length $l + 1$. By Lemma 1.1 and the above dimension counting, this is equivalent to $(t^{-1}\mathcal{L}_1)/\mathcal{L}$ being a cyclic $k[t]$-module. This is equivalent to $\mathcal{L}_1/t\mathcal{L}$ being a cyclic $k[t]$-module.

From this follows (ii): By the cyclicity requirement we must have $a_1 = 0$ or $a_2 = 0$. Assume $a_1 = 0$. Then $t^{-1}v_1 \in \mathcal{L} - \mathcal{L}_1$ and its image modulo $t^{l+1}\mathcal{L}_0$ lies in $L - N_1$. 

\[
\begin{tikzcd}
\mathfrak{Flag}_{P,V,l}(k) \ar[r] \ar[u] & X^{SL_2}(k) \ar[u] \\
\ar[r]^{S(k)} & \mathfrak{Flag}_{l+1,l+2}(N)(k) \ar[u] \\
Y(k) \ar[r] & \end{tikzcd}
\]
To prove the converse, assume that \( \mathcal{L}_1/t\mathcal{L} \) is not cyclic. We must have \( a_1, a_2 \neq 0 \) (otherwise \( v_1 \) or \( v_2 \) would induce a cyclic generator of \( \mathcal{L}_1/t\mathcal{L} \)). Thus \( (v_1, v_2)_{k[t]} \supseteq t\mathcal{L} \). This implies \( \mathcal{L}_1 = (v_1, v_2)_{k[t]} \supseteq \mathcal{L} \), which is equivalent to \( N_1 \supseteq L \).

Thus, by Definition, the set \( \mathfrak{F}_{P,V,l}(k) \) is exactly the subset of the set on the right hand side in (4.1), consisting of all chains \( (L \subseteq L') \in S(k) \) with \( L \subseteq N_1 \):

\[
\mathfrak{F}_{P,V,l}(k) = \{(L \subseteq L') \in S(k) : N/L, N/L' \text{ are cyclic as } k[t]\text{-modules}, L \not\subseteq N_1\}.
\]

**Third step.** We have the following

**Lemma 4.6.**

\[
\mathfrak{F}_{P,V,l}(k) = \left\{ L \subseteq L' \subseteq N : \exists v = e_2 + a_0e_1 + a_1te_1 + \ldots + a_l t^lee_1, L = \langle v \rangle_{k[t]}, L' = L \oplus \langle t^lee_1 \rangle_k, a_0, \ldots, a_l \in k \right\}.
\]

The vector \( v \) is uniquely determined by the chain \( L \subseteq L' \).

**Proof of Lemma.** To prove the inclusion ‘\( \subseteq \)’ let \( L \subseteq L' \subseteq N \) be an element of \( \mathfrak{F}_{P,V,l}(k) \) as described in (4.2). In particular \( L \not\subseteq N_1 \). Then, there exists a \( v = e_2 + w \in L \) with \( w \in N_1 \). Since \( L \) is \( t \)-invariant, we can by successive replacing of \( v \) by \( v - c_it^iv \) with a suitable \( c_i \in k \) for \( 1 \leq i \leq l \) assume that \( v \) is of the following form: \( v = e_2 + a_0e_1 + a_1te_1 + \ldots + a_l t^lee_1 \). This vector \( v \) is unique in \( L \): this follows from \( \dim_k L = l + 1 \) and linear independence of \( v, tv, \ldots, t^lv \). Further \( L' \) is uniquely determined by \( L \): \( L'/L \) is a one-dimensional \( t \)-invariant submodule of \( N/L = \langle e_1, \ldots, t^lee_1 \rangle_k \), so \( L' = L \oplus \langle t^lee_1 \rangle_k \).

The inverse inclusion is easy: \( v \in L - N_1 \) and \( e_1 \) generates the \( k[t] \)-modules \( N/L \) and \( N/L' \). \( \square \)

**Fourth step.** Finally, we prove the existence of a locally closed subscheme \( \mathfrak{F} \) of \( S \), isomorphic to \( \mathbb{A}^{l+1}_{k} \) such that \( \mathfrak{F}(\bar{k}) = \mathfrak{F}_{P,V,l}(\bar{k}) \).

The partial flag manifold \( \mathfrak{flag}_{l+1,t+2}(N) \) has ([Bo] 10.3) the open subscheme \( Y \), defined over \( k \), whose \( \bar{k} \)-valued points are:

\[
Y(\bar{k}) = \left\{ E_0 \subseteq E_1 \subseteq N : \dim_k E_0 = l + 1, \dim_k E_1 = l + 2, E_0 \cap \langle e_1 \rangle_{k[t]} = 0, E_1 \cap \langle e_1, te_1, \ldots, t^{l-1}e_1 \rangle_{\bar{k}} = 0 \right\}.
\]

As a \( k \)-scheme, it is isomorphic to \( \mathbb{A}^{l+1}_{k} \) with the affine coordinate ring \( k[a_{ij}, b_p] : 0 \leq i, j < l + 1, 0 \leq p < l \) where a parametrization is given by:

\[
E_0 = \langle t^i e_2 + \sum_j a_{ij}t^j e_1 : i = 0, \ldots, l \rangle_k, \\
E_1 = E_0 \oplus \langle t^i e_1 + b_0 e_1 + \ldots + b_{l-1} t^{l-i} e_1 \rangle_k.
\]

All the arguments in the steps one, two and three will also work if we replace \( k \) by \( \bar{k} \). Therefore from Lemma 4.6 immediately follows that \( \mathfrak{F}_{P,V,l}(\bar{k}) \subset Y(\bar{k}) \). Moreover, the closed subscheme \( \mathfrak{F} \) of \( Y \) defined by the ideal

\[
(a_{ij} - a_{0,j-i}, a_{p,j'}, b_p : 0 \leq i < j < l + 1, 0 \leq j' < l', 0 \leq p < l),
\]

of \( k[a_{ij}, b_p] : 0 \leq i, j < l + 1, 0 \leq p < l \) satisfies \( \mathfrak{F}(\bar{k}) = \mathfrak{F}_{P,V,l}(\bar{k}) \) and hence \( \mathfrak{F}_{P,V,l}(\bar{k}) \) is a locally closed subset of \( X^{SL_2}(\bar{k}) \). Further, we have: \( \mathfrak{F} \cong \mathbb{A}^{l+1}_{\bar{k}} \). The uniqueness of \( \mathfrak{F} \) follows.
from the fact that a reduced \( k \)-subscheme of \( X^{SL_2} \) is uniquely determined by its \( \overline{k} \)-valued points.

In the following, we denote the scheme \( \mathfrak{F} \) by \( \mathfrak{F}_{P,V,l} \). Thus we have:

\[
\mathfrak{F}_{P,V,l} = \text{Spec} k[a_{00}, ..., a_{0l}] \cong A_{k}^{l+1}.
\]

**Fifth step.** Now let \( V \) be arbitrary. We can assume that \( C_{M}^{0} \in V \). Then \( \mathfrak{F}_{P,V,l}(\overline{k}) \subseteq \mathfrak{F}_{P,C_{M}^{0},l}(\overline{k}) \). Consider the morphism of \( k \)-schemes

\[
\beta : \mathfrak{F}_{P,C_{M}^{0},l} = \text{Spec} k[a_{00}, ..., a_{0l}] \to \mathfrak{F}_{P,C_{M}^{0},0} = \text{Spec} k[a_{00}]
\]

which is defined by using the coordinate rings from the last step:

\[
\beta^{0} : k[a_{00}] \to k[a_{00}, ..., a_{0l}], a_{00} \mapsto a_{00}.
\]

One sees easily that this morphism sends an alcove \( D \in \mathfrak{F}_{P,C_{M}^{0},l}(\overline{k}) \) to the first alcove of the minimal gallery, stretched from \( P \) to \( D \). Now an easy computation shows that there is an (unique) open subscheme of \( \mathfrak{F}_{P,C_{M}^{0},0} \), isomorphic to \( P_{1}^{1} - V \) whose \( \overline{k} \)-rational points are \( \mathfrak{F}_{P,V,0}(\overline{k}) \).

We clearly have

\[
\mathfrak{F}_{P,V,l}(\overline{k}) = \beta^{-1}(\mathfrak{F}_{P,V,0}(\overline{k})),
\]

(as sets) and thus \( \mathfrak{F}_{P,V,l}(\overline{k}) \) is a locally closed subset of \( X^{SL_2}(\overline{k}) \) and there is a unique open subscheme of \( \mathfrak{F}_{P,C_{M}^{0},l} \), whose \( \overline{k} \)-valued points are \( \mathfrak{F}_{P,V,l}(\overline{k}) \). We denote this subscheme by \( \mathfrak{F}_{P,V,l} \). It fits into the following Cartesian diagram:

\[
\begin{array}{ccc}
\mathfrak{F}_{P,V,l} & \longrightarrow & \mathfrak{F}_{P,V,0} \\
\downarrow & & \downarrow \\
\mathfrak{F}_{P,C_{M}^{0},l} & \longrightarrow & \mathfrak{F}_{P,C_{M}^{0},0},
\end{array}
\]

where the lower map is \( \beta \). Now \( \mathfrak{F}_{P,C_{M}^{0},0} \cong A_{k}^{1} \), \( \mathfrak{F}_{P,V,0} \cong P_{1}^{1} - V \), \( \mathfrak{F}_{P,C_{M}^{0},l} \cong A_{k}^{l+1} \), and since \( \beta \) is just the projection on the last factor and \( \mathfrak{F}_{P,V,0} \hookrightarrow \mathfrak{F}_{P,C_{M}^{0},0} \) is the inclusion \( P_{1}^{1} - V \hookrightarrow P_{1}^{1} - \{pt\} = A_{k}^{1} \), we have:

\[
\mathfrak{F}_{P,V,l} \cong A_{k}^{l} \times (P_{1}^{1} - V).
\]

**Comparison of the two proofs.** We go back to the assumption that \( P \) is represented by \( \mathfrak{L}_{0} = \mathfrak{o} \oplus \mathfrak{o}, V = \{C_{M}^{0}\} \). In the first four steps we essentially proved that the open Schubert cell associated to \( v \in W_{a} \) with \( vC_{M}^{0} = C_{M}^{-(l+1)} \) is isomorphic to \( A_{k}^{l+1} \). There exists also an other proof of this fact which works similarly as in the finite case (for the finite case compare [Bo], 14.12, the first theorem). This proof uses affine root subgroups of \( SL_2 \). They are given by \( (n \in \mathbb{Z}) \):

\[
U_{(\alpha, n)} = \left\{ \begin{pmatrix} 1 & ct^{n} \\ 0 & 1 \end{pmatrix} : c \in \overline{k} \right\}, \quad U_{(-\alpha, n)} = \left\{ \begin{pmatrix} 1 & 0 \\ ct^{n} & 1 \end{pmatrix} : c \in \overline{k} \right\},
\]
where $\alpha$ denotes the unique positive (finite) root of $SL_2$. We have:

$$(\beta, n) \text{ is positive if and only if } \begin{cases} \beta = \alpha & \text{and } n \geq 0, \\
\beta = -\alpha & \text{and } n > 0. \end{cases}$$

Thus $(\beta, n)$ is positive if and only if $U_{(\beta, n)} \subset ISL_2$. The affine Weyl group acts on the set of all affine roots.

Consider for every $v \in W_a$ the morphism of varieties (the left hand side carries a natural variety structure):

$$\psi: \prod_{(\beta, n) > 0, v^{-1}(\beta, n) < 0} U_{(\beta, n)} \to ISL_2 v ISL_2 / ISL_2, \ (x_n)_n \mapsto (\prod_n x_n) v ISL_2.$$  

In the other proof one shows that $\psi$ is an isomorphism.

If $l + 1 = 2s > 0$ is even (the other cases are essentially the same) and $v = \begin{pmatrix} t^s & 0 \\
0 & t^{-s} \end{pmatrix}$ then

$$\mathfrak{Z}_{P,C^0, l}(k) = ISL_2 v ISL_2 / ISL_2$$

and we have the following morphisms of varieties:

$$\prod_{0 \leq n < l + 1} U_{(\alpha, n)} \xrightarrow{\psi} ISL_2 v ISL_2 / ISL_2 = \mathfrak{Z}_{P,C^0, l}(k) \to Spec \ k[a_0, \ldots, a_{l}]$$

where the variety on the right hand side is the closed subvariety of $Y$ constructed in our proof. We also constructed the map on the right and proved it to be an isomorphism.

The image of a $k$-valued point $(c_n)_{n=0}^l$ on the left, under the composition of the two morphisms, is a chain of subspaces of $N \otimes k$ (where $N = \mathfrak{L}_0 / tl^1 \mathfrak{L}_0$) which is determined by the alcove

$$\psi((c_n)_{n=0}^l) = \begin{pmatrix} t^s & t^{-s}c_0 + \ldots + t^{s-1}c_l \\
0 & t^{-s} \end{pmatrix} C^0_M.$$  

With notations as in the proof, choosing representatives inside $\mathfrak{L}_0$ of appropriate codimensions and dividing out $tl^1 \mathfrak{L}_0$, gives the representing chain

$$\langle e_2 + c_0 e_1 + c_1 t e_1 + \ldots + c_l t^l e_1 \rangle_k[t] \subseteq \langle e_2 + c_0 e_1 + c_1 t e_1 + \ldots + c_l t^l e_1 \rangle_{k[t]} \oplus \langle t^l e_1 \rangle_k$$

inside $N \otimes k$.

The vector $v = e_2 + c_0 e_1 + c_1 t e_1 + \ldots + c_l t^l e_1$ is exactly the same as in Lemma 4.3. In the fourth step $\mathfrak{Z}_{P,V,l}$ was parametrized by the coordinates of this unique vector $v$. Thus the composition of the two morphisms in (4.4) is given on $k$-valued points by $(c_n)_{n=0}^l \mapsto (c_n)_{n=0}^l$.

**Corollary 4.7.**

(i) Let $P$ be a vertex in $\mathfrak{W}_1, n \geq 0$ and $V = \mathbb{P}^1_k(k)$ be the set of all alcoves in $\mathfrak{W}_1$, having $P$ as a vertex, considered as a finite variety over $k$. Then there is a locally closed reduced subscheme of $X^{SL_2}$, defined over $k$, whose $k$-valued points are exactly $D^n_{\mathfrak{W}_1}(P) = \mathfrak{Z}_{P,V,n-1}(k)$. We denote this scheme again by $D^n_{\mathfrak{W}_1}(P)$. There is an isomorphism of $k$-schemes:

$$D^n_{\mathfrak{W}_1}(P) \cong \mathbb{A}^{n-1} \times (\mathbb{P}^1_k - \mathbb{P}^1_k(k)).$$
(ii) Let $P$ be a vertex in $A_M, n \geq 0$ and $V$ be the set of all alcoves in $A_M$, having $P$ as a vertex (thus $V$ has two elements). Then there is a locally closed reduced subscheme of $X^{SL_2}$, defined over $k$, whose $\bar{k}$-valued points are exactly $D^n_{A_M}(P) = \mathcal{F}_{P,V,n-1}(\bar{k})$. We denote this scheme again by $D^n_{A_M}(P)$. There is an isomorphism of $k$-schemes:

$D^n_{A_M}(P) \cong \mathbb{A}^{n-1}_k \times (\mathbb{P}_k^1 \setminus \{0, \infty\})$.

(iii) Let $m \in \{0,1\}$ and $P_m$ be the vertex represented by $o \oplus t^m o$. Then there is a locally closed reduced subscheme of $X^{SL_2}$, defined over $k$, whose $\bar{k}$-valued points are exactly $D^n_{C_M}(P_m) = \mathcal{F}_{P_m,C_M,n-1}(\bar{k})$. We denote this scheme by $D^n_{C_M}(P_m)$. There is an isomorphism of $k$-schemes:

$D^n_{C_M}(P_m) \cong \mathbb{A}^n_k$.

Proof. It follows directly from the Proposition. \hfill \Box

In particular, the varieties $D^n_{\mathcal{B}_1}(P), D^n_{A_M}(P), D^n_{C_M}(P_i)$ are all smooth and irreducible.

4.2. Some preliminaries before stating the results.

Recall that $H_b = \text{Ker}(v_L \circ \text{det} : J_b \rightarrow \mathbb{Z})$ and $J^{SL_2}_b = J_b \cap SL_2(L)$. We introduce the following notation:

**Notation 4.8.** Let $b \in GL_2(L)$. Let $m \in \{0,1\}$, and let $P_m$ be the vertex of $\mathcal{B}_\infty$ represented by $o \oplus t^m o$. Set

\[
K^{SL_2,(m)}_b = \text{Stab}_{J^{SL_2}_b}(P_m) \quad \text{and} \quad K^{(m)}_b = \text{Stab}_{H_b}(P_m),
\]

where $\text{Stab}_{J^{SL_2}_b}(P_m)$ resp. $\text{Stab}_{H_b}(P_m)$ denotes the stabilizer of the vertex $P_m$ under the action of $J^{SL_2}_b$ resp. $H_b$.

For the rest of the work we also use the following notation: for $m \in \{0,1\}$ we set

$g_m = \begin{pmatrix} 1 & 0 \\ 0 & t^m \end{pmatrix}$

(i.e. $g_0 = 1$). Easy computations give:

**Remark 4.9.**

(i) Let $b = 1$. Then for $m \in \{0,1\}$:

\[
K^{SL_2,(m)}_1 = g_m SL_2(o_F) g_m^{-1} \quad \text{and} \quad K^{(m)}_1 = g_m GL_2(o_F) g_m^{-1}.
\]

(ii) Let $b = \begin{pmatrix} 1 & 0 \\ 0 & t^\alpha \end{pmatrix}$ with $\alpha > 0$. Then for $m \in \{0,1\}$:
\[ K^\text{SL}_2,(m) = T^\text{SL}_2(\sigma_F) = \{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \sigma_F^\times \} \quad \text{and} \]

\[ K^m = T(\sigma_F) = \{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \sigma_F^\times \}. \]

For every \( b \in \text{GL}_2(L) \) and \( m \in \{0, 1\} \), we have:

\[ K^\text{SL}_2,(m) \subseteq J^\text{SL}_2 \subseteq H \subseteq J, \quad \text{and} \]

\[ K^m \subseteq H. \]

**Lemma 4.10.**

(i) Let \( b = 1 \). The group \( H_1 \) resp. \( J_1^\text{SL}_2 \) acts transitively on \( \mathfrak{B}^{(m)}_1 \), and the stabilizer of \( P_m \) is the subgroup \( K^{(m)}_1 \) resp. \( K_1^\text{SL}_2,(m) \).

(ii) Let \( b = \begin{pmatrix} 1 & 0 \\ 0 & t^a \end{pmatrix} \). The group \( H_b \) resp. \( J_b^\text{SL}_2 \) acts transitively on \( A^{(m)}_M \), and the stabilizer of \( P_m \) is the subgroup \( K^m_b \) resp. \( K_b^\text{SL}_2,(m) \).

**Proof.** That \( K^{(m)}_b \) resp. \( K_b^\text{SL}_2,(m) \) is the stabilizer of \( P_m \) under these actions, follows directly from Definition 4.8. The only thing to show is the transitivity. Since \( H_b \supseteq J_b^\text{SL}_2 \), it is enough to show this for \( J_b^\text{SL}_2 \).

In the first case \( (b = 1) \) we have \( J_1^\text{SL}_2 = \text{SL}_2(F) \). Let \( P \in \mathfrak{B}^{(m)}_1 \) be represented by a \( \sigma_F \)-lattice \( \mathfrak{L} \) in \( F^2 \) with \( \wedge^2 \mathfrak{L} = t^m \sigma_F \) and let \( \{v_1, v_2\} \) be an \( \sigma_F \)-basis of it such that the determinant of the matrix, having \( v_1, v_2 \) as columns, is \( t^m \). If now \( \{e_1, e_2\} \) is the standard basis of \( F^2 \), then \( P_m \) is represented by the lattice with basis \( \{e_1, t^m e_2\} \). Now the matrix \( x \) sending \( \{e_1, t^m e_2\} \) in \( \{v_1, v_2\} \) has determinant 1. We have \( xP_m = P \) and thus \( J_1^\text{SL}_2 \) acts transitively on \( \mathfrak{B}^{(m)}_1 \). In the second case the proof is a similar straightforward computation. \( \square \)

**Notation 4.11.** For \( m \in \{0, 1\} \) we set:

\[ I^{(m)} = \text{Stab}_{H_1}(C^m_M). \]

Thus,

\[ I^{(m)} = g_m(I \cap \text{GL}_2(F))g_m^{-1} \]

(in particular, \( I^{(0)} = I \cap \text{GL}_2(F) \)) or more explicitly:

\[ I^{(0)} = \begin{pmatrix} \sigma_F^\times & \sigma_F \\ p_F & \sigma_F^\times \end{pmatrix}, \quad I^{(1)} = \begin{pmatrix} \sigma_F^\times & p_F^{-1} \\ p_F^2 & \sigma_F^\times \end{pmatrix}. \]

**4.3. The first case: \( X^\text{SL}_2(1) \) and \( X_w(1) \).**

We have \( J_1^\text{SL}_2 = \text{SL}_2(F) \) and \( J_1 = \text{GL}_2(F) \).

**Proposition 4.12.**
(i) We have:

\[ X_{1}^{SL_2}(1) \cong \coprod_{J_{1}^{SL_2}/(I_{1}^{SL_2} \cap SL_2(F))} \{pt\} \quad \text{and} \quad X_{1}(1) \cong \coprod_{J_{1}/I^{(0)}} \{pt\} \]

as \( k \)-varieties, and \( J_{1}^{SL_2} \) resp. \( J_1 \) acts on the set of these connected components by left multiplication on the index set.

Let now \( 1 \neq w \in W_a \) such that \( X_w(1) \neq \emptyset \) and \( m \in \{0, 1\} \) as in Proposition 3.7(ii). Then

\[ X_w^{SL_2}(1) \cong \coprod_{J_{1}^{SL_2}/K_1^{SL_2}(m)} A_k^{\ell(w)-1} \times (\mathbb{P}_k^1 - \mathbb{P}_k^1) \]

as \( k \)-varieties, and \( J_{1}^{SL_2} \) resp. \( J_1 \) acts on the set of these connected components by left multiplication on the index set.

**Proof.** We prove (i). From Proposition 3.7(i) follows that the variety \( X_{1}^{SL_2}(1) \) is a disjoint union of points, \( J_{1}^{SL_2} \) acts transitively on the set of these points, and the stabilizer of one of them (we take \( C_{M}^0 \)) is \( I_{1}^{SL_2} \cap SL_2(F) \). Thus the assertion about \( X_{1}^{SL_2}(1) \) follows.

For \( X_{1}(1) \) we have to replace \( J_{1}^{SL_2} \) by \( H_1 \). The stabilizer of \( C_{M}^0 \) inside \( H_1 \) is \( I^{(0)} \). Thus \( X_{1}^{SL_2}(1) = \coprod_{H_1/I^{(0)}} \{pt\} \). The assertion about \( X_{1}(1) \) follows now from Proposition 2.23:

\[ X_{1}(1) = \coprod_{J_{1}/H_1} \coprod_{H_1/I^{(0)}} \{pt\} = \coprod_{J_{1}/I^{(0)}} \{pt\}, \]

where the \( J_1 \)-action on the set of the connected components is given by left multiplication on the index set.

To prove (ii), let \( w \neq 1 \) such that \( X_w^{SL_2}(1) \neq \emptyset \) (see 3.7(ii)) and let \( m \in \{0, 1\} \) be as in Proposition 3.7(ii). Let \( i = \frac{\ell(w)+1}{2} \). For every \( P \in \mathcal{B}_1^{(m)} \) the variety \( D_{\mathcal{B}_1}^i(P) \) is connected (Corollary 3.7). \( J_{1}^{SL_2} \) acts on the set of the connected components of \( X_w^{SL_2}(1) = \coprod_{P \in \mathcal{B}_1^{(m)}} D_{\mathcal{B}_1}^i(P) \) by translating them: thus the action is given by permuting the vertices of departure. They lie in \( \mathcal{B}_1^{(m)} \) and \( J_{1}^{SL_2} \) acts transitively on them (Lemma 4.10). The vertex of departure of the connected component \( D_{\mathcal{B}_1}^i(P_m) \) of \( X_w^{SL_2}(1) \) is \( P_m \), and its stabilizer is \( K_1^{SL_2,(m)} \). The assertion about \( X_w^{SL_2}(1) \) follows thus from Proposition 3.7(ii) and Corollary 4.7.

For \( H_1 \) instead of \( J_{1}^{SL_2} \) we have completely analogously:

\[ X_w^{SL_2}(1) = \coprod_{H_1/K_1^{(m)}} A_k^{\ell(w)-1} \times (\mathbb{P}_k^1 - \mathbb{P}_k^1), \]

and thus from Proposition 2.23 follows

\[ X_{1}(1) = \coprod_{J_{1}/H_1} X_w^{SL_2}(1) = \coprod_{J_{1}/H_1} \coprod_{H_1/K_1^{(m)}} A_k^{\ell(w)-1} \times (\mathbb{P}_k^1 - \mathbb{P}_k^1) = \coprod_{J_{1}/K_1^{(m)}} A_k^{\ell(w)-1} \times (\mathbb{P}_k^1 - \mathbb{P}_k^1), \]
where \( J_1 \) acts on the set of the connected components by left multiplication on the index set.

\[ \square \]

4.4. **The second case: \( X_w^{SL_2}(b) \) and \( X_w(b) \) for diagonal \( b \neq 1 \).**

Let \( b = \begin{pmatrix} 1 & 0 \\ 0 & \ell^\alpha \end{pmatrix} \) with \( \alpha > 0 \). Then \( J_b = T(F), J_b^{SL_2} = T^{SL_2}(F) \). Recall that we have \( K_b^{(m)} = T(\mathfrak{o}_F) \) and \( K_b^{SL_2,(m)} = T^{SL_2}(\mathfrak{o}_F) \).

**Proposition 4.13.**

(i) Let \( w \in W_a \) with \( \ell(w) = \alpha \). We have:

\[
X_w^{SL_2}(b) \cong \coprod_{J_b^{SL_2}/K_b^{SL_2,(0)}} \{pt\} \quad \text{and} \quad X_w(b) \cong \coprod_{J_b/K_b^{(0)}} \{pt\}
\]

as \( k \)-varieties, and \( J_b^{SL_2} \) resp. \( J_b \) acts on the set of these connected components by left multiplication on the index set.

(ii) Let \( w \in W_a \) such that \( \ell(w) > \alpha \) and \( X_w(1) \neq \emptyset \). Let \( m \in \{0,1\} \) as in Proposition 4.11(iii). Then

\[
X_w^{SL_2}(b) \cong \coprod_{J_b^{SL_2}/K_b^{SL_2,(m)}} A_k^{\ell(w)-\alpha-1} \times (\mathbb{P}_k^1 - \{0, \infty\}) \quad \text{and}
\]

\[
X_w(b) \cong \coprod_{J_b/K_b^{(m)}} A_k^{\ell(w)-\alpha-1} \times (\mathbb{P}_k^1 - \{0, \infty\})
\]

as \( k \)-varieties, and \( J_b^{SL_2} \) resp. \( J_b \) acts on the set of these connected components by left multiplication on the index set.

**Proof.** To prove (i), notice that from Proposition 3.11, the variety \( X_w^{SL_2}(b) \) is a disjoint union of points if \( \ell(w) = \alpha \). If \( wC_M^0 = C_M^0 \) resp. \( wC_M^{-\alpha} = C_M^{-\alpha} \), the group \( J_1^{SL_2} \) acts transitively on the set of these points, \( C_M^0 \) resp. \( C_M^{-\alpha} \) is one of them and the stabilizer of \( C_M^0 \) resp. \( C_M^{-\alpha} \) is \( K_b^{SL_2,(0)}(F) \). Thus the assertion about \( X_w^{SL_2}(b) \) follows. The assertion about \( X_w(b) \) follows analogously (as in Proposition 4.12(i)) from Proposition 2.23 since \( K_b^{(0)} \) is the stabilizer of \( C_M^0 \) resp. \( C_M^{-\alpha} \) under the action of \( H_b \).

The proof of (ii) is the same as the proof of Proposition 4.12(ii) (one has to replace \( \mathfrak{Z}_1^{(m)} \) by \( A_M^{(m)} \) and use Proposition 3.11 instead of Proposition 3.7). \( \square \)

**Remark 4.14.** Recall that \( J_b/K_b^{(m)} = T(F)/T(\mathfrak{o}_F) \cong \mathbb{Z}^2 \). Thus if \( X_w(b) \) is non-empty, then \( \pi_0(X_w(b)) = \mathbb{Z}^2 \).

4.5. **The third case: \( X_w^{SL_2}(b_1) \) and \( X_w(b_1) \).**

Let \( E \) denote the unramified quadratic extension of \( F \) contained in \( L \). Then (compare Example (iii) in 2.4):

\[
J_{b_1} = \left\{ \begin{pmatrix} a & \sigma(c) \\ tc & \sigma(a) \end{pmatrix} : a, c \in E, a\sigma(a) - \sigma(c)t \in F^\times \right\}
\]
We understand the affine space of dimension zero to be a point. Recall from Proposition 3.15 that $X_w(b_1)$ is non-empty if and only if $\ell(w)$ is even.

**Proposition 4.15.** Let $w \in W_a$ such that $X_w(b_1) \neq \emptyset$. Then

$$X_w^{SL_2}(b_1) \cong \mathbb{A}_k^{\ell(w)}$$

as $k$-varieties. Further:

$$X_w(b_1) \cong \coprod_{J_{b_1} / H_{b_1}} \mathbb{A}_k^{\ell(w)}$$

as $k$-varieties, and $J_{b_1}$ acts on the set of these connected components by left multiplication on the index set.

**Proof.** The claim about $X_w^{SL_2}(b_1)$ follows directly from Proposition 3.15 and Corollary 4.7. The claim about $X_w(b_1)$ follows now from Proposition 2.23.

**Remark 4.16.**

(i) Recall that $J_{b_1} / H_{b_1} \cong \mathbb{Z}$. Thus if $X_w(b_1)$ is non-empty, then $\pi_0(X_w(b_1)) = \mathbb{Z}$.

(ii) The next Lemma says that $H_{b_1} = K_{b_1}^{(m)}$. Thus we can write $X_w(b_1)$ more similarly to the two previous cases. In fact, let $w \in W_a$ such that $X_w(b_1)$ is non-empty and let $m \in \{0,1\}$ be as in Proposition 3.15. Then:

$$X_w(b_1) \cong \coprod_{J_{b_1} / K_{b_1}^{(m)}} \mathbb{A}_k^{\ell(w)}.$$

**Lemma 4.17.** We have $H_{b_1} = K_{b_1}^{(m)}$.

**Proof.** By Definition we have $K_{b_1}^{(m)} \subseteq H_{b_1}$, hence it is enough to prove that $H_{b_1} \subseteq K_{b_1}^{(m)}$. This is equivalent to saying that $H_{b_1}$ stabilizes $P_0$ and $P_1$. Thus we have to prove that $H_{b_1} \subseteq I$. Let $E$ be the unramified quadratic extension of $F$, contained in $L$, and assume $x = \begin{pmatrix} a & \sigma(c) \\ tc & \sigma(a) \end{pmatrix} \in H_{b_1}$ with $a, c \in E$ (compare Example (iii) in 2.4).

Since $x \in H_{b_1}$, we have $0 = v_L(\det(x)) = v_L(a\sigma(a) - tc\sigma(c))$. But $v_L(a\sigma(a))$ is even and $v_L(tc\sigma(c))$ is odd. Thus we have $0 = \min\{v_L(a\sigma(a)), v_L(tc\sigma(c))\}$. From this follows $a \in \mathfrak{o}^\times$ and $c \in \mathfrak{o}$. Hence $tc \in \mathfrak{p}$ and $\sigma(c) \in \mathfrak{o}$. From all these follows $x \in I$.

**5. The cohomology of $X_w(b)$**

Extend the scalars to $\bar{k}$. Let

$$\overline{X}_w(b) = X_w(b) \times_{\text{Spec} k} \text{Spec} \bar{k}.$$

The group $J_b$ acts on $X_w(b)$ and thus on $\overline{X}_w(b)$ and this action induces an action on the cohomology groups with compact support $H^*_c(\overline{X}_w(b), \mathbb{Q}_l)$ where $l \neq p$ is a prime. In this section we will compute the cohomology groups of $\overline{X}_w(b)$ and the induced representations of $J_b$ on them. All representations we consider are $\mathbb{Q}_l$-vector spaces. The group $J_b$ is in
every case locally profinite and the representations we get are smooth (in the sense of [BH] 2.1).

Beside $J_b$, also the Galois group

$$\Gamma = \text{Gal}(\bar{k}/k)$$

of the extension $\bar{k}/k$ acts on the cohomology $H_c^r(X_{\mu}(b), \mathbb{Q}_l)$. If $X$ is a representation of a group $G$ and $n \in \mathbb{Z}$, then we write $X(n)$ for the twisted representation of $G \times \Gamma$, which is isomorphic to $X$ as $G$-representation, and on which the topological generator $\sigma$ of $\Gamma$ acts by multiplication with $q^n$. By abuse of notation we write sometimes $X$ instead of $X(0)$.

If $G$ is a locally profinite group, then the (compact) induction of smooth representations from an open subgroup $K$ of $G$ (see sect. 5.1 below or [BH] 2.4-5 for Definitions) commutes by construction with this twisting: if $X$ is a smooth representation of $K$, then

$$\text{c} - \text{Ind}_G^K X(n) = (\text{c} - \text{Ind}_G^K X)(n)$$

and

$$\text{Ind}_G^K X(n) = (\text{Ind}_G^K X)(n).$$

We will identify these terms, and only use the left side.

To simplify notation we write in the future $\mathbb{P}^n$ and $\mathbb{A}^n$ for $\mathbb{P}^n_{\bar{k}}$ and $\mathbb{A}^n_{\bar{k}}$. Further, we will write $\mathbb{P}^1 - \mathbb{P}^1(k)$ for the scheme $(\mathbb{P}^1_{\bar{k}} - \mathbb{P}^1(k)) \times_{\text{Spec} \bar{k}} \text{Spec} \bar{k}$, which is the projective line over $\bar{k}$ with the $q + 1$ points defined already over $k$ deleted.

5.1. Three Lemmas.

Here we will prove three easy facts which we use later on.

**Lemma 5.1.** Let $X$ be a connected smooth projective curve over $\bar{k}$, and let $G$ be a group acting on $X$. Then the induced action of $G$ on $H_c^r(X, \mathbb{Q}_l)$ for $r = 0, 2$ is trivial.

**Proof.** This follows from the fact, that the induced action of $G$ on the constant sheaf is trivial and from Poincaré duality. □

Now we will prove two group-theoretic Lemmata. If $G \rightarrow H$ is a homomorphism, then $\text{inf}_H^G$ denotes the inflation of representations from $H$ to $G$. If $\pi$ is a representation of a group $G$, and $K \subseteq G$ is a subgroup, then we also write $\pi$ for the restriction of $\pi$ to $K$ if it causes no ambiguity.

The induced representation $\text{Ind}_K^G \pi$ of a smooth representation $(\pi, V)$ of a closed subgroup $K$ of a locally profinite group $G$ is the set of functions $f : G \rightarrow V$, satisfying the conditions

1. $f(kg) = \pi(k)f(g)$ for all $g \in G$;
2. there exists a compact open subgroup $P$ of $G$ such that $f(gp) = f(g)$ for all $g \in G, p \in P$,

and on which $G$ operates by $(gf)(x) := f(xg)$. The compactly induced representation is the subspace $\text{c} - \text{Ind}_K^G \pi$ of $\text{Ind}_K^G \pi$ which consists of functions $f : G \rightarrow V$ with the additional property

3. the image of the support of $f$ in the set of right cosets of $G$ modulo $K$ is compact.

**Lemma 5.2.** Let $\alpha : G \rightarrow K$ be a continuous, surjective, open homomorphism of locally profinite groups, $N$ an open subgroup of $K$ and $R = \alpha^{-1}(N)$. Let $(\pi, V)$ be a smooth
representation of \(N\). Then we have an isomorphism of smooth \(G\)-representations:

\[
\inf_K^G \text{Ind}_N^K \pi \cong \text{Ind}_R^G \inf_N^R \pi.
\]

**Proof.** Induction from open subgroups and inflation respect smoothness. Consider the two maps:

\[
\begin{align*}
\inf_K^G \text{Ind}_N^K \pi &\to \text{Ind}_R^G \inf_N^R \pi, \quad f \mapsto (\tilde{f} : G \to V), \quad \tilde{f}(g) = f(\alpha(g)) \quad \text{and} \\
\text{Ind}_R^G \inf_N^R \pi &\to \inf_K^G \text{Ind}_N^K \pi, \quad \tilde{f} \mapsto (f : K \to V), \quad f(k) = \tilde{f}(g), \quad \text{for some } g \in \alpha^{-1}(k).
\end{align*}
\]

It is straightforward to show now that these maps are well-defined (i.e., the functions in the image satisfy the conditions (1) and (2) and the second map is independent of the choice of the preimage of \(k\)), inverse to each other and both \(G\)-equivariant. \(\square\)

**Lemma 5.3.** Let \(H\) be a locally profinite group, and \(K, N\) two open subgroups such that \(H = K \cdot N\) where \(K \cdot N = \{kn : k \in K, n \in N\}\). Let \((\pi, V)\) be a smooth \(K\)-representation. Then there is the following isomorphism of smooth \(N\)-representations:

\[
\text{Ind}_K^H \pi \cong \text{Ind}_N^K \text{Ind}_{K \cap N}^N \pi.
\]

**Proof.** Induction from an open subgroup respects smoothness. Consider the two maps:

\[
\begin{align*}
\text{Ind}_K^H \pi &\to \text{Ind}_N^K \text{Ind}_{K \cap N}^N \pi, \quad \tilde{f} \mapsto \tilde{f}|_N, \quad \text{and} \\
\text{Ind}_N^K \text{Ind}_{K \cap N}^N \pi &\to \text{Ind}_K^H \pi, \quad f \mapsto \tilde{f}, \quad \text{where } \tilde{f}(kn) = \pi(k)f(n), \quad \text{for all } k \in K, n \in N.
\end{align*}
\]

It is straightforward to show now that these maps are well-defined (i.e., the functions in the image satisfy the conditions (1) and (2), and the second map is independent of the choice of \(k\) and \(n\)), inverse to each other and both \(N\)-equivariant. \(\square\)

### 5.2. The first case: cohomology of \(X_w(1)\).

Let \(b = 1\). We will use the following notation:

\[
G := GL_2(F) = J_1.
\]

Recall that for \(m \in \{0,1\}\), we denoted by \(P_m\) the vertex in \(B_\infty\), represented by the lattice chain \(\mathfrak{o} \oplus t^m \mathfrak{o}\), by \(g_m\) the matrix \(\begin{pmatrix} 1 & 0 \\ 0 & t^m \end{pmatrix}\), and finally we had the stabilizer of \(P_m\) under the action of \(H_1\):

\[
K_1^{(m)} = g_mGL_2(\mathfrak{o}_F)g_m^{-1}.
\]

For any group \(N\), let \(1_{\mathbb{Q}_l}\) denote the trivial representation of \(N\) on the one-dimensional \(\mathbb{Q}_l\)-vector space. Let \(\mathcal{B}\) denote the subgroup of upper triangular matrices of \(GL_2(k)\). Then the Steinberg representation \(\text{St}_{GL_2(k)}\) of \(GL_2(k)\) is defined by the following exact sequence:

\[
0 \longrightarrow 1_{\mathbb{Q}_l} \longrightarrow \text{Ind}_{\mathcal{B}}^{GL_2(k)} 1_{\mathbb{Q}_l} \longrightarrow \text{St}_{GL_2(k)} \longrightarrow 0
\]

where the image of the first map is the set of the constant functions. This sequence is split and we have

\[
1_{\mathbb{Q}_l} \oplus \text{St}_{GL_2(k)} \cong \text{Ind}_{\mathcal{B}}^{GL_2(k)} 1_{\mathbb{Q}_l}.
\]
The representation $\text{St}_{GL_2(k)}$ is irreducible. For all these facts about $\text{St}_{GL_2(k)}$, we refer to [BH] 6.3.

For $m \in \{0, 1\}$ the map

$$\pi_m : K_1^{(m)} \to GL_2(k), x \mapsto g_m^{-1}xg_m \text{ mod } \left( 1 + \left( \begin{array}{cc} p_F & p_F \\ p_F & p_F \end{array} \right) \right)$$

is continuous, surjective (for $m = 0$ it is just the projection on $GL_2(k)$ induced from the projection $\phi_F \to \phi_F/p_F = k$ and for $m = 1$ first conjugation by $g_1^{-1}$ and then this projection) and open ($GL_2(k)$ is finite). Recall that $I^{(m)} = \text{Stab}_{H_1}(C_M^m)$ (Notation 4.11). We have $I^{(m)} = \pi_m^{-1}(\overline{\text{St}})$. For $m \in \{0, 1\}$ set:

$$\overline{\text{St}} := \text{inf}_{GL_2(k)}^{K_1^{(m)}} \text{St}_{GL_2(k)},$$

where we inflate with respect to $\pi_m$. Since $\text{St}_{GL_2(k)}$ is irreducible, $\overline{\text{St}}$ is as well. As an inflation of a representation from a finite group, $\overline{\text{St}}$ clearly is smooth. From Lemma 5.2 we have the exact sequence for $m \in \{0, 1\}$:

$$0 \to \overline{l_{\mathbb{Q}_l}} \to \text{Ind}_{K^{(m)}_1(I^{(m)})}^{K_1^{(m)}} \text{St}_{I^{(m)}} \to \overline{\text{St}} \to 0,$$

where the image of the map on the left side is the set of the constant functions. Notice that the index of $I^{(m)}$ in $K_1^{(m)}$ is $q + 1$, thus $\overline{\text{St}}$ is $q$-dimensional, and that $\overline{l_{\mathbb{Q}_l}} \oplus \overline{\text{St}} = \text{Ind}_{I^{(m)}}^{K_1^{(m)}} l_{\mathbb{Q}_l}$.

**Proposition 5.4.** Let $w \in W_a$ such that $X_w(1) \neq \emptyset$ (Proposition 3.7) and if $w \neq 1$ let $m$ be as in Proposition 3.7. There are the following isomorphisms of $G \times \Gamma$-modules:

(i) if $w = 1$,

$$H_c^r(X_1(1), \overline{\mathbb{Q}_l}) \cong \begin{cases} c - \text{Ind}_{I^{(0)}}^{G_{I^{(0)}}} l_{\mathbb{Q}_l} & \text{if } r = 0, \\ 0 & \text{else.} \end{cases}$$

(ii) if $w \neq 1$,

$$H_c^r(X_w(1), \overline{\mathbb{Q}_l}) \cong \begin{cases} c - \text{Ind}_{K_1^{(m)}}^{G_{K_1^{(m)}}} \overline{\text{St}}_{\ell(w) - 1} & \text{if } r = \ell(w), \\ c - \text{Ind}_{K_1^{(m)}}^{G_{K_1^{(m)}}} l_{\mathbb{Q}_l} \ell(w) - 3 & \text{if } r = \ell(w) + 1, \\ 0 & \text{else.} \end{cases}$$

**Proof.** Here the cohomology with compact support of a disjoint union of schemes of finite type over $k$ is defined as the direct sum of the cohomology with compact support of these schemes. With this definition, the cohomology with compact support commutes with colimits. For $w = 1$, we get from Proposition 4.12

$$H_c^r(X_1(1), \overline{\mathbb{Q}_l}) = H_c^r(\coprod_{G/I^{(0)}} \{pt\}, \overline{\mathbb{Q}_l}) = c - \text{Ind}_{I^{(0)}}^{G_{I^{(0)}}}(H_c^r(\{pt\}, \overline{\mathbb{Q}_l})).$$

The cohomology of a point in positive degrees vanishes and for $r = 0$ we have: $H_c^0(\{pt\}, \overline{\mathbb{Q}_l}) = 1_{\overline{\mathbb{Q}_l}}$. The action of $\Gamma$ on the zero cohomology group with coefficients in a constant sheaf is trivial. Hence (i) follows.
Assume now \( w \neq 1 \) such that \( X_w(1) \neq \emptyset \) and \( m \in \{0, 1\} \) as in Proposition 3.7. Then from Proposition 4.12 follows:

\[
H^r_c(\mathbb{A}^n \times Y, \mathcal{Q}_{\ell}) = H^r_c(\mathbb{A}^{\ell(w)-1}_{\mathbb{Q}} \times (\mathbb{P}^1 - \mathbb{P}^1(k)), \mathcal{Q}_{\ell})
\]

\[
= c \cdot \text{Ind}_{K_1(m)}(H^r_c(\mathbb{A}^{\ell(w)-1}_{\mathbb{Q}} \times (\mathbb{P}^1 - \mathbb{P}^1(k)), \mathcal{Q}_{\ell})).
\]

The group \( K_1(m) \) acts trivially on the first factor of the Künneth-formula of the right hand side, thus the \( K_1(m) \)-actions on both sides in the following equation (applied to \( Y = \mathbb{P}^1 - \mathbb{P}^1(k) \)) coincide. For any scheme \( Y \) over \( \bar{k} \), we have:

\[
(5.2) \quad H^r_c(\mathbb{A}^n \times Y, \mathcal{Q}_{\ell}) = H^r_c(\mathbb{A}^{n-2}(Y, \mathcal{Q}_{\ell}(n))).
\]

From this follows

\[
(5.3) \quad H^r_c(\mathbb{A}^n, \mathcal{Q}_{\ell}) = c \cdot \text{Ind}_{K_1(m)}(H^r_c(\mathbb{A}^{\ell(w)-1}_{\mathbb{Q}} \times (\mathbb{P}^1 - \mathbb{P}^1(k)), \mathcal{Q}_{\ell}(\mathbb{P}^1 - \mathbb{P}^1(k)))).
\]

Now it suffices to determine the action of \( K_1(m) \) and of \( \Gamma \) on the cohomology groups of this variety. Thus we have reduced the computation of the cohomology of \( \mathbb{X}_w(1) \) to the cohomology of the Drinfeld upper halfplane \( \Omega^2_k = \mathbb{P}^1 - \mathbb{P}^1(k) \), which occurs in the theory of finite Deligne-Lusztig varieties: \( \Omega^2_k \) is isomorphic to the Deligne-Lusztig variety corresponding to \( SL_2(k) \) and to the unique nontrivial element of the Weyl group of \( SL_2(k) \).

In fact, \( \mathbb{P}^1 - \mathbb{P}^1(k) \) is one-dimensional, thus all cohomology groups in degrees \( r > 2 \) vanish. Recall that

\[
(5.4) \quad H^r_c(\mathbb{P}^1, \mathcal{Q}_{\ell}) = \begin{cases} \mathcal{Q}_{\ell} & \text{if } r = 0, \\ \mathcal{Q}_{\ell}(-1) & \text{if } r = 2, \\ 0 & \text{else}. \end{cases}
\]

Consider now the Mayer-Vietoris long exact cohomology sequence for the cohomology with compact supports arising from the decomposition \( \mathbb{P}^1 = (\mathbb{P}^1 - \mathbb{P}^1(k)) \cup \mathbb{P}^1(k) \):

\[
0 \longrightarrow H^0_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathcal{Q}_{\ell}) \longrightarrow \mathcal{Q}_{\ell} \longrightarrow H^0_c(\mathbb{P}^1, \mathcal{Q}_{\ell}) \longrightarrow 0
\]

\[
H^1_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathcal{Q}_{\ell}) \longrightarrow 0 \longrightarrow 0
\]

\[
H^2_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathcal{Q}_{\ell}) \longrightarrow \mathcal{Q}_{\ell}(-1) \longrightarrow 0.
\]

Since \( \mathbb{P}^1 - \mathbb{P}^1(k) \) and \( \mathbb{P}^1(k) \) are both stable under \( K_1(m) \), the \( \mathcal{Q}_{\ell} \)-vector spaces in the above sequence are \( K_1(m) \)-representations and the morphisms are \( K_1(m) \)-equivariant. \( \mathbb{P}^1(k) \) represents the alcoves of \( \mathfrak{B}_1 \) around \( P_m \), and \( K_1(m) \) acts transitively on them. Hence the group \( H^0_c(\mathbb{P}^1(k), \mathcal{Q}_{\ell}) \) as \( K_1(m) \)-representation is the induced representation from the trivial representation of the stabilizer of one alcove (we take \( C_{\mathfrak{M}}^{(m)} \)). From Lemma 5.1 follows:
$H^0_c(\mathbb{P}^1, \mathbb{Q}_l) = 1_{\mathbb{Q}_l}$ as $K^{(m)}_1$-representation. The map on the right in the first line is the inclusion in (5.1). Thus

$$H^0_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathbb{Q}_l) = 0, \quad \text{and}$$
$$H^1_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathbb{Q}_l) = \text{St}.$$

as $K^{(m)}_1$-representations. As a representation of $\Gamma$ the latter is trivial. Further, $H^2_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathbb{Q}_l) = H^2_c(\mathbb{P}^1, \mathbb{Q}_l) = 1_{\mathbb{Q}_l}(-1)$ is trivial as $K^{(m)}_1$-representation by Lemma 5.1. For any scheme $Y$ over $\bar{k}$ and for any $n$ we have:

$$(5.5) \quad H^r_c(Y, \mathbb{Q}_l(n)) = H^r_c(Y, \mathbb{Q}_l) \otimes \mathbb{Q}_l(n).$$

Hence

$$H^r_c(\mathbb{P}^1 - \mathbb{P}^1(k), \mathbb{Q}_l(\frac{\ell(w) - 1}{2})) = \begin{cases} 
\text{St}(\frac{\ell(w) - 1}{2}) & \text{if } r = 1, \\
1_{\mathbb{Q}_l}(\frac{\ell(w) - 3}{2}) & \text{if } r = 2, \\
0 & \text{else}.
\end{cases}$$

From this and (5.3) follows:

$$H^{\ell(w)}_c(X_w(1), \mathbb{Q}_l) = c - \text{Ind}^G_{K^{(m)}_1} \text{St}(\frac{\ell(w) - 1}{2}),$$
$$H^{\ell(w)+1}_c(X_w(1), \mathbb{Q}_l) = c - \text{Ind}^G_{K^{(m)}_1} 1_{\mathbb{Q}_l}(\frac{\ell(w) - 3}{2}),$$

and $H^r_c(X_w(1), \mathbb{Q}_l) = 0$, for $r \neq \ell(w), \ell(w) + 1$. \qed

In particular the representations $H^r_c(X_w(1), \mathbb{Q}_l)$ are smooth. We will further investigate them in subsection 5.4.

### 5.3. Some representation theory of $GL_2(F)$

Here we recall briefly some aspects of the representation theory of the locally profinite group $G := GL_2(F)$. For a more detailed discussion we refer to [BH], paragraph 9. We assume here all representations to be smooth. By a character of a locally profinite group $N$ we always mean a continuous homomorphism $N \to \mathbb{Q}_l^\times$. To give such a character is equivalent to giving a (smooth) one-dimensional representation of $N$. For $m \in \{0, 1\}$ the subgroups $K^{(m)}_1$ of $G$ are compact and open.

Let $B$ denote the Borel subgroup of upper triangular matrices in $G$. Recall further that $T(F)$ is the diagonal torus of $G$ contained in $B$. Thus

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G : a, c \in F^\times, b \in F \right\} \quad \text{and} \quad T(F) = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \in G : a, c \in F^\times \right\}.$$  

The map $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$ gives a projection from $B$ to $T(F)$. Since $T(F)$ is abelian, any irreducible representation $\chi$ of $T(F)$ is a character. We can inflate it to a character of
B by using this projection \( B \rightarrow T(F) \). This inflated character of \( B \) will also be denoted by \( \chi \).

If \( \phi \) is a character of \( F^\times \), then set

\[
\phi_G := \phi \circ \det.
\]

Then \( \phi_G \) is a character of \( G \). If \( \phi \) ranges over all characters of \( F^\times \), then \( \phi_G \) ranges over all characters of \( G \) ( [BH] 9.2 Proposition).

We can also associate to \( \phi \) the character \( \phi_T \) of \( T(F) \), defined by

\[
\phi_T \left( \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \right) := \phi(a)\phi(b).
\]

In other words \( \phi_T = \phi \otimes \phi \). The (twisted) Steinberg representation \( \phi \cdot \text{St}_G \) of \( G \) is defined by the following exact sequence:

\[
(5.6) \quad 0 \longrightarrow \phi_G \longrightarrow \text{Ind}^G_B \phi_T \longrightarrow \text{St}_G \longrightarrow 0.
\]

The class of all irreducible smooth representations of \( G \) can be divided in two disjoint subclasses: the principal series (or noncuspidal) representations and the cuspidal representations. Now, following ( [BH], 9.11) we list all isomorphism classes of irreducible noncuspidal representations of \( G \):

1. the irreducible induced representations \( \text{Ind}^G_B \chi \) where \( \chi \) ranges over the characters of \( T(F) \);
2. the one-dimensional representations \( \phi_G \), where \( \phi \) ranges over the characters of \( F^\times \);
3. the special representations \( \phi \cdot \text{St}_G \), where \( \phi \) ranges over the characters of \( F^\times \).

**Definition 5.5.** A character \( \chi \) of \( T(F) \) resp. of \( F^\times \) is called unramified if \( \chi \) is trivial on the subgroup \( T(o_F) \) resp. on \( o^F_X \).

**Proposition 5.6** (Frobenius Reciprocity). Let \( H \) be a locally profinite group and \( K \) an open subgroup. Let \( \pi \) be a smooth representation of \( H \), and \( \chi \) a smooth representation of \( K \). Then there are the following isomorphisms which are functorial in both variables:

(i) \( \text{Hom}_H(c \text{-} \text{Ind}^H_K \chi, \pi) = \text{Hom}_K(\chi, \pi) \).

(ii) \( \text{Hom}_H(\pi, \text{Ind}^H_K \chi) = \text{Hom}_K(\pi, \chi) \).

**Proof.** For the proof we refer to [BH] 2.4-5. \( \square \)

5.4. The representations \( H^*_c(\mathcal{X}_w(1), \mathbb{Q}_l) \).

If \( V, W \) are two \( G \times \Gamma \)-representations then the \( \mathbb{Q}_l \)-vector space \( \text{Hom}_G(V, W) \) is in a natural way a \( \Gamma \)-representation: \( (\gamma f)(v) := \gamma f(\gamma^{-1}v) \). We see all representations \( \text{Ind}^G_B \chi, \phi_G, \phi \cdot \text{St}_G \) of the last section as trivial \( \Gamma \)-representations. Thus the \( \mathbb{Q}_l \)-vector spaces \( \text{Hom}_G(H^*_c(\mathcal{X}_w(1), \mathbb{Q}_l), \text{Ind}^G_B \chi) \), ... are \( \Gamma \)-representations.

**Theorem 5.7.** Let \( 1 \neq w \in W_o \) such that \( X_w(1) \neq \emptyset \).
(i) Let $\chi$ be a character of $T(F)$. Then
\[
\text{Hom}_G(H_c^{(w)+1}(X_w(1), \overline{Q_l}), \text{Ind}^G_B \chi) = \begin{cases} 
\overline{Q_l}(\frac{3-\ell(w)}{2}) & \text{if } \chi \text{ unramified,} \\
0 & \text{else.}
\end{cases}
\]

(ii) Let $\phi$ be a character of $F^\times$. Then
\[
\text{Hom}_G(H_c^{(w)+1}(X_w(1), \overline{Q_l}), \phi) = \begin{cases} 
\overline{Q_l}(\frac{3-\ell(w)}{2}) & \text{if } \phi \text{ unramified,} \\
0 & \text{else.}
\end{cases}
\]

(iii) Let $\phi$ be a character of $F^\times$. Then $\text{Hom}_G(H_c^{(w)+1}(X_w(1), \overline{Q_l}), \phi \ast \text{St}_G) = 0$.

(iv) Let $\pi$ be a cuspidal representation of $G$. Then $\text{Hom}_G(H_c^{(w)+1}(X_w(1), \overline{Q_l}), \pi) = 0$.

Proof. Fix a $w \in W_a$ such that $X_w(1) \neq \emptyset$. The Frobenius morphism $\sigma \in \Gamma$ acts on $H_c^{(w)+1}(X_w(1), \overline{Q_l})$ by multiplication with $q^{\frac{\ell(w)-3}{2}}$ (see Proposition 5.3), and trivial on all representations standing on the right side in $\text{Hom}_G(\cdot, \cdot)$. Thus it acts by multiplication with $q^{\frac{\ell(w)}{2}}$ on the Hom-spaces.

Let now $m \in \{0, 1\}$ be as in Proposition 3.7. Then $B$ and $K_1^{(m)}$ are open subgroups of $G$, and they satisfy $B \cdot K_1^{(m)} = G$. Hence from Proposition 5.4 we get with Frobenius reciprocity and Lemma 5.3
\[
\text{Hom}_G(H_c^{(w)+1}(X_w(1), \overline{Q_l}), \text{Ind}^G_B \chi) = \text{Hom}_G(c - \text{Ind}^G_{K_1^{(m)}} 1_{\overline{Q_l}}, \text{Ind}^G_B \chi) = \\
= \text{Hom}_{K_1^{(m)}}(1_{\overline{Q_l}}, \text{Ind}^G_{B \cap K_1^{(m)}} \chi) = \text{Hom}_{B \cap K_1^{(m)}}(1_{\overline{Q_l}}, \chi) = \begin{cases} 
\overline{Q_l} & \text{if } 1_{\overline{Q_l}} = \chi|_{B \cap K_1^{(m)}}, \\
0 & \text{else.}
\end{cases}
\]

But
\[
B \cap K_1^{(m)} = \left( \sigma_F^\times, t^{-m} \sigma_F, 0, \sigma_F^\times \right)
\]
is mapped onto $T(\sigma_F)$ under the projection from $B$ to $T(F)$, and hence $1_{\overline{Q_l}} = \chi|_{B \cap K_1^{(m)}}$ if and only if $\chi$ is unramified. Hence (i) follows. We prove (ii) similarly:

\[
\text{Hom}_G(H_c^{(w)+1}(X_w(1), \overline{Q_l}), \phi) = \text{Hom}_G(c - \text{Ind}^G_{K_1^{(m)}} 1_{\overline{Q_l}}, \phi) = \\
= \text{Hom}_{K_1^{(m)}}(1_{\overline{Q_l}}, \phi) = \begin{cases} 
\overline{Q_l} & \text{if } 1_{\overline{Q_l}} = \phi \circ \det|_{K_1^{(m)}}, \\
0 & \text{else.}
\end{cases}
\]

But $\det(K_1^{(m)}) = \sigma_F^\times$. Thus $\phi|_{K_1^{(m)}} = 1_{\overline{Q_l}}$ if and only if $\phi$ is unramified.

To prove (iii) and (iv), notice that if $\pi$ is some smooth representation of $G$ with $\text{Hom}_G(c - \text{Ind}^G_{K_1^{(m)}} 1_{\overline{Q_l}}, \pi) \neq 0$ then $\pi$ has a $K_1^{(m)}$-stable vector:

\[
\pi^{K_1^{(m)}} = \text{Hom}_{K_1^{(m)}}(1_{\overline{Q_l}}, \pi) = \text{Hom}_G(c - \text{Ind}^G_{K_1^{(m)}} 1_{\overline{Q_l}}, \pi) \neq 0.
\]
To prove (iv), notice that if $\pi \neq 0$ contains the trivial character on $I(0) \subseteq K_1^{(m)}$ and is not cuspidal by [BH] 14.3 Proposition. To prove (iii), it is enough to show that $(\phi \cdot \text{St}_G)^{K_1^{(m)}} = 0$. Every smooth $K_1^{(m)}$-representation is semisimple since $K_1^{(m)}$ is compact ([BH] 2.2 Lemma). Thus we have from (5.6):

(5.9) \[
    \phi_G|_{K_1^{(m)}} \oplus \phi \cdot \text{St}_G|_{K_1^{(m)}} = (\text{Ind}_B^G \phi_T)|_{K_1^{(m)}}.
\]

Hence:

$$\text{Hom}_{K_1^{(m)}}(1_{\overline{Q}_l}, \phi_G) \oplus \text{Hom}_{K_1^{(m)}}(1_{\overline{Q}_l}, \phi \cdot \text{St}_G) = \text{Hom}_{K_1^{(m)}}(1_{\overline{Q}_l}, \text{Ind}_B^G \phi_T).$$

The proofs of (i) and (ii) give:

$$\text{Hom}_{K_1^{(m)}}(1_{\overline{Q}_l}, \text{Ind}_B^G \phi_T) = \text{Hom}_{K_1^{(m)}}(1_{\overline{Q}_l}, \phi_G) = \begin{cases} 
    \overline{Q}_l & \text{if } \phi \text{ unramified,} \\
    0 & \text{else.}
\end{cases}$$

In both cases ($\phi$ unramified and $\phi$ not unramified), by dimension counting, it follows

$$\text{Hom}_{K_1^{(m)}}(1_{\overline{Q}_l}, \phi \cdot \text{St}_G) = 0. \quad \square$$

**Theorem 5.8.** Let $1 \neq w \in W_a$ such that $X_w(1) \neq \emptyset$.

(i) Let $\chi$ be a character of $T(F)$. Then

$$\text{Hom}_G(H_c^{\ell}(w) \overline{X}_w(1), \overline{Q}_l), \text{Ind}_B^G \chi) = \begin{cases} 
    \overline{Q}_l(1 - \ell(w)) & \text{if } \chi \text{ unramified,} \\
    0 & \text{else.}
\end{cases}$$

(ii) Let $\phi$ be a character of $F^\times$. Then $\text{Hom}_G(H_c^{\ell}(w) \overline{X}_w(1), \overline{Q}_l), \phi_G) = 0$.

(iii) Let $\phi$ be a character of $F^\times$. Then

$$\text{Hom}_G(H_c^{\ell}(w) \overline{X}_w(1), \overline{Q}_l), \phi \cdot \text{St}_G) = \begin{cases} 
    \overline{Q}_l(1 - \ell(w)) & \text{if } \phi \text{ unramified,} \\
    0 & \text{else.}
\end{cases}$$

(iv) Let $\pi$ be a cuspidal representation of $G$. Then $\text{Hom}_G(H_c^{\ell}(w) \overline{X}_w(1), \overline{Q}_l), \pi) = 0$.

**Proof.** Fix a $w \in W_a$ such that $X_w(1) \neq \emptyset$. Analogous to the last theorem, $\sigma \in \Gamma$ acts on $H_c^{\ell}(w) \overline{X}_w(1), \overline{Q}_l)$ by multiplication with $q^{\frac{\ell(w) - 1}{2}}$, and trivially on all representations occurring on the right hand side in $\text{Hom}_G(\cdot, \cdot)$. Thus it acts by multiplication with $q^{\frac{1 - \ell(w)}{2}}$ on the $\text{Hom}$-spaces.

Let now $m \in \{0, 1\}$ be as in Proposition 3.7. Set:

$$\omega^{(m)} := \begin{pmatrix} 0 & t^{-m} & 0 \\
0 & t^{-m} & 0 \\
0 & 0 & 0 
\end{pmatrix} \in G,
U^{(m)} := \omega^{(m)} I^{(m)} (\omega^{(m)})^{-1} \cap I^{(m)}.$$

Let further $\overline{T} = \begin{pmatrix} k \times & 0 \\
0 & k \times 
\end{pmatrix}$ be the diagonal torus in $GL_2(k)$. Then $U^{(m)}$ is exactly the preimage of $\overline{T}$ in $K_1^{(m)}$ under $\pi_m$ and we have:
Lemma 5.9. We have the following isomorphism of $\overline{B}$-representations:

$$\text{St}_{GL_2(k)} |_{\overline{B}} \cong \text{Ind}_{F}^{\overline{B}} 1_{\overline{qI}}.$$ 

Proof of Lemma. Let $\omega = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The decomposition of $GL_2(k)$ in double cosets modulo $\overline{B}$ is given by

$$GL_2(k) = \overline{B} \cup \overline{B} \omega \overline{B},$$

compare [BH] 5.2. Further, $\omega \overline{B} \omega^{-1}$ is the subgroup of lower triangular matrices in $GL_2(k)$ and $\overline{B} \cap \omega \overline{B} \omega^{-1} = \overline{T}$. Hence the Mackey formula, [We] 4.2 implies:

$$1_{\overline{qI}} \oplus \text{St}_{GL_2(k)} |_{\overline{B}} = (\text{Ind}_{B}^{GL_2(k)} 1_{\overline{qI}}) |_{\overline{B}} \cong 1_{\overline{qI}} \oplus \text{Ind}_{B \cap \omega B \omega^{-1}}^{\overline{B}} 1_{\overline{qI}} = 1_{\overline{qI}} \oplus \text{Ind}_{F}^{\overline{B}} 1_{\overline{qI}}.$$ 

From this follows $\text{St}_{GL_2(k)} |_{\overline{B}} \cong \text{Ind}_{F}^{\overline{B}} 1_{\overline{qI}}$. \hfill \square

By commutativity of induction and inflation (Lemma 5.2), we get from the last Lemma:

$$\overline{\text{St}} |_{I^{(m)}} \cong \text{Ind}_{U^{(m)}}^{I^{(m)}} 1_{\overline{qI}}.$$ 

Now we prove (i). The subgroups $B, K_1^{(m)}$ of $G$ are open, and they satisfy $B \cdot K_1^{(m)} = G$. Hence we get from Proposition 5.3 with Frobenius reciprocity and Lemma 5.3

$$\text{Hom}_G(H^1_c(\chi, \overline{\text{St}}), \text{Ind}_B^{\overline{B}} \chi) = \text{Hom}_G(c - \text{Ind}_{K_1^{(m)}}^{G} \overline{\text{St}}, \text{Ind}_B^{\overline{B}} \chi) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \text{Ind}_{B \cap K_1^{(m)}}^{K_1^{(m)}} \chi) = \text{Hom}_{B \cap K_1^{(m)}}(\overline{\text{St}}, \chi).$$

But $B \cap K_1^{(m)} = \begin{pmatrix} \phi_F^\times & t^{-m} \phi_F \\ 0 & \phi_F^\times \end{pmatrix} = B \cap I^{(m)}$, and $U^{(m)} \cdot (B \cap I^{(m)}) = I^{(m)}$, hence from (5.11) follows with Frobenius reciprocity and Lemma 5.3

$$\text{Hom}_{B \cap K_1^{(m)}}(\overline{\text{St}}, \chi) = \text{Hom}_{B \cap U^{(m)}}(\text{Ind}_{U^{(m)}}^{I^{(m)}} 1_{\overline{qI}}, \chi) = \text{Hom}_{B \cap U^{(m)}}(\text{Ind}_{B \cap U^{(m)}}^{I^{(m)}} 1_{\overline{qI}}, \chi) = \text{Hom}_{B \cap U^{(m)}}(1_{\overline{qI}}, \chi) = \begin{cases} \overline{qI} & \text{if } \chi \text{ unramified,} \\ 0 & \text{else,} \end{cases}$$

since $B \cap U^{(m)}$ is mapped onto $T(\phi_F)$ under the projection from $B$ to $T(F)$ (compare (5.10)).

Now we prove (ii): let $\phi$ be a character of $F^\times$. Consider

$$\text{Hom}_G(c - \text{Ind}_{K_1^{(m)}}^{G} \overline{\text{St}}, \phi_G) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \phi_G).$$

Now $\overline{\text{St}}$ and $\phi_G |_{K_1^{(m)}}$ are both irreducible and $\dim_{\overline{qI}} \overline{\text{St}} = q$, $\dim_{\overline{qI}} \phi_G = 1$. Thus there are no morphisms between them. Thus (ii) follows.
In particular, $\text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \phi_G) = 0$ and thus from (5.9) follows:

$$\text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \phi \cdot \text{St}_G) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \phi_G) \oplus \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \phi \cdot \text{St}_G) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \text{Ind}_B^G \phi_T).$$

From Frobenius reciprocity and the proof of part (i) follows now:

$$\text{Hom}_G(c - \text{Ind}_{K_1^{(m)}}^G(\overline{\text{St}}, \phi \cdot \text{St}_G) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \phi \cdot \text{St}_G) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \text{Ind}_B^G \phi_T)$$

$$= \begin{cases} \overline{\mathbb{Q}_l} & \text{if } \phi \text{ unramified}, \\ 0 & \text{else}, \end{cases}$$

since $\phi_T$ is unramified if and only if $\phi$ is.

Now it remains to prove (iv).

**Lemma 5.10.** We have: $\text{Ind}_{I^{(m)}}^{K_1^{(m)}}(1) = \text{Ind}_{I^{(0)}}^{K_1^{(m)}}(1)$. 

**Proof.** We recall from the proof of Proposition 5.4 that $\text{Ind}_{I^{(m)}}^{K_1^{(m)}}(1)$ is the $K_1^{(m)}$-representation, which one obtains by letting $K_1^{(m)}$ act (transitively) on the alcoves in $\mathfrak{B}_1$ having $P_m$ as a vertex, and then taking its action on the zero cohomology group of $\mathbb{P}^1(k)$; it is the induced representation of the trivial representation of the stabilizer of one of these alcoves. But $C_M^0$ has both $P_0$ and $P_1$ as a vertex and thus $\text{Ind}_{I^{(m)}}^{K_1^{(m)}}(1) = \text{Ind}_{I^{(0)}}^{K_1^{(m)}}(1)$. \[\square\]

Thus we have: $1_{\overline{\mathbb{Q}_l}} \oplus \overline{\text{St}} = \text{Ind}_{I^{(m)}}^{K_1^{(m)}}(1) = \text{Ind}_{I^{(0)}}^{K_1^{(m)}}(1)$. Now consider any smooth representation $\pi$ of $G$ with $\text{Hom}_G(c - \text{Ind}_{K_1^{(m)}}^G(\overline{\text{St}}, \pi) \neq 0$. Then:

$$0 \neq \text{Hom}_G(c - \text{Ind}_{K_1^{(m)}}^G(\overline{\text{St}}, \pi) = \text{Hom}_{K_1^{(m)}}(\overline{\text{St}}, \pi) \subseteq \text{Hom}_{K_1^{(m)}}(\text{Ind}_{I^{(0)}}^{K_1^{(m)}}(1), \pi) = \text{Hom}_{I^{(0)}}(1, \pi).$$

Thus $\pi$ would contain the trivial character of $I^{(0)} = I \cap GL_2(F)$. Then $\pi$ is not cuspidal by [BH] 14.3 Proposition. \[\square\]

The next Corollary shows that $H_c^0(X_1(1), \overline{\mathbb{Q}_l})$ contains no new information.

**Corollary 5.11.** As $G \times \Gamma$-modules we have:

$$H_c^0(X_1(1), \overline{\mathbb{Q}_l}) \cong c - \text{Ind}_{K_1^{(0)}}^G(1) \oplus c - \text{Ind}_{K_1^{(0)}}^G(\overline{\text{St}}).$$

(i.e. the $\Gamma$-action is trivial).

(i) Let $\chi$ be a character of $T(F)$. Then

$$\text{Hom}_G(H_c^0(X_1(1), \overline{\mathbb{Q}_l}), \text{Ind}_{I^{(0)}}^G \chi) = \begin{cases} \overline{\mathbb{Q}_l} & \text{if } \chi \text{ unramified}, \\ 0 & \text{else}. \end{cases}$$

(ii) Let $\phi$ be a character of $F^\times$. Then

$$\text{Hom}_G(H_c^0(X_1(1), \overline{\mathbb{Q}_l}), \phi_G) = \begin{cases} \overline{\mathbb{Q}_l} & \text{if } \phi \text{ unramified}, \\ 0 & \text{else}, \end{cases}$$
The second case: cohomology of $\overline{X}_w(b)$ with diagonal $b \neq 1$.

Let $b = \begin{pmatrix} 1 & 0 \\ 0 & \ell^a \end{pmatrix}$ with $\alpha > 0$. Recall that $J_b = T(F)$ and $K_b^{(m)} = T(\mathfrak{p}_F)$ for $m \in \{0, 1\}$. In particular all these groups are abelian.

**Proposition 5.12.** Let $w \in W_a$ such that $X_w(b) \neq \emptyset$ (compare Proposition 3.11), and if $\ell(w) > \alpha$, let $m$ be as in Proposition 3.11. There are the following isomorphisms of $J_b \times \Gamma$-modules:

(i) If $\ell(w) = \alpha$,

$$H^r_c(\overline{X}_w(b), \overline{\mathbb{Q}_l}) = \begin{cases} c - \text{Ind}_{K_b^{(0)}}^{J_b} 1_{\mathbb{Q}_l} & \text{if } r = 0, \\ 0 & \text{else.} \end{cases}$$

(ii) If $\ell(w) > \alpha$,

$$H^r_c(\overline{X}_w(b), \overline{\mathbb{Q}_l}) = \begin{cases} c - \text{Ind}_{K_b^{(m)}}^{J_b} 1_{\mathbb{Q}_l} (\ell(w) - \alpha - 1) / 2 & \text{if } r = \ell(w) - \alpha, \\ c - \text{Ind}_{K_b^{(m)}}^{J_b} 1_{\mathbb{Q}_l} (\ell(w) - \alpha - 3) / 2 & \text{if } r = \ell(w) - \alpha + 1, \\ 0 & \text{else.} \end{cases}$$

**Proof.** We prove (i). If $\ell(w) = \alpha$, then we get from Proposition 4.13

$$H^r_c(\overline{X}_w(b), \overline{\mathbb{Q}_l}) = H^r_c(\bigoplus_{J_b/K_b^{(0)}} (pt), \overline{\mathbb{Q}_l}) = c - \text{Ind}_{K_b^{(0)}}^{J_b} (H^r_c((pt), \overline{\mathbb{Q}_l})).$$

The cohomology of a point in positive degrees vanishes and for $r = 0$ we have: $H^0_c((pt), \overline{\mathbb{Q}_l}) = 1_{\mathbb{Q}_l}$ as $K_b^{(m)}$-representation. The action of $\Gamma$ on the zero cohomology group with coefficients in $\mathbb{Q}_l$ is trivial. Hence (i) follows.
Consider now a \( w \in W_a \) with \( \ell(w) > \alpha \) such that \( X_w(b) \neq \emptyset \), and let \( m \in \{0,1\} \) be as in Proposition 3.11. Then from Proposition 4.13 follows:

\[
(5.12) \quad H^\ell_c(\overline{X}_w(b), \overline{\mathbb{Q}_l}) = H^\ell_c\left( \bigtimes_{J_0/K_b^{(m)}} \overline{\mathbb{A}}_{(\ell(w)-\alpha-1)/2} \times (\mathbb{P}^1 - \{0, \infty\}), \overline{\mathbb{Q}_l}\right) \]

\[
= c - \text{Ind}^{J_0}_{K_b^{(m)}}(H^\ell_c(\overline{\mathbb{A}}_{(\ell(w)-\alpha-1)/2} \times (\mathbb{P}^1 - \{0, \infty\}), \overline{\mathbb{Q}_l})).
\]

Now (5.2) implies:

\[
(5.13) \quad H^\ell_c(\overline{\mathbb{A}}_{(\ell(w)-\alpha-1)/2} \times (\mathbb{P}^1 - \{0, \infty\}), \overline{\mathbb{Q}_l}) = H^\ell - (\ell(w)+\alpha+1)(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}(\ell(w) - \alpha - 1)),
\]

Hence it suffices to determine the action of \( K_b^{(m)} \) and of \( \Gamma \) on the cohomology of \( \mathbb{P}^1 - \{0, \infty\} \). First of all \( \mathbb{P}^1 - \{0, \infty\} \) is one-dimensional, hence all cohomology groups in degrees \( r > 2 \) vanish.

Using (5.4), we get the Mayer-Vietoris long exact cohomology sequence for the cohomology with compact supports arising from the decomposition \( \mathbb{P}^1 = (\mathbb{P}^1 - \{0, \infty\}) \cup \{0, \infty\} \), where \( \{0, \infty\} \) represents the two alcoves of \( A_M \) having \( P_m \) as a vertex:

\[
0 \longrightarrow H^0_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}) \longrightarrow \overline{\mathbb{Q}_l} \longrightarrow H^0_c(\{0, \infty\}, \overline{\mathbb{Q}_l}) \longrightarrow 0.
\]

\[
\longrightarrow H^1_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}) \longrightarrow 0 \longrightarrow 0.
\]

\[
\longrightarrow H^2_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}) \longrightarrow \overline{\mathbb{Q}_l}(-1) \longrightarrow 0.
\]

Since \( \mathbb{P}^1 - \{0, \infty\} \) and \( \{0, \infty\} \) are both stable under \( K_b^{(m)} \), the \( \overline{\mathbb{Q}_l} \)-vector spaces in the above sequence are \( K_b^{(m)} \)-representations and the morphisms are \( K_b^{(m)} \)-equivariant. But \( K_b^{(m)} = T(\sigma_F) \) stabilizes every alcove in \( A_M \), and thus acts trivially on \( \{0, \infty\} \). Hence the group \( H^0_c(\{0, \infty\}, \overline{\mathbb{Q}_l}) \) is isomorphic to \( 1_{\overline{\mathbb{Q}_l}} \oplus 1_{\overline{\mathbb{Q}_l}} \) as \( K_b^{(m)} \)-representation. The map on the right in the first line is the diagonal embedding \( 1_{\overline{\mathbb{Q}_l}} \rightarrow 1_{\overline{\mathbb{Q}_l}} \oplus 1_{\overline{\mathbb{Q}_l}} \). Thus

\[
H^0_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}) = 0, \quad \text{and}
\]

\[
H^1_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}) = 1_{\overline{\mathbb{Q}_l}},
\]

as \( K_b^{(m)} \)-representations. The latter is trivial as a \( \Gamma \)-representation. Further,

\[
H^2_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}) = H^2_c(\mathbb{P}^1, \overline{\mathbb{Q}_l}) = 1_{\overline{\mathbb{Q}_l}}(-1)
\]

is trivial as \( K_b^{(m)} \)-representation by Lemma 5.1. From (5.5) we have:

\[
H^\ell_c(\mathbb{P}^1 - \{0, \infty\}, \overline{\mathbb{Q}_l}(\ell(w) - \alpha - 1)) = \begin{cases} 
1_{\overline{\mathbb{Q}_l}}(\ell(w) - \alpha - 1) & \text{if } r = 1, \\
1_{\overline{\mathbb{Q}_l}}(\ell(w) - \alpha - 3) & \text{if } r = 2, \\
0 & \text{else}
\end{cases}
\]
From this and \((5.12), (5.13)\) follows:

\[
H_c^\ell(w) - \alpha(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}) = c - \text{Ind}_{K_1(w)}^J 1_{\mathbb{Q}_l}(\frac{\ell(w) - \alpha - 1}{2}), \quad \text{and}
\]

\[
H_c^\ell(w) - \alpha + 1(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}) = c - \text{Ind}_{K_1(w)}^J 1_{\mathbb{Q}_l}(\frac{\ell(w) - \alpha - 3}{2}),
\]

and \(H_c^r(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}) = 0\) if \(r \neq \ell(w) - \alpha, \ell(w) - \alpha + 1\).

In particular, these representations are smooth, since they are compactly induced from the trivial representation of an open subgroup. We have the exact sequence of abelian groups:

\[
0 \longrightarrow T(\sigma_F) \longrightarrow T(F) \longrightarrow \mathbb{Z}^2 \longrightarrow 0.
\]

Let \(R\mathbb{Z}^2\) denote the regular representation of \(\mathbb{Z}^2\). This means \(\{e_{a,b}: a, b \in \mathbb{Z}\}\) is a \(\overline{\mathbb{Q}_l}\)-basis of \(R\mathbb{Z}^2\), and \(\mathbb{Z}^2\) operates by translation on these basis vectors: \((c, d), e_{a,b} = e_{a+c, b+d}\).

We assume all occurring representations and characters of \(T(F)\) to be smooth. Further, since \(T(F)\) is abelian, every irreducible representation of \(T(F)\) is a character.

**Theorem 5.13.** Let \(w \in W_a\) such that \(X_w(b) \neq \emptyset\). If \(r\) is such that \(H_c^r(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}) \neq 0\) (i.e. \(r = 0\) if \(\ell(w) = \alpha\), and \(r = \ell(w) - \alpha, \ell(w) - \alpha + 1\) if \(\ell(w) > \alpha\)), then we have:

\[
H_c^r(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}) \cong \text{inf}_{\mathbb{Z}^2}^{T(F)} R\mathbb{Z}^2,
\]

as \(T(F)\)-representations. Let now \(\chi\) be a character of \(T(F)\).

(i) If \(\ell(w) = \alpha\), then

\[
\text{Hom}_{T(F)}(H_c^0(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}), \chi) = \begin{cases} \overline{\mathbb{Q}_l} & \text{if } \chi \text{ unramified}, \\ 0 & \text{else.} \end{cases}
\]

(ii) If \(\ell(w) > \alpha\), then

\[
\text{Hom}_{T(F)}(H_c^\ell(w) - \alpha(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}), \chi) = \begin{cases} \overline{\mathbb{Q}_l}(\frac{\alpha + 1 - \ell(w)}{2}) & \text{if } \chi \text{ unramified,} \\ 0 & \text{else.} \end{cases}
\]

\[
\text{Hom}_{T(F)}(H_c^\ell(w) - \alpha + 1(\mathbb{X}_w(b), \overline{\mathbb{Q}_l}), \chi) = \begin{cases} \overline{\mathbb{Q}_l}(\frac{\alpha + 3 - \ell(w)}{2}) & \text{if } \chi \text{ unramified,} \\ 0 & \text{else.} \end{cases}
\]

**Proof.** All statements about the \(\Gamma\)-action follow from Proposition 5.12 similarly as in Theorem 5.7. Further, all non-zero cohomology groups are isomorphic to \(c - \text{Ind}_{T(\sigma_F)}^{T(F)} 1_{\mathbb{Q}_l}\) as \(T(F)\)-representations. For any character \(\chi\) of \(T(F)\), Frobenius reciprocity gives:

\[
\text{Hom}_{T(F)}(c - \text{Ind}_{T(\sigma_F)}^{T(F)} 1_{\mathbb{Q}_l}, \chi) = \text{Hom}_{T(\sigma_F)}(1_{\mathbb{Q}_l}, \chi) = \begin{cases} \overline{\mathbb{Q}_l} & \text{if } \chi \text{ unramified,} \\ 0 & \text{else.} \end{cases}
\]

It remains to show that \(c - \text{Ind}_{T(\sigma_F)}^{T(F)} 1_{\mathbb{Q}_l} \cong \text{inf}_{\mathbb{Z}^2}^{T(F)} R\mathbb{Z}^2\). Since \(T(F)\) is abelian, Lemma 5.14 below shows that \(T(\sigma_F)\) operates trivially on \(c - \text{Ind}_{T(\sigma_F)}^{T(F)} 1_{\mathbb{Q}_l}\). Hence \(c - \text{Ind}_{T(\sigma_F)}^{T(F)} 1_{\mathbb{Q}_l}\) is equal to an inflation of a representation from \(T(F)/T(\sigma_F) = \mathbb{Z}^2\). Now the matrices
$t^{(a,b)} = \begin{pmatrix} t^a & 0 \\ 0 & t^b \end{pmatrix}$ with $a, b \in \mathbb{Z}$ represent the left cosets in $T(F)/T(\mathfrak{o}_F)$ and by [BH] 2.5

Lemma a basis of $c - \text{Ind}^{T(F)}_{T(\mathfrak{o}_F)} 1_{\mathcal{Q}_l}$ is given by the functions

$$e_{a,b}: T(F) \rightarrow \mathcal{Q}_l, \quad e_{a,b}(x) = \begin{cases} 1 & \text{if } x \in t^{(-a,-b)}T(\mathfrak{o}_F), \\ 0 & \text{else.} \end{cases}$$

For any $x \in T(\mathfrak{o}_F)$ and $c, d \in \mathbb{Z}$, the element $t^{(c,d)}_x$ operates on $e_{a,b}$ by translation: $(t^{(c,d)}_x e_{a,b})(y) = e_{a,b}(t^{(c,d)}_x y) = e_{a+c,b+d}(y)$ for any $y \in T(F)$. From this the result follows.

**Lemma 5.14.** Let $K \subseteq H$ be a normal subgroup of a locally profinite group. Then $\text{Ind}^H_K 1_{\mathcal{Q}_l}$ is trivial as $K$-representation.

**Proof.** Let $k \in K$ and $f: H \rightarrow \mathcal{Q}_l$ in $\text{Ind}^H_K 1_{\mathcal{Q}_l}$. Then we have for all $x \in H$: $(kf)(x) = f(k'x) = k'f(x) = f(x)$ for some $k' \in K$.

5.6. The third case: cohomology of $\mathcal{X}_w(b_1)$.

**Proposition 5.15.** Let $w \in W_a$ be such that $X_w(b_1) \neq \emptyset$ (compare Proposition 3.15). Then we have the following isomorphisms of $J_{b_1} \times \Gamma$-modules:

$$H^r_c(\mathcal{X}_w(b_1), \mathcal{Q}_l) = \begin{cases} c - \text{Ind}^{J_{b_1}}_{H_{b_1}} 1_{\mathcal{Q}_l}(\ell(w)/2) & \text{if } r = \ell(w), \\ 0 & \text{else.} \end{cases}$$

**Proof.** From Proposition 4.15 we have:

$$H^r_c(\mathcal{X}_w(b_1), \mathcal{Q}_l) = H^r_c(\prod_{J_{b_1}/H_{b_1}} A^{\ell(w)/2}, \mathcal{Q}_l) = c - \text{Ind}^{J_{b_1}}_{H_{b_1}} H^r_c(A^{\ell(w)/2}, \mathcal{Q}_l).$$

Now (5.2) implies:

$$H^r_c(A^{\ell(w)/2}, \mathcal{Q}_l) = H^r_c(A^{\ell(w)}(\{pt\}, \mathcal{Q}_l(\ell(w)/2))) = \begin{cases} 1_{\mathcal{Q}_l}(\ell(w)/2) & \text{if } r = \ell(w), \\ 0 & \text{else.} \end{cases}$$

In particular all these representations are smooth, since they are compactly induced from an open subgroup (or zero).

5.7. The representations $H^r_c(\ell(w), \mathcal{X}_w(b_1), \mathcal{Q}_l)$. At first, we recall briefly some facts about the multiplicative group $J_{b_1}$ of the quaternion algebra over $F$. For all facts presented here we refer to [BH] 53, 54. Recall that $E$ denotes the unramified extension of $F$ of degree two contained in $L$. Let

$$D = \left\{ \begin{pmatrix} a & \sigma(c) \\ \sigma(a) & \sigma(c) \end{pmatrix} : a, c \in E \right\}$$
be the corresponding quaternion algebra. Thus we have \( J_{b_1} = D^\times = D \setminus \{0\} \). The reduced norm on \( D \) is given by the determinant:

\[
\text{Nrd} = \det : D \to F, \quad \left( \begin{array}{cc} a & \sigma(c) \\ tc & \sigma(a) \end{array} \right) \mapsto a\sigma(a) - tc\sigma(c).
\]

Its restriction to \( D^\times \) gives a surjective homomorphism \( \det : D^\times \to F^\times \). The map \( v_D : x \mapsto v_D(x) := v_L(\det(x)) \)
defines a discrete valuation on \( D \) and we have the corresponding valuation ring and the group of units in it:

\[
\mathcal{O} = \{ x \in D : v_D(x) \geq 0 \} \quad \text{and} \quad U_D = \mathcal{O}^\times = \{ x \in D : v_D(x) = 0 \}.
\]

The group \( U_D \) is normal, compact and open subgroup of \( D^\times \). By definition, we have:

\[
H_{b_1} = U_D.
\]

If we speak of a representation of \( D^\times \) or \( F^\times \), we mean a smooth representation. Let \( \chi \) be a character of \( F^\times \). Then \( \chi_D := \chi \circ \det \) is a character of \( D^\times \). If \( \chi \) ranges over all characters of \( F^\times \), then \( \chi_D \) ranges over all characters of \( D^\times \) ( [BH] 53.5). The characters \( \chi_D \) are exactly the one-dimensional representations of \( D^\times \). Further, if \( \pi \) is an irreducible representation of \( D^\times \), then \( \pi \) is finite-dimensional ( [BH] 54.1).

We have the projection \( D^\times \to D^\times / U_D \cong \mathbb{Z} \). Let \( RZ \) denote the regular representation of \( \mathbb{Z} \).

**Lemma 5.16.** Let \( w \in W_a \) such that \( X_w(b_1) \neq \emptyset \). Then

\[
H_c^{(w)}(\overline{X}_w(b_1), \overline{Q}_l) \cong \inf_{\mathbb{Z}} D^\times \ RZ
\]
as \( D^\times \)-representations. Further if \( \pi \) is an irreducible representation of \( D^\times \) with

\[
\text{Hom}_{D^\times}(H_c^{(w)}(\overline{X}_w(b_1), \overline{Q}_l), \pi) \neq 0,
\]
then \( U_D \subseteq \text{Ker}(\pi) \). In particular \( \pi \) is one-dimensional and unramified.

**Proof.** Since \( U_D \) is normal in \( D^\times \), Lemma 5.14 shows that \( c - \text{Ind}_{U_D}^{D^\times} 1_{Q_l} \) is trivial as \( U_D \)-representation. A similar computation as in the proof of Theorem 5.13 shows that

\[
H_c^{(w)}(\overline{X}_w(b_1), \overline{Q}_l) \cong \inf_{\mathbb{Z}} D^\times \ RZ.
\]

If \( \alpha : c - \text{Ind}_{U_D}^{D^\times} 1_{Q_l} \to \pi \) is some non-zero homomorphism, then \( \alpha \) is surjective, since \( \pi \) is irreducible. Hence \( \pi \) must be trivial on \( U_D \). Hence \( \pi \) is the inflation of some irreducible representation of \( \mathbb{Z} \). Since \( \mathbb{Z} \) is abelian, this representation must be one-dimensional. Thus \( \pi \) is one-dimensional. \( \square \)

Summarizing the results, we get the following

**Theorem 5.17.** Let \( w \in W_a \) such that \( X_w(b_1) \neq \emptyset \). Then

\[
H_c^{(w)}(\overline{X}_w(b_1), \overline{Q}_l) \cong \inf_{\mathbb{Z}} D^\times \ RZ
\]
as \( D^\times \)-representations.
(i) Let $\chi$ be a character of $F^\times$. Then
\[
\text{Hom}_{D^\times}(H^\ell_{c}(w)(X_w(b_1), Q_l), \chi) = \begin{cases} 
Q_l(-\ell(w)/2) & \text{if $\chi$ unramified,} \\
0 & \text{else.}
\end{cases}
\]

(ii) Let $\pi$ be an irreducible representation of $D^\times$ of dimension $\geq 2$. Then
\[
\text{Hom}_{D^\times}(H^\ell_{c}(w)(X_w(b_1), Q_l), \pi) = 0.
\]

**Proof.** The first statement was already proven in Lemma 5.16. All statements about the $\Gamma$-action follow from Proposition 5.15 as in Theorem 5.7. To prove (i), notice that Frobenius reciprocity implies:
\[
\text{Hom}_{D^\times}(H^\ell_{c}(w)(X_w(b_1), Q_l), \chi) = \text{Hom}_{D^\times}(c - \text{Ind}_{U_D^c}^{D^\times}(1_{Q_l}, \chi_D)) = \text{Hom}_{U_D^c}(1_{Q_l}, \chi_D) =
\begin{cases} 
Q_l & \text{if $\chi\circ\det\mid_{U_D} = 1_{Q_l}$,} \\
0 & \text{else.}
\end{cases}
\]

But since $\det$ is surjective and $U_D = \det^{-1}(\sigma_F^X)$, we have $\det(U_D) = \sigma_F^X$. This implies: $\chi\circ\det\mid_{U_D} = 1_{Q_l}$ if and only if $\chi$ is unramified.

(ii) follows directly from Lemma 5.16. \[\square\]

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