ON THE SMOOTHNESS OF NORMALISERS AND THE SUBALGEBRA STRUCTURE OF MODULAR LIE ALGEBRAS

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Abstract. We investigate aspects of the Lie subalgebra structure of the Lie algebra \( g \) associated to a reductive algebraic group \( G \) over an algebraically closed field of characteristic \( p \). Firstly, we revisit work of O.K. Ten on maximal non-semisimple subalgebras of \( g \) in case \( G \) is simple of classical type. Secondly, we give bounds on \( p \) which guarantee that normalisers of subalgebras of \( g \) in \( G \) are smooth, i.e. so that their Lie algebras coincide with the infinitesimal normalisers. We apply these results to obtain information about maximal and maximal solvable subalgebras of \( g \). One of our main tools is to exploit cohomology vanishing of small dimensional modules. Along the way, we obtain complete reducibility results for small dimensional modules in the spirit of similar results due to Jantzen and Serre.

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1. Introduction

A celebrated theorem of Borel and Tits gives detailed information on the lattice of non-reductive subgroups of reductive algebraic groups. Specifically, let $G$ be a reductive algebraic group over any algebraically closed field and let $U$ be a closed unipotent subgroup contained in a Borel subgroup (that is, a maximal, closed, connected, solvable subgroup) of $G$. Then there exists a parabolic subgroup $P$ of $G$ with $U$ contained in its unipotent radical $R_u(P)$ and the normaliser $N_G(U) \leq P$. Thus if $H \leq G$ is a closed subgroup with unipotent radical $R_u(H) = U$ then we have $H \leq N_G(U) \leq P$. An immediate consequence of this theorem is that a maximal closed subgroup of $G$ is either a maximal parabolic subgroup, or reductive.

Parabolic subgroups, i.e. those subgroups containing Borel subgroups, enjoy certain nice (well-known) properties. To start with, all Borel subgroups are conjugate—a consequence of Borel’s famous fixed point theorem. More generally, parabolic subgroups fall into only a few conjugacy classes, corresponding bijectively with the $2^r$ subsets of the Dynkin diagram of $G$ where $r$ is the rank of $G$.

The existence of the exponential and logarithmic maps in characteristic zero readily show that the same structure exists for Lie algebras of reductive groups over fields of characteristic zero, where one may define a Borel subalgebra as a maximal solvable subalgebra, or what can be shown to be equivalent, the Lie algebra of a Borel subgroup. See also [Bou05, Ch. VIII, §10, Cor. 2.] for a direct, Lie-algebraic proof of this.

It would clearly be nice to have similar information about the subalgebra structure of modular Lie algebras; that is, Lie algebras over fields of positive characteristic $p > 0$. Unfortunately, little is true in general; this can often be traced back to the failure of Lie’s theorem in the modular case. For example, let $\mathfrak{h}$ be the three-dimensional Heisenberg algebra with basis $\{x, y, z\}$ whose elements commute, excepting the relation $[x, y] = z$. Clearly $\mathfrak{h}$ is nilpotent, yet it has an irreducible representation of dimension $p$, over any field of characteristic $p > 0$. Thus there exist solvable subalgebras of $\mathfrak{gl}(V)$ for any $p \leq \dim V$ which are not contained in the Lie algebra of any Borel subgroup. The paper [YC12] establishes that there are two classes of maximal solvable subalgebras of the $p$-dimensional simple Witt algebra $W_1$ (rather than one). Even when one considers classical (algebraic) Lie algebras, there can be many classes of maximal solvable subalgebras in the modular case.

In the algebraic case, that is where $\mathfrak{g} = \text{Lie}(G)$ for $G$ an algebraic group, one can express this aberrant behaviour as the statement that ‘normalisers are not smooth’. Conversely, when all normalisers of subalgebras are smooth, the Borel–Tits theorem does hold: see Theorem 10.1.

There are several ways to define the smoothness of normalisers. If we think of the linear algebraic group $G$ as a group scheme, then any closed (scheme-theoretic) subgroup $N$ of $G$ is smooth if $\dim N = \dim \text{Lie}(N)$ [Jan03, I.7.17]. One can show further that if $H$ is a closed subgroup of $G$, the (scheme-theoretic) normaliser $N_G(\mathfrak{h})$ is smooth if and only if $\dim(N_G(\mathfrak{h})) = \dim \mathfrak{n}_G(\mathfrak{h})$ where $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(H)$ and $\mathfrak{n}_G(\mathfrak{h})$ is the Lie-theoretic normaliser of $\mathfrak{h}$ in $\mathfrak{g}$. This last characterisation makes sense in the more classical theory where linear algebraic groups $G = G(k)$ are considered as a group of points. There it can be replaced by the statement $\text{Lie}(N_G(H)(k)) = \mathfrak{n}_G(\mathfrak{h})$.

Our main theorem is

**Theorem A.**  
(i) Let $G = \text{GL}(V)$ and let $\mathfrak{g}$ be its Lie algebra. Then the normalisers $N_G(\mathfrak{h})$ of all subalgebras $\mathfrak{h}$ are smooth if and only if $p > \dim V + 1$. 


(ii) Let $V$ be the minimal-dimensional faithful module for a simple algebraic group $G$ and suppose $p > \dim V + 1$. Then the normalisers $N_G(\mathfrak{h})$ of all subalgebras $\mathfrak{h}$ are smooth. In particular, the conclusion holds when $G$ is classical and $p > h + 1$.

(iii) Let $p > 2h - 2$ for the reductive group $G$. Then the normalisers $N_G(\mathfrak{h})$ of all subspaces $\mathfrak{h}$ of $\mathfrak{g}$ are smooth. Moreover, for any $p \leq 2n - 2$, there are non-smooth normalisers of subspaces of $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$.

Note that while item (ii) gives better bounds for the case $G$ classical, item (iii) gives better bounds for the case $G$ exceptional. This theorem complements results in [BMRT10], [Her13] where exact conditions were given for all centralisers of subgroup schemes in reductive groups to be smooth.

There are a number of consequences of Theorem A which we discuss below. However, note first that if all normalisers of subalgebras of $\mathfrak{g}$ are smooth and $\mathfrak{h}$ is a maximal subalgebra, then $\mathfrak{h}$ is its own normaliser, hence is an algebraic subalgebra of $\mathfrak{g}$, i.e. $\mathfrak{h} = \text{Lie}(H)$ for some connected closed (smooth) subgroup of $G$, which must clearly be maximal amongst such subgroups. Thus the possibilities for $H$ (and $\mathfrak{h}$) are given by work of Liebeck, Seitz and Testerman: see [Sei87] and [LS04]. In particular, maximal non-semisimple subalgebras of $\mathfrak{g}$ are parabolic.

This last corollary had been tackled earlier in the announcement [Ten87] where the author classified all maximal non-semisimple subalgebras of simple algebraic Lie algebras of type $A$–$D$ in characteristic $p > 3$. We provide proofs for all the statements in that paper correcting some lemmas. While we do not have access to the proofs intended by Ten, we note our proofs also work in the case $p = 3$. Thus we show

**Theorem B.** Let $\mathfrak{g} = \text{Lie}(G)$ for $G$ a simple, simply-connected classical algebraic group over an algebraically closed field $k$ of characteristic $p > 2$. Let $\mathfrak{m}$ be a maximal non-semisimple subalgebra of $\mathfrak{g}$. Then $\mathfrak{m}$ is maximal; and precisely one of the following holds:

(i) $G$ is of type $B, C$ or $D$ and $\mathfrak{m}$ is parabolic;
(ii) $G = \text{SL}_n$, $p | n$ and $\mathfrak{m}$ is parabolic;
(iii) $G = \text{SL}(V)$, $\dim(V) = n = lp^m$ with $m > 0$ and $(l, p) = 1$. Write $V \cong U \otimes O_r$ where $O_r \cong k[X_1, \ldots, X_r]/(X_1^p, \ldots, X_r^p)$ for $1 \leq r \leq m$ is the truncated polynomial ring. Then one of the following holds:
   (a) $\mathfrak{m}/\mathfrak{j}(\mathfrak{g}) \subseteq \mathfrak{psl}(V)$ is semisimple;
   (b) $\mathfrak{m}$ is parabolic;
   (c) $\mathfrak{m}$ is a semidirect product $1 \otimes W_r + \mathfrak{gl}(U) \otimes O_r$, where $W_r$ is the Witt algebra acting on $O_r$ by derivations, $O_r$ acts on $O_r$ by multiplication;
   (d) $l = 1$, $r = m$ and $\mathfrak{m}$ is a semidirect product $\mathfrak{sp}_{2r} + \mathcal{H}_{2r+1}$ where $\mathcal{H}_{2r+1}$ is the $2r + 1$-dimensional Heisenberg algebra.

The proof of the theorem in case $p$ is very good is done in §3, while the more technical results dealing with the cases in (iii) are consigned to the Appendix.

We mine the examples in the above theorem to show that no ‘obvious’ Borel–Tits type structure theory can hold whenever $p < \rk(G)$. On the other hand, we prove in Theorem 10.1 that when all normalisers of subalgebras are smooth, the conclusion of the Borel–Tits theorem does hold. We can also use Theorem B to weaken the hypotheses needed in Theorem A to reach some of the conclusions discussed in the preceding paragraph. We have the following two results, the first immediate from Theorem B. (Note that if $p$ is very good and $\pi : G_{sc} \to G$ is the simply connected cover of $G$, then $\pi$ induces an isomorphism $\text{Lie}(G_{sc}) \cong \text{Lie}(G)$.)
Corollary C. Suppose $G$ is a simple algebraic group of classical type. If $p$ is a very good prime for $G$ then any maximal non-semisimple subalgebra of $\mathfrak{g}$ is parabolic.

Theorem D.  
(a) Suppose $G$ is a simple algebraic group of classical type. Then the following are equivalent:

(i) All maximal solvable subalgebras of $\mathfrak{g}$ are conjugate.

(ii) All maximal solvable subalgebras of $\mathfrak{g}$ are Borel subalgebras.

(iii) $G$ is of type $A$ and $p > \text{rk}(G) + 1$ or $G$ is of type $B$, $C$ or $D$ and $p > \text{rk}(G)$.

(b) Suppose $G$ is any connected reductive algebraic group and $p > 2h - 2$. Then the statements (i) and (ii) above are true.

In fact the proof of Theorem A, given in §7, §8 and §9 uses our minor improvement of Ten’s result, Theorem B, together with a number of other results, possibly on independent interest. Building on work of Bendel–Nakano–Pillen, Jantzen and Serre, we give vanishing results on the restricted and ordinary cohomology of algebraic semisimple Lie algebras. We summarise our results when $G$ is simple into the following. (The extensions to the case $G$ is semisimple or reductive can be found in §4.)

Theorem E. Suppose $G$ is a simple algebraic group and let $G_r$ be its $r$-th Frobenius kernel and $\mathfrak{g}$ its Lie algebra. Let $V$ be a module for $G$ with dim $V \leq p$. Then:

(a) $V$ is completely reducible for $G_r$ unless dim $V = p$, and either $G$ is of type $A_1$ or $p = 2$ and $G$ is of type $C_n$. In the exceptional cases, $V$ is known explicitly.

(b) Suppose $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then either $V$ is completely reducible for $\mathfrak{g}$ or dim $V = p$, $G$ is of type $A_1$ and $V$ is known explicitly.

(c) Let $p > h$. Then $H^2(\mathfrak{g}, L(\mu)) = 0$, for all $\mu \in C_{\mathbb{Z}}$ unless $G$ is of type $A_1$ and $\mu = (p - 2)$ or $G$ is of type $A_2$ and $\mu = (p - 3, 0)$ or $(0, p - 3)$.

(d) Suppose $V$ and $W$ are semisimple $\mathfrak{g}$-modules with dim $V + \text{dim } W < p + 2$. Then $V \otimes W$ is semisimple and $H^2(\mathfrak{g}, V \otimes W) = 0$.  

All these ingredients go together to prove Theorem A(i) and (iii), by exponentiating $n := n_\mathfrak{g}(\mathfrak{h})$ in a somewhat piecemeal fashion and finding a closed subgroup of $N_G(\mathfrak{h})(k)$ whose Lie algebra contains $n$.

For Theorem A (ii) we use a further tool, which we believe is also of independent interest. We need a definition due to Richardson: Suppose that $(G', G)$ is a pair of reductive algebraic groups such that $G \subseteq G'$ is a closed subgroup. We say that $(G', G)$ is a reductive pair provided there is a subspace $\mathfrak{m} \subseteq \text{Lie}(G')$ such that Lie$(G')$ decomposes as a $G$-module into a direct sum Lie$(G') = \text{Lie}(G) + \mathfrak{m}$.

Adapting a result from [Her13] we show

Proposition F. Let $(G', G)$ be a reductive pair and let $H \leq G$ be a closed subgroup scheme. Then if $N_{G'}(H)$ is smooth, $N_G(H)$ is smooth too.

Clearly one would like to include the exceptional types in Corollary C. If $\mathfrak{g}$ is of exceptional type, then Theorem A tells us at least that whenever $p > 2h - 2$, the conclusions of Corollary C hold

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1After a first draft of this paper had been prepared, G. Röhrle pointed out the paper [Del13], in which it is proved that if $V$ and $W$ are semisimple modules for a group scheme, under dim $V + \text{dim } W < p + 2$, then $V \otimes W$ is semisimple. The semisimplicity part of Theorem E(d) can be deduced from this. In fact, in part of Corollary 4.13 we do prove the semisimplicity statement for arbitrary Lie algebras. Our proof is different to Deligne’s, relying on the cohomological calculations of §4.
and that maximal solvable subalgebras are Borels, see §10.2. But if \( g = E_8 \), this would mean a bound \( p > 58 \). It seems unlikely this would be best possible. However, the conclusions of Theorem B seem to arise for very different reasons in type \( B \), \( C \) and \( D \) so it is not really clear exactly which tight bound to expect.

**Question 1.1.** Let \( g = \text{Lie}(G) \) be of exceptional type. Suppose \( p \) is a good prime: does it follow that maximal non-semisimple subalgebras are parabolic? Suppose \( p > \text{rk}(g) \): is it true that all maximal solvable subalgebras are Borel subalgebras?

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2. **Notation and preliminaries**

Let \( k \) be a field of characteristic \( p \geq 0 \) and let \( G \) be an algebraic group defined over \( k \). Unless otherwise noted, \( k \) will assumed to be algebraically closed. For all aspects to do with the representation theory of a reductive algebraic group \( G \) we keep notation compatible with [Jan03]. In particular, \( R \) is the root system of \( G \), and \( h \) is the associated Coxeter number.

Let \( g \) be a Lie algebra. Sometimes but not all the time, we will have \( g = \text{Lie}(G) \) for \( G \) an algebraic group, in which case we refer to \( g \) as algebraic; in this case, \( g \) will carry the structure of a restricted Lie algebra. More generally, all restricted Lie algebras are of the form \( \text{Lie}(H) \), where \( H \) is an infinitesimal group scheme of height one over \( k \). Under this correspondence, the restricted subalgebras of \( g = \text{Lie}(G) \) correspond to height one subgroup schemes of \( G \). If \( Z(g) = 0 \), then a Lie algebra \( g \) has at most one restricted structure. In particular, if two semisimple restricted Lie algebras are isomorphic as Lie algebras, they are isomorphic as restricted Lie algebras.

If \( g \) is a restricted Lie algebra, a representation \( V \) is called restricted provided it is given by a morphism of restricted Lie algebras \( g \to \text{gl}(V) \). The following fact follows e.g. from the Kac–Weisfeiler conjecture (see [Pre95, Cor. 3.10]): if \( G \) is a simple, simply connected group defined in very good characteristic, and if \( V \) is an irreducible \( g \)-module with \( \dim V < p \), then \( V \) is restricted. In particular, it is well-known that \( V \) is then obtained via differentiating a simple restricted rational representation of \( G \).

When \( g \) is a Lie algebra, \( \text{Rad}(g) \) is the solvable radical of \( g \) and \( N(g) \) is the nilradical of \( g \). If \( g \subseteq \text{gl}(V) \) there is also the radical of \( V \)-nilpotent elements \( \text{Rad}_V(g) \). When \( g \) is restricted, \( \text{Rad}_p(g) \) is the \( p \)-radical of \( g \), defined to be the biggest \( p \)-nilpotent ideal. Further, \( g \) is \( p \)-reductive if the radical \( \text{Rad}_p(g) \) is zero. Recall the following properties from [SF88, §2.1]:

**Lemma 2.1.** (a) \( \text{Rad}_p(g) \) is contained in the nilradical \( N(g) \) and hence in the solvable radical of \( g \). In particular, semisimple Lie algebras are \( p \)-reductive.

(b) \( \text{Rad}_p(g) \) is the maximal \( p \)-nil (that is, consisting of \( p \)-nilpotent elements) ideal of \( g \).

(c) \( g/\text{Rad}_p(g) \) is \( p \)-reductive.

In particular, by part (b), if \( g \subseteq \text{gl}(V) \) is a restricted subalgebra then \( \text{Rad}_p(g) = \text{Rad}_V(g) \). If \( g \subseteq \text{gl}(V) \) is a restricted Lie subalgebra and \( G_1 \) is the height-one subgroup scheme of \( GL(V) \) associated to \( g \), then \( g \) is \( p \)-reductive if and only if \( G_1 \) is reductive in the sense that it has no connected normal nontrivial unipotent subgroup schemes. For the usual notion of reductivity of
smooth algebraic groups only smooth unipotent subgroups are considered. The relation between these two concepts is as follows:

**Proposition 2.2** ([Vas05]). Let $G$ be a connected reductive algebraic group. Then $G$ has no non-trivial connected normal unipotent subgroup schemes, except if both $p = 2$ and $G$ contains a factor $SO_{2n+1}$ for some $n \geq 1$.

Since there are a number of possible definitions, let us be clear on the following: We define a Borel subalgebra (resp. parabolic subalgebra, resp. Levi subalgebra) of $\mathfrak{g}$ to be $\text{Lie}(B)$ (resp. $\text{Lie}(P)$, resp. $\text{Lie}(L)$), where $B$ (resp. $P$, resp. $L$) is a Borel (resp. parabolic, resp. Levi subgroup of a parabolic) subgroup of $G$.

By $P = LQ$ we will denote a parabolic subgroup of $G$ with unipotent radical $Q$ and Levi factor $L$. We will usually write $p = \text{Lie}(P) = 1 + q$. A fact that we will use continually during this paper, without proof, is that if $H$ (resp. $\mathfrak{h}$) is a subgroup (resp. subalgebra) of $P$ (resp. $\mathfrak{p}$), such that the projection to the Levi is in a proper parabolic of the Levi, then there is a strictly smaller parabolic $P_1 < P$ (resp. $\mathfrak{p}_1 < \mathfrak{p}$) such that $H \leq P_1$ (resp. $\mathfrak{h} \leq \mathfrak{p}_1$). See [BT65, Prop. 4.4(c)].

We also use the following fact: If $t \subseteq \mathfrak{gl}_n$ is a torus, then $C_{\mathfrak{gl}_n}(t)$ is a Levi subgroup (this follows e.g. from the construction of a torus $T \subseteq \mathfrak{gl}_n$ in [Die52, Prop. 2] with $C_{\mathfrak{gl}_n}(t) = C_{\mathfrak{gl}_n}(T)$).

Let $V$ be an $\mathfrak{g}$-module and let $\lambda : V \times V \to k$ be a bilinear form on $V$. We say $\mathfrak{g}$ preserves $\lambda$ if $\lambda(x(v), w) = -\lambda(v, x(w))$ for all $x \in \mathfrak{g}$, $v, w \in V$.

Let $k$ be an algebraically closed field. We recall definitions of the algebraic simple Lie algebras of classical type: those with root systems of types $A$–$D$. Then $\mathfrak{o}(V)$ is the set of elements $x \in \mathfrak{gl}(V)$ preserving the form $\lambda(v, w) = v^t w$. $\mathfrak{o}(V)$ is the subset of traceless matrices of $\mathfrak{o}(V)$. On the other hand when $\dim V$ is even, $\mathfrak{sp}(V)$ is the set of elements preserving the form $\lambda(v, w) = v^t J w$ with $J = [[0, -I_n], [I_n, 0]]$. If $\text{char } k \neq 2$ then $\mathfrak{sp}(V)$ and $\mathfrak{o}(V)$ are simple (see below).

We say $\mathfrak{sp}(V)$ is of type $C_n$ with $2n = \dim V$; $\mathfrak{o}(V)$ is of type $B_n$ when $\dim V = 2n + 1$, or type $D_n$ when $\dim V = 2n$. One fact that we shall use often in the sequel is that that for types $B, D$, parabolic subalgebras are the stabilisers of totally singular subspaces. (See for example, [Kan79].) Furthermore recall that if $G$ is simple, then $\mathfrak{g}$ is simple at least whenever $p$ is very good. See [Hog82, Cor. 2.7] for a more precise statement. This means in particular that $\mathfrak{sl}(V)$ is simple unless $p | \dim V$, in which case the quotient $\mathfrak{psl}(V) = \mathfrak{sl}(V)/kI$ is simple; we refer to such algebras as type $A_n$ classical Lie algebras, where $\dim V = n + 1$. In all cases, we refer to $V$ as the natural module for the algebra in question.

We make extensive use of the current state of knowledge of cohomology in this paper, especially in §4. Importantly, recall that the group $\text{Ext}^1_A(V, W)$ (with $A$ either an algebraic group or a Lie algebra) corresponds to the equivalence classes of extensions $E$ of $A$-modules $0 \to W \to E \to V \to 0$, and that $H^2(A, V)$ measures the equivalence classes of central extensions $B$ of $V$ by $A$, equivalence classes of exact sequences $0 \to V \to B \to A \to 0$, where $B$ is either an algebraic group or a Lie algebra. We remind the reader that for restricted Lie algebras, two forms of cohomology are available—the ordinary Lie algebra cohomology, denoted $H^i(A, V)$ or the restricted Lie algebra cohomology (where modules respectively morphisms are assumed to be restricted). Since the latter can always be identified with $H^i(A, V)$ for $A$ the height-one group scheme associated to $\mathfrak{g}$, we shall always use the associated group scheme when we mean restricted cohomology.
Finally, we record the following theorem of Strade which is a central tool in our study of small-dimensional representations:

**Theorem 2.3** ([Str73, Main theorem]). Let \( \mathfrak{g} \) be a semisimple Lie subalgebra of \( \mathfrak{gl}(V) \) over an algebraically closed field \( k \) of characteristic \( p > 2 \) with \( p > \dim V \). Then \( \mathfrak{g} \) is either a direct sum of algebraic Lie algebras or \( p = \dim V + 1 \) and \( \mathfrak{g} \) is the \( p \)-dimensional Witt algebra \( W_1 \).

3. **Non-semisimple subalgebras of classical Lie algebras**

Suppose \( \operatorname{char} k > 2 \) for this section.

This section provides proofs for some of the claims made in [Ten87]. Here we tackle the proof of Theorem B, parts (i) and (ii), hence give a proof of Corollary C.

**Proposition 3.1** (see [SF88, §5.8, Exercise 1]). Let \( \mathfrak{g} \leq \mathfrak{gl}(V) \) be a Lie algebra acting irreducibly on an \( \mathfrak{g} \)-module \( V \) such that \( \mathfrak{g} \) preserves a non-zero bilinear form. Then \( \mathfrak{g} \) is semisimple.

**Proof.** Assume otherwise. Then \( \operatorname{Rad}(\mathfrak{g}) \neq 0 \) and we can find an abelian ideal \( 0 \neq J \triangleleft \mathfrak{g} \). Take \( x \in J \). As \( [x^p, y] = \text{ad}(x)^p y \in J^{(1)} = 0 \), \( x^p \) centralises \( \mathfrak{g} \) and we have that \( v \mapsto x^p v \) is a \( \mathfrak{g} \)-homomorphism \( V \rightarrow V \). Since \( k \) is algebraically closed and \( V \) is irreducible, Schur’s lemma implies that \( x^p v = \alpha(x)v \) for some map \( \alpha : J \rightarrow k \).

Since \( \lambda \neq 0 \) there are \( v, w \) with \( \lambda(v, w) = 1 \). Now \( \alpha(x) = \lambda(x^p v, w) = -\lambda(v, x^p w) = -\alpha(x) \) so \( \alpha(x) = 0 \). Thus \( x^p v = 0 \) for all \( x \in J \). Hence \( J \) acts nilpotently on \( V \) and so Engel’s theorem gives an element \( 0 \neq v \in V \) annihilated by \( J \). Since \( V \) is irreducible, it follows that \( JV = J(gv) \leq \mathfrak{g}Jv = 0 \). Thus \( J = 0 \) and \( \mathfrak{g} \) is semisimple.

Since any subalgebra of a classical simple Lie algebra of type \( B, C \) or \( D \) preserves the associated (non-degenerate) form we get

**Corollary 3.2.** If \( \mathfrak{h} \) is a non-semisimple subalgebra of a classical simple Lie algebra \( \mathfrak{g} \) of type \( B, C \) or \( D \) then \( \mathfrak{h} \) acts reducibly on the natural module \( V \) for \( \mathfrak{g} \).

**Remark 3.3.** If \( \mathfrak{g} = \mathfrak{g}_2 \) (resp. \( \mathfrak{f}_4, \mathfrak{e}_7, \mathfrak{e}_8 \)) then a subalgebra acting irreducibly on the self-dual modules \( V_7 \) (resp. \( V_{26}, V_{25} \) if \( p = 3 \), \( V_{56}, V_{248} = \mathfrak{e}_8 \)) is semisimple. Here \( V_n \) refers to the usual irreducible module of dimension \( n \).

A subalgebra is **maximal rank** if it is proper and contains a Cartan subalgebra (CSA) of \( \mathfrak{g} \). (Note that CSAs of classical Lie algebras are tori.) Call a subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) an **\( R \)-subalgebra** if \( \mathfrak{h} \) is contained in a maximal rank subalgebra of \( \mathfrak{g} \).

For the following, notice that a subalgebra \( \mathfrak{h} \) of \( \mathfrak{psl}(V) \) is an \( R \)-subalgebra of \( \mathfrak{psl}(V) \) if and only if its preimage \( \pi^{-1}\mathfrak{h} \) under \( \pi : \mathfrak{sl}(V) \rightarrow \mathfrak{psl}(V) \) is an \( R \)-subalgebra. We say \( \mathfrak{h} \) acts reducibly on \( V \) if \( \pi^{-1}\mathfrak{h} \) does.

**Proposition 3.4.** Let \( \mathfrak{g} \) be a classical simple Lie algebra of classical type and let \( \mathfrak{h} \leq \mathfrak{g} \) act reducibly on the natural module \( V \) for \( \mathfrak{g} \). Then \( \mathfrak{h} \) is an \( R \)-subalgebra unless \( \mathfrak{g} = \mathfrak{so}(V) \) with \( \dim V = 2n \) with \( \mathfrak{g} \leq \mathfrak{so}(W) \times \mathfrak{so}(W') \) stabilising a decomposition of \( V \) into two odd-dimensional, non-degenerate subspaces \( W \) and \( W' \) of \( V \).
Proof. Let \( V \) be the natural module for \( \mathfrak{g} \) and let \( W \leq V \) be a minimal \( \mathfrak{h} \)-submodule, so that \( \mathfrak{h} \leq \text{Stab}_\mathfrak{g}(W) \). If \( \mathfrak{g} \) is of type \( A \) then \( \text{Stab}_\mathfrak{g}(W) \) is \( \text{Lie}(P) \) for a (maximal) parabolic \( P \) of \( \text{SL}(V) \). Hence \( \mathfrak{h} \) is an \( R \)-subalgebra of \( \mathfrak{g} \).

If \( \mathfrak{g} \) is of type \( B, C \) or \( D \), then consider \( U = W \cap W^\perp \); this is the subspace of \( W \) whose elements \( v \) satisfy \( \lambda(v, w) = 0 \) for every \( w \in W \). Since \( M \) preserves \( \lambda \), this is a submodule of \( W \), hence we have either \( U = 0 \) or \( U = W \) by minimality of \( W \). If the latter, \( W \) is totally singular. Thus \( \text{Stab}_\mathfrak{g} W \) is a parabolic subgroup of the associated algebraic group.

On the other hand, \( U = 0 \) implies that \( W \) is non-degenerate. Then \( V = W \oplus W^\perp \) is a direct sum of \( \mathfrak{g} \)-modules and we see that \( \text{Stab}_\mathfrak{g} W \) is isomorphic to

(i) \( \mathfrak{sp}_{2r} \times \mathfrak{sp}_{2s} \) in case \( L \) is of type \( C \), \( \dim W = 2s \) and \( 2r + 2s = \dim V \)
(ii) \( \mathfrak{so}_r \times \mathfrak{so}_s \) in case \( L \) is of type \( B \) or \( D \), \( \dim W = s \) and \( r + s = \dim V \).

Note that by [Bou05, VII, §2, No. 1, Prop. 2] the dimensions of the CSA of a direct product is the sum of the dimensions of the CSAs of the factors. In case (i), the subalgebra described has the \((r + s)\)-dimensional CSA arising from the two factors. In case (ii), if \( \dim V = 2n + 1 \) is odd then one of \( r \) and \( s \) is odd. If \( r \) is odd then \( \mathfrak{so}_r \) has a CSA of dimension \((r − 1)/2\), and \( \mathfrak{so}_s \) has a CSA of dimension \( s/2 \), so that the two together give a CSA of dimension \( s/2 + (r − 1)/2 = n \).

(Similarly if \( s \) is odd.) Otherwise \( \dim V = 2n \) is even. If \( \dim W \) is even then \( \text{Stab}_\mathfrak{g} W \) contains a CSA of dimension \( r/2 + s/2 = n \). If \( \dim W \) is odd then we are in the exceptional case described in the proposition.

Remark 3.5. In the exceptional case, note that \( \mathfrak{so}_{2r+1} \times \mathfrak{so}_{2s+1} \) contains a CSA of dimension \( r + s \), whereas \( \mathfrak{so}_{2n+2} \) contains a CSA of dimension \( n + 1 = r + s + 1 \).

Corollary 3.6. Let \( \mathfrak{g} \) be of type \( B, C \) or \( D \). If \( \mathfrak{h} \) is a maximal non-semisimple subalgebra of \( \mathfrak{g} \), then \( \mathfrak{h} \) is \( \text{Lie}(P) \) for \( P \) a maximal parabolic of \( G \). In particular, if \( \mathfrak{h} \) is any non-semisimple subalgebra of \( \mathfrak{g} \), it is an \( R \)-subalgebra.

Proof. Assume otherwise. Then \( \mathfrak{h} \) fixes no singular subspace on \( V \). Suppose \( \mathfrak{h} \) preserves a decomposition \( V = V_1 \perp V_2 \perp \cdots \perp V_n \) on \( V \) with \( n \) as large as possible, with the \( V_i \) all non-degenerate. Then \( \mathfrak{h} \leq \mathfrak{g}_1 = \mathfrak{so}(V_1) \times \cdots \times \mathfrak{so}(V_n) \) or \( \mathfrak{h} \leq \mathfrak{g}_1 = \mathfrak{sp}(V_1) \times \cdots \times \mathfrak{sp}(V_n) \). Since \( \mathfrak{h} \) is non-semisimple, the projection \( \mathfrak{h}_1 \) of \( \mathfrak{h} \) in \( \mathfrak{so}(V) \) or \( \mathfrak{sp}(V) \), say, is non-semisimple. Then Proposition 3.1 shows that \( \mathfrak{h} \) acts reducibly on \( V_1 \). Since \( \mathfrak{h} \) stabilises no singular subspace, the proof of Proposition 3.4 shows that \( \mathfrak{h} \) stabilises a decomposition of \( V_1 \) into two non-degenerate subspaces, a contradiction of the maximality of \( n \).

Let \( \mathfrak{h} \) be a restricted Lie algebra, \( I \leq \mathfrak{h} \) an abelian ideal and \( V \) an \( \mathfrak{h} \)-module. Let \( \lambda \in I^\ast \). Recall from [S-F, §5.7] that \( \mathfrak{h}^\lambda = \{ x \in \mathfrak{h} | \lambda([x, y]) = 0 \text{ for all } y \in I \} \) and \( V^\lambda = \{ v \in V | x.v = \lambda(x)v \text{ for all } x \in I \} \).

Proposition 3.7. Let \( \mathfrak{h} \) be a non-semisimple subalgebra of \( \mathfrak{sl}(V) \) with \( V \) irreducible for \( \mathfrak{h} \). Then \( p | \dim V \).

Proof. Let \( \mathfrak{h} \) be as described and let \( I \) be a nonzero abelian ideal of \( \mathfrak{h} \). If \( \mathfrak{h}_p \) denotes the closure of \( \mathfrak{h} \) under the \( p \)-mapping, then by [SF88, 2.1.3(2),(4)], \( I_p \) is an abelian \( p \)-ideal of \( \mathfrak{h}_p \). Thus \( \text{Rad} \mathfrak{h}_p \neq 0 \) and \( \mathfrak{h}_p \) is non-semisimple. Hence we may assume from the outset that \( \mathfrak{h} = \mathfrak{h}_p \) is restricted with nonzero abelian ideal \( I \).
Since $\mathfrak{h}$ acts irreducibly on $V$, by [SF88, Corollary 5.7.6(2)] there exist $S \in \mathfrak{h}^*$, $\lambda \in I^*$ such that
\[ V \cong \text{Ind}_{\mathfrak{h}^\lambda}^{\mathfrak{h}}(V^\lambda, S). \]

If $\lambda$ is identically 0 on $I$ then $V^\lambda$ is an $\mathfrak{h}$-submodule. We cannot have $V^\lambda = 0$ (or else $V = 0$) so $V^\lambda = V$ and $I$ acts trivially on $V$, a contradiction since $I \leq \mathfrak{sl}(V)$.

Hence $\lambda(x) \neq 0$ for some $x \in I$. Suppose $V^\lambda = V$. Then as $x \in \mathfrak{sl}(V)$, we have $\text{tr}_V(x) = \dim V \cdot \lambda(x) = 0$ and thus $p|\dim V$ and we are done. If $\dim V^\lambda < \dim V$, then by [SF88, Prop. 5.6.2] we have $\dim V = p^{\dim L/\lambda} \cdot \dim V^\lambda$. Thus again $p|\dim V$, proving the theorem. \qed

**Corollary 3.8.** If $p \nmid \dim V$ then any non-semisimple subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(V)$ acts reducibly on $V$. Hence it is contained in $\text{Lie}(P)$ for $P$ a maximal parabolic of $\text{SL}(V)$. In particular $\mathfrak{h}$ is an $R$-subalgebra.

Putting together Corollaries 3.6 and 3.8, this proves Theorem B in the case that $p$ is very good for $G$, and proves Corollary C.

As a first application, the following lemma uses Corollary C to reduce from $p$-reductive to strongly $p$-reductive Lie algebras. Recall that a restricted Lie algebra is *strongly $p$-reductive* if it is the direct sum of a central $p$-torus and a semisimple ideal.

**Lemma 3.9.** Let $\mathfrak{h} \subseteq \mathfrak{gl}_n$ be a subalgebra and let $p > n$. If $\mathfrak{h}$ is $p$-reductive, it is strongly $p$-reductive.

**Proof.** Take $\mathfrak{p} = \mathfrak{l} + \mathfrak{q}$ a minimal parabolic subalgebra with $\mathfrak{h} \leq \mathfrak{p}$. Set $\mathfrak{h}_I$ to be the image of $\mathfrak{h}$ under the projection $\pi : \mathfrak{p} \to \mathfrak{l}$. Since $p > n$, we have $I \cong \mathfrak{gl}(W_1) \times \cdots \times \mathfrak{gl}(W_s) \cong \mathfrak{sl}(W_1) \times \cdots \mathfrak{sl}(W_s) \times \mathfrak{z}$, where $\mathfrak{z}$ is a torus. Let $\mathfrak{s}_I$ be the projection of $\mathfrak{h}_I$ to $\mathfrak{sl}(W_i)$, and let $\mathfrak{z}'$ be the projection of $\mathfrak{h}_I$ to $\mathfrak{z}$. If the projection of $\text{Rad}(\mathfrak{h}_I)$ to $\mathfrak{sl}(W_i)$ is non-trivial, then $\mathfrak{s}_I$ is not semisimple. By Corollary C, $W_i$ is not irreducible for $\mathfrak{s}_I$. Thus $\mathfrak{p}$ is not minimal subject to containing $\mathfrak{h}$, a contradiction, proving that all the $\mathfrak{s}_I$ are semisimple. Moreover, $\mathfrak{z}' = Z(\mathfrak{h}_I)$, as the projection of $\mathfrak{z}$ to each $\mathfrak{sl}(W_i)$ must vanish. This forces $\mathfrak{h}_I \subseteq \mathfrak{s}_I \times \cdots \times \mathfrak{s}_s \times Z(\mathfrak{h}_I)$ to be strongly $p$-reductive. As $\mathfrak{h}$ is $p$-reductive, we have that $\pi$ is injective on $\mathfrak{h}_I$, and hence $\mathfrak{h} \cong \mathfrak{h}_I$ is strongly $p$-reductive. \qed

4. Complete reducibility and low-degree cohomology for classical Lie algebras: Proof of Theorem E

Let $G$ be a reductive algebraic group with root system $R$ and let $G_r \triangleleft G$ be the $r$th Frobenius kernel. It is well-known that the representation theory of $G_1$ and $\mathfrak{g}$ are very closely related. In this section we recall results on the cohomology of small $G_r$-modules and use a number of results of Bendel, Nakano and Pillen to prove that small $G_r$-modules are completely reducible with essentially one class of exceptions. We do this by examining $\text{Ext}^1_{G_1}(L(\lambda), L(\mu))$ for two simple modules $L(\lambda)$ and $L(\mu)$ of bounded dimension or weight. While we are at it, we also get information about $H^2(G_1, L(\lambda))$. In a further subsection, we then go on to use this to prove the analogous statements for $\mathfrak{g}$-modules. One crucial difference we notice is with central extensions: $H^2(\mathfrak{g}, k)$ tends to be zero, whereas $H^2(G_1, k)$ is almost always non; c.f. Corollary 4.2 and Theorem 4.9.

All the notation in this section is as in [Jan03, List of Notations, p. 569]: In particular, for a fixed maximal torus $T \leq G$, we denote by $R$ the corresponding root system, by $R^+$ a choice of positive roots with corresponding simple roots $S \subseteq R^+$, by $X(T)_+ \subseteq X(T)$ the dominant weights inside the character lattice, by $L(\lambda)$ the simple $G$-module of highest weight $\lambda \in X(T)_+$, by $H^0(\lambda)$ the module
induced from \( \lambda \) with socle \( L(\lambda) \), by \( C_Z \) (resp. \( \tilde{C}_Z \)) the dominant weights inside the lowest alcove (respectively, in the closure of the lowest alcove). If \( G \) is simply connected, we write \( \omega_i \in X(T)_+ \) for the fundamental dominant weight corresponding to \( \alpha_i \in S = \{\alpha_1, \ldots, \alpha_l\} \).

Let us recall some results from [McN02] which show the interplay between the conditions that, relative to \( p \), (i) modules are of small dimension; (ii) their high weights are small; and (iii) the Coxeter number is small.

**Proposition 4.1** ([McN02, Prop. 5.1]). Let \( G \) be simple, simply connected, let \( L \) be a simple non-trivial restricted \( G \)-module with highest weight \( \lambda \in X(T)_+ \) and suppose that \( \dim L \leq p \). Then

(i) We have \( \lambda \in \tilde{C}_Z \).
(ii) We have \( \lambda \in C_Z \) if and only if \( \dim L < p \).
(iii) We have \( p \geq h \). If moreover \( \dim L < p \) then \( p > h \).
(iv) If \( R \) is not of type \( A \) and \( \dim L = p \) then \( p > h \). If \( p = h \) and \( \dim L = p \) then \( R = A_{p-1} \) and \( \lambda = \omega_i \) with \( i \in \{1, p-1\} \).

### 4.1 Cohomology and complete reducibility for small \( G_1 \)-modules

We need values of \( H^i(G_1, H^0(\mu)) \) for \( \mu \in \tilde{C}_Z \) and \( i = 1 \) or 2. Thus \( H^0(\mu) = L(\mu) \).

**Proposition 4.2.** Let \( G \) be simple and simply connected and suppose \( L = L(\mu) \) with \( \mu \in \tilde{C}_Z \) and \( p \geq 3 \). Then:

(i) we have \( H^1(G_1, L) = 0 \) unless \( G \) is of type \( A_1 \), \( L = L(p-2) \) and in that case \( H^1(G_1, L)^{-1} \simeq L(1) \);

(ii) suppose \( p > h \). Then we have \( H^2(G_1, L) = 0 \) unless: \( L = k \) and \( H^2(G_1, k)^{-1} \simeq g^* \); or \( G = SL_3 \), with \( H^2(G_1, L(p-3,0))^{-1} \simeq L(0,1) \) and \( H^2(G_1, L(0,p-3))^{-1} \simeq L(1,0) \).

**Proof.** Part (i) is immediate from [BNP02, Corollary 5.4 B(i)]. The \( A_1 \) result is well known. Part (ii) requires some argument. If \( H^2(G_1, H^0(\mu)) \neq 0 \) then since \( p > h \) we may assume \( \mu \in C_Z \). Now, the values of \( H^2(G_1, H^0(\mu))^{-1} \) are known from [BNP07, Theorem 6.2]. It suffices to find those that are non-zero for which \( \mu \in \tilde{C}_Z \setminus \{0\} \). All of these have the form \( \mu = w.0 + p\lambda \) for \( l(w) = 2 \) and \( \lambda \in X(T)_+ \). Now, if \( l(w) = 2 \), we have \( -w.0 = \alpha + \beta \) for two distinct roots \( \alpha, \beta \in R^+ \) (cf. [BNP07, p. 166]). To have \( w.0 + p\lambda \) in the lowest alcove, one needs \( \langle w.0 + p\lambda + \rho, \alpha_0^\vee \rangle < p \). Now \( \langle p\lambda, \alpha_0^\vee \rangle \geq p \) so \( \langle w.0 + \rho, \alpha_0^\vee \rangle < p \). Thus \( m := \langle \alpha + \beta, \alpha_0^\vee \rangle > h - 1 \). Now one simply considers the various cases. If \( G \) is simply-laced, then the biggest value of \( \langle \alpha, \alpha_0^\vee \rangle \) is 2, when \( \alpha = \alpha_0 \) and 1 otherwise, thus \( m > h - 1 \) implies \( h \leq 3 \). Thus we get \( G = SL_3 \), and this case is calculated in [Ste12, Prop. 2.5]. If \( G = G_2 \) we have \( m \) at most 5, giving \( h \) at most 5, a contradiction. If \( G \) is type \( B, C \) or \( F \), then \( m \) is at most 4, so \( G = Sp_4, p \geq 5 \) and this is calculated in [Ibr12, Prop. 4.1]. One checks that all \( \mu \) such that \( H^2(G_1, L(\mu)) \neq 0 \) have \( \mu \not\in C_Z \). \( \square \)

**Remark 4.3.** All the values of \( H^2(G_1, H^0(\lambda))^{-1} \) are known for all \( \lambda \) by [BNP07, Theorem 6.2] \((p \geq 3)\) and [Wri11] \((p = 2)\). For example, \( H^2(G_1, k)^{-1} \simeq g^* \) also when \( G \) is of type \( A_1 \) and \( p = 2 \). Even for \( \lambda = 0 \) there are quite a few exceptional cases when \( p = 2 \): see [Wri11, C.1.4]. There are also two exceptional cases for \( p = 3 \), for \( A_2 \) and \( G_2 \), see [BNP07, Theorem 6.2].

One can go further in the case of 1-cohomology to include extensions between simple modules:

**Lemma 4.4** ([BNP02, Corollary 5.4 B(i)]). Let \( G \) be a simple, simply connected algebraic group not of type \( A_1 \). If \( p > 2 \) then \( Ext^1_G(L(\lambda), L(\mu)) = 0 \) for all \( \lambda, \mu \in \tilde{C}_Z \).
We will use the above result to show that small $G_r$-modules are completely reducible, but we must first slightly soup it up before we use it.

**Lemma 4.5.** Let $G$ be a simple, simply connected algebraic group not of type $A_1$ and $p > 2$.

(i) We have $\text{Ext}^1_{G_r}(L(\lambda)^[s],L(\mu)^[t]) = 0$ for all $\lambda, \mu \in \widehat{\mathbb{Z}}$ and $s, t \geq 0$.

(ii) For $\lambda, \mu \in X_+(T)$, let $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^{r-1}\lambda_{r-1}$ and $\mu = \mu_0 + p\mu_1 + \cdots + p^{r-1}\mu_{r-1}$ be their $p$-adic expansions. Suppose we have $\lambda_i, \mu_i \in \mathbb{C}_\mathbb{Z}$ for each $i$. Then $\text{Ext}^1_{G_r}(L(\lambda),L(\mu)) = 0$.

**Proof.** (i) Clearly we may assume $s, t < r$. When $r = 1$ the result follows from Lemma 4.4. So assume $r > 1$. Without loss of generality (dualising if necessary) we may assume $s \leq t$. Suppose $s > 0$ and consider the following subsequence of the five-term exact sequence of the LHS spectral sequence applied to $G_s \ltimes G_r$ (see [Jan03, I.6.10]):

$$0 \to \text{Ext}^1_{G_{r-s}}(L(\lambda),L(\mu)^{[t-s]}) \to \text{Ext}^1_{G_r}(L(\lambda)^[s],L(\mu)^[t]) \to \text{Hom}_{G_{r-s}}(L(\lambda),\text{Ext}^1_{G_s}(k,k)^{[-s]} \otimes L(\mu)^{[t-s]}) \to 0.$$ 

Since $\text{Ext}^1_{G_s}(k,k) = 0$, we have $\text{Ext}^1_{G_{r-s}}(L(\lambda),L(\mu)^{[t-s]}) \cong \text{Ext}^1_{G_r}(L(\lambda),L(\mu))$, and the left-hand side vanishes by induction, so we may assume $s = 0$. There is another exact sequence

$$0 \to \text{Ext}^1_{G_{r-1}}(k,\text{Hom}_{G_1}(L(\lambda),L(\mu)^{[t]}[−1])) \to \text{Ext}^1_{G_r}(L(\lambda),L(\mu)^[t]) \to \text{Hom}_{G_{r-1}}(k,\text{Ext}^1_{G_1}(L(\lambda),L(\mu)^{[t]}[−1])) = 0,$$

where the last term vanishes by induction. If $t = 0$ then as $\lambda \neq \mu$, the first term of the sequence vanishes and we are done. So we may assume $t > 0$. Now we can rewrite the first term as $\text{Ext}^1_{G_{r-1}}(k,\text{Hom}_{G_1}(L(\lambda),k)^{[−1]} \otimes L(\mu)^{[t−1]})$. If this expression is non-trivial we have $\lambda = 0$ and $\text{Ext}^1_{G_{r-1}}(k,L(\mu)^{[t-1]})$ vanishes by induction, which completes the proof.

(ii) Suppose $i$ is the first time either $\lambda_{i-1}$ or $\mu_{i-1}$ is non-zero. Without loss of generality, $\lambda_{i-1} \neq 0$. Write $\lambda = \lambda^i + p^i\lambda'$ and take a similar expression for $\mu$. Then there is an exact sequence

$$0 \to \text{Ext}^1_{G_{r-i}}(L(\lambda^i),\text{Hom}_{G_1}(L(\lambda^i),L(\mu^i)^{[−i]} \otimes L(\mu^i)) \to \text{Ext}^1_{G_r}(L(\lambda),L(\mu)) \to \text{Hom}_{G_{r-i}}(L(\lambda^i),\text{Ext}^1_{G_1}(L(\lambda^i),L(\mu^i)^{[−i]} \otimes L(\mu^i)).$$

We have $L(\lambda^i) = L(\lambda_{i-1})^{[i]}$ and $L(\mu^i) = L(\mu_{i-1})^{[i]}$. Hence the right-hand term vanishes by part (i). The left-hand term is non-zero only if $\lambda^i = \mu^i$ and then we get $\text{Ext}^1_{G_{r-i}}(L(\lambda^i),L(\mu^i)) \cong \text{Ext}^1_{G_r}(L(\lambda),L(\mu))$. Thus the result follows by induction on $r$. \hfill \Box

We put these results together to arrive at an analogue of Jantzen’s well-known result [Jan97] that $G$-modules for which dim $V \leq p$ are completely reducible.

**Proposition 4.6.** Let $G$ be a simple, simply connected algebraic group and let dim $V \leq p$ be a $G_r$-module. Then exactly one of the following holds:

(i) $V$ is a semisimple $G_r$-module;

(ii) $G$ is of type $A_1$, $p > 2$, $r = 1$, dim $V = p$ and $V$ is uniserial, with composition factors $L(p−2−r)$ and $L(r)$;

(iii) $G$ is of type $C_n$ with $n \geq 1$, $p = 2$ and $V$ is uniserial with two trivial composition factors.
Proof. Assume $V$ has only trivial composition factors. We have $\Ext^1_{G_r}(k,k) \neq 0$ if and only if $p = 2$ and $G$ is of type $C_n$, in which case $\Ext^1_{G_r}(k,k)^{ [-r] } \cong L(\omega_1)$; [Jan03, II.12.2]. This is case (iii).

Otherwise, $p > 2$ and $\Ext^1_{G_r}(L(\lambda),L(\lambda)) = 0$ for all $\lambda \in X_r(T)$.

Assume $G$ is not of type $A_1$. By assumption, $V$ has a non-trivial composition factor with $\dim V \leq p$. Then $p > 2$ and the hypotheses of Lemma 4.4 hold. Since $\dim V \leq p$, by Proposition 4.1 any $G_r$-composition factor $L(\lambda)$ of $V$ has a $p$-adic expansion $\lambda = \lambda_0 + \cdots + p^{r-1}\lambda_r$ with each $\lambda_i \in \hat{C}_Z$. If there were a non-split extension $0 \to L(\lambda) \to V \to V/L(\lambda) \to 0$ then there would be a non-split extension of $L(\lambda)$ by $L(\mu)$ for $L(\mu)$ a composition factor of $V$, also of the same form. But by Lemma 4.5(ii) we have $\Ext^1_{G_r}(L(\lambda),L(\mu)) = 0$, hence this is impossible and $L(\lambda)$ splits off as a direct summand. Induction on the direct complement completes the proof in this case.

If $G$ is of type $A_1$ then the $G_r$-extensions of simple modules are well known. If $r > 1$ with $\lambda,\mu \in X_r(T)$ then $\dim \Ext^1_{G_r}(L(\lambda),L(\mu)) = \dim \Ext^1_{G}(L(\lambda),L(\mu))$ and this must vanish whenever $\dim L(\lambda) + \dim L(\mu) \leq p$. If $r = 1$, then the only pairs of $G_1$-linked weights are $r$ and $p-2-r$ with $\Ext^1_{G_1}(L(r),L(p-2-r)) \cong L(1)^{[1]}$ as $G$-modules. Here we have $\dim L(r) + \dim L(p-r-2) = p$ giving case (ii). □

The following two corollaries are immediate, in the first case, the passage from $G$ being simple to being reductive is trivial (consider the cover of $G$ by the product of the radical and the simply connected cover of the derived group).

Corollary 4.7. Let $G$ be a reductive algebraic group and let $V$ be a $G_r$-module with $p > \dim V$. Then $V$ is semisimple.

Corollary 4.8. Let $G$ be reductive and $G_r \leq \GL(V)$ with $\dim V \leq p$. Then either $G_r$ is completely reducible on $V$ or $\dim V = p$, $G$ is of type $A_1$, $r = 1$ and $G_r$ is maximal reductive in a maximal parabolic of $\GL(V)$ acting on $V$ as described in case (ii) of Proposition 4.6.

Moreover, if $\mathfrak{g}$ is a $p$-reductive subalgebra of $\GL(V)$ with $\dim V < p$ then $\mathfrak{g}$ acts semisimply on $V$.

Proof. If $G$ is not simple, it can be written as $HK$ with $H$ and $K$ non-trivial mutually centralising reductive subgroups with tori $S$ and $T$ say. The Frobenius kernels $H_1, K_1 \leq G_1 \leq G_r$ are also mutually centralising, so that $H_1$ is in the centraliser of $T_1$. Now the centraliser of $T_1$ is a proper Levi subgroup of $\GL(V)$, hence restriction of $V$ to $H_r$ has at least one trivial direct factor, with direct complement $W$ say, $\dim W < p$. Thus by Corollary 4.7, $W$ is completely reducible for $H_r$ and by symmetry, for $K_r$. Thus $W$ is completely reducible for $K_rH_r = G_r$.

Otherwise, $G$ is simple and Proposition 4.6 gives the result (note that case (iii) does not occur due to dimension restrictions).

For the last part, Lemma 3.9 implies that $\mathfrak{g}$ is the direct sum of a semisimple ideal and a torus, and we may hence assume that $\mathfrak{g}$ is a semisimple restricted subalgebra of $\mathfrak{gl}(V)$. If $\mathfrak{g}$ is not irreducible on $V$, then by Theorem 2.3 there exists a semisimple group $\tilde{G}$ with $\tilde{\mathfrak{g}} = \Lie(\tilde{G})$. Now the result follows from the case $G_1$ above. □

4.2. Cohomology and complete reducibility for small $\mathfrak{g}$-modules. We now transfer our results to the ordinary Lie algebra cohomology for $\mathfrak{g}$.
Recall the exact sequence [Jan03, I.9.19(1)]:
\[
0 \to H^1(G_1, L) \to H^1(g, L) \to \text{Hom}^s(g, L^g)
\]
(1)
\[
\to H^2(G_1, L) \to H^2(g, L) \to \text{Hom}^s(g, H^1(g, L))
\]

The following theorem is the major result of this section.

**Theorem 4.9.** Let \( g = \text{Lie}(G) \) be semisimple. Then:

(a) If \( p > h \) with \( \mu \in C_\mathbb{Z} \) then either \( H^2(g, L(\mu)) = 0 \), or one of the following holds: (i) \( g \) contains a factor of type \( \text{sl}_3 \) and \( L(\mu) \) contains a tensor factor of \( L(p-3,0) \) or \( L(0,p-3) \) for this \( \text{sl}_3 \); (ii) \( g \) contains a factor of type \( \text{sl}_2 \) and \( L(\mu) \) has a tensor factor \( L(p-2) \) for this \( \text{sl}_2 \).

(b) If \( p > 2 \) is very good for \( G \) then \( H^2(g, k) = 0 \).

(c) If \( p > 2 \) is very good for \( G \) and \( \lambda, \mu \in C_\mathbb{Z} \) we have \( \text{Ext}^1_g(L(\lambda), L(\mu)) = 0 \), or \( G \) contains a factor of type \( A_1 \), \( L(\lambda) \) and \( L(\mu) \) are simple modules for that factor, \( \lambda = r < p - 1 \), \( \mu = p - 2 - r \) and we have \( \text{Ext}^1_g(L(\lambda), L(\mu))[-1] \cong L(1) \).

**Proof.** We may assume that \( G \) is simply connected, since the condition on \( p \) implies that \( g = g_1 \times g_2 \cdots \times g_s \). Now one can reduce to the case that \( G \) is simple using a Künneth formula. To begin with, any simple module \( L(\lambda) \) for \( g = g_1 \times g_2 \times \cdots \times g_s \) is a tensor product of simple modules \( L(\lambda_1) \otimes \cdots \otimes L(\lambda_s) \) for the factors. Then by the Künneth formula \( \dim \text{Ext}^1_g(L(\lambda), L(\mu)) \neq 0 \) implies that \( \lambda_i = \mu_i \) for all \( i \neq j \), some \( 1 \leq j \leq s \) and \( \text{Ext}^1_g(L(\lambda), L(\mu)) \cong \text{Ext}^1_{g_j}(L(\lambda_j), L(\mu)) \). This means we may assume \( G \) to be simple in (c). For \( H^2(g, L(\lambda)) \) to be non-zero one must have all \( \lambda_i = 0 \) for all \( i \neq j, k \) some \( 1 \leq j < k \leq s \) and then

\[
H^2(g, L(\lambda)) = H^2(g_j, L(\lambda_j)) \otimes H^0(g_k, L(\lambda_k)) \oplus H^1(g_j, L(\lambda_j)) \otimes H^1(g_k, L(\lambda_k))
\]

\[
\oplus H^0(g_j, L(\lambda_j)) \otimes H^2(g_k, L(\lambda_k)).
\]

Now first suppose that both \( \lambda_j \) and \( \lambda_k \) are non-trivial. Then only the second direct summand in \( H^2(g, L(\lambda)) \) survives, and by (1) it coincides with the tensor product of the 1-cohomology groups of the corresponding Frobenius kernels. By Proposition 4.2, non-vanishing would force \( \lambda_j = p - 2 = \lambda_k \) and \( g_j = g_k = \text{sl}_2 \) giving one exceptional case.

Next we treat the case \( \lambda_k = 0 \) and \( \lambda_j \) non-trivial. Again by (1) and Proposition 4.2, we obtain \( H^2(g, L(\lambda)) = H^2(g_j, L(\lambda_j)) \), and we are in the case where \( G \) is simple and \( L(\lambda) \) non-trivial. In case \( g = \text{sl}_2 \), the result follows from [Dzh92]. So suppose \( g \neq \text{sl}_2 \). Setting \( L = L(\mu) \) in (1) we see that if \( \mu \neq 0 \) we have \( H^1(g, L) \cong H^1(G_1, L) \) and the right-hand side is zero by Lemma 4.4. Thus we also have \( H^2(g, L) \cong H^2(G_1, L) \) and the latter is zero by Proposition 4.2 unless \( g = \text{sl}_3 \) and the exception is as in the statement of the Theorem, since we have excluded the \( A_1 \) case.

Finally, the case \( \lambda_j = \lambda_k = 0 \) reduces by the above to the case \( G \) simple, \( L = k \) and the claim that \( H^2(g, k) = 0 \). Here we have \( H^1(g, k) \cong (g/[g,g])^* \) and this is zero since \( p \) is very good and \( g \) is semisimple. We also have \( H^2(G_1, k)[-1] \cong g^* \). The injective map \( \text{Hom}^s(g, L^g) \to H^2(G_1, L) \) is hence an isomorphism, which forces \( H^2(g, k) = 0 \) in the sequence (1). This also proves (b).

Now we prove the statement (c) under the assumption that \( G \) is simple. We have an isomorphism \( \text{Ext}^1_g(L(\lambda), L(\mu)) \cong H^1(g, L(\mu) \otimes L(\lambda)^*) \). Let \( M = L(\mu) \otimes L(\lambda)^* \). If \( \lambda \neq \mu \), then applying the exact sequence (1) to \( M \) yields \( H^1(g, M) \cong H^1(G_1, M) \) and the latter is zero by Lemma 4.4 if \( G \) is not of type \( A_1 \) and well-known if \( G \) is of type \( A_1 \). Hence we may assume \( \lambda = \mu \). The assignation of \( L \)
to the sequence (1) is functorial, thus, associated to the $G$-map $k \to M \cong \text{Hom}_k(L, L)$, there is a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & H^1(\mathfrak{g}, k) = 0 & \longrightarrow & \text{Hom}^\alpha(\mathfrak{g}, k^\theta) \cong (\mathfrak{g}^*)[1] & \cong \longrightarrow & H^2(G_1, k) \\
& & \downarrow & & \cong \downarrow & \theta & , \\
0 & \longrightarrow & H^1(\mathfrak{g}, M) & \longrightarrow & \text{Hom}^\alpha(\mathfrak{g}, M^\theta) \cong (\mathfrak{g}^*)[1] & \zeta \longrightarrow & H^2(G_1, M)
\end{array}
$$

where the natural isomorphism $k^\theta \to M^\theta$ induces the middle isomorphism and the top right isomorphism has been discussed already. We want to show that $\zeta$ is injective, since then it would follow that $H^1(\mathfrak{g}, M) = 0$. To do this it suffices to show that $\theta$ is an injection $(\mathfrak{g}^*)[1] \to H^2(G_1, M)$ and for this, it suffices to show that the simple $G$-module $(\mathfrak{g}^*)[1]$ does not appear as a submodule of $H^1(G_1, M/k)$. Now since $\lambda \in \tilde{C}_\mathbb{Z}$ we have $L(\lambda) \cong H^0(\lambda)$ and so by [Jan03, II.4.21], $M$ has a good filtration. The socle of any module $H^0(\mu)$ with $\mu \in X^+$ is simple. Thus the submodule $k \leq M$ constitutes a section of this good filtration, with $M/k$ also having a good filtration.

The $G$-modules $H^1(G_1, H^0(\mu))$ have been well-studied by Jantzen [Jan91] and others. In order to have $(\mathfrak{g}^*)[1]$ a composition factor of $H^1(G_1, H^0(\mu))$, we would need $\mathfrak{g} \cong \mathfrak{g}^* \cong H^0(\omega_\alpha)$ where $\mu = p\omega_\alpha - \alpha$ and $\alpha$ is a simple root with $\omega$ the corresponding fundamental dominant weight; [BNP04, Theorem 3.1(A,B)]. Now for type $A_n$, with $p | n+1$, we have $\mathfrak{g} = \mathfrak{L}(2\omega_1)$ if $n = 1$ and $\mathfrak{g} = \mathfrak{L}(\omega_1 + \omega_n)$ else; and for type $B_2$, we have $\mathfrak{g} = \mathfrak{L}(2\omega_2)$, ruling these cases out. For the remaining types, we have

| Type $\mathfrak{g} \cong \mathfrak{L}(\omega_\alpha)$ for $\omega_\alpha = \langle p\omega_\alpha - \alpha, \alpha_0^\vee \rangle$ | $B_n, C_n$ | $D_n$ | $E_6$ | $E_7$ | $E_8$ | $F_4$ | $G_2$ |
|---|---|---|---|---|---|---|---|
| $\omega_2$ | $\omega_2$ | $\omega_2$ | $\omega_1$ | $\omega_8$ | $\omega_1$ | $\omega_2$ |
| $2p$ | $2p$ | $2p - 1$ | $2p - 1$ | $2p - 1$ | $2p - 1$ | $3p$ |

On the other hand, since $\lambda \in \tilde{C}_\mathbb{Z}$ it satisfies $\langle \lambda + \rho, \alpha_0^\vee \rangle \leq p$, i.e. $\langle \lambda, \alpha_0^\vee \rangle \leq p - h + 1$. Hence any high weight $\mu$ of $M = \mathfrak{L} \otimes L^*$ satisfies $\langle \mu, \alpha_0^\vee \rangle \leq 2p - 2h + 2$. Looking at the above table, it is easily seen that this is a contradiction. Thus $(\mathfrak{g}^*)[1]$ is not a composition factor of $H^1(G_1, M/k)$ and the result follows.

Remarks 4.10. (i) When $\lambda \neq \mu$ in the proof of the above proposition, one also sees that there is an isomorphism $\text{Ext}^2_{G_1}(L(\lambda), L(\mu)) \cong \text{Ext}^2_{\mathfrak{g}}(L(\lambda), L(\mu))$ but we do not use this fact in the sequel.

(ii) The conclusion of the theorem is incorrect if $G$ is reductive but not semisimple. For example, if $G$ is a torus, then $\mathfrak{g}$ is an abelian Lie algebra, and $H^1(\mathfrak{g}, k)$ is non-trivial. For instance the two-dimensional non-abelian Lie algebra is a non-direct extension of $k$ by $k$. One also has $H^2(k \times k, k) \neq 0$ by the Künneth formula: for example the Heisenberg Lie algebra is a non-split extension of $k$ by $k \times k$.

(iii) When $p = 3$ and $G = \text{SL}_3$, then $H^2(G_1, k)[-1] \cong \mathfrak{g}^* \oplus L(\omega_1) \oplus L(\omega_2)$, by [BNP07, Theorem 6.2]. Thus the same argument shows that $H^2(\mathfrak{g}, k) \cong L(\omega_1) \oplus L(\omega_2)$. It follows from the Künneth formula that if $G$ is a direct product of $n$ copies of $\text{SL}_3$ then $H^2(\mathfrak{g}, k) \cong [L(\omega_1) \oplus L(\omega_2)]^\otimes n$.

(iv) In part (a) of the theorem, one can be more specific. If $\mathfrak{g} = \mathfrak{s}\mathfrak{l}_2$ then [Dzh02] shows that $H^2(\mathfrak{g}, L(p - 2))$ is isomorphic to $L(1)^{[1]}$ as a $G$-module. If $\mathfrak{g} = \mathfrak{s}\mathfrak{l}_2 \times \cdots \times \mathfrak{s}\mathfrak{l}_2 \times \mathfrak{h}$ then one can show moreover that $H^2(\mathfrak{g}, L(\mu))$ is non-zero only if

$$L(\mu) \cong L(\mu_1) \otimes \cdots \otimes L(\mu_n) \otimes L(\mu_{n+1})$$
with each $\mu_i \in \{0, p - 2\}$ and $\mu_{n+1} = 0$. Let $r$ be the number of times $\mu_i = p - 2$. Then, the Künneth formula shows that

$$\dim H^2(\mathfrak{g}, L(\mu)) = \begin{cases} 
0 & \text{if } r=0; \\
2 & \text{if } r=1; \\
4 & \text{if } r=2; \\
0 & \text{otherwise.}
\end{cases}$$

We use the theorem above to get analogues of Corollary 4.8 for Lie algebra representations.

**Proposition 4.11.** Let $G$ be a simple algebraic group with $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and let $\dim V \leq p$ be a $\mathfrak{g}$-module. Then exactly one of the following holds:

(i) $V$ is a semisimple $\mathfrak{g}$-module;

(ii) $G$ is of type $A_1$, $\dim V = p$ and $V$ is uniserial, with composition factors $L(p - 2 - r)$ and $L(r)$.

**Proof.** The proof is similar to Proposition 4.6. Since $\dim V \leq p$, any composition factor of $V$ is a restricted simple $\mathfrak{g}$-module, or $V$ is simple. Since $\text{Ext}^1_{\mathfrak{g}}(k, k) = H^1(\mathfrak{g}, k) \cong (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* = 0$, if $V$ consists only of trivial composition factors then $V$ is semisimple. Thus we may assume that $\mathfrak{g}$ contains a non-trivial composition factor $L$. Then either $\dim L = p$ and $V$ is simple, or $p > h$ by Lemma 4.1(iii). By the condition on $V$, any two distinct composition factors, $L(\lambda)$ and $L(\mu)$ satisfy $\lambda, \mu \in C_\mathbb{Z}$ by Lemma 4.1(ii). If $G$ is not of type $A_1$, then $\text{Ext}^1_{\mathfrak{g}}(L(\lambda), L(\mu)) = 0$ by Theorem 4.9 and the exceptional case, where $G = A_1$, is well known. □

As before there is a corollary:

**Corollary 4.12.** Let $G$ be a semisimple algebraic group and let $V$ be a $\mathfrak{g}$-module with $p > \dim V$. Assume that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Then $V$ is semisimple.

The next corollary uses a famous result of Serre on the semisimplicity of tensor products to extend our results a little further. This result will be crucial for showing the splitting of certain non-semisimple Lie algebras.

**Corollary 4.13.** Let $\mathfrak{g}$ be a Lie algebra and $V, W$ two semisimple $\mathfrak{g}$-modules with $\dim V + \dim W < p + 2$. Then $V \otimes W$ is semisimple.

Furthermore, let $\mathfrak{g} = \text{Lie}(G)$ for $G$ a semisimple algebraic group with $p > 2$ and $p$ very good. Then $H^2(\mathfrak{g}, V \otimes W) = 0$ unless $\mathfrak{g}$ contains a factor $\mathfrak{sl}_2$ and $V \otimes W$ contains a composition factor of the $\mathfrak{sl}_2$-module $L(p - 2)$. Also $H^1(\mathfrak{g}, V \otimes W) = 0$, unless one of $V$ and $W$ is isomorphic to $k$ and we are in one of the exceptional case of Theorem 4.9.

**Proof.** For the first statement, we begin with some reductions as in [Ser94]. If $W = 0$ or $k$ there is nothing to prove. If $W$ is at least 2-dimensional, then either $p = 2$ and $V$ is trivial (so that the result holds), or both $\dim V$ and $\dim W < p$. We may assume that both $V$ and $W$ are simple. Further, we may replace $\mathfrak{g}$ by the restricted algebra generated by its image in $\mathfrak{gl}(V \oplus W)$. As $V \oplus W$ is a semisimple module, we may thus assume $\mathfrak{g}$ is $p$-reductive. Now $\mathfrak{g} \subset \mathfrak{gl}(V) \times \mathfrak{gl}(W) = \mathfrak{sl}(V) \times \mathfrak{sl}(W) \times \mathfrak{z}$, where $\mathfrak{z}$ is a torus, and where the projections of $\mathfrak{g}$ onto the first two factors are irreducible, hence semisimple by Corollary C. We thus may assume $\mathfrak{g} \subset \mathfrak{sl}(V) \times \mathfrak{sl}(W)$ is a semisimple restricted subalgebra.
By Theorem 2.3, either (i) \( g \) has a factor \( W_1 \), the first Witt algebra and \( V \) is the \((p-1)\)-dimensional irreducible module for \( W_1 \); or (ii) \( g \) is \( \text{Lie}(G) \) for a direct product of simple algebraic groups, and \( V \) and \( W \) are (the differentials of) \( p \)-restricted modules for \( G \). In case (i), as \( p > 2 \), we would have \( W \cong k \oplus k \) for \( W_1 \) and the result holds. So we may assume that (ii) holds. Now [Ser94, Prop. 7] implies that \( V \otimes W \) is the direct sum of simple modules with restricted high weights \( \lambda \) satisfying \( \lambda \in C_\mathbb{Z} \). Since each of these composition factors is simple also for \( g \), \( V \otimes W \) is semisimple with those same composition factors.

For the remaining statements, let \( h \) be the image of \( g \) in \( \mathfrak{gl}(V \oplus W) \), so that \( g = h \oplus s \) with \( s \) acting trivially. Let \( h \) be the coxeter number of \( h \). Now if \( W = k \), say, then since \( p \) is very good for \( g \) we can have \( p = \dim V \) by Proposition 4.1 only for \( p > h \), so otherwise \( \dim V < p \). And if \( \dim V > 1 \) then \( \dim V < p \) also. Now \( \dim V < p \) also implies by Proposition 4.1 that \( p > h \). Also a summand \( L(\lambda) \) of \( V \otimes W \) has \( \lambda \in C_\mathbb{Z} \). Now Theorem 4.9 implies that \( H^1(g, V \otimes W) = H^2(g, V \otimes W) = 0 \), unless we are in the exceptional cases described. However, if \( g = \mathfrak{sl}_3 \) then the module \( L(3, 0, 0) \) or its dual has dimension \( (p-1)(p-2)(p-3)/2 > ((p+1)/2)^2 \) hence it cannot appear as a composition factor of \( V \otimes W \).

\[ \square \]

Remark 4.14. If \( g = W_1 \) the conclusion of the second part is false, since \( H^1(g, V) \neq 0 \) when \( V \) is the irreducible \((p-1)\)-dimensional module for \( g \).

**Proof of Theorem E**: We must just give references for the statements made. For (a), see Proposition 4.6; for (b), see Proposition 4.11; for (c), see Theorem 4.9; for (d), see Corollary 4.13. This completes the proof of Theorem E. \[ \square \]

## 5. Decomposability: the existence of Levi factors

Let \( h \) be a restricted subalgebra of \( \mathfrak{gl}(V) \) with \( p > \dim V \). In this section we show, in Theorem 5.2, a strong version of the Borel–Tits Theorem in this context. Briefly, \( h \) is in a parabolic subalgebra \( p = l + q \) so that its \( p \)-radical \( r \) is contained in \( q \) and \( h \) decomposes as a semidirect product \( s + r \) with \( s \leq l \).

Let \( G \) be reductive. Recall, say from [ABS90] that if \( p = l + q \) is a parabolic subalgebra of \( g = \text{Lie} G \) then \( q \) has a central filtration such that successive quotients have the structure of \( l \)-modules. We record a specific case:

**Lemma 5.1.** In case \( G = \text{GL}_n \), a parabolic \( p = l + q \) has that \( l \) is a direct product \( \mathfrak{gl}(V_1) \times \mathfrak{gl}(V_2) \times \cdots \times \mathfrak{gl}(V_r) \) and \( q \) has a central filtration with successive factors being modules of the form \( V_i \otimes V_j^* \).

**Theorem 5.2.** Let \( h \) be a restricted Lie subalgebra of \( \mathfrak{gl}(V) \) with \( \dim V < p \), and let \( r = \text{Rad}_p(h) \) (= \( \text{Rad}_V(h) \)).

Then there is a parabolic subalgebra \( p = l + q \), with \( r \leq q \) and containing a complement \( s \) to \( r \) in \( h \), with \( s \leq l \) and \( h = s + r \) as a semidirect product. Furthermore, \( s \) acts completely reducibly on \( V \) and is the direct sum of a torus and a semisimple ideal.

**Proof.** As in the proof of Lemma 3.9 we take a minimal parabolic subalgebra \( p = l + q \) containing \( h \), which implies the projection \( h l := \pi(h) \) of \( h \) to the Levi subalgebra \( l \) is strongly \( p \)-reductive and we may write \( h l = h_s \oplus z \) where \( h_s \) is semisimple.

Now by Theorem 2.3, either \( h_s = W_1 \), \( h = h_0 \), \( p = l = \mathfrak{gl}(V) \) and we are done; or \( h_s \) is isomorphic to a direct product of classical Lie algebras \( s_i \) and \( z \). Furthermore \( h_l \) acts completely reducibly on \( V \).
Let $\pi' : \mathfrak{h} \to \mathfrak{h}_s$ be the composition of $\pi$ with the projection onto $\mathfrak{h}_s$. By [SF88, Lemma 2.4.4(2)], $\ker(\pi') = \pi^{-1}(\mathfrak{z}) = \mathfrak{z} + \mathfrak{r}$, where $\mathfrak{z} \leq \mathfrak{h}$ is a (restricted) torus. We obtain $\mathfrak{z} \leq \mathfrak{z}' + \mathfrak{r} \leq \mathfrak{h}$, and hence may take $\mathfrak{z} = \mathfrak{z}'$. Let $\mathfrak{h}' \subseteq \mathfrak{h}$ be a vector space complement to $\ker(\pi')$. Then $\mathfrak{r} + \mathfrak{h}' \leq \mathfrak{h}$ is a subalgebra, and we have an exact sequence $0 \to \mathfrak{r} \to \mathfrak{h}' \xrightarrow{\pi'} \mathfrak{h}_s \to 0$.

We show this sequence is split. By Lemma 5.1, the nilpotent radical $\mathfrak{q}$ of $\mathfrak{l}$ has a filtration $\mathfrak{q} = \mathfrak{q}_1 \supseteq \mathfrak{q}_2 \supseteq \cdots \supseteq \mathfrak{q}_m = 0$ with each $\mathfrak{q}_i/\mathfrak{q}_{i+1}$ having the structure of an $\mathfrak{l}$-module $M_i \otimes N_i$ with $M_i$ and $N_i$ irreducible modules for the projections of $\mathfrak{h}_l$ to distinct factors of the Levi.

Since $\dim M_i + \dim N_i < p$, we have by Corollary 4.13 that $M_i \otimes N_i$ is a direct sum of irreducible modules for $\mathfrak{h}_s$ with $H^2(\mathfrak{h}_s, M_i \otimes N_i) = 0$. By intersecting with $\mathfrak{r}$, we get a filtration $\mathfrak{r} = \mathfrak{r}_1 \supseteq \mathfrak{r}_2 \supseteq \cdots \supseteq \mathfrak{r}_m = 0$ by $\mathfrak{h}_s$-modules so that each $\mathfrak{r}_i/\mathfrak{r}_{i+1}$ is a submodule of $M_i \otimes N_i$, hence also a semisimple module with $H^2(\mathfrak{h}_s, \mathfrak{r}_i/\mathfrak{r}_{i+1}) = 0$. By an obvious induction on the length $m$ of the filtration $\{\mathfrak{r}_i\}$ we now see that the sequence

$$0 \to \mathfrak{r} \to \mathfrak{h}' + \mathfrak{r} \to \mathfrak{h}_s \to 0$$

is split. Thus we may set $\mathfrak{h}'_s$ a complement to $\mathfrak{r}$ in $\mathfrak{h}' + \mathfrak{r}$.

We would like to set $\mathfrak{s} = \mathfrak{h}'_s + \mathfrak{z}$, however it is not clear if this vector space would be a subalgebra of $\mathfrak{g}$.

Write $\mathfrak{r} = \mathfrak{c}_\mathfrak{t}(\mathfrak{z}(l)) + [\mathfrak{r}, \mathfrak{z}(l)] \cap \mathfrak{r}$. (This can be done, for instance by [SF88, Lemma 2.4.4(1)].) Any element $h$ of $\mathfrak{h}'$ can be written as $h_1 + r_1 + r_2$ for $h_1 \in \mathfrak{h}$, $r_1$ in $\mathfrak{c}_\mathfrak{t}(\mathfrak{z}(l))$ and $r_2 \in [\mathfrak{r}, \mathfrak{z}(l)] \cap \mathfrak{r}$ as $\mathfrak{r} \leq \mathfrak{h}$, we have the element $h' = h_1 + r_1 \in \mathfrak{h}$. By truncating in this way, we may form the subspace $\mathfrak{h}'_s \leq \mathfrak{h}$ with $\mathfrak{h}'_s \leq \mathfrak{r} + \mathfrak{c}_\mathfrak{t}(\mathfrak{z}(l))$.

Using that $\mathfrak{h}'_s \leq \mathfrak{h}$ is a subalgebra, we see that (using the Jacobi identity several times) $\mathfrak{c}_\mathfrak{t}(\mathfrak{z}(l)) \cap \mathfrak{r}$ is $l = \mathfrak{c}_\mathfrak{gl}(V)(\mathfrak{z}(l))$-invariant and, by considering the weights of $\mathfrak{z}(l)$ on $[\mathfrak{r}, \mathfrak{z}(l)]$, that $[\mathfrak{r}, \mathfrak{z}(l)] \cap \mathfrak{r}$ is an ideal in $\mathfrak{r}$. Further, one checks that this $\mathfrak{h}'_s$ is indeed a subalgebra of $\mathfrak{gl}(V)$, with $\mathfrak{h}'_s$ also a complement to $\mathfrak{r}$ in $\mathfrak{h}'_s + \mathfrak{r}$. Now we have guaranteed that $\mathfrak{s} = \mathfrak{h}'_s + \mathfrak{z}$ is a subalgebra of $\mathfrak{h}$, a complement to $\mathfrak{r}$ in $\mathfrak{h}$.

Now, by Corollary 4.12, $\mathfrak{h}'_s$ acts completely reducibly. Also, since $\mathfrak{z}$ is a restricted torus, $\mathfrak{z}$ is linearly reducible, hence also acts completely reducibly. Thus $\mathfrak{s}$ is completely reducible on $V$. In particular, we may replace $\mathfrak{l}$ with a Levi subalgebra of $\mathfrak{p}$ that contains $\mathfrak{s}$, which finishes the proof. □

6. ON EXPONENTIATION AND NORMALISING

Let $G$ be a connected reductive group. We recall the existence of exponential and logarithm maps for $p$ big enough, see [Ser98, Thm. 3]. We fix a maximal torus $T$ and a Borel subgroup $B = T \ltimes U$ containing $T$.

**Theorem 6.1.** Assume that $p > h$ ( $p \geq h$ for $G$ simply connected), where $h$ is the Coxeter number. Then there exists a unique isomorphism of varieties $\log : G^u \to \mathfrak{g}_{\text{nilp}}$, whose inverse we denote by $\exp : \mathfrak{g}_{\text{nilp}} \to G^u$, with the following properties:

(i) $\log \circ \sigma = d\sigma \circ \log$ for all $\sigma \in \text{Aut}(G)$;

(ii) the restriction of $\log$ to $U$ is an isomorphism of algebraic groups $U \to \text{Lie}(U)$, whose tangent space is the identity; here the group law on $\text{Lie}(U)$ is given by the Hausdorff formula;

(iii) $\log(x_\alpha(a)) = aX_\alpha$ for every root $\alpha$ and $a \in k$, where $X_\alpha = dx_\alpha(1)$.

The uniqueness implies that for $G = \text{GL}(V)$, $p \geq \dim V$, $\exp$ and $\log$ are the usual truncated series.
Recall (cf. [Ser98]) that for a $G$-module $V$, the number $n(V)$ is defined as $n(V) = \sup_\lambda n(\lambda)$, where $\lambda$ ranges over all $T$-weights of $V$, and where $n(\lambda) = \sum_{\alpha \in R^+} \langle \lambda, \alpha^\vee \rangle$. For the adjoint module $\mathfrak{g}$, one obtains $n(\mathfrak{g}) = 2h - 2$.

**Proposition 6.2.** Let $\rho : G \to \text{GL}(V)$ be a rational representation of $G$. Suppose that $p > h$ and $p > n(V)$. Let $x \in \mathfrak{g}$ be a nilpotent element. Then

$$\rho(\exp_G x) = \exp_{\text{GL}}(d\rho(x)).$$

In particular, if $p > 2h - 2$, then $\text{Ad}(\exp_G x) = \exp_{\text{GL}}(\text{ad}(x))$.

**Proof.** Consider the homomorphism $\phi : G_a \to \text{GL}(V)$ given by $\phi(t) = \rho(\exp_G(t.x))$. Under our assumptions, it follows from [Ser98, Thm. 5] that $\phi$ is a morphism of degree $< p$, (i.e. the matrix entries of $\phi$ are polynomials of degree less than $p$ in $t$). Moreover, $d\phi(1) = d\rho(x)$. By [Ser94, §4], this implies that $d\rho(x)^p = 0$ and that $\phi$ agrees with the homomorphism $t \mapsto \exp_{\text{GL}}(t.d\rho(x))$. The claim follows.

**Lemma 6.3.** Let $X \in \mathfrak{gl}(V)$ be a nilpotent element satisfying $X^n = 0$ for some integer $n \leq p$. Let $l, r \in \text{End}(\mathfrak{gl}(V))$ be left multiplication with $X$, respectively right multiplication with $-X$. Set $W = W_p(l, r) \in \text{End}(\mathfrak{gl}(V))$, where $W_p(l, y)$ is the the image of $\frac{1}{n}(l(l + y)^p - l^p - y^p) \in \mathbb{Z}[x, y]$ in $k[x, y]$. Let $\mathfrak{h}$ be a subset of $\mathfrak{gl}(V)$ normalised (resp. centralised) by $X$. Suppose that $\mathfrak{h} \subseteq \ker(W)$. Then $\exp(X) \in \text{GL}(V)$ normalises (resp. centralises) $\mathfrak{h}$.

In particular, if $p \geq 2n - 1$, then $W = 0$ and so $\exp(X)$ normalises (resp. centralises) every subspace that is normalised (resp. centralised) by $X$.

**Proof.** Since the nilpotence degree of $X$ is less than $p$, the exponential $\exp(X) = 1 + X + X^2/2 + \ldots$ gives a well-defined element of $\text{GL}(V)$. Moreover, for each $Y \in \mathfrak{h}$ we have the equality

$$\text{Ad}(\exp(X))(Y) = \exp(\text{ad}(X))(Y) = Y + \text{ad}(X)(Y) + \text{ad}(X)^2(Y)/2 + \cdots \in \mathfrak{gl}(V).$$

Indeed, we have $\text{ad}(X) = l + r$, and $\text{Ad}(\exp(X)) = \exp(l)\exp(r)$. Now by [Ser94, (4.1.7)],

$$\exp(l)\exp(r) = \exp(l + r - W).$$

Since $l$ and $r$ commute with $W$, we deduce $(l + r - W)^m(Y) = (l + r)^m(Y)$ for each $m \geq 0$. Thus $\text{Ad}(\exp(X))(Y) = \exp(l + r)(Y) = \exp(\text{ad}(X))(Y)$, as claimed. Hence $\exp(X)$ is contained in $\mathfrak{n}_{\text{GL}(V)}(\mathfrak{h})$ whenever $X \in \mathfrak{n}_{\mathfrak{gl}(V)}(\mathfrak{h})$ and $\exp(X) \in C_{\text{GL}(V)}(\mathfrak{h})$ whenever $X \in \mathfrak{c}_{\mathfrak{gl}(V)}(\mathfrak{h})$.

Moreover, $W_p(l, r) = \sum_{i=1}^{p-1} c_i l^i r^{p-i}$ for certain non-zero coefficients $c_i \in k$. In particular, this expression vanishes for $p \geq 2n - 1$.

**Corollary 6.4.** Let $\mathfrak{p} = \mathfrak{q} + l \subseteq \mathfrak{gl}(V)$ be a parabolic subalgebra, and suppose that $p \geq \dim V$. If $X \in \mathfrak{q}$ normalises a subset $\mathfrak{h} \subseteq \mathfrak{p}$, then so does $\exp(X)$.

**Proof.** By Lemma 6.3, it suffices to show that $\mathfrak{p} \subseteq \ker(W)$. Let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = V$ be a flag with the property

$$\mathfrak{p} = \{ Y \in \mathfrak{gl}(V) \mid YV_i \subseteq V_i \}$$

$$\mathfrak{q} = \{ Y \in \mathfrak{gl}(V) \mid YV_i \subseteq V_{i-1} \}.$$ 

By assumption, we have $p \geq m$, and therefore all products $X_1 \cdots X_{p+1}$ with all $X_i \in \mathfrak{p}$ and all but one $X_i \in \mathfrak{q}$ vanish on $V$. In particular $l^p r^{p-i}(Y) = 0$ for all $Y \in \mathfrak{p}$ and hence $W(Y) = 0$. □
Lemma 6.5. Suppose $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ generated as a $k$-algebra by a set of nilpotent elements $\{X_i\}$ of nilpotence degree less than $p$, and let $G = \langle \exp(t.X_i) \rangle$ be the closed subgroup of $\mathfrak{gl}(V)$ generated by $\exp(t.X_i)$ for each $t \in k$. Then $\mathfrak{g} \leq \text{Lie}(G)$.

Proof. Since $\text{Lie}(G)$ contains the element $d/dt \exp(t.X_i)|_{t=0}$ it contains each element $X_i$. Since $\mathfrak{g}$ is generated by the elements $X_i$, we are done. $\square$

7. Smoothness of normalisers of $\text{GL}_n$: Proof of Theorem A(1)

In this section let $G = \text{GL}(V)$ and let $p > \dim V + 1$. Here we use the results of the previous sections to take the normaliser $\mathfrak{n} := \mathfrak{n}_g(\mathfrak{h})$ of a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and lift it to a smooth algebraic group which also normalises $\mathfrak{h}$.

Observe that $\mathfrak{n}$, being a normaliser, must be a restricted subalgebra of $\mathfrak{g}$. By Theorem 5.2 $\mathfrak{n}$ decomposes as a semidirect product $\mathfrak{s} + \mathfrak{r}$ where $\mathfrak{r}$ is its $p$-radical and $\mathfrak{s} = \mathfrak{t} + \mathfrak{z}$ is the direct sum of simple algebraic Lie algebras and a central torus $\mathfrak{z}$. We lift each component, $\mathfrak{t}$, $\mathfrak{z}$ and $\mathfrak{r}$ in three separate stages. We first deal with $\mathfrak{z}$:

Lemma 7.1. Let $G$ be a connected reductive group and let $\mathfrak{h} \subseteq \mathfrak{g}$ be any subset of elements of $\mathfrak{g}$. Suppose that $\mathfrak{c} \subseteq \mathfrak{g}$ is a restricted torus normalising $\mathfrak{h}$. Then $\mathfrak{c} = \text{Lie}(C)$ for a torus $C \subseteq N_G(\mathfrak{h})$.

Proof. Suppose that $G \subseteq \text{GL}_n$ is a closed subgroup. There is a torus $T \subseteq G$ such that $\mathfrak{c} \subseteq \text{Lie}(T)$, cf. [Hum67, Thm. 13.3]. Moreover, we may assume that $T$ and $\text{Lie}(T)$ consist of diagonal matrices in $\text{GL}_n$, respectively $\mathfrak{gl}_n$.

So since $\mathfrak{c}$ is restricted, it has a basis defined over $\mathbb{F}_p$ of elements $Z_1, \ldots, Z_s$ with $Z_i = \text{diag}(z_{i1}, \ldots, z_{in})$ and each $z_{ij} \in \mathbb{F}_p$. We take the torus $C = \langle t_i(t) \rangle_{i=1, \ldots, s; t \in k^*} \leq \text{GL}_n$ where $t_i(t) = \text{diag}(t^{z_{i1}}, \ldots, t^{z_{in}})$. Then $C \subseteq T \subseteq G$ is a torus with $\text{Lie}(C) = \mathfrak{c}$, see [Die52, Prop. 2]. It remains to show that $C$ normalises $\mathfrak{h}$.

Since $k$ is algebraically closed, we may take a decomposition of $\mathfrak{h}$ into generalised ‘eigensets’ for $\mathfrak{c}$. We have $\mathfrak{h} = \mathfrak{h}_0 \oplus \bigoplus_\alpha \mathfrak{h}_\alpha$ where $\mathfrak{h}_0$ is some set of elements commuting with $\mathfrak{c}$, $\alpha$ is a non-trivial linear functional $\mathfrak{c} \to k$ and each $\mathfrak{h}_\alpha$ is a scalar-closed subset of $\mathfrak{gl}_n$ with $[c, X] = \alpha(c)X$ for $c \in \mathfrak{c}$ $X \in \mathfrak{h}_\alpha$.

Let $X \in \mathfrak{h}$ and write $X = X_0 + \sum_\alpha X_\alpha$. Then $[c, X] = X_0 + \sum_\alpha \alpha(c)X_\alpha$. Extend $\mathfrak{c}$ to the full diagonal Cartan subalgebra $\mathfrak{m}$ of $\mathfrak{gl}_n$. Then for $X \in \mathfrak{h}_\alpha$ we can write $X = \sum_\beta X_\beta$ where the sum is over all roots $\beta$ for $\mathfrak{m}$ and $X_\beta$ is in the $\beta$ root space. Let $B$ denote the set of roots on which $X$ is supported. Since $[c, X]$ is a multiple of $\alpha(c)$ of $X$, and since $[c, X_\beta]$ is a multiple of $\beta(c)$ of $X_\beta$ we must have $[c, X] = \beta(c)X = \alpha(c)X$, so that $\beta(c) = \alpha(c)$ for any $\beta \in B$ and $c \in \mathfrak{c}$.

Now consider $\text{Ad}(t_i(t)).X_\beta$. One checks $\text{Ad}(t_i(t)).X_\beta = t^{\beta(Z_i)}X_\beta$. (If $X_\beta = e_{rs}$, this is just $t^{z_{is}}e_{rs} = t^{\beta(Z_i)}$.) So for $X \in \mathfrak{h}_\alpha$ we have $\text{Ad}(t_i(t)).X = t^{\alpha(Z_i)}X$, a multiple of $X$, hence in $\mathfrak{h}$. Similarly if $X$ is in $\mathfrak{h}_0$ then $\mathfrak{c}$ commutes with $X$, hence $X$ is in the Levi subalgebra of $\mathfrak{gl}_n$ centralising $\mathfrak{c}$, which is the same Levi subalgebra centralising $C$. $\square$

Corollary 7.2. Let $G$ be a connected reductive group and let $\mathfrak{h} \subseteq \mathfrak{g}$ be any subset of elements of $\mathfrak{g}$. Let $T \subseteq G$ be a torus. Then $N_T(\mathfrak{h})$ is smooth.

Proof. Let $\mathfrak{c} = \mathfrak{n}_{\text{Lie}(T)}(\mathfrak{h})$. By definition, this is a restricted toral subalgebra of $\text{Lie}(T)$. By Lemma 7.1 and its proof, we have $\mathfrak{c} \subseteq \text{Lie}(N_T(\mathfrak{h}))$ and the claim follows. $\square$
We are now in a position to prove the first statement of Theorem A.

Proof of Theorem A(i). Observe that the Witt algebra in its $p-1$-dimensional representation gives an example of a subalgebra of $\mathfrak{sl}_p$ which is self-normalising. We get $W_1 \leq \mathfrak{gl}_{p-1} \oplus \mathfrak{gl}_{p-r+1} \leq \mathfrak{gl}_r$, for $r \geq p-1$, with $W_1$ contained in the first factor. Now the normaliser of $W_1$ in $\mathfrak{gl}_1$ is $W_1 \oplus \mathfrak{gl}_{p-r+1}$ with $\mathfrak{z} = \mathfrak{z}(\mathfrak{gl}_{p-1})$. Now this is not an algebraic Lie algebra, hence we have proved the implication $\Rightarrow$ of Theorem A(i).

Now assume $p > \dim V + 1$. Since any element normalising $\mathfrak{h}$ also normalises its $p$-closure, we may assume $\mathfrak{h}$ is restricted. Let $n = n_q(\mathfrak{h})$. By Theorem 5.2 we may decompose both $\mathfrak{n}$ and $\mathfrak{h}$. Let $n = n_l + n_q \leq p = l + q$ with $n_l \leq l$ and $n_q \leq q$, with $n_l = n_s + \mathfrak{z}$, $\mathfrak{z}$ a torus and $n_q$ is by Theorem 2.3 isomorphic to a direct product of classical Lie algebras acting completely reducibly on $V$; also set $\mathfrak{h}_q = \mathfrak{h} \cap \mathfrak{q}$ and $\mathfrak{h}_l = \pi(\mathfrak{h})$ the projection to $\mathfrak{l}$. Since the complement to $\mathfrak{h}_q$ in $\mathfrak{h}$ obtained by Theorem 5.2 is completely reducible on $V$ and hence conjugate into $\mathfrak{l}$, we may assume that $\mathfrak{h} = \mathfrak{h}_q + \mathfrak{h}_l$ is this splitting.

Now $\mathfrak{h}_l \leq n_l = n_s + \mathfrak{z}$ has $\mathfrak{z}(\mathfrak{h}_l) \leq \mathfrak{z}$ or else $n_s$ would have an abelian ideal. Following the proof of Lemma 3.9 we have $\mathfrak{h}_l$ is strongly $p$-reductive. Write $\mathfrak{h}_l = \mathfrak{h}_s \oplus \mathfrak{z}(\mathfrak{h}_l)$. Furthermore, $\mathfrak{h}_s \leq n_s$ is a normal subalgebra of a direct product of simple subalgebras, hence is a direct product of some subset of those simples.

Since $V$ has dimension less than $p$, $V|_{n_s}$ is a restricted module for $n_s$. Hence there is a connected algebraic group $N_s$ with $\text{Lie}(N_s) \cong n_s$, $N_s \leq \text{GL}(V)$ and $V|_{\text{Lie}(N_s)} \cong V|_{n_s}$. Hence, replacing $N_s$ by a conjugate if necessary, we have $\text{Lie}(N_s) = n_s$. Moreover if $L$ is a Levi subgroup of $\text{GL}(V)$ chosen so that $\text{Lie}(L) = \mathfrak{l}$ then we may produce $N_s \leq L$. Clearly $N_s$ normalises $n_l = n_s + \mathfrak{z}$, but $\mathfrak{h}_l = \mathfrak{h}_s \oplus \mathfrak{z}(\mathfrak{h}_l)$ for $\mathfrak{h}_s$ a direct sum of some of the simple factors of $n_s$, so $N_s$ normalises $\mathfrak{h}_l$.

Now, since the $l$-composition factors of $\mathfrak{q}$ are all of the form $W_1 \otimes W_2$ for $\dim W_1 + \dim W_2 < p$ and $W_1$, $W_2$ irreducible for $n_s$, [Ser94, Prop. 7] implies that $\mathfrak{q}$ is a restricted semisimple module for $n_s$ and $n_s$. Since $n_s$ normalises $\mathfrak{h}_q = \mathfrak{h} \cap \mathfrak{q}$, this space also appears as an $n_s$-submodule in $\mathfrak{q}$, hence $N_s$ normalises $\mathfrak{h}_q$.

Let $Z$ be a torus of $G$ with $\text{Lie} Z = \mathfrak{z}$ normalising $\mathfrak{h}$, which is guaranteed to exist by Lemma 7.1.

It remains to construct a unipotent algebraic group $N_q$ such that $\text{Lie} N_q = n_q$ with $N_q$ normalising $\mathfrak{h}$. For this we use Corollary 6.4. Let $N_q = \langle \exp x : x \in n_q \rangle$. Then $N_q$ is a closed subgroup, which by Corollary 6.4 consists of elements normalising $\mathfrak{h}$. By Lemma 6.5, $n_q \leq \text{Lie}(N_q)$.

Let $N$ be the smooth algebraic group given by $N = \langle N_s, Z, N_q \rangle$. We have shown that $N$ normalises $\mathfrak{h}$ and that $n \subseteq \text{Lie} N$. Since also $\text{Lie} N \subseteq n$ we are done. □

8. Smoothness of normalisers of subspaces: Proof of Theorem A(iii)

In this section we show that tightening further the condition on the characteristic, we can prove a stronger statement that the normalisers of all subspaces of the Lie algebra are smooth.

Proof of Theorem A(iii). Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ and let $n = n_q(\mathfrak{h})$ be the Lie-theoretic normaliser of $\mathfrak{h}$ in $\mathfrak{g}$.

As before, by definition, $n$ is a restricted subalgebra of $\mathfrak{g}$. Hence, applying the Jordan decomposition for restricted Lie algebras, we see that $n$ is generated by its nilpotent and semisimple elements. Let
\{x_1, \ldots, x_r, y_1, \ldots, y_s\} be such a generating set with \(x_1, \ldots, x_r\) nilpotent and \(y_1, \ldots, y_s\) semisimple. To show that \(N_G(\mathfrak{h})\) is smooth, it suffices to show that all the elements \(x_i\) and \(y_j\) belong to the Lie algebra of \(N_G(\mathfrak{h})(k)\).

For a nilpotent generator \(x_i\), consider the closed subgroup \(M_i = \{\exp(t x_i) \mid t \in k\}\) of \(G\). By Proposition 6.2, \(M_i \subseteq N_G(\mathfrak{h})(k)\) and hence \(x_i \in \text{Lie}(M_i) \subseteq \text{Lie}(N_G(\mathfrak{h})(k))\), as required.

It remains to consider the semisimple generators \(y_i\). Let \(t_i := (y_i)_p \leq \mathfrak{n}\) be the torus generated by the \(p\)-powers of \(y_i\). By Lemma 7.1 we can find a torus \(T_i \leq N_G(\mathfrak{h})\) such that \(\text{Lie}(T_i) = t_i\). In particular \(y_i \in \text{Lie}(N_G(\mathfrak{h})(k))\). This finishes the proof. \(\square\)

With the following remark we prove the last statement of Theorem A (iii).

**Remark 8.1.** If \(p < 2n - 1\), normalisers of subspaces of \(\mathfrak{gl}_n\) (or \(\mathfrak{sl}_n\)) are not necessarily smooth. In fact, let \(p = 2n - 3\) and let \(\mathfrak{h} = \mathfrak{sl}_2 = \text{Lie} H\) with \(H = \text{SL}_2\) over a field \(k\) of characteristic \(p\). Then the action of \(H\) on the simple module \(L((p + 1)/2)\) gives an (irreducible) embedding \(H \to \text{GL}_n\).

Restricting the adjoint representation of \(\mathfrak{gl}_n\) on itself to \(H\) gives a module \(L((p + 1)/2) \otimes L((p + 1)/2)^* \cong T(p + 1) \oplus M\), where \(M\) is a direct sum of irreducibles for \(H\) (and \(\mathfrak{h}\)) and \(T(p + 1)\) is a tilting module, uniserial with successive composition factors \(L(p - 3)|L(p + 1)|L(p - 3)\).

Now for the algebraic group \(H = \text{SL}_2\) we have \(L(p + 1) \cong L(1) \otimes L(1)^{[1]}\) by Steinberg’s tensor product formula. Restricting to \(\mathfrak{h}\), \(L(p + 1)\) is isomorphic to \(L(1) \oplus L(1)\). Now it is easy to show the restriction map \(\text{Ext}^1_G(L(p + 1), L(p - 3)) \to \text{Ext}^1_{\mathfrak{h}}(L(1), L(p - 3)) \oplus \text{Ext}^1_{\mathfrak{h}}(L(1), L(p - 3))\) is injective. Hence \(T(p + 1)|_{\mathfrak{h}}\) contains a submodule \(M\) isomorphic to \(L(1)/L(p - 3)\).

Now, the Lie theoretic normaliser of \(M\) contains \(\mathfrak{h}\) but the scheme-theoretic stabiliser does not contain \(H\). It follows that the normaliser of this subspace is not smooth.

Indeed, as \(\mathfrak{h}\) acts irreducibly on the \(n\)-dimensional natural representation for \(\mathfrak{gl}_n\), it is in no parabolic of \(\mathfrak{gl}_n\) (or \(\mathfrak{sl}_n\)). However, the set of \(k\)-points \(N_H(M)(k) = N_{\text{GL}_n}(M)(k) \cap H\) is in a parabolic of \(H\), hence in a parabolic of \(\text{GL}_n\).

9. **Reductive pairs: Proofs of Proposition F and Theorem A(ii)**

The following definition is due to Richardson [Ric67].

**Definition 9.1.** Suppose that \((G', G)\) is a pair of reductive algebraic groups such that \(G \subseteq G'\) is a closed subgroup. Let \(\mathfrak{g}' = \text{Lie}(G), \mathfrak{g} = \text{Lie}(G)\). We say that \((G', G)\) is a **reductive pair** provided there is a subspace \(\mathfrak{m} \subseteq \mathfrak{g}'\) such that \(\mathfrak{g}'\) decomposes as a \(G\)-module into a direct sum \(\mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{m}\).

With \(p\) sufficiently large, reductive pairs are easy to find.

**Lemma 9.2** ( [BHMR11, Thm. 3.1]). Suppose \(p > 2 \dim V - 2\) and \(G\) is a reductive subgroup of \(\text{GL}(V)\). Then \((\text{GL}(V), G)\) is a reductive pair.

We need a compatibility result for normalisers of subgroup schemes of height one.

**Lemma 9.3.** Let \(H \subseteq G\) be a closed subgroup scheme of height one, with \(\mathfrak{h} = \text{Lie}(H)\). Then \(N_G(H) = N_G(\mathfrak{h})\) (scheme-theoretic normalisers).
Proof. We have a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(H, H) & \longrightarrow & \text{Hom}_{p-Lie}(\mathfrak{h}, \mathfrak{h}) \\
\downarrow & & \downarrow \\
\text{Hom}(H, G) & \longrightarrow & \text{Hom}_{p-Lie}(\mathfrak{h}, \mathfrak{g}),
\end{array}
\]

where the horizontal arrows are given by differentiation and are bijective (cf. [DG70, II, §7, Thm. 3.5]). Now if \( x \in N_G(\mathfrak{h}) \), the map \( \text{Ad}(x)_{\mathfrak{h}} \) in the bottom right corner may be lifted via the top right corner to a map in \( \text{Hom}(H, H) \). The commutativity of the diagram shows that conjugation by \( x \) stabilises \( H \), and hence \( x \in N_G(H) \). This works for points \( x \) with values in any \( k \)-algebra, and hence proves the containment of subgroup schemes \( N_G(\mathfrak{h}) \subseteq N_G(H) \). The reverse inclusion is clear. \( \square \)

We show that the smoothness of normalisers descends along reductive pairs.

**Proposition 9.4.** Let \((G', G)\) be a reductive pair and let \( H \subseteq G \) be a closed subgroup scheme. If \( N_{G'}(H) \) is smooth, then so is \( N_G(H) \).

In particular, if \( \mathfrak{h} \subseteq \mathfrak{g} \) is a restricted subalgebra and if \( N_{G'}(\mathfrak{h}) \) is smooth, then so is \( N_G(\mathfrak{h}) \).

**Proof.** The last assertion follows from Lemma 9.3.

Let \( H \subseteq G \) be a closed subgroup scheme. We follow the proof of [Her13, Lem. 3.6]. Let \( \mathfrak{g}' = \mathfrak{g} \oplus \mathfrak{m} \) be a decomposition of \( G \)-modules.

By [DG70, II, §5, Lem. 5.7], we have

\[
\dim \text{Lie}(N_{G'}(H)) = \dim \mathfrak{h} + \dim (\mathfrak{g}'/\mathfrak{h})^H = \dim \mathfrak{h} + \dim (\mathfrak{g}/\mathfrak{h})^H + \dim \mathfrak{m}^H
\]

\[
= \dim \text{Lie}(N_G(H)) + \dim \mathfrak{m}^H \geq \dim N_G(H) + \dim \mathfrak{m}^H.
\]

On the left hand side, as \( N_{G'}(H) \) is smooth by assumption, we have \( \dim N_{G'}(H) = \dim \text{Lie}(N_{G'}(H)) \). Thus to show that \( N_G(H) \) is smooth, it suffices to show that \( \dim N_{G'}(H) - \dim N_G(H) \leq \dim \mathfrak{m}^H \).

Now as in [Her13, Lem. 3.6], one shows that there is a monomorphism of quotient schemes \( N_{G'}(H)/N_G(H) \hookrightarrow (G'/G)^H \), and that the tangent space on the right hand side identifies as \( T_e(G'/G)^H \cong \mathfrak{m}^H \). The claim follows. \( \square \)

**Proof of Theorem A(ii).** Let \( G \) be a simple algebraic group and with minimal dimensional representation \( V \). Then since \( p > \dim V \), \( (GL(V), G) \) is a reductive pair. Indeed, the assumption on \( p \) guarantees that the trace form associated to \( V \) is non-zero, see [Gar09, Fact 4.4]. This implies the reductive pair property (cf. the proof of [Gar09, Prop. 8.1]). Now Theorem A(ii) follows from Theorem A(i) in combination with Proposition 9.4. \( \square \)

We finish this section by proving that the bound in Theorem A(ii) is tight for \( \mathfrak{sp}_{2n} \). We do that in the following proposition.

**Proposition 9.5.** The \( p \)-dimensional Witt algebra \( W_1 \) is a maximal subalgebra of \( \mathfrak{sp}_{p-1} \).

**Proof.** Since \( W_1 \) stabilises the element

\[
X \wedge X^{p-1} + \frac{1}{2}X^2 \wedge X^{p-2} + \frac{1}{3}X^3 \wedge X^{p-3} + \cdots + \frac{2}{p-1}X^{(p-1)/2} \wedge X^{(p+1)/2} \in \bigwedge^2 V
\]

We show that the smoothness of normalisers descends along reductive pairs.
we find that $W_1$ is contained in $\mathfrak{sp}_{p-1}$, acting irreducibly on the $p-1$-dimensional module. Exponentiating a set of nilpotent generators of the Witt algebra as in the proof of Theorem A(iii) gives an irreducible subgroup $W \leq \mathfrak{sp}_{p-1}$. We claim that we must have equality. From this claim it follows that $W_1$ is in no proper classical algebraic subalgebra of $\mathfrak{sp}_{p-1}$, hence, by Theorem 2.3, is maximal.

To prove the claim, suppose $W$ is a proper subgroup of $G = \mathfrak{sp}_{p-1}$. Since $W$ is irreducible on the $p-1$-dimensional module, $W$ is it no parabolic of $G$. Thus it is in a reductive maximal subgroup $M$. We must have that $M$ is simple, or else $W_1$ would be in a parabolic of $G$. Now since the lowest dimensional non-trivial representation of $W_1$ is $p-1$, it follows that $M$ can have no lower-dimensional non-trivial representation. Since $p > 2$, $\mathfrak{sp}_{p-1}$ has no simple maximal rank subgroup. All classical groups of rank lower than $\frac{p-1}{2}$ have natural modules of smaller dimension than $p-1$, so $M$ is of exceptional type. The lowest dimensional representations of the exceptional types are 6 ($p = 2$), 7, 25 ($p = 3$), 26, 27, 56 and 248. The only time one of these is $p-1$ is when $p = 57$ and $M = E_7$. But if $p = 57$ then $p > 2h - 2$ for $E_7$, thus by Theorem A(iii) all maximal subalgebras are algebraic and so $W_1$ is not a subalgebra of $E_7$. This proves the claim, hence gives the proposition.

Now for $2n > p-1$, we have $W_1 \leq \mathfrak{sp}_{p-1} \oplus \mathfrak{sp}_{2n-p+1}$ with $W_1$ sitting in the first factor. Then its normaliser is evidently $W_1 \oplus \mathfrak{sp}_{2n-p+1}$ which is not smooth. Thus normalisers of subalgebras of $\mathfrak{sp}_{2n}$ are smooth if and only if $p > h + 1$.

10. Consequences

10.1. A variant of the Borel–Tits Theorem. From now on let $G$ denote a connected reductive algebraic group with Lie algebra $\mathfrak{g}$.

**Theorem 10.1.** Suppose $p > 2$ and all normalisers of subalgebras of $\mathfrak{g}$ are smooth. Then the conclusion of the Borel–Tits Theorem holds. More precisely, if $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ whose $p$-closure has $p$-radical $\mathfrak{r}$ then there is a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{q}$ with $\mathfrak{r} \leq \mathfrak{q}$ and $\mathfrak{h} \leq \mathfrak{g}(\mathfrak{r}) \leq \mathfrak{p}$.

**Proof.** Suppose $\mathfrak{h} \leq \mathfrak{g}$ is a restricted subalgebra with $p$-radical $\mathfrak{r}$. Then $\mathfrak{h}$ is contained in the Lie theoretic normaliser $\mathfrak{g}(\mathfrak{r})$. Now since the normaliser is smooth, there is an algebraic group $N_G(\mathfrak{r})$ whose Lie algebra is $\mathfrak{n}_G(\mathfrak{r})$. Put $N_G(\mathfrak{r})$ in a minimal parabolic subalgebra $P = \mathfrak{L} \mathfrak{Q}$ with Lie algebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{q}$ containing $\mathfrak{g}(\mathfrak{r})$. It remains to show $\mathfrak{r} \leq \mathfrak{q}$. Let $\pi : P \to \mathfrak{L}$ be the projection, and let $R \leq N_G(\mathfrak{r})$ be the normal, unipotent, height one subgroup scheme corresponding to $\mathfrak{r}$. If $\pi(R) \neq 1$, then the image of $N_G(\mathfrak{r})$ is non-reductive by Proposition 2.2, hence contained in a proper parabolic subgroup of $\mathfrak{L}$. This contradicts the minimality of $P$. Hence $d\pi(\mathfrak{r}) = 0$ and $\mathfrak{r} \leq \mathfrak{q}$. 

**Definition 10.2.** Let $\mathfrak{h} \subseteq \mathfrak{g} = \text{Lie}(G)$ be a subalgebra. Then $\mathfrak{h}$ is called $G$-completely reducible provided that whenever $\mathfrak{h}$ is contained in a parabolic subalgebra $\mathfrak{p}$ of $G$, it is contained in a Levi subalgebra of $\mathfrak{p}$.

Note that $\mathfrak{h}$ is completely reducible if and only if its $p$-closure is, so one may assume $\mathfrak{h}$ to be restricted.

**Corollary 10.3.** Suppose all normalisers of subalgebras of $\mathfrak{g}$ are smooth. Let $\mathfrak{h} \leq \mathfrak{g}$ be a restricted subalgebra. If $\mathfrak{h}$ is $G$-cr, it is $p$-reductive.
If, in addition, \((GL(V), G)\) is a reductive pair and \(p > \dim V\), then any \(p\)-reductive subalgebra \(h \leq g\) is \(G\)-cr.

**Proof.** The first assertion directly follows from Theorem 10.1. Under the further assumptions, \(h\) is \(GL(V)\)-cr by Corollary 4.8, and hence \(G\)-cr by [BMR05, Cor. 3.36]. \(\square\)

For simple algebraic groups \(G\) of classical types in characteristics bigger than two, we find a reductive pair with the natural module, and hence complete reducibility on the natural module is equivalent to being \(G\)-cr by [BMR05, Cor. 3.36]. On the other hand, a restricted completely reducible subalgebra of \(\mathfrak{gl}(V)\) is automatically \(p\)-reductive. Thus we get

**Corollary 10.4.** Let \(G\) be a simple classical algebraic group with natural module \(V\) such that \(\dim V < p\). Let \(h\) be a restricted Lie subalgebra of \(g = \text{Lie}(G)\). Then \(h\) is \(G\)-cr if and only if \(h\) is \(p\)-reductive.

If \(g\) is simple, then maximal subalgebras are self-normalising, hence algebraic if normalisers are smooth. This yields the following easy consequence.

**Corollary 10.5.** Let \(g = \text{Lie}(G)\) be simple and suppose that all normalisers of subalgebras are smooth. Then every maximal subalgebra in \(g\) is the Lie algebra of a maximal subgroup of \(G\).

### 10.2. Maximal solvable subalgebras.

If \(p > 3\) and \(b\) is a maximal solvable subalgebra of \(g\) containing a Cartan subalgebra of \(g\) then by [Hum67, 14.4], \(b\) is a Borel subalgebra of \(g\). However in general it is far from true that all maximal solvable subalgebras are Borel subalgebras. In this subsection we prove Theorem D and recover the equivalence with a constraint on the characteristic. Note that our constraint is strict in the classical case.

**Proof of Theorem D.** We first prove part (a). First assume \(p\) satisfies the hypotheses given in the statement of Theorem D(a)(iii), and let \(b\) be a maximal solvable subalgebra. We wish to show (ii) holds, i.e. that \(b\) is a Borel subalgebra. Let \(r = \text{rk}(G)\) and assume the result is proved for all classical algebraic groups of rank less than \(r\). Now since \(p\) is very good for \(G\), Corollary C shows that \(b\) is in a parabolic \(p\) of \(g\). Pick \(p\) minimal with that property. Since \(g\) is classical, its Levi subalgebra is of the form \(l' \oplus z\) where \(l' = s_1 \oplus \cdots \oplus s_n\) is a direct product of simple Lie algebras of classical type and \(z\) is a torus. Projection \(\pi(b)\) to the Levi subalgebra \(l\) of \(p\) yields a solvable subalgebra of \(l\). Further projection to each \(s_i\) yields a solvable subalgebra of a simple classical Lie algebra, hence, by induction is contained in some Borel subalgebra of \(s_i\). Since \(p\) is minimal, this forces \(l\) to be a torus and \(p\) to be a Borel subalgebra, and thus \(b = p\). We have proved (iii) \(\Rightarrow\) (ii).

The implication (ii) \(\Rightarrow\) (i) in Theorem D(a) is clear, since all Borel subgroups are conjugate. See for instance [Hum67, 14.4].

Next, assume that \(p\) does not satisfy the hypotheses given. Then \(g\) has a Levi subalgebra \(l\) with a factor of type \(A_{p-1}\) (\(l\) is \(g\) itself if \(g\) is of type \(A_{p-1}\)). We exhibit a solvable subalgebra not contained in a Borel subalgebra. For this, note that the 3-dimensional nilpotent Heisenberg algebra has an irreducible representation of dimension \(p\). This gives a solvable subalgebra \(b\) of \(sl_p\) (hence of \(psl_p\)) which is irreducible on the \(p\)-dimensional natural module. Thus \(b\) acts on the natural module for \(g\) with composition factors which are not all 1-dimensional, hence cannot be any Borel subalgebra of \(g\).

This proves part (a) of Theorem D.
For part (b), we invoke Theorem A(iii) to see that Theorem 10.1 applies, and we may proceed as in the proof of (a), (iii)⇒(ii). By maximality, any maximal solvable subalgebra b of g is p-closed. Let u =Radₚb be its p-radical. If u = 0 then b is a maximal solvable subalgebra. Otherwise, by Theorem 10.1, b is contained in a proper parabolic subalgebra p = l + q. Choose p minimal. As the projection π(b) to l is solvable, by induction it is contained in a Borel subalgebra of l. By minimality, l is a torus and b = p is a Borel subalgebra.

11. APPENDIX. THE MAXIMAL NON-SEMISIMPLE SUBALGEBRAS OF TYPE $A_n$, FOR $p|n + 1$: CONCLUSION OF THE PROOF OF THEOREM B.

In this section we prove the remainder of Theorem B, reproving and clarifying results from [Ten87]. We again assume $p > 2$ throughout this section.

For $r ≥ 1$ let $O_r = k[x_1, . . . , x_r]/(x_1^p, . . . , x_r^p)$ be the truncated polynomial algebra. Let $W_r = \{\sum f_i \partial_i \mid f_i ∈ O_r\}$ be the Lie algebra of its derivations, where $\partial_i = \frac{\partial}{\partial x_i}$. Since $W_r$ acts on $O_r$, by derivations, one can form the semidirect product $W_r + O_r$ of Lie algebras. Moreover, let $U$ be any vector space. As $O_r$ is abelian, one can form a further semidirect product $1 \otimes W_r + gl(U) \otimes O_r$, acting on the space $V = U \otimes O_r$, where $O_r$ acts on itself by multiplication.

**Lemma 11.1.** Let $V = U \otimes O_r$, where $U$ is a k-vector space. Then $h = 1 \otimes W_r + gl(U) \otimes O_r$ is a maximal subalgebra of $sl(V)$.

**Proof.** Let $h ≤ g ≤ sl(V)$ be a subalgebra that properly contains $h$. We claim that $g = sl(V)$. We proceed in steps.

(1) We first show that $g$ contains $1 \otimes \partial^b$ for all $b$ with $|b| = 2$.

Indeed, let $\varphi = \sum_{a,b} \psi_{a,b} \otimes x^a \partial^b ∈ g \setminus h$, where the sum runs over all $0 ≤ a, b ≤ (p − 1, . . . , p − 1)$ and where $\psi_{a,b} ∈ gl(U)$. Note that $\varphi ∈ h$ precisely if all $b$ satisfy $|b| ≤ 1$ and $\psi_{a,b} ∈ k$ for $|b| = 1$.

For $1 ≤ i ≤ r$ we have $[\partial_i, x^a] = a_i x^{a−\varepsilon_i}$ in $gl(O_r)$, and hence

$$[1 \otimes \partial_i, \varphi] = \sum_{a,b} a_i \psi_{a,b} \otimes x^{a−\varepsilon_i} \partial^b ∈ g.$$  

After repeated application of this formula we may assume that all $a_i = 0$, i.e. $\varphi = \sum_b \psi_b \otimes \partial^b$. Now from $[\partial^b, x_i] = b_i \partial^{b−\varepsilon_i}$ we deduce that

$$[1 \otimes x_i, \varphi] = − \sum_b b_i \psi_b \otimes \partial^{b−\varepsilon_i} ∈ g.$$  

As long as the result is still not contained in $h$, we may repeatedly apply this formula for various $i$. We may thus assume that $\varphi = \sum_b \psi_b \otimes \partial^b$ where all $b$ satisfy $|b| ≤ 2$ and $\psi_b ∈ k$ whenever $|b| = 2$.

Now we have

$$[1 \otimes x_i \partial_j, \varphi] = − \sum_b b_i \psi_b \otimes \partial^{b−\varepsilon_i+\varepsilon_j} ∈ g.$$  

Applying this formula for $i = j$ deletes all terms with $b_j = 0$, while all other terms survive. For some $j$, say $j = 1$, the result will still not lie in $h$, hence we may assume that

$$\varphi = \psi \otimes \partial_l + \sum_{i=1}^r \psi_i \otimes \partial_l \partial_i,$$  

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with $\psi_i \in k$. Then for $i \neq 1$,
\[
[1 \otimes x_i \partial_1, \varphi] = -\psi_i \otimes \partial_i^2,
\]
So we either may assume that $\varphi = 1 \otimes \partial_i^2$, or that $\varphi = \psi \otimes \partial_1 + \psi_1 \otimes \partial_i^2$, with $\psi \notin k$, $\psi_1 \in k$. In the latter case, we find $\psi' \in \mathfrak{g}(U)$ such that $\psi'' := [\psi', \psi] \notin k$. Then $[\psi' \otimes 1, \varphi] = \psi'' \otimes \partial_1 \notin \mathfrak{h}$, hence we may assume that $\varphi = \psi \otimes \partial_1$ for some $\psi \notin k$. We want reduce again to the the case $1 \otimes \partial_i^2 \in \mathfrak{g}$.

Since $\psi \notin k$, we must have $\dim U > 1$. Taking commutators $[\psi' \otimes 1, \varphi] = [\psi', \psi] \otimes \partial_1$ for various $\psi' \in \mathfrak{g}(U)$, we may obtain $\psi \otimes \partial_1 \in \mathfrak{g}$ for any $\psi \in \mathfrak{sl}(U)$. Picking $\psi'$ non-commuting with $\psi$, we obtain $[\psi \otimes \partial_1, \psi' \otimes \partial_1] = [\psi, \psi'] \otimes \partial_i^2 \in \mathfrak{g}$, and $[\psi' \otimes x_1, \psi \otimes \partial_1] = [\psi', \psi] \otimes x_1 \partial_1 - \psi \psi' \otimes 1$. Proceeding similarly, we obtain all $\psi \otimes \partial_i^2$, and $\psi \otimes x_1 \partial_1$ in $\mathfrak{g}$, for $\psi \in \mathfrak{sl}(U)$. Since dim $U > 1$, we may find a pair $\psi, \psi' = \psi^{-1} \in \mathfrak{sl}(U)$. Then $[\psi' \otimes x_1 \partial_1, \psi \otimes \partial_1^2] = [\psi', \psi] \otimes x_1 \partial_1 - 2\psi \psi' \otimes \partial_i^2 = -2 \otimes \partial_i^2 \in \mathfrak{g}$, which finally reduces again to $1 \otimes \partial_i^2 \in \mathfrak{g}$.

For $i \neq j$ we have $[x_i \partial_j, \partial_i \partial_j] = -\partial_j^2$ and $[x_i \partial_j, \partial_i^2] = -2 \partial_i \partial_j$, whence the first claim follows.

(2) We have $1 \otimes x^a \partial^b \in \mathfrak{g}$ for all $(a, b) \neq ((p-1, \ldots, p-1), (p-1, \ldots, p-1))$. This is clear for $|b| \leq 1$, since these elements are in $\mathfrak{h}$.

We first assume that $1 \otimes \partial^b \in \mathfrak{g}$ for some fixed $b$, and that $\mathfrak{g}$ contains the span $\mathfrak{h}_{|b|}$ of all $1 \otimes x^a \partial^b'$, where $b'$ satisfies $|b'| < |b|$. Then using the general formula
\[
\partial^b x^c_i = b! \cdot \sum_{l=0}^{b_i} \frac{1}{l!} \binom{c_i}{b_i - l} x_i^{c_i - b_i + l} \partial_i^l,
\]
we compute that
\[
[1 \otimes x_i^c \partial_i, 1 \otimes x^a \partial^b] = (a_i - c_i b_i) 1 \otimes x^a (c_i - 1) \partial^b \mod \mathfrak{h}_{|b|}.
\]
Starting with $1 \otimes x^a \partial^b = 1 \otimes \partial^b \in \mathfrak{g}$, where all $a_i = 0$, we may thus obtain all elements $1 \otimes x^a \partial^b \in \mathfrak{g}$ which satisfy $a_i \leq (p-2)$.

In particular, we have $1 \otimes x_i \partial^b \in \mathfrak{g}$ for all $i$, and hence by (1), $[1 \otimes \partial_i^2, 1 \otimes x_i \partial^b] = 2 \otimes \partial^b + \varepsilon_1 \in \mathfrak{g}$.

Again by (1), it thus suffices to show that we may obtain all $1 \otimes x^a \partial^b$ for our fixed $b$.

Then starting from $a_i = p-2$, $c_i = 2$, and applying the above formula again we may increase $a_i$ to $p-1$ as long as $b_i \leq p-2$. This finishes the case $r = 1$.

Now assume $r > 1$ and, say, $b_1 = p_1 - 1$. By assumption, we have $1 \otimes x^a \partial^{b_1 - \varepsilon_1} \in \mathfrak{g}$ even for $a_1 = p-1$. Applying the commutator with $1 \otimes \partial_1^2$ if necessary, we may assume that $b_2 > 0$. Then
\[
[1 \otimes x_2 \partial_1, 1 \otimes x^a \partial^{b_1 - \varepsilon_1}] = -b_2 \otimes x^a \partial^{b_1 - \varepsilon_2} \in \mathfrak{g},
\]
and this expression contains $x_1^{p-1} \partial_1^{p-1}$. We may continue in this manner to modify the other positions, as long as we still have one free position to swap with. This finishes the claim.

(3) We have $\psi \otimes x^a \partial^b \in \mathfrak{g}$, whenever this expression has zero trace (this finishes the proof of the Lemma).
Note that the trace computes as
\[
\text{tr}(\psi \otimes x^a \partial^b) = \text{tr}(\psi) \cdot \text{tr}(x^a \partial^b)
\]
\[
= \text{tr}(\psi) \cdot \delta_{a,b} \cdot a! \sum_{0 \leq c \leq (p-1, \ldots, p-1)} \left( \frac{c}{a} \right)
\]
\[
= \text{tr}(\psi) \cdot \delta_{a,b} \cdot a! \left( \frac{p}{a_1 + 1} \right) \cdots \left( \frac{p}{a_r + 1} \right),
\]
which is non-zero precisely for \( \text{tr}(\psi) \neq 0 \) and \( a = b = (p-1, \ldots, p-1) \). So let us first assume that \((a,b)\) is not this exceptional pair of tuples, and thus \( \psi \in \mathfrak{gl}(U) \) may be arbitrary. The element \( 1 \otimes x^a \partial^b \) is in \( \mathfrak{g} \), by (2).

If \( b = 0 \), then \( \psi \otimes x^a \in \mathfrak{h} \leq \mathfrak{g} \), for all \( \psi \in \mathfrak{gl}(U) \). If \( b > 0 \), say \( b_i > 0 \), then
\[
[\psi \otimes x_i, 1 \otimes x^a \partial^b] = -b_i \psi \otimes x^a \partial^{b-e_i} \in \mathfrak{g},
\]
and so \( \psi \otimes x^a \partial^{b-e_i} \in \mathfrak{g} \), in particular \( \psi \otimes x_i \partial_i \in \mathfrak{g} \). Then on the other hand, also
\[
[\psi \otimes x_i \partial_i, 1 \otimes x^a \partial^b] = \psi \otimes (a_i - b_i)x^a \partial^b \in \mathfrak{g},
\]
which takes care of all cases where \( a \neq b \); on the other hand, all cases \( a = b < (p-1, \ldots, p-1) \) are covered by \( \psi \otimes x^a \partial^{b-e_i} \) above.

Thus it remains to show that for all \( \psi \in \mathfrak{sl}(U) \), we have \( \psi \otimes x^a \partial^b \in \mathfrak{g} \), where \( a = b = (p-1, \ldots, p-1) \).

Let \( A = E_{l+1,l+1}, B = E_{l,l+1} \in \mathfrak{gl}(U) \). By the above, the elements \( A \otimes \partial_1 \) and \( B \otimes x^a \partial^b \) belong to \( \mathfrak{g} \), where \( b'_i = p - 1 \) for \( i > 1 \) and \( b'_1 = p - 2 \). But \( AB = 0 \), whereas \( BA = B \), and thus
\[
[A \otimes \partial_1, B \otimes x^a \partial^b] = -B \otimes x^a \partial^b \in \mathfrak{g}.
\]
Since \( B = E_{l,l+1} \) may vary with \( l \), and since we may form commutators with elements of the form \( \psi' \otimes 1, \psi' \in \mathfrak{gl}(U) \), we obtain any matrix in \( \mathfrak{sl}(U) \) as a first component, which finishes the proof. \( \square \)

Recall that the Heisenberg \( H_n \) algebra of dimension \( 2n + 1 \) is defined as the finite-dimensional vector space with generators \( \{x_1, \ldots, x_n, y_1, \ldots, y_n, z\} \) with relations \([x_i, x_j] = [y_i, y_j] = [x_i, z] = [y_i, z] = 0 \) and \([x_i, y_j] = \delta_{ij}z\).

The following lemma is [Dix96, 4.6.2] and should replace [Ten87, Lemma 2] whose statement is at best confusing. While [Dix96] concentrates on characteristic 0, the proof is seen to be valid over any field.

**Lemma 11.2.** Let \( \mathfrak{n} \) be a nilpotent Lie algebra. Assume that every commutative characteristic ideal of \( \mathfrak{n} \) is of dimension \( \leq 1 \). Then either \( \mathfrak{n} \) is trivial or a Heisenberg algebra.

**Proof.** Let \( \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}] \). If \( \dim \mathfrak{n}' > 1 \), let \( n_0 \) be minimal with \( \mathfrak{n}^{n_0+2} = 0 \) and set \( \mathfrak{n}_2 = \mathfrak{n}^{n_0}, \mathfrak{n}_1 = \mathfrak{n}^{n_0+1} \). Then \( \mathfrak{n}_1 \leq \mathfrak{n}_2 \leq \mathfrak{n}' \) and we have

\[
[\mathfrak{n}', \mathfrak{n}_2] \subset [ [\mathfrak{n}, \mathfrak{n}_2], \mathfrak{n} ] + [ [\mathfrak{n}_2, \mathfrak{n}], \mathfrak{n} ] \subset [ \mathfrak{n}_1, \mathfrak{n} ] = 0.
\]

Hence the (characteristic) centre of \( \mathfrak{n}' \) is of dimension \( > 1 \), a contradiction. Thus \( \dim \mathfrak{n}' \leq 1 \). If \( \mathfrak{n}' = 0 \) then \( \mathfrak{n} = \mathfrak{j}(\mathfrak{n}) \) is commutative and so \( \mathfrak{n} \cong k \) so that \( \mathfrak{n} \) is a Heisenberg algebra (with \( n = 0 \)).

Otherwise \( \mathfrak{n}' = \{z\} \), say and we may define an alternating bilinear form \( B \) on \( \mathfrak{n} \) such that \([x, y] = B(x, y)z \). The radical of \( B \) is \( \mathfrak{j}(\mathfrak{n}) \), so that \( B \) is non-degenerate on \( \mathfrak{n}/\mathfrak{j}(\mathfrak{n}) \). Now one takes a lift to \( \mathfrak{n} \) of a symplectic basis of \( \mathfrak{j}(\mathfrak{n}) \); together with the element \( z \) this basis of \( \mathfrak{n} \) can be seen to satisfy the defining relations for the Heisenberg algebra. \( \square \)
It follows from Lemma 11.2 that a non-trivial nonsemisimple subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is one of the following types:

Type I: $\mathfrak{h}$ has an abelian ideal of dimension greater than one;
Type II: $\mathfrak{h}$ has nilradical isomorphic to a Heisenberg algebra.

We will show that if $\mathfrak{h}$ is an irreducible maximal nonsemisimple subalgebra of $\mathfrak{sl}(V)$ of type I then $\mathfrak{h}$ is of the form described by Lemma 11.1.

To do this we quote some results from [Blo73], which must be interpreted for Lie subalgebras of $\mathfrak{sl}(V)$. We will consider the case that $A$ is a finite-dimensional associative $k$-algebra. Recall, in the notation of [Blo73] that a $k$-linear mapping $t : V \to V$ of a module $V$ over $A$ is a (k-linear) differential transformation if there is an endomorphism $d$ of $A$ such that $ta - at = d(a)$ considered as elements of $\text{End}_k(V)$. If $V$ is faithful it follows that $d$ is a derivation of $A$. The module $V$ is differentiably irreducible or d.i. if $AV \neq 0$ and there is a set of differential transformations such that no proper submodule $W$ of $V$ is stable under this set.

In the case that $A$ is a nonzero Lie-subalgebra of $\mathfrak{gl}(V)$, we see that $V$ is d.i. for the associative algebra $\langle A \rangle$ if the Lie normaliser $n_{\mathfrak{gl}(V)}(A)$ acts irreducibly on $V$.

For any $k$-algebra $S$ we denote by $S_{[n]}$ the algebra $S[T_1, \ldots, T_n]/(T_1^n, \ldots, T_n^n)$. Then $O_n = k_{[n]}$ and $S_{[n]} = S \otimes O_n$. If $V$ is an $S$-module, we denote by $V_{[n]} = V \otimes O_n$ the corresponding $S_{[n]}$-module.

**Theorem 11.3** ([Blo73, Thm. 1]). Suppose $A$ is a finite-dimensional associative $k$-algebra and that $V$ is a faithful and d.i. $A$-module. Then either $A$ is simple or $A$ has a simple subalgebra $S$ and $sV$ has a submodule $U$ such that $A = S_{[r]}$ and $V = U_{[r]}$ for some $r > 0$.

**Lemma 11.4.** Let $A$ be a Lie subalgebra of $\mathfrak{g} = \mathfrak{gl}(V)$ generating $\langle A \rangle$ as an associative subalgebra. Then

$$n_\mathfrak{g}(A) \subset n_\mathfrak{g}(\langle A \rangle).$$

**Proof.** Take $g \in n_\mathfrak{g}(A)$ and $a, b \in A$. We have $[g, ab] = gab - agb + a(gb) - abg = [g, a]b + a[g, b]$, so that $n_\mathfrak{g}(A) \subset n_\mathfrak{g}(A^2)$. Similarly $n_\mathfrak{g}(A^n) \subset n_\mathfrak{g}(A^{n+1})$, proving the claim by induction. □

**Corollary 11.5.** If $A$ is a solvable Lie subalgebra of $\mathfrak{gl}(V)$ such that its Lie normaliser $n_{\mathfrak{gl}(V)}(A)$ acts irreducibly on $V$ then the associative subalgebra $\langle A \rangle$ generated by $A$ has a subalgebra $S$ and $V$ has an $S$-submodule $U$ such that $\langle A \rangle = S_{[r]}$ and $V = U_{[r]}$.

In particular if $A$ is abelian, then $\langle A \rangle$ is an abelian subalgebra isomorphic to $k_{[r]} = O_r$ and $V \cong U \otimes O_r$ with $\langle A \rangle$ acting as $1 \otimes O_r$ on $V$.

**Proof.** The first part follows from Theorem 11.3. It is clear that if $A$ is abelian then $\langle A \rangle$ is. In this case, a simple $k$-subalgebra $S$ must equal $k$ (as it is a finitely generated algebra that is a field over $k$, and $k$ is algebraically closed). □

**Lemma 11.6.** Suppose $V = V_1 \otimes V_2$ is a vector space and $\mathfrak{h} \leq \mathfrak{gl}(V_2)$ is a subalgebra. Then

$$n_{\mathfrak{gl}(V)}(1 \otimes \mathfrak{h}) = \mathfrak{gl}(V_1) \otimes \mathfrak{c}_{\mathfrak{gl}(V_2)}(\mathfrak{h}) + 1 \otimes n_{\mathfrak{gl}(V_2)}(\mathfrak{h}).$$

**Proof.** Let $1, \varphi_1, \ldots, \varphi_n$ be a basis of $\mathfrak{gl}(V_1)$, and suppose that

$$\varphi = 1 \otimes \psi + \sum_{i=1}^n \varphi_i \otimes \psi_i \in n_{\mathfrak{gl}(V)}(1 \otimes \mathfrak{h}),$$

where

$$\psi = \sum_{i=1}^n \varphi_i \otimes \psi_i \in n_{\mathfrak{gl}(V_2)}(\mathfrak{h}).$$
where $\psi, \psi_i \in \mathfrak{gl}(V_2)$. Then for any $1 \otimes g \in 1 \otimes \mathfrak{h}$ we have
\[
[\varphi, 1 \otimes g] = 1 \otimes [\psi, g] + \sum_{i=1}^{n} \varphi_i \otimes [\psi_i, g].
\]
Comparing coefficients with an element in $1 \otimes \mathfrak{h}$ yields $[\psi, g] \in \mathfrak{h}$ and $[\psi_i, g] = 0$ for all $g \in \mathfrak{h}$, $i = 1, \ldots, n$. This proves that
\[
\mathfrak{n}_{\mathfrak{gl}(V)}(1 \otimes \mathfrak{h}) \subseteq \mathfrak{gl}(V_1) \otimes \mathfrak{gl}(V_2)(\mathfrak{h}) + 1 \otimes \mathfrak{n}_{\mathfrak{gl}(V_2)}(\mathfrak{h}).
\]
The reverse inclusion is clear. \hfill \Box

**Lemma 11.7.** We have
\[
\mathcal{O}_{\mathfrak{gl}(O)}(O_r) = O_r,
\]
\[
\mathfrak{n}_{\mathfrak{gl}(O)}(O_r) = W_r + O_r.
\]

*Proof.* Follows from the identity
\[
[\varphi, x_i] = \sum_{a,b} \alpha_{a,b} x^a \partial^b - \varepsilon_i,
\]
for $\varphi = \sum_{a,b} \alpha_{a,b} x^a \partial^b \in \mathfrak{gl}(O_r)$. \hfill \Box

**Proposition 11.8.** Suppose $\dim V = lp^m$ with $(l, p) = 1$ and $m \geq 1$. Then each maximal irreducible non-semisimple subalgebra $\mathfrak{m}$ of Type I of $\mathfrak{sl}(V)$ has the form
\[
\mathfrak{m} = 1 \otimes W_r + \mathfrak{gl}(U_r) \otimes O_r
\]
where $V = U_r \otimes O_r$ for $1 \leq r \leq m$. In particular, $\mathfrak{m}$ is a maximal subalgebra of $\mathfrak{sl}(V)$.

*Proof.* Suppose $A$ is an abelian ideal in the irreducible subalgebra $\mathfrak{m}$ of dimension at least 2. By Corollary 11.5, there is $r \geq 1$ such that $V = U_r \otimes O_r$ for some vector space $U_r$ and $(A)$ acts as $1 \otimes O_r$. By Lemmas 11.6 and 11.7, we have $\mathfrak{n}_{\mathfrak{gl}(V)}(1 \otimes O_r) = 1 \otimes W_r + \mathfrak{gl}(U_r) \otimes O_r$, which is an irreducible non-semisimple subalgebra in $\mathfrak{sl}(V)$. As $\mathfrak{m}$ is maximal this forces $\mathfrak{m} = \mathfrak{n}_{\mathfrak{gl}(V)}(1 \otimes O_r)$ by Lemma 11.4. By Lemma 11.1, $\mathfrak{m}$ is a maximal subalgebra. \hfill \Box

**Theorem 11.9 ( [Pan83, Prop. 3]).** Let $\mathfrak{h}$ be an irreducible subalgebra of $\mathfrak{sl}(V)$ of type II. Let $\mathfrak{r} \leq \mathfrak{h}$ be its nilradical, a Heisenberg algebra. Then
\[
V = U \otimes W,
\]
where $W$ is an irreducible $\mathfrak{r}$-representation and $\mathfrak{r}$ acts as $1 \otimes \mathfrak{r}$.

The following Lemma is proved in [Str04, p. 149]:

**Lemma 11.10.** Let $\mathcal{H}_m \leq \mathfrak{sl}(W)$ be an irreducible Heisenberg algebra with $\dim \mathcal{H}_m = 2m + 1$, $m \geq 0$. Then $\dim W = p^m$. Moreover, let $A = \langle z, x_i : 1 \leq i \leq m \rangle$. Then $V$ is induced from $A$ and of the form
\[
V \cong u(\mathcal{H}_m, \chi) \otimes u(A, \chi | A) k_\lambda = \bigoplus_{0 \leq a_i \leq p - 1} k y_1^{a_1} \cdot y_m^{a_m} \otimes 1,
\]
for some $\chi \in \mathcal{H}_m^*$ and some eigenvalue function $\lambda$, in the terminology of loc. cit.

Write $\mathcal{H}_m = V_0 \oplus k \cdot z$, with $V_0 = \langle x_i, y_i : 1 \leq i \leq m \rangle$. Let $B$ be the alternating form on $V_0$ defined by $B(v, w)z = [v, w]$. Then $\mathfrak{sp}_{2m} = \mathfrak{sp}(V_0, B)$ acts naturally on $\mathcal{H}_m$ via $\langle x, (v, \alpha z) = (xv, 0) \rangle$ and we may form the semidirect product $\mathfrak{sp}_{2m} \ast \mathcal{H}_m$. 29
Lemma 11.11. In $\mathfrak{gl}(O_m)$, we have $\mathfrak{n}_{\mathfrak{gl}(O_m)}(\mathcal{H}_m) \cong \mathfrak{sp}_{2m} + \mathcal{H}_m$, where $\mathcal{H}_m = \langle 1, x_i, \partial_i : 1 \leq i \leq m \rangle$.

Proof. Recall that $\mathfrak{gl}(O_m)$ has basis $\{x^a \partial^b : a, b \in \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p\}$, where $x^{(a_1, \ldots, a_m)} = x_1^{a_1} \cdots x_m^{a_m}$ and similarly for $\partial^b$. We identify $\mathcal{H}_m = \langle 1, x_i, \partial_i : 1 \leq i \leq m \rangle$ as a Heisenberg algebra, with only nontrivial products given by $[\partial_i, x_i] = 1$.

Now let $\varphi = \sum_{a,b} \alpha_{a,b} x^a \partial^b \in \mathfrak{gl}(O_m)$. Using the identities
\[
[\varphi, x_i] = \sum_{a,b} \alpha_{a,b} x^a \partial^b - \varepsilon_i, \\
[\varphi, \partial_i] = -\sum_{a,b} \alpha_{a,b} x^a - \varepsilon_i \partial^b,
\]
we find that $\mathfrak{n}_{\mathfrak{gl}(O_m)}(\mathcal{H}_m)$ is generated by $\mathcal{H}_m$ together with the $2m^2 + m$-dimensional space $M'$ of monomials of degree 2 in $x_1, \ldots, x_m, \partial_1, \ldots, \partial_m$.

Let $M$ be the subspace with same generator as $M'$, except with $x_i \partial_i + 1/2$ instead of $x_i \partial_i$. This is a complement to $\mathcal{H}_m$ in $\mathfrak{n}_{\mathfrak{gl}(O_m)}(\mathcal{H}_m)$ which can be seen to form a subalgebra.

Moreover, it is symplectic in its adjoint action on $V_0 = \langle x_i, \partial_i : 1 \leq i \leq m \rangle$. The claim follows. □

Proposition 11.12. Let $\mathfrak{h} \leq \mathfrak{sl}(V)$ be an irreducible maximal nonsemisimple subalgebra of Type II. Then either the image of $\mathfrak{h}$ in $\mathfrak{psl}(V)$ is semisimple, or $\dim V = p^m$ for some $m \geq 1$, and $\mathfrak{h}$ is isomorphic to $\mathfrak{sp}_{2m} + \mathcal{H}_m$.

Proof. Suppose that the image of $\mathfrak{h}$ in $\mathfrak{psl}(V)$ is not semisimple. Let $\mathcal{H}_m \leq \mathfrak{h}$ be the Heisenberg nilradical of $\mathfrak{h}$.

By Theorem 11.9 and Lemma 11.10, we may write $V = U \otimes W$ for some irreducible $\mathcal{H}_m$-representation $W$ of dimension $p^m$. If $m = 0$, then $W$ is one-dimensional and $\mathcal{H}_m \cong k$ acts by scalar multiplication on $V$, thus $\mathcal{H}_m = Z(\mathfrak{sl}(V))$. Now if $\mathfrak{h}/\mathcal{H}_m$ had an abelian ideal $I$, then $\mathfrak{h}$ would have an abelian ideal $I + \mathcal{H}_m$ contained in the nilradical. Thus $I = 0$ and $\mathfrak{h}/\mathcal{H}_m \leq \mathfrak{psl}(V)$ is semisimple, a contradiction.

Hence $m \geq 1$ and $\dim W > 1$. We have $\mathfrak{c}_{\mathfrak{gl}(W)}(\mathcal{H}_m) = k$, and by Lemma 11.6
\[
\mathfrak{h} \leq \mathfrak{n}_{\mathfrak{gl}(V)}(1 \otimes \mathcal{H}_m) = \mathfrak{gl}(U) \otimes 1 + 1 \otimes \mathfrak{n}_{\mathfrak{gl}(W)}(\mathcal{H}_m) \subseteq \mathfrak{gl}(U) \otimes 1 + 1 \otimes \mathfrak{sl}(W) \subseteq \mathfrak{sl}(V).
\]

Now $\mathfrak{gl}(U) \otimes 1 + 1 \otimes \mathfrak{sl}(W)$ is a proper subalgebra of $\mathfrak{gl}(V)$ unless $U$ is one-dimensional. Moreover, it is non-semisimple with one-dimensional central nilradical and hence with semisimple image in $\mathfrak{psl}(V)$. Since $\mathfrak{h}$ is maximal non-semisimple, we would have $\mathfrak{h} = \mathfrak{gl}(U) \otimes 1 + 1 \otimes \mathfrak{sl}(W)$, contradicting our assumption.

Hence $U \cong k$, $V \cong W$ and $\mathfrak{h} \leq \mathfrak{n}_{\mathfrak{gl}(W)}(\mathcal{H}_m) = \mathfrak{sp}_{2m} + \mathcal{H}_m$ by Lemma 11.11. □

We conclude the

Proof of Theorem B: The case $p$ very good was treated in the last section, showing that one of (i) or (ii) holds, or $G = SL(V)$ with $\dim V = p^m$. Now a maximal non-semisimple subalgebra is either parabolic, giving case (iii)(a) of the Theorem, or is irreducible. Then it is either of Type I, in which case it is described by Proposition 11.8 or it is Type II, in which case it is described by Proposition 11.12. This concludes the proof of Theorem B.
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