Harmonic Lattice Approximation and Density Correlations in the Calogero-Sutherland Model

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Abstract

For a one-dimensional model in which the two-body interactions are long-range and strong, the system almost crystallizes. The harmonic modes of such a lattice can be used to compute the ground state wave function and the dynamical density-density correlations. We use this method to calculate the density correlations in the Calogero-Sutherland model. We show numerically that the correlations obtained are quite accurate even if the coupling is not very large. Our method is considerably simpler than the ones used to derive the exact results, and yields expressions for the correlations which are easily plotted. Our equal-time correlations can be expanded in powers of the inverse coupling; we show that the two leading order terms agree with the exact results which are known for integer values of the coupling.

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I. INTRODUCTION

The Calogero-Sutherland-Moser model (CSM) [1-3] has received much attention recently due to its connection with a variety of interesting problems. Some examples are random matrix theory [4], quantum spin chains with long-range interactions [5, 6], Luttinger liquids [7], Gaussian conformal field theories [8], edge states in a quantum Hall system [9], generalized exclusion statistics [10-16], and nonlinear internal waves in a stratified fluid [17].

Although the quantum mechanical energy spectrum of the CSM has been known for a long time [1-3], its collective properties [18-22] and dynamical density-density correlations [23-28] are not yet expressed in an easily accessible form. The correlations are best understood for three particular values of the coupling parameter $\lambda = 1/2$, 1 and 2 which are closely related to random matrix theory [23, 24]. An impressive amount of information is also available for all integer [25-27] and rational [28] values of $\lambda$. However, even though these are all exact results, the mathematical expressions are not very convenient for plotting, say, the space-time dependence of the correlations, unless $\lambda$ (or its inverse) is equal to a fairly small integer.

It therefore seems useful to pursue simple and approximate methods for obtaining the correlations for general values of $\lambda$. In Sec. II, we present a method for doing this which is actually exact in the limit of large $\lambda$. In that limit, the particles almost freeze into a lattice [3, 8]; the harmonic modes about such a lattice may then be used to compute the ground state energy, wave function and correlations. We use this idea to obtain the first two terms in an expansion in $1/\lambda$ for the ground state energy and the asymptotic
expression for the equal-time density-density correlation; these agree with the known exact results [25]. We show that the equal-time correlations given by our method agree reasonably well with the exact result for all distances even if $\lambda$ is not very large. The dynamical correlations are also obtained with almost equal ease by our method, and are explicitly given for any space-time point. We discuss the low-temperature specific heat, and make a suggestion for improving the correlations by using a slightly modified phonon dispersion. In Sec. III, we briefly consider a more general model in which the two-body interaction decays as the power $\alpha$ of the distance. We find that the system behaves as a Luttinger liquid if $\alpha > 1$, and as a Wigner crystal if $\alpha < 1$; the Coulomb problem with $\alpha = 1$ lies on the border between the two possibilities [29].

II. CALOGERO-SUTHERLAND-MOSER MODEL

We will consider the form of the CSM in which particles on an infinite line interact pairwise through an inverse-square potential. The model may also be defined on a circle with periodic boundary condition [4]; the two versions of the model have identical physical properties in the thermodynamic limit in which the number of particles $N$ and the length $L$ of the line (or circle) are simultaneously taken to infinity keeping the particle density $\rho_0 = N/L$ fixed. The Hamiltonian for particles on a line is given by

$$H = \sum_n \frac{p_n^2}{2m} + \frac{\hbar^2 \lambda (\lambda - 1)}{m} \sum_{l<n} \frac{1}{(x_l - x_n)^2},$$

where the dimensionless coupling $\lambda \geq 0$. To make the problem well-defined quantum mechanically, we have to add the condition that the wave functions
go to zero as $|x_l - x_n|^\lambda$ whenever two particles $l$ and $n$ approach each other. For $\lambda = 0$ and 1, the model describes free bosons and free fermions respectively. Since the two-body potential is singular enough to prevent particles from crossing each other, we can choose the wave functions to be either symmetric (bosonic) or antisymmetric (fermionic). The energy spectrum is of course the same in the two descriptions. Two of the exactly known results for this model are as follows [2, 18]. In the thermodynamic limit, the ground state energy $E_0$ is given by

$$ E_0 N = \frac{\pi^2 \hbar^2 \lambda^2 \rho_0^2}{6m} . $$

The sound velocity $v$ is given by the group velocity of long-wavelength modes (wavelengths much bigger than the average particle spacing $1/\rho_0$),

$$ v = \frac{\pi \hbar \lambda \rho_0}{m} . $$

### A. Ground State Energy and Wave Function

Let us now consider the limit in which the coupling $\lambda \rightarrow \infty$. Then the potential energy term in (1) dominates over the kinetic term. The potential energy is minimized if the particles sit at the sites of a lattice, so that the mean position of the $n$-th particle is $\langle x_n \rangle = n/\rho_0$ [3, 18]. The total energy at this order in $\lambda$ is then given by

$$ E_0 N = \frac{\pi^2 \hbar^2 \lambda (\lambda - 1) \rho_0^2}{6m} . $$

To the next order in $\lambda$, we have to consider the harmonic oscillations about the mean positions. If we denote the fluctuation by $\eta_n = x_n - n/\rho_0$, the
Hamiltonian for the phonons is obtained by Taylor expanding the potential energy in (1) to quadratic order. Thus

$$H_{ph} = \sum_n \frac{p_n^2}{2m} + \frac{3\hbar^2 \lambda (\lambda - 1) \rho_0^4}{m} \sum_{l<n} \frac{(\eta_l - \eta_n)^2}{(l-n)^4},$$

where we have ignored higher order anharmonic terms. We now find the dispersion relation $\omega_q$ for phonons with wavenumber $q$ to be

$$\omega_q^2 = \frac{12 \hbar^2 \lambda (\lambda - 1) \rho_0^4}{m^2} \sum_{n=1}^\infty \frac{1 - \cos(qn/\rho_0)}{n^4},$$

$$= \frac{\hbar^2 \lambda (\lambda - 1)}{m^2} \left( \pi \rho_0 |q| - \frac{q^2}{2} \right)^2, \quad |q| \leq \pi \rho_0.$$

(Note that $q$ can vary from $-\pi \rho_0$ to $\pi \rho_0$ in units of $2\pi/L$, so that the total number of modes is $N$). The sound velocity is given by $v = (\partial \omega_q/\partial q)_{q=0} = \pi \hbar \rho_0 \sqrt{\lambda (\lambda - 1)}/m$. We see that this agrees with Eq. (3) to leading order in $\lambda$; we will therefore simply replace $\lambda (\lambda - 1)$ by $\lambda^2$ in the expression (6) for $\omega_q^2$.

Namely, we use the exactly known sound velocity to correct the coefficient of our phonon dispersion. Henceforth we take

$$\omega_q = \frac{\hbar \lambda}{m} \left( \pi \rho_0 |q| - \frac{q^2}{2} \right).$$

The zero point energy of the phonons is given by

$$\sum_q \frac{\hbar \omega_q}{2} = L \int_{-\pi \rho_0}^{\pi \rho_0} dq \frac{\hbar \omega_q}{2}.$$

When this is added to (4), we precisely recover Eq. (2); thus the harmonic lattice approximation gives the ground state energy correctly up to order $\lambda^2$ and $\lambda$. To get the next order term in the energy, i.e. order 1, we would have to use the original dispersion (3) as well as consider some of the anharmonic terms beyond $H_{ph}$; we will not pursue this here.
We now use the harmonic approximation to write down the ground state wave function following Ref. [30]. The unnormalized wave function for a collection of decoupled phonons is given by
\[
\Psi_0 = \exp \left( -\frac{m}{2\hbar} \sum_q \omega_q \eta_q \eta_{-q} \right),
\]
\[
\eta_q = \sum_{n=-\infty}^{\infty} \eta_n e^{i q n / \rho_0}.
\] (9)

Using the dispersion (7), we find that
\[
\Psi_0 = \exp \left( -\frac{\lambda \rho_0^2}{2} \sum_{l<n} \frac{(\eta_l - \eta_n)^2}{(l - n)^2} \right). \tag{10}
\]

We see that this is equivalent to the exact ground state wave function of the Hamiltonian (1)
\[
\Psi_0 = \prod_{l<n} \left| \rho_0(x_l - x_n) \right|^\lambda, \tag{11}
\]
if we Taylor expand
\[
\log(|\rho_0|x_l - x_n|) = \log |l - n| + \frac{\rho_0(\eta_l - \eta_n)}{l - n} - \frac{\rho_0^2(\eta_l - \eta_n)^2}{2(l - n)^2} + \cdots. \tag{12}
\]

Before ending this subsection, let us write down the second quantized form for the Heisenberg operators \(x_n(t)\),
\[
x_n(t) = \frac{n}{\rho_0} + \int_{-\pi \rho_0}^{\pi \rho_0} dq \frac{2\pi}{f_q} \left[ a_q e^{i(\frac{2\pi}{\rho_0} - \omega_q) t} + a_q^\dagger e^{-i(\frac{2\pi}{\rho_0} - \omega_q) t} \right],
\]
\[
f_q = \left( \frac{\hbar}{2m\rho_0 \omega_q} \right)^{1/2}. \tag{13}
\]

Here
\[
[a_q, a_{q'}^\dagger] = 2\pi \delta(q - q'). \tag{14}
\]

We can verify that \(p_n(t) = m dx_n(t)/dt\) satisfies the equal-time commutation relation \([x_l, p_n] = i\hbar \delta_{ln}\).
B. Dynamical Correlation Function

We now use the harmonic lattice approximation to derive the dynamical density-density correlations defined as

\[ g(x, t) = \langle 0 | \rho(x, t) \rho(0, 0) | 0 \rangle , \]

\[ \rho(x, t) = \sum_n \delta(x - x_n(t)) , \]  

(15)

where \( x_n(t) \) is the Heisenberg operator given in (13). To compute this, we write the \( \delta \)-functions in (15) as

\[ \delta(x - x_n(t)) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iq(x - x_n(t))} . \]  

(16)

We then evaluate expectation values of the form \( \langle 0 | \exp(i(qx_n(t) - q'x(0))) | 0 \rangle \) using the Baker-Campbell-Hausdorff formula,

\[ e^{A + B} = e^A e^{[B, A]/2} \]  

(17)

if \([B, A] \) commutes with both \( A \) and \( B \). Due to the logarithmic divergence at small momenta of an integral of the form \( \int dk/\omega_k \), we find that this expectation value vanishes unless \( q = q' \). We finally obtain

\[ g(x, t) = \rho_0 \int_{-\infty}^{\infty} \frac{dq}{2\pi} \sum_{n=\infty}^{\infty} \exp \left[ \frac{iq}{\rho_0} (x - n) - \frac{q^2}{\rho_0^2} F(n, t) \right] , \]  

(18)

where

\[ F(n, t) = \int_0^{\pi \rho_0} \frac{dk}{\pi} \frac{\hbar \rho_0}{2m\omega_k} \left[ 1 - e^{-i\omega_k t} \cos(\frac{kn}{\rho_0}) \right] . \]  

(19)

It can be shown that the real part of \( F(n, t) \) is always positive, unless both \( n \) and \( t \) are zero. We can therefore perform the \( q \) integration in (18) to get

\[ g(x, t) = \rho_0^2 \sum_{n=-\infty}^{\infty} \left( \frac{1}{4\pi F(n, t)} \right)^{1/2} \exp \left[ -\frac{(x\rho_0 - n)^2}{4F(n, t)} \right] . \]  

(20)
The expansion in (18) can be understood in a physically intuitive way as follows. Each lattice point \( n \) contributes a Gaussian (correctly normalized to unity) to \( g(x, t) \); the squared width of the Gaussian is given by the mean square fluctuation

\[
\frac{2F(n, t)}{\rho_0^2} = \langle 0 | \left( x_n(t) - x_0(0) - \frac{n}{\rho_0} \right)^2 | 0 \rangle .
\] (21)

Let us now study the equal-time correlations \( g(x, 0) \equiv g(x) \) in some detail. From (13), we see that \( F(n, 0) \) is purely real, and \( F(0, 0) = 0 \). We therefore obtain

\[
g(x) = \rho_0 \delta(x) + \rho_0^2 \sum_{n \neq 0} \left( \frac{1}{4\pi F(n, 0)} \right)^{1/2} \exp \left[ -\frac{(x\rho_0 - n)^2}{4F(n, 0)} \right] ,
\] (22)

where

\[
\lambda F(n, 0) = \frac{1}{2\pi^2} \int_0^\pi dy \frac{1 - \cos (yn)}{y - y^2/2\pi} = \frac{1}{2\pi^2} \int_0^{2\pi} dy \frac{1 - \cos (yn)}{y} .
\] (23)

This gives us (31)

\[
\lambda F(n, 0) = \frac{1}{2\pi^2} \left[ \log (2\pi e\gamma n) - \text{Ci} (2\pi n) \right] ,
\] (24)

where \( \gamma \approx 0.57722 \) is Euler’s constant, and \( \text{Ci} \) denotes the cosine integral. For large integer \( n \), (24) has an asymptotic expansion beginning as

\[
\lambda F(n, 0) = \frac{1}{2\pi^2} \log (Cn) + \frac{1}{8\pi^4 n^2} + O\left(\frac{1}{n^4}\right) ,
\] (25)

where

\[
C = 2\pi e\gamma \approx 11.191 .
\] (26)

The expansion in (25) converges extremely rapidly; for \( n = 1 \) and 2, the difference between (24) and the first two terms in (25) is only about 0.1%
and 0.01% respectively. For numerical purposes, therefore, we will use (23) to compute \( g(x) \) from (22).

We show a plot of \( g(x)/\rho_0^2 \) vs. \( x\rho_0 \) for \( \lambda = 2 \) in Fig. 1; the solid line denotes the exact result [2], while the dotted line shows our lattice approximation (22). The agreement appears to be reasonable even though \( \lambda \) is not very large; for instance, the root mean square fluctuation for nearest neighbor particles is given by (21) and (25) to be \( \sqrt{0.247/\lambda/\rho_0} \), which is as large as 35% of the lattice spacing \( 1/\rho_0 \) for \( \lambda = 2 \). In Fig. 2, we show the lattice approximation for \( g(x)/\rho_0^2 \) for \( \lambda = 4 \). It is clear that the system shows an increasing tendency to crystallize as \( \lambda \) increases. We should note that the \( \delta \)-function at the origin has not been shown in Figs. 1 and 2; thus the \( g(x) \) shown in the figures integrates to one hole,

\[
\lim_{n \to \infty} 2 \int_0^{n/2} dx \ g(x) = \rho_0 \ (n - 1) . \tag{27}
\]

We also note that in the lattice approximation, \( g(x) \) does not quite vanish at \( x = 0 \) as it should in an exact calculation; however \( g(0) \) does become small rapidly as \( \lambda \) increases.

We can Fourier transform \( g(x) \) to obtain the static form factor

\[
S(q) = \int_{-\infty}^{\infty} dx \ e^{-iqx} \ g(x) . \tag{28}
\]

Since \( g(x) \to 1 \) for large \( x \), \( S(q) \) has a \( \delta \)-function at the origin. Hence it is easier to compute

\[
S(q) - 2\pi \rho_0^2 \ \delta(q) = \rho_0 + \rho_0^2 \int_0^{\infty} dx \ \cos (qx) \left[ \sum_{n \neq 0} \exp \left[ -\frac{(x\rho_0 - n)^2}{4F(n,0)} \right] - 2 \right] . \tag{29}
\]
We show a plot of $S(q)/\rho_0$ vs. $q/\rho_0$ in Fig. 3 for $\lambda = 2$ (the solid and dotted lines again denote the exact result and the lattice approximation respectively), and in Fig. 4 for $\lambda = 4$. We will comment on the divergences at $q/\rho_0 = 2\pi$ in subsection C. (The small wiggles in Fig. 4 are due to numerical inaccuracies; the integral in (29) converges very slowly if $\lambda$ is large. The power-law which governs the convergence will also be derived in the next subsection).

We now consider the dynamical correlations $g(x,t)$. It is convenient to introduce a dimensionless time variable

$$\tilde{t} = vt\rho_0 .$$

Then

$$\lambda F(n, t) = \frac{1}{2\pi^2} \int_0^\pi dy \left( 1 - \cos (yn) \exp \left[ -i\tilde{t}(y - y^2/2\pi) \right] \right).$$

In the lattice approximation, plotting $g(x,t)$ is almost as easy as plotting $g(x)$. In Figs. 5 and 6, we show $g(x,t)/\rho_2$ vs. $x\rho_0$ for $\tilde{t} = 0.2$ and $\tilde{t} = 2$ respectively, both for $\lambda = 2$. The solid and dotted lines show the real and imaginary parts respectively. Note that the $\delta$-function at the origin which was not shown for $g(x)$ in Fig. 1 has now spread out and become visible at finite time in Figs. 5 and 6; hence there is no hole in the real part of $g(x,t)$.

For a small value of time $\tilde{t}$ and for $x$ close to the origin (Fig. 5), the $\delta$-function at the origin spreads out as a free particle with dispersion $\omega_q = \hbar q^2/2m$; this explains the large oscillations in both the real and imaginary parts. For a larger value of time (Fig. 6), and also for larger values of $x$ for any time, the particle feels the harmonic force of the lattice. Hence the behavior of $g(x,t)$
is different from the spreading of a free particle; in particular, the large oscillations get damped out. Finally, note that for \( \tilde{t} = 2 \), that \( g(x, t) \) can differ appreciably from \( g(x, 0) \) only in the region \( x\rho_0 = 0 \) to 2 since phonons can only propagate that far in that time. This is most strikingly visible in Fig. 5 in the imaginary part of \( g(x, t) \) which vanishes rapidly beyond \( x\rho_0 = 2 \).

Given \( g(x, t) \), we can Fourier transform the \( x \) coordinate to obtain a function which we denote by

\[
S(q, t) = \rho_0 \sum_{n=-\infty}^{\infty} \exp \left[ - \frac{iqn}{\rho_0} - \frac{q^2}{2\rho_0^2} F(n, t) \right]. \tag{32}
\]

Thus the function \( S(q) \) given in (28) is equal to \( S(q, t = 0) \). If we now Fourier transform (32) in time, we obtain the dynamical structure factor

\[
S(q, \omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} S(q, t). \tag{33}
\]

This quantity can be written in terms of all the excited states \(|\nu\rangle\) of the system,

\[
S(q, \omega) = \rho_0 \sum_{\nu \neq 0} 2\pi \delta(\omega - E_\nu + E_0) \ |\langle \nu | \rho(q) | 0 \rangle|^2,
\]

\[
\rho(q) = \int dx \ \rho(x) e^{-iqx} = \sum_n e^{-iqx_n}. \tag{34}
\]

It is useful to consider the moments of \( S(q, \omega) \) \[23\]

\[
I_n(q) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^n S(q, \omega). \tag{35}
\]

These can be numerically obtained in a straightforward way from (32), since \( I_n(q) = i^n (\partial^n S(q, t)/\partial t^n)_{t=0} \). We have already considered \( I_0(q) = S(q) \) above. Since \( F(0, 0) = 0 \) and \( (\partial F(n, t)/\partial t)_{t=0} = i\delta_{n,0} \hbar \rho_0^2/2m \), we find that the first
moment
\[ I_1(q) = \frac{\hbar q^2 \rho_0}{2m} \]  
(36)
is independent of \( \lambda \). This is an exact result following from the velocity independence of the two-body interactions, and it is called the \( f \)-sum rule.

The second moment \( I_2(q) \) is shown in Fig. 6 for \( \lambda = 2 \); we have actually plotted \( I_2(q)m^2/(h^2 \rho_0^2 q^3) \) which is dimensionless and goes to a constant at \( q = 0 \).

At this point, we observe the phenomenon of saturation by sound modes for small momenta \([23]\). Namely, for \( q \ll 2\pi \rho_0 \), \( S(q, \omega) \) is dominated by values of \( \omega \) close to the phonon energy \( \omega_q \) given by (7); thus \( I_n(q)/I_0(q) \) approaches \( \omega^n_q \) as \( q \to 0 \). Hence the \( y \)-coordinate in Fig. 6 should approach \( \pi \lambda/2 \) at \( q = 0 \) which it does. For the same reason, the curves in Figs. 3 and 4 are linear near the origin with a slope of \( 1/(2\pi \lambda) \), since \( S(q)\rho_0 \) approaches \( \hbar q^2/(2m \omega_q) \).

The saturation by phonons at low energies also has a bearing on low-temperature thermodynamic properties of the CSM like the specific heat and pressure. We use the dispersion (7) to compute the free energy per unit length \( f \) taking the phonons to have zero chemical potential. Thus
\[ \beta f = \int_{-\infty}^{\infty} \frac{dq}{2\pi} \ln \left( 1 - e^{-\beta \hbar \omega_q} \right), \]  
(37)
where \( \beta = 1/k_B T \). After evaluating this, we obtain the specific heat per unit length \( C_V = -T \partial^2 f/\partial T^2 \) to second order in \( T \). We find
\[ C_V = \frac{\pi k_B^2 T}{3h\nu} + \frac{6\zeta(3)\lambda}{\pi} \frac{k_B^3 T^2}{h\nu^3}, \]
\[ \zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3}. \]  
(38)
On comparing this with the exact result in Ref. [15], we see that we have the
correct coefficient for the term of order $T$; this is related to having the right
phonon velocity at $q = 0$. However we have a coefficient $\lambda$ instead of $\lambda - 1$
for the term of order $T^2$ in (38). A different approach to the CSM based on
collective field theory actually gives a dispersion [22]

$$\omega_q = \frac{\hbar}{m} \left( \pi \rho_0 \lambda |q| - (\lambda - 1) \frac{q^2}{2} \right), \quad (39)$$

which leads to the correct coefficient $\lambda - 1$ for the $T^2$ term. All this suggests
that we may get more accurate low-momentum or long-distance correlations
if we use the dispersion (39) instead of (7). Let us therefore use (39) to
calculate $F(n, 0)$ and then $g(x, 0)$ following Eqs. (19) and (20). The result is
plotted and compared to the exact results for $\lambda = 2$ in Fig. 8. On contrasting
this with Fig. 1 which is based on the dispersion (7), we see an improved
agreement for large values of $x$, and, somewhat unexpectedly, for very small
$x$ also. This analysis again shows the close connection between the phonon
dispersion and the correlations in the lattice formalism.

C. Large-$x$ Asymptotics for $g(x)$

We will now analytically evaluate $g(x)$ in (22) for large values of $x$, and
show that the order 1 and $1/\lambda$ terms agree with the exact results given in
Ref. [23]. We use the Poisson resummation formula

$$\sum_{n=-\infty}^{\infty} \phi(n) = \sum_{p=-\infty}^{\infty} \int_{-\infty}^{\infty} dy \ e^{i2\pi yp} \phi(y). \quad (40)$$

Given a function $\phi(n)$ defined for integer values, there are clearly many ways
to interpolate it to a function of a real variable $\phi(y)$. Eq. (40) says that
different interpolations on the right hand side (assuming that the integrals and sum converge) must give the same answer for the left hand side. We will use this freedom to interpolate $F(n, 0)$ to $F(y, 0)$ taking the function in (25), rather than the one in (24). This is a convenient choice because the function in (25) is smooth, unlike (24) which contains oscillations. Thus, we use (40) with
\[
\phi(y) = \left( \frac{\lambda}{4\pi f(y)} \right)^{1/2} \exp \left[ - \frac{\lambda(y - x_0^2)}{4f(y)} \right],
\]
where $f(y)$ and its first two derivatives have the form
\[
\begin{align*}
f(y) &= \frac{1}{2\pi^2} \log (Cy) + \frac{1}{8\pi^4}y^2 + O\left(\frac{1}{y^4}\right), \\
f'(y) &= \frac{1}{2\pi^2}y - \frac{1}{4\pi^4}y^3 + O\left(\frac{1}{y^5}\right), \\
f''(y) &= -\frac{1}{2\pi^2}y^2 + \frac{3}{4\pi^4}y^4 + O\left(\frac{1}{y^6}\right).
\end{align*}
\]
We now Taylor expand $\phi(y)$ around $y = x_0^2$, and do the integrals over $y$ on the right hand side of (40). Upto order $1/\lambda$, this yields the general expression
\[
\frac{g(x)}{\rho_0^2} = 1 + \frac{f''(x)}{\lambda} + 2 \sum_{p=1}^{\infty} \exp \left( - \frac{4\pi^2 p^2 f(x)}{\lambda} \right) \left[ \left( 1 + \frac{f''(x)}{\lambda} \right) \cos(2\pi px) \right.
\]
\[
\left. - \frac{4\pi pf'(x)}{\lambda} \sin(2\pi px) \right].
\]
(43)
We now use the explicit forms for $f(y)$ and its derivatives in Eq. (42) to obtain, upto order $1/x^2$,
\[
\frac{g(x)}{\rho_0^2} = 1 - \frac{1}{2\lambda \pi^2 x^2} + 2 \sum_{p=1}^{\infty} \left( Cxe^{1/4\pi^2 x^2} \right)^{2p^2/\lambda} \left[ \left( 1 - \frac{1}{2\lambda \pi^2 x^2} \right) \cos(2\pi px) \right.
\]
\[
\left. + \frac{1}{\lambda \pi^2 x^2} \sin(2\pi px) \right].
\]
\begin{equation}
\frac{2p}{\lambda \pi x} \sin(2\pi px) \right].
\end{equation}

(On the right hand sides of Eqs. (43-44), we have written $x$ instead of $x\rho_0$ for convenience). Eq. (44) agrees with the asymptotic expression given in Ref. [25] unto $O(\frac{1}{\lambda}, \frac{1}{x^2})$. We should make two comments here. First, the sum over $p$ runs up to infinity in (44), while it only goes up to $p = \lambda$ in the exact result for integer $\lambda$; however, the difference between the two is exponentially small for large $\lambda$. Secondly, by going up to order $1/\lambda^2$, we find terms involving $\log x$ which are not present in the exact result. This shows that the harmonic lattice approximation fails at order $1/\lambda^2$, and that anharmonic terms must be included to recover the correct asymptotics at that order.

The oscillatory terms in (44) imply that the Fourier transform $S(q)$ diverges at momenta $q = 2\pi p\rho_0$ for all nonzero integers $p$ satisfying $p^2 \leq \lambda/2$. The divergence has a power-law form $|q - 2\pi p\rho_0|^{(2p^2/\lambda) - 1}$ if $p^2 < \lambda/2$, and is logarithmic if $p^2 = \lambda/2$. We indeed see a logarithmic divergence in Fig. 3 for $\lambda = 2$, and a square-root divergence in Fig. 4 for $\lambda = 4$, both at $q = 2\pi \rho_0$.

The asymptotic analysis can be extended to the dynamical correlations $g(x, t)$. Let us define

\begin{equation}
s = \rho_0 \left\{ x^2 - v^2 t^2 \right\}^{1/2},
\end{equation}

and assume that $s >> 1$. Then we can show that the leading term in (19) is given by

\begin{equation}
F(n, t) = \frac{1}{2\pi^2 \lambda} \log (Cs).
\end{equation}
Following the above arguments, we find that
\[ g(x, t) = \rho_0^2 \left[ 1 + 2 \sum_{p=1}^{\infty} (Cs)^{-2p^2/\lambda} \cos(2\pi xp\rho_0) \right] \]  
(47)
to leading order in the $1/\lambda$ expansion.

The various power-law exponents derived above are in accordance with arguments based on the idea of Luttinger liquids \[7\] and conformal field theory \[8\]. In general, if the phonons in a one-dimensional system have a low-momentum dispersion of the form $\omega_q \simeq vq$, the arguments presented here show that the prefactors of the oscillatory terms in (43) must decay with the exponents $2\pi\hbar\rho_0 p^2/(mv)$. In the next section, we will study the conditions under which a class of long-range interactions can lead to Luttinger liquid-like behavior.

### III. OTHER LONG-RANGE MODELS

We consider a one-dimensional system in which the two-body interactions decay as the power $\alpha$ of the distance. Using the lattice approximation, we find that the long-distance correlations are quite different depending on whether $\alpha > 1$, $\alpha = 1$ (the Coulomb problem), or $0 < \alpha < 1$ \[29\]. This directly follows from the phonon dispersion which has the form
\[ \omega_q^2 \sim \sum_{n=1}^{\infty} \frac{1 - \cos(qn)}{n^{\alpha+2}}. \]  
(48)
(We set the density $\rho_0 = 1$ in this section). For low momenta, $q \ll \pi$, we see that
\[ \omega_q \sim |q| \text{ if } \alpha > 1, \]
\[ \omega_q \sim |q| \sqrt{\log |q|} \text{ if } \alpha = 1, \]
The asymptotics of the correlations \( g(x) \) are governed by the long-distance behavior of

\[
F(n) \sim \int_0^{\pi} \frac{dq}{\omega_q} \left[ 1 - \cos(qn) \right].
\]

This diverges as \( \log n \) if \( \alpha > 1 \) and as \( \sqrt{\log n} \) if \( \alpha = 1 \), but it does not diverge if \( \alpha < 1 \). Correspondingly, the prefactors of the oscillatory terms in \( g(x) \) decay as powers of \( x \) if \( \alpha > 1 \) (Luttinger liquid) and as exponentials of \( -\sqrt{\log x} \) if \( \alpha = 1 \), but they do not decay if \( \alpha < 1 \) (Wigner crystal) \[29\]. It is amusing that the Coulomb system, which is physically the most interesting of the three cases, lies exactly in between the Luttinger liquid and Wigner crystal scenarios.

\[ \omega_q \sim |q|^{(\alpha + 1)/2} \quad \text{if} \quad 0 < \alpha < 1. \tag{49} \]

\[ \text{IV. DISCUSSION} \]

We have seen that the harmonic lattice approximation is an useful technique from which many properties of the CSM can be derived readily without having to solve the Schrödinger equation (1) in great detail. We should however point out two problems which are either not clear or are beyond the reach of this approach.

a) We have been unable to derive the one-particle Green’s function, either bosonic or fermionic, from the phonon wave functions. This is because the symmetry (or antisymmetry) of the particle wave function does not enter the lattice formulation in a natural way; the lattice site and the particle labels are distinct from each other. This may be purely a technical difficulty.

(b) It would be useful to systematically go to higher orders in the \( 1/\lambda \) expa-
sion by starting from the original phonon dispersion (6) and including the anharmonic terms perturbatively. This may provide a justification for replacing the dispersion (6) by either (7) or (39), and may enable us to extend our results to smaller values of $\lambda$.

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Note Added:

After writing this paper, we learnt that much of this work has been published earlier [32, 33]; we thank P. J. Forrester for pointing this out. Ref. [32] has gone much further by presenting expressions for $g(x, t)$ at finite temperature, for the singularities in $S(q, \omega)$, and for the equal-time Green’s function.

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Figure Captions

1. Equal-time correlation $g(x)/\rho_0^2$ for $\lambda = 2$. The solid and dotted lines denote the exact result and the lattice approximation respectively.
2. Correlation $g(x)/\rho_0^2$ for $\lambda = 4$ obtained by the lattice approximation.
3. Static form factor $S(q)/\rho_0$ for $\lambda = 2$. The solid and dotted lines denote the exact result and the lattice approximation respectively.
4. $S(q)/\rho_0$ for $\lambda = 4$ obtained by the lattice approximation.
6. Dynamical correlation $g(x,t)/\rho_0^2$ for $\lambda = 2$ and $vt\rho_0 = 0.2$ obtained by the lattice approximation. The solid and dotted lines denote the real and imaginary parts respectively.
6. Correlation $g(x,t)/\rho_0^2$ for $\lambda = 2$ and $vt\rho_0 = 2$ obtained by the lattice approximation. The solid and dotted lines denote the real and imaginary parts respectively.
7. Second moment of dynamical structure factor $I_2(q)m^2/(\hbar^2\rho_0^2q^3)$ for $\lambda = 2$ obtained by the lattice approximation.
8. Equal-time correlation $g(x)/\rho_0^2$ for $\lambda = 2$. The solid and dotted lines denote the exact result and the lattice approximation using the modified dispersion relation (39) respectively.
Fig. 1

\[ \lambda = 2 \]

\[ \frac{g(x)}{\rho_0^2} \]

\[ x\rho_0 \]
Fig. 2

$\frac{g(x)}{\rho_0^2}$ vs. $x\rho_0$

$\lambda = 4$
Fig. 3
Fig. 4
Fig. 5

\[ \frac{g(x,t)}{\rho_0^2} \] vs. \( x\rho_0 \)

\( \lambda = 2 \quad vt\rho_0 = 0.2 \)
\[
\frac{g(x,t)}{\rho_0^2}
\]

\[
\begin{align*}
\lambda &= 2 \\
vt\rho_0 &= 2
\end{align*}
\]

Fig. 6
Fig. 7

\[ I_2(q) \frac{m^2}{(h q)^2 q^3} \]

\[ \lambda = 2 \]
