REMARKS ON THE SCHUR–HOWE–SERGEEV DUALITY

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Abstract. We establish a new Howe duality between a pair of two queer Lie superalgebras \((q(m), q(n))\). This gives a representation theoretic interpretation of a well-known combinatorial identity for Schur \(Q\)-functions. We further establish the equivalence between this new Howe duality and the Schur–Sergeev duality between \(q(n)\) and a central extension \(\tilde{H}_k\) of the hyperoctahedral group \(H_k\). We show that the zero-weight space of a \(q(n)\)-module with highest weight \(\lambda\) given by a strict partition of \(n\) is an irreducible module over the finite group \(H_n\) parameterized by \(\lambda\). We also discuss some consequences of this Howe duality.

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1. Introduction

This is a sequel to \([2]\), in which we studied the Howe duality between two general linear Lie superalgebras and other closely related multiplicity-free actions of a general linear Lie superalgebra, which generalize and unify several classical results, cf. Howe \([3, 4]\).

In this Letter, we construct a new Howe duality involving the queer Lie superalgebra \(q(n)\). The queer Lie superalgebra \(q(n)\) (cf., e.g., \([7, 9]\)) can be regarded as a true super analog of the general linear Lie algebra. We show that there is a mutually centralizing action of \(q(m)\) and \(q(n)\) on the symmetric algebra of \(C^{mn|mn}\). A multiplicity-free decomposition of the \(q(m) \times q(n)\)-module \(S(C^{mn|mn})\) is explicitly obtained. To achieve this, we use a remarkable duality, due to Sergeev \([11]\) between \(q(n)\) and a finite group \(\tilde{H}_k\), which generalizes the celebrated Schur duality. Here \(\tilde{H}_k\) is a central extension of the hyperoctahedral group.

On the other hand, we show the \((q(m), q(n))\) Howe duality can be used to re-derive Sergeev duality as well. We also show that the zero-weight space of a \(q(n)\)-module with highest weight \(\lambda\), given by a strict partition of \(n\), is an irreducible module over the finite group \(H_n\) parameterized by \(\lambda\). All these are very much analogous to the classical picture in the general linear Lie algebra case, cf. \([4]\).

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†In this Letter, we will freely suppress the term super. So in case when a superspace is involved, the terms symmetric, commute etc. mean supersymmetric, supercommute etc unless otherwise specified.
As is well known, there is no unique notion of a Weyl (super)group for a Lie superalgebra. Our results suggest that $\tilde{H}_n$ may be regarded as a Weyl supergroup for $q(n)$ in an appropriate sense.

It has been known [11] that the characters of the irreducible $q(n)$-modules under consideration in this letter are essentially the Schur $Q$-functions (cf. [8]). We remark that the difficult question of finding the character of a general finite-dimensional irreducible $q(n)$-module has been solved recently by Penkov and Serganova [10]. The $(q(m), q(n))$-duality can now be interpreted as a representation theoretic realization of the following well known identity for Schur $Q$-functions (cf. Macdonald [8]):

$$\prod_{i,j=1}^{\infty} \frac{1+x_i y_j}{1-x_i y_j} = \sum_{\lambda} 2^{-l(\lambda)} Q_\lambda(x) Q_\lambda(y),$$

where $x = (x_1, x_2, \ldots)$, $y = (y_1, y_2, \ldots)$ and the summation is over all strict partitions.

In the case when $n = 1$, the $(q(m), q(n))$ Howe duality essentially tells us that the $k$-th symmetric algebra of the natural representation of $q(n)$ is irreducible of highest weight $(k, 0, \ldots, 0)$. When $m = n$, the $(q(m), q(n))$ Howe duality implies the existence of a distinguished basis for the center of the universal enveloping algebra of $q(n)$ (also compare [12]) parameterized by strict partitions of length not exceeding $n$. It is a very interesting question to give a more precise description of this basis and its relation with symmetric functions as $m$ goes to infinity. The results of [2] and the present work also suggest that there are other Howe dual pairs involving various Lie superalgebras (cf. [7]) which deserve further study.

The plan of the letter is as follows. In Section 2 we recall some representation-theoretic background of the queer Lie superalgebras with emphasis on Schur–Sergeev duality. Section 3 is devoted to establishing the $(q(m), q(n))$-duality and to the study of its consequences.

## 2. The Schur–Sergeev Duality

Let $\mathbb{C}^{m|n}$ denote the complex vector superspace of dimension $m|n$, and $\mathfrak{gl}(m|n)$ the Lie superalgebra of linear transformations of $\mathbb{C}^{m|n}$ (see, e.g., [4]). Choosing a homogeneous basis of $\mathbb{C}^{m|n}$ we may regard $\mathfrak{gl}(m|n)$ as the space of complex $(m + n) \times (m + n)$ matrices. In the case when $m = n$ consider an odd automorphism $P : \mathbb{C}^{m|m} \to \mathbb{C}^{m|m}$ with $P^2 = -1$. The linear transformations of $\mathfrak{gl}(m|m)$ preserving $P$ is a subalgebra of $\mathfrak{gl}(m|m)$, denoted by $q(m)$. We have $q(m) = q(m)_0 \oplus q(m)_1$, with $q(m)_0$ isomorphic to the general linear Lie algebra $\mathfrak{gl}(m)$ and $q(m)_1$ isomorphic to the adjoint module of $\mathfrak{gl}(m)$ (cf. [2], [3]). Choosing $P$ to be the $2m \times 2m$ matrix

$$
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
$$

(2.1)
with $I$ denoting the identity $m \times m$ matrix, we may identify $q(m)$ inside $\mathfrak{gl}(m|m)$ with the space of complex $2m \times 2m$ matrices of the form:

\[
(2.2) \quad \begin{pmatrix} A & B \\ B & A \end{pmatrix},
\]

where $A$ and $B$ are arbitrary complex $m \times m$ matrices. Of course the even elements of $q(m)$ are those for which $B = 0$, while the odd elements are those for which $A = 0$.

We will below recall some aspects of finite-dimensional irreducible representations of $q(m)$ (cf. [4]). Let $\mathfrak{B}$ be the Borel subalgebra consisting of those matrices in (2.2) with $A$ and $B$ upper triangular. Furthermore, let $\mathfrak{N}$ be the nilpotent subalgebra of $\mathfrak{B}$ consisting of those matrices in (2.2) with $A$ and $B$ strictly upper triangular. The Cartan subalgebra $\mathfrak{h}$ is the subalgebra of $q(m)$ consisting of those matrices in (2.2) with $A$ and $B$ diagonal. Taking a linear form $\lambda$ on $\mathfrak{h}_0$ we may consider the symmetric bilinear form on $\mathfrak{h}_1$ defined by $(a|b)_{\lambda} := \lambda([a,b])$, $a, b \in \mathfrak{h}_1$.

Now if $\mathfrak{h}' \subset \mathfrak{h}$ is a maximal isotropic subspace with respect to this bilinear form, we may extend $\lambda$ to a one-dimensional representation of $\mathfrak{h}_0 + \mathfrak{h}'$ in a trivial way. Inducing from this we obtain an irreducible $\mathfrak{h}$-module. This module has an odd automorphism if and only if the dimension of the quotient space $\mathfrak{h}_1/\ker(\cdot|\cdot)_{\lambda}$ is odd. We now can extend this irreducible $\mathfrak{h}$-module to an irreducible $\mathfrak{B}$-module by letting $\mathfrak{N}$ act trivially. This way one obtains all finite-dimensional irreducible $\mathfrak{B}$-modules. Inducing further we obtain the Verma module of $q(m)$ associated to the linear form $\lambda \in \mathfrak{h}_0^*$ and, thus, an irreducible $q(m)$-module of highest weight $\lambda$ by dividing by its maximal proper submodule. We note that an odd automorphism of an irreducible $\mathfrak{h}$-module descends to an odd automorphism of this irreducible quotient. Thus one may associate to an $m$-tuple $\lambda = (a_1, \ldots, a_m)$ of complex numbers an irreducible representation $U^\lambda_m$ of highest weight $\lambda$, which is finite-dimensional if and only if $a_i - a_{i+1} \in \mathbb{Z}_+$ and $a_i = a_{i+1}$ implies that $a_i = 0$ for all $i = 1, \ldots, m - 1$, cf. [4].

Recall a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of length $l$ is called strict if $\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0$. We will identify $\lambda$ with $(\lambda_1, \lambda_2, \ldots, \lambda_l, 0, \ldots, 0)$ by adding zeros in the end and denote by $|\lambda|$ the sum $\lambda_1 + \cdots + \lambda_l$. We see that a partition $\lambda$ may be regarded as a highest weight of a finite-dimensional irreducible $q(m)$-module if and only if $\lambda$ is strict with length $l(\lambda)$ not exceeding $m$. Furthermore, we have [4]:

\[
(2.3) \quad \dim \left( \text{Hom}_{q(m)}(U^\lambda_m, U^{\mu}_m) \right) = \delta_{\lambda\mu}2^{\delta(l(\lambda))},
\]

where $\text{Hom}$ is to be understood in the $\mathbb{Z}_2$-graded sense, and the number $\delta(l(\lambda))$ is 0 for $l(\lambda)$ even and 1 otherwise. More precisely, $\text{Hom}_{q(m)}(U^\lambda_m, U^{\mu}_m)$ is isomorphic to $\mathbb{C}$ in the case when $l(\lambda)$ is even and it is isomorphic to a Clifford superalgebra in one odd variable in the case $l(\lambda)$ is odd. We also recall that the character
The symmetric group $S_k$ acts on $\Pi_k$ via permutations of the elements $a_1, \ldots, a_k$ and we may thus form the semidirect product $S_k \rtimes \Pi_k$. We denote this semidirect product (again a finite group) by $\widetilde{H}_k$ which is naturally $\mathbb{Z}_2$-graded by putting $p(a_i) = 1$, $p(z) = 0$, and $p(\sigma) = 0$ for $\sigma \in S_k$. Thus $\widetilde{H}_k$ is a finite supergroup in the sense of [3], i.e. this is a group with a subgroup of index two, whose elements we call even and by definition all other elements are odd (see Remark 2.1). In this letter we will only concern about the $\mathbb{Z}_2$-graded spin modules of $\widetilde{H}_k$ (i.e. those $\mathbb{Z}_2$-graded modules on which $z$ acts as $-1$) which are equivalently modules over the group superalgebra $\mathfrak{H}_k = \mathbb{C}[\widetilde{H}_k]/\langle z = -1 \rangle$.

**Remark 2.1.** For a supergroup $G$ by $G$-module homomorphisms we will mean linear maps commuting with the respective group action. Explicitly this means that for a homogeneous linear map $f : V \to W$, where $V$ and $W$ are supermodules, to be a $G$-homomorphism we must have $f(g \cdot v) = (-1)^{p(g)p(v)}g \cdot f(v)$, for all $v \in V$. Recall [3] that given two modules $V$ and $W$ over a supergroup $G$, the space $\text{Hom}_G(V, W)$ is naturally a $G$-module via the action $(g \cdot T)(v) := (-1)^{p(g)p(T)}gT(g^{-1}v)$, which amounts to giving $V \otimes W$ a $G$-module structure via the action $g \cdot (v \otimes w) = (-1)^{p(g)p(v)+p(w)}) gw \otimes gw$. Thus in what follows the action on a tensor product of two $G$-modules will always be given this $G$-module structure. By $G$-invariants inside $V \otimes W$ we shall always mean the usual invariants, i.e. $(V \otimes W)^G = \{ \gamma \in V \otimes W \mid g \cdot \gamma = \gamma, \forall g \in G \}$.

According to [11] and [3], the ($\mathbb{Z}_2$-graded) irreducible spin modules of $\widetilde{H}_k$ are also parameterized by strict partitions. For strict partitions $\lambda$ and $\mu$ let $T^\lambda_k$ and $T^\mu_k$ denote the corresponding irreducible spin modules over $\widetilde{H}_k$. We have ([11], [6])

$$\dim(\text{Hom}_{\widetilde{H}_k}(T^\lambda_k, T^\mu_k)) = \delta_{\lambda\mu} \delta(l(\lambda)), \tag{2.4}$$

where $\text{Hom}$ is again to be understood in the $\mathbb{Z}_2$-graded sense. Furthermore it is known (cf. [4]) that the character value of $T^\lambda_k$ is real and thus $T^\lambda_k$ is self-contragredient.

Let us now consider the natural action of $q(m)$ on $\mathbb{C}^{m|m}$. We may form the $k$-fold tensor product $\bigotimes^k \mathbb{C}^{m|m}$, on which $q(m)$ acts naturally. In addition we have an action of the finite supergroup $\widetilde{H}_k$: the symmetric group in $k$ letters acts on $\bigotimes^k \mathbb{C}^{m|m}$ by permutations of the tensor factors with appropriate signs (corresponding to the permutations of odd elements in $\mathbb{C}^{m|m}$). However, we also

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2In [3, 6] these are called negative supermodules.
have an action of \( a_i \) on \( \bigotimes^k \mathbb{C}^{m|m} \) by means of exchanging the parity of \( i \)-th copy of \( \mathbb{C}^{m|m} \) via the odd automorphism of \( \mathbb{C}^{m|m} \) given by the matrix \( P \) of (2.1). More explicitly, \( a_i \) transforms the vector \( v_1 \otimes \ldots \otimes v_{i-1} \otimes v_i \otimes \ldots \otimes v_k \) in \( \bigotimes^k \mathbb{C}^{m|m} \) into \((-1)^{p(v_1)+\ldots+p(v_{i-1})} v_1 \otimes \ldots \otimes v_{i-1} \otimes P(v_i) \otimes \ldots \otimes v_k \).

The following remarkable theorem is due to Sergeev [11], which will be referred to as (Schur–)Sergeev duality throughout the Letter. We refer the reader to [8] for definitions and properties of the Schur \( Q \)-functions \( Q_\lambda \).

**Theorem 2.1 (Sergeev).** The actions of \( q(m) \) and \( \tilde{H}_k \) on the space \( \bigotimes^k \mathbb{C}^{m|m} \) commute and \( \bigotimes^k \mathbb{C}^{m|m} \) is completely reducible over \( q(m) \times \tilde{H}_k \). Explicitly we have

\[
\bigotimes^k \mathbb{C}^{m|m} \cong \sum_\lambda 2^{-\delta(l(\lambda))} U_{\lambda}^m \otimes T_{\lambda}^k,
\]

where \( \lambda \) is summed over all strict partitions with \( |\lambda| = k \) and \( l(\lambda) \leq m \). Furthermore, the character \( \text{ch} U_{\lambda}^m \) is given by

\[
2^{\frac{\delta(l(\lambda)) - l(\lambda)}{2}} Q_\lambda(x).
\]

**Remark 2.2.** The expression \( 2^{-\delta(l(\lambda))} U_{\lambda}^m \otimes T_{\lambda}^k \) above has the following meaning. Suppose \( A \) and \( B \) are two superalgebras and \( V_A \) and \( V_B \) are irreducible modules over \( A \) and \( B \) such that \( \text{Hom}_A(V_A, V_A) \) and \( \text{Hom}_B(V_B, V_B) \) are both isomorphic to the Clifford superalgebra in one odd variable. We remark that in the language of [11] (respectively [3]) this is to say that \( V_A \) and \( V_B \) are irreducible, but not absolutely irreducible (respectively are of \( Q \)-type). It is known that \( V_A \otimes V_B \) as a module over \( A \otimes B \) is not irreducible, but decomposes into a direct sum of two isomorphic copies (via an odd isomorphism) of the same irreducible representation (see, e.g., [11, 3]). In our particular setting when \( l(\lambda) \) is odd both \( T_{\lambda}^k \) and \( U_{\lambda}^m \) are such modules by (2.3) and (2.4). So in this case we mean to take one copy inside their tensor product.

3. **The \((q(m),q(n))\) Howe Duality**

Recall that \( \Pi_k \) acts on \( \bigotimes^k \mathbb{C}^{m|m} \) and \( \bigotimes^k \mathbb{C}^{n|n} \) and hence the diagonal subgroup \( \Delta \Pi_k \subset \Pi_k \times \Pi_k \) acts on their tensor product

\[
\bigotimes^k \mathbb{C}^{m|m} \otimes \bigotimes^k \mathbb{C}^{n|n} \cong \bigotimes^k (\mathbb{C}^{m|m} \otimes \mathbb{C}^{n|n})
\]

(see Remark [2.3]). So does the symmetric group \( S_k \). This gives rise to the diagonal action of \( \tilde{H}_k \).

**Lemma 3.1.** As a module over \( q(m) \times q(n) \), we have

\[
\bigotimes^k (\mathbb{C}^{m|m} \otimes \mathbb{C}^{n|n}) \Delta(\tilde{H}_k) \cong S^k (\mathbb{C}^{mn|mn}),
\]
where \((\cdot)^{\Delta(H_k)}\) denotes the space \(\Delta(H_k)\)-invariants.

**Proof.** Recall that \(P\) is the odd automorphism given by the matrix \((2.1)\). Consider first \((C^m|_m \otimes C^n|_n)^{\Delta P}\), that is, the \(\Delta(P)(= P \times P)\)-invariants in \(C^m|_m \otimes C^n|_n\).

Letting \(v_m \in C^m|_m\) and \(v_n \in C^n|_n\), it is clear that this space consists of elements of the form \(v_m \otimes v_n + P(v_m) \otimes P(v_n)\) and \(v_m \otimes P(v_n) + P(v_m) \otimes v_n\), and hence is isomorphic to \(C^{mn}|_{mn}\). Therefore, since \(\Delta \Pi_k\) is a subgroup (since all elements are now even) of \(\Delta(S_k \rtimes \Pi_k)\) generated by the \(k\) copies of \(\Delta(P)\)'s, we have

\[
\bigotimes_{i=1}^k (C^m|_m \otimes C^n|_n)^{\Delta(S_k \rtimes \Pi_k)} \blacktriangleright \sim (\bigotimes_{i=1}^k (C^m|_m \otimes C^n|_n)^{\Delta \Pi_k})^{\Delta S_k} \blacktriangleright \sim (\bigotimes_{i=1}^k (C^m|_m \otimes C^n|_n)^{\Delta P})^{\Delta S_k} \blacktriangleright \sim (\bigotimes_{i=1}^k C^{mn}|_{mn})^{S_k} \blacktriangleright \sim S^k(C^{mn}|_{mn}).
\]

Let \(x_1^i, \ldots, x_m^i, \xi_1^i, \ldots, \xi_m^i\), for \(i = 1, \ldots, n\) denote the standard coordinates of \(C^{mn}|_{mn}\). We may then identify \(S(C^{mn}|_{mn})\) with the polynomial algebra generated by \(x_i\) and \(\xi_j\). Introduce the following first order differential operators:

\[
A_{pq} = \sum_{i=1}^m (x_i^p \frac{\partial}{\partial x_i^q} + \xi_i^p \frac{\partial}{\partial \xi_i^q}), \quad 1 \leq p, q \leq n,
\]

\[
B_{pq} = \sum_{i=1}^m (x_i^p \frac{\partial}{\partial \xi_i^q} - \xi_i^p \frac{\partial}{\partial x_i^q}), \quad 1 \leq p, q \leq n,
\]

\[
A_{pq} = \sum_{j=1}^n (x_j^p \frac{\partial}{\partial x_j^q} + \xi_j^p \frac{\partial}{\partial \xi_j^q}), \quad 1 \leq p, q \leq m,
\]

\[
B_{pq} = \sum_{j=1}^n (x_j^p \frac{\partial}{\partial \xi_j^q} + \xi_j^p \frac{\partial}{\partial x_j^q}), \quad 1 \leq p, q \leq m.
\]

The following lemma can be proved directly.

**Lemma 3.2.** The operators \(A_{pq}\) and \(B_{pq}\), for \(1 \leq p, q \leq m\), form a copy of \(q(m)\), while \(A_{pq}\) and \(B_{pq}\), for \(1 \leq p, q \leq n\), form a copy of \(q(n)\). Furthermore, they define a commuting action of \(q(m)\) and \(q(n)\) in \(S(C^{mn}|_{mn})\).
Theorem 3.1. The action of \( q(m) \times q(n) \) on \( S(\mathbb{C}^{mn|mn}) \) is multiplicity-free. More precisely we have the following decomposition:

\[
S^k(\mathbb{C}^{mn|mn}) \cong \sum_{\lambda} 2^{-\delta(l(\lambda))} U^\lambda_m \otimes U^\lambda_n,
\]

where \( \lambda \) is summed over all strict partitions of length not exceeding \( \min(m, n) \).

Proof. By Schur–Sergeev duality (Theorem 2.1) we have

\[
\bigotimes^k \mathbb{C}^{m|m} \cong \sum_{\lambda} 2^{-\delta(l(\lambda))} U^\lambda_m \otimes T^\lambda_k,
\]
as \( q(m) \times \tilde{H}_k \) module, where the summation is over strict partitions \( \lambda \) with length \( l(\lambda) \leq m \). Therefore combined with Lemma 3.1 this gives us

\[
S^k(\mathbb{C}^{mn|mn}) \cong \sum_{\lambda} 2^{-\delta(l(\lambda))} U^\lambda_m \otimes \left( \sum_{\mu} \delta(l(\mu)) \right) U^\mu_m \otimes T^\lambda_k \triangle(S_k \ltimes \Pi_k)
\]

Now by (2.4) and the fact that irreducible \( \tilde{H}_k(= S_k \ltimes \Pi_k) \)-modules are self-contragredient we have

\[
S^k(\mathbb{C}^{mn|mn}) \cong \sum_{\lambda} 2^{-\delta(l(\lambda))} U^\lambda_m \otimes U^\lambda_m \otimes \left( \sum_{\mu} \delta(l(\mu)) \right) U^\mu_m \otimes T^\mu_k \triangle(S_k \ltimes \Pi_k)
\]

where the summation is over all strict partitions of length \( l(\lambda) \leq \min(m, n) \). This identity of Schur \( Q \)-functions is well known (see e.g. [8]) and Theorem 3.1 provides a representation-theoretic interpretation of it.
The next corollary is immediate from Theorem 3.1.

**Corollary 3.1.** The images of the universal enveloping algebras of $q(m)$ and $q(n)$ in the endomorphism algebra of $S^k(C^m|m)$ are mutual centralizers.

When $n = 1$ the $(q(m), q(n))$-duality reads

$$S^k(C^m|m) \cong \frac{1}{2}(U_m^{(k)} \otimes U_1^{(k)}),$$

where $(k)$ above denotes the one-part partition. Since $U_1^{(k)}$ is a two-dimensional module, the right-hand side is exactly $U_m^{(k)}$ as a $q(m)$-module. Hence we have established the following.

**Proposition 3.1.** The $k$-th symmetric tensor of $C^m|m$ is the irreducible $q(m)$-module $U_m^{(k)}$, associated to the one-part partition $(k)$.

**Remark 3.1.** Another proof of Proposition 3.1 goes as follows. Recalling that $S^k(C^m|n) = \bigoplus_{i=0}^k S_i C^m \otimes \Lambda^{k-i} C^n$, we have therefore $\text{ch} S^k(C^m|m) = \sum_{i=0}^k h_i e_{k-i} = q_k$ (see [3] pp. 261 for notation), which coincides with $Q_{(k)}$. Here $h_i$ and $e_i$ denote the $i$-th complete and respectively elementary symmetric functions. But $\text{ch} U_m^{(k)} = Q_{(k)}$ by [4] and so $S^k(C^m|m) \cong U_m^{(k)}$.

**Remark 3.2.** Given a left module $M$ over a Lie superalgebra $\mathfrak{g}$, we can make $M$ into a right module by defining $m \cdot x := (-1)^{\rho(m)p(x)} xm$, for $m \in M$ and $x \in \mathfrak{g}$. Now if $M$ in addition has a left module structure over another Lie superalgebra $\mathfrak{g}'$ such that the action of $\mathfrak{g}$ and $\mathfrak{g}'$ commute, then the so induced right action of $\mathfrak{g}$ on $M$ will not commute with the left action of $\mathfrak{g}$ (in the super-sense), but rather they will commute with each other in the usual sense. Thus the induced right action of $q(n)$ above has the result that it commutes with the left action of $q(m)$ in the usual sense.

Let $GL(n)$ be the Lie group whose Lie algebra is the even part of the Lie superalgebra $q(n)$ and let $A_n$ denote its diagonal torus. Given a $q(n)$-module (or a $GL(n)$-module) $U$, we call the subspace $U^{A_n, \text{det}}$ inside $U$, which transforms under the action of $A_n$ by the determinant character, the zero-weight space of $U$.

**Theorem 3.2.** The $(q(m), q(n))$ Howe duality implies the Sergeev duality.

**Theorem 3.3.** Given a strict partition $\lambda$ of $n$, the zero weight space of $U_n^\lambda$ admits a natural action of the finite group $\tilde{H}_n$, and it is isomorphic to the irreducible module $T_n^\lambda$.

**Proof.** We will establish Theorem 3.2 and Theorem 3.3 together. The argument we will present follows closely the one used in [4] to derive the Schur duality from the $(\mathfrak{sl}(m), \mathfrak{sl}(n))$-duality.
The \((q(m), q(n))\)-duality says that \(S(\mathbb{C}^{m|n} \otimes \mathbb{C}^n) \cong \sum_\lambda 2^{-\delta(l(\lambda))} U^\lambda_m \otimes U^\lambda_n\), where the summation is over all strict \(\lambda\) with \(l(\lambda) \leq \min(m, n)\). Observe that the space \(\bigotimes^n \mathbb{C}^{m|n} \cong \mathbb{C}^{m|n} \otimes \mathbb{C}^n\) may be identified with the zero-weight space \(S(\mathbb{C}^{m|n} \otimes \mathbb{C}^n)_{A_n, \text{det}}\). Putting these together we have

\[
\bigotimes^n \mathbb{C}^{m|n} \cong \sum_\lambda 2^{-\delta(l(\lambda))} U^\lambda_m \otimes (U^\lambda_n)_{A_n, \text{det}},
\]

where \(\lambda\) runs over all strict partitions \(n\). So to recover Theorem 3.3 it suffices to show that \((U^\lambda_n)_{A_n, \text{det}}\) is isomorphic to the irreducible \(\tilde{H}_n\)-module \(T^\lambda_n\), which is the contents of Theorem 3.3.

First note that the normalizer of \(A_n\) acts on the \(A_n\)-weight spaces by permutation. Since the determinant character is invariant under permutation of weights, we see that \((U^\lambda_n)_{A_n, \text{det}}\) is invariant under the action of \(S_n\). Now the induced right action (see Remark 3.2) of the operators \(B^{ii}\) (see (3.1)) when acting on \((U^\lambda_n)_{A_n, \text{det}}\) satisfy the commutation relations of the \(a_i\)'s in \(\tilde{H}_n\), for \(i = 1, \ldots, n\). This action of the \(B^{ii}\)’s combined with the action of \(S_n\) then gives a right action of \(\tilde{H}_n\) on \((U^\lambda_n)_{A_n, \text{det}}\).

Set \(n = m\) in the remainder of the proof. (However, it is convenient to continue making the distinction between \(n\) and \(m\).) Consider

\[
(\bigotimes^n \mathbb{C}^{m|m})_{A_m, \text{det}} \cong \sum_\lambda 2^{-\delta(l(\lambda))} (U^\lambda_m)_{A_m, \text{det}} \otimes (U^\lambda_n)_{A_n, \text{det}}.
\]

Recall that \(x_1^i, \ldots, x_m^i, \xi_1^i, \ldots, \xi_m^i\) \((i = 1, \ldots, n)\) are the standard coordinates of \(\mathbb{C}^{m|n} \cong \mathbb{C}^{m|m} \otimes \mathbb{C}^n\). It is not difficult to see that the space \((\bigotimes^n \mathbb{C}^{m|m})_{A_m, \text{det}}\) inside \(S(\mathbb{C}^{m|m} \otimes \mathbb{C}^n)_{A_n, \text{det}}\), which is \(S(\mathbb{C}^{m|n} \otimes \mathbb{C}^n)_{A_m \times A_n, \text{det} \times \text{det}}\), may be identified with the space spanned by vectors of the form \(v_1^{\sigma_1} v_2^{\sigma_2} \ldots v_n^{\sigma_n}\), where \(v\) denotes either \(x\) or \(\xi\) and \(\sigma\) is a permutation of \(\{1, \ldots, n\}\). Thus this space is in bijection with the space \(\mathcal{B}_n = \mathbb{C}[\tilde{H}_n]/\langle z = -1 \rangle\). On this space the normalizer of \(A_m\) acts, and so we obtain an action of the symmetric group \(S_m\). We also have an action of \(B_{jj}\) (see (3.2)), which gives rise to the action of \(a_j\) in \(\tilde{H}_m\), for \(j = 1, \ldots, m\). This action combined with that of \(S_m\) gives a left action of \(\tilde{H}_m\) on \((U^\lambda_m)_{A_m, \text{det}}\). Furthermore the induced right action of \(\tilde{H}_n\) above and this action commute in the usual sense according to Remark 3.2.

Also our left action of \(S_n\) permutes the upper indices of the vector \(v_1^{\sigma_1} v_2^{\sigma_2} \ldots v_n^{\sigma_n}\), whereas the left action of the \(a_i\)'s changes the parity of \(v_i^{\sigma_i}\). Thus this is the left regular action of \(\tilde{H}_n\). Our right action of \(S_n\), on the other hand, permutes the lower indices of \(v_1^{\sigma_1} v_2^{\sigma_2} \ldots v_n^{\sigma_n}\), whereas our right action of \(a_i\) changes the parity of \(v_i^{\sigma_i^{-1}}\). Thus our right action is the right regular representation of \(\tilde{H}_n\). From the general theory of finite supergroup \(3.3\), the left-hand side of (3.4), which is isomorphic to \(\mathcal{B}_n\) under the left and right actions of \(\tilde{H}_n\), is equal to the summation
\[ \sum_{\lambda} 2^{-\delta(l(\lambda))} T_{n}^{\lambda} \otimes T_{n}^{\lambda} \] over all strict partitions of \( n \). Decomposing \((U_{m}^{\lambda})^{A_{m}, \text{det}}\) in the right-hand side of (3.4) into a direct sum of irreducible \( \widehat{H}_{m} \)-modules, we see that this is only possible when each \((U_{m}^{\lambda})^{A_{m}, \text{det}}\) itself is irreducible as a \( \widehat{H}_{m} \)-module. Comparing with (3.3), we see that this irreducible module is isomorphic to \( T_{n}^{\lambda} \).

In [12] Sergeev computed the center of the universal enveloping algebra \( U(q(m)) \) of the Lie superalgebra \( q(m) \). We will see that a different description of it can also be obtained from the \((q(m), q(n))\)-duality.

Recall that for a finite-dimensional Lie superalgebra \( g \) we have \( S(g)^{g} \cong Z(U(g)) \) as a \( g \)-module (cf. [12]), the center of the universal enveloping algebra of \( g \). Now replacing \( q(n) \) and \( \mathbb{C}^{n|m} \) in the \((q(m), q(n))\)-duality by \( q(m) \) and \( \mathbb{C}^{m|m*} \), the action contragredient to the natural action of \( q(m) \), we have

\[ S^{k}(\mathbb{C}^{m|m} \otimes \mathbb{C}^{m|m*})^{\Pi_{k}} \cong 2^{-\delta(l(\lambda))} \sum_{\lambda} U_{m}^{\lambda} \otimes U_{m}^{\lambda*}. \]

However, from our earlier description of \((\mathbb{C}^{m|m} \otimes \mathbb{C}^{m|m*})^{\Delta^{P}}\), we see that, as a \( q(m) \)-module, it is isomorphic to the adjoint representation of \( q(m) \). Thus

\[ S(q(m))^{q(m)} \cong \sum_{\lambda} 2^{-\delta(l(\lambda))} (U_{m}^{\lambda} \otimes U_{m}^{\lambda*})^{\Delta q(m)} \]

\[ \cong \sum_{\lambda} 2^{-\delta(l(\lambda))} \text{Hom}_{q(m)}(U_{m}^{\lambda}, U_{m}^{\lambda}), \]

where the summation is over all strict partitions \( \lambda \) with \( l(\lambda) \leq m \). Thus combined with (2.3), we have proved the following proposition.

**Proposition 3.2.** The center of the universal enveloping algebra of \( q(m) \) admits a distinguished basis parameterized by strict partitions of length less than or equal to \( m \).

**Acknowledgment.** After we completed this work, we came across a preprint of Sergeev, ‘An analog of the classical invariant theory for Lie superalgebras’, math.RT/9810113, where he independently obtained the \((q(m), q(n))\) Howe duality (i.e. Theorem 3.1 in this Letter). The other results of this Letter seem to be new.

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