ON OPEN-OPEN GAMES OF UNCOUNTABLE LENGTH.

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Abstract. The aim of this note is to investigate the open-open game of uncountable length. We introduce a cardinal number $\mu(X)$, which says how long the Player I has to play to ensure a victory. It is proved that $c(X) \leq \mu(X) \leq c(X)^+$. We also introduce the class $C_\kappa$ of topological spaces that can be represented as the inverse limit of $\kappa$-complete system $\{X_\sigma, \pi_\sigma^\rho, \Sigma\}$ with $w(X_\sigma) \leq \kappa$ and skeletal bonding maps. It is shown that product of spaces which belong to $C_\kappa$ also belongs to this class and $\mu(X) \leq \kappa$ whenever $X \in C_\kappa$.

1. Introduction

The following game is due to P. Daniels, K. Kunen and H. Zhou [4]: two players take turns playing on a topological space $X$; a round consists of Player I choosing a nonempty open set $U \subseteq X$; and Player II choosing a nonempty open set $V \subseteq U$; a round is played for each natural number. Player I wins the game if the union of open sets which have been chosen by Player II is dense in $X$. This game is called the open-open game.

In this note, we consider what happens if one drops restrictions on the length of games. If $\kappa$ is an infinite cardinal and rounds are played for every ordinal number less then $\kappa$, then this modification is called the open-open game of length $\kappa$. The examination of such games is a continuation of [9], [10] and [11]. A cardinal number $\mu(X)$ is introduced such that $c(X) \leq \mu(X) \leq c(X)^+$. Topological spaces, which can be represented as an inverse limit of $\kappa$-complete system $\{X_\sigma, \pi_\sigma^\rho, \Sigma\}$ with $w(X_\sigma) \leq \kappa$ and each $X_\sigma$ is a $T_0$ space and skeletal bonding map $\pi_\sigma^\rho$, are listed as the class $C_\kappa$. If $\mu(X) = \omega$, then $X \in C_\omega$. There exists a space $X$ with $X \not\in C_{\mu(X)}$. The class $C_\kappa$ is closed under any Cartesian product.
product. In particular, the cellularity number of $X'$ is equal $\kappa$ whenever $X \in C_\kappa$. This implies Theorem of D. Kurepa that $c(X') \leq 2^\kappa$, whenever $c(X) \leq \kappa$. Undefined notions and symbols are used in accordance with books [3], [5] and [8]. For example, if $\kappa$ is a cardinal number, then $\kappa^+$ denotes the first cardinal greater than $\kappa$.

2. When games favor Player I

Let $X$ be a topological space. Denote by $\mathcal{T}$ the family of all non-empty open sets of $X$. For an ordinal number $\alpha$, let $\mathcal{T}^\alpha$ denotes the set of all sequences of the length $\alpha$ consisting of elements of $\mathcal{T}$. The space $X$ is called $\kappa$-favorable whenever there exists a function $s : \bigcup\{(\mathcal{T})^\alpha : \alpha < \kappa\} \to \mathcal{T}$ such that for each sequence $\{B_\alpha : \alpha < \kappa\} \subseteq \mathcal{T}$ with $B_1 \subseteq s(\emptyset)$ and $B_{\alpha+1} \subseteq s(\{B_\gamma : \gamma < \alpha\})$, for each $\alpha < \kappa$, the union $\bigcup\{B_\alpha : \alpha < \kappa\}$ is dense in $X$. We may also say that the function $s$ is witness to $\kappa$-favorability of $X$. In fact, $s$ is a winning strategy for Player I. For abbreviation we say that $s$ is $\kappa$-winning strategy. Sometimes we do not precisely define a strategy. Just give hints how a player should play. Note that, any winning strategy can be arbitrary on steps for limit ordinals.

A family $\mathcal{B}$ of open non-empty subset is called a $\pi$-base for $X$ if every non-empty open subset $U \subseteq X$ contains a member of $\mathcal{B}$. The smallest cardinal number $|\mathcal{B}|$, where $\mathcal{B}$ is a $\pi$-base for $X$, is denoted by $\pi(X)$.

**Proposition 1.** Any topological space $X$ is $\pi(X)$-favorable.

*Proof.* Let $\{U_\alpha : \alpha < \pi(X)\}$ be a $\pi$-base. Put $s(f) = U_\alpha$ for any sequence $f \in \mathcal{T}^\alpha$. Each family $\{B_\gamma : B_\gamma \subseteq U_\gamma \text{ and } \gamma < \pi(X)\}$ of open non-empty sets is again a $\pi$-base for $X$. So, its union is dense in $X$. □

According to [5, p. 86] the cellularity of $X$ is denoted by $c(X)$. Let sat($X$) be the smallest cardinal number $\kappa$ such that every family of pairwise disjoint open sets of $X$ has cardinality $< \kappa$, compare [6]. Clearly, if sat($X$) is a limit cardinal, then sat($X$) = $c(X)$. In all other cases, sat($X$) = $c(X)^+$. Hence, $c(X) \leq \text{sat}(X) \leq c(X)^+$. Let $\mu(X) = \min\{\kappa : X \text{ is a } \kappa\text{-favorable and } \kappa \text{ is a cardinal number}\}$.

Proposition 1 implies $\mu(X) \leq \pi(X)$. The next proposition gives two natural strategies and gives more accurate estimation than $c(X) \leq \mu(X) \leq c^+(X)$.
Proposition 2. $c(X) \leq \mu(X) \leq \text{sat}(X)$.

Proof. Suppose $c(X) > \mu(X)$. Fix a family $\{U_\xi : \xi < \mu(X)^+\}$ of pairwise disjoint open sets. If Player II always chooses an open set, which meets at most one $U_\xi$, then he will not lose the open-open game of the length $\mu(X)$, a contradiction.

Suppose sets $\{B_{\gamma+1} : \gamma < \alpha\}$ are chosen by Player II. If the set

$$X \setminus \overline{\bigcup\{B_{\gamma+1} : \gamma < \alpha\}}$$

is non-empty, then Player I choses it. Player I wins the open-open game of the length $\text{sat}(X)$, when he will use this rule. This gives $\mu(X) \leq \text{sat}(X)$. □

Note that, $\omega_0 = c(\{0,1\}^\kappa) = \mu(\{0,1\}^\kappa) \leq \text{sat}(\{0,1\}^\kappa) = \omega_1$, where $\{0,1\}^\kappa$ is the Cantor cube of weight $\kappa$. There exists a separable space $X$ which is not $\omega_0$-favorable, see A. Szymański [14] or [4, p.207-208]. Hence we get

$$\omega_0 = c(X) < \mu(X) = \text{sat}(X) = \omega_1.$$

3. On inverse systems with skeletal bonding maps

Recall that, a continuous surjection is skeletal if for any non-empty open sets $U \subseteq X$ the closure of $f[U]$ has non-empty interior. If $X$ is a compact space and $Y$ is a Hausdorff space, then a continuous surjection $f : X \to Y$ is skeletal if and only if $\text{Int} f[U] \neq \emptyset$, for every non-empty and open $U \subseteq X$, see J. Mioduszewski and L. Rudolf [13].

Lemma 3. A skeletal image of $\kappa$-favorable space is a $\kappa$-favorable space.

Proof. A proof follows by the same method as in [1, Theorem 4.1]. In fact, repeat and generalize the proof given in [11, Lemma 1]. □

According to [3], a directed set $\Sigma$ is said to be $\kappa$-complete if any chain of length $\leq \kappa$ consisting of its elements has the least upper bound in $\Sigma$. An inverse system $\{X_\sigma, \pi^\sigma_\alpha, \Sigma\}$ is said to be a $\kappa$-complete, whenever $\Sigma$ is $\kappa$-complete and for every chain $A \subseteq \Sigma$, where $|A| \leq \kappa$, such that $\sigma = \sup A \in \Sigma$ we get

$$X_\sigma = \lim_{\leftarrow} \{X_\alpha, \pi^\sigma_\alpha, A\}.$$ 

In addition, we assume that bonding maps are surjections.
For $\omega$-favorability, the following lemma is given without proof in [4, Corollary 1.4]. We give a proof to convince the reader that additional assumptions on topology are unnecessary.

**Lemma 4.** If $Y \subseteq X$ is dense, then $X$ is $\kappa$-favorable if and only if $Y$ is $\kappa$-favorable.

**Proof.** Let a function $\sigma_X$ be a witness to $\kappa$-favorability of $X$. Put

$$\sigma_Y(\emptyset) = \sigma_X(\emptyset) \cap Y.$$ 

If Player II chooses open set $V_1 \cap Y \subseteq \sigma_Y(\emptyset)$, then put

$$V_1' = V_1 \cap \sigma_X(\emptyset) \subseteq \sigma_X(\emptyset).$$

We get $V_1' \cap Y = V_1 \cap Y \subseteq \sigma_Y(\emptyset)$, since $V_1 \cap Y \subset \sigma_X(\emptyset) \cap Y$. Then we put

$$\sigma_Y(V_1 \cap Y) = \sigma_X(V_1') \cap Y.$$ 

Suppose we have already defined

$$\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \gamma\}) = \sigma_X(\{V'_{\alpha+1} : \alpha < \gamma\}) \cap Y,$$

for $\gamma < \beta < \kappa$. If Player II chooses open set $V_{\beta+1} \cap Y \subseteq \sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \beta\})$, then put

$$V_{\beta+1}' = V_{\beta+1} \cap \sigma_X(\{V'_{\alpha+1} : \alpha < \beta\}) \subseteq \sigma_X(\{V'_{\alpha+1} : \alpha < \beta\}).$$

Finally, put

$$\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha \leq \beta\}) = \sigma_X(\{V'_{\alpha+1} : \alpha \leq \beta\}) \cap Y$$

and check that $\sigma_Y$ is witness to $\kappa$-favorability of $Y$.

Assume that $\sigma_Y$ is a witness to $\kappa$-favorability of $Y$. If $\sigma_Y(\emptyset) = U_0 \cap Y$ and $U_0 \subseteq X$ is open, then put $\sigma_X(\emptyset) = U_0$. If Player II chooses open set $V_1 \subseteq \sigma_X(\emptyset)$, then $V_1 \cap Y \subseteq \sigma_Y(\emptyset)$. Put $\sigma_X(V_1) = U_1$, where $\sigma_Y(V_1 \cap Y) = U_1 \cap Y$ and $U_1 \subseteq X$ is open. Suppose

$$\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \gamma\}) = U_\gamma \cap Y \quad \text{and} \quad \sigma_X(\{V_{\alpha+1} : \alpha < \gamma\}) = U_\gamma$$

have been already defined for $\gamma < \beta < \kappa$. If II Player chooses open set $V_{\beta+1} \subseteq \sigma_X(\{V_{\alpha+1} : \alpha < \beta\})$, then put $\sigma_X(\{V_{\alpha+1} : \alpha < \beta + 1\}) = U_{\beta+1}$, where open set $U_{\beta+1} \subseteq X$ is determined by $\sigma_Y(\{V_{\alpha+1} \cap Y : \alpha < \beta + 1\}) = U_{\beta+1} \cap Y$. □

The next Theorem is similar to [2, Theorem 2]. We replace a continuous inverse system with indexing set being a cardinal, by $\kappa$-complete inverse system, and also $c(X)$ is replaced by $\mu(X)$. Let $\kappa$ be a fixed cardinal number.
Theorem 5. Let \( X \) be a dense subset of the inverse limit of the \( \kappa \)-complete system \( \{ X_\sigma, \pi^\sigma_\sigma, \Sigma \} \), where \( \kappa = \sup \{ \mu(X_\sigma) : \sigma \in \Sigma \} \). If all bonding maps are skeletal, then \( \mu(X) = \kappa \).

Proof. By Lemma 4, one can assume that \( X = \varprojlim \{ X_\sigma, \pi^\sigma_\sigma, \Sigma \} \). Fix functions \( s_\sigma : \mathcal{T}^{< \kappa}_\sigma \rightarrow \mathcal{T}_\sigma \), each one is a witness to \( \mu(X_\sigma) \)-favorability of \( X_\sigma \). This does not reduce the generality, because \( \mu(X_\sigma) \leq \kappa \) for every \( \sigma \in \Sigma \). In order to explain the induction, fix a bijection \( f : \kappa \rightarrow \kappa \times \kappa \) such that:

1. If \( f(\alpha) = (\beta, \zeta) \), then \( \beta, \zeta \leq \alpha \);
2. \( f^{-1}(\beta, \gamma) < f^{-1}(\beta, \zeta) \) if and only if \( \gamma < \zeta \);
3. \( f^{-1}(\gamma, \beta) < f^{-1}(\zeta, \beta) \) if and only if \( \gamma < \zeta \).

One can take as \( f \) an isomorphism between \( \kappa \) and \( \kappa \times \kappa \), with canonical well-ordering, see [3]. The function \( f \) will indicate the strategy and sets that we have taken in the following induction.

We construct a function \( s : \mathcal{T}^{< \kappa} \rightarrow \mathcal{T} \) which will provide \( \kappa \)-favorability of \( X \). The first step is defined for \( f(0) = (0, 0) \). Take an arbitrary \( \sigma_1 \in \Sigma \) and put

\[
s(\emptyset) = \pi^{-1}_{\sigma_1}(s_{\sigma_1}(\emptyset)).
\]

Assume that Player II chooses non-empty open set \( B_1 = \pi^{-1}_{\sigma_2}(V_1) \subseteq s(\emptyset) \), where \( V_1 \subseteq X_{\sigma_2} \) is open. Let

\[
s(\{B_1\}) = \pi^{-1}_{\sigma_1}(s_{\sigma_1}(\{\text{Int cl } \pi_{\sigma_1}(B_1) \cap s_{\sigma_1}(\emptyset)\}))
\]

and denote \( D^0_0 = \text{Int cl } \pi_{\sigma_1}(B_1) \cap s_{\sigma_1}(\emptyset) \). So, after the first round and the next respond of Player I, we know: indexes \( \sigma_1 \) and \( \sigma_2 \), the open set \( B_1 \subseteq X \) and the open set \( D^0_0 \subseteq X_{\sigma_1} \).

Suppose that sequences of open sets \( \{B_{\alpha+1} \subseteq X : \alpha < \gamma\} \), indexes \( \{\sigma_\alpha+1 : \alpha < \gamma\} \) and sets \( \{D^\varphi_\zeta : f^{-1}(\varphi, \zeta) < \gamma\} \) have been already defined such that:

if \( \alpha < \gamma \) and \( f(\alpha) = (\varphi, \eta) \), then

\[
B_{\alpha+1} = \pi_{\sigma_\alpha+2}(V_{\alpha+1}) \subseteq s(\{B_{\xi+1} : \xi < \alpha\}) = \pi^{-1}_{\sigma_{\alpha+1}}(s_{\sigma_{\varphi+1}}(\{D^\varphi_\zeta : \nu < \eta\})),
\]

where \( D^\varphi_\zeta = \text{Int cl } \pi_{\sigma_{\varphi+1}}(B_{f^{-1}(\varphi, \nu)+1}) \cap s_{\sigma_{\varphi+1}}(\{D^\varphi_\zeta : \zeta < \nu\}) \) and \( V_{\alpha+1} \subseteq X_{\sigma_{\alpha+2}} \) are open.

If \( f(\gamma) = (\theta, \lambda) \) and \( \beta < \lambda \), then take

\[
D^\theta_\beta = \text{Int cl } \pi_{\sigma_{\theta+1}}(B_{f^{-1}(\theta, \beta)+1}) \cap s_{\sigma_{\theta+1}}(\{D^\theta_\zeta : \zeta < \beta\})
\]
and put
\[ s(\{B_{\alpha+1} : \alpha < \gamma\}) = \pi_{\sigma_{\theta+1}}^{-1}(s_{\sigma_{\theta+1}}(\{D_{\alpha}^\theta : \alpha < \lambda\})). \]
Since \( \Sigma \) is \( \kappa \)-complete, one can assume that the sequence \( \{\sigma_{\alpha+1} : \alpha < \kappa\} \) is increasing and \( \sigma = \sup\{\sigma_{\xi+1} : \xi < \kappa\} \in \Sigma. \)

We shall prove that \( \bigcup_{\alpha<\kappa} B_{\alpha+1} \) is dense in \( X \). Since \( \pi_{\sigma}^{-1}(\pi_{\sigma}(B_{\alpha+1})) = B_{\alpha+1} \) for each \( \alpha < \kappa \) and \( \pi_{\sigma} \) is skeletal map, it is sufficient to show that \( \bigcup_{\alpha<\kappa} \pi_{\sigma}(B_{\alpha+1}) \) is dense in \( X_{\sigma} \). Fix arbitrary open set \( (\pi_{\sigma_{\xi+1}})^{-1}(W) \) where \( W \) is an open set of \( X_{\xi+1} \). Since \( s_{\sigma_{\xi+1}} \) is winning strategy on \( X_{\sigma_{\xi+1}} \), there exists \( D_{\alpha}^\xi \) such that \( D_{\alpha}^\xi \cap W \neq \emptyset \), and \( D_{\alpha}^\xi \subseteq \text{Int cl} \pi_{\sigma_{\xi+1}}(B_{f^{-1}(\xi,\alpha)+1}) \). Therefore we get
\[ (\pi_{\sigma_{\xi+1}}(W)) \cap \pi_{\sigma}(B_{\delta+1}) \neq \emptyset, \]
where \( \delta = f^{-1}(\xi,\alpha) \). Indeed, suppose that \( (\pi_{\sigma_{\xi+1}})^{-1}(W) \cap \pi_{\sigma}(B_{\delta+1}) = \emptyset \). Then
\[ 0 = \pi_{\sigma_{\xi+1}}[(\pi_{\sigma_{\xi+1}})^{-1}(W) \cap \pi_{\sigma}(B_{\delta+1})] = W \cap \pi_{\sigma_{\xi+1}}[\pi_{\sigma}(B_{\delta+1})] = W \cap \pi_{\sigma_{\xi+1}}(B_{\delta+1}). \]

Hence we have \( W \cap \text{Int cl} \pi_{\sigma_{\xi+1}}(B_{\delta+1}) = \emptyset \), a contradiction. \( \Box \)

**Corollary 6.** If \( X \) is dense subset of an inverse limit of \( \mu(X) \)-complete system \( \{X_{\sigma}, \pi_{\sigma}^\alpha, \Sigma\} \), where all bonding map are skeletal, then
\[ c(X) = \sup\{c(X_{\sigma}) : \sigma \in \Sigma\}. \]

**Proof.** Let \( X = \lim_{\xleftarrow{\longrightarrow} \Sigma} \{X_{\sigma}, \pi_{\sigma}^\alpha, \Sigma\} \). Since \( c(X) \geq c(X_{\sigma}) \), for every \( \sigma \in \Sigma \), we shall show that
\[ c(X) \leq \sup\{c(X_{\sigma}) : \sigma \in \Sigma\}. \]

Suppose that \( \sup\{c(X_{\sigma}) : \sigma \in \Sigma\} = \tau < c(X) \). Using Proposition 2 and Theorem 5 check that
\[ \mu(X) = \sup\{\mu(X_{\sigma}) : \sigma \in \Sigma\} \leq \sup\{c(X_{\sigma})^+ : \sigma \in \Sigma\} \leq \tau^+ \leq c(X). \]
So, we get \( \mu(X) = c(X) = \tau^+ \). Therefore, there exists a family \( \mathcal{R} \), of size \( \tau^+ \), which consists of pairwise disjoint open subset of \( X \). We can assume that
\[ \mathcal{R} \subseteq \{\pi_{\sigma}^{-1}(U) : U \text{ is an open subset of } X_{\sigma} \text{ and } \sigma \in \Sigma\}. \]
Since \( \{X_{\sigma}, \pi_{\sigma}^\alpha, \Sigma\} \) is \( \mu(X) \)-complete inverse system and \( |\mathcal{R}| = \mu(X) \), there exists \( \beta \in \Sigma \) such that
\[ \mathcal{R} \subseteq \{\pi_{\beta}^{-1}(U) : U \text{ is an open subset of } X_{\beta}\}, \]
a contradiction with \( c(X_{\beta}) < \tau^+ \). \( \Box \)
The above corollary is similar to [2, Theorem 1], but we replaced a continuous inverse system, whose indexing set is a cardinal number by \( \kappa \)-complete inverse system.

4. Classes \( \mathcal{C}_\kappa \)

Let \( \kappa \) be an infinite cardinal number. Consider inverse limits of \( \kappa \)-complete system \( \{X_\sigma, \pi_\sigma^\rho, \Sigma\} \) with \( w(X_\sigma) \leq \kappa \). Let \( \mathcal{C}_\kappa \) be a class of such inverse limits with skeletal bonding maps and \( X_\sigma \) being \( T_0 \)-space. Now, we show that the class \( \mathcal{C}_\kappa \) is stable under Cartesian products.

**Theorem 7.** The Cartesian product of spaces from \( \mathcal{C}_\kappa \) belongs to \( \mathcal{C}_\kappa \).

**Proof.** Let \( X = \prod \{X_s : s \in S\} \) where each \( X_s \in \mathcal{C}_\kappa \). For each \( s \in S \), let \( X_s = \lim \{X_\sigma, s_\rho^\sigma, \Sigma_s\} \) be a \( \kappa \)-complete inverse system with skeletal bonding map such that each \( T_0 \)-space \( X_\sigma \) has the weight \( \leq \kappa \). Consider the union

\[
\Gamma = \bigcup \{ \prod \Sigma_s : A \in [S]^{\kappa}\}.
\]

Introduce a partial order on \( \Gamma \) as follows:

\[
f \preceq g \iff \text{dom}(f) \subseteq \text{dom}(g) \quad \text{and} \quad \forall a \in \text{dom}(f), f(a) \preceq_a g(a),
\]

where \( \preceq_a \) is the partial order on \( \Sigma_a \). The set \( \Gamma \) with the relation \( \preceq \) is upward directed and \( \kappa \)-complete.

If \( f \in \Gamma \), then \( Y_f \) denotes the Cartesian product

\[
\prod \{X_{f(a)} : a \in \text{dom}(f)\}.
\]

If \( f \preceq g \), then put

\[
p_f^g = \left( \prod_{a \in \text{dom}(f)} a^{g(a)}_{f(a)} \right) \circ \pi_{\text{dom}(f)},
\]

where \( a^{g(a)}_{\text{dom}(f)} \) is the projection of \( \prod \{X_{g(a)} : a \in \text{dom}(g)\} \) onto \( \prod \{X_{g(a)} : a \in \text{dom}(f)\} \) and \( \prod_{a \in \text{dom}(f)} a^{g(a)}_{f(a)} \) is the Cartesian product of the bonding maps \( a^{g(a)}_{f(a)} : X_{g(a)} \rightarrow X_{f(a)} \). We get the inverse system \( \{Y_f, p_f^g, \Gamma\} \) which is \( \kappa \)-complete, bonding maps are skeletal and \( w(Y_f) \leq \kappa \). So, we can take \( Y = \lim \{Y_f, p_f^g, \Gamma\} \).

Now, define a map \( h : X \rightarrow Y \) by the formula:

\[
h(\{x_s\}_{s \in S}) = \{x_f\}_{f \in \Gamma},
\]
where \( x_f = \{ x_{f(a)} \}_{a \in \text{dom}(f)} \in Y_f \) and \( f \in \prod \{ \Sigma_a : a \in \text{dom}(f) \} \) and \( \text{dom}(f) \in [S]^\kappa \). By the property
\[
\{ x_s \}_{s \in S} = \{ t_s \}_{s \in S} \Leftrightarrow \forall s \in S \forall \sigma \in \Sigma_s \ x_s = t_s \Leftrightarrow \forall f \in \Gamma \ x_f = t_f,
\]
the map \( h \) is surjection. Indeed, let \( \{ b_f \}_{f \in \Gamma} \in Y \). For each \( s \in S \) and each \( \sigma \in \Sigma_s \) we fix \( f^s_\sigma \in \Gamma \) such that \( s \in \text{dom}(f^s_\sigma) \) and \( f^s_\sigma(s) = \sigma \). Let \( \pi_{f(s)} : Y_f \to X_{f(s)} \) be a projection for each \( f \in \Gamma \).

For each \( t \in S \) let define \( b_t = \{ b_\sigma \}_{\sigma \in \Sigma_t} \), where \( b_\sigma = \pi_{f^s_\sigma(t)}(b_{f^s_\sigma}) \). We shall prove that an element \( b_t \) is a thread of the space \( X_t \). Indeed, if \( \sigma \geq \rho \) and \( \sigma, \rho \in \Sigma_t \), then take functions \( f^s_\sigma \) and \( g^s_\rho \). For abbreviation, denote \( f = f^s_\sigma \) and \( g = g^s_\rho \). Define a function \( h : \text{dom}(f) \cup \text{dom}(g) \to \bigcup \{ \Sigma_t : t \in \text{dom}(f) \cup \text{dom}(g) \} \) in the following way:
\[
h(s) = \begin{cases} g(s) & \text{if } s \in \text{dom}(g) \setminus \text{dom}(f) \\ f(s) & \text{if } s \in \text{dom}(f). \end{cases}
\]
The function \( h \) is element of \( \Gamma \) and \( f, g \preceq h \). Note that \( h|\text{dom}(f) = f \) and \( h|\text{dom}(g) \setminus \{ t \} = g|\text{dom}(g) \setminus \{ t \} \). Since
\[
\{ b_{g(s)} \}_{s \in \text{dom}(g)} = b_g = p^h_g(b_h) = \left( \prod_{s \in \text{dom}(g)} s^{h(s)}_{g(s)} \right) \left( \pi_{\text{dom}(g)}(b_h) \right) = \\
\left( \prod_{s \in \text{dom}(g)} s^{h(s)}_{g(s)} \right) \left\{ b_{h(s)} \right\}_{s \in \text{dom}(g)} = \left\{ s^{h(s)}_{g(s)}(b_{h(s)}) \right\}_{s \in \text{dom}(g)}
\]
we get
\[
b_\rho = b_{g(t)} = s^{h(t)}_{g(t)}(b_{h(t)}) = s^{f(t)}_{g(t)}(b_{f(t)}) = s^\sigma_{f(t)}(b_\sigma).
\]
It is clear that \( h(\{ a_t \}_{t \in S}) = \{ b_f \}_{f \in \Gamma} \).

We shall prove that the map \( h \) is continuous. Take an open subset \( U = \prod_{s \in \text{dom}(f)} A_{f(s)} \subseteq Y_f \) such that
\[
A_{f(s)} = \begin{cases} V, & \text{if } s = s_0; \\ X_{f(s)}, & \text{otherwise}, \end{cases}
\]
where \( V \subseteq X_{f(s_0)} \) is open subset. A map \( p_f \) is projection from the inverse limit \( Y \) to \( Y_f \). It is sufficient to show that :
\[
h^{-1}((p_f)^{-1}(U)) = \prod_{s \in S} B_s
\]
where
\[ B_s = \begin{cases} 
W, & \text{if } s = s_0; \\
X_s, & \text{otherwise,}
\end{cases} \]
and \( W = \pi_{f(s_0)}^{-1}(V) \) and \( \pi_{f(s_0)} : Y_f \to X_{s_0} \) is the projection and \( f(s_0) = \sigma_0 \). We have
\[
\{x_s\}_{s \in S} \in h^{-1}((p_f)^{-1}(U)) \iff p_f(h(\{x_s\}_{s \in S})) \in U \\
\iff p_f(\{x_f\}_{f \in F}) = x_f \in U \iff x_f(s_0) \in V \\
\iff x_{s_0} \in W \iff x \in \prod_{s \in S} B_s \subseteq \prod_{s \in S} X_s = X
\]
Since the map \( h \) is bijection and
\[
(p_f)^{-1}(U) = h(h^{-1}((p_f)^{-1}(U))) = h(\prod_{s \in S} B_s)
\]
for any subbase subset \( \prod_{s \in S} B_s \subseteq X \), the map \( h \) is open. \( \square \)

In the case \( \kappa = \omega \) we have well know results that I-favorable space is productable (see [4] or [9])

**Corollary 8.** Every I-favorable space is stable under any product.

If \( D \) is a set and \( \kappa \) is cardinal number then we denote \( \bigcup_{\alpha < \kappa} D^\alpha \) by \( D^{<\kappa} \).

The following result probably is known but we give a proof for the sake of completeness.

**Theorem 9.** Let \( \kappa \) be an infinite cardinal and let \( T \) be a set such that \( |T| \geq \kappa^{\kappa} \). If \( A \in [T]^{\kappa} \) and \( f_\delta : T^{<\kappa} \to T \) for all \( \delta < \kappa^{<\kappa} \) then there exists a set \( B \subseteq T \) such that \( |B| \leq \tau \) and \( A \subseteq B \) and \( f_\delta(C) \in B \) for every \( C \in B^{<\kappa} \) and every \( \delta < \kappa^{<\kappa} \), where
\[
\tau = \begin{cases} 
\kappa^{<\kappa} & \text{for regular } \kappa \\
\kappa^{\kappa} & \text{otherwise.}
\end{cases}
\]

Proof. Assume that \( \kappa \) is regular cardinal. Let \( A \in [T]^{\kappa} \) and let \( f_\delta : \bigcup_{\alpha < \kappa} T^{\alpha} \to T \) for \( \delta < \kappa^{<\kappa} \). Let \( A_0 = A \). Assume that we have defined \( A_\alpha \) for \( \alpha < \beta \) such that \( |A_\alpha| \leq \kappa^{[\alpha]} \). Put
\[
A_\beta = \left( \bigcup_{\alpha < \beta} A_\alpha \right) \cup \{ f_\delta(C) : C \in \left( \bigcup_{\alpha < \beta} A_\alpha \right)^{<\beta} \text{ and } \delta < \kappa^{[\beta]} \}.
\]
Calculate the size of the set \( A_\beta \):
\[ |A_\beta| \leq \left| \left( \bigcup_{\alpha<\beta} A_\alpha \right) \right| \leq \kappa |\beta| \leq \kappa |\beta| = \kappa |\beta|. \]

Let \( B = \bigcup_{\beta<\kappa} A_\beta \), so we get \( |B| \leq \kappa\). Fix a sequence \( \langle b_\alpha : \alpha < \beta \rangle \subseteq B \) and \( f_\gamma \). Since \( cf(\kappa) = \kappa \) there exists \( \delta < \kappa \) such that \( C = \{ b_\alpha : \alpha < \beta \} \subseteq A_\delta \) and \( f_\gamma(C) \in A_{\sigma+1} \) for some \( \sigma < \kappa \).

In the second case \( cf(\kappa) < \kappa \), we proceed the above induction up to \( \beta = \kappa \). Let \( B = A_\kappa \), so we get \( |B| \leq \kappa \) and \( B = \bigcup_{\beta<\kappa} A_\beta \). Similarly to the first case we get that \( B \) is closed under all function \( f_\delta, \delta < \kappa\).

**Theorem 10.** If \( X \) belongs to the class \( C_\kappa \) then \( c(X) \leq \kappa \).

**Proof.** If \( X \in C_\kappa \) then by Theorems 5 and 2 we get \( c(X) \leq \mu(X) \leq \kappa \).

We apply some facts from the paper [10]. Let \( \mathcal{P} \) be a family of open subset of topological space \( X \) and \( x, y \in X \). We say that \( x \sim_\mathcal{P} y \) if and only if \( x \in V \iff y \in V \) for every \( V \in \mathcal{P} \). The family of all sets \( [x]_\mathcal{P} = \{ y : y \sim_\mathcal{P} x \} \) we denote by \( X/\mathcal{P} \). Define a map \( q: X \to X/\mathcal{P} \) as follows \( q[x] = [x]_\mathcal{P} \). The set \( X/\mathcal{P} \) is equipped with topology \( \mathcal{T}_\mathcal{P} \) generated by all images \( q[V] \) where \( V \in \mathcal{P} \).

Recall Lemma 1 from paper [10]: If \( \mathcal{P} \) is a family of open set of \( X \) and \( \mathcal{P} \) is closed under finite intersection then the mapping \( q : X \to X/\mathcal{P} \) is continuous. Moreover if \( X = \bigcup \mathcal{P} \) then the family \( \{ q[V] : V \in \mathcal{P} \} \) is a base for the topology \( \mathcal{T}_\mathcal{P} \).

Notice that if \( \mathcal{P} \) has a property

\[
(\text{seq}) \quad \forall(W \in \mathcal{P}) \exists \{ V_n : n < \omega \} \subseteq \mathcal{P} \exists \{ U_n : n < \omega \} \subseteq \mathcal{P} \\
W = \bigcup_{n<\omega} U_n \text{ and } \forall(n < \omega) U_n \subseteq X \setminus V_n \subseteq U_{n+1}
\]

then \( \bigcup \mathcal{P} = X \) and by [10] Lemma 3 the topology \( \mathcal{T}_\mathcal{P} \) is Hausdorff. Moreover if \( \mathcal{P} \) is closed under finite intersection then by [10] Lemma 4 the topology \( \mathcal{T}_\mathcal{P} \) is regular. Theorem 5 and Lemma 9 [10] yield.

**Theorem 11.** If \( \mathcal{P} \) is a set of open subset of topological space \( X \) such that:

1. is closed under \( \kappa \)-winning strategy, finite union and finite intersection,
2. has property \( (\text{seq}) \),

\[ (\bigwedge_{\alpha<\beta} A_\alpha) \leq \kappa |\beta| \]
then $X/P$ with topology $T_P$ is completely regular space and $q : X \to X/P$ is skeletal.

If a topological space $X$ has the cardinal number $\mu(X) = \omega$ then $X \in C_{\omega_1}$, but for $\mu(X)$ equals for instance $\omega_1$ we do not even know if $X \in C_{\omega_1\omega}$.

**Theorem 12.** Each Tikhonov space $X$ with $\mu(X) = \kappa$ can be dense embedded into inverse limit of a system $\{X_\sigma, \pi_\sigma, \Sigma\}$, where all bonding map are skeletal, indexing set $\Sigma$ is $\tau$-complete each $X_\sigma$ is Tikhonov space with $w(X_\sigma) \leq \tau$ and

$$\tau = \begin{cases} \kappa^{<\kappa} & \text{for regular } \kappa \\ \kappa^\kappa & \text{otherwise} \end{cases}$$

**Proof.** Let $B$ be a $\tau$-base for topological space $X$ consisting of cozero-sets and $\sigma : \bigcup \{B^\alpha : \alpha < \kappa\} \to B$ be a $\kappa$-winning strategy. We can define a function of finite intersection property and finite union property as following: $g(B_0, B_1, \ldots, B_n) = B_0 \cap B_1 \cap \ldots \cap B_n$ and $h(B_0, B_1, \ldots, B_n) = B_0 \cup B_1 \cup \ldots \cup B_n$. For each cozero-set $V \in B$ fix a continuous function $f_V : X \to [0, 1]$, $V = f_V^{-1}((0, 1])$. Put $\sigma_2n(V) = f_V^{-1}(\left(\frac{1}{n}, 1\right])$ and $\sigma_{2n+1}(V) = f_V^{-1}(\left[0, \frac{1}{n}\right])$. By Theorem 9 for each $R \in [B]^\kappa$ and all functions $h, g, \sigma_n, \sigma$ there is subset $P \subseteq B$ such that:

1. $|P| \leq \tau$ where
   $$\tau = \begin{cases} \kappa^{<\kappa} & \text{for regular } \kappa \\ \kappa^\kappa & \text{otherwise} \end{cases}$$
2. $R \subseteq P$,
3. $P$ is closed under $\kappa$-winning strategy $\sigma$, function of finite intersection property and finite union property ,
4. $P$ is closed under $\sigma_n$, $n < \omega$, hence $P$ holds property (seq).

Therefore by Theorem 11 we get skeletal mapping $q_P : X \to X/P$. Let $\Sigma \subseteq [B]^\leq\tau$ be a set of families which satisfies above condition (1), (2), (3), (4). If $\Sigma$ is directed by inclusion. It is easy to check that $\Sigma$ is $\tau$-complete. Similar to [10, Theorem 11] we define a function $f : X \to Y$ as following $f(x) = \{f_P(x)\}$, where $f(x)_P = q_P(x)$ and $Y = \lim_{\leftarrow} X/R, q_{\Sigma}^R, \mathcal{C}$. If $R, P \in \mathcal{C}$ and $P \subseteq R$, then $q_{\Sigma}^R(f(x)_R) = f(x)_P$. Thus $f(x)$ is a thread, i.e. $f(x) \in Y$. It easy to see that $f$ is homeomorphism onto its image and $f[X]$ is dense in $Y$, compare [10, proof of Theorem 11]
The Theorem 12 suggests a question:

Does each space $X$ belong to $\mathcal{C}_{\mu(X)}$?

Fleissner [7] proved that there exists a space $Y$ such that $c(Y) = \aleph_0$ and $c(Y^3) = \aleph_2$. Hence we get $\mu(Y) = \aleph_1$, by Theorem 5 and Corollary 8. Suppose that $Y \in \mathcal{C}_{\mu(X)}$ then $c(Y^3) \leq \aleph_1$, by Theorem 10, a contradiction.

**Corollary 13.** If $X$ is topological space with $\mu(X) = \kappa$ then $c(X^I) \leq \tau$ and

$$\tau = \begin{cases} 
\kappa^\kappa & \text{for regular } \kappa \\
\kappa^\kappa & \text{otherwise.}
\end{cases}$$

**Proof.** By Theorem 9 we get $X^I \in \mathcal{C}_\tau$. Hence by Theorem 10 and 7 we have $c(X^I) \leq \tau$. □

By above Corollary we get the following

**Corollary 14.** [12] Kurepa If $\{X_s : s \in S\}$ is a family of topological spaces and $c(X_s) \leq \kappa$ for each $s \in S$, then $c(\prod\{X_s : s \in S\}) \leq 2^\kappa$. □

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