A no-pure-boost uncertainty principle from spacetime noncommutativity

Giovanni Amelino-Camelia, Giulia Gubitosi, Antonino Marcianò, Pierre Martinetti, and Flavio Mercati

Dipartimento di Fisica
Università di Roma “La Sapienza”
and Sez. Roma1 INFN
P.le A. Moro 2, 00185 Roma, Italy

Abstract

We study boost and space-rotation transformations in κ-Minkowski noncommutative spacetime, using the techniques that some of us had previously developed (hep-th/0607221) for a description of translations in κ-Minkowski, which in particular led to the introduction of translation transformation parameters that do not commute with the spacetime coordinates. We find a similar description of boosts and space rotations, which allows us to identify some associated conserved charges, but the form of the commutators between transformation parameters and spacetime coordinates is incompatible with the possibility of a pure boost.
I. INTRODUCTION

A rather sizeable literature has been devoted over these past few years to the study of the so-called $\kappa$-Minkowski noncommutative spacetime [1, 2], with the characteristic noncommutativity

\[ [x_j, x_0] = i\lambda x_j \]

\[ [x_k, x_j] = 0, \]

where $x_0$ is the time coordinate, $x_j$ are space coordinates ($j, k \in \{1, 2, 3\}$), and $\lambda$ is a length scale, usually expected to be of the order of the Planck length. For most researchers involved in these studies the key source of motivation comes from some technical observations that appear to suggest that the symmetries of $\kappa$-Minkowski should be described by a Hopf-algebra, the so-called “$\kappa$-Poincaré” Hopf algebra [1, 2, 3]. However, the task of understanding the physical implications of these Hopf-algebra $\kappa$-Poincaré symmetries has turned out to be very difficult. In particular, after more than a decade of study and hundreds of papers devoted to the $\kappa$-Minkowski/$\kappa$-Poincaré framework, even the existence of some conserved charges associated to these Hopf-algebra spacetime symmetries was only established very recently, and only for the translations sector of $\kappa$-Poincaré, in the analysis reported by some of us in Ref. [4]. This recently-developed tool of Noether analysis of some relevant Hopf-algebra symmetries, which at least allows us to contemplate some physical observables of the theory (the conserved charges), could of course provide valuable elements for the debate on the physical implications of the framework, if all of its potentialities are exploited.

With this goal in mind, we here intend to extend the analysis reported in Ref. [4] to the full $\kappa$-Poincaré Hopf algebra, thereby including also the Lorentz sector of space-rotations and boosts. While for the description of pure translation transformations it is necessary (as shown in Ref. [4]) to introduce transformation parameters that do not commute with the $\kappa$-Minkowski spacetime coordinates, we shall show that for the case of pure space rotations it is possible to introduce commutative transformation parameters. We find however that the necessity of noncommutative transformation parameters is encountered once again in the description of boost transformations, and it takes a rather striking form: when the boost parameters are not set to zero then also the space-rotation parameters must not all be zero and both the boost parameters and the space-rotation parameters must satisfy some nontrivial commutation relations with the $\kappa$-Minkowski spacetime coordinates. This feature could be described as a “no-pure-boost uncertainty principle”, since it is incompatible with the possibility of a symmetry transformation in which the only nonzero transformation parameters are boost parameters.

This key part of our analysis is reported in the next section (Section 2). In Section 3 we show that the transformations we introduce are genuine symmetries of the theory, even allowing the derivation of some associated conserved charges. In the closing Section 4 we offer a perspective on the possible implications of our findings and consider some further studies that could take our analysis as starting point.

II. NONCOMMUTATIVE TRANSFORMATION PARAMETERS

The analysis we here report of space-rotations and boosts will be guided by the description of translations proposed in Ref. [4]. After the failures of several other attempts, with that description of translation transformations it was finally possible to bring to completion a Noether analysis. We shall therefore assume that the criteria adopted in Ref. [4] for the description of translation transformations should be applied also to the case of space-rotations and boosts.

In Ref. [4] the action of a translation transformation on a function $f(x)$ of the $\kappa$-Minkowski spacetime coordinates was parametrized as follows:

\[ df(x) = ie^\mu P_\mu f(x), \]
where $\epsilon_\mu$ are the transformation parameters, and $P_\mu$ are the Majid-Ruegg\footnote{As conventional in the $\kappa$-Minkowski literature, we only give explicitly the actions on a basis of exponentials of the noncommutative coordinates. The action on a generic function of the $\kappa$-Minkowski coordinate is then induced by linearity through a Fourier-transform structure. For example, from\cite{4} one deduces\cite{5}} translation generators, with classical action on “time-to-the-right-ordered” exponentials:

$$P_\mu \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \right) = k_\mu e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} . \quad (4)$$

The properties of the transformation parameters $\epsilon_\mu$ were derived\cite{4} by imposing Leibniz rule on the differential\cite{5},

$$d(f(x)g(x)) = (df(x))g(x) + f(x)(dg(x)) , \quad (5)$$

which, as a result of the observation (“coproduct”\cite{5}) that from\cite{4} it follows that

$$P_\mu \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} e^{i\vec{q} \cdot \vec{x}} e^{-iq_0 x_0} \right) = \left( k_\mu + e^{-\lambda k_0 (1-\delta_{00})} q_\mu \right) \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} e^{i\vec{q} \cdot \vec{x}} e^{-iq_0 x_0} \right) , \quad (6)$$

amounts to the following requirement for the $\epsilon_\mu$:

$$\left( f(x) \epsilon_\mu - \epsilon_\mu e^{-\lambda P_\mu} f(x) \right) P_\mu g(x) = 0 . \quad (7)$$

Clearly this equation implies that, unlike the corresponding transformation parameters for classical Minkowski spacetime, the $\epsilon_\mu$ cannot be simply some real numbers. Ref.\cite{4} introduced the concept of “noncommutative transformation parameters” as the most conservative generalization of the concept of transformation parameters that would allow to find solutions for\cite{7}. These noncommutative transformation parameters were required to still act only by (associative) multiplication on the spacetime coordinates, but were allowed to be subject to nontrivial rules of commutation with the spacetime coordinates. Within this generalization of the concept of transformation parameters Eq.\cite{5} does admit a solution, characterized by the following rules of commutation with the spacetime coordinates:

$$[\epsilon_j, x_0] = i\lambda \epsilon_j , \quad [\epsilon_j, x_k] = 0 , \quad [\epsilon_0, x_\mu] = 0 . \quad (8)$$

We then intend to parametrize pure space-rotation transformations in the following way:

$$d_R f(x) = i\sigma_j R_j f(x) , \quad (9)$$

where $R_j$ are the classical-action space-rotation generators\cite{1, 5},

$$R_j \left( e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \right) = \epsilon_{jk} x_k \epsilon_{il} x_l e^{i\vec{k} \cdot \vec{x}} e^{-ik_0 x_0} , \quad (10)$$

and the properties of the space-rotation transformation parameters $\sigma_j$ are again to be deduced from the enforcement of Leibniz rule for $d_R$. But the well-known fact that the $R_j$ are truly classical space-rotation generators (not only classical action but also classical “co-action”\cite{1, 5}) here manifests itself in the fact that the condition for enforcing Leibniz rule for $d_R$ is trivial:

$$\sigma_j (R_j f(x)) g(x) + \sigma_j f(x) R_j g(x) = \sigma_j (R_j f(x)) g(x) + f(x) \sigma_j R_j g(x) . \quad (11)$$
Therefore, for pure space-rotation transformations the \( \sigma_j \) are ordinary (commutative) transformation parameters:

\[
[\sigma_j, x_\mu] = 0 .
\] (12)

A new level of complexity is encountered in dealing with boosts. It is well established that the requirement that the boost generators combine with the space-rotation and translation generators to close on a Hopf algebra does not allow the introduction of classical-action boost generators. And this requirement of a close Hopf algebra leads to the Majid-Ruegg boost generators \( N_j \):

\[
N_j \left( e^{i\vec{k} \cdot \vec{x}} e^{-i k_0 x_0} \right) = -k_j e^{i\vec{k} \cdot \vec{x}} e^{-i k_0 x_0} x_0 + \left[ x_j \left( 1 - \frac{e^{-2\lambda k_0}}{2\lambda} + \frac{\lambda}{2} |\vec{k}|^2 \right) - \lambda x_l k_l k_j \right] e^{i\vec{k} \cdot \vec{x}} e^{-i k_0 x_0} .
\] (13)

We should therefore set up the description of a pure boost as follows:

\[
d_B f(x) = i\tau_j N_j f(x),
\] (14)

in terms of the Majid-Ruegg boost generators \( N_j \) and of some transformation parameters \( \tau_j \) such to enforce Leibniz rule on \( d_B \). But the form of the Majid-Ruegg boost generators \( N_j \) is such that the Leibniz-rule requirement,

\[
[f(x)\tau_j - \tau_j e^{-\lambda P_0} f(x)] N_j g(x) = \lambda \tau_j \epsilon_{jkl}(P_k f(x))(R_l g(x)) ,
\] (15)

does not admit any acceptable solution \( \tau_j \). \([15]\) would require the \( \tau_j \) to be operators with highly nontrivial action on functions of the spacetime coordinates, rather than being “noncommutative parameters” that act by simple (associative) multiplication on the spacetime coordinates.

We conclude that whereas pure translations and pure space rotations are allowed in \( \kappa \)-Minkowski, the possibility of pure boosts is excluded.

It is natural then to wonder whether boosts are at all allowed: one cannot have a pure boost, but can one have transformations which combine boosts and other transformations? We found that this is allowed and the way in which the formalism allows it is rather intriguing. It is sufficient to contemplate a transformation that involves both a boost and a space rotation:

\[
d_L f(x) = i\tau_j N_j f(x) + i\sigma_k R_k f(x) ,
\] (16)

since remarkably the Leibniz-rule requirement for this “pure-Lorentz differential”:

\[
[\tau_j (e^{-\lambda P_0} f) - f \tau_j] N_j g + \left[ \lambda \epsilon_{jkl} \tau_j (P_k f) + [\sigma_l, f] \right] (R_l g) = 0 ,
\] (17)

does admit solutions. It is however necessary that both the \( \tau_j \) and the \( \sigma_j \) be “noncommutative parameters”:

\[
\begin{aligned}
[\tau_j, x_k] &= 0, & [\sigma_j, x_k] &= i\lambda \epsilon_{jkl} \tau_l, \\
[\tau_j, x_0] &= i\lambda \tau_j, & [\sigma_j, x_0] &= 0.
\end{aligned}
\] (18)

From the broader perspective of these general Lorentz transformations one also acquires a better understanding of what emerged for pure space rotations and pure boosts. This is codified in the commutation relations \([\sigma_j, x_k] = i\lambda \epsilon_{jkl} \tau_l\), which admit the case of a pure space rotation, with commutative transformation parameters \( \tau_l = 0 \rightarrow [\sigma_j, x_k] = 0 \), but are incompatible with the case of a pure boost \( \tau_l \neq 0 \rightarrow [\sigma_j, x_k] \neq 0 \rightarrow \sigma_j \neq 0 \).
III. NOETHER ANALYSIS

In order to give some substance to our claim that the transformations we constructed in the previous section are good candidates as symmetries of theories in $\kappa$-Minkowski spacetime, we now derive associated conserved charges for the most studied [2, 3, 6] theory formulated in $\kappa$-Minkowski: a theory for a massless scalar field $\Phi(x)$ governed by the Klein-Gordon-like equation of motion

$$\Box_\lambda \Phi(x) \equiv \bar{P}_\mu \bar{P}^\mu \Phi \equiv \left[ - \left( \frac{2}{\lambda} \right)^2 \sinh^2 \left( \frac{\lambda P_0}{2} \right) + e^{\lambda P_0} |\bar{P}|^2 \right] \Phi(x) = 0, \quad (19)$$

where the operator $\Box_\lambda$ is the “mass Casimir” of the $\kappa$-Poincaré Hopf algebra\(^2\) and we introduced the convenient notation $\bar{P}_\mu$,

$$\bar{P}_0 = \left( \frac{2}{\lambda} \right) \sinh(\lambda P_0/2) \quad \bar{P}_j = e^{\lambda P_0/2} P_j, \quad (20)$$

The equation of motion (19) is indeed invariant\(^3\) ($\delta(\Box_\lambda \Phi) = \Box_\lambda \delta \Phi = 0$) under our transformations with $x_\mu \to x_\mu + i e^\nu P_\nu x_\mu + i \tau_j N_j x_\mu + i \sigma_k R_k x_\mu$ and $\Phi \to \Phi + \delta \Phi = \Phi - d_{\text{tot}} \Phi \equiv \Phi - i \left[ e^\mu P_\mu + \sigma_j R_j + \tau_k N_k \right] \Phi$ (valid when the field is a scalar under the coordinate transformation).

Our Noether analysis takes as starting point the action

$$S = \frac{1}{2} \int d^4 x \Phi(x) \Box_\lambda \Phi(x), \quad (21)$$

from which the equation of motion (19) can be obtained variationally\(^4\). And we focus on the Lorentz sector derived here

$$\delta \Phi = - d_L \Phi \equiv [i \sigma_j R_j + i \tau_k N_k] \Phi, \quad (22)$$

since for the translations one can of course follow\(^4\) the Noether analysis reported in detail in Ref. [4].

The result of a variation of the action (21) under our general space-rotation and boost transformation is:

$$\delta S = \frac{1}{2} \int d^4 x \bar{P}_\mu \left\{ \bar{P}_\mu \left( e^{\lambda P_0} \Phi \right) \delta \Phi \right\} - 2 \left( e^{\lambda P_0} \bar{P}_\mu \Phi \right) e^{\lambda P_0/2} \delta \Phi, \quad (23)$$

where $\delta \Phi$ is given in (22), we already specialized to fields that are solutions of the equation of motion, and we used the following property of the operators $\bar{P}_\mu$:

$$\bar{P}_\mu [f(x)g(x)] = \left[ \bar{P}_\mu f(x) \right] \left[ e^{\lambda P_0} g(x) \right] + \left[ e^{\lambda P_0} f(x) \right] \left[ \bar{P}_\mu g(x) \right]. \quad (24)$$

Using the fact that from the rules of commutation (18) between transformation parameters and spacetime coordinates it follows that, for a generic function of the coordinates $f(x)$, one has $f(x) \tau_j = \tau_j (e^{-\lambda P_0} f(x))$ and $[f(x), \sigma_j] = \lambda e_{jkl} \tau_l (P_k f(x))$, and the observation [4]

$$\int d^4 x e^{\lambda P_0} \left[ f(x) \right] = \int d^4 x f(x) \quad \forall \xi, \quad (25)$$

\(^2\) Eq. (19) reduces to the Klein-Gordon equation in the $\lambda \to 0$ limit, and its form was proposed (see, e.g., Refs. [2, 3, 6]) using as guidance the idea that it should involve an operator that commutes with all the generators in the $\kappa$-Poincaré Hopf algebra.

\(^3\) Note that the mass Casimir $\Box_\lambda$ commutes with all the generators $P_\mu, R_j, N_j$ of the Hopf algebra and with the transformation parameters $\epsilon_\mu, \sigma_j, \tau_j$.

\(^4\) Since we established in the previous section that both pure translation transformations and pure Lorentz-sector (space-rotation/boost) transformations are allowed, one can indeed treat these two types of transformations separately. But of course, they can also be analyzed simultaneously, at the only cost of writing longer formulas.
one can rewrite $\delta S$ in the following form:

$$
\delta S = \int d^4 x \left( i \tau_\mu P_\mu J^\mu_j + i \sigma_j P_\mu K^\mu_j \right),
$$

where:

$$
J^\mu_j(x) = \frac{1}{2} \left( \bar{P}^\mu \Phi e^{\frac{1}{2} R_0} N_j \Phi - e^{-\frac{1}{2} R_0} \Phi \bar{P}^\mu N_j \Phi \right) +
\frac{\lambda}{2} \epsilon_{ijkl} \left( e^{\lambda R_0} \bar{P}^\mu P_k \Phi e^{\frac{1}{2} R_0} R_l \Phi - e^{\frac{1}{2} R_0} P_k \Phi \bar{P}^\mu R_l \Phi \right),
$$

$$
K^\mu_j(x) = \frac{1}{2} \left( e^{\lambda R_0} \bar{P}^\mu \Phi e^{\frac{1}{2} R_0} R_j \Phi - e^{\frac{1}{2} R_0} \Phi \bar{P}^\mu R_j \Phi \right).
$$

And by spatial integration of the $J^0_j(x)$ and the $K^0_j(x)$

$$
Q^N_j \equiv \int d^3 x J^0_j(x), \quad Q^R_j \equiv \int d^3 x K^0_j(x),
$$

we do obtain six time-independent (conserved) charges. The verification of this time independence

Analogously to the notation $\tilde{q}_i = \frac{1}{\sqrt{2}} \sinh \left( \frac{1}{2} q_i \right)$, we also use the notation $\tilde{q}_i = e^{\frac{1}{2} q_i} q_i$ to write more compactly some frequently occurring combinations of Fourier parameters.

The details of this derivation will be reported elsewhere.\[3\]
In light of the findings reported in the previous section, we should stress that, as the careful reader can easily verify, one can perform a Noether analysis of pure space rotations ($\tau_j = 0$ from the onset of the Noether analysis) working with ordinary (commutative) transformation parameters $\sigma_j$, and obtain exactly the charges $Q^R_j$. Therefore these charges can be meaningfully interpreted as charges associated with the space-rotation symmetries of the theory. Instead, since pure boosts are not allowed, one should perhaps be cautious in characterizing the charges $Q^N_j$ as resulting from the invariance under boosts.

IV. CLOSING REMARKS

The understanding of the physical significance of Hopf-algebra spacetime symmetries must still be considered “work in progress”. The recently acquired [4, 8, 9] capability to bring to completion some Noether analyses should accelerate this progress, but several corollary issues must be solved in order to fully exploit this new tool. Combining some findings reported in Ref. [4] and some of the observations reported here, we are proposing a criterion for the search of a suitable description of transformation parameters: these parameters should be allowed to be “noncommutative” (allowed to be subject to nontrivial rules of commutation with the spacetime coordinates) but should still act only by (associative) multiplication on the spacetime coordinates. While of course in this young subject it is appropriate to continue to probe the robustness of all ideas, this criterion has passed already some nontrivial tests, most notably leading to successful Noether analyses in all applications attempted so far.

Concerning the time-independent quantities, the charges, that our Noether analyses produce a lot remains to be understood in order to establish whether these time-independent quantities really provide an acceptable formalization of quantities we measure in a laboratory. The best way to illustrate our concerns is to consider the charge associated with time-translation symmetry. How is that charge related to the observable measured by, say, a calorimeter? The way in which a calorimeter works depends very strongly on the law of conservation of energy-momentum in particle collisions, and this is an aspect of $\kappa$-Minkowski theories that our Noether-analysis techniques still do not allow us to master. Some of these issues could be addressed within a symmetry analysis of quantum fields in $\kappa$-Minkowski, whereas here we only considered classical fields.

Still, the fact that within our line of analysis we encountered an obstruction for pure boosts could be rather valuable. One of the primary motivations for considering spacetime noncommutativity comes from the desire to develop some intuition for the implications of non-classical, “fuzzy”, spacetimes, but at least in the (not isolated) case of $\kappa$-Minkowski, very little has been accomplished toward a physical characterization of the fuzzyness. For example, attempts to describe the observable “distance between two $\kappa$-Minkowski spacetime points” have shown very little promise. One might perhaps, and the results reported here could provide a starting point for that, attempt to characterize spacetime fuzzyness in terms of “fuzzy limitations” for symmetry transformations, rather than directly in terms of observables such as distance, area and volume.
While we were in the final stages of preparation of this manuscript, the paper in Ref. [10] was posted on the arXiv, also reporting an analysis of space-rotation and boost symmetries of some theories in $\kappa$-Minkowski spacetime, but within a setup which is completely different form the one we adopted here and with interest directed toward different aspects of the theories. The equation of motion (and the action) considered in Ref. [10] are different from ours, and they are analyzed following a different technical scheme. In particular, Ref. [10] does not provide a formula for the charges carried by classical fields in the noncommutative spacetime, but rather expresses the charges in terms of some operators for the “number of particles” to be used in a corresponding theory of quantum fields. Most importantly for us, Ref. [10] does not discuss the possibility of limitations on the “purity” of Lorentz-sector transformations. In Section III of Ref. [10] there is a discussion of Leibniz rule for a Lorentz-sector differential transformation, but no comments are offered on possible limitations to the types of pure transformation that are allowed. And actually we find that (probably just as a result of a mere typographical error) the equations reported in Section III of Ref. [10] do not implement Leibniz rule for the relevant differential transformation. The transformation parameters $\omega_{\mu \nu}$ introduced in Section III of Ref. [10] would be “noncommutative parameters” of the type we advocate here, as implicitly codified in the equation $e^{i \vec{k} \cdot \vec{x}} e^{-ik_0 x_0} \omega_{\mu \nu} = \omega_{\alpha \beta} K_{\mu \nu}^{\alpha \beta} e^{i \vec{k} \cdot \vec{x}} e^{-ik_0 x_0}$, but the properties later attributed to the matrix $K_{\mu \nu}^{\alpha \beta}$, which are $K_{\mu \nu}^{\alpha \beta} = \delta_{\mu}^{\alpha} k_{\nu}^{\beta}$ with $k_{\nu}^{\beta} = \begin{pmatrix} 1 & \vec{k} \\ 0 & 1 e^{-k_0} \end{pmatrix}$, would amount to a “no-pure-space-rotation” uncertainty principle and would not comply with the Leibniz-rule requirement.