Quasi exactly solvable (QES) equations with multiple algebraisations

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Abstract
We review three examples of quasi exactly solvable (QES) Hamiltonians which possess multiple algebraisations. This includes the most prominent example, the Lamé equation, as well as recently studied many-body Hamiltonians with Weierstrass interaction potential and finally, a $2 \times 2$ coupled channel Hamiltonian.

1 Introduction
Quasi exactly solvable (QES) operators can be formed by considering elements of the enveloping algebra of some Lie Algebras realised in terms of differential operators preserving a finite dimensional vector space of smooth functions. Some of them can be transformed into Schrödinger operators after performing a suitable change of variable and a suitable change of function (also called a “gauge transformation”). These Schrödinger operators admit a finite number of algebraic eigenvectors and are thus called QES. For certain QES Schrödinger operators there exist several choices of the “gauge functions” which then lead to different algebraisation of the corresponding eigenvalue equation. If this is possible, we speak of “multiple algebraisations”. The

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2 Quasi exact solvability (QES)

On the one hand, we have a Hamiltonian $H(\vec{x}) = -\Delta d + V(\vec{x})$ with $\vec{x} \in \mathbb{R}^d$ and the corresponding eigenvalue equation $H\psi(\vec{x}) = E\psi(\vec{x})$. On the other hand, we have a Lie Algebra $G$ of dimension $n_G$ realised in terms of operators $J_k = J_k(\vec{y}, \vec{\nabla} \vec{y})$, $\vec{y} \in \mathbb{R}^d$, $k = 1, ..., n_G$ which preserves a $m$-dimensional vector space $\mathcal{V}$ of smooth functions.

Now, if there exists a) a change of variables $\vec{x} = \vec{x}(\vec{y})$, b) an element $\mathcal{H}(\vec{y})$ of the enveloping algebra of $G$ as well as c) an invertible scalar function $f(\vec{x})$ such that $H(\vec{x}) = f(\vec{x})\mathcal{H}(\vec{y})f^{-1}(\vec{x})|_{\vec{y}=\vec{g}(\vec{x})}$, then the Hamiltonian $H$ preserves the vector space $f\mathcal{V}$: $Hf\mathcal{V} \subseteq f\mathcal{V}$. As a consequence, the restriction of the operator $H$ to the vector space $\mathcal{V}$ leads to an algebraic equation [1]. This equation allows to construct $m$ eigenvectors algebraically. In this construction, the factor $f$ which is called “gauge factor” plays a crucial role. The above procedure is usually called an algebraisation of the Schrödinger problem.

In this contribution, we will put the main emphasis on quantum Hamiltonians $H$ which possess multiple, i.e. several algebraisations. More specifically, more than one “gauge factor” $f$, say $f_a$, $a = 1, 2, 3...$ exists. This allows to write $H(\vec{x}) = f_a(\vec{x})\mathcal{H}(\vec{y})f_a^{-1}(\vec{x})|_{\vec{y}=\vec{g}(\vec{x})}$ where $J_k,a$ are now the generators of the Lie Algebra $G_a$. Correspondingly, the invariant vector spaces $\mathcal{V}_a$ of the generators $J_{k,a}$ can also be of different nature.

3 Hamiltonians with multiple algebraisations

3.1 The Lamé equation

A natural example of a QES equation for which multiple algebraisations exists is the Lamé equation:

$$\left(-\frac{d^2}{dx^2} + N(N+1)k^2sn^2(x,k)\right)\psi(x) = E\psi(x) \quad (1)$$

where $sn(x,k)$ is the Jacobi elliptic function of period $4K(k)$ with $K(k)$ being the complete elliptic integral of first type. The potential in this problem is periodic with period $2K(k)$ and the parameter $N$ determines the height of the potential. For the equation $(1)$, four different algebraisations ($a = 1, 2, 3, 4$) exist. For the case of even $N = 2n$, $n \in \mathbb{N}$, the gauge factors are given by:

$$f_1 = 1 \ , \ f_2 = sn(x,k)cn(x,k) \ , \ f_3 = sn(x,k)dn(x,k) \ , \ f_4 = dn(x,k)cn(x,k) \ . \quad (2)$$
The relevant change of variable is \( y(x) = sn^2(x, k) \). Then \( \mathcal{H}_a = f_a H f_a^{-1} \) can be written in terms of the operators

\[
J_+ = y^2 \frac{d}{dy} - n_a y \ , \ J_0 = y \frac{d}{dy} - \frac{n_a}{2} \ , \ J_- = \frac{d}{dy} ,
\]

where \( n_1 = n \), \( n_2 = n_3 = n_4 = n - 1 \) and \( \mathcal{V}_1 = \mathcal{P}(n) \), \( \mathcal{V}_2 = \mathcal{V}_3 = \mathcal{V}_4 = \mathcal{P}(n - 1) \). Here \( \mathcal{P}(n) \) denotes the set of polynomials of degree less or equal to \( n \) in \( y \). The four underlying algebras \( \mathcal{G}_a \) are in this case equal to \( sl(2) \).

### 3.2 Many-body Hamiltonians

In this section, we will discuss how the idea of multiple algebratisation can be applied to many-body Hamiltonians with \( N \) degrees of freedom. The Hamiltonians considered here are generalisations of Olshanetsky-Perelomov-[2] and Inozemtsev-[3]-type. More specifically, the Hamiltonian is given by:

\[
H_N(\vec{x}) = -\Delta_N + a(a - 1) \sum_{j,k=1 \atop j \neq k}^{N} (P(x_j + x_k) + P(x_j - x_k))
\]

\[
+ \quad 4b(b - 1) \sum_{k=1}^{N} P(2x_k) + c \sum_{k=1}^{N} P(x_k + i\beta) . \quad (4)
\]

\( P(z,g_2,g_3) \) denotes the Weierstrass function of parameters \( g_2, g_3 \) which are related to the periods on the real and imaginary axes of the complex plane, \( 2\alpha \) and \( 2i\beta \), respectively. \( a, b, c \) are coupling constants. For later use, we recall the following identities: (i) \( \left( \frac{dP(z)}{dz} \right)^2 = 4P^3(z) - g_2P(z) - g_3 \) and (ii) \( P(x + i\beta) = e_3 + (e_2 - e_3)sn^2(\sqrt{e_1 - e_3}x, k) \) with \( k = \frac{e_2 - e_3}{e_1 - e_3} \). The singularity of the Weierstrass function at the origin \( P(\varepsilon << 1) = \frac{1}{\varepsilon^2} + O(\varepsilon^2) \) renders the potential singular for \( |x_k - x_l| \to 0 \) such that the eigenvalue problem can be restricted to the domain \( D = \{(x_1,x_2,...x_N)| \ 0 < x_1 < x_2 < ... < x_N < \alpha \} \). The condition that the wave function vanishes on the boundary of \( D \) leads to a discrete spectrum of \( H_N \).

Before addressing the main result of [4], we give the following definition of symmetric polynomials [5]:

\[
\tau_1 = z_1 + z_2 + ... + z_N , \ \tau_2 = z_1z_2 + z_1z_3 + ... + z_{N-1}z_N , \ .... , \ \tau_N = z_1z_2z_3...z_N \quad (5)
\]

and the definition of the vector space

\[
\mathcal{V}_m = \text{span}\{\tau_1^{l_1} \tau_2^{l_2} ... \tau_N^{l_N} | \sum_{j=1}^{N} l_j \leq m \} , \ \text{dim}(\mathcal{V}_m) = C_{m}^{N+m} \quad (6)
\]

where \( C \) denotes the combinatoric symbol. It is well known [5] that the space \( \mathcal{V}_m \) is preserved by a set of operators realising the Lie Algebra \( sl(N + 1) \).
Table 1: Specific limits of the five parameter Hamiltonian $H(a, b, c, e_2, e_3)$

| Parameters   | # of algebraic eigenvectors | Remark               |
|--------------|-----------------------------|----------------------|
| $(a, b, c, e_2, e_3)$ | 0                           |                      |
| $(a, b, c_m, e_2, e_3)$ | $C_{N^{+m}}$                 | see [4]              |
| $(a, 0, c_m, e_2, e_3)$ | $C_{N^{+m}} + 3C_{N^{1+m}}$ | see further text     |
| $(0, 0, c_m, e_2, e_3)$ | $(4m + 1)^N$                 | $N$ decoupled Lamé equations |
| $(0, 0, c_m, e_2, e_2)$ | infinite                    | $N$ free oscillators  |

**Proposition 1**: Let

(i) $\mu(\vec{x}) = \prod_{j<k}^{N} [P(x_j + i\beta) - P(x_k + i\beta)]^a \prod_{k=1}^{N} [P'(x_k + i\beta)]^b$

(ii) $z_k = P(x_k + i\beta)$

(iii) $H_N(\vec{z}) = \mu^{-1}(\vec{x})H_N(\vec{x})|_{\vec{x} = \vec{z}}$

(i) is the “gauge factor” and (ii) the new variable. Then $H_N(\vec{z})$ depends only on the symmetric variables $\tau_1, ..., \tau_N$ and $H_N(\vec{z})\mathcal{V}_m \subseteq \mathcal{V}_m$ iff

$$c \equiv c_m = [2m + 2a(N - 1) + 4b] \cdot [2m + 1 + 2a(N - 1) + 2b], m \in \mathbb{N}. \quad (7)$$

If in the above expression for the coupling constant $c$ the parameter $m$ is an integer the total number of algebraically obtainable eigenvectors is $C_{m^{+m}}$.

The five parameter Hamiltonian $H(a, b, c, e_2, e_3)$ has well-known limits for specific choices of the parameters as is denoted in Table 1.

It should be remarked that out of the $(4m + 1)^N$ eigenvectors for the case $a = b = 0$ a total number of $C_{N^{+m}} + 3C_{N^{-1+m}}$ are completely symmetric under permutations of the coordinates and as such can be written in terms of the $\tau$-variables. This number contrasts with the number of algebraic states existing in [5]. This gives a hint to the fact that further algebraisations might be possible to construct by introducing different “gauge factors”.

In [6], we introduced these additional factors of the form

$$\tilde{\mu}(\vec{z}) = \prod_{k=1}^{N} (z_k - e_1)^{\nu_1}(z_k - e_2)^{\nu_2}(z_k - e_3)^{\nu_3} \quad (8)$$

such that $\tilde{H}_N(\vec{z}) = \tilde{\mu}^{-1}(\vec{z})H_N(\vec{z})\tilde{\mu}(\vec{z})$. This then leads to the following

**Proposition 2**: If the exponents $\nu_i, i = 1, 2, 3$ in (8) are chosen according to $\nu_i = 0$
or \( \nu_i = \frac{1}{2} - b \), then \( \tilde{H}_N(z) \mathcal{M}_{\tilde{m}} \subseteq \mathcal{M}_{\tilde{m}} \) where \( \tilde{m} := m + (b - \frac{1}{2})n_f \).

In the above definition of \( \tilde{m} \), \( n_f \) denotes the number of non-zero values chosen for the exponents \( \nu_i \), that is to say \( n_f = 0 \) or \( 1 \) or \( 2 \) or \( 3 \). This proposition results into the existence of seven possible new algebraisations. The algebraisations studied in [5] correspond to the choice \( n_f = 0 \). The choices \( n_f = 1 \) or \( n_f = 2 \) lead to three new algebraisations according to the possible permutation of the non-trivial factor. Finally, the choice \( n_f = 3 \) obviously leads to only one new algebraisation. In order to avoid singularities of all possible “gauge factors”, we will restrict \( b \) to \( 0 \leq b < \frac{1}{2} \).

It should be remarked that the previous condition \( m \in \mathbb{N} \) is now replace by the condition \( \tilde{m} \in \mathbb{N} \). Then, for fixed \( b \) and \( n_f \) the parameter \( m \) defining the coupling constant \( c \) has to be chosen appropriately. Though \( m \) is still quantised, the number of possible values of the coupling constant \( c \equiv c_m \) is considerably increased. To finish this section, we investigate for which fixed values of the coupling constants \( (a, b, c, e_2, e_3) \) multiple algebraisations exist. For generic values of \( b \) one (resp. three) algebraisations occur for the case when \( \tilde{m} \) is an integer and \( n_f = 0 \) or \( n_f = 3 \) (resp. \( n_f = 2 \) or \( 3 \)). For \( b = 0 \) four algebraisations coexist corresponding either to (a) \( n_f = 0 \) (with invariant subspace \( \mathcal{M}_m \)) and \( n_f = 2 \) (with invariant subspace \( \mathcal{M}_{m-1} \)) or to (b) \( n_f = 1 \) and \( n_f = 3 \). Finally, the case \( b = 1/6 \) leads to two algebraisation with \( n_f = 0 \) and \( n_f = 3 \) with respective invariant spaces \( \mathcal{M}_m \) and \( \mathcal{M}_{m-1} \).

### 3.3 Coupled channel equation

Apart from studying many body equations with algebraic properties also systems of coupled Schrödinger equations of the form

\[
H_{2 \times 2} = -\frac{d^2}{dx^2} I_2 + \begin{pmatrix} V_{11}(x) & V_{12}(x) \\ V_{12}^*(x) & V_{22}(x) \end{pmatrix}
\]

(9)

can be studied. In [7] a classification of equations of this type which admit multiple algebraisations was achieved. Here, we will study one example which has a physical motivation. The potential reads:

\[
V = sn^2(x, k) \begin{pmatrix} A - 2b & 0 \\ 0 & A + 2b \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \theta sn(x, k)cn(x, k) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

(10)

where \( k \in [0 : 1], b \in \mathbb{R}, m \in \mathbb{N} \) and \( A = \frac{k^2}{4}(4m^2 + 2m + 1), \theta = 4b^2 - k^4(4m + 1)^2 \). Using again the change of variable \( y = sn^2(x, k) \) and an appropriate “gauge factor” which is now a \( 2 \times 2 \) matrix, we were able to show that the Hamiltonian \( H_{2 \times 2} \) has a quadruple algebraisation. Here we present two of the algebraisations and refer the reader for the remaining ones to [7]:

\[
(F^{-1}HF) \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m) \end{pmatrix} \subseteq \begin{pmatrix} \mathcal{P}(m-1) \\ \mathcal{P}(m) \end{pmatrix},
F = \begin{pmatrix} sn & 0 \\ 0 & cn \end{pmatrix} \begin{pmatrix} 1 & \kappa_1 \\ 0 & 1 \end{pmatrix}
\]

5
\((G^{-1}HG) \binom{\mathcal{P}(m-1)}{\mathcal{P}(m-1)} \subseteq \binom{\mathcal{P}(m-1)}{\mathcal{P}(m-1)}\), \(G = \begin{pmatrix} \sin & \cos & \delta_n & 0 \\ 0 & \delta_n & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{\theta}{k^2} \\ 0 & 1 \end{pmatrix}\)

where \(\kappa_1 = -\theta/(2b+k^2(1+4m))\). Note that in suitable realisations (which we will not present here) the operator \(F^{-1}HF\) can be interpreted as an element of the Super-Lie Algebra \(osp(2,2)\), while \(G^{-1}HG\) is an element of the Lie Algebra \(sl(2) \times sl(2)\). This provides an example of multiple algebraisations where the different algebraisations are related to distinct Lie Algebras.

The limit \(k = 0\) of these coupled equations is non-trivial and leads to the following operator:

\[
H_{2 \times 2} = -\frac{d^2}{dx^2} I_2 + 2b \begin{pmatrix} \cos^2(x) - \frac{1}{2} & \cos(x) \sin(x) \\ \cos(x) \sin(x) & \sin^2(x) - \frac{1}{2} \end{pmatrix}, \quad x \in [0, 2N\pi].
\] (11)

In this limit, the dependence of the potential on the integer \(m\) disappears and the system becomes exactly solvable. In order to construct the flag of invariant vector spaces, it is useful to introduce the vectors

\[
E(p) \equiv \left( \begin{array}{c} \cos \frac{px}{N} \\ \sin \frac{px}{N} \end{array} \right), \quad G(p) \equiv \left( \begin{array}{c} -\sin \frac{px}{N} \\ \cos \frac{px}{N} \end{array} \right), \quad p = 0, \pm 1, \pm 2, \cdots
\] (12)

It is easy to show that the following vector spaces are left invariant by the operator (11):

\[
\begin{align*}
V_0 &= \text{Span}\{E(N)\}, \quad \tilde{V}_0 = \text{Span}\{G(N)\} \\
V_k &= \text{Span}\{E(N+k), E(N-k)\}, \quad k = 1, 2, 3, \ldots \\
\tilde{V}_k &= \text{Span}\{G(N+k), G(N-k)\}, \quad k = 1, 2, 3, \ldots
\end{align*}
\] (13)

The operator can be diagonalized on any of these vector spaces and leads to the following sets of eigenvalues. Using natural notations, we find

\[
\omega^2(N,k,\pm) = 1 - 2b + \frac{k^2}{N^2} \pm \sqrt{b^2 + 4 \frac{k^2}{N^2}}, \quad k = 0, 1, 2, \ldots
\] (14)

\[
\tilde{\omega}^2(N,k,\pm) = \omega^2(N,k,\pm), \quad k = 1, 2, 3, \ldots
\] (15)

This is reminiscent to the situation in the elliptic Perelomov model: the elliptic model is not solvable for generic values of the parameters determining the period but in the limits corresponding to rational or trigonometric potentials the model becomes completely solvable.

The above system is related to the normal mode analysis of the soliton appearing in the two dimensional Goldstone-model with periodic boundary condition of the space variable \([8]\). We would like to point out that coupled-channel Hamiltonians which posses a multiple algebraisation are rare, very specific and difficult to construct but many of them are related to the normal mode analysis of classical solutions (solitons or sphaleron) of simple field theories. These algebraic properties can be used as testing ground for more ambitious models.
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