POSITIVITY OF ANTICANONICAL DIVISORS IN ALGEBRAIC FIBRE SPACES

CHI-KANG CHANG

Abstract. Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties and $F$ be a general fibre of $f$. We prove an Iitaka-type inequality $\kappa(X, -K_X) \leq \kappa(F, -K_F) + \kappa(Y, -K_Y)$ under some mild conditions. We also obtain results relating to the positivity of $-K_X$ and $-K_Y$.

1. Introduction

Given a surjective projective morphism $f : X \to Y$ between normal projective varieties, a natural and important problem is to compare the canonical divisors of $X$ and $Y$. In particular, the well-known Iitaka conjecture asserts that

$$\kappa(X) \geq \kappa(F) + \kappa(Y),$$

where $F$ denotes a general fibre. The main purpose of the conjecture is to relate the ”positivities” of $K_X$, $K_Y$ and $K_F$. Additionally, it is also very natural and important to compare the positivities of $-K_X$ and $-K_Y$ especially in the case of Fano varieties and related varieties.

To study the above problems, recall that we have the weakly positivity theorem developed and generalized by [C04, F78, F17, K81, V83]. These theorems show that for sufficiently divisible positive integer $m$, the sheaf $f_*\mathcal{O}_X(m(K_X/Y))$ is a weakly positive sheaf in the sense of [EG19] (or [V83]), where $K_{X/Y} = K_X - f^*K_Y$. Thus, we should expect that the positivity of $K_Y$ will affect the positivity of $K_X$. Consequently, we should expect the positivity of $-K_X$ will affect the positivity of $-K_Y$.

The main purpose of this paper is to study the aspects of positivity of the anti-canonical divisors in algebraic fibre spaces. The main theorem below can be thought of as an analog of the Iitaka conjecture of the anti-canonical Iitaka dimension.

Theorem 1.1 (cf. Theorem 4.1). Let $f : X \to Y$ be an algebraic fibre space between normal projective $\mathbb{Q}$-Gorenstein varieties, and $F$ be a general fibre of $f$. Suppose $X$ has at worst klt singularities, and $-K_X$ is effective with stable base locus $B(-K_X)$ which does not dominate $Y$. Then we have

$$\kappa(X, -K_X) \leq \kappa(F, -K_F) + \kappa(Y, -K_Y).$$

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Note that the strict inequality in Theorem 1.1 can happen:

**Example 1.2.** Let $X$ be an elliptic K3 surface, with a morphism $f : X \to \mathbb{P}^1$ whose general fibres are smooth elliptic curves. Then $0 = \kappa(X, -K_X) < \kappa(F, -K_F) + \kappa(Y, -K_Y) = 1$, which shows that $X$ is an example of strict inequality in Theorem 1.1.

The proof of Theorem 1.1 relies on the following theorem:

**Theorem 1.3** (cf. Theorem 3.8). Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Let $\Delta$ an effective $\mathbb{Q}$-Weil divisor on $X$ such that $(X, \Delta)$ is klt, and $D$ a $\mathbb{Q}$-Cartier divisor on $Y$. Suppose that $-(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective with the stable base locus $B_-(-(K_X + \Delta) - f^*D)$ that does not surject to $Y$. Then $-K_Y - D$ is $\mathbb{Q}$-effective.

The motivation of our investigation traces back to the following question asked by Demailly-Peternell-Schneider in [DPS01]:

**Question 1.4.** Let $f : X \to Y$ be a surjective morphism between normal projective $\mathbb{Q}$-Gorenstein varieties. Suppose that $-K_X$ is pseudo-effective and the non-nef locus of $-K_X$ does not dominate $Y$. Is $-K_Y$ pseudo-effective?

We recall some previous results towards answering the above question:

- In [CZ13 Main Theorem], Chen and Zhang proved that if there is a log canonical pair $(X, \Delta)$ such that if $-(K_X + \Delta)$ is nef, then $-K_Y$ is pseudo-effective. Also, [CZ13 Example 1.5] shows that if the non-log canonical locus of $(X, \Delta)$ surjects onto $Y$, then $-(K_X + \Delta)$ being nef does not imply that $-K_Y$ is pseudo-effective.
- In [Den17 Theorem D], by using analytic methods, Deng proved that if there is a log canonical pair $(X, \Delta)$ such that $-(K_X + \Delta)$ is pseudo-effective and the non-nef locus of $-(K_X + \Delta)$ does not surject onto $Y$ via $f$, then $-K_Y$ is pseudo-effective.
- In [EG19 Theorem 3.1], Ejiri and Gongyo generalized Chen and Zhang's theorem, by showing that even if $(X, \Delta)$ is not log canonical, as long as $(F, \Delta|_F)$ is lc for general fibres $F$, then $-K_Y$ is still pseudo-effective. The proof is algebraic and can be generalized to the positive characteristic.

Using the method in the proof of [EG19 Theorem 3.1], with additional ideas from [CZ13 Main Theorem], we can give an algebraic proof of Theorem 3.1, which generalized [EG19 Theorem 3.1] in characteristic zero. Furthermore, Theorem 3.1 simplifies in the following theorem which generalizes [CZ13 Main Theorem].

**Theorem 1.5** (cf. Theorem 3.1). Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties. Suppose the following conditions hold:

1. There is a log pair $(X, \Delta)$ which is log canonical;
2. $Y$ is $\mathbb{Q}$-Gorenstein and there is a $\mathbb{Q}$-Cartier divisor $D$ on $Y$ such that $L := -(K_X + \Delta) - f^*D$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor;
3. The restricted base locus $B_-(L)$ does not surject onto $Y$ via $f$.

Then $-K_Y - D$ is pseudo-effective.
Note that our result does not cover [Den17, Theorem D(a)], since for varieties with singularities worse than klt singularities, it is not confirmed if $\operatorname{Nef}(\mathbb{Q}) = B_+(\mathbb{Q})$ for $\mathbb{Q}$-Cartier divisors (cf. [BBP13, Conjecture 1.7], [CDiB13, Theorem 1.2]). Using this theorem, we can give an algebraic proof of a bigness criterion of anti-canonical divisor:

**Theorem 1.6** (cf. Theorem 3.7). Under the same notation and assumption of Theorem 1.5. Assume $L$ is big, and one of the following conditions holds:

1. $B_+(L)$ does not dominate $Y$;
2. $B_-(L)$ does not surject onto $Y$, and $(X, \Delta)$ is klt.

Then $-K_Y - D$ is big.

In the above theorems, the assumption that the asymptotic base locus does not surject onto $Y$ is essential as the following example will show.

**Example 1.7.** Let $Y$ be a smooth curve with genus $g \geq 2$, and consider the ruled surface $X = \mathbb{P}_Y(\mathcal{O} \oplus \mathcal{O}(-K_Y - D))$, where $D$ is an ample divisor on $Y$ with $\deg D = d > 2g - 2$. Then $K_X = -2C_0 - f^*D$, where $f$ is the structure morphism $f : X \to Y$ and $C_0$ is the distinguished section. Note that $-K_X \geq f^*D + C_0 \geq 0$ and $(f^*D + C_0)^2 = d + 2 - 2g > 0$, which implies $-K_X$ is big, but $-K_Y$ is anti-ample. Hence the conclusions of the above theorems fail. Note that in this case, we have $B_-(K_X) = B(-K_X) = B_+(K_X) = \operatorname{Supp}C_0$, which is surject onto $Y$.

The structure of this paper is as follows. In Section 2, we give a brief review of asymptotic invariants for $\mathbb{Q}$-Cartier divisor, klt-trivial fibration, moduli $b$-divisors, and weakly positive sheaves. In Section 3, we generalize [EG19, Theorem 3.1] into Theorem 3.1. Then we apply Theorem 3.1 to prove Theorem 3.7. This is then followed by an application of the theory of moduli $\mathbb{Q}$-$b$-divisor and [A04, Theorem 3.3] to prove Theorem 3.8. This leads to Corollary 3.9 where we describe some asymptotic invariants of the relative anti-canonical divisor $-K_{X/Y}$. In Section 4, we prove Theorem 4.1 by first using the methodology of [EG19] to show the inequality for $\kappa(Y, -K_Y) = 0$ then using the Iitaka fibrations to obtain the general case. In section 5, we discuss related questions and possible generalizations.

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2. Preliminaries

In this paper, we work over an algebraically closed field $k$ with characteristic zero. A morphism between normal varieties $f : X \to Y$ is called an *algebraic fibre space* if it is a surjective projective morphism with connected fibres. In particular, if $f$ is an algebraic fibre space, then $f_* \mathcal{O}_X = \mathcal{O}_Y$. We say a $\mathbb{Q}$-divisor $D$ on a variety is $\mathbb{Q}$-Cartier (resp. $\mathbb{Q}$-effective) if there is a positive integer $m$ such that $mD$ is a Cartier divisor (resp. linearly equivalent to an effective divisor.) In particular, a $\mathbb{Q}$-Cartier divisor $D$ is $\mathbb{Q}$-effective if and only if its Iitaka dimension $\kappa(X, D) \geq 0$.

For a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ on $X$, we will denote

$$D^+ := \sum_{d_i > 0} d_i D_i, \quad D^- := \sum_{d_i < 0} d_i D_i,$$

and that

$$D^h := \sum_{f(\text{Supp } D_i) = Y} d_i D_i, \quad D^v := \sum_{f(\text{Supp } D_i) \neq Y} d_i D_i.$$

We called $D = D^+ - D^-$ the *effective decomposition*, and $D = D^h + D^v$ the *vertical decomposition* of $D$. For other standard notions of birational geometry, such as singularities of pairs, we refer the readers to [KM98] for more details.

2.1. Asymptotic invariants of $\mathbb{Q}$-Cartier divisors. In this subsection, we will give a brief review of the basic properties of asymptotic invariants of $\mathbb{Q}$-divisors. Further details of this topic can be found in [BBP13], [BKKMSUL5], and [ELMNP06].

**Definition 2.1.** Let $D$ be a Cartier divisor on a normal variety, the stable base locus of $D$, denoted by $B(D)$, is defined as follows:

$$B(D) := \cap_{m \in \mathbb{Z}_{>0}} \text{Bs}(mD).$$

If $D$ is a $\mathbb{Q}$-Cartier divisor, and $l$ is a natural number such that $lD$ is Cartier, then we define $B(D)$ to be the stable base locus of $lD$.

Moreover, by fixing an ample divisor $A$, we can define the following sets

$$B_+(D) := \cap_{\varepsilon > 0} B(D - \varepsilon A);$$

$$B_-(D) := \cup_{\varepsilon > 0} B(D + \varepsilon A).$$

It is well-known that both $B_+(D)$ and $B_-(D)$ are independent of the choice of $A$. We call $B_+(D)$ the augmented base locus of $D$, and $B_-(D)$ the restricted base locus (or diminished base locus) of $D$. 

Clearly, from the definition, we have

\[ B_+(D) \subset B(D) \subset B_-(D). \]

It is easy to see that \( B_+(D) \) and \( B_-(D) \) depend only on the numerical equivalent class of \( D \) and hence they are numerical invariants, however \( B(D) \) is not a numerical invariant (cf. [ELMNP06, Example 1.1]).

Remark 2.2. By the definition, both \( B_+(D) \) and \( B_-(D) \) are Zariski closed subsets of \( X \). But by [Le13], \( B_-(D) \) can be a countably infinite union of Zariski closed subsets.

From the theory of the cone of divisors (cf. [La04, Chapter 1.4, Chapter 2.2]), we immediately have the following correspondence between these asymptotic invariants and positivity of \( \mathbb{Q} \)-divisor (cf. [BKKMSU15, section 4]):

- \( D \) is pseudo-effective \( \iff B_+(D) \neq X \),
- \( D \) is effective \( \iff B(D) \neq X \),
- \( D \) is big \( \iff B_+(D) \neq X \),
- \( D \) is nef \( \iff B_-(D) = \emptyset \),
- \( D \) is semiample \( \iff B(D) = \emptyset \),
- \( D \) is ample \( \iff B_+(D) = \emptyset \).

2.2. The klt trivial fibration and moduli (b-)divisor. In this subsection, we will give a brief review of the klt-trivial fibration and the moduli b-divisors. Details of these topics can be found in [A04, A05, F12, FG14, K97, and K98].

Definition 2.3. Let \( D \) be a divisor on a normal variety \( X \). A b-divisor \( D \) contains a family of divisors \( \{D_{X'}\} \), where \( X' \) is taken over all higher birational models \( \pi : X' \to X \) such that \( \pi \) is a proper birational morphism, \( D_{X'} \) is a divisor on \( X' \) such that \( \pi_*(D_{X'}) = D \), and \( \pi_*(D_{X''}) = D_{X'} \) for any birational morphism \( \pi' : X'' \to X' \). If \( D \) is a \( \mathbb{Q} \)-divisor, then the \( \mathbb{Q} \)-b-divisor is defined in the same way.

Definition 2.4. A klt-trivial fibration (which is equivalent to the lc-trivial fibration in the sense of [A05]) is an algebraic fibre space \( f : (X, B) \to Y \) between normal varieties with a sub log pair \( (X, B) \) such that

1. \( (X, B) \) is subklt over the generic point of \( Y \);
2. \( \text{rank } f_*\mathcal{O}_X([A(X, B)]) = 1 \);
3. \( K_X + B \sim_{\mathbb{Q}} f^*D \) for some \( \mathbb{Q} \)-Cartier divisor \( D \) on \( Y \).

Here, the discrepancy \( \mathbb{Q} \)-b-divisor \( A(X, B) = \{A_{X'}\} \) is defined by the formula

\[ K_{X'} = \pi^*(K_X + B) + A_{X'}. \]

Next, we recall the definition of the moduli \( \mathbb{Q} \)-b-divisors and the discriminant \( \mathbb{Q} \)-b-divisors.
Definition 2.5. Let \( f : (X, B) \to Y \) be a klt-trivial fibration, we define the discriminant \( \mathbb{Q} \)-divisor \( B_Y \) of \( f : (X, B) \to Y \) in the following way: Let \( P \) be a prime divisor on \( Y \), which is Cartier in a neighborhood of its generic point, then we define
\[
b_P := \max \{ t \in \mathbb{Q} | (X, B + tf^*P) \text{ is sublc over the generic point of } P \},
\]
and set
\[
B_Y := \sum P (1 - b_P) P,
\]
where \( P \) runs over all prime divisor of \( Y \). We set \( M_Y = D - K_Y - B_Y \) and call \( M_Y \) the moduli \( \mathbb{Q} \)-divisor of \( f : (X, B) \to Y \).

The moduli \( \mathbb{Q} \)-b-divisor \( M = \{ M_{Y'} \} \) and the discriminant \( \mathbb{Q} \)-b-divisor \( B = \{ B_{Y'} \} \) is defined in the following way: For a proper birational morphism \( \mu : Y' \to Y \), let \( X' \) be a normalization of the main component of \( X \times_Y Y' \) such that the induced morphism \( \pi : X' \to X \) is proper and birational. Define \( B_{X'} \) by
\[
K_{X'} + B_{X'} = \pi^*(K_X + B),
\]
then \( f' : (X', B_{X'}) \to Y' \) is also a klt-trivial fibration. Thus, let \( M_{Y'} \) and \( B_{Y'} \) be the moduli \( \mathbb{Q} \)-divisor and discriminant \( \mathbb{Q} \)-divisor of \( f' : (X', B_{X'}) \to Y' \). Then we set \( M_{Y'} = M_{Y'} \) and \( B_{Y'} = B_{Y'} \).

By the definition, it is easy to see that if \( B \) is \( \mathbb{Q} \)-effective, then so is \( B_Y \).

2.3. Weakly positive sheaves. In this subsection, we will give a brief review of the definition and basic properties of weakly positive sheaves. We adopt the convention and results in [EG19, Section 2.2] which will be necessary. More details of weakly positive sheaves can be found in [C04, Section 4.2], [EG19, Section 2.2], and [V83, Section 1].

Definition 2.6. Let \( X \) be a normal quasi-projective variety, \( \mathcal{G} \) be a coherent sheaf on \( X \), and \( A \) be a fixed ample divisor on \( X \).

(1) We say that \( \mathcal{G} \) is generically globally generated if the natural map \( H^0(X, \mathcal{G}) \otimes \mathcal{O}_X \to \mathcal{G} \) is surjective over the generic point of \( X \).

(2) We say that \( \mathcal{G} \) is weakly positive if for any natural number \( n \), there is a natural number \( m \) such that the sheaf \( (S^n(-))^\ast \otimes \mathcal{O}_X(mA) \) is generically globally generated, where \( S^n(-) \) denotes the \( n \)-th symmetric power, and \( (-)^\ast \) denotes the double dual.

Note that the definition is independent of the choice of \( A \), and when \( \mathcal{G} = \mathcal{O}_X(D) \) is a line bundle, \( \mathcal{G} \) is weakly positive if and only if \( D \) is a pseudo-effective divisor. Moreover, if \( \mathcal{G}|_U \) is weakly positive on \( U \) for some open dense subset \( U \subset X \) such that \( X - U \) has codimension at least 2, then \( \mathcal{G} \) is weakly positive on \( X \).

Here we recall two useful lemmas about the weakly positive sheaves.

Lemma 2.7. ([EG19, Lemma 2.4]) Let \( f : Y' \to Y \) be a surjective projective morphism between geometrically normal quasi-projective varieties over a field, let \( \mathcal{G} \) be a torsion-free coherent sheaf on \( Y' \):
(1) If there is no $f$-exceptional divisor on $Y'$, and $G$ is weakly positive, then $f^*G$ is also weakly positive. Here a prime divisor $E$ on $Y'$ is called $f$-exceptional if $f(E)$ has codimension at least 2 in $Y$.

(2) If $f^*G \otimes O_{Y'}(E)$ is weakly positive for some effective $f$-exceptional divisor $E$ on $X$, then $G$ is weakly positive.

Lemma 2.8. ([EG19, Lemma 2.5]) Let $F \rightarrow G$ be a generically surjective morphism between coherent sheaves on a normal quasi-projective variety over a field. If $F$ is weakly positive, so is $G$.

3. Positivity of the anti-canonical divisor of the image

We first prove a pseudo-effectiveness criterion of the anti-canonical divisor, which is a generalization of [EG19, Theorem 3.1] in characteristic zero. The proof follows the original argument of Ejiri and Gongyo closely.

Theorem 3.1. Let $f : X \rightarrow Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Let $\Delta = \Delta^+ - \Delta^-$ be a $\mathbb{Q}$-Weil divisor on $X$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, $(F, \Delta^+|_F)$ is log canonical for general fibre $F$ of $f$, and $f(Supp\Delta^-) \neq Y$. Suppose that there is a $\mathbb{Q}$-Cartier divisor $D$ on $Y$ such that $L := -(K_X + \Delta) - f^*D$ is a pseudo-effective $\mathbb{Q}$-Cartier divisor such that $B - (L)$ does not surject onto $Y$ via $f$. Then, for $l \in \mathbb{N}$ such that $l(K_X + \Delta)$ and $l(K_Y + D)$ are Cartier and $l\Delta^-$ is integral, we have that

$$\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^- + B))$$

is weakly positive for some effective $f$-exceptional $\mathbb{Q}$-divisor $B$. Moreover, if $Y$ has at worst canonical singularities, then we can take $B = 0$.

In particular, if $\Delta$ is effective, then $-K_Y - D$ is pseudo-effective by Lemma 2.7(2).

Proof. The proof is based on the methods in the proof of [EG19, Theorem 3.1], modified with ideas in [CZ13, Main Theorem]. First, we prove the case where $f$ is equidimensional $\mathbb{1}$. Let

$$\mathcal{F} := \mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^-)),$$

and $A$ be an ample divisor on $X$, then by definition, it suffices to show that for all $n \in \mathbb{N}$, there is some $m \in \mathbb{N}$ such that the sheaf $(\mathcal{F}^n \otimes \mathcal{O}_X(mlA))|_U$ is weakly positive on some Zariski open subset $U$ of $X$ with codim$(X - U) \geq 2$. Since $L$ is pseudo-effective, for any $n \in \mathbb{N}$, the $\mathbb{Q}$-Cartier divisor $G_n := L + \frac{1}{n}A$ is big with $B(G_n) \subset B_-(L)$. Therefore, $B(G_n)$ is a proper Zariski closed subset that does not dominate $Y$. Let $\pi : X' \rightarrow X$ be a birational morphism such that $\pi^{-1}(SuppB(G_n) \cup Supp\Delta) \cup Exc(\pi)$ is a snc divisor. Let $f' := f \circ \pi : X' \rightarrow Y$ and write

$$K_{X'} + \Delta' = \pi^*(K_X + \Delta) + E,$$

$\mathbb{1}$In this case, since the pullback of $\mathbb{Q}$-Weil divisors is well-defined, we do not need to assume $-K_Y$ is $\mathbb{Q}$-Cartier.
where $\Delta'$ is the proper transform of $\Delta$. The effective decomposition yields $(\Delta')^\pm = (\Delta^\pm)'$ and we can write $E = E^+ - E^-$. Let $F$ be a general fibre of $f$ and $F' = \pi^{-1}(F)$, then every component of $E^-(F)$ has coefficient at most 1 since $(F, \Delta|_F)$ is log canonical. Pick $\Gamma_n \sim \pi^*G_n$ be a general divisor such that $(F', (\Delta^+ + \Gamma_n + E^-)|_{F'})$ is log canonical, then by [C04, Theorem 4.13], for $k \gg 0$ the sheaf
\[
f^*(f^*_Y(O_nml(K_X' + \Delta^+ + \Gamma_n + E^-)) \otimes O_Y(-klK_Y))
\]
is locally free and weakly positive on a smooth open subvariety $Y_0 \subset Y$ with codim$(Y - Y_0) \geq 2$, hence the sheaf is weakly positive on $Y$. Therefore, for $m \gg 0$, we have the following generically surjective morphisms
\[
f^*(f^*_Y(O_{nml(K_X' + \Delta^+ + \Gamma_n + E^-)} \otimes O_Y(-nmK_Y)))
\]
isomorphism to the ample line bundle $O_X(mlA)$, and $\text{Supp}(\Delta^-) \cup \text{Supp}(f^*D)$ does not dominate $Y$. As $f$ is equidimensional, there is no $f$-exceptional divisor, thus $\mathcal{G}$ is weakly positive by Lemma 2.7(1) and Lemma 2.8.

Let $\mathcal{G}_1 := (\pi_*O_{nml(K_X' + \Delta^- - \pi^*f^*D + 1/n\pi*A)) \otimes f^*O_Y(-nmK_Y))$. Since $E^+$ is effective and $\pi$-exceptional, there is a smooth open subvariety $U \subset X$ with codim$(X - U, X) \geq 2$ such that $\mathcal{F}$ and $\mathcal{G}$ are locally free on $U$ and $\mathcal{G}|_U \cong \mathcal{G}_1|_U$.

Therefore, $\mathcal{G}_1$ is weakly positive since codim$(X - U, X) \geq 2$. By the projection formula, we have
\[
\mathcal{G}_1|_U \cong (O_X(nml(\Delta^- - f^*D + 1/nA)) \otimes f^*O_Y(-nmK_Y))|_U
\]
where the last isomorphism is due to $\mathcal{F}|_U$ being a line bundle by our choice of $U$. $(\mathcal{F}^{\otimes nm} \otimes O_X(mlA))|_U$ is weakly positive on $U$. It follows that $\mathcal{F}^{\otimes nm} \otimes O_X(mlA)$ is weakly positive on $X$ since codim$(X - U, X) \geq 2$. Lastly, $n$ can be taken to be arbitrarily large, thus $\mathcal{F}$ is weakly positive.

The proof for the general case follows verbatim to the argument of [EG19, Theorem 3.1]. For the convenience of the readers, we reproduce the proof here.

By the flattening theorem in [A000] 3.3, flattening lemma, there is a normal birational modification $\mu : Y' \rightarrow Y$ such that let and $X'$ be the normalization of the main component
of $X \times_Y Y'$, then $\pi : X' \to X$ is proper birational and the induced morphism $f' : X' \to Y'$ is equidimensional

\[
\begin{array}{c}
X' \xrightarrow{\pi} X \\
\downarrow f' \downarrow f \\
Y' \xrightarrow{\mu} Y.
\end{array}
\]

Now we define $\Delta'$ by

\[
\Delta' = \pi^*(K_X + \Delta),
\]

and write $\Delta' = \Delta^+ - \Delta^-$, then we have

\[
-(K_{X'} + \Delta') - \pi^* f'^* D = \pi^*(-(K_X + \Delta) - f^* D)
\]
is pseudo-effective with the restricted base locus does not surject onto $Y'$. Therefore, $\mathcal{O}_{X'}(l(f^*(-K_{Y'} - \mu^* D) + \Delta^-))$ is weakly positive since $f'$ is equidimensional. Now write $K_{Y'} = \mu^* K_Y + E$, and $E = E^+ - E^-$, then we have

\[
\mu^*(-K_Y) + E^- = -K_{Y'} + E^+ \geq -K_{Y'}.
\]

Thus $\mathcal{O}_{X'}(l(f^*(\mu^*(-K_Y) + E^- - \mu^* D) + \Delta^-))$ is also weakly positive, and so does $\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^- + \pi_*(f^* E^-)))$. Now, define $B := \pi_*(f^* E^-)$, then since $f'$ is equidimensional, we have $f_* B = \mu_* E^- = 0$. Also, if $Y$ has at worst canonical singularities, then $E^- = 0$. This completes the proof.

By Theorem 3.1, we generalize [CZ13, Corollary 2.3].

**Corollary 3.2.** Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Suppose there exists a log pair $(X, \Delta)$ such that $-(K_X + \Delta)$ is pseudo-effective and $\mathbb{Q}$-Cartier with the restricted base locus $B_-(-(K_X + \Delta))$ not surject onto $Y$, and for general fibre $F$ of $f$, $(F, \Delta|_F)$ is log canonical. Then either $Y$ is uniruled, or $K_Y \sim \mathbb{Q}0$.

**Proof.** The proof follows from Theorem 3.1 by repeating the original proof in [CZ13, Corollary 2.3]. Suppose $Y$ is not uniruled, then $K_Y$ is pseudo-effective by [BDPP13, Theorem 2.6]. However, by Theorem 3.1, $-K_Y$ is pseudo-effective. This implies that $K_Y \sim \mathbb{Q}0$ by [N04, Corollary 4.9].

**Corollary 3.3.** Let $f : (X, \Delta) \to Y$ be an algebraic fibre space between normal projective varieties such that $(X, \Delta)$ is klt and $Y$ is $\mathbb{Q}$-Gorenstein. Suppose furthermore that $-(K_X + \Delta)$ is pseudo-effective with the non-nef locus $\text{NNef}(-(K_X + \Delta))$ not surject onto $Y$, then $-K_Y$ is pseudo-effective.

**Proof.** It directly follows from [CDiB13, Theorem 1.2] and Theorem 3.1.

**Corollary 3.4.** The following theorems [EG19, Theorem 4.2, Proposition 4.4, Corollary 4.5, Corollary 4.7, and Corollary 5.1] still hold under the assumption $L$ is pseudo-effective with $B_-(L)$ not surject onto $Y$ (instead of being nef).
Proof. Pseudo-effectiveness and $B_-(\cdot)$ depends on the numerical equivalence classes only. So, in the proofs of all the above-mentioned theorems in [EG19], we can replace the nefness of $L$ in [EG19] by $L$ being pseudo-effective with $B_-(L)$ not surject onto $Y$. Using Theorem 3.1 instead of [EG19, Theorem 3.1], then the same conclusions hold.

As a corollary of Theorem 3.1, we have a bigness criterion for $-K_Y - D$, whose statement is very similar to [FG12, Theorem 3.1], but they cover different cases.

**Corollary 3.5.** Keep the same notation as in Theorem 3.1. Suppose there is a big $\mathbb{Q}$-Cartier divisor $H$ on $Y$ such that $-(K_X + \Delta) - f^*(D + H)$ is pseudo-effective with $B_-(-(K_X + \Delta) - f^*(D + H))$ not surject onto $Y$. Then for any $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, there is a rational number $\varepsilon > 0$ such that for sufficiently divisible positive integer $l$, the sheaf

$$\mathcal{O}_X(l(f^*(-K_Y - D - \varepsilon D_0) + \Delta^- + B))$$

is weakly positive for some effective $f$-exceptional $\mathbb{Q}$-divisor $B$. We can take $B = 0$ if $Y$ has at worst canonical singularities.

**Proof.** This statement follows immediately from Theorem 3.1.

**Remark 3.6.** In Corollary 3.5, if $\Delta \geq 0$, this means for any $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, there is a rational number $\varepsilon > 0$ such that $-K_Y - D - \varepsilon D_0$ is pseudo-effective, hence $-K_Y - D \in \text{Int}(\text{Eff}(Y)) = \text{Big}(Y)$ is big.

As another application of Theorem 3.1, we have another bigness criterion of $-K_Y$ in the following. The reader can compare our results with [Den17, Theorem E] and [EIM20, Corollary 3.5 and Remark 3.6].

**Theorem 3.7.** Under the same notation and assumption as in Theorem 3.1. Suppose furthermore that $L := -(K_X + \Delta) - f^*D$ is big, and one of the following holds:

1. $B_+(L)$ does not dominate $Y$;
2. $B_-(L)$ does not surject onto $Y$, and $(F, \Delta_F^\pm)$ is klt.

Then for any $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, there is a rational number $\varepsilon > 0$ such that for sufficiently large and divisible integer $l$, the sheaf

$$\mathcal{O}_X(l(f^*(-K_Y - D - \varepsilon D_0) + \Delta^- + B))$$

is weakly positive for some effective $f$-exceptional $\mathbb{Q}$-divisor $B$. We can take $B = 0$ if $Y$ has at worst canonical singularities.

In particular, if $\Delta \geq 0$, then $-K_Y - D$ is big.

**Proof.** For case (1), let $A$ be an ample divisor on $X$ such that $A - f^*D_0$ is ample and effective. Since $B_+(L)$ does not surject onto $Y$, there exists $0 < \varepsilon \ll 1$ such that $B(L - \varepsilon A)$ does not surject onto $Y$. We have

$$B(L - \varepsilon A) \supseteq B(L - \varepsilon A + \varepsilon(A - f^*D_0)) = B(L - \varepsilon f^*D_0) \supseteq B_-(L - \varepsilon f^*D_0).$$
Therefore, $B_-(L - \varepsilon f^*D_0)$ does not surject onto $Y$. By Theorem 3.1, we conclude that the sheaf $\mathcal{O}_X(l(f^*(-K_Y - D - \varepsilon D_0) + \Delta^- + B))$ is weakly positive for sufficiently divisible integer $l$ and some effective $f$-exceptional $\mathbb{Q}$-divisor $B$.

For (2), it suffices to show that $B_-(-(K_X + \Delta') - f^*D - \varepsilon f^*D_0)$ does not surject over $Y$ for some $\mathbb{Q}$-divisor $\Delta'$ where $(F, \Delta_F^+)$ is log canonical.

We consider $L_n := L - \frac{1}{n}f^*D_0$. Note that $L$ is big and hence so is $L_n$ for $n \gg 0$. Let $G$ be an effective $\mathbb{Q}$-divisor such that $L_n - G$ is ample. Since we assume that $B_-(L)$ does not surject onto $Y$, it follows that $B(L + \delta A)$ does not surject onto $Y$ for any $\delta > 0$ and ample divisor $A$. Hence for any $p \in \mathbb{N}$,

$$B(pL - \frac{1}{n}f^*D_0 + \delta A - G) = B((p - 1)L + \delta A + L_n - G) \subset B((p - 1)L + \delta A)$$

does not surject onto $Y$. As a result, neither does $B_-(pL - \frac{1}{n}f^*D_0 - G) = B_-(Lpn - \frac{1}{p}G)$.

Now since $(F, \Delta_F^+)$ is klt, by letting $\Delta' := \Delta + \frac{1}{p}G$, we have $(F, \Delta_F^+)$ is still klt for $p \gg 0$. Thus,

$$B_-(Lpn - \frac{1}{p}G) = B_-(-(K_X + \Delta') - f^*D - \frac{1}{np}f^*D_0)$$

does not surject onto $Y$. By Theorem 3.1, there is an effective $f$-exceptional $\mathbb{Q}$-divisor $B$ on $X$ such that $\mathcal{O}_X(l(f^*(-(K_Y + D) - \frac{1}{np}D_0) + \Delta^- + B))$ is weakly positive for sufficiently divisible integer $l$.

Observing Theorem 3.1 and Theorem 3.7(1), it is reasonable to expect analogous statements under the assumption that $-K_X$ is $\mathbb{Q}$-effective.

**Theorem 3.8.** Let $f : X \to Y$ be an algebraic fibre space between normal projective varieties such that $Y$ is $\mathbb{Q}$-Gorenstein. Let $\Delta = \Delta^+ - \Delta^-$ be a $\mathbb{Q}$-divisor on $X$ such that $(K_X + \Delta)$ is $\mathbb{Q}$-Cartier, $f(Supp\Delta^-) \neq Y$ and $(F, \Delta_F^+)$ is klt for general fibres $F$ of $f$. Let $D$ be a $\mathbb{Q}$-Cartier divisor on $Y$ such that $L := -(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective with the stable base locus $B(L)$ does not dominate $Y$, then we have that $f^*(-K_Y - D) + \Delta^- + E_X$ is $\mathbb{Q}$-effective for some effective $f$-exceptional $\mathbb{Q}$-divisor $E_X$. Moreover, we can take $E_X = 0$ if one of the following assumptions holds:

1. $f$ is equidimensional;
2. $\Delta$ is effective;
3. $Y$ has at worst canonical singularities.

**Proof.** The idea is similar to the proof of [A05, Theorem 4.1]. At first, we prove that $f : (X, B) \to Y$ is a klt-trivial fibration in the sense of [FG14], where $B := \Delta + L$. Since $L$ is $\mathbb{Q}$-effective with stable base locus $B(L)$ which does not dominate $Y$, then up to $\mathbb{Q}$-linear equivalence, we may assume that $\Delta_F + L|_F$ is effective and klt for general fibres $F$. Let $B := \Delta + L$, then by the above discussion, we have Nklt$(\Delta^+ + L)$ does not surject onto $Y$, so $(X, B)$ is subklt over the generic point of $Y$.

Let $\pi : X' \to X$ be any birational morphism such that $X'$ is smooth. By the formula $K_{X'} = \pi^*(K_X + B) + A_{X'}$, we have $A_{X'} = -B' + E$, where $B'$ is the proper transform of
Now we let $A$ be a Cartier divisor on $X$ such that $\pi^*f^*A \geq [-B']$ (such $A$ exists since $[-B']$ is vertical for $f \circ \pi$), then by the projection formula and [Deb01, Lemma 7.11 and 7.12], we have
\[
\begin{align*}
    f_*\pi_*\mathcal{O}_{X'_*}(A_{X'_*}) & \leq f_*\pi_*\mathcal{O}_{X'_*}(\pi^*f^*A + [E^+]) \\
    & = f_*\left(\mathcal{O}_X(f^*A) \otimes \pi_*\mathcal{O}_{X'_*}([E^+])\right) \\
    & \cong f_*\left(\mathcal{O}_X(f^*A) \otimes \mathcal{O}_X\right) = \mathcal{O}_Y(A).
\end{align*}
\]
In particular, $f_*\pi_*\mathcal{O}_{X_*}(A_{X_*})$ is of rank 1 for any birational morphism $\pi : X' \to X$ with $X'$ smooth, hence $f_*\mathcal{O}_X([A(X, B)])$ is of rank 1. Moreover, we have
\[
    K_X + B = K_X + \Delta + L \sim_{\mathbb{Q}} f^*(-D).
\]
Therefore, $f : (X, B) \to Y$ is a klt-trivial fibration in the sense of [FG14], which is also equivalent to the lc-trivial fibration in the sense of [A05], so we can apply the construction of moduli divisor on $f' : X' \to Y'$ to get
\[
f^*(-D) \sim_{\mathbb{Q}} K_X + B \sim_{\mathbb{Q}} f^*(K_Y + M_Y + B_Y),
\]
and hence $(-K_Y - D) \sim_{\mathbb{Q}} M_Y + B_Y$.

Thus, to show the $\mathbb{Q}$-effectiveness of $f^*(-K_Y - D) + \Delta^- + E$ for some $f$-exceptional divisor $E$, it remains to show that $f^*(M_Y + B_Y) + \Delta^- + E$ is $\mathbb{Q}$-effective. Similar to Theorem 3.41 we first prove the theorem under the assumption that $f$ is equi-dimensional. Since $B^h$ is effective, we can apply [A05 Theorem 3.3] to conclude that the moduli-$b$-divisor $\mathcal{M}$ is nef and abundant. There is a resolution $\mu : Y' \to Y$ such that $M_{Y'}$ is nef $\mathbb{Q}$-effective. Note that in general $M_Y$ may not be $\mathbb{Q}$-Cartier, but it is $\mathbb{Q}$-Cartier over $Y_0$, where $Y_0$ is the smooth locus of $Y$. Since $\mu_* (M_{Y'}) = M_Y$, over the open dense subset $\mu^{-1}(Y_0)$ of $Y'$ we have the equality $\mu^* M_Y = M_{Y'} + E^+ - E^-$ for $E^+, E^-$ be some effective $\mu$-exceptional $\mathbb{Q}$-divisors on $Y'$. Hence
\[
    \mu^* M_Y + E^- \geq M_{Y'},
\]
is $\mathbb{Q}$-effective over $\mu^{-1}(Y_0)$. Thus, $\mu^* M_Y$ is $\mathbb{Q}$-linear equivalent to a divisor which is effective except the exceptional set over $\mu^{-1}(Y_0)$. It follows that $M_Y$ is $\mathbb{Q}$-effective over $Y_0$. Since $Y$ is normal, we have $Y - Y_0$ has codimension at least 2, which implies $M_Y$ is $\mathbb{Q}$-effective over $Y$. Next, let $B^v$ be the vertical part of $B$. By the definition of $B$, we have $B + \Delta^- = \Delta^+ + L$ is effective. Hence $B^v + \Delta^- = (\Delta^+ + L)^v$ is effective since $\Delta^-$ is vertical. Also, by the definition of $B_Y$, the negative part of $f^*B_Y - B^v$ is $f$-exceptional. Since $f$ is equidimensional, there is no $f$-exceptional divisor on $X$ and hence $f^*B_Y \geq B^v$. Thus, we have $f^*(M_Y + B_Y) \geq B^v$ since $M_Y$ is an effective $\mathbb{Q}$-divisor, which implies $f^*(M_Y + B_Y) + \Delta^- \geq B^v + \Delta^- \geq 0$. 

The proof for the general case again follows verbatim to the argument of [EG19, Theorem 3.1]. By the flattening theorem, there is a normal birational modification \( \mu : Y' \rightarrow Y \) such that let \( X' \) be the normalization of the main component of \( X \times_Y Y' \), then \( \pi : X' \rightarrow X \) is proper birational and the induced morphism \( f' : X' \rightarrow Y' \) is equidimensional.

\[
\begin{array}{c}
X' \xrightarrow{\pi} X \\
\downarrow{f'} \downarrow{f} \\
Y' \xrightarrow{\mu} Y
\end{array}
\]

Now we define \( \Delta' \) by

\[ K_{X'} + \Delta' = \pi^*(K_X + \Delta), \]

and write \( \Delta' = \Delta'^+ - \Delta'^- \), then we have

\[ -(K_{X'} + \Delta') - \pi^*f^*D = \pi^*(-(K_X + \Delta) - f^*D) \]

is \( \mathbb{Q} \)-effective with the stable base locus not dominant over \( Y' \). Therefore, \( f^*(-K_{Y'} - \mu^*D) + \Delta'^- \) is \( \mathbb{Q} \)-effective since \( f' \) is equidimensional. Now write \( K_{Y'} = \mu^*K_Y + E \), and \( E = E^+ - E^- \), then we have

\[ \mu^*(-K_Y) + E^- = -K_Y, E^+ \geq -K_Y. \]

Thus \( f^*(\mu^*(-K_Y) + E^- - \mu^*D) + \Delta'^- \) is also \( \mathbb{Q} \)-effective, and so does

\[ \pi_*(f^*(\mu^*(-K_Y) + E^- - \mu^*D) + \Delta'^-) = f^*(-K_Y - D) + \Delta^- + \pi_*(f^*E^-). \]

Now, define \( E_X := \pi_*(f^*E^-) \), then since \( f' \) is equidimensional, we have \( f_*(E_X) = \mu_*E^- = 0 \), hence \( E_X \) is \( f \)-exceptional. Also, if \( Y \) has at worst canonical singularities, then \( E^- = 0 \), hence \( E_X = 0 \). Finally, if \( \Delta \) is effective, then \( \Delta^- = 0 \) and hence the \( \mathbb{Q} \)-effectiveness of \( f^*(-K_Y - D) + E_X \) implies \( f^*(-K_Y - D) \) is \( \mathbb{Q} \)-effective by Lemma [5.6] (no matter \( Y \) has at worst canonical singularities or not). \( \square \)

By the above theorems in this section, we can give more properties of the relative anti-canonical divisor, generalizing [Deb01, Theorem 3.12] and [EIM20, Corollary 3.7].

**Corollary 3.9.** Let \( f : X \rightarrow Y \) be an algebraic fibre space between normal projective varieties such that \( Y \) is \( \mathbb{Q} \)-Gorenstein and is not a point. Let \( (X, \Delta) \) be a log pair such that \( (F, \Delta_F) \) is log canonical for general fibres \( F \) of \( f \), then:

1. Let \( D_0 \) be a pseudo-effective \( \mathbb{Q} \)-Cartier divisor on \( Y \) that is not numerically trivial. Then \( B_-(-(K_{X/Y} + \Delta + f^*D_0)) \) and \( B_-(-(K_{X/Y} + \Delta + f^*D_0)) \) surjects onto \( Y \).
2. \( B_-(-(K_{X/Y} + \Delta)) \) always surjects onto \( Y \). Moreover, if \( -(K_{X/Y} + \Delta) \) is big, and \( (F, \Delta_F) \) is klt for general fibres \( F \). Then \( B_-(-(K_{X/Y} + \Delta)) \) and \( B_-(-(K_{X/Y} + \Delta)) \) surjects onto \( Y \).

**Proof.** For (1), let \( D = -K_Y + D_0 \), then since \( -K_Y - D = -D_0 \) is not pseudo-effective, \(- (K_X + \Delta) - f^*D = -(K_{X/Y} + \Delta + f^*D_0) \) must not satisfies the condition of Theorem 3.1. Hence \( B_-(-(K_{X/Y} + \Delta + f^*D_0)) \) should surject onto \( Y \). For (2), let \( D = -K_Y \).
in Theorem 3.7, then \(-K_Y - D = 0\) is not big, so we can get the result by the same method. \(\square\)

4. THE ANTI-CANONICAL IITAKA DIMENSION.

Based on Theorem 3.8, we have the following result, which can be thought of as the anti-canonical version of the Iitaka conjecture.

**Theorem 4.1.** Let \(f : (X, \Delta) \to Y\) be an algebraic fibre space between normal projective varieties such that \(\Delta\) is effective, \(K_X + \Delta\) is \(\mathbb{Q}\)-Cartier, and \(Y\) is \(\mathbb{Q}\)-Gorenstein. Suppose there is a \(\mathbb{Q}\)-Cartier divisor \(D\) on \(Y\) such that \(L := -(K_X + \Delta) - f^*D\) is \(\mathbb{Q}\)-effective and \(B(L)\) does not dominate \(Y\). Suppose furthermore that \((F, \Delta_F)\) is klt for general fibres \(F\) of \(f\), where \(\Delta_F\) is defined by \(-(K_F + \Delta_F) = -(K_X + \Delta)|_F = F\). Then we have

\[
\kappa(X, L) \leq \kappa(F, L_F) + \kappa(Y, -K_Y - D).
\]

In particular, if \(X\) has at worst klt \(\mathbb{Q}\)-Gorenstein singularities, and \(-K_X\) is effective with stable base locus \(B(-K_X)\) does not dominate \(Y\), then we have

\[
\kappa(X, -K_X) \leq \kappa(F, -K_F) + \kappa(Y, -K_Y).
\]

To prove Theorem 4.1, we need the following proposition, which is a variant of [EG19, Proposition 4.4].

**Proposition 4.2.** Let \(f : X \to Y\) be an algebraic fibre space between normal projective varieties such that \(Y\) is \(\mathbb{Q}\)-Gorenstein. Let \(\Delta = \Delta^+ - \Delta^-\) be a \(\mathbb{Q}\)-divisor on \(X\) such that \((K_X + \Delta)\) is \(\mathbb{Q}\)-Cartier, \(f(\text{Supp} \Delta^-) \neq Y\) and \((F, \Delta^+_F)\) is klt for general fibres \(F\) of \(f\). Let \(D\) and \(E\) be \(\mathbb{Q}\)-Cartier divisors on \(Y\) such that \(L := -(K_X + \Delta) - f^*D\) is \(\mathbb{Q}\)-effective with the stable base locus \(B(L)\) does not dominate \(Y\). Suppose furthermore that there is an effective \(\mathbb{Q}\)-divisor \(\Gamma\) such that \(L - g^*E \sim \mathbb{Q} \Gamma \geq 0\). Moreover, assume one of the following three conditions holds:

1. \(f\) is equidimensional;
2. \(\Delta\) is effective;
3. \(Y\) has at worst canonical singularities.

Then for \(0 < \epsilon < 1\), \(f^*(-K_Y - D - \epsilon E) + \Delta^-\) is \(\mathbb{Q}\)-effective.

**Proof.** The proof follows the argument of [EG19, Proposition 4.4] closely. We consider

\[
\Delta_\epsilon := \Delta + \epsilon \Gamma; \quad D_\epsilon := D + \epsilon E; \quad L_\epsilon := -(K_X + \Delta_\epsilon) - f^*D_\epsilon.
\]

Then \(L_\epsilon = L - \epsilon(\Gamma + f^*E) \sim \mathbb{Q} (1 - \epsilon)L\). Thus, for \(\epsilon < 1\), \(L_\epsilon\) is \(\mathbb{Q}\)-effective with the stable base locus \(B(L_\epsilon)\) that does not dominate \(Y\). For \(0 < \epsilon < 1\), \((F, (\Delta_\epsilon)|_F)\) is still klt for general fibre \(F\). Therefore, applying Theorem 3.8 on \(L_\epsilon\), \(f^*(-K_Y - D_\epsilon) + \Delta^-\) is \(\mathbb{Q}\)-effective, hence so does \(f^*(-K_Y - D - \epsilon E) + \Delta^- \geq f^*(-K_Y - D_\epsilon) + \Delta^-\). \(\square\)

In fact, it is not difficult to generalize Proposition 4.2 to the situation that \(f\) is an almost holomorphic fibration, by following the same proof in [EG19, Proposition 4.4]. We leave the details to the readers.
The next theorem is a variant of Ejiri and Gongyo’s injectivity theorem \[ EG19 \] Theorem 1.2. This is an application of Proposition 4.2 and plays a key role in the proof of Theorem 4.1.

**Theorem 4.3.** Use the notation and assumption in Proposition 4.2. Suppose there exists a \( \mathbb{Q} \)-Cartier divisor \( P \) on \( X \) such that \( P \geq \Delta^- \), \( f(\text{Supp} P) \neq Y \), and \( \kappa(X, f^*(-K_Y - D) + P) = 0 \). Then for any general fibre \( F \) of \( f \), the following morphism between graded ring defined by restriction

\[
\bigoplus_{m \geq 0} H^0(X, [mL]) \to \bigoplus_{m \geq 0} H^0(F, [mL_F])
\]

is injective. In particular, this implies \( \kappa(X, L) \leq \kappa(F, L_F) \).

**Proof.** We closely follow the proof of \[ EG19 \] Theorem 4.2, Corollary 4.7. Consider the map between graded rings defined by restriction

\[
\bigoplus_{m \geq 0} H^0(X, [mL]) \to \bigoplus_{m \geq 0} H^0(F, [mL_F]).
\]

As in the proof of \[ EG19 \] Corollary 4.7, the kernel of this map are the sections corresponding to the effective divisors \( N \in |mL| \) such that \( \text{Supp} N \supset F \). So it suffices to show that for every effective \( \mathbb{Q} \)-divisor \( N \sim_Q L \), \( \text{Supp} N \) does not contains \( F \).

Since \( \kappa(X, f^*(-K_Y - D) + P) = 0 \), there is a unique effective \( \mathbb{Q} \)-Cartier divisor \( M \sim_Q f^*(-K_Y - D) + P \). Let \( F \) be a normal irreducible fibre of \( f \) such that \( y = f(F) \) is a smooth point on \( Y \), \( f \) is flat over an open dense neighborhood of \( y \), \( F \cap \text{Supp} P = \emptyset \), and \( F \not\subseteq \text{Supp} M \). Now define \( \pi : X' \to X \) (resp. \( \mu : Y' \to Y \)) to be the blow-up of \( X \) (resp. \( Y \)) with respect to \( F \) (resp. \( y \)). Then as in the proof of \[ EG19 \] Theorem 4.2, since \( f \) is flat over an open neighborhood of \( y \), we have \( X' = X \times_Y Y' \) by \[ Stacks \] Lemma 31.32.3. Also, since \( y \) is a smooth point on \( Y \), we have \( Y' \) is still normal and \( \mathbb{Q} \)-Gorenstein and \( X' \) is still normal since \( f' \) is flat with normal fibres over a neighborhood of exceptional locus, where \( f' \) is the induced morphism \( f' : X' \to Y' \). Now we have the following commutative diagram:

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{\mu} & & \downarrow{f} \\
Y' & \xrightarrow{\mu} & Y.
\end{array}
\]

Write \( K_{Y'} = \mu^*K_Y + aE \), where \( E \) is the exceptional divisor of \( \mu \) on \( Y' \), and \( a = \dim Y - 1 \). Let \( G \) be the exceptional divisor of \( \pi \) (where \( G \cong F \times E \)), then by the flatness over \( F \), we have \( G = f^*E \). Also, since \( F \cap \text{Supp} \Delta^- = \emptyset \), we can write \(-\pi^*(K_X + \Delta) = -(K_{Y'} + \Delta') + bG \) for some \( b \leq \text{codim}(F, X) - 1 = \dim Y - 1 = a \), where \( \Delta' \) is the proper transform of \( \Delta \) on \( X' \). Let

\[
L' := -(K_{X'} + \Delta') - f^*(\mu^*D - aE) = -(K_{X'} + \Delta') - \pi^*f^*D + aG = \pi^*L + (a-b)G \geq \pi^*L,
\]

where \( \pi^*f^*D \geq \pi^*L \).
which is also effective with stable base locus that does not dominate $Y'$.

Now, suppose there exists an effective $\mathbb{Q}$-divisor $N \sim \mathbb{Q} L$ with $\text{Supp} N \supset F$. Then we have $\text{Supp}(\pi^* N) \supset \text{Supp} G$. Now define

$$N' := \pi^* N + (a - b)G \sim \mathbb{Q} \pi^* L + (a - b)G = L',$$

then $N'$ is effective with $\text{Supp} N' \supset \text{Supp} G$. Therefore, there exists a rational number $0 < \delta \ll 1$, such that $\Gamma := N' - \delta G \geq 0$, then

$$L' - \delta f^* E = L' - \delta G \sim \mathbb{Q} N' - \delta G = \Gamma \geq 0.$$ 

So we can apply Proposition 4.2 on $X', Y', \mu^* D - aE, L', \delta E$, and $\Gamma$ to conclude that $f^*(-K_{Y'} - \mu^* D + aE - \varepsilon \delta E) + \pi^* P$ is $\mathbb{Q}$-Cartier and $\mathbb{Q}$-effective. Note that

$$f^*(-K_{Y'} - \mu^* D + aE - \varepsilon \delta E) + \pi^* P = \pi^*(f^*(-K_Y - D) + P) - \varepsilon \delta G.$$

Therefore, $G \subset \text{Supp}(\pi^* M)$ by the uniqueness of $M$, thus $F \subset \text{Supp} M$, but this is a contradiction to our choice of $F$, hence the proof is completed. \hfill $\Box$

**Proof of Theorem 4.1.** First, we prove the theorem under the assumption that $Y$ is smooth. By Theorem 3.8, we have $-K_Y - D$ is $\mathbb{Q}$-effective, and by Theorem 4.3, we only need to consider the case of $\kappa(Y, -K_Y - D) > 0$. Consider the following commutative diagram

$$
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow f' & & \downarrow f \\
Y' & \xrightarrow{\mu} & Y \\
\downarrow g' & & \downarrow g \\
Z' & \xrightarrow{\nu} & Z.
\end{array}
$$

Where

1. $g : Y \dashrightarrow Z$ is the rational map defined by $m(-K_Y - D)$ for some sufficiently divisible $m \gg 0$ such that $\dim(\text{Im}(g)) = \kappa(Y, -K_Y - D)$ and $\text{Bs}(m(-K_Y - D)) = \mathcal{B}(-K_Y - D)$.

2. $g' : Y' \to Z'$ is a minimal resolution of Iitaka fibration induced by $m(-K_Y - D)$. More precisely, consider the following diagram

$$
\begin{array}{ccc}
Y_0 & \xrightarrow{\mu_0} & Y \\
\downarrow g_0 & & \downarrow g \\
Z' & \xrightarrow{\nu} & Z.
\end{array}
$$

Here $g_0 : Y_0 \to Z'$ is the Iitaka fibration induced by $m(-K_Y - D)$. Note that $Y_0$ is possibly singular, but by the smoothness of $Y$, the singularity of $Y_0$ must contained in $\text{Exc}(\mu_0)$. Now we let $\mu' : Y' \to Y_0$ be a minimal resolution of singularities.

3. $\mu := \mu_0 \circ \mu' : Y' \to Y$ and $g' := g_0 \circ \mu' : Y' \dashrightarrow Z'$ are the induced maps, note that $\mu$ is birational and $g'$ is an algebraic fibre space.
(4) $X'$ is the normalization of the main component of $X \times_Y Y'$, and $\pi : X' \to X$ and $f' : X' \to Y'$ are the induced morphism. Note that we have $f'(\text{Exc}(\pi)) \subset \text{Exc}(\mu)$.

(5) $F$ is a general fibre of $f$ (by the construction of $X'$ and $f'$, $F$ is also a general fibre of $f'$).

(6) $G'$ is a general fibre of $g'$. Note that $G'$ is smooth by the smoothness of $Y'$.

(7) $W' := f'^{-1}(G')$ is a general fibre of $g' \circ f'$.

Here $X'$ and $f'$ are constructed by the following way: Let $X_0$ be the main component of $X \times_Y Y'$ with projections $f_0 : X_0 \to Y'$ and $\pi_0 : X_0 \to X$. Then we have $\text{Exc}(\pi_0) \subset f_0^{-1}(\text{Exc}(\mu))$ and $\pi_0(\text{Exc}(\pi_0)) \subset f^{-1}(\text{Exc}(\mu))$. Since $X$ is normal, $X_0 - \text{Exc}(\pi_0) \cong X - \pi_0(\text{Exc}(\pi_0))$ is normal. Hence for the normalization $\pi' : X' \to X_0$ of $X_0$, $\pi'$ is an isomorphism over $X_0 - \text{Exc}(\pi_0)$. Thus, we have

$$\text{Exc}(\pi) \subset \pi'^{-1}(\text{Exc}(\pi_0)) \subset f'^{-1}(\text{Exc}(\mu)),$$

where $f' = f_0 \circ \pi'$ and $\pi := \pi_0 \circ \pi'$.

By the easy addition formula [Mo85 Corollary 1.7], we have

$$\kappa(X', \pi^* L) \leq \kappa(W', (\pi^* L)|_{W'}) + \dim Z'.$$

Since $\kappa(X', \pi^* L) = \kappa(X, L)$ and $\dim Z' = \kappa(Y, -K_Y - D)$, this implies

$$\kappa(X, L) \leq \kappa(W', (\pi^* L)|_{W'}) + \kappa(Y, -K_Y - D).$$

Thus, it remains to show that $\kappa(W', (\pi^* L)|_{W'}) \leq \kappa(F, L_F)$. To show this, let $B := \pi^*(K_{X/Y} + \Delta) - K_{X'/Y'}$, then

$$\mu^*(-K_Y - D) = -K_{Y'} - (\mu^*(K_Y + D) - K_{Y'}),$$

$$\pi^* L = -(K_{X'} + B) - f'^*(\mu^*(K_Y + D) - K_{Y'}).$$

Next, write

$$K_{Y'} = \mu^* K_Y + \sum a_i E_i, \quad K_{X'} = \pi^* (K_X + \Delta) + \sum b_j P_j,$$

then $B = \sum a_i f'^* E_i - \sum b_j P_j$. Since $\Delta$ is effective, if $b_j > 0$ then $P_j$ must be $\pi$-exceptional, which implies $f'(P_j) \subset \text{Exc}(\mu)$. Therefore, $f'(\text{Supp}(B^{-})) \subset \text{Exc}(\mu)$ and hence $\text{Supp}(B^{-}) \subset f'^{-1}\text{Exc}(\mu)$. Note that by the construction of the Iitaka fibration, we may assume $\mu_0(\text{Exc}(\mu_0))$ is contained in $B(-K_Y - D)$, which implies $\text{Exc}(\mu_0) \subset \mu^{-1}_0B(-K_Y - D)$. Note that by the smoothness of $Y$, we have the singularity of $Y_0$ must contained in $\text{Exc}(\mu_0)$, which implies $\text{Exc}(\mu) \subset \mu^{-1}\text{Exc}(\mu_0)$ and hence

$$\text{Exc}(\mu) \subset \text{Exc}(\mu') \cup \mu'^{-1}(\text{Exc}(\mu_0)) = \mu'^{-1}(\text{Exc}(\mu_0)) \subset \mu^{-1}(B(-K_Y - D)).$$

Thus, for sufficiently divisible $m \gg 0$, $\text{Supp}(\mu^*(m(-K_Y - D))) \supset \text{Exc}(\mu)$, which implies

$$\text{Supp}(f'^* \mu^*(m(-K_Y - D))) \supset f'^{-1}(\text{Exc}(\mu)) \supset \text{Supp}(B^{-}).$$

Therefore, $P_{mn} := f'^* \mu^*(mn(-K_Y - D)) \geq B^-$ for $n \gg 0$. Consider the morphism $f'|_{W'} : W' \to G'$, whose general fibre is also a general fibre $F$ of $f$ by our construction.
Then we have
\[ \kappa(W', (f^*\mu^*(-K_Y - D))|_{W'}) = \kappa(W', f^*\mu^*(-K_Y - D)|_{W'}) \]
\[ = \kappa(G', \mu^*(-K_Y - D)|_{G'}) \]
\[ = \kappa(G_0, \mu_0^*(-K_Y - D)|_{G_0}) = 0, \]
where the last equality is from the structure of Iitaka fibration.

Now, consider the morphism \( f'|_{W'} : W' \to G' \) and the divisors
\[ -K_{G'} - (\mu^*(K_Y + D))|_{G'} - K_{G'} = \mu^*(-K_Y - D)|_{G'}, \]
\[ - (K_{W'} + B_{W'}) - f'^*(\mu^*(K_Y + D))|_{G'} - K_{G'} = (\pi^*L)|_{W'}. \]
Here \( B_{W'} \) is defined by \( (K_{X'} + B)|_{W'} = K_{W'} + B_{W'} \). Note that we have
\[ (K_{W'} + B_{W'})|_F = (K_{X'} + B)|_F \]
\[ = (\pi^*(K_X + \Delta) - f'^*(K_{Y'} - \mu^*K_Y))|_F \]
\[ = (\pi^*(K_X + \Delta))|_F \]
\[ \cong (K_X + \Delta)|_F \]
\[ = K_F + \Delta_F, \]
where the isomorphism follows from the fact that \( \pi \) is an isomorphism over general fibres \( F \). In particular, let \( B_F \) be defined by \( (K_{W'} + B_{W'})|_F = K_F + B_F \), then \( B_F = \Delta_F \) under the natural isomorphism \( \pi^{-1}(F) \cong F \), hence \( (F, B_F) \) is klt. Hence we can apply Theorem 4.13 (since \( G' \) is smooth) to conclude that
\[ \kappa(W', (\pi^*L)|_{W'}) \leq \kappa(F, -(K_{W'} + B_{W'})|_F) = \kappa(F, -(K_X + \Delta)|_F) = \kappa(F, L_F). \]
This completes our proof when \( Y \) is smooth.

For the general case, consider the following diagram
\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{\mu} & & \downarrow{f} \\
Y' & \xrightarrow{\mu} & Y.
\end{array}
\]
Where \( Y' \) is a resolution of singularities of \( Y \), and \( X' \) is a normalization of the main component of \( X \times_Y Y' \). Similar to the above argument, we may assume \( f'(\text{Exc}(\pi)) \subseteq \text{Exc}(\mu) \). Now we let \( L = -(K_{X'} + \Delta') - f^*D \), and write \( \pi^*L = -(K_{X'} + \Delta') + E^+ - E^- - f'^*\mu^*D \), where \( \Delta' \) is the proper transform of \( \Delta \) on \( X' \), and \( E^+, E^- \) are effective \( \pi \)-exceptional \( \mathbb{Q} \)-divisors defined by the equality \( K_{X'} + \Delta' = \pi^*(K_X + \Delta) + E^+ - E^- \). Note that by the construction of \( X' \), both \( E^+ \) and \( E^- \) are vertical with \( f(\text{Supp}E^+) \), \( f(\text{Supp}E^-) \subseteq \text{Exc}(\mu) \), hence there exists an effective \( \mu \)-exceptional divisor \( N \) on \( Y' \) such that \( f'^*N \geq E^+ \geq E^+ - E^- \). Now, since the theorem holds if the base is smooth, we have
\[
\kappa(X, L) = \kappa(X', \pi^*L) \leq \kappa(X', -(K_{X'} + \Delta') - f'^*\mu^*D + f'^*N) \leq \kappa(F, -(K_F + \Delta_F)) + \kappa(Y', -K_{Y'} - \mu^*D + N). \]
Note that $-(K_X + \Delta') - f^*\mu^*D + f^*N = \pi^*L + (f^*N - (E^+ - E^-))$ is still effective with the stable base locus does not dominate $Y'$. Now, we write $K_{Y'} = \mu^*K_Y + B^+ - B^-$ for some $\mu$-exceptional divisors $B^+, B^-$. Then we have
\[
\kappa(Y', -K_{Y'} - \mu^*D + N) = \kappa(Y', \mu^*(-K_Y - D) + N + B^- - B^+)
\leq \kappa(Y', \mu^*(-K_Y - D) + N + B^-)
= \kappa(Y', \mu^*(-K_Y - D))
= \kappa(Y, -K_Y - D)
\]
since $N$ and $B$ are $\mu$-exceptional, hence the result holds for general case. \qed

The following corollaries directly follow from Theorem 4.1.

**Corollary 4.4.** Let $(X, \Delta)$ be a log pair on $X$ and $f : X \to Y$ be an algebraic fibre space between normal projective varieties. Suppose $Y$ is $\mathbb{Q}$-Gorenstein and $(F, \Delta_F)$ is klt for general fibres $F$. Then:

1. Let $D$ be a $\mathbb{Q}$-Cartier divisor on $X$ such that $-(K_X + \Delta) - f^*D$ is $\mathbb{Q}$-effective and the stable base locus $B(-(K_X + \Delta) - f^*D)$ does not dominate $Y$, then
   \[
   \dim Y - \kappa(Y, -K_Y - D) \leq \dim X - \kappa(X, -(K_X + \Delta) - f^*D).
   \]

2. Suppose $-(K_{X/Y} + \Delta)$ is $\mathbb{Q}$-effective with $B(-(K_{X/Y} + \Delta))$ that does not dominate $Y$, then for any effective $\mathbb{Q}$-Cartier divisor $D_0$ on $Y$, we have
   \[
   \kappa(X, -(K_{X/Y} + \Delta) + f^*D_0) \leq \kappa(F, -(K_F + \Delta_F)) + \kappa(Y, D_0).
   \]

**Remark 4.5.** There is another way to prove Theorem 4.1. First, we prove the theorem under the assumption that $f$ is equidimensional. Under this assumption, $f'$ is also equidimensional, hence so does $W' \to G'$. Therefore, we can prove the inequality holds for equidimensional morphism by the same steps of the above proof. Then for the general case, we can let $X' \to Y'$ be the normalization of the flattening $f$, and use the same argument to prove the inequality holds for $f$ by the fact the inequality holds for $f'$.

### 5. Further discussions and questions

#### 5.1. Asymptotic invariants of the anti-canonical divisor.

In [EIM20, Corollary 3.5, Remark 3.6], Ejiri, Iwai, and Matsumura describe the asymptotic base locus of $-K_Y - D$ when $Y$ has at worst canonical singularities. In Theorem 3.1, Theorem 3.7, and Theorem 3.8 we proved the positivity of $-K_Y - D$ without assuming that $Y$ has at worst canonical singularities. Thus, it is natural to ask the following question:

**Question 5.1.** Can we describe the asymptotic base locus of $-K_Y - D$ without assuming that $Y$ has at worst canonical singularities?

It seems not easy to show that [EIM20, Corollary 3.5, Remark 3.6] is still true when $Y$ has klt singularities or worse (if it is true). Naively thinking, if $Y$ has at worst canonical singularities, then for any resolution $\mu : Y' \to Y$, we have $\mu^*(-K_Y) \geq -K_Y$. Therefore,
if we can compute the intersection number and/or asymptotic invariants of \(-K_Y\) on some birational model \(Y'\) (for example, as the proof of [FG12, Theorem 4.1], which uses [K98 Theorem 2] to compute \((-K_Y.C)\) on a higher birational model), then it will give a lot of information about asymptotic invariants of \(-K_Y\). But this approach does not work without assuming \(Y\) has canonical singularities.

Also, comparing [EG19, Corollary 4.7], [CCM19, Theorem 1.2] and [EIM20, Theorem 3.9] to our results in Section 4, it seems that the behavior of nef (relative) anti-canonical divisors and effective (relative) anti-canonical divisor with the stable base locus that does not dominate \(Y\) are very similar, so we can also ask the following question:

**Question 5.2.** Does the inequality of Theorem 4.1 still hold if \(L\) is \(\mathbb{Q}\)-effective and \(B_-(L)\) does not surject onto \(Y\) (instead of the assumption \(B(L)\) does not dominate \(Y\))?

5.2. **Rational map.** Comparing [EG19, Section 4 and Section 5] to Theorem 4.1 it seems natural to ask whether Theorem 4.1 holds if \(f\) is an almost holomorphic fibration. Unfortunately, we have the following counterexample:

**Example 5.3.** Let \(X \coloneqq \mathbb{P}^2 \times \mathbb{P}^1\), and \(p_1 : X \rightarrow \mathbb{P}^2\) be the first projection. Then let \(\mu : Y \rightarrow \mathbb{P}^2\) be the blow-up of 13 general points \(P_1, \ldots, P_{13}\) on \(\mathbb{P}^2\), and denote \(E_1, \ldots, E_{13}\) be the corresponding exceptional divisors. Then we have \(E_i^2 = -1\) and \((E_i, E_j) = 0\) if \(i \neq j\). Since 14 general points define a quartic plane curve, for a general point \(P\) on \(\mathbb{P}^2\), there is a quartic \(C_0\) passing \(P\) and all \(P_i\) with multiplicity 1. Now, let \(C\) be the proper transform of \(C_0\) on \(Y\). Then we have \((C.E_i) = 1\) for all \(i\), and

\[
(-K_Y.C) = (-\pi^*K_{\mathbb{P}^2} - \sum E_i.\pi^*C_0 - \sum E_i) = -1.
\]

Therefore, \(-K_Y\) is not almost nef, hence not pseudo-effective by [BKKMSU15, Proposition 4.2 and 4.5]. Let \(f : X \dashrightarrow Y\) be the induced rational map, which is almost holomorphic since the fibre is well-defined and closed over \(Y - \text{Exc}(\mu)\). Then \(-K_X\) is ample, but \(-K_Y\) is not effective, hence the inequality in Theorem 4.1 does not hold for the map \(f\).

This example also shows that the last assertion of Theorem 3.1, Theorem 3.7, and Theorem 3.8 can not be generalized to the situation that \(f\) is only an almost holomorphic fibration even if \(D = 0\). However, by Lemma 2.7 we have the following result, which is a generalization of [Den17, Lemma 4.1]:

**Proposition 5.4.** Let \(f : X \dashrightarrow Y\) be a birational map between normal projective varieties. Let \(X_0, Y_0\) be the maximal open sets of \(X, Y\) such that \(f|_{X_0} : X_0 \rightarrow Y_0\) is a morphism. Let \(\Delta\) be a \(\mathbb{Q}\)-divisor such that both \(K_X + \Delta\) and \(K_Y + f_*\Delta\) are \(\mathbb{Q}\)-Cartier. Suppose that \(\pm(K_X + \Delta)\) is pseudo-effective (resp. effective, big), and \(Y - Y_0\) has codimension at least 2, then \(\pm(K_Y + f_*\Delta)\) is pseudo-effective (resp. effective, big). In particular, this result holds if \(f\) is either a divisorial contraction or a \((K_X + \Delta)\)-flip.

**Proof.** We work on the case of anti-canonical divisors, and the case of canonical divisors can be derived in the same way. Let \(g : W \rightarrow Y\) be a resolution of indeterminacy of \(f\) such that \(W\) is normal and denote \(\pi : W \rightarrow X\) be the corresponding birational morphism.
Suppose that \(-(K_X + \Delta)\) is pseudo-effective, then we write
\[ K_W + \Delta_W = \pi^*(K_X + \Delta) + E_X, \]
where \(\Delta_W\) is the proper transform of \(\Delta\) on \(W\), and \(E_X\) is \(\pi\)-exceptional. Then we have
\[ -\pi^*(K_X + \Delta) = -K_W - \Delta_W + E_X \]
Next, let \((f_*\Delta)_W\) be the proper transform of \(f_*\Delta\) on \(W\). Then there is a \(g\)-exceptional divisor \(E_Y\) such that
\[ K_W + (f_*\Delta)_W = g^*(K_Y + f_*\Delta) + E_Y. \]
Therefore,
\[ -g^*(K_Y + f_*\Delta) = (-K_W - \Delta_W + E_X) + (\Delta_W - (f_*\Delta)_W) + E_Y - E_X. \]
Note that \((\Delta_W - (f_*\Delta)_W)\) is \(g\)-exceptional, and since \(Y - Y_0\) has codimension at least 2, every \(\pi\)-exceptional divisor is \(g\)-exceptional. So
\[ -g^*(K_Y + f_*\Delta) = (-K_W - \Delta_W + E_X) + (g\text{-exceptional divisors}). \]
Hence by Lemma 2.7(2), \(-(K_Y + f_*\Delta)\) is pseudo-effective.

Suppose now that \(-(K_X + \Delta)\) is effective. Given a birational morphism \(\pi : X' \to X\), if \(\pi^*D\) is effective outside the exceptional locus, then \(D\) is effective itself. Thus, under the assumption that \(-(K_X + \Delta)\) is effective, we can use the same argument of the pseudo-effective case to show that \(-(K_Y + f_*\Delta)\) is effective, by using this fact at where we use Lemma 2.7 in the pseudo-effective case.

Suppose now that \(-(K_X + \Delta)\) is big. Since \(Y - Y_0\) has codimension at least 2, for any \(\mathbb{Q}\)-divisor \(G\) on \(Y\), there is a \(\mathbb{Q}\)-divisor \(G'\) on \(X\) such that \(f_*G' = G\). Since \(-(K_X + \Delta)\) is big, we have
\[ -(K_X + \Delta + \varepsilon G') = \text{pseudo-effective for } \varepsilon \text{ sufficiently small.} \]
Thus, replacing \(\Delta\) by \(\Delta + \varepsilon G'\), the pseudo-effective case implies
\[ -(K_Y + f_*(\Delta + \varepsilon G')) = -(K_Y + f_*\Delta) - \varepsilon G \]
is pseudo-effective. Since \(G\) is arbitrary, this implies \(-(K_Y + f_*\Delta)\) is big.

By the above proposition, it is natural to ask whether we can generalize Theorem 3.1, Theorem 3.7, and Theorem 3.8 to almost holomorphic fibration with \(Y - Y_0\) has codimension at least 2. To answer this question, first, we define the pullback of divisors under rational maps:

**Definition 5.5.** (cf. [Ma14, Definition 1.2]) Let \(f : X \dashrightarrow Y\) be a rational map. Then for a \((\mathbb{Q}\text{-})\)divisor \(D\) on \(Y\), we can define the pullback \(f^*D\) in the following way: Let \(\pi : X' \to X\) be a resolution of indeterminacy on \(f\), and \(f' : X' \to Y\) be the corresponding resolution. Then we define \(f^*D := \pi_*f'^*D\). Note that this definition does not depend on the choice of resolution because the push-forward of exceptional divisors is zero.

Also, we need the following lemma:

**Lemma 5.6.** Let \(f : X \to Y\) be an algebraic fibre space between normal varieties. Let \(E\) be an effective \(f\)-exceptional \(\mathbb{Q}\)-divisor on \(X\), and \(D\) be a \(\mathbb{Q}\)-divisor on \(Y\). Suppose \(f^*D + nE\) is \(\mathbb{Q}\)-effective for some \(n \in \mathbb{N}\), then \(D\) is \(\mathbb{Q}\)-effective.

**Proof.** Let \(m \in \mathbb{N}\) sufficiently divisible such that \(mD\) and \(mE\) has integer coefficients, and \(m(f^*D + nE)\) is effective. Since \(f\) is an algebraic fibre space and \(f(\text{Supp\,}E)\) has codimension at least 2 in \(Y\), letting \(Y_0 := Y - f(\text{Supp\,}E)\), \(X_0 := X - f^{-1}f(\text{Supp\,}E)\), we have the following natural isomorphisms:
\[ H^0(X_0, f^*(mD)|_{X_0}) \cong H^0(Y_0, (mD)|_{Y_0}) \cong H^0(Y, mD) \cong H^0(X, f^*(mD)).\]
In particular, if \((f^*(mD))|_{X_0}\) is effective, then so does \(mD\). Thus, if \(m(f^*D + nE)\) is effective, then \((f^*(mD))|_{X_0}\), and hence \(mD\) itself is effective. \(\square\)

Now we have the following generalization:

**Proposition 5.7.** Let \(f : X \rightarrow Y\) be an almost holomorphic fibration between normal projective varieties, with \(X_0, Y_0\) be the maximal open subsets of \(X\) and \(Y\) such that \(f|_{X_0} : X_0 \rightarrow Y_0\) is a morphism. Suppose that \(Y - Y_0\) has codimension at least 2, then the conclusions of Theorem 3.1, 3.7, and 3.8 still hold by replacing the assumption on \(B\) (resp. \(B_+\), \(B_-\)) with the one that \(B_-\) \(X_0\) (resp. \(B_+(L) \cap X_0, B(L) \cap X_0\) does not surject onto \(Y_0\).

**Proof.** Let \(\pi : X' \rightarrow X\) be a resolution of indeterminacy, \(f' : X' \rightarrow Y\) be the corresponding resolution. We can write \(-(K_{X'} + \Delta') - f^*D = \pi^*L + E^+ - E^-\) for some effective \(\pi\)-exceptional divisor \(E^+, E^-\), where \(\Delta'\) is the proper transform of \(\Delta\) on \(X'\). Since \(f\) is almost holomorphic, we may assume \(f'(\text{Exc}(\pi)) \subset Y - Y_0\). In particular, every \(\pi\)-exceptional divisors are \(f'\)-exceptional.

Now, suppose \(L\) is pseudo-effective and \((B_-(L) \cap X_0)\) does not surject onto \(Y_0\). Then \(B_-(-(K_{X'} + \Delta' + E^+ - E^-) - f^*D) = B_-(\pi^*L)\) is not surject onto \(Y\) since \(B_-(\pi^*L) \subset \pi^{-1}(B_-(L))\). In fact, let \(H\) be an ample Cartier divisor on \(X\) and \(A\) be an ample Cartier divisor on \(X'\) such that \(A - \pi^*H\) is also ample. Then for any \(0 < \varepsilon \ll 1\), we have

\[
B(\pi^*L + \varepsilon A) = B(\pi^*L + \varepsilon H + \varepsilon(A - \pi^*H)) \subset B(\pi^*L + \varepsilon H) \subset \pi^{-1}(B(L + \varepsilon H)) \subset \pi^{-1}(B_-(L)).
\]

Hence \(B_-(\pi^*L) \subset \pi^{-1}(B_-(L))\) follows from the definition of \(B_-(\cdot)\). Since \(f(\text{Supp}E) \neq Y\), we can apply Theorem 3.1 to conclude that for sufficiently divisible positive integer \(l\), \(\mathcal{O}_{X'}(l(f^*(-K_Y - D) + \Delta^+ + E^- + B))\) is weakly positive for some effective \(f'\)-exceptional divisor \(B\), hence so is \(\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^- + \pi_*B))\). In particular, since \(E^-\) is also \(f'\)-exceptional, by the weakly positivity of \(\mathcal{O}_{X'}(l(f^*(-K_Y - D) + \Delta^+ + E^- + B))\) and Lemma 2.7(2), we conclude that if \(\Delta\) is effective, then \(-K_Y - D\) is pseudo-effective, which generalizes Theorem 3.1 (and hence Theorem 3.7). Using a similar method, and applying Lemma 5.6, we can also generalize Theorem 3.8. \(\square\)

**Remark 5.8.** The above proof shows that even if we only assume \(f\) is almost holomorphic, \(\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^+ + \pi_*B))\) is still weakly positive. But without the assumption of \(Y - Y_0\) having codimension at least 2, then the pseudo-effectiveness (resp. effectiveness, bigness) of \(\mathcal{O}_X(l(f^*(-K_Y - D) + \pi_*B))\) and \(\mathcal{O}_{X'}(l(f^*(-K_Y - D) + E^- + B))\) is not enough to imply the pseudo-effectiveness (resp. effectiveness, bigness) of \(-K_Y - D\) even if \(\Delta\) is effective. However, if \(f\) is only a rational map, then \(\text{Supp}(E^-)\) may map onto \(Y\) and hence the weak positivity of \(\mathcal{O}_X(l(f^*(-K_Y - D) + \Delta^+ + \pi_*B))\) may not be true.

**Proposition 5.9.** The inequality of Theorem 4.1 holds if \(f\) is an almost holomorphic fibration with \(Y - Y_0\) has codimension at least 2.
Proof. The proof is similar to the argument generalizing Theorem 4.1 from smooth cases to the general case. Consider the following diagram

\[
X' \xrightarrow{\pi} X \quad \text{and} \quad Y' \xrightarrow{\mu} Y.
\]

Where \( Y' \) is log resolution of \((Y, Y - Y_0)\), and \( X' \) is a normalization of the resolution of indeterminacy of \( X \to Y' \). We may assume for the induced morphisms \( \pi : X' \to X \) and \( f' : X' \to Y' \), \( \text{Exc}(\pi) \) is purely codimension 1 and \( f'(\text{Exc}(\pi)) \subset \text{Exc}(\mu) \). Now we let \( L = -(K_X + \Delta) - f^*D \), and write \( \pi^*L = -(K_{X'} + \Delta') + E^+ - E^- - f'^*\mu^*D \), where \( \Delta' \) is the proper transform of \( \Delta \) on \( X' \), and \( E^+, E^- \) are effective \( \pi \)-exceptional \( \mathbb{Q} \)-divisors defined by the equality \( K_{X'} + \Delta' = \pi^*(K_X + \Delta) + E^+ - E^- \). Note that by the construction of \( X' \), both \( E^+ \) and \( E^- \) are vertical with \( f(\text{Supp}E^+) \subset \text{Exc}(\mu) \), hence there exists an effective \( \mu \)-exceptional divisor \( N \) on \( Y' \) such that \( f'^*N \geq E^+ \geq E^- \). Now, since the theorem holds if \( f \) is an algebraic fibre space, we have

\[
\kappa(X, L) = \kappa(X', \pi^*L) \leq \kappa(X', -(K_{X'} + \Delta') - f'^*\mu^*D + f'^*N)
\]

\[
\leq \kappa(F, -(K_F + \Delta_F)) + \kappa(Y', -K_{Y'} - \mu^*D + N).
\]

Note that \( -(K_{X'} + \Delta') - f'^*\mu^*D + f'^*N = \pi^*L + (f'^*N - (E^+ - E^-)) \) is still effective with the stable base locus does not dominant \( Y' \). Now, we write \( K_{Y'} = \mu^*K_Y + B^+ - B^- \) for some \( \mu \)-exceptional divisors \( B^+, B^- \). Then we have

\[
\kappa(Y', -K_{Y'} - \mu^*D + N) = \kappa(Y', \mu^*(-K_Y - D) + N + B^- - B^+)
\]

\[
\leq \kappa(Y', \mu^*(-K_Y - D) + N + B^-)
\]

\[
= \kappa(Y', \mu^*(-K_Y - D))
\]

\[
= \kappa(Y, -K_Y - D)
\]

since \( N \) and \( B \) are \( \mu \)-exceptional, hence the proof completes. \( \square \)

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