Circular-polarization sensitive metamaterial based on triple quantum-dot molecules

Panagiotis Kotetes\textsuperscript{1,2}, Pei-Qing Jin\textsuperscript{3}, Michael Marthaler\textsuperscript{1}, and Gerd Schön\textsuperscript{1,2}

\begin{flushleft}
\textsuperscript{1}Institut für Theoretische Festkörperphysik, Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany
\textsuperscript{2}DFG Center for Functional Nanostructures (CFN), Karlsruhe Institute of Technology, 76128 Karlsruhe, Germany and
\textsuperscript{3}Institute of Logistics Engineering, Shanghai Maritime University, Shanghai 201306, China
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We propose a new type of a chiral metamaterial based on an ensemble of artificial molecules formed by three identical quantum-dots in a triangular arrangement. A static magnetic field oriented perpendicular to the plane breaks mirror symmetry, rendering the molecules sensitive to the circular polarization of light. By varying the orientation and magnitude of the magnetic field one can control the polarization and frequency of the emission spectrum. We identify a threshold frequency, $\Omega$, above which we find strong birefringence. In addition, a Kerr rotation and circularly polarized lasing action can be implemented. We investigate the single-molecule lasing properties for different energy-level arrangements and demonstrate the possibility of circular polarization conversion.

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Light with circular polarization (CP) has a broad range of applications, e.g., for spintronics devices requiring coherent spin control, or for optical communication in the context of spin-based information processing. Efficient generation, manipulation, and detection of CP requires materials with broken mirror symmetry, either due to the structure or to the violation of time-reversal symmetry ($T$). So far studied CP sensitive materials include chiral semiconducting nanostructures employed as CP light emitters, photonic metamaterials involving gold helices, and spin lasers based on III-V type semiconducting quantum dots. However, tuning the CP emission characteristics, which would open perspectives for further applications, remains challenging. E.g., the structurally chiral materials can not be manipulated due to their built-in handedness, while the functionality of spin-lasers is limited by the spin-injection efficiency.

Progress along this direction can be made by using quantum-dot systems, which combine the versatility of nano-electronics with the advantages of structurally chiral metamaterials. Specifically we suggest to use ensembles of quantum-dot molecules, each of which consists of three identical quantum-dots arranged in a triangular fashion. Such quantum-dot molecules recently became accessible. An external magnetic field can influence the quantum-dot molecules by coupling to both spin and orbital degrees of freedom. In particular, linear molecules are currently investigated for Zeeman-splitting based applications, including quantum engineering based on the exchange-qubit protocol, as well as for spin-blockade effects. The triangular quantum-dot (TQD) molecules display additional orbital effects, related to their topological arrangement. They acquire a chirality when a magnetic field oriented perpendicular to the plane is applied. So far, the influence of the magnetic flux on the TQD molecules has been considered in the context of spin-chirality-coded qubits, Aharonov-Bohm oscillations and transport properties.

In this work we address a yet unexplored aspect of TQD molecules, namely their flux-tunable CP sensitivity, which is fundamental for the operation of the chiral metamaterial that we propose. Each TQD molecule exhibits CP birefringence and thus can serve as a Kerr rotator. CP birefringence becomes strong above a threshold frequency, $\Omega$, for which the TQD molecule becomes completely transparent to one of the two CPs. Furthermore, we propose the TQD molecule as an active medium for CP lasing action. A broadband pumping field can create a population inversion, and light with controlled circular polarization is emitted. The energy level hierarchy of the TQD molecule, which can be manipulated via the magnetic flux, determines which CP dominates for each molecular transition. This allows for a tunable switching of the CP of the emitted light, by experimentally accessible magnetic fields.
For our description, we consider three identical quantum-dots which are located at $R_1 = (a,0)$ and $R_{1/3} = (-a/2, \pm \sqrt{3}a/2)$ (see Fig. 1). Here we assume that each dot is characterized by a single energy level $\epsilon$, and consider interdot hopping of strength $t$. We only take into account a single spin species and work with a dot basis $\{ |1\rangle, |2\rangle, |3\rangle \}$. This is justified since we focus on single-electron effects leading to a spin-independent emission spectrum, while spin-mixing terms, such as spin-orbit coupling, are weak and can be neglected. In the absence of the magnetic field, the system possesses a $C_{3v}$ point group symmetry, generated by a $C_3$ $(2\pi/3)$ counterclockwise rotation of the system about the $\hat{z}$ axis and a mirror operation $\sigma_y (y \leftrightarrow -y)$. The $C_3$ symmetry suggests to switch to the more appropriate chiral basis

$$\ket{\lambda} = \frac{1}{\sqrt{3}} \left( 1, e^{-\lambda 2\pi i/3}, e^{\lambda 2\pi i/3} \right)^T \quad \text{with} \quad \lambda = 0, \pm, (1)$$

where the superscript $T$ denotes matrix transposition. Note that the chiral basis states satisfy $C_3 |\lambda\rangle = e^{\lambda 2\pi i/3} |\lambda\rangle$ with $\lambda = 0, \pm$. Mirror symmetry implies that the states ($|\pm\rangle$) are degenerate, thus additionally reflecting the preservation of time reversal symmetry ($T$).

In the presence of a perpendicular magnetic field $B = B_\perp \hat{z}$, the Hamiltonian in the dot basis $\{|1\rangle, |2\rangle, |3\rangle\}$ reads

$$\hat{H}_{\text{TQD}} = \begin{pmatrix} -t e^{-i\phi} - te^{i\phi} & \epsilon & \epsilon \\ -te^{-i\phi} & -t e^{i\phi} & \epsilon \\ -te^{i\phi} & -te^{-i\phi} & \epsilon \end{pmatrix}, \quad (2)$$

where $\phi = 2\pi \nu/3$ and $\nu = \Phi/\Phi_0$ denotes the normalized flux ($\Phi_0 = \hbar/e$) piercing the triangular area $A = 3\sqrt{3}a^2/4$ of the TQD molecule. The Hamiltonian is diagonal in the chiral basis with eigenenergies, $E_\lambda = \epsilon - 2t \cos[2\pi (\nu - \lambda)/3]$. For half-integer multiples of three flux quanta, mirror and $T$ symmetries are restored, leading to the aforementioned degeneracy between the $|\pm\rangle$ states. For the remaining half-integer multiples of $\Phi_0$, additional degeneracies appear between the $|0\rangle$ and the $|\pm\rangle$ levels. If the dots are considered as ideal zero-dimensional objects, $\epsilon$ and $t$ are flux-independent, rendering the energy spectrum periodic in the flux with a period of three flux quanta. On the other hand, for finite-size dots, $\epsilon$ and $t$ depend on the flux due to the orbital effects of the applied magnetic field. The latter dependence can be obtained via a microscopic continuum model (see Sup. A). As a result, the flux periodicity is broken, while the degeneracies persist.

The flux dependence of the eigenenergies of the TQD molecule, measured from $\epsilon$, is depicted in Fig. 2. Apart from the degeneracies occurring at half-integer values of $\nu$, we observe that the energy differences, $E_\lambda - \epsilon$, decrease with increasing flux as a result of the field dependence of the interdot tunneling strength $t$. Essentially the latter sets the range of the Bohr frequencies $\omega_{\lambda, \nu} \equiv (E_\lambda - E_\nu)/\hbar$ and thus the frequencies of the emitted or absorbed light. Furthermore, the level hierarchy determines the selection of CP for each molecular transition. Its tunability, via the flux dependence of the spectrum, is crucial for the CP of the emission spectrum. Finally, we obtain an overall increase of $E_\lambda$ with the enclosed flux due to the diamagnetic shift of the single-electron energy $\epsilon$ (see Sup. A).

The CP-sensitivity of the TQD molecule becomes manifest in its dipole coupling to light, which is described by $\hat{H}_{\text{dip}} = -\hat{P} \cdot \hat{E}$, with polarization operator $\hat{P} = -e \sum_{n=1,2,3} |n\rangle \langle n| R_n$ and elementary charge $e > 0$. In the chiral basis, we have

$$\hat{P}_x = -\frac{ea}{2} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \hat{P}_y = -\frac{ea}{2} \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}. \quad (3)$$

It is more convenient to rewrite the dipole coupling in the CP basis, i.e., $\hat{H}_{\text{dip}} = -\hat{P}_+ \hat{\xi}_- - \hat{P}_- \hat{\xi}_+$, with $\hat{P}_\pm = (\hat{P}_x \pm i \hat{P}_y)/\sqrt{2}$ and $\hat{\xi}_\pm = (\hat{\xi}_x \pm i \hat{\xi}_y)/\sqrt{2} = \hat{\xi}(\hat{a}_+ + \hat{a}_-)$, where $+/-$ corresponds to right/left-handed CP (RHCP/LHCP). Here we expressed the quantized electric field in terms of the photonic operators, $\hat{a}_\pm = (\hat{a}_x \pm i \hat{a}_y)/\sqrt{2}$. The prefactor $\hat{\xi} = \sqrt{\hbar \omega/(2\pi)}$, depends on the dielectric constant $\varepsilon$ and the effective mode volume $v$. In the chiral basis, the dipole coupling becomes

$$\hat{H}_{\text{dip}} = \hbar g \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} (\hat{a}_+ + \hat{a}_-) + \text{H.c.}, \quad (4)$$

with coupling strength $g = e\varepsilon\alpha/\sqrt{2}\hbar$, which depends on the interdot distance and the frequency domain where the system operates.
When the state \(|0\rangle\) is occupied and lies higher in energy than \(|0\rangle\) (as illustrated in Fig. 3(a)), the process \(|+\rangle \rightarrow |0\rangle\) is accompanied by the emission of a right-handed photon (\(\hat{a}_+\)). In comparison the absorption of a left-handed photon (\(\hat{a}_-\)), which involves a fast oscillation, is strongly suppressed and ignored in the rotating-wave approximation [27]. When the energy level arrangement for the levels is inverted (as illustrated in Fig. 3(b)), now with the \(|0\rangle\) state occupied, left-handed photons are emitted. Consequently, by controlling the energy level arrangement via the magnetic flux, one can select the polarization of photons which are emitted by the TQD molecule. Based on these properties, the TQD molecule can be used for CP-conversion. Specifically, by irradiating the TQD molecule with photons of a particular CP and frequency equal to the largest TDQ Bohr frequency (see Fig. 3), we can collect photons of the opposite CP, via coupling to a cavity which is resonant to one of the two remaining TDQ Bohr frequencies (see Fig. 3).

The properties of the TQD molecules open new perspectives in CP-sensitive applications. Three examples are described below: (i) CP-birefringence, (ii) tunable Kerr rotation, both considered when in the presence of a static magnetic field \(B_z\), a classical electromagnetic wave \(E(x, t) = E_0 e^{i(qz - \omega t)}\) is normally incident to the TQD \(xy\)-plane, and (iii) lasing action with a TQD-based metamaterial as an active medium, when the latter is coupled to a cavity supporting CP, e.g., a vertical cavity used in a spin-laser [26]. The measurement of the CP-dependent reflectivities, \(R_{\pm}\), and the resulting Kerr angle, \(\theta_K\), can serve as a diagnostic tool for the field-induced chirality of each TQD molecule. To obtain the reflectivities and the Kerr angle we calculate the conductivity (\(\sigma\)) and dielectric (\(\varepsilon\)) tensors (see [28], Supps. [13] and [17] which provide the complex refractive index. The two tensors are related via \(\varepsilon(\omega) = 1 + i\sigma(\omega) / \omega \varepsilon_0\), where \(\varepsilon_0\) defines the permittivity of vacuum. For the calculations, we consider zero temperature and assume that only the state \(|0\rangle\) is initially occupied, allowing transitions to \(|\pm\rangle\) with equal excitation and relaxation rates \(\Gamma_0\). The Kerr angle becomes sizeable for \(\omega > \omega_{\pm,0}\), shows a sharp peak for \(\omega \approx \omega_{\pm,0}\) and remains large beyond. Here we choose \(\nu = 1/20\) which yields \(\hbar \omega_{\pm,0} \approx 44/50 \mu eV\). (c) Pumping scheme for a CP-tunable laser. A broadband pumping field induces transitions between states \(|\pm\rangle\) and \(|0\rangle\) (with equal excitation and relaxation rates \(\Gamma_p\), while the lasing transition involves the states \(|+\rangle\) and \(|0\rangle\). A population inversion occurs when the relaxation from state \(|\pm\rangle\) (with rate \(\Gamma_1\)) is stronger than that from state \(|+\rangle\) (with rate \(\Gamma_2\)).

For a given energy-level hierarchy, each transition between the dot levels becomes CP-filtered. For illustration, we focus on the transition between the states \(|0\rangle\) and \(|+\rangle\), for which the dipole interaction reads

\[
\hat{H}^{(0,+)} = \hbar g \left( \hat{a}_+ + \hat{a}_- \right) |0\rangle \langle + | + H.c. \tag{5}
\]

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sorption lines $\omega_{\pm,0}$. In this frequency region, the TDQ molecule becomes highly reflective, with a small difference $\delta R \equiv R_- - R_+$. For photon frequencies above the threshold frequency, $\Omega \approx (\omega_{+,0}^2 + \omega_{+,0}^4)/2(\omega_{+,0}^2 - \omega_{-,0}^2)$ (see Sup. C), $\delta R$ changes dramatically and strong birefringence sets in. The sharp dip at $\omega = \Omega$ occurs due to complete transparency of the TDQ molecule to RHCP, i.e. $R_+ = 0$, while at the same time the LCP is completely reflected, i.e. $R_- \approx 1$. The reason for this sharp dip is that the dielectric function becomes $\varepsilon_\pm(\Omega) = 1$, since at this frequency $\text{Re} \sigma_\Omega(\omega) = \text{Im} \sigma(\omega)$. Obviously, this feature requires a finite Hall conductivity and thus emerges only in the presence of a magnetic field.

**Tunable Kerr effect.** The TDQ can additionally produce a Kerr angle $\theta_K$ of measurable size, reaching a few mrads even for weak magnetic fields. In Fig. 1(b) we show results for $\nu = 1/20$ with $\hbar \omega_{+,0} = 44\mu eV$ and $\hbar \omega_{-,0} = 50\mu eV$. The Kerr angle is in principle observable for $\omega > \omega_{+,0}$ and is characterized by a sharp peak-like feature for $\omega \approx \omega_{-,0}$. For $\omega \gg \omega_{0,0}$ the Kerr angle increases monotonically and acquires quite high values. As a matter of fact, in the latter region, where both $\delta R$ and $\theta_K$ become significant and experimentally accessible, the TDQ molecule can be employed as a sensitive CP-filter, rotator and detector.

**CP-sensitive lasing.** The strategy for lasing action based on TDQ molecules is to operate close to an energy degeneracy point and to create a population inversion between the two nearly degenerate states. This allows controlling the handedness of the emission CP and inverting it by a slight modification of the magnetic field. For instance, a lasing transition between the states $|+\rangle$ and $|0\rangle$ at $\nu = 0.40$ couples to the photons with right-handed CP (RHCP), as indicated in Fig. 2. For an interdot distance of 200nm an increase of the magnetic field of about 50 mT can bring the system to $\nu = 0.64$, where the photons with the same frequency ($\omega \approx 2\pi \times 2 \text{ GHz}$) but left-handed CP (LHCP) couple to the lasing states.

In order to create the population inversion, $\tau_0$ (see [27] and Sup. [1]), we propose a pumping scheme based on the intrinsic three levels of the TDQ. For definiteness we consider the level arrangement of Fig. 2(a). An unpolarized broadband radiation pumps from $|0\rangle$ to $|0\rangle$, while the lasing transition involves the states $|+\rangle$ and $|0\rangle$ (Fig. 2(c)). For simplicity we assume that the pumping by a classical field induces up- and down-transitions with equal rates $\Gamma_\nu$. Population inversion occurs when the relaxation rate $\Gamma_1$ from the state $|0\rangle$ to $|0\rangle$ is larger than the rate $\Gamma_2$ from $|+\rangle$ to $|0\rangle$, in combination with the condition $\Gamma_\nu > \Gamma_1^2/(\Gamma_1^2 - \Gamma_\nu^2)$. Since in the particular case all the molecular transitions occur with equal matrix elements (Eq. 4) a difference between $\Gamma_1,\Gamma_2$ can be achieved via an asymmetric coupling to the environment, which arises if $\omega_{+,0} \neq \omega_{-,0}$. In a realistic situation, the broadband pumping field may also effect the transition between $|0\rangle$ and $|+\rangle$ due to their closeness in frequencies, while the state $|0\rangle$ also has a finite relaxation rate $\Gamma_3$ to the state $|0\rangle$. Both sources of population inversion can be avoided by an appropriate choice of the pumping field frequency. Note also that the aforementioned conditions are not that restrictive, as lasing can occur even without population inversion [29].

Another aspect which is important for the CP-sensitive applications is the coupling to the cavity, represented by $g$ in Eq. 4. Strong coupling has been demonstrated for both gate-defined and self-assembled quantum-dots. Specifically, for gate-defined quantum dots coupled to a microwave transmission line cavity, the coupling strength $g$ can reach tens of MHz for an interdot distance of about a hundred nanometers [30, 31]. In addition, strong coupling of the order of 10 GHz was recently reported for a single self-assembled quantum dot coupled to a fiber Fabry-Perot cavity [32]. The experimentally feasible strong coupling to the cavity allows satisfying a further condition for lasing with TDQ molecules, namely the inequality $g \geq \sqrt{\kappa(\Gamma_\nu + \Delta^2)/(2\gamma_0 \Gamma_\nu^2)}$, which requires a strong coupling to overcome the total dephasing of the quantum-dots with rate $\Gamma_\nu$ and the damping of the cavity with rate $\kappa$, as well as the frequency detuning $\Delta$ between the dot states and the cavity [33].

The operational frequency range of the proposed device is determined by the energy splitting of the TDQ molecule, which may reach few tens of GHz. In addition, the CP of the emission spectrum depends on the energy-level arrangement, which for the typical realization can be manipulated and switched by changing a weak magnetic field by a few tens of mT. Further miniaturization of the interdot distance, down to the order of twenty nanometers, opens the door for applications and lasing operation in the lower far infrared regime. However, the degree of CP-tunability is reduced in this case, since the small TDQ area requires a magnetic field above 5T to change the flux quanta by ten percent.

In summary, we proposed a new highly tunable CP-sensitive metamaterial based on arrays of artificial molecules. Each molecule consists of three triangularly arranged quantum-dots. The particular platform is in principle accessible in the lab and can be readily employed for CP-sensitive applications in the microwave regime, taking advantage of the high-tunability provided by the quantum-dot technology. The system can show strong signatures of CP-birefringence, which become evident in the reflectivity and the Kerr rotation, rendering the setup a CP-filter and detection device. Finally, we showed that lasing action accompanied by CP-conversion is realizable in this frequency domain and can be finely tuned using weak magnetic fields.

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Supplementary Material

Appendix A: Estimation of the interdot tunneling strength

We estimate the tunneling strength by adopting a microscopic model of a lateral double-dot system occupied by one electron, which allows for simple analytical results. We assume the quantum dots are located at $R_{L/R} = (\pm \sqrt{3} a/2, 0)$. A perpendicular magnetic field, $B = B_z \hat{z}$, is present, for which we adopt a symmetric gauge $A(r) = \frac{B_z}{2}(-y, x, 0)$. The total Hamiltonian can be written as a sum of an orbital part, $H_{\text{orb}} = \frac{1}{2m} [\mathbf{P} + e\mathbf{A}(r)]^2 + V_{\text{ovl}}(r)$, and a Zeeman term, $H_Z$. Here we adopt an overall confinement $\frac{2}{\alpha}$,

$$V_{\text{ovl}}(r) = \frac{m \omega_0^2}{2} \left[ \frac{1}{3\alpha^2} \left( x^2 - 3a^2/4 \right)^2 + y^2 \right],$$

which reduces to a parabolic potential well $V_{L/R}(r) = m \omega_0^2 \left( (x \pm \sqrt{3}a/2)^2 + y^2 \right)/2$ around each dot center $R_{L/R}$. We employ a tight-binding model with the Hamiltonian of each isolated quantum dot given by

$$H = \frac{1}{2m} \left( \mathbf{P} + e\mathbf{A} \right)^2 + \nu,$$

which approximates to a parabolic potential well. The overlap of the electronic orbitals is of the order of the Bohr radius $a$, and the interdot distance is of the same order of the Bohr radius $B$. The total Hamiltonian can be written as a sum of an orbital part, $H_{\text{orb}} = \frac{1}{2m} [\mathbf{P} + e\mathbf{A}(r)]^2 + V_{\text{ovl}}(r)$, and a Zeeman term, $H_Z$. Here we adopt an overall confinement $\frac{2}{\alpha}$.

Appendix B: Conductivity tensor

For the calculation of the conductivity tensor we consider the related current-current correlation function and take into account only the lowest order bubble diagram:

$$\sigma_{ss'}(\omega) = \sum_{\lambda, \lambda'} \frac{i \langle \lambda | \mathcal{J}_s | \lambda' \rangle \langle \lambda' | \mathcal{J}_{s'} | \lambda \rangle [f(E_\lambda) - f(E_{\lambda'})]}{v \hbar \omega (\omega_{\lambda, \lambda'} + \omega + i\eta)},$$

where $\mathcal{J}_s$ is the current operator, $f(E)$ is the Fermi-Dirac function, and $\omega_{\lambda, \lambda'}$ is the Matsubara frequency. The conductivity is given by

$$\sigma_{ss'}(\omega) = \frac{e^2}{\pi \hbar} \sum_{\lambda, \lambda'} \frac{1}{\omega_{\lambda, \lambda'} + \omega + i\eta},$$

where $e$ is the electron charge and $\hbar$ is the reduced Planck constant.

We estimate the tunneling strength by adopting a microscopic model of a lateral double-dot system occupied by one electron, which allows for simple analytical results. We assume the quantum dots are located at $R_{L/R} = (\pm \sqrt{3} a/2, 0)$. A perpendicular magnetic field, $B = B_z \hat{z}$, is present, for which we adopt a symmetric gauge $A(r) = \frac{B_z}{2}(-y, x, 0)$. The total Hamiltonian can be written as a sum of an orbital part, $H_{\text{orb}} = \frac{1}{2m} [\mathbf{P} + e\mathbf{A}(r)]^2 + V_{\text{ovl}}(r)$, and a Zeeman term, $H_Z$. Here we adopt an overall confinement $\frac{2}{\alpha}$.

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Appendix B: Conductivity tensor

For the calculation of the conductivity tensor we consider the related current-current correlation function and take into account only the lowest order bubble diagram:

$$\sigma_{ss'}(\omega) = \sum_{\lambda, \lambda'} \frac{i \langle \lambda | \mathcal{J}_s | \lambda' \rangle \langle \lambda' | \mathcal{J}_{s'} | \lambda \rangle [f(E_\lambda) - f(E_{\lambda'})]}{v \hbar \omega (\omega_{\lambda, \lambda'} + \omega + i\eta)},$$

where $\mathcal{J}_s$ is the current operator, $f(E)$ is the Fermi-Dirac function, and $\omega_{\lambda, \lambda'}$ is the Matsubara frequency. The conductivity is given by

$$\sigma_{ss'}(\omega) = \frac{e^2}{\pi \hbar} \sum_{\lambda, \lambda'} \frac{1}{\omega_{\lambda, \lambda'} + \omega + i\eta},$$

where $e$ is the electron charge and $\hbar$ is the reduced Planck constant.
where $f(E)$ denotes the Fermi-Dirac distribution, $s, s' = x, y$ and $\eta \to 0^+$. Here we assumed thermal equilibrium. We introduced the effective volume of the TQD molecule, $v = \pi a^2w$, with $w \sim 10$nm corresponding to the length of the quantum-dots along the $z$-axis[21]. In Eq. (B1), $\mathbf{J}$ defines the current operator, which is related to the polarization operator of the TQD molecule through the relation $\mathbf{J} = \partial \mathbf{P} / \partial t$. The current operator is directly obtained based on the expression $\mathbf{J} = i [\mathbf{H}_{\text{TQD}}, \mathbf{P}] / \hbar$ yielding the matrix elements $\langle \lambda | \mathbf{J} | \lambda' \rangle = i \hbar \omega_{\lambda, \lambda'} (\lambda \mathbf{P} | \lambda' \rangle$ with $\lambda, \lambda' = 0, \pm$. The eigenenergies differences are compactly written
\[
h\omega_{\lambda, \lambda'} = E_{\lambda} - E_{\lambda'} = -4t \sin [(\lambda - \lambda')\pi/3] \sin [\phi - (\lambda + \lambda')\pi/3]. \tag{B2}
\]
The conductivity tensor satisfies $\sigma_{xx}(\omega) = \sigma_{yy}(\omega) \equiv \sigma(\omega)$ and $\sigma_{xy}(\omega) = -\sigma_{yx}(\omega) \equiv \sigma_H(\omega)$, which denotes the Hall conductivity. The general expressions for the conductivities, when $\omega > 0$ and $\eta \to 0^+$ (for the calculations we used $\hbar\eta = 0.3\mu$eV), become
\[
\sigma(\omega) = -\frac{e^2}{wh} \sum_{(\lambda, \lambda')}^{\mathcal{O}} f(E_{\lambda}) - f(E_{\lambda'}) \omega_{\lambda, \lambda'}^3 \frac{\eta + i(\omega - |\omega_{\lambda, \lambda'}|)}{\omega + |\omega_{\lambda, \lambda'}|^2 + \eta^2}, \tag{B3}
\]
\[
\sigma_H(\omega) = -\frac{e^2}{wh} \sum_{(\lambda, \lambda')}^{\mathcal{O}} f(E_{\lambda}) - f(E_{\lambda'}) \omega_{\lambda, \lambda'}^3 \frac{\omega - |\omega_{\lambda, \lambda'}| - i\eta}{\omega + |\omega_{\lambda, \lambda'}|^2 + \eta^2}, \tag{B4}
\]
where $\mathcal{O} \equiv \{(0, +), (+, -), (-, 0)\}$ denotes the pairs of $(\lambda, \lambda')$ that have to be taken into account. If we assume zero-temperature and therefore consider that only the $(0)$ is occupied, we obtain the Bohr frequencies $h\omega_{\lambda, 0} \equiv E_{\lambda} - E_0 = 2\sqrt{\mathcal{V}t} \sin (\pi/3 - \lambda\phi) = 2\sqrt{\mathcal{V}t} \sin [(1 - 2\lambda\nu)\pi/3] \geq 0$ and the conductivities
\[
\sigma(\omega) \approx \frac{e^2}{wh} \sum_{\lambda = \pm}^{\mathcal{O}} \omega_{\lambda, 0}^3 \frac{\eta - i(\omega_{\lambda, 0} - \omega)}{\omega_{\lambda, 0} - \omega^2 + \eta^2}, \quad \sigma_H(\omega) \approx \frac{e^2}{wh} \sum_{\lambda = \pm}^{\mathcal{O}} \frac{\omega_{\lambda, 0}^3}{\omega_{\lambda, 0} + \omega} \left(\omega_{\lambda, 0} - \omega + i\eta \right). \tag{B5}
\]
Note that $\sigma_H(\omega) = 0$ for $\nu = 3k/2$ with $k \in \mathbb{Z}$, since $\omega_{+, 0} = \omega_{-, 0}$, implying that $T$ is restored. For frequencies $\omega \gg \omega_{\pm, 0}$, the conductivities become
\[
\sigma(\omega) \approx \frac{e^2}{wh} \frac{i}{\omega^3} \left(\omega_{-, 0}^3 + \omega_{+, 0}^3\right), \quad \sigma_H(\omega) \approx \frac{e^2}{wh} \frac{\omega_{+, 0}^2 - \omega_{-, 0}^2}{\omega^2}. \tag{B6}
\]

**Appendix C: Dielectric tensor and CP birefringence**

Starting from the conductivity tensor we can calculate the dielectric tensor $\varepsilon$ through the defining relation
\[
\varepsilon(\omega) = 1 - \frac{\sigma(\omega)}{i\omega\varepsilon_0} = 1 - \frac{\text{Im} \sigma(\omega)}{\varepsilon_0} + i \frac{\text{Re} \sigma(\omega)}{\varepsilon_0}, \tag{C1}
\]
where we have introduced the permittivity of vacuum $\varepsilon_0$ and considered electromagnetic waves of the following form $E(z,t) = \mathbf{E} e^{\sqrt{q^2 - \omega^2} t}$. By introducing
\[
\varepsilon(\omega) = 1 - \frac{\text{Im} \sigma(\omega)}{\omega\varepsilon_0} + i \frac{\text{Re} \sigma(\omega)}{\omega\varepsilon_0} \quad \text{and} \quad \varepsilon_H(\omega) = \frac{\sigma_H(\omega)}{\omega\varepsilon_0}, \tag{C2}
\]
we can diagonalize the dielectric tensor and obtain the dispersions
\[
\omega = \frac{\varepsilon q}{\sqrt{\varepsilon(\omega)}}, \tag{C3}
\]
with $\varepsilon_{\pm}(\omega) = \varepsilon(\omega)\pm\varepsilon_H(\omega)$ and complex refractive index, $N_{\pm}(\omega) = \sqrt{\varepsilon_{\pm}(\omega)}$, corresponding to the circular polarizations $\mathcal{E}_{\pm}$. Consequently the TQD exhibits optical birefringence and we expect a finite Kerr angle for the reflected beam of an incident linearly polarized beam onto the TQD. For $\omega \gg \omega_{\pm, 0}$ the dielectric takes the approximate form
\[
\varepsilon_{\pm}(\omega) \approx 1 + \frac{1}{\omega\varepsilon_0} \left[ \pm \text{Re} \sigma_H(\omega) - \text{Im} \sigma(\omega) \right] = 1 + \frac{e^2}{\omega\varepsilon_0 \hbar w} \left[ \pm \frac{\omega_{+, 0}^2 - \omega_{-, 0}^2}{\omega^2} - \frac{\omega_{-, 0}^3 + \omega_{+, 0}^3}{\omega^3} \right]. \tag{C4}
\]
Note that $\varepsilon_+ = 1$ for approximately the frequency $\Omega = (\omega_{-, 0}^2 + \omega_{+, 0}^2) / (\omega_{-, 0}^2 - \omega_{+, 0}^2)$ leading to complete transparency for the right-handed CP, while the other is almost perfectly reflected.
Appendix D: Reflectivity and Kerr angle

We consider that an electromagnetic wave of the form
\[ \mathcal{E}_i(z, t) = \mathcal{E}_{i+} e^{i(qz - \omega t)} + \mathcal{E}_{i-} e^{i(qz - \omega t)}, \] (D1)
is incident on the TQD with \( cq = \omega \). Here \( \mathcal{E}_{i\pm} \) correspond to the unit vectors for the respective CPL. Due to the interface to the TQD there is a reflected and transmitted beam of the following form
\[ \mathcal{E}_r(z, t) = -\mathcal{E}_{r+} e^{-i(qz + \omega t)} - \mathcal{E}_{r-} e^{-i(qz + \omega t)}, \] (D2)
\[ \mathcal{E}_t(z, t) = +\mathcal{E}_{r+} e^{i(qz + \omega t)} + \mathcal{E}_{r-} e^{i(qz + \omega t)}, \] (D3)
with \( cq_{\pm} = \omega \sqrt{\varepsilon_{\pm}(\omega)} = \omega N_{\pm}(\omega) \). By introducing the reflection and transmission coefficients \( \mathcal{E}_{r\pm} = r_{\pm} \mathcal{E}_{i\pm} \) and \( \mathcal{E}_{t\pm} = t_{\pm} \mathcal{E}_{i\pm} \) respectively and considering the continuity of the fields at the interface \( z = 0 \) we obtain
\[ 1 - r_{\pm} = t_{\pm} \quad \text{and} \quad 1 + r_{\pm} = t_{\pm} N_{\pm}(\omega), \] (D4)
which yields
\[ r_{\pm}(\omega) = \frac{N_{\pm}(\omega) - 1}{N_{\pm}(\omega) + 1} \quad \text{and} \quad t_{\pm}(\omega) = \frac{2}{N_{\pm}(\omega) + 1}. \] (D5)
The reflectivity reads \( R_{\pm}(\omega) = |r_{\pm}(\omega)|^2 \). The Kerr angle \( \theta_K \) is given by the expression: \( \theta_K = (\theta_+ - \theta_-)/2 \), with the related angles defined as
\[ \theta_l = \tan^{-1}\left[ \frac{\text{Im} r_l(\omega)}{\text{Re} r_l(\omega)} \right], \quad \text{with} \quad l = \pm. \] (D6)

Appendix E: Lasing and population inversion

The dynamics of the system is described by the master equation for the density matrix \( \rho \) \[27\],
\[ \dot{\rho} = -\frac{i}{\hbar}[H_{\text{sys}}, \rho] + \sum_i \mathcal{L}_i \rho, \] (E1)
where \( H_{\text{sys}} \) denotes the Hamiltonian for the coupled dot-cavity system, and \( \mathcal{L}_i \) the Liouville superoperators for dissipation. Here we assume the system is weakly coupled to the environment and adopt the Born-Markovian approximation where the dissipative dynamics are described in the Lindblad form,
\[ \mathcal{L}_i \rho = \frac{\Gamma_i}{2} \left( 2L_i \rho L_i^\dagger - L_i^\dagger L_i \rho - \rho L_i^\dagger L_i \right). \] (E2)

For definiteness, we consider the situation in Fig 4(c) where the cavity is resonant to the transition between the states \( |+\rangle \) and \( |0\rangle \), while the pumping field connects the states \( |\rangle \) and \( |0\rangle \). In this case, the relaxation and excitation induced by the incoherent pumping are described by \( L_\uparrow = |\rangle \langle 0| \) and \( L_\downarrow = |0\rangle \langle \rangle \) with the same rate \( \Gamma_p \), and the relaxation processes \( |\rangle \rightarrow |+\rangle \) and \( |\rangle \rightarrow |0\rangle \) by \( L_1 = |+\rangle \langle \rangle \) and \( L_2 = |0\rangle \langle +| \) with rates \( \Gamma_{1,2} \), respectively. The population inversion, \( \tau_0 \), can then be obtained from the steady state solution of the master equation with vanishing coupling to the cavity, which is given by
\[ \tau_0 = \frac{\gamma_1 \gamma_p - \gamma_1 - \gamma_p}{\gamma_1 \gamma_p + \gamma_1 + \gamma_p}. \] (E3)
with rates \( \gamma_{1,p} = \Gamma_{1,p}/\Gamma_2 \). From Eq. [E3], a positive population inversion is achieved when the relaxation from the state \( |\rangle \) is stronger than that from \( |+\rangle \), namely, \( \gamma_1 > 1 \), in combination with a strong pumping field satisfying \( \gamma_p > \gamma_1/(\gamma_1 - 1) \).