Abstract. Let $G$ be a semisimple algebraic group defined over a number field $K$, $T$ a maximal $K$-split torus of $G$, $S$ a finite set of valuations of $K$ containing the archimedean ones, $O$ the ring of $S$-integers of $K$ and $K_S$ the direct product of the completions $K_v$, $v \in S$. Let $G = G(K_S)$, $T = T(K_S)$ and $\Gamma$ be an $S$-arithmetic subgroup of $G$. For the action of $T$ on $G/\Gamma$ by left translations we describe the closures of the locally divergent non-closed $T$-orbits. If $\#S = 2$ such a closure is a union of finitely many $T$-orbits stratified in terms of parabolic subgroups of $G \times G$. Therefore it is never homogeneous contradicting a conjecture of Margulis. If $\#S > 2$ and $K$ is not a CM-field the closure is homogeneous if $G = \text{SL}_n$ (confirming the conjecture) and squeezed between two closed orbits of reductive groups of equal semisimple ranks, in general. As an application, we prove that if $f = (f_v)_{v \in S} \in K_S[x_1, \cdots, x_n]$, where $f_v$ are non-pairwise proportional decomposable homogeneous forms over $K$, then $f(O^n)$ is dense in $K_S$.

1. Introduction

Let $G$ be a semisimple algebraic group defined over a number field $K$. Let $S$ be a finite set of (normalized) valuations of $K$ containing all archimedean ones and $O$ the ring of $S$-integers of $K$. We denote by $K_v$, $v \in S$, the completion of $K$ with respect to $v$ and by $K_S$ the direct product of the topological fields $K_v$. Put $G = G(K_S)$. The group $G$ is naturally identified with the direct product of the locally compact groups $G_v = G(K_v)$, $v \in S$, and $G(K)$ is diagonally imbedded in $G$. Let $\Gamma$ be an $S$-arithmetic subgroup of $G$, that is, $\Gamma \cap G(O)$ have finite index in both $\Gamma$ and $G(O)$. Recall that $\Gamma$ is a lattice in $G$ meaning that the homogeneous space $G/\Gamma$ has finite volume with respect to the Haar measure. Every closed subgroup $H$ of $G$ acts on $G/\Gamma$ by left translations

$$h\pi(g) \overset{\text{def}}{=} \pi(hg),$$

where $\pi : G \to G/\Gamma$ is the quotient map. An orbit $H\pi(g)$ is called divergent if the orbit map $H \to G/\Gamma, h \mapsto h\pi(g)$ is proper, i.e. if
\( \{ h \pi(g) \} \) leaves compacts of \( G/\Gamma \) whenever \( \{ h \} \) leaves compacts of \( H \). It is clear that every divergent orbit is closed. Recall that the closure \( \overline{H \pi(g)} \) of \( H \pi(g) \) in \( G/\Gamma \) is homogeneous if \( \overline{H \pi(g)} = L \pi(g) \) where \( L \) is a closed subgroup of \( G \).

Fix a maximal \( K \)-split torus \( T \) of \( G \) and, for every \( v \in S \), a maximal \( K_v \)-split torus \( T_v \) of \( G \) containing \( T \). Recall that, given a field extension \( F/K \), the \( F \)-rank of \( G \), denoted by \( \text{rank}_F G \), is the common dimension of the maximal \( F \)-split tori of \( G \). So, \( \text{rank}_{K_v} G \geq \text{rank}_K G \) and \( \text{rank}_{K_v} G = \text{rank}_K G \) if and only if \( T = T_v \). Let \( T_v = T_v(K_v) \) and \( T = \prod_{v \in S} T_v \subset G \). An orbit \( T \pi(g) \) is called locally divergent if \( T_v \pi(g) \) is divergent for every \( v \in S \).

The locally divergent orbits, in general, and the closed locally divergent orbits, in particular, are completely described by the following theorem:

**Theorem 1.1.** (cf. [T1, Theorem 1.4 and Corollary 1.5]) With the above notation, we have:

(a) An orbit \( T_v \pi(g) \) is divergent if and only if

1. \( \text{rank}_{K_v} G = \text{rank}_K G \)

and

2. \( g \in Z_G(T_v)G(K) \),

where \( Z_G(T_v) \) is the centralizer of \( T_v \) in \( G \). So, \( T \pi(g) \) is locally divergent if and only if (1) and (2) hold for all \( v \in S \);

(b) An orbit \( T \pi(g) \) is locally divergent orbit and closed if and only if (1) holds for all \( v \in S \) and

\( g \in N_G(T)G(K) \),

where \( N_G(T) \) is the normalizer of \( T \) in \( G \).

Theorem 1.1 implies easily:

**Corollary 1.2.** A locally divergent and non-closed \( T \)-orbit exists if and only if \( \text{rank}_K G > 0 \), \( \# S \geq 2 \), and (1) is valid for all \( v \in S \).

The proof Theorem 1.1 had been preceded by [T2], [We], [T-We], and a result of G.A. Margulis describing the divergent orbits for the action of the group of diagonal matrices on \( \text{SL}_n(\mathbb{R})/\text{SL}_n(\mathbb{Z}) \) (cf. [T-We, Appendix]).

From now on we denote by \( T \pi(g) \) a locally divergent non-closed orbit. Our main goal is to describe the closure \( \overline{T \pi(g)} \) of \( T \pi(g) \) in \( G/\Gamma \). Let us note that the ideas involved in the present paper are new in comparison with [T-We], [We] and [T2]. The cases \( \# S = 2 \)
and \( \#S > 2 \) behave in drastically different ways. When \( \#S = 2 \) the structure of \( \overline{T\pi(g)} \) is similar to that of a toric variety. Namely, in this case \( T\pi(g) \) is open in \( \overline{T\pi(g)} \) and \( \overline{T\pi(g)} \) is a union of finitely many locally divergent \( T \)-orbits all stratified in terms of parabolic subgroups of \( G \times G \) (Theorem 1.3). So, if \( \#S = 2 \) the closure of \( T\pi(g) \) is never homogeneous (Corollary 1.6). On the other hand, if \( \#S > 2 \) and \( K \) is not a CM-field then the closure \( \overline{T\pi(g)} \) is “almost” homogeneous in the sense that \( H_1\pi(g) \supset \overline{T\pi(g)} \supset H_2\pi(g) \) where \( H_1\pi(g) \) and \( H_2\pi(g) \) are closed orbits of reductive subgroups \( H_1 \) and \( H_2 \) containing \( T \) of equal semi-simple ranks (Theorem 1.8). We show that \( H_1 = H_2 \) (that is, \( \overline{T\pi(g)} \) is homogenous) if \( G = \text{SL}_n \) (Corollary 1.9) and \( H_1 \neq H_2 \), in general (see §7).

Note that during the recent years difficult problems from the Diophantine approximation of numbers have been reformulated in terms of tori actions on \( G/\Gamma \) and, subsequently, successfully tackled (cf. [M2], [E-K-L], [E-Kl], [Sha]). It is worth mentioning that the dynamics of the action of maximal split tori on \( G/\Gamma \) is closely related to the dynamics of the action of unipotent groups on \( G/\Gamma \). Both actions have many complementary features but the latter is much better understood and motivates problems and conjectures about the former. For example, the unipotent flows are always recurrent (see [M6] and [L]) and, therefore, never divergent. As another example, recall that if \( H \) is a subgroup of \( G \) generated by 1-parameter unipotent subgroups then \( \overline{H\pi(g)} \) is always homogenous. (See Margulis’ proof of the Oppenheim Conjecture [M1], followed by [DMI], M.Ratner’s results for arbitrary real Lie groups [Ra1] and [Ra2], and the corresponding results in \( S \)-adic setting [BP], [MT], [Ra3] and [To4].) It was believed up to recently that \( T\pi(g) \) is homogenous whenever \( T \) is maximal (even higher dimensional) split torus and \( G/\Gamma \) does not admit rank 1 \( T \)-invariant factors ([M3: Conjecture 1]). Sparse counter-examples to this conjecture have been given in [Main] for \( G = \text{SL}_n(\mathbb{R}), n \geq 6 \), and the action of multi-dimensional but non-maximal \( T \), in [Sha] (see also [L-Sha, Theorem 1.5]) for \( \text{SL}_3(\mathbb{R})/\text{SL}_3(\mathbb{Z}) \) and maximal \( T \), and in [T3] for direct products of \( r \geq 2 \) copies of \( \text{SL}_2(\mathbb{R}) \) or \( \text{SL}_2(\mathbb{C}) \) and the action of both maximal and non-maximal \( T \). The main results from [T3] are extended in this paper to all semi-simple algebraic \( K \)-groups.

Let us give the precise formulations of our results for \( \#S = 2 \). So, let \( S = \{v_1, v_2\} \) and \( g = (g_{v_1}, g_{v_2}) \in G \) be such that \( T\pi(g) \) is locally divergent (Theorem 1.1(a)). We will need some basic concepts from the theory of linear algebraic groups (cf. [B] or §2.3). Let \( \Pi \) be a system of simple \( K \)-roots with respect to the maximal \( K \)-split torus.
T. Given $\Psi \subset \Pi$, we denote by $P_{\Psi}$ the corresponding to $\Psi$ standard parabolic subgroup and by $P_{\Psi}^{-}$ the opposite to $P_{\Psi}$ parabolic subgroup. It is well known that $P_{\Psi}$ (respectively, $P_{\Psi}^{-}$) is a semidirect product of its unipotent radical $V_{\Psi}$ (respectively, $V_{\Psi}^{-}$) and the Levy subgroup $Z_G(T_{\Psi}) = P_{\Psi} \cap P_{\Psi}$. Put

$$P_{\Psi}(g) = \{ \omega_1P_{\Psi}^{-}\omega_1^{-1} \times \omega_2P_{\Psi}^{-}\omega_2^{-1} | \omega_1, \omega_2 \in N_G(T), g_{\omega_1}g_{\omega_2}^{-1} \in \omega_1V_{\Psi}P_{\Psi}^{-}\omega_2^{-1} \}$$

and

$$P(g) = \bigcup_{\Psi \subset \Pi} P_{\Psi}(g).$$

The parabolic subgroups from the finite set $P(g)$ are called *admissible with respect to* $g$. It is clear that $P_{\Pi}(g) = \{ G \times G \}$ and $P_{\emptyset}(g)$ consists of minimal parabolic $K$-subgroups of $G \times G$.

To every $P \in P(g)$ we associate a locally divergent $T$-orbit as follows. If $P = \omega_1P_{\Psi}^{-}\omega_1^{-1} \times \omega_2P_{\Psi}^{-}\omega_2^{-1}$ and $g_{\omega_1}g_{\omega_2}^{-1} = \omega_1v_{\Psi}z_{\Psi}v_{\Psi}^{-1}\omega_2^{-1}$, where $v_{\Psi} \in V_{\Psi}$, $z_{\Psi} \in Z_G(T_{\Psi})$ and $v_{\Psi} \in V_{\Psi}$, we put

$$P(g) \bigcap (3) \text{ Orb}_g(P) \overset{\text{def}}{=} T(\omega_1(v_{\Psi})^{-1}, \omega_2v_{\Psi}v_{\Psi}^{-1})_\pi(g).$$

It is easy to see that $\text{Orb}_g(P)$ is a well-defined locally divergent orbit (Lemma 5.1). Note that $\text{Orb}_g(G \times G) = T\pi(g)$.

**Theorem 1.3.** With the above notation, let $P \in P(g)$. Then

$$\text{Orb}_g(P) = \bigcup_{P' \in P(g), P' \subset P} \text{Orb}_g(P').$$

In particular,

$$T\pi(g) = \bigcup_{P \in P(g)} \text{Orb}_g(P).$$

Theorem 1.3 implies immediately:

**Corollary 1.4.** Let $P \in P(g)$. Choose a $\tilde{P} \in P(g)$ with $\text{Orb}_g(\tilde{P}) = \text{Orb}_g(P)$ and being minimal with this property. Then

$$\text{Orb}_g(P) \setminus \text{Orb}_g(P) = \bigcup_{P' \in P(g), P' \subset \tilde{P}} \text{Orb}_g(P').$$

In particular, $T\pi(g)$ is open in its closure.

The closed $T$-orbits in $T\pi(g)$ are parameterized by the minimal parabolic subgroups of $G \times G$ belonging to $P(g)$. More precisely, we have:

**Corollary 1.5.** If $P$ is minimal in $P(g)$ then $\text{Orb}_g(P)$ is closed and $P$ is a minimal parabolic subgroup of $G \times G$. Moreover, $P_{\emptyset}(g) \neq \emptyset$ and \{Orb$_g(P)$ : $P \in P_{\emptyset}(g)$\} is the set of all closed $T$-orbits in $T\pi(g)$. 
We get the following refinement of Theorem 1.1 (b):

**Corollary 1.6.** The following conditions are equivalent:

(a) $T\pi(g)$ is closed,
(b) $T\pi(g)$ is homogenous,
(c) $g \in \mathcal{N}_G(T)G(K),$
(d) $T\pi(g) = \text{Orb}_g(P)$ for some minimal $P \in \mathcal{P}(g),$
(e) $T\pi(g) = \text{Orb}_g(P)$ for every $P \in \mathcal{P}(g).$

The map $P \mapsto \text{Orb}_g(P), P \in \mathcal{P}(g),$ is not injective, in general, but it becomes injective if $g_1g_2^{-1}$ belongs to a non-empty Zariski dense subset of $G.$

**Corollary 1.7.** Denote by $n_\Psi$ the number of parabolic subgroups containing $T$ and conjugated to $P_\Psi.$

(a) The number of different $T$-orbits in $\overline{T\pi(g)}$ is bounded from above by $\sum_{\Psi \subset \Pi} n_\Psi^2$ and the number of different closed $T$-orbits in $\overline{T\pi(g)}$ is bounded from above by $n_0^2 = |W|^2.$
(b) There exists a non-empty Zariski dense subset $\Omega \subset G(K)$ such that if $g_1, g_2^{-1} \in \Omega$ then the map $\text{Orb}_g(\cdot)$ is injective and $\overline{T\pi(g)}$ is a union of exactly $\sum_{\Psi \subset \Pi} n_\Psi^2$ pairwise different $T$-orbits and among them exactly $n_0^2$ are closed.

Now suppose that $\#S > 2.$ Denote $S = \{v_1, \ldots, v_r\}$ and $g = (g_{v_1}, \ldots, g_{v_r}) \in G.$ Recall that the semi-simple $K$-rank of a reductive $K$-group $H,$ denoted by $\text{s.s.rank}_K(H),$ is equal to $\text{rank}_K \mathcal{D}(H)$ where $\mathcal{D}(H)$ is the derived subgroup of $H.$ Also recall that $K$ is a CM-field if it is a quadratic extension of a totally real number field which is totally imaginary.

**Theorem 1.8.** Let $\#S > 2,$ $K$ be not a CM-field and $T\pi(g)$ be a locally divergent non-closed orbit. Then there exist $h_1$ and $h_2 \in \mathcal{N}_G(T)G(K)$ and reductive $K$-subgroups $H_1$ and $H_2$ of $G$ such that

$$\text{s.s.rank}_K(H_1) = \text{s.s.rank}_K(H_2) > 0,$$

and

$$h_2H_2\pi(e) \subset \overline{T\pi(g)} \subset h_1H_1\pi(e),$$

where $H_1 = H_1(K_S), H_2 = H_2(K_S)$ and the orbits $h_1H_1\pi(e)$ and $h_2H_2\pi(e)$ are closed and $T$-invariant.

It is shown on the example of a non-split $K$-group of type $A_3$ (Theorem 7.1) that the reductive subgroups $H_1$ and $H_2$ in the formulation of the theorem might be different. Nevertheless, in the important case when $G = \text{SL}_n$ we have:
Corollary 1.9. Let $G = \text{SL}_n$. We suppose that $\#S > 2$ and $K$ is not a CM-field. Then $T\pi(g) = H\pi(g)$, where $H$ is a closed subgroup of $G$. Moreover, $H = G$ if and only if $\bigcap_{i=1}^{r-1} Z_T(\omega_ig_{v_i}(g_{v_i})^{-1})$ is finite for all choices of $\omega_i \in N_G(T)$.

Corollary 1.9 is not true for CM-fields, cf. [T3, Theorem 1.8].

We apply Corollary 1.9 to describe the closures of the values at $S$-integral points of families of decomposable homogeneous forms over $K$. Let $K_S[\vec{x}]$ be the ring of polynomials in $n \geq 2$ variables $\vec{x} = (x_1, \ldots, x_n)$ with coefficients from the topological ring $K_S$. Note that $K_S[\vec{x}] = \prod_{v \in S} K_v[\vec{x}]$. The ring $K[\vec{x}]$ is identified with its diagonal imbedding in $K_S[\vec{x}]$. Let $f(\vec{x}) = (f_v(\vec{x}))_{v \in S} \in K_S[\vec{x}]$. We suppose that every $f_v(\vec{x}) = l_{1}^{(v)}(\vec{x}) \cdots l_{m}^{(v)}(\vec{x})$, where $l_{1}^{(v)}(\vec{x}), \ldots, l_{m}^{(v)}(\vec{x})$ are linearly independent over $K_v$ linear forms in $K_v[\vec{x}]$. So, $m \leq n$. It is easy to see that if $f(\vec{x}) = c \cdot h(\vec{x})$, where $h(\vec{x}) \in K[\vec{x}]$ and $c \in K_S$, then $f(\mathcal{O}^n)$ is discrete in $K_S$. In fact, the opposite is also valid: the discreteness of $f(\mathcal{O}^n)$ in $K_S$ implies that $f_v(\vec{x}), v \in S$, are all proportional to a polynomial $h$ with coefficients from $K$ (Theorem 1.8). It is natural to ask what is the closure of $f(\mathcal{O}^n)$ in $K_S$ if $f_v(\vec{x}), v \in S$, are not all proportional to a polynomial from $K[\vec{x}]$.

The following conjecture is plausible:

Conjecture 1. With $f$ as above, let $\#S > 2$ and $K$ be not a CM-field. Suppose that there is not a non-zero polynomial $h(\vec{x})$ with coefficients from $K$ such that every $f_v(\vec{x}), v \in S$, is proportional to $h(\vec{x})$. Then $f(\mathcal{O}^n)$ is dense in $K_S$.

Corollary 1.9 is used to prove:

Theorem 1.10. Conjecture 1 is true if for every $v \in S$ the linear forms $l_{1}^{(v)}(\vec{x}), \ldots, l_{m}^{(v)}(\vec{x})$ are all with coefficients from $K$.

It is shown in §8 that, with $f$ as in the formulation of the theorem, $f(\mathcal{O}^n)$ is not always dense in $K_S$ if $\#S = 2$ or $\#S \geq 2$ and $K$ is a CM-field.

2. Preliminaries: notation and some basic concepts

2.1. Numbers. As usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the non-negative integer, integer, rational, real and complex numbers, respectively. Also, $\mathbb{N}_+ = \{x \in \mathbb{N} : x > 0\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

In this paper $K$ is a number field, that is, a finite extension of $\mathbb{Q}$. All valuations of $K$ which we consider are supposed to be normalized.
(see [CF] ch.2, §7) and, therefore, pairwise non-equivalent. If \( v \) is a valuation of \( K \) then \( K_v \) is the completion of \( K \) with respect to \( v \) and \( | \cdot |_v \) is the corresponding norm on \( K_v \). Recall that if \( K_v = \mathbb{R} \) (respectively, \( K_v = \mathbb{C} \)) then \( | \cdot |_v \) is the absolute value on \( \mathbb{R} \) (respectively, the square of the absolute value on \( \mathbb{C} \)). If \( v \) is non-archimedean then \( \mathcal{O}_v = \{ x \in K_v : | x |_v \leq 1 \} \) is the ring of integers of \( K_v \).

We fix a finite set \( S = \{ v_1, \cdots, v_r \} \) of valuations of \( K \) containing all archimedean valuations of \( K \). The archimedean valuations in \( S \) will be denoted by \( S_\infty \) (or, simply, \( \infty \) if this does not lead to confusion). We also denote \( S_f = S \setminus S_\infty \).

Sometimes we will write \( K_i \) instead of \( K_{v_i} \) and \( | \cdot |_i \) instead of \( | \cdot |_{v_i} \).

We denote by \( \mathcal{O} \) the ring of \( S \)-integers of \( K \), i.e., \( \mathcal{O} = K \cap (\bigcap_{v \in S} \mathcal{O}_v) \). Also, \( \mathcal{O}_\infty = K \cap (\bigcap_{v \in S_\infty} \mathcal{O}_v) \) is the ring of integers of \( K \).

Let \( K_S = \prod_{v \in S} K_v \). The field \( K \) is a dense subfield of the topological ring \( K_S \) and \( \mathcal{O} \) is a lattice in \( K_S \).

As usual, if \( R \) is a ring \( R^* \) denotes the multiplicative group of units of \( R \).

2.2. Groups. Further on, we use boldface letters to denote the \( K \)-algebraic groups. Let \( H \) be a \( K \)-algebraic group. As usual, \( \mathcal{R}_u(H) \) (respectively, \( \text{Lie}(G) \)) stands for the unipotent radical (respectively, the Lie algebra) of \( H \). Given \( v \in S \), we write \( H_v \overset{\text{def}}{=} H(K_v) \) or simply \( H_i \) if \( S = \{ v_1, \cdots, v_r \} \) and \( v = v_i \). Put \( H \overset{\text{def}}{=} H(K_S) \). On every \( H_v \) we have Zariski topology induced by the Zariski topology on \( H \) and Hausdorff topology induced by the Hausdorff topology on \( K_v \). The formal product of the Zariski (resp., the Hausdorff) topologies on \( H_v, v \in S \), is the Zariski (respectively, the Hausdorff) topology on \( H \). In order to distinguish the two topologies, all topological notions connected with the first one will be used with the prefix "Zariski".

The algebraic groups in this paper are supposed to be linear. Every \( K \)-algebraic group \( H \) is a Zariski closed \( K \)-subgroup of \( \text{SL}_l \) for some \( l \in \mathbb{N}_+ \). The group \( \text{SL}_l \) itself is identified with \( \text{SL}_l(\Omega) \) where \( \Omega \) is a universal domain, i.e. \( \Omega \) is an algebraically closed field of infinite transcendental degree over \( \mathbb{Q} \) containing \( K \) and all \( K_v \). We denote by \( \text{GL}_1 \) the 1-dimensional \( \mathbb{Q} \)-split torus and by \( D_l \) the subgroup of diagonal matrices in \( \text{SL}_l \). So, \( D_l \) is isomorphic over \( \mathbb{Q} \) to \( \text{GL}_1^{l-1} \). Moreover, \( H(\mathcal{O}) = \text{SL}_l(\mathcal{O}) \cap H \). A subgroup \( \Delta \) of \( H \) is called \( S \)-arithmetic if \( \Delta \) and \( H(\mathcal{O}) \) are commensurable, that is, if \( \Delta \cap H(\mathcal{O}) \) has finite index in both \( \Delta \) and \( H(\mathcal{O}) \). Recall that if \( H \) is semisimple then \( \Delta \) is a lattice in \( H \), i.e. \( H/\Delta \) has finite Haar measure.
The Zariski connected component of the identity \( e \in H \) is denoted by \( H^\bullet \). In the case of a real Lie group \( L \) the connected component of the identity is denoted by \( L^\circ \).

If \( A \) and \( B \) are subgroups of an abstract group \( C \) then \( \mathcal{N}_A(B) \) (resp., \( \mathcal{Z}_A(B) \)) is the normalizer (resp., the centralizer) of \( B \) in \( A \).

2.3. \( K \)-roots. In this paper \( G \) is a connected, semisimple, \( K \)-isotropic algebraic group and \( T \) is a maximal \( K \)-split torus in \( G \). The imbedding of \( G \) in \( SL_l \) (see §2.2) is chosen in such a way that \( T = G \cap D_l = G^\circ \cap D_l \) and \( T(O) = G \cap D_l(O) \sim \mathbb{O}^{* \cdot \text{rank}_K G} \).

We denote by \( \Phi(\equiv \Phi(T,G)) \) the system of \( K \)-roots with respect to \( T \). Let \( \Phi^+ \) be a system of positive \( K \)-roots in \( \Phi \) and \( \Pi \) be the set of simple roots in \( \Phi^+ \). (We refer to [B, §21.1] for the standard definitions related to the \( K \)-roots.)

If \( \chi \in \Phi \) we let \( g_{\chi} \) be the corresponding root-space in \( \text{Lie}(G) \). For every \( \alpha \in \Pi \) we define a projection \( \pi_\alpha : \Phi \to \mathbb{Z} \) by \( \pi_\alpha(\chi) = n_{\alpha} \) where \( \chi = \sum_{\beta \in \Pi} n_{\beta} \beta \).

Let \( \Psi \subset \Pi \) and \( T_\Psi \overset{\text{def}}{=} (\bigcap_{\alpha \in \Psi} \ker(\alpha))^\bullet \). We denote by \( P_\Psi \) the (standard) parabolic subgroup corresponding to \( \Psi \) and by \( P^-_\Psi \) the opposite parabolic subgroup corresponding to \( \Psi \). The centralizer \( Z_G(T_\Psi) \) is a common Levi subgroup of \( P_\Psi \) and \( P^-_\Psi \), \( P_\Psi = Z_G(T_\Psi) \rtimes R_u(P_\Psi) \) and \( P^-_\Psi = Z_G(T_\Psi) \rtimes R_u(P^-_\Psi) \). We will often use the simpler notation \( V_\Psi \overset{\text{def}}{=} R_u(P_\Psi) \) and \( V^-_\Psi \overset{\text{def}}{=} R_u(P^-_\Psi) \). Recall that

\[
\text{Lie}(V_\Psi) = \bigoplus_{\exists \alpha \in \Pi \setminus \Psi, \pi_\alpha(\chi) > 0} g_{\chi},
\]

\[
\text{Lie}(V^-_\Psi) = \bigoplus_{\exists \alpha \in \Pi \setminus \Psi, \pi_\alpha(\chi) < 0} g_{\chi},
\]

and

\[
\text{Lie}(Z_G(T_\Psi)) = \text{Lie}(Z_G(T)) \oplus \bigoplus_{\forall \alpha \in \Pi \setminus \Psi, \pi_\alpha(\chi) = 0} g_{\chi}.
\]

It is well known that the map \( \Psi \mapsto P_\Psi \) is a bijection between the subsets of \( \Pi \) and the parabolic subgroups of \( G \) containing \( B \), cf. [B, §21.11]. Note that \( P_\emptyset, P^-_\emptyset \) are minimal parabolic subgroups and \( G = P_\Pi = P^-_\Pi \).

Given \( \alpha \in \Phi \) we let \( (\alpha) \) be the set of roots which are positive multiple of \( \alpha \). Then \( g_{(\alpha)} \overset{\text{def}}{=} \bigoplus_{\beta \in (\alpha)} g_{\beta} \) is the Lie algebra of a unipotent group denoted by \( U_{(\alpha)} \). Given \( \Psi \subset \Pi \), let \( \Psi' \) be the set of all non-divisible positive roots \( \chi \) such that \( \exists \alpha \in \Delta \setminus \Psi, \pi_\alpha(\chi) > 0 \). Then the product
morphism (in any order) $\prod_{\chi \in \Psi'} U_{\chi} \to V_\Psi$ is an isomorphism of $K$-varieties \cite{B 21.9].

It follows from the above definitions that $\Psi_1 \subset \Psi_2 \iff P_{\psi_1} \subset P_{\psi_2} \iff V_{\psi_1} \supset V_{\psi_2} \iff Z_G(T_{\psi_1}) \subset Z_G(T_{\psi_2})$. Let $V_{[\psi_2 \setminus \psi_1]} \defeq Z_G(T_{\psi_2}) \cap V_{\psi_1}$ and $V_{[\psi_2 \setminus \psi_1]}^{-1} \defeq Z_G(T_{\psi_2}) \cap V_{\psi_1}^{-1}$. It is easy to see that

$$V_{\psi_1} = V_{\psi_2} V_{[\psi_2 \setminus \psi_1]} = V_{[\psi_2 \setminus \psi_1]} V_{\psi_2}. \tag{7}$$

Recall that the Weyl group $\mathcal{W} \defeq N_G(T)/Z_G(T)$ acts by conjugation simply transitively on the set of all minimal parabolic $K$-subgroups of $G$ containing $T$. When this does not lead to confusion, we will identify the elements from $\mathcal{W}$ with their representatives from $N_G(T)$. It is easy to see that $\mathcal{W}_\Psi = N_{Z_G(T_\Psi)}(T)/Z_G(T)$ is the Weyl group of $Z_G(T_\Psi)$. Note that $\mathcal{W} = \mathcal{W}_0$.

We will denote by $\omega_0$ the element from $\mathcal{W}$ such that $\omega_0 P_0 \omega_0^{-1} = P_0^-$.

3. ON THE GROUP OF UNITS OF $O$

Recall that $S = \{v_1, \ldots, v_r\}$, $r \geq 2$, $K_i = K_{v_i}$ and $K_S = \prod K_i$. By the $S$-adic version of Dirichlet’s unit theorem, the $Z$-rank of $O^*_i$ is equal to $r - 1$.

Moreover, if $K_S^1 = \{(x_1, \ldots, x_r) \in K_S^* : |x_1|_1 \cdots |x_r|_r = 1\}$ then $O^*$ is a lattice of $K^*_S$.

For every $m \in \mathbb{N}_+$, we denote $O^*_m = \{\xi^m | \xi \in O^*\}$. The next proposition follows easily from the compactness of $K^*_S/O^*_m$.

**Proposition 3.1.** For every $m \in \mathbb{N}_+$ there exists a constant $\kappa_m > 1$ such that given $(a_i) \in K^*_S$ there exists $\xi \in O^*_m$ satisfying

$$\frac{1}{\kappa_m} \leq |\xi a_i|_i \leq \kappa_m$$

for all $1 \leq i \leq r$.

Let $S_\infty = \{v_1, \ldots, v_r\}$ and $S_f = \{v_{r+1}, \ldots, v_r\}$. So, $K_1 = \mathbb{R}$ or $\mathbb{C}$. In the next proposition $p : K^*_S \to K^*_1$ is the natural projection and $L \defeq p(O^*)$.

**Proposition 3.2.** With the above notation, we have:

1. If $r = 2$ then $L^o = \{1\}$;
2. Let $r \geq 3$. We have:
   a. $L^o \neq \{1\}$. In particular, $L^o = \mathbb{R}_+$ if $K_1 = \mathbb{R}$;
   b. Let $K_1 = \mathbb{C}$.
      i. If $L^o = \mathbb{R}_+$ then $K$ is a CM-field;
(ii) Suppose that $K$ is not a CM-field and $L \neq \mathbb{C}^*$. Then $L^\circ$ coincides with the unit circle group in $\mathbb{C}$ unless $r = 3$.

**Proof.** (1) follows easily from the compactness of $K^1_S/O^*_m$.

(2) If $r \geq 3$ in view of Dirichlet’s unit theorem $O^*$ contains a subgroup of $\mathbb{Z}$-rank 2. Therefore $p(O^*)$ is not discrete, proving that $L^\circ \neq \{1\}$.

Let $K_1 = \mathbb{C}$. Suppose that $L^\circ = \mathbb{R}_+$. Therefore $L$ is a finite extension of $L^\circ$. Hence there exists $m$ such that $p(O^m)$ is a dense subgroup of $L^\circ$. Let $F$ be the number field generated over $\mathbb{Q}$ by $O^*_m$. Then $F$ is proper subfield of $K$ and its unit group has the same $\mathbb{Z}$-rank as that of $K$, i.e. $K$ has a ”unit defect”. It is known that the fields with ”unit defect” are exactly the CM-fields (cf. [Re]).

It remains to consider the case when $K$ is not a CM-field, $L \neq \mathbb{C}^*$ and $r > 3$. Since $L^\circ$ is a 1-dimensional subgroup of $\mathbb{C}^*$ we need to prove that $L^\circ$ couldn’t be a spiral. This will be deduced from the following six exponentials theorem due to Siegel: if $x_1, x_2, x_3$ are three complex numbers linearly independent over $\mathbb{Q}$ and $y_1, y_2$ are two complex numbers linearly independent over $\mathbb{Q}$ then at least one of the six numbers $\{e^{x_i y_j} : 1 \leq i \leq 3, 1 \leq j \leq 2\}$ is transcendental.

Now, suppose by the contrary that $L^\circ$ is a spiral, that is, $L^\circ = \{e^{(a+ib)} : t \in \mathbb{R}\}$ for some $a$ and $b \in \mathbb{R}^*$. Since $r > 3$ there exist $\xi_1, \xi_2$ and $\xi_3 \in p(O^*)$ which are multiplicatively independent over the integers. We may suppose that $\xi_1 = e^{a+i}, \xi_2 = e^{u(a+ib)}$ and $\xi_3 = e^{v(a+ib)}$ where $u$ and $v \in \mathbb{R}^*$ and $i = \sqrt{-1}$. Remark that $\{1, u, v\}$ are linearly independent over $\mathbb{Q}$, $\{a+ib, ib\}$ are linearly independent over $\mathbb{Q}$, and the six numbers $\xi_1, \xi_2, \xi_3, \frac{\xi_1}{|\xi_1|}, \frac{\xi_2}{|\xi_2|}, \frac{\xi_3}{|\xi_3|}$ are all algebraic. This contradicts the six exponentials theorem.

\[ \square \]

If $K_1 = \mathbb{C}$ and $K$ is not a CM-field, it is not difficult to give examples when $L^\circ$ is the circle group and when $L = \mathbb{C}^*$.

**Examples.**

1) For every $n \geq 1$, let $f_n(x) = (x^2 - (\sqrt{n^2 + 1} + n)x + 1)(x^2 - (\sqrt{n^2 + 1} + n)x + 1)$. Then $f_n(x)$ is an irreducible polynomial in $\mathbb{Q}[X]$ with two real and two (conjugated) complex roots. Let $K_1 = \mathbb{Q}(\alpha_n)$ where $\alpha_n$ is one of the complex roots of $f_n(x)$. Then $L^\circ$ is the circle group of $\mathbb{C}^*$.

2) It is easy to see that if $K$ is a totally imaginary, Galois, non-CM-number field of degree $\geq 6$ then $L = \mathbb{C}^*$.

Finally, the following is quite plausible:

**Conjecture 2.** $L^\circ$ is never a spiral.
In response to a question of the author, Federico Pellarin observed that Conjecture 2 follows from the still open four exponentials conjecture. The use of the six exponentials theorem in our proof of Proposition 3.2 is inspired by Pellarin's argument.

4. ACCUMULATIONS POINTS FOR LOCALLY DIVERGENT ORBITS

As in the introduction $\Gamma$ is an $\mathcal{S}$-arithmetic subgroup of $G$, $T = T(K_S)$ and $T$ acts on $G/\Gamma$ by left translations. In the next lemma $T(\mathcal{O})$ is identified with $(\mathcal{O}^*)^\text{rank}_K G$ via the isomorphism from §2.3.

Lemma 4.1. Let $h \in G(K)$. The following assertions hold:

(a) There exists a positive integer $m$ such that $\xi \pi(h) = \pi(h)$ for all $\xi \in \mathcal{O}^*_m G$.
(b) If $h_i$ is a sequence in $G$ such that $\{\pi(h_i)\}$ converges to an element from $G/\Gamma$ then the sequence $\{\pi(h_i h)\}$ admits a converging to an element from $G/\Gamma$ subsequence.

The lemma is an easy consequence from the commensurability of $\Gamma$ and $h \Gamma h^{-1}$.

4.1. Main proposition. We need the following general

Proposition 4.2. Let $n \in N_G(T)$ and $\Psi \subset \Pi$. The following conditions are equivalent:

(i) $n \in w_0 \mathcal{W}_\Psi$;
(ii) $V_0 w_0 n P_\Psi$ is Zariski dense in $G$;
(iii) $w_0 n V_\Psi(w_0 n)^{-1} \subset V_0$.

Proof. The implications (i) ⇒ (ii) and (i) ⇒ (iii) are easy.

Let (ii) holds. Then $n^{-1} V_0 n P_\Psi$ is Zariski dense. Since $n^{-1} V_0 n$ and $P_\Psi$ are $T$-invariant

\[ \text{Lie}(n^{-1} V_0 n) = \text{Lie}(V_0^-) + \text{Lie}(n^{-1} V_0 n \cap P_\Psi). \]

Therefore

\[ n^{-1} V_0 n = V_0^- (n^{-1} V_0 n \cap P_\Psi). \]

Since $n^{-1} V_0 n$ is a product of root groups, if $n^{-1} V_0 n \cap V_\Psi \neq \{e\}$ then, in view of (8), $n^{-1} V_0 n$ contains two opposite root groups which is not possible. Therefore

\[ n^{-1} V_0 n \cap V_\Psi = \{e\}. \]

This implies that $n^{-1} V_0 n \cap P_\Psi = n^{-1} V_0 n \cap Z_G(T_\Psi)$. In view of (8) $n^{-1} V_0 n \cap Z_G(T_\Psi)$ is a maximal unipotent subgroup of $Z_G(T_\Psi)$. Let
$n' \in \mathcal{W}_\Psi$ be such that $n'(n^{-1}V_\Psi n \cap Z_G(T_\Psi))n'^{-1} \subset V^-_\Psi$. Since $n'$ normalizes $V^-_\Psi$, it follows from (ii) that
\[ n'n^{-1}V_\Psi nn'^{-1} = V^-_\Psi \]
which implies (i).

Suppose that (iii) holds. Then $(w_0 n)^{-1}P_\Psi w_0 n \supset V_\Psi$. Hence, $w_0 n \in P_\Psi$ (cf. [B, 14.22]). Therefore $w_0 n \in \mathcal{W}_\Psi$, proving (i).

Further on, $g = (g_1, g_2, \cdots, g_r) \in G$ where $g_i \in G_i$. If $g_i \in G(K)$ and $h = (h_1, \cdots, h_r) \in G$, writing $\pi(hg_i)$ we mean that $g_i$ is identified with its the diagonal imbedding in $G$, so that, $hg_i = (h_1g_i, \cdots, h_r g_i)$.

Our main proposition is the following:

**Proposition 4.3.** Let $\#S \geq 2$, $\text{rank}_K G = \text{rank}_{K_1} G = \text{rank}_{K_2} G$, $g_1$ and $g_2 \in G(K)$ and $\Psi$ be a proper subset of $\Pi$. Let $(s_n, t_n, e, \cdots, e) \in T$ be a sequence and $C > 1$ be a constant such that for all $n$ we have:

\[ |\alpha(s_n)|_1 > \frac{1}{C} \text{ for all } \alpha \in \Pi, \quad |\alpha(t_n)|_2 \to 0 \text{ for all } \alpha \in \Pi \setminus \Psi \quad \text{and} \quad \frac{1}{C} < |\alpha(t_n)|_2 < C \text{ for all } \alpha \in \Psi. \]

Then the sequence $(s_n, t_n, e, \cdots, e)\pi(g)$ is bounded in $G/\Gamma$ if and only if the following conditions are satisfied:

1. $g_1 g_2^{-1} \in V^-_\Psi P_\Psi$, and
2. there exists a constant $C' > 1$ such that $\frac{1}{C'} < |\alpha(s_n)|_1 \cdot |\alpha(t_n)|_2 < C'$ for all $\alpha \in \Delta$ and all $n$.

**Proof.** $\Leftarrow$ Suppose that (i) and (ii) hold. Then $g_1 = vpg_2$ where $v \in V^-_\Psi(K)$ and $p \in P_\Psi(K)$. It follows from (ii), Lemma 4.1 and Proposition 3.1 that there exists a sequence $d_n \in \text{Stab}_T\{\pi(pg_2)\}$ such that the sequence $(s_n d_n^{-1}, t_n d_n^{-1}, d_n^{-1}, \cdots, d_n^{-1})$ is bounded in $T$.

Now\[
(s_n, t_n, e, \cdots, e) \cdot (g_1, g_2, \cdots, g_r)\pi(e) = \]
\[
(s_n, t_n, e, \cdots, e) \cdot (vpg_2, p^{-1}pg_2, g_3 \cdots, g_r)\pi(e) = \]
\[
(s_n v s_n^{-1}, t_n p^{-1} t_n^{-1}, e, \cdots, e) \cdot (s_n, t_n, e, \cdots, e) \cdot \]
\[
(e, e, g_3(pg_2)^{-1}, \cdots, g_n(pg_2)^{-1})\pi(pg_2) = \]
\[
(s_n v s_n^{-1}, t_n p^{-1} t_n^{-1}, e, \cdots, e) \cdot (s_n d_n^{-1}, t_n d_n^{-1}, e, \cdots, e) \cdot \]
\[
(e, e, g_3(pg_2)^{-1} d_n^{-1}, \cdots, g_n(pg_2)^{-1} d_n^{-1})\pi(pg_2). \]

Note that $t_n p^{-1} t_n^{-1}$ is bounded in $G_2$. By (ii) $|\alpha(s_n)|_1 \to \infty$ for all $\alpha \in \Pi \setminus \Psi$. Therefore $s_n v s_n^{-1} \to e$ in $G_1$. Now using the choice of $d_n$ we conclude that $(s_n, t_n, e, \cdots, e)\pi(g)$ is bounded in $G/\Gamma$.

$\Rightarrow$ Let $(s_n, t_n, e, \cdots, e)\pi(g)$ be bounded. By Bruhat decomposition $g_1 g_2^{-1} = v^{-1}w_0 np$ where $v^{-1} \in V^-_\Psi$, $n \in N_G(T)$, $p \in P_\Psi$ and $w_0 V^-_\Psi w_0^{-1} = V$. Suppose that $V^-_\Psi w_0 n P_\Psi$ is not Zariski dense in $G$. In view of Proposition 4.2 there exists a root $\chi$ such that $U_\chi \subset V_\Psi$ and $\chi \circ
Int\((w_0\alpha)^{-1}\) is a negative root. Let \(d_n \in T \cap \text{Stab}_G\{pg_2\}\) be such that \(\{t_n d_n^{-1}\}\) is bounded in \(G_2\) and \(\{d_n\}\) is bounded in every \(G_i, i \geq 3\). Then \(|\chi((w_0\alpha)^{-1} s_n w_0^\alpha)|_1\) is bounded from above and \(|\chi(d_n^{-1})|_1 \to 0\). Hence

\[
|\chi((w_0\alpha)^{-1} s_n w_0^\alpha)\chi(d_n^{-1})|_1 \to 0.
\]

But

\[
(s_n, t_n, e, \cdots, e)\pi(g) = (s_n, t_n, e, \cdots, e) \cdot (v w_0^\alpha, p^{-1}, g_3 (pg_2)^{-1}, \cdots, g_r (pg_2)^{-1}) \pi(pg_2) = ((s_n v^{-1} s_n w_0^\alpha)((w_0\alpha)^{-1} s_n w_0^\alpha) d_n^{-1}, (t_n p^{-1} t_n^{-1}) t_n d_n^{-1}, g_3 (pg_2)^{-1} d_n^{-1}, \cdots, g_r (pg_2)^{-1} d_n^{-1}) \pi(pg_2).
\]

It follows from \([10]\) and from the choice of \(\{d_n\}\) that the sequence \(\{s_n v^{-1} s_n w_0^\alpha\}(w_0\alpha)^{-1} s_n w_0^\alpha d_n^{-1}\})\) is unbounded in \(G_1\) and the sequences \(\{t_n p^{-1} t_n^{-1}\} t_n d_n^{-1}\), \(\{g_3 (pg_2)^{-1} d_n^{-1}\}, \cdots, \{g_r (pg_2)^{-1} d_n^{-1}\}\) are bounded in \(G_2, G_3, \cdots, G_r\), respectively. Since \(T_1 \pi(pg_2)\) is divergent, we get that \((s_n, t_n, e, \cdots, e)\pi(g)\) is unbounded which is a contradiction. Therefore \(V_{\emptyset}^- w_0^\alpha P_{\Psi} = V_{\emptyset}^- P_{\Psi} = V_{\emptyset}^- P_\Psi\), proving \((i)\).

Let \(g_1 = v^- p g_2\), where \(v^- \in V_{\emptyset}^-\). Then

\[
(s_n, t_n, e, \cdots, e)\pi(g) = ((s_n v^{-1} s_n^{-1})(s_n d_n^{-1}), (t_n p^{-1} t_n^{-1}) t_n d_n^{-1}, g_3 (pg_2)^{-1} d_n^{-1}, \cdots, g_r (pg_2)^{-1} d_n^{-1}) \pi(pg_2).
\]

Using that \(s_n v^{-1} s_n^{-1}\) is bounded in \(G_1\), \(t_n p^{-1} t_n^{-1}\) and \(t_n d_n^{-1}\) are both bounded in \(G_2\) and the projections of \(d_n^{-1}\) into \(G_i, i \geq 3\), are all bounded, it follows from the assumptions that \((s_n, t_n, e, \cdots, e)\pi(g)\) is bounded in \(G/\Gamma\) and \(T_1 \pi(g)\) is divergent that \(s_n d_n^{-1}\) is bounded in \(G_1\). Hence there exists \(C_1 > 1\) such that \(\frac{1}{C_1} < |\alpha(s_n d_n^{-1})|_1 \cdot |\alpha(t_n d_n^{-1})|_2 < C_1\) for all \(\alpha \in \Pi\). By Artin’s product formula \(\prod_{\nu \in \mathcal{W}} |\alpha(d_n)|_\nu = 1\) where \(\mathcal{W}\) is the set of all normalized valuations of \(K\). This implies \((ii)\). □

The above proposition implies:

**Corollary 4.4.** Let \(s_n \in T_1\) and \(t_n \in T_2\) be such that for every \(\alpha \in \Pi\) each of the sequences \(|\alpha(s_n)|_1\) and \(|\alpha(t_n)|_2\) converges to an element from \(\mathbb{R} \cup \infty\). We suppose that \(g_1 \in G(K)\) and that \((s_n, t_n, e, \cdots, e)\pi(g)\) converges in \(G/\Gamma\). Then there exist \(\Psi \subset \Pi\) and \(\omega_1, \omega_2 \in \mathcal{W}\) with the following properties:

1. \(\omega_1^{-1} P_{\Psi} \omega_1(K_1) \times \omega_2^{-1} P_{\Psi} \omega_2(K_2) = \{ (x, y) \in G_1 \times G_2 : \text{Int}(s_n, t_n)(x, y) \text{ is bounded in } G_1 \times G_2 \}\),
2. \(g_1 g_2^{-1} \in \omega_1^{-1} V_{\Psi} P_{\Psi} \omega_2\).
(iii) If \(g_1g_2^{-1} = \omega_1^{-1}v_{\Psi}z_{\Psi}v_{\Psi}\omega_2\), where \(v_{\Psi} \in V_{\Psi}(K)\), \(z_{\Psi} \in Z_G(T_{\Psi})(K)\) and \(v_{\Psi} \in V_{\Psi}(K)\), then
\[
\lim_n (s_n, t_n, e, \cdots, e)\pi(g) = (d_1\omega_1^{-1}(v_{\Psi})^{-1}(d_0\omega_0^{-1}v_{\Psi}^{-1}g_2), d_0\omega_0^{-1}v_{\Psi}^{-1}g_2, e, \cdots, e)\pi(g),
\]
where \(d_1 \in T_1\) and \(d_2 \in T_2\).

**Proof.** There exists a parabolic \(K\)-subgroup \(P\) containing \(T\) such that \(P(K_1) = \{ x \in G(K_1) : \text{Int}(s_n)x \text{ is bounded}\}\). Let \(\Psi \subset \Pi\) and \(\omega_1 \in \mathcal{W}\) be such that \(\omega_1^{-1}P_1\omega_1 = P_1\). Similarly, we find \(\Psi' \subset \Pi\) and \(\omega_2 \in \mathcal{W}\) such that \(\omega_2\Psi\omega_2^{-1}(K_2) = \{ x \in G(K_2) : \text{Int}(t_n)x \text{ is bounded}\}\).

Put \((\tilde{s}_n, \tilde{t}_n, e, \cdots, e)\pi(\tilde{g})\) converges, \(P_{\Psi}(K_1) = \{ x \in G(K_1) : \text{Int}(s_n)x \text{ is bounded}\}\) and \(P_{\Psi'}(K_2) = \{ x \in G(K_2) : \text{Int}(t_n)x \text{ is bounded}\}\). So, there exists a constant \(C > 1\) such that \(C > |\alpha(\tilde{s}_n)|_1 > \frac{1}{C}\) for all \(\alpha \in \Psi\), \(|\alpha(\tilde{s}_n)|_1 \to \infty\) for all \(\Pi \setminus \Psi\), \(|\alpha(\tilde{t}_n)|_2 \to 0\) for all \(\Pi \setminus \Psi'\) and \(C > |\alpha(\tilde{t}_n)|_2 > \frac{1}{C}\) for all \(\alpha \in \Psi'\). Passing to a subsequence and replacing if necessary \(C\) by a larger constant we may suppose that for every \(\alpha \in \Phi\) either \(C > |\alpha(\tilde{t}_n)|_2 > \frac{1}{C}\) for all \(n\) or \(|\alpha(\tilde{t}_n)|_2\) is converging to 0 or \(\infty\). It follows from Proposition 4.3(ii) that \(\frac{1}{C} < |\alpha(\tilde{s}_n)|_1 \cdot |\alpha(\tilde{t}_n)|_2 < C\) for all \(\alpha \in \Pi\) and \(n\). This implies easily that \(\Psi = \Psi'\). In view of Proposition 4.3(i) \(g_1g_2^{-1} \in \omega_1 V_{\Psi} P_{\Psi} \omega_2^{-1}\). Hence (i) and (ii) hold.

Let \(\omega_1^{-1}g_1 = v_{\Psi}z_{\Psi}v_{\Psi}^{-1}g_2\) as in the formulation of the corollary. Then writing
\[
(\tilde{s}_n, \tilde{t}_n, e, \cdots, e)\pi(\tilde{g}) =(\tilde{s}_n, \tilde{t}_n, e, \cdots, e)(v_{\Psi}z_{\Psi}, v_{\Psi}, \cdots, g_r(v_{\Psi}^{-1}g_2)^{-1})\pi(v_{\Psi}^{-1}g_2),
\]
a similar calculation as in the proof of Proposition 4.3 shows that
\[
\lim (\tilde{s}_n, \tilde{t}_n, e, \cdots, e)\pi(\tilde{g}) = \pi(d_1'v_{\Psi}^{-1}g_2, d_2'v_{\Psi}^{-1}g_2, g_3, \cdots, g_r)
\]
where \((d_1', d_2') \in T_1 \times T_2\). This implies (iii). \(\square\)

5. **Locally divergent orbits for \#S = 2**

In this section \(g = (g_1, g_2)\) and \(T_\pi(g)\) is a locally divergent orbit. We use the notation \(P(g), P_{\Psi}(g)\) and \(\text{Orb}_g(P)\) as defined in the Introduction.

---

1Remark that \(P_{\Psi}\) and \(P_{\Psi}\) are not always conjugated and, therefore, in the definition of \(P_{\Psi}(g)\) we can not replace \(P_{\Psi}\) by \(P_{\Psi}\). Indeed, let \(G\) be a simple \(K\)-split algebraic group of type \(D_l, l \geq 4\), and \(\alpha \in \Pi\) be such that \(\omega_0(\alpha) \neq -\alpha\). Then choosing \(\Psi = \{\alpha\}\) it is easy to see that \(P_{\Psi}\) is not conjugated to \(P_{\Psi}\).
Lemma 5.1. Let $P = \omega_1 P_{\varphi}^{-1} \times \omega_2 P_{\varphi}^{-1} \in P(\varphi(g))$ and $g_1 g_2^{-1} = \omega_1 v_{\varphi}^z\varphi v_{\varphi}^{-1}$, where $v_{\varphi} \in V_{\varphi}$, $z_{\varphi} \in Z_G(T_{\varphi})$, $v_{\varphi} \in V_{\varphi}$, and $\omega_1$ and $\omega_2$ are $G(T)$. We have:

(a) $(\omega_1 v_{\varphi}^z\varphi)^{-1} g_1 \omega_2 v_{\varphi} \omega_2^{-1} g_2) \in \bigcap_{v \in S} Z_G(T_{\varphi}) G(K)$, that is, the orbit is well-defined and locally divergent;

(b) If $w \in W \times W$ then $w P w^{-1} \in P(wg)$ and

\[ w \text{Orb}_{g}(P) = \text{Orb}_{wg}(wPw^{-1}). \]

Proof. The assertion (a) of the lemma is invariant under multiplication of $(g_1, g_2)$ from the left by elements from $Z_G(T)$. In view of Theorem 1.1(a), this reduces the proof to the case when $g_1$ and $g_2 \in G(K)$. Since $N_G(T) = N_G(T)(K)Z_G(T)$, we have $\omega_i = \tilde{\omega}_i a_i$ where $\tilde{\omega}_i \in N_G(T)(K)$ and $a_i \in G(T)$. So,

\[ \tilde{\omega}_i^{-1} g_1 g_2^{-1} = a_1 v_{\varphi}^z\varphi v_{\varphi}^{-1} = \tilde{v}_{\varphi}^z\varphi \tilde{v}_{\varphi} \in G(K), \]

where $\tilde{v}_{\varphi} \in V_{\varphi}$, $z_{\varphi} \in Z_G(T_{\varphi})$, and $\tilde{v}_{\varphi} \in V_{\varphi}$. Since the product map $V_{\varphi}(K) \times Z_G(T_{\varphi})(K) \times V_{\varphi}(K) \rightarrow (V_{\varphi} Z_G(T_{\varphi})) V_{\varphi}(K)$ is bijective, we get that $\tilde{v}_{\varphi}, z_{\varphi},$ and $\tilde{v}_{\varphi}$ are $K$-rational. It remains to note that $\omega_1(v_{\varphi}^{-1})^{-1} \omega_1^{-1} = \tilde{\omega}_1(v_{\varphi}^{-1})^{-1} \omega_1^{-1}$ and $\omega_2 v_{\varphi} \omega_2^{-1} = \tilde{\omega}_2 \tilde{v}_{\varphi} \tilde{\omega}_2$.

The part (b) follows from the definition of the orbit $\text{Orb}_{g}(P)$ by a simple computation.

5.1. Proof of Theorem 1.3. In view of (11), it is enough to prove the theorem for $P = P_{\varphi}^{-1} \times P_{\varphi}$. In this case $g_1 g_2^{-1} = v_{\varphi} z_{\varphi} v_{\varphi}$, where $v_{\varphi} \in V_{\varphi}$, $z_{\varphi} \in Z_G(T_{\varphi})$, and $v_{\varphi} \in V_{\varphi}$, and

\[ \text{Orb}_{g}(P) = T(z_{\varphi}, e) \pi(v_{\varphi} g_2). \]

The orbit $Z_G(T_{\varphi}) \pi(v_{\varphi} g_2)$ is closed, it contains $\text{Orb}_{g}(P)$, and $Z_G(T_{\varphi})$ is a reductive $K$-algebraic group. Since $Z_G(T_{\varphi}) \pi(v_{\varphi} g_2)$ is homeomorphic to $Z_G(T_{\varphi})/\Delta$, where $\Delta$ is an $S$-arithmetic subgroup of $Z_G(T_{\varphi})$, the $T$-orbits on $Z_G(T_{\varphi}) \pi(v_{\varphi} g_2)$ contained in $\text{Orb}_{g}(P)$ are described by Corollary 1.4 applied to $Z_G(T_{\varphi})$. Let $Tm$ be such an orbit. There exists $\Psi' \subset \Psi$ such that up to a conjugation of $z_{\varphi}$ by an element from $W_{\varphi}$ we have $z_{\varphi} = v_{\varphi} z_{\varphi}' \pi(v_{\varphi} z_{\varphi}')$, where $v_{\varphi} z_{\varphi}' \in V_{\varphi}(\varphi')$, $z_{\varphi} \in Z_G(T_{\varphi}')$, and $v_{\varphi} z_{\varphi}' \in V_{\varphi}(\varphi')$ (see (7)), and

\[ Tm = T(z_{\varphi}', e) \pi(v_{\varphi} z_{\varphi}') v_{\varphi} g_2). \]

It is clear that $P_{\varphi} \times P_{\varphi} \in P(g)$ and

\[ Tm = \text{Orb}_{g}(P_{\varphi} \times P_{\varphi}), \]

proving the theorem.
5.1.1. Proof of Corollaries 1.5 and 1.6 By (11) and Theorem 1.1(a) we may (and will) suppose that $g_1$ and $g_2 \in \mathbf{G}(K)$ and $P = P^\psi_\emptyset \times P^\psi_\emptyset$. If $g_1g_2^{-1} = v_\psi z_\psi v_\psi$ where $v_\psi \in V^\psi_\emptyset$, $z_\psi \in Z^\psi_\mathbf{G}(T)$ and $v_\psi \in V_\psi$ then
\begin{equation}
\text{Orb}_g(P) = T(z_\psi, e)\pi(v_\psi g_2).
\end{equation}
If $P$ is minimal among the subgroups in $\mathcal{P}(g)$ then $\text{Orb}_g(P)$ is closed in view of Theorem 1.3. It follows from Theorem 1.1(b) that $z_\psi \in \mathcal{W}_\psi$. So,
\begin{equation}
g_1g_2^{-1} = v_\psi(z_\psi v_\psi z_\psi^{-1})z_\psi \in P^\psi_\emptyset P^\psi_\emptyset z_\psi
\end{equation}
and $P^\psi_{\emptyset} \times z_\psi^{-1}P^\psi_\emptyset z_\psi \in \mathcal{P}(g)$. Since $P$ is minimal in $\mathcal{P}(g)$ and $P^\psi_{\emptyset} \times z_\psi^{-1}P^\psi_\emptyset z_\psi \subset P^\psi_{\emptyset} \times P^\psi_{\emptyset}$ we get that $P$ is a minimal parabolic subgroup of $\mathbf{G} \times \mathbf{G}$. This complete the proof of Corollary 1.5.

Concerning the proof of Corollary 1.6 it is easy to see that $(a) \iff (b)$ in view of Theorem 1.3, $(a) \iff (c)$ in view of Theorem 1.1 and $(a) \iff (d)$ in view of Corollary 1.5.

5.1.2. Proof of Corollary 1.7. The part (a) of the corollary is a direct consequence from Theorem 1.3.

Let us prove (b). Let $\Omega_1 = \bigcap_{(\omega_1, \omega_2) \in \mathcal{W} \times \mathcal{W}} \omega_1^{-1}P^\psi_{\emptyset}P^\psi_\emptyset \omega_2$. Then $\Omega_1$ is $W$-invariant, Zariski open, non-empty and $\mathcal{P} = \mathcal{P}(g)$ if and only if $g_1g_2^{-1} \in \Omega_1$.

Let $P$ and $P' \in \mathcal{P}(g)$ and $g_1g_2^{-1} \in \Omega_1$. Suppose that the set of minimal parabolic subgroups of $P$ containing $T$ coincides with the set of minimal parabolic subgroups of $P'$ containing $T$. We may suppose that $P = P^\psi_\emptyset \times P^\psi_{\emptyset}$. Since $P^\psi_{\emptyset} \times P^\psi_\emptyset \subset P'$ we get that $P' = P^\psi_{\emptyset} \times P^\psi_{\emptyset}$ for some $\Psi' \subset \Psi$. The group $\mathcal{W}_\psi$ (respectively, $\mathcal{W}_\psi'$) acts simply transitively on the minimal parabolic subgroups of $P^\psi_{\emptyset}$ (respectively, $P^\psi_{\emptyset'}$) containing $T$. Therefore $\mathcal{W}_\psi = \mathcal{W}_\psi'$. But $P^\psi_{\emptyset} = P^\psi_{\emptyset} P^\psi_\emptyset$ and $P^\psi_{\emptyset'} = P^\psi_{\emptyset} W^\psi_{\emptyset} P^\psi_\emptyset$. Hence $P^\psi_{\emptyset} = P^\psi_{\emptyset'}$ and $P = P'$, i.e. each $P \in \mathcal{P}(g)$ is uniquely determined by its minimal parabolic subgroup. Therefore the map $\text{Orb}_g(\cdot)$ is injective if and only if its restriction to the set of minimal parabolic subgroups is injective.

Let $\Delta = g_2 \Gamma g_2^{-1}$. It is well known that the product map $V^\psi_{\emptyset} \times Z^\psi_\mathbf{G}(T) \times V^\psi_\emptyset \to \mathbf{G}$ is a $K$-rational isomorphism. Let $p : \mathbf{G} \to V^\psi_{\emptyset}$ be the natural projection. Choose a non-archimedean completion $F$ of $K$ different from $K_1$ and $K_2$. Since $p$ is $K$-rational the closure $p(\Delta)$ of $p(\Delta)$ in $V^\psi_{\emptyset}(F)$ (for the Hausdorff topology on $V^\psi_{\emptyset}(F)$ induced by the topology on $F$) is compact. Therefore there exists a non-empty, $\mathcal{W}$-invariant, open (for the Hausdorff topology on $\mathbf{G}(F)$) subset $\Omega_2 \subset \mathbf{G}(F)$ with the following properties: if $x \in \Omega_2$, $\omega \in \mathcal{W} \setminus \{e\}$ and
\[ \omega x \omega^{-1} = v^- z v, \] where \( v^- \in V^- \) for any \( \omega \in \mathcal{Z}_G(T)(F) \) and \( v \in V \). Then \( p(\omega^{-1} z x \omega) \notin p(\Delta) \).

Let \( \Omega = \Omega_1 \cap \Omega_2 \). It is clear that \( \Omega \) is non-empty and Zariski dense in \( G \). Let \( g_1 \in \Omega g_2 \). Then we get (3) where \( c \in \mathcal{V} \). Therefore \( t = 2 v \). This implies \( t_1 z v = v_1^{-1} z_1 v_2^{-1} \omega \delta \) and \( t_2 v = v_1^{-1} v_2 \omega \delta \).

This implies \( \omega_1^{-1} \omega_2 = (t_1 z t_2^{-1})(\omega_2^{-1} z_1 \omega_2) \in \mathcal{Z}_G(T) \).

Therefore \( \omega_1 = \omega_2 = \omega \). Using (13)

\[ (\omega^{-1} \omega)^{-1} t_2 v \in \Delta. \]

Finally, in view of the choice of \( \Omega_2 \), we get \( \omega = e \), i.e. \( P = P' \). \( \square \)

6. Locally divergent orbits for \( \#S > 2 \)

Further on we suppose that \( \text{rank}_K G = \text{rank}_{K_v} G \) for all \( v \in S \).

6.1. Horospherical subgroups. Let \( t \in T_v, v \in S \). We set

\[ W^+(t) = \{ x \in G_v : \lim_{n \to +\infty} t^{-n} x t^n = e \}, \]

\[ W^-(t) = \{ x \in G_v : \lim_{n \to +\infty} t^n x t^{-n} = e \} \]

and

\[ Z(t) = \{ x \in G_v : t^n x t^{-n}, n \in \mathbb{Z}, \text{ is bounded} \}. \]

Then \( W^+(t) \) (respectively, \( W^-(t) \)) is the positive (respectively, negative) horospherical subgroup of \( G_v \) corresponding to \( t \).

The following proposition is well known and easy to prove.

**Proposition 6.1.** With the above notation, there exist opposite parabolic \( K \)-subgroups \( P \) and \( P^- \) containing \( T \) such that \( W^+(t) = \mathcal{R}_u(P)(K_v) \), \( W^-(t) = \mathcal{R}_u(P^-)(K_v) \) and \( Z(t) = (P \cap P^-)(K_v) \).

**Lemma 6.2.** Let \( \Psi \subset \Pi, \sigma \in G(K) \) and \( 1 \leq s_1 < s_2 \leq r \) where \( \#S = r \). There exists a sequence \( s_n \in T(K) \cap \sigma F \sigma^{-1} \) with the following properties:

(a) if \( \alpha \in \Psi \) then \( \alpha(s_n) = 1 \) for all \( n \);
Proposition 6.3. Let

\( \sigma, g, s_1, s_2 \) be as in the formulation of Lemma 6.2. Passing to a subsequence we suppose that the projection of the sequence \( t_n \) in \( T_i \) is convergent for every \( i > s_2 \). Since \( t_n \pi(\sigma) = \pi(\sigma) \) and in view of Lemma 6.2, we get

\[
\lim_{n} t_n \pi(u_1, \ldots, u_{s_2}, g_{s_2+1}, \ldots, g_r) = (e, \ldots, e, h_{s_2+1}, \ldots, h_r) \pi(\sigma),
\]

where \( h_i = \lim_{n} t_n g_i \sigma^{-1} t_n^{-1} \), \( i > s_2 \). Using once again the convergence of \( t_n \) in every \( T_i \), \( i > s_2 \), we get

\[
\lim_{n} t_n^{-1}(e, \ldots, e, h_{s_2+1}, \ldots, h_r) \pi(\sigma) = \pi(\sigma, \ldots, g_{s_2+1}, \ldots, g_r).
\]

Lemma 6.4. Let \( \Psi \subset \Pi \) and \( g \in G(K) \).

(a) We have \( g = \omega z v_+ v_- \), where \( \omega \in \mathcal{N}_G(T)(K), z \in \mathcal{Z}_G(T)(K), v_+ \in V_\Psi^+(K) \) and \( v_- \in V_\Psi^-(K) \). Moreover,

\[
\mathcal{Z}_{T_\Psi}(g) = \mathcal{Z}_{T_\Psi}(v_-) \cap \mathcal{Z}_{T_\Psi}(v_+) \cap \mathcal{Z}_{T_\Psi}(\omega).
\]

(b) With \( g = \omega z v_+ v_- \) as in (a), suppose that \( \dim \mathcal{Z}_{T_\Psi}(g) \geq \dim \mathcal{Z}_{T_\Psi}(\theta g) \) for every \( \theta \in \mathcal{N}_G(T) \). Then

\[
\mathcal{Z}_{T_\Psi}(\omega)^* \supset \mathcal{Z}_{T_\Psi}(g)^* = (\mathcal{Z}_{T_\Psi}(v_-) \cap \mathcal{Z}_{T_\Psi}(v_+))^*.
\]

Proof. The part (a) follow freely from the existence and the uniqueness of the Bruhat decomposition for reductive \( K \)-algebraic groups (cf. [B] Theorem 21.15]). The part (b) follows immediately from (a). \( \square \)

Lemma 6.5. Let \( g \in G(K) \) be such that \( \dim \mathcal{Z}_{T}(g) \geq \dim \mathcal{Z}_{T}(\theta g) \) for all \( \theta \in \mathcal{N}_G(T) \). Let \( \Lambda \) be a subset of \( \Phi(T, G) \) and \( S = (\bigcap_{\alpha \in \Lambda} \ker \alpha)^* \).

There exist systems of simple roots \( \Pi \) and \( \Pi' \) in \( \Phi \) and subsets \( \Psi \subset \Pi \) and \( \Psi' \subset \Pi' \) with the following properties:

(a) \( S = T_\Psi = T_{\Psi'} \).
LOCALLY DIVERGENT ORBITS

(b) $g = \omega z v_+ v_- = \omega' z' v'_+ v'_-$, where $\omega$ and $\omega' \in N_G(T)(K)$, $z$ and $z' \in Z_G(S)(K)$, $v_+ \in V_\Psi(K)$, $v_- \in V_\Phi(K)$, and $v'_+ \in V_{\Psi'}(K)$, and

$Z_S(g)^* = Z_S(v_+)^* = Z_S(v'_+)^*$.

Proof. Fix $v \in S$. We choose $t \in S(K_v)$ such that $|\alpha(t)|_v \neq 1$ for every root $\alpha$ which is not a linear combination of roots from $\Lambda$. Applying Proposition 6.1 we associate to $t$ a system of simple roots $\Pi$ and a subset $\Psi$ of $\Pi$ such that $S = T_\Psi$, $W^+(t) = V_\Psi(K_v)$, $W^-(t) = V_\Phi(K_v)$ and $Z_G(t) = Z_G(T_\Psi)$. With these $\Pi$ and $\Psi$, let $g = \omega z v_+ v_-$ as given by Lemma 6.4. Now we suppose that $t$ is chosen in such a way that dim $Z_S(v_+)$ is minimal. In view of Lemma 6.4(b), it is enough to prove that $Z_S(v_+)^* \subset Z_S(v_-)^*$. Suppose by the contrary that $Z_S(v_+)^* \not\subset Z_S(v_-)^*$. Pick a $t' \in Z_S(v_+)^*$ such that the subgroup generated by $t'$ is Zariski dense in $Z_S(v_+)^*$ and for every $K$-root $\beta$ either $\beta(t') = 1$ or $|\beta(t)|_v \neq 1$. Then $v_- = w_+ w_0 w_- \in W^+(t')$, $w_\in W^-(t')$ and $w_0 \in Z_G(t')$. Since $Z_S(v_+)^* \not\subset Z_S(v_-)^*$ either $w_+ \neq e$ or $w_- \neq e$. Replacing $t'$ by $t'^{-1}$ if necessary, we may suppose that $w_+ \neq e$. Let $\bar{t} = tt^n$ where $n \in \mathbb{N}$. After choosing $n$ sufficiently large, we get that $|\alpha(\bar{t})|_v \neq 1$ for every root $\alpha$ which is not a linear combination of roots from $\Lambda$, $v_+ w_+ \in W^+(\bar{t})$ and $w_0 w_- \in W^-(\bar{t})$. But $Z_S(v_+ w_+) = Z_S(v_+) \cap Z_S(w_+)$. Since $w_+ \neq e$ we obtain that dim $Z_S(v_+ w_+) < \text{dim} Z_S(v_+)$ which contradicts the choice of $t$. Therefore $Z_S(v_+)^* \subset Z_S(v_-)^*$ proving the claim.

The existence of $\Pi'$ and $\Psi' \subset \Pi'$ as in the formulation of the lemma is proved by virtually the same argument. \qed

6.2. Definition of $H_1$ and reduction of the proof of Theorem 1.8. Let us define the subgroup $H_1$ of $G$ as in the formulation of Theorem 1.8. So, let $g = (g_1, \cdots, g_r) \in G$ be such that $T \pi(g)$ is a locally divergent orbit. Since $g_i \in Z_{G_i}(T_i)G(K)$ for all $i$ (Theorem 1.1) the proof is reduced to the case when every $g_i \in G(K)$.

Next choose $\omega_i \in N_G(T)(K), 1 \leq i \leq r - 1$, in such a way that $\dim \bigcap_{i=1}^{r-1} Z_T(\omega_i g_i g_i^{-1})$ is maximal. Let $H'_1 = (Z_G(\bigcap_{i=1}^{r-1} Z_T(\omega_i g_i g_i^{-1})))^*$. We put

$H_1 = g_r^{-1} H'_1 g_r$.

It is easy to see that

$T \pi(g) \subset h_1 H_1 \pi(e)$,

where $h_1 = (\omega_1^{-1} g_r, \cdots, \omega_{r-1}^{-1} g_r, g_r)$ and $H_1 = H_1(K_S)$. Note that $h_1 H_1 \pi(e)$ is closed and $T$-invariant.
Furthermore, we specify the choice of $\omega_i$ as follows. Since $H_i' = (\mathcal{Z}_{G}(\prod_{i=1}^{r-1} \mathcal{Z}_{T}(\omega_i g_i g_i^{-1})))^*$ for all $\omega_i' \in N_{H_i'}(T)$, we choose $\omega_i$ in such a way that $\dim \mathcal{Z}_{T}(\omega_i g_i g_i^{-1}) \geq \dim \mathcal{Z}_{T}(\omega' g_i g_i^{-1})$ for all $\omega' \in N_{H_i'}(T)$.

Note that $H_1$ is a reductive $K$-subgroup, $g_i^{-1} T g_i \subset H_1$, $g_i^{-1} \omega_i g_i \in H_1$ for all $i$, and

$$T \pi(g) = h_1(g_i^{-1} T g_i) \pi(g_i^{-1} \omega_i g_i, \cdots, g_i^{-1} \omega_r g_i, e).$$

Therefore replacing $G$ by the quotient of $H_1$ by its center and $T$ by the projection of $g_i^{-1} T g_i$ in this quotient, we reduce the proof of Theorem 1.8 to the following case:

(*) all $g_i \in G(K)$, $g_r = e$, $\bigcap \mathcal{Z}_{T}(\omega_i g_i)$ is finite for all choices of $\omega_i \in N_{G}(T)(K)$ and $\dim \mathcal{Z}_{T}(g_i) \geq \dim \mathcal{Z}_{T}(\omega_i g_i)$ for all $1 \leq i \leq r$ and all $\omega \in N_{G}(T)(K)$.

Assuming (*), it is enough to prove that there exists a semisimple subgroup $H$ of $G$ and $h \in G(K)$ such that $\text{rank}_K(G) = \text{rank}_K(H)$ and $\overline{T \pi(g)} \supset h H \pi(e)$,

where $H = H(K_S)$ and $h H \pi(e)$ is $T$-invariant.

6.3. Special elements in $\overline{T \pi(g)}$. In view of the reductions from §6.2, we will suppose up to the end of this and the next sections that the conditions of (*)& are fulfilled.

**Proposition 6.6.** For every $j$, $1 \leq j \leq r$, $\overline{T \pi(g)}$ contains an element of the form $\omega(e, \cdots, e, u_j, e, \cdots, e) \pi(h)$, where $h \in G(K)$, $\omega \in N_{G}(T)$, $u_j$ belongs to a unipotent subgroup of $G(K)$ normalized by $T(K)$ and $\mathcal{Z}_{T}(u_j)$ is finite.

**Proof.** First consider the case when there exists $i$ such that $\mathcal{Z}_{T}(g_i)$ is finite. Suppose for simplicity that $i = 1$. By Lemma 6.5 and Lemma 6.4 there exists a system of simple roots $\Pi$ such that every $g_i$ can be written in the form $g_i = z_i u_i^+ u_i^-$, where $u_i^+ \in V_0(K)$, $u_i^- \in V_0^\perp(K)$, and $z_i \in N_{G}(T)(K)$, and, moreover, $\mathcal{Z}_{T}(u_i^\pm)$ is finite. Shifting $g$ from the left by an appropriate element from $N_{G}(T)$ we may suppose that all $z_i = e$. Since

$$\pi(g) = (g_1(u_{r-1}^-)^{-1}, \cdots, g_{r-2}(u_{r-1}^-)^{-1}, u_{r-1}^+, (u_{r-1}^-)^{-1}) \pi(u_{r-1}^-),$$

applying Proposition 6.3 we get

$$(g_1(u_{r-1}^-)^{-1}, \cdots, g_{r-2}(u_{r-1}^-)^{-1}, e, e) \pi(u_{r-1}^-) \in \overline{T \pi(g)}.$$

Repeating the argument $r - 2$ times (or using induction on $r$) we prove that $\overline{T \pi(g)}$ contains an element $(u_1^+ v^-, e, \cdots, e) \pi(h)$, where $h \in G(K)$.
and \( v^- \in V_{\emptyset}(K) \). By Lemma 6.5 there exist opposite minimal parabolic \( K \)-subgroups \( \overline{P}_0^+ \) and \( \overline{P}_0^- \) containing \( T \) such that \( u_1^+ v^-=zw^-w^+ \), where \( z \in N_G(T)(K), \; w^- \in R_u(\overline{P}_\emptyset^-)(K), \; w^+ \in R_u(\overline{P}_\emptyset^+)(K) \) and \( Z_T(w^+) \) is finite. We may suppose that \( z = e \). Given \( 1 < j < r \), since 
\[
(w^-w^+, e, \cdots, e)\pi(h) = (w^-w^+, e, \cdots, (w^+)^{-1}w^+, e, \cdots, e)\pi(h),
\]
Proposition 6.3 implies that \( \overline{T}\pi(g) \) contains \( (w^+, e, \cdots, e)\pi(h) \) and, therefore, it contains \( (e, \cdots, w^+, \cdots, e)\pi(h) \) too. This completes the proof of the proposition when \( Z_T(g_i) \) is finite for some \( i \).

It easy to see that the proof of the proposition may be reduced to the particular case considered above if we prove that \( \overline{T}\pi(g) \) contains an element \( \pi(g'_1, g'_2, g'_3, \cdots, g'_r) \) such that \( g'_1 \) and \( g'_2 \in G(K) \), \( \dim Z_T(g'_1) \geq \dim Z_T(\omega g'_1) \) for all \( \omega \in N_G(T) \), and
\[
Z_T(g'_1)^* = (Z_T(g_1) \cap Z_T(g_2))^*.
\]
There is nothing to prove if \( Z_T(g_1)^* \subset Z_T(g_2) \). Suppose that \( Z_T(g_1)^* \notin Z_T(g_2) \). By Lemmas 6.4 and 6.5 there exist a system of simple roots \( \Pi \) and \( \Psi \subset \Pi \) such that \( T_\Psi = Z_T(g_1)^* \), \( g_2 = \omega z v_- v_+ \), where \( \omega \in N_G(T)(K), \; z \in Z_G(T_\Psi)(K), \; v_+ \in V_\Psi(K) \) and \( v_- \in V_\Psi(K) \), and
\[
Z_{T_\Psi}(g_2)^* = Z_{T_\Psi}(v_+)^*.
\]
Representing \( \pi(g) \) in the form \((g_1 v_+)^{-1}v_+, \omega z v_- v_+, \cdots, g_r)\pi(e) \), Proposition 6.3 implies that \( \overline{T}\pi(g) \) contains \((g_1 v_+, \omega z v_+, g_3, \cdots, g_r)\pi(e) \). It is clear that
\[
Z_T(g_1 v_+)^* = (Z_T(g_1) \cap Z_T(v_+))^* = (Z_T(g_1) \cap Z_{T_\Psi}(g_2))^* = (Z_T(g_1) \cap Z_T(g_2))^*,
\]
completing the proof.

We need the following specification of Proposition 6.6.

**Corollary 6.7.** With the notation and assumptions of Proposition 6.6, \( \overline{T}\pi(g) \) contains an element of the form \( \pi(uh, h, \cdots, h) \), where \( h \) and \( u \in G(K) \), \( u \) belongs to an abelian unipotent subgroup of \( G \) normalized by \( T \) and \( Z_T(u) \) is finite.

First we establish the following:

**Lemma 6.8.** Consider the \( \mathbb{Q} \)-vector space \( \mathbb{Q}^n \) endowed with the standard scalar product: \((x_1, \cdots, x_n), (y_1, \cdots, y_n) \) def\( = \sum x_i y_i \). Let \( v_1, \cdots, v_m \)
be pairwise non-proportional vectors in \( \mathbb{Q}^n \) and \( v \in \mathbb{Q}^n \) be such that
\[(v_i, v) > 0 \text{ for all } 1 \leq i \leq m. \] Put \( \mathcal{C} = \{ \sum_{i=1}^{m} \alpha_i v_i | a_i \in \mathbb{Q}, a_i \geq 0 \} \).

Suppose that \( m > n \) and the interior of \( \mathcal{C} \) with respect to the topology on \( \mathbb{Q}^n \) induced by \((., .)\) is not empty. Then there exist \( 1 \leq i_0 \leq m \) and \( w \in \mathbb{Q}^n \) such that \( (w, v_{i_0}) < 0 \), \( (w, v_i) > 0 \) if \( i \neq i_0 \) and \( \{v_i | i \neq i_0 \} \) contains a basis of \( \mathbb{Q}^n \).

**Proof.** Let \( \mathbb{Q}v_i \cap \mathcal{C}, 1 \leq i \leq m \), be the edges of the cone \( \mathcal{C} \). Then \( m_1 \geq n \) and \( v_1, \ldots, v_n \) is bases of \( \mathbb{Q}^n \). If \( m_1 = n \) and \( v_{n+1} = \sum_{i=1}^{n} c_i v_i \) one of \( c_i > 0 \). We suppose that \( c_1 > 0 \). Let

\[ \mathcal{C}' = \{ \sum_{i=2}^{m} a_i v_i | a_i \in \mathbb{Q}, a_i \geq 0 \}. \]

It is easy to see that in both cases \( m_1 > n \) and \( m_1 = n \) the interior of the cone \( \mathcal{C}' \) is nonempty and \( \mathcal{C}' \) contains all \( v_i \) but \( v_1 \). Therefore there exists \( w \in \mathbb{Q}^n \) such that \( (w, v_1) < 0 \) and \( (w, v_i) > 0 \) for all \( i > 1 \). \( \square \)

**Proof of Corollary 6.7.** Let \( V \) be minimal \( T \)-invariant unipotent \( K \)-subgroup of \( G \) containing \( u \). There exists a system of positive roots \( \Phi^+ \) such that the corresponding to \( \Phi^+ \) maximal unipotent \( K \)-subgroup contains \( V \). Let \( \Phi^+_{nd} \) be the set of non-divisible roots and \( \{\alpha_1, \ldots, \alpha_m\} = \{\alpha \in \Phi^+_{nd} : U_{(\alpha)} \cap V \neq \{e\}\} \) where \( U_{(\alpha)} \) is the corresponding to \( \alpha \) root group. Put \( V_{\alpha_i} = U_{(\alpha_i)} \cap V \). Then \( V \) is directly spanned by \( V_{\alpha_i} \) taken in any order (cf. [B 21.9]). It follows from the minimality assumption in the definition of \( V \) and the fact that every \( \{\alpha_i\} = \{\alpha_i\} \) or \( \{\alpha_i, 2\alpha_i\} \) that all \( V_{\alpha_i} \) are abelian.

We will complete the proof by induction on \( \dim V \). There is nothing to prove if \( V \) is abelian. Suppose that the derived subgroup \( \mathfrak{D}(V) \) of \( V \) is not trivial. Let \( u = u_1 \cdots u_m \) where \( u_i \in V_i(K) \). There exists \( 1 \leq l \leq m \) such that after a rearrangement of \( \{\alpha_1, \ldots, \alpha_m\} \) we have \( u_i \notin \mathfrak{D}(V) \) if and only if \( 1 \leq i \leq l \). Then every \( \alpha_j, j > l \), is a linear combination with positive coefficients of at least two roots in \( \{\alpha_1, \ldots, \alpha_l\} \). Since \( \mathcal{Z}_T(u) \) is finite and \( V \) is not abelian, \( \{\alpha_1, \ldots, \alpha_l\} \) contains a basis of the \( \mathbb{Q} \)-vector space \( X(T) \otimes \mathbb{Q} \) and \( m > l \). Put

\[ \mathcal{C} = \{ \sum_{i=1}^{m} a_i \alpha_i | a_i \in \mathbb{Q}, a_i \geq 0 \}. \]

By Lemma 6.8 there exist \( 1 \leq i \leq l \), say \( i = 1 \), and \( t \in T(K) \) such that \( \lim_{n \to \pm \infty} t^n u_1 t^{-n} = e \) in \( G_1 \) and \( \lim_{n \to \pm \infty} t^n u_1 t^{-n} = e \) in \( G_1 \) for all \( i > 1 \). Put \( u' = u_2 \cdots u_m \). It follows from Proposition 6.3 that \( \overline{T\pi(g)} \) contains \( \pi(u'h, h, \ldots, h) \). Since \( \mathcal{Z}_T(u') \) is finite and \( u' \) is contained in a proper \( T \)-invariant \( K \)-subgroup of \( V \) the corollary is proved. \( \square \)
6.4. Unipotent orbits on $T\pi(g)$. Further on some propositions formulated in $S$-adic setting will be deduced from their archimedean analogs when $S = S_\infty$. For this purpose the following lemma is needed.

**Lemma 6.9.** Let $V$ be a unipotent $K$-algebraic group and $U$ be its $K$-subgroup. Put $U = U(K_S)$ and $U_\infty = U(K_\infty)$. Let $M$ be a subset of $U_\infty$ such that $M V(\mathcal{O}_\infty) = U_\infty V(\mathcal{O}_\infty)$. Then

$$M V(\mathcal{O}) = U V(\mathcal{O}).$$

**Proof.** By the strong approximation for unipotent groups (see, for example, [PR, §7.1, Corollary]), we have that $U = U_\infty U(\mathcal{O})$. Using that $U(\mathcal{O}) \subset V(\mathcal{O})$, $V(\mathcal{O}_\infty) \subset V(\mathcal{O})$ and $U V(\mathcal{O})$ is closed, we get

$$U V(\mathcal{O}) = U_\infty V(\mathcal{O}) = U_\infty V(\mathcal{O}_\infty) V(\mathcal{O}) = M V(\mathcal{O}_\infty) V(\mathcal{O}) = M V(\mathcal{O}).$$

$\square$

**Proposition 6.10.** We suppose that $r > 2$, $K$ is not a CM-field and the completion $K_1$ is archimedean. Let $V$ be an abelian unipotent $K$-subgroup of $G$ normalized by $T$. Let $u \in V(K)$ and $Z_T(u)$ be finite. Then there exists a $K$-subgroup $U$ of $V$ which is $T$-invariant, contains $u$, and

$$U \pi(e) = \{(t u t^{-1}, e, \ldots, e) \pi(e) : t \in T(\mathcal{O})\},$$

where $U = U(K_S)$

As in §3, we denote by $L$ the closure of the projection of $\mathcal{O}_\infty^*$ in $K_1^*$. There are two possibilities: either $L = K_1^*$ or $L \neq K_1^*$. Since $K$ is not a CM-field, Proposition 3.2 implies that in the latter case $K_1 = \mathbb{C}$, dim $L = 1$ and $L \neq \mathbb{R}_+$.  

**Proof of Proposition 6.10** when $L = K_1^*$. Since $V$ is normalized by $T$ there exists an order $\Phi^+$ of the set of $K$-roots with respect to $T$ such that $V \subset V_{\Phi}$. Therefore, identifying $T(K_1)$ with $(K_1^*)^{\dim T}$, the map $T(K_1) \to V(K_1), t \mapsto tut^{-1}$, coincides with the restriction to $(K_1^*)^{\dim T}$ of a polynomial map $K_1^{\dim T} \to V(K_1)$. Let $\pi_\infty : V_\infty \mapsto V_\infty / V(\mathcal{O}_\infty)$, where $V_\infty = V(K_\infty)$, be the natural projection. By the polynomial measure rigidity for tori (cf. [Weyl] or [Sh, Corollary 1.2] for a more general result) there exists a $T$-invariant $K$-subgroup $U$ of $V$ such that

$$U_\infty \pi_\infty(e) = \{(t u t^{-1}, e, \ldots, e) \pi_\infty(e) : t \in T(K_1)\},$$

where $U_\infty = U(K_\infty)$. Now (14) follows from Lemma 6.9. $\square$

**Proof of Proposition 6.10** when $K_1 = \mathbb{C}$ and dim $L = 1$. Up to a subgroup of finite index there are two possibilities for $L$: there exists
\( \alpha \in \mathbb{R}^* \) such that either (a) \( L \) is a direct product of the unit circle group \( S^1 \) and an infinite cyclic group, i.e. \( L = \{ e^{2\pi i n t} : n \in \mathbb{Z}, 0 \leq t < 2\pi \} \), where \( i^2 = -1 \), or (b) \( L \) is a spiral, i.e. \( L = \{ e^{(\alpha+i)(t+2\pi n)} : n \in \mathbb{Z}, 0 \leq t < 2\pi \} \). The case \( \alpha < 0 \) being analogous to the case \( \alpha > 0 \), further on we will suppose that \( \alpha > 0 \). In order to treat the cases (a) and (b) simultaneously, note that \( L = \{ e^{2\pi i n t} : n \in \mathbb{Z}, 0 \leq t < 2\pi \} \) where \( \tilde{\alpha} = 0 \) in case (a) and \( \tilde{\alpha} = \alpha \) in case (b). (We use the equality \( e^{2\pi i n t} = e^{(\alpha+i)(t+2\pi n)} \)).

Given \( \theta \in [0, 2\pi) \) and \( b < c \), we denote \( [b,c]_{\theta} = \{ re^{i\varphi} : a < r < b \} \) and \( \mathbb{R}_{\theta} = \{ re^{i\theta} : r \in \mathbb{R} \} \). As usual, GL\(_1\) stands for the 1-dimensional \( \mathbb{Q}\)-split torus.

Returning to the proof of Proposition 6.10, note that \( V \) is a \( K\)-vector space and \( V = \bigoplus_i V_{\lambda_i} \) where \( V_{\lambda_i} \) are different weight spaces for the action of \( T \) on \( V \). Since \( u \in V(K) \) we have \( u = \sum_i u_i \) where \( u_i \in V_{\lambda_i}(K) \). Next using that the projection of \( \mathcal{O} \) into \( \prod_{v \in S \setminus \{ v_1 \} } K_v \) is dense, we get that \( U \) as in the formulation of the proposition is spanned by the vectors \( u_i \). This allows to replace \( T \) by its one dimensional sub-torus \( T' \) such that the restrictions of \( \lambda_i \) to \( T' \) are pairwise different. Therefore Proposition 6.10 follows immediately from the following:

**Lemma 6.11.** Let GL\(_1\) act \( K\)-rationally on a finite dimensional \( K\)-vector space \( V \) and \( V = \bigoplus_i V_{\lambda_i} \) be the decomposition of \( V \) as a sum of one-dimensional weight sub-spaces with weights \( \lambda_i(t) = t^{n_i} \). Suppose that \( r > 2 \), \( K \) is not a CM-field, and \( n_i \) are pairwise different positive integers. Let \( u = \sum_i u_i \) where \( u_i \in V_{\lambda_i}(K) \setminus \{ 0 \} \) for all \( i \). Then for every real \( C > 1 \), we have

\[
V = \{(\sum_i \lambda_i(a) u_i, 0, \cdots, 0) + V(\mathcal{O}) : a \in L, |a|_1 \geq C\},
\]

where \( V = V(K_S) \).

**Proof.** We will (as we may) suppose that \( V(K) = K^t \) and \( u_i = (0, \cdots, 1_i, \cdots, 0) \) for all \( i \). Since the projection of \( K_1 \) into \( K_S/\mathcal{O} \) is dense, it follows from [Sh. Corollary 1.2] (or [Wey]) that for every \( C > 1 \)

\[
K^t_S = \{(\sum_i \lambda_i(a), 0, \cdots, 0) + \mathcal{O}^t : a \in K_1, |a|_1 \geq C\}.
\]
In view of Lemma 6.9, we need to prove that

\[ V_\infty = \{(\sum_i \lambda_i(a)u_i, 0, \cdots, 0) + V(O_\infty) : a \in L, |a|_1 \geq C\}, \]

where \( V_\infty = V(K_\infty) \).

Let \( 0 < n_1 < n_2 < \cdots < n_t \). For every \( i \) we introduce a parametric curve \( f_i : [0, 2\pi) \to \mathbb{C}_1, t \mapsto e^{(\alpha+i)n_it} \). Since \( O_\infty \) is a group of finite type and \( \mathbb{R}_0 + O_\infty, 0 \leq \theta < 2\pi \), is a subspace of the real vector space \( K_\infty \), the set of all \( 0 \leq \theta < 2\pi \) such that \( \mathbb{R}_0 + O_\infty \subset K_\infty \) is countable.

The tangent line at \( t \) of the curve \( f_i(t) \) runs over all directions when \( 0 \leq t < 2\pi \). Therefore, there exist \( 0 < \psi < 2\pi \) such that if the tangent line at \( \psi \) of the parametric curve \( f_i \) is parallel to \( \mathbb{R}_0, 0 \leq \theta < 2\pi \), then \( \mathbb{R}_0 + O_\infty = K_\infty \) for all \( 1 \leq i \leq n \).

For every \( n \in \mathbb{N}_+ \), let

\[ F_n : [0, 2\pi) \to K^{l_\infty}_\infty, t \mapsto ((e^{2\pi n_1 \alpha} f_1(t), \cdots, 0), \cdots, (e^{2\pi n_t \alpha} f_t(t), \cdots, 0)) \]

A subset \( M \) of \( K^{l_\infty}_\infty/O^{l_\infty}_\infty \) will be called \( \varepsilon \)-dense if the \( \varepsilon \)-neighborhood of any point in \( K^{l_\infty}_\infty/O^{l_\infty}_\infty \) contains an element from \( M \). (As usual, \( K^{l_\infty}_\infty/O^{l_\infty}_\infty \) is endowed with a metric induced by the standard metrics on \( K_\infty \) considered as a real vector space.)

Now the lemma follows from the next

**Claim.** With \( \psi \) and \( F_n \) as above, let \( \varepsilon > 0 \). There exist reals \( A_\varepsilon > 0 \) and \( b_\varepsilon > 0 \) such that if \( \psi - b_\varepsilon \leq c < d \leq \psi + b_\varepsilon \) and \( e^{2\pi n_1 (d - c)} > A_\varepsilon \) for some \( c \) and \( d \in \mathbb{R} \) and \( n \in \mathbb{N}_+ \), then

\[ \{F_n(t) + O^{l_\infty}_\infty | c \leq t \leq d\} \]

is \( \varepsilon \)-dense in \( K^{l_\infty}_\infty/O^{l_\infty}_\infty \).

We will prove the claim by induction on \( l \). Let \( l = 1 \), i.e. \( F_n : [0, 2\pi) \to K_\infty, t \mapsto (e^{2\pi n_1 \alpha} f_1(t), 0, \cdots, 0) \), where \( f_1(t) = e^{(\alpha+i)n_1t} \). It follows from the choice of \( \psi \) that there exists a real \( B_\varepsilon > 0 \) such that the projection of \([0, B_\varepsilon]_{\theta_1}\) into \( K_\infty/O_\infty \) is \( \frac{\varepsilon}{2} \)-dense. Hence every shift of \([0, B_\varepsilon]_{\theta_1} + O_\infty \) by an element from \( K_\infty \) is \( \frac{\varepsilon}{2} \)-dense in \( K_\infty/O_\infty \) too. Choosing \( A_\varepsilon \) sufficiently large and \( b_\varepsilon \) sufficiently small we get that if \( n \) is such that \( e^{2\pi n_1 (2b_\varepsilon)} > A_\varepsilon \) then the length of the curve \( \{F_n(t) | \psi - b_\varepsilon \leq t \leq \psi + b_\varepsilon \} \) is greater than \( B_\varepsilon \) and if \( I \) is any connected piece of this curve of length \( B_\varepsilon \) then \( I \) is \( \frac{\varepsilon}{2} \)-close (with respect to the Hausdorff metrics on \( \mathbb{C} \)) to a shift of \([0, B_\varepsilon]_{\theta_1}\). This implies the claim for \( l = 1 \).

Now suppose that \( l > 1 \) and the claim is valid for \( l - 1 \). Let

\[ \tilde{F}_n(t) = ((e^{2\pi n_1 \alpha} f_1(t), \cdots, 0), \cdots, (e^{2\pi n_{l-1} \alpha} f_{l-1}(t), \cdots, 0)) \].
It follows from the induction hypothesis for \( l - 1 \) and from the validity of the Claim for \( l = 1 \) that for every \( \varepsilon > 0 \) there exist positive reals \( A_\varepsilon \) and \( b_\varepsilon \) such that if \( \psi - b_\varepsilon \leq c < d \leq \psi + b_\varepsilon \) and \( e^{2\pi n_1}(d - c) \geq A_\varepsilon \) for some \( n \in \mathbb{N}_+ \) then \( \{ \tilde{F}_n(t) + \mathcal{O}_l^{-1} | c \leq t \leq d \} \) is \( \varepsilon \)-dense in \( K_l^{-1}/\mathcal{O}_l^{-1} \) and

\[
\{(e^{2\pi n_1}\alpha f_i(t), 0 \cdots, 0) + \mathcal{O}_\infty | c \leq t \leq d' \}
\]
is \( \varepsilon \)-dense in \( K_\infty/\mathcal{O}_\infty \) whenever \( c < c' < d' < d \) and \( e^{2\pi n_1}(d' - c') \geq A_\varepsilon \).

Further on, given \( c_* < d_* \), we define the length of the parametric curve \( \{ \tilde{F}_n(t)|c_* \leq t \leq d_* \} \subseteq K_l^{-1} \) as the maximum of the lengths of the curves \( \{e^{2\pi \alpha f_i(t)}|c_* \leq t \leq d_* \} \subseteq \mathbb{C}, 1 \leq i \leq l - 1 \). With \( b_\varepsilon, A_\varepsilon \) and \( n \) as above, let \( x \in K_l^{-1}/\mathcal{O}_l^{-1} \). There exist \( c_{x,n} \) and \( d_{x,n} \) such that \( c < c_{x,n} < d_{x,n} < d \) and \( \{ F_n(t) + \mathcal{O}_l^{-1} | c_{x,n} \leq t \leq d_{x,n} \} \) is of length \( \frac{\varepsilon}{2} \) and contained in an \( \varepsilon \)-neighborhood of \( x \). In view of the definition of \( \tilde{F}_n \), there exists \( \delta \) not depending on \( x \) and \( n \) such that \( e^{2\pi n_1}(d_{x,n} - c_{x,n}) \geq \delta \). Now, since \( n_t > n_{l-1} \), choosing \( n \) large enough we get that \( e^{2\pi n_1}(d_{x,n} - c_{x,n}) \geq A_\varepsilon \) completing the proof of the claim. \( \square \)

6.5. **A refinement of Jacobson-Morozov lemma.** We will need the following known lemma (cf. [E-L, Lemma 3.1]):

**Lemma 6.12.** Let \( L \) be a semisimple group over a field \( F \) of characteristic 0, \( T \) be a maximal \( F \)-split torus in \( L \), \( \alpha \) be an indivisible root with respect to \( T \) and \( U_\alpha \) be the corresponding to \( \alpha \) root group. Denote

\[
U = \{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in F \}, \quad D = \{ \begin{pmatrix} 0 & 0 \\ b & b^{-1} \end{pmatrix} : b \in F^* \}.
\]

Let \( u \in U_\alpha(F) \). Suppose that \( u = \exp(v) \) where \( v \) belongs to the root space \( g_\alpha \) or \( g_{2\alpha} \). Then there exists an \( F \)-morphism \( f : \text{SL}_2 \rightarrow L \) such that \( u \in f(U) \) and \( f(D) \subset T(F) \).

6.6. **Actions of epimorphic subgroups on homogeneous spaces in \( S \)-adic setting.** In this section \( G \) is any \( K \)-isotropic semisimple \( K \)-group and \( S \) is \( \subset \neq \infty \). We have \( G = G_\infty \times G_f \) where \( G_\infty = \prod_{v \in S_\infty} G_v \) and \( G_f = \prod_{v \in S_f} G_v \). Let \( H \) be a closed subgroup of \( G_\infty \) which have finite index in its Zariski closure. Recall that a subgroup \( B \) of \( H \) is called **epimorphic** if all \( B \)-fixed vectors are \( H \)-fixed for every rational linear representation of \( H \). For example, the parabolic subgroups in \( H \) are epimorphic.

The following proposition is deduced from [Sh-W, Theorem 1].
Proposition 6.13. Let $H$ be a subgroup of $G_\infty$ generated by 1-parameter unipotent subgroups and $B$ be an epimorphic subgroup of $H$. Then any closed $B$-invariant subset of $G/\Gamma$ is $H$-invariant.

Proof. We need to prove that $B\pi(g)$ is $H$-invariant for every $g \in G$. Since $g^{-1}Bg$ is an epimorphic subgroup of $g^{-1}Hg$ it is enough to prove that $B\pi(e)$ is $H$-invariant, that is, $B\pi(e) = H\pi(e)$. Let $G_{f,n}$ be a decreasing sequence of compact subgroups of $G_f$ such that $\bigcap_n G_{f,n} = \{e\}$. Let $G_n = G_\infty \times G_{f,n}$ and $\Gamma_n = \Gamma \cap G_n$. Let $\phi_n : G \to G_\infty$ be the natural projection and $\Gamma_{n,\infty} = \phi_n(\Gamma_n)$.

In view of [Sh-W Theorem 1]

$$BT_{n,\infty} = H\Gamma_{n,\infty}.$$  

By the topological rigidity for unipotent groups [Ra1], for every $n$ there exists a connected subgroup $L_n$ of $G_\infty$ which contains $H$ and

$$HT_{n,\infty} = L_n\Gamma_{n,\infty}.$$  

But $L_n \cap \Gamma_{n,\infty}$ is a lattice in $L_n$ and $\Gamma_n$ has finite index in $\Gamma_{n+1}$. It follows from the connectedness of $L_n$ that all $L_n$ coincide, i.e. $L_n = L$. So, $\phi_n(BT_n) = BT_{n,\infty} = L\Gamma_{n,\infty}$. Hence for every $x \in L$ there exists $a_n \in G_{f,n}$ such that $xa_n \in BT_n$. Since $xa_n$ converges to $x$ in $G$, we get that $x \in BT$, i.e. $L \subset BT$. In view of the inclusions $B \subset H \subset L$ it is obvious that

$$BT = HT,$$

proving the proposition. \qed

6.7. Proof of Theorem 1.8. We keep the assumptions from §6.2. Shifting $\pi(g)$ from the left by an appropriate element from $N_G(T)$ we may suppose, in view of Propositions 6.6 and 6.10, that there exists a unipotent subgroup $U$ defined over $K$ and normalized by $T$ such that $Z_T(U)$ is finite and $\overline{T\pi(g)} \supset U\pi(h)$ where $h \in G(K)$ and $U = U(K_S)$. Let $H$ be a Zariski connected $K$-subgroup of $G$ with the properties: $U \cup T \subset H$ and $H\pi(h) \subset \overline{T\pi(e)}$ where $H = H(K_S)$. We choose $H$ to be maximal with the above properties. Let $u \in (U(\alpha) \cap H)(K)$, $u \neq e$, and $u = \exp(v)$ where $\alpha \in \Phi_{nd}^+$ and $v \in g_\alpha \cup g_{2\alpha}$. By Lemma 6.12 there exists a $K$-morphism $f : SL_2 \to G$ such that if $B$ is the group of upper triangular matrices in $SL_2(K)$ then $u \in f(B)$ and $f(B) \subset H$. Using Proposition 6.13 we conclude that $H\pi(h)$ is invariant under the action of the subgroup spanned by $f(SL_2(K_S))$ and $H$. In view of the maximality in the choice of $H$ we get that $f(SL_2) \subset H$. Therefore $H$ is a reductive subgroup of maximal $K$-rank. Since $U \subset H$ and $Z_T(U)$ is finite, $H$ is semisimple. \qed
6.8. Proof of Corollary 1.9. We suppose that \( G = SL_{n+1}, n \geq 1 \).
As usual, \( SL_{n+1} = SL(W) \) where \( W \) is the standard \( K \)-vector space
with \( W(K) = K^{n+1} \) and \( W(\mathcal{O}) = \mathcal{O}^{n+1} \).

Corollary 1.9 follows immediately from the next proposition.

**Proposition 6.14.** Let \( H \) be a proper Zariski connected reductive \( K \)-
subgroup of \( SL_{n+1} \) of \( K \)-rank \( n \). Then \( H \) is not semisimple and, moreover,
the following holds:

(a) there exists a direct sum decomposition \( W = W_1 \oplus \cdots \oplus W_r \),
where \( r \geq 2 \) and \( W_i \) are non-zero \( K \)-subspaces of \( W \), such that
\( H = \{ g \in SL_{n+1} : gW_i = W_i \text{ for all } i \} \);

(b) if \( H' \) is a reductive \( K \)-subgroup of \( SL_{n+1} \) such that \( H' \supset H \) and
\( s.s.rank_K(H) = s.s.rank_K(H') \) then \( H = H' \).

**Proof.** By Borel-De Siebenthal theory [BD] every maximal connected
subgroup of \( SL_{n+1} \) containing \( H \) has a one dimensional center.
Therefore the (Zariski) connected component \( Z \) of the center of \( H \) is
not trivial. Let \( W = W_1 \oplus \cdots \oplus W_r \), \( r \geq 2 \), where \( W_i \) are the weight
subspaces for the action of \( Z \) on \( W \). Each \( W_i \) is \( H \)-invariant and \( H \) is
an almost direct product of \( Z \) and \( SL(W_1) \cap H \times \cdots \times SL(W_r) \cap H \).
Since \( H \) has maximal \( K \)-rank we get that the \( K \)-rank of \( SL(W_i) \cap H \) is
equal to \( \dim W_i - 1 \) for every \( i \). Therefore \( SL(W_i) \subset H \). This implies
that \( H = \{ g \in SL_{n+1} : gW_i = W_i \text{ for all } i \} \).

In order to prove (b) note that \( H = \mathcal{G}(Z) \) and \( 2(H) = SL(W_1) \times \cdots \times SL(W_r) \). Hence if \( H' \) is a reductive \( K \)-subgroup of \( SL_{n+1} \) con-
taining \( H \) and \( s.s.rank_K(H) = s.s.rank_K(H') \) then \( H = H' \).

\[ \square \]

7. Non-homogeneous \( T \)-orbits closures when \( r > 2 \)

The theorem in this section provide examples showing that, in general,
the groups \( H_1 \) and \( H_2 \) as in the formulation of Theorem 1.8 are
different.

In the next theorem \( K \) is a totally real number field, \( \mathcal{S} = \infty \) and
\( r > 2 \) (equivalently, the degree of \( K \) is \( > 3 \) ). Let \( \Delta \) be a quaternion
skew field over \( K \) such that \( \Delta \otimes_K K_v \) is a skew field for every \( v \in \infty \).
In the sequel \( \Lambda \) is a fixed \( \mathcal{O} \)-order in \( \Delta \), that is, \( \Lambda \) is a sub-ring of
\( \Delta \) which is a finitely generated \( \mathcal{O} \)-module. There exists a unique \( K \)-
algebraic group \( G \) such that \( G(K) = SL_2(\Delta), G = \prod_{v \in \infty} SL_2(\Delta \otimes_K K_v) \)
and \( \Gamma = SL_2(\Lambda) \). Note that \( G \) is of \( K \)-rank 1 and \( G \) is of \( \mathbb{R} \)-rank \( r \).
We will use the notation \( u^-(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \) and \( u^+(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \)
for any \( x \in \Delta \) or \( \Delta \otimes_K K_v \).
Theorem 7.1. With the above notation and assumptions, let \( g = (u^{-1}(\beta)u^{+}(\alpha), e, \cdots, e) \in G \) where \( \alpha \in K^* \) and \( \beta \in \Delta \setminus K \). Then the following holds:

(i) \( \overline{T\pi(g)} \) is not dense in \( G/\Gamma \),

(ii) \( \overline{T\pi(g)} \) contains a closed orbit \( L\pi(e) \) where \( L = SL_2(K_{\infty}) \).

(iii) \( \overline{T\pi(g)} \setminus \overline{T\pi(g)} \) is not contain in a union of countably many closed orbits of proper subgroups of \( G \).

In particular, \( \overline{T\pi(g)} \) is not an orbit of a closed subgroup of \( G \).

Proof. A direct calculation shows that \( u^{-1}(\beta)u^{+}(\alpha) = du^{+}(\alpha_1)u^{-}(\beta_1) \)

where \( \alpha_1 = (1+\alpha\beta)\alpha, \beta_1 = (1+\alpha\beta)^{-1}\beta \) and \( d = \begin{pmatrix} (1+\alpha\beta)^{-1} & 0 \\ 0 & 1+\alpha\beta \end{pmatrix} \).

Denote by \( L_1 \) and \( L_2 \) the \( K \)-algebraic subgroups of \( G \) such that \( L_1(K) = \{ \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in G(K) : x,y,z,t \in K \} \) and \( L_2(K) = \{ \begin{pmatrix} x & y\beta^{-1}_1 \\ z\beta_1 & t \end{pmatrix} \in G(K) : x,y,z,t \in K \} \). It is clear that \( L_1 \) and \( L_2 \) are both \( K \)-groups isomorphic to \( SL_2 \). Since \( \beta \in \Delta \setminus K \) we get that \( \beta_1 \in \Delta \setminus K \) and , therefore, \( L_1 \cap L_2 = T \). Denote \( L_1 = L_1(K_{\infty}) \) and \( L_2 = L_2(K_{\infty}) \). Then \( L_1\pi(g) \) and \( L_2\pi(g) \) are closed.

It easy to see that \( \overline{T\pi(g)} \subset \bigcup_{0 \leq \mu \leq 1} \{ (u^{-}(\mu\beta), \cdots, e)\pi L_1(e) \} \bigcup \bigcup_{0 \leq \nu \leq 1} \{ d(u^{+}(\nu\alpha_1), \cdots, e)\pi L_2(e) \}. \)

Since the right hand side of the above inclusion is a closed proper subset of \( G/\Gamma \), \( T\pi(g) \) is not dense in \( G/\Gamma \) proving (i). Let \( U_1^+ \) be the unipotent subgroup of \( L_1 \) of upper triangular matrices and \( U_2^- \) be the unipotent subgroup of \( L_2 \) of lower triangular matrices. Put \( U_1^+ = U_1^+(K_{\infty}) \) and \( U_2^- = U_2^-(K_{\infty}) \). It follows from part Proposition 3.2(2a) and the density of the projection of \( O \) into \( \prod_{i=2}^r K_i \) that \( \overline{T\pi(g)} \supset U_1^+\pi(e) \cup dU_2^-\pi(e) \). In view of Proposition 6.13 \( \overline{T\pi(g)} \supset L_1\pi(e) \cup dL_2\pi(e) \) which proves (ii).

Let \( a \) be a real transcendental number. Using again Proposition 3.2 we obtain that \( \overline{T\pi(g)} \) contains \( \pi(\tilde{g}) \) where \( \tilde{g} = (u^{-}(a^{-2}\beta)u^{+}(a^2\alpha), \cdots, e) \).

Remark that \( \overline{T\pi(g)} = \overline{T\pi(g)} \). Since \( a \) is transcendental and \( \Delta \otimes K \)

\( K_v, v \in \infty, \) is a quaternion skew field, \( \pi(\tilde{g}) \notin \pi(g) \). Suppose by the contrary that \( \overline{T\pi(g)} \setminus \overline{T\pi(g)} \subset \bigcup_i Q_i\pi(g_i) \) where \( Q_i \) are connected closed subgroups of \( G \) and \( Q_i\pi(g_i) \) are closed orbits. It follows from Baire’s category theorem that there exists \( i_0 \) such that \( T \subset Q_{i_0} \) and \( \pi(\tilde{g}) \in Q_{i_0}\pi(g_{i_0}) \). Then \( L_1 \cup dL_2d^{-1} \subset Q_{i_0} \). Since \( G \) is spanned by \( L_1 \) and \( L_2 \) we get that \( Q_{i_0} = G \) and (iii) is proved. \( \square \)
The main point in the above theorem is the property (iii). It contrasts with the construction in [Sha] where \( \overline{T\pi(g)} \setminus T\pi(g) \) is included in a union of two closed orbits of proper subgroups in \( SL(3,\mathbb{R}) \) (see [L-Sha] Theorem 1.5)).

8. A NUMBER THEOREICAL APPLICATION

8.1. Reduction of the proof of Theorem 1.10 to the case \( m = n \). We need the following simple

**Proposition 8.1.** Let \( M_1, \ldots, M_r \) be finite subsets of the vector space \( K^n \) with the following properties: each \( M_i \) consists of \( m \) linearly independent vectors and there exist \( w \in M_1 \) and \( j \geq 2 \) such that no vector from \( M_j \) is proportional to \( w \). Then there exists a linear map \( \phi : K^n \to K^m \) such that every \( \phi(M_i) \) consists of \( m \) linearly independent vectors and no vector from \( \phi(M_j) \) is proportional to \( \phi(w) \).

**Proof.** It easy to see that the conditions on \( \phi \) define a non-empty Zariski open subset of the vector space of all linear maps from \( K^n \) to \( K^m \). This implies the proposition. \( \square \)

In order to reduce the proof of Theorem 1.10 to the case \( m = n \) we apply Proposition 8.1 to the vector space \( Kx_1 + \cdots + Kx_n \) and the subsets \( M_v = \{ l_i^{(v)}(\bar{x}), \ldots, l_m^{(v)}(\bar{x}) \} \), \( v \in S \). There exists a basis \( \{ y_1, \ldots, y_n \} \) of \( Kx_1 + \cdots + Kx_n \) such that the map \( \phi \), as in the formulation of Proposition 8.1, is given by \( \phi(a_1 y_1 + \cdots + a_n y_n) = a_1 y_1 + \cdots + a_n y_n \). Let \( l_i^{(v)}(\bar{x}) = l_i^{(v)}(\bar{y}) \) for all \( i \) and \( v \). Then the linear forms \( l_i^{(v)}(y_1, \ldots, y_m, 0, \cdots, 0), \ldots, l_m^{(v)}(y_1, \ldots, y_m, 0, \cdots, 0) \) are linearly independent over \( K \) and the polynomials \( \prod_{i=1}^m l_i^{(v)}(y_1, \ldots, y_m, 0, \cdots, 0), v \in S \), are not pairwise proportional. This completes the reduction.

8.2. Proof of Theorem 1.10. Let \( G = SL_{n+1}, G = SL_{n+1}(K_S) \) and \( \Gamma = SL_{n+1}(O) \). The group \( G \) is acting on \( K_S[\bar{x}] \) according to the law \( (\sigma \phi)(\bar{x}) = \phi(\sigma^{-1} \bar{x}) \), where \( \sigma \in G \) and \( \phi \in K_S[\bar{x}] \). We denote \( f_0(\bar{x}) = x_1 x_2 \ldots x_{n+1} \). Let \( f(\bar{x}) \) be as in the formulation of the theorem and \( m = n \). There exists \( g = (g_v)_{v \in S} \in G \) such that every \( g_v \in G(K) \) and \( f(\bar{x}) = \alpha(g^{-1} f_0)(\bar{x}) \) where \( \alpha \in K_S \). Since \( f_v(\bar{x}), v_n \in S \), are not pairwise proportional the orbit \( T\pi(g) \) is locally divergent but non-closed (Theorem 1.1). Note that \( f(\bar{x}) = \alpha(w g f_0)(\bar{x}) \) for every \( w \in N_G(T) \). In view of Theorem 1.8 and its Corollary 1.9 there exist a reductive \( K \)-subgroup \( H \) with \( T \not\subset H \) and \( \sigma \in G(K) \) such that \( T\pi(g) = H\pi(\sigma) \) where \( H = H(K_S) \). By Proposition 6.14 there exists a direct sum decomposition \( W = W_1 \oplus \cdots \oplus W_r \) such that \( H = \)
\( \{ g \in \text{SL}_{n+1} : g \mathbf{W}_i = \mathbf{W}_i \text{ for all } i \} \). (We use that \( \text{SL}_{n+1} \) is identified with \( \text{SL}(W) \).) Therefore given \( a = (a_v)_{v \in S} \) there exist \( z \in \mathcal{O}^{n+1} \) and \( h = (h_v)_{v \in S} \in H \) such that \( f_0(h \sigma(z)) = a \). Since \( H \sigma \Gamma = Tg \Gamma \) there exist \( t_i \in T \) and \( \gamma_i \in \Gamma \) such that
\[
\lim_i t_i g \gamma_i = h \sigma.
\]
Therefore
\[
\lim_i f(\gamma_i z) = a,
\]
proving the theorem. \( \square \)

8.3. The cases when \( \#S = 2 \) or \( \#S \geq 2 \) and \( K \) is a CM-field. As mentioned in the introduction the assertion of Theorem 1.10 does not hold if \( \#S = 2 \) or \( K \) is a CM-field and \( \#S \geq 2 \). Here we provide the necessary counter-examples assuming for simplicity that \( S = \infty \) and \( m = n = 2 \).

The following is a particular case of [T3, Theorem 1.10]:

**Theorem 8.2.** Let \( \#S = 2 \). Then \( \overline{f(\mathcal{O}^2)} \cap K_S^* \) is a countable set. In particular, \( \overline{f(\mathcal{O}^2)} \) is not dense in \( K_S^* \).

Now let \( K \) be a CM-field. There exists a totally real subfield \( F \subset K \) and \( d \in F \) such that \( K = F(\sqrt{-d}) \). We denote by \( \mathcal{O}_F \) the ring of integers of \( F \). Note that the map which associates to every \( v \in S \) its restriction to \( F \) is injective. We will also use the notation \( v \) for the restriction of \( v \) to \( F \).

**Theorem 8.3.** We suppose that \( \#S \geq 2 \), \( K \) is a CM-field and all \( l^{(v)}_i(x) \) are with coefficients from \( F \). Then there exists a real \( C > 0 \) such that for every \( z \in \mathcal{O}^2 \) such that \( f(z) \in K_S^* \) either
\[
\prod_{v \in S} |f_v(z)|_v \geq C
\]
(15)
or there exists \( w \in \mathbb{C} \) such that
\[
f_v(z) \in \mathbb{R}w \text{ for all } v \in S.
\]
In particular, \( \overline{f(\mathcal{O}^2)} \) is not dense in \( K_S^* \).

**Proof.** Choose \( d \in F \) such that \( \sqrt{-d} \in \mathcal{O} \). Put \( l = [\mathcal{O} : \mathcal{O}_F(\sqrt{-d})] \). If \( S = \{v_1, \ldots, v_r\} \) we use the simpler notation: \( K_j := K_{v_j}, |.|_j := |.|_{v_j}, f_j := f_{v_j} \) and \( l^{(j)}_i := l^{(v_j)}_i \). Let \( l^{(j)}_i(x_1, x_2) = h^{(j)}_{11} x_1 + h^{(j)}_{12} x_2 \) where \( j \in \{1, \ldots, r\} \) and \( i \in \{1, 2\} \). Put \( h^{(j)} := \left( \begin{array}{cc} h^{(j)}_{11} & h^{(j)}_{12} \\
 h^{(j)}_{21} & h^{(j)}_{22} \end{array} \right) \). After
multiplying $f_j$ by appropriate elements from $F^*$ we may (and will) suppose that $h^{(j)} \in \text{SL}_2(F)$ for all $j$. Further on, if $\vec{w}_1 = (w_{11}, w_{12}) \in \mathbb{C}^2$ and $\vec{w}_2 = (w_{21}, w_{22}) \in \mathbb{C}^2$ we denote by $\det(\vec{w}_1, \vec{w}_2)$ the determinant of

\[
\begin{pmatrix}
w_{11} & w_{12} \\
w_{21} & w_{22}
\end{pmatrix}.
\]

Also, given $w = \begin{pmatrix} w_{11} & w_{12} \\
w_{21} & w_{22} \end{pmatrix}$ and $\vec{a} = (a_1, a_2) \in \mathbb{C}^2$ we write $w(\vec{a}) = (w_{11}a_1 + w_{12}a_2, w_{21}a_1 + w_{22}a_2)$.

With $C = \left( \frac{d}{4t} \right)^r$ we will prove that given $\vec{z} \in O^2$ either (15) or (16) holds. Let $\vec{z} = \vec{\gamma} + \sqrt{-d}\vec{\delta} \in O^2$ where $\vec{\gamma} = (\gamma_1, \gamma_2) \in F^2$ and $\vec{\delta} = (\delta_1, \delta_2) \in F^2$. By the choice of $l$, $(l\gamma_1, l\gamma_2) \in O^2_F$ and $(l\delta_1, l\delta_2) \in O^2_F$. It is well known that if $\alpha \in O_F$ then $\prod_j |\alpha_j| \in \mathbb{N}$ (cf. [CF, ch. 2, Theorem 11.1]). Therefore

\[
\prod_j |\det(h^{(j)}(\vec{\gamma}), h^{(j)}(\vec{\delta}))| = \prod_j |\det(\vec{\gamma}, \vec{\delta})| \in \left( \frac{d}{4t} \right)^{\frac{r}{4}} \mathbb{N}.
\]

We have

\[
h^{(j)}(\vec{z}) = h^{(j)}(\vec{\gamma}) + \sqrt{-d}h^{(j)}(\vec{\delta}) = (r_1^{(j)} e^{i\phi_1^{(j)}}, r_2^{(j)} e^{i\phi_2^{(j)}}) \in \mathbb{C}^2,
\]

where $r_1^{(j)}$ and $r_2^{(j)}$ are the absolute values and $\phi_1^{(j)}$ and $\phi_2^{(j)}$ are the arguments of the complex coordinates of $h^{(j)}(\vec{z})$. Since $f_j(\vec{z}) = r_1^{(j)} r_2^{(j)} e^{i(\phi_1^{(j)} + \phi_2^{(j)})}$, we get

\[
\det(h^{(j)}(\vec{\gamma}), h^{(j)}(\vec{\delta})) = \frac{f_j(\vec{z})}{\sqrt{d}} e^{-i(\phi_1^{(j)} + \phi_2^{(j)})} \det \begin{pmatrix}
cos \phi_1^{(j)} & \sin \phi_1^{(j)} \\
cos \phi_2^{(j)} & \sin \phi_2^{(j)}
\end{pmatrix}.
\]

Therefore

\[
\prod_j |f_j(\vec{z})| \geq \left( \frac{d}{4} \right)^r \prod_j |\det(\vec{\gamma}, \vec{\delta})| \in \left( \frac{d}{4t} \right)^{\frac{r}{4}} \mathbb{N}.
\]

So, (15) holds unless $\det(\vec{\gamma}, \vec{\delta}) = 0$. In the latter case $\vec{\gamma}$ and $\vec{\delta}$ are proportional which implies (16). \qed

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