Poor man’s holography: how far can it go?

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Abstract
Almost a century ago, Einstein, after Newton, shed new light on gravity by claiming that gravity is geometry. There has been no deeper insight beyond that later on except the recent suspicion that gravity may also be holographic, dual to some sort of quantum field theory living on the boundary with one less dimension. Such a suspicion has been supported mainly by a variety of specific examples from string theory. This paper is intended to purport the holographic gravity from a different perspective. Namely, we shall show that such a holography can actually be observed by working merely within the context of Einstein’s gravity through promoting Brown–York’s formalism, where neither is the spacetime required to be asymptotically AdS nor the boundary to be located at conformal infinity, which also conforms to the spirit inherited from Wilson’s effective field theory. In particular, we show that our holography works remarkably well at least at the level of thermodynamics and hydrodynamics, where a perfect matching between the bulk gravity and boundary fluid is found for entropy and its production by the conserved current method.

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1. Introduction

Although string theory provides an explicit implementation of quantum gravity in a holographic way, now dubbed as AdS/CFT correspondence, it is worthwhile to keep in mind that there are various hints from within the context of Einstein’s gravity towards the speculation that gravity is essentially holographic, where neither supersymmetry nor strings as well as branes are involved. Here we would like to list four of them as follows.

Brown–Henneaux’s asymptotic symmetry analysis for three-dimensional gravity \cite{1}.
Brown–York’s surface tensor formulation of quasilocal energy and conserved charges [2].
Black hole thermodynamics [3].
Bousso’s covariant entropy bound [4].

In particular, Brown–York’s surface tensor formulation bears a strong resemblance to the
recipe in the dictionary for AdS/CFT correspondence especially when one is brave enough to
declare that Brown–York’s surface tensor is not only for the purpose of the bulk side but also
for some sort of system living on the boundary. In this sense, Brown–York’s tensor formulation
implies that gravity is holographic. Actually, such a holographic interpretation can be exactly
proven, at least at the level of thermodynamics and hydrodynamics. This is the main purpose
of this paper.

Let us first promote such a formulation in a holographic way by the following
dictionary, i.e.
\[
\int_{\phi_0} D\phi \exp(-S_{\text{bulk}}[\phi]) = \left\langle \exp \left( - \int d^d x \sqrt{-\gamma} O \phi_0 \right) \right\rangle_{\text{bdry}},
\]
where \(\phi_0\) plays a dual role, namely, serves as the boundary condition for the bulk path integral
over \(\phi\) on the left-hand side and as the external source of the dual boundary operator \(O\) on
the right-hand side. When the spacetime is asymptotically AdS with the conformal boundary,
the above dictionary recovers the standard AdS/CFT correspondence. But here we do not
require the spacetime to be asymptotically AdS, or the boundary to be located at the conformal
infinity. Furthermore, in the saddle point approximation, the expectation value of the dual
operator is given by
\[
\langle O \rangle = \frac{1}{\sqrt{-\gamma}} \frac{\delta S_{\text{classical}}}{\delta \phi_0}.
\]

Two examples are of special interest. One is the case of \(\phi\) to be the bulk metric \(g_{ab}\) with \(\phi_0\)
the induced metric \(\gamma_{ab}\) on the boundary, in which the dual operator is simply the stress–energy
tensor of the boundary system, and its expectation value is given by
\[
t^{ab} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S_{\text{classical}}}{\delta \gamma_{ab}}.
\]
The other is the case of \(\phi\) to be the electromagnetic potential \(A_a\) with \(\phi_0\) the pull back of \(A_a\)
on the boundary, in which the dual operator is just the electric current with its expectation value
given by
\[
j_a = \frac{1}{\sqrt{-\gamma}} \frac{\delta S_{\text{classical}}}{\delta A_a}.
\]

In particular, if the bulk actions for gravity and electromagnetic fields are given by the Einstein–
Hilbert action plus Gibbons–Hawking term and Maxwell action, respectively, i.e.
\[
S_{\text{GR}} = \frac{1}{16\pi} \left[ \int d^{d+1} x \sqrt{-g} (R - 2\Lambda) + 2 \int d^d x \sqrt{-\gamma} K \right],
\]
\[
S_{\text{EM}} = -\frac{1}{16\pi} \int d^{d+1} x \sqrt{-g} F_{ab} F^{ab},
\]
we have
\[
t^{ab} = \frac{1}{8\pi} (K\gamma^{ab} - K^{ab} - C\gamma^{ab}), \quad j^a = -\frac{1}{4\pi} n_b F^{ba},
\]
where \(K = \gamma^{ab} K_{ab}\) is the trace of extrinsic curvature, \(K_{ab} = \gamma_{ab} \nabla_c n^c\) with \(n_b\) the outward
normal vector to the boundary, and \(C\) is the constant from some sort of renormalization.
2. Equilibrium state version of correspondence: thermodynamics

First of all, let us build up the equilibrium state version of our correspondence by considering the Schwarzschild AdS black hole in the bulk, i.e.

\[
ds_{d+1}^2 = \frac{dr^2}{f(r)} - f(r)\, dt^2 + r^2 \, d\Omega_i^2, \quad f(r) = \varepsilon + \frac{r^2}{L^2} - \frac{2M}{r^d - 2},
\]

where \(d\Omega_i^2\) can be the metric on the sphere, plane or hyperbola for \(\varepsilon = 1, 0, -1\), respectively.

Then by the standard calculation, the entropy and temperature of black hole are given by

\[
S_{BH} = \frac{\ell_P^{d-1}\Omega_{d}}{4}, \quad T_H = \frac{f'(r_h)}{4\pi},
\]

with \(r_h\) the location of horizon satisfying \(f(r_h) = 0\).

Now the boundary can be any hypersurface of \(r = r_c\) outside the horizon, with the induced metric

\[
ds_4^2 = -f_c \, dt^2 + r_c^2 \, d\Omega_i^2, \quad f_c = f(r_c).
\]

The nice thing is that one can easily show that the boundary stress–energy tensor has a form of ideal fluid, i.e.

\[
\tau^{ab} = \epsilon u^a u^b + p(u^a u^b + \gamma^{ab}),
\]

with the fluid four velocity \(u^a = \frac{1}{\sqrt{f_c}}(\frac{3}{d})^d\) on the boundary, and the energy density as well as pressure given by

\[
\epsilon = -\frac{(d-1)\sqrt{f_c}}{8\pi r_c} + C,
\]

\[
p = \frac{(d-2)\sqrt{f_c}}{8\pi r_c} + \frac{f_c'}{16\pi \sqrt{f_c}} - C.
\]

Note that the volume of the boundary system is

\[
V = r_c^{d-1}\Omega_d.
\]

So the total energy is given by

\[
E = \varepsilon V = \left(-\frac{(d-1)\sqrt{f_c}r_c^{d-2}}{8\pi} + Cr_c^{d-1}\right)\Omega_d.
\]

If our bulk/boundary correspondence is right, then we must have the black hole entropy identified as the entropy for the boundary fluid, i.e.

\[
S_{BF} = S_{BH}.
\]

With this identification, we can express \(E\) as a function of \(S_{BF}\) and \(V\), which further gives rise to

\[
\frac{\partial E}{\partial S_{BF}} = T_c, \quad \frac{\partial E}{\partial V} = -p,
\]

where \(T_c = \frac{\ell_P}{\sqrt{f_c}}\) is the temperature for the boundary fluid, redshifted as it should be the case.

So we have a well-defined first law of thermodynamics for the boundary fluid, i.e.

\[
dE = T_c \, dS_{BF} - p \, dV.
\]
3. Physical process version of correspondence: hydrodynamics

If the boundary system is perturbed by some sort of external sources away from the equilibrium state, then the transport process will intend to bring the system back to a new equilibrium state, which generically causes entropy production. In particular, when the boundary system is perturbed by the electromagnetic field $A_{\alpha\beta} = 0$ and gravitational field satisfying $h_{\alpha\beta} = 0$ as well as $h_{00} = 0$, the rate for entropy production is given by

$$\Sigma = \frac{1}{T_c} j^\alpha E_\alpha - \frac{1}{T_c} \langle D^{(1)}_{\alpha} u_\alpha + D_{u_\beta} u^{(1)}_\beta \rangle.$$  

(3.1)

Here the superscript 1 denotes the first-order variation induced by the gravitational perturbation $h_{\alpha\beta}$. For instance,

$$D^{(1)}_{\alpha} u_\alpha + D_{u_\beta} u^{(1)}_\beta = -\Gamma^{(1)}_{\alpha\beta\gamma} u^\gamma + D_{\gamma} (h_{\alpha\beta} u^\gamma) = \sqrt{\gamma} (\Gamma^{(1)}_{\alpha\beta\gamma} - D_{\gamma} h_{\alpha\beta}).$$  

(3.2)

where we have used the fact that $D^{(1)}_{\alpha} u_\alpha$ comes essentially from the first-order variation of the Christoffel symbol, i.e.

$$D^{(1)}_{\alpha} u^\beta = \Gamma^{(1)}_{\alpha\beta\gamma} u^\gamma = \frac{1}{2} \gamma^{\beta\gamma} (D_{\gamma} h_{\alpha\beta} + D_{\beta} h_{\alpha\gamma} - D_{\alpha} h_{\beta\gamma}).$$  

(3.3)

From the bulk point of view, such perturbations on the boundary should propagate towards the black hole and be absorbed. Eventually, the black hole will settle down to a new static final state with an increase in the area of the black hole horizon, or put it another way, with an increase of black hole entropy. If our bulk/boundary correspondence is right, the increase of black hole entropy should be precisely equal to the aforementioned entropy production on the boundary. As we shall prove shortly, this is actually the case. The basic idea for such a proof is to relate the boundary to the bulk by the conserved current, which can be best presented by considering first the electromagnetic perturbation.

Let us start with the stress–energy tensor for the electromagnetic waves, i.e.

$$T^{ab}_{\rm EM} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\rm EM}}{\delta g^{ab}} = \frac{1}{4\pi} \left( F^{ac} F^b_c - \frac{1}{4} g^{ab} F_{cd} F^{cd} \right),$$  

(3.4)

which is conserved $\nabla_a T^{ab}_{\rm EM} = 0$. So one can construct the conserved current as follows,

$$J^a = T^{a\beta}_{\rm EM} \xi^\beta,$$  

(3.5)

associated with the time-like Killing vector field $\xi = \frac{\partial}{\partial t}$. Now, suppose the non-equilibrium region has compact support on the boundary, which naturally gives rise to the corresponding compact support for both of the perturbed bulk and perturbed horizon. Then integrating $\nabla_a J^a = 0$ over the perturbed bulk with the perturbed horizon as the inner boundary and using the Gauss law, we end up with

$$\int_H T^{ab}_{\rm EM} n_a \xi^b = \int_{\text{bdy}} T^{ab}_{\rm EM} n_a \xi^b,$$  

(3.6)

where $H$ is the horizon. To relate the left-hand side with the increase in the black hole entropy in a simple way, we would like to make the null geodesic generators of the event horizon of the perturbed black hole coincide with the null geodesic generators of the unperturbed black hole by using our diffeomorphism freedom [9]. With this, the perturbation in the horizon location vanishes and $\delta S \propto \xi$ on the horizon. Then the Raychaudhuri equation implies [10, 11]

$$T_H \delta S_{\rm BH} = \int_H T^{ab}_{\rm EM} n_a \xi^b.$$  

(3.7)

4 With such a setup, actually the second term consists of two contributions, namely the entropy production induced by the inhomogeneous temperature [6–8], and the one produced by the shear as well as bulk viscosity.
On the other hand, with the electric field felt by the boundary fluid as $E_c = F_{cb}b^b$, we have

$$\int_{\text{bdry}} T_{EBm}n_a\xi_b = \sqrt{f_c} j^a E_a,$$

(3.8)

which gives rise to

$$\delta S_{\text{BH}} = \frac{j^a E_a}{T_c} = \delta S_{\text{BF}}.$$  

(3.9)

Next we consider the case for the entropy production induced by the gravitational perturbation on the boundary. To proceed, let us first expand the bulk Einstein equation on the black hole background to second order, i.e.

$$G^{ab} + \Lambda g^{ab} = 0,$$

(3.10)

$$G^{(1)ab}[h] - \Lambda h^{ab} = 0,$$

(3.11)

$$G^{(1)ab}[q] - \Lambda q^{ab} = 8\pi T_{GW}^{ab}[h] = -\left[G^{(2)ab}[h] + \Lambda h^{ab}h^{ab}\right],$$

(3.12)

where the metric is expanded as $g_{ab} + \epsilon h_{ab} + \epsilon^2 q_{ab}$ with the indices raised or lowered by the background metric $g_{ab}$. Furthermore, it follows from the Bianchi identity that the energy–momentum tensor is conserved for the gravitational waves propagating on the background, i.e.

$$\nabla_a T_{GW}^{ab} = 0,$$

(3.13)

which, as before, gives rise to

$$\delta S_{\text{BH}} = \frac{1}{T_H} \int_H T_{GW}^{ab}n_a \xi_b = \frac{1}{T_H} \int_{\text{bdry}} T_{GW}^{ab}n_a \xi_b.$$  

(3.14)

So now the task boils down into whether one can express the above flux across the boundary in terms of entropy production on the boundary, which can actually be achieved by a straightforward but lengthy calculation. But here we would like to present a shortcut towards the final result by taking advantage of the dual role played by the gravitational waves. Namely, as demonstrated in equations (3.11) and (3.12), the gravitational waves, albeit treated as sort of matter waves like light, are essentially ripples in the fabric of spacetime. Thus, we can relate the aforementioned flux to the quantities for the dual system on the boundary by the Gauss–Codazzi equation, which, expanded to second order, gives

$$D_a\gamma^{ac} = -\frac{1}{8\pi} G^{ab} n_b \gamma^c = 0,$$

(3.15)

$$D_a\Phi^{(1)ac} + D_d^{(1)ac} = -\frac{1}{8\pi} G^{(1)ab}[h] n_b \gamma^c = 0,$$

(3.16)

$$D_a\Phi^{(2)ac} + D_d^{(1)ac} + D_d^{(2)ac} = -\frac{1}{8\pi} G^{(2)ab}[h] n_b \gamma^c = T_{GW}^{ab} n_a \gamma^c,$$

(3.17)

where $D_d^{(2)}$ is determined by the second-order Christoffel symbol, i.e.

$$D_a^{(2)} v^b = \Gamma^{(2)bc}_a v^c = -\frac{1}{2} h^{bd} (D_a h_{cd} + D_c h_{ad} - D_d h_{ac}) v^c.$$

(3.18)

5 Obviously, in order to guarantee the increase of entropy, one is forced to impose the natural boundary condition for the perturbation on our cutoff surface by requiring that the conserved current flux be allowed only from the outside to the interior bulk.
Then one can show
\[
\delta S_{\text{BH}} = - \frac{\sqrt{\epsilon}}{T_c} \int_{\text{bdry}} b^d \left( \frac{1}{2} D_d \phi_{1d}^{(1)} + \Gamma^{(1)1}_{cd} \phi_{cd}^{(1)} + \Gamma^{(2)0}_{cd} \phi_{cd}^{(1)} \right)
\]
\[
= - \frac{\sqrt{\epsilon}}{T_c} \int_{\text{bdry}} b^d \left( \frac{1}{12} D_d \phi_{1d}^{(1)} + \frac{1}{2} \Gamma^{(1)1}_{cd} \phi_{cd}^{(1)} + \frac{1}{2} \Gamma^{(2)0}_{cd} \phi_{cd}^{(1)} + \frac{1}{2} \Gamma^{(1)1}_{cd} \phi_{cd}^{(1)} + \frac{1}{2} \Gamma^{(2)0}_{cd} \phi_{cd}^{(1)} \right)
\]
\[
= - \frac{\sqrt{\epsilon}}{T_c} \int_{\text{bdry}} b^d \left( \frac{1}{2} h^{01}_{\phi} D_d \phi d + \frac{1}{2} \Gamma^{(1)1}_{cd} \phi_{cd}^{(1)} + \frac{1}{2} \Gamma^{(2)0}_{cd} \phi_{cd}^{(1)} \right)
\]
\[
= - \frac{\sqrt{\epsilon}}{T_c} \int_{\text{bdry}} b^d \left( \frac{1}{2} h^{01}_{\phi} D_d \phi d + \frac{1}{2} \Gamma^{(1)1}_{cd} \phi_{cd}^{(1)} + \frac{1}{2} \Gamma^{(2)0}_{cd} \phi_{cd}^{(1)} \right)
\]
\[
= \delta S_{\text{BF}},
\]
where we have thrown away all the total derivative terms at each step, and employed \( D_c e_{ad} = 0 \) as well as \( h_{ad} e_{ad} = \frac{1}{2} \eta \) in the second last step.

4. Discussion

We have provided Brown–York’s formalism with the holographic interpretation. In particular, we have demonstrated that such a holographic formulation works very well, at least at the level of thermodynamics and hydrodynamics, where a perfect matching between the bulk gravity and boundary system is exactly derived for entropy and its production. Although we are working only with the Schwarzschild AdS black hole, it can be shown that our discussion can be applied to the charged AdS black hole, where the calculation is a little bit involved due to the fact that the electromagnetic and gravitational perturbations are coupled to each other [8]. Furthermore, it is obvious that our procedure can actually be applied not only to asymptotically flat and de Sitter charged black holes but also to the spacetime patch associated with the Rindler horizon in the flat spacetime or de Sitter horizon in the de Sitter spacetime [8].

Compared to the standard AdS/CFT correspondence, our holography is more general in the following sense. First, we do not require the spacetime to be asymptotically AdS. Second, our boundary is not required to be located at conformal infinity. Actually, an echo can be found for such a relaxation in the membrane paradigm [12–16]. In addition, such a generalization is also consistent with Wilson’s modern interpretation of quantum field theory, where quantum field theory can be defined up to some finite energy scale no matter whether there exists a UV completion or whatever the would-be UV completion is. So it is intriguing to refine such a connection based on the recently developed Wilsonian formulation of holographic renormalization [17, 18].

On the other hand, to our best knowledge, the conserved current method we have used in the proof of our bulk/boundary correspondence is totally novel in the context of holography. This method further suggests another natural local correspondence between the black hole horizon and boundary by the integral curves of conserved current. Such a local correspondence seems more reasonable than the conjectured one defined by the null geodesics in the previous literature such as [19–21].

We conclude with various other issues worthy of further investigation. For one thing, we have worked simply to second-order perturbation so far. It is interesting to see whether the

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6 It is noteworthy that in our holography what we are concerned with is the entropy production while in the previous literature the concerned quantity is the so-called entropy current whose divergence gives rise to the entropy production. There are at least two reasons for us to prefer the entropy production to the entropy current. For one thing, although the entropy current is sort of elegant object, its definition has some kind of ambiguity in nature. For another, as far as we know, the entropy current has never shown up in any kind of dynamics except its divergence, namely the entropy production.
whole procedure can be performed to any higher order. For another, we have worked merely within the context of Einstein’s gravity. It is worthwhile to see whether our holography can also be valid for other higher derivative gravity theories, where the entropy is given by the Wald formula [22, 23]. We hope to address these issues elsewhere.

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