\( C^N \)-Smorodinsky-Winternitz system in a constant magnetic field

Hovhannes Shmavonyan\(^1\,*\)

\(^1\)Yerevan Physics Institute, 2 Alikhanian Brothers St., Yerevan 0036 Armenia

We propose the superintegrable generalization of Smorodinsky-Winternitz system on the \( N \)-dimensional complex Euclidian space which is specified by the presence of constant magnetic field. We find that in addition to \( 2N \) Liouville integrals the system has additional functionally independent constants of motion, and compute their symmetry algebra. We perform the Kustaanheimo-Stiefel transformation of \( C^2 \)-Smorodinsky-Winternitz system to the (three-dimensional) generalized MICZ-Kepler problem and find the symmetry algebra of the latter one. We observe that constant magnetic field appearing in the initial system has no qualitative impact on the resulting system.

I. INTRODUCTION

The one-dimensional singular oscillator is a textbook example of a system which is exactly solvable both on classical and quantum levels. The sum of its \( N \) copies, i.e. \( N \)-dimensional singular isotropic oscillator is, obviously, exactly solvable as well. It is given by the Hamiltonian

\[
H = \sum_{i=1}^N I_i, \quad \text{with} \quad I_i = \frac{p_i^2}{2} + \frac{g_i^2}{2x_i^2} + \frac{\omega^2 x_i^2}{2}, \quad \{p_i, x_j\} = \delta_{ij}, \quad \{p_i, p_j\} = \{x_i, x_j\} = 0 \tag{1}
\]

It is not obvious that in addition to Liouville Integrals \( I_i \) this system possesses supplementary series of constants of motion, and is respectively, maximally superintegrable, i.e. possesses \( 2N - 1 \) functionally independent constants of motion. All these constants of motion are of the second order on momenta, which guarantee the separability of variables in few coordinate systems and cause degeneracy of the energy spectrum in quantum level. Seems that this was first noticed by Smorodinsky and Winternitz, who then investigated superintegrability properties of this system in great detail \cite{1}. For this reason this model is sometimes called Smorodinsky-Winternitz system and we will use this name as well. For sure, such a simple and internally reach system was obliged to attract wide attention in mathematical and theoretical physics society, so that the number of publications devoted to its study and further generalizations is enormous, among other ones, we should mention the references \cite{2-4, 5} (see also the recent PhD thesis on this subject with expanded list of references \cite{6}). Notice also that Smorodinsky-Winternitz system is a simplest case of the generalized Calogero model (with oscillator potential) associated an arbitrary Coxeter root system \cite{6}. Thus, one hopes that observations done in this simple model could be somehow extended to the Calogero models.

There is well-known superintegrable generalization of the oscillator to sphere, which is known as Higgs oscillator \cite{7}. Smorodinsky-Winternitz model admits superintegrable generalization of the sphere as well \cite{8}. It may sound strange, but it was later rediscovered by many authors, e.g. \cite{9}, though was actually suggested by Rosochatius in XIX century (without noticing its superintegrability) \cite{10}. Superintegrable generalization of Calogero model on the sphere also exists \cite{11}.

In this paper we consider simple generalization of the Smorodinsky-Winternitz system interacting with constant magnetic field. It is defined on the \( N \)-dimensional complex Euclidian space parameterized by the coordinates \( z^a \) by the Hamiltonian

\[
\mathcal{H} = \sum_{a=1}^N \left( \pi_a \bar{\pi}_a + \frac{g_a^2}{z^a \bar{z}^a} + \omega^2 z^a \bar{z}^a \right), \quad \text{with} \quad \{\pi_a, z^b\} = \delta_{ab}, \quad \{\pi_a, \bar{\pi}_b\} = iB\delta_{ab} \tag{2}
\]

The (complex) momenta \( \pi_a \) have nonzero Poisson brackets due to the presence of magnetic field with magnitude \( B \). We will refer this model as \( C^N \)-Smorodinsky-Winternitz system. For sure, in the absence of magnetic field this model could be easily reduced to the conventional Smorodinsky-Winternitz model, but the presence of magnetic field could have nontrivial impact which will be studied in this paper. So, our main goal is to investigate the whole symmetry algebra of this system. Notice that this is not only for academic interest: the matter is that \( C^1 \)-Smorodinsky-Winternitz system is a popular model for the qualitative study of the so-called quantum ring \cite{12}, and the study

*Electronic address: shmavonyanhov@gmail.com
of its behaviour in external magnetic field is quite a natural task. Respectively, \( \mathbb{C}^N \)-Smorodinsky-Winternitz could be viewed as an ensemble of \( N \) quantum rings interacting with external magnetic field. So that investigation of its symmetry algebra is of the physical importance.

It is well-known for many years that the energy surface of two-/three-/five-dimensional Coulomb system could be transformed to those of two-/four-/eight-dimensional oscillator by the use of so-called Levi-Civita-Bohlin/Kustaanheimo-Stiefel/Hurvitz transformation. More generally, reducing the two-/four-/eight-dimensional oscillator (-like) models by the action of \( \mathbb{Z}_2/U(1)/SU(2) \) group action, we can get the two-/three-/five-dimensional Coulomb-like systems specified by the presence of \( \mathbb{Z}_2/\text{Dirac/Yang}\)monopole \[13\]. Since \( \mathbb{C}^2 \)-Smorodinsky-Winternitz system is manifestly invariant with respect to \( U(1) \) group action, we can then perform its Kustaanheimo-Stiefel transformation, in order to obtain three-dimensional Coulomb-like system. It was done some ten years ago \[14\], but in the absence of magnetic field in initial system. Repeating this transformation for the system with constant magnetic field we get unexpected result, that it has no qualitative impact in the resulting system, which was referred in \[15\] as "generalized MICZ-Kepler system". In addition, we obtain, in this way, the explicit expression of its symmetry generators and their symmetry algebra, which was not constructed before, to our best knowledge.

We already mentioned that both oscillator and Smorodinsky-Winternitz system admit superintegrable generalizations to the spheres. On the other hand the isotropic oscillator on \( \mathbb{C}^N \) admits the superintegrable generalization on the complex projective space, moreover, the inclusion of constant magnetic field preserves all symmetries of that system \[16\]. It will be shown that introduction of a constant magnetic field doesn’t change these properties of the \( \mathbb{C}^N \)-Smorodinsky-Winternitz system. Thus, presented model could be viewed as a first step towards the construction of the analog of Smorodinsky-Winternitz system on \( \mathbb{C}^P N \).

The paper is organized as follows.

In the Section 2 we review the main properties of the conventional \((\mathbb{R}^N \text{-})\)Smorodinsky-Winternitz system, presenting explicit expressions of its symmetry generators, as well as wavefunctions and Energy spectrum. We also write down symmetry algebra in a very simple, and seemingly new form via redefinition of symmetry generators.

In the Section 3 we present \( \mathbb{C}^N \)-Smorodinsky-Winternitz system in a constant magnetic field, find the explicit expressions of its constants of motion. We compute their algebra and find that it is independent from the magnitude of constant magnetic field. Then we quantize a system and obtain wavefunctions and energy spectrum. We notice that the \( \mathbb{C}^N \)-Smorodinsky-Winternitz system has the same degree of degeneracy as \( \mathbb{R}^N \)-one, due to the lost part of additional symmetry.

In the Section 4 we perform Kustaanheimo-Stiefel transformation of the \( \mathbb{C}^2 \)-Smorodinsky-Winternitz system in constant magnetic field and obtain, in this way, the so-called "generalized MICZ-Kepler system". We find that constant magnetic field appearing in the initial system, does not lead to any changes in the resulting one.

In the Section 5 we discuss the obtained results and possibilities of further generalizations. Possible extensions of discussed system include supersymmetrization and quaternionic generalization as well as generalization of these systems in curved background.

II. SMORODINSKY-WINTERNITZ SYSTEM ON \( \mathbb{R}^N \)

Smorodinsky-Winternitz system is defined as a sum of \( N \) copies of one-dimensional singular oscillators \[(1)\], each of them defined by generators \( I_i \), which obviously form its Liouville integrals \( \{I_i, I_j\} = 0 \). Some fifty years ago it was noticed that this system possesses additional set of constants of motion given by the expressions \[1\]

\[
I_{ij} = L_{ij}L_{ji} - \frac{g_i^2 x_j^2}{x_i^2} - \frac{g_j^2 x_i^2}{x_j^2}, \quad \{I_{ij}, H\} = 0, \tag{3}
\]

where \( L_{ij} \) are the generators of \( SO(N) \) algebra,

\[
L_{ij} = p_i x_j - p_j x_i : \quad \{L_{ij}, L_{kl}\} = \delta_{ik} L_{jkl} + \delta_{jl} L_{ik} - \delta_{il} L_{jik} - \delta_{jk} L_{il}. \tag{4}
\]

The generators \( I_{ij} \) provides additional \( N - 1 \) functionally independent constants of motions and so this system is maximally superintegrable. These generators define highly nonlinear symmetry algebra,

\[
\{I_i, I_{jk}\} = \delta_{ij} S_{ik} - \delta_{ik} S_{ij}, \quad \{I_i, I_{kl}\} = \delta_{ik} T_{ijl} + \delta_{il} T_{jik} - \delta_{jl} T_{ik} - \delta_{jk} T_{iil} \tag{5}
\]

where

\[
S_{ij}^2 = -16(I_{ij}I_{ij} + I_{ij}^2 g_j^2 - I_{ij}^2 g_i^2 + \omega^2 I_{ij}^2 - g_i^2 g_j^2 \omega^2), \quad T_{ijk}^2 = -16(I_{ij}I_{jk}L_{ik} + g_k^2 I_{ij}^2 + g_j^2 I_{ik}^2 + g_i^2 I_{jk}^2 - 4g_i^2 g_j^2 g_k^2). \tag{6}
\]
The generators $S_{ij}^2$ and $T_{ijk}^2$ are of the sixth-order in momenta and antisymmetric over $i, j, k$ indices. The above symmetry algebra could be written in a compact form if we introduce the notation

$$M_{ij} = I_{ij}, \quad M_{0i} = I_i, \quad M_{ii} = g_i^2, \quad M_{00} = \frac{\omega^2}{4}, \quad R_{ijk} = T_{ijk}, \quad R_{ij0} = S_{ij}. \quad (7)$$

Then one can introduce capital letters which will take values from 0 to $N$. It is worth to mention that $M_{ij}$ is a symmetric, whereas $R_{ijk}$ is antisymmetric with respect to all indices. In this terms the whole symmetry algebra of Smorodinsky-Winternitz system reads

$$\{M_{ij}, M_{KL}\} = \delta_{JK} R_{IJK} + \delta_{IK} R_{JKL} - \delta_{IL} R_{IKJ} \quad (8)$$

where

$$R_{IJK}^2 = -16(M_{IJ} M_{JK} M_{IK} + M_{I,J}^2 M_{KK} + M_{I,K}^2 M_{JJ} + M_{K,L}^2 M_{II} - 4 M_{II} M_{JJ} M_{KK}) \quad (9)$$

One important fact should be mentioned, although in this algebra on the right side we have sum of many terms (square roots) only one term always survives, since in case of three indices are equal, the result is automatically 0. Consequently in this algebra we always have one square root on the right hand side. Quantum-mechanically the maximal superintegrability is reflected in the dependence of its energy spectrum on the single, “principal” quantum number only. Having in mind that in Cartesian coordinates the system decouples to the set of one-dimensional singular oscillators, we can immediately extract the expressions for its wavefunctions and spectrum from the standard textbooks on quantum mechanics, e.g., [17],

$$E_{n|\omega} = \hbar \omega \left(2n + 1 + \sum_{i=1}^{N} \sqrt{\frac{1}{4} + \frac{g_i^2}{\hbar^2}}\right), \quad \Psi = \prod_{i=1}^{N} \psi(x_i, n_i), \quad n = \sum_{i=1}^{N} n_i \quad (10)$$

where

$$\psi(x_i, n_i) = F\left(-n_i, 1 + \sqrt{\frac{1}{4} + \frac{g_i^2}{\hbar^2}}, \frac{\omega x_i^2}{\hbar}\right) \left(\frac{\omega x_i^2}{\hbar}\right)^{1+\frac{1+4g_i^2/\hbar^2}{2}} e^{-\frac{\omega x_i^2}{2\hbar}} \quad (11)$$

Here $F$ is the hypergeometric function. With these expressions at hands we are ready to study Smorodinsky-Winternitz system on complex Euclidean space in the presence of constant magnetic field.

### III. $C^N$-SMORODINSKY-WINTERNITZ SYSTEM

Now let us study $2N$-dimensional analog of Smorodinsky-Winternitz system interacting with constant magnetic field. It is defined by (2) and could be viewed as an analog of Smorodinsky-Winternitz system on complex Euclidian space $\mathbb{C}^N, ds^2 = \sum_{a=1}^{N} dz^a d\bar{z}^a$. Thus, we will refer it as $\mathbb{C}^N$-Smorodinsky-Winternitz system. The analog of SW-system which respects the inclusion of constant magnetic field is defined as follows,

$$\mathcal{H} = \sum_a I_a, \quad I_a = \pi_a \bar{\pi}_a + \frac{g_a^2}{z^a \bar{z}^a} + \frac{\omega^2}{2} z^a \bar{z}^a, \quad (12)$$

where $z^a, \pi_a$ are complex (phase space) variables with the following non-zero Poisson bracket relations

$$\{\pi_a, z^b\} = \delta_{ab}, \quad \{\bar{\pi}_a, \bar{z}^b\} = \delta_{ab}, \quad \{\pi_a, \bar{\pi}_b\} = iB \delta_{ab}. \quad (13)$$

For sure, it can be interpreted as a sum of $N$ two-dimensional singular oscillators interacting with constant magnetic field perpendicular to the plane. It is obvious that in addition to $N$ commuting constants of motion $I_a$ this system has another set of $N$ constants of motion defining manifest $(U(1))^N$ symmetries of the system

$$L_{a\bar{a}} = i(\pi_a \bar{z}^a - \bar{\pi}_a z^a) - Bz^a \bar{z}^a + {\mathcal{L}}_{a\bar{a}}, \quad \{L_{a\bar{a}}, \mathcal{H}\} = 0 \quad (14)$$

and supplementary, non-obvious, set of constants of motion defined in complete analogy with those of conventional Smorodinsky-Winternitz system:

$$I_{ab} = L_a \bar{L}_b - \left(\frac{g_a^2}{z^a \bar{z}^a} \frac{z^b}{\bar{z}^b} + \frac{g_b^2}{z^b \bar{z}^b} \frac{z^a}{\bar{z}^a}\right), \quad \{I_{ab}, \mathcal{H}\} = 0, \quad a \neq b \quad (15)$$
with $L_{a\bar{b}}$ being generators of $SU(N)$ algebra

$$L_{a\bar{b}} = i(\pi_a z^b - \bar{\pi}_b z^a) - B z^a z^b : \{L_{a\bar{b}}, L_{c\bar{d}}\} = i\delta_{ac}L_{\bar{d}b} - i\delta_{\bar{d}b}L_{ac}.$$  \hfill (16)

These symmetry generators, and $I_a$ obviously commute with $L_{a\bar{a}}$ due to manifest $U(1)^N$ symmetry

$$\{L_{a\bar{a}}, I_b\} = \{L_{a\bar{a}}, I_{b\bar{c}}\} = \{L_{a\bar{a}}, L_{b\bar{b}}\} = \{I_a, I_b\} = 0$$  \hfill (17)

The rest Poisson brackets between them are highly nontrivial

$$\{I_a, I_{b\bar{c}}\} = \delta_{ab}S_{ac} - \delta_{ac}S_{ab}, \quad \{I_{ab}, I_{cd}\} = \delta_{bc}T_{a\bar{d}} + \delta_{ac}T_{b\bar{d}} - \delta_{\bar{d}b}T_{a\bar{c}} - \delta_{\bar{c}a}T_{b\bar{d}},$$  \hfill (18)

where

$$S_{ab}^2 = 4I_{ab}I_aI_b - (L_{a\bar{a}}I_b + L_{b\bar{b}}I_a)^2 - 4g^2_{ab}I_a^2 - 4g^2_{a\bar{b}}I_a^2 - 4\omega^2I_{ab}(I_{a\bar{a}}L_{b\bar{b}}) + 4\omega^2g^2_{a\bar{a}}L^2_{a\bar{a}} + 4g^2_{a\bar{a}}\omega^2L^2_{a\bar{a}} + 16g^2_{a\bar{a}}g^2_{b\bar{b}}\omega^2$$

$$- 2B(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})(I_{a\bar{a}}I_b + L_{b\bar{b}}I_a) - B^2(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})^2 + 4B(g^2_{ab}I_aL_{a\bar{a}} + g^2_{a\bar{b}}I_bL_{b\bar{b}}) + 4B^2g^2_{a\bar{a}}g^2_{b\bar{b}}$$

$$T_{abc}^2 = 2(I_{ab} - L_{a\bar{a}}L_{b\bar{b}})(I_{bc} - L_{b\bar{b}}L_{c\bar{c}})(I_{ac} - L_{a\bar{a}}L_{c\bar{c}}) + 2I_{ab}I_{ac}I_{b\bar{c}} + L^2_{a\bar{a}}L^2_{b\bar{b}}L^2_{c\bar{c}} - (L^2_{b\bar{b}}L^2_{a\bar{a}} + L^2_{a\bar{a}}L^2_{c\bar{c}} + L^2_{b\bar{b}}L^2_{c\bar{c}})$$

$$- 4(g^2_{ab}I_{ab}(I_{a\bar{a}}L_{b\bar{b}}) + g^2_{ac}I_{bc}(I_{a\bar{c}}L_{b\bar{c}}) + g^2_{bc}I_{bc}(I_{a\bar{c}}L_{b\bar{c}})) + 4g^2_{a\bar{a}}g^2_{b\bar{b}}L^2_{a\bar{a}} + 4g^2_{a\bar{a}}g^2_{b\bar{b}}L^2_{a\bar{a}} + 4g^2_{a\bar{a}}g^2_{b\bar{b}}L^2_{a\bar{a}} + 16g^2_{a\bar{a}}g^2_{b\bar{b}}g^2_{c\bar{c}}.$$  \hfill (19)

To write the symmetry algebra in a simpler form we can redefine the generators

$$M_{a\bar{a}} = L^2_{a\bar{a}} + 4g^2_{a\bar{a}}, \quad M_{ab} = I_{ab} - \frac{1}{2}L_{a\bar{a}}L_{b\bar{b}}, \quad M_{a\bar{a}} = I_{a\bar{a}} - \frac{B}{2}L_{a\bar{a}}, \quad M_{b\bar{b}} = 4\omega^2 + B^2.$$  \hfill (21)

Since $L_{a\bar{a}}$ commute with all other generators Poisson brackets of $M$ will exactly coincide with the Poisson brackets of $I_{ab}$ and $I_a$. Similarly the $R$ tensor is defined as in the real case. So the algebra will have the following form

$$\{M_{a\bar{b}}, M_{c\bar{d}}\} = \delta_{bc}T_{a\bar{d}} + \delta_{ac}T_{b\bar{d}} - \delta_{\bar{d}b}T_{a\bar{c}} - \delta_{\bar{c}a}T_{b\bar{d}}, \quad \{M_{a\bar{a}}, M_{ab}\} = \delta_{ab}S_{ac} - \delta_{ac}S_{ab}.$$  \hfill (22)

where

$$S_{ab}^2 = 4M_{ab}M_{a\bar{a}}M_{b\bar{b}} + \left(\omega^2 + \frac{B^2}{4}\right)(M_{aa}M_{bb} - 4M_{ab}^2) - M_{a\bar{a}}^2M_{a\bar{a}} - M_{b\bar{b}}^2M_{b\bar{b}}.$$  \hfill (23)

$$T_{abc}^2 = 4M_{ab}M_{bc}M_{ac} - M_{a\bar{a}}^2M_{cc} - M_{a\bar{a}}^2M_{bb} - M_{b\bar{b}}^2M_{aa} + \frac{1}{4}M_{aa}M_{bb}M_{cc}.$$  \hfill (24)

Needless to say that $L_{a\bar{a}}$ commute with all the other constants of motion. Finally the full symmetry algebra then reads

$$\{M_{AB}, M_{CD}\} = \delta_{BC}R_{ABD} + \delta_{AC}R_{BCD} - \delta_{BD}R_{ACD} - \delta_{AD}R_{ABC}$$  \hfill (25)

where

$$R_{ABC}^2 = 4M_{AB}M_{BC}M_{AC} - M_{A\bar{A}}^2M_{CC} - M_{A\bar{A}}^2M_{BB} - M_{B\bar{B}}^2M_{AA} + \frac{1}{4}M_{AA}M_{BB}M_{CC}.$$  \hfill (26)

Again capital letters take values from 0 to $N$. In the complex case $R_{ABC}$ and $M_{AB}$ are again respectively antisymmetric and symmetric as in the real case. Up to multiplication by a constant this has the same form as the symmetry algebra for the real case.
Quantization

Quantization will be done using the fact that $C^N$-Smorodinsky-Winternitz system is a sum of two dimensional singular oscillators. This allows to write the wave function as a product of $N$ wave functions and total energy of the system as a sum of the energies of its subsystems. So the initial problem reduces to two-dimensional one.

$$\hat{I}_a \Psi_a(z_a, \bar{z}_a) = E_a \Psi_a(z_a, \bar{z}_a), \quad \hat{H} \Psi_{tot} = E_{tot} \Psi_{tot}, \quad \Psi_{tot} = \prod_{a=1}^{N} \Psi_a(z_a, \bar{z}_a), \quad E_{tot} = \sum_{a} E_a. \quad (27)$$

After this reduction, complex indices can be temporarily dropped. Now it is obvious to introduce the momenta system as a sum of the energies of its subsystems. So the initial problem reduces to two-dimensional one.

$$\hat{\pi} = -i(\hbar \partial + \frac{B}{2} \bar{z}), \quad \hat{\pi} = -i(\hbar \partial - \frac{B}{2} z) \quad [\pi, \bar{\pi}] = \hbar B, \quad [\pi, z] = -i\hbar \quad (28)$$

Schrödinger equation can be written down

$$\left[ -\hbar^2 \partial \partial + \left( \omega^2 + \frac{B^2}{4} \right) z \bar{z} - \hbar \frac{B}{2} (\bar{z} \partial - \partial z) + \frac{g^2}{\bar{z} z} \right] \Psi(z, \bar{z}) = E \Psi(z, \bar{z}). \quad (29)$$

Even in this two-dimensional system additional separation of variables can be done if one writes this system in a polar coordinates using the fact that $z = r e^{i\phi}$.

$$\left[ \frac{\partial^2}{\partial \phi^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{2}{\hbar^2} \left( E - \frac{\hbar^2 m^2}{2 r^2} - \frac{2 g^2}{r^2} - \frac{1}{2} \left( \omega^2 + \frac{B^2}{4} \right) r^2 - \frac{B m}{2} \right) \right] \Psi(r, \phi) = 0. \quad (30)$$

Further separation of variables can be done and one can use the fact that $L$ is a constant of motion.

$$\Psi(r, \phi) = R(r)\Phi(\phi), \quad \hat{L} \Phi = \hbar m \Phi. \quad (31)$$

Using the explicit form of the $U(1)$ generator, normalized solution can be written

$$\hat{L} = -i\hbar \frac{\partial}{\partial \phi}, \quad \Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{i m \phi}. \quad (32)$$

This result allows to write the equation (30) in the following form

$$\left[ \frac{d^2}{dt^2} + \frac{1}{r} \frac{d}{dr} + \frac{2}{\hbar^2} \left( E - \frac{\hbar^2 m^2}{2 r^2} - \frac{2 g^2}{r^2} - \frac{1}{2} \left( \omega^2 + \frac{B^2}{4} \right) r^2 - \frac{B m}{2} \right) \right] R(r) = 0. \quad (33)$$

Solution of this kind of Schrödinger equation can be written down. The final result for the wave functions of two-dimensional system and the energy spectrum are as follows

$$\psi(z, \bar{z}, n, m) = \frac{1}{\sqrt{2\pi}} (\sqrt{z / \bar{z}})^m F \left( -n, \sqrt{m^2 + \frac{4 g^2}{\hbar} + 1}, \frac{2 \sqrt{\omega^2 + \frac{B^2}{4} \bar{z} z}}{\hbar} \right) \left( \frac{2 \sqrt{\omega^2 + \frac{B^2}{4} \bar{z} z}}{\hbar} \right)^{1/2} e^{\sqrt{\omega^2 + \frac{4 g^2}{\hbar} - \bar{z} z}} \quad (34)$$

$$E = \hbar \sqrt{\omega^2 + \frac{B^2}{4} \left( 2n + 1 + \sqrt{m^2 + \frac{4 g^2}{\hbar}} \right) + \frac{B m}{2}}. \quad (35)$$

Finally the indices of $C^N$ can be recovered. The total wave function is a product of the wavefunctions and the total energy is the sum of the energies of two-dimensional subsystems

$$\Psi(z, \bar{z}) = \prod_{a=1}^{N} \psi(z_a, \bar{z}_a, n_a, m_a) \quad (36)$$

$$E_{tot} = \sum_{a=1}^{N} E_{n_a, m_a} = \hbar \sqrt{\omega^2 + \frac{B^2}{4} \left( 2n + 1 + \sum_{a=1}^{N} \sqrt{m_a^2 + \frac{4 g^2}{\hbar}} \right) + \frac{B}{2} \sum_{a=1}^{N} m_a} \quad \text{with} \quad n = \sum_{a=1}^{N} n_a \quad (37)$$

In contrast to the real case the energy spectrum of the $C^N$-Smorodinsky-Winternitz system depends on $N + 1$ quantum numbers.
IV. KUSTAAHIEMO-STIEFEL TRANSFORMATION

There is a well-known procedure reducing two-/four-/eight-dimensional oscillator to the two-/three-/five-dimensional Coulomb system. It is related with the Hopf maps and assumes the reduction by $\mathbb{Z}_2 / U(1) / SU(2)$--group action. In general case it results in the Coulomb like systems specified by the presence of $\mathbb{Z}_2$-/Dirac-/Yang-monopole\cite{13}. Since $\mathbb{C}^N$-Smorodinsky-Winternitz system has manifest $U(1)$ invariance, we could apply its respective reduction procedure related with first Hopf map $S^3/S^1 = S^2$, which is known as Kustaanheimo-Stiefel transformation, for the particular case of $N = 2$. Such a reduction was performed decade ago \cite{14} and was found to be resulted in the so-called “generalized MICZ-Kepler problem” suggested by Mardoyan a bit earlier \cite{15}. However the initial system was considered, it was not specified by the presence of constant magnetic field, furthermore, the symmetry algebra of the reduced system was not obtained there. Hence, it is at least deductive to perform Kustaanheimo-Stiefel transformation to the $\mathbb{C}^2$-Smorodinsky-Winternitz system with constant magnetic field in order to find its impact (appearing in the initial system) in the resulting one. Furthermore, it is natural way to find the constants of motion of the “generalized MICZ-Kepler system” and construct their algebra.

So, let us perform the reduction of $\mathbb{C}^2$-Smorodinsky-Winternitz system by the $U(1)$-group action given by the generator

\[
J_0 = L_{11} + L_{22} = i(z\pi - \bar{z}\bar{\pi}) - Bzz\bar{z}
\]

For this purpose we have to choose six independent functions of initial phase space variables which commute with that generators,

\[
q_k = z\sigma_k\bar{z}, \quad p_k = \frac{z\sigma_k\pi + \bar{z}\pi\sigma_k}{2z\bar{z}}, \quad k = 1, 2, 3
\]

where $\sigma_k$ are standard $2 \times 2$ Pauli matrices. Then, we calculate their Poisson brackets and fix the value of the $U(1)$-generator $J_0 = 2s$. As a result we get the reduced Poisson brackets

\[
\{q_k, q_l\} = 0, \quad \{p_k, q_l\} = \delta_{kl}, \quad \{p_k, p_l\} = s\epsilon_{klm} \frac{q_m}{|q|^2}
\]

Expressing the Hamiltonian via $q_l, p_l, J_0$ and fixing the value of the latter one, we get

\[
H_{SW} = 2|q|\left[\frac{p^2}{2} + \frac{s^2}{2|q|^2} + \frac{Bz}{2|q|} + \frac{1}{2} \left(\frac{B^2}{4} + \omega^2\right) + \frac{|q|^2}{|q||q| + q_3} + \frac{g^2_3}{|q||q| - q_3}\right]
\]

So, we reduced the $\mathbb{C}^2$-Smorodinsky-Winternitz Hamiltonian to the three-dimensional system. To get the Coulomb-like system we fix the energy surface or reduced Hamiltonian, $H_{SW} - E_{SW} = 0$ and divide it on $2|q|$. This yields the equation

\[
\mathcal{H}_{MICZ} - \mathcal{E} = 0, \quad \text{with} \quad \mathcal{E} \equiv -\frac{\omega^2 + B^2/4}{2}
\]

and

\[
\mathcal{H}_{gMICZ} = \frac{p^2}{2} + \frac{s^2}{2|q|^2} + \frac{g^2_1}{|q||q| + q_3} + \frac{g^2_2}{|q||q| - q_3} - \frac{\gamma}{|q|} \quad \text{with} \quad \gamma = \frac{E_{SW} - Bs}{2}
\]

The latter expression defines the Hamiltonian of “generalized MICZ-Kepler problem”. Hence, we transformed the energy surface of the reduced $\mathbb{C}^2$-Smorodinsky-Winternitz Hamiltonian to those of (three-dimensional) “Generalized MICZ-Kepler system”. Additionally it has an inverse square potential and this system has an interaction with a Dirac monopole magnetic field which affects the symplectic structure.

Surprisingly, the reduced system contains interaction with Dirac monopole field only, i.e. the constant magnetic field in the original system does not contribute in the reduced one.

Now this reduction can be done for constants of motion. Before doing that it is convenient to present the initial generators of $u(2)$ algebra given by (16) in the form

\[
J_0 = i(z\pi - \bar{z}\bar{\pi}) - Bzz\bar{z}, \quad J_k = \frac{i}{2}(z\sigma_k\pi - \bar{z}\pi\sigma_k) - \frac{Bz\sigma_k\bar{z}}{2} : \{J_0, J_i\} = 0, \quad \{J_i, J_j\} = \epsilon_{ijk}J_k.
\]

After reduction we get $J_0 = 2s$. The rest $su(2)$ generators after the reduction result in the generators of the $so(3)$ rotations of three-dimensional Euclidian space with the Dirac monopole placed in the beginning of Cartesian coordinate frame,

\[
J_k = \epsilon_{klm}p_lq_m - \frac{s}{|q|}q_k
\]
Then the symmetry generators for the “generalized MICZ-Kepler system” can be written down,
\[
\mathcal{I} = \frac{I_1 - I_2}{2} + \frac{B}{4}(L_{22} - L_{11}) = p_1J_2 - p_2J_1 + \frac{x_3\gamma}{r} + \frac{g^2(r - x_3)}{r(r + x_3)} - \frac{g^2(r + x_3)}{r(r - x_3)}.
\]
\[
\mathcal{L} = \frac{1}{2}(L_{22} - L_{11}) = J_3 = p_1q_2 - p_2q_1 - \frac{sg_3}{|g|}, \quad \mathcal{J} = I_{12} = J^2 + J^2 + \frac{g^2(r - q_3)}{r + q_3} + \frac{g^2(r + q_3)}{r - q_3}.
\]

It is important to notice that \(\mathcal{I}\) is a generalization of the \(z\)-component of the Runge-Lenz vector.

The relation of the initial system and the reduced one will allow to find the symmetry algebra of the final system using the previously obtained result for the complex Smorodinsky-Winternitz system. First of all the constants of motion in the initial system will also commute with the reduced Hamiltonian.

\[
\{\mathcal{H}_{gMICZ}, \mathcal{I}\} = \{\mathcal{H}_{gMICZ}, \mathcal{J}\} = \{\mathcal{H}_{gMICZ}, \mathcal{L}\} = 0
\]

Moreover since in the initial system \(L_{aa}\) generators commute with all the other constants of motion one can write.

\[
\{\mathcal{L}, \mathcal{J}\} = \{\mathcal{L}, \mathcal{I}\} = 0
\]

There is only one non-trivial commutator

\[
\{\mathcal{I}, \mathcal{J}\} = \{\mathcal{I}, \mathcal{L}\} = S
\]

\(S\) here coincides with \(S_{12}\) of \(C^2\)-Smorodinsky-Winternitz system and can be written using the generators of the reduced system.

\[
S^2 = 2\mathcal{H}_{gMICZ} \left[ 4\left(\mathcal{J} + \frac{1}{2}(\mathcal{L}^2 - s^2)\right)^2 - \left(4g^2 + (\mathcal{L} + s)^2\right)\left(4g^2 + (\mathcal{L} - s)^2\right)\right] - \left(4g^2 + (\mathcal{L} + s)^2\right)\left(\mathcal{I} + \gamma\right)^2 - \left(4g^2 + (\mathcal{L} - s)^2\right)\left(\mathcal{I} - \gamma\right)^2 - 4\left(\mathcal{J} + \frac{1}{2}(\mathcal{L}^2 - s^2)\right)\left(\mathcal{I} - \gamma\right)\left(\mathcal{I} + \gamma\right)
\]

There is a crucial fact that should be mentioned. Although the initial system had an interaction with magnetic field, after reduction we don’t have any dependence on \(B\) both in symplectic structure and in generators of the symmetry algebra, at least in classical level. In other words the reduced system does not feel the magnetic field of the initial system.

V. DISCUSSION AND OUTLOOK

In this paper we formulated, on the \(N\)-dimensional complex Euclidian space \(\mathbb{C}^N\) the analog Smorodinksy-Winternitz system interacting with a constant magnetic field. We found it has \(3N - 1\) functionally independent constants of motion and derived the symmetry algebra of this system. Quantization of these systems is also discussed. While for the real Smorodinsky-Winternitz system energy spectrum is totally degenerate and depends on single (“principal”) quantum number, the \(\mathbb{C}^N\)-Smorodinsky-Winternitz energy spectrum depends on \(N + 1\) quantum numbers. Then we performed Kustaanheimo-Stiefel transformation of the \(C^2\)-Smorodinsky-Winternitz system and reduced it to the so-called "generalized MICZ-Kepler problem”. We obtained the symmetry algebra of the latter system using the result obtained for the initial ones. Moreover we have shown that the presence of constant magnetic field in the initial problem does not affect the reduced system.

There are several generalizations one can perform for this system. Straightforward task is the construction of a quaternionic (\(\mathbb{H}^N\)) analog of this system. While complex structure allows to introduce constant magnetic field without violating the superintegrability, quaternionic structure should allow to introduce interaction with \(SU(2)\) instanton. It seems that one can also introduce the superintegrable analogs of the \(\mathbb{C}^N\)-/\(\mathbb{H}^N\)-Smorodinsky-Winternitz systems on the complex/quaternionic projective space \(\mathbb{C}P^N/\mathbb{H}P^N\), having in mind the existence of such generalization for the \(\mathbb{C}^N\)-/\(\mathbb{H}^N\) oscillator [16, 18]. We expect that the inclusion of a constant magnetic/instanton field does not cause any qualitative changes for this system. These generalizations will be discussed later on.
I am grateful to Armen Nersessian for his impact on this work, which includes suggestion of the problem and introduction to subject, numerous discussions and help in preparation of manuscript. Thanks to his great support and encouragement this work is finally completed. I am also thankful to Lusine Goroyan and Ruben Hasratyan for helping to fix mistakes.

This work was done within ICTP Affiliated Center programs AF-04 and Regional Doctoral Program on Theoretical and Experimental Particle Physics sponsored by Volkswagen Foundation. I also acknowledge partial financial support within research grant from the ANSEF-Armenian National Science and Education Fund based in New York, USA.

[1] I. Fris, V. Mandrosov, Ya. A. Smorodinsky, M. Uhlir, and P. Winternitz, On higher symmetries in quantum mechanics, Phys. Lett. 16 (1965) 354;
P. Winternitz, Ya. A. Smorodinsky, M. Uhlir, I. Fris, Symmetry groups in classical and quantum mechanics, Soviet J. Nuclear Phys. 4 (1967), 444;
A. A. Makarov, Ya. A. Smorodinsky, Kh. Valiev, and P. Winternitz, A systematic search for non-relativistic system with dynamical symmetries, Nuovo Cim. A 52 (1967) 1061.

[2] N. W. Evans, Superintegrability in classical mechanics, Phys. Rev. A 41 (1990) 5666-5676; Group theory of the Smorodinsky-Winternitz system. J. Math. Phys. 32 (1991) 3369-3375

[3] E. G. Kalnins, G. C. Williams, W. Miller Jr. and G. S. Pogosyan, Superintegrability in three-dimensional Euclidean space, J. Math. Phys. 40 (1999) 708-725

C. Grosche, G. S. Pogosyan, A. N. Sissakian, Path Integral Discussion for Smorodinsky-Winternitz Potentials: I. Two- and Three Dimensional Euclidean Space. Fortschr. der Physik 43 (1995) 53-521

[4] M. F. Hoque, I. Marquette and Y. Z. Zhang, Recurrence approach and higher rank cubic algebras for the N-dimensional superintegrable systems, J. Phys. A 49 (2016) no.12, 125201 [arXiv:1511.03331 [math-ph]]; Quadratic algebra structure and spectrum of a new superintegrable system in N-dimension, J. Phys. A 48 (2015) no.18, 185201.

[5] M. F. Hoque, Superintegrable systems, polynomial algebra structures and exact derivations of spectra arXiv:1802.08410 [math-ph]

[6] M. A. Olshanetsky and A. M. Perelomov, Classical integrable finite dimensional systems related to Lie algebras, Phys. Rept. 71 (1981) 313

[7] P. W. Higgs, Dynamical Symmetries in a Spherical Geometry, J J. Phys. A 12 (1979) 309.

H. I. Leemon, Dynamical Symmetries in a Spherical Geometry, J. Phys. A 12 (1979) 489.

[8] C. Grosche, G. S. Pogosyan, A. N. Sissakian, Path Integral Discussion for Smorodinsky-Winternitz Potentials: I. Two- and Three Dimensional Euclidean Sphere. Fortschr. der Physik 43 (1995) 523-563

[9] J. Harada and O. Yermolayaeva, Superintegrability, Lax matrices and separation of variables, CRM Proc. Lect. Notes 37 (2004) 65 [nlin/0309009 [nlin.SI]]. A. Galajinsky, A. Nersessian and A. Saghatelian, Superintegrable models related to near horizon extremal Myers-Perry black hole in arbitrary dimension, JHEP 1306 (2013) 002 [arXiv:1303.4901 [hep-th]].

[10] E. Rosochatius, Über die Bewegung eines Punktes, Doctoral dissertation, University of Göttingen, 1877.

[11] T. Hakobyan, O. Lechtenfeld and A. Nersessian, Superintegrability of generalized Calogero models with oscillator or Coulomb potential, Phys. Rev. D 90 (2014) no.10, 101701 [arXiv:1409.8288 [hep-th]].

F. Correa, T. Hakobyan, O. Lechtenfeld and A. Nersessian, Spherical Calogero model with oscillator/Coulomb potential: quantum case, Phys. Rev. D 93 (2016) no.12, 125009 [arXiv:1604.00027 [hep-th]]; Spherical Calogero model with oscillator/Coulomb potential: classical case, Phys. Rev. D 93 (2016) no.12, 125008 [arXiv:1604.00026 [hep-th]].

[12] T. Chakraborty, P. Pietilinen, Electron-electron interaction and the persistent current in a quantum ring Physical Review B 50 (1994) 8460-8468;

W.-C. Tan, J.C. Inkson, Electron states in a two-dimensional ring- an exactly soluble model Semiconductor. Sci. Technology, 11 (1996) 1635-1641;

J. Simonin, C. R. Proetto, Z. Barticevic, G. Fuster, Single-particle electronic spectra of quantum rings: A comparative study, Phys. Rev. B 70 (2004), 205305

[13] A. Nersessian, V. Ter-Antonian and M. M. Tsulaia, A Note on quantum Bohlin transformation, Mod. Phys. Lett. A 11 (1996) 1605 [hep-th/9604197].

A. Nersessian and V. Ter-Antonian, 'Charge dyon' system as the reduced oscillator, Mod. Phys. Lett. A 9 (1994) 2431 [hep-th/9406130]; Quantum oscillator and a bound system of two dyons, Mod. Phys. Lett. A 10 (1995) 2633 [hep-th/9508137];

L. G. Mardoyan, A. N. Sisakian and V. M. Ter-Antonian, 8-D oscillator as a hidden SU(2) monopole Phys. Atom. Nucl. 61 (1998) 1746 [hep-th/9712235].

[14] L. G. Mardoyan and M. G. Petrosyan, 4D singular oscillator and generalized MIC-Kepler system, Phys. Atom. Nucl. 70 (2007) 572 [quant-ph/0604127].

[15] L. Mardoyan, The Generalized MIC-Kepler system, J. Math. Phys. 44 (2003) 4981 [quant-ph/0306168]; Spheroidal analysis of the generalized MIC-Kepler system Phys. Atom. Nucl. 68 (2005) 1746 [quant-ph/0310143].
[16] S. Bellucci and A. Nersessian, \textit{(Super)oscillator on CP**N and constant magnetic field}, Phys. Rev. D \textbf{67} (2003) 065013 [hep-th/0211070];
S. Bellucci, A. Nersessian and A. Yeranyan \textit{Quantum oscillator on CP**N in a constant magnetic field} Phys.Rev. D70 (2004) 085013 [hep-th/0406184]

[17] L.D. Landau, L.M. Lifshitz \textit{Quantum Mechanics (Volume 3 of A Course of Theoretical Physics)} Pergamon Press 1965

[18] S. Bellucci, S. Krivonos, A. Nersessian and V. Yeghikyan, \textit{Isospin particle systems on quaternionic projective spaces}, Phys. Rev. D \textbf{87} (2013) no.4, 045005 [arXiv:1212.1663 [hep-th]].