Graph approach to entanglement generation by boson subtractions

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Entanglement is at the heart of quantum information science in the fundamental and practical aspects. A priority for studying and utilizing entanglement is to find reliable procedures to generate entangled states. In this work, we propose a graph method to systematically search for schemes that obtains genuine entanglement in arbitrary $N$-partite boson systems without postselection.

Our physical setup is based on the sculpting protocol, which converts the bosonic symmetrization into entanglement through an indeterministic $N$ boson subtraction operator. This protocol can be realized as heralded schemes of many-boson systems. We show that our graph picture of the sculpting protocol provides an organized strategy to find suitable sculpting protocols for various genuinely entangled states. We have found general schemes for qubit $N$-partite GHZ and W states which are much more efficient than former schemes with sculpting protocol. We also have found a qudit $N$-partite GHZ state generation scheme, which shows our approach provides a significantly powerful insight into finding simple solutions for complicated entangled states. As proof of concept that our theoretical schemes can be realized in many-boson systems, we propose a Bell state generation scheme in linear optical systems with polarization qubit encoding and heralded detections.

I. INTRODUCTION

Entanglement is an essential quantum feature that plays a role of the key ingredient in the EPR paradox \cite{EPR}, investigation of which led to new foundational insights about nature \cite{Bell}. From a more practical side, entanglement is known to be an important resource in quantum information processing, with potential applications in, among others, cryptography \cite{QKD} and computation \cite{QC}.

To study and utilize entanglement, it is a prerequisite to find reliable procedures to construct entangled quantum systems. One of the promising approaches to this task is to exploit the indistinguishability of quantum particles \cite{Wang2020}. Various researches suggested theoretical and experimental entanglement generation schemes based on the indistinguishability of particles and postselection. Along these lines, refs. \cite{Keesling2019,BN2019} showed that two spatially overlapped indistinguishable particles can carry bipartite entanglement. The quantitative relation of particle indistinguishability and spatial overlap to the bipartite entanglement was rigorously analyzed in Refs. \cite{Chir2020}. For the case of multipartite entanglement, schemes for GHZ and W states with identical particles have been theoretically suggested \cite{Chir2020} and experimented \cite{Peyronneau2021,Chir2022}. Ref. \cite{Chir2022} presented a comprehensive graph-theoretic approach to embrace the schemes to generate the entanglement of identical particles in linear quantum networks (LQNs) with postselection.

On the other hand, considering that the postselected schemes are highly sensitive to particle loss in circuits \cite{Pea2020,Bag2022} and the multipartite correlations can be created by the postselection bias \cite{Bag2022,Bag2023}, there have been several attempts to generate the entanglement of identical particles \textit{without postselection}. Specifically, heralded generation of entangled states of photons was studied for bipartite \cite{Polzik2000,Polzik2001} and multipartite systems \cite{Chir2020,Chir2022}. Unlike postselected schemes, heralded ones allow for sorting out the experimental runs in which the target state was generated prior to measuring it. While this property of heralded operations renders more tolerable schemes from photon loss \cite{Bag2022}, it is usually more challenging to find proper circuits to obtain the her-

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tanglement generation schemes for an arbitrary \(N\) to overcome the difficulty of obtaining heralded entanglement and the postselected ones \[38\].

In this work, we introduce a systematic method to overcome the difficulty of obtaining heralded entanglement generation schemes for an arbitrary \(N\)-partite system. Our method employs the sculpting protocol introduced in Ref. \[40\], which generates an \(N\)-partite entangled state by taking an \(N\) boson subtraction operator (which we name “sculpting operator”) to a 2\(N\) boson initial state. By setting the initial state to have the even distribution of the bosons in different 2\(N\) states (Fig. 1), the indeterministic sculpting operation generates the \(N\)-partite entanglement. And the variation of the sculpting operators result in different entangled states. Since LQNs with heralding detectors can realize boson subtraction operators \[11, 14\], we can design \(N\)-partite genuine entanglement generation schemes with this theoretical process.

In the sculpting protocol, the difficulty of designing a circuit for an \(N\)-partite entangled state is transformed to the difficulty of finding a suitable sculpting operator to chisel the state. However, former research on the sculpting protocol \[40, 45\] lack any systematic way of linking the features of sculpting operators to the expected final states. Our work shows that a graph picture of the sculpting protocol provides an organized strategy of finding proper sculpting operators for entanglement. We map multi-boson systems with sculpting operators into bipartite graphs (bigraphs), for which we develop techniques to understand key properties of the entanglement generation process. Our list of correspondence relations between sculpting protocols and graphs is a variation of that given in Ref. \[14\], which provided a systematic method to analyze and design LQNs for obtaining entanglement with postselection.

With our graph-theoretic approach, we present sculpting operators that generate qubit \(N\)-partite GHZ and W states, which are more efficient than those given in Ref. \[40\]. Particularly, our W state generation scheme needs 2\(N\) + 1 bosons and one sculpting operation, which is a major improvement from the scheme in Ref. \[40\] that needs 4\(N\) bosons and two steps of sculpting operations. To top it off, by generalizing the bigraph used to obtain qubit GHZ states, we also present a qudit \(N\)-partite GHZ state generation scheme. To our knowledge, there has been no scheme to generate such states without postselection. Our theoretical schemes can, in principle, be realized in any many-boson system, e.g., linear optical systems with polarization qubit encoding and heralded detections.

Our work is organized as follows: Sec. II reviews the sculpting protocol introduced in Ref. \[40\]. Sec. III explains our dictionary of mapping the sculpting protocol to bigraphs. We also show that the perfect matchings (PMs) of bigraphs determine the final state after the sculpting operation. Sec. IV gives sculpting operators that generate qubit \(N\)-partite GHZ and W states. Using the qubit GHZ generation graph, Sec. V presents a qudit GHZ state generation scheme. Sec. VI explains how linear optical systems with polarization qubit encoding and heralding detectors can build our sculpting schemes. To showcase our method, we present a simple Bell state generation example. Sec. VII summarizes the significance of our results and discusses possible follow-up researches.

II. SCULPTING PROTOCOL FOR QUBIT ENTANGLEMENT

In this section, we formalize the sculpting protocol \[40\] that converts the boson identity into entanglement. While \(N\)-partite entangled state was constructed in Ref. \[40\] based on 2\(N\) modes with the dual-rail qubit encoding, we re-explain it based on \(N\) spatial modes and consider the qubit state as a two-dimensional internal degree of freedom of bosons. This way of expression not only embraces the dual-rail encoding, but also provides a more intuitive description of qubit states in the system.

Since in our setup each boson in \(j\)th spatial mode \((j \in \{1, 2, \cdots, N\})\) has a two-dimensional internal degree of freedom \(s \in \{0, 1\}\), boson creation and annihilation operators are denoted as \(\hat{a}_{j, 0}^\dagger\) and \(\hat{a}_{j, s}\) respectively. Then, as an input state, we distribute 2\(N\) bosons into \(N\) spatial modes so that each mode has two bosons with orthogonal internal states (0 and 1, see Fig. 1). Therefore, the initial state is given by

\[
|\text{Sym}_N\rangle = \hat{a}_{1, 0}^\dagger \hat{a}_{1, 1}^\dagger \hat{a}_{2, 0}^\dagger \hat{a}_{2, 1}^\dagger \cdots \hat{a}_{N, 0}^\dagger \hat{a}_{N, 1}^\dagger |\text{vac}\rangle = \prod_{j=1}^N (\hat{a}_{j, 0}^\dagger \hat{a}_{j, 1}^\dagger) |\text{vac}\rangle. \tag{1}
\]

Following the former works \[40, 45\], we call it the maximally symmetric state of 2\(N\) bosons. Rewriting \(|\text{Sym}_N\rangle\) in the mode occupation representation as

\[
|\text{Sym}_N\rangle = |(1, 1), (1, 1), \cdots (1, 1)\rangle \tag{2}
\]

and the particle number distribution in the 2\(N\) states as a vector, we see that \(|\text{Sym}_N\rangle\) is majorized by all the other Fock states of 2\(N\) bosons. Several researches showed that this kind of state is very
pressed in the most general form as operator, which we name sculpting operator \( \hat{S} \) specifically in several bosonic experimental setups \([41–44]\).

An operation has been implemented indeterministically, so that the final state of order \( N - 2 \), then the final state is given by

\[
|\Psi\rangle_{\text{fin}} = \hat{O}_{N-2} \prod_{j \neq k} \hat{a}_{j,0}^\dagger \hat{a}_{j,1}^\dagger |\text{vac}\rangle + \cdots. \tag{6}
\]

We see that the first term on RHS has at least one mode with two bosons in it. Therefore, we need to adjust \( k_{j,s}^{(l)} \) so that \( \hat{A}_N \) excludes all the terms of the form \( \hat{a}_{k,s} \hat{a}_{k,l} \hat{O}_{N-2} \). We call such a restriction the no-bunching condition \([4]\).

For a later convenience, we rewrite \( \hat{A}^{(l)} \) as

\[
\hat{A}^{(l)}(l) = \sum_{j=1}^{N} (k_{j,0}^{(l)} \hat{a}_{j,0} + k_{j,1}^{(l)} \hat{a}_{j,1})
\]

\[
= \sum_{j=1}^{N} \alpha_j^{(l)} \hat{a}_{j,\psi_j^{(l)}}
\]

where \( \alpha_j^{(l)} \in \mathbb{C} \) with \( \sum_j |\alpha_j^{(l)}|^2 = 1 \) and \( |\psi_j^{(l)}\rangle \) a normalized qubit state.

All things considered, we summarize the sculpting protocol as follows:

**Sculpting protocol**

1. **Initial state:** We prepare the maximally symmetric state \( |\text{Sym}_N\rangle \) of \( 2N \) bosons, i.e., each boson has different states (either spatial or internal) with each other as Eq. (30) (see Fig. 1).

2. **Operation:** We apply the sculpting operator \( \hat{A}_N \) of the form (6) to the initial state \( |\text{Sym}_N\rangle \). The sculpting process must satisfy the no-bunching condition.

3. **Final state:** The final state can be fully separable, partially separable, or genuinely entangled.

Most of the technical difficulty to find \( \hat{A}_N \) for a specific entanglement state comes from Step 2, for it is critical to control the probability amplitudes so that the final state satisfies the no-bunching restriction. There have been no systematic technique.

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1 In the dual rail encoding setup \([40, 45]\), the no-bunching condition appears as a seemingly different form. In that setup, repetitive annihilations on the same mode naturally vanish. However, since two modes combine to constitute one subsystem for the case, valid final states are only restricted to those with one boson per two modes. This exactly corresponds to the no-bunching restriction in our setup.
III. GRAPH PICTURE OF BOSON SYSTEMS WITH SCULPTING OPERATORS

In this section, we present a list of correspondence relations between the fundamental elements of the sculpting protocol and those of graphs. With the mapping, we can replace key physical properties and restrictions on the sculpting operators with those on graphs, which renders a handy guideline to the operator-finding process for genuinely entangled states.

Ref. [23] proposed a list of correspondence relations of linear quantum networks (LQNs) to graphs for providing a systematic method to analyze and design networks for obtaining entanglement. Since our sculpting protocol also consists of linear transformations of boson annihilation operators, a similar graph mapping dictionary can be imposed to find a suitable \( A_N \) that generates genuine entanglement. Indeed, we can map indeterministic annihilation operators into graph elements with a variation of the correspondence relations in Ref. [23], which leads to a practical graph-theoretic method to analyze our system.

The correspondence relations of elements between bosonic systems with sculpting operators and bigraphs can be enumerated as follows:

| Boson systems with sculpting operator | Bipartite Graph \( G_b = (U \cup V, E) \) |
|--------------------------------------|-----------------------------------------------|
| Spatial modes                        | Labelled vertices \( \in U \)                |
| \( A^{(l)} (l \in \{1, 2, \cdots, N\}) \) | Unlabelled vertices \( \in V \)               |
| Superposed modes of \( A^{(l)} \)    | Edges between \( U \) and \( V \in E \)      |
| Probability amplitude \( \alpha_j^{(l)} \) | Edge weight \( \alpha_j^{(l)} \)            |
| Internal state \( \psi_j^{(l)} \)    | Edge weight \( \psi_j^{(l)} \)               |

In the above table, \( \alpha_j^{(l)} \) and \( \psi_j^{(l)} \) are defined as in Eq. (7). In our graph picture, an annihilation operator \( A^{(l)} = \sum_{j=1}^{N} \alpha^{(l)}_j a_{\beta, \psi_j^{(l)}} \) is drawn on an unlabelled vertex in \( V \) that is connected to labelled vertices \( j \) in \( U \). A brief glossary in graph theory can be found in Ref. [23], Appendix A.

As a proof of concept, we analyze the simplest \( N = 2 \) example with \( A_2 = \hat{A}^{(1)} \hat{A}^{(2)} \). The two spatial modes are denoted as labelled non-identical vertices \( (\in U) \) and the annihilation operators are denoted as unlabelled identical dots \( (\in V) \).

Let us write \( \hat{A}^{(1)} \) and \( \hat{A}^{(2)} \) as

\[
\hat{A}^{(1)} = \alpha_1 \hat{a}_{1, \psi_1} + \alpha_2 \hat{a}_{2, \psi_2}, \\
\hat{A}^{(2)} = \beta_1 \hat{a}_{1, \phi_1} + \beta_2 \hat{a}_{2, \phi_2}.
\]

Then \( \hat{A}^{(1)} \) applied to the system is mapped to a bipartite graph (bigraph)

Now, by taking \( \hat{A}^{(2)} \), the total sculpting operator \( \hat{A}_2 \) corresponds to

Note that the exchange of two identical vertices in \( V \) (dots) does not change the physics, i.e., \( A_2 \) can be also expressed as

This represents nothing but the commutation relation \([\hat{A}^{(1)}, \hat{A}^{(2)}] = 0\).

When \( \hat{A}_2 \) is expanded as

\[
\hat{A}_2 = \hat{A}^{(1)} \hat{A}^{(2)} = \alpha_1 \beta_1 \hat{a}_{1, \psi_1} \hat{a}_{1, \phi_1} + \alpha_1 \beta_2 \hat{a}_{1, \psi_1} \hat{a}_{2, \phi_2} + \alpha_2 \beta_1 \hat{a}_{2, \psi_2} \hat{a}_{1, \phi_1} + \alpha_2 \beta_2 \hat{a}_{2, \psi_2} \hat{a}_{2, \phi_2}.
\]
achieve such a sculpting operator by setting
are taken to
\[ \hat{A}_2 \]
that the first two bigraphs in (13) vanish when they
bunching condition, we must set the amplitudes so
\[ A \]
i.e., \( \hat{A}_2 \) is a superposition of the above four bigraphs.

For the sculpting operator \( \hat{A}_2 \) to obey the no-bunching condition, we must set the amplitudes so that the first two bigraphs in (13) vanish when they are taken to \( |\text{Sym}_2\rangle = a_{1,0}^\dagger a_{2,0}^\dagger a_{1,2}^\dagger a_{2,1}^\dagger |\text{vac}\rangle \). We can achieve such a sculpting operator by setting
\[ \alpha_j = \beta_j = \frac{1}{\sqrt{2}} \quad (j \in \{1, 2\}) \]
\[ |\psi_1\rangle = |\phi_2\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \equiv |+\rangle, \]
\[ |\psi_2\rangle = |\phi_1\rangle = \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \equiv |-\rangle. \quad (14) \]

Then it is direct to check that
\[ \hat{A}_2|\text{Sym}_{2,2}\rangle \]
\[ = \frac{1}{2}(\hat{a}_{1,+}^\dagger \hat{a}_{2,+}^\dagger + \hat{a}_{1,-}^\dagger \hat{a}_{2,-}^\dagger)|\text{vac}\rangle, \quad (15) \]
i.e., by fixing amplitudes as Eq. (14), we obtain a Bell state as the final state.

From the above \( N = 2 \) example, we can understand the role of the no-bunching condition in the graph picture. Since the bigraph expression of \( \hat{A}_N \) such as (10) is expanded with a summation of all the possible paths of annihilation operators as Eq. (13), we have to control the complex weights of the edges so that any bigraph with more than two edges in the same vertex does not contribute to the final state. This restriction on bigraphs results in the following useful property:

For a specific sculpting operator \( \hat{A}_N \), the final state \( |\Psi\rangle_{\text{fin}} = \hat{A}_N|\text{Sym}_N\rangle \) must be fully determined by the summation of the perfect matchings (PMs) of the sculpting bigraph corresponding to \( \hat{A}_N \).

Indeed, we see that the two bigraphs in Eq. (15) are the two perfect matchings of the sculpting graph (10). The above property plays an essential role in finding sculpting operators that generates entanglement for the following reasons:

First, for a given bigraph that corresponds to a sculpting operator, we can immediately read the possible final state from the PMs of the bigraph. For the \( N = 2 \) example, we can expect from (13) that \( \hat{A}_2 \) has the potential to generate the Bell state before fixing the amplitudes.

Second, we can apply all the PM diagram techniques developed in Ref. [23] to our system. Since the final states in both approaches correspond to the summation of PMs in a given bigraph, necessary conditions for a bigraph to carry genuine entanglement in LQNs (see Theorem 1 of Ref. [23]) are also valid to our protocol.

Third, in the same context as the second reason, we can consider bigraphs that generate entanglement in LQNs [23] as strong candidates for sculpting

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2 Here, we define a path as a possible connection of dots to vertices. In a path, each dot is uniquely connected to one vertex. For example, the \( N = 2 \) bigraph (10) has four paths as denoted in (13).
operators that generate entanglement in our protocol.

Based on these advantages, we can build a strategy to find sculpting operator for a genuinely entangled state [49].

Sculpting-operator-finding strategy

1. Write down all the states that consist of the entangled state that we want to generate. Draw the PMs that correspond to the states.

2. Draw a bigraph that has the above PMs. We choose a bigraph with minimal edges so that it has minimal paths.

3. Examine whether we can set the edge weights so that only PMs among the paths contribute to the final state.

4. If we can find such an edge weight solution, it corresponds to the sculpting operators that generates the entangled state we expect. If we cannot, we try other bigraph with the same PMs but more edges.

A bigraph that corresponds to a sculpting operator is called a sculpting bigraph from now on. In Step 2, we can use, e.g., a method suggested in Ref. [23], Sec. III.B to find bigraphs for a specific set of PMs. Note that Step 2 provides a significant benefit since reducing edges in bigraphs means reducing the number of possible no-bunching restrictions that we have to consider. Furthermore, a sculpting operator found in that way usually can be constructed more efficiently since a smaller number of edges implies a small amount of resource to create operator superpositions.

IV. QUBIT GHZ AND W STATES

In this section, we present sculpting operators that generate qubit $N$-partite GHZ and W states using the strategy to find sculpting operators presented in Sec. [III]. Our operator solutions are more efficient than those given in Ref. [40], especially for the W state case.

The following identities will be repeatedly used in this section:

\[
\hat{a}_{j,\pm}\hat{a}_{j,0}^{\dagger}\hat{a}_{j,1}^{\dagger} = \pm\hat{a}_{j,\pm},
\]

\[
\hat{a}_{j,\pm}\hat{a}_{j,0}\hat{a}_{j,1}^{\dagger} = 0, \quad \forall j \in \{1, 2, \cdots, N\}
\] (16)

where $|\pm\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

Also, since we will deal with only four qubit states \{$|0\rangle, |1\rangle, |+\rangle, |\rightarrow\rangle$\} in this section, the edge weight for the internal state is expressed directly as edge colors \{Black, Gray, Red, Blue\} respectively.

A. Qubit GHZ state

The sculpting bigraph that generates the $N$-partite GHZ state is given by

\[
\begin{array}{c}
1 \quad 1 \\
2 \quad 1 \\
3 \quad 1 \\
N \quad 1
\end{array}
\]

where the edge weights represent the probability amplitudes and edge colors Red and Blue represent the internal states $|+\rangle$ and $|-\rangle$. This bigraph was also used in Ref. [23] to obtain the GHZ state in LQNs (see bigraph (30) of Ref. [23]).

Since the above bigraph has two PMs

\[
\begin{array}{c}
1 \quad 1 \\
2 \quad 1 \\
3 \quad 1 \\
N \quad 1
\end{array}
\]

One can understand the edge number as the coherence number [50] of a quantum state, which is a coherence monotone that quantifies the amount of coherence in a quantum system. Therefore, we can consider in a general sense that a system corresponding to a bigraph with more edges needs more quantum resource.
and

\begin{equation}
\hat{A}_N = \frac{1}{\sqrt{2^N}} \left( \prod_{j=1}^{N} \hat{a}_{j,+} + \prod_{j=1}^{N} \hat{a}_{j,-} \right) \cdot \left( \prod_{l=1}^{N} \hat{a}_{l,-} + \hat{a}_{1,+} \right) \cdot \left( \prod_{l=1}^{N} \hat{a}_{l,+} - \hat{a}_{1,-} \right) \cdot \cdots
\end{equation}

\begin{equation}
= \frac{1}{\sqrt{2^N}} \prod_{j=1}^{N} (\hat{a}_{j,+} + \hat{a}_{j,N_{1,-}}),
\end{equation}

where \( \oplus_N \) in the last line is defined as the addition mod \( N \). In the expansion of the above operator, any term of the form \( \hat{a}_{j,-} \oplus_{j,N_{1,+}} \hat{O}_{N_{-2}} \) vanish when taken to \( |\text{Sym}_N\rangle \) by the second identity of Eq. (16). Therefore, the only terms that contribute to the final state are \( \prod_{j=1}^{N} \hat{a}_{j,+} \) and \( \prod_{j=1}^{N} \hat{a}_{j,-} \), which exactly correspond to the two PMs (18).

Then the final state by the sculpting bigraph (17) is given by

\begin{equation}
\hat{A}_N |\text{Sym}_N\rangle = \frac{1}{\sqrt{2^N}} \left( \prod_{j=1}^{N} \hat{a}_{j,+} + \prod_{j=1}^{N} \hat{a}_{j,-} \right) |\text{Sym}_N\rangle
\end{equation}

\begin{equation}
= \frac{1}{\sqrt{2^N}} \left( \prod_{j=1}^{N} \hat{a}_{j,+} + \prod_{j=1}^{N} \hat{a}_{j,+} \right) |\text{vac}\rangle
\end{equation}

\begin{equation}
= \frac{1}{\sqrt{2^{N-1}}} |\text{GHZ}_{N,2}\rangle,
\end{equation}

where the second equality is obtained by the first identity of Eq. (16). From the normalization factor, we directly see that the success probability becomes \( 1/2^{N-1} \).

Note that we can find other sculpting bigraphs for the GHZ state based on (17). While the GHZ state is invariant under the permutation of spatial modes, the bigraph (17) is not. Therefore, any bigraph with the permuted vertex labels of (17) also generates the GHZ state, i.e.,

\begin{equation}
\hat{A}_N = \frac{1}{\sqrt{2^N}} \prod_{l=1}^{N} \left( \sum_{j=1}^{N} \hat{a}_{j,0} + \sum_{j=1}^{N} e^{2\pi i (j-l)} \hat{a}_{j,1} \right).
\end{equation}

Most importantly, the consecutive subtractions in (23) do not remove particles from orthogonal modes. Hence they are very challenging to realize with experimental setups. In contrast, the procedure (20) is based on orthogonal modes, so that a single unitary change of basis is sufficient to prepare all the modes from which a single particle is to be removed. On top of that, each mode in (23) is a weighted superposition of all the initial ones. Understanding the operator from the graph picture, (23) corresponds to a bigraph with \( N^2 \) edges. On the other hand, (17) corresponds to a bigraph with only \( 2N \) edges. Therefore, the scheme described by (17) is more effective in the sense that each annihilation operator used there is constructed by superposing just two modes with internal state basis changes.
B. Qubit W state

A sculpting bigraph for \(N\)-partite W state can be conceived with one ancillary mode as

\[
\begin{array}{c}
A \\
\sqrt{N} \\
\alpha \\
\beta \\
\sqrt{N} \\
N \\
\sqrt{N} \\
3 \\
\sqrt{N} \\
2 \\
\sqrt{N} \\
1 \\
\end{array}
\]

we can see that it is a candidate to generate W state.

Let us directly check (24) indeed generates \(N\)-partite W state. The sculpting operator corresponding to (24) is given by

\[
\hat{A}_{N+1} = \left(\alpha \hat{a}_{1+} + \beta \hat{a}_{A0}\right)\left(\alpha \hat{a}_{2+} + \beta \hat{a}_{A0}\right) \cdots \left(\alpha \hat{a}_{N+} + \beta \hat{a}_{A0}\right) \times \frac{1}{\sqrt{N}} \left(\hat{a}_{1-} + \hat{a}_{2-} + \cdots + \hat{a}_{N-}\right)
\]

\[
= \frac{1}{\sqrt{N}} \left(\prod_{j=1}^{N} (\alpha \hat{a}_{j+} + \beta \hat{a}_{A0})\right) \sum_{k=1}^{N} \hat{a}_{k-}.
\]

We prepare the initial state as

\[
|\text{Sym}_{N+A}\rangle \equiv \left( \prod_{m=1}^{N} \hat{a}_{m0}^{\dagger} \hat{a}_{m1}^{\dagger}\right) \hat{a}_{A0}^{\dagger} |\text{vac}\rangle.
\]

By expanding Eq. (26) in the order of \(\hat{a}_{A0}\), it is direct to see that all the terms with \((\hat{a}_{A0})^2\) or higher order vanish when taken to \(|\text{Sym}_{N+A}\rangle\). And the terms without \(\hat{a}_{A0}\) are written as

\[
\prod_{j=1}^{N} \hat{a}_{j+} \sum_{k=1}^{N} \hat{a}_{k-},
\]

which also vanish when taken to \(|\text{Sym}_{N+A}\rangle\) by the second identity of Eq. (16). Therefore, the only surviving terms correspond to the \(N\) PMs (25).
Then, the final state is given by
\[
\hat{A}_{N+A}|\text{Sym}_{N+A}\rangle = \frac{\alpha^{N-1}\beta}{\sqrt{N}} (\hat{a}_{1-} \hat{a}_{2+} \cdots \hat{a}_{N+} + \hat{a}_{1+} \hat{a}_{2-} \cdots \hat{a}_{N-}) \\
+ \cdots + \hat{a}_{1+}\hat{a}_{2+} \cdots \hat{a}_{N-})|\text{vac}\rangle \\
\times \hat{a}_{A0}'\hat{a}_{A0}' \prod_{m=1}^{N} \hat{a}_{m0}'\hat{a}_{m1}'|\text{vac}\rangle \\
= \frac{\alpha^{N-1}\beta}{\sqrt{N}} (\hat{a}_{1-} \hat{a}_{2+} \cdots \hat{a}_{N+} + \hat{a}_{1+} \hat{a}_{2-} \cdots \hat{a}_{N+}) \\
+ \cdots + \hat{a}_{1+}\hat{a}_{2+} \cdots \hat{a}_{N-})|\text{vac}\rangle \\
= \alpha^{N-1}\beta |W_N\rangle.
\]
(29)

The success probability is \(|\alpha^{N-1}\beta|^2\), whose maximal value becomes \(\frac{(N-1)^{N-1}}{N^{N-1}}\) when \(|\alpha| = \sqrt{\frac{N-1}{N}}\) and \(|\beta| = \frac{1}{\sqrt{N}}\).

We can also check that this bigraph can be used to generate W state in LQNs with postselection. Indeed, by drawing a bigraph that corresponds to the schemes suggested in Ref. [17, 18], we obtain the same form of bigraph with (25).

Comparing with the W state generation scheme suggested in Ref. [10], we can easily see that our current scheme has accomplished an outstanding improvement. The scheme in Ref. [40] starts from 4N bosons in 2N modes and goes through two steps of sculpting to generate the final N-partite W state. On the other hand, using the graph mapping technique, we have obtained a much more efficient N-partite W-state generation scheme just with 2N + 1 bosons in N + 1 modes and one simple step of sculpting.

V. QUDIT GHZ STATE

Our graph picture also provides a useful insight to find sculpting operators for the general qudit systems. We will present in this section a sculpting bigraph for the qudit N-partite GHZ state, which has a generalized form of the qubit GHZ bigraph (33).

The qudit state is represented by a d-dimensional internal degree of freedom \(s \in \{0,1,\ldots,d\}\) of bosons. To construct N partite qudit genuinely entangled states, we initially distribute \(dN\) bosons into N modes so that exactly d bosons with mutually orthogonal internal states belong to a spatial mode (see Fig. 2). Hence, the initial state is given by
\[
|\text{Sym}_{N,d}\rangle \equiv \prod_{j=1}^{N} (\hat{a}_{j0} \hat{a}_{j1} \cdots \hat{a}_{j,d})|\text{vac}\rangle.
\]
(30)

FIG. 2: The initial state \(|\text{Sym}_{N,d}\rangle\) of \(dN\) bosons in \(N\) spatial modes. Each mode has \(d\) bosons, which have mutually orthogonal internal states \(|0\rangle, |1\rangle, \ldots, |d\rangle\).

Here \(|\text{Sym}_{N,d}\rangle\) denotes the \(N\) mode maximally symmetric state with a \(d\)-dimensional internal degree of freedom.

The sculpting operator
\[
\hat{A}_N = \prod_{l=1}^{(d-1)N} \hat{A}^{(l)}
\]
(31)
must be set to extract \((d - 1)\) bosons per mode so that one boson per mode in the final state determines the qudit state of each subsystem. All in all, the sculpting protocol is modified for qudits as follows:

**Sculpting protocol of qudits**

1. Initial state: We prepare the maximally symmetric state \(|\text{Sym}_{N,d}\rangle\) of \(dN\) bosons, i.e., each boson has different states (either spatial or internal) with each other as Eq. (30). See Fig. 2.

2. Operation: We apply the sculpting operator \(\hat{A}_N\) to the initial state \(|\text{Sym}_{N,d}\rangle\). The sculpting operator must be set to extract \((d - 1)\) bosons per mode.

3. Final state: The final state can be fully separable, partially separable, or genuinely entangled.

Now we provide a sculpting operator that generates the N-partite GHZ state of \(d\)-level systems, denoted as \(|\text{GHZ}_{N,d}\rangle\), by generalizing the qubit GHZ sculpting operator in Sec. IV A.

First, by generalizing the \(d = 2\) basis set \(|+\rangle, |-\rangle\) for the internal states of the sculpting operators, we choose the arbitrary \(d\)-dimensional basis set \(|\tilde{k}\rangle\) for \(k = 0, \ldots, (d-1)\) where
\[
|\tilde{k}\rangle = \frac{1}{\sqrt{d}} \left( |0\rangle + \omega^k|1\rangle + \omega^{2k}|2\rangle + \cdots + \omega^{(d-1)k}|d-1\rangle \right)
\]
(32)
\((\omega = e^{i \frac{2\pi}{d}})\) for the internal states of the sculpting operators.
Second, we use an overlap of \((d - 1)\) copies of the graph \((17)\) for the sculpting bigraph, i.e., the following bigraph corresponds to the sculpting operator for the GHZ state:

![Graph](image)

where a gray circle represents a group of \((d - 1)\) identical vertices that have the same edges. For example, when \(N = 3\) and \(d = 4\), the above graph is explicitly drawn as

![Graph](image)

The sculpting operator that corresponds to the bigraph \((33)\) is given by

\[
\hat{A}_{N,d} = \left( \frac{1}{\sqrt{2}} \right)^{(d-1)N} (\hat{a}_{1,\tilde{0}} - \hat{a}_{2,\tilde{d}-1})^{d-1} (\hat{a}_{2,\tilde{0}} - \hat{a}_{3,\tilde{d}-1})^{d-1} \times (\hat{a}_{3,\tilde{0}} - \hat{a}_{4,\tilde{d}-1})^{d-1} \ldots \times (\hat{a}_{N,\tilde{0}} - \hat{a}_{1,\tilde{d}-1})^{d-1}.
\]

(35)

We verify that the above operator constructs the qudit \(N\)-partite GHZ state in Appendix A. To catch the sense of how the graph \((33)\) works, we explicitly explain the qutrit case \((d = 3)\) here.

### A. Qutrit GHZ state

For a qutrit system, the basis set \((32)\) is given by

\[
\begin{align*}
\{|\tilde{0}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \\
\{|\tilde{1}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + e^{i\frac{2\pi}{3}}|1\rangle + e^{i\frac{4\pi}{3}}|2\rangle), \\
\{|\tilde{2}\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + e^{i\frac{4\pi}{3}}|1\rangle + e^{i\frac{2\pi}{3}}|2\rangle). \\
\end{align*}
\]

(36)

Then, we directly check that the following identities hold:

\[
\begin{align*}
(\hat{a}_{j,\tilde{0}})^2\hat{a}_{j,0}\hat{a}_{j,1}\hat{a}_{j,2} &= \frac{2}{\sqrt{3}}\hat{a}_{j,0}^4, \\
\hat{a}_{j,\tilde{0}}\hat{a}_{j,\tilde{2}}\hat{a}_{j,0}\hat{a}_{j,2} &= -\frac{1}{\sqrt{3}}\hat{a}_{j,1}^4, \\
(\hat{a}_{j,\tilde{2}})^2\hat{a}_{j,0}\hat{a}_{j,1}\hat{a}_{j,2} &= \frac{2}{\sqrt{3}}\hat{a}_{j,2}^4.
\end{align*}
\]

(37)

(note that the above identities are also obtained from Eq. \((A1)\) with \(d = 3\)). From the second identity of the above, we can see that

\[
(\hat{a}_{j,\tilde{0}})^2\hat{a}_{j,\tilde{2}}\hat{a}_{j,0}\hat{a}_{j,1}\hat{a}_{j,2} = \hat{a}_{j,\tilde{0}}(\hat{a}_{j,\tilde{2}})^2\hat{a}_{j,0}\hat{a}_{j,1}\hat{a}_{j,2} = 0
\]

(38)

also holds. The graph \((33)\) for \(d = 3\) is now drawn as

![Graph](image)

(39)
Then, the final state is given by
\[ \hat{A}_{N,3}|\text{Sym}_{N,3}\rangle = \left( \frac{1}{\sqrt{2}} \right)^N \left( \prod_{r=1}^N \hat{a}_{r,\bar{0}}^{\dagger} \hat{a}_{s,\bar{2}}^{\dagger} \right) \left( \prod_{t=1}^N \hat{a}_{t,\bar{2}}^2 \right) \left( \prod_{p=1}^N \hat{a}_{p,0}^{\dagger} \hat{a}_{p,1}^{\dagger} \right) |\text{vac}\rangle \]

The second line is obtained by Eq. (38) and the third by Eq. (37). The success probability is \( 1/3^{N-1} \).

VI. HERALDED SCHEME OF SCULPTING PROTOCOLS IN LINEAR OPTICS: BELL STATE EXAMPLE

We can realize the sculpting protocol in various multi-boson systems. There are general schemes in optics \cite{41,42} and trapped-ions \cite{51} to establish indeterministic annihilation operators of bosons. Based on such methods, Ref. \cite{40} proposed an optical scheme for constructing sculpting operators. More recently, Ref. \cite{45} suggested an experimental scheme with arithmetic annihilations of trapped ions to generate the GHZ state with the sculpting operator \( (23) \) in Ref. \cite{40}. Both works are based on the dual-rail qubit encoding.

Here, we propose an alternative optical scheme to execute the sculpting protocol with polarization qubit encoding and heralding detections. In this setup, we set the internal boson states \( \{ |0\rangle, |1\rangle \} \) as the polarization of photons \( \{ |H\rangle, |V\rangle \} \), with
\[
|+\rangle \rightarrow |D\rangle = \frac{1}{\sqrt{2}} (|H\rangle + |V\rangle), \quad
|\rangle \rightarrow |A\rangle = \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle). \tag{42}
\]

And the initial state is given by
\[ |\text{Sym}_N\rangle = \prod_{j=1}^N \hat{a}_{j,H}^{\dagger} \hat{a}_{j,V}^{\dagger}. \tag{43} \]

All our theoretical schemes for qubit entanglement in this work sculpt out photons of internal states \( \{ |+, |\rangle \} \) (see sculpting bigraphs \( (17) \) and \( (24) \)). We can realize such annihilations by rotating the polarization of photons to 90° and send photons indeterministically to heralding detectors. We attach a heralding detector per mode to sort out the cases when the extraction fits in with the sculpting operator.

As the simplest nontrivial example, we present a linear optical circuit to generate the Bell state. The
optical elements used in the circuit can be applied to more general multipartite entangled states. Our circuit design is based on the sculpting bigraph [10] by fixing edge weights as in Eq. (14). Fig. 3 shows our experimental scheme. The initial state is prepared as

$$\hat{a}_H^\dagger \hat{a}_V^\dagger \hat{b}_H^\dagger \hat{b}_V^\dagger |\text{vac}\rangle,$$

where $\hat{a}_H^\dagger$ and $\hat{b}_H^\dagger$ are creation operators for two spatial modes. To take the sculpting operator

$$\hat{A}_2 = \frac{1}{2}(a_D + \hat{b}_A)(\hat{a}_A + \hat{b}_D),$$

we rotate the initial state into the $D/A$ direction with half-wave plates (HWP) and send them through polarizing beam splitters (PBS). We place PBSs so that the photon path follows the sculpting bigraph [10]. By selecting only the cases when each detector detects a single photon, we generate Bell states as expected. Appendix [13] explains the photon state evolution in detail.

Comparing with the Bell state scheme with dual rail encoding designed in Ref. [40], we can state that our current scheme is experimentally much more feasible. The scheme in Ref. [40] needs high transmittivity beam splitters for sculpting photons and uses several phase rotators which are difficult to control. On the other hand, our circuit only uses HWPs and PBSs to construct photon annihilation operators.

VII. DISCUSSIONS

We have proposed an organized strategy to find entanglement generation schemes based on heralding. It proceeds in two steps: first, we find a theoretical sculpting operator that generates an entangled state. Second, we construct a concrete experimental circuit for such a sculpting operator. In the process of finding sculpting operators, we have exploited graph techniques by imposing the correspondence relations of bosonic systems to bigraphs. We have shown that the graph picture of bosonic systems facilitates a powerful tool to find proper sculpting operators. The theoretical schemes can be realized in any many-boson systems. As the simplest example, we have presented a Bell state circuit in linear optics with polarized photons, which can be generalized to other schemes that we have proposed.

Our work suggests two research directions: First, as a next step towards the realization of our theoretical schemes, it is essential to design experimental circuits of qubit GHZ and W states with polarized photons based on the optical elements given in Sec. VI. While most of former works on generating genuine entangled states remain in tripartite cases [20, 25], optical circuits based on our theoretical scheme will be generalized to an arbitrary $N$-partite case. For the case of qudit GHZ state, one can encode qudit states with the orbital angular momentum (OAM) of photons [52]. Second, our sculpting protocol can generate other interesting multipartite entangled states. Since our graph approach provides a handy guideline to coming up with useful sculpting operators, we expect that it could be used to find heralded schemes for other entangled states, such as graph states [53], $N$-particle $N$-level singlet states [54], and cyclically symmetric states [55].

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**Appendix A: Qudit GHZ state**

To show that $\hat{S}_{GHZ,d}|\text{Sym}_{N,d}\rangle$ constructs the GHZ state, we use the following identities:

\[
\begin{align*}
(\hat{a}_0)^l(\hat{a}_{d-1})^{d-1-l} \prod_{s=0}^{d-1} \hat{a}_s^\dagger &= (-1)^{(d-1-l)} \frac{l!(d-1-l)!}{\sqrt{d^{d-2}}} \hat{a}_{d-1-l}^\dagger \\
\prod_{s=0}^{d-1} \hat{a}_s^\dagger &= 0
\end{align*}
\]

(A1)

In the above equations, we can check that the second identity is directly obtained by taking $\hat{a}_0$ or $\hat{a}_{d-1}$ to the first identity.
FIG. 4: Linear optical circuit for generating the Bell state by 5 steps

Then the sculpting operator \( \hat{S}_{\text{GHZ}} \) is expanded as

\[
\hat{S}_{\text{GHZ}} = \left( \frac{1}{\sqrt{2}} \right)^{(d-1)N} (\hat{a}_{1,0} - \hat{a}_{2,0} - \hat{a}_{3,0} - \cdots - \hat{a}_{N,0})^{d-1} \cdots (\hat{a}_{1,d-1} - \hat{a}_{2,d-1} - \cdots - \hat{a}_{N,d-1})^{d-1} = \left( \frac{1}{\sqrt{2}} \right)^{(d-1)N} \sum_{l_1=0}^{d-1} \left( \begin{array}{c} d-1 \nonumber \end{array} \right) (\hat{a}_{1,0})^{l_1} (\hat{a}_{2,0})^{d-1-l_1} \cdots \left( \sum_{l_N=0}^{d-1} \left( \begin{array}{c} d-1 \nonumber \end{array} \right) (\hat{a}_{N,0})^{l_N} (\hat{a}_{0,0})^{d-1-l_N} \right). \tag{A2}
\]

Then, using identities (A1), we have

\[
\hat{S}_{\text{GHZ}} \mid_{\text{Sym}_d} = \left( \frac{1}{\sqrt{2}} \right)^{(d-1)N} \sum_{l_1=0}^{d-1} \left( \begin{array}{c} d-1 \nonumber \end{array} \right) (\hat{a}_{1,0})^{l_1} (\hat{a}_{2,0})^{d-1-l_1} \cdots \left( \sum_{l_N=0}^{d-1} \left( \begin{array}{c} d-1 \nonumber \end{array} \right) (\hat{a}_{N,0})^{l_N} (\hat{a}_{0,0})^{d-1-l_N} \right) = \left( \frac{(d-1)!}{\sqrt{2^{d-1} \cdot d^{d-2}}} \right)^N (|\tilde{0}, \tilde{0}, \cdots, \tilde{0}\rangle + \cdots + |\tilde{d-1}, \tilde{d-1}, \cdots, \tilde{d-1}\rangle), \tag{A3}
\]

i.e., the qudit GHZ state in the quantum Fourier transformed basis.

Appendix B: Bell state generation in optics

In this section, we explain the photon state evolution of our experimental scheme presented in Sec. VI. The state evolution consists of 5 steps as denoted in Fig. 4:

- **Step 1**

\[
\hat{a}^\dagger_H \hat{a}^\dagger_V \hat{b}^\dagger_H \hat{b}^\dagger_V \mid \text{vac} \rangle \rightarrow \hat{a}^\dagger_D \hat{a}^\dagger_A \hat{b}^\dagger_D \hat{b}^\dagger_A \mid \text{vac} \rangle \tag{B1}
\]

- **Step 2**

\[
\frac{1}{4}(\hat{a}^\dagger_H + \hat{a}^\dagger_{2V})(\hat{a}^\dagger_{1H} - \hat{a}^\dagger_{2V})(\hat{b}^\dagger_H + \hat{b}^\dagger_{2V})(\hat{b}^\dagger_{1H} - \hat{b}^\dagger_{2V}) \mid \text{vac} \rangle \tag{B2}
\]

- **Step 3**

\[
\frac{1}{4}(\hat{a}^\dagger_{1D} + \hat{a}^\dagger_{2A})(\hat{a}^\dagger_{1D} - \hat{a}^\dagger_{2A})(\hat{b}^\dagger_{1D} + \hat{b}^\dagger_{2A})(\hat{b}^\dagger_{1D} - \hat{b}^\dagger_{2A}) \mid \text{vac} \rangle \tag{B3}
\]
• Step 4

\[
\frac{1}{16} \left( \left( \hat{a}_{1H}^\dagger + \hat{a}_{2V}^\dagger \right) \left( \left( \hat{a}_{1H}^\dagger + \hat{a}_{2V}^\dagger \right) - \left( \hat{b}_{2H}^\dagger - \hat{b}_{1V}^\dagger \right) \right) \right. \\
\times \left. \left( \left( \hat{b}_{1H}^\dagger + \hat{b}_{2V}^\dagger \right) + \left( \hat{a}_{2H}^\dagger - \hat{a}_{1V}^\dagger \right) \right) \left( \left( \hat{b}_{1H}^\dagger + \hat{b}_{2V}^\dagger \right) - \left( \hat{a}_{2H}^\dagger - \hat{a}_{1V}^\dagger \right) \right) \right) |\text{vac}\rangle \\
= \frac{1}{16} \left( \hat{a}_{1H}^\dagger^2 + 2 \hat{a}_{1H}^\dagger \hat{a}_{2V}^\dagger + \hat{a}_{2V}^\dagger^2 - \hat{b}_{2H}^\dagger^2 + 2 \hat{b}_{2H}^\dagger \hat{b}_{1V}^\dagger - \hat{b}_{1V}^\dagger^2 \right) |\text{vac}\rangle \\
\times \left( \hat{b}_{1H}^\dagger^2 + 2 \hat{b}_{1H}^\dagger \hat{b}_{2V}^\dagger + \hat{b}_{2V}^\dagger^2 - \hat{a}_{2H}^\dagger^2 + 2 \hat{a}_{2H}^\dagger \hat{a}_{1V}^\dagger - \hat{a}_{1V}^\dagger^2 \right) |\text{vac}\rangle \right)^{\dagger} \tag{B4}
\]

• Step 5

\[
\frac{1}{4} \left( \left( \hat{a}_{1H}^\dagger \hat{b}_{1H}^\dagger + \hat{a}_{1V}^\dagger \hat{b}_{1V}^\dagger \right) \left( \hat{a}_{2H}^\dagger \hat{b}_{2H}^\dagger + \hat{a}_{2V}^\dagger \hat{b}_{2V}^\dagger \right) + \left( \hat{a}_{1H}^\dagger \hat{b}_{1V}^\dagger + \hat{a}_{1V}^\dagger \hat{b}_{1H}^\dagger \right) \left( \hat{a}_{2H}^\dagger \hat{b}_{2H}^\dagger - \hat{a}_{2V}^\dagger \hat{b}_{2V}^\dagger \right) \right) |\text{vac}\rangle \tag{B5}
\]

At Step 5, we sort out the states that click one of the two detectors in each PBS. When the two heralding detectors receive photons of the same polarization, we obtain a Bell state \(\frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)\). When the two heralding detectors receive photons of the opposite polarization, we obtain a different Bell state \(\frac{1}{\sqrt{2}}(|HH\rangle - |VV\rangle)\).