Quantum effects near future singularities

John D. Barrow\textsuperscript{a},\textsuperscript{1}\hspace{1em}Antônio B. Batista\textsuperscript{b},\textsuperscript{2}\hspace{1em}Júlio C. Fabris\textsuperscript{b},\textsuperscript{3}\hspace{1em}Mahouton J.S. Houndjo\textsuperscript{c},\textsuperscript{4}\hspace{1em}and\hspace{1em}Giuseppe Dito\textsuperscript{d},\textsuperscript{e},\textsuperscript{5}

\textsuperscript{a} DAMTP, Centre for Mathematical Sciences, University of Cambridge, UK
\textsuperscript{b} Departamento de Física, Universidade Federal do Espírito Santo, ES, Brazil
\textsuperscript{c} Instituto de Física, Universidade Federal da Bahia, Ba, Brazil
\textsuperscript{d} Instituto de Matemática, Universidade Federal da Bahia, Ba, Brazil
\textsuperscript{e} Institut de Mathématiques de Bourgogne, Université de Bourgogne, Dijon, France

General relativity allows a variety of future singularities to occur in the evolution of the universe. At these future singularities, the universe will end in a singular state after a finite proper time and geometrical invariants of the space time will diverge. One question that naturally arises with respect to these cosmological scenarios is the following: can quantum effects lead to the avoidance of these future singularities? We analyze this problem considering massless and conformally coupled scalar fields in an isotropic and homogeneous background leading to future singularities. It is shown that near strong, big rip-type singularities, with violation of the energy conditions, the quantum effects are very important, while near some milder classes of singularity like the sudden singularity, which preserve the energy conditions, quantum effects are irrelevant.

PACS number: 98.80-k

1 Introduction

It is general believed that today the Universe is in a stage of accelerated expansion. The primary evidence for this accelerated phase of cosmic evolution came from the use of the supernova type Ia as standard candles to measure distances in the universe. Supernova are very bright objects that can be seen to great distances. Presently, measurements of supernova type Ia up to $z \sim 1.8$

\begin{itemize}
  \item [1] \textsuperscript{1}E-mail: jdb34@damtp.cam.ac.uk
  \item [2] \textsuperscript{2}E-mail: abralib918@gmail
  \item [3] \textsuperscript{3}E-mail: fabris@pq.cnpq.br
  \item [4] \textsuperscript{4}E-mail: sthoundjo@yahoo.fr
  \item [5] \textsuperscript{5}E-mail: giuseppe.dito@u-bourgogne.fr
\end{itemize}
are available. The fact that these very distant supernova appear dimmer than would be expected in a pure matter-dominated universe led to the conclusion that the universe is accelerating. Much speculation has arisen as to the source of this acceleration and even its reality has been questioned. But, the fact that the spectrum of anisotropy of the cosmic microwave background radiation indicates that the spatial section of the geometry of the universe must be almost flat, while the observations of the virialized system (galaxies, clusters of galaxies, etc.) indicate a low density of the universe, implies indirectly the acceleration of the universe. In fact, in order to complete the cosmic energy budget and have flat spatial sections, it is necessary to include a component that does not agglomerate locally, but remains as a smooth component of universe. To have this feature, this component must have negative pressure, and consequently it must dominate the matter content of the universe asymptotically, driving the accelerated expansion in the later phases of the cosmic evolution.

In brief, to explain the acceleration of the universe, an exotic component in the cosmic budget exhibiting negative pressure is needed. This exotic component is named dark energy. We must remember that observations require also a second non-baryonic component, called dark matter, with zero pressure, necessary to explain conveniently the formation of structures in the universe and the dynamics of local, virialized system, like galaxies and clusters of galaxies. However, while there exists a lot of reasonable candidates to represent dark matter (neutralinos, axions, sterile neutrinos, primordial black holes etc – for a review, see reference [1]), it is not clear what kind of fluid or field would constitute dark energy. The first natural candidate to be evoked has been the cosmological constant, seen as a phenomenological manifestation of the vacuum energy of quantum fields existing in the universe. However, this very attractive possibility is plagued with fine-tunings and poorly understood issues, (see the classical references [2]). In spite of this, the cosmological constant remains the most popular candidate to represent dark energy, leading to the so-called ΛCDM model, highly supported by observations. A simple extension of general relativity, created by restricting the variational principle for deriving the Einstein equations to causal variations, leads to a general prediction that there is a cosmological constant with the observed value and a prediction that the value of the dimensionless curvature will be observed to be -0.0055 [3].

The cosmological constant implies an equation of state such that \( \alpha_x = p_x/\rho_x = -1 \), where \( \rho_x \) is the density and \( p_x \) is the pressure (the subscript \( x \) denotes a dark energy component). Observations will never give an exactly precise result for the equation of state of dark energy (or any component of the universe), due to statistical and systematic errors. But, if these dispersions in the evaluation of the dark energy equation of state can be reduced substantially, a clear case for the simple cosmological constant can emerge. We are far from this situation. But, in the search for the determination of \( \alpha_x \) some curious results have appeared: while the estimations lead to values near \( \alpha_x = -1 \), more negative values are highly admitted. This fact has raised a dramatic speculation: could dark energy have an equation of state such that \( \alpha_x < -1 \)? In fact, for example, if \( \alpha_x \) is constant and the spatial sections of the universe is flat, the
recent results of 7-years WMAP observations indicate that $\alpha_x = -1.10 \pm 0.14$ at 1$\sigma$.

If $\alpha_x < -1$, all energy conditions are violated, in particular the null energy condition, which requires $p_x + p_x \geq 0$. From the conservation of the energy-momentum tensor in an expanding universe, we obtain the equation,

$$\dot{\rho}_x + 3\frac{\dot{a}}{a}(\rho_x + p_x) = 0,$$

where $a$ is the scale factor of the universe, and dots mean derivatives with respect to the cosmic time. From this expression, it comes out that the violation of the null energy condition implies that the energy density of the dark energy fluid grows with the expansion of the universe, instead of decreasing. A remarkable consequence of such behaviour is that any universe dominated by such an exotic fluid will inevitably hit a singularity in its future evolution, after a finite proper time. At this future singularity, the expansion scale factor $a$ and the energy density $\rho_x$ both diverge. This highly singular state in the future evolution of the universe has been called *big rip singularity* [5], and it is a remarkable and plausible example of what are called *future cosmological singularities*.

However, the situation is more complicated, and future singularities can exist even if the energy conditions are not violated. The possibility of future singularities was pointed out for the first time in reference [6] and their occurrence does not necessarily require the violation of the energy conditions. There is also a later discussion in reference [7]. A recent example of future singularities which does not violate the null energy condition is the 'big brake' singularity, which emerges from the DBI action [8, 9]. The big brake singularity has the curious property that it can be traversed by a pointlike particle [10]. A milder type of finite time singularity is the so-called "*sudden singularity*" [11]. The sudden future singularity occurs without violating any energy condition (so $\rho + p \geq 0$ and $\rho + 3p \geq 0$ at all times). This singularity is characterized by a finite value for the scale factor, its first time derivative, and the density, while the second derivative of the scale factor and the pressure diverge at finite time [11]. They are singularities of the weak sort discussed by Tipler [12] and Krolak [13]. They cannot occur if $dp/d\rho$ is continuous.

In general, it is believed that the fate of the universe near any singularity (past or future) must be affected by quantum effects arising in the extreme conditions that exist in its spacetime neighborhood. In the case of the *big rip*, this problem has been treated for example, in references [14] [15] [16]. In these investigations, it was found that the quantum effects are important. But, the conclusions concerning the back reaction of the quantum effects on the evolution of the universe were harder to decide unambiguously. In the case of the *sudden* singularity, quantum effects were studied in references [17] [18] [19], and the results indicated that quantum effects do not change the evolution near the singularity. These results were obtained for massive and massless scalar fields.

Here, the fate of quantum effects near future singularities is reviewed. We begin, in next section, by describing the some general class of future singularities. In section 3 the evolution of a scalar field in the background of *big rip* and *sudden*
future singularities is determined. In section 4, the structure of the background solution is shown, and the solution for the scalar field equation is found. In section 5, quantum effects near future singularities are studied. In section 6 we present our conclusions.

2 Future singularities

Generally, a singularity in geometric theories of gravitation can be characterized by the divergence of some curvature invariants or, alternatively, by the incompleteness of the geodesic trajectories. Cosmology is commonly studied using a homogeneous and isotropic space-time. In this case, all dynamics is encoded in the behavior of a single function, the scale factor $a(t)$, determined by some matter density $\rho(t)$ endowed with a pressure $p(t)$. In this case, singularities appear as a divergence in the Hubble factor $H = \frac{\dot{a}}{a}$ and in the density $\rho(t)$. A traditional example is the initial big bang singularity, for which $H(t) \to \infty$ and $\rho(t) \to \infty$ as $t \to 0$. Of course, at the big bang singularity the curvature scalars diverge and the geodesics "begin" in the singularity, implying the the space-time is geodesically incomplete.

The future singularities which occur at the end of a cosmological evolution, have in principle many of the features as the big bang singularity, including the divergence of the curvature invariants, which are displayed by many types of singularity. However, geodesic incompleteness is not a universal characteristic of these singularities and nor is a divergence in the Hubble parameter or in the density. The big rip singularity, which requires the violation of all energy conditions, bears a close resemblance to the big bang singularity, since, besides the divergence of curvature invariants, it is incomplete geodesically, and the energy density and the Hubble parameter both diverge. This is direct consequence of the violation of the null energy condition. As already remarked in the introduction, since $\rho + p$ is negative when the null energy condition is violated, the energy density grows as the universe expands, in contrast to the usual situation. As consequence, $R \to \infty$, $H \to \infty$ and $\rho \to \infty$ after a finite time $t_s$. Moreover, the geodesics terminate at this singularity, and are inextendible.

The case of the big rip singularity can be described in a simple way by assuming a fluid (named from now on phantom fluid) with an equation of state of the type

$$p = \alpha \rho, \quad \alpha < -1.$$  \hspace{1cm} (2)

Solving the conservation equation, the energy density scales as

$$\rho \propto a^{-3(1+\alpha)}. \hspace{1cm} (3)$$

Since, $\alpha < -1$, $\rho$ grows as $a$ increases.

This singularity occurs in a finite proper time. This can be seen more simply in the following way. In terms of the cosmic time, the scale factor behaves as

$$a \propto t^\frac{2}{3(1+\alpha)}. \hspace{1cm} (4)$$
Since $1 + \alpha$ is negative, this solution can represent an expanding universe if $-\infty < t < 0$. It must be remarked that at $t \to -\infty$ there is no singularity, as we will see below. If we live in a moment $t_0 < 0$, the time to elapse from $t = t_0$ to $t = 0$ is obviously finite. If there is any other form of "normal" matter, the phantom fluid will always dominate asymptotically, since the density of normal forms of matter decreases with the expansion. In reference [4] a simple model including pressureless matter and phantom fluid has been described. On the other hand, since $R \propto \frac{1}{t^2}$, $\rho \propto \frac{1}{t^2}$ and $H \propto \frac{1}{t}$, all these quantities diverge as $t \to 0$. It is in this sense that the big rip singularity can be considered as the reverse of the big bang singularity, with a Minkowskian asymptotic space-time as $a \to 0$ and a singular state as $a \to \infty$.

There are other classes of mild future singularities. We understand a "mild singularity" to be a singularity that exhibits a divergence in the curvature invariants (which is a requirement), but perhaps with no divergence in the density. If the density does not diverge then Einstein’s equations imply that the Hubble function also does not diverge at the singularity. At same time, since density is connected with the Hubble function, it must reach a finite value at singularity. However, the Ricci scalar is given by

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right].$$

In order to have a divergence in $R$, we see that $\ddot{a}$ must diverge. Through Einstein’s equations, this implies that the pressure must diverge. This curious structure may allow that the geodesics to be continued through the singularity.

One example of these class of "mild future singularities" is the sudden singularity, which can be described by the following expression for the scale factor [11, 20, 21]:

$$a(t) = \left(\frac{t}{t_s}\right)^q (a_s - 1) + 1 - \left(1 - \frac{t}{t_s}\right)^n,$$

where $t_s$ is the time where the sudden singularity occurs, and $a_s$ is the value of the scale factor at this moment. Moreover, $0 < q \leq 1$ and $1 < n < 2$ where $q$ and $n$ are free constants and no specific relation is assumed between the pressure $p$ and the density $\rho$.

The sudden singularity does not require any violation of the energy conditions. Pressure can remain positive. The only requirement is the divergence of pressure in a given finite time. One difficult with such scenario is to find a type of matter that could satisfy these requirements. While this seems to be difficult using ordinary types of fluids, non-standard self-interacting scalar fields may fulfill the requirements needed for a sudden singularity scenario.

In fact, a self-interacting scalar field which presents non-standard coupling structures may lead to new and unexpected scenarios. One example is given by the DBI action,

$$\mathcal{L} = \sqrt{-g}V(T)\sqrt{1 - T \rho T'},$$

where $T$ is a tachyonic scalar field, and $V(T)$ is a potential term. This action can emerge from some specific configuration of string theory. The choice of
the potential \( V(T) \) may lead to very new kind of cosmological scenario. One example has been given in [7], where a new kind of future singularity has been exhibited: the *big brake singularity*. In this singularity, the Hubble parameter vanishes on the singularity, while the second derivative of the scale factor goes to \(-\infty\), from which the name *big brake*. Geodesics can also traverse the *big brake* singularity.

From now on, we will concentrate our analysis on the *big rip* and *sudden* singularities.

### 3 The master equation

We will consider the quantum creation of particles near a given future cosmological singularity. Two types of future singularity will be analyzed: the *big rip* singularity and the *sudden* singularity. Let us initially consider a general massive, non-minimally coupled scalar field \( \phi \), giving by the following Lagrangian:

\[
L = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} m^2 \phi^2 + \frac{\xi}{2} R \phi^2.
\]  

The conformal coupling corresponds to \( \xi = \frac{1}{6} \). From the Lagrangian (8) we can deduce the following field equation:

\[
\Box \phi + m^2 \phi - \xi R \phi = 0.
\]  

For a flat FLRW metric, this equation reduces to

\[
\phi'' + \frac{2}{a} \phi' + \left\{ k^2 + m^2 a^2 + 6 \xi \left( \frac{a''}{a} \right) \right\} \phi = 0.
\]  

In general, at future singularities the second derivative of the scale factor diverges, while the first one remains finite. Hence, it is convenient to transform equation (10) in order to eliminate this possibly singular term. This can be achieved by defining \( \phi = a^{-6\xi} \chi \). This transformation leads to the equation

\[
\chi'' + 2 \left( 1 - 6\xi \right) \frac{a'}{a} \chi' + \left\{ k^2 + m^2 a^2 + 6 \xi \left( \frac{a''}{a} \right) \right\} \chi = 0.
\]  

The equation (11) simplifies considerably for a conformal coupling, \( \xi = \frac{1}{6} \). If, a massless field is also considered, equation (11) takes the form of an harmonic oscillator equation.

Particle production near the *big rip* has been analyzed the references [13, 14, 15], considering only massless scalar particle, while the corresponding analysis for the *sudden* singularity has been performed in reference [16]. In reference [17] the massive case with conformal coupling in the background of the *sudden* singularity has been studied. We will review the results obtained in these references.
later. We must remark that the non-minimal coupling should strengthen any quantum effects near the singularity, and also introduces some technical features that leads to exact solutions for the problem.

Under variations with respect to the metric, the Lagrangian (8) gives the following momentum-energy tensor:

\[ T_{\mu\nu} = (1 - 2\xi)\phi_{,\mu}\phi_{,\nu} + \left(2\xi - \frac{1}{2}\right)g_{\mu\nu}\phi_{,\rho}\phi^{,\rho} + \frac{1}{2}m^2g_{\mu\nu}\phi^2 + \xi G_{\mu\nu}\phi^2 - 2\xi\phi\left(\phi_{,\mu,\nu} - g_{\mu\nu}\Box\phi\right), \]  

\[ (12) \]

where

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R \]  

\[ (13) \]
is the Einstein tensor. For a conformal coupling \((\xi = \frac{1}{6})\), this expression reduces to,

\[ T_{\mu\nu} = \frac{2}{3}\phi_{,\mu}\phi_{,\nu} - \frac{1}{6}g_{\mu\nu}\phi_{,\rho}\phi^{,\rho} + \frac{1}{2}m^2g_{\mu\nu}\phi^2 - \frac{1}{3}\phi\left(\phi_{,\mu,\nu} - g_{\mu\nu}\Box\phi\right) + \frac{1}{6}G_{\mu\nu}\phi^2. \]

\[ (14) \]

In the case of minimally coupled massive field \((\xi = 0)\), the energy-momentum tensor reduces to

\[ T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\rho}\phi^{,\rho} + \frac{1}{2}m^2g_{\mu\nu}\phi^2. \]

\[ (15) \]

From now on, we will consider two main cases: \(m \neq 0\) and \(\xi = \frac{1}{6}\) for the massive conformally coupled scalar field, and \(m = 0\) and \(\xi = 0\), for the massless minimally coupled scalar field.

4 The cosmological background and the solutions of the master equation

The first task in analyzing the quantization of the scalar field equations is to determine the background where the quantum field evolves. As stated above, two cases will be considered: the big rip and the sudden singularity.

4.1 The big rip background

Let us consider the flat Friedmann equation and the conservation law for a fluid with an equation of state \(p = \alpha\rho\):

\[ \left(\frac{a'}{a}\right)^2 = \frac{8\pi G}{3}\rho a^2, \]

\[ (16) \]

\[ \rho' + 3(1 + \alpha)\rho = 0. \]

\[ (17) \]
The general solution for the scale factor is given by \( a = a_0 |\eta|^{\frac{2}{1 + 3 \alpha}} \) \((\eta \text{ is the conformal time defined by } dt = a(\eta) d\eta)\), while the density behaves as \( \rho = \rho_0 a^{-3(1 + \alpha)} \). If \( \alpha < -\frac{1}{3} \) then accelerated expansion occurs. Moreover, if \( \alpha < -1 \) (phantom regime) then the matter density grows during this accelerated expansion, leading to a big rip. One of the main aspects of these accelerating solutions is that the conformal time takes values in the interval \(-\infty > \eta > 0\). Note that for \( \eta \to -\infty \) the scale factor goes to zero, and the Minkowski space-time is asymptotically approached in the phantom regime, since \( \rho \to 0 \) in this regime. This property will be important in order to fix the initial conditions.

4.2 **Sudden singularity: two cosmological eras**

For the sudden singularity, the situation is more complex due to the complicated form of the scale factor. But, the scale factor (6) admits two asymptotic forms, which we will call the primordial phase and the singular phase, respectively.

- **Primordial phase, \( t \to 0 \):**
  \[
  a \to \left( \frac{t}{t_s} \right)^q (a_s - 1),
  \]
  \[
  \dot{a} \to \frac{q}{t_s} \left( \frac{t}{t_s} \right)^{q-1} (a_s - 1),
  \]
  \[
  \ddot{a} \to \frac{q}{t_s^2} (q-1) \left( \frac{t}{t_s} \right)^{q-2} (a_s - 1).
  \]

- **Singular phase, \( t \to t_s \):**
  \[
  a \to a_s,
  \]
  \[
  \dot{a} \to \frac{q}{t_s} (a_s - 1),
  \]
  \[
  \ddot{a} \to -\frac{n}{t_s^2} (n-1) \left( 1 - \frac{t}{t_s} \right)^{n-2}.
  \]

There will be a radiation-dominated primordial phase if \( q = 1/2 \). On the other hand, in the singular phase, the scale factor and its first derivative approach constant values, and the second derivative, \( \ddot{a} \), diverges as \( t \to t_s \), since \( n < 2 \).

In terms of the conformal time, \( d\eta = a^{-1} dt \), we have for the scale factor evolution to leading order:

- **Radiation-dominated primordial phase:**
  \[
  a = a_0 \eta.
  \]

- **Singular phase:**
  \[
  a = a_s.
  \]
The scale factor and its first derivative must be continuous during the transition from one phase to the other. If $a_s$ is the scale factor value at the moment of the transition, and $H_0$ the corresponding Hubble parameter value, then the transition moment is given by $\eta_t = 1/(H_0 a_s)$ and $a_0 = H_0 a_s^2$.

The isotropic and homogeneous form we have assumed for the cosmological evolution of the scale factor, $a(t)$, towards a sudden singularity captures the essential features of the general solution of the Einstein equations near such a singularity. For some more general scenarios, including anisotropies, see refs [21, 22, 23, 24, 25, 26].

We are ready now to determine the solution for the master equations for some relevant configurations.

4.3 **Big rip: massless, minimally coupled case**

When $\xi = m = 0$ it is more practical to solve directly the equation for the field $\phi$. The Klein-Gordon equation assumes the form,

$$\phi'' + 2\frac{a'}{a}\phi' + k^2\phi = 0,$$

where from now on $k$ denotes the wave number related to the Fourier decomposition of the spatial dependence of the scalar field. When the scale factor is given by $a = a_0|\eta|^{-\frac{2}{1+3\alpha}}$, representing the big rip scenario, the solution is:

$$\phi_k(\eta, \vec{x}) = c_1 \eta^\nu H^{(1)}_\nu(k\eta)e^{i\vec{k} \cdot \vec{x}},$$

where $c_1$ is an integration constant, independent of $k$, $\nu = -\frac{3(1-\alpha)}{2(1+3\alpha)}$ and $H^{(1)}_\nu(x)$ is the Hankel function of the first kind of order $\nu$. This form of the final solution, which should contain in principle two independent functions and coefficients with dependence on $k$, is dictated by the imposition of an initial Bunch-Davies vacuum state [27].

4.4 **Sudden singularity: massless, minimally coupled case**

Let us return to the Klein-Gordon equation (26). For the sudden singularity, we have previously considered two phases, a radiative phase and a singular phase. The initial conditions, corresponding to the Bunch-Davies vacuum state, are imposed at the beginning of the radiative phase, where the solution of the Klein-Gordon equation may be expresses in the form of plane waves:

$$\phi_k(\eta) = c_1 e^{ik\eta},$$

But, the main interested is in the behavior of the quantum field near the singularity. In this phase, the solution of equation (26) reads,

$$\phi(\eta, \vec{x}) = e^{-\eta} \left\{ A_+ e^{i(\omega_\eta - \vec{k} \cdot \vec{x})} + A_- e^{-i(\omega_\eta + \vec{k} \cdot \vec{x})} \right\},$$

where $c_1$ is an integration constant, independent of $k$, and $\nu = -\frac{3(1-\alpha)}{2(1+3\alpha)}$. This form of the final solution, which should contain in principle two independent functions and coefficients with dependence on $k$, is dictated by the imposition of an initial Bunch-Davies vacuum state [27].
where $\omega = \sqrt{k^2 - 1}$ and the constants $A_\pm$ are fixed by the matching conditions across the two phases:

$$A_\pm = \frac{1}{2\omega} \sqrt{\frac{1}{2k}} e^{i(k+\omega)+1}(\omega \pm k).$$  \\ (30)

### 4.5 Sudden singularity: massive, conformally coupled case

If the mass is non-zero, and $\xi = \frac{1}{6}$, the resulting Klein-Gordon equation takes the following form during the singular phase:

$$\chi'' + (k^2 + m^2a^2)\chi = 0.$$  \\ (31)

where $\chi = a\phi$. During the radiative phase it is reasonable to consider the massless approximation, since we are in the regime where $a \to 0$. During the singular phase, the equation is

$$\chi'' + (k^2 + m^2a_0^2\eta^2)\chi = 0.$$  \\ (32)

which again has plane wave solutions with a modified frequency. Hence, the solutions during the two phases can be written as follows:

$$\phi_k(\eta) = \frac{e^{ik\eta}}{\sqrt{2k}} \quad \text{(primordial phase)},$$

$$\phi_k(\eta) = \xi_{01}e^{i\tilde{\omega}\eta} + \xi_{02}e^{-i\tilde{\omega}\eta} \quad \text{(singular phase)},$$  \\ (33)

where $\xi_{01,02}$ are constants, to be fixed by the matching conditions, and $\tilde{\omega} = \sqrt{k^2 + m^2a_0^2}$.

### 5 The energy of the created particles and the regularization procedure

The $0-0$ component of the energy-momentum tensor gives the energy density of a given configuration. The solutions shown in the previous section give the expression for the scalar field (massive or massless) for a given mode $k$. Inserting this expression in the energy-momentum tensor, the corresponding $0-0$ component gives the energy associated with this mode. The total energy is obtained by integrating over all modes. In general, this integration leads to a divergent quantity, a common problem in quantum field theory. Hence, in order to make sense of the energy associated with the field, a renormalisation procedure must be employed. The renormalisation can be interpreted as a redefinition of the fundamental constants present in the original problem. In order for this procedure to be physically meaningful, it is essential that the final result does not depend on how the renormalisation is performed.

Hence, our problem here is to compute the expression

$$\rho = \int \rho_k d^3k,$$  \\ (35)
with \( \rho_k = T_0^0 \). More precisely, we are interested in the final expression for the renormalized energy, \( \rho^{ren} \). It is this expression that will give the energy associated with any particles created by the quantum processes near the singularity, and which may be relevant to the computation of any associated back-reaction phenomena in the evolution of the universe.

In order to regularize the expression of the energy, we use the \( n \)-wave method exposed in the reference [28]. This method is based on the Pauli-Villars technique used for quantum field theory in Minkowski space-time. First, let us write the energy as,

\[
\rho = \int_0^\infty \rho_k(k,m)k^2 \, dk. \tag{36}
\]

Let us define,

\[
\rho_k^{(n)} = \frac{1}{n} \rho_k(nk, nm), \tag{37}
\]

where \( n \) is a parameter that characterizes the order of the divergence. From this expression we construct the quantities,

\[
E_k^p = \lim_{n \to \infty} \frac{\partial^p \rho_k^{(n)}}{\partial(n^2)^p}. \tag{38}
\]

The expression for the regularised energy is given by,

\[
\rho_{k}^{ren} = \rho_k - E_k^0 - E_k^1 - \frac{1}{2} E_k^2, \tag{39}
\]

where \( E_k^0 \) eliminates the logarithmic divergence, \( E_k^1 \) the quadratic divergence, and \( E_k^2 \) the quartic divergence – all those that are normally present in the energy-momentum tensor. This regularization of the energy corresponds to a full renormalisation of the coupling constants, as described in [29, 30].

The goal now is to determine the final expression for the regularized energy-momentum tensor, especially for the energy density that corresponds to the component \( 0 - 0 \) of this tensor.

### 5.1 Big rip: the massless, minimally coupled case

Using the solution (27) and the expression for the energy given by the \( 0 - 0 \) component of the energy momentum tensor [18] with \( m = 0 \), we obtain,

\[
\rho_k = A \eta^{\gamma - 3} x^2 \left[ H_{\nu - 1}^{(1)}(x) H_{\nu - 1}^{(2)}(x) \right. \\
\left. + H_{\nu}^{(1)}(x) H_{\nu}^{(2)}(x) \right]. \tag{40}
\]

The integration on all \( k \) modes reveals the existence of logarithmic, quadratic and quartic divergences. Using the \( n \)-wave regularization scheme, we write

\[
\rho^{ren} = \int_0^\infty x^2 (\rho_k - E_k^0 - E_k^1) \, dx \\
- \int_{1/\sigma}^\infty x^2 E_k^{log} \, dx. \tag{41}
\]
The last term is a modification of usual method in order to include the logarithmic divergence in the massless case, see [13]. The result is:

\[ \rho^{\text{ren}} = \bar{A}_1 \eta^{\gamma-3} I_1 = \bar{A}_1 \eta^{\frac{12(1+\alpha)}{1+3\alpha}} I_1, \]

where \( I_1 \) is a number. This expression describes how the energy of the created particles evolves with time.

Taking the ratio between created particles and the background phantom fluid \( \rho_x \), we find

\[ \frac{\rho^{\text{ren}}}{\rho_x} \propto \eta^{-6\left(\frac{1+\alpha}{1+3\alpha}\right)}, \quad (42) \]

which diverges as the singularity at \( \eta = 0^- \) is approached. This may indicate that the evolution of the universe is modified by quantum effects and the singularity is avoided. But, in order to verify such effect, a full back-reaction analysis is necessary. Similar studies for the big rip singularity have been made in the context of quantum cosmologies, with inconclusive results, see references [31, 32, 33, 34].

### 5.2 Sudden singularity for the massless, minimally coupled case

Let us turn now to the sudden singularity case. As in the previous analysis for the big rip, we consider a massless, minimally coupled scalar field. In contrast to the big rip analysis above, an approximation considering the two phases is now necessary, and the initial vacuum condition is imposed in the first phase, which is that of a radiation-dominated universe. Using the solutions for the Klein-Gordon for the two phase given above, we find that the total energy is given by the following integral:

\[ \rho = \bar{A}e^y \int_0^\infty dk \frac{k}{\omega^2} \left\{ (2k^2 - 1)k^2 - \cos \omega y + \omega \sin \omega y \right\}, \quad (43) \]

where \( y = 2(1 - \eta) \) and \( \omega = \sqrt{k^2 - 1} \). The background constants are fixed such the singularity occurs at \( \eta = 1, y = 0 \).

Employing the \( n \)-wave regularization scheme, the regularized energy can be determined [15]:

\[ \rho^{\text{ren}} = \bar{A}e^y \left\{ \chi(-y) + \frac{\cosh y}{y} \right\}, \quad (44) \]

where \( \chi \) denotes the hyperbolic cosine integral function. The regularized energy decreases as the singularity is approached \( (y \to 0) \). Hence, the quantum effects are ineffective in preventing the singularity, at least for the massless scalar field case.
5.3 **Sudden** singularity for the massive, conformal coupling

In this case, as for the previous one, two phases were considered. The initial vacuum state is fixed during the first phase, and the energy is computed during the second phase, in order to evaluate the possibility of a back-reaction effect on the evolution of the universe.

Using the solutions (33,34), the matching conditions and the expression for the energy-momentum tensor (14) we obtain for the energy of the $k$ mode:

$$\rho_k = \frac{k}{4} \left( 1 - \frac{k}{\tilde{\omega}} \right).$$

(45)

An integration over all $k$-modes gives,

$$\rho = \int_0^\infty \rho_k d^3k = \pi \int_0^\infty k^2 \tilde{\omega} \left( 1 - \frac{k}{\tilde{\omega}} \right)^2 dk.$$ 

(46)

This expression clearly diverges so it is necessary to regularise it. But, heuristically, since it is a polynomial expression, it seems clear that after regularisation we must obtain zero. Hence, the particle production should not contribute to the energy-momentum tensor and the **sudden** singularity is unaffected by these quantum effects.

Note that the integral (46) admits an analytical solution:

$$\int \rho_k d^3k = \pi \int k^2 \tilde{\omega} \left( 1 - \frac{k}{\tilde{\omega}} \right)^2 dk$$

$$= \pi \left\{ k \sqrt{k^2 + \tilde{m}^2} \left( \frac{k^2}{2} - \frac{\tilde{m}^2}{4} \right) - \frac{k^4}{2} \right\} + \frac{\tilde{m}^2}{4} \ln \left[ 2 \left( k + \sqrt{k^2 + \tilde{m}^2} \right) \right],$$

(47)

with $\tilde{m} = m a_0$. There is no infrared divergence, but there is a logarithmic divergence when $k \to \infty$ (ultraviolet limit).

We have,

$$\rho_k = \sqrt{k^2 + \tilde{m}^2} - 2k + \frac{k^2}{\sqrt{k^2 + \tilde{m}^2}}.$$ 

(48)

It follows that

$$\rho_k^{(n)} = \rho_k.$$ 

(49)

Hence, only the zero-order term survives, and leads to,

$$\rho_k^{\text{ren}} = \rho_k - E^0_k = \rho_k - \rho_k = 0.$$ 

(50)

As we suspected, the renormalized energy is zero. There is no effect, and the quantum phenomena associated with the cosmological dynamics do not change the character of the **sudden** singularity or prevent its occurrence.
6 Conclusions

It is known that quantum effects may play an important rôle near classical singularities, for example, for the big bang cosmological scenario. Even if the back reaction of these quantum effects on the classical evolution is still an open question, there are at least hints that quantum effects may lead to a dramatic deviation from the classical behavior [27].

Recently, new kinds of singularities have been identified in cosmology, the so-called future singularities. While the big bang singularity occurs in the origin of the universe, future singularities may mark the end of the universe. The interest in this kind of singularity has increased recently because of the unusual possibility that the presently observed accelerated expansion of the universe may be driven by a phantom fluid, which violates all energy conditions. However, it had been shown some time earlier that such future singularities may occur even if the energy conditions are not violated.

Concerning the big rip, the quantum effects may be relevant, but a more careful analysis of the back reaction process is necessary to decide under what conditions the big rip can be evaded by quantum effects [16]. In the case of the sudden singularity, the results obtained so far indicate that this singularity is robust against quantum effects because they are negligible in its vicinity.

It must be remarked however that such studies has been carried out using special configurations of scalar fields. More general quantum fields must be analyzed and, of course, the problem of the back reaction must also be treated in greater generality.

Acknowledgements: We thank CNPq (Brasil) for partial financial support. JDB thanks S. Cotsakis, S.Z.W. Lip, C.G. Tsagas and A. Tsokaros for their active collaboration.

References

[1] G. Bertone, D. Hooper and J. Silk, Phys. Rep. 405, 279(2005).

[2] J.D. Barrow and F.J. Tipler, The Anthropic Cosmological Principle, Oxford UP, Oxford (1986); S. Weinberg, Rev. Mod. Phys. 61, 1(1989); R. Bousso, Gen. Rel. Gravit. 40, 607 (2008).

[3] J.D. Barrow and D.J. Shaw, Phys. Rev. 83, 04351 (2010); J.D. Barrow and D.J. Shaw, Phys. Rev. Lett. 106, 101302 (2011); J.D. Barrow and D.J. Shaw, Gen. Rel. Gravit. 43, 2555 (2011).

[4] E. Komatsu et al., Astrophys. J. Suppl. 192, 18 (2011).

[5] R.R. Caldwell, Phys. Lett. B545, 23 (2002).
[6] J.D.Barrow, G.J. Galloway and F.J. Tipler, Mon. Not. Roy. astron. Soc. 223, 835 (1986).

[7] A.A. Starobinsky, Grav. & Cosmol. 6, 157(2000); Y. Shtanou and V. Sahni, Class. Quantum Grav. 19, L101 (2002).

[8] V. Gorini, A.Y. Kamenshchik, U. Moschella and V. Pasquier, Phys. Rev. D69, 123512(2004).

[9] Z. Keresztes, L.Á. Gergely, V. Gorini, U. Moschella and A. Yu. Kamenshchik, Phys. Rev. D79, 083504 (2009).

[10] Z. Keresztes, L.A. Gergely, A.Y. Kamenshchik, V. Gorini and D. Polarski, Phys. Rev. D82, 123534(2010)

[11] J.D. Barrow, Class. Quantum Grav. 21, L79 (2004).

[12] F. J. Tipler, Phys. Lett. A64, 8 (1977).

[13] A. Królik, Class. Quantum Grav. 3, 267 (1986).

[14] A.B. Batista, J.C. Fabris, S. Houndjo, Grav. & Cosmol. 14, 140 (2008).

[15] F.G. Alvarenga, A.B. Batista, J.C. Fabris and S. Houndjo, Grav. & Cosmol. 16, 105 (2010).

[16] J.D. Bates and P.R. Anderson, Phys. Rev. D82, 024018 (2010).

[17] J.D. Barrow, A.B. Batista, J.C. Fabris and S. Houndjo, Phys. Rev. D78, 123508 (2008).

[18] J.D. Barrow, A.B. Batista, G. Dito, J.C. Fabris and S. Houndjo, Phys. Rev. D84, 123518 (2011).

[19] S.J.M. Houndjo, Europhys. Lett. 92, 10004 (2010).

[20] J.D.Barrow, Class. Quantum Grav. 21, 5619 (2004).

[21] J.D. Barrow and C.G. Tsagas, Class. Quantum Grav. 22, 1563 (2005).

[22] J.D. Barrow, S. Cotsakis and A. Tsokaros, Class. Quantum Grav. 27, 165017 (2010).

[23] J.D. Barrow and S.Z.W. Lip, Phys. Rev. D80, 043518, (2009).

[24] H. Stefancic Phys. Rev. D71, 084024 (2005).

[25] E.J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod. Phys. D15, 1753 (2006).

[26] S. Cotsakis and I. Klaoudatou, J. Geom. Phys. 57, 1303 (2007).

[27] N.D. Birrell and P.C.W. Davies, Quantum fields in Curved Space, Cambridge University Press, Cambridge (1982).
[28] Ya. B. Zel’dovich and A. A. Starobinsky, Sov. Phys. JETP 34, 1159 (1972).

[29] A.A. Grib, S.G. Mamayev and V.M. Mostepanenko, Vacuum quantum effects in strong fields, Friedmann, St. Petersburg (1994).

[30] M. Bordag, J. Lindig and V.M. Mostepanenko, Class. Quantum Grav. 15, 581 (1998).

[31] M.P. Dabrowski, C. Kiefer and B. Sandhofer, Phys. Rev. D74, 044022(2006).

[32] A. Kamenshchik, C. Kiefer and B. Sandhofer, Phys. Rev. D76, 064032(2007).

[33] E.M. Barboza Jr. and N.A. Lemos, Gen. Rel. Grav. 38, 1609(2006).

[34] N. Pinto-Neto and D.M. Pantoja, Phys. Rev. D80, 083509(2009).