1. Introduction

Consider the free Schrödinger equation
\begin{equation}
  i\partial_t u(x, t) = -\Delta u(x, t) \quad x \in \mathbb{R}^n, t \in \mathbb{R}
  
  u(x, 0) = f(x).
\end{equation}

It is well known that $e^{it\Delta} f$ is the unique solution to the initial value problem (1.1). The following remarkable estimate for the solution to the initial value problem (1.1) is first obtained by Strichartz [13] in connection with Fourier restriction theory:

**Theorem 1.1.** Let $f \in L^2(\mathbb{R}^n)$. If $p, q \geq 1$ satisfying $(p, q, n) \neq (1, \infty, 2)$ and $\frac{2}{p} + \frac{n}{q} = n$ then $e^{it\Delta} f \in L^2_t L^2_x(\mathbb{R} \times \mathbb{R}^n)$ and satisfies the inequality
\begin{align*}
  &\int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} |(e^{it\Delta} f)(x)|^{2q} \, dx \right)^{\frac{p}{q}} \, dt \\
  &\leq C \left( \int_{\mathbb{R}^n} |f(x)|^2 \, dx \right)^{\frac{p}{q}}.
\end{align*}

The above inequality have been substantially generalized for a system of orthonormal functions by Frank-Lewin-Lieb-Seiringer [3] and Frank-Sabin [4].

**Theorem 1.2.** [3, 4] Assume that $p, q, n \geq 1$ such that
\begin{align*}
  &1 \leq q < \frac{n + 1}{n - 1} \quad \text{and} \quad \frac{2}{p} + \frac{n}{q} = n.
\end{align*}

For any (possibly infinite) system $u_j$ of orthonormal functions in $L^2(\mathbb{R}^n)$ and any coefficients $(n_j) \subset \mathbb{C}$, we have
\begin{align*}
  &\int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \left| \sum_j n_j |(e^{it\Delta} u_j)(x)|^2 \right|^{\frac{q}{2}} \, dx \right)^{\frac{p}{q}} \, dt \\
  &\leq C_{n,q}^p \left( \sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{p(q+1)}{2q}}.
\end{align*}
where $C_{n,q}$ is a universal constant which only depends on $n$ and $q$.

Further, Theorem 1.1 has been extended for the Schrödinger equation associated with the Hermite operator $H = -\Delta + |x|^2$:

\begin{equation}
\label{Schrodinger}
\begin{aligned}
i\partial_t u(x, t) &= H u(x, t) \quad x \in \mathbb{R}^n, t \in \mathbb{R} \\
u(x, 0) &= f(x).
\end{aligned}
\end{equation}

Assuming $f \in L^2(\mathbb{R}^n)$, the solution of the initial value problem (1.2) is given by $u(x, t) = e^{-itH}f(x)$. The Strichartz inequality in this case has been proved by Koch-Tataru [6] or Nandakumaran-Ratnakumar [8] resulting in the following.

**Theorem 1.3.** Let $f \in L^2(\mathbb{R}^n)$. If $p, q \geq 1$ satisfying $(p, q, n) \neq (1, \infty, 2)$ and $\frac{2}{p} + \frac{n}{q} = n$, then

\[
\|e^{-itH}f\|_{L^2_t L^p_z(T \times \mathbb{R}^n)} \leq C\|f\|_2.
\]

The above inequality is further generalized for a system of orthonormal functions in [2, 7].

**Theorem 1.4.** [2, 7] (Strichartz inequality for orthonormal functions for Hermite operator) Let $p, q, n \geq 1$ such that

\[
1 \leq q < \frac{n + 1}{n - 1} \quad \text{and} \quad \frac{2}{p} + \frac{n}{q} = n.
\]

For any (possibly infinite) system $(u_j)$ of orthonormal functions in $L^2(\mathbb{R}^n)$ and any coefficients $(n_j) \subset \mathbb{C}$, we have

\begin{equation}
\label{Strichartz}
\int_{-\pi}^\pi \left( \int_{\mathbb{R}^n} \left| \sum_j n_j (e^{-itH}u_j)(x) \right|^q \, dx \right)^{\frac{p}{q}} \, dt \leq C_{n,q}^p \left( \sum_j |n_j|^{2q} \right)^{\frac{p(q + 1)}{2pq}},
\end{equation}

where $C_{n,q}$ is a universal constant which only depends on $n$ and $q$.

The Strichartz estimate for the Schrödinger equation associated with special Hermite operator $\mathcal{L}$ (defined in section 2) on $L^2(\mathbb{C}^n)$ has been considered by Ratnakumar [9] in the following initial value problem:

\begin{equation}
\label{SpecialSchr}
\begin{aligned}
i\partial_t u(z, t) &= \mathcal{L} u(z, t) \quad z \in \mathbb{C}^n, t \in \mathbb{R} \\
u(z, 0) &= f(z).
\end{aligned}
\end{equation}

For $f \in L^2(\mathbb{C}^n)$, the solution of the initial value problem (1.4) is given by $u(z, t) = e^{-it\mathcal{L}}f(z)$ and satisfies the following Strichartz estimate.

**Theorem 1.5.** [9] Let $f \in L^2(\mathbb{C}^n)$. If $1 < p < \infty$, $\frac{1}{p} \geq n \left( 1 - \frac{1}{q} \right)$ or $\frac{1}{2} \leq p \leq 1$, $1 \leq q < \frac{n}{n - 1}$ then

\[
\|e^{-it\mathcal{L}}f\|_{L^2_t L^p_z(T \times \mathbb{C}^n)} \leq C\|f\|_2.
\]
The main aim of this paper is to obtain the following Strichartz estimate for system of orthonormal functions associated with the special operator with respect to the special Hermite transform. To the best of our knowledge the study on restriction theorem with respect to the special Hermite transform has not been considered in the literature so far.

**Theorem 1.6.** Let \( q, n \geq 1 \) and \( p > 1 \) such that
\[
1 \leq q \leq 1 + \frac{1}{n} \quad \text{and} \quad \frac{1}{p} + \frac{n}{q} = n.
\]
For any (possibly infinite) system \((u_j)\) of orthonormal functions in \( L^2(\mathbb{C}^n) \) and any coefficients \((n_j) \subset \mathbb{C}\), there exists a constant \( C > 0 \) such that
\[
\left\| \sum_j n_j |e^{-it\mathcal{L}u_j}|^2 \right\|_{L^p_t L^q_z(\mathbb{T} \times \mathbb{C}^n)} \leq C \left( \sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{2q}}.
\]

Let \( f \in L^1(\mathbb{C}^n) \). Define the special Hermite transform of \( f \) by
\[
\hat{f}(\mu, \nu) = \int_{\mathbb{C}^n} f(z) \Phi_{\mu\nu}(z) \, dz, \quad \mu, \nu \in \mathbb{N}_0^n
\]
where \( \mathbb{N}_0 \) denotes the set of all non-negative integers and \( \Phi_{\mu\nu} \)'s are the special Hermite functions (defined in section 2) on \( \mathbb{C}^n \). If \( f \in L^2(\mathbb{C}^n) \) then \( \{\hat{f}(\mu, \nu)\} \in \ell^2(\mathbb{N}_0^n) \) and satisfies the Plancherel formula
\[
\|f\|^2 = \sum_{(\mu,\nu) \in \mathbb{N}_0^n} |\hat{f}(\mu, \nu)|^2.
\]
The inverse special Hermite transform is given by
\[
f(z) = \sum_{(\mu,\nu) \in \mathbb{N}_0^n} \hat{f}(\mu, \nu) \Phi_{\mu\nu}(z).
\]
Given a discrete surface \( S \) in \( \mathbb{N}_0^{2n} \), we define the restriction operator \((R_S f) := \{\hat{f}(\mu, \nu)\}_{\mu,\nu \in S}\) and the operator dual to \( R_S \) (called the extension operator) as
\[
E_S(\{\hat{f}(\mu, \nu)\}) := \sum_{\mu,\nu \in S} \hat{f}(\mu, \nu) \Phi_{\mu\nu}.
\]
We consider the following problem:

**Problem 1:** For which exponents \( 1 \leq p \leq 2 \), the sequence of special Hermite transforms of a function \( f \in L^p(\mathbb{C}^n) \) belongs to \( \ell^2(S) \)?

This question can be reframed to the boundedness of the operator \( E_S \) from \( \ell^2(S) \) to \( L^{p'}(\mathbb{C}^n) \), where \( p' \) is the conjugate exponent of \( p \). Since \( E_S \) is bounded from \( \ell^2(S) \) to \( L^{p'}(\mathbb{C}^n) \) if and only if \( T_S := E_S(E_S)^* \) is bounded from \( L^p(\mathbb{C}^n) \) to \( L^{p'}(\mathbb{C}^n) \), Problem 1 can be re-written as follows:
Problem 2: For which exponents $1 \leq p \leq 2$, the operator $T_S := \mathcal{E}_S(\mathcal{E}_S)^*$ is bounded from $L^p(C^n)$ to $L^{p'}(C^n)$?

To address this problem we introduce an analytic family of operators $(T_z)$ defined on the strip $a \leq \Re z \leq b$ in the complex plane such that $T_S = T_c$ for some $c \in (a, b)$ and show that the operator $W_1T_3W_2$ belongs to a Schatten class for $W_1, W_2 \in L^{2\pi}(\mathbb{T} \times C^n)$, which is more general result $L^p - L^{p'}$ boundedness of $T_S$.

Such problems are often considered in the literature. For example, on $\mathbb{R}^n$, the celebrated Stein-Tomas Theorem (see [12, 16, 17]) gives an affirmative answer to Fourier restriction problem for compact surfaces with non-zero Gaussian curvature if and only if $1 \leq p \leq \frac{2(n+1)n+3}{n+3}$. For quadratic surfaces, Strichartz [13] gave a complete solution to Fourier restriction problem, when $S$ is a quadratic surface given by $S = \{x \in \mathbb{R}^n : R(x) = r\}$, where $R(x)$ is a polynomial of degree two with real coefficients and $r$ is a real constant. Further the Stein-Tomas Theorem is generalized to a system of orthonormal functions with respect to the Fourier transform by Frank-Lewin-Lieb-Seiringer [4] and Frank-Sabin [3].

The schema of the paper apart from introduction is as follows: In Section 2, we discuss the spectral theory of the Hermite operator and the kernel estimates for the special Hermite semigroup. In section 3, we obtain the duality principle in terms of Schatten bounds of the operator $W e^{-it\mathcal{E}} (e^{-it\mathcal{E}})^* W$ and prove the Strichartz estimate for $1 \leq q \leq 1 + \frac{1}{n}$, for the system of orthonormal functions associated with the special Hermite operator as the restriction of the special Hermite transform to the discrete surface $S = \{ (\mu, \nu, \lambda) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{Z} : \lambda = 2|\nu| + n \}$.

2. Preliminary

In this section we discuss some basic definitions and provide necessary background information about the special Hermite semigroup.

2.1. Hermite Operator and Special Hermite functions. Let $\mathbb{N}_0$ be the set of all non-negative integers. Let $H_k$ denote the Hermite polynomial on $\mathbb{R}$, defined by

$$H_k(x) = (-1)^k \frac{d^k}{dx^k} (e^{-x^2}) e^{x^2}, \quad k \in \mathbb{N}_0$$

and $h_k$ denote the normalized Hermite functions on $\mathbb{R}$ defined by

$$h_k(x) = (2\pi k!)^{-\frac{1}{2}} H_k(x) e^{-\frac{1}{2}x^2}, \quad k \in \mathbb{N}_0.$$  

The higher dimensional Hermite functions denoted by $\Phi_\alpha$ are obtained by taking tensor product of one dimensional Hermite functions. Thus for any multi-index $\alpha \in \mathbb{N}_0^n$ and $x \in \mathbb{R}^n$, we define $\Phi_\alpha(x) = \prod_{j=1}^n h_{\alpha_j}(x_j)$.

For each multi-index $\mu, \nu$ and $\zeta \in \mathbb{C}$, we define the special Hermite functions $\Phi_{\mu\nu}$ by

$$\Phi_{\mu\nu}(\zeta) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \Phi_\mu (\xi + \frac{y}{2}) \Phi_\nu (\xi - \frac{y}{2}) d\xi, \quad \zeta = x + iy \in \mathbb{C}^n.$$
The family of functions \( \{ \Phi_{\mu\nu} \} \) form an orthonormal basis for \( L^2(\mathbb{C}^n) \). The special Hermite functions are the eigenfunctions of the special Hermite operator \( L \) (or the twisted Laplacian) defined by

\[
L = \frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j),
\]

where \( Z_j = \frac{\partial}{\partial \zeta_j} + \frac{1}{2} \zeta_j \), \( \bar{Z}_j = -\frac{\partial}{\partial \bar{\zeta}_j} + \frac{1}{2} \bar{\zeta}_j \), \( j = 1,2,\ldots n \) with eigenvalues \( 2|\nu| + n \).

The special Hermite operator \( L \) is self-adjoint and admits a spectral decomposition in terms of special Hermite functions. Given \( f \in L^2(\mathbb{C}^n) \) the expansion

\[
f = \sum_{\mu,\nu \in \mathbb{N}_0^n} \langle f, \Phi_{\mu\nu} \rangle \Phi_{\mu\nu}
\]

converges to \( f \) in \( L^2(\mathbb{C}^n) \). The above expansion also can be written as

\[
f = \sum_{k=0}^{\infty} P_k f,
\]

where

\[
P_k = \sum_{\mu,|\nu|=k} \langle \cdot, \Phi_{\mu\nu} \rangle \Phi_{\mu\nu}
\]

is the orthogonal projection of \( L^2(\mathbb{C}^n) \) onto the eigenspace spanned by \( \{ \Phi_{\mu\nu} : |\nu| = k \} \).

For each \( k \in \mathbb{N} \), the spectral decomposition of \( L \) can be written as

\[
L f = \sum_{k=0}^{\infty} (2k + n) P_k f.
\]

The twisted convolution of two functions \( f \) and \( g \) on \( \mathbb{C}^n \) is defined by

\[
f \times g(\zeta) = \int_{\mathbb{C}^n} f(\zeta - w)g(w)e^{\frac{i}{2} \text{Im}(\zeta \cdot \bar{w})} dw,
\]

where \( \text{Im} \) denotes the imaginary part. The family \( \{ \Phi_{\mu\nu} \} \) satisfies the following orthogonality properties

\[
\Phi_{\mu\nu} \times \Phi_{\alpha\beta} = \begin{cases} (2\pi)^{n/2} \Phi_{\mu\beta} & \text{if } \nu = \alpha, \\ 0 & \text{otherwise.} \end{cases}
\]

Let \( L^k_\alpha \) denote the Laguerre polynomial of degree \( k \) and of order \( \alpha > -1 \), defined by the generating function identity (see [9])

\[
\sum_{k=0}^{\infty} L^k_\alpha(x) \omega^k = (1 - \omega)^{-\alpha-1} e^{-\frac{1}{2} \omega + \frac{i}{2} x \omega}, \quad |\omega| < 1
\]

and let \( \phi_k(z) = L^{n-1}_k \left( \frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2} \) be the Laguerre function of order \( n-1 \). The special Hermite functions \( \Phi_{\mu\nu} \) are related to the Laguerre functions \( \phi_k \) by the following relation

\[
(2\pi)^{n/2} \sum_{|\nu| = k} \Phi_{\nu\nu} = \phi_k.
\]
Now taking twisted convolution on both sides of (2.1) with $\Phi_{\alpha\alpha}$ and using the orthogonality property (2.2), we have

\[ f \times \Phi_{\alpha\alpha} = (2\pi)^{n/2} \sum_{\mu} \langle f, \Phi_{\mu\alpha} \rangle \Phi_{\mu\alpha}. \tag{2.4} \]

Summing both sides of (2.4) with respect to all $\alpha$ such that $|\alpha| = k$ and using (2.3), the spectral projection $P_k$ has the simpler representation

\[ P_k f(\zeta) = (2\pi)^{-n} \sum_{|\alpha|=k} f \times \Phi_{\alpha\alpha}(\zeta) = (2\pi)^{-n} f \times \varphi_k(\zeta). \]

Then the special Hermite expansion takes the compact form

\[ f(\zeta) = (2\pi)^{-n} \sum_k f \times \varphi_k(\zeta). \]

The operator $L$ defines a semigroup, called the special Hermite semigroup and denoted by $e^{-tL}$, $t > 0$, by the expansion

\[ e^{-tL} f = (2\pi)^{-n/2} \sum_{k=0}^{\infty} e^{-(2k+n)t} f \times \varphi_k \]

for $f \in L^2(\mathbb{C}^n)$. For the auxiliary complex semigroup $\{e^{-\eta L}\}$, $\eta = r + it$, $r > 0$, we write

\[ e^{-\eta L} f(\zeta) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-\eta(2k+n)} f \times \varphi_k(\zeta). \]

Thus, $e^{-\eta L}$ is a twisted convolution operator

\[ e^{-\eta L} f(\zeta) = \int_{\mathbb{C}^n} f(\zeta - w) K_\eta(w) e^{\frac{i}{2} \text{Im}(\zeta \cdot \bar{w})} dw \]

with kernel (see [9])

\[ K_\eta(\zeta) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-\eta(2k+n)} \phi_k(\zeta) = (2\pi)^{-n} e^{-\eta(1 - \omega)} e^{-\frac{1+i\omega |\zeta|^2}{4}}, \]

where $\omega = e^{-2\eta}$. So $K_{r+it}(\zeta) = K_{r+i(t+2\pi)}(\zeta)$, and

\[ |K_\eta(\zeta)| \leq \frac{2}{|\sin t|^n}, \quad \eta = r + it, \quad \zeta \in \mathbb{C}^n. \tag{2.5} \]

We refer to [9] for a detailed study on special Hermite semigroup.
2.2. Schatten class and the duality principle. Let \( \mathcal{H} \) be a complex and separable Hilbert space in which the inner product is denoted by \( \langle \cdot, \cdot \rangle_\mathcal{H} \). Let \( T: \mathcal{H} \to \mathcal{H} \) be a compact operator and let \( T^* \) denotes the adjoint of \( T \). For \( 1 \leq r < \infty \), the Schatten space \( \mathcal{G}^r(\mathcal{H}) \) is defined as the space of all compact operators \( T \) on \( \mathcal{H} \) such that

\[
\sum_{n=1}^{\infty} (s_n(T))^r < \infty,
\]

where \( s_n(T) \) denotes the singular values of \( T \), i.e., the eigenvalues of \( |T| = \sqrt{T^*T} \) counted according to multiplicity. For \( T \in \mathcal{G}^r(\mathcal{H}) \), the Schatten \( r \)-norm is defined by

\[
\|T\|_{\mathcal{G}^r} = \left( \sum_{n=1}^{\infty} (s_n(T))^r \right)^{\frac{1}{r}}.
\]

An operator belongs to the class \( \mathcal{G}^1(\mathcal{H}) \) is known as Trace class operator. Also, an operator belongs to \( \mathcal{G}^2(\mathcal{H}) \) is known as Hilbert-Schmidt operator.

3. Strichartz inequality for system of orthonormal functions

In order to obtain the Strichartz inequality for the system of orthonormal functions we need a duality principle lemma in our context. We refer to Proposition 1 and Lemma 3 of [1] with appropriate modifications to obtain the following two results:

**Proposition 3.1.** Let \( (T_s) \) be an analytic family of operators on \( \mathbb{T} \times \mathbb{C}^n \) in the sense of Stein defined on the strip \( -\lambda_0 \leq \text{Re} z \leq 0 \) for some \( \lambda_0 > 1 \). Assume that we have the following bounds

\[
\|T_is\|_{L^2(\mathbb{T} \times \mathbb{C}^n) \to L^2(\mathbb{T} \times \mathbb{C}^n)} \leq M_0 e^{a|s|},
\]

(3.1)

\[
\|T_{-\lambda_0 + is}\|_{L^1(\mathbb{T} \times \mathbb{C}^n) \to L^\infty(\mathbb{T} \times \mathbb{C}^n)} \leq M_1 e^{b|s|}
\]

(3.2)

for all \( s \in \mathbb{R} \) and for some \( a, b, M_0, M_1 \geq 0 \). Then, for all \( W_1, W_2 \in L^{2\lambda_0}(\mathbb{T} \times \mathbb{C}^n, \mathbb{C}) \) the operator \( W_1 T_{-1} W_2 \) belongs to \( \mathcal{G}^{2\lambda_0}(L^2(\mathbb{T} \times \mathbb{C}^n)) \) and we have the estimate

\[
\|W_1 T_{-1} W_2\|_{\mathcal{G}^{2\lambda_0}(L^2(\mathbb{T} \times \mathbb{C}^n))} \leq M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L_t^{2\lambda_0} L^2(\mathbb{T} \times \mathbb{C}^n)} \|W_2\|_{L_t^{2\lambda_0} L^{2\lambda_0}(\mathbb{T} \times \mathbb{C}^n)}.
\]

(3.3)

**Lemma 3.2.** (Duality principle) Let \( p, q \geq 1 \) and \( \alpha \geq 1 \). Let \( Af(t, w) = e^{-it\xi} f(w) \). Then the following statements are equivalent.

1. There is a constant \( C > 0 \) such that

\[
\|WAA^* W\|_{\mathcal{G}^\alpha(L^2(\mathbb{T} \times \mathbb{C}^n))} \leq C \|W\|_{L_t^{\frac{2q}{2q-\alpha}} L_w^{\frac{2p}{2p-\alpha}}(\mathbb{T} \times \mathbb{C}^n)}^2
\]

(3.4)

for all \( W \in L_t^{\frac{2q}{2q-\alpha}} L_w^{\frac{2p}{2p-\alpha}}(\mathbb{T} \times \mathbb{C}^n) \), where the function \( W \) is interpreted as an operator which acts by multiplication.
(2) For any orthonormal system \((f_j)_{j \in J}\) in \(L^2(\mathbb{C}^n)\) and any sequence \((n_j)_{j \in J} \subset \mathbb{C}\), there is a constant \(C' > 0\) such that

\[
\left\| \sum_{j \in J} n_j |A f_j|^2 \right\|_{L^q(L^p(T \times \mathbb{C}^n))} \leq C' \left( \sum_{j \in J} |n_j|^\alpha' \right)^{1/\alpha'}.
\]

If \(f \in L^2(\mathbb{C}^n)\), the solution of the initial value problem (1.4) can be realized as the extension operator of some function \(f\) on \(T \times \mathbb{C}^n\).

Let \(S\) be the discrete surface

\[
S = \{(\mu, \nu, \lambda) \in \mathbb{N}_0^3 \times \mathbb{N}_0 \times \mathbb{Z} : \lambda = 2|\nu| + n\}
\]

with respect to counting measure. Then for all \(f\) such that \(\hat{f} \in \ell^1(S)\) and for all \((t, \zeta) \in [-\pi, \pi] \times \mathbb{C}^n\), the extension operator can be written as

\[
E_S f(t, \zeta) = \sum_{\mu, \nu, \lambda \in S} \hat{f}(\mu, \nu, \lambda) \Phi_{\mu\nu}(\zeta) e^{-it\lambda},
\]

where \(\hat{f}(\mu, \nu, \lambda) = \int_{\mathbb{C}^n} \int_T f(t, w) \Phi_{\mu\nu}(w) e^{it\lambda} dtdw\). Using the fact that

\[
f \times \Phi_{\mu\mu} = (2\pi)^{\frac{\alpha}{2}} \sum_\nu (f, \Phi_{\mu\nu}) \Phi_{\mu\nu}(\zeta)
\]

and choosing

\[
\hat{f}(\mu, \nu, \lambda) = \begin{cases} 
(2\pi)^n \hat{u}(\mu, \nu) & \text{if } \lambda = 2|\nu| + n, \\
0 & \text{otherwise}
\end{cases}
\]

for some \(u : \mathbb{C}^n \to \mathbb{C}\) in (3.6), we get

\[
E_S f(t, \zeta) = (2\pi)^n \sum_{\mu, \nu \in S} \hat{u}(\mu, \nu) \Phi_{\mu\nu}(\zeta) e^{-it(2|\nu|+n)}
\]

\[
= (2\pi)^n \sum_\nu \left( \sum_\mu (f, \Phi_{\mu\nu}) \Phi_{\mu\nu}(\zeta) \right) e^{-it(2|\nu|+n)}
\]

\[
= (2\pi)^{\frac{\alpha}{2}} \sum_\nu e^{-it(2|\nu|+n)} u \times \Phi_{\nu\nu}(\zeta)
\]

\[
= (2\pi)^{\frac{\alpha}{2}} \sum_{k=0}^{\infty} e^{-it(2k+n)} \left( u \times \sum_{|\nu|=k} \Phi_{\nu\nu}(\zeta) \right)
\]

\[
= \sum_{k=0}^{\infty} e^{-it(2k+n)} u \times \phi_k(z) = e^{-itL} u(\zeta).
\]

For \(-1 < \Re z \leq 0\), define the analytic family of generalized functions

\[
G_z(\mu, \nu, \lambda) = \frac{1}{\Gamma(z+1)} (\lambda - (2|\nu| + n))^z_+,
\]

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where
\[
(\lambda - (2|\nu| + n))^\frac{z}{\pi} = \begin{cases} 
(\lambda - (2|\nu| + n))^z & \text{for } \lambda - (2|\nu| + n) > 0, \\
0 & \text{for } \lambda - (2|\nu| + n) \leq 0.
\end{cases}
\]

For Schwartz class functions $\phi$ on $\mathbb{N}_0^{2n} \times \mathbb{Z}$, using the discrete Taylor series expansion (see [14]), we have
\[
(3.8) \quad \langle \phi(z), \phi \rangle = \sum_{\xi \in \mathbb{N}_0^{2n} \times \mathbb{Z}} \phi(z) = \sum_{\xi \in \mathbb{N}_0^{2n} \times \mathbb{Z}} \phi(z) = \sum_{\xi \in \mathbb{N}_0^{2n} \times \mathbb{Z}} \phi(z) = \sum_{\xi \in \mathbb{N}_0^{2n} \times \mathbb{Z}} \phi(z) = \sum_{\xi \in \mathbb{N}_0^{2n} \times \mathbb{Z}} \phi(z).
\]

The above formula is valid for $z \neq -1, -2, \cdots$, regularizing (3.8). Notice that (3.9) shows that $\langle \phi(z), \phi \rangle$ is treated as a function of $z$ with simple poles at $z = -1, -2, \cdots$. Thus for Schwartz class functions $\phi$ on $\mathbb{N}_0^{2n} \times \mathbb{Z}$, we have
\[
\lim_{z \to -1} \langle G_z, \phi \rangle = \lim_{z \to -1} \frac{1}{\Gamma(z+1)} \sum_{\mu, \nu} \phi(\mu, \nu, \lambda)(\lambda - (2|\nu| + n))^\frac{z}{\pi} = \sum_{(\mu, \nu, \lambda) \in S} \phi(\mu, \nu, \lambda).
\]

We refer to [5] for the distributional calculus of $(\lambda - (2|\nu| + n))^\frac{z}{\pi}$. Thus $G_{\lambda-1} = \delta_S$. To prove our main result we need to prove the following proposition.

**Proposition 3.3.** Let $-1 < \text{Re } z < 0$. Then the series $\sum_{k=0}^{\infty} k^\frac{z}{\pi} e^{-itk}$ is the Fourier series of an integrable function on $[-\pi, \pi]$ which is of class $C^\infty$ on $[-\pi, \pi] \setminus \{0\}$. Near origin this function has the same singularity as the function whose values are $\Gamma(z+1)(it)^{-z-1}$, i.e.,
\[
(3.10) \quad \sum_{k=0}^{\infty} k^\frac{z}{\pi} e^{-itk} \sim \Gamma(z+1)(it)^{-z-1} + b(t),
\]
where $b \in C^\infty[-\pi, \pi]$.

**Proof.** For $\tau > 0$, we calculate the inverse Fourier transform of $u^\frac{z}{\pi} e^{-\tau u}$.
\[
\mathcal{F}^{-1}[u^\frac{z}{\pi} e^{-\tau u}](x) = \int_{\mathbb{R}} u^\frac{z}{\pi} e^{-\tau u} e^{-isu} du = \int_{0}^{\infty} u^\frac{z}{\pi} e^{-isu} du,
\]
where $s = x - it$ so that $-\pi < \arg s < 0$. Then $u^\frac{z}{\pi} e^{-\tau u}$ converges to $u^\frac{z}{\pi}$ in the sense of distributions as $\tau \to 0$. Also, the inverse Fourier transform of $u^\frac{z}{\pi} e^{-\tau u}$ converges to the inverse Fourier transform of $u^\frac{z}{\pi}$.

Using the change of variable $isu = \xi$ and proceeding as in page 170 of [5], we get
\[
(3.11) \quad \mathcal{F}^{-1}[u^\frac{z}{\pi} e^{-\tau u}](x) = \frac{1}{(is)^{z+1}} \int L \xi^\frac{z}{\pi} e^{-\xi} d\xi = \frac{\Gamma(z+1)}{(is)^{z+1}},
\]
where the contour $L$ of the integral is a ray from origin to infinity whose angle with respect to the real axis is given by $\arg \xi = \arg s + \frac{\pi}{2}$. Letting $\tau \to 0$ in (3.11), we have

$$
(3.12) \quad \mathcal{F}^{-1}[u^+_n](x) = \frac{\Gamma(z + 1)}{(ix)^{z+1}}.
$$

We use the idea given in Theorem 2.17 of [11] to prove (3.10). To make the paper self contained, we will only indicate the main steps. Let us consider a function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that $\eta(x) = 1$ if $|x| \geq 1$, and vanishes in a neighborhood of the origin. Let $F(x) = \eta(x)x^+_n$ for $x \in \mathbb{R}$. Writing $F(x) = x^+_n + (\eta(x) - 1)x^+_n$, using (3.12) and denoting $f$ to be the inverse Fourier transform of $F$ in the sense of distributions, we have

$$
\hat{f}(x) = \Gamma(z + 1)(ix)^{-z-1} + b_1(x),
$$

where $b_1 \in \mathcal{C}^\infty(\mathbb{R})$ and $f \in L^1(\mathbb{R})$.

Applying Poisson summation formula (see page 250 of [11]) to the function $f$ and using the fact $\hat{f} = F$, we get

$$
\sum_{k=0}^{\infty} k^z e^{-itk} = \sum_{k \in \mathbb{Z}} F(k) e^{-itk}
\sim \sum_{k \in \mathbb{Z}} f(2k\pi + t)
= f(t) + \sum_{|k| > 0} f(2k\pi + t)
= \Gamma(z + 1)(it)^{-z-1} + b_1(t) + \sum_{|k| > 0} f(2k\pi + t)
= \Gamma(z + 1)(it)^{-z-1} + b(t),
$$

where $b(t) = b_1(t) + \sum_{|k| > 0} f(2k\pi + t) \in \mathcal{C}^\infty[-\pi, \pi]$.

Now we are in a position to prove the following Strichartz inequality for the diagonal case.

**Theorem 3.4.** Let $n \geq 1$. For any (possibly infinite) system $(u_j)$ of orthonormal functions in $L^2(\mathbb{C}^n)$ and any coefficients $(n_j) \subset \mathbb{C}$, we have

$$
(3.13) \quad \left\| \sum_j n_j |e^{it\xi_j}u_j|^2 \right\|_{L^{\frac{2(n+1)}{2n+1}}(\mathbb{T} \times \mathbb{C}^n)} \leq C \left( \sum_j |n_j|^{\frac{2(n+1)}{2n+1}} \right)^{\frac{(2n+1)}{(2n+1)}},
$$

where $C$ is a constant depends on $n$. 

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Proof. In order to prove (3.13), by Lemma 3.2, it is enough to show

\[ \|W_1 T S W_2\|_{L^2(\mathbb{T} \times \mathbb{C}^n)} \leq C \|W_1\|_{L^2(n+1)(\mathbb{T} \times \mathbb{C}^n)} \|W_2\|_{L^2(n+1)(\mathbb{T} \times \mathbb{C}^n)} \]

for all \(W_1, W_2 \in L^2(n+1)(\mathbb{T} \times \mathbb{C}^n)\), where \(T_S := \mathcal{E}_S(\mathcal{E}_S)^*\). For \(-1 < \text{Re} \ z \leq 0\), define the operator \(T_z\) (on Schwartz class functions on \(\mathbb{T} \times \mathbb{C}^n\)) by

\[ T_z g(t, w) = \sum_{\mu, \nu, \lambda} \hat{g}(\mu, \nu, \lambda) G_z(\mu, \nu, \lambda) \Phi_{\mu \nu}(w) e^{-i\lambda t}, \]

where \(G_z\) is defined in (3.17). When \(\text{Re} \ z = 0\), we have

\[ \|T_{is}\|_{L^2(\mathbb{T} \times \mathbb{C}^n) \rightarrow L^2(\mathbb{T} \times \mathbb{C}^n)} = \|G_{is}\|_{L^\infty(\mathbb{T} \times \mathbb{C}^n)} \leq \left| \frac{1}{\Gamma(1+is)} \right| \leq Ce^{\pi|s|/2}. \]

Further, using (3.15), we have

\[ T_z g(t, w) = \sum_{\mu, \nu, \lambda} \int_T (\mathcal{F}_3^{-1} \hat{g})(\mu, \nu)(s) G_z(\mu, \nu, \lambda) \Phi_{\mu \nu}(w) e^{-i\lambda(t-s)} ds, \]

where \(\mathcal{F}_3^{-1} \hat{g}\) denotes the inverse Fourier transform of \(\hat{g}\) with respect to third variable. Then using (3.7), Proposition 3.3 and the distributional calculus of \((\lambda - (2|\nu| + n))^z\), we get

\[
    T_z g(w, t) = \sum_{\mu, \nu, \lambda} \int_T \langle g(\cdot, \cdot, s), \Phi_{\mu \nu}\rangle \Phi_{\mu \nu}(w) G_z(\mu, \nu, \lambda) e^{-i\lambda(t-s)} ds \\
    = \sum_{\mu, \nu} \int_T \langle g(\cdot, \cdot, s), \Phi_{\mu \nu}\rangle \Phi_{\mu \nu}(w) \sum_{\lambda} G_z(\mu, \nu, \lambda) e^{-i\lambda(t-s)} ds \\
    = \frac{1}{\Gamma(z+1)} \sum_{\mu, \nu} \int_T \langle g(\cdot, \cdot, s), \Phi_{\mu \nu}\rangle \Phi_{\mu \nu}(w) e^{-i(\lambda-s)(2|\nu|+n)} \sum_{\kappa=0}^{\infty} \kappa^z e^{-\kappa(t-s)} ds \\
    = -i e^{-iz\frac{t}{2}} \sum_{\mu, \nu} \int_{[-\pi, \pi]} \langle g(s, \cdot, \cdot), \Phi_{\mu \nu}\rangle \Phi_{\mu \nu}(w)(t-s)^{-z} e^{-i(t-s)(2|\nu|+n)} ds \\
    + \frac{b(t)}{\Gamma(z+1)} \sum_{\mu, \nu} \int_{[-\pi, \pi]} \langle g(s, \cdot, \cdot), \Phi_{\mu \nu}\rangle \Phi_{\mu \nu}(w) e^{-i(t-s)(2|\nu|+n)} ds.
\]

We ignore the second term in (3.17) as it vanishes as \(z\) tends to \(-1\) in our calculation. Now

\[
    \sum_{\mu, \nu} \langle g(\cdot, \cdot, s), \Phi_{\mu \nu}\rangle \Phi_{\mu \nu}(w) e^{-i(t-s)(2|\nu|+n)} \\
    = (2\pi)^{-\frac{n}{2}} \sum_{\nu} g(\cdot, s) \times \Phi_{\nu \nu}(w) e^{-i(t-s)(2|\nu|+n)}
\]

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\[
= (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-i(t-s)(2k+n)} g(\cdot, \cdot, s) \times \phi_k(w)
= (2\pi)^{-n} g(\cdot, \cdot, s) \times \sum_{k=0}^{\infty} e^{-i(t-s)(2k+n)} \phi_k(w).
\]

Thus from (3.17), we have
\[
T_z g(t, w) = -i(2\pi)^{-n}e^{-iz\frac{\pi}{2}} \int_{\mathbb{T}} g(\cdot, \cdot, s) \times (t-s)^{-z-1} \sum_{k=0}^{\infty} e^{-i(t-s)(2k+n)} \phi_k(u) ds
= -i(2\pi)^{-n}e^{-iz\frac{\pi}{2}} \int_{\mathbb{T}} \int_{\mathbb{C}^n} g(u, s) H(u, w, t-s) e^{-\frac{i}{2} \text{Im}(u \cdot \bar{w})} ds du
= -i(2\pi)^{-n}e^{-iz\frac{\pi}{2}} \int_{\mathbb{C}^n} (g(u, \cdot) \ast H(u, w, \cdot))(t) e^{-\frac{i}{2} \text{Im}(u \cdot \bar{w})} du,
\]
where
\[
H(u, w, t-s) = (t-s)^{-z-1} \sum_{k=0}^{\infty} e^{-i(t-s)(2k+n)} \phi_k(u - w).
\]

When \(z = -z_0 + is\), using (2.5), we get
\[
|T_z g(t, w)| \leq (2\pi)^{-n}e^{-iz\frac{\pi}{2}} \int_{\mathbb{C}^n} \|g(u, \cdot)\|_1 \sup_{t \in [-\pi, \pi]} |H(u, w, t)| du.
\]

Thus \(T_z\) is bounded from \(L^1(\mathbb{T} \times \mathbb{C}^n)\) to \(L^\infty(\mathbb{T} \times \mathbb{C}^n)\) if and only if \(\sup_{t \in [-\pi, \pi]} |H(u, w, t)|\) is bounded for each \(u, w \in \mathbb{C}^n\). But from (2.5), we get
\[
(3.18) \quad |H(u, w, t)| \sim \frac{C}{|t| \text{Re}(z+1+n)}.
\]

So for each \(w, u \in \mathbb{C}^n\), \(|H(u, w, t)|\) is bounded if and only if \(\text{Re}(z) = -(n+1)\). Thus
\[
(3.19) \quad \|T_{-z_0+is}\|_{L^1(\mathbb{T} \times \mathbb{C}^n) \to L^\infty(\mathbb{T} \times \mathbb{C}^n)} \leq C e^{-\frac{s\pi}{2}}.
\]

By (3.16) and (3.19), the family of operators \((T_z)\) satisfy (3.1) and (3.2). The conclusion of the theorem follows by choosing \(z_0 = n + 1\) in Proposition 3.1 and the identity \(T_S = T_{-1}\).

3.1. **Proof of Theorem 1.6.** Using the fact that the operator \(e^{-it\mathcal{L}}\) is unitary, triangle inequality gives (1.5) for the pair \((p, q) = (\infty, 1)\). Equivalently, the operator
\[
W \in L_1^\infty L_2^2(\mathbb{T} \times \mathbb{C}^n) \mapsto W e^{-itH} (e^{-itH})^* \bar{W} \in \mathcal{G}_1^\infty
\]
is bounded by Lemma 3.2. Similarly, by (3.14), the operator
\[
W \in L_1^{2(n+1)} L_2^2(\mathbb{T} \times \mathbb{C}^n) \mapsto W e^{-itH} (e^{-itH})^* \bar{W} \in \mathcal{G}_1^{2(n+1)}
\]
is bounded. Applying the complex interpolation method \[1\] (Chapter 4), the operator 
\[ W \in L^\frac{2q}{2-q} L^{\frac{2p}{2-p}} (\mathbb{T} \times \mathbb{C}^n) \mapsto We^{-itH}(e^{-itH})^*W \in G^\alpha \]
is bounded for \( 2 \leq \frac{2q}{2-q} \leq 2(n+1) \) and \( 2(n+1) \leq \frac{2p}{2-p} \leq \infty \). Again, applying Lemma \[3.2\] the inequality \[1.5\] holds for the range \( 1 \leq q \leq 1 + \frac{1}{n} \).

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