Overdetermined boundary value problems for the ∞-Laplacian

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Abstract: We consider overdetermined boundary value problems for the ∞-Laplacian in a domain Ω of R^n and discuss what kind of implications on the geometry of Ω the existence of a solution may have. The classical ∞-Laplacian, the normalized or game-theoretic ∞-Laplacian and the limit of the p-Laplacian as p → ∞ are considered and provide different answers, even if we restrict our domains to those that have only web-functions as solutions.

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1 Motivation

Suppose that Ω ⊂ R^n is connected and bounded, with boundary at least of class C^1, and that u ∈ C^1(Ω) is a positive solution of the overdetermined boundary value problem

\[- \Delta_p u_p := - \text{div} (|\nabla u_p|^{p-2} \nabla u_p) = 1 \quad \text{in } \Omega, \quad (1.1)\]
\[u_p = 0 \quad \text{on } \partial \Omega, \quad (1.2)\]
\[- \frac{\partial u_p}{\partial \nu} = a \quad \text{on } \partial \Omega, \quad (1.3)\]

where p ∈ (1, ∞) and a is a positive constant. Does this have consequences on the geometry of Ω? This question was answered in 1971 for p = 2 by Serrin [17] and Weinberger [18], and for general p in 1987 by Garofalo and Lewis [6]. See also Farina and Kawohl [5] for related results. In both cases the domain Ω must be a ball of fixed radius related to a. This result leads us to the question: what happens if the p-Laplacian is replaced by the infinity Laplacian?
The answer depends on how we define the $\infty$-Laplacian and the notion of solution. In case of equation (1.1) and $p = 2$ Serrin and Weinberger had classical $C^2(\Omega)$ solutions in mind, while for general $p \in (1, \infty)$ the solutions were weak in the sense that

$$\int_{\Omega} |\nabla u_p|^{p-2} \nabla u_p \nabla v \, dx = \int_{\Omega} v \, dx \quad \text{for every } v \in W^{1,p}_0(\Omega).$$

2 The classical $\infty$-Laplacian

The classical $\infty$-Laplacian operator is usually defined as $\Delta_\infty u := \langle D^2 u Du, Du \rangle$, with $Du$ denoting the gradient and $D^2 u$ the Hessian matrix of $u$. For functions in $C^2$ the second directional derivative in direction $\nu$ is given by $\langle D^2 u \nu, \nu \rangle$. If $\nu$ denotes the direction $-Du/|Du|$ of steepest descent of $u$, the equation $-\Delta_\infty u = 1$ can be rewritten as

$$-u_{\nu\nu} |u_\nu|^2 = 1,$$

and if $\Omega$ should happen to be a ball of radius $R$ centered at zero, $u(x)$ is necessarily a radial function. In fact, then

$$u(r) = \frac{3^{1/3}}{4} (R^{4/3} - r^{4/3}) \quad \text{and} \quad u_r(R) = -(3R)^{1/3}$$

imply that $R$ must be equal to $a^{4/3}$ to match both boundary conditions. Notice that this function is exactly of class $C^{1,1/4}$, which is the conjectured optimal regularity for $\infty$-harmonic functions $v$, that is for functions satisfying $\Delta_\infty v = 0$.

Therefore we cannot expect classical solutions. Since the equation is not in divergence form, we cannot expect a notion of weak solution either. Instead we define a viscosity solution $u$ of the equation

$$F(Du, D^2 u) := -\langle D^2 u Du, Du \rangle - 1 = 0$$

as a continuous function which is both a viscosity sub- and viscosity supersolution. A viscosity subsolution has the property that $F(D\varphi, D^2 \varphi)(x) \leq 0$ whenever $\varphi$ is a $C^2$-function such that $\varphi - u$ has a local minimum at $x$. A viscosity supersolution has the property that $F(D\psi, D^2 \psi)(x) \geq 0$ whenever $\psi$ is a $C^2$-function such that $\psi - u$ has a local maximum at $x$, see for instance [2]. In our autonomous case we may also assume that $\varphi$ touches $u$ from above at $x$ if we check the definition of subsolutions, and that $\psi$ touches from below at $x$ if we check supersolutions.

Let us see that the explicit radial function $c - k r^{4/3}$, with $k = 3^{4/3}/4$ is a viscosity solution of $F(Du, D^2 u) = 0$ at $x = 0$. If $\varphi$ is a smooth function touching $u$ from above, then $\nabla \varphi(0) = 0$, so $\varphi_\nu = 0$ and $F(D\varphi, D^2 \varphi) = -1$, which is less or equal to zero, as required for subsolutions. For supersolutions
the set of test functions $\psi$ that touch $u$ from below in the origin is empty, so that the condition for a supersolution is trivially satisfied. Effects like this happen quite often when viscosity solutions are not smooth. Checking the property of sub- or supersolution is somehow easier in points where the solutions loose smoothness.

Now suppose that $\Omega$ is not necessarily a ball, but a more general smooth domain.

**Remark 2.1** From every point $x_0$ on $\partial \Omega$ we can follow the line of steepest ascent, parametrized as $x(t)$ by solving the initial value problem

$$x(0) = x_0, \quad \frac{dx_i}{dt} = u_{x_i} \text{ for small but positive } t.$$ (2.2)

A simple calculation shows, assuming that $u$ is locally of class $C^2$,

$$\frac{d}{dt} \left( \left| \frac{dx}{dt} \right|^2 \right) = 2u_{x_i} u_{x_i x_j} u_{x_j} = -2,$$ (2.3)

so that upon integration from 0 to $t$

$$\left| \frac{dx}{dt} \right|^2 = |\nabla u(x(t))|^2 = a^2 - 2t.$$ (2.4)

Note that this works until $t$ reaches $a^2/2$, at which time $\nabla u = 0$. Subsequently we get the estimate

$$|x(t) - x_0| = \left| \int_0^t x_i(s) \, ds \right| \leq \frac{1}{3} \left( a^3 - (a^2 - 2t)^{3/2} \right) \leq \frac{a^3}{3}. \quad (2.5)$$

This shows that our trajectories can never reach a distance greater than $a^3/3$ from the boundary of $\Omega$ and that any critical point of $u$ that can be approached this way has at most distance $a^3/3$ from $\partial \Omega$.

Notice that the radial solution on a ball is a web-function in the sense of [3], i.e. a function, whose value depends only on the distance to $\partial \Omega$. From now on we assume that a solution of (2.1) (1.2) (1.3) happens to be a webfunction for a general domain as well. This may be justified via the Cauchy-Kowalewski Theorem or by using the remark above, but we could not give a precise proof. Under this assumption we can interpret equation (2.1) as an ordinary differential equation for a function $u(d)$ that depends only on the distance $d = d(x, \partial \Omega)$ to the boundary, with initial condition (1.2) and (1.3) at $d = 0$. Then we arrive after the first integration at

$$u_s'(d) - a^3 = -3d \text{ or } -u_\nu = (a^3 - 3d)^{1/3} \text{ and after a second integration at}$$

$$u(d) = \int_0^d (a^3 - 3t)^{1/3} \, dt = \frac{1}{4} \left[ a^4 - (a^3 - 3d)^{4/3} \right].$$
Clearly the integrations are only justifiable for sufficiently small $d$ and as long as $d$ is locally of class $C^{1,1}$. When $d = a^3/3$, the gradient of $u$ vanishes and we have reached the peak on our way uphill from the boundary. This shows that $\Omega$ has an inradius of exactly $a^3/3$. Incidentally, the points in

$$M(\Omega) := \{ y \in \Omega \mid d(y, \partial \Omega) = \max_{x \in \Omega} d(x, \partial \Omega) \}$$

belong to the ridge of $\Omega$ or cut locus of $\partial \Omega$, which is defined as follows. Let $G$ be the largest open subset of $\Omega$ such that every point $x$ in $G$ has a unique closest point on $\partial \Omega$. Then we call

$$\mathcal{R}(\Omega) := \Omega \setminus G$$

the ridge $\mathcal{R}(\Omega)$. In $G$, the distance $d(x, \partial \Omega)$ to the boundary is at least of class $C^1$, and also smooth, i.e., of class $C^2$ or $C^{k,\alpha}$ with $k \geq 2$ and $\alpha \in (0, 1)$ provided $\partial \Omega$ is of the same class, see [4, 11]. It is remarkable that even for a convex plane domain the ridge can have positive measure, see pages 10 and 11 in [14]. Simple examples such as an ellipse or a rectangle show that in general $M(\Omega)$ is a genuine subset of the ridge, but there are many domains with the property $M(\Omega) = \mathcal{R}(\Omega)$.

Examples of such domains are for instance a stadium domain (convex hull of two balls of same radius and different center), an annulus, or plane domains which are generated as follows. Let $\gamma$ be a compact $C^{1,1}$ curve with curvature not exceeding $K$ in modulus and $\Omega = U_b(\gamma) = \{ x \in \mathbb{R}^2 \mid d(x, \gamma) < b \}$ with $b < 1/K$. Then $M(\Omega) = \mathcal{R}(\Omega)$, see Figure 1.

![Figure 1: A domain satisfying $M(\Omega) = \mathcal{R}(\Omega)$](image)

**Theorem 2.2** Suppose that $\partial \Omega$ is of class $C^2$. Then a webfunction $u \in C^1(\Omega)$ is a viscosity solution of [2.1] [1.2] [1.3] if and only if $M(\Omega) = \mathcal{R}(\Omega)$ and every $x \in \partial \Omega$ has distance $a^3/3$ to $\mathcal{R}(\Omega)$. 
Proof. In fact, if $M(\Omega) = R(\Omega)$, then the function
\[
u(x) = \frac{1}{4} \left[ a^4 - \left( a^3 - 3d(x, \partial \Omega) \right)^{4/3} \right]
\]
is well defined and differentiable everywhere in $\Omega$. Moreover, according to [4], it is of class $C^2(\Omega \setminus R(\Omega))$ and solves (2.1) in $\Omega \setminus R(\Omega)$ in the classical (and a fortiori in the viscosity) sense. Finally on $M(\Omega) = R(\Omega)$ we can argue as in the radial case to see that $u$ is a viscosity solution there as well. This shows that the geometric constraint $M(\Omega) = R(\Omega)$ is sufficient for the existence of solutions to (2.1) (1.2) (1.3).

To prove necessity, suppose that $M(\Omega)$ is a genuine subset of $R(\Omega)$, so that there exists a $z \in R(\Omega) \setminus M(\Omega)$. But then $d(z, \partial \Omega) < a^3/3$ and $d(z, \partial \Omega)$ has a kink in the sense that some directional derivative of $d$, and subsequently of $u$, is discontinuous at $z$. This is incompatible with being a viscosity solution, because one can then find an admissible test function $\varphi \in C^2(\Omega)$ for which $F(D\varphi, D^2\varphi)$ fails to satisfy the proper inequality. To be precise, suppose that $\Omega$ is essentially a rectangle (with rounded corners to make it smooth) or an ellipse. Then $z$ lies on a line segment and $d(x, \partial \Omega)$ is concave near $z$ and has one-sided nonzero derivatives in direction $\eta$ orthogonal to the ridge in $z$. But then one can choose a $C^2$ function $\varphi$, touching $u$ from above in $z$ such that $\nabla \varphi(z) \neq 0$ points in direction $\eta$ and $\varphi_{\eta\eta}(z) < -K$, where $K$ is an arbitrarily large number. Therefore $F(D\varphi, D^2\varphi)(z) > 0$, which contradicts the requirement for subsolutions. There is a similar reasoning using supersolutions, if $\Omega$ is essentially $L$-shaped and $u$ is convex and nondifferentiable on parts of its ridge. \hfill \Box

3 The normalized or game-theoretic $\infty$-Laplacian

Recently the following operator has received considerable attention (see for instance [13, 16, 9, 12, 13, 20]) in the PDE community
\[
\Delta_N^\infty u = \langle D^2u Du, Du \rangle |Du|^{-2}.
\]
Here $u(x)$ denotes the (unique) running costs in a differential game called “tug of war”, see [20]. Let us therefore study the differential equation
\[
- u_{\nu\nu} = 1 \quad \text{in } \Omega
\]
under boundary conditions (1.2) and (1.3). A simple integration shows that certainly for a ball of radius $R = a$ this overdetermined problem has the explicit solution $u(r) = (a^2 - r^2)/2$, provided we can live with the ambiguity that $\nu$ is not properly defined at the origin. Fortunately the notion of viscosity solution allows us to do so. A viscosity solution $u$ of
\[
G(Du, D^2u)(x) := - \frac{\langle D^2u Du, Du \rangle}{|Du|^2} (x) - 1 = 0 \quad \text{in } \Omega
\]
is a viscosity subsolution of $G_* (Du, D^2 u) = 0$ and a viscosity supersolution of $G^* (Du, D^2 u) = 0$. Here $G_*$ and $G^*$ are the upper and lower semicontinuous envelopes of $G$, see Remark 6.3 in [2]. Thus $u \in C(\Omega)$ is a viscosity subsolution of (3.1) or (3.2), if for every $x \in \Omega$ and every smooth test function $\varphi$, that touches $u$ from above (only) in $x$, the following relations hold:

$$
\begin{cases}
G(\nabla \varphi(x), D^2 \varphi(x)) \leq 0 & \text{when } \nabla \varphi(x) \neq 0, \\
-\Lambda(D^2 \varphi(x)) - 1 \leq 0 & \text{when } \nabla \varphi(x) = 0.
\end{cases}
$$

(3.3)

In a similar fashion viscosity supersolutions $u \in C(\Omega)$ of (3.1) are characterized by the fact that

$$
\begin{cases}
G(\nabla \psi(x), D^2 \psi(x)) \geq 0 & \text{when } \nabla \psi(x) \neq 0, \\
-\lambda(D^2 \psi(x)) - 1 \geq 0 & \text{when } \nabla \psi(x) = 0.
\end{cases}
$$

(3.4)

for every smooth test function $\psi$ that touches $u$ from below (only) in $x$. Here $\Lambda(X)$ and $\lambda(X)$ denote the maximal and minimal (nonnegative) eigenvalue of the symmetric matrix $X$.

For a more general $\Omega$, if we interpret (3.1) again as an ODE and (1.2) and (1.3) as initial data on $\partial \Omega$, then an integration like in the previous section along lines of steepest ascent of $u$ leads to the local representation

$$
u(x) = \frac{d(x, \partial \Omega)}{2} \left( 2a - d(x, \partial \Omega) \right) \text{ in } \Omega \setminus R(\Omega).$$

**Theorem 3.1** Suppose that $\partial \Omega$ is of class $C^2$. Then a webfunction $u \in C^1(\Omega)$ is a viscosity solution of (3.1) (1.2) (1.3) if and only if $M(\Omega) = R(\Omega)$ and every $x \in \partial \Omega$ has distance $a$ to $R(\Omega)$.

The proof parallels the one of Theorem 2.2 and is left to the reader.

**Remark 3.2** Notice that annuli provide examples of domains (other than balls) for which a smooth solution of this problem (but not of Serrin’s and Weinberger’s original problem) exists.

## 4 The limit of $u_p$

It is well-known, that $p$-harmonic functions or viscosity solutions of $\Delta_p u = 0$ converge to the viscosity solution of $\Delta_\infty u = 0$ as $p \to \infty$. Therefore one is inclined to believe that solutions $u_p$ of the inhomogeneous equation (1.1) should converge to those of (2.1). This is not the case, and in the present section we investigate this limit. For $\Omega$ a ball in $\mathbb{R}^n$ the solutions of (1.1), (1.2) were explicitly calculated and shown to converge uniformly to $d(x, \partial \Omega)$ in [10]. Let us demonstrate that this behaviour happens for any connected domain, even for a nonsmooth one. First one has to note that $u_p$ on $\Omega$ can
be estimated in $L^q$ for any $q \in [0, \infty]$ by the corresponding solution $U_p$ on a ball $\Omega^*$ of same volume as $\Omega$, so that the $u_p$ are uniformly bounded in $L^\infty(\Omega)$ as $p \to \infty$. Furthermore $u_p$ minimizes the functional

$$J_p(v) = \int_\Omega \left[ \frac{1}{p} |\nabla v(x)|^p - v(x) \right] dx \quad \text{on } W^{1,p}_0(\Omega).$$

In particular

$$J_p(u_p(x)) \leq J_p(d(x, \partial \Omega)) = \frac{1}{p} |\Omega| - \int_\Omega d(x, \partial \Omega) dx,$$

the right hand of which is negative for sufficiently large $p$. Thus

$$\int_\Omega |\nabla u_p|^p dx \leq p \int_\Omega u_p dx,$$

or for $p > q$ and $q$ large enough

$$\int_\Omega |\nabla u_p|^q dx \leq \left( \int_\Omega |\nabla u_p|^p dx \right)^{q/p} |\Omega|^{1-q/p} \leq \left( p \int_\Omega u_p dx \right)^{q/p} |\Omega|^{1-q/p}.$$

But this implies $||\nabla u_p||_q \leq p^{1/p} |u_p|^{1/p} |\Omega|^{1/q}$, so that the family $\{u_p\}_{p \to \infty}$ is uniformly bounded in every $W^{1,q}(\Omega)$ and converges uniformly to some limit $u_\infty$ with Lipschitz constant 1.

Therefore $|\nabla u_\infty| \leq 1$ a.e. in $\Omega$, and this implies not only that $u_\infty(x) \leq d(x, \partial \Omega)$ in $\Omega$, but it (almost) proves the first half of our following result.

**Theorem 4.1** The limit $u_\infty$ is a viscosity solution of the eikonal equation $|Du(x)| - 1 = 0$ in $\Omega$ under the Dirichlet boundary condition $u = 0$ on $\partial \Omega$.

**Remark 4.2** Since this Hamilton-Jacobi equation has a unique viscosity solution, see e.g. [2], we obtain $u_\infty := d(x, \partial \Omega)$ as a Lipschitz solution for a highly overdetermined boundary value problem. It satisfies not only $|Du| - 1 = 0$ in $\Omega$ but also $-\Delta u_\infty = 0$ in $\Omega \setminus R(\Omega)$, and not only $u = 0$ on $\partial \Omega$ but also $-\frac{\partial u}{\partial \nu} = 1$ on differentiable parts of $\partial \Omega$.

**Remark 4.3** Notice that the statement $M(\Omega) = R(\Omega)$ is conspicuously missing in Theorem 4.1. Under the additional assumption $M(\Omega) = R(\Omega)$, however, the function $u_\infty$ is moreover (up to multiplication by a constant) the unique eigenfunction for the $\infty$–Laplacian operator, i.e. it satisfies in addition

$$\min \{-\langle D^2 u_\infty(x) Du_\infty(x), Du_\infty(x) \rangle, -|Du(x)| + \Lambda_\infty u(x)\} = 0 \quad \text{in } \Omega$$

in the viscosity sense, see [7, 19]. Here $\Lambda_\infty$ is the inverse of the inradius of $\Omega$. Without this assumption, as demonstrated in [8] there is nonuniqueness of this eigenfunction.
Proof of Theorem 4.1. Let us first realize that $|Du_\infty| \leq 1$ a.e. in $\Omega$ implies $|Du_\infty| - 1 \leq 0$ in the viscosity sense. Otherwise there would be a function $\varphi \in C^2$ touching $u$ from above in some $x_0$ such that $|Du(x_0)| \geq 1 + \gamma$, with $\gamma > 0$, and $|Du(x)| \geq 1 + \gamma/2$ in a neighbourhood $B_\varepsilon(x_0)$. But then $u(x_0) - u(x) \geq \varphi(x_0) - \varphi(x) \geq (1 + \gamma/2)|x_0 - x|$ for a suitable $x \in B_\varepsilon(x_0)$. This contradicts the fact that $u_\infty$ has Lipschitz constant 1.

To show the reverse inequality, it is instructive to follow ideas in [7, 1] and to identify the limiting equation. Suppose that $\varphi$ is a $C^2$-function such that $\varphi - u_\infty$ has a local minimum at $x_0 \in \Omega$. Then without loss of generality we may assume that $\varphi - u_\infty \geq \delta > 0$ on $\partial B_\varepsilon(x_0) \subset \Omega$. Moreover, for $p$ large enough, $\varphi - u_p$ has a local minimum at some $x_p \in B_\varepsilon(x_0)$ and $x_p \to x_0$ as $p \to \infty$. Since $u_p$ is a viscosity subsolution of \eqref{1.1} it follows

$$-|Du|^p - 2 \left( \text{tr}(D^2u) + (p - 2) \frac{D^2uDu, Du}{|Du|^2} \right) - 1 = 0 \quad \text{in } \Omega, \quad (4.1)$$

Now either $|D\varphi(x_0)| \leq 1$ or otherwise there exists a positive constant $\gamma$ independent of $p$, such that $|D\varphi(x_0)| > 1 + \gamma$ for large $p$. Upon division of the last inequality by $(p - 2)|D\varphi(x)|^{p-4}$ one sees that in this case the first term on the left and the right hand side in

$$-\frac{1}{p-2} |D\varphi(x_0)|^2 \text{tr} D^2\varphi(x_0) - \langle D^2\varphi(x_0)D\varphi(x_0), D\varphi(x_0) \rangle \leq \frac{1}{p-2} |D\varphi(x_0)|^{4-p}$$

converge to zero as $p \to \infty$, so that $-\langle D^2\varphi(x_0)D\varphi(x_0), D\varphi(x_0) \rangle \leq 0$. This proves that $u_\infty$ is a viscosity subsolution of

$$\min \{ |Du| - 1, -\langle D^2uDu, Du \rangle \} = 0 \quad \text{in } \Omega. \quad (4.2)$$

A similar reasoning holds for supersolutions. Since $u_p$ is a viscosity supersolution of \eqref{1.1}, we have

$$-|D\psi(x_p)|^{p-2} \left( \text{tr}(D^2\psi(x_p)) + (p - 2) \frac{D^2\psi(x_p)D\psi(x_p), D\psi(x_p)}{|D\psi(x_p)|^2} \right) \geq 1$$

for testfunctions $\psi \in C^2$ such that $u - \psi$ has a local maximum at $x_0$ and $u_p - \psi$ has a local maximum at $x_p$. This time we can rule out that $D\psi(x_p) = 0$, otherwise the last inequality cannot hold. Arguing as before, the inequality

$$-\frac{1}{p-2} |D\psi(x_p)|^2 \text{tr} D^2\psi(x_p) - \langle D^2\psi(x_p)D\psi(x_p), D\psi(x_p) \rangle \geq \frac{1}{p-2} |D\psi(x_0)|^{4-p}$$

follows and leads to $|D\psi(x_0)| \geq 1$, because else the right hand side would explode for $p \to \infty$, as well as to $-\langle D^2\psi(x_0)D\psi(x_0), D\psi(x_0) \rangle \geq 0$. This
shows that \( u_\infty \) is also a viscosity supersolution of (4.2). In particular \( u_\infty \) satisfies \( |Du| \geq 1 \) in the viscosity sense, and this completes the proof of Theorem 4.1. \( \square \)

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