Fourier transforms from strongly complementary observables

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Abstract—In this work, strongly complementary observables are shown to provide an abstract characterisation of Fourier transform and representation theory in Categorical Quantum Mechanics. Our starting point is the observation that the Fourier transform can be seen as an isomorphism of internal monoids in \(\text{fdHilb}\) between convolution and pointwise multiplication. We proceed to generalise the necessary tools of representation theory from \(\text{fdHilb}\) to arbitrary \(\dagger\)-compact monoidal categories: we define groups, characters and representations, and we prove their relation to strong complementarity. We define the Fourier transform in terms of pairs of strongly complementary observables, in both the abelian and non-abelian case. We draw the connection with Pontryagin duality and the theory of unitary symmetries of systems, linking observables to the conserved quantities associated with dynamics. Finally, our work finds application in the novel characterisation of the Fourier transform for the category \(\text{Rel}\) of sets and relations. This is a result of great interest for the study of categorical quantum algorithms, as the usual construction of the quantum Fourier transform in terms of Hadamard matrices is shown to fail in \(\text{Rel}\).

1. Background

Classical structures, aka special commutative \(\dagger\)-Frobenius algebras (\(\dagger\)-SCFAs), play a central role in Categorical Quantum Mechanics (CQM) as the abstract incarnation of non-degenerate observables. The operational aspect of \(\dagger\)-SCFAs is extensively covered in \([1]\) and \([2]\), where they are interpreted as models for the classical data operations of copy, deletion, and comparison.

Their key connection with non-degenerate observables in quantum mechanics is provided by \([3]\), where it is proven that \(\dagger\)-SCFAs in \(\text{fdHilb}\) canonically correspond to orthonormal bases (their unique basis of copyable, or \textit{classical}, states), and can thus be used to model a basis of eigenstates; more generally, commutative \(\dagger\)-Frobenius algebras (\(\dagger\)-CFAs) correspond to orthogonal bases.

In order to model possibly degenerate observables, \([1]\) introduces spectra as coalgebras for the comonoid part of a classical structure. Here classical states are labels for the eigenspace projectors. While classical structures already provide most of the framework necessary to talk about observables, grasping notions of complementary observables requires further tools. This is done by viewing measurements as invariants for certain symmetries.

Strongly complementary pairs of classical structures appear in \([4]\) \([5]\) to model non-locality in terms of non-commutative non-degenerate observables, while in \([6]\) they are shown to correspond to finite abelian groups in \(\text{fdHilb}\).

Some of the material on representation theory in this paper already appears in \([7]\) and \([8]\) in relation to the hidden subgroup algorithm and the group homomorphism identification algorithm, respectively. Finally, the upcoming \([9]\) and \([10]\) provide a comprehensive reference for many structures and results used here.

2. Introduction

In this work, we explore the connection between strong complementarity and finite-dimensional representation theory. In Section 3 we cast the usual treatment of the Fourier transform into the language of monoids in \(\text{fdHilb}\), and show the transform to be a particular monoid isomorphism.

In Section 4 we introduce strong complementarity. This allows a generalization beyond \(\text{fdHilb}\) to strong complementarity of pairs of a quasi-Special \(\dagger\)-Frobenius Algebra (\(\dagger\)-qSFA or \(\dagger\)-qSCFA if commutative) and a \(\dagger\)-SCFA. We use this generalization to embed finite groups and their multiplicative characters in arbitrary dagger symmetric monoidal categories.

In Section 5 we make contact with the monadic dynamics programme of \([11]\) and connect the coalgebraic formulation of spectra to the corresponding algebraic formulation of group actions, proving that observables are exactly the invariants associated with unitary symmetries, where eigenspaces are labelled by the characters of the symmetry group.

In Section 6 the results of Section 3 are abstractly reformulated in terms of general Pontryagin duality,
proving that the existence of the Fourier transform is equivalent to multiplicative characters forming an orthogonal basis of copyables for the *qSCFA of the strongly complementary pair.

In Section 4 these results are extended to the non-abelian case, where matrix elements of irreducible representations give a (non-canonical) orthogonal basis of copyables for the *qSFA. In analogy with Section 5 we prove that the irreducible characters for the symmetry group label the eigenspaces for the observables. The Gelfand-Naimark construction is then presented as the natural generalisation of the Fourier transform using its Pontryagin duality formulation.

In Section 8 we use the classification of strongly complementary observables in Rel [12] to define the appropriate Fourier transform in sets and relations. This provides an example construction of the Fourier transform in a new dagger compact category.

Finally, a note on nomenclature. In this work, the term classical structure will always mean *SCFA, and will be synonymous with abelian non-degenerate (normalised) observable; as this work will also concern itself with the general case of possibly degenerate observables, the qualification in italic cannot in general be omitted. It is to be noted, however, that the term observable finds widespread use in the CQM literature as a synonym to classical structure: to ease the transition, Section 3 will follow this custom, and leave the qualification as understood.

3. The Fourier transform as a monoid isomorphism in fdHilb

We start by recalling some basic facts from group theory. In what follows, \((G, +, 0)\) is a finite abelian group of order \(N\) and we denote the multiplicative group of non-zero complex numbers as \(\mathbb{C}^\times = \mathbb{C} \setminus \{0\}\).

**Definition 3.1.** A (multiplicative) character of \(G\) is a group homomorphism \(\chi : G \to \mathbb{C}^\times\). For finite \(G\), \(\chi\) maps into the unit complex numbers \(S^1\).

**Definition 3.2.** The set of characters with pointwise multiplication \((\chi \cdot \psi)(x) := \chi(x)\psi(x)\) forms a group called the Pontryagin dual or dual group of \(G\), denoted \(G^\wedge\). For finite groups \(G^\wedge \cong G\), albeit not canonically.

Let \(L^2(G)\) denote the space of functions \(f : G \to \mathbb{C}\). These functions are necessarily square-integrable as \(G\) is finite, and thus \(L^2(G)\) is an \(N\)-dimensional complex Hilbert space. In its usual definition, the Fourier transform is a bijection between complex functions on \(G\) and complex functions on its Pontryagin dual \(G^\wedge\).

**Definition 3.3.** The Fourier transform over a finite abelian group \(G\) is the map \(\mathcal{F} : L^2(G) \to L^2(G^\wedge)\) defined by:

\[
(f : G \to \mathbb{C}) \mapsto \left(\hat{f} : G^\wedge \to \mathbb{C}\right)
\]

such that for any \(\chi : G \to \mathbb{C}\)

\[
\hat{f}(\chi) := \sum_{g \in G} \chi^*(g) f(g).
\]

We now show that the Fourier transform can alternatively be viewed as a monoid isomorphism between the convolution monoid and the pointwise multiplication monoid.

**Definition 3.4.** The orthonormal computational basis for \(L^2(G)\) is given by the delta functions \((\delta_g)_{g \in G}\), defined by:

\[
\delta_g(h) = \begin{cases} 1, & \text{if } h = g. \\ 0, & \text{otherwise.} \end{cases}
\]

**Theorem 3.5 (The convolution monoid).** There is a monoid structure \((L^2(G^\wedge), \ast, \delta_0)\) on the computational basis defined by the following convolution operation:

\[
(\delta_g \ast \delta_h) = \delta_{g+h}
\]

**Proof.** From the general definition of convolution we have that

\[
(\delta_g \ast \delta_h)(k) = \sum_{j \in \mathbb{C}} \delta_g(j) \delta_h(-j + k) = \delta_h(-g + k)
\]

\[
= \begin{cases} 1, & \text{if } k = g + h. \\ 0, & \text{otherwise.} \end{cases}
\]

\[
= \delta_{g+h}(k).
\]

Closure of the group then implies closure of the convolution monoid. That \(\delta_0\) acts as the identity is obvious. \(\square\)

The Fourier transforms of the computational basis define the Fourier basis as follows.

**Definition 3.6.** The orthonormal basis formed by \((\hat{\delta}_g)_{g \in G}\) is called the Fourier basis.

\[
\delta_g(\chi) := \sum_{h \in G} \chi^*(h) \hat{\delta}_h(h) = \chi^*(g)
\]

**Theorem 3.7 (The pointwise monoid).** There is a monoid structure \((\{L^2(G^\wedge)\}, \cdot, 1)\) on the Fourier basis given by pointwise multiplication

\[
\left(\hat{\delta}_g \cdot \hat{\delta}_h\right)(\chi) := \hat{\delta}_g(\chi) \hat{\delta}_h(\chi) = \chi^*(g)\chi^*(h)
\]

**Proof.** We first show that the monoid operation is closed. Note that \(\chi^*(g) = (\chi(g))^{-1} = \chi(g^{-1})\) since characters of finite groups are valued in \(S^1\); so \(\chi^*\) is a character for any character \(\chi\). As \(\chi^*\) is a group homomorphism \(\chi^*(g)\chi^*(h) = \chi(g + h)\), and we get

\[
\left(\hat{\delta}_g \cdot \hat{\delta}_h\right)(\chi) = \chi^*(g)\chi^*(h)
\]

\[
= \chi^*(g + h) = \hat{\delta}_{g+h}(\chi).
\]

\[
\left(\hat{\delta}_g \cdot \hat{\delta}_h\right)(\chi) = \chi^*(g)\chi^*(h)
\]

\[
= \chi^*(g + h) = \hat{\delta}_{g+h}(\chi).
\]
We check that $\hat{\delta}_0$ is the identity for pointwise multiplication using the definition of the Fourier transform:

$$\hat{\delta}_0(\chi) = \sum_{g \in G} \chi^*(g)\delta_0(g) = \chi^*(0) = 1,$$

(3.10)

where the last step again follows from the fact that $\chi^*$ is a group homomorphism. Thus

$$(\hat{\delta}_0 \cdot \hat{\delta}_g)(\chi) = \hat{\delta}_0(\chi)\hat{\delta}_g(\chi)$$

(3.11)

$$= \chi^*(0)\chi^*(g) = \chi(g)$$

(3.12)

and likewise $$(\hat{\delta}_g \cdot \hat{\delta}_0) = \hat{\delta}_g.$$ (3.13)

\textbf{Definition 3.8.} The inverse Fourier transform $F^{-1}$ takes a function $f : G \to \mathbb{C}$ to a function $F : G \to \mathbb{C}$ defined as:

$$f(g) = \frac{1}{N} \sum_{\chi : G \to \mathbb{C}} \hat{f}(\chi)(g).$$ (3.14)

\textbf{Theorem 3.9 (Fourier transform monoid iso).} The Fourier transform $F$ is a monoid isomorphism

$$F : (L^2[G], \ast, \delta_0) \to (L^2[G^\wedge], \cdot, \delta_0).$$ (3.15)

\textbf{Proof.} That $F$ has $\delta_0 \mapsto \hat{\delta}_0$ and $F^{-1} : \delta_0 \mapsto \delta_0$ is clear. We then demonstrate that monoid multiplication is mapped appropriately in both directions.

$$F[\delta_g \ast \delta_h] = F[\delta_{g+h}] = \hat{\delta}_g \cdot \hat{\delta}_h = \hat{\delta}_{g+h}$$

(3.16)

$$\Leftrightarrow$$

$$F^{-1}[\hat{\delta}_g \cdot \hat{\delta}_h] = F^{-1}[\delta_{g+h}] = \delta_g \ast \delta_h = \delta_{g+h}$$

(3.17)

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\section{4. Classical groups}

To generalise the constructions of Section \ref{sec:3}, we start by considering how to “embed” finite abelian groups into an arbitrary $\dag$-compact symmetric monoidal category.

\textbf{Definition 4.1.} A quasi-special $\dag$-Frobenius algebra $(\mathbb{A}, \mathbb{I}, \mathbb{M}, \mathbb{N})$ is one that satisfies the following equation:

$$\mathbb{N} = \mathbb{M}$$

(4.1)

for some invertible scalar $N$. We will often refer to them as non-degenerate observables, and use the shorthand $\dag$-qSFA.

\textbf{Definition 4.2.} A pair of quasi-special $\dag$-Frobenius algebras $(\mathbb{A}, \mathbb{I}, \mathbb{M}, \mathbb{N})$ and $(\mathbb{V}, \mathbb{U}, \mathbb{W}, \mathbb{O})$ is strongly complementary if it satisfies the following bialgebra equations:

$$\psi$$

(4.2)

$$\psi$$

(4.3)

We use strong complementarity, together with Theorem \ref{thm:3.5} to embed groups in arbitrary $\dag$-SMCs.

\textbf{Definition 4.3.} A classical group, denoted by $(G, A, B, C, D)$, is a strongly complementary pair of:

(i) a $\dag$-qSFA $(\mathbb{A}, \mathbb{I}, \mathbb{M}, \mathbb{N})$, the group structure,

(ii) a $\dag$-SCFA $(\mathbb{V}, \mathbb{U}, \mathbb{W}, \mathbb{O})$, the point structure.

on the same object $G$ of a $\dag$-SMC. The multiplication and unit for the group structure are called \textbf{group multiplication} and \textbf{group unit}, and the antipode $\mathbb{N}$ for the pair is called the \textbf{group inverse}. An \textbf{abelian classical group} is one where the group structure is commutative.

In this definition, the group structures is a non-degenerate observable while the point structure is an abelian non-degenerate normalised observable. This is a choice dictated by convenience: all the results to follow can be easily adapted to the case where the point structure is not normalised (i.e. is also a $\dag$-qSCFA).

\textbf{Definition 4.4.} The copyable states (or copyables) for a comonoid $(G, C, D)$ in a $\dag$-SMC are the maps $\psi : I \to G$ such that

$$\psi$$

(4.4)

\textbf{Theorem 4.5.} If $(G, A, B, C, D)$ is an (abelian) classical group in any $\dag$-SMC, then $(A, B)$ acts as an (abelian) group $G$ on the set of copyables of $(C, D)$, henceforth the \textbf{group elements}. Furthermore, this correspondence is an equivalence between the category of (abelian) classical groups in $\text{fdHilb}$ and the category of finite (abelian) groups.
Proof. The group action is an easy check: multiplication of copyables for \((\uparrow, \bullet)\) is still a copyable by Equation (4.2) and the unit \(\bullet\) is a copyable by the first equation in 5.3. Section 7.2 of [6] proves that in \(\text{fdHilb}\) the copyables are exactly the group elements, forming an orthogonal basis. A morphism between classical groups is a morphism \(f : G \to H\) of the \(\dagger\)-SMC such that \(f\) is a covariant homomorphism of the point structures and \(f^\dagger\) is a contravariant homomorphism of the group structures. In \(\text{fdHilb}\), the covariant part of the statement is equivalent to \(f\) acting as a classical map of group elements, while the contravariant part is equivalent to \(f^\dagger\) being a group homomorphism. \(\square\)

In \(\text{fdHilb}\), the point structure \((\uparrow, \bullet)\) clearly characterises the group elements \(|g\rangle_{G,G}\) as an orthonormal basis for \(G \cong L^2(G)\). These are the same basis of delta functions from Definition 3.4 in the previous section, with vector \(|g\rangle : L^2[G]\) corresponding to \(\delta_g\). If \(f : G \to \mathbb{C}\) is a (square-integrable) function, then we have the following relation between the corresponding vector \(|f\rangle : L^2[G]\) and the basis of group elements:

\[
|f\rangle = \sum_{g \in G} f(g)|g\rangle \tag{4.5}
\]

The multiplicative fragment \((\uparrow, \Rightarrow)\) of the group structure corresponds to the convolution monoid of Theorem 3.5.

**Definition 4.6.** A **multiplicative character** for an internal monoid \((\Lambda, \bullet)\) in a \(\dagger\)-SMC is an effect \(\Lambda\) satisfying the following two equations:

\[
\begin{align*}
\Lambda \circ \bullet &= \Lambda \circ \Lambda \tag{4.6} \\
\Lambda \circ \Lambda &= \Lambda \circ \Lambda \tag{4.7}
\end{align*}
\]

i.e. it can be seen as a monoid homomorphism from \((\Lambda, \bullet)\) to the canonical monoid on the trivial object induced by the unitors.

**Theorem 4.7.** If \((G, \Lambda, \bullet, \delta', \delta)\) is a classical group in a \(\dagger\)-SMC, then the copyables of the group structure are exactly the (adjoints of its) multiplicative characters. In the case of \(\text{fdHilb}\), the group structure of an abelian classical group thus characterises the (group theoretic) multiplicative characters of \(G\) as an orthogonal basis for the **linear dual space** \(G^*\), all characters having norm \(N\) from Equation (4.7).

Proof. The first part is immediate, the second follows from the results in [3]. \(\square\)

The multiplicative characters defined here live in \(L^2(G)^*\), while those from Definition 5.1 are functions \(G \to \mathbb{C}\). The relation between the two is that the multiplicative characters of Definition 4.6 coincide, when evaluated on the basis of group elements for \(L^2(G)\), with those of Definition 5.1.

**Theorem 4.8.** If \((\mathcal{G}, \Lambda, \Lambda, \delta', \delta)\) is a classical group in a \(\dagger\)-SMC then the diagonal map \(\delta'\) acts as a group, the **pointwise multiplication** group, on the multiplicative characters, with the **trivial character** \(\delta\) as its unit. Furthermore, multiplicative characters are always closed under conjugation, with the group inverse conjugating them.

Proof. The group action follows from strong complementarity. Closure under conjugation is due to the fact that, for any \(\dagger\)-Frobenius algebra, the conjugate of the multiplication is its composition with a swap. The action of the group inverse is an easy graphical check. \(\square\)

The group \(\hat{G}\) given by \((\uparrow, \bullet)\) acting on the multiplicative characters is the Pontryagin dual of the group \(G\) given by \((\Lambda, \bullet)\) acting on the group elements (it is the same group of Definition 3.2).

Finally, it is worth clarifying that the pointwise monoid of Theorem 3.7 is different from the pointwise multiplication of Theorem 4.8: the former is a pointwise product of functions of characters, and thus corresponds to the monoid \((\Lambda, \bullet)\) (because the group comultiplication duplicates multiplicative characters), while the latter is pointwise product of functions of group elements, and thus corresponds to the monoid \((\Lambda, \bullet)\) (which duplicates group elements).

### 5. Dynamics and observables

In the language of [11], dynamics are actions of internal monoids on systems, i.e. algebras of a certain monad. On the other hand, in the language of [1], observables are defined to be coalgebras of a certain comonad. These two perspectives are dual.

**Definition 5.1.** The **dynamics** for a classical group \((\mathcal{G}, \Lambda, \Lambda, \delta', \delta)\) in a \(\dagger\)-SMC are defined to be the morphisms \(\Lambda : H \otimes \mathcal{G} \to H\) satisfying the following two equations:

\[
\Lambda \circ \Lambda' = \Lambda \circ \Lambda \tag{5.1}
\]
Furthermore the dynamics are said to be **unitary** if they are controlled unitaries for the point structure, i.e. if they satisfy the following two additional equations:

\[ \begin{align*}
\text{(5.2)} \\
\text{(5.3)}
\end{align*} \]

An equivalent formulation of unitarity for dynamics is given by Equation 5.5 below.

**Definition 5.2.** The observables for a classical group \((G, \cdot, \ast, \varepsilon, \lambda)\) in a \(\dagger\)-SMC are defined to be the maps \(\mathcal{H} \to \mathcal{H} \otimes G\) which are self-adjoint, idempotent and complete with respect to the group structure:

\[ \begin{align*}
\text{(5.5)} \\
\text{(5.6)}
\end{align*} \]

1. Note that these observables are possibly degenerate, possibly un-normalized, and possibly non-abelian.

**Theorem 5.3.** The unitary dynamics for a classical group \((G, \cdot, \ast, \varepsilon, \lambda)\) in a \(\dagger\)-SMC, are exactly the adjoints of the observables. In particular, this holds for the dynamic and the (non-degenerate) observable.

**Proof.** Equation follows from controlled unitarity (Equation 5.3), Equation 5.1 and the Hopf law. In the other direction, Equation is proven from Equation 5.3 using Equation and pushing the antipodes down (it is useful to note that the antipode introduces a swap as it is pushed up/down through the group multiplication).

6. **Abelian Fourier transform**

We can now connect the Fourier transform of Section to the strong complementarity of (abelian non-degenerate) observables, via Pontryagin duality. We begin with a technical lemma, showing that the existence of a resolution of the identity in terms of multiplicative characters is equivalent to the characters providing the spectral decomposition for observables, i.e. labelling their complete set of projectors.

In the abelian case, the existence of such a resolution of the identity is trivial, as it follows from the existence of an orthogonal basis of characters; in the non-abelian case, on the other hand, things will get more interesting.

**Theorem 6.1.** Given an abelian classical group \((G, \cdot, \ast, \varepsilon, \lambda)\) in \(\text{fdHilb}\), the following three facts are equivalent:

(i) there is an orthonormal family \((\chi)\chi\) of multiplicative characters which forms a partition of the counit, i.e. which satisfies \(\sum \chi = \cdot\).

(ii) any space \(\mathcal{H}\) endowed with a unitary dynamic \(\mathcal{H}\) is covered by a complete family of projectors, labelled by the multiplicative characters, via the character-valued spectrum.

(iii) the 1-dimensional projectors corresponding to the multiplicative characters form a resolution of the identity.
implies a resolution (iii) of the identity for a family of projectors applied to the group (co)multiplication identity by Equation 5.7. The existence (ii) of a complete projector, and these projectors form a resolution of the identity is in turn equivalent to the multiplicative characters forming an orthogonal basis, which finally implies that the multiplicative characters form a partition (i) of the counit as the multiplicative characters are a basis indeed the operator associated with each copyable will be self-adjoint and idempotent by Equations 5.5 and 5.6, i.e. for any abelian classical group $G = (L^2[G], \hat{\cdot}, \cdot, \hat{\chi}, \chi)$ in fdHilb, the Fourier transform is defined to be the map:

\[
\hat{f} \rightarrow \sum_{\chi} \frac{1}{N} \chi f \chi(g) = \int \chi f \chi(g) d\mu(g)
\]  

The Fourier transform as defined here is a (canonical) isomorphism $L^2[G] \xrightarrow{\cong} L^2[G^\wedge] = L^2[G]^*$.

The resolution of the cap in Equation 6.3 corresponds to the resolution of the identity in Equation 6.1 and takes into account the dual nature of characters and group elements: in this light, Theorem 6.1 can be re-interpreted to state that the existence of the Fourier transform is equivalent to the spectral decomposition of observables in terms of characters.

Armed with the results of Section 4, we introduce Pontryagin duality within our framework. The use of the $L^2[G]$ notation is consistent with the fact that $L^2$-spaces over finite groups are exactly finite-dimensional Hilbert spaces that come with a canonical choice of basis (the group elements) and a group operation over them. We have identified $L^2[G]\cong L^2[G]^*$ as the multiplicative characters are a basis of $L^2[G]^*$.

**Theorem 6.2. (Pontryagin duality)**

In fdHilb every classical group takes the form $G = (L^2[G], \hat{\cdot}, \cdot, \hat{\chi}, \chi)$ for some finite group $G$. If $G$ is abelian, then the classical abelian group corresponding to its Pontryagin dual is $G^\wedge = (L^2[G]^*, \hat{\cdot}, \cdot, \hat{\chi}, \chi)$. Thus the Pontryagin dual $(G^\wedge)^\wedge$ of the Pontryagin dual $G^\wedge$ of a classical group $G$ is the classical group $G$ itself.

**Proof.** The form of $G$ follows from Section 4 and so does the form corresponding to the Pontryagin dual $G^\wedge$. The final statement is just the observation that, in terms of classical groups, Pontryagin duality consist of exchanging $L^2[G]$ with $L^2[G]^*$ and $(\hat{\cdot}, \cdot)$ with $(\hat{\chi}, \chi)$, and hence it is trivially an involution.

Pontryagin duality is usually formulated as the statement that there is a canonical group isomorphism $G \cong \hat{G}$:

\[
g \mapsto (\chi \mapsto \chi(g))
\]

In the context of Theorem 6.2, this canonical isomorphism corresponds to the canonical inner product between group elements (living in $L^2[G]$) and the multiplicative characters (living in $L^2[G]^*$), and manifests itself as the identity being a canonical isomorphism of classical groups $G \cong (G^\wedge)^\wedge$.

Having established Pontryagin duality, we can pick up where Section 3 left off and extend the Definition 3.8 of the Fourier transform to classical groups in fdHilb.

**Definition 6.3.** For any abelian classical group $G = (L^2[G], \hat{\cdot}, \cdot, \hat{\chi}, \chi)$ in fdHilb, the Fourier transform is defined to be the map:

\[
\hat{f} \rightarrow \sum_{\chi} \frac{1}{N} \chi f \chi(g) = \int \chi f \chi(g) d\mu(g)
\]  

The Fourier transform as defined here is a (canonical) isomorphism $L^2[G] \xrightarrow{\cong} L^2[G^\wedge] = L^2[G]^*$.

The resolution of the cap in Equation 6.3 corresponds to the resolution of the identity in Equation 6.1 and takes into account the dual nature of characters and group elements: in this light, Theorem 6.1 can be re-interpreted to state that the existence of the Fourier transform is equivalent to the spectral decomposition of observables in terms of characters.

In fdHilb, the group structure $(\cdot)$ characterises the multiplicative characters $|\chi\rangle_{\chi:G^\wedge}$ as an orthogonal basis for $L^2[G]^* = L^2[G^\wedge]$. If $f : G^\wedge \rightarrow \mathbb{C}$ is a the Fourier transform of a (square-integrable) function $f : G \rightarrow \mathbb{C}$, then we have a nice relation between the vector corresponding to its Fourier transform $\langle f \rangle : L^2[G]^*$ and the basis of multiplicative characters:

\[
\langle f \rangle = \sum_{\chi:G^\wedge} \langle \chi | f \rangle \chi(g)
\]

This corresponds to Equation 3.2. The fundamental point here is that Fourier transform in fdHilb manifests itself canonically as a change of basis, and Theorem 6.4, the key result of Section 3, is seen to correspond to the following result.

**Theorem 6.4.** The Fourier transform from Definition 6.3 is an isomorphism of monoid structures $(L^2[G], \hat{\cdot}, \cdot) \rightarrow (L^2[G]^*, \hat{\cdot}, \cdot)$. 

**Proof.** The Fourier transform from Equation 6.3 is just transposition. Transposing $(L^2[G], \hat{\cdot}, \cdot)$ does not exactly yield $(L^2[G]^*, \hat{\cdot}, \cdot)$, as there is an additional
swap on the output wires of \( \otimes \), but \( \otimes \) is commutative and the swap can be removed. \( \square \)

Finally, the Fourier transform presented here has some key differences with respect to the usual presentation of the quantum Fourier transform, which relies on the Hadamard transform defined below.

**Definition 6.5.** A **Hadamard transform** is a monoid isomorphism \( h : (\mathbb{1}, \mathbb{1}) \rightarrow (\mathbb{1}, \mathbb{1}) \) s.t. \( h^\dagger \) is also a monoid isomorphism \( h^\dagger : (\mathbb{1}, \mathbb{1}) \rightarrow (\mathbb{1}, \mathbb{1}) \). In \( \text{fdHilb} \) this is any group isomorphism mapping the group elements to the multiplicative characters.

The traditional approach to the quantum Fourier transform proceeds as follows:

1. prepare a state in the computational basis,
2. apply a Hadamard transform (which is equivalent to a non-canonical choice of isomorphism \( G \cong G^\wedge \)),
3. do something interesting,
4. measure the state in the computational basis.

The Fourier transform from Definition 6.3 on the other hand, corresponds to the following approach to quantum Fourier transform:

1. prepare a state in the computational basis,
2. apply the Fourier transform (which is just transposition),
3. do something interesting,
4. measure the state in the character basis.

Aside from being more in line with the usual theory on Fourier transform, the latter approach has the advantage of being canonical, i.e. of not involving any arbitrary choice of isomorphism \( G \cong G^\wedge \). It is true that (subject to a choice of isomorphism) the Hadamard transform can be used to “implement” the Fourier transform in \( \text{fdHilb} \), but the fundamental operational point is that the Fourier transform maps classical points of one structure to unbiased points of the the dual structure. We will see that only this latter point of view survives the transfer to the category \( \text{Rel} \) of sets and relations.

## 7. Generalised Fourier transform

For non-abelian groups, the multiplicative characters, while still orthogonal, never form a basis for the dual space. As a consequence, the second part of Theorem 4.7 as well as the whole of Theorem 6.1 and Theorem 6.3 fail. In order to cover the representation theory of non-abelian groups, we have to take a step back.

In \( \text{fdHilb} \), the dynamics for a group are exactly its (necessarily unitary) complex linear representations. We see that transposing the \( \mathcal{H} \) input wires into output wires in Equations 5.1 and 5.2 yields the same graphical definition for representations of a group given in [7].

**Definition 7.1.** The **representations** of a classical group \( (\mathcal{G}, \mathcal{H}, \otimes, \mathcal{H}^\dagger, \mathcal{H}) \) in a \( \dagger\text{-SMC} \) are defined to be the morphisms \( \rho : \mathcal{G} \rightarrow \mathcal{H} \otimes \mathcal{H}^\dagger \) satisfying the following two equations:

\[
\begin{align*}
\rho & = \rho \\
\rho & = \rho
\end{align*}
\]

A representation is said to be **unitary** if

\[
\rho = \rho
\]

It is thus of no surprise that:

**Theorem 7.2.** The unitary representations of a classical group \( (\mathcal{G}, \mathcal{H}, \otimes, \mathcal{H}^\dagger, \mathcal{H}) \) in a \( \dagger\text{-SMC} \) are exactly the maps obtained from unitary dynamics by transposing the space input wire to an output wire.

**Definition 7.3.** A **character** of a classical group \( (\mathcal{G}, \mathcal{H}, \otimes, \mathcal{H}^\dagger, \mathcal{H}) \) in a \( \dagger\text{-SMC} \) is the trace of a representation:

\[
\chi_\rho := \rho
\]

A character is **unitary** if the representation is. In particular, multiplicative characters are both unitary representations and characters.

The following is the non-abelian result corresponding to the second part of Theorem 4.7 and the third part of Theorem 6.1. While in the abelian case the existence of a resolution of the identity was an easy fact, in the non-abelian case things are more complicated.

**Theorem 7.4.** *(finite-dim Peter-Weyl)*

If \( (L^2[\mathcal{G}], \mathcal{H}, \otimes, \mathcal{H}^\dagger, \mathcal{H}) \) is a classical group in...
fdHilb, then there is a finite orthogonal family \( \text{Irr}[G] \) of unitary representations \( \rho : G \to V_\rho \otimes V_\rho^* \), the irreps (or irreducible representations) for the group, which form a resolution of the identity:

\[
\forall \rho, \rho' : \text{Irr}[G], \quad \sum_{\rho \in \text{Irr}[G]} \frac{d_\rho}{N} = 1 \quad (7.5)
\]

Equivalently, the matrix elements of the representations in \( \text{Irr}[G] \) form an orthogonal basis of \( G^* \) (but not a canonical one, as any choice of orthonormal bases for the \( V_\rho \) and the \( V_\rho^* \) will yield one such family of orthogonal matrix elements). The norm of a matrix element of \( a \) is \( 1/\sqrt{d_\rho} \), where \( d_\rho \) is the dimensionality of \( V_\rho \).

**Proof.** This is the finite-dimensional special case of the Peter-Weyl theorem. See, for example, [13] and [14]. Note that in both equations the double wires are dependent on the representation. However, the RHS of Equation 7.5 is well defined because of the existence and uniqueness, in Ab-enriched categories, of the zero map between any two spaces. \( \square \)

The following result will be key to irreducible characters replacing multiplicative characters in the non-abelian case.

**Definition 7.5.** A comonoid \((x, y)\) is a matching for a set \( S \) if for all states \( x, y : S \) we have the following equation

\[
\begin{cases}
  x, & \text{if } x = y, \\
  0, & \text{otherwise.}
\end{cases} \quad (7.7)
\]

**Theorem 7.6.** The irreducible characters (the characters of the irreps from Theorem 7.4) of a classical group \((L^2[G] \ , \ \rho \ , \ , \ , \ , \ , \ \rho)\) in fdHilb can be scaled to give a family \((\frac{d_\rho}{N})_{\rho \in \text{Irr}[G]}\) for which \((x, y)\) is a matching, exactly like multiplicative characters in the abelian case.

**Proof.** Follows from Equation 7.6 (applied above the \( \rho \)), Equation 7.1 (to eliminate \( \rho \)), and Equation 7.5. \( \square \)

The following theorem is the non-abelian result corresponding to the first two parts of Theorem 6.1. As a consequence of Theorem 7.4 irreducible characters are seen to provide the spectral decomposition for observables in the non-abelian case.

**Theorem 7.7.** Given a classical group \((L^2[G] \ , \ , \ , \ , \ , \ \rho)\) in fdHilb, the irreducible characters \((\frac{d_\rho}{N})_{\rho \in \text{Irr}[G]}\) form an orthogonal partition of the counit, i.e. satisfy \( \sum_{\rho} \frac{d_\rho}{N} = 1 \). As a consequence, any space \( \mathcal{H} \) endowed with a unitary dynamic \( \rho \) is covered by a complete family of projectors, labelled by the irreducible characters via the character-valued spectrum \( \rho \).

**Proof.** Partition of the counit follows from Equation 7.6 and Equation 7.2 The rest goes as Theorem 6.1. \( \square \)

Lastly, the following is the non-abelian form of the Fourier transform corresponding to that presented in Definition 6.3 and yielding the non-abelian Fourier theory of [14]. A decomposition in terms of a basis of (normalised) matrix coefficients for \( L^2[G]^* \), analogous to the decomposition in terms of a basis of multiplicative characters of Definition 6.3 can be obtained by fixing orthonormal bases for all the \( V_\rho \) and \( V_\rho^* \) spaces.

\[
\mathfrak{f} \quad \hat{f} \quad \text{def} \quad \sum_{\rho} \frac{d_\rho}{N} = \frac{d_\rho}{N} \quad (7.8)
\]

Fourier transform is then, in the most general case, a change of basis from the group elements to a basis of (normalised) matrix elements for the irreps of the group. This is not, however, an exact generalisation of Theorems 6.3 and 6.2 as it misses the point of algebraic duality between \( L^2[G]^\land \) and \( L^2[G]^* \). The correct general statement goes through the following GNS construction.

**Theorem 7.8.** *(finite-dim Gelfand-Naimark)*

Let \((L^2[G] \ , \ , \ , \ , \ , \ \rho)\) be a classical group in fdHilb, and \( \text{Irr}[G] \) its family of irreps. Let \( V := \bigoplus_{\rho \in \text{Irr}[G]} V_\rho \) and let \( \iota_\rho : V_\rho \hookrightarrow V \) be the injections of subspaces. Then
the following map is a unitary representation of the classical group, and an isometry.

\[
\bigoplus_{\rho \in \text{Irr}(G)} \rho := \sum_{\rho \in \text{Irr}(G)} \frac{d_{\rho}}{\sqrt{N}} \sum_{\mu} \rho_{\mu} \otimes \rho_{\mu}^* \rho \in L^2(G)
\]  

(7.9)

**Proof.** Immediate to see that it is linear and a representation (i.e. a representation of unital Banach algebras), and furthermore a unitary representation (i.e. a representation of unital C*-algebras). Isometry can be proven on the orthonormal group elements basis, by considering Equations 7.1 and 7.3 remembering that inner product \(\langle A|B \rangle\) in the algebra of operators is the trace \(\text{Tr}[A^*B]\) and applying the first part of Theorem 7.2 (the fact that irreducible characters form a partition of the count).

\[\square\]

### 8. The Fourier transform in the category \(\text{Rel}\) of sets and relations

The abstract correspondence between strongly complementary observables and Fourier transforms in Definition 6.3 means that a characterization of strongly complementary observables in any dagger compact category could be viewed as a construction of the Fourier transform in that category. In this section, we apply this idea to \(\text{Rel}\), the category of sets and relations. In this setting, the relevant classical structures are no longer groups but are groupoids\(^4\) as the following theorem from \(\cite{12}\) shows.

**Theorem 8.1.** Classical structures in the category of sets and relations correspond to abelian groupoids. In particular, the multiplication of a classical structure on some set \(A\) acts as the multiplication \(\mu\) of a groupoid on the elements of \(A\) (the elements of the groupoid):

\[
a \cdot \mu b = \begin{cases} 
\text{some singleton \{a \cdot b\}} & \text{if } \text{defined} \\
\emptyset & \text{otherwise}
\end{cases}
\]  

(8.1)

The unit of the classical structure is the set \(\eta \subseteq A\) comprising all the identities for the groupoid, i.e. all the \(u : A\) such that

\[
u \cdot \mu a = a = u \cdot \mu u
\]

(8.2)

whenever the multiplication is defined (i.e. does not return the empty set).

In \(\cite{10}\), it is shown that complementarity in \(\text{Rel}\) can be understood as follows.

**Theorem 8.2.** The following are equivalent for groupoids \(G\) and \(H\) with the same set \(A\) of morphisms:

(a) their dagger special Frobenius monoids are complementary;

(b) the map \(A \rightarrow \text{Ob}(G) \times \text{Ob}(H)\) given by \(a \mapsto (\text{dom}_G(a), \text{dom}_H(a))\) is a bijection.

Furthermore, all such complementary groupoids\(^4\) are of the particular form \(\cite{10}\) given by the following theorem.

**Theorem 8.3.** Let \(G\) and \(H\) be nontrivial groups. Let \(G\) be the groupoid with objects \(G\) and homsets \(G(g,g) = H\) and no morphisms between distinct objects, and let \(H\) be the groupoid with objects \(H\) and homsets \(H(h,h) = G\) and no morphisms between distinct objects. Then \(G\) and \(H\) give rise to complementary Frobenius algebras.

The fact that these groupoids are not only complementary, but also strongly complementary is expressed in the following theorem, also from \(\cite{10}\).

**Theorem 8.4.** Any two complementary special dagger Frobenius structures in \(\text{Rel}\) form a bialgebra.

In particular, the bialgebra of our complementary classical structures - as \(\dagger\)-SCFAs - means that they will be strongly complementary, by Definition 4.2.

This latter point, taken together with Definition 6.3 motivates the following construction. Consider groupoids \(Z\) and \(X\) such that \(Z = \bigoplus_{|H|} G\) and \(X = \bigoplus_{|G|} H\), where \(H\) and \(G\) are abelian groups. These two groupoids will be strongly complementary by Theorem 8.3 and thus they will define a abelian classical group in \(\text{Rel}\) by Definition 4.3 where the classical “group” multiplication is the groupoid multiplication on \(Z\).

In particular, \(Z\) will act as a group (not a groupoid) on the classical points of \(X\), and \(X\) will act as a group on the classical points of \(Z\); in contrast to what happened in \(\text{fdHilb}\), the two groups will in general be non-isomorphic. This means that there is no Hadamard transform in \(\text{Rel}\) under Definition 6.5.

In \(\text{Rel}\), points are simply subsets, as they are morphisms (i.e. relations) from the monoidal unit \(\{\cdot\}\) to a set. Seen this way, subsets are the relational analogs of vectors in \(\text{fdHilb}\), with orthogonality of vectors corresponding to disjointness of subsets. Copyables for observables are characterized in the following way.

**Theorem 8.5.** Given a groupoid \(Z = \bigoplus_{|H|} G\), Theorem 8.2 shows that we can use \(G\) and \(H\) to index its underlying set

\(^4\) It was shown in \(\cite{15}\) that this correspondence also extends to the noncommutative case, i.e. non-commutative groupoids correspond to \(\dagger\)-special Frobenius algebras.

\(^5\) This holds when we allow ourselves to consider the groups \(G\) and \(H\) up to isomorphism.
The copyable points of a groupoid $Z$ are exactly the subsets of $A$ in the form $G_h$ for some $h : H$, i.e. the sets of elements of each individual copy of group $G$ making up $Z$.

If $Z = \bigoplus^{|H|} G$ and $X = \bigoplus^{|G|} H$, then we can equivalently write $A \cong \bigcup_{h:H} G_h$ or $A \cong \bigcup_{g:G} H_g$, where $H_g$ is defined dually to $G_h$. The copyable points of $Z$ are pairwise disjoint, and always form a partition of $A$; so do the copyable points of $X$.

**Example 8.6.** For the four element groupoid $\mathbb{Z}_4$ there is only a single copyable point: $\{0, 1, 2, 3\}$. For its complement groupoid $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, there are four copyable points: $\{0\}, \{1\}, \{2\}, \{3\}$.

By virtue of Theorem 6.7 we are now able to understand the copyable points of $Z$ as multiplicative characters for the groupoid structure. This yields a relational analogue to spectral decomposition as long as the multiplicative characters form a “character basis” in the sense of Theorem 6.1 (iii), i.e. by yielding a resolution of the identity. We restrict this possibility by showing that the copyable points of a groupoid $Z$ form a resolution of the identity if and only if the groupoid satisfies a certain condition.

**Theorem 8.7.** The copyable points of $Z$ form a resolution of the identity for the underlying set if and only if $Z = \bigoplus^{|H|} \mathbb{Z}_1$.

**Proof.** In Rel the scalars are 0 or 1 and summation is given by set union. Thus a resolution of the identity must satisfy the following equation, where each $\chi$ is a copyable point:

$$
\bigcap_{\bigcup_x \chi \subseteq \mathbb{Z}} \chi = 1
$$

In the specific case of $Z = \bigoplus^{|H|} \mathbb{Z}_1$, we have that copyable points are in the form $\chi = \{(\ast, h)\}$ and Equation 8.3 reads

$$
\bigcup_{h:H} \{(\ast, h)\} \circ \{(h, \ast)\} = \bigcup_{h:H} \{(h, h)\} = \text{id}
$$

When $Z$ is of the generic form $Z = \bigoplus^{|H|} G$, the copyable points are in the form $\chi = G_h$, and Equation 8.3 reads

$$
\bigcup_{h:H} \{(\ast, g')|g' : G_h\} \circ \{(g, \ast)|g : G_h\} = \bigcup_{h:H} \{(g, g')|g, g' : G_h\} = \text{id}
$$

which cannot be the identity if $|G_h| > 1$.

Define the span $\langle s_j \rangle_{j:J}$ of a family of points $s_j \subseteq A$ to be the set of all points $r \subseteq A$ which can be obtained as the union $r = \bigcup_{j:J} s_j$ of some subfamily $(s_j)_{j:J}$ of the points. If the $(s_j)_{j:J}$ are pairwise disjoint, then the points $r : \langle s_j \rangle_{j:J}$ are exactly those that descend to boolean functions over the set $\{s_j \text { s.t. } j:J\}$.

**Theorem 8.8.** Let $Z = \bigoplus^{|H|} G$ and $X = \bigoplus^{|G|} H$ be strongly complementary groupoids on some set $A$. If $r \subseteq A$ is a point of $A$ in the span $\langle G_h \rangle_{h:H}$ of the copyable points of $X$, then the following holds:

$$
\bigcap_{g:G} G_g \bigcup_{h:H} H_h = \bigcup_{S_G \subseteq G} \bigcup_{S_H \subseteq H} (S_G \cap S_H) = \bigcup_{S \subseteq A} \bigcup_{\{0, 1\}^S} (S \cap (\mathcal{F}(S) \cup \mathcal{F}(\overline{S})))
$$

If $r \subseteq A$ is a point of $A$ in the span $\langle G_h \rangle_{h:H}$ of the copyable points of $Z$, then the following holds:

$$
\bigcap_{S_G \subseteq G} G_g \bigcup_{h:H} H_h = \bigcup_{S_H \subseteq H} (S_H \cap (\mathcal{F}(S_H) \cup \mathcal{F}(\overline{S_H})))
$$

**Proof.** By definition of the span, contraction of $r$ with a copyable point will either give the copyable point itself or the empty set. Furthermore, the copyable points form a partition of $A$ into disjoint subsets. Thus $r$ is reconstructed as the union of exactly all the copyable points that have non-vanishing intersection with it.

The set $\langle H_g \rangle_{g:G}$, seen as the set $\{0, 1\}^G$ of boolean functions on $\langle H_g \text { s.t. } g : G\rangle$, plays the role in Rel that $L^2[G]$ played in fHilb, and similarly $\langle G_h \rangle_{h:H}$, seen as $\{0, 1\}^H$, is the analogue of $L^2[H]$. Furthermore, on $\langle G_h \rangle_{h:H}$, the map $r \mapsto \hat{r}$ is self-inverse (taking appropriate adjoints), so we would expect some analogue of Theorem 6.3 stating that the Fourier transform in Rel corresponds to a (canonical) isomorphism $\{0, 1\}^G \cong \{0, 1\}^H$. Unfortunately, this is not the case.

**Theorem 8.9.** Let $Z = \bigoplus^{|H|} G$ and $X = \bigoplus^{|G|} H$ be strongly complementary groupoids on some set $A$. Then

$$
\langle H_g \rangle_{g:G} \cap \langle G_h \rangle_{h:H} = \{0, A\}
$$

In particular, the Fourier transform does not correspond to an isomorphism $\{0, 1\}^G \cong \{0, 1\}^H$, nor can be restricted to an isomorphism $S_G \cong S_H$ for any $S_G \subseteq \{0, 1\}^G$ and $S_H \subseteq \{0, 1\}^H$ containing non-constant functions. Thus the best analogue to Theorem 6.3 is the trivial statement that

$$
\{0_G, 1_G\} \cong \{0_H, 1_H\}
$$

where $0, 1$ are the constant functions in $\{0, 1\}^G$ and in $\{0, 1\}^H$.

In fHilb, there are three equivalent ways of seeing the Fourier transform:
1. as quantum Fourier transform, implemented by application of a Hadamard transform (subject to a non-canonical choice of isomorphism $G \cong G^\otimes$) followed by measurement in the computational basis.

2. in the sense of Pontryagin duality, as a canonical isomorphism $L^2[G] \cong L^2[G^\otimes]$.

3. again as quantum Fourier transform, but implemented by measurement in the strongly complementary character basis.

In Rel, things are very different:

1. except in the case where $Z = \bigoplus G$ and $X = \bigoplus G$ (isomorphic, but different), no Hadamard transform can exist in Rel.

2. the Fourier transform does not give, in Rel, an isomorphism $\{0,1\}^G \cong \{0,1\}^H$ between the spaces of boolean-valued functions on the group elements / multiplicative characters.

3. preparing a state in the group element span $\{H_g\}_{g \in G}$ and then measuring it in the strongly complementary character family $(G_h)_h$, on the other hand, is a well-defined operation in Rel.

The following examples give explicit examples of quantum Fourier transforms in Rel.

**Example 8.10.** Take $G = \mathbb{Z}_2 = \{0,1\}$, $H = \mathbb{Z}_4 = \{\ast\}$, $Z = G \oplus G = \{0,1\} \times \{0,1\}$ and $X = H \oplus H = \{0,1\} \times \{0,1\}$. The computational basis is the family $(H_g)_{g \in G}$ of copyable points for $X$, i.e. $H_0 = \{(0,0)\}$ and $H_1 = \{(1,1)\}$. The character family used for the quantum Fourier transform consists a single copyable point $G_* = \{(0,0), (1,1)\}$ for $Z$. In this case all states can be prepared in the computational basis, but the measurement in the character family will be trivial.

**Example 8.11.** Take $G = \mathbb{Z}_2 = \{0,1\}$, $H = \mathbb{Z}_2 = \{a,b\}$, $Z = G \oplus G = \{0,1\} \times \{0,1\}$ and $X = H \oplus H = \{0,1\} \times \{0,1\}$. The computational basis is the family $(H_g)_{g \in G}$ of copyable points for $X$, i.e. $H_0 = \{(0,0),(0,1)\}$ and $H_1 = \{(1,0),(1,1)\}$. The character family used for the quantum Fourier transform is the family $(G_h)_h$ of copyable points for $Z$, i.e. $G_a = \{(a,0),(a,1)\}$ and $G_b = \{(b,0),(b,1)\}$.

It part of the CQM perspective that the operational features of quantum theory can be modelled categorically, and that any category sharing features with fdHilb should be considered, at least in principle, as a potential model of quantum mechanics. The category Rel is then one example model.

Hadamard matrices do not generalise well outside fdHilb, and certainly fail to implement a quantum Fourier transform in Rel, but our treatment of Fourier theory based on strong complementarity goes through unharmed. As a consequence, categorical quantum algorithms where the quantum Fourier transform is implemented via strong complementarity will straightforwardly generalise to Rel. This is a good indicator that our approach to Fourier theory is appropriate.

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