Corvino’s construction using Brill waves

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Abstract

For two-black-hole time-symmetric initial data we consider the Corvino construction of gluing an exact Schwarzschild end. We propose to do this by using Brill waves. We address the question of whether this method can be used to reduce the overall energy, which seems to relate to the question of whether it can reduce the amount of ‘spurious’ gravitational radiation. We find a positive answer at first order in the inverse gluing radius.

1 Introduction

Processes involving collisions and mergers of black holes are amongst the most promising candidates regarding the generation of sufficiently strong gravitational waves. Hence much analytical and numerical effort went into the modeling of such processes as Cauchy problems for the matter-free Einstein equations. Initial data consist of a triple \((\sigma, h, K)\), where \(\Sigma\) is a three manifold (the Cauchy hypersurface) and \(h, K\) are two symmetric second-rank tensor fields satisfying a system of four underdetermined-elliptic differential equations, the constraints. \(h\) is a Riemannian metric for \(\Sigma\) and \(K\) essentially its time derivative. Here we shall only be concerned with data representing two black holes at the moment of time symmetry. The latter condition is equivalent to \(K = 0\). Physically this corresponds to the most simple situation where two uncharged and unspinning black holes are placed at some distance, without relative velocity and without any overall linear or angular momentum. We will even assume equal hole masses so that the only free parameters are the mass and the mutual distance.

Data representing two holes at the moment of time symmetry have been known for long and can be written down analytically (see e.g. [5] for a recent review). One might think that the really hard part is their time evolution by means of a sophisticated combination of analytical and numerical techniques. However, there are also unsolved...
conceptual and analytical problems regarding just the initial data. One of them is the problem of 'spurious' gravitational radiation, which we address here.

**Figure 1:** Conformal diagram of spacetime with two spacelike hypersurfaces $\Sigma$ and $\Sigma'$ ending at spacelike infinity $i_0$ and future lightlike infinity $J^+$ respectively. The difference between the ADM-energy of $\Sigma$ (computed at $i_0$) and the Bondi mass of $\Sigma'$ (computed at the intersection 2-sphere between $\Sigma'$ and $J^+$, here denoted by the point $b$) must be due to gravitational radiation escaping between $i_0$ and $b$. This radiation originates form the causal past $P$ of the region $i_0b \subset J^+$, whose intersection with $\Sigma$ is the region $a\bar{a}_0$. In this sense we say that the data on $\Sigma$ contained radiation in that region. Any gravitational radiation emerging from an (quasi localized) event $e$, e.g. the formation of a black hole due to neutron-star mergers, cannot register before $c$, the intersection of the future light-cone at $e$ with $J^+$.

It has recently been shown to some level of rigour that even the most simple two black-hole collision data contain radiation in any neighbourhood of spacelike infinity $i^0$; see Figure 1. This was done by approximately evaluating the Bondi mass on a cut of $J^+$ close to $i^0$ for time symmetric, conformally flat data. The result was

$$m_{\text{Bondi}} = m_{\text{ADM}} + \sum_{k=-2}^{k=2} |G_k|^2 \left( \frac{\sqrt{2}}{u} \right)^7 + O(1/u^8),$$

(1)

where the $G_k$ are the Newman-Penrose constants and where $u$ is the Bondi parameter along $J^+$, which tends to $-\infty$ as one approaches $i^0$. Hence $m_{\text{Bondi}} \leq m_{\text{ADM}}$ with equality at $i^0$, as it must be. If some of the Newman-Penrose constants do not vanish, we have a strict inequality off $i^0$, meaning that some gravitational radiation has reached $J^+$ in between. If this is the case, one says that the initial data contained radiation in a neighbourhood of (spacelike) infinity. This radiation is spurious in the sense that it cannot be caused by the collision process. Hence one would like to remove it. If we were in a linear theory, e.g. electrodynamics, the meaning of ‘remove’ would be to subtract the appropriate solution to the homogeneous equations. However, in a non-linear theory, like General Relativity, it is a priori unclear how to analytically achieve a ‘removal’. Is it possible at all to get rid of the spurious radiation without touching the interior data?
The question addressed here relates to the problem of analytically understanding the region where null infinity touches spacelike infinity. The first step in this direction was taken by Friedrich [4], who used the conformal field equations to analyse this region. In this approach, spacelike infinity is blown up to a cylinder $S^2 \times [0, 1]$, whose two boundary components connect to future and past null infinity ($\mathcal{I}^\pm$) respectively. The Cauchy surface, $\Sigma$, intersects the cylinder in a 2-sphere cross section, $\mathcal{I}^0$, from which certain transport equations govern the propagation of initial data along the cylinder to its boundary. This propagation will generically lead to logarithmic divergences at these intersections and, consequently, to a non-smooth null infinity. Whether, and to what degree, a non-smooth null infinity is physically objectionable is currently unclear. Some necessary conditions for the avoidance of such singularities are known [9], but so far no theorem establishes a sufficient set. Instead, the following conjecture was launched [9]:

**Conjecture:** If an initial data set which is time-symmetric and conformally flat in a neighbourhood $B_\alpha(i_0)$ of the point $i_0$ yields a development with a smooth null infinity, then the initial data are, in fact, Schwarzschild in $B_\alpha(1)$.

If true, the conjecture tells us that gravitational radiation is excluded near spacelike infinity. At this point one might fear that the condition of being asymptotically Schwarzschild might exclude most data of physical interest. But fortunately this is not true, thanks to the work of Corvino [3], who proved that is possible to glue a Schwarzschild metric along an annulus to an asymptotically flat, conformally flat three metric if the gluing radius is chosen sufficiently large and if the mass of the Schwarzschild metric is chosen appropriately. That is, there do exist spacetimes which are Schwarzschild at infinity (therefore ensuring a smooth null-infinity) and non-static in the inside.

For example, we could glue a Brill-Lindquist initial data set for the collision of two black holes to a Schwarzschild metric. This would clearly remove the ‘spurious’ radiation at spacelike infinity. However, it might at the same time introduce ‘spurious’ radiation through the modified gluing metric on the annulus, which is generally very complicated. In particular, it is not conformally flat. This motivates to ask about the mass of the glued Schwarzschild solution, depending on the specific gluing instructions.

Information about the mass is not easy to obtain: the gluing function is hard to control and not easily calculated explicitly. What we need is a physical idea of how the gluing function might look like. Here we propose to specialize to Brill waves. They are axisymmetric (as is Brill-Lindquist) and not conformally flat. Is it possible to construct a spacetime which is Brill-Lindquist to some radius, has Brill-form on an annulus, and is Schwarzschild outside? It is the purpose of this paper to explore this idea a little further, based on the diploma thesis by one of us [7].

### 2 Corvino’s construction

We mentioned that the Brill-Lindquist data seem to contain spurious gravitational radiation near spacelike infinity and, intimately related to this fact, cannot lead to an asymptotically simple spacetime. Hence we wish to modify them near infinity. The
techniques of Corvino use the underdetermined ellipticity of the constraint equations to glue a Schwarzschild end to an arbitrary asymptotically flat manifold.

Consider a smooth gluing function $\beta \geq 0$, $|\beta| \leq 1$, where

$$\beta = \begin{cases} 1 & \text{for } x \leq 1 \\ 0 & \text{for } x \geq 2 \end{cases}$$

and an arbitrary asymptotically flat metric $g_{AF}$ together with a Schwarzschild metric $g_{m,\vec{c}}$ with ADM-energy $m$ and center $\vec{c}$. Then, in an asymptotically flat chart, the metric

$$g_{\text{glue}} = \beta \left(\frac{|x|}{R}\right) g_{AF} + \left(1 - \beta \left(\frac{|x|}{R}\right)\right) g_{m,\vec{c}}$$

(for constant $R$) is smooth and glues $g_{AF}$ to $g_{m,\vec{c}}$. However, the constraint equation of vanishing scalar curvature, $R(g_{\text{glue}}) = 0$, is now violated on the annulus, $R < |x| < 2R$, hence $g_{\text{glue}}$ is not a time-symmetric initial datum for Einstein’s equation. Corvino now shows that it is possible to choose the gluing radius $R$ large enough to find a metric $h$, having support only on the annulus, such that $R(g_{\text{glue}} + h) = 0$ everywhere. To make this construction work, the variables $m$ and $\vec{c}$ have to be chosen appropriately. In this way Corvino’s construction provides us with time-symmetric initial data, isometric to Schwarzschild outside some finite radius and non-trivial in the inside. But his construction is not very explicit. We have no method at hand to construct the metric explicitly, because we do not know much more about the perturbation $h$ on the annulus than its existence. This leaves important physical questions unanswered: Can we give a physical interpretation for the gluing construction? One is tempted to ask the following ‘physical’ questions: does the construction actually remove spurious gravitational-radiation energy, or does it at best just lead to a redistribution of energy from infinity to the annulus region? The answer does not seem to be known.

3 Brill waves and the gluing construction

3.1 Brill-Lindquist data

We propose to phrase these questions in a more explicit environment. Consider Brill-Lindquist data describing the head-on collision of two black holes:

$$g_{ij} = \left(1 + \frac{a_1}{|x - \vec{c}_1|} + \frac{a_2}{|x - \vec{c}_2|}\right)^4 \delta_{ij}.$$  

(4)

We further simplify by choosing $a_1 = a_2$, corresponding to equal masses for the holes and $\vec{c}_1 = -\vec{c}_2 = \vec{c} := (0, 0, d/2)$, so that the holes lie symmetrically about the origin on the z-axis:

$$g_{ij} = \left(1 + \frac{a}{|x - \vec{c}|} + \frac{a}{|x + \vec{c}|}\right)^4 \delta_{ij}.$$  

(5)

These data are axisymmetric. We shall equip them with a Schwarzschild end at infinity.
3.2 Brill waves

Our key assumption is that the metric on the annulus is a conformally transformed Brill wave \[^1\]. In cylindrical coordinates it corresponds to the regular 3-metric

\[
g_{\text{Brill}} = e^{2q(\rho,z)} \left( d\rho^2 + dz^2 \right) + \rho^2 d\theta^2. \tag{6}
\]

Here \(q\) is a function of \(\rho\) and \(z\) satisfying the following conditions.

\[
q = 0 \quad \text{if} \quad \rho = 0 \quad \text{(on the z-axis)},
\]
\[
\frac{\partial}{\partial \rho} q = 0 \quad \text{if} \quad \rho = 0 \quad \text{(on the z-axis)}, \tag{7}
\]
\[
q \to 0 \quad \text{faster than} \quad \frac{1}{r} \quad \text{for} \quad r \to \infty.
\]

The second condition is immediately obvious: If \(q\) is to represent an axisymmetric solution, it must have an extremum on the axis in the \(\rho\) direction. The first condition ensures that there is no conical singularity on the z-axis. The third condition will be satisfied automatically because our \(q\) will have compact support.

To solve the constraint equation \(R = 0\), we need to perform a conformal transformation of the Brill metric. We ask if there exists a conformal factor, \(\psi\), with the following properties

\[
R(\psi^4 g_{\text{Brill}}) = 0, \\
\psi > 0 \quad \text{everywhere}, \tag{8}
\]
\[
\psi \to 1 \quad \text{at} \quad \infty.
\]

Using the behaviour of the Ricci scalar under conformal transformations, \(R(\psi^4 g) = -8\psi^{-5} (\triangle g - \frac{1}{8}R(g)\psi)\), we end up with the equation

\[
8 \triangle_g \psi - R(g)\psi = 0, \tag{9}
\]

where \(\psi > 0\) everywhere and \(\psi \to 1\) at \(\infty\). In the past much work has been done to derive existence theorems for solutions \(\psi\) given \(q\). Here we will take a slightly different viewpoint.

3.3 Our construction

Combining the spirit of Corvino’s construction with our physical idea of a Brill wave on the annulus, we define a conformal factor

\[
\psi = \left( 1 + \frac{a}{|x-c|} + \frac{a}{|x+c|} \right) \beta \left( \frac{|x|}{R}, \theta \right) + \left( 1 - \beta \left( \frac{|x|}{R}, \theta \right) \right) \left( 1 + \frac{A}{|x|} \right). \tag{10}
\]

We have generalized the gluing function \(\beta\) to be axisymmetric on the annulus:

\[
\beta = \begin{cases} 
1 & \text{for } x \leq 1 \text{ and all } \theta \\
0 & \text{for } x \geq 2 \text{ and all } \theta 
\end{cases} \tag{11}
\]

and \(\beta^{(n)}(1) = \beta^{(n)}(2) = 0\) for all \(n \geq 1\).
Next we use the fourth power of the function $\psi$ in (10) as conformal factor to define the physical initial-data metric:

$$g = \psi^4 g_{\text{Brill}} = \psi^4 \left( e^{2q(\rho,z)} (d\rho^2 + dz^2) + \rho^2 d\theta^2 \right).$$ \hspace{1cm} (12)

Here $q$ is, by definition, a function with support on the annulus, vanishing derivatives on its boundary\(^1\) and which satisfies the Brill-conditions (7). Taking all this into account we observe that

- $g$ is Brill-Lindquist for $r < R$,
- $g$ is Schwarzschild for $r > 2R$ (with ADM-energy $M = \frac{A}{2}$),
- $g$ is a complicated, conformally transformed Brill wave on $r \in [R, 2R]$.

Now, the constraint equation $R(g) = 0$ is satisfied if

$$\left( \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} \right) q(\rho,z) = \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) q(r,\theta) = -\frac{\Delta^{(3)} \psi}{\psi},$$ \hspace{1cm} (13)

Since $\psi$ stands for the expression (10), this equation should be read as relating the gluing function $\beta$ to the Brill-function $q$. It turns into an identity ($0 = 0$) for $r < R$ and $r > 2R$. The mathematical structure of the equation is discussed in the next section.

### 3.4 The mathematical problem

Equation (13) can be understood as an elliptic equation on the two-dimensional plane. However, the boundary conditions are not of the usual Dirichlet or Neumann type. Note that the right hand side—for any gluing function $\beta$—is of bump-type, i.e. vanishes with all derivatives at $r = R$ and $r = 2R$. Hence we have a large freedom in the choice of the right hand side by exploiting the freedom of the gluing function $\beta$, of which only the boundary values (including all derivatives) are fixed. On the other hand, there are boundary conditions on $q$ and its derivatives that have to be taken into account.

As it turns out, the problem can be formulated in the following way: Given the Poisson equation

$$\Delta^{(2)} q = f = -\frac{\Delta^{(3)} \psi}{\psi}$$ \hspace{1cm} (14)

in the half-annular region $H$ sketched in Figure 2. Is it possible to choose the right hand side of (14) such that $q$ simultaneously satisfies Dirichlet and Neumann conditions for the given region $H$? This we will call a DN-solution. Note that imposing Dirichlet and Neumann conditions guarantees the Brill-conditions (7) on the $z$-axis, as well as the vanishing of all radial derivatives on the arches of $H$. Indeed, if $q$ and its first derivative vanish on the arches, then, by equation (14), all derivatives of $q$ will vanish there since $f$ vanishes with all derivatives on the arches. Although one can easily write down some $f$’s admitting a DN-solution, it is not clear if these $f$ can be obtained by $-\frac{\Delta^{(3)} \psi}{\psi}$ with an appropriate choice of $\beta$—the root of the problem being that the inhomogeneity is highly non-linear in $\beta$.

\(^1\)In the sequel a function with these properties will sometimes be called a “function of bump-character” (although it can become negative, of course).
3.5 Approximate results

Corvino’s construction only works for large gluing radii $R$. We could therefore try to solve the DN-problem to first order in the inverse gluing radius. To do this we expand the inhomogeneity to first order in $1/R$, making two further choices:

1. The first is related to the possibility of reducing the energy. The further outside the gluing is performed, the smaller the difference of the Schwarzschild and the Brill-Lindquist data will be. In other words, the difference will scale in some way with the inverse gluing radius—the exact form being dependent on the details of the gluing procedure. Recall equation (10), introduce the scaling $|x| \rightarrow |x|R$ and expand in the inverse gluing radius, $1/R$:

$$
\psi = 1 + \beta(r,\theta) \left( \frac{2a - A}{Rr} + \frac{a d^2}{2R^3r^3} + O \left( \frac{1}{R^5} \right) \right)
$$

(15)

Note that we have no $\frac{1}{R}^2$-(dipole)-terms due to the choice of equal masses for the two Brill-Lindquist holes and equal mass-centres for the Brill-Lindquist system and the Schwarzschild hole. We choose the scaling to be

$$
2a - A = \frac{b}{R^2}
$$

(16)

where $b$ takes real values in a small interval around 0 (eventually we will be interested in $b > 0$).

2. The second choice is an ansatz for the gluing function $\beta$:

$$
\beta(r,\theta) = \alpha(r) + \mu(r) \sin^2 \theta.
$$

(17)

Here $\alpha(r)$ is a radial gluing function ($\alpha(1) = 1$, $\alpha(2) = 0$, all derivatives vanish at these two points) and $\mu(r)$ is a function with bump character on the annulus.

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If we chose $2a - A = \frac{b}{R}$ we would, at the lowest order, have the same situation as for the gluing of two Schwarzschild black holes of ADM-energies $2a$ and $A$ respectively. On account of the Penrose inequality one expects this not to work with mass reduction. Note that for large gluing radii the conformally transformed Brill wave will certainly not introduce a new apparent horizon. In fact, in can be shown that the recursion technique presented below fails for the Schwarzschild-Schwarzschild case.

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2
(μ(1) = 0, μ(2) = 0, all derivatives vanish at these two points) but otherwise arbitrary. For the expansion of \( ψ \) we explicitly introduce the gluing radius \( R \) by scaling \( r \to Rr \) in the formulae. This makes the gluing occur on the interval \([1, 2]\).

The right hand side can then be written as

\[
\frac{\Delta ψ}{ψ} = \frac{1}{R^5} \left( \alpha''(r) + \frac{2}{r} \alpha'(r) + \frac{4}{r^2} \mu(r) \right) \left( \frac{b}{r} + \frac{a d^2}{2r^3} \right) + \frac{2}{r} \alpha'(r) \left( \frac{b}{r^2} - \frac{3a d^2}{2r^4} \right) + \sin^2 θ \left[ \left( \mu''(r) + \frac{2}{r} \mu'(r) - \frac{6}{r^2} \mu(r) \right) \left( \frac{b}{r} + \frac{a d^2}{2r^3} \right) - \frac{3a d^2}{4r^3} \left( \alpha''(r) + \frac{2}{r} \alpha'(r) \right) \right.
\]

\[
+ 2\mu'(r) \left( -\frac{b}{r^2} - \frac{3a d^2}{2r^4} \right) + \frac{9a d^2}{2r^4} \left( \alpha'(r) + \frac{2}{r} \mu'(r) \right) \left( \frac{b}{r^2} - \frac{3a d^2}{2r^4} \right) \left( \frac{a d^2}{r^3} - \frac{9a d^2}{2r^4} \right) + \frac{6a d^2}{r^5} \mu(r) \]
\]

\[
+ O(R^{-6}) \tag{18}
\]

the right-hand side of which may be written in short-hand:

\[
\frac{1}{R^5} \left( A_0(\alpha', \alpha'', \mu) + A_2(\alpha', \alpha'', \mu, \mu', \mu'') \right) \sin^2 θ \left[ A_2(\mu', \mu'', \mu, \mu', \mu'') \sin^2 θ + A_4(\mu, \mu', \mu'') \sin^2 θ \right] + O(R^{-6}). \tag{19}
\]

We make the following ansatz for \( q(r, θ) \):

\[
q(r, θ) = \frac{1}{R^5} \left( B_1(r/R) \sin^2 θ + B_2(r/R) \sin^4 θ \right) \tag{20}
\]

The functions \( B_1 \) are of bump-character in \([1, 2]\). This ansatz satisfies the three Brill-conditions for \( q(r, θ) \). The expression will be of order \( \frac{1}{R^5} \) when acted upon by the Laplacian. If we apply the Laplacian and perform the scaling with the gluing radius afterwards, we obtain the following left hand side:

\[
\text{LHS} = \frac{1}{R^5} \left[ \frac{2B_1(r)}{r^2} + \sin^2 θ \left( B_1''(r) + \frac{2}{r} B_1'(r) - \frac{4}{r^2} B_1(r) + \frac{12B_2(r)}{r^2} \right) \right.
\]

\[
\left. + \sin^4 θ \left( B_2''(r) + \frac{1}{r} B_2'(r) - \frac{16B_2(r)}{r^2} \right) \right]. \tag{21}
\]

The right-hand side is given by minus the expansion (19) with

\[
A_0 = \left( \alpha''(r) + \frac{4}{r^2} \mu(r) \right) \left( \frac{b}{r} + \frac{a d^2}{2r^3} \right) - \frac{2}{r} \alpha'(r) \frac{a d^2}{r^4} \tag{22}
\]

\[
A_2 = \left( \mu''(r) - \frac{6}{r^2} \mu(r) \right) \left( \frac{b}{r} + \frac{a d^2}{2r^3} \right) \right.
\]

\[
- \frac{3a d^2}{4r^3} \left( \alpha''(r) + \frac{4a d^2}{r^2} \alpha'(r) + \frac{12}{r^2} \mu(r) \right) - 2\mu'(r) \frac{a d^2}{r^4} \tag{23}
\]

\[
A_4 = -\frac{3a d^2}{4r^3} \left( \mu''(r) + \frac{2}{r} \mu'(r) - \frac{6}{r^2} \mu(r) \right) + \frac{9a d^2}{2r^4} \mu'(r) + \frac{6a d^2}{r^5} \mu(r) \tag{24}
\]
The Dirichlet-Neumann problem for this part of the inhomogeneity is solved if the following differential equations hold (Note the additional minus sign since the right hand side is $-\frac{\partial \psi}{\partial n}$):

\[
\frac{2}{r^2} B_2(r) = -A_0(\alpha', \alpha'', \mu)
\]

\[
B_2''(r) + \frac{1}{r} B_2'(r) - \frac{4}{r^2} B_2(r) + \frac{12B_4(r)}{r^2} = -A_2(\alpha', \alpha'', \mu, \mu', \mu'')
\]

\[
B_4''(r) + \frac{1}{r} B_4'(r) - 16\frac{B_4(r)}{r^2} = -A_4(\mu, \mu', \mu'')
\]

We interpret the first two equations as defining equations for $B_2$ and $B_4$ given any $\alpha$ and $\mu$. The functions $\alpha$ and $\mu$ are then determined by the third differential equation: We relate $\alpha$ and $\mu$ in such a way that the third differential equation holds. In this case $q$ (constructed from $B_2$, $B_4$) is a solution of the Dirichlet-Neumann problem.

Carrying out the calculation explicitly we arrive at

\[
\frac{ad^2 + 2br^2}{24r} \mu^{(4)}(r) + \frac{2br^2 - 7ad^2}{24r^2} \mu^{(3)}(r) + \frac{55ad^2 - 34br^2}{24r^3} \mu''(r)
\]

\[
- \frac{85ad^2 - 32br^2}{12r^4} \mu'(r) - \frac{245ad^2}{12r^5} \mu(r)
\]

\[
= -\frac{1}{48} r \left( ad^2 + 2br^2 \right) \alpha^{(6)}(r) - \frac{1}{24} \left( -ad^2 + 10br^2 \right) \alpha^{(5)}(r)
\]

\[
+ \left( \frac{ad^2}{6r} - \frac{5br^2}{24} \right) \alpha^{(4)}(r) + \left( \frac{15b}{8} - \frac{7ad^2}{6r^2} \right) \alpha^{(3)}(r)
\]

\[
+ \frac{3}{8r^3} \left( 8ad^2 - 5br^2 \right) \alpha^{(2)}(r) - \frac{3ad^2}{r^4} \alpha'(r).
\]

What we have obtained is a differential equation relating the functions $\alpha$ and $\mu$. Can we find a gluing function $\alpha$ and a function $\mu$ of bump-character related by (25)? We interpret this equation as an inhomogeneous ordinary differential equation for $\mu$. The right hand side— independent of the choice of the gluing function— will be a function of bump-character. If we set initial conditions $\mu^{(n)}(1) = 0$ for $n = 0, 1, 2, 3$ all derivatives of $\mu$ will vanish at the point $r = 1$ by the properties of the right hand side. The only additional condition $\mu$ has to satisfy is that the function and all its derivatives should also vanish at the point $r = 2$. Can we choose a right hand side, such that $\mu$ and all its derivatives vanish also at $r = 2$? The answer is provided by the solution formula for inhomogeneous ordinary differential equations. We multiply (28) by $\frac{24}{ad^2+2br^2} \neq 0$ and obtain an equation of the following type

\[
\mu^{(4)}(r) + f(r)\mu^{(3)}(r) + g(r)\mu^{(2)}(r) + h(r)\mu'(r) + k(r)\mu(r) = \mathcal{F}[\alpha](r)
\]

Here $f(r), g(r), h(r), k(r)$ are well behaved, bounded, rational functions in the interval $[1, 2]$. They also depend on the parameter $b$, which for notational simplicity we suppress to write out explicitly most of the time. We will restore the argument $b$ when it becomes important, as it will be below. (The functions also depend on the other parameters $a, d$. But this dependence does not interest us anyway.) We recall that $\mathcal{F}[\alpha](r)$ is defined by the right hand side of (28) multiplied by $\frac{24}{ad^2+2br^2} \neq 0$ and hence depends
on the first to sixth derivatives of \( \alpha \). All derivatives of \( \mathcal{F}[\alpha](r) \) with respect to \( r \) vanish at the boundary \( r = 1, 2 \). The dependence of \( \mathcal{F} \) on \( b \) is also notationally suppressed at this moment.

We transform the system to a system of four ordinary differential equations of first order in the usual way:

\[
\begin{pmatrix}
\mu(r) \\
u(r) \\
v(r) \\
w(r)
\end{pmatrix}' = 
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
-k(r) & -g(r) & -f(r) & -h(r) \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\mu(r) \\
u(r) \\
v(r) \\
w(r)
\end{pmatrix} + 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

This we will write as

\[
\ddot{x}(r) = A(r)\dot{x}(r) + \bar{r}(\alpha(r)) \quad (31)
\]

with the obvious identifications. Derivatives are always taken with respect to \( r \). We impose the boundary condition \( \ddot{x}(1) = 0 \).

The following standard arguments about ODEs are taken from [2]. If \( \Phi(r) \) is a fundamental matrix\(^4\) of the homogeneous system, then

\[
\dot{x}(r) = \Phi(r) \int_1^r \Phi^{-1}(t)\bar{r}(\alpha(t)) \, dt \quad (32)
\]

is a solution of the inhomogeneous system satisfying the boundary condition \( \ddot{x}(1) = 0 \) imposed above. It then automatically follows from (32) that \( \dot{x} \) has altogether vanishing derivatives at \( r = 1 \). We also need to impose the condition that \( \dot{x} \) and all its derivatives vanishes at the other boundary at \( r = 2 \). How this can be achieved will be discussed next. But before we shall make the following observation: A priori our equations were real, so that \( \Phi \) is a priori a real \( 4 \times 4 \) matrix. However, it will turn out to be useful to complexify our system. Note that even if we use complex fundamental matrices we will get a real solution \( \dot{x} \) of (32). This follows from the fact that if \( \Phi \) is some (complex) fundamental matrix, any other fundamental matrix is given by \( \Phi C \) with some appropriate constant complex matrix \( C \) (compare Thm. 2.3 of [2]). Obviously (32) is invariant under \( \Phi \mapsto \Phi C \).

Now we turn to imposing the second boundary condition: \( \ddot{x}(2) = 0 \). Again we observe that this boundary condition together with (32) implies that all derivatives, too, vanish at \( r = 2 \). Requiring two boundary conditions for an ODE of first order is clearly an overdetermination. However, in our case, the inhomogeneity in (31) is not fixed. Hence we read \( \ddot{x}(r = 2) = 0 \) as equation restricting the choice of the function \( \alpha \). If we define \( \ddot{v} \) to be the vector whose components are the fourth column of the matrix \( \Phi^{-1} \), i.e. \( v_i := \Phi^{-1}_{4i} \), then the boundary condition is equivalent to

\[
\int_1^2 \ddot{v}(t)\mathcal{F}[\alpha](t) \, dt = 0. \quad (33)
\]

Before we continue we remark that under a redefinition \( \Phi \mapsto \Phi C \) we have \( \Phi^{-1} \mapsto C^{-1}\Phi^{-1} \) and hence \( \ddot{v} \mapsto C^{-1}\ddot{v} \), showing that (33) is satisfied for one choice of \( \Phi \) if and only if it is satisfied for any other.

\(^3\)We exclude values \( b \leq -at^2/8 \).

\(^4\)A matrix, whose \( n \) columns are the \( n \) linearly independent solutions of the homogeneous system.
To get an idea what may go wrong in trying to satisfy (33), we use the explicit expression for \( F \) and get, after some integrations by parts:

\[
\int_1^2 dr \alpha'(r) \left[ \frac{1}{2} r^2 \bar{v}(b, r) \right]^{(5)} + \frac{a d^2 - 10 b r^2}{a d^2 + 2 b r^2} \bar{v}(b, r) \right]^{(4)} - \left( \frac{4 a d^2 - 5 b r^2}{a d^2 + 2 b r^2} \bar{v}(b, r) \right)^{(3)} + \left( \frac{135 b r^2 - 28 a d^2}{r a d^2 + 2 b r^2} \bar{v}(b, r) \right)^{(2)} \]

\[= \bar{v}(b, r) \right]^{(1)} + \left( \frac{9 d a d^2 - 5 b r^2}{r^2 a d^2 + 2 b r^2} \bar{v}(b, r) \right)^{(0)} + \left( \frac{72 a d^2}{r^5 a d^2 + 2 b r^2} \right) = 0. \]

This clearly cannot be satisfied if one of the components functions in the square bracket is constant, since

\[
\int_1^2 \alpha'(r) dr = \alpha(2) - \alpha(1) = 1. \]

More general, (34) cannot be satisfied if after left multiplication with some \( C^{-1} \) at least one of the component functions is constant. However, in the sequel we shall prove that at least for small parameters \( b \) this cannot happen. The idea is to prove this directly for \( b = 0 \) and then use an inverse-function type argument to extend this to small \( b \). Hence we first need to consider the case \( b = 0 \) in some detail.

### 3.6 The case \( b = 0 \)

In the case \( b = 0 \) the system (28) reduces to

\[
\mu^{(4)}(r) - \frac{7}{r} \mu^{(3)}(r) + \frac{55}{r^2} \mu''(r) - \frac{170}{r^3} \mu'(r) - \frac{490}{r^4} \mu(r) =
\]

\[- \frac{1}{2} r^2 \alpha^{(6)}(r) + r \alpha^{(5)}(r) + 4 \alpha^{(4)}(r) - \frac{28}{r} \alpha^{(3)}(r) + \frac{72}{r^2} \alpha^{(2)}(r) - \frac{72}{r^3} \alpha^{(1)}(r). \]

The homogeneous part is an equation of Euler-type, i.e. solved by simple powers \( r^{n_i} \), so that the fundamental matrix is readily computed:

\[
\Phi(r) = \begin{pmatrix}
  r^{n_1} & r^{n_2} & r^{n_3} & r^{n_4} \\
  (r^{n_1})' & (r^{n_2})' & (r^{n_3})' & (r^{n_4})' \\
  (r^{n_1})'' & (r^{n_2})'' & (r^{n_3})'' & (r^{n_4})'' \\
  (r^{n_1})''' & (r^{n_2})''' & (r^{n_3})''' & (r^{n_4})''' 
\end{pmatrix}
\]

The exponents \( n_i \in \mathbb{C} \) are determined by the fourth-order polynomial obtained by inserting \( \mu(r) = r^{n_i} \) into the left-hand side of (36) and equating this to zero. The
solutions are
\[ n_1 = 7 \]
\[ n_2 = 2 - \frac{11}{(-72 + \sqrt{6515})^{\frac{1}{3}}} + \left(-72 + \sqrt{6515}\right)^{\frac{1}{3}} \]  
(38)
\[ n_3 = 2 + \frac{11 \left(1 + i \sqrt{3}\right)}{2 \left(-72 + \sqrt{6515}\right)^{\frac{1}{3}}} - \frac{\left(1 - i \sqrt{3}\right) \left(-72 + \sqrt{6515}\right)^{\frac{1}{3}}}{2} \]  
(39)
\[ n_4 = 2 + \frac{11 \left(1 - i \sqrt{3}\right)}{2 \left(-72 + \sqrt{6515}\right)^{\frac{1}{3}}} - \frac{\left(1 + i \sqrt{3}\right) \left(-72 + \sqrt{6515}\right)^{\frac{1}{3}}}{2} \]  
(40)

Note that \( n_1, n_2 \) are real and \( n_3, n_4 \) are complex conjugates. Of course we could construct two real solutions from the complex ones but, as already argued, we may continue to use a complex \( \phi \). This is advisable since the complex form (37), whose entries do not contain sums, has a simpler inverse. The fourth column of the inverse is the vector \( \vec{v} \). It reads:

\[ \vec{v}(r) = \begin{pmatrix} r^{3-n_1} \\ \frac{n_1-n_3}{(n_1-n_4)(n_1-n_2)} \\ \frac{n_2-n_3}{(n_2-n_4)(n_2-n_1)} \\ \frac{n_3-n_4}{(n_3-n_1)(n_3-n_2)} \\ \frac{n_4-n_3}{(n_4-n_1)(n_4-n_2)} \end{pmatrix} \]  
(42)

We can now explicitly evaluate the integral condition (33). This leads to

\[ \int_1^2 dr \ r^{3-n_i} \left( -\frac{1}{2} r^2 \alpha^{(6)}(r) + r \alpha^{(5)}(r) + 4 \alpha^{(4)}(r) \right. \]
\[ \left. - \frac{28}{r} \alpha^{(3)}(r) + \frac{72}{r^2} \alpha^{(2)}(r) - \frac{72}{r^3} \alpha^{(1)}(r) \right) = 0 \]  
(43)

which is to be read as a condition for \( \alpha \), where the \( n_i \) take the values computed above.

Performing the partial integrations explicitly we can state the conditions in the form

\[ \int_1^2 dr \ w_i(r) \alpha'(r) = 0 \quad \text{or} \quad \int_1^2 w_i'(r) \alpha(r) = -w_i(1) \]  
(44)

where the \( w_i \) are four real, non constant, linearly independent functions on \([1, 2]\), whose exact form need not be spelled out here. Regarding our functions as elements of the Hilbert space \( L^2([1, 2], dr) \) this states that we need to choose \( \alpha \) with prescribed orthogonal projections onto the four \( w_i \). This can always be achieved due to their linear independence (for linear dependent \( w_i \) the projection conditions might turn out to be contradictory), while still leaving an infinite freedom for \( \alpha \). Hence we have shown that for \( b = 0 \) the functions \( \alpha \) and \( \mu \) can be related such that the DN-problem is solved at first order. This we summarize as
Lemma: The DN-problem is solvable at the first order in the inverse glueing radius for the choice \( b = 0 \). In this case the glued and original data have the same overall ADM-energy.

3.7 The case \( b > 0 \)

We now wish to extend the result to small positive \( b \), which corresponds to a reduction of energy. The idea is to set up an implicit-function theorem. For this it is convenient to explicitly display the relevant dependencies on the parameter \( b \), which we suppressed so far.

Pick a glueing function \( \alpha_0(t) \) which satisfies the four integral conditions (44). Define the Banach space of bump-functions

\[
\mathcal{B} := \{ \beta \in C^\infty([1,2]) \mid \beta^{(n)}(1) = \beta^{(n)}(2) = 0 \quad \forall n \in \mathbb{N}_0 \},
\]

equipped with the \( \max \)-norm on \([1,2]\). Next consider a map \( f : \mathbb{R} \times \mathcal{B} \to \mathbb{R}^4 \), defined by:

\[
f : (b, \beta) \mapsto \Phi(b, r = 2) \int_1^2 \mathrm{d}t \, \bar{v}(b, t) \mathcal{F}[b, \alpha_0 + \beta](t). \tag{46}
\]

As already argued, this map indeed maps into \( \mathbb{R}^4 \) even if we use complex fundamental matrices. Note also that the function \( \alpha_0(t) + \beta(t) \) is a gluing function. The map (46) is a smooth map of Banach spaces, which is linear in \( \alpha_0 + \beta \).

We have shown above that \( f(0,0) = 0 \). We wish to show that for any small \( \tilde{b} \in \mathbb{R} \) there exists a \( \tilde{\beta} \in \mathcal{B} \) such that \( f(\tilde{b}, \tilde{\beta}) = 0 \). This is ensured if the differential

\[
D_2 f(0,0) : \mathcal{B} \to \mathbb{R}^4
\]

is surjective. Since the map \( f \) is linear in \( \alpha_0 + \beta \), we arrive at

\[
D_2 f(0,0) : \tilde{\beta} \mapsto \Phi(0, r = 2) \int_1^2 \mathrm{d}r \, \bar{v}(\tilde{\beta})(0, \tilde{\beta})(r), \tag{47}
\]

where \( \bar{v}(0, r) = \bar{v}(r) \) is explicitly given by the expression in (42). This indeed shows surjectivity and we have

Theorem: For sufficiently small \( b \in \mathbb{R} \) the Dirichlet-Neumann-problem can be solved to first order in the inverse glueing radius. Hence the gluing can be used to reduce the ADM-energy.

4 Conclusion

Motivated by recent results concerning the radiation content at spacelike infinity of the most simple, time symmetric, conformally flat two-black-hole data (Brill-Lindquist data), we considered modifications of that data which render then Schwarzschild at infinity by using Corvino’s gluing technique. We used Brill waves for the gluing and have given a perturbative argument (for large glueing radii) that this technique may be used to reduce the ADM-energy. We consider this as being a first small step towards making Corvino’s technique more concrete, which also sheds some light onto the problem of how to remove ‘spurious’ radiation at spacelike infinity. A natural extension
of this work would obviously consist in giving a rigorous proof for the possibility of energy reduction beyond finite-order approximation techniques.

More generally, an interesting question to ask is what happens if one goes to smaller and smaller gluing radii. Could one reasonably expect this procedure to lead to a minimal overall ADM-energy for given hole-masses and -separations, so as to correspond to the ideal situation of no ‘spurious’ radiation? Or should one rather expect such a minimum not to exist? We do not know the answer to this question.

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