Functional peaks-over-threshold analysis

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Abstract
Peaks-over-threshold analysis using the generalised Pareto distribution is widely applied in modelling tails of univariate random variables, but much information may be lost when complex extreme events are studied using univariate results. In this paper, we extend peaks-over-threshold analysis to extremes of functional data. Threshold exceedances defined using a functional $r$ are modelled by the generalised $r$-Pareto process, a functional generalisation of the generalised Pareto distribution that covers the three classical regimes for the decay of tail probabilities, and that is the only possible continuous limit for $r$-exceedances of a properly rescaled process. We give construction rules, simulation algorithms and inference procedures for generalised $r$-Pareto processes, discuss model validation and apply the new methodology to extreme European windstorms and heavy spatial rainfall.

KEYWORDS
functional regular variation, peaks-over-threshold analysis, rainfall, $r$-Pareto process, spatial statistics, statistics of extremes, windstorm

1 | INTRODUCTION

Extreme value theory provides a mathematical framework for the description and modelling of tails of statistical distributions that can be used to extrapolate beyond observed events. This theory has been studied extensively (Beirlant et al., 2004; Coles & Tawn, 1991; Embrechts et al., 1997; Fisher & Tippett, 1928; Gnedenko, 1943; Heffernan & Tawn, 2004; Pickands, 1975) and...
is widely used in applications (Coles, 2001; Hosking & Wallis, 1987; Katz et al., 2002). However, many complex phenomena cannot be modelled using univariate or multivariate methods, so richer approaches to the analysis of high-dimensional data have been explored over the past decade.

Max-stable processes (de Haan, 1984; de Haan & Ferreira, 2006, section 9.2) provide a functional extension of the classical extreme value distributions and have successfully been used to model maxima, but are difficult to fit in high dimensions (Huser & Davison, 2013). Moreover, they conflate individual extremal events and hence discard information, making it difficult to detect phenomena such as mixtures of tail behaviours. For example, in some regions rainfall events are either convective, and hence locally very intense, or cyclonic, with larger spatial accumulations of water but lower local intensities. Although driven by different weather patterns, both may lead to flooding, and, as suggested by Figure 1, the marginal distributions of their tails and their spatiotemporal structures may differ greatly. Even though large-scale events may also be very damaging, such events tend to be overlooked when focusing on maxima.

In the one-dimensional case, the analysis of threshold exceedances is often preferred to that of block maxima. The approach originated in hydrology under the name of ‘peaks over threshold’ (POT) or ‘partial duration series’ analysis (NERC, 1975; Todorovic & Rousselle, 1971; Todorovic & Zelenhasic, 1970), its goal being to include all large individual events and thus access more information than can be extracted from block, typically annual, maxima. This approach is particularly
valuable when the data are limited and there is an appreciable seasonal component. A probabilistic basis for threshold modelling was provided by Balkema and de Haan (1974), Pickands (1975) and Leadbetter (1991) and statistical aspects were developed by Davison (1984), Smith (1984) and Davison and Smith (1990). The basic idea is to fit the generalised Pareto distribution to the exceedances of a variable such as river flow or pollution level over a threshold. A large literature has built on this early work and the method and its many variants have been applied in numerous other contexts.

In some applications, it is helpful to reduce multivariate data to scalar ‘structure variables’ (Coles & Tawn, 1994) that can be analysed using POT or other univariate methods, but this approach gives no insight into the combinations of variables yielding a rare event. Moreover, the tail behaviours of different structure variables may vary widely owing to such factors as the presence of several underlying physical processes. Functional peaks-over-threshold analysis modifies this approach to give different perspectives on the dependence structure and provides a theoretical foundation for the detection of mixtures of tail behaviours through definitions of functional extremes tailored to particular types of events, as illustrated in Figure 1.

Existing functional peaks-over-threshold procedures rely on particular types of exceedances (Ferreira & de Haan, 2014) or are limited to settings where the data must have unbounded support and share the same polynomial-type tail decay (Dombry & Ribatet, 2015). Observations can be transformed to have a common marginal distribution, such as the unit Fréchet (Coles & Tawn, 1991, section 5) or unit Pareto (Klüppelberg & Resnick, 2008), and exceedances may be defined on this transformed scale, but, as many extreme phenomena are most naturally characterised on the scale of the original data, the use of transformations can impose a trade-off of interpretability against mathematical rectitude (e.g., de Fondeville & Davison, 2018). In univariate extreme value theory, the generalised Pareto distribution provides a single framework for modelling the original data in any of the classical Weibull, Gumbel or Fréchet regimes. The present paper provides a similar unified formulation for functional peaks-over-threshold analysis under the assumptions that the process has the same rate of tail decay over its domain and that its limiting tail distribution presents some level of dependence. The tail decay restriction is needed to define the exceedances directly on the original process, as otherwise the region or location with the heaviest tail would dominate the limit distribution, leading to unrealistic models. We extend the work of Dombry and Ribatet (2015) by introducing the generalised $r$-Pareto process, allowing more flexible definitions of rare events and generalised Pareto margins for tails. The generalised $r$-Pareto process is the only continuous limit of exceedances of a properly rescaled process. For some definitions of exceedances, it allows the Monte Carlo simulation of events with a fixed intensity, that is, events for which the level of risk has a prescribed return level. These results rely on a specific type of convergence that excludes independence of the limiting tail distribution; although our results can be generalised to deal with this, we leave this for future work.

Section 2 reviews classical univariate results and introduces functional peaks-over-threshold analysis. We derive convergence results for the three tail decay regimes, define and characterise the generalised $r$-Pareto process, present simulation algorithms and discuss the scope of our models. Section 3 introduces a general model for functional exceedances. In Section 4, we discuss statistical inference, and in Section 5, we describe methods for model validation. In Section 6, we use our ideas to develop a stochastic weather generator for windstorms over Europe, and Section 7 illustrates the importance of risk definition when studying potential flooding in the city of Zurich. Technical details, proofs of the main results and additional figures are given in the Supplementary Material.
2 | MODELLING THRESHOLD EXCEEDANCES

2.1 | Univariate exceedances

If a scalar random variable $X$ has distribution function $F$ and there exist sequences of constants $a_n > 0$ and $b_n$ such that

$$n \Pr \left( \frac{X - b_n}{a_n} > x \right) \to -\log G(x), \quad n \to \infty,$$

where $G$ is a non-degenerate distribution function, then $X$ is said to belong to the max-domain of attraction of $G$ (Resnick, 1987, p. 12). For a large enough threshold $u < \inf \{x: F(x) = 1\}$ and $x > 0$, the tail behaviour of $X$ can be described using a generalised Pareto distribution,

$$\Pr(X > x + u | X > u) \approx H_{\xi,\sigma}(x) = \begin{cases} (1 + \frac{\xi x}{\sigma})^{-1/\xi}, & \xi \neq 0, \\ \exp(-\frac{x}{\sigma}), & \xi = 0, \end{cases}$$

where $\sigma = \sigma(u) > 0$ and, here and below, $a_+ = \max(a, 0)$ for real $a$. The shape parameter $\xi$ is also called the tail index. If $\xi$ is negative then $X - u$ lies in the interval $[0, -\sigma/\xi]$, and otherwise $X - u$ can take any positive value. The random variable $X$ is said to belong to the Weibull, Gumbel or Fréchet domains of attraction if $\xi$ is, respectively, negative, zero or positive. The max-domain of attraction conditions are broadly but not universally satisfied (e.g., Beirlant et al., 2004, pp. 59, 62, 72).

Davison and Smith (1990) use Equation (2) as the basis of the approximation

$$F(x) \approx 1 - \zeta_u H_{\xi,\sigma}(x - u), \quad x > u,$$

where $\zeta_u$ denotes the probability that $X$ exceeds the threshold $u$. This offers a general, flexible, unified and widely applied model for the distribution tails that has been extended to multivariate settings (Rootzén & Tajvidi, 2006; Rootzén et al., 2018a,b) and to continuous processes (Dombry & Ribatet, 2015; Ferreira & de Haan, 2014).

2.2 | Functional exceedances

Let $S$ be a compact subset of $\mathbb{R}^D$, let $\mathcal{F}$ denote the space of real-valued continuous functions on $S$ equipped with norm $\| \cdot \|$, and let $\mathcal{F}_+$ denote the subset of $\mathcal{F}$ containing only non-negative functions that are not everywhere zero; this excludes the zero function and hence avoids the appearance of degenerate probability measures when taking limits.

Exceedances for a random function $X = \{X(s) : s \in S\}$ can be defined using risk functionals and $r$-exceedances. A risk functional $r$ is defined as a continuous mapping from $\mathcal{F}$ into $\mathbb{R}$ and an $r$-exceedance is defined to be an event of the form $\{r(X) \geq u\}$ for some $u \geq 0$, that is, an event for which the scalar $r(X)$ exceeds a threshold $u$. This definition was introduced by Dombry and Ribatet (2015) for homogeneous ‘cost functionals’ on $\mathcal{F}_+$, i.e., functionals for which there exists $\kappa > 0$ such that $r(ay) = a^\kappa r(y)$ when $y \in \mathcal{F}_+$ and $a > 0$. The term ‘radial aggregation function’ was earlier used by Opitz (2013b), but in our view the term ‘risk functional’ better reflects how $r(X)$ measures the severity of $X$ in terms of the risk summarised by $r$. 
Ferreira and de Haan (2014) studied threshold exceedances for continuous processes using the functional $r(X) = \sup_{s \in S} X(s)$, but this only treats as extreme all events with an exceedance at least one point in $S$. Coles and Tawn (1996) had earlier modelled areal rainfall via large values of $\int_S X(s)\,ds$, and other functionals such as $\int_S X^2(s)\,ds$ for a proxy of the energy inside a climatic system (Powell & Reinhold, 2007), $X(s_0)$ for risks impacting a specific location $s_0$, and so forth, may arise in applications. Likewise, one might project climate data onto scalar signals of particular weather patterns and examine the behaviour of their $r$-exceedances. The motivation behind the present paper is to define risk functionals tailored to specific types of physical processes, and this may yield different models based on different functionals. If a single model that merges different notions of risk is required, consistency between definitions can be enforced in our framework by studying

$$r(X) = \max \{ r_1(X) - u_1, \ldots, r_M(X) - u_M \}.$$  

where $r_1, \ldots, r_M$ are functionals of interest and $u_1, \ldots, u_M$ the corresponding thresholds.

Below we generalise $r$-exceedances under minimal assumptions on the risk functional and derive limit distributions for the three tail decay regimes.

### 2.3 Functional $r$-exceedances

#### 2.3.1 Notation, assumptions and convergence

Let $\xi$ be a scalar shape parameter, and let $a \equiv a(s) > 0$ and $b \equiv b(s)$ be continuous functions defined for $s \in S$, with $\equiv$ denoting equivalent notations. For given $\xi$, $a$ and $b$, we define the set

$$P^{\xi,a,b} = \begin{cases} 
F_+ - (b - \xi^{-1}a), & \xi > 0, \\
F, & \xi = 0, \\
(b - \xi^{-1}a) - F_+, & \xi < 0.
\end{cases}$$  

When $\xi \neq 0$ this is the positive quadrant in $P$, shifted by $b - a/\xi$, and also reflected when $\xi < 0$. Figure 2 illustrates (5) in the three possible tail regimes: $P^{\xi,a,b}$ is bounded below by $b - \xi^{-1}a$ when $\xi > 0$, is unbounded when $\xi = 0$, and is bounded above by $b - \xi^{-1}a$ when $\xi < 0$.

In this section, $X$ denotes a stochastic process with sample paths in $P$ for which there exist a real number $\xi$ and sequences $\{a_n\}_{n=1}^\infty > 0$ and $\{b_n\}_{n=1}^\infty$ of continuous functions such that the conditions for the univariate approximation (2) are satisfied for each $s \in S$, that is

$$\lim_{n \to \infty} n \Pr \left\{ \frac{X(s) - b_n(s)}{a_n(s)} > x \right\} = \begin{cases} 
(1 + \xi x)_+^{-1/\xi}, & \xi \neq 0, \\
\exp(-x), & \xi = 0.
\end{cases}$$  

In a functional setting, it is natural to extend this by assuming the existence of a boundedly finite non-zero measure $\Lambda$ on the space of non-negative and non-zero functions over $S$ such that

$$\lim_{n \to \infty} n \Pr \left\{ \left( 1 + \xi \left( \frac{X - b_n}{a_n} \right) \right)^{1/\xi} \in \cdot, \right\} = \Lambda(\cdot),$$

(7)
Assumption (7) is weak in general, and any functional model using approximation (3) should be linked to some limiting measure \( \Lambda \). Here we assume that \( \Lambda \) is non-zero only on the set of continuous functions \( F^+ \), which rules out some types of extremal dependence; see Section 2.4. In this case, Equation (7) involves a specific type of convergence described in the Appendices and defines a general form of functional regular variation (Hult & Lindskog, 2005) introduced by Ferreira and de Haan (2014); we write \( X \in \text{GRV}(\xi, a_n, b_n, \Lambda) \), where GRV stands for generalised regular variation. The limiting measure \( \Lambda \) is homogeneous of order \(-1\): \( \Lambda(tA) = t^{-1}\Lambda(A) \) for any positive scalar \( t > 0 \) and Borel set \( A \subset F^+ \) (Lindskogetal., 2014, theorem 3.1).

Equation (7) requires that \( \xi \) is constant over \( S \). As we wish to compute the risk directly from \( X \), useful limiting results are obtained only if the shape parameter is constant—if \( \xi \) varies then either those locations with the highest values of \( \xi \) or those with the highest upper bound determine the asymptotic tail behaviour and the limiting dependence cannot be modelled. For environmental applications, \( \xi \) can be considered as stemming from the physical process, such as convective rainfall, that is characterised by the functional \( r \). Imposing constant \( \xi \) could be avoided by transforming the data to have a common rate of tail decay throughout \( S \), for example by studying the limiting \( r \)-exceedances of the rescaled processes \( \{1 + \xi(X - b_n)/a_n\}^{1/\xi}_+ \) with \( \xi \) a function that varies smoothly over \( S \), as in Ferreira and de Haan (2014), but typically this entails losing the physical interpretation of the risk in terms of the original data.

We also suppose that there exists a sequence of real numbers \( a_n^\prime \) and a continuous strictly positive function \( A \) on \( S \) such that

\[
\limsup_{n \to \infty} \sup_{s \in S} \left| \frac{a_n(s)}{a_n^\prime} - A(s) \right| = 0, \tag{8}
\]

so \( a_n(s) \approx a_n^\prime A(s) \) for large \( n \). A similar assumption was used in Ferreira et al. (2012) and Engelke et al. (2019) and seems reasonable in many applications. For instance, assuming that the marginal distributions belong to a location-scale family \( F[(x(s) - B(s))/A(s)] \) that describes the behaviour
of the underlying physical process characterised by the risk functional \( r \) implies both a common limiting shape parameter \( \xi \) and that we can choose \( a_n(s) = a_n' A(s) \) and \( b_n(s) = b_n' A(s) + B(s) \) with real sequences \( a_n' > 0 \) and \( b_n' \). We also assume that the support of \( X \) is bounded below for \( \xi > 0 \) and above for \( \xi < 0 \).

A risk functional \( r : \mathcal{F} \to \mathbb{R} \) is said to be valid for the process \( X \in \text{GRV}(\xi, a_n, b_n, \Lambda) \) if the set of \( r \)-exceedances of the limiting process has positive finite measure. The properties needed for \( r \) to be valid depend on whether the risk is defined directly in terms of the original process \( X \) or using some transformation such as \( (X - b_n)/a_n \), with scale and location functions \( a_n \) and \( b_n \). Expressions (7) and (8) imply that any risk functional \( r \) applied to the shifted and rescaled quantity \( (X - b_n)/a_n \) will be valid if it satisfies the condition

\[
\begin{align*}
&\text{if } \xi > 0, \text{ then } r(-A\xi^{-1}) < 0, \\
&\text{if } \xi \leq 0, \text{ then } r(x) \to -\infty \text{ as } x \to -\infty;
\end{align*}
\]  

(9)

with this transformation, the function \( A \) is uniformly equal to 1. If we further assume that \( r \) is monotonic and that there exists a positive finite real scalar \( \alpha \) such that

\[
\lim_{n \to \infty} \frac{r(a_n)}{a_n'} = \alpha,
\]  

(10)

then one can replace the function \( a_n \) by the scalar \( r(a_n) \), i.e., apply \( r \) to \( (X - b_n)/a_n \). In this case, \( A \) may be non-constant and we can rescale \( a_n' \) and \( A \) so that \( \alpha = 1 \); then \( a_n' \) can be replaced by \( r(a_n) \) in Equation (8). We shall do this below. Expression (10) applies for instance to 1-homogeneous functionals, for which \( r \) can be applied directly to \( X - b_n \). For linear functionals, non-degenerate limits will arise when \( r \) is computed directly on the original process.

We focus on \( r \)-exceedances of \( (X - b_n)/r(a_n) \), the most technical case; linear functionals are treated in the Supplementary Material. Generalisations to \( r \)-exceedances of \( (X - b_n)/a_n \) are obtained by replacing \( r(a_n) \) by \( a_n \) and setting \( A \equiv 1 \) everywhere below. With the maxima taken pointwise and \( x \in \mathcal{F} \), we let

\[
[x] = \begin{cases} 
\max(x, -A\xi^{-1}), & \xi > 0, \\
x, & \xi \leq 0.
\end{cases}
\]  

(11)

Our first main result is the following.

**Theorem 1** Let \( X \) be a stochastic process whose sample paths are continuous functions on \( S \). If \( u \geq 0 \), the risk functional \( r \) is valid for \( X \in \text{GRV}(\xi, a_n, b_n, \Lambda) \) and (8)–(10) hold, then

\[
\Pr \left[ \frac{X - b_n}{r(a_n)} \in \cdot \mid r \left( \frac{X - b_n}{r(a_n)} \right) \geq u \right] \to \Pr(P \in \cdot), \quad n \to \infty,
\]  

(12)

where \( P \) is a generalised \( r \)-Pareto process with tail index \( \xi \), scale function \( A \), location function zero and measure \( \Lambda \).

Theorem 1 states that generalised \( r \)-Pareto processes, which we discuss in Section 2.3.2, appear as limits for any properly rescaled stochastic process \( X \) that is regularly varying in the sense of Equation (7), conditional on \( r \)-exceedances of \( (X - b_n)/r(a_n) \). Whether or not these exceedances
are themselves large will depend on the choice of scaling functions \(a_n\) and \(b_n\). For linear risk functionals, which satisfy \(r(x + y) = r(x) + r(y)\) for any \(x, y \in \mathcal{F}\), the conditioning event in Equation (12) simplifies to \(r(X) \geq u_n\) with \(u_n = ur(a_n) + r(b_n)\), that is generalised \(r\)-Pareto processes appear as the tail limit processes of increasingly large \(r\)-exceedances of \(X\).

The linear transformation \(x \mapsto (x - b_n)/r(a_n)\) required in Theorem 1 before characterising the risk is both simpler and closer to the original data than classical marginal transforms (Klüppelberg & Resnick, 2008), as it does not modify the tail decay regime. For homogeneous functionals and \(\xi > 0\), we can choose \(b_n = 0\); then Theorem 1 retrieves the work of Dombry and Ribatet (2015), which describes the limiting distribution of \(X\) for increasingly high thresholding of \(r(X)\).

### 2.3.2 Generalised \(r\)-Pareto processes

We now describe generalised \(r\)-Pareto processes, give their properties, describe simulation algorithms and link them to max-stable processes. For a given shape parameter \(\xi \in \mathbb{R}\) and positive function \(a \equiv a(s)\), let \(A = a/r(a)\) and define the set of positive functions

\[
\mathcal{A}_r = \begin{cases} 
\{ y \in \mathcal{F}_+ : r(A^{\frac{\xi - 1}{\xi}}) \geq 0 \}, & \xi \neq 0, \\
\{ y \in \mathcal{F}_+ : r(A \log y) \geq 0 \}, & \xi = 0, 
\end{cases}
\]

which contains possible sample paths for \(P\) in Equation (12) after transformation to a scale with the same marginal tail distribution.

**Definition 1** Let \(a > 0\) and \(b\) be continuous functions on \(S\), let \(r : \mathcal{F} \to \mathbb{R}\) be a valid risk functional and let \(\Lambda\) be a \((-1)\)-homogeneous measure on \(\mathcal{F}_+\). The generalised \(r\)-Pareto process \(P\) associated to the measure \(\Lambda\) and tail index \(\xi \in \mathbb{R}\) is the stochastic process taking values in \(\{ x \in \mathcal{F}_{\xi,a,b} : r((x - b)/r(a)) \geq 0 \}\) and defined as

\[
P = \begin{cases} 
a(Y_r^\xi - 1)/\xi + b, & \xi \neq 0, \\
a \log Y_r + b, & \xi = 0, 
\end{cases}
\]

where \(Y_r\) is the stochastic process on \(\mathcal{A}_r\) with probability measure \(\Lambda(\cdot)/\Lambda(\mathcal{A}_r)\).

Generalised \(r\)-Pareto processes are thus closely related to the stochastic processes \(Y_r\). A standard approach to dependence modelling, the use of copulas, requires that all the components of a random vector be transformed to follow uniform distributions. Similarly, marginal properties and dependence are typically handled separately in extreme-value modelling, with the marginal variables standardised to a distribution such as the unit Pareto. Here we use \(Y_r\), whose margins lie in the Fréchet domain of attraction with tail index \(\xi = 1\), as the process of reference. Other standardisations are possible, using for instance a Gumbel domain of attraction (e.g., Rootzén et al., 2018b), but we focus on the Fréchet case to keep the exposition concise.

Following Dombry and Ribatet (2015) and de Fondeville and Davison (2018), there is a polar decomposition

\[
Y_r \overset{\text{D}}{=} RW \quad r[A\xi^{-1}((RW)^\xi - 1)] \geq 0.
\]
where \( R \) and \( W \) are independent, the scalar \( R \) is unit Pareto and \( W \) is a stochastic process with state space \( S \) and taking values in \( S = \{ y \in \mathcal{P}_+ : \| y \|_1 = 1 \} \) with probability measure

\[
\sigma_0(x) = \frac{\Lambda \{ y \in \mathcal{P}_+ : y/\| y \|_1 \in \cdots, \| y \|_1 \geq 1 \}}{\Lambda \{ y \in \mathcal{P}_+ : \| y \|_1 \geq 1 \}}. \tag{16}
\]

where \( \| \cdot \|_1 \) denotes the 1-norm on \( \mathcal{P}_+ \). The equality (15) holds in distribution, denoted by \( \overset{\text{D}}{=} \), and is convenient because it allows the Monte Carlo simulation of \( W \) at many locations for certain common models (Dombry et al., 2016; Thibaud & Opitz, 2015).

One desirable feature of generalised \( r \)-Pareto processes is that for each \( s_0 \in S \), \( P(s_0) \) has a generalised Pareto distribution after suitable conditioning: if for a threshold \( u_0 \geq 0 \) sufficiently high that for any \( t > 1 \),

\[
x \in \mathcal{A}_{u_0} \Rightarrow tx \in \mathcal{A}_{u_0},
\]

where \( \mathcal{A}_{u_0} = \{ x \in \mathcal{F}^{\xi,a,b} : x(s_0) \geq u_0, r(\{ x - b \}/r(a)) \geq 0 \} \), then

\[
\Pr \{ P(s_0) > x_0 | P(s_0) > u_0 \} = \{ 1 + \xi(x_0 - u_0)/\sigma(s_0) \}^{-1/\xi}, \quad x_0 > u_0, \tag{17}
\]

where \( \sigma(s_0) = r(a)A(s_0) + \xi \{ u_0 - b(s_0) \} \). Unfortunately, there is no simple general expression for the distribution of \( r(P - b)/r(a) \), but if necessary it can be estimated using Monte Carlo methods. If the risk functional is linear, generalised \( r \)-Pareto processes also admit a pseudo-polar decomposition and the distribution of the risk \( r(P) \) above \( r(b) \) is generalised Pareto, with shape and scale parameters \( \xi \) and \( r(a) \); see the Supplementary Material. In univariate extreme-value theory, the marginal assumptions of Equation (6) are equivalent to convergence of rescaled block maxima toward the generalised extreme value (GEV) distribution. The analogous link between generalised \( r \)-Pareto processes and the functional extensions of GEV variables known as max-stable processes is described in the Supplementary Material.

2.3.3 Simulation

The pseudo-polar decomposition (15) is key to the construction of generalized \( r \)-Pareto processes and to their simulation. Simple algorithms to draw samples from \( Y_r \) are available for risk functionals such as \( r_1(x) = \| x \|_1 \) or \( r_2(x) = \sup_{s \in S} x(s) \); see Asadi et al. (2015), for example. We generalise the principle of de Fondeville and Davison (2018, section 2.3) to develop an accept–reject algorithm for the generalised \( r \)-Pareto process when \( \xi \neq 0 \); modification for \( \xi = 0 \) is straightforward. If we can find a threshold \( u > 0 \) such that

\[
\mathcal{A}_r \subset \{ y \in \mathcal{F}_+ : \| y \|_1 \geq u \}, \tag{18}
\]

then Algorithm 1 enables simulation of \( P \) when an algorithm for \( Y_r \) with \( r(\cdot) = \| \cdot \|_1 \) is available. In the algorithm, every unit Pareto variable is independent of every other and has distribution function \( 1 - 1/v \) for \( v \geq 1 \). Its efficiency is determined by the capacity to find the largest possible \( u \), \( u_{\text{sup}}, \) say, such that (18) is satisfied, and its acceptance rate is the ratio of the measures of the two sets in (18). Simulated generalised \( r \)-Pareto processes on \( [0,1] \) are displayed in Figure 2 for three different risk functionals: for \( \xi > 0 \), exceedances are defined as positive values of \( \sup_{s \in [0,1]} x(s) −... \)
b(s); for \( \xi = 0 \), they are functions that are large at \( s_0 = 0.1 \); and for negative tail index exceedances are functions with exceptionally high integrals over \([0, 1]\). When the risk functional is linear, an alternative algorithm in the Supplementary Material allows simulation of such processes with a pre-determined risk \( r(P) \).

Algorithm 1: Simulation of generalized \( r \)-Pareto process, \( P \)

- **Inputs:** scaling functions \( a \), \( b \), \( A \), threshold function \( u \), and scalar shape parameter \( \xi \).
- Set \( Y_0 = 0 \);
- while \( r[A \xi^{-1} \{(Y_r)^\xi - 1\}] < 0 \) do
  - generate a unit Pareto random variable \( R \);
  - generate \( W \) on \( S \) with probability measure \( \sigma_0 \) in (16);
  - set \( Y_r = uRW \);
- end
- Set \( P = a \xi^{-1} \{(Y_r)^\xi - 1\} + b \);

2.4 Limitations on the asymptotic dependence regime

The derivations above presuppose the existence of a limiting measure \( \Lambda \) in Equation (7) with non-zero mass on the space of continuous functions. This precludes asymptotic independence (Ledford & Tawn, 1996) throughout \( S \), but mixed regimes, in which asymptotic independence replaces asymptotic dependence at distances greater than some finite radius, are possible. The methodology could be extended to asymptotic independence throughout \( S \) by assuming positivity of \( \Lambda \) on more general functional spaces. For instance, asymptotic independence would be covered by considering measures that place positive mass on the set of functions that are non-zero only at a specific location. The study of such functional spaces would require more general notions of convergence than in Hult and Lindskog (2005) and has not been undertaken, so far as we know.

The positivity of \( \Lambda \) on \( P_+ \) implies homogeneity of order \(-1\), that is dependence at ‘low’ levels of intensity extrapolates further into the tail. In practice, this implies that the average size of a region on which the threshold is locally exceeded is independent of the intensity of the event. Decreased dependence at high intensities has, however, been observed in numerous environmental phenomena, for which the asymptotic models described in this paper may over-estimate dependence for high intensities. Sub-asymptotic models with decreasing dependence have recently been investigated (Huser & Wadsworth, 2019), but they correspond to processes that are asymptotically independent throughout \( S \) and thus may underestimate dependence at extreme levels especially for close locations. In general, the choice of asymptotic dependence regime should be determined by the investigator’s tolerance of risk. Asymptotically dependent sub-asymptotic models fitting into the above framework could provide more realistic alternatives than asymptotic models, but do not yet exist, so far as we are aware. The current methodology is for now the only functional approach for risk-averse policy makers: simulations of generalised \( r \)-Pareto processes provide scenarios whose extent may be pessimistic but that can be fed into impact models to assess the potential for damage to infrastructure.

Wadsworth and Tawn (2019) propose an approach that encompasses both asymptotic independence and asymptotic dependence, can be applied in high dimensions and involves conditioning on the process being extreme at any one of a number of locations, but is based
3 FUNCTIONAL PEAKS-OVER-THRESHOLD MODELLING

We now describe a general approach to modelling \( r \)-exceedances over a high threshold. Theorem 1 suggests that in principle the choice of risk functional should not impact the model parameters, but in practice this choice does affect what events are considered to be extreme, especially when there is a mixture in the tail behaviour, c.f. Figure 1. If so, the user can focus on one component of a mixture by incorporating domain-specific considerations into the risk functional, while improving sub-asymptotic behaviour by fitting the model using only the data most relevant to the type of event selected.

Suppose we have a valid risk functional \( r \) whose exceedances occur for a single physical process, such as cyclonic rainfall, and that for such events it is reasonable to use a uniform tail index \( \xi \). More specifically, let \( X \in \text{GRV}(\xi, a_n, b_n, \Lambda) \) and suppose that the marginal distributions of \( X(s) \) form a location-scale family with continuous positive scale function \( A(s) \), continuous real location function \( B(s) \), and distribution function \( F \) satisfying Equation (1) with real-valued sequences \( a'_n > 0 \) and \( b'_n \). If so, the normalising functions \( a_n(s) \) and \( b_n(s) \) for \( X(s) \) satisfy

\[
a_n(s) = A(s)a'_n, \quad b_n(s) = B(s) + A(s)b'_n, \quad s \in S,
\]

yielding the asymptotic decomposition implied by Equation (8).

We impose a parametric structure on the extremal dependence of \( X \) and on the marginal scale and location functions \( A \) and \( B \), which are assumed to belong to parametric families of functions \( A_{\theta_A} \) and \( B_{\theta_B} \). The limiting measure \( \Lambda_{\theta_W} \) is supposed to be parametrised by the distribution of \( W \), which depends on parameters \( \theta_W \).

The dependence properties of the limiting generalised \( r \)-Pareto process are determined by the angular process \( W \), which takes values in \( S = \{ y \in F_+ : ||y||_1 = 1 \} \). To characterise and compare angular process models, we need a measure of dependence, but classical measures such as the covariance function or the semi-variogram

\[
\gamma(h) = \frac{1}{2} \text{var}\{X(s') - X(s)\}
\]

rely on the existence of moments and may be undefined in our setting. A more suitable dependence measure is (de Fondeville & Davison, 2018)

\[
\pi_r(s', s) = \lim_{q \to 1} \Pr\{X(s') > u_q(s') \mid \{X(s) > u_q(s)\} \cap \{r(X) \geq u\}\}, \quad s, s' \in S,
\]

where \( u_q(s) \) denotes the \( q \) quantile of \( X(s) \) and \( u \geq 0 \). Equation (20) summarises the pairwise extremal dependence between \( X(s) \) and \( X(s') \); it extends the extremogram (Davis & Mikosch, 2009) to \( r \)-exceedances and generalises the extremal dependence coefficient \( \chi \) (Ledford & Tawn, 1996) to processes. Expression (20) matches the extremogram for high enough \( q \), that is if the risks \( r(X) \) for all those \( X \) for which \( X(s) > u_q(s) \) and \( X(s') > u_q(s') \) also exceed \( u \), then the additional condition \( r(X) \geq u \) in Equation (20) has no theoretical impact, though in practice it allows one to disentangle tail mixtures and thus to identify any differences in tail dependence regimes.
Although other dependence measures exist (Cooley et al., 2006; Smith, 1990), we prefer \( \pi_r \) for its interpretability. The literature on max-stable processes suggests several parametric models for \( W \). The Gaussian extreme value process (Smith, 1990) relies on deterministic Gaussian kernels randomly shifted in space and is attractive for its computational tractability and relative simplicity, but it yields unrealistic random fields. Under the Brown and Resnick (1977) model, the angular process \( W \) is a log-Gaussian random function whose underlying Gaussian process has stationary increments and semi-variogram function \( \gamma \), and (20) reduces to
\[
2 \left( 1 - \Phi \left[ \frac{\gamma(h)}{2} \right]^{1/2} \right),
\]
where \( h = s - s' \) and \( \Phi \) denotes the standard normal cumulative distribution function. The Brown–Resnick model is particularly attractive because many standard semi-variograms furnish models for extremal dependence. The behaviour of \( \gamma \) as \( h \to 0 \) determines the smoothness of the generalised \( r \)-Pareto process and its behaviour as \( h \to \infty \) determines the extremal dependence regime. Indeed, if the semi-variogram is bounded, as is the case for strictly stationary Gaussian processes, then \( \pi_r(h) > 0 \) for any \( h > 0 \), whereas if \( \gamma \) is unbounded then we obtain near-independence, \( \pi_r(h) \to 0 \), for large \( h \); see Figure 3. Use of a log-Gaussian \( W \) implies that for any linear \( r \), \( \Lambda(\partial A_r) = 0 \), where \( \partial A_r \) is the boundary of the set \( A_r \) defined in Equation (13).

An alternative model, for which \( \Lambda(\partial A_r) \neq 0 \), is the extremal-\( t \) process (Opitz, 2013a)
\[
W(s) \propto \max \{ G(s), 0 \}^\nu, \quad s \in S, \nu > 0,
\]
where \( G \) is a strictly stationary Gaussian process with correlation function \( C \). The maximum in this definition induces non-zero measure on the boundary of \( T^{s,a,b} \), making the model improper when \( \xi < 0 \), as then \( \Pr[X(s) = -\infty] > 0 \). Its extremogram,
\[
2 \left( 1 - t_{v+1} \left[ (v + 1)^{1/2} \left\{ \frac{1 - C(h)}{1 + C(h)} \right\}^{1/2} \right] \right),
\]
must exceed \(2\left[1 - t_{i+1}\{(v + 1)^{1/2}\}\right]\) for positive correlation functions, so when \(v\) is low, the model can only produce strong dependence. This limitation weakens as \(v\) increases and the Brown–Resnick model is approached.

In the next section, we describe an approach to joint inference on the complete parameter vector \(\theta = (\xi, a_n', b_n', \theta_A, \theta_B, \theta_W)\). Identifiability issues that may arise with the parametric models for \(A\) and \(B\) can be solved for instance by ensuring that \(r(A) = 1\), and, for a linear functional, \(r(B) = 0\); see Engelke et al. (2019), for example.

### 4 | STATISTICAL INFERENCE

In this section, we suppose for simplicity of exposition that the risk functional is linear; inference for general risk functionals essentially involves replacing the process \(X\) by the shifted and scaled version \((X - b_n)/r(a_n)\), with some further additional minor changes. Difficulties that might arise for general functionals are discussed in the Supplementary Material.

Statistical inference for \(r\)-exceedances of a stochastic process \(X \in \text{GRV} (\xi, a_n, b_n, \Lambda)\) is based on the approximation

\[
\operatorname{Pr}(X \in R) = \operatorname{Pr} \{r(X) \geq u_n\} \times \operatorname{Pr} \{X \in R \mid r(X) \geq u_n\},
\]

where \(R \subset R(u_n) = \{x \in F^{\xi,a_n,b_n} : r(x) \geq u_n\}\) and \(u_n = r(b_n)\) is a high quantile of \(r(X)\).

Let \(x_1, \ldots, x_n \in F\) be independent realisations of a generalised regularly varying stochastic process \(X\) observed at locations \(s_1, \ldots, s_L \in S\). The log-likelihood function for Equation (21) based on the \(r\)-exceedances over the threshold \(u_n\) among \(x_1, \ldots, x_n\) is

\[
\mathcal{L}_{\text{Thres}}(\theta) = \sum_{j \in K_{u_n}} \log \operatorname{Pr} \{r(x_j) \geq u_n; \theta\} + \sum_{j \in K_{u_n}} \log f^r(x_j; \theta),
\]

where \(K_{u_n} = \{j \in 1, \ldots, n : r(x_j) \geq u_n\}\) contains the indexes of the \(n_{u_n}\) \(r\)-exceedances over \(u_n\), and \(f^r\) denotes the finite-dimensional density function of a generalised \(r\)-Pareto process observed at \(s_1, \ldots, s_L\), i.e.,

\[
\frac{\lambda_{\theta_w}}{\Lambda_{\theta_w}(A_r)} \left[1 + \xi \frac{(x(s_1:L) - b_n(s_1:L))/a_n(s_1:L)}{1/\xi}\right] L \\
\prod_{l=1}^L a_n(s_l)^{-1} \left\{(1 + \xi \frac{x(s_l) - b_n(s_l)}{a_n(s_l)})^{1/\xi - 1}\right\},
\]

where \(x(s_1:L) = \{x(s_1), \ldots, x(s_L)\}\), and the \(L\)-dimensional intensity function \(\lambda_{\theta_w}\) is given by

\[
\Lambda_{\theta_w}(A_{\max}(z)) = \int_{R^L \setminus \{0,z\}} \lambda_{\theta_w}(y) dy,
\]

with \(A_{\max}(z) = \{y \in A_r : \max_{l=1,..,L} y(s_l)/z(s_l) \geq 1\}\). The second term of (23) is the Jacobian for the marginal transformations from the generalised Pareto scale used for the data to the unit Fréchet scale on which the dependence model is defined.

A model must be specified for the probabilities that \(r(x_j) \geq u_n\). In similar contexts Wadsworth and Tawn (2014) and Engelke et al. (2015) use a Poisson distribution suggested by the relationship with block maxima, which yields log-likelihood
\[ L_{\text{Poiss}}(\theta) = n_u \log \Lambda_{\theta_W}(A_r) - \Lambda_{\theta_W}(A_r) + \sum_{j \in k_{n_u}} \log f^r(x_j; \theta), \]  

(25)

when the exceedance events are identically distributed, but the Pareto methodology accommodates other possibilities. Thibaud and Opitz (2015), for instance, suppose that the corresponding random variable \( N_u \) is fixed and use a binomial distribution, which is easily linked to the Poisson point process model. Such approaches presuppose that the probability of an exceedance does not depend on explanatory variables, but logistic regression could be used to model the probability of observing an extreme event if not; see Section 6.4.

Maximisation of Equations (22) or (25) can be difficult and we suggest first estimating the marginal parameters \( \xi, a'_n, A, b'_n \) and \( B \) and then fitting a dependence model by fixing them at their estimates. The marginal parameters can be estimated by maximising the independence log-likelihood,

\[
\sum_{j=1}^{n} \sum_{l=1}^{L} \{ x_j(s_l) \geq b_n(s_l), r(x_j) \geq u_n \} \log \Pr\{ x_j(s_l) \geq b_n(s_l) \} \times 
\log \left[ \frac{1}{a_n(s_l)} \left\{ \frac{x_j(s_l) - b_n(s_l)}{a_n(s_l)} \right\}^{\frac{1}{\xi} - 1} \right],
\]  

(26)

under the constraint \( r(b_n) = u_n \), with parameter uncertainty assessed by resampling the \( x_j \). Any other inference procedure allowing a common value of \( \xi \) could be used instead.

One way to estimate the dependence parameters is to minimise the function

\[
\sum_{l,r=1,...,L} \left\{ \hat{\pi}(s_l, s_r) - \pi_{\theta_W}(s_l, s_r) \right\}^2,
\]  

(27)

where \( \hat{\pi} \) denotes an estimate of Equation (20), such as that obtained by replacing exceedance probabilities by the corresponding frequencies (Davis et al., 2013),

\[
\hat{\pi}(s_l, s_r) = \frac{\sum_{j=1}^{n} \{ x_j(s_l') \geq b_n(s_l'), x_j(s_l) \geq b_n(s_l), r(x_j) \geq u_n \}}{\sum_{j=1}^{n} \{ x_j(s_l) \geq b_n(s_l), r(x_j) \geq u_n \}}.
\]

This approach is robust and can be tailored to the situation at hand, for example by weighting summands to improve spatial prediction at ranges of particular interest or to reduce the computational burden when \( L \) is very large. The approach ensures that the fitted model has the same average number of locations jointly exceeding the location function \( b_n \) as in the data, but uncertainty quantification for the resulting estimates typically involves resampling and may be time-consuming, though this allows uncertainty for both marginal and dependence aspects to be readily combined.

Maximum likelihood estimation of \( \theta_W \) has been studied for specific risk functionals but can perform poorly because the limiting process is misspecified for finite \( u_n \) (Engelke & Malinowski, 2014; Huser et al., 2016). Alternatives involve censoring of low components (e.g., Wadsworth & Tawn, 2014), composite likelihoods (Castruccio et al., 2016; Huser & Davison, 2013; Padoan et al., 2010) or M-estimation using pairwise tail indexes (Einmahl et al., 2016a,b). All are more robust to mis-specification but can be used only for specific risk functionals and are dimensionally limited, either by the computational burden due to the numerical evaluation of the normalising constant.
\[ \Lambda_{\theta_w}(A_r) \] and the censoring, or, for pairwise procedures, by combinatorial considerations. Efficient algorithms for censored likelihood are available (de Fondeville, 2016) and tractable for \( L \) up to a few hundred for the Brown–Resnick and extremal \( t \) models. Gradient scoring (de Fondeville & Davison, 2018) can be applied to a large class of risk functionals and avoids the computation of \( \Lambda_{\theta_w}(A_r) \), making inference tractable for \( L \) in the thousands; for log-Gaussian random functions, its numerical complexity is that of matrix inversion. These approaches could also be used to estimate the entire parameter vector \( \theta \) simultaneously, thereby allowing a full quantification of the uncertainties, for instance by resampling. More details can be found in the Supplementary Material.

5 | MODEL VALIDATION

Suppose that we have an estimate \( \hat{\theta} \) of the parameters and a measure of its uncertainty and we wish to check the quality of the fitted model.

The marginal tail behaviour at each sampled location \( s_1, \ldots, s_L \) can be checked by comparing the observations with the fitted marginal model. If \( u_q(s_i) \) denotes the empirical \( q \) quantile of the \( r \)-exceedances at \( s_i \), estimated using only observations \( x_j \) for which \( r(x_j) \geq u_n \), where \( q \) has been chosen such that (17) holds, and if \( n_q \) denotes the number of \( x_j \) exceeding \( u_q(s_i) \), then we can check the marginal fits using the approximation

\[
\text{Pr}\{X(s_l) - u_q(s_l) \geq x \mid X(s_l) \geq u_q(s_l)\} \approx H_{\hat{\xi}, \hat{\sigma}}(s_l)(x), \quad x \geq 0,
\]

with \( \hat{\sigma}(s_l) = \hat{\alpha}_n(s_l) + \hat{\xi}\{u_q(s_l) - \hat{b}_n(s_l)\} \). Pointwise confidence intervals for quantile-quantile plots can be obtained by resampling: we draw \( m \) samples of size \( n_q \), \((Z_{l1}^1, \ldots, Z_{l1}^{n_q}), \ldots, (Z_{lm}^1, \ldots, Z_{lm}^{n_q})\) from the fitted distribution and let \( Z_{l1}^1, \ldots, Z_{lm}^j \) denote the \( j \)th order statistic of each sample. A 95% confidence interval for the generalised Pareto fit is then defined as the 2.5 and 97.5 empirical percentiles of \( Z_{l1}^1, \ldots, Z_{lm}^j \). When the estimator used to obtain \( \hat{\theta} \) is asymptotically normal, estimation uncertainty can be taken into account to some extent by drawing the \( m \) samples from different generalised Pareto distributions whose parameters \((\xi, \log \sigma)\) are normally distributed with mean \((\hat{\xi}, \log \hat{\sigma}(s_l))\) and covariance matrix corresponding to the uncertainty of \( \hat{\theta} \). When the risk functional is linear, a similar marginal check can be performed for the exceedances of \( r(X) \).

The dependence model can be assessed by comparing the fitted extremogram with the empirical version of Equation (20). If the model is stationary and isotropic, then \( \pi \) depends only the distance \( h \) between two locations, and \( \pi \) can be plotted as a function of the distance, and perhaps the orientation, of pairs of locations. For an anisotropic model it is preferable to map how the dependence varies with the spatial coordinates. More general dependence measures based on aggregation (Engelke et al., 2019) could be considered.

Model comparison can be performed using the Akaike or composite likelihood information criteria (Davison & Gholamrezaee, 2012), and formal comparison of nested models can be based on scoring rules (Dawid et al., 2016; de Fondeville & Davison, 2018). A relative root mean squared error or the continuous ranked probability score (Gneiting & Raftery, 2007) can be used to assess the predictive performance of the model. If \( S \) has a temporal component, then an empirical probabilistic forecast is available by simulating from the fitted model at future times conditioned on
currently available observations. When the angular process is log-Gaussian, this is equivalent to conditional simulation of a Gaussian process, followed by a marginal transformation.

We describe applications of these ideas in the next two sections.

6 | MODELLING EXTREME EUROPEAN WINDSTORMS

6.1 | Motivation

One of the severest extra-tropical cyclones ever observed, windstorm Daria, struck the United Kingdom on 25 January 1990. Over that day and the next, 97 deaths were reported and damage valued at around 8.2 billion US dollars occurred. The strongest measured gusts were 47.2 ms\(^{-1}\), equivalent to a category 1 hurricane. Figure 4 shows the maximum speed over 3-h intervals of the wind gusts sustained for at least 3s for the 24 h during which the storm peaked over the United Kingdom. To give an idea of the severity of this storm, damaging windspeeds are considered to start at 25 ms\(^{-1}\) (Roberts et al., 2014). About ten years later, on 26 December 1999, storm Lothar swept across western and central Europe. A wind speed of 46.9 ms\(^{-1}\) was recorded in Paris, and the weather station at the summit of ‘La Dole’ in Switzerland recorded a maximum wind gust of 55.9 ms\(^{-1}\). Lothar, equivalent to a category 2 hurricane, caused losses of 8 billion US dollars and more than 100 deaths.

These two events illustrate why estimating the risk linked to such natural hazards has become a major question in recent decades, especially as the possible influence of global warming on them is far from understood.

6.2 | Risk estimation for extreme windstorms

Risk estimation for extreme windstorms has generally been limited to the use of historical catalogues of events to test the resilience of infrastructure (Haylock, 2011; Pinto et al., 2012), but unfortunately such storms are rare and the catalogues usually span only a few decades. Further events can be generated by statistical perturbation of the wind field intensity, shape and location (Hall & Jewson, 2008) or by detecting storms in multiple numerical climate outputs (Della-Marta et al., 2010). In both cases, the same storms may be re-cycled but with differing climatological indexes because of different hypotheses and approximations used by the models. Yiou (2014) proposed creating new storms from historical catalogues by reordering time steps based on spatial analogues. Uncertainties and bias linked to all these approaches may be large and difficult to estimate, and studies on climatological projections have stressed their inability to accurately reproduce extreme events (e.g., Weller et al., 2013). All these methods generate storms whose tail behaviour cannot be extrapolated to still rarer events.

Extreme value theory was applied to the problem by Della Marta and Mathis (2008) and by Mornet et al. (2017), who performed a POT analysis on univariate summaries characterising extreme windstorms, but did not model spatial dependence. Ferreira and de Haan (2014) suggest how historical windstorm records might be up-scaled to higher intensities using Pareto processes, but their approach cannot generate new storms. Economou and David (2014) used Bayesian hierarchical models of extra-tropical cyclones, but included dependence using covariates such as mean sea level pressure, which limits the capacity of the model to generate new patterns and intensities. The existing work closest to ours is by Sharkey et al. (2020), who use
a Lagrangian approach to model the tracks and severity of European windstorms. Their model for storm tracks is more detailed than ours, but their dependence structure uses a non-extremal model and neglects the temporal element.

We propose an approach based on generalised $r$-Pareto processes, which extends the Della Marta and Mathis (2008) approach to allow not only local risk estimation but also the generation of new extreme storms that are spatially and temporally consistent.

### 6.3 Data set and region of study

To build our stochastic weather generator, we follow the methodology of the extreme windstorms (XWS) catalogue (Roberts et al., 2014), which provides historical records of the 50 most extreme storms over Europe for winters from 1979 to 2014; more precisely it contains maps of 72-h max-
Figure 5  Left: Study region $E$ (coloured cells) for modelling extreme windstorms over Europe. Mountainous regions were removed to avoid the systematic bias of the reanalysis model. The green cells show the region $E_{ALLP}$ containing Amsterdam, Brussels, London and Paris. Estimated location and scale functions $b_n$ (middle) and $a_n$ (right) (both in ms$^{-1}$) of the generalized $r$-Pareto process [Colour figure can be viewed at wileyonlinelibrary.com]

imum wind gusts over northern Europe. In this catalogue, the ‘extreme storms’ are chosen to focus on events with high impact on infrastructure; indeed, the storms with the highest maximum wind speeds may not cause the most damage overall unless they cross inhabited areas. To apply our methods, we must define univariate summaries that characterise the most damaging events.

The XWS catalogue tracks storms in the ERA-Interim reanalysis (Dee et al., 2011), a real-time climate model whose records start in 1979 and that provides time series for many climatological indexes. The model is run every 6 h on a grid whose cells are squares with sides that can be chosen between 3° and 0.125°; the native size is 0.75° and other resolutions are obtained by interpolation. In addition to the 6-hourly fields obtained by data assimilation, which constrains the grid values to station measurements, 256-h forecasts are generated each day at 00UTC and 12UTC. These forecasts can be combined with the assimilated data to obtain a three-hourly database of the maximum speed of the wind gusts sustained for at least 3 s, as shown in Figure 4. Most European winter storms evolve quickly and last only for a day or so, so fine time-resolution is necessary. The link between reanalysis outputs and station measurements is unclear, so for simplicity we treat observations as instantaneous rather than as temporal aggregates.

Our study focuses on the coloured region $E$ in the left-hand panel of Figure 5. The reanalysis model is known to be systematically biased and to have a different dependence regime over regions with rapid variations in altitude (Donat et al., 2011), so we exclude mountainous regions such as the Pyrenees and the Alps, leaving 605 cells based on the native resolution of 0.75°. Similar to the XWS catalogue methodology, we combined the maximum wind gust sustained for at least 3 s from the reanalysis with the forecasts to obtain a three-hourly spatial time series. Extra-tropical windstorms over Europe occur only during the winter, so we take our study period $T$ to be the months October–March over the years 1979 to 2014.

6.4  Storm definition and exceedance probability modelling

Following Roberts et al. (2014) and Vautard et al. (2019), we consider storms that give exceedances of the spatial average over a region with very dense infrastructure during a 24-h period. For this
application, we write $S = E \times [0, 24]$ with $E \subset \mathbb{R}^2$ denoting the region of Europe considered. The spatiotemporal process $X(s, t)$ represents the wind field at location $s \in E$ and time $t \in T$. We take the risk functional $r$ at time $t$ to be the spatial average of an observed wind field $x(s, t)$,

$$r(x)(t) = |E_{ABLPL}|^{-1} \int_{E_{ABLPL}} x(s, t) \, ds, \quad t \in T,$$

where $E_{ABLPL}$, the green region shown in Figure 5, includes Amsterdam, Brussels, London and Paris. To suppress the temporal clustering of high values of $r(x)(t)$, we centre the time frame on the largest spatial average for each event and keep only events that are at least 48 h apart, yielding $n = 1561$ observations. Storm Daria corresponds to a maximum intensity of $r(x) = 32.1 \text{ ms}^{-1}$.

The choice of the declustering algorithm influences the distribution of the events and must be taken into account in the model and estimation procedures; in this work, the model described in Sections 5 and 6 does not allow temporal variation of the dependence structure but ensures unbiased estimation.

The approximation (21) requires models for the probability that $r(X) \geq u_n$, for the margins, including a tail index $\xi$ and the functions $a_n$ and $b_n$, and for the dependence structure of the generalised $r$-Pareto process $P$.

A natural choice for $u_n = r(b_n)$ is a high quantile of the observed $r(x)$. In order to include most of the XWS storms in our set of exceedances, we take $u_n = q_{0.96}\{r(x)\} = 24 \text{ ms}^{-1}$, yielding 63 events in the study period. The value 0.96 lies within a range of quantiles over which the estimated tail index for $r(x)$ is stable. The risk functional, $r$-exceedances and XWS storms (Figure 1 of the Supplementary Material) show that the 63 selected events coincide with most of the windstorms from the XWS catalogue, so the exceedances $r(x) \geq u_n$ well characterise extreme windstorms that strike $E_{ABLPL}$.

The temporal distribution of the selected events is non-stationary. Donat et al. (2010) and Pfahl (2014) have established that climatic circulation patterns such as the North Atlantic Oscillation index (NAO) influence windstorms, and we use logistic regression to model this. We extracted the 3-hourly mean sea level pressure from the ERA–Interim reanalysis and computed the NAO using its definition in terms of empirical orthogonal functions (EOF) (Blessing et al., 2005), as the first eigenvalue of the mean sea level pressure anomaly at a given time $t$. We likewise computed the Antarctic Oscillation index (AAO) and created indexes for temperature anomalies. Time was also included as a potential covariate. Analysis of deviance reveals that the NAO index and the first and third eigenvalues of the temperature anomaly affect the occurrence of winter storms at the 0.1% significance level; see Figures 2–6 of the Supplementary Material.

### 6.5 Marginal model

Fitting the marginal model involves the estimation of a tail index $\xi$ and the functions $a_n$ and $b_n$ under the assumptions of Section 3. In general, a parametric model for $a_n$ and $b_n$ might be necessary, as in Engelke et al. (2019), but for simplicity we here set $a_n(s_l) = a_l > 0$ and $b_n(s_l) = b_l \in \mathbb{R}$ for each of the $L = 605$ locations $s_l$.

With the model for the probability of storm occurrence described in Section 4, the parameter $b'_n = r(b_n)$ is fixed to the empirical 0.96 quantile of the observed $r(x)$. The threshold-stability of generalised Pareto distributions does not allow us to identify the function $b_n$ without further
assumptions, so we choose $b_i$ to equal the empirical $q'$ quantile $u_q'\{x(s_i)\}$ of the $r$-exceedances above threshold $u_n$ at the location $s_i$, with $q'$ chosen so that $r(b_n) = b'_n$. We obtain $q' = 0.675$, yielding 184 marginal excesses and estimated location function $\hat{b}_n$ shown in the central panel of Figure 5.

For tractability, we first fit the marginal model, estimating the tail index $\xi$ and the positive scale parameters $a_1, \ldots, a_L$ by maximising the independence log-likelihood (26). For a given tail index $\xi$, the likelihood for the exceedances above the threshold $b_i$ is optimised independently for each location $s_i$. We treat storms as independent events, and account for strong temporal dependence within each of them by weighting each log-likelihood contribution inversely proportionally to the number of exceedances in the storm from which it arises, so that each storm affects the estimates roughly equally. This yields the maximum independence likelihood estimate $\hat{\xi} = -0.150_{0.01}$, close to the average of the locally estimated tail indexes; the corresponding estimated scale function $\hat{a}_n$ is shown in the right-hand panel of Figure 5. Standard errors for $\hat{b}_n$, $\hat{a}_n$ and $\hat{\xi}$ are obtained by resampling. Estimates of $A$ and $B$ can then be deduced using Equation (19) with $a'_n = r(a)$.

The overall fit of the marginal model is convincing, and the fit for $r(x)$ above the threshold $u_n$ seems to be adequate; see Figure 7 of the Supplementary Material.

### 6.6 Dependence model

Following Equation (21), we model the storms by a generalised $r$-Pareto process with state space $S = E \times [0, 24]$ and whose dependence structure must be specified. For the angular component $W$, we choose a process with log-Gaussian random functions and Whittle–Matérn (Matern, 1960; Whittle, 1954, 1963) semi-variogram

$$\gamma(s, s', t, t') = \kappa \{ 1 - ||h|| K_v(||h||) \}, \quad \kappa, v > 0,$$

where $K_v$ is the modified Bessel function of the second kind of order $v$, and (Gelfand et al., 2010, pp. 428, 432)

$$||h|| = ||h(s, s', t, t')|| = \left\{ \left[ \frac{\Omega(s' - s) - V(t' - t)}{\tau_s} \right]^2 + \left[ \frac{t' - t}{\tau_t} \right]^2 \right\}^{1/2},$$

for $s, s' \in E$ and $t, t' \in [0, 24]$, with positive scale parameters $\tau_s$ and $\tau_t$ for the space and time dependence, a wind vector $V \in \mathbb{R}^2$ that models the average displacement of the storm in a 3-h period, and an anisotropy matrix

$$\Omega = \begin{bmatrix} \cos \eta & -\sin \eta \\ a \sin \eta & a \cos \eta \end{bmatrix}, \quad \eta \in \left( -\frac{\pi}{4}, \frac{\pi}{4} \right), \quad a > 0,$$

that allows the spatial dependence in Equation (29) to decrease faster in a direction determined by the angle $\eta$. Estimation of $v$ is known to be difficult, so we set $v = 1$, to foreshadow our planned use of more flexible non-stationary models such as that of Fuglstad et al. (2015). Indeed, further exploratory analysis reveals that dependence varies over space, so more complex models would ideally be considered.
TABLE 1  Semi-variogram parameter estimates obtained by minimizing (27) and using the gradient score. The standard errors (subscripts) are obtained using a block jackknife.

|       | $\kappa$ | $\tau_s$(km) | $\tau_t$(h) | $\alpha$ | $\eta$($^\circ$) | $V_1$(km.h)$^{-1}$ | $V_2$(km.h)$^{-1}$ |
|-------|----------|--------------|-------------|----------|-----------------|-------------------|-------------------|
| Least squares | 3.5 | 614 | 23.8 | 1.41 | $-$4.12 | 51.3 | 14.4 |
| Gradient score | 2.85$^{0.01}$ | 337$^{11.6}$ | 9.6$^{2.8}$ | 1.32$^{0.01}$ | 21.2$^{0.1}$ | 50.4$^{2.9}$ | 12.5$^{1.7}$ |

The semi-variogram function (28) is motivated by an exploratory analysis in which the space-time extremogram

$$\pi(h_s, h_t) = \Pr\{X(s', t') \geq u' | X(s, t) \geq u\}, \quad h_s = s' - s, \quad h_t = t' - t,$$

with thresholds $u, u'$ at local 0.675 empirical quantiles of the set of $r$-exceedances, is estimated as described in Section 4.

We used both least squares and gradient scoring procedures to estimate the parameters of Equation (28), the latter using a composite approach with 100 random subsets and the same 50 locations for every storm, since we found this to be more robust than including all locations. In general, we recommend using subsets whose size roughly equals the number of chosen events.

Table 1 and Figure 6 summarise the resulting fits, which agree on the strength of dependence at long distances and are overall consistent with the empirical values, but differ for the anisotropy: least squares picks out the long-range north-east anisotropy but the gradient score fit captures the short-range south-east anisotropy. This change in direction cannot be captured by our over-simple model. The estimated wind vectors $\hat{V}$ for the two fits are similar and agree with the observation that storms are born over the Atlantic and usually move towards the North Sea in an east-north-easterly direction. The fits look reasonable, though the scoring approach may slightly under-estimate the temporal dependence.

6.7  | Simulations

The usefulness of our model can be checked by simulating extreme storms from it, using a version of Algorithm 1 modified to ensure that the maximum spatial average occurs at $t = 12$ h, consistent with our definition of an extreme storm. We first simulate the angular component of the spatial process at time $t = 12$, and then simulate the remaining time steps by successively generating the spatial process at times $t = 9, 6, 3, 0, 15, 18, 24$ conditionally on the variables already simulated. If a new time step yields a spatial average greater than its value at time $t = 12$, the sample is rejected and the procedure is repeated until a suitable candidate is found.

For an angular process with log-Gaussian random functions, such a simulation algorithm is equivalent to conditional simulation of multivariate Gaussian random vectors. Figure 8 of the Supplementary Material shows a simulated storm with intensity $r(x) = 29.1$ ms$^{-1}$, similar to that of Daria. The images are rougher than those in Figure 4 but nevertheless the higher windspeeds at sea, the general scale of spatial dependence and the directionality seem credible.
7 | FLOOD RISK ASSESSMENT

7.1 | Motivation

In August 2005, the city of Zurich suffered from heavy floods that led to estimated property damage of around 3 billion Swiss francs and six deaths (Bezzola & Hegg, 2007). Zurich is especially risk-prone, as it lies at the foot of a lake and is traversed by several rivers, including the Sihl, which flows under the city’s main railway station. Although the 2005 event was not caused by an unusually high level of the Sihl (Jaun et al., 2008), it triggered an overall assessment of flood risk for the city. An extreme discharge of this river could cause hundreds of millions of francs of losses by damaging infrastructure and by preventing half a million commuters from travelling. A good
understanding of the risk related to high levels of the Sihl is thus crucial when considering potential mitigation measures. Below we use our ideas to construct a stochastic generator of extreme rainfall over the Sihl river basin, in order to create a catalogue of events for input to hydrological models. Cloke and Pappenberger (2009) review similar approaches based on climate models.

### 7.2 Data set and region of study

Figure 7 shows the region of study, a rectangle south-west of Zurich that includes the Sihl river basin. Any rain falling in the green area can be expected to flow under Zurich main station. Rainfall is the result of various physical processes, including cyclonic and convective regimes, which can usually only be distinguished using high resolution data such as radar measurements. In this study, we use the CombiPrecip data set produced by MeteoSwiss (Gabella et al., 2017; Panzier et al., 2018; Sideris et al., 2014), which estimates the hourly accumulated rainfall for a grid over Switzerland from 2013 to 2018. Owing to changes in 2013, earlier measurements are inconsistent with more recent data, but even with this reduced period the data set includes \( n = 52,413 \) radar images. The Sihl river basin is orographically homogeneous and is located at a reasonable distance from the radar, so the estimated rain fields are thought to be fairly reliable. CombiPrecip provides discrete measures of rain accumulation that result from post-processing, and this particularity would require specific treatment, for instance using a discrete generalised Pareto distribution (Anderson, 1970; Krishna & Singh Pundir, 2009; Prieto et al., 2014). Here we aim to illustrate the flexibility and advantages of functional peaks-over-threshold analysis, so we leave the discreteness to future work. To ensure good behaviour of rank-based procedures such as the computation of the empirical extremogram, the original discrete measurements are jittered by adding a small noise.

### 7.3 Risk definition and model formulation

Following de Fondeville and Davison (2018), we model both locally intense and large spatial accumulations of rainfall, but rather than use unnatural risk functions based on standardised data, we here first defined the risk in terms of the jittered measurements \( x_1, \ldots, x_n \) through the functionals

\[
 r_1(x) = |S|^{-1} \int_S x(s) ds, \quad r_2(x) = \max_{s \in S} x(s),
\]

where \( r_1 \) represents a volume of water and thus has a direct hydrological interpretation. In order to use \( r_1 \) and \( r_2 \) to entirely separate these different types of events, we must choose the thresholds \( u_1, u_2 > 0 \) so high that only the eleven most intense events are used for inference; see Figure 1. In order to use more events and to illustrate the flexibility of the functional POT methodology, we study a modified spatial average risk functional,

\[
 r'_1(x) = |S|^{-1} \int_S x(s) ds \times \frac{||\hat{x}_1||}{||\hat{x}||},
\]

where \( ||\hat{x}|| \) denotes the norm of the two-dimensional discrete Fourier transform of \( x \) and \( ||\hat{x}_1|| \) denotes the norm of its first component. This focuses the risk on events with large spatial average...
rainfall that are also spatially widespread and discards ‘hybrid’ events: deeper exploratory analysis suggests that more than two types of rain are encountered in this region.

The \( r'_1(x_i) \) are highly correlated with \( r_1(x_i) \), especially in the tail, but using \( r'_1(x_i) \) allows us to lower the threshold enough to retain 132 events; see Figure 9 of the Supplementary Material. It also illustrates the use of a risk functional that has a non-linear part and shows that image processing ideas can help to characterise extreme rain types. Another way to discriminate between types of extremes would be to project the database onto specific weather regimes obtained via EOF analysis (Braud et al., 1993) or via a methodology tailored for extremes (Cooley & Thibaud, 2019), which could help in studying weather patterns such the North American winter dipole (Wang et al., 2015).

When building a model for rainfall, it is important to be able to handle dry grid cells, for which \( x(s) = 0 \). Below we use a log-Gaussian generalised r-Pareto process, as presented in Section 3, but the model must accommodate zeros. In our region we can treat the distribution of dry cells as homogeneous, so we suppose that zero rainfall corresponds to a negative value of the process, which we treat as left-censored at zero. A simple modification to allow for variation in the distribution of dry cells would be to construct a new data set \( x' \) by adding a positive function \( c \equiv c(s) \) to the original data \( x \), to treat \( x'(s) \) as left-censored if it equals \( c(s) \), and to let \( c(s) \) increase with the frequency of dry events. This does not affect the model fit, as \( x' - b'_n = x - b_n \) if \( b_n \) and \( b'_n \) are local empirical quantiles of \( x \) and \( x' \), though the censoring must be accommodated.

To fit the model, we first estimate the marginal tail behaviour and then the dependence model. For the margins we proceed as in Section 5: \( \hat{b}_1 \) and \( \hat{b}_2 \) are defined for \( r_1 \) and \( r_2 \) separately as local empirical quantiles of the exceedances for these risk functionals, with the levels chosen such...
that $r_1'(b_n^1) = u_1$ and $r_2(b_n^2) = u_2$. The corresponding tail indexes and scale parameters are then estimated using the independence likelihood (26).

We considered the Matérn and the Bernstein (Schlather & Moreva, 2017) semi-variogram models for the dependence. Matérn semi-variograms are bounded above, whereas the Bernstein model bridges the two regimes illustrated in Figure 3. In the case of the spatial maximum functional $r_2$, we use censored likelihood estimation with thresholds $\hat{b}_n^1 > c$. For the spatial average functional $r_1'$, no dry cells were observed for any of the 132 events yielding exceedances, and we can use a gradient score approach to estimate the dependence model unhampered by dry cells. In both cases, we found composite approaches to be more stable, so we estimated the dependence using 1000 random sets of 30 locations for $r_1'$ and 100 random sets of 10 locations for $r_2$, for which the number of subsets was reduced for tractability. We again observed that composite procedures with subsets of size roughly $n$ gave fairly stable estimates.

### 7.4 Estimated models

The marginal model fits for both risk functionals were checked using QQ-plots and were found to be good everywhere. The estimated models, summarised in Figure 8, have different tail behaviours. Events corresponding to exceedances for $r_1'$ have estimated tail index $\hat{\xi}_1 = -0.2_{0.04}$, and those for $r_2$ have $\hat{\xi}_2 = 0.02_{0.02}$: the rough standard errors shown as subscripts were obtained by resampling. The estimates suggest that spatially widespread accumulations of rainfall are

| Tail Index | Dependence Models | Observations | Simulations |
|------------|------------------|--------------|-------------|
| $\hat{\xi} = -0.2_{0.04}$ | ![](image1) | ![image2] | ![image3] |
| $\hat{\xi} = 0.2_{0.02}$ | ![image4] | ![image5] | ![image6] |

\[ r_1'(x) = |S|^{-1} \int_S x(s) \, ds / |S| \]

\[ r_2(x) = \max_{s \in S} x(s) \]

**Figure 8** Fitted models for extremes of the modified spatial average (top) and spatial maxima (bottom). Left: estimated tail index and fitted extremogram. Center: largest observed events. Right: simulated events

[Colour figure can be viewed at wileyonlinelibrary.com]
bounded above, whereas the tail decay estimate for locally heavy rain lies in the Fréchet regime, which gives no upper bound. While one could argue that events for $r_2$-exceedances will dominate in the limit, other types of event are nevertheless of interest, especially if we consider more complex definitions of extremes. In this application there appears to be a worst-case scenario for large widespread rainfall over the Sihl river basin that could be used in deriving mitigation procedures, above which we need to focus only on locally intense rainfall events.

For $r'_1$, the lower score was obtained with a Matérn model, while the Bernstein semi-variogram gave a higher likelihood for $r_2$-exceedances. The fitted models show much weaker extremal dependence for $r_2$, whereas the theoretical extremogram does not drop below 0.7 for $r'_1$, highlighting the importance of suitable definitions of risk. The illustrative simulations in Figure 8 appear consistent with the data. The model estimated for $r_2$ seems to over-estimate extremal dependence compared to the data: as the threshold increases, the estimated extremogram decreases. This decrease in dependence at high levels is not accommodated by our model. Huser et al. (2017) and Huser and Wadsworth (2019) have proposed spatial models in which dependence decreases that could be extended to our setting.

8 | DISCUSSION

Peaks-over-threshold methods are widely used for modelling the tails of univariate distributions, but a more general approach is needed to take advantage of complex data. This paper extends peaks-over-threshold analysis to continuous stochastic processes. Exceedances are defined in terms of a real-valued functional $r$, and modelled with the generalised $r$-Pareto process, which appears as the limit for $r$-exceedances of a properly rescaled process and is the functional generalisation of generalised Pareto variables. We derive construction rules for such processes, give simulation algorithms, highlight their link to max-stable processes, and propose inference and model validation procedures. The ideas are illustrated by applications to windstorms and spatial rainfall.

The minimal assumptions under which one can derive the convergence of conditional $r$-exceedances are quite weak: if the marginal distributions are assumed to have generalised Pareto tails, then the existence of a non-zero joint limit should naturally be considered. If the assumptions are unrealistic, then the need for a functional model might be questioned. If one assumes that such a limit is a continuous function, then generalised $r$-Pareto processes arise naturally. A consequence is that the convergence results presented here do not allow asymptotic independence throughout $S$, which would involve the appearance of discontinuous functions in the limit. More general convergence notions, as yet undeveloped, are needed to provide a fully unified peaks-over-threshold analysis for functions.

The stochastic windstorm generator obtained in Section 6 produces events consistent with historical records. The simulated windstorms peak at the centre of the 24-h time window, but this is not accommodated in the estimation process; ideas of Coles et al. (1994) or Smith et al. (1997) might be adapted to deal with this in future work. Moreover, the underlying model does not capture the full complexity of the spatiotemporal structure of extreme windstorms, whose dependence changes over space. Oesting et al. (2017) show that the potential types of non-stationarity are limited, but models with varying local anisotropy, such as in Fuglstad et al. (2015) or Fouedjio et al. (2016), would be a natural extension. The realism of simulated storms might be improved by using the methodology of Lindgren et al. (2011) to build physically-inspired non-stationary spatiotemporal dependence structures, using for instance the diffusion equation,
and this would be computationally efficient and perhaps more realistic. Our windstorm model introduces non-stationarity by allowing the probability that a windstorm will occur to depend on explanatory variables, but the distribution of conditional $r$-exceedances does not vary, and this may be too restrictive. The methodology is flexible enough to allow explanatory variables to influence the generalised $r$-Pareto process, if necessary.

The rainfall application in Section 7 highlights the importance of an appropriate definition of risk by illustrating how it impacts the tail behaviour of the selected events and showing how $r$-exceedances allow one to disentangle mixtures of extremes. The approximation provided by the asymptotic framework would be questionable if, for instance, marginal shape parameter estimates varied strongly over the region and disagreed with that for the risk functional. Sub-asymptotic models for which extremal dependence diminishes with intensity would then be preferable, as this phenomenon is commonly observed with rainfall.

Another notion of complexity for extremes is linked with compound events. Let $(X^1, X^2)$ be a bivariate continuous stochastic process and let $r^1$ and $r^2$ be suitable risk functionals. Then under conditions similar to those above, the functional

$$r(X^1, X^2) = \min \{ r^1(X^1) - u^1, r^2(X^2) - u^2 \}$$

can be used to characterise extremes of both types and could be applied when studying infrastructure that is vulnerable to different sources of risk. This differs from (4), which concerns multiple risks for a single process.

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