Graviton propagator asymptotics and the classical limit of ELPR/FK spin foam models

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Abstract

We study the classical limit of the ELPR/FK spin foam models by analyzing the large-distance asymptotics of the corresponding graviton propagators. This is done by examining the large-spin asymptotics of the Hartle-Hawking wavefunction which is peaked around a classical flat spatial geometry. By using the stationary phase method we determine the wavefunction asymptotics. The obtained asymptotics does not give the desired large-distance asymptotics for the corresponding graviton propagator. However, we show that the ELPR/FK vertex amplitude can be redefined such that the corresponding Hartle-Hawking wavefunction gives the desired asymptotics for the graviton propagator.

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I. INTRODUCTION

Loop quantum gravity is a candidate for a realistic quantum theory of gravity and it represents a nonperturbative and background independent way of quantizing general relativity \[1\]. However, one of its main problems is finding the classical limit. This is difficult to do in the canonical formulation, because there are no appropriate solutions of the Hamiltonian constraint. But even if one had such a solution, it would be a complicated expression, and showing that its transform to the triad representation has a semiclassical limit which implies the Einstein equations is a daunting task, see \[2\]. In the covariant formulation, i.e. the spin foam formalism, one can compute the transition amplitudes between the spin network states, from which one can infer the spin network wavefunction. However, this will be again a complicated expression, and it will be difficult to compute the classical limit.

In spite of these difficulties Rovelli found a way to study the semiclassical limit indirectly \[3\]. His idea was to consider the graviton propagator within the spin foam formalism and to study the semiclassical limit by analyzing the large distance asymptotics of the propagator. By using an assumption that the flat-space wavefunction has a specific Gaussian form in the spin network basis, Rovelli was able to show that the graviton propagator had the correct large distance asymptotics. For more detailed studies and further developments see \[4–7\].

In \[6\] it was pointed out that the Gaussian wavefunction which had been used to calculate the graviton propagator asymptotics does not satisfy the Hamiltonian constraint, also see \[8\]. Although the physical wavefunction \(\Psi_\gamma(j, j_0)\) is not a Gaussian, one will still obtain the desired propagator asymptotics if \(\Psi_\gamma(j, j_0)\) is approximated by the Rovelli’s Gaussian wavefunction for large spins, i.e.

\[
\Psi_\gamma(j; j_0) \approx \mathcal{A}(j_0) \exp \left[ -\frac{1}{2j_0} \sum_{a,b} \alpha_{ab}(j_a - j_0)(j_b - j_0) + i \sum_a \theta_a j_a \right].
\]  
(1)

Here \(\gamma\) denotes the spin network graph, \(j_a\) is a spin of a link of \(\gamma\), \(\theta_a\) are arbitrary constants and \(\alpha\) is a numerical matrix. The parameter \(j_0\) determines the scale of a triangle area in the spin network and can be related to the boundary background metric, see \[6\].

However, nobody has investigated whether any viable candidate for the flat-space wavefunction has the Gaussian asymptotic form \(1\). Note that such an analysis has been recently performed in the case of canonical Euclidean loop quantum gravity (LQG) theory \[9\]. It was shown that the wavefunction has a Gaussian asymptotics, but it is not of the form
This result then implies that the Euclidean LQG graviton propagator does not have the desired large-distance asymptotics. In order to be sure about the implications of this result for physics, one needs to perform the same analysis in the Lorentzian case. This can be done by using ELPR/FK spin foam models [10, 11], since these are the only spin foam models that have a Lorentzian formulation and give rise to a LQG theory on the spin foam boundary.

In order to satisfy the Hamiltonian constraint, we will consider a boundary spin network wavefunction obtained from the spin foam state sum for a spin foam with a spin network boundary. This is a spin foam analog of the Hartle-Hawking wavefunction [12], and it is known that a Hartle-Hawking wavefunction satisfies the Hamiltonian constraint. Guided by the construction of the flat-space wavefunction in the Euclidean LQG case [2], we will introduce the edge insertions in the boundary spin network in order to simulate the presence of the boundary background metric. The large-spin asymptotics of the boundary wavefunction will be studied by using the stationary phase method. We will determine the conditions necessary for the asymptotics to be of the form (1), see Eq. (20). Since the graviton propagator asymptotics is determined by the $j_0$-dependence of the exponent in (1), we will focus our attention on the coefficient $S$ in (20), which is determined by the Hessian matrix for the logarithm of the spin-foam amplitude for the boundary wavefunction.

The method to determine the $j_0$-dependence of $S$ relies on a nontrivial mathematical result formulated in Theorem 1. We obtain that $S = O(1)$, rather than the desired result $S = O(1/j_0)$, see Eq. (27). The result $S = O(1)$ implies that the graviton propagator behaves as the distance to the fourth power in the limit of large distances. We will also show that the $S = O(1)$ asymptotics is a direct consequence of the vertex amplitude asymptotics (B1), which is a common feature of all known spin foam models.

This paper is organized as follows. In section II we introduce the boundary spin-network wavefunction with insertions. In section III we rewrite the wavefunction in the form suitable for the asymptotic analysis. Section IV is devoted to the analysis of the critical points of the wavefunction, which play a major role in the asymptotic analysis. We discuss the properties of the stationary point equations and outline a method that can be used to solve them. However, it is not necessary to solve explicitly the stationary point equations since it is sufficient to use certain properties of the critical points. In section V we apply the extended stationary phase method to determine the asymptotic behavior of the wavefunction...
in the large-spin limit. A detailed analysis shows that if certain reasonable assumptions are satisfied, the wavefunction will have a Gaussian asymptotics. The width of the Gaussian is determined by a complex matrix which is essentially the Schur complement of the Hessian of the logarithm of the integrand. It depends in a nontrivial way on the scaling parameter \( j_0 \). In order to be able to compare the wavefunction asymptotics with the Gaussian from (1), we need to determine the scaling of the Schur complement in the limit \( j_0 \to \infty \), which is done in section VI. An explicit calculation of the Schur complement will not be possible, but it will be possible to determine its scaling dependence on \( j_0 \). Surprisingly, one finds that in the leading order the Schur complement scales as a constant in the limit \( j_0 \to \infty \), in contrast to the assumed \( 1/j_0 \) scaling in (1). This implies that the corresponding graviton propagator does not have the distance scaling corresponding to a graviton propagator from general relativity. In the final section VII we discuss the possible ways to solve this problem and to recover the desired scaling of the propagator. It turns out that the most promising method is to redefine the vertex amplitude of the spin foam model, and we propose two ways to do that. The appendices A, B, C, D and E contain derivations of the results that were used in the main text.

II. THE BOUNDARY WAVEFUNCTION

A boundary state \(|\Psi\rangle\) for an ELPR/FK spin foam model can be constructed in the following way. We expand \(|\Psi\rangle\) in the spin network basis \(|\gamma, j_l, \iota_p\rangle\), where \( \gamma \) is the boundary spin network graph, \( j_l \) are the spins of the edges of \( \gamma \) and \( \iota_p \) are the corresponding intertwiners. We then expand each \(|\gamma, j_l, \iota_p\rangle\) in the coherent state basis \(|\gamma, j_l, \vec{n}_{pl}\rangle\), see (13), so that

\[
|\Psi\rangle = \sum_{\gamma} \sum_{j_l} \int \prod_{(pl)} d^2 \vec{n}_{pl} \ \Psi_\gamma(j_l, \vec{n}_{pl}; j_0) |\gamma, j_l, \vec{n}_{pl}\rangle.
\]

The coefficients \( \Psi_\gamma(j, \vec{n}; j_0) \) are constructed as boundary spin-network wavefunctions with edge insertions. The edge insertions are introduced in order to provide the wavefunction with the information about the boundary background metric. This is done through the background spin parameter \( j_0 \), so that

\[
\Psi_\gamma(j_l, \vec{n}_{pl}; j_0) = \prod_{l \in \gamma} \mu_l(j_l; j_0) d_f(j_l) \sum_{k_f} \int_{(ef)} \prod_{(ef)} d^2 \vec{n}_{ef} \prod_{f} d_f(k_f) \prod_{v} W_v(k_f, \vec{n}_{ef}, j_l, \vec{n}_{pl}), \quad (2)
\]
where $\sigma$ is a 2-complex whose boundary one-complex is $\gamma$. The face labels $k$ and the edge-face labels $\vec{n}$ of the corresponding spin foam are fixed to be $j_l, \vec{n}_{pl}$ at the boundary spin network. $k_f$ is a non-boundary spin, which labels a face $f$, while a unit vector $\vec{n}_{ef}$ labels an edge $e$ and the face $(ef)$ adjacent to $e$ in the 2-complex $\sigma$. One can also include a sum over various 2-complexes $\sigma$ that have the fixed boundary $\gamma$ in (2), thereby implementing the “sum over triangulations” idea. However, this will not affect our analysis, so that we will work with a single $\sigma$ for a given $\gamma$. The expressions for the face and the vertex amplitudes $d_f$ and $W_v$ can be found in [10, 11, 14, 15] and we do not write them explicitly because we will need only their asymptotic form for large spins.

The motivation for the introduction of the edge insertions $\mu_l(j_l, j_0)$ comes from the construction of the flat-space wavefunction in the Euclidean canonical LQG [2]. This wavefunction solves the Hamiltonian constraint and it is given by a state-sum similar to (2). The parameter $j_0$ is proportional to the areas of the triangles determined by the background geometry triads. The boundary spin network insertions are arbitrary functions of the edge spins and $j_0$. Therefore the expression (2) is a natural generalization of the Euclidean wavefunction from [2] to the Lorentzian geometry case. Furthermore, since (2) is constructed as a Hartle-Hawking wavefunction for a boundary spin network for the ELPR/FK spin foam model, it will satisfy automatically the corresponding Hamiltonian constraint and the insertions will insure that it is peaked around a flat spatial geometry. Since the insertions $\mu_l(j_l, j_0)$ can be arbitrary functions, one can try to choose them such that the asymptotics (1) is obtained.

The vectors $\vec{n}_{ef}$ are in general defined up to arbitrary phase factors. These phase factors can be chosen such that they insure nice gluing properties of neighboring simplices in the triangulation dual to $\sigma$. As discussed in [16], such a choice will fix the phase factors on the spin foam boundary $\gamma$, and thus give rise to the phase term in (1). However, these phase factors disappear when the graviton propagator is calculated in the standard canonical formalism, see [6], and therefore their values will not be important for our purposes.

**III. ASYMPTOTIC ANALYSIS**

The wavefunction (2) does not necessarily have the large-spin asymptotic form (1). In what follows, we are going to study its large-spin asymptotics, in order to find out is there
a choice of the insertions such that the asymptotics is obtained. If the asymptotics is indeed of the form , the wavefunction can be a good candidate for a flat-space wavefunction.

We begin the analysis of the large-spin asymptotics of by defining the large-spin limit. Namely, we are interested in the limit

\[ j_l = j_0 \tilde{j}_l, \quad j_0 \to \infty. \]  

(3)

Here \( \tilde{j}_l \in \mathbb{N}_0/2 \) are spins which are fixed, while \( j_0 \) is the large parameter.

It is important to note that the scaling of boundary spins \( j_l \) via the parameter \( j_0 \) will induce a similar scaling in some of the internal spins \( k_f \), due to the triangle inequalities built in the vertex amplitude \( W_v \). However, not all internal spins need to be scaled, depending on the combinatorics of the two-complex \( \sigma \). The domain of summation in will contain sectors where all spins are scaled and sectors where only some of them are scaled. Those internal spins which must scale do so by a prescription analogous to (3).

The first step in finding the asymptotic behavior of is to approximate the sums over the internal spins \( k_f \) with integrals. The wavefunction can be then approximated as

\[ \Psi_\gamma(j_l, \vec{n}_{pl}; j_0) \approx I_\gamma(j_l, \vec{n}_{pl}; j_0) = \int_D \prod_f dk_f \int \prod_{(ef)} d^2 \vec{n}_{ef} e^{j_0 F(j, k, \vec{n}; j_0)}, \]  

(4)

where the function \( F \) is given by

\[ F(j, k, \vec{n}; j_0) = \frac{1}{j_0} \sum_l \log \mu_l(j_l; j_0) d_l(j_l) + \]  

\[ + \frac{1}{j_0} \sum_{f \neq t} \log d_f(k_f) + \frac{1}{j_0} \sum_v \log W_v(j, k, \vec{n}). \]  

(5)

\( D \) is the domain of integration over spins \( k \) and the form is suitable for the stationary phase approximation. Note that the vertex amplitude \( W_v \) is complex-valued in general, so that the logarithm is defined up to a multiple of \( 2\pi i \). However, this constant factor does not influence the subsequent analysis and we can ignore it. Also note that the insertion functions \( \mu_l \) depend explicitly on \( j_0 \), while \( d_f \) and \( W_v \) may depend on \( j_0 \) only through boundary spins \( j \) and those internal spins \( k \) that are constrained to scale via triangle inequalities.
We will use the extended stationary phase method \cite{17} in order to approximate the integral (4). The method will be applicable if the function (5) satisfies
\[ F(j, k, \vec{n}; j_0) = O(1), \] for \( j_0 \to \infty \). This condition will be satisfied on a subset of \( D \) where the asymptotic formulae for the ELPR/FK vertex amplitude \( W_v \), derived in \cite{16, 20, 21}, are valid. See the appendix \ref{app:explicit-expr} for the explicit expressions.

When the boundary spins \( j_l \) are large, i.e. \( j_l = O(j_0) \), then the integration domain \( D \) will contain spin foams whose spins are all large. \( D \) will also contain spin foams where some of the spins are large and other are small. This structure is a consequence of the triangular inequalities among the spins which form a spin-foam vertex (rules for the addition of angular momenta). Let \( D_{\text{ndg}} \) be the set of spin foams in \( D \) such that each spin foam from \( D_{\text{ndg}} \) contains at least one vertex with all spins large. Then \( D_{\text{dg}} = D \setminus D_{\text{ndg}} \) is the set of spin foams where every vertex in a spin foam from \( D_{\text{dg}} \) contains a small spin. Consequently
\[ I_{\gamma} = I_{\gamma}^{\text{ndg}} + I_{\gamma}^{\text{dg}}, \]
where \( I_{\gamma}^{\text{ndg}} \) and \( I_{\gamma}^{\text{dg}} \) are defined by taking the integral (4) over the domains \( D_{\text{ndg}} \) and \( D_{\text{dg}} \), respectively.

It is not known whether the function \( F \) satisfies the condition (6) on \( D_{\text{dg}} \), since the asymptotic formulae for \( W_v \) when some of the vertex spins are large and the other are small are not known. On the other hand, the asymptotic formula for \( W_v \) in the case when all the vertex spins are large is known, see (B1) and (B2), so that it can be shown that \( F \) satisfies the condition (6) on \( D_{\text{ndg}} \). This is true because every spin foam from \( D_{\text{ndg}} \) contains at least one vertex with nondegenerate asymptotics, and therefore the contribution of such a vertex to \( F \) is given by
\[
\frac{1}{j_0} \log W_v^{\text{ndg}} \approx \frac{1}{j_0} \log \left( N_+^{(a)} e^{i \alpha S_R^{(v)}} + N_-^{(a)} e^{-i \alpha S_R^{(v)}} \right) + O \left( \frac{\ln j_0}{j_0} \right).
\]
Since \( N_\pm \neq 0 \), then
\[
\frac{1}{j_0} \log W_v^{\text{ndg}} \approx i \alpha \frac{S_R^{(v)}}{j_0} + \frac{1}{j_0} \log \left( N_+^{(a)} + N_-^{(a)} e^{-2i \alpha S_R^{(v)}} \right) + O \left( \frac{\ln j_0}{j_0} \right).
\] According to (A2) the Regge action \( S_R^{(v)} \) is of \( O(j_0) \), so that the first term in (7) is of \( O(1) \). Since the coefficients \( N_\pm^{(a)} \) are of \( O(1) \), the second term in (7) is of \( O(j_0^{-1}) \). Therefore, a nondegenerate vertex gives an \( O(1) \) contribution to the function (5).
A degenerate vertex from $D_{ndg}$ can give a contribution to $F$ of $O(1)$ or lower, depending on the type of degeneracy of each particular vertex. The sum over the insertion functions $\mu_l$ in $F$ can be chosen such that it is of $O(1)$. One particularly useful choice for the insertion functions is

$$\mu_l(j_l; j_0) = \exp \left[ -\frac{(j_l - j_0)^2}{j_0} \right].$$

This choice is very natural for our purposes, since it enforces the flat background metric in the boundary state and it gives an $O(1)$ contribution to $F$. As far as the the sum over the face amplitudes in $F$ is concerned, it is of $O(j_0^{-1} \ln j_0)$, which is subleading to $O(1)$. This is because $d_f(j)$ is of $O(j_0^q)$, where $q = 1$ or $q = 2$, see [15] for a discussion of the various proposals for $d_f(j)$. Therefore $F = O(1)$ on $D_{ndg}$, provided that there is no cancellation of $O(1)$ terms. Hence one can use the stationary phase approximation for the integral $I_{ndg}$.

As far as the order of $F$ on $D_{dg}$ is concerned, it can be of $O(1)$ if the choice (8) is used. However, the stationary phase approximation cannot be made because of the absence of the asymptotic formulas for the degenerate vertices.

Also note that the extended stationary phase method is directly applicable only if $F$ is a Morse function, which means that its Hessian matrix does not have zero eigenvalues at the critical points. However, in our case the Hessian of $F$ may happen to be degenerate, so that we need to take this fact into account when applying the stationary phase method. This will be discussed in detail in section [V].

IV. CRITICAL POINTS

The idea of the stationary phase method is to approximate the integrand in the nondegenerate piece of $I_\gamma$ as a sum of Gaussian functions, where each Gaussian is centered around a stationary point $(j^*, k^*, \vec{n}^*)$ of $e^{j_0 F}$. As $j_0 \to \infty$, only the immediate neighborhoods of the stationary points will contribute to the integral [17]. Furthermore, only the stationary points for which

$$\text{Re } F(j^*, k^*, \vec{n}^*; j_0) = 0,$$

will give a noticeable contribution. The stationary points which satisfy (9) are called the critical points.

Note that the stationary points of $e^{j_0 F}$ are the same as the stationary points of $F$. 
Therefore, the stationary point equations are given by

$$\frac{\partial F}{\partial j_t} = 0, \quad \frac{\partial F}{\partial k_f} = 0, \quad \frac{\partial F}{\partial \vec{n}_{\text{ef}}} = 0.$$  \hspace{1cm} (10)

The geometric interpretation of these equations has been studied extensively in [18] for the Euclidean theory and in [19] for both Euclidean and Lorentzian versions of the theory. In this paper we do not need to go into the details, just let us mention that the condition (9) and the stationary point equation for $\vec{n}_{\text{ef}}$ are satisfied only if one considers a spin foam which is dual to the triangulation of a Regge geometry. As far as the $j$ and $k$ equations are concerned, we can unify them by using a common label $x_a$ for $j_t$ and $k_f$. Then $\frac{\partial F}{\partial x_a} = 0$ gives

$$\sum_{v \in \text{ndg}} \frac{N_+^{(a)} e^{i\alpha S_R^{(v)}} - N_-^{(a)} e^{-i\alpha S_R^{(v)}}}{N_+^{(a)} e^{i\alpha S_R^{(v)}} + N_-^{(a)} e^{-i\alpha S_R^{(v)}}} i\alpha \delta_{a \in v} \Theta_a^{(v)} +$$

$$+ \sum_{v \in \text{ndg}} \frac{e^{i\alpha S_R^{(v)}} \partial_a N_+^{(a)} + e^{-i\alpha S_R^{(v)}} \partial_a N_-^{(a)}}{N_+^{(a)} e^{i\alpha S_R^{(v)}} + N_-^{(a)} e^{-i\alpha S_R^{(v)}}} +$$

$$+ \sum_{v \in \text{dg}} \partial_a \log W_v + \sum_l \partial_a \log \mu_l + \sum_f \partial_a \log d_f = 0,$$  \hspace{1cm} (11)

where we have used \[A3\]. When $x_a = k_f$ then the $\mu_l$ terms are absent in (11), while for $x_a = j_t$ the last sum in (11) goes only over the boundary $f$’s.

The $j_0$-dependence of (11) is the following. The first sum in (11) is of $O(1)$, and it gives the dominant contribution to the equation. The second sum in (11) is of $O(1/j_0)$ since the factors $N_\pm^{(a)}$ are of $O(1)$ and consequently their derivatives with respect to $j$ and $k$ are of $O(1/j_0)$. The fourth and the fifth sum in (11) are of $O(1/j_0)$, while the third sum is at most of $O(1)$. Given this, the critical point equations (9) and (10) can be solved by restricting to a Regge triangulation and writing a stationary point $x^*$ as

$$x^* = c_0 j_0 + d_a + O(1/j_0),$$  \hspace{1cm} (12)

where $c_0$ and $d_a$ are coefficients to be determined. The equations can be then expanded into a power series in $1/j_0$ and solved order by order for $c$ and $d$. This has been done explicitly in [9] for the case of Euclidean LQG flat-space wavefunction. In that case the vertex amplitude is the $6j$ symbol, and some explicit solutions can be found.

However, for the purposes of this paper, it is not necessary to construct explicit solutions of (9) and (11). Rather, we only need to assume that they have at least one nontrivial
solution \((j^*, k^*, \vec{n}^*)\) which is a critical point. If there are no such solutions, the integral \(I^{\text{ndg}}_\gamma\) will be of \(o(1/j_0^3)\) for all \(n > 0\), and thus it will not have the asymptotic form \(1\).

Note that \(\Theta^{(v)}_f = 0\) for all \(f\) is a leading-order solution of \(1\), since if we neglect \(O(1/j_0)\) terms we obtain

\[
\sum_{v \in \text{ndg}} \frac{N_+^{(a)} e^{i\alpha S_R^{(v)}} - N_-^{(a)} e^{-i\alpha S_R^{(v)}}}{N_+^{(a)} e^{i\alpha S_R^{(v)}} + N_-^{(a)} e^{-i\alpha S_R^{(v)}}} i\alpha \delta_{a \in v} \Theta^{(v)}_a = 0.
\]

However, such solutions have to be discarded because the corresponding \((j^*, k^*, \vec{n}^*)\) do not satisfy the triangular inequalities. The reason is that the angles \(\Theta^{(v)}_f\) are exterior dihedral angles of a 4-simplex dual to \(v\). Since a 4-simplex is a convex body, its exterior dihedral angles cannot all be equal to zero.

V. EXTENDED STATIONARY PHASE METHOD

We are going to determine the large-spin asymptotics of \(I^{\text{ndg}}_\gamma\) by using the extended stationary phase method. As explained in the previous section, we will assume that there is a dominant critical point and such a point must satisfy

\[
\Theta^{(v)}_f \neq 0,
\]

for some \(f\) and all \(v\).

As we have already pointed out, the function \(F\) may not be a Morse function, and consequently we cannot apply directly the well known results \[17\] so we will perform the calculation step by step.

Let \(J = (j_l, \vec{n}_{pl}), K = (k_f, \vec{n}_{ef})\) and \(x = (j_l, k_f, \vec{n}_{pl}, \vec{n}_{ef})\). We first approximate the integrand \(e^{j_0 F}\) with a sum of Gaussian functions, each centered around a critical point \(x^*\). The corresponding exponents are obtained by expanding \(j_0 F\) into a power series around each \(x^*\) up to quadratic terms. The integral \(I^{\text{ndg}}_\gamma\) then becomes

\[
I^{\text{ndg}}_\gamma(J; j_0) \approx \sum_{x^*} e^{j_0 F^*} \int dK \ e^{\frac{1}{2}(x-x^*)^T \Delta(x^*)(x-x^*)} = \sum_{x^*} e^{j_0 F^*} I^*(J, x^*).
\]

Here \(F^* = F(x^*) = i \text{Im} F(x^*)\) is the evaluation of \(F\) at the critical point \(x^*\), the sum goes over the set of all distinct critical points, and \(\Delta\) is the Hessian matrix of \(j_0 F\) evaluated at
\( x^* \)

\[
\Delta_{ab} \equiv j_0 \frac{\partial^2 F}{\partial x_a \partial x_b} \bigg|_{x^*}.
\]

The scaling parameter \( j_0 \) has been absorbed into \( \Delta \) for convenience.

In order to perform the Gaussian integrations in (14), we will split the \( \Delta \) matrix into \( JJ \), \( JK \) and \( KK \) blocks, which will be denoted as \( A, N \) and \( M \), respectively

\[
\Delta = \begin{bmatrix}
A & N \\
N^T & M
\end{bmatrix}.
\]

Let us rewrite the exponent in a Gaussian integral as

\[
\frac{1}{2} (x - x^*)^T \Delta (x - x^*) = \frac{1}{2} (J - J^*)^T A (J - J^*) + \\
\frac{1}{2} (K - K^*)^T M (K - K^*) + (J - J^*)^T N (K - K^*)
\]

The first term is independent of \( K \) and can be moved in front of the integral. By making a change of variables \( K = Q \hat{K} \) and by making a suitable choice of the matrix \( Q \), the matrix \( Q^T MQ \) becomes diagonal. Then

\[
I^* = \prod_a I_a^* = \prod_a \int_{D_a} d\hat{K}_a e^{\frac{1}{2} m_a \hat{K}_a^2 + n_a \hat{K}_a}.
\]

where \( m_a \) are the eigenvalues of the matrix \( M \), \( n_a = [(J - J^*)^T N Q]_a \), and \( D_a \) is the one-dimensional domain of integration, determined by \( D \) and the change of variables \( K = Q \hat{K} \).

Let us now discuss the integral \( I_a^* \).

- If \( m_a \neq 0 \) and \( \text{Re } m_a \leq 0 \), the integral converges. Since \( m_a = O(j_0) \), in the limit \( j_0 \to \infty \) the result is independent of the domain \( D_a \) in the leading order, and can be written as

\[
I_a = e^{-\frac{n_a^2}{2 m_a}} \sqrt{\frac{2 \pi}{-m_a}} \left[ 1 + O \left( \frac{1}{j_0} \right) \right].
\]

Note that quadratic dependence on \( n_a \) generates a term of type \((J - J^*)^2\) in the exponent, which gives us the desired Gaussian asymptotics.

- If \( m_a \neq 0 \) and \( \text{Re } m_a > 0 \), the integral diverges exponentially in the limit \( j_0 \to \infty \), so that in this case the Gaussian asymptotics cannot be obtained.

- If \( m_a = 0 \) and \( n_a \neq 0 \), the integral might or might not converge, depending on whether the domain \( D_a \) is compact or not. However, even when it converges, the result will be a non-Gaussian function of \( J - J^* \).
If $m_a = 0$ and $n_a = 0$, the integral converges if $D_a$ is compact. Most importantly, in this case the result is independent of $J - J^*$. Namely, if we denote $D_a = [\alpha_a, \beta_a]$, we have

$$I_a = \beta_a - \alpha_a \equiv A_a.$$  

The integrals of this type do not influence the propagator asymptotics.

Therefore the integral $I^*$ will be a Gaussian function of $J - J^*$, if the following conditions are satisfied:

- all nonzero eigenvalues of $M$ have their real part negative or zero,
- the matrix $N$ is projected to zero on the kernel of $M$, and
- the domain of integration over the kernel space of $M$ is compact.

These conditions imply that we can always make a change of variables such that the matrices $M$ and $N$ are given by

$$M = \begin{bmatrix} \bar{M} & 0 \\ 0 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} \bar{N} & 0 \end{bmatrix},$$  

where the matrix $\bar{M}$ is invertible and has a negative-definite real part, and the horizontal dimension of the zero-block in $N$ is equal to the corresponding dimension of the zero-block in $M$.

We note here that these zero blocks appear because the function $F$ has continuous symmetries, which give rise to manifolds of stationary points instead of discrete sets of stationary points. As we have shown, integrating over these manifolds does not affect the propagator asymptotics, as long as they are compact (otherwise the integral will diverge). A similar situation was encountered when applying the extended stationary phase method to the case of an Euclidean spin foam model [18], as well as to the case of a spin foam consisting of a single vertex [16, 20, 21].

Consequently we obtain

$$I^\mathrm{ndg}_{\gamma}(J; j_0) \approx \sum_{x^*} A(x^*, j_0) e^{\frac{i}{2} (J - J^*)^T S(x^*, j_0)(J - J^*)},$$  

where

$$A(x^*, j_0) = e^{j_0F^*} \sqrt{\frac{(-2\pi)^r}{\det M}} \prod_a A_a \left[ 1 + O \left( \frac{1}{j_0} \right) \right].$$  

(17)
is the dimension of $\tilde{M}$, while the product is taken over $a$ for which $m_a = 0$. Also,

$$\tilde{S} = A - \tilde{N}\tilde{M}^{-1}\tilde{N}^T$$

(18)

is the Schur complement [22] of the (regular minor of the) Hessian matrix $\Delta$.

Note that the asymptotic form (17) is a sum of many Gaussian functions, while the desired asymptotics (1) is just a single Gaussian function. The expression (17) can yield a single Gaussian if there is a dominant critical point $x_0^*$ such that $|A(x_0^*)|$ dominates any other $|A(x^*)|$ when $j_0 \to \infty$. This point can be determined as the one for which the dimension $r$ of $\tilde{M}$ is minimal, i.e. when $M$ is maximally degenerate. Consequently

$$I_{\gamma}^{ndg}(J; j_0) \approx A(j_0) e^{\frac{1}{2}(j-j^*)^T \tilde{S}(j_0)(j-j^*)}$$

(19)

when $j_0 \to \infty$. Finally, the quadratic form in the exponent of (19) can be decomposed into a sum of $jj$, $j\bar{n}$ and $\bar{n}\bar{n}$ terms. Since $j = O(j_0)$ and $\bar{n} = O(1)$ then the $jj$ terms will be dominant in the limit $j_0 \to \infty$. Therefore

$$I_{\gamma}^{ndg}(j; j_0) \approx A(j_0) e^{\frac{1}{2}(j-j^*)^T S(j_0)(j-j^*)},$$

(20)

where $S$ is the $jj$ block of the Schur matrix $\tilde{S}$. Note that (12) implies $j^* = O(j_0)$, so that (20) can be written as

$$I_{\gamma}^{ndg}(j; j_0) \approx A(j_0) \exp \left[ \frac{1}{2} \sum_{a,b} S_{ab}(j_0)(j_a - c_a j_0)(j_b - c_b j_0) \right].$$

(21)

From (20) it follows that

$$\Psi_\gamma \approx A(j_0) e^{\frac{1}{2}(j-j^*)^T S(j_0)(j-j^*)} + I_{\gamma}^{dg}.$$ 

(22)

In order to obtain a single Gaussian asymptotics we need to assume that $I_{\gamma}^{dg}$ has a subleading asymptotics to that of $I_{\gamma}^{ndg}$. Since we do not know how to calculate the asymptotics of $I_{\gamma}^{dg}$, there is a possibility that $I_{\gamma}^{dg}$ has the right asymptotics which is dominant with respect to $I_{\gamma}^{ndg}$. However, we will argue that such a possibility would give a classical limit whose spacetime geometry has the curvature which varies greatly on small scales, while the corresponding propagator is that for a flat spacetime, see section VII. Therefore we will assume that (22) implies

$$\Psi_\gamma \approx A(j_0) e^{\frac{1}{2}(j-j^*)^T S(j_0)(j-j^*)}.$$ 

(23)

However, the asymptotics (23) is still not the desired asymptotics (2). We need to to determine whether or not $S = O(1/j_0)$. We will analyze this problem in the next section.
VI. CALCULATION OF THE EXPONENT FACTOR

The asymptotic form (23) will give the desired asymptotics if \( S = O(1/j_0) \). Note that it is very difficult to calculate the matrix \( S \) explicitly. However, we only need to calculate the leading \( j_0 \)-order of \( S \). This can be done by using the following theorem

**Theorem 1.** If the matrix \( S \) is nonzero, and if the leading order contribution to \( \Delta \) comes from \( W_v \) terms, we have

\[
S = O(\Delta)
\]

for \( j_0 \to \infty \). If the matrix \( S \) is zero, the wavefunction asymptotics is non-Gaussian.

The proof is essentially based on the Schur determinant formula, \( \det \Delta = \det S \det M \), see [22], and is given in Appendix C (see also [9]).

The asymptotic dependence of \( \Delta \) on \( j_0 \) can be determined quite easily, if the large spin asymptotics of the vertex amplitude \( W_v \) is known. From (5) and (15) it follows that

\[
\Delta_{ab} = \sum_V \frac{\partial^2 \log A_V}{\partial x_a \partial x_b},
\]

where \( V \in \{l, f, v\} \), \( A_l = \mu_l \), \( A_f = d_f \), \( A_v = W_v \), and the derivatives are evaluated at the critical point \( x^* \). Each term in (24) contributes to the asymptotics of \( \Delta \) with some power of \( j_0 \), so that the leading order asymptotics of \( \Delta \) will be determined by the highest power of \( j_0 \).

The insertion functions can be chosen arbitrarily and therefore can give any desired contribution of \( O(j_0^q) \). However, they only contribute to the diagonal elements of \( \Delta \), since each insertion function \( \mu_l \) depends only on the spin of its link, \( j_l \). For the choice (8) one easily gets from (24)

\[
\frac{\partial^2 \log \mu_l}{\partial x_a \partial x_b} = -\frac{2\delta_{ab}\delta_{al}}{j_0} = O\left(\frac{1}{j_0}\right).
\]

The face amplitude \( d_f \) is commonly chosen to be \( d_f(j) = 2j_f + 1 \), see [15]. Substituting into (24), we obtain

\[
\frac{\partial^2 \log d_f}{\partial x_a \partial x_b} = -\frac{4\delta_{ab}\delta_{af}}{(2x_f + 1)^2} = O\left(\frac{1}{j_0}\right),
\]

and this is also a contribution to the diagonal elements of \( \Delta \). Note that other choices for \( d_f(j) \) have also been proposed in the literature, see for example [11]. However, all the proposed choices satisfy \( d_f = O(j_f^q) \), where \( q \geq 1 \), so that one obtains an \( O(j_0^{-2}) \) contribution.
Finally, the main nontrivial contribution comes from the vertex amplitude $W_v$. The asymptotics of the degenerate configurations of the vertex amplitude is unknown, and such vertices can in general give a contribution to $\Delta$ of order $O(1)$ or smaller. However, the asymptotics of the nondegenerate vertices is well studied, see Appendix B. Furthermore, each spin foam in $D_{ndg}$ contains at least one nondegenerate vertex. By a straightforward calculation one obtains from (24) and (B2)

$$\frac{\partial^2 \log W_v}{\partial x_a \partial x_b} = \Delta^{(0)}_{vab} + \Delta^{(1)}_{vab} + \Delta^{(2)}_{vab},$$

where the three terms on the right-hand side represent contributions of order $O(1)$, $O(j^{-1})$, and $O(j^{-2})$. In the non-Regge cases the contributions are of $O(1/j^2)$ or subleading. The leading term is

$$\Delta^{(0)}_{vab} = \left[ \frac{N^+(\alpha)e^{i\alpha S^+(\gamma)} - N^-(\alpha)e^{-i\alpha S^-(\gamma)}}{N^+(\alpha)e^{i\alpha S^+(\gamma)} + N^-(\alpha)e^{-i\alpha S^-(\gamma)}} \right]^2 - 1 \alpha^2 \delta_{a,b \in \gamma} \Theta^v_a \Theta^v_b,$$

while explicit expressions for $\Delta^{(1)}_{vab}$ and $\Delta^{(2)}_{vab}$ are given in Appendix E.

The equation (25) is evaluated at some critical point $x^*$. As we have discussed in Section IV there are no critical points where all the angles $\Theta^v_a$ are zero. Moreover, for the ELPR/FK vertex amplitude the coefficients $N^+(\alpha)$ and $N^-(\alpha)$ are also always different from zero, so the square bracket in (26) is also nonzero. Therefore we see that the term (26) is always nonzero. Hence the dominant contribution to the Hessian $\Delta$ comes from the vertex amplitudes whose spins form a Regge geometry, and it is of order $O(1)$. Consequently, the assumptions of Theorem 1 are satisfied, so that

$$S = O(1).$$

The implication of (27) for the large-distance asymptotics of the graviton propagator can be seen from the following result. Consider a generalized Rovelli asymptotics for the wavefunction

$$\Psi_\gamma(j; j_0) \approx A(j_0) \exp \left[ -\frac{1}{2j_0^p} \sum_{a,b} \alpha_{ab}(j_a - c_a j_0)(j_b - c_b j_0) \right],$$

where $p \geq 0$. Note that the obtained result (27) corresponds to $p = 0$ while the Rovelli ansatz corresponds to $p = 1$. The propagator asymptotic scaling with spacetime distance
$|x - y|$ can be determined by repeating the calculation done in [3], also see [6]. Therefore one obtains for large distances

$$G(x, y) \approx \frac{\text{const}}{|x - y|^{4-2p}}, \quad (29)$$

where $G$ denotes the diagonal components of the graviton propagator.

The equation (29) gives for $p = 1$ the propagator asymptotics consistent with general relativity, while for $p = 0$ it gives

$$G(x, y) \approx \frac{\text{const}}{|x - y|^4}. \quad (30)$$

The asymptotics (30) is not consistent with general relativity.

VII. DISCUSSION AND CONCLUSIONS

The result (27) has been derived under certain assumptions, so that one would like to know is it possible to relax the assumptions such that the desired classical limit is obtained. The first thing one can try is to change the insertion functions $\mu_l$, since these functions can be chosen freely. The insertion functions could be chosen such that they cancel the $O(1)$ terms in the Hessian $\Delta$. However, these functions can only change the diagonal elements of $\Delta$, while the off-diagonal elements will still have the $O(1)$ terms. Note that we have introduced the insertion functions in the simplest possible way, namely as multiplicative factors for the amplitude of each link on the boundary spin network. In the most general case a $\mu_l$ can be a matrix function, see [2], so that this gives an additional possibility to change the $O(1)$ behavior. This possibility should be explored, but the problem is that it is difficult to analyze.

Note that we have assumed that the dominant contribution to the asymptotics of the wavefunction comes from a non-degenerate spin foam. The reason was that only in that case we know how to calculate the asymptotics. Hence there is a possibility that the dominant contribution comes from a degenerate spin foam and that this contribution is such that it gives the desired propagator asymptotics. However, there is a problem with this. Namely, a degenerate spin foam is such that its every vertex has at least one small spin. This means that the corresponding spacetime geometry has the curvature which grateley varies at small scales, which is not consistent with the propagator asymptotics for a flat spacetime.
The only remaining possibility is to modify the ELPR/FK vertex amplitude $W(j, \vec{n})$. Note that the $O(1)$ contribution to $\Delta$ is given by (26), and it vanishes if one of the coefficients $N_{\pm}^{(\alpha)}$ is zero. Consequently, if the modified vertex amplitude $\tilde{W}(j, \vec{n})$ had the asymptotic behavior

$$\tilde{W}(j, \vec{n}) \approx \frac{e^{i\alpha S_R^{(v)}(j)}}{V(j)},$$

where $V(j)$ is the function from (B2), then it is easy to show from (24) that

$$S = O(1/j_0).$$

By using (8) for the insertion functions, one would then obtain the correct graviton propagator asymptotics. Note that $\tilde{W}$ gives a state sum which for large spins looks like a path integral for Regge discretization of general relativity, because $\tilde{W}$ has the asymptotics (31). This explains why $\tilde{W}$ gives a graviton propagator with a good asymptotics. On the other hand, the presence of the complex conjugate term $e^{-i\alpha S_R^{(v)}}$ in (B2) gives an unnatural path integral, so that it is not a surprise that the corresponding propagator has wrong asymptotics.

Note that all known spin foam models have the vertex amplitude asymptotics which is a linear combination of $e^{\pm i\alpha S_R^{(v)}}$ terms, see [16] for the Euclidean ELPR/FK model or [23, 24] for the Barret-Crane model. Consequently one will obtain $S = O(1)$ for the large-spin asymptotics of the boundary wavefunction, because the calculation is the same as the one presented in this paper. It is also instructive to compare our result with the results of the similar calculation for the Euclidean theory done in [18]. This is done in Appendix F and supports our result (27).

There are two ways to interpret the result (27). One way is to say that the choice (2) for the boundary wavefunction is not the most general one, and there may exist another solution which could give the correct asymptotics. This is of course a possibility, and it is an open problem for future research. However, such an interpretation of our result essentially brings us back to the original problem of finding a wavefunction which satisfies the Hamiltonian constraint and has the asymptotics (1). However, it is difficult to see what would be an alternative construction to the one we used.

The other possibility is to use the same construction for the wavefunction and to modify the ELPR/FK vertex amplitude, so that the result (27) is circumvented. As discussed above, the way to achieve this is to construct a new vertex amplitude which would have the
asymptotics of the type (31). For example, if \( N_+ \neq N_- \) then one can define the new vertex amplitude \( \tilde{W}(j, \vec{n}) \) as
\[
\tilde{W} = \frac{N_+ W - N_- W^*}{N_+^2 - N_-^2},
\]
where \( W^* \) is the complex-conjugate of the ELPR/FK vertex amplitude \( W \). The new amplitude will have the asymptotics (31). A more general redefinition, valid for \( N_+ = N_- \) case, is given by
\[
\tilde{W} = \frac{1}{2N_+} \left( W + \sqrt{W^2 - \frac{4N_+N_-}{V^2}} \right).
\]
This expression also gives the asymptotics (31). Hence the spin foam model defined by the new amplitude \( \tilde{W} \) will give the correct propagator asymptotics and it will represent a good candidate for a spin foam model whose classical limit is general relativity.

Note that the correct asymptotics of the graviton propagator does not guarantee that the classical limit of a spin foam model is general relativity. Namely, the graviton propagator for a boundary state is defined as a 2-point correlation function. However, in order to determine the corresponding semiclassical equations of motion one needs the effective action, which is the generating functional for all \( n \)-point correlation functions. Knowing just the 2-point correlation function is not sufficient, so that one needs to compute the effective action and to show that its classical limit is the Einstein-Hilbert action.

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Appendix A: The Regge action for a 4-simplex

The Lorentzian Regge action for a 4-simplex dual to vertex \( v \) is given as
\[
S_R^{(v)}(k) = \sum_{f \in v} k_f \Theta_f^{(v)}(k)
\]
(A1)
Here \( k_f \) are 10 spins labeling the faces, while each \( \Theta_f^{(v)}(k) \) is the exterior dihedral angle between two tetrahedra of the simplex dual to \( v \) which share the triangle dual to \( f \).
If all spins $k_f$ are uniformly scaled as $k_f = j_0 \tilde{k}_f$, in the limit $j_0 \to \infty$ the Regge action scales as

$$S^{(v)}_R(k) = O(j_0),$$

(A2)
since $k_f = O(j_0)$ and $\Theta_f^{(v)}(k) = O(1)$.

Also, if we take the derivative of the Regge action with respect to some spin $k_a$, we obtain

$$\frac{\partial S^{(v)}_R}{\partial k_a} = \sum_{f \in v} \delta_{a,f} \Theta_f^{(v)} + \sum_{f \in v} k_f \frac{\partial \Theta_f^{(v)}}{\partial k_a}.$$

The first sum reduces to $\Theta_a^{(v)}$ if $a \in v$, and is zero otherwise. The second sum is identically zero due to the Schl"afli identity, so we have

$$\frac{\partial S^{(v)}_R}{\partial k_a} = \delta_{a \in v} \Theta_a^{(v)}.$$

(A3)

Note that this derivative scales as $O(1)$ in the limit $j_0 \to \infty$. Also note that for the nondegenerate 4-simplex all dihedral angles $\Theta_f^{(v)}$ are different from zero, since the 4-simplex is always convex. These properties are essential for the derivation of our results.

**Appendix B: Asymptotics of the ELPR/FK vertex amplitude**

The asymptotic properties of the ELPR/FK vertex amplitude $W_v$ were investigated in depth in [16, 20, 21], and neatly summarized in [25].

A single vertex amplitude $W_v$ is a function of 10 spins $k_f$ and 20 normals $\vec{n}_{ef}$. Some of the spins may be scaled as $k_f = j_0 \tilde{k}_f$, while others do not scale. In the limit $j_0 \to \infty$, the asymptotic behavior of $W_v$ can be split into several cases, based on the possible choices of these variables. These are

1. **The nondegenerate case**

In this case we assume all 10 spins scale with $j_0$, and the boundary of the corresponding 4-simplex has the Regge-like geometry. In the case of the Lorentzian version of the theory, the vertex amplitude has the asymptotic formula

$$W(j_0 k, n) \approx \frac{1}{j_0^2} \left[ N_+^{(\alpha)} e^{i \alpha j_0 S_R^{(v)}(k)} + N_-^{(\alpha)} e^{-i \alpha j_0 S_R^{(v)}(k)} \right].$$

(B1)

Here $\alpha = 1$ for the 4-simplex with a Euclidean geometry on the boundary, while $\alpha = \gamma$ in the Lorentzian boundary case, where $\gamma$ is the Immirzi parameter. The constants
\(N_{\pm}^{(\alpha)}\) are different from zero and \(S_{R}^{(v)}(k)\) is the Euclidean/Lorentzian Regge action (A1).

2. The degenerate cases of zero 4-volume

These are the cases when all 10 spins scale with \(j_0\), but the boundary of the 4-simplex does not have Regge-like geometry. The vertex asymptotics was analyzed in [21] where it was determined that it has the form

\[
W_v \approx \frac{N(k)}{j_0^{1/2}}
\]

if the boundary is a 3D vector geometry, while

\[
W_v = o(j_0^{-K}), \quad \forall K \geq 0,
\]

in all other situations with zero 4-volume. All these cases contribute with zero measure in the integral (4) and can be ignored.

3. The degenerate cases of non-zero 4-volume

These are the cases when only some of the 10 spins scale with \(j_0\), while others are kept fixed. These situations have not been analyzed so far, and the vertex asymptotics in these cases is still unknown. Note that such configurations contribute with non-zero measure in the integral (4), and thus cannot be ignored.

It is important to emphasize that explicit dependence of the asymptotic formula on normals \(\vec{n}_{ef}\) is lost in (B1), and the asymptotic expression on the right-hand side of (B1) depends only on 10 spins \(k_f\). Namely, the assumption of Regge-like geometry of the 4-simplex implies that its triangle areas \(k_f\) and its normals \(\vec{n}_{ef}\) are fully determined by its 10 edge lengths \(l_i\), which also induce the Lorentzian/Euclidean signature of the metric in the 4-simplex. However, given that the number of triangles in a 4-simplex is equal to its number of edges, the functions \(k_f(l_i)\) can in a generic situation be inverted, and edge lengths regarded as functions of the triangle areas. This is possible always except in some particular cases where the Jacobian of the transformation is singular. Nevertheless, these singular cases contribute with zero measure in the integral (4) and thus cannot be ignored. Given the inverted functions \(l_i(k_f)\), one can also express the normals \(\vec{n}_{ef}(l_i)\) as functions of \(k_f\), which therefore remain the only independent variables in (B1).
Note that the asymptotics (B1) can be rewritten as
\[ W(j, n) \approx \frac{1}{V(j)} \left[ N_+^{(\alpha)} e^{i\alpha S_+^{(v)}(j)} + N_-^{(\alpha)} e^{-i\alpha S_-^{(v)}(j)} \right], \tag{B2} \]
for \( j \to \infty \) where \( V(j) = O(j^{12}) \).

Appendix C: Proof of Theorem 1

Here we give a proof of Theorem 1 used in the main text. Let us repeat the statement of the theorem, for completeness.

**Theorem 1.** If the matrix \( S \) is nonzero, and if the leading order contribution to \( \Delta \) comes from \( W_v \) terms, we have
\[ S = O(\Delta) \]
for \( j_0 \to \infty \). If the matrix \( S \) is zero, the wavefunction asymptotics is non-Gaussian.

The proof goes as follows. Begin by noting that the Hessian matrix \( \Delta \) is non-diagonal. Namely, looking at its definition (15) and the action (3), we see that the insertion functions \( \mu_l \) and face amplitudes \( d_f \) contribute only to diagonal terms in \( \Delta \), since each of them is a function of a single variable. In contrast to this, the vertex amplitudes \( W_v \) are functions of 10 or 30 variables each, according to the combinatorics of the spin foam 2-complex and the possible degeneracy of \( W_v \). Therefore, each vertex amplitude will contribute to the diagonal terms of \( \Delta \), and in addition also to some off-diagonal terms on each side of the main diagonal, in such a way that in every row and column there will be some nonzero non-diagonal elements present.

We want to discuss the dependence of \( \det \Delta \) on \( j_0 \) in the limit \( j_0 \to \infty \). For simplicity, in what follows we shall assume that \( \det \Delta \neq 0 \), and we shall discuss the singular case later.

The determinant of \( \Delta \) is by definition given as
\[ \det \Delta = \sum_p \sgn(p) \Delta_{1p(1)} \Delta_{2p(2)} \cdots \Delta_{Rp(R)}, \]
where \( p \) is the permutation of indices 1 \( \ldots \) \( R \), and \( R \) is the rank (and simultaneously the dimension) of \( \Delta \). In this sum, there will be some terms which contain diagonal terms of \( \Delta \), and terms which do not contain any diagonal element. The first set of terms will have contributions of \( \mu_l, d_f \) and \( W_v \), while the second set of terms will be determined solely by
amplitudes $W_v$. Given the assumption that the leading order of $\Delta$ comes from $W_v$, we have that the determinant of $\Delta$ will scale with $j_0$ as:

$$\det \Delta = O(\Delta^R).$$

Namely, the scaling of terms with diagonal elements in the determinant cannot be established without the detailed knowledge of its dependence on $\mu_l$ and $d_f$ terms. However, the scaling of each off-diagonal component $\Delta_{kp(k)}$ (where $p(k) \neq k$, $k = 1, \ldots, R$) will be determined only by the vertex amplitude $W_v$, and is dominant by assumption. As a consequence, the terms in $\det \Delta$ which do not contain any diagonal elements are dominant and scale as $O(\Delta^R)$, while the terms which do contain diagonal elements may scale with smaller power in $j_0$ and can be neglected in the limit $j_0 \to \infty$.

Once the scaling of $\det \Delta$ has been established, we can employ some well-known results about the Schur complement matrix in order to establish the scaling of $S$. These results are summarized and proved in the form of Lemma 1 in Appendix D.

Let the Hessian matrix $\Delta$, its submatrix $M$ and its Schur complement $\tilde{S}$ scale as

$$\Delta = O \left( \frac{1}{j_0^d} \right), \quad M = O \left( \frac{1}{j_0^d} \right), \quad \tilde{S} = O \left( \frac{1}{j_0^s} \right).$$

Note that $M$, being the submatrix of $\Delta$, scales with $j_0$ with the same power $-d$ as $\Delta$. However, this cannot be assumed for the Schur complement $\tilde{S}$ since there might be nontrivial cancellations between the leading terms in $A$ and $NM^{-1}N^T$ in (18). Consequently, the scaling power of $\tilde{S}$ is $-s$. What we need to prove is that these cancellations do not happen, and that in fact $s = d$.

Denote the ranks of $M$ and $\tilde{S}$ matrices as $r$ and $\rho$, respectively. By part (b) of the Lemma 1, we have

$$\det \Delta = \det M \det \tilde{S}.$$ 

Calculating the scaling order of the left-hand and right-hand sides, and using the usual properties of determinants, we easily see that

$$\frac{1}{j_0^{Rd}} = \frac{1}{j_0^{rd}} \frac{1}{j_0^{ps}},$$

which gives

$$Rd = rd + ps.$$
By the part (a) of the Lemma, we have $R = r + \rho$. Using this to eliminate both $R$ and $r$, the above equation reduces to

$$\rho(s - d) = 0.$$  

Finally, by assumption of the theorem, matrix $\tilde{S}$ is nonzero, which means that its rank $\rho$ is nonzero. Therefore we conclude that $s = d$, which actually means that $\tilde{S} = O(\Delta)$. As matrix $S$ is a $jj$ submatrix of $\tilde{S}$, it scales in the same way as $\tilde{S}$. Consequently,

$$S = O(\Delta),$$

which proves the theorem in the case when $\Delta$ is nondegenerate.

If $\Delta$ has zero eigenvalues, the determinant equation above vanishes identically. However, in this case we can repeat the whole analysis in the same way, except that we need to use part (c) of the Lemma instead of part (b), bearing in mind that $O(B_4) = 1$ (see Remark 3 in Appendix D). Namely, instead of analyzing the determinants of $\Delta$, $M$ and $\tilde{S}$, we can rotate the basis to represent these three matrices in the form

$$\Delta = \begin{bmatrix} 0 & 0 \\ 0 & M_\Delta \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & \tilde{M} \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} 0 & 0 \\ 0 & M_\tilde{S} \end{bmatrix},$$

and repeat the whole proof using the regular minors $M_\Delta$, $\tilde{M}$ and $M_\tilde{S}$ instead. Note that as a consequence of the part (a) of the Lemma, the sum of dimensions of the zero-blocks of $M$ and $\tilde{S}$ must be equal to the dimension of the zero-block of $\Delta$. These zero-blocks represent the kernel of $\Delta$, and as discussed in the main text, appear as a consequence of continuous symmetries of the action (5). As was shown in section V they may safely be integrated out, and the Schur complement (18) constructed from the regular part of $M$, i.e. the minors $\tilde{M}$ and $\tilde{N}$.

Again, since $S$ is a submatrix of $\tilde{S}$, if it is nonzero it scales with the same power as $\tilde{S}$, so consequently we have

$$S = O(\Delta),$$

in the degenerate case as well. This completes the proof of the theorem.

**Appendix D: Properties of the Schur complement**

Here we establish some properties of the Schur complement that we have used in the proof of Theorem 1. These results can be found in [22]. However, one of the results, the
Lemma 1. Let $\Delta$ be a symmetric complex matrix of type $n \times n$ and let $R$ be its rank. Let us split $\Delta$ into blocks as

$$\Delta = \begin{bmatrix} A & N \\ N^T & M \end{bmatrix},$$

where $A$ is a $J \times J$ matrix, $N$ is a $J \times r$ matrix, $M$ is a $r \times r$ matrix and $n = J + r$. We will also assume that $M$ is invertible, and that real and imaginary parts of $\Delta$ commute.

Let us construct the Schur complement $\tilde{S}$ (see [22]), which is a $J \times J$ matrix

$$\tilde{S} = A - NM^{-1}N^T.$$

Denote the rank of $\tilde{S}$ as $\rho$. Then

(a) $R = r + \rho$ (Guttman rank additivity);

(b) $\det \Delta = \det \tilde{S} \det M$ (Schur determinant formula);

(c) if $0 < \rho < J$, then

$$\det M_\Delta (\det B_4)^2 = \det M \det M_{\tilde{S}}. \quad (D1)$$

Here $M_\Delta$ and $M_{\tilde{S}}$ are invertible $R \times R$ and $\rho \times \rho$ matrices, respectively. They are obtained by using orthogonal transformations which put $\Delta$ and $\tilde{S}$ into a block-diagonal form

$$\Delta = \begin{bmatrix} 0 & 0 \\ 0 & M_\Delta \end{bmatrix}, \quad \tilde{S} = \begin{bmatrix} 0 & 0 \\ 0 & M_{\tilde{S}} \end{bmatrix},$$

The $B_4$ matrix will be explicitly constructed in the proof below.

Proof. We start from the Aitken block diagonalization formula [22] and from now on we use $I$ to denote a unit matrix of any size appropriate for its position in an equation:

$$\begin{bmatrix} I - NM^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A & N \\ N^T & M \end{bmatrix} \begin{bmatrix} I & 0 \\ -M^{-1}N^T & I \end{bmatrix} = \begin{bmatrix} \tilde{S} & 0 \\ 0 & M \end{bmatrix}. \quad (D2)$$

This equation can be verified by a direct multiplication of the left-hand side. Denoting the first matrix on the left as $C$, we can rewrite this identity in a compact form $C\Delta C^T = \tilde{S} \oplus M$. The rank of the right-hand side is the sum of ranks of $\tilde{S}$ and $M$, which amounts to $\rho + r$. 
Since the rank of $C$ is equal to its dimension $n$, the total rank of the product on the left-hand side is equal to the rank of $\Delta$, so we easily obtain
\[ R = r + \rho, \]
which completes the proof of part (a).

Next, we take the determinant of (D2). Since $C$ is block-triangular, its determinant is a product of determinants of blocks on the diagonal, so we obtain $\det C = 1$. The left-hand side is thus the product of determinants, $\det C \det \Delta \det C^T$, and it is equal to $\det \Delta$ because $\det C^T = \det C = 1$. On the right-hand side we have a block-diagonal matrix, so that its determinant is equal to $\det \tilde{S} \det M$. Hence,
\[ \det \Delta = \det \tilde{S} \det M, \]
which completes the proof of part (b).

In order to prove (c), let $O$ be a $J \times J$ orthogonal matrix which transforms $\tilde{S}$ into a block-reduced form,
\[ O\tilde{S}O^T = 0 \oplus M_{\tilde{S}}. \]
Since $\rho \neq 0$, matrix $\tilde{S}$ has exactly $\rho$ nonzero eigenvalues, which constitute $M_{\tilde{S}}$. Given that the eigenvalues of $M_{\tilde{S}}$ are nonzero, it is invertible. The zero-block is of type $\nu \times \nu$, where $\nu = J - \rho$ is the dimension of the null-space of $\tilde{S}$. By using $O$ one can construct an orthogonal $n \times n$ matrix $P = O \oplus I$ such that
\[ P \left( \tilde{S} \oplus M \right) P^T = 0 \oplus M_{\tilde{S}} \oplus M. \]  
(D3)

By using an analogous argument one can always construct an orthogonal $n \times n$ matrix $Q^T$ such that
\[ Q^T \Delta Q = 0 \oplus M_\Delta, \]
which can be solved for $\Delta$:
\[ \Delta = Q \left( 0 \oplus M_\Delta \right) Q^T. \]  
(D4)
The zero block comes from the null-space of $\Delta$. It is of the size $n - R$, which is also equal to $\nu$, since $n = J + r$ and $R = r + \rho$ according to the part (a).

Consider (D2), and multiply it by $P$ from the left and by $P^T$ from the right, and use (D3) and (D4) to rewrite it in the form
\[ PCQ \left( 0 \oplus M_\Delta \right) Q^T C^T P^T = 0 \oplus M_{\tilde{S}} \oplus M. \]  
(D5)
Let us introduce the matrix $B \equiv PCQ$ and write it in the block form as

$$B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix},$$

where the blocks $B_1, B_2, B_3$ and $B_4$ are $\nu \times \nu, \nu \times R, R \times \nu$ and $R \times R$ matrices, respectively. Substituting this into the left-hand side of (D5) yields

$$PCQ (0 \oplus M_\Delta) Q^T C^T P^T \equiv B \begin{bmatrix} 0 & 0 \\ 0 & M_\Delta \end{bmatrix} B^T =$$

$$= \begin{bmatrix} B_2 M_\Delta B_2^T & B_2 M_\Delta B_4^T \\ B_4 M_\Delta B_2^T & B_4 M_\Delta B_4^T \end{bmatrix}. \tag{D6}$$

By comparing (D6) to the right-hand side of (D5), we obtain

$$\begin{bmatrix} B_2 M_\Delta B_2^T & B_2 M_\Delta B_4^T \\ B_4 M_\Delta B_2^T & B_4 M_\Delta B_4^T \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & M_\tilde{S} & 0 \\ 0 & 0 & M \end{bmatrix}. \tag{D7}$$

Note that the zero-block of (D7) is a $\nu \times \nu$ matrix, which is also the $B_2 M_\Delta B_2^T$ block. We then read off the following equations

$$B_4 M_\Delta B_4^T = M_\tilde{S} \oplus M, \tag{D8}$$

$$B_2 M_\Delta B_4^T = 0, \tag{D9}$$

$$B_2 M_\Delta B_2^T = 0. \tag{D10}$$

By taking the determinant of (D8), we finally obtain

$$\det M_\Delta (\det B_4)^2 = \det M \det M_\tilde{S}.$$ 

This establishes (D11) and completes the proof of part (c).

Given that $M, M_\tilde{S}$ and $M_\Delta$ are all invertible, we have $\det B_4 \neq 0$ which means that $B_4$ is also invertible. By multiplying (D9) by $(B_4^T)^{-1}M_\Delta^{-1}$ from the right, we obtain

$$B_2 = 0.$$ 

26
The equation 10 now vanishes and does not provide any additional constraint. Therefore, the matrix $B$ has the following form

$$B \equiv PCQ = \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix}. \quad (D11)$$

*End of proof.*

**Remark 1.** The $\Delta$ matrix from the main text has the form

$$\Delta = \begin{bmatrix} A & N & 0 \\ N^T & M & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which differs from the one in Lemma 1 by an additional zero-block. However, these additional zeroes are integrated out before the lemma is applied, and hence they do not affect any statements of lemma.

**Remark 2.** The result (c) is a generalization of the result (b) to the case when $\Delta$ is a singular matrix. While the part (b) is in fact valid for singular matrices, it merely states that $0 = 0$ and provides no information about nonsingular principal minors of $\Delta$. The result (c) is more fine-grained, and provides precisely this nontrivial information about $\Delta$.

It was assumed in the part (c) that $0 < \rho < J$. If $\rho = J$ then $\Delta$ is a regular matrix, and hence the result (b) can be used. If $\rho = 0$, then $\tilde{S} = 0$, $\nu = J$, and instead of (D8) we obtain

$$B_4 M \Delta B_4^T = M,$$

and consequently

$$\det M \Delta (\det B_4)^2 = \det M.$$

In this case we can set $P = I$ and obtain

$$B \equiv CQ = \begin{bmatrix} B_1 & 0 \\ B_3 & B_4 \end{bmatrix}$$

for the matrix $B$.

**Remark 3.** In Appendix C we use the results (b) and (c) to determine the leading $j_0$-order of the Schur complement $\tilde{S}$, knowing $O(\Delta)$. However, it is necessary to show that $B_4$ is of order $O(1)$. In order to do this, note that

$$\det B = \det P \det C \det Q = \pm 1,$$
since \( P \) and \( Q \) are orthogonal matrices. On the other hand, from (D11) we know that 
\[ \det B = \det B_1 \det B_4, \] so that we have 
\[ \det B_1 \det B_4 = \pm 1. \quad (D12) \]

Let us now assume that the blocks \( B_1 \) and \( B_4 \) are of order \( k \) and \( m \) in \( 1/j_0 \), respectively 
\[ B_1 = \frac{D}{j_0^k} + O \left( \frac{1}{j_0^{k+1}} \right), \quad B_4 = \frac{E}{j_0^m} + O \left( \frac{1}{j_0^{m+1}} \right), \]
\[ k, m \geq 0, \quad D, E \sim O(1). \]

The numbers \( k \) and \( m \) cannot be negative since the whole \( B \) matrix must be of order \( O(1) \). Namely, the matrices \( P \) and \( Q \) are orthogonal, and consequently all their elements are bounded above by 1. Thus \( P \) and \( Q \) are \( O(1) \). The matrix \( C \) is also \( O(1) \), since \( \Delta \) and consequently \( M, N, M^{-1} \) are all of the same order. Therefore, \( B = PCQ \sim O(1) \).

Since \( B_1 \) is a \( \nu \times \nu \) matrix and \( B_4 \) is a \( R \times R \) matrix, we have 
\[ \det B_1 = \frac{1}{j_0^{k\nu}} \det D + O \left( \frac{1}{j_0^{k+1}} \right), \quad \det B_4 = \frac{1}{j_0^{mR}} \det E + O \left( \frac{1}{j_0^{m+1}} \right). \quad (D13a) \]
\[ (D13b) \]

Substituting (D13) back into (D12) we obtain the consistency equation 
\[ k\nu + mR = 0. \]

Since both \( \nu, R > 0 \) while \( k, m \geq 0 \), the only solution of this equation is \( k = m = 0 \). Therefore 
\[ \det B_4 \sim B_4 \sim O(1). \]

In the case when \( \nu = 0 \) the \( \Delta \) matrix is regular and instead of the part (c) we use the part (b) of Lemma 1. However, the part (b) does not involve \( \det B_4 \), so that we need the above result only for \( \nu > 0 \).

**Appendix E: Vertex amplitude contribution to the Hessian matrix**

Here we give the explicit formulae for the terms on the right-hand side of (25). These terms are calculated by directly substituting the vertex asymptotics (B1) into equation (24).
and differentiating. It is important to note that in the expression \([31]\) the \(x\)-dependence is in the coefficients \(N_+\) and \(N_-\), as well as in the Regge action \(S_R\). However, the scaling of \(N_\pm\) is different than that of \(S_R\). The former scale as \(O(1)\) while the latter scales as \(O(j_0)\) in the limit \(j_0 \to \infty\).

The \(O(1)\) term in \([25]\) has already been quoted in the text in equation \([26]\), and we repeat it here for completeness:

\[
\Delta_{vab}^{(0)} = \left[ \left( \frac{N_+^{(a)} e^{iaS_R^{(v)}} - N_-^{(a)} e^{-iaS_R^{(v)}}}{N_+^{(a)} e^{iaS_R^{(v)}} + N_-^{(a)} e^{-iaS_R^{(v)}}} \right)^2 - 1 \right] \alpha^2 \delta_{a,b\in\mathbb{V}} \Theta_a^{(v)} \Theta_b^{(v)}.
\]

The \(O(j_0^{-1})\) term is given as:

\[
\Delta_{vab}^{(1)} = \frac{N_+^{(a)} e^{iaS_R^{(v)}} - N_-^{(a)} e^{-iaS_R^{(v)}}}{N_+^{(a)} e^{iaS_R^{(v)}} + N_-^{(a)} e^{-iaS_R^{(v)}}} i\alpha \delta_{a,b\in\mathbb{V}} \partial_a \Theta_b^{(v)} + 2 \frac{N_+^{(a)} \partial_b N_-^{(a)} - N_-^{(a)} \partial_b N_+^{(a)}}{N_+^{(a)} e^{iaS_R^{(v)}} + N_-^{(a)} e^{-iaS_R^{(v)}}} 2 i\alpha \delta_{a,b\in\mathbb{V}} \Theta_a^{(v)}.
\]

The \(O(j_0^{-2})\) term is given as:

\[
\Delta_{vab}^{(2)} = \frac{e^{iaS_R^{(v)}} \partial_a \partial_b N_+^{(a)} + e^{-iaS_R^{(v)}} \partial_a \partial_b N_-^{(a)}}{N_+^{(a)} e^{iaS_R^{(v)}} + N_-^{(a)} e^{-iaS_R^{(v)}}} \delta_{a,b\in\mathbb{V}} - \frac{\left( e^{iaS_R^{(v)}} \partial_a N_+^{(a)} + e^{-iaS_R^{(v)}} \partial_a N_-^{(a)} \right) \left( e^{iaS_R^{(v)}} \partial_b N_+^{(a)} + e^{-iaS_R^{(v)}} \partial_b N_-^{(a)} \right)}{\left( N_+^{(a)} e^{iaS_R^{(v)}} + N_-^{(a)} e^{-iaS_R^{(v)}} \right)^2} \delta_{a,b\in\mathbb{V}}.
\]

**Appendix F: Comparison with the Euclidean results**

It is instructive to compare our results with the asymptotic analysis of the Euclidean ELPR/FK state-sum kernel given in \([18]\). There are several differences in the setting between the approach of \([18]\) and the one taken in this paper. Since they do not consider the boundary wavefunction, they do not have the insertion functions. Next, they integrate the state sum over all variables except the spins, and obtain

\[
Z_\Delta = \sum_{j_f} \prod_f d_{j_f^+} d_{j_f^-} W_\Delta(j_f),
\]

where \(W_\Delta(j_f)\) is the stat-sum kernel, \(\Delta\) is the triangulation dual to \(\sigma\) and the face amplitude is a product of two \(d_f\) terms.
One of the main results of [18] is the asymptotic large-spin expression for the nondegenerate part of this kernel

\[ W_{\Delta}^{\text{ndg}}(N j_f) = \frac{c_{\Delta}(j_f)}{N^\frac{d}{2}} \cos(N S_R), \quad N \to \infty, \]

where \( N \) is the large parameter, \( c_{\Delta} \) is the function of spins, but not of \( N \) and \( S_R \) is

\[ S_R = (\gamma^+ + \gamma^-) \sum_f j_f \Theta_f, \quad \Theta_f = \sum_{e \in f} \theta_{ef}, \]

see equations (87) and (101) in [18]. Note that \( \Theta_f \) is the sum of all dihedral angles around a face \( f \). The action \( S_R \) is constructed for the triangulation \( \Delta \), with non-scaled spins, and the large parameter \( N \) is written explicitly in front of it in \( W_{\Delta} \). In our notation, the above expression can be rewritten as

\[ W_{\Delta}^{\text{ndg}}(j) = A(j) \cos(S_R), \quad j \to \infty, \]

where now \( S_R \) is constructed with the scaled spins \( j \), while the amplitude is denoted simply by \( A(j) \).

Despite all the differences in the two setups, there is a rather simple generic relation between the kernel \( W_{\Delta}(j) \) and our boundary wavefunction [2]. The quantity that corresponds to [2] can be constructed from the kernel \( W_{\Delta}(j) \) in the following way. First we choose the triangulation \( \Delta \) so that it has a boundary. The dual of \( \Delta \) will be the 2-complex \( \sigma \), while the dual of the boundary will be a 1-complex \( \gamma = \partial \sigma \). Next, we split the face labels \( j_f \) into the boundary and internal labels, and denote them \( j_f \) and \( k_f \), respectively. Then the wavefunction is given as

\[ \Psi_{\gamma}^{\text{ndg}}(j) = \sum_{k_f} \prod_f (d_+ d_-) W_{\Delta}^{\text{ndg}}(j, k). \]

Here we have not introduced the insertion functions \( \mu_l \) on the boundary, and the face amplitude is quadratic in spins.

In the limit where both internal and boundary spins are large, one can approximate the sum with an integral and write the wavefunction as

\[ \Psi_{\gamma}^{\text{ndg}}(j) = \int dk \prod_f (d_+ d_-) A(j, k) \cos(S_R(j, k)). \quad (F1) \]

In order to evaluate the integral over \( k \) in the large-spin limit, we would like to approximate the cosine with a Gaussian in the neighborhood of each of its stationary points, and employ
the stationary-point method. This technique was used for the asymptotic analysis of the wavefunction for the Euclidean canonical LQG [9], which is given by a similar state sum as (F1). The cosine has infinitely many stationary points, but we can assume that the amplitude \( A(j, k) \) is peaked around only one of them (otherwise the asymptotics will never be a Gaussian function; also one could introduce the insertion functions which would single out one stationary point). Denote this stationary point as \((j^*, k^*)\). In the neighborhood of this point, the cosine can be approximated with a Gaussian via the formula

\[
\cos S_R(x) = e^{\log \cos S_R(x)} \approx e^{-\frac{1}{2}(x-x^*)^T \Delta(x-x^*)}, \quad x \to x^*.
\]

where \( x = (j, k) \) and \( \Delta_{ab} \equiv \frac{\partial^2}{\partial x_a \partial x_b} \log \cos S_R(x) \bigg|_{x^*} \). Using the fact that \( x^* \) is the stationary point of the cosine and the fact that the variation of \( S_R \) w.r.t. the angles \( \Theta_f \) is identically zero, an explicit evaluation of \( \Delta_{ab} \) gives

\[
\Delta_{ab} = -\Theta_a \Theta_b = O(1).
\]

At this point one can split the matrix \( \Delta \) into \( jj \), \( jk \) and \( kk \) blocks, like in Eq. (16) and perform the integration over \( k \)-spins. The result will be a Gaussian over the remaining \( j \)-spins,

\[
\Psi_{\gamma}^{ndg}(j) = A(j^*) e^{-\frac{1}{2}(j-j^*)^T S(j-j^*)},
\]

where \( S \) is the Schur complement of \( \Delta \). Since the order of \( \Delta \) is \( O(1) \), so will be the order of the matrix \( S \), confirming our general result (27).
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