An optimal algorithm for Bisection for bounded-treewidth graphs

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Abstract. The maximum/minimum bisection problems are, given an edge-weighted graph, to find a bipartition of the vertex set into two sets whose sizes differ by at most one, such that the total weight of edges between the two sets is maximized/minimized. Although these two problems are known to be NP-hard, there is an efficient algorithm for bounded-treewidth graphs. In particular, Jansen et al. (SIAM J. Comput. 2005) gave an $O(2^t n^3)$-time algorithm when given a tree decomposition of width $t$ of the input graph, where $n$ is the number of vertices of the input graph. Eiben et al. (ESA 2019) improved the running time to $O(8^t t^5 n^2 \log n)$. Moreover, they showed that there is no $O(n^{2-\varepsilon})$-time algorithm for trees under some reasonable complexity assumption.

In this paper, we show an $O(2^{t(\ln n)^2})$-time algorithm for both problems, which is asymptotically tight to their conditional lower bound. We also show that the exponential dependency of the treewidth is asymptotically optimal under the Strong Exponential Time Hypothesis. Moreover, we discuss the (in)tractability of both problems with respect to special graph classes.

Keywords: Bisection · Fixed-Parameter Tractable · Treewidth.

1 Introduction

Let $G = (V, E)$ be a graph and let $w : E \to \mathbb{R}$ be an edge-weight function. For disjoint subsets $X, Y$ of $V$, we denote by $w(X, Y)$ the total weight of edges between $X$ and $Y$. A bisection of $G$ is a bipartition of $V$ into two sets $A$ and $B$ such that $-1 \leq |A| - |B| \leq 1$. The size of a bisection $(A, B)$ is defined as the number of edges between $A$ and $B$. We also consider bisections of edge-weighted graphs. In this case, the size of a bisection $(A, B)$ is defined as $w(A, B)$.

In this paper, we consider the following two problems: MIN BISECTION and MAX BISECTION.

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MIN BISECTION

Input: a graph $G = (V, E)$ and $w : E \to \mathbb{R}_+$.
Goal: Find a bisection $(A, B)$ of $G$ that minimizes $w(A, B)$.

MAX BISECTION

Input: a graph $G = (V, E)$ and $w : E \to \mathbb{R}_+$.
Goal: Find a bisection $(A, B)$ of $G$ that maximizes $w(A, B)$.

If the edge weight is allowed to be arbitrary, these problems are equivalent. These problems are well-known variants of MINCUT and MAXCUT problem. If every edge has non-negative weight, MINCUT, which is the problem of minimizing $w(A, B)$ for all bipartition $(A, B)$ of $V$, is solvable in polynomial time. For MAXCUT, which is the maximization version of MINCUT, the problem is NP-hard in general [12] and trivially solvable in polynomial time for bipartite graphs. Orlova and Dorfman [16] and Hadlock [9] proved that MAXCUT can be solved in polynomial time for planar graphs with non-negative edge weights, and Shih et al. [19] finally gave a polynomial time algorithm for planar graphs with arbitrary edge weights. However, the complexity of bisection problems is quite different from MINCUT and MAXCUT. MAX BISECTION is known to be NP-hard even for planar graphs [11] and unit disk graphs [5]. For MIN BISECTION, it is known to be NP-hard [7] even for $d$-regular graphs for fixed $d \geq 3$ [3] and unit disk graphs [7]. It is worth noting that MIN BISECTION for planar graphs is still open.

On bounded-treewidth graphs, MIN BISECTION and MAX BISECTION are solvable in polynomial time. More precisely, given an input graph $G$ of $n$ vertices and a tree decomposition of $G$ of width $t$, Jansen et al. [11] proved that MAX BISECTION can be solved in time $O(2^t n^3)$. This algorithm also works on graphs with arbitrary edge-weights, which means MIN BISECTION can be solved within the same running time. Very recently, Eiben et al. [5] improved the polynomial factor of $n$ with a sacrifice of the exponential factor in $t$ in the running time. The running time of their algorithm is $O(8^t t^5 n^2 \log n)$. They also discussed a conditional lower bound on the running time: They showed that for any $\varepsilon > 0$, MIN BISECTION cannot be solved in $O(n^{2-\varepsilon})$ for $n$-vertex trees unless $(\min, +)$-CONVOLUTION (defined in Section 3.3) has an $O(n^{2-\delta})$-time algorithm for some $\delta > 0$. Since trees are a precisely connected graphs of treewidth at most one, this lower bound also holds for bounded-treewidth graphs. Therefore, there is still a gap between the upper and (conditional) lower bound on the running time for bounded-treewidth graphs.

In this paper, we give an “optimal” algorithm for MIN BISECTION and MAX BISECTION on bounded-treewidth graphs. The running time of our algorithm is $O(2^t (tn)^2)$, provided that a width-$t$ tree decomposition of the input graph is given as input. The polynomial factor in $n$ matches the conditional lower bound given by Eiben et al. [5]. We also show that MAX BISECTION cannot be solved in
time $(2 - \varepsilon)^n n^{O(1)}$ for any $\varepsilon > 0$ unless the Strong Exponential Time Hypothesis (SETH) [10] fails. These facts imply that the exponential dependency with respect to $t$ and the polynomial dependency with respect to $n$ in our running time are asymptotically optimal under these well-studied complexity-theoretic assumptions.

We also investigate the complexity of Min Bisection and Max Bisection on special graph classes. From the hardness of MaxCut, we immediately have several complexity results for Max Bisection and Min Bisection. The most notable case is that both problems are NP-hard even on unweighted bipartite graphs, on which MaxCut can be trivially solved in polynomial time. Apart from these complexity results, we show that Max Bisection can be solved in linear time on line graphs.

2 Preliminaries

Let $G = (V, E)$ be a graph, which is simple and undirected. Throughout the paper, we use $n$ to denote the number of vertices of an input graph. For a vertex $v \in V$, we denote by $N(v)$ the set of neighbors of $v$ in $G$. For two disjoint subsets $X, Y \subseteq V$, we denote by $E(X, Y)$ the set of edges having one end in $X$ and the other end in $Y$. We write $w(X, Y)$ to denote the total weight of edges in $E(X, Y)$ (i.e., $w(X, Y) = \sum_{e \in E(X, Y)} w(e)$). A bipartition $(A, B)$ of $V$ is called a cut of $G$. The size of a cut is the number of edges between $A$ and $B$, that is, $|E(A, B)|$. For edge-weighted graphs, the size is measured by the total weight of edges between $A$ and $B$. A cut is called a bisection if $-1 \leq |A| - |B| \leq 1$.

In the next section, we work on dynamic programming based on tree decompositions. A tree decomposition of $G$ is a pair of a rooted tree $T$ with vertex set $I$ and a collection $\{X_i : i \in I\}$ of subsets of $V$ such that

- $\bigcup_{i \in I} X_i = V$;
- for each $\{u, v\} \in E$, there is an $i \in I$ with $\{u, v\} \subseteq X_i$;
- for each $v \in V$, the subgraph of $T$ induced by $\{i \in I : v \in X_i\}$ is connected.

We refer to the vertices of $T$ as nodes to distinguish them from vertices of $G$. The width of $T$ is defined as $\max_{i \in I} |X_i| - 1$. The treewidth of $G$ is the minimum integer $k$ such that $G$ has a tree decomposition of width $k$.

To facilitate dynamic programming on tree decompositions, several types of “special” tree decompositions are known. Jansen et al. [11] used the well-known nice tree decomposition for solving Max Bisection. Eiben et al. [6] improved the dependency on $n$ by means of “shallow tree decompositions” due to Bodlaender and Hagerup [11]. In this paper, we rather use nice tree decompositions as well as Jansen et al. [11], and the algorithm itself is in fact identical with theirs.

We say that a tree decomposition $T$ is nice if for every non-leaf node $i$ of $T$, either

- Introduce node $i$ has an exactly one child $j \in I$ such that $X_i = X_j \cup \{v\}$ for some $v \in V \setminus X_j$. 

for each S compute $bs_i$

**Forget node** $i$ has an exactly one child $j \in I$ such that $X_i = X_i \cup \{v\}$ for some $v \notin V \setminus X_i$, or

**Join node** $i$ has exactly two children $j, k \in I$ such that $X_i = X_j = X_k$.

**Lemma 1 (Lemma 13.1.3 in [13])**. Given a tree decomposition of $G$ of width $t$, there is an algorithm that converts it into a nice tree decomposition of width at most $t$ in time $O(t^2n)$. Moreover, the constructed nice tree decomposition has at most $4tn$ nodes.

3 Bounded-treewidth graphs

Let $G = (V,E)$ be an edge-weighted graph with weight function $w : E \to \mathbb{R}$. Note that we do not restrict the weight function to take non-negative values. Given this, **MIN BISECTION** is essentially equivalent to **MAX BISECTION**. Therefore, in this section, we will only consider the maximization counterpart.

3.1 An $O(2^t n^3)$-time algorithm

In this subsection, we quickly review the algorithm of Jansen et al. [11]. Let $T$ be a nice tree decomposition of width at most $t$. For each node $i \in I$, we use $V_i$ to denote the set of vertices of $G$ that is contained in $X_i$ or $X_j$ for some descendant $j \in I$ of $i$.

Let $i \in I$ be a node of $T$. For each $S \subseteq X_i$ and $0 \leq d \leq |V_i|$, we compute the value $bs(i, S, d)$ which is the maximum size of a bisection $(A_i, B_i)$ of $G[V_i]$ such that $A_i \cap X_i = S$ and $|A_i| = d$.

**Leaf node**: Let $i \in I$ be a leaf of $T$. For each $S \subseteq X_i$, $bs(i, S, d) = w(S, X_i \setminus S)$ if $d = |S|$. Otherwise we set $bs(i, S, d) = -\infty$.

**Introduce node**: Let $i \in I$ be an introduce node of $T$ and let $v \in X_i \setminus X_j$ be the vertex introduced at $i$, where $j \in I$ is the unique child of $i$. Since the neighborhood of $v$ in $G[V_i]$ is entirely contained in $X_i$, we can compute $bs(i, S, d)$ as

$$bs(i, S, d) = \begin{cases} bs(j, S \setminus \{v\}, d - 1) + w(\{v\}, X_i \setminus S) & \text{if } v \in S \\ bs(j, S, d) + w(\{v\}, S) & \text{otherwise,} \end{cases}$$

for each $S \subseteq X_i$ and $0 \leq d \leq |V_i|$.

**Forget node**: Let $i \in I$ be a forget node of $T$ and let $v \in X_j \setminus X_i$ be the vertex forgotten at $i$, where $j \in I$ is the unique child of $i$. As $G[V_i] = G[V_j]$, we can compute $bs(i, S, d)$ as

$$bs(i, S, d) = \max(bs(j, S, d), bs(j, S \cup \{v\}, d))$$

for each $S \subseteq X_i$ and $0 \leq d \leq |V_i|$.
Join node: Let \( i \in I \) be a join node of \( T \) with children \( j, k \in I \). By the definition of nice tree decompositions, we have \( X_i = X_j = X_k \). For \( S \subseteq X_i \) and \( 0 \leq d \leq |V_i| \),

\[
bs(i, S, d) = \max_{|S| \leq d'} (bs(j, S, d') + bs(k, S, d - d' + |S|) - w(S, X_i \setminus S)). \tag{1}
\]

Note that the edges between \( S \) and \( X_i \setminus S \) contribute to both \( bs(j, S, d') \) and \( bs(k, S, d - d' + |S|) \). Thus, we subtract \( w(S, X_i \setminus S) \) in the recurrence (1).

Running time: For each Leaf/Introduce/Forget node \( i \), we can compute \( bs(i, S, d) \) in total time \( O(2^t|V_i|) \) for all \( S \subseteq X_i \) and \( 0 \leq d \leq |V_i| \). For join node \( i \), the recurrence (1) can be evaluated in \( O(|V_i|) \) time for each \( S \subseteq X_i \) and \( 0 \leq d \leq |V_i| \), provided that \( bs(j, S, +) \), \( bs(k, S, +) \), and \( w(S, X_i \setminus S) \) are stored in the table. Therefore, the total running time for a join node \( i \) is \( O(2^t|V_i|^2) \). Since \( |V_i| = O(n) \) and \( T \) has \( O(n) \) nodes, the total running time of the entire algorithm is \( O(2^t n^3) \).

Theorem 1 ([11]). Given a tree decomposition of \( G \) of width \( t \), Max Bisection can be solved in \( O(2^t n^3) \) time.

3.2 A refined analysis for join nodes

For a refined running time analysis, we reconsider the recurrence (1) for join nodes. This can be rewritten as

\[
bs(i, S, d) = \max_{|S| \leq d', d' \leq d} (bs(j, S, d') + bs(k, S, d - d' + |S|) - w(S, X_i \setminus S)) = \max_{d', d'', d'' \leq d + |S|} (bs(j, S, d') + bs(k, S, d'') - w(S, X_i \setminus S)).
\]

Since \( d' \) and \( d'' \) respectively run over \( 0 \leq d' \leq |V_j| \) and \( 0 \leq d'' \leq |V_k| \), we can compute \( bs(i, S, d) \) in total time \( O(2^t|V_i| \cdot |V_k|) \) for all \( S \) and \( d \).

For each node \( i \in I \), we let \( n_i = \sum_{j \in X_i} |X_j| \), where the summation is taken over all descendants \( j \) of \( i \) and \( i \) itself. Clearly, \( n_i \geq |V_i| \) and hence the total running time of join nodes is upper bounded by

\[
\sum_{i: \text{join node}} O(2^t n_i n_k) = O \left( 2^t \cdot \sum_{i: \text{join node}} n_i n_k \right).
\]

Note that we abuse the notations \( n_i \) and \( n_k \) for different join nodes \( i \), and the children nodes \( j \) and \( k \) are defined accordingly. We claim that \( \sum_{i: \text{join node}} n_i n_k \) is \( O((tn)^{1/2}) \). To see this, let us consider the term \( n_i n_k \) for a join node \( i \). For each node \( i \) of \( T \), we label all the vertices contained in \( X_i \) by distinct labels \( v_1^i, v_2^i, \ldots, v_{|X_i|}^i \). Note that some single vertex can receive two or more labels since a vertex can be contained in more than one nodes in the tree decomposition. From now on, we regard such a vertex as distinct labeled vertices and hence \( n_i \) corresponds to the number of labeled vertices appeared in the node \( i \) or some
descendant node of \( i \). Now, the term \( n_j n_k \) can be seen as the number of pairs of labeled vertices \((l, r)\) such that \( l \) is a labeled vertex contained in the subtree rooted at the left child \( j \) and \( r \) is a labeled vertex contained in the subtree rooted at the right child \( k \). A crucial observation is that for any pair of labeled vertices, there is at most one join node \( i \) counted in the term \( n_j n_k \). This implies that \( \sum_{i: \text{join node}} n_j n_k \) is at most the number of pairs of distinct labeled vertices. Since each node of \( T \) contains at most \( t + 1 \) vertices and \( T \) contains \( O(n) \) nodes, it is \( O((tn)^2) \). Therefore, the total running time of the algorithm is \( O(2^t (tn)^2) \).

**Theorem 2.** Given a tree decomposition of \( G \) of width \( t \), MAX BISECTION can be solved in time \( O(2^t (tn)^2) \).

### 3.3 Optimality of our algorithm

Eiben et al. [6] proved that if the following \((\min, +)\)-CONVOLUTION does not admit \( O(n^{2-\delta})\)-time algorithm for some \( \delta > 0 \), there is no \( O(n^{2-\epsilon})\)-time algorithm for MIN BISECTION on trees for any \( \epsilon > 0 \).

\((\min, +)\)-CONVOLUTION

**Input:** Two sequences of numbers \((a_i)_{1 \leq i \leq n}, (b_i)_{1 \leq i \leq n}\)

**Goal:** Compute \( c_i = \min_{1 \leq j \leq i} (a_j + b_{i-j+1}) \) for all \( 1 \leq i \leq n \)

This conditional lower bound matches the dependency on \( n \) in the running time of our algorithm.

In terms of the dependency on treewidth, we can prove that under the Strong Exponential Time Hypothesis (SETH) [10], and hence the exponential dependency on \( t \) is asymptotically optimal.

**Theorem 3 (\([15]\)).** Unless SETH fails, there is no algorithm for the unweighted maximum cut problem that runs in time \( 2^{t-\epsilon} n^{O(1)} \) for any \( \epsilon > 0 \) even if a width-\( t \) tree decomposition of the input graph is given as input for some \( t \).

The known reduction (implicitly appeared in [3]) from the unweighted MAX-CUT to MAX BISECTION works well for our purpose. Specifically, let \( G \) be an unweighted graph and let \( n \) be the number of vertices of \( G \). We add \( n \) isolated vertices to \( G \) and the obtained graph is denoted by \( G' \). It is easy to see that \( G \) has a cut of size at least \( k \) if and only if \( G' \) has a bisection of size at least \( k \). Moreover, \( tw(G') = tw(G) \). Therefore, we have the following lower bound.

**Theorem 4.** Unless SETH fails, there is no algorithm for MAX BISECTION that runs in time \( 2^{t-\epsilon} n^{O(1)} \) for any \( \epsilon > 0 \) even if a width-\( t \) tree decomposition of the input graph is given as input for some \( t \).

### 4 Hardness on graph classes

In this section, we discuss some complexity results for MIN BISECTION and MAX BISECTION. In Section 3.3, we have seen that there is a quite simple reduction from MAX-CUT to MAX BISECTION. We formally describe some immediate consequences of this reduction as follows. Let \( \mathcal{C} \) be a graph class such that
MAXCut is NP-hard even if the input graph is restricted to be in \( \mathcal{C} \) and
- for every \( G \in \mathcal{C} \), a graph \( G' \) obtained from \( G \) by adding arbitrary number of isolated vertices is also contained in \( \mathcal{C} \).

The reduction shows that MAX BISECTION is NP-hard for every graph classes \( \mathcal{C} \) that satisfies the above conditions.

**Theorem 5.** MAX BISECTION is NP-hard even for split graphs, comparability graphs, AT-free graphs, and claw-free graphs.

It is known that MAXCut is NP-hard even for split graphs \([2]\) and comparability graphs \([17]\), and co-bipartite graphs \([2]\) that is a subclass of AT-free graphs and claw-free graphs. If \( \mathcal{C} \) is the class of co-bipartite graphs, the second condition does not hold in general. However, we can prove the hardness of MAX BISECTION on co-bipartite graphs, which will be discussed in the last part of this section.

Suppose the input graph \( G \) has 2\( n \) vertices. Let \( \overline{G} \) is the complement of \( G \). It is easy to see that \( G \) has a bisection of size at least \( k \) if and only if \( \overline{G} \) has a bisection of size at most \( n^2 - k \). This immediately gives the following theorem from Theorem 5

**Theorem 6.** MIN BISECTION is NP-hard even for split graphs and co-comparability graphs.

For bipartite graphs, MAX CUT is solvable in polynomial time. However, we show that MIN BISECTION and MAX BISECTION are NP-hard even on bipartite graphs.

**Theorem 7.** MIN BISECTION is NP-hard even for bipartite graphs.

**Proof.** We prove the theorem by performing a polynomial-time reduction from MIN BISECTION on 4-regular graphs, which is known to be NP-hard \([3]\).

Let \( G = (V, E) \) be a 4-regular graph. We can assume that \( G \) has 2\( n \) vertices since the reduction given by \([3]\) works on graphs of even number of vertices. For each edge \( e = (u, w) \in E \), we split \( e \) by introducing a new vertex \( v_e \) and replacing \( e \) with two edges \( \{u, v_e\} \) and \( \{v_e, w\} \). Then, for each \( v \in V \), we add \( n^3 \) pendant vertices and make adjacent them to \( v \). We denote by \( V_E \) the set of vertices newly added for edges, by \( V_p \) the set of pendant vertices, and by \( G' = (V \cup V_E \cup V_p, E') \) the graph obtained from \( G \) as above (see Fig. 1). As \( G \) is 4-regular, we have \( |V_E| = |E| = 4n \) and \( |V \cup V_E \cup V_p| = 2n + 4n + 2n^4 = 2n^4 + 6n \). Moreover, \( G' \) is bipartite. In the following, we show that \( G \) has a bisection of size at most \( k \) if and only if so does \( G' \).

Suppose \( G \) has a bisection \( (V_1, V_2) \) of size at most \( k \). Since \( |V| = 2n \), it holds that \( |V_1| = |V_2| \). For \( i = 1, 2 \), we set \( V_i = V_i \cup \{v_e \mid e \subseteq V_i \} \cup V_p \), where \( V_p \) is the set of pendant vertices such that its unique neighbor is contained in \( V_i \). Note that there are no edges between \( V_1 \) and \( V_2 \) in \( G' \) and \( |V_1'| = |V_2'| \) so far. Observe that for every \( e \in E(V_1, V_2) \), exactly one of the incidental edges \( \{u, v_e\} \) and \( \{v_e, w\} \) of the corresponding vertex \( v_e \) contributes to its size no matter whether
\( \nu_e \) is included in either \( V'_1 \) or \( V'_2 \). Therefore, we can appropriately distribute the remaining vertices \( \{ \nu_e : e \in E(V_1, V_2) \} \) to obtain a bisection of size at most \( k \).

Suppose that \( G' \) has a bisection \( \{ V'_1, V'_2 \} \) of size at most \( k \). Let \( V_1 = V'_1 \cap V \) and \( V_2 = V'_2 \). We claim that \( |V_1| = |V_2| \). Suppose for contradiction that \( |V_1| > |V_2| \). As \( |V_1 \cup V_2| = 2n \), we have \( |V_1| \geq n + 1 \). Since \( \{ V'_1, V'_2 \} \) is a bisection of \( G' \), it holds that \( |V'_1| = n^3 + 3n \). Thus, there are at least \( n^3 - 2n + 1 \) pendant vertices in \( V'_2 \) whose neighbor is contained in \( V'_1 \). For \( n \geq 5 \), it holds that \( n^3 - 2n + 1 > 4n^2 > k \), contradicting to the assumption that \( \{ V'_1, V'_2 \} \) is a bisection of size at most \( k \). Moreover, for every edge \( e = \{ u, w \} \in E(V_1, V_2) \) in \( G \), at least one of \( \{ u, \nu_e \} \) or \( \{ \nu_e, w \} \) contributes to the cut \( \{ V'_1, V'_2 \} \) in \( G' \). Therefore, we conclude that the size of the cut \( \{ V_1, V_2 \} \) in \( G \) is at most \( k \).

\[ \square \]

Interestingly, the same construction works well for proving the hardness of Max Bisection for bipartite graphs.

**Theorem 8.** Max Bisection is NP-hard even for bipartite graphs.

**Proof.** The proof of this theorem is similar to that of Theorem 7. Let \( G = (V, E) \) be a 4-regular graph and let \( G' = (V \cup V_E \cup V_p, E') \) be the bipartite graph described in the proof of Theorem 7. In the following, we prove that \( G \) has a bisection of size at most \( k \) if and only if \( G' \) has a bisection of size at least \( 2n^4 + 8n - k \).

Suppose first that \( G \) has a bisection \( \{ V_1, V_2 \} \) of size \( k \). Then, we set \( V'_i \) for \( i \in \{1, 2\} \) as:

- \( V_i \subseteq V'_i \);
- If \( v \in V_i \), all the pendant vertices \( w \) with \( N(w) = \{ v \} \) are contained in \( V'_{3-i} \);
- For each \( e \in E \) with \( e \subseteq V_i \), \( \nu_e \) is contained in \( V'_{3-i} \).

For each remaining \( e \in E(V_1, V_2) \), we add \( \nu_e \) to arbitrary side \( V'_i \) so that \( \{ V'_1, V'_2 \} \) becomes a bisection of \( G' \). This can be done since \( G \) is 4-regular, which means \( G[V_i] \) contains exactly \( 2n - k \) edges for each \( i \in \{1, 2\} \). Let us note that \( E'(V'_1, V'_2) \) has \( 2n^4 + 8n - 2k \) edges and for \( \nu_e \in E \) with \( e \in E(V_1, V_2) \), exactly one of the incident edges of \( \nu_e \) contributes to the size of the bisection no matter which \( V'_i \) includes \( \nu_e \). This implies that the size of bisection \( \{ V'_1, V'_2 \} \) is \( 2n^4 + 8n - k \).
For the converse, suppose that \( G' \) has a bisection \( (V'_1, V'_2) \) of size at least \( 2n^4 + 8n - k \). For each \( i = 1, 2 \), we let \( V_i = V'_i \cap V \). Then, we claim that \( (V_1, V_2) \) is a bisection of \( G \). To see this, we assume for contradiction that \( |V_1| > |V_2| \).

Clearly, \( V_1 \) contains at least \( n + 1 \) vertices. As \( |V'_1| = |V'_2| \) and \( G' \) has \( 2n + 2n \cdot n^3 + 4n = 2n^4 + 6n \) vertices, we have \( |V'_2| = n^4 + 3n \). Since \( V_1 \) has at least \( n + 1 \) vertices, at least \( (n + 1)n^3 - |V'_2| = n^3 - 3n \) pendant vertices adjacent to some vertex in \( V_1 \) are included in \( V_1 \). Therefore, at most \( |E'| - (n^3 - 3n) = 2|E| + 2n \cdot n^3 - (n^3 - 3n) = 2n^4 - n^3 + 12n \) edges can belong to \( E'(V'_1, V'_2) \).

For \( n \geq 3 \), we have \( 2n^4 - n^3 + 12n < 2n^4 + 8n - 4n < 2n^4 + 8n - k \), which contradicts to the fact that the size of \( (V'_1, V'_2) \) is at least \( 2n^4 + 8n - k \). Note that \( k \leq 4n \).

Now, we show that the bisection \( (V_1, V_2) \) of \( G \) has size at most \( k \). Since there are \( 2n^4 \) pendant edges in \( G' \), at least \( 8n - k \) edges of \( G'[V \cup E] \) belong to \( E'(V'_1, V'_2) \). Note that as \( V \) and \( V_E \) are respectively independent sets in \( G' \), such edges are in \( E'(V, V_E) \). Moreover, there are \( 8n \) edges in \( E'(V, V_E) \). If there are at least \( k + 1 \) vertices \( \nu_e \) in \( V_E \) having neighbors both in \( V_1 \) and in \( V_2 \), the size of \( E'(V'_1 \cap V \cup V_E), V'_2 \cap (V \cup V_E) \) is at most \( 8n - k - 1 \) since exactly one of the incident edges of \( \nu_e \) does not contribute to the cut. Thus, the number of such vertices is at most \( k \). Since each \( \nu_e \in V_E \) having neighbors both in \( V_1 \) and in \( V_2 \) corresponds to a cut edge of \( (V_1, V_2) \) in \( G \), the size of the bisection \( (V_1, V_2) \) of \( G \) is at most \( k \).

Since both Min Bisection and Max Bisection are NP-hard on bipartite graphs, by the same argument with Theorem 6, we have the following corollary.

**Corollary 1.** Min Bisection and Max Bisection are NP-hard even for co-bipartite graphs.

## 5 Line graphs

Guruswami [8] showed that MAXCUT can be solved in linear time for unweighted line graphs. The idea of the algorithm is to find a cut satisfying a certain condition using an Eulerian tour of the underlying graph of the input line graph. In this section, we show that his approach works well for MAX BISECTION.

Let \( G = (V, E) \) be a graph. The **line graph** of \( G \), denoted by \( L(G) = (V_L, E_L) \), is an undirected graph with \( V_L = E \) such that two vertices \( e, f \in V_L \) are adjacent to each other if and only if \( e \) and \( f \) share their end vertex in \( G \). We call \( G \) an underlying graph of \( L(G) \). Note that from a line graph, its underlying graph is not uniquely determined. However, it is sufficient to take an arbitrary one to discuss our result. Guruswami gave the following sufficient condition for MAXCUT and showed that every line graph has a cut satisfying this condition.

**Lemma 2 ([8]).** Let \( G = (V, E) \) be a (not necessarily line) graph and let \( C_1, C_2, \ldots, C_k \) be edge disjoint cliques with \( \bigcup_{1 \leq i \leq k} C_i = E \). If there is a cut \( (A, B) \) of \( G \) such that \( -1 \leq |A \cap C_i| - |B \cap C_i| \leq 1 \) for every \( 1 \leq i \leq k \), then \( (A, B) \) is a maximum cut of \( G \).
Since the maximum size of a bisection is at most the maximum size of a cut, we immediately conclude that every bisection satisfying the condition in Lemma 2 is a maximum bisection. The construction of a bipartition \((A, B)\) of \(V \cup \{r\}\) is as follows.

Let \(L(G) = (V_L, E_L)\) be a line graph whose underlying graph is \(G\). We make \(G\) an even-degree graph by putting a vertex \(r\) and make adjacent \(r\) to each vertex of odd degree. Let \(G'\) be the even-degree graph obtained as above. Suppose first that \(G'\) is connected. Fix an Eulerian tour starting from \(r\) and alternately assign labels \(a\) and \(b\) to each edge along with the Eulerian tour. Let \(A\) and \(B\) be the set of edges having label \(a\) and \(b\), respectively. Observe that the bipartition \((A \cap V_L, B \cap V_L)\) of \(V_L\) satisfies the sufficient condition in Lemma 2. To see this, consider a vertex \(v\) of \(G\). Since the set of edges \(C_v\) adjacent to \(v\) forms a clique in \(L(G)\). Moreover, it is known that, in line graphs, the edges of cliques \(\{C_v : v \in G\}\) partitions the whole edge set \(E_L\). Every vertex \(v\) of \(G'\) except for \(r\) has an equal number of incidental edges with label \(a\) and those with label \(b\) in \(G'\), which implies that every clique \(C_v\) satisfies 
\[-1 \leq |C_v \cap (A \cap V_L) - |C_v \cap (B \cap V_L)| \leq 1.\]

Now, we show that the bipartition \((A \cap V_L, B \cap V_L)\) of \(V_L\) is also a bisection of \(L(G)\). Consider the labels of the edges incident to \(r\) in \(G'\). Observe that every two consecutive edges in the Eulerian tour except for the first and last edge have difference labels. Moreover, the first edge has label \(a\). If the last edge has label \(a\), we have \(|A| = |B| + 1\) and hence \(|A \cap V_L| + 1 = |B \cap V_L|\). Otherwise, the last edge has label \(b\), we have \(|A| = |B|\) and hence \(|A \cap V_L| = |B \cap V_L|\). Therefore, \((A \cap V_L, B \cap V_L)\) is a bisection of \(L(G)\).

If \(G'\) has two or more connected components, we apply the same argument to each connected component and appropriately construct a bipartition of \(V_L\). It is not hard to see that this bipartition also satisfies the condition in Lemma 2.

Since, given a line graph, we can compute its underlying graph \([14,18]\) and an Eulerian tour in linear time, \textsc{Max Bisection} on line graphs can be solved in linear time.

### 6 Conclusion

In this paper, we show that there is an \(O(2^t(n^2)^2)\) time algorithm for \textsc{Min Bisection} and \textsc{Max Bisection}, provided that a width-\(t\) tree decomposition is given as input. This running time matches the conditional lower bound given by Eiben et al. \([6]\) based on \((\min, +)\)-\textsc{Convolution}. We also show that the exponential dependency of treewidth in our running time is asymptotically optimal under the Strong Exponential Time Hypothesis.

For unweighted graphs, Eiben et al. showed that the polynomial dependency can be slightly improved: They gave an \(O(8^t n^{O(1)} \log n)\)-time algorithm for \textsc{Min Bisection} using an extension of the fast \((\min, +)\)-\textsc{Convolution} technique due to Chan et al. \([4]\). It would be interesting to know whether a similar improvement can be applied to our case.

We also show that \textsc{Min Bisection} and \textsc{Max Bisection} are NP-hard even for several restricted graph classes. In particular, both problems are NP-hard even on
unweighted bipartite graphs, which is in contrast with the tractability of MINCUT and MAXCUT on this graph classes. However, there are several open problems related to these results. One of the most notable open questions would be to reveal the complexity of MIN BISECTION on planar graphs.

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