STAR VERSIONS OF LINDELÖF SPACES

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Abstract. A space $X$ is said to be set star-Lindelöf (resp., set strongly star-Lindelöf) if for each nonempty subset $A$ of $X$ and each collection $U$ of open sets in $X$ such that $\overline{A} \subseteq \bigcup U$, there is a countable subset $V$ of $U$ (resp., a countable subset $F$ of $A$) such that $A \subseteq \operatorname{St}(\bigcup V, U)$ (resp., $A \subseteq \operatorname{St}(F, U)$). The classes of set star-Lindelöf spaces and set strongly star-Lindelöf spaces lie between the class of Lindelöf spaces and the class of star-Lindelöf spaces. In this paper, we investigate the relationship among set star-Lindelöf spaces, set strongly star-Lindelöf spaces, and other related spaces by providing some suitable examples and study the topological properties of set star-Lindelöf and set strongly star-Lindelöf spaces.

1. Introduction and Preliminaries

A cardinal function $sL$ defined from a set of topological spaces to a cardinal number $sL(X)$. Arhangel’skii [1] defined a cardinal number $sL(X)$ of $X$: the minimal infinite cardinality $\tau$ such that for every subset $A \subseteq X$ and every open cover $U$ of $A$, there is a subfamily $V \subseteq U$ such that $|V| \leq \tau$ and $A \subseteq \bigcup V$. If $sL(X) = \omega$, then the space $X$ is called $sL$-Lindelöf space. Following this idea, Kočinac and Konca [5] introduced and studied the new types of selective covering properties called set-covering properties. A space $X$ is said to have the set-Menger [5] property if for each nonempty subset $A$ of $X$ and each sequence $(U_n : n \in \mathbb{N})$ of collections of open sets in $X$ such that for each $n \in \mathbb{N}$, $\overline{A} \subseteq \bigcup U_n$, there is a sequence $(V_n : n \in \mathbb{N})$ such that for each $n \in \mathbb{N}$, $V_n$ is a finite subset of $U_n$ and $A \subseteq \bigcup_{n \in \mathbb{N}} \bigcup V_n$. The author [9] noticed that the set-Menger property is nothing but another view of Menger covering property. Recently, the author [8] defined and studied set star-compact and set strongly starcompact spaces (also see [10]).

In this paper, we apply the above study of Kočinac, Konca, and Singh to defined a new subclass of star-Lindelöf spaces called set star-Lindelöf and set strongly star-Lindelöf spaces. With the help of some suitable example, we investigate the relationship among set star-Lindelöf, set strongly star-Lindelöf, and other related spaces. A space having a dense Lindelöf subspace is star-Lindelöf (see [11]). We show that this result is not true if we replace star-Lindelöf space with a set star-Lindelöf space.

If $A$ is a subset of a space $X$ and $U$ is a collection of subsets of $X$, then $\operatorname{St}(A, U) = \bigcup \{ U \in U : U \cap A \neq \emptyset \}$. We usually write $\operatorname{St}(x, U) = \operatorname{St}(\{x\}, U)$.

Throughout the paper, by “a space” we mean “a topological space”, $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{Q}$ denotes the set of natural numbers, set of real numbers, and set of rational numbers, respectively, the cardinality of a set is denoted by $|A|$. Let $\omega$ denote the first infinite
cardinal, $\omega_1$ the first uncountable cardinal, $\kappa$ the cardinality of the set of all real numbers. An open cover $\mathcal{U}$ of a subset $A \subset X$ means elements of $\mathcal{U}$ open in $X$ such that $A \subseteq \bigcup \mathcal{U} = \bigcup \{U : U \in \mathcal{U}\}$.

We first recall the classical notions of spaces that are used in this paper.

**Definition 1.1.** [3] A space $X$ is said to be

1. starcompact if for each open cover $\mathcal{U}$ of $X$, there is a finite subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = \text{St}(\bigcup \mathcal{V}, \mathcal{U})$.
2. strongly starcompact if for each open cover $\mathcal{U}$ of $X$, there is a finite subset $\mathcal{F}$ of $X$ such that $X = \text{St}(\mathcal{F}, \mathcal{U})$.

**Definition 1.2.** [8, 10] A space $X$ is said to be

1. set starcompact if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $A \subseteq \bigcup \mathcal{U}$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subseteq \text{St}(\bigcup \mathcal{V}, \mathcal{U})$.
2. set strongly starcompact if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $A \subseteq \bigcup \mathcal{U}$, there is a countable subset $\mathcal{F}$ of $X$ such that $A \subseteq \text{St}(\mathcal{F}, \mathcal{U})$.

**Definition 1.3.** A space $X$ is said to be

1. star-Lindelöf [3] if for each open cover $\mathcal{U}$ of $X$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $X = \text{St}(\bigcup \mathcal{V}, \mathcal{U})$.
2. strongly star-Lindelöf [3] if for each open cover $\mathcal{U}$ of $X$, there is a countable subset $\mathcal{F}$ of $X$ such that $X = \text{St}(\mathcal{F}, \mathcal{U})$.

Note that the star-Lindelöf spaces have a different name such as 1-star-Lindelöf and $1^{\uparrow}$-star-Lindelöf in different papers (see [3, 6]) and the strongly star-Lindelöf space is also called star countable in [6, 13]. It is clear that, every strongly star-Lindelöf space is star-Lindelöf.

Recall that a collection $\mathcal{A} \subseteq P(\omega)$ is said to be almost disjoint if each set $A \in \mathcal{A}$ is infinite and the sets $A \cap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. For an almost disjoint family $\mathcal{A}$, put $\psi(\mathcal{A}) = \mathcal{A} \cup \omega$ and topologize $\psi(\mathcal{A})$ as follows: for each element $A \in \mathcal{A}$ and each finite set $F \subset \omega$, $\{A\} \cup (A \setminus F)$ is a basic open neighborhood of $A$ and the natural numbers are isolated. The spaces of this type are called Isbell-Mrówka $\psi$-spaces [2, 7] or $\psi(A)$ space. For other terms and symbols we follow [4].

The following result was proved in [10].

**Theorem 1.4.** [10] Every countably compact space is set strongly starcompact.

2. **Set Star-Lindelöf and Related Spaces**

In this section, we give some examples showing the relationship among set star-Lindelöf spaces, set strongly star-Lindelöf spaces, and other related spaces. First we define our main definition.

**Definition 2.1.** A space $X$ is said to be

1. set star-Lindelöf if for each nonempty subset $A$ of $X$ and each collection $\mathcal{U}$ of open sets in $X$ such that $\overline{A} \subseteq \bigcup \mathcal{U}$, there is a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $A \subseteq \text{St}(\bigcup \mathcal{V}, \mathcal{U})$.  

(2) set strongly star-Lindelöf if for each nonempty subset $A$ of $X$ and each collection $U$ of open sets in $X$ such that $\overline{A} \subseteq \bigcup U$, there is a countable subset $F$ of $\overline{A}$ such that $A \subseteq \text{St}(F, U)$.

The following result shows that the class of set strongly star-Lindelöf spaces is big enough.

**Theorem 2.2.** For a space $X$, the following statements are hold:

1. If $X$ is a Lindelöf space, then $X$ is set strongly star-Lindelöf.
2. If $X$ is a countably compact space, then $X$ is set strongly star-Lindelöf.

**Proof.**

(1). Let $A \subseteq X$ be a nonempty set and $U$ be a collection of open sets such that $A \subseteq \bigcup U$. Since space $X$ is Lindelöf, closed subset $\overline{A}$ of $X$ is also Lindelöf. Thus there exists a countable subset $V$ of $\bigcup U$ such that $A \subseteq \overline{A} \subseteq \bigcup V$. Choose $x \in \overline{A} \cap V$ for each $V \in V$. Let $F = \{x \in \overline{A} \cap V : V \in V\}$. Then $F$ is a countable subset of $\overline{A}$ and $A \subseteq \bigcup V \subseteq \text{St}(F, U)$. Therefore $X$ is set strongly star-Lindelöf.

(2). Since every set strongly starcompact space is set strongly star-Lindelöf, by Theorem 1.4, $X$ is set strongly star-Lindelöf.

We have the following diagram from the definitions and Theorem 2.2. However, the following examples shows that the converse of these implications are not true.

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set strongly starcompact \rightarrow \text{set starcompact} \\
\downarrow \downarrow \\
\text{Lindelöf} \rightarrow \text{set strongly star-Lindelöf} \rightarrow \text{set star-Lindelöf} \\
\downarrow \downarrow \\
\text{strongly star-Lindelöf} \rightarrow \text{star-Lindelöf}
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**Example 2.3.** There exists Tychonoff set strongly star-Lindelöf (hence, set star-Lindelöf) space which is not set starcompact (hence, not set strongly starcompact).

**Proof.** Let $X = \omega$ be the discrete space. Then $X$ is set strongly star-Lindelöf but not set starcompact space.

**Example 2.4.** Let $X = [0, \omega_1)$. Then $X$ is countably compact space but not Lindelöf and not separable. Thus $X$ is set strongly star-Lindelöf space which is not Lindelöf.

The following example shows that the converse of Theorem 2.2(2) is not true.

**Example 2.5.** Let $Y$ be a discrete space with cardinality $c$. Let $X = Y \cup \{y^*\}$, where $y^* \notin Y$. Define topology on $X$, each $y \in Y$ is an isolated point and a set $U$ containing $y^*$ is open if and only if $X \setminus U$ is countable. Then $X$ is Lindelöf, hence set strongly star-Lindelöf.

Now to show $X$ is not countably compact. Let $y^* \in U$ such that $X \setminus U = \{y_\alpha : \alpha < \omega\}$, where $y_\alpha \in Y$. Then $U$ is open in $X$ and $U = \{U\} \cup \{y_\alpha : \alpha < \omega\}$ is a countable cover of $X$, which does not have a finite subcover. Thus $X$ is not countably compact.

**Example 2.6.** There exists a Tychonoff strongly star-Lindelöf space which is not a set strongly star-Lindelöf.
2.7. Let $X = \psi(A) = A \cup \omega$ be the Isbell-Mrówka space with $|A| = \omega_1$. Then $X$ is strongly star-Lindelöf Tychonoff pseudocompact space, since $X$ is separable. Now we prove that $X$ is not set strongly star-Lindelöf. Let $A = A = \{a_\alpha : \alpha < \omega_1\}$. Then $A$ is closed subset of $X$. For each $\alpha < \omega_1$, let $U_\alpha = \{a_\alpha\} \cup (a_\alpha)$. Let $U = \{U_\alpha : \alpha < \omega_1\}$. Then $U$ is an open cover of $A$. It is enough to show that there exists a point $a_\beta \in A$ such that $a_\beta \notin \text{St}(F, U)$, for any countable subset $F$ of $A$. Let $F$ be any countable subset of $A$. Then there exists $\alpha' < \omega_1$ such that $U_\alpha \cap F = \emptyset$, for each $\alpha > \alpha'$. Pick $\beta > \alpha'$, then $U_\beta \cap F = \emptyset$. Since $U_\beta$ is the only element of $U$ containing the point $a_\beta$. Thus $a_\beta \notin \text{St}(F, U)$, which shows that $X$ is not set strongly star-Lindelöf.

The following lemma was proved by Song [11].

**Lemma 2.7.** ([11], Lemma 2.2) A space $X$ having a dense Lindelöf subspace is star-Lindelöf.

The following example shows that the Lemma 2.7 does not hold if we replace star-Lindelöf space by a set star-Lindelöf space.

**Example 2.8.** There exists a Tychonoff space $X$ having a dense Lindelöf subspace such that $X$ is not set star-Lindelöf.

**Proof.** Let $D(\mathfrak{c}) = \{d_\alpha : \alpha < \mathfrak{c}\}$ be a discrete space of cardinality $\mathfrak{c}$ and let $Y = D(\mathfrak{c}) \cup \{d^*\}$ be one-point compactification of $D(\mathfrak{c})$. Let $X = (Y \times [0, \omega)) \cup (D(\mathfrak{c}) \times \{\omega\})$ be the subspace of the product space $Y \times [0, \omega]$. Then $Y \times [0, \omega]$ is a dense Lindelöf subspace of $X$ and by Lemma 2.7, $X$ is star-Lindelöf.

Now we show that $X$ is not set star-Lindelöf. Let $A = D(\mathfrak{c}) \times \{\omega\}$. Then $A$ is the closed subset of $X$. For each $\alpha < \mathfrak{c}$, let $U_\alpha = \{d_\alpha\} \times [0, \omega]$. Then $U_\alpha \cap U_{\alpha'} = \emptyset$ for $\alpha \neq \alpha'$.

Let $U = \{U_\alpha : \alpha < \mathfrak{c}\}$. Then $U$ is an open cover of $A$. It is enough to show that there exists a point $\langle d_\beta, \omega \rangle \in A$ such that $\langle d_\beta, \omega \rangle \notin \text{St}(\bigcup V, U)$ for any countable subset $V$ of $U$. Let $V$ be any countable subset of $U$. Then there exists $\alpha' < \mathfrak{c}$ such that $U_\alpha \notin V$ for each $\alpha > \alpha'$. Pick $\beta > \alpha'$. Then $U_\beta \cap (\bigcup V) = \emptyset$, but $U_\beta$ is the only element of $U$ containing $\langle d_\beta, \omega \rangle$. Thus $\langle d_\beta, \omega \rangle \notin \text{St}(\bigcup V, U)$. Therefore $X$ is not set star-Lindelöf.

**Example 2.9.** There exists a $T_1$ set star-Lindelöf space $X$ that is not set strongly star-Lindelöf.

**Proof.** Let $X = A \cup B$, where $A = [0, \mathfrak{c})$ and $B = \{b_n : n \in \omega\}$ and for each $n \in \omega$, $b_n \notin A$. Topologize $X$ as follows: for each $\alpha \in A$ and each finite subset $F \subset B$, $\{\alpha\} \cup (B \setminus F)$ is a basic open neighborhood of $\alpha$ and for each $n \in \omega$, $b_n$ is isolated. Then $X$ is a $T_1$-space. Let $C$ be any nonempty subset of $X$ and $U$ be an open cover of $C$. 


First we show that \( X \) is set star-Lindelöf space. For this we have three possible cases:

Case (i): If \( C \subset A \). Then for each \( \alpha \in C \), there exists \( U_\alpha \in \mathcal{U} \) such that \( \alpha \in U_\alpha \). Then for each \( \alpha \in C \), we can find a finite set \( F_\alpha \) such that \( \{ \alpha \} \cup (B \setminus F_\alpha) \subseteq U_\alpha \). It is clear that for each \( \alpha \neq \alpha' \), \( U_\alpha \cap U_{\alpha'} \neq \emptyset \). Let \( \mathcal{V}' = \{ U_\alpha \} \). Then \( C \subset \text{St}(\bigcup \mathcal{V}', \mathcal{U}) \).

Case (ii): If \( C \subset B \). Since \( B \) is countable, thus \( B \) is set star-Lindelöf. Hence we have a countable subset \( \mathcal{V}'' \) of \( \mathcal{U} \) such that \( C \subset \text{St}(\bigcup \mathcal{V}'', \mathcal{U}) \).

Case (iii): If \( C = C_1 \cup C_2 \) such that \( C_1 \subset A \) and \( C_2 \subset B \). Choose \( \mathcal{V}' \) and \( \mathcal{V}'' \) from Case (i) and Case (ii), respectively such that \( C_1 \subset \text{St}(\bigcup \mathcal{V}', \mathcal{U}) \) and \( C_2 \subset \text{St}(\bigcup \mathcal{V}'', \mathcal{U}) \). Let \( \mathcal{V} = \mathcal{V} \cup \mathcal{V}'' \). Then \( \mathcal{V} \) is a countable subset of \( \mathcal{U} \) and \( C \subset \text{St}(\bigcup \mathcal{V}', \mathcal{U}) \).

Thus \( X \) is a set star-Lindelöf space.

Now we prove that \( X \) is not set strongly star-Lindelöf space. Since \( A = [0, c) \) is a closed subset of \( X \). For each \( \alpha < c \), let \( U_\alpha = \{ a_\alpha \} \cup \{ a_\alpha \} \). Then \( \mathcal{U} = \{ U_\alpha : \alpha < c \} \) is an open cover of \( A \). It is enough to show that there exists a point \( \beta \in A \) such that \( \beta \notin \text{St}(F, \mathcal{U}) \) for any countable subset \( F \) of \( A \). Let \( F \) be a countable subset of \( A \). Then there exists \( \alpha' < \epsilon \) such that \( U_\alpha \cap F = \emptyset \). Pick \( \beta > \alpha' \), then \( U_\beta \cap F = \emptyset \). Since \( U_\beta \) is the only element of \( \mathcal{U} \) containing the point \( \beta \). Thus \( \beta \notin \text{St}(F, \mathcal{U}) \), which shows that \( X \) is not set strongly star-Lindelöf.

\(\square\)

Remark 2.10. (1) In [8], Singh gave an example of a Tychonoff set starcompact space \( X \) that is not set strongly starcompact.

(2) It is known that there are star-Lindelöf spaces that are not strongly star-Lindelöf (see [3], Example 3.2.3.2 and [3], Example 3.3.1).

Now we give some conditions under which star-Lindelöfness coincide with set star-Lindelöfness and strongly star-Lindelöfness coincide with set strongly star-Lindelöfness.

Recall that a space \( X \) is paraLindelöf if every open cover \( \mathcal{U} \) of \( X \) has a locally countable open refinement.

Song and Xuan [12] proved the following result.

Theorem 2.11. [[12], Theorem 2.24] Every regular paraLindelöf star-Lindelöf spaces are Lindelöf.

We have the following theorem from Theorem 2.11 and the diagram.

Theorem 2.12. If \( X \) is a regular paraLindelöf space, then the following statements are equivalent:

(1) \( X \) is Lindelöf;
(2) \( X \) is set strongly star-Lindelöf;
(3) \( X \) is set star-Lindelöf;
(4) \( X \) is strongly star-Lindelöf;
(5) \( X \) is star-Lindelöf.

A space is said to be metaLindelöf if every open cover of it has a point-countable open refinement.

Xuan and Shi [13] proved the following result.
Theorem 2.13. [[13], Proposition 3.12] Every strongly star-Lindelöf meta-Lindelöf spaces are Lindelöf.

We have the following theorem from Theorem 2.13 and the diagram.

Theorem 2.14. If $X$ is a meta-Lindelöf space, then the following statements are equivalent:

1. $X$ is Lindelöf;
2. $X$ is set strongly star-Lindelöf;
3. $X$ is strongly star-Lindelöf.

3. Properties of set star-Lindelöf spaces and set strongly star-Lindelöf spaces

In this section, we study the topological properties of set star-Lindelöf and set strongly star-Lindelöf spaces.

Theorem 3.1. If $X$ is a set star-Lindelöf space, then every open and closed subset of $X$ is set star-Lindelöf.

Proof. Let $X$ be a set star-Lindelöf space and $A \subseteq X$ be an open and closed set. Let $B$ be any subset of $A$ and $\mathcal{U}$ be a collection of open sets in $(A, \tau_A)$ such that $\text{Cl}_A(B) \subseteq \bigcup \mathcal{U}$. Since $A$ is open, then $\mathcal{U}$ is a collection of open sets in $X$. Since $A$ is closed, $\text{Cl}_A(B) = \text{Cl}_X(B)$. Applying the set star-Lindelöfness property of $X$, there exists a countable subset $\mathcal{V}$ of $\mathcal{U}$ such that $B \subseteq \text{St}(\bigcup \mathcal{V}, \mathcal{U})$. Hence $A$ is a set star-Lindelöf.

Similarly, we can prove the following.

Theorem 3.2. If $X$ is a set strongly star-Lindelöf space, then every open and closed subset of $X$ is set strongly star-Lindelöf.

Consider the Alexandorff duplicate $A(X) = X \times \{0, 1\}$ of a space $X$. The basic neighborhood of a point $(x, 0) \in X \times \{0\}$ is of the form $(U \times \{0\}) \cup (U \times \{1\}) \setminus \{(x, 1)\}$, where $U$ is a neighborhood of $x$ in $X$ and each point $(x, 1) \in X \times \{1\}$ is a isolated point.

Theorem 3.3. If $X$ is a $T_1$-space and $A(X)$ is a set star-Lindelöf space. Then $e(X) < \omega_1$.

Proof. Suppose that $e(X) \geq \omega_1$. Then there exists a discrete closed subset $B$ of $X$ such that $|B| \geq \omega_1$. Hence $B \times \{1\}$ is an open and closed subset of $A(X)$ and every point of $B \times \{1\}$ is an isolated point. Thus $A(X)$ is not set star-Lindelöf, by Theorem 3.1, every open and closed subset of a set star-Lindelöf space is set star-Lindelöf and $B \times \{1\}$ is not set star-Lindelöf.

Corollary 3.4. If $X$ is a $T_1$-space and $A(X)$ is a set strongly star-Lindelöf space. Then $e(X) < \omega_1$.

Theorem 3.5. Let $X$ be a space such that the Alexandorff duplicate $A(X)$ of $X$ is set star-Lindelöf (resp., set strongly star-Lindelöf). Then $X$ is a set star-Lindelöf (resp., set strongly star-Lindelöf) space.
Theorem 3.6. A continuous image of set star-Lindelöf space is set star-Lindelöf.

Proof. Let $X$ be a set star-Lindelöf space and $f : X \rightarrow Y$ be a continuous mapping from $X$ onto $Y$. Let $B = f^{-1}(Y)$ and $\mathcal{U}$ be any open cover of $\overline{B}$. Let $A = f^{-1}(B)$. Since $f$ is continuous, $\mathcal{U} = \{f^{-1}(V) : V \in \mathcal{V}\}$ is the collection of open sets in $X$ with $A = \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B}) \subseteq f^{-1}(\bigcup \mathcal{V}) = \bigcup \mathcal{U}$. As $X$ is set star-Lindelöf, there exists a countable subset $\mathcal{U}'$ of $\mathcal{U}$ such that

$$A \subseteq \text{St}(\bigcup \mathcal{U}', \mathcal{U}).$$

Let $\mathcal{V}' = \{V : f^{-1}(V) \in \mathcal{U}'\}$. Then $\mathcal{V}'$ is a countable subset of $\mathcal{V}$ and $B = f(A) \subseteq f(\text{St}(\bigcup \mathcal{U}', \mathcal{U})) \subseteq \text{St}(\bigcup f(f^{-1}(V) : V \in \mathcal{V}')), \mathcal{V}) = \text{St}(\bigcup \mathcal{V}', \mathcal{V})$. Thus $Y$ is set star-Lindelöf space. \hfill \square

On the images of set star-Lindelöf spaces, we have the following result.

Theorem 3.7. A continuous image of a set strongly star-Lindelöf space is set strongly star-Lindelöf.

Proof. Let $X$ be a set strongly star-Lindelöf space and $f : X \rightarrow Y$ be a continuous mapping from $X$ onto $Y$. Let $B$ be any nonempty subset of $X$ and $\mathcal{U}$ be an open cover of $\overline{B}$. Let $C = B \times \{0\}$ and $A(\mathcal{U}) = \{U \times \{0,1\} : U \in \mathcal{U}\}$.

Then $A(\mathcal{U})$ is an open cover of $\overline{C}$. Since $A(X)$ is set star-Lindelöf, there is a countable subset $A(\mathcal{V})$ of $A(\mathcal{U})$ such that $C \subseteq \text{St}(\bigcup A(\mathcal{V}), A(\mathcal{U}))$. Let

$$\mathcal{V} = \{U \in \mathcal{U} : U \times \{0,1\} \in A(\mathcal{V})\}.$$ 

Then $\mathcal{V}$ is a countable subset of $\mathcal{U}$. Now we have to show that

$$B \subseteq \text{St}(\bigcup \mathcal{V}, \mathcal{U}).$$

Let $x \in B$. Then $\langle x, 0 \rangle \in \text{St}(\bigcup A(\mathcal{V}), A(\mathcal{U}))$. Choose $U \times \{0,1\} \in A(\mathcal{U})$ such that $\langle x, 0 \rangle \in U \times \{0,1\}$ and $U \times \{0,1\} \cap (\bigcup A(\mathcal{V})) \neq \emptyset$, which implies $U \cap (\bigcup \mathcal{V}) \neq \emptyset$ and $x \in U$. Therefore $x \in \text{St}(\bigcup \mathcal{V}, \mathcal{U})$, which shows that $X$ is set star-Lindelöf space. \hfill \square

Next, we turn to consider preimages of set strongly star-Lindelöf and set star-Lindelöf spaces. We need new concepts called nearly set strongly star-Lindelöf and nearly set star-Lindelöf spaces. A space $X$ is said to be nearly set strongly star-Lindelöf (resp., nearly set star-Lindelöf) in $X$ if for each subset $Y$ of $X$ and each open cover $\mathcal{U}$ of $X$, there is a countable subset $F$ of $X$ (resp., a countable subset $\mathcal{V}$ of $\mathcal{U}$) such that $Y \subseteq \text{St}(F, \mathcal{U})$ (resp., $Y \subseteq \text{St}(\bigcup \mathcal{V}, \mathcal{U})$).

Theorem 3.8. Let $f : X \rightarrow Y$ be an open, closed, and finite-to-one continuous mapping from a space $X$ onto a set strongly star-Lindelöf space $Y$. Then $X$ is nearly set strongly star-Lindelöf.
Proof. Let $A \subseteq X$ be any nonempty set and $\mathcal{U}$ be an open cover of $X$. Then $B = f(A)$ is a subset of $Y$. Let $y \in \overline{B}$. Then $f^{-1}\{y\}$ is a finite subset of $X$, thus there is a finite subset $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}\{y\} \subseteq \bigcup \mathcal{U}_y$ and $U \cap f^{-1}\{y\} \neq \emptyset$ for each $U \in \mathcal{U}_y$. Since $f$ is closed, there exists an open neighborhood $V_y$ of $y$ in $Y$ such that $f^{-1}(V_y) \subseteq \bigcup\{U : U \in \mathcal{U}_y\}$. Since $f$ is open, we can assume that

$$V_y \subseteq \bigcap\{f(U) : U \in \mathcal{U}_y\}.$$  

Then $\mathcal{V} = \{V_y : y \in \overline{B}\}$ is an open cover of $\overline{B}$. Since $Y$ is set strongly star-Lindelöf, there exists a countable subset $F$ of $\overline{B}$ such that $B \subseteq \text{St}(\mathcal{F}, \mathcal{V})$. Since $f$ is finite-to-one, then $f^{-1}(F)$ is a countable subset of $X$. We have to show that

$$A \subseteq \text{St}(f^{-1}(F), \mathcal{U}).$$

Let $x \in A$. Then there exists $y \in B$ such that $f(x) \in V_y$ and $V_y \cap F \neq \emptyset$. Since

$$x \in f^{-1}(V_y) \subseteq \bigcup\{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f(U)$. Thus $U \cap f^{-1}(F) \neq \emptyset$. Hence $x \in \text{St}(f^{-1}(F), \mathcal{U})$. Therefore $X$ is nearly set strongly star-Lindelöf. \hfill $\Box$

**Theorem 3.9.** If $f : X \to Y$ is an open and perfect continuous mapping and $Y$ is a set star-Lindelöf space, then $X$ is nearly set star-Lindelöf.

Proof. Let $A \subseteq X$ be any nonempty set and $\mathcal{U}$ be an open cover of $X$. Then $B = f(A)$ is a subset of $Y$. Let $y \in \overline{B}$. Then $f^{-1}\{y\}$ is a compact subset of $X$, thus there is a finite subset $\mathcal{U}_y$ of $\mathcal{U}$ such that $f^{-1}\{y\} \subseteq \bigcup \mathcal{U}_y$. Let $U_y = \bigcup \mathcal{U}_y$. Then $V_y = Y \setminus f(X \setminus U_y)$ is a neighborhood of $y$, since $f$ is closed. Then $\mathcal{V} = \{V_y : y \in \overline{B}\}$ is an open cover of $\overline{B}$. Since $Y$ is set star-Lindelöf, there exists a countable subset $\mathcal{V}'$ of $\mathcal{V}$ such that

$$B \subseteq \text{St}(\bigcup \mathcal{V}', \mathcal{V}).$$

Without loss of generality, we may assume that $\mathcal{V}' = \{V_{y_i} : i \in N' \subseteq N\}$. Let $\mathcal{W} = \bigcup_{i \in N'} \mathcal{U}_{y_i}$. Since $f^{-1}(V_{y_i}) \subseteq \bigcup\{U : U \in \mathcal{U}_{y_i}\}$ for each $i \in N'$. Then $\mathcal{W}$ is a countable subset of $\mathcal{U}$ and

$$f^{-1}(\bigcup \mathcal{V}') = \bigcup \mathcal{W}.$$  

Next, we show that

$$A \subseteq \text{St}(\bigcup \mathcal{W}, \mathcal{U}).$$

Let $x \in A$. Then there exists a $y \in B$ such that

$$f(x) \in V_y \text{ and } V_y \cap (\bigcup \mathcal{V}') \neq \emptyset.$$  

Since

$$x \in f^{-1}(V_y) \subseteq \bigcup\{U : U \in \mathcal{U}_y\},$$

we can choose $U \in \mathcal{U}_y$ with $x \in U$. Then $V_y \subseteq f(U)$. Thus $U \cap f^{-1}(\bigcup \mathcal{V}') \neq \emptyset$. Hence $x \in \text{St}(f^{-1}(\bigcup \mathcal{V'}), \mathcal{U})$. Therefore $x \in \text{St}(\bigcup \mathcal{W}, \mathcal{U})$, which shows that $A \subseteq \text{St}(\bigcup \mathcal{W}, \mathcal{U})$. Thus $X$ is nearly set star-Lindelöf. \hfill $\Box$

It is known that the product of star-Lindelöf space and compact space is a star-Lindelöf (see [3]).
Problem 3.10. Does the product of set star-Lindelöf space and a compact space is set star-Lindelöf?

However, the product of two set strongly star-Lindelöf spaces need not be set strongly star-Lindelöf. The following well-known example shows that the product of two countably compact (hence, set strongly star-Lindelöf) spaces need not be set star-Lindelöf. Here we give the roughly proof for the sake of completeness.

Example 3.11. There exist two countably compact spaces $X$ and $Y$ such that $X \times Y$ is not set star-Lindelöf (hence, not set strongly star-Lindelöf).

Proof. Let $D(\alpha)$ be a discrete space of the cardinality $\mathfrak{c}$. We can define $X = \bigcup_{\alpha < \omega_1} E_\alpha$ and $Y = \bigcup_{\alpha < \omega_1} F_\alpha$, where $E_\alpha$ and $F_\alpha$ are the subsets of $\beta(D(\alpha))$ which are defined inductively to satisfy the following three conditions:

1. $E_\alpha \cap F_\beta = \emptyset$ if $\alpha \neq \beta$;
2. $|E_\alpha| \leq \mathfrak{c}$ and $|F_\alpha| \leq \mathfrak{c}$;
3. every infinite subset of $E_\alpha$ (resp., $F_\alpha$) has an accumulation point in $E_{\alpha+1}$ (resp., $F_{\alpha+1}$).

Those sets $E_\alpha$ and $F_\alpha$ are well-defined since every infinite closed set in $\beta(D(\alpha))$ has the cardinality $2^\mathfrak{c}$ (see [14]). Then, $X \times Y$ is not set star-Lindelöf, and the diagonal $\{(d, d) : d \in D(\alpha)\}$ is a discrete open and closed subset of $X \times Y$ with the cardinality $\mathfrak{c}$. Thus $X \times Y$ is not set star-Lindelöf, since the open and closed subset of set star-Lindelöf space is set star-Lindelöf and the diagonal $\{(d, d) : d \in D(\alpha)\}$ is not set star-Lindelöf.

Remark 3.12. Example 3.11, shows that the product of set star-Lindelöf space and countably compact space need not be set star-Lindelöf.

van Douwen-Redis-Roscoe-Tree ([3], Example 3.3.3] gave an example of a countably compact $X$ (hence, set star-Lindelöf) and a Lindelöf space $Y$ such that $X \times Y$ is not strongly star-Lindelöf. Now we use this example to show that $X \times Y$ is not set star-Lindelöf.

Example 3.13. There exists a countably compact (hence, set strongly star-Lindelöf) space $X$ and a Lindelöf space $Y$ such that $X \times Y$ is not set star-Lindelöf.

Proof. Let $X = [0, \omega_1]$ with the usual order topology. Let $Y = [0, \omega_1]$ with the following topology. Each point $\alpha < \omega_1$ is isolated and a set $U$ containing $\omega_1$ is open if and only if $Y \setminus U$ is countable. Then, $X$ is countably compact and $Y$ is Lindelöf. It is enough to show that $X \times Y$ is not star-Lindelöf, since every set star-Lindelöf space is star-Lindelöf.

For each $\alpha < \omega_1$, $U_\alpha = X \times \{\alpha\}$ is open in $X \times Y$. For each $\beta < \omega_1$, $V_\beta = [0, \beta] \times (0, \omega_1]$ is open in $X \times Y$. Let $\mathcal{U} = \{U_\alpha : \alpha < \omega_1\} \cup \{V_\beta : \beta < \omega_1\}$. Then $\mathcal{U}$ is an open cover of $X \times Y$. Let $\mathcal{V}$ be any countable subset of $\mathcal{U}$. Since $\mathcal{V}$ is countable, there exists $\alpha' < \omega_1$ such that $U_\alpha \notin \mathcal{V}$ for each $\alpha > \alpha'$. Also, there exists $\alpha'' < \omega_1$ such that $V_\beta \notin \mathcal{V}$ for each $\beta > \alpha''$. Let $\beta = \sup\{\alpha', \alpha''\}$. Then $U_\beta \cap (\bigcup \mathcal{V}) = \emptyset$ and $U_\beta$ is the only element containing $\langle \beta, \alpha \rangle$. Thus $\langle \beta, \alpha \rangle \notin \text{St}(\bigcup \mathcal{V}, \mathcal{U})$, which shows that $X$ is not star-Lindelöf.

van Douwen-Redis-Roscoe-Tree ([3], Example 3.3.6] gave an example of Hausdorff regular Lindelöf spaces $X$ and $Y$ such that $X \times Y$ is star-Lindelöf. Now we use this example and show that the product of two Lindelöf spaces is not set star-Lindelöf.
Example 3.14. There exists a Hausdorff regular Lindelöf spaces $X$ and $Y$ such that $X \times Y$ is not set star-Lindelöf.

Proof. Let $X = \mathbb{R} \setminus \mathbb{Q}$ have the induced metric topology. Let $Y = \mathbb{R}$ with each point of $\mathbb{R} \setminus \mathbb{Q}$ is isolated and points of $\mathbb{Q}$ having metric neighborhoods. Hence both spaces $X$ and $Y$ are Hausdorff regular Lindelöf spaces and first countable too, so $X \times Y$ Hausdorff regular and first countable. Now we show that $X \times Y$ is not set star-Lindelöf. Let $A = \{(x,x) \in X \times Y : x \in X\}$. Then $A$ is uncountable closed and discrete set (see [3], Example 3.3.6). For $(x,x) \in A$, $U_x = X \times \{x\}$ is open subset of $X \times Y$. Then $\mathcal{U} = \{U_x : (x,x) \in \overline{A}\}$ is an open cover of $\overline{A}$. Let $\mathcal{V}$ be any countable subset of $\mathcal{U}$. Then there exists $(a,a) \in A$ such that $(a,a) \notin \bigcup \mathcal{V}$ and thus $(\bigcup \mathcal{V}) \cap U_a = \emptyset$. But $U_a$ is the only element of $\mathcal{U}$ containing $(a,a)$. Thus $(a,a) \notin \text{St}(\bigcup \mathcal{V}, U_a)$, which completes the proof. \hfill $\Box$

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