On constant $U_q(sl_2)$–invariant R–matrices

A. G. Bytsko
Steklov Mathematics Institute
Fontanka 27, 191023, St.-Petersburg, Russia

Abstract

The spectral resolution of a $U_q(sl_2)$–invariant solution $R$ of the constant Yang–Baxter equation in the braid group form is considered. It is shown that, if the two highest coefficients in this resolution are not equal, then $R$ is either the Drinfeld R–matrix or its inverse.

§1. Introduction

Recall that the algebra $U_q(sl_2)$ is generated by the generators $X^+, X^-, q^H, q^{-H}$ satisfying the relations

$$[X^+, X^-] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}, \quad q^H X^\pm = q^{\pm 1} X^\pm q^H, \quad q^{\pm H} q^{\mp H} = 1.$$  

(1)

The homomorphism $\Delta$ which is defined on the generators as follows

$$\Delta(X^\pm) = X^\pm \otimes q^{-H} + q^H \otimes X^\pm, \quad \Delta(q^\pm H) = q^\pm H \otimes q^\pm H,$$

(2)

turns $U_q(sl_2)$ into a bialgebra (moreover, a Hopf algebra [Sk]).

We will consider the standard finite dimensional representation $\pi_s$ of the algebra $U_q(sl_2)$ in which the generators act on the basis vectors $\omega_k$ of a module $V_s$ ($\dim V_s = (2s+1)$, $2s \in \mathbb{N}$) as follows

$$\pi_s(X^\pm) \omega_k = \sqrt{[s \pm k][s \pm k + 1]} \omega_{k \pm 1}, \quad \pi_s(q^\pm H) \omega_k = q^{\pm k} \omega_k,$$

(3)

where $[t] \equiv (q^t - q^{-t})/(q - q^{-1})$ and $k = -s, -s+1, \ldots, s$.

The universal R–matrices for the algebra (1)–(2) are given by [D1]

$$R^\pm = q^{\pm H \otimes H} \sum_{n=0}^{\infty} q^{\frac{1}{2}(n^2 - n)} \prod_{k=1}^{n} [k]_q (\pm (q - q^{-1}) X^\mp \otimes X^\pm)^n q^{\mp H \otimes H}.$$  

(4)

Let $\mathbb{P}$ denote the operator which permutes the tensor components in $U_q(sl_2)^{\otimes 2}$. Then the operator $R \equiv \mathbb{P} R^+ = (R^-)^{-1} \mathbb{P}$ satisfies the Yang–Baxter equation in the braid group form:

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}.  
$$

(5)

The spectral resolution of $R$ in the representation $\pi_s$ is given by [KR1]

$$R \equiv \pi_s^{\otimes 2} (R) = \sum_{k=0}^{2s} \xi_k \mathbb{P}^{2s-k},$$

(6)
Its entries are expressed in terms of the $6–j$ symbols of the algebra $U_q(sl_2)$ as follows:

$$A_{kk'}^{(s,n)} = (-1)^{2s-n} \sqrt{[4s-2k+1]_q [4s-2k'+1]_q} \begin{bmatrix} s & s & 2s-k \\ s & 3s-n & 2s-k' \end{bmatrix}_q.$$
The statement that the Yang–Baxter equation (5) holds when it is reduced onto the subspace $W_n^{(s)}$ is equivalent to the following equality

$$(D_0^{(n)} A^{(s,n)})^3 = (A^{(s,n)} D_0^{(n)})^3.$$  \hspace{1cm} (14)

Actually, however, a stronger statement holds: the r.h.s. and the l.h.s. of (14) are equal up to a multiplicative constant to the identity operator on $W_n^{(s)}$. This follows from the following statement (which is a $q$–analogue of Lemma 3 in [B2]):

**Lemma 1** For all $n = 0, \ldots, |3s|$, the following relation holds:

$$A^{(s,n)} D_0^{(n)} A^{(s,n)} = \theta_n (D_0^{(n)})^{-1} A^{(s,n)} (D_0^{(n)})^{-1},$$  \hspace{1cm} (15)

where $\theta_n \equiv (-1)^n q^{\rho(3s-n)-3\rho(s)}$.

The proof of this and other lemmas is given in the Appendix.

The statement of Lemma 1 can be written in the following form:

$$(R_{12} R_{23} R_{12})|_{W_n^{(s)}} = (R_{23} R_{12} R_{23})|_{W_n^{(s)}} = \theta_n A^{(s,n)}.$$

For $q = 1$ this relation turns into $(P_{13})|_{W_n^{(s)}} = (-1)^n A^{(s,n)}$.

From (16) and (12) it follows that

$$(R_{12} R_{23})^3|_{W_n^{(s)}} = ((R_{23} R_{12})^3)|_{W_n^{(s)}} = q^{2\rho(3s-n)-6\rho(s)}.$$  \hspace{1cm} (17)

Let us note that

$$(R_{12} R_{23} R_{12})^2 = (R_{23} R_{12} R_{23})^2 = (R_{12} R_{23})^3 = (R_{23} R_{12})^3$$

$$= \pi_s^\otimes 3 \left( (R_{12} R_{23} R_{12}^{-1} R_{12}^+ R_{12}^3) \right) = \pi_s^\otimes 3 (\chi_1 \chi_2 \chi_3 \Delta(2)(\chi^{-1})), $$

where the element $\chi$ is constructed in the following way: write the $R$–matrix (4) as $R^+ = \sum r_a^{(1)} \otimes r_a^{(2)}$, and let $S$ stand for the antipode operation, then $\chi = q^{2H} (\sum_a S(r_a^{(2)}) r_a^{(1)})$. It is known [D2] that the element $\chi$ is central, $\pi_s(\chi) = q^{-2\rho(s)}$, and $\chi_1 \chi_2 \Delta(\chi^{-1}) = (R^{-1})^{-1} R^+$. The last relation allows us to derive the last equality in (19) (and its generalization for $\Delta^{(N)}(\chi^{-1})$, see the proof of Lemma 1 in [B3]). Thus, relation (16) can be regarded as the definition of a certain square root of the operator given by the r.h.s. of (19).

§3. Yang–Baxter equation on $W_n^{(s)}$

We will prove Proposition 1 using the following statement (a $q$–analogue of Lemma 4 in [B2]).

**Lemma 2** Let $0 \leq \overline{m} \leq n \leq 2s$, where $\overline{m} \equiv (2s-m)$. The reductions of the operators $P_{12}^m$, $P_{23}^m$, and $R_{12}^{\pm 1}$, $R_{23}^{\pm 1}$ on $W_n^{(s)}$ satisfy the following relations

$$R_l R_{l'} R_l = R_{l'} R_{l'} R_l,$$  \hspace{1cm} (20)

$$P_l^m R_{l'}^m P_l^m = \eta_{\overline{m}, \overline{m}}^2 P_l^m,$$

$$P_{l'}^m R_{l'}^{\pm 1} R_l^m = (\theta m \xi_{\overline{m}}^{2 \pm 1})_{\eta_{\overline{m}, \overline{m}}} P_l^m, \hspace{1cm} R_{l'}^{\pm 1} P_l^m = (\theta m \xi_{\overline{m}}^{-2 \pm 1})_{\eta_{\overline{m}, \overline{m}}} R_{l'}^{\mp 1} P_{l'}^m, \hspace{1cm} (21)$$

$$P_{l'}^m P_l^m R_{l'}^{\pm 1} = (\theta m \xi_{\overline{m}}^{-2 \pm 1})_{\eta_{\overline{m}, \overline{m}}} P_l^m R_{l'}^{\mp 1} P_{l'}^m, \hspace{1cm} R_{l'}^{\pm 1} P_{l'}^m P_l^m = (\theta m \xi_{\overline{m}}^{2 \pm 1})_{\eta_{\overline{m}, \overline{m}}} R_{l'}^{\mp 1} P_{l'}^m.$$  \hspace{1cm} (22)

where $l = \{12\}$, $l' = \{23\}$ or $l = \{23\}$, $l' = \{12\}$, and $\eta_{\overline{m}, \overline{m}} = A^{(s,n)}_{\overline{m}, \overline{m}}$. 
Let us remark that not all relations in Lemma 2 are independent. For instance, the second relation in (21) follows from (22); the first relation in (21) and the second relation in (20) can be derived from each other with the help of (22).

Let us remark also that, for \( q = 1 \), the operators \( R^\pm \) coincide with the permutation operator \( P \), and relations (20)–(22) become the relations of the Brauer algebra [Br] (taking into account the additional relation \( P^2 = E \), where \( E \) is the identity operator). For \( q \neq 1 \), the reductions of the operators \( R^\pm \) onto \( W^{(s)}_{1} \) can be represented as linear combinations of \( P^m \) and the identity operator \( E \). As a consequence, relations (20)–(22) for \( n = 1 \) can be derived from the second relation in (20), which is the defining relation for the Temperley–Lieb algebra [TL]. For \( n \geq 2 \), relations (20)–(22) are the relations that hold in the Birman–Wenzl–Murakami algebra [BW, Mu]. However, in this algebra an additional relation must also hold, which in our case holds only for \( n = 2 \) (the operator \( R^{-1} \) being reduced onto \( W^{(s)}_{2} \) can be represented as a linear combination of the operators \( R, P, \) and \( E \)).

Returning to consideration of the spectral resolution (8), let us note that without a loss of generality we can set \( r_0 = \xi_0 \). Then \( R' \) can be represented in the following form:

\[
R' = R + g P^{2s-n} + \ldots ,
\]

where \( n \geq 1 \) and \( \ldots \) stands for the sum involving projectors of ranks smaller than the rank of \( P^{2s-n} \).

Substitute the ansatz (23) in the Yang–Baxter equation and consider its reduction onto \( W^{(s)}_{n} \) for \( n \leq 2s \). With the help of relations of Lemma 2, it can be verified that the Yang–Baxter equation for \( R' \big|_{W^{(s)}_{n}} \) is equivalent to the following matrix equation

\[
g J + (\theta_n^2 \xi_n^2 \eta_{n,n} g^2 + \eta_{n,n}^2 g^3) G + (\theta_n \xi_n^{-1} \eta_{n,n} g^2) H = 0 ,
\]

where

\[
\begin{align*}
G &= (P^{2s-n} - P^{2s-n}^{12}) \big|_{W^{(s)}_{n}} = \pi^{(n)} - A^{(s,n)} \pi^{(n)} A^{(s,n)}, \\
J &= (R_{12} P^{2s-n}^{12} R_{12} - R_{23} P^{2s-n}^{12} R_{23}) \big|_{W^{(s)}_{n}} \\
&= D^{(n)}_0 A^{(s,n)} \pi^{(n)} A^{(s,n)} D^{(n)}_0 - 2 \eta_n^2 (D^{(n)}_0)^{-1} A^{(s,n)} \pi^{(n)} A^{(s,n)} (D^{(n)}_0)^{-1}, \\
H &= (P^{2s-n} R_{23} - P^{2s-n}^{12} R_{23} - P^{2s-n} R_{12} - R_{12} P^{2s-n}^{12}) \big|_{W^{(s)}_{n}} \\
&= \theta_n^{-1} \xi_n (\pi^{(n)} A^{(s,n)} D^{(n)}_0 + D^{(n)}_0 A^{(s,n)} \pi^{(n)}) \\
&- A^{(s,n)} \pi^{(n)} A^{(s,n)} (D^{(n)}_0)^{-1} - (D^{(n)}_0)^{-1} A^{(s,n)} \pi^{(n)} A^{(s,n)}.
\end{align*}
\]

Here \( \pi^{(n)} \) is a matrix such that \( (\pi^{(n)})_{kk'} = \delta_{kn} \delta_{k'n} \).

**Lemma 3** i) For \( n = 1 \), the following relations hold:

\[
J = (q^2 \xi_0^2 - \xi_0^2 - \xi_0^2) G = (q^{4s(s-1)} - q^{4s^2}) G , \quad H = 2 \xi_0^{-1} G = 2q^{-2s^2} G .
\]

ii) For \( n = 2 \), the matrices \( J \) and \( G \) are linearly independent, and the following relation holds:

\[
\xi_0 \xi_1 H = (\xi_0 + \xi_1) G + (\xi_0 + \xi_1)^{-1} J .
\]

iii) For \( n \geq 3 \), the matrices \( J, G, H \) are linearly independent, and \( J \neq 0 \).
Substituting relations (25) in (24), we infer that, for \( n = 1 \), the coefficient \( g \) must be a root of the following equation:

\[
\eta_{1,1}^2 g^3 + \eta_{1,1} \theta_1 \xi_1^{-1}(\xi_1^{-1} + 2 \xi_0^{-1}) g^2 + (\theta_1^2 \xi_0^{-2} \xi_1^{-2} - \xi_0^2) g = 0.
\]

Hence, taking into account that \( \eta_{1,1} = (q^{2s} + q^{-2s})^{-1} \), we find that, for \( n = 1 \), the coefficient \( g \) can take one of the following values: \( g = 0 \), \( g = q^{2s(s-2)}(1 - q^{8s}) \), \( g = q^{2s(s-2)}(1 - q^{4s}) \). In the first and second cases, the spectral resolution of \( R' \) coincides in the two highest orders with that of \( R \) and \( q^{4s^2}R^{-1} \), respectively. In the third case, we have \( r_1 = r_0 \).

For \( n = 2 \), substitute relations (26) in (24) and eliminate \( H \). It is easy to check that the resulting coefficients at \( J \) and \( G \) vanish if either \( g = 0 \) or

\[
\eta_{1,1} g = -\theta_2 \xi_0^{-1} \xi_1^{-1} \xi_2^{-1} (\xi_0 \xi_1 \xi_2^{-1} + \xi_0 + \xi_1) = -\theta_2^{-1} \xi_0 \xi_1 \xi_2 \xi_0 + \xi_1).
\]

However, the last equality cannot hold because \( \xi_0^2 \xi_1^2 \xi_2^2 = \theta_2^2 \) (see (34)).

For \( n \geq 3 \), the coefficient at \( J \) in (24) vanishes only if \( g = 0 \). Thus, the coefficient \( g \) in (23) must be zero if \( n \geq 2 \). Therefore, if \( R' \) coincides with \( R \) in the two highest orders, then \( R' = R \). An analogous statement can be established if we consider the ansatz (23) with \( R \) being replaced by \( R^{-1} \). Thus, Proposition 1 is proven.

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**Appendix**

**Proof of Lemma 1.**

The 6–j symbols of the algebra \( U_q(sl_2) \) satisfy the following \( q \)–analogue of the Racah identity [KR1, No]:

\[
\sum_p \left( \frac{(-1)^p}{2p+1} \right)_q \left( \begin{array}{lll} r_1 & r_2 & l \\ r_3 & r_4 & p \end{array} \right)_q q^{\rho(p) - \rho(r_1) - \rho(r_4) - \rho(r_3)} \left( \begin{array}{ll} r_1 & r_2 \\ r_3 & r_4 \end{array} \right)_q = \left( \frac{(-1)^{l+l'}}{2l' + 1} \right)_q q^{\rho(r_2) - \rho(l')} \left( \begin{array}{ll} r_3 & r_1 \\ r_2 & r_4 \end{array} \right)_q q^{\rho(r_3) - \rho(l')}.
\]

(Note that the identity remains true if we set \( \rho(t) = -t(t+1) \), since the 6–j symbols are self–dual with respect to the replacement \( q \rightarrow q^{-1} \).)

Consider the matrix entry \( (kk') \) of equality (15). Using formula (10) and taking into account that \( A^{(s,n)} \) is a symmetric matrix, we obtain:

\[
\sum_m (-1)^m A_{km}^{(s,n)} q^{\rho(2s-m) - 2\rho(s)} A_{k'm}^{(s,n)} = (-1)^{n+k+k'} q^{\rho(3s-n) + \rho(s) - \rho(2s-k) - \rho(2s-k')} A_{kk'}^{(s,n)}.
\]

Now, taking into account formula (13), it is easy to see that relation (28) follows from the identity (27) if we set \( r_1 = r_2 = r_3 = s \), \( r_4 = 3s-n \), \( l = 2s-k \), \( l' = 2s-k' \), \( p = 2s-m \).

**Proof of Lemma 2.**

We will prove those relations of Lemma 2 that contain \( R \) on the l.h.s. Their counterparts with \( R^{-1} \) on the l.h.s. can be proven similarly.
The second relation in (20):
\[
\pi(\overline{m})\pi(\overline{m})\pi(\overline{m}) = \pi(\overline{m})A^{(s,n)}\pi(\overline{m})A^{(s,n)}\pi(\overline{m}) = (A^{(s,n)})^2\pi(\overline{m}).
\]
Here and below we denote \(\hat{\pi}(\overline{m}) \equiv A^{(s,n)}\pi(\overline{m})A^{(s,n)}\).

Relations (21):
\[
\pi(\overline{m})\hat{D}_0^{(n)}\pi(\overline{m}) = \pi(\overline{m})A^{(s,n)}D_0^{(n)}A^{(s,n)}\pi(\overline{m}) = \theta_n\pi(\overline{m})(D_0^{(n)})^{-1}A^{(s,n)}(D_0^{(n)})^{-1}\pi(\overline{m})
\]
\[
\equiv \theta_n\xi^2\pi(\overline{m})A^{(s,n)}\pi(\overline{m}),
\]
\[
D_0^{(n)}\pi(\overline{m}) = D_0^{(n)}A^{(s,n)}\pi(\overline{m})A^{(s,n)}D_0^{(n)} = \xi^2 D_0^{(n)}A^{(s,n)}D_0^{(n)}A^{(s,n)}D_0^{(n)}
\]
\[
\equiv \theta_n^2\pi(\overline{m})A^{(s,n)}(D_0^{(n)})^{-1}A^{(s,n)}\pi(\overline{m})A^{(s,n)}(D_0^{(n)})^{-1}A^{(s,n)}
\]
\[
= \theta_n^2\pi(\overline{m}A^{(s,n)}\pi(\overline{m})(D_0^{(n)})^{-1}A^{(s,n)}(D_0^{(n)})^{-1}.
\]
The first relation in (22) (the second can be proven similarly):
\[
\pi(\overline{m})\pi(\overline{m})D_0^{(n)} = \pi(\overline{m})A^{(s,n)}\pi(\overline{m})A^{(s,n)}D_0^{(n)} = A^{(s,n)}\pi(\overline{m})A^{(s,n)}D_0^{(n)}(A^{(s,n)})^2
\]
\[
\equiv \theta_n A^{(s,n)}\pi(\overline{m})(D_0^{(n)})^{-1}A^{(s,n)}(D_0^{(n)})^{-1}A^{(s,n)}
\]
\[
= \theta_n\pi(\overline{m}A^{(s,n)}\pi(\overline{m})(D_0^{(n)})^{-1}.
\]

Proof of Lemma 3.
For \(n = 1\), the matrices \(G, H, J\) are of the size \(2 \times 2\) and relations (25) can be verified straightforwardly using the explicit form of the matrix \(A^{(s,1)}\) (see eq. (73) in [B1]).

In order to examine the case \(n \geq 2\), let us write down explicitly the matrix entries of \(G, H,\) and \(J:\)
\[
G_{kk'} = \delta_{kk'} A^{(s,n)}_k A^{(s,n)}_{k'},
\]
\[
H_{kk'} = \theta^{-1} n(\delta_{kn} A^{(s,n)}_{k'} + \delta_{kk'} A^{(s,n)}_k) - (\xi_k^{-1} + \xi_{k'}^{-1}) A^{(s,n)}_k A^{(s,n)}_{k'},
\]
\[
J_{k'k} = (\xi_k A^{(s,n)}_k - \theta^2 n^2 \xi_k A^{(s,n)}_k A^{(s,n)}_k A^{(s,n)}_k).
\]

Recall that \(k, k' = 0, 1, \ldots, n\).

Considering (31) for \(k = 0\) and \(k' = 0, 1\), it is easy to infer that \(J \neq 0\) (since \(\xi_0^2 \neq \xi_1^2\)).

Assume that the following relation holds
\[
\alpha G + \beta J - \gamma H = 0,
\]
where \(\alpha \beta \gamma \neq 0\). Using formulae (29)–(31), write down the matrix entries of (32) for \((k, k') = (0, 0), (0, 1), (1, 1)\) dividing them by \(A^{(s,n)}_k A^{(s,n)}_{k'}\) (note that \(A^{(s,n)}_k \neq 0\) for all \(k\), see eq. (97) in [B1]):
\[
-\alpha + (\xi_0^2 - \theta^2 n^2 \xi_0^2) \beta + 2\xi_0^2 \gamma = 0,
\]
\[
-\alpha + (\xi_0 \xi_1 - \theta^2 n^2 \xi_0 \xi_1) \beta + (\xi_0^2 + \xi_1^2) \gamma = 0,
\]
\[
-\alpha + (\xi_1^2 - \theta^2 n^2 \xi_1^2) \beta + 2\xi_1^2 \gamma = 0.
\]
The determinant of this system of equations is \(d = (\xi_0^2 - \xi_1^2)^3(\theta^2 n^2 \xi_0^2 - \xi_0^2 \xi_1^2)\). Since \(\xi_0 \neq \xi_1\) then the equality \(d = 0\) can be satisfied only if
\[
\theta_n^2 = \xi_0^2 \xi_1^2 \xi_n^2,
\]
which is equivalent to the following condition: $\rho(3s-n) + 3\rho(s) - \rho(2s) - \rho(2s-1) = 2s(2-n) = 0$. Thus, relation (32) cannot hold for $n \geq 3$.

For $n = 2$, a solution of the system (33) is given by: $\alpha = \beta^{-1} = \xi_0 + \xi_1$, $\gamma = \xi_0\xi_1$. A direct check, using the explicit form of the matrix $A^{(s,2)}$ (see eq. (74) in [B1]), shows that relation (32) with such coefficients holds indeed. Since system (33) has no solution for $\gamma = 0$, we conclude that $G$ and $J$ are linearly independent.

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