AUGMENTED CUP PRODUCTS

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ABSTRACT. In this paper, we present Tate’s theory of augmented cup products in profinite cohomology in a modern constructive style. As an application, we interpret pairings between groups associated to curves constructed by Lichtenbaum, in terms of augmented cup products.

1. Introduction

The notion of augmented cup products was introduced by Tate in the early days of Galois cohomology (in the 1950’s). These are $G$-bilinear maps between cohomology groups. The augmented cup products have similar properties to those of the standard cup product, but they are adjusted and especially useful in situations where one has reciprocity laws, as will be described below.

Recall that given a profinite group $G$, non-negative integers $r$, $s$, and a $G$-bilinear pairing of discrete $G$-modules $A \times B \to C$, the cup product is a bilinear map

$$\cup : H^r(G, A) \times H^s(G, B) \to H^{r+s}(G, C).$$

The augmented cup products are bilinear maps

$$\cup_{\text{aug}} : H^r(G, A') \times H^s(G, B') \to H^{r+s+1}(G, C),$$

where $A'$, $B'$, $C$ come from a Tate product $(A, B, C)$. Here a Tate product $(A, B, C)$ is a data consisting of two short exact sequences of discrete $G$-modules

$$0 \to A' \to A \to A'' \to 0,$$
$$0 \to B' \to B \to B'' \to 0,$$

and two $G$-bilinear maps

$$A' \times B \to C, \quad A \times B' \to C,$$

coinciding on $A' \times B'$. 
The reciprocity mentioned above is expressed in the following example of a Tate product. Suppose that we have an embedding of discrete $G$-modules $A' \hookrightarrow A$, and a $G$-bilinear map of discrete $G$-modules

$$\circ : A' \times A \rightarrow C.$$ 

Suppose that for every $f, g \in A'$ there is a reciprocity law

$$f \circ i(g) = g \circ i(f).$$

Then $(A, A, C)$ forms a Tate product, which gives rise for every $r, s$ to an augmented cup product

$$\cup_{\text{aug}} : H^r(G, A/A') \times H^s(G, A/A') \rightarrow H^{r+s+1}(G, C).$$

An example of such reciprocity is Weil reciprocity law [Lan83, Page 172]: Consider a curve over a separably closed field, and elements $f, g$ in its function field. Let $\text{div}$ be the divisor map mapping each function to its divisor. Suppose that the associated principal divisors $\text{div}(f)$ and $\text{div}(g)$ have disjoint supports. Weil reciprocity law states that

$$f(\text{div}(g)) = g(\text{div}(f)).$$

This will be described in detail in Chapter 5 and will induce the Tate product and the augmented cup product for our geometrical applications.

In general, in various situations with a similar reciprocity (see for example [Lan96, Chapter X, Section 3]), the theory of augmented cup products can be applied to construct pairings between cohomology groups and to study their properties.

The notion of augmented cup products was presented in a 1966 book by Lang [Lan96]. On the other hand, other canonical textbooks of Galois cohomology such as [Ser97], [Koc02], [NSW08], do not mention the augmented cup products at all. In [Mil06] there is a short description of the augmented cup product, but without comprehensive explanations.

In the first part of this paper we give a constructive description of the augmented cup products and develop their general theory in detail. We prove their natural properties such as functoriality, (graded) commutativity, compatibility with group action, and compatibility with other morphisms.

In his book, Lang presents the augmented cup products using a non-constructive approach. He defines the notion for Tate products in an abelian category equipped with an abstract $\delta$-functor to another abelian category. This machinery of $\delta$-functors and dimension shifting is the core of the proofs of the various properties of the augmented cup products in
Lang’s book. These proofs are based on the abstract uniqueness theorem \cite{Lan96}, Chapter 1, Section 1, which enables to deduce isomorphisms between cohomology groups in higher dimensions from those in dimension zero, without explicit calculations. By contrast, in this paper we use a more explicit approach. We prove the various properties of the augmented cup products using detailed calculations. For example, the compatibility of the augmented cup product with the restriction and the corestriction maps, which is proved in \cite{Lan96} using the abstract uniqueness theorem, is proved here very explicitly.

The second part of this paper deals with constructions made by Lichtenbaum in his important 1969 paper \cite{Lic69}. In this paper, Lichtenbaum constructs three canonical pairings between groups associated to curves over fields, and proves that over $p$-adic fields these pairings are perfect. He also mentions, in a few words and without any explanations, that one of those pairings is a special case of the augmented cup product. Our main goal in this part of the paper is to explain this statement. We apply the theory developed in the first part to interpret Lichtenbaum’s pairings in the general context of augmented cup products in Galois cohomology.

More specifically, consider a proper, smooth, geometrically connected curve $X$ over a field $k$ with separable closure $\bar{k}$. Consider the extension by scalars $\bar{X} = X \otimes_k \bar{k}$ of $X$ to $\bar{k}$. Let $G = \text{Gal}(\bar{k}/k)$ be the absolute Galois group of $k$. Let $\text{Pic}(X)$ be the Picard group of $X$, $\text{Pic}_0(X)$ the subgroup of $\text{Pic}(X)$ consisting of the divisor classes of degree zero, and $\text{Br}(X)$ the Brauer group of $X$. Lichtenbaum constructs pairings:

\begin{align*}
(1) & \quad H^0(G, \text{Pic}_0(\bar{X})) \times H^1(G, \text{Pic}_0(\bar{X})) \to \text{Br}(k), \\
(2) & \quad \text{Pic}_0(X) \times H^1(G, \text{Pic}(X)) \to \text{Br}(k), \\
(3) & \quad \text{Pic}(X) \times \text{Br}(X) \to \text{Br}(k),
\end{align*}

and proves their compatibility, using the geometrical properties of the elements involved and computations in the level of cochains.

We show that the pairing (1) is indeed an augmented cup product. Further, we give different proofs, based on the general theory of the augmented cup products, to the existence of the pairing (2), and to the compatibility of the pairing (3) with the former two pairings. We also generalize some of Lichtenbaum’s results by using general cohomological arguments proved in the first part.

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2. Cohomological preliminaries

2.1. Basic profinite cohomology.

We recall some basic cohomological definitions and notations. Standard references for profinite cohomology are [Koc02, Mil06, NSW08, Ser97]. Let \( G \) be a profinite group. Let \( A \) be a discrete \( G \)-module, and let \( r \) be a non-negative integer.

The (non-homogeneous) \( r \)-cochains are the continuous maps \( G^r \to A \).

The group of all \( r \)-cochains is denoted by \( C^r(G, A) \).

The coboundary maps \( d_r : C^r(G, A) \to C^{r+1}(G, A) \) are defined by:

\[
d_r f(x_1, \ldots, x_{r+1}) = x_1 \cdot f(x_2, \ldots, x_{r+1}) + \sum_{i=1}^{r} (-1)^i \cdot f(x_1, \ldots, x_i \cdot x_{i+1}, \ldots, x_{r+1}) + (-1)^{r+1} \cdot f(x_1, \ldots, x_r).
\]

One has \( d_{r+1} \circ d_r = 0 \), i.e., we have a complex:

\[
C^0(G, A) \xrightarrow{d_0} C^1(G, A) \xrightarrow{d_1} C^2(G, A) \xrightarrow{d_2} \cdots
\]

The \( r \)-cocycles are the elements of \( Z^r(G, A) = \text{Ker} \ d_r \).

The \( r \)-coboundaries are the elements of \( B^r(G, A) = \text{Im} \ d_{r-1} \).

The \( r \)th cohomology group of \( G \) (with values in \( A \)) is

\[
H^r(G, A) = Z^r(G, A) / B^r(G, A).
\]

We denote the cohomology class of a cocycle \( f \) by \([f]\).

2.2. Induced morphisms of cochains and cohomology groups.

Let \( G_1 \) and \( G_2 \) be profinite groups. Let \( A_1 \) be a discrete \( G_1 \)-module, and let \( A_2 \) be a discrete \( G_2 \)-module. Suppose that we have a morphism of profinite groups \( \phi : G_2 \to G_1 \), and a morphism of abelian groups \( \psi : A_1 \to A_2 \) such that \((\phi, \psi)\) is a morphism of discrete modules (i.e. \( \psi(\phi(g_2) \cdot a_1) = g_2 \cdot \psi(a_1) \), for \( a_1 \in A_1, g_2 \in G_2 \)). Then for every \( r \) we get an induced morphism of cochains [Koc02, §3.1]:

\[
(\phi, \psi)^* : C^r(G_1, A_1) \to C^r(G_2, A_2)
\]

\[
f \mapsto \psi \circ f \circ \phi.
\]
Here, $\psi \circ f \circ \phi$ denotes the map given by
$$(\sigma_1, \ldots, \sigma_r) \mapsto \psi f(\phi(\sigma_1), \ldots, \phi(\sigma_r)).$$

The induced morphism $(\phi, \psi)^*$ is compatible with the coboundary maps, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
C^r(G_1, A_1) & \xrightarrow{(\phi, \psi)^*} & C^r(G_2, A_2) \\
\downarrow d_r & & \downarrow d_r \\
C^{r+1}(G_1, A_1) & \xrightarrow{(\phi, \psi)^*} & C^{r+1}(G_2, A_2).
\end{array}
$$

Hence, it induces a morphism of cohomology groups
$$(\phi, \psi)^*: H^r(G_1, A_1) \to H^r(G_2, A_2)$$
$[f] \mapsto [\psi \circ f \circ \phi].$

In the special case $G_1 = G_2 = G, \phi = \text{id}_G,$ we abbreviate the induced morphisms by
$$\psi_*: C^r(G, A_1) \to C^r(G, A_2), \quad \psi_*: H^r(G, A_1) \to H^r(G, A_2).$$

For the rest of this paper, let $G, G_1, G_2$ be profinite groups.

For the following proposition we consider a short exact sequence of discrete $G$-modules:
$$0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0,$$
and the short exact sequence of cochains that it induces:
$$0 \to C^r(G, A') \xrightarrow{i_*} C^r(G, A) \xrightarrow{j_*} C^r(G, A'') \to 0.$$

This short exact sequence implies that $C^r(G, A')$ may be identified with the kernel of $j_*.$

**Proposition 2.1.** For the setting as above:

(a) If $f \in C^r(G, A)$ and $j_* f \in Z^r(G, A''),$ then $d_r f \in C^{r+1}(G, A').$

(b) If $f \in C^r(G, A)$ and $j_* f \in B^r(G, A''),$ then $f \in C^r(G, A')$ up to a coboundary, i.e. there exists:
$$\tilde{f} \in C^{r-1}(G, A)$$
such that $f - d_{r-1} \tilde{f} \in C^r(G, A').$

**Proof.** We have the following commutative diagram with exact rows:
2.3. Cup products.
Suppose that we have a $G$-bilinear map of discrete $G$-modules

$$A \times B \to C$$

$$(a, b) \mapsto a \times b.$$  

Namely, the map is bilinear and for $g \in G$ one has

$$g \cdot (a \times b) = g \cdot a \times g \cdot b.$$  

Let $r, s$ be non-negative integers. The cup product of cochains

$$\cup : C^r(G, A) \times C^s(G, B) \to C^{r+s}(G, C),$$

is defined by:

$$\tag{2.2} (f \cup g)(x_1, \ldots, x_{r+s}) := f(x_1, \ldots, x_r) \times x_1 \cdots x_r \cdot g(x_{r+1}, \ldots, x_{r+s}),$$

for every $f \in C^r(G, A), \ g \in C^s(G, B)$.

The "Leibnitz rule" in cohomology [NSW03, Proposition 1.4.1] is

$$\tag{2.3} d_{r+s}(f \cup g) = d_r f \cup g + (-1)^r f \cup d_s g.$$  

If $f$ and $g$ are cocycles then $f \cup g$ is a cocycle as well, since

$$\tag{2.4} d_{r+s}(f \cup g) = 0 \cup g + (-1)^r f \cup 0 = 0 + 0 = 0.$$
If \( f \) is a coboundary and \( g \) is a cocycle, or conversely, then \( f \cup g \) is a coboundary. To see this, take \( \tilde{f} \in C^{r-1}(G, A) \) such that \( d_{r-1} \tilde{f} = f \), and \( g \in Z^s(G, B) \). We have,

\[
d_{r+s-1}(\tilde{f} \cup g) = d_{r-1} \tilde{f} \cup g + (-1)^{r-1} \tilde{f} \cup d_s g = f \cup g + (-1)^{r-1} \tilde{f} \cup 0 = f \cup g,
\]
as required. The case \( f \in Z^r(G, A), g \in B^s(G, B) \), is proved in a similar way.

The cup product of cochains induces the **cup product of cohomology classes**

\[
\cup : H^r(G, A) \times H^s(G, B) \to H^{r+s}(G, C),
\]
defined as follows:

Given \( \alpha \in H^r(G, A), \beta \in H^s(G, B) \), we choose representatives \( f \in Z^r(G, A), g \in Z^s(G, B) \) such that \( \alpha = [f], \beta = [g] \), and set:

\[
(2.6) \quad \alpha \cup \beta := [f \cup g].
\]

It is well defined since \((2.4)\) implies that \( f \cup g \) is indeed a cocycle, and \((2.5)\) implies that the definition is independent of the choice of the representatives.

The cup product of cohomology classes is bilinear, functorial, associative, and graded commutative, in the sense that

\[
\alpha \cup \beta = (-1)^{rs} \cdot \beta \cup \alpha
\]

[Koc02, Theorems 3.25, 3.26, 3.27].

The cup product is functorial in the following sense: Suppose that we have a \( G_1 \)-bilinear map of discrete \( G_1 \)-modules \( A_1 \times B_1 \to C_1 \), and a \( G_2 \)-bilinear map of discrete \( G_2 \)-modules \( A_2 \times B_2 \to C_2 \). Suppose that \( \phi : G_2 \to G_1 \) is a morphism of profinite groups, and that we have morphisms

\[
\psi_A : A_1 \to A_2, \quad \psi_B : B_1 \to B_2, \quad \psi_C : C_1 \to C_2,
\]
such that \((\phi, \psi_A), (\phi, \psi_B), (\phi, \psi_C)\) are morphisms of discrete modules. Suppose that the above morphisms of discrete modules preserve bilinear maps. Namely, that the diagram

\[
(2.7) \quad \begin{array}{ccc}
A_1 & \times & B_1 \\
\downarrow \psi_A & & \downarrow \psi_B \\
A_2 & \times & B_2
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\psi_C \\
\longrightarrow \\
C_1 \\
C_2
\end{array}
\]
is commutative.
Proposition 2.2. In the above setting:

(a) The diagram

\[
\begin{array}{ccc}
C^r(G_1, A_1) & \times & C^s(G_1, B_1) \\
\downarrow_{(\phi, \psi_A)^*} & & \downarrow_{(\phi, \psi_B)^*} \\
C^r(G_2, A_2) & \times & C^s(G_2, B_2)
\end{array}
\xrightarrow{\cup} \begin{array}{c}
C^r+s(G_1, C_1) \\
C^r+s(G_2, C_2)
\end{array}
\]

is commutative.

(b) The diagram

\[
\begin{array}{ccc}
H^r(G_1, A_1) & \times & H^s(G_1, B_1) \\
\downarrow_{(\phi, \psi_A)^*} & & \downarrow_{(\phi, \psi_B)^*} \\
H^r(G_2, A_2) & \times & H^s(G_2, B_2)
\end{array}
\xrightarrow{\cup} \begin{array}{c}
H^r+s(G_1, C_1) \\
H^r+s(G_2, C_2)
\end{array}
\]

is commutative.

Proof. See [NSW08, Proposition 1.4.2].

Suppose that in addition to the setting above we have morphisms

\[\chi : G_1 \to G_2, \quad \xi : B_2 \to B_1,\]

such that \((\chi, \xi)\) is a morphism of discrete modules. Suppose that \(\chi \circ \phi = \text{id}_{G_2}\), and that the diagram

\[
\begin{array}{ccc}
A_1 & \times & B_1 \\
\downarrow_{\psi_A} & & \uparrow_{\xi} \\
A_2 & \times & B_2
\end{array}
\xrightarrow{\psi_C} \begin{array}{c}
C_1 \\
C_2
\end{array}
\]

is commutative.

Proposition 2.3. In the above setting:

(a) The following diagram commutes:

\[
\begin{array}{ccc}
C^r(G_1, A_1) & \times & C^s(G_1, B_1) \\
\downarrow_{(\phi, \psi_A)^*} & & \uparrow_{(\chi, \xi)^*} \\
C^r(G_2, A_2) & \times & C^s(G_2, B_2)
\end{array}
\xrightarrow{\cup} \begin{array}{c}
C^r+s(G_1, C_1) \\
C^r+s(G_2, C_2)
\end{array}
\]

(b) The following diagram commutes:

\[
\begin{array}{ccc}
H^r(G_1, A_1) & \times & H^s(G_1, B_1) \\
\downarrow_{(\phi, \psi_A)^*} & & \uparrow_{(\chi, \xi)^*} \\
H^r(G_2, A_2) & \times & H^s(G_2, B_2)
\end{array}
\xrightarrow{\cup} \begin{array}{c}
H^r+s(G_1, C_1) \\
H^r+s(G_2, C_2)
\end{array}
\]
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Proof. (a) Take $f \in C^r(G_1, A_1)$ and $g \in C^s(G_2, B_2)$. We have on the one hand
\[(\phi, \psi_C)^*(f \cup (\chi, \xi)^* g) = (\phi, \psi_C)^*(f \cup (\xi \circ g \circ \chi)) = \psi_C \circ (f \cup (\xi \circ g \circ \chi)) \circ \phi,\]
and on the other hand
\[(\phi, \psi_A)^*(f) \cup g = (\psi_A \circ f \circ \phi) \cup g.\]
We now show that both cochains are equal. For $(\sigma_1, \ldots, \sigma_{r+s}) \in G_{2+r+s}$ we have
\[(\psi_C \circ (f \cup (\xi \circ g \circ \chi)) \circ \phi)(\sigma_1, \ldots, \sigma_{r+s}) =
\psi_C \circ (f \cup (\xi \circ g \circ \chi))(\phi \sigma_1, \ldots, \phi \sigma_{r+s}) =
\psi_C \circ (f(\phi \sigma_1, \ldots, \phi \sigma_{r}) \times \phi \sigma_r \times (\chi \circ g \circ \chi)(\phi \sigma_{r+1}, \ldots, \phi \sigma_{r+s})).\]
Since $(\chi, \xi)$ is a morphism of discrete modules, the latter expression is equal to
\[\psi_C \circ (f(\phi \sigma_1, \ldots, \phi \sigma_{r}) \times \chi \circ (\chi \phi \sigma_1 \cdots \chi \phi \sigma_{r} \circ g(\chi \phi \sigma_{r+1}, \ldots, \chi \phi \sigma_{r+s}))) =
\psi_C \circ (f(\phi \sigma_1, \ldots, \phi \sigma_{r}) \times \chi \circ (\sigma_1 \cdots \sigma_r \circ g(\sigma_{r+1}, \ldots, \sigma_{r+s})),\]
which by the commutativity of diagram (2.8), is
\[\psi_A \circ f(\phi \sigma_1, \ldots, \phi \sigma_{r}) \times \sigma_1 \cdots \sigma_r \circ g(\sigma_{r+1}, \ldots, \sigma_{r+s}) =
(\psi_A \circ f \circ \phi)(\sigma_1, \ldots, \sigma_r) \times \sigma_1 \cdots \sigma_r \circ g(\sigma_{r+1}, \ldots, \sigma_{r+s}) =
((\psi_A \circ f \circ \phi) \cup g)(\sigma_1, \ldots, \sigma_{r+s}).\]
(b) This follows immediately from (a) and the definition of cup products between cohomology classes \((2.6)\).

Corollary 2.4. For a fixed profinite group $G$, and discrete $G$-module morphisms $\psi_A : A_1 \to A_2, \psi_B : B_2 \to B_1, \psi_C : C_1 \to C_2$, if the diagram
\[
\begin{array}{ccc}
A_1 \times B_1 & \to & C_1 \\
\downarrow \psi_A & & \downarrow \psi_C \\
A_2 \times B_2 & \to & C_2 \\
\end{array}
\]
is commutative, then the diagrams
\[
\begin{array}{ccc}
C^r(G, A_1) \times C^s(G, B_1) & \cup & C^{r+s}(G, C_1) \\
\downarrow \psi_A & & \downarrow \psi_C \\
C^r(G, A_2) \times C^s(G, B_2) & \cup & C^{r+s}(G, C_2),
\end{array}
\]
is also commutative.
3. Definition of the augmented cup product

3.1. Tate Products.

In order to define the augmented cup product we need to define one more notion, and that is the notion of a Tate product.

**Definition 3.1.** A **Tate product** is a data consisting of two short exact sequences of discrete $G$-modules:

\[
0 \to A' \xrightarrow{i} A \xrightarrow{j} A'' \to 0,
\]

\[
0 \to B' \xrightarrow{i} B \xrightarrow{j} B'' \to 0,
\]

and two $G$-bilinear maps:

\[
A' \times B \to C, \quad A \times B' \to C,
\]

coinciding on $A' \times B'$. Such data is denoted by $(A, B, C)$.

This definition can be generalized by considering $A, B, C$ as objects of any abelian category. For our purposes it is sufficient to restrict the definition to the category of $G$-modules as above.

**Example 3.2.** Let $A' \leq A$ and $C$ be discrete $G$-modules. If we have a $G$-bilinear map $\circ : A' \times A \to C$, which commutes on $A' \times A'$, then $(A, A, C)$ forms a Tate product. To see this, consider the short exact sequence

\[
0 \to A' \to A \to A/A' \to 0,
\]

and the pairings

\[
A' \times A \to C, \quad A \times A' \to C
\]

\[
(a', a) \mapsto a' \circ a, \quad (a, a') \mapsto a' \circ a.
\]

The Tate products form a category. Namely, consider Tate products $(A_1, B_1, C_1)$ over a profinite group $G_1$, and $(A_2, B_2, C_2)$ over a profinite group $G_2$. A **morphism of Tate products**

\[
(\phi, \psi) : (A_1, B_1, C_1) \to (A_2, B_2, C_2),
\]
consists of a morphism of profinite groups $\phi : G_2 \to G_1$, and morphisms of abelian groups

$$
\psi_{A'} : A'_1 \to A'_2, \quad \psi_A : A_1 \to A_2, \quad \psi_{A''} : A''_1 \to A''_2,
$$

$$
\psi_{B'} : B'_1 \to B'_2, \quad \psi_B : B_1 \to B_2, \quad \psi_{B''} : B''_1 \to B''_2,
$$

$$
\psi_C : C_1 \to C_2,
$$
such that:

1. The diagrams

$$
\begin{align*}
0 & \to A'_1 \xrightarrow{i} A_1 \xrightarrow{j} A''_1 \to 0 \\
0 & \to A'_2 \xrightarrow{i} A_2 \xrightarrow{j} A''_2 \to 0
\end{align*}
$$

(3.1)

$$
\begin{align*}
0 & \to B'_1 \xrightarrow{i} B_1 \xrightarrow{j} B''_1 \to 0 \\
0 & \to B'_2 \xrightarrow{i} B_2 \xrightarrow{j} B''_2 \to 0
\end{align*}
$$

(3.2)

are exact and commutative.

2. The morphisms

$$
(\phi, \psi_{A'}), (\phi, \psi_A), (\phi, \psi_{A''}), (\phi, \psi_{B'}), (\phi, \psi_B), (\phi, \psi_{B''}), (\phi, \psi_C)
$$

are morphisms of discrete modules.

3. The bilinear maps are preserved. Namely, the following diagrams are commutative:

$$
\begin{array}{ccc}
A'_1 \times B_1 & \to & C_1 \\
\downarrow \psi_{A'} & & \downarrow \psi_C \\
A'_2 \times B_2 & \to & C_2
\end{array}
\qquad
\begin{array}{ccc}
A_1 \times B'_1 & \to & C_1 \\
\downarrow \psi_A & & \downarrow \psi_C \\
A_2 \times B'_2 & \to & C_2
\end{array}
$$

**Remark 3.3.** The $G$-bilinear maps $A' \times B \to C$ and $A \times B' \to C$, give rise to the cup products:

$$
\cup : C^r(G, A') \times C^s(G, B) \to C^{r+s}(G, C),
$$

$$
\cup : C^r(G, A) \times C^s(G, B') \to C^{r+s}(G, C),
$$

$$
\cup : H^r(G, A') \times H^s(G, B) \to H^{r+s}(G, C),
$$

$$
\cup : H^r(G, A) \times H^s(G, B') \to H^{r+s}(G, C).
$$
3.2. The augmented cup product.

For the rest of this paper, we fix a Tate product \((A, B, C)\) over \(G\).

Recall that by the notations in Section 2.2, the morphisms

\[
(3.3) \quad j : A \to A'', \quad j : B \to B'',
\]

induce the morphisms of cochains

\[
(3.4) \quad j_* : C^r(G, A) \to C^r(G, A''), \quad j_* : C^r(G, B) \to C^r(G, B'').
\]

Also recall that \(A'\) and \(B'\) may be identified with the kernels of the morphisms in \((3.3)\), respectively, and that \(C^r(G, A')\) and \(C^r(G, B')\) may be identified with the kernels of the morphisms in \((3.4)\), respectively.

**Definition 3.4.** We define the augmented cup product

\[
\cup_{\text{aug}} : H^r(G, A'') \times H^s(G, B'') \to H^{r+s+1}(G, C)
\]

as follows:

Given \(\alpha'' \in H^r(G, A'')\), \(\beta'' \in H^s(G, B'')\), we choose \(f'' \in Z^r(G, A'')\), \(g'' \in Z^s(G, B'')\) such that \(\alpha'' = [f'']\), \(\beta'' = [g'']\). We choose \(f \in C^r(G, A)\), \(g \in C^s(G, B)\) such that \(j_* f = f''\), \(j_* g = g''\), and define:

\[
\alpha'' \cup_{\text{aug}} \beta'' := [d_r f \cup g + (-1)^r f \cup d_s g].
\]

**Proposition 3.5.** The augmented cup product \(\cup_{\text{aug}}\) is well defined, i.e.:

(a) The cup products in the definition above have meanings.

(b) \(d_r f \cup g + (-1)^r f \cup d_s g\) is a cocycle, so it indeed has a cohomology class.

(c) The construction is independent of the choice of the representatives.

**Proof.** (a) By Proposition 2.1(a), \(d_r f \in C^{r+1}(G, A')\) and \(d_s g \in C^{s+1}(G, B')\).

By Remark 3.3, the cup products \(d_r f \cup g, f \cup d_s g\) have meanings.

(b) By \((2.3)\),

\[
d_{r+s+1}(d_r f \cup g) = d_{r+1}(d_r f) \cup g + (-1)^r d_r f \cup d_s g = (-1)^{r+1} d_r f \cup d_s g,
\]

and

\[
d_{r+s+1}(f \cup d_s g) = d_r f \cup d_s g + (-1)^r f \cup d_{s+1}(d_s g) = d_r f \cup d_s g.
\]

Therefore we have:

\[
d_{r+s+1}(d_r f \cup g + (-1)^r f \cup d_s g) = (-1)^{r+1} d_r f \cup d_s g + (-1)^r d_r f \cup d_s g = 0.
\]

(c) Consider \(f_1, f_2 \in C^r(G, A)\) with \(j_*(f_1), j_*(f_2) \in Z^r(G, A'')\) and \([j_*(f_1)] = [j_*(f_2)] = \alpha''\). Then \(j_*(f_1 - f_2) \in B^r(G, A'')\). By Proposition 2.1(b), \(f_1 - f_2\) is in \(C^r(G, A')\) up to a coboundary. Thus,

\[
f_1 - f_2 = f' + h,
\]
for some \( f' \in C^r(G, A') \) and \( h \in B^r(G, A) \).

Hence,

\[
[d_r f_1 \cup g + (-1)^r f_1 \cup d_s g] - [d_r f_2 \cup g + (-1)^r f_2 \cup d_s g] \\
= [d_r (f_1 - f_2) \cup g + (-1)^r (f_1 - f_2) \cup d_s g] \\
= [d_r (f' + h) \cup g + (-1)^r (f' + h) \cup d_s g] \\
= [d_r f' \cup g + (-1)^r f' \cup d_s g] + [d_r h \cup g + (-1)^r h \cup d_s g].
\]

The separation in the last row to two different cohomology classes is possible, since the corresponding cochains are indeed cocycles (and even coboundaries) as we now show. In view of Remark 3.3, there is a cup product \( f' \cup g \), and (2.3) gives,

\[
[d_r f' \cup g + (-1)^r f' \cup d_s g] = 0.
\]

Further, \( d_r h = 0 \), and by Proposition 2.5, \( h \cup d_s g \) is a coboundary (as a cup product of two coboundaries). Hence,

\[
[d_r h \cup g + (-1)^r h \cup d_s g] = 0.
\]

A similar argument shows that the construction is independent of the choice of different representatives \( g_1, g_2 \in C^s(G, B) \).

**Remark 3.6.** If there exists a \( G \)-bilinear map \( A \times B \rightarrow C \) which induces the maps \( A' \times B \rightarrow C \) and \( A \times B' \rightarrow C \), then the augmented cup product is 0. Indeed, in this case, for \( f, g \) as in Definition 3.4 we have a cup product \( f \cup g \). Hence, by (2.3),

\[
d_{r+s}(f \cup g) = d_r f \cup g + (-1)^r f \cup d_s g.
\]

Thus,

\[
\alpha'' \cup_{\text{aug}} \beta'' = [d_r f \cup g + (-1)^r f \cup d_s g] = [d_{r+s}(f \cup g)] = 0.
\]

**Example 3.7.** We compute explicitly

\[
\cup_{\text{aug}} : H^0(G, A'') \times H^0(G, B'') \rightarrow H^1(G, C).
\]

Take \( \alpha'' \in H^0(G, A'') = (A'')^G \) with representative \( a \in C^0(G, A) = A \), such that \( \alpha'' = [j_\sigma a] \). Similarly, take \( \beta'' \in H^0(G, B'') \) with representative \( b \in B \), such that \( \beta'' = [j_\sigma b] \). We show that:

\[
\alpha'' \cup_{\text{aug}} \beta'' = [h],
\]

where \( h \in Z^1(G, C) \) is defined for \( \sigma \in G \) by:

\[
h(\sigma) = (\sigma a - a) \times \sigma b + a \times (\sigma b - b).
\]

The multiplications on the right are those of the bilinear maps \( A' \times B \rightarrow C \), \( A \times B' \rightarrow C \).
By definition, 
\[ \alpha'' \cup_{\text{aug}} \beta'' = [d_0a \cup b + a \cup d_0b], \]
where the 1-cocycle in the RHS is:
\[ (d_0a \cup b + a \cup d_0b)(\sigma) = (d_0a \cup b)(\sigma) + (a \cup d_0b)(\sigma) = d_0a(\sigma) \times \sigma b + a \times d_0b(\sigma) = (\sigma a - a) \times \sigma b + a \times (\sigma b - b). \]

Note that since
\[ j(\sigma a - a) - ja = \sigma(ja) - ja = ja - ja = 0, \]
\( (\sigma a - a) \) is an element of \( A' = \text{Ker } j \). Similarly, \( (\sigma b - b) \in B' \).

For the rest of this paper, when we write that a cochain \( f \) is a representative of a cohomology class \( \alpha'' \), we mean that it satisfies
\[ \alpha'' = [j_*f]. \]

4. Properies of the augmented cup product

4.1. Graded commutativity.

**Proposition 4.1.** For every \( \alpha'' \in H^r(G, A'') \), \( \beta'' \in H^s(G, B'') \), we have
\[ \alpha'' \cup_{\text{aug}} \beta'' = (-1)^{rs} \cdot \beta'' \cup_{\text{aug}} \alpha''. \]

**Proof.** Let \( f, g \) be representatives of \( \alpha'', \beta'' \) respectively, as in the definition of the augmented cup product. Then:
\[ \alpha'' \cup_{\text{aug}} \beta'' = [d_r f \cup g + (-1)^r f \cup d_s g], \]
\[ \beta'' \cup_{\text{aug}} \alpha'' = [d_s g \cup f + (-1)^s g \cup d_r f]. \]

By the graded commutativity of the cup product [Koc02 Theorem 3.27],
\[ \alpha'' \cup_{\text{aug}} \beta'' = \left[ (-1)^{(r+1)s} g \cup d_r f + (-1)^r (s+1) (-1)^r d_s g \cup f \right] \]
\[ = \left( -1 \right)^{rs} \left[ (-1)^s g \cup d_r f + (-1)^{2r} d_s g \cup f \right] \]
\[ = \left( -1 \right)^{rs} \left[ d_s g \cup f + (-1)^s g \cup d_r f \right] = \left( -1 \right)^{rs} \cdot \beta'' \cup_{\text{aug}} \alpha''. \]

\[ \square \]
4.2. **Compatibility with the \( G \)-action.**

For a discrete \( G \)-module \( A \), define an action of \( G \) on \( C^r(G, A) \) by:

\[
(\sigma \cdot f)(\tau_1, \ldots, \tau_r) = \sigma f(\sigma^{-1}\tau_1, \ldots, \sigma^{-1}\tau_r).
\]

This action is compatible with the coboundary maps, hence it induces an action on \( H^r(G, A) \):

\[
\sigma \cdot [f] = [\sigma \cdot f].
\]

Further, the action commutes with morphisms of discrete modules, and hence with morphisms of cochains. Namely, consider a morphism of profinite groups \( \phi : G_2 \to G_1 \) and a morphism \( \psi : A_1 \to A_2 \) such that \((\phi, \psi)\) is a morphism of discrete modules. For the induced morphism of cochains

\[
(\phi, \psi)^* : C^r(G_1, A_1) \to C^r(G_2, A_2),
\]

the diagram

\[
\begin{array}{ccc}
C^r(G_1, A_1) & \xrightarrow{(\phi, \psi)^*} & C^r(G_2, A_2) \\
\downarrow{\sigma} & & \downarrow{\sigma} \\
C^r(G_1, A_1) & \xrightarrow{(\phi, \psi)^*} & C^r(G_2, A_2)
\end{array}
\]

is commutative. In particular, recall that the Tate product \((A, B, C)\) consists of a \( G \)-module morphism \( j : A \to A'' \). For the induced morphism of cochains \( j_* : C^r(G, A) \to C^r(G, A'') \) we have:

\[
\sigma \cdot (j_* f) = j_* (\sigma \cdot f).
\]

The following proposition shows that the \( G \)-action is compatible with the augmented cup product.

**Proposition 4.2.** For every \( \sigma \in G \), \( \alpha'' \in H^r(G, A'') \), \( \beta'' \in H^s(G, B'') \) we have:

\[
\sigma \cdot (\alpha'' \cup_{\text{aug}} \beta'') = \sigma \cdot \alpha'' \cup_{\text{aug}} \sigma \cdot \beta''.
\]

**Proof.** Let \( f, g \) be representatives of \( \alpha'', \beta'' \), respectively. Since the cup product is \( G \)-bilinear, and the \( G \)-action is compatible with the coboundary maps,

\[
\sigma \cdot (\alpha'' \cup_{\text{aug}} \beta'') = \sigma \cdot [d_r f \cup g + (-1)^r f \cup d_sg]
\]

(4.1)

\[
= [\sigma \cdot (d_r f) \cup \sigma \cdot g + (-1)^r \sigma \cdot f \cup \sigma \cdot (d_sg)]
\]

\[
= [d_r(\sigma \cdot f) \cup \sigma \cdot g + (-1)^r \sigma \cdot f \cup d_s(\sigma \cdot g)].
\]

Since

\[
\sigma \cdot \alpha'' = \sigma \cdot [j_* f] = [\sigma \cdot (j_* f)] = [j_* (\sigma \cdot f)],
\]

\[
\sigma \cdot \beta'' = \sigma \cdot [j_* g] = [\sigma \cdot (j_* g)] = [j_* (\sigma \cdot g)].
\]

\[
\sigma \cdot \alpha'' \cup_{\text{aug}} \sigma \cdot \beta''
\]

\[
= \sigma \cdot (\alpha'' \cup_{\text{aug}} \beta'')
\]

\[
= [\sigma \cdot \alpha'' \cup_{\text{aug}} \sigma \cdot \beta'']
\]

\[
= \sigma \cdot \alpha'' \cup_{\text{aug}} \sigma \cdot \beta''.
\]
σ · f is a representative of σ · α″. Similarly, σ · g is a representative of σ · β″.
Hence, the final expression in (4.1) is
\[ σ · α″ \cup_{\text{aug}} σ · β″. \]

4.3. Functoriality.

By the discussion in Section 3.1, given a morphism of Tate products \((φ, ψ) : (A_1, B_1, C_1) \to (A_2, B_2, C_2)\), we have induced morphisms of cochains and cohomology groups for any two corresponding modules of the Tate products. Namely,
\[
\begin{align*}
(φ_A)^* : C^r(G_1, A'_1) &\to C^r(G_2, A'_2), & H^r(G_1, A'_1) &\to H^r(G_2, A'_2), \\
(φ_A)^* : C^r(G_1, A_1) &\to C^r(G_2, A_2), & H^r(G_1, A_1) &\to H^r(G_2, A_2), \\
(φ_{A''})^* : C^r(G_1, A''_1) &\to C^r(G_2, A''_2), & H^r(G_1, A''_1) &\to H^r(G_2, A''_2), \\
(φ_{B'})^* : C^s(G_1, B'_1) &\to C^s(G_2, B'_2), & H^s(G_1, B'_1) &\to H^s(G_2, B'_2), \\
(φ_{B''})^* : C^s(G_1, B''_1) &\to C^s(G_2, B''_2), & H^s(G_1, B''_1) &\to H^s(G_2, B''_2), \\
(φ_C)^* : C^{r+s}(G_1, C_1) &\to C^{r+s}(G_2, C_2), & H^{r+s}(G_1, C_1) &\to H^{r+s}(G_2, C_2).
\end{align*}
\]
Furthermore, by Proposition 2.2 and condition (3) in the definition of a morphism of Tate products in Section 3.1, the diagrams
\[
\begin{align*}
C^r(G_1, A'_1) \times C^s(G_1, B_1) &\xrightarrow{(φ_A)^* \times (φ_B)^*} C^{r+s}(G_1, C_1), \\
C^r(G_2, A'_2) \times C^s(G_2, B_2) &\xrightarrow{(φ_A)^* \times (φ_B)^*} C^{r+s}(G_2, C_2), \\
C^r(G_1, A_1) \times C^s(G_1, B'_1) &\xrightarrow{(φ_A)^* \times (φ_{B'})^*} C^{r+s}(G_1, C_1), \\
C^r(G_2, A_2) \times C^s(G_2, B'_2) &\xrightarrow{(φ_A)^* \times (φ_{B'})^*} C^{r+s}(G_2, C_2),
\end{align*}
\]
are commutative.

The following proposition shows the functoriality of the augmented cup product.
Proposition 4.3. For a morphism of Tate products \((\phi, \psi) : (A_1, B_1, C_1) \to (A_2, B_2, C_2)\), the diagram

\[
\begin{array}{ccc}
H^r(G_1, A_1'') \times H^s(G_1, B_1'') & \xrightarrow{\cup_{\text{aug}}} & H^{r+s+1}(G_1, C_1) \\
\downarrow^{(\phi, \psi A'')} & & \\
H^r(G_2, A_2'') \times H^s(G_2, B_2'') & \xrightarrow{\cup_{\text{aug}}} & H^{r+s+1}(G_2, C_2)
\end{array}
\]

is commutative.

Proof. Take \(\alpha'' \in H^r(G_1, A_1'')\) and \(\beta'' \in H^s(G_1, B_1'')\) with representatives \(f \in C^r(G_1, A_1)\) and \(g \in C^s(G_1, B_1)\), respectively (namely, \(\alpha'' = [j_* f], \beta'' = [j_* g]\)).

Note that \((\phi, \psi A'')^*(\alpha'') = (\phi, \psi A'')^*([j_* f]) = [\psi A'' \circ j \circ f \circ \phi].\)

By the commutativity of diagram (3.1), the latter expression is equal to \([j \circ \psi_A \circ f \circ \phi] = [j_* (\phi, \psi A)^*(f)].\)

Thus, \((\phi, \psi A)^*(f)\) is a representative of \((\phi, \psi A'')^*(\alpha'')\). Similarly, \((\phi, \psi B)^*(g)\) is a representative of \((\phi, \psi B'')^*(\beta'')\).

By the definition of the augmented cup product,

\[(\phi, \psi C)^*(\alpha'' \cup_{\text{aug}} \beta'') = (\phi, \psi C)^*[d_r f \cup g + (-1)^r f \cup d_s g].\]

By the commutativity of diagrams (4.2) and (4.3), the latter expression is equal to

\[
[(\phi, \psi A)^*(d_r f) \cup (\phi, \psi B)^*(g) + (-1)^r (\phi, \psi A)^*(f) \cup (\phi, \psi B)^*(d_s g)],
\]

which, by the commutativity of diagram (2.1), is equal to

\[
[d_r ((\phi, \psi A)^*(f)) \cup (\phi, \psi B)^*(g) + (-1)^r (\phi, \psi A)^*(f) \cup d_s ((\phi, \psi B)^*(g))] = (\phi, \psi A'')^* \alpha'' \cup_{\text{aug}} (\phi, \psi B'')^* \beta''.
\]

Suppose that we have the following setting: A fixed profinite group \(G\), Tate products \((A_1, B_1, C_1), (A_2, B_2, C_2)\) over \(G\), and discrete \(G\)-module morphisms

\[
\begin{array}{ccc}
\psi_{A'} & i & \psi_A & j & \psi_{A''} \\
\downarrow & \psi_A & \downarrow & \psi_{A''} & \\
0 & A_1' & A_1 & A_1'' & 0 \\
0 & A_2' & A_2 & A_2'' & 0
\end{array}
\]

(4.4)
\[ 0 \to B_1' \xrightarrow{i} B_1 \xrightarrow{j} B_1'' \to 0 \]
\[ 0 \to B_2' \xrightarrow{i} B_2 \xrightarrow{j} B_2'' \to 0, \]
\[ C_1 \xrightarrow{\psi_C} C_2, \]
such that diagrams (4.4) and (4.5) are exact and commutative.

The following proposition shows the functoriality of the augmented cup product in this setting.

**Proposition 4.4.** For the setting as above, if the diagrams

\[ A_1' \times B_1 \to C_1 \quad A_1 \times B_1' \to C_1 \]
\[ A_2' \times B_2 \to C_2, \quad A_2 \times B_2' \to C_2 \]
are commutative, then the diagram

\[ H^r(G, A_1'') \times H^s(G, B_1'') \xrightarrow{\cup_{\text{aug}}} H^{r+s+1}(G, C_1) \]
\[ H^r(G, A_2'') \times H^s(G, B_2'') \xrightarrow{\cup_{\text{aug}}} H^{r+s+1}(G, C_2) \]
is commutative.

**Proof.** By Corollary 2.4, the diagrams

(4.6)
\[ C^r(G, A_1') \times C^s(G, B_1) \xrightarrow{\cup} C^{r+s}(G, C_1) \]
\[ C^r(G, A_2') \times C^s(G, B_2) \xrightarrow{\cup} C^{r+s}(G, C_2), \]
are commutative.

Take \( \alpha'' \in H^r(G, A_1'') \) and \( \beta'' \in H^s(G, B_2'') \) with representatives \( f \in C^r(G, A_1) \) and \( g \in C^s(G, B_2) \), respectively (namely, \( \alpha'' = [j_* f], \beta'' = [j_* g] \)). Note that

\[ \psi_{A''_*} (\alpha'') = \psi_{A''_*} ([j_* f]) = [\psi_{A''} \circ j \circ f]. \]
By the commutativity of diagram (3.1), the latter expression is equal to
\[ [j \circ \psi_A \circ f] = [j_\ast \psi_A \ast (f)]. \]
Thus, \( \psi_A \ast (f) \) is a representative of \( \psi_A \ast (\alpha'') \). Similarly, \( \psi_B \ast (g) \) is a representative of \( \psi_B \ast (\beta'') \).

By the definition of the augmented cup product,
\[ \psi_C \ast (\alpha'' \cup_{\text{aug}} \psi_B \ast (\beta'')) = \psi_C \ast [d_r f \cup \psi_B \ast (g) + (-1)^r f \cup d_s (\psi_B \ast (g))] \]
By the commutativity of diagrams (4.6) and (4.7), the latter expression is equal to
\[ [\psi_A \ast (d_r f) \cup g + (-1)^r \psi_A \ast (f) \cup d_s g] = \]
\[ [d_r (\psi_A \ast (f)) \cup g + (-1)^r \psi_A \ast (f) \cup d_s g] = \psi_A \ast (\alpha'') \cup_{\text{aug}} \beta''. \]

4.4. Compatibility with the connecting homomorphism.
Recall that a short exact sequence of discrete \( G \)-modules
\[ 0 \rightarrow A' \xrightarrow{j} A \xrightarrow{\delta} A'' \rightarrow 0, \]
induces a long exact sequence of cohomology groups
\[ \cdots \rightarrow H^r(G, A') \rightarrow H^r(G, A) \rightarrow H^r(G, A'') \rightarrow H^{r+1}(G, A') \rightarrow H^{r+1}(G, A) \rightarrow \cdots \]
The connecting homomorphism \( \delta \) is defined as follows: Given \( \alpha'' \in H^r(G, A'') \),
take \( f'' \in Z^r(G, A'') \) such that \( \alpha'' = [f''] \). Choose \( f \in C^r(G, A) \) such that \( j_\ast f = f'' \), and set
\[ \delta(\alpha'') = [d_r f] \in H^{r+1}(G, A'). \]
[NSW08, Theorem 1.3.2] shows that \( \delta \) is well defined.

Proposition 4.5. In the following diagram:
\[
\begin{array}{ccc}
H^r(G, A) \times & H^{r+1}(G, B') & \xrightarrow{\cup} & H^{r+s+1}(G, C) \\
\downarrow j_\ast & \delta & \uparrow & \\
H^r(G, A'') \times & H^s(G, B'') & \xrightarrow{\cup_{\text{aug}}} & H^{r+s+1}(G, C) \\
\downarrow \delta & \uparrow & \downarrow j_\ast & \\
H^{r+1}(G, A') \times & H^s(G, B) & \xrightarrow{\cup} & H^{r+s+1}(G, C),
\end{array}
\]

(1) The upper square is commutative of character \((-1)^r\).
(2) The lower square is commutative.
Proof. (1) Take $\alpha \in H^r(G, A), \beta'' \in H^s(G, B'')$. We show that
\[ j_*(\alpha) \cup_{\text{aug}} \beta'' = (-1)^r \alpha \cup \delta(\beta''). \]
Take $f \in Z^r(G, A)$ such that $\alpha = [f]$, and let $g \in C^s(G, B)$ be a representative with $[j_* g] = \beta''$. Since $d_r f = 0$, we have:
\[ j_*(\alpha) \cup_{\text{aug}} \beta'' = [j_* f] \cup_{\text{aug}} [j_* g] = [d_r f \cup g + (-1)^r f \cup d_s g] \]
\[ = ((-1)^r f \cup d_s g) = (-1)^r [f] \cup [d_s g] = (-1)^r \alpha \cup \delta(\beta''). \]

(2) Take $\alpha'' \in H^r(G, A''), \beta \in H^s(G, B)$. We show that
\[ \alpha'' \cup_{\text{aug}} j_*(\beta) = \delta(\alpha'') \cup \beta. \]
Take $g \in Z^s(G, B)$ such that $\beta = [g]$, and let $f \in C^r(G, A)$ be a representative with $[j_* f] = \alpha''$. Since $d_s g = 0$, we have:
\[ \alpha'' \cup_{\text{aug}} j_*(\beta) = [j_* f] \cup_{\text{aug}} [j_* g] = [d_r f \cup g + (-1)^r f \cup d_s g] \]
\[ = [d_r f \cup g] = [d_r f] \cup [g] = \delta(\alpha'') \cup \beta. \]

4.5. Induced pairings.
Suppose that the short exact sequence $0 \to B' \to B \to B'' \to 0$, from the Tate product $(A, B, C)$, can be extended to an exact and commutative diagram of discrete $G$-modules
\[
\begin{array}{ccccccccc}
0 & \rightarrow & B' & \rightarrow & B & \rightarrow & D & \rightarrow & E & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & B'' & \rightarrow & \phi & \rightarrow & D'' & \rightarrow & \psi & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & & 0 & & 0 & & 0 & & 0 & \\
\end{array}
\]
Then we have, for $s \geq 1$, the exact sequences
\[
\begin{align*}
(4.9) \quad H^{s-1}(E) & \xrightarrow{\delta} H^s(B'') \xrightarrow{\phi_*} H^s(D'') \xrightarrow{\psi_*} H^s(E), \\
(4.10) \quad H^{s-1}(E) & \xrightarrow{\delta} H^s(B) \to H^s(D) \to H^s(E), \\
(4.11) \quad H^{s-1}(B'') & \xrightarrow{\delta} H^s(B') \to H^s(B) \to H^s(B''). 
\end{align*}
\]
(in these statements we omit the references to $G$). We use the compatibility of the augmented cup product with the connecting homomorphism (Proposition 4.5), to show that in this case the augmented cup product induces an additional pairing. This will be useful for the geometrical applications is Section 5.

**Theorem 4.6.** Suppose that we have a Tate product $(A, B, C)$ and a commutative diagram as in $(4.8)$. Then:

1. The restricted pairing
   $$\cup_{\text{aug}} : \text{Im}(H^r(A) \xrightarrow{j} H^r(A')) \times \text{Ker}(H^s(B') \xrightarrow{\phi} H^s(D')) \to H^{r+s+1}(C)$$
   is trivial.

2. The augmented cup product $\cup_{\text{aug}}$ induces a bilinear map:
   $$\text{Im}(H^r(A) \xrightarrow{j} H^r(A')) \times \text{Im}(H^s(B') \xrightarrow{\phi} H^s(D')) \to H^{r+s+1}(C).$$

**Proof.** (1) Take $\alpha'' \in \text{Im}(j_* \subseteq H^r(A'))$ and $\alpha \in H^r(A)$ such that $j_*(\alpha) = \alpha''$. Take $\beta'' \in \text{Ker}(\phi_*) \subseteq H^s(B')$. By the exactness of $(4.9)$, there exists $\gamma \in H^{s-1}(E)$ such that $\delta(\gamma) = \beta''$. By Proposition 4.5 (1),

$$\alpha'' \cup_{\text{aug}} \beta'' = (-1)^r \alpha \cup \delta(\beta'') = (-1)^r \alpha \cup \delta(\delta(\gamma)).$$

By diagram $(4.8)$ and the functoriality of the connecting homomorphism [NSW08 Proposition 1.3.3], we have the commutative diagram with an exact column:

$$
\begin{array}{ccc}
H^{s-1}(E) & \xrightarrow{\delta} & H^s(B) \\
\downarrow & & \downarrow \\
H^{s-1}(E) & \xrightarrow{\delta} & H^s(B') \\
& \downarrow{\delta} & \\
& H^{s+1}(B'). & 
\end{array}
$$

Hence, the composition
$$H^{s-1}(E) \xrightarrow{\delta} H^s(B') \xrightarrow{\delta} H^{s+1}(B')$$

is zero. Specifically, $\delta(\delta(\gamma)) = 0$, and we get $(4.12)$

$$\alpha'' \cup_{\text{aug}} \beta'' = 0.$$ 

(2) This follows from (1) and the bilinearity of the augmented cup product. \qed
Corollary 4.7. In the above setting, in the case that $E = \mathbb{Z}$ (with trivial $G$-action), the augmented cup product $\cup_{\text{aug}}$ induces a bilinear map:

$$\text{Im}(H^r(A) \overset{j^*}{\to} H^r(A')) \times H^1(D'') \to H^{r+2}(C).$$

Proof. The group $H^1(\mathbb{Z})$ consists of the continuous homomorphisms from $G$ to $\mathbb{Z}$. Since $G$ is compact and $\mathbb{Z}$ is discrete and torsion-free, there are no such homomorphisms except the trivial homomorphism. Hence, $H^1(\mathbb{Z}) = 0$.

Thus, by the exactness of (4.9), $H^1(B'') \overset{\phi^*}{\to} H^1(D'')$ is surjective, and by Theorem 4.6 (2), we get the requested bilinear map. □

4.6. Compatibility with the restriction and the corestriction maps.

Using the discussion in [Ser97, Chapter I, §2.5], we give the following interpretation of the restriction and the corestriction maps.

Let $H$ be a closed subgroup of $G$, and let $C$ be a discrete $H$-module. The (co-)induced module $\text{Ind}_G^H(C)$ is the group of continuous maps $c^*$ from $G$ to $C$ such that $c^*(hx) = h \cdot c^*(x)$ for every $h \in H$, and $x \in G$. The $G$-module structure of $\text{Ind}_G^H(C)$ is given for $c^* \in \text{Ind}_G^H(C)$ and $g, x \in G$ by

$$(gc^*)(x) = c^*(xg).$$

The functor $\text{Ind}_G^H$ is an exact functor from the category of $H$-modules to the category of $G$-modules [Ser97, Chapter I, §2.5].

The discrete $H$-module morphism

$$e = e_C : \text{Ind}_G^H(C) \to C$$

$$c^* \mapsto c^*(1),$$

together with the inclusion map $\text{incl} : H \hookrightarrow G$, induce the cohomology group homomorphism

$$e^* = (\text{incl}, e)^* : H^r(G, \text{Ind}_G^H(C)) \to H^r(H, C).$$

By Shapiro’s lemma [Ser97, Ch. I, §2.5, Proposition 10], $e^*$ is an isomorphism.

Given a bilinear map of discrete $H$-modules $C_1 \times C_2 \to C_3$, we have a natural bilinear map of discrete $G$-modules

$$\text{Ind}_G^H(C_1) \times \text{Ind}_G^H(C_2) \to \text{Ind}_G^H(C_3),$$

defined by

$$(c_1^* \times c_2^*)(x) = c_1^*(x) \times c_2^*(x).$$
Lemma 4.8. The following diagram commutes:

\[
\begin{array}{ccc}
\text{Ind}_G^H(C_1) \times \text{Ind}_G^H(C_2) & \longrightarrow & \text{Ind}_G^H(C_3) \\
\downarrow e_{C_1} & & \downarrow e_{C_2} \\
C_1 \times C_2 & \longrightarrow & C_3.
\end{array}
\]

Proof. Take \(c_1^* \in \text{Ind}_G^H(C_1)\) and \(c_2^* \in \text{Ind}_G^H(C_2)\). We have
\[
e_{C_3}(c_1^* \times c_2^*) = (c_1^* \times c_2^*)(1) = c_1^*(1) \times c_2^*(1) = e_{C_1}(c_1^*) \times e_{C_2}(c_2^*).
\]

There is a discrete \(H\)-module morphism

\[
i : C \to \text{Ind}_G^H(C)
\]

\[c \mapsto i(c) : G \to C
\]

\[x \mapsto x \cdot c.
\]

It induces the cohomology group homomorphism

\[
i_* : H^r(G, C) \to H^r(G, \text{Ind}_G^H(C)).
\]

The map \(i_*\) coincides with the restriction map \(\text{res} : H^r(G, C) \to H^r(H, C)\), in the sense that the diagram

\[
\begin{array}{ccc}
H^r(G, C) & \overset{i_*}{\longrightarrow} & H^r(G, \text{Ind}_G^H(C)) \\
\overset{\text{res}}{\longrightarrow} & & \overset{\sim}{\longrightarrow} \\
H^r(H, C)
\end{array}
\]

is commutative.

If \(H\) is open in \(G\) (hence of finite index \(n = [G : H]\)), then there is a discrete \(H\)-module morphism

\[
\pi : \text{Ind}_G^H(C) \to C
\]

\[c^* \mapsto \sum_{x \in G/H} x \cdot c^*(x^{-1}),
\]

where \(x\) varies through representatives of the cosets of \(H\) in \(G\). It induces the cohomology group homomorphism

\[
\pi_* : H^r(G, \text{Ind}_G^H(C)) \to H^r(G, C).
\]

The map \(\pi_*\) coincides with the corestriction map \(\text{cor} : H^r(H, C) \to H^r(G, C)\), in the sense that the diagram

\[
\begin{array}{ccc}
H^r(H, C) & \overset{\sim}{\longrightarrow} & H^r(G, \text{Ind}_G^H(C)) \\
\overset{\text{cor}}{\longrightarrow} & & \overset{\pi_*}{\longrightarrow} \\
H^r(G, C)
\end{array}
\]

is commutative.
Lemma 4.9. Given a bilinear map of discrete $H$-modules $C_1 \times C_2 \to C_3$, the following diagram commutes:

$$
\begin{array}{ccc}
\text{Ind}_G^H(C_1) & \times & \text{Ind}_G^H(C_2) \\
\downarrow \pi_{C_1} & & \downarrow \pi_{C_2} \\
C_1 & \times & C_2 \\
\end{array}
\rightarrow
\begin{array}{c}
\text{Ind}_G^H(C_3) \\
\end{array}
$$

Proof. Take $c_1^* \in \text{Ind}_G^H(C_1)$ and $c_2 \in C_2$. We have

$$
\pi_{C_3}(c_1^* \times c_2) = \sum_{x \in G/H} x \cdot (c_1^* \times c_2)(x^{-1}) = \sum_{x \in G/H} (x \cdot c_1^* (x^{-1}) \times x \cdot c_2)(x^{-1})
$$

$$
= \sum_{x \in G/H} (x \cdot c_1^* (x^{-1}) \times x \cdot c_2) = \sum_{x \in G/H} x \cdot c_1^* (x^{-1}) \times c_2 = \pi_{C_1}(c_1^*) \times c_2. \quad \square
$$

Consider a Tate product of discrete $H$-modules $(A, B, C)$. Since $\text{Ind}_G^H$ is an exact functor, we have short exact sequences

$$
0 \to \text{Ind}_G^H(A') \to \text{Ind}_G^H(A) \to \text{Ind}_G^H(A'') \to 0,
$$

$$
0 \to \text{Ind}_G^H(B') \to \text{Ind}_G^H(B) \to \text{Ind}_G^H(B'') \to 0.
$$

Further, we have natural pairings

$$
\text{Ind}_G^H(A') \times \text{Ind}_G^H(B) \to \text{Ind}_G^H(C), \quad \text{Ind}_G^H(A) \times \text{Ind}_G^H(B') \to \text{Ind}_G^H(C),
$$

which coincide on $\text{Ind}_G^H(A') \times \text{Ind}_G^H(B')$, as described above. Thus, we get an induced Tate product of discrete $G$-modules $(\text{Ind}_G^H(A), \text{Ind}_G^H(B), \text{Ind}_G^H(C))$, which gives rise to an augmented cup product

$$
\cup_{\text{aug}} : H^r(G, \text{Ind}_G^H(A')) \times H^s(G, \text{Ind}_G^H(B'')) \to H^{r+s+1}(G, \text{Ind}_G^H(C)).
$$

Theorem 4.10. For an open subgroup $H \leq G$, the diagram

$$
\begin{array}{ccc}
H^r(H, A'') & \times & H^s(H, B'') \\
\downarrow \text{cor} & & \downarrow \text{res} \\
H^r(G, A'') & \times & H^s(G, B'') \\
\end{array}
\rightarrow
\begin{array}{c}
H^{r+s+1}(H, C) \\
\end{array}
$$

is commutative.
Proof. By (4.13) and (4.14), the desired commutativity is equivalent to the commutativity of

\[
\begin{array}{ccc}
H^r(H, A'') & \times & H^s(H, B'') \\
\downarrow_{(e_A^*)^{-1}} & e_B^* \uparrow & \downarrow_{(e_C^*)^{-1}} \\
H^r(G, \text{Ind}_G^H(A'')) & \times & H^s(G, \text{Ind}_G^H(B'')) \\
\downarrow_{\pi_A*} & \uparrow_{i_{B*}} & \downarrow_{\pi_C*} \\
H^r(G, A'') & \times & H^s(G, B'') \\
\longrightarrow & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad "
Chapter 2] for detailed definitions). Let $\bar{X}$ be the extension of $X$ by scalars to $\bar{k}$ (i.e. $\bar{X} = X \otimes_k \bar{k}$), and $\bar{K}$ the function field of $\bar{X}$. The divisor group of a curve $X$, denoted by $\text{Div}(X)$, is the free abelian group on the points of $X$. Namely, its elements are the formal sums $D = \sum_{P \in X} n_P P$, where $n_P \in \mathbb{Z}$, and $n_P = 0$ for all but finitely many points. The support $|D|$ of the divisor $D$ is $\{ P \in X \mid n_P \neq 0 \}$. We have the degree homomorphism
\[ \text{deg} : \text{Div}(X) \to \mathbb{Z} \]
\[ D \mapsto \sum_{P \in X} n_P [k(P) : k]. \]
Here, $k(P)$ is the residue field of $P$. Denote by $\text{Div}_0(X)$ the kernel of $\text{deg}$.

For a curve $\bar{X}$ over $\bar{k}$, there exists $P \in \bar{X}$ with $\bar{k}(P) = \bar{k}$. \cite[Chapter 3, Proposition 2.20]{Liu06}. Hence the degree homomorphism $\text{deg} : \text{Div}(\bar{X}) \to \mathbb{Z}$ is surjective, and we have a short exact sequence
\[ (5.1) \quad 0 \to \text{Div}_0(\bar{X}) \to \text{Div}(\bar{X}) \xrightarrow{\text{deg}} \mathbb{Z} \to 0. \]

Back to the case of a curve $X$ over a general field $k$ (not necessarily separably closed). Consider the divisor map
\[ \text{div} : K^* \to \text{Div}(X) \]
\[ f \mapsto \sum_{P \in X} v_P(f) P. \]
Here, $v_P$ is a valuation on $K$ corresponding to the point $P$. It is a non-trivial discrete valuation which is trivial on $k$.

A divisor $D$ is called a principal divisor if $D \in \text{Im}(\text{div})$. The Picard group $\text{Pic}(X)$ of $X$ is the cokernel of the map $\text{div}$. We have
\[ (1) \quad \text{Ker}(\text{div}) = k^* \quad \text{and} \]
\[ (2) \quad \text{deg} \circ \text{div} = 0 \]
\cite[Chapter 2, Proposition 3.1]{Sil09}. As a consequence, the group $\text{Im}(\text{div})$ of principal divisors is actually a subgroup of $\text{Div}_0(X)$. Let $\text{Pic}_0(X) = \text{Div}_0(X) / \text{Im}(\text{div})$ be the quotient group. Further, we get a well-defined induced degree homomorphism
\[ \text{deg} : \text{Pic}(X) \to \mathbb{Z} \]
\[ E \mapsto \text{deg}(D), \]
where $D$ is a divisor in the class of $E$, and a well-defined induced divisor map
\[
\text{div} : K^*/k^* \to \text{Div}_0(X)
\]
\[
\tilde{f} \mapsto \text{div}(f),
\]
where $f \in K^*$ is in the class of $\tilde{f}$. Thus we have an exact and commutative diagram:
\[
\begin{array}{ccc}
0 & 0 & \\
\downarrow & & \\
0 \to K^*/k^* \xrightarrow{\text{div}} \text{Div}_0(X) \xrightarrow{\text{Pic}_0(X)} \text{Pic}(X) & & 0
\end{array}
\]
(5.2)

By (5.1), we have a short exact sequence
\[
0 \to \text{Pic}_0(X) \to \text{Pic}(X) \xrightarrow{\text{deg}} \mathbb{Z} \to 0,
\]
which induces the exact sequence of cohomology groups
\[
(5.3) \quad H^1(G, \text{Pic}_0(\bar{X})) \to H^1(G, \text{Pic}(\bar{X})) \to H^1(G, \mathbb{Z}).
\]
Recall that $H^1(G, \mathbb{Z}) = 0$ (see the proof of Corollary 4.7). Thus the map $H^1(G, \text{Pic}_0(\bar{X})) \to H^1(G, \text{Pic}(\bar{X}))$ is surjective.

The evaluation of a function $f \in K^*$ at a point $P \in X$ is the result of the substitution of the coordinates of $P$ in the variables of $f$ (recall that $f$ is simply a quotient of two polynomials with coefficients in $k$). Hence we have $f(P) \in k \cup \{\infty\}$. The evaluation of a function $f \in K^*$ at a divisor $D = \sum_{P \in X} n_P P$ with $f(P) \notin \{0, \infty\}$ for every $P$ in the support $|D|$ of $D$ is defined by
\[
f(D) = \prod_{P \in |D|} f(P)^{n_P} \in k^*.
\]
By Weil reciprocity law ([Lan83, Page 172], [Sil09, Exercise 2.11]), for $f, g \in K^*$ with $|\text{div}(f)| \cap |\text{div}(g)| = \emptyset$ we have
\[
f(\text{div}(g)) = g(\text{div}(f)).
\]

The group $G$ acts on $P = (x_1, \ldots, x_n) \in \bar{X}$ by the Galois action on each coordinate, i.e.
\[
\sigma \cdot P = (\sigma(x_1), \ldots, \sigma(x_n)),
\]
and acts on \( D = \sum_{P \in \bar{X}} n_P P \in \text{Div}(\bar{X}) \) by 
\[
\sigma \cdot D = \sum_{P \in \bar{X}} n_P (\sigma \cdot P).
\]
This makes \( \text{Div}(\bar{X}) \) a \( G \)-module.

The Brauer group \( \text{Br}(X) \) of a curve \( X \) is defined in [Lic69, Section 1] as the kernel of the induced homomorphism 
\[
\text{div}_*: H^2(G, \bar{K}^*) \to H^2(G, \text{Div}(\bar{X})).
\]
In [Lic69, Appendix] it is shown that this definition of the Brauer group is equivalent to the usual one using étale cohomology.

By diagram (5.2) and the functoriality of the connecting homomorphism, we have an exact and commutative diagram:
\[
\begin{array}{cccccc}
H^1(G, \text{Div}_0(\bar{X})) & \to & H^1(G, \text{Pic}_0(\bar{X})) & \delta & H^2(G, \bar{K}^*/\bar{k}^*) & \to & H^2(G, \text{Div}_0(\bar{X})) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^1(G, \text{Div}(\bar{X})) & \to & H^1(G, \text{Pic}(\bar{X})) & \delta & H^2(G, \bar{K}^*/\bar{k}^*) & \to & H^2(G, \text{Div}(\bar{X})).
\end{array}
\]
As pointed out in [Lic69, Section 2], \( H^1(G, \text{Div}(\bar{X})) = 0 \), hence the following diagram is exact and commutative:
\[
\begin{array}{cccccc}
0 & & 0 & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Br}(X) & & H^1(G, \text{Pic}(\bar{X})) & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(G, \bar{K}^*) & & H^2(G, \bar{K}^*/\bar{k}^*) & & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^2(G, \text{Div}(\bar{X})) & & H^2(G, \text{Div}(\bar{X})). & & & \\
\end{array}
\]
Let \( \phi: \text{Br}(X) \to H^2(G, \bar{K}^*/\bar{k}^*) \) be the composition of the embedding \( \text{Br}(X) \hookrightarrow H^2(G, \bar{K}^*) \) and the induced homomorphism \( H^2(G, \bar{K}^*) \to H^2(G, \bar{K}^*/\bar{k}^*) \).

By diagram (5.5), for every element in the image of \( \phi \) there exists a unique preimage in \( H^1(G, \text{Pic}(X)) \), and we get an induced homomorphism
\( h : \text{Br}(X) \to H^1(G, \text{Pic}(\bar{X})) \), making the left triangle in the following diagram commutative:

\[
\begin{array}{ccc}
\text{Br}(X) & \xrightarrow{h} & H^1(G, \text{Pic}(\bar{X})) \\
\downarrow{\phi} & & \downarrow{\delta} \\
H^2(G, \bar{K}^*/\bar{k}^*) & \xrightarrow{\delta} & H^2(G, \bar{K}^*/\bar{k}^*).
\end{array}
\]

(5.6)

By diagram (5.4), the right square of diagram (5.6) is also commutative.

**Lemma 5.1.** The image of \( \phi : \text{Br}(X) \to H^2(G, \bar{K}^*/\bar{k}^*) \) is contained in the image of \( \delta : H^1(G, \text{Pic}(\bar{X})) \to H^2(G, \bar{K}^*/\bar{k}^*) \).

**Proof.** It follows from diagram (5.6) and the surjectivity of the map

\[
H^1(G, \text{Pic}_0(\bar{X})) \to H^1(G, \text{Pic}(\bar{X})).
\]

(5.3) \( \square \)

5.2. **The pairing \( \rho_0 \) - Lichtenbaum’s construction.**

Consider the following setting. Let \( X \) be a proper, smooth, geometrically connected curve over a field \( k \), with separable closure \( \bar{k} \) and absolute Galois group \( G \). Let \( \bar{X} \) be the extension of \( X \) by scalars to \( \bar{k} \). Recall that the Brauer group \( \text{Br}(k) \) is canonically isomorphic to \( H^2(G, \bar{k}^*/k^*) \).

Lichtenbaum’s construction of the pairing

\[
\rho_0 : H^0(G, \text{Pic}_0(\bar{X})) \times H^1(G, \text{Pic}_0(\bar{X})) \to \text{Br}(k)
\]

is as follows: Take \( x \in H^0(G, \text{Pic}_0(\bar{X})) \) and \( \alpha \in H^1(G, \text{Pic}_0(\bar{X})) \). Choose \( y \in Z^0(G, \text{Pic}_0(\bar{X})) \) representing \( x \), and choose \( E \in C^0(G, \text{Div}_0(\bar{X})) = \text{Div}_0(\bar{X}) \) mapping onto \( y \). Choose \( a_\sigma \in Z^1(G, \text{Pic}_0(\bar{X})) \) representing \( \alpha \), and choose \( b_\sigma \in C^1(G, \text{Div}_0(\bar{X})) \) mapping onto \( a_\sigma \). The coboundaries of these representatives, namely \( d_0 E \in C^1(G, \text{Div}_0(\bar{X})) \) and \( d_1 b_\sigma \in C^2(G, \text{Div}_0(\bar{X})) \), are in the image of the induced morphism \( \text{div}_* \). Namely, there exist \( f_{\sigma,\tau} \in C^2(G, \bar{K}^*/\bar{k}^*) \) and \( g_{\sigma} \in C^1(G, \bar{K}^*/\bar{k}^*) \) such that \( \text{div}_*(g_{\sigma}) = d_0 E \) and \( \text{div}_*(f_{\sigma,\tau}) = d_1 b_\sigma \). It follows from Proposition 2.1(a). Now define

\[
\rho_0(x, \alpha) = [g_{\sigma}(\sigma b_{\tau}) \cdot f_{\sigma,\tau}(E)].
\]

Lichtenbaum shows directly that this is independent of the various choices made and that \( g_{\sigma}(\sigma b_{\tau}) \cdot f_{\sigma,\tau}(E) \) is indeed an element of \( Z^2(G, \bar{k}^*) \), so it has a cohomology class which is in \( H^2(G, \bar{k}^*) = \text{Br}(k) \).
5.3. The pairing $\rho_0$ - interpretation in terms of the augmented cup product.

Let $\bar{K}$ be the field of rational functions on $\bar{X}$. Consider the short exact sequence of $G$-modules,

$$0 \to \bar{K}^*/\bar{k}^* \overset{\text{div}}{\to} \text{Div}_0(\bar{X}) \overset{j}{\to} \text{Pic}_0(\bar{X}) \to 0.$$ 

We have a natural pairing:

$$\bar{K}^*/\bar{k}^* \times \text{Div}_0(\bar{X}) \to \bar{k}^*$$

(5.7)

$$(f, D) \mapsto f(D).$$

Due to Weil reciprocity law, this pairing commutes on $\bar{K}^*/\bar{k}^* \times \bar{K}^*/\bar{k}^*$. Thus, in view of Example 3.2, we get a Tate product, and can define for every non-negative integers $r, s$ an augmented cup product:

$$\cup_{\text{aug}} : H^r(\bar{G}, \text{Pic}_0(\bar{X})) \times H^s(\bar{G}, \text{Pic}_0(\bar{X})) \to H^{r+s+1}(\bar{G}, \bar{k}^*).$$

In particular, we get an augmented cup product:

$$\cup_{\text{aug}} : H^0(\bar{G}, \text{Pic}_0(\bar{X})) \times H^1(\bar{G}, \text{Pic}_0(\bar{X})) \to H^2(\bar{G}, \bar{k}^*) = \text{Br}(k).$$

We now show that Lichtenbaum’s $\rho_0$ is exactly this augmented cup product. Consider $x \in H^0(\bar{G}, \text{Pic}_0(\bar{X}))$ and $\alpha \in H^1(\bar{G}, \text{Pic}_0(\bar{X}))$ as before. In order to compute $x \cup_{\text{aug}} \alpha$, we should choose representatives $E \in C^0(\bar{G}, \text{Div}_0(\bar{X}))$ such that $x = [j_*E]$, and $b \in C^1(\bar{G}, \text{Div}_0(\bar{X}))$ such that $\alpha = [\bar{b}].$ Note that these are the same representatives from Lichtenbaum’s construction. As shown in Proposition 2.1(a), the coboundaries of $b$ and $E$ lie in the kernel of $j_*$. Use the following commutative diagram with exact rows to see this:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(\bar{G}, \bar{K}^*/\bar{k}^*) & \overset{\text{div}}{\longrightarrow} & C^0(\bar{G}, \text{Div}_0(\bar{X})) & \overset{j_*}{\longrightarrow} & C^0(\bar{G}, \text{Pic}_0(\bar{X})) & \longrightarrow & 0 \\
 & & \downarrow{d_0} & & \downarrow{d_0} & & \downarrow{d_0} & & \\
0 & \longrightarrow & C^1(\bar{G}, \bar{K}^*/\bar{k}^*) & \overset{\text{div}}{\longrightarrow} & C^1(\bar{G}, \text{Div}_0(\bar{X})) & \overset{j_*}{\longrightarrow} & C^1(\bar{G}, \text{Pic}_0(\bar{X})) & \longrightarrow & 0 \\
 & & \downarrow{d_1} & & \downarrow{d_1} & & \downarrow{d_1} & & \\
0 & \longrightarrow & C^2(\bar{G}, \bar{K}^*/\bar{k}^*) & \overset{\text{div}}{\longrightarrow} & C^2(\bar{G}, \text{Div}_0(\bar{X})) & \overset{j_*}{\longrightarrow} & C^2(\bar{G}, \text{Pic}_0(\bar{X})) & \longrightarrow & 0
\end{array}
$$

Hence, we can take $f \in C^2(\bar{G}, \bar{K}^*/\bar{k}^*)$ with $\text{div}_*(f) = d_1b$, and $g \in C^1(\bar{G}, \bar{K}^*/\bar{k}^*)$ with $\text{div}_*(g) = d_0E$. Since $\text{div}_*$ is an embedding, we may identify $f$ with $d_1b$, and $g$ with $d_0E$.

By the definition of the augmented cup product, we have:

$$x \cup_{\text{aug}} \alpha = [d_0E \cup b + (-1)^0 \cdot E \cup d_1b] = [g \cup b + E \cup f].$$
Computing the cup products by its definition, and passing from additive to multiplicative writing (for the multiplicative group $\bar{k}^*$), we get:

$$x \cup_{\text{aug}} \alpha = [(g \cup b)(\sigma, \tau) \cdot (E \cup f)(\sigma, \tau)] = [(g \sigma \times \sigma b \tau) \cdot (E \times f_{\sigma, \tau})] = [g \sigma (\sigma b \tau) \cdot f_{\sigma, \tau}(E)] = \rho_0(x, \alpha).$$

Thus, $\rho_0$ is indeed an example of an augmented cup product, and it is well defined by the general theory (Proposition 3.5).

5.4. The pairing $\rho$ - Lichtenbaum’s construction.

We now study the pairing

$$\rho : \text{Pic}_0(X) \times H^1(G, \text{Pic}(\bar{X})) \to \text{Br}(k).$$

Lichtenbaum defines this pairing as follows. He considers the short exact sequence of discrete $G$-modules

$$0 \to \text{Pic}_0(\bar{X}) \to \text{Pic}(\bar{X}) \xrightarrow{\text{deg}} \mathbb{Z} \to 0,$$

and the exact sequence of cohomology groups it induces:

$$(5.9) \quad \mathbb{Z} = H^0(G, \mathbb{Z}) \xrightarrow{\delta} H^1(G, \text{Pic}_0(\bar{X})) \xrightarrow{\phi} H^1(G, \text{Pic}(\bar{X})) \to H^1(G, \mathbb{Z}) = 0.$$

Since $\text{Pic}_0(X) \subseteq H^0(G, \text{Pic}_0(\bar{X}))$ and $\phi$ is surjective, by proving that the restricted pairing

$$\rho_0 : \text{Pic}_0(X) \times \text{Ker}(\phi) \to \text{Br}(k)$$

is trivial, he gets that $\rho_0$ induces a pairing $\rho$ as above. Lichtenbaum’s proof of the triviality of the restricted pairing is by computations based on the geometrical properties of the divisors groups (see [Lic69, Section 4] for details).

5.5. The pairing $\rho$ - interpretation in terms of the augmented cup product.

We now show that this construction is just a specific example of a more general result from the theory of the augmented cup product that we have developed in Chapter 4. The essence of the generalization is to prove that the restricted pairing is trivial, using general cohomological arguments as in Theorem 4.6 (1). Recall that $\rho_0$ is an augmented cup product, and that the short exact sequence of the Tate product which gives rise to it is:

$$0 \to \bar{k}^*/\bar{k}^* \to \text{Div}_0(\bar{X}) \to \text{Pic}_0(\bar{X}) \to 0.$$
This short exact sequence can be extended to an exact and commutative diagram

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\bar{K}^*/\bar{k}^* & \bar{K}^*/\bar{k}^* \\
\downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
0 & \text{Div}_0(\bar{X}) & \text{Div}(\bar{X}) & \text{deg} & \mathbb{Z} & 0 \\
\downarrow & \downarrow & \downarrow & \| & \| & \| \\
0 & \text{Pic}_0(\bar{X}) & \text{Pic}(\bar{X}) & \text{deg} & \mathbb{Z} & 0 \\
\end{array}
\]

Hence, the assumptions of Corollary 4.7 are satisfied, and we get that the augmented cup product

\[
\cup_{\text{aug}} : H^0(G, \text{Pic}_0(\bar{X})) \times H^1(G, \text{Pic}_0(\bar{X})) \to H^2(G, \bar{k}^*) = \text{Br}(k),
\]

induces a bilinear map

\[
\text{Im}(H^0(G, \text{Div}_0(\bar{X})) \to H^0(G, \text{Pic}_0(\bar{X}))) \times H^1(G, \text{Pic}(\bar{X})) \to \text{Br}(k).
\]

Since \( H^0(G, \text{Div}_0(X)) = \text{Div}_0(X) \), the image of \( H^0(G, \text{Div}_0(X)) \to H^0(G, \text{Pic}_0(\bar{X})) \) is \( \text{Pic}_0(X) \). Thus, we get the requested pairing:

\[
\rho : \text{Pic}_0(X) \times H^1(G, \text{Pic}(\bar{X})) \to \text{Br}(k).
\]

Note that this is the same induced pairing as in Lichtenbaum’s paper.

5.6. **The pairing** \( \psi \).

This pairing is not a special case of the augmented cup product as the former two, but we show its compatibility with the previous pairings using the general theory from Chapter 4. The pairing is

\[
\psi : \text{Div}(X) \times \text{Br}(X) \to \text{Br}(k)
\]

\[(D, \alpha) \mapsto [f_{\alpha, \tau}(D)],\]
where \( f_{\sigma,\tau} \in Z^2(G, \bar{K}^*) \) is a 2-cocycle representing \( \alpha \). Lichtenbaum shows that \( \psi \) is well-defined [Lic69, Section 3], and that \( \psi \) vanishes when \( D \) is a principal divisor [Lic69, Theorem 1]. Hence, \( \psi \) induces a pairing:

\[
\psi : \text{Pic}(X) \times \text{Br}(X) \to \text{Br}(k)
\]

\[(E, \alpha) \mapsto [f_{\sigma,\tau}(D)].\]

Here \( D \) is a divisor mapping onto \( E \), and \( f_{\sigma,\tau} \) is as above.

### 5.7. Compatibility of \( \psi \) with \( \rho \)

Lichtenbaum proves [Lic69, Section 4] that \( \psi \) and \( \rho \) are compatible, in the sense of the following proposition. We prove this compatibility using the general theory of the augmented cup products.

**Proposition 5.2.** Let \( h : \text{Br}(X) \to H^1(G, \text{Pic}(\bar{X})) \) be the homomorphism from diagram (5.6). The following diagram is commutative:

\[
\begin{array}{ccc}
\rho : & \text{Pic}_0(X) \times H^1(G, \text{Pic}(\bar{X})) & \longrightarrow \text{Br}(k) \\
& \downarrow & \uparrow h & \| \\
\psi : & \text{Pic}(X) \times \text{Br}(X) & \longrightarrow \text{Br}(k).
\end{array}
\]

**Proof.** First note that the pairing (5.7), which gives rise to our augmented cup product \( \rho_0 \), induces a cup product

\[
\cup : \text{Div}_0(X) = H^0(G, \text{Div}_0(\bar{X})) \times H^2(G, \bar{K}^*/\bar{k}^*) \to H^2(G, \bar{k}^*) = \text{Br}(k)
\]

\[(D, [f_{\sigma,\tau}]) \mapsto [f_{\sigma,\tau}(D)].\]

By the definitions of the pairings (5.12) and (5.14), the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Div}_0(X) \times H^2(G, \bar{K}^*/\bar{k}^*) & \xrightarrow{\cup} \text{Br}(k) \\
\downarrow & \phi & \| \\
\text{Pic}(X) \times \text{Br}(X) & \xrightarrow{\psi} \text{Br}(k).
\end{array}
\]

Further, by (5.11) and (5.8), the diagram

\[
\begin{array}{ccc}
\text{Pic}_0(X) \times H^1(G, \text{Pic}(\bar{X})) & \longrightarrow \text{Br}(k) \\
\downarrow & \| \\
H^0(G, \text{Pic}_0(\bar{X})) \times H^1(G, \text{Pic}_0(\bar{X})) & \xrightarrow{\rho_0 \cup_{\text{aug}}} \text{Br}(k)
\end{array}
\]
is commutative. Finally, by Proposition 4.5 (1), the diagram

\[
\begin{array}{ccc}
H^0(G, \text{Pic}_0(\bar{X})) & \times & H^1(G, \text{Pic}_0(\bar{X})) \\
\uparrow & & \downarrow \delta \\
\text{Div}_0(X) & \times & H^2(G, K^*/\bar{k}^*) \\
\downarrow & & \uparrow \\
\text{Pic}(X) & \times & \text{Br}(X)
\end{array}
\] (5.17)

is also commutative. Combining (5.15), (5.16) and (5.17), we get a commutative diagram:

\[
\begin{array}{ccc}
\text{Pic}_0(X) & \times & H^1(G, \text{Pic}(\bar{X})) \\
\downarrow & & \downarrow \rho \\
H^0(G, \text{Pic}_0(\bar{X})) & \times & H^1(G, \text{Pic}_0(\bar{X})) \\
\uparrow & & \uparrow \\
\text{Div}_0(X) & \times & H^2(G, K^*/\bar{k}^*) \\
\downarrow & & \uparrow \phi \\
\text{Pic}(X) & \times & \text{Br}(X)
\end{array}
\] (5.18)

We call a 4-tuple of elements in

\[ \text{Pic}_0(X) \times H^0(G, \text{Pic}_0(\bar{X})) \times \text{Div}_0(X) \times \text{Pic}(X) \]

compatible if its elements map one to each other under the relevant maps in the left column of (5.18). Note that every element of \( \text{Pic}_0(X) \) can be completed to a compatible 4-tuple (since the map \( \text{Div}_0(X) \to \text{Pic}_0(X) \) is surjective (diagram (5.2))).

Similarly, we call a 4-tuple of elements in

\[ H^1(G, \text{Pic}(\bar{X})) \times H^1(G, \text{Pic}_0(\bar{X})) \times H^2(G, K^*/\bar{k}^*) \times \text{Br}(X) \]

compatible if its elements map one to each other under the relevant maps in the middle column of (5.18). By Lemma 5.1, every element of \( \text{Br}(X) \) can be completed to a compatible 4-tuple. By the commutativity of (5.6), composing the maps in the middle column of (5.18) is equivalent to applying the homomorphism \( h \), namely we have a commutative diagram:

\[ \text{Br}(X) \xrightarrow{\delta} H^2(G, K^*/\bar{k}^*) \xleftarrow{\delta} H^1(G, \text{Pic}_0(\bar{X})) \xrightarrow{\rho} H^1(G, \text{Pic}(\bar{X})). \]

Now given elements of \( \text{Pic}_0(X) \) and \( \text{Br}(X) \), we complete them into compatible 4-tuples as above, and then use the commutativity of (5.18) to conclude that (5.13) commutes, as desired. \( \square \)
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