Effective perihelion advance and potentials in a conformastatic background with magnetic field

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An Exact solution of the Einstein-Maxwell field equations for a conformastatic metric with magnetized sources is studied. In this context, effective potential are studied in order to understand the dynamics of the magnetic field in galaxies. We derive the equations of motion for neutral and charged particles in a spacetime background characterized by this class of solutions. In this particular case, we investigate the main physical properties of equatorial circular orbits and related effective potentials. In addition, we obtain an effective analytic expression for the perihelion advance of test particles. Our theoretical predictions are compared with the observational data calibrated with the ephemerides of the planets of the Solar system and the Moon (EPM2011). We show that, in general, the magnetic punctual mass predicts values that are in better agreement with observations than the values predicted in Einstein gravity alone.

I. INTRODUCTION

Magnetics field are extensively studied in literature and its influence on the dynamics are on currently field of research, e.g, on the understanding the galactic jets and inner process of “active” galaxy core, neutron stars dynamics \[1\]. A interesting review can be found in \[2\]–[6] and/or movement of charged particles in spacetimes \[7\]–\[9\] or neutral particles in charged galactic halo \[10\]–\[12\]. The Einstein Maxwell equations have been revealed to be an important tool to deal with this problem and helping us on the understanding the dynamics of magnetic fields in galaxies. Important approaches are the relativistic models with disk like configurations and relativistic disk accretion models proposed in recent years, e.g, \[13\]–\[15\] and references therein. In a recent publication \[15\], we studied the behaviour of a test particle submitted to a magnetic field in a relativistic galaxy disk model and on how its influence may affect its dynamics. In this paper, we investigate effective potentials using Einstein-Maxwell equations motivated by the necessity to understand how the dynamics of a galaxy responds to flattening and how the magnetic field can be related to this process.

The present paper is divided in sections. In the second section, we study the basic framework of a conformastatic background and investigate some applications using the isothermal-sphere logarithm potential and Toomre-Kuzmin-like potential, which are compatible with axisymmetric systems. In the third section, we obtain an expression for the perihelion advance of a charged test particle in a generic conformastatic spacetime in the presence of a magnetic field and perform a comparison between our results, the results from Einstein gravity alone and the values observed for the secular perihelion precession of some inner planets and minor objects of the Solar System. In the conclusion section, we make the final considerations.

II. THE CONFORMASTATIC BACKGROUND

Considering the background of a conformastatic gravitational source in presence of a magnetic field described by the line element in standard cylindrical coordinates \[14\],

\[
dS^2 = -e^{2\phi}dt^2 + e^{-2\phi}(dr^2 + dz^2 + r^2d\phi^2),
\]  

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where the metric potential \( \phi \) depend only on the \( r \) and \( z \). We use the term conformastatic in the sense of Synge [19]. The vacuum Einstein-Maxwell equations in geometrized units such that \( c = 8 \pi G = \mu_0 = \epsilon_0 = 1 \), are given by

\[
G_{\alpha\beta} = E_{\alpha\beta}, \quad (2a)
\]

\[
F^{\alpha\beta}_{\ ;\beta} = 0, \quad (2b)
\]

where \( F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \) and \( E_{\alpha\beta} \) is the electromagnetic energy-momentum tensor

\[
E_{\alpha\beta} = \frac{1}{4\pi} \left\{ F_{\alpha\gamma}F_{\beta}^{\gamma} - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \right\}.
\]

The Greek indices run from 1 to 4.

With the electromagnetic potential \( A_\alpha = (0, 0, 0, A_\phi(r, z)) \) and the line element in Eq.(1) the Einstein-Maxwell equations in Eq.(2) are equivalent to the system of equations

\[
\nabla \cdot \left( r^{-2} e^{2\phi} \nabla A_\phi \right) = 0, \quad (3a)
\]

\[
\nabla^2 \phi - \nabla \phi \cdot \nabla \phi = 0, \quad (3b)
\]

\[
\phi_r^2 - \frac{1}{2r^2} e^{2\phi} A_{r,z}^2 = 0, \quad (3c)
\]

\[
\phi_z^2 - \frac{1}{2r^2} e^{2\phi} A_{r,z}^2 = 0, \quad (3d)
\]

\[
\phi_r \phi_z + \frac{1}{2r^2} e^{2\phi} A_{r,r} A_{\phi,z} = 0. \quad (3e)
\]

By using the procedure to obtain solutions of the Einstein-Maxwell equations presented in [14], a suitable solutions of the system in Eq.(3) can be displayed as

\[
\phi = \frac{1}{1 - U}, \quad (4a)
\]

\[
A_{\phi,r} = \sqrt{2r} U_{,z}, \quad (4b)
\]

\[
A_{\phi,z} = -\sqrt{2r} U_{,r}, \quad (4c)
\]

where \( U(r, z) \) is a solution of the Laplace’s equation.

### A. Motion of test charged particles

The motion of a test particle of charge \( q \) and mass \( m \) moving in a magnetized background is described by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} m u_\mu u^\mu + q A_\alpha u^\alpha, \quad (5)
\]

where \( u^\mu = dx^\mu/ds \), being \( s \) and arbitrary parameter. The corresponding Hamiltonian of the particle is

\[
\mathcal{H} = \frac{1}{2m} (p^\mu - q A^\mu)(p_\mu - q A_\mu), \quad (6)
\]

where the canonical momentum is given by \( p_\mu = mu_\mu + q A_\mu \). The motion equations are given by

\[
u^\mu = \frac{\partial \mathcal{H} g^{\mu\nu}}{\partial p^\nu} \quad (7a)
\]

\[
dp^\mu ds = - \frac{\partial \mathcal{H} g^{\mu\nu}}{\partial p^\nu} \quad (7b)
\]
where $\mathcal{H}_c \equiv p_\mu p^\mu/(2m)$. Accordingly, by introducing Eq.\((6)\) into Eq.\((7)\) we obtain

\[
\begin{align*}
\frac{dp^t}{ds} &= 0, \quad (8a) \\
\frac{dp^\phi}{ds} &= 0, \quad (8b) \\
\frac{dp^r}{ds} &= \frac{p^\mu p_\mu}{2m} \frac{\partial g^{rr}}{\partial r}, \quad (8c) \\
\frac{dp^z}{ds} &= \frac{p^\mu p_\mu}{2m} \frac{\partial g^{zz}}{\partial z}. \quad (8d)
\end{align*}
\]

From Eq.\((6)\) and the normalization condition $u^\mu u_\mu = -\varepsilon$ (with $\varepsilon = (1, 0, -1)$ for space-like, null and time-like curves) we have the condition

\[
\mathcal{H} = -\frac{1}{2} m \varepsilon. \quad (9)
\]

On the another hand, from Eq.\((8a)\) and Eq.\((8b)\) we have,

\[
p^t = \text{constant} \equiv -E, \quad (10)
\]

and also

\[
p^\phi = \text{constant} \equiv L, \quad (11)
\]

respectively. Whereas, from Eq.\((8c)\) and Eq.\((8d)\) we obtain

\[
\begin{align*}
\ddot{r} &= W \frac{\partial g^{rr}}{\partial r}, \quad (12a) \\
\ddot{z} &= W \frac{\partial g^{zz}}{\partial z}. \quad (12b)
\end{align*}
\]

where

\[
W \equiv \frac{1}{2} \left( \varepsilon + \frac{q^2}{m^2} A_\phi A^\phi - \frac{2qL}{m^2} A_\phi \right). \quad (13)
\]

We can write the last system in the form

\[
\begin{align*}
\ddot{r} &= -\frac{\partial \Phi_{eff}}{\partial r}, \quad (14a) \\
\ddot{z} &= -\frac{\partial \Phi_{eff}}{\partial z}. \quad (14b)
\end{align*}
\]

where

\[
d\Phi_{eff} = W_r \frac{\partial g^{rr}}{\partial r} dr + W_z \frac{\partial g^{zz}}{\partial z} dz.
\]

$\Phi_{eff}$ is called the “effective potential” (See equations (3.68) Pg. 160 in [16]). In terms of the solution in Eq.\((4)\), one obtains

\[
\begin{align*}
\ddot{r} &= -\frac{\partial \Phi_{eff}(U)}{\partial r}, \quad (15a) \\
\ddot{z} &= -\frac{\partial \Phi_{eff}(U)}{\partial z}. \quad (15b)
\end{align*}
\]
where

\[ d\Phi_{eff}(U) = -\frac{h(r,z)}{(1-U)^3}dU, \quad (16) \]

\[ h(r,z) \equiv \varepsilon + \frac{2}{m^2r^2(1-U)^2} \left( \frac{\partial}{\partial z} \int_0^r Urdr \right)^2 - \frac{2\sqrt{2qL}}{m^2} \frac{\partial}{\partial z} \int_0^r Urdr. \quad (17) \]

Thus the three-dimensional motion of the particle in an axis-symmetric potential can be reduced to the two-dimensional motion of the particle in a “Newtonian potential” \( U(r,z) \).

**B. Circular Motion in the plane \( z = 0 \)**

To study the circular motion of the test charged particle we start with the conditions

\[ \dot{r} = 0, \quad \frac{\partial \Phi_{eff}}{\partial r} = 0. \quad (18) \]

Then, from the first of these equations, Eqs. 6 and 9, we have the energy of the particle

\[ E^2 = -g^{tt}(\varepsilon m^2 + g_{\phi\phi}(L - qA_\phi g^{\phi\phi})^2). \quad (19) \]

From the second condition in Eq. (18) we have

\[ \ddot{r} = W \frac{\partial g^{rr}}{\partial r} = -\frac{\partial \Phi_{eff}}{\partial r} = 0. \quad (20) \]

Notice that if \( W = 0 \), from Eqs. 19 and 13, we obtain

\[ E^2 = g_{zz} L^2. \quad (21) \]

Thus, by introducing the corresponding metric coefficients of the line element in Eq. 1, such as

\[ E^2 = -r^2e^{-4\phi} L^2, \quad (22) \]

which lacks of physical meaning. Hence, the condition \( \frac{\partial \Phi_{eff}}{\partial r} = 0 \) is equivalent to

\[ \frac{\partial g^{rr}}{\partial r} = 0, \quad W \neq 0. \quad (23) \]

The minimum radius for stable circular orbit occurs in the inflection points of the effective potential. Thus we must solve the equation

\[ \ddot{r} = \frac{\partial^2 \Phi_{eff}}{\partial r^2} = 0. \quad (24) \]

or, equivalently, to solve the equation

\[ \frac{\partial^2 g^{rr}}{\partial r^2} = 0. \quad (25) \]

On the other hand, by calculating the derivative respect to \( z \) in both sides of Eq. (19) we obtain for the angular moment

\[ L = qA_\phi g^{\phi\phi} + \frac{l}{(g^{tt}g_{\phi\phi})z}, \quad (26) \]

where

\[ l \equiv qA_\phi g^{tt}g_{\phi\phi}g^{\phi\phi}z \pm \sqrt{\left(qA_\phi g^{tt}g_{\phi\phi}g^{\phi\phi}z\right)^2 - \varepsilon m^2 g^{tt}z(g^{tt}g_{\phi\phi})z}, \]
and we have used the Einstein-Maxwell equation $\phi_r^2 = \frac{1}{2} x^2 A_r^2 z$. By substituting this value for $L$ in Eq. (19) we obtain the energy of the particle

$$E^2 = -g_{tt} \left( \varepsilon m^2 + g_{\phi\phi} \left[ \frac{l^2}{g_{tt} g_{\phi\phi} r^2} \right] \right).$$

(27)

Since the Lagrangian in Eq. (5) does not depend explicitly on the variables $t$ and $\phi$, and one can obtain the following two conserved quantities

$$p_t = -mc c^2 \hat{t} \equiv -\frac{E}{c},$$

(28)

and also

$$p_\phi = m r^2 c^{-2} \dot{\phi} + \frac{q}{c} A_\phi \equiv L,$$

(29)

where $E$ and $L$ are, respectively, the energy and the angular momentum of the particle as measured by an observer at rest at infinity. Furthermore, the momentum $p_\alpha$ of the particle can be normalized so that $g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = -\Sigma$. Accordingly, for the metric in Eq. (11) we have

$$-e^{2\phi} \dot{t}^2 + e^{-2\phi} (\dot{r}^2 + \dot{z}^2 + r^2 \dot{\phi}^2) = -\Sigma,$$

(30)

where, with $c = 1$, the notation $\Sigma = 1, 0, -1$ denotes space-like, null and time-like curves, respectively.

As an application of Eq. (16), we use an axial bi-dimensional isothermal potential, which has the form

$$U(r) = 1 - v_0^2 \ln(r^2 + z^2),$$

and straightforwardly, we get the expression

$$d\Phi_{eff} = \frac{4}{v_0} \left\{ \epsilon + a \left( \frac{z}{r} \right)^2 + b \ln(z^2 + r^2) \right\} \frac{rdr + zdz}{(z^2 + r^2) \ln(z^2 + r^2)^3} .$$

Hence, integrating the former expression, it is necessary to obtain a convergence of the integral away from origin, we use a Laurent expansion $\sum_{k=1}^{\infty} = \frac{1}{2} \sim \frac{z^2}{4v_0^2}$. Finally, after a long algebra, we can write the form of the effective potential felt by charged particle with mass moving with velocity $v_0$ and total angular momentum $L$

$$\Phi_{eff}(r, z) = \frac{\epsilon}{4v_0^2} \left( \frac{\ln(z^2 + r^2) - 4 \ln z}{\ln z \ln(z^2 + r^2)} \right) + \frac{bz}{4v_0^2} \left[ \frac{\ln(z^2 + r^2) - 2 \ln z}{\ln z \ln(z^2 + r^2)} \right]$$

$$-\frac{a z^2}{4v_0^4 \ln^2(z^2 + r^2)} + \frac{a z^2 \pi^2}{24v_0^4 \ln^2(z^2 + r^2)}$$

where we denote $a = \frac{2 \phi^2}{m}$ and $b = \frac{2 \phi^2}{m} qLv_0^2$. We notice that a small value induces to outgoing lines from the center as expected. In the center figure, we notice that for time-like curves, the magnetic lines distort the path of a test charged particle away from the center of the galaxy.

In the same sense, we investigate a Toomre-Kuzmin-like potential since we are dealing with an axisymmetric system, which has the form

$$U(r) = 1 - \frac{\alpha}{\sqrt{(r^2 + z^2)}},$$

where $\alpha$ is a unitary free parameter to guarantee the correct units, and straightforwardly, one can get the expression

$$d\Phi_{eff} = \frac{2 r}{\alpha^2} \left\{ \epsilon + \frac{2}{m^2 r^2} \left( z - \sqrt{(r^2 + z^2)} \right)^2 + \frac{2 \sqrt(2) q \alpha L}{m} \right\} (r^2 + z^2) (rdr + zdz),$$

And, after a long algebra, we can write the form of the effective potential felt by charged particle with mass moving
FIG. 1. The figures are made respectively with $(\epsilon = 1, 0, -1)$ with fixed parameter $v_0 = 0.02$ and unitary value for mass and charge with 50 contour lines in the ranges $r[-90,90]$ and $z[0, 120]$.

with total angular momentum $L$

$$\Phi_{\text{eff}}(r, z) = P(z)r^3 + Ur^5 - K(z) \ln |z| + V(r, z)\sqrt{r^2 + z^2} + I(r, z) + C(z) \ln |r + \sqrt{r^2 + z^2}|.$$ 

where we denote the following terms

$$\begin{align*}
P(z) &= \frac{2\epsilon}{3\alpha^2}z^2 + \frac{4}{m^2\alpha^3}z^2 + \frac{3\epsilon}{2}\frac{1}{m^2\alpha^3}z^2 - \frac{3\sqrt{2}}{m\alpha}qL - \frac{4\sqrt{2}}{3m\alpha}qLz^2 \\
U &= \frac{1}{5} \left( \frac{2\epsilon}{\alpha^2} + \frac{4}{m^2\alpha^3} \right) - \frac{4}{5}\frac{\sqrt{2}}{m\alpha}qL \\
K(z) &= \frac{3}{m^2\alpha^3}z^5 + \frac{\sqrt{2}}{2m\alpha}qLz^3 \\
V(r, z) &= \frac{4r}{m^2\alpha^3} (r^2 + z^2) + \frac{3}{m^2\alpha^3}z^3 + \frac{\sqrt{2}}{m\alpha}rz (2r^2 + z^2), \\
I(r, z) &= \frac{8r}{\alpha^3m^2}z^4 \\
C(z) &= \frac{3}{m^2\alpha^3}z^5 - \frac{\sqrt{2}}{2}\frac{qL}{m\alpha}z^3
\end{align*}$$

FIG. 2. The figures are made respectively with $(\epsilon = 1, 0, -1)$ with fixed parameters $m = 1, q = 1, L = 1, \text{and } \alpha = 1$, for 30 contour lines in ranges $r[-50,50]$ and $z[-1,10]$. 
In this situation, we do not have any considerable difference between the figures and around the origin it is possible to check the singularity and the lines away to the center.

On the other hand, we can express the effective potential directly related to energy. In doing so, we use the relations in Eqs. (28), (29) and (30) that give three linear differential equations, involving the four unknowns \( \dot{x}^{\alpha} \). It is possible to study the motion of test particles with only these relations, if we limit ourselves to the particular case of equatorial trajectories, i.e. \( z = 0 \). Indeed, since the gravitational configuration is symmetric with respect to the equatorial plane, a particle with initial state \( z = 0 \) and \( \dot{z} = 0 \) will remain confined to the equatorial plane which is, therefore, a geodesic plane. Substituting the conserved quantities of Eqs. (28) and (29) into Eq. (30), we find

\[
\dot{r}^2 + \Phi_{\text{eff}} = \frac{E^2}{m^2c^2},
\]

where

\[
\Phi_{\text{eff}}(r) = \frac{L^2}{m^2r^2} \left( 1 - \frac{qA_{\phi}}{Lc} \right)^2 e^{4\phi} + \Sigma e^{2\phi}
\]

is an effective potential. We assume the convention that the positive value of the energy corresponds to the positivity of the solution \( E_{\pm} = \pm mc\Phi_{\text{eff}}^{1/2} \). Consequently, \( E_+ = -E_- = mc\Phi_{\text{eff}}^{1/2} \).

The motion of charged test particles is governed by the behavior of the effective potential in Eq. (32). The radius of circular orbits and the corresponding values of the energy \( E \) and angular momentum \( L \) are given by the extrema of the function \( \Phi_{\text{eff}} \). Therefore, the conditions for the occurrence of circular orbits are

\[
\frac{d\Phi_{\text{eff}}}{dr} = 0, \quad \Phi_{\text{eff}} = \frac{E^2}{m^2c^2}.
\]

Thus, by calculating the condition in Eq. (33) for the effective potential in Eq. (32), we find the angular momentum of the particle in circular motion

\[
L_{\pm} = \frac{qA_{\phi}}{c} + \frac{qrA_{\phi,r}e^{\phi} \pm \sqrt{(qrA_{\phi,r}e^{\phi})^2 - 4\Sigma e^{2\phi} r^3 (2r\phi_r - 1)}}{2ce^{\phi}(2r\phi_r - 1)}.
\]

Conventionally, we can associate the plus and minus signs in the subscript of the notation \( L_{\pm} \) to dextrorotation and levorotation, respectively. Moreover, by inserting the value of the angular momentum in Eq. (34) into the second equation of Eq. (33), we obtain the energy \( E_{\pm} \) of the particle in a circular orbit as

\[
E_{\pm} = \pm mc e^{\phi} \left( \Sigma + \xi(\pm) \right)^{1/2},
\]

where

\[
\xi(\pm) = \frac{\left( qrA_{\phi,r}e^{\phi} \pm \sqrt{(qrA_{\phi,r}e^{\phi})^2 - 4\Sigma e^{2\phi} r^3 (2r\phi_r - 1)} \right)^2}{4m^2c^2 r^2 (2r\phi_r - 1)^2}.
\]

Therefore, each sign of the value of the energy corresponds to two kind of motions (dextrorotation and levorotation) indicated in Eqs. (35) and (36) by the superscripts \((\pm)\).

III. PERIHELION ADVANCE IN A CONFORMASTATIC MAGNETIZED SPACETIME

One of the most important tests of general relativity and modified theories of gravitation in astrophysical scale is the perihelion advance of celestial objects. In this section, we present the analytic expressions which determine the perihelion advance of charged test particle, moving in a conformastatic spacetime under the presence of a magnetic field. Starting with the first integral in Eq. (30), we restrict the analysis to the motion of a particle on the plane with
\[ \left( \frac{dr}{d\varphi} \right)^2 = -r^2 \left[ 1 + \frac{m^2 r^2}{(L - \frac{q}{c}A_r)^2} \left( \Sigma (1 - U)^2 - \frac{E^2}{m^2 c^2} (1 - U)^4 \right) \right], \] (37)

where all the quantities are evaluated at \( z = 0 \) and we have used the expressions for the energy and angular momentum of the particle given by Eqs. (28) and (29), respectively. With the change of variable \( u = 1/r \), Eq. (37) can be transformed into

\[ \frac{d^2 u}{d\varphi^2} + u^2 = F(u) \] (38)

where

\[ F(u) \equiv \frac{1}{2} \frac{dG}{du}, \] (39)

and

\[ G(u) \equiv \frac{1}{(1 - \frac{qA_r}{cL})^2} \left[ \frac{E^2}{c^2 L^2} (1 - U)^4 - \frac{\Sigma m^2}{L^2} (1 - U)^2 \right]. \] (40)

Accordingly, by following the procedure proposed in [21], we have for the resulting perihelion advance

\[ \delta \varphi = \pi \left( \frac{dF}{du} \right)_{u = u_0}, \] (41)

where \( u_0 \) is the radius of a nearly circular orbit, which is given by the roots of the equation \( F(u_0) = u_0 \). In Eq. (41), we have shown the procedure to obtain an expression for the perihelion advance of a charged test particle in a generic conformastatic spacetime with a magnetic field. We now illustrate the results by considering a particular conformastatic spacetime generated from the harmonic potential of a punctual mass

\[ U(r, z) = -\frac{GM}{c^2 R}, \quad R^2 = r^2 + z^2. \] (42)

Thus, by inserting Eq. (42) into Eq. (39) we obtain for \( F(u) \)

\[ F(u) = \frac{\left[ \frac{2E^2 GM}{c^2 L^2} \left( 1 + \frac{GM}{c^2 u} \right)^3 - \frac{\Sigma m^2 GM}{c^2 L^2} \left( 1 + \frac{GM}{c^2 u} \right) \right]}{\left( 1 - \frac{q\sqrt{G}M}{cL} \right)^2}. \] (43)

Accordingly, the perihelion advance of a particle in this spacetime is given by

\[ \delta \varphi = \pi \left( \frac{6E^2 G^2 M^2}{c^6 L^2} x_0^2 - \frac{\Sigma m^2 G^2 M^2}{c^6 L^2} \right) \left( 1 - \frac{q\sqrt{G}M}{cL} \right)^2, \] (44)

where the term

\[ x_0 \equiv 1 + \frac{GM}{c^2 u_0} \] (45)
satisfies the equation

\[ 2E^2G^2M^2x_0^3 - \left[ \Sigma m^2G^2M^2c^2 + c^6L^2 \left( 1 - \frac{q\sqrt{GM}}{cL} \right)^2 \right] x_0 \]

\[ + c^6L^2 \left( 1 - \frac{q\sqrt{GM}}{cL} \right)^2 = 0. \]  \hspace{1cm} (46)

Thus, by inserting the real solution of Eq. (46) into Eq. (43), we find that the perihelion advance of the test particle orbit is given by

\[ \delta \varphi = \frac{\pi \psi_0 - k_2^2}{Q^2}, \]  \hspace{1cm} (47)

where

\[ \psi_0 = \frac{6 \left( Q^2 + k_2^2 \right) + \left[ 54Q^2k_1 \left( -1 + \sqrt{1 - \frac{6(Q^2+k_2^2)}{81Q^2k_1^2}} \right) \right]^2 \left[ 6 \left[ 54Q^2k_1 \left( -1 + \sqrt{1 - \frac{6(Q^2+k_2^2)}{81Q^2k_1^2}} \right) \right]^2 \right]^2}{6 \left[ 54Q^2k_1 \left( -1 + \sqrt{1 - \frac{6(Q^2+k_2^2)}{81Q^2k_1^2}} \right) \right]^2} \]

with

\[ k_1^2 = \frac{E^2G^2M^2}{c^6L^2}, \quad k_2^2 = \frac{\Sigma m^2G^2M^2}{c^4L^2}, \]

and also

\[ Q^2 = \left( 1 - \frac{q\sqrt{GM}}{cL} \right)^2. \]

Notice that when \( q = 0 \) (and, consequently \( Q = 1 \)) we get the case in which Eq. (44) describes the perihelion advance of a neutral particle. Actually, we restrict ourselves to the neutral case, since objects like planets, asteroids and comets are neutral on average and the consideration of a charge is hardly significant. Moreover, this neutrality is essentially due to the influence of the solar wind, but a global net charge, e.g. in stars, is still on discussion [22].

In order to get a real use of Eq. (47), we follow the procedure presented in [21]. First, we rewrite both the angular momentum (34) and the energy (35), which depend on the radial distance \( r \), in terms of the parameters that describe the orbit of rotating test particles. For the radial distance, one can use the ellipse formula in the Euclidean plane as

\[ r = \frac{s(1 - \epsilon^2)}{1 + \epsilon \cos \varphi}, \]  \hspace{1cm} (48)

where \( s \) is the semimajor axis and \( \epsilon \) the eccentricity of the orbit. Moreover, we can rewrite Eq. (47) by using physical units related to observations as

\[ \delta \varphi^* = \frac{\pi \gamma^* \left( \psi_0 - k_2^2 \right) s^2}{Q^2M_\odot T^2}, \]  \hspace{1cm} (49)

where we have introduced the solar mass \( M_\odot \) and the period \( T \) of the rotating body. The parameter \( \gamma^* = \frac{180}{\pi^*}T \) allows us to transform units from radians to (secular) degrees. Moreover, in order to obtain a real effective advance \( \delta \varphi_{eff} \) and to alleviate the error propagation, we define a deviation formulae away from general relativity standard result \( \delta \varphi_{eins} \) induced by the coupled Einstein-Maxwell fields as

\[ \delta \varphi_{eff} = \delta \varphi_{eins} \pm \beta_0 \delta \varphi^*, \]  \hspace{1cm} (50)

where a dimensionless parameter \( \beta_0 \) measures the tiny variation of the orbits through time. As we have checked in the studied cases in table (I), a variation of \( \beta_0 \) must not exceed 10% of the ratio between the Einstein-Maxwell
TABLE I. Comparison between the values for secular precession of inner planets in units of arcsec/century (′′.cy⁻¹) of the standard (Einstein) perihelion precession $\delta \phi_{\text{eins}}$ [24] for neutral test particles (planets, asteroids/comets) in the conformastatic magnetized spacetime of a punctual mass $\delta \phi_{\text{eff}}$. The data for $\delta \phi_{\text{obs}}$ stands for the secular observed perihelion precession in units of arcsec/century adapted from [23] by adding a supplementary precession correction from EPM2011 [25, 26]. In addition, the results for the NEOs 433 Eros, 3200 Phaethon and 2p/Encke comet are also presented. The mass of the 2p/Encle comet as $m = 3.85 \times 10^{13}$ kg was estimated with a bulk density $\rho = 0.5$ g.cm⁻³ as shown in [27].

| Object      | $\delta \phi_{\text{obs}}$ | $\delta \phi_{\text{eins}}$ | $\delta \phi_{\text{eff}}$ | $\beta_0$        |
|-------------|----------------------------|-------------------------------|----------------------------|------------------|
| Mercury     | 43.098 ± 0.503             | 42.97817                      | 42.9782                    | 0.7605×10⁻⁴      |
| Venus       | 8.026 ± 5.016              | 8.62409                       | 8.62425                    | 0.1426×10⁻²      |
| Earth       | 5.00019 ± 1.00038          | 3.83848                       | 3.83944                    | 0.4375×10⁻²      |
| Mars        | 1.36238 ± 0.000537         | 1.35086                       | 1.36980                    | 0.3729×10⁻¹      |
| 433 Eros    | 1.60                       | 1.57317                       | 1.58668                    | 0.2906×10⁻¹      |
| 3200 Phaethon| 10.1                      | 10.1201                      | 10.1213                    | 0.3499×10⁻²      |
| 2p/Encke    | 1.9079                     | 1.868                         | 1.92833                    | 0.5623×10⁻¹      |

When applied to the observational data [23] plus a supplementary precession corrections from EPM2011 [25, 26], one can test Eq.(50). Thus, we obtain the results presented in table [I] for the perihelion precession of inner planets of the Solar system, two NEO’s asteroids named 433 Eros and 3200 Phaethon, and NEO 2p/Encke comet. The data for the astrophysics parameters of planets like semimajor axis, eccentricity, period and mass, can be found in JPL solar system dynamics (http://ssd.jpl.nasa.gov/?planets) and for asteroids and comets, in JPL small body database (http://ssd.jpl.nasa.gov/sbdb.cgi). The orbital periods are in units of years.

As shown in table [I], the theoretical results match the observations, and a slight improvement is obtained as compared to the standard Einstein gravity which turns our model closer to the observations. We conclude that the gravitational interaction generated by the magnetic field of the central body can play an important role in astrophysical observations. It is worth to say that the values of $\alpha$ seem to be sensitive to the variation of the eccentricity of the orbits and the mass of the object as seen in the studied cases and the values have a close resemblance to PPN parameters that have a bound $|2\gamma - \beta - 1| < 3 \times 10^{-3}$ [28].

In addition, some other considerations must be noted. The constant $\Sigma$ enters explicitly the expression for the perihelion advance in Eq.(30) and it represents null, time-like and space-like curves. For $\Sigma = 0$, we do not have a solution since Eq.(50) diverges. For time-like trajectories, $\Sigma = -c^2$ no physical results are obtained, because in the corresponding Newtonian limit a differential equation is obtained, whose solution implies that $r$ is negative. Moreover, no significant differences were found for different values of the charge of the order $q/m \sim 10^{-3}$, which is the value where the behavior of the energy and angular momentum becomes affected by the presence of the effective charge. In the same sense, no differences could be found when using both solutions for the angular momentum $L_c \pm$ and energy $E_c \pm$.

CONCLUSION

In this work we have shortly shown the characteristics of the motion of a charged particle along circular orbits in a spacetime described by a conformastatic solution of the Einstein-Maxwell equations. As a particular example we have considered the case of a charged particle moving in the gravitational field of a punctual source placed at the origin of coordinates. Our analysis is based on the study of the behavior of an effective potential that determines the position and stability properties of circular orbits. We also have investigated the behaviour of effective potentials.
In addition, we have also calculated an expression for the perihelion advance of a test particle in a general magnetized conformastatic spacetime obtaining a good agreement with the observed values for the perihelion of inner Solar planets and some selected NEO asteroids. It is worth noting that all results presented were obtained with the initial assumption of a neutral particle, in accordance with the fact that planets are largely neutral. Specifically, in the perihelion drift, we find that the differences between a neutral particle and a charged particle are slightly small, when realistic values for the effective charge are used. This means that the electromagnetic interaction between the charge and the central magnetized body does not seriously affect the value of the perihelion advance. Nevertheless, the magnetic field enters explicitly the metric components and, consequently, affects the motion of neutral test particles through the gravitational interaction. This explains why the numerical predictions of the perihelion advance generated by a punctual magnetized mass are in better agreement with observations than the predictions of Einstein’s theory alone. As a future prospect, we will apply the Poincaré surface-of-section method for analyzing weakly perturbed Hamiltonian conformastatic systems.

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