ON ORTHOGONAL $p$-ADIC WAVELET BASES

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Abstract. A variety of different orthogonal wavelet bases has been found in $L_2(\mathbb{R})$ for the last three decades. It appeared that similar constructions also exist for functions defined on some other algebraic structures, such as the Cantor and Vilenkin groups and local fields of positive characteristic. In the present paper we show that the situation is quite different for the field of $p$-adic numbers. Namely, it is proved that any orthogonal wavelet basis consisting of band-limited (periodic) functions is a modification of Haar basis. This is a little bit unexpected because from the wavelet theory point of view, the additive group of $p$-adic numbers looks very similar to the Vilenkin group where analogs of the Daubechies wavelets (and even band-limited ones) do exist. We note that all $p$-adic wavelet bases and frames appeared in the literature consist of Schwartz-Bruhat functions (i.e., band-limited and compactly supported ones).

1. Introduction

In the early nineties a general scheme for constructing wavelets of real argument was developed. This scheme is based on the concept of multiresolution analysis (MRA in the sequel) introduced by Meyer and Mallat [6], [7] (see also, [9], [20]). The theory allowed to construct orthogonal wavelet bases essentially different from the Haar basis, and, in particular, the Daubechies wavelets [9] which are actively implemented in signal processing and other engineering areas.

It appeared that similar constructions also exist for functions defined on some other algebraic structures, such as the Cantor and Vilenkin groups, local fields of positive and zero characteristic, and adele rings (see, e.g., [15], [16], [17], [18], [19], [12], [1], [4]).

In the $p$-adic setting, the situation is as follows. In 2002 the first $p$-adic wavelet basis for $L_2(\mathbb{Q}_p)$ where $\mathbb{Q}_p$ is the $p$-adic field, was found in [8]. It is an analog of the Haar basis for reals and consists of translations by elements of the $p$-adic interval

$$I_p = \left\{ \frac{k}{p^n} \in \mathbb{Q}_p : k = 0, \ldots, p^n - 1, \ n = 0, 1 \ldots \right\}$$

and $p$-adic dilations, of $p - 1$ functions (wavelet functions in the sequel). In [12] the notion of $p$-adic MRA was introduced and a general scheme for its construction was described. Following Meyer and Mallat one starts with a scaling function, i.e. a function $\varphi$ whose $I_p$-translations form an orthonormal system, and the sequence of spaces

$$V_m = \text{span} \left\{ \varphi \left( \frac{x}{p^m} - a \right) : a \in I_p \right\}, \ m \in \mathbb{Z}.$$
is increasing and dense in $L^2(\mathbb{Q}_p)$. The scheme was used to construct the $p$-adic Haar MRA where the characteristic function of the ring of $p$-adic integers $\mathbb{Z}_p$ was taken as a scaling function. This leads to the wavelet basis constructing in [8].

Next it was interesting to find other MRAs in order to obtain essentially different wavelet bases. However all these attempts were unsuccessful. Although some other scaling functions were found in [13], it appeared that all of them lead to the same Haar MRA. An explanation of this failure was given in [1], where the authors proved that the Haar MRA is a unique MRA in the sense of the definition given in [12], that is generated by a test (Schwartz-Bruhat) scaling function. So, it is not possible to find essentially new orthogonal $p$-adic wavelet bases as MRA-based ones. (It is interesting to note that as it was shown in [2], [3], there exist infinitely many non-orthogonal MRAs in the sense of the definition given in [1], i.e. those for which the $I_p$-translations of the scaling function do not form an orthogonal system).

Now the following question arises: do there exist orthogonal wavelet bases not generated by the Haar MRA? In the present paper we answer to this question in the negative restricting ourselves to the bases consisting of band-limited functions (the class of band-limited functions contains the test functions, and coincides with the class of periodic functions in $L^2(\mathbb{Q}_p)$). Moreover, it is proved that any orthogonal wavelet basis consisting of test functions is equivalent in a natural sense to the Haar basis constructed in [8].

Let us discuss what the words "basis is generated by the Haar MRA" mean. Following the standard scheme for constructing MRA-based wavelets, one defines the Haar wavelet spaces $W_m$ (the orthogonal complement of $V_{m+1}$ in $V_m$) and find a set of wavelet functions whose $I_p$-translations form an orthonormal basis for $W_0$. We call such a set standard; the corresponding wavelet basis for $L^2(\mathbb{Q}_p)$ consisting of $I_p$-translations and $p$-adic dilations of its elements, is called a standard Haar basis. Any standard set of wavelet functions consists of $p-1$ elements belonging to $W_0$. Moreover, a set of functions that is unitary equivalent to a standard set is standard too, but there are standard sets which are not unitary equivalent to each other. All standard sets consisting of test functions were described in [14].

There exists, however, a lot of non-standard orthogonal wavelet bases generated by the Haar MRA. Indeed, it is easy to see that each standard wavelet function $\psi$ can be replaced by $p$ functions (not belonging to $W_0$) the set of $I_p$-translations and $p$-adic dilations of which, coincides with that of $\psi$ (see Section 2 for details). So, the same standard wavelet basis can be generated by a non-standard set of wavelet functions. Moreover, if one applies a unitary transform to the latter set, then a new orthogonal wavelet basis different from a standard one, can be obtained. Of course these two operations (and their inverses) can be repeated several times, and any basis obtained in this way is actually generated by the Haar MRA. It is natural to call it a "damaged " Haar basis. We observe that all non-standard orthogonal $p$-adic wavelet bases we saw in the literature (see, e.g., [10], [11]) are of that type. In the present paper we prove that any orthogonal $p$-adic wavelet basis whose elements are band-limited is such a "damaged " Haar basis.

The paper is organized as follows. Notation and basic facts on $p$-adic analysis we need, are concentrated in Section 2. Section 3 contains basic definitions and statements of main results. In Section 4 we prove auxiliary results on general and periodic vector-functions generating orthonormal wavelet systems and especially orthonormal wavelet bases. Section 5 contains the proofs of Theorems 1, 2 and 3.
2. Notations and basic facts

Here and in what follows, we systematically use the notation and results from [21]. As usual, by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the ring of rational integers and the fields of rational, real and complex numbers, respectively.

The field $\mathbb{Q}_p$ of $p$-adic numbers is the completion of the field $\mathbb{Q}$ with respect to the $p$-adic norm $|\cdot|_p$ defined as follows:

$$|x|_p = \begin{cases} p^{-\gamma}, & \text{if } x = p^\gamma \frac{m}{n} \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

where $\gamma \in \mathbb{Z}$ and $m, n$ are integers not divisible by $p$. The extension of this norm to $\mathbb{Q}_p$ is also denoted by $|\cdot|_p$.

The norm $|\cdot|_p$ is non-Archimedean, i.e., satisfies the strong triangle inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p), \quad x, y \in \mathbb{Q}_p.$$ 

Any $p$-adic number $x \neq 0$ can uniquely be written in the form

$$(2.1) \quad x = \sum_{j=\gamma}^{\infty} x_j p^j$$

where $\gamma \in \mathbb{Z}$ and $x_j \in \{0, 1, \ldots, p-1\}$ with $x_\gamma \neq 0$. The fractional part $\{x\}_p$ of the number $x$ equals by definition $\sum_{j=\gamma}^{\infty} x_j p^j$. We also set $\{0\}_p = 0$.

The ring of $p$-adic integers $\mathbb{Z}_p$ and the $p$-adic interval $I_p$ are defined by

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : \{x\}_p = 0\}, \quad I_p = \{x \in \mathbb{Q}_p : \{x\}_p = x\}.$$ 

We observe that the translations of $\mathbb{Z}_p$ by the elements of $I_p$ are mutually disjoint and the union of them equals $\mathbb{Q}_p$. Besides, $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$.

The additive character $\chi_p$ of the field $\mathbb{Q}_p$ is defined by

$$\chi_p(x) = e^{2\pi i x_p}, \quad x \in \mathbb{Q}_p.$$ 

The field $\mathbb{Q}_p$ is locally compact. Denote by $dx$ the normalized Haar measure on it. By definition this measure is positive, invariant under translations, i.e., $d(x + a) = dx$ for all $a \in \mathbb{Q}_p$, and satisfies the condition $\int_{\mathbb{Z}_p} dx = 1$. Moreover,

$$d(ax) = |a|_p ^\ell dx, \quad a \in \mathbb{Q}_p \setminus \{0\}.$$ 

The Hilbert space of all complex-valued functions on $\mathbb{Q}_p$ summing with square with respect to the measure $dx$, is denoted by $L^2(\mathbb{Q}_p)$.

Denote by $\mathcal{D}$ the $p$-adic Bruhat-Schwartz space, i.e. the linear space of locally-constant compactly supported functions defined on $\mathbb{Q}_p$ (so-called test functions). This space is a $p$-adic analog of the Schwartz space in the real analysis.

The Fourier transform of a function $f \in \mathcal{D}$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{Q}_p} \chi_p(\xi x) f(x) dx, \quad \xi \in \mathbb{Q}_p,$$

This yields a linear isomorphism taking $\mathcal{D}$ onto $\mathcal{D}$. It can uniquely be extended to a linear isomorphism of $L^2(\mathbb{Q}_p)$. Moreover, the Plancherel equality holds

$$\int_{\mathbb{Q}_p} f(x) \overline{g(x)} dx = \int_{\mathbb{Q}_p} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi, \quad f, g \in L^2(\mathbb{Q}_p).$$
A function \( f \in L_2(\mathbb{Q}_p) \) is said to be band-limited if its Fourier transform \( \hat{f} \) is compactly supported. Given \( m \in \mathbb{Z} \), the function \( \hat{f} \) is supported on \( p^{-m}\mathbb{Z}_p \) if and only if \( f \) is \( p^m \)-periodic, i.e.

\[
f(x + p^m) = f(x), \quad x \in \mathbb{Q}_p.
\]

Thus the spaces of band-limited and periodic functions coincide. Moreover, the \( p \)-adic Bruhat-Schwartz space \( D \) equals the space consisting of all both band-limited (periodic) and compactly supported functions in \( L_2(\mathbb{Q}_p) \).

Below for \( f \in L_2(\mathbb{Q}_p) \), \( m \in \mathbb{Z} \) and \( a \in \mathbb{Q}_p \), we set

\[
(2.2) \quad f_{m,a}(x) = p^{m/2} f \left( \frac{x}{p^m} - a \right), \quad x \in \mathbb{Q}_p.
\]

Let \( \varphi \) be a characteristic function of the set \( \mathbb{Z}_p \). For every integer \( m \), the functions \( \varphi_{m,a}, a \in I_p \), form an orthonormal system. The sampling spaces \( V_m \) of the Haar MRA are defined by

\[
(2.3) \quad V_m = \text{span} \{ \varphi_{m,a} : a \in I_p \}, \quad m \in \mathbb{Z}.
\]

The union of all spaces \( V_m \) is dense in \( L_2(\mathbb{Q}_p) \), and \( V_m \subset V_{m+1} \) for all \( m \) (see the \( p \)-adic MRA theory [12]). Any function \( \varphi_{m,a} \) is \( p^m \)-periodic. Moreover,

\[
V_m = \{ f \in L_2(\mathbb{Q}_p) : f \text{ is } p^m \text{-periodic} \}.
\]

The wavelet spaces \( W_m \) are defined by

\[
(2.4) \quad W_m = V_{m+1} \ominus V_m, \quad m \in \mathbb{Z},
\]

and we have the following orthogonal decomposition

\[
(2.5) \quad L_2(\mathbb{Q}_p) = \bigoplus_{m \in \mathbb{Z}} W_m.
\]

Finally, let us define a translation operator \( T \) on \( L_2(\mathbb{Q}_p) \) by

\[
(2.6) \quad (Tf)(x) = f(x - 1), \quad x \in \mathbb{Q}_p.
\]

Then the operator \( T \) is unitary and the spaces \( V_j \) and \( W_j \) are \( T \)-invariant. Moreover, \( Tf = f_{0,1} \) in accordance with (2.2).

3. Main results

Let \( r \) be a positive integer and \( \Psi = (\psi^{(1)}, \ldots, \psi^{(r)})^T \) where \( \psi^{(\nu)} \in L_2(\mathbb{Q}_p) \) for all \( \nu = 1, \ldots, r \).

**Definition 3.1.** The system of functions

\[
\{ \psi^{(\nu)}_{m,a} : m \in \mathbb{Z}, a \in I_p, \nu = 1, \ldots, r \}
\]

is called *wavelet system* generated by the vector-function \( \Psi \) (or by its components \( \psi^{(\nu)} \) called *wavelets*). If this system is an orthonormal basis for \( L_2(\mathbb{Q}_p) \), one says that \( \Psi \) generates an *orthonormal wavelet basis* (ONWB in the sequel).

The number \( r = \text{rk}(\Psi) \) is called the *rank* of \( \Psi \). The vector-function \( \Psi \) is called *eigen* if every component of \( \Psi \) is an eigenfunction of the translation operator (2.6).

Two vector-functions \( \Psi \) and \( \Psi' \) are said to be *unitary equivalent* if there exists a unitary matrix \( U \) such that \( \Psi = U \Psi' \). Evidently, if \( \Psi \) generates an ONWB and \( \Psi' \) is unitary equivalent to \( \Psi \), then \( \Psi' \) generates an ONWB.
Definition 3.2. A vector-function generating an ONWB is called standard Haar if it is of rank $p - 1$ and all its components are in $W_0$. The ONWB is called standard Haar basis in this case.

It is well known that the vector-function $\Theta = (\theta^{(1)}, \ldots, \theta^{(p-1)})^T$ where
\[
\theta^{(\nu)}(x) = \varphi(x)\chi_p \left( \frac{\nu x}{p} \right), \quad \nu = 1, \ldots, p - 1,
\]
is standard Haar (see [12]). It is called the basic Haar vector-function. Moreover, $\Theta$ is eigen because
\[
\theta^{(\nu)}(x - 1) = \chi_p \left( -\frac{\nu}{p} \right) \theta^{(\nu)}(x).
\]

Given a standard Haar vector-function, one can easily "damage" it to obtain another vector-function (which is non-standard) generating the same ONWB. Using such a technique together with replacing some vector-functions by unitary equivalent ones, new orthogonal wavelet bases can be obtained. This is illustrated as follows.

Example 3.3. Let $p = 2$, $\psi = \theta^{(1)}$. Then $\psi$ can be treated as a standard Haar vector-function consisting of one function $\psi$. Set
\[
\psi^{(1)}(x) = \sqrt{2}\psi(x/2), \quad \psi^{(2)}(x) = \sqrt{2}\psi((x - 1)/2).
\]
Evidently, the sets $\{\psi^{(\nu)}_{m,a} : m \in \mathbb{Z}, a \in I_p, \nu = 1, 2\}$ and $\{\psi_{m,a} : m \in \mathbb{Z}, a \in I_p\}$ coincide. So the wavelet system generated by $\psi^{(1)}, \psi^{(2)}$ is a standard Haar basis, but $\psi^{(1)}, \psi^{(2)}$ is not a standard Haar vector-function. Similarly, the functions $\psi^{(1)}, \psi^{(2,1)}, \psi^{(2,2)}$, where
\[
\psi^{(2,1)}(x) = \sqrt{2}\psi^{(2)}(x/2), \quad \psi^{(2,2)}(x) = \sqrt{2}\psi^{(2)}((x - 1)/2),
\]
generate the same standard Haar basis. Set
\[
\tilde{\psi}^{(1)} = \frac{1}{\sqrt{2}}(\psi^{(1)} + \psi^{(2,1)}), \quad \tilde{\psi}^{(2)} = \frac{1}{\sqrt{2}}(\psi^{(1)} - \psi^{(2,1)}), \quad \tilde{\psi}^{(3)} = \psi^{(2,2)}.
\]
These functions generate an ONWB which is not a standard Haar basis, but the vector-function $\tilde{\Psi} = (\tilde{\psi}^{(1)}, \tilde{\psi}^{(2)}, \tilde{\psi}^{(3)})^T$ is unitary equivalent to a vector-function generating a standard Haar basis. Next set
\[
\tilde{\psi}^{(1,1)}(x) = \sqrt{2}\tilde{\psi}^{(1)}(x/2), \quad \tilde{\psi}^{(1,2)}(x) = \sqrt{2}\tilde{\psi}^{(1)}((x - 1)/2).
\]
It is not difficult to see that the vector-function $\tilde{\Psi}' = (\tilde{\psi}^{(1,1)}, \tilde{\psi}^{(1,2)}, \tilde{\psi}^{(2)}, \tilde{\psi}^{(3)})^T$ generates an ONWB, but $\tilde{\Psi}'$ does not generate a standard Haar basis, and $\tilde{\Psi}'$ is not unitary equivalent to a vector-function generating a standard Haar basis.

Definition 3.4. Two vector-functions $\Psi, \Psi'$ are said to be wavelet equivalent if there exist vector-functions $\Psi_0, \ldots, \Psi_N$ such that $\Psi_0 = \Psi$, $\Psi_N = \Psi'$, and for every $j > 0$ either $\Psi_j$ is unitary equivalent to $\Psi_{j-1}$ or $\Psi_j$ and $\Psi_{j-1}$ generate the same wavelet system (as a set).

Evidently, the property of vector-functions to be wavelet equivalent is preserved under changing the order of components in one of them. So, we can use the term "wavelet equivalent" for two sets of functions, or to say that a vector-function is wavelet equivalent to a set of functions (which can be considered as the components of another vector-function).
We note also that the wavelet equivalence relation is reflexive, symmetric and transitive. Moreover, if $\Psi$ generates an ONWB and $\Psi'$ is unitary equivalent to $\Psi$, then $\Psi'$ generates an ONWB.

It is easy to see that any standard Haar basis consists of periodic functions. So, if a vector-function is wavelet equivalent to a standard Haar vector-function, then all its components are periodic. It appears that the converse statement is also true.

**Theorem 1.** Any periodic vector-function $\Psi$ generating an ONWB is wavelet equivalent to a standard Haar vector-function which can be taken eigen. If, moreover, $\Psi$ is compactly supported, then it is wavelet equivalent to the basic Haar vector-function $\Theta$.

As a byproduct of Theorem 1 we obtain the following statement.

**Theorem 2.** The rank $r$ of a periodic vector-function generating an ONWB is divisible by $p - 1$, in particular, $r \geq p - 1$.

When considering Example 3.3, we see how to find ONWBs which are not standard Haar bases. Every vector-function constructed in this way is wavelet equivalent to a standard Haar one. However, really the situation is more specific. Namely, at each step, where the vector-functions $\Psi_j$ and $\Psi_{j-1}$ generate the same wavelet basis, in fact $\Psi_j$ is obtained from $\Psi_{j-1}$ by changing one of its components for two functions. To describe all bases that can be constructed in such a way for an arbitrary $p$, we say that $\Psi'$ is reducible to $\Psi$ if there exist vector-functions $\Psi_0, \ldots, \Psi_N$ such that $\Psi_0 = \Psi$, $\Psi_N = \Psi'$, and for every $j > 0$, either $\Psi_j$ is unitary equivalent to $\Psi_{j-1}$ or $\Psi_{j-1}$ is obtained from $\Psi_j$ by changing one of its components, say $\psi^{(\nu)}_j(x)$, for $p$ functions $\psi^{(\nu)}_j(x - k/p)$, $k = 0, \ldots, p - 1$.

It would be attractive to replace the words "wavelet equivalent" by "reducible" in Theorem 1. However this is impossible due to the following statement.

**Theorem 3.** There exists a vector-function generating an ONWB that cannot be reduced to a standard Haar vector-function.

### 4. Auxiliary results

**Lemma 4.** Let $m \geq 0$ be an integer and $\psi \in L_2(\mathbb{Q}_p)$. If $\psi = \sum_{a \in I_p} c_a \varphi_{m,a}$ with $c_a \in \mathbb{C}$, then

$$
(4.1) \quad \sum_{\nu=0}^{p^m-1} \sum_{j=0}^{\infty} \sum_{b \in I_p} |\langle f^{(\nu)}, \psi_{-j,b} \rangle|^2 = \frac{p}{p - 1} \sum_{a \in I_p} |c_a|^2
$$

where $f^{(\nu)} = \varphi_{m,\nu/p^m}$.

**Proof.** Set

$$
A_0 = \{0\} \cup \left(I_p \cap (1 - \frac{1}{p^m}, 1)\right), \quad A_\nu = I_p \cap \left(\frac{\nu - 1}{p^m}, \frac{\nu}{p^m}\right], \quad \nu = 1, \ldots, p^m - 1.
$$

Then evidently, $I_p = \bigcup_{\nu=0}^{p^m-1} A_\nu$ and $A_\nu \cap A_\mu = \emptyset$ for $\nu \neq \mu$. 

Let \( \nu \neq 0, j \geq 0 \) and \( b \in I_p \). Then

\[
\langle f^{(\nu)}, \psi_{-j,b} \rangle = \sum_{a \in I_p} c_a \int_{\mathbb{Q}_p} p^{\nu/2} \varphi \left( \frac{x}{p^m} - \frac{\nu}{p^m} \right) p^{(m-j)/2} \varphi \left( \frac{x}{p^m} - \frac{b}{p^m} - a \right) dx = p^{-j/2} \sum_{a \in I_p} c_a \int \varphi(x) \varphi \left( p^j x - \left( a + \frac{b}{p^m} - \frac{\nu}{p^m} \right) \right) dx = p^{-j/2} c_{(\nu-b)/p^m}.
\]

Since \( (\nu - b)/p^m \in A_\nu \) for all \( b \in I_p \), and each \( a \in A_\nu \) can be represented in this form, we have

\[
\sum_{b \in I_p} |\langle f^{(\nu)}, \psi_{-j,b} \rangle|^2 = p^{-j} \sum_{a \in A_\nu} |c_a|^2.
\]

Now let \( \nu = 0 \). Then similarly, for \( j \geq 0 \) we have \( \langle f^{(0)}, \psi_{-j,b} \rangle = p^{-j/2} c_{0} \) and \( \langle f^{(0)}, \psi_{-j,b} \rangle = p^{-j/2} c_{1} \) if \( b \in I_p \setminus \{0\} \). This yields

\[
\sum_{b \in I_p} |\langle f^{(0)}, \psi_{-j,b} \rangle|^2 = p^{-j} \sum_{a \in A_0} |c_a|^2.
\]

Adding (4.2) and (4.3) for all \( \nu = 0, \ldots, p^m - 1 \) and \( j = 0, 1, \ldots \) we obtain (4.1).}

**Proposition 5.** Let functions \( \psi^{(1)}, \ldots, \psi^{(r)} \in V_m, m \geq 0 \), generate an orthonormal wavelet system. Then

1. \( r \leq (p - 1)p^{m-1} \); in particular, the wavelet system generated by \( \psi \in V_0 \) cannot be orthogonal.
2. If all functions \( \psi^{(1)}, \ldots, \psi^{(r)} \) are in \( W_{m-1}, m > 0 \), and generate an ONWB, then \( r = (p - 1)p^{m-1} \).

**Proof.** Any function \( \psi^{(\nu)} \) is in \( V_m \), so it can be expanded as \( \psi^{(\nu)} = \sum_{a \in I_p} c^{(\nu)} a \varphi_{ma} \).

Let \( f^{(\mu)} \) be the function from Lemma 4. Then using the Bessel inequality and (4.1), we have

\[
p^m = \sum_{\mu=0}^{p^m-1} |f^{(\mu)}|^2 \geq \sum_{\mu=0}^{p^m-1} \sum_{\nu=0}^{p^m-1} \sum_{j \in \mathbb{Z}} \sum_{b \in I_p} |\langle f^{(\mu)}, \psi^{(\nu)}_{-j,b} \rangle|^2 \geq \sum_{\nu=0}^{p^m-1} \sum_{\mu=0}^{p^m-1} \sum_{j=0}^{b \in I_p} |\langle f^{(\mu)}, \psi^{(\nu)}_{-j,b} \rangle|^2 = \sum_{\nu=0}^{p^m-1} \sum_{a \in I_p} \sum_{\mu=0}^{p^m-1} |c^{(\nu)} a|^2 = \sum_{\nu=0}^{p^m-1} \sum_{a \in I_p} |c^{(\nu)} a|^2 = \frac{p}{p-1} \sum_{\nu=0}^{p^m-1} \|\psi^{(\nu)}\|^2 = \frac{rp}{p-1},
\]

which proves statement (1).

Now if \( \psi^{(\nu)} \in W_{m-1} \), then \( \psi^{(\nu)}_{-j,b} \) is orthogonal to \( V_m \) for all \( j > 0, b \in I_p \). Hence, \( \langle f^{(\mu)}, \psi^{(\nu)}_{-j,b} \rangle = 0 \) whenever \( j < 0 \). It follows that the second inequality in (4.4) can be replaced by equality. If, moreover, the functions \( \psi^{(1)}, \ldots, \psi^{(r)} \) generate an ONWB, then due to the Parseval equality the first inequality in (4.4) also can be replaced by equality. This proves statement (2).

To formulate Proposition 7 which is a driver of the Theorem 1 proof, we need the following simple observation to be also used several times in what follows.

**Lemma 6.** Let \( m, n \) be positive integers, \( m \geq n \), and \( f \in V_{m-n} \). Then there exist functions \( f^{(k)} \in V_m, k = 0, \ldots, p^n - 1 \), the wavelet system generated by which
coincides with that generated by \( f \). The functions \( f^{(k)} \) can be given by
\[
(4.5) \quad f^{(k)}(x) := f_{n,k/p^n}(x) = p^{n/2}f \left( \frac{x - k}{p^n} \right), \quad k = 0, \ldots, p^n - 1.
\]

**Proof.** The functions defined in (4.5) are what we need because
\[
f^{(k)} \left( \frac{x}{p^j} - b \right) = p^{n/2}f \left( \frac{x}{p^{j+n}} - \frac{k + b}{p^n} \right)
\]
for all \( j \in \mathbb{Z} \) and \( b \in \mathbb{Q}_p \), and
\[
I_p = \bigcup_{k=0}^{p^n-1} \left\{ \frac{k + b}{p^n} : b \in I_p \right\}. \quad \Box
\]

Below by a *non-trivial* linear combination of functions \( \psi^{(1)}, \ldots, \psi^{(r)} \) we mean any function \( \alpha_1\psi^{(1)} + \cdots + \alpha_r\psi^{(r)} \) such that \( \alpha_\nu \neq 0 \) for some \( \nu = 1, \ldots, r \).

**Proposition 7.** Let a vector-function \( \Psi \) of rank \( r \) with all components in \( V_m \) where \( m > 0 \), generate an ONWB. Suppose that there exists a non-trivial linear combination of the components which is in \( V_{m-1} \). Then \( \Psi \) is wavelet equivalent to a vector-function of rank \( r + p - 1 \), all components of which are in \( V_m \).

**Proof.** Let \( \Psi = (\psi^{(1)}, \ldots, \psi^{(r)})^T \) and let \( \alpha_1\psi^{(1)} + \cdots + \alpha_r\psi^{(r)} \) be a non-trivial linear combination belonging to \( V_{m-1} \). Set \( \Psi' = U\Psi \), where \( U \) is a unitary matrix whose first row is the normalized vector \((\alpha_1, \ldots, \alpha_r)\). The vector-function \( \Psi' \) is unitary equivalent to \( \Psi \), which yields that \( \text{rk}(\Psi') = r \) and \( \Psi' \) generates an ONWB. Since the first component of \( \Psi' \) is in \( V_{m-1} \) and the other components are in \( V_m \), by Lemma 6 there exists a vector function \( \tilde{\Psi} \) with all components in \( V_m \), such that \( \text{rk}(\tilde{\Psi}) = r + p - 1 \) and the wavelet systems generated by \( \Psi' \) and \( \tilde{\Psi} \) coincide. \( \Box \)

**Proposition 8.** If a periodic vector-function \( \Psi = (\psi^{(1)}, \ldots, \psi^{(r)})^T \) generates an ONWB, then
\[
\Psi(x - 1) = \sum_{j=-\infty}^{n} A_j \Psi(p^{-j}x), \quad x \in \mathbb{Q}_p,
\]
where \( n \geq 0 \) and \( A_j \) is an \( r \times r \) matrix with complex entries.

**Proof.** The functions \( \psi^{(\nu)}_j \), \( a \in I_p \), \( j \in \mathbb{Z} \), \( \nu = 1, \ldots, r \), form an orthonormal basis for \( L_2(\mathbb{Q}_p) \). Therefore, for every \( \nu \) we have
\[
\psi^{(\nu)}(x - 1) = \psi^{(\nu)}_0(x) = \sum_{j \in \mathbb{Z}} \sum_{a \in I_p} \sum_{\nu=1}^{r} \langle \psi^{(\nu)}_0, \psi^{(\nu)}_j \rangle \psi^{(\nu)}_j(x).
\]
If \( j \in \mathbb{Z} \) and \( a \in I_p \setminus \{0\} \), then there exists a rational number \( b = b(j, a) \), such that \( 1 - b \in I_p \), \( a - p^{-j}b \in I_p \), whence
\[
\langle \psi^{(\nu)}_0, \psi^{(\nu)}_j \rangle = \int_{\mathbb{Q}_p} \psi^{(\nu)}(x - 1)\overline{\psi^{(\nu)}_j(p^{-j}x - a)} \, dx =
\]
\[
p^{j/2} \int_{\mathbb{Q}_p} \psi^{(\nu)}(x - (1 - b))\overline{\psi^{(\nu)}_j(p^{-j}x - (a - p^{-j}b))} \, dx = 0,
\]
Then every component of which belongs to some $m$. Proposition 10. Let a vector-function pairwise orthogonal and use statement (2).

\[ \lambda \] corresponds to the eigenvalue $V$ (4.8)

\[ W \] prove statement (3) it suffices to note that by (2.5) the spaces subspace of $V$

the space $V$ whenever eigenvalues are orthogonal, the spaces $l$

Lemma 9. orthogonal. Moreover, by statement (1) any space $V_m$ can be decomposed into the orthogonal sum of its eigenspaces (see e.g. [5, Ch.1]):

\[ V_m = \bigoplus_{l=0}^{p^m-1} V_{m,l}. \]

(Indeed, if $f \in V_m$, then $f = \sum f_l$ where $f_l := p^{-m} \sum_j e^{2\pi i \frac{l}{p^m}} T^j f$ is in $V_{m,l}$).

Moreover, by statement (1) any space $V_m$ with $l$ coprime to $p$ is orthogonal to the space $V_{m-1} = \bigoplus_{l=0}^{p^{m-1}-1} V_{m,l,p}$ whereas any space $V_{m,l}$ with $l$ dividing $p$ is a subspace of $V_{m-1}$. Thus statement (2) follows from the definition of $W_{m-1}$. To prove statement (3) it suffices to note that by (2.5) the spaces $W_j$, $j \in \mathbb{Z}$, are pairwise orthogonal and use statement (2).

Proposition 10. Let a vector-function $\Psi$ generate an ONWB. Suppose that for some $m > 0$ any non-trivial linear combination of its components is in $V_m \setminus V_{m-1}$. Then $r(\Psi) = (p-1)p^m$ and $\Psi$ is unitary equivalent to an eigen vector-function every component of which belongs to $W_{m-1}$.
Due to Proposition 8, identity (4.6) holds with some \( n \geq 0 \). Assume that \( n > 0 \) and \( A_n \) is a non-zero matrix. Then by the hypothesis at least one component of the vector-function \( A_n \Psi(p^{-n}x) \) is in \( V_{m+n} \setminus V_{m+n-1} \), whereas all components of the left-hand side of (4.6) and of the sum \( \sum_{j=-\infty}^{n-1} A_j \Psi(p^{-j}x) \) are in \( V_{m+n-1} \), a contradiction. Therefore, \( n = 0 \), i.e.,

\[
\Psi(x - 1) = \sum_{j=-\infty}^{0} A_j \Psi(p^{-j}x), \quad x \in \mathbb{Q}_p.
\]

(4.9)

Iterating this identity \( p^m \) times and taking into account that \( \Psi(x - p^m) = \Psi(x) \) for all \( x \in \mathbb{Q}_p \), we obtain that every component of the vector-function \( (I - A_0^{p^m}) \Psi \) belongs to \( V_{m-1} \), where \( I \) is the \( r \times r \) identity matrix with \( r = \text{rk}(\Psi) \). But if \( I \neq A_0^{p^m} \), then by the hypothesis at least one component of this vector-function is in \( V_m \setminus V_{m-1} \), a contradiction. Thus,

\[
A_0^{p^m} = I.
\]

(4.10)

It follows from (4.9) and the Parseval equality that the Euclidean norm of each row of \( A_0 \) does not exceed 1. On the other hand, by the Adamic inequality the product of these norms is not less than \( |\det A_0| \) which equals 1 due to (4.10). Thus, \( A_0 \) is a unitary matrix. Again using the Parseval equality, we obtain that \( A_j = 0 \) for all \( j < 0 \), i.e.,

\[
\Psi(x - 1) = A_0 \Psi(x), \quad x \in \mathbb{Q}_p.
\]

(4.11)

Let \( \lambda_1, \ldots, \lambda_r \) be eigenvalues of \( A_0 \), and let \( D \) be the \( r \times r \) diagonal matrix with \( \lambda_1, \ldots, \lambda_r \) on the diagonal. There exists a unitary matrix \( U \) such that \( A_0 = UDU^{-1} \). Set \( \bar{\Psi} = U\Psi \), and rewrite (4.11) as

\[
\bar{\Psi}(x - 1) = D\bar{\Psi}(x), \quad x \in \mathbb{Q}_p.
\]

So every component of \( \bar{\Psi} \) is an eigenfunction of \( T \). It cannot belong to \( V_{m-1} \) by the hypothesis, thus it is in \( W_{m-1} \) by statement (2) of Lemma 9. \( \Box \)

**Proposition 11.** Let \( \Psi \) be an eigen vector-function generating an ONWB. Suppose that \( r(\Psi) = (p - 1)p^{m-1} \) and every component of \( \Psi \) belongs to \( W_{m-1} \). Then \( \Psi \) is wavelet equivalent to an eigen standard Haar vector-function.

**Proof.** Let \( \Psi = (\psi^{(1)}, \ldots, \psi^{(M)})^T \) where \( M = (p - 1)p^{m-1} \). By the hypothesis and statement (2) of Lemma 9 we have \( \psi^{(\nu)} \in V_{m,l_\nu} \) for all \( \nu = 1, \ldots, M \) where \( l_\nu \in S_m \). If the mapping

\[
F : \{1, \ldots, M\} \to S_m, \quad \nu \mapsto l_\nu
\]

is not a surjection, then there exists \( l \in S_m \) not belonging to the image of \( F \). So by statement (3) of Lemma 9, any function in the space \( V_{m,l} \neq \{0\} \) is orthogonal to the wavelet system generated by \( \Psi \) which is a basis for \( L_2(\mathbb{Q}_p) \) by the hypothesis, a contradiction. Since \( \#S_m = M \) we conclude that \( F \) is a bijection.

For \( \mu = 1, \ldots, p - 1 \) set

\[
f^{(\mu)}(x) = p^{1-m} \sum_{\nu \in T_\mu} \psi^{(\nu)}(p^{m-1}x), \quad x \in \mathbb{Q}_p,
\]

where

\[
T_\mu = F^{-1}(S_{m,\mu}) \quad \text{with} \quad S_{m,\mu} = \{l \in S_m : l \equiv \mu \pmod{p}\}.
\]
We have \( \|f^{(\mu)}\| = 1 \) for all \( \mu \) because \( \#T_\mu = #S_{m,\mu} = p^{m-1} \). Moreover,
\[
f^{(\mu)}(x - 1) = c \sum_{\nu \in T_\mu} \psi^{(\nu)}(p^{m-1}x - p^{m-1}) = c \sum_{\nu \in T_\mu} e^{-2\pi i \frac{\nu}{p^{m-1}}} \psi^{(\nu)}(p^{m-1}x) = c \sum_{\nu \in T_\mu} e^{-2\pi i \frac{\nu}{p} \psi^{(\nu)}(p^{m-1}x)} = e^{-2\pi i \frac{\mu}{p} f^{(\mu)}(x)}
\]
(4.12)
where \( c = p^{1-m} \).

By Lemma 6 the wavelet system generated by a function \( f^{(\mu)} \) coincides with that generated by the functions \( f^{(\mu)}_{m-1,k/p^{m-1}} \), \( k = 0, \ldots, p^{m-1} - 1 \), where
\[
f^{(\mu)}_{m-1,k/p^{m-1}}(x) = p^{(m-1)/2} f^{(\mu)} \left( \frac{x - k}{p^{m-1}} \right).
\]
On the other hand, since \( S_{m,\mu} = \{ l = \mu + pj : j = 0, \ldots, p^{m-1} - 1 \} \) we have
\[
f^{(\mu)}_{m-1,k/p^{m-1}}(x) = c^{1/2} \sum_{\nu \in T_\mu} \psi^{(\nu)}(x - k) = c^{1/2} \sum_{\nu \in T_\mu} e^{-2\pi i \frac{\nu}{p^{m-1}}} \psi^{(\nu)}(x) =
\]
\[
e^{\frac{n}{2} i \nu k \psi^{(\nu)}(x)}
\]
where \( \nu_j, \nu_l = F^{-1}(l) \) for \( l = \mu + pj \). However it is well known that the matrix
\[
\left(p^{n/2} e^{-2\pi i \frac{k}{p^{m-1}}}ight)_{j,k=0}^{p^{m-1} - 1}
\]
is unitary for any integer \( n \geq 0 \). So the system of functions
\[
\{ f^{(\mu)}_{m-1,k/p^{m-1}} : k = 0, \ldots, p^{m-1} - 1 \}
\]
is unitary equivalent to the system \( \{ \psi^{(\nu)}_{j,p^{m-1}} : j = 0, \ldots, p^{m-1} - 1 \} = \{ \psi^{(\nu)} : \nu \in T_\mu \} \). Since the sets \( T_\mu \) are pairwise disjoint and the union of them equals \( \{1, \ldots, M\} \), we conclude that the vector-function \( \Psi' = (f^{(1)}, \ldots, f^{(p-1)}) \) is wavelet equivalent to \( \Psi \). The components of \( \Psi \) are in \( W_{m-1} \), so the components of \( \Psi' \) are in \( W_0 \).
Thus, \( \Psi' \) is a standard Haar vector function. It is eigen due to (4.12).

5. Proof of main results

Proof of Theorems 1 and 2. In what follows, we say that a vector-function is in \( V_j \) if all its components are in \( V_j \). Let \( \Psi = \Psi_0 \) be a \( p^{m} \)-periodic vector-function generating an ONWB. Then \( \Psi_0 \) is in \( V_m \) and \( m > 0 \) because of Proposition 5. Assume that some non-trivial linear combination of components of \( \Psi_0 \) is in \( V_m \). Due to Proposition 7, \( \Psi_0 \) is wavelet equivalent to a vector-function \( \Psi_1 \) in \( V_m \), such that \( \text{rk}(\Psi_1) = \text{rk}(\Psi_0) + p - 1 \). Similarly, if some non-trivial linear combination of components of \( \Psi_1 \) is in \( V_{m-1} \), then \( \Psi_1 \) is wavelet equivalent to a vector-function \( \Psi_2 \) in \( V_m \), such that \( \text{rk}(\Psi_2) = \text{rk}(\Psi_1) + p - 1 \). Continue this process while possible. At each step we obtain a new vector-function belonging to \( V_m \) and generating an ONWB. The rank of the vector-functions strictly increases. However by Proposition 5, it does not exceed \( (p - 1) p^{m-1} \). Hence the process will stop after a finite
number of steps, say $N$, i.e., $\Psi$ is wavelet equivalent to a vector-function $\Psi_N$ such that any non-trivial linear combination of its components is in $V_m \setminus V_{m-1}$ (in particular, it may happen that $N = 0$). By Proposition 10, $\text{rk}(\Psi_N) = (p - 1)p^{m-1}$. So
\[
\text{rk}(\Psi) = \text{rk}(\Psi_0) \equiv \text{rk}(\Psi_1) \equiv \cdots \equiv \text{rk}(\Psi_N) \equiv 0 \pmod{p - 1}
\]
whence Theorem 2 follows. Besides, Propositions 10 and 11 imply that $\Psi_N$ (and hence $\Psi$) is wavelet equivalent to an eigen standard Haar vector-function, which proves the first part of Theorem 1.

Now suppose that every component of $\Psi$ is compactly supported. By above without loss of generality we can also assume that $\Psi = (\psi(1), \ldots, \psi(p-1))^T$ is a standard Haar vector function. Every function $\psi(\mu)$ is in $W_0$. So it can be expanded on the basis $\theta_{0,a}^{(\nu)}$, $a \in I_p$, $\nu = 1, \ldots, p - 1$, of this space where $\theta^{(\nu)}$ is the $\nu$-th component of the basic Haar vector-function $\Theta$. Since $\psi(\mu)$ is compactly supported, the expansion is finite. Therefore, there exists an integer $n \geq 0$ such that
\[
(5.1) \quad \psi(\mu) = \sum_{\nu=1}^{p-1} \sum_{k=0}^{p^n-1} c^{\mu}_{\nu,k} \theta^{(\nu)}_{0,k/p^n}, \quad \mu = 1, 2, \ldots, p-1,
\]
where $c^{\mu}_{\nu,k}$ is a complex number.

Let us denote by $\Psi'$ (resp. $\Theta'$) the vector-function of rank $(p - 1)p^n$ with components numerated by pairs $(\nu, k)$ where $\nu = 1, \ldots, p - 1$ and $k = 0, \ldots, p^n - 1$, such that the $(\nu, k)$-th component equals $\theta^{(\nu)}_{n,k/p^n}$ (resp. $\psi^{(\nu)}_{n,k/p^n}$). Due to Lemma 6, the vector-functions $\Psi'$ and $\Psi$ (as well as $\Theta'$ and $\Theta$) generate the same ONWB. So, it only remains to check that $\Psi'$ and $\Theta'$ are unitary equivalent.

To do this, from (5.1) we derive that
\[
\psi^{(\mu)}_{n,l/p^n} = \sum_{\nu=1}^{p-1} \sum_{k=0}^{p^n-1} c^{\mu}_{\nu,k} \theta^{(\nu)}_{n,k+l/p^n}, \quad l = 0, \ldots, p^n - 1,
\]
and taking into account that $\theta^{(\nu)}_{n,(a+p^n)/p^n} = \chi_p \left( -\frac{x}{p^n} \right) \theta^{(\nu)}_{n,a/p^n}$ for all $a \in \mathbb{Q}_p$. Due to (3.1), we conclude that each function $\psi^{(\mu)}_{n,l/p^n}$ is a linear combination of the functions $\theta^{(\nu)}_{n,k/p^n}$, $\nu = 1, \ldots, p - 1$, $k = 0, \ldots, p^n - 1$. Hence,
\[
\Psi' = U \Theta'
\]
where $U$ is a matrix with complex entries. But the matrix $U$ is unitary because the components of each of the vector-functions $\Psi'$ and $\Theta'$ form an orthonormal system of the same rank.$\checkmark$

**Proof of Theorem 3.** Let $p = 2$, and $\theta = \theta^{(1)}$. For $k = 0, 1$ set
\[
f^{(k)}(x) = \frac{\sqrt{2}}{2} \sum_{l=0}^{1} e^{-2\pi i \frac{(l+1)(l+2)}{4}} \theta_{1,l/2}(x) = \sum_{l=0}^{1} e^{-2\pi i \frac{(l+1)(l+2)}{4}} \theta \left( \frac{x - l}{2} \right).
\]
The vector-function $(f^{(0)}, f^{(1)})$ generates an ONWB because it is unitary equivalent to $(\theta_{1,0}, \theta_{1,1/2})$ and by Lemma 6 the latter generates the same ONWB as the basic vector-function $\Theta = (\theta)$. Note that $f^{(0)}, f^{(1)} \in W_1$ and
\[
(5.2) \quad f^{(0)}(x - 1) = if^{(0)}(x), \quad f^{(1)}(x - 1) = -if^{(1)}(x).
\]
For $k = 0, 1, 2, 3$ set
\[
g^{(k)}(x) = \frac{1}{2} \sum_{l=0}^{3} e^{-2\pi i \frac{(l+4k)}{16}} f^{(0)}_{2^{-l+4}}(x) = \sum_{l=0}^{3} e^{-2\pi i \frac{(l+4k)}{16}} f^{(0)}(x - \frac{l}{4}).
\]

We note that $g^{(k)} \in W_3$. Moreover, using (5.2) one can easily check that
\[
g^{(k)}(x - 1) = e^{2\pi i \frac{3+4k}{16}} g^{(k)}(x).
\]
The vector-function $(f^{(0)}, g^{(0)}, g^{(1)}, g^{(2)}, g^{(3)})^T$ generates an ONWB because from (5.3) it follows that it is unitary equivalent to $(f^{(0)}, f^{(1)}_{2,0}, f^{(1)}_{2,1/4}, f^{(1)}_{2,1/2}, f^{(1)}_{2,3/4})^T$ and by Lemma 6 the latter generates the same ONWB as $(f^{(0)}, f^{(1)})^T$.

Let $h^{(0)} = af^{(0)} + bg^{(2)} + cg^{(3)}$ where $a, b, c \in \mathbb{C}$. Set
\[
h^{(1)}(x) = h^{(0)}(x - 1).
\]
Then $h^{(1)} = \zeta^4 af^{(0)} + \zeta^{11} bg^{(2)} + \zeta^{15} cg^{(3)}$ by (5.2) and (5.4) where $\zeta = e^{2\pi i/16}$. So
\[
(h^{(0)}, h^{(1)}) = \zeta^{12} |a|^2 + \zeta^5 |b|^2 + |c|^2
\]
Take $a, b, c$ so that $|a|^2 + |b|^2 + |c|^2 = 1$ and the right-hand side of the last equality equal 0 (e.g., $a = \lambda, b = \lambda \sqrt{\cos \pi/8}, c = \lambda \sqrt{\sin \pi/8}$, where $\lambda$ is a positive real).

Then there exists a unitary matrix of the form
\[
U = \begin{pmatrix} a & b & c \\ \zeta^4 a & \zeta^{11} b & \zeta^{15} c \\ \alpha & \beta & \gamma \end{pmatrix}.
\]
Set $h^{(3)} = \alpha f^{(0)} + \beta g^{(2)} + \gamma g^{(3)}$. Then $(h^{(0)}, h^{(1)}, h^{(2)})^T = U(f^{(0)}, g^{(2)}, g^{(3)})^T$. So the vector function $(g^{(0)}, g^{(1)}, h^{(0)}, h^{(1)}, h^{(2)})^T$ generates an ONWB. Finally, set
\[
h(x) = \sqrt{\frac{7}{2}} h^{(0)}(2x).
\]
Then $h^{(0)} = h_{1,0}$, $h^{(1)} = h_{1,1/2}$, and consequently by Lemma 6 the vector function
\[
\Psi = (g^{(0)}, g^{(1)}, h^{(2)}, h)^T
\]
also generates an ONWB. We observe that $g^{(0)}, g^{(1)}, h^{(2)} \in V_4$ and $h \in V_3$. Moreover, since $(\beta, \gamma) \neq (0, 0)$, the $W_3$-parts of the functions $g^{(0)}, g^{(1)}, h^{(2)}$ with respect to decomposition (2.5) span a linear space of dimension 3.

Suppose that $\Psi = \Psi_0$ is reduced to a standard Haar vector-function $(\psi) = \Psi_N$ in $N$ steps. There are two kind of steps: unitary ones for which $\text{rk}(\Psi_j) = \text{rk}(\Psi_{j-1})$, and decreasing ones for which $\text{rk}(\Psi_j) = \text{rk}(\Psi_{j-1}) - 1$. Without loss of generality it can be assumed that they alternate, the first step is unitary and the last is decreasing. Since $\text{rk}(\Psi_0) = 4$ and $\text{rk}(\Psi_N) = 1$, we have $N = 6$, $\text{rk}(\Psi_4) = 4$ and $\text{rk}(\Psi_2) = 3$. Moreover, $\Psi_1$ is in $V_4$, $\Psi_2$ is in $V_3$ and the $W_3$-parts of components of $\Psi_1$ span a linear space of dimension at most 2, by the definition of decreasing step. So $\Psi_0$ as unitary equivalent to $\Psi_1$, also has this property. However, the dimension of the corresponding space for $\Psi$ equals 3 by above, a contradiction. \(\Box\)
References

1. S. Albeverio, S. Evdokimov and M. Skopina, p-adic multiresolution analysis and wavelet frames, J. Fourier Anal. Appl., Vol. 16, Number 5, (2010), 693–714.
2. S.A. Evdokimov and M.A. Skopina, 2-Adic wavelet bases, Proceedings of Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences, 15 (2009), N 1, 135–146 (Russian).
3. S. Albeverio, S. Evdokimov, and M. Skopina, p-Adic Nonorthogonal Wavelet Bases, Proceedings of the Steklov Institute of Mathematics, 265 (2009), 1–12.
4. Evdokimov, S. Haar multiresolution analysis and Haar bases on the ring of rational adeles. Zap. Nauchn. Sem. POMI, 400 (2012), 158-165 (Russian); translation in Journal of Mathematical Sciences, 192 (2013), pp. 215–219.
5. J.-P. Serre, Linear representations of finite groups. Graduate Texts in Mathematics, vol. 42, Springer-Verlag, New York, Heidelberg, Berlin, 1977
6. Mallat, S. (1988). Multiresolution representation and wavelets, Ph. D. Thesis, University of Pennsylvania, Philadelphia, PA.
7. Meyer, Y. (Décembre 1986). Ondelettes et fonctions splines, Séminaire EDP. Paris.
8. S.V. Konyrev, Wavelet analysis as a p-adic spectral analysis, Izvestia Akademii Nauk, Seria Math. 66 no. 2 (2002) 149–158.
9. Daubechies I. Ten Lectures on wavelets, CBMS-NSR Series in Appl. Math., SIAM, 1992.
10. J.J. Benedetto, and R.L. Benedetto, A wavelet theory for local fields and related groups, The Journal of Geometric Analysis 3 (2004) 423–456.
11. Khrennikov, A. Yu. and Shelkovich, V. M. Non-Haar p-adic wavelets and their application to pseudo-differential operators and equations. Appl. Comput. Harmon. Anal. 28 (2010), no. 1, 1-23.
12. V. M. Shelkovich and M. Skopina, p-Adic Haar multiresolution analysis and pseudo-differential operators, J. Fourier Analysis and Appl., 15 (2009), N 3, 366-393.
13. A. Yu. Khrennikov, V.M. Shelkovich, M, Skopina, p-Adic refinable functions and MRA-based wavelets. J. Approx. Theory 161 (2009) 226-238.
14. A. Yu. Khrennikov, V.M. Shelkovich, M, Skopina, p-Adic Orthogonal Wavelet Bases, p-Adic Numbers, Ultrametric Analysis and Applications, 1 (2009), No 2, 145-156.
15. Lang, W. C., Orthogonal wavelets on the Cantor dyadic group. SIAM J. Math. Anal. 27 (1996), no. 1, 305312.
16. Protasov, V. Yu.; Farkov, Yu. A. Dyadic wavelets and scaling functions on a half-line. Mat. Sb. 197 (2006), no. 10, 129–160 (Russian); translation in Sb. Math. 197 (2006), no. 9-10, 1529-1558.
17. Farkov, Yu. A., Orthogonal wavelets on direct products of cyclic groups. Mat. Zametki 82 (2007), no. 6, 934–952 (Russian); translation in Math. Notes 82 (2007), no. 5-6, 843-859.
18. Behera, B., Jahan, Q., Multiresolution analysis on local fields and characterization of scaling functions. Adv. Pure Appl. Math. 3 (2012), no. 2, 181202.
19. Behera, B., Jahan, Q., Wavelet packets and wavelet frame packets on local fields of positive characteristic. J. Math. Anal. Appl. 395 (2012), no. 1, 114.
20. Novikov, I. Ya., Protasov, V. Yu., Skopina, M. A., Wavelet Theory. AMS, Translations Mathematical Monographs, V. 239 (2011).
21. V.S. Vladimirov, I.V. Volovich and E.I. Zelenov, p-Adic analysis and mathematical physics. World Scientific, Singapore, 1994.