Stationary Stochastic Viscosity Solutions of SPDEs

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Summary. In this paper we aim to find the stationary stochastic viscosity solutions of a parabolic type SPDEs through the infinite horizon backward doubly stochastic differential equations (BDSDEs). For this, we study the existence, uniqueness and regularity of solutions of the corresponding infinite horizon BDSDEs as well as the “perfection procedure” applied to the solutions of BDSDEs. At last the “perfect” stationary stochastic viscosity solutions of SPDEs constructed by solutions of corresponding BDSDEs are obtained.

Keywords: stochastic partial differential equations, backward doubly stochastic differential equations, stochastic viscosity solutions, stationary solutions, random dynamical systems.

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1 Introduction

The pathwise stationary solution of a stochastic dynamical system is one of the fundamental concepts in the study of the long time behaviour of the stochastic dynamical systems. It describes the pathwise invariance of the stationary solution, over time, along the measurable and $P$-preserving transformation $\theta_t : \Omega \rightarrow \Omega$ and the pathwise limit of the solutions of the random dynamical systems:

$$u(t, Y(\omega), \omega) = Y(\theta_t \omega) \quad t \geq 0, \ a.s.,$$

where $u : [0, \infty) \times U \times \Omega \rightarrow U$ is a measurable random dynamical system on a measurable space $(U, B)$ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \geq 0})$ and $Y : \Omega \rightarrow U$ is a $\mathcal{F}$-measurable stationary solution. Needless to say that the “one-force, one-solution” setting is a natural extension of the equilibrium or fixed point in the theory of the deterministic dynamical systems to stochastic counterparts. Such a random fixed point consists of infinitely many randomly moving invariant surfaces on the configuration space due to the random external force pumped to the system constantly. Therefore, in contrast to the deterministic dynamical systems, the existence and stability of stationary solutions of stochastic dynamical systems, generated e.g. by SDEs or SPDEs, are a difficult and subtle problem.

In many works on random dynamical systems the existence of stationary solutions is a basic assumption, e.g. in the study of stability (Has'minskii [11]) and in the theory of stable and unstable manifolds (Arnold [1], Mohammed, Zhang and Zhao [15], Duan, Lu and Schmalfuss [10]). These theories gave neither the existence of stationary solutions, nor a way to find them. Although in [15], Mohammed, Zhang and Zhao introduced an integral equation of infinite horizon for the stationary
solutions of certain stochastic evolution equations, the existence of the solutions of such stochastic integral equations in general is far from clear.

Besides, from a pathwise stationary solution we can construct an invariant measure for the skew product of the metric dynamical system and the random dynamical system. The invariant measure describes the invariance of a certain solution in law when time changes, therefore it is a stationary measure of the Markov transition probability. It is well known that an invariant measure gives a stationary solution when it is a random Dirac measure. Although an invariant measure of a random dynamical system on $\mathbb{R}^1$ gives a stationary solution, in general, this is not true unless one considers an extended probability space. However, considering the extended probability space, one essentially regards the random dynamical system as noise as well, so the dynamics is different. In fact, the pathwise stationary solution gives the support of the corresponding invariant measure, so reveals more detailed information than an invariant measure.

In spite of the importance of stationary solution, the difficulties, arising mainly from random external force, prevent researchers from finding a method universal to the stationary solutions of SPDEs with great generalities. Some works on stationary solutions of certain types of SPDEs usually under additive or linear noise include Sinai [20], [21] for stochastic Burgers’ equations with periodic or random forcing, Caraballo, Kloeden, Schmalfuss [8] for stochastic evolution equations with small Lipschitz constant. If one notices the solutions of infinite horizon backward stochastic differential equations (BSDEs) give a classical or viscosity solution of elliptic type PDEs (Poisson equations) from the works of Peng [19] and Pardoux [16], then it would be natural to conjecture the stationary solutions of SPDEs can be represented as the solutions of infinite horizon backward doubly stochastic differential equations (BDSDEs). Inspired by this idea, Zhang and Zhao in [22] proved that under the Lipschitz and monotone conditions, the $L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^1) \otimes L^2_{\rho}(\mathbb{R}^d; \mathbb{R}^d)$ valued solution of an infinite horizon BDSDE exists and gives the stationary weak solution of the corresponding parabolic SPDE. Zhang and Zhao further considered this problem under the linear growth and monotone conditions in [23]. It is easy to see the solutions of elliptic type PDEs give the stationary solutions of the corresponding parabolic type PDEs, however, for SPDEs of the parabolic type, such kind of connection does not exist, so in this sense BDSDEs (or BSDEs) can be regarded as more general SPDEs (or PDEs).

The stochastic viscosity solution of SPDE was first put forward by Lions and Souganidis in [14] through stochastic characteristics to remove the stochastic integrals in the SPDE. Then Buckdahn and Ma in [15]-[17] gave their definition through the Doss-Sussmann transformation. After that a few works on stochastic viscosity solutions of SPDEs emerge using Buckdahn and Ma’s definition and corresponding BDSDEs, such as Boufoussi, Van Casteren and Mrhardy [4] for the SPDEs with Neumann boundary conditions, Boufoussi and Mrhardy [3] for the multivalued SPDEs. Then an interesting question arises: can we also find the stationary solution of some SPDE in the sense of stochastic viscosity solution? This paper gives this question a positive answer. By adopting Buckdahn and Ma’s definition and using its connection with BDSDE we can find the stationary stochastic viscosity solution of the following SPDE:

$$v(t,x) = v(0,x) + \int_0^t [\mathcal{L}v(s,x) + f(x,v(s,x),\sigma^*(s,x)Dv(s,x))]ds + \int_0^t (g(x,v(s,x)),dB_s) + \int_0^t \langle g(x,v(s,x)),dB_s \rangle. \quad (1.2)$$

Here $(B_t)_{t \geq 0}$ is a Brownian motion with values in $\mathbb{R}^l$; $f, g$ satisfy the condition $(A.1)$-$(A.3)$ in Section 2; $\mathcal{L}$ is the infinitesimal generator of the diffusion process $X^{t,x}$ generated by the SDE as follows:

$$\begin{align*}
\int dX^{t,x}_s &= b(X^{t,x}_s)ds + \sigma(X^{t,x}_s)dW_s, \quad s > t \\
X^{t,x}_s &= x, \quad 0 \leq s \leq t,
\end{align*} \quad (1.3)$$
Here the integral w.r.t. $\hat{v}$ forward and backward Itô’s integral. Our purpose is to prove that, for arbitrary $T > 0$ and all $s \in \mathbb{R}^1$, the integral w.r.t. $\hat{B}$ is a backward Itô’s integral (see [22] for details and the relationship between the forward and backward Itô’s integral). Our purpose is to prove that, for arbitrary $T > 0$ and $0 < t \leq T$, $v(t,x)(\omega) = Y_{T-t,x}^T(\omega)$ is a stationary stochastic viscosity solution of SPDE (1.2). Five sections are organized in this paper for this purpose. In next section we give brief introduction to the notion of stochastic viscosity solutions of SPDEs and the connection between SPDEs and BDSDEs in the sense of stochastic viscosity solution. In Section 3 under the assumption of the existence, uniqueness and regularity of solution to infinite horizon BDSDE, we study its stationary property, in which the general version “perfection procedure” plays an important role. The existence, uniqueness and regularity of solution to infinite horizon BDSDE are proved in Section 4. In Section 5 we deduce the stationary property for the stochastic viscosity solutions of SPDEs constructed by the solutions of infinite horizon BDSDEs.

As far as we know, the connection between the pathwise stationary stochastic viscosity solutions of SPDEs and infinite horizon BDSDEs in this paper is new. By the techniques as we dealt with the weak solutions of PDEs or SPDEs in [23] and [24], we believe this connection can be extended to studying the stationary stochastic viscosity solutions of more general parabolic SPDEs such as those with linear or polynomial growth nonlinear terms, more types of noises etc., but in this paper we only study Lipschitz continuous nonlinear term and finite dimensional noise for simplicity in order to initiate this method to the case of stationary stochastic viscosity solutions of SPDEs. Finally we would like to point out that the uniqueness of the stationary solution of SPDE (1.2) is still an open problem due to its complexity.

2 Definition and Results for Stochastic Viscosity Solutions of SPDEs

The main purpose of this paper is to find the stationary stochastic viscosity solution of SPDE (1.2). As shown in [22] and [23], under appropriate conditions, for $T \geq t \geq 0$, defining $u(t,x) \equiv v(T-t,x)$, we can obtain the time reverse version of SPDE (1.2) on $[0,T]$: $u(t,x) = u(T,x) + \int_t^T [\mathcal{L}u(s,x) + f(x,u(s,x), (\sigma^* \nabla u)(s,x))]ds - \int_t^T \langle g(x,u(s,x)), d\hat{B}_s \rangle$. (2.1)
The BDSDE on \([t, T]\) associated with SPDE (2.1) has the following form:

\[ Y_{s,x}^t = Y_{T,x}^t + \int_s^T f(X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr - \int_s^T \langle g(X_r^{t,x}, Y_r^{t,x}), dB_r \rangle - \int_s^T \langle Z_r^{t,x}, dW_r \rangle. \]  

(2.2)

For \(k, l \geq 0\), we denote by \(C_{b}^{k,l}\) the set of \(C^{k,l}\)-functions whose partial derivatives of order for the first variable less than or equal to \(k\) and for the second variable less than or equal to \(l\) are bounded. We assume

(A.1). Functions \(f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}\) and \(g : \mathbb{R}^d \times \mathbb{R}^1 \to \mathbb{R}^d\) are \(\mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{\mathbb{R}^1} \otimes \mathcal{B}_{\mathbb{R}^d}\) measurable respectively, and there exist constants \(C_0, C_1, C \geq 0\) s.t. for any \((x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d\),

\[
|f(x_1, y_1, z_1) - f(x_2, y_2, z_2)|^2 \leq C_0 |x_1 - x_2|^2 + C_1 |y_1 - y_2|^2 + C |z_1 - z_2|^2,
\]

\[
|g(x_1, y_1) - g(x_2, y_2)|^2 \leq C_0 |x_1 - x_2|^2 + C |y_1 - y_2|^2;
\]

(A.2). \(g(\cdot, \cdot) \in C_{b}^{2,3}(\mathbb{R}^d \times \mathbb{R}^1; \mathbb{R}^d);\)

(A.3). There exist constants \(K \in \mathbb{R}^+, p > d + 2, K < K' < 2K\) and \(\mu > 0\) with \(2\mu - \frac{\mu}{2} K' - \frac{p(p+1)}{2} C > 0\) s.t. for any \(y_1, y_2 \in \mathbb{R}^1, x, z \in \mathbb{R}^d,\)

\[(y_1 - y_2)(f(x, y_1, z) - f(x, y_2, z)) \leq -\mu |y_1 - y_2|^2;\]

(A.4). Functions \(b(\cdot) : \mathbb{R}^d \to \mathbb{R}^d, \sigma(\cdot) : \mathbb{R}^d \to \mathbb{R}^{d \times d}\) are globally Lipschitz continuous with Lipschitz constant \(L\) and for \(p, K\) in (A.3), \(K - pL - \frac{p(p-1)}{2} L^2 > 0\).

Denote the set of \(C^{0,0}\)-functions with linear growth by \(C_{l}^0\). Buckdahn and Ma proved that if \(u(T, \cdot) \in C_{l}^0(\mathbb{R}^d; \mathbb{R}^1)\) is given, the solution \(Y_{t,x}^u\) of BDSDE (2.2), \((t, x) \in [0, T] \times \mathbb{R}^d\), is a stochastic viscosity solution of SPDE (2.1) under Conditions (A.1), (A.2) and (A.4), therefore it gives the stochastic viscosity solution of SPDE (1.2) through the time reversal argument. To benefit the reader, we include briefly Buckdahn and Ma’s definition of stochastic viscosity solution of SPDE (1.2) through the Doss-Sussmann transformation in [3]-[7].

Let \(\mathcal{N}\) be the class of \(\mathcal{P}\) null measurable sets of \(\mathcal{F}\). For any process \((\eta_t)_{t \geq 0}, \mathcal{F}_{s,t}^\eta \triangleq \sigma \{\eta_r - \eta_s; 0 \leq s < r \leq t\} \setminus \mathcal{N}\), \(\mathcal{F}_s^\eta \triangleq \mathcal{F}_{0,s}^\eta, \mathcal{F}_{t,\infty}^\eta \triangleq \bigcap_{T \geq 0} \mathcal{F}_{t,T}^\eta\). Let \(\mathbb{E}\) and \(\mathbb{F}\) be the generic Euclidean spaces, then we denote

- \(\mathcal{M}_{[0,T]}\) to be all the \(\{\mathcal{F}_t^B\}_{t \geq 0}\) stopping times \(\tau\) such that \(0 \leq \tau \leq T\) a.s., where \(T > 0\) is some fixed time horizon;

- for any sub-\(\sigma\)-field \(\mathcal{G} \subseteq \mathcal{F}_t^B\) and real number \(p \geq 0\), \(L^p(\mathcal{G}; \mathbb{E})\) to be \(\mathbb{E}\)-valued, \(\mathcal{G}\)-measurable random variables \(\xi\) such that \(E[|\xi|^p] < \infty;\)

- for any sub-\(\sigma\)-field \(\mathcal{G} \subseteq \mathcal{F}_t^B, C^{k,l}(\mathcal{G}, [0, T] \times \mathbb{E}; \mathbb{F})\) to be the space of all \(C^{k,l}([0, T] \times \mathbb{E}; \mathbb{F})\)-valued random variables that are \(\mathcal{G} \otimes \mathcal{B}_{[0,T]} \otimes \mathcal{B}_\mathbb{F}\)-measurable;

- \(C^{k,l}(\{\mathcal{F}_t^B\}_{t \geq 0}, [0, T] \times \mathbb{E}; \mathbb{F})\) to be the space of all random fields \(\varphi \in C^{k,l}(\mathcal{F}_t^B, [0, T] \times \mathbb{E}; \mathbb{F}),\) such that for fixed \(x \in \mathbb{E},\) the mapping \((t, \omega) \to \varphi(t, x, \omega)\) is \(\mathcal{F}_t^B\)-progressively measurable.

The definition of stochastic viscosity solution depends heavily on the following stochastic flow \(\lambda \in C^{0,0}([\mathcal{F}_t^B]_{t \geq 0}, [0, T] \times \mathbb{R}^d \times \mathbb{R}^1; \mathbb{R}^1),\) defined as the unique solution of the following SDE

\[\lambda(t, x, y) = y + \frac{1}{2} \int_0^t \langle g(D_y g)(x, \lambda(s, x, y)), dB_s \rangle + \int_0^t \langle g(x, \lambda(s, x, y)), dW_s \rangle.\]
Under Condition (A.2), \(\lambda(t, x, y)\) is a stochastic flow, i.e. for fixed \(x\), the random field \(\lambda(t, x, y)\) is continuously differentiable in the variable \(y\), and the mapping \(y \mapsto \lambda(t, x, y)\) defines a diffeomorphism for all \((t, x)\), \(P\text{-a.s.}\). Denote the inverse of \(\lambda\) by \(\zeta(t, x, y) = (\lambda(t, x, \cdot))^{-1}(y)\).

**Definition 2.1** (\([5]\)) A random field \(w \in C^{0,0}([\mathcal{F}_t^B]_{t \geq 0}, [0, T] \times \mathbb{R}^d; \mathbb{R}^1)\) is called a stochastic viscosity subsolution (resp. supersolution) of SPDE (1.2), if \(w(0, x) \leq \) (resp. \(\geq\)) \(v(0, x)\), \(\forall x \in \mathbb{R}^d\); and if for any \(\tau \in \mathcal{M}_0, \xi \in L^0(\mathcal{F}_T; \mathbb{R}^d)\), and any random field \(\phi \in C^{1,2}(\mathcal{F}_T; [0, T] \times \mathbb{R}^d; \mathbb{R}^1)\) satisfying

\[
\begin{align*}
\lambda(t, x, \tau, \xi, \phi) - \phi(t, x) - D_t \phi(t, x) - \mathcal{L} \phi(t, x) &\geq \text{(resp.} \geq\text{) 0} = w(t, \xi) - \lambda(t, \xi, \phi(t, \xi)),
\end{align*}
\]

for all \((t, x)\) in a neighborhood of \((\tau, \xi)\), \(P\text{-a.e.}\) on the set \(\{0 < \tau < T\}\), it holds that

\[
\begin{align*}
\mathcal{L} \phi(t, \xi, \tau, \xi) + f(\xi, \phi(t, \xi), \sigma^*(\xi)D \phi(t, \xi)) &\geq \text{(resp.} \leq\text{) } D_t \lambda(t, \xi, \phi(t, \xi)) D_t \phi(t, \xi),
\end{align*}
\]

\(P\text{-a.e.}\) on \(\{0 < \tau < T\}\), where \(\psi(t, x) \triangleq \lambda(t, x, \phi(t, x))\).

A random field \(w \in C^{0,0}([\mathcal{F}_t^B]_{t \geq 0}, [0, T] \times \mathbb{R}^d; \mathbb{R}^1)\) is called a stochastic viscosity solution of SPDE (1.2), if it is both a stochastic viscosity subsolution and a supersolution.

By Doss-Sussmann transformation, SPDE (1.2) can be converted to the following PDE

\[
\begin{align*}
\tilde{v}(t, x) &= \tilde{v}(0, x) + \int_0^t \left[\mathcal{L} \tilde{v}(s, x) + \tilde{f}(s, x, \tilde{v}(s, x), \sigma^*(x)D \tilde{v}(s, x))\right] ds,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{f}(t, x, y, z) = \frac{1}{D_y \lambda(t, x, y)} &\left(f(x, \lambda(t, x, y), \sigma^*(x)D_x \lambda(t, x, y) + D_y \lambda(t, x, y)z)\right) \\
&+ \mathcal{L} \lambda(t, x, y) + \langle \sigma^*(x)D_{xy} \lambda(t, x, y), z \rangle + \frac{1}{2}D_{yy} \lambda(t, x, y)|z|^2,
\end{align*}
\]

and the stochastic viscosity solutions of (1.2) and (2.3) have a kind of relationship like \(\tilde{v}(t, x) = \zeta(t, x, v(t, x))\). The Doss-Sussman transformation plays a big role in the notion of the stochastic viscosity solution of SPDE (1.2). For more details, see Buckdahn and Ma [3-7].

Define

\[
\mathcal{F}_{t,T} \triangleq \mathcal{F}_t^B \bigvee_{S} \mathcal{F}_t^W, \text{ for } 0 \leq t \leq T; \quad \mathcal{F}_t \triangleq \mathcal{F}_{t,\infty}^B \bigvee_{S} \mathcal{F}_t^W, \text{ for } t \geq 0.
\]

For \(q \geq 2\), we define some useful solution spaces.

**Definition 2.2** Let \(S\) be a Banach space with norm \(\|\cdot\|_S\) and Borel \(\sigma\)-field \(\mathcal{F}\). For \(K \in \mathbb{R}^+\), we denote by \(M^{q-K}([0, \infty); S)\) the set of \(\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F} / \mathcal{F}\) measurable random processes \(\{\phi(s)\}_{s \geq 0}\) with values in \(S\) satisfying

(i) \(\phi(s) : \Omega \to S\) is \(\mathcal{F}_s\) measurable for \(s \geq 0\);

(ii) \(E[\int_0^\infty e^{-Ks} \|\phi(s)\|_S ds] < \infty\).

Also we denote by \(S^{q-K}([0, \infty); S)\) the set of \(\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F} / \mathcal{F}\) measurable random processes \(\{\psi(s)\}_{s \geq 0}\) with values in \(S\) satisfying

(i) \(\psi(s) : \Omega \to S\) is \(\mathcal{F}_s\) measurable for \(s \geq 0\) and \(\psi(\cdot, \omega)\) is continuous \(P\text{-a.s.}\);

(ii) \(E[\sup_{s \geq 0} e^{-Ks} \|\psi(s)\|_S] < \infty\).

Similarly, for \(0 \leq t \leq T < \infty\), we define \(M^{q,0}([t, T]; S)\) and \(S^{q,0}([t, T]; S)\) on a finite time interval.
**Definition 2.3** Let $\mathbb{S}$ be a Banach space with norm $\| \cdot \|_{\mathbb{S}}$ and Borel $\sigma$-field $\mathcal{F}$. We denote by $M^{q,0}([t,T];\mathbb{S})$ the set of $\mathcal{B}_{[t,T]} \otimes \mathcal{F}/\mathcal{F}$ measurable random processes $\{\phi(s)\}_{t \leq s \leq T}$ with values in $\mathbb{S}$ satisfying

(i) $\phi(s) : \Omega \rightarrow \mathbb{S}$ is $\mathcal{F}_{s,T} \vee \mathcal{F}_{T,\infty}$ measurable for $t \leq s \leq T$;
(ii) $E[\int_{t}^{T} \|\phi(s)\|_{\mathbb{S}}^q ds] < \infty$.

Also we denote by $S^{q,0}([t,T];\mathbb{S})$ the set of $\mathcal{B}_{[t,T]} \otimes \mathcal{F}/\mathcal{F}$ measurable random processes $\{\psi(s)\}_{t \leq s \leq T}$ with values in $\mathbb{S}$ satisfying

(i) $\psi(s) : \Omega \rightarrow \mathbb{S}$ is $\mathcal{F}_{s,T} \vee \mathcal{F}_{T,\infty}$ measurable for $t \leq s \leq T$ and $\psi(\cdot, \omega)$ is continuous $P$-a.s.;
(ii) $E[\sup_{t \leq s \leq T} \|\psi(s)\|_{\mathbb{S}}^q] < \infty$.

The following Buckdahn and Ma’s result established the connection between the solution of BDSDE (2.2) and the stochastic viscosity solution of SPDE (1.2) on finite time interval $[0, T]$.

**Theorem 2.4** (\cite{BuckdahnMa}) Assume Conditions (A.1), (A.2), (A.4) are satisfied and the function $v(0, \cdot) \in C_{1}^{0}(\mathbb{R}^{d})$ is given. Then $v(t, x) = u(T - t, x) = Y_{T-t}^{T-x}$, where $Y_{T-t}^{T-x} \in S^{2,0}([0, T]; \mathbb{R}^1)$ is the solution of BDSDE (2.2), is a stochastic viscosity solution of SPDE (1.2) on finite time interval $[0, T]$.

**Remark 2.5** From the argument of Buckdahn and Ma we can see if we replace the condition $v(0, \cdot) \in C_{1}^{0}(\mathbb{R}^{d})$ in Theorem 2.4 by that $v(0, x)$ is continuous w.r.t. $x$ and $E[\|v(0, X_{T-t}^{T-x})\|_{2}^2] < \infty$, then the conclusion of Theorem 2.4 remains true since $E[\|v(0, X_{T-t}^{T-x})\|_{2}^2] = E[\|Y_{t}^{T-x}\|_{2}^2] < \infty$ guarantees the corresponding BDSDE has a square-integrable terminal value.

### 3 Stationary Property of Solutions of BDSDEs

The purpose of this section is to study the stationary property of the solution to infinite horizon BDSDE (1.4). In order to show the main idea, we first assume that there exists a unique solution $(Y_{t}, \mathcal{F}, Z_{t}) \in S_{p,K}([-\infty, \infty); \mathbb{R}^1) \cap M_{2-K}([-\infty, \infty); \mathbb{R}^1) \times M_{2-K}([-\infty, \infty); \mathbb{R}^d)$ to BDSDE (1.4) and $(t, x) \rightarrow Y_{t}^{x}$ is a.s. continuous. The study of the existence, uniqueness and regularity of solution to BDSDE (1.4) will be deferred to next section.

We now construct the measurable metric dynamical system through defining a measurable and measure-preserving shift. Let $\hat{\theta}_{t} : \Omega \rightarrow \Omega$, $t \geq 0$, be a measurable mapping on $(\Omega, \mathcal{F}, P)$, defined by $\hat{\theta}_{t} : \Omega \rightarrow \Omega$, $t \geq 0$, be a measurable mapping on $(\Omega, \mathcal{F}, P)$, defined by

$$\hat{\theta}_{t} : \Omega \rightarrow \Omega, \quad \hat{\theta}_{t} = B_{s+t} - \tilde{B}_{t}, \quad \hat{\theta}_{t} = W_{s} - W_{t}.$$ Then for any $s, t \geq 0,$

(i) $P \cdot \hat{\theta}^{-1}_{t} = P$;
(ii) $\hat{\theta}_{0} = I$, where $I$ is the identity transformation on $\Omega$;
(iii) $\hat{\theta}_{s} \circ \hat{\theta}_{t} = \hat{\theta}_{s+t}$.

Also for an arbitrary $\mathcal{F}$ measurable random variable $\phi$, set

$$\hat{\phi} \circ \hat{\theta}(\omega) = \phi(\hat{\phi}(\omega)).$$

For any $r \geq 0$, $s \geq t$, $x \in \mathbb{R}^d$, applying $\hat{\theta}_{r}$ to SDE (1.3), we have
\[
\hat{\theta}_r \circ X_s^{t,x} = x + \int_{t+\theta}^{s+\theta} b(\hat{\theta}_r \circ X_u^{t,x})du + \int_{t+\theta}^{s+\theta} \sigma(\hat{\theta}_r \circ X_u^{t,x})dW_u.
\]

So under Condition (A.4), by the uniqueness of the solution, we have for any \( r, t \geq 0, x \in \mathbb{R}^d \),
\[
\hat{\theta}_r \circ X_s^{t,x} = X_{s+\theta}^{t+r,x}, \quad \text{for all } s \geq 0 \text{ a.s.}
\]  
(3.1)

For \( Y \in \mathbb{R}^1, Z \in \mathbb{R}^d \), let
\[
\hat{f}(T, Y, Z) = f(X_s^{t,x}, Y, Z), \quad \hat{g}(T, Y, Z) = g(X_s^{t,x}, Y, Z).
\]

Here we take \( T = (s, t) \) as a dual time variable (\( t \) is fixed). Using (3.1) we can verify that \( \hat{f} \) and \( \hat{g} \) satisfy the stationary conditions in Proposition 2.5 in [22] for any \( \hat{\theta}_r (r \geq 0), T, Y \) and \( Z \), then using a similar argument as in Theorem 2.12 in [22] we can deduce the following proposition by the uniqueness of BDSDE [14]:

**Proposition 3.1** Assume BDSDE (1.4) has a unique solution \((Y_t^{t,x}, Z_t^{t,x}) \in S^{(1, \infty)}(\mathbb{R}^1) \cap M^{2,-K}([0, \infty); \mathbb{R}^1) \times M^{2,-K}([0, \infty); \mathbb{R}^d)\), then under Condition (A.4), \((Y_t^{t,x}, Z_t^{t,x})_{s \geq 0}\) satisfies the following stationary property w.r.t. \( \hat{\theta}_r \): for any \( r, t \geq 0, x \in \mathbb{R}^d \),
\[
\hat{\theta}_r \circ Y_s^{t,x} = Y_{s+\theta}^{t+r,x}, \quad \hat{\theta}_r \circ Z_s^{t,x} = Z_{s+\theta}^{t+r,x} \quad \text{for all } s \geq 0 \text{ a.s.}
\]

In particular, for any \( r, t \geq 0, x \in \mathbb{R}^d \),
\[
\hat{\theta}_r \circ Y_t^{t,x} = Y_{t+\theta}^{t+r,x} \quad \text{a.s.}
\]
(3.2)

If we regard \( Y_t^{t,x} \) as a function of \((t, x)\), (3.2) gives a “very crude” stationary property of \( Y \). Borrowing the idea of perfecting crude cocycles in [1] and [2], we then prove the following theorem which makes the “very crude” stationary property of \( Y \) “perfect”.

**Theorem 3.2** Let \((\Omega, \mathscr{F}, \mathbb{P})\) be a probability space and \( \mathbb{H} \) be a separable Hausdorff topological space with \( \sigma \)-algebra \( \mathscr{H} \). Assume \( Y(t, x, \omega) : [0, \infty) \times \mathbb{R}^d \times \Omega \rightarrow \mathbb{H} \) is \( \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathscr{F} \) measurable, a.s. continuous w.r.t. \( t \), \( x \) and satisfies the “very crude” stationary property w.r.t. \( \hat{\theta}_r \), i.e., for any \( t, r \geq 0, x \in \mathbb{R}^d \)
\[
\hat{\theta}_r \circ Y(t, x, \omega) = Y(t + r, x, \omega) \quad \text{a.s.}
\]
(3.3)

Then there exists a \( \tilde{Y}(t, x, \omega) \) which is an indistinguishable version of \( Y(t, x, \omega) \) s.t. \( \tilde{Y}(t, x, \omega) \) is \( \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathscr{F} \) measurable, continuous w.r.t. \( t \), \( x \) for all \( \omega \) and satisfies the “perfect” stationary property w.r.t. \( \hat{\theta}_r \):
\[
\hat{\theta}_r \circ \tilde{Y}(t, x, \omega) = \tilde{Y}(t + r, x, \omega) \quad \text{for all } t, r \geq 0, x \in \mathbb{R}^d \text{ a.s.}
\]
(3.4)

**Proof.** From the continuity of \( Y(t, x, \omega) \) w.r.t. \( t \), \( x \) and using a standard argument, we easily see that for any \( r \geq 0 \),
\[
\hat{\theta}_r \circ Y(t, x, \omega) = Y(t + r, x, \omega) \quad \text{for all } t \geq 0, x \in \mathbb{R}^d \text{ a.s.}
\]
(3.5)

Define
Obviously, \( A(r,t,x,\omega) \) is measurable w.r.t. \( \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{F} \). If we denote by \( Q \) and \( \tilde{Q} \) the normalized Lebesgue measure on \( \mathbb{R}^+ \) and \( \mathbb{R}^d \) respectively such that \( Q(\mathbb{R}^+) = 1 \) and \( \tilde{Q}(\mathbb{R}^d) = 1 \), then by \( 3.5 \),

\[
Q \otimes Q \otimes \tilde{Q} \otimes P(A^{-1}(0)) = \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \int_{\Omega} I_{A^{-1}(0)}(r,t,x,\omega)dPd\tilde{Q}dQdQ = 1,
\]

where  \( I \) is the indicator function in \((\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^d \times \Omega, \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{B}_{\mathbb{R}^d} \otimes \mathcal{F})\). It is easy to see that

\[
M = \{(r, \omega) : \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} I_{A^{-1}(0)}(r,t,x,\omega)d\tilde{Q}dQ = 1\} \in \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}.
\]

And by \( 3.6 \), we have

\[
Q \otimes P(M) = Q \otimes P(\{(r, \omega) : \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} I_{A^{-1}(0)}(r,t,x,\omega)d\tilde{Q}dQ = 1\}) = 1.
\]

Similarly, we can also know

\[
\Omega = \{\omega : \int_{\mathbb{R}^+} I_M(r,\omega)dQ = 1\} \in \mathcal{F}
\]

and

\[
P(\Omega) = P(\{\omega : \int_{\mathbb{R}^+} I_M(r,\omega)dQ = 1\}) = 1.
\]

Moreover, the measurability of \( \Omega^* \) can be seen easily as

\[
\Omega^* = \{\omega : \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} I_M(r,\tilde{\theta}_u \omega)dQdQ = 1\} \in \mathcal{F}.
\]

And since \( \tilde{\Omega} \) has full measure,

\[
P(\Omega^*) \geq P(\{\omega : Y(t+r,x,\tilde{\theta}_u \omega) = Y(t,x,\tilde{\theta}_r \circ \tilde{\theta}_u \omega) \text{ for a.e. } r \text{ and } u, \text{ and all } t, x \} \cap \tilde{\Omega})
\]

\[
= P(\{\omega : Y(t+r+u,x,\omega) = Y(t,x,\tilde{\theta}_{r+u} \omega) \text{ for a.e. } r \text{ and } u, \text{ and all } t, x \} \cap \tilde{\Omega})
\]

\[
= P(\{\omega : Y(t+r',x,\omega) = Y(t,x,\tilde{\theta}_{r'} \omega) \text{ for a.e. } r', \text{ and all } t, x \} \cap \tilde{\Omega})
\]

\[
= P(\tilde{\Omega})
\]

\[
= 1.
\]

One can prove \( \tilde{\theta}_u \Omega^* \subset \Omega^* \) for any \( u \geq 0 \). Indeed, for any \( \omega \in \tilde{\theta}_u \Omega^* \), there exists \( \hat{\omega} \in \Omega^* \) s.t. \( \omega = \hat{\theta}_u \hat{\omega} \) and \( \hat{\theta}_r \hat{\omega} \in \tilde{\Omega} \) for a.e. \( r \geq 0 \). But \( \hat{\theta}_r \omega = \hat{\theta}_{r+u} \hat{\omega} \in \tilde{\Omega} \) for a.e. \( r \geq 0 \), so \( \omega \in \Omega^* \). That is to say \( \tilde{\theta}_u \Omega^* \subset \Omega^* \).

Define

\[
\begin{cases}
\hat{Y}(t,x,\omega) = Y(t-r,x,\tilde{\theta}_r \omega), & \text{where } r \in [0,t] \text{ with } \tilde{\theta}_r \omega \in \tilde{\Omega}, \text{ if } \omega \in \Omega^*, \\
\hat{Y}(t,x,\omega) = 0, & \text{if } \omega \in \Omega^c.
\end{cases}
\]

\[Q. \] Zhang
An important fact is that if \( \omega \in \Omega^* \), then for an arbitrary \( r \in [0,t] \) with \( \hat{\theta}_r \omega \in \hat{\Omega} \), \( Y(t-r,x,\hat{\theta}_r \omega) \) is independent of \( r \) and
\[
 Y(t-r,x,\hat{\theta}_r \omega) = Y(t,x,\omega). \tag{3.7}
\]

To see this, as \( \hat{\theta}_r \omega \in \hat{\Omega} \), so there exists \( u \geq r \) s.t. \( (u, \hat{\theta}_r \omega) \in M \) and \( (u-r, \hat{\theta}_r \omega) \in M \). If not, it means for a.e. \( r \) there doesn’t exist \( u \) satisfying \( (u, \hat{\theta}_r \omega) \in M \) and \( (u-r, \hat{\theta}_r \omega) \in M \). Then one can easily get the measure of \( \{ u : (u, \hat{\theta}_r \omega) \notin M \} \) is positive. That is a contradiction. So such a \( u \) certainly exists and satisfies
\[
 \hat{\theta}_u Y(t-r,x,\hat{\theta}_r \omega) = Y(t-r+u,x,\hat{\theta}_r \omega) = Y(t,x,\hat{\theta}_u \omega).
\]

So
\[
 Y(t-r,x,\hat{\theta}_r \omega) = \hat{\theta}_u^{-1} Y(t,x,\hat{\theta}_u \omega) = Y(t,x,\omega).
\]

Therefore \( (3.7) \) is true and \( \hat{Y}(t,x,\omega) \) doesn’t depend on the choice of \( r \). That is to say \( \hat{Y}(t,x,\omega) \) is well defined. Moreover \( (3.7) \) implies that \( Y(t,x,\omega) = \hat{Y}(t,x,\omega) \) for all \( t \geq 0, x \in \mathbb{R}^d \) on a full measure set \( \Omega^* \), thus \( Y(t,x,\omega) \) and \( \hat{Y}(t,x,\omega) \) are indistinguishable. Define
\[
 \begin{cases} 
 B(r,t,x,\omega) = Y(t-r,x,\hat{\theta}_r \omega), & \text{if } r \in [0,t], \ \hat{\theta}_r \omega \in \hat{\Omega}, \ \text{and } \omega \in \Omega^*, \\
 B(r,t,x,\omega) = 0, & \text{otherwise}.
\end{cases}
\]

Then \( B(r,t,x,\omega) \) is \( B_{\mathbb{R}^+} \otimes B_{\mathbb{R}^d} \otimes \mathcal{F} \) measurable. By the definition of \( \Omega^* \), if \( \omega \in \Omega^* \), then for a.e. \( 0 \leq r \leq t, \hat{\theta}_r \omega \in \hat{\Omega} \). We denote \( L(r) \) the Lebesgue measure in \([0,t]\). Since the countable base of \( H \) generates \( \mathcal{H} \) and separates points, \( (H, \mathcal{H}) \) is isomorphic as a measurable space to a subset of \([0,1]\).

Consequently, for all \( t,x,\omega \),
\[
 \hat{Y}(t,x,\omega) = \int_0^t B(r,t,x,\omega)dL(r).
\]

So by Fubini’s theorem, \( \hat{Y}(t,x,\omega) \) is \( B_{\mathbb{R}^+} \otimes B_{\mathbb{R}^d} \otimes \mathcal{F} \) measurable. \( \hat{Y}(t,x,\omega) \) is a.s continuous w.r.t. \( t, x \) due to the a.s continuity of \( Y(t-r,x,\omega) \). But there exists a null measure set \( N \in \mathcal{F} \) s.t. \( \{ \omega : \hat{Y}(t,x,\omega) \text{ is not continuous w.r.t. } t, x \} \subset N \). Let \( \tilde{Y}(t,x,\omega) \) on \( N \) equal 0. We still denote this new version of \( \hat{Y}(t,x,\omega) \) by \( \hat{Y}(t,x,\omega) \), then this version of \( \hat{Y}(t,x,\omega) \) is continuous for all \( \omega \).

The remaining work is to check \( \hat{Y}(t,x,\omega) \) satisfies the “perfect” stationary property \( (3.4) \). For \( \omega \in \Omega^* \) and all \( r \geq 0 \), \( \hat{\theta}_r \omega \in \hat{\theta}_r \Omega^* \subset \Omega^* \). Pick a \( u \) s.t. \( \hat{\theta}_u \omega \in \hat{\Omega}, \hat{\theta}_u + r \omega \in \hat{\Omega} \), then by \( (3.7) \) we have
\[
 \hat{Y}(t,x,\hat{\theta}_r \omega) = Y(t-u,x,\hat{\theta}_u + r \omega) = Y(t+r-u,r,\hat{\theta}_u + r \omega) = Y(t+r,x,\omega) = \hat{Y}(t+r,x,\omega). \tag{3.8}
\]

The theorem is proved.

From now on, we neglect the difference between two distinguishable random processes. Then with Proposition \( (3.1) \) and Theorem \( (3.2) \) it follows immediately that

**Theorem 3.3** If BDSDE \( (3.4) \) has a unique solution \( Y^{t,x} \) \( Z^{t,x} \) \( Y^{t,x} \in S^{p_0}(0,\infty) ; \mathbb{R}^1 \ \cap M_{2-K}^2([0,\infty); \mathbb{R}^d) \) and \( t,x \to Y^{t,x} \) is a.s. continuous, then under Condition \((A.4)\), \( Y^{t,x} \) satisfies the “perfect” stationary property w.r.t. \( \hat{\theta} \), i.e.
\[
 \hat{\theta}_r \circ Y^{t,x} = Y^{t+r,x}_r \quad \text{for all } r \geq 0, \ x \in \mathbb{R}^d \ \text{a.s.}
\]
4 Infinite Horizon BDSDEs

In this section we first prove the assumption in Theorem 3.3 that BDSDE (1.4) has a unique solution $(Y, Z) \in S^{p-k} \cap M^{2-k}(\Omega; \mathbb{R}) \times M^{2-k}([0, \infty); \mathbb{R}^d)$ is obtainable and reasonable under Conditions (A.1)-(A.4). To begin with, we briefly introduce the pioneering work by Pardoux and Peng in [18] for the following finite horizon BDSDE:

\[ Y_s = Y_T + \int_s^T f(r, Y_r, Z_r)dr - \int_s^T \langle g(r, Y_r, Z_r), d^r B_r \rangle - \int_s^T \langle Z_r, dW_r \rangle. \]  

(4.1)

Here we only consider $\mathbb{R}^1$-valued BDSDE for our purpose. One can also refer to [18] for multi-dimensional BDSDE if interested. Assume

(A.1). Functions $f : \Omega \times [0, T] \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^1$ and $g : \Omega \times [0, T] \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^d$ are $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}^1 \otimes \mathcal{B}^d$ measurable, and for any $(y, z) \in \mathbb{R}^1 \times \mathbb{R}^d$, $f(\cdot, y, z) \in M^{2,0}([0, T]; \mathbb{R}^1)$ and $g(\cdot, y, z) \in M^{2,0}([0, T]; \mathbb{R}^d)$, moreover there exist constants $C \geq 0$ and $0 \leq \alpha < 1$ s.t. for any $r \in [0, T]$, $(y_1, z_1), (y_2, z_2) \in \mathbb{R}^1 \times \mathbb{R}^d$,

\[ |f(r, y_1, z_1) - f(r, y_2, z_2)|^2 \leq C|y_1 - y_2|^2 + C|z_1 - z_2|^2, \]
\[ |g(r, y_1, z_1) - g(r, y_2, z_2)|^2 \leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2. \]

Theorem 4.1 ([18]) Under Condition (A.1), for any given $\mathcal{F}_T \cup \mathcal{F}_{T,\infty}$ measurable $Y_T \in L^2(\Omega)$, BDSDE (4.1) has a unique solution

\[ (Y, Z) \in S^{2,0}([0, T]; \mathbb{R}) \otimes M^{2,0}([0, T]; \mathbb{R}^d). \]

In [18], Pardoux and Peng also discussed a type of forward BDSDE, a special case of BDSDE (4.1),

\[ Y^{t,x}_s = h(X^{t,x}_T) + \int_s^T f(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r)dr - \int_s^T \langle g(X^{t,x}_r, Y^{t,x}_r, Z^{t,x}_r), d^r B_r \rangle - \int_s^T \langle Z^{t,x}_r, dW_r \rangle, \]

(4.2)

where $(X^{t,x}_r)_{t \leq r \leq T}$ is the solution of SDE (1.1). Assume

(A.2). Functions $f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^1$ and $g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^d$ are $\mathcal{B}^d \otimes \mathcal{B} \otimes \mathcal{B}^d$ measurable, and there exist constants $C \geq 0$ and $0 \leq \alpha < 1$ s.t. for any $(x, y_1, z_1), (x, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d$,

\[ |f(x, y_1, z_1) - f(x, y_2, z_2)|^2 \leq C|x_1 - x_2|^2 + C|y_1 - y_2|^2 + C|z_1 - z_2|^2, \]
\[ |g(x, y_1, z_1) - g(x, y_2, z_2)|^2 \leq C|x_1 - x_2|^2 + C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2. \]

For BDSDE (4.2), it is not difficult to deduce from Theorem 4.1 that

Theorem 4.2 Under Condition (A.2), for each $x \in \mathbb{R}^d$ and any given $\mathcal{F}_T \cup \mathcal{F}_{T,\infty}$ measurable $h$ satisfying $h(X^{t,x}_T) \in L^2(\Omega)$, BDSDE (4.2) has a unique solution

\[ (Y^{t,x}_t, Z^{t,x}_t) \in S^{2,0}([t, T]; \mathbb{R}) \otimes M^{2,0}([t, T]; \mathbb{R}^d). \]

In [18], for the first time, Pardoux and Peng associated the classical solution of SPDE, if any, with the solution of BDSDE (4.2). They proved that under some strong smoothness conditions of $h$, $b$, $\sigma$, $f$ and $g$ (for details see [18]), $u(t, x) = Y^{t,x}_t$, where $Y$ is the unique solution of BDSDE (4.2),
Applying Itô’s formula to $e^{Kt}Y_t$, we assume that

\begin{equation}
H.3.
\end{equation}

**Proof** as in Pardoux [16].

Now let’s turn to the existence and uniqueness of solution to the following infinite horizon BDSDE:

\begin{equation}
B(\text{s.t. for any } (\omega, t) \in \Omega \times [0, \infty), \ (y_1, z_1), \ (y_2, z_2) \in \mathbb{R}^1 \times \mathbb{R}^d, \ 

\left| f(t, y_1, z_1) - f(t, y_2, z_2) \right|^2 \leq C_1 |y_1 - y_2|^2 + C|z_1 - z_2|^2;

\left| g(t, y_1, z_1) - g(t, y_2, z_2) \right|^2 \leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2;

\end{equation}

\begin{equation}
H.2.
\end{equation}

There exist constants $K \in \mathbb{R}^+$, $p > d + 2$, $K < K' < 2K$ and $\mu > 0$ with $2\mu - K' - \frac{p(p+1)}{2}C > 0$ s.t. for any $(\omega, t) \in \Omega \times [0, \infty)$, $y_1$, $y_2 \in \mathbb{R}^1$, $z \in \mathbb{R}^d$,

\begin{equation}
(\text{H).3.}) (y_1 - y_2)(f(t, y_1, z) - f(t, y_2, z)) \leq -\mu |y_1 - y_2|^2;
\end{equation}

**Theorem 4.3** Under Conditions (H.1)–(H.3), BDSDE (4.3) has a unique solution

\begin{equation}
(Y, Z) \in S^{p-K} \cap M^{2-K}((0, \infty); \mathbb{R}^1) \otimes M^{2-K}((0, \infty); \mathbb{R}^d),
\end{equation}

where the norm in $S^{p-K}((0, \infty); \mathbb{R}^1) \cap M^{2-K}((0, \infty); \mathbb{R}^1) \otimes M^{2-K}((0, \infty); \mathbb{R}^d)$ is defined as

\begin{equation}
\left( \left( E\sup_{t \geq 0} e^{-Kt} \cdot |\nu|^2 \right)^{\frac{1}{2}} + E\left[ \int_0^\infty e^{-Kr} \cdot |\nu^1|^2 dr \right] + E\left[ \int_0^\infty e^{-Kr} \cdot |\nu^2|^2 dr \right] \right)^\frac{1}{2};
\end{equation}

as in Pardoux [10].

**Proof.** Uniqueness. Let $(Y_1^t, Z_1^t)$ and $(Y_2^t, Z_2^t)$ be two solutions of BDSDE (4.1). Define

\begin{equation}
\bar{Y}_t = Y_1^t - Y_2^t, \quad \bar{Z}_t = Z_1^t - Z_2^t, \quad t \geq 0.
\end{equation}

Applying Itô’s formula to $e^{-Kt}|\bar{Y}_s|^2$, we have
Taking $K'$ as in Condition (H.2) and noting $2\mu - K' - 3C > 0$ as well, we can see that (4.4) remains true when $K$ replaced by $K'$. Therefore, we have

$$E[e^{-K't}|\bar{Y}_t|^2] \leq e^{-(K'-K)T}E[e^{-KT}|\bar{Y}_T|^2].$$

(4.5)

Since $\sup_{T \geq 0} E[e^{-KT}|\bar{Y}_T|^2] < \infty$, taking the limit as $T \to \infty$ in (4.5), we have

$$E[e^{-K't}|\bar{Y}_t|^2] = 0.$$

Then the uniqueness is proved.

**Existence.** For each $n \in \mathbb{N}$, we define a sequence of BDSDEs as follows

$$Y^n_t = \int_t^T f(s, Y^n_s, Z^n_s)ds - \int_t^T \langle g(s, Y^n_s, Z^n_s), d\hat{B}_s \rangle - \int_t^T \langle Z^n_s, dW_s \rangle.$$  

(4.6)

Let $(Y^n_t, Z^n_t)_{t \geq n}$ be the solution of BDSDE (4.6) and according to Theorem 4.1, BDSDE (4.6) has a unique solution $(Y^n, Z^n) \in S^{2-K} \cap M^{2-K}([0, \infty); \mathbb{R}^1) \otimes M^{2-K}([0, \infty); \mathbb{R}^n)$. Also under Conditions (H.1)–(H.3), we can prove $Y^n \in S^{p-K}([0, \infty); \mathbb{R}^1)$ in the following lemma.

**Lemma 4.4** Let $(Y^n_t)_{t \geq 0}$ be the solution of BDSDE (4.6), then under Conditions (H.1)–(H.3), $Y^n \in S^{p-K}([0, \infty); \mathbb{R}^1)$.

**Proof.** Let

$$\psi_M(x) = x^2I_{\{x \leq M\}} + 2M(x-M)I_{\{x \geq M\}} - 2M(x+M)I_{\{x \leq -M\}}$$

$$\varphi_{N,p}(x) = x^2I_{\{0 \leq x < N\}} + \frac{p}{2}N^{2-p}I_{\{x \geq N\}}.$$

Applying generalized Itô’s formula (c.f. Elworthy, Truman and Zhao [12]) to $e^{-Kr}\varphi_{N,p}(\psi_M(Y^n_r))$ to have the following estimation

$$e^{-Ks}\varphi_{N,p}(\psi_M(Y^n_s)) - K \int_s^ne^{-Kr}\varphi_{N,p}(\psi_M(Y^n_r))dr$$

$$+ \frac{1}{2} \int_s^ne^{-Kr}\varphi''_{N,p}(\psi_M(Y^n_r))|\psi'(M(Y^n_r))|^2Z^n_r^2dr$$

$$+ \int_s^ne^{-Kr}\varphi_{N,p}(\psi_M(Y^n_r))I_{\{s \leq Y^n_r < M\}}|Z^n_r|^2dr$$

$$\leq \int_s^ne^{-Kr}\varphi'_{N,p}(\psi_M(Y^n_r))\psi'(M(Y^n_r)f(r, Y^n_r, Z^n_r)dr$$

$$+ \int_s^ne^{-Kr}\varphi''_{N,p}(\psi_M(Y^n_r))I_{\{s \leq Y^n_r < M\}}|g(r, Y^n_r, Z^n_r)|^2dr$$

$$+ \frac{1}{2} \int_s^ne^{-Kr}\varphi''_{N,p}(\psi_M(Y^n_r))|\psi'(M(Y^n_r)|^2|g(r, Y^n_r, Z^n_r)|^2dr$$

$$- \int_s^ne^{-Kr}\varphi'_{N,p}(\psi_M(Y^n_r))\psi'(M(Y^n_r)g(r, Y^n_r, Z^n_r, d\hat{B}_r)$$

$$- \int_s^ne^{-Kr}\varphi'_{N,p}(\psi_M(Y^n_r))\psi'(M(Y^n_r)Z^n_r, dW_r).$$

(4.7)
As \((Y^n, Z^n) \in S^{2-K} \cap M^{2-K}([0, \infty); \mathbb{R}^1) \otimes M^{2-K}([0, \infty); \mathbb{R}^d)\) and \(\varphi'_{N,p}(\psi_M(Y^n))\psi'_M(Y^n)\) is bounded, taking the expectation on both sides, we know that all the stochastic integrals have zero expectation. Using Conditions (H.1)-(H.3) and taking first the limit as \(M \to \infty\), then the limit as \(N \to \infty\), by the monotone convergence theorem, we have

\[
(p\mu - K - \frac{p(p+1)}{2} - (3 + \frac{p(p-1)}{2})\varepsilon)E[\int_s^\infty e^{-Kr}|Y^n|^p_r] + \frac{p}{4}(2p - 3 - (2p - 2)\alpha - (2p - 2)\varepsilon)E[\int_s^\infty e^{-Kr}|Y^n|^{p-2}_r|Z^n|^2_r]
\leq C_p E[\int_0^\infty e^{-Kr}|f(r, 0, 0)|^p dr] + C_p E[\int_0^\infty e^{-Kr}|g(r, Y^n, Z^n)|^p dr] < \infty. \tag{4.8}
\]

Note that here and in the following the constant \(\varepsilon\) can be chosen to be sufficiently small and \(C_p\) is a generic constant. Due to Conditions (H.1), (H.2) and the arbitrariness of \(\varepsilon\), all the terms on the left hand side of (4.8) are positive. Furthermore, by the B-D-G inequality, Cauchy-Schwartz inequality and Young inequality, from (4.7) we have

\[
E[\sup_{s \geq 0} e^{-Ks}|Y^n|^p_s] \
\leq C_p E[\int_0^\infty e^{-Kr}|Y^n|^p_r |Z^n|^2_r dr] + C_p E[\int_0^\infty e^{-Kr}|Y^n|^p_r dr] \
+ C_p E[\int_s^\infty (e^{-Kr}\varphi'_{N,p}(\psi_M(Y^n))|\psi'_M(Y^n)|^2)(e^{-Kr}\varphi'_{N,p}(\psi_M(Y^n))|g(r, Y^n, Z^n)|^2) dr] \
+ C_p E[\int_s^\infty (e^{-Kr}\varphi'_{N,p}(\psi_M(Y^n))|\psi'_M(Y^n)|^2)(e^{-Kr}\varphi'_{N,p}(\psi_M(Y^n))|Z^n|^2) dr] \
\leq C_p E[\int_0^\infty e^{-Kr}|Y^n|^p_r |Z^n|^2_r dr] + C_p E[\int_0^\infty e^{-Kr}|Y^n|^p_r dr] \
+ \varepsilon E[\sup_{s \geq 0} (e^{-Ks}\varphi'_{N,p}(\psi_M(Y^n))|\psi'_M(Y^n)|^2) + C_p E[\int_0^\infty e^{-Kr}\varphi'_{N,p}(\psi_M(Y^n))|g(r, Y^n, Z^n)|^2 dr] \
+ C_p E[\int_0^\infty e^{-Kr}\varphi'_{N,p}(\psi_M(Y^n))|Z^n|^2 dr]. \tag{4.9}
\]

Taking the limits as \(M, N \to \infty\) and applying the monotone convergence theorem, we have

\[
E[\sup_{s \geq 0} e^{-Ks}|Y^n|^p_s] \leq C_p E[\int_0^\infty e^{-Kr}|Y^n|^p_r |Z^n|^2_r dr] + C_p E[\int_0^\infty e^{-Kr}|Y^n|^p_r dr]. \tag{4.10}
\]

By (1.8), \(Y^n \in S^{p-K}([0, \infty); \mathbb{R}^1)\). Lemma 4.4 is proved.

\[\square\]

**Remark 4.5** The proof of Lemma 4.4 also works with \(p\) replaced by 2. Note that if \(f(\cdot, 0, 0) \in M^{p-K}([0, \infty); \mathbb{R}^1)\), then by Hölder inequality, it turns out that \(f(\cdot, 0, 0) \in M^{2-K}([0, \infty); \mathbb{R}^1)\) and \(g(\cdot, 0, 0) \in M^{2-K}([0, \infty); \mathbb{R}^d)\). So it is easy to see in (4.8) with \(p\) replaced by 2 that

\[(Y^n, Z^n) \in M^{2-K}([0, \infty); \mathbb{R}^1) \otimes M^{2-K}([0, \infty); \mathbb{R}^d).\]

For the rest of our paper, we will leave out the similar localization argument as in the proof of Lemma 4.4 when applying Itô’s formula to save the space of this paper.
Define \( Y_{t}^{m,n} = Y_{t}^{m} - Y_{t}^{n}, \bar{Z}_{t}^{m,n} = Z_{t}^{m} - Z_{t}^{n} \).

(i) When \( n \leq t \leq m, \)

\[
Y_{t}^{m,n} = Y_{t}^{m} - Y_{t}^{n} = \int_{t}^{m} f(s, Y_{s}^{m}, Z_{s}^{m}) ds - \int_{t}^{m} \langle g(s, Y_{s}^{m}, Z_{s}^{m}), d\hat{B}_{s} \rangle - \int_{t}^{m} \langle Z_{s}^{m}, dW_{s} \rangle.
\]

Some similar calculations as in (4.8) and (4.10) lead to

\[
E[\sup_{n \leq t \leq m} e^{-Kt}|Y_{t}^{m}|^{p}] \leq C_{p}E[\int_{n}^{m} e^{-Kr}|Y_{r}^{m}|^{p-2}|Z_{r}^{m}|^{2} dr] + C_{p}E[\int_{n}^{m} e^{-Kr}|Y_{r}^{m}|^{p} dr] + C_{p}E[\int_{n}^{m} e^{-Kr}(|f(r,0,0)|^{p} + |g(r,0,0)|^{p}) dr] \to 0, \text{ as } n, m \to \infty.
\]

(ii) When \( 0 \leq t \leq n, \)

\[
\bar{Y}_{t}^{m,n} = Y_{t}^{m} + \int_{t}^{m} f(r, Y_{r}^{m}, Z_{r}^{m}) - f(r, Y_{r}^{n}, Z_{r}^{n}) - \int_{t}^{m} \langle g(r, Y_{r}^{m}, Z_{r}^{m}) - g(r, Y_{r}^{n}, Z_{r}^{n}), d\hat{B}_{r} \rangle - \int_{t}^{m} \langle \bar{Z}_{r}^{m,n}, dW_{r} \rangle.
\]

Applying Itô’s formula to \( e^{-Kr}|Y_{r}^{m,n}|^{p} \) and following a similar calculation as in (4.7) and (4.8), we have for \( s \leq n, \)

\[
E[\int_{0}^{n} e^{-Kr}|\bar{Y}_{r}^{m,n}|^{p-2} |\bar{Z}_{r}^{m,n}|^{2} dr] + E[\int_{0}^{n} e^{-Kr}|\bar{Y}_{r}^{m,n}|^{p} dr] \leq C_{p}E[e^{-Kn}|Y_{n}^{m}|^{p}].
\]

From (i), the right hand side of the above inequality converges to 0 as \( n, m \to \infty. \) By some similar calculations as in (4.10), we have

\[
E[\sup_{0 \leq t \leq n} e^{-Kt}|\bar{Y}_{t}^{m,n}|^{p}] \leq C_{p}E[e^{-Kn}|Y_{n}^{m}|^{p}] \to 0 \text{ as } n,m \to \infty.
\]

From (i) (ii), we have for \( m, n \in \mathbb{N}, \)

\[
\lim_{n,m \to \infty} E[\sup_{t \geq 0} e^{-Kt}|Y_{t}^{m} - Y_{t}^{n}|^{p}] = 0.
\]

It is easy to see that the above arguments also hold for \( p = 2 \) in (4.11) and (4.12). Noting Remark 4.3, we have as \( n, m \to \infty \)

\[
E[\int_{0}^{\infty} e^{-Kr}|\bar{Y}_{r}^{m,n}|^{2} dr] + E[\int_{0}^{\infty} e^{-Kr}|\bar{Z}_{r}^{m,n}|^{2} dr] \to 0.
\]

Therefore, \((Y^{m}, Z^{n})\) is a Cauchy sequence in the Banach space \( S^{p,-K}((0, \infty); \mathbb{R}^{1}) \cap M^{2,-K}((0, \infty); \mathbb{R}^{1}) \otimes \mathbb{M}^{2,-K}((0, \infty); \mathbb{R}^{d})\).

We take \((Y_{t}, Z_{t})_{t \geq 0}\) as the limit of \((Y_{t}^{m}, Z_{t}^{n})_{t \geq 0}\) in \( S^{p,-K}((0, \infty); \mathbb{R}^{1}) \cap M^{2,-K}((0, \infty); \mathbb{R}^{1}) \otimes \mathbb{M}^{2,-K}((0, \infty); \mathbb{R}^{d})\) and then show that \((Y_{t}, Z_{t})_{t \geq 0}\) is the solution of BDSDE (4.2). First note that for \( t \leq n, \)

(4.9) is equivalent to
\[
e^{-\frac{K_r}{r}Y^n_t} = \int_t^n e^{-\frac{K_r}{r}f(s, Y^n_s, Z^n_s) ds} + \int_t^n K_r \frac{K_r}{2} e^{-\frac{K_r}{r}Y^n_s ds}
- \int_t^n e^{-\frac{K_r}{r}g(s, Y^n_s, Z^n_s), d^\prime \hat{B}_s} - \int_t^n e^{-\frac{K_r}{r}(Z^n_s, dW_s)}. \tag{4.13}
\]

Actually BDSDE (4.13) converges to BDSDE (4.3) in \(L^2(\Omega)\) as \(n \to \infty\). To see this, we verify the convergence term by term. For the first term,

\[
E[ |e^{-\frac{K_r}{r}Y^n_t} - e^{-\frac{K_r}{r}Y^n_t}|^2] \leq E[\sup_{t \geq 0} e^{-\frac{K_r}{r}|Y^n_t - Y^n_t|^2}] \to 0.
\]

For the second term, by H"older inequality,

\[
E[| \int_t^n e^{-\frac{K_r}{r}f(s, Y^n_s, Z^n_s) ds} - \int_t^n e^{-\frac{K_r}{r}f(s, Y^n_s, Z^n_s) ds}|^2]
\leq 2E[\int_t^n e^{-(K' - K)s} ds \int_t^n e^{-Ks}|f(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z^n_s)|^2 ds]
+ 2E[\int_t^n e^{-(K' - K)s} ds \int_t^n e^{-Ks}|f(s, Y^n_s, Z^n_s)|^2 ds] \to 0.
\]

We can deal with the third term similarly as above and deal with two stochastic integration terms by Itô’s isometry. Thus \((Y_t, Z_t)_{t \geq 0}\) is the solution of BDSDE (4.3) and the proof of Theorem 4.3 is completed.

Then we consider the existence and uniqueness of solution to the following infinite horizon forward BDSDE:

\[
e^{-\frac{K_r}{r}Y^{t,x}_s} = \int_s^\infty e^{-\frac{K_r}{r}r} f(x, Y^{t,x}_r, Z^{t,x}_r) dr + \int_s^\infty K_r \frac{K_r}{2} e^{-\frac{K_r}{r}Y^{t,x}_r dr}
- \int_s^\infty e^{-\frac{K_r}{r}r} g(x, Y^{t,x}_r, Z^{t,x}_r), d^\prime \hat{B}_r) - \int_s^\infty e^{-\frac{K_r}{r}r}(Z^{t,x}_r, dW_r), \quad s \geq 0.
\tag{4.14}
\]

We replace Condition (A.1) by

(A.1)*. Functions \(f : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^1\) and \(g : \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d \to \mathbb{R}^l\) are \(\mathcal{B}_{\mathbb{R}^1} \otimes \mathcal{B}_{\mathbb{R}^1} \otimes \mathcal{B}_{\mathbb{R}^d}\) measurable, and there exist constants \(C_0, C_1, C \geq 0\) and \(0 \leq \alpha < \frac{1}{2}\) s.t. for any \((x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^d \times \mathbb{R}^1 \times \mathbb{R}^d,\)

\[
|f(x_1, y_1, z_1) - f(x_2, y_2, z_2)|^2 \leq C_0|x_1 - x_2|^2 + C_1|y_1 - y_2|^2 + C|z_1 - z_2|^2,
|g(x_1, y_1, z_1) - g(x_2, y_2, z_2)|^2 \leq C_0|x_1 - x_2|^2 + C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2.
\]

**Proposition 4.6** Under Conditions (A.1)*, (A.3), (A.4), BDSDE (4.14) has a unique solution \((Y^{t,x}, Z^{t,x}) \in \mathcal{S}^{p,-K}_{\mathbb{R}^1}(0, \infty) \cap \mathcal{M}^{2,-K}_{\mathbb{R}^1}(0, \infty) \otimes \mathcal{M}^{2,-K}_{\mathbb{R}^d}(0, \infty) \otimes \mathcal{R}^d).\)

**Remark 4.7** For \(s \in [0, t]\), BDSDE (4.14) is equivalent to the following BDSDE

\[
e^{-\frac{K_r}{r}Y^{t,x}_s} = \int_s^\infty e^{-\frac{K_r}{r}r} f(x, Y^{t,x}_r, Z^{t,x}_r) dr + \int_s^\infty K_r \frac{K_r}{2} e^{-\frac{K_r}{r}Y^{t,x}_r dr}
- \int_s^\infty e^{-\frac{K_r}{r}r} g(x, Y^{t,x}_r, Z^{t,x}_r), d^\prime \hat{B}_r) - \int_s^\infty e^{-\frac{K_r}{r}r}(Z^{t,x}_r, dW_r).
\]

To unify the notation, we define \((Y^{t,x}_s, Z^{t,x}_s) = (Y^x_s, Z^x_s)\) when \(s \in [0, t]\).
Proof of Proposition 4.6 Let
\[ \hat{f}(s, y, z) = f(X_{s}^{t,x}, y, z), \quad \hat{g}(s, y, z) = g(X_{s}^{t,x}, y, z). \]
We need to verify that \( \hat{f}, \hat{g} \) satisfy Conditions (H.1)–(H.3) in Theorem 4.3. It is obvious that \( \hat{f}, \hat{g} \) satisfy (H.1) and (H.2), so we only need to show that \( \hat{f}, \hat{g} \) satisfy (H.3) as well, i.e.
\[ E[\int_{0}^{\infty} e^{-Ks}|\hat{f}(s, 0, 0)|^{p}ds] < \infty \quad \text{and} \quad E[\int_{0}^{\infty} e^{-Ks}|\hat{g}(s, 0, 0)|^{p}ds] < \infty. \]
Since
\[ E[\int_{0}^{\infty} e^{-Ks}|\hat{f}(s, 0, 0)|^{p}ds] \leq C_{p}E[\int_{0}^{\infty} e^{-Ks}C_{0}|X_{s}^{t,x}|^{p}ds] + C_{p}E[\int_{0}^{\infty} e^{-Ks}|f(0, 0)|^{p}ds], \]
we only need to prove \( E[\int_{0}^{\infty} e^{-Ks}|X_{s}^{t,x}|^{p}ds] < \infty. \) Now applying Itō’s formula to \( e^{-Kt}|X_{t}^{t,x}|^{p} \) and noticing Condition (A.4), we have
\[ E[\int_{t}^{\infty} e^{-Kr}|X_{r}^{t,x}|^{p}dr] \leq e^{-Kt}E[|X_{t}^{t,x}|^{p}] + C_{p}E[\int_{t}^{\infty} e^{-Kt}(|b(0)|^{p} + ||\sigma(0)||^{p})dr] < \infty. \]
Taking the limit of \( s \) and noting that \( (X_{s}^{t,x})_{s<t} = x \), we have \( E[\int_{0}^{\infty} e^{-Kr}|X_{r}^{t,x}|^{p}dr] < \infty. \) So \( E[\int_{0}^{\infty} e^{-Ks}|\hat{f}(s, 0, 0)|^{p}ds] < \infty. \) Similarly, \( E[\int_{0}^{\infty} e^{-Ks}|\hat{g}(s, 0, 0)|^{p}ds] < \infty. \)

Now we prove the other assumption in Theorem 4.3, i.e. the regularity of solutions of infinite horizon BDSDEs. An simple application of stochastic flow property proved in [13] leads to

Lemma 4.8 Under Condition (A.A), for arbitrary \( T \) and \( t, t' \in [0, T] \), \( x, x' \) belonging to an arbitrary bounded set in \( \mathbb{R}^{d} \), the diffusion process \( (X_{s}^{t,x})_{s \geq 0} \) defined in SDE (1.3) satisfies
\[ E[\int_{0}^{\infty} e^{-Kr}|X_{r}^{t',x'} - X_{r}^{t,x}|^{p}dr] \leq C_{p}(|x' - x|^{p} + |t' - t|^{\frac{p}{2}}) \quad \text{a.s.} \]

We concentrate ourselves on the regularity of infinite horizon BDSDE (1.4), which is a simpler form of BDSDE (1.1). For arbitrary given terminal time \( T \), the form of BDSDE (1.4) on \([t, T]\) is (2.2).

Proposition 4.9 Under Conditions (A.1)–(A.A), let \( (Y_{s}^{t,x})_{s \geq 0} \) be the solution of BDSDE (1.4), then for arbitrary \( T \) and \( t \in [0, T], x \in \mathbb{R}^{d}, (t, x) \rightarrow Y_{t}^{t,x} \) is a.s. continuous.

Proof. For \( t, t', r \geq 0 \), let
\[ \hat{Y}_{r} = Y_{r}^{t',x'} - Y_{r}^{t,x}, \quad \hat{Z}_{r} = Z_{r}^{t',x'} - Z_{r}^{t,x}. \]
Applying Itō's formula to \( e^{-\frac{pK'}{2}r}|\hat{Y}_{r}|^{p} \) and following a similar calculation as in (1.7), we have for \( 0 \leq s \leq T, \)
\[ e^{-\frac{pK'}{2}s}|\hat{Y}_{s}|^{p} + (p\mu - \frac{pK'}{2} - \frac{p(p + 1)}{2}C - \varepsilon) \int_{s}^{T} e^{-\frac{pK'}{2}r}|\hat{Y}_{r}|^{p}dr \]
\[ + \frac{p(2p - 3)}{4} \int_{s}^{T} e^{-\frac{pK'}{2}r}|\hat{Y}_{r}|^{p} \hat{Z}_{r}^{2}dr \]
\[ \leq e^{-\frac{pK'}{2}T}|\hat{Y}_{T}|^{p} + C_{p} \int_{s}^{T} e^{-\frac{pK'}{2}T}|\hat{X}_{r}|^{p}dr - p \int_{s}^{T} e^{-\frac{pK'}{2}r}|\hat{Y}_{r}|^{p - 2}\hat{Y}_{r} \langle \hat{g}_{r}, d\hat{B}_{r} \rangle \]
\[ - p \int_{s}^{T} e^{-\frac{pK'}{2}r}|\hat{Y}_{r}|^{p - 2}\hat{Y}_{r} \langle \hat{Z}_{r}, dW_{r} \rangle. \]  
(4.15)
Noticing Condition (A.3), for $0 \leq s \leq T$, we have

$$E[e^{-\frac{pK'}{2}}]\bar{Y}_s^p] + E[\int_s^T e^{-\frac{pK'}{2}}|\bar{Y}_r|^p dr] + E[\int_s^T e^{-\frac{pK'}{2}}|\bar{X}_r|^p dr]
\leq C_pE[e^{-\frac{pK'}{2}}|\bar{Y}_T|^p] + C_pE[\int_s^T e^{-\frac{pK'}{2}}|\bar{X}_r|^p dr].$$

Since $E[e^{-\frac{pK'}{2}}|\bar{Y}_T|^p] \leq E[\sup_{s \geq 0} e^{-Ks}|\bar{Y}_s|^p] < \infty$, by the Lebesgue's dominated convergence theorem, we have

$$\lim_{T \to \infty} E[e^{-\frac{pK'}{2}}|\bar{Y}_T|^p] = E[\lim_{T \to \infty} e^{-\frac{pK'}{2}}|\bar{Y}_T|^p] = 0.\tag{4.17}$$

So taking the limit of $T$ in (4.10), by Lemma 4.8 and the monotone convergence theorem, we have

$$E[\int_0^\infty e^{-\frac{pK'}{2}}|\bar{Y}_r|^p dr] + E[\int_0^\infty e^{-\frac{pK'}{2}}|\bar{X}_r|^p dr] \leq C_pE[\int_0^\infty e^{-Ks}|\bar{Y}_r|^p dr].$$

From (4.10), by B-D-G inequality and (4.17), we have

$$E[\sup_{s \geq 0} e^{-pKs}|\bar{Y}_s|^p] \leq C_pE[\int_0^\infty e^{-\frac{pK'}{2}}|\bar{X}_r|^p dr] + C_pE[\int_0^\infty e^{-\frac{pK'}{2}}|\bar{Y}_r|^p dr] + C_pE[\int_0^\infty e^{-\frac{pK'}{2}}|\bar{Y}_r|^p dr].$$

By the above inequality, Lemma 4.8 and (4.18), for arbitrary $T > 0$, $t$, $t' \in [0, T]$, $x$, $x'$ belonging to an arbitrary bounded set in $\mathbb{R}^d$, we have

$$E[\sup_{s \geq 0} e^{-pKs}|\bar{Y}_s|^p] \leq C_pE[\int_t^{t'} e^{-\frac{pK'}{2}}|\bar{X}_r|^p dr] \leq C_p(|x' - x|^p + |t' - t|^p).\tag{4.19}$$

Noting $p > d + 2$ in (4.19), by Kolmogorov Lemma (see e.g. [13]), we have $Y_t^{t',x}$ has a continuous modification for $t \in [0, T]$ and $x$ belonging to an arbitrary bounded set in $\mathbb{R}^d$ under the norm $\sup_{s \geq 0} e^{-Ks}|Y_{s,t'}^{t',x}|$. In particular,

$$\lim_{t' \to t} e^{-Kt'}|Y_t^{t',x} - Y_{t'}^{t',x}| = 0.$$  

Thus we have a.s.

$$\lim_{t' \to t} e^{-Kt'}Y_{t'}^{t',x} - e^{-Kt}Y_t^{t,x} \leq \lim_{t' \to t} (|e^{-Kt'}Y_{t'}^{t',x} - e^{-Kt}Y_t^{t,x}| + |e^{-Kt'}Y_{t'}^{t,x} - e^{-Kt}Y_t^{t,x}|) = 0.$$  

The convergence of the second term follows from the continuity of $Y_{t'}^{t,x}$ in $s$. That is to say $e^{-Kt}Y_t^{t,x}$ is a.s. continuous, therefore $Y_t^{t,x}$ is continuous w.r.t. $t \in [0, T]$ and $x$ belonging to an arbitrary bounded set in $\mathbb{R}^d$.

Denote by $\tilde{B}(0, R)$ the closed ball in $\mathbb{R}^d$ of radius $R$ centered at 0. It is obvious that $\bigcup_{R=1}^\infty \tilde{B}(0, R) = \mathbb{R}^d$. $Y_t^{t,x}$ is continuous w.r.t. $t \in [0, T]$ and $x \in \tilde{B}(0, R)$ on $\Omega$. Take $\tilde{\Omega} = \bigcap_{R=1}^\infty \Omega^R$, then $P(\tilde{\Omega}) = 1$. Now for any $t \in [0, T]$ and $x \in \mathbb{R}^d$, there exists an $R$ s.t. $x \in \tilde{B}(0, R)$. On the other hand, for all $\omega \in \tilde{\Omega}$, it is obvious that $\omega \in \Omega$. So $Y_t^{t,x}$ is continuous w.r.t. $t \in [0, T]$ and $x \in \mathbb{R}^d$ on $\tilde{\Omega}$. Proposition 4.9 is proved.

$\diamondsuit$
5 Stationary Property of Stochastic Viscosity Solutions of SPDEs

With the regularity of solution of BDSDE (1.4), for arbitrary given $T$, we can obtain a stochastic viscosity solution of SPDE (1.2) on the time interval $[0, T]$ through BDSDE (1.4).

**Theorem 5.1** Under Conditions (A.1)–(A.4), for arbitrary given $T$ and $t \in [0, T]$, $x \in \mathbb{R}^d$, let $v(t, x) \equiv \hat{Y}_{T-t}^{T-t, x}$, where $(Y_t^{t, x}, Z_t^{t, x})$ is the solution of BDSDE (1.4) with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geq 0$. Then $v(t, x)$ is continuous w.r.t. $t$ and $x$ and is a stochastic viscosity solution of SPDE (1.2) on the time interval $[0, T]$.

**Proof.** Notice that Condition (A.1) is stronger than (A.2), so by Theorem 4.9 BDSDE (1.4) has a unique solution $(Y_t^{t, x}, Z_t^{t, x}) \in S^{p,K}(0, \infty) \cap \mathcal{M}_{2,K}(0, \infty) \otimes \mathcal{M}_{2,K}(0, \infty)$. On $[t, T]$, BDSDE (1.4) has a form of (2.2) which can be associated with SPDE (1.2) on $[0, T]$ through time reversal transformation in (2.1). First note that by Proposition 4.9 $v(t, x)$ defined by $Y_{T-t}^{T-t, x}$ is a.s. continuous w.r.t. $t \in [0, T]$ and $x \in \mathbb{R}^d$. Moreover, since $X_{s}^{T, X_{s}^{t, x}} = X_{s}^{t, x}$ for $s \geq T$, by the uniqueness of BDSDE (1.4) we have $Y_{T-t}^{T-t, x} = Y_{T-t}^{t, x}$ a.s., where $Y_{T-t}^{t, x}$ is the solution of BDSDE (1.4) when the diffusion process $X$ defined in (1.3) starts at time $T$ and point $x \in \mathbb{R}^d$. Therefore $E[|v(0, X_{0}^{t, x})|^2] = E[|Y_{T-t}^{T-t, x}^{t, x}|^2] = E[|Y_{T-t}^{t, x}|^2] < \infty$. By Theorem 2.4 and Remark 2.5, we know that $v(t, x)$ is a stochastic viscosity solution of SPDE (1.2) on the time interval $[0, T]$. Theorem 5.1 is proved. \(\diamondsuit\)

In the following, we show that the $v(t, x)$ constructed in Theorem 5.1 is a stationary solution of SPDE (1.2). For this, we need first prove a claim that $v(t, x)(\omega) = Y_{T-t}^{T-t, x}(\hat{\omega})$ is independent of the choice of $T$. This independence can be proved by a similar argument as in [22] (Page 186-187) since it is unrelated to which kind of solution (weak solution or stochastic viscosity solution) $v$ is. Therefore, for any $T \geq t \geq 0$, $Y_{T-t}^{T-t, x}(\hat{\omega}) = Y_{T-t}^{T-t, x}(\hat{\omega}')$ when $0 \leq t \leq T$, where $\hat{\omega}(s) = B_{T-s} - B_T$ and $\hat{\omega}'(s) = B_{T-s} - B_T$.

On the probability space $(\Omega, \mathcal{F}, P)$, we define $\theta_t = (\hat{\theta}_t)^{-1}$, $t \geq 0$. Actually $\hat{B}$ is a two-sided Brownian motion, so $(\hat{\theta}_t)^{-1} = \hat{\theta}_{-t}$ is well defined (see [1]). It is easy to see that $\theta_t$ is a shift w.r.t. $B$ satisfying

(i) $P \cdot (\theta_t)^{-1} = P$;
(ii) $\theta_0 = I$;
(iii) $\theta_s \circ \theta_t = \theta_{s+t}$;
(iv) $\theta_t \circ B_s = B_{s+t} - B_t$.

By Theorem 3.3 and the relationship between $\theta$ and $\hat{\theta}$, we have

$$\theta_r v(t, x)(\omega) = \hat{\theta}_{-r} Y_{T-t}^{T-t, x}(\hat{\omega}) = \hat{\theta}_{-r} \theta_r Y_{T-t}^{T-t, x}(\hat{\omega}) = Y_{T-t-r}^{T-t, x}(\hat{\omega}) = v(t+r, x)(\omega),$$

for all $r \geq 0$ and $T \geq t + r$, $x \in \mathbb{R}^d$ a.s. In particular, let $Y(x, \omega) = v(0, x)(\omega) = Y_{T-t}^{T-t, x}(\hat{\omega})$, then the above formula implies (1.3):

$$\theta_t Y(x, \omega) = Y(x, \theta_t \omega) = v(t, x)(\omega) = v(t, x, v(0, x)(\omega))(\omega) = v(t, x, Y(x, \omega))(\omega),$$

for all $t \geq 0$, $x \in \mathbb{R}^d$ a.s. That is to say $v(t, x)(\omega) = Y(x, \theta_t \omega) = Y_{T-t}^{T-t, x}(\hat{\omega})$ is a stationary solution of SPDE (1.2) w.r.t. $\theta$.

Therefore we have the following conclusion.
Theorem 5.2 Under Conditions (A.1)–(A.4), for arbitrary $T$ and $t \in [0,T]$, let $v(t,x) \triangleq Y^T_{T-t,x} - Y^T_{T-t,x}$, where $(Y^t_{s,x}, Z^t_{s,x})$ is the solution of BDSDE (1.4) with $\hat{B}_s = B_{T-s} - B_T$ for all $s \geq 0$. Then $v(t,x)$ is a “perfect” stationary stochastic viscosity solution of SPDE (1.3).

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