ω-Rudin spaces, well-filtered determined spaces and first-countable spaces✩

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Abstract
We investigate some versions of d-space, well-filtered space and Rudin space concerning various countability properties. The main results include: (i) if the sobrification of a T0 space X is first-countable, then X is an ω-Rudin space; (ii) every ω-well-filtered space is sober if its sobrification is first-countable; (iii) if a T0 space is second-countable or first-countable and with a countable underlying set, then it is a ω-Rudin space; (iv) every first-countable T0 space is well-filtered determined; (v) every irreducible closed subset in a first-countable ω-well-filtered space is countably-directed; (vi) every first-countable ω-well-filtered ω∗-d-space is sober.

Keywords: First-countability; Sober space; ω-Rudin space; ω-Well-filtered space; ω∗-d-Space; Countably-directed set
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1. Introduction
In [17, 21, 21], we introduced and studied the Rudin spaces, well-filtered determined spaces and ω-well-filtered spaces. Some relationships and links among these new non-Hausdorff topological properties and the well studied sobriety and well-filteredness were uncovered. In [21], it was proved that in a first-countable ω-well-filtered space X, every irreducible closed subset of X is directed under the specialization order of X. It follows immediately that every first-countable ω-well-filtered d-space is sober.

In the current paper, we continue studying some aspects of d-space, well-filtered space and Rudin spaces concerning countability. Employing countably-directed sets, we define the ω∗-d-spaces and ω∗-well-filtered spaces. It is proved that if the sobrification of a T0 space X is first-countable, then X is an ω-Rudin space. Therefore, every ω-well-filtered space is sober if it has a first countable sobrification. From these, we obtain that if a T0 space X is second-countable or first-countable with a countable underlying set, then X is an ω-Rudin space, and X is sober if it is additionally ω-well-filtered. Another major result obtained is that every first-countable T0 space is well-filtered determined. In each first-countable ω-well-filtered space, every irreducible closed subset is proved to be countably-directed, hence every first-countable ω-well-filtered ω∗-d-space is sober. We also prove that a T0 space Y is ω∗-well-filtered iff its Smyth power space is ω∗-well-filtered iff its Smyth power space is an ω∗-d-space. The work presented here enriched the theory of non-Hausdorff topological spaces and lead to some nontrivial open problems for further investigation.

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2. Preliminary

In this section, we briefly recall some fundamental concepts and notations that will be used in the paper. Some basic properties of irreducible sets and compact saturated sets are presented.

For a poset \( P \) and \( A \subseteq P \), let \( \downarrow A = \{ x \in P : x \leq a \text{ for some } a \in A \} \) and \( \uparrow A = \{ x \in P : x \geq a \text{ for some } a \in A \} \). For \( x \in P \), we write \( \downarrow x \) for \( \downarrow \{x\} \) and \( \uparrow x \) for \( \uparrow \{x\} \). A subset \( A \) is called a lower set (resp., an upper set) if \( A = \downarrow A \) (resp., \( A = \uparrow A \)). Let \( P^{(\omega)} = \{ F \subseteq P : F \) is a nonempty finite set \( \} \). For a set \( X \), \( |X| \) will denote the cardinality of \( X \). Let \( \mathbb{N} \) denotes the set of all natural numbers with the usual order and \( \omega = |\mathbb{N}| \).

A nonempty subset \( D \) of a poset \( P \) is directed if every two elements in \( D \) have an upper bound in \( D \). The set of all directed subsets of \( P \) is denoted by \( \mathcal{D}(P) \). A subset \( I \subseteq P \) is called an ideal of \( P \) if \( I \) is a directed lower subset of \( P \). Let \( \text{Id}(P) \) be the poset (with the order of set inclusion) of all ideals of \( P \). Dually, we define the notion of filters and denote the poset of all filters of \( P \) by \( \text{Filt}(P) \). A poset \( P \) is called a directed complete poset, or dcpo for short, if for any \( D \in \mathcal{D}(P) \), \( \bigvee D \) exists in \( P \).

As in [3], the upper topology on a poset \( Q \), generated by the complements of the principal ideals of \( Q \), is denoted by \( v(Q) \). A subset \( U \) of \( Q \) is Scott open if (i) \( U = \uparrow U \) and (ii) for any directed subset \( D \subseteq Q \) with \( \bigvee D \) existing, \( \bigvee D \in U \) implies \( D \cap U \neq \emptyset \). All Scott open subsets of \( Q \) form a topology, called the Scott topology on \( Q \) and denoted by \( \sigma(Q) \). The space \( \Sigma Q = (Q, \sigma(Q)) \) is called the Scott space of \( Q \). The upper sets of \( Q \) form the (upper) Alexandroff topology \( \alpha(Q) \).

For a \( T_0 \) space \( X \), we use \( \leq_X \) to denote the specialization order on \( X \): \( x \leq_X y \) iff \( x \in \overline{\{y\}} \). In the following, when a \( T_0 \) space \( X \) is considered as a poset, the partial order always means the specialization order provided otherwise indicated. Let \( \mathcal{O}(X) \) (resp., \( \mathcal{C}(X) \)) be the set of all open subsets (resp., closed subsets) of \( X \), and let \( \mathcal{S}^+(X) = \{ \uparrow x : x \in X \} \). Let \( \mathcal{S}_0(X) = \{ \{x\} : x \in X \} \) and \( \mathcal{P}_c(X) = \{ \bigvee D : D \in \mathcal{D}(X) \} \).

A nonempty subset \( A \) of \( X \) is irreducible if for any \( \{F_1, F_2\} \subseteq \mathcal{C}(X) \), \( A \subseteq F_1 \cup F_2 \) implies \( A \subseteq F_1 \) or \( A \subseteq F_2 \). Denote by \( \text{Irr}(X) \) (resp., \( \text{Irr}_c(X) \)) the set of all irreducible (resp., irreducible closed) subsets of \( X \). Every directed subset of \( X \) is irreducible. \( X \) is called sober, if for any \( F \in \text{Irr}(X) \), there is a unique point \( a \in X \) such that \( F = \overline{\{a\}} \).

Remark 2.1. In a \( T_0 \) space \( X \), if \( x \in X \) and \( A \subseteq X \) such that \( \overline{\{x\}} = A \), then \( \bigvee A \) exists in \( X \) and \( x = \bigvee A \).

The following two lemmas on irreducible sets are well-known.

Lemma 2.2. Let \( X \) be a space and \( Y \) a subspace of \( X \). Then the following conditions are equivalent for a subset \( A \subseteq Y \):

1. \( A \) is an irreducible subset of \( Y \).
2. \( A \) is an irreducible subset of \( X \).
3. \( \text{cl}_X A \) is an irreducible subset of \( X \).

Lemma 2.3. If \( f : X \to Y \) is continuous and \( A \subseteq \text{Irr}(X) \), then \( f(A) \subseteq \text{Irr}(Y) \).

A \( T_0 \) space \( X \) is called a d-space (or monotone convergence space) if \( X \) (with the specialization order) is a dcpo and \( \mathcal{O}(X) \subseteq \sigma(X) \) (cf. [3, 18]).

Definition 2.4. [21] A \( T_0 \) space \( X \) is called a directed closure space, \( \text{DC} \) space for short, if \( \text{Irr}_c(X) = \mathcal{D}_c(X) \), that is, for each \( A \in \text{Irr}_c(X) \), there exists a directed subset of \( X \) such that \( A = \overline{\{x\}} \).

For any topological space \( X \), \( \mathcal{G} \subseteq 2^X \) and \( A \subseteq X \), let \( \Diamond_{\mathcal{G}} A = \{ G \in \mathcal{G} : G \cap A \neq \emptyset \} \) and \( \Box_{\mathcal{G}} A = \{ G \in \mathcal{G} : G \subseteq A \} \). The symbols \( \Diamond_{\mathcal{G}} \) and \( \Box_{\mathcal{G}} \) will be simply written as \( \Diamond \) and \( \Box \) respectively if no ambiguity occurs. The lower Vietoris topology on \( \mathcal{G} \) is the topology that has \( \{ \Diamond U : U \in \mathcal{O}(X) \} \) as a subbase, and the resulting space is denoted by \( P_H(\mathcal{G}) \). If \( \mathcal{G} \subseteq \text{Irr}(X) \), then \( \Diamond \mathcal{G} U : U \in \mathcal{O}(X) \) is a topology on \( \mathcal{G} \). The space \( P_H(\mathcal{C}(X) \setminus \{\emptyset\}) \) is called the Hoare power space or lower space of \( X \) and is denoted by \( P_H(X) \) for short (cf. [16]). Clearly, \( P_H(X) = (\mathcal{C}(X) \setminus \{\emptyset\}, v(\mathcal{C}(X) \setminus \{\emptyset\})) \). So \( P_H(X) \) is always sober (see [24], Corollary 4.10) or [21, Proposition 2.9]). The upper Vietoris topology on \( \mathcal{G} \) is the topology that has \( \{ \Box_{\mathcal{G}} U : U \in \mathcal{O}(X) \} \) as a base, and the resulting space is denoted by \( P_S(\mathcal{G}) \).
Remark 2.5. Let $X$ be a $T_0$ space.

(1) If $S_c(X) \subseteq G$, then the specialization order on $P_I(G)$ is the set inclusion order, and the canonical mapping $\eta_X : X \rightarrow P_I(G)$, given by $\eta_X(x) = \{x\}$, is an order and topological embedding (cf. [3, 6, 10]).

(2) The space $X^* = P_I(br_c(X))$ with the canonical mapping $\eta_X : X \rightarrow X^*$ is the sobrification of $X$ (cf. [3, 4]).

A subset $A$ of a space $X$ is called saturated if $A$ equals the intersection of all open sets containing it (equivalently, $A$ is an upper set in the specialization order). We shall use $K(X)$ to denote the set of all nonempty compact saturated subsets of $X$ and endow it with the Smyth preorder, that is, for $K_1, K_2 \in K(X)$, $K_1 \sqsubseteq K_2$ iff $K_2 \subseteq K_1$. The space $P_S(K(X))$, denoted shortly by $P_S(X)$, is called the Smyth power space or upper space of $X$ (cf. [7, 10]). It is easy to verify that the specialization order on $P_S(X)$ is the Smyth order (that is, $\leq_{P_S(X)} = \sqsubseteq$). The canonical mapping $\xi_X : X \rightarrow P_S(X), x \mapsto \uparrow x$, is an order and topological embedding (cf. [3, 6, 10]). Clearly, $P_S(S^w(X))$ is a subspace of $P_S(X)$ and $X$ is homeomorphic to $P_S(S^w(X))$.

Lemma 2.6. ([10, 16]) Let $X$ be a $T_0$ space. For any nonempty family $\{K_i : i \in I\} \subseteq K(X)$, $\bigvee_{i \in I} K_i$ exists in $K(X)$ iff $\bigwedge_{i \in I} K_i \in K(X)$. In this case $\bigvee_{i \in I} K_i = \bigwedge_{i \in I} K_i$.

Lemma 2.7. ([10, 16]) Let $X$ be a $T_0$ space.

(1) If $K \in K(P_S(X))$, then $\bigcup K \in K(X)$.

(2) The mapping $\bigcup : P_S(P_S(X)) \rightarrow P_S(X), K \mapsto \bigcup K$, is continuous.

A $T_0$ space $X$ is called well-filtered if it is $T_0$, and for any open set $U$ and filtered family $K \subseteq K(X)$, $\bigcap K \subseteq U$ implies $K \subseteq U$ for some $K \in K$.

Remark 2.8. The following implications are well-known (cf. [3]):

sobriety $\Rightarrow$ well-filteredness $\Rightarrow$ $d$-space.

3. Topological Rudin’s Lemma, Rudin spaces and well-filtered determined spaces

Rudin’s Lemma is a useful tool in topology and plays a crucial role in domain theory (see [1, 3-9, 20-21, 23]). Heckmann and Keimel [8] presented the following topological variant of Rudin’s Lemma.

Lemma 3.1. (Topological Rudin’s Lemma) Let $X$ be a topological space and $A$ an irreducible subset of the Smyth power space $P_S(X)$. Then every closed set $C \subseteq X$ that meets all members of $A$ contains a minimal irreducible closed subset $A$ that still meets all members of $A$.

Applying Lemma 3.1 to the Alexandroff topology on a poset $P$, one obtains the original Rudin’s Lemma (see [10, 15]).

Corollary 3.2. (Rudin’s Lemma) Let $P$ be a poset, $C$ a nonempty lower subset of $P$ and $F \in \text{Fin} P$ a filtered family with $F \subseteq \Diamond C$. Then there exists a directed subset $D$ of $C$ such that $F \subseteq \Diamond D$.

For a $T_0$ space $X$ and $K \subseteq K(X)$, let $M(K) = \{A \in C(X) : K \cap A \neq \emptyset \text{ for all } K \in K\}$ (that is, $K \subseteq \Diamond A$) and $m(K) = \{A \in C(X) : A$ is a minimal member of $M(K)\}$. By the proof of [8, Lemma 3.1], we have the following result.

Lemma 3.3. Let $X$ be a $T_0$ space and $K \subseteq K(X)$. If $C \in M(K)$, then there is a closed subset $A$ of $C$ such that $C \in m(K)$.

In [17, 21], based on topological Rudin’s Lemma, Rudin spaces and well-filtered determined spaces were introduced and studied.
Definition 3.4. ([17, 21]) Let $X$ be a $T_0$ space. A nonempty subset $A$ of $X$ is said to have the Rudin property, if there exists a filtered family $K \subseteq K(X)$ such that $\overline{A} \in m(K)$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $K$). Let $\text{RD}(X) = \{ A \in \mathcal{C}(X) : A$ has Rudin property $\}$. The sets in $\text{RD}(X)$ will also be called Rudin sets. The space $X$ is called a Rudin space, $\text{RD}$ space for short, if $\text{Irr}_c(X) = \text{RD}(X)$, that is, every irreducible closed set of $X$ is a Rudin set. The category of all Rudin spaces with continuous mappings is denoted by $\text{Top}_r$.

Definition 3.5. ([21]) A subset $A$ of a $T_0$ space $X$ is called a well-filtered determined set, WD set for short, if for any continuous mapping $f : X \to Y$ to a well-filtered space $Y$, there exists a unique $y_A \in Y$ such that $f(A) = \{ y_A \}$. Denote by $\text{WD}(X)$ the set of all closed well-filtered determined subsets of $X$. $X$ is called a well-filtered determined space, WD space for short, if all irreducible closed subsets of $X$ are well-filtered determined, that is, $\text{Irr}_c(X) = \text{WD}(X)$.

Proposition 3.6. ([21]) Let $X$ be a $T_0$ space. Then $\mathcal{D}_c(X) \subseteq \text{RD}(X) \subseteq \text{WD}(X) \subseteq \text{Irr}_c(X)$.

Corollary 3.7. ([21]) Sober $\Rightarrow$ DC $\Rightarrow$ RD $\Rightarrow$ WD.

A topological space $X$ is locally hypercompact if for each $x \in X$ and each open neighborhood $U$ of $x$, there is $\uparrow F \in \text{Fin}_X$ such that $x \in \text{int} \uparrow F \subseteq \uparrow F \subseteq U$ (cf. [2]). A space $X$ is called core-compact if $(\mathcal{O}(X), \subseteq)$ is a continuous lattice (cf. [3]).

Theorem 3.8. ([3]) Let $X$ be a locally hypercompact $T_0$ space and $A \in \text{Irr}(X)$. Then there exists a directed subset $D \subseteq \downarrow A$ such that $\overline{A} = \overline{\bigvee D}$. Therefore, $X$ is a DC space, and hence a Rudin space and a WD space.

Theorem 3.9. ([21]) Every locally compact $T_0$ space is a Rudin space.

Theorem 3.10. ([21]) Every core-compact $T_0$ space is well-filtered determined.

By Theorem 3.10 we immediately deduce the following.

Corollary 3.11. ([13, 21]) Every core-compact well-filtered space is sober.

At the moment, it is still not sure whether every core-compact $T_0$ space is a Rudin space.

4. $\omega$-$d$-Spaces and $\omega$-well-filtered spaces

For a $T_0$ space $X$, let $\mathcal{D}^\omega(X) = \{ D \subseteq X : D$ is countable and directed $\}$ and $\mathcal{D}^\omega_c(X) = \{ \overline{D} : D \in \mathcal{D}^\omega(X) \}$.

Definition 4.1. ([21]) A poset $P$ is called an $\omega$-depo, if for any $D \in \mathcal{D}^\omega(P)$, $\bigvee D$ exists.

Lemma 4.2. ([21]) Let $P$ be a poset and $D \in \mathcal{D}^\omega(P)$. Then there exists a countable chain $C \subseteq D$ such that $D = \downarrow C$. Hence, $\bigvee C$ exists and $\bigvee C = \bigvee D$ whenever $\bigvee D$ exists.

By Lemma 4.2 a poset $P$ is an $\omega$-depo iff for any countable chain $C$ of $P$, $\bigvee C$ exists.

Definition 4.3. ([21]) Let $P$ be a poset. A subset $U$ of $P$ is called $\omega$-Scott open if (i) $U = \uparrow U$, and (ii) for any countable directed set $D$, $\bigvee D \in U$ implies that $D \cap U \neq \emptyset$. All $\omega$-Scott open sets form a topology on $P$, denoted by $\sigma_\omega(P)$ and called the $\omega$-Scott topology. The space $\Sigma_\omega = (P, \sigma_\omega(P))$ is called the $\omega$-Scott space of $P$.

Clearly, $\sigma(P) \subseteq \sigma_\omega(P)$. The converse need not be true, see Example 7.5 in Section 7.

Definition 4.4. ([21]) A $T_0$ space $X$ is called an $\omega$-$d$-space (or an $\omega$-monotone convergence space) if for any $D \in \mathcal{D}^\omega(X)$, the closure of $D$ has a generic point, equivalently, if $\mathcal{D}^\omega_c(X) = \mathcal{S}_c(X)$.

Some characterizations of $\omega$-$d$-spaces were given in [21, Proposition 3.7]
Definition 4.5. (20) A $T_0$ space $X$ is called $\omega$-well-filtered, if for any countable filtered family $\{K_i : i < \omega\} \subseteq \mathcal{K}(X)$ and $U \in \mathcal{O}(X)$, it holds that

$$\bigcap_{i<\omega} K_i \subseteq U \Rightarrow \exists i_0 < \omega, K_{i_0} \subseteq U.$$  

By Lemma 4.12 we have the following result.

Proposition 4.6. (20) A $T_0$ space $X$ is $\omega$-well-filtered iff for any countable descending chain $K_0 \supseteq K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$ of compact saturated subsets of $X$ and $U \in \mathcal{O}(X)$, the following implication holds:

$$\bigcap_{i<\omega} K_i \subseteq U \Rightarrow \exists i_0 < \omega, K_{i_0} \subseteq U.$$  

It is easy to check that every $\omega$-well-filtered space is an $\omega$-d-space (see [20, Proposition 3.11]). In [20], we introduced and studied two new classes of closed subsets in $T_0$ spaces - $\omega$-Rudin sets and $\omega$-well-filtered determined closed sets. The $\omega$-Rudin sets lie between the class of all closures of countable directed subsets and that of $\omega$-well-filtered determined closed sets, and $\omega$-well-filtered determined closed sets lie between the class of all $\omega$-Rudin subsets and that of irreducible closed subsets.

Definition 4.7. (20) Let $X$ be a $T_0$ space. A nonempty subset $A$ of $X$ is said to have the $\omega$-Rudin property, if there exists a countable filtered family $\mathcal{K} \subseteq \mathcal{K}(X)$ such that $\overline{A} \in \mathcal{m}(\mathcal{K})$ (that is, $\overline{A}$ is a minimal closed set that intersects all members of $\mathcal{K}$). Let $\text{RD}_\omega(X) = \{A \in \mathcal{C}(X) : A$ has $\omega$-Rudin property$\}$. The sets in $\text{RD}_\omega(X)$ will also be called $\omega$-Rudin sets. The space $X$ is called $\omega$-Rudin space, if $\text{Irr}_c(X) = \text{RD}_\omega(X)$, that is, all irreducible closed subsets of $X$ are $\omega$-Rudin sets.

Definition 4.8. (20) A subset $A$ of a $T_0$ space $X$ is called an $\omega$-well-filtered determined set, $\omega$-$\mathcal{D}_\omega$ set for short, if for any continuous mapping $f : X \longrightarrow Y$ to an $\omega$-well-filtered space $Y$, there exists a unique $y_A \in Y$ such that $f(A) = \{y_A\}$. Denote by $\omega$-$\mathcal{D}_\omega$ the set of all closed $\omega$-well-filtered determined subsets of $X$. The space $X$ is called $\omega$-well-filtered determined, $\omega$-$\mathcal{D}$ space for short, if $\text{Irr}_c(X) = \omega$-$\mathcal{D}$, that is, all irreducible closed subsets of $X$ are $\omega$-well-filtered determined.

Proposition 4.9. (20) Let $X$ be a $T_0$ space. Then $S_c(X) \subseteq \omega$-$\mathcal{D}$ $(\subseteq \text{RD}_\omega(X) \subseteq \omega$-$\mathcal{D}) \subseteq \text{Irr}_c(X)$.

By Proposition 4.9, every $\omega$-Rudin space is $\omega$-well-filtered determined.

Definition 4.10. A $T_0$ space $X$ is called a countable directed closure space, $\omega$-$\mathcal{D}$ space for short, if $\text{Irr}_c(X) = \omega$-$\mathcal{D}$, that is, for each $A \in \text{Irr}_c(X)$, there exists a countable directed subset of $X$ such that $A = \overline{D}$.

Theorem 4.11. (20) For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is sober.
2. $X$ is an $\omega$-$\mathcal{D}$ and $\omega$-d-space.
3. $X$ is an $\omega$-$\mathcal{D}$ and $\omega$-well-filtered space.
4. $X$ is an $\omega$-Rudin and $\omega$-well-filtered space.
5. $X$ is an $\omega$-well-filtered determined and $\omega$-well-filtered space.

5. $\omega^*$-Scott topologies and $\omega^*$-d-spaces

We now introduce and study two new types of spaces.

Definition 5.1. A nonempty subset $D$ of a poset $P$ is called countably-directed if every nonempty countable subset of $D$ have an upper bound in $D$. The set of all countably-directed sets of $P$ is denoted by $\mathcal{D}^{\omega^*}(P)$. The poset $P$ is called a countably-directed complete poset, or $\omega^*$-dcpo for short, if for any $D \in \mathcal{D}^{\omega^*}(P)$, $\bigvee D$ exists in $P$. 

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Clearly, \( \{ \downarrow x : x \in P \} \subseteq D^{\omega}(P) \subseteq D(P) \).

**Example 5.2.** For the countable chain \( \mathbb{N} \) (with the usual order of natural numbers), \((\mathbb{N}^{<\omega}, \subseteq)\) is directed in \(2^n\), but not countably-directed.

**Definition 5.3.** A subset \( U \) of a poset \( P \) is \( \omega^*\)-Scott open if (i) \( U = \uparrow U \) and (ii) for any countably-directed subset \( D \) with \( \bigvee D \) existing, \( \bigvee D \in U \) implies \( D \cap U \neq \emptyset \). All \( \omega^*\)-Scott open subsets of \( P \) form a topology, called the \( \omega^*\)-Scott topology on \( Q \) and denoted by \( \sigma_{\omega^*}(Q) \). Let \( \sigma_{\omega^*}(Q) = (Q, \sigma_{\omega^*}(Q)) \).

Clearly, \( \nu(P) \subseteq \sigma(P) \subseteq \sigma_{\omega^*}(P) \subseteq \alpha(P) \). In general, \( \sigma(P) \neq \sigma_{\omega^*}(P) \) as shown in Example 5.5 below. It is straightforward to verify the following two results (cf. \[ \text{Proposition II-2.1]}.)

**Proposition 5.4.** A continuous function \( f : X \to Y \) from an \( \omega^*\)-d-space \( X \) to any \( T_0 \) space \( Y \) preserves countably-directed supersets in the specialization orders.

**Proposition 5.5.** Let \( P, Q \) be posets and \( f : P \to Q \). Then the following two conditions are equivalent:
1. \( f : \Sigma_{\omega^*}P \to \Sigma_{\omega^*}Q \) is continuous.
2. \( f \) preserves countably-directed supersets, that is, for every \( D \in D_{\omega^*}(P) \) for which \( \bigvee D \) exists, \( \bigvee f(D) \) exists and \( f(\bigvee D) = \bigvee f(D) \).

**Definition 5.6.** A \( T_0 \) space \( X \) is said to be an \( \omega^*-d \)-space (or an \( \omega^*\)-monotone convergence space), if every subset \( D \) countably-directed relative to the specialization order of \( X \) has a sup, and the relation \( D \subseteq U \) for any open set \( U \) implies \( D \cap U \neq \emptyset \).

For a \( T_0 \) space \( X \) (endowed with the specialization order), let \( D^\omega(X) = \{ \overline{T} : D \in D^\omega(X) \} \). Then \( S_\omega(X) \subseteq D_{\omega^*}(X) \subseteq D(X) \).

**Proposition 5.7.** For a \( T_0 \) space \( X \), the following conditions are equivalent:
1. \( X \) is an \( \omega^*\)-d-space.
2. \( D^\omega(X) = S_\omega(X) \), that is, for any \( D \in D^\omega(X) \), the closure of \( D \) has a (unique) generic point.
3. \( X \) (with the specialization order \( \leq_X \)) is an \( \omega^*-\text{depo} \) and \( \mathcal{O}(X) \subseteq \sigma_{\omega^*}(X) \).
4. For any \( D \in D^\omega(X) \) and \( U \in \mathcal{O}(X) \), \( \bigcap_{d \in D} \uparrow d \subseteq U \) implies \( \uparrow d \subseteq U \) (i.e., \( d \in U \)) for some \( d \in D \).
5. For any \( D \in D^\omega(X) \) and \( A \in \mathcal{C}(X) \), if \( D \subseteq A \), then \( A \cap \bigcap_{d \in D} \uparrow d \neq \emptyset \).
6. For any \( D \in D^\omega(X) \) and \( A \in \mathcal{Ir}_e(X) \), if \( D \subseteq A \), then \( A \cap \bigcap_{d \in D} \uparrow d \neq \emptyset \).
7. For any \( D \in D^\omega(X) \), \( \overline{D \cap \bigcap_{d \in D} \uparrow d} \neq \emptyset \).

**Proof.** (1) \( \Rightarrow \) (2): Let \( D \in D^\omega(X) \). Then by (1), \( \bigvee D \) exists and the relation \( \bigvee D \subseteq U \) for any open set \( U \) of \( X \) implies \( D \cap U \neq \emptyset \). Therefore, \( \overline{D} = \{ \sup D \} \).

(2) \( \Rightarrow \) (3): For each \( D \in D^\omega(X) \) and \( A \in \mathcal{C}(X) \) with \( D \subseteq A \), by condition (2) there is \( x \in X \) such that \( \overline{D} = \{ x \} \), and consequently, \( \bigvee D = x \) and \( \bigvee D \in A \) since \( \overline{D} \subseteq A \). Thus \( X \) is an \( \omega^*-\text{depo} \) and \( \mathcal{O}(X) \subseteq \sigma_{\omega^*}(X) \).

(3) \( \Rightarrow \) (4): Suppose that \( D \in D^\omega(X) \) and \( U \in \mathcal{O}(X) \) with \( \bigcap_{d \in D} \uparrow d \subseteq U \). Then by condition (3), \( \bigvee D \subseteq \bigcap_{d \in D} \uparrow d \subseteq U \subseteq \sigma_{\omega^*}(X) \). Therefore, \( \bigvee D \subseteq U \), and whence \( d \in U \) for some \( d \in D \).

(4) \( \Rightarrow \) (5): If \( A \cap \bigcap_{d \in D} \uparrow d = \emptyset \), then \( \bigcap_{d \in D} \uparrow d \subseteq X \setminus A \). By condition (4), \( \uparrow d \subseteq X \setminus A \) for some \( d \in D \), which is in contradiction with \( D \subseteq A \).

(5) \( \Rightarrow \) (6) \( \Rightarrow \) (7): Trivial.

(7) \( \Rightarrow \) (1): For each \( D \in D^\omega(X) \) and \( A \in \mathcal{C}(X) \) with \( D \subseteq A \), by condition (6), we have \( \overline{D \cap \bigcap_{d \in D} \uparrow d} \neq \emptyset \). Select an \( x \in \overline{D \cap \bigcap_{d \in D} \uparrow d} \). Then \( D \subseteq \downarrow x \subseteq \overline{D} \), and hence \( \overline{D} = \downarrow x \). Thus \( X \) is an \( \omega^*-d \)-space. \( \square \)
By Proposition 5.11 every $d$-space is an $\omega$-$d$-space, and for any $\omega^*\text{-dcpo } P$, $\Sigma_{\omega^*} P$ is an $\omega^*\text{-d-space}$. Let $Q = (\mathbb{N}(\leq), \subseteq)$. Then $Q$ is an $\omega^*\text{-dcpo but not a dcpo.}$ If $P$ is any $\omega^*\text{-dcpo but not a dcpo, then } \Sigma_{\omega^*}(P)$ is an $\omega^*\text{-d-space but not a d-space.}$

**Definition 5.8.** A $T_0$ space $X$ is called a countable-directed closure space, or $\omega^*\text{-DC space}$ for short, if $\text{Irr}_c(X) = D_c^*(X)$, that is, for each $A \in \text{Irr}_c(X)$, there exists a countably-directed subset $D$ of $X$ such that $A = \overline{D}$.

Now we introduce another type of weak-filtered spaces.

**Definition 5.9.** A $T_0$ space $X$ is called $\omega^*$-well-filtered, if for any countable-filtered family $\{K_i : i \in I\} \subseteq K(X)$ (that is, $\{K_i : i \in I\} \in D^{\omega^*}(K(X))$) and $U \in \mathcal{O}(X)$, it satisfies

$$\bigcap_{i \in I} K_i \subseteq U \Rightarrow \exists j \in I, K_j \subseteq U.$$  

Clearly, every well-filtered space is $\omega^*$-well-filtered. The converse implications does not hold in general, as shown by the following example.

**Example 5.10.** Let $P = (\mathbb{N}(\leq), \subseteq)$ and $X = \Sigma_{\omega^*} P$. It is easy to verify that any countably-directed subset of $P$ has a largest element. Therefore, $\sigma_{\omega^*}(P) = \sigma(P)$ and $K(X) = \{F : F \in \mathbb{N}(\leq)\}$, and hence $X$ is $\omega^*$-well-filtered as any countable-filtered family $\{K_i : i \in I\} \subseteq K(X)$ has a least element. Since $P$ is not a dcpo ($P$ is directed but has not a largest element), $X$ is not a $d$-space, and hence not well-filtered.

**Proposition 5.11.** Every $\omega^*$-well-filtered space is an $\omega^*$-d-space.

**Proof.** Let $X$ be an $\omega^*$-well-filtered space and $D \in D^{\omega^*}(X)$. Then $\{\uparrow d : d \in D\} \in D^{\omega^*}(K(X))$. By the $\omega^*$-well-filteredness of $X$, we have $\bigcap_{d \in D} \uparrow d \not\subseteq X \setminus \overline{D}$ or, equivalently, $\bigcap_{d \in D} \uparrow d \cap \overline{D} \neq \emptyset$. Therefore, there is $x \in \bigcap_{d \in D} \uparrow d \cap \overline{D},$ and hence $\overline{D} = \{x\}$. \hfill $\square$

In the following, using the topological Rudin’s Lemma, we prove that a $T_0$ space $X$ is $\omega^*$-well-filtered iff the Smyth power space of $X$ is $\omega^*$-well-filtered iff the Smyth power space of $X$ is $\omega^*$-d-space. The corresponding results for well-filteredness and $\omega^*$-well-filteredness are proved in [19, 20, 21].

**Theorem 5.12.** For a $T_0$ space $X$, the following conditions are equivalent:

1. $X$ is $\omega^*$-well-filtered.
2. $P_S(X)$ is an $\omega^*$-d-space.
3. $P_S(X)$ is $\omega^*$-well-filtered.

**Proof.** (1) $\Rightarrow$ (2): Suppose that $X$ is an $\omega^*$-well-filtered space. For any countable-filtered family $\mathcal{K} \subseteq K(X)$, by the $\omega^*$-well-filteredness of $X$, $\bigcap_{i \in I} K_i \subseteq U$ for any $U \in \mathcal{O}(X)$. Thus $P_S(X)$ is an $\omega^*$-d-space.

(2) $\Rightarrow$ (3): Suppose that $\{K_i : i \in I\} \subseteq K(P_S(X))$ is countable-filtered, $U \in \mathcal{O}(P_S(X))$, and $\bigcap_{i \in I} K_i \subseteq U$. If $K_i \subseteq U$ for all $i \in I$, then by Lemma 5.11 $K(X) \setminus U$ contains an irreducible closed subset $\mathcal{A}$ that still meets all $K_i$ ($i \in I$). For each $i \in I$, let $K_i = \bigcup_{j \in K(K_i)} (A \cap K_i) = \bigcup_{j \in K(K_i)} (A \cap K_i)$. Then by Lemma 2.7 $\{K_i : i \in I\} \subseteq K(X)$ is countable-filtered, and $K_i \in \mathcal{A}$ for all $i \in I$ since $A = \downarrow_{K(K_i)} A$. Let $K = \bigcap_{i \in I} K_i$. Then $K \in K(X)$ and $K = \bigvee_{K(K_i)} \{K_i : i \in I\} \in \mathcal{A}$ by Lemma 2.7 and condition (2). We claim that $K \in \bigcap_{i \in I} K_i$.

Suppose, on the contrary, that $K \not\subseteq \bigcap_{i \in I} K_i$. Then there is a $j \in I$ such that $K \not\subseteq K_j$. Select a $G \in \mathcal{A} \cap K_j$. Then $K \not\subseteq G$ (otherwise, $K \in \downarrow_{K(K_i)} K_j \subseteq K_j$, being a contradiction with $K \not\subseteq K_j$), and hence there is a $g \in K \setminus G$. It follows that $g \in K_i = \bigcup_{j \in K(K_i)} (A \cap K_i)$ for all $i \in I$ and $G \not\subseteq \downarrow_{K(K_i)} \{g\}$. For each $i \in I$, by $g \in K_i = \bigcup_{j \in K(K_i)} (A \cap K_i)$, there is a $K_i^g \in \mathcal{A} \cap K_i$ such that $g \in K_i^g$, and hence $K_i^g \in \downarrow_{K(K_i)} \{g\} \cap \mathcal{A} \cap K_i$. Thus $\downarrow_{K(K_i)} \{g\} \cap \mathcal{A} \cap K_i \neq \emptyset$ for all $i \in I$. By the minimality of $\mathcal{A}$, we have $\mathcal{A} = \downarrow_{K(K_i)} \{g\} \cap \mathcal{A}$, and
consequently, \( G \in \mathcal{A} \cap \mathcal{K}_j = \mathcal{O}_{\mathcal{K}(\mathcal{K})}[g] \cap \mathcal{A} \cap \mathcal{K}_j \), which is a contradiction with \( G \not\in \mathcal{O}_{\mathcal{K}(\mathcal{K})}[g] \). Thus \( K \in \bigcap_{i \in I} \mathcal{K}_i \subseteq U \subseteq \mathcal{K}(X) \setminus \mathcal{A} \), being a contradiction with \( K \in \mathcal{A} \). Therefore, \( P_S(X) \) is \( \omega^* \)-well-filtered.

(3) \implies (1): Suppose that \( \mathcal{K} \subseteq \mathcal{K}(X) \) is countable-filtered, \( U \in \mathcal{O}(X) \), and \( \bigcap \mathcal{K} \subseteq U \). Let \( \tilde{\mathcal{K}} = \{ \uparrow_{\mathcal{K}(X)} K : K \in \mathcal{K} \} \). Then \( \tilde{\mathcal{K}} \subseteq \mathcal{K}(P_S(X)) \) is countable-filtered and \( \bigcap \tilde{\mathcal{K}} \subseteq \mathcal{K}(X) \). By the \( \omega^* \)-well-filteredness of \( P_S(X) \), there is a \( \mathcal{K} \in \mathcal{K} \) such that \( \uparrow_{\mathcal{K}(X)} \mathcal{K} \subseteq \mathcal{O}(U) \), and whence \( \mathcal{K} \subseteq U \), proving that \( X \) is \( \omega^* \)-well-filtered. \( \square \)

**Definition 5.13.** A \( T_0 \) space \( X \) is called a countably-directed closure space, \( \omega^*-\text{DC} \) space for short, if \( \text{Irr}_c(X) = D_{\omega^*}(X) \), that is, for each \( A \in \text{Irr}_c(X) \), there exists a countably-directed subset of \( X \) such that \( A = \overline{D} \).

By Remark 2.6 Proposition 5.7 and Proposition 5.11 we get the following result.

**Proposition 5.14.** For any \( T_0 \) space \( X \), the following conditions are equivalent:

1. \( X \) is sober.
2. \( X \) is an \( \omega^*-\text{DC} \) and \( \omega^* \)-well-filtered space.
3. \( X \) is an \( \omega^*-\text{DC} \) and \( \omega^*-d \)-space.

**6. First-countability of sobrifications and \( \omega \)-Rudin spaces**

In this section, we prove that if the sobrification of a \( T_0 \) space \( X \) is first-countable, then \( X \) is a \( \omega \)-Rudin space. Hence every \( \omega \)-well-filtered space having a first-countable sobrification is sober.

We first prove two useful lemmas.

**Lemma 6.1.** Let \( X \) be a \( T_0 \) space and \( A \in \mathcal{C}(X) \). For \( \{ U_n : n \in \mathbb{N} \} \subseteq \mathcal{O}(X) \) with \( U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq U_{n+1} \supseteq \cdots \), if \( A \in m(\{ U_n : n \in \mathbb{N} \} \) and \( x_n \in U_n \cap A \) for each \( n \in \mathbb{N} \), then every subset of \( \{ x_n : n \in \mathbb{N} \} \) is compact.

**Proof.** Suppose \( E \subseteq \{ x_n : n \in \mathbb{N} \} \) and \( \{ V_i : i \in I \} \) is an open cover of \( E \), that is, \( E \subseteq \bigcup_{i \in I} V_i \).

Case 1. \( E \cap (X \setminus V_j) \) is finite for some \( j \in I \).

Then there is \( I_j \in \lambda(\omega) \) such that \( E \cap (X \setminus V_j) \subseteq \bigcup_{i \in I_j} V_i \), and hence \( E \subseteq V_j \cup \bigcup_{i \in I_j} V_i \).

Case 2. \( E \cap (X \setminus V_j) \) is infinite for all \( j \in I \).

For each \( i \in I \), since \( U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq U_{n+1} \supseteq \cdots \) and \( x_n \in U_n \cap A \) for each \( n \in \mathbb{N} \), we have that \( U_n \cap A \cap (X \setminus V_i) \neq \emptyset \) for all \( n \in \mathbb{N} \), and hence \( A \cap (X \setminus V_i) = A \) by the minimality of \( A \). It follows that \( A \subseteq \bigcap_{i \in I} (X \setminus V_i) = X \setminus \bigcup_{i \in I} V_i \). Therefore, \( E \subseteq A \cup \bigcup_{i \in I} V_i = \emptyset \).

By Case 1 and Case 2, \( E \) is compact. \( \square \)

**Lemma 6.2.** Let \( X \) be a \( T_0 \) space and \( A \in \text{Irr}_c(X) \). For any open neighborhood base \( \{ \mathcal{O} U_i : i \in I \} \) of \( A \) in \( X^* \), \( A \in m(\{ U_i : i \in I \}) \).

**Proof.** Clearly, \( A \in M(\{ U_i : i \in I \}) \). Suppose \( B \in \mathcal{C}(X) \) and \( B \subseteq A \). If \( B \neq A \), then \( A \cap (X \setminus B) \neq \emptyset \), and hence \( A \in \mathcal{O}(X \setminus B) \). Since \( \{ \mathcal{O} U_i : i \in I \} \) is an open neighborhood base at \( A \) in \( X^* \), there is \( j \in I \) such that \( \mathcal{O} U_j \subseteq \mathcal{O}(X \setminus B) \) or, equivalently, \( U_j \subseteq X \setminus B \). Therefore, \( U_j \cap B = \emptyset \). So \( B \notin M(\{ U_i : i \in I \}) \). Thus \( A \in m(\{ U_i : i \in I \}) \). \( \square \)

**Proposition 6.3.** For a \( T_0 \) space \( X \), the following two conditions are equivalent:

1. \( X \) is second-countable.
2. \( X^* \) is second-countable.

**Proof.** For any \( \{ U_i : i \in I \} \subseteq \mathcal{O}(X) \), it is easy to verify that \( \{ U_i : i \in I \} \) is a base of \( X \) iff \( \{ \mathcal{O} U_i : i \in I \} \) is a base of \( X^* \).

Since the first-countability is a hereditary property, by Remark 2.5 we have the following result. 


Proposition 6.4. For a $T_0$ space $X$, if $X^*$ is first-countable, then $X$ is first-countable.

The converse of Proposition 6.4 does not hold in general, as shown in Example 6.13 below.

Proposition 6.5. If a $T_0$ space $X$ is first-countable and $|X| \leq \omega$, then $X^*$ is first-countable.

Proof. Let $X = \{x_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, since $X$ is first-countable, there is a countable base $\{U_m(x_n) : m \in \mathbb{N}\}$ at $x_n$. For any $A \in \mathcal{I}(X)$, it straightforward to verify that $\{\bigcap U_m(x_n) : (m, n) \in \mathbb{N} \times \mathbb{N} \text{ and } A \cap U_m(x_n) \neq \emptyset\}$ is a countable base at $A$ in $X^*$. Thus $X^*$ is first-countable. \hfill \qed

The following example shows that the Scott topology on a countable complete lattice may not be first-countable.

Example 6.6. (26) Let $L = \{\bot\} \cup (\mathbb{N} \times \mathbb{N}) \cup \{\top\}$ and define a partial order $\leq$ on $L$ as follows:

(i) $\forall(n, m) \in \mathbb{N} \times \mathbb{N}$, $n \leq (n, m) \leq \top$;

(ii) $(n_1, m_1), (n_2, m_2) \in \mathbb{N} \times \mathbb{N}$, $(n_1, m_1) \leq (n_2, m_2)$ iff $n_1 = n_2$ and $m_1 \leq m_2$.

Then $(L, \sigma(L))$ does not have any countable base at $\top$. Assume, on the contrary, there exists a countable base $\{U_n : n \in \mathbb{N}\}$ at $\top$. Then for each $n \in \mathbb{N}$, as

\[\bigvee\{(n) \times \mathbb{N}\} = \top \in U_n,\]

there exists $m_n \in \mathbb{N}$ such that $(n, m_n) \in U_n$. Let $U = \bigcup_{n \in \mathbb{N}} (n, m_n + 1)$. Then $U \in \sigma(L)$. But for each $n \in \mathbb{N}$, $(n, m_n) \in U_n \setminus U$, which contradicts that $\{U_n : n \in \mathbb{N}\}$ is a base at $\top$. Therefore, $(L, \sigma(L))$ is not first-countable. One can easily check that $(L, \sigma(L))$ is sober.

Theorem 6.7. For a $T_0$ space $X$, if $X^*$ is first-countable, then $X$ is an $\omega$-Rudin space.

Proof. Let $A \in \mathcal{I}(r(X))$. By the first-countability of $X^*$, there is an open neighborhood base $\{\diamond U_n : n \in \mathbb{N}\}$ of $A$ such that

\[\diamond U_1 \supseteq \diamond U_2 \supseteq \ldots \supseteq \diamond U_n \supseteq \ldots,\]

or equivalently, $U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots$. By Lemma 6.12, $A \in M(\{U_n : n \in \mathbb{N}\})$. For each $n \in \mathbb{N}$, choose an $x_n \in U_n \cap A$, and let $K_n = \uparrow\{x_n : m \geq n\}$. Then $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots$, and $\{K_n : n \in \mathbb{N}\} \subseteq \mathcal{K}(X)$ by Lemma 6.13. Clearly, $A \in M(\{K_n : n \in \mathbb{N}\})$. For any $B \in \mathcal{C}(X)$, if $B$ is a proper subset of $A$, that is, $A \cap (X \setminus B) = A \setminus B \neq \emptyset$, then $A \in \diamond (X \setminus B) \in \mathcal{O}(X^*) = \mathcal{O}(\mathcal{I}(r(X)))$. Therefore, $\diamond U_n \subseteq \diamond (X \setminus B)$ for some $m \in \mathbb{N}$, and hence $U_n \subseteq X \setminus B$ or, equivalently, $U_n \setminus B = \emptyset$. Thus $B \notin M(\{K_n : n \in \mathbb{N}\})$, proving $A \in m(\{K_n : n \in \mathbb{N}\})$. So $X$ is an $\omega$-Rudin space. \hfill \qed

Corollary 6.8. Every second-countable $T_0$ space is an $\omega$-Rudin space.

Corollary 6.9. Every countable first-countable $T_0$ space is an $\omega$-Rudin space.

Theorem 6.10. Every $\omega$-well-filtered space with a first-countable sobrification is sober.

Proof. For $A \in \mathcal{I}(r(X))$, by Theorem 6.7 and its proof (or Proposition 4.10), there is a decreasing sequence $\{K_n : n \in \mathbb{N}\} \subseteq \mathcal{K}(X)$ such that $A \in m(\{K_n : n \in \mathbb{N}\})$. Since $X$ is $\omega$-well-filtered, $\bigcap_{n \in \mathbb{N}} K_n \subseteq X \setminus A$, that is, $\bigcap_{n \in \mathbb{N}} K_n \cap A \neq \emptyset$. Choose $x \in \bigcap_{n \in \mathbb{N}} K_n \cap A$. Then $\{x\} \in M(\{K_n : n \in \mathbb{N}\})$ and $\{x\} \subseteq A$. By the minimality of $A$, we have $A = \{x\}$. Thus $X$ is sober. \hfill \qed

Corollary 6.11. Every second-countable $\omega$-well-filtered space is sober.

Corollary 6.12. Every countable first-countable $\omega$-well-filtered space is sober.

In Theorem 6.7 and Theorem 6.10, the first-countability of $X^*$ can not be weakened to that of $X$ as shown in the following example (see also Example 7.3 in Section 7).
Lemma 7.1. \( \Sigma \) \( \omega \) is a continuous mapping. Then for any sup of a countable family of countable ordinal numbers is still a countable ordinal number, \( \Sigma \) \( [0, \omega_1] \) has no countable base at the point \( \omega_1 \).

(c) \( \Sigma P \) is first-countable.

(d) \( (\Sigma P)^* \) is not first-countable. In fact, it is easy to verify that \( (\Sigma P)^* \) is homeomorphic to \( \Sigma [0, \omega_1] \). Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, \( \Sigma [0, \omega_1] \) has no countable base at the point \( \omega_1 \).

(e) \( P \) is an \( \omega \)-dcpo but not a dcpo. So \( \Sigma P \) is an \( \omega \)-d-space but not a d-space, and hence not a sober space.

(f) \( K(\Sigma P) = \{ \uparrow x : x \in P \} \). For \( K \in K(\Sigma P) \), we have inf \( K \in K \), and hence \( K = \uparrow \inf K \).

(g) \( \Sigma P \) is a Rudin space. One can easily check that \( \text{Irr}_c(\Sigma P) = \{ \downarrow x : x \in P \} \cup \{ P \} \). Clearly, \( \downarrow x \) is a Rudin set for each \( x \in P \). Now we show that \( P \) is a Rudin set. First, \( \{ \uparrow s : s \in P \} \) is filtered. Second, \( P \in M(\{ \uparrow s : s \in P \}) \). For a closed subset \( B \) of \( \Sigma P \), if \( B \neq P \), then \( B = \uparrow t \) for some \( t \in P \), and hence \( \uparrow (t + 1) \cap \uparrow t = \emptyset \). Thus \( B \notin M(\{ \uparrow s : s \in P \}) \), proving that \( P \) is a Rudin set.

(h) \( \Sigma P \) is not an \( \omega \)-Rudin space. We prove that the irreducible closed set \( P \) is not an \( \omega \)-Rudin set. For any countable filtered family \( \{ \uparrow \alpha_n : n \in \mathbb{N} \} \subseteq K(\Sigma P) \), let \( \beta = \sup \{ \alpha_n : n \in \mathbb{N} \} \). Then \( \beta \) is still a countable ordinal number. Clearly, \( \downarrow \beta \in M(\{ \uparrow \alpha_n : n \in \mathbb{N} \} \) \( \notin \downarrow \beta \). Therefore, \( P \notin \mathcal{g}(\{ \uparrow \alpha_n : n \in \mathbb{N} \}) \). Thus \( P \) is not an \( \omega \)-Rudin set, and hence \( \Sigma P \) is not an \( \omega \)-Rudin space.

(i) \( \Sigma P \) is \( \omega \)-well-filtered. If \( \{ \uparrow x_n : n \in \mathbb{N} \} \subseteq K(\Sigma P) \) is countable filtered family and \( U \in \sigma(P) \) with \( \bigcap_{n \in \mathbb{N}} \uparrow x_n \subseteq U \), then \( \{ x_n : i \in \mathbb{N} \} \) is a countable subset of \( P = [0, \omega_1] \). Since sup of a countable family of countable ordinal numbers is still a countable ordinal number, we have \( \beta = \sup \{ n : n \in \mathbb{N} \} \), and hence \( \uparrow \beta = \bigcap_{n \in \mathbb{N}} \uparrow x_n \subseteq U \). Therefore, \( \beta \in U \), and consequently, \( x_n \in U \) for some \( n \in \mathbb{N} \) or, equivalently, \( \uparrow x_n \subseteq U \), proving that \( \Sigma P \) is \( \omega \)-well-filtered.

7. First-countability and well-filtered determined spaces

In this section, we show that any first-countable \( T_0 \) space is well-filtered determined. In [24] it was shown that in a first-countable \( \omega \)-well-filtered \( T_0 \) space \( X \), all irreducible closed subsets of \( X \) are directed (see [24] Theorem 4.1]). In the following we will strengthen this result by proving that in a first-countable \( \omega \)-well-filtered space \( X \), every irreducible closed subset of \( X \) is countably-directed.

Lemma 7.1. Suppose that \( X \) is a first-countable \( T_0 \) space, \( Y \) is an \( \omega \)-well-filtered space and \( f : X \to Y \) is a continuous mapping. Then for any \( A \in \text{Irr}(X) \) and \( \{ \alpha_n : n \in \mathbb{N} \} \subseteq X \), \( \bigcap_{n \in \mathbb{N}} \uparrow f(\alpha_n) \cap \overline{f(A)} \neq \emptyset \).

Proof. For each \( x \in X \), since \( X \) is first-countable, there is an open neighborhood base \( \{ U_n(x) : n \in \mathbb{N} \} \) of \( x \) such that

\[
U_1(x) \supseteq U_2(x) \supseteq \ldots \supseteq U_k(x) \supseteq \ldots,
\]

that is, \( \{ U_n(x) : n \in \mathbb{N} \} \) is a decreasing sequence of open subsets.

For each \( (n, m) \in \mathbb{N} \times \mathbb{N} \), since \( \alpha_n \in \overline{A} \) and \( A \in \text{Irr}(X) \), we have \( A \cap \bigcap_{i=1}^{m} U_i(\alpha_k) \cap A \neq \emptyset \) for all \( \{(l, k) : (l, k) \in \mathbb{N} \times \mathbb{N} : 1 \leq i \leq m \} \).

Choose \( c_1 \in U_1(\alpha_1) \cap \overline{A} \). Now suppose we already have a set \( \{ c_1, c_2, \ldots, c_{n-1} \} \) such that for each \( 2 \leq i \leq n - 1 \),

\[
c_i \in \bigcap_{j=1}^{i-1} U_i(c_j) \cap \bigcap_{j=1}^{i} U_i(a_j) \cap A.
\]

Note that above condition implies that for any positive integer \( 1 \leq k \leq n - 1 \),

\[
\bigcap_{j=1}^{k} U_k(c_j) \cap \bigcap_{j=1}^{k} U_k(a_j) \cap A \neq \emptyset,
\]
Thus \( \bigcap_{j=1}^{n-1} U_n(e_j) \cap \bigcap_{j=1}^{n} U_n(a_j) \cap A \neq \emptyset \).

So we can choose \( c_n \in \bigcap_{j=1}^{n-1} U_n(e_j) \cap \bigcap_{j=1}^{n} U_n(a_j) \cap A \). By \( A \in \text{Irr}(X) \) again, we have

\[
\bigcap_{j=1}^{n} U_n(e_j) \cap \bigcap_{j=1}^{n} U_n(a_j) \cap A \neq \emptyset.
\]

By induction, we can obtain a set \( \{c_n : n \in \mathbb{N}\} \) for each \( n \in \mathbb{N} \).

**Claim 1:** \( \forall n \in \mathbb{N}, K_n \in K(X) \).

Suppose \( \{V_i : i \in I\} \) is an open cover of \( K_n \), i.e., \( K_n \subseteq \bigcup_{i \in I} V_i \). Then there is \( i_0 \in I \) such that \( c_{n} \in V_{i_0} \), and thus there is \( m \geq n \) such that \( c_{n} \in U_m(c_{n}) \subseteq V_{i_0} \). It follows that \( c_{k} \in U_m(c_{k}) \subseteq V_{i_0} \) for all \( k \geq m \).

Thus \( \{c_k : k \geq m\} \subseteq V_{i_0} \). For each \( c_k \), where \( n \leq k < m \), choose a \( V_{i_k} \) such that \( c_k \in V_{i_k} \). Then the finite family \( \{V_{i_k} : n \leq k < m\} \cup \{V_{i_0}\} \) covers \( K_n \). So \( K_n \) is compact.

**Claim 2:** \( \{\uparrow f(K_n) : n \in \mathbb{N}\} \subseteq \text{K}(Y) \) and \( \uparrow f(K_1) \supseteq \uparrow f(K_2) \supseteq \uparrow f(K_3) \supseteq \ldots \supseteq \uparrow f(K_{n+1}) \supseteq \ldots \).

For each \( n \in \mathbb{N} \), since \( K_m \in \mathbb{K}(X) \) and \( f \) is continuous, we have \( \uparrow f(K_m) \in \mathbb{K}(Y) \). Clearly, \( K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq K_{n+1} \supseteq \ldots \) and hence \( \uparrow f(K_1) \supseteq \uparrow f(K_2) \supseteq \ldots \supseteq \uparrow f(K_{n+1}) \).

**Claim 3:** \( \bigcap_{i \in \mathbb{N}} \uparrow f(K_n) = \bigcap_{i \in \mathbb{N}} \uparrow f(c_n) \).

Clearly, \( \bigcap_{n \in \mathbb{N}} \uparrow f(c_n) \subseteq \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \). Now we show \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \subseteq \bigcup_{n \in \mathbb{N}} \uparrow f(c_n) \) for all \( m \in \mathbb{N} \). Suppose \( V \in \mathbb{O}(Y) \) with \( f(c_m) \in V \). Then \( c_m \in f^{-1}(V) \subseteq \mathbb{O}(X) \), and whence \( U_n(m)(c_m) \subseteq f^{-1}(V) \) for some \( n(m) \in \mathbb{N} \). For any \( l \geq \max \{m, n(m)\} + 1 \), we have \( K_l \subseteq U_l(c_m) \subseteq U_n(m)(c_m) \subseteq f^{-1}(V) \), and consequently, \( \uparrow f(K_l) \subseteq V \).

It follows that \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n \in \mathbb{N}} \uparrow f(c_n) \).

**Claim 4:** \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n \in \mathbb{N}} \uparrow f(a_n) \).

For \( m \in \mathbb{N} \) and \( W \in \mathbb{O}(Y) \) with \( f(a_m) \in W \). Then \( a_m \in f^{-1}(W) \subseteq \mathbb{O}(X) \), and whence \( U_k(m)(a_m) \subseteq f^{-1}(W) \) for some \( k(m) \in \mathbb{N} \). For any \( l \geq \max \{m, k(m)\} \), we have \( K_l \subseteq U_l(a_m) \subseteq U_k(m)(a_m) \subseteq f^{-1}(W) \), and consequently, \( \uparrow f(K_l) \subseteq W \). It follows that \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n \in \mathbb{N}} \uparrow f(a_n) \).

Thus \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \subseteq \bigcap_{n \in \mathbb{N}} \uparrow f(a_n) \).

**Claim 5:** \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \in \text{K}(Y) \) and \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset \).

By Claim 2 and the \( \omega \)-well-filteredness of \( Y \), \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \in \text{K}(Y) \). Now we show \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset \). Assume, on the contrary, that \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} = \emptyset \) or, equivalently, \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \subseteq Y \setminus \overline{f(A)} \).

Then by the \( \omega \)-well-filteredness of \( Y \) and Claim 2, \( \uparrow f(K_n) \subseteq Y \setminus \overline{f(A)} \), which is in contradiction with \( \{c_m : m \geq n\} \subseteq A \cap K_n \). Therefore, \( \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \cap \overline{f(A)} \neq \emptyset \).

**Corollary 7.2:** In a first-countable \( \omega \)-well-filtered space \( X \), every irreducible closed subset of \( X \) is countably-directed. Therefore, \( X \) is a \( \omega^*\)-DC space.

By Remark 2.8 Proposition 5.7 and Corollary 7.2 we get the following result.

**Theorem 7.3:** For a first-countable \( T_0 \) space \( X \), the following conditions are equivalent:

1. \( X \) is a sober space.
2. \( X \) is a \( \omega \)-well-filtered space.
3. \( X \) is an \( \omega \)-well-filtered \( d \)-space.
4. \( X \) is an \( \omega \)-well-filtered \( \omega^*\)-\( d \)-space.

A first-countable \( d \)-space may not be sober as shown in the following example.

**Example 7.4:** Let \( X \) be a countably infinite set and \( X_{cof} \) the space equipped with the \( co\)-finite topology (the empty set and the complements of finite subsets of \( X \) are open). Then

(a) \( C(X_{cof}) = \{\emptyset, X\} \cup X^{(\leq \omega)} \), \( X_{cof} \) is \( T_1 \) and hence a \( d \)-space.

(b)
(b) \( K(X_{cof}) = 2^X \setminus \{\emptyset\} \).
(c) \( X_{cof} \) is first-countable.
(d) \( X_{cof} \) is locally compact and hence a Rudin space by Theorem 3.9
(e) \( X_{cof} \) is non-sober. \( K_X = \{X \setminus F : F \in X^{(\omega)}\} \subseteq K(X_{cof}) \) is countable filtered and \( \bigcup K_X = X \setminus X = \emptyset \), but \( X \setminus F \neq \emptyset \) for all \( F \in X^{(\omega)} \). Thus \( X_{cof} \) is not \( \omega \)-well-filtered, and consequently, \( X_{cof} \) is not sober by Theorem 7.3.

The following example shows that a first-countable \( \omega \)-well-filtered space need not to be sober.

**Example 7.5.** Let \( L \) be the complete chain \([0, \omega_1]\). Then

(a) \( \Sigma_n L \) is a first-countable.
(b) \( \sigma(P) \neq \sigma(L) \). Since sups of all countable families of countable ordinal numbers are still countable ordinal numbers, we have that \( \{\omega_1\} \in \sigma(L) \) but \( \omega_1 \notin \sigma(L) \) (note that \( \omega_1 = \sup \{0, \omega_1\} \)).
(c) \( \sigma(L) \neq \sigma^*(L) \). It is easy to check that \([\omega, \omega_1] \in \sigma^*(L) \) but \([\omega, \omega_1] \notin \sigma(L) \) (note that \( \omega = \sup N \)).
(d) \( K(\Sigma_n L) = \{\alpha : \alpha \in [0, \omega_1]\} \). For \( K \in K \), we have \( \inf K \in K \), and hence \( K = \{inf K\} \).
(e) \( \Sigma_n L \) is not an \( \omega \)-Rudin space. It is easy to check that \( [0, \omega_1] \in \mathsf{Irr}(\Sigma_n L) \) (note that \( \{\omega_1\} \in \sigma(L) \)). If \( [0, \omega_1] \in \mathsf{RD}(\Sigma_n L) \), then by (d), there is a countable subset \( \{\alpha_n : n \in N\} \subseteq [0, \omega_1] \) such that \( [0, \omega_1] \in M(\{\alpha_n : n \in N\}) \). Let \( \beta = \sup \{\alpha_n : n \in N\} \). Then \( \beta \in [0, \omega_1] \), and hence \( \downarrow \beta \in C(\Sigma_n L) \) and \( \downarrow \beta \in M(\{\alpha_n : n \in N\}) \), which is in contradiction with \( [0, \omega_1] \in M(\{\alpha_n : n \in N\}) \).
(f) \( \Sigma_n L \) is \( \omega \)-well-filtered. Suppose that \( \{\alpha_n : n \in N\} \subseteq K(\Sigma_n L) \) is countable filtered and \( U \in \sigma(L) \) with \( \bigcap_{n \in N} \alpha_n \subseteq U \). Let \( \alpha = \sup \{\alpha_n : n \in N\} \). Then \( \alpha \in [0, \omega_1] \) is a countable directed subset of \( L \) and \( \alpha \in U \) since \( \alpha = \bigcap_{n \in N} \alpha_n \subseteq U \). It follows that \( \alpha \in U \) or, equivalently, \( \alpha \in U \) for some \( n \in N \).

Thus \( \Sigma_n L \) is \( \omega \)-well-filtered, and hence an \( \omega \)-d-space.

Since \( \Sigma_n L \) is not well-filtered, it is non-sober. So in Theorem 7.3 condition (4) (and so condition (3)) cannot be weakened to the condition that \( X \) is only an \( \omega \)-well-filtered space.

**Definition 7.6.** Let \( X \) be a first-countable \( T_0 \) space, \( A \in \mathsf{Irr}(X) \) and \( \{a_n : n \in N\} \). The countable family \( \{K_n : n \in N\} \subseteq K(X) \) obtained in the proof of Lemma 7.1 is called a decreasing sequence of compact saturated subsets related to \( \{a_n : n \in N\} \). 

**Theorem 7.7.** Suppose that \( X \) is a first-countable \( T_0 \) space, \( Y \) is an \( \omega \)-well-filtered space and \( f : X \rightarrow Y \) is a continuous mapping. Then for any \( A \in \mathsf{Irr}(X) \), \( f(A) \in \mathsf{RD}(Y) \).

**Proof.** Let \( K_A = \{\bigcap_{n \in N} \downarrow f(K_n) : K_n : n \in N\} \) is a decreasing sequence of compact saturated subsets related to a countable set \( \{a_n : n \in N\} \subseteq A \). Then by the proof of Lemma 7.1 we have

1. \( K_A \neq \emptyset \) and \( K_A \subseteq K(Y) \).
2. \( f(A) \in K_A \) for all \( A \in A \).
3. \( f(A) \) is filtered.

Suppose that \( \{K_n : n \in N\} \) and \( \{G_n : n \in N\} \) are decreasing sequences of compact saturated subsets related to countable sets \( \{a_n : n \in N\} \subseteq A \) and \( \{b_n : n \in N\} \subseteq A \), respectively. By the proof of Lemma 7.1 for each \( n \in N \), \( K_n = \downarrow \{c_m : m \geq n\} \) and \( G_n = \downarrow \{d_m : m \geq n\} \), where \( \{c_m : n \in N\} \subseteq A \) and \( \{d_n : n \in N\} \subseteq A \) are obtained by the choice procedures (by induction) in the proof of Lemma 7.1 related to
\{a_n : n \in \mathbb{N}\} \subseteq A and \{b_n : n \in \mathbb{N}\} \subseteq A, respectively. Consider \{s_n : n \in \mathbb{N}\} = \{c_1, d_1, c_2, d_2, \ldots, c_n, d_n, \ldots\} \subseteq A, that is,

\[
s_n = \begin{cases} 
c_k & n = 2k + 1 
d_k & n = 2k.
\end{cases}
\]

Then by the proof of Lemma 7.1 we can inductively choose a countable set \{t_n : n \in \mathbb{N}\} such that

\[
t_n \in \bigcap_{j=1}^{n-1} U_n(t_j) \cap \bigcup_{j=1}^{n} U_n(s_j) \cap A \text{ for all } n \in \mathbb{N}.
\]

For each \(n \in \mathbb{N}\), let \(H_n = \uparrow \{t_m : m \geq n\}\). Then by Claim 1 and Claim 2 in the proof of Lemma 7.1, \(\uparrow f(H_n) \in K_A\). By Claim 3 and Claim 4 in the proof of Lemma 7.1, we have \(\bigcap_{n \in \mathbb{N}} \uparrow f(H_n) = \bigcap_{n \in \mathbb{N}} \uparrow f(t_n) \subseteq \bigcap_{n \in \mathbb{N}} \uparrow f(s_n) = \bigcap_{n \in \mathbb{N}} \uparrow f(c_n) \cap \bigcap_{n \in \mathbb{N}} \uparrow f(d_n) = \bigcap_{n \in \mathbb{N}} \uparrow f(K_n) \cap \bigcap_{n \in \mathbb{N}} \uparrow f(G_n)\). Thus \(K_A\) is filtered.

4. \(\overline{f(A)} \in M(K_A)\).

By Claim 5 in the proof of Lemma 7.1, \(f(A) \in M(K_A)\).

5. \(\overline{f(A)} \in m(K_A)\).

If \(B\) is a closed subset with \(B \subseteq M(K_A)\), then for each \(a \in A\), by 2°, we have \(\uparrow f(a) \cap B \neq \emptyset\), and hence \(f(a) \in B\). It follows that \(f(A) \subseteq B\). Thus \(\overline{f(A)} \subseteq m(K_A)\).

By 1°, 3° and 5°, \(\overline{f(A)} \in RD(Y)\).

**Theorem 7.8.** Every first-countable \(T_0\) space is a well-filtered determined space.

**Proof.** Let \(X\) be a first-countable \(T_0\) space and \(A \subseteq \text{Irr}_c(X)\). We need to show \(A \in \text{WD}(X)\). Suppose that \(f : X \rightarrow Y\) is a continuous mapping from \(X\) to a well-filtered space \(Y\). By Theorem 7.4, \(\overline{f(A)} \subseteq \text{RD}(Y)\), and whence by the well-filteredness of \(Y\) and Proposition 5.6, there is a (unique) \(y_A \in Y\) such that \(\overline{f(A)} = \{y_A\}\). Thus \(A \in \text{WD}(X)\).

In [14], Example 4.15, a well-filtered space \(X\) but not a Rudin space was given. It is straightforward to check that \(X\) is not first-countable.

Finally, based on Theorem 6.7 and Theorem 7.8, we pose the following two natural problems.

**Problem 7.9.** Is every first-countable \(T_0\) space a Rudin space?

**Problem 7.10.** Is every first-countable \(T_0\) space \(\omega\)-well-filtered determined?

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