Analysis of the (CR) equation in higher dimensions

T. Buckmaster, P. Germain, Z. Hani, J. Shatah

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Abstract

This paper is devoted to the analysis of the continuous resonant (CR) equation, in dimensions greater than 2. This equation arises as the large box (or high frequency) limit of the nonlinear Schrodinger equation on the torus, and was derived in a companion paper by the same authors. We initiate the investigation of the structure of (CR), its local well-posedness, and the existence of stationary waves.

1 Introduction

The purpose of this paper is to investigate various properties of the continuous resonant (CR) equation, derived in the companion paper [1]. We start with a brief description of this equation.

1.1 Presentation of the equation

The (CR) equation, derived in [1] (following [3]) reads

\[
\begin{aligned}
&i\partial_t g(t, \xi) = T(g, g, g)(t, \xi) & (t, \xi) \in \mathbb{R} \times \mathbb{R}^d \\
&T(g, g, g)(\xi) = \iiint_{(\mathbb{R}^d)^3} g(\xi_1)g(\xi_2)g(\xi_3)\delta_{\mathbb{R}^d}(\xi_1 - \xi_2 + \xi_3 - \xi)\delta_{\mathbb{R}^d}(\Omega) \, d\xi_1 d\xi_2 d\xi_3, \\
\end{aligned}
\]  

(CR)

where

\[
\Omega = |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi|^2.
\]

This equation describes the effective dynamics of the nonlinear Schrödinger equation

\[
i\partial_t u - \Delta u = |u|^2 u,
\]  

(NLS)

set on a box of size \(L\) (with periodic boundary conditions) in the limit of large \(L\) and small initial data. Equivalently, it also gives the effective dynamics of the same equation posed on the unit torus \(\mathbb{T}^d\) in the high-frequency limit. Thus, (CR) is one of the few simpler models which could help understand of the dynamics of (NLS) on compact domains.
In fact, the companion paper [1] derives such continuous resonant effective models for (NLS) with any analytic power nonlinearity (where \(|u|^2u\) is replaced by \(|u|^{2p}u\) for any positive integer \(p\)). We shall restrict our analysis in this paper to the cubic case \(p = 1\), mainly for concreteness purposes. We should note that this equation was derived for \(d = 2\) in [3], and later analyzed in [6, 7]. The pair \((d, p) = (2, 1)\) (where \(p\) is the power of the nonlinearity \(|u|^{2p}u\)) corresponds to the mass-critical case for (NLS). In this case, which also holds for \((d, p) = (1, 2)\), the equation (CR) has very special and surprising properties, like being invariant under the Fourier transform and “commuting” with the quantum harmonic oscillator [3, 6]. Moreover, the same equation turns out to give the effective dynamics for various physical systems such as NLS with harmonic trapping [8], the dispersion-managed NLS equation, or lowest-Landau-level dynamics [5]. It is an interesting question to understand which of the rich properties that hold for (CR) in \(d = 2\) generalize to higher dimensions. This is one of the purposes of this paper.

1.2 Properties

We shall see that (CR) is Hamiltonian with energy functional given by

\[
\mathcal{H}(g) = \frac{(2\pi)^{d-1}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |e^{is\Delta} \hat{g}|^4 \, dx \, ds.
\]

Moreover, the equation satisfies a handful of symmetries, some are inherited from (NLS), and some are new. These translate into conservation laws. In contrast to the 2D case, the kinetic energy

\[
\int |\nabla_\xi g(t, \xi)|^2 \, d\xi,
\]

which actually corresponds to the quantity \(\int x^2 |e^{-it\Delta} u|^2 \, dx\) for (NLS), is not conserved for \(d \geq 3\). In fact, we shall see that it satisfies the following “virial-type” identity

\[
\frac{d}{dt} \|\nabla g\|_{L^2}^2 = 2(2 - d)(2\pi)^{d-1} \int \int s|e^{is\Delta} \hat{g}|^4 \, dx \, ds.
\]

This is proved in Section 2.3.

In terms of well-posedness, this is directly related to the boundedness properties of the operator \(T\) in (CR). In Proposition 5 we study such properties for a bunch of function spaces. This directly translates into local well-posedness in those function spaces. It should be noted that this includes the function spaces which were used in our companion paper [1] to prove the rigorous approximation result between the (CR) dynamics and that of (NLS).

1.2.1 Stationary solutions

It was noticed in [3, 6], equation (CR) enjoys a wealth of stationary solutions when \(d = 2\). This motivated us to study stationary solutions for this equation in higher dimensions. Here,
we construct them by variational methods as maximizers of the Hamiltonian functional with fixed mass and/or weighted $L^2$ norm. We should point out that we only took the first steps in this study and many remaining interesting questions remain open for investigation.

1.3 Notations

The Fourier transform and the Fourier transform of a function on $\mathbb{R}^d$ are given by, respectively,

$$\mathcal{F} f(\xi) = \hat{f}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx$$

$$\mathcal{F}^{-1} f(x) = \check{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \hat{f}(\xi) \, d\xi.$$  

The function spaces $L^{p,s} = L^{p,s} (\mathbb{R}^d)$, $\dot{L}^{p,s} = \dot{L}^{p,s} (\mathbb{R}^d)$, $W^{p,s} = W^{p,s} (\mathbb{R}^d)$, $X^{\ell,N} = X^{\ell,N} (\mathbb{R}^d)$ are given by their norms

$$\| f \|_{L^{p,s}} = \| \langle x \rangle^s f \|_{L^p}$$

$$\| f \|_{\dot{L}^{p,s}} = \| |x|^s f \|_{L^p}$$

$$\| f \|_{W^{p,s}} = \| \langle D \rangle^s f \|_{L^p}$$

$$\| f \|_{X^{\ell,N} (\mathbb{R}^d)} = \sum_{0 \leq |\alpha| \leq N} \| \nabla^\alpha f \|_{L^{\infty,\ell}}.$$  

2 Structure of the equation

2.1 The Hamiltonian

In this subsection, we remain at a formal level, and discuss the structure of (CR). Notice first that it is Hamiltonian: it can be written

$$i\dot{g} = \frac{\partial \mathcal{H}(g)}{\partial g}$$

with

$$\mathcal{H}(g) = \frac{1}{2} \iiint_{(\mathbb{R}^d)^4} g(\xi_1)g(\xi_2)g(\xi_3)g(\xi_4) \delta_{\mathbb{R}^d}(\xi_1 - \xi_2 + \xi_3 - \xi_4) \delta_{\mathbb{R}^d}(\xi_1 - \xi_2 + \xi_3 - \xi_4) \, d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$  

We will also use the polarized version

$$\mathcal{H}(g_1, g_2, g_3, g_4) = \frac{1}{2} \iiint_{(\mathbb{R}^d)^4} g_1(\xi_1)g_2(\xi_2)g_3(\xi_3)g_4(\xi_4) \delta_{\mathbb{R}^d}(\xi_1 - \xi_2 + \xi_3 - \xi_4) \delta_{\mathbb{R}^d}(\xi_1 - \xi_2 + \xi_3 - \xi_4) \, d\xi_1 d\xi_2 d\xi_3 d\xi_4.$$  

First notice that, viewed in physical space, the Hamiltonian is nothing but the $L^4$ Strichartz norm.
Proposition 1. The Hamiltonian and its polarized version can be written

\[ H(g) = \frac{(2\pi)^{d-1}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |e^{i \Delta \hat{g}}|^4 \, dx \, ds \]

\[ H(g_1, g_2, g_3, g_4) = \frac{(2\pi)^{d-1}}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i \Delta \hat{g}_1(x)} e^{i \Delta \hat{g}_2(x)} e^{i \Delta \hat{g}_3(x)} e^{i \Delta \hat{g}_4(x)} \, dx \, ds. \]

As for the trilinear operator \( T \), it reads

\[ T(g_1, g_2, g_3) = (2\pi)^{d-1} \mathcal{F} \int_{\mathbb{R}^d} e^{-i \Delta} \left( e^{i \Delta \hat{g}_1(x)} e^{i \Delta \hat{g}_2(x)} e^{i \Delta \hat{g}_3(x)} \right) \, ds. \]

Proof. It suffices to prove the identity for the polarized Hamiltonian function. Using that
\[ (2\pi)^{-1} \int_{\mathbb{R}} e^{i x \xi} \, d\xi = \delta_{x=0}, \]
we obtain that
\[ \int_{\mathbb{R}} \int_{\mathbb{R}^d} e^{i \Delta \hat{g}_1(x)} e^{i \Delta \hat{g}_2(x)} e^{i \Delta \hat{g}_3(x)} e^{i \Delta \hat{g}_4(x)} dx \, ds = (2\pi)^{-2d} \int_{\mathbb{R}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{4n}} e^{i x (\eta_1 - \eta_2 + \eta_3 - \eta_4)} e^{-i s (\eta_1^2 - \eta_2^2 + \eta_3^2 - \eta_4^2)} g_1(\eta_1) g_2(\eta_2) g_3(\eta_3) g_4(\eta_4) \, d\eta_1 \ldots d\eta_4 \, dx \, ds = (2\pi)^{-d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^{4d}} e^{-i s (\eta_1^2 - \eta_2^2 + \eta_3^2 - \eta_4^2)} g_1(\eta_1) g_2(\eta_2) g_3(\eta_3) g_4(\eta_4) \delta_{\mathbb{R}^d}(\eta_1 - \eta_2 + \eta_3 - \eta_4) \, d\eta_1 \ldots d\eta_4 \, dx \, ds.
\]

2.2 Symmetries and conserved quantities

Proposition 2 (Symmetries). The following symmetries leave the Hamiltonian \( H \) invariant:

(i) Phase rotation \( g \mapsto e^{i \theta} g \) for all \( \theta \in \mathbb{R} \).

(ii) Translation: \( g \mapsto g(\cdot + \xi_0) \) for any \( \xi_0 \in \mathbb{R} \).

(iii) Modulation: \( g \mapsto e^{i \xi : x_0} g \) for any \( x_0 \in \mathbb{R}^d \).

(iv) Quadratic modulation: \( g \mapsto e^{i \tau |\xi|^2} g \) for any \( \tau \in \mathbb{R} \).

(v) Rotation: \( g \mapsto g(O \cdot) \) for any \( O \) in the orthogonal group \( O(d) \).

(vi) Scaling: \( g \mapsto \mu^{|x|/|\xi|} g(\mu \cdot) \) for any \( \mu > 0 \).

The first five symmetries are canonical transformations (except in dimension 2, where the scaling is also canonical) which, by Noether’s theorem, are associated to conserved quantities.

Proposition 3 (Conserved quantities). The following quantities are conserved by the flow of \( (\mathbb{C}_R) \):

\[ \]
(i) Mass: $\int |g(\xi)|^2 d\xi$.
(ii) Position: $\int x |\dot{g}(x)|^2 dx$.
(iii) Momentum: $\int \xi |g(\xi)|^2 d\xi$.
(iv) Kinetic energy: $\int \xi^2 |g(\xi)|^2 d\xi$.
(v) Angular momentum: $\int (\xi_i \partial_{\xi_j} - \xi_j \partial_{\xi_i}) g(\xi) \overline{g(\xi)} d\xi$ for $i \neq j$.

Notice further that the first four symmetries of the Hamiltonian naturally translate into symmetries of the set of solutions of the equation. Furthermore, the following transformation leaves the set of solutions of (CR) invariant:

$$g(t, \xi) \mapsto \lambda g(\lambda^2 \mu^2 t, \mu x),$$
for $\mu, \lambda > 0$.

**Remark 1.** In the 2D cubic case and the 1D quintic case, the (CR) equation enjoys a much bigger group of symmetries and more conserved quantities due the mass critical nature of the nonlinearity. We refer to [3] for details.

### 2.3 Virial formula

The virial formula is a classical tool in the study of the nonlinear Schrödinger equation. It has an analog for the (CR) equation.

**Proposition 4.** If $g$ is a solution of the (CR) equation,

$$\frac{1}{2} \frac{d}{dt} \|\nabla g\|^2_{L^2} = 2(2 - d)(2\pi)^{d-1} \int s|e^{is\Delta} \tilde{g}|^4 dx ds. $$

**Remark 2.** The above theorem shows a sharp distinction between the special dimension $d = 2$ and higher dimensions. Indeed, this conservation was observed in [3] for $d = 2$, but what seems interesting is that the $H^1$ norm is not conserved in higher dimensions.

**Proof.** By Proposition [1], and using that $\mathcal{F}$ is an isometry on $L^2$,

$$\frac{1}{2} \frac{d}{dt} \|\nabla g\|^2 d\xi = (2\pi)^{d-1} \text{Re} \int i\mathcal{F} e^{-is\Delta} \left( e^{is\Delta} \tilde{g} \overline{e^{is\Delta} \tilde{g}} \right) \Delta \phi d\xi ds$$

$$= (2\pi)^{d-1} \text{Im} \int e^{is\Delta} \tilde{g} e^{is\Delta} \tilde{g} e^{is\Delta} \tilde{g} \overline{e^{is\Delta} \tilde{g}} dx ds.$$  

Using the identity

$$e^{is\Delta} x^2 \phi = x^2 e^{is\Delta} \phi + 2isde^{is\Delta} \phi + 4isx \cdot \nabla e^{is\Delta} \phi - 4s^2 \Delta e^{is\Delta} \phi,$$

the above becomes

$$\frac{1}{2} \frac{d}{dt} \int |\nabla g|^2 dx$$

$$= (2\pi)^{d-1} \text{Im} \int e^{is\Delta} \tilde{g} e^{is\Delta} \tilde{g} e^{is\Delta} \tilde{g} \left[ 2isde^{is\Delta} \tilde{g} + 4isx \cdot \nabla e^{is\Delta} \tilde{g} - 4s^2 \Delta e^{is\Delta} \tilde{g} \right] dx ds$$

$$= I + II + III$$

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Through integrations by parts, it is easy to see that

\[ I = -(2\pi)^{d-1} 2d \int s |e^{is\Delta} \tilde{g}|^4 \, dx \, ds \]
\[ II = (2\pi)^{d-1} d \int s |e^{is\Delta} \tilde{g}|^4 \, dx \, ds \]
\[ III = (2\pi)^{d-1} 2 \int s |e^{is\Delta} \tilde{g}|^4 \, dx \, ds, \]

from which the desired result follows.

\[ \square \]

3 Local and global well-posedness

The first step is to establish boundedness properties of the trilinear operator \( T \).

**Proposition 5.** The trilinear operator \( T \) is bounded from \( X \times X \times X \) to \( X \) for the following Banach spaces \( X \):

(i) \( X = \dot{L}^{2,d-2} \)

(ii) \( X = L^{2,s} \) for \( s \geq \frac{d-2}{2} \).

(iii) \( X = L^{\infty,s} \), for \( s > d - 1 \).

(iv) \( X = L^{p,s} \) for \( p \geq 2 \) and \( s > d - 1 - \frac{d}{p} \).

(v) \( X = X^{\sigma,N} \) for any \( \sigma > d - 1 \) and \( N \geq 0 \).

The borderline spaces for well-posedness in the above proposition are \( \dot{L}^{p,s} \), with \( s = d - 1 - \frac{d}{p} \). They share the same scaling, and are also scale-invariant for the cubic NLS \( i\partial_t u - \Delta u = |u|^2 u \) set in \( \mathbb{R}^d \) (when viewed as spaces for \( \hat{u} \)).

**Proof.** The assertions (i) and (ii) follow from applying successively Proposition \( \mathbb{1} \) the Minkowski inequality, the fractional Leibniz rule, Strichartz’ inequality, and Sobolev embedding (cf. Appendix A, \[12\]): indeed,

\[ \| T(f_1, f_2, f_3) \|_{L^{2,s}} \leq (2\pi)^{d-1} \left\| e^{it\Delta} \tilde{f}_1(x)e^{it\Delta} \tilde{f}_2(x)e^{it\Delta} \tilde{f}_3(x) \right\|_{L_t^1L_x^{2s}} \]
\[ \lesssim \sum_{j=1}^3 \left\| e^{it\Delta} \tilde{f}_j \right\|_{L_t^{2(d+2)}W^{s-\frac{d}{d+2}}_{d+2}} \prod_{k \neq j} \left\| e^{it\Delta} \tilde{f}_j \right\|_{L_t^{2(d+2)}L_x^{2(d+2)}} \lesssim \prod_{j=1}^3 \| \tilde{f}_j \|_{H^{s}}, \]

provided \( s \geq \frac{d-2}{2} \).

The assertion (iv) is obtained by interpolating between (ii) and (iii); therefore, it only remains to prove (iii). It is equivalent to proving that, for \( s > d - 1 \),

\[ \sup_{\xi} \iiint_{(\mathbb{R}^d)^3} \frac{\langle \xi \rangle^s}{\langle \xi_1 \rangle^s \langle \xi_2 \rangle^s \langle \xi_3 \rangle^s} \delta_\mathbb{R}^d(\xi_1 - \xi_2 + \xi_3 - \xi) \delta_\mathbb{R}(\Omega) \, d\xi_1 d\xi_2 d\xi_3 < \infty. \]
Observe that the first $\delta$ in the above integrand imposes $\xi_2 = \xi_1 + \xi_3 - \xi$. Switching to the integration variables $y = \frac{\xi_1 + \xi_3 - 2\xi}{2}$ and $z = \frac{\xi_1 - \xi_3}{2}$, the above becomes

$$\sup_{\xi} \iint_{(\mathbb{R}^d)^2} \frac{\langle \xi \rangle^s}{(\xi + y + z)^s(\xi + y - z)^s(\xi + 2y)^s} \delta(y^2 - z^2) \, dy \, dz < \infty.$$ 

Since the $\delta$ function in the above integrand forces $|y| = |z|$, either $\langle \xi \rangle \lesssim \langle \xi + y + z \rangle$ or $\langle \xi \rangle \lesssim \langle \xi + y - z \rangle$. Assuming without loss of generality that the former occurs, it suffices to show that

$$\sup_{\xi} \iint_{(\mathbb{R}^d)^2} \frac{1}{(\xi + y + z)^s(\xi + 2y)^s} \delta(y^2 - z^2) \, dy \, dz < \infty.$$ 

We will now rely on the simple estimate: if $A \in \mathbb{R}^d$, $r > 0$, and $s > d - 1$,

$$\int_{S^{d-1}} \frac{1}{(A + r\omega)^s} \, d\omega \lesssim \langle |A| - r \rangle^{d-1-s} \langle r \rangle^{1-d}$$

(in order to check this estimate, observe first that $\int_{S^{d-1}} \frac{1}{(A + r\omega)^s} \, d\omega \sim \int_0^\pi \frac{(\sin \phi)^{d-2}}{(|A|^2 + r^2 - 2A |\cos \phi + 1|)^{s/2}} \, d\phi$, and then that, close to $\phi = 0$, this becomes $\sim \int_0^1 \frac{1}{(|A| - r)^{2-d} + |A| r \phi^{d-2}} \, d\phi$). Changing the integration variables to $y = r\omega$ and $z = r\phi$, and using the inequality above twice gives the desired result:

$$\sup_{\xi} \iint_{(\mathbb{R}^d)^2} \frac{1}{(\xi + y - z)^s(\xi + 2y)^s} \delta(y^2 - z^2) \, dy \, dz$$

$$= \sup_{\xi} \int_0^\infty \int_{S^{d-1}} \frac{1}{(\xi + r\omega - r\phi)^s(\xi + 2r\omega)^s} r^{2d-3} \, d\omega \, d\phi \, dr$$

$$\lesssim \sup_{\xi} \int_0^\infty \int_{S^{d-1}} \frac{1}{(\xi + 2r\omega)^s} r^{d-2} \, d\omega \, dr$$

$$\lesssim \sup_{\xi} \int_0^\infty \frac{1}{\langle r \rangle \langle |\xi| - 2r \rangle^{s-d+1}} \, dr < \infty.$$ 

Finally, part (v) follows from part (iii) and Leibniz rule.

Local and global regularity properties of (CR) follow easily:

**Corollary 1.** (i) Local well-posedness: For $X$ any of the spaces given in Proposition 3, the Cauchy problem (CR) is locally well-posed in $X$. More precisely, for any $g_0$ in $X$, there exists a time $T > 0$, and a solution in $C^\infty([0,T],X)$, which is unique in $L^\infty([0,T],X)$, and depends continuously on $g_0$ in this topology.

(ii) Global well-posedness for finite mass: if $d = 2$ and $g_0 \in L^2$, the local solution can be prolonged into a global one. More precisely: there exists a unique solution $g \in C^\infty([0,\infty),L^2)$ which, for any $T$, is unique in $L^\infty([0,T],L^2)$, and depends continuously on $g_0$ in this topology.

(iii) Global well-posedness for finite kinetic energy: if $d = 2,3,4$ and $g_0 \in L^{2,1}$, the local solution can be prolonged into a global one.
(iv) Propagation of regularity: assume \( g_0 \in L^{2, \frac{d-2}{2}} \), and let \( g \) be the solution given in (i). If in addition \( g_0 \in L^2 \) for \( \sigma \geq \frac{d-2}{2} \) then \( g \in C^\infty([0, T), L^{2, \sigma}) \).

Proof. The proof of (i) is immediate since (CR) is an ODE in \( X \). Combining this local well-posedness result with conservation laws classically gives global well-posedness, leading to (ii) and (iii). Finally, (iv) is classical. \qed

4 Stationary waves

4.1 Basic properties

We will discuss here the existence of solutions of the type
\[
g(t, \xi) = e^{-i(\mu + \lambda|\xi|^2 + \nu \cdot \xi)t} \psi,
\]
where \( \lambda, \mu \in \mathbb{R} \), and \( \nu \in \mathbb{R}^d \). For \( g \) to solve (CR), it suffices that \( \psi \) solves
\[
(\lambda|\xi|^2 + \mu + \nu \cdot \xi)\psi = T(\psi, \psi, \psi).
\]
Notice that \( g \) defined above oscillates in Fourier space, but it actually travels in physical space, as can be seen by taking its inverse Fourier transform:
\[
\hat{\tilde{g}}(t, x) = e^{-i(\mu - \lambda \Delta)t} \hat{\psi}(x - \nu t).
\]
The conservation of position (part (ii) of Proposition 3) gives a restriction on the relation between \( \nu \) and \( \lambda \). Indeed, using the identity \([x, e^{it\Delta}] = -2it \nabla e^{it\Delta}\), one obtains that
\[
\int x|\hat{\tilde{g}}|^2 dx = \int x|\hat{\psi}|^2 dx + t \left( \nu M(\hat{\psi}) - 2\lambda P(\hat{\psi}) \right)
\]
where \( M(\hat{\psi}) = \int |\hat{\psi}|^2 dx \) and \( P(\hat{\psi}) = i \int \nabla \hat{\psi} \overline{\hat{\psi}} dx \in \mathbb{R} \). Since \( \int x|\tilde{g}|^2 dx \) is conserved, one must have that
\[
\nu = \frac{2P(\hat{\psi})}{M(\hat{\psi})\lambda}.
\]
By invariance of \( T \) under translations, we can define
\[
\phi(\xi) = \psi(\xi - \frac{\nu}{2\lambda})
\]
which solves
\[
(\lambda|\xi|^2 + \mu)\phi = T(\phi, \phi, \phi); \tag{1}
\]
we will from now on focus on this equation.
Lemma 1 (Energy and Pohozaev identities). Assume that $\phi$ solves (1), and that $H(\phi) < \infty$. If furthermore $\phi \in L^{2,1}$, it satisfies the energy identity
\[ \lambda \| \xi \phi \|^2_{L^2} + \mu \| \phi \|^2_{L^2} = H(\phi). \] (2)
If furthermore $\phi \in L^2$ and $\xi \nabla \phi \in L^2$, it satisfies the Pohozaev identity
\[ \lambda \left( \frac{d}{2} - 1 \right) \| \xi \phi \|^2_{L^2} + \mu \frac{d}{2} \| \phi \|^2_{L^2} = \left( \frac{1}{2} + \frac{d}{4} \right) H(\phi). \] (3)

Proof. The energy identity follows immediately from taking the $L^2$ scalar product of (1) with $\phi$. As for the Pohozaev identity, take first the inverse Fourier transform to obtain
\[ (-\lambda \Delta + \mu) \hat{\phi} = (2\pi)^{d-1} \int e^{-is\Delta} \left( e^{is\Delta} \hat{\phi} \overline{e^{is\Delta} \phi} e^{is\Delta} \phi \right) ds. \]
Pairing this identity with $x \cdot \nabla \phi$ and subsequently taking the real part gives three terms, which after integrations by parts read
\[ \Re \lambda \int \Delta \phi x \cdot \nabla \phi dx = \lambda \left( 1 - \frac{d}{2} \right) \int |\nabla \phi|^2 dx \]
\[ \Re \mu \int \phi x \cdot \nabla \phi dx = -\mu \frac{d}{2} \int |\phi|^2 dx \]
\[ \Re (2\pi)^{d-1} \int e^{-is\Delta} \left( e^{is\Delta} \phi \overline{e^{is\Delta} \phi} e^{is\Delta} \phi \right) ds x \cdot \nabla \phi dx = -(2\pi)^{d-1} \left( \frac{1}{2} + \frac{d}{4} \right) \iint |e^{it\Delta} \phi|^4 dx ds. \]
This gives the desired identity! \[ \square \]

Linear combinations of (2) and (3) reveal that, under the hypotheses of Lemma 1
\[ \lambda \| \xi \phi \|^2_{L^2} = \frac{d-2}{4} H(\phi) \]
\[ \mu \| \phi \|^2_{L^2} = \frac{6-d}{4} H(\phi). \]
In particular, still under the hypotheses of Lemma 1 necessary conditions for (1) to admit a solution are
- If $d = 2$, $\lambda = 0$ and $\mu > 0$.
- If $3 \leq d \leq 5$, $\lambda > 0$ and $\mu > 0$.
- If $d = 6$, $\lambda > 0$ and $\mu = 0$.
- If $d \geq 7$, $\lambda > 0$ and $\mu < 0$.

If $2 \leq d \leq 6$, the existence of a solution is ensured by the following theorem, but in less smooth classes of solutions.

Remark 3. We construct solutions to those equations below, but in slightly less smooth classes. It would be interesting to prove regularity of such solutions to bridge the gap.
4.2 The variational problem

Theorem 1. The following variational problems admit nonzero maximizers:

- \[ \sup_{\|g\|_{L^2} = 1} \mathcal{H}(g) \] if \( d = 2 \).
- \[ \sup_{\|g\|_{L^2}^2 + \|\xi g\|_{L^2}^2 = 1} \mathcal{H}(g) \] if \( 3 \leq d \leq 5 \).
- \[ \sup_{\|\xi g\|_{L^2} = 1} \mathcal{H}(g) \] if \( d = 6 \).

Furthermore, maximizing sequences are compact modulo the symmetries of the equation. In dimension 2, maximizers are given by Gaussians.

This variational problem makes sense due to the Strichartz estimates

\[ \mathcal{H}(g) = \|e^{it\Delta}g\|_{L^4_t} \lesssim \|g\|_{L^2} \] if \( d = 2 \)
\[ \mathcal{H}(g) = \|e^{it\Delta}g\|_{L^4_t} \lesssim \|g\|_{L^2,1} \] if \( 3 \leq d \leq 5 \)
\[ \mathcal{H}(g) = \|e^{it\Delta}g\|_{L^4_t} \lesssim \|g\|_{\dot{H}^{-1}} \] if \( d = 6 \).

Therefore, these variational problems belong to the class which arises from Fourier restriction functionals. The cases \( d = 2 \) and \( d = 6 \), as extremizers of the Strichartz [4, 9, 3] and Sobolev-Strichartz inequalities [11] are already known; as we will see, the case \( 3 \leq d \leq 5 \), which is somewhat simpler since it is not critical, will follow from an application of the concentration-compactness technology (as in [10, 2]).

Of great interest is the question of the uniqueness (modulo symmetries) of extremizers, but we can only establish this in a handful of cases, such as the case \( d = 2 \) of the above theorem.

The Euler-Lagrange equations satisfied by the maximizers of these variational problems read

- \( \lambda g = \mathcal{T}(g, g, g) \) (for a Lagrange multiplier \( \lambda \)) if \( d = 2 \).
- \( \lambda [g + |\xi|^2 g] = \mathcal{T}(g, g, g) \) (for a Lagrange multiplier \( \lambda \)) if \( 3 \leq d \leq 5 \).
- \( \lambda |\xi|^2 g = \mathcal{T}(g, g, g) \) (for a Lagrange multiplier \( \lambda \)) if \( d = 6 \).

These three equations should be understood as equalities in \( L^2 \), \( H^{-1} \), and \( \dot{H}^{-1} \) respectively. Since the maximizers are nonzero, testing the above equations against \( g \) reveals that \( \lambda > 0 \); up to scaling it can be taken equal to 1. Therefore, we find nonzero solutions of

- \( g = \mathcal{T}(g, g, g) \) if \( d = 2 \).
- \( g + |\xi|^2 g = \mathcal{T}(g, g, g) \) if \( 3 \leq d \leq 5 \).
- \( |\xi|^2 g = \mathcal{T}(g, g, g) \) if \( d = 6 \).

We now turn to the proof of the theorem.
Proof. The case $d = 2$ was proved in [9, 11], see also [3]; the case $d = 6$ can be found in [11]. There remains $d = 3, 4, 5$, to which we now turn. It is more convenient to take the Fourier transform of the above and consider the variational problem

$$
\sup_{\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 = 1} \| e^{it\Delta} f \|_{L^4_{t,x}}^4.
$$

(4)

Step 1: scaling of the problem. For $A > 0$, let

$$
I(A) = \sup_{\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 = A} \| e^{it\Delta} f \|_{L^4_{t,x}}^4.
$$

It is clear that

$$
I(A) = A^2 I(1).
$$

Step 2: profile expansion. As is well-known, the lack of compactness of this variational problem can be overcome through a profile expansion. The one needed for this particular variational problem is very similar to the one in [2], and it can be proved along the same lines. The statement is as follows: consider $(f_n)$ a bounded sequence in $H^1$. Then there exists a subsequence, also denoted $(f_n)$, a second sequence $(\psi^j)$, and doubly indexed subsequences $(t^j_n)$ and $(x^j_n)$ giving for any $J$ the decomposition

$$
f_n(x) = \sum_{j=1}^J e^{it^j_n \Delta} f(x + x^j_n) + r^J_n.
$$

Furthermore

- The expansion is orthogonal in the Strichartz norm:

$$
\lim_{J \to \infty} \limsup_{n \to \infty} \left[ \| e^{it\Delta} f_n \|_{L^4_{t,x}}^4 - \sum_{j=1}^J \| e^{it\Delta} \psi_j \|_{L^4_{t,x}}^4 \right] = 0.
$$

- The expansion is orthogonal in $L^2$: for any $J$,

$$
\lim_{n \to \infty} \left[ \| f_n \|_{L^2}^2 - \sum_{j=1}^J \| \psi_j \|_{L^2}^2 - \| r^J_n \|_{L^2}^2 \right] = 0.
$$

- The expansion is orthogonal in $\dot{H}^1$: for any $J$,

$$
\lim_{n \to \infty} \left[ \| \nabla f_n \|_{L^2}^2 - \sum_{j=1}^J \| \nabla \psi_j \|_{L^2}^2 - \| \nabla r^J_n \|_{L^2}^2 \right] = 0.
$$

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Step 3: maximizing sequence. Pick \( (f_n) \) a maximizing sequence for the variational problem (4). Then, due to the orthogonality property in the Strichartz norm,

\[
I(1) = \lim_{n \to \infty} \| e^{it\Delta} f_n \|_{L^4_t L^4_x}^4 = \sum_{j=1}^{\infty} \| e^{it\Delta} \psi_j \|_{L^4_t L^4_x}^4.
\]

By the scaling property of the variational problem, the above implies that

\[
I(1) \leq \sum_{j=1}^{\infty} I(\| \psi_j \|_{L^2}^2 + \| \nabla \psi_j \|_{L^2}^2) \leq I(1) \sum_{j=1}^{\infty} [\| \psi_j \|_{L^2}^2 + \| \nabla \psi_j \|_{L^2}^2]^2,
\]

which implies in turn that

\[
1 \leq \sum_{j=1}^{\infty} [\| \psi_j \|_{L^2}^2 + \| \nabla \psi_j \|_{L^2}^2]^2 \quad \text{while} \quad \sum_{j=1}^{\infty} \| \psi_j \|_{L^2}^2 + \| \nabla \psi_j \|_{L^2}^2 \leq 1.
\]

This is only possible if only one of the \( (\psi_j) \) is nonzero, say \( \psi_1 \). In other words, the maximizing sequence is compact (modulo symmetries), and \( \psi_1 \) is the desired maximizer.

4.3 Decay for the Euler-Lagrange problem

**Proposition 6** (Fourier decay of \( g \)). (i) If \( d = 2 \), a solution \( g \in L^2 \) of \( g = T(g,g,g) \)
belongs to \( L^{2,s} \) for all \( s > 0 \).

(ii) If \( 3 \leq d \leq 5 \), a solution \( g \in L^{2,1} \) of \( g + |\xi|^2 g = T(g,g,g) \)
belongs to \( L^{2,s} \) for all \( s > 0 \).

**Proof.** The case \( d = 2 \) was treated in \[3\]. In fact, much more can be said there and one can prove exponential decay and analyticity of stationary solutions \[6\].

If \( d = 3 \), we know that \( g \in L^{2,1} \). By Proposition \[5\] this implies that \( T(g,g,g) \in L^{2,1} \). The equation satisfied by \( g \) implies that \( g \in L^{2,3} \), and iterating this argument gives the desired result.

The case \( d = 4 \) is similar.

If \( d = 5 \), we will show by duality that \( \| T(g,g,g) \|_{L^2} \lesssim \| g \|_{L^{2,1}}^3 \), from which the iterative process can be started. Indeed, using Hölder’s inequality, the Strichartz estimate and Sobolev embedding,

\[
|\langle T(g,g,g) , F \rangle| = (2\pi)^{d-1} \left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{is\Delta} F \overline{e^{is\Delta} g} |e^{is\Delta} g|^2 e^{is\Delta} g \, dx \, ds \right|
\leq \| e^{it\Delta} \hat{g} \|_{L_t^4 L_x^4}^3 \| e^{it\Delta} F \|_{L_t^4 L_x^{8/2}}^3
\lesssim \| \hat{g} \|_{H^1} \| F \|_{L^2} = \| g \|_{L_t^{2,1}}^3 \| F \|_{L^2}.
\]
Although we don’t pursue it here, we remark that the issue of regularity of those solutions is very interesting.

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