Time-development of energy spectra in the simulation of quantum turbulence

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Abstract. Bradley et al. studied experimentally the emission of vortex rings by a vibrating grid in superfluid $^3$He-B.¹ They observed a sharp transition from ballistic propagation of vortex rings at low grid velocities to a cloud of quantum turbulence at higher velocities, the turbulence being generated by coalescence of the rings. This behaviour is consistent with the results of a full Biot-Savart numerical simulation with the vortex filament model.² Bradley et al suggested that in the quantum turbulent regime a Kolmogorov energy spectrum develops at small wave numbers (presumably less than $2\pi/\ell$, where $\ell$ is the vortex line spacing), and they suggested that the observed rate of free decay of the turbulence is consistent with this idea. In this work we have studied numerically the time-development of the energy spectrum. For the separated rings the spectrum contains very little energy at small wave numbers. After the transition to turbulence the energy at small wavenumbers increases, but it remains much less than would be the case for a Kolmogorov spectrum. We consider why the assumptions underlying the numerical simulations do not lead to the generation of a Kolmogorov spectrum.

1. Introduction

Quantum turbulence means turbulence of a superfluid, motion in the superfluid component taking the form of a tangle of quantized vortices (vortex line spacing $\ell$) [1]. In this work we confine ourselves to the case of zero temperature at which the fraction of normal fluid is negligible. Correlations in the vortex tangle can be classified into two types [2]. In the correlated tangle turbulent energy is concentrated in the "classical" range of length scales, larger than $\ell$, where it exhibits a Kolmogorov spectrum. The energy is transferred to smaller scales by a Richardson cascade, and the total vortex line density, $L$, decays as $t^{-3/2}$, if we turn off the excitation sustaining the turbulence. In the uncorrelated tangle the turbulent energy is associated with a random vortex tangle of spacing $\ell$ and is concentrated in the "quantum" range of length scales, $\leq \ell$. Then $L$ decays as $t^{-1}$. The type of vortex tangle is therefore characterized by its energy spectrum and by the time-dependence of the decay of $L$.

Bradley et al [3, 4] showed that a vibrating grid in superfluid $^3$He-B at a very low temperature emits vortex rings. At low grid velocities these rings are observed to propagate ballistically and independently of one another. At higher grid velocities a higher density of vortex rings leads to interactions and coalescence, and to a sharp transition to a quantum turbulent regime. This

¹ D. I. Bradley et al., Phys. Rev. Lett. 95, 035302 (2005); Phys. Rev. Lett. 96, 035301 (2006)
² S. Fujiyama et al., Phys. Rev. B 81, 180512(R) (2010).
transition is well understood from full Biot-Savart numerical simulations of the vortex filament model [5]. The present paper uses simulations of the same type to obtain the energy spectrum for this type of quantum turbulence.

Figure 1. Snapshot of vortices at \( t = 0.6 \) sec. The simulation box is 200 \( \mu m \times 200 \mu m \times 600 \mu m \). Periodic boundary conditions are used for all three directions. Vortex rings of radius 10 \( \mu m \) are injected from the left-hand side of the cell at a time interval \( \tau_i = 2 \) msec.

Figure 2. Time development of line length density for the simulation of Fig.1.

Figure 3. Time development of energy spectra for the simulation of Fig.1. See the text.

2. Numerical results

The vortex dynamics is described by the vortex filament model and based on the full Biot-Savart law [6]. A filament is represented in parametric form \( s(\xi, t) \), where \( \xi \) refers to one-dimensional coordinate along filaments. Once the configuration of vortices is determined from the dynamics, the energy spectrum \( E(k) \) is obtained from [7]

\[
E(k) = \frac{\kappa^2}{2V(2\pi)^3} \int \frac{d\Omega_k}{|k|^2} \int_{\xi_1} \int_{\xi_2} d\xi_1 d\xi_2 s'(\xi_1) \cdot s'(\xi_2) e^{-ik \cdot (s(\xi_1) - s(\xi_2))}.
\]
The numerical simulation of the vortex dynamics is similar to that in Ref. 5. The numerical time resolution is $5 \times 10^{-5}$msec and the space resolution, $\Delta \xi$ is 0.5$\mu$m. The simulation cell is a box of the cross section $200 \mu$m $\times$ $200 \mu$m and the length 600 $\mu$m. Here we use periodic boundary conditions for all three directions, in contrast to the simulations of Ref. 5 where the periodic boundary conditions were imposed only for the transverse directions. Vortex rings of radius $10 \mu$m are injected from the left-hand side of the cell at a regular time intervals $\tau_i$, at random positions and at random angles within a 20 degree cone around the forward direction. Vortex loops and other structures that are smaller than the numerical space resolution are eliminated, which simulates an effective dissipation.

At low injection rates the rings propagate independently with their self-induced velocity. At higher injection rates, the vortices start to collide and reconnect, establishing a vortex tangle, as shown in Fig. 1: this is consistent with the observations. Figure 2 shows how the line length density $L$ increases with time. As long as the injected rings do not interact, $L$ increases linearly with time. The first departure from linearity signals the onset of interactions and the formation of a vortex tangle. Eventually $L$ reaches a constant value, indicating a statistically steady state, sustained by a balance between vortex injection and numerical elimination of the smallest structures.

Time-development of the energy spectra for the case corresponding to Fig.1 is shown in Fig. 3. The three black vertical lines at wave numbers $k_{\Delta \xi}$, $k_{2R}$ and $k_D$ correspond to the spatial resolution $\Delta \xi$, the diameter $2R$ of an injected ring and the box size $D$. The wave number of each colored vertical line refers to $2\pi/\ell$ at each instant. We see that at first the spectrum has a sharp peak close to $k_{2R}$, as for the independent rings, but that it gradually increases at both lower and higher wavenumbers. Eventually it reaches a statistically steady state, corresponding to the constant value of $L$. The spectra for $k > 2\pi/\ell$ show a $k^{-1}$ behavior, reflecting the short-range velocity field of each vortex. At smaller wave numbers the steady-state energy spectrum is seen to bend over, with a maximum at a value of $k$ that tends to be at about $2\pi/\ell$ for large times. There is no sign of the development of a Kolmogorov energy spectrum for $k < 2\pi/\ell$; the spectrum always decreases with decreasing $k$ in this range of wave numbers. We deduce that the vortex tangle produced by the coalescence of vortex rings can be classed as uncorrelated (or quantum) rather than correlated (or classical).

The way in which $L$ decays in time after the injection of vortex rings is turned off, shown in Fig. 4, confirms this picture. We see that $L$ decays as $t^{-1}$, until the vortex density becomes very small, such that the line spacing is less than about 1/4 of the width of the box, when the time dependence of $L$ changes to $t^{-0.7}$. The observed decay of the energy spectrum is shown in Fig. 5, where we see that the spectrum decays uniformly over the whole range of wave numbers.

It should be emphasized that in the simulations reported here we used periodic boundary conditions in all three directions, in contrast to those, relating to the same experiments, reported earlier [5]. In this earlier report, we discussed only the way in which the line density, $L$, behaved. We have since examined the energy spectra relating to these earlier simulations, and they do not differ significantly from those reported here. We conclude that the results we report are not sensitive to these precise boundary conditions.

3. Conclusions and comparison with experiment

We have performed numerical simulations that relate to recent experiments on the production of quantum turbulence by a vibrating grid in superfluid $^3$He-B [3, 4], and we have shown that such turbulence is expected to be of the uncorrelated (quantum) type, with no sign of the development of a Kolmogorov spectrum. This result is in apparent contrast to the experimental results [4], according to which the decay at large times goes as $t^{-3/2}$, characteristic of correlated turbulence, the size the the largest (classical) eddies having been estimated as about 1.5 mm (much larger than the mesh of the grid). However, it is important to notice that there is an
important difference between the simulations and the experiments. In the former case the "grid" (i.e the plane at which the vortex rings are generated) extends over the whole of the cross-section of the confining box, whereas in the latter case the grid occupies only a relatively small fraction of this cross-section, the rest of the cross-section being open. In this latter case vibration of the grid can be expected to generate not only the vortex rings but also a large-scale backflow round the sides of the grid. We suspect that this backflow, which is hard to quantify, has an important effect on the form of turbulence produced by the vibrating grid.

**References**

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