TWO WEIGHT BUMP CONDITIONS FOR MATRIX WEIGHTS

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ABSTRACT. In this paper we extend the theory of two weight, $A_p$ bump conditions to the setting of matrix weights. We prove two matrix weight inequalities for fractional maximal operators, fractional and singular integrals, sparse operators and averaging operators. As applications we prove quantitative, one weight estimates, in terms of the matrix $A_p$ constant, for singular integrals, and prove a Poincaré inequality related to those that appear in the study of degenerate elliptic PDEs.

1. INTRODUCTION

In this paper we extend the theory of $A_p$ bump conditions to matrix weights. To put our results into context we first briefly review the theory in the case of scalar weights. A scalar weight $w$ (i.e., a non-negative, locally integrable function) satisfies the Muckenhoupt $A_p$ condition, $1 < p < \infty$, if

$$[w]_{A_p} = \sup Q \int_Q w \, dx \left( \int_Q w^{1-p'} \, dx \right)^{p-1} < \infty,$$

where here and below the supremum is taken over all cubes $Q$ with edges parallel to the coordinate axes. It is well known that this condition is sufficient for a wide variety of classical operators (e.g., the Hardy-Littlewood maximal operator, singular integral operators) to be bounded on $L^p(w)$. (Cf. [12, 16].)

This condition naturally extends to pairs of weights: we say $(u, v) \in A_p$ if

$$[u, v]_{A_p} = \sup Q \int_Q u \, dx \left( \int_Q v^{1-p'} \, dx \right)^{p-1} < \infty.$$ 

However, unlike in the one weight case, while this condition is often necessary for an operator to map $L^p(v)$ into $L^p(u)$, it is almost never sufficient. (See [6] and...)

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2010 Mathematics Subject Classification. Primary 42B20, 42B25, 42B35.

Key words and phrases. Matrix weights, $A_p$ bump conditions, maximal operators, fractional integral operators, singular integral operators, sparse operators, Poincaré inequalities, $p$-Laplacian.

The first author is supported by NSF Grant DMS-1362425 and research funds from the Dean of the College of Arts & Sciences, the University of Alabama. The second and third authors are supported by the Simons Foundation.
the references it contains.) Therefore, for many years, the problem was to find a similar condition that was sufficient. The idea of \( A_p \) bump conditions originated with Neugebauer \[27] but was fully developed by Pérez \[28, 29, 31]\. (See also Sawyer and Wheeden \[34]\.) If we rewrite the two weight \( A_p \) condition as

\[
\sup_Q |Q|^{-1} \| u^{\frac{1}{p}} \|_{\Phi, Q} \| v^{-\frac{1}{p}} \|_{\Psi, Q} < \infty,
\]

where \( \| \cdot \|_{\Phi, Q} \) denotes the localized \( L^p \) norm with respect to measure \( |Q|^{-1} \chi_Q \, dx \), then a “bumped” \( A_p \) condition is gotten by replacing the \( L^p \) and/or \( L^{p'} \) norms with a slightly larger norm in the scale of Orlicz spaces.

We recall a few properties of Orlicz spaces; for more details see \[6\]. Let \( \Phi : [0, \infty) \to [0, \infty) \) be a Young function: convex, increasing, \( \Phi(0) = 0 \), and \( \Phi(t)/t \to \infty \) as \( t \to \infty \). Given \( \Phi \), its associate function is another Young function defined by

\[
\bar{\Phi}(t) = \sup_{s > 0} \{ st - \Phi(s) \}.
\]

If \( \Phi(t) = t^p \), \( \bar{\Phi}(t) \approx t^{p'} \). Given \( 1 < p < \infty \), we say that \( \Phi \) satisfies the \( B_p \) condition, denoted by \( \Phi \in B_p \), if

\[
\int_1^\infty \Phi(t) \frac{dt}{t^p} < \infty.
\]

Given a cube \( Q \) we define the localized Orlicz norm \( \| f \|_{\Phi, Q} \) by

\[
\| f \|_{\Phi, Q} = \inf \left\{ \lambda > 0 : \int_Q \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\} < \infty.
\]

The pair \( \Phi, \bar{\Phi} \) satisfy the generalized Hölder inequality in the scale of Orlicz spaces:

\[
\int_Q |f(x)g(x)| \, dx \leq 2 \| f \|_{\Phi, Q} \| g \|_{\bar{\Phi}, Q}.
\] (1.1)

Pérez proved that if the term on the right in the two weight \( A_p \) condition is “bumped” in the scale of Orlicz spaces, then the maximal operator satisfies a two weight inequality. Recall that the Hardy-Littlewood maximal operator is defined by

\[
M f(x) = \sup_Q \int_Q |f(y)| \, dy \cdot \chi_Q(x).
\]

**Theorem 1.1.** Given \( 1 < p < \infty \), suppose \( \Phi \) is a Young function such that \( \Phi \in B_p \). If \( (u, v) \) is a pair of weights such that

\[
\sup_Q \| u^{\frac{1}{p}} \|_{\Phi, Q} \| v^{-\frac{1}{p}} \|_{\bar{\Phi}, Q} < \infty,
\]

then \( M : L^p(v) \to L^p(u) \).
**Remark 1.2.** For instance, if we take \( \Phi(t) = t^p \log(e + t)^{\rho - 1 + \delta}, \delta > 0, \) then \( \Phi(t) \approx t^p \log(e + t)^{-1 - \epsilon}, \epsilon > 0, \) and \( \Phi \in B_p. \) Orlicz functions of this kind are referred to as “log bumps.”

It was conjectured (see [9]) that a comparable result held for Calderón-Zygmund singular integral operators if both terms in the two weight \( A_p \) condition were bumped. After a number of partial results, this was proved by Lerner [22]. Recall that a Calderón-Zygmund singular integral is an operator \( T : L^2 \to L^2 \) such that if \( f \in C_c^\infty(\mathbb{R}^d) \), then for \( x \notin \text{supp}(f) \),

\[
T f(x) = \int_{\mathbb{R}^d} K(x, y) f(y) \, dy,
\]

where the kernel \( K : \mathbb{R}^d \times \mathbb{R}^d \setminus \Delta \to \mathbb{C} \) (\( \Delta = \{(x, x) : x \in \mathbb{R}^d\} \)), satisfies

\[
|K(x, y)| \leq \frac{C}{|x - y|^d},
\]

and

\[
|K(x, y) - K(x, y + h)| + |K(x, y) - K(x + h, y)| \leq C \frac{|h|^\delta}{|x - y|^{d + \delta}},
\]

for some \( \delta > 0 \) and \( |x - y| > 2|h| \).

**Theorem 1.3.** Given \( 1 < p < \infty \), suppose \( \Phi \) and \( \Psi \) are Young functions such that \( \Phi \in B_p \) and \( \Psi \in B_{p'} \). If \( (u, v) \) is a pair of weights such that

\[
\sup_Q \|u^{\frac{1}{p}}\|_\Phi,Q \|v^{-\frac{1}{p'}}\|_\Psi,Q < \infty,
\]

and if \( T \) is a Calderón-Zygmund singular integral, then \( T : L^p(v) \to L^p(u) \).

Analogous results hold for the fractional maximal operator \( M_\alpha \), and the fractional integral operator \( I_\alpha \), \( 0 < \alpha < d \), defined by

\[
M_\alpha f(x) = \sup_Q |Q|^{\frac{\alpha}{d}} \int_Q |f(y)| \, dy \cdot \chi_Q(x),
\]

and

\[
I_\alpha f(x) = \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d - \alpha}} \, dy.
\]

For these operators we are interested in off-diagonal inequalities, when \( 1 < p \leq q < \infty \). The corresponding two weight condition is

\[
[u, v]_{A_{p', q}} = \sup_Q |Q|^{\frac{\alpha}{d} - \frac{1}{p} + \frac{1}{p'} + \frac{1}{q} + \frac{1}{p}} \|u^{\frac{1}{p}}\|_{\Psi,Q} \|v^{-\frac{1}{p'}}\|_{\Phi',Q} < \infty.
\]
(In the one weight case, which requires \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d} \), the weight \( w \) satisfies \( u = w^q, v = w^p \). See [6] for details.) Again, this condition is itself not sufficient, but if the norms are bumped a sufficient condition is gotten. For the off-diagonal inequalities (i.e., when \( p < q \)) we replace the \( B_p \) condition by the weaker \( B_{p,q} \) condition: we say a Young function \( \Phi \in B_{p,q} \) if

\[
\int_1^\infty \frac{\Phi(t)^{\frac{q}{p}}}{t} \, dt < \infty.
\]

It was shown in [7] that \( B_p \subset B_{p,q} \) when \( p < q \). The following two results were first proved by Pérez [30] with the stronger \( B_p \) condition; they were improved to use the \( B_{p,q} \) condition in [7].

**Theorem 1.4.** Given \( 1 < p \leq q < \infty \) and \( 0 < \alpha < d \), suppose \( \Phi \) is a Young function such that \( \Phi \in B_{p,q} \). If \( (u, v) \) is a pair of weights such that

\[
\sup_Q |Q|^{\frac{\alpha}{d} - \frac{1}{q} + \frac{1}{p}} \|u\|_{q,Q} \|v\|_{p,Q}^{\frac{1}{p}} < \infty,
\]

then \( M_\alpha : L^p(v) \to L^q(u) \).

**Theorem 1.5.** Given \( 1 < p \leq q < \infty \) and \( 0 < \alpha < d \), suppose \( \Phi \) and \( \Psi \) are Young functions such that \( \Phi \in B_{p,q} \) and \( \Psi \in B_{q',p'} \). If \( (u, v) \) is a pair of weights such that

\[
\sup_Q |Q|^{\frac{\alpha}{d} - \frac{1}{q} + \frac{1}{p}} \|u\|_{q',Q} \|v\|_{p',Q}^{\frac{1}{p'}} < \infty,
\]

then \( I_\alpha : L^p(v) \to L^q(u) \).

The primary goal of this paper is to generalize Theorems 1.1 through 1.5 to the setting of matrix weights. To state our results we first give some basic information on matrix weights. For more details, see [8, 15, 32]. A matrix weight \( U \) is an \( n \times n \) self-adjoint matrix function with locally integrable entries such that \( U(x) \) is positive definite for a.e. \( x \in \mathbb{R}^d \). For a matrix weight we can define \( U^r \) for any \( r \in \mathbb{R} \), via diagonalization. Given an exponent \( 1 \leq p < \infty \) and an \( n \times n \) matrix weight \( U \) on \( \mathbb{R}^d \) we define the matrix weighted space \( L^p(U) \) to be the set of measurable, vector-valued functions \( f : \mathbb{R}^d \to \mathbb{C}^n \) such that

\[
\|f\|_{L^p(U)} = \left( \int_{\mathbb{R}^d} |U(x)^{\frac{1}{2}} f(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.
\]
Given a matrix weight $U$ and $x \in \mathbb{R}^d$, define the operator norm of $U(x)$ by

$$|U(x)|_{op} = \sup_{e \in \mathbb{C}^n \atop ||e||=1} |U(x)e|.$$  

For brevity, given a norm $\| \cdot \|$ on a some scalar valued Banach function space (e.g., $L^p$), we will write $\|U\|$ for $\||U||_{op}$ and $\|Ue\|$ for $\||Ue||$.

Given two matrix weights $U$ and $V$, a linear operator $T$ satisfies

$$T : L^p(V) \to L^q(U)$$

if and only if

$$U^{\frac{1}{p}}TV^{-\frac{1}{q}} : L^p(\mathbb{R}^d, \mathbb{C}^n) \to L^q(\mathbb{R}^d, \mathbb{C}^n),$$

and it is in this form that we will prove matrix weighted norm inequalities. However, this approach no longer works for sublinear operators such as maximal operators. Following the approach introduced in [4, 15] we define a matrix weighted fractional maximal operator. Given matrix weights $U$ and $V$ and $0 < \alpha < d$, we define

$$M_{\alpha,U,V}f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{\frac{1}{1-\frac{\alpha}{d}}}} \int_Q |U(x)\frac{1}{q}V(y)\frac{1}{p}f(y)| dy. \quad (1.2)$$

When $U = V$, this operator was first considered in [20].

Our first result give sufficient conditions on the matrices $U$ and $V$ for $M_{\alpha,U,V}$ to be bounded from $L^p(\mathbb{R}^d, \mathbb{C}^n)$ to $L^q(\mathbb{R}^d, \mathbb{C}^n)$.

**Theorem 1.6.** Given $0 \leq \alpha < d$ and $1 < p \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d}$, suppose $\Phi$ is a Young function with $\tilde{\Phi} \in B_{p,q}$. If $(U, V)$ is a pair of matrix weights such that

$$[U, V]_{p,q,\Phi} = \sup_Q |Q|^{\frac{1}{q} + \frac{1}{p} - \frac{\alpha}{d}} \left( \int_Q \|U(x)\frac{1}{q}V(x)\frac{1}{p}\Phi(x) dx \right)^{\frac{1}{q}} < \infty, \quad (1.3)$$

then $M_{\alpha,U,V} : L^p(\mathbb{R}^d, \mathbb{C}^n) \to L^q(\mathbb{R}^d, \mathbb{C}^n)$.

**Remark 1.7.** In the scalar case (i.e., when $n = 1$) Theorem 1.6 immediately reduces to Theorem 1.1 when $\alpha = 0$ and Theorem 1.4 when $\alpha > 0$.

**Remark 1.8.** Theorem 1.6 generalizes two results known in the one weight case (i.e., when $U = V$). When $p = q$ and $\alpha = 0$, if we take $\Phi(t) = t^{p'}$, then the condition (1.3) reduces to the matrix $A_p$ condition,

$$[U]_{A_p} = \sup_Q \int_Q \left( \int_Q |U(x)\frac{1}{q}U(y)\frac{1}{p}\Phi_{op} dy \right)^{\frac{1}{p'}} dx < \infty, \quad (1.4)$$

which is sufficient for $M_U = M_{0,U,U}$ to be bounded on $L^p(\mathbb{R}^d, \mathbb{C}^n)$: see [4, 15].
Similarly, when \( \alpha > 0 \) and \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d} \), and we again take \( \Phi(t) = t^{p'} \), then (1.3) becomes the matrix \( A_{p,q} \) condition,

\[
[U]_{p,q} = \sup_Q \int_Q \left( \int_Q |U(x)^{1/q} U(y)^{-1/p} \|_{op}^q dy \right)^{\frac{1}{p'}} dx < \infty,
\]

introduced in [20], where they showed this condition is sufficient for \( M_{\alpha,U} = M_{\alpha,U,U} \) to map \( L^p(\mathbb{R}^d, \mathbb{C}^n) \) into \( L^q(\mathbb{R}^d, \mathbb{C}^n) \).

**Remark 1.9.** In Theorem 1.6 the restriction on \( p \) and \( q \) that \( \frac{1}{p} - \frac{1}{q} \geq \frac{\alpha}{d} \) is natural. For if the opposite inequality holds, given matrix weights \( U \) and \( V \) such that \( M_{\alpha,U,V} : L^p(\mathbb{R}^d, \mathbb{C}^n) \to L^q(\mathbb{R}^d, \mathbb{C}^n) \), then \( U(x) = 0 \) almost everywhere. See Proposition 3.1 below. In the scalar case, this was first proved by Sawyer [33].

Our second result gives sufficient conditions on the matrices \( U \) and \( V \) for \( I_{\alpha} \) to map \( L^p(V) \) to \( L^q(U) \). Here and in Theorem 1.14, by \( \| \cdot \|_{\Phi,\Psi} \) we mean that the Orlicz norm is taken with respect to the \( y \) variable. We define \( \| \cdot \|_{\Psi,x,\Psi} \) similarly.

**Theorem 1.10.** Given \( 0 < \alpha < d \) and \( 1 < p \leq q < \infty \) such that \( \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d} \), suppose that \( \Phi \) and \( \Psi \) are Young functions with \( \bar{\Phi} \in B_{p,q} \) and \( \bar{\Psi} \in B_{q'} \). If \( (U, V) \) is a pair of matrix weights such that

\[
[U, V]_{p,q,\Phi,\Psi} = \sup_Q |Q|^{\frac{1}{q'} + \frac{1}{p} - \frac{1}{q}} \left\| U(x)^{\frac{1}{q}} V(y)^{-\frac{1}{p}} \|_{\Phi,y,\Psi} \right\|_{\Psi,x,\Psi} < \infty,
\]

then \( I_{\alpha} : L^p(V) \to L^q(U) \).

**Remark 1.11.** In the scalar case, Theorem 1.10 reduces to a special case of Theorem 1.5 in that we do not recapture the weaker hypothesis \( \bar{\Psi} \in B_{q',p'} \). This is a consequence of our proof; we conjecture that this result remains true with this weaker hypothesis.

**Remark 1.12.** In the one weight case, it was proved in [20] that if \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{d} \) and \( U \in A_{p,q} \), then \( I_{\alpha} : L^p(U) \to L^q(U) \).

**Remark 1.13.** As for the fractional maximal operator, the restriction that \( \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d} \) is natural. In the scalar case (i.e., when \( n = 1 \)), if the opposite inequality holds, then, since \( M_{\alpha} f(x) \lesssim I_{\alpha} (\|f\|)(x) \), we have that the weights are trivial. See [33] for details.

Our third result gives sufficient conditions on the matrices \( U \) and \( V \) for a Calderón-Zygmund operator \( T \) to map \( L^p(V) \) to \( L^p(U) \).
Theorem 1.14. Given $1 < p < \infty$, suppose $\Phi$ and $\Psi$ are Young functions with $\bar{\Phi} \in B_p$ and $\bar{\Psi} \in B_{p'}$. If $(U, V)$ is a pair of matrix weights such that

$$[U, V]_{p, \Phi, \Psi} = \sup_Q \|U(x)^{\frac{1}{p}} V(y)^{-\frac{1}{p'}}\|_{\Phi, Q} \|V_{x, Q} < \infty,$$

(1.6)

and if $T$ is a Calderón-Zygmund operator, then $T : L^p(V) \rightarrow L^p(U)$.

**Remark** 1.15. Theorem 1.14 also holds if $T$ is a Haar shift operator or a paraproduct. See the discussion in Section 5 below.

As a corollary to Theorem 1.14 we can prove quantitative one weight estimates for Calderón-Zygmund operators. To state our result, recall that if $W$ is in matrix $A_p$, then for every $e \in \mathbb{C}^n$, $|W^{\frac{1}{p}} e|^p$ is a scalar $A_p$ weight, and

$$[W^{\frac{1}{p}} e]^p_{A_p} \leq [W]_{A_p}.$$

Thus, following [26], we can then define the “scalar $A_\infty$” constant of $W$ by

$$[W]_{A_p}^{\infty} = \sup_{e \in \mathbb{C}^n} [W^{\frac{1}{p}} e]^p_{A_\infty}.$$

(We will make precise our definition of $A_\infty$ in Section 5.)

**Corollary 1.16.** Given $1 < p < \infty$, suppose $W$ is a matrix $A_p$ weight. If $T$ is a Calderón-Zygmund operator, then

$$\|T\|_{L^p(W)} \lesssim [W]_{A_p}^{\frac{1}{p}} [W^{\frac{1}{p'}}]_{A_{p'}^{\infty}}^{\frac{1}{p'}} [W^{\frac{1}{p}}]_{A_{p}^{\infty}} \lesssim [W]_{A_{p}}^{1+\frac{1}{p-1}-\frac{1}{p}}.$$

**Remark** 1.17. Corollary 1.16 appears to be the first quantitative estimate for matrix weighted inequalities for singular integrals for all $p$, $1 < p < \infty$. Qualitative one weight, matrix $A_p$ estimates for Calderón-Zygmund operators were first proved in [4, 15]. Bickel, Petermichl and Wick [2] proved that for the Hilbert transform $H$, $\|H\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}} \log([W]_{A_2})$. This result was improved by Nazarov, et al. [26] and Culiuc, di Plinio and Ou [11] and extended it to all Calderón-Zygmund operators $T$, getting $\|T\|_{L^2(W)} \lesssim [W]_{A_2}^{\frac{3}{2}}$. (In fact, in [26] they prove a stronger result which we will discuss below.) Corollary 1.16 reduces to this estimate when $p = 2$.

We doubt that our estimate is sharp: it is reasonable to conjecture that the sharp exponent for matrix weights is the same as in the scalar case: $\max\{1, \frac{1}{p-1}\}$. We do note that in the scalar case, our exponent is sharper than what would be gotten from Rubio de Francia extrapolation, which starting from the exponent $\frac{3}{2}$ when $p = 2$ is $\frac{3}{2} \max\{1, \frac{1}{p-1}\}$. In particular, it is asymptotically sharp as $p \to \infty$. 
We now consider the two weight matrix $A_{p,q}$ condition,

$$[U, V]_{A_{p,q}} = \sup_Q |Q|^{\frac{\alpha}{q} + \frac{1}{q} - \frac{1}{p}} \left( \int_Q \left( \int_Q |U(x)\frac{1}{q}V(y)^{-\frac{1}{p'}}|_{op} dy \right)^\frac{1}{q} dx \right) \frac{1}{q} < \infty. \quad (1.7)$$

By the properties of Orlicz norms, we have that $[U, V]_{A_{p,q}}$ is dominated by $[U, V]_{p,q,\Phi}$ and $[U, V]_{p,q,\Phi,\Psi}$. As we noted in Remark 1.8 above, this condition is sufficient in the one weight case for the strong type, two weight norm inequalities for maximal and fractional integrals. However, even in the scalar case this condition is not sufficient for two weight norm inequalities for fractional maximal or integral operators [5]. It is known to be necessary and sufficient for averaging operators to map $L^p(u)$ into $L^p(u)$ [1] and for the fractional maximal operator to map $L^p(v)$ into $L^{q,\infty}(u)$ [6]. We give two generalizations of these results to the matrix setting. Since these results include endpoint estimates, we extend the definition of $A_{p,q}$ to the case $p = 1$: given matrix weights $U$ and $V$, define

$$[U, V]_{1, q} = \sup_Q |Q|^{\frac{\alpha}{q} + \frac{1}{q} - 1} \text{ess sup}_{y \in Q} \left( \int_Q \left| U_{\frac{1}{q}}(x)V^{-\frac{1}{p'}}(y) \right|_{op} dx \right)^\frac{1}{q} < \infty. \quad (1.8)$$

Our first result concerns averaging operators. For $0 \leq \alpha < d$, given a cube $Q$, define

$$A^\alpha_Q f(x) = |Q|^\alpha \int_Q f(y) dy \cdot \chi_Q(x).$$

More generally, given a family $Q$ of disjoint cubes, define

$$A^\alpha_Q f(x) = \sum_{Q \in Q} A^\alpha_Q f(x).$$

**Theorem 1.18.** Given $0 \leq \alpha < d$, $1 \leq p \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d}$, and a pair of matrix weights $(U, V)$, the following are equivalent:

1. $(U, V) \in A^\alpha_{p,q}$;
2. Given any set $Q$ of pairwise disjoint cubes in $\mathbb{R}^d$,
   $$\|A^\alpha_Q f\|_{L^q(U)} \lesssim [U, V]_{A^\alpha_{p,q}} \|f\|_{L^p(V)},$$
   where the constant is independent of $Q$.

**Remark** 1.19. In the one weight, scalar case when $p = q$ Theorem 1.18 was implicit in Jawerth [21]; for the general result in the scalar case, see Bereznoi [1]. In the one weight matrix case, again when $p = q$, Theorem 1.18 was proved in [8].
Remark 1.20. As a corollary to Theorem 1.18 we prove two weight estimates for convolution operators and approximations of the identity, generalizing one weight results from [8]. See Corollary 6.1 below.

Our second result is a weak type inequality for a two weight variant of the so-called auxiliary maximal operator introduced in [4,15]. Given $0 \leq \alpha < d$ and matrix weights $U$ and $V$, define

$$M'_{\alpha,U,V} f(x) = \sup_Q |Q|^{\frac{d}{p}} \int_Q |U_q^\alpha V^{-\frac{1}{q}}(y) f(y)| \, dy \cdot \chi_Q(x), \quad (1.9)$$

where $U_q^\alpha$ is the reducing operator associated with the matrix $U$. (For a precise definition, see Section 2 below.) Given any cube $Q$, the associated averaging operator is

$$B_\alpha^Q f(x) = |Q|^{\frac{d}{p}} \int_Q |U_q^\alpha V^{-\frac{1}{p}}(y) f(y)| \, dy \cdot \chi_Q(x).$$

**Theorem 1.21.** Given $0 \leq \alpha < d$, $1 \leq p \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d}$, and a pair of matrix weights $(U,V)$, the following are equivalent:

1. $(U,V) \in A_{p,q}^\alpha$;
2. $M'_{\alpha,U,V} : L^p \to L^{q,\infty}$;
3. For every cube $Q$, $B_\alpha^Q : L^p \to L^{q,\infty}$ with norm independent of $Q$.

Remark 1.22. It is very tempting to conjecture that Theorem 1.21 remains true with the auxiliary maximal operator replaced by $M_{\alpha,U,V}$, but we have been unable to prove this. We can prove 1.21 because the auxiliary maximal operator is much easier to work with when considering weak type inequalities.

Finally, as a corollary to Theorem 1.10 we prove a “mixed” Poincaré inequality involving both scalar and matrix weights.

**Theorem 1.23.** Given $1 \leq p \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{d}$, suppose that $\Phi$ and $\Psi$ are Young functions with $\bar{\Phi} \in B_{p,q}$ and $\bar{\Psi} \in B_{q'}$. If $u$ is a scalar weight and $V$ is a matrix weight such that

$$\sup_Q |Q|^{\frac{d}{p} + \frac{1}{q'}} \|u\|^\bar{\Phi}_{q,Q} \|V^{-\frac{1}{p}}\|_{\Phi,Q} < \infty, \quad (1.10)$$

then given any open convex set $E \subset \mathbb{R}^d$ with $u(E) < \infty$, and any scalar function $f \in C^1(E)$,

$$\left( \int_E |f(x) - f_{E,u}|^q u(x) \, dx \right)^{\frac{1}{q}} \lesssim \left( \int_E |V^\frac{1}{q}(x) \nabla f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad (1.11)$$
where \( f_{E,u} = u(E)^{-1} \int_E f(x)u(x) \, dx \). The implicit constant is independent of \( E \).

**Remark 1.24.** Poincaré inequalities of this kind play a role in the study of degenerate elliptic equations. See, for instance, [23–25, 35, 36]. As an immediate consequence of Theorem 1.23 we can use the main result in [10] to prove the existence of weak solutions to a Neumann boundary value problem for a degenerate \( p \)-Laplacian. See Corollary 7.1 below.

The remainder of this paper is organized as follows. In Section 2 we gather together some preliminary results about the so called reducing operators associated with matrix weights. Reducing operators play a major role in all of our proofs.

In Section 3 we prove Theorem 1.6 and Proposition 3.1 In Section 4 we prove Theorem 1.10. In our proofs of these two theorems we make extensive use of the theory of dyadic approximations for fractional maximal and integral operators; for the scalar theory, see [5].

In Section 5 we prove Theorem 1.14 and Corollary 1.16. In our proof we use the recent result of Nazarov, et al. [26], who extended dyadic approximation theory for singular integrals to the matrix setting, and showed that to prove matrix weighted estimates for Calderón-Zygmund operators it is enough to prove them for sparse operators.

In Section 6 we prove Theorems 1.18 and 1.21, and prove Corollary 6.1 about convolution operators. Finally, in Section 7 we prove Theorem 1.23, and prove Corollary 7.1 giving weak solutions to a degenerate \( p \)-Laplacian.

Throughout this paper notation is standard or will be defined as needed. If we write \( X \lesssim Y \), we mean that \( X \leq cY \), where the constant \( c \) can depend on the dimension \( d \) of the underlying space \( \mathbb{R}^d \), the dimension \( n \) of our vector functions, the exponents \( p \) and \( q \) in the weighted Lebesgue spaces, and the underlying fractional maximal or integral operators (i.e., on \( \alpha \)) or on the underlying Calderón-Zygmund operator. The dependence on the matrix weights will always be made explicit. If we write \( X \approx Y \), then \( X \lesssim Y \) and \( Y \lesssim X \).

## 2. Reducing operators

Given a matrix weight \( A \), a Young function \( \Psi \), and a cube \( Q \), we can define a norm on \( \mathbb{C}^n \) by \( \|Ae\|_{\Psi, Q}, \, e \in \mathbb{C}^n \). The following lemma yields a very important tool in the study of matrix weights, the so-called reducing operator, which lets us replace this norm by a norm induced by a constant positive matrix. The following result was proved by Goldberg [15, Proposition 1.2].
Lemma 2.1. Given a matrix weight \( A \), a Young function \( \Psi \), and a cube \( Q \), there exists a (constant positive) matrix \( \mathcal{A}_Q^\Psi \), called a reducing operator of \( A \), such that for all \( e \in \mathbb{C}^n \),

\[
|\mathcal{A}_Q^\Psi e| \approx \|Ae\|_{\Psi,Q},
\]

where the implicit constants depend only on \( d \).

As a consequence of Lemma 2.1, we get the following result for the norms of reducing operators. These estimates are implicit in the literature, at least for \( L^p \) norms; we prove them for the convenience of the reader.

Proposition 2.2. Given matrix weights \( A \) and \( B \), Young functions \( \Phi \) and \( \Psi \), a cube \( Q \), and reducing operators \( \mathcal{A}_Q^\Psi \) and \( \mathcal{B}_Q^\Phi \), then for all \( e \in \mathbb{C}^n \),

\[
|\mathcal{A}_Q^\Psi|_{\text{op}} \approx \|A\|_{\Psi,Q},
\]

\[
|\mathcal{A}_Q^\Psi \mathcal{B}_Q^\Phi|_{\text{op}} \approx \|A(x)\mathcal{B}_Q^\Phi\|_{\Psi_x,Q} \approx \left\|\|A(x)B(y)\|_{\Phi_y}\right\|_{\Psi_x,Q}.
\]

In both cases the implicit constants depend only on \( d \).

Remark 2.3. As will be clear from the proof, the first estimate in (2.2) is true if \( \mathcal{B}_Q^\Phi \) is replaced with any constant matrix.

Proof. To prove (2.1) fix an orthonormal basis \( \{e_j\}_{j=1}^n \) of \( \mathbb{C}^n \). Then by the definition of the operator norm and of reducing operators,

\[
|\mathcal{A}_Q^\Psi|_{\text{op}} \approx \sum_{j=1}^n |A_Q^\Psi e_j| \approx \sum_{j=1}^n \|Ae_j\|_{\Psi,Q} \approx \|A\|_{\Psi,Q}.
\]

The proof of (2.2) is similar, but we exploit the fact that while matrix products of self-adjoint matrices do not commute, they have the same operator norm:

\[
|\mathcal{A}_Q^\Psi \mathcal{B}_Q^\Phi|_{\text{op}} \approx \sum_{j=1}^n |A_Q^\Psi B_Q^\Phi e_j|
\]

\[
\approx \sum_{j=1}^n \|A(x)B_Q^\Phi e_j\|_{\Psi_x,Q}
\]

\[
\approx \|A(x)\mathcal{B}_Q^\Phi\|_{\Psi_x,Q}
\]

\[
= \|\mathcal{B}_Q^\Phi A(x)\|_{\Psi_x,Q}
\]

\[
\approx \sum_{j=1}^n \|\mathcal{B}_Q^\Phi A(x)e_j\|_{\Psi_x,Q}
\]
\[ \approx \sum_{j=1}^{n} \left\| B(y)A(x)e_j \right\|_{\Phi_y, Q} \]

\[ \approx \left\| B(y)A(x) \right\|_{\Phi_y, Q} \]

\[ = \left\| A(x)B(y) \right\|_{\Phi_y, Q} \]

As a consequence of Proposition 2.2 we can restate all of the weight conditions in our theorems in terms of reducing operators. Given matrix weights \( U \) and \( V \), Young functions \( \Psi \) and \( \Phi \), and \( 1 \leq p \leq q < \infty \), let \( U_Q^{\Psi, \Phi} \) and \( V_Q^{p, \Phi} \) be the reducing operators

\[ |U_Q^{\Psi, \Phi}| \approx \| U^\frac{1}{p} e \|_{\Phi, Q}, \quad |V_Q^{p, \Phi}| \approx \| V^{-\frac{1}{p}} e \|_{\Phi, Q}. \]

If \( \Psi(t) = t^q \) or \( \Phi(t) = t^{p'} \) then we will write \( U_Q^q, V_Q^{p, p'} \) (or more simply, \( U_Q^q, V_Q^{p, p'} \)).

With this definition, we have the following equivalences: in Theorem 1.6,

\[ [U, V]_{p, q, \Phi} \approx \sup_{Q} |Q|^{\frac{1}{q} + \frac{1}{p} - \frac{1}{q}} |U_Q^{p, \Psi} V_Q^{p, \Phi}|_{op}; \tag{2.3} \]

in Theorem 1.10,

\[ [U, V]_{p, q, \Phi, \Psi} \approx \sup_{Q} |Q|^{\frac{1}{q} + \frac{1}{p} - \frac{1}{q}} |U_Q^{p, \Psi} V_Q^{p, \Phi}|_{op}; \tag{2.4} \]

in Theorem 1.14,

\[ [U, V]_{p, \Phi, \Psi} \approx \sup_{Q} |U_Q^{p, \Psi} V_Q^{p, \Phi}|_{op}. \tag{2.5} \]

When \( p > 1 \) we can restate the two weight \( A_{p, q} \) condition (1.7) as

\[ [U, V]_{A_{p, q}} \approx \sup_{Q} |Q|^{\frac{1}{q} + \frac{1}{p} - \frac{1}{q}} |U_Q^{p, \Psi} V_Q^{p'}|_{op}, \tag{2.6} \]

and when \( p = 1 \) by

\[ [U, V]_{A_{p, q}} \approx \sup_{Q} |Q|^{\frac{1}{q} + \frac{1}{q} - 1} \text{ess sup}_{y \in Q} |U_Q^{p} V^{-1}(y)|_{op}. \tag{2.7} \]

Finally, we will need the following lemma in the proof of Corollary 1.16. It is a quantitative version of a result proved in Roudenko [32, Corollary 3.3]. It follows at once if we use (2.6) to restate the definitions of one weight matrix \( A_p \) and \( A_{p'} \) from (1.4).

**Lemma 2.4.** Given \( 1 < p < \infty \) and a matrix weight \( W \), if \( W \in A_p \), then \( W^{-\frac{p'}{p}} \in A_{p'} \) and

\[ [W]_{A_p}^{\frac{1}{p'}} \approx [W^{-\frac{p}{p'}}]_{A_{p'}}^{\frac{1}{p'}}. \]
3. Proof of Theorem 1.6

We first prove that in Theorem 1.6 we may assume without loss of generality that \( \frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d} \).

**Proposition 3.1.** Given \( 0 < \alpha < d \), matrix weights \( U, V \) and \( 1 < p < q < \infty \) such that \( \frac{1}{p} - \frac{1}{q} > \frac{\alpha}{d} \), suppose that \( M_{\alpha,U,V} : L^p \to L^q \). If \( V_{\frac{\alpha}{d}} \) is locally integrable, then \( U(x) = 0 \) for almost every \( x \in \mathbb{R}^d \).

**Proof.** Fix \( Q \) and a vector \( e \), and let \( f(y) = V_{\frac{\alpha}{d}}(y)e_{\chi_Q}(y) \). Then for \( x \in Q \),

\[
M_{\alpha,U,V}f(x) \geq |Q|^{\frac{\alpha}{d}} \int_Q |U_{\frac{\alpha}{d}}(x)e| \, dy = |Q|^{\frac{\alpha}{d}}|U_{\frac{\alpha}{d}}(x)e|.
\]

Therefore,

\[
|Q|^{\frac{\alpha}{d}} \left( \int_Q |U_{\frac{\alpha}{d}}(x)e|^q \, dx \right)^{\frac{1}{q}} \leq \|M_{\alpha,U,V}f\|_{L^q} \lesssim \|f\|_{L^p} = \left( \int_Q |V_{\frac{\alpha}{d}}(x)e|^p \, dx \right)^{\frac{1}{p}},
\]

which in turn implies that

\[
\left( \int_Q |U_{\frac{\alpha}{d}}(x)e|^q \, dx \right)^{\frac{1}{q}} \lesssim |Q|^{\frac{\alpha}{d} - \frac{1}{q} - \frac{1}{p}} \left( \int_Q |V_{\frac{\alpha}{d}}(x)e|^p \, dx \right)^{\frac{1}{p}}.
\]

Let \( x_0 \) be any Lebesgue point of the functions \( |U_{\frac{\alpha}{d}}(x)e|^q \) and \( |V_{\frac{\alpha}{d}}(x)e|^p \) and let \( Q_k \) be an sequence of cubes centered at \( x_0 \) that shrink to this point. By the Lebesgue differentiation theorem, since \( \frac{1}{p} - \frac{1}{q} - \frac{\alpha}{d} > 0 \), the righthand side of the above inequality tends to 0. Therefore, \( |U_{\frac{\alpha}{d}}(x_0)e|^q = 0 \). Since this is true for every vector \( e \), we have that \( |U(x_0)|_{op} = |U_{\frac{\alpha}{d}}(x_0)|_{op} = 0 \). Hence, \( U(x_0) = 0 \). \( \square \)

To prove Theorem 1.6 we will first reduce the problem to the corresponding dyadic maximal operator. We recall some facts from the theory of dyadic operators. We say that a collection of cubes \( \mathcal{D} \) in \( \mathbb{R}^d \) is a dyadic grid if

1. if \( Q \in \mathcal{D} \), then \( \ell(Q) = 2^k \) for some \( k \in \mathbb{Z} \).
2. If \( P, Q \in \mathcal{D} \), then \( P \cap Q \in \{P, Q, \emptyset\} \).
3. For every \( k \in \mathbb{Z} \), the cubes \( \mathcal{D}_k = \{Q \in \mathcal{D} : \ell(Q) = 2^k\} \) form a partition of \( \mathbb{R}^d \).

We can approximate arbitrary cubes in \( \mathbb{R}^d \) by cubes from a finite collection of dyadic grids. (For a proof, see [5, Theorem 3.1].)

**Proposition 3.2.** For \( t \in \{0, \pm \frac{1}{3}\}^d \) define the sets

\[
\mathcal{D}^t = \{2^{-k}([0,1]^d + m + t) : k \in \mathbb{Z}, m \in \mathbb{Z}^d\}.
\]
Then each $\mathcal{D}^t$ is a dyadic grid, and given any cube $Q \subset \mathbb{R}^d$, there exists $t$ and $Q_t \in \mathcal{D}^t$ such that $Q \subset Q_t$ and $\ell(Q_t) \leq 3\ell(Q)$.

Given $0 \leq \alpha < d$, matrix weights $U$ and $V$ and a dyadic grid $\mathcal{D}$, define the dyadic maximal operator $M_{\alpha,U,V}^\mathcal{D}$ as in (1.2) but with the supremum taken over all cubes $Q \in \mathcal{D}$ containing $x$. Then the following result follows at once from Proposition 3.2 (cf. [5, Proposition 3.2]).

**Proposition 3.3.** Given $0 \leq \alpha < d$, matrix weights $U$ and $V$, let $\mathcal{D}^t$ be the dyadic grids from Proposition 3.2. Then for all $x \in \mathbb{R}^d$, $M_{\alpha,U,V}^\mathcal{D} f(x) \lesssim \sum_{t \in \{0, \pm 1/2\}^d} M_{\alpha,U,V}^{\mathcal{D}^t} f(x)$.

As a consequence of Proposition 3.3, to prove Theorem 1.6 it will suffice to prove it for $M_{\alpha,U,V}^\mathcal{D}$, where $\mathcal{D}$ is any dyadic grid. For the remainder of this section, fix a dyadic grid $\mathcal{D}$.

Our proof is adapted from the proof of the boundedness of the one weight maximal operator in [15]. We begin with two lemmas. For brevity, we will write $\mathcal{V}_Q^\Phi$ for the reducing operator $\mathcal{V}_Q^{\Phi}$. The first gives a norm inequality for an auxiliary fractional maximal operator, analogous to the operator $M'_W$ introduced in [4, 15].

**Lemma 3.4.** Given $0 \leq \beta < d$, let $1 \leq p \leq q < \infty$ be such that $\frac{\beta}{d} = \frac{1}{p} - \frac{1}{q}$. Let $\Phi$ be a Young function such that $\overline{\Phi} \in B_{p,q}$. Given a matrix weight $V$, define the auxiliary maximal operator

$$M_{\beta,V}^\mathcal{D} f(x) = \sup_{Q \in \mathcal{D}} |Q|^{\frac{\beta}{d}} \int_Q |\mathcal{V}_Q^\Phi|^{-1}V(y)^{-\frac{1}{p}} f(y)\, dy \cdot \chi_Q(x).$$

Then $M_{\beta,V}^\mathcal{D} : L^p(\mathbb{R}^d, \mathbb{C}^n) \to L^q(\mathbb{R}^d, \mathbb{C}^n)$.

**Proof.** Define the Orlicz fractional maximal operator

$$M_{\beta,\Phi} f(x) = \sup_{Q} |Q|^{\frac{\beta}{d}} \| f \|_{\Phi,Q} \cdot \chi_Q(x);$$

if $\beta = 0$, we write $M_{\Phi} = M_{0,\Phi}$. It was shown in [7] that

$$M_{\beta,\Phi} : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d).$$

(3.1)

Now fix $x \in \mathbb{R}^d$ and $Q \in \mathcal{D}$ containing $x$. Then by the generalized H"older inequality (1.1) we have that

$$|Q|^{\frac{\beta}{d}} \int_Q |\mathcal{V}_Q^\Phi|^{-1}V(y)^{-\frac{1}{p}} f(y)\, dy \lesssim \| \mathcal{V}_Q^\Phi \|_{\Phi,Q} |Q|^{\frac{\beta}{d}} \| f \|_{\Phi,Q}.$$
By the first inequality in (2.2) (which holds if we replace the reducing operator $B^\phi_Q$ by any matrix), we have that for all cubes $Q$,

$$\|(V_Q^\phi)^{-1}V^{-\frac{1}{p}}\|_{\Phi,Q} = \|V^{-\frac{1}{p}}(V_Q^\phi)^{-1}\|_{\Phi,Q} \lesssim |V_Q^\phi(V_Q^\phi)^{-1}|_{op} = 1.$$ 

Therefore, if we combine these two inequalities and take the supremum over all cubes $Q$ containing $x$, we get that $M_{\beta,V}f(x) \lesssim M_{\beta,\bar{\phi}}(|f|)(x)$. The desired norm inequality follows at once.

For the second lemma, given a cube $Q \in \mathcal{D}$, let $\mathcal{D}(Q) = \{P \in \mathcal{D} : P \subset Q\}$ and define the maximal type operator

$$N_Q(x) = \sup_{R \in \mathcal{D}(Q)} |Q|^{\beta + \frac{1}{q} - \frac{1}{p}} |U(x)|^{\frac{1}{q}} |V_R^\phi|_{op} \chi_R(x). \quad (3.2)$$

**Lemma 3.5.** Given a pair of matrix weights $U, V$ that satisfy (1.3), then

$$\sup_{Q \in \mathcal{D}} \int_Q N_Q(x)^q \, dx < \infty. \quad (3.3)$$

Lemma 3.5 is actually an immediate consequence of Lemma 4.1 which we will need to prove Theorem 1.10, and so its proof is deferred to the next section: see Remark 4.2.

**Proof of Theorem 1.6.** Fix $\beta$ such that $\frac{\beta}{d} = \frac{1}{p} - \frac{1}{q}$. Note that by our assumption on $p$ and $q$, $\beta \geq 0$. Given any cube $Q$,

$$|Q|^{\frac{\beta}{d}} \int_Q |U(x)|^{\frac{1}{q}} V(y)^{-\frac{1}{p}} f(y) \, dy$$

$$= |Q|^{\frac{\beta}{d} + \frac{1}{q} - \frac{1}{p}} |U(x)|^{\frac{1}{q}} \int_Q |V_Q^\phi(V_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} f(y) \, dy$$

$$\leq |Q|^{\frac{\beta}{d} + \frac{1}{q} - \frac{1}{p}} |U(x)|^{\frac{1}{q}} |V_Q^\phi|_{op} |Q|^{\frac{\beta}{d}} \int_Q |(V_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} f(y) \, dy.$$ 

For every $x \in \mathbb{R}^d$ there exists $Q = Q_x \in \mathcal{D}$ such that

$$M_{\alpha,U,V}^\phi f(x) \leq 2|Q|^{\frac{\beta}{d} + \frac{1}{q} - \frac{1}{p}} |U(x)|^{\frac{1}{q}} |V_Q^\phi|_{op} |Q|^{\frac{\beta}{d}} \int_Q |(V_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} f(y) \, dy.$$ 

There exists a unique $j = j_x \in \mathbb{Z}$ such that

$$2^j < |Q_x|^{\frac{\beta}{d}} \int_{Q_x} |(V_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} f(y) \, dy \leq 2^{j+1}. \quad (3.4)$$
Now for each $j \in \mathbb{Z}$, let $S_j$ be the collection of cubes $Q = Q_x$ that are maximal with respect to (3.4). Note that the cubes in $S_j$ are disjoint. Then for each $x \in \mathbb{R}^d$ there exists $j \in \mathbb{Z}$ and $S \in S_j$ such that $x \in Q \subset S$ and

$$M_{\alpha,U,V}^Q f(x) \leq 2|Q|^\frac{d}{q} + \frac{1}{q} |U(x)|^\frac{1}{p} |V_x|_{\text{op}} |Q|^\frac{1}{q} \int_Q |(V_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} f(y)| dy \leq 2^{j+2} N_S(x).$$

Moreover, we have that

$$\bigcup_{S \in S_j} S \subset \{ x \in \mathbb{R}^d : M_{\beta,V}^Q f(x) > 2^j \}.$$

Hence, by Lemmas 3.4 and 3.5,

$$\int_{\mathbb{R}^d} |M_{\alpha,U,V}^Q f(x)|^q dx \leq \sum_{j \in \mathbb{Z}} 2^{jq} \sum_{S \in S_j} \int_S N_S(x)^q dx \leq \sum_{j \in \mathbb{Z}} 2^{jq} \sum_{S \in S_j} |S| = \sum_{j \in \mathbb{Z}} 2^{jq} \left| \bigcup_{S \in S_j} S \right| \leq \sum_{j \in \mathbb{Z}} 2^{jq} \left| \{ x : M_{\beta,V}^Q f(x) > 2^j \} \right| \leq \int_{\mathbb{R}^d} M_{\beta,V}^Q f(x)^q dx \lesssim \left( \int_{\mathbb{R}^d} |f(x)|^p dx \right)^\frac{q}{p}.$$

\[\square\]

### 4. Proof of Theorem 1.10

Throughout this section, for brevity we will write $U_Q^\Psi = U_Q^\psi$ and $V_Q^\Psi = V_Q^\phi$. We begin with a lemma that extends [15, Lemma 3.3] to the scale of Orlicz spaces.

**Lemma 4.1.** Given a pair of matrix weights $U$, $V$ that satisfy (1.5), then

$$\sup_{Q \in \mathcal{D}} \| N_Q \|_{\psi,Q} < \infty$$

**Remark** 4.2. Since $\Psi \in B_{q'}$, we have that $\Psi(t) \lesssim t^{q'}$ and so $t^q \lesssim \Psi(t)$. Therefore, for any cube $Q \in \mathcal{D}$, $\| N_Q \|_{q,Q} \lesssim \| N_Q \|_{\psi,Q}$ (cf. [6]), and so Lemma 3.5 follows immediately from Lemma 4.1.
Proof of Lemma 4.1. Fix a cube $Q \in \mathcal{D}$. We first claim that there exists $C > 0$ sufficiently large such that if $\{R_j\}$ is the collection of maximal dyadic subcubes $R$ of $Q$, if any, satisfying
\[ |R|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}} |\nabla^\phi R_u^\psi_Q|_{\text{op}} > C, \]
then
\[ \left| \bigcup_j R_j \right| \leq \frac{1}{2} |Q|. \] (4.1)
To see this, note that since $\frac{2}{d} \geq \frac{1}{p} - \frac{1}{q}$, by inequality (2.2) we have that
\[ C < |R_j|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}} |\nabla^\phi R_u^\psi_Q|_{\text{op}} \leq |Q|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}} |\nabla^\phi R_u^\psi_Q|_{\text{op}} \leq C' |Q|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}} \|V^{-\frac{1}{p}} u^\psi_Q\|_{\Phi,R_j}, \]
where $C' > 1$ depends only on $n$. Therefore, by the definition of the Luxemburg norm,
\[ \int_{R_j} \Phi \left( \frac{|V(y)^{-1} U_u^\psi_Q|_{\text{op}}}{(C' |Q|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}})^{-1} C} \right) dy > 1. \]
Now set $C = 2C' \|(U, V)\|'$, where by (2.2), (2.4) and our assumption on the weights $U$ and $V$,
\[ \|(U, V)\|' = \sup_Q |Q|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}} \|V^{-\frac{1}{p}} u^\psi_Q\|_{\Phi,Q} \lesssim [U, V]_{p,q,\Phi,\Psi} < \infty. \]
Since the cubes $\{R_j\}$ are disjoint and $\Phi$ is convex, we get
\[ \sum_j |R_j| \leq \sum_j \int_{R_j} \Phi \left( \frac{|V(y)^{-1} U_u^\psi_Q|_{\text{op}}}{(C' |Q|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}})^{-1} C} \right) dy \]
\[ \leq \frac{|Q|}{2} \int_Q \Phi \left( \frac{|V(y)^{-1} U_u^\psi_Q|_{\text{op}}}{\|V^{-\frac{1}{p}} u^\psi_Q\|_{\Phi,Q}} \right) dy \leq \frac{|Q|}{2}; \]
this proves (4.1).
To complete the proof we will use an approximation argument. For $m \in \mathbb{N}$ such that $2^{-m} < \ell(Q)$, define the truncated operator
\[ N^m_Q(x) = \sup_{R \in \mathcal{D}(Q)} \sup_{x \in R} |R|^{\frac{2}{p} + \frac{1}{q} - \frac{1}{p}} |U(x)^{\frac{1}{q}} \nabla^\phi R|_{\text{op}}. \]
We will prove that
\[ \int_Q \Psi \left( \frac{N^m_Q(x)}{CC''} \right) dx \leq 3, \] (4.2)
where
\[ C'' = \sup_Q \| U^\frac{1}{q} (U_Q^\Psi)^{-1} \|_{\psi,Q} \lesssim 1. \]
(The last inequality follows from (2.2).) Then by convexity and the definition of the Luxemburg norm we will have that \( \| N_Q^m \|_{\psi,Q} \leq 3CC'' \), and the desired inequality follows from Fatou’s lemma as \( m \to \infty \).

To prove (4.2), let \( G_Q = \bigcup_j R_j^1 \). If \( x \in Q \setminus G_Q \), then for any dyadic cube \( R \in \mathcal{D}(Q) \) containing \( x \) such that \( \ell(R) > 2^{-m} \) we have
\[
| R |^{\delta + \frac{1}{q} - \frac{1}{p}} | U(x) \frac{1}{q} V_R^\Psi |_{\text{op}} = | R |^{\delta + \frac{1}{q} - \frac{1}{p}} | U(x) \frac{1}{q} (U_Q^\Psi)^{-1} U_R^\Psi V_R^\Psi |_{\text{op}} \\
\leq | R |^{\delta + \frac{1}{q} - \frac{1}{p}} | U(x) \frac{1}{q} (U_Q^\Psi)^{-1} |_{\text{op}} | U_Q^\Psi V_R^\Psi |_{\text{op}} \leq C | U(x) \frac{1}{q} (U_Q^\Psi)^{-1} |_{\text{op}}.
\]

Let \( F_j = \{ x \in R_j^1 : N_Q^m(x) \neq N_{R_j^1}(x) \} \). Then by the maximality of the cubes \( \{ R_j^1 \} \) and the previous estimate, we have that if \( x \in F_j \), \( N_Q^m(x) \leq C | U(x) \frac{1}{q} (U_Q^\Psi)^{-1} |_{\text{op}}. \)

We can now estimate as follows:
\[
\int_Q \Psi \left( \frac{N_Q^m(x)}{CC''} \right) \, dx \\
\leq \int_{Q \setminus G_Q} \Psi \left( \frac{| U^\frac{1}{q} (U_Q^\Psi)^{-1} |_{\text{op}}}{C''} \right) \, dx + \sum_j \int_{F_j} \Psi \left( \frac{| U^\frac{1}{q} (U_Q^\Psi)^{-1} |_{\text{op}}}{C''} \right) \, dx \\
+ \sum_j \int_{R_j^1 \setminus F_j} \Psi \left( \frac{N_Q^m(x)}{CC''} \right) \, dx \\
\leq 2|Q| + \sum_j \int_{R_j^1} \Psi \left( \frac{N_Q^m(x)}{CC''} \right) \, dx.
\]

To estimate the last term we iterate this argument. For each \( j \) form the collection \( \{ R_k^2 \} \) of maximal dyadic cubes, if any, \( R \in \mathcal{D}(R_j^1) \) such that
\[
| R |^{\delta + \frac{1}{q} - \frac{1}{p}} | U_Q^\Psi V_R^\Psi |_{\text{op}} > C.
\]

Then we can repeat the first argument above to show that for each \( j \),
\[
\sum_{k : R_k^2 \subset R_j^1} | R_k^2 | \leq \frac{1}{2} |R_j^1|. \quad (4.3)
\]
Thus, repeating the second argument we get
\[
\sum_j \int_{R_j^1} \Psi \left( \frac{N_{R_j^1}(x)}{CC''} \right) dx \leq \sum_j \sum_{k: R_k^1 \subset R_j^1} |R_k^1| + \int_{R_k^2} \Psi \left( \frac{N_{R_k^2}(x)}{CC''} \right) dx
\]
\[
\leq \frac{1}{2} |Q| + \sum_j \sum_{k: R_k^2 \subset R_j^1} \int_{R_k^2} \Psi \left( \frac{N_{R_k^2}(x)}{CC''} \right) dx.
\]

We continue with this argument on each integral on the right-hand side. However, by (4.3), the cubes \( R_k^2 \) are properly contained in the cubes \( R_j^1 \). But for this argument we are assuming that all the cubes have side length greater than \( 2^{-m} \). Therefore, after \( k \) iterations, where \( k \geq m + \log_2(\ell(Q)) \), the resulting collection of cubes \( \{R_k^2\} \) must be empty so the final sum in the estimate vanishes. So if we sum over the \( k \) steps, we get
\[
\int_Q \Psi \left( \frac{N_Q(x)}{CC''} \right) dx \leq 3 - 2^{-k} \leq 3.
\]
This gives us (4.2) and our proof is complete. \( \square \)

**Proof of Theorem 1.10.** We will prove that \( U_{\frac{1}{2}} I_a V_{-\frac{1}{p}} : L^p(\mathbb{R}^d, \mathbb{C}^n) \to L^p(\mathbb{R}^d, \mathbb{C}^n) \). By a standard approximation argument, it will suffice to prove that
\[
\left| \left\langle U_{\frac{1}{2}} I_a V_{-\frac{1}{p}} f, g \right\rangle \right|_{L^2} \lesssim \|f\|_{L^p} \|g\|_{L^{p'}},
\]
where \( f, g \) are bounded functions of compact support. In [20, Lemma 3.8] it was shown that
\[
\left| \left\langle U_{\frac{1}{2}} I_a V_{-\frac{1}{p}} f, g \right\rangle \right|_{L^2} \lesssim \sum_{t \in \{0, \pm \frac{1}{2} \}^d} \sum_{Q \in \mathcal{D}^t} |Q|^{\frac{1}{p}} \int_Q \int_Q \left| \left\langle V(y)^{-\frac{1}{p}} f(y), U(x)^{\frac{1}{p}} g(x) \right\rangle \right| \ dx \ dy,
\]
where the dyadic grids \( \mathcal{D}^t \) are defined as in Proposition 3.2. Therefore, to complete the proof, it suffices to fix a dyadic grid \( \mathcal{D} \) and show that the inner sum is bounded by \( \|f\|_{L^p} \|g\|_{L^{p'}} \). Our argument adapts to the matrix setting the scalar, two weight argument originally due to Pérez [29] (see also [5]).

First note that by the generalized Hölder’s inequality in the scale of Orlicz spaces, inequality (2.2) and the definition of \( \mathcal{V}_{Q}^\psi \),
\[
\sum_{Q \in \mathcal{D}} |Q|^{\frac{1}{p}} \int_Q \int_Q \left| \left\langle V(y)^{-\frac{1}{p}} f(y), U(x)^{\frac{1}{p}} g(x) \right\rangle \right| \ dx \ dy
\]
\[
\leq \sum_{Q \in \mathcal{D}} |Q|^d \left( \int_{Q} \left| (\nabla Q^{-1} V(y) - \frac{1}{n} f(y) \right| \, dy \right) \left( \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx \right)
\]
\[
\leq \sum_{Q \in \mathcal{D}} |Q|^d \| (\nabla Q^{-1} V \Phi Q) - f \Phi Q \|_{\Phi Q} \left( \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx \right)
\]
\[
\leq \sum_{Q \in \mathcal{D}} |Q|^d \| f \Phi Q \|_{\Phi Q} \left( \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx \right).
\]

Fix $a > 2^{d+1}$ and define the collection of cubes

\[
\mathcal{Q}^k = \{ Q \in \mathcal{D} : a^k < \| f \Phi Q \| \leq a^{k+1} \},
\]
and let $\mathcal{S}^k$ be the disjoint collection of $Q \in \mathcal{D}$ that are maximal with respect to the inequality $\| f \Phi Q \| > a^k$. Set $\mathcal{S} = \bigcup_k \mathcal{S}^k$. We now continue the above estimate:

\[
\sum_{k} \sum_{Q \in \mathcal{Q}^k} |Q|^d \| f \Phi Q \|_{\Phi Q} \left( \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx \right)
\]
\[
\leq \sum_{k} a^{k+1} \sum_{Q \in \mathcal{Q}^k} |Q|^d \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx
\]
\[
= \sum_{k} a^{k+1} \sum_{P \in \mathcal{S}^k} \sum_{Q \subset P} |Q|^d \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx.
\]

Fix a cube $P \in \mathcal{S}^k$; then we can estimate the inner most sum:

\[
\sum_{Q \subset P} |Q|^d \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx
\]
\[
\leq \sum_{Q \subset P} |Q|^d \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx
\]
\[
= \sum_{j=0}^{\infty} \sum_{\ell(Q) = 2^{-j} \ell(P)} |Q|^d \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx
\]
\[
= |P|^d \sum_{j=0}^{\infty} 2^{-j\alpha} \sum_{\ell(Q) = 2^{-j} \ell(P)} \int_{Q} \left| \nabla Q U(x) \frac{1}{n} g(x) \right| \, dx.
\]
TWO WEIGHT BUMP CONDITIONS FOR MATRIX WEIGHTS

\[ \lesssim |P|^{\frac{\beta}{d}} \int_{P} N_{P}(x)|g(x)|\,dx, \]

where \( \frac{\beta}{d} = \frac{1}{p} - \frac{1}{q} \), \( \beta \geq 0 \) by our hypotheses, and \( N_{P} \) is defined by (3.2).

If we insert this estimate into the above inequality, then by the generalized Hölder inequality (1.1) and Lemma 4.1,

\[
\sum_{k} a^{k+1} \sum_{P \in S_{k}} |P|^{\frac{\beta}{d}} \int_{P} N_{P}(x)|g(x)|\,dx
\]

\[
\leq a \sum_{k} \sum_{P \in S_{k}} |P|(|P|^{\frac{\beta}{d}}\|f\|_{\Phi,P}) \left( \int_{P} N_{P}(x)|g(x)|\,dx \right)
\]

\[
\leq a \sum_{k} \sum_{P \in S_{k}} |P|(|P|^{\frac{\beta}{d}}\|f\|_{\Phi,P}) (\|N_{P}\|_{\Psi,P}\|g\|_{\Psi,P})
\]

\[
\leq a \sum_{k} \sum_{P \in S_{k}} |P| \inf_{x \in P} M_{\Phi}^{\beta} f(x) M_{\Psi} g(x).
\]

For each \( Q \in S \), define

\[ E_{Q} = Q \setminus \bigcup_{Q' \subset Q} Q'. \]

Then by [6, Proposition A.1], the sets \( E_{Q} \) are pairwise disjoint and \( |E_{Q}| \geq \frac{1}{2}|Q| \).

Given this, we can continue the above estimate:

\[ \sum_{k} \sum_{P \in S_{k}} |P| \inf_{x \in P} M_{\Phi}^{\beta} f(x) M_{\Psi} g(x) \]

\[ \leq 2 \sum_{Q \in S} |E_{Q}| \inf_{x \in Q} M_{\Phi}^{\beta} f(x) M_{\Psi} g(x) \]

\[ \leq 2 \sum_{Q \in S} \int_{E_{Q}} M_{\Phi}^{\beta} f M_{\Psi} g \,dx \]

\[ \leq 2 \int_{\mathbb{R}^{d}} M_{\Phi}^{\beta} f M_{\Psi} g \,dx \]

\[ \leq 2 \|M_{\Phi}^{\beta} f\|_{L^q} \|M_{\Psi} g\|_{L^{q'}} \]

\[ \lesssim \|f\|_{L^p} \|g\|_{L^{q'}}. \]

The last inequality follows from (3.1). If we combine all of the above inequalities, we get the desired result. \( \Box \)
5. **Proof of Theorem 1.14 and Corollary 1.16**

Throughout this section, for brevity we will write $U_Q^\Psi = U_Q^{p,\Psi}$ and $V_Q^\Phi = V_Q^{p,\Phi}$. In order to prove our results about Calderón-Zygmund operators we introduce the concept of sparse operators. For complete details, see [5]. Given a dyadic grid $\mathcal{D}$, a set $S \subseteq \mathcal{D}$ is sparse if for each cube $Q \in S$, there exists a set $E_Q \subset Q$ such that $|E_Q| \geq \frac{1}{2}|Q|$ and the collection of sets $\{E_Q\}$ is pairwise disjoint. Define the dyadic sparse operator $T^S_\alpha$ by

$$T^S_\alpha f(x) = \sum_{Q \in S} |Q|^{\frac{\beta}{d}} \int_Q f(y) \, dy \cdot \chi_Q(x).$$

Note that in the proof of Theorem 1.10 the set of cubes $S$ is sparse, and the sums being approximated can be viewed as the integrals of sparse operators. By modifying this proof we can prove the following result.

**Theorem 5.1.** Given $0 \leq \alpha < d$ and $1 < p \leq q < \infty$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{\alpha}{d}$, suppose that $\Phi$ and $\Psi$ are Young functions with $\bar{\Phi} \in B_{p,q}$ and $\bar{\Psi} \in B_{q'}$. If $(U, V)$ is a pair of matrix weights satisfy the bump condition (1.5), then $T^S_\alpha : L^p(V) \to L^q(U)$.

**Remark 5.2.** In the one weight case, a quantitative version of Theorem 5.1 was proved in [3,19] when $p = q = 2$ and $\alpha = 0$.

**Proof.** The proof is virtually identical to the proof of Theorem 1.10 above, except that, since we start an operator defined over a sparse family $S$, we may omit the argument used to construct the set $S$. This was the only part of the proof of Theorem 1.10 where we used the assumption that $\alpha > 0$; everywhere else in the proof we may take $\alpha = 0$.

Because of these similarities, we only sketch the main steps:

$$\left| \left\langle U_{\frac{\alpha}{d}}^\Psi T^S_\alpha V^{-\frac{\alpha}{d}} f, g \right\rangle \right|_2 \leq \sum_{Q \in S} |Q|^{1 + \frac{\alpha}{d}} \int_Q \int_Q \left| \left\langle V(y)^{-\frac{\alpha}{d}}(y)f(y), U(x)^{\frac{\alpha}{d}}(x)g(x) \right\rangle \right|_\infty \, dx \, dy \leq \sup_Q |Q|^{\frac{\alpha}{d} + \frac{\beta}{d} - \frac{\alpha}{d}} |V_Q^\Phi U_Q^\Psi|_{op} \times \sum_{E_Q \subseteq S} |E_Q| \left( \int_Q |(V_Q^\Phi)^{-1} V^{-\frac{\alpha}{d}} f| \, dx \right) \left( \int_Q |(U_Q^\Psi)^{-1} U^{\frac{\alpha}{d}} g| \, dx \right) \lesssim \| M_{\Phi}^2 f \|_{L^p} \| M_{\Psi} g \|_{L^{q'}} \lesssim \| f \|_{L^p} \| g \|_{L^{q'}}.$$
We will now use Theorem 5.1 with \( p = q \) and \( \alpha = 0 \) (or more precisely, its proof) to prove Theorem 1.14 and Corollary 1.16. To do so, we must first describe the recent results of Nazarov, et al. [26] on convex body domination. Fix a cube \( Q \) and a \( \mathbb{C}^n \) valued function \( f \in L^1(Q) \). Define

\[
\langle f \rangle_Q = \left\{ \int_Q \varphi f \, dx : \varphi : Q \to \mathbb{R}, \|\varphi\|_{L^\infty(Q)} \leq 1 \right\};
\]

Then \( \langle f \rangle_Q \) is a symmetric, convex, compact set in \( \mathbb{C}^n \). If \( T \) is a CZO (or a Haar shift or a paraproduct) then for \( f \in L^1(Q) \), \( Tf \) is dominated by a sparse convex body operator. More precisely, there exists a sparse collection \( S \) such that for some constant \( C \) independent of \( f \), and a.e. \( x \in \mathbb{R}^d \),

\[
Tf(x) \in C \sum_{Q \in S} \langle f \rangle_Q \chi_Q(x),
\]

(5.1)

where the sum is an infinite Minkowski sum of convex bodies.

As a consequence of this fact, to prove norm inequalities for a CZO, it is enough to prove uniform estimates for the generalized sparse operators of the form

\[
T^S f(x) = \sum_{Q \in S} \int_Q \varphi_Q(x, y) f(y) \, dy,
\]

where for each \( Q \), \( \varphi_Q \) is a real valued function supported on \( Q \) as a function of \( y \) and such that for each \( x \), \( \|\varphi_Q(x, \cdot)\|_{\infty} \leq 1 \). Note that it is not clear from [26] whether \( \varphi_Q(x, y) \) can be chosen as a measurable function of \( x \), though this is not important for us (and is unlikely to be important for the further study of matrix weighted norm inequalities.)

**Proof of Theorem 1.14.** Since

\[
\left| \left\langle U_1^{\frac{1}{2}} V^{-\frac{1}{2}} f(x), g(x) \right\rangle_{\mathbb{C}^n} \right| = \left| \sum_{Q \in S} \chi_Q(x) \left\langle U_Q^{\frac{1}{2}} \nu_Q^{\frac{1}{2}} \int_Q \varphi_Q(x, \cdot) (\nu_Q^{-1})^{1/2} f, (U_Q^{\frac{1}{2}})^{-1} U_1^{\frac{1}{2}} g(x) \right\rangle_{\mathbb{C}^n} \right| \leq \sup_{Q} |\nu_Q^{\frac{1}{2}} U_Q^{\frac{1}{2}}|_{\text{op}} \sum_{Q \in S} \left( \int_Q |(\nu_Q^{-1})^{1/2} V^{1/2} f(x)| \right) \left( \chi_Q(x) |(U_Q^{\frac{1}{2}})^{-1} U_1^{\frac{1}{2}} g(x)| \right)
\]

the proof now continues exactly as in the proof of Theorem 5.1. \( \square \)
To prove Corollary 1.16 we first need a few additional facts about scalar weights and Orlicz maximal operators due to Hytönen and Pérez. We say that a weight $w \in A_\infty$ if it satisfies the Fujii-Wilson condition

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w \chi_Q)(x) \, dx < \infty.$$  

(There are several other definitions of the $A_\infty$ condition; see [13]. This definition, which seems to yield the smallest constant, has proved to be the right choice in the study of sharp constant inequalities for CZOs.) In [17] they showed that if $w \in A_\infty$, then it satisfies a sharp reverse Hölder inequality: for any cube $Q$, $w \in RH_s$: i.e.,

$$\left( \int_Q w^s \, dx \right)^{\frac{1}{s}} \leq 2 \int_Q w \, dx,$$

where $s = 1 + \frac{1}{2d+11[w]_{A_\infty}}$.

They also proved a quantitative version of inequality (3.1): in [18] they showed that given a Young function $\Phi$,

$$\|M_\Phi\|_{L^p} \leq c(n) \left( \int_0^\infty \left( \frac{t}{\Phi(t)} \right)^p \, d\Phi(t) \right)^{\frac{1}{p}}.$$  

In particular, if we let $\Phi(t) = t^{r \rho'}, r > 1$, then a straightforward computation shows that

$$\|M_\Phi\|_{L^p} \lesssim (r')^{\frac{1}{p}}.$$  

(5.2)

**Proof of Corollary 1.16.** By the argument in the proof of Theorem 1.14 it is enough to prove this estimate for sparse operators. Fix a dyadic grid $\mathcal{D}$ and a sparse set $S \subset \mathcal{D}$ and let $W$ be a matrix $A_p$ weight. As we noted in the introduction, for every $e \in \mathbb{C}^n$, $|W^{\frac{1}{p}}e|$ is a scalar $A_p$ weight with uniformly bounded constant [15, Corollary 2.2]. Using the Fujii-Wilson condition, we define

$$[W]_{A_p,\infty} = \sup_{e \in \mathbb{C}^n} [W^{\frac{1}{p}}e]^p_{A_\infty}.$$  

By the sharp reverse Hölder inequality, if we let

$$s = 1 + \frac{1}{2d+11[W]_{A_p,\infty}}, \quad r = 1 + \frac{1}{2d+11[W^{\frac{1}{p}}]_{A_p,\infty}},$$

(5.3)

then for every $e \in \mathbb{C}^n$, $|W^{\frac{1}{p}}e|^p \in RH_s$ and $|W^{\frac{1}{p}}e|^{p'} \in RH_r$. 

Define $\Psi(t) = t^{sp}$ and $\Phi(t) = t^{rp'}$. Then $\Psi \in B_{p'}$ and $\Phi \in B_p$. Moreover, we claim that

$$[W, W]_{p, \Phi, \Psi} \lesssim [W]_{A_p}^{\frac{1}{p}}.$$ 

To see this, we argue as in the proof of Proposition 2.2. Let $\{e_j\}_{j=1}^n$ be an orthonormal basis in $\mathbb{C}^n$. Then by (2.5) (with $p = q$ and $U = V = W$), and the reverse Hölder inequality,

$$[W, W]_{p, \Phi, \Psi} \approx \sup_Q |U_Q^\Phi V_Q^\Psi|_{op} \approx \sup_Q \sum_{j=1}^n \left( \int_Q |W^{-\frac{1}{r'}}(x)U_Q^\Phi e_j|^{rp'} dx \right)^{\frac{1}{p'}} \leq 2 \sup_Q \sum_{j=1}^n \left( \int_Q |W^{-\frac{1}{r'}}(x)U_Q^\Phi e_j|^{rp'} dx \right)^{\frac{1}{p'}} \lesssim \sup_Q |U_Q^\Phi V_Q^\Psi|_{op}.$$ 

If we repeat this argument again, exchanging the roles of $U$ and $V$, we get that

$$[W, W]_{p, \Phi, \Psi} \lesssim \sup_Q |U_Q^\Phi V_Q^\Psi|_{op} \lesssim \sup_Q |U_Q^\Phi V_Q^\Psi|_{op} \lesssim [W]_{A_p}^{\frac{1}{p}}. \quad (5.4)$$

Therefore, we can apply Theorem 1.14 with the pair of weights $(W, W)$. A close examination of the proof of this result (i.e., the proof of Theorem 5.1) shows that

$$\|T\|_{L^p(W)} \lesssim [W, W]_{p, \Phi, \Psi} \|M_\Phi\|_{L^p} \|M_\Psi\|_{L^{p'}}.$$ 

But by (5.4) and by (5.2) combined with (5.3) we get

$$[W, W]_{p, \Phi, \Psi} \|M_\Phi\|_{L^p} \|M_\Psi\|_{L^{p'}} \lesssim [W]_{A_p}^{\frac{1}{p}} [W]_{A_p}^{\frac{1}{p}} [W^{\frac{1}{r'}}]_{A_{p, \infty}}^{\frac{1}{p'}} [W^{\frac{1}{r'}}]_{A_{p', \infty}}.$$ 

This gives us the first estimate in Corollary 1.16; the second follows from this one, Lemma 2.4 and the fact that

$$[W]_{A_{p, \infty}} \leq \sup_{e \in \mathbb{C}^n} [W^{\frac{1}{r}} e]^p_{A_p} \leq [W]_{A_p};$$

see [15, Corollary 2.2].

**Remark** 5.3. In [26] they proved that the sparse matrix domination inequality (5.1) holds if $T$ is a Haar shift or a paraproduct. Consequently, Theorem 1.14 and Corollary 1.16 hold for these operators. Additionally, they proved a slightly stronger result when $p = 2$, assuming that a pair of matrix weights $[U, V]$ satisfy the two weight $A_p$ condition, and each of $U$ and $V$ satisfy the appropriate scalar $A_\infty$ condition. We can
immediately extend our proofs to give the analog of this result for all $1 < p < \infty$. Details are left to the interested reader.

6. Proof of Theorems 1.18 and 1.21

For brevity, in this section if $\alpha = 0$ we will write $A_{p,q} = A_{p,q}^0$; if $p = q$ we will write $A_p^\alpha$ or $A_p$ if $\alpha = 0$.

Proof of Theorem 1.18. We first prove the sufficiency of the $A_{\alpha}^{p,q}$ condition. When $p > 1$ we estimate using Hölder’s inequality and (2.6):

$$\|A_Q f\|_{L^q(U)}^q = \int_{\mathbb{R}^d} |U_q^{\frac{1}{p}}(x) A_Q f(x)|^q dx$$

$$= \int_{\mathbb{R}^d} \left| \sum_{Q \in \mathcal{Q}} |Q|^{\frac{\alpha}{p}} \int_Q \chi_Q(x) U_q^{\frac{1}{p}}(x) V_q^{\frac{1}{-p}}(y) V_q^{\frac{1}{p}}(y) f(y) dy \right|^q dx$$

$$\leq \int_{\mathbb{R}^d} \sum_{Q \in \mathcal{Q}} |Q|^{\frac{\alpha}{p}} \chi_Q(x) \left( \int_Q |U_q^{\frac{1}{p}}(x) V_q^{\frac{1}{-p}}(y) |_{L^p}^p dy \right)^{\frac{q}{p}} \left( \int_Q |V_q^{\frac{1}{p}}(y) f(y) |^p dy \right)^{\frac{q}{p}} dx$$

$$= \sum_{Q \in \mathcal{Q}} |Q|^{\frac{\alpha}{p} + 1} \left( \int_Q |U_q^{\frac{1}{p}}(x) |_{L^p}^p dy \right)^{\frac{q}{p}} \left( \int_Q |V_q^{\frac{1}{p}}(y) f(y) |^p dy \right)^{\frac{q}{p}} dx$$

$$\lesssim [U, V]_{A_{p,q}^0}^q \left( \sum_{Q \in \mathcal{Q}} \int_Q |V_q^{\frac{1}{p}}(y) f(y) |^p dy \right)^{\frac{q}{p}}$$

$$\leq [U, V]_{A_{p,q}^0}^q \|f\|_{L^p(V)}^q.$$

When $p = 1$ we can argue as above, except that instead of Hölder’s inequality we use Fubini’s theorem and (2.7).

To prove necessity when $p > 1$, fix a cube $Q$ and let $e \in C^n$ be such that $|e| = 1$. Then, assuming averaging operators are uniformly bounded with norm at most $K$, we have by duality that there exists $g \in L^p(V)$, $\|g\|_{L^p(V)} = 1$, such that

$$|V_q^{\frac{1}{p}} U_q^a e| \approx \left( \int_Q |V_q^{\frac{1}{p}}(y) U_q^a e|^{\frac{1}{p'}} dy \right)^{\frac{1}{p}}$$

$$= |Q|^{\frac{1}{p'}} \|U_q^a e\|_{L^{p'}(V_q^{\frac{1}{p'}})}$$

$$= |Q|^{\frac{1}{p'}} \int_Q \langle U_q^a e, g(x) \rangle dx.$$
\[\begin{align*}
&= |Q|^\frac{1}{p} \left\langle e, U_Q^q \int_Q g(x) \, dx \right\rangle_{C^\alpha} \\
&\leq |Q|^\frac{1}{p} \left| U_Q^q \int_Q g(x) \, dx \right| \\
&\approx |Q|^\frac{1}{p} \left( \int_Q \left| U_q^\frac{1}{q} (y) A_Q g(y) \right|^q \, dy \right)^\frac{1}{q} \\
&= |Q|^{\frac{1}{p} - \frac{1}{q} + \frac{\alpha}{q}} \| A_Q^\alpha g \|_{L^q(U)} \\
&\leq K |Q|^{\frac{1}{p} - \frac{1}{q} + \frac{\alpha}{q}} \| g \|_{L^p(V)} \\
&\leq K |Q|^{\frac{1}{p} - \frac{1}{q} + \frac{\alpha}{q}}.
\end{align*}\]

If we now rearrange terms and take the supremum over all \(Q\) we get that
\[
\sup_Q |Q|^{\frac{1}{p} + \frac{1}{q} - \frac{1}{q'}} |U_Q^q V_Q^{q'}|_{op} = \sup_Q |Q|^{\frac{1}{p} + \frac{1}{q} - \frac{1}{q'}} |V_Q^{q'} U_Q^q|_{op} \lesssim K,
\]
and so \((U, V) \in A^\alpha_{p,q'}\).

When \(p = 1\) we cannot use duality, so we argue as follows. Since \(A_Q\) is linear, given \(f \in L^1(Q)\) we can rewrite our assumption to get

\[
\| A_Q f \|_{L^q(Q)} = \left( \int_Q |Q|^{\frac{1}{p}} \int_Q U_Q^\frac{1}{q} (x) V^{-1}(y) f(y) \, dy \right)^\frac{1}{q} \leq C \| f \|_{L^1(Q)}.
\]

Therefore, given any \(S \subseteq Q\) with \(|S| > 0\), if we let \(f(x) = \chi_S(x)e\), where \(e \in \mathbb{C}^n\) and \(|e| = 1\), then

\[
|S| |Q|^{\frac{1}{p} - \frac{1}{q} + \frac{\alpha}{q}} \left( \int_Q U_Q^\frac{1}{q} (x) \left( \int_S V^{-1}(y) e \, dy \right) \right)^\frac{1}{q} \leq K |S|.
\]

Thus, by the definition of \(V_Q^q\) we get that

\[
|Q|^{\frac{1}{p} - \frac{1}{q} + \frac{\alpha}{q}} \left| U_Q^q \left( \int_S V^{-1}(y) e \, dy \right) \right| \lesssim K.
\]

But then by the Lebesgue differentiation theorem it follows that

\[
|Q|^{\frac{1}{p} + \frac{1}{q} - 1} \operatorname{ess} \sup_{y \in Q} |U_Q^q V^{-1}(y)|_{op} \lesssim K.
\]

By (2.2) it follows that \((U, V) \in A^\alpha_{p,q'}\).

As a corollary to Theorem 1.18 we have the uniform boundedness of convolution operators and the convergence of approximate identities.
Corollary 6.1. Given $1 \leq p < \infty$ and a pair of matrix weights $(U, V)$ in $A_p$, let $\varphi \in C^\infty_c(B(0, 1))$ be a non-negative, radially symmetric and decreasing function with $\|\varphi\|_1 = 1$, and for $t > 0$ let $\varphi_t(x) = t^{-n}\varphi(x/t)$. Then

$$\sup_{t>0} \|\varphi_t * f\|_{L^p(U)} \leq C \|f\|_{L^p(V)}.$$  

Moreover, we have that

$$\lim_{t \to 0} \|\varphi_t * f - f\|_{L^p(U)} = 0.$$  

This was proved in the one weight case in [8, Theorem 4.9]. The proof is essentially the same, bounding the convolution operator by averaging operators and then applying Theorem 1.18. Details are left to the interested reader, except for the following result which is of independent interest.

Recall that if $(u, v)$ is a pair of scalar $A_p$ weights, then it is immediate by the Lebesgue differentiation theorem that $u(x) \leq [u, v]_{A_p}v(x)$ a.e. The following result is the matrix analog.

Proposition 6.2. Given $1 \leq p < \infty$, if $(U, V) \in A_p$, then

$$|U^\frac{1}{p}(x)V^{-\frac{1}{p}}(x)|_{op} \lesssim [U, V]_{A_p}^\frac{1}{p}.$$  

Remark 6.3. In the proof of Corollary 6.1, this is used to prove that the $L^p(U)$ norm of a function is dominated by the $L^p(V)$ norm:

$$\|f\|_{L^p(U)} \leq \left( \int_{\mathbb{R}^d} |U^\frac{1}{p}(x)V^{-\frac{1}{p}}(x)|_{op} |V^\frac{1}{p}(x)f(x)|\,dx \right) \lesssim [U, V]_{A_p}^\frac{1}{p} \|f\|_{L^p(V)}.$$  

Proof. We first consider the case when $p > 1$. Since $U$ is locally integrable, we have for a.e. $x \in \mathbb{R}^d$ that

$$\lim_{m \to \infty} \left| U(x) - \int_{Q_m^x} U(y)\,dy \right|_{op} = \lim_{m \to \infty} \left| U^\frac{1}{p}(x) - \int_{Q_m^x} U^\frac{1}{p}(y)\,dy \right|_{op} = 0,$$

and that the same holds for $V, V^{-\frac{1}{p}}$, and the scalar function $|U|_{op}$; here $\{Q_m^x\}$ is a sequence of nested cubes whose intersection is $\{x\}$ and whose side-length tends to zero. Thus by Hölder’s inequality, for any $e \in \mathbb{C}^n$ we have

$$|U^\frac{1}{p}(x)e|^p = \lim_{m \to \infty} \left| \int_{Q_m^x} U^\frac{1}{p}(y)e\,dy \right|^p \leq \limsup_{m \to \infty} \left( \int_{Q_m^x} |U^\frac{1}{p}(y)e|\,dy \right)^p \leq \limsup_{m \to \infty} \int_{Q_m^x} |U^\frac{1}{p}(y)e|^p\,dy \approx \limsup_{m \to \infty} |U^p_{Q_m^x}e|^p.$$
On the other hand,

\[
\limsup_{m \to \infty} |\mathcal{U}_{Q_m}^p|^p_{op} \approx \limsup_{m \to \infty} \sum_{j=1}^n \int_{Q_m^x} |U_{p}^{\frac{1}{\tau}}(y)e_j|^p \, dy \leq \limsup_{m \to \infty} \int_{Q_m^x} |U(y)|_{op} \, dy = |U(x)|_{op};
\]

in particular, \(\{|\mathcal{U}_{Q_m}^p|^p_{op}\}\) is bounded. Then we can argue as we did above above to get that for any \(e \in \mathbb{C}^n\),

\[
\limsup_{m \to \infty} |V^{-\frac{1}{\tau}}(x)\mathcal{U}_{Q_m}^p e|^p_{\tau'} \lesssim \limsup_{m \to \infty} |\mathcal{V}_{Q_m}^{p'} \mathcal{U}_{Q_m}^p e|^{p'}.
\]

Hence, we get that

\[
|U^{\frac{1}{\tau}}(x)V^{-\frac{1}{\tau}}(x)|_{op} \approx \left( \sum_{j=1}^n |U_{p}^{\frac{1}{\tau}}(x)V^{-\frac{1}{\tau}}(x)e_j|^p \right)^{\frac{1}{p}}
\]

\[
\lesssim \limsup_{m \to \infty} \left( \sum_{j=1}^n |\mathcal{U}_{Q_m}^p V^{-\frac{1}{\tau}}(x)e_j|^p \right)^{\frac{1}{p}}
\]

\[
\approx \limsup_{m \to \infty} |V^{-\frac{1}{\tau}}(x)\mathcal{U}_{Q_m}^p|^p_{op}
\]

\[
\approx \limsup_{m \to \infty} \left( \sum_{j=1}^n |V^{-\frac{1}{\tau}}(x)\mathcal{U}_{Q_m}^p e_j|^p \right)^{\frac{1}{p}}
\]

\[
\lesssim \limsup_{m \to \infty} |\mathcal{V}_{Q_m}^{p'} \mathcal{U}_{Q_m}^p|^p_{op}
\]

\[
\lesssim |U, V|_{\Lambda_p}^{\frac{1}{p}}.
\]

\[\square\]

**Proof of Theorem 1.21.** We first prove (1) implies (2). Given \((U, V) \in A_{\alpha, p, q}^\tau\), we will prove that \(M'_{\alpha, U, V} : L^p \to L^{q, \infty}\). Arguing exactly as we did in Section 3 using Proposition 3.2, it will suffice to fix a dyadic grid \(D\) and prove that \(M'_{\alpha, U, V, D} : L^p \to L^{q, \infty}\), where \(M_{\alpha, U, V, D}\) is defined as in (1.9) but with the supremum restricted to cubes in \(D\).

Fix \(\lambda > 0\) and let \(f \in L^p(\mathbb{R}^d, \mathbb{C}^n)\). Then for any cube \(Q \in D\) we have by (2.6) that

\[
|Q|^{\frac{1}{p}} \int_Q |\mathcal{U}_Q^p V^{-\frac{1}{\tau}}(y)f(y)| \, dy \lesssim |Q|^{\frac{1}{p} - \frac{1}{\tau}} |\mathcal{U}_Q^p \mathcal{V}_Q^{p'}|^p_{op} \|f\|_{L^p} \lesssim |Q|^{-\frac{1}{\tau}} |U, V|_{\Lambda_{p, q}} \|f\|_{L^p}.
\]
The right-hand side tends to 0 as $|Q| \to \infty$, so (see [6, Proposition A.7]) there exists a collection $\{Q_j\}$ of maximal, disjoint cubes in $D$ such that

$$|Q_j|^\frac{n}{q} \int_{Q_j} \mathcal{U}_{Q_j}^\alpha V^{-\frac{1}{p'}}(y) f(y) \, dy > \lambda$$

and

$$\bigcup_j Q_j = \{x : M'_{\alpha,U,V,\mathcal{D}} f(x) > \lambda\}.$$ 

But then we can estimate as follows: by Hölder’s inequality and the definition of $V_{_Q}^{p'}$,

$$|\{x : M'_{\alpha,U,V,\mathcal{D}} f(x) > \lambda\}|$$

$$= \sum_j |Q_j|$$

$$\leq \frac{1}{\lambda^q} \sum_j \left( |Q_j|^{-1 + \frac{q}{p} + \frac{n}{q}} \int_{Q_j} \mathcal{U}_{Q_j}^\alpha V(y)^{-\frac{1}{p}} f(y) \, dy \right)^q$$

$$\leq \frac{1}{\lambda^q} \sum_j \left( |Q_j|^{-1 + \frac{q}{p} + \frac{n}{q}} \right)^q \left( \int_{Q_j} \mathcal{U}_{Q_j}^\alpha V(y)^{-\frac{1}{p}} f(y) \, dy \right)^{\frac{q}{p}} \left( \int_{Q_j} |f(y)|^p \, dy \right)^{\frac{q}{p}}$$

$$\lesssim \frac{1}{\lambda^q} \sum_j \left( |Q_j|^{-1 + \frac{q}{p} + \frac{n}{q}} \right)^q |\mathcal{U}_{Q_j}^\alpha V_{_Q}^{p'},|_{L^q} \left( \int_{Q_j} |f(y)|^p \, dy \right)^{\frac{q}{p}};$$

by (2.6),

$$\leq [U, V]_{A_{p,q}} \lambda^{-q} \sum_j \left( \int_{Q_j} |f(y)|^p \, dy \right)^{\frac{q}{p}}$$

$$\leq [U, V]_{A_{p,q}} \lambda^{-q} \|f\|_{L^p}^q;$$

the last inequality holds since $q \geq p$ (so by convexity we may pull the power outside the sum), and since the cubes $\{Q_j\}$ are disjoint. This completes the proof that (1) implies (2).

The proof that (2) implies (3) is immediate: given a cube $Q$, $B_Q^\alpha f(x) \leq M'_{\alpha,U,V} f(x)$.

Finally, we prove that (3) implies (1). It follows at once from the definition of the $L^{q,\infty}$ norm that for any $e \in \mathbb{C}^n$, $|Q|^{-\frac{n}{q}} \int_{Q} |\chi_Q e|^{\frac{1}{q}} \, d\lambda = |e|$. First suppose that $p > 1$. 

Then using this identity, duality, and (1.7), we have that

\[
\begin{align*}
\sup_Q \sup_{\|f\|_{L^p}} \left\| \chi_Q \frac{U_q^a V^{-\frac{1}{p}}(y)}{f(y)} dy \right\|_{L^{q,\infty}} &= \sup_Q \sup_{\|f\|_{L^p}} \left| Q^{\frac{1}{2}} \int_Q U_q^a V^{-\frac{1}{p}}(y) f(y) dy \right| \\
&= \sup_Q \sup_{\|f\|_{L^p}} \sup_{|e|=1} \left| Q^{\frac{1}{2}} \int_Q \left\langle U_q^a V^{-\frac{1}{p}}(y), e \right\rangle_{C^n} dy \right| \\
&= \sup_Q \sup_{|e|=1} \sup_{\|f\|_{L^p}} \left| Q^{\frac{1}{2}} \int_Q \left\langle f(y), \chi_Q V^{-\frac{1}{p}}(y) U_q^a e \right\rangle_{C^n} dy \right| \\
&= \sup_Q \sup_{|e|=1} \left| Q^{\frac{1}{2}} \int_{\mathbb{R}^d} \left\langle f(y), \chi_Q V^{-\frac{1}{p}}(y) U_q^a e \right\rangle_{C^n} dy \right| \\
&\approx \sup_Q \left| Q^{\frac{1}{2}} \int_{\mathbb{R}^d} \left\langle f(y), \chi_Q V^{-\frac{1}{p}}(y) U_q^a e \right\rangle_{C^n} dy \right| \\
&\approx [U, V]_{A_{p,q}}.
\end{align*}
\]

When \( p = 1 \) the proof is nearly the same, except that instead of using duality to get the \( L^{p'} \) norm, we take the operator norm of the matrices and use (2.7). This completes the proof that (3) implies (1). \( \square \)

### 7. Proof of Theorem 1.23

The proof of Theorem 1.23 is really a corollary of the proof of Theorems 1.10 and 5.1. First, we will show that it will suffice to assume \( |E| < \infty \) and prove (1.11) with the left-hand side replaced by

\[
\frac{1}{u(E)} \int_E |f(x) - f_E|^q u(x) \, dx,
\]

where \( f_E = f_E f(x) \) dx. For by Hölder’s inequality,

\[
\int_E |f(x) - f_E|^q u(x) \, dx \\
\lesssim \int_E |f(x) - f_E|^q u(x) \, dx + \int_E |f - f_E|^q u(x) \, dx \\
= \int_E |f(x) - f_E|^q u(x) \, dx + u(E) \left| \frac{1}{u(E)} \int_E (f(x) - f_E) u(x) \, dx \right|^q
\]

\]
To get (1.11) with $E$ such that $|E| = \infty$, replace $E$ by $E' = E \cap B_R(0)$. Then $E'$ is convex and $v(E')$, $|E'| < \infty$. The desired inequality follows from Fatou’s lemma if we let $R \to \infty$.

Next, recall that for convex sets $E$, we have the following well-known inequality (see [14]): for scalar functions $f \in C^1(E)$ and $x \in E$,

$$|f(x) - f_E| \lesssim \int_E \frac{|\nabla f(y)|}{|x - y|^{d-1}} dy = I_1(\chi_E|\nabla f|)(x).$$

Therefore, it will be enough to prove that given any vector-valued function $g$,

$$\|u^\frac{1}{2} I_1(|V^{-\frac{1}{p}} g|)\|_{L^q} \lesssim \|g\|_{L^p}.$$  \hspace{1cm} (7.1)

For in this case, if we let let $g = \chi_E V^\frac{1}{p} \nabla f$, then combining the above inequalities we get inequality (1.11).

To prove (7.1) we argue as in the proof of Theorems 1.10 and 5.1, so here we only sketch the main ideas. Define the matrix $U$ to be the diagonal matrix $u(x)I_d$, where $I_d$ is the $d \times d$ identity matrix. Let $\mathcal{U}_Q^\phi$ and $\mathcal{V}_Q^\phi$ be the reducing operators associated to $U$ and $V$ as in Section 4. Fix a vector function $g$ and a scalar function $h \in L^{q'}$; without loss of generality we may assume $g$ and $h$ are bounded functions of compact support. By the scalar theory of domination by sparse operators for the fractional integral (see [5]), we have that

$$\left|\left\langle u^\frac{1}{2} I_1(|V^{-\frac{1}{p}} g|), h \right\rangle \right|_{L^2} \lesssim \sum_{t \in \{0, \pm \frac{1}{2}\}^d} \sum_{Q \in S^t} |Q|^{\frac{1}{2}} \int_Q \int_Q |V(y)^{-\frac{1}{p}} g(y)||u(x)^{\frac{1}{2}} h(x)| dx dy,$$

where each $S^t$ is a sparse set contained in the dyadic grid $D^t$ which is defined as in Proposition 3.2. Therefore, we need to fix a sparse set $S$ and show that the inner sum is bounded by $\|g\|_{L^p}\|h\|_{L^{q'}}$.

Let $\{e_j\}$ be any orthonormal basis of $\mathbb{C}^n$. Then

$$\sum_{Q \in S^t} |Q|^{\frac{1}{2}} \int_Q \int_Q |V(y)^{-\frac{1}{p}} g(y)||u(x)^{\frac{1}{2}} h(x)| \; dx \; dy \leq \sum_{Q \in S} |Q|^{\frac{1}{2}} \int_Q \int_Q |(\mathcal{V}_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} g(y)||\mathcal{V}_Q^\phi |_{op} |u(x)^{\frac{1}{2}} h(x)| \; dx \; dy$$

$$\lesssim \sum_{Q \in S} \sum_{j=1}^n |Q|^{1+\frac{1}{2}} \int_Q |(\mathcal{V}_Q^\phi)^{-1} V(y)^{-\frac{1}{p}} g(y)| \; dy \int_Q |\mathcal{V}_Q^\phi U(x)^{\frac{1}{2}} h(x)e_j| \; dx.$$
\[
\leq \sup_Q |Q|^{\frac{d}{p} + \frac{1}{p} - \frac{1}{q}} |\mathcal{V}_Q^\Phi u_Q^\Psi|_{\text{op}} \\
\times \sum_{j=1}^n \sum_{Q \in S} |E_Q| |Q|^{\frac{d}{q} + \frac{1}{p} - \frac{1}{q}} \int_Q (|V_Q^\Phi|^{-\frac{1}{p}} - 1 V_Q^\Phi f(y)) dy \int_Q (U_Q^\Psi)^{-\frac{1}{q}} U_Q^\Psi h(x)e_j dx.
\]

The middle inequality holds since \(u\) and \(h\) are scalars and \(|V_Q^\Phi|_{\text{op}} \approx \sum |V_Q^\Phi e_j|\).

The proof now continues exactly as before. To estimate the supremum in the last inequality, note that by (2.2) it is equivalent to (1.10) which is finite by assumption.

Finally, we use Theorem 1.23 to prove the existence of a weak solution of a degenerate \(p\)-Laplacian equation. In a recent paper [10] it was shown that the existence of a weak solution was equivalent to the existence of a \((p, p)\) Poincaré inequality. For brevity, we refer the reader to [10] for precise definitions of a weak solution, which is technical in the degenerate case.

**Corollary 7.1.** Fix \(1 < p < \infty\) and a bounded, convex, open set \(E \subset \mathbb{R}^d\). Let \(u\) be a scalar weight and \(A\) a matrix weight such that \(|A|_{op}^{\frac{p}{2}} \in L_{\text{loc}}^1(E)\). Suppose that there exist Young functions \(\Phi\) and \(\Psi\), \(\Phi \in B_p\) and \(\Psi \in B_{p'}\), such that

\[
\sup_Q |Q|^{\frac{d}{p} + \frac{1}{p} - \frac{1}{q}} \|u^\frac{1}{p}\|_{\Phi, Q} \|A^{-\frac{1}{2}}\|_{\Phi, Q} < \infty. \tag{7.2}
\]

Then for every \(f \in L^p(u; E)\) there exists a weak solution \(g\) to the degenerate \(p\)-Laplacian Neumann problem

\[
\begin{aligned}
\text{div} \left( |\sqrt{A(x)}\nabla g(x)|^{p-2} A(x) \nabla g(x) \right) &= |f(x)|^{p-2} f(x) u(x) \text{ in } E \\
n \cdot A(x) \nabla u &= 0 \text{ on } \partial E,
\end{aligned}
\tag{7.3}
\]

where \(n\) is the outward unit normal vector of \(\partial E\).

**Remark 7.2.** In the statement of Corollary 7.1 there seems to be an implicit assumption on the regularity of \(\partial E\) so that \(n\) exists. This is not the case, but we refer the reader to [10] for details.

**Proof.** Define the matrix weight \(V\) by \(A^{\frac{1}{2}} = V^{\frac{1}{p}}\). Then (7.2) is equivalent to (1.10). Therefore, by Theorem 1.23 we have the Poincaré inequality

\[
\int_E |f(x) - f_{E,u}|^p u(x) dx \lesssim \int_E |A^{\frac{1}{2}}(x) \nabla f(x)|^p dx.
\]

But by the main result in [10], this is equivalent to the existence of a weak solution to (7.3). \(\Box\)
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