Two congruences involving Andrews-Paule’s broken 3-diamond partitions and 5-diamond partitions

By Xinhua XIONG

Department of Mathematics, China Three Gorges University, 443002 YiChang, P. R. China

(Communicated by Shigefumi MORI, M.J.A., April 12, 2011)

Abstract: In this note, we will prove two congruences involving broken 3-diamond partitions and broken 5-diamond partitions. The two congruences were conjectured by Peter Paule and Silviu Radu in 2009.

Key words: Broken diamond partitions; congruences; modular forms.

1. Introduction. In 2007 George E. Andrews and Peter Paule [1] introduced a new class of combinatorial objects called broken k-diamond partitions. Let \( \Delta_k(n) \) denote the number of broken k-diamond partitions of \( n \), they showed that
\[
\sum_{n=0}^{\infty} \Delta_k(n) q^n = \prod_{n=1}^{\infty} \frac{(1-q^{2n})(1-q^{2k+1+2n})}{(1-q^n)(1-q^{k+2n})}.
\]
In 2008 Song Heng Chan [3] proved an infinite family of congruences when \( k = 2 \). In 2009 Peter Paule and Silviu Radu [10] gave two non-standard infinite families of broken 2-diamond congruences. Moreover they stated four conjectures related to broken 3-diamond partitions and 5-diamond partitions. In this note we show that their first conjecture and the third conjecture are true:

**Theorem 1.1** (Conjecture 3.1 of [10]).
\[
\prod_{n=1}^{\infty} (1-q^n)^4 (1-q^{2n})^6 \equiv 6 \sum_{n=0}^{\infty} \Delta_3(7n+5)q^n \pmod{7}.
\]

**Theorem 1.2** (Conjecture 3.3 of [10]).
\[
E_4(q^2) \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^2 \equiv 8 \sum_{n=0}^{\infty} \Delta_2(11n+6)q^n \pmod{11}.
\]

The techniques in [7,8] are adapted here to prove Theorem 1.1 and Theorem 1.2.

2. Preliminaries. Let \( H \) denote the upper half of the complex plane, for a positive integer \( N \), the congruence subgroup \( \Gamma_0(N) \) of \( SL_2(\mathbb{Z}) \) is defined by
\[
\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, c \equiv 0 \pmod{N} \right\}.
\]
\( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) acts on the upper half of the complex plane by the linear fractional transformation \( \gamma z := \frac{az + b}{cz + d} \). If \( f(z) \) is a function on \( H \), which satisfies \( f(\gamma z) = \chi(d)(cz + d)^k f(z) \), where \( \chi \) is a Dirichlet character modulo \( N \), and \( f(z) \) is holomorphic on \( H \) and meromorphic at all the cusps of \( \Gamma_0(N) \), then we call \( f(z) \) a weakly holomorphic modular form of weight \( k \) with respect to \( \Gamma_0(N) \) and character \( \chi \). Moreover, if \( f(z) \) is holomorphic on \( H \) and at all cusps of \( \Gamma_0(N) \), then we call \( f(z) \) a holomorphic modular form of weight \( k \) with respect to \( \Gamma_0(N) \) and character \( \chi \). The set of all holomorphic modular forms of weight \( k \) with respect to \( \Gamma_0(N) \) and character \( \chi \) is denoted by \( M_k(\Gamma_0(N), \chi) \).

Dedekind’s eta function is defined by \( \eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1-q^n), \) where \( q = e^{2\pi i z} \) and \( \text{Im}(z) > 0 \). A function \( f(z) \) is called an eta-product if it can be written in the form of \( f(z) = \prod_{\delta \in \mathbb{N}} \eta^\delta(\delta z) \), where \( N \) and \( \delta \) are natural numbers and \( r_\delta \) is an integer. The following Proposition 2.1 obtained by Gordon-Hughes [4] and Newman [11] is useful to verify whether an eta-product is a weakly holomorphic modular form.

**Proposition 2.1** ([9], p.18 Thm 1.64). If \( f(z) = \prod_{\delta \in \mathbb{N}} \eta^\delta(\delta z) \) is an eta-product with \( k := \frac{1}{2} \sum_{\delta | N} r_\delta \) satisfying the conditions:
\[
\sum_{\delta | N} \delta r_\delta \equiv 0 \pmod{24}, \quad \sum_{\delta | N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24},
\]
then \( f(z) \) is a weakly holomorphic modular form of weight \( k \) with respect to \( \Gamma_0(N) \) with the character \( \chi \),

\[2000 \text{ Mathematics Subject Classification. Primary 11F33, 11P83.}\]
do: 10.3792/pjaa.87.65
©2011 The Japan Academy
here $\chi$ is defined by $\chi(d) = \frac{(-1)^d}{d}$ and $s$ is defined by $s := \prod_{d \mid N} d^\alpha$.

The following Proposition obtained by Ligozat [6] gives the analytic order of an eta-product at a cusp of $\Gamma_0(N)$.

**Proposition 2.2** ([9], p.18 Thm 1.65). Let $c,d$ and $N$ be positive integers with $d|N$ and $(c,d) = 1$. If $f(z)$ is an eta-product satisfying the conditions in Proposition 2.1 for $N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\operatorname{Ord}_M(f) = \inf\{n \mid a(n) \neq 0 \text{ mod } M\}.$$

The following Proposition obtained by Ligozat [6] gives the analytic order of an eta-product at a cusp of $\Gamma_0(N)$.

**Proposition 2.2** ([9], p.18 Thm 1.65). Let $c,d$ and $N$ be positive integers with $d|N$ and $(c,d) = 1$. If $f(z)$ is an eta-product satisfying the conditions in Proposition 2.1 for $N$, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is

$$\operatorname{Ord}_M(f) = \inf\{n \mid a(n) \neq 0 \text{ mod } M\}.$$

Let $p$ be a prime, and $f(q) = \sum_{n \geq mn_0} a(n) q^n$ be a formal power series, we define $f(q)U_p = \sum_{n \geq mn_0} a(pm)q^n$. If $f(z) \in \mathcal{M}_k(\Gamma_0(N), \chi)$, then $f(z)$ has an expansion at the point $i\infty$ of the form $f(z) = \sum_{n=n_0} a(n) q^n$ where $q = e^{2\pi i z}$ and $\operatorname{Im}(z) > 0$. We call this expansion the Fourier series of $f(z)$. Moreover we define $f(z)U_p$ to be the result of applying $U_p$ to the Fourier series of $f(z)$. When $U_p$ acts on spaces of modular forms and $p|N$, we have

$$U_p : \mathcal{M}_k(\Gamma_0(N), \chi) \to \mathcal{M}_k(\Gamma_0(N), \chi)\text{.}$$

The $U_p$ operator has the property that

$$\left[ \sum_{n=0}^\infty b(n)q^n \right] U_p = \left[ \sum_{n=0}^\infty a(n)q^n \right] \left[ \sum_{n=0}^\infty b(n)q^n \right].$$

In [12] Sturm proved the following criterion to determine whether two modular forms are congruent, this reduces the proof of a conjectured congruence to a finite calculation. In order to state his theorem, we introduce the notion of the $M$-adic order of a formal power series. Let $M$ be a positive integer and $f = \sum_{n \geq N} a(n) q^n$ be a formal power series in the variable $q$ with rational integer coefficients. The $M$-adic order of $f$ is defined by

$$\operatorname{Ord}_M(f) = \inf\{n \mid a(n) \neq 0 \text{ mod } M\}.$$
As before, applying \( q \prod_{n=4}^{\infty} (1 - q^n)^{12} (1 - q^{2n})^6 \) (mod 7).

Where we used the elementary fact

\[
\frac{\eta^2(7z)}{\eta^4(z)} = \prod_{n=1}^{\infty} \frac{(1 - q^n)^2}{(1 - q^{3n})^2} \equiv 1 \quad \text{(mod 7)}.
\]

We note that our Theorem 1.1 is equivalent to the congruence:

\[
q \prod_{n=1}^{\infty} (1 - q^n)^{12} (1 - q^{2n})^6 \equiv q \prod_{n=1}^{\infty} \Delta_3(7n + 5) \eta^n \prod_{n=1}^{\infty} (1 - q^n)^6 \quad \text{(mod 7),}
\]

i.e.

\[
G(z) \equiv 6F(z)|U_7 \quad \text{(mod 7)}.
\]

Using Sturm’s theorem 2.3, it suffices to verify the congruence above holds for the first \( \frac{N}{12} \cdot |SL_2(\mathbb{Z}) : \Gamma_0(88)| + 1 = 25 \) terms, which is easily completed by using Mathematica 6.0. \( \square \)

4. Proof of Theorem 1.2. The proof of Theorem 1.2 is similar. The difference is that we need to construct two eta-products to represent the left hand side of the equation in Theorem 1.2 up to a factor by using Proposition 2.4.

Proof. Define

\[
H(z) := \frac{\eta(2z)\eta^3(11z)}{\eta^3(z)\eta(22z)},
\]

setting \( N = 88 \), we find that \( H(z) \) satisfies the conditions of Proposition 2.1 and \( H(z) \) is holomorphic at all cusps of \( \Gamma_0(88) \) by Proposition 2.2, so \( H(z) \) is in \( \mathcal{M}_2(\Gamma_0(88), \chi) \), where \( \chi(d) = (\frac{d}{2}) \) is a Dirichlet character modulo 88. We note that

\[
H(z) = q^5 \prod_{n=1}^{\infty} \frac{(1 - q^{3n})(1 - q^{11n})^{13}}{(1 - q^n)^3 (1 - q^{22n})},
\]

and

\[
\sum_{n=0}^{\infty} \Delta_5(n) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 - q^{11n})}{(1 - q^n)^3 (1 - q^{22n})}.
\]

As before, applying \( U_{11} \) operator on \( H(z) \), we find that

\[
H(z)|U_{11} = q^5 \prod_{n=1}^{\infty} \frac{(1 - q^{2n})(1 - q^{11n})^{13}}{(1 - q^n)^3 (1 - q^{22n})}|U_{11} = q \sum_{n=0}^{\infty} \Delta_5(n) q^n \prod_{n=1}^{\infty} (1 - q^{11n})^{12}|U_{11}.
\]

We define another two eta-products by

\[
L_1(z) := \frac{\eta^2(2z)\eta^2(11z)}{\eta(2z)\eta^2(11z)} \quad \text{and} \quad L_2(z) := \frac{\eta^2(2z)\eta^2(11z)}{\eta^2(2z)\eta^2(11z)}.
\]

Setting \( N = 88 \), it is easy to verify that both \( L_1(z) \) and \( L_2(z) \) satisfy the conditions in Proposition 2.1 and both are holomorphic at all the cusps of \( \Gamma_0(88) \) by using Proposition 2.2, hence both \( L_1(z) \) and \( L_2(z) \) are in \( \mathcal{M}_2(\Gamma_0(88), \chi) \), where \( \chi(d) = (\frac{d}{2}) \) is a Dirichlet character modulo 88. So \( L(z) := L_1(z) + 2^9 L_2(z) \) is in \( \mathcal{M}_2(\Gamma_0(88), \chi) \). On the other hand,\n
\[
L(z) \equiv 8H(z)|U_{11} \quad \text{(mod 11)}.
\]

We find that Theorem 1.2 is equivalent to the following congruence of modular forms by using the expressions (3) and (4):

\[
L(z) \equiv 8H(z)|U_{11} \quad \text{(mod 11)}.
\]

Using Sturm’s criterion i.e Proposition 2.3, it suffices to verify the congruence above holds for the first \( \frac{N}{12} \cdot |SL_2(\mathbb{Z}) : \Gamma_0(88)| + 1 = 61 \) terms, which is easily completed by using Mathematica 6.0. \( \square \)

References

[1] G. E. Andrews and P. Paule, MacMahon’s partition analysis. XI. Broken diamonds and modular forms, Acta Arith. 126 (2007), no. 3, 281–294.
[2] G. E. Andrews, P. Paule and A. Riese, MacMahon’s partition analysis. VIII. Plane partition diamonds, Adv. in Appl. Math. 27 (2001), no. 2–3, 231–242.

[3] S. H. Chan, Some congruences for Andrews-Paule’s broken 2-diamond partitions, Discrete Math. 308 (2008), no. 23, 5735–5741.

[4] B. Gordon and K. Hughes, Ramanujan congruences for \( q(n) \), in Analytic number theory (Philadelphia, Pa., 1980), 333–359, Lecture Notes in Math., 899, Springer, Berlin.

[5] M. D. Hirschhorn and J. A. Sellers, On recent congruence results of Andrews and Paule for broken \( k \)-diamonds, Bull. Austral. Math. Soc. 75 (2007), no. 1, 121–126.

[6] G. Ligozat, Courbes modulaires de genre 1, 43, Soc. Math. France, Paris, 1975.

[7] J. Lovejoy, Ramanujan-type congruences for three colored Frobenius partitions, J. Number Theory 85 (2000), no. 2, 283–290.

[8] K. Ono, Congruences for Frobenius partitions, J. Number Theory 57 (1996), no. 1, 170–180.

[9] K. Ono, The web of modularity: arithmetic of the coefficients of modular forms and \( q \)-series, CBMS Regional Conference Series in Mathematics, 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[10] P. Paule and S. Radu, Infinite families of strange partition congruences for broken 2-diamonds, Ramanujan J. 23 (2010), no. 1–3, 409–416.

[11] M. Newman, Construction and application of a class of modular functions. II, Proc. London Math. Soc. (3) 9 (1959), 373–387.

[12] J. Sturm, On the congruence of modular forms, in Number theory (New York, 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin.