Loop Gas Model for Open Strings

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The open string with one-dimensional target space is formulated in terms of an SOS, or loop gas, model on a random surface. We solve an integral equation for the loop amplitude with Dirichlet and Neumann boundary conditions imposed on different pieces of its boundary. The result is used to calculate the mean values of order and disorder operators, to construct the string propagator and find its spectrum of excitations. The latter is not sensible neither to the string tension Λ nor to the mass μ of the “quarks” at the ends of the string. As in the case of closed strings, the SOS formulation allows to construct a Feynman diagram technique for the string interaction amplitudes.

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1. Introduction

The open string theories always attracted a considerable attention of the physicists, not only from the point of view of critical strings but also as a possible source of field theoretical applications. For example, the idea to formulate the multicolor QCD as a theory of noninteracting strings (random surfaces) has been fascinating the minds of some theoretical physicists in the 80’s. This string theory should involve both closed strings describing glueballs and open strings describing the $q\bar{q}$ bound states (mesons).

Now, after more than ten years of study, we know how to formulate and solve the simplest theory of random surfaces - the (noncritical) closed bosonic string. In order to go further, an obligatory exercise to do is to extend the solution to the case of open strings.

It is clear that the physics of open strings should be more complicated than that of closed strings, since it depends on the choice of the boundary conditions at the ends of the string. The string amplitudes will depend now on two dimensionful parameters: the string tension $\Lambda$ coupled to the area of the world sheet (the “bulk cosmological constant” in the language of 2d gravity) and the mass $\mu$ at the ends of the string coupled to the length of the boundary of the world surface (the “boundary cosmological constant”).

The open bosonic string is well defined for embedding spaces with effective dimension (“the central charge of the matter fields”) $-\infty < C \leq 1$; otherwise the vacuum would be unstable due to the tachyonic excitation. The field theory of open strings with no embedding ($C = 0$) has been formulated as a random matrix model in [1]. This model was then solved in the double scaling limit in [2] (see also [3]). Further, the theory of $C = 1$ open strings (embedding space $\mathbb{R}$) was considered as a solution of matrix quantum mechanics in [4], [5] and [6]. Open strings with $C = -2$ and $C = 1$ have been also considered in [7].

On the other hand, the noncritical open strings have been studied by means of the Liouville theory [8], [9], [10], [11]. The continuum approach is based on a free field theory (the Liouville potential is treated as a perturbation) and therefore cannot be used to evaluate the full string interaction amplitudes. This approach is sufficient to study the so-called bulk amplitudes which obey the conservation of the Liouville energy.

In this paper we propose a systematic approach to the open noncritical strings with $-\infty < C \leq 1$ which can be used to find the exact string interaction amplitudes. A very convenient framework for this purpose is provided by the loop gas (or SOS-, or heights-) model on a random surface [2], [8], [14], [15], [16]. The target space in this model is the infinite discretized line $\mathbb{Z}$. It is sometimes called loop gas model because the domain walls between the regions of constant height form a configuration of nonintersecting loops on the world sheet. In the case of Dirichlet boundary condition the points along a connected boundary have the same height and therefore the domain walls cannot end at the boundary.

Below we are going to adapt this model for the case of Neumann boundary conditions corresponding to free endpoints of the open string. In this case the domain walls are always orthogonal to the edge of the world sheet. This means that the loops are repulsed from the boundary but domain walls can approach it at right angle. We will solve the loop equation for the amplitude of a disk with a boundary divided into two parts with Dirichlet and Neumann boundary conditions respectively. Knowing this amplitude we can further calculate the open string propagator and the string interactions following the strategy applied in the case of the closed SOS string [15], [16]. The eigenstates diagonalizing the string propagator are different for different choices of the parameters $\Lambda$ and $\mu$ of the open string. However, the diagonalized propagator is universal and is in fact identical to the one of the closed SOS string.

In order to explore the whole range of effective dimensions of the target space $-\infty < C \leq 1$ we will introduce, following the Coulomb gas picture, a distributed background momentum (“electric charge”) proportional to the curvature of the world sheet metric.
The momentum conservation (the electric charge neutrality) is assured by introducing pointlike electric charges at the critical points of the embedding of the world sheet. On the lattice this construction has been elaborated in [16] and [17].

In the present paper we avoid introducing lattice discretization of the world sheet in order to keep closer connection with the continuum theory. A derivation of the basic loop equation for a discretized surface is presented in the Appendix.

2. Definition of the model

2.1. Coulomb gas picture

The dynamical fields in the Polyakov formulation of the string path integral [18] are the position field \( x(\xi) \) and the intrinsic metric \( G_{ab}(\xi), a, b = 1, 2 \) of the world sheet (which we will denote by \( \mathcal{M} \)). We will assume that \( \mathcal{M} \) has the topology of a disk. The boundary \( \partial \mathcal{M} \) is divided into \( 2n \) pieces as is shown in fig. 1 on which we impose alternatively boundary conditions of Dirichlet

\[
\partial_\parallel x(\xi) \equiv t^a(\xi)\partial_a x(\xi) = 0, \quad t_a(\xi) = \text{unit tangent vector} \tag{2.1}
\]

and Neumann

\[
\partial_\perp x(\xi) \equiv n^a(\xi)\partial_a x(\xi) = 0, \quad n_a(\xi) = \text{unit normal vector} \tag{2.2}
\]

We will denote the Dirichlet boundary by \( \partial \mathcal{M}^{(D)} \) and the Neumann boundary by \( \partial \mathcal{M}^{(N)} \). Each kind of boundary consists of \( n \) connected pieces

\[
\partial \mathcal{M}^{(D)} = \partial \mathcal{M}_1^{(D)} + \ldots + \partial \mathcal{M}_n^{(D)}, \\
\partial \mathcal{M}^{(N)} = \partial \mathcal{M}_1^{(N)} + \ldots + \partial \mathcal{M}_n^{(N)}, \\
\partial \mathcal{M} = \partial \mathcal{M}^{(D)} + \partial \mathcal{M}^{(N)} \tag{2.3}
\]

The Dirichlet boundary condition (2.1) is appropriate for the initial and final string states; it describes a boundary which occupies a single point of the embedding space. The Neumann boundary condition means that the flow of energy across the boundary is zero; it should be imposed along the edges of the world sheet representing the endpoints of the open string.

The world sheet with \( n \) pairs of boundaries describes the interaction of \( n \) open strings. The corresponding amplitude will depend on the intrinsic geometry of the world sheet only through the gauge invariant quantities: the total area of the world sheet

\[
A = \int_{\mathcal{M}} dA(\xi); \quad dA(\xi) = d^2\xi \sqrt{\det G(\xi)} \tag{2.4}
\]

and the total lengths of the Dirichlet and Neumann boundaries

\[
\ell_k = \int_{\partial \mathcal{M}_k^{(D)}} d\ell(\xi), \quad \bar{\ell}_k = \int_{\partial \mathcal{M}_k^{(N)}} d\ell(\xi), \quad k = 1, \ldots, n; \quad d\ell(\xi) = t_a(\xi) d\xi^a \tag{2.5}
\]
An effective dimension $C < 1$ can be achieved by introducing a coupling $p_0$ ("distributed electric charge" in the Coulomb gas terminology) between the field $x$ and the intrinsic geometry of the world sheet. The world sheet action then reads

$$A[x, G_{ab}] = A'[x, G_{ab}] + A''[x, G_{ab}]$$

$$A'[x, G_{ab}] = \frac{g}{4\pi} \int_M dA(\xi) G^{ab} \partial_a x(\xi) \partial_b x(\xi)$$

$$A''[x, G_{ab}] = ip_0 \left[ \frac{1}{4\pi} \int_M dA(x) x(\xi) \hat{R}(\xi) + \frac{1}{2\pi} \int_{\partial M} d\ell(x) x(\xi) \hat{K}(\xi) \right]$$

where $\hat{R}(\xi)$ is the intrinsic Gaussian curvature at the point $\xi \in M$ and $\hat{K}(\xi)$ is the geodesic curvature at the point $\xi \in \partial M$. The factor $g$ known as the Coulomb gas coupling constant can be eliminated by rescaling $x$. We fix the normalization of $x$ to have

$$g = 1 + p_0. \quad (2.7)$$

The two curvatures are normalized so that the Gauss-Bonnet formula reads

$$\int_M d^2 \xi \sqrt{\det G} \hat{R}(\xi) + 2 \int_{\partial M} d\ell(x) \hat{K}(\xi) = 4\pi \quad (2.8)$$

The boundary term in $(2.6)$ is introduced in order to be able to satisfy the momentum conservation (the "electric charge neutrality"). It is clear from the Gauss-Bonnet formula $(2.8)$ that the zero mode $x(\xi) = x_0$ of the $x$-field produces only a factor $\exp(-ip_0 x_0)$ and can be neutralized by introducing a background momentum $-p_0$ at some point of the boundary.

Eq. $(2.6)$ defines the standard Coulomb gas description of the $C \leq 1$ strings. In this paper we propose a modified version of the Coulomb gas approach in which the electric charge neutrality is required in a stronger sense. We introduce a system of pointlike electric charges associated with the points where the string picture changes. These are the critical points of the map $M \to \mathbb{R}$

$$dx(\xi) = \partial_a x d\xi^a = 0 \quad (2.9)$$

shown in fig. 2. We distinguish four kinds of critical points $\xi^*$ which will be characterized by a weight $\chi(\xi^*)$ taking values $1, -1, 1/2, -1/2$.

For the critical points in the interior of the world sheet (cases $a, b$) we define

$$\chi(\xi^*) = \text{sgn} \det \| \partial_a \partial_b x(\xi) \|_{\xi=\xi^*}, \quad \xi^* \in M \quad (2.10)$$

For the critical points along the edge of the world sheet (cases $c, d$) we define

$$\chi(\xi^*) = \frac{1}{2} \text{sgn} \partial^2_{\parallel} x(\xi)_{\xi=\xi^*} \quad (2.11)$$

The sum over the weights of all critical points gives the Euler characteristics of the world sheet

$$\sum_{\xi^*} \chi(\xi^*) = \chi \quad (2.12)$$
Therefore, if we associate with each critical point $\xi^*$ a charge $-p_0\chi(\xi^*)$, the electric charge neutrality will be fulfilled. The factor

$$\prod_{\xi^*} e^{-ip_0\chi(\xi^*)}$$

(2.13)

can be taken into account by adding to the standard action (2.6) a second linear term

$$A'' = -p_0 \int_{\mathcal{M}} d^2\xi \: x(\xi)\rho(\xi); \: \rho(\xi) = \sum_{\xi^*} \chi(\xi^*) \delta_{\xi,\xi^*}$$

(2.14)

The density $\rho(\xi)$ can be expressed through the vector field with unit norm $\hat{n}(\xi)$

$$\rho(\xi) = \frac{1}{2\pi} \int_{\mathcal{M}} d^2\xi x(\xi) \varepsilon^{abc} \varepsilon_{cde} \partial_a \hat{n}_c \partial_b \hat{n}_d$$

(2.15)

$$\hat{n}_a(\xi) = \frac{\partial_a x(\xi)}{\sqrt{\partial_a x(\xi) \partial^a x(\xi)}}$$

(2.16)

Consider the functional integral

$$Z(A; \tilde{\ell_i}, \ell_i, x_i, i = 1, 2, ..., n) = \int [dx(\xi)][dG_{ab}(\xi)] e^{-A[x, G_{ab}]}$$

(2.17)

$$A[x, G_{ab}] = A' + A'' + A'''$$

(2.18)

where the integral over intrinsic geometries is restricted to surfaces with fixed area $A$ and lengths $\ell_i$ and $\tilde{\ell_i}$ of the Dirichlet and Neumann boundaries, correspondingly. The integral depends also on the positions $x_i$ of the Dirichlet boundaries in the embedding space. The interaction amplitude of $n$ open strings with momenta $p_1, ..., p_n$ and lengths $\ell_1, ..., \ell_n$ is defined by the Laplace transform

$$v(p_k, \ell_k; k = 1, ..., n) =$$

$$\int_0^\infty dA e^{-AA} \prod_{k=1}^n \int_0^\infty d\tilde{\ell}_k e^{-\mu\tilde{\ell}_k + ip_kx_k} Z(A; \tilde{\ell_i}, \ell_i, x_i, i = 1, 2, ..., n)$$

(2.19)

Here $x_k$ denotes the position of the $k$-th Dirichlet boundary $\partial \mathcal{M}_k^{(D)}$. The string tension $\Lambda$ coupled to the area of the world sheet is called sometimes cosmological constant, since this functional integral can be also considered as the partition function for two-dimensional quantum gravity. Similarly, the mass $\mu$ of the ends of the string can be also interpreted as a boundary cosmological constant since it is coupled to the length of the Neumann boundary.

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1 Strictly speaking, this term is not linear in $x$ because of the charge density $\rho(\xi)$ depending on the embedding $\xi \rightarrow x(\xi)$. Note also that each connected Dirichlet boundary contributes a factor $-p_0$ if it is a closed loop and $-p_0/2$ if it is an open interval.
The amplitude (2.19) is nonzero only if the sum of all momenta is equal to the background momentum \((1 - n/2)p_0\)

\[ p_1 + ... + p_n = (1 - \frac{n}{2})p_0 \]  

(2.20)

It is convenient to introduce the variables \(z_k\) dual to the lengths \(\ell_k\) and consider the Laplace image of (2.19)

\[ \hat{v}(p_1, z_1; ...; p_n, z_n) = \int_0^\infty d\ell_1 ... \int_0^\infty d\ell_n \ e^{-\ell_1 z_1 - ... - \ell_n z_n} v(p_1, \ell_1; ...; p_n, \ell_n) \]  

(2.21)

The presence of the background momentum \(p_0\) diminishes the effective dimension of the embedding space (the conformal anomaly due to the matter field) from one to

\[ C = 1 - 6p_0^2/g = 1 - 6(g - 1)^2/g \]  

(2.22)

and restricts the spectrum of allowed momenta to

\[ p = k \frac{p_0}{2}, \quad k \in \mathbb{Z} \]  

(2.23)

The local operators in the theory are those creating microscopic closed and open strings. Sometimes they are called bulk and boundary operators\[ ]10. The spectrum of the bulk operators can be fixed with self-consistency arguments known as David-Distler-Kawai analysis\[ ]19, based on the assumption that at distances large compared to the cutoff but small compared to the size of the world sheet, the fluctuations of the metric \(G_{ab}\) are described by a gaussian field.

Below we present a sketch of these arguments mainly to help the reader to become familiar with our normalization which is not the standard one used in the string theory.

After introducing a conformal gauge

\[ G_{ab} = e^{2\nu(\xi)} G^0_{ab}(\xi) \]  

(2.24)

where \(G^0_{ab}\) is some fiducial metric, and taking account of the conformal anomaly we arrive at an effective action depending on a two-component gaussian field \((x, \phi)\):

\[ \mathcal{A}_{\text{Liouville}}[x, \phi] = \frac{1}{4\pi} \int d\xi \sqrt{\det \hat{G}^0(\xi)} [g G^{0ab}(\partial_a x(\xi) \partial_b x(\xi) \right.

\left. - \partial_a \phi(\xi) \partial_b \phi(\xi)) + (ip_0 x(\xi) - \epsilon_0 \phi(\xi)) \hat{R}^0(\xi)] \]  

(2.25)

The vertex operator creating a momentum \(p\), dressed by the fluctuations of the metric is\[ ]***

\[ \mathcal{V}_{(p, \epsilon)}(\xi) = e^{i(p - p_0)x(\xi)} e^{-(\epsilon(p) - \epsilon_0)\phi(\xi)} \]  

(2.26)

In particular, the puncture operator \(\mathcal{P} = -\partial/\partial \Lambda\) which marks a point on a surface is represented by

\[ \mathcal{P}(\xi) = e^{-(\epsilon(p_0) - \epsilon_0)\phi(\xi)} = e^{2\nu(\xi)} \]  

(2.27)

\[ ^2 \text{Here we write explicitly the compensating charge } -p_0 \text{ associated with the puncture} \]
The condition of the absence of conformal anomaly yields

\[ C_{\text{tot}} = C_{(x)} + C_{(\phi)} - 26 \equiv 1 - \frac{6\epsilon_0^2}{g} + 1 + 6\epsilon_0^2 - 26 = 0 \Rightarrow \epsilon_0^2 - p_0^2 = 4g \]  

(2.28)

We choose the positive solution thus fixing a positive direction in the \( \phi \)-space

\[ \epsilon_0 = g + 1, \quad p_0 = g - 1 \]  

(2.29)

The condition that the conformal dimension of the operator (2.26) is one

\[ \Delta_x + \Delta_\phi \equiv \frac{p^2 - p_0^2}{4g} - \frac{\epsilon(p)^2 - \epsilon_0^2}{4g} = 1 \]  

(2.30)

combined with (2.28) leads to the relation

\[ \epsilon(p)^2 - p^2 = 0 \]  

(2.31)

which can be interpreted as a mass-shell condition for the 2-momentum \((p, \epsilon)\). All physical operators correspond to positive Liouville energies

\[ \epsilon(p) = |p| \]  

(2.32)

The Liouville charge of the identity operator equals to \(\epsilon_0 - \epsilon(p_0) = 2\nu\) where

\[ \nu = \frac{1}{2}(g + 1 - |g - 1|) \]  

(2.33)

The gravitational dimensions of the vertex operators coherent with the background momentum \(p_0 = |g - 1|\) are

\[ \delta_{rs} = 1 - \frac{\epsilon_0 - \epsilon(p_{rs})}{\epsilon_0 - \epsilon(p_0)} = \frac{|r - gs| - |g - 1|}{g + 1 - |g - 1|} \]  

(2.34)

Finally, the string susceptibility exponent \(\gamma_{\text{str}}\) giving the dimension of the string interaction constant is equal to

\[ \gamma_{\text{str}} = -\frac{\epsilon(p_0)}{\nu} = -\frac{2|g - 1|}{g + 1 + |g - 1|}; \quad \nu(2 - \gamma_{\text{str}}) = \epsilon_0 \]  

(2.35)

The above arguments can be easily generalized to the boundary operators [10]. However, as it has been noticed in [13]-[16], the semiclassical analysis is not always applicable at the boundary.

2.2. Formulation as an SOS model on the world sheet

Let us now try to find a link between this continuous formulation of the path integral and the so-called SOS model in which the \(x\) field is restricted to take only discrete values (heights) \(x/\pi \in \mathbb{Z}\). At large distances the configurations of such field should look as continuous; this is achieved by the condition that the \(x\) field can jump only with a step...
\( \pm \pi \). The domain walls separating the domains on the world surface where \( x \) takes constant value form a pattern of nonintersecting lines. In the case of a surface without boundary all these lines should be closed loops. Across each domain wall the height \( x \) jumps by \( \pi : x \rightarrow x \pm \pi \). The sign can be taken into account by assigning an orientation to the domain wall.

If we consider a Dirichlet boundary, the above picture holds unchanged. Since the height \( x \) is not changing along the boundary, the whole boundary belongs to a single domain. Note however that the loops are allowed to touch the boundary and this should be taken into account when writing the loop equations for the closed string [12].

On the other hand, in presence of a Neumann boundary the above geometric picture changes drastically. The domain walls are not only loops but also lines ending at the boundary (fig. 3). The condition that the normal derivative of the \( x \) field is zero in the vicinity of the boundary means that all these lines meet the boundary at right angle. In addition, the closed loops are not allowed to approach the boundary.

The integration over the \( x \) field can be replaced by a sum over all loop configurations on the world sheet and a subsequent sum over all allowed values of \( x \) in the domains bounded by these loops:

\[
\int \prod_{\xi} dx(\xi) \ldots \rightarrow \sum_{\text{loop configurations}} \sum_{x(\text{domains})} \ldots
\]

(2.36)

Suppose that the integral over the world-sheet intrinsic geometries is regularized, say, by a discretization using planar graphs. Then (2.36) can serve as a microscopic definition of the string path integral.

Let us “derive” the Boltzmann weights of the domain-wall configurations from the continuum action (2.18). We can imagine that the SOS configuration is regularized so that the map \( M \rightarrow \pi \mathbb{Z} \) is obtained as a limit of a smooth map \( M \rightarrow \mathbb{R} \). Then the contribution of the last term in (2.18) comes only from the vicinity of the domain walls. Notice that \( d\phi(\xi) = \hat{n}(\xi) \times \partial_a \hat{n}(\xi) d\xi^a \) is the infinitesimal angle swept by the unit vector \( n_a(\xi) \) along the interval \( d\xi \).

Let us consider an SOS configuration of the field \( x(\xi) \) described by a system of domains and domain walls and evaluate the action (2.18).

The term \( \partial_a x \partial^a x \) in the integrand is just a square of the invariant gradient of the \( x \)-field, which is zero everywhere except of the domain walls, where it is an (infinite) positive constant along the wall. Being integrated over the world surface it yields the total length of the domain walls times a (cut-off dependent) positive factor. Thus its contribution to the action is

\[
\mathcal{A}' = K_0 \int_{\text{domain walls}} d\ell(\xi) = K_0 \ell_{\text{total}}
\]

(2.37)

where \( d\ell(\xi) \) is the length element along a domain wall and \( \ell_{\text{total}} \) is the total length of the domain walls on the world sheet.

Now let us demonstrate that he contribution of the last two terms in (2.6) depends only on the topology of the configuration of domains and domain walls. Let us consider a world sheet with the topology of a disk and a system of domain walls separating the domains \( D_1, D_2, \ldots, D_k, \ldots \) on it. The domain \( D_k \) is bounded by domain walls (loops and open lines) and pieces of the boundary of the world sheet. Two neighbour domains can be separated either by a closed loop or by an open line. For each two domains having a common boundary one of them is surrounded by the other; therefore the system of domains has a tree-like structure. The boundary of each domain \( D_k \) consists of \( c_k \) connected components. The \( c_k - 1 \) internal boundaries are all closed loops. The external boundary is
made out of $n_k$ open lines separated by pieces of the boundary of the world sheet. $n_k = 0$ means that the external boundary is also a closed loop.

Now we can express the contribution of the term in (2.10) proportional to the gaussian curvature in terms of the heights $x_k$ of the domains $D_k$ and the numbers $c_k, n_k$ characterizing the topology of the domain wall configuration. Applying the Gauss-Bonnet formula to each domain domain $D_k$ we find

$$A'' = \frac{i p_0}{2\pi} \int_{\partial M} d\ell(x) x(\xi) \hat{K}(\xi)$$

$$= \frac{i p_0}{4\pi} \int_{M} dA(x) x(\xi) \hat{R}(\xi)$$

$$= \frac{i p_0}{4\pi} \sum_k x_k \int_{D_k} dA(\xi) \hat{R}(\xi)$$

$$= -\frac{i p_0}{4\pi} \sum_k x_k \left[ 2 \left( \int_{\partial D_k} d\ell(\xi) \hat{K}(\xi) + \pi n_k \right) + 4\pi (h_k + c_k - 2) \right]$$

where $\hat{K}(\xi)d\ell(\xi)$ is the infinitesimal angle swept by the normal vector $n_a(\xi)$ along the boundary $\partial D_k$, and $h_k$ denotes the enclosed genus (number of handles) in the domain $D_k$. Since we are considering the topology of the disk, $h_k = 0$. The boundary integral is understood as a sum of the integrals over the smooth pieces of the boundary. The last term on the r.h.s. is due to the most external connected component of the boundary $\partial D_k$ of the domain $D_k$ having $2n_k$ edges with angle $\pi/2$; their contribution to the global geodesic curvature is $\pi n_k$.

Now let us consider the third term $A'''$. After integrating by parts the integrand in (2.15) turns to $dx(x) \wedge d\phi(x)$. It is easy to see that the contribution of each domain wall is $\frac{i p_0}{2\pi} (x_{\text{right}} - x_{\text{left}}) \varphi_{\text{global}}$ where $\varphi_{\text{global}}$ is the angle swept by the normal vector $\hat{n}(\xi)$ along the domain wall (it is equal to the integral of the geodesic curvature). Adding the contributions of all domain walls we find

$$A''' + \frac{i p_0}{2\pi} \int_{\partial M} d\ell(x) x(\xi) \hat{K}(\xi) = \frac{i p_0}{4\pi} \int_{M} d^2\xi \partial_a x(\xi) \varepsilon^{a\beta} \varepsilon^{c\delta} \hat{n}_c \partial_b \hat{n}_d$$

$$= \frac{i p_0}{2\pi} \sum_k \int_{\partial D_k} d\ell(\xi) \hat{K}(\xi)x(\xi)$$

Collecting the three terms (2.37), (2.38) and (2.39) we arrive at the following action for given loop configuration

$$\mathcal{A} = \mathcal{A}' + \mathcal{A}'' + \mathcal{A}''' = K_0 \ell_{\text{tot}} + ip_0 \sum_k x_k [2 - c_k - \frac{1}{2} n_k]$$

(2.40)

Thus the Boltzmann weight $e^\mathcal{A}$ of each domain wall configuration depends only on its topology and the total length of the loops. The action (2.40) can be simplified further by performing the sum over the heights $x_k$ of the domains $D_k$. We have to calculate the sum

$$\Omega = \sum_{x_k} e^{ip_0 \sum_k x_k [2 - c_k - \frac{1}{2} n_k]}$$

(2.41)
The calculation is performed in the same way as in the case of the closed string \cite{12,13}. We will exploit the fact that the system of domain loops and open lines on the disk has a tree-like structure. Let us start with a domain $D_k$ on the top of the tree, i.e., a simply connected one. It is represented by a vertex with a single line (tadpole) of the corresponding graph. Consider first the case when $n_k = 0, c_k = 1$ when the boundary is a closed loop. Then the sum over $x_{\text{inside}} = x_k$ yields

$$\sum_{x_{\text{inside}} = x_{\text{outside}} \pm \pi} e^{ip_0 x_{\text{inside}}} = 2 \cos(\pi p_0) e^{ip_0 x_{\text{outside}}}$$

(2.42)

But $x_{\text{outside}}$ is the $x$ coordinate of the surrounding domain. Therefore the result of the summation is a factor $2 \cos(\pi p_0)$ and a reduction by one of the number of connected boundaries ($c \rightarrow c - 1$) of the surrounding domain. Proceeding in the same way we can eliminate one by one all loops until we arrive at a configuration (a “rainbow diagram”) containing only open lines ending at the boundary of the world surface. Each domain $D_k$ is characterized by the number $n_k$ of the domain walls along its boundary ($c_k = 1$). This configuration has again a structure of a tree and we can sum over $x$ as before starting with the domains on the top of the tree, i.e., those whose boundary is formed by a single line ($c_k = 1, n_k = 1$). The sum over the $x$ coordinate of such domain yields a factor $2 \cos(1/2 \pi p_0)$ and eliminates the term associated with its boundary

$$\sum_{x_{\text{inside}} = x_{\text{outside}} \pm \pi} e^{\frac{1}{2} i p_0 x_{\text{inside}}} = 2 \cos\left(\frac{1}{2} \pi p_0\right) e^{\frac{1}{2} i p_0 x_{\text{outside}}}$$

(2.43)

After repeating this procedure several times we eliminate all domain lines. Thus the sum over the embeddings produces the following weight of each configuration of domain walls

$$\Omega = \left(2 \cos(\pi p_0)\right)^{\# \text{loops}} \left(2 \cos\left(\frac{1}{2} \pi p_0\right)\right)^{\# \text{open lines}}$$

(2.44)

The sum over the last coordinate yields an infinite factor which is the volume of the embedding space.

Summarizing, we arrived at a modified loop gas model on the random surface. Its partition function is a sum over configurations of nonintersecting loops and open lines ending at the boundary

$$Z = \sum_{\text{surfaces}} \sum_{\text{loop configurations}} e^{-2 K_0 \ell_{\text{tot}} \left(2 \cos(\pi p_0/2)\right)^{\# \text{open lines}} \left(2 \cos(\pi p_0)\right)^{\# \text{loops}}}$$

(2.45)

The construction of the generalized loop gas can be made more explicit by discretizing the measure over random surfaces as prescribed in \cite{12}. The only difference is that the curvature is concentrated at the sites of the lattice and the Gauss-Bonnet theorem degenerates to the Euler formula.

3. Loop equations for the open string

In order to exploit the definition (2.43) we have to give a meaning of the functional integral over surfaces. It is convenient to take the two sums in (2.43) in the opposite order:
first to fix the topology of the configuration and the lengths of all lines, and then perform
the sum over the geometries of the connected pieces of the surface (the “windows”). Each
“window” contributes a factor depending only on the length of its boundary. Finally we
integrate over the lengths of the lines and sum over all topologies. This sum can be most
easily performed using equations of Dyson-Schwinger type [12]. Below we will use the
continuum formulation of the Dyson-Schwinger equations proposed in [20].

In order to obtain a closed loop equation we have to consider a disk with only one
pair of Dirichlet and Neumann boundaries with lengths \( \ell \) and \( \tilde{\ell} \). It seems that the only
consistent way to avoid loops touching the Neumann boundary is to have an open line end
at each point. If we are using a lattice regularization, this means that a line is ending at
the middle of each bond forming the Neumann boundary (see Appendix A).

Let \( V(\tilde{\ell}, \ell) \) be the partition function of the disk with such mixed boundary conditions.
It is related to the functional integral (2.17) with \( n = 1 \) by

\[
V(\tilde{\ell}, \ell) = \int_0^\infty dA e^{-\Lambda^2} Z(A; \tilde{\ell}, \ell, x)
\]  

(3.1)

An infinitesimal deformation of the Neumann boundary at its endpoint (one of the points
separating the two boundary conditions) singles out the line starting from this point which
splits the world surface into two pieces. The loop equation follows from the geometrical
decomposition of the disk shown in fig. 4

\[
\frac{\partial}{\partial \tilde{\ell}} V(\tilde{\ell}, \ell) = 2 \cos(\pi p_0/2) \int_0^{\tilde{\ell}} d\ell' \int_0^{\infty} d\ell'' e^{-2K \ell'} V(\tilde{\ell}', \ell') V(\tilde{\ell} - \ell', \ell + \ell')
\]

(3.2)

Eq. (3.2) has a clear geometrical meaning. It sums up the rainbow diagrams with an
additional structure: the space between its lines is occupied by surfaces with loops. Note
that this loop equation determines only the dependence on \( L \); therefore it has to be com-
plemented with another equation specifying the dynamics of closed loops. The missing
information can be supplied by fixing \( W(\ell) = V(0, \ell) \) which is exactly the partition func-
tion of a disk with Dirichlet boundary conditions. It satisfies a loop equation [12] - [16] of
the type

\[
-U' \left( \frac{\partial}{\partial \ell} \right) W(\ell) = \int_0^\ell d\ell' W(\ell')W(\ell - \ell') + 2 \cos(\pi p_0) \int_0^{\infty} d\ell'' e^{-2K \ell'} W(\ell')W(\ell + \ell')
\]

(3.3)

where \( U' \left( \frac{\partial}{\partial \ell} \right) \) is some local (differential) operator describing an infinitesimal deformation
of the boundary of the disk.

As usual, in order to turn the convolution in (3.2) into a product, we introduce the
Laplace transform

\[
\hat{V}(T, t) = \int_0^{\infty} dL \int_0^{\infty} d\ell e^{-T\tilde{\ell} - t\ell} V(\tilde{\ell}, \ell)
\]

(3.4)

\( ^3 \) We denote this amplitude by \( V \) saving the letter \( v \) for the corresponding renormalized
amplitude

\( ^4 \) When \( n = 1 \) the open string amplitude does not depend on the position \( x \) of its only Dirichlet
boundary
and eq. (3.2) turns to

\[
T \dot{V}(T, t) - \dot{W}(t) = \frac{2 \cos(\pi p_0/2)}{2\pi i} \oint_{t - t'} dt' \dot{V}(T, t') \dot{V}(T, 2K_0 - t') \tag{3.5}
\]

where \( T \) plays the rôle of bare mass of the “quarks” at the ends of the open string and

\[
\dot{W}(t) = \int_0^\infty dT \dot{V}(T, t) = \int_0^\infty d\ell e^{-t\ell} W(\ell) \tag{3.6}
\]

The contour integral in (3.4) goes around the singularities of \( \dot{W}(T, t) \) and leaves outside the singularities of \( \dot{W}(T, 2K_0 - t) \). Similarly, the Laplace transform of (3.3) reads

\[
\dot{W}(t) = \frac{1}{2\pi i} \oint \frac{dt'}{t - t'} \dot{W}(t')^2 [U'(t') - 2 \cos(\pi p_0) \dot{W}(2K_0 - t')] \tag{3.7}
\]

All these loop equations can be derived in a rigorous way starting from the lattice version of the model (see Appendix A).

The loop amplitude \( V(\ell, T) \) can be considered as the classical background field in an open string field theory. It satisfies a mean-field type equation which is equivalent to eq. (3.2). This equation is derived by cutting the world sheet along the most internal open lines as shown in fig. 3. In this way the amplitude \( V \) can be expressed as an integral of the product of a \( W \)-amplitude and a number of \( V \) amplitudes

\[
\dot{V}(t, T) = \sum_{n=0}^{\infty} \int_0^\infty d\ell e^{-t\ell} \prod_{k=1}^n \left( \frac{d\ell_k}{T} e^{-2K_0 \ell_k} 2 \cos(\pi p_0/2) V(\ell_k, T) \right) W(\ell + \ell_1 + \ldots + \ell_n) \tag{3.8}
\]

It is known [14] [16] [21] that depending on the explicit form of the operator \( U(\partial/\partial \ell) \) one can achieve different critical regimes at the critical temperature \( K^* \) of the loop gas on the random surface. Here we will consider in details the so-called dense phase corresponding to the simplest choice \( U'(\partial/\partial \ell) = \partial/\partial \ell \). In this phase the loops fill the world surface densely, without leaving space between them. One of the peculiarities of the dense phase is that the fractal dimension of the Dirichlet boundary is larger than one: \( 1/\nu = 1/(1 - |p_0|) = 1/g \). The dilute phase of the loop gas corresponds to (multi)critical potential \( U \) [16] [21]. The potential is tuned so that the area of the world surface not occupied by loops also diverges. The equation for the loop amplitudes is the same for both phases but the scaling of the cosmological constant is different. In the dilute phase the fractal dimension of the Dirichlet boundary is \( 1/\nu = 1 \). In the Coulomb gas picture the dense and dilute phases are related by a duality transformation \( g \to 1/g \).

---

5 By \( V(\ell, T) \) we denote the Laplace image of (3.1) w.r. to \( \ell \); it depends on \( T \) through the factor \( \exp(\# \text{open lines}) \).
4. Solution of the loop equation in the scaling limit

We will follow the method worked out in \cite{12,16} for solving eq. (3.3) directly in the continuum limit, and apply it to our master equation (3.5).

Let us first recall the solution of eq. (3.3). We expect that $W(t)$ has a cut $t_L, t_R$ along the real axis of the $t$-plane (on the first sheet of its Riemann surface). The contour of integration in (3.7) goes around this cut. If we replace in (3.7) $t$ with $2K_0 - t$ the integrand will not change, but the contour of integration will envelop the cut $[2K_0 - t_R, 2K_0 - t_L]$ of the function $W(2K_0 - t)$. Therefore, adding these two equations, we integrate along both contours which form together a contour surrounding all singularities of the integrand. Applying the Cauchy theorem we find the following functional equation for $\hat{W}(t)$ (we consider the simplest differential operator $U'(t) = t$)

$$\hat{W}(t)^2 + \hat{W}(2K_0 - t)^2 + 2\cos(\pi p_0)\hat{W}(t)\hat{W}(2K_0 - t) = t\hat{W}(t) + (2K_0 - t)\hat{W}(2K_0 - t) - 2$$

(4.1)

Taking the imaginary part of (4.1) and knowing that $\text{Im} W(t) \neq 0$ along the cut, we arrive at a linear Cauchy-Riemann problem:

$$\text{Re} \hat{W}(t) + \cos(\pi p_0)\text{Im} \hat{W}(2K_0 - t) = t/2, \quad t \in [t_L, t_R]$$

$$\text{Im} \hat{W}(t) = 0, \quad t \notin [t_L, t_R]$$

(4.2)

If we take eq. (4.1) at the symmetry point $t = K_0$ we obtain

$$\hat{W}(K_0) = \frac{2}{K_0 + \sqrt{K_0^2 - 4(1 + \cos(\pi p_0))}}$$

(4.3)

In the dense phase the temperature $2K_0$ of the loop gas is also the bare cosmological constant since the total length of the loops is equal to the area of the surface. The singularity of eq. (4.3) gives its critical value

$$K_0 \rightarrow K_0^* = 2\sqrt{1 + \cos \pi p_0} = 2\sqrt{2}\sin(\pi g/2)$$

(4.4)

At that point the two cuts touch each other:

$$t_R^* = 2K_0^* - t_L^* = K_0^*$$

(4.5)

In order to explore the vicinity of the critical point we blow up, as usual, the infinitesimal vicinity of the point $t = K_0^*$ by introducing a cutoff parameter $a$ playing the role of elementary length along the boundary

$$t = K_0^* + az, \quad t_R = K_* - aM$$

(4.6)

The parameter $z$ is coupled to the renormalized length of the boundary (a boundary cosmological constant) and $M$ is the renormalized position of the cut. Note that the characteristic length of a loop grows near the critical point as $(Ma)^{-1}$. 

Since the singular part of the loop amplitude behaves for $M = 0$ as
\[ \hat{W}(t) \sim (az)^g, \quad g = 1 - |p_0| \] (4.7)
we define the scaling part of $\hat{W}$ as
\[ \hat{W}(t) = W^* + A_g a^g \hat{w}(z) \] (4.8)
where $W^*$ is the critical value of $\hat{W}$ at $t = K_0^*$, $K_0 = K_0^*$ and $A_g$ is a constant factor depending on the normalization of $\hat{w}$.

Finally, we introduce the renormalized cosmological constant
\[ K_0 = K_0^* + B_g a^{2\nu} \Lambda \] (4.9)
where $B_g$ is an appropriate constant and $\nu$ has the meaning of the inverse fractal dimension of the Dirichlet boundary, if the dimension of the world sheet is assumed to be 2. Since $\Lambda$ is the only parameter in the theory, we expect that $M^{2\nu} \sim \Lambda$. To determine $\nu$ we note that from (4.6)-(4.8) (for $1/2 < g < 1$)
\[ \hat{W}(K_0) = \hat{W}^* + A_g a^g \hat{w}(0) \]
\[ = \hat{W}^* - (2/K_0^*)^{3/2} \sqrt{B_g \Lambda} \] (4.10)
and, since $\hat{w}(0) \sim M^g$, we find $\nu = g$.

The renormalized loop amplitude $\hat{w}(z)$ has a cut $-\infty < z < -M$ and satisfies the following equation which is a direct consequence of (4.2)
\[ \text{Re } \hat{w}(z) - \cos(\pi g) \hat{w}(-z) = 0, \quad z \leq -M \]
\[ \text{Im } \hat{w}(z) = 0, \quad z \geq -M \] (4.11)
If we parametrize $z$ by means of a new variable $\tau$
\[ z = M \cosh \tau \] (4.12)
the reflection $z \to -z \pm i0$ corresponds to $\tau \to \tau \pm i\pi$ and (4.11) is replaced by
\[ [e^{i\pi \partial/\partial \tau} + e^{-i\pi \partial/\partial \tau} - 2 \cos(\pi g)]\hat{w}(z) = 0, \quad g = 1 - |p_0| \] (4.13)
with an evident solution \[ \hat{w}(z) = -M^g \frac{\cosh(g \tau)}{\cos(g \pi/2)} \]
\[ = -\frac{(z + \sqrt{z^2 - M^2})^g + (z + \sqrt{z^2 - M^2})^g}{2 \cos(g \pi/2)} \] (4.14)
\footnote{In this interval $K_0 = K_0^*$ up to terms of higher than linear order in the cutoff $a$}

\footnote{This normalization corresponds to $A_g = 2^{3g/2 - 1/2}(1 - g)^{g - 1} |\sin(\pi g/2)|^{-g - 1}$}
We have normalized the solution so to have

\[ \hat{w}(0) = -M^g \]  

(4.15)

Then by (4.10), with \( B_g = [K_0^2/2]^3 A_g^2 \), the relation between \( M \) and \( \Lambda \) is just

\[ \Lambda = M^{2g}, \quad 0 < g < 1 \]  

(4.16)

The function (4.14) has a cut \([−∞, −M]\) on its physical sheet whereas the cut \([M, ∞]\) appears only on the next sheets.

In the same way we can analyze the dilute phase of the loop gas (\( g > 1 \)). We would obtain the same expression (4.14) for the loop amplitude but the scaling of the cosmological constant will be different: \( \Lambda = M^{2g} [14][16][21] \). The scaling of \( \Lambda \) in both phases of the loop gas is determined by the dimension \( d_D = 1/\nu \) of the Dirichlet boundary, with \( \nu \) given by (2.33)

\[ \Lambda = M^{2\nu}, \quad \nu = \begin{cases} g, & \text{if } g < 1 \\ 1, & \text{if } g > 1 \end{cases} \]  

(4.17)

Let us now consider the loop equation (3.5) for the open string. We choose to work in the dense phase, but all calculations can be easily extended to both phases of the loop gas. Again, after symmetrization w.r.t. the reflection \( t \rightarrow 2K_0 - t \)

\[ T[\hat{V}(T, t) + \hat{V}(T, 2K_0 - t)] = \hat{W}(t) + \hat{W}(2K_0 - t) \]
\[ + 2 \cos(\pi p_0/2) \hat{V}(T, t) \hat{V}(T, 2K_0 - t) \]  

(4.18)

We have assumed that \( \hat{V}(t) \) has the same cut as \( \hat{W}(t) \).

In order to determine the critical value of \( T \) we consider eq. (4.18) at the point \( t = K_0 \) where it becomes algebraic

\[ 2T\hat{V}(T, K_0) = 2\hat{W}(K_0) + 2 \cos(\pi p_0/2)\hat{V}^2(T, K_0) \]  

(4.19)

Using (4.10) and dropping all powers of the cutoff \( a \) higher than \( a^{g/2} \) we write its solution in the vicinity of the critical point as

\[ \hat{V}(T, K_0) = \frac{T - \sqrt{T^2 - 2\hat{W}(K_0)2 \cos(\pi p_0/2)}}{2 \cos(\pi p_0/2)} \]  

(4.20)

\[ \approx \hat{V}^* - a^{g/2} \sqrt{2 \cos(\pi p_0/2)} \sqrt{A_g(\mu + 2\sqrt{\Lambda})} \]

where

\[ \hat{V}^* = T_* = \sqrt{\frac{2W^*}{2 \cos(\pi p_0/2)}} \]  

(4.21)

is the critical value of the open string amplitude and

\[ 2a^g \mu = A_g[T^2 - T_*^2] \]  

(4.22)

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is the renormalized mass at the endpoints of the open string (the parameter coupled to the length of the Dirichlet boundary).

As before, we retain in the scaling limit only the singular part of $\hat{W}(t)$:

\[
\hat{V}(T, t) = \hat{V}^* + a^\alpha \sqrt{\frac{A_g}{2 \cos(\pi p_0/2)}} \hat{v}(\mu, z) \quad (4.23)
\]

Comparing (4.23) and (4.20) we find

\[
\hat{v}(\mu, 0) = -\sqrt{2(\mu + \sqrt{\Lambda})} \quad (4.24)
\]

Then, throwing away the higher order terms we obtain from (4.18) $\alpha = g/2$ and

\[
\hat{w}(z) + \hat{w}(-z) + \hat{v}(\mu, z)\hat{v}(\mu, -z) = 2\mu \quad (4.25)
\]

At $z = 0$ this equation reproduces (4.24).

This equation is compatible with the integral equation (3.8) in the scaling limit. Indeed, introducing the scaling variables according to (4.6) (4.8) and (4.23) we find the integral equation

\[
\hat{v}(z, \mu) = \int_M^{\infty} \frac{dz_1}{\pi(z + z_1)} \frac{\text{Im}w(-z_1)}{v(z_1, \mu)} \quad (4.26)
\]

Taking the imaginary part of (4.26) along the cut we find

\[
\text{Im}\hat{v}(z, \mu) = -\frac{\text{Im}\hat{w}(z)}{\hat{v}(z, \mu)}, \quad z < -M \quad (4.27)
\]

which gives the imaginary part of (4.25).

Exactly the same equation can be obtained for the dilute phase of the loop gas on the world sheet which corresponds to the choice $g > 1$. The only difference is that the cosmological constant is replaced by $\Lambda = M^2$. All further arguments are valid for both regimes.

If we parametrize $z$ by (4.12) and $\mu$ by

\[
\mu = M^g \cosh(g\sigma); \quad \hat{v}(\mu, z) = v(\sigma, \tau) \quad (4.28)
\]

eq. (4.23) becomes

\[
v(\sigma, \tau + i\pi)v(\sigma, \tau) = M^g \left( \frac{\cosh[g(\tau + i\pi)] + \cosh(g\tau)}{\cos(g\pi/2)} + 2\cos(g\sigma) \right) \quad (4.29)
\]

After shifting $\tau$ to $\tau + i\pi/2$, eq. (4.24) becomes

\[
v(\tau + i\pi/2)v(\tau - i\pi/2) = 4M^g \cosh[\frac{g}{2}(\tau + \sigma)] \cosh[\frac{g}{2}(\tau - \sigma)] \quad (4.30)
\]
In the limit $\Lambda = 0, \mu = 0$ the solution of (4.30) is
\[ \hat{v}(z) = (2z)^{g/2} \quad (4.31) \]

For for nonzero $\Lambda$ and $\mu$ but for some particular values of $\sigma$
\[ \sigma = \pm i\pi/2, \pm i\pi/2 \pm i\pi/g \quad (4.32) \]
eq (4.30) has a solution in elementary functions
\[ v(\pm i\pi/2, \tau) = -2M^{g/2} \cosh(g\tau/2) \]
\[ v(\pm i\pi/2 \pm i\pi/g, \tau) = -2M^{g/2} \sinh(g\tau/2) \quad (4.33) \]

In order to solve eq. (4.28) in the general case let us take the logarithm of both sides to obtain a linear equation on
\[ u(\sigma, \tau) = \log v(\sigma, \tau) \quad (4.34) \]
of the form
\[ (e^{i\pi/2} \partial + e^{-i\pi/2} \partial)u(\sigma, \tau) = \log[2 \cosh(g(\tau + \sigma)/2)] + \log[2 \cosh(g(\tau - \sigma)/2)] \quad (4.35) \]
It is easy to solve it by performing a Fourier transform which gives an integral representation for $u(\sigma, \tau)$
\[ u(\sigma, \tau) = u(\tau, \sigma) = f(\tau + \sigma) + f(\tau - \sigma) \quad (4.36) \]
\[ f(\tau) = f(-\tau) = -\frac{1}{4} \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{e^{i\omega\tau}}{\cosh(\pi\omega/2) \sinh(\pi\omega/g)} \quad (4.37) \]
The ambiguity due to the singularity at $\omega = 0$ is lifted by imposing the condition $f(\tau) \to g|\tau|/4$ when $\tau \to \infty$. By deforming the contour of integration and applying the Cauchy theorem we can write the integral (4.37) as the following formal series which makes sense for $\text{Re} \tau$ positive
\[ f(\tau) = \frac{g}{4} \tau + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{e^{-gn\tau}}{\cos(\pi gn/2)} + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \frac{e^{-(2k-1)\tau}}{\sin[(2k-1)\pi/g]} \quad (4.38) \]
When $g$ is rational, this series for $df/d\tau$ can be easily summed up.
Consider for example the case when $g = p/q$ with $p, q$ co-primes and $p$ even. In this case $q$ is automatically odd.
Representing the summation indices in (4.38) as
\[ n = 2qN + \bar{n}, \quad N = 0, 1, 2, \ldots; \bar{n} = 1, 2, \ldots, 2q \]
\[ k = pK + \bar{k}, \quad K = 0, 1, 2, \ldots; \bar{k} = 1, 2, \ldots, p \quad (4.39) \]
we arrive at the following expression
\[ \frac{df(\tau)}{d\tau} = \frac{1}{4 \sinh(p\tau)} \left( g \cosh(p\tau) + g \sum_{\bar{n}=1}^{2q-1} (-)^{\bar{n}-p/2} \frac{\cosh[g(q - \bar{n})\tau]}{\cos[g(q - \bar{n})\pi/2]} \right. 
\left. + 2 \sum_{\bar{k}=1}^{p} (-)^{\bar{k}} \frac{\cosh[(p + 1 - 2\bar{k})\tau]}{\sin[(p + 1 - 2\bar{k})\pi/g]} \right) \quad (4.40) \]
In the simplest case \( g = 2 \) \((p = 2, q = 1)\) this expression reproduces the result

\[
f(\tau) = \log(\cosh \frac{\tau}{2}); \quad \frac{df(\tau)}{d\tau} = \frac{1}{2} \text{th}(\tau/2) \tag{4.41}
\]

which can be easily obtained directly from the integral \([1.37]\).

If \( g = p/q \) with \( p \) odd, some of the coefficients in this series become infinite. This happens when \( ng = 2k - 1 \). It is easy to see that the divergent coefficients appear in pairs and the contribution of each such pair is finite:

\[
\left( -1 \right)^{n-1} \frac{e^{-g n \tau}}{n} + \left( -1 \right)^{k-1} \frac{e^{-(2k-1)\tau}}{2k-1} \sin[(2k-1)\pi/g] \rightarrow \left( -1 \right)^{n+k} \frac{\tau}{n} e^{-(2k-1)\tau} \tag{4.42}
\]

The r.h.s. represents the limit of the l.h.s. when \( g \rightarrow (2k-1)/n \). For example, for \( g = 1 \) we obtain:

\[
f(\tau) = \frac{1}{4} \log(\cosh \tau) + \frac{\tau}{\pi} \arctg e^{-\tau} - \frac{1}{2\pi} \sum_{0}^{\infty} \frac{e^{-(2n+1)\tau}}{(2n+1)^2} \tag{4.43}
\]

This result is already inexpressible in terms of elementary functions, unlike the formula for the derivative \( df/d\tau \).

Let us note that our disk amplitude \( v(\sigma, \tau) \) being represented in the integral form \([4.37]\) is remarkably similar to the \( S \)-matrix of the \( O(n) \)-vector model with \( n = -2 \cos(\pi g) \) on the regular lattice presented in the paper \([23]\). Our \( \tau \)-parameter corresponds to the rapidity parameter in the two-particle \( S \)-matrix. Eq. \([4.31]\) is analogous to the unitarity condition on the \( S \)-matrix. This \( S \)-matrix was first calculated in \([24]\) in terms of an infinite product of gamma functions, which we can use for our amplitude as well. Expanding \( \cosh \frac{\pi \omega}{2} \) in \([4.37]\) in the exponents and performing the integration we obtain:

\[
v(\tau, 0) = e^{2f(\tau)} = e^{\frac{4\tau}{\pi}} \prod_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2} + g \frac{k+3}{4} + \frac{\tau q}{2\pi i}) \Gamma(\frac{1}{2} + g \frac{k+3}{4} - \frac{\tau q}{2\pi i}) \Gamma^{2}(\frac{1}{2} + g \frac{k+1}{4})}{\Gamma(\frac{1}{2} + g \frac{k+1}{4} + \frac{\tau q}{2\pi i}) \Gamma(\frac{1}{2} + g \frac{k+1}{4} - \frac{\tau q}{2\pi i}) \Gamma^{2}(\frac{1}{2} + g \frac{k+3}{4})} \tag{4.44}
\]

The exponential factor in front of the product depends on how we treat the singularity at \( \omega = 0 \) in this integral. It is defined through the asymptotics \([4.31]\) of \( v(\tau) \).

It is not clear whether this coincidence is accidental or it reflects some deep relationship between the \( O(n) \)-vector field in the flat and fluctuating metric of two-dimensional space, respectively. May be the representation of the model in the flat space in terms of the effective Sine-Gordon theory presented in \([23]\) can shed some light on this strange fact.

5. Boundary operators

It is very convenient to regard the loop amplitudes as expectation values of operators creating boundaries on the world sheet. For this purpose we are going to introduce the following notations. Denote by \( \mathcal{O}(\ell) \) the operator creating a closed Dirichlet boundary of length \( \ell \) on the world sheet. The expectation value of this operator is nothing but loop amplitude \([4.14]\)

\[
\hat{w}(z) = -\frac{\partial}{\partial z} \langle \mathcal{O}(z) \rangle, \quad \mathcal{O}(z) = \int_{0}^{\infty} d\ell \ e^{-z\ell} \mathcal{O}(\ell) \tag{5.1}
\]

notice a misprint there: the factor \( 1/k \) was missing there in the integral.
(The derivative comes from the loop amplitude being defined with a marked point on it.)

Similarly, by \( \tilde{O}(\tilde{\ell}) \) we denote the operator creating closed Neumann boundary of length \( \tilde{\ell} \). Its expectation value is the loop amplitude with Neuman boundary condition

\[
\tilde{\mathcal{O}}(\mu) = \int_0^\infty d\tilde{\ell} \ e^{-i\mu \tilde{\ell}} \tilde{\mathcal{O}}(\tilde{\ell})
\] (5.2)

Once a Dirichlet boundary exists, one can define a boundary operator \( \tilde{\mathcal{C}}(\tilde{\ell}) \) creating an open Neumann boundary of length \( \tilde{\ell} \) at some point. In a similar manner we define the operator \( \mathcal{C}(\ell) \) creating open Dirichlet boundary of length \( \ell \) at some point on the Neuman boundary. By construction

\[
\tilde{\mathcal{C}}(\tilde{\ell}) \tilde{\mathcal{O}}(\ell) = \mathcal{C}(\ell) \tilde{\mathcal{O}}(\tilde{\ell})
\] (5.3)

The disk amplitude \( \hat{v}(z, \mu) \) with Dirichlet-Neumann boundary conditions is the expectation value of any of the products \( \tilde{\mathcal{C}}(\tilde{\ell}) \tilde{\mathcal{O}}(\ell) \)

\[
\hat{v}(z, \mu) = \langle \tilde{\mathcal{C}}(\mu) \tilde{\mathcal{O}}(z) \rangle = \langle \tilde{\mathcal{C}}(\mu) \mathcal{O}(z) \rangle
\] (5.4)

Here we used the notations

\[
\mathcal{C}(\ell) = \int_0^\infty d\tilde{\ell} \ e^{-z\tilde{\ell}} \tilde{\mathcal{C}}(\tilde{\ell}), \quad \tilde{\mathcal{C}}(\mu) = \int_0^\infty d\tilde{\ell} \ e^{-\mu \tilde{\ell}} \tilde{\mathcal{C}}(\tilde{\ell})
\] (5.5)

The operator \( \mathcal{C}(\ell) \) (resp. \( \tilde{\mathcal{C}}(\tilde{\ell}) \)) creating open Dirichlet (resp. Neumann) boundary can be expanded as an infinite series of local boundary operators in the same way as the loop operator in the closed string is expanded as a series of operators creating microscopic loops [25]. (The boundary operators in the framework of the Liouville theory have been studied in [10].)

Consider first the limit of large \( \mu \) corresponding to small Neumann boundary and expand \( \hat{v}(\mu, z) \) in negative powers of \( \mu \)

\[
v(\tau, \sigma) = M^{g/2} e^{g\sigma/2} \left[ 1 + \frac{\cosh(g\tau)}{\cos(g\pi/2)} e^{-g\sigma} + \frac{\cosh \tau}{\sin(\pi/g)} e^{-\sigma} + ... \right]
\]

\[
= \sum_{k,n=0}^\infty \tilde{\mathcal{C}}_{k,m}(\Lambda, z) \mu^{\frac{k}{g} - n/g}
\] (5.6)

with

\[
\tilde{\mathcal{C}}_{0,0} = 1, \quad \tilde{\mathcal{C}}_{1,0} = \frac{M^g \cosh(g\tau)}{\cos(g\pi/2)} = \tilde{w}(z), \quad \tilde{\mathcal{C}}_{0,1} = \frac{M \cosh \tau}{\sin(\pi/g)} \sim z, ...
\] (5.7)

The coefficients in (5.6) can be interpreted as expectation values of local boundary operators

\[
\tilde{\mathcal{C}}_{k,m}(\Lambda, z) = \langle \tilde{\mathcal{C}}_{k,m}(\Lambda) \tilde{\mathcal{O}}(z) \rangle
\] (5.8)

The operators \( \mathcal{C}_{0,0}, \tilde{\mathcal{C}}_{0,1}, ... \) are not, strictly speaking, local operators. They are “boundary operators” for the Dirichlet boundary.
so that (5.6) would imply the following expansion of the operator \( \tilde{C}(\mu) \) creating Neuman boundary as an infinite series of local scaling operators

\[
\tilde{C}(\mu) = \sum_{k,n} \tilde{C}_{k,n} \mu^{\frac{1}{2} - k - \frac{n}{g}}
\]

The leading nontrivial coefficient in (5.6) is just the amplitude \( \hat{w}(z) \) of the closed string and the corresponding boundary operator is

\[
\tilde{C}_{1,0} = -\frac{\partial}{\partial z}
\]

Let us define the dimension \( \delta \) of a local boundary operator \( \tilde{C} \) acting at a point of the Dirichlet boundary. The mean value \( \langle \tilde{C} \hat{O}(z) \rangle = F(\Lambda, z, \mu) \) of such an operator is assumed to have the following scaling property

\[
F(\rho^2 \Lambda, \rho^{d_D} z, \rho^{d_N} \mu) = \rho^\alpha F(\Lambda, z, \mu)
\]

where

\[
\alpha = (2 - \gamma_{str}) \chi - d_D (1 - \tilde{\delta})
\]

Note that we measure the dimension of the operator \( C \) in units of dimension of the Dirichlet boundary and not the world sheet. Otherwise we would have an additional factor of \( d_D/2 \). Let us recall that the dimensions of the Dirichlet boundary is \( d_D = 1/\nu \) and the dimension of the Neummann boundary is \( d_N = 1/\tilde{\nu} = g/\nu \) if the dimension of the world sheet is 2. The term \( d_D \) (resp. \( d_N \)) comes from the fact that marking a point on the boundary breaks the cyclic symmetry and produces a factor of length. In the case of the topology of a disk (Euler characteristic \( \chi = 1 \)), eq. (2.35) yields \( \alpha = \frac{g + \delta}{\nu} \) and the dimension of the operator \( \tilde{C} \) is related to \( \tilde{\delta} \) by \( \tilde{\delta} = \alpha - g/\nu \).

Now let us examine the mean values \( \tilde{C}_{k,n} \). It is easy to check that they satisfy the scaling (5.11) with \( \alpha = (kg + n)/\nu \). Therefore the (boundary) dimensions of the corresponding local operators are

\[
\delta_{k,n} = (k - 1)g + n
\]

The dimension of the operator \( \tilde{C}_{1,0} \) is zero, as expected.

Let us consider the opposite limit \( z \to \infty \) corresponding to small Dirichlet boundary. The corresponding expansion of \( \hat{v} \) is obtained from (4.36) and (4.38) with \( \sigma > \tau > 0 \). From (4.36) and (4.38), assuming that \( \tau > \sigma > 0 \), we find

\[
v(\tau, \sigma) = M^{g/2} e^{g\tau/2} [1 + \frac{\cosh(g\sigma)}{\cos(g\pi/2)} e^{-g\tau} + \frac{\cosh \sigma}{\sin(\pi/g)} e^{-\tau} + ...]
\]

\[
= \sum_{k,m=0}^{\infty} C_{k,m}(\Lambda, \mu) \mu^{\frac{1}{2} - k - m - n}
\]

with

\[
C_{0,0} = 1, \quad C_{1,0} = \frac{M^g \cosh(g\sigma)}{\cos(g\pi/2)} \sim \mu, \quad C_{0,1} = \frac{M \cosh \sigma}{\sin(\pi/g)} = \tilde{w}(\mu), ...
\]
Analogously to the previous case we define the dimension $\delta$ of a local operator at some point on the Neumann boundary by the scaling properties of $\langle C \tilde{O}(\mu) \rangle = F(\Lambda, z, \mu)$. In this case the power $\alpha$ in (5.11) is related to the dimension of $C$ by

$$\alpha = (2 - \gamma_{str})\chi - d_N(1 - \delta) = (1 + g\delta)/\nu$$

(5.16)

We consider the coefficients $C_{k,n}$ as mean values of microscopic operators $O_{k,n}$ at the Neumann boundary which implies the expansion

$$C(z) = \sum_{k,n} C_{k,n} z^{g(\frac{1}{2} - k) - n}$$

(5.17)

The coefficient $C_{k,n}$ obeys the scaling (5.11) with $\alpha = (gk + n)/\nu$ and we find by (5.16)

$$\delta_{k,n} = k + \frac{n - 1}{g}$$

(5.18)

The identity operator is $\tilde{O}_{1,0} = -\partial/\partial z$. Its expectation value $C_{0,1}$ gives the loop amplitude with Neumann boundary condition

$$\tilde{w}(\mu) = C_{0,1} = \frac{[\mu + \sqrt{\mu^2 - M^2 g}]^{1/g} + [\mu - \sqrt{\mu^2 - M^2 g}]^{1/g}}{2 \sin(\pi/g)}$$

(5.19)

Let us make the following remark. The duality transformation of the functional integral with the gaussian action (2.6) leads to a similar action but with $g$ replaced by $1/g$ and the Dirichlet and Neumann boundary conditions exchanged. The symmetry of the function $u(\sigma, \tau)$ is a manifestation of this duality symmetry. In this sense the dense ($g < 1$) and the diluted ($g > 1$) phases of the loop gas are dual to each other. In the dense phase the Neumann boundary has the classical dimension $d_N = g/\nu = 1$ while the Dirichlet boundary has anomalous dimension $d_D = 1/\nu = 1/g > 1$. In the dilute phase the Dirichlet boundary has classical dimension $d_D = 1/\nu = 1$ while the Neumann boundary has anomalous dimension $d_N = g/\nu = g > 1$. The loop amplitude with Neumann boundary condition is related to that with Dirichlet boundary condition by $z \leftrightarrow \mu, g \leftrightarrow 1/g$. Therefore the quasiclassical treatment (see, for example [26]) is applicable for the Neumann boundary when $g < 1$ and for the Dirichlet boundary when $g > 1$ but not for both in the same time.

The operators involved in the expansions (5.9) and (5.17) are not the only boundary operators presented in the theory. Each kind of boundary allows its special boundary operators.

Consider first the Dirichlet boundary. Since all points have the same $x$-coordinate, it is kinematically impossible to introduce an order operator exp$(ipx)$. However, we can define a disorder operator $\tilde{X}[m]$ with magnetic charge $m$ representing a discontinuity $\Delta x = m\pi$ at some point of the Dirichlet boundary. Geometrically this operator is represented by a source of $m$ domain lines starting at the same point at the boundary. The expectation value of such an operator can be calculated by decomposing the world sheet along the $m$
\langle \tilde{\chi}[m]C(z)\mathcal{O}(\mu) \rangle = \int_0^\infty d\ell_0 \int_0^\infty d\ell_1 \ldots \int_0^\infty d\ell_{m+1} e^{-\left(\ell_0 + \ell_{m+1}\right)z} v(\ell_0 + \ell_1, \mu) v(\ell_1 + \ell_2, \mu) \ldots v(\ell_m + \ell_{m+1}, \mu) (5.20)

Since \( \hat{v}(z, \mu) \sim z^{g/2} \), the whole integral scales as \( z^{(m+1)g/2 - 1} \). On the other hand, the amplitude with a marked point on the Dirichlet boundary scales as \( \partial \hat{v}(z, \mu) / \partial z \sim z^{g/2 - 1} \). Comparing the two powers we find the dimension of the disorder operator on the boundary

\[ \tilde{\delta}_{[m]} = mg/2 \] (5.21)

With the Neumann boundary the things stay in the opposite way. The disorder operators do not make sense because there is already a discontinuity at each point of the boundary. However, the order operator \( \mathcal{V}(p) \) introducing a factor \( \exp\left[i\left(p - \frac{1}{2}p_0\right)x\right] \) at some point \( \xi \) of the boundary can be defined perfectly well. Going to the Fourier space and distributing the exponential factor among the domain lines crossing the way between the point \( \xi \) and the Dirichlet boundary, we arrive at the following geometrical description of the order operator with electric charge (momentum) \( p \). The expectation value

\[ \hat{v}(p)(z, \mu) = \langle \mathcal{V}(p)C(\mu)\mathcal{O}(z) \rangle \] (5.22)

is equal to the statistical sum for the mixed Dirichlet-Neumann loop amplitude with the fugacity of some of the domain lines modified. Namely, all domain lines surrounding the point \( \xi \) are taken with a factor \( \cos(\pi p)/ \cos(\frac{1}{2}\pi p_0) \). (We remind that the factor of \( \cos(\frac{1}{2}\pi p_0) \) has been absorbed in \( T \))

The loop amplitude (5.22) satisfies the following integral equation

\[ \hat{v}(p)(z, \mu) = \int_M^\infty \frac{dz'}{\pi} \frac{\cos(p)}{\cos(\frac{1}{2}\pi p_0)} \frac{\hat{v}(p)(z, \mu)\text{Im}[\hat{w}(z)\hat{v}(p)(-z, \mu)]}{\hat{v}(z, \mu)^2} \] (5.23)

which can derived in the same way as eq. (3.8). Eq. (5.23) can be considered as the dispersion integral for an analytic function with a cut \( M < z < \infty \). Therefore along the cut we have

\[ \text{Im}\hat{v}(p)(z, \mu) = -\frac{\cos(\pi p)}{\cos(\frac{1}{2}\pi p_0)} \frac{\text{Im}[\hat{w}(z)\hat{v}(p)(-z, \mu)]}{\hat{v}(z, \mu)^2}, \quad z < M \] (5.24)

Inserting the relation

\[ \text{Im}[\hat{v}(z, \mu)] + \frac{\text{Im}[\hat{w}(z)]}{\hat{v}(-z, \mu)} = 0 \] (5.25)
in (5.24) we find
\[ \text{Im}[\hat{v}_p(z, \mu)] = -\frac{\cos(\pi p)}{\cos(\frac{1}{2}\pi p_0)} \text{Im}[\hat{v}(z, \mu)] \hat{v}_p(-z, \mu), \quad z < -M \] (5.26)

For \( z \) large \( v(z, \mu) \sim z^{g/2} \) and therefore
\[ \frac{\text{Im}[\hat{v}(z, \mu)]}{\hat{v}(-z, \mu)} \rightarrow \sin(\frac{1}{2} \pi g) = \cos(\frac{1}{2} \pi p_0), \quad z \rightarrow -\infty \] (5.27)

Therefore at large \( z \) the amplitude (5.22) satisfies the functional equation
\[ \text{Im}[\hat{v}_p(z, \mu)] = \cos(\pi p) \hat{v}_p(-z, \mu) \] (5.28)

whose solution is any power \( z^\alpha \) with \( \alpha = \pm (p - \frac{1}{2}) + \) even integer. The leading power at \( z \rightarrow \infty \) large can be fixed by the requirement that when the momentum \( p \) coincides with the background momentum \( \frac{1}{2}p_0 \), the amplitude (5.22) coincides with the expectation value of the identity operator \(-\partial/\partial \mu\)
\[ \hat{v}(\frac{1}{2}p_0)(z, \mu) = -\frac{\partial}{\partial \mu} \hat{v}(z, \mu) \] (5.29)

Therefore \( \hat{v}_p(z, \mu) \) behaves for \( z \) large as
\[ \hat{v}_p(-z, \mu) \sim z^{\left| p - \frac{1}{2}p_0 \right| - \frac{g}{2}} \] (5.30)

Comparing this with the asymptotics \( \partial \hat{v}/\partial \mu \sim z^{g/2 - g} \) we find
\[ \delta(p) = \frac{|p - p_0/2|}{g} \] (5.31)

For finite \( z \) (5.22) is given by the infinite series
\[ \hat{v}_p(z, \mu) = \sum_{k,n} C^{(p)}_{k,n}(\mu, \Lambda) z^{\left| p - \frac{1}{2}p_0 \right| - g(\frac{1}{2} + k) - n} \] (5.32)

6. Open string propagator and the spectrum of momenta

All amplitudes involving closed and open strings are defined by imposing appropriate Dirichlet and Neumann boundary conditions on the boundaries of a world sheet with given topology.
Any string amplitude can be decomposed into elementary pieces (propagators and vertices) following the logic of refs. [15] and [16].
In this paper we concentrate ourselfs on the calculation of the open string propagator. It will be obtained following the same steps as in the case of the closed string propagator [15]. Contrary to our expectations, the case of open strings turned out to be technically
much more difficult than the case of closed strings. We have found the spectrum of the propagator but we were not able to obtain the explicit form of the eigenstates.

Before considering the open string, let us repeat the major steps of the calculation of the closed string propagator, using the SOS model. One has to calculate a string-string amplitude with the world sheet configuration of the loops as shown in fig. 7, with non-contractable domain walls going around the cylindric surface. In this way we take into account the possibility for the closed string to propagate in the \( x \)-space.

Let \( x \) and \( x' \) be the coordinates of the two (Dirichlet) boundaries of the cylinder. If we pass to the momentum space the factor \( e^{i\pi p(x-x')} \) can be written as a product of factors \( e^{\pm i\pi p} \) associated with the domain walls wrapping the cylinder. Taking into account the two different orientations, each such domain wall acquires a factor \( 2 \cos(\pi p) \). Further, the amplitude of each elementary cylinder between two subsequent noncontractable domain walls is \( [12][16] \)

\[
G_0(\ell, \ell') = \sqrt{\ell} e^{-M(\ell+\ell')} \sqrt{\ell'}
\]  

(6.1)

This amplitude describes the deformation of the closed string from the ”in” state of length \( \ell \) to the ”out” state with a length \( \ell' \), without a change of the \( x \)-space position. The whole propagator \( G(p; \ell, \ell') \) in the momentum space can be obtained by sewing such elementary cylinders:

\[
G(p; \ell, \ell') = \sum_{n=0}^{\infty} \int_0^\infty \ldots \int_0^\infty \frac{d\ell_1}{\ell_1} \ldots \frac{d\ell_n}{\ell_n} [2 \cos(\pi p)]^n G_0(\ell, \ell_1)G_0(\ell_1, \ell_2) \ldots G_0(\ell_n, \ell')
\]  

(6.2)

To calculate it we have to diagonalize \( G_0(\ell, \ell') \) in the \( \ell \)-space. This was done in \([25]\)

\[
G_0(\ell, \ell') = \int_0^\infty dE \langle \ell | E \rangle \frac{1}{2 \cos(\pi E)} \langle E | \ell' \rangle
\]  

(6.3)

where

\[
\langle \ell | E \rangle = \frac{2}{\pi} \sqrt{\pi E \sinh(\pi E)} K_i(E M \ell) \approx (\ell M)^{iE}, \ell \to 0
\]  

(6.4)

form a complete set of delta-function normalized eigenstates

\[
\int_0^\infty \frac{d\ell}{\ell} \langle \ell | E \rangle \langle E' | \ell \rangle = 2\pi \delta(E, E')
\]  

(6.5)

It is convenient to introduce the Liouville variable \( \phi = \log \ell \); then the integration measure becomes uniform:

\[
\frac{d\ell}{\ell} = d\phi; \quad \ell = e^\phi
\]  

(6.6)

The wave functions behave as plane waves in the limit \( \ell \to 0 \) (\( \phi \to -\infty \)) and decay rapidly when \( \phi \sim \log(1/M) \). Therefore the \( \delta \)-function is produced only by the small-\( \ell \) behavior and the normalization coefficient is not affected by the form of the eigenstates for \( \ell \sim 1/M \). This important feature of the half-space quantum mechanics was emphasised and well explained in \([25]\). It can help to calculate directly the spectrum of the kernel \( G_0 \)
from its small-$\ell$ behaviour. Indeed, let us expand the r.h.s. of (6.1) in $\ell/\ell'$, assuming that $\ell < \ell'$. One finds, writing the series as the result of a contour integration

$$G_0(\ell, \ell') = \sum_0^\infty (-)^n \left( \frac{\ell}{\ell'} \right)^{n+1/2} = \int_{-\infty}^{\infty} \frac{dE}{2 \cosh(\pi E)} \left( \frac{\ell}{\ell'} \right)^{iE}$$  \hspace{1cm} (6.7)

Once the irreducible part $G_0$ is diagonalized, the r.h.s. of (6.2) can be evaluated immediately

$$G(p; \ell, \ell') = \int_0^\infty dE \langle \ell|E\rangle \frac{1}{\cosh(\pi E) - \cos(\pi p)} \langle E|\ell'\rangle$$  \hspace{1cm} (6.8)

The quantum number $E$ plays the role of the momentum of an additional "Liouville" dimension.

The propagator (6.8) is universal in the sense that it does not depend on the background momentum: it is the same for any closed string theory with effective dimension (central charge of the matter in the language of 2d gravity) less than 1.

The poles of the propagator define the possible Liouville energies corresponding to given momentum $p$

$$iE_n(p) = \pm p + 2k, \ k \in \mathbb{Z}$$  \hspace{1cm} (6.9)

If we consider a theory with a background momentum $p_0 = |g - 1|$ the allowed momenta are $p = np_0, \ n \in \mathbb{Z}$. For each value $p$ of the momentum together with the lowest energy states $E = \pm p$ (the creation and annihilation operators of a closed string "tachyon") there is an infinite discrete set of states which describe infinitesimal deformations of the boundary at this point. A boundary of finite length can be expanded as an infinite series of such operators.

Now let us calculate the propagator for the open string. A typical configuration of the loops on the world sheet is shown in fig. 8. The generic loop configuration involves three kinds of domain walls: closed loops, lines ending at the same boundary and lines connecting two different boundaries of the world sheet. This last kind of domain walls describes the propagation of the open string in $x$-space. If we denote by $\Gamma_0(\ell, \ell')$ the amplitude of propagation between two consequent domain walls with the lengths $\ell$ and $\ell'$, then the full propagator is given by

$$\Gamma(p; \ell, \ell') = \sum_{n=0}^\infty \int_0^\infty d\ell_1...d\ell_n \left( \frac{2 \cos(\pi p)}{2 \cos(\pi p_0/2)} \right)^n \Gamma_0(\ell, \ell_1)\Gamma_0(\ell_1, \ell_2)...\Gamma_0(\ell_n, \ell')$$  \hspace{1cm} (6.10)

i.e., by the same expression as (6.2), with $G_0$ replaced by $\Gamma_0/2 \cos(\pi p_0/2)$ and the measure $d\ell/\ell$ replaced by $d\ell$. The difference between the two measures of integration is due to the cyclic symmetry of the closed string which is absent for the open string. Now let us calculate $\Gamma_0(\ell, \ell')$. This is the loop amplitude for a disk with a boundary divided into four segments having alternatively Dirichlet and Neumann boundary conditions. Such an amplitude can be decomposed as a convolution of an amplitude with Dirichlet boundary and a number of amplitudes with Dirichlet-Neumann boundaries. The decomposition can be performed by cutting the world surface along the most internal open lines. Summing
over the numbers \( m \) and \( n \) of such lines at the two opposite boundaries we find

\[
\Gamma_0(\ell, \ell') = \sum_{k,m=0}^{\infty} \int_0^\infty \frac{2\cos(\pi p_0/2)}{T} \prod_{i=1}^k d\ell_i e^{-2K_0\ell_i} V(\ell_i) \prod_{j=1}^m d\tilde{\ell}_j e^{-2K_0\tilde{\ell}_j} V(\tilde{\ell}_j) W(\ell + \ell' + \sum_{i=1}^k \ell_i + \sum_{j=1}^m \tilde{\ell}_j)
\]

\[
= \sum_{m,k=0}^{\infty} \int \frac{dt}{2\pi i} e^{t(\ell+\ell')} \tilde{W}(t)[\tilde{W}(2K_0 - t)]^{k+m} \frac{2\cos(\pi p_0/2)}{T} \]

\[
= \int \frac{dt}{2\pi i} e^{t(\ell+\ell')} \frac{\tilde{W}(t)}{[1 - 2\cos(\pi p_0/2) \tilde{W}(2K_0 - t)/T]^2}
\]

where product from 1 to 0 is assumed 1. The contour of integration goes around the cut \([t_L, t_R]\) of \( \tilde{W}_0(t) \), leaving outside the cut \([2K_0 - t_R, 2K_0 - t_L]\) of the denominator in the integrand. Note that \( \Gamma_0 \) depends only on the sum \( \ell + \ell' \) of its arguments.

In the scaling limit we replace the quantities in the integrand by their singular parts according to eqs. (4.8) and (4.23). The regular part in the denominator cancels and we obtain the following scaling limit of the irreducible part of the string propagator

\[
\frac{1}{2\cos(\pi p_0/2)} \Gamma_0(\ell, \ell') = \int_M^\infty dz \ e^{-z(\ell+\ell')} \Gamma_0(z)
\]

(6.11)

\[
\Gamma_0(z) = \frac{1}{2\cos(\pi p_0/2)} \frac{\text{Im} \hat{\omega}(-z)}{[\hat{\nu}(z)]^2} = \frac{1}{2\cos(\pi p_0/2)} \frac{\text{Im} [\hat{\nu}(-z, \mu)]}{\hat{\nu}(z, \mu)} \sim 1, \quad z \to \infty
\]

(6.12)

The asymptotic value at \( z \to \infty \) follows from the large \( z \) asymptotics of the open string background amplitude \( \hat{\nu}(z) \sim z^{g/2} \).

Before proceeding further let us notice that at small lengths \( \Gamma_0(\ell + \ell') \) is identical to its analog (6.1) for the closed string, up to the factor \( \sqrt{\ell'} \)

\[
\frac{1}{2\cos(\pi p_0/2)} \Gamma_0(\ell, \ell') = \frac{e^{-M(\ell + \ell')}}{\ell + \ell'}
\]

(6.13)

(6.14)

The factor \( \sqrt{\ell'} \) will appear if we replace the open string integration measure \( d\ell = e^\phi d\phi \) with \( d\ell/\ell = d\phi \). Hence one can expect that the spectrum of the open string propagator is the same as the one of the closed string, since the spectrum is defined by the small \( l \) asymptotics of the propagator. However, the eigenfunctions will be certainly different.

A rigorous proof of this statement can be done by demonstrating that the traces of all powers of the two propagators are the same.

Let us consider the trace of the open string propagator (6.10) and perform the \( \ell \) integrations using the form (6.12) of \( \Gamma_0 \). The result is

\[
\text{Tr} \Gamma(p) = \int_0^\infty d\ell G(p; \ell, \ell)
\]

\[
= \sum_{n=1}^{\infty} [2\cos(\pi p)]^n \int_M^{1/a} \ldots \int_M^{1/a} dz_1 \ldots dz_n \frac{\Gamma_0(z_1) \ldots \Gamma_0(z_n)}{(z_1 + z_2)(z_2 + z_3) \ldots (z_n + z_1)}
\]

(6.15)
The integral is logarithmically divergent for large $z$ and has to be cut off at $z \sim 1/a$. If we plug the large $z$ expansion
\[
\Gamma_0(z) = 1 + (\text{const.})z^{-g} + \ldots
\] (6.16)
in (6.15), the constant term reduces to exactly the same convolutive integral as in the case of the closed string propagator. This term diverges as $\log(1/a)$. All subdominant terms will produce finite corrections. Therefore
\[
\text{Tr}\Gamma(p) = \lim_{a \to 0} \log\left(\frac{1}{a}\right) \int_0^\infty \frac{dE}{\cosh(\pi E) - \cos(\pi p)} + \text{finite terms}
\] (6.17)

One can repeat the same argument for any integer power of the propagator. This means that the open string propagator can be written in a form similar to (6.8)
\[
\Gamma(p; \ell, \ell') = \int_0^\infty dE \langle \ell|E\rangle_{\Lambda,\mu} \frac{1}{\cosh(\pi E) - \cos(\pi p)} \langle E|\ell'\rangle_{\Lambda,\mu}
\] (6.18)

All dependence on the cosmological constant $\Lambda$ and the mass $\mu$ at the ends of the string is absorbed in the eigenstates of the propagator
\[
\langle E|\ell\rangle_{\Lambda,\mu} = \langle V_i E \tilde{C}(\mu) O(\ell)\rangle_{\Lambda}
\] (6.19)

Of course, the next terms of the asymptotics also contain some universal information about the open string theory, and they will be important for the calculation of the string interactions. Note that the on-mass-shell (microscopic) states $E = \pm ip$ are the order parameter amplitudes (5.22).

Once the propagator is known, the amplitudes involving string interactions can be calculated by decomposing the world sheet into irreducible pieces (vertices and propagators) in the same way as it has been done in the case of the closed string. The dependence on the external momenta is only through the propagators. For the three-string amplitude this is illustrated by fig. 9. The vertices $\Gamma_n(\ell_1, \ell_2, \ldots, \ell_n+2, \tilde{\ell}_1, \ldots, \tilde{\ell}_n+2)$ represent amplitudes for a disk with $n + 2$ pairs of Neumann and Dirichlet boundaries with lengths $\ell_k$ and $\tilde{\ell}_k$, $k = 1, 2, \ldots, n+2$. It is easy to see that these vertices will depend only on the total lengths $\ell = \ell_1 + \ldots + \ell_{n+2}$ and $\tilde{\ell} = \tilde{\ell}_1 + \ldots + \tilde{\ell}_{n+2}$ of the Dirichlet and Neumann boundaries. Therefore the Laplace image of $\Gamma_n$ (we denote it by the same letter) will depend only on two variables $z$ and $\mu$. It is easy to establish the following integral equation which follows from geometrical decomposition of the world sheet shown in fig. 10

\[
\Gamma_n(z, \mu) = \int_M^\infty \frac{\text{Im}\hat{w}(-z)}{[\hat{b}(z, \mu)]^{n+2}}
\] (6.20)

Therefore
\[
\text{Im}\Gamma_n(z, \mu) = \frac{\text{Im}\hat{w}(z)}{[\hat{b}(z, \mu)]^{n+2}} \quad \text{along the cut } z < M
\] (6.21)

In particular, the tadpole vertex ($n = -1$) is the basic open string amplitude
\[
\Gamma_{-1}(z, \mu) = \hat{v}(z, \mu)
\] (6.22)
and the integral representation (6.20) becomes a closed equation which is the continuum limit of (3.7).

The spectrum of excitations of the open string is fixed by the set of allowed target space momenta \( p \). Since the background momentum \( p_0/2 \) of the open string is twice less than that of the closed string, its spectrum will be twice denser

\[
p = \frac{1}{2} np_0, \quad n \in \mathbb{Z}
\]

(6.23)

7. Conclusion

We have demonstrated in this paper an approach to the open string theory which allow us not only to calculate the scaling dimensions of the boundary operators, but also to obtain, in principle, any given amplitude for the open strings in the dimensions less than 1, with arbitrary in- and out- momenta. The propagator of two open strings presented here is the simplest possible example. Technically, this question is not simple, since one has to know the eigenfunctions of this propagator, for which we know only the functional equation.

Another interesting possibility is to try to solve the whole field theory for the open strings, which seems now to be a difficult but not a hopeless task. This might shed some light on the string picture in the multicoulor QCD according to an observation made in [27]. This might be an object of further study.

Finally, let us note that the spectrum of the open string excitations is not related to the value of the parameter \( \mu \) which can be considered as the mass of the “quarks” at the ends of the open string.

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Appendix A.

To give a more precise meaning to the derivation of the eq. (3.2) for the open string amplitude let us use its definition by means of a three-positioned graphs as shown in the fig. [11]. A world sheet of the string amplitude \( V_{\tilde{m},m} \) looks like a \( \phi^3 \) planar Feynman diagram with \( \tilde{m} \) legs occupied by the ends of open lines (thick lines) and \( m \) nonoccupied legs (thin lines) at the boundary. \( \tilde{m} \) and \( m \) are the lengths of the Neumann and Dirichlet boundaries, correspondingly. To every link on the graph occupied by a loop or open line one subscribes a weight \( 1/(2K_0) \).

If we pick up one leg at an edge of the Neumann boundary, the corresponding open line decomposes a graph of the amplitude into two similar ones, only with the different lengths of the Neumann and Dirichlet boundaries, and this geometrically obvious decomposition allows us to write the loop equation in the form:

\[
V_{\tilde{m},m} = \sum_{k=2}^{\tilde{m}} \sum_{p,q=0}^{\infty} W_{k-2,q} W_{\tilde{m}-k,p+m} \frac{(p+q)!}{p!q!} (2K_0)^{-p-q-1}
\]

(A.1)
If we introduce the generating function

\[ \hat{V}(T, t) = \sum_{\tilde{m}, m=0}^{\infty} T^{-\tilde{m}-1} t^{-m-1} V_{\tilde{m}, m} \]  

(A.2)

this equation transforms in the following way

\[
\hat{V}(T, t) - \frac{1}{T} V_0(t) = \sum_{\tilde{m}=2}^{\infty} T^{-\tilde{m}-1} \sum_{m=0}^{\infty} t^{-m-1} \sum_{n=2}^{\infty} \oint \frac{d\sigma}{2\pi i} \oint \frac{d\tau}{2\pi i} V_{n-2}(\sigma) V_{\tilde{m}-n}(\tau) \sum_{p, q=0}^{\infty} \frac{(p+q)!}{p!q!} (2K_0)^{-p-q-1} \sigma^p \tau^q \]  

(A.3)

where \( V_k(t) \) is the amplitude with \( k \) legs on the Neumann boundary (\( k \) is even, of course) as a function of the spectral parameter \( t \) of the Dirichlet boundary. In particular, \( V_0(t) = \hat{W}(t) \) is the amplitude with pure Dirichlet boundary. Performing all the sums and the integration over \( \sigma \) in (A.3), we arrive to the following equation

\[
T \hat{V}(T, t) = \hat{W}(t) + \oint \frac{d\tau}{2\pi i} \frac{1}{t - \tau} \hat{V}(T, \tau) \hat{V}(T, 2K_0 - \tau) 
\]  

(A.4)

which is identical (up to a normalization of \( \hat{V}(T, t) \)) to our main equation (3.5).


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**Figure Captions**

Fig. 1. The geometry of the world sheet with alternative Dirichlet-Neumann boundary conditions. The thick lines represent the Neumann boundaries.

Fig. 2. Critical points of the embedding of the world sheet. a) Creation (annihilation) of a closed string state. b) Splitting (joining) of closed strings. c) Creation (annihilation) of open string state. d) Splitting (joining) of open strings.

Fig. 3. A configuration of domain walls for a world sheet with Dirichlet-Neumann boundary conditions.

Fig. 4. The geometry of the loop equation for the Neumann-Dirichlet disk. The fat line represents the Neumann boundary.

Fig. 5. The geometry of the classical field equation for the open string tadpole.

Fig. 6. The geometrical description of a disorder operator and the corresponding decomposition of the world surface.

Fig. 7. Loop configuration for the closed string propagator.

Fig. 8. A typical configuration of domain walls for the open string propagator.

Fig. 9. Decomposition of the world sheet for the three string amplitude.

Fig. 10. The decomposition of the world sheet producing the integral representation for the vertex $\Gamma_1$.

Fig. 11. Discretized world sheet of the open string amplitude in the loop gas representation.