Bounds for the energy of a complex unit gain graph

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Abstract

A T-gain graph, Φ = (G, ϕ), is a graph in which the function ϕ assigns a unit complex number to each orientation of an edge, and its inverse is assigned to the opposite orientation. The associated adjacency matrix A(Φ) is defined canonically. The energy E(Φ) of a T-gain graph Φ is the sum of the absolute values of all eigenvalues of A(Φ). We study the notion of energy of a vertex of a T-gain graph, and establish bounds for it. For any T-gain graph Φ, we prove that 2τ(G) − 2c(G) ≤ E(Φ) ≤ 2τ(G)√∆(G), where τ(G), c(G) and ∆(G) are the vertex cover number, the number of odd cycles and the largest vertex degree of G, respectively. Furthermore, using the properties of vertex energy, we characterize the classes of T-gain graphs for which E(Φ) = 2τ(G) − 2c(G) holds. Also, we characterize the classes of T-gain graphs for which E(Φ) = 2τ(G)√∆(G) holds. This characterization solves a general version of an open problem. In addition, we establish bounds for the energy in terms of the spectral radius of the associated adjacency matrix.

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1 Introduction

In a simple undirected graph G with vertex set V(G) = \{v_1, \ldots, v_n\} and edge set E(G), if two vertices v_p and v_q are adjacent in G, then we write v_p ∼ v_q, and the edge in between them is denoted by e_{p,q}. The number of vertices adjacent with the vertex v_p, the degree of v_p, is denoted by d(v_p) (or simply d_p). ∆(G) denotes the maximum vertex degree of G. A directed graph (or digraph) X is an order pair (V(X), E(X)), where

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*This paper is dedicated to Professor Ravindra Bhalchandra Bapat on the occasion of his 65th birthday with much admiration.

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\[ V(X) = \{v_1, v_2, \ldots, v_n\} \] is the vertex set and \( E(X) \) is the directed edge set. A directed edge from the vertex \( v_p \) to the vertex \( v_q \) is denoted by \( \overrightarrow{e_{p,q}} \). If \( \overrightarrow{e_{p,q}} \in E(X) \) and \( \overrightarrow{e_{p,q}} \in E(X) \), then the pair \( \{v_p, v_q\} \) is called a digon of \( X \). The Hermitian adjacency matrix \( H \) of a digraph \( X \) is denoted by \( H(X) \) and is defined as follows:

\[
(p,q)\text{th entry of } H(X) = h_{p,q} = \begin{cases} 
1 & \text{if both } \overrightarrow{e_{p,q}} \text{ and } \overrightarrow{e_{q,p}} \in E(X), \\
i & \text{if } \overrightarrow{e_{p,q}} \in E(X) \text{ and } \overrightarrow{e_{q,p}} \notin E(X), \\
-i & \text{if } \overrightarrow{e_{p,q}} \notin E(X) \text{ and } \overrightarrow{e_{q,p}} \in E(X), \\
0 & \text{otherwise.} 
\end{cases}
\]

The Hermitian adjacency matrix can be thought of as the adjacency matrix of a \( \mathbb{T} \)-gain graph with the gains are from the set \( \{1, \pm i\} \). A digraph is said to be an oriented graph if it has no digons. A graph contains both directed and undirected edges is called a mixed graph and it is denoted by \( D_G \), where \( G \) is the underlying simple graph. When we consider Hermitian adjacency matrix, \( H(D_G) \) of a mixed graph \( D_G \), the undirected edges are treated as digons.

From a simple graph \( G \), by orienting each undirected edge \( e_{p,q} \in E(G) \) in two opposite directions, namely \( \overrightarrow{e_{p,q}} \) and \( \overrightarrow{e_{q,p}} \), we get a digraph. Let \( \overrightarrow{E(G)} = \{\overrightarrow{e_{p,q}}, \overrightarrow{e_{q,p}} : e_{p,q} \in E(G)\} \) and \( \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} \). A complex unit gain graph (simply, \( \mathbb{T} \)-gain graph) on a simple graph \( G \) is a pair \((G, \varphi)\), where \( \varphi : \overrightarrow{E(G)} \to \mathbb{T} \) is a mapping such that \( \varphi(\overrightarrow{e_{p,q}}) = \varphi(\overrightarrow{e_{q,p}})^{-1} \). A \( \mathbb{T} \)-gain graph \((G, \varphi)\) is denoted by \( \Phi \). For more details about the \( \mathbb{T} \)-gain graphs, we refer to \([10, 11, 12, 13, 18]\).

The adjacency matrix of \( \Phi \) is the Hermitian matrix \( A(\Phi) = (a_{p,q})_{n \times n} \) defined as follows:

\[
a_{p,q} = \begin{cases}
\varphi(\overrightarrow{e_{p,q}}) & \text{if } v_p \sim v_q, \\
0 & \text{otherwise.}
\end{cases}
\]

Let \( \{\lambda_1, \ldots, \lambda_n\} \) be the spectrum of \( A(\Phi) \) (or the spectrum of \( \Phi \)), and is denoted by \( \text{spec}(\Phi) \). The energy of \( \Phi \), denoted by \( \mathcal{E}(\Phi) \), is defined by \( \sum_{j=1}^{n} |\lambda_j| \).

For a vertex \( v_j \) of \( G \), the energy of the vertex \( v_j \), denoted by \( \mathcal{E}_G(v_j) \), is defined by \( \mathcal{E}_G(v_j) = |A(G)|_{jj} \), where \( |A(G)| = (A(G)A(G)^*)^{\frac{1}{2}} \) and \( |A(G)|_{jj} \) is the \((j,j)\)-th entry of \( |A(G)| \). Then \( \mathcal{E}(G) = \sum_{j=1}^{n} \mathcal{E}_G(v_j) \) \([11]\). In Section 3, we establish bounds for \( \mathcal{E}_\varphi(v_j) \), the vertex energy of a \( \mathbb{T} \)-gain graph, in terms degree of the vertex \( v_j \), and characterize the classes of graphs for which the bounds are sharp. As a consequence of these bounds, we provide a couple of bounds for the energy of a \( \mathbb{T} \)-gain graph in terms of the energy of the underlying graph and the number of vertices of the graph.

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A matching in a graph $G$ is a set of edges of $G$ such that no two edges are incident with the same vertex. The cardinality of a matching with the maximum number of edges is the matching number of $G$, and is denoted by $\mu(G)$. A matching that saturates all the vertices of $G$ is known as a perfect matching of $G$. A vertex cover $U$ of a graph $G$ is a subset of $V(G)$ such that every edge of $G$ is incident with at least one vertex of $U$. The cardinality of a vertex cover with the minimum number of vertices is the vertex cover number of $G$, and is denoted by $\tau(G)$. For any $T$-gain graph $\Phi = (G, \varphi)$, the matching number, and the vertex cover number of $\Phi$ are the matching number and the vertex cover number of the underlying graph $G$, respectively.

In [15], the authors derived a lower bound for the energy of an undirected graph in terms of the vertex cover number and the number of odd cycles.

**Theorem 1.1** ([15, Theorem 4.2]). If $G$ is a graph with the vertex cover number $\tau(G)$ and the number of odd cycle $c(G)$, then $E(G) \geq 2\tau(G) - 2c(G)$. Equality occurs if and only if each component of $G$ is a complete bipartite graph with perfect matching together with some isolated vertices.

In [16], the authors extended Theorem 1.1 for Hermitian adjacency matrices of mixed graphs.

**Theorem 1.2** ([16, Theorem 4.5]). Let $D_G$ be a mixed graph with vertex cover number $\tau(G)$ and number of odd cycles $c(G)$. Then $E_H(D_G) \geq 2\tau(G) - 2c(G)$. Equality occurs if and only if $D_G$ is switching equivalent to its underlying graph $G$, where each component of $G$ is either a complete bipartite graph with equal partition size or isolated vertices.

Further extensions of Theorem 1.1 are discussed in [14, 17].

In Section 4, we obtain lower bounds for $E(\Phi)$ in terms of the gains of fundamental cycles [Theorem 4.1 and Theorem 4.4]. We show that a connected $T$-gain bipartite graph has exactly one positive eigenvalue if and only if it is the balanced complete bipartite graph [Theorem 4.2]. We establish a bound for the energy of a $T$-gain graph in terms of the spectral radius of $\Phi$, and characterize the sharpness of the inequality [Theorem 4.3]. Further, we establish lower bounds for $E(\Phi)$ in terms of the vertex cover number, the number of odd cycles, and the matching number [Theorem 4.7 and Theorem 4.8]. After completion of this work, we learned that Theorem 4.7 has been proved in [8] independently. However, our proof uses the properties of vertex energy of $T$-gain graphs, and hence the proof is different from the proof given in [8].
In [15], the authors established an upper bound of the energy of an undirected graph in terms of the vertex cover number and the largest vertex degree.

**Theorem 1.3.** [15, Theorem 3.1] If $G$ is an undirected graph with vertex cover number $\tau(G)$ and maximum vertex degree $\Delta(G)$, then $\mathcal{E}(G) \leq 2\tau(G)\sqrt{\Delta(G)}$. Equality occurs if and only if $G$ is the disjoint union of $\tau(G)$ copies of $K_{1,\Delta(G)}$ together with some isolated vertices.

In [16], the authors extended this inequality for a mixed graph and proposed the equality part as an open problem.

**Theorem 1.4** ([16, Theorem 4.9]). Let $D_G$ be a mixed graph with vertex cover number $\tau(G)$ and largest vertex degree $\Delta(G)$. Then

$$\mathcal{E}_H(D_G) \leq 2\tau(G)\sqrt{\Delta(G)}. \quad (1)$$

In Section 5, we extend Theorem 1.4 for the $T$-gain graphs [Theorem 5.1].

**Problem 1.1** ([16 Problem 4.1]). Characterize all mixed graphs which make the equality in (1) hold.

We solve this problem for the $T$-gain graphs [Theorem 5.2]. The Hermitian adjacency matrices of mixed graphs are particular cases of the adjacency matrices of the $T$-gain graphs. Also, in a recent manuscript [8], the author mentioned the difficulties in extending Theorem 1.4 and characterizing the graphs for which equality hold for the $T$-gain graphs.

This article is organized as follows: In Section 2 we collect needed known definitions and results. In Section 3 we extend the notion of vertex energy for $T$-gain graphs, and establish some of the properties. In Section 4 we establish various lower bounds for the energy of $T$-gain graphs, and Section 5 is devoted to upper bounds for the energy of $T$-gain graphs.

## 2 Definitions, notation and preliminary results

In this section we recall some of the needed graph theory and linear algebra terminologies and some of the basic results. A subgraph $H$ of a graph $G$ is an **induced subgraph** if two vertices of $H$ are adjacent in $G$, then they are adjacent in $H$. For an induced subgraph $H$ of $G$ the **complement** of $H$ in $G$, denoted by $G - H$, defined as the induced subgraph of $G$ with vertex set $V(G) \setminus V(H)$. The subgraphs $H$ and $G - H$ are called **complementary induced subgraphs** in $G$. If $E$ is any edge set of $G$, then $G - E$ denotes the spanning subgraph of $G$ with edge set $E(G) \setminus E$ and vertex set $V(G)$. A **cut** of a graph $G$ is a partition of the vertex set $V(G)$ into
two sets $U$ and $W$. A cut set of $G$ is a set of edges $\{e_{p,q} \in E(G) : v_p \in U, v_q \in W\}$, where $U$ and $V$ partition the vertex set $V(G)$. Suppose $E$ is a cut set, then there are two induced subgraphs $H$ and $G - H$ complement to each other such that each edge of $E$ is incident to a vertex of $H$ and to another vertex of $G - H$ [3]. Then we denote $H \oplus (G - H) = G - E$.

Let $e_{p,q} \in E(G)$. To avoid confusion, we denote $G - \lfloor e_{p,q} \rfloor$ as an induced subgraph of $G$ whose vertex set is $V(G) \setminus \{v_p, v_q\}$. If $K$ is a spanning subgraph of $G$, then for any edge $e \in E(G) \setminus E(K)$, $K + e$ denotes a spanning subgraph of $G$ with the edge set $E(K) \cup \{e\}$. If $G$ is a connected graph and $T$ is a spanning tree of $G$, then any edge $e \in E(G) \setminus E(T)$ induces a unique cycle in $T + e$. This is called a fundamental cycle in $G$ with respect to $T$.

The adjacency matrix of a simple graph $G$, denoted by $A(G)$, is the symmetric $n \times n$ matrix whose $(p,q)$th entry is defined by $a_{p,q} = 1$ if $v_p \sim v_q$, and $a_{p,q} = 0$ otherwise. The energy of the graph $G$, denoted by $\mathcal{E}(G)$, is the sum of the absolute values of the eigenvalues of $A(G)$.

**Lemma 2.1 ([3, Theorem 3.6]).** Let $L$ and $M$ be two complementary induced subgraph of a graph $G$ and $E$ be the cut set in between them. If $E$ is not empty and all edges in $E$ are incident to one and only one vertex in $M$, then $\mathcal{E}(G - E) < \mathcal{E}(G)$.

Let $\Phi = (G, \varphi)$ be any $T$-gain graph, and $H$ be a subgraph of $G$. We call $(H, \xi)$ a subgraph of $\Phi$ if the function $\xi$ is the restriction of $\varphi$ on $\overline{E(H)}$, and is denoted by $(H, \varphi)$ (instead of $(H, \xi)$). If $H$ is an induced subgraph of $G$ and $E$ is any edge set of $G$, then similar to undirected graphs we can define $\Phi - H$ and $\Phi - E$.

The adjacency matrix of $\Phi = (G, \varphi)$, denoted by $A(\Phi)$, is defined as the Hermitian matrix whose $(p,q)$-th element is $\varphi(e_{p,q})$ if $v_p \sim v_q$ and, zero otherwise. The spectrum of $\Phi$, denoted by $\text{spec}(\Phi)$, is the spectrum of $A(\Phi)$. The spectral radius of $\Phi$ is denoted by $\rho(\Phi)$. The energy of $\Phi$, denoted by $\mathcal{E}(\Phi)$, is defined as $\mathcal{E}(\Phi) = \sum_{j=1}^{n} |\lambda_j|$, where $\lambda_j$ are the eigenvalues of $\Phi$. Two $T$-gain graphs $\Phi = (G, \varphi)$ and $\Phi' = (G, \varphi')$ are switching equivalent if there exists a unitary diagonal matrix $U$ such that $A(\Phi') = UA(\Phi)U^*$. If $\Phi$ and $\Phi'$ are switching equivalent, then it is denoted by $\Phi \sim \Phi'$.

A directed cycle is called an oriented cycle if all of its edges are directed such that each edge is traversed in the same direction. An undirected cycle of $k$ vertices $C \equiv v_1 - v_2 - \ldots - v_k - v_1$ has two oriented cycles. If one of the orientation, say $v_1 \rightarrow v_2 \rightarrow \ldots v_k \rightarrow v_1$, is denoted by $\overrightarrow{C}$, then opposite oriented cycle is denoted by $\overleftarrow{C}$. The gain of an oriented cycle $\overrightarrow{C}$ is defined as $\varphi(\overrightarrow{C}) = \varphi(e_{1,2})\varphi(e_{2,3})\cdots\varphi(e_{k,1})$. Therefore, $\varphi(\overrightarrow{C}) = \{\varphi(\overrightarrow{C})\}^{-1}$. For any complex number $\lambda$, $\text{Re}(\lambda)$ denotes the real part of $\lambda$. If $\varphi(\overrightarrow{C}) = \varphi(\overrightarrow{C}^*) = 1$, then we simply
write \( \varphi(C) = 1 \). Similarly, for any cycle \( C \), \( \text{Re}(\varphi(\vec{C})) = \text{Re}(\varphi(\vec{C}^*)) \). Thus, we simple write \( \text{Re}(\varphi(\vec{C})) \).

A \( T \)-gain graph \( \Phi = (G, \varphi) \) is called \textit{balanced} if \( \varphi(\vec{C}) = 1 \), for any cycle \( C \) in \( G \). If \( \Phi \) is balanced, then \( \Phi \sim (G, 1) \). Some of the properties of \( T \)-gain graphs are collected in the next couple of results.

**Theorem 2.1** (\cite{10} Lemma 4.1, Theorem 4.4). Let \( \Phi = (G, \varphi) \) be any \( T \)-gain graph on a connected graph \( G \). Then \( \rho(\Phi) \leq \rho(G) \). Equality occur if and only if either \( \Phi \) or \( -\Phi \) is balanced.

**Theorem 2.2** (\cite{10} Theorem 4.5). Let \( G \) be a connected graph. Then we have the following:

1. If \( G \) is bipartite, then whenever \( \Phi \) is balanced implies \( -\Phi \) is balanced.

2. If \( \Phi \) is balanced implies \( -\Phi \) is balanced for some gain, then \( G \) is bipartite.

**Lemma 2.2** (\cite{13} Corollary 3.2). Let \( \Phi_1 = (G, \varphi_1) \) and \( \Phi_2 = (G, \varphi_2) \) be two \( T \)-gain graphs on a connected graph \( G \) with \( n \) vertices and \( m \) edges. Let \( \{C_1, C_2, \ldots, C_{m-n+1}\} \) be the fundamental cycles of \( G \) with respect to a normal spanning tree of \( G \). Then \( \Phi_1 \sim \Phi_2 \) if and only if \( \varphi_1(\vec{C}_j) = \varphi_2(\vec{C}_j) \), for all \( j = 1, 2, \ldots, (m-n+1) \).

Let \( C_n \) denote the cycle on \( n \) vertices.

**Theorem 2.3** (\cite{11} Theorem 6.1). Let \( \Phi = (C_n, \varphi) \) be a \( T \)-gain graph with \( \varphi(\vec{C}_n) = e^{\theta i} \). Then

\[
\text{spec}(\Phi) = \left\{ 2 \cos \left( \frac{\theta + 2\pi j}{n} \right) : j = 0, 1, \ldots, (n-1) \right\}.
\]

**Lemma 2.3** (\cite{6} Theorem 1.13). Let \( \Phi = (G, \varphi) \) be any connected \( T \)-gain graph. Then

\[
2 \max_{V_0} \mu(G - V_0) \leq r(G, \varphi) \leq 2\mu(G) + b(G),
\]

where \( V_0 \) is any proper subset of \( V(G) \) such that \( G - V_0 \) is acyclic and \( b(G) \) is the minimum integer \( |U| \) such that \( G - U \) is bipartite, \( U \subset V(G) \).

In \cite{1}, the authors studied the notion of vertex energy of a graph.

**Definition 2.1** (\cite{1} Definition 2.1). Let \( G \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Then the energy of a vertex \( v_j \), denoted by \( E_G(v_j) \), is defined as 

\[
E_G(v_j) = |A(G)|_{jj}, \text{ where } |A(G)| = (A(G)A(G)^*)^{1/2}.
\]
Next, we recall a few results related to the vertex energy.

**Lemma 2.4** ([1, Lemma 2.2]). Let $G$ be an undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then
\[
\mathcal{E}_G(v_i) = \sum_{j=1}^n Q_{ij} |\lambda_j|, \text{ for } i = 1, 2, \ldots, n. \tag{3}
\]
where $Q_{ij} = q_{ij}^2$ and $Q = (q_{ij})$ is the orthogonal matrix whose columns are the eigenvectors of $G$ and $\lambda_j$ is the $j$-th eigenvalue of $G$.

**Lemma 2.5** ([1, Theorem 3.3]). If $G$ is a connected graph on $n$ vertices with at least one edge, then
\[
\mathcal{E}_G(v_j) \geq d_j \Delta(G), \text{ for all } v_j \in V(G). \tag{4}
\]
Equality occurs if and only if $G$ is a complete bipartite graph with equal partition size.

Let $G$ and $G_1$ be two simple graphs. Let $D_G$ be a mixed graph on $G$. The mixed Kronecker product, denoted by $D_G \otimes G_1$, is the Kronecker product of the Hermitian adjacency matrix of $G$ and the adjacency matrix of the simple graph $G_1$ [16].

**Lemma 2.6** ([16, Lemma 2.7]). Let $\{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ be the spectrum of $G_1$, and $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be the spectrum of $D_G$ (with respect to the Hermitian adjacency matrix), then the spectrum of a mixed Kronecker product $D_G \otimes G_1$ is $\{\lambda_i \gamma_j : 1 \leq i \leq s, 1 \leq j \leq t\}$

The Hermitian energy of a mixed graph $D_G$ is the sum of the absolute values of the eigenvalues of $H(D_G)$, and is denoted by $\mathcal{E}_H(D_G)$.

Let us collect a few results on energy in terms of matching number.

**Lemma 2.7** ([15, Lemma 4.1]). For any bipartite graph $G$, $\mathcal{E}(G) \geq 2\mu(G)$. Equality occurs if and only if each component of $G$ is complete bipartite graph with perfect matching together with some isolated vertices.

**Theorem 2.4** ([17, Theorem 1.1]). Let $G$ be a graph with matching number $\mu(G)$. Then $\mathcal{E}(G) \geq 2\mu(G)$. If all cycles (if any) of $G$ are pairwise vertex disjoint, then equality holds if and only if each component of $G$ is either an edge or 4-cycle or an isolated vertices.

**Theorem 2.5** ([16, Theorem1.1, Theorem 1.2]). Let $D_G$ be a mixed graph with matching number $\mu(G)$, then $\mathcal{E}_H(D_G) \geq 2\mu(G)$. Equality occurs if and only if $D_G$ is switching equivalent to its underlying graph $G$, where each component of $G$ is either a complete bipartite graph with equal partition size or isolated vertices.
Lemma 2.8 ([16] Lemma 3.8). Let $D_G$ be a mixed graph on a connected non bipartite graph $G$. Then $\mathcal{E}_H(D_G) > 2\mu(G)$.

Lemma 2.9 ([16] Lemma 3.6). Let $D_G$ be a mixed graph without isolated vertices. If $\mathcal{E}_H(D_G) = 2\mu(G)$, then $G$ has a perfect matching.

A graph $G$ is bipartite graph if its vertex set $V(G)$ can be partitioned into two sets, $X$ and $Y$ such that every edge of $G$ joins a vertex of $X$ with a vertex of $Y$. If every vertex in $X$ is adjacent to every vertex in $Y$, then the graph $G$ is called a complete bipartite graph. If $G$ is a complete bipartite graph with $|X| = p$ and $|Y| = q$, then $G$ is denoted by $K_{p,q}$. For instance, $K_{p,p}$ is a complete bipartite graph with a perfect matching. A graph $G$ is called an $r$-regular graph (or regular graph) if every vertex of $G$ has the same degree $r$. A graph $G$ is called a semiregular bipartite graph with parameter $(n_a, n_b, r_a, r_b)$ if $G$ is a bipartite graph with $|X| = n_a$ and $|Y| = n_b$ such that all the vertices of $X$ have the same degree $r_a$, and the vertices of $Y$ have the same degree $r_b$.

Theorem 2.6 ([5] Theorem 3]). If $G$ is a $d$-regular graph of $n$ vertices, then $\mathcal{E}(G) \geq n$. Equality holds if and only if each component is isomorphic to $K_{d,d}$.

Let $G$ be a semiregular bipartite graph with partition size $n_a$ and $n_b$, and the vertex degree of each vertex of first and second partition is $r_a$ and $r_b$, respectively. The next result provides a bound of $\mathcal{E}(G)$.

Theorem 2.7 ([5] Theorem 5]). If $G$ is a semiregular graph with the parameter $(n_a, n_b, r_a, r_b)$. Then $\mathcal{E}(G) \geq n_a \sqrt{\frac{r_a}{r_b}} + n_b \sqrt{\frac{r_b}{r_a}}$ and equality occur if and only if every component of $G$ is $K_{r_a,r_b}$.

For an $n \times n$ complex square matrix $A$, trace($A$) denotes the trace of the matrix $A$. The next result is known as the von Neumann’s trace theorem.

Theorem 2.8 ([7]). Let $A$ and $B$ be two square complex matrices with singular values $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ and $\lambda_1(B) \geq \lambda_2(B) \geq \cdots \geq \lambda_n(B)$, respectively. Then

$$\text{Re}(\text{trace}(AB)) \leq \sum_{j=1}^{n} \lambda_j(A) \lambda_j(B). \quad (5)$$

Theorem 2.9 ([3] Corollary 2.4]). If $A = \begin{bmatrix} B & X \\ Y & C \end{bmatrix}$ is any partition matrix with $A$ and $B$ are the square matrices, then $\mathcal{E}(A) \geq \mathcal{E}(B)$. Equality occurs if and only if $X, Y$ and $C$ are all zero matrices.
Theorem 2.10 (Theorem 2.2). Let \( A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \) be a complex block matrix such that both the diagonal blocks are square matrices. Then \( \mathcal{E}(A_{11}) + \mathcal{E}(A_{22}) \leq \mathcal{E}(A) \), where \( \mathcal{E}(A) \) is the sum of the singular values of \( A \). Equality occurs if and only if there exist unitary matrices \( U \) and \( V \) such that \( \begin{bmatrix} UA_{11} & UA_{12} \\ VA_{21} & VA_{22} \end{bmatrix} \) is positive semidefinite.

Theorem 2.11. Let \( C, C_1 \) and \( C_2 \) be three square complex matrices of order \( n \) such that \( C = C_1 + C_2 \). If \( S_j(\cdot) \) is the \( j \)-th singular value of corresponding matrix, then \( \sum_p S_p(C) \leq \sum_p S_p(C_1) + \sum_p S_p(C_2) \).

3 Energy of a vertex of \( \mathbb{T} \)-gain graphs

The energy of a vertex in an undirected graph is studied in [1]. In this section, first we extend this notion for the \( \mathbb{T} \)-gain graphs, and establish some of the properties.

Definition 3.1. The energy of a vertex \( v_i \) of a \( \mathbb{T} \)-gain graph \( \Phi \) is denoted by \( \mathcal{E}_\Phi(v_i) \) and is defined by

\[
\mathcal{E}_\Phi(v_i) = |A(\Phi)|_{ii}, \quad \text{for } i = 1, 2, \ldots, n,
\]

where \( |A(\Phi)|_{ii} \) is the \((i, i)\)-th entry of \((A(\Phi)A(\Phi)^*)^\frac{1}{2}\).

It is easy to see that, the energy of a \( \mathbb{T} \)-gain graph can be expressed as the sum of the energies of vertices of \( \Phi \). That is,

\[
\mathcal{E}(\Phi) = \mathcal{E}_\Phi(v_1) + \mathcal{E}_\Phi(v_2) + \cdots + \mathcal{E}_\Phi(v_n). \tag{6}
\]

Energy of a vertex of a \( \mathbb{T} \)-gain graph can be obtained from the eigenvalues and the eigenvectors of \( \Phi \). This is done in the next Lemma, and this result is an extension of Lemma 2.4 for the \( \mathbb{T} \)-gain graphs.

Lemma 3.1. Let \( \Phi \) be a \( \mathbb{T} \)-gain graph with the vertex set \( \{v_1, v_2, \ldots, v_n\} \). Then

\[
\mathcal{E}_\Phi(v_i) = \sum_{j=1}^{n} Q_{ij} |\lambda_j|, \quad \text{for } i = 1, 2, \ldots, n.
\]

where \( Q_{ij} = |q_{ij}|^2 \) and \( Q = (q_{ij}) \) is the unitary matrix whose columns are the eigenvectors of \( \Phi \) and \( \lambda_j \) is the \( j \)-th eigenvalue of \( \Phi \).
Proof. Since $A(\Phi)$ is Hermitian, so there exists a unitary matrix $Q = (q_{ij})$ such that $A(\Phi) = QDQ^*$, where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. Therefore, the columns of $Q$ are eigenvectors of $A(\Phi)$. Now, it is easy to see that

$$E_\Phi(v_i) = \sum_{j=1}^{n} Q_{ij} |\lambda_j|, \text{ for } i = 1, 2, \ldots, n.$$  

where $Q_{ij} = |q_{ij}|^2$.  

Let $\mathbb{C}_{n \times n}$ denote the set of all $n \times n$ complex matrices. Consider the function $\Omega_i : \mathbb{C}_{n \times n} \to \mathbb{C}$ such that $\Omega_i(B) = b_{i,i}$, for $i = 1, 2, \ldots, n$, where $b_{i,i}$ is the $(i, i)$-th entry of $B$. Let $\Phi = (G, \varphi)$ be a $T$-gain graph on $n$ vertices with adjacency matrix $A(\Phi)$. Then, $\Omega_i(|A(\Phi)|) = E_\Phi(v_i), i = 1, 2, \ldots, n$. Now, it is clear that, for any two complex matrices $B$ and $C$, $|\Omega_i(BC)| \leq \Omega_i(|BC|)$. Since $\Omega_i$ is a positive linear functional, so the Hölder inequality holds, see [1]. That is, if $0 < s, t \leq \infty$ with $\frac{1}{s} + \frac{1}{t} = 1$, then

$$\Omega_i(|BC|) \leq \Omega_i(|B|^s)^{\frac{1}{s}} \Omega_i(|C|^t)^{\frac{1}{t}}. \quad (7)$$

**Lemma 3.2.** Let $\Phi = (G, \varphi)$ be a $T$-gain graph on $G$ of $n$ vertices and at least one edge. If $r \geq 2$, $0 < s, t < \infty$ such that $\frac{1}{s} + \frac{1}{t} = 1$, then

$$\left(\frac{\Omega_p(|A(\Phi)|^r)}{\Omega_p(|A(\Phi)|^{s(r-1)+1})}\right)^{\frac{t}{2}} \leq E_\Phi(v_p), \quad p = 1, 2, \ldots, n. \quad (8)$$

Proof. Let $B = |A(\Phi)|^{r-\frac{1}{t}}$ and $C = |A(\Phi)|^{\frac{1}{t}}$. Then, by the Hölder inequality (7), we have

$$\Omega_p(|A(\Phi)|^r) = \Omega_p\left(|A(\Phi)|^{r-\frac{1}{t}} |A(\Phi)|^{\frac{1}{t}}\right) \leq \left(\Omega_p(|A(\Phi)|^{s(r-1)+1})\right)^\frac{1}{s} \Omega_p\left(|A(\Phi)|^r\right)^{\frac{1}{t}}$$

That is,

$$\left(\Omega_p(|A(\Phi)|^r)\right)^{\frac{t}{2}} \leq \left(\Omega_p(|A(\Phi)|^{s(r-1)+1})\right)^\frac{1}{s} \Omega_p(|A(\Phi)|^r)$$

Since $0 < s, t < \infty$, and $\frac{1}{s} + \frac{1}{t} = 1$, so $rs - \frac{r}{t} = s(r - 1) + 1$. Therefore,

$$\left(\frac{\Omega_p(|A(\Phi)|^r)}{\Omega_p(|A(\Phi)|^{s(r-1)+1})}\right)^{\frac{t}{2}} \leq \Omega_p(|A(\Phi)|^r) = E_\Phi(v_p), \quad p = 1, 2 \ldots, n. \quad \square$$
Let $M_k(\Phi, p)$ denote the sum of the gains of directed $k$-walk from the vertex $v_p$ to itself, in the $\mathbb{T}$-gain graph $\Phi$. In the next result, we establish a bound of the vertex energy $E_\Phi(v_p)$ in terms of $M_k(\Phi, p)$ and the vertex degree.

**Lemma 3.3.** Let $\Phi = (G, \varphi)$ be any $\mathbb{T}$-gain graph of $n$ vertices with at least one edge. Then

$$\frac{d_j^3}{M_4(\Phi, p)} \leq E_\Phi(v_p), \text{ for all } p = 1, 2, \ldots, n.$$ 

**Proof.** In the Inequality [8], we substitute $r = 2, s = 3$ and $t = \frac{3}{2}$. Then we get

$$\frac{\Omega_p(A(\Phi)^2)^{\frac{3}{2}}}{\Omega_p(A(\Phi)^4)^{\frac{1}{2}}} \leq E_\Phi(v_p), \quad p = 1, 2, \ldots, n.$$ 

Since $\Omega_p(A(\Phi)^4) = M_4(\Phi, p)$ and $\Omega_p(A(\Phi)^2) = d_p$, So the corollary follows. 

Now, we establish a bound for $E_\Phi(v_j)$ for $\mathbb{T}$-gain graph in terms of vertex degree of $v_j$ and the largest vertex degree $\Delta(G)$. For undirected graph $G$, these results are presented in [1].

**Theorem 3.1.** Let $\Phi = (G, \varphi)$ be any connected $\mathbb{T}$-gain graph with at least one edge. Then

$$E_\Phi(v_j) \geq \sqrt{\frac{d_j}{\Delta(G)}}, \quad \text{for all } v_j \in V(G). \quad (9)$$

Equality holds if and only if $\Phi \sim (K_{d_j, \Delta(G)}, 1)$.

**Proof.** Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ be the vertex set of $G$. Let the degree of the vertex $v_j$ be $d_j$. Set $d_j = d$. Then the following three types of directed 4-walks, starting from the vertex $v_j$ to itself, are possible:

1. $v_j \rightarrow v_i \rightarrow v_j \rightarrow v_k \rightarrow v_j$;
2. $v_j \rightarrow v_i \rightarrow v_s \rightarrow v_i \rightarrow v_j$, where $v_j \neq v_s$;
3. $v_j \rightarrow v_i \rightarrow v_s \rightarrow v_k \rightarrow v_j$, where four vertices are mutually distinct.

Now the maximum value of the sum of the gains of the walks of type 1 is $d^2$. Similarly, for the type 2, the maximum value is $d(\Delta(G) - 1)$, and for the type 3, the maximum value is $2 \sum_{t=1}^p \cos(\theta_t)$, where $p \leq \frac{d(\Delta(G) - 1)(d - 1)}{2}$ and $\varphi(C_m) = e^{i\theta m}$, $C_m$ is a 4-cycle formed by this
walk. Thus the maximum value is \(d(\Delta(G) - 1)(d - 1)\), and hence \(M_4(\Phi, j) \leq d^2 \Delta(G)\). Now, by Lemma 3.3, we have \(E_{\Phi}(v_j) \geq \sqrt{\frac{d_j}{\Delta(G)}}\).

If equality occurs in (9), then \(M_4(\Phi, j) = d^2 \Delta(G)\). Therefore, \(G = K_{d, \Delta(G)}\). Again from the equality \(M_4(\Phi, j) = d^2 \Delta(G)\), we have \(\varphi(\overrightarrow{C_n}) = 1\), for all cycle passing through the vertex \(v_j\). Thus, by the Lemma 4.1 \(\Phi\) is balanced. Hence \(\Phi \sim (K_{d, \Delta(G)}, 1)\). Converse is easy to verify.

\[\square\]

**Corollary 3.1.** Let \(\Phi = (G, \varphi)\) be any connected \(\mathbb{T}\)-gain graph on a \(r\)-regular graph \(G\).

Then,

\[E_{\Phi}(v_i) \geq 1, \text{ for all } v_i \in V(G)\]

Equality occurs if and only if \(\Phi = (K_{r,r}, 1)\).

**Proof.** Since \(G\) is a connected \(r\)-regular graph with vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\), so the degree of each vertex is same. For \(i = 1, 2, \ldots, n\), \(d_i = r = \Delta(G)\). Then by the Theorem 3.1 we have \(E_{\Phi}(v_i) \geq 1\), for all \(v_i \in V(G)\). Equality occur if and only if \(\Phi \sim (K_{r,r}, 1)\) \[\square\]

In the next lemma, we show that the energy of a vertex is invariant under the switching equivalence of \(\mathbb{T}\)-gain graphs.

**Lemma 3.4.** Let \(\Phi_1\) and \(\Phi_2\) be any two switching equivalent \(\mathbb{T}\)-gain graphs on a graph \(G\) with the vertex set \(V(G) = \{v_1, v_2, \ldots, v_n\}\). Then for each \(i\),

\[E_{\Phi_1}(v_i) = E_{\Phi_2}(v_i)\]

**Proof.** Since \(\Phi_1 \sim \Phi_2\), so \(\text{spec}(\Phi_1) = \text{spec}(\Phi_2)\) and there is a diagonal unitary matrix \(U\) such that \(A(\Phi_1) = UA(\Phi_2)U^*\). Hence, by the Lemma 3.1 we have \(E_{\Phi_1}(v_i) = E_{\Phi_2}(v_i)\) for each \(i\). \[\square\]

In the next lemma, we provide a sufficient condition for the vertex energy of a \(\mathbb{T}\)-gain graph equals to the vertex energy of its underlying graph.

**Lemma 3.5.** Let \(\Phi = (G, \varphi)\) be any \(\mathbb{T}\)-gain graph such that either \(\Phi\) is balanced or \(-\Phi\) is balanced. Then \(E_{\Phi}(v_i) = E_{-\Phi}(v_i) = E_G(v_i)\).

**Proof.** If \(\Phi\) is balanced, then \(\Phi \sim G\). Thus by the Lemma 3.4 \(E_{\Phi}(v_i) = E_G(v_i)\), for \(i = 1, 2, \ldots, n\). Let \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\) be the spectrum of \(\Phi\). Let \(D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)\). Then there exist an unitary matrix \(Q\) such that \(A(\Phi) = QDQ^*\). Thus \(A(-\Phi) = -A(\Phi) = Q(-D)Q^*\). Therefore, by the Lemma 3.1 \(E_{\Phi}(v_i) = E_{-\Phi}(v_i)\), for \(i = 1, 2, \ldots, n\). Thus \(E_{\Phi}(v_i) = E_{-\Phi}(v_i) = E_G(v_i)\). If \(-\Phi\) is balanced then we can prove the statement similarly. \[\square\]
In the next Theorem, we provide a lower bound for the vertex energy of a $\mathbb{T}$-gain graph in terms of the degree of the vertex and the maximum vertex degree of the underlying graph.

**Theorem 3.2.** Let $\Phi = (G, \varphi)$ be any connected $\mathbb{T}$-gain graph with at least one edge. Then

$$E_\Phi(v_i) \geq \frac{d_i}{\Delta(G)}, \quad \text{for all } v_i \in V(G).$$

Equality occurs if and only if $\Phi \sim (K_{d,d}, 1)$, for some $d$.

*Proof.* Let $\Phi = (G, \varphi)$ be any $\mathbb{T}$-gain graph on $G$. Let $\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1$ be the eigenvalues of $\Phi$. By Theorem 2.1, $\max\{\lambda_1, -\lambda_n\} = \rho(\Phi) \leq \rho(G) \leq \Delta(G)$. Hence $\lambda_i \in [-\Delta(G), \Delta(G)]$ for all $i$. Therefore, $|\frac{\lambda_i}{\Delta(G)}| \leq 1$. Then $|\frac{\lambda_i}{\Delta(G)}|^2 \geq |\frac{\lambda_i}{\Delta(G)}|^2$ and equality occurs if and only if $\lambda_i \in \{-\Delta(G), 0, \Delta(G)\}$. Using Lemma 3.1, we have

$$E_\Phi(v_i) = \sum_{j=1}^{n} |q_{ij}|^2 |\lambda_j| \geq \sum_{j=1}^{n} |q_{ij}|^2 \frac{\lambda_j^2}{\Delta(G)} = \frac{d_i}{\Delta(G)}, \quad \text{for all } v_i \in V(G),$$

where $Q = (q_{ij})$ is the unitary matrix whose columns are eigenvectors of $\Phi$. Since $G$ has at least one edge so there is a vertex $v_j$ such that $E_\Phi(v_j) > 0$. Therefore, if equality occurs then either $\Delta(G)$ or $-\Delta(G)$ must be an eigenvalue of $\Phi$.

Now $\Delta(G) = \rho(\Phi) \leq \rho(G) \leq \Delta(G)$, so $\rho(G) = \rho(\Phi)$. Thus, by Theorem 2.1 either $\Phi$ is balanced or $-\Phi$ is balanced.

**Case-I:** If $\Phi$ is balanced, then by Lemma 3.4, $E_\Phi(v_i) = E_G(v_i)$ for all $v_i \in V(G)$. Then $E_G(v_i) = \frac{d_i}{\Delta(G)}$. Therefore, by Lemma 2.5, $G$ is isomorphic to $K_{d,d}$, for some $d$. Hence $\Phi \sim (K_{d,d}, 1)$.

**Case-II:** If $-\Phi$ is balanced, then, similar to case-I, $-\Phi \sim (K_{d,d}, 1)$. Since the underlying graph is bipartite and $-\Phi$ is balanced, so, by Theorem 2.2 $\Phi$ is balanced. Thus $\Phi \sim (K_{d,d}, 1)$.

Using the above theorem, we prove that the energy of the complete bipartite $\mathbb{T}$-gain graph $K_{n,n}$ is always greater than or equal to the energy of the underlying graph.

**Theorem 3.3.** If $\Phi = (K_{n,n}, \varphi)$ is any $\mathbb{T}$-gain graph on the complete bipartite graph $K_{n,n}$, then $E(\Phi) \geq E(K_{n,n}) = 2n$, and equality holds if and only if $\Phi \sim (K_{n,n}, 1)$.

*Proof.* Let $V(G) = \{v_1, v_2, \ldots, v_{2n}\}$ be the set of vertices of $\Phi$. Then, $E(\Phi) = E_\Phi(v_1) + E_\Phi(v_2) + \cdots + E_\Phi(v_{2n})$. By Theorem 3.2, we have

$$E(\Phi) = \sum_{j=1}^{2n} E_\Phi(v_j) \geq \sum_{j=1}^{2n} \frac{d_j}{\Delta(G)} = 2n = E(K_{n,n}).$$

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It is easy to see that, equality occur if and only if $\Phi$ is balanced.

If $G$ is a $r$-regular graph of $n$ vertices, then $E(G) \geq n$ and equality occur if and only if each component of $G$ is $K_{r,r}$ [Theorem 2.6]. Next corollary is an extension of the above result for the $T$-gain graph.

**Corollary 3.2.** Let $\Phi = (G, \varphi)$ be any $r$-regular $T$-gain graph on $n$ vertices, where $r > 0$. Then $E(\Phi) \geq n$ and equality occur if and only if each component of $\Phi$ is switching equivalent to $(K_{r,r}, 1)$.

**Proof.** Let $G_1, G_2, \ldots, G_k$ be the connected components of $G$. Then $E(\Phi) = \sum_{j=1}^{k} E((G_j, \varphi))$. By the definition 3.1 and the equation (6), $E((G_j, \varphi)) = \sum_{v \in V(G_j)} E(G_j, \varphi)(v)$, for each $j = 1, 2, \ldots, k$. Now, by Corollary 3.1, $E((G_j, \varphi)(v)) \geq 1$, for any $v \in V(G_j)$. Then $E(G_j, \varphi) \geq |V(G_j)|$, for all $j = 1, 2, \ldots, k$. Thus $E(\Phi) \geq n$. If $E(\Phi) = n$, then $E\Phi(u) = 1$, for all $u \in V(G)$. Therefore, by Corollary 3.1 each component of $\Phi$ is switching equivalent to $(K_{r,r}, 1)$. \qed

Let $G$ be a semiregular bipartite graph with parameter $(n_a, n_b, r_a, r_b)$. A semi regular bipartite $T$-gain graph with parameter $(n_a, n_b, r_a, r_b)$ is a $T$-gain graph whose underlying graph is a semiregular bipartite graph of parameter $(n_a, n_b, r_a, r_b)$. The next bound is the generalization of a Theorem 2.7 for the $T$-gain graphs.

**Corollary 3.3.** Let $\Phi = (G, \varphi)$ be any semiregular bipartite $T$-gain graph with parameter $(n_a, n_b, r_a, r_b)$. Then $E(\Phi) \geq n_a \sqrt{\frac{r_a}{r_b}} + n_b \sqrt{\frac{r_b}{r_a}}$. Equality occur if and only if each component is switching equivalent to $(K_{r_a, r_b}, 1)$.

**Proof.** Let $G_1, G_2, \ldots, G_k$ be the connected components of $G$. Then each $G_j$ is connected semiregular bipartite graph. Now, by applying Theorem 3.1 to each component $(G_j, \varphi)$, we get the result. \qed

**Remark 3.1.** Hermitian adjacency matrices of mixed graphs are particular case of adjacency matrices of $T$-gain graphs. Therefore all the above results for energy of a vertex holds true for mixed graphs.

### 4 Lower bounds of energy of $T$-gain graphs

In this section, we establish several lower bounds for the energy of $T$-gain graphs. We begin this section with the following theorem which gives a lower bound for the energy of a $T$-gain graph in terms of the gain of the real parts of the fundamental cycles.
Theorem 4.1. Let $\Phi = (G, \varphi)$ be any connected $T$-gain graph on $n$ vertices. Let $T$ be a normal spanning tree of $G$, and $\{C_1, C_2, \ldots, C_l\}$ be the collection of all fundamental cycles in $G$ with respect to $T$. Then,

$$E(\Phi) \geq 2 \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) + (5n - n^2 - 4).$$

(10)

The inequality is sharp.

Proof. Let $\Phi = (G, \varphi)$ be any connected $T$-gain graph. Let $T$ be a normal spanning tree of $G$, and $\{C_1, C_2, \ldots, C_l\}$ be the collection of all fundamental cycles in $G$ with respect to $T$. Define a new $T$-gain graph $\Phi'$ on $G$ such that $\varphi'(\overrightarrow{e}) = 1$ for all $e \in E(T)$ and $\varphi(C_i) = \varphi'(C_i)$ for all $i$. So, by Lemma 2.2, the $T$-gain graphs $\Phi$ and $\Phi'$ are switching equivalent. Therefore,

$$\sum_{i,j} \varphi'(\overrightarrow{e}_{i,j}) = \sum_{j=1}^{l} \{\varphi'(\overrightarrow{C}_j) + \varphi'(\overrightarrow{C}_j)^{-1}\} + 2(n - 1) = 2 \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) + 2(n - 1).$$

(11)

Now, $\text{Re}(\text{trace}(A(K_n)A(\Phi'))) = \sum_{i,j} \varphi'(\overrightarrow{e}_{i,j})$. Let $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$ be the singular values of $\Phi'$. Since $\text{spec}(K_n) = \{-1^{(n-1)}, (n - 1)^{(1)}\}$, by Theorem 2.8, we have $\text{Re}(\text{trace}(A(K_n)A(\Phi'))) \leq (n - 1)|\lambda_1| + \sum_{j=2}^{n-1} |\lambda_j|$, and hence $\sum_{i,j} \varphi'(\overrightarrow{e}_{i,j}) \leq (n - 2)|\lambda_1| + E(\Phi')$.

So, by equation (11), we have $2 \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) + 2(n - 1) \leq (n - 2)|\lambda_1| + E(\Phi')$. Now, using Theorem 2.1 we get $|\lambda_1| = \rho(\Phi') = \rho(\Phi) \leq \rho(G) \leq \Delta \leq (n - 1)$. Thus

$$\sum_{j=1}^{l} \text{Re}(\varphi(C_j)) + 2(n - 1) \leq (n - 2)(n - 1) + E(\Phi),$$

and hence

$$E(\Phi) \geq 2 \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) + (5n - n^2 - 4).$$

Now, if $\Phi \sim (K_n, 1)$, then $2 \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) = n^2 - 3n + 2$ and hence equality holds in equation (10).

If $G$ is a complete bipartite graph, then $A(G)$ has exactly one positive eigenvalue. Also, if $G$ is any non-complete bipartite graph on more than 4 vertices, then it contains $P_4$ as an induced subgraph, and hence $A(G)$ has at least two positive eigenvalues. So, if $G$ is a bipartite graph on more than 4 vertices, then $G$ is complete bipartite if and only if $A(G)$
has exactly one positive eigenvalue. Our next objective is to study the counter part of this property for the \(T\)-gain graphs. The following lemma is a key in the proof of Theorem 4.2.

This gives a sufficient condition a \(T\)-gain graph to be balanced.

**Lemma 4.1.** Let \( \Phi = (G, \varphi) \) be any \(T\)-gain graph on a complete bipartite graph \( G \). If every 4-cycle which passes through the vertex \( v \), for some vertex \( v \) of \( G \), has gain 1, then \( \Phi \) is balanced.

**Proof.** Let \( \Phi = (G, \varphi) \) be any \(T\)-gain graph on a complete bipartite graph \( G \). Let \( v \) be a vertex in \( G \) such that the gain of any 4-cycle passing through the vertex \( v \) is 1. First let us show that gain of any four cycle in \( G \) is 1. Let \( C_4 \equiv v_1 - v_2 - v_3 - v_4 - v_1 \) be any 4-cycle in \( \Phi \) such that \( C_4 \) does not contain the vertex \( v \). Without loss of generality, let us assume that \( v_2 \sim v \). Now, consider the two 4-cycles: \( C_4(v) \equiv v_1 - v_2 - v - v_4 - v_1 \) and \( C_4'(v) \equiv v_2 - v_3 - v_4 - v \). Then \( \varphi(C_4) = \varphi(C_4(v))\varphi(C_4'(v)) = 1 \).

Let \( C_{2p} \) be any cycle in \( G \) on \( 2p \) vertices. Without loss of generality, let us assume that \( C_{2p} \equiv v_1 - v_2 - v_3 - v_4 - \cdots - v_{(2p-1)} - v_2 - v_1 \). Then

\[
\varphi(C_{2p}) = \varphi(e_{1,2}) \varphi(e_{2,3}) \varphi(e_{3,4}) \cdots \varphi(e_{(2p-1),2p}) \varphi(e_{2p,1})
\]

\[
= \{ \varphi(e_{1,2}) \varphi(e_{2,3}) \varphi(e_{3,4}) \varphi(e_{4,1}) \}
\]

\[
= \{ \varphi(e_{1,2}) \varphi(e_{4,5}) \varphi(e_{5,6}) \varphi(e_{6,1}) \}
\]

\[
= \cdots
\]

\[
= \{ \varphi(e_{1,(2p-2)}) \varphi(e_{(2p-2),(2p-1)}) \varphi(e_{(2p-1),2p}) \varphi(e_{(2p,1)}) \}
\]

\[
= 1.
\]

Thus \( \Phi \) is balanced.

\[ \square \]

**Theorem 4.2.** Let \( \Phi = (G, \varphi) \) be any \(T\)-gain graph on a connected bipartite graph \( G \). Then \( \Phi \) has exactly one positive eigenvalue if and only if \( \Phi \) is a balanced complete bipartite graph.

**Proof.** Let \( \Phi = (G, \varphi) \) have exactly one positive eigenvalue. If the number of vertices of \( G \) is two or three, then \( G \) must be \( K_2 \) or \( K_{1,2} \), respectively. Therefore, in both the cases, \( \Phi \) is a balanced complete bipartite graph. Now we consider a graph \( G \) with \( |V(G)| \geq 4 \). Suppose that \( P_4 \) is an induced subgraph of \( G \). So \( (P_4, \varphi) \) is an induced \(T\)-gain subgraph of \( \Phi \). As \( P_4 \) is a tree, so the spectrum of \( P_4 \) with respect to \( \varphi \) is same as that of \( \text{spec}(P_4) \). Thus \( P_4 \) has two positive eigenvalue with respect to \( \varphi \). Therefore, by the interlacing theorem, \( \Phi \) has at least two positive eigenvalue, a contradiction. Thus \( G \) can not have \( P_4 \) as an induced subgraph.
and hence the diameter of $G$ is at most 2. Now it is easy to see that any two non adjacent vertices have the same neighbors. Thus $G$ is complete multipartite. But $G$ is bipartite, so $G$ is complete bipartite.

Now we consider the following two cases to show that $\Phi$ is balanced.

**Case 1:** If $G$ does not contain any cycles, then $G$ must be a star, and hence $\Phi$ is balanced.

**Case 2:** If $G$ contains cycles, then it must contains an induced $C_4$. Let $\varphi(\overrightarrow{C_4}) = e^{i\theta}, \theta \in [0, 2\pi)$.

Let $C = (C_4, \varphi)$ be an induced subgraph of $\Phi$ whose underlying graph is $C_4$. Therefore, by Theorem 2.3, we have

$$\text{spec}(C) = \left\{ 2 \cos \left( \frac{\theta}{4} \right), 2 \cos \left( \frac{\theta + \pi}{4} \right), 2 \cos \left( \frac{\theta + 3\pi}{4} \right) \right\}$$

Let $x = \frac{\theta}{4} \in [0, \frac{\pi}{2})$. It is easy to see that $\text{spec}(C)$ has two positive and two negative eigenvalues if and only if $x \in (0, \frac{\pi}{2})$. Hence $\text{spec}(C)$ has exactly one positive eigenvalue if and only if $x = 0$. Now, by interlacing theorem, $\Phi$ cannot have any induced 4-cycle $C_4$ such that $\varphi(\overrightarrow{C_4}) = e^{i\theta}$, where $\theta \in (0, 2\pi)$. Therefore, for any induced 4-cycle $C$ in $G$, we have $\varphi(C) = 1$. Thus, by Lemma 4.1, $\Phi$ is balanced. Conversely, if $\Phi$ is a balanced complete bipartite $\mathbb{T}$-gain graph, then $\Phi$ has exactly one positive eigenvalue. \qed

Next result gives a lower bound of energy of $\mathbb{T}$-gain graph in terms of spectral radius.

**Theorem 4.3.** If $\Phi = (G, \varphi)$ be any $\mathbb{T}$-gain graph on a connected graph $G$. Then $E(\Phi) \geq 2\rho(\Phi)$. If $G$ is bipartite then equality occurs if and only if $\Phi \sim (K_{p,q}, 1)$ for some $p, q$.

**Proof.** Let $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be the spectrum of $\Phi$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Now $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 0$. Therefore, $2|\lambda_1| \leq |\lambda_1| + |\lambda_2| + \cdots + |\lambda_n|$. Thus $E(\Phi) \geq 2\rho(\Phi)$.

Let $G$ be bipartite. Since $|\lambda_1| = |\lambda_2| + \cdots + |\lambda_n|$ holds if and only if all of $\lambda_j$’s, for $j = 2, 3, \ldots, n$ are of the same sign. Therefore, equality occur if and only if $\Phi$ has only one positive eigenvalue. Hence, by the Theorem 4.2, equality holds if and only if $\Phi \sim (K_{p,q}, 1)$. \qed

Let $J$ denote the all 1’s matrix of appropriate size. The following two theorems provide a lower bound for energy of $\mathbb{T}$-gain graph in terms of the number of vertices and the gains of fundamental cycles.

**Theorem 4.4.** Let $\Phi = (G, \varphi)$ be any connected $\mathbb{T}$-gain graph with $n$ vertices and $\{C_1, C_2, \ldots, C_l\}$ be the collection of all fundamental cycles in $G$ with respect to a normal spanning tree $T$. Then

$$E(\Phi) \geq 4 + \frac{4}{n} \left\{ \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) - 1 \right\}.$$  \hspace{1cm} (12)
The inequality is sharp.

Proof. Let \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be the spectrum of \( \Phi \) such that \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \). Define a new \( T \)-gain \( \Phi' \) on \( G \) such that \( \varphi'(e) = 1 \) for all \( e \in E(T) \) and \( \varphi(C_i) = \varphi'(C_i) \) for all \( i \). So, by Lemma 2.2, the \( T \)-gain graphs \( \Phi \) and \( \Phi' \) are switching equivalent. Then

\[
\sum_{i,j} \varphi'(e_{ij}) = 2 \sum_{j=1}^l \text{Re}(\varphi(C_j)) + 2(n-1). \]

By Theorem 2.8, we have

\[
\text{Re}(\text{trace}(A(\Phi)J)) \leq n|\lambda_1|,
\]

and hence

\[
2(n-1) + 2 \sum_{j=1}^l \text{Re}(\varphi(C_j)) \leq n|\lambda_1|.
\]

As \( |\lambda_1| \leq |\lambda_2| + \cdots + |\lambda_n| \), so \( |\lambda_1| \leq \frac{E(\Phi)}{n} \). Therefore,

\[
E(\Phi) \geq 4 + \frac{4}{n} \left\{ \sum_{j=1}^l \text{Re}(\varphi(C_j)) - 1 \right\}.
\]

Let us take \( \Phi \sim (G, 1) \), where \( G = K_{r,r,\ldots,r} \) is a connected complete \( p \)-partite graph on \( m \) edges and \( n \) vertices. Then the right hand side expression \((\ref{eq:12})\) becomes \( 4 + \frac{4}{n} (m - n + 1 - 1) \), which is \( \frac{4m}{n} \). Since \( G \) is a complete multipartite graph, \( G \) has exactly one positive eigenvalue. Thus, if \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of \( \Phi \), then \( |\lambda_1| = |\lambda_2| + \cdots + |\lambda_n| \), and hence \( E(\Phi) = 2|\lambda_1| = 2\lambda_1 \). Also the spectral radius of \( \Phi \) is \( (r-1) \), the degree of each vertex in \( G \), and the degree of each vertex of \( G \) is \( \frac{2m}{n} \). Therefore, \( E(\Phi) = \frac{4m}{n} \). Hence the inequality is sharp.

If the underlying graph is a bipartite graph, then we can completely characterize the classes for which equality holds in \((\ref{eq:12})\).

**Corollary 4.1.** Let \( \Phi = (G, \varphi) \) be any connected \( T \)-gain graph on a bipartite graph \( G \) with \( n \) vertices. Then

\[
E(\Phi) \geq 4 + \frac{4}{n} \left\{ \sum_{j=1}^l \text{Re}(\varphi(C_j)) - 1 \right\}.
\]

Equality occurs if and only if \( \Phi \sim (K_{n/2+1, n/2+1, 1}) \).

**Proof.** Let \( \Phi = (G, \varphi) \) be any connected \( T \)-gain graph on a bipartite graph \( G \) with \( m \) edges and \( n \) vertices. Let \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) be the spectrum of \( \Phi \) such that \( |\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n| \). Let \( \{C_1, C_2, \ldots, C_l\} \) be the fundamental cycles of \( G \) with respect to a normal spanning tree \( T \). Now the inequality is clear from the Theorem 4.4. Let us consider the equality,

\[
E(\Phi) = 4 + \frac{4}{n} \left\{ \sum_{j=1}^l \text{Re}(\varphi(C_j)) - 1 \right\}.
\]
Then from the proof of the Theorem 4.4, we have
\[ E(\Phi) \geq 2|\lambda_1| \geq 4 + \frac{4}{n} \left( \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) - 1 \right). \]

Since \( \Phi \) is a bipartite \( \mathbb{T} \)-gain graph, so \( \Phi \) must satisfy the following equality.

(i) \( E(\Phi) = 2|\lambda_1| = 2\rho(\Phi) \)

(ii) \( 2\lambda_1 = 2|\lambda_1| = 4 + \frac{4}{n} \left( \sum_{j=1}^{l} \text{Re}(\varphi(C_j)) - 1 \right) \)

Now by the Theorem 4.3, the equation (i) is satisfied if and only if \( \Phi \sim (K_{p,q}, 1) \), for some \( p \) and \( q \). Then \( \Phi \) is balanced, so from equation (ii), we have \( \lambda_1 = \sqrt{pq} \), \( n = p + q \) and \( m = pq \). Therefore, we have \( \lambda_1 = \sqrt{pq} = \frac{2pq}{p+q} \). That is \( p = q = \frac{n}{2} \). Thus \( \Phi \sim (K_{\frac{n}{2}, \frac{n}{2}}, 1) \). Converse is easy to verify.

The following lemma is about the change in the energy of a graph obtained from a graph by removing a cut set. This will be useful in the proof of some of the following results.

**Lemma 4.2.** Let \( \Phi = (G, \varphi) \) be a \( \mathbb{T} \)-gain graph and \( E \) be a cut set of \( \Phi \). Then \( E(\Phi - E) \leq E(\Phi) \).

**Proof.** For any cut set \( E \) of \( G \), there exist two induced sub graphs \( L \) and \( M \) complement to each other in \( G \) such that \( G - E = L \oplus M \). Then \( \Phi - E = (L, \varphi) \oplus (M, \varphi) \). Now \( A(\Phi) \) can be expressed as \( A(L, \varphi) X X^* A(M, \varphi) \). Therefore by the Theorem 2.10 \( E(\Phi) \geq E(A(L, \varphi)) + E(A(M, \varphi)) = E(\Phi - E) \). □

In the next result, we establish a connection between the gain energy and the matching number of a graph. This result is a counter part (for the \( \mathbb{T} \)-gain graphs) of Lemma 2.7 and Theorem 2.4 for undirected graph, a main result in [14] for skew energy of oriented graph, and Theorem 2.5 for mixed graph.

**Theorem 4.5.** Let \( \Phi = (G, \varphi) \) be a \( \mathbb{T} \)-gain graph, and let \( \mu(G) \) be the matching number of \( G \). Then \( E(\Phi) \geq 2\mu(G) \).

**Proof.** Let \( \Phi = (G, \varphi) \) be any \( \mathbb{T} \)-gain graph with matching number \( \mu(G) \). We prove the result by induction on \( \mu(G) \). If \( \mu(G) = 0 \), then \( E(\Phi) = 2\mu(G) = 0 \). If \( \mu(G) = 1 \), then \( G \) must be \( K_{1,p} \), for some \( p \), together with some isolated vertices. Therefore, \( \Phi \sim (G, 1) \). Thus \( E(\Phi) = 2\sqrt{p} \geq 2 = 2\mu(G) \). Let us assume that for any \( \mathbb{T} \)-gain graph \( \Psi = (H, \psi) \)
with matching number \( \mu(H) < \mu(G) \), \( \mathcal{E}(\Psi) \geq 2\mu(H) \). Let \( M \) be a maximum matching of \( G \) and \( e \in M \). Now consider an induced subgraph \( G - [e] \). Then \( \mu(G - [e]) = \mu(G) - 1 \). By induction, we have \( \mathcal{E}((G - [e], \varphi)) \geq 2\mu(G - [e]) \). Let \( E \) be the set of edges in \( G \) which are incident with the edge \( e \). Then \( E \) is a cut set, and \( (G - E) = (G - [e]) \oplus K_2 \). By the Lemma 4.2, \( \mathcal{E}(\Phi) \geq \mathcal{E}(\Phi - E) \). Now \( \mathcal{E}(\Phi) \geq \mathcal{E}(\Phi - E) = \mathcal{E}((G - [e], \varphi)) + \mathcal{E}((K_2, \varphi)) \geq 2\mu(G) - 2 + 2 = 2\mu(G) \). Hence the result.

We shall discuss the sharpness of the inequality in the above bound in Theorem 4.7.

The following lemma is the counter part of Lemma 2.1 for the \( T \)-gain graphs.

**Lemma 4.3.** Let \( \Phi = (G, \varphi) \) be a \( T \)-gain graph. If \( E \) is a cut set in \( G \) such that \( V(G) = V_1 \cup V_2 \) and all the edges of \( E \) are from the vertices of \( V_1 \) to a fixed vertex of \( V_2 \), then \( \mathcal{E}(\Phi - E) < \mathcal{E}(\Phi) \).

**Proof.** Let \( E \) be a cut set, and \( L \) and \( M \) be two complementary induced subgraphs in \( G \) corresponding to \( E \). Let us assume that the edges of \( E \) are incidence with a single vertex \( v \) of \( M \). After a suitable relabeling of vertices, we can express \( A(\Phi) = \begin{bmatrix} A((L, \varphi)) & X \\ X^* & A((M, \varphi)) \end{bmatrix} \) such that the first column of the matrix \( X \), say \( y \), corresponds to the vertex \( v \). Hence all the entries of the matrix \( X \) are zero, except the first column. Now, by Lemma 4.2, \( \mathcal{E}(\Phi - E) \leq \mathcal{E}(\Phi) \). Suppose that \( \mathcal{E}(\Phi - E) = \mathcal{E}(\Phi) \). Then, by Theorem 2.10, there exists two unitary matrices \( P \) and \( Q \), such that \( \begin{bmatrix} PA((L, \varphi)) & PX \\ QX^* & QA((M, \varphi)) \end{bmatrix} \) is positive semi definite. As \( (PX)^* = QX^* \), we have \( Q = \begin{bmatrix} \beta & 0 \\ 0 & Q_1 \end{bmatrix} \) with \( |\beta| = 1 \), and \( Q_1 \) is unitary matrix. Let \( A((M, \varphi)) = \begin{bmatrix} 0 & z^* \\ z & N \end{bmatrix} \). Then \( QA((M, \varphi)) = \begin{bmatrix} 0 & \beta z^* \\ Q_1 z & Q_1 N \end{bmatrix} \) is positive semi definite. So \( Q_1 z = 0 \) and \( \beta z^* = 0 \). That is \( z = 0 \). Hence \( A((M, \varphi)) = \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} \). Therefore \( A(\Phi) = \begin{bmatrix} A((L, \varphi)) & y & 0 \\ y^* & 0 & 0 \\ 0 & 0 & N \end{bmatrix} \). Now \( \mathcal{E}(A((L, \varphi))) + \mathcal{E}(N) = \mathcal{E}(A((L, \varphi))) + \mathcal{E}(A(M, \varphi))) = \mathcal{E}(\Phi - E) = \mathcal{E}(\Phi) = \mathcal{E}(A((L, \varphi))) + \mathcal{E}(N) \). That is \( \mathcal{E}(A((L, \varphi))) = \mathcal{E}(A((L, \varphi))) \). Hence, by Lemma 2.9, \( y = 0 \). Thus \( E \) is empty. Which is a contradiction.

The following lemma provides a (spectral) sufficient condition for a graph to have perfect matching. This is a counter part of Lemma 2.9 for the \( T \)-gain graphs.
Lemma 4.4. If $\Phi = (G, \varphi)$ is a connected $\mathbb{T}$-gain graph and $\mathcal{E}(\Phi) = 2\mu(G)$, then $G$ has a perfect matching.

Proof. Suppose that $G$ has no perfect matching. Let $M$ be any maximum matching of $G$. Since $G$ is a connected graph, so there exist a vertex $u$ which is not adjacent with any edges in $M$. Then $\mu(G) = \mu(G - u)$. Let $K_1$ be the graph which is an isolated vertex $u$. Let $E$ be the set of all edges incident with the vertex $u$ in $G$. Then, $E$ is a cut set, and $\Phi - E = (\Phi - u) \oplus K_1$. Therefore, by Lemma 4.3 and Theorem 4.5, $\mathcal{E}(\Phi) > \mathcal{E}(\Phi - E) = \mathcal{E}(\Phi - u) + 0 = 2\mu(G - u) = 2\mu(G)$. That is, $\mathcal{E}(\Phi) > 2\mu(G)$, a contradiction. Thus $G$ has a perfect matching.

Now, let us establish a couple of lemmas about the energy of a $\mathbb{T}$-gain graph in terms of the matching number of the underlying graph.

Lemma 4.5. Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph with a pendant vertex. If $G$ is not $K_2$, then $\mathcal{E}(\Phi) > 2\mu(G)$.

Proof. Let $v$ be a pendant vertex of $G$, $u$ be its unique neighbor vertex, and $e$ be the edge between them. Then the induced subgraphs $(G - [e])$ and $K_2$ are complement to each other in $G$. Let $E$ be the collection of all edges between the vertex $u$ and the vertices of $G - \{u, v\}$. Then $G - E = (G - [e]) \oplus K_2$. By Lemma 4.3, $\mathcal{E}(\Phi) > \mathcal{E}(\Phi - E) = \mathcal{E}((G - [e], \varphi)) + \mathcal{E}((K_2, \varphi))$. Also $\mu(G - [e]) = \mu(G) - 1$. Therefore, by Theorem 4.5, $\mathcal{E}(\Phi) > 2\mu(G)$.

Lemma 4.6. Let $\Phi = (G, \varphi)$ be a connected $\mathbb{T}$-gain graph and $L$ be an induced subgraph of $G$. If $\mathcal{E}((L, \varphi)) > 2\mu(L)$ and $\mu(G) = \mu(L) + \mu(G - L)$. Then $\mathcal{E}(\Phi) > 2\mu(G)$.

Proof. Since $L$ is an induced subgraph of $G$, so $(G - L)$ is the complementary induced subgraph of $G$. Let $E$ be a cut set of $G$ such that $(G - E) = (G - L) \oplus L$. Then $(\Phi - E) = (G - L, \varphi) \oplus (L, \varphi)$. Since $E$ is a cut set of $G$, so, by Lemma 4.2, $\mathcal{E}(\Phi) \geq \mathcal{E}(\Phi - E) = \mathcal{E}((L, \varphi)) + \mathcal{E}((G - L, \varphi))$. Now, by Theorem 4.5 and the hypothesis, we have $\mathcal{E}(\Phi) > 2\mu(L) + 2\mu(G - L) = 2\mu(G)$. Hence, $\mathcal{E}(\Phi) > 2\mu(G)$.

Lemma 4.7. Let $\Phi = (G, \varphi)$ be any $\mathbb{T}$-gain graph on a connected graph $G$ which is given in the figure 4. Then $\mathcal{E}(\Phi) > 2\mu(G)$.

Proof. Let $E$ be the cut set consist of the set of edges which are incidence with the edge $e$ in Figure 1. Then $G - E = K_2 \oplus P_4$. Now, by Lemma 4.2, $\mathcal{E}(\Phi) \geq \mathcal{E}(\Phi - E) = \mathcal{E}((K_2, \varphi)) + \mathcal{E}((P_4, \varphi))$. Since $(K_2, \varphi) \sim (K_2, 1)$ and $(P_4, \varphi) \sim (P_4, 1)$, so by the Lemma 4.5, we have $\mathcal{E}(\Phi) > 2 + 2\mu(P_4) = 2\mu(G)$. 

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In the next theorem, we characterize the class of bipartite $T$-graphs for which equality holds in Theorem 4.5. Define $N(u) = \{x \in V(G) : u \sim x\}$.

**Theorem 4.6.** Let $\Phi = (G, \varphi)$ be any connected $T$-gain bipartite graph with $n$ vertices. Then $E(\Phi) = 2\mu(G)$ if and only if $\Phi \sim (K_{\frac{n}{2}, \frac{n}{2}}, 1)$.

**Proof.** First let us show that $G$ is complete bipartite using induction on the number of vertices. Let $|V(G)| = 2$. If $E(\Phi) = 2\mu(G)$, then it is clear that $G = K_{1,1}$. Let us assume that for any connected bipartite $T$-gain graph $(H, \psi)$ with $|V(H)| < n$, if $E((H, \psi)) = 2\mu(H)$, then $H$ is a complete bipartite graph with same partition size. Let $\Phi = (G, \varphi)$ be any connected bipartite $T$-gain graph with $n$ vertices such that $E(\Phi) = 2\mu(G)$. By the Lemma 4.4, $G$ has perfect matching, $M$ (say). Let $X$ and $Y$ be the vertex partition of $G$ such that $|X| = |Y| = \frac{n}{2}$.

**Claim 1:** For any vertex $u \in X$, $N(u) = Y$.

Suppose that $N(u)$ is a proper subset of $Y$. Let $v' \in Y \setminus N(u)$. Then there exists vertices $u' \in X$ and $v \in Y$ such that the edges $(u, v)$ and $(u', v')$ are in $M$.

Let $P$ be an induced subgraph formed by the vertices $\{u, v, u', v'\}$. The vertices $u'$ and $v$ are not adjacent in $G$. Suppose they are adjacent. Then $P$ is isomorphic to $P_1$. If $|V(G)| = 4$, then, by Lemma 4.5, $E(\Phi) > 2\mu(G)$, a contradiction. Thus $G = K_{2,2}$.

If $|V(G)| > 4$, then it is clear that $\mu(G) = \mu(P) + \mu(G - P)$. By Lemma 4.5, $E((P, \varphi)) > 2\mu(P)$. Then, by Lemma 4.6, $E(\Phi) > 2\mu(G)$, which is again a contradiction. Thus $u' \sim v$.

Let $Q = (G - P)$. Then $Q$ is the complementary induced subgraph of $P$ in $G$. Therefore, $\mu(G) = \mu(P) + \mu(Q) = 2 + \mu(Q)$. Now, we have $2\mu(G) = E(\Phi) \geq E((P, \varphi)) + E((Q, \varphi)) \geq 2(2 + \mu(Q)) = 2\mu(G)$. Thus, $E((Q, \varphi)) = 2\mu(Q)$. Then, by induction hypothesis, $Q$ is complete bipartite graph with partition $X'$ and $Y'$ such that $|X'| = |Y'|$. Then $X = X' \cup \{u, u'\}$ and $Y = Y' \cup \{v, v'\}$.

**sub claim:** For every $x \in X'$ the vertices $x$ and $v$ are adjacent, and for every $y \in Y'$ the vertices $y$ and $u$ are adjacent.
Since $G$ is connected, so at least one of the vertices of $u$ or $v$ is adjacent with the vertices in $Y'$ or $X'$, respectively. Without loss of generality, let us assume that $u \sim y$ for some $y \in Y'$. Now, every vertex of $X'$ is adjacent with $v$. Otherwise, there is a vertex $x \in X'$ such that $x \sim v$. Then the induced underlying subgraph $H_1$ (say) formed by the vertices \{u, v, y, x\} is isomorphic to $P_4$ and $\mu(G) = \mu(H_1) + \mu(G - H_1)$. Therefore, by Lemma 4.6, $\mathcal{E}(\Phi) > 2\mu(G)$, a contradiction. Thus every vertex in $X'$ is adjacent with $v$.

Suppose $u$ is not adjacent to some of the vertices of $Y'$. Let $y_1 \in Y'$ such that $y_1 \sim u$. Let $x_1 \in X'$. Then the induced subgraph formed by the vertices \{y_1, x_1, u, v\} is isomorphic to $P_4$. Then by an argument similar to above, we can show that $\mathcal{E}(\Phi) > 2\mu(G)$. Again we get a contradiction. Therefore, $X' \subset N(v)$ and $Y' \subset N(u)$. Consider $x_3 \in X'$ and $y_3 \in Y'$. Take the induced subgraph $H_2$ formed by the vertices \{u, v, u', v', x_3, y_3\} which is given in the Figure 2 and of the form shown in Figure 1. Then $\mu(G) = \mu(H_2) + \mu(G - H_2)$. Also by the Lemma 4.7, $\mathcal{E}((H_2, \varphi)) > 2\mu(H_2)$. Therefore, by the Lemma 4.6, $\mathcal{E}(\Phi) > 2\mu(G)$. Which is a contradiction. Thus $N(u) = Y$. Therefore, $G = K^{n/2, n/2}$.

Claim 2: $\varphi = 1$.

Since $G = K^{n/2, n/2}$, so $\mu(G) = \frac{n}{2}$. Then $\Phi = (K^{n/2, n/2}, \varphi)$ with $\mathcal{E}(\Phi) = n = \mathcal{E}(K^{n/2, n/2})$. Therefore, by the Theorem 3.3 $\Phi \sim (K^{n/2, n/2}, 1)$.

A $k$-walk (or simply walk) in an undirected graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_s\}$ is an alternative sequence of vertices and edges. We simply denote $v_1 - v_2 - \cdots - v_r$ as a $r$-walk from the vertex $v_1$ to $v_r$, where the vertices and edges in this walk may or may not be distinct. We call a walk $v_1 - v_2 - \cdots - v_r$, a path if all the edges in this walk are distinct. If there is a path in between the vertices $v_x$ and $v_y$, then we call $v_x$ and $v_y$ is connected and denoted by $v_x \leftrightarrow v_y$. 23
Let $G_1$ and $G_2$ be two undirected graph with $V(G_1) = \{v_1, v_2, \ldots, v_s\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_t\}$. To avoid the confusion, in definition of Kronecker product, we use the following notation. If $v_i \sim v_j$, then the undirected edge in between them is denoted by $v_iv_j$ and the oriented edge from the vertex $v_i$ to $v_j$ is denoted by $(v_i \rightarrow v_j)$. Let $\Phi = (G_1, \varphi)$ be any $T$-gain graph. A $T$-gain Kronecker product of $\Phi$ and a simple graph $G_2$ is defined as a $T$-gain graph, $\Phi \otimes G_2 = (G_1 \otimes G_2, \psi)$ on an underlying graph $G_1 \otimes G_2$ with vertex set $V(G_1 \otimes G_2) = \{(v_p, u_q) : p = 1, 2, \ldots, s, \text{ and } q = 1, 2, \ldots, t\}$ and edge set $E(G_1 \otimes G_2) = \{(v_p, u_q)(v_a, u_b) : v_p \sim v_a \text{ and } u_q \sim u_b\}$ such that $\psi((v_p, u_q)(v_a, u_b)) = \varphi(v_p v_a)$. The $T$-gain graph $\Phi \otimes K_2$ is called $T$-gain bipartite double, where $K_2$ is a complete graph of 2 vertices. We illustrate the following example of a $T$-gain bipartite double.

**Example 4.1.** Let $G$ be a triangle with vertex set $V(G) = \{v_1, v_2, v_3\}$ and $V(K_2) = \{x, y\}$. Let $\Phi = (G, \varphi)$ be a $T$-gain graph. Then $\Phi \otimes K_2$ is a $T$-gain bipartite double. See Figure 3.

![Figure 3: T-gain bipartite double of $\Phi$ and $K_2$](image)

For any two matrices $P = (p_{ij})_{r_1 \times r_2}$ and $Q = (q_{st})_{s_1 \times s_2}$, the Kronecker product of the matrices $P$ and $Q$ are defined as $P \otimes Q = (p_{ij}Q)_{r_1s_1 \times r_2s_2}$. Now, it is easy to see that $A(\Phi \otimes G_2) = A(\Phi) \otimes A(G_2)$.

The following lemma is an extension of Lemma 2.6 for the $T$-gain graphs.

**Lemma 4.8.** Let $\Phi \otimes G$ be a $T$-gain Kronecker product of a $T$-gain graph $\Phi = (G_1, \varphi)$ and an undirected graph $G$. If $\text{spec}(\Phi) = \{\lambda_1, \lambda_2, \ldots, \lambda_s\}$ and $\text{spec}(G) = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$. Then $\text{spec}(\Phi \otimes G) = \{\lambda_i \gamma_j : i = 1, 2, \ldots, s, j = 1, 2, \ldots, t\}$.

The following lemma is an extension of Lemma 2.8 for the $T$-gain graphs.
Lemma 4.9. If $\Phi = (G, \varphi)$ be any connected $T$-gain graph on a non bipartite graph $G$, then $\mathcal{E}(\Phi) > 2\mu(G)$.

Proof. Let $\Phi = (G, \varphi)$ be a connected $T$-gain graph on a non bipartite graph $G$ with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $m$ edges. If possible let $\mathcal{E}(\Phi) = 2\mu(G)$. Then, by Lemma 4.4, $G$ has perfect matching, say $M$. Let $v_iv_j$ denote the edge between the vertices $v_i$ and $v_j$, if it exists. Let $M = \{v_1v_2, v_3v_4, \ldots, v_{n-1}v_n\}$ be a perfect matching in $G$. Then $\mu(G) = |M| = \frac{n}{2}$. Therefore, $\mathcal{E}(\Phi) = 2\mu(G) = n$. Let $V(K_2) = \{x, y\}$.

Now we consider a $T$-gain Kronecker product $\Phi \otimes K_2$. Then $|V(\Phi \otimes K_2)| = 2n$ and $|E(\Phi \otimes K_2)| = 2m$. It is easy to see that $\Phi \otimes K_2$ is a bipartite graph with a perfect matching $\{(v_1, x)(v_2, y), (v_1, y)(v_2, x), \ldots, (v_{n-1}, x)(v_n, y), (v_{n-1}, y)(v_n, x)\}$. Then $\mu(\Phi \otimes K_2) = n$. Now, by Lemma 4.8, $\mathcal{E}(\Phi \otimes K_2) = 2\mathcal{E}(\Phi) = 2n = 2\mu(\Phi \otimes K_2)$. That is, $\mathcal{E}(\Phi \otimes K_2) = 2\mu(\Phi \otimes K_2)$.

Claim: $\Phi \otimes K_2$ is connected.

The vertex set of $\Phi \otimes K_2$ is $V(\Phi \otimes K_2) = \{(v_1, x), (v_2, x), \ldots, (v_n, x), (v_1, y), (v_2, y), \ldots, (v_n, y)\}$. Since $G$ is connected, so for any pair of vertices $v_i$ and $v_j$, there is a path in between them, $v_i = v_{i_0} - v_{i_1} - \cdots - v_{i_t} = v_j$, (say). Now $v_i$ and $v_j$ is corresponds with the four vertices, $S = \{(v_i, x), (v_i, y), (v_j, x), (v_j, y)\}$ in $V(\Phi \otimes K_2)$. We show that any pair of two vertices in that four vertices set is connected. If $t$ is even, then we have two paths in $(\Phi \otimes K_2)$, $(v_i, x) = (v_{i_0}, y) - (v_{i_1}, y) - \cdots - (v_{i_t}, y) = (v_j, y)$ and $(v_i, y) = (v_{i_0}, y) - (v_{i_1}, x) - \cdots - (v_{i_t}, y) = (v_j, y)$. Thus $(v_i, x) \leftrightarrow (v_j, x)$ and $(v_i, y) \leftrightarrow (v_j, y)$. If $t$ is odd then similarly, $(v_i, x) \leftrightarrow (v_j, y)$ and $(v_i, y) \leftrightarrow (v_j, x)$. Therefore, it is enough to show that $(v_i, x) \leftrightarrow (v_i, y)$. Since $G$ is connected non bipartite graph, so we can always find a walk from $v_i$ to $v_i$ of odd length (walk travels an odd cycle). Then similar to above, $(v_i, x) \leftrightarrow (v_i, y)$. Therefore, $(v_i, x)$ is connected with other three vertices of $S$. Since $v_i$ and $v_j$ are arbitrary pair of vertices of $G$, so any two vertices of $\Phi \otimes K_2$ are connected. Thus $\Phi \otimes K_2$ is connected.

Since $\Phi \otimes K_2$ is a connected bipartite $T$-gain graph of $2n$ vertices with $\mathcal{E}(\Phi \otimes K_2) = 2\mu(\Phi \otimes K_2)$, so by Lemma 4.8, $\Phi \otimes K_2 \sim (K_{n,n}, 1)$. Thus $2m = |K_{n,n}| = n^2$. That is, $|E(G)| = m = \frac{n^2}{2} > \frac{n(n-1)}{2} = |E(K_n)|$. Which is a contradiction. Hence the result.

Lemma 4.10. Let $\Phi = (G, \varphi)$ be any connected $T$-gain graph on $n$ vertices with the matching number $\mu(G)$. If $\mathcal{E}(\Phi) = 2\mu(G)$, then $\Phi \sim (K_{\frac{n}{2}, \frac{n}{2}}, 1)$.

Proof. Since $\mathcal{E}(\Phi) = 2\mu(G)$, so by the Lemma 4.9, $G$ must be bipartite. Therefore, $\Phi = (G, \varphi)$ is a connected bipartite $T$-gain graph. Now, applying the Theorem 4.6 we have $\Phi \sim (K_{\frac{n}{2}, \frac{n}{2}}, 1)$.

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Let $A_1, A_2, \ldots, A_t$ be $t$ square complex matrices. Then we denote $A_1 \oplus A_2 \oplus \cdots \oplus A_t$ as a block diagonal matrix with diagonal blocks are $A_1, A_2, \ldots, A_t$. That is, $\bigoplus_{j=1}^t A_j = \text{diag}(A_1, A_2, \ldots, A_t)$.

In the next theorem, we characterize the class of $T$-gain graphs for which equality holds in Theorem 4.5.

**Theorem 4.7.** Let $\Phi = (G, \varphi)$ be any $T$-gain graph with matching number $\mu(G)$. Then $E(\Phi) = 2\mu(G)$ if and only if each component of $\Phi$ is a balanced complete bipartite $T$-gain graph with a perfect matching together with some isolated vertices.

*Proof.* Let $G_1, G_2, \ldots, G_p, G_{p+1}, \ldots, G_{p+r}$ be the connected components of $G$. Without loss of generality, let us assume that the last $r$ components are the only isolated vertices. Then $\mu(G) = \mu(G_1) + \cdots + \mu(G_p)$. It is clear that $A(\Phi) = \bigoplus_{j=1}^{p+r} A((G_j, \varphi))$, so $E(\Phi) = \sum_{j=1}^p E((G_j, \varphi))$. Therefore, by the Theorem 4.5, we have,

$$2\mu(G) = E(\Phi) = \sum_{j=1}^p E((G_j, \varphi)) \geq 2 \sum_{j=1}^p \mu(G_j) = 2\mu(G)$$

(13)

Thus $E((G_j, \varphi)) = 2\mu(G_j)$, for each $j = 1, 2, \ldots, s$. Now, using the Lemma 4.10, we can derive the result.

As an application of the above theorem, we can establish a relationship among the energy of $T$-gain graph, the vertex cover number and the number of odd cycles. This result generalizes one of the main results of [16]. Let $\Phi = (G, \varphi)$ be a $T$-gain graph with vertex set $V(G)$. Let $u \in V(G)$. Then $(\Phi - u)$ denotes an induced subgraph of $\Phi$ with vertex set $V(G) \setminus \{u\}$.

**Theorem 4.8.** Let $\Phi = (G, \varphi)$ be any $T$-gain graph on $G$ with $c(G)$ number of odd cycles and vertex cover number $\tau(G)$. Then

$$E(\Phi) \geq 2\tau(G) - 2c(G).$$

Equality occurs if and only if each component of $\Phi$ is a balanced complete bipartite $T$-gain graph with a perfect matching together with some isolated vertices.

*Proof.* Let $\Phi = (G, \varphi)$ be any $T$-gain graph with $c(G)$ number of odd cycles. Let us prove the bound using induction on the number of odd cycles $c(G)$. If $c(G) = 0$, then $G$ is bipartite. Therefore $\mu(G) = \tau(G)$. Now, by Theorem 4.5 we have $E(\Phi) \geq 2\mu(G) = 2\tau(G) - 2c(G)$.
Assume that the statement is true for any $T$-gain graph with the number of odd cycles is at most $(c(G) - 1)$. Consider $\Phi$ with $c(G) \geq 1$ number of odd cycles. Let $u$ be a vertex in an odd cycle of $G$. Then the number of odd cycles, say $c'$, of $\Phi - u$ is at most $(c(G) - 1)$. Thus, by induction hypothesis, $E(\Phi - u) \geq 2\tau(G - u) - 2c'$. Since $u$ is an isolated vertex, so, by Lemma 4.3, $E(\Phi) > E(\Phi - u)$.

It is easy to see that $\tau(G - u) \geq \tau(G) - 1$. Therefore, $E(\Phi) > E(\Phi - u) \geq 2\tau(G) - 2c(G)$. Now, let $E(\Phi) = 2\tau(G) - 2c(G)$. If $c(G) \geq 1$, then, by the above observation, $E(\Phi) > 2\tau(G) - 2c(G)$. which is a contradiction. That is $c(G) = 0$. Therefore, $G$ is bipartite and $\mu(G) = \tau(G)$. Thus $E(\Phi) = 2\mu(G)$. Now, by Theorem 4.6, $\Phi$ is the disjoint union of some balanced complete bipartite $T$-gain graphs with a perfect matching together with some isolated vertices.

5 Upper bound of energy of $T$-gain graph in terms of vertex cover number and largest vertex degree

In this section, our main objective is to obtain an upper bound for the energy of a $T$-gain graph in terms of the vertex cover number and the largest vertex degree. This result is the counter part of the corresponding known result about undirected graph [Theorem 1.3] and mixed graph [Theorem 1.4]. Furthermore, we characterize all $T$-gain graphs for which the upper bound is attained. This characterization completely solve one of the open problem [16].

Theorem 5.1. Let $\Phi = (G, \varphi)$ be any $T$-gain graph with the vertex cover number $\tau(G)$, and maximum vertex degree $\Delta(G)$. Then,

$$E(\Phi) \leq 2\tau(G)\sqrt{\Delta(G)}.$$  \hspace{1cm} (14)

Proof. Let $\Phi = (G, \varphi)$ be any $T$-gain graph with vertex cover number $\tau(G)$. We prove the result by induction on $\tau(G)$. If $\tau(G) = 1$, then $G$ must be $K_{1,r}$, for some $r$ together with some isolated vertices. Therefore, $\Phi$ is balanced. Now $E(\Phi) = E(K_{1,r}) = 2\sqrt{r} = 2\tau(G)\sqrt{\Delta(K_{1,r})}$.

Let us assume that for any $T$-gain graph $\Psi = (G_1, \psi)$ with $\tau(G_1) < \tau(G)$, we have $E(\Psi) \leq 2\tau(G_1)\sqrt{\Delta(G_1)}$. Let $U$ be a minimum vertex cover of $G$. Then $|U| = \tau(G) \geq 2$. Let $x \in U$. Let $S$ be an induced subgraph of $G$ which is formed by removing the vertex $x$, and the edges incident with $x$ from $G$. That is $S = G - x$. Then $\tau(S) = \tau(G) - 1$. Therefore, by the induction hypothesis, $E(\Phi - x) = E((S, \varphi)) \leq 2\tau(S)\sqrt{\Delta(S)}$. After a suitable relabeling
of vertices, we can express $A(\Phi)$ as

$$A(\Phi) = \begin{bmatrix} 0 & v^* & 0 \\ v & A_1 & Y^* \\ 0 & Y & A_2 \end{bmatrix} = \begin{bmatrix} 0 & v^* & 0 \\ v & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & A_1 & Y^* \\ 0 & Y & A_2 \end{bmatrix}$$

Here the first column and the first row are associated with the vertex $x$. Let the degree of $x$ be $d$. Then $\begin{bmatrix} 0 & v^* \\ v & 0 \end{bmatrix}$ and $\begin{bmatrix} A_1 & Y^* \\ Y & A_2 \end{bmatrix}$ are the adjacency matrices of the $T$-gain subgraphs $(K_{1,d}, \varphi)$ and $(S, \varphi)$, respectively. By Theorem 2.11 we have

$$\mathcal{E}(\Phi) = 2\tau(G)\sqrt{\Delta(G)}. \quad (15)$$

**Theorem 5.2.** Let $\Phi = (G, \varphi)$ be any $T$-gain graph on $G$ with vertex cover number $\tau(G)$ and maximum vertex degree $\Delta(G)$. Then

$$\mathcal{E}(\Phi) = 2\tau(G)\sqrt{\Delta(G)} \quad (16)$$

if and only if $\Phi$ is the disjoint union of $\tau(G)$ copies of balanced $T$-gain graph $(K_{1,\Delta(G)}, 1)$ together with some isolated vertices.

**Proof.** First let us show that all the vertices of $U$ have the same vertex degree, $\Delta(G)$. Let $x \in U$ be any vertex in $U$ (as in Theorem 5.1). Since $\mathcal{E}(\Phi) = 2\tau(G)\sqrt{\Delta(G)}$, so all the inequalities of (15) become equations. So $\mathcal{E}((S, \varphi)) = 2\tau(S)\sqrt{\Delta(S)}$, and $d = \Delta(S) = \Delta(G)$. As $x$ is arbitrary, so all the vertices of $U$ are of degree $\Delta(G)$.

Now we claim that the underlying graph $G$ is bipartite. Let $W = V(G) \setminus U$. It is clear that $U \setminus \{x\}$ is a minimum vertex cover of the induced subgraph $S$. Also, we have $\mathcal{E}((S, \varphi)) = 2\tau(S)\sqrt{\Delta(S)}$. Now, applying the argument to $S$. Therefore, all the vertices of $U \setminus \{x\}$ in $S$ is of degree $\Delta(S)$. Also we know that $\Delta(S) = \Delta(G)$. Since $d = \Delta(G)$, so there is no edge between the vertex $x$ and the vertices of $U \setminus \{x\}$. As $x$ is arbitrary, so we get no two vertices of $U$ are adjacent. Now $U$ is a minimum vertex cover of $G$, so no two vertices of $W$ are adjacent. Hence $G$ is a bipartite graph with vertex partition sets $U$ and $W$.

Let $G_1, G_2, \ldots, G_p$ be the only nontrivial components of $G$ (That is components contain at least one edge). Then,

$$2\tau(G)\sqrt{\Delta(G)} = \mathcal{E}(\Phi) = \sum_{j=1}^{p} \mathcal{E}((G_j, \varphi)) \leq \sum_{j=1}^{p} 2\tau(G_j)\sqrt{\Delta(G_j)} \leq 2\tau(G)\sqrt{\Delta(G)}. \quad (16)$$
From the above expression, we get \( E((G_j, \varphi)) = 2\tau(G_j)\sqrt{\Delta(G_j)} \) and \( \Delta(G_j) = \Delta(G) \), for \( j = 1, 2, \ldots, p \).

Now let us show that the rank of each component \((G_j, \varphi)\) is 2. Let \( r_j \) be the rank of \((G_j, \varphi)\). Since \((G_j, \varphi)\) is bipartite, so its spectrum is symmetric with respect to origin. Thus \( r_j \) is an even number and \( r_j \geq 2 \). Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{r_j} \) be the nonzero eigenvalues of \((G_j, \varphi)\). Suppose that \( r_j > 2 \). Then \( \lambda_1 \geq \lambda_2 > 0 \). Therefore, by the Cauchy-Schwartz inequality

\[
E((G_j, \varphi)) = \sum_{t=1}^{j} |\lambda_t| < \sqrt{r_j} \sqrt{\sum_{t=1}^{j} \lambda_t^2} = \sqrt{2|E(G_j)|r_j}.
\]

For any \( \mathbb{T}\)-gain graph \( \Psi = (B, \psi) \) on a bipartite graph \( B \), we know that \( |E(B)| \leq \tau(B)\Delta(B) \). By Lemma \( \ref{lem:rank} \) we have \( \text{rank}(\Psi) \leq 2\mu(B) = 2\tau(B) \). Hence \( E((G_j, \varphi)) < 2\tau(G_j)\sqrt{\Delta(G_j)} \), a contradiction (as for each component, \( E((G_j, \varphi)) = 2\tau(G_j)\sqrt{\Delta(G_j)} \)). Hence the rank of \((G_j, \varphi)\) is 2 for \( j = 1, 2, \ldots, p \).

Since each nontrivial component \((G_j, \varphi)\) is bipartite and of rank 2. Now \((G_j, \varphi)\) is of rank 2 if and only if it has exactly one positive eigenvalue. Therefore, by Lemma \( \ref{lem:eigenvalue} \), \((G_j, \varphi) \sim (K_{a,b}, 1)\). Without loss of generality, consider \( a \leq b \). Then \( \tau(G_j) = a \) and \( \Delta(G_j) = b \). Now \( E((G_j, \varphi)) = 2\tau(G_j)\sqrt{\Delta(G_j)} = 2\tau(G_j)\sqrt{\Delta(G_j)} = 2a\sqrt{b} \). On the other hand \( E((G_j, \varphi)) = E((K_{a,b}, 1)) = 2\sqrt{ab} \). Thus \( 2a\sqrt{b} = 2\sqrt{ab} \). Thus \( a = 1 \), and hence \( b = \Delta(G_j) = \Delta(G) \). Therefore, for each \( j = 1, 2, \ldots, p \), \((G_j, \varphi) \sim (K_{1, \Delta(G)}, 1)\).

\[\square\]

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