THE SIGNATURE OF A MANIFOLD

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Abstract

Let us consider a compact oriented riemannian manifold \( M \) without boundary and of dimension \( n = 4k \). The signature of \( M \) is defined as the signature of a given quadratic form \( Q \). Two different products could be used to define \( Q \) and they render equivalent definitions: the exterior product of \( 2k \)-forms and the cup product of cohomology classes. The signature of a manifold is proved to yield a topological invariant. Additionally, using the metric, a suitable Dirac operator can be defined whose index coincides with the signature of the manifold. This second version includes corrections and many examples.
1 Introduction

In differential geometry we can integrate forms and only forms. The degree of the form is given by the dimension of the domain of integration. Say, if we have a three dimensional density of electric charge, the total charge is its integral: the charge density shall be encoded as a 3-form. Likewise, if we want to reproduce the fact that the line integral of a force gives a work, we encode forces as 1-forms. On the other hand, Force = Electric field \times charge, so that the electric field could be encoded as a 1-form also. The Lorentz force represents the capability of the electromagnetic field to modify the world: it is a 1-form given by $\vec{F} = k\vec{E} + c\vec{B} \times \vec{v}$, where $\vec{E}$ is the electric field, a 1-form, and $\vec{B}$ is the magnetic field that must be encoded as a 2-form that operates over a pair of vectors, velocity $\vec{v}$ and displacement.

Let us imagine now a rubber surface filled in electric charges with a given density. Clearly, if we deform the surface, the total electric charge is conserved. Thus, thanks to electric charges we have an invariant. The question is whether or not we can somehow balance the role of the densities of charge in such a way that we get a charge-free invariant, i.e., an invariant that depends only on the manifold. In the following we present a positive partial answer to this question at whose root we find cohomology, which is a byproduct of derivation.

1 Differential forms. The machinery for derivation and integration in differential geometry is built around forms that are defined over a smooth manifold $M$.

Let $T_pM$ be the tangent space to $M$ at $p$. At any point $p \in M$, a $k$-form $\beta$ defines an alternating multilinear map from $k$ factors of $T_pM$ into $\mathbb{R}$:

$$\beta_p: T_pM \times \cdots \times T_pM \to \mathbb{R}$$

An element in the domain can be understood as a parallelepiped localized at $p$ and the form measures its content, say, of electric charge.

The set of all differential $k$-forms on a manifold $M$ at $p$ is a vector space denoted $\Lambda^k_p(M)$. The set of all differential $k$-forms on a manifold $M$ is a vector space denoted $\Omega^k(M)$. So, a charge density is a 3-form, i.e., an element of $\Omega^3(M)$. The space of all forms is denoted as $\Omega(M)$ (Nakahara, [7] 1990; Frankel, [2] 1997).

2
2 Definition. Let us consider a closed riemannian manifold with a metric $(M, g)$. The inner product of the two $p$–forms, $\omega, \eta$, is defined as

$$(\omega, \eta) = (1/p!) \int_M \omega_{\mu_1\ldots \mu_p} \eta^{\mu_1\ldots \mu_p} \sqrt{g} dx^1 \ldots dx^n.$$ 

3 Definition. The Hodge-star operator $\ast$ associates to anyone $p$–form $\eta$ the one and only $(n-p)$–form $\ast \eta$ such that for whatever $p$–form $\omega$ we have:

$$(\omega, \eta) = \int \omega \wedge \ast \eta.$$ 

4 Derivatives. Let $d_k$ denote the exterior derivative of $k$–forms and $\delta_{k+1}$ its adjoint according to the scalar product $(\alpha, \beta) = \int_\Omega \alpha \wedge \ast \beta$. We have that $d_{k+1} \circ d_k = 0$. The subindex of the exterior derivative rarely is explicitly written and $d_k$ is simplified into $d$ and $d_{k+1} \circ d_k = 0$ is noted simply as $d^2 = 0$. The same abuse of notation will be applied to other operators that appear in the sequel. A form $\omega$ is called closed if $d\omega = 0$, coclosed if $\delta\omega = 0$, exact if there exist $\psi$ such that $d\psi = \omega$, coexact if there exists $\phi$ such that $\delta \phi = \omega$.

We know that the divergence of a rotational is zero, and that the rotational of a gradient is also zero. These facts lead to the general law $d^2 = 0$ which more exactly means $d_{k+1} \circ d_k = 0$.

5 Cohomology. In cohomology we take closed forms to investigate whether or not they are exact.

Since $d_{k+1} \circ d_k = 0$ then $\text{Im}(d_k) \subseteq \ker(d_{k+1})$ and moreover both are vector spaces. The quotient $H^k = \ker(d_{k+1})/\text{Im}(d_k)$ is a vector space that is called the de Rham $k$-cohomology vector space.

How are the classes of $H^k$? If $\omega_1$ and $\omega'_1$ belong to the same cohomology class then they belong in the kernel of $d_{k+1}$, therefore $d_{k+1} \omega = d_{k+1} \omega' = 0$ and moreover they differ by an exact form $d_k \tau$, so $\omega_1 = \omega'_1 + d_k \tau$ with $d_{k+1} \circ d_k \tau = 0$. On the other hand, the operator $d$ is antisymmetric. Hence, $d_k \omega = 0$ for $k > n$ given that $n$ is the dimension of the manifold. This implies that $H^k(M) = 0$ for $k > n$. 

3
We have presented cohomology in terms of classes of closed forms modulo exact differential forms, that operate locally. But cohomology can also be given in terms of holes, great topological items. For instance, to say that $H^k(M) = 0$ for $k > n$ is equivalent to saying that holes cannot have a dimension higher than $n$.

6 Example: the circle vs. $\mathbb{R}$.

Let us consider $\mathbb{R}$ and a closed smooth trajectory $\gamma$ over it that begins and ends in $a$ together with a 1-form $\omega = f(x)dx$ with $f$ a continuous function. Then, there exist $F$ such that $dF = f$ and $\int_\gamma \omega = \int_a^a dF = F(a) - F(a) = 0$. By contrast, let us turn now to the circle. It has a problem: trajectories appear that are closed and smooth but that does not un-walk what was walked. So, the form that measures arc length, $d\theta$, has the property : $\int_a^a d\theta \neq 0$. This seems to contradict the proposition saying that $\int_\gamma d\theta = \theta(a) - \theta(a) = 0$. The veto to this and similar reasonings about all kinds of holes in higher dimensions is cohomology. This mathematical concept is an abstraction that is embedded in our world: the existence of electric motors is due to the fact that $H^1(\mathbb{R}^2 - \{0\})$ is not trivial. Similar effects are observed in the theory of fundamental interactions.

The cohomology of the circle in terms of differential forms reads as follows: differential forms of degree 0 consist on functions $h(\theta)$ that take on real values. A real function $f$ is closed if $df = 0$, i.e., if $f$ is constant. Constant functions $= H^0$ forms a vector space of dimension 1. At the other hand, Differential forms of degree 1 are of the form $\omega = f(\theta)d\theta$. They are always closed because $d\omega = (\partial f/\partial \theta)d\theta \wedge d\theta = 0$. If one has two differential closed forms $\omega_1 = f(\theta)d\theta$ and $\omega_2 = g(\theta)d\theta$ then, $\omega_2 = \omega_1 + (g(\theta) - f(\theta))d\theta = \omega_1 + dF$ if $f$ and $g$ are continuous. Therefore $H^1$ also has dimension 1 and is generated by arc-length or by whatever other not null 1-form.

The general duality among differential forms and holes is consecrated by the Stokes’ theorem. Thinking of the electric field would help us to understand the situation: if an electric field has a net flux across a closed surface that borders a spatial region $M$, it is because inside $M$ there shall be electric charges that function like sources or sinks of field. So, the flux that is an integral over the border of $M$ must be equal to the net balance of creation
vs. annihilation of field inside $M$ that is related to the integration over $M$ of charge densities. Now, an electric charge represents a hole in the domain of the field because there is no manner of defining it to extend continuity and differentiability. Formally:

7 **Stokes’ theorem.** If a field can be represented by a differential form $\omega$, then the net flux of the field across the border $\partial M$ of an orientable manifold $M$ must be equal to the net birth-death balance of field that happens at the interior of $M$ and that is measured by $d\omega$:

$$
\int_{\partial M} \omega = \int_M d\omega
$$

The theorem literally states a truth for a differential form $\omega$ but indeed it is also true for its cohomology class:

$$
\int_{\partial M} [\omega] = \int_M d[\omega]
$$

Proof:

$$
\int_{\partial M} [\omega] = \int_{\partial M} \omega + d\phi = \int_M d(\omega + d\phi) = \int_M d[\omega] = \int_M d\omega + d^2\phi = \int_M d\omega.
$$

8 **Poincaré’s lemma and Betti numbers.**

We use to understand cohomology in the light of the **Poincaré’s lemma**: a closed form that is defined over a domain that is contractile into a point is also exact. Thus, a differential form operates locally but its exactness depends on large topological properties. In that way, cohomology connects local and global properties. In fact, Poincaré’s lemma allows us to see the de Rham cohomology vector space as an obstruction to the global exactness of closed forms. The dimension of a cohomology vector space is finite and is known as the **Betti number** of the vector space and therefore could be seen as a measure of the variability of the global inexactness of closed forms.

Thus, Betti numbers measures obstructions to contractibility to a point, i.e., holes and disconnections in the domain:

1. $b_0$ is the number of connected components.
2. $b_1$ is the number of one-dimensional or “circular” holes.

3. $b_2$ is the number of two-dimensional holes or “voids”.

4. $b_n$ is the number of $n$-dimensional holes.

A sphere $S$ has one large $n$-hole and nothing else. So, all Betti numbers are zero except $b_0$ (which is 1 for $n > 0$ and 2 for $n = 0$ because $S^0$ consists in two points) and $b_n = 1$.

Besides, we have the following three statements:

1. If $M$ is contractile to a point then, by Poincaré’s lemma, then all closed forms are exact. Therefore, $H^p(M; \mathbb{R}) = 0$.

2. If $M$ is a compact, connected, orientable manifold, and $\dim M = n$, then $H^n(M; \mathbb{R}) = \mathbb{R}$. This applies to all spheres: $H^n(S^n; \mathbb{R}) = \mathbb{R}$.

3. If $M$ is a compact, connected, non-orientable manifold, and $\dim M = n$, then $H^n(M; \mathbb{R}) = 0$. For $M = \text{Möbius strip}$, $H^2(M; \mathbb{R}) = 0$.

4. If $M$ is a non-compact, connected, orientable or non-orientable manifold, and $\dim M = n$, then $H^n(M; \mathbb{R}) = 0$. All $\mathbb{R}^n$ have $H^n(\mathbb{R}^n; \mathbb{R}) = 0$.

5. Forms that measure lengths, areas, volumes, ..., are not exact on a compact manifold.

We will illustrate the theory with some specific calculations over spheres, toruses and projective spaces. So, let us review some material about them.

9. **Our notation for spheres is as follows:**

   The $n$-sphere of $\mathbb{R}^{n+1}$ is $S^n = \{x \in \mathbb{R}^{n+1} | d(x, 0) = 1\}$. $S^n$ is a compact orientable manifold of dimension $n$.

   The $n$-disc of $\mathbb{R}^n$ is $D^n = \{x \in \mathbb{R}^n | d(x, 0) \leq 1\}$. The $n$-disc $D^n$ is a compact orientable manifold of dimension $n$.

   The frontier of $D^n$ is the $n-1$-sphere: $\partial D^n = S^{n-1}$. The frontier has one less dimension.

   The $n$-cell $e^n$ is a set that is homeomorphic to the open disc $D^n - \partial D^n$. By definition, $e^0$ is a point.
10 The algebraic topology decomposition. Our intuition suffers too much in spaces of higher dimensions, so we recur to algebraic trickery. We can see how it functions if we pay attention to a very simple case: $S^2$.

If we punch $S^2$ at a point, we get the sphere without a point: it is an open set that can be covered by $\mathbb{R}^2$ as a single chart. So, it is also equivalent to the open cell $e^2$. We say that the punctured $S^2$ retracts to $e^2$.

We can specify how to reconstruct $S^2$: we invert the retracting process of the punctured $S^2$ towards $e^2$. We use the jargon: glue the border of $e^2$ to the point. Algebraic topologists say:

$$S^2 = e^0 \cup e^2.$$ 

The same procedure is valid for every sphere.

One can transmit the same information using the Poincaré polynomial that results from combining Betti number, $b_k$, as coefficient and the order of cohomology as power $k$ in the polynomial:

$p(t) = \Sigma b_k t^k$

For the sphere $S^2$:

$p(t) = 1 + t^2$.

This says that constant functions span $H^0$, that the order one cohomology of the sphere is zero, since every laze over the sphere can retract to a point. At last, $S^2$ has the volume element that is closed but not exact. So, the order 2 cohomology of the sphere is not zero. Now, since the sphere has only one 2-hole, its $H^2$ must have only one generator.

We will use Poincaré polynomials as Ansatz to ease calculations. To see how this works, let us consider the Torus $T = S^1 \times S^1$: if we punch it, a normal donut, we find two non contractible circles, one horizontal and the other vertical, and a two dimensional cover. So, for $S^1 \times S^1$ we have

$p(t) = 1 + 2t + t^2$.

that says that the Torus results from gluing a point to two circles to a 2-cell that serves as cover:

$$T = e^0 \cup 2e^1 \cup e^2.$$ 

Let us observe now that we can find that polynomial as the square of $q(t) = 1 + t$, the polynomial of $S^1$:

$p(t) = (1 + t)^2 = 1 + 2t + t^2$. 

7
So, our guess is that the polynomial associated to the cartesian product of two (compact, without boundary, oriented) manifolds is the product of the polynomials associated to the manifolds.

11 Example. Cohomology of a cylinder.

An open cylinder is a surface that has two 1-forms in cylindrical coordinates: \( d\theta \) and \( dz \). The first one is a generator of \( H^1 \) while the second is not because all pure-\( z \) closed curves are contractile to a point.

12 Quotient spaces.

We can infer how to work with quotient spaces if we look at the circle: it is a manifold in its own but it can be seen as a quotient space: the circle is the winding of \( \mathbb{R} \) that is furnished with the equivalence relation \( x \sim y \) iff \( y = x + 2k\pi \). Let us notice that winding preserves orientability. To see this, let us imagine that \( \mathbb{R} \) is a trajectory that winds over the circle. One chooses a tangent vector at a given point over \( \mathbb{R} \) and then one notices that the corresponding winding always observes the same direction: the circle is orientable and its orientation can be given by an orientation over \( \mathbb{R} \).

The element of volume in the circle is \( d\theta \) which can be seen as the differential form of coordinate \( \theta \), the argument over the circle, else as the coordinate of \( \mathbb{R} \) in the base space of the quotient space \( \mathbb{R} / \sim \). We have that \( d\theta \) over \( \mathbb{R} \) is exact but over the circle is not because there it measures arc length which is \( 2\pi r \) over the whole circle over a path with start and final points coincident. There is no contradiction because \( \theta \) to be a (uni-valued) chart function needs to be restricted to the open set \((0, 2\pi)\) and cannot be extended any further.

13 The decomposition of the real projective plane.

Real projective space \( \mathbb{RP}^n \) is defined to be the space of all lines through the origin in \( \mathbb{R}^{n+1} \). Formally: \( \mathbb{RP}^n \) is the quotient space of \( \mathbb{R}^{n+1} - \{0\} \) under the equivalence relation \( v \sim \lambda v \) for scalars \( \lambda \neq 0 \).
The algebraic topology decomposition of the real projective space is found by induction.

\( \mathbb{RP} = \mathbb{RP}^1 \) can be seen as follows:

1. \( \mathbb{RP} \subset \mathbb{R}^2 \) is the set of lines that pass through the origin. This space has dimension one: every such a line can be represented by the two points of intersection of the line with the circle, the 1-sphere \( S = S^1 \subset \mathbb{R}^2 \), but with antipodal points identified. One can imagine it as the superior hemicircle with its terminal antipodal points identified. (So, \( \mathbb{RP} \) looks like \( S^1 \).)

2. That superior hemicircle can be smashed into a disc \( D^1 \subset \mathbb{R}^1 \) to get that \( \mathbb{RP} \) is the quotient space of a disc \( D^1 \) with terminal points \( \partial D^1 \) identified. We get a circle: \( \mathbb{RP}^1 = S^1 = e^0 \cup e^1 \) that means that \( S^1 \) is just the open interval \((0, 1)\) joined to a point. The topologies of \( S^1 \) and \( \mathbb{RP}^1 \) are the same but the winding number is multiplied by two in \( \mathbb{RP}^1 \) that means that while you make a round trip in \( S^1 \) you gives two round trips in \( \mathbb{RP}^1 \).

If we pass to the next dimension, we get:

1. \( \mathbb{RP}^2 \subset \mathbb{R}^3 \) is just the sphere \( S^2 \subset \mathbb{R}^3 \) with antipodal points identified. One can imagine it as the superior hemisphere with the antipodal points of the bottom circle identified. (This circle functions as a single point, so \( \mathbb{RP}^2 \) looks like \( S^2 \).)

2. That superior hemisphere can be smashed into a disc \( D^2 \subset \mathbb{R}^2 \) to get that \( \mathbb{RP}^2 \) is the quotient space of a hemidisc \( D^2 \) with antipodal points of its border \( S^1 = \partial D^2 \) identified.

3. Since \( \partial D^2 = S^1 \) with antipodal points identified is just \( \mathbb{RP}^1 \), we see that \( \mathbb{RP}^2 \) is obtained by attaching a 2-cell \( e^2 \) to \( S^1 = \mathbb{RP}^1 \). So, we get \( \mathbb{RP}^2 = S^1 \cup e^2 = e^0 \cup e^1 \cup e^2 \)

In general, we have:

1. \( \mathbb{RP}^n \subset \mathbb{R}^{n+1} \) is just the sphere \( S^n \) with antipodal points identified. One can imagine it as the superior hemisphere with the antipodal points of the bottom hyper-circle identified.
2. This hemisphere can be smashed into a disc to get that $\mathbb{RP}^n$ is the quotient space of a hemidisc $D^n \subset \mathbb{R}^n$ with antipodal points of its border $\partial D^n$ identified.

3. Since $\partial D^n$ with antipodal points identified is just $\mathbb{RP}^n$, we see that $\mathbb{RP}^n$ is obtained by attaching an $n$-cell $e^n$ to $\mathbb{RP}^{n-1}$ but taking care of gluing the structure of lower dimension to the border of the higher one.

Therefore, the decomposition of $\mathbb{RP}^n$ is $e^0 \cup e^1 \cup \ldots \cup e^n$ with one cell $e^i$ in each dimension $i \leq n$. Because we have one hole in each dimension, we might predict that all Betti number are one. Nevertheless, one must take care of orientability:

1. $\mathbb{RP} = \mathbb{S}$ is orientable.

2. $\mathbb{RP}^2 = S^2$ with antipodal points identified is not orientable: if one walks over the sphere $S^2$ through a maximal circle and one carries in parallel transport a frame $\vec{t}, \vec{j}$, then while one remains in the upper hemisphere, the corresponding frame in $\mathbb{RP}^2$ has the same orientation. But if one passes to the lower hemisphere, the corresponding frame in $\mathbb{RP}^2$ reverses orientation and so we cannot have a single valued orientation from the different parts of $S^2$: $\mathbb{RP}^2$ is not orientable. Hence, its 2-Betti number is zero.

3. In general, $\mathbb{RP}^n$ is orientable for odd dimensions and non-orientable for even dimensions.

4. The Betti numbers cannot be read from the cell structure in even dimensions because this depends on gluing maps whose output depends on orientability.

**Remark.** $\mathbb{RP}^n$ can be understood as the compactification of $\mathbb{R}^n$. To fix ideas, let us show that $\mathbb{RP}^2$ can be seen as the compactification of the plane $\mathbb{R}^2$. To begin with, $\mathbb{RP}^2$ contains a copy of the plane $\mathbb{R}^2$. In fact, every point of the plane $z = 1$ defines a unique line that passes through the origin of $\mathbb{R}^3$: that is why we say that the plane $\mathbb{R}^2$ is contained in $\mathbb{RP}^2$. Nevertheless, $\mathbb{RP}^2$ is bigger than the plane $\mathbb{R}^2$. To get $\mathbb{RP}^2$, we need to add the set of horizontal lines. That set is homeomorphic to $S^1$ with antipodal points identified. Now, the plane $z = 1$ is homeomorphic to the upper part of $S^2$ which in its turn is homeomorphic to $e^2$, the open disc of the plane $z = 0$. 
So, the compactification of the plane \( \mathbb{C}^2 \) is made by \( S^1 \) with antipodal points identified, which functions as a point. At last, \( \mathbb{R}P^2 \) looks like a sphere.

15 The decomposition of the complex projective plane.

Complex projective space \( \mathbb{C}P^n \) is the space of complex lines through the origin in \( \mathbb{C}^{n+1} \). Formally, \( \mathbb{C}P^n \) is the quotient space of \( \mathbb{C}^{n+1} - \{0\} \) under the equivalence relation \( v \sim \lambda v \) for \( \lambda \neq 0 \). To fix ideas, let us consider the point \( P = (1, 0, ..., 0) \in \mathbb{C}P^n \). The complex line through \( P \) is the set composed of all points of the form \( \lambda P = (\lambda, 0, ..., 0) \) for \( \lambda \in \mathbb{C} \). This set is just a copy of \( \mathbb{C} \) of complex dimension 1 and real dimension 2. The same happens for every other line.

What is the essence of \( \mathbb{C}P^n \) from a topological point of view? The following construction shows us that \( \mathbb{C}P^n \) can best be understood as a generalized Riemann sphere with a recursive construction, i.e., as the compactification of \( \mathbb{C}^n \) that uses a simple constructive algorithm. This results from the way as one builds \( \mathbb{C}P^n \) from \( \mathbb{C}P^{n-1} \):

1. \( \mathbb{C}P^n \subset \mathbb{C}^{n+1} \) has complex dimension \( n \). A line in \( \mathbb{C}^n \) can be denoted by any non zero point in it. Let us denote the line in \( \mathbb{C}^n \) through point \((z_1, ..., z_{n+1})\) as \([z_1, ..., z_{n+1}]\). Let \( j \) be one of those coordinates that are not zero. We have \([z_1, ..., z_{n+1}] = [(\frac{z_1}{z_j}, ..., 1, ..., \frac{z_{n+1}}{z_j})] \leftrightarrow [(w_1, ..., w_n)] \) with \( w_i \in \mathbb{C}^n \). We conclude that \( \mathbb{C}P^n \) can be covered with \( n \) patches each one a copy of \( \mathbb{C}^n \). We refer to this system of coordinates as homogeneous.

2. \( \mathbb{C}P^n \) is compact. In aforementioned coordinates \( \mathbb{C}P^n \) looks unbounded, so it is convenient to think that it inherits from a sphere a metric and a topology that makes it a compact manifold. In fact, the unit sphere \( S^{2n+1} \) of \( \mathbb{C}^{n+1} = \mathbb{R}^{2n+2} \) is another cover of \( \mathbb{C}P^n \); every line that passes through the origin has a point of norm one. The corresponding association defines a continuous function from the sphere, which is compact, onto \( \mathbb{C}P^n \). Since the image of a compact subset is compact, we have that \( \mathbb{C}P^n \) is compact. The topology of \( \mathbb{C}P^n \) is that of the quotient space \( S^{2n+1} \) by the \( \lambda \) relation.
3. \( \mathbb{CP}^n \) can be understood as the compactification of \( \mathbb{C}^n \). In first place, \( \mathbb{C}^n \) can be embedded into \( \mathbb{CP}^n \) and, in second place, the points of \( \mathbb{CP}^n \) that are not in the embedding can be understood as the points in infinity of \( \mathbb{C}^n \). This is shown as follows. The embedding of \( \mathbb{C}^n \) into \( \mathbb{CP}^n \) is given by

\[
(z_1, ..., z_n) \leftrightarrow [(1, z_1, ..., z_n)]
\]

Equivalently, we observe \( \mathbb{CP}^n \) through the hyper-plane \( z_0 = 1 + 0i \) of real dimension \( 2n \).

Let us notice now that the points of the form \( [(1, z_1, ..., z_{n+1})] \) do not cover \( \mathbb{CP}^n \) completely: we lack those points of the form \( [(0, z_1, ..., z_{n+1})] \). Let us see now why these points can be understood as points at infinity in \( \mathbb{C}^n \). In fact, we can represent a point of \( \mathbb{C}^n \) in infinite as \( (\lambda z_1, ..., \lambda z_n) \) for \( \lambda \to \infty \).

The embedding gives

\[
[(1, \lambda z_1, ..., \lambda z_n)] = [(\frac{1}{\lambda}, z_1, ..., z_n)] \to [0, z_1, ..., z_n]
\]

This shows that \( \mathbb{CP}^n \) becomes compact by adding the points at infinity of \( \mathbb{C}^n \). Nevertheless, our procedure seems to produce two discrete units. To remedy this trouble, we use the sphere as intermediary: every line that passes through the origin also can be represented over the sphere \( S^{2n+1} \). Actually, the demi-sphere on the side of \( (1, 0, ..., 0) \) suffices, whose border represents the points at infinity. Concretely, the border is \( S^{2n-1} \) because it is the sphere in \( \mathbb{C}^n = \mathbb{R}^{2n} \). So, this border with the \( \lambda \) equivalence is what serves for the compactification of \( \mathbb{C}^n \).

4. The topological structure of \( \mathbb{CP}^n \) is given by \( \mathbb{CP}^n = e^0 \cup e^2 \cup ... \cup e^{2n} \).

To see this, let us observe that the points at infinity \( [(0, z_1, ..., z_n)] \) are stable under elongation, as it should be. But to be stable under elongation is the trade mark of projective spaces. That is why the points at infinity represent an embedding of \( \mathbb{CP}^{n-1} \) into \( \mathbb{CP}^n \). From this we deduce the way to form \( \mathbb{CP}^n \) from \( \mathbb{CP}^{n-1} \): we must paste \( \mathbb{CP}^{n-1} \) to the infinite of (the embedding of) \( \mathbb{C}^n \), which must be understood as an open set, which in its turn is homeomorphic to \( e^{2n} \). By recursion we get \( \mathbb{CP}^n = e^0 \cup e^2 \cup ... \cup e^{2n} \). This construction proposes that its Poincaré polynomial is

\[
p(t) = 1 + t^2 + ... + t^{2n}.
\]
16 **Remark.** Our representation of projective spaces by means of quotient spaces of spheres allows us to see that these spaces are bounded and that every path contained in any one of them always leads to an interior point. So, projective spaces are closed and without boundary. Officially, they are boundaryless, compact manifolds. This representation also allows us to associate a volume form and a volume to the entire manifold if it is orientable. The volume of the manifold would be equal to just the volume of the cell of highest dimension contained in the given quotient space. Glued parts of lower dimensions have measure zero. Since apart from our representation there exist many others (just change the radius of the representing sphere), we will say that the volume of the quotient manifold is an unspecified positive number $c$ and our propositions will state relations with that number $c$.

17 **Electric motors and cohomology.**

We know from experience that electromagnetism exercises forces over charged particles that eventually can be in movement. A classical description of this activity is given by the Lorentz force $\vec{F} = k\vec{E} + c\vec{v} \times \vec{B}$, where $E$ es the electric field, $\vec{B}$ is the magnetic field and $\vec{v}$ is the velocity of the charged particle. This equation allows us to understand how an electric motor functions:

1. The term $\vec{F} = c\vec{v} \times \vec{B}$ says that if the velocity of an electron is perpendicular to a magnetic field, then it is deflected sideways. If instead of one electron we consider an electric current along a wire, the deflection of electrons will be passed to the wire because electrons can move along the wire but as they try to leave the wire, a separation of charges of different polarity occurs and so the wire is attracted and dragged by the electrons.

2. If the wire forms a square, it will begin to spin. In fact, if one of its sides deflects in one direction, the opposite side will deflect in contrary direction because corresponding currents have opposite directions.

3. The spinning force can be made to endure forever if an appropriate switching mechanism is endowed to cause the direction of current to reverse in agreement with the position of the wire (Nave, 2013).
Now, classic electromagnetism and electric motors cannot be considered without cohomology. Let us see why.

There are many versions of electric motors but all rely on magnets. Some depend on the magnetic field created by currents along a wire, other depend on permanent magnets.

In regard with electromagnets, magnetic field lines form circles around a (straight, infinite) wire carrying an electric current. The magnitude of this field grows as one approaches the wire. That is why the magnetic field cannot be defined directly over the wire because it would be not univalued. Therefore, the domain of definition of the magnetic field has a hole composed by the wire which is equivalent to a hole in a plane.

On the other side, permanent magnet results from atoms that function as tiny magnets whose fields are aligned. One can imagine in classical mechanics that these atoms have an unbalanced electron that spins around the nucleus and that this movement generates a magnetic field. That field is well defined everywhere with exception of the trajectory of the electron: we end with the same cohomology: that of a plane with a hole.

The next reasoning shows that the non nil cohomology is not just a classical effect but that it is endemic to electromagnetism.

In the depiction of electromagnetism given by the Lorentz force, it is a collage of two items: electricity and magnetism. The unification of these two fields was given by relativity: electricity and magnetism are two particular folds of a single entity that populates space-time, the electromagnetic field.

The electromagnetic field is described by a 2-form $T$. It happens that $T$ can be expressed as a differential: $T = dA$ where $A$ is a 1-form, the vector potential. But, beware, $T$ is an experimental not nil object. This implies that $A$ itself cannot be exact, i.e. there is no scalar function $\alpha$ such that $A = d\alpha$. Otherwise $T = dA = d^2\alpha = 0$. In other words, we are declaring that the cohomology vector space $H^1$ is not nil. Therefore, the tensor field cannot have global definition: it must have holes else be defined by sectors. De facto, the electromagnetic field cannot be defined in the points that are occupied by electric charges. A charge defines a punctual hole in space but a line in space-time. So, its cohomology in space-time is just that of a plane with a punctual hole. That is why we say that the existence of electric motors is due to the fact that $H^1(\mathbb{R}^2 - \{0\})$ is not trivial.

A very soft introduction to these themes can be found in Rodriguez ([9], 2008).
2 The signature quadratic form

We define a bilinear form and find the constraints under which it becomes symmetrical.

18 Example to show the whole idea.

The Torus \( S^1 \times S^1 \) is compact, without boundary, orientable manifold of real dimension 2. Its Poincaré polynomial is:

\[
p(t) = (1 + t)^2 = 1 + 2t + t^2
\]

This polynomial says that a torus can be decomposed as a point to which two circles are glued to which a covering 2-cell must be attached. This means that \( H^1 \) has 2 generators, \( d\theta \) and \( d\phi \), while \( H^2 \) has 1, \( d\theta d\phi \).

We can define a quadratic form as follows:

\[
Q : H^1(M) \otimes H^1(M) \to \mathbb{R}
\]

\[
Q([\omega_1], [\omega_2]) = \int_M \omega_1 \wedge \omega_2 = \int \omega_1 \wedge \omega_2
\]

This bilinear form can be represented by a \( 2 \times 2 \) matrix that we also call \( Q \), whose entries represent respectively

\[
\int d\theta \wedge d\theta = 0,
\int d\theta \wedge d\phi = c, \text{ in the same class as the area-form.}
\int d\phi \wedge d\theta = -c,
\int d\phi \wedge d\phi = 0
\]

So,

\[
Q = \begin{pmatrix}
d\theta & d\phi \\
d\phi & \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}
\end{pmatrix}
\]

The characteristic polynomial of this matrix is:

\[
p(\lambda) = \lambda^2 + c^2.
\]

The roots of this polynomial are imaginary and cannot be compared with zero. The problem is just that matrix \( Q \) is not symmetric, a failure that in
its turn hangs on the lack of commutativity of the wedge product. So, \( Q \) will be a bilinear symmetric form when the wedge product is commutative, a property that holds in dimensions that are multiple of 4.

19 Definition. Let \( M \) be a compact oriented riemannian manifold without boundary and of dimension \( n=4k \). The signature quadratic form of \( M \) is defined to be the bilinear form

\[
Q : H^{2k}(M) \otimes H^{2k}(M) \to \mathbb{R}
\]

\[
Q([\omega_1],[\omega_2]) = \int_M \omega_1 \wedge \omega_2 = \int \omega_1 \wedge \omega_2
\]

20 Theorem. The signature quadratic form is well defined, i.e., the integral on the right does not depend on the representatives of the cohomology classes.

To prove claimed result, let us take another representative of \([\omega_1]\) say \( \omega'_1 = \omega_1 + d\tau \). Now, whatever representative one uses, one gets the same result:

\[
\int \omega'_1 \wedge \omega_2 = \int (\omega_1 + d\tau) \wedge \omega_2 = \int \omega_1 \wedge \omega_2 + \int d\tau \wedge \omega_2
\]

\[
= \int \omega_1 \wedge \omega_2 + \int d(\tau \wedge \omega_2) = \int \omega_1 \wedge \omega_2 + 0
\]

We have used the fact that \( d(\tau \wedge \omega_2) = d\tau \wedge \omega_2 \pm \tau \wedge d\omega_2 = d\tau \wedge \omega_2 \) because \( \omega_2 \) is closed. And that

\[
\int d(\tau \wedge \omega_2) = \int_M d(\tau \wedge \omega_2) = \int_{\partial M} \tau \wedge \omega_2 = 0
\]

because our manifolds have no boundary, like the spheres. Extending the reasoning to the general form of representatives of \([\omega_2]\) we obtain the independence of the bilinear form from representatives and so \( Q \) is defined over pairs of cohomology classes of order \( 2k \).

Let us prove now that when \( n = 4k \), we get a symmetric bilinear form:
21 Theorem. The signature quadratic form $Q$ is symmetric iff $n = 4k$. Therefore, its eigenvalues are real.

Proof: For a $p$-form $\omega$ and a $q$-form $\nu$ we have the general rule:
\[
\omega \wedge \nu = (-1)^{pq} \nu \wedge \omega
\]
For our case we have $p = q$ then $(-1)^{pq} = (-1)^{p^2}$. But $p^2 = p \mod 2$, i.e., $p^2 - p = p(p - 1) = 2m$, i.e., either $p$ or $p - 1$ is even and hence $p(p - 1)$ is always even. So, $(-1)^{p^2} = (-1)^p = 1$ iff $p$ is even iff $2p$, the dimension of the manifold, is multiple of 4.

22 Definition. When $Q$ is symmetric, its eigen-values are real and can be compared with zero. In that case, we can compute the signature of the quadratic form $Q$ as the number of positive eigenvalues minus the number of negative ones.

2.1 The cup product

Since the wedge product is defined for pairs of forms of any order we can verify that the vector space $H^* = H^*(M) = \bigoplus_{p \geq 0} H^p M$ can be endowed with a ring structure as follows: for $[\omega] \in H^p$ and $[\nu] \in H^q$ we define the cup product as
\[
\sqcup : H^* \times H^* \to H^*
\]
\[
[\omega] \sqcup [\nu] = [\omega \wedge \nu]
\]
Let us check that this is a well defined product in the set of cohomology classes: if $[\omega] \in H^p$ and $[\nu] \in H^q$ then $\omega \wedge \nu \in \Omega^{p+q}$ and $[\omega \wedge \nu] \in H^{p+q}$ but we shall show that if we take other representatives their wedge product is still in $[\omega \wedge \nu]$.

If $\omega, \omega' \in [\omega]$ in $H^p$ then $d\omega = d\omega' = 0$ and $\omega' = \omega + d\tau$. If $\nu \in H^q$ then $d\nu = 0$. Therefore $\omega' \wedge \nu = (\omega + d\tau) \wedge \nu = \omega \wedge \nu + d\tau \wedge \nu = \omega \wedge \nu + d(\tau \wedge \nu)$ because $d(\tau \wedge \nu) = d\tau \wedge \nu + \tau \wedge d\nu = d\tau \wedge \nu$. Henceforth, $\omega' \wedge \nu$ and $\omega \wedge \nu$ differ by an element of the form $d(\alpha)$ where $\alpha = \tau \wedge \nu$ and so they belong to the same cohomology class.

Restricting the cup product to $H^{2k}(M) \otimes H^{2k}(M)$, the signature quadratic form $Q$ takes the form:
\[ Q([\omega_1], [\omega_2]) = \int [\omega_1] \cup [\omega_2] = \int \omega_1 \wedge \omega_2 \]

We see that the cup product is commutative when restricted to \( H^{2k}(M) \otimes H^{2k}(M) \) because \( Q \) is symmetric when \( n = 4k \).

### 2.2 The signature of a manifold, \( \sigma(M) \)

Since the space \( H^{2k} \) is finite dimensional with dimension \( b_{2k} \), called the 2k-th Betti number, we can take a basis \( E = \{[h_i]\} \) with respect to which the quadratic form \( Q \) has an associated matrix, \( Q[E] \), whose entries are \( Q[E]_{ij} = \int [h_i] \wedge [h_j] \). This matrix can be diagonalized to a matrix with eigenvalues \( \lambda_1, \lambda_2, ... \lambda_s \), where \( s = \dim H^{2k} = b_{2k} \). Below we will see a concrete basis in which \( Q \) is diagonal.

Let us recall that the signature of a real symmetric matrix is the number of positive eigenvalues minus the number of negative eigenvalues. The signature of a matrix is invariant under changes of basis with the same orientation. This happens because the signature of a real symmetric matrix measure an intrinsic property. In fact, a matrix represents a linear transformation whose number of positive eigenvalues is the dimension of the maximal vector subspace over which it is positive-definite. All these results enable the next

#### 23 Definition. The signature of compact, without boundary, orientable manifold, \( \sigma(M) \), is the signature of the quadratic form \( Q \).

Let us find the signature over some examples. To this aim, we will follow the general procedure stated above to calculate the signature of a riemannian boundariless manifold and of dimension \( 4k \): it is the number of positive eigenvalues minus the number of negative eigenvalues of the finite dimensional matrix of \( Q \) in any basis of \( H^{2k} \) and in any system of coordinates, where \( Q \) is the bilinear form

\[ Q : H^{2k}(M) \otimes H^{2k}(M) \to \mathbb{R} \]

\[ Q([\omega_1], [\omega_2]) = \int \omega_1 \wedge \omega_2 \]
24 Example. Let us calculate the signature of $S^4$.

$S^4$ contains only one big hole, so its decomposition is $S^4 = e^0 \cup e^4$
and its Poincaré polynomial is
$p(t) = 1 + t^4$.
There is no terms with intermediate powers because the sphere has no low dimensional holes.
Thus, to calculate the signature of $S^4$, we must consider the space $H^2(S^4) \otimes H^2(S^4)$ but $H^2$ is zero. So, the signature is zero.

25 Example. Let us find the signature of the 4-torus $T^4 = S^1 \times S^1 \times S^1 \times S^1$.

We consider $T^4$ as a compact manifold over the reals of dimension 4. Its Poincaré polynomial is:

\[ p(t) = (1 + t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4 \]

or

\[ T^4 = e^0 \cup 4e^1 \cup 6e^2 \cup 4e^3 \cup e^4 \]

This means that $H^2$ has 6 generators while $H^4$ has 1. Thus, $Q$ is a $6 \times 6$ matrix. Specifically, $H^2$ is generated by $d\theta_1 d\theta_2$, $d\theta_1 d\theta_3$, $d\theta_1 d\theta_4$, $d\theta_2 d\theta_3$, $d\theta_2 d\theta_4$, $d\theta_3 d\theta_4$. On the other hand, $d\theta_1 d\theta_2 d\theta_3 d\theta_4$ is the generator of $H^4$ which is in the same class as the form that measures the 4-area of $T^4$. Let its integral be $c$. So the matrix $Q$ of the integrals of wedge products is:

\[
Q = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & c \\
0 & 0 & 0 & 0 & -c & 0 \\
0 & 0 & 0 & c & 0 & 0 \\
0 & 0 & c & 0 & 0 & 0 \\
0 & -c & 0 & 0 & 0 & 0 \\
c & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
For instance, $\int d\theta_3 d\theta_1 d\theta_2 = - \int d\theta_3 d\theta_1 d\theta_4 d\theta_2 = + \int d\theta_1 d\theta_3 d\theta_4 d\theta_2 = - \int d\theta_1 d\theta_3 d\theta_2 d\theta_4 = + \int d\theta_1 d\theta_2 d\theta_3 d\theta_4 = c$

The characteristic polynomial is:

$$p(\lambda) = \lambda^6 - 3c^2 \lambda^4 + 3c^4 \lambda^2 + c^6 = (\lambda - c)^3(\lambda + c)^3.$$  

This polynomial has six roots, three of them are $c$ and the other 3 are $-c$.

The number of positive eigenvalues equals that of negative ones. Therefore, the signature of $T^4$ is zero.

26 Example of a sophism about the signature of $\mathbb{C}P^2$. The clarification of this sophism will be found below.

We consider $\mathbb{C}P^2$ as a compact manifold over the reals of dimension 4. It can be decomposed as $e^0 + e^2 + e^4$. Its Poincaré polynomial is:

$$p(t) = 1 + t^2 + t^4$$

which means that $b_2 = b_4 = 1$. So, $H^2$ and $H^4$ both have one generator. Thus, $Q$ is a $1 \times 1$ matrix. Nevertheless, the only entry of $Q$ is zero because every $Q$ matrix has zeros in its diagonal due to the fact that in the diagonal appear squares of forms that necessarily have repeated terms. Therefore, $Q$ is the zero matrix, a result that predicts that the only eigenvalue of $Q$ is zero. Hence, the signature of $\mathbb{C}P^2$ is zero.

3 Invariance

A diffeomorphism represents a change of coordinates and we will show that the signature of a manifold is independent of them so, it is an intrinsic object. A direct demonstration of this result over the ring $H^*$ is instructive and can be carried out thanks to the pull-back technology. When in calculus one says “change of variable”, in differential geometry one says “pull-back” of $p$–forms, which is built upon the notion of differential of a function:

27 Definitions. Let $\phi : M \rightarrow N$ be a map of manifolds and let $\phi(x) = y$. Let $T_x M$ and $T_{\phi(x)} N$ be the tangent spaces at $x$ over $M$ and, respectively, at $\phi(x)$ over $N$. We define the differential $\phi_*$ of $\phi$ as the linear isomorphism $\phi_* : T_x M \rightarrow T_{\phi(x)} N$ such that for any vector $v \in T_x M$ and any scalar function $f : N \rightarrow \mathbb{R}$ we have $[\phi_*(v)](f) = v(\phi \circ f)$. A map is called differentiable
or smooth if its differential exists. For a smooth map $\phi : M \rightarrow N$ the **pull-back** $\phi^* : \Lambda^1_{\phi(x)} N \rightarrow \Lambda^1 x M$ is a linear transformation such that for $x \in M$ we get $\phi^*(\omega)(v) = \omega(\phi_*(v))$ for all vectors $v \in T_x M$ and 1-forms $\omega$.

The pull-back observes the following properties, the first of which allows to extend the pull-back of 1-forms to the entire space of forms:

1. $\phi^*(\alpha \wedge \beta) = \phi^*(\alpha) \wedge \phi^*(\beta)$
2. $\phi^*(d\alpha) = d(\phi^*(\alpha))$
3. $(\phi \circ \psi)^* = \psi^* \circ \phi^*$
4. $(\phi^{-1})^* = (\phi^*)^{-1}$, if $\phi^{-1}$ exists.

**28 Lemma.** Let $\phi : M \rightarrow N$ be an orientation preserving diffeomorphism between manifolds and let $H^*(M)$ and $H^*(N)$ the respective cohomology rings, then $\phi$ induces a ring contravariant isomorphism

$$F^*_\phi : H^*(N) \rightarrow H^*(M) \text{ defined by } F^*_\phi([\eta]) = [\phi^* (\eta)].$$

**Proof:** Let us prove that $F^*_\phi$ is well defined, with an inverse which is the ring homomorphism induced by $\phi^{-1}$ and that

$$F^*_\phi([\eta_1] \sqcup [\eta_2]) = F^*_\phi([\eta_1]) \sqcup F^*_\phi([\eta_2]).$$

To see that $F^*_\phi$ is well defined we need to prove that two representatives of the same cohomology class in the domain are transformed into members of the same cohomology class in the range. Two elements are in the same cohomology class if they differ by an exact form, i.e., if $[\eta_1] = [\eta_2]$ then $\eta_1 = \eta_2 + d\tau$. In this case,

$$\phi^*(\eta_1) = \phi^*(\eta_2) + \phi^*(d\tau) = \phi^*(\eta_2) + d(\phi^*(\tau))$$

this means that the members of a class are transformed into elements that differ by an exact form, i.e., they are members of the same class:

if $[\eta_1] = [\eta_2]$ then $[\phi^*(\eta_1)] = [\phi^*(\eta_2)]$ so that $F^*_\phi([\eta_1]) = F^*_\phi([\eta_2])$ is well defined.

Let us verify now that if $F^*_\phi([\eta]) = [\phi^* (\eta)]$, the inverse of $F^*_\phi$ is the ring homomorphism $G^*_\phi^{-1}$ induced by $\phi^{-1}$, i.e., if $G^*_\phi^{-1}([\omega]) = [(\phi^{-1})^*(\omega)]$ then $G^*_\phi^{-1}(F^*_\phi([\eta])) = [\eta]$ and $F^*_\phi(G^*_\phi^{-1}([\omega])) = [\omega]$. For the first part we have:
\[ G^*_{\phi^{-1}}(F^*_\phi([\eta])) = G^*_{\phi^{-1}}((\phi^*)([\eta])) = ((\phi^{-1})^*(\phi^*(\eta))) = ((\phi^{-1})^*(\phi^*(\eta))) = [I_\eta] = [\eta]. \] The proof of the second part is similar.

Let us show that \( F^*_\phi \) is indeed a ring homomorphism:
\[ F^*_\phi([\eta_1] \sqcup [\eta_2]) = F^*_\phi([\eta_1 \land \eta_2]) = [\phi^*(\eta_1 \land \eta_2)] = [\phi^*(\eta_1) \land \phi^*(\eta_2)] \]
\[ = [\phi^*(\eta_1)] \sqcup [\phi^*(\eta_2)] = F^*_\phi([\eta_1]) \sqcup F^*_\phi([\eta_2]). \]

In conclusion, \( F^*_\phi \) is a ring homomorphism:
\[ F^*_\phi([\eta_1] \sqcup [\eta_2]) = F^*_\phi([\eta_1]) \sqcup F^*_\phi([\eta_2]). \]

We also say that \( F^*_\phi \) is orientation preserving, in the sense that if we choose a basis \( E \) of \( H^{2k}(N) \) then \( F^*_\phi(E) \) will be a basis of \( H^{2k}(M) \) that observes the same orientation.

29 **Convention.** The ring isomorphism \( F^*_\phi \) is denoted as \( \phi^* \).

30 **Theorem.** The quadratic signature form \( Q \) is invariant under diffeomorphisms.

Proof. Let \( \phi : M \to N \) be a diffeomorphism between manifolds, and let \( F^*_\phi \) be its induced isomorphism between \( H^*(M) \) and \( H^*(N) \). The theorem of change of variables reads:
\[ \int_{N=\phi(M)} \eta = \int_M \phi^*(\eta). \]

Let us apply this theorem to the signature quadratic form:
\[ \int_{N=\phi(M)}[\eta_1] \sqcup [\eta_2] = \int_{N=\phi(M)} \eta_1 \land \eta_2 = \int_M \phi^*(\eta_1 \land \eta_2) = \int_M \phi^*\eta_1 \land \phi^*\eta_2 = \int_M [\phi^*\eta_1 \land \phi^*\eta_2] = \int_M [\phi^*(\eta_1)] \sqcup [\phi^*(\eta_2)] = F^*_\phi([\eta_1]) \sqcup F^*_\phi([\eta_2]). \]

Thus, we have proved that \( \int_{N=\phi(M)}[\eta_1] \sqcup [\eta_2] = \int_M F^*_\phi([\eta_1]) \sqcup F^*_\phi([\eta_2]) \)
and this means that two diffeomorphic manifolds have the same quadratic signature form:
\[ Q([\eta_1], [\eta_2]) = \int_{N=\phi(M)}[\eta_1] \sqcup [\eta_2] = \int_M F^*_\phi([\eta_1]) \sqcup F^*_\phi([\eta_2]) = Q([\phi^*\eta_1], [\phi^*\eta_2]). \]

31 **Corollary.** Diffeomorphic oriented manifolds have the same signature.

The following illustration explains the whole idea. Let us suppose, as in an example above, that the signature quadratic form in \( N \) with respect to basis \( \{d\theta, d\phi\} \) has matrix
and that we have a diffeomorphism \( \phi : M \rightarrow N \) that preserves orientation. Now, the preimage of any basis \( E \) of \( H^{2k}(N) \) through \( \phi^* \) is also a basis, otherwise \( \phi \) would not be a diffeomorphism. Therefore, the matrix of the signature form in \( M \) with respect to basis \( \{ \phi^*(d\theta), \phi^*(d\phi) \} \) has matrix

\[
Q = \begin{pmatrix}
d\theta & d\phi \\
\phi^*(d\theta) & \phi^*(d\phi)
\end{pmatrix}
\]

We see that the entries of the matrix are conserved. In particular, signs are conserved as a result of the orientation preserving property of \( \phi \). Changes affect only the labels of the matrix and therefore the structure of eigenvalues and eigenvectors is untouched as it is also its signature.

### 3.1 Invariance of \( \sigma(M) \) under homotopy equivalence

Conceptually, smooth manifolds are inseparable from smooth functions: What does happen with the signature if the manifold is smoothly deformed?

**3.2 Definition.** Two functions with the same domain and codomain, \( h_0, h_1 : Z \rightarrow W \), are homotopically equivalent, \( h_0 \sim h_1 \), if there exists a smooth map \( F : Z \times [0,1] \rightarrow W \) such that \( F|_{Z \times \{0\}} = h_0 \) and \( F|_{Z \times \{1\}} = h_1 \).

Remarks: Homotopic equivalence is a topological concept so, the closed interval \([0,1]\) is endowed with the relative or subspace topology inherited from \((-\epsilon, 1+\epsilon)\). Intuitively, two functions are homotopically equivalent if there exists a continuous deformation of \( h_0(Z) \) into \( h_1(Z) \) or just imagine yourself taking a bar of plasticine and deforming it continuously from an initial state into a final one. Now, we changed continuity for smoothness because we deal with smooth manifolds and so we demand from \( F \) to be also smooth. The functions shall not be onto and the images of the two functions could be disjoint.
33 **Definition.** Let $M$ and $N$ be two oriented smooth compact manifolds. We say that $M$ and $N$ are (strongly) homotopically equivalent if there exist two orientation preserving smooth maps

$$g : N \to M$$

$$f : M \to N$$

such that $f \circ g \sim id_N$ and $g \circ f \sim id_M$.

To understand the meaning of this concept let us imagine that $M$ and $N$ represent plasticine figures. If $M$ can deform itself smoothly into its image $g(f(M))$ and if $N$ also can do the equivalent in its side, then we say that the two manifolds are homotopically equivalent.

34 **Question.** Let us consider the following two manifolds, the first a projective space, the second a torus: $\mathbb{CP}^3$ and $S^2 \times S^4$. These two manifolds seem very similar according to certain descriptors. Let us see.

To begin with, both have real dimension 6. Moreover, they both have the same associated polynomial:

From our construction of $\mathbb{CP}^n$, we have the following decomposition:

$$\mathbb{CP}^3 = e^0 \cup e^2 \cup e^4 \cup e^6 \leftrightarrow p(t) = 1 + t^2 + t^4 + t^6.$$ 

On the other hand, the decomposition of the super-torus $S^2 \times S^4$ can be calculated thanks to Poincaré polynomials:

$$S^2 \times S^4 \leftrightarrow (1 + t^2)(1 + t^4) = 1 + t^2 + t^4 + t^6 \leftrightarrow e^0 \cup e^2 \cup e^4 \cup e^6.$$ 

We see that these two spaces have the same Poincaré polynomial, so they share the same cohomology. Does this means that they are homotopy equivalent? To answer this question, we will show that homotopy deformations generate isomorphic cohomology rings. As a consequence, the signature of a manifold is conserved under homotopy equivalence. In conclusion, our two manifolds will be not homotopy equivalent if they have non isomorphic cohomology rings.
To begin with, let us prove that the antitransport of closed forms through two homotopy equivalent functions belong in the same cohomology class, i.e., that they differ by an exact form.

35 Lemma. Let $f, g : M \rightarrow N$ be smooth maps that are homotopic to each other. If $\omega \in \Omega^k(N)$ is a closed form, the difference of the pull-back images is exact:

$$f^* \omega - g^* \omega = d\psi$$

where $\psi \in \Omega^{k-1}(M)$ and $f^*$ and $g^*$ are the pull-backs of $f$ and $g$ respectively.

Proof. Since $f \sim g$, there exists a smooth map $F : M \times [0, 1] \rightarrow N$ such that

$$F|_{M \times \{0\}} = f \text{ and } F|_{M \times \{1\}} = g,$$

i.e., $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$ for $x \in M$.

Now we will be involved in a game that uses $F$ and the fundamental theorem of calculus: to prove that $f^* \omega - g^* \omega = d\psi$ for some $\psi$, we will prove a rigorous version of the following idea: $f^* \omega - g^* \omega = d \int F^* \omega$. In what follows in this lemma, the integral will be replaced by the operator $P$.

Let us consider a $k$-form $\eta \in \Omega^k(M \times [0, 1])$. $\eta$ takes the form

$$\eta = a_{i_1...i_k}(x,t)dx^{i_1} \wedge ... dx^{i_k} + b_{j_1...j_{k-1}}(x,t)dt \wedge dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}$$

where $x \in M, t \in [0, 1]$. The second term is of degree $k$ but it shall include $dt$ so it has only $k - 1$ degrees of freedom to choose its components.

Define a map $P : \Omega^k(M \times I) \rightarrow \Omega^{k-1}(M)$ by

$$P(\eta) = (\int_0^1 b_{j_1...j_{k-1}}(x,s)ds)dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}$$

Let $f_t$ be a map $f_t : M \rightarrow M \times I$ such that $f_t(p) = (p, t)$

We have

$$f_t^* \eta = a_{i_1...i_k}(x,t)dx^{i_1} \wedge ... dx^{i_k} \in \Omega^k(M)$$

since $f_t^*(dt \wedge dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}) = 0$.

Let us prove now that

$$d(P(\eta)) + P(d\eta) = f_1^*(\eta) - f_0^* \eta.$$

Indeed, if we calculate each term in the lhs we get

$$d(P(\eta)) = d(\int_0^1 b_{j_1...j_{k-1}}(x,s)ds)dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}$$

$$= (\int_0^1 (\partial b_{j_1...j_{k-1}}(x,s)/\partial x^{j_{k}})ds)dx^{j_{k}} \wedge dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}$$

25
On the other hand,
\[ P(d\eta) = P[d(a_{i_1...i_k}(x,t)dx^{i_1} \wedge ... dx^{i_k} + b_{j_1...j_{k-1}}(x,t)dt \wedge dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}] \\
= P[(\partial a_{i_1...i_k}(x,t)/\partial x^{i_{k+1}})dx^{i_{k+1}} \wedge dx^{i_1} \wedge ... \wedge dx^{i_k} \\
+ \partial a_{i_1...i_k}(x,t)/\partial \eta dt \wedge dx^{i_1} \wedge ... \wedge dx^{i_k}] \\
+ (\partial b_{j_1...j_{k-1}}(x,t)/\partial x^{j_k})dx^{j_k} \wedge dt \wedge dx^{j_1} \wedge ... \wedge dx^{j_{k-1}}) \\
= (\int_0^1 (\partial a_{i_1...i_k}(x,s)/\partial s)ds)dx^{i_1} \wedge ...dx^{i_k} \\
- (\int_0^1 (\partial b_{j_1...j_{k-1}}(x,s)/\partial x^{j_k})ds)dx^{j_k} \wedge dx^{j_1} \wedge ... \wedge dx^{j_{k-1}} \\
\]

Summing up these two terms we get
\[ d(P(\eta)) + P(d\eta) = (\int_0^1 (\partial a_{i_1...i_k}(x,s)/\partial s)ds)dx^{i_1} \wedge ...dx^{i_k} \]
Applying the fundamental theorem of calculus for a continuous function we have
\[ d(P(\eta)) + P(d\eta) = [a_{i_1...i_k}(x,1) - a_{i_1...i_k}(x,0)]dx^{i_1} \wedge ...dx^{i_k} \]
\[ d(P(\eta)) + P(d\eta) = f_1^*(\eta) - f_0^*\eta \]
Let us apply this identity to the pull-back of a closed form \( \omega \in \Omega^k(N) \) and \( \eta = F^*\omega \in \Omega^k(M \times [0,1]) \): so \( d(P(\eta)) + P(d\eta) \) becomes
\[ d(P(F^*\omega)) + P(dF^*\omega) = f_1^*(F^*\omega) - f_0^*(F^*\omega) \]
recalling that \( (FG)^* = G^*F^* \) we get
\[ d(P(F^*\omega)) + P(dF^*\omega) = (Ff_1)^*\omega - (Ff_0)^*\omega \]
but \( f_1(x) = (x,0) \) and moreover \( f \sim g \) so that \( F(f_0(x)) = F(x,0) = f(x) \)
and \( F(f_1(x)) = F(x,1) = g(x) \)
Hence
\[ d(P(F^*\omega)) + P(dF^*\omega) = f^*\omega - g^*\omega. \]
Recalling now that \( \omega \) was chosen to be closed, \( d\omega = 0 \), we get \( F^*d\omega = 0. \)
We can rewrite this as \( d(F^*\omega) = 0 \) because the pull-back and the exterior derivative commute. Integrating with \( P \) between 0 and 1 we have \( P(dF^*\omega) = 0. \)
Replacing
\[ d(P(F^*\omega)) + P(dF^*\omega) = d(P(F^*\omega)) = f^*\omega - g^*\omega \]
Explicitly \( f^*\omega - g^*\omega = d\psi \) where \( \psi = P(F^*\omega) \) proving that the difference of the pull-backs of closed forms throughout homotopy equivalent functions is exact.

### 36 Corollary

Let \( f, g : M \to N \) be maps which are homotopic to each other. Then, the pull-back maps \( f^*, g^* : \H^*(N) \to \H^*(M) \) defined on the de Rham cohomology rings are identical, i.e., for a closed form \( \omega \in \Omega^k(N) \), we have \( [f^*\omega] = [g^*\omega] \).
In fact,
\[ [f^*\omega] - [g^*\omega] = [f^*\omega - g^*\omega] = [d\psi] = 0 \]
because the zero class of \( H^*(M) \) is conformed by the forms that are exact.

37 Theorem. If \( M \) and \( N \) are homotopy equivalent, oriented, even dimensional, compact manifolds, then \( \sigma(M) = \sigma(N) \).

Proof. Since \( M \) and \( N \) are homotopy equivalent manifolds, there exist two orientation preserving smooth maps
\[ f : M \to N \]
\[ g : N \to M \]
such that \( f \circ g \sim id_N \) and \( g \circ f \sim id_M \). We can now apply the result of the previous corollary: when two functions are homotopic to each other, their pullbacks are identical. So, on one hand we have:
\[ (f \circ g)^* = g^* \circ f^* = (id_N)^* = id_{H^*(N)} \]
and on the other
\[ (g \circ f)^* = f^* \circ g^* = (id_M)^* = id_{H^*(M)}. \]

These system of equalities says us that as \( f^* \) as \( g^* \) are isomorphisms or that \( M \) and \( N \) are diffeomorphic. Since they are oriented, their ring structure is also isomorphic and henceforth they have the same signature.

Next definitions and ensuing comment teach us how to simplify manifolds to its most fundamental cores without distorting their differential structure. Say, a torus \( S^1 \times S^1 \) can be considered as the simplification of all those surfaces into which it can be smoothly deformed. Compare with Zeeman ([10], 1966).

38 Definition. Let \( R \) be a, not empty, topological subspace of \( M \). If there exists a continuous map \( f : M \to R \) such that \( f|_R = id_R \), \( R \) is called a retract of \( M \) and \( f \) a retraction.

A retraction is our formalization of a curvilinear projection.

39 Definition. Let \( R \) be a, not empty, topological subspace of \( M \) and \( f \) a retraction of \( M \) over \( R \). \( R \) is said to be a deformation retract if \( id_M \) and \( f \) are homotopically equivalent and \( R \) is point by point invariant in the deformation.
We shall highlight the fact that our retractions eliminate homotopic redundantances but holes are not eliminated. So, a circle cannot be joined to its north pole by a deformation retraction.

**Disclaimer.** Two manifolds could be inequivalent and yet they can have the same cohomology vector spaces and henceforth the same signature. Example: take as $M = S^2 \times S^4$ and as $N = \mathbb{CP}^3$. To prove that they are homotopically inequivalent, we will show that they do not expand isomorphic cohomology rings. In fact, they are dissimilar: $\mathbb{CP}^3$ is recursive up to 3 complex dimensions, i.e., 6 real ones, while $S^2 \times S^4$ is recursive only up to 4 real dimensions. Thus products vanish above 4 terms in the last case with the exception of that product that corresponds to the volume form, while we can rise up to 6 in the former one.

**40 Summary.** The maximal subspaces of forms over which $Q$ is definite-positive have the same dimension for oriented, boundariless, compact, finite dimensional $= 4k$, homotopically equivalent manifolds.

### 3.2 The Hodge star operator

In the sequel we will present an elaboration of a nice observation regarding our matrices $Q$: the only products that matter are those whose output complete the volume form. Thus, a question arises: can we define an operator that associates to any given form what it lacks to complete the volume form? The solution to this problem has shown to be very rich if we consider a closed riemannian manifold with a metric $(M, g)$, where we can define the Hodge-$*$ operator as above. We need some few properties of this operator.

**41 Properties of $*$:**

We need the following fundamental properties of the Hodge-$*$ operator, where $vol$ is the volume form (Dray, [1] 1999; Ivancevic et al, [3] 2011):

1. $\alpha \wedge * \beta = (\alpha, \beta)vol$.

2. The star operator provides what a form lacks to be the volume form:

   $\alpha \wedge * \alpha = ||\alpha||^2 vol$

   where $||\alpha||^2 = (\alpha, \alpha)$
3. $** = (-1)^{p(n-p)}$

and hence

$$**(-1)^{p(n-p)} = 1$$

This implies that

$$*-1 = (-1)^{p(n-p)}*$$

4. We also need $*$, the adjoint of $*$ for the scalar product between p-forms. That product reads:

$$(\alpha, \beta) = \int_{\Omega} \alpha \wedge *\beta.$$  

From the property

$$(*\alpha, *\beta) = (\alpha, \beta)$$

we get

$$(*\alpha, *\beta) = (\alpha, *\beta) = (\alpha, \beta)$$

Hence $** = I$. This implies that

$$** = -1 = (-1)^{p(n-p)}* = *$$
42 Example. Let us calculate $*$ over some forms of $\mathbb{R}^4$.

To calculate $*$, we use the next trick: $(\ast \alpha)$ must complement $\alpha$ to fill in the volume form $dxdydzdv$ and the sign must be adjusted accordingly.

Examples:

* $dxdy = dzdv$ because $(dxdy)(dzdv) = dxdydzdv$.
* $dxdzdv = dy$ because $dxdzdvdy = -dxdydzdv = dx dydzdv$.
* $dydzdv = -dx$ because $dydzdv(-dx) = -dydzdvdx = dydzdxvd = -dydxdzdv = dx dydzdv$.

43 Example. Let us inquire in $\mathbb{R}^4$ over the eigenvectors of $*$.

Since $*$ completes forms to fill in the volume form, possible eigenvectors of $*$ must be a linear combination of terms that complete one another. So, let us prove that $\omega = dxdy + dzdv$ is an eigenvector of $*$. In fact:

\[
\ast \omega = \ast (dxdy + dzdv) = dzdv + dxdy = \omega
\]

Thus, $\omega$ is an eigenvector with eigenvalue 1. To fabricate an eigenvector with eigenvalue -1, let us try $\eta = dxdz + dydv$:

\[
\ast \eta = \ast (dxdz + dydv) = -dydv - dxdz = -\eta.
\]

3.3 $\delta$: the adjoint of the derivative $d$

We define $\delta = d^*$, as the adjoint of $d$ which is defined by the equation

\[
(d\alpha, \beta) = (\alpha, \delta \beta).
\]

It is found that $\delta = (-1)^{n(p+1)+1} \ast d^*$. When $n$ is even, $\delta = - \ast d^*$
Theorem. $\delta^* = d \ (\text{for } n \ \text{even}).$

To prove this theorem, we begin with

$$\delta = - * d*$$

To get

$$\delta^* = -(d* *) = -** \delta^* = -(1)^p(n-p) (-1)^p(n-p) * \delta^* = ** d ** = d.$$

3.4 The Euler characteristic

Because $** = (-1)^k(n-k)$, when operating over $k$–forms, there is a natural bijection between $H^k$ and $H^{n-k}$ known as Poincaré’ duality. Example $dx$ and $dydz$ single out one to another in $\mathbb{R}^3$. Let the k-Betti number $b_k$ be defined by $b_k = \dim H^k$. Define the Euler characteristic of $M$ as $\chi(M) = \sum (-1)^k b_k$.

44 Lemma. Let $\dim M = 4k$ then the Euler characteristic of $M$ and the dimension of $H^{2k}$ have the same parity, i.e., $\chi(M) = b_{2k} \mod 2$.

Proof. $\chi(M) = \sum_{0}^{1k} (-1)^k b_k = \sum_{0}^{2k-1} (-1)^k b_k + (-1)^{2k} b_{2k} + \sum_{2k+1}^{4k} (-1)^k b_k.$

Using the Poincaré duality, this can be rewritten as

$$\chi(M) = 2 \sum_{0}^{2k-1} (-1)^k b_k + (-1)^{2k} b_{2k}$$

$$\chi(M) - b_{2k} = 2 \sum_{0}^{2k-1} (-1)^k b_k$$

that shows that at both sides of this equation we are dealing with even numbers. Or, equivalently, $\chi(M) = b_{2k} \mod 2$.

4 Harmonic forms

It is shown here that there is a suitable basis for the calculation of the signature of a manifold.
46 Definitions. The Dirac operator $D$ is $D = d + \delta$ and the Laplacian $
abla = D^2 = (d + \delta)^2 = (d + \delta)(d + \delta) = d^2 + d\delta + \delta d + \delta^2 = d\delta + \delta d$. A form that satisfies the Laplace equation $\nabla \omega = 0$ is called harmonic. The space of harmonic forms of degree $k$ is denoted as $\text{Harm}^k(M)$.

47 Example. Let us exhibit the harmonic representatives of $H^k$ over $S^1$.

For $S^1$, $b_0 = 1$, since it has just one connected component, and $b_1 = 1$ since it has just one 1-dimensional hole. We will look for solutions to the equation $\nabla \alpha = (d\delta + \delta d)(\alpha)$. Because $S^1$ has dimension 1, we need the basic definition of $\delta = (-1)^{n(p+1)+1} * d *$ that for $n = 1$ becomes $\delta = (-1)^{p+2} * d * = (-1)^p * d *$. So, we are looking for solutions of

$$\nabla \alpha = (-1)^p (d\delta + \delta d)(\alpha) = (-1)^p (d * d * + * d * d)(\alpha) = 0.$$

or of

$$(d * d * + * d * d)(\alpha) = 0.$$

The obvious candidates for harmonic forms are: the constant function 1, a 0-form, and $d\theta$, a 1-form. Let us inquire whether or not they are harmonic. We shall use the fact that $* 0 = 0$ because $* 0 = 0 d\theta = 0$.

Let us test the 0-form 1:

$$(d * d * + * d * d)1 = d * d * 1 + d * d 1 = d * d\theta + * d * 0 = d * 0 + 0 = 0$$

So, 1 is harmonic. What happens with the 1-form $1 d\theta$?

$$(d * d * + * d * d)(1 d\theta) = d * d * (1 d\theta) + * d * d(1 d\theta) = d * d 1 + 0 = 0$$

So, $d\theta$ is harmonic.

48 Example. Let us exhibit the harmonic representatives of $H^k$ over $S^2$. We consider the usual coordinates $\theta$ and $\phi$ in that order.

$S^2$ has one connected component, so $b_0 = 1$. It has one 2-hole, so $b_2 = 1$ and every closed curve over it is contractile to a point so, $b_1 = 0$. In
consequence, let us show that the constant function 1 and the volume form $d\theta d\phi$ are harmonic. With 1 we have:

$$(-d^*d^* - dd^*)1 = -d^*d^*1 - d^*d1 = -d^*dd\theta d\phi - d^*0 = -d^*0 - 0 = 0$$

So, 1 is harmonic. What happens with $1d\theta d\phi$?

$$(-d^*d^* - dd^*)(1d\theta d\phi) = -d^*d^*(1d\theta d\phi) - d^*d(1d\theta d\phi) = -d^*d1 - 0 = 0$$

So, $d\theta d\phi$ is harmonic.

49 Theorem. $D^* = D$.

We have:

$$D = d + \delta$$

so

$$D^* = d^* + \delta^* = \delta + d = D.$$ 

50 Theorem. $\Delta$ is selfadjoint.

Proof: $(\Delta \omega, \psi) = (D^2 \omega, \psi) = (D\omega, D^*\psi) = (D\omega, D\psi) = (\omega, D^*D\psi) = (\omega, D^2\psi) = (\omega, \Delta \psi)$.

51 Remark. We have proved that $(\Delta \omega, \psi) = (\omega, \Delta \psi)$ but only if that makes sense. So, we need to give a glance at the domain of definition of involved operators. Basically, we have two items: integration and derivation of differential forms. Since at last these are reduced to integration of functions involving partial derivatives of real valued functions that are defined over open sets of $\mathbb{R}^n$, domains must be referred to them. Moreover, our arguments strongly rely on duality so, we must resort to Sobolev spaces. In the case of the Dirac operator we must think of Sobolev space of order 1, and of Sobolev space of order 2 for the Laplacian. Sobolev spaces are complete so, the usual inner product makes then into Hilbert Spaces (Paycha, [8] 1997). Now, to define Sobolev spaces intrinsically, we must consider patching local definitions thanks to partitions of unity. Actually, this trickery is already present in the integration machinery.
52 Theorem. \( \triangle \) is positive. In other words: \( (\triangle \omega, \omega) \geq 0 \) where \( \omega \in \Omega(M) \).

Proof. \( (\triangle \omega, \omega) = ((d\delta + \delta d)\omega, \omega) = (d\delta \omega, \omega) + (d\omega, \delta \omega) = \|d\delta \omega\|^2 + \|d\omega\|^2 \geq 0 \)

53 Corollary. \( (\triangle \omega, \omega) = 0 \) iff \( \delta \omega = d \omega = 0 \), i.e., harmonic forms are both closed and coclosed.

54 Theorem. Poisson’s equation \( \triangle \psi = \omega \) has a solution iff \( \omega \) is orthogonal to \( \text{Harm}(M) \). The solution \( \psi \) is noted as \( \psi = \triangle^{-1} \omega \).

Proof. Let us assume that there exists \( \psi \) such that \( \triangle \psi = \omega \). Then, by taking \( \gamma \) from \( \text{Harm}(M) \) we get:

\[
(\omega, \gamma) = (\triangle \psi, \gamma) = (\psi, \triangle \gamma) = (\psi, 0) = 0
\]

which shows that the orthogonality of \( \omega \) to \( \text{Harm}(M) \) is necessary for the existence of a solution of the Poisson’s equation. On the other hand, if \( \omega \) is orthogonal to \( \text{Harm}(M) \), its preimage cannot be in \( \text{Harm}(M) \) because it is the kernel of \( \triangle \). Hence, every preimage must have a component that belongs in \( \text{Harm}(M)^\perp \), the orthogonal complement of \( \text{Harm}(M) \). Let us consider the restriction of \( \triangle \) to \( \text{Harm}(M)^\perp \), i.e., tolerating an abuse of notation, we consider \( \triangle : \text{Harm}(M)^\perp \to \text{Harm}(M)^\perp \): let us prove that this operator is one to one and onto. It is in this sense that \( \triangle \) is a bijection with an inverse that can be denoted as \( \triangle^{-1} \).

To see why \( \triangle : \text{Harm}(M)^\perp \to \text{Harm}(M)^\perp \) is one to one, let us take \( \phi \) and \( \psi \) in \( \text{Harm}(M)^\perp \). In this case, \( \phi - \psi \) is also in \( \text{Harm}(M)^\perp \). Suppose now that \( \triangle \phi = \triangle \psi \). Then, \( \triangle (\phi - \psi) = 0 \). So, \( \phi - \psi \) is in \( \text{Harm}(M) \). Now, the only element that is in both \( \text{Harm}(M) \) and \( \text{Harm}(M)^\perp \) is the zero element. Henceforth, \( \phi = \psi \).

To prove that \( \triangle : \text{Harm}(M)^\perp \to \text{Harm}(M)^\perp \) is onto and that therefore \( \triangle \psi = \omega \) has a solution, we apply the Riesz’ Representation Theorem for continuous linear functionals on Hilbert spaces. In fact, the equation \( \triangle \psi = \omega \) implies

\[
(\triangle \psi, \phi) = (\psi, \Delta \phi) = (\omega, \phi) \text{ for } \phi.
\]

Now, because \( \omega \) is a fixed element, the expression \( (\omega, \phi) \) defines a linear functional that can be denoted as \( l_\omega(\phi) = (\omega, \phi) \). So, we can define over \( \text{Harm}(M)^\perp \) the inner product \( [[.,.]] \) given by:

\[
[[\psi, \phi]] = (\psi, \Delta \phi)
\]
The Riesz' Representation Theorem allows us to represent the functional $l_\omega(\phi) = (\omega, \phi)$ by a single element $\Omega$ that operates through the inner product $[\cdot, \cdot]$:

\[
(\omega, \phi) = l_\omega(\phi) = [\Omega, \phi] = (\Omega, \Delta \phi) = (\Delta \Omega, \phi)
\]

Since this is certain for every $\phi$, we conclude that $\Delta \Omega = \omega$ and that therefore the Poisson’s equation has a solution when $\omega$ is orthogonal to $\text{Harm}(M)$. Now, a lot of analysis over Sobolev spaces is necessary to fill in the details in regard with continuity, see (Min Ru, [5], 2000).

55 **Theorem.** Harmonic, Exact and co-exact forms are mutually orthogonal spaces.

Proof. Since $d^2 = 0$ then $d^2 \alpha_{k-1}, \beta_{k+1} = (d \alpha_{k-1}, \delta \beta_{k+1}) = 0$, showing that exact and coexact forms are orthogonal.

Since harmonic forms are closed then if $\gamma_k$ is harmonic then $d \gamma_k = 0$ and $(\beta_{k+1}, d \gamma_k) = (\delta \beta_{k+1}, \gamma_k) = 0$ showing the mutual orthogonality between coexact and harmonic forms.

Likewise, harmonic forms are closed, i.e., $\delta \gamma_k = 0$. Then $(\alpha_{k-1}, \delta \gamma_k) = (d \alpha_{k-1}, \gamma_k) = 0$ showing the mutual orthogonality between exact and harmonic forms.

56 **Hodge Decomposition Theorem.** Harmonic, Exact and co-exact forms generate $\Omega^k(M)$: $\Omega(M) = \text{Ker}\Delta \oplus \text{Kerd} \oplus \text{Kerd}^*$.

Proof. Let $P : \Omega^k(M) \to \text{Harm}^k(M)$ be the orthogonal projection operator generated by the scalar product of forms, then for any $\omega \in \Omega^k(M)$ we have that $\omega - P \omega$ is orthogonal to $\text{Harm}^k(M)$. Hence, Poisson’s equation

$\Delta \psi = \omega - P \omega$ has a solution that can be written as $\psi = \Delta^{-1}(\omega - P \omega)$

In conclusion, it makes sense to write

$\omega = \Delta \psi + P \omega = (d \delta + \delta d) \psi + P \omega = d(\delta \psi) + \delta(d \psi) + P \omega$

that reads: any form can be orthogonally decomposed as a sum of an exact form plus a coexact form plus a harmonic form.

57 **Hodge Theorem.** $H^k(M) = \text{Harm}^k(M)$, in other words, any cohomology class is the class of a harmonic form and every harmonic form is a nontrivial member of $H^k(M)$. 

35
Proof. Our universe is the space of closed forms, since $H^k(M)$ is the space of equivalence classes of closed forms defined by the relation $\omega \sim \omega'$ if there exists $d\phi$ such that $\omega = \omega' + d\phi$.

Let us show that there is an isomorphism between $H^k(M)$ and $Harm^k(M)$ induced by the projection operator over harmonic forms $P : \Omega^k(M) \rightarrow Harm^k(M)$. The isomorphism is $P([\omega]) = [P\omega]$.

Decomposing $\omega$ in its components:
$$\omega = d(\delta\psi) + \delta(d\psi) + P\omega$$

but $d\omega = 0$ hence applying $d$ to both sides of this equation
$$d\omega = d^2(\delta\psi) + d\delta(d\psi) + dP\omega = d\delta(d\psi) + dP\omega$$

but $dP\omega = 0$ because $P\omega$ is harmonic and harmonic forms are closed, so
$$d\omega = d\delta(d\psi) = 0$$

hence $(d\delta(d\psi), d\psi) = (\delta(d\psi), \delta(d\psi)) = 0$ hence $\delta d\psi = 0$.

Therefore
$$\omega = d(\delta\psi) + P\omega$$

Thus, any closed form can be decomposed into an exact and a harmonic form. Moreover, we can clearly specify $\psi$: since $\omega - P\omega = d(\delta\psi)$ is exact, it is orthogonal to harmonic forms, and so the corresponding Poisson’s equation has a solution. This decomposition can be translated into the language of cohomology classes:

The closed form $\omega$ is in its class $[\omega]$ and another representative reads $\omega' = \omega + d\phi$. Proceeding as before, we find
$$\omega + d\phi = d(\delta\psi') + P(\omega + d\phi)$$

But $d\phi$ is exact, so it has no harmonic component:
$$P(\omega + d\phi) = P\omega + Pd\phi = P(\omega)$$

which means that if two closed forms belong in the same cohomology class, they have the same harmonic component. Therefore, the projector is well defined over cohomology classes.

Let us now prove that different cohomology classes correspond with different harmonic forms.

Let us suppose that $P([\omega]) = P([\eta])$. Let us decompose both forms:
$$\omega = d(\delta\psi) + P\omega$$
$$\eta = d(\delta\rho) + P\eta$$

then
$$\omega - \eta = d(\delta\psi) + P\omega - d(\delta\rho) - P\eta = d(\delta\psi) - d(\delta\rho) = d(\delta\psi - \delta\rho)$$
and we have discovered that if two forms have the same harmonic component, then they are related by an exact form, and so they belong in the same cohomology class. Nice.

Let us show now that any harmonic form is a nontrivial member of $H^k(M)$ that is, for $\gamma \in \text{Harm}^k(M)$ we have $d\gamma = 0$ and $\exists \psi$ such that $\gamma = d\psi$. The first requirement is automatically fulfilled because any harmonic form is closed and the second relies on the fact that harmonic forms and exact forms are orthogonal one to another.

58 Theorem. $\text{DimHarm}^k(M) = b_k < \infty$.

The space of cohomology classes $H^k = \text{Ker}(d_{k+1})/\text{Im}(d_k)$ is a vector space that is equal to $\text{Harm}^k(M)$, the space of harmonic forms. So, both have the same dimension. This implies that if one of them has finite dimension then the other also. That of $H^k(M)$ is $b_k$, which counts holes and components. Now, $b_k$ is finite because a compact manifold cannot have an infinite number of holes or an infinite number of components: otherwise it would be possible to construct in both cases an open covering that has no finite subcovering. Another proof, based on analysis, that $H^k(M)$ has finite dimension can be found in Michor(2008, pag 153, [4]).

59 Commutation properties for $\ast, d, \delta$

We can decompose $\delta$ as $\delta = (-1)^{n(p+1)+1} \ast d\ast$. In our case, $n$ is even, so $\delta = - \ast d\ast$. Observe that $\delta$ takes $k-$forms and produces $k-1-$ forms. In this respect, it is similar to integration. Indeed, over $p-$forms $\ast$ produces an $(n-p)$-form over which $d$ produces a $(n-p+1)$-form and hence $\delta = - \ast d\ast$ produces a $n-(n-p+1)$-form or a $p-1$ form. Henceforth, recalling that over $p$-forms $** = (-1)^{(n-p)p}$ we have that over $(p-1)$-forms it reads $** = (-1)^{(n-p+1)(p-1)}$. Hence

$$\ast\delta = - \ast \ast d\ast = (-1)^{(n-p+1)(p-1)+1} d\ast$$

On the other hand, $\delta \ast = - \ast d \ast \ast = - \ast d(-1)^{p(n-p)} = (-1)^{p(n-p)+1} \ast d$. Or, $\delta \ast = (-1)^{p(n-p)+1} \ast d$. Therefore, multiplying by the sign in the right side, we get

$$\ast d = (-1)^{p(n-p)+1} \delta \ast.$$

With the help of these identities we can prove the expected result:
60 **Theorem.** The wedge product of harmonic forms is harmonic and the Hodge star of a harmonic form is also harmonic.

Proof: Let us take two harmonic forms, \( \gamma_1, \gamma_2 \). Both are closed and coclosed, i.e., \( d\gamma_1 = d\gamma_2 = \delta \gamma_1 = \delta \gamma_2 = 0 \). Let us calculate the Laplacian of their wedge product:

\[
\Delta (\gamma_1 \wedge \gamma_2) = (d \delta + \delta d)(\gamma_1 \wedge \gamma_2) =
\]

\[
d\delta (\gamma_1 \wedge \gamma_2) + (\delta d)(\gamma_1 \wedge \gamma_2) =
\]

\[
d(\delta \gamma_1 \wedge \gamma_2 \pm \gamma_1 \wedge \delta \gamma_2) + \delta (d \gamma_1 \wedge \gamma_2 \pm \gamma_1 \wedge d \gamma_2) = 0.
\]

We have thus proved that the product of two harmonic forms is harmonic.

It rests to prove now that if \( \gamma \) is harmonic, so is \( \ast \gamma \), i.e., if \( \Delta (\gamma) = 0 \) then \( \Delta (\ast \gamma) = 0 \).

\[
\Delta (\ast \gamma) = (\delta d + d \delta)(\ast \gamma)
\]

\[
= (\delta d + d \delta)(-1)^{n-p+1}d(1) \gamma + d(1)(\delta d + d \delta)(1) \gamma
\]

\[
= (\delta d + d \delta)(-1)^{n-p+1}d(1) \gamma + d(1)(\delta d + d \delta)(1) \gamma
\]

\[
= (-1)^{2(n-p+1)}d \delta + \delta d(-1)^{2(p-n)+2} \gamma
\]

\[
= (\delta d + d \delta)(\gamma) = \ast \Delta (\gamma) = 0.
\]

61 **Theorem.** Let \( E \) be the basis of \( \text{Harm}^{2k} \) formed with those elements of \( H^{2k} \) that are harmonic. Then, the signature quadratic form \( Q \) is block diagonal in \( \{ E \} \) and the signature of \( M \) fulfills

\[
\sigma(M) = \dim(\text{Harm}^{2k}_+) - \dim(\text{Harm}^{2k}_-)
\]

where \( \text{Harm}^{2k}_+ \), \( \text{Harm}^{2k}_- \) are the eigenspaces of \( \ast \) with \( \pm 1 \) eigenvalues.

Proof. One can combine \( 2k \) forms that complete one another in a manifold of dimension \( 4k \) to get eigenvectors of the Hodge * operator. Since, \( \ast \) satisfies \( \ast \ast = 1 \), \( \ast \) has eigenvalues \( \pm 1 \). Let eigenvectors be such \( \ast \omega^+ = \omega^+ \) and \( \ast \omega^- = -\omega^- \). Then:

\[
\int \omega^+ \wedge \omega^+ = \int \omega^+ \wedge \ast \omega^+ = (\omega^+, \omega^+) > 0
\]

\[
\int \omega^- \wedge \omega^- = -\int \omega^- \wedge \ast \omega^- = -(\omega^-, \omega^-) < 0
\]

Using the fact that \( \alpha \wedge \ast \beta = \beta \wedge \ast \alpha \) and that in our case the wedge product is commutative, we have

\[
\int \omega^+ \wedge \omega^- = -\int \omega^+ \wedge \ast \omega^- = -\int \omega^- \wedge \ast \omega^+ = -\int \omega^- \wedge \omega^+ = 0
\]
So, the entries of $Q$ that are outside the diagonal are all zero.

These results show that the matrix of $Q$ evaluated over a basis of harmonic forms is block diagonal, say, $D$. This allows to alternatively define the signature of a manifold using the following idea: for a real number $r$, we can define its sign as $\text{sign}(r) = r / \sqrt{|r^2|}$. For selfadjoint diagonal matrices we also have $\text{sign}(D) = D(\sqrt{|D^2|})^{-1}$. In a general case $\text{sign}(A) = A(\sqrt{|A^\ast A|})^{-1}$ which displays a diagonal filled in $\pm 1$. Thus

$$
\sigma(M) = \text{Trace}(\text{sign}(Q))
$$

where $Q$ is any matrix representing the signature quadratic form.

When some eigenvalues of a matrix are zero, they don’t enter to define the signature of a given matrix. So, it would be nice to face up at this point the following intrigue: Is $Q$ degenerate allowing the existence of null eigenvalues?

62 Theorem. $Q$ is not degenerate over $H^\ast$.

Proof. It is enough to restrict to harmonic representatives, which are both closed and coclosed with the star version also harmonic. Let us suppose now that

$$
\int (\gamma_1 \wedge \gamma_2) = 0, \forall \gamma_2.
$$

If that is true, then we can take $\gamma_2 = * \gamma_1$. In that case we get

$$
\int (\gamma_1 \wedge * \gamma_1) = (\gamma_1, \gamma_1) = 0
$$

from which we conclude that $\gamma_1 = 0$, which means that $Q$ is not degenerate and that has no zero eigenvalues.

63 Lemma. Let $\dim M = 4k$ then the dimension of $H^{2k}$ and the signature of $M$ have the same parity, i.e., $b_{2k} = \sigma(M) \mod 2$.

Proof: Since the harmonic forms $\text{Harm}^{2k}$ can be decomposed into the direct sum

$$
\text{Harm}^{2k} = \text{Harm}_+^{2k} \oplus \text{Harm}_-^{2k}
$$

thus $b_{2k} = \dim(\text{Harm}_+^{2k}) + \dim(\text{Harm}_-^{2k})$ and moreover, $\sigma(M) = \dim(\text{Harm}_+^{2k}) - \dim(\text{Harm}_-^{2k})$
Therefore
\[ b_{2k} = \dim(H_{2k}^+ \cap \Omega^+_{2k}) + \dim(H_{2k}^-) + \dim(H_{2k}^-) \]
\[ b_{2k} = \sigma(M) + 2 \dim(H_{2k}^-) \]
\[ b_{2k} - \sigma(M) = 2 \dim(H_{2k}^-) \]
or \[ b_{2k} = \sigma(M) \mod 2. \]

Since “to have the same parity” is transitive, we can conclude

**64 Corollary.** Let \( \dim M = 4k \) then the signature of \( M \) and the Euler characteristic has the same parity, i.e., \( \sigma(M) = \chi(M) \mod 2. \)

These results allow us to correct a sophism above.

**65 Example: clarification of a sophism on the signature of \( \mathbb{C}P^2. \)**

We consider \( \mathbb{C}P^2 \) as a compact, oriented manifold over the reals of dimension 4. It can be decomposed as \( e^0 + e^2 + e^4 \). Its Poincaré polynomial is:
\[ p(t) = 1 + t^2 + t^4 \]
which means that \( b_2 = b_4 = 1 \). So, \( H^2 \) and \( H^4 \) both have one generator. Thus, \( Q \) is a \( 1 \times 1 \) matrix. Since \( Q \) cannot be degenerate, the only entry of \( Q \) cannot be zero. In the sophism above it was argued that because of duplication of terms every \( Q \) matrix has zeros in its diagonal and that therefore the signature of \( \mathbb{C}P^2 \) should be zero. This generalization is in general false: when \( Q \) is computed over harmonic forms, it gets a diagonal form with all eigenvalues different than zero.

On the other hand, we have said that \( H^2 \) and \( H^4 \) both have one generator. This seems to be contradictory: \( e^4 \) can be parameterized in polar coordinates and so \( H^4 \) would be generated by \( d\theta_1 d\theta_2 d\theta_3 d\theta_4 \). This would imply that \( H^2 \) would have six generators instead of one: \( d\theta_1 d\theta_2, d\theta_1 d\theta_3, d\theta_1 d\theta_4, \ldots \). This trouble is resolved by arguing that there is only one generator in \( H^2 \) and that the other terms play the role of the \( dz \) of a cylinder that generates no cohomology at all. Let us blame complex numbers for that deficit because they include rotations in their algebraic structure. The use of harmonic forms follows:

The cell decomposition technology allows us to model \( \mathbb{C}P^2 \) over \( e^4 \) whose \( H^4 \) is generated by \( d\theta_1 d\theta_2 d\theta_3 d\theta_4 \). We know that \( H^2 \) has only one generator and that it has a harmonic representative. Let us name the coordinates of
in such a way that the following form is a representative of the only one class of harmonic 2-forms of $\mathbb{CP}^2$:

$$\nu = d\theta_1 d\theta_2 + d\theta_3 d\theta_4$$

Let us check that $\nu = d\theta_1 d\theta_2 + d\theta_3 d\theta_4$ is harmonic, i.e., that

$$\Delta \nu = (d\delta + \delta d)(\nu) = 0$$

Taking $n = 4$ and $p = 2$ in $\delta = (-1)^{n(p+1)+1} \ast d\ast = - \ast d\ast$, we get

$$\Delta \nu = (d \ast d \ast + \ast d \ast)(\nu) = (d \ast d \ast + \ast d \ast)(d\theta_1 d\theta_2 + d\theta_3 d\theta_4) = (d \ast d\ast)(d\theta_1 d\theta_2 + d\theta_3 d\theta_4) + (\ast d \ast)(d\theta_1 d\theta_2 + d\theta_3 d\theta_4)$$

We have that $\ast(d\theta_1 d\theta_2 + d\theta_3 d\theta_4) = d\theta_1 d\theta_2 + d\theta_3 d\theta_4$. So:

$$\Delta \nu = (d \ast d)(d\theta_1 d\theta_2 + d\theta_3 d\theta_4) + (\ast d \ast)(d\theta_1 d\theta_2 + d\theta_3 d\theta_4)$$

Now, $d(d\theta_1 d\theta_2 + d\theta_3 d\theta_4) = d(1(d\theta_1 d\theta_2 + d\theta_3 d\theta_4)) = 0$. Therefore,

$$\Delta \nu = 0.$$ 

Computed over $\nu$, matrix $Q$ takes the diagonal form:

$$Q = \left[ \int \nu \wedge \nu \right] = \left[ \int (d\theta_1 d\theta_2 + d\theta_3 d\theta_4) \wedge (d\theta_1 d\theta_2 + d\theta_3 d\theta_4) \right] = \left[ \int 2d\theta_1 d\theta_2 d\theta_3 d\theta_4 \right] = [2c] = [c]$$

This value is different than zero because the integrand is in the same class of the volume form, so it is positive. Therefore, matrix $Q$ has one positive eigenvalue and nothing else. As a consequence, the signature of $\mathbb{CP}^2$ is 1.

Our result is backed by the prediction given by the Euler Characteristic $\chi(\mathbb{CP}^2) = \sum (-1)^k b_k$, which becomes $\chi(\mathbb{CP}^2) = (-1)^0 + (-1)^2 + (-1)^4 = 3$. Because the Euler characteristic and the signature have the same parity, the signature is predicted to be odd, such as it was found.

**66 Example.** Let us battle with the complex hyper-torus $K = \mathbb{CP}^2 \times \mathbb{CP}^2$.

Since the Poincaré polynomial of $\mathbb{CP}^2$ is $q(t) = 1 + t^2 + t^4$, the Poincaré polynomial of $K = \mathbb{CP}^2 \times \mathbb{CP}^2$ is:
\[ p(t) = (1 + t^2 + t^4)^2 = 1 + t^4 + t^8 + 2t^2 + 2t^4 + 2t^6 = 1 + 2t^2 + 3t^4 + 2t^6 + t^8 \]

This polynomial predicts that \( H^4 \) has 3 generators, that \( H^8 \) has one and that therefore the \( Q \) matrix is \( 3 \times 3 \). The Euler characteristic reads:
\[ \chi(K) = (-1)^0 + 2(-1)^2 + 3(-1)^4 + 2(-1)^6 + (-1)^8 = 9 \]
so the signature is expected to be odd. From \( \mathbb{CP}^2 \) to \( K \) there is nothing special, so we can make calculations in whatever basis without too much trouble:

The generator of \( H^8 \) comes from the hypertorus \( e^4 \times e^4 \) so, the volume form in \( K \) is in the class of \( d\omega = d\theta_1 d\theta_2 d\theta_3 d\theta_4 d\phi_1 d\phi_2 d\phi_3 d\phi_4 \). On the other hand, we know that \( H^4 \) has 3 generators, whose precedence is the following: \( 3t^4 = 2t^2 + t^2 t^2 \). This means that we have two forms that come from \( 2e^4 \) and another that comes from \( e^2 \times e^2 \). In consequence, let \( dg_1 = d\theta_1 d\theta_2 d\theta_3 d\theta_4 \) be the first generator of \( H^4 \) and let the second be \( dg_2 = d\phi_1 d\phi_2 d\phi_3 d\phi_4 \). And, what about the generator that comes from the torus \( e^2 \times e^2 \)? Let it be \( dg_3 = d\eta_1 d\eta_2 d\nu_1 d\nu_2 \).

From this we can compute the \( Q \) matrix of the integrals of wedge products. Since we are dealing with a dimension that is multiple of 4, matrix \( Q \) is symmetric:

\[
Q = \begin{pmatrix}
dg_1 & dg_2 & dg_3 \\
dg_1 & 0 & c & d \\
dg_2 & c & 0 & e \\
dg_3 & d & e & 0 \\
\end{pmatrix}
\]

where \( d \) and \( e \) are volumes of 8-dimensional shapes others than \( e^4 \times e^4 \) and that we were unable to predict at the start. By assuming that \( c = d = e \), we get:

\[
Q = \begin{pmatrix}
dg_1 & dg_2 & dg_3 \\
dg_1 & 0 & c & c \\
dg_2 & c & 0 & c \\
dg_3 & c & c & 0 \\
\end{pmatrix}
\]

The characteristic polynomial of this matrix is
\[ p(x) = -\lambda^3 + 3c^2 \lambda + 2c^3 \]
We verify that \(-c\) is a root: \( -c^3 + 3c^3 - 2c^3 = 0 \). Hence,
\[ p(x) = (\lambda + c)(\lambda + c)(\lambda - 2c). \]
This implies that the eigenvalues are \(-c, -c, 2c\). The signature of \( K \) is, therefore, \(-1\).
67 Observation. The signature of a manifold is the number of positive eigenvalues minus the number of negative eigenvalues of the matrix of the signature quadratic form in whatever basis. It also can be rewritten as

$$\sigma(M) = \dim(H_{2k}^+) - \dim(H_{2k}^-)$$

where the subtraction refers to finite numbers.

68 Definition. The index of a Fredholm operator $P$ is

$$\text{ind}(P) = \dim(\text{Ker}P) - \dim(\text{coKer}P) = \dim(\text{Ker}P) - \dim(\text{Ker}P^*)$$

where $P^*$ is the adjoint of $P$. An operator is Fredholm when it has a kernel and cokernel of finite dimensions and its range is closed. The cokernel is the codomain quotient the image. Example: If $K$ is compact (the image of a bounded set has a compact closure), $I + K$ is Fredholm (MIT, [6], 2000).

69 Suspicion. A Fredholm operator must exist whose index is precisely the signature of the manifold.

In the following we will construct that operator.

5 The signature operator

We introduce the chirality operator which together with the Dirac operator will allow us to define the signature operator, whose tremendous importance will be exhibited in a suitable index theorem.

The following properties of the Hodge-star $*$ operator will be used in this section:

1. $** = (-1)^{(n-p)}$ when * operates over a $p$-form. Hence, over $(p+1)$-forms $** = (-1)^{(p+1)(n-p-1)}$ while over $(p-1)$-forms $** = (-1)^{(p-1)(n-p+1)}$.

2. Let us define the chirality operator $J$ over $p$-forms by $J = i^p_{\frac{n}{2}+(p-1)}$. It becomes $J = i^p_{\frac{n}{2}+(p+1)}$ over $(p+1)$-forms and $J = i^p_{\frac{n}{2}+(p-1)(p-2)}$ over $(p-1)$-forms.
3. Recall that over $p$–forms we have
\[ *\delta = -*d* = (-1)^{(n-p+1)(p-1)+1} d* \]

4. \[ *d = (-1)^{p(n-p)+1} \delta*. \]

These identities will be used to change the relative position of $*$ with respect to $d$ and $\delta$.

70 Theorem. If $D$ is defined as $D = d + \delta$ then $J$ and $D$ anticommute: $JD = -DJ$, where $J = i^{n/2+p(p-1)} *$.

Proof: $JD = J(d + \delta) = Jd + J\delta$. Recall that $d$ rises the order of a form by one unit, while $\delta$ lowers the order of a form by one. This must be kept in mind when operating with $J$: $\text{Proof: } JD + J\delta = i^{n/2+(p+1)(p)} * d + i^{n/2+(p-1)(p-2)} * \delta$.

Changing the relative position of $*$ we have:
\[
JD = i^{n/2+(p+1)(p)}(-1)^{p(n-p)+1}\delta* + i^{n/2+(p-1)(p-2)}(-1)^{(n-p+1)(p-1)+1} d*
\]
\[
= i^{n/2+(p-1)(p)}[-i^{2p}(-1)^{p(n-p)+1}\delta + i^{-2(p-1)}(-1)^{(n-p+1)(p-1)+1} d] *
\]
\[
= [(-1)^{p+p(n-p)+1}\delta + (-1)^{-(p-1)+(n-p+1)(p-1)+1} d] J
\]

Now, let us see that if $n$ is even then the exponents in this last expression are odd. To see this, recall that $p(p-1)$ is always even, that if $n$ is even then so is $np$ and that $p^2$ always behave like $p$, i.e., module two we have the following equalities:
\[
p + p(n-p) + 1 = p + np - p^2 + 1 = p + 0 - p + 1 = 1 \text{(odd)}.
\]
\[
-(p-1) + (n-p+1)(p-1) + 1 = (p-1)(-1 + n - p + 1) + 1 = (p-1)(n-p) + 1 = np - p^2 - n + p + 1 = 0 - p + 0 + p + 1 = 1 \text{(odd)}.
\]

Putting all together:
\[
JD = -\delta d J = -DJ \text{, and so, } J \text{ and } D \text{ anticommute.}
\]

71 Example. Let us illustrate over $\mathbb{R}^4$ the property $JD = -DJ$. Variables are denoted $x, y, z, v$, an order that defines the orientation.

Our definitions and identities for $p$–forms are:
\[
\delta = - *d* \\
D = d + \delta = d - *d* \\
J = i^{n/2+p(p-1)} *
\]
To test the relation of $DJ$ and $JD$ we make a try with $\omega = f dxdy$, where $f$ is a real function of the coordinates. We have:

$$d\omega = (\sum \frac{\partial f}{\partial x_i} dx_i) dxdy = \frac{\partial f}{\partial z} dz dxdy + \frac{\partial f}{\partial v} dv dxdy$$

$$= \frac{\partial f}{\partial z} dx dy dz + \frac{\partial f}{\partial v} dx dy dv$$

Since $\delta = -* d*$, we get:

$$\delta \omega = -* d*(f dxdy) = -* d(fdzdv) = -* (\frac{\partial f}{\partial x} dxdzdv + \frac{\partial f}{\partial y} dy dzdv)$$

$$\delta \omega = -\frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx$$

To calculate $*$, we use the next trick: $(*)$ must complement $\alpha$ to fill in the volume form $dxdydzdv$ and the sign must be adjusted accordingly. Examples over $\mathbb{R}^4$:

$$*f dxdy = fdzdv$$

$$*f dxdzdv = fdy$$

$$*hdydzdv = -hdx$$

$$dydzdv(-dx) = -dydzdvdx = dydzdxv = -dydxdv = dxdydzdv.$$  

Plugging results into $D = d + \delta$, we get:

$$D\omega = d\omega + \delta \omega = \frac{\partial f}{\partial z} dx dy dz + \frac{\partial f}{\partial v} dx dy dv - \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx$$

Besides, we have that in $\mathbb{R}^4$, the operator $J = i^{\frac{q}{2}+(p-1)}$ takes over $p$-forms the next values:

For 0-forms: $J = i^{\frac{2}{2}+0} = i^2 = -*$

For 1-forms: $J = i^{\frac{2}{2}+1(1-1)} = i^2* = -*$

For 2-forms: $J = i^{\frac{2}{2}+2(2-1)} = i^4 = *$

For 3-forms: $J = i^{\frac{2}{2}+3(3-1)} = i^8 = *$

For 4-forms: $J = i^{\frac{2}{2}+4(4-1)} = i^{14} = -*$

Applying $J$ to $D\omega$ we get:

$$JD\omega = J \left( \frac{\partial f}{\partial z} dx dy dz + \frac{\partial f}{\partial v} dx dy dv - \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx \right)$$  

45
\[ JD\omega = * \frac{\partial f}{\partial z} \, dz \, dy \, dv + * \frac{\partial f}{\partial w} \, dw \, dx \, dv - (-*) \frac{\partial f}{\partial z} \, dz \, dy + (-*) \frac{\partial f}{\partial y} \, dy \] 
\[ JD\omega = \frac{\partial f}{\partial z} \, dz \, dy \, dv + * \frac{\partial f}{\partial w} \, dw \, dx \, dv - \frac{\partial f}{\partial z} \, dz \, dy - * \frac{\partial f}{\partial y} \, dy \, dz \, dv \]

This result must be compared with \( DJ\omega \):

\[ DJ\omega = D(J(f \, dx \, dy)) = D(*f \, dx \, dy) = D(f \, dz \, dv) = (d + \delta)(f \, dz \, dv) = d(f \, dz \, dv) + \delta(f \, dz \, dv) = d(f \, dz \, dv) - *d(*f \, dz \, dv) \]
\[ = \frac{\partial f}{\partial x} \, dx \, dz \, dv + \frac{\partial f}{\partial y} \, dy \, dz \, dv - *d(f \, dx \, dy) \]
\[ = \frac{\partial f}{\partial x} \, dx \, dz \, dv + \frac{\partial f}{\partial y} \, dy \, dz \, dv - (\frac{\partial f}{\partial z} \, dz \, dx \, dv + \frac{\partial f}{\partial w} \, dw \, dx \, dv) \]
\[ = \frac{\partial f}{\partial x} \, dx \, dz \, dv + \frac{\partial f}{\partial y} \, dy \, dz \, dv - (\frac{\partial f}{\partial z} \, dz \, dx \, dv + \frac{\partial f}{\partial w} \, dw \, dx \, dv) \]
\[ = \frac{\partial f}{\partial x} \, dx \, dz \, dv + \frac{\partial f}{\partial y} \, dy \, dz \, dv - \frac{\partial f}{\partial z} \, dz \, dx \, dv - \frac{\partial f}{\partial w} \, dw \, dx \, dv \]
\[ = \frac{\partial f}{\partial x} \, dx \, dz \, dv + \frac{\partial f}{\partial y} \, dy \, dz \, dv - \frac{\partial f}{\partial z} \, dz \, dx \, dv + \frac{\partial f}{\partial w} \, dw \, dx \, dv \]

Proof. We know that if \( J = i^{n/2+p(p-1)} \) operates over \( p \)-forms, it produces \( n-p \)-forms, so a new application of \( J \) operates over \( (n-p) \)-forms and hence \( J \) takes the form \( J = i^{n/2+(n-p)(n-p-1)} \). Thus
\[ J^2 = i^{n/2+(n-p)(n-p-1)} \cdot i^{n/2+2p(p-1)} = i^{n/2+(n-p)(n-p-1)+n/2+p(p-1)} \]

Recalling that over \( p \)-forms \(* = (-1)^{p(n-p)} = (i)^{2p(n-p)} \) we get:
\[ J^2 = i^{n/2+(n-p)(n-p-1)+n/2+2p(p-1)} \]
\[ = i^{n/2+(n-p)(n-p-1)+n/2+2p(p-1)+2p(n-p)} = i^{n+(n-p)(n-p-1)+2p(p-1)+2p(n-p)} \]
\[ = i^{n+(n-p)(n-p-1)+2p(p-1)+2p(n-p)} = i^{n+2+(n-p)+2p(p-1)+2p(n-p)} \]
\[ = i^{n+(n-p)(n-p-1)+2p(p-1)+2p(n-p)} = i^{n+2-(n-p)(n-p-1)+2p(p-1)+2p(n-p)} \]

Simplifying we obtain
\[ J^2 = i^2 = i^4k^2 = (i^4)k^2 = 1. \]

72 Theorem. \( J^2 = 1 \) if \( n = 2k \)

73 Corollary. When \( p = n-p, p = \frac{n}{2} \) and \( J \) sends \( \frac{n}{2} \)-forms into \( \frac{n}{2} \)-forms. If \( J \) has eigenvalues, these must be \( \pm 1 \) and for the corresponding eigenvector \( \omega, J\omega = \pm \omega \). In that case, if \( J\omega = \lambda \omega \) then \( J^2\omega = \lambda J\omega = \lambda^2 \omega = \omega \) then \( \lambda^2 = 1 \) and \( \lambda = \pm 1 \).

74 Lemma. If \( J\omega = \omega \) then \( JD\omega = -D\omega \), and viceversa if \( J\omega = -\omega \) then \( JD\omega = D\omega \). So, \( J \) functions as an inversion over \( D\omega \) when \( \omega \) is a
positive eigenvector of $J$, while it functions as the identity over $D\omega$ when $\omega$ is a negative eigenvector of $J$.

Proof: If $J\omega = \omega$ then $DJ\omega = D\omega$, but $DJ\omega = -JD\omega$ because these operators anticommute. Hence $-JD\omega = D\omega$ or $JD\omega = -D\omega$. Likewise, if $J\omega = -\omega$ then $DJ\omega = -JD\omega = -D\omega$ or $JD\omega = D\omega$.

75 Example. Let us illustrate over $\mathbb{R}^4$ the fact that if $J\omega = \omega$ then $JD\omega = -D\omega$, and vice-versa if $J\omega = -\omega$ then $JD\omega = D\omega$. Variables are denoted $x, y, z, v$.

Let us consider the 2 form $\alpha = fdx dy + f dz dv$. Recalling that for $n = 4$ and $p = 2$, $J = \ast$, we have:

$J\alpha = \ast f dx dy + \ast f dz dv = f dz dv + f dx dy$

So, $\alpha$ is in the positive sector of $J$. Let us verify that $JD\alpha = -D\alpha$.

Borrowing results from the previous example, we have that for $\omega = f dx dy$:

$D\omega = d\omega + \delta \omega = \frac{\partial f}{\partial z} dx dy dz + \frac{\partial f}{\partial v} dx dy dv - \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dx$

$JD\omega = \frac{\partial f}{\partial z} dv - \frac{\partial f}{\partial x} dz - \frac{\partial f}{\partial x} dx dz dv - \frac{\partial f}{\partial y} dy dz dv$

By the same token, if $\theta = f dz dv$, and since $D = d - \delta = d - \ast d\ast$, we get:

$D\theta = (d - \ast d\ast)(\theta) = d\theta - \ast d \ast \theta = d(f dz dv) - \ast d \ast (f dz dv)$

$= \frac{\partial f}{\partial x} dx dz dv + \frac{\partial f}{\partial y} dy dz dv - \ast (\frac{\partial f}{\partial x} dx dy dz + \frac{\partial f}{\partial y} dx dy dv)$

$= \frac{\partial f}{\partial x} dx dz dv + \frac{\partial f}{\partial y} dy dz dv - \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial v} dv$

and

$JD\theta = \ast \frac{\partial f}{\partial x} dx dz dv + \ast \frac{\partial f}{\partial y} dy dz dv + \ast \frac{\partial f}{\partial z} dz - \ast \frac{\partial f}{\partial v} dv$

$= \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial z} dx dy dz - \frac{\partial f}{\partial v} dx dy dv$

Therefore
\[ D\alpha = D\omega + D\theta = \frac{\partial f}{\partial z} dxdydz + \frac{\partial f}{\partial y} dxdydv - \frac{\partial f}{\partial x} dy + \frac{\partial f}{\partial y} dxdzdv + \frac{\partial f}{\partial z} dydzdv - \frac{\partial f}{\partial y} dxdydv \]

while

\[ JD\alpha = JD\omega + JD\theta = \frac{\partial f}{\partial z} dv - \frac{\partial f}{\partial x} dxdzdv - \frac{\partial f}{\partial y} dydzdv + \frac{\partial f}{\partial x} dy - \frac{\partial f}{\partial y} dx - \frac{\partial f}{\partial z} dxdydz - \frac{\partial f}{\partial y} dxdydv \]

We have verified that \( JD\alpha = -D\alpha \), given that \( J\alpha = \alpha \).

By contrast, \( \beta = gdxdz + gdydx \) is in the negative sector of \( J \) and so one shall expect that \( JD\beta = D\beta \).

We have used the signature of the manifold to infer the index of the signature operator because they are equal one to another. Nevertheless, it is interesting to calculate the index of the signature operator from raw definitions.

**76 Definition.** Let \( E \) be the positive sector of \( J \) and \( F \) the negative sector. The **signature operator** \( D_s \) of \( D \) is the restriction of \( D \) to \( E \) which goes onto \( F \):

\[ D_s = D/E : E \rightarrow F \]

\( D_s \) cannot be selfadjoint because its domain is different than the codomain. Besides, we have:

\[ D_s^* = D/F : F \rightarrow E \]

\[ D_s^* D_s = \triangle_+ / E : E \rightarrow E \]

\[ D_s D_s^* = \triangle_- / F : F \rightarrow F \]

where \( \triangle_{\pm} \) are the restrictions to the \( \pm \) eigenspaces of \( J \) of the Laplace operator \( \triangle \).

**77 Lemma.** The sectors of \( J \) allow the decomposition of \( \triangle \).
In regard with the sectors of $J$, we can write:

$$\Delta = \begin{pmatrix} \Delta_+ & 0 \\ 0 & \Delta_- \end{pmatrix} = \begin{pmatrix} D_s^* D_s & 0 \\ 0 & D_s D_s^* \end{pmatrix}$$

Because $\Delta = D^2$, this amounts to a decomposition of the Dirac operator:

$$D = \begin{pmatrix} 0 & D_s^* \\ D_s & 0 \end{pmatrix}$$

78 The Signature Index Theorem. The index of the signature operator is equal to the signature of $M$. Thus, the signature of $M$, which is a global topological invariant, can be calculated analytically by means of the index of a suitable differential operator which operates locally.

Proof: Taking $2k$–forms and the corresponding operators we have:

$$\text{Ind}(D_s) = \dim(\text{Ker}D_s) - \dim(\text{Ker}D_s^*)$$

Recalling that for an operator $L$ we have $\text{ind}(L) = \text{ind}(L^*)$, we can add to the right side

$$0 = \dim(\text{Ker}D_s^*) - \dim(\text{Ker}D_s^*) = \dim(\text{Ker}D_s^*) - \dim(\text{Ker}D_s)$$

and we have

$$\text{Ind}(D_s) = \dim(\text{Ker}D_s) - \dim(\text{Ker}D_s^*) + \dim(\text{Ker}D_s^*) - \dim(\text{Ker}D_s) = \dim(\text{Ker}D_s^*) + \dim(\text{Ker}D_s) - (\dim(\text{Ker}D_s) + \dim(\text{Ker}D_s^*))$$

but $\dim(\text{Ker}AB) = \dim(\text{Ker}A) + \dim(\text{Ker}B)$, hence

$$\text{Ind}(D_s) = \dim(\text{Ker}D_s^* D_s) - \dim(\text{Ker}D_s D_s^*) = \dim(\text{Ker} \Delta_+) - \dim(\Delta_-) = \dim(\text{Harm}_{2k}^+) - \dim(\text{Harm}_{2k}^-) = \sigma(M)$$

79 Corollary. All eigenvalues of matrix $Q$ are real when $n = 4k$.

80 Example. Let us find by direct calculation the index of the signature operator of $S^4$. Polar coordinates: $\theta_1, \theta_2, \theta_3, \theta_4$ with $\rho = 1$:

To begin with we must determine the positive sector of $J = i^{\frac{n}{2}+p(p-1)} \star$.

In this case, $n = 4$ and $J$ takes over p-forms the next values:

For 0-forms: $J = i^{\frac{4}{2}+0} \star = i^2 \star = - \star$

For 1-forms: $J = i^{\frac{4}{2}+1(1-1)} \star = i^2 \star = - \star$

For 2-forms: $J = i^{\frac{4}{2}+2(2-1)} = i^4 = \star$
For 3-forms: $J = i^4 + 3(3-1) = i^8 = \ast$
For 4-forms: $J = i^4 + 4(4-1) = i^{14} = -\ast$

The candidates for eigenvectors of $J$ are:

- $f d\theta_1 d\theta_2 d\theta_3 d\theta_4$
- $g_1 d\theta_1 + g_1 d\theta_2 d\theta_3 d\theta_4$
- $g_2 d\theta_2 + g_2 d\theta_1 d\theta_3 d\theta_4$
- $g_3 d\theta_3 + g_3 d\theta_1 d\theta_2 d\theta_4$
- $g_4 d\theta_4 + g_4 d\theta_1 d\theta_2 d\theta_3$
- $h_1 d\theta_1 d\theta_2 + h_1 d\theta_3 d\theta_4$
- $h_2 d\theta_1 d\theta_3 + h_2 d\theta_2 d\theta_4$
- $h_3 d\theta_1 d\theta_4 + h_3 d\theta_2 d\theta_3$

Now:

$$J(f + f d\theta_1 d\theta_2 d\theta_3 d\theta_4) = -\ast f - * f d\theta_1 d\theta_2 d\theta_3 d\theta_4 = - f d\theta_1 d\theta_2 d\theta_3 d\theta_4 - f,$$

$$J(g_1 d\theta_1 + g_1 d\theta_2 d\theta_3 d\theta_4) = -* g_1 d\theta_1 + * g_1 d\theta_2 d\theta_3 d\theta_4 = - g_1 d\theta_2 d\theta_3 d\theta_4 - g_1 d\theta_1,$$

$$J(g_2 d\theta_2 + g_2 d\theta_1 d\theta_3 d\theta_4) = -* g_2 d\theta_2 + * g_2 d\theta_1 d\theta_3 d\theta_4 = g_2 d\theta_1 d\theta_3 d\theta_4 + g_2 d\theta_2,$$

$$J(g_3 d\theta_3 + g_3 d\theta_1 d\theta_2 d\theta_4) = -* g_3 d\theta_3 + * g_3 d\theta_1 d\theta_2 d\theta_4 = - g_3 d\theta_1 d\theta_2 d\theta_4 - g_3 d\theta_3,$$

$$J(g_4 d\theta_4 + g_4 d\theta_1 d\theta_2 d\theta_3) = -* g_4 d\theta_4 + * g_4 d\theta_1 d\theta_2 d\theta_3 = g_4 d\theta_1 d\theta_2 d\theta_3 + g_4 d\theta_4,$$

$$J(h_1 d\theta_1 d\theta_2 + h_1 d\theta_3 d\theta_4) = * h_1 d\theta_1 d\theta_2 + * h_1 d\theta_3 d\theta_4 = h_1 d\theta_3 d\theta_4 + h_1 d\theta_1 d\theta_2,$$

$$J(h_2 d\theta_1 d\theta_3 + h_2 d\theta_2 d\theta_4) = * h_2 d\theta_1 d\theta_3 + * h_2 d\theta_2 d\theta_4 = - h_2 d\theta_2 d\theta_4 - h_2 d\theta_1 d\theta_3,$$

$$J(h_3 d\theta_1 d\theta_4 + h_3 d\theta_2 d\theta_3) = * h_3 d\theta_1 d\theta_4 + * h_3 d\theta_2 d\theta_3 = h_3 d\theta_2 d\theta_3 + h_3 d\theta_1 d\theta_4.$$
We see that the positive sector of $J$ is generated by

\[ g_2 d\theta_2 + g_4 d\theta_3 d\theta_4 \]
\[ g_4 d\theta_4 + g_3 d\theta_1 d\theta_2 d\theta_3 \]
\[ h_1 d\theta_1 d\theta_2 + h_1 d\theta_3 d\theta_4 \]
\[ h_3 d\theta_1 d\theta_4 + h_3 d\theta_2 d\theta_3 \]

We must apply $D = d - \delta = d - \ast d\ast$ over these forms to see whether or not they are in the kernel of $D$:

\[
D(g_2 d\theta_2 + g_2 d\theta_1 d\theta_3 d\theta_4) = (d - \ast d\ast)(g_2 d\theta_2 + g_2 d\theta_1 d\theta_3 d\theta_4)
\]
\[= dg_2 d\theta_2 + dg_2 d\theta_1 d\theta_3 d\theta_4 - \ast d\ast g_2 d\theta_2 - \ast d\ast g_2 d\theta_1 d\theta_3 d\theta_4\]
\[= \frac{\partial g_2}{\partial \theta_1} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_2 + \frac{\partial g_2}{\partial \theta_4} d\theta_2 + \frac{\partial g_2}{\partial \theta_2} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_3 d\theta_4 + \ast d\ast \frac{\partial g_2}{\partial \theta_1} d\theta_2 + \ast d\ast \frac{\partial g_2}{\partial \theta_3} d\theta_3 d\theta_4\]
\[= \frac{\partial g_2}{\partial \theta_1} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_2 + \frac{\partial g_2}{\partial \theta_4} d\theta_2 - \frac{\partial g_2}{\partial \theta_2} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_3 d\theta_4 - \frac{\partial g_2}{\partial \theta_1} d\theta_2 \ast d\ast d\theta_3 d\theta_4\]
\[= \frac{\partial g_2}{\partial \theta_1} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_2 + \frac{\partial g_2}{\partial \theta_4} d\theta_2 - \frac{\partial g_2}{\partial \theta_2} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_3 d\theta_4 - \frac{\partial g_2}{\partial \theta_1} d\theta_2 \ast d\ast d\theta_3 d\theta_4\]
\[= \frac{\partial g_2}{\partial \theta_1} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_2 + \frac{\partial g_2}{\partial \theta_4} d\theta_2 - \frac{\partial g_2}{\partial \theta_2} d\theta_2 + \frac{\partial g_2}{\partial \theta_3} d\theta_3 d\theta_4 - \frac{\partial g_2}{\partial \theta_1} d\theta_2 \ast d\ast d\theta_3 d\theta_4\]

This is different than zero, so $g_2 d\theta_2 + g_2 d\theta_1 d\theta_3 d\theta_4$ is not in the kernel of $D$. 

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Let us investigate now what happens with $g_4d\theta_4 + g_4d\theta_1d\theta_2d\theta_3$:

$$D(g_4d\theta_4 + g_4d\theta_1d\theta_2d\theta_3) = (d - *d)(g_4d\theta_4 + g_4d\theta_1d\theta_2d\theta_3)$$

$$= dg_4d\theta_4 + dg_4d\theta_1d\theta_2d\theta_3 - *d*g_4d\theta_4 - *d*g_4d\theta_1d\theta_2d\theta_3$$

$$= \frac{\partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \frac{\partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \frac{\partial g_4}{\partial \theta_3}d\theta_3d\theta_4$$

$$= \frac{\partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \frac{\partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \frac{\partial g_4}{\partial \theta_3}d\theta_3d\theta_4$$

$$\overline{= \partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \partial g_4}{\partial \theta_3}d\theta_3d\theta_4}$$

$$= \frac{\partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \frac{\partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \frac{\partial g_4}{\partial \theta_3}d\theta_3d\theta_4$$

$$\overline{= \partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \partial g_4}{\partial \theta_3}d\theta_3d\theta_4}$$

$$= \frac{\partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \frac{\partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \frac{\partial g_4}{\partial \theta_3}d\theta_3d\theta_4$$

$$\overline{= \partial g_4}{\partial \theta_1}d\theta_1d\theta_4 + \partial g_4}{\partial \theta_2}d\theta_2d\theta_4 + \partial g_4}{\partial \theta_3}d\theta_3d\theta_4}$$

Since this is different than zero, $g_4d\theta_4 + g_4d\theta_1d\theta_2d\theta_3$ is not in the kernel of $D$.

On the other hand, $h_1d\theta_1d\theta_2 + h_1d\theta_3d\theta_4$ cannot be in the kernel of $D = d - \delta$ because $d$ rises the degree of the form by one while $\delta$ diminishes it by one. So, $D$ would produce the sum of a 1-form and a 3-form. The same applies to $h_3d\theta_1d\theta_4 + h_3d\theta_2d\theta_3$.

We have proved that $\text{dim}(\text{Ker}(D_s)) = 0$.

Since the index of the signature operator $D_s$ is
\[ \text{ind}(D_s) = \dim(Ker(D_s)) - \dim(Ker(D^*_s)) \]

to proceed further we need to calculate \(Ker(D^*_s)\). We have:

\[ D = d - \delta = d - *d* \]

so

\[ D^* = d^* - \delta^* = \delta - (*d)^* = \delta - *\delta^* \]

Recalling that

\[ *^* = (-1)^{p(n-p)} * \]

we get

\[ D^* = \delta - *\delta^* = \delta - * * d * * = \delta - d = -D. \]

Hence \(Ker(D_s)^* = 0\) and \(\Delta^* = (D^2)^* = D^2 = \triangle\) so its index is zero. We conclude by direct calculation that

\[ \text{ind}(D_s) = \dim(Ker(D_s)) - \dim(Ker(D^*_s)) = 0 - 0 = 0 \]

We have verified that for \(S^4\) the index of the Signature Operator is equal to its signature that is zero.

**81 Example.** Let us verify by direct calculation that the index of \(D_s\) over the 4-torus \(T^4 = S^1 \times S^1 \times S^1 \times S^1\) is zero.

Let us notice that \(S^4\) and \(T^4\) both have the same leading cell \(e^4\):

\[ S^4 = e^0 \cup e^4 \]

while

\[ T^4 = e^0 \cup 4e^1 \cup 6e^2 \cup 4e^3 \cup e^4 \]

Therefore, we can reuse the calculations made for \(S^4\).
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