Regularity results for a class of Semilinear Parabolic Degenerate Equations and Applications

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Abstract

We prove some regularity results for viscosity solutions to strongly degenerate parabolic semilinear problems. These results apply to a specific model used for pricing Mortgage-Backed Securities and allow a complete justification of use of the classical Ito's formula.

AMS subject classifications: 35K55, 35K65, 35B65, 60H30, 91B70.

1 Introduction

In this paper we investigate the second order regularity of bounded viscosity solutions to the following semilinear parabolic equation of degenerate type

$$\partial_t u + H(x, t, u, \nabla u, \nabla^2 u) = 0,$$

where the Hamiltonian function is defined by the expression,

$$H(x, t, u, p, X) = -\frac{1}{2} tr(\sigma \sigma^T (t) X) + \langle \mu(x, t), p \rangle + \lambda(u)|\sigma^T p|^2 + \eta(u)|\sigma^T (t)p, w(x, t)| + f(x, t, u),$$

for every \((x, t) \in \mathbb{R}^N \times (0, T), a < u < b, p \in \mathbb{R}^N, X \in \mathcal{S}^N,\) where \(\mathcal{S}^N\) is the space of \(N \times N\) symmetric matrices endowed with the usual ordering. In (1.2), \(tr, |.|\) and \(\langle ., . \rangle\) denote the trace of a square matrix, the Euclidean norm and inner product, respectively. Moreover if \(\mathcal{M}_{N \times d}(\mathbb{R})\) is the space of real \(N \times d\) matrices, with \(N \geq d,\) then we assume that, \(\mu : \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}^N,\) \(\sigma : [0, T) \rightarrow \mathcal{M}_{N \times d}(\mathbb{R}), w : \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}^d,\) and \(f : \mathbb{R}^N \times [0, T) \times (a, b) \rightarrow \mathbb{R},\) are continuous functions.

Actually our main motivation comes from the following semilinear equation
\[
\partial_t U - \frac{1}{2} tr(\sigma^T \nabla^2 U) - \langle \mu, \nabla U \rangle + \rho \frac{|\sigma^T \nabla U|^2}{U + h + \xi(t)} + r(U + h) - \tau h = 0, \quad (1.3)
\]

in \(\mathbb{R}^N \times (0, T)\), \(\rho > 0\), \(\tau, T > 0\), where \(\xi = \xi(t)\) and \(h = h(x, t)\) are regular functions of their variables. Equation (1.3) has been proposed in [31] as a differential model used for pricing some widely traded American financial instruments, the Mortgage-Backed Securities (MBS); in particular the model was derived following the outline of X. Gabaix in [13].

Let \(U\) be a viscosity solution of (1.3), then setting \(u = U + h + \xi\), \(u\) solves a differential equation of type (1.1). The arbitrage pricing principle applies to financial instruments whose cash flows are related to the values of some economic factors, such as the interest rates \((r = r(t))\). Using that principle the value of these securities can be expressed as a conditional expectation over the probability space \((\Omega)\) of the underlying factors that determine the instrument’s price \((V_t)\) with respect to a particular measure \((Q)\) defined over that space. As a consequence the knowledge of second order regularity properties of \(u\), and therefore of \(U\), plays a fundamental role in order to close in a rigorous way the financial argument used for deriving equation (1.3). If we have sufficient conditions about \(U\), for applying the classical Ito’s rule (see for instance [23], or [1]), then, as we will show in section 5, there exists a probability measure called the Equivalent Martingale Measure, \(Q\), which has the required empiric and statistic representation proposed in [13], such that the following probabilistic equality holds:

\[
U(X_t, T-t) = \mathbb{E}_t^Q \left[ \int_t^T (\tau - r(T-s)) e^{-\int_s^T r(T-\kappa)d\kappa} h(X_s, T-s) ds \right], \quad \text{a.s. (1.4)}
\]

\[
dX_t = \mu(X_t, T-t)dt + \sigma(T-t)dW_t, \quad 0 < t \leq T. \quad (1.5)
\]

There \(\mathbb{E}_t^Q[\cdot]\) denotes the conditional mean value up to the time \(t\), taken with respect to \(Q\), and \(\xi(t) = e^{\int_0^t r(s)ds}, U(\cdot, 0) \equiv 0, h(\cdot, 0) \equiv 0, r\) is a deterministic function which represents the interest rate variable. The process \(W_t\) is a \(d\)-dimensional standard Brownian Motion over the probability space \(\Omega\), while \(X_t\) is a \(N\)-dimensional Ito process which describes the factors which affect the value of a Mortgage-Backed security. In this model \(h\) models the remaining principal during the life of the pool of mortgages (see [13], [33]).

The equation (1.4) represents the conclusive statement of our works about the pricing equation introduced in [31] and then studied from the point of view of the existence and uniqueness. Actually that formula gives the formalization of the existence of the market price of risk proposed by X. Gabaix in [13], which describes the martingale measure such that the value of the security can be expressed as the solution of the Hamilton Jacobi equation (1.3).

Our main result concerns the semiconvexity/concavity property, at a fixed time \(t \in [0, T)\), of the viscosity solution \(u\) of the equation (1.1). Applying our comparison result presented in [32], it can be proved the existence of a unique viscosity solution for the equation (1.1), with an assigned continuous and bounded initial datum \(u_0\), valued in the interval \((a, b)\). Moreover, using the same arguments of Theorem 4.5 in [32], it can be also deduced that, if the coefficients \(\mu, \lambda, \eta, f\)
and the initial datum are \( t \)-uniformly Lipschitz continuous functions then, also the viscosity solution has the same regularity at a fixed time. Keeping in mind these features, we start our considerations from the assumption of the existence of a \( t \)-uniformly Lipschitz continuous solution \( u \) for the equation (1.1) and, then, we prove the semiconvexity/concavity property of it.

The first result shows that \( u(t) \) is a semiconvex function over the whole space. In the recent literature of viscosity solutions for Hamilton-Jacobi equations, this kind of property is not still proved for a class of problems which could include our differential model. In [21], the authors only prove the semiconcavity of the solution for the Bellman equations, but the type of nonlinearity of their equation, does not include our Hamiltonian (1.2). There is another important paper, by Y. Giga, S. Goto, H. Ishii, and M.H. Sato, [14], where it was proved the convexity preserving property of the solution. But also in that case the authors study a fully nonlinear equation, whose Hamiltonian does not depend on the unknown \( u \). Therefore, although the initial datum in the original financial model (1.3) is constant, their technique is not suitable for treating our equations (1.1), (1.3).

Using our results about the semiconvexity and semiconcavity and assuming the same regularity of the coefficient used in (2), we deduce the \( W^{2,\infty} \) regularity, uniformly in \( t \in [0, T) \).

In Theorem 2.5, we conclude next, with a global result of regularity which contains also the time regularity of the solution. In particular, this last result will follow through an application of the Lipschitz continuity of \( u \) with respect to the time variable, which we will present in section 4.

The general framework for deriving a financial pricing equation consists in the application of the classical Ito’s rule. Therefore, we should consider classical solutions which have continuous derivatives. However, there are some works about some generalizations of that formula, see for instance in [17]. Here it is proved that the Ito’s formula holds in arbitrary dimensions for \( f \in W^{2,1,\infty}(\mathbb{R}^N \times (0, T)) \), if the equation

\[
df(X_t, t) = \mathcal{L}f(X_t, t)dt + \nabla^\top f(X_t, t)\sigma(t) \cdot dW_t, \quad \text{a.s.,} \tag{1.6}
\]

where

\[
\mathcal{L}f(x, t) = \partial_t f(x, t) + \langle \mu(x, T - t), \nabla f(x, t) \rangle \\
+ \frac{1}{2} tr(\sigma \sigma^\top (T - t) \nabla^2 f(x, T - t)),
\]

are interpreted appropriately using the generalized Hessian.

In section 5 we derive some mathematical and financial consequences concerning the model (1.3). In particular we present a result which combines the regularity of the solution of the model (1.3) and the results of [17] to state the relation (1.4). To reach this objective we propose a technical condition about the process \( X_t \) which allows us to extend the Ito’s rule as a conditional expected value representation, see Lemma 5.4 later. The technical results proposed in section 5 allow to consider the degeneration of the process \( X_t \), produced by the inequality \( N > d \). That degeneration was studied in literature to give sufficient conditions to state the existence of densities for the solutions of stochastic differential equations, see for instance [5], which contain a probabilistic form of Hörmander’s. Our condition is of a different nature and applies to many practical cases, like for constant coefficients, where the results of [5] do not hold.
2 Main Results

In this section we present our main results about the regularity of the solutions, the Section 3 being devoted to their proofs.

Here we leave out the definition of a viscosity sub/super solution of a differential problem like (1.1), and refer the reader to some classical works about the viscosity theory, such as [10] or [25].

**Definition 2.1.** A function $g$ in $\mathbb{R}^N$ is said to be semiconvex with constant $L > 0$, if

$$g(x + h) + g(x - h) - 2g(x) \geq -L|h|^2,$$

for every $x, h \in \mathbb{R}^N$.

In a same way we will say that $g$ is semiconcave with constant $L > 0$ if $-g$ is semiconvex with constant $L$.

Over $\mathbb{R}^m$, we will denote $\langle \cdot, \cdot \rangle$ the usual standard scalar product, while, over the space, $\mathcal{M}_{m,n}(\mathbb{R})$, of matrices with real coefficients and $m$-rows and $n$-columns, we will consider the usual norm,$$
\|A\| = \sup_{x \in \mathbb{R}^n} |Ax|, \quad \forall A \in \mathcal{M}_{m,n}(\mathbb{R}).$

Moreover we will denote as $\text{Im}(A)$ the range of the linear map defined by the matrix $A$.

If $g : \mathbb{R}^N \times [0, T) \times I \to \mathbb{R}$, where $I$ is a closed interval in $\mathbb{R}$, then we will use the following notations,$$
\|g\|_\infty = \text{esssup}_{(x, t, u) \in \mathbb{R}^N \times [0, T) \times I} |g(x, t, u)|,
$$
$$
\|g(t)\|_\infty = \text{esssup}_{(x, u) \in \mathbb{R}^N \times I} |g(x, t, u)|, \quad \forall t \in [0, T).
$$

We will denote as $W^{k, \infty}(\mathbb{R}^N)$, $k = 1, 2$, the usual Sobolev Space of bounded functions with weakly bounded derivatives in $\mathbb{R}^N$ of order less or equal to $k$. Moreover $W^{2,1,\infty}(\mathbb{R}^N \times (0, T))$ denotes the Sobolev space of functions $u$, with weakly derivatives $\partial_t u$, $\partial_i u$, $\partial_{ij} u \in L^\infty(\mathbb{R}^N \times (0, T))$, for $i, j = 1, \ldots, N$. These spaces are respectively endowed with the following norms

$$
\|u\|_{W^{k, \infty}} = \sum_{i=0}^{k} \|\nabla^i u\|_\infty,
$$

$$
\|u\|_{W^{2,1,\infty}} = \|\partial_t u\|_\infty + \|\nabla u\|_\infty + \|\nabla^2 u\|_\infty.
$$

If the functions take values in $\mathbb{R}^M$, we refer the same properties to the single components, and a similar notation holds for functions which depend on another variable in $I$.

In the following we shall assume the existence of a viscosity solution $u$ of (1.1), such that $u(\cdot, t) \in W^{1,\infty}(\mathbb{R}^N)$ and whose norm is uniformly bounded in $t \in [0, T)$. 

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Theorem 2.2. Let the viscosity solution $u$ of the equation (1.1) be valued in a bounded closed subinterval $I$ of the domain $(a,b)$, and let $\mu(\cdot,t)$, $w(\cdot,t) \in W^{2,\infty}(\mathbb{R}^N)$, $f(\cdot,t,\cdot) \in W^{2,\infty}(\mathbb{R}^N \times I)$, uniformly in time.

Assume:

i) $\lambda \in C((a,b))$, $\eta \in C((a,b)) \cap C^2(I)$;

ii) For every $(x,t) \in \mathbb{R}^N \times [0,T)$, $w(x,t) \in \text{Im}(\sigma^T(t))$.

If $u_0$ is semiconvex, then there are positive constants $C$, $M_0$, $C_0$, such that

$$u(x+h,t) + u(x-h,t) - 2u(x,t) \geq -(M_0e^{Ct} + C_0)|h|^2,$$

(2.8)

holds for every $x, h \in \mathbb{R}^N$, and $t \in [0,T)$. Therefore, for every $t \in [0,T)$, $u(\cdot,t)$ is semiconvex.

We have also the analogous result for the semiconcavity of $u$.

Theorem 2.3. Let the viscosity solution $u$ of the equation (1.1) be valued in a bounded closed subinterval $I$ of the domain $(a,b)$, and let $\mu(\cdot,t)$, $w(\cdot,t) \in W^{2,\infty}(\mathbb{R}^N)$, $f(\cdot,t,\cdot) \in W^{2,\infty}(\mathbb{R}^N \times I)$, uniformly in time. Assume:

i) $\lambda \in C((a,b))$, $\eta \in C((a,b)) \cap C^2(I)$;

ii) For every $(x,t) \in \mathbb{R}^N \times [0,T)$, $w(x,t) \in \text{Im}(\sigma^T(t))$.

If $u_0$ is semiconcave, then there are positive constants $C$, $M_0$, $C_0$, such that

$$u(x+h,t) + u(x-h,t) - 2u(x,t) \leq (M_0e^{Ct} + C_0)|h|^2,$$

(2.9)

holds for every $x, h \in \mathbb{R}^N$, and $t \in [0,T)$. Therefore, for every $t \in [0,T)$, $u(\cdot,t)$ is semiconcave.

As a consequence, we will deduce the announced result about the regularity of the solution $u$.

Theorem 2.4. If $\mu$, $w$, $f$, $\lambda$ and $\eta$ satisfy the assumptions of Theorems 2.2 and 2.3 and $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, then $u(t) \in W^{2,\infty}(\mathbb{R}^N)$, uniformly in time.

Theorem 2.5. If $\sigma$ is Lipschitz continuous, $\mu$, $w \in W^{2,1,\infty}(\mathbb{R}^N \times (0,T))$, $f \in W^{2,1,\infty}(\mathbb{R}^N \times (0,T) \times I)$, $w(x,t) \in \text{Im}(\sigma^T(t))$ for every $(x,t) \in \mathbb{R}^N \times [0,T)$, $u_0 \in W^{2,\infty}(\mathbb{R}^N)$, then

$$u \in W^{2,1,\infty}(\mathbb{R}^N \times (0,T)).$$

(2.10)

To prove Theorems 2.2, 2.3 and 2.4, we need of some technical results, see the Propositions 3.1 and 3.2, in section 3, which state the same thesis but use some structural and regularity assumptions on the function $\lambda$.

Then, these conditions on $\lambda$, can be removed by a particular compatibility between the second order linear term and the first order term in the equation (1.1). The proof of these facts is based on the classical property about the preservation of the notion of viscosity solution through a global, increasing, change of the variable $u$. 

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3 Proof of the Results

This part is devoted to the presentation of the technical results which are useful for proving Theorems 2.2, 2.3 and 2.4, illustrated in the previous section.

**Proposition 3.1.** Consider a viscosity solution \( u \) of problem (1.1) valued in a bounded closed subinterval \( I \) of the domain \((a, b)\), such that \( u(\cdot, t) \in \mathbb{W}^{1,\infty}(\mathbb{R}^N) \), with a norm uniformly bounded in the time \( t \in [0, T) \). Assume a semiconvex initial datum \( u_0 \) (see definition 2.1) with a constant \( L_0 > 0 \). Suppose that, \( \mu(\cdot, t), w(\cdot, t) \) are \( \mathbb{W}^{2,\infty}(\mathbb{R}^N) \) functions and, \( f(\cdot, t, \cdot) \in \mathbb{W}^{2,\infty}(\mathbb{R}^N \times I) \), uniformly in time. Moreover assume:

i) \( \lambda, \eta \in C((a, b)) \cap C^2(I) \).

ii) \( \lambda < 0, \lambda' > 0, \lambda \lambda'' - 2(\lambda')^2 > 0 \), over \( I \).

iii) For every \((x, t) \in \mathbb{R}^N \times [0, T)\), \( w(x, t) \in \text{Im}(\sigma^T(t)) \).

Then, there are positive constants \( C \), which depends on \( \text{sup}_{t \in [0, T]} \| u(t) \|_{\mathbb{W}^{1,\infty}} \) and \( C_0 \), which depends on \( L_0 \) and \( \text{Lip}(u_0) \), such that

\[
u(x + h, t) + u(x - h, t) - 2u(x, t) \geq -C_0 e^{C_1 |h|^2} \]

holds for every \((x, h, t) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T)\). In particular \( u(\cdot, t) \) is semiconvex, for every \( t \in [0, T) \).

The equivalent result for the semiconcave property of the solution, is the following.

**Proposition 3.2.** Consider a viscosity solution \( u \) of problem (1.1) valued in a bounded closed subinterval \( I \) of the domain \((a, b)\), such that \( u(\cdot, t) \in \mathbb{W}^{1,\infty}(\mathbb{R}^N) \), with a norm uniformly bounded in the time \( t \in [0, T) \). Assume a semiconcave initial datum \( u_0 \) (see definition 2.1) with a constant \( L_0 > 0 \). Suppose that, \( \mu(\cdot, t), w(\cdot, t) \) are \( \mathbb{W}^{2,\infty}(\mathbb{R}^N) \) functions and, \( f(\cdot, t, \cdot) \in \mathbb{W}^{2,\infty}(\mathbb{R}^N \times I) \), uniformly in time. Moreover assume:

i) \( \lambda, \eta \in C((a, b)) \cap C^2(I) \).

ii) \( \lambda > 0, \lambda' > 0, \lambda \lambda'' - 2(\lambda')^2 > 0 \), over \( I \).

iii) For every \((x, t) \in \mathbb{R}^N \times [0, T)\), \( w(x, t) \in \text{Im}(\sigma^T(t)) \).

Then, there are positive constants \( C \), which depends on \( \text{sup}_{t \in [0, T]} \| u(t) \|_{\mathbb{W}^{1,\infty}} \) and \( C_0 \), which depends on \( L_0 \) and \( \text{Lip}(u_0) \), such that

\[
u(x + h, t) + u(x - h, t) - 2u(x, t) \leq C_0 e^{C_1 |h|^2} \]

holds for every \((x, h, t) \in \mathbb{R}^N \times \mathbb{R}^N \times [0, T)\). In particular \( u(\cdot, t) \) is semiconcave, for every \( t \in [0, T) \).
Remark 3.3. We can remark the analogies between the two formulations 3.1, 3.2. Actually the hypothesis \( \lambda' > 0 \), is the same in both the Propositions. This can be interpreted as a consequence of the comparison principle, which requires a monotonicity for the Hamiltonian with respect to \( u \), (see [14], pag. 462). While of course, the others expressions in \( \text{ii} \) have exactly an opposite sign, showing a perfect symmetry between the two formulations.

The proof of Proposition 3.2 follows by argues as Proposition 3.1 and then it is omitted.

Proof of Proposition 3.1. Consider the function \( v = ue^{-Ct} \), where \( C \) is a nonnegative constant. Then \( v \) is an \( x \)-Lipschitz continuous function uniformly with respect to the time \( t \in [0, T) \). Moreover \( v \) is a continuous viscosity solution of the following equation

\[
\partial_t v - \frac{1}{2} \text{tr}(\sigma \nabla^2 v) + \langle \mu, \nabla v \rangle + \lambda (ve^{Ct})e^{Ct}|\sigma \nabla v|^2 + \eta (ve^{Ct})\langle \sigma^T v, w \rangle + e^{-Ct}f(x, t, ve^{Ct}) + Cv = 0, 
\]

(3.13)

where \((x, t) \in \mathbb{R}^N \times (0, T)\), and \( v_0 = u_0 \). We now will go to prove that, under the assumptions made on \( \lambda \) and \( \eta \), we have the semiconvexity of the function \( v \), and therefore the semiconvexity of the function \( u \). In particular we are going to show that,

\[
v(x, t) + v(y, t) - 2v(z, t) \geq -M(\|x - z\|^4 + \|y - z\|^4 + \|x + y - 2z\|^2)^{1/2} \tag{3.14}
\]

for every \( x, y, z \in \mathbb{R}^N \), \( t \in [0, T) \), where \( M = \frac{\sqrt{2}}{2} \max(L_0, \text{Lip}(u_0)) \). By the assumptions on \( u \), the initial datum is Lipschitz continuous, therefore this constant is well defined; this obviously yields the assertion on \( v \), by plugging \( x = z + h \) and \( y = z - h \). It is easy to see that the above inequality is equivalent to the following one:

\[
v(x, t) + v(y, t) - 2v(z, t) \geq -M[\delta + \frac{1}{\delta}(\|x - z\|^4 + \|y - z\|^4 + \|x + y - 2z\|^2)], \tag{3.15}
\]

Remark 3.4. We observe that if a function \( g \) is a semiconvex as in the Definition 2.1 and is a Lipschitz continuous function, then, for all \( x, y, z \), the following inequality holds,

\[
g(x) + g(y) - 2g(z) \geq -L(\|x - z\|^2 + \|y - z\|^2) - \text{Lip}(g)|x + y - 2z|.
\]

Where \( \text{Lip}(g) \) is the Lipschitz constant of \( g \).

Proof of Proposition 3.1. Consider the function \( v = ue^{-Ct} \), where \( C \) is a nonnegative constant. Then \( v \) is an \( x \)-Lipschitz continuous function uniformly with respect to the time \( t \in [0, T) \). Moreover \( v \) is a continuous viscosity solution of the following equation

\[
\partial_t v - \frac{1}{2} \text{tr}(\sigma \nabla^2 v) + \langle \mu, \nabla v \rangle + \lambda (ve^{Ct})e^{Ct}|\sigma \nabla v|^2 + \eta (ve^{Ct})\langle \sigma^T v, w \rangle + e^{-Ct}f(x, t, ve^{Ct}) + Cv = 0, 
\]

(3.13)

where \((x, t) \in \mathbb{R}^N \times (0, T)\), and \( v_0 = u_0 \). We now will go to prove that, under the assumptions made on \( \lambda \) and \( \eta \), we have the semiconvexity of the function \( v \), and therefore the semiconvexity of the function \( u \). In particular we are going to show that,

\[
v(x, t) + v(y, t) - 2v(z, t) \geq -M(\|x - z\|^4 + \|y - z\|^4 + \|x + y - 2z\|^2)^{1/2} \tag{3.14}
\]

for every \( x, y, z \in \mathbb{R}^N \), \( t \in [0, T) \), where \( M = \frac{\sqrt{2}}{2} \max(L_0, \text{Lip}(u_0)) \). By the assumptions on \( u \), the initial datum is Lipschitz continuous, therefore this constant is well defined; this obviously yields the assertion on \( v \), by plugging \( x = z + h \) and \( y = z - h \). It is easy to see that the above inequality is equivalent to the following one:

\[
v(x, t) + v(y, t) - 2v(z, t) \geq -M[\delta + \frac{1}{\delta}(\|x - z\|^4 + \|y - z\|^4 + \|x + y - 2z\|^2)], \tag{3.15}
\]
for every $\delta > 0$, $\forall x, y, z \in \mathbb{R}^N$.

Hence, fix $\varepsilon, \delta, \gamma > 0$, and consider the following test function

$$
\Psi(x, y, z, t) = v(x, t) + v(y, t) - 2v(z, t) + M \left[ \delta + \frac{1}{\delta} (|x - z|^4 + |y - z|^4 + |x + y - 2z|^2) \right] + \varepsilon |x|^2 + \varepsilon |y|^2 + \varepsilon |t|^2, \quad (3.16)
$$

defined for $(x, y, z, t) \in U = \mathbb{R}^m \times [0, T)$. The assertion $(3.15)$ is equivalent to proving that, for every $\delta, \gamma > 0$, there exists $\varepsilon_0 = \varepsilon_0(\delta, \gamma) > 0$, such that for every $0 < \varepsilon < \varepsilon_0$, the following holds:

$$
\inf_{U} \Psi \geq 0. \quad (3.17)
$$

Actually if $(3.17)$ holds, then fixing a point over $U$, we can send $\varepsilon$ to zero in the inequality $\Psi \geq 0$, and then sending also $\gamma$ to zero we obtain $(3.15)$. Thus we limit us to consider the assertion $(3.17)$.

We assume as usual that $(3.17)$ is false, and will get a contradiction. Therefore exist $\delta_0, \gamma_0 > 0$ and a sequence $\varepsilon_j \to 0$, as $j \to \infty$, such that

$$
\inf_{U} \Psi < 0, \quad (3.18)
$$

with $\delta = \delta_0, \gamma = \gamma_0$, and $\varepsilon = \varepsilon_j$, for every integer $j > 0$. If we consider a minimizing sequence $(x_k, y_k, z_k, t_k) \in U$ for $\Psi$. By $(3.18)$, the definition $(3.16)$ and the boundness of $v$, we see that $(x_k, y_k, z_k)$ must be bounded, so we can extract a convergent subsequence, which converges to some point $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t}) \in U$, which, by the continuity of $v$, is a global minimum point for $\Psi$ over $U$; moreover $\tilde{t}$ (which is obviously less than $T$) is strictly positive. In fact if $\tilde{t} = 0$, by $(3.18)$ and the Remark 3.4, we obtain,

$$
0 > \Psi(\tilde{x}, \tilde{y}, \tilde{z}, 0) \geq u_0(\tilde{x}) + u_0(\tilde{y}) - 2u_0(\tilde{z}) + M \left[ \delta + \frac{1}{\delta} (|\tilde{x} - \tilde{z}|^4 + |\tilde{y} - \tilde{z}|^4 + |\tilde{x} + \tilde{y} - 2\tilde{z}|^2) \right] \geq -L_0(|\tilde{x} - \tilde{z}|^2 + |\tilde{y} - \tilde{z}|^2) - Lip(u_0)|\tilde{x} + \tilde{y} - 2\tilde{z}| + +M \left[ \delta + \frac{1}{\delta} (|\tilde{x} - \tilde{z}|^4 + |\tilde{y} - \tilde{z}|^4 + |\tilde{x} + \tilde{y} - 2\tilde{z}|^2) \right] \geq [-L_0 + \frac{2}{\sqrt{3}} M]|\tilde{x} - \tilde{z}|^2 + |\tilde{y} - \tilde{z}|^2) + +[-Lip(u_0) + \frac{2}{\sqrt{3}} M]|\tilde{x} + \tilde{y} - 2\tilde{z}| \geq 0.
$$

where in the last two inequalities we have used the following relations

$$
\sqrt{x_1^2 + x_2^2 + x_3^2} \geq \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3), \quad \forall x_1, x_2, x_3 \geq 0,
$$

$$
x_1^2 + x_2^2 \geq 2x_1x_2, \quad \forall x_1, x_2 \in \mathbb{R}
$$

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Hence by the previous contradiction we deduce that \( \hat{t} > 0 \). So the minimum point is an interior stationary point of \( \Psi \). Setting the function

\[
g(x, y, z, t) = -v(x, t) - v(x, t) + 2v(z, t),
\]

\[
\Phi(x, y, z, t) = M [\delta + \frac{1}{2}(|x - z|^4 + |y - z|^4 + |x + y - 2z|^2)]
\]

\[
+ \varepsilon |x|^2 + \varepsilon |y|^2 + \varepsilon |y|^2 + \frac{\gamma}{T - t}
\]

we have that \( g - \Phi = -\Psi \) has a global interior maximum point at \((\hat{\xi}, \hat{t}) = (\hat{x}, \hat{y}, \hat{z}, \hat{t})\), therefore we can apply the classical Theorem about the maximum principle for semicontinuous functions of M.G. Crandall and H. Ishii, in [9], to deduce that for \( \kappa = \frac{1}{\varepsilon} > 0 \) there exist \((b_i, X_i) \in \mathbb{R} \times S^N\), for \( i = 1, 2, 3 \), such that,

\[
(-b_i, -\Phi_i - X_i) \in P_{2-}^v, \ i = 1, 2
\]

\[
\left(\frac{b_3}{2}, \frac{\Phi_3}{2}, \frac{1}{2} X_3\right) \in P_{2+}^v, \quad \text{(3.22)}
\]

and, if \( O \) denotes the null \( N \times N \) matrix, we have

\[
\begin{pmatrix}
X_1 & 0 & 0 \\
0 & X_2 & 0 \\
0 & 0 & X_3
\end{pmatrix}
\leq \nabla^2 \Phi(\hat{\xi}, \hat{t}) + \kappa |\nabla^2 \Phi(\hat{\xi}, \hat{t})|^2, \quad \text{(3.23)}
\]

\[
b_1 + b_2 + b_3 = \partial_t \Phi(\hat{\xi}, \hat{t}). \quad \text{(3.24)}
\]

Where for simplifying notations we have set \( v_1, v_2, v_3 \) for \( v(\hat{x}, \hat{t}), v(\hat{y}, \hat{t}), v(\hat{z}, \hat{t}) \), respectively, and in a same way \( \Phi_1, \Phi_2, \Phi_3 \), for the partial derivatives of \( \Phi \) evaluated at the considered maximum point. Now we compute the derivatives of \( \Phi \). Set

\[
p = \frac{2M}{\delta}(\hat{x} - \hat{z})|\hat{x} - \hat{z}|^2
\]

\[
q = \frac{2M}{\delta}(\hat{y} - \hat{z})|\hat{y} - \hat{z}|^2
\]

\[
m = \frac{2M}{\delta}(\hat{x} + \hat{y} - 2\hat{z}).
\]

Then

\[
\partial_t \Phi(\hat{\xi}) = \frac{\gamma}{(T - t)^2} \geq \frac{\gamma}{T^2}
\]

\[
\Phi_1 = 2\varepsilon \hat{x} + 2p + m
\]

\[
\Phi_2 = 2\varepsilon \hat{y} + 2q + m
\]

\[
\Phi_3 = 2\varepsilon \hat{z} - 2p - 2q - 2m.
\]

Moreover, if \( I \) denotes the \( N \times N \) identity matrix, we have
\[ \nabla^2 \Phi(\xi, \hat{t}) = \left( \frac{2M}{\delta} \right)^{\frac{3}{2}} \left\{ \begin{array}{ccc} |p| \hat{t} I & 0 & -|p| \hat{t} I \\ 0 & |q| \hat{t} I & -|q| \hat{t} I \\ -|p| \hat{t} I & -|q| \hat{t} I & (|p| \hat{t} + |q| \hat{t}) I \end{array} \right\} + 4 \left\{ \begin{array}{ccc} \frac{\rho \otimes p}{|p|^2} & 0 & -\frac{\rho \otimes p}{|p|^2} \\ 0 & \frac{\rho \otimes q}{|q|^2} & -\frac{\rho \otimes q}{|q|^2} \\ -\frac{\rho \otimes p}{|p|^2} & \frac{\rho \otimes q}{|q|^2} & \frac{\rho \otimes p}{|p|^2} + \frac{\rho \otimes q}{|q|^2} \end{array} \right\} \right\} + \frac{2M}{\delta} \left( \begin{array}{ccc} I & I & -2I \\ I & I & -2I \\ -2I & -2I & 4I \end{array} \right) + 2\varepsilon \left( \begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{array} \right) \] (3.29)

By (3.21), and the equation (3.13), we have

\[ b_i \leq \frac{1}{2} tr(\sigma \sigma^T X) - \langle \mu_i, \Phi_i \rangle + \lambda(v_i e^{C^T} e^{C^T} |\sigma^T \Phi_i|^2 \right) \\
-\eta(v_i e^{C^T} |\sigma^T \Phi_i, w_i \rangle + Cv_i + e^{C^T} f_i, \quad i = 1, 2 \]

\[ b_3 \leq \frac{1}{2} tr(\sigma \sigma^T X_i) - \langle \mu_3, \Phi_3 \rangle - 2\lambda(v_3 e^{C^T} e^{C^T} |\sigma^T \Phi_3|^2 \right) \\
-\eta(v_3 e^{C^T} |\sigma^T \Phi_3, w_3 \rangle - 2e^{C^T} f_3 - 2Cv_3. \] (3.30)

Where we have used the same notation of (3.21) for denoting the functions \( \mu, w, f \), and we have also omitted the dependency of the matrix \( \sigma \) by \( \hat{t} \). Adding inequalities (3.30) and using (3.23), we compute

\[ \partial_t \Phi(\xi, \hat{t}) \leq \frac{1}{2} tr(\sigma \sigma^T (X_1 + X_2 + X_3)) - [\langle \mu_1, \Phi_1 \rangle + \langle \mu_2, \Phi_2 \rangle + \langle \mu_3, \Phi_3 \rangle] + \\
+ [\lambda(v_1 e^{C^T} e^{C^T} |\sigma^T \Phi_1|^2 - \eta(v_1 e^{C^T} |\sigma^T \Phi_1, w_1 \rangle \\
+ \lambda(v_2 e^{C^T} e^{C^T} |\sigma^T \Phi_2|^2 - \eta(v_2 e^{C^T} |\sigma^T \Phi_2, w_2 \rangle \\
- 2\lambda(v_3 e^{C^T} e^{C^T} |\sigma^T \Phi_3|^2 - \eta(v_3 e^{C^T} |\sigma^T \Phi_3, w_3 \rangle \right] + \\
+ C(v_1 + v_2 - 2v_3) + e^{C^T} [f_1 + f_2 - 2f_3]. \] (3.31)

Now we use (3.28) and (3.29) for estimating the single part in the brackets of the inequality (3.31). Let \( \sigma^{(l)} \) be the \( l \) column of \( \sigma \), and define

\[ \Sigma^{(l)} = \begin{pmatrix} \sigma^{(l)} \\ \sigma^{(l)} \\ \sigma^{(l)} \end{pmatrix} \]

for \( l = 1, \ldots, d \), then by (3.23), we have
$$\text{tr}(\sigma \sigma^T (X_1 + X_2 + X_3)) = \sum_{l=1}^{d} (\langle X_1 \sigma^{(l)}, \sigma^{(l)} \rangle + \langle X_2 \sigma^{(l)}, \sigma^{(l)} \rangle + \langle X_3 \sigma^{(l)}, \sigma^{(l)} \rangle)$$

\[
\leq \sum_{l=1}^{d} (\|\nabla^2 \Phi(\tilde{\xi}, \tilde{\tau}) \Sigma^{(l)} \| + \kappa \| \nabla^2 \Phi(\tilde{\xi}, \tilde{\tau}) \Sigma^{(l)} \|^2).
\quad (3.32)
\]

Using (3.29), it is easy to compute that

$$\nabla^2 \Phi(\tilde{\xi}, \tilde{\tau}) \Sigma^{(l)} = 2\varepsilon \Sigma^{(l)}.$$  

(3.33)

So introducing (3.33) in (3.32), we deduce

$$\text{tr}(\sigma \sigma^T (X_1 + X_2 + X_3)) \leq 18\varepsilon \sum_{l=1}^{d} |\sigma^{(l)}|^2 = O(\varepsilon).$$  

(3.34)

By condition (3.18), we have

$$\varepsilon |\tilde{x}|, \varepsilon |\tilde{y}|, \varepsilon |\tilde{z}| \leq O(\sqrt{\varepsilon}),$$

(3.35)

$$\frac{1}{2} (\frac{\delta}{2M}) \gamma (|p| \gamma + |q| \gamma) + \frac{\delta}{2M} |m|^2 + M\delta \leq 2v_3 - v_1 - v_2.$$ 

Therefore, using the notations introduced in (3.25)-(3.28), (3.35) and the regularity assumptions on $\mu$, we obtain

$$|\langle \mu_1, \Phi_1 \rangle + \langle \mu_2, \Phi_2 \rangle + \langle \mu_3, \Phi_3 \rangle| \leq$$

\[
\leq |\langle \mu_1, 2p + m \rangle + \langle \mu_2, 2q + m \rangle - \langle \mu_3, 2p + 2q + 2m \rangle + 2\varepsilon |(\mu_1, \tilde{x}) + (\mu_2, \tilde{y}) + (\mu_3, \tilde{z})| \leq
\]

\[
\leq 2 |\mu_1 - \mu_3, p \rangle + 2 |\mu_2 - \mu_3, q \rangle + |\mu_1 + \mu_2 - 2\mu_3, m \rangle + O(\sqrt{\varepsilon}) \leq
\]

\[
\leq \sup_{t \in [0, T]} \| \mu(t) \|_{W^{2, \infty}} \left[ 2 (\frac{\delta}{2M})^{\frac{1}{4}} (|p|^{\frac{1}{4}} + |q|^{\frac{1}{4}}) + \frac{\delta}{2M} |m|^2 \right] + O(\sqrt{\varepsilon}) \leq
\]

\[
\leq \sup_{t \in [0, T]} \| \mu(t) \|_{W^{2, \infty}} \left[ 4(2v_3 - v_1 - v_2) + \frac{1}{2} (\frac{\delta}{2M})^{\frac{1}{4}} (|p|^{\frac{1}{4}} + |q|^{\frac{1}{4}}) + \frac{\delta}{2M} |m|^2 \right] + \frac{\delta}{4M} |m|^2 \right] + O(\sqrt{\varepsilon}) \leq
\]

\[
\leq 6 \sup_{t \in [0, T]} \| \mu(t) \|_{W^{2, \infty}} (2v_3 - v_1 - v_2) + O(\sqrt{\varepsilon}),
\quad (3.36)
\]

where in the last inequality we have again used (3.35).
For estimating the nonlinear part we consider the function
\[ G(u, \theta) = \lambda(u)e^{Ct}[\sigma^T\theta]^2 - \eta(u)\langle \sigma^T\theta, w_3 \rangle, \] (3.37)
which depends on \((u, \theta) \in [a, b] \times \mathbb{R}^N\), and for every \(u_1, u_2, \theta_1, \theta_2\), define
\[ \Delta^2G(u_1, u_2, \theta_1, \theta_2) = G(u_1, \theta_1) + G(u_2, \theta_2) - 2G\left(\frac{u_1 + u_2}{2}, \frac{\theta_1 + \theta_2}{2}\right). \] (3.38)

Let \(\eta\) denotes the value of \(\eta\) at the point \(v_ie^{Ct}\), for \(i = 1, 2, 3\), then using notations (3.37), (3.38) and (3.25)-(3.28), we have
\[
\begin{align*}
\lambda(v_1e^{Ct})e^{Ct}[\sigma^T\Phi_1]^2 - \eta(v_1e^{Ct})\langle \sigma^T\Phi_1, w_1 \rangle + \lambda(v_2e^{Ct})e^{Ct}[\sigma^T\Phi_2]^2 \\
- \eta(v_2e^{Ct})\langle \sigma^T\Phi_2, w_2 \rangle - 2\eta(v_3e^{Ct})e^{Ct}\frac{1}{2}\sigma^T\Phi_3^2 - \eta(v_3e^{Ct})\langle \sigma^T\Phi_3, w_3 \rangle
\end{align*}
\]
\[= \Delta^2G(v_1e^{Ct}, v_2e^{Ct}, 2p + m, 2q + m) + 2\left[\frac{G(v_1 + v_2)e^{Ct}}{2}, p + q + m\right] - G(v_3e^{Ct}, p + q + m) + \eta_1(\langle \sigma^T(2p + m), w_3 - w_1 \rangle + \langle \sigma^T q, w_3 - w_2 \rangle)
+ \eta_2(\langle \sigma^T(2q + m), w_3 - w_2 \rangle) + O(\sqrt{\varepsilon}). \] (3.39)

In the last passages we have used the inequalities (3.35) for estimating the residual terms which involve \(\varepsilon\).

\[\eta_1(\langle \sigma^T(2p + m), w_3 - w_1 \rangle + \langle \sigma^T (2q + m), w_3 - w_2 \rangle =
= [\eta_1 - \eta_3][\langle \sigma^T(2p + m), w_3 - w_1 \rangle + \langle \sigma^T(2q + m), w_3 - w_2 \rangle
+ 2\eta_3(\langle \sigma^T p, w_3 - w_1 \rangle + \langle \sigma^T q, w_3 - w_2 \rangle)
- \eta_3(\langle \sigma^T m, w_1 + w_2 - 2w_3 \rangle. \] (3.40)

Hence, using the Lipschitz regularity of the function \(u\), the regularity of \(w\), jointly with (3.35), we deduce,
\[
\begin{align*}
[\eta_1 - \eta_3][\langle \sigma^T(2p + m), w_3 - w_1 \rangle \leq & \|\eta\|_{\infty}\sigma^T \|_{\infty} \sup_{t \in [0,T]} \|w(t)\|_{W^{2,\infty}} [4\|u\|_{\infty}(\frac{\delta}{2M})\|p\| + \frac{1}{2} \sup_{t \in [0,T]} \|u(t)\|_{W^{1,\infty}} \\
& + \frac{\delta}{2M} \|p\| + \frac{\delta}{2M} \|m\|] \leq 
\leq & \|\eta\|_{\infty}\sigma^T \|_{\infty} \sup_{t \in [0,T]} \|w(t)\|_{W^{2,\infty}} [4\|u\|_{\infty}(\frac{\delta}{2M})\|p\| + \frac{1}{2} \sup_{t \in [0,T]} \|u(t)\|_{W^{1,\infty}} \\
& + \frac{\delta}{2M} \|p\| + \frac{\delta}{2M} \|m\|] \leq C_1 \left[\frac{1}{2}(\frac{\delta}{2M})\|p\| + \frac{1}{4}(\frac{\delta}{2M})\|m\|\right]. \] (3.42)
\]

In the last inequality we have used the notation
\[C_1 = 2\|\eta\|_{\infty}\sigma^T \|_{\infty} \sup_{t \in [0,T]} \|w(t)\|_{W^{2,\infty}} \max(8\|u\|_{\infty}, \sup_{t \in [0,T]} \|u(t)\|_{W^{1,\infty}}) \] (3.43)
We can repeat the argument for estimating \([\eta_2 - \eta_3](\sigma^T (2q + m), w_3 - w_2)\), obtaining, again using the second relation in (3.35), the inequality

\[
[\eta_1 - \eta_3](\sigma^T (2p + m), w_3 - w_1) + [\eta_2 - \eta_3](\sigma^T (2q + m), w_3 - w_2) \leq C_1 \left[\frac{1}{2} \frac{\delta}{2M} |p| + |q| + \frac{1}{2} \frac{\delta}{2M} |m| \right] \leq C_1 (2v_3 - v_1 - v_2) \quad (3.44)
\]

By (3.35) and the following inequality,

\[
|w_1 + w_2 - 2w_3| \leq \sup_{t \in [0, T]} \|w(t)\| \sup_{t \in [0, T]} \|w(t)\| w_2 \infty \left[\frac{\delta}{2M} |p| + \frac{\delta}{2M} |q| + \frac{\delta}{2M} |m| \right]^{\frac{1}{2}},
\]

then it is easy to estimate the last two terms in the brackets \([\]\) in (3.40), through the expression \(C_2(2v_3 - v_1 - v_2)\), where

\[
C_2 = 5\|\eta\| \|\sigma^T\| \sup_{t \in [0, T]} \|w(t)\| w_2 \infty.
\quad (3.45)
\]

Now we proceed with the estimates of the first two terms in (3.39). We observe that, by the assumption \(ii\), we can consider the positive constant,

\[
C_3 = \frac{1}{2} \frac{\|\eta\|}{\|w\|} \|w\| \infty \frac{1}{4 \min \lambda^2},
\quad (3.46)
\]

and also we can write

\[
\partial_u G(u, \theta) \geq -C_3 e^{-C_3}, \ \forall (u, \theta) \in [a, b] \times \mathbb{R}^N.
\quad (3.47)
\]

Therefore by (3.47) and \(2v_3 - v_1 - v_2 > 0\), the following holds,

\[
G\left(\frac{v_1 + v_2}{2}, p + q + m\right) - G(v_3 e^{C_3}, p + q + m) \leq C_3 (2v_3 - v_1 - v_2). \quad (3.48)
\]

By \(iii\), we can set \(w_3 = \sigma^T b_3\), for some \(N\)-dimensional vector \(b_3\). Moreover

\[
\begin{align*}
\partial_{\sigma}^2 G &= e^{C_3} \lambda^2 \sigma^T \theta^2 - \eta'' \langle \sigma^T \theta, w_3 \rangle, \\
\partial_{\sigma, u}^2 G &= \sigma \sigma^T J, \\
\partial_{\sigma, \sigma}^2 G &= 2e^{C_3} \lambda \sigma^T,
\end{align*}
\quad (3.49)
\]

where \(J = 2e^{C_3} \lambda \theta - \eta'' b_3\). For every \(X = (k, h) \in \mathbb{R} \times \mathbb{R}^N\) consider the orthogonal matrix \(A\) such that \(A \sigma A^T\) is diagonal, with entries \(S_i \geq 0, i = 1, \ldots, N\). Set \(\overline{h} = Ah, \overline{J} = AJ\), and define

\[
C_4 = \frac{\|\eta'' - 2\lambda \eta'\| \|w\|}{4 \min (2 \frac{\|w\|}{\lambda} - \lambda'' \|w\| \infty)} - \frac{1}{2 \max (\lambda)} \|\eta''\| \|w\| \infty, \quad (3.50)
\]

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by (3.49), ii), and (3.50), we have

\[ \langle \nabla^2 G - C_4 e^{-C \hat{t}} \left( \begin{array}{cc} I & 0 \\ 0 & 0 \end{array} \right) \rangle X, X \rangle = \]

\[ = (\partial^2_u G - C_4 e^{-C \hat{t}})k^2 + \sum_{i=1}^N (2e^{C \hat{t}} \lambda_i^2 + 2kJ_i \hat{t}_i)S_i \leq \]

\[ \leq [\partial^2_u G - C_4 e^{-C \hat{t}} - \sum_{i=1}^N \frac{e^{-C \hat{t}}}{2\lambda} S_i |J_i|^2]k^2 = \]

\[ = [e^{C \hat{t}} \lambda_\parallel |\sigma^\top \theta|^2 + \eta'' \langle \sigma^\top \theta, w_3 \rangle, -C_4 e^{-C \hat{t}} - \frac{e^{-C \hat{t}}}{2\lambda} 2e^{C \hat{t}} \lambda_\parallel |\sigma^\top \theta + \eta' w_3|^2]k^2 \]

\[ = [e^{C \hat{t}} (\lambda'' - 2\frac{(\lambda')^2}{\lambda}) |\sigma^\top \theta|^2 + (\eta'' - 2\lambda' \eta') \langle \sigma^\top \theta, w_3 \rangle - \]

\[ - \frac{e^{-C \hat{t}}}{2\lambda} \lambda'' |w_3|^2 - C_4 e^{-C \hat{t}}]k^2 \leq \]

\[ \leq [e^{C \hat{t}} \|\eta'' - 2\lambda' \eta'\|_\infty^2 \|w\|_\infty^2 \]

\[ \frac{4 \min(2\frac{\lambda''}{\lambda} - \lambda'')}{4 \max(\lambda)} \|\eta'' - 2\lambda' \eta'\|_\infty^2 \|w\|_\infty^2 - C_4 e^{-C \hat{t}}]k^2 \leq 0. \quad (3.51) \]

Set the constant

\[ C_5 = \frac{1}{2} \left( 1 + \left[ \frac{1}{2M^2} - 1 \right]_+ \right) \sup_{t \in [0, T)} \|u(t)\|^2_{W^{1, \infty}}, \quad (3.52) \]

where \([\cdot]_+\) denote the positive part of a real number; therefore by (3.51), (3.25), (3.35), the Lipschitz regularity of \(u\), and using the Young inequality with exponent 2 we can write,
\[
\Delta^2 G(v_1 e^{Ct}, v_2 e^{Ct}, 2p + m, 2q + m) = \\
\quad \Delta^2 (G - C_4 e^{-Ct}u^2) + \frac{1}{4} C_4 e^{Ct} |v_2^2 - v_1^2| \\
\leq \frac{1}{4} C_4 e^{Ct} |v_1 - v_2| \\
= \frac{1}{4} C_4 e^{Ct} |u_1 - u_2| \\
\leq \frac{1}{2} C_4 (|u_1 - u_3|^2 + |u_2 - u_3|^2) \\
\leq \frac{1}{2} C_4 \sup_{t \in [0, T)} \|u(t)\|_{W^1, \infty}^2 \left[ \frac{\delta |p|^2}{2M} + \frac{\delta |q|^2}{2M} \right] \\
\leq \frac{1}{2} C_4 \sup_{t \in [0, T)} \|u(t)\|_{W^1, \infty}^2 \left( \frac{1}{2} \left[ \frac{\delta |p|^2}{2M} + \frac{\delta |q|^2}{2M} \right] \right) \\
\leq \frac{1}{2} C_4 \sup_{t \in [0, T)} \|u(t)\|_{W^1, \infty}^2 \left( 2v_3 - v_1 - v_2 + \delta M \left[ \frac{1}{2M^2} - 1 \right] \right) \\
\leq C_4 C_5 (2v_3 - v_1 - v_2). \tag{3.53}
\]

Introducing estimates (3.40), (3.41), (3.44), (3.48), (3.53), in (3.39), we finally obtain

\[
\left[ \lambda(v_1 e^{Ct}) e^{Ct} |\sigma^T \Phi_1|^2 - \eta(v_1 e^{Ct}) \langle \sigma^T \Phi_1, w_1 \rangle + \lambda(v_2 e^{Ct}) e^{Ct} |\sigma^T \Phi_2|^2 \\
- \eta(v_2 e^{Ct}) \langle \sigma^T \Phi_2, w_2 \rangle - 2\lambda(v_3 e^{Ct}) e^{Ct} \frac{1}{2} \sigma^T \Phi_3|^2 - \eta(v_3 e^{Ct}) \langle \sigma^T \Phi_3, w_3 \rangle \right] \leq \leq (C_1 + C_2 + C_3 + C_4 C_5) (2v_3 - v_1 - v_2) + O(\sqrt{\varepsilon}). \tag{3.54}
\]

Consider now the last term in (3.31), by the regularity assumptions on the function \( f \), the same argument of (3.53) used for estimating \( |u_1 - u_2| \) and again (3.35), we have,

\[
f_1 + f_2 - 2f_3 \leq \sup_{t \in [0, t)} \|f(t)\|_{W^2, \infty} \left[ \left( \frac{\delta |p|^2}{2M} + \frac{\delta |q|^2}{2M} \right) \right] + \left( \frac{\delta |p|^2}{2M} + \frac{\delta |q|^2}{2M} \right)^2 |n|^2 \right] \\
+ |u_1 - u_3|^2 + |u_2 - u_3|^2 + |u_1 + u_2 - 2u_3| \\
\leq \sup_{t \in [0, t)} \|f(t)\|_{W^2, \infty} \left\{ 2v_3 - v_1 - v_2 + \frac{1}{2} \delta M \left[ \frac{1}{4M^2} - 1 \right] \right\} + \left. \left. 2 \sup_{t \in [0, T)} \|u(t)\|_{W^1, \infty}^2 C_5 (2v_3 - v_1 - v_2) + e^{Ct} (2v_3 - v_1 - v_2) \right\} \leq \leq C_6 (2v_3 - v_1 - v_3), \tag{3.55}
\]

where we consider

\[
C_6 = \sup_{t \in [0, t)} \|f(t)\|_{W^2, \infty} \left( 1 + \frac{1}{4M^2} - 1 \right) + 2 \sup_{t \in [0, T)} \|u(t)\|_{W^1, \infty}^2 C_5 + e^{Ct} \right) \tag{3.56}
\]

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Finally using (3.34), (3.36), (3.54), and (3.55) in (3.31), we have

\[
\frac{\gamma}{T^2} \leq [6 \sup_{t \in (0,T)} \| \mu(t) \|_W + C_1 + C_2 + C_3 + C_4 C_5 \\
+ C_6 e^{-C \tilde{t}} - C |(2v_3 - v_1 - v_3) + O(\sqrt{\varepsilon})].
\] (3.57)

From the definition (3.56), we see that \( C_6 e^{-C \tilde{t}} \) is bounded as a function of \( C > 0 \), so if we choose \( C \) sufficiently great, we obtain a contradiction letting \( \varepsilon \to 0 \). This proves the result.

Now we use these Propositions to eliminate the conditions on \( \lambda \) and to obtain the assertion of the Theorems 2.2 and 2.3.

**Proof of Theorem 2.2.** We go to build a change of variable such that the new differential equation will have the required structural properties of Proposition 3.1.

Let \( c, \Lambda \) be respectively the infimum of the closed interval \( I \) where \( u \) take its values, and the primitive of \( \lambda \) with \( \Lambda(c) = 0 \). Then consider the solution \( Q = Q(\tau) \) of the following ordinary Cauchy problem

\[
\begin{cases}
\frac{dQ}{d\tau} = \exp(4\sqrt{\tau + 1} + 2\Lambda(Q)), \\
Q(0) = c.
\end{cases}
\] (3.58)

The problem admits an increasing local solution. Moreover by the continuity of \( \lambda \), \( Q \) is also \( C^2 \). We prove that \( Q \) maps the interval \( I \). Consider the following cases

**Case 1, \((b = \infty)\).** Let \([0, \tau^*)\) be the maximal interval of existence for \( Q \), and denote with \( \overline{Q} \) the limit for \( \tau \to \infty \). If \( \tau^* = \infty \), then \( \overline{Q} = \infty \). Actually by the equation (3.58), we have,

\[
Q'(\tau) \geq \exp \left( \inf_{u \in [c, \overline{Q}]} \Lambda(u) + 4 \right), \quad \forall \tau > 0.
\] (3.59)

So integrating (3.59) from 0 to \( \tau > 0 \) and letting \( \tau \) to infinity we obtain the assertion. If \( \tau^* < \infty \), then for definition of maximal interval \( Q \) blow-ups at \( \tau^* \). Otherwise since that \( \lambda \) is defined in \((a, \infty)\), the solution \( Q \) could be extended.

**Case 2, \((b < \infty)\).** Consider again the maximal interval of existence. Then if \( \tau^* = \infty \) and \( \overline{Q} \) is strictly less than \( b \), then, again using (3.59), we obtain a contradiction. If \( \tau^* < \infty \), then and \( \overline{Q} < b \), then the solution \( Q \) can be extended because \( \lambda \) is continuous in \((\overline{Q}, b)\).

In each case the function reaches \( b \) in the limit sense; in particular \( Q \) maps \( I \). Moreover \( Q \) can be defined in an open interval \( V \subset (-\varepsilon_0, \infty) \), \( \varepsilon_0 > 0 \), and \( I \subset Q(V) \subset (a, b) \). By the increasing property of \( Q \), it admits a \( C^2(Q(V); V) \) inverse, which we denote \( P \). We use \( Q \) as a transformation for a global change
of the variable $u$. The function $\tau = P \circ u$, is a bounded, $t$-uniformly Lipschitz continuous viscosity solution of

\[
\partial_t \tau - \frac{1}{2} tr(\sigma \sigma^\top \tau) + \langle \mu, \nabla \tau \rangle - \frac{|\sigma^\top \nabla \tau|^2}{\sqrt{\tau + 1}} + \eta(Q(\tau))\langle \sigma^\top \nabla \tau, w \rangle + f(x, t, Q(\tau)) = 0, \quad (x, t) \in \mathbb{R}^N \times (0, T),
\]

where the initial datum is $\tau_0 = P \circ u_0$, and which takes values in the closed interval $P(I)$. It is easy to verify the structural hypothesis $i)$, $ii)$ of Proposition 3.1, where $\lambda$ and $\eta$ are substituted, respectively, by the functions $-(1 + \tau)^{-\frac{1}{2}}$ and $\eta \circ Q$, over the interval $V$, with regularity properties over $P(I)$. So applying Proposition 3.1, we deduce that there exist positive constants $C, K_0 > 0$, such that,

\[
\tau(x + h, t) + \tau(x - h, t) - 2\tau(x, t) \geq -e^{Ct}K_0|h|^2, \quad \forall x, h \in \mathbb{R}^N, t \in [0, t), (3.61)
\]

where the constant $K_0$ depends on $P$ and, on $L_0$ and $\text{Lip}(u_0)$. Therefore for every $x, h \in \mathbb{R}^N, t \in [0, t)$, and some $s^+, s^- \in [0, 1]$, denoting

\[
\begin{align*}
\tau^+ &= \tau(x, t) + s^+(\tau(x + h, t) - \tau(x, t)), \\
\tau^- &= \tau(x, t) + s^- (\tau(x - h, t) - \tau(x, t)), \\
\tau &= \tau(x, t),
\end{align*}
\]

we can write,

\[
\begin{align*}
u(x + h, t) + u(x - h, t) - 2u(x, t) = Q'(\tau^+)(\tau(x + h, t) - \tau(x, t)) \\
+ Q'(\tau^-)(\tau(x - h, t) - \tau(x, t)) \\
= Q'(\tau) (\tau(x + h, t) + \tau(x - h, t) - 2\tau(x, t)) + s^+ Q''(\tau^+)(\tau(x + h, t) \\
- \tau^2(x, t) + s^- Q''(\tau^-)(\tau(x - h, t) - \tau^2(x, t)) \\
\geq -[K_0C_0 + C_0]|h|^2.
\end{align*}
\]

(3.63)

for some $\tau^+ \in [\min(\tau, \tau^+), \max(\tau, \tau^+)], \tau^- \in [\min(\tau, \tau^-), \max(\tau, \tau^-)]$, where $C_0$ is a positive constant depending on $Q'$, while $C_0$ depends on the Lipschitz constant of the solution $u$ and on $Q''$. This prove the assertion of the Theorem.

\[\blacksquare\]

The equivalent result for the semiconcavity property can be then obtained with same arguments. So we limit us to give some outlines in the following proof.

**Proof of Theorem 2.3.** By the same notations used for proving Theorem 2.2, we observe that, choosing the increasing transformation $u = Q(\tau)$, where $Q$, is implicitly defined as the solution of the ordinary Cauchy problem,

\[
\left\{ \begin{array}{l}
\frac{dQ}{d\tau} = \exp(-\frac{2}{t+1}(\tau + 1)^{l+1} + 2\Lambda(Q)), \\
Q(0) = c,
\end{array} \right.
\]

(3.64)

where $l$, is chosen as bigger than 3, then we obtain, as in the semiconvexity case a new equation for $\tau$, which satisfies the structural hypothesis required for applying Proposition 3.2.
As an immediate consequence of Theorems 2.2 and 2.3, and by the definition of the space $W^{2,\infty}$, it follows the proof of the Theorem 2.4. This conclusive fact allows us to have a second-order regularity result for the solution of problem (1.1), with regular initial data.

4 Time regularity

In this section we use the spatial regularity of the solution $u$, obtained through the Theorem 2.4, to prove the time regularity of it.

The result which we show here, see Theorem 4.1, is not stated for second order Hamilton-Jacoby equations which have the structure of (1.1), since the lack of regularity in the spatial variable. So we present it as a possible interesting extension of previous works, in the framework of the viscosity theory.

In order to simplify the notations, in the following, for every function $g$ defined in $\mathbb{R}^N \times [0, T) \times (a, b)$, and $T > h > 0$, we set $g_h(x, t, u) = g(x, t+h, u)$, where $0 \leq t < T - h$, and $x \in \mathbb{R}^N$, $u \in (a, b)$. Moreover, if $g$ is a Lipschitz continuous function over $[0, T)$, uniformly with respect to the other variables, then we shall denote $L(g)$ the constant defined as:

$$L(g) = \sup_{t, s \in [0, T), \ t \neq s} \frac{|g(x, t, u) - g(x, s, u)|}{|t - s|},$$

for $(x, u) \in \mathbb{R}^N \times (a, b)$.

**Theorem 4.1.** Let $u(t) \in W^{2,\infty}(\mathbb{R}^N)$, for every $t \in [0, T)$. Assume that $\sigma, \mu, w, f$ are Lipschitz continuous functions of the time, uniformly with respect to the other variables, and $\mu, w, f$ are bounded, then $u(x, \cdot)$ is a Lipschitz continuous function uniformly in $\mathbb{R}^N$.

This result is based on the next Lemma, which shows how the solution $u$ tends to the initial datum, when we send the time at zero.

**Lemma 4.2.** If $u_0 \in W^{2,\infty}$, and $\mu, w, f$ are bounded, then there exists a constant $C_0 > 0$, such that

$$|u(x, t) - u_0(x)| \leq C_0 t,$$

holds for every $(x, t) \in \mathbb{R}^N \times [0, T)$.

**Proof.** We limit us to prove the inequality,

$$u - u_0 \geq -C_0 t,$$

choosing,

$$C_0 = \sup \{H(x, t, u, p, X) : (x, t, u) \in \mathbb{R}^N \times [0, T) \times I, |p| \leq \|\nabla u_0\|_{\infty}, \|X\| \leq \|\nabla^2 u_0\|_{\infty}\}.$$  

(4.67)

The other follows with same arguments. Consider the function,

$$\Phi_{\gamma, \varepsilon}(x, t) = u(x, t) - u_0^\varepsilon(x) + C_0 t + \varepsilon |x|^2 + \frac{\gamma}{t - T},$$

(4.68)
For every $\gamma, \varepsilon > 0$, $(x, t) \in \mathbb{R}^N \times [0, T)$. Where $u_0^\gamma$ is the convolution between the standard mollifier and $u_0$. The inequality (4.66) is satisfied if for every $\gamma > 0$, there is $\varepsilon_0 > 0$, such that, for every $0 < \varepsilon < \varepsilon_0$,

$$\inf_{\mathbb{R}^N \times (0, T)} \Phi_{\gamma, \varepsilon} \geq 0. \quad (4.69)$$

Actually if (4.69) holds, setting a point $(y, s) \in \mathbb{R}^N \times (0, T)$, we have

$$u(y, s) - u_0^\gamma(y) + C_0s + \varepsilon|y|^2 + \frac{\gamma}{T-s} \geq 0.$$

So we send $\gamma, \varepsilon$ to zero and obtain (4.66).

Suppose that (4.69) would be false. Hence by the same observation used for proving the Proposition 3.1, there is $\gamma_0 > 0$, and a sequence $\varepsilon_j \to 0$, as $j \to \infty$, such that $\Phi_{\gamma, \varepsilon}$ has a global minimum point, $(\hat{x}, \hat{t})$, and $\Phi_{\gamma, \varepsilon}(\hat{x}, \hat{t}) < 0$, for $\gamma = \gamma_0$, $\varepsilon = \varepsilon_j$. If $\hat{t} = 0$, then

$$0 > \Phi_{\gamma, \varepsilon}(\hat{x}, 0) \geq -\varepsilon\|u_0\|_{W^{1, \infty}} + \frac{\gamma}{T}.$$

Hence for large $j$, $\hat{t} > 0$, and the minimum point is an interior point; moreover, $u_0^\gamma - C_0t - \varepsilon|x|^2 - \frac{\lambda}{T-t}$ is a test function for $\mathcal{P}^2 - u$ at $(\hat{x}, \hat{t})$. By (4.68), we deduce that

$$\varepsilon|x| = \sqrt{\varepsilon^2|x|^2} = O(\varepsilon^{\frac{1}{2}}). \quad (4.70)$$

Moreover by the regularity assumption on $u_0$, we have $\|\nabla u_0^\gamma\|_\infty \leq \|\nabla u_0\|_\infty$ and $\|\nabla^2 u_0^\gamma\|_\infty \leq \|\nabla^2 u_0\|_\infty$, where $\nabla u_0$, $\nabla^2 u_0$, denote the weakly derivatives of $u_0$.

Introducing the derivatives of the test function in the equation (1.1) and using the assumption of boundness of the coefficients, we can write

$$C_0 + \frac{\gamma}{T^2} \leq H(\hat{x}, \hat{t}, u(\hat{x}, \hat{t}), \nabla u_0^\gamma(\hat{x}, \hat{t}), \nabla^2 u_0^\gamma(\hat{x}, \hat{t})) + O(\varepsilon^{\frac{1}{2}}). \quad (4.71)$$

By (4.67), letting $j \to \infty$ in (4.71), we obtain the contradiction.

---

**Proof of Theorem 4.1.** Consider $u_h$, for $T > h > 0$, $u_h(t) \in \mathcal{W}^{2, \infty}(\mathbb{R}^N)$ uniformly in time $t < T - h$, and is a viscosity solution of the problem

$$\partial_t u_h - \frac{1}{2} \text{tr}(\sigma^T(t + h)\nabla^2 u_h) + \langle \mu_h, \nabla u_h \rangle + \lambda(u_h)|\sigma^T(t + h)\nabla u_h|^2$$

$$+ \eta(u_h)|\sigma^T(t + h)\nabla u_h, \omega_h| + f_h(x, t, u_h) = 0, \quad (4.72)$$

in $\mathbb{R}^N \times (0, T - h)$. For every $(x, t) \in \mathbb{R}^N \times [0, T)$, define

$$u^h(x, t) = u(x, t) + (C_0 + \alpha(t))h,$$

$$\alpha(t) = \frac{e^{B_1t} - 1}{B_1}(C_0B_1 + B_2). \quad (4.73)$$
Then $u, p, X$ have, for every element $(s, p, X) \in \mathcal{R}$ on the domain we have
\[|u| \leq \|u\|_{L^1}\|u\|_{w^1}\|u\|_{w^1} + L(f).\]

(4.74)

\[B_2 = \left(\frac{1}{2}N^2\|u(t)\|_{w^2} + \|\lambda\|_\infty \sup_{t \in [0, T]} \|u(t)\|^2_{w^1}\right) + \sup_{t \in [0, T]} \|u(t)\|_{w^1}(L(\sigma^T)|u|_{w^1} + L(\sigma^T)|u|_{w^1} + L(\mu)).\]

Then $u^h$ is a viscosity solution of the following problem
\[
\partial_t u^h - \alpha' h - \frac{1}{2}tr(\sigma^T \nabla^2 u^h) + (\mu, \nabla u^h) + \lambda(u^h - C_0h - \alpha h)\sigma^T \nabla u^h|^2 + \eta(u^h - C_0h - \alpha h)(\sigma^T \nabla u^h, w) + \mu(x, t, u^h - C_0h - \alpha h) = 0. \quad (4.75)
\]

Recall the Hamiltonian $H$ given by (1.2), then
we observe that by the regularity assumption on $u(t)$ for every time $t$, we have, for every element $(s, p, X) \in \mathcal{P}_{s, p, X}^2(x_0, t_0)$, for $(x_0, t_0) \in \mathbb{R}^N \times (0, T)$, $|p| \leq \|u(t_0)\|_{w^1}$, while $|X| \leq C\|u(t_0)\|_{w^2}$. The same kind of observation holds for $u^h$. Moreover using the last observations, and introducing the constants (4.74), we can estimate (4.72) and (4.75), obtaining
\[
\partial_t u^h + H(x, t, u^h, \nabla u^h, \nabla^2 u^h) - B_2 h \leq 0,
\]
\[
\partial_t u^h + H(x, t, u^h, \nabla u^h, \nabla^2 u^h) + B_1(\alpha + C_0)h - \alpha' h \geq 0, \quad (4.76)
\]
on the domain $\mathbb{R}^N \times (0, T - h)$, in a viscosity sense. By the definition (4.73), we have
\[
0 \leq \partial_t u^h + H(x, t, u^h, \nabla u^h, \nabla^2 u^h) + B_1(\alpha + C_0)h - \alpha' h
\]
\[
= \partial_t u^h + H(x, t, u^h, \nabla u^h, \nabla^2 u^h) - B_2 h. \quad (4.77)
\]

Hence $u_h$ and $u^h$ are subsolution and supersolution of the same problem, respectively, and by Lemma 4.2, $u_h(x, 0) = u(x, h) \leq u_0 + C_0 h = u^h(x, 0)$, for every $x \in \mathbb{R}^N$. Using the comparison principle, which holds applying the comparison principle which we have stated in [32], we deduce that $u(x, t+h) - u(x, t) \leq C_0 h + \alpha(t)h$, for every $(x, t) \in \mathbb{R}^N \times [0, T)$. Using the same argument, with the same function $\alpha$, and using again Lemma 4.2, we have $u(x, t+h) \geq u(x, t) - C_0 - \alpha(t)h$, for each point $(x, t) \in \mathbb{R}^N \times [0, T - h)$. These two inequalities imply that, for every $x \in \mathbb{R}^N$, $t_1, t_2 \in (0, T)$,
\[
|u(x, s_2) - u(x, s_1)| \leq C_T|s_2 - s_1|. \quad (4.78)
\]

Where the constant $C_T$ depends on the final time $T$, and $B_1, B_2 C_0$. 

By Theorem 2.4 and Theorem 4.1, then it easily follows Theorem 2.5.
5 Regularity and the Ito’s formula

In this section, we derive some consequences from the results of the previous sections. Let us recall first from [6] a useful result about Sobolev spaces $W^{2,1}_p(\mathbb{R}^N \times (0,T))$, i.e.: the space of the functions $u$, such that $u$, $\nabla u$, $\nabla^2 u$, $\partial_t u$ $\in L^p_{loc}(\mathbb{R}^N \times (0,T))$.

We start by an observation about the Lebesgue points.

**Remark 5.1.** If $g \in L^p_{loc}(\mathbb{R}^{N+1})$, for every $p \in [1, \infty)$, then for a.e. $(\tilde{x}, \tilde{t})$,

$$\lim_{r \to 0^+} \frac{1}{|\Omega_r(\tilde{x}, \tilde{t})|} \int_{\Omega_r(\tilde{x}, \tilde{t})} |f - f(\tilde{x}, \tilde{t})|^p \, dx \, dt = 0,$$

where $\Omega_r(x, t) = B_r(x) \times (t-r^2, t)$, and $B_r(x)$ denotes the open ball in $\mathbb{R}^N$ with ray $r$, and centre at $x$. This follows from a version of Vitali’s covering Theorem by replacing the balls with $\Omega_r$, see e.g. Remark I.3.1 in [16].

**Proposition 5.2.** Let $p > \frac{N+2}{2}$, $N \geq 2$, and $u \in W^{2,1}_p(\mathbb{R}^N \times (0,T))$. Let $(\tilde{x}, \tilde{t})$ be a Lebesgue point (in the sense of (5.1)) of $u$, $\nabla u$ (in $L^1$), and $\nabla^2 u$, $\partial_t u$ (in $L^p$). Then as $(x,t) \to (\tilde{x}, \tilde{t})$, we have

$$|u(x, t) - (u(\tilde{x}, \tilde{t}) + \partial_t u(\tilde{x}, \tilde{t})(t-\tilde{t}) + (\nabla u(\tilde{x}, \tilde{t}), x - \tilde{x})) + \frac{1}{2}(\nabla^2 u(\tilde{x}, \tilde{t}))(x - \tilde{x}), (x - \tilde{x}))| \leq o(|x - \tilde{x}|^2 + |t - \tilde{t}|). \quad (5.79)$$

For the proof we refer the reader to [6]. If the assumptions of Theorem 2.5 are satisfied, then we can apply Proposition 5.2 to $u$, obtaining that for a.e. $(x,t) \in \mathbb{R}^N \times (0,T)$, we have

$$\partial_t u(x, t) + H(x, t, u(x, t), \nabla u(x, t), \nabla^2 u(x, t)) = 0, \quad (5.80)$$

where $\partial_t u$, $\nabla u$, $\nabla^2 u$, represent the weakly derivatives of $u$. This regularity of the solution $u$ has a useful application when we use the Ito’s rule along a stochastic process.

As we have noted in the Introduction, the representation of financial derivatives can be expressed as a conditional expectation over the probability space of the underlying factors that determine the instrument’s price.

In the equality (1.4), we introduce a relation between this conditional expected value and the solution of (1.3). This is a consequence of the regularity of $u = U + h + \xi$, which follows by applying the results of the previous sections to (1.3). In the model proposed by X. Gabaix in [13], the measure $Q$ depends in a nonlinear way by the price of the Mortgage-Backed Security and its volatility; this feature produces the quadratic nonlinear term in the equation (1.3). To rigorously derive the equation (1.4), we need to apply the Ito’s formula to $u(X_t, t)$, where $X_t$ is the underlying stochastic process. When $u$ is less regular than $C^{2,1}$, a first result was established by Krylov [25]. In a more recent work, Haussmann [17] describes a result in this direction, which shows that the Ito’s formula holds for $u \in W^{2,1,\infty}(\mathbb{R}^N \times (0,T))$, provided that it is interpreted appropriately, using the generalized Hessian.

We use the regularity of the solution $U$, which gives the property (5.80), combining it with the result of Haussmann to obtain the equality (1.4).
In the next we shall assume \( \mu \in \mathbb{W}^{2,1,\infty}(\mathbb{R}^N \times [0, T]) \) and for sake of simplicity we limit us to treat in detail the case of a constant matrix \( \sigma \) and consider the case \( N \geq d \).

As we have already pointed out in the Introduction, equation (1.3) is equivalent to the general problem (1.1). Actually, after the change \( u = U + h + \xi \), \( u \) solves a problem which has the same structure of (1.1), where in particular \( w = \sigma^\top \nabla h \). Hence, following the comparison principle and the Lipschitz regularity for viscosity solutions of (1.3) proved in [32], and assuming \( h \) to be a smooth function, as in [32], by Theorem 2.5 is straightforward to assume the existence of a unique viscosity solution \( U \in \mathbb{W}^{2,1,\infty}(\mathbb{R}^N \times [0, T]) \) of the equation (1.3) with \( U(x, 0) \equiv 0 \), such that \( U + h + \xi > 0 \).

**Theorem 5.3.** Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \((W_t, \mathcal{F}_t)_{t\in[0,T]}\) be a d-dimensional continuous standard Brownian motion over \( \Omega \) and let \( X_t \) be the Ito process defined as the solution of the stochastic differential equation (1.5), for some initial datum \( X_0 \in \mathbb{R}^N \). Define

\[
\gamma_s = \frac{\sigma^\top (T - s) \nabla U(X_s, T - s)}{U(X_s, T - s) + h(X_s, T - s) + \xi(T - s)}, \quad \text{a.s.}
\]

\[
\hat{W}_s = W_s + \int_0^s \gamma(\kappa) d\kappa,
\]

\[
\frac{dQ}{dP} = e^{-\int_0^T \gamma_s d\kappa - \frac{1}{2} \int_0^T |\gamma_s|^2 ds},
\]

(5.81)

with \( 0 < \rho < 1 \) a parameter. If \( N = d \) we do not need any assumption. Otherwise for the case \( N > d \), assume that the projection-process \( \pi_t \), defined as the projection of \( X_t \) over the Kernel of \( \sigma^\top \), satisfies

\[
A \rightarrow P (\pi_s \in A | X_t), \quad A \subset \mathbb{R}^m \text{ is } \mathcal{L}^m\text{-absolutely continuous}, \quad (5.82)
\]

for every \( s > t \), where \( \mathcal{L}^m \) denotes the Lebesgue measure over \( \mathbb{R}^m \) and \( m = N - \text{rank}(\sigma^\top) \). Then

\[
U(X_t, T - t) = \mathbb{E}_t^Q \left[ \int_t^T (\tau - r(T - s)) e^{-\int_r^\tau (T - s)ds} h(X_s, T - s) ds \right], \quad (5.83)
\]

holds a.s. for every \( 0 < t \leq T \).

We remark that a classical result, the Girsanov’s Theorem, state that \( \hat{W}_s \) is a Brownian motion with the same filtration of \( W_s \), see for instance [1] or [23]. To prove the relation (5.83) for the solution \( U \) to the model (1.3), we shall use the results of [17]. Actually the assertion of Haussmann interprets the Ito’s rule through some processes which substitute the usual derivatives of the function evaluated at the process \((X_t, t)\), which are equal if the underlying process belongs to some set of full Lebesgue measure. In order to state our formula we need to neglect the term which corresponds to an integration over the paths of \( X \) which fall in a set of null measure. Therefore the assumption (5.82) can be explained by that purpose. Actually the degeneration of the volatility matrix plays a crucial role: in general along the components of the vector \( X \) which correspond to the directions of the kernel of \( \sigma^\top \), we have almost-deterministic trajectories which surely form a set of zero measure; this feature, in general, could not allow
to justify the previous assertion (5.83) about $U$, using the Haussmann’s Theorem, unless the drift coefficient has some compatibility condition with $\sigma$; in fact this feature is expressed through the condition on the projection-process $\pi_t$, see the next remark 5.6.

As we have pointed out in the Introduction, there are some works such as [5], where the existence of densities for the solutions of stochastic differential equations under the Hörmander’s condition is proved. In particular Theorem 3.1 of chapter 3 by [5] gives a sufficient condition, which involves the use of the Lie bracket between the set of vector fields represented by the drift coefficient and the columns of the volatility matrix, to prove the existence of an absolutely continuous distribution with respect to the Lebesgue measure. Although that condition can be used in practice to investigate differential equations with a complicated form, it does not allow to consider a time dependence of the coefficients, and it requires also more regularity for computing the Lie brackets of the vector fields. Moreover, while for financial purposes the coefficients $\mu$ and $\sigma$ can be taken as constants, the probabilistic Hörmander’s condition used by D.R. Bell in [5] does not hold in this case. Therefore we propose an alternative assumption, contained in (5.82), which covers the cases of interest for our applications. Actually let us notice that case $N = d$, the assumption (5.82) can be removed via an approximation of the underlying process $X$. The proof of the following technical Lemma motivates of condition (5.82).

**Lemma 5.4.** Let $Q$ be a probability measure equivalent to $P$ over $\Omega$ and let $B$ be a set of zero Lebesgue measure over $\mathbb{R}^N \times [t, T]$, for $0 \leq t < T$. Set

$$Z_{t,T} = \int_t^T p_s 1_B(X_s, s)ds + q_s 1_B(X_s, s)dW_s,$$

(5.84)

where $p_s, q_s$ are $(\mathcal{F}_s)_{t \leq s \leq T}$-adapted bounded processes. Then, under the assumption (5.82), it holds

$$E_t^Q[|Z_{t,T}|] = 0.$$

**Remark 5.5.** Let us recall the following important fact. For every integrable random variable $\mathcal{F}_T$-measurable there holds

$$E_t^Q[X] = \frac{E_t \left[ \frac{dQ}{dP} X \right]}{E_t \left[ \frac{dQ}{dP} \right]},$$

where $E_t[\cdot]$ denote the conditional mean with respect to the measure $P$. To see this relation between the conditional means taken through two equivalent probability measure, we refer the reader to [30].

**Proof of Lemma 5.4.** By remark 5.5, the boundness of the processes $p_s, q_s$ and

$$E_t \left[ \int_t^T q_s 1_B(X_s, s)dW_s \right] \leq E \left[ \left( \int_t^T q_s 1_B(X_s, s)dW_s \right)^2 \right]^{\frac{1}{2}}$$

$$= E \left[ \int_t^T |q_s|^2 1_B(X_s, s)ds \right]^{\frac{1}{2}},$$

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where in the last passage we have used the classical Ito isometry, we are reduced
to prove
\[
\mathbb{E}_t \left[ \int_t^T 1_B(X_s, s) \, ds \right] = 0.
\] (5.55)

**Case** \( N > d \). Let \( \{b_1, \ldots, b_m\} \) be an orthonormal basis of the kernel of \( \sigma^\top \),
then consider an \( N \times N \) invertible matrix \( M \), such that
\[
M^\top e_i = b_i, \quad i = 1, \ldots, m,
\]
and define the process \( Y_s = MX_s \). Therefore
\[
Y_s = (\pi_s^M, G_s) := \left( (M^\top)^{-1}_{\ker(\sigma^\top)} \pi_s, G_s \right),
\]
and
\[
\begin{align*}
\quad d\pi_s^M &= \mu_{\pi}(Y_s, s) \, ds, \quad \pi_s^M \in \mathbb{R}^m, \\
\quad dG_s &= \mu_G(Y_s, s) \, ds + \sigma_G \cdot dW_s, \quad G_s \in \mathbb{R}^{N-m}.
\end{align*}
\] (5.86)

Moreover
\[
\mu_{\pi}(y, s) = (\langle \mu(M^{-1}y, T-s), b_1 \rangle, \ldots, \langle \mu(M^{-1}y, s), b_m \rangle),
\] (5.87)
and \( \sigma_G \in \mathcal{M}_{N-m,d}(\mathbb{R}) \) is a matrix of rank \( N-m \). Let
\[
\tilde{M} = \begin{pmatrix}
0 & \cdots & 0 \\
M & \cdots & 0 \\
0 & \cdots & 1
\end{pmatrix}, \quad \tilde{B} = \tilde{M}B.
\]

Noting that \( \tilde{B} \) has zero Lebesgue measure, (5.55) is equivalent to
\[
\mathbb{E}_t \left[ \int_t^T 1_{\tilde{B}}(Y_s, s) \, ds \right] = 0.
\] (5.88)

Fix \( \varepsilon > 0 \) and \( t^+ > t/(1 - \varepsilon) \), then consider the approximating process \( Y_s^\varepsilon = (\pi_s^{M, \varepsilon}, G_s^\varepsilon) \), where
\[
\begin{align*}
\quad \pi_s^{M, \varepsilon} &= \pi_s^M + \int_t^s(1-\varepsilon) \mu_{\pi}(Y_\lambda, \lambda) \, d\lambda \\
\quad G_s^\varepsilon &= G_t + \int_t^s(1-\varepsilon) \mu_G(Y_\lambda, \lambda) \, d\lambda + \sigma_G \cdot (W_s - W_t).
\end{align*}
\] (5.89)

It easy to see that this process converges in \( L^1 \) to \( Y_s \), uniformly in time, as \( \varepsilon \to 0 \). The \( F_{s(1-\varepsilon)} \)-conditional density of the random variable \( G_s^\varepsilon \) is normal
with mean
\[
E_G := G_t + \int_t^s(1-\varepsilon) \mu_G(Y_\lambda, \lambda) \, d\lambda
\]
and covariance matrix

\[ T_G(s, t) := (s - t)\sigma_T \sigma_T^\top \geq 0. \]

Consider \( \delta > 0 \), then there exists a countable collection of sets \( Q_j \subset \mathbb{R}^N \times [t^+, T] \) which, without loss of generality, can be supposed to be closed \( N+1 \)-cubes, such that the projection of \( Q_j \) over the coordinates \( x_{m+1}, \ldots, x_N \) lies in a bounded set of \( A_2 \) (this is not a restriction because we could always take the intersection of \( \hat{B} \) with an increasing sequence of sets whose projection over these coordinates is bounded), the intersection of a couple of cubes is a set of zero Lebesgue measure and

\[ \hat{B} \cap (\mathbb{R}^N \times [t^+, T]) \subset \bigcup_j Q_j \]

and

\[ \sum_j \mathcal{L}^{N+1}(Q_j) < \delta. \]

We proceed at first with an estimate of the expected value (5.88) where \( \hat{B} \) is substituted by \( Q_j \) and \( Y_s^\varepsilon \) by \( Y_s^\varepsilon \). By the distributional property of the approximating process we have

\[
\int_{t^+}^T \mathbb{E}_t \left[ 1_{Q_j}(Y_s^\varepsilon, s) \right] ds = \int_{t^+}^T \mathbb{E}_t \left[ \mathbb{E}_{s(1-\varepsilon)} \left[ 1_{Q_j}(Y_s^\varepsilon, s) \right] \right] ds = \int_{t^+}^T \mathbb{E}_t \left[ P \left( (Y_s^\varepsilon, s) \in Q_j | F_{s(1-\varepsilon)} \right) \right] ds.
\]

We shall use a diagonalization procedure to separate the variables in the following integration. Hence consider the orthogonal matrix \( H \), such that \( H^T T_G H = \text{Diag}(\lambda_1, \ldots, \lambda_{N-m}) \), where \( \lambda_1, \ldots, \lambda_{N-m} \) are the eigenvalues of \( T_G \).

If \( Q_i(a), B_i(a) \) respectively denote the cube of centre \( a \) and side of length \( t > 0 \) and the open ball of radius \( t \) and centre \( a \) in \( \mathbb{R}^{N-m} \), then again by the properties of \( G_s^\varepsilon \), we have

\[
P \left( G_s^\varepsilon \in Q_i(a) | F_{s(1-\varepsilon)} \right) \leq \int_{B_i(\sqrt{N-m})} \frac{1}{\sqrt{2\pi N-m}} \exp \left( -\frac{1}{2} T_G^{-1}(s, t)(x - E_G)^2 \right) dx \]

\[
= \int_{B_i(\sqrt{N-m})} \frac{1}{\sqrt{2\pi (s-t)^{N-m}}} \sqrt{\lambda_1 \cdots \lambda_{N-m}} \exp \left( -\frac{1}{2(s-t)} \sum_i \lambda_i^{-1} y_i^2 \right) dy \]

\[
\leq \int_{Q_i(\sqrt{N-m})} \frac{1}{\sqrt{2\pi (s-t)^{N-m}}} \lambda_1 \cdots \lambda_{N-m} \exp \left( -\frac{1}{2(s-t)} \sum_i \lambda_i^{-1} y_i^2 \right) dy \]

\[
= \Pi_{i=1}^{N-m} \left[ \Phi \left( \frac{t\sqrt{N-m} + \overline{a}_i}{\sqrt{\lambda_i(s-t)}} \right) - \Phi \left( -\frac{t\sqrt{N-m} + \overline{a}_i}{\sqrt{\lambda_i(s-t)}} \right) \right] \]

\[
\leq \frac{(2t\sqrt{N-m})^{N-m}}{\sqrt{2\pi (t^+ - t)^N \text{det}(\sigma_T \sigma_T^\top)}} = C(N, m, t, t^+, \sigma_T) \mathcal{L}^{N-m}(Q_i(a)), \quad (5.91)
\]
where $\overline{\pi} = H^T(a - E_G)$, and $\Phi$ denotes the standard Gaussian distribution.

Writing $Q_j = Q_j^1 \times Q_j^2 \times I_j$, for some cubes $Q_j^1 \subset \mathbb{R}^m$, $Q_j^2 \subset A_2$, and $I_j \subset [t^+, T]$, (5.90) can be continued using the estimate (5.91) in the following way

$$
= \int_{I_j} \mathbb{E}_t \left[ 1_{Q_j^1} \left( \pi^s_{M,\tau} \right) P \left( G^s_{\tau} \in Q_j^2 | F_{s(1-\varepsilon)} \right) \right] ds \leq C(N, m, t, t^+, \sigma_G) \mathcal{L}^{N-m}(Q_j^2) \int_{I_j} \mathbb{E}_t \left[ 1_{Q_j^1} \left( \pi^s_{M,\tau} \right) \right] ds. \quad (5.92)
$$

By the condition (5.82), there exists the density $d_\pi(x; \tau, \omega, t) \geq 0$ $P$-a.s. of the measure $P(\tau_{-1}(\cdot) | X_t)$ with respect to $\mathcal{L}^m$ which, by the regularity of $X_t$, is a continuous function of $\tau \in [t^+, T]$. By this property, (5.87) and denoting $\tilde{M}$ the restriction of $M^T$ over the kernel of $\sigma^T$, we obtain

$$
\mathbb{E}_t \left[ 1_{Q_j^1} \left( \pi^s_{M,\tau} \right) \right] = P \left( \pi^s_{M(1-\varepsilon)} \in Q_j^1 | F_t \right) = P \left( \pi_{s(1-\varepsilon)} \in \tilde{M}Q_j^1 | F_t \right) = \int_{\tilde{M}Q_j^1} d_\pi(x; s(1-\varepsilon), \omega, t) dx. \quad (5.93)
$$

Introducing (5.93) in (5.92), we have

$$
\int_{t^+}^T \mathbb{E}_t \left[ 1_{Q_j^1}(Y^s_{\varepsilon}, s) \right] ds \leq C(N, m, t, t^+, \sigma_G) \mathcal{L}^{N-m}(Q_j^2) \int_{I_j} \int_{\tilde{M}Q_j^1} d_\pi(x; s(1-\varepsilon), X_t, t) dx ds
$$

$$
= C(N, m, t, t^+, \sigma_G) \int_{\tilde{M}Q_j^1 \times Q_j^2 \times I_j} 1_{A_2}(y) d_\pi(x; s(1-\varepsilon), \omega, t) dx dy ds, \quad (5.94)
$$

By the properties of the collection $\{Q_j\}_j$, we can write

$$
\mathcal{L}^{N+1} \left( \cup_j \tilde{M}Q_j^1 \times Q_j^2 \times I_j \right) = |det \tilde{M}| \mathcal{L}^{N+1}(\cup_j Q_j) < |det \tilde{M}| \delta. \quad (5.95)
$$

For $P$-a.s., the measure $\nu(\cdot) = \nu(\cdot; \omega, t)$, defined through the density function $(x, y, s) \mapsto 1_{A_2}(y) d_\pi(x; s(1-\varepsilon), \omega)$ over $\mathbb{R}^N \times [t^+, T]$, is an absolutely continuous measure with respect to $N + 1$-dimensional Lebesgue measure. Therefore set $\beta > 0$ and choose $\delta$ so that for every set $A$ with $\mathcal{L}^{N+1}(A) < \delta |det \tilde{M}|$, it holds $\nu(A) \leq \beta$. Hence using (5.95), we deduce

$$
\mathbb{E}_t \left[ \int_{t^+}^T 1_{\tilde{B}}(Y^s_{\varepsilon}, s) ds \right] \leq \nu \left( \cup_j \tilde{M}Q_j^1 \times Q_j^2 \times I_j \right) \leq \beta. \quad (5.96)
$$

Then the assertion in the case $N > d$ follows by the arbitrary choice of $\beta$ and letting $\varepsilon \to 0$ and then $t^+ \to t$ in (5.96).
Case $N = d$. Introducing the approximation

$$X^\varepsilon_t = X_t + \int_t^{s(1-\varepsilon)} \mu(X_{\lambda}, T - \lambda) d\lambda + \sigma^\varepsilon(W_s - W_t),$$

(5.97)

where $\sigma^\varepsilon \to \sigma$, as $\varepsilon \to 0$ and $\text{det}(\sigma^\varepsilon) \neq 0$, then we can repeat the same arguments used for the previous statement substituting $X^\varepsilon_s$ at the process $Y^\varepsilon_s$ of the previous case. Therefore we obtain

$$\mathbb{E}_t \left[ \int_{t^+}^T 1_B(X^\varepsilon_s, s) ds \right] \leq C(\sigma^\varepsilon)\delta,$$

(5.98)

for arbitrary $\delta > 0$. Hence the limits with respect to $\varepsilon$ and $t^+$ is taken.

Remark 5.6. The condition (5.82) in some sense in optimally. Actually in the case $N > d$, the strong degeneration of the quadratic form $\sigma \sigma^\top$ could make false the property (5.84). For instance if $N = 2$ and $d = 1$, choosing $\sigma^\top = (0, 1)$, $\mu = 0$, $t = 0$, $B = \{0\} \times (-\infty, 0) \times [0, T]$, then with $X_0 = 0,$

$$X_t = (0, W_t)$$

and

$$\mathbb{E} \left[ \int_0^T 1_B(X_s, s) ds \right] = \frac{T}{2} > 0$$

which contradicts the assertion of Lemma 5.4.

Proof of Theorem 5.3. Let $\Sigma(t) = \sigma(T - t)$, $\mu^\circ(x, t) = \mu(x, T - t)$, $h^\circ = h(x, T - t)$ and $\xi^\circ(t) = \xi(T - t)$, $r^\circ(t) = r(t)$ for every $(x, t) \in \mathbb{R}^N \times (0, T]$. Then define the function $U^\circ(x, t) = U(x, T - t)$. $U^\circ$ is a viscosity solution of the equation

$$\partial_t U^\circ + \frac{1}{2} tr(\Sigma^\top \nabla^2 U^\circ) + \langle \mu^\circ, \nabla U^\circ \rangle - \rho U^\circ + h^\circ + \xi^\circ(t) =$$

$$= r^\circ(U^\circ + h^\circ) - \tau h^\circ \quad \text{and} \quad U^\circ(x, T) \equiv 0.$$

Moreover the equality (5.99) holds for a.e. $(x, t) \in \mathbb{R}^N \times (0, T)$ with the weakly derivatives of $U^\circ$. Applying Theorem 3.1 pg. 733, in [17], to the function $(x, s) \mapsto e^{-\int_t^s r^{\circ}(\kappa) d\kappa} U^\circ(x, s)$ relatively to the time interval $[t, T)$ there exist processes $\beta_0, \beta, \alpha$, such that

$$0 = e^{-\int_t^T r^{\circ}(\kappa) d\kappa} U^\circ(X_T, T) = U^\circ(X_t, t) + \int_t^T [\beta_0(\omega, s) + \langle \mu^\circ(X_s, s), \beta(\omega, s) \rangle +$$

$$+ \frac{1}{2} \langle \Sigma^\top(s) \alpha(\omega, s) \rangle] ds + \int_t^T \beta^\top(\omega, s) \Sigma(s) \cdot dW_s, \quad \text{a.s.} \quad (5.100)$$
Moreover there exists a set $A \subset \mathbb{R}^N \times (t, T]$ of full Lebesgue measure such that the usual derivatives of $U^\circ$ exist in $A$ and

$$
\begin{align*}
\beta_0(\omega, s) &= e^{-\int_t^s r^\circ(\kappa) d\kappa} \langle \partial_t U^\circ(X_s(\omega), s) - r^\circ(s) U^\circ(X_s(\omega), s) \rangle, \\
\beta(\omega, s) &= e^{-\int_t^s r^\circ(\kappa) d\kappa} \nabla U^\circ(X_s(\omega), s), \\
\alpha(\omega, s) &= e^{-\int_t^s r^\circ(\kappa) d\kappa} \nabla^2 U^\circ(X_s(\omega), s),
\end{align*}
$$

whenever $(X_s(\omega), s) \in A$. Without loss of generality we can also assume that the equation (5.99) holds over $A$. The application of (5.99), (5.100) and (5.101) yields

$$0 = U^\circ(X_t, t) + \int_t^T \left[ \beta_0(s) + \langle \mu^\circ(X_s, s), \beta(s) \rangle + \frac{1}{2} tr(\Sigma \Sigma^T(s)) \alpha(s) \\
- \beta^T(s) \Sigma(s) \gamma(s) \right] 1_A(X_s, s) ds + \int_t^T \beta^T(\omega, s) \Sigma(s) \cdot d\hat{W}_s + Z_{t,T}
$$

$$= U^\circ(X_t, t) + \int_t^T e^{-\int_t^s r^\circ(\kappa) d\kappa} \left[ r^\circ(s) (U^\circ(X_s, s) + h^\circ(X_s, s)) - \tau h^\circ(X_s, s) \\
- r^\circ(s) U^\circ(X_s, s) \right] ds + \int_t^T \beta^T(\omega, s) \Sigma(s) 1_A(X_s, s) \cdot d\hat{W}_s + Z_{t,T}, \quad \text{a.s. (5.102)}
$$

By the boundness of the processes $\beta_0, \beta, \alpha$, we recognize that the remaining term $Z_{t,T}$ has the same structure of (5.84) in Lemma 5.4, therefore, by the definition of $Q$, (5.81), as a measure equivalent to $P$ and Lemma 5.4, $Z_{t,T}$ has a null conditional $Q$-expected value.

Using this remark and taking the conditional expected value in both the left and right side of the equation (5.102), we obtain

$$U^\circ(X_t, t) = \mathbb{E}^Q_t \left[ \int_t^T e^{-\int_t^s r^\circ(\kappa) d\kappa} (\tau - r^\circ(s)) h^\circ(X_s, s) 1_A(X_s, s) ds \right], \text{a.s. (5.103)}
$$

By the full measure of $A$ and noting that the functions $h$ and $r$ are defined everywhere and are in particular continuous, the assertion is proved.

Finally we observe that the same conclusions can be obtained for a process $X_t$ such that satisfies

$$dX_t = \mu(X_t, T - t) X_t dt + \sigma(T - t) X_t dW_t, \quad X_t(0) > 0, \ 1 = 1, \ldots, N.
$$

In this case can is possible to treat the problem using the same previous arguments simply making the change $y = \log(x)$ in the equation (1.3).

Also if the drift coefficient depends in a linear way by $x$, and the diffusion coefficient is constant with respect to $x$, then we could repeat the same argument used before to state the regularity of $u$. Actually, in that case, we have a linear increasing with respect to the variable $x$, only in the first order linear term of the equation.
References

[1] P. Baldi, Equazioni differenziali stocastiche ed applicazioni, Quaderni Unione Matematica Italiana 28, Bologna, U.M.I., (1984).

[2] G. Barles, B. Perthame, Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbation of degenerated elliptic equations, Appl. Math. Optim. 21 (1990), 21-44.

[3] G. Barles, “Solutions de viscosité des équations de Hamilton-Jacobi”, Mathématiques et Applications, Springer, Paris, 1994.

[4] G. Barles, J. Busca, Existence and Comparison Results for Fully Nonlinear Degenerate Elliptic Equations without Zeroth-Order Term. First version.

[5] D.R. Bell, “Degenerate Stochastic Differential Equations and Hypoellipticity”, Harlow. Longman, 1995.

[6] M.G. Crandall, K. Fok, M. Kocan, A. Swiech, Remarks on Nonlinear Uniformly Parabolic Equations, Indiana University Mathematics Journal 47 (1998), No.4.

[7] M.G. Crandall, M. Kocan, A. Swiech, $L^p$-theory for fully nonlinear uniformly parabolic equations, Preprint: Version May 13, (1999).

[8] M.G. Crandall, P.L. Lions, Quadratic growth of solutions of fully nonlinear second order equations in $\mathbb{R}^m$, Differential and Integral Equations 3 (1990), 601-616.

[9] M.G. Crandall, H. Ishii, The maximum principle for semicontinuous functions, Differential and Integral Equations 3 (1990), 1001-1014.

[10] M.G. Crandall, H. Ishii, P.L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992), 1-67.

[11] D. Duffie, “Dynamic asset pricing theory”, Princeton University Press, New Jersey, 1996.

[12] W.H. Fleming, H.M. Soner, “Controlled Markov Processes and Viscosity Solutions”, Springer, New York, 1993.

[13] X. Gabaix, O. Vigneron, The valuation of Mortgage-Backed Securities: theory and evidence, Harvard University, University of Chicago (1998).

[14] Y. Giga, S. Goto, H. Ishii, M.H. Sato, Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains, Indiana University Mathematics Journal 40 (1991), 444-469.

[15] D. Gilbarg, N.S. Trudinger, “Elliptic Partial Differential equations of second-order”, Springer, New-York, 1983.

[16] M. Guzman, Differentiation of integrals in $\mathbb{R}^N$, Lecture Notes in Math. 481 (1975), Springer-Verlag, New York.
[17] U.G. Haussmann, Generalized solutions of the Hamilton-Jacobi equation of stochastic control, *SIAM J. Control and Optimization* **32** (1994), 728-743.

[18] H. Ishii, Perron’s method for Hamilton-Jacobi equations, *Duke Mathematical Journal* **55** (1987), 369-384.

[19] H. Ishii, On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDEs, *Comm. Pure Appl. Math.* **42** (1989), 15-45.

[20] H. Ishii, K. Kobayasi, On the uniqueness and existence of solutions of fully nonlinear parabolic PDEs under the Osgood type condition, *Diff. Int. Eq.* **7** (1994), 909-920.

[21] H. Ishii, P.L. Lions, Viscosity Solutions of Fully Nonlinear Second-Order Elliptic Partial Differential Equations, *Journal of Differential Equations* **83** (1990), 26-78.

[22] R. Jensen, P.L. Lions, P.E. Souganidis, A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations, *Proc. Amer. Math. Sci.* **102** (1987), 975-978.

[23] I. Karatzas, S. Shreve, “Brownian motion and stochastic calculus”, Springer Verlag, New-York, 1998.

[24] I. Karatzas, S.E. Shreve, “Methods of mathematical finance”, Springer Verlag, New York, 1998.

[25] N.V. Krilov, “Controlled Diffusion Processes”, Springer Verlag, New York, 1980.

[26] O.A. Ladyzhenskaya, N.N. Ural’Tseva, “Linear and Quasilinear Elliptic Equations”, Academic Press, 1968.

[27] P.L. Lions, P.E. Souganidis, Fully nonlinear stochastic partial differential equations: nonsmooth equations and applications, *C. R. Acad. Sci. Paris, Série I* **327** (1998), 735-741.

[28] P.L. Lions, P.E. Souganidis, Fully nonlinear stochastic pde with semilinear stochastic dependence, *C. R. Acad. Sci. Paris, Série I* **331** (2000), 617-624.

[29] P.L. Lions, P.E. Souganidis, Equations aux derivees partielles stochastiques non lineaires et solutions de viscosite, Preprint.

[30] B. Oksendal, “Stochastic Differential equations”, Springer Verlag, 1995, fourth edition.

[31] M. Papi, Analysis of a Financial Model for valuing Mortgage-Backed Securities, Master Thesis, Istituto per le Applicazioni del Calcolo (IAC-CNR), Italy, Rome, 1999.

[32] M. Papi, A Generalized Osgood Condition for Viscosity Solutions to Fully Nonlinear, Quaderno IAC 2000, Italy, Rome, 1999. Preprint.

[33] R. Stanton, Rational prepayment and the valuation of Mortgage-Backed Securities, *The Review of financial studies* **8** (1995), 677-708.
[34] D. Tavella, C. Randall, “Pricing Financial Instruments”, John Wiley and Sons, Inc., New-York, 2000.