Baryons in QCD\textsubscript{AS} at Large $N_c$: A Roundabout Approach

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(Dated: December 2009)

QCD\textsubscript{AS}, a variant of large $N_c$ QCD in which quarks transform under the color two-index antisymmetric representation, reduces to standard QCD at $N_c=3$ and provides an alternative to the usual large $N_c$ extrapolation that uses fundamental representation quarks. Previous strong plausibility arguments assert that the QCD\textsubscript{AS} baryon mass scales as $N_c^2$; however, the complicated combinatoric problem associated with quarks carrying two color indices impedes a complete demonstration. We develop a diagrammatic technique to solve this problem. The key ingredient is the introduction of an effective multi-gluon vertex: a “traffic circle” or “roundabout” diagram. We show that arbitrarily complicated diagrams can be reduced to simple ones with the same leading $N_c$ scaling using this device, and that the leading contribution to baryon mass does, in fact, scale as $N_c^2$.

PACS numbers: 11.15.Pg, 14.20.-c

I. INTRODUCTION

In 1974 ’t Hooft proposed \cite{1} a generalization of QCD from three to an arbitrary number $N_c$ of colors. Many aspects of QCD simplify in the large $N_c$ limit: Inasmuch as the $N_c \rightarrow \infty$ and $N_c = 3$ worlds are qualitatively similar, one can learn about QCD by starting with the large $N_c$ limit and expanding physical quantities systematically in powers of $1/N_c$. This approach has provided a powerful tool to study QCD and other strongly coupled gauge theories. It is implemented by generalizing the gauge group of QCD with coupling $g$ from SU(3) to SU($N_c$), and taking $N_c \rightarrow \infty$ while keeping $g^2 N_c$ and the number of flavors $N_f$ fixed. In this limit, each quark loop is suppressed by a factor of $N_c^{-1}$, while each nonplanar contribution to a diagram leads to suppression by a factor of $N_c^{-2}$, and thus the number of diagrams one must consider is radically reduced. In 1+1 dimensions this diagrammatic simplification is sufficient to allow direct calculation of the spectrum at leading order in $1/N_c$ \cite{2}, but unfortunately is not sufficient in higher dimensions. Nevertheless, by studying the $N_c$ scaling behavior of correlation functions, one can deduce the scaling of various quantities with $N_c$.

The study of large $N_c$ baryons suffers the complication that the physical operators acting upon them are not independent of $N_c$. However, as observed by Witten \cite{3}, one can employ a combinatoric analysis to deduce the scaling of observables associated with baryons. Moreover, a spin-flavor symmetry for baryons emerges at large $N_c$, allowing one to relate various observables with relative errors of $O(1/N_c)$, or in some cases $O(1/N_c^2)$ \cite{3, 4, 5, 6, 7}. These relations tend to describe the real world quite well; for early examples, see Refs. \cite{6, 8, 9, 10, 11, 12}. Thus, large $N_c$ QCD and the $1/N_c$ expansion provide both qualitative insight and semiquantitative predictions.

However, it has long been recognized that the extrapolation from $N_c = 3$ to large $N_c$ is not unique \cite{13}. While assuming that quarks at any $N_c$ transform in the fundamental (F) representation of SU($N_c$) seems natural, it is not mandatory. Corrigan and Ramond \cite{15} (CR) noted that using quarks transforming in the two-index antisymmetric (AS) representation of SU($N_c$) provides an equally valid extrapolation to large $N_c$. At $N_c = 3$ the AS representation is just the 3, and thus simply swapping all AS quark and antiquark labels reduces the theory to standard QCD. The original motivation for this construction was a scheme in which baryons have three quarks at any $N_c$: one AS quark and two F quarks.

The study of large $N_c$ QCD containing quarks in the AS representation has been revived in the past few years and has received considerable attention. The motivation is a profound new insight by Armoni, Shifman, and Veneziano \cite{13}, that an “orientifold equivalence” exists at large $N_c$ between QCD with quarks in the AS representation and QCD with quarks in the adjoint. One immediate consequence follows in the case of one massless quark flavor: The equivalent theory is supersymmetric, and all of the considerable formal power of SUSY can be brought to bear on the problem. A key issue is whether one can use this connection to learn information—even approximate information—about QCD with more than one flavor, as occurs in the physical world.

One strategy to deal with the real-world case of multiple flavors is to focus on the chiral limit and allow one quark flavor to transform in the AS representation while the other flavors transform as F, thus maintaining supersymmetry at large $N_c$. In point of fact, this approach is
just the CR scheme and gives 3-quark baryons at any $N_c$. Unfortunately, it also suffers the major defect of breaking flavor symmetry at any $N_c \neq 3$, which can then only be restored by summing the all-orders expansion in $1/N_c$. Since flavor is a critical part of QCD dynamics, truncating the expansion at any low order of $1/N_c$ is highly problematic. This issue is discussed in some detail in Ref. [15].

Given this difficulty, one may consider a different implementation in which all flavors of quark transform in the AS representation. We refer to this theory as QCD$_{AS}$ and distinguish it from the usual theory QCD$_F$ in which the quarks transform in the F representation. QCD$_{AS}$ has the disadvantage of losing the equivalence with a supersymmetric theory, but it correctly retains the flavor symmetries, including chiral symmetry, that are essential to reproduce correct low-energy QCD dynamics. QCD$_{AS}$ and QCD$_F$ coincide at $N_c = 3$, but extrapolate to large $N_c$ in different ways; while the two large $N_c$ extrapolations have much in common, they also differ in important respects. While nonplanar diagrams are suppressed in both the QCD$_F$ and QCD$_{AS}$ limits, quark loops are not suppressed in the QCD$_{AS}$ limit because AS quarks, like gluons, carry two color indices. This feature alters the nature of large $N_c$ scaling in the theory.

A priori, it is not obvious whether the $1/N_c$ expansion based on QCD$_F$ or the $1/N_c$ expansion based on QCD$_{AS}$ gives a superior description of the $N_c = 3$ world. Presumably the answer depends upon which observable is considered. A recent comparison of baryon mass splittings [15] from the two expansions suggests that both do a good job.

However, this comparison depends upon understanding the nature of baryons in the two limits. The case of baryons in QCD$_F$ is well known: The baryon mass was shown by Witten to scale as $N_c^2$ [3]. Witten’s reasoning was somewhat heuristic; it was based on a study of QCD with heavy (non-relativistic) quarks and an argument that QCD with light quarks should behave analogously. A more formal version of this argument valid for light quarks was developed in Ref. [16]. Both of these treatments are based upon a diagrammatic analysis and thus are not strictly rigorous; the analyses depend upon the assumption that the $N_c$ scaling can be deduced from the contributions of diagrams. That is, they assume that fundamentally nonperturbative effects do not alter the leading $N_c$ scaling. However, up to this assumption, the analysis is reliable. The crux of the analysis is the demonstration that the contribution to the mass due to the interaction of $n$ quarks scales with $N_c$ in the same way as the contribution from diagrams containing $n$ quarks, all of which are connected by gluons, summed over all possible combinations of $n$ quarks (out of $N_c$ total) that can contribute.

The second critical part of the analysis for QCD$_F$ is the demonstration that the contributions from $n$ connected quarks all scale as $N_c^{-1}$ or less; this demonstration is quite straightforward. Consider the simplest case: one-gluon exchange between a pair of quarks. The contribution is proportional to $g^2 \sim N_c^{-1}$ times an $O(N_c^2)$ combinatoric factor that specifies the number of distinct quark pairs to which the gluons couple, giving a total of $O(N_c^1)$. It is easy to see that this is scaling is unaltered if the two quarks are connected by a more complicated set of (planar) gluon exchanges. Moreover, it is also easy to see that the counting is $O(N_c^1)$ for 3-quark interactions; in effect, adding a third quark costs a factor of $N_c^{-1}$ due to the additional factor of $g^2$ needed to couple it to the other quarks but brings in an additional combinatoric factor of $N_c^2$. This analysis can be extended to the interactions of an arbitrary number of quarks, and for all cases yields a leading scaling of $N_c^2$.

The situation for QCD$_{AS}$ is more complicated. Bolognesi [17] has shown that the operator creating the lightest color-singlet state with nonzero baryon number in QCD$_{AS}$ with $N_c$ odd contains a product of $N_c(N_c-1)/2 \sim N_c^2$ quark operators, one carrying each color combination $c_1c_2$, where $c_1 \neq c_2$. The associated states are naturally identified with baryons. This observation suggests that baryon masses scale as $N_c$ instead of $N_c^2$, as in the case of QCD$_F$. Such a conclusion seems particularly plausible since Skyrme-type models in QCD$_{AS}$, in which baryons are topological solitons of QCD$_{AS}$ meson fields, produce baryon masses scaling as $N_c^2$ [18]. It is important to show that this argument is more than plausible, and is in fact correct. The natural approach is simply to extend the analysis used in Refs. [3] [16] for QCD$_F$ to the case of QCD$_{AS}$. As noted above, this analysis contains two key elements: The first is a demonstration that the contribution to the mass due to the interaction of $n$ quarks scales with $N_c$ in the same way as the contribution from diagrams containing $n$ quarks, all of which are connected by gluons, summed over all possible combinations of $n$ quarks (out of $N_c$ total) that can contribute; this demonstration is easily extended to the case of QCD$_{AS}$. The second is an analysis of the scaling of contributions from $n$ connected quarks. For the case of QCD$_{AS}$ one expects the scaling $\sim N_c^2$. This second part is where the complexity lies.

Indeed, as discussed in Ref. [18], an apparent paradox arises in computing the scaling of the contribution from $n$ connected quarks. Consider, for example, the case of one-gluon exchange between two quarks and use reasoning analogous to that of Ref. [3]. Again, this contribution has two coupling constants that contribute a scaling factor of $1/N_c$. Again, one computes a combinatoric factor: Assuming that the gluon can couple to any of the $N_c^2$ quarks, one apparently obtains a factor of $N_c^4$, yielding an overall contribution of $N_c^3$. This result contradicts the expectation that the mass scales as $N_c^2$. Moreover, similar reasoning leads one to conclude that the contribution of 3-quark clusters scales as $N_c^4$, and more generally that $n$-quark clusters scale as $N_c^{2n+1}$.

The paradox is resolved by a more careful treatment of the combinatorics. The wave function of a QCD$_F$ baryon consists of a set of terms in which each color ap-
pears associated with precisely one quark (with the full wave function fully antisymmetrized in order to form a color singlet). A gluon exchange between any two quarks switches their colors, leading to a final state again containing each color once, and again corresponding to a color singlet. Thus, the exchange of a gluon between any two quarks maintains the color-singlet structure, and one obtains a combinatoric factor given by the total number of pairs of quarks in the state. In contrast, consider what happens in QCD AS: in this case a baryon again contains each color combination once, by which is meant a pair of distinct fundamental color indices (e.g., red-blue). A typical gluon exchange between two quarks exchanges one of the two colors, yielding two quarks with different color combinations than the initial two. For example, consider two quarks $q_{ab}$ and $q_{cd}$ where $a, b, c, d$ are the individual color indices, and assume that all four of these colors are distinct (We use this specific example later, in Sec. III). Suppose a gluon exchange swaps two of these colors (say $b$ and $c$); following this exchange the quarks become $q_{ac}$ and $q_{bd}$, which are different from the original quarks. However, by construction these quarks inhabit a baryon in which the color-singlet nature of the state requires that each color combination occurs once and only once. The exchange described above is inconsistent with this constraint: After the exchange, one has two copies of $q_{ac}$ and $q_{bd}$ and no copies of $q_{ab}$ or $q_{cd}$. Thus, in a color-singlet baryon the exchange described above cannot occur. In order for a one-gluon exchange diagram to appear within a baryon state, it must occur between quarks that share a color. Thus, for example, the quarks $q_{ab}$ and $q_{ca}$ can exchange a gluon that swaps $b$ and $c$, yielding $q_{ac}$ and $q_{ba}$, consistent with the color-singlet constraint. This need for a repeated color label means that the combinatoric factor does not scale as $N^2_c$ but only $N^3_c$, which, in combination with the $1/N_c$ factor from coupling constants, yields a contribution of $N^2_c$—precisely as expected.

Reference [18] considered many classes of diagram. In all cases the restrictions imposed by the condition that the color combinations of the initial quarks and the final quarks must be identical yielded overall contributions no higher than $O(N^2_c)$. While these results greatly strengthen one's confidence that the mass does indeed scale as $N^2_c$, Ref. [18] nevertheless provided no general approach to prove that this result holds for all classes of diagram; each class was analyzed separately. The purpose of this paper is to remedy this situation by providing a general demonstration that the contribution of any $n$-quark cluster to the baryon mass in QCD AS scales as $N^2_c$ (or less) for any class of diagram.

Our strategy is to introduce a diagrammatic simplification by replacing diagrams of arbitrary complexity with effective diagrams that preserve the connectivity between incoming and outgoing lines and the $N_c$ counting of the original diagram. These effective diagrams resemble traffic circles or roundabouts. Thus, despite the fact that this approach is quite direct, it is aptly described as a “roundabout approach.” In what follows we first illustrate the power of the roundabout approach for the simple and known case of baryons in QCD F (Sec. II), and then turn to the case of interest, QCD AS (Sec. III).

In the following demonstrations, we draw representative Feynman diagrams. When it is important to illustrate the color flow, we follow ’t Hooft and use diagrams in which gluons appear as two oppositely directed color lines. Quarks in QCD F are represented by single color lines, while quarks in QCD AS are represented by doubled color lines pointing in the same direction, in order to reflect the fact that quarks now carry two color indices. The double-line representation for quarks in QCD AS is used in both Feynman diagrams and color-flow diagrams. Whenever necessary, colors are labeled with lower-case letters.

II. BARYONS IN FUNDAMENTAL QCD

Each QCD F quark carries one color index, and an operator in QCD F carrying unit baryon quantum number contains $N_c$ quarks, each with a distinct color. Consider a general Feynman diagram with $n$ quarks connected irreducibly (i.e., one cannot obtain two distinct interacting subdiagrams by cutting all of the quark lines without also cutting a gluon line). As demonstrated by ’t Hooft, diagrams with nonplanar gluons are suppressed, as are quark loops. Furthermore, neither connecting additional gluons to pre-existing gluons nor connecting additional quarks increases the $N_c$ counting. All irreducible diagrams built this way are known to scale as $N^2_c$ [3].

The only other diagrammatic possibility is the case of multiple gluon connections to a single quark, an example of which is shown in Fig. 1. Notice that distorting the internal quark lines gives a diagram (Fig. 2) that, while distinct as a Feynman diagram, is equivalent in terms of color flow to one in which the quark effectively connects to only a single gluon, with any additional gluons connected only to other gluons.

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{Fig1.png}
\caption{Multiple gluon connections to a single quark and the corresponding color flow diagram.}
\end{figure}

Since this color reduction can be done to any of the $n$ quarks in the general Feynman diagram, the resulting equivalent diagram is one in which a single gluon connects each of the $n$ quarks to a general gluon diagram. As demonstrated in Ref. [3], $n$-quark diagrams all scale as $N^2_c$ (or less). If one now divides by $N_c$ to re-
move the combinatoric factors associated with each connected quark, one concludes that the gluon interaction must scale as $1/N_c^{n-1}$. The critical observation is that this $1/N_c^{n-1}$ scaling does not depend upon the details of the interaction.

An alternate method of deriving the $N_c$ dependence of the $n$-gluon vertex follows from using the rescaled QCD Lagrangian,

$$\mathcal{L} = \frac{N_c}{g_s^2} \times \left[ -\frac{1}{4} (F_{\mu\nu})^a_b (F^{\mu\nu})^b_a + \bar{\psi}_a (i\slashed{D})^a_b \psi^b - m \bar{\psi}_a \psi^a \right],$$

where the fields $A^a$ and $\psi$ each absorb a factor $g_s/\sqrt{N_c}$ compared to the conventional definition. This form provides the most convenient starting point not only for proving the finiteness of the large $N_c$ limit, but along the way shows that 3-point vertices scale as $1$, and that the gluon 4-point vertex scales as $1/N_c$. Using the same approach, if the Lagrangian is manipulated to allow for effective multi-gluon vertices (such as appear in the passage from Fig. 1 to Fig. 2), then the $n$-gluon vertex scales as $N_c^{1-n}/2$. Including the $n$ vertices where the gluons terminate on quark lines brings in another $(1/\sqrt{N_c})^n$, making for a complete interaction that scales as $N_c^{1-n}$, exactly as obtained above.

This observation motivates the introduction of the term roundabout, or traffic circle, to depict such multi-gluon interactions, in which each color line flows in at one location and out at an adjacent location (which is allowed in the roundabout, so far as $N_c$ scaling is concerned. In fact, the roundabout represents a highly nonlocal QCD operator.

The initial step in reducing an $n$-quark diagram is therefore to make all possible reductions of the form shown in Fig. 2 color flow on any given quark line then exits via a single equivalent gluon that ultimately enters a roundabout. With this notion in mind, it is easy to see that any coupled quark cluster can be reduced to the form of Fig. 3 (in that specific case, for 4 quarks), with one effective gluon leaving each quark and the effective gluons meeting at a roundabout. While such substitutions alter the Feynman diagrams and hence the dynamics, they fully capture the color flow and hence the $N_c$ counting.

With the idea of a roundabout in hand, generating Witten’s argument that $n$-quark Feynman diagrams scale as $N_c^1$ or less becomes trivial. Since $n$ quarks are connected, the combinatoric contribution to the $N_c$ counting is $N_c^n$, while the interaction contribution is at most $1/N_c^{n-1}$. Thus the total counting is at most $N_c^n/N_c^{n-1}$, or simply $N_c$.

As seen in this section, the notion of the roundabout diagram simplifies the derivation of the $N_c$ scaling for baryons in QCD$_F$. However, the problem is sufficiently simple that the construction is not really essential. In the following section, we apply the roundabout diagram approach to the more complicated case of QCD$_{AS}$, in which it plays a pivotal role to make the counting tractable.

### III. BARYONS IN ANTISYMMETRIC QCD

Each QCD$_{AS}$ quark has two color indices, and an operator in QCD$_{AS}$ creating a baryon contains $N_c(N_c-1)/2 \sim N_c^2$ quarks. Consider a general Feynman diagram with $n$ connected quarks. We demonstrate that the interaction energy contribution to the baryon mass scales at most as $N_c^n$ in the large $N_c$ limit. It is therefore sufficient to show that all leading-order diagrams scale as $N_c^3$.

As noted in the Introduction, at first glance this scaling rule appears to be violated by one-gluon exchange (Fig. 4). The four distinct color indices provide a combinatoric contribution of $N_c^4$, while the interaction (with two gluon vertices) scales as $1/N_c$, thus giving a total dependence of $N_c^3$. However, as demonstrated in Ref. [13], one-gluon exchange in QCD$_{AS}$ does not contribute unless
one restricts the exchange to quarks that have one color index in common (Fig. 5). This constraint is necessary in order to enforce Bolognesi’s condition that a color-singlet baryon operator contains a quark carrying any given pair of colors precisely once 

It is important to note that the one-gluon exchange in Fig. 4 remains physically valid even when all four colors involved are distinct. The Feynman diagram as given, with a particular specified set of colors for the quarks, is perfectly legitimate. The key point is that, if the incident set of quarks as drawn is part of a color singlet and the other quarks not pictured maintain their color identities, then the final set does not form a color singlet. One obtains a color singlet by forming an appropriate linear combination of quark color combinations, as appears in the baryon operator. However, color is conserved and moreover, the color singlet contains each pair of colors once and only once. One-gluon exchange diagrams that yield quarks with different color combinations in the initial and final states can contribute only to color-nonsinglet processes. Summing over the initial color combinations to yield a one-particle color-singlet state necessarily causes such contributions to cancel completely. The net contribution of any Feynman diagram in which the set of color pairs of quarks in the initial and final states differ must vanish in correlation functions of color-singlet operators.

The upshot of this argument is that the set of color pairs of quarks in the final state must be a permutation of the set in the initial state in order to contribute to correlation functions of color-singlet operators. The only ways for one-gluon exchange to respect this condition are either if the quarks share one color in common while exchanging the other, or if the gluon couples to two quark lines carrying the same color, as in Fig. 5. Note, however, that the restriction of having a pair of colors in common reduces the combinatoric factor from \( N_c^4 \) to \( N_c^3 \). Combining this \( N_c^3 \) with the \( 1/N_c \) from the two coupling constants yields a total scaling of \( N_c^2 \), as desired.

![FIG. 4: One-gluon exchange that does not contribute to color-singlet correlators in QCD\(_A\)S baryons.](image)

The question we consider is whether the restriction that all leading-order contributions to correlation functions of color-singlet operators require the sets of quark color pairs in the initial and final states to be the same up to permutation is sufficient to ensure that all classes of diagram contribute no faster than \( N_c^2 \). In Ref. [18], many classes of diagrams were considered, and in fact all contributed at \( O(N_c^2) \) or less. The question we address here is whether this result is completely general, holding for any diagram and with any number of repeated color indices. We claim that it does and provide a general demonstration below.

Before addressing this general question, let us note that one’s naive intuition about which classes of diagrams can contribute at leading order can easily be wrong. Consider for example the role played by interactions of quarks that have a color in common among the pair of colors specifying each quark. Naively it might seem that such interactions are unimportant at large \( N_c \). After all, the odds that any two quarks share a common color is \( \sim 1/N_c \), and one might expect the effect of such interactions to be suppressed by a factor of \( 1/N_c \). However, as seen in the case of one-gluon exchange, such interactions occur at leading order, i.e., \( O(N_c^2) \). In the case of one-gluon exchange the reason is rather clear: Quarks with one color in common do appear at relative \( O(1/N_c) \) compared to the typical case with all distinct colors; however, the typical case happens not to contribute to color singlets. In a similar way, one might think that gluons that change the color of the quark line with which they interact necessarily dominate over those that do not (i.e., those that are diagonal in color space), since they outnumber the color-diagonal ones by a factor of \( N_c \). We refer to the latter as Cartan gluons since they are associated with the generators of the Cartan subgroup. Again one finds that Cartan gluons can also contribute at leading order, \( N_c^2 \).

Of course, the critical question is not whether one can find leading-order contributions, but rather whether one can show that no “superleading” contributions scaling faster than \( N_c^2 \) occur. However, the fact that repeated color indices and Cartan gluons contribute at leading order means that they are as important as the apparently more typical examples. Hence, they need to be included in a demonstration that no superleading contributions exist. This fact greatly complicates the analysis.

The strategy to demonstrate that no class of diagram scales faster than \( N_c^2 \) has three main steps. The first two
remove certain types of gluons from the diagram, which breaks the set of interacting quarks into smaller clusters of interacting quarks. We show that the leading \( N_c \) scaling of the contribution to a color-singlet correlator of the combined diagram is no larger than the \( N_c \) scaling of the contribution of the smaller clusters. We call the smallest of these irreducible clusters. The final step exploits the roundabout diagrams to show that irreducible clusters contribute at \( O(N_c^2) \) or less.

We illustrate the procedure on a particular diagram (Fig. 6): however, all of the individual steps apply to any arbitrarily complicated diagram. One can check that this diagram does indeed contribute, since it preserves the color combinations between initial and final states. Furthermore, by counting the number of distinct indices (12) and the number of gluon vertices (20), this diagram evidently scales as \( N_c^{12}/N_c^{10} = N_c^2 \), making it a leading-order diagram. While obtaining this result is simple in this particular case, the key question is how to break up the generically complicated combinatoric problem into pieces that can be simply analyzed.

The first step of the simplification is to disconnect the general diagram by removing any Cartan gluons, keeping track of the number removed and where they were connected. Since this type of connection results in a color line that interacts but does not change its color value, the removal of Cartan gluons can completely separate clusters of quarks without changing color flow (In the case of \( QCD_{AS} \), one of the two color lines of a particular quark is allowed not to interact as long as the other one does, thus coupling the whole quark to the diagram). In our example, this step decouples only the \( ag' \) quark from the diagram (Fig. 7). Notice that if an interaction occurred between the \( d' \) from the \( dd' \) quark and the \( d' \) from the \( ed' \) quark, this Cartan gluon would also have been removed in this step. However, such an extra interaction was not included because it would have clearly made the diagram subleading in \( 1/N_c \). The distinction is that this gluon does not connect to a quark in the diagram that carries a noninteracting line (such as \( g' \)) that provides a combinatoric factor \( N_c \) to compensate the additional vertex \( 1/\sqrt{N_c} \) factors.

This step is important because all of the remaining gluons change quark colors, allowing one to develop a connection between color flow and \( N_c \) scaling. The removal of a Cartan gluon either separates the diagram into two distinct interacting clusters or it does not. If not, then the diagram was necessarily subleading: An identical diagram without it would have the same color flow and combinatorics but one pair fewer of coupling constants and hence one factor less of \( 1/N_c \). One can thus dismiss this case as uninteresting in establishing whether or not any class of diagrams contributes at a superleading order. If the removal breaks the diagram into two clusters, then suppose further that one can show the contribution from any connected cluster not containing a Cartan gluon scales no faster than \( N_c^2 \). In that case, one easily sees that connecting two such clusters also leads to \( N_c^2 \) scaling: Each cluster contributes \( N_c^2 \), the two coupling constants together contribute \( 1/N_c \), and a combinatoric suppression factor of \( 1/N_c \) combines with them to give a total scaling of \( N_c^2 \). The origin of this last factor is easy to see: In order for the two clusters to be coupled by a Cartan gluon, they must share a color, suppressing the combinatoric possibilities compared to two clusters with completely distinct colors. Of course, this argument is completely general and does not require the diagram to have the form of Fig. 6. The problem is thus reduced to showing that graphs containing no Cartan gluons scale as \( N_c^2 \) or less.

The clusters created in the first step above are now analyzed individually. Each cluster may further consist of some number of separate quark groups, each of which shares no color indices with another group, nevertheless connected to one another (by means of colors passed back and forth between the two). The second step is to remove the gluons that connect these groups, again keeping track of the number removed and where they were connected. This action results in a set of what we call irreducible
clusters, irreducible because no quark can be removed since each quark interacts, changes color, and is coupled to every other quark in a sequence. In our example, the cluster consisting only of the $ag'$ quark is itself an irreducible cluster, while the other cluster can be separated into two irreducible clusters by removing the two gluons connecting the $cc'$ quark to the $dd'$ quark (Fig. 8).

![Feynman diagram](image)

FIG. 8: The Feynman diagram of Fig. 6 after the second step, the separation of regular clusters into irreducible clusters. Three irreducible clusters remain.

It is easy to see that the leading scaling of a cluster made up of irreducible clusters must be $N_c^2$, provided that an irreducible cluster also scales as $N_c^2$. By construction, the gluons connecting two irreducible clusters must first shift color from one irreducible cluster to the other and then back, which requires at least two gluon exchanges, or four coupling constants, which contribute a factor of $1/N_c^2$. Combining with a factor of $N_c^2$ from each irreducible cluster yields $N_c^2$, as required. A simple inductive argument shows that one can then combine any number of irreducible clusters together in just this manner while retaining the $N_c^2$ scaling behavior. Thus the problem is now reduced to showing that irreducible clusters scale as $N_c^2$.

The third and final step is the demonstration that any irreducible cluster, regardless of its size and the details of its interactions, scales at most as $N_c^2$. Roundabout diagrams greatly aid in this step.

Note that, in all of the irreducible clusters of Fig. 8, the two color lines in each quark are divided into two classes: those with an unprimed letter written on the left (L indices) and those with a primed letter written on the right (R indices). This distinction is a priori artificial, since the quarks are antisymmetric in the indices. It is noteworthy, however, that the gluons in Fig. 8 only connect among quark L indices or among quark R indices, meaning that L indices of the various quarks exchange colors exclusively amongst themselves, and similarly for R indices. Thus, while the question of which index one labels as L or R is arbitrary, the connections break up into two distinct classes of connected color flow for the type of diagram seen here. Of course, one can consider diagrams where the color flow does not break up into two separate classes. However, it can be shown that such graphs are necessarily subleading. In the following argument we assume that this is true, deferring until the end a demonstration, for even the case of irreducible diagrams with separate color flow among the L and R indices remains more complicated than the situation in QCD$_F$.

A principal reason for this complication is that an irreducible cluster can still have any number of repeated L or R indices. However, it is possible to simplify the analysis by exploiting the $n$-quark roundabout diagrams corresponding to each interaction. In order to describe quark lines that do not interact, we consider a single non-interacting quark line to form a trivial roundabout (or cul de sac; see Fig. 6 for examples). Although more than one roundabout can appear in an arbitrary diagram, the roundabouts themselves look exactly like those in the case of QCD$_F$. The L indices and the R indices interchange separately, and for each of L and R one requires one or more roundabouts (some of which may be trivial) to describe any interaction.

We now show that the leading-order diagrams maximize the number of complete roundabouts for a given number of repeated indices. This crucial feature follows from the fact that replacing one roundabout with two (whenever it is valid to do so, given the color connections between quarks required by the gluons present in the original diagram) effectively contributes an additional factor of $N_c$. When roundabouts split, an additional color line can interact (introducing its combinatoric factor of $N_c$) without the cost of an additional factor of $1/N_c$. In other words, a roundabout with $n = n_1 + n_2$ lines scales as $N_c^{n_1-n_1-n_2}$, while the combination of roundabouts with $n_1$ and $n_2$ lines scales as $N_c^{1-n_1} N_c^{1-n_2} = N_c^{2-n_1-n_2}$. A roundabout with repeated indices is subleading compared to one in which the repeated indices are separated into two roundabouts, because the former loses a combinatoric factor of $N_c$ compared to the latter. In summary, a leading-order diagram is one in which none of the roundabouts contain more than one instance of a given index, whether or not that index is repeated elsewhere. In our example, we have drawn the roundabout diagrams corresponding to each irreducible cluster (Fig. 8). The first irreducible cluster consists of two trivial roundabouts, one from L lines and one from R lines. The second consists of two ordinary roundabouts, one L and one R. The third consists of one ordinary roundabout from the L lines and one ordinary plus one trivial roundabout from the R lines.
We now demonstrate that the number of possible complete roundabouts has an upper limit \( N_r + 2 \), where \( N_r \) is the number of redundant indices (separately counted for L and for R lines), by which we mean the total number of color indices minus the number of distinct color values carried by the indices. We then show how this result implies that no irreducible cluster can scale with \( N_c \) faster than \( N_r^2 \).

The simplest case (in which each quark line has a different color index, and all of the indices permute) is composed of two roundabouts, one encompassing all L indices and one encompassing all R indices; it is consistent with the general result because \( N_r = 0 \). Now consider a general irreducible cluster with some number \( N_r \) of redundant indices. By turning a previously distinct color index into a repeat of one already present, one creates a diagram with \( N_r + 1 \) redundant indices. As argued above, the diagram gains a power of \( N_c \) if the roundabout in which this new redundancy occurs is replaced with two smaller ones, for a gain of one roundabout; repeating this argument for each of the \( N_r \) redundant indices shows that leading-order diagrams occur when the number of roundabout interactions is \( N_r + 2 \).

With this result in hand, we now directly compute the maximum \( N_c \) dependence of any irreducible cluster. Since two color indices are provided by each AS quark, and each distinct color index adds a combinatoric factor of \( N_c \), the combinatoric contribution is \( N_c^{2n-2N_r} \), where \( n \) is the number of quarks connected in the irreducible cluster. Each roundabout scales as \( N_c^{1-n_i} \), where \( n_i \) is the number of color lines appearing in the \( i^{th} \) roundabout, and each of the \( n \) quarks (which has an L and an R color line) contributes to two roundabouts. The product of these factors over the \( N_r + 2 \) roundabouts is just \( N_c^{N_r+2-2n} \). Thus the total \( N_c \) dependence is \( N_c^{2n-2N_r} \times N_c^{N_r+2-2n} \), which reduces to just \( N_c^2 \). Of course, the \( N_c \) scaling can be less than this value if the total number of roundabouts does not reach the upper limit of \( N_r + 2 \).

In our example (Fig. 9), the first irreducible cluster consists of two trivial roundabouts, each with an interaction contribution \( 1/N_c^0 = N_c^0 \) and a combinatoric contribution \( N_c^1 \). Thus the total \( N_c \) scaling is indeed \( N_c^2 \). The second irreducible cluster also scales as \( N_c^2 \), with combinatoric contribution \( N_c^0 \) and interaction contribution \( 1/N_c^2 \). The third irreducible cluster has combinatoric contribution \( N_c^5 \) and interaction contribution \( 1/N_c^3 \), for a total \( N_c \) dependence of \( N_c^2 \).

The basic demonstration that no class of diagram scales faster than \( N_c^2 \) is now complete. However, it was based on the assumption that no leading-order diagrams mix L and R indices. To prove this claim, consider any connected diagram among \( n \) quarks featuring an interaction between the two sets—namely, an LR gluon—and then consider the connected \( n \)-quark diagram in which the LR gluons are simply deleted. Note that the diagram remains connected because an LR mismatch does not appear unless both the L line of the second quark and the R line of the first quark are required to connect to other quark lines of the corresponding handednesses elsewhere in the diagram. However, deleting a given LR gluon line gains a power of \( N_c \) from the two now-absent trilinear vertices, and since every quark remains connected to the diagram, this deletion loses no combinatoric factors of \( N_c \). Thus, the effect of deleting the LR gluons is to increase the \( N_c \) scaling, indicating that the original graph was subleading.

One might worry that possible factors of \( N_c \) could also be lost in this deletion, arising from color loops that the LR gluon might have completed. However, the act of reducing all gluon interactions to roundabouts means that all color lines enter the diagram at an initial point, pass through a roundabout, and leave through a final point. Reminding the reader that the color loop in each roundabout diagram is merely a representational device, one sees that no color loops remain in a diagram converted to roundabout form, and hence no loop factors of \( N_c \) are lost in this deletion. In fact, one can further show, once the Cartan gluons are removed, that diagrams with LR gluons are suppressed by a factor of at least \( N_c^{-2} \) compared to diagrams without them: Since non-Cartan gluons exchange colors on the quark lines, exchanging color values
between the L and R sides and switching them back again requires at least two LR gluon exchanges, and hence induces a suppression of $1/N_c^2$ compared to diagrams without this color exchange. One concludes that the set of diagrams with L color lines and R color lines separated always includes diagrams with the leading-order $N_c$ counting. As mentioned above, this separation is present in our example.

Thus we have completed our demonstration that no class of connected $n$-quark diagrams contributes to color-singlet correlators with power greater than $N_c^2$ (provided the number of gluons in the graph is finite). Following the arguments of Refs. [3, 16], this conclusion implies that the QCDAS baryon mass scales as $N_c^2$. While the arguments sketched here are informal, they could be converted into a rigorous theorem with no essential difficulty. However, one should recall that the arguments in Refs. [3, 16] are themselves not rigorous: They depend upon the assumption that one can deduce the baryon mass scaling behavior from the behavior of Feynman diagrams, which are intrinsically perturbative in nature. While extremely unlikely, it is logically possible that fundamentally nonperturbative effects not diagrammatically expressible could alter this result. The conclusion one reaches from such considerations is that the claim that the baryon mass scales as $N_c^2$ in QCDAS is now as solid as the claim that the baryon mass scales as $N_c^1$ in QCD$_F$.

Acknowledgements. We thank the organizers and participants of the INT workshop “New Frontiers in Large N Gauge Theories,” where this work was initiated, and also the INT and the University of Washington for their hospitality. T.D.C. acknowledges the support of the U.S. Dept. of Energy under Grant No. DE-FG02-93ER-40762. R.F.L. acknowledges the support of the NSF under Grant No. PHY-0757394.

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