Integrable Sigma-models and Drinfeld-Sokolov Hierarchies

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Abstract

Local commuting charges in sigma-models with classical Lie groups as target manifolds are shown to be related to the conserved quantities appearing in the Drinfeld-Sokolov (generalized mKdV) hierarchies. Conversely, the Drinfeld-Sokolov construction can be used to deduce the existence of commuting charges in these and in wider classes of sigma-models, including those whose target manifolds are exceptional groups or symmetric spaces. This establishes a direct link between commuting quantities in integrable sigma-models and in affine Toda field theories.

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1 Introduction

Some recent work [1] established the existence of infinite families of local, conserved, commuting charges in each two-dimensional principal chiral model (PCM) with target space a compact, classical Lie group \( G \). The currents underlying these charges are defined using totally symmetric \( G \)-invariant tensors \( k_{a_1 a_2 \ldots a_m}^{(m)} \), where indices \( a \) refer to a basis for the Lie algebra \( g \) corresponding to \( G \). The \( k \)-tensors and their currents are given by the formulas

\[
K_m = k_{a_1 a_2 \ldots a_m}^{(m)} j^{a_1} j^{a_2} \ldots j^{a_m} \quad (1.1)
\]

\[
det(1 - \mu j^a t_a)^{s/h} |_{\mu + 1} , \quad s = m - 1 , \quad (1.2)
\]

where \( j^a t_a \) is a Noether current (arising from a global \( G \) symmetry of the model) which takes values in the defining representation of the classical algebra \( g \) with generators \( t_a \) (other conventions will be given below). The tensors \( k^{(m)} \) defined by (1.2) are non-vanishing precisely when the spin of the corresponding charge, \( s = m - 1 \), is equal to an exponent of \( G \) modulo its Coxeter number, \( h \).

One motivation for the work just mentioned was the appearance of certain common features [2] of exact \( S \)-matrices for PCMs [3] and affine Toda field theories (ATFTs) (see [4] and references therein). The implications of commuting charges with spins as given above have been thoroughly studied in the Toda case, as reviewed in [4], and the appearance of analogous charges in PCMs indeed offers a natural explanation for the otherwise mysterious similarities displayed by their \( S \)-matrices [1]. Subsequently, it was shown that similar families of commuting charges can still be constructed if a Wess-Zumino (WZ) term is added to the PCM [5], or if the target space is some compact symmetric space rather than a Lie group [6]. Analogous results also hold for supersymmetric models [5]. A number of important questions remain unanswered, however.

The most obvious problem is to extend the results of [1, 5, 6] to all Lie groups, by including the exceptional cases along with the classical families. It is natural to expect that this should be possible, and yet the formula (1.2) utilizes the defining representation for each classical algebra, and so it has no unambiguous interpretation for the exceptional cases. One can also ask, at a more general level: what is the deeper mathematical significance of the \( k \)-tensors, and what is their relationship—if any—to more familiar mathematical structures? It is reasonable to suppose that an answer to this would help in passing from classical to exceptional groups. Finally, while it was successfully shown in [1] that there are classical commuting charges with identical patterns of spins in both PCMs and ATFTs,
one may ask whether it is possible to go further and explicitly relate the charges in the two models.

In this paper we shall give answers to each of the three questions posed above. The next section consists mostly of introductory material on sigma-models and their currents, but it finishes with a useful lemma relevant to the existence of commuting charges. We then examine, in section 3, whether the formula (1.2) gives satisfactory results when applied to exceptional groups: it proves entirely adequate for $G_2$ but not for the more complicated cases. In the remainder of the paper we develop a more systematic approach which involves a direct relationship between sigma-model and Toda charges; this turns out to be the key to understanding all three of the problems mentioned above.

Section 4 summarizes some facts about Toda theories and their associated Drinfeld Sokolov/modified KdV (DS/mKdV) hierarchies of commuting charges [7, 8]. The connection with sigma-models is particularly natural from the physical perspective of gauged WZW models [9], although our interest here is to relate PCMs and affine Toda theories, rather than WZW models and conformal Toda theories. The links between these various points of view will be mentioned below.

In section 5 we describe an explicit correspondence between commuting charges in PCMs on the one hand, and DS hierarchies or commuting charges in ATFTs on the other. This shows that the $k$-tensors of (1.2) are indeed closely-related to well-known mathematical structures, and the correspondence allows us to deduce the existence of $k$-tensors and commuting charges in PCMs based on all Lie groups, including the exceptional cases. In section 6 we show that similar results can be derived for sigma-models based on compact symmetric spaces too.

Appendix A collects together useful data concerning Lie groups and symmetric spaces, while appendix B contains some arguments which supplement remarks made in section 3.

2 Sigma-models revisited

We first recall some well-known facts about the classical dynamics and canonical structure of a PCM, with or without a WZ term. It is convenient to use a light-cone canonical formalism, where the (real) light-cone coordinates $x = x^0 + x^1$ and $\bar{x} = x^0 - x^1$ play the role of ‘space’ and ‘time’ respectively; the corresponding derivatives will be written $\partial = \partial_x$.
and \( \bar{\partial} = \partial \). A sigma-model with target manifold \( G \) can be described in terms of a current with components \((j^a, \bar{j}^a)\) taking values in the Lie algebra \( \mathfrak{g} \) and satisfying equations of motion

\[
\bar{\partial} j^a = -\partial \bar{j}^a = \kappa f_{bc}^a j^b \bar{j}^c
\]

for some constant \( \kappa \). The structure constants are those appearing in the commutation relations \([t_a, t_b] = f_{abc} t_c\) for the generators of \( \mathfrak{g} \), and all Lie algebra indices are raised and lowered using the invariant inner-product \( \eta_{ab} \). (If \( G \) is non-compact or if \( G \) is compact but our basis is not orthonormal then the positions of the Lie algebra indices are important.)

The equal ‘time’ Poisson brackets can be written

\[
\{ j^a(x), j^b(y) \} = f_{ab}^c \bar{j}^c(x) \delta(x-y) + \eta^{ab} \delta'(x-y)
\]

where \( \bar{j}^a \) is a certain linear combination of \( j^a \) and \( \bar{j}^a \). If the coefficient of the WZ term is assigned the critical value necessary to define a WZW model then \( \bar{j}^a \) is proportional to \( j^a \) alone and (2.2) becomes a classical Kac-Moody algebra. More details concerning the general case can be found in e.g. [5].

It is a simple consequence of the equations of motion (2.1) that any current defined by (1.1) is conserved provided that \( k_{(m)}^{(a_1...a_m)} \) is an invariant tensor:

\[
k_{(a_1...a_{m-1})}^{(m)} f_{b)dc} = 0 \quad \Rightarrow \quad \bar{\partial} K_m = 0 .
\]

Now consider the Poisson bracket of two charges constructed from such currents

\[
\left\{ \int dx \, K_m(x), \int dy \, K_n(y) \right\} = \int dx \int dy \, k_{(m)}^{(a_1...a_m)} k_{(n)}^{(b_1...b_n)} \{ j^{a_1}(x) \ldots j^{a_m}(x), j^{b_1}(y) \ldots j^{b_n}(y) \} .
\]

In calculating this using (2.2), all contributions involving the term \( \delta(x-y) \) and the structure constants ultimately vanish, by invariance of each \( k \)-tensor. The \( \delta'(x-y) \) terms contribute a non-trivial integrand, however, and it is easy to check that this becomes a total derivative, implying that the charges commute, if and only if

\[
k_{(a_1...a_{m-1})}^{(m)} c k_{(b_1...b_{n-2})}^{(n)} b_{n-1} c = k_{(a_1...a_{m-1})}^{(m)} c k_{(b_1...b_{n-2})}^{(n)} b_{n-1} c .
\]

The principal result established in [1] is that this condition holds when the tensors \( k_{(m)}^{(a)} \) are defined by the formula (1.2) for each compact classical group. We emphasize that

\[\text{In [5] we considered explicitly only compact groups, and used a canonical formalism with } x^0 \text{ as time. These differences entail only very minor modifications of the discussion, however.}\]

\[\text{Actually, more general families are allowed for the groups } B_n \text{ and } C_n, \text{ in which } h \text{ is replaced by an arbitrary real parameter [1]. This possibility will not be of much concern to us here.}\]
both the conservation of the currents and the commutation of their charges are completely independent of the presence or absence of a WZ term in the model.

From this point on we will take $G$ to be compact unless we explicitly state otherwise (we will discuss briefly in the next section some aspects of Hamiltonian reduction, for which $G$ is required to be maximally non-compact). Our aim in the remainder of this section is to show that the validity, or otherwise, of the condition (2.4), and hence the question of commuting charges, can be settled essentially by restricting attention to a Cartan subalgebra. Although the restriction lemma which we shall establish is quite simple, it will prove very useful throughout the remainder of the paper. To explain the arguments properly, we first need to recall some standard results [10, 11] and introduce some notation.

Let $g$ be a compact Lie algebra and $g_0$ a Cartan subalgebra (CSA). We assume, with no loss of generality, that our basis $\{t_a\}$ for $g$ can be partitioned into bases $\{t_i\}$ for $g_0$ and $\{t_\alpha\}$ for its orthogonal complement $g^\perp_0$. Allowing complex linear combinations of generators, the latter may be chosen to consist of the usual Cartan-Weyl step operators corresponding to the non-zero roots of $g$. Recall that any $X = X^a t_a \in g$ is conjugate to some member of our chosen CSA, i.e. there exists $g \in G$ such that $gXg^{-1} \in g_0$. The remnant of $G$ which fixes the Cartan subalgebra (under the adjoint action) is the Weyl group, $W(G)$.

For a tensor $d^{(m)}$ of degree $m$ on $g$, and any $U, V, \ldots, Z \in g$ we write

$$d^{(m)}(U, V, \ldots, Z) = d^{(m)}_{a_1a_2\ldots a_m} U^{a_1} V^{a_2} \ldots Z^{a_m}$$

so that the components of the tensor can be expressed

$$d^{(m)}_{a_1a_2\ldots a_m} = d^{(m)}(t_{a_1}, t_{a_2}, \ldots, t_{a_m}) \quad (2.5)$$

(we shall not always indicate the degree of the tensor explicitly). Such a tensor is $G$-invariant if

$$d([T, U], V, \ldots, Z) + d(U, [T, V], \ldots, Z) + \ldots + d(U, V, \ldots, [T, Z]) = 0 \quad (2.6)$$

for all $T \in g$ (which coincides with the condition written earlier in (2.3)); or equivalently

$$d(gUg^{-1}, gVg^{-1}, \ldots, gZg^{-1}) = d(U, V, \ldots, Z) \quad (2.7)$$

for all $g \in G$. If $d$ is totally symmetric, then it is completely determined by specifying $d(X, \ldots, X)$ for all $X \in g$. From our earlier remarks, it follows that any symmetric
The restricted tensor is determined by its restriction to the CSA, this restricted tensor having components
\[ d_{t_{1}t_{2}...t_{m}}^{(m)} = d^{(m)}(t_{1}, t_{2}, \ldots, t_{m}) \]  
(2.8)
The restricted tensor is invariant under the Weyl group \( W(G) \).

A useful observation is that any symmetric invariant tensor satisfies
\[ d_{\alpha^{1}\alpha^{m-1}} = 0 \quad \text{or} \quad d(Y, X_{1}, \ldots, X) = 0 \quad \text{for} \quad X \in g_{0}, \quad Y \in g_{0}^{\perp}. \]  
(2.9)

This can be understood in a number of ways— for example, it is a consequence of the usual \( Z \)-grading of \( g \) defined by the choice of CSA and a set of simple roots: a particular component of an invariant tensor must vanish unless the total grade associated with all the indices it carries is zero.\(^4\) Alternatively, consider some specific \( Y = t_{\alpha} \in g_{0} \) corresponding to a non-zero root of \( g \). For each such element, we can choose \( T = t_{i} \in g_{0} \) (so \([T, X] = 0\)) such that \([T, Y] = \lambda Y \) with \( \lambda \neq 0 \). The invariance condition (2.6) with \( U = Y \) and \( V = \ldots = Z = X \) implies that (2.9) holds for this particular \( Y \), and the result follows for general \( Y \in g_{0}^{\perp} \) by linearity.

Following these remarks on general symmetric invariant tensors, we now focus specifically on the property necessary to construct commuting charges, and establish the following. 

Restriction lemma: totally symmetric, \( G \)-invariant tensors \( k^{(m)} \) and \( k^{(n)} \) on \( g \) satisfy the condition (2.4) if and only if their restrictions to the CSA \( g_{0} \) obey the analogous condition:
\[ k_{(i_{1}...i_{m-1})}^{(m)} \ell \quad k_{j_{1}...j_{n-2})j_{n-1} \ell}^{(n)} = k_{(i_{1}...i_{m-1})}^{(m)} \ell \quad k_{j_{1}...j_{n-1}}^{(n)} \ell, \]  
(2.10)
where \( \ell \) is a CSA index.

Proof: (2.4) holds iff
\[ k_{(a_{1}...a_{m-1})}^{(m)} c \quad k_{b_{1}...b_{n-2}b_{n-1}}^{(n)} c X^{a_{1}} \ldots X^{a_{m-1}} X^{b_{1}} \ldots X^{b_{n-2}} Y^{b_{n-1}} = k_{(a_{1}...a_{m-1})}^{(m)} c \quad k_{b_{1}...b_{n-2}b_{n-1}}^{(n)} c X^{a_{1}} \ldots X^{a_{m-1}} X^{b_{1}} \ldots X^{b_{n-2}} Y^{b_{n-1}} \]  
for any \( X, Y \in g \). Because the tensors are invariant, this is equivalent to
\[ k_{(i_{1}...i_{m-1})}^{(m)} c \quad k_{j_{1}...j_{n-2}b_{n-1}}^{(n)} c X^{i_{1}} \ldots X^{i_{m-1}} X^{j_{1}} \ldots X^{j_{n-2}} Y^{b_{n-1}} = k_{(i_{1}...i_{m-1})}^{(m)} c \quad k_{j_{1}...j_{n-2}b_{n-1}}^{(n)} c X^{i_{1}} \ldots X^{i_{m-1}} X^{j_{1}} \ldots X^{j_{n-2}} Y^{b_{n-1}} \]

\(^4\) The grade of any element of \( g \) is its eigenvalue under commutation with a specific element \( M \in g_{0} \); all members of the CSA therefore have grade zero, while step operators for the positive/negative simple roots are assigned grades \( \pm 1 \), by construction. The result follows from (2.6) with \( T = M \).
with $X \in g_0$ and $Y \in g$. Now (2.9) implies that one or other of the tensor factors on each side of this equation will vanish unless $Y \in g_0$ and the summation over $c$ is also restricted to the CSA. Hence, the original condition holds iff

$$k^{(m)}_{i_1 \ldots i_{m-1}} \ell k^{(n)}_{j_1 \ldots j_{n-1}} X^{i_1} \ldots X^{i_{m-1}} X^{j_1} \ldots X^{j_{n-2}} Y^{j_{n-1}}$$

$$= k^{(m)}_{(i_1 \ldots i_{m-1}} \ell k^{(n)}_{j_1 \ldots j_{n-2})j_{n-1}} X^{i_1} \ldots X^{i_{m-1}} X^{j_1} \ldots X^{j_{n-2}} Y^{j_{n-1}}$$

for all $X, Y \in g_0$. This is equivalent to (2.10), completing the proof.

3 Exceptional groups: a direct approach

In this section we investigate whether the formula (1.2) might be of use when applied to exceptional groups. To formulate this question properly, one must first choose a representation for the group to which the generators appearing in (1.2) will belong. To answer it, one must then determine whether the resulting currents $K_{s+1}$ share the properties of their counterparts for the classical algebras: (i) that they vanish identically unless $s$, the spin of the conserved charge, is equal to an exponent of the algebra modulo its Coxeter number $h$; (ii) that these conserved charges commute.

It is not obvious how to carry out calculations for an exceptional group $G$ in the same manner as was done for the classical families in [1]. We can, however, make use of these earlier results by combining them with the restriction lemma proved at the end of the last section. This lemma tells us that it is sufficient to calculate the quantities $K_m$ and their Poisson brackets when the underlying Lie algebra variables $j^a t_a$ are restricted to a CSA of $g$. More conveniently, we can choose a classical subgroup of maximal rank $H \subset G$, with Lie subalgebra $h \subset g$, and carry out all calculations assuming $j^a t_a$ belongs to $h$. Because $h$ contains a CSA of $g$, the restriction lemma ensures that these results will reveal all the information we seek. We must, of course, take into account that our chosen representation of $G$ will in general decompose into various irreducible representations of $H$.

Of the five exceptional groups, $G_2$ is certainly the simplest and also the one which most nearly possesses something like a defining representation, of dimension 7. A classical subgroup of maximal rank is $SU(3) \subset G_2$, with respect to which this representation decomposes $7 = 3 \oplus 3^{*} \oplus 1$. Following the strategy explained above, we consider a current

5The appropriate definition of $G_2$ is the automorphism group of the octonions, with the 7-dimensional space of pure-imaginary octonions furnishing the representation [11].
$j^a t_a$ belonging to the SU(3) subalgebra. Let us introduce the notation

$$A(x, \mu) = \det(1 - \mu j^a t_a) \quad \text{for } j^a t_a \text{ in } 3 \text{ of } SU(3),$$

$$= 1 - \mu^2 a(x) - \mu^3 b(x)$$

(3.1)

which defines $A(x, \mu)$, $a(x)$ and $b(x)$; consequently

$$A(x, -\mu) = \det(1 - \mu j^a t_a) \quad \text{for } j^a t_a \text{ in } 3^* \text{ of } SU(3),$$

$$= 1 - \mu^2 a(x) + \mu^3 b(x).$$

(3.2)

Now define

$$B(x, \mu) = \det(1 - \mu j^a t_a) \quad \text{for } j^a t_a \text{ in } 7 \text{ of } G_2,$$

$$= A(x, \mu) A(x, -\mu)$$

$$= (1 - \mu^2 a)^2 - \mu^6 b^2$$

(3.3)

where the second equality follows from the decomposition of representations given above. The Coxeter number for $G_2$ is $h = 6$, and so the formula (1.2) becomes

$$\mathcal{K}_{s+1} = B(x, \mu)^{s/6} \bigg|_{\mu^{s+1}}$$

(3.4)

where the expansion is to be taken in ascending powers of $\mu$. Now let us consider whether these currents have the desired properties.

Since $B(x, \mu)$ is an even polynomial in $\mu$, $\mathcal{K}_{s+1}$ is non-zero only if $s$ is an odd integer. In order to investigate in more detail which of these expressions are non-vanishing, it is convenient to write

$$B(x, \mu)^{s/6} = (1 - \mu^2 a)^{s/3} \left[1 - \mu^6 b^2 (1 - \mu^2 a)^{-2}\right]^{s/6}$$

$$= (1 - \mu^2 a)^{s/3} \sum_{p \geq 0} c_p \mu^{6p} b^{2p} (1 - \mu^2 a)^{-2p}$$

(3.5)

where $c_p$ are certain binomial coefficients, and the brackets appearing in each term of the sum are yet to be expanded as power series in $\mu$. Now, if $s = 6n + 3$ for some integer $n$, then

$$B(x, \mu)^{s/6} = \sum_{p \geq 0} c_p \mu^{6p} b^{2p} (1 - \mu^2 a)^{2n+1+2p},$$

(3.6)

and $\mathcal{K}_{s+1}$ is, by definition, the coefficient of $\mu^{s+1} = \mu^{6n+4}$. Because of the factor of $\mu^{6p}$ in this sum, those terms with $p > n$ produce powers of $\mu$ which are never less than $6n + 6$.

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6 In the notation of [1]: $2a = \mathcal{J}_2 = \text{Tr}(j^2)$ and $3b = \mathcal{J}_3 = \text{Tr}(j^3)$ for $j = j^a t_a$ an SU(3) current.
and so never contribute to $K_{s+1}$. On the other hand, for terms with $p \leq n$ the power of the bracket $(1 - \mu^2a)$ is a positive integer, and the highest power of $\mu$ which arises after its expansion is therefore $2(2n + 1 - 2p) + 6p = 4n + 2p + 2 < 6n + 4$, so these terms cannot contribute either. We conclude that $K_{s+1}$ vanishes for $s = 6n + 3$, while in general it will be non-vanishing for $s = 6n + 1$ or $s = 6n + 5$. This is precisely what we require, because the exponents of $G_2$ are 1 and 5. The formula (1.2) for the 7 representation of $G_2$ has therefore passed the first test.

The second test, that the non-trivial conserved charges commute, is even more stringent. We will now calculate the relevant Poisson brackets using similar methods to those of [1, 9]. It was shown in [1] (equation (4.14)) that the function $A(x, \mu)$ introduced above has Poisson brackets

$$\{A(x, \mu), A(y, \nu)\} = \mu^2\nu^2 \left[ \frac{\partial_x - \partial_y}{\mu - \nu} + \frac{\partial_\mu \partial_\nu}{3} \right] A(x, \mu) \partial_x (A(x, \nu) \delta(x-y))$$

$$+ \frac{\mu^2\nu^2}{(\mu - \nu)^2} \left[ \partial_x A(x, \mu) A(x, \nu) - A(x, \mu) \partial_x A(x, \nu) \right] \delta(x-y).$$

(3.7)

From this we can calculate

$$\{B(x, \mu), B(y, \nu)\} = \frac{2\mu^2
u^2}{\mu^2 - \nu^2} \left[ (\mu \partial_\mu - \nu \partial_\nu) B(x, \mu) \partial_x (B(x, \nu) \delta(x-y)) \right.$$

$$+ \frac{\mu^2 + \nu^2}{\mu^2 - \nu^2} \left( \partial_x B(x, \mu) B(x, \nu) - B(x, \mu) \partial_x B(x, \nu) \right) \delta(x-y) \right]$$

$$+ \frac{\mu^2\nu^2}{3} C(x, \mu) \partial_x (C(x, \nu) \delta(x-y)).$$

(3.8)

where

$$C(x, \mu) = \partial_\mu A(x, \mu) A(x, -\mu) - A(x, \mu) \partial_\mu A(x, -\mu)$$

$$= -2b\mu^2(3 - a\mu^2).$$

(3.9)

With the exception of the additional $C$-terms, the Poisson bracket (3.8) is again familiar from [1] (equation (4.25) with $\mu$ and $\nu$ replaced by $\mu^2$ and $\nu^2$ respectively).

To find the Poisson brackets of two conserved charges, we must evaluate

$$\int dx \int dy \left\{ B(x, \mu)^{s/6}, B(y, \nu)^{r/6} \right\}$$

(3.10)

in sufficient detail to extract the coefficient of $\mu^{s+1}\nu^{r+1}$. Following precisely the same arguments as in section 4.3 of [1], it can be shown that the $B$-terms on the right-hand-side
of (3.8) will not contribute to this result. The additional $C$-terms, however, produce an expression proportional to
\[
\int dx \mu^2 \nu^2 B(x, \mu)^{s/6-1} C(x, \mu) \partial_x (B(x, \nu)^{r/6-1} C(x, \nu)) .
\] (3.11)

Using (3.9), and extracting the relevant powers of $\mu$ and $\nu$, we find that the integrand contains a factor
\[
3B(x, \mu)^{(s-6)/6} \bigg|_{\mu^{s-3}} - a(x)B(x, \mu)^{(s-6)/6} \bigg|_{\mu^{s-5}} .
\] (3.12)

But this can be shown to vanish by expanding each term in the form (3.5) (with $s$ replaced by $s - 6$) and noting that
\[
(1 - a\mu^2)^{(2n-1)/3} \bigg|_{\mu^{2n+2}} = \frac{a}{3} (1 - a\mu^2)^{(2n-1)/3} \bigg|_{\mu^{2n}}
\] (3.13)

for any integer $n$ (if the general term in each expansion is labelled by $p$, we have set $s - 5 - 6p = 2n$). Hence, the charges commute.

We have shown that the formula (1.2), taking the seven-dimensional representation of $G_2$, defines currents with all the properties we require. The remaining exceptional groups can be investigated similarly. For example, there are convenient classical subgroups of maximal rank: $\text{SO}(9) \subset F_4$ and $\text{SO}(16) \subset E_8$. In these two instances, the representations of smallest dimension and their decompositions are $26 = 9 \oplus 16 \oplus 1$ and $248 = 120 \oplus 128$ respectively. In both these cases, however, the formula (1.2) fails the first test, because it is easy to check that there are non-trivial currents $K_{s+1}$ for which $s$ is not congruent (mod $h$) to an exponent. Furthermore, explicit computations for $F_4$ show that the currents $K_6$ and $K_8$ obtained from this formula do not yield commuting charges.

In summary, although the formula (1.2) works perfectly for $G_2$, the same cannot be said for the other exceptional groups. It would be interesting to investigate other representations of these groups, or possible modifications of the formula, but these are not issues that we shall pursue here. Some insight into the special nature of $G_2$ can be gained by regarding it as the subgroup of $\text{SO}(8)$ invariant under outer automorphisms. By exploiting the fact that these automorphisms become inner when $\text{SO}(8)$ is embedded in $F_4$, we can even use the $G_2$ tensors defined above to construct commuting quantities in the $F_4$ model. Since these arguments lie somewhat outside the main development of ideas in this paper, we shall present them in appendix B. It is, in any case, convenient to delay such discussion until after we have explained the connection between sigma-models and Toda theories, which will lead ultimately to a more uniform understanding of both the classical and exceptional cases.
4 Toda theories and DS/mKdV hierarchies

We shall be concerned with conformal Toda field theories (CTFTs) based on finite dimensional Lie algebras \( \mathfrak{g} \), and affine Toda field theories (ATFTs) associated with (untwisted) affine Kac-Moody algebras \( \hat{\mathfrak{g}} \). In either case, the Toda fields \( \phi^i \) take values in the CSA \( \mathfrak{g}_0 \subset \mathfrak{g} \). They obey classical equations of motion of the form

\[
\bar{\partial} \partial \phi^i = m^2 \sum_{\alpha \in R} \alpha^i \exp(\alpha \cdot \phi) \tag{4.1}
\]

where \( R \) is a certain subset of the roots (a dot denotes the inner-product \( \eta_{ij} \) on the CSA). For a CTFT, \( R \) is precisely the set of simple roots of \( \mathfrak{g} \), while for an ATFT, \( R \) contains in addition the lowest root of \( \mathfrak{g} \) (these are the simple roots of \( \hat{\mathfrak{g}} \) projected onto the CSA of the ‘horizontal’ subalgebra \( \mathfrak{g} \)). The conformal symmetry present in the former case means that the mass parameter \( m \) can be removed by shifting the fields, but for an ATFT the effect of the additional term involving the lowest root is to produce a minimum in the potential, resulting in a massive theory. We use the same light-cone canonical formalism as before, with \( \bar{x} \) as ‘time’ and \( x \) as ‘space’. The canonical Poisson brackets are identical in both CTFT and ATFT: introducing the variables \( u^i = \partial \phi^i \), the brackets are

\[
\{u^i(x), u^j(y)\} = \eta^{ij} \delta'(x-y). \tag{4.2}
\]

Each CTFT contains an infinite number of conserved currents which are differential polynomials in the quantities \( u^i \). They take the form

\[
\mathcal{W}_m = d_{i_1 i_2 \ldots i_m} u^{i_1} u^{i_2} \ldots u^{i_m} + \text{(derivative terms)} \quad \text{with} \quad \bar{\partial} \mathcal{W}_m = 0 \tag{4.3}
\]

where the ‘derivative terms’ are lower-order in the fields \( u \) but may also involve \( \partial u, \partial^2 u \), and so on. Under the Poisson bracket, these conformal currents form a classical \( \mathcal{W} \)-algebra.

Much progress in understanding the \( \mathcal{W} \)-algebra structure of CTFTs has come from regarding them as constrained WZW models [9], and we now recall in outline how this is done.

The construction starts with a WZW model based on a maximally non-compact group \( G \). Its Lie algebra is the real span of a set of Cartan-Weyl generators, and it can be decomposed according to the associated integer grading: \( \mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_- \), where \( \mathfrak{g}_\pm \) are the nilpotent subalgebras consisting of elements of positive/negative grade respectively. The corresponding nilpotent subgroups of \( G \) will be denoted by \( G_\pm \). The currents \( j \) in the WZW model satisfy \( \bar{\partial} j = 0 \) and obey a Kac-Moody algebra.
The WZW model is then modified by the addition of gauge fields so as to ensure invariance under an enlargement of the Kac-Moody symmetry, corresponding to gauging the nilpotent subgroups $G_{\pm}$. As in any gauge theory, only quantities invariant under the gauge transformations now have intrinsic meaning. A particularly important class of such objects are the gauge-invariant differential polynomials in the Kac-Moody currents, which take the form

$$d_{a_1a_2...a_m} j^{a_1} j^{a_2} ... j^{a_m} + \text{(derivative terms)} \quad (4.4)$$

It is shown in [9] that the tensor $d_{a_1a_2...a_m}$ must be $G$-invariant (even though we are considering gauge transformations involving just the nilpotent subgroups $G_{\pm}$). The additional derivative terms have a complicated structure which is also dictated by gauge invariance.

The Toda description of such a modified WZW model emerges on making a particular gauge choice which allows the sigma-model target-space to be parameterized by fields $\phi^i$ living in the CSA. In terms of these Toda fields, the expressions (4.4) reduce to the currents (4.3). An important consequence of this is that the tensor $d_{i_1i_2...i_m}$ is invariant under the Weyl group of $G$.

The holomorphic currents (4.3) can also be constructed directly in the CTFT. An elegant approach is to introduce a (pseudo)differential Lax operator $\mathcal{L}(u, \partial)$, of order $n$ say, which can be expanded in descending powers of derivatives:

$$\mathcal{L} = \sum_{m \geq 0} W_m \partial^{n-m} \quad (W_0 = 1) . \quad (4.5)$$

The operator is constructed so as to obey

$$[\bar{\partial}, \mathcal{L}] = 0 \quad \Rightarrow \quad \bar{\partial} W_m = 0 \quad (4.6)$$

and hence the coefficients in its expansion yield the desired conserved currents. For CTFTs based on certain algebras there is a particularly simple formula for the Lax operator [12]:

$$\mathcal{L} = \prod_{\lambda \in W} (\partial + \lambda \cdot u) \quad \text{for} \quad A_r, B_r, C_r \quad \text{and} \quad G_2 , \quad (4.7)$$

where $W$ is the set of weights of the defining representation, or the seven-dimensional representation for $G_2$, and the product is taken in the order lowest to highest weights from left to right. For other algebras, such an ordering is ambiguous and this complication manifests itself through the appearance of inverse powers of $\partial$. Thus, we have [12]

$$\mathcal{L} = \prod_{\lambda \in W_-} (\partial + \lambda \cdot u) \frac{1}{\partial} \prod_{\lambda \in W_+} (\partial + \lambda \cdot u) \quad \text{for} \quad D_r , \quad (4.8)$$
where $W_\pm$ are the strictly positive/negative weights of the fundamental representation, and each product is again ordered lowest to highest. Similar Lax operators have apparently not been calculated for the remaining exceptional algebras, although there is no obstacle to doing so in principle [12, 9].

We have now outlined two approaches to the construction of the infinite set of conserved currents in each TCFT. An important finite subset of these currents consists of the conformal primary fields for the W-algebra [12, 9]. These currents have spins $s+1$ where $s$ is an exponent of the Lie algebra $g$. Another important set of currents, less familiar from the point of view of conformal field theory perhaps, are those which give rise to commuting charges. It is known from the work of Drinfeld and Sokolov [7, 8] that there are infinitely many such currents; from our point of view these correspond to very special choices of the $d$-tensors in (4.3). Let us denote these currents

$$\mathcal{H}_m = h_{i_1 i_2 \ldots i_m} u^{i_1} u^{i_2} \ldots u^{i_m} + \text{(derivative terms)} .$$

(4.9)

Drinfeld and Sokolov establish the existence of a maximal set of commuting charges

$$H_s = \int dx \mathcal{H}_{s+1}$$

(4.10)

where the spins $s$ take values equal to the exponents of $g$ modulo the Coxeter number $h$. Note that there is a finite subset of commuting charges whose spins are exactly the exponents, but their currents do not coincide, in general, with the primary field currents of the W-algebra.

Thus far we have discussed conserved quantities in CTFT corresponding to a finite-dimensional algebra $g$. The ATFT based on $\hat{g}$ is obviously closely related: it has the same field content, and the classical equations of motion differ only by the addition of one term involving the lowest root. This extra term has the dramatic consequence of making the ATFT a massive theory and so the conformal conservation laws (4.3) of the CTFT will not survive in general. There are still infinitely many conserved quantities in each ATFT, however, and they involve precisely the quantities $\mathcal{H}_m$ introduced in (4.9) above. In ATFT these satisfy the modified conservation equations

$$\bar{\partial} \mathcal{H}_m + \partial \bar{\mathcal{H}}_m = 0$$

(4.11)

for certain $\mathcal{H}_m$ which will be complicated functions of the fields involving, in particular, exponentials of the lowest root. Fortunately, the exact expressions need not concern us, because in the light-cone canonical formalism $\mathcal{H}_m$ plays the role of the ‘space’ component
of the current, and hence the conserved charge constructed from (4.11) is given by precisely the same formulas (4.10) and (4.9) as before.

To summarize: the set of conserved charges in \( \hat{g} \) ATFT commute with one another, and they can be identified in a direct way with the maximal set of commuting charges contained within the W-algebra of \( g \) CTFT. The charges \( H_s \) regarded as functions of the fields \( u \) via (4.10) and (4.9) constitute precisely the Drinfeld-Sokolov generalizations of the mKdV hierarchy (mKdV being the simplest example, associated to the algebra \( A_1 \)). For summaries of this and much related material see e.g. [14].

Formulas for many of the DS currents \( H_m \) were derived in the original works [7, 8] and they can be expressed in terms of the Lax operators \( \mathcal{L} \) for CTFT given in (4.7) and (4.8). For each ATFT based on a classical group or on \( G_2 \), we introduce a related Lax operator \( \hat{\mathcal{L}}(u, \partial) \) of order \( h \) (the Coxeter number of the algebra) in terms of which

\[
H_{s+1} = \text{Res}(\hat{\mathcal{L}}^{s/h}).
\]  

(4.12)

The fractional power must be defined by means of an expansion in descending powers of the operator \( \partial \), and Res is an instruction to extract the residue, meaning the coefficient of \( \partial^{-1} \). (For background on such techniques for general pseudo-differential operators, see e.g. [13]). Specifically, we have for each algebra

\[
\hat{\mathcal{L}} = \mathcal{L} \quad \text{for} \quad A_r, C_r; \quad \hat{\mathcal{L}} = \mathcal{L} \partial^{-1} \quad \text{for} \quad B_r, D_r, G_2.
\]  

(4.13)

These definitions, in conjunction with (4.12), provide concrete expressions for all the commuting charges which arise in these models, with the exception of those associated with the Pfaffian-type invariants of \( D_r \). (Analogous expressions for these currents are apparently not known—see [7, 8].)

We conclude this section with one simple deduction from the beautiful results of Drinfeld and Sokolov. With the currents written in the form (4.9), let us consider the leading term, involving the largest number of fields \( u \) and the lowest number of derivatives, which arises when we calculate directly the Poisson bracket \( \{H_{m-1}, H_{n-1}\} \) using (4.2). Since we know this bracket vanishes, the entire integrand must be a total derivative, and its leading term must be a total derivative by itself (it is the unique term containing the largest possible number of fields \( u \)). It is straightforward to see that this requires

\[
h_{(i_1 \ldots i_{m-1})}^{(m)} h_{j_1 \ldots j_{n-1}}^{(n)} = h_{(i_1 \ldots i_{m-1})}^{(m)} h_{j_1 \ldots j_{n-2} j_{n-1}}^{(n)} \ell.
\]  

(4.14)

This is just the condition (2.10) encountered earlier. We shall make the connection explicit in the next section.
5 DS hierarchies and PCM/WZW models

In each PCM or WZW model, we have conserved commuting charges based on $G$-invariant $k$-tensors satisfying (2.4). We showed that this condition is valid iff it holds when the tensors are restricted to the Cartan subalgebra. From direct calculation, the condition is known to be satisfied when the $k$-tensors are defined by (1.2) for each of the classical algebras (in their defining representations) and for $G_2$ (in its seven-dimensional representation). In conformal or affine Toda theory, on the other hand, we have a set of mutually commuting charges given by the Drinfeld-Sokolov construction for any Lie algebra $g$. The $h$-tensors on the CSA which appear in (4.9) are Weyl-invariant, and must satisfy (4.14) in order that the corresponding charges commute.

There is an obvious way to relate the two pictures: extend the $h$-tensors of the DS hierarchies to tensors $h_{a_1a_2...a_m}$ on each compact Lie algebra $g$. Specifically, for any $X \in g$, we choose $g \in G$ such that $gXg^{-1} \in g_0$ and set

$$h(X, \ldots, X) = h(gXg^{-1}, \ldots, gXg^{-1})$$

(5.1)

to define a totally symmetric extension of the tensor from $g_0$ to $g$. This extension is unambiguous because the choice of $g$ is unique up to elements which fix the CSA, but the (adjoint) actions of such elements on $g_0$ constitute the Weyl group, under which each $h$-tensor is invariant. The extended $h$-tensor is $G$-invariant on $g$, by construction. By the restriction lemma, (4.14) is sufficient to ensure that our extended tensors can be used to define commuting charges in the PCM based on $G$, with currents $h_{a_1a_2...a_m}j^{a_1}j^{a_2}...j^{a_m}$.

In this manner, the DS/mKdV hierarchies ensure the existence of commuting sets of charges in any sigma-model with target space a compact Lie group $G$, whether classical or exceptional. It remains to reconcile this new definition with our old definition, in terms of $k$-tensors given by (1.2), however. To achieve this we must show that any concrete expressions which are available for both the $k$-tensors and $h$-tensors agree up to irrelevant overall constants.

In the formulas (4.12) the differential operators $\partial$ in every factor of the Lax operator generate a large number of terms. But to extract the $h$-tensor for each current we are concerned only with the leading, non-derivative terms, as written in (4.9). Discarding the derivative terms is equivalent to neglecting the action of each operator $\partial$ on all fields $u$ standing to its right. This can be achieved simply by replacing $\partial$ in the definition by a
parameter 1/µ, say, so that

\[ h_{i_1 i_2 \ldots i_n} u^{i_1} u^{i_2} \ldots u^{i_n} = \hat{L}(u, 1/\mu)^{s/h}|_{\mu} \]  

(5.2)

The choice of parameter is natural if we recall that the expansion inherent in the definition (4.12) must be carried out in descending powers of \( \partial \), which now corresponds to ascending powers of \( \mu \).

The formulas (4.13) simplify and unify after replacing \( \partial \) by 1/\( \mu \): for each classical algebra and for \( G_2 \), we find that

\[ \hat{L}(u, 1/\mu) = \mu^{-h} \prod_{\lambda \in W} (1 + \mu \lambda \cdot u) \]  

(5.3)

where \( W \) is the set of weights of the defining representation. Notice that, because the factors no longer involve operators, their ordering in the product is now irrelevant. Notice also that the overall power of \( \mu \) corresponds to the fact that the order of \( \hat{L} \) is always \( h \).

Substituting this expression into the previous formula above, and taking account of the overall power of \( \mu \) that results, we find

\[ h_{i_1 i_2 \ldots i_n} u^{i_1} u^{i_2} \ldots u^{i_n} = \left( \prod_{\lambda \in W} (1 + \mu \lambda \cdot u) \right)^{s/h}|_{\mu^{s+1}} \]

\[ = \det(1 + \mu u^{i} t_{i})^{s/h}|_{\mu^{s+1}} \]  

(5.4)

The last equality follows because the weights are, by definition, the eigenvalues of the CSA generators (in the defining representation in this case). This final formula for the \( h \)-tensors clearly coincides, up to some irrelevant overall constants, with the definition (1.2) for the \( k \)-tensors when \( j \) is restricted to the CSA.

6 Symmetric space sigma-models

The last topic we shall discuss is that of sigma-models on compact symmetric spaces \( G/H \). It was shown in [6] that commuting families of charges, with characteristic patterns of spins, exist for each such model with \( G \) and \( H \) classical groups. The approach we have developed in this paper is sufficient to extend the analysis to all symmetric space sigma-models, and we now indicate briefly how this can be done (some routine details will be omitted in view of the strong similarities with the preceding discussions of Lie groups).

Most of the equations discussed above for PCM/WZW models carry over immediately to symmetric space sigma-models with appropriate re-interpretations of symbols. For a
symmetric space $G/H$, we have an orthogonal decomposition of the Lie algebra $g = h + p$, say. The dynamical variables of the $G/H$ sigma-model are currents $j^a t_a$, where $\{t_a\}$ is a basis for $p$. The conserved currents in this theory take the familiar form $k_{a_1...a_m} j^{a_1} \cdots j^{a_m}$ but the symmetric $k$-tensor which appears must now be $H$-invariant on $p$. (In general, an $H$-invariant tensor $d$ on $p$ satisfies (2.6) for any $T \in h$ and $U, V, \ldots, Z \in p$.) The condition for the corresponding conserved charges to commute is then still given by (2.4).

The analysis of [6] relied on the observation that when $G$ and $H$ are classical groups, every $H$-invariant tensor on $p$ arises as the restriction of some $G$-invariant tensor on $g$. This is not true for symmetric spaces involving exceptional groups, however [15] (see also [16], where this point proved relevant). To deal with these exceptional cases, we will follow the route developed in this paper for Lie groups: starting from Weyl-invariant tensors on a Cartan subalgebra and then extending them in an invariant fashion to construct the desired commuting charges.

A CSA for a compact symmetric space $G/H$ is a maximal set of commuting generators $p_0 \subset p$; this space is unique up to conjugation by elements of $H$ and the rank of $G/H$ is defined to be the dimension of $p_0$. Let $\{t_i\}$ be a basis for $p_0$, and $\{t_a\}$ a basis for $p_0^\perp$, its orthogonal complement in $p$. Any $X \in p$ is conjugate by some $h \in H$ to a member of this CSA: $hXh^{-1} \in p_0$ [10]. The residual $H$-transformations which fix $p_0$ constitute the Weyl group, which we shall denote $W(G/H)$.

As in the case of Lie groups, one can introduce the idea of a root system for a symmetric space, and then summarize much of this information by means of a diagram which encodes the properties of a basis of simple roots. It turns out that the diagram for any symmetric space $G/H$ coincides with the Dynkin diagram for some simple Lie group $K$, say [10]. Moreover, rank($K$) = rank($G/H$) and $W(K) = W(G/H)$. We list the compact symmetric spaces $G/H$ and their diagrams, as given by $K$, in appendix A.

Now, any $H$-invariant symmetric tensor on $p$ is clearly determined by its restriction to the CSA, $p_0$. Furthermore, any tensor on $p_0$ which is invariant under $W(G/H)$ can be extended uniquely to an $H$-invariant tensor on $p$. A family of Weyl-invariant tensors $h_{i_1...i_m}$ on $p_0$ is provided by the DS construction for the group $K$, and we can therefore extend these to $H$-invariant tensors on $p$ to define conserved currents in the $G/H$ sigma-model. It remains to show that the conserved charges constructed in this manner really commute. We know that the $h$-tensors satisfy (4.14), but we must promote this to the condition (2.4) on $p$ which means that we must generalize the restriction lemma of section 2 from Lie
groups to symmetric spaces.

Following the same method as before, the proof of the restriction lemma will generalize to symmetric spaces if each $H$-invariant tensor $d$ on $p$ satisfies

$$d_{i_1...i_{n-1}i_0} = 0 \quad \text{or} \quad d(X, ..., X, Y) = 0 \quad \text{for} \quad X \in p_0, \ Y \in p_0^\perp. \quad (6.1)$$

It is shown in [10] (Chapter 7, Lemma 2.3) that one can choose pairs of generators $t_\alpha \in p_0^\perp$ (the basis introduced above) and $s_\alpha \in h$ such that, for each $X \in p_0$, $[X, t_\alpha] = \lambda s_\alpha$ and $[X, s_\alpha] = \lambda t_\alpha$ for some number $\lambda(X, \alpha)$. Now fix $X$ and consider $Y = t_\alpha$. If $\lambda(X, \alpha) \neq 0$, then (6.1) follows from (2.6) with $U = V = ... = Z = X$ and $T = s_\alpha$. But if $\lambda(X, \alpha) = 0$, there exists some other $X' \in p_0$ with $\lambda(X', \alpha) \neq 0$ (because $p_0$ is a CSA) and (6.1) then follows from (2.6) with $U = X'$, $V = ... = W = X$ and $T = s_\alpha$. This completes the proof.

7 Summary and Comments

Using the results of Drinfeld and Sokolov, we have shown that there exist commuting charges in any PCM (or WZW model) based on a compact Lie group $G$, and that these charges have spins given by the exponents of the group modulo its Coxeter number. We have also established analogous results for each sigma-model based on a compact symmetric space $G/H$, with the spins of the conserved charges given by the exponents of a related Lie group $K$ whose Dynkin diagram also encodes the root structure of $G/H$ (see appendix A). These results extend the work of [1, 6] for classical groups and symmetric spaces to include all the exceptional cases.

Our construction involves a direct algebraic correspondence between the conserved, commuting charges in ATFTs, as given by the Drinfeld-Sokolov hierarchies, and those appearing in PCMs. Although this correspondence is formally very similar to the process of Hamiltonian reduction using gauged WZW models (reviewed briefly in section 4) we should be clear about how these procedures differ. Unlike the well-known WZW-CTFT connection, our construction neither involves nor requires any dynamical relationship between PCMs (or WZW models) and ATFTs. It is possible that some such relationship might be established, by carrying out a non-conformal reduction of WZW models for instance. Investigations of this kind have already been considered in the literature, but their precise status remains rather unclear at present [17].

As part of our account, we have compared the detailed expressions (4.12), given by
Drinfeld and Sokolov for ATFT charges, with the formula (1.2), originally introduced in [1] for sigma-models, and found complete agreement in all cases where both are applicable. This includes all conserved charges for models based on classical groups or on $G_2$, with the exception of those charges associated with the Pfaffian invariant and its generalizations for the groups $D_n$. In a sigma-model, the Pfaffian-type currents can still be extracted from (1.2) provided this formula is interpreted appropriately (see the account in [1]) but there are apparently no explicit results for the corresponding Drinfeld-Sokolov or ATFT currents. For the other exceptional groups, beyond $G_2$, there are no known formulas of either type. It would be interesting to rectify this.

One can regard sigma-models and Toda theories as two rather different ways of introducing interactions amongst sets of free fields whilst maintaining integrability. An important message which is already familiar from hamiltonian reduction is that these two broad classes of models are much more closely related than might initially be supposed. Our results reinforce this in a precise sense: they reveal that the local commuting charges in both sigma-models (PCMs, WZW models, or symmetric space models) and Toda theories (conformal or affine) are based on precisely the same sets of Weyl-invariant tensors.

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Appendix A: Data for Lie groups and symmetric spaces

Table 1: Lie groups, exponents and Coxeter numbers

| Lie group $G$ or Lie algebra $g$ | exponents                  | Coxeter number $h$ |
|---------------------------------|----------------------------|--------------------|
| $A_n = \text{SU}(n+1)$          | $1, 2, \ldots, n$          | $n+1$              |
| $B_n = \text{SO}(2n+1)$         | $1, 3, \ldots, 2n-1$       | $2n$               |
| $C_n = \text{Sp}(2n)$           | $1, 3, \ldots, 2n-1$       | $2n$               |
| $D_n = \text{SO}(2n)$           | $1, 3, \ldots, 2n-3; n-1$  | $2n-2$             |
| $E_6$                           | $1, 4, 5, 7, 8, 11$         | $12$               |
| $E_7$                           | $1, 5, 7, 9, 11, 13, 17$    | $18$               |
| $E_8$                           | $1, 7, 11, 13, 17, 19, 23, 29$ | $30$       |
| $F_4$                           | $1, 5, 7, 11$              | $12$               |
| $G_2$                           | $1, 5$                     | $6$                |
Table 2(a): Symmetric spaces based on classical groups

| Symmetric space $G/H$ | Simple root system of type $K$ |
|-----------------------|-------------------------------|
| $SU(n+m)/S(U(n) \times U(m))$ $(n < m)$ | $B_n$ |
| $SU(2n)/S(U(n) \times U(n))$ | $C_n$ |
| $SO(n+m)/SO(n) \times SO(m)$ $(n < m)$ | $B_n$ |
| $SO(2n)/SO(n) \times SO(n)$ | $D_n$ |
| $Sp(2n+2m)/Sp(2n) \times Sp(2m)$ $(n < m)$ | $B_n$ |
| $Sp(4n)/Sp(2n) \times Sp(2n)$ | $C_n$ |
| $SU(n)/SO(n)$ | $A_{n-1}$ |
| $Sp(2n)/U(n)$ | $C_n$ |
| $SO(4n)/U(2n)$ | $C_n$ |
| $SO(4n + 2)/U(2n + 1)$ | $B_n$ |
| $SU(2n)/Sp(2n)$ | $A_{n-1}$ |
Table 2(b): Symmetric spaces based on exceptional groups

| Symmetric space $G/H$ | Simple root system of type $K$ |
|-----------------------|-------------------------------|
| $E_6/\text{Sp}(8)$    | $E_6$                         |
| $E_6/\text{SU}(6)\times\text{SU}(2)$ | $F_4$ |
| $E_6/\text{SO}(10)\times\text{U}(1)$ | $G_2$ |
| $E_6/F_4$             | $A_2$                         |
| $E_7/\text{SU}(8)$    | $E_7$                         |
| $E_7/\text{SO}(12)\times\text{SU}(2)$ | $F_4$ |
| $E_7/E_6\times\text{U}(1)$ | $C_3$ |
| $E_8/\text{SO}(16)$   | $E_8$                         |
| $E_8/E_7\times\text{SU}(2)$ | $F_4$ |
| $F_4/\text{Sp}(6)\times\text{SU}(2)$ | $F_4$ |
| $F_4/\text{SO}(9)$    | $A_1$                         |
| $G_2/\text{SU}(2)\times\text{SU}(2)$ | $G_2$ |
Appendix B: Folding, $G_2$, SO(8) and $F_4$

Consider a Lie algebra $\mathfrak{g}$ and an automorphism $\sigma$ of order $n$. There is a homomorphism $\pi : \mathfrak{g} \to \mathfrak{g}^\sigma$, the $\sigma$-invariant subalgebra, given by $\pi(X) = X + \sigma(X) + \ldots + \sigma^{n-1}(X)$. Suppose $\sigma$ represents an outer automorphism and so corresponds to a non-trivial symmetry of the Dynkin diagram of $\mathfrak{g}$ with $n = 2$ or 3. Identifying simple roots of $\mathfrak{g}$ under this symmetry yields the Dynkin diagram for $\mathfrak{g}^\sigma$, a process that is commonly referred to as ‘folding’ [4]. Applying this to simply-laced algebras of types A or D using their outer automorphisms of order 2 yields non-simply-laced algebras of types B or C. Folding $D_4 = \text{SO}(8)$ using an automorphism of order 3 yields $G_2$, the only exceptional group which can be constructed in this fashion.

There is a one-to-one correspondence between (Ad)-invariant tensors on $\mathfrak{g}^\sigma$ and (Ad)-invariant tensors on $\mathfrak{g}$ which are additionally invariant under $\sigma$. In one direction this correspondence is given by restricting the tensor to the subalgebra (pulling back by the inclusion map) while in the other direction it is given by composing the tensor with the map $\pi$ in the obvious sense (pulling back using $\pi$). Furthermore, if we have a family of such $\sigma$-invariant tensors $k^{(m)}$ on $\mathfrak{g}$ which satisfy the key condition (2.4), then it is not difficult to show (using arguments similar to those in the proof of the restriction lemma in section 2) that the corresponding tensors on $\mathfrak{g}^\sigma$ satisfy (2.4) on this subalgebra, and vice versa.

Now consider $G_2$ obtained by folding $\text{SO}(8)$ using $\sigma$ of order 3. Our calculations in section 3 establish the existence of a family of tensors $k^{(m)}$ on $G_2$ satisfying (2.4) with $m = 2, 6 \pmod{6}$. By the remarks above, these can also be regarded as $\sigma$-invariant tensors on $\text{SO}(8)$. (They do not coincide with the $\text{SO}(8)$-tensors defined by (1.2), however, which are not $\sigma$-invariant in general.) But $\text{SO}(8)$ is a subgroup of $F_4$ of maximal rank; moreover $F_4$ is the minimal group in which the outer automorphisms of $\text{SO}(8)$ become inner, and $W(F_4)$ is a semi-direct product of $W(\text{SO}(8))$ with the permutation group $S_3$ of outer automorphisms [11]. Since (Ad)-invariant tensors are determined by their values on a CSA, and the remnant of the (Ad)-action on the CSA is the Weyl group, we see that the tensors $k^{(m)}$ on $\text{SO}(8)$ can further be identified, via the common CSA, with (Ad)-invariant tensors on $F_4$. The restriction lemma of section 2 ensures that these tensors still obey (2.4) and thus define commuting charges in the $F_4$ model. Notice also that the degrees of these tensors can be re-written as $m = 2, 6, 8, 12 \pmod{12}$, exactly as expected. This provides us with a convenient method for constructing the tensors on $F_4$ that we seek, although it does not alter the fact that these apparently cannot be derived by applying (1.2) directly.
The folding construction is widely used in Toda theory [4] and the outer automorphisms, or Dynkin diagram symmetries, correspond directly to discrete symmetries of the Toda lagrangian. The Lax operators of the simple form (4.7) and (4.13) for G₂ can be derived directly from those for SO(8) by using this observation.

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