Mean Square Stability of Model Based Networked Control Systems with Multiplicative Noise

Wataru Toriumi and Yasumasa Fujisaki
Department of Information and Physical Sciences,
Graduate School of Information Science and Technology,
Osaka University,
1–5 Yamadaoka, Suita, Osaka 565–0871, Japan
E-mail: {wataru-toriumi, fujisaki}@ist.osaka-u.ac.jp

Abstract

This paper presents a necessary and sufficient condition of mean square stability of a model based networked control systems (MB-NCS) with multiplicative noise, where a linear discrete time plant is controlled by using a model state feedback. The lifting approach is employed, and the original time varying system is recast as an equivalent time invariant system whose mean square stability is investigated. The necessary and sufficient condition derived here is given as a set of a linear matrix inequality and linear equations, which enjoys an advantage on numerical computation.

1 Introduction

Communication cost is a crucial issue for control systems over networks. If sensor data is sent through the network at each sampling time, the network bandwidth required for control could be very large. Thus, the other operations over the network could be interrupted and/or deteriorated due to packet collisions and networked induced delays. In order to obtain a better trade-off between control and the other operations, we need a networked control system having low sampling rate. This motivates us to utilize the model based networked control systems (MB-NCS) [1], where the sensor data is sent not every time but once every $h$ times. The missing data is generated in the controller by using a plant model, and thus the control input is applied for the plant every time. The concept of MB-NCS is shown in Fig.1.

In this paper, we investigate the state feedback MB-NCS under the situation that stochastic noises exist. In particular, here we introduce a multiplicative noise in the problem setup, which is consistent with several practical settings [2], where large signal contains large noise, while small signal contains small noise. Notice also that the existing literature [1, 3] only treats the deterministic (noise free) case and/or a standard additive noise case whose stability analysis is rather straightforward.

We first represent the MB-NCS having time varying nature as a time invariant system with the help of the so-called lifting technique. We then define mean square stability of this system. The main result of this paper is a necessary and sufficient condition for mean square stability of the system, where the condition is given as a set of a linear matrix inequality and linear equations. Thus we can easily analyze stability by standard numerical computational software. The proof of the result basically follows the literature [2, 4] which treats multiplicative noise, though several detailed points of the proof are different in order to handle a compact and transparent representation of stability condition.

2 Problem Formulation

Let us consider a linear discrete time plant

$$x(k + 1) = Ax(k) + Bu(k) + Dw(k), \quad k = 0, 1, \ldots ,$$

where $x(k) \in \mathbb{R}^n$ is the plant state, $u(k) \in \mathbb{R}^m$ is the control input, and $w(k) \in \mathbb{R}^r$ is the stochastic noise defined later.

For this plant, we employ a controller which consists of a plant model, a model state update based on the plant state measurement, and a model state feedback, i.e.,

$$\begin{align*}
\hat{x}(\ell h + j) &= \hat{A} \hat{x}(\ell h + j - 1) + \hat{B} u(\ell h + j - 1), \\
\hat{x}(\ell h) &= x(\ell h), \quad j = 1, 2, \ldots , h - 1, \quad \ell = 0, 1, \ldots , \\
u(\ell h) &= K \hat{x}(\ell h)
\end{align*}$$

where $\hat{x}(k) \in \mathbb{R}^n$ is the plant model state and $h \in \mathbb{N}$ is the update period. That is, the sensor data is sent not every time but once every $h$ times, which means that low sampling rate
is realized for sensor networks. The matrices $A$ and $B$ describe the plant, while the matrices $\hat{A}$ and $\hat{B}$ describe the plant model. That is, the difference of these matrices represents a modeling error. We remark that, if $h = 1$, it turns out that $u(k) = Kx(k)$ where $K$ is the state feedback gain. That is, the model state feedback (2) works as the standard state feedback in the ideal situation.

The above is the basic set-up of the state feedback MB-NCS, where $w(k)$ has been assumed to be zero or additive noise [1]. That is, the deterministic case and a simple stochastic case have been considered. However, the noise could be multiplicative in practical situations as is pointed out in the context of finite-signal-to-noise model [2]. In fact, a signal with large energy carries noise with large energy, which is a natural setting from an engineering point of view.

The objective of this paper is to establish a stability condition for the state feedback MB-NCS having multiplicative and additive stochastic noises. Here we define the stochastic noise $w(k)$ as

$$w(k) = p(k)x(k) + q(k), \quad k = 0, 1, \ldots, \tag{3}$$

where $p(k) \in \mathbb{R}$ denotes multiplicative noise and $q(k) \in \mathbb{R}^n$ denotes additive noise. They are independent and identically distributed random variables with

$$\mathbb{E}[p(k)] = 0, \quad \mathbb{E}[p(k)^2] = \sigma^2 \geq 0,$$
$$\mathbb{E}[q(k)] = 0, \quad \mathbb{E}[q(k)q^T(k)] = Q \geq 0,$$
$$\mathbb{E}[p(k) + iq(k)] = 0$$

for any $i$ and $k$. The initial state $x(0)$ is independent of the processes $p$ and $q$, where we define

$$X_0 = \mathbb{E}[x(0)x^T(0)] \geq 0.$$

We remark that the noise $w(k)$ becomes the standard additive noise if we set $\sigma = 0$. That is, the situation treated in the existing literature [1] is a special case of the setting that we consider here.

### 3 Main Result

Since the controller contains the model state update once every $h$ times, the state feedback MB-NCS is a linear periodically time varying system. In fact, the plant model state can be represented as

$$\hat{x}(th + j) = (\hat{A} + \hat{B}K)^j x(th), \quad j = 0, 1, \ldots, h - 1,$$

and the overall system can be described as a linear time invariant system if we take the so-called lifting approach.

We first define the lifted signals

$$\chi_j(t) = x(th + j), \quad w_j(t) = w(th + j), \quad j = 0, 1, \ldots, h - 1.$$

Then we have the lifted system which is a linear time invariant system described by

$$\chi_j(t + 1) = F_h \chi_j(t) + \sum_{j = 0}^{h-1} G_{h-1-j} w_j(t),$$
$$\chi_j(t) = F_i \chi_j(t) + \sum_{j = 0}^{i} G_{i-j} w_j(t), \quad i = 0, 1, 2, \ldots, h - 1, \quad \ell = 0, 1, \ldots, \tag{4}$$

where

$$F_i = A^i + \sum_{j = 0}^{i-1} A^{i-j} BK (\hat{A} + \hat{B}K)^j, \quad G_i = A^i D,$$

in which the summation with no summands is zero. That is, $x(th)$ is now the state of this time invariant system, while we regard the other $x(th + i)$, $i = 1, 2, \ldots, h - 1$, as the outputs.

Let us define the correlation matrices of the state and outputs of the lifted system as

$$X(t) = \mathbb{E}[\chi_j(t)\chi_j^T(t)], \quad Z_j(t) = \mathbb{E}[\chi_j(t)w_j^T(t)], \quad j = 1, 2, \ldots, h - 1.$$

We also use the definition

$$Z_0(t) = X(t)$$

for simplifying expressions. Note that we have the correlation matrices of the noise $w_j(t)$ as

$$\mathbb{E}[w_j(t)w_j^T(t)] = \sigma^2 Z_j(t) + Q, \quad j = 0, 1, \ldots, h - 1.$$

Then the correlation matrices $X(t)$ and $Z_j(t), i = 0, 1, \ldots, h - 1$, satisfy a difference equation define by a set of matrix equations

$$X(t + 1) = F_h X(t)F_h^T + \sum_{j = 0}^{h-1} G_{h-1-j} \left( \sigma^2 Z_j(t) + Q \right) G_{h-1-j}^T,$$
$$Z_j(t) = F_i X(t)F_i^T + \sum_{j = 0}^{i-1} G_{i-j} \left( \sigma^2 Z_j(t) + Q \right) G_{i-j}^T, \quad j = 0, 1, \ldots, h - 1,$$
$$X(0) = X_0, \quad i = 0, 1, \ldots, h - 1. \tag{5}$$

We say that the lifted system (4) is mean square stable if the steady correlation matrix of $\chi_j(t)$ exists and is finite regardless of $\chi_0(0) = x(0)$, i.e., there exists $X_0$ such that

$$\lim_{t \to \infty} X(t) = X_0$$

for any $X_0 \geq 0$. Notice here that (5) implies that, if such a $X_0$ exists, there also exist

$$\lim_{t \to \infty} Z_j(t) = Z_j, \quad j = 0, 1, \ldots, h - 1.$$

That is, if the lifted system (4) is mean square stable, we see that the correlation matrices of the state of the original system (1), (2), and (3) are bounded for any $k$.

Here we state the main result of this paper.
Theorem 1 The lifted system (4) is mean square stable if and only if there exists a set of symmetric and positive definite/semidefinite matrices $X = X^T > 0$ and $Z_i = Z_i^T \geq 0$, $i = 0, 1, \ldots, h - 1$, such that

$$-X + F_n X F_n^T + \sigma^2 \sum_{j=0}^{h-1} G_{h-j} Z_j G_{h-j}^T < 0,$$

$$-Z_i + F_i X F_i^T + \sigma^2 \sum_{j=0}^{i-1} G_{i-j} Z_j G_{i-j}^T = 0,$$

$i = 0, 1, \ldots, h - 1.$ 

(6)

Proof: Notice here that the matrix equations in Theorem 1 can be solved successively for a given $X$. In fact, $Z_i$ is determined by $X$ and $Z_j$, $j = 0, 1, \ldots, i - 1$, where $Z_0 = X$. Since all of $Z_i$ can be expressed by $X$, we can obtain a condition equivalent to (6) as the form

$$-X + \sum_{j=1}^{2^h} H_j X H_j^T = -S,$$

(7)

where $H_j$, $j = 1, 2, \ldots, 2^h$ are appropriate matrices and $S$ is a certain symmetric and positive definite matrix.

With the same $H_i$, we can also rewrite the difference equation (5) as

$$X(\ell + 1) = \sum_{j=1}^{2^h} H_j X(\ell) H_j^T + V, \quad X(0) = X_0,$$

where $V$ is a symmetric and positive semidefinite matrix. If the lifted system (4) is mean square stable, the steady correlation matrix $X_\infty \geq 0$ exists for any $X_0 \geq 0$ and it satisfies

$$X_\infty = \sum_{j=1}^{2^h} H_j X_\infty H_j^T + V.$$

Then we see that $\tilde{X}(\ell) = X(\ell) - X_\infty$ satisfies a difference equation

$$\tilde{X}(\ell + 1) = \sum_{j=1}^{2^h} H_j \tilde{X}(\ell) H_j^T.$$  

(8)

We further see that

$$\lim_{\ell \to \infty} \tilde{X}(\ell) = 0$$

(9)

for any $X_0 \geq 0$, where $\tilde{X}(0) = X_0 - X_\infty$. The above gives another representation of mean square stability of the lifted system (4).

We first prove that the condition (7) is necessary for mean square stability of the lifted system (4).

Suppose that the lifted system (4) is mean square stable. Then we can define

$$X = \sum_{j=0}^{\infty} \tilde{X}(i).$$

Summing up (8) from $\ell = 0$ to infint, we have

$$X - \tilde{X}(0) = \sum_{j=1}^{2^h} H_j X H_j^T,$$

that is,

$$-X + \sum_{j=1}^{2^h} H_j X H_j^T = -\tilde{X}(0).$$

We therefore see that this $X$ satisfies (7) and is symmetric and positive definite for $\tilde{X}(0) > 0$.

We next prove that the condition (7) is sufficient for mean square stability of the lifted system (4).

Let us introduce one more difference equation

$$\tilde{Y}(\ell + 1) = \sum_{j=1}^{2^h} H_j^T \tilde{Y}(\ell) H_j.$$  

(10)

We see that (8) achieves (9) for any symmetric $\tilde{X}(0) \in \mathbb{R}^{n \times n}$ if and only if (10) achieves

$$\lim_{\ell \to \infty} \tilde{Y}(\ell) = 0$$

(11)

for any symmetric $\tilde{Y}(0) \in \mathbb{R}^{n \times n}$. In fact, both difference equations can be represented as

$$\text{vec}(\tilde{X}(\ell + 1)) = R \text{vec}(\tilde{X}(\ell)), \quad \text{vec}(\tilde{Y}(\ell + 1)) = R^T \text{vec}(\tilde{Y}(\ell)).$$  

(12) \hspace{1cm} (13)

$$R = \sum_{j=1}^{2^h} D_n^*(H_j \otimes H_j) D_n,$$

where $\otimes$ is the Kronecker product and $\text{vec}$ and $D_n$ are define with the standard vec operator. That is, for a given matrix $A \in \mathbb{R}^{n \times n}$, we define

$$\text{vec}(A) = \begin{bmatrix} a_{11} & \cdots & a_{1n} & a_{12} & \cdots & a_{2n} & \cdots & a_{nn} \end{bmatrix}^T \in \mathbb{R}^{n^2}.$$

Similarly, for a given symmetric matrix $P = P^T \in \mathbb{R}^{n \times n}$, we define

$$\text{vec}(P) = \begin{bmatrix} p_{11} & \cdots & p_{1n} & p_{21} & \cdots & p_{2n} & \cdots & p_{nn} \end{bmatrix}^T \in \mathbb{R}^{n(n+1)/2}.$$

Then we introduce a duplicate matrix $D_n \in \mathbb{R}^{n^2 \times n(n+1)/2}$ as

$$\text{vec}(P) = D_n \text{vec}(P).$$

Its pseudo-inverse $D_n^* \in \mathbb{R}^{n(n+1)/2 \times n^2}$ has the properties:

$$\text{vec}(P) = D_n^* \text{vec}(P), \quad D_n^* D_n = I \in \mathbb{R}^{n(n+1)/2 \times n(n+1)/2},$$

$$\text{rank}(D_n) = \text{rank}(D_n^*) = \frac{n(n + 1)}{2}.$$
In the rest of this proof, we show that (10) achieves (11) for any symmetric and positive semidefinite $\tilde{Y}(0)$ if (7) has a symmetric and positive definite solution $X$. It should be noted that stability of (10) for symmetric $\tilde{Y}(0)$ is equivalent to stability for $\tilde{Y}(0) \geq 0$ due to linearity. Similarly, stability of (8) for symmetric $\tilde{X}(0)$ is equivalent to stability for $X_0 \geq 0$ with $\tilde{X}(0) = X_0 - X_\infty$.

Let $X = X^T > 0$ satisfy (7). We introduce a candidate of Lyapunov function

\[ V(Y) = \text{tr} YX \]

which is positive for all nonzero, symmetric, and positive semidefinite $Y$. Since $\tilde{Y}(0) \geq 0$, we see $\tilde{Y}(\ell) \geq 0$ for all $\ell$. Then we have

\[ V(\tilde{Y}(\ell + 1)) = \text{tr} \left( \sum_{j=1}^{2^\ell} H_j^T \tilde{Y}(\ell) H_j \right) X = \text{tr} \tilde{Y}(\ell) \left( \sum_{j=1}^{2^\ell} H_j X H_j^T \right) \]

\[ = \text{tr} \tilde{Y}(\ell) (X - S) \]

\[ \leq (1 - \epsilon) \text{tr} \tilde{Y}(\ell) X = (1 - \epsilon) V(\tilde{Y}(\ell)) \]

for nonzero $\tilde{Y}(\ell)$, where $\epsilon \in (0, 1]$ is a number which satisfies $S \geq \epsilon X$. Such an $\epsilon$ always exists. Since $0 \leq (1 - \epsilon) < 1$, we see that

\[ \lim_{\ell \to \infty} V(\tilde{Y}(\ell)) = 0, \]

that is, (10) achieves (11) for any $\tilde{Y}(0) \geq 0$. \hfill $\Box$

### 4 Numerical Example

We show a numerical example which illustrates stable and unstable trajectories of the state of an MB-NCS. We set the coefficient matrices of the plant and the model as

\[ A = \begin{bmatrix} 1.5 & 0 \\ 1 & 0.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \]

\[ \hat{A} = A, \quad \hat{B} = B. \]

The parameters of the stochastic noises are

\[ \sigma^2 = 0.01, \quad Q = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}. \]

We select the update period as $h = 5$.

For this system, we apply the following two matrices as the state feedback gain $K$. One is $K_1 = \begin{bmatrix} -0.5 & -0.3 \end{bmatrix}$, and the other is $K_2 = \begin{bmatrix} -0.15 & -0.3 \end{bmatrix}$. Both gains stabilize the nominal system, i.e., all of the absolute values of the eigenvalues of $A + BK_1$ and $A + BK_2$ are strictly less than one. However, only the gain $K_1$ satisfies the stability condition of Theorem 1. The gain $K_2$ does not meet the requirement.

We performed numerical simulations under the setting above, where we computed the state trajectories 100 times for both cases with random samples of the noises. The results are given in Fig. 2 and Fig. 3. As we see, the variance of the state is bounded in the stable case, while its variance diverges in unstable case.

### 5 Concluding Remarks

In this paper, we have derived a necessary and sufficient condition of stability for MB-NCS with multiplicative noise. The condition is given as a set of linear matrix inequality and linear matrix equations, and thus we can easily check stability of the system by standard numerical computational software[4].

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