ON RATIOS OF HARMONIC FUNCTIONS

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Abstract. Following the recent work of Dan Mangoubi we consider harmonic functions in \( \mathbb{R}^n \) sharing the same set of zeros \( Z \). We show that the ratio of this functions is always well-defined and is a real-analytic function, it satisfies the maximum and minimum principles. In dimension \( n = 3 \) we also obtain the Harnack inequality for the ratios of harmonic functions and then generalize the gradient estimates obtained by Mangoubi in dimension two.

1. Introduction

1.1. Motivation and main results. Let \( u \) and \( v \) be harmonic functions in a ball \( B \) of \( \mathbb{R}^n \) that vanish at exactly the same set \( Z \subset B \), we call this set the nodal set of \( v \) (and \( u \)). We study the ratio \( u/v \). For general real analytic functions having the same set of real zeros the ratio is not always well-defined, there are also examples when the ratio is a continuous but not differentiable function. The situation changes when we assume that the functions are harmonic. Their real zeros determine to some extent their complex zeros and the ratio turns out to be a real analytic function. A connected component \( \Omega \) of \( B \setminus Z \) is called a nodal domain. In dimension two \( \partial \Omega \cap B \) can be represented locally as a graph of a Lipschitz function and one can actually apply the boundary Harnack principle for Lipschitz domains (see for example [2]) and see that \( u = fv \) for some locally bounded function \( f \) that does not change the sign. In higher dimensions the geometry of the nodal domains can be much more complicated. We give an example illustrating that already in dimension three the nodal domains may violate the Harnack chain condition. Thus there exists a harmonic function \( v \) such that \( B \setminus Z \) has components that are not NTA domains (see [4] for the definition). However, we prove that local division is possible, the ratio \( f \) is always defined and is real-analytic. Then we establish Maximum principle for the ratios of harmonic functions.

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Our interest in the ratios of harmonic functions grew up from studying the recent work of Dan Mangoubi, [8]. The following result was obtained in [8] in dimension two.

**Theorem (Mangoubi).** Let $Z \subset B_2 = \{ x : |x| < 2 \} \subset \mathbb{R}^2$, denote
\[
\mathcal{F}(Z) = \{ u : B_2 \to \mathbb{R}, \Delta u = 0, Z(u) = \{ u = 0 \} = Z \}.
\]
Then for any $u, v \in \mathcal{F}(Z)$ the ratio $f = u/v$ extends to a smooth nowhere vanishing function in $B_2$ and there exists a constant $C_Z > 0$ such that $|\nabla \log |f|| \leq C_Z$ in $B_1$.

We refer the reader to [8] for motivation of the problem, its connection to Li-Yau’s gradient estimate, and a list of examples of harmonic functions sharing the zero set. As we mentioned above in dimension two the ratio function $f$ is a positive function that satisfies the Harnack inequality. Our proof of the local division result shows also that the Harnack property for the ratios implies the gradient estimate in any dimension. It looks like the main problem is to establish the Harnack inequality for ratios in higher dimensions. We succeeded only in dimension three, for this case the structure of the critical set of harmonic function (where a harmonic function and its gradient simultaneously vanish) is less complicated than in higher dimensions. We prove the following result.

**Theorem 1.1.** Assume that $w$ is a harmonic function in $B \subset \mathbb{R}^3$. For any compact subset $K \subset B_2$ there exists a constant $C$ that depends on $w$ and $K$ only such that for any harmonic functions $u, v$ in $B$ such that $Z(u) = Z(v) = Z(w)$ and any points $x, y \in K$ we have
\[
\left| \frac{u(x)}{v(x)} \right| \leq C \left| \frac{u(y)}{v(y)} \right|.
\]

This result combined with the local division argument gives the following generalization of Mangoubi’s theorem.

**Theorem 1.2.** Suppose $u$ and $v$ are harmonic functions in $\Omega \subset \mathbb{R}^3$. If $Z(v) = Z(u) = Z$, then there exist a real-analytic function $f$ such that $u = vf$. If we fix $x_0 \in \Omega$ and assume $\frac{u}{v}(x_0) = 1$, then for any compact set $K \subset \Omega$ there exist positive numbers $A$ and $R$ depending only on $K, Z$ and $\Omega$ such that for all $x \in K$ and any multi-index $\alpha$
\[
\frac{|D^\alpha(u/v)|}{\alpha!} (x) \leq AR^{\alpha}.
\]

It would be interesting to know if Theorem 1.1 remains true in $\mathbb{R}^n, n \geq 4$. If it is true then our argument shows that higher dimensional analog of Theorem 1.2 would follow.
The extension of Magoubi’s estimate of $|\nabla \log |f||$ to dimension three immediately follows from Theorem 1.1 and Theorem 1.2.

1.2. Outline of the proof. In order to study the ratios of harmonic functions we need to understand the local behavior of a harmonic function near its zero point. In higher dimensions the structure of the zero set of a harmonic function could be very complicated. However the following key observations still hold: locally the zero set resembles that of a harmonic polynomial (at least in some sense, see Lemma 2.2 and also Counterexample 4.1); if $P$ and $Q$ are homogeneous harmonic polynomials satisfying $Z(Q) \subset Z(P)$ then $Q|P$ as a polynomial. Our first step is division of harmonic polynomials with common set of zeros. We note that the division of harmonic polynomials is a classical topic, the special feature of homogeneous harmonic polynomials is that all their non-trivial factors change sign. The lemma we need follows form the results of B.H. Murdoch [10], it was also recently discussed by Joan Mateu, Joan Orobitg, and Joan Verdera and applied for estimates of the maximal singular operators in [9], some facts on divisibility similar to what we need are proved in [9, Section 5]. We show then that a formal power series for the ratio $f$ is convergent locally, so $f$ is a real analytic function. Next step is to prove Theorem 1.1 we establish the maximum principle for ratios of harmonic functions and then show that in dimension three the nodal set is locally a boundary of a Lipschitz domain near most of its points. A combination of the Boundary Harnack Principle (BHP) and the Maximum Principle (MP) gives required Harnack inequality for the ratio.

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2. Local division of power series of harmonic functions

In this section we discuss local division of harmonic functions with common set of zeros in $\mathbb{R}^n$. We show that $u(x) = v(x)f_a(x)$ near any
\(a \in B\) as a formal power series. We may assume that \(a = 0\) to simplify the notation and that \(u\) and \(v\) are defined in some neighborhood of the origin. Clearly, the statement is false for real analytic functions, we use some special properties of harmonic polynomials.

2.1. Division of Harmonic Polynomials and Formal Power Series. Let \(P\) and \(Q\) be polynomials in \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). We are interested in conditions on \(P\) and \(Q\) ensuring the divisibility of \(P\) by \(Q\). If \(P\) is divisible by \(Q\), then surely \(Z(Q) \subset Z(P)\). The converse statement is false in general but it appears to be true if \(Q\) is a homogeneous harmonic polynomial.

**Lemma 2.1** (Division Lemma). Suppose \(Q\) is a homogeneous harmonic polynomial and \(P\) is a polynomial such that \(Z(Q) \subset Z(P)\). Then \(P = QR\) for some \(R \in \mathbb{R}[x_1, x_2, \ldots, x_n]\)

Lemma 2.1 follows from Theorem 2 and Lemma 4 in [10]. We outline a proof in the last section for reader’s convenience.

We extend the division to a general case and divide a real analytic function by a harmonic function. For the rest of this subsection we suppose that a real analytic function \(u\) and a harmonic function \(v\) are given. Represent \(u\) and \(v\) as infinite sums of homogeneous polynomials:

\[
u = \sum_{i=k}^{\infty} u_i, \quad v = \sum_{i=l}^{\infty} v_i,
\]

where \(u_i\) and \(v_i\) denote homogeneous polynomials of degree \(i\) and \(u_k\) and \(v_l\) are non-zero polynomials.

**Lemma 2.2.** If \(Z(v) \subset Z(u)\), then \(Z(v_l) \subset Z(u_k)\).

**Proof.** Assume the contrary: let \(y\) be a point such that \(u_k(y) \neq 0\) and \(v_l(y) = 0\). We may assume \(u_k(y) > 0\), so there is an open convex cone \(\Gamma\) containing \(y\) and \(\varepsilon > 0\) such that \(u_k(x) > \varepsilon |x|^k\) for any \(x \in \Gamma\). Since \(u(x) = u_k(x) + o(|x|^k)\) near the origin, there exists \(r > 0\) such that for any \(x \in \Gamma\) with \(|x| < r\) the inequality \(u(x) > 0\) holds.

Clearly, \(v_l\) is a harmonic polynomial. By the maximum and minimum principle there exist \(y_+, y_-\) arbitrarily close to \(y\) with \(v_l(y_+) > 0\) and \(v_l(y_-) < 0\), take \(y_+, y_-\) within \(\Gamma\). Consider \(ty_+\) and \(ty_-\), where \(t\) is a positive real number. If \(t\) is small enough, then \(v(ty_+) > 0, v(ty_-) < 0, |ty_+| < r\) and \(|ty_-| < r\). Choose \(t\) so that the previous four inequalities hold, then there exists \(x\) in the segment connecting \(ty_+\) and \(ty_-\) such that \(v(x) = 0\). It is clear that \(x \in \Gamma\) and \(|x| < r\), therefore \(u(x) > 0\). Thus we obtained a contradiction with \(Z(v) \subset Z(u)\). \(\square\)
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Now, we are in position to divide an analytic function by a harmonic function as Taylor series.

**Lemma 2.3.** If $Z(v) \subset Z(u)$, then there exists a formal power series $f$ such that the equality of power series $u = vf$ holds.

**Proof.** By Lemma 2.2 $Z(v) \subset Z(u)$ implies $Z(v_l) \subset Z(u_k)$ and by Lemma 2.1 $u_k$ is divisible by $v_l$. The ratio of $u_k$ and $v_l$ is a homogeneous polynomial of degree $k - l$ and we denote it by $f_{k-l}$; put $\tilde{u} := u - vf_{k-l}$. Note that $Z(v) \subset Z(\tilde{u})$ and the degree of the first non-zero polynomial in the Taylor representation of $\tilde{u}$ is at least $k + 1$. Using similar division step for $\tilde{u}$ and $v$ (instead of $u$ and $v$) we can find a polynomial $f_{k-l+1}$ such that $\tilde{u}_{k+1} = f_{k-l+1}v_l$, put $\tilde{u} := \tilde{u} - vf_{k-l+1}$ and see that the degree of the first polynomial in representation of $\tilde{u}$ is at least $k + 2$. Applying this division step infinitely many times we obtain formal equality of power series $u = vf$, where $f = \sum_{j=0}^{\infty} f_{k-l+j}$. □

2.2. Estimates of Formal Power Series. In the previous subsection we obtained an equality of power series $u = vf$. Now, we are going to show that $f$ is not only a formal power series but a real analytic function (in some neighborhood of zero) with estimates on the growth of its partial derivatives. We use usual multi-index notation, $\alpha = (\alpha_1, ..., \alpha_n), \alpha_j \in \mathbb{Z}_+, x^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}...x_n^{\alpha_n}$. We write $\gamma \leq \beta$ if $\gamma_i \leq \beta_i$ for any $i$: $1 \leq i \leq n$.

**Lemma 2.4.** Let $u = \sum_{\alpha} u_\alpha x^\alpha, v = \sum_{\alpha} v_\alpha x^\alpha, f = \sum_{\alpha} f_\alpha x^\alpha$ be formal power series such that $u = vf$. Suppose that $|v_\alpha| \leq ar^{||\alpha||}, |u_\alpha| \leq ar^\alpha$ for any $\alpha$ and some positive $a$ and $r$. Assume also $|v_{(k,0,\ldots,0)}| = c > 0$ and $v = \sum_{|\alpha| \geq k} u_\alpha$. Then there exist $A$ and $R = (R_1, R_2, \cdots, R_n)$ depending only on $a, c, r, k, n$ such that

$$(1) \quad |f_\beta| \leq AR^\beta \quad (R^\beta := R_1^{\beta_1}R_2^{\beta_2}\cdots R_n^{\beta_n})$$

for any multi-index $\beta$. Hence $f$ represents a real analytic function near zero.

Denote $(k,0,\ldots,0)$ by $k$. By the equality of formal power series $u = vf$ we have

$$(2) \quad u_{\beta+k} = \sum_{\gamma \leq \beta+k, |\gamma| \leq |\beta|} f_\gamma v_{\beta+k-\gamma} \quad \text{for any multi-index } \beta.$$  

We need an auxiliary proposition which will help to estimate $|f_\beta|$.

**Proposition 2.5.** For any $a_0, r > 0$ there exist $A = A(a_0, r)$ and $R = (R_1, R_2, \ldots, R_n) = R(a_0, r)$ such that for any multi-index $\beta$
We postpone the proof of the Proposition. First, we show that Lemma 2.4 holds with $A = A(ac^{-1}, r)$ and $R = R(ac^{-1}, r)$ from Proposition 2.5. We prove (1) by induction with respect to some lexicographic order on multi-indices.

Consider the set of multi-indices $A := \{\alpha = (\alpha_1, \ldots, \alpha_n) : \alpha_i \in \mathbb{Z}_+\}$ with the order $\prec$ defined by

$$
\gamma \prec \beta \iff \begin{cases} 
\gamma_n < \beta_n \\
\gamma_n = \beta_n, \; \gamma_{n-1} < \beta_{n-1} \\
\vdots \\
\gamma_n = \beta_n, \; \gamma_{n-1} = \beta_{n-1} \; \cdots \; \gamma_2 = \beta_2, \; \gamma_1 < \beta_1.
\end{cases}
$$

Then $(A, \prec)$ is a well-ordered set.

**Proof of Lemma 2.4.** Denote by $S$ the set of multi-indices $\alpha$ with $|f_\alpha| > AR^\alpha$. Our goal is to show that $S$ is an empty set. Suppose $S$ is not empty, then $S$ has a least element in the ordering $\prec$, denote it by $\beta$. Let us write $\star$ instead of summation condition $\gamma \leq \beta + \tilde{k}, |\gamma| \leq |\beta|, \gamma \neq \beta$, it is clear that this condition implies $\gamma \prec \beta$. Then (2) can be written as

$$v_\beta f_\beta = u_\beta + \sum_\star f_\gamma v_{\beta + \tilde{k} - \gamma}$$

Note that $|f_\gamma| \leq AR^\gamma$ for any $\gamma \prec \beta$. Keeping in mind that $|v_\tilde{k}| = c > 0$ we obtain

$$|f_\beta| \leq c^{-1}|u_{\beta + \tilde{k}}| + c^{-1} \sum_\star |f_\gamma v_{\beta + \tilde{k} - \gamma}| \leq c^{-1}AR|\beta + \tilde{k}| + c^{-1} \sum_\star AR^\gamma ar|\beta + \tilde{k} - \gamma| \leq AR^\beta.$$ 

Therefore $|f_\beta| \leq AR^\beta$ and $\beta \notin S$. Thus $S$ is an empty set and the proof is completed. $\square$

**Proof of Proposition 2.5.** We use notation $r^\alpha$ for $r^{\sum \alpha_i}$, where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \in \mathbb{Z}$. Divide both sides of inequality (3) by $AR^\beta$:

$$a_0 r^\beta + a_0 r^k \sum_\star \frac{R^\gamma r^{\beta - \gamma}}{R^\beta} \leq 1$$
We can easily make the first summand above to be less than $1/2$ for all $\beta$ if we take any $A$ and $R$ sufficiently large so that
\[
\frac{a_0 r^k}{A} \leq 1/2 \text{ and } R_i \geq r \text{ for all } i \in [1, n].
\]
Therefore it suffices to achieve
\[
\sum_\star \frac{R^\gamma r^{\beta-\gamma}}{R^\beta} \leq \frac{1}{2a_0 r^k}
\]
to make the inequality (4) true. By $\star$ we have $\beta_i \geq \gamma_i$ for any $i \in [2, n]$ and $|\beta| \geq |\gamma|$, denote $\beta_i - \gamma_i$ by $\delta_i$ and $|\beta| - |\gamma|$ by $\delta$. It’s easy to see that
\[
(5) \quad \sum_\star \frac{R_i^{\gamma} r^{\beta-\gamma}}{R_i^\beta} \leq \sum_\bullet \left( \frac{R_1}{R_2} \right)^{\delta_2} \left( \frac{R_1}{R_3} \right)^{\delta_3} \ldots \left( \frac{R_1}{R_n} \right)^{\delta_n} \left( \frac{r}{R_1} \right)^{\delta},
\]
where $\bullet$ is the following condition:
\[
\delta, \delta_2, \delta_3, \ldots, \delta_n \in \mathbb{Z}_+, \quad \delta + \sum_{i \geq 2} \delta_i > 0.
\]
Note that the right hand side of (5) is the product of geometric progressions without the first term 1. Therefore
\[
\sum_\bullet \left( \frac{R_1}{R_2} \right)^{\delta_2} \left( \frac{R_1}{R_3} \right)^{\delta_3} \ldots \left( \frac{R_1}{R_n} \right)^{\delta_n} \left( \frac{r}{R_1} \right)^{\delta} = \frac{1}{1 - \frac{R_1}{R_2}} \frac{1}{1 - \frac{R_1}{R_3}} \ldots \frac{1}{1 - \frac{R_1}{R_n}} \frac{1}{1 - \frac{r}{R_1}} - 1.
\]
And the last expression can be done arbitrarily small by a proper choice of $R$ (we can take $r << R_1, R_1 << R_2 = R_3 = \ldots = R_n$).

2.3. Division by a harmonic function. Lemma 2.4 together with Lemma 2.3 gives us the following theorem.

**Theorem 2.6.** Suppose $u$ is a real-analytic function and $v$ is a harmonic function, both functions are defined in some domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. If $Z(v) \subset Z(u)$, then there exist a real-analytic function $f$ in $\Omega$ such that $u = vf$.

**Proof.** Indeed, for any $x_0 \in \Omega$ we know that the Taylor series at $x_0$ of $v$ is a divisor of a Taylor series at $x_0$ of $u$. And the only obstacle is to show that formal power series $f = u/v$ is absolutely convergent. And here Lemma 2.4 comes into play. Represent $v$ as a sum of monomials: $v = \sum v_\alpha (x - x_0)^\alpha$. If $v$ is not identically zero, we can take $k \geq 0$ such that $v_\alpha = 0$ for any multi-index $\alpha$ with $|\alpha| < k$ and there is $\alpha$ with...
\( |\alpha| = k: v_\alpha \neq 0 \). After that we can rotate the coordinate lines to obtain \( v_{(k,0,\ldots,0)} \neq 0 \) and apply Lemma 2.4. Finally the estimate (1) implies the absolute convergence of the power series of \( f \) in some neighborhood of \( x_0 \).

2.4. Maximum and Minimum Principle for Harmonic Fractions. Suppose that \( u \) and \( v \) are harmonic in \( \Omega \) functions such that \( Z(u) \supset Z(v) \). We already know that there exists a real analytic in \( \Omega \) function \( f \) such that \( u = fv \).

**Theorem 2.7.** Harmonic fraction \( f \) enjoys the maximum and minimum principle, i.e. for any subdomain \( O \Subset \Omega \)

\[
\max_{\partial O} f = \max_O f \quad \& \quad \min_{\partial O} f = \min_O f.
\]

**Proof.** Let \( M = \max_{\partial O} f \) and \( m = \min_{\partial O} f \). Let \( D \subset \Omega \) be any nodal domain of \( v \) that intersects \( O \). Let \( \Gamma_0 \) denote \( Z \cap \partial D \) and \( \Gamma_1 \) denote \( \partial D \cap \partial O \). We may assume \( v \) is positive in \( D \). It’s clear that \( mv \leq u \leq Mv \) on \( \Gamma_1 \) and surely \( mv \leq u \leq Mv \) on \( \Gamma_0 \) because \( u \) and \( v \) vanish on \( \Gamma_0 \). Therefore \( mv \leq u \leq Mv \) on \( D \cap O \) by the standard maximum principle. Hence we have \( m \leq f \leq M \) everywhere in \( \bar{O} \).

Thus \( \max_{\partial O} f = \max_O f \) and \( \min_{\partial O} f = \min_O f \).

**Remark 2.8.** The maximum principle for harmonic fractions is strict, i.e. a local maximum or minimum can not be attained at an interior point unless \( f \) is a constant function. In order to prove it one can show that if \( f = 0 \) at some interior point, then \( f \) changes a sign. The last claim can be proved with the help of Proposition 4.2.

3. Harnack Inequality for Harmonic Fractions in \( \mathbb{R}^3 \) and Gradient Estimate

In this section we prove Theorem 1.1 and then deduce Theorem 1.2. We fix a harmonic function \( w \) in a subdomain \( \Omega \) of \( \mathbb{R}^3 \).

3.1. Structure of the nodal set of harmonic function in dimension three. Let \( Z = Z(w) \subset \Omega \). We say that a point \( x \in Z \) is a good point if for any nodal domain \( \Omega_i \) with \( \partial \Omega_i \ni x \) the following holds: there exists a neighborhood \( W \) of \( x \) such that \( \partial \Omega_i \cap W \) can be parametrized by a graph of a Lipschitz function, i.e. \( \partial \Omega_i \) is Lipschitz in some neighborhood of \( x \). We say that a point \( x \in Z \) is a bad point if it is not good.

We have \( Z = Z_0 \cup Z_1 \), where \( Z_0 = \{ x : w(x) = 0, \nabla w(x) \neq 0 \} \) and \( Z_1 = \{ x : w(x) = 0, \nabla w(x) = 0 \} \). In some neighborhood of each point of \( Z_0 \) the nodal set is a smooth surface and all points of \( Z_0 \) are
good; $Z_1$ is the critical set of $w$ it is locally a finite union of analytic curves and a discrete set of points. We refer here to a general structure theorem for real analytic varieties of Lojasiewicz, see [7] or [6, Chapter 6.3]. Consider any analytic curve $\Gamma$ from $Z_1$. Denote by $d(x)$ the depth of zero at $x$, i.e. $d(x)$ is a degree of a first non-zero homogeneous polynomial in the Taylor series of $w$ at $x$. Suppose there is a sequence of points $\{x_i\}_{i=1}^\infty$ from $\Gamma$ converging to an interior point $x_\infty$ of $\Gamma$ such that $d(x_i) \geq k$ for some $k \in \mathbb{N}$, then $d(x) \geq k$ for any $x \in \Gamma$. The last claim implies that there exists $k \in \mathbb{N}$ such that $d(x) = k$ for all $x \in \Gamma$ except at most a countable set of points from $\Gamma$ with at most two accumulation points at the ends of the curve (see the proof of Lemma 2.4 in [5] for a similar decomposition of the critical set).

**Lemma 3.1.** Let $x$ be an interior point of $\Gamma$ and let $U$ be a neighborhood of $x$ such that for any $y \in \Gamma \cap U$: $d(y)=k$. Then $x$ is a good point.

The main point in the proof is that we know the first non-zero term of the Taylor expansion of $w$ at each point $y \in \Gamma \cap U$, it is a homogeneous harmonic polynomial of degree $k$ of two variables in the plane orthogonal to $\Gamma$ at $y$ and the gradient of the this term restricted on the plane is bounded from below by $(k - 1)$ st power of the distance to $y$. Probably there exists a diffeomorphism $H : U \to B$ such that $w(x, y, z) = g_k(H(x, y, z))$, where $g_k(x, y, z) = \Re(x + iy)^k$ that can be constructed like in the proof of Kuiper-Kuo theorem in [3], the difference is that one needs to apply it to an analytic one-parametric family of functions, we don’t construct it here, but show that each nodal domain is Lipschitz near the given point $x$.

**Proof.** Let $\Gamma : [-r, r] \to \mathbb{R}^3$ be parametrized such that $\Gamma(0) = 0, \Gamma(t) = (x(t), y(t), t)$, $\Gamma'(0) = (0, 0, 1)$. Assume further that at each point of $\Gamma(-r, r)$ the function $w$ vanishes with all its derivatives up to order $k - 1$ but not all derivatives of order $k$ vanish. Let $p_t(x, y, z)$ be the $k$th Taylor polynomial of $w$ at the point $\gamma(t)$,

$$w(x, y, z) = p_t(x - x(t), y - y(t), z - t) + q_t(x - x(t), y - y(t), z - t),$$

where $|q_t(X)| \leq C|X|^{k+1}$ and $|\nabla q_t(X)| \leq C|X|^k$; by $X$ we denote $(x, y, z)$. Clearly, $p_t$ is a homogeneous harmonic polynomial of degree $k$ whose coefficients are real analytic in $t$.

Fix some point $t_0$ and let $v_0 = v(t_0) = (x'(t_0), y'(t_0), 1)$ be the tangent vector to $\Gamma$ at $t_0$. Let $f$ be some partial derivative of $w$, $f = \partial^\alpha w$, of order $k-1$ or less, then $g(t) = f(\Gamma(t)) = 0$ and $(\nabla f(\Gamma(t_0)), v_0) = 0$. On the other hand $(\nabla f(\Gamma(t_0)) = \nabla(\partial^\alpha p_{t_0})(0)$, but all partial derivatives of order $k$ of $p(t_0)$ are constants. Hence $\partial^\alpha p_{t_0}(sv_0) = 0$ for any $s \in \mathbb{R}$ and any $\alpha$ with $|\alpha| \leq k - 1$. In addition we know that $p_{t_0}(\xi)$ is a
homogeneous harmonic polynomial or order $k$. It does not depend on $(\xi, v_0)$. Then there exists an orthogonal basis \{a(t_0), b(t_0), v(t_0)\} such that $p(t_0)(X) = c(t_0)\Re\{(X, a(t_0)) + i(X, b(t_0)))^k\}, c(t_0) \neq 0$.

For small $t$ we may choose $a(t), b(t), c(t)$ to be real analytic functions in $t$, with $|c(t)| > c_0 > 0$. We note that $v(t) = (x'(t), y'(t), 1)$ and $|v(t)| < 1 + \delta$ for small $t$. We consider projections of $a(t)$ and $b(t)$ onto the plane $\{z = 0\}$ and call them $a_1(t), b_1(t)$. We will also need the matrix $A(t) \in M_2$ which is the inverse of the matrix $[a_1(t), b_1(t)]$, it exists when $t$ is small enough and depends analytically on $t$.

Our aim is to show that each nodal domain of $w$ near the origin is a Lipschitz domain. We will perform some diffeomorphic changes of variables to simplify the geometry of the nodal set near zero. From this point we don’t use the fact that the function $w$ harmonic.

First, let us consider the map $F(x, y, z) = (x + x(z), y + y(z), z)$ defined on some neighborhood $U \subset \mathbb{R}^2 \times [-r, r]$ of the origin, clearly it is a diffeomorphism. We consider $w_1 = w \circ F$. Then $w_1$ vanishes on the $z$-axis with all its derivatives up to order $k - 1$. It easy to see that the $k$-th Taylor polynomial of $w_1$ at $(0, 0, t)$ is $P_t(X) = c(t)\Re\{(X, a_1(t)) + i(X, b_1(t)))^k\}$.

Next let $G : (x, y, z) = (A(z)(x, y), z)$, it is a diffeomorphism in a neighborhood of the origin, we consider $w_2(x, y, z) = c^{-1}(z)(w_1 \circ G)(x, y, z)$. Then $w_2$ vanishes on $(0, 0, z)$ for small $z$ with all its derivatives of order up to $k - 1$ and $k$th Taylor polynomial of $w_2$ at each point $(0, 0, z)$ is $\Re(x + iy)^k$. It is enough to show that the nodal domains of $w_2$ are Lipschitz.

Let us fix $z_0$ and consider the plane $(x, y, z_0)$, the restriction of $w_2$ to this plane has the form $\Re(x + iy)^k + q_{z_0}(x, y)$ and $|\nabla q_{z_0}(x, y)| \leq C(x^2 + y^2)^{k/2}$, while the gradient of the main term is greater than or equal to $c(x^2 + y^2)^{(k-1)/2}$. It means that the gradient of $w_2$ does not vanish in $(B \setminus \{0\}) \times \mathbb{R}$, where $B$ is a small enough two-dimensional ball around the origin.

We consider the domain $\Omega = \{(x, y, z) : |z| < r_0, x \tan \phi_1 < |y| < x \tan \phi_2\}$, where $0 < \phi_1 < \pi/k < \phi_2 < \pi/(k - 1))$. For $x > 0$ small enough we have
\[ u(x, x \tan \phi, z) = x^k(\cos \phi)^{-k} \cos k\phi + O(|x|^{k+1}) > 0, \quad |\phi| \leq \phi_1, \quad \text{and} \]
\[ u(x, \pm x \tan \phi_2, z) = x^k(\cos \phi_2)^{-k} \cos k\phi_2 + O(|x|^{k+1}) < 0. \]

Further, we have $\partial_x u(x, y, z) \geq c(|x|^2 + |y|^2)^{(k-1)/2}$ in $\Omega$. Then there are exactly $2k$ nodal domains in $B_{\varepsilon}(0)$, where $\varepsilon$ is sufficiently small. Let $D_0$ be the one that contains points $(s, 0, 0)$ for $s$ small enough. The inequalities above imply that $(\partial D_0) \setminus \{(0, 0, z)\) \subset \Omega$. Further, $\partial D_0$ is
a graph over the plane \((0, y, z)\), and by the implicit function theorem if \(\partial D_0\) is given by \((g(y, z), y, z)\) then
\[
|\nabla g(y, z)| \leq |\nabla u(g(y, z), y, z)| / |\partial_x u(g(y, z), y, z)| \leq C, \quad \text{when } y \neq 0.
\]
Then \(g(y, z)\) is a continuous function differentiable everywhere except for the line \(\{y = 0\}\) with uniformly bounded derivative, hence it is Lipschitz. \(\square\)

3.2. Proof of Theorem 1.1. First, Lemma 3.1 implies

Corollary 3.2. The set of bad points in \(Z \cap V\) is at most countable set with a finite number of accumulation points.

This corollary will be used in the proof of the following theorem.

Theorem 3.3. Let \(u\) and \(v\) be any harmonic in \(\Omega\) functions with \(Z(u) = Z(v) = Z\) let \(x_0\) be a point in \(Z\). Let \(f\) be a harmonic fraction of \(u\) and \(v\). There exists a positive constant \(C = C(Z, x_0)\) and a ball \(B_r(x_0) \subset \Omega\) with center \(x_0\) and some radius \(r\) such that
\[
C \cdot \inf_{B_r(x_0)} |f| \geq \sup_{B_r(x_0)} |f|.
\]

Proof. We already know from Theorem 2.6 that \(f\) is a continuous function in \(\Omega\). Corollary 3.2 implies that there exists a ball layer \(Q := B_R(x_0) \setminus \overline{B_r(x_0)}\) with \(R > r \geq 0\) and \(\overline{B_r(x_0)} \subset \Omega\) such that any \(x \in Q \cap Z\) is a good point. Consider a sphere of radius \(\frac{r + R}{2}\) with center \(x_0\), denote it by \(S\). Let \(\Omega_i\) be any nodal domain with non-empty intersection with \(S\) and let \(S_i\) denote \(S \cap \overline{\Omega_i}\). Note that \(S_i\) is compact subset of \(Q\). By the boundary Harnack principle for Lipschitz domains (we refer the reader to \([1]\) and the references therein), there exists a constant \(C_i: \max_{S_i} |f| \leq C_i \min_{S_i} |f|\). Put \(C := \prod_{\Omega_i} \prod_i C_i\). It can be easily checked by induction on a number of \(\Omega_i\) that
\[
\max_S |f| \leq C \min_S |f|.
\]
By the maximum and minimum principle for harmonic fractions we have \(\sup_{B_r} |f| \leq \max_S |f|\) and \(\inf_{\overline{S}} |f| \leq \inf_{B_r} |f|\). Thus
\[
\sup_{B_r(x_0)} |f| \leq C \inf_{B_r(x_0)} |f|.
\]
\(\square\)

Now, Theorem 1.1 follows from the previous theorem and standard compactness argument.
3.3. **Proof of Theorem [1,2]** Let $Z$ be a zero set of some harmonic function $w$ in $\Omega \subset \mathbb{R}^3$. Let $x_0 \in \Omega \setminus Z$ and define $F_0(Z) = \{u : \Omega \to \mathbb{R} : \Delta u = 0, Z(u) = Z, u(x_0) = w(x_0)\}$. Clearly for any $u$ with $Z(u) = Z$ there exists a constant $c_u$ such that $c_u u \in F_0(Z)$.

**Lemma 3.4.** Consider a point $y \in \Omega_0$. There exist a neighborhood $V_y$ of $y$, $V_y \subset \Omega_0$, and positive constants $A_y, R_y$ such that for any $x \in V_y$ the inequality

$$\frac{|D^\alpha(f)|}{\alpha!}(x) \leq A_y R_y^{||\alpha||}$$

holds whenever $f = u/v$ for some $u, v \in F_0(Z)$.

**Proof.** To simplify the notation let $y = 0$. By Theorem 1.1 there exist a constant $C = C(y, w)$ and a neighborhood $V$ of 0 such that $\frac{1}{C} \leq |\frac{u}{v}|(x) \leq C$ and $\frac{1}{C} \leq |\frac{u}{w}|(x) \leq C$ for any $x \in V$. Let $M = \sup_V |w|$, then $|u|$ and $|v|$ are bounded by $CM$ in $V$. By the standard Cauchy estimates, there exist positive numbers $a = a(y, w)$ and $r = r(y, w)$ such that

$$|u| \leq ar^{|\alpha|} \text{ and } |v| \leq ar^{|\alpha|}.$$ 

Let $w = \sum_{i=k}^{\infty} w_i$ be a representation of $w$ as a sum of homogeneous harmonic polynomials and let $w_k$ be the first non-zero polynomial with degree $k$ in this sum. Let us rotate the coordinate lines to make $w_{(k,0,...,0)} \neq 0$. Let $u = \sum_{i=l}^{\infty} u_i$ and $v = \sum_{i=m}^{\infty} v_i$ be analogous sums for $u$ and $v$. By Lemma 2.2 we have $Z(w_k) = Z(u_l) = Z(v_m)$, then Lemma 2.1 implies $k = l = m$ and $w_k = c_1 w_k, v_k = c_2 w_k$, where $c_1, c_2$ are non-zero constants. By l’Hôpital’s rule $\frac{v'_{w_{(k,0,...,0)}}}{w_{(k,0,...,0)}} = \frac{w'}{w}(0)$. Since $\frac{1}{C} \leq |\frac{w}{w}|(x) \leq C$, we have $|\tilde{v}_{(k,0,...,0)}| \geq C^{-1} |w_k, 0, ..., 0|$, where $\tilde{v} = C v$. Now we are in position to apply Lemma 2.4 the constants $A_y, R_y$ depend on $y, w$ but does not depend on $u$ and $v$.

Now our main result is a straightforward consequence of the previous lemma and a standard compactness argument.

**Proof of Theorem 1.2.** Real analyticity of $f$ was proved in Theorem 2.6 Lemma 2.3 claims that for any $y$ there exist $A_y$ and $R_y$ such that $\frac{|D^\alpha(f)|}{\alpha!} \leq A_y R_y^{||\alpha||}$ for any multi-index $\alpha$. Since $f$ is real-analytic, then there exist a neighborhood of $y$ denoted by $V_y$ such that $\frac{|D^\alpha(f)|}{\alpha!} \leq \tilde{A}_y (\tilde{R}_y)^{||\alpha||}$ for any $x \in V_y$. Note that $K \subset \bigcup_{y \in K} V_y$. Since $K$ is a compact
set, there exist a finite set \( \{ y_1, y_2, \ldots, y_m \} \) such that \( K \subset \bigcup_{i \in \{1, \ldots, m\}} V_{y_i} \).

Take \( A := \max_i \{ \tilde{A}_{y_i} \} \) and \( R := \max_i \{ \tilde{R}_{y_i} \} \).

\[ \square \]

4. Concluding remarks

4.1. Nodal sets of harmonic functions and harmonic polynomials. It is an interesting question to which extend the nodal set of a harmonic function (or more generally of a solution to some elliptic equation) resembles the nodal set of its first non-zero homogeneous polynomial. We refer the reader to [5] and references therein. In dimension two the nodal set of harmonic functions and solutions to elliptic equations locally look like regular intersections of curves. In higher dimensions we implicitly used some information on the nodal sets to divide harmonic functions sharing the same zeros in \( \mathbb{R}^n \) and prove that most of the points of the nodal set in dimension three are good. However the following example shows that starting from dimension three the nodal sets may have complicated local geometry.

**Example 4.1.** Consider harmonic polynomial \( H(x, y, z) = x^2 - y^2 + z^3 - 3x^2z \). If we look at its nodal set on each plane \( \{0, 0, z\} \). We will see that it is a union of two orthogonal lines for \( z = 0 \) and two hyperbolas for \( z \neq 0 \). There are only two nodal domains \( \Omega_1 \) and \( \Omega_2 \) (not four like for \( x^2 - y^2 \)) and those nodal domains are not Lipschitz. Moreover the Harnack chain condition does not hold for \( \Omega_{1,2} \) (see [1, 4] for the definition). We don’t know if the boundary Harnack principle is valid.

4.2. Differential equation for the ratio. One can think about the ratio \( f \) as a positive solution of the following second order degenerate elliptic equation

\[ \text{div}(v^2 \nabla f) = 0. \]

Unfortunately the coefficient is very singular, \( v^2 \) does not belong to the Muckenhoupt class \( A_2 \) when \( v \) changes sign. It would be interesting to see if one can use harmonicity of \( v \) to obtain Harnack inequality for positive solutions of such equations in \( \mathbb{R}^n \). More delicate equations for the log \( f \) were used in [8] in dimension 2.

Another interesting question is when the equation above admits any positive solutions and how big this family may be.

4.3. Zeroes and Division of Real-valued Polynomials in Several Variables. We suggest a proof of Lemma 2.1 in this subsection. The following division follows from general results in algebraic geometry, we borrowed it from [9, Chapter 5].
Lemma (Division Lemma, Mateu, Orobitg, Verdera). Let $Q$ and $P$ be polynomials in $\mathbb{R}[x_1, x_2, \ldots, x_n]$. Suppose that $H^{n-1}(Z(P) \cap Z(Q)) > 0$ and $Q$ is irreducible. Then there is $R \in \mathbb{R}[x_1, x_2, \ldots, x_n]$ such that $P = QR$.

We are going to replace irreducible polynomial $Q$ by a homogeneous harmonic one to prove Lemma 2.1.

We write $S \subseteq T$ in case sets $S, T \subset \mathbb{R}^n$ satisfy $H^{n-1}(S \setminus T) = 0$. Lemma 2.1 follows from two propositions below.

**Proposition 4.2.** If $Q$ is a non-zero homogeneous harmonic polynomial and $Q_1$ is a non-constant divisor of $Q$, then $Q_1$ changes sign and $H^{n-1}(Z(Q_1)) > 0$.

**Proposition 4.3.** Suppose polynomials $P$ and $Q$ enjoy the following properties:

1. $Z(Q) \subseteq Z(P)$,
2. If $Q_1$ is a non-constant divisor of $Q$, then $Q_1$ changes sign.

Then $P$ is divisible by $Q$.

*Proof of Proposition 4.2.* If $Q_1$ changes sign, then $H^{n-1}(Z(Q_1)) > 0$ (see also the dimension lemma in [9, Chapter 5]). We may therefore assume $Q_1 \geq 0$ and try to obtain a contradiction. Let $Q = Q_1Q_2$ then clearly, $Q_2Q = q_1Q_2^2 \geq 0$. The degree of $Q_2$ is strictly less than the degree of $Q$. Since a homogeneous harmonic polynomial is orthogonal on sphere to any polynomial of smaller degree, $\int_{S_r} Q_2(x)Q(x)d\sigma(x) = 0$,

where $S_r$ is the $(n-1)$-dimensional sphere with center 0 and some radius $r$, $\sigma$ is the surface Lebesgue measure. Keeping in mind that $Q_2Q \geq 0$ we obtain $Q_2Q = 0$ a.e. on $S_r$. Since $r$ is an arbitrary positive number, $Q_2Q \equiv 0$. We therefore have $Q_1 \equiv 0$ and a contradiction is obtained ($Q_1$ is a non-constant polynomial).

*Proof of Proposition 4.3.* If $P$ or $Q$ is a constant function, then statement is trivial. We argue by induction on the degree of $Q$. Consider any irreducible non-constant divisor of $Q$ and denote it by $Q_1$. We know that $Z(Q_1) \subset Z(Q) \subseteq Z(P)$ and that $H^{n-1}(Z(Q_1)) > 0$, hence $H^{n-1}(Z(P) \cap Z(Q_1)) > 0$. Applying Lemma 4.2 for $P$ and $Q_1$ we see that $P$ is divisible by $Q_1$. Put $\tilde{P} := P/Q_1$ and $\tilde{Q} := Q/Q_1$. It’s clear that $\tilde{Q}$ enjoys the property 2.

Now, we show that $Z(\tilde{Q}) \subseteq Z(\tilde{P})$. Assume it is not true, i.e., $H^{n-1}(Z(\tilde{Q}) \setminus Z(\tilde{P})) > 0$. Clearly $Z(\tilde{P}) = Z(\tilde{P}) \cup Z(Q_1)$ and the property 1, $H^{n-1}(Z(Q) \setminus Z(P)) = 0$. Hence $H^{n-1}(Z(Q \cap Q_1) \cap Z(Q_1)) > 0$. 
Then by Lemma 4.3, $Q_1|Q/Q_1$ and $Q$ has a divisor $Q_2$, which contradicts the property 2.

We see that $\tilde{P}$ and $\tilde{Q}$ enjoy the properties 1 and 2, since the degree of $\tilde{Q}$ is less than the degree of $\tilde{Q}$ we obtain $\tilde{P} = \tilde{Q}R$ and then $P = QR$. □

Remark 4.4. A mind-reader might see that in Theorem 2.6 we can replace $Z(v) \subset Z(u)$ by $Z(v) \subseteq Z(u)$:

Suppose $u$ is a real-analytic function and $v$ is a harmonic function, both functions are defined in some domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. If $Z(v) \subseteq Z(u)$, then there exist a real-analytic function $f$ in $\Omega$ such that $u = vf$.

If we apply the last claim several times we can obtain the following theorem:

**Theorem 4.5.** Suppose $u$ is a real-analytic function and $v_1, v_2, \ldots, v_k$ are harmonic functions, all functions are defined in some domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$. If $Z(v_i) \subseteq Z(u)$ for any $i \in [1, k]$ and $H^{n-1}(Z(v_i) \cap Z(v_j)) = 0$ for $i \neq j$, then there exist a real-analytic function $f$ in $\Omega$ such that $u = f \prod_{i=1}^{k} v_i$.

4.4. **Real and Complex zeros of harmonic functions.** Our results show that if harmonic functions $u$ and $v$ have the same zero set $Z$ in a ball $B \subset \mathbb{R}^n$, then there zero sets in $\mathbb{C}^n$ coincide at least at some complex neighborhood of a smaller real ball $b$. If $n = 2$ or 3 Theorem 1.2 implies that this neighborhood can be chosen to depend on $Z$ only and not on $u$ and $v$, i.e. the real zeros of a harmonic function uniquely determine its complex zeros in some complex neighborhood. It would be interesting to prove this directly and see if this neighborhood can be chosen to depend on $Z$ only in higher dimensions as well.

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