Laminar Tight Cuts in Matching Covered Graphs✩

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Abstract

An edge cut $C$ of a graph $G$ is tight if $|C \cap M| = 1$ for every perfect matching $M$ of $G$. Barrier cuts and 2-separation cuts are called ELP-cuts, which are two important types of tight cuts in matching covered graphs. Edmonds, Lovász and Pulleyblank proved that if a matching covered graph has a nontrivial tight cut, then it also has a nontrivial ELP-cut. Carvalho, Lucchesi, and Murty made a stronger conjecture: given any nontrivial tight cut $C$ in a matching covered graph $G$, there exists a nontrivial ELP-cut $D$ in $G$ which does not cross $C$. We confirm the conjecture in this paper.

Keywords: tight cut, ELP-cut, perfect matching, matching covered graph

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1. Introduction

All graphs considered in this paper are finite and may contain multiple edges, but no loops. We will generally follow the notation and terminology used by Bondy and Murty in [1].

Let $G$ be a graph with vertex set $V$ and edge set $E$. For any $X \subseteq V$, let $N_G(X)$ be the set of vertices of $V - X$ which are adjacent to vertices of $X$ and let $\overline{X} = V - X$ be the complement of $X$ in $V$. The set of coboundary edges of $X$, denoted by $\partial(X)$, is the set of edges with exact one end in $X$ and one end in $\overline{X}$. A cut $\partial(X)$ is trivial if $|X| = 1$ or $|\overline{X}| = 1$.

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Let $C := \partial(X)$ be a cut of $G$. We denote by $G/(X \to x)$ the graph obtained from $G$ by contracting $X$ to a single vertex $x$ (and removing any resulting loops). The graphs $G/(X \to x)$ and $G/(\overline{X} \to \overline{x})$ are the two $C$-contractions of $G$.

Let $C := \partial(X)$ and $D := \partial(Y)$ be two cuts of $G$. We say that $C$ and $D$ cross if all the four sets $X \cap Y, X \cap \overline{Y}, \overline{X} \cap Y$ and $\overline{X} \cap \overline{Y}$ are nonempty, and are laminar otherwise. So, two cuts $C$ and $D$ are laminar if and only if one of the two shores of $C$ is a subset of one of the shores of $D$. A collection of cuts is laminar if no two of its cuts cross.

A graph is called matching covered if it is connected, has at least one edge and each of its edges is contained in some perfect matching. Suppose that our graph $G$ is matching covered. A cut $C$ of $G$ is tight if $|C \cap M| = 1$ for every perfect matching $M$ of $G$. Clearly every trivial cut is a tight cut. We call a matching covered graph which is free of nontrivial tight cuts a brace if it is bipartite, and a brick otherwise.

Let $S$ be a set of vertices of $G$. A component of $G - S$ is odd if it consists of an odd number of vertices, is even if it consists of an even number of vertices, and is trivial if it consists of only one vertex. We denote by $o(G - S)$ the number of odd components of $G - S$.

We shall make use of the following known facts about matching covered graphs and tight cuts.

**Theorem 1.1 (Tutte [2]).** A graph $G$ has a perfect matching if and only if $o(G - S) \leq |S|$ for every subset $S$ of $V(G)$.

**Corollary 1.2.** Let $G$ be a matching covered graph and let $S$ be a subset of $V(G)$. Then, $o(G - S) \leq |S|$, with equality only if $S$ is independent and $G - S$ has no even components.

**Proof:** A matching covered graph has perfect matchings, hence the asserted inequality holds, by Theorem 1.1. Suppose that $S$ is not independent, let $e := vw$ be an edge of $G$ having both ends in $S$. As $G$ is matching covered, it has a perfect matching, $M$, that contains the edge $e$. Let $T := S - \{v, w\}$. Then, $M - e$ is a perfect matching of $H := G - \{v, w\}$. By Theorem 1.1 $o(G - S) = o(H - T) \leq |T| = |S| - 2$. The inequality is thus strict in this case. Now suppose that $G - S$ has an even component, $K$. As $G$ is connected, $K$ has a vertex, $x$, which is adjacent to a vertex, $y$ of $S$. Let $U := S + x$. Every odd component of $G - S$ is an odd component of $G - U$. In addition, $K - x$ has at least one odd component which is not a component of $G - S$. Moreover, the set $U$ is not independent, hence $o(G - S) \leq o(G - U) - 1 \leq |U| - 3 = |S| - 2$. □

**Proposition 1.3 ([3]).** Let $G$ be a matching covered graph and let $\partial(X)$ and $\partial(Y)$ be two tight cuts such that $|X \cap Y|$ is odd. Then $\partial(X \cap Y)$ and $\partial(X \cup Y)$ are also tight in $G$. Furthermore, no edge connects $X \cap \overline{Y}$ to $\overline{X} \cap Y$. □

**Proposition 1.4 ([4]).** Every matching covered graph on four or more vertices is 2-connected. □
Proposition 1.5 ([5]). Let $G$ be a matching covered graph, and let $C$ be a tight cut of $G$. Then both $C$-contractions are matching covered. Moreover, if $G'$ is a $C$-contraction of $G$ then a tight cut of $G'$ is also a tight cut of $G$. Conversely, if a tight cut of $G$ is a cut of $G'$ then it is also tight in $G'$.

Corollary 1.6. Let $G$ be a matching covered graph, and let $C = \partial(X)$ be a tight cut of $G$. Then, both shores $X$ and $\overline{X}$ of $C$ induce connected graphs.

Proof: The $C$-contraction $G' := G/(X \to x)$ of $G$ is matching covered, hence it is 2-connected, by Proposition 1.4. Thus, $G' - x$ is connected. In other words, $X$ induces a connected subgraph of $G$. Likewise, $X$ also induces a connected subgraph of $G$.

If $C$ is a tight cut of $G$, then both $C$-contractions of $G$ are matching covered. A nontrivial tight cut may help us to reduce a matching covered graph to smaller matching covered graphs. We may apply to $G$ a procedure, called a tight cut decomposition of $G$, which produces a list of bricks and braces. If $G$ itself is a brick or a brace then the list consists of just $G$. Otherwise, $G$ has a nontrivial tight cut, $C$. Then, both $C$-contractions of $G$ are matching covered. One may recursively apply the tight cut decomposition procedure to each $C$-contraction of $G$, and then combine the resulting lists to produce a tight cut decomposition of $G$ itself.

Theorem 1.7 (Lovász [5]). Any two applications of the tight cut decomposition procedure to $G$ produce the same list of bricks and braces, up to multiple edges.

In particular, any two applications of the tight cut decomposition procedure yield the same number of bricks, which is called the brick number of $G$ and denoted by $b(G)$.

The graph $G$ is bicritical if $G - S$ has a perfect matching, for each pair $S$ of vertices of $G$. If $G$ is bicritical then, by Tutte’s Theorem, every barrier of $G$ is trivial.

1.1. ELP-cuts

There are two types of tight cuts that play a critical role in studying matching theory. A barrier of $G$ is a nonempty set of vertices of $G$ such that $o(G - B) = |B|$. Moreover, $B$ is a nontrivial barrier if $|B| \geq 2$. By Tutte’s theorem, each barrier $B$ of $G$ is an independent set, and all components of $G - B$ are odd components. A cut $C$ is called a barrier-cut if there exists a barrier $B$ and a component $H$ of $G - B$ such that $C = \partial(V(H))$. Clearly, a barrier-cut is a tight cut.

A 2-separation of $G$ is a pair $S$ of vertices of $G$ such that $G - S$ is not connected and each of the components of $G - S$ is even. Let $\{u, v\}$ be a 2-separation of $G$, let $\{G_1, G_2\}$ be a partition of $G - \{u, v\}$. Each of the two cuts $C' := \partial(V(G_1) + u)$ and $C'' := \partial(V(G_1) + v)$ is a 2-separation cut of $G$ and the pair $\{C', C''\}$ is a 2-separation cut pair of $G$. 

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Barrier-cuts and 2-separation cuts are particular types of tight cuts and are called ELP cuts, named after Edmonds, Lovász, and Pulleyblank, who proved the following fundamental result.

**Theorem 1.8 (The ELP Theorem [3]).** Every matching covered graph that has a nontrivial tight cut has either a nontrivial barrier or a 2-separation. □

Their proof is based on linear programming techniques. Szigeti [6] gave a purely graph theoretical proof. Carvalho, Lucchesi and Murty [7] provided an alternative proof.

1.2. Ultimately ELP cuts

We now define a special type of tight cut decomposition procedure. We may apply to $G$ a procedure, called an ELP cut decomposition of $G$, which produces a list of bricks and braces. If $G$ itself is a brick or a brace then the list consists of just $G$. Otherwise, by the ELP Theorem, $G$ has a nontrivial ELP cut, $C$. Then, both $C$-contractions of $G$ are matching covered. One may recursively apply the ELP cut decomposition procedure to each $C$-contraction of $G$, and then combine the resulting lists to produce an ELP cut decomposition of $G$ itself. Each cut used in the ELP cut decomposition procedure is said to be ultimately an ELP cut. It is easy to see that the following conjecture implies that every nontrivial tight cut of $G$ is ultimately an ELP cut.

**Conjecture 1.9.** [Carvalho, Lucchesi and Murty [7]] Let $C$ be a nontrivial tight cut of a matching covered graph $G$. Then, $G$ has a nontrivial ELP cut which is laminar with $C$.

In 2002, the three authors proved a particular and very important case of the Conjecture, in which $G$ is a brick and $e$ is an edge of $G$ such that $G - e$ is matching covered and $b(G - e) = 2$ [8]. In [7] they also proved the validity of the Conjecture for bicritical graphs.

In this paper we present a proof of a result, our Main Theorem, that implies Conjecture 1.9. To state the Main Theorem we need one more definition and a simple result.

Let $G$ be a matching covered graph, let $C$ be a tight cut of $G$ and let $S$ be a set of vertices of $G$ which is either a barrier or a 2-separation. The set $S$ is $C$-sheltered if $S$ is a subset of a shore of $C$, and is $C$-avoiding if, each ELP cut associated with $S$ is laminar with $C$. If $S$ is $C$-sheltered then, as each shore of $C$ induces a connected subgraph $H$ of $G$ (Corollary 1.6), one of the components of $G - S$ is a supergraph of $H$. It follows that if $S$ is $C$-sheltered then $S$ is $C$-avoiding. We record this result for later reference.

**Proposition 1.10.** Let $S$ be either a 2-separation or a barrier of a matching covered graph $G$ and let $C$ be a tight cut of $G$. If $S$ is $C$-sheltered then some cut associated with $S$ has a shore that is a superset of a shore of $C$, say, $X$, and all the other cuts associated with $S$ have a shore that is a subset of $X$. Consequently, if $S$ is $C$-sheltered then it is $C$-avoiding. □
We now state our result, which, in view of the Proposition above, implies Conjecture 1.9.

**Theorem 1.11 (Main Theorem).** Let $C$ be a nontrivial tight cut of a matching covered graph $G$. Then either $G$ has a $C$-sheltered nontrivial barrier or a 2-separation cut which is laminar with $C$ (see Example 1.12).

**Example 1.12.** Consider the graph depicted in Figure 1. The tight cut $C$ is laminar with the cut $D$, which is a 2-separation cut associated with the pair $\{u_1, u_2\}$. The cut $C$ is also laminar with the 2-separation cut $F$, which is associated with the pair $\{1, b_3\}$. The barriers $\{b_1, b_2, b_3\}$ and $\{2, 3\}$ are $C$-sheltered, whereas the barrier $\{1, 2, 3\}$ is not $C$-avoiding.

The proof of the Main Theorem will be given in Section 3, after we present some necessary results in Section 2.

**2. Ingredients**

In this section we prove three Lemmas, which play a crucial role in the proof of the Main Theorem.

**Lemma 2.1.** Let $C := \partial(X)$ be a 2-separation cut of a matching covered graph $G$, associated with a 2-separation $\{u_1, u_2\}$, where $u_1 \in X$ and $u_2 \in \overline{X}$. Let $S_H$ denote either a 2-separation or a barrier of the $C$-contraction $H := G/(\overline{X} \rightarrow \overline{\kappa})$ of $G$ and let

$$S := \begin{cases} S_H, & \text{if } \overline{\kappa} \notin S_H, \\ (S_H - \overline{\kappa}) + u_2, & \text{if } \overline{\kappa} \in S_H. \end{cases}$$

Then the following properties hold (See Figure 2):
Figure 2: Illustration for Lemma 2.1

(i) every component of $H - S_H$ that does not contain vertices in $\{u_1, \overline{x}\}$ is a component of $G - S$,

(ii) at most one component of $H - S_H$ contains vertices in $\{u_1, \overline{x}\}$, and

(iii) if $S_H$ is a barrier of $H$ then $S$ is a barrier of $G$ and if $S_H$ is a 2-separation of $H$ then $S$ is a 2-separation of $G$.

Proof: Let $H$ denote the collection of components of $H - S_H$ and let $H_0$ denote the collection of components of $H - S_H$ that contain a least one vertex in $\{u_1, \overline{x}\}$.

(i) Let $H[Y]$ be a component in $H - H_0$. By definition of $H_0$, $Y \subseteq X - u_1$, hence $\partial_G(Y) = \partial_H(Y)$. We shall now prove that every edge of $\partial(Y)$ is incident with a vertex of $S$.

Let $e$ be an edge of $\partial(Y)$. Then $e$ joins a vertex $u$ in $Y$ to a vertex $v$ in $S_H$. If $v \neq \overline{x}$ then $v \in S$. We may thus assume that $v = \overline{x}$, which implies that $e \in C$. Every edge of $C$ is incident in $G$ with a vertex in $\{u_1, u_2\}$. Thus, either $u_1$ is an end of $e$ in $X$ or $u_2$ is an end of $e$ in $\overline{X}$. As $Y \subseteq X - u_1$, the end of $e$ in $Y$ is not $u_1$. It follows that $e$ is not incident with $u_1$, hence $e$ is incident with $u_2$ in $G$. In both alternatives we conclude that $e$ is incident with a vertex of $S$. This conclusion holds for each edge $e \in \partial(Y)$. We deduce that $H[Y]$ is a component of $G - S$.

(ii) The graph $G$ is 2-connected (Proposition 1.4), hence the vertices $u_1$ and $\overline{x}$ are adjacent in the graph $H$.

2.1.1. $|H_0| \leq 1$, with equality if $S_H$ is a barrier of $H$.

Proof: If neither $u_1$ nor $\overline{x}$ is in $S_H$ then, as $u_1$ and $\overline{x}$ are adjacent, it follows that $|H_0| = 1$. Assume thus that at least one of $u_1$ and $\overline{x}$ is in $S_H$. In that case, the asserted inequality holds. Moreover, suppose that $S_H$ is a barrier then, by Corollary 1.2, $S_H$ is independent. It follows that precisely one of the vertices in $\{u_1, \overline{x}\}$ is in $S_H$, hence equality holds.

(iii) By (i), all the components in $H - H_0$ are components of $G - S$. By 2.1.1, $|H_0| \leq 1$, with equality if $S_H$ is a barrier of $H$.

Consider first the case in which $S_H$ is a barrier of $H$. Each of the $|S| - 1$ (odd) components in $H - H_0$ is a component of $G - S$. By parity, $G - S$ has at
least $|S|$ odd components. By Corollary 1.2, $G - S$ has precisely $|S|$ components, all of which are odd. Indeed, $S$ is a barrier of $G$.

Finally, suppose that $S_H$ is a 2-separation. The collection $\mathcal{H} - \mathcal{H}_0$ contains a graph, $K$, which is an even component of $H - S_H$. Thus, $K$ is an even component of $G - S$. Moreover, $K$ is a subgraph of $G[X - u_1]$, the subgraph of $G$ induced by $X - u_1$, hence $G - S$ has two or more components. As $K$ is even and $G$ is matching covered, it follows that $G - S$ has only even components (Corollary 1.2). Indeed, $S$ is a 2-separation of $G$. □

Let $G$ be a matching covered graph, let $C := \partial(X)$ be a tight cut of $G$, let $B$ denote a barrier of $G$ and let $\mathcal{H}$ denote the set of components of $G - B$. For each shore $Z$ of $C$, let $\mathcal{H}_Z$ be the set of those components $H \in H$ such that $|V(H) \cap Z|$ is odd. Clearly, $|\mathcal{H}_X| + |\mathcal{H}_X^-| = |\mathcal{H}| = |B|$. See Figure 3.

![Figure 3](image_url)

**Figure 3:** A barrier $B := \{b_1, b_2, b_3\}$, where $|\mathcal{H}_X| = 2$ and $|\mathcal{H}_X^-| = 1$

**Lemma 2.2.** Let $G$ be a matching covered graph, let $C := \partial(X)$ be a tight cut of $G$, let $B$ denote a barrier of $G$ and let $K$ be a component of $\mathcal{H}_X$ that contains a vertex adjacent to a vertex in $B \cap X$. The following properties hold:

(i) $|B \cap X| = |\mathcal{H}_X| - 1$ and $|B \cap X^-| = |\mathcal{H}_X^-| + 1$.

(ii) each component in $\mathcal{H}_X^-$ is a subgraph of $G[X]$ and has no vertex adjacent to a vertex of $B \cap X$.

(iii) $B \cap X$ is a (C-sheltered, possibly trivial) barrier of $G$, and

(iv) if $C$ is nontrivial and $B$ is $C$-avoiding and nontrivial, then $B \cap X$ is a (C-sheltered) nontrivial barrier of $G$.

**Proof:** The following simple statement is important in the proof of the Lemma.

**2.2.1.** Let $Z$ be a shore of $C$, let $H \in \mathcal{H}_Z$, let $e$ be an edge in $\partial(V(H))$ incident with a vertex of $B \cap \overline{X}$ and let $M$ be a perfect matching of $G$ that contains edge $e$. Then, the edge of $M \cap C$ has at least one end in $H$. 


Proof: Let \( e := vw, v \in V(H), w \in B \cap Z \). If \( v \in Z \) then clearly \( e \) is the edge of \( M \cap C \). Assume thus that \( v \in Z \). As \( W := V(H) \cap Z \) is even, it follows that \( M \cap \partial(W) \) has another edge, \( f \). The edge \( f \) cannot have an end in \( B \), because \( H \) is matched by \( M \) to the end \( w \) of \( e \). Thus, \( f \) has an end in \( V(H) \cap Z \), hence \( f \) has both ends in \( H \) and is the edge of \( M \cap C \). \( \square \)

**(i)** By hypothesis, \( K \) has a vertex, \( v \), which is adjacent to a vertex, \( w \), of \( B \cap X \).
Let \( M \) be a perfect matching of \( G \) that contains edge \( vw \). By (2.2.1), the edge of \( M \cap C \), say, \( f \), has at least one end in \( K \). Clearly, either \( f \) has both ends in \( K \) or it is incident with a vertex of \( B \). It follows that except for \( K \), all the other components of \( H_X \) are matched by \( M \) to vertices of \( B \cap X \) and every component of \( H_{\overline{X}} \) is matched by \( M \) to a vertex of \( B \cap \overline{X} \). The asserted equality holds.

**(ii)** Assume, to the contrary, that \( H_{\overline{X}} \) has a component that contains a vertex adjacent to a vertex in \( B \cap X \). From (i) with the roles of \( X \) and \( \overline{X} \) interchanged, we deduce that

\[
|B \cap \overline{X}| = |H_{\overline{X}}| - 1 \quad \text{and} \quad |B \cap X| = |H_X| + 1.
\]

This is a contradiction to property (i). Let \( H \) be a component in \( H_{\overline{X}} \) and assume, to the contrary, that \( H \) has vertices in \( X \). As \( H \) has an odd number of vertices in \( \overline{X} \), it has an even number of vertices in \( X \), hence \( V(H) \cap X \) is a proper subset of \( X \). The subgraph \( G[X] \) of \( G \) is connected (Corollary 1.6), hence some vertex of \( V(H) \cap X \) is adjacent to a vertex of \( B \cap X \), a contradiction.

**(iii)** Let \( H \) be a component in \( H_{\overline{X}} \). From (ii) we deduce that \( H \) is a subgraph of \( G[X] \). Moreover, every edge of \( \partial(V(H)) \) is incident with a vertex of \( B \cap \overline{X} \).
We conclude that \( H \) is a component of \( G - (B \cap \overline{X}) \). This conclusion holds for each \( H \in H_{\overline{X}} \), from (ii) we infer that the \( |B \cap \overline{X}| - 1 \) components in \( H_{\overline{X}} \) are (odd) components of \( G - (B \cap \overline{X}) \). By parity and Corollary 1.2, \( B \cap \overline{X} \) is a barrier of \( G \).

**(iv)** From (iii) we infer that \( B \cap \overline{X} \) is a \((C\text{-sheltered})\) barrier of \( G \). Suppose that \( C \) is nontrivial and \( B \) is \( C\text{-avoiding} \) and nontrivial. Assume, to the contrary, that \( B \cap \overline{X} \) is trivial, let \( v \) denote the only vertex of \( B \cap \overline{X} \). By (i) \( H = H_X \).

By hypothesis, \( C \) is nontrivial, thus \( \overline{X} - v \) is not empty. Consequently, some component of \( G - B, H \), contains vertices of \( \overline{X} - v \). Moreover, as \( H = H_X \), the component \( H \) contains vertices in \( X \). In sum, \( V(H) \) contains vertices in both shores of \( C \).

By hypothesis, \( B \) is nontrivial. As \( v \) is the only vertex of \( B \) in \( \overline{X} \), it follows that \( B \) has vertices in \( X \). Thus, \( V(H) \) is not a superset of \( X \). As \( v \), a vertex of \( \overline{X} \), is in \( B \), it follows that \( V(H) \) is not a superset of \( \overline{X} \). In sum, \( V(H) \) is neither a subset nor a superset of any shore of \( C \). Consequently, the cuts \( C \) and \( \partial(V(H)) \) cross, in a contradiction to the hypothesis that \( B \) is \( C\text{-avoiding} \). We conclude that \( B \cap \overline{X} \) is nontrivial. \( \square \)
Lemma 2.3. Let $C := \partial(X)$ be a nontrivial tight cut of a matching covered graph $G$ and let $t$ be a vertex of $\overline{X}$. Suppose that the assertion of the Main Theorem holds for every graph having $|V(G)|$ or fewer vertices. Then one of the following properties holds:

(i) either the graph $G$ has a 2-separation that does not contain the vertex $t$, or
(ii) $G$ has a 2-separation, $S$, that contains the vertex $t$, associated with a cut $D := \partial(Y)$, such that $Y \subseteq \overline{X}$, or
(iii) the graph $G$ has a $C$-sheltered nontrivial barrier.

Proof: By induction on $|V(G)|$. By hypothesis, the assertion of the Main Theorem holds for $G$. If $G$ has a $C$-sheltered nontrivial barrier then alternative (iii) of the statement of the Lemma holds. We may thus assume that $G$ has a 2-separation $T$, and an associated cut $F := \partial(Z)$ such that $Z$ is a subset of a shore of $C$. If $t \notin T$ then the alternative (i) of the assertion holds. We may thus assume that $t \in T$. If $Z \subseteq \overline{X}$ then the alternative (ii) holds, with $Y := Z$ and $D := F$. It now remains the case in which $Z \cap X$ and $t \notin T$.

Let $u$ be the vertex of $T - t$. One of $u$ and $t$ is in $X$. As $t \notin X$, it follows that $u \in Z$. Let $Z' := (Z - u) + t$. The cuts $F$ and $F' := \partial(Z')$ are members of a 2-separation cut pair of $G$. Let $H$ be the $F'$-contraction $G/(Z' \rightarrow z')$ of $G$, let $X_H := (X - Z) + u$ and let $C_H := \partial(X_H)$. See Figure 4.

We plan now to apply the induction hypothesis, with $H$, $X_H$, $C_H$ and $z'$ playing respectively the roles of $G$, $X$, $C$ and $t$. The intersection of the shores $X$ and $\overline{X}$ is odd and equal to the shore $X_H$ of $C_H$. The cut $C_H$ is tight in $G$ (Proposition 1.3). As $C_H$ and $F'$ are laminar, $C_H$ is tight in $H$ (Proposition 1.5). Moreover, as $Z \subseteq X$, it follows that $C_H$ is a nontrivial tight cut of $H$. As $|V(H)| < |V(G)|$, we may infer, by hypothesis, that the assertion of the Main Theorem holds for every graph having $|V(H)|$ or fewer vertices. Moreover, $z' \in \overline{X}_H$. We now apply the induction hypothesis to $H$, $X_H$, $C_H$ and $z'$ playing respectively the roles of $G$, $X$, $C$ and $t$. We consider separately the three possible alternatives.

Case 1. The graph $H$ has a 2-separation which does not contain the vertex $z'$.

Let $S$ be a 2-separation of $H$ that does not contain the vertex $z'$. By Lemma 2.1, $S$ is a 2-separation of $G$. Clearly, $S$ does not contain the vertex $t$. The alternative (i) of the assertion holds.

Case 2. The graph $H$ has a 2-separation $S_H$ that contains the vertex $z'$, associated with a 2-separation cut $D_H := \partial(Y_H)$, such that $Y_H \subseteq \overline{X}_H$.

Let $W_H := Y_H - S_H$. As $z' \in S_H$, the vertex $z'$ is not in $W_H$. As $Y_H \subseteq \overline{X}$, the vertex $u$ is not in $W_H$. In sum, $W_H$ and $\{u, z'\}$ are disjoint. Let $S := (S_H - z') + t$ and let $Y := W_H + t = (Y_H - S_H) + t$. By Lemma 2.1, every (even) component of $H[W_H]$ is a component of $G - S$ and $S$ is a 2-separation of $G$. Moreover, the vertex $t$ is in $S$, the cut $D := \partial(Y)$ is a 2-separation cut of $G$ associated with $S$ and its shore $Y$ is a subset of $\overline{X}$. The alternative (iii) of the assertion holds.
Case 3. The graph $H$ has a $C_H$-sheltered nontrivial barrier.

Let $S_H$ be a $C_H$-sheltered nontrivial barrier of $H$. We now apply Lemma 2.1. Let $S$ be as defined in the statement of that Lemma. Then, $S$ is a (nontrivial) barrier of $G$. If $S_H$ is a subset of $X_H$ then $S = S_H$ and $S$ is a subset of $X$. If $S_H$ is a subset of $\overline{X}_H - z'$ then $S = S_H$ and $S$ is a subset of $\overline{X}$. Finally, if $S_H$ is a subset of $\overline{X}$ and $z' \in S_H$ then $S$ is equal to $(S_H - z') + t$ and is a subset of $\overline{X}$. In all alternatives, $S$ is a $C$-sheltered nontrivial barrier of $G$, hence alternative (iii) of the assertion of the Lemma holds.

The proof of the Lemma is complete. \qed
3. Proof of the Main Theorem

Proof: Let \( G \) be a matching covered graph and let \( C := \partial(X) \) be a nontrivial tight cut of \( G := (V, E) \). Assume, as the induction hypothesis, that the assertion holds for every matching covered graph having fewer than \( |V| \) vertices. We shall now prove that either \( G \) has a 2-separation cut having a shore that is a subset of a shore of \( C \) or \( G \) has a \( C \)-sheltered nontrivial barrier.

As \( C \) is nontrivial and tight, we infer from the ELP Theorem (Theorem 1.8) that \( G \) either has a 2-separation or a nontrivial barrier. We consider these possibilities separately.

Case 1. The graph \( G \) does not have 2-separations.

In this case, the graph \( G \) has a nontrivial barrier, \( B \). If \( B \) is \( C \)-avoiding then, by Lemma 2.2(iv) one of \( B \cap X \) and \( B \cap \overline{X} \) is a nontrivial (\( C \)-sheltered) barrier of \( G \). We may thus assume that \( G \) is free of 2-separations. (In fact, the cut \( \partial((X \cap Y) + u) \) is a 2-separation cut of \( G \) associated with \( \{u, v\} \).)

We now apply Lemma 2.3 with \( H, X \cap Y, I \) and \( \overline{y} \) playing, respectively, the roles of \( G, X, C \) and \( t \). We then deduce that one of the following possibilities holds:

(i) either the graph \( H \) has a 2-separation that does not contain the vertex \( \overline{y} \),

(ii) \( H \) has a 2-separation, \( S_H \), that contains the vertex \( \overline{y} \), associated with a cut \( D_H := \partial(Y_H) \), such that \( Y_H \subseteq (X \cap Y) + \overline{y} \), or

(iii) the graph \( H \) has a nontrivial \( I \)-sheltered barrier, \( B_H \).
We shall now eliminate the two first possibilities. Assume, to the contrary, that \( H \) has a 2-separation, \( S_1 \), that does not contain the vertex \( \overline{y} \). One of the (even) components of \( H - S_1, K_1 \), does not contain the vertex \( \overline{y} \) and is a proper subgraph of \( H \). In that case, \( K_1 \) is an even component of \( G - S_1 \). By Corollary 1.2, \( G - S_1 \) has no odd components hence \( S_1 \) is a 2-separation of \( G \). This is a contradiction to the hypothesis that \( G \) is free of 2-separations.

Assume, to the contrary, that \( H \) has a 2-separation, \( S_H \), that contains the vertex \( \overline{y} \), associated with the cut \( D_H := \partial(Y_H) \), where \( Y_H \subseteq (\overline{X} \cap Y) + \overline{y} \). Let \( v \) denote the vertex of \( S_H - \overline{y} \) and let \( K \) be a component of \( H[Y_H - S_H] \). Necessarily, \( K \) is even. The set \( V(K) \) is a subset of \( \overline{X} \cap Y \), hence \( \partial_H(V(K)) = \partial_G(V(K)) \). Let \( e \) be an edge of \( \partial(V(K)) \) that is not incident with \( v \). Then, \( e \) is incident with \( \overline{y} \) in \( H \), hence, in \( G \), \( e \) joins a vertex of \( \overline{X} \cap Y \) to a vertex, \( w \), of \( \overline{y} \). Necessarily \( w = u \), hence \( K \) is an even component of \( G - \{u,v\} \). By Corollary 1.2 \( \{u,v\} \) is a 2-separation of \( G \), again a contradiction to the hypothesis that \( G \) is free of 2-separations.

We deduce that \( H \) has a nontrivial I-sheltered barrier, \( B_H \). If \( B_H \) is a subset of \( X \cap Y \) or of \( \overline{X} \cap Y \) then \( B_H \) is \( C \)-sheltered, and the assertion of the Theorem holds. We may thus assume that \( B_H \) is a subset of \( (\overline{X} \cap Y) + \overline{y} \) that contains the vertex \( \overline{y} \). In that case, the set \( B_G := (B_H - \overline{y}) \cup B \) is a barrier of \( G \). One of the components of \( H - B_H \), say, \( K \), contains all the vertices of \( X \cap Y \), by Proposition 1.10. As \( \overline{y} \in B_H \), \( K \) is also a component of \( G - B_G \) and \( V(K) \cap X = X \cap Y \), hence \( K \) contains an odd number of vertices in \( X \). Moreover, as \( H \) is 2-connected, \( K \) has vertices adjacent to at least one vertex of \( B_H - \overline{y} \), which is a vertex of \( B_G \cap \overline{X} \). Finally, \( B_G \cap \overline{X} \) is nontrivial. By Lemma 2.4(iii) \( B_G \cap \overline{X} \) is a nontrivial \( C \)-sheltered barrier of \( G \).

The analysis of Case 1 is complete. We may thus assume that

\[
G \text{ has 2-separations}.
\]

If \( G \) has a \( C \)-sheltered 2-separation then the assertion holds, by Proposition 1.10. We may thus assume that

each 2-separation of \( G \) contains a vertex in each shore of \( C \).

Let \( S \) be a 2-separation of \( G \) and let \( K \) be a component of \( G - S \). As \( K \) is even, it follows that \( |X \cap V(K)| \equiv |\overline{X} \cap V(K)| \mod 2 \). We say that \( K \) is balanced if \( |X \cap V(K)| \) is even and unbalanced, otherwise.

**Proposition 3.2.** Let \( S \) be a 2-separation of \( G \) such that \( G - S \) has an unbalanced component. Then, \( G - S \) has precisely two components, both of which are unbalanced. Moreover, every edge of \( C \) has both ends in a component of \( G - S \) (See Figure 2).

**Proof.** Suppose that \( G - S \) has an unbalanced component, \( K_1 := G[L_1] \). The shore \( X \) of \( C \) is odd and it contains precisely one vertex of \( S \). Moreover, \( |L_1 \cap X| \) is also odd. Thus, \( |X - L_1 - S| \) is odd. It follows that \( G - S \) has an unbalanced component, \( K_2 := G[L_2] \), distinct from \( K_1 \).
Assume, to the contrary, that $G - S$ contains a component, $K_3 := G[L_3]$, not necessarily unbalanced, but distinct from both $K_1$ and $K_2$. Let $e$ be an edge of $\partial(L_3)$ and let $M$ be a perfect matching of $G$ that contains edge $e$. Necessarily, $e$ is incident with a vertex of $S$. As $L_3$ is even, $M$ contains also an edge in $\partial(L_3)$ which is incident with the other vertex of $S$. For $i = 1, 2$, as $|X \cap L_i|$ is odd, the cut $\partial(X \cap L_i)$ contains an edge in $M$ and that edge is not incident with a vertex of $S$, hence it is an edge having both ends in $L_i$, therefore it is an edge of $C$. We conclude that $M$ contains more than one edge in $C$, a contradiction to the hypothesis that $C$ is tight. Indeed, $K_1$ and $K_2$ are the only two components of $G - S$ and they are both unbalanced.

Let $u_1$ and $u_2$ be two vertices of $S$. Let $Y := L_1 + u_1$ be the shore of a cut associated with $S$ such that $|X \cap Y|$ is odd. We have assumed that $S$ has one vertex in each shore of $C$, hence $u_1 \in \overline{X}$ and $u_2 \in X$. Let $D := \partial(Y)$ (See Figure 5). The cuts $C$ and $D$ are tight and cross. No edge of $G$ joins a vertex in $(X \cap L_1) + u_1$ to a vertex in $(X \cap L_2) + u_2$ (Proposition 3.3). By symmetry, no edge of $G$ joins a vertex in $(X \cap L_1) + u_2$ to a vertex in $(X \cap L_2) + u_1$. □

Let $S$ be a 2-separation of $G$. A component $K$ of $G - S$ is good if either $K$ is balanced or each vertex of $S$ is adjacent to two or more vertices of $K$. Let $\mathcal{S}$ be the collection of 2-separations of $G$. For each $S \in \mathcal{S}$, let $\mathcal{F}(S)$ be the set of good components of $G - S$. Let

$$\mathcal{F} := \bigcup_{S \in \mathcal{S}} \mathcal{F}(S).$$

**Case 2.** The collection $\mathcal{F}$ is empty.

Let $S$ be a 2-separation of $G$. The hypothesis of the case implies that the components of $G - S$ are unbalanced. By Proposition 3.2, $G - S$ consists of precisely two components, $K_i := G[L_i]$, $i = 1, 2$. We have assumed that each shore of $C$ contains a vertex of $S$. Let $u_1$ be the vertex of $S$ in $\overline{X}$ and let $u_2$ be the vertex of $S$ in $X$ (Figure 5).
The hypothesis of the case also implies that for \( i = 1, 2 \), one of \( u_1 \) and \( u_2 \) is adjacent only to one vertex of \( L_i \). Adjust notation so that \( u_1 \) is adjacent to only one vertex of \( L_1 \), says \( v_1 \).

**Proposition 3.3.** The vertex \( u_1 \) is adjacent only to one vertex of \( L_2 \).

**Proof:** Suppose, to the contrary, that \( u_1 \) is adjacent to more than one vertex in \( L_2 \). By the hypothesis of the case, \( u_2 \) is adjacent only to one vertex in \( L_2 \), say, \( v_2 \). Let \( T := \{v_1, v_2\} \). No edge of \( G - T \) joins a vertex of \( Z_1 := (L_1 - v_1) + u_2 \) to a vertex of \( Z_2 := (L_2 - v_2) + u_1 \). Thus, \( G - T \) has two or more components.

Let us now prove that each component of \( G - T \) is even. If \( G - T \) has an odd component then, by parity, it has at least two odd components. In that case, by Corollary 1.2, \( G - T \) has precisely two components, both odd. But if \( G - T \) has only two components then they are \( G[Z_1] \) and \( G[Z_2] \), both even. We conclude that each component of \( G - T \) is even.

Let us now prove that each component of \( G - T \) has an even number of vertices in each shore of \( C \). If \( G - T \) has only two components then they are \( G[Z_1] \) and \( G[Z_2] \), and both have an even number of vertices in each shore of \( C \). Alternatively, if \( G - T \) has more than two components then again each of these components has an even number of vertices in each shore of \( C \), by Proposition 3.2.

As each component of \( G - T \) is even, the pair \( T \) is a 2-separation of \( G \). Each component of \( G - T \) has an even number of vertices in each shore of \( C \). We conclude that \( \mathcal{F} \) is nonempty, in contradiction to the hypothesis of the case. \( \square \)

In sum, for \( i = 1, 2 \), the vertex \( u_1 \) is adjacent to only one vertex of \( L_i \), say, \( v_i \). Clearly, \( \{v_1, v_2\} \) is a \( C \)-sheltered nontrivial barrier of \( G \).

**Case 3.** The collection \( \mathcal{F} \) is nonempty.

Let \( K_1 \) be a minimal component in \( \mathcal{F} \), let \( L_1 := V(K_1) \) and let \( S \) be the associated 2-separation of \( G \). Let \( u_1 \) and \( u_2 \) be two vertices of \( S \). Let \( Y := L_1 + u_1 \) be the shore of a cut associated with \( S \) such that \( |X \cap Y| \) is odd and let \( D := \partial(Y) \). We have assumed that \( S \) has one vertex in each shore of \( C \). If \( K_1 \) is unbalanced then \( u_1 \in X \) and \( u_2 \in X \) (See Figure 3). Alternatively, if \( K_1 \) is balanced then \( u_1 \in X \) and \( u_2 \in X \) (See Figure 4). Let \( H := G[(\overline{Y} \rightarrow \overline{Y})] \) and let \( I := \partial(X \cap Y) \). The cuts \( C \) and \( D \) cross. By Proposition 1.3 the cut \( I \) is tight and no edge joins a vertex of \( X \cap Y \) to a vertex of \( X \cap Y \).

**Case 3.1.** The cut \( I \) is trivial.

Suppose that \( I \) is trivial. If \( K_1 \) is unbalanced, then, as \( K_1 \) is good, it has two or more vertices adjacent to vertices of \( S \), hence \( I \) is nontrivial. We deduce that \( K_1 \) is balanced. In that case, \( X \cap Y = \{u_1\} \) and every edge of \( \partial(X \cap Y) \) is incident with a vertex in \( S \). It follows that the shore \( Y' := (X \cap Y) + u_2 \) of the associated cut \( \partial(Y') \) is a subset of \( X \). The assertion of the Theorem holds in this case. We may thus assume that

The cut \( I \) is nontrivial.
The cuts $D$ and $I$ are laminar, hence $I$ is tight in $H$ (Proposition 1.5). We now apply the induction hypothesis, with $H$ and $I$ playing respectively the roles of $G$ and $C$.

Case 3.2. The graph $H$ has a nontrivial $I$-sheltered barrier.

Let $S_H$ be a nontrivial $I$-sheltered barrier of $H$. Let us now apply Lemma 2.1 to $H$, $Y$ and $\overline{y}$ playing respectively the roles of $G$, $X$ and $x$. Let $S$ be the set defined in the statement of that Lemma. By the item (iii) of the Lemma, $S$ is a (nontrivial) barrier of $G$. If $\overline{y} \notin S_H$ then $S$ is equal to $S_H$ and is a subset of $X \cap Y$ or of $\overline{X} \cap Y$, hence $S$ is $C$-sheltered. We may thus assume that $\overline{y} \in S_H$, in which case $S = (S_H - \overline{y}) + u_2$ and $S_H$ is a subset of $(X \cap Y) + \overline{y}$. If $K_1$ is balanced then $u_2 \in X$ (Figure 6), hence $S$ is a subset of $\overline{X}$. We may thus assume that $K_1$ is unbalanced.

In $H$, some component of $H - S_H$, say, $W$, contains all the vertices of $X \cap Y$, by Proposition 1.10. By the 2-connectivity of $H$, $W$ contains a vertex adjacent to a vertex of $S_H \cap X \cap Y$. Moreover, by Lemma 2.1, $W$ is a component of $G - S$. By Lemma 2.2 (iii) the set $S - u_2$, which is equal to $S \cap \overline{X}$, is a barrier of $G$. If $S \cap \overline{X}$ is nontrivial then the assertion of the Theorem holds.

We may thus assume that $S \cap \overline{X}$ is trivial, let $v$ be its only vertex. Let $L$ be a component of $S - S_H$ distinct from $W$. Then, $V(L) \subset \overline{X} \cap Y$. The only vertex of $\overline{X} \cap Y$ adjacent to $\overline{y}$ is the vertex $u_1$, hence $L$ contains $u_1$. We deduce that $L$ and $W$ are the only two components of $H - S_H$. As $K_1$ is unbalanced and good, $u_1$ is adjacent to two or more vertices of $\overline{X} \cap L_1$, hence $L$ is nontrivial. The graph $L - u_1$ is a proper nonempty subgraph of $G$, hence the graph $G - \{u_1, v\}$ is not connected. By Corollary 1.2, $\{u_1, v\}$ is either a barrier or a 2-separation of $G$. As $\{u_1, v\}$ is $C$-sheltered, the assertion of the Theorem holds, by Proposition 1.10. The analysis of Case 3.2 is complete.

Case 3.3. The graph $H$ has a 2-separation cut, $D_H := \partial(Z_H)$, associated with a 2-separation $S_H$, such that $Z_H$ is a subset of a shore of $I$. 

Figure 6: The components $G[L_1]$ of $G - \{u_1, u_2\}$ is balanced
Let us now apply Lemma 2.1 to $H$, $Y$ and $\overline{y}$ playing respectively the roles of $G$, $X$ and $\overline{x}$. Let $S$ be the set defined in the statement of that Lemma. By the item (iii) of the Lemma, $S$ is a 2-separation of $G$. We have assumed that one of the vertices of $S$ is in $X$, the other is in $\overline{X}$. Let $w_1$ be the vertex of $S$ in $X$, and let $w_2$ be the vertex of $S$ in $\overline{X}$.

Let $W_H := Z_H - S_H$. Then, $W_H$ is a proper subset of a shore of $I$. If $W_H$ is a subset of a shore of $C$, then, as $S$ meets both shores of $C$, it follows that one of the 2-separation cuts of $G$ associated with $S$ has a shore that is a subset of $C$. We may thus assume that $W_H$ meets both shores of $I$. As $W_H$ is a subset of a shore of $I$, it follows that $\overline{y} \in W_H$ and $W_H$ is a subset of the shore $(\overline{X} \cap Y) + \overline{y}$. Then, $S = S_H$.

If $K_1$ is unbalanced then, by definition of $K_1$, the vertex $u_2$ is adjacent to at least two vertices of $X \cap Y$. Thus, both vertices of $S_H$ are in $X \cap Y$, hence $S$ is $C$-sheltered, a case already considered. We may thus assume that $K_1$ is balanced. The subgraph of $G$ induced by $\overline{y}$ is connected (Corollary 1.4). One of the components of $G - S$ is the graph $G[W]$, where $W = (W_H - \overline{y}) \cup \overline{y}$. As $W_H$ is even, then so too is $W$. The vertices $u_1$ and $\overline{y}$ are adjacent in $H$, therefore $u_1 \in S_H \cup W_H$, hence $u_1 \in S \cup W$. We conclude that every component of $G - S$ distinct from $G[W]$ is a proper subgraph of $K_1$. Moreover, the set $T_1 := W_H - \overline{y}$ is an odd subset of $\overline{X} \cap Y$, and $T_2 := \overline{X} \cap \overline{y}$ is also odd. Clearly, $W \cap \overline{X} = T_1 \cup T_2$, hence $W \cap \overline{X}$ is even. That is, $W$ is balanced. By Proposition 3.2, every component of $G - S$ distinct from $W$ are proper subgraphs of $K_1$, in contradiction to the minimality of $K_1$.

The analysis of Case 3.3 concludes the analysis of Case 3 and completes the proof of the Main Theorem. □

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