EXISTENCE AND CONCENTRATION OF SEMICLASSICAL SOLUTIONS FOR HAMILTONIAN ELLIPTIC SYSTEM

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Abstract. In this paper, we study the following Hamiltonian elliptic system with gradient term
\[
\begin{align*}
-\epsilon^2 \Delta \psi + \epsilon b \cdot \nabla \psi + \psi + V(x) \varphi &= K(x)f(|\eta|) \varphi \quad \text{in } \mathbb{R}^N, \\
-\epsilon^2 \Delta \varphi - \epsilon b \cdot \nabla \varphi + \psi + V(x) \psi &= K(x)f(|\eta|) \psi \quad \text{in } \mathbb{R}^N,
\end{align*}
\]
where \( \eta = (\psi, \varphi) : \mathbb{R}^N \to \mathbb{R}^2, V, K \in C(\mathbb{R}^N, \mathbb{R}), \epsilon \) is a small positive parameter and \( b \) is a constant vector. Suppose that \( V(x) \) is sign-changing and has at least one global minimum, and \( K(x) \) has at least one global maximum, we prove the existence, exponential decay and concentration phenomena of semiclassical ground state solutions for all sufficiently small \( \epsilon > 0 \).

1. Introduction. We study the following Hamiltonian elliptic system with gradient term
\[
\begin{align*}
-\epsilon^2 \Delta \psi + \epsilon b \cdot \nabla \psi + \psi + V(x) \varphi &= K(x)f(|\eta|) \varphi \quad \text{in } \mathbb{R}^N, \\
-\epsilon^2 \Delta \varphi - \epsilon b \cdot \nabla \varphi + \psi + V(x) \psi &= K(x)f(|\eta|) \psi \quad \text{in } \mathbb{R}^N,
\end{align*}
\]
\( (P_\epsilon) \)
where \( \eta = (\psi, \varphi) : \mathbb{R}^N \to \mathbb{R}^2, \epsilon \) is a small positive parameter, \( b \) is a constant vector, \( V \) can be sign-changing and \( \inf K > 0 \). In this paper, we are concerned with the existence, exponential decay and concentration phenomena of semiclassical ground state solutions of problem \( (P_\epsilon) \).

Systems \( (P_\epsilon) \) or similar to \( (P_\epsilon) \) were studied by a number of authors. But most of them focused on the case \( b = 0 \). For example, see [3, 4, 5, 9, 16, 22, 25, 27, 30, 32, 34, 35, 37, 38, 39, 40, 42, 43, 45, 47] and the references therein. When \( b \neq 0 \)

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and $\epsilon = 1$, there are not many works on elliptic systems with gradient term. Zhao and Ding [41] considered the following system
\begin{align*}
\begin{cases}
-\Delta \psi + b(x) \cdot \nabla \psi + V(x)\psi = H_\varphi(x, \psi, \varphi) & \text{in } \mathbb{R}^N, \\
-\Delta \varphi - b(x) \cdot \nabla \varphi + V(x)\varphi = H_\psi(x, \psi, \varphi) & \text{in } \mathbb{R}^N,
\end{cases}
\end{align*}
(1.1)

where $b = (b_1, \cdots, b_N) \in C^1(\mathbb{R}^N, \mathbb{R}^N)$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $H \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$. In this case, the appearance of the gradient term in this system will bring some difficulties, and the variational framework for the case $b = 0$ cannot work any longer. Hence the authors first established suitable variational framework through the studying of the spectrum of operator, and obtained the multiplicity of solution for the non-periodic asymptotically quadratic case by applying the theorems of Bartsch and Ding [6]. Moreover, without the assumption that $H(x, \eta)$ is even in $\eta$, infinitely many geometrically distinct solutions for the periodic asymptotically quadratic case were obtained by using a reduction method. For the periodic superquadratic case, Zhang et al.[44] proved the existence of ground state solution for system (1.1). Recently, Yang et al.[36] considered the non-periodic superquadratic system
\begin{align*}
\begin{cases}
-\Delta \psi + b \cdot \nabla \psi + \psi = H_\varphi(x, \psi, \varphi) & \text{in } \mathbb{R}^N, \\
-\Delta \varphi - b \cdot \nabla \varphi + \varphi = H_\psi(x, \psi, \varphi) & \text{in } \mathbb{R}^N,
\end{cases}
\end{align*}
(1.2)

with a constant vector $b$. Since the problem is set in unbounded domain with non-periodic nonlinearities, the $(C)_c$-condition does not hold in general. To overcome the difficulties, they first considered certain limit problem related to system (1.2) which is autonomous, and constructed linking levels of the variational functional and proved $(C)_c$-condition.

For small $\epsilon > 0$ the solutions (standing waves) of $(P_\epsilon)$ are referred to as semiclassical states, which describes the transition from quantum mechanics to classical mechanics when the parameter $\epsilon$ goes to zero, and possess an important physical interest. To the best of our knowledge, there is only a little works concerning the existence and concentration phenomenon of semiclassical states. Very recently, Zhang et al.[46] considered the following singularly perturbed system
\begin{align*}
\begin{cases}
-\epsilon^2 \Delta \psi + \epsilon b \cdot \nabla \psi + \psi = P(x)|\eta|^{p-2}\varphi & \text{in } \mathbb{R}^N, \\
-\epsilon^2 \Delta \varphi - \epsilon b \cdot \nabla \varphi + \varphi = P(x)|\eta|^{p-2}\psi & \text{in } \mathbb{R}^N,
\end{cases}
\end{align*}
(1.3)

with $p \in (2, 2^*)$, where $2^*$ is the usual critical exponent. The authors proved that the semiclassical solution concentrates around the maxima point of nonlinear potential $P(x)$ as $\epsilon \to 0$. Since the phenomenon of concentration is very interesting for both mathematicians and physicists. So, in the present paper, we shall continue to study the existence, exponential decay and concentration phenomena of semiclassical ground state solutions for system $(P_\epsilon)$ with the general nonlinearities. Here, it is worth mentioning that, as to the concentration of semiclassical ground state solutions of system $(P_\epsilon)$, there is competition between the linear potential $V(x)$ and the nonlinear potential $K(x)$, i.e., $V(x)$ want to attract ground state solutions to its minimum points but $K(x)$ want to attract ground state solutions to its maximum points. Also the solutions depend not only on the linear potential but also on the nonlinear potential. Hence, compared with the study of system (1.3), the study of system $(P_\epsilon)$ is much more complicated, since the effect of the competing potentials, and the present argument seems to be more delicate.
Let us now describe the results of the present paper. For notational convenience, let
\[ J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
and \( S_\epsilon = -\epsilon^2 \Delta + 1 \). We denote
\[ A_\epsilon := S_\epsilon J_0 + \epsilon b \cdot \nabla J = \begin{pmatrix} 0 & 0 \\ -\epsilon^2 \Delta + \epsilon b \cdot \nabla + 1 & 0 \end{pmatrix}. \]
Then system \( (\mathcal{P}_\epsilon) \) can be rewritten as
\[ A_\epsilon \eta + V(x) \eta = K(x)f(\| \eta \|) \eta. \] (1.4)
Furthermore, set
\[ \nu = \min V, \quad \mathcal{V} = \{ x \in \mathbb{R}^N : V(x) = \nu \}, \quad \nu_\infty = \liminf_{|x| \to \infty} V(x), \]
\[ \kappa = \max K, \quad \mathcal{K} = \{ x \in \mathbb{R}^N : K(x) = \kappa \}, \quad \kappa_\infty = \limsup_{|x| \to \infty} K(x). \]
Before stating our results, we first make the following assumptions:
(A0) \( V, K \in C^1(\mathbb{R}^N, \mathbb{R}), a := \max |V| < 1 \) and \( d := \inf K > 0 \);
(A1) \( \nu < \nu_\infty \), and there is \( x_v \in \mathcal{V} \) such that \( K(x_v) \geq K(x) \) for all \( |x| \geq R \) and some large \( R > 0 \);
(A2) \( \kappa > \kappa_\infty \), and there is \( x_k \in \mathcal{K} \) such that \( V(x_k) \leq V(x) \) for all \( |x| \geq R \) and some large \( R > 0 \);
(F0) \( f \in C^1(\mathbb{R}^+, \mathbb{R}^+), f(0) = 0 \) and \( f'(s) \geq 0 \) for \( s > 0 \), \( \mathbb{R}^+ = [0, \infty) \);
(F1) there exist \( c_0 > 0 \) and \( p \in (2, 2^*) \) such that \( f(s) \leq c_0 (1 + s^{p-2}) \) for \( s \geq 0 \);
(F2) there exist \( \theta > 2 \) such that \( 0 < \theta F(|z|) \leq f(|z|)|z|^2 \) if \( z \neq 0 \), where \( F(|z|) = \int_0^{|z|} f(s)ds \).

**Remark 1.** The assumption \( (A_0) \) implies that the linear potential \( V \) may change sign. Here we say that \( V \) is sign-changing if \( V(x_1) < 0 < V(x_2) \) for some \( x_1, x_2 \in \mathbb{R}^N \).

Observe that, in case \( (A_1) \), we may assume \( K(x_v) = \max_{x \in \mathcal{V}} K(x) \) and in case \( (A_2) \), we may assume \( V(x_k) = \min_{x \in \mathcal{K}} V(x) \). Motivated by [12, 13], in order to describe some concentration phenomena of semiclassical solution, we define the following sets which will be used later
\[ \mathcal{A}_v := \{ x \in \mathcal{V} : K(x) = K(x_v) \} \cup \{ x \notin \mathcal{V} : K(x) > K(x_v) \}, \]
and
\[ \mathcal{A}_k := \{ x \in \mathcal{K} : V(x) = V(x_k) \} \cup \{ x \notin \mathcal{K} : V(x) < V(x_k) \}. \]
Then \( \mathcal{A}_v \) and \( \mathcal{A}_k \) are bounded since \( V \) and \( K \) are bounded. Moreover, \( \mathcal{A}_v = \mathcal{A}_k = \mathcal{V} \cap \mathcal{K} \) if \( \mathcal{V} \cap \mathcal{K} \neq \emptyset \).

We denote energy of a solution \( \eta \neq 0 \) of \( (\mathcal{P}_\epsilon) \) by
\[ \Phi_\epsilon(\eta) := \int_{\mathbb{R}^N} \left( \frac{1}{2} A_\epsilon \eta \cdot \eta + \frac{1}{2} V(x)|\eta|^2 - K(x)f(\| \eta \|) \right) dx. \]
Set
\[ \ell_\epsilon := \inf \{ \Phi_\epsilon(\eta) : \eta \neq 0 \ \text{is a solution of} \ (\mathcal{P}_\epsilon) \}, \]
a solution \( \eta_0 \neq 0 \) with \( \ell_\epsilon = \Phi_\epsilon(\eta_0) \) is called a ground state solution. Let \( \mathcal{L}_\epsilon \) denote the set of all ground state solutions of \( (\mathcal{P}_\epsilon) \). Our main result is the following theorem.
Theorem 1.1. Assume that $|b| \leq 2$, $(A_0)$ and $(F_0)-(F_2)$ are satisfied.

(I) Suppose that $(A_1)$ holds. Then for all sufficiently small $\epsilon > 0$, $(P_\epsilon)$ has a
ground state solution $\eta_\epsilon$. If additionally $\nabla V$ and $\nabla K$ are bounded, one has that

(i) $\mathcal{L}_\epsilon$ is compact in $H^2(\mathbb{R}^N, \mathbb{R}^2)$.

(ii) There exists a maximum point $x_\epsilon$ of $|\eta_\epsilon(x)|$ with $\lim_{\epsilon \to 0} \text{dist}(x_\epsilon, A_\epsilon) = 0$, such

that, for some $c, C > 0$

$$|\eta_\epsilon(x)| \leq C \exp \left(-\frac{c}{\epsilon} |x - x_\epsilon| \right).$$

(iii) Setting $v_\epsilon(x) := \eta_\epsilon(\epsilon x + x_\epsilon)$, for any sequence $x_\epsilon \to x_0$ as $\epsilon \to 0$, $v_\epsilon$
converges in $H^2(\mathbb{R}^N, \mathbb{R}^2)$ to a ground state solution of

$$\left\{ \begin{array}{ll}
-\Delta \psi + b \cdot \nabla \psi + \psi + V(x_0)\varphi = K(x_0)f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\
-\Delta \varphi - b \cdot \nabla \varphi + \varphi + V(x_0)\psi = K(x_0)f(|\eta|)\psi & \text{in } \mathbb{R}^N.
\end{array} \right.$$ 

If particularly $\mathcal{V} \cap \mathcal{X} \neq \emptyset$ then $\lim_{\epsilon \to 0} \text{dist}(x_\epsilon, \mathcal{V} \cap \mathcal{X}) = 0$, and $v_\epsilon$
converges in $H^2(\mathbb{R}^N, \mathbb{R}^2)$ to a ground state solution of

$$\left\{ \begin{array}{ll}
-\Delta \psi + b \cdot \nabla \psi + \psi + \nu \varphi = \kappa f(|\eta|)\varphi & \text{in } \mathbb{R}^N, \\
-\Delta \varphi - b \cdot \nabla \varphi + \varphi + \nu \psi = \kappa f(|\eta|)\psi & \text{in } \mathbb{R}^N.
\end{array} \right.$$ 

(II) Suppose that $(A_2)$ holds. Then, replacing $A_\epsilon$ with $A_\kappa$, all the conclusions of
(I) remain true.

For the proof of our results, we do not handle the system $(P_\epsilon)$ directly, but instead
we handle an equivalent system to $(P_\epsilon)$. For this purpose, set $z(x) = (u(x), v(x)) = (\psi(\epsilon x), \varphi(\epsilon x)) = \eta(\epsilon x)$, $V_\epsilon(x) = V(\epsilon x)$ and $K_\epsilon(x) = K(\epsilon x)$. Then the system $(P_\epsilon)$
is equivalent to the following:

$$\left\{ \begin{array}{ll}
-\Delta u + b \cdot \nabla u + u + V_\epsilon(x)v = K_\epsilon(x)f(|z|)v & \text{in } \mathbb{R}^N, \\
-\Delta v - b \cdot \nabla v + v + V_\epsilon(x)u = K_\epsilon(x)f(|z|)u & \text{in } \mathbb{R}^N.
\end{array} \right. \quad (P'_\epsilon)$$

Moreover, system $(P'_\epsilon)$ can be expressed as

$$A z + V_\epsilon(x)z = K_\epsilon(x)f(|z|)z, \tag{1.5}$$

where

$$A = \begin{pmatrix}
0 & -\Delta - b \cdot \nabla + 1 \\
-\Delta + b \cdot \nabla + 1 & 0
\end{pmatrix}.$$ 

Clearly, (1.4) is equivalent to (1.5).

As a motivation we recall that there is a large number of articles devoted to the study on the
semiclassical states of Schrödinger equations

$$- \epsilon^2 \Delta u + V(x)u = f(x, u), \quad u \in H^1(\mathbb{R}^N). \tag{1.6}$$

It was shown, under suitable assumptions of course, that for all $\epsilon > 0$, (1.6) possesses
a ground state $u_\epsilon$ which concentrates on the set of minimum points of $V(x)$
as $\epsilon \to 0$. For example, see [2, 7, 8, 12, 19, 20, 23, 24, 28, 29, 33]. For semiclassical
Dirac equation, we refer the readers to [11, 13, 14, 15] and the references therein.
Note that, since the Schrödinger operator $-\Delta + V$ is bounded from below, techniques
based on the mountain pass theorem are well applied to the investigation. However, for our problem, the mountain pass structure no longer satisfies since the corresponding energy functional is strongly indefinite, the classical critical point
theory cannot be applied directly. Hence our problem poses more challenges in the
calculus of variation in nature.
Our argument is variational, the semiclassical solutions are obtained as critical points of the energy functional \( \Phi_\epsilon \) associated to \( (P_\epsilon) \). As a result, the functional \( \Phi_\epsilon \) is strongly indefinite, hence possesses an infinite-dimensional linking structure instead of a mountain pass. Our arguments will be based on a suitable functional analytic framework. The linking structure yields a minimax value \( c_\epsilon \) for \( \Phi_\epsilon \). Since the problem is posed on the whole space \( \mathbb{R}^N \), \( \Phi_\epsilon \) does not satisfy the general Palais-Smale condition, and so it cannot be directly concluded that \( c_\epsilon \) is a critical value. Inspired by Ackermann [1], our strategy is to use a reduced method to overcome the strongly indefiniteness of the energy functional \( \Phi_\epsilon \). Hence, we define a reduced functional \( I_\epsilon \) of \( \Phi_\epsilon \) and Nehari manifold \( \mathcal{N}_\epsilon \). Moreover, by applying limit problem and some techniques, we prove the value \( c_\epsilon \) is the minimum of \( I_\epsilon \) on \( \mathcal{N}_\epsilon \). In addition, we also prove the concentration phenomenon of semiclassical solutions. Finally, for proving the exponential decay, the Kato’s inequality seems not work well since the presence of the gradient term in system \( (P_\epsilon) \), we handle, instead of \( |z| \) in Kato’s inequality, the square of \( |\Delta z| \), that is \( \Delta |z|^2 \), and then describe the decay at infinity in a subtle way.

2. Variational setting and linking structure. Below by \( |\cdot|_q \) we denote the usual \( L^q \)-norm, \( \langle \cdot, \cdot \rangle_2 \) denote the usual \( L^2 \)-inner product; \( c, c_i \) or \( C_i \) stand for different positive constants. Denote by \( \sigma(A) \) and \( \sigma_e(A) \) the spectrum and the essential spectrum of the operator \( A \), respectively. In order to establish suitable variational framework for system \( (P_\epsilon') \), we have the following lemmas, which are two special cases in [41].

**Lemma 2.1** ([41]). The operator \( A \) is a selfadjoint operator on \( L^2(\mathbb{R}^N, \mathbb{R}^2) \) with domain \( \mathcal{D}(A) := H^2(\mathbb{R}^N, \mathbb{R}^2) \).

**Lemma 2.2** ([41]). The following two conclusions hold:
(1) \( \sigma(A) = \sigma_e(A) \), i.e., \( A \) has only essential spectrum;
(2) \( \sigma(A) \subset \mathbb{R} \setminus (-1, 1) \) and \( \sigma(A) \) is symmetric with respect to origin.

It follows from Lemma 2.1 and Lemma 2.2 that the space \( L^2 := L^2(\mathbb{R}^N, \mathbb{R}^2) \) possesses the orthogonal decomposition
\[
L^2 = L^- \oplus L^+ = z^- + z^+
\]
such that \( A \) is negative definite (resp. positive definite) in \( L^- \) (resp. \( L^+ \)). Let \( |A| \)
denote the absolute value of \( A \) and \( |A|^\frac{1}{2} \) be the square root of \( |A| \). Let \( E = \mathcal{D}(|A|^\frac{1}{2}) \)
be the Hilbert space with the inner product
\[
\langle z, w \rangle = \langle |A|^\frac{1}{2} z, |A|^\frac{1}{2} w \rangle_2
\]
and norm \( \|z\| = \langle z, z \rangle^\frac{1}{2} \). There is an induced decomposition
\[
E = E^- \oplus E^+, \text{ where } E^\pm = E \cap L^\pm,
\]
which is orthogonal with respect to the inner products \( \langle \cdot, \cdot \rangle_2 \) and \( \langle \cdot, \cdot \rangle \).

**Lemma 2.3** ([41]). \( \| \cdot \| \) and \( \cdot \cdot \cdot \|_{H^1} \) are equivalent norms. Therefore, \( E \) embeds continuously into \( L^p := L^p(\mathbb{R}^N, \mathbb{R}^2) \) for any \( p \in [2, 2^*) \) and compactly into \( L^p_{loc} := L^p_{loc}(\mathbb{R}^N, \mathbb{R}^2) \) for any \( p \in [2, 2^*) \), and there exists constant \( \pi_p \) such that
\[
\pi_p \|z\|_p \leq \|z\|, \text{ for all } z \in E, p \in [2, 2^*].
\]
In virtue of the assumptions $(F_0)$–$(F_2)$, for any $\varepsilon > 0$, there exist positive constants $r_\varepsilon$, $C_\varepsilon$ and $C'_\varepsilon$ such that

$$\begin{align*}
f(s) &\leq \varepsilon \text{ for all } 0 \leq s \leq r_\varepsilon, \\
F(|z|) &\leq \varepsilon|z|^2 + C_\varepsilon |z|^p \text{ for all } z \in \mathbb{R}^2, \ p \in (2, 2^*), \\
F(|z|) &\geq C'_\varepsilon |z|^\theta - \varepsilon|z|^2 \text{ for all } z \in \mathbb{R}^2, \ \theta > 2.
\end{align*}$$

(2.2)

On $E$ we define the following functional

$$\Phi_\varepsilon(z) = \frac{1}{2} \left(\|z^+\|^2 - \|z^-\|^2\right) + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon(x)|z|^2 - \int_{\mathbb{R}^N} K_\varepsilon(x)F(|z|).$$

(2.3)

Lemma 2.2 implies that $\Phi_\varepsilon$ is strongly indefinite. Moreover, our hypotheses imply that $\Phi_\varepsilon \in C^1(E, \mathbb{R})$, and a standard argument shows that critical points of $\Phi_\varepsilon$ are solutions of problem $(P'_\varepsilon)$ (see \cite{10, 48}). For convenience, let

$$\Psi_\varepsilon(z) = \int_{\mathbb{R}^N} K_\varepsilon(x)F(|z|).$$

**Lemma 2.4.** $\Psi_\varepsilon$ is weakly sequentially lower semi-continuous. $\Psi'_\varepsilon$ are weakly sequentially continuous.

Using the Lemma 2.3, one can check easily the above lemma, here we omit the details (see, for example \cite{10} and \cite{48}).

Set, for $r > 0$, $B^+_r = \{z \in E^+ : \|z\| \leq r\}$ and $S^+_r = \{z \in E^+ : \|z\| = r\}$, and for $e \in E^+$, $E_c := E^- \oplus \mathbb{R}^+ e$ with $\mathbb{R}^+ = [0, \infty)$. Now we discuss the linking structure of $\Phi_\varepsilon$.

**Lemma 2.5.** There exist $r > 0$ and $\rho > 0$ both independent of $\varepsilon$ such that $\Phi_\varepsilon|_{B^+_r}(z) \geq 0$ and $\Phi_\varepsilon|_{S^+_r}(z) \geq \rho$.

**Proof.** Observe that, $\pi_2 = 1$ by Lemma 2.2 (2). For any $z \in E^+$, by (2.1) and (2.2) we have

$$\Phi_\varepsilon(z) = \frac{1}{2} \|z\|^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_\varepsilon(x)|z|^2 - \int_{\mathbb{R}^N} K_\varepsilon(x)F(|z|)$$

$$\geq \frac{1}{2} \|z\|^2 - \frac{1}{2} a \|z\|^2 - \varepsilon \kappa |z|^2 - C_\varepsilon \kappa |z|^p$$

$$\geq \frac{1}{2} \|z\|^2 - \frac{1}{2} a \|z\|^2 - \varepsilon \kappa |z|^2 - C_\varepsilon \kappa \pi_2^{-p} |z|^p$$

$$= \left(1 - \frac{1}{2} a - \varepsilon \kappa \right) \|z\|^2 - C_\varepsilon \kappa \pi_2^{-p} \|z\|^p.$$

Since $p \in (2, 2^*)$, choosing suitable $\varepsilon, r > 0$ we see that the desired conclusion holds. \hfill \square

**Lemma 2.6.** For any $e \in E^+ \setminus \{0\}$, there exist $R_e > 0$ and $C = C_e > 0$ both independent of $e$ such that $\Phi_\varepsilon(z) < 0$ for all $z \in E_c \setminus B_{R_e}$ and $\max \Phi_\varepsilon(E_c) \leq C$. 
Proof. For any $z \in E_{\epsilon}$, that is $z = t\epsilon + v$ for some $t \geq 0$ and $v \in E^-$. By Young inequality and (2.2) we have
\[
\Phi_\epsilon(z) = \frac{1}{2}(t^2\|e\|^2 - \|v\|^2) + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon(x)|t\epsilon + v|^2 - \int_{\mathbb{R}^N} K_\epsilon(x)F(|t\epsilon + v|)
\leq \frac{1}{2}t^2 \left(\|e\|^2 + a|e|^2_2\right) + \frac{1}{2} \left(\|v\|^2 - \|\epsilon v\|^2_2\right) + d\epsilon|t\epsilon + v|^2_2 - dC_\epsilon'|te + v|^2_\theta
\leq t^2 \left(\frac{1}{2} \left(\|e\|^2 + a|e|^2_2\right) + \frac{1}{2} \left(\|v\|^2 - \|\epsilon v\|^2_2\right) + \frac{1}{2} \left(\|v\|^2_2 - \|\epsilon v\|^2_2\right) + d\epsilon|v|^2_2\right)
- t^\theta dC_\epsilon'(1 - (\theta - 1)\gamma)|\epsilon v|^\theta_\theta + \gamma^{-(\theta - 1)}dC_\epsilon'|v|^\theta_\theta
\]
for some $0 < \gamma < \frac{1}{\theta - 1}$. Since $\theta > 2$, choosing large $R_\epsilon > 0$ we see that the desired conclusion holds.

Similar to a argument in [31], we define the following minimax value
\[
c_{\epsilon} := \inf_{\epsilon \in E^+ \setminus \{0\}} \max_{z \in E_{\epsilon}} \Phi_\epsilon(z).
\]
As a consequence of Lemmas 2.5 and 2.6, we have

Lemma 2.7. There is $C > 0$ independent of $\epsilon$ such that $\rho \leq c_{\epsilon} < C$.

Recall that a sequence $\{z_n\} \subset E$ is said to be a (PS)$_c$ sequence for functional $\Phi_\epsilon$ if $\Phi_\epsilon(z_n) \rightarrow c$ and $\Phi_\epsilon'(z_n) \rightharpoonup 0$, and $\Phi_\epsilon$ is said to satisfy the (PS) condition if any (PS)$_c$ sequence for $\Phi_\epsilon$ has a convergent subsequence. With Lemmas 2.4, 2.5 and 2.6 and by a standard linking argument it follows that $\Phi_\epsilon$ has a (PS)$_{c_{\epsilon}}$ sequence (see [10] and [31]). Obviously, if $\Phi_\epsilon$ satisfies the (PS)$_{c_{\epsilon}}$ condition, then $c_{\epsilon}$ is a critical value of $\Phi_\epsilon$. Unfortunately, since there is no compactly embedding from $H^1$ into $L^p$ for $2 \leq p < 2^*$, then (PS)$_{c_{\epsilon}}$ condition does not hold in general, we have to go through more analysis.

Following Ackermann [1], for a fixed $z \in E^+$ we introduce $\phi_{\epsilon,z} : E^- \rightarrow \mathbb{R}$ defined by
\[
\phi_{\epsilon,z}(w) = \Phi_\epsilon(z + w).
\]
A direct computation gives, for any $w, \varphi \in E^-$,
\[
\phi''_{\epsilon,z}(w)[\varphi, \varphi] = -\|\varphi\|^2 + \int_{\mathbb{R}^N} V_\epsilon(x)|\varphi|^2 - \Psi''_\epsilon(z + w)[\varphi, \varphi]
\leq -(1 - a)\|\varphi\|^2 - \int_{\mathbb{R}^N} K_\epsilon(x) \left(f'(\|z + w\|)\|z + w\|\varphi|^2 + f(\|z + w\|)\varphi^2\right).
\]
By $(A_0)$ and $(F_0)$, it is easy to see that $\phi''_{\epsilon,z}(w)[\varphi, \varphi] < 0$ for $\varphi \neq 0$. In addition,
\[
\phi_{\epsilon,z}(w) \leq \frac{1}{2}(1 + a)\|z\|^2 - \left(\frac{1}{2} - \frac{a}{2}\right)\|w\|^2 \rightarrow -\infty \text{ as } \|w\| \rightarrow \infty.
\]
Therefore, there is a unique $h_\epsilon(z) \in E^-$ such that
\[
\phi_{\epsilon,z}(h_\epsilon(z)) = \max_{w \in E^-} \phi_{\epsilon,z}(w)
\]
and
\[
w \neq h_\epsilon(z) \Leftrightarrow \Phi_\epsilon(z + w) < \Phi_\epsilon(z + h_\epsilon(z)).
\]
Lemma 2.10. For any \( w \in E^- \), \( 0 = \phi'_{e,z}(h_e(z))w \). For any \( z \in E^+ \) and \( w \in E^- \), setting \( v = w - h_e(z) \) and \( g(t) = \phi_{e,z}(h_e(z) + tv) \), one has \( g(1) = \phi_{e,z}(w), g(0) = \phi_{e,z}(h_e(z)) \) and \( g'(0) = 0 \). Thus
\[
g(1) - g(0) = \int_0^1 (1 - t)g''(t)dt.
\]
This implies that
\[
\phi_{e,z}(w) - \phi_{e,z}(h_e(z)) = \int_0^1 (1 - t)\phi''_{e,z}(h_e(z) + tv)[v, v]dt
\]
\[
= -\int_0^1 (1 - t) \left[ \|v\|^2 - \int_{\mathbb{R}^N} V_e(x)|v|^2 
+ \int_{\mathbb{R}^N} K_e(x) \left( f'(|z + h_e(z) + tv|) \frac{(z + h_e(z) + tv)w}{|z + h_e(z) + tv|} + f(|z + h_e(z) + tv|)|v|^2 \right) \right] dt
\]
\[
= -\frac{1}{2} \left( \|v\|^2 - \int_{\mathbb{R}^N} V_e(x)|v|^2 \right) - \int_{\mathbb{R}^N} H_e(x),
\]
where
\[
H_e(x) = \int_0^1 (1 - t)K_e(x) \left( f'(|z + h_e(z) + tv|) \frac{(z + h_e(z) + tv)w}{|z + h_e(z) + tv|} 
+ f(|z + h_e(z) + tv|)|v|^2 \right) dt.
\]
Hence
\[
\Phi_e(z + h_e(z)) - \Phi_e(z + w) = \frac{1}{2} \left( \|v\|^2 - \int_{\mathbb{R}^N} V_e(x)|v|^2 \right) + \int_{\mathbb{R}^N} H_e(x). \tag{2.4}
\]
Now, define the reduced functional \( I_e : E^+ \to \mathbb{R} \) by
\[
I_e(z) = \Phi_e(z + h_e(z)) = \frac{1}{2} \left( \|z\|^2 - \|h_e(z)\|^2 \right) + \frac{1}{2} \int_{\mathbb{R}^N} V_e(x)|z + h_e(z)|^2 - \Psi_e(z + h_e(z)),
\]
and its Nehari manifold by
\[
\mathcal{N}_e := \{ z \in E^+ \setminus \{0\} : I'_e(z)z = 0 \}.
\]
Then critical points of \( I_e \) and \( \Phi_e \) are in one to one correspondence via the injective map \( z \mapsto z + h_e(z) \) from \( E^+ \) into \( E \).

Lemma 2.8. For any \( z \in E^+ \setminus \{0\} \), there is a unique \( t = t(z) > 0 \) such that \( t(z)z \in \mathcal{N}_e \).

For the proof of the previous lemma, we refer to [1, 17]. Here we omit the details.

Lemma 2.9. We define \( d_e := \inf_{z \in \mathcal{N}_e} I_e(z) \), and \( d_e = c_e \).

Proof. Indeed, given \( e \in E^+ \), if \( z = w + se \in E_e \) with \( \Phi_e(z) = \max_{v \in E_e} \Phi_e(v) \) then the restriction \( \Phi_e|_{E_e} \) of \( \Phi_e \) on \( E_e \) satisfies \( (\Phi_e|_{E_e})'(z) = 0 \) which implies \( w = h_e(se) \) and \( I'_e(se) = \Phi'_e(z)(se) = 0 \), i.e., \( se \in \mathcal{N}_e \). Thus \( d_e \leq c_e \). On the other hand, if \( z \in \mathcal{N}_e \) then \( (\Phi_e|_{E_e})'(z + h_e(z)) = 0 \) so \( c_e \leq \max_{v \in E_e} \Phi_e(v) = I_e(z) \). Thus \( d_e \geq c_e \). This proves \( d_e = c_e \). \( \square \)

Lemma 2.10. For any \( e \in E^+ \setminus \{0\} \), there is \( T_e > 0 \) independent of \( e > 0 \) such that \( t_e \leq T_e \) for \( t_e > 0 \) satisfying \( t_e \in \mathcal{N}_e \).
Proof. Since $I'_e(t,e)(t,e) = 0$, one gets
\[ \Phi_e(t,e + h_e(t,e)) = \max_{z \in K_e} \Phi_e(z). \quad (2.5) \]
This, together with Lemmas 2.6 and 2.9, implies that
\[ c_\epsilon \leq \Phi_e(t,e + h_e(t,e)) \leq c_1 t_\epsilon^2 - c_2 t_\epsilon^2 + c_3, \]
from which one can show the desired conclusion. \( \square \)

Let $K_e := \{ z \in E \setminus \{0\} : \Phi'_e(z) = 0 \}$ be the critical set of $\Phi_e$. Since critical points of $I_e$ and $\Phi_e$ are in one-to-one correspondence via the injective map $z \mapsto z + h_e(z)$ from $E^+$ into $E$. From Lemma 2.9, it is easy to see that if $K_e \neq \emptyset$ then
\[ c_\epsilon = \inf\{ \Phi_e(z) : z \in K_e \}. \]
Using the standard bootstrap argument (see, e.g., [17, 18] for the iterative steps) one obtains easily the following result.

**Lemma 2.11.** If $z \in K_e$ with $|\Phi_e(z)| \leq C_1$ and $|u|_2 \leq C_2$, then, for any $r \geq 2$, $z \in H^{2,r}$ and $\|z\|_{H^{2,r}} \leq \Lambda_r$, where $\Lambda_r$ depends only on $C_1$, $C_2$ and $r$.

Let $\mathcal{L}_e$ be the set of all least energy solutions of $\Phi_e$. If $z \in \mathcal{L}_e$ then $\Phi_e(z) = c_\epsilon$ and a standard argument shows that $\mathcal{L}_e$ is bounded in $E$, hence, $|z|_2 \leq C_2$ for $z \in \mathcal{L}_e$, some $C_2 > 0$ independent of $\epsilon$. Therefore, as a consequence of Lemmas 2.9 and 2.11 we see that, for each $r \geq 2$, there is $C_r > 0$ independent of $\epsilon$ such that $\|z\|_{H^{2,r}} \leq C_r$ for all $z \in \mathcal{L}_e$. This, together with the Sobolev embedding theorem, implies that there is $C_\infty > 0$ independent of $\epsilon$ with
\[ |z|_\infty \leq C_\infty \text{ for all } z \in \mathcal{L}_e. \quad (2.6) \]

**Lemma 2.12.** Assume that $|b| \leq 2$, there is $C_0 > 0$ independent of $x$, $z \in \mathcal{L}_e$ and $\epsilon > 0$ such that
\[ |z(x)| \leq C_0 \left( \int_{B_1(x)} |z(y)|^2 dy \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^N, \quad z \in \mathcal{L}_e, \quad (2.7) \]
where $B_1(x) = \{ y : |y - x| \leq 1 \}$.

**Proof.** Let $z = (u, v)$ be a solution of $(P'_e)$. Recall that $|z|^2 = u^2 + v^2$. Then
\[ D_i(|z|^2) = 2z|D_i(|z|)| = 2\left| \frac{u D_i u + v D_i v}{z} \right| = 2(u D_i u + v D_i v) \]
and
\[ D_{ii}(|z|^2) = 2D_i(u D_i u + v D_i v) = 2(D_i u D_i u + u D_{ii} u + D_i v D_i v + v D_{ii} v) \]
for $i = 1, 2, \cdots, N$. This yields that
\[ \Delta |z|^2 = \sum_{i=1}^N D_{ii}(|z|^2) = 2 \sum_{i=1}^N (D_i u D_i u + u D_{ii} u + D_i v D_i v + v D_{ii} v) \]
\[ = 2(|\nabla u|^2 + |\nabla v|^2 + u \Delta u + v \Delta v) = 2(|\nabla z|^2 + z \Delta z), \]
where $\nabla z = (\nabla u, \nabla v)$. Since $z = (u, v)$ is a solution of $(P'_{\tau})$, then
\[
\Delta |z|^2 = 2(|\nabla z|^2 + z\Delta z)
\]
\[
= 2(|u|^2 + |v|^2 + 2V_\tau(x)uv - 2K_\tau(x)f(|z|)uv + b \cdot \nabla uu - b \cdot \nabla vv + |\nabla z|^2)
\]
\[
\geq 2(|z|^2 - a(|u|^2 + |v|^2) - \kappa f(|z|)(|u|^2 + |v|^2) + b \cdot \nabla uu - b \cdot \nabla vv + |\nabla z|^2)
\]
\[
\geq 2(|z|^2 - a|z|^2 - \kappa f(|z|)|z|^2 - b|z|^2 - b|\nabla z|^2 + |\nabla z|^2)
\]
\[
= 2\left(|z|^2 - a|z|^2 - \kappa f(|z|)|z|^2 - \frac{|b|}{2}|\nabla z|^2 \right)
\]
\[
\geq 2\left(|z|^2 - a|z|^2 - \kappa f(|z|)|z|^2 - \frac{|b|}{2}|z|^2 \right)
\]  \hspace{1cm} (2.8)

since $|b| \leq 2$. On the other hand, by some similar arguments in [9], we know that $|z(x)| \to 0$ as $|x| \to \infty$. Thus, for any $\alpha > 0$, it follows from $(F_1)$ that there is $R > 0$ such that
\[
f(|z|) \leq \alpha, \quad |x| \geq R.
\]  \hspace{1cm} (2.9)

Therefore, by (2.8) and (2.9), there exists $\delta > 0$ such that
\[
\Delta |z|^2 \geq -\delta|z|^2, \quad x \in \mathbb{R}^N,
\]  \hspace{1cm} (2.10)

which implies that $|z|^2$ is a sub-solution of the equation $(-\Delta - \delta)z = 0$. Moreover, by the sub-solution estimate [21], we have
\[
|z(x)| \leq C_0 \left(\int_{B_1(x)} |z(y)|^2 \, dy\right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^N,
\]

with $C_0 > 0$ independent of $x$, $z \in \mathcal{L}_\tau$ and $\epsilon > 0$. \hspace{1cm} \Box

3. The constant coefficients problem and the auxiliary problem. We will make use of the constant coefficients problem for proving our main result. To this end, we first discuss in this section the existence and some properties of the ground state solutions of the constant coefficients problem.

Consider the constant coefficients problem
\[
\begin{cases}
-\Delta u + b \cdot \nabla u + u + \mu v = \tau f(|z|)v & \text{in } \mathbb{R}^N, \\
-\Delta v - b \cdot \nabla v + v + \mu u = \tau f(|z|)u & \text{in } \mathbb{R}^N.
\end{cases}
\]  \hspace{1cm} (P_{\mu\tau})

It is well known that the solutions of problem $(P_{\mu\tau})$ are critical points of the functional
\[
\Phi_{\mu\tau}(z) = \frac{1}{2}(||z^+||^2 - ||z^-||^2) + \frac{\mu}{2} \int_{\mathbb{R}^N} |z|^2 - \tau \int_{\mathbb{R}^N} F(|z|)
\]
\[
:= \frac{1}{2}(||z^+||^2 - ||z^-||^2) + \frac{\mu}{2} \int_{\mathbb{R}^N} |z|^2 - \Psi_{\tau}(z)
\]
defined for $z = z^- + z^+ \in E = E^- \oplus E^+$. Denote the critical set, the least energy, and the set of ground state solutions of $\Phi_{\mu\tau}$ as follows
\[
\mathcal{K}_{\mu\tau} := \{ z \in E : \Phi_{\mu\tau}'(z) = 0 \},
\]
\[
c_{\mu\tau} := \inf\{ \Phi_{\mu\tau}(z) : z \in \mathcal{K}_{\mu\tau} \setminus \{0\} \},
\]
\[
\mathcal{L}_{\mu\tau} := \{ z \in \mathcal{K}_{\mu\tau} : \Phi_{\mu\tau}(z) = c_{\mu\tau} \}.
\]
Lemma 3.1. There hold the following conclusions:
(i) \((P_{\mu\tau})\) has at least one ground state solutions, i.e., \(\mathcal{K}_{\mu\tau} \neq \emptyset\), and \(c_{\mu\tau} > 0\);
(ii) \(c_{\mu\tau}\) is attained, and \(\mathcal{L}_{\mu\tau}\) is compact in \(H^2(\mathbb{R}^N, \mathbb{R})\);
(iii) there exist \(C, c > 0\) such that
\[
|z(x)| \leq C \exp(-c|x|), \quad x \in \mathbb{R}^N, \ z \in \mathcal{L}_{\mu\tau}.
\]

Proof. For the proof of (i) and (ii), we refer to the proof in [36, Theorem 1.1]. Now, we show that (iii) holds. Let \(z = (u, v) \in \mathcal{L}_{\mu\tau}\). By (ii), \(|z(x)| \to 0\) as \(|x| \to \infty\) uniformly in \(z \in \mathcal{L}_{\mu\tau}\). In fact, if not, then there exist \(\varrho > 0\), \(\{z_j\} \subset \mathcal{L}_{\mu\tau}\) and \(\{x_j\} \subset \mathbb{R}^N\) with \(|x_j| \to \infty\) such that
\[
\varrho \leq |z_j(x_j)| \quad \text{for all} \ j.
\]
Without loss of generality, we may assume that \(z_j \to z\) in \(H^2\). By (2.7),
\[
\varrho \leq |z_j(x_j)| \leq C_0 \left( \int_{B_1(x_j)} |z_j|^2 \right)^{\frac{1}{2}} + C_0 \left( \int_{B_1(x_j)} |z|^2 \right)^{\frac{1}{2}} \to 0,
\]
a contradiction. Therefore, by (2.8) and (2.9), there are \(R > 0\) and \(\delta > 0\) such that
\[
\Delta |z|^2 \geq \delta |z|^2 \quad \text{for} \ |x| \geq R.
\]
Let \(\Gamma(y)\) be a fundamental solutions to \(-\Delta \Gamma + \delta \Gamma = 0\). Using the uniform boundedness, we may choose \(\Gamma(y)\) so that \(|z(y)|^2 \leq \Gamma(y)\) for \(|y| = R\). Set \(w = |z|^2 - \Gamma\), then
\[
\Delta w = \Delta |z|^2 - \Delta \Gamma \geq \delta (|z|^2 - \Gamma) = \delta w.
\]
By the maximum principle we can conclude that \(w(y) \leq 0\) for \(|y| \geq R\), i.e., \(|z(y)|^2 \leq \Gamma(y)\) for \(|y| \geq R\). It is well known that there are \(c', C' > 0\) such that
\[
\Gamma(y) \leq C' \exp(-c'|y|) \quad \text{for} \ |y| \geq 1.
\]
Therefore, there are \(c, C > 0\) such that
\[
|z(x)|^2 \leq C \exp(-c|x|) \quad \text{for} \ x \in \mathbb{R}^N,
\]
that is,
\[
|z(x)| \leq \sqrt{C} \exp(-\frac{c}{2} |x|) \quad \text{for} \ x \in \mathbb{R}^N.
\]

As before we introduce the following notations:
\[
h_{\mu\tau} : E^+ \to E^-, \Phi_{\mu\tau}(z + h_{\mu\tau}(z)) = \max_{w \in E^-} \Phi_{\mu\tau}(z + w);
\]
\[
I_{\mu\tau} : E^+ \to \mathbb{R}, I_{\mu\tau}(z) = \Phi_{\mu\tau}(z + h_{\mu\tau}(z));
\]
\[
\mathcal{N}_{\mu\tau} := \{z \in E^+ \setminus \{0\} : I'_{\mu\tau}(z) = 0\}.
\]
Plainly, critical points of \(I_{\mu\tau}\) and \(\Phi_{\mu\tau}\) are in one-to-one correspondence via the injective map \(z \to z + h_{\mu\tau}(z)\) from \(E^+\) into \(E\). Similar to Lemma 2.8, it is easy to check that, for each \(z \in E^+ \setminus \{0\}\), there is a unique \(t = t(z) > 0\) such that \(tz \in \mathcal{N}_{\mu\tau}\) and
\[
c_{\mu\tau} = \inf \{I_{\mu\tau}(z) : z \in \mathcal{N}_{\mu\tau}\} = \inf_{c \in E^+ \setminus \{0\}} \max_{z \in E^-} \Phi_{\mu\tau}(z).
\]
Lemma 3.2. Let \( z \in \mathcal{N}_{\mu_T} \) be such that \( I_{\mu_T}(z) = c_{\mu_T} \) and set \( E_z = E^- \oplus \mathbb{R}z \). Then
\[
\max_{w \in E_z} \Phi_{\mu_T}(w) = I_{\mu_T}(z).
\]

Proof. Clearly, since \( z + h_{\mu_T}(z) \in E_z \),
\[
I_{\mu_T}(z) = \Phi_{\mu_T}(z + h_{\mu_T}(z)) \leq \max_{w \in E_z} \Phi_{\mu_T}(w).
\]
On the other hand, for any \( w = v + sz \in E_z \),
\[
\Phi_{\mu_T}(w) = \Phi_{\mu_T}(v + sz) \leq \Phi_{\mu_T}(sz + h_{\mu_T}(sz)) = I_{\mu_T}(sz).
\]
Thus, since \( z \in \mathcal{N}_{\mu_T} \),
\[
\max_{w \in E_z} \Phi_{\mu_T}(w) \leq \max_{s \geq 0} I_{\mu_T}(sz) = I_{\mu_T}(z).
\]
The proof is complete. \( \square \)

Lemma 3.3. Let \( \mu_i \in (-1, 1) \) and \( \tau_i > 0 \) for \( i = 1, 2 \), with \( \min\{\mu_2 - \mu_1, \tau_1 - \tau_2\} \geq 0 \).
Then \( c_{\mu_1 \tau_1} \leq c_{\mu_2 \tau_2} \). If additionally \( \max\{\mu_2 - \mu_1, \tau_1 - \tau_2\} > 0 \), then \( c_{\mu_1 \tau_1} < c_{\mu_2 \tau_2} \).
In particular, \( c_{\mu_1 \tau_1} < c_{\mu_2 \tau_2} \) if \( \mu_1 < \mu_2 \), and \( c_{\mu_1 \tau_1} > c_{\mu_2 \tau_2} \) if \( \tau_1 < \tau_2 \).

Proof. Let \( z \in \mathcal{L}_{\mu_2 \tau_2} \) with \( \Phi_{\mu_2 \tau_2}(z) = c_{\mu_2 \tau_2} \) and set \( e = z^+ \). Then
\[
c_{\mu_2 \tau_2} = \Phi_{\mu_2 \tau_2}(z) = \max_{w \in E_z} \Phi_{\mu_2 \tau_2}(w).
\]
Let \( z_0 \in E_e \) be such that \( \Phi_{\mu_1 \tau_1}(z_0) = \max_{w \in E_e} \Phi_{\mu_1 \tau_1}(w) \). One has
\[
c_{\mu_2 \tau_2} = \Phi_{\mu_2 \tau_2}(z_0) \geq \Phi_{\mu_2 \tau_2}(z_0)
= \Phi_{\mu_1 \tau_1}(z_0) + \frac{1}{2} (\mu_2 - \mu_1) |z_0|^2 + (\tau_1 - \tau_2) \int_{\mathbb{R}^N} F(|z_0|)
\geq c_{\mu_1 \tau_1} + \frac{1}{2} (\mu_2 - \mu_1) |z_0|^2 + (\tau_1 - \tau_2) \int_{\mathbb{R}^N} F(|z_0|),
\]
this implies \( c_{\mu_2 \tau_2} \geq c_{\mu_1 \tau_1} \). \( \square \)

In the following, we study the auxiliary problem and give some auxiliary results. Assume that the sequence of functions \( \hat{V}_\epsilon, \hat{K}_\epsilon \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), satisfy
\[
(\hat{A}) \hat{a} := \sup_{x \in \mathbb{R}} |\hat{V}_\epsilon(x)| < 1, \hat{d} := \inf_{x \in \mathbb{R}} \hat{K}_\epsilon(x) > 0; \hat{V}_\epsilon(x) \to \mu \text{ and } \hat{K}_\epsilon(x) \to \tau \text{ uniformly on bounded sets of } x \text{ as } \epsilon \to 0.
\]
Now, we consider the following auxiliary problem
\[
\begin{cases}
-\Delta u + b \cdot \nabla u + u + \hat{V}_\epsilon(x) v = \hat{K}_\epsilon(x) f(|z|) v & \text{in } \mathbb{R}^N, \\
-\Delta v + b \cdot \nabla v + v + \hat{V}_\epsilon(x) u = \hat{K}_\epsilon(x) f(|z|) u & \text{in } \mathbb{R}^N,
\end{cases}
\]
and the associated energy functional
\[
\Phi_\epsilon(z) = \frac{1}{2} (||z^+||^2 - ||z^-||^2) + \frac{1}{2} \int_{\mathbb{R}^N} \hat{V}_\epsilon(x) ||z^+||^2 - \int_{\mathbb{R}^N} \hat{K}_\epsilon(x) F(|z|).
\]
In the sequel, we use the associated notations \( \hat{h}_\epsilon, \hat{I}_\epsilon, \mathcal{\hat{N}}_\epsilon, \hat{c}_\epsilon \) as before. Clearly,
\[
\phi_\epsilon(z) = \Phi_{\mu_T}(z) + \frac{1}{2} \int_{\mathbb{R}^N} V_\epsilon^0(x) ||z^+||^2 + \int_{\mathbb{R}^N} K_\epsilon^0(x) F(|z|),
\]
(3.1)
where \( V_\epsilon^0(x) = \hat{V}_\epsilon(x) - \mu \) and \( K_\epsilon^0(x) = \tau - \hat{K}_\epsilon(x) \).
We need the following asymptotic estimate for \( \hat{c}_\epsilon \) as \( \epsilon \to 0 \), which is the key ingredient for existence result.

Lemma 3.4. \( \limsup_{\epsilon \to 0} \hat{c}_\epsilon \leq c_{\mu_T} \).
Proof. Let \( z \in \mathcal{L}_{\mu \tau} \) be a ground state solution to \((P_{\mu \tau})\) and \( e = z^+ \). Obviously, \( e \in \mathcal{N}_{\mu \tau}, z^- = h_{\mu \tau}(e) \) and \( I_{\mu \tau}(e) = c_{\mu \tau} \). There is a unique \( t_e \) such that \( t_e e \in \mathcal{N}_e \), and hence

\[
\hat{c}_e \leq \hat{I}(t_e). \tag{3.2}
\]

By Lemma 2.10, \( \{t_e\} \) is bounded. Without loss of generality, we may assume that \( t_e \to t_0 \) as \( \epsilon \to 0 \). Set

\[
z_e = t_e e + \hat{h}_e(t_e), \quad w_e = t_e e + h_{\mu \tau}(t_e)
\]

and

\[
v_e = w_e - z_e = h_{\mu \tau}(t_e) - \hat{h}_e(t_e).
\]

Next, we show that \( v_e \to 0 \) in \( E \). Similar to (2.4), we have

\[
\hat{\Phi}_e(z_e) - \hat{\Phi}_e(w_e) = \frac{1}{2} \left( \|v_e\|^2 - \int_{\mathbb{R}^N} V_e(x)|v_e|^2 \right) + \int_{\mathbb{R}^N} \hat{H}_e(x)
\]

\[
\geq \frac{1}{2} \left( \|v_e\|^2 - \int_{\mathbb{R}^N} V_e(x)|v_e|^2 \right)
\]

and

\[
\Phi_{\mu \tau}(w_e) - \Phi_{\mu \tau}(z_e) = \frac{1}{2} \left( \|v_e\|^2 - \mu \int_{\mathbb{R}^N} |v_e|^2 \right) + \tau \int_{\mathbb{R}^N} G_e(x)
\]

\[
\geq \frac{1}{2} \left( \|v_e\|^2 - \mu \int_{\mathbb{R}^N} |v_e|^2 \right), \tag{3.4}
\]

where

\[
\hat{H}_e(x) = \int_0^1 (1 - t) \hat{K}_e(x) \left( f'(|z_e + tv_e|) \frac{|(z_e + tv_e)v_e|^2}{|z_e + tv_e|} + f(|z_e + tv_e|)|v_e|^2 \right) dt
\]

and

\[
G_e(x) = \int_0^1 (1 - t) \left( f'(|w_e - tv_e|) \frac{|(w_e - tv_e)v_e|^2}{|w_e - tv_e|} + f(|w_e - tv_e|)|v_e|^2 \right) dt.
\]

From (3.3), (3.4) and the mean value theorem we know that

\[
\|v_e\|^2 - \frac{1}{2} \int_{\mathbb{R}^N} (\mu + V_e(x))|v_e|^2 \leq \hat{\Phi}_e(z_e) - \hat{\Phi}_e(w_e) + \Phi_{\mu \tau}(w_e) - \Phi_{\mu \tau}(z_e)
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} V_e^0(x)(|z_e|^2 - |w_e|^2) + \int_{\mathbb{R}^N} K_e^0(x)(F(|z_e|) - F(|w_e|))
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^N} V_e^0(x)|v_e|^2 - \int_{\mathbb{R}^N} V_e^0(x)w_ev_e - \int_{\mathbb{R}^N} K_e^0(x)f(w_e)w_ev_e + \int_{\mathbb{R}^N} K_e^0(x)G_e(x)
\]

\[
\leq \frac{1}{2} \int_{\mathbb{R}^N} V_e^0(x)|v_e|^2 - \int_{\mathbb{R}^N} V_e^0(x)w_ev_e - \int_{\mathbb{R}^N} K_e^0(x)f(w_e)w_ev_e + \tau \int_{\mathbb{R}^N} G_e(x).
\]
It follows from \((\hat{A})\) and \(\left( F_1 \right) \) that
\[
\| v_e \|^2 - \hat{a} |v_e|^2 \leq \int_{\mathbb{R}^N} |V^0_e(x)||w_e| + \int_{\mathbb{R}^N} |K^0_e(x)||f(w_e)||w_e| + \int_{\mathbb{R}^N} |K^0_e(x)||w_e||v_e|
\]
\[
\leq \int_{\mathbb{R}^N} |V^0_e(x)||w_e| + c_1 \int_{\mathbb{R}^N} |K^0_e(x)||w_e||v_e|
\]
\[
+ c_1 \int_{\mathbb{R}^N} |K^0_e(x)||w_e|^{p-1}|v_e|
\]
\[
\leq c_2 \left( \int_{\mathbb{R}^N} (|V^0_e(x)| + |K^0_e(x)|)^2 |w_e|^2 \right)^{1/2} |v_e|_2
\]
\[
+ c_1 \left( \int_{\mathbb{R}^N} |K^0_e(x)|^{p/(p-1)}|w_e| |w_e|^p \right)^{(p-1)/p} |v_e|_p.
\]
(3.5)

Since \(t_e \to t_0\), it is clear that \(\{z_e\}, \{w_e\}\) and \(\{v_e\}\) are bounded. Moreover, by the exponential decay of \(e\) and the continuity of \(h_{\mu \tau}\), we have for \(q \in [2, 2^*] \)
\[
\limsup_{R \to \infty} \int_{|z|>R} |w_e|^q = \limsup_{R \to \infty} \int_{|z|>R} |t_e e + h_{\mu \tau}(t_e e)|^q = 0.
\]

By \((\hat{A})\) again, it follows that
\[
\int_{\mathbb{R}^N} (|V^0_e(x)| + |K^0_e(x)|)^2 |w_e|^2 = \left( \int_{|x| \leq R} + \int_{|x| > R} \right) (|V^0_e(x)| + |K^0_e(x)|)^2 |w_e|^2
\]
\[
= \int_{|x| \leq R} (|V^0_e(x)| + |K^0_e(x)|)^2 |w_e|^2 + c_3 \int_{|x| > R} |w_e|^2
\]
\[
= o(1)
\]
(3.6)
as \(e \to 0\), and similarly
\[
\int_{\mathbb{R}^N} |K^0_e(x)|^{p/(p-1)}|w_e|^p = o(1)
\]
(3.7)
as \(e \to 0\). Thus, by (3.5), (3.6), (3.7) and \((\hat{A})\) we know that \(\|v_e\| = \|h_{\mu \tau}(t_e e) - \hat{h}_e(t_e e)\| \to 0\), that is, \(\hat{h}_e(t_e e) \to h_{\mu \tau}(t_0 e)\). Consequently,
\[
\int_{\mathbb{R}^N} V^0_e(x)|z_e|^2 \to 0 \text{ and } \int_{\mathbb{R}^N} K^0_e(x)F(|z_e|) \to 0
\]
as \(e \to 0\). This, jointly with (3.1), implies
\[
\hat{\Phi}_e(z_e) = \Phi_e(t_e e + \hat{h}_e(t_e e)) = \Phi_{\mu \tau}(t_0 e + \hat{h}_e(t_0 e)) + o(1)
\]
\[
= \Phi_{\mu \tau}(t_0 e + h_{\mu \tau}(t_0 e)) + o(1),
\]
that is,
\[
\hat{I}_e(t_e e) = I_{\mu \tau}(t_0 e) + o(1)
\]
as \(e \to 0\). Recall that by Lemma 3.2
\[
I_{\mu \tau}(t_0 e) \leq \max_{w \in E_e} \Phi_{\mu \tau}(w) = I_{\mu \tau}(e) = c_{\mu \tau}.
\]

Now, using (3.2) we obtain
\[
\lim_{e \to 0} \hat{c}_e \leq \lim_{e \to 0} \hat{I}_e(t_e e) = I_{\mu \tau}(t_0 e) \leq c_{\mu \tau}.
\]
This ends the proof.
Proof. Given Lemma 4.2. the Ekeland variational principle we can assume that \( \{ \)
We use the notations \( N \) and \( \) and let \( \) Hence, by Lemma 3.4, one has
Moreover, observe that
This, together with Lemma 3.4, shows that
In particular,
if \( V(0) \leq \mu \) and \( K(0) \geq \tau \).

4. Proof of the main result. In this section we give the proof of the main results.

4.1. The case with \( (A_0) \) and \( (A_1) \). We consider firstly the situation that \( (A_0) \) and \( (A_1) \) are satisfied. We start with observing that, for any \( x_0 \in Y \), setting \( \) if \( \bar{z}(x-x_0) \) solves \( \) Thus, without loss of generality, we can assume that \( K(x_v) \leq \max_{x \in Y} K(x) \) and \( x_v = 0 \in Y \) \( (x_v = 0 \in Y \cap \mathcal{H} \text{ if } Y \cap \mathcal{H} \neq \emptyset) \). Then \( \nu = V(0) \) and \( \iota := K(0) \geq K(x) \) for all \( |x| \geq R \) with large enough.

**Lemma 4.1.** \( \limsup_{\epsilon \to 0} c_{\epsilon} \leq c_{\nu} \). In particular, \( \lim_{\epsilon \to 0} c_{\epsilon} = c_{\nu} \) if additionally \( Y \cap \mathcal{H} \neq \emptyset \).

**Proof.** Take \( \mu = \nu \) and \( \tau = \kappa \). Then \( V^\mu(x) = V(x), K^\tau(x) = K(x), \nu = V^\mu(0) \) and \( \iota = K^\tau(0) \leq \kappa \). Hence \( c_{\epsilon} = c_{\epsilon}^{\mu \tau} \) and the conclusion follows directly from (3.9) and (3.10).

**Lemma 4.2.** \( c_{\epsilon} \) is attained for all small \( \epsilon > 0 \).

**Proof.** Given \( \epsilon > 0 \), let \( \{ z_n \} \subset \mathcal{N}_\epsilon \) be a minimizing sequence: \( \) By the Ekeland variational principle we can assume that \( \{ z_n \} \) is, in addition, a \( (PS)_{c_{\epsilon}} \) sequence for \( I_{\epsilon} \) on \( \mathcal{N}_\epsilon \). A standard argument shows that \( \{ z_n \} \) is in fact a \( (PS)_{c_{\epsilon}} \) sequence for \( I_{\epsilon} \) on \( \mathcal{N}_\epsilon \) (see [48]) and \( \{ z_n \} \) is bounded. Hence \( \{ z_n + h_{\epsilon}(z_n) \} \) is a bounded \( (PS)_{c_{\epsilon}} \) sequence for \( \Phi_{\epsilon} \) on \( E \). Then we can assume, without loss of generality, that \( z_n + h_{\epsilon}(z_n) := w_n \to w_{\epsilon} = w_1^\tau + w_2^\tau \) in \( E \) with \( \Phi_{\epsilon}^\tau(w_\epsilon) = 0 \). For complete the proof, we only need to show that \( w_{\epsilon} \neq 0 \) for all small \( \epsilon > 0 \). Assume by contradiction that there is a sequence \( \epsilon_j \to 0 \) with \( w_{\epsilon_j} = 0 \). Then \( w_n = z_n + h_{\epsilon_j}(z_n) \to 0 \) in \( E \), \( z_n \to 0 \) in \( L_0^q \), for \( q \in [2, 2^*] \), and \( w_n(x) \to 0 \) a.e. on \( \mathbb{R}^N \). Choose \( \nu < \mu < \nu_\infty \) and \( \tau = \iota \). Let \( \iota_n > 0 \) be such that \( \iota_n z_n \in \mathcal{N}^{\mu \tau}_{\iota_n} \). We see
Lemma 4.3.

Lemma 4.4.

Proof. Assume by contradiction that, for some \( j \), \( \nu < \mu \) and in virtue of Lemma 4.1, letting \( j \to \infty \),

\[
\Phi_{\epsilon_j}(t_n z_n + h_{\epsilon_j}^\mu(t_n z_n)) \leq \Phi_{\epsilon_j}(t_n z_n + h_{\epsilon_j}(t_n z_n)) = I_{\epsilon_j}(t_n z_n) \leq I_{\epsilon_j}(z_n).
\]

Then, we obtain

\[
c_{\epsilon_j}^\mu \leq I_{\epsilon_j}^\mu(t_n z_n) = \Phi_{\epsilon_j}(t_n z_n + h_{\epsilon_j}^\mu(t_n z_n))
\]

\[
= \Phi_{\epsilon_j}(t_n z_n + h_{\epsilon_j}^\mu(t_n z_n)) + \frac{1}{2} \int_{\mathbb{R}^N} (V_{\epsilon_j}(x) - V_{\epsilon_j}(x))|t_n z_n + h_{\epsilon_j}^\mu(t_n z_n)|^2 + \int_{\mathbb{R}^N} (K_{\epsilon_j}(x) - K_{\epsilon_j}^\mu(x))F(|t_n z_n + h_{\epsilon_j}^\mu(t_n z_n)|)
\]

\[
\leq I_{\epsilon_j}(z_n) + \frac{1}{2} \int_{A_{\epsilon_j}} (\mu - V_{\epsilon_j}(x))|t_n z_n + h_{\epsilon_j}^\mu(t_n z_n)|^2 + \int_{A_{\epsilon_j}} (K_{\epsilon_j}(x) - \tau)F(|t_n z_n + h_{\epsilon_j}^\mu(t_n z_n)|)
\]

\[
= c_{\epsilon_j} + o(1)
\]

as \( n \to \infty \), hence, \( c_{\epsilon_j}^\mu \leq c_{\epsilon_j} \). By (3.9), \( c_{\mu} \leq c_{\epsilon_j}^\mu \), hence \( c_{\mu} \leq c_{\epsilon_j} \). Recalling that \( \tau = \iota \) and in virtue of Lemma 4.1, letting \( j \to \infty \), we obtain

\[
c_{\mu} \leq c_{\nu},
\]

which contradicts with \( c_{\nu} < c_{\mu} \) since \( \nu < \mu \).

In the same way, we obtain

**Lemma 4.3.** \( \mathcal{L}_\epsilon \) is compact for all small \( \epsilon > 0 \).

Proof. Assume by contradiction that, for some \( \epsilon_j \to 0 \), \( \mathcal{L}_{\epsilon_j} \) is not compact in \( E \). Thus, for each \( j \), there exists a sequence \( \{z_j^k\} \subset \mathcal{L}_{\epsilon_j} \) so that it has no convergent subsequence. But \( \{z_n^j\} \) is bounded. So, we may assume, without loss of generality, that \( z_n^j \to 0 \) as \( n \to \infty \). As done for proving the Lemma 4.2, one gets a contradiction.

**Lemma 4.4.** Assume \( z \in \mathcal{L}_\epsilon \), \( \nabla V \) and \( \nabla K \) are bounded. Then there is a maximum point \( y \) of \( |z(x)| \) such that \( \lim_{\epsilon \to 0} \text{dist}(ey, \mathcal{A}_\epsilon) = 0 \), and for any sequence \( y \epsilon \to \gamma \), \( w(x) := z(x + y) \) converges to a ground state solution of

\[
\begin{cases}
-\Delta u + b \cdot \nabla u + u + V(y) v = K(y) f(|z|) v & \text{in } \mathbb{R}^N,
-\Delta v + b \cdot \nabla v + v + V(y) u = K(y) f(|z|) u & \text{in } \mathbb{R}^N,
\end{cases}
\]

in \( H^2(\mathbb{R}^N) \). If moreover \( V \cap \mathcal{K} \neq \emptyset \) then \( \lim_{\epsilon \to 0} \text{dist}(ey, V \cap \mathcal{K}) = 0 \), and, up to a subsequence, \( w \) converges in \( H^2(\mathbb{R}^N) \) to a ground state solution of

\[
\begin{cases}
-\Delta u + b \cdot \nabla u + u + \nu v = \kappa f(|z|) v & \text{in } \mathbb{R}^N,
-\Delta v - b \cdot \nabla v + v + \nu u = \kappa f(|z|) u & \text{in } \mathbb{R}^N.
\end{cases}
\]

Proof. Let \( \epsilon_j \to 0 \), \( z_j \in \mathcal{L}_j \) where \( \mathcal{L}_j \) is \( \mathcal{L}_\epsilon \). Then \( \{z_j\} \) is bounded. A concentration argument shows that there exist a sequence \( \{y_j^j\} \subset \mathbb{R}^N \) and constants \( r > 0 \), \( \vartheta > 0 \) such that

\[
\liminf_{j \to \infty} \int_{B_r(y_j^j)} |z_j|^2 \geq \vartheta.
\]
Set

\[ w_j(x) = z_j(x + y_j'). \]

Then \( w_j = (u_j, v_j) \) is a ground state solution to the following system

\[
\begin{align*}
-\Delta u_j + b \cdot \nabla u_j + u_j + \tilde{V}_j(x) v_j &= \hat{K}_j(x) f(|w_j|) v_j \quad \text{in } \mathbb{R}^N, \\
-\Delta v_j - b \cdot \nabla v_j + v_j + \tilde{V}_j(x) u_j &= \hat{K}_j(x) f(|w_j|) u_j \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

where \( \tilde{V}_j(x) = V(\epsilon_j(x + y_j')) \) and \( \hat{K}_j(x) = K(\epsilon_j(x + y_j')) \). Hence, for any \( \varphi \in E, \)

\[
\tilde{\Phi}'_{\epsilon_j}(w_j) \varphi = (w_j^+, \varphi^+) - (w_j^-, \varphi^-) + \int_{\mathbb{R}^N} \tilde{V}_j(x) w_j \varphi - \int_{\mathbb{R}^N} \hat{K}_j(x) f(|w_j|) w_j \varphi = 0 \tag{4.4}
\]

and the least energy (using the notations of the previous section)

\[
\hat{c}_{\epsilon_j} = \tilde{\Phi}_{\epsilon_j}(w_j)
\]

\[
= \frac{1}{2}(||w_j^+||^2 - ||w_j^-||^2) + \frac{1}{2} \int_{\mathbb{R}^N} \tilde{V}_j(x) |w_j|^2 - \int_{\mathbb{R}^N} \hat{K}_j(x) F(|w_j|)
\]

\[
= \tilde{\Phi}_{\epsilon_j}(w_j) - \frac{1}{2} \tilde{\Phi}'_{\epsilon_j}(w_j) w_j
\]

\[
= \int_{\mathbb{R}^N} \hat{K}_j(x) \hat{F}(|w_j|),
\]

where

\[
\hat{F}(|z|) = \frac{1}{2} f(|z|)|z|^2 - F(|z|).
\]

Moreover,

\[
\hat{c}_{\epsilon_j} = \tilde{\Phi}_{\epsilon_j}(w_j) = \Phi_{\epsilon_j}(z_j) = c_{\epsilon_j}.
\]

After extracting a subsequence, we may assume that \( w_j \to w \) in \( E \) and \( w_j \to w \) in \( L^q_{\text{loc}} \) for \( q \in [2, 2^*) \) and \( w_j(x) \to w(x) \) a.e. on \( \mathbb{R}^N \).

Now, the rest of the proof is divided into several steps.

Step 1: \( w_j \to w \) in \( E \). Since \( V \) and \( K \) are bounded, we can assume without loss of generality that \( V(\epsilon_j y_j') \to V_0 \) and \( K(\epsilon_j y_j') \to K_0 \) as \( j \to \infty \). By the mean value theorem and the boundedness of \( \nabla V \) and \( \nabla K \), we know that

\[
\tilde{V}_j(x) \to V_0 \quad \text{and} \quad \hat{K}_j(x) \to K_0 \quad \text{as} \quad j \to \infty \tag{4.6}
\]

uniformly on bounded sets. It then follows from (4.4) that, for any \( \varphi \in C_0^\infty \),

\[
\Phi'_{\epsilon_j K_0}(w) \varphi
\]

\[
= (w^+, \varphi^+) - (w^-, \varphi^-) + \int_{\mathbb{R}^N} V_0 w \varphi - \int_{\mathbb{R}^N} K_0 f(|w|) w \varphi
\]

\[
= \lim_{j \to \infty} \left( (w_j^+, \varphi^+) - (w_j^-, \varphi^-) + \int_{\mathbb{R}^N} \tilde{V}_j(x) w_j \varphi - \int_{\mathbb{R}^N} \hat{K}_j(x) f(|w_j|) w_j \varphi \right)
\]

\[
= 0,
\]

consequently, \( w = (u, v) \) solves

\[
\begin{align*}
-\Delta u + b \cdot \nabla u + u + V_0 v &= K_0 f(|w|) v \quad \text{in } \mathbb{R}^N, \\
-\Delta v - b \cdot \nabla v + v + V_0 u &= K_0 f(|w|) u \quad \text{in } \mathbb{R}^N,
\end{align*}
\]

\[
\begin{align*}
\hat{c}_{\epsilon_j} &= \Phi_{\epsilon_j K_0}(w) \\
&= \Phi_{\epsilon_j K_0}(w) \\
&= \Phi_{\epsilon_j K_0}(w)
\end{align*}
\]
with the energy
\[
\Phi_{V_0K_0}(w) = \frac{1}{2}(\|w^+\|^2 - \|w^-\|^2) + \int_{\mathbb{R}^N} V_0|w|^2 - \int_{\mathbb{R}^N} K_0\tilde{F}(|w|) = \int_{\mathbb{R}^N} K_0\tilde{F}(|w|) \geq c_{V_0K_0},
\]
where \(c_{V_0K_0}\) denotes the least energy of (4.8). By Fatou’s lemma,
\[
\int_{\mathbb{R}^N} K_0\tilde{F}(|w|) \leq \lim_{j \to \infty} \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x)\tilde{F}(|w_j|)
\]
which, jointly with Lemma 3.4, implies that
\[
\Phi_{V_0K_0}(w) \leq \lim_{j \to \infty} \epsilon_j \leq c_{V_0K_0}.
\]
Therefore,
\[
\lim_{j \to \infty} \epsilon_j = \Phi_{V_0K_0}(w) = c_{V_0K_0}
\]
and
\[
\lim_{j \to \infty} \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x)\tilde{F}(|w_j|) = \int_{\mathbb{R}^N} K_0\tilde{F}(|w|) = c_{V_0K_0}.
\]

Let \(\chi : [0, \infty) \to [0, 1]\) be a smooth function satisfying \(\chi(t) = 1\) if \(t \leq 1\), \(\chi(t) = 0\) if \(t \geq 2\). Define \(\tilde{w}_j := \chi(2|x|/j)w(x)\). Then
\[
\|w - w_j\| \to 0 \quad \text{and} \quad |w - w_j|_q \to 0 \quad \text{as} \quad j \to \infty
\]
for \(q \in [2, 2^*]\). Setting \(\xi_j = w_j - \tilde{w}_j\), it is not difficult to verify that along a subsequence,
\[
\lim_{j \to \infty} \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x)(F(|w_j|) - F(|\xi_j|) - F(|\tilde{w}_j|)) = 0
\]
and
\[
\lim_{j \to \infty} \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x)(f(|w_j|)w_j - f(|\xi_j|)\xi_j - f(|\tilde{w}_j|)\tilde{w}_j) \varphi = 0
\]
uniformly in \(\varphi \in E\) with \(|\varphi| \leq 1\) (see [1, 10]). Using the exponentially decay of \(w\), the Hölder inequality, (4.6) and (4.12), we have
\[
\left| \int_{\mathbb{R}^N} \hat{V}_{\epsilon_j}(x)w_j\tilde{w}_j - V_0|w|^2 \right|
\]
\[
= \left| \int_{\mathbb{R}^N} \hat{V}_{\epsilon_j}(x)w_j\tilde{w}_j - \hat{V}_{\epsilon_j}(x)w_jw + \hat{V}_{\epsilon_j}(x)w_jw - V_0|w|^2 \right|
\]
\[
\leq C|w_j|_2|w - \tilde{w}_j|_2 + \left| \int_{\mathbb{R}^N} \hat{V}_{\epsilon_j}(x)w_jw - V_0|w|^2 \right|
\]
\[
\leq \left( \int_{|x| \leq R} + \int_{|x| > R} \right) |\hat{V}_{\epsilon_j}(x) - V_0||w_j||w|
\]
\[
+ \left( \int_{|x| \leq R} + \int_{|x| > R} \right) |V_0||w_j - w||w| + o(1)
\]
\[
= o(1).
\]
Similarly,
\[
\int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x)F(|\tilde{w}_j|) \to \int_{\mathbb{R}^N} K_0F(|w|).
\] (4.16)

Consequently, by (4.11), (4.13), (4.15) and (4.16), we obtain
\[
\hat{\Phi}_{\epsilon_j}(\xi_j) = \hat{\Phi}_{\epsilon_j}(w_j) - \Phi_{\nu_0K_0}(w)
\]
\[
+ \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x) \left( F(|w_j|) - F(|\xi_j|) - F(|\tilde{w}_j|) \right) + o(1)
\]
\[
= o(1)
\]
as \( j \to \infty \), which implies that \( \hat{\Phi}_{\epsilon_j}(\xi_j) \to 0 \). Similarly,
\[
\hat{\Phi}'_{\epsilon_j}(\xi_j) \varphi = \hat{\Phi}'_{\epsilon_j}(w_j) \varphi - \Phi'_{\nu_0K_0}(w) \varphi
\]
\[
+ \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x) \left( F(|w_j|) - F(|\xi_j|) - F(|\tilde{w}_j|) \right) \varphi + o(1)
\]
\[
= o(1)
\]
as \( j \to \infty \) uniformly in \( \|\varphi\| \leq 1 \), which implies that \( \hat{\Phi}'_{\epsilon_j}(\xi_j) \to 0 \). Then,
\[
o(1) = \hat{\Phi}_{\epsilon_j}(\xi_j) - \frac{1}{2} \hat{\Phi}'_{\epsilon_j}(\xi_j) \xi_j = \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x) \hat{F}(|\xi_j|).
\] (4.17)

Observe that, by \((F_2)\), it holds that
\[
\hat{F}(z) = \frac{1}{2} f(|z|)|z|^2 - F(|z|) \geq \left( \frac{1}{2} - \frac{1}{\theta} \right) f(|z|)|z|^2.
\]

This, together with (4.17), shows that
\[
\int_{\mathbb{R}^N} \hat{K}_{\epsilon_j}(x) f(|\xi_j||\xi_j|)^2 \to 0.
\] (4.18)

Therefore, it follows from (4.18) that
\[
(1 - a)\|\xi_j\|^2 \leq \|\xi_0\|^2 - a|\xi_j|^2
\]
\[
\leq \|\xi_0\|^2 + \int_{\mathbb{R}^N} \hat{V}_{\epsilon_j} \xi_j (\xi_j^+ - \xi_j^-)
\]
\[
\leq \hat{\Phi}'_{\epsilon_j}(\xi_j) (\xi_j^+ - \xi_j^-) + \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j} f(|\xi_j||\xi_j|) (\xi_j^+ - \xi_j^-)
\]
\[
\leq \hat{\Phi}'_{\epsilon_j}(\xi_j) (\xi_j^+ - \xi_j^-) + \int_{\mathbb{R}^N} \hat{K}_{\epsilon_j} f(|\xi_j||\xi_j|) (\xi_j^+ - \xi_j^-)
\]
\[
= o(1),
\]
that is, \( \|\xi_j\| \to 0 \). This, together with (4.12), implies that \( w_j \to w \) in \( E \) as \( j \to \infty \).

Step 2: \( w_j \to w \) in \( H^2(\mathbb{R}^N) \). In fact, using the same notations as in (1.5), we have
\[
Aw_j = \left( \hat{K}_{\epsilon_j}(x) f(|w_j|) - \hat{V}_{\epsilon_j}(x) \right) w_j
\]
and
\[
Aw = (K_0(x) f(|w|) - V_0) w.
\]
Therefore, by (4.6) and the fact that \( w_j \to w \) in \( E \), we have
\[
|A(w_j - w)|^2 = \int_{\mathbb{R}^N} \left| \tilde{K}_{\varepsilon_j}(x)f(|w_j|)w_j - K_0(x)f(|w|)w + V_0w - \tilde{V}_{\varepsilon_j}(x)w_j \right|^2
\leq \int_{\mathbb{R}^N} \left| \tilde{K}_{\varepsilon_j}(x)f(|w_j|)w_j - K_0(x)f(|w|)w \right|^2
+ \int_{\mathbb{R}^N} \left| V_0w - \tilde{V}_{\varepsilon_j}(x)w_j \right|^2
= o(1),
\]
which implies that \( w_j \to w \) in \( H^2(\mathbb{R}^N) \).

Step 3: \( w_j \to 0 \) as \( |x| \to \infty \) uniformly in \( j \in \mathbb{N} \). Assume by contradiction that the conclusion of the lemma does not hold. Then by (2.7) there exist \( \sigma > 0 \) and \( x_j \in \mathbb{R}^N \) with \( |x_j| \to \infty \) such that
\[
\sigma \leq |w_j(x_j)| \leq C_0 \left( \int_{B_1(x_j)} |w_j|^2 \right)^{1/2}
\leq C_0 \left( \int_{\mathbb{R}^N} |w_j - w|^2 \right)^{1/2} + C_0 \left( \int_{B_1(x_j)} |w_j|^2 \right)^{1/2}
\to 0,
\]
since \( w_j \to w \) in \( H^2(\mathbb{R}^N) \), a contradiction.

Step 4: \( \{\varepsilon_jy_j\} \) is bounded. If not, up to a subsequence, \( \varepsilon_j|y_j| \to \infty \). Then
\[
\nu < \nu_\infty \leq \lim_{j \to \infty} V(\varepsilon_j(x + y_j)) = \lim_{j \to \infty} \tilde{V}_{\varepsilon_j} = V_0
\]
and
\[
K_0 = \lim_{j \to \infty} \tilde{K}_{\varepsilon_j} = \lim_{j \to \infty} K(\varepsilon_j(x + y_j)) \leq \kappa_\infty \leq \iota,
\]
and hence
\[
c_{\varepsilon_j} \to c_{V_0K_0} \leq c_\nu.
\]
This contradicts with Lemma 3.3. Therefore, without loss of generality, we may assume that \( \varepsilon_jy_j \to y_0 \),
\[
V_0 = V(y_0), \quad K_0 = K(y_0).
\]
This is to say that \( w \) is a ground state solution to (4.2). Now by Step 3 it is easy to see that one may assume that \( y_j = y_j' \) is a maximum point of \( |z_j| \).

Step 5: \( \{\varepsilon_jy_j\} \) is bounded for all small \( \epsilon > 0 \), where \( y_\epsilon \) is a maximum point of \( |z_\epsilon| \).
Suppose to the contrary that there is \( \epsilon_j \to 0 \) with \( \epsilon_j|y_j| \to \infty \), where \( y_j := y_{\epsilon_j} \) and \( y_j \) is a maximum point of \( |z_j| := |z_{\epsilon_j}| \). Using the above arguments of the step 3, we know that the associate \( y_j' \) and \( w_j = (x + y_j') \) satisfies that \( w_j(x) \to 0 \) as \( |x| \to \infty \) uniformly in \( j \in \mathbb{N} \). It follows from the step 4 that \( \{\varepsilon_jy_j'\} \) is bounded, and hence \( \varepsilon_j|y_j - y_j'| \geq \epsilon_j|y_j| - \epsilon_j|y_j'| \to \infty \). This yields that \( |y_j - y_j'| \to \infty \). Then,
\[
\max |z_j| = |z_j(y_j)| = |w_j(y_j - y_j')| \to 0,
\]
a contradiction.

Step 6: \( \lim_{\nu \to 0} \text{dist}(\varepsilon_jy_j, A\nu) = 0 \). It is sufficient to show that \( y_0 \in A\nu \). Suppose to the contrary that \( y_0 \notin A\nu \). From the definition of \( A\nu \), we denote \( A := \{ x \in \mathcal{Y} : K(x) = K(x_0) \} \) and \( \mathcal{B} := \{ x \notin \mathcal{Y} : K(x) > K(x_0) \} \), where \( K(x_0) = \max_{x \in \mathcal{Y}} K(x) \). Then \( y_0 \in (\mathcal{Y} \setminus A) \cup (\mathcal{Y} \setminus \mathcal{B}) \). Recall that \( x_0 = 0 \in \mathcal{Y} \) and \( K(0) = \iota \). If \( y_0 \in (\mathcal{Y} \setminus A) \), then \( V(y_0) = \nu \) and \( K(y_0) < \iota \), and hence \( c_{\nu \iota} < c_{V(y_0)K(y_0)} \) by Lemma 3.4. If
Lemma 4.7. \( y_0 \in (\mathcal{V}^c \setminus \mathcal{R}) \), then \( \nu < V(y_0) \) and \( K(y_0) \leq \iota \) and we obtain \( c_{\nu \iota} < c_{V(y_0)K(y_0)} \) again. Jointly with (4.10) and Lemma 4.1, implies
\[
\lim_{\epsilon \to 0} c_{\epsilon} \leq c_{\nu \iota} < c_{V(y_0)K(y_0)} = \lim_{\epsilon \to 0} c_{\epsilon},
\]
a contradiction. Finally, if assume additionally \( \mathcal{V} \cap \mathcal{K} \neq \emptyset \), one has \( \mathcal{A}_\epsilon = \mathcal{V} \cap \mathcal{K} \), so \( \lim_{\epsilon \to 0} \text{dist}(\mathcal{A}_\epsilon, \mathcal{V} \cap \mathcal{K}) = 0 \) and \( w_\epsilon \) converges in \( H^2(\mathbb{R}^N) \) to a least energy solution of (4.3). The proof is complete. \( \square \)

Lemma 4.5. There are \( c, C > 0 \) and a maximum point of \( |z_\epsilon(x)| \) such that
\[
|z_\epsilon| \leq C \exp\left(-\frac{c}{2} |x - y_\epsilon|\right).
\]

Proof. It follows from (2.8), (F1) and the conclusion of the step 3 that, there are \( R > 0 \) and \( \delta > 0 \) such that
\[
\Delta |w_\epsilon|^2 \geq \delta |w_\epsilon|^2
\]
for all \( |x| \geq R \) and \( \epsilon > 0 \) small. Let \( \Gamma(y) \) be a fundamental solutions to \(-\Delta \Gamma + \delta \Gamma = 0\). Using the uniform boundedness, we may choose \( \Gamma(y) \) so that \( |w_\epsilon(y)|^2 \leq \Gamma(y) \) holds on \( |y| = R \) for all \( \epsilon > 0 \) small. Set \( v_\epsilon = |w_\epsilon|^2 - \Gamma \), then
\[
\Delta v_\epsilon = \Delta |w_\epsilon|^2 - \Delta \Gamma \geq \delta |w_\epsilon|^2 - \Gamma = \delta v_\epsilon.
\]
By the maximum principle we can conclude that \( v_\epsilon(y) \leq 0 \) for \( |y| \geq R \), i.e., \( |w_\epsilon(y)|^2 \leq \Gamma(y) \) for \( |y| \geq R \). It is well known that there are \( c', C' > 0 \) such that
\[
\Gamma(y) \leq C' \exp(-c'|y|) \text{ for } |y| \geq 1.
\]
Therefore, there are \( c, C > 0 \) such that
\[
|w_\epsilon(x)|^2 \leq C \exp(-c|x|)
\]
for all \( x \in \mathbb{R}^N \) and \( \epsilon > 0 \) small, that is,
\[
|z_\epsilon(x)| \leq \sqrt{C} \exp\left(-\frac{c}{2} |x - y_\epsilon|\right)
\]
for all \( x \in \mathbb{R}^N \) and \( \epsilon > 0 \) small. \( \square \)

Proof of Theorem 1.1(I). Going back to \((P_\epsilon)\) with the variable substitution \( x \mapsto x/\epsilon, \eta_\epsilon(x) := z_\epsilon(x/\epsilon) \) is a semiclassical ground state solution of \((P_\epsilon)\) for all \( \epsilon > 0 \) small by Lemma 4.2. Lemma 4.3 shows that the conclusion (i) holds. Finally, it is clear that \( x_\epsilon := \epsilon y_\epsilon \) is a maximum point of \( |\eta_\epsilon(x)| \), and the conclusions (ii) and (iii) follow from Lemmas 4.4 and 4.5, respectively.

4.2. The case with \((A_0)\) and \((A_2)\). Now we assume that the conditions \((A_0)\) and \((A_2)\) hold. As above, without loss of generality, we can assume that \( x_k = 0 \in \mathcal{K} \). Then \( \kappa = K(0) \) and \( V(x) \geq \pi := V(0) \geq \nu \) for all \( |x| \geq R \) with large enough. In particular, if \( \mathcal{V} \cap \mathcal{K} \neq \emptyset \), then \( 0 \in \mathcal{V} \cap \mathcal{K} \). Similar to Lemma 4.1, one can check the following lemma

Lemma 4.6. \( \limsup_{\epsilon \to 0} c_\epsilon \leq c_{\pi \kappa} \). In particular, \( \lim_{\epsilon \to 0} c_\epsilon = c_{\pi \kappa} \) if additionally \( \mathcal{V} \cap \mathcal{K} \neq \emptyset \).

Combining Lemma 4.2 with Lemma 4.6, we can prove the following existence result.

Lemma 4.7. \( c_\epsilon \) is attained for all small \( \epsilon > 0 \).
Finally, arguing along the lines carried out in the proofs of Lemmas 4.4 and 4.5 with small modifications one gets the following lemma.

**Lemma 4.8.** We have the following conclusions.

(i) Assume \( z_\epsilon \in \mathcal{L}_\epsilon, \nabla V \text{ and } \nabla K \text{ are bounded. Then there is a maximum point } y_\epsilon \text{ of } |z_\epsilon| \text{ such that } \lim_{\epsilon \to 0} \text{dist}(e_\epsilon, \partial K) = 0, \text{ and for any sequence } e_\epsilon \to y_0, \)

\[
\begin{align*}
-\Delta u + b \cdot \nabla u + u + V(y_\epsilon)v = K(y_\epsilon)f(|z|)v \quad &\text{in } \mathbb{R}^N, \\
-\Delta v + b \cdot \nabla v + v + V(y_\epsilon)u = K(y_\epsilon)f(|z|)u \quad &\text{in } \mathbb{R}^N,
\end{align*}
\]

in \( H^2(\mathbb{R}^N). \) If moreover \( \mathcal{V} \cap \mathcal{K} \neq \emptyset \) then \( \lim_{\epsilon \to 0} \text{dist}(e_\epsilon, \mathcal{V} \cap \mathcal{K}) = 0, \) and, up to a subsequence, \( w_\epsilon \text{ converges in } H^2(\mathbb{R}^N) \) to a ground state solution of

\[
\begin{align*}
-\Delta u + b \cdot \nabla u + u + \nu v = \kappa f(|z|)v \quad &\text{in } \mathbb{R}^N, \\
-\Delta v - b \cdot \nabla v + v + \nu u = \kappa f(|z|)u \quad &\text{in } \mathbb{R}^N.
\end{align*}
\]

(ii) There are \( c,C > 0 \) such that

\[|z_\epsilon| \leq C \exp\left(-\frac{c}{2} |x - y_\epsilon|\right)\]

for all \( x \in \mathbb{R}^N. \)

**Proof of Theorem 1.1(II).** Going back to (\( P_\epsilon \)) with the variable substitution \( x \mapsto x/\epsilon, \eta_\epsilon(x) := z_\epsilon(x/\epsilon) \) is a semiclassical ground state solution of (\( P_\epsilon \)) for all \( \epsilon > 0 \) small by Lemma 4.7. Lemma 4.3 shows that the conclusion (i) holds. Finally, it is clear that \( x_\epsilon := e_\epsilon \) is a maximum point of \( |\eta_\epsilon(x)|, \) and the conclusions (ii) and (iii) follow from Lemma 4.8. \( \square \)

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