Robust Reductions

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Abstract

We continue the study of robust reductions initiated by Gavaldà and Balcázar. In particular, a 1991 paper of Gavaldà and Balcázar [7] claimed an optimal separation between the power of robust and nondeterministic strong reductions. Unfortunately, their proof is invalid. We re-establish their theorem.

Generalizing robust reductions, we note that robustly strong reductions are built from two restrictions, robust underproductivity and robust overproductivity, both of which have been separately studied before in other contexts. By systematically analyzing the power of these reductions, we

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explore the extent to which each restriction weakens the power of reductions.

We show that one of these reductions yields a new, strong form of the Karp-Lipton Theorem.

1 Introduction

The study of the relative power of reductions has long been one of central importance in computational complexity theory. Reductions are the key tools used in complexity theory to compare the difficulty of problems. When we say that \( A \) reduces to \( B \), we informally interpret this to mean “\( A \) is roughly easier than \( B \),” where the “roughly” regards a certain tolerance that reflects the power or flexibility of the reduction. To understand precisely the complexity of a problem, we must understand the nature of this tolerance, and thus we must understand the relative power of reductions.

Beyond that, reductions play a central role in countless theorems of complexity theory, and to understand the power of such theorems we must understand the relationships between reductions. For example, Karp and Lipton [14] proved that if SAT Turing-reduces to some sparse set then the polynomial hierarchy collapses. A more careful analysis reveals that the same result applies under the weaker hypothesis that SAT robustly-strong-reduces to some sparse set. In fact, the latter result is simply a relativized version of the former result [14], though the first proofs of the latter result were direct and quite complex [1,13]. As another example, in the present paper—but not by simply asserting relativization—we will note that various theorems, among them the Karp-Lipton Theorem, indeed hold for certain reductions that are even more flexible than robustly strong reductions.

In this paper, we continue the investigation of robust reductions started by Gavaldà and Balcázar [7]. We now briefly mention one way of defining strong reduction [14,7] and robustly strong reduction [7]. Definition 2.1 provides a formal definition of the same notions in terms of concepts that are central to this paper. We say that a nondeterministic Turing machine is a nondeterministic polynomial-time Turing machine (NPTM) if there is a polynomial \( p \) such that, for each oracle \( A \) and for each integer \( n \), the nondeterministic runtime of \( N^A \) on inputs of size \( n \) is bounded by \( p(n) \). (Requiring that the polynomial upper-bounds the runtime independent of the oracle is superfluous in the definition of \( \leq_{SN} \), but may be a nontrivial restriction in the definition of \( \leq_{RS} \); see the discussion of this point in Section 5. The definitions used here agree with those in the previous literature.) Consider NPTMs with three possible outcomes on each path: \textbf{acc}, \textbf{rej}, and \textbf{?}. We say \( A \) \textit{strong-reduces to} \( B \), \( A \leq_{RS} B \), if there is an NPTM \( N \) such that, for every input \( x \), it holds that (a) if \( x \in A \) then \( N^B(x) \) has at least one \textbf{acc} path and no \textbf{rej} paths, and (b) if \( x \notin A \),
then $N^B(x)$ has at least one rej path and no acc paths. (Note that in either case the machine may also have some ? paths.) Furthermore, we say $A$ robustly strong-reduces to $B$, $A \leq_{RS}^T B$, if there is an NPTM $N$ such that $A \leq_{SN}^T B$ via $N$ (in the sense of the above definition) and, moreover, for every oracle $O$ and every input $x$, $N^O(x)$ is strong, i.e., it either has at least one acc path and no rej paths, or has at least one rej path and no acc paths. This paper is concerned with the relative power of these two reductions, and with reductions whose power is intermediate between theirs.

In particular, it is claimed in [7] that the following strong separation holds with respect to the two reductions:

For every recursive set $A \not\in \text{NP} \cap \text{coNP}$, there is a recursive set $B$ such that $A$ strong-reduces to $B$ but $A$ does not robustly strong-reduce to $B$ [7, Theorem 11].

Unfortunately, there is a subtle but apparently fatal error in their proof. One of the main contributions of this paper is that we re-establish their sweeping theorem. Note that the zero degrees of these reducibilities are identical, namely the class NP $\cap$ coNP [7]. Thus, in a certain sense, the above claim of Gavaldà and Balcázar is optimal (if it is true, as we prove it is), as if $A \in \text{NP} \cap \text{coNP}$ then $A$ strong-reduces to every $B$ and $A$ also robustly strong-reduces to every $B$.

Section 3 presents our proof of the above claim of Gavaldà and Balcázar. The proof is delicate, and is carried out in three stages: First, we establish the result for all $A \in \text{EXP} - (\text{NP} \cap \text{coNP})$, where $\text{EXP} = \bigcup_{k>0} \text{DTIME}[2^{n^k}]$. Here, the set $B$ produced from the proof is not necessarily recursive. Second, we remove the restriction of $A \in \text{EXP}$, by showing that if the result fails for $A \not\in \text{EXP}$ then indeed $A \in \text{EXP}$, yielding a contradiction. The proof so far only establishes the existence of some $B$, which is not necessarily recursive. Finally, with the certainty that some $B$ exists, we can recast the proof and show that for every recursive $A$ a recursive $B$ can be constructed. We mention to the reader the referee’s comment that [20] is an antecedent of our proof approach.

The notion of “robustly strong” is made up of two components—one stating that for all sets and all inputs the reducing machine has at least one non-? path, and the other stating that for all sets and all inputs the reducing machine does not simultaneously have acc and rej paths. Each component has been separately studied before in the literature, in different contexts (see Section 4). By considering each of these two requirements in conjunction with strong reductions, we obtain two natural new reductions whose power falls between that of strong reductions and that of robustly strong reductions. Section 4 studies the relative power of Turing reductions, of strong reductions, of robustly strong reductions, and of our two new reductions. In some cases we prove absolute separations. In other cases, we see that the relative computation power is tied to the $P = \text{NP}$ question. Curiously, the two
new reductions are deeply asymmetric in terms of what is currently provable about their properties. For one of the new reductions, we show that if it differs from Turing reductions then $P \neq \text{NP}$. For the other, we prove that the reduction does differ from Turing reductions.

In Section 5, we discuss some issues regarding what collapses of the polynomial time hierarchy occur if sparse sets exist that are hard or complete for NP with respect to the new reductions. One of the new reductions extends the reach of hardness results.

2 Two New Reducibilities

For each NPTM $N$ and each set $D \subseteq \Sigma^*$, define $\text{out}_{ND}(x) = \{y \mid y \in \{\text{acc, rej, ?}\} \wedge \text{some computation path of } N^D(x) \text{ has outcome } y\}$. As is standard, for each nondeterministic machine $N$ and each set $D \subseteq \Sigma^*$, let $L(N^D)$ denote the set of all $x$ for which $\text{acc} \in \text{out}_{ND}(x)$. For each nondeterministic machine $N$ and each set $D \subseteq \Sigma^*$, let $L_{\text{rej}}(N^D)$ denote the set of all $x$ for which $\text{rej} \in \text{out}_{ND}(x)$. A computation $N^D(x)$ is called underproductive if $\{\text{acc, rej}\} \not\subseteq \text{out}_{ND}(x)$. That is, $N^D(x)$ does not have as outcomes both acc and rej. $N^D$ is said to be underproductive if, for each string $x$, $N^D(x)$ is underproductive. That is, $L(N^D) \cap L_{\text{rej}}(N^D) = \emptyset$.

Underproductive machines were introduced by Buntrock [4]. Allender et al. [2] have shown underproductivity to be very useful in the study of almost-everywhere complexity hierarchies for nondeterministic time classes.

A computation $N^D(x)$ is called overproductive if $\text{out}_{ND}(x) \neq \{?\}$. A machine $N^D$ is said to be overproductive if, for each string $x$, $N^D(x)$ is overproductive. Equivalently, $L(N^D) \cup L_{\text{rej}}(N^D) = \Sigma^*$.

We say that $N$ is robustly overproductive if for each $D \subseteq \Sigma^*$ it holds that $N^D$
is overproductive. We say that $N$ is \textit{robustly underproductive} if for each $D \subseteq \Sigma^*$ it holds that $N^D$ is underproductive.

Using underproductivity, overproductivity, and robustness, we may now define strong and robustly strong reductions, which have been previously studied. We also introduce two intermediate reductions, obtained by limiting the robustness to just the overproductivity or the underproductivity.

\textbf{Definition 2.1}

1. \cite{17}, see also \cite{19} ("strong reductions") $A \leq_{SN}^T B$ if there is an NPTM $N$ such that $N^B$ is overproductive, $N^B$ is underproductive, and $A = L(N^B)$.

2. \cite{7} ("robustly strong reductions") $A \leq_{RS}^T B$ if $A \leq_{SN}^T B$ via an NPTM $N$, and $N$ is both robustly overproductive and robustly underproductive.

3. ("strong and robustly underproductive reductions" or, for short, "U-reductions") $A \leq_{U}^T B$ if $A \leq_{SN}^T B$ via an NPTM $N$ that is robustly underproductive.

4. ("strong and robustly overproductive reductions" or, for short, "O-reductions") $A \leq_{O}^T B$ if $A \leq_{SN}^T B$ via an NPTM $N$ that is robustly overproductive.

The trivial containment relationships are shown in Proposition 2.3 and Figure 1. In this paper we ask whether some edges of the diamond picture in Figure 1 might collapse, and in particular we seek necessary conditions and sufficient conditions for such collapses.

\textbf{Notation 2.2} For each well-defined reduction $\leq_a^b$, let $\leq_a^b$ denote $\{(A, B) \mid A \leq_a^b B\}$.

\textbf{Proposition 2.3} $\leq_{P}^T \subseteq \leq_{RS}^T \subseteq \leq_{U}^T \subseteq \leq_{O}^T \subseteq \leq_{SN}^T$.

Using different terminology, robust underproductivity (though not $\leq_{U}^T$) has been introduced into the literature by Beigel (\cite{3}, see also \cite{9}), and the following theorem will be of use in the present paper.

\textbf{Theorem 2.4} (\cite{3}, see also \cite{9}) If NPTM $N$ is robustly underproductive, then

$$(\forall A)(\exists L \in P^{SAT\oplus A})[L_{rej}(N^A) \subseteq L \subseteq \overline{L}(N^A)]$$

\footnote{The literature contains various notations for strong reductions (also known as strong nondeterministic reductions). We adopt the notation of Long’s paper \cite{17}, i.e., $\leq_{SN}^T$. However, we note that some papers use other notations, such as $\leq_{SN}$, $\leq_{SN}^T$, and $\leq_{SN}^{P\cap coNP}$. For the other reductions we discuss, we replace the SN with a mnemonic abbreviation. For robustly strong we follow Gavaldà and Balcázar \cite{6} and use RS. For brevity, we use O as our abbreviation for our “strong and robustly overproductive” reductions, and we use U as our abbreviation for our “strong and robustly underproductive” reductions.}
Theorem 2.4 says that if a machine is robustly underproductive, then for every oracle there is a relatively simple set that separates its acceptance set from its $L_{\text{rej}}$ set. In particular, if $P = NP$ and $N$ is a robustly underproductive machine, then for every oracle $A$ it holds that $L(N^A)$ and $L_{\text{rej}}(N^A)$ are $P^A$-separable.

As is standard, we say that a set $S$ is \textit{sparse} if there is a polynomial $r$ such that, for each $n$, $|S^{\leq n}| \leq r(n)$. Using different terminology, “robust with respect to sparse oracles”-overproductivity (though not $\leq^O_T$) has been introduced into the literature by Hartmanis and Hemachandra \cite{9}, and the following theorem will be of use in the present paper.

**Theorem 2.5** \cite{9} If NPTM $N$ is such that for each sparse set $S$ it holds that $N^S$ is overproductive, then for every sparse set $S$ there exists a predicate $b$ computable in $FP^{SAT \oplus S}$ such that, for all $x$, $\{x \mid b(x)\} \subseteq L(N^S)$ and $\{x \mid \neg b(x)\} \subseteq L_{\text{rej}}(N^S)$, where $FP$ denotes the polynomial-time computable functions.

Theorem 2.5 says that if a machine is “robustly with respect to sparse oracles”-overproductive, then for every sparse oracle there is a relatively simple function that for each input correctly declares either that the machine has accepting paths or that the machine has rejecting paths. Crescenzi and Silvestri \cite{6} show via Sperner’s Lemma that Theorem 2.5 fails when the sparseness condition is removed, and their proof approach will be of use in this paper.

It is known that SN reductions and RS reductions have nonuniform characterizations. In particular, for every reducibility $\leq_b$ and every class $C$, let

$$R^b_a(C) = \{A \mid (\exists B \in C)[A \leq_b^a B]\}.$$  

Gavaldà and Balcázar proved the following result.

**Theorem 2.6** \cite{7}  

1. $R^{SN}_O(\text{SPARSE}) = \text{NP}/\text{poly} \cap \text{coNP}/\text{poly}$.  
2. $R^{RS}_U(\text{SPARSE}) = (\text{NP} \cap \text{coNP})/\text{poly}$.

We note in passing that the downward closures of the sparse sets under our two new reductions have analogous characterizations, albeit somewhat stilted ones. We say $A \in \text{NP}/\text{poly} \cap \text{coNP}/\text{poly}$ via the pair $(M, N)$ of NPTMs if there is a sparse set $S$ such that $A = L(M^S)$ and $\overline{A} = \overline{L(N^S)}$. Hartmanis and Hemachandra \cite{9} defined \textit{robustly $\Sigma^*$-spanning pairs of machines} $(M, N)$ to be pairs having the property $L(M^X) \cup L(N^X) = \Sigma^*$ for every oracle $X$, and \textit{robustly disjoint pairs} to be pairs having the property $L(M^X) \cap L(N^X) = \emptyset$ for every oracle $X$. Using these notions we note the following characterizations. $A \in R^O_T(\text{SPARSE})$ if and only if $A \in \text{NP}/\text{poly} \cap \text{coNP}/\text{poly}$ via some robustly $\Sigma^*$-spanning pair $(M, N)$ of NPTMs. $A \in R^U_T(\text{SPARSE})$ if and only if $A \in \text{NP}/\text{poly} \cap \text{coNP}/\text{poly}$ via some robustly disjoint pair $(M, N)$ of NPTMs.
A Strong Separation of $\leq^\text{SN}_T$ and $\leq^\text{RS}_T$

It follows from each of Section 4's Theorems 4.2 and 4.8, both of which have relatively simple proofs, that the reducibilities $\leq^\text{SN}_T$ and $\leq^\text{RS}_T$ are distinct. However, more can be said. The separation of these two reductions turns out to be extremely strong, namely, for every recursive set $A \not\in \text{NP} \cap \text{coNP}$, there exists a recursive set $B$ such that $A$ is strongly reducible to $B$ but $A$ is not robustly strongly reducible to $B$. This is Theorem 3.4. As noted in Section 1, this claim cannot be generalized to include $\text{NP} \cap \text{coNP}$ since $\text{NP} \cap \text{coNP}$ is the zero degree of $\leq^\text{RS}_T$, as has been pointed out by Gavaldà and Balcázar [7].

Theorem 3.4 was first stated in Gavaldà and Balcázar’s 1991 paper [7]. The diagonalization proof given there correctly establishes $A \not\leq^\text{RS}_T B$, but it fails to establish $A \leq^\text{SN}_T B$. The main error is the following: In the proof there is a passage [7, p. 6, lines 21–25] where a certain word $x$ is searched for. If such an $x$ is found, then $B$ is augmented by some suitably chosen word (triple). Now it is true that such an $x$ must always exist. However, it might be huge, and then between this $x$ and the previous one, say $x'$, no coding has been done, i.e., for all $z$ between $x'$ and $x$, no triple $(z, y, 0)$ or $(z, y, 1)$ with $|z| = |y|$ has been added to $B$. Thus, the condition “(i)” of [7, p. 6], which is intended to guarantee $A \leq^\text{SN}_T B$, is violated.

We now turn towards the proof of Theorem 3.4. However, we first prove the following claim.

**Theorem 3.1** $(\forall$ recursive $A \not\in \text{NP} \cap \text{coNP})(\exists B)[A \leq^\text{SN}_T B \land A \not\leq^\text{RS}_T B]$.

**Proof** We distinguish two cases: Case 1: $A \in \text{EXP}$ and Case 2: $A \not\in \text{EXP}$. In the first case, we will show that if no such $B$ exists, then $A \in \text{NP} \cap \text{coNP}$. In the second case, we will show that if no such $B$ exists for our $A$, then in fact $A \in \text{EXP}$, thus generating a contradiction to $A \not\in \text{EXP}$.

**Case 1:** Suppose $A \in \text{DTIME}[2^{nk}]$ for some constant $k$. Our set $B$, as finally constructed, will satisfy the following condition.

**Condition 3.2** For each $x \in \Sigma^*$,

1. $x \in A \longrightarrow \big((\exists y)[|y| = |x|^k \land (x, y, 1) \in B] \land (\exists y')[|y'| = |x|^k \land (x, y', 2) \in B)\big)$, and

2. $x \not\in A \longrightarrow \big((\exists y)[|y| = |x|^k \land (x, y, 1) \in B] \land (\exists y')[|y'| = |x|^k \land (x, y', 2) \in B)\big)$.

A subset of $\Sigma^*$ is identified with its characteristic sequence according to lexicographic order. The $n$-segment, $B^{<n}$, of a set $B$ is defined to be the initial segment of this sequence including all words of length less than $n$.

A set $B$ satisfying Condition 3.2 is called *admissible*. Each initial segment of an admissible set is also called admissible. A set $C$ is called a *consistent extension* of an admissible initial $n$-segment $I$ if $C$ is admissible and $C^{<n} = I$. 

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For each $B$ satisfying Condition 3.2, it clearly holds that $A \leq^{SN} B$. We will define $B$ in such a way that no NP machine with three final states—acc, rej, and $?$—can robustly strong-reduce $A$ to $B$. Assume a list $\hat{L}$ of acc/rej/$?$-final-state NPTMs created by pairing every acc/rej/$?$-final-state nondeterministic Turing machine with every clock of the form $n\hat{k} + \hat{k}$, $\hat{k} > 0$. Note that each $n\hat{k} + \hat{k}$ is a monotonically increasing function on the natural numbers. Initially, $i = 1$, $n = 0$, and $B^{<0} = \emptyset$. Note that this is admissible. Suppose now $i \geq 1$, and that $B$ has been determined up to $B^{<n}$, and that $B^{<n}$ is admissible.

Stage $i$ Let $N$ be the $i$th machine on the list $\hat{L}$. Let $p$ be the polynomial of the (clearly attached) clock that (for all oracles) upper-bounds the running time of $N$. Define $n_0$ to be the smallest number $m > n$ such that:

$$(3.a) \quad (\forall b \in \{1, 2\})(\forall x, y \in \Sigma^*)(|\langle x, y, b \rangle| \geq m \land |y| = |x|^k) \longrightarrow p(|x|) < 2|x|^k.$$ 

Extend $B^{<n}$ to an admissible segment $B^{<n_0}$. (This can always be done, as the very fact that $B^{<n}$ is an admissible $n$-segment implies that there is an admissible set $B$ of which it is a prefix, so the $n_0$-segment of that set $B$ is an extension of $B^{<n}$ and must also be admissible.)

Now we consider the following question. (Recall that we have defined above the notion of a consistent extension of an admissible initial segment.)

**Question 3.3** Is there a consistent extension $C$ of $B^{<n_0}$ such that either $(\exists x \in A)[N^C(x) \text{ has a rejecting computation}]$ or $(\exists x \not\in A)[N^C(x) \text{ has an accepting computation}]$?

If the answer to Question 3.3 is “yes,” then we fix such an extension $C$ for the lexicographically smallest applicable $x$, determine $m$ by $m = \max\{n_0 + 1, p(|x|)\}$, and extend $B^{<n_0}$ to $B^{<m}$, which is chosen to be the $m$-segment of $C$. This choice of the extension preserves all rejecting paths (if the first case of Question 3.3 occurs) and all accepting paths (if the second case of Question 3.3 occurs). If the answer to Question 3.3 is “no,” then nothing further will be done in Stage $i$. Notice that we do not claim that the answer to Question 3.3 is recursively computable. However, this is not a problem, as Theorem 3.1 does not require $B$ to be recursive.

Clearly, Question 3.3 is answered “yes” infinitely often. ($\hat{L}$, as any reasonable enumeration of Turing machines does, contains infinitely many machines $N$ such that for all oracles $C$, $L(N^C) \not\subseteq A$ or $L_{\text{rej}}(N^C) \not\subseteq \overline{A}$.) Since Question 3.3 is answered “yes” infinitely often, clearly the length to which $B$ is constructed increases unboundedly. Thus $B$, as finally constructed, is admissible (and not a finite initial segment). Since $B$ is admissible, it follows that $A$ is strongly reducible to $B$.

It remains to show that $A$ is not robustly strong reducible to $B$. We prove this by contradiction. Suppose $A \leq^{RS}_T B$ via some machine $N$ from list $\hat{L}$. (If $A \leq^{RS}_T B$
via some machine not on list $\hat{L}$ then certainly $A \leq^{\text{RS}}_{T} B$ via some machine on $\hat{L}$, so the assumption that $N$ is chosen from $\hat{L}$ can be made without loss of generality.) Let $p$ be the polynomial enforced by the clock of machine $N$; recall that this is a monotonically increasing polynomial that, independent of the oracle, upper-bounds the running time of $N$. Assume that $N$ was considered in Stage $i$ and let $B^{<n_0}$ be the initial segment of $B$ that was constructed at the beginning of Stage $i$ as admissibly extended as described right after Equation 3.a. We prove the following claim.

**Claim** If $A \leq^{\text{RS}}_{T} B$ (where $B$ is as just constructed), then $A \in \text{NP} \cap \text{coNP}$. We will describe an NPTM $M$ that witnesses the membership $A \in \text{NP} \cap \text{coNP}$. The machine $M$ remembers the finite initial segment $B^{<n_0}$. On each input $x$, $M$ will simulate without an oracle the computation of $N$ on $x$ using a certain oracle that actually depends on $x$. For each $x$, let the integer $m_x$ be defined by

$$m_x = \max\{n \mid 2^n k \leq p(|x|)\},$$

and let

$$B_x = B^{<n_0} \cup \{\langle x', 0^{|x'|^k}, 1 \rangle \mid |\langle x', 0^{|x'|^k}, 1 \rangle| \geq n_0 \land |x'| \leq m_x \land x' \in A\}$$

$$\cup \{\langle x', 0^{|x'|^k}, 2 \rangle \mid |\langle x', 0^{|x'|^k}, 2 \rangle| \geq n_0 \land |x'| \leq m_x \land x' \not\in A\}.$$

Now we describe $M$. On input $x$, $M$ simulates $N^{B_x}(x)$, where queries are handled as follows: Short queries (i.e., of length $< n_0$) are answered directly by the finite set $B^{<n_0}$. Each other query is answered “no,” unless it is of the form $\langle x', 0^{|x'|^k}, 1 \rangle$ or $\langle x', 0^{|x'|^k}, 2 \rangle$, where $|x'| \leq m_x$. For each such query, use the EXP algorithm for $A$ to determine membership of $x'$ in $A$. This algorithm runs in time at most $2^{|x'|^k} \leq p(|x|)$, since $|x'| \leq m_x$. If $x' \in A$ then answer “yes” to $\langle x', 0^{|x'|^k}, 1 \rangle$ and answer “no” to $\langle x', 0^{|x'|^k}, 2 \rangle$. If $x' \not\in A$ then do exactly the opposite.

This shows:

$M$ runs in nondeterministic polynomial time and, for each $x$, simulates the work of $N^{B_x}(x)$.

Since we assumed that $N$ is robustly strong, it follows that, for all $x$ and for all oracle sets, $N$ on $x$ using the oracle yields either some accepting computation and no rejecting computation, or some rejecting computation but no accepting computation. This is true even though the oracle $B_x$ used by $N$ depends on $N$’s input, $x$. Thus the only issue is whether the decision made along each accepting or rejecting path is correct.

Suppose some accepting or rejecting path is incorrect. Thus, for some $x$, either $x \in A$ and yet some path as described above rejects, or $x \not\in A$ and yet some path as described above accepts. However, for this particular $x$ and a particular such
path we can extend \( B^{<\alpha_0} \) (in fact, can extend \( B_x \) in light of the comments later in this paragraph regarding the pairing function) consistently in such a way that such a rejecting or accepting computation path is preserved. The key point is that the number of queries is at most \( p(|x|) \), which is strictly less than \( 2^{|\bar{x}|} \), the number of available \( y \)'s for any \( \bar{x} \) with \( |\bar{x}| > m_x \). To claim that all these \( y \)'s are actually available, we note that we are assuming certain properties of the pairing function, namely, that if \( |a| = |a'|, |b| = |b'|, c \in \{1, 2\} \), and \( c' \in \{1, 2\} \), then \( |\langle a, b, c \rangle| = |\langle a', b', c' \rangle| \). We also assume, as is standard, that if one increases the length of any one input to the pairing function the length of the output does not decrease. We do not require that the pairing function be onto, though we do require it be 1-to-1. Thus, a consistent extension of \( B_x \) can be found that preserves the particular path.

However, if that is the case, then the answer to the initial question, Question 3.3, during the construction of \( B \) must have been “yes,” and the construction of \( B \) would have explicitly ruled out the possibility that the machine \( N \) accepts \( A \) with oracle \( B \). This contradicts our supposition regarding there being some incorrect path. Thus, it must be the case that for all \( x \) the described computation of \( M \) is such that (a) if \( x \in A \) then some path accepts and no path rejects, and (b) if \( x \not\in A \) then some path rejects and no path accepts. This is a proof that \( A \in \text{NP} \cap \text{coNP} \). Thus our assumption (for the current case—\( A \in \text{EXP} \)) that the \( B \) constructed does not satisfy Theorem 3.1 leads to a contradiction, as in fact \( A \not\in \text{NP} \cap \text{coNP} \). This concludes the proof of Case 1 (\( A \in \text{EXP} \)).

**Case 2:** Suppose \( A \not\in \text{EXP} \). The proof structure is as follows: Similarly to Case 1 we construct a set \( B \) such that \( A \leq^T_S B \). However, if \( A \leq^R_S B \) then we show that this implies that \( A \in \text{EXP} \), yielding a contradiction to \( A \not\in \text{EXP} \).

We require that \( B \) satisfy Condition 3.2 for \( k = 1 \):

\[
x \in A \longrightarrow \left( (\exists y)[|y| = |x| \land \langle x, y, 1 \rangle \in B] \land (\exists z)[|z| = |x| \land \langle x, z, 2 \rangle \in B] \right),
\]

\[
x \not\in A \longrightarrow \left( (\exists y)[|y| = |x| \land \langle x, y, 1 \rangle \in B] \land (\exists z)[|z| = |x| \land \langle x, z, 2 \rangle \in B] \right).
\]

The construction of \( B \) is as in Case 1. Thus we get an admissible \( B \), and hence \( A \leq^S_B \).

The crucial claim is the following:

Claim: If \( A \leq^R_T B \) then \( A \in \text{EXP} \).

Let us assume \( A \leq^R_T B \) via \( N \). For reasons analogous to those discussed in Case 1, we may without loss of generality assume that \( N \) is chosen from the list \( \hat{L} \). Let \( p \) be the (nondecreasing) polynomial clock upper-bounding, independent of the oracle, the running time of \( N \). Assume that \( N \) was considered in Stage \( i \) and let \( B^{<\alpha_0} \) be the initial segment of \( B \) that has been constructed at the beginning of Stage \( i \) and admissibly extended as described right after Equation 3.3.
For each $x$, let the integer $m_x$ be defined by

$$m_x = \max \{ n \mid 2^n \leq p(|x|) \}.$$ 

Note that $m_x = \mathcal{O}(\log |x|)$, and an upper-bound on the constant of the $\mathcal{O}$ can be seen immediately given the polynomial $p$. For each $x$, let

$$B_x = B^{<n_0} \cup \{ \langle x', 0|x'|, 1 \rangle \mid |\langle x', 0|x'|, 1 \rangle| \geq n_0 \land |x'| \leq m_x \land x' \in A \}$$

$$\cup \{ \langle x', 0|x'|, 2 \rangle \mid |\langle x', 0|x'|, 2 \rangle| \geq n_0 \land |x'| \leq m_x \land x' \not\in A \}.$$ 

We will describe a deterministic computation that for any given $x$ simulates $N^{B_x}(x)$. For a given $x$, the algorithm either will use table lookup or will examine $2^p(|x|)$ nondeterministic paths of $N(x)$ in turn.

Since $m_x = \mathcal{O}(\log |x|)$, there are at most a finite number of inputs $x$ for which $|x| \leq m_x$. On input $x$, if $|x| \leq m_x$ then use finite table lookup to determine the correct answer. Now suppose $|x| > m_x$.

Let $p'(n) = p(n) + n^2$. Then $p'(n) \geq p(n)$, $p'(n) \geq n^2$ and, like $p(n)$, $p'(n)$ is monotonic increasing. We will inductively assume that for all $x'$ such that $|x'| < |x|$, “$x' \in A$?” can be decided in time $2^{p'(|x'|)}$. Upon input $x$, $|x| = n$, we first compute $\chi_A(x')$ for all $|x'| < |x|$. This takes time at most $\sum_{i=0}^{n-1} 2^i 2^{p(i)}$, which is bounded by $2^{p(n-1)+n}$, by the fact that $p$ is monotonic and thus this is a geometric series. Next we write down $p(n)$ bits, as nondeterministic moves of $N(x)$. We will cycle through all such $p(n)$ bits. For a particular sequence of $p(n)$ bits, we simulate the computation of $N$ on $x$ as follows: Whenever $N(x)$ makes a query $q$ of length less than $n_0$, we answer it according to $B^{<n_0}$. If a query $q$ has length $|q| \geq n_0$, and is of the relevant form $\langle x', 0|x'|, 1 \rangle$ or $\langle x', 0|x'|, 2 \rangle$, and $|x'| \leq m_x$ (note that this implies that $|x'| \leq m_x < |x|$), then we answer according to $\chi_A(x')$. For all other queries, the answer is “no.” Thus, the computation for $x$ with $|x| = n$ takes time at most

$$2^{p(n-1)+n} + 2^{p(n)}p(n).$$

We have $p(n) < 2^n$ since $n = |x| > m_x$. It is easy to verify that $p(n) + n \leq p'(n) - 1$ as well as $p'(n-1) + n \leq p'(n) - 1$, by the definition of $p'(n)$, and the monotonicity of $p(n)$. Thus the time taken to simulate the computation of $N$ on $x$ is at most $2^{p(n)}$, completing the induction. This completes the proof of the Claim, namely $A \in \text{EXP}$, and thus we reach a contradiction in Case 2.

The proof of Theorem 3.1 is complete. \(\Box\)

Now we prove the main claim of this section.

**Theorem 3.4** (\(\forall \text{ recursive } A \not\in \text{NP } \cap \text{coNP}) (\exists \text{ recursive } B)[A \leq_T^{\text{SN}} B \land A \not\leq_T^{\text{RS}} B].
**Proof** For an initial $n$-segment $I$ we define $X \ast I = (X \setminus X^{<n}) \cup I$. We make the following three preliminary claims. As they are clear, we state them without proof.

Claim 1: For each initial segment $I$: If $A \leq_{SN}^T B$ and $C = B \ast I$, then $A \leq_{SN}^T C$.

Claim 2: Let $A$ and $B$ be chosen according to Theorem 3.1 and let $C = B \ast I$ for some initial segment $I$. Then every machine $N$ strongly reducing $A$ to $C$ must fail to be strong for some extension $E$ of $I$.

Claim 3: Every admissible initial segment $I$ has a consistent extension $C$ such that $A \leq_{SN}^T C$, but no machine reducing $A$ strongly to $C$ is strong for all extensions of $I$.

Now we prove Theorem 3.4. We modify the construction of $B$. Let us assume that at the beginning of Stage $i$ the admissible initial segment $I$ is available as the result of the previous steps. Let the machine $N$ be considered in Stage $i$.

We check by a systematic search whether

1. there exists a finite initial segment that is an admissible extension of $I$ that witnesses that $A$ is not correctly reduced to this extension by $N$ (recall that the definition of an admissible initial segment means each such initial segment is an initial segment of some admissible set), or

2. there exists a finite initial segment that is an extension of $I$ that witnesses that $N$ fails to be strong using this extension as its oracle. (Here we can allow even inadmissible extensions.)

Call these two cases type-1 and type-2. If a type-1 extension is found first, then extend $B$ in an admissible way to preserve a computation witnessing a contradictory computation (i.e., an accepting path if $x \notin A$, and a rejecting path if $x \in A$). If a type-2 extension is found first, then nothing is done in this stage.

*Claim: One of the two cases must happen.*

Let $N$ be strong for all extensions of $I$. Since $I$ is admissible, Claim 3 yields an admissible $C$ such that $A \leq_{SN}^T C$, but no machine strongly reducing $A$ to $C$ can be robustly strong for all extensions of $I$. Since $N$ is strong for all extensions of $I$, from Claim 3 we can conclude that $N$ does not strong-reduce $A$ to $C$. So $N$ does not reduce $A$ to some particular finite initial segment of $C$ (one long enough to witness the non-reduction of $A$ to $C$ via $N$). Thus a type-1 extension will be found unless a type-2 extension is encountered sooner.

Suppose now that $N$ is not strong for some extension of $I$. Then this will become apparent at some finite length for some (not necessarily consistent) finite extension of $I$ on some input. Hence a type-2 extension is found unless a type-1 extension is found sooner. This modification of the construction of $B$ shows that $B$ is the union of a growing sequence of effectively constructed finite initial segments. Thus, $B$ is recursive. $\square$

More generally, the proof actually shows that a $B$ recursive in $A$ can be found to satisfy the theorem.
One can ask whether the difference of $\leq_{ST}^{SN}$ and $\leq_{RT}^{RS}$ is so strong that the following statement holds: $(\forall$ recursive $B \not\in NP \cap coNP)(\exists$ recursive $A)[A \leq_{ST}^{SN} B \land A \not\leq_{RT}^{RS} B]$. This can be reformulated in terms of reducibility downward closures: $(\forall$ recursive $B \not\in NP \cap coNP)[R_{RT}^{RS}(B) \subset R_{RT}^{SN}(B)]$. However, this claim is false. Intuitively, if $B$ is chosen to be of appropriately great structural richness, the differences between the two reductions may be too fine to still be distinguishable in the presence of $B$. For instance, if $B$ is an EXPSPACE-complete set, and thus is certainly not contained in $NP \cap coNP$, then for every $A \in R_{RT}^{SN}(B) = NP^B \cap coNP^B = EXPSPACE$ we have $A \leq_{m}^p B$ and hence $A \leq_{RT}^{RS} B$, i.e., $R_{RT}^{SN}(B) = R_{RT}^{RS}(B)$.

4 Comparing the Power of the Reductions

Long [17] proved that strong and Turing polynomial-time reductions differ. More precisely, he proved the following result.

**Theorem 4.1** [17] $(\forall$ recursive $A \not\in P)(\exists$ recursive $B)[A \leq_{ST}^{SN} B \land A \not\leq_{RT}^{P} B]$.

Consequently, at least one of the edges in Figure 1 must represent a strict inclusion. Indeed, we can show that strong reductions differ from both overproductive and underproductive reductions.

**Theorem 4.2**

1. $(\exists$ recursive $A)(\exists$ recursive $B)[A \leq_{ST}^{SN} B \land A \not\leq_{RT}^{O} B]$. Indeed, we may even achieve this via a recursive sparse set $B$ and a recursive tally set $A$.

2. $(\exists$ recursive $A)(\exists$ recursive $B)[A \leq_{ST}^{SN} B \land A \not\leq_{RT}^{U} B]$. Indeed, we may even achieve this via a recursive sparse set $B$ and a recursive tally set $A$.

**Proof** For a given set $B$ define $A$ (implicitly, $A_B$) by

\[(4.b) \quad A = \{0^i \mid (\exists y)[|y| = i \land \langle 0^i, y, 1 \rangle \in B]\}.
\]

$B$ is constructed by diagonalization, and we make sure that

\[(4.c) \quad (\forall i)(\exists \text{ exactly one } u \in \{0, 1\})(\exists \text{ exactly one } y)[\langle 0^i, y, u \rangle \in B \land |y| = i].
\]

Assume that $B$ has the property of Equation [4.c] and that $A$ is defined by Equation [4.b]. Then $A$ strongly reduces to $B$ via a machine $N$ that is described as follows (this is simply making explicit what is implicitly clear from Equation [4.b]):

An input $x$ is rejected if it is not an element of $\{0\}^*$. Otherwise, $x$ is an element of $\{0\}^*$, say $x = 0^i$. Then $N$ nondeterministically generates all $y$ such that $|y| = i$, and using the oracle $B$, for each $y$ and $u \in \{0, 1\}$, it finds
out whether \(\langle 0^i, y, u \rangle \in B\). A path accepts if on this path \(\langle 0^i, y, 1 \rangle \in B\) is determined, and it rejects if \(\langle 0^i, y, 0 \rangle \in B\) is determined. In all remaining cases, the path is a no-comment (i.e., ?) path.

By Equation 4.4, on each input \(N\) reaches on exactly one path the correct answer to the question of whether \(x \in A\), and all remaining paths are no-comment paths.

For the construction of \(B\) we assume an effective enumeration, \(E\), of all nondeterministic polynomial-time oracle machines. \(B\) is constructed to be the union \(B = \bigcup_{i=0}^{\infty} B_i\), where \(B_i\) is determined (in a recursively computable way) in Stage \(i\) of the construction, and \(B_0 \subseteq B_1 \subseteq B_2 \subseteq \ldots\) are finite initial segments of \(B\). This ensures that \(B\) is recursive, which ensures that \(A\) also is recursive. Set \(B_0 = \emptyset\) and let \(i\) initially equal 1. As a formal choice to avoid problems in Stage 1, we act as if the \(\#n\) of Stage 0 were \(-1\).

**Start of stage construction**

**Stage \(i\).** Let \(M\) be the \(i\)th machine in our enumeration \(E\). Let the polynomial \(p\) be a bound on the computation time of \(M\). Choose \(n\) sufficiently large (so large that the lengths of all queries occurring in all previous stages are less than \(n\) and that \(p(n) < 2^n\)). To help maintain Equation 4.4, we will do some coding now. For each \(\tilde{n}\) that is strictly less than the \(\#n\) we just set, but strictly greater than the \(\#n\) that was set in Stage \(i - 1\), put into \(B\) exactly one as-yet-untouched string of the form: \(\langle 0^n, \tilde{w}, b \rangle\), with \(|\tilde{w}| = \tilde{n}\) and \(b \in \{0, 1\}\). Such strings can be found (as, by our choice of the \(\#n\) of Stage \(i - 1\), the simulation we ran during Stage \(i - 1\) cannot touch enough strings to prevent this).

Let \(x = 0^n\).

**Case 1:** \(M^{B_{i-1}}(x)\) has a rejecting path. We choose one such path \(\alpha\) and freeze it, i.e., we make sure that all queries answered negatively on \(\alpha\) must remain outside of all the \(B_j\). There exist \(2^\#n\) triples of the form \(\langle 0^n, y, 1 \rangle\) where \(|y| = n\), but only \(p(n)\) can occur on \(\alpha\). Thus, for some \(z\) for which \(\langle 0^n, z, 1 \rangle\) is not queried on \(\alpha\), we set \(B_i = B_{i-1} \cup \{\langle 0^n, z, 1 \rangle\}\). Note that Equation 4.4 is maintained at this length.

**Case 2:** \(M^{B_{i-1}}(x)\) has an accepting path. In this case we freeze an accepting path and put some \(\langle 0^n, w, 0 \rangle\) in \(B\). Note that Equation 4.4 is maintained at this length.

**Case 3:** \(M^{B_{i-1}}(x)\) has only no-comment paths. Note that \(M\) is in this case clearly not robustly overproductive. Add to \(B\) exactly one string of the form: \(\langle 0^n, w, b \rangle\), with \(|w| = n\) and \(b \in \{0, 1\}\). Note that Equation 4.4 is maintained at this length.

**End of stage construction**

Assume \(A \leq^0_1 B\) via machine \(M\), and let \(i\) be such that this machine was considered in Stage \(i\). Let \(n\) be the input length chosen in Stage \(i\) and let \(x = 0^n\). Let us consider the implications based on which of the three cases applied to \(M^{B_{i-1}}(x)\). If Case 1 held, a rejecting path \(\alpha\) was frozen, and thus, since later stages will not interfere, \(\alpha\) in fact will be a rejecting path of \(M^B(x)\). By adding \(\langle 0^n, z, 1 \rangle\) to \(B\), we
obtained \( x \in A \) by Equation \([1.3]\). There are two cases:

1. \( M^B(x) \) has no accepting path.
2. \( M^B(x) \) has at least one accepting path.

In the former case, \( L(M^B) \neq A \), since \( x \in A \), but \( M^B \) does not accept it. In the latter case, \( M^B \) on input \( x \) fails to be strong, but this is impossible because \( M \) reduces \( A \) strongly to \( B \) (by the assumption that \( A \leq^O B \) via \( M \)). Case 2 is analogous. If Case 3 happened, then \( M \) is not robustly overproductive, so \( A \leq^O B \) is certainly not implemented by \( M \).

So, in all three cases, \( M \) does not reduce \( A \) to \( B \) in the sense of \( \leq^O \). So, due to our stage construction, we have that \( A \nleq^T B \). This completes the proof of Part 1.

The proof of Part 2 of the theorem is very similar to that of Part 1, so we simply briefly sketch the differences. At the start of Stage \( i \), where one sets the \( n \) that will be used in Stage \( i \), we choose \( n \) so large that it not only satisfies the conditions used in Part 1, but also so large that the “Party Lemma” applies (see below). The three cases of Part 1 are then replaced by the following argument flow. If there is any length \( n \) string \( y \) for which either

1. \( M(B_{i-1} \cup \{0^n, y, 1\}^n)(x) \) does not both have at least one accepting path and no rejecting paths, or
2. \( M(B_{i-1} \cup \{0^n, y, 0\}^n)(x) \) does not both have at least one rejecting path and no accepting paths,

we are easily done, via adding the obvious string to taint the reduction. On the other hand, if for each length \( n \) string \( y \) we have that

1. \( M(B_{i-1} \cup \{0^n, y, 1\}^n)(x) \) has at least one accepting path and no rejecting paths, and
2. \( M(B_{i-1} \cup \{0^n, y, 0\}^n)(x) \) has at least one rejecting path and no accepting paths,

then (again, assuming \( n \) was chosen appropriately large) by the “Party Lemma” (page 104) there exist strings \( y \) and \( y' \) (possibly the same string) of length \( n \) such that \( M(B_{i-1} \cup \{0^n, y, 0\}^n \cup \{0^n, y', 1\}^n)(x) \) has both accepting and rejecting paths, and thus the machine considered at Stage \( i \) in fact is not robustly underproductive. (Of course, we do not in this case add both \( \{0^n, y, 0\} \) and \( \{0^n, y', 1\} \) to our oracle as this would taint the promise of Equation 4.3. Rather, the very fact that some oracle causes overproductivity is enough.) So, in this case, we add any single coding string of the form \( \{0^n, y'', b\} \), with \( b \in \{0, 1\} \) and \( |y''| = n \), to maintain Equation 4.3, and we move on to the next stage.

Next we consider the relationship between \( \leq^O \) and \( \leq^{RS} \). Let \( M \) be an NPTM. By interchanging the accept and the reject states of \( M \) we get a new NPTM machine \( N \) such that \( L_{rej}(M) = L(N) \). If \( M \) is robustly strong, we have \( L(M^A) = \overline{L(N^A)} \).
for every oracle $A$. The pair $(M, N)$ is what Hartmanis and Hemachandra call a robustly complementary pair of machines. For such a pair, the following is known.

**Theorem 4.3** \[9\] If $(M, N)$ is a robustly complementary pair of machines, then

$$(\forall A)[L(M^A) \in P^{\text{SAT} \oplus A}].$$

Gavaldà and Balcázar \[7\] noted that, in view of the preceding discussion, one gets as an immediate corollary the following.

**Corollary 4.4** \[7\] $$(\forall A, B)[A \leq_{RS} B \rightarrow A \in P^{\text{SAT} \oplus B}].$$

In fact, the proof of Theorem 4.3 still works if $M$ is an underproductive machine reducing $A$ to $B$. Thus, we have the following.

**Theorem 4.5** $$(\forall A, B)[A \leq_{U} B \rightarrow A \in P^{\text{SAT} \oplus B}].$$

Not only is the proof of Theorem 4.3 not valid for $\leq_{OT}$, but indeed the statement of Theorem 4.5 with $\leq_{U}$ replaced by $\leq_{OT}$ is outright false. This follows as a corollary to a proof of Crescenzi and Silvestri \[6, Theorem 3.1\] in which they give a very nice application of Sperner’s Lemma.

**Theorem 4.6** $$(\exists A, E)[A \leq_{OT} E \land A \not\in P^{\text{SAT} \oplus E}].$$

**Proof** The proof follows from a close inspection of the proof of \[4\], Theorem 3.1. Crescenzi and Silvestri prove the existence of a $\Sigma^*$-spanning pair $(N_0, N_1)$ of machines and the existence of an oracle $E$ with the property that $(\forall 0\text{-}1 \text{ valued function } f \in \text{FP}^{\text{SAT} \oplus E})(\exists x)[x \not\in L(N^E_{f(x)})].$ We need some preliminaries. The standard triangulation of size $n$ is the triangle, $\Delta$, in a Euclidean $x, y$-plane with the corners $A = (0, 0), B = (n - 1, 0)$ and $C = (0, n - 1)$ that is triangulated by the lines $x = i, (i = 0, \ldots, n - 2), y = i, (i = 0, \ldots, n - 2)$ and $x + y = i, (i = 1, \ldots, n - 1)$. The vertices of this triangulation are exactly the points $(i, j)$ with natural $i$ and $j$, and $i + j \leq n - 1$. A coloring of the standard triangulation is a mapping that associates with each vertex one of three given colors, 1, 2, and 3. Such a coloring is called $c$-admissible if the vertices on the border of $\Delta$ satisfy a certain condition pair, namely the following.

1. $A$, $B$, and $C$ are colored 1, 2, and 3 respectively.
2. If $P$ is a vertex on a side joining the corners colored $i$ and $j$, then $P$ must be colored by $i$ or $j$.

Sperner’s Lemma guarantees that each $c$-admissible coloring of a standard triangulation contains at least one three-colored triangle.

A standard triangulation of size $n$ has $\frac{n(n+1)}{2}$ vertices. So a coloring can be encoded by $n(n+1)$ bits, since a color can easily be encoded using two bits. If $l_n$ is the smallest
natural number \( l \) such that \( n(n + 1) \leq 2^l \), and if a certain ordering of the vertices of \( \Delta \) is fixed, for instance \((0, 0), (1, 0), \ldots, (n-1, 0), (0, 1), (1, 1), \ldots, (n-2, 1), \ldots, (0, n-1)\), then a coloring certainly can always be encoded by the first \( n(n + 1) \) bits of a word of length \( 2^l \), which means by a subset \( U \) of \( \{0, 1\}^{|x|} \). In order to recover the coloring from \( U \), assume \( \{0, 1\}^{|x|} \) to be ordered in the usual lexicographic way—\( 0^{|x|} < 0^{|x|-1}1 < \ldots < 1^{|x|} \)—and determine the values of the characteristic function of \( U \) for the first \( n(n + 1) \) words. So, the first two values determine the color of \((0, 0)\), the next two values determine the color of \((1, 0)\), and so on.

Let \( s_n = \max\{k \mid k(k + 1) \leq 2^n\} \). Now the nondeterministic oracle machines \( N_0 \) and \( N_1 \) are defined in such a way that for arbitrary oracle \( C \) and input \( x \):

1. \( N_0^C(x) \) has accepting paths if and only if \( C \cap \Sigma^{|x|} \) encodes a coloring of a standard triangulation of size \( s_{|x|} \) containing a 3-colored triangle.
2. \( N_1^C(x) \) has accepting paths if and only if \( C \cap \Sigma^{|x|} \) encodes a non-c-admissible coloring of a standard triangulation of size \( s_{|x|} \).

Clearly, both machines work in polynomial time. We define a new machine \( N \) that, on input \( x \), nondeterministically transfers the input to both \( N_0 \) and \( N_1 \) and

1. On a given path simulating a path of \( N_1 \), \( N \) ends in the state \texttt{acc} if \( N_1 \) accepts and otherwise ends in the no-comment state.
2. On a given path simulating a path of \( N_0 \), \( N \) ends in the state \texttt{rej} if \( N_0 \) accepts and otherwise ends in the no-comment state.

By Sperner’s Lemma, \((N_0, N_1)\) is a robustly \( \Sigma^* \)-spanning pair, and this means that \( N \) is robustly overproductive.

The oracle \( E \) is defined by diagonalization such that for the \( i \)th deterministic polynomial-time oracle machine \( T_i \) and a suitably chosen \( n_i \) it holds that:

1. \( T_i^{\text{SAT} \oplus E}(0^{n_i}) = 1 \) and \( E \cap \Sigma^{n_i} \) encodes a \( c \)-admissible coloring of a standard triangulation of size \( s_{n_i} \), or
2. \( T_i^{\text{SAT} \oplus E}(0^{n_i}) = 0 \) and \( E \cap \Sigma^{n_i} \) encodes a coloring of a standard triangulation of size \( s_{n_i} \) without three-colored triangles.

The crucial point is that \( E \) is in fact constructed in such a way that it does not encode colorings that both contain a three-colored triangle and are non-\( c \)-admissible. The fact that such an \( E \) can be constructed follows from [3, Remark 1]. This has the important consequence that \( N \) with oracle \( E \) is strong. Thus, defining \( A = L(N^E) \), we have \( A \leq_T \mathcal{O} \) \( E \) via \( N \), since \( N \) is robustly overproductive. Finally, we observe that \( A \not\in \text{PSAT} \oplus E \). Why? If not, then we have some \( T_i \) with \( A = L(T_i^{\text{SAT} \oplus E}) = L(N^E) \). However, from the definition of \( N \) and \( E \) (in particular in the stage where \( T_i \) was
considered) we conclude $0^m \in L(T_i^{\text{SAT}\oplus E}) \iff 0^m \not\in L(N^E)$. This contradiction shows $A \not\in P^{\text{SAT}\oplus E}$.

As mentioned earlier, Theorem 4.6 follows from the proof of [6, Theorem 3.1], but not from the theorem itself.

The preceding two theorems have the consequence of showing a deep asymmetry between $\leq^O_T$ and $\leq^U_T$. This asymmetry—that $\leq^O_T \neq \leq^P_T$, yet to prove the analog for $\leq^U_T$ would resolve the $P \neq \text{NP}$ question—contrasts with the seemingly symmetric definitions of these two notions. We now turn to some results that will lead to the proof of this asymmetry.

**Theorem 4.7** Overproductive and underproductive reductions differ in such a way that

$$\leq^O_T \not\subset \leq^U_T.$$

**Proof** By Theorem 4.6 we have sets $X$ and $Y$ such that

$$X \leq^O_T Y \land X \not\in P^{\text{SAT}\oplus Y},$$

and because of Theorem 4.3 for these sets the statement

$$X \leq^U_T Y \implies X \in P^{\text{SAT}\oplus Y}$$

is true. The conjunction of these two statements is equivalent to

$$X \leq^O_T Y \land X \not\leq^U_T Y \land X \not\in P^{\text{SAT}\oplus Y},$$

from which we conclude $\leq^O_T \not\subset \leq^U_T$.

An immediate consequence of Theorem 4.7 is the following.

**Theorem 4.8** $\leq^{\text{RS}}_T \neq \leq^O_T$.

We conjecture that Theorem 4.8 can be stated in the much stronger form of Theorem 4.4, where $\leq^{\text{SN}}_T$ is replaced with $\leq^O_T$. From Theorem 4.8 it follows that $\leq^O_T \neq \leq^P_T$. It is interesting to note that, although we know that $\leq^O_T$ and $\leq^P_T$ differ, it may be extremely hard to prove them to differ with a sparse set on the right-hand side. More precisely, we have the following.

**Theorem 4.9** $(\exists B \in \text{SPARSE})[R^O_T(B) \neq R^p_T(B)] \implies P \neq \text{NP}$.

**Proof** Assume $P = \text{NP}$ and $A \leq^O_T B$ with a sparse set $B$. This certainly implicitly gives a robustly $\Sigma^*$-spanning pair $(N_0, N_1)$ of machines and, say, $A = L(N_1^B)$. By [8, Theorem 2.7], there exists a 0-1 function $b \in \text{FP}^{\text{SAT}\oplus B}$ such that $(\forall x)[x \in L(N_b^B(x))]$. Since $N$ is strong for $B$, the sets $L(N_0^B)$ and $L(N_1^B)$ are disjoint. Hence $x \in L(N_0^B)$ if and only if $x \not\in L(N_1^B)$, from which we conclude

$$x \in A \iff b(x) = 1.$$
From the assumption $P = NP$, it follows that $b \in \text{FP}^B$ and thus $A \leq^p_T B$. 

The fact $\leq^O_T \neq \leq^p_T$, stated above, sharply contrasts with the following.

**Theorem 4.10** $\leq^U_T \neq \leq^p_T \rightarrow P \neq NP$.

**Proof** Assume $P = NP$ and $A \leq^U_T B$. By Theorem 4.13 we have $A \in P^{\text{SAT} \oplus B}$, and because of our $P = NP$ assumption, this means $A \leq^p_T B$. 

Theorem 4.10 strengthens in two ways the statement, noted by Gavaldà and Balcázar [7], that if $\leq^{RS}_T$ differs from $\leq^p_T$ anywhere on the recursive sets then $P \neq NP$. In particular, we have these two improvements of that statement of Gavaldà and Balcázar: (a) we improve from $\leq^{RS}_T$ to $\leq^U_T$, and (b) we remove the “on the recursive sets” scope restriction.

Below, we use $X \not\subset Y$ to denote that it is not the case that $X \subset Y$.

**Corollary 4.11**

1. $\leq^U_T \not\subset \leq^O_T \rightarrow P \neq NP$.

2. $\leq^U_T \neq \leq^{RS}_T \rightarrow P \neq NP$.

**Proof** By Theorem 4.10, $P = NP$ implies $\leq^U_T = \leq^p_T$ and thus $\leq^U_T \subset \leq^O_T$. In light of Theorem 4.17 we even have $\leq^U_T \not\subset \leq^O_T$. 

So proving $\leq^U_T \neq \leq^p_T$, $\leq^U_T \neq \leq^{RS}_T$, or $\leq^U_T \not\subset \leq^O_T$ amounts to proving $P \neq NP$. In particular, we cannot hope to strengthen Theorem 3.4 so that it is valid for $\leq^U_T$ rather than $\leq^{SN}_T$.

Although we know that $\leq^O_T \not\subset \leq^U_T$, it is also difficult to show that they differ with respect to a sparse set on the right hand side, because we have

$$(\exists B \in \text{SPARSE})[R^O_T(B) \not\subset R^U_T(B)] \rightarrow P \neq NP,$$

which is a consequence of Theorem 4.9.

**Theorem 4.12**

1. $\leq^{RS}_T \neq \leq^p_T \rightarrow P \neq NP$.

2. $\leq^{RS}_T = \leq^p_T \rightarrow P = NP \cap \text{coNP}$.

**Proof** Let $P = NP$. Then from $A \leq^{RS}_T B$ we have, by Corollary 4.4, $A \in P^{\text{SAT} \oplus B} = P^B$, i.e., $A \leq^p_T B$. If $\leq^{RS}_T \leq^p_T$, then their zero degrees coincide, so $P = NP \cap \text{coNP}$.
5 Overproductive Reductions and the Classic Hardness Theorems

The polynomial hierarchy is defined as follows: (a) \( \Sigma^p_0 = P \); (b) for each \( i \geq 0 \), \( \Sigma^p_{i+1} = \text{NP}^{\Sigma^p_i} \); (c) for each \( i \geq 0 \), \( \Pi^p_i = \{ L \mid \overline{L} \in \Sigma^p_i \} \); and (d) \( \text{PH} = \bigcup_{i \geq 0} \Sigma^p_i \) [21]. \( \Theta^p_2 = \{ L \mid L \leq^p_{tt} \text{SAT} \} \) (see [22]), where \( \leq^p_{tt} \) denotes polynomial-time truth-table reduction. \( \text{ZPP} \) denotes expected polynomial time [8]. It is well-known that \( \text{NP} \subseteq \Theta^p_2 \subseteq \text{P} \text{NP} \subseteq \text{ZPP} \text{NP} \subseteq \Sigma^p_2 \).

It is very natural to ask whether the existence of sparse hard or complete sets with respect to our new reductions would imply collapses of the polynomial hierarchy similar to those that are known to hold for \( \leq^p_T \). That is, are our reductions useful in extending the key standard results? To study this question, we must first briefly review what is known regarding the consequences of the existence of sparse NP-hard sets. The classic result in this direction was obtained by Karp and Lipton, and more recent research has yielded three increasingly strong extensions of their result.

Theorem 5.1

1. \( \text{NP} \subseteq \text{R}^{\text{SPARSE}}_T \rightarrow \text{PH} \subseteq \Sigma^p_3 \).
2. (implicit in [14], see [18] and the discussion in [10]; explicit in [1,13]) \( \text{NP} \subseteq \text{R}^{\text{RS}}_T \rightarrow \text{PH} \subseteq \Sigma^p_3 \).
3. \( \text{NP} \subseteq \text{R}^{\text{RS}}_T \rightarrow \text{PH} \subseteq \text{ZPP} \text{NP} \).
4. If \( A \) is self-reducible and \( A \in (\text{NP}^B \cap \text{coNP}^B) / \text{poly} \), then \( \text{ZPP}^{\text{NP}^A} \subseteq \text{ZPP}^{\text{NP}^B} \).
5. If \( A \) has self-computable witnesses and \( A \in (\text{NP}^B \cap \text{coNP}^B) / \text{poly} \), then \( \text{ZPP}^{\text{NP}^A} \subseteq \text{ZPP}^{\text{NP}^B} \).

We mention that Köbler and Watanabe [16] state part 3 in the form \( \text{NP} \subseteq (\text{NP} \cap \text{coNP}) / \text{poly} \rightarrow \text{PH} \subseteq \text{ZPP} \text{NP} \), which is equivalent to the statement of part 3 in light of Theorem 2.6. Both part 4 and part 5 extend part 3.

It remains open whether parts 3, 4, or 5 of Theorem 5.1 can be extended from robustly strong reductions to overproductive reductions. However, as Theorem 5.2 we extend part 2 of Theorem 5.1 to overproductive reductions. As a consequence, there is at the present time no single strongest theorem on this topic; Theorem 5.2 seems to be incomparable in strength relative to either of the final two parts of Theorem 5.1.

Theorem 5.2 \( \text{NP} \subseteq \text{R}^{\text{SPARSE}}_T \rightarrow \text{PH} \subseteq \Sigma^p_2 \).

Proof Our proof will in effect extend the approach of Hopcroft’s [11] proof of the Karp-Lipton Theorem (Theorem 5.1, part 1) in a way that allows the proof to work even when the machine involved is one implementing an overproductive reduction.
We will centrally use the fact that such machines are also underproductive for the specific set to which the reduction maps.

Assume \( \text{NP} \subseteq \text{R}_{1}^{\Pi_{2}^{p}}(\text{SPARSE}) \). Then there is a sparse set \( S \) such that \( \text{SAT} \leq_{p}^{F} S \), and let \( M \) be a machine certifying the reduction. That is, NPTM \( M \) is robustly overproductive, \( M^{S} \) is underproductive, and \( \text{SAT} = L(M^{S}) \). Let \( p_{S} \) bound the sparseness of \( S \), i.e., for each \( m \), \( ||S^{\leq m}|| \leq p_{S}(m) \).

Let \( L \) be an arbitrary \( \Pi_{2}^{p} \) set. So for some NPTM \( N \) we have \( \overline{T} = L(N^{\text{SAT}}) \). We will describe a \( \Sigma_{2}^{p} \) algorithm for \( L \). Say the runtime of \( N \) (respectively, \( M \)) is upper-bounded (without loss of generality, for all oracles) by \( p_{N} \) (respectively, \( p_{M} \)). Our \( \Sigma_{2}^{p} \) algorithm will be implemented by an NPTM \( \widetilde{N} \) with \( \text{SAT} \) as its oracle. Since \( \text{SAT} \) is NP-complete, we will act as if \( \widetilde{N} \) had two different NP sets (\( A \) and \( B \), defined below) as its oracle; implicitly, each when called is implemented via a reduction to \( \text{SAT} \).

We now describe \( \widetilde{N} \). For each \( y \) that is a boolean formula with at least one variable, let \( y_{T} \) denote \( y \) with its first variable set to true, and let \( y_{F} \) denote \( y \) with its first variable set to false. Let the function “coding” be such that given any finite set \( R \), \( \text{coding}(R) \) is a standard, easily decodable encoding of \( R \). For a computation path \( \rho \) (of some NPTM implementing an overproductive reduction), let \( \text{outcome}(\rho) \) denote the outcome of the path \( \rho \) (which will be one of \( \text{acc} \), \( \text{rej} \), or \( ? \)). On input \( x \), \( |x| = n \), \( \widetilde{N} \) nondeterministically guesses each subset, \( R \), of \( \Sigma_{2}^{p}(p_{M}(p_{N}(n))) \) containing at most \( p_{S}(p_{M}(p_{N}(n))) \) elements. \( \widetilde{N} \) then asks \( \langle x, \text{coding}(R) \rangle \) to the NP set \( A \) implicitly defined by the following. \( \langle x, H \rangle \in A \) if and only if there is an \( R \), with \( H = \text{coding}(R) \) such that, for each string \( y \) satisfying \( |y| \leq p_{N}(|x|) \), the following conditions hold:

1. if \( y \) is a legal formula with at least one variable then
   \[
   \begin{align*}
   (\forall \rho_{1} : \rho_{1} \text{ is a path of } M^{R}(y) \text{ and } \text{outcome}(\rho_{1}) \in \{\text{acc, rej}\}) \\
   (\forall \rho_{2} : \rho_{2} \text{ is a path of } M^{R}(y_{T}) \text{ and } \text{outcome}(\rho_{2}) \in \{\text{acc, rej}\}) \\
   (\forall \rho_{3} : \rho_{3} \text{ is a path of } M^{R}(y_{F}) \text{ and } \text{outcome}(\rho_{3}) \in \{\text{acc, rej}\}) \\
   [\text{outcome}(\rho_{1}) = \text{acc} \iff \text{outcome}(\rho_{2}) = \text{acc} \lor \text{outcome}(\rho_{3}) = \text{acc}],
   \end{align*}
   \]

2. if \( y \) is a legal formula with no variables then
   \[
   (\forall \rho : \rho \text{ is a path of } M^{R}(y)) \\
   [(y \equiv \text{true} \Rightarrow \text{outcome}(\rho) \neq \text{rej}) \land (y \equiv \text{false} \Rightarrow \text{outcome}(\rho) \neq \text{acc})].
   \]

Crucially, note that for each \( x \) it will hold that for at least one \( R \) the query \( \langle x, \text{coding}(R) \rangle \) that \( \widetilde{N} \) asks will be such that \( \langle x, \text{coding}(R) \rangle \notin A \). For each \( R \) satisfying \( \langle x, \text{coding}(R) \rangle \notin A \), note that by the definition of \( A \) we have that (a) \( M^{R}(y) \) is underproductive for all \( y \) satisfying \( |y| \leq p_{N}(|x|) \), and (b) \( \text{SAT} \leq_{p}^{N}(p_{N}(n)) = \left(L(M^{R})\right)^{\leq_{p}^{N}(n)} \). (Additionally, recall that \( M \) is robustly overproductive.) The key point here is that if \( M^{R} \) actually overproduces (has both accepting and rejecting paths) on some \( y \) with \( |y| \leq p_{N}(|x|) \), the test will be effected by this in such a way
that we will have \( \langle x, \text{coding}(R) \rangle \in A \). In particular, the set \( \overline{A} \) is essentially, regarding internal nodes of \( \text{SAT}'s \) disjunctive self-reducibility tree, testing that each triple of non-\( ? \) outputs, one each from a node and its children, is such that the three outputs are consistent with a correct self-reduction; if a machine overproduces anywhere (in the range considered) it will fail this test.

\( \overline{N} \), on each guessed path (that is, each guess of \( R \)) that gets the answer \( \langle x, \text{coding}(R) \rangle \in A \) simply rejects. \( \overline{N} \), on each guessed path that gets the answer \( \langle x, \text{coding}(R) \rangle \notin A \) asks the query \( \langle x, \text{coding}(R) \rangle \) to the set \( B \in \text{NP} \), and accepts if and only if the answer to this query is \( \langle x, \text{coding}(R) \rangle \notin B \). \( B \) is defined as follows. \( \langle x, H \rangle \in B \) if and only if there is an \( R \) with \( H = \text{coding}(R) \) such that nondeterministically simulating \( N^{L(M^R)}(x) \) yields at least one accepting path (of \( N \)), where by “simulating” we mean simulating \( N \) and, each time an oracle call to \( L(M^R) \) is made, nondeterministically guessing a path \( \rho' \) of \( M^R(w) \) and (a) continuing the simulation of \( N \) with the answer yes (respectively, no) if the outcome of \( \rho' \) is acc (respectively, rej), and (b) halting and rejecting (on the current path—recall that \( N \) is a standard Turing machine whose paths thus each either accept or reject, and by definition the machine accepts an input exactly if there is some accepting path on that input) if \( \rho' \) has \( ? \) as its outcome. The crucial point here is that, for those \( R \) on which it is actually called on actual runs of \( \overline{N} \), \( B \)'s use of \( R \) will correctly simulate \( \text{SAT} \).

We have shown that each \( \Pi_2^p \) set has a \( \Sigma_2^p \) algorithm, and thus have proved our theorem. \( \square \)

The above proof does not work for the case of underproductive reductions, and indeed it remains open whether Theorem 5.2 can in some way be extended to underproductive reductions. An analog for strong nondeterministic reductions is implicitly known, but has a far weaker conclusion.

**Theorem 5.3 (implicit in [16])** \( \text{NP} \subseteq \text{R}^{\text{SN} \text{(SPARSE)} \rightarrow \text{PH} \subseteq \text{ZPP}^{\Sigma_2^p} \).

**Proof** We start out from Yap’s theorem [23] in its strengthened form found by Köbler and Watanabe [16], namely, \( \text{coNP} \subseteq \text{NP}/\text{poly} \rightarrow \text{PH} \subseteq \text{ZPP}^{\Sigma_2^p} \). The following two statements show that the theorem to be proved is simply an equivalent reformulation of Yap’s theorem: \( \text{NP} \subseteq \text{R}^{\text{SN} \text{(SPARSE)}} \leftrightarrow \text{coNP} \subseteq \text{R}^{\Sigma_2^p} \text{(SPARSE)} \) and (recalling Theorem 2.4) \( \text{coNP} \subseteq \text{R}^{\text{SN} \text{(SPARSE)}} \rightarrow \text{coNP} \subseteq \text{NP}/\text{poly}. \) \( \square \)

In contrast with the above results regarding sparse hard sets for \( \text{NP} \), in the case of sparse complete sets for \( \text{NP} \) we have just as strong a collapse for \( \leq_{\text{SN}} \text{-reductions} \) as we have for \( \leq_{\text{TT}} \text{-reductions} \).

**Theorem 5.4 ([12])** \( \text{NP} \subseteq \text{R}^{\text{SN} \text{(SPARSE} \cap \text{NP})} \rightarrow \text{PH} = \Theta_2^p. \)

**Proof** Kadin [12] proved: If there exists a set \( S \subseteq \text{NP} \cap \text{SPARSE} \) such that \( \text{coNP} \subseteq \text{NP}^S \), then \( \text{PH} = \Theta_2^p \). This is equivalent to Theorem 5.4, because \( \text{coNP} \subseteq \text{NP}^S \).
NP^S is equivalent to NP ⊆ NP^S ∩ coNP^S, and this, in turn, is equivalent to NP ⊆ R^{SN}(S).

As mentioned earlier, we leave as an open problem whether one can establish the collapse PH ⊆ \Sigma_2^P (or, better still, PH ⊆ ZPP^{NP}) under the assumption NP ⊆ R^{SN}(SPARSE), or even under the stronger assumption that NP ⊆ R_U^{SPARSE}. We conjecture that no such extension is possible.

6 Conclusions and Open Problems

We mention as open problems the issues of finding equivalent conditions for \leq_U^T \subseteq \leq_O^T, \leq_{RS}^T = \leq_{P}^T, and \leq_U^T \subseteq \leq_{RS}^T.

Define the runtime of a nondeterministic machine on a given input to be the length of its longest computation path. (Though in most settings this is just one of a few equivalent definitions, we state it explicitly here as for the about-to-be-defined notion of local-polynomial machines, it is not at all clear that this equivalence remains valid.) Recall that we required that NPTMs be such that for each NPTM, N, it holds that there exists a polynomial p such that, for each oracle D, the runtime of N^D is bounded by p. Call such a machine “global-polynomial” as there is a polynomial that globally bounds its runtime. Does this differ from a requirement that for a machine N it holds that, for each oracle D, there is a polynomial p (which may depend on D) such that the runtime of N^D is bounded by p? Call such a machine “local-polynomial” as, though for every oracle it runs in polynomial time, the polynomial may depend on the oracle.

In general, these notions do differ, notwithstanding the common wisdom in complexity theory that one may “without loss of generality” assume enumerations of machines come with attached clocks independent of the oracle. (The subtle issue here is that the notions in fact usually do not differ on enumerations of machines that will be used with only one oracle.) The fact that they in general differ is made clear by the following theorems. These theorems show that there is a language transformation that can be computed by a local-polynomial machine, yet each global-polynomial machine will, for some target set, fail almost everywhere to compute the set’s image under the language transformation. We write A =* B if A and B are equal almost everywhere, i.e., if (A − B) ∪ (B − A) is a finite set.

Theorem 6.1 There is a function \( f_N : \Sigma^* \to \Sigma^* \) (respectively, \( f_D : \Sigma^* \to \Sigma^* \)) such that

1. there is a nondeterministic (respectively, deterministic) local-polynomial Turing machine \( \bar{M} \) such that for each oracle A it holds that \( L(\bar{M}^A) = f_N(A) \) (respectively, \( L(\bar{M}^A) = f_D(A) \)), and
2. for each NPTM, i.e., each nondeterministic global-polynomial Turing machine $M$ (respectively, DPTM, i.e., each deterministic global-polynomial Turing machine $M$) it holds that there is a set $A \subseteq \Sigma^*$ such that $L(M^A) = \ast f_N(A)$ (respectively, $L(M^A) = \ast f_D(A)$).

Though this claim may at first seem counterintuitive, its proof is almost immediate if one is given $f_N$ and $f_D$, and so we simply give functions $f_N$ and $f_D$ satisfying the theorem. In particular, we can use $f_N(A) = \{ x \mid (\exists y)[(|y| \leq \log |x|) \land (y \text{ is the lexicographically first string in } A) \land (\exists z)[|z| = |x| \land xz \in A]) \}$ and $f_D(A) = \{ x \mid (\exists y)[(|y| \leq \log |x|) \land (y \text{ is the lexicographically first string in } A) \land (\exists z)[|z| = \text{ one of the } |x|\text{-length strings in } \Sigma^* \land xz \in A]) \}$.

The difference between global-polynomial machines and local-polynomial machines in general mappings, as just proven, may make one wonder whether the fact that robust strong reduction is defined in terms of global-polynomial (as opposed to local-polynomial) machines makes a difference and, if so, which definition is more natural. Regarding the former issue, we leave it as an open question. (The above theorems do not resolve this issue, as they deal with language-to-language transformations defined specifically over all of $2^{\Sigma^*}$, but in contrast a robustly strong reduction must accept a specific language only for one oracle, and for all others merely has to be underproductive and overproductive, plus it must have the global-polynomial property.) That is, the open question is: Does there exist a pair of sets $A$ and $B$ such that $A \not\leq_{RS}^T B$ (which by definition involves a global-polynomial machine) and yet there exists a nondeterministic local-polynomial Turing machine $N$ such that $L(N^B) = A$ and $(\forall D \subseteq \Sigma^*)[N^D \text{ is both underproductive and overproductive}]$? Regarding the question of naturalness, this is a matter of taste. However, we point out that the global-polynomial definition is exactly that of Gavaldà and Balcázar [7], and that part 2 of Theorem 2.6, Gavaldà and Balcázar’s [8] natural characterization of robustly strong reductions to sparse sets in terms of the complexity class $(\text{NP} \cap \text{coNP})/\text{poly}$, seems to depend crucially on the fact that one’s machines are global-polynomial.

On the other hand Theorem 5.2, though its proof seems on its surface to be dependent on the fact that $\leq_{O^T}$ is defined via global-polynomial machines, in fact remains true even if $\leq_{O^T}$ is redefined via local-polynomial machines. The trick here is that we modify the proof to clock the key local-polynomial machine (implementing the overproductive reduction) with the clock that applies for the sparse oracle to which the reduction actually reduces it, and then in the simulations of the proof if we detect that a path is about to exceed that clock, we know that we are dealing with a bad oracle $R$, and so in our simulation of that too-long path we truncate the path and “cap” it with two leaves, one an acc leaf and one a rej leaf. This, in effect, rules
out that potential oracle, as it will seem to be overproductive.

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