Finite-time dissipative control for time-delay Markov jump systems with conic-type non-linearities under guaranteed cost controller and quantiser

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Abstract
For a class of conic-type non-linear time-delay Markov jump systems, the asynchronous dissipative output feedback controller based on the guaranteed cost control and quantiser is designed in this study. In real applications, the system and the controller modes are always non-synchronous, so we introduce the hidden Markov model to solve this problem. Furthermore, we define three novel auxiliary variables and use quantisers to accomplish the output feedback controller design. Then, the finite-time boundedness and strict dissipativity of the closed-loop systems are guaranteed by sufficient conditions, and the controller also meets the guaranteed cost-control performance. By solving a set of linear matrix inequalities, we get the controller gains, the guaranteed cost control performance index $J^*$, and the dissipative performance index $\alpha$. Finally, the correctness and feasibility of this designed approach are demonstrated by a given example.

1 | INTRODUCTION

In the 1960s, the conception of Markov jump systems (MJSs) [1] has been proposed. With the random jumping structure, MJSs can be seen as a class of hybrid systems and have always received considerable attention. Due to the randomness in data and structure, MJSs have been widely applied, such as multi-agent systems [2], electrohydraulic servo systems [3], medical prognosis [4] and networked systems [5]. But in engineering applications, it is difficult to synchronise the system and the controller modes due to some unavoidable errors and delays. For this asynchronous phenomenon different from synchronisation [6], the hidden Markov model (HMM) might be helpful which introduces a random process to estimate the Markov process [7]. In [8], the finite-time boundedness (FTB) and $H_\infty$ performance of time-varying MJSs were investigated by designing an HMM-based controller. For time-delay MJSs (TDMJSs), the asynchronous controller was designed to guarantee the stochastic stability and $H_\infty$ performance [9]. For fuzzy MJSs, the asynchronous filtering was designed to guarantee the FTB and $H_\infty$ performance [10]. In [11], the authors designed an HMM-based sliding mode controller to ensure the stochastic stability and dissipative performance for TDMJSs.

On the other hand, many achievements have been made in the design of robust stabilising controllers [12]. Although it can make the system stable, the upper bound of the controller performance cannot be guaranteed. For this problem, the guaranteed cost control (GCC) strategy has been put forward [13]. For discrete-time switched singular systems, the robust GCC problem was investigated [14]. For fuzzy MJSs, the GCC performance was guaranteed by designing a quantised asynchronous controller [15]. The event-triggered GCC strategy was investigated in [16, 17].

In practical applications, because of disturbances and errors, the non-linear characteristics are indispensable. As a special kind of non-linear dynamics, conic-type non-linearities have great representativeness and they are widely used in engineering, such as locally sinusoidal non-linearities, dead-zone non-linearities and so forth. In fact, we can consider Lipschitz non-linearity as a special conic-type non-linearity. For discrete-time conic non-linear MJSs [18], the authors investigated the FTB and $H_\infty$ performance by designing an asynchronous controller with a
DC-motor model to show the feasibility. The stability analysis of the conic non-linear systems was studied [19, 20]. In [21], the fault detection of conic non-linear systems was investigated. 

The dissipative theory [22], since its proposal by Willems, has received much attention and has been widely used, for example, for discrete-time TDMJSs was studied. In [38], the HMM-based dissipative control scheme was designed for asynchronous time-delay controller to investigate the dissipativity of the closed-loop systems, and the upper bound of the controller performance is also guaranteed.

In practice, the packet loss and time delay will reduce the stability and performance of the system. Due to the limited transmission rate of network, and the ability to map continuous signals to discrete sets, quantiser is needed and it is cheaper, more reliable and convenient. For the results of quantiser, the readers can refer [9, 10, 37].

To our best knowledge, the asynchronous dissipative control problem for TDMJSs with conic-type non-linearities under guaranteed cost controller and quantiser has not been fully studied. In this study, we introduced the HMM and quantiser to design the output feedback controller for a class of TDMJSs with conic-type non-linearities. The following points reflect the main contributions of this study:

1. In order to make the matrix inequalities solvable and to reduce the computational complexity, we defined the three novel auxiliary variables to accomplish the controller design.
2. By solving a set of linear matrix inequalities (LMIs), sufficient conditions are given to guarantee the FTB and strict dissipativity of the closed-loop systems, and the upper bound of the controller performance is also guaranteed.
3. By the given liquid monopropellant rocket motor model with a pressure feeding system, the correctness and feasibility of the designed strategy are guaranteed.

In the following, Table 1 introduces the presented notations in this study.

### TABLE 1 The notations

| Notation   | Denotes                                                                 |
|------------|-------------------------------------------------------------------------|
| $E\{\cdot\}$ | The mathematical expectation operator                                   |
| $\max_{\{\cdot\}}$ | The maximum eigenvalue of $A$                                           |
| $\min_{\{\cdot\}}$ | The minimum eigenvalue of $A$                                           |
| $\mathbb{R}^n$ | $n$-dimensional Euclidean space                                           |
| $\mathbb{R}^{m\times n}$ | $m\times n$ real matrix                                                  |
| $\text{diag}\{A,B\}$ | Block-diagonal matrix of $A$ and $B$                                      |
| $I$         | Unit matrix                                                             |
| $A^{-1}$    | Matrix inverse                                                          |
| $A^T$       | Matrix transpose                                                        |
| $*$         | Symmetric matrix                                                        |
| $\text{Her}(\cdot)$ | The sum of $A$ and transposition of $A$                                  |

2 | SYSTEM DESCRIPTION AND PRELIMINARIES

Consider the following TDMJSs with conic-type non-linearities:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), x(t-\tau), \omega(t)) + D_r(t)u(t), \\
\zeta(t) &= E_r(t)x(t) + F_r(t)x(t-\tau) + G_r(t)\omega(t),
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^m$ is the controlled input, $\omega(t) \in \mathbb{R}^q$ is the external disturbance with $\omega(t) \leq \varpi(t)$, $\zeta(t) \in \mathbb{R}^p$ is the controlled output. $\dot{x}(x(t), x(t-\tau), \omega(t))$ is an unknown non-linear function by the following dynamics conic sector:

$$
\begin{aligned}
\|f(x(t), x(t-\tau), \omega(t)) - [A_r(t)x(t) + B_r(t)x(t-\tau) + C_r(t)\omega(t)]\| &
\leq \|A_{ar}(t)x(t) + A_{br}(t)x(t-\tau) + A_{cr}(t)\omega(t)\|. 
\end{aligned}
$$

The values of the Markov stochastic process $\{r(t), t \geq 0\}$ are in a finite set $\mathcal{L} = \{1, 2, ..., L\}$ with the transition rate matrix $\Pi = [\lambda_{ij}]$ given by

$$
\Pi_{r(t) + \Delta t} = P[r(t) = i \mid r(t) = j] = \begin{cases}
\hat{\lambda}_i \Delta t + o(\Delta t), & s \neq l \\
1 + \hat{\lambda}_i \Delta t + o(\Delta t), & s = l
\end{cases}
$$

where $\Delta t$ satisfies $\lim_{\Delta t \to 0} o(\Delta t) = 0$, $\hat{\lambda}_i \geq 0$ represents the jump rate from mode $i$ at time $t$ to mode $l$ at time $t + \Delta t$ and $\lambda_{i} = -\sum_{j \neq i} \lambda_{ij}$.

Letting $r(t) = s$ and combining inequality in Equation (2), we get the following TDMJSs:

$$
\begin{aligned}
\dot{x}(t) &= A_{s}x(t) + B_{s}x(t-\tau) + C_{s}\omega(t) + D_{s}u(t) \\
+ g_{s}(x(t), x(t-\tau), \omega(t)), \\
\zeta(t) &= E_{s}x(t) + F_{s}x(t-\tau) + G_{s}\omega(t),
\end{aligned}
$$

where

$$
g_{s}(x(t), x(t-\tau), x(t-\tau), \omega(t)) = f(x(t), x(t-\tau), \omega(t)) = [A_{s}x(t) + B_{s}x(t-\tau) + C_{s}\omega(t)].
$$

Then, the following inequality holds:

$$
\|g_{s}(x(t), x(t-\tau), \omega(t))\|^2 \leq \|A_{as}x(t) + A_{bs}x(t-\tau) + A_{cs}\omega(t)\|^2
$$

In this study, the HMM-based controller is designed by

$$
g(t) = K_{s}(t)\zeta(t),
$$
where $K_\delta(t) \in \mathbb{R}^{n \times n}$ is the controller gain to be designed and the stochastic jump process $\delta(t)$ is under the range of $O = \{1, 2, \ldots, O\}$. The conditional probability matrix $\Phi = [\Phi_m]$ is shown as

$$\mathcal{N}(\omega) = \{N_1(\omega_1) \ldots N_i(\omega_i)\}$$

where $i$ means the all quantisers, $q_i$ means the $i$th component, and $b_{ij} = l_i/l_j$ with $0 < l_i < 1$. $b_{ij} > 0$ are the input and output of quantisers, respectively. The relationships between $t_i$ and $\chi_i$ are given by $\chi_i = \frac{1 - l_i}{1 + l_i}$. In addition, $-\chi_i e \leq N_i(e) - e \leq \chi_i e$ shows the boundedness of the quantisation error, which can be expressed as

$$N_i(e) - e = \Omega_i e \quad \Omega_i e \in [-\chi_i \chi_i]$$

It should be noted that Equation (10) always holds for $t$. Combining Equations (8) and (10), we get

$$N_i(q_i(e)) = (I + \Delta_i(t))q_i(e),$$

where $\Delta_i(t) = \text{diag}(\Omega_1(t), \ldots, \Omega_m(t))$ with $\Omega_m(t) \in [-\chi_m, \chi_m]$, $m = \{1, 2, \ldots, i\}$. By controller in Equation (6) and quantiser in Equation (11), we can get the following controller:

$$u(t) = (I + \Delta_i(t))q_i(e).$$

Substituting controller in Equation (12) into TDM[J]s in Equation (4), we obtain the following closed-loop TDM[J]s:

$$\begin{align*}
\dot{x}(t) &= (A_i + D_i(I + \Delta_i(t))K_i E_i)x(t) + (B_i + D_i(I + \Delta_i(t))K_i G_i)\omega(t) + G_i \omega(t), \\
\dot{\chi}(t) &= E_i x(t) + F_i x(t - \tau) + G_i \omega(t).
\end{align*}$$

Remark 1. The mode manipulation of Markov chain is significant in engineering application of M[J]s, but the controller cannot acquire the mode $r(t)$ and some inaccuracy will be caused.

In this study, we bring $\delta(t)$ as the controller mode to solve the non-synchronous phenomenon, and Equation (7) indicates the relationships between $\delta(t)$ and $r(t)$. In controller design, $K_\delta(t)$ and $\delta(t)$ are only related to $\Delta_i$, which can reflect the hidden information. Then, the closed-loop TDM[J]s in Equation (13) can be regarded as a double random process.

The following GCC performance index is introduced to design the controller in Equation (12):

$$J = \int_0^T x^T(t)R_1 x(t) + u^T(t)R_2 u(t)dt$$

where $R_1$ and $R_2$ are the given positive-definite matrices and $J < J^*$ holds. $J^*$ represents the minimal upper bound of the GCC performance index.

The energy supply function of the closed-loop TDM[J]s (13) is described by

$$f(z(t), \omega(t), T) = \int_0^T E[x^T(t), \omega(t)]dt.$$
3 | MAIN RESULTS

Here, we set the controller gain as \([8, 40]\)

\[ K_v = W_v H_v^{-1}, \tag{19} \]

where \(W_v\) and \(H_v\) are unknown matrices to be designed.

Then, we define three novel auxiliary matrices as

\[ \theta(t) = Y_i^{-1} x(t), \tag{20} \]
\[ \theta(t - \tau) = Y_i^{-1} x(t - \tau), \tag{21} \]
\[ \xi(t) = E_i \theta(t) + F_i \theta(t - \tau) - H_i^{-1} z(t), \tag{22} \]

where \(Y_i \in \mathbb{R}^{p \times n}\) are a set of positive-definite symmetric matrices.

Considering Equations (19) to (22), we can get

\[ \dot{x}(t) = (A_i Y_i + D_i (I + \Delta_i(t))) W_i E_i \theta(t) \]
\[ + (B_i Y_i + D_i (I + \Delta_i(t))) W_i F_i \theta(t - \tau) + C_i \omega(t) \]
\[ - D_i (I + \Delta_i(t)) W_i \xi(t) + g_s(x(t), x(t - \tau), \omega(t)). \tag{23} \]

Then, we will propose sufficient conditions to ensure the FTB of the closed-loop TDMJSs in Equation (13) and investigate the GCC performance.

**Theorem 1.** Under the given scalars \(\gamma_i > 0\), the FTB of the closed-loop TDMJSs in Equation (13) with \((a_1, a_2, \mathcal{T}, S, d)\) is guaranteed, and the GCC performance index holds \(J = x^T(0) Y_i^{-1} x(0) + d\), if for any \(s \in \mathcal{L}\) and \(r \in \mathcal{O}\), there exists a set of mode-dependent scalars \(\rho_{\eta}, \sigma_i > 0\) and \(Y_i > 0\) satisfying the following LMIs:

\[ \Psi < 0, \tag{24} \]
\[ \Sigma < 0, \tag{25} \]
\[ S < Y_i^{-1} < \sigma_i S, \tag{26} \]
\[ \phi_i^T \sigma, a_1 + \frac{a_2}{\gamma_i} (1 - \phi_i^T) < a_2, \tag{27} \]

where

\[ \Psi = \begin{bmatrix} N_1 & N_2 \\ * & N_3 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix}, \]
\[ N_1 = \begin{bmatrix} Z_1 + Z_2 - Y_i \gamma_i & Z_3 \\ * & -Y_i \gamma_i \end{bmatrix}, \]
\[ N_2 = \begin{bmatrix} C_i & Y_i A_i^T \gamma_i & Q_1 \\ 0 & Y_i A_i^T 0 & 0 \\ -\rho_{\eta} G_i & 0 & 0 \end{bmatrix}, \]
\[ N_3 = \begin{bmatrix} -I & A_i^T 0 & 0 \\ * & -\varepsilon^{-1} I & 0 \\ * & * & Q_2 \end{bmatrix}, \]
\[ N_4 = \begin{bmatrix} Z_1 + Z_2 & Z_3 & Z_4 & C_i \\ * & -Y_i & Z_5 & 0 \\ * & * & Z_6 & -\rho_{\eta} G_i \\ * & * & * & 0 \end{bmatrix}, \]
\[ Z_1 = \begin{bmatrix} Z_1 + Z_2 & Z_3 & Z_4 & C_i \\ * & -Y_i & Z_5 & 0 \\ * & * & Z_6 & -\rho_{\eta} G_i \\ * & * & * & 0 \end{bmatrix}, \]
\[ Z_2 = \begin{bmatrix} Q_3 & Y_i A_i^T & Q_1 \\ Q_4 & 0 & Y_i A_i^T & 0 \\ Q_5 & 0 & 0 & 0 \\ 0 & 0 & A_i^T & 0 \end{bmatrix}, \]
\[ Z_3 = \begin{bmatrix} -\varepsilon^{-1} & 0 & 0 & 0 \\ 0 & -\varepsilon^{-1} & 0 & 0 \\ 0 & 0 & -\varepsilon^{-1} & 0 \\ 0 & 0 & 0 & \end{bmatrix}, \]
\[ Z_1 = (\lambda_{\eta} + 1) Y_i + \varepsilon^{-1}, \]
\[ Z_2 = H_\mathcal{O} (\sum_{i=1}^{\mathcal{O}} \phi_i (A_i Y_i + D_i (I + \Delta_i(t))) W_i E_i)), \]
\[ Z_3 = B_i Y_i + \sum_{i=1}^{\mathcal{O}} \phi_i (A_i Y_i + D_i (I + \Delta_i(t))) W_i F_i), \]
\[ Z_4 = \rho_{\eta} (E_i^T H_i^T - Y_i E_i^T) - \sum_{i=1}^{\mathcal{O}} \phi_i (D_i (I + \Delta_i(t))) W_i), \]
\[ Z_5 = \rho_{\eta} (E_i^T H_i^T - Y_i E_i^T), Z_6 = -\rho_{\eta} (H_i)), \]
\[ Q_1 = Y_i \sqrt{\lambda_{\eta,1}, \ldots, \sqrt{\lambda_{\eta,n+1}, \ldots, \sqrt{\lambda_{\eta,n}}},} \]
\[ Q_2 = -\text{diag}(Y_i \cdots Y_{i-1}, Y_{i+1} \cdots Y_{\mathcal{O}}), \]
\[ Q_3 = E_i^T W_i (I + \Delta_i(t))^T, Q_4 = F_i^T W_i (I + \Delta_i(t))^T, \]
\[ Q_5 = -W_i (I + \Delta_i(t))^T. \]

**Proof.** We construct the stochastic Lyapunov functional candidate as

\[ V(x(t)) = x^T(t) Y_i^{-1} x(t) + \int_{-\tau}^{0} x^T(t + T) Y_i^{-1} x(t + T) dT. \tag{28} \]
We define $\Lambda V^\prime$ as the weak infinitesimal generator, and get

$$
\Lambda V^\prime(x(t)) = x^T(t)\Sigma_{j=1}^m A_j Y_j^{-1} x(t) + 2\Sigma_{j=1}^m \phi_j^T(t) Y_j^{-1} x(t) + x^T(t) Y_j^{-1} x(t) - x^T(t - \tau) Y_j^{-1} x(t - \tau).
$$

(29)

Considering Equations (19) to (22), Equation (29) can be written as

$$
\Lambda V^\prime(x(t)) = \theta^T(t) Y_{\theta}(\Sigma_{j=1}^m A_j Y_j^{-1}) \theta(t) + 2\Sigma_{j=1}^m \phi_j^T(t) W_j^T(t) \theta(t) + (B_j Y_j + D_j(I + \Delta_j(t))) W_j^T(t) \theta(t - \tau) + C_j \omega(t)
$$

$$
- D_j(I + \Delta_j(t)) W_j^T(t) \eta(t)
$$

$$
+ g_r(x(t), x(t - \tau), \omega(t))^T \theta(t)
$$

$$
+ x(t)^T Y_j^{-1} x(t) - x(t - \tau)^T Y_j^{-1} x(t - \tau)
$$

and the following relationship holds:

$$
0 = H_j \theta(t) + H_j F_j \theta(t - \tau) - H_j \xi(t) - \zeta(t)
$$

$$
= (H_j E_j - E_j Y_j) \theta(t) + (H_j F_j - F_j Y_j) \theta(t - \tau) - H_j \xi(t)
$$

$$
- G_j \omega(t).
$$

(31)

By Lemma 1 and inequality in Equation (5), we get

$$
2\xi^T(t) \omega(t) \eta(t) \leq \xi(t)^T \theta(t)
$$

$$
+ \xi(t) \omega(t) \eta(t) + A_{yi} Y_i \theta(t) + A_{yi} Y_i \theta(t - \tau)
$$

$$
+ A_{yi} \omega(t)^T A_{yi} Y_i \theta(t) + A_{yi} Y_i \theta(t - \tau) + A_{yi} \omega(t).
$$

(32)

For the given scalars $\gamma_i > 0$, we define

$$
J_1(t) = \mathbb{E}[\Lambda V^\prime(x(t)) - \gamma_i V(x(t)) - \omega^T(t) \omega(t)].
$$

(33)

By Equations (30) to (33), the following relationship holds:

$$
J_1(t) = \mathbb{E}[\Lambda V^\prime(x(t)) - \gamma_i V(x(t)) - \omega^T(t) \omega(t)]
$$

$$
+ 2\rho \xi^T(t) [(H_j E_j - E_j Y_j) \theta(t) + (H_j F_j - F_j Y_j) \theta(t - \tau) - H_j \xi(t) - G_j \omega(t)]
$$

$$
\leq \eta^T(t) \Psi_i \eta(t) + \xi |A_{yi} Y_i \theta(t) + A_{yi} Y_i \theta(t - \tau)
$$

$$
+ A_{yi} \omega(t)^T A_{yi} Y_i \theta(t) + A_{yi} Y_i \theta(t - \tau) + A_{yi} \omega(t)].
$$

(34)

where $\eta^T(t) = [\theta^T(t) \theta^T(t - \tau) \xi^T(t) \omega^T(t)]$,

$$
\Psi_i =
\begin{bmatrix}
Z_2 - Z_2 - \gamma_i Y_i & Z_3 & Z_4 & C_i \\
* & -Y_i & Z_5 & 0 \\
* & * & Z_6 & -\rho \omega \omega_C \\
* & * & * & -I
\end{bmatrix},
$$

$$
Z_7 = Y_j (\Sigma_{j=1}^m A_j Y_j^{-1}) Y_j + Y_j + \epsilon^{-1}.
$$

By using Schur complement for inequality in Equation (34) and combining inequality in Equation (24), we can ensure $J_1(t) < 0$. Then, we have

$$
\mathbb{E}[\Lambda V^\prime(x(t))] < \gamma_i V(x(t)) + \omega^T(t) \omega(t).
$$

(35)

Multiplying inequality in Equation (35) by $e^{-\gamma_i t}$ and taking integration from 0 to $t$, we obtain

$$
e^{-\gamma_i t} \mathbb{E}[V(x(t))] - \mathbb{E}[V(0)] < \int_0^t e^{-\gamma_i \tau} \omega^T(t) \omega(t) \, d\tau.
$$

(36)

By $\gamma_i > 0$ and $t \in [0, T]$, we get

$$
\mathbb{E}[V(x(t))] < e^{\gamma_i t} \mathbb{E}[V(0)] + \gamma_i \int_0^t e^{-\gamma_i \tau} \omega^T(t) \omega(t) \, d\tau.
$$

(37)

Then, the following relationship holds:

$$
\mathbb{E}[\xi^T(t) \mathcal{S}(x(t))]
$$

$$
\leq \gamma_i T \xi_{\max}(S_{\ell}^{-\frac{1}{2}} Y_{\ell}^{-1} S_{\ell}^{-\frac{1}{2}} Y_{\ell}^{-1} \mathcal{S}(0) \mathcal{S}(x(0))) + \frac{\gamma_i}{\xi_{\min}(S_{\ell}^{-\frac{1}{2}} Y_{\ell}^{-1} S_{\ell}^{-\frac{1}{2}})} (1 - e^{-\gamma_i T}).
$$

(38)

From inequalities in Equations (26) and (27), we can ensure $\xi_{\max}(S_{\ell}^{-\frac{1}{2}} Y_{\ell}^{-1} S_{\ell}^{-\frac{1}{2}}) < \sigma_i$ and $\xi_{\min}(S_{\ell}^{-\frac{1}{2}} Y_{\ell}^{-1} S_{\ell}^{-\frac{1}{2}}) > 1$, which gives $\mathbb{E}[\xi^T(t) \mathcal{S}(x(t)) < \sigma_i]$. From Definition 1, the FTB of the closed-loop TDMJSs in Equation (13) is guaranteed.

Then, we will investigate the GCC performance and define

$$
J_2(t) = \Lambda V^\prime(x(t)) + x^T(t) R_1 x(t) + u^T(t) R_2 u(t) - \omega^T(t) \omega(t).
$$

(39)

Substituting Equation (12) into Equation (39) and combining Equations (30) to (52), we obtain $J_2(t) < 0$ by inequality in Equation (25). Then, taking integration it from 0 to $T$ and
recalling to Definition 1, we get
\[ J < J^* = x^T(0)Y_x^{-1}x(0) + d. \]  
(40)

The proof is completed.

In the next theorem, we will ensure the strict dissipativity of the closed-loop TDMJSs in Equation (13).

**Theorem 2.** The FTB and strict dissipativity of the closed-loop TDMJSs in Equation (13) is guaranteed, and the controller meets the GCC performance, if for any \( s \in \mathcal{L} \) and \( r \in \mathcal{O} \), there exists a set of mode-dependent scalars \( P_{rr}, Y_t > 0 \) satisfying Equations (24) to (27) and the following matrix inequality:
\[ \Xi < 0, \]  
(41)

where
\[
\Xi = \begin{bmatrix}
\Gamma_1 & \Gamma_2 \\
\Gamma_3 & \Xi_1
\end{bmatrix},
\Gamma_1 = \begin{bmatrix}
Z_1 + Z_2 & \text{Z}_3 & \text{Z}_4 \\
* & \text{Z}_5 & \text{Z}_6
\end{bmatrix},
\Xi_1 = \begin{bmatrix}
Z_8 & \text{Y}_sA_{ts}^T & \text{Q}_6 & \text{Q}_1 \\
\text{Z}_9 & \text{Y}_sA_{ts}^T & \text{Q}_7 & 0 \\
-\rho_{g}s & 0 & 0 & 0 \\
\end{bmatrix},
\text{Z}_8 = C_t - Y_sE_{ts}^T G_s, \text{Z}_9 = -Y_sE_{ts}^T G_s, \text{Z}_{10} = C_t - Y_sE_{ts}^T G_s,
\text{Q}_6 = \{\sqrt{\phi_1}Y_sE_{ts}^T \text{U}_t, \sqrt{\phi_2}Y_sE_{ts}^T \text{U}_t, \ldots, \sqrt{\phi_s}Y_sE_{ts}^T \text{U}_t\},
\text{Q}_7 = \{\sqrt{\phi_1}Y_sE_{ts}^T \text{U}_t, \sqrt{\phi_2}Y_sE_{ts}^T \text{U}_t, \ldots, \sqrt{\phi_s}Y_sE_{ts}^T \text{U}_t\},
\text{Q}_8 = \{\sqrt{\phi_1}G_{ts}^T \text{U}_t, \sqrt{\phi_2}G_{ts}^T \text{U}_t, \ldots, \sqrt{\phi_s}G_{ts}^T \text{U}_t\}.
\]

**Proof.** We define
\[
J_3(t) = \mathbb{E}\{\text{A}V(\varsigma(t))\} - S(\varsigma(t), \omega(t)) + \alpha \omega^T(t) \omega(t).
\]  
(42)

Combining Equations (20) to (22), the supply rate can be written as
\[
S(\varsigma(t), \omega(t)) = \left[F_sE_{ts} \theta(t) + F_tY_t \theta(t - \tau) + G_r \omega(t)\right]^T \times \text{U}_t \left[F_sE_{ts} \theta(t) + F_tY_t \theta(t - \tau) + G_r \omega(t)\right] + 2[F_sE_{ts} \theta(t) + F_tY_t \theta(t - \tau) + G_r \omega(t)^T \omega(t)] + 2[F_sE_{ts} \theta(t) + F_tY_t \theta(t - \tau) + G_r \omega(t)]^T \text{G}_s \omega(t) + \alpha \omega^T(t) \text{U}_t \omega(t). \]  
(43)

From inequalities in Equations (30) to (32) and Equations (42) and (43), we have
\[
J_3(t) = \mathbb{E}\{\text{A}V(\varsigma(t))\} - S(\varsigma(t), \omega(t)) + \alpha \omega^T(t) \omega(t) + 2 \rho_g \xi^T(t) (H_tE_t - E_tY_t) \theta(t) + (H_tF_t - F_tY_t) \theta(t - \tau) - H_t \xi(t) - G_r \omega(t)] \leq \eta^T(t) \Xi \eta(t) + \varepsilon [A_{aw}Y_t \theta(t) + A_{aw}Y_t \theta(t - \tau) + A_{aw} \omega(t)],
\]  
(44)

where
\[
\Xi = \begin{bmatrix}
Z_7 + Z_2 - Z_{11} & Z_3 - Z_{12} & Z_4 & Z_8 - Z_{13} \\
* & -Y_t - Z_{14} & Z_5 & Z_9 - Z_{15} \\
* & * & Z_6 & -\rho_{g} G_s \\
* & * & Z_{10} - Z_{16}
\end{bmatrix},
\]

\[
Z_{11} = Y_sE_{ts}^T \text{U}_t E_t Y_t, Z_{12} = Y_sE_{ts}^T \text{U}_t F_t Y_t, Z_{13} = Y_sE_{ts}^T \text{U}_t G_s, Z_{14} = Y_sE_{ts}^T \text{U}_t F_t Y_t, Z_{15} = Y_sE_{ts}^T \text{U}_t G_s, Z_{16} = G_s^T \text{U}_t G_s.
\]

By using Schur complement for inequality in Equation (44) and combining inequality in Equation (41), we can ensure \( J_3(t) < 0 \). Then, integrating it with zero initial conditions, we get
\[
\mathbb{E}\{V(\varsigma(t))\} - \int_0^T S(\varsigma(t), \omega(t)) dt + \int_0^T \alpha \omega^T(t) \omega(t) dt \leq 0.
\]  
(45)

Considering \( V(\varsigma(t)) > 0 \), we obtain
\[
\int_0^T \mathbb{E}\{S(\varsigma(t), \omega(t))\} dt > \alpha \int_0^T \omega^T(t) \omega(t) dt.
\]  
(46)
By inequality in Equation (17), the FTB and the strict dissipativity of the closed-loop TDMJSs in Equation (13) are guaranteed, and the controller meets the GCC performance. The proof is completed.

Remark 3. In order to avoid the existence of the $Y_s$ and $Y_{s-1}$ in the matrix inequalities at the same time, we define two novel auxiliary variables in Equations (20) and (21). Then, in order to guarantee that LMI in Equations (24), (25) and (41) are established, we need to make the principal diagonal be negative. We define the novel auxiliary variable in Equation (22). For the unsolvable form $Y_s \{ \Sigma l \sum \lambda_s Y_{s-1} \}$, we use Schur complement to transform it into $Q_1$ and $Q_2$ shown in LMI in Equations (24), (25) and (41).

4 SIMULATION EXPERIMENTS

In this section, we will introduce the liquid monopropellant rocket motor model with a pressure-feeding system [41, 42] to demonstrate the correctness and feasibility of the designed approach shown as

\[
\begin{align*}
\dot{x}_1(t) &= (\beta_1 - 1)x_1 - \beta_2 x_1(t - \tau) + x_3(t - \tau), \\
\dot{x}_2(t) &= \frac{1}{\pi f_{11}}[-x_4(t) + u(t) + \omega_2(t)], \\
\dot{x}_3(t) &= \frac{1}{(1 - \pi)f_{11}}[-x_3(t) + x_4(t) - f_2 x_1(t)], \\
\dot{x}_4(t) &= \frac{1}{f_{31}}[x_2(t) - x_3(t) + \omega_4(t)],
\end{align*}
\]

where $\beta_1 = 0.5, \beta_2 = 0.5, \pi = 0.5, f_{11} = 10, f_{12} = 6.67, f_{21} = 1, f_{22} = 1.2, f_{31} = 3.3, f_{32} = 2.5, \tau = 1s$. Assuming there exists the non-linearities, the parameters of the system are given by

\[
A_1 = \begin{bmatrix}
-0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.2 \\
-0.2 & 0 & -0.2 & -0.2 \\
0 & 0.3 & -0.3 & 0
\end{bmatrix}
\]
The system states $X_1, X_2, X_3, X_4$.

**FIGURE 2** The state trajectories of the closed-loop time-delay Markov jump systems (TDMJSs; Equation 13)

\[ A_2 = \begin{bmatrix} -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.3 \\ -0.3 & 0 & -0.3 & -0.3 \\ 0 & 0.4 & -0.4 & 0 \end{bmatrix}, \]

\[ A_{d1} = A_{d2} = A_{d1} = A_{d2} = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ A_{c1} = A_{c2} = \begin{bmatrix} -0.1 \\ 0 \end{bmatrix}, \]

\[ B_1 = \begin{bmatrix} -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \]

\[ C_1 = \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0.3 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 \\ 0.3 \\ 0 \\ 0.4 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0.3 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0.2 \\ 0 \\ 0.3 \end{bmatrix}, \]

\[ E_1 = E_2 = [0.1, 0, 0.2, 0], \]

\[ F_1 = F_2 = [0.1, -0.1, 0, 0], \quad G_1 = G_2 = 0.1, \]

\[ \omega(t) = e^{-t} \times \sin(0.1 \times t), g(x(t), x(t - \tau)), \]

\[ \omega(t) = 0.01 \times (|x_0| + 0.1) + |x_0 - 0.1|. \]

The transition rate $\Pi_{ij}$ and the conditional probability $\Phi$ are:

\[ \Pi_{d1} = \begin{bmatrix} -4 & 4 \\ 5 & -5 \end{bmatrix}, \Phi = \begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{bmatrix}. \]

The dissipative parameters are given by $\mathcal{U} = -I, \mathcal{G} = I, \mathcal{V} = I, R_1 = R_2 = 0.2I$.

Here are the quantisation densities $t_1 = 0.7$ and $t_2 = 0.8$.

By computation, we get $\chi_1 = 0.176$ and $\chi_2 = 0.11$. Then, the quantisation errors are assumed to be $\Delta_1(t) = 0.176 \times \sin(t)$ and $\Delta_2(t) = 0.11 \times \sin(t)$. 
By solving LMI s in Equations (24) to (27) and (41), we obtain the dissipative performance index $\alpha = 0.6076$ and the GCC performance index $J^\ast = 0.3607$. The controller gains are: $K_1 = 6.3674 \times 10^{-3}$, $K_2 = 1.4387 \times 10^{-3}$.

The system and the controller modes are shown in Figure 1, where $r(t)$ represents the system mode, $s(t)$ represents the controller mode. With $x_0 = \left[ -0.3 \ 0.3 \ -0.3 \ 0.3 \right]^T$, the state trajectories of the closed-loop TDMJSs in Equation (13) are shown in Figure 2, where $X_1, X_2, X_3$ and $X_4$ represent the system-state components, respectively. We find that all the state trajectories are inclined to zero, which indicates the designed controller is effective.

Due to the fact that the controller is based on the HMM, we will analyse the following three cases: Synchronous case, partially asynchronous case, and asynchronous case. The results are shown in Table 2. We find that the dissipative performance index $\alpha$ and the GCC performance index $J^\ast$ become smaller with the increase in asynchrony, which means that the control performance shows better.

For the quantiser, we will analyse the situation of different $t$ by setting $t_1 = t_2$ with different numbers. The simulation results are shown in Table 3. We find that $\alpha$ increases with increasing $t$, but $J^\ast$ will be different. Therefore, the balance between the system requirements and the quantiser accuracy is of great significance in practical applications.

Remark 4. The considered model is a more practical and complex example, where $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$ are the non-dimensional instantaneous (NDI) pressure in the combustion chamber, the NDI mass-flow upstream of the capacitance, the NDI mass-rate of injected propellant and the NDI pressure at the place in the feeding line, respectively. It is more practical to study its transient performance, which can be seen as FTB. Furthermore, dissipativity with its ability to absorb energy greater than supply is significant in system analysis and synthesis. In order to attain disturbance attenuation performance, the dissipativity of the controlled system needs to be ensured. These motivate us to consider finite-time dissipative control for this model.

Remark 5. In [37], the asynchronous dissipative control of MJJs under quantiser is studied and the closed-loop system is stable at near 140 s, but the performance under different conditional probabilities and different quantisation densities are not studied. In [15], the closed-loop system under asynchronous quantised controller is stable at near 55 steps, and the performance under different conditional probabilities and different quantisation densities are studied. But in this study, the closed-loop system is stable at 12 s. We also study the performance under different conditional probabilities and different quantisation densities. Furthermore, the GCC performance indexes are lower than any case in [15], which means that the proposed method is efficient.

5 | CONCLUSION

In this study, the finite-time asynchronous dissipative output feedback-based GCC and quantiser for TDMJSs with conic-type non-linearities have been investigated. For the non-synchronous phenomenon between the system and the controller, the HMM is effectively introduced. Based on the sufficient conditions by using the LMI s technique, the FTB and the strict dissipative performance have been guaranteed, and the controller meets the GCC performance. Finally, the correctness and feasibility of this designed approach have been verified by a given practical example.

Remarkably, the conditional probability “$\phi_{mv}$” in Equation (7) is considered to be completely known. But in practice, it may be difficult and costly. On the other hand, in this study, the GCC performance index depends on the initial mode and state. It is obvious that GCC performance will be less conservative when it does not depend on the initial state and mode. A potentially interesting work may be associated with the asynchronous GCC under partially known conditional probability and the GCC performance which is independent of initial modes and states in the near future.

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