Improved Confidence Bounds for the Linear Logistic Model and Applications to Linear Bandits

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Abstract

We propose improved fixed-design confidence bounds for the logistic model. Our bounds significantly improve upon the state-of-the-art bound by Li et al. (2017) via recent developments of the self-concordant analysis of the logistic loss (Faury et al., 2020). Specifically, our confidence width avoids a direct dependence on $1/\kappa$, where $\kappa$ is the worst-case variance of the sample, that scales exponentially with the norm of the unknown linear parameter $\theta^*$. Instead, our bound depends directly on the variance induced by $\theta^*$ and the arm. We present two applications of our novel bounds on two logistic bandit problems: regret minimization and pure exploration. Our analysis shows that the new confidence bounds improve upon previous state-of-the-art performance guarantees.

1 Introduction

Multi-armed bandits algorithms offer a principled approach to solve explore/exploit problems and date back to Thompson (1933). They mix exploration and exploitation strategies in an optimal manner to maximize cumulative gain, or, adaptively discover the best among a set of items (Lattimore and Szepesvári, 2020). Such algorithms are widely deployed in industry, with applications spanning news recommendation (Li et al., 2010), ads (Sawant et al., 2018), online retail (Teo et al., 2016), website design (Ding et al., 2019), and drug discovery (Kazerouni and Wein, 2019).

The workhorse of bandit methods is contextual bandit algorithms which use an underlying ML model to incorporate contextual and content features in the decision-making process. Various underlying models have been leveraged in the bandit space, ranging from simple linear models (Auer, 2002; Dani et al., 2008), to complex non-linear neural networks (Riquelme et al., 2018). What one gains in ease of interpretation in linear models, losses for higher expressiveness in neural models. Generalized Linear Models (GLM) emerge as a middle ground, where the link function acts as a non-linear layer on top of an interpretable linear model (Filippi et al., 2010; Ding et al., 2020).

Let $d$ be the number of features, $x \in \mathbb{R}^d$ a feature vector representing an action, and $y$ the target variable. A GLM assumes the existence of an unknown parameter $\theta^* \in \mathbb{R}^d$, and a fixed, strictly increasing link function $\mu : \mathbb{R} \rightarrow \mathbb{R}$, such that $E[y|x] = \mu(x^\top \theta^*)$. One recovers the linear case by having $\mu(x) = x$. Most industrial use-cases require a binary target $y$, especially in the web and e-commerce domains where click data is prevalent. One is often interested in clicks, subscriptions, disease detection, or other binary conversion events. One advantage of GLMs is that it allows the representation of both continuous and categorical variables. To model binary targets $y \in \{0,1\}$, one can simply use a logistic or probit link function $\mu$, such that: $y \sim \text{Bernoulli}(\mu(x^\top \theta^*))$.

However, logistic bandits are not commonly used in industrial settings, perhaps due to a poor understanding of the impact of the non-linearity on algorithms. Specifically, consider the finite-armed contextual bandit setting with the time horizon $T$. At each time step $t$, the learner is given an arm set $\mathcal{X}_t \subset \mathbb{R}^d$ with $|\mathcal{X}_t| \leq K$, they choose an arm $x_t$ from $\mathcal{X}_t$, and receive a reward associated with $x_t$. In this setting, the state-of-the-art regret bound is $\hat{O}(\kappa^{-1} T \sqrt{\log(K)})$ where $\hat{O}$ hides polylogarithmic factors in $T$ and $\kappa^{-1}$ is a problem-dependent parameter that characterizes the degree of non-linearity of the link function (Li et al., 2017). $\kappa^{-1}$ is especially large for binary-target GLM link functions, as these classifiers are highly non-linear. In the logistic bandit case where $\mu(z) = 1/(1 + \exp(-z))$, the value $\kappa^{-1}$ scales exponentially with the norm $\|\theta^*\|$.

Such an issue was partially resolved in Faury et al. (2020) with a novel Bernstein self-normalized martingale tail-inequality, which, coupled with the generalized self-concordant property of the log-loss and a local analysis of the non-linearities of the reward function, reduces the regret bound of the logistic bandit to $O(d\sqrt{T})$. The dependence on $\kappa^{-1}$ is thus removed from the leading term (i.e., the $\sqrt{T}$ term). How-
ever, this regret bound is only tight for the worst-case arm set. It is not able to match the lower regret of $O(\sqrt{dT \log(K)})$ which has a smaller dependence on $d$ when $K$ is not exponential in $d$.

To achieve this lower regret, one may attempt to extend their regret analysis. However, their analysis is based on constructing a confidence set for $\theta^*$ independently of the arms, i.e., an adaptive design concentration inequality, which inherently carries the dependence on $\sqrt{d}$ in the confidence width over to the regret bound. Therefore, one must turn to a fundamentally different argument to derive a tighter concentration of measure. Specifically, one may turn to a fixed design counterpart to their adaptive design argument to derive a tighter concentration inequality, which inherently carries the dependence on $\kappa$ in the confidence width that carries over to the regret bound.

The main contribution of this paper is to propose fixed design concentration inequalities that do not involve $d$ nor $\kappa^{-1}$ in the confidence width. Let $\hat{\theta}_t$ be the MLE of $\theta^*$. Our bound takes the form of

$$
P\left(x^\top(\hat{\theta}_t - \theta^*) \leq O(\|x\|_{H^{-1}(\hat{\theta}_t)} \sqrt{\log(1/\delta)}) \right) \geq 1 - \delta,$$

where $H_i(\hat{\theta}_t)$ is an estimate of the hessian of the log likelihood at $\theta^*$, defined later in Eq. (2). To see the improvement upon Li et al. (2017), their bound takes the form of $O(\kappa^{-1} \|x\|_{V^{-1}} \sqrt{\log(1/\delta)})$ where $V_t$ satisfies $\kappa^2 V_t \preceq \kappa V_t \preceq c H_i(\hat{\theta}_t)$ for some absolute constant $c$ as we show later. Our bounds are the first of their type, complement those in (Faury et al., 2020) as a fixed design counterpart to their adaptive design version, and can be applied in any situation where inference from a logistic model is needed.

To demonstrate the power of our confidence interval, we focus on two applications in the bandit space. The first is the aforementioned contextual regret bandit setting where we prove a regret bound of $O(\sqrt{dT})$, improving on a previous regret bound of $O(\kappa^{-1} \sqrt{dT})$ by (Li et al., 2017). This also improves the previous result of (Faury et al., 2020) in the finite arm setting, by removing an extra $\sqrt{d}$ factor. Interestingly, our regret bound does not involve any dependence on the norm $\|\theta^*\|$ in the leading term. We present our results on contextual bandits in Section 3.

Maybe replace by shorter paragraph: Our second application is an important problem that has received little attention: pure exploration (aka best-arm identification) for logistic bandits. In this setting, we seek to identify the best arm, with high confidence, in the fewest samples possible. One prominent application is over dueling bandits for pairwise comparisons (Yue and Joachims, 2009), which can be modeled naturally as $Beroulli(\mu(x_1 - x_2)^\top \theta^*)$. Section 4 explores this context. (Kazerouni and Wein, 2019) gave a sample complexity that depends on $\kappa^{-1}$ multiplicatively. Our improved sample complexity, enabled by our confidence bound, depends on a variance induced by $\theta^*$, and only depends on $\kappa^{-1}$ additively.

Finally, we conclude our work in Section 5 with exciting future research directions.

## 2 Improved Confidence Intervals for the Linear Logistic MLE

In this paper, we focus on the logistic generalized model with link function,

$$
\mu(x^\top \theta) = \frac{1}{1 + e^{-x^\top \theta}}.
$$

Assume that we have a fixed $\theta^* \in \mathbb{R}^d$ and consider a set of measurements $\{(x_s, y_s)\}_{s=1}^t \subset \mathbb{R}^d \times \mathbb{R}$, where each $y_s \in \{0, 1\}$ and $P(y_s = 1) = \mu(x_s^\top \theta^*)$. Let $\eta_s = y_s - \mu(x_s^\top \theta^*)$. Denote by $\tilde{\mu}(z)$ the first order derivative of $\mu(z)$. Let $L$ and $M$ be the upper bounds on the first and the second derivative of $\mu(z)$. We choose $L = M = 1/4$. Define $\kappa = \min_{s: \|x_s\| \leq 1} \tilde{\mu}(x_s^\top \theta^*)$.

The maximum likelihood estimate is the maximizer of the regularized log-likelihood:

$$
\hat{\theta} = \arg\max_{\theta \in \mathbb{R}^d} \sum_{s=1}^t y_s \log \mu(x_s^\top \theta) + (1 - y_s) \log(1 - \mu(x_s^\top \theta)) 
$$

(1)

We also define the Hessian of the log-likelihood at $\theta$ as

$$
H_i(\theta) = \sum_{s=1}^t \tilde{\mu}(x_s^\top \theta^*) x_s x_s^\top.
$$

(2)

We now introduce our improved confidence interval for the linear logistic model.

**Theorem 1.** Let $\delta \leq e^{-1}$ and fix $x \in \mathbb{R}^d$ such that $\|x\| \leq 1$. Define $\gamma(d) = (10(d + \sqrt{\ln(12/\delta)})^2$. Let $\hat{\theta}_t$ be the solution of Eq. (2). Suppose $:\max_{s \in [t]} \|x_s\|_{H_t^{-1}(\theta^*)}^2 \leq \frac{1}{\gamma(d)}$. Then,

$$
P\left(x^\top(\hat{\theta}_t - \theta^*) \leq 4.2 \cdot \|x\|_{H_t^{-1}(\theta^*)} \sqrt{\ln(12/\delta)} \right) \geq 1 - \delta.
$$

Furthermore,

$$
P\left(x^\top(\hat{\theta}_t - \theta^*) \leq 6 \cdot \|x\|_{H_t^{-1}(\theta^*)} \sqrt{\ln(12/\delta)} \right) \geq 1 - \delta.
$$

Let $V_t = \sum_{s=1}^t x_s x_s^\top$. Our theorem significantly improves upon Li et al. (2017) in that their bound depends on $\frac{1}{\kappa} \|x\|_{V_t^{-1}}$, which can be much larger than $\|x\|_{H_t^{-1}(\theta^*)}$, depending on $x$. Specifically, the main use of the confidence interval is to set $x = x_i - x_j$ to distinguish between the arms $i$ and $j$. If such an $x$ is orthogonal to $\theta^*$ and if we have pulled arms a lot in the direction of $x$, then we do not pay $1/\kappa$ in the
confidence width. In contrast, their confidence bound pays for $1/\kappa$ at all times.

Another improvement is at the requirement on the minimum eigenvalue in $\kappa$. For comparison, we can use $\frac{1}{d^2} \geq \lambda_{\min}(H_t(\theta_t)) \geq \frac{1}{d} \lambda_{\min}(V_t)$ to have a more strict version of our requirement as $\lambda_{\min}(V_t) = \Omega(\frac{1}{d^2})$, which improves upon theirs: $\lambda_{\min}(V_t) = \Omega(\frac{1}{\kappa^2})$. Note that the dependence on the dimension is the same; i.e., $\lambda_{\min}(V_t) = \Omega(d^2)$.

Suppose $\|\theta\| \leq S_s$ where $S_s$ is known to us. Faury et al. (2020, Lemma 3) have shown the following confidence bound: w.p. at least $1 - \delta$, \forall t \geq 1, \forall x \in \mathbb{R}^d,
\begin{align*}
\|x^\top (\hat{\theta}_t - \theta^*)\| \leq \Theta \left( \|x\|_{H_t(\theta^*)}^{-1} S_s^{3/2} \sqrt{(d + \ln(t/\delta))} \right).
\end{align*}
While this bound does not directly depend on $\kappa$ like ours, it differs from ours in that the bound works for all $x$ simultaneously without union bounds. Furthermore, their bound is anytime (i.e., holds with any $t$) by construction. In contrast, ours requires union bounds to make it work for multiple $x$’s or to derive an anytime version. On the other hand, their bound has a factor $\sqrt{d}$ because it relies on the confidence set construction on $\theta^*$ independently of $x$. Furthermore, their bound introduces a factor $S_s^{3/2}$ in the confidence width absent in ours.\footnote{We are not sure whether this is necessary for the L2-regularized MLE in logistic models. In the linear model, the dependence on $S_s$ only appears in the logarithmic term, if tuned (Abbasi-Yadkori et al., 2011, Theorem 2).} In contrast, our bound does not depend on $S_s$ in the confidence width.

**Proof of Theorem 1.** We provide a sketch of the proof for the first statement and defer the full proof to the supplementary. The novelty is to exploit the variance term without introducing $\kappa$ explicitly in the confidence width. We follow the main decomposition of Li et al. (2017, Theorem 1) but deviate from their proof by employing: (i) Bernstein’s inequality rather than Hoeffding’s to directly depend on $H_t(\theta^*)$ in the bound (as opposed to $\kappa^{-1} V_t$); and (ii) the self concordant analysis by Faury et al. (2020) to significantly relax the requirements on $H_t(\theta^*)$ w.r.t. $\kappa$, especially via $\xi_t$ rather than $1/\lambda_{\min}(H_t(\theta^*))$. Since $\xi_t \leq 1/\lambda_{\min}(H_t(\theta^*))$, the requirement w.r.t. $\xi_t$ is weaker.

Let $\alpha_s(\hat{\theta}_t, \theta^*) := \frac{\mu(x^\top \hat{\theta}_t) - \mu(x^\top \theta^*)}{x^\top (\hat{\theta}_t - \theta^*)}$. Let := $\sum_s x_s$ and := $\sum_s \alpha_s(\hat{\theta}_t, \theta^*) x_s x^\top_s$. By the optimality condition of $\hat{\theta}_t$, we have:
\begin{align*}
z = &\sum_s (\mu(x^\top \hat{\theta}_t) - \mu(x^\top \theta^*)) x_s \\
= &\sum_s \alpha_s(\hat{\theta}_t, \theta^*) x_s x^\top_s (\hat{\theta}_t - \theta^*) \\
= &G(\hat{\theta}_t - \theta^*). \quad (3)
\end{align*}

We use the shorthand $H := H_t(\theta^*)$ and define := $G - H$. The main decomposition is
\begin{align*}
x^\top (\hat{\theta}_t - \theta^*) = &x^\top (H + E)^{-1} z \\
= &x^\top H^{-1} z - x^\top H^{-1} E(H + E)^{-1} z . \quad (4)
\end{align*}
We bound $x^\top H^{-1} z$ by $O(||x||_{H^{-1} \sqrt{\log(1/\delta)}}$ (omitting a lower order term) that uses Bernstein’s inequality and the assumption on $\xi_t^2$. Since this is achieves the target order, it remains to control the other term:
\begin{align*}
-x^\top H^{-1} E(H + E)^{-1} z \\
\leq ||x||_{H^{-1}/2} E(H + E)^{-1} H^{1/2} ||z||_{H^{-1}} .
\end{align*}
We are able to control $||H^{-1/2} E(H + E)^{-1} H^{1/2} || = O \left( \frac{\sqrt{d + \log(1/\delta)}}{d + \log(1/\delta)} \right)$ where the self-concordant control lemma (Faury et al., 2020, Lemma 9) plays a key role. This concludes the proof.

Note that controlling $||H^{-1/2} E(H + E)^{-1} H^{1/2} || = O \left( \frac{1}{\kappa^2} \right)$ may seem to work as well, but this results in requiring a larger $\gamma(d)$ of the order $d^2 + d \ln(1/\delta)$, introducing a factor of $d$ to $\ln(1/\delta)$. This can make a significant difference as $\delta \rightarrow 0$.\hfill \Box

### 3 Finite-Armed Logistic Linear Contextual Bandits

Consider the contextual bandit setting where at each time step $t$ the environment presents the learner with an arm set $\chi_t = \{x_{t,1}, \ldots, x_{t,K}\} \subset \mathbb{R}^d$ (Auer, 2002). In this setting, while the arm set is changing over time, it is determined by the environment independently from the behavior of the learner. In each round $t$, the learner chooses an arm index $a_t \in [K]$ and then receives its associated reward $y_t \in \{0,1\}$ drawn from a Bernoulli distribution with mean $\mu(x_{t,a_t}^\top \theta^*)$, where $\theta^*$ is unknown to the learner. Let $x_{t,a^*} = \arg \max_{x \in \chi_t} \mu(x_{t,a}^\top \theta^*)$. The goal is to minimize the cumulative (pseudo-)regret over the time horizon $T$:
\begin{align*}
\text{Reg}_T = \sum_{t=1}^T \mu(x_{t,a_t}^\top \theta^*) - \mu(x_{t,a^*}^\top \theta^*). \quad (5)
\end{align*}
When the reward structure is Bernoulli, the best known regret upper bound is $O(\frac{1}{\kappa} \sqrt{dT \ln(K)})$ where $\tilde{O}$ hides poly-logarithmic factors in $T$. However, the factor $1/\kappa$ is exponential w.r.t. $\|\theta^*\|$, which makes the regret impractically large. Leveraging our new confidence bound, we propose a new algorithm SupGLM that removes $1/\kappa$ from the leading term:
We remark that this setting is slightly different from that of Faury et al. (2020), which allows an infinite arm set but has higher regret $O(d\sqrt{T})$. One can view our setting as an accelerated regret rate that adapts to the arm set size.

We assume that $\|x_{t,a}\| \leq 1, \forall t \in [T], a \in [K]$, and that $\|\theta^*\| \leq S$, where $S_t$ is known to the learner. We follow (Li et al., 2017) and assume that there exists $\sigma^2$ such that $\min(\mathbb{E}[(\frac{1}{T}\sum_{a \in [K]} x_{t,a}x_{t,a}^\top)] \geq \sigma^2$, which is used to characterize the length of the burn-in sampling rounds in our theorem.

We describe SupGLM in Algorithm 1, which follows the standard mechanism for maintaining independent samples (Auer, 2002). As the confidence bound is not available until enough samples are accrued, we follow (Li et al., 2017) and perform $\tau$ rounds of burn-in sampling and then spread the samples across the buckets $\Psi_1, \ldots, \Psi_S, \Phi$ equally. Our burn-in sampling is different from Li et al. (2017), which we discuss in our supplementary. In each round $t$, we loop through the buckets until we find an arm that satisfies the criteria.

For each iteration of the while loop indexed by $s \in [S]$, we compute $\hat{\theta}_{t-1}^{(s)}$, the MLE given in Eq. (22), using the samples in the bucket $\Psi_s(t-1)$. We compute $\hat{\Phi}$ in the same way using $\hat{\Phi}$. Let $X_t = x_{t,a}$. For any $\theta$, define

$$H_t^{(s)}(\theta) := \sum_{u \in \Psi_s(t)} \hat{\mu}(X_u^\top \theta)X_uX_u^\top. \quad (6)$$

The algorithm computes the mean estimate and the confidence width of each arm $a \in [K]$ as follows:

$$m_{t,a} := \langle x_{t,a}, \theta_{t-1}^{(s)} \rangle, \quad w_{t,a} := \alpha \sqrt{2\|x_{t,a}\|H_t^{(s)}(\theta_{t-1}^{(s)})^{-1}}. \quad (7)$$

For each $s \in [S]$, we check if there is an underexplored arm (step (a)) and pull it. Otherwise, we check if all the arms are sufficiently explored and, if so, pull the arm with the highest empirical mean. Finally, we filter arms whose empirical means are sufficiently far from the highest empirical mean and go to the next iteration.

The reason for the complicated sample bucketing is to maintain the validity of the concentration inequality in the analysis, which requires that the context vectors and rewards are conditionally independent. To see why, suppose we use both $\Psi_1$ and $\Psi_2$ to compute the mean estimator (say $\hat{\theta}$) in iteration $s = 2$. The decision to include a data point $(x, y)$ in $\Psi_2$ (step (a)) is a function of the context vectors in $\Psi_1$ and their rewards, because $(x, y)$ passed the filtering in step (c) in $s = 1$. This means that in later rounds, estimator $\hat{\theta}$ is computed based on $\Psi_1 \cup \Psi_2$, which contains samples collected based on the rewards in itself, breaking the conditional independence.

Algorithm 1: SupLogistic

Input: time horizon $T$, burn-in length $\tau$, and exploration rate $\alpha$

1: initialize $S = \lfloor \log_T T \rfloor$
2: initialize Bucket $\Psi_1 = \cdots = \Psi_S = \Psi_{S+1} = \emptyset$
3: for $t \in [\tau]$ do
4: Choose $a_t \in [K]$ uniformly at random.
5: Add $a_t$ to the set $\Psi_{(t-1) \mod S+1+1}$
6: end for
7: initialize $\Psi_0 = \emptyset, \Phi = \Psi_{S+1}$
8: for $t = \tau + 1, \tau + 2, \cdots, T$ do
9: initialize $A_1 = [K], s = 1, a_t = \emptyset$
10: while $a_t \in \emptyset$ do
11: Compute $m_{t,a}$ and $w_{t,a} \quad \triangleright \text{use Eq } (7)\triangleright$
12: if $w_{t,a} > 2^{-\gamma}, \exists a \in A_s$ then
13: $a_t = a$ \quad \triangleright \text{ (a)}
14: $\Psi_s \leftarrow \Psi_s \cup \{t\}$
15: else if $w_{t,a} \leq 1/\sqrt{T}, \forall a \in A_s$ then
16: $a_t = \arg \max_{a \in A_s} m_{t,a}$ \quad \triangleright \text{ (b)}
17: $\Psi_0 \leftarrow \Psi_0 \cup \{t\}$
18: else if $w_{t,a} \leq 2^{-\gamma}, \forall a \in A_s$ then
19: $A_{s+1} = \{a \in A_s : m_{t,a} \geq \max_{j \in A_s} m_{t,j} - 2 \cdot 2^{-\gamma}\}$
20: $s \leftarrow s + 1$
21: end if
22: end while
23: end while
24: end for

Algorithm 1 differs from SupCB-GLM of Li et al. (2017) in its confidence bound, and in having an extra bucket $\Phi$. Our tight concentration inequality requires the confidence width to depend on an estimate of $\theta^*$ (see Theorem 1). If we do not use $\Phi$ and replace $\theta_4$ in (7) with $\theta_{t-1}$, we lose the conditional independence in each bucket, as we collect samples into buckets as a function of the rewards from the same bucket.

We present our regret bound in the following theorem, whose proof can be found in our supplementary.

Theorem 2. Let $\tau = \sqrt{dT}$ and $\alpha = 4.2 \sqrt{\ln \frac{12.25STK}{\delta}}$. Then, there exists

$$T_0 = \Theta(Z \ln^4(Z)) \quad \text{where} \quad Z = \frac{d^3}{\kappa^2} + \frac{\ln^2(K/\delta)}{d\kappa^2}$$

such that $\forall T \geq T_0$, we have, w.p. at least $1 - \delta$,

$$\text{Reg}_T \leq 10\alpha \sqrt{dT \ln(T/d) \log_2(T)} + O\left(\frac{\alpha^2}{\kappa} \ln \left(\frac{\alpha^2 T}{\kappa}\right)\right)$$

Our bound improves upon SupCB-GLM (Li et al., 2017) by removing the factor $1/\kappa = \Theta(\exp(S_\delta))$ in the leading term (i.e., $\sqrt{T}$ term). This improvement parallels that of Faury et al. (2020) with $O(d\sqrt{T})$ upon UCB-GLM (Li et al., 2017) with $O(\frac{1}{d}d\sqrt{T})$. While Faury et al. (2020) manage to avoid an exponential
dependence on $S_z$, their regret still has a factor $S^{1.5}$ in the leading term. In contrast, our bound does not depend on $S_z$ in the leading term at all. We are not aware of any other logistic bandit regret bounds with such an asymptotically norm-free regret. For linear bandits, Lattimore and Szepesvári (2020, the Remark after Exercise 19.3) show that a careful analysis leads to avoiding a polynomial dependence on $S_z$ in the leading term as well. But a logarithmic dependence on $S_z$ in the leading term is still present.

4 Transductive Pure Exploration Logistic Bandits

As the second application of our confidence bound, we consider the pure exploration problem where we want to identify the best arm rather than minimizing regret. Specifically, we assume access to finite arm subsets $X, Z \subseteq \mathbb{R}^d$ which are known, an unknown parameter vector $\theta^* \in \mathbb{R}$, and a chosen confidence $\delta \in (0, 1)$. Our goal is to, with probability greater than $1 - \delta$, discover $z^* = \arg \max_{z \in Z} z^\top \theta^*$ using as few measurements from $X$ as possible. This generalization setup is known as the transductive setting (Fiez et al., 2019). In each round, an algorithm chooses an arm $x_t$, which is measurable with respect to the history $F_{t-1} = \{(x_s, y_s)_{s < t}\}$, and observes a reward $y_t \in \{0, 1\}$. It stops at a random stopping time $\tau$ and recommends $\hat{z} \in Z$, where $\tau$ and $\hat{z}$ are both measurable w.r.t. the history $F_\tau$. We assume that $\mathbb{P}(y_t = 1|x_t, F_{t-1}) = \mu(x_t^\top \theta^*)$. Let $\mathbb{P}_{\theta^*}, \mathbb{E}_{\theta^*}$ denote the probability and corresponding expectation induced by the actions and rewards when the true parameter is $\theta^*$. Formally, we define a $\delta$-PAC algorithm as follows:

**Definition 1.** An algorithm for the logistic-transductive-bandit problem is $\delta$-PAC for $(X, Z)$ if $\mathbb{P}(z_{\tau} \neq z^*) \leq \delta, \forall \theta \in \mathbb{R}^d$.

**Example: Pairwise Judgements.** As a concrete and natural example of the transductive linear bandit problem, consider an e-commerce application where the goal is to recommend an item from a set of items based on relative judgements by the user. For example, the user may be repeatedly shown two items to compare, and must choose one. The use of relative judgements is natural when trying to build a user preference profile (Jain et al., 2016) or in adaptive interactive search (Biswas et al., 2019).

To model this as a transductive logistic bandit, denote the items by $Z$, and assume an underlying parameter vector $\theta^*$. In each round, we choose two items $z, z' \in Z$, and observe the binary user preference of item $z$ or item $z'$. A natural model is to give each item $z \in Z$ a score $z^\top \theta^*$, with the goal of finding $z^* = \max_{z \in Z} z^\top \theta^*$. The probability the user prefers item $z$ over $z'$ is given by $\mathbb{P}(z > z') = \mu((z - z')^\top \theta^*)$. To discover $z^*$, the bandit selects arm $x_t$ from $X = \{z - z' : z, z' \in Z\}$, and shows the corresponding items $z_t$ and $z'_t$ to the user for pairwise judgment.

Although the dueling bandit problem has been considered in the multi-armed bandit literature (Yue et al., 2012), as far as we are aware, this is the first work to propose the transducing bandit problem as a natural extension of the transductive linear bandit setting under a logistic noise model.

**Related Work.** Soare et al. (2014) first proposed the problem of pure exploration in linear bandits with Gaussian noise and $X = Z$, and provided a lower bound. (Fiez et al., 2019) introduced the general transductive setting, providing the RAGE elimination based method, and is the main motivation for our algorithm. RAGE achieves the lower bound up to logarithmic factors with excellent empirical performance. Other works include (Degenne et al., 2020; Karnin et al., 2013), which achieve the lower bound asymptotically. Finally we mention (Katz-Samuels et al., 2020), which follows a similar approach to (Fiez et al., 2019) but uses empirical process theory to replace the union bound over the number of arms with a Gaussian width, and proposes a computationally efficient algorithm.

Despite its importance in many real-life settings, the logistic pure-exploration bandit has received little attention. The only work we are aware of is (Kazerouni and Wein, 2019) which defines the problem and provides an algorithm motivated by (Xu et al., 2018). We contrast our method with theirs in the next section.

**Notation.** Let $\Delta_X = \{\lambda \in \mathbb{R}^{|X|}, \lambda \geq 0, \sum_{x \in X} \lambda_x = 1\}$ be the probability simplex over $X$. Given a design $\lambda \in \Delta_X$, define:

$$H(\lambda, \theta) := \sum_{x \in X} \lambda_x \mu(x^\top \theta) xx^\top, \quad A(\lambda) := \sum_{x \in X} \lambda_x xx^\top.$$  

Define $\kappa_0 = \min_{x \in X} \mu(x^\top \theta^*)$, i.e. the smallest derivative of the link function among elements in $X$.

4.1 Algorithm

Algorithm 6 proceeds in rounds. In each round, it maintains a set of active arms $Z_t$. It computes an

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**Algorithm 2 BurnIn**

**Input:** $X, \kappa_0$

1. initialize $\lambda_0 = \arg \min_{\lambda \in \Delta_X} \max_{x \in X} \|x\|^2_{A(\lambda)^{-1}}$

2. initialize $n_0 = (1 + \epsilon) d \log(12|X|/\delta)$

3. $x_1, \cdots, x_{n_0} \leftarrow \text{round}(n_0, \lambda_0, \epsilon)$

4. Observe associated rewards $y_1, \cdots, y_{n_0}$

5. return associated rewards $y_1, \cdots, y_{n_0}$
Algorithm 3 RAGE-GLM

Input: $\epsilon$, $\delta$, $\mathcal{X}$, $Z$, $\kappa_0$, effective rounding procedure $\text{round}(n, \epsilon, \lambda)$
1: initialize $t = 1$, $Z_1 = Z$, $r(\epsilon) = d^2/\epsilon$
2: $\theta_0 \leftarrow \text{BurnIn}(\mathcal{X}, \kappa_0)$ \quad $\triangleright$ Burn-in phase
3: while $|Z_t| > 1$ do \quad $\triangleright$ Elimination phase
4: \hspace{1em} $f(\lambda) := \max \gamma(d) \max_{x \in Z \mathcal{X}} \|x\|_2^2 H(\lambda, \theta_{t-1})^{-1}$
5: \hspace{2em} $\lambda_t = \arg \min_{\lambda \in \Delta_{\mathcal{X}}} f(\lambda)$
6: \hspace{2em} $r_t = f(\lambda_t)$
7: \hspace{2em} $n_t = \max\{(1+\epsilon)r_t \log(\max\{|Z|, |\mathcal{X}|^2\}t^2/\delta), r(\epsilon)\}$
8: \hspace{2em} $x_1, \ldots, x_{n_t} \leftarrow \text{round}(n_t, \epsilon, \lambda)$
9: \hspace{2em} Observe rewards $y_1, \ldots, y_{n_t} \in \{0, 1\}$
10: \hspace{2em} Compute the unregularized MLE $\hat{\theta}_t$
11: \hspace{2em} $\hat{z}_t = \arg \max_{z \in Z} \theta_t^\top z$
12: \hspace{2em} $Z_{t+1} \leftarrow Z_t \setminus \{z \in Z_t : \theta_t^\top (z_t - z) \geq 2^{-t}\}$
13: \hspace{2em} $t \leftarrow t + 1$
14: end while
15: return $\hat{z}_t$

Experimental Design. The naive approximation: As we have yet to know any information on $\max \hat{\theta}_t$, we cannot have an upper bound on $\kappa_0$, which is equivalent to having an upper bound on $S^*$. The second component of line 4 minimizes $\max_{z \in Z} \|x - z\|_2^2 H(\lambda, \theta_{t-1})^{-1}$. Similarly, the burn-in phase. This guarantees that we satisfy the conditions needed to use the confidence interval in Theorem 1. In addition it guarantees that the estimate $\hat{\theta}_t$ is sufficiently close to $\theta^*$ for all directions in $\mathcal{X}$. Combining this with self-concordance, $|\hat{\mu}| \leq \hat{\mu}$, we show that $H(\lambda, \theta^*)$ is within a constant factor of $H(\lambda, \hat{\theta}_t)$ (see Supplementary). Finally, we stop the algorithm once $|Z_t| = 1$ and return the remaining arm.

Theorem 3 (Sample Complexity). Define

$$\beta_t = \min_{\lambda \in \Delta_{\mathcal{X}}} \max_{z : z^\top \in \mathcal{S}_t} \|x - z\|_2^2 H(\lambda, \theta^*)^{-1},$$

$$\gamma(d) \max_{x \in \mathcal{X}} \|x\|_2^2 H(\lambda, \theta^*)^{-1}$$

Algorithm 6 returns $z^*$ with probability greater than $1 - 2\delta$ in a number of samples no more than

$$O\left(\sum_{t=1}^{[\log_2(2/\Delta_{\text{min}})]} \beta_t \log(\max(|Z|, |\mathcal{X}|^2) t^2/\delta) + r(\epsilon) \log(1/\Delta_{\text{min}}) + \kappa_0^{-1} d \gamma(d) (1 + \epsilon) \log(|\mathcal{X}|^2/\delta))\right)$$

where $\Delta_{\text{min}} = \min_{z \neq z^* : z^\top \in \mathcal{S}_t} (\theta^*, z^* - z)$.

Comparison to past work. Define $\rho_t := \min_{\lambda \in \Delta_{\mathcal{X}}} \max_{z : z^\top \in \mathcal{S}_t} \|x - z\|_2^2 H(\lambda, \theta^*)^{-1}$. Ignoring the burn-in samples, and samples for ensuring a valid confidence interval, the scaling of the main term of the Wolfowitz theorem (Soare et al., 2014). Finally, note that for the burn-in phase we are assuming we have an upper bound on $\kappa_0$, which is equivalent to having an upper bound on $S^*$.

Experimental Design. Following the burn-in phase, in each round, line 4 of Algorithm 6 computes an experimental design that minimizes two objectives simultaneously. The primary objective, up to constants, is

$$\min_{\lambda \in \Delta_{\mathcal{X}}} \max_{z : z^\top \in \mathcal{S}_t} \|z - z^\star\|_2^2 H(\lambda, \theta_{t-1})^{-1}.$$
sample complexity of Theorem 3 is
\[
\sum_{t=1}^{\log_2(2/\Delta_{\min})} 2^{2t} \rho_t \log(|Z|^2 t^2/\delta).
\]
Most importantly, this depends on \(H(\theta^*)\) instead of a loose bound based on \(\kappa^{-1}\).

This is reminiscent of a similar quantity that arises in the sample complexity of pure exploration linear bandits (Fiez et al., 2019), where they demonstrate a sample complexity of
\[
\tilde{O}\left(\sum_{t=1}^{\log_2(2/\Delta_{\min})} 2^{2t} \min_{\lambda \in \Delta_X} \max_{z,z' \in S_t} \|z - z'\|^2_{A(\lambda)^{-1}}\right),
\]
and show it is within a logarithmic factor of the information theoretic lower bound for this problem.

Next, we compare to the result of (Kazerouni and Wein, 2019). Using a variant of the UGapE algorithm for linear bandits (Xu et al., 2018), they demonstrate a sample complexity \(\tilde{O}(n/\Delta_{\min})\) in the setting where \(X = Z\). This sample complexity scales with the number of arms, and only captures a dependency on the smallest gap. We note that one can improve on their sample complexity by using a naive passive algorithm that uses a fixed G-optimal design, along with the trivial bound \(H(\lambda, \theta^*) \geq \kappa_0 A(\lambda)\), resulting in \(\tilde{O}(d/(\kappa_0 \Delta_{\min}^2))\) (Soare et al., 2014)\(^3\). In contrast, the bound of Theorem 3 only depends on the number of arms logarithmically, captures a local dependence on \(\theta^*\), and has a better gap dependence.

**Extra samples.** Nevertheless, Algorithm 6 has two weaknesses, the burn-in period at the start of the algorithm, and the need to potentially take extra samples (i.e., the second argument of the max in line 4) in each round to ensure the confidence interval is available. On problem instances where the minimum gap is very small, the first component of \(\kappa_t\) will eventually dominate the second component \(\tilde{O}(\gamma(d) \max_{x \in X} \|x\|^2_{H(\lambda, \theta^*)^{-1}})\). In Section 4.4, we will remove the extra samples needed in each round.

### 4.2 Lower Bounds for Pure Exploration Logistic Bandits

To explain our sample complexity a bit more consider the following lemma.

**Lemma 1.**
\[
\sum_{t=1}^{\log_2(2/\Delta_{\min})} 2^{2t} \min_{\lambda \in \Delta_X} \max_{z,z' \in S_t} \|z - z'\|^2_{H(\theta^*)} \leq \log\left(\frac{2}{\Delta_{\min}}\right) \min_{\lambda \in \Delta_X} \max_{z,z' \in Z} \frac{\|z^* - z\|^2_{H(\theta^*)^{-1}}}{\langle z^* - z, z' - z \rangle^2} = \frac{1}{4} \log\left(\frac{2}{\Delta_{\min}}\right) \left(\max_{\lambda \in \Delta_X} \min_{z,z' \in Z} \|\theta^* - z\|^2_{H(\lambda, \theta^*)} \right)^{-1}
\]
where for a fixed \(\lambda, \theta, z := \min_{\theta \in \mathbb{R}^d; \theta^T(z^* - z) \leq 0} \|\theta - \theta^*\|^2_{H(\lambda, \theta^*)} \).

The lemma shows that the main term of the sample complexity of Theorem 3 given in (9) is bounded by a natural experimental design arising from the true problem parameter.

Finally, we provide the following information theoretic lower bound for any PAC-\(\delta\) algorithm. Define,
\[
\beta(a, b) = \int_0^1 (1 - t) \mu(a + t(b - a)) dt
\]
and analogous to \(H(\lambda, \theta)\) we define two additional matrix valued functions,
\[
G(\lambda, \theta_1, \theta_2) = \sum_{x \in X} \lambda_x \alpha(x, \theta_1, \theta_2) xx^\top (10)
\]
\[
K(\lambda, \theta_1, \theta_2) = \sum_{x \in X} \lambda_x \beta(x, \theta_1, \theta_2) xx^\top (11)
\]

**Theorem 4.** Any PAC-\(\delta\) algorithm for the pure exploration logistic bandits problem has a stopping time \(\tau\) satisfying,
\[
\mathbb{P}[\tau] \geq \min_{\lambda \in \Delta_x} \max_{z \in X} \frac{1}{\max_{\theta \in \mathbb{R}^d; \theta^T(z^* - z) \leq 0}} \sum_{x \in X} \lambda_x K(\nu_x, \theta^*) \log\left(\frac{1}{2.4 \delta}\right) = c(\lambda)^{-1} \log\left(\frac{1}{2.4 \delta}\right),
\]
\[
c(\lambda) = \max_{\lambda \in \Delta_X} \min_{z \in Z} \|z - z^*\|^2_{H(\lambda, \theta^*)},
\]
where firstly, \(\nu_x, \theta^* \sim \text{Bernoulli}(\langle x, \theta^* \rangle)\), and secondly \(\theta^* := \min_{\theta \in \mathbb{R}^d; \theta^T(z^* - z) \leq 0} \|\theta - \theta^*\|^2_{K(\nu_x, \theta^*)}\) and is given explicitly as the solution to the fixed-point equation
\[
\theta^*_x = \theta^* - \frac{(z^* - z)^\top \nu_x G(\lambda, \theta^*_x, \theta^*)^{-1}(z^* - z)}{\|z^* - z\|^2_{G(\lambda, \theta^*_x, \theta^*)^{-1}}}
\]
In general, it is not clear how to compare our upper bound from Theorem 6 to this lower bound due to the non-explicit nature of \(G(\lambda, \theta^*_x, \theta^*)\). The quantity, \(\max_{\lambda \in \Delta_X} \min_{z \in Z} \|z^* - \theta^*\|^2_{H(\lambda, \theta^*)}\) can be interpreted as a lower bound arising from a quadratic approximation of the KL-divergence in the first line of the lower bound by the Fisher information matrix.

### 4.3 Interpreting the Upper Bound

In this section we consider the sample complexity in a concrete example.

**Example 1.** Consider a simple setting where \(Z = X = \{e_1, e_2\} \subset \mathbb{R}^2\), and \(\theta^* = (r, r - \epsilon)\), for \(r \geq 0\). In this case, \(\kappa_0^{-1} = \max_{\lambda \in [1,2]} \|z^*_\lambda\|_{\theta^*_\lambda}^{-1} \leq e^\epsilon\). Thus in the burn-in phase, we take roughly \(\tilde{O}(e^\epsilon)\) sam-
ples. Now, for small $\epsilon$, the minimizer of $\min_{\lambda \in \Delta_X} \|e_1 - e_2\|_{H(\theta)}^{-1}$ places roughly equal mass on $e_1$ and $e_2$, giving an objective value that is roughly bounded above by $e^\epsilon$. Thus the sample complexity of Algorithm 6 scales like $O\left(\sum_{i=1}^{\log_2(1/\delta)} 2^i \epsilon^2 \log(1/\delta)\right) \approx e^{\epsilon^2}$.

Note this problem is equivalent to a standard best-arm identification algorithm with two Bernoulli arms (Kaufmann et al., 2015). Standard results in Pure Exploration show that a lower bound on this problem is given by the KL-divergence $KL(Bernoulli(\mu(\theta^2z_2)), Bernoulli(\mu(\theta^2z_2)))^{-1} \approx \frac{1}{2\epsilon^2}$ for sufficiently small $\epsilon$. This shows that our bound is at least no worse than the well-studied unstructured case.

**Example 2.** We extend the above setting and consider $\mathcal{X} = \{e_1, e_2, e_1 - e_2\}$, $\mathcal{Z} = \{e_1, e_2\}$ and the same $\theta^*$. As above, the burn-in phase requires roughly $\kappa^{-1} \approx e^\epsilon$ samples. Starting from the first round, our computed experimental design will place all of its samples on the third arm. In this case, $\min_\lambda \|e_1 - e_2\|_{H(\theta_\lambda)}^{-1} = 1/\mu(\epsilon) \leq C^4$, for small $\epsilon$. Focusing on the main term of the sample complexity, we have

$$O\left(\sum_{s=1}^{\log_2(1/\Delta_{\min})} 2^s \log(1/\delta)\right) \leq O\left(\frac{1}{\epsilon^2} \log(1/\delta)\right).$$

Hence ignoring burn-in or the additional samples we take in each round to guarantee the confidence interval, the total sample complexity would be $O\left(\frac{1}{\epsilon^2}\right)$. This is exponentially smaller than in Example 1 and demonstrates the power of an informative arm in reducing the sample complexity.

On the other hand, the burn-in phase, common to all logistic bandit algorithms based on the MLE (no regularization), may potentially take a number of samples exponential in $r$. This example demonstrates the need for further work on understanding the precise dependence of $\kappa$ in pure exploration. In the next section, we take a first stab at this by removing the additional samples needed to ensure the confidence interval is valid in each round.

### 4.4 Removing extra samples GLM-RAGE

In practice, Algorithm 6 suffers a large amount of samples to ensure that the confidence interval is valid in each round. In this section, we avoid this by using regularization and invoking a concentration bound from (Faury et al., 2020) for the regularized MLE.

**Estimator.** We quickly introduce the estimator and confidence intervals of (Faury et al., 2020). Define,

$$\hat{\theta}_t = \arg \min_{\theta} \|g_t(\theta) - g_t(\tilde{\theta}_t^{MLE})\|$$

where $\tilde{\theta}_t^{MLE}$ is the regularized MLE, with regularization parameter $\eta$, in the $t$-th round of the algorithm (see Supplementary for more details). Define

$$\gamma_t(\delta) = \sqrt{\delta}(S_* + 1/2) + \frac{2}{\sqrt{\eta}} \log(1/\delta) + \frac{2d}{\sqrt{\eta}} \log(2(1 + \frac{n_t}{d\eta}))$$

Then Lemma 11 of (Faury et al., 2020) shows that with probability greater than $1 - \delta$,

$$\theta^* \in \{\theta \in \mathbb{R}^d : \|\theta\| \leq S_*, \|\theta^* - \tilde{\theta}_t\|_{H(\theta)} \leq (2 + 4S_*)\gamma_t(\delta)\}.$$

We obtain a confidence interval for any $x \in \mathbb{R}^d$ as follows. By the Cauchy-Schwarz inequality, $\forall x \in \mathbb{R}^d$:

$$x^\top(\hat{\theta}_t - \theta^*) \leq 2(1 + 2S_*)\|x\|_{H(\theta^*)} \cdot \|\hat{\theta}_t - \theta^*\|_{H(\theta^*)} \leq 2(1 + 2S_*)\|x\|_{H(\theta^*)} \cdot \gamma_t(\delta).$$

Note that this confidence interval does not require a burn-in number of samples, and in addition, we do not have to pay for additional union bounds over the set of all $z$’s. However, we do pay for an extra factor of $d$ that is not present in Theorem 1. Algorithm 8 modifies Algorithm 6 to utilize the confidence interval of Equation (13). The Burn-in and Elimination phases are very similar, with the notable exception that we use $\hat{\theta}_t$ in each round rather than an estimate of $\hat{\theta}_{t-1}$.

**Theorem 5 (Sample Complexity).** Algorithm 8, returns $z^*$ with probability greater than $1 - 2\delta$ in a number of samples no more than

$$\sum_{r=1}^{\log_2(\frac{1}{\Delta_{\min}})} 2^{2r} \rho_r(2S_* + 1)^3[d\log\left(\frac{1}{\Delta_{\min}}\rho_t\right) + \log(1/\delta)]$$

$$+ \kappa_0^{-1}d(1 + \epsilon) \log(|\mathcal{X}|/\delta) + r(\epsilon) \log(1/\delta)$$

where $\mathcal{S}_t = \{z \in \mathcal{Z} : (z^* - z)^\top \theta_* \leq 2 \cdot 2^{-t}\}$ for some absolute constant $c$.

Note that we have an extra factor of $d$ compared to
RAGE-GLM due to the looseness from the concentration inequality, as \( \delta \to \infty \) the extra factor of \( d \) becomes negligible, and \( \log(1/\delta) \) term leads to an asymptotic sample complexity of Equation (9). However, for a fixed value of \( \delta \), the sample complexity potentially suffers a multiplicative factor of \( d \) and a factor of \( \|\theta^*\|^2 \) compared to the bound in Theorem 3. As discussed previously, in each round the number of samples needed to use the confidence bound scales with \( \gamma(d) = O(d^2) \) and in practice these samples can dominate the sample complexity. Hence, even though the bound in Theorem 3 may be better for certain problem instances, Algorithm 8, could potentially perform better even though we are paying by a factor of \( d \) multiplicatively.

**Remark.** Employing the confidence intervals from (Faury et al., 2020) could allow us to recycle samples between rounds and make the algorithm anytime. We do this in the supplementary.

## 5 Future work

Our confidence bound utilizes self-concordance and local analysis to improve upon the existing state of the art results for the logistic MLE. We remove a direct dependence on \( \kappa^{-1} \) in the confidence width and relax the requirement on the minimum sample size for the bound to be valid. However, there are improvements to be made. We would like to reduce the dependence on the minimum sample requirement stated w.r.t. \( \xi^2 \) from \( O(d^2 + \ln(1/\delta)) \) to \( O(d + \ln(1/\delta)) \). Furthermore, an extension to regularized MLEs is an interesting direction. This includes understanding whether regularization can significantly alleviate the minimum sample requirement without inflating the confidence width.

For contextual bandits, the fact that SupLogistic (and its ancestors like (Auer, 2002)) have to maintain independent buckets and cannot share the samples across buckets is a major road block to developing practical algorithms. It would be interesting to develop new algorithms that do not waste samples without increasing the regret bound. Foster and Rakhlin (2020) has proposed such an algorithm but its dependence on the number of arms is sub-optimal.

Last but not least, pure exploration for linear logistic models is largely under-explored, although its applications are abundant. Exploiting the local nature of the logistic loss and closely working with nonuniform variances that naturally arise from the model is crucial in sample-efficient design of experiments. While our bound improves upon existing bounds, we do not achieve the asymptotic lower bound. Furthermore, we believe finding practical algorithms that works well, and do not suffer a large number of burn-in samples, in the finite-time regime is important even at the price of not achieving optimality.

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F Proof of Theorem 1

We state the full version of our concentration result.

Theorem 6. Let $\delta \leq e^{-1}$. Fix $x \in \mathbb{R}^d$ such that $\|x\| \leq 1$. Define $= (10(d + \sqrt{\ln(12/\delta)})^2$. Let $\hat{\theta}$ be the solution
of Eq. (1) in the main text. Suppose := \max_{s\in[t]} \|x_s\|_{H_t(\theta^*)}^2 \leq \frac{1}{\gamma(\delta)}$. Then,
\[\mathbb{P}\left( |x^\top(\hat{\theta}_t - \theta^*)| \leq 4.2 \cdot \|x\|_{H_t(\theta^*)} \sqrt{\ln(12/\delta)} \right),\]
\[\frac{1}{\sqrt{2}} \|x\|_{(H_t(\theta^*))^{-1}} \leq \|x\|_{(H_t(\hat{\theta}_t))^{-1}} \leq \sqrt{2} \|x\|_{(H_t(\theta^*))^{-1}} \geq 1 - \delta.\]
which implies the following empirical variance bound:
\[\mathbb{P}\left( |x^\top(\hat{\theta}_t - \theta^*)| \leq 6 \cdot \|x\|_{H_t(\hat{\theta}_t)^{-1}} \sqrt{\ln(12/\delta)} \right) \geq 1 - \delta.\]

To improve the concentration inequality from (Li et al., 2017), we follow their analysis closely but exploit the variance term whenever possible.

We define the following:

- Let \(H_t(\theta^*) = \sum_{s=1}^t \mu(x_s^\top \theta^*)x_s x_s^\top.\)
- Define \(\eta_s := \sum_{s=1}^t \eta_s x_s.\)
- Let \(\alpha(x, \theta_1, \theta_2) = \frac{u(x^\top \theta_1) - \mu(x^\top \theta_2)}{x^\top (\theta_1 - \theta_2)}\). We use the shorthand := \(\alpha(x_s, \hat{\theta}_t, \theta^*).\)
- Let \(\zeta_t = \sum_s (\mu(x_s^\top \hat{\theta}_t) - \mu(x_s^\top \theta^*)x_s x_s^\top(\hat{\theta}_t - \theta^*) = G(\hat{\theta}_t - \theta^*)\).

Define := \(\sum_{s=1}^t \mu(x_s^\top \theta)x_s.\) The following identity is well-known (e.g., (Filippi et al., 2010, Proposition 1)):
\[\|\hat{\theta}_t - \theta^*\|_{G} = \|g_t(\hat{\theta}_t) - g_t(\theta^*)\|_{G^{-1}.\}
\]

Let \(G = H.\)

First, we assume the following event:
\[:= \left\{ \exists Q \in (0,1) : \forall s \in [t], \left| \frac{\alpha_s(\hat{\theta}_t, \theta^*) - \tilde{\mu}(x_s^\top \theta^*)}{\tilde{\mu}(x_s^\top \theta^*)} \leq Q \right\},\]
which we will show is true later under suitable stochastic events.

The main decomposition: We use the following decomposition based on (14) and tackle those two terms separately.
\[|x^\top(\hat{\theta}_t - \theta^*)| = |x^\top G^{-1} \zeta_t| = |x^\top (H + E)^{-1} \zeta_t| = |x^\top H^{-1} \zeta_t| \leq |x^\top H^{-1} \zeta_t| + |x^\top H^{-1} E(H + E)^{-1} \zeta_t|\]

We bound the two terms separately.

**Term 1:** \(|x^\top H^{-1} \zeta_t| = \left| \sum_s \langle x, H^{-1} x_s \rangle \eta_s \right|

Note that \(H^{-1}\) is deterministic (unlike \(G^{-1}\)) conditioning on \(\{x_1, \ldots, x_t\}\), so we can apply the standard argument for the concentration inequality. With the following Bernstein’s inequality in mind, we assume the event \(E_1\) defined below.

**Lemma 2.** Let \(\delta \leq e^{-1}\) and define
\[:= \left\{ |x^\top H^{-1} \zeta_t| \leq 2\|x\|_{H^{-1}} \sqrt{\ln(2/\delta) + \|x\|_{H^{-1}} \xi_t \ln(2/\delta)} \right\}.\]
Then, \(\mathbb{P}(E_1) \geq 1 - \delta.\)

**Proof.** The proof can be found in Section F.3.

**Term 2:** \(|x^\top H^{-1} E(H + E)^{-1} \zeta_t|\)
We have
\[ |x^\top H^{-1} E(H + E)^{-1} z_t| \leq \|x\|_{H^{-1}} \|H^{-1/2} E(H + E)^{-1} H^{1/2}\| z_t \|_{H^{-1}}. \]
In Li et al. (2017), derivations between Eq. (24) and Eq. (25) therein show that, if \( \|H^{-1/2} E H^{-1/2}\| < 1 \),
\[ \|H^{-1/2} E(H + E)^{-1} H^{1/2}\| \leq \frac{\|H^{-1/2} E H^{-1/2}\|}{1 - \|H^{-1/2} E H^{-1/2}\|}. \]
Let us study the term \( \|H^{-1/2} E H^{-1/2}\| \). For a matrix \( A \), we have
\[ \|A\| = \max\{ \max_{x: \|x\| \leq 1} x^\top A x, \max_{x: \|x\| \leq 1} x^\top (-A)x \}. \]
With this, we need to study both \( x^\top H^{-1/2} E H^{-1/2} x \) and \( x^\top H^{-1/2} (-E) H^{-1/2} x \). Let \( x \) satisfy \( \|x\| \leq 1 \).
\[ \max\{x^\top H^{-1/2} E H^{-1/2} x, \ x^\top H^{-1/2} (-E) H^{-1/2} x\} \]
\[ \leq x^\top H^{-1/2} \left( \sum_s |\alpha_s(\hat{\theta}_t, \theta^*) - \hat{\mu}(x^s, \theta^*)| x_s x^\top \right) H^{-1/2} x \]
\[ \leq x^\top H^{-1/2} \left( \sum_s Q |\hat{\mu}(x^s, \theta^*)| x_s x^\top \right) H^{-1/2} x \]
\[ = Q \|x\| \leq Q. \]
The event \( \mathcal{E}_0 \) implies that \( Q < 1 \). Then, we have
\[ \|H^{-1/2} E(H + E)^{-1} H^{1/2}\| \leq \frac{\|H^{-1/2} E H^{-1/2}\|}{1 - \|H^{-1/2} E H^{-1/2}\|} \leq \frac{Q}{1 - Q}. \]
Therefore,
\[ |x^\top H^{-1} E(H + E)^{-1} z_t| \leq \frac{Q}{1 - Q} \|x\|_{H^{-1}}, \|z_t\|_{H^{-1}}. \]
To bound \( \|z_t\|_{H^{-1}} \), we will rely on the following concentration result via covering argument, which is based on the proof of Li et al. (2017, Lemma 7). The difference is that we use the Bernstein’s inequality instead of Hoeffding’s.

**Lemma 3.** Recall \( \xi_t = \max_{s \in [t]} \|x_s\|_{H^{-1}} \). Let \( \delta \leq e^{-1} \). Define
\[ \mathcal{E}_2 := \left\{ \sum_{s} \eta_s x_s \right\} \leq \sqrt{d + \ln(6/\delta)} + \xi_t(d + \ln(6/\delta)) =: \right\}. \]
Then, \( P(\mathcal{E}_2) \geq 1 - \delta. \)

**Proof.** See Section F.3.

Let us assume \( \mathcal{E}_2 \). To summarize, assuming (16) and \( \mathcal{E}_1 \cap \mathcal{E}_2 \), we have
\[ |x^\top (\hat{\theta}_t - \theta^*)| \]
\[ \leq 2 \|x\|_{H^{-1}} \sqrt{\ln(2/\delta)} + \|x\|_{H^{-1}} \xi_t \ln(2/\delta) + \frac{Q}{1 - Q} \|x\|_{H^{-1}} \left( 2 \sqrt{d + \ln(6/\delta)} + \xi_t(d + \ln(6/\delta)) \right). \]
(17)
Note that we have \( d \) on the RHS, which is not desirable.

It remains to figure out a sufficient condition on \( \xi_t \) for ensuring \( \mathcal{E}_0 \) is true and controlling \( Q/(1 - Q) \) to remove the dependence on \( d \), and also show \( \frac{1}{\sqrt{d}} \|x\|_{(H_t(\theta^*))^{-1}} \leq \|x\|_{(H_t(\theta^*))^{-1}} \leq \sqrt{2} \|x\|_{(H_t(\theta^*))^{-1}} \).

**F.1 Finding a Sufficient Condition for \( \mathcal{E}_0 \)**

Assume \( \mathcal{E}_2 \). The goal is to show that we can satisfy \( \mathcal{E}_0 \) by finding some small enough \( Q \). This will allow us to control the term involving \( d \).
Let \( \max_{s \leq t} |x_s^T(\hat{\theta}_t - \theta^*)| \). One can show that the self concordance control lemma (Faury et al., 2020, Lemma 9) implies the following, which we use to motivate our choice \( Q \).

\[
\left| \frac{\alpha_s(\hat{\theta}_t, \theta^*) - \hat{\mu}(x_s^T \theta^*)}{\hat{\mu}(x_s^T \theta^*)} \right| \leq \max \left\{ \frac{D}{1 + D}, \frac{e^D - 1 - D}{D} \right\} =: .
\]

Note that the max crosses over at \( D \approx 0.6130 \), and if \( D, Q \leq 1.7932 \) then \( Q \leq D \).

Examining the order of (17), one might aim to control \( \frac{Q}{1 - Q} \leq 1/\sqrt{d} \). However, in our attempt, this led to a condition where \( \xi_t \) is at most \( \Theta\left(\frac{1}{\sqrt{d} + \ln(1/\delta)}\right) \), which has a factor \( d \) for \( \ln(1/\delta) \). This dependence is suboptimal when \( \delta \to 0 \). To get around this issue, we aim to control

\[
\frac{Q}{1 - Q} \leq \frac{\sqrt{d} + \ln(6/\delta)}{d + \sqrt{\ln(6/\delta)}} =: .
\]

where the RHS goes to 1 as \( \delta \to 0 \) unlike the naive aim of \( 1/\sqrt{d} \) above. Note that, using \( d \geq 1 \) and \( \delta \leq e^{-1} \), we have \( A \leq 1 \). Our aim above is equivalent to \( Q \leq A/(1 + A) \). We claim that the following condition is sufficient to satisfy \( Q \leq A/(1 + A) \):

\[
D \leq \frac{1}{2} A .
\]

(18)

To see this, because \( D \leq \frac{1}{2} A \leq \frac{A}{1 + A} \leq 1 \leq 1.7932 \), we have \( Q \leq D \), which implies that \( Q \leq D \leq \frac{A}{1 + A} \). Therefore, we aim to find a sufficient condition for (18), which automatically implies \( \mathcal{E}_0 \) as well.

Using the self-concordance control lemma (Faury et al., 2020, Lemma 9), we can relate \( G \) and \( H \) as a function of \( D \). If \( A \) and \( B \) are matrices, then we use \( A \geq B \) to mean that \( A - B \) is positive semi-definite.

**Lemma 4.** Let \( D = \max_{s \leq t} |x_s^T(\hat{\theta}_t - \theta^*)| \).

\[
G \geq \frac{1}{1 + D} \cdot H
\]

where \( A \geq B \) means that \( A - B \) is positive semi-definite.

**Proof.** We first note that, by the self-concordance control lemma (Faury et al., 2020, Lemma 9),

\[
\alpha_s(\theta_1, \theta_2) \geq \frac{\hat{\mu}(x_s^T \theta_2)}{1 + |x_s^T(\hat{\theta}_1 - \theta_2)|} .
\]

Then,

\[
G = \sum_s \alpha_s(x_s, \theta_1, \theta_2)x_s x_s^T \geq \frac{1}{1 + \max_{s \leq t} |x_s^T(\hat{\theta}_1 - \theta_2)|} \sum_s \hat{\mu}(x_s^T \theta_2)x_s x_s^T .
\]

Notice that the sum on the RHS is \( H_{\ell}(\theta_2) \).

\[\square\]

First, we know that

\[
D = \max_s |x_s^T(\hat{\theta}_t - \theta^*)| \leq \xi_t \|\hat{\theta}_t - \theta^*\|_H .
\]

Here, the second factor is troublesome.

Recall the definition \( g_t = \sum_{s=1}^t \mu(x_s^T \theta)x_s \). Note that the optimality condition (14) tells us that \( z_t = g_t(\hat{\theta}_t) - g_t(\theta^*) \). We would want to relate \( \|\hat{\theta}_t - \theta^*\|_H \) to \( \|g_t(\hat{\theta}_t) - g_t(\theta^*)\|_{H^{-1}} = \|z_t\|_{H^{-1}} \), which can be controlled by Lemma 3. The key idea is that we can connect \( \|\hat{\theta}_t - \theta^*\|_H \) to \( \|\hat{\theta}_t - \theta^*\|_G = \|g_t(\hat{\theta}_t) - g_t(\theta^*)\|_{G^{-1}} \) (the equality is by (15)) then to \( \|g_t(\hat{\theta}_t) - g_t(\theta^*)\|_{H^{-1}} \).

Using Lemma 4,

\[
D^2 \leq \xi_t^2 \|\hat{\theta}_t - \theta^*\|_H^2 \leq \xi_t^2 (1 + D) \|\hat{\theta}_t - \theta^*\|_G^2
\]

\[
= \xi_t^2 (1 + D) \cdot \|g_t(\hat{\theta}_t) - g_t(\theta^*)\|_{G^{-1}}^2
\]

\[
\leq \xi_t^2 (1 + D)^2 \|g_t(\hat{\theta}_t) - g_t(\theta^*)\|_{H^{-1}}^2
\]

\[
\leq (1 + D)^2 \xi_t^2 \beta_t \quad \text{(by (14) and } \mathcal{E}_2 \text{)}
\]
$$\Rightarrow D \leq (1 + D)\xi_t \sqrt{B_t}$$

$$\Rightarrow D \leq \frac{\sqrt{B_t} \xi_t}{1 - \sqrt{B_t} \xi_t}.$$  

where the last line requires an assumption that $1 - \sqrt{B_t} \xi_t > 0$. In fact, let us assume $\sqrt{B_t} \xi_t \leq 3/4$, which we verify soon under a suitable condition. Then,

$$D \leq 4\sqrt{B_t} \xi_t.$$  

Recall that we wanted to find a sufficient condition to control the RHS above to be at most $\frac{1}{2}A$. For this, we make the following claim:

**Claim:**  
\[\xi_t \leq \frac{0.1}{d + \sqrt{B}} \Rightarrow 4\sqrt{B_t} \xi_t \leq \frac{1}{2}A.\]

We prove the claim by contradiction. Suppose $4\sqrt{B_t} \xi_t > \frac{1}{2}A$. Let $\xi := \ln(6/\delta)$. Then,

$$\xi_t > \frac{A}{8\sqrt{B_t}} = \frac{1}{8} \cdot \frac{A}{2\sqrt{d + B} + \xi_t \cdot (d + B)}.$$  

This is a quadratic inequality. Solving it, we have \(\xi > -\frac{1}{d + \sqrt{B}} \cdot 1 + \sqrt{1 + \frac{1}{2}\frac{\delta}{d + \sqrt{B}}} \geq \frac{0.1}{d + \sqrt{B}}\), which is a contradiction.

One can also check that if $\xi_t \leq \frac{0.1}{d + \sqrt{B}}$, then $\sqrt{B_t} \xi_t \leq 3/4$, which was previously promised to be verified. Thus, we can conclude that

$$\xi_t \leq \frac{0.1}{d + \sqrt{B}} \Rightarrow D \leq \frac{1}{2}A \Rightarrow \frac{Q}{1 - Q} \leq A.$$  

To summarize, we have shown that the condition on $\xi_t$ implies $\mathcal{E}_0$ under $\mathcal{E}_2$.

### F.2 Deriving the Final Form

Let us now assume $\mathcal{E}_1$ and work out the final form of the confidence interval, we first show that:

$$\frac{Q}{1 - Q} \sqrt{B} \leq \frac{\sqrt{d + B}}{d + \sqrt{B}} \cdot \left(2\sqrt{d + B} + \left(\frac{0.1}{d + \sqrt{B}}\right) \cdot (d + B)\right)$$

$$\leq \frac{d + B}{d + \sqrt{B}} + 0.1 \cdot \frac{\sqrt{d + B} \cdot d + B}{d + \sqrt{B}}$$

$$\leq 2\frac{d + B}{d\sqrt{B} + B} \cdot \sqrt{B} + 0.1 \cdot \sqrt{B}$$

$$\leq 2.1\sqrt{B}$$

where (a) is by $\frac{\sqrt{d + B}}{d + \sqrt{B}} \leq \frac{\sqrt{d + B}}{d + \sqrt{B}} \leq 1$ and $\frac{d + B}{d + \sqrt{B}} \leq \frac{d + \sqrt{B}}{d + \sqrt{B}} \leq \frac{\sqrt{d + B}}{d + \sqrt{B}} \leq \sqrt{B}$ and (b) for the same reason. Furthermore,

$$\|x\|_{H^{-1}, \xi_t, B} \leq \|x\|_{H^{-1}/0.1} \cdot \frac{1}{d + \sqrt{B}} \cdot \sqrt{B} \cdot \sqrt{B} \leq \|x\|_{H^{-1}/0.1\sqrt{B}}.$$  

With the two results above, we work out the final form of the confidence interval:

$$\|x^T (\hat{\theta}_t - \theta^*)\|$$

$$\leq 2\|x\|_{H^{-1}/0.1\sqrt{\ln(2/\delta)}} + \|x\|_{H^{-1}, \xi_t, \ln(2/\delta)} + \frac{Q}{1 - Q} \cdot \|x\|_{H^{-1}/\xi_t, \sqrt{B}}$$

$$\leq 4.1\|x\|_{H^{-1}/0.1\sqrt{\ln(2/\delta)}} + \|x\|_{H^{-1}, \xi_t, \ln(2/\delta)}$$

$$\leq 4.2\|x\|_{H^{-1}/0.1\sqrt{\ln(2/\delta)}}.$$  

Furthermore, the following lemma shows that the empirical variance is within a constant factor of the true variance. One can easily check that the condition therein is satisfied under $\mathcal{E}_2$ and Theorem 6’s assumption on $\xi_t$ because we just showed $D \leq \frac{1}{2}A \leq 1/2$ above.
Lemma 5. Suppose $D = \max_{x \in [t]} |x^\top_s (\hat{\theta}_t - \theta^*)| \leq 1$. Then, for all $x$,
\[ \frac{1}{\sqrt{2D + 1}} \|x\|_{(H_t(\theta^*))^{-1}} \leq \|x\|_{(H_t(\delta_t))^{-1}} \leq \sqrt{2D + 1} \|x\|_{(H_t(\theta^*))^{-1}} . \]

Proof. See Section F.3. \qed

To conclude the proof, we use Lemma 2 and 3 to have $\mathbb{P}(E_1 \cap E_2) \geq 1 - 2\delta$. Substituting $\delta \leftarrow \delta/2$ concludes the proof.

F.3 Proof of Auxiliary Results

Proof of Lemma 2. One can easily extend Faury et al. (2020, Lemma 7) to show the following: if a random variable $Z$ is centered and bounded ($|Z| \leq R$ for some $R > 0$) with variance $\sigma^2$, then we have $\mathbb{E}[\exp(\phi Z - \phi^2 \sigma^2)] \leq 1$ for any deterministic $|\phi| \leq 1/R$. Note that Faury et al. (2020, Lemma 7) is a special case of $R = 1$.

Note that scaling $Z$ by a constant $c$ would increase $R$ to $cR$ and $\sigma^2$ to $c^2 \sigma^2$. This observation leads to: for all $\phi$ with $|\phi| \leq \frac{1}{\max_{x \in [t]} |\langle x, H^{-1} x \rangle|}$,
\[ \mathbb{E} \left[ \exp \left\{ \phi \sum_{s=1}^{t} \langle x, H^{-1} x_s \rangle \eta_s - \phi^2 \sum_{s=1}^{t} \langle x, H^{-1} x_s \rangle^2 \sigma^2_s \right\} \right] \leq 1 . \]

where the equality is by the definition $H = \sum_s \sigma^2_s x_s x_s^\top$. Thus, by Markov’s ineq., we have, w.p. at least $1 - \delta$,
\[ x^\top H^{-1} z_t \leq \phi \|x\|_{H^{-1}}^2 + \frac{1}{\phi} \ln(1/\delta) . \]

Then, one can tune $\phi = \frac{\ln(1/\delta)}{\|x\|_{H^{-1}}^2} \wedge \frac{1}{\max_{x \in [t]} |\langle x, H^{-1} x \rangle|}$ to show that
\[ x^\top H^{-1} z_t \leq 2 \|x\|_{H^{-1}} \sqrt{\ln(1/\delta)} + \max_{s \in [t]} |\langle x, H^{-1} x_s \rangle| \ln(1/\delta) \]
\[ \leq 2 \|x\|_{H^{-1}} \sqrt{\ln(1/\delta)} + \|x\|_{H^{-1}} \xi_t \ln(1/\delta) . \]

Using $|\langle x, H^{-1} z_t \rangle| \leq \max\{ \langle x, H^{-1} z_t \rangle, - \langle x, H^{-1} z_t \rangle \}$, one can make the same argument for $- \langle x, H^{-1} z_t \rangle$, substitute $\delta$ with $\delta/2$, and then apply the union bound. \qed

Proof of Lemma 3. Let $B$ be the Euclidean ball of radius 1 and be a 1/2-cover of $B(1)$. It is well-known that one can find a cover $\tilde{B}(1)$ of cardinality $6^d$ (Pollard, 1990, Lemma 4.1). In this proof, we use the shortcut $H := H_1(\theta^*)$.

Note that $\|z_t\|_{H^{-1}} = \|H^{-1/2} z_t\|_2 = \sup_{a \in \tilde{B}(1)} \langle a, H^{-1/2} z_t \rangle$. Fix $x \in \mathbb{R}^d$. Let $\hat{x}$ be the closes point to $x$ in the cover $\tilde{B}(1)$. Then,
\[ \langle x, H^{-1/2} z_t \rangle = \langle \hat{x}, H^{-1/2} z_t \rangle + \langle x - \hat{x}, H^{-1/2} z_t \rangle \]
\[ = \langle \hat{x}, H^{-1/2} z_t \rangle + \|x - \hat{x}\| \langle \frac{x - \hat{x}}{\|x - \hat{x}\|}, H^{-1/2} z_t \rangle \]
\[ \leq \langle \hat{x}, H^{-1/2} z_t \rangle + \frac{1}{2} \sup_{a \in \tilde{B}(1)} \langle a, H^{-1/2} z_t \rangle \]
\[ = \langle \hat{x}, H^{-1/2} z_t \rangle + \frac{1}{2} \|z_t\|_{H^{-1}} . \]

Taking sup over $x \in B(1)$ on both sides, we have
\[ \|z_t\|_{H^{-1}} \leq \langle \hat{x}, H^{-1/2} z_t \rangle + \frac{1}{2} \|z_t\|_{H^{-1}} \implies \|z_t\|_{H^{-1}} \leq 2\langle \hat{x}, H^{-1/2} z_t \rangle . \]
This implies that
\[
\mathbb{P}(\|z_t\|_{H^{-1}} > t) \leq \mathbb{P}(2 \sup_{z \in \mathcal{B}(1)} \langle \hat{x}, H^{-1/2} z_t \rangle > t) \leq \sum_{z \in \mathcal{B}(1)} \mathbb{P}(\langle \hat{x}, H^{-1/2} z_t \rangle > t/2) .
\] (19)

It remains to bound \(\mathbb{P}(\langle \hat{x}, H^{-1/2} z_t \rangle > t/2)\) for any \(\hat{x}\) and then apply the union bound. Let \(\phi > 0\) and
\[
M_t = \exp \left( \phi \sum_s \langle \hat{x}, H^{-1/2} x_s \rangle \eta_s - \phi^2 \left( \sum_s \langle \hat{x}, H^{-1/2} x_s \rangle^2 \sigma_s^2 \right) \right).
\]
\[
= \exp \left( \phi \sum_s \langle \hat{x}, H^{-1/2} x_s \rangle \eta_s - \phi^2 \|\hat{x}\|^2 \right).
\]

Let \(M_0 = 1\) as a convention. One can show that \(M_t\) is supermartingale for all \(|\phi| \leq \frac{1}{\max_{z \in \mathcal{B}(1), x \in [t]} \langle \hat{x}, H^{-1/2} x_s \rangle}\).

In fact, for simplicity, we require a tighter condition on \(\phi\) which is \(|\phi| \leq \frac{1}{\max_{z \in \mathcal{B}} \|z\|_{H^{-1}}} = \frac{1}{\xi}\). Thus, w.p. at least \(1 - \delta\), we have
\[
\sum_s \langle \hat{x}, H^{-1/2} x_s \rangle \eta_s \leq \phi \|\hat{x}\|^2 + \frac{1}{\phi} \ln(1/\delta) .
\]

Applying \(\|\hat{x}\| \leq 1\) and the usual tuning of \(\phi = \sqrt{\ln(1/\delta)} \land \frac{1}{\xi}\) leads to the RHS being \(2\sqrt{\ln(1/\delta)} + \ln(1/\delta)\).

Replacing \(\delta\) above with \(\frac{\delta}{\xi^2}\) and taking the union bound at (19) conclude the proof. 

**Proof of Lemma 5.** Let \(\alpha(z_1, z_2) = \frac{\mu(z_1) - \mu(z_2)}{z_1 - z_2}\). Let \(s \in [t]\). Because \(\alpha(z_1, z_2) = \alpha(z_2, z_1)\), Faury et al. (2020, Lemma 9) implies
\[
\hat{\mu}(x^\top \theta^*) \frac{1 - \exp(-D)}{D} \leq \alpha(x^\top \theta^*, x^\top \hat{\theta}_t) \leq \hat{\mu}(x^\top \theta^*) \frac{\exp(D) - 1}{D} .
\]

and
\[
\hat{\mu}(x^\top \hat{\theta}_t) \frac{1 - \exp(-D)}{D} \leq \alpha(x^\top \theta^*, x^\top \hat{\theta}_t) \leq \hat{\mu}(x^\top \hat{\theta}_t) \frac{\exp(D) - 1}{D} .
\]

Then,
\[
\hat{\mu}(x^\top \hat{\theta}_t) \geq \frac{D}{\exp(D) - 1} \cdot \alpha(x^\top \theta^*, x^\top \hat{\theta}_t) \geq \frac{D}{\exp(D) - 1} \cdot \frac{1 - \exp(-D)}{D} \cdot \hat{\mu}(x^\top \theta^*) \geq \frac{1}{2D + 1} \hat{\mu}(x^\top \theta^*)
\]

where (a) is due to the following fact: using \(z \leq 1 \implies e^z \leq z^2 + z + 1\), we have
\[
\frac{D}{\exp(D) - 1} \frac{1 - \exp(-D)}{D} = \frac{D}{\exp(D) - 1} \frac{1}{e^D - 1} = \frac{1}{e^D} \geq \frac{1}{2D + 1} \geq \frac{1}{2D + 1} .
\]
This implies that \(H_t(\hat{\theta}_t) \geq \frac{1}{2D + 1} H_t(\theta^*)\). This concludes the proof of the second inequality. One can prove the other inequality similarly. 

**G Comments on Li et al. (2017)**

Li et al. (2017) collects the burn-in samples in a different way from our SupLogistic. They collect burn-in samples (denoted by \(\Phi\) here) for the first \(\tau = \sqrt{dT}\) rounds, and the buckets \(\Psi_1, \ldots, \Psi_S\) are empty at the beginning of time step \(\tau + 1\). Then, at time \(t > \tau\), when they compute the estimate \(\hat{\theta}^{(s)}_t\), they use both the samples from \(\Psi_s\) and \(\Phi\). However, we claim that this scheme invalidates the concentration inequality. We explain below how this happens with the help of Figure 1.

- At time \(\tau + 1\), we choose \(X_{\tau + 1}\) in \(s = 1\) with step (a).
- At time \(\tau + 2\), we pass \(s = 1\) and then choose \(X_{\tau + 2}\) in \(s = 2\) with step (a). Note that the set of arms that has survived \(s = 1\) and passed onto \(s = 2\) are dependent on \(\hat{\theta}^{(1)}_{\tau + 1}\) that is a function of \(y_1, \ldots, y_\tau\); see the orange thick line in Figure 1.
- At time \(\tau + 3\), we pass \(s = 1\), arrive at \(s = 2\) step (c), and we perform the arm rejections using \(\hat{\theta}^{(2)}_{\tau + 2}\). At this point, we cannot apply our concentration inequality in the analysis, as we describe below.

The main reason is that our concentration inequality requires that, for all \(t \in \Psi_2\), rewards \(y_t\) follows \(\text{Bernoulli}(\mu(x_t^\top \theta^*))\) independently, conditioning on \(\{X_t\}_{t \in \Psi_2}\). Since our reward model assumption was made
Figure 1: A diagram showing the dependency of the variables in SupCB-GLM of Li et al. (2017). The troublesome dependency is colored orange with thick lines. Note that we did not show all the dependencies here to avoid clutter. For example, $X_{\tau+1}$ depends on $X_1, \ldots, X_\tau$.

To check this, let $\mu \in \mathbb{R}^d$ denote both the PDF and PMF. Note that the inequality $\|\mu(X_1^T \theta^*) - \mu(X_3^T \theta^*)\|$ is not conditionally independent from $X_3$. Thus, the concentration inequality involving $\hat{\theta}_{\tau+2}^{(s)}$ to filter arms. Proceeds to $s=2$ step (c). At this point, there is another challenge in dealing with the confidence width that depends on $\theta_{\tau+2}^{(s)}$ due to our novel and tight variance-dependent concentration inequality.

### H Proofs for SupLogistic

Our proof closely follows the standard SupLinRel-type analysis (Auer, 2002; Li et al., 2017), but deviate from them by applying our new confidence interval, which provides a nontrivial challenge. Specifically, our confidence widths $\{w_{t,\alpha}^{(s)}\}$ now rely on the MLE estimates $\hat{\theta}_{t}^{(s)}$ yet the algorithm uses the widths using $\theta_{t}^{(s)}$ computed based on the bucket $\Phi$. Such a design was necessary (as far as we stick to the SupLinRel type) because using the confidence widths based on $\theta_{t}^{(s)}$ would introduce a dependency issue similar to what we described in Section G and prevent us from using the confidence bound.

While our regret bound improves upon the dependence on $\|\theta^*\|$ in the leading term, we believe it should also be possible to incorporate recent developments of SupLinRel-type algorithms by Li et al. (2019) to shave off some logarithmic factors. The focus our paper is, however, to show the impact of our novel confidence bounds.

We first define our notations for the proof.

- We define a shorthand $x_{t,\alpha_t}$ for the arm chosen at time step $t$.  
- Denote by the set of time steps at which the pulled arm $a_t$ was included to the bucket $s$ up to (and including) time $t$. In other words, $\Psi_s(t)$ is the variable $\Psi_s$ in the pseudocode of SupLogistic at the end of time step $t$.
- Define $= \sum_{u \in \Psi_s(t)} \hat{\mu}(X_u^T \theta)X_uX_u^T, \forall s \in [S]$, and $:= \sum_{u \in \Phi(\tau)} \hat{\mu}(X_u^T \theta)X_uX_u^T$. We remark that this definition depends on three variables: bucket index, time step, and the parameter $\theta$ for computing the variance $\hat{\mu}(X_u^T \theta)$. Note that the bucket $\Phi$ is never updated after the time step $\tau$ and, so we often use the notation $\Phi(\tau)$.

We present the proof by a bottom-up approach:
Lemma 6. Fix δ > 0. Consider SupLogistic with τ = \sqrt{dT} and \alpha = 4.2 \sqrt{\ln\left(\frac{12.2STK}{\delta}\right)\ln(S)}L. Let H_t^{(s+1)}(\theta^*) := H_t^s(\theta^*)'. Define the following event:

\begin{align*}
\mathcal{E}(t,a) := & \left\{ \forall t \in \{\tau + 1, \ldots, T\}, a \in [K], s \in [S], |x_{t,a} - x_{t,a}^\ast| \leq \alpha \|x_{t,a}\|_{H_t^{(s)}(\theta^*)}^{-1},
\frac{1}{\sqrt{2}} \|x_{t,a} \|_{H_t^{(s)}(\theta^*)}^{-1} \leq \|x\|_{H_t^{(s)}(\theta^*)}^{-1} \leq \sqrt{2} \|x\|_{H_t^{(s)}(\theta^*)}^{-1} \right\} \\
:= & \left\{ \forall s \in [S + 1], \sqrt{\lambda_{\min}(H_t^s(\theta^*))} \geq 10 \sqrt{d^2 + \ln(12.2STK/\delta)} \right\}.
\end{align*}

Then, there exists

T_0 = \Theta(\sqrt{d} \ln^4(Z)) \text{ where } Z = \frac{d^3}{\kappa^2} + \frac{\ln^2(K/\delta)}{d\kappa^2},

such that \forall T \geq T_0, \mathbb{P}(\mathcal{E}_{\text{mean}}, \mathcal{E}_{\text{diversity}}) \geq 1 - \delta. Note: both events rely on T since they rely on \tau that is a function of T.

Proof. To avoid clutter, let us fix s and drop the superscript from H_t^s(\theta^*) and use H_t(\theta^*). Note that each bucket has at least \left\lceil \frac{\tau - 1}{S + 1} \right\rceil samples. Since \lambda_{\min}(H_t(\theta^*)) \geq \kappa \lambda_{\min}(V_t) where = \sum_{u=1}^\kappa X_u X_u^\top, to ensure \mathcal{E}_{\text{diversity}}, it suffices to show that \lambda_{\min}(V_t) \geq 10 \frac{d^2 + \ln(12.2STK/\delta)}{\kappa} := \kappa. Recall our stochastic assumption on the context vectors x_{t,a}, the definition of \Sigma, and our assumption \lambda_{\min}(\Sigma) \geq \sigma_0^2. By Li et al. (2017, Proposition 1), there exists C_1 and C_2 such that if

\begin{align*}
\frac{\tau}{S + 1} \geq & \frac{\tau}{S + 1} - 1 = \frac{\sqrt{dT}}{S + 1} - 1 \geq \frac{C_1 \sqrt{d} + C_2 \sqrt{\ln(2STK/\delta)}\kappa}{\sigma_0^2} + \frac{2}{\sigma_0^2} \cdot F, \\
\geq & \frac{C_1 \sqrt{d} + C_2 \sqrt{\ln(2STK/\delta)}\kappa}{\lambda_{\min}(\Sigma)} + \frac{2}{\lambda_{\min}(\Sigma)} \cdot F,
\end{align*}

then \mathbb{P}(\lambda_{\min}(V) \geq 1) \geq 1 - \frac{4}{2STK}. It remains to find the smallest T that satisfies the inequality above. Omitting the dependence on \sigma_0^2, one can show that it suffices to find a sufficient condition for T such that

\begin{align*}
T \geq \left( C_3 \cdot \frac{d^3}{\kappa^2} + C_4 \cdot \frac{1}{d\kappa^2} \cdot \ln^2(K/\delta) \right) \ln^4(T)
\end{align*}

for some absolute constants C_3 and C_4. Using the fact that T < Z \ln^4(T) implies T < \Theta(\sqrt{d} \ln^4(Z)) along with a proof by contradiction, one can conclude that there exists T_0 = \Theta(\sqrt{d} \ln^4(Z)) such that \mathbb{P}(\mathcal{E}_{\text{diversity}}) \geq 1 - \frac{4}{2STK} \cdot (S + 1) via union bounds.

When \mathcal{E}_{\text{diversity}} is true, it is easy to see that the condition on \xi_t in Theorem 6 is satisfied if we set \delta \leftarrow \delta/(2STK). Thus, by the union bound,

\begin{align*}
\mathbb{P}(\mathcal{E}_{\text{mean}} \mid \mathcal{E}_{\text{diversity}}) \geq 1 - \frac{\delta}{2STK} \cdot STK.
\end{align*}
Note that \( \mathbb{P}(A \cup B) = \mathbb{P}(A \cap B) + \mathbb{P}(B) \leq \mathbb{P}(A \mid B) + \mathbb{P}(B) \). Setting \( A = \xi_{\text{mean}} \) and \( B = \xi_{\text{diversity}} \), we have
\[
\mathbb{P}(\xi_{\text{mean}} \cup \xi_{\text{diversity}}) \leq \frac{\delta}{2STK} \cdot STK + \frac{\delta}{2STK} (S + 1) \leq \delta
\]
where the last inequality is by \( K \geq 2 \).

**Lemma 7.** Take \( \tau \) and \( \alpha \) from Lemma 6. Suppose \( \mathcal{E}_{\text{mean}} \). Consider the time step \( t \geq \tau + 1 \). Let \( s_t \) be the while loop counter \( s \) at which the arm \( a_t \) is chosen. Let \( a_t^* = \arg \max_{a \in [K]} \mu(x_{t,a}^* \theta^*) \) be the best arm at time \( t \). Then, the best arm \( a_t^* \) survives through \( s_t \), i.e., \( a_t^* \in A_s \) for all \( s \leq s_t \). Furthermore, we have
\[
\mu(x_{t,a}^*, \theta^*) - \mu(x_{t,a}^*, \theta^*) \\
\leq \begin{cases} 
\hat{\mu}(x_{t,a}^*, \theta^*) 8 \cdot 2^{-s_t} + M \cdot 64 \cdot 2^{-2s_t} & \text{if } a_t \text{ is selected in step (a)} \\
n(x_{t,a}^*, \theta^*) 2T^{-1/2} + M \cdot 4 \cdot T^{-1} & \text{if } a_t \text{ is selected in step (b)} 
\end{cases}
\]

**Proof.** This proof is adapted from Li et al. (2017, Lemma 6) while keeping the dependence on the variance \( \hat{\mu}(x_{t,a}, \theta^*) \) to avoid introducing \( \kappa^{-1} \) explicitly. Fix \( t \). To avoid clutter, let us omit the subscript \( t \) from \( \{x_{t,a}, a_t^*\} \) and use \( \{x_a, a^*\} \) instead, respectively. We also drop the subscript \( t \) from \( H_{s+1}(\theta) \). Let us refer to the iteration index of the while loop as level. We use the notation \( m_{a^*(s)} \) to denote \( m_{a^*, t} \) at level \( s \).

We prove the first part of the lemma by induction. For the base case, we trivially have \( a^* \in A_1 \). Suppose that \( a^* \) has survived through the beginning of the \( s \)-th level (i.e., \( a^* \in A_s \)). We want to prove \( a^* \in A_{s+1} \). Since the algorithm proceeds to level \( s + 1 \), we know from step (a) at \( s \)-th level that, \( \forall a \in A_s \),
\[
|m_{a^*}^{(s)} - x_{a^*}^\top \theta^*| \leq \alpha \|x_a\|_{(H_{s+1}(\theta^*))^{-1}} \leq \alpha \sqrt{2}\|x_a\|_{(H_{s+1}(\theta^*))^{-1}} \leq 2^{-s}
\]
where both inequalities are due to \( \mathcal{E}_{\text{mean}} \). Specifically, it holds for \( a = a^* \) because \( a^* \in A_s \) by our induction step. Then the optimality of \( a^* \) implies that, \( \forall a \in A_s \),
\[
m_{a^*}^{(s)} \geq x_{a^*}^\top \theta^* - 2^{-s} \geq x_{a^*}^\top \theta^* - 2^{-s} \geq m_{a^*}^{(s)} - 2 \cdot 2^{-s}.
\]
Thus we have \( a^* \in A_{s+1} \) according to step (c).

For the second part of the lemma, suppose \( a_t \) is selected at level \( s_t \) in step (a). If \( s_t = 1 \), obviously the lemma holds because \( \mu(z) \in (0, 1), \forall z \). If \( s_t > 1 \), since we have proved \( a^* \in A_{s_t} \), step (a) at level \( s_t - 1 \) implies that for \( a \in \{a_t, a^*\} \),
\[
|m_{a}^{(s_t-1)} - x_{a}^\top \theta^*| \leq 2^{-s_t+1}.
\]
Step (c) at level \( s_t - 1 \) implies
\[
m_{a^*}^{(s_t-1)} - m_{a}^{(s_t-1)} \leq 2 \cdot 2^{-s_t+1}.
\]

Combining the two inequalities above, we get
\[
x_{a_t}^\top \theta^* \geq m_{a_t}^{(s_t-1)} - 2^{-s_t+1} \geq m_{a}^{(s_t-1)} - 3 \cdot 2^{-s_t+1} \geq x_{a^*}^\top \theta^* - 4 \cdot 2^{-s_t+1}.
\]
Recall that \( M = 1/4 \) is an upper bound on \( \hat{\mu}(z) \). The inequality above implies that
\[
\mu(x_{a_t}^\top \theta^*) - \mu(x_{a^*}, \theta^*) \leq \alpha (x_{a_t}^\top \theta^*, x_{a_t}^\top \theta^*) \cdot (x_{a^*} - x_{a_t})^\top \theta^* \\
\leq \left( \hat{\mu}(x_{a_t}^\top \theta^*) + M \cdot (x_{a_t} - x_{a_t})^\top \theta^* \right) \cdot (x_{a^*} - x_{a_t})^\top \theta^* \quad \text{(Taylor's theorem)} \\
= \hat{\mu}(x_{a_t}^\top \theta^*) (x_{a^*} - x_{a_t})^\top \theta^* + M \cdot ((x_{a^*} - x_{a_t})^\top \theta^*)^2 \\
\leq \hat{\mu}(x_{a_t}^\top \theta^*) 8 \cdot 2^{-s_t} + M \cdot 64 \cdot 2^{-2s_t}.
\]
When \( a_t \) is selected in step (b), since \( m_{a_t}^{(s_t)} \geq m_{a^*}^{(s_t)} \), we have
\[
x_{a_t}^\top \theta^* \geq m_{a_t}^{(s_t)} - 1/\sqrt{T} \geq m_{a^*}^{(s_t)} - 1/\sqrt{T} \geq x_{a^*}^\top \theta^* - 2/\sqrt{T}.
\]
With a similar reasoning as above, we have
\[
\mu(x_{a_t}^\top \theta^*) - \mu(x_{a^*}, \theta^*) \leq \hat{\mu}(x_{a_t}^\top \theta^*) 2T^{-1/2} + M \cdot 4 \cdot T^{-1}.
\]
\]
Lemma 8 (Regret per bucket). Assume $\mathcal{E}_{\text{mean}}$ and take $\alpha$ from Lemma 6. Then, $\forall s \in [S]$, 
\[
\sum_{t \in \Psi_s(T) \setminus \tau} \mu(x_{t,s}^\top \theta^*) - \mu(x_{t,s}^\top \theta^*) \leq 16\sqrt{L} \cdot \alpha \sqrt{\Psi_s(T)d \ln(LT/d)} + \frac{128M\alpha^2}{\kappa}d \ln(LT/d) .
\]

Proof. By Lemma 7 and the fact that $\mu(z) \in (0, 1), \forall z$, we have 
\[
\sum_{t \in \Psi_s(T)} \mu(x_{t,s}^\top \theta^*) - \mu(x_{t,s}^\top \theta^*) \leq \sum_{t \in \Psi_s(T)} 1 \wedge \left( \hat{\mu}(X_{t,s}^\top \theta^*) \cdot 8 \cdot 2^{-s} + 64M \cdot 2^{-2s} \right)
\]
\[
\leq \left( \sum_{t \in \Psi_s(T)} 1 \wedge \hat{\mu}(X_{t,s}^\top \theta^*) \cdot 8 \cdot 2^{-s} \right) + \left( \sum_{t \in \Psi_s(T)} 1 \wedge 64M \cdot 2^{-2s} \right)
\]
where the last inequality is true by $1 \wedge (a + b) \leq 1 \wedge a + 1 \wedge b$. For the first summation, we use $w_{t,a_t}^{(s)} > 2^{-s}$ due to step (a) of the algorithm:
\[
\left( \sum_{t \in \Psi_s(T)} 1 \wedge \hat{\mu}(X_{t,s}^\top \theta^*) \cdot 8 \cdot 2^{-s} \right) \leq \sum_{t \in \Psi_s(T)} 1 \wedge \hat{\mu}(X_{t,s}^\top \theta^*) \cdot 8 \cdot w_{t,a_t}^{(s)}
\]
\[
= \sum_{t \in \Psi_s(T)} 1 \wedge \hat{\mu}(X_{t,s}^\top \theta^*) \cdot 8 \cdot \alpha \sqrt{2} \|X_t\|_{(H_{t-1}^\top(\theta^*))}^{-1} \quad \text{(Def'ns of $w_{t,a_t}^{(s)}$)}
\]
\[
\leq \sum_{t \in \Psi_s(T)} 1 \wedge \hat{\mu}(X_{t,s}^\top \theta^*) \cdot 16 \cdot \alpha \|X_t\|_{(H_{t-1}^\top(\theta^*))}^{-1} \quad \text{($\mathcal{E}_{\text{mean}}$)}
\]
\[
\leq \sum_{t \in \Psi_s(T)} 1 \wedge 16\alpha \sqrt{L} \|\sqrt{\hat{\mu}(X_{t,s}^\top \theta^*)X_t\|_{(H_{t-1}^\top(\theta^*))}^{-1}} \leq \sqrt{L}
\]
\[
\leq \sqrt{\Psi_s(T)} \cdot d \ln (LT/d)
\]
\[
\leq \sqrt{\Psi_s(T)} \cdot \sqrt{\Psi_s(T)} \cdot d \ln (LT/d)
\]
where (a) by the Cauchy-Schwarz inequality and (b) by Lemma 9, with the fact that $(16\alpha \sqrt{L})^2 \geq \frac{1}{2}$, and $\lambda_{\text{min}}(H_{t-1}(\theta^*)) \geq 1$.

The second summation follows a similar derivation:
\[
\sum_{t \in \Psi_s(T)} 1 \wedge 64M \cdot 2^{-2s} \leq \sum_{t \in \Psi_s(T)} 1 \wedge 64M \cdot \alpha^2 \|X_t\|_{(H_{t-1}(\theta^*))}^{-1}
\]
\[
\leq \sum_{t \in \Psi_s(T)} 1 \wedge 64M \cdot \alpha^2 \frac{\hat{\mu}(X_{t,s}^\top \theta^*)}{\hat{\mu}(X_{t,s}^\top \theta^*)} \|X_t\|_{(H_{t-1}(\theta^*))}^{-1}
\]
\[
\leq \sum_{t \in \Psi_s(T)} 1 \wedge \frac{64M\alpha^2}{\kappa} \cdot 2 \cdot \left| \sqrt{\hat{\mu}(X_{t,s}^\top \theta^*)X_t\|_{(H_{t-1}(\theta^*))}^{-1}} \right|
\]
\[
\leq \frac{128M\alpha^2}{\kappa}d \ln(LT/d)
\]
where (a) is by Lemma 9 and $\frac{128M\alpha^2}{\kappa} \geq 128M^2 \alpha^2 / L \geq 1/2$, and $\lambda_{\text{min}}(H_{t-1}(\theta^*)) \geq 1$.

\[\Box\]

Theorem 7 (Regret of SupLogistic). Consider SupLogistic with $\tau$, $\alpha$, and $T_0$ from Lemma 6. Then, if $T \geq T_0$, then
\[
R_T \leq 10\alpha \sqrt{dT \ln(T/d) \log_2(T)} + O \left( \frac{\alpha^2}{\kappa} d \cdot \ln(T/d) \cdot \ln T \right) .
\]

Proof. Recall that $\Psi_0$ contains the time step indices at which the choice $a_t$ happened in step (b). Recall that
we set \(\tau = \sqrt{dT}\). Let:= \(\mu(x_{t,s}^T, \theta^*) - \mu(x_{t,\theta_0}^T, \theta^*)\). Then,

\[
R_T = \sum_{t=1}^{\tau} \Delta_t + \sum_{t=\tau+1}^{T} \Delta_t \\
\leq \sqrt{dT} + \sum_{t\in \Psi_0(T)} + \sum_{s=1}^{S} \sum_{t\in \Psi_s(T)\setminus[t]} \Delta_t .
\]

For the first term, using Lemma 7,

\[
\sum_{t\in \Psi_0(T)} \Delta_t \leq T \cdot \left(2\mu(X_t^T, \theta^*) \cdot \frac{2M}{T} + \frac{4M}{T} \right) \leq 2L \sqrt{T} + 4M .
\]

For the second term, using Lemma 8, using the Cauchy-Schwarz inequality,

\[
\sum_{s=1}^{S} \sum_{t\in \Psi_s(T)\setminus[t]} \Delta_t \leq \sum_{s=1}^{S} \left(16\alpha \sqrt{L|\Psi_s(T)|d\ln(LT/d)} + \frac{128M\alpha^2}{\kappa} d\ln(LT/d) \right) \\
\leq 16\alpha \sqrt{Ld\ln(LT/d)} \left( \sum_{s=1}^{S} |\Psi_s(T)| + S \cdot \frac{128M\alpha^2}{\kappa} d\ln(LT/d) \right) \\
\leq 16\alpha \sqrt{Ld\ln(LT/d)} \cdot \sqrt{T \log_2(T) + \log_2(T) \cdot \frac{128M\alpha^2}{\kappa} d\ln(LT/d) .
\]

Using \(L = 1/4\), the terms involving \(\sqrt{T}\) is: \(\sqrt{dT} + 2L \sqrt{T} + 16\alpha \sqrt{LdT\ln(LT/d)\log_2(T)} \leq 10\alpha \sqrt{dT\ln(T/d)\log_2(T)} .\) This concludes the proof.

\[ \square \]

### H.1 Auxiliary Results

**Lemma 9** (Elliptical potential). Let \(s \in [S]\) and \(F > 0\). Then,

\[
\sum_{t\in \Psi_s(T)} \min_{t\in \Psi_s(T)} \left\{ 1, F \parallel \sqrt[2]{\mu(X_{t,s}^T, \theta^*)} X_t \parallel_{(H^*(\Psi_{s(t-1)}))^{-1}}^2 \right\} \leq (2F \lor 1) \cdot d\ln \left( \frac{L|\Psi_s(T)|}{d\mu_{\min}(H^*(\Psi_s(T)))} \right) .
\]

**Proof.** Using Lemma 3 of Jun et al. (2017), we have that \(\forall q, x > 0, \min\{q, x\} \leq \max\{2, q\} \ln(1 + x)\). Thus,

\[
\min \left\{ 1, F \parallel \sqrt[2]{\mu(X_{t,s}^T, \theta^*)} X_t \parallel_{(H^*(\Psi_{s(t-1)}))^{-1}}^2 \right\} = F \min \left\{ 1, \parallel \sqrt[2]{\mu(X_{t,s}^T, \theta^*)} X_t \parallel_{(H^*(\Psi_{s(t-1)}))^{-1}}^2 \right\} \\
\leq F \cdot \max \left\{ 2, \frac{1}{F} \right\} \ln \left( 1 + \parallel \sqrt[2]{\mu(X_{t,s}^T, \theta^*)} X_t \parallel_{(H^*(\Psi_{s(t-1))^{-1}}) \right)
\]

where the last inequality is by \(\max\{2, 1/F\} = 2\). Then,

\[
\sum_{t\in \Psi_s(T)} \min_{t\in \Psi_s(T)} \left\{ 1, F \parallel \sqrt[2]{\mu(X_{t,s}^T, \theta^*)} X_t \parallel_{(H^*(\Psi_{s(t-1)}))^{-1}}^2 \right\} \leq (2F \lor 1) \sum_{t\in \Psi_s(T)} \ln \left( 1 + \parallel \sqrt[2]{\mu(X_{t,s}^T, \theta^*)} X_t \parallel_{(H^*(\Psi_{s(t-1))^{-1}}) \right)
\]

\[
= (2F \lor 1) \ln \left( \frac{|\Psi_s(T)|}{|H^*(\Psi_s(T))|} \right) \\
\leq (2F \lor 1) \cdot d\ln \left( \frac{L|\Psi_s(T)|}{d\mu_{\min}(H^*(\Psi_s(T)))} \right)
\]

where the last inequality is by the arithmetic-geometric mean inequality. \[ \square \]

### I Proofs for GLM-Rage

**Burn-In Results**

**Lemma 10.** With probability greater than \(1 - \delta\), for all \(\lambda \in \Delta_X\), \(\frac{1}{2}H(\lambda, \theta^*) \leq H(\lambda, \hat{\theta}_0) \leq 4H(\lambda, \theta^*)\)
With this, we apply Lemma 20 to conclude the proof.

Firstly note that,

\[ H(\lambda_0, \theta^*) \geq \sum_{x \in \mathcal{X}} \lambda_{0,x} \kappa_0 x x^\top \geq \kappa_0 A(\lambda_0) \]

So for any \( x \in \mathcal{X} \),

\[ \|x\|^2_{H(\lambda_0, \theta^*)^{-1}} \leq \kappa_0^{-1} \|x\|^2_{A(\lambda_0)^{-1}} \]

Thus at the end of the burn-in phase,

\[ \max_{x \in \mathcal{X}} \|x\|^2_{H_{\kappa_0^{-1}}(\theta^*)} \leq \frac{(1 + \epsilon) n_0}{n_0} \max_{x \in \mathcal{X}} \|x\|^2_{H_{\kappa_0^{-1}}(\lambda_0, \theta^*)} \]

\[ \leq \frac{(1 + \epsilon) \kappa_0^{-1} \max_{x \in \mathcal{X}} \|x\|^2_{A^{-1}(\lambda_0)}}{n_0} \]

\[ = \frac{1}{\gamma(d) \log(12|\mathcal{X}|/\delta)} \]

where we have employed the Kiefer-Wolfowitz theorem (Lattimore and Szepesvári, 2020, Theorem 21.1), which states that \( \min_{\lambda \in \Delta_{\mathcal{X}}} \max_{x \in \mathcal{X}} \|x\|^2_{A(\lambda)^{-1}} = d \). In particular this implies using Theorem 6,

\[ |x^\top (\theta^* - \hat{\theta}_0)| \leq 4.2 \sqrt{\log(12|\mathcal{X}|/\delta) \|x\|^2_{(H_{\kappa_0}(\theta^*))^{-1}}} \]

\[ \leq 4.2 \sqrt{\frac{\log(12|\mathcal{X}|/\delta)}{(10(d + \sqrt{\log(12/\delta)}))^2 \log(12|\mathcal{X}|/\delta)}} \]

\[ \leq 1 \]

With this, we apply Lemma 20 to conclude the proof.

---

**Algorithm 5 BurnIn**

**Input:** \( \mathcal{X}, \kappa_0 \)

1. initialize \( \lambda_0 = \arg \min_{\lambda \in \Delta_{\mathcal{X}}} \max_{x \in \mathcal{X}} \|x\|^2_{A(\lambda)^{-1}} \)

2. initialize \( n_0 = \frac{(1 + \epsilon) d^2 (\log(12|\mathcal{X}|/\delta))}{\delta_{\kappa_0}} \)

3. \( x_1, \cdots, x_{n_0} \leftarrow \text{round}(n_0, \hat{\lambda}_0, \epsilon) \)

4. Observe associated rewards \( y_1, \cdots, y_{n_0} \)

5. return \( \text{MLE } \hat{\theta}_0 \) \( \triangleright \) Use Eq (22)

---

**Algorithm 6 RAGE-GLM**

**Input:** \( \epsilon, \delta, \mathcal{X}, \mathcal{Z}, \kappa_0 \), effective rounding procedure \( \text{round}(n, \epsilon, \lambda) \)

1. initialize \( t = 1, \mathcal{Z}_1 = \mathcal{Z}, r(\epsilon) = d^2/\epsilon \)

2. \( \theta_0 \leftarrow \text{BurnIn}(\mathcal{X}, \kappa_0) \) \( \triangleright \) Burn-in phase

3. while \( |\mathcal{Z}_t| > 1 \) do

4. \( f(\lambda) := \max \left[ \gamma(d) \max_{x \in \mathcal{X}} \|x\|^2_{H(\lambda, \hat{\theta}_{t-1})^{-1}}, 2^{2t+2} \cdot (4.2)^2 \max_{z,z^\prime \in \mathcal{Z}_t} \|z - z^\prime\|^2_{H(\lambda, \hat{\theta}_{t-1})^{-1}} \right] \)

5. \( \lambda_t = \arg \min_{\lambda \in \Delta_{\mathcal{X}}} f(\lambda) \)

6. \( r_t = f(\lambda_t) \)

7. \( n_t = \max\{1 + \epsilon r_t \log(12 \max\{1, t\}) d^2/\delta, r(\epsilon)\} \)

8. \( x_1, \cdots, x_{n_t} \leftarrow \text{round}(n, \epsilon, \lambda) \)

9. Observe rewards \( y_1, \cdots, y_{n_t} \in \{0, 1\} \)

10. Compute the unregularized MLE \( \hat{\theta}_t \)

11. \( \hat{z}_t = \arg \max_{z \in \mathcal{Z}_t, \hat{\theta}_t^\top z} \)

12. \( \mathcal{Z}_{t+1} = \mathcal{Z}_t \setminus \{z \in \mathcal{Z}_t : \hat{\theta}_t^\top (\hat{z}_t - z) \geq 2^{-t}\} \)

13. \( t \leftarrow t + 1 \)

14. end while

15. return \( \hat{z}_t \)
Define the events

\[ \mathcal{R}_t = \{ \frac{1}{4} H(\lambda, \theta^*) \leq H(\lambda, \hat{\theta}_t) \leq 4H(\lambda, \theta^*), \forall \lambda \in \Delta_X \}, t \geq 0 \]

and

\[ \mathcal{E}_{2,t} = \{ \forall z \in \mathcal{Z}_t, |\langle z^* - z, \hat{\theta}_t - \theta^* \rangle| \leq 2^{-t}, t \geq 1 \}. \]

In addition, define \( \mathcal{E}_1 = \cap_{t=0}^{\infty} \mathcal{R}_t \) and \( \mathcal{E}_2 = \cap_{t=1}^{\infty} \mathcal{E}_{2,t} \).

**Lemma 11** (Closeness of \( \theta_t \)). We have that \( \mathbb{P}(\mathcal{R}_t | \mathcal{R}_{t-1}, \cdots, \mathcal{R}_0) \geq 1 - 2\delta \), i.e. for all \( t \geq 1 \), \( \frac{1}{4} H(\lambda_t, \theta^*) \leq H(\lambda_t, \hat{\theta}_{t-1}) \leq 4H(\lambda_t, \theta^*) \)

**Proof.** We proceed by induction. The base case of \( t = 0 \), is handled by Lemma 10 above. Assume that the event \( \mathcal{R}_{t-1} \) holds. On this event, for \( t > 1 \), we first verify that \( \max_{x \in \mathcal{X}} \| x \|^2_{H_t(\theta^*)^{-1}} \leq 1/\gamma(d) \)

\[
\max_{x \in \mathcal{X}} \| x \|^2_{H_t(\theta^*)^{-1}} \leq \frac{(1 + \epsilon)}{n_t} \max_{x \in \mathcal{X}} \| x \|^2_{H_t(\lambda, \theta^*)^{-1}} \leq 4\frac{(1 + \epsilon)}{n_t} \max_{x \in \mathcal{X}} \| x \|^2_{H_t(\lambda_t, \theta^*)^{-1}} \leq \frac{4}{n_t} \gamma(d) \log(12t^2|\mathcal{X}|/\delta)
\]

Thus with probability greater than \( 1 - \delta/(t^2|\mathcal{X}|) \) conditioned on \( \mathcal{R}_{t-1} \)

\[
|\langle x^\top (\theta^* - \hat{\theta}_{t-1}) \rangle| \leq 4.2 \sqrt{(1 + \epsilon) \log(12t^2|\mathcal{X}|/\delta)} \| x \|^2_{H_t(\theta^*)^{-1}} \leq 4.2 \sqrt{(1 + \epsilon) \log(12t^2|\mathcal{X}|/\delta)} \| x \|^2_{H_t(\lambda_t, \theta^*)^{-1}} \leq 4.2 \sqrt{\frac{4(1 + \epsilon) \log(12t^2|\mathcal{X}|/\delta)}{4.2^2(1 + \epsilon) \gamma(d) \log(12t^2|\mathcal{X}|/\delta)}} \leq 1
\]

Then, union bounding over \( \mathcal{X} \) gives that conditioned on \( \mathcal{R}_{t-1} \), we have the event \( \cup_{x \in \mathcal{X}} \{ |\langle x^\top (\hat{\theta}_{t-1} - \theta^*) \rangle| \leq 1 \} \) is true with probability greater than \( 1 - \delta/t^2 \). In particular, now applying Lemma 20 proves the claim.

**Lemma 12** (Concentration). In round \( t \), if we take \( n_t \) samples as specified in the algorithm, then

\[ |\langle z - z \rangle^\top (\hat{\theta}_t - \theta^*)| \leq 2^{-t} \]

for all \( z \in \mathcal{Z}_t \) with probability greater than \( 1 - \frac{\delta}{t^2} \) given \( \mathcal{R}_s, \mathcal{E}_{2,s} \}_{s=1}^{t-1} \cap \mathcal{R}_0 \), or in other words \( \mathbb{P} (\mathcal{E}_{2,t} | \mathcal{R}_s, \mathcal{E}_{2,s} )_{s=1}^{t-1} \cap \mathcal{R}_0 \) \( \geq 1 - \frac{\delta}{t^2} \)

**Proof.** In the previous lemma we showed that conditioned on \( \mathcal{R}_s, \mathcal{E}_{2,s} )_{s=1}^{t-1} \max_{x \in \mathcal{X}} \| x \|^2_{H_t(\theta^*)^{-1}} \leq 1/\gamma(d) \) .

Given \( \mathcal{Z}_t \) (a random set), we can apply Theorem 6, to calculate for any \( z \in \mathcal{Z}_t \)

\[
|\langle z - z \rangle^\top (\hat{\theta}_t - \theta^*)| \leq 4.2 \sqrt{|\mathcal{Z}| \log(12t^2|\mathcal{Z}|^2/\delta))} \leq 4.2 \sqrt{\frac{4(1 + \epsilon) |\mathcal{Z}| \log(12t^2|\mathcal{Z}|^2/\delta)}{n_t}}
\]
Proof. Firstly note for any set of events \( A_t \),
\[
\mathbb{P}(A_t \cup \bigcup_{j=t}^{\infty} A_j) \leq \sum_{t=1}^{\infty} \mathbb{P}(A_t \cup (\bigcup_{j=t}^{\infty} A_j)) \leq \sum_{t=1}^{\infty} \mathbb{P}(A_t | (\bigcup_{j < t} A_j)) \cdot
\]

Then, with \( A_t = R_t \cup E_{2,t} \), we have, using Lemma 10, Lemma 12, and Lemma 13,
\[
P(E_{2,t} \cup \overline{E_{2,t}}) \leq \mathbb{P}(\bigcup_{t=1}^{\infty} (R_t \cup E_{2,t} \cup R_0))
\leq \sum_{t=1}^{\infty} \mathbb{P}(R_t \cup E_{2,t} | R_{t-1}, E_{2,t-1}, \ldots, R_1, E_2, 1) + \mathbb{P}(R_0)
\leq \sum_{t=1}^{\infty} \mathbb{P}(E_{2,t} | R_{t-1}, E_{2,t-1}, \ldots, R_1, E_2, 1) \cdot \sum_{t=1}^{\infty} \mathbb{P}(R_t | R_{t-1}, E_{2,t-1}, \ldots, R_1, E_2, 1) + \delta
\leq \frac{2\delta}{t^2} + \delta
\leq 3\delta.
\]

For the following we will assume that event \( E_1 \cup E_2 \) holds. Now we argue that \( z^* \) will never be eliminated. Indeed for any \( z \in Z_t \), note that
\[
\langle z - z^*, \theta_t \rangle = \langle z - z^*, \theta_t - \theta^* \rangle + \langle z - z^*, \theta^* \rangle
\leq 2^{-t} + \langle z - z^*, \theta^* \rangle
\leq 2^{-t} ,
\]

implying that \( z^* \) is not kicked out. Finally, if \( \langle z^* - z, \theta^* \rangle \geq 2 \times 2^{-t} \), then
\[
\langle z^* - z, \theta_t \rangle = \langle z^* - z, \theta_t - \theta^* + \theta^* \rangle
= \langle z^* - z, \theta^* \rangle + \langle z^* - z, \theta_t - \theta^* \rangle
\geq 2 \times 2^{-t} - 2^{-t}
\geq 2^{-t} .
\]

Finally, we have \( \langle z^* - z, \theta^* \rangle = \langle z^* - z, \theta^* - \theta_t \rangle + \langle z^* - z, \theta_t \rangle \leq 2^{-t} + 2^{-t} \), which concludes the proof. \( \square \)
**Theorem 8** (Sample Complexity). Define $\mathcal{S}_t = \{z \in \mathcal{Z} : (z^* - z)^\top \theta_s \leq 2 \cdot 2^{-(t-1)}\}$, and take $\epsilon \leq 1/2$. Algorithm 6 returns $z^*$ with probability greater than $1 - 3\delta$ in a number of samples no more than

$$O\left((1 + \epsilon) \sum_{t=1}^{[\log_2(2/\Delta_{\min})]} \min_{\lambda \in \Delta_Y} \max_{z,z' \in \mathcal{S}_t} \|z - z'\|_2^2 \gamma(d) \max_{x \in \mathcal{X}} \|x\|_{H(\lambda, \theta^*)}^{-1} \log(\max(|\mathcal{X}|, |\mathcal{Z}|) t^2/\delta) + \kappa_0^{-1}(1 + \epsilon) d \gamma(d) \log(|\mathcal{X}|/\delta) + r(\epsilon) \log_2\left(\frac{1}{\Delta_{\min}}\right)\right).$$

**Proof.** For the remainder of the proof we will assume that $\mathcal{E}_1 \cap \mathcal{E}_2$ holds.

By Lemma 13 on $\mathcal{E}_2$, we have that $\mathcal{Z}_t \subseteq \mathcal{S}_t$, in particular this implies that when $2 \times 2^{-t} \leq \Delta_{\min}$, we have $|\mathcal{Z}_t| = 1$, so this implies that the algorithm will terminate in a number of rounds not exceeding $[\log_2(2/\Delta_{\min})]$.

By Lemma 11 on $\mathcal{E}_1$, we have that $H(\lambda_t, \hat{\theta}_t) \geq \frac{1}{4} H(\lambda_t, \theta^*)$. Thus, in each round,

$$\min_{\lambda \in \Delta_Y} \max_{z,z' \in \mathcal{S}_t} \|z - z'\|_2^2 \gamma(d) \max_{x \in \mathcal{X}} \|x\|_{H(\lambda, \theta^*)}^{-1} \leq 4 \min_{\lambda \in \Delta_Y} \max_{z,z' \in \mathcal{S}_t} \|z - z'\|_2^2 \gamma(d) \max_{x \in \mathcal{X}} \|x\|_{H(\lambda, \theta^*)}^{-1} \log(\max(|\mathcal{X}|, |\mathcal{Z}|) t^2/\delta) + c \log_2(\Delta_{\min}^{-1}) r(\epsilon).$$

Let $c$ be an absolute constant. Our final sample complexity is given by

$$n_0 + \sum_{t=1}^{[\log_2(2/\Delta_{\min})]} n_t \leq \kappa_0^{-1}(1 + \epsilon) d \log(12|\mathcal{X}|/\delta) + c(1 + \epsilon) \sum_{t=1}^{[\log_2(2/\Delta_{\min})]} \min_{\lambda \in \Delta_Y} \max_{z,z' \in \mathcal{S}_t} \|z - z'\|_2^2 \gamma(d) \max_{x \in \mathcal{X}} \|x\|_{H(\lambda, \theta^*)}^{-1} \log(\max(|\mathcal{X}|, |\mathcal{Z}|) t^2/\delta) + c \log_2(\Delta_{\min}^{-1}) r(\epsilon).$$

**Lemma 14.** Define, $\mathcal{S}_t = \{z \in \mathcal{Z} : (z^* - z)^\top \theta_s \leq 2 \cdot 2^{-(t-1)}\}$.

$$\sum_{t=1}^{\log_2(2/\Delta_{\min})} 2t \min_{\lambda \in \Delta_Y} \max_{z,z' \in \mathcal{S}_t} \|z - z'\|_2^2 H(\lambda, \theta^*) \leq \log\left(\frac{1}{\Delta_{\min}}\right) \min_{\lambda \in \Delta_Y} \max_{z \in \mathcal{Z} \setminus z^*} \frac{\|z^* - z\|_2^2 H(\lambda, \theta^*)^{-1}}{\langle \theta^*, z^* - z \rangle^2} \leq \frac{1}{4} \log\left(\frac{2}{\Delta_{\min}}\right) \left(\min_{\lambda \in \Delta_Y} \|\theta^* - \theta\|_{H(\lambda, \theta^*)}^2\right)^{-1}$$

where $\mathcal{C} = \{\theta \in \mathbb{R}^d : \exists z \in \mathcal{Z} \setminus z^*, \theta^\top (z^* - z) \leq 0\}$

**Proof.** Note that,

$$\log\left(\frac{2}{\Delta_{\min}}\right) \min_{\lambda \in \Delta_Y} \max_{z \in \mathcal{Z} \setminus z^*} \frac{\|z^* - z\|_2^2 H(\lambda, \theta^*)^{-1}}{\langle \theta^*, z^* - z \rangle^2} = \log\left(\frac{2}{\Delta_{\min}}\right) \min_{\lambda \in \Delta_Y} \max_{z \in \mathcal{Z} \setminus z^*} \frac{\|z^* - z\|_2^2 H(\lambda, \theta^*)^{-1}}{\langle \theta^*, z^* - z \rangle^2} \leq \log\left(\frac{2}{\Delta_{\min}}\right) \min_{\lambda \in \Delta_Y} \max_{t \leq \log_2(2/\Delta_{\min})} 2^{-2t+4} \max_{z \in \mathcal{S}_t \setminus z^*} \|z^* - z\|_2^2 H(\lambda, \theta^*)^{-1} \leq \frac{1}{16} \sum_{t=1}^{[\log_2(2/\Delta_{\min})]} 2t \min_{\lambda \in \Delta_Y} \max_{z \in \mathcal{S}_t \setminus z^*} \|z^* - z\|_2^2 H(\lambda, \theta^*)^{-1} \leq \frac{1}{4} \sum_{t=1}^{[\log_2(2/\Delta_{\min})]} 2t \min_{\lambda \in \Delta_Y} \max_{z \in \mathcal{S}_t \setminus z^*} \|z^* - z\|_2^2 H(\lambda, \theta^*)^{-1}$$
where \(a\) is replacing a max with an average and \(b\) is using \(\max_{z,z'\in S_t} \|z-z'\|^2_{H(\theta^*)^{-1}} = \max_{z,z'\in S_t} \|z-z'\|^2_{H(\theta^*)^{-1}} + \|z'-z\|^2_{H(\theta^*)^{-1}} - 2\|z'-z\|_{H(\theta^*)^{-1}} \|z-z\|_{H(\theta^*)^{-1}} \leq 4 \max_{z\in S_t} \|z-z\|^2_{H(\theta^*)^{-1}}.

We now tackle the second equality in the theorem statement. Define \(C_t = \{\theta \in \mathbb{R}^d : \theta^T(z^*-z) \leq 0\}\). Note that,

\[
\max_{\lambda \in \Delta_s} \min_{\theta \in \mathcal{C}_s} \|\theta^* - \theta\|^2_{H(\lambda,\theta^*)} = \max_{\lambda \in \Delta_s} \min_{z \in \mathcal{Z}\setminus \mathcal{C}_s} \|\theta^* - \theta\|^2_{H(\lambda,\theta^*)}
\]

For a fixed \(\lambda\), standard computation with Lagrange multipliers (as in Theorem 11) shows that the projection,

\[
\hat{\theta} := \arg \min_{\theta \in \mathcal{C}_s} \|\theta^* - \theta\|^2_{H(\lambda,\theta^*)} = \theta^* - \frac{(z^*-z)^T H(\lambda,\theta^*)^{-1}(z^*-z)}{{\|z^*-z\|}^2_{H(\lambda,\theta^*)^{-1}}}
\]

Thus,

\[
\|\theta^* - \hat{\theta}\|^2_{H(\lambda,\theta^*)} = \frac{(z^*-z)^T H(\lambda,\theta^*)^{-1}(z^*-z)}{{\|z^*-z\|}^2_{H(\lambda,\theta^*)^{-1}}}
\]

and the result follows.

\[\square\]

J RAGE-GLM-2

J.1 Review of confidence bounds of (Faury et al., 2020)

Assume that we have observed a sequence of samples \((x_s, y_s)_{s=1}^T\), where, \(\{x_s\}_{s=1}^T \in \mathcal{X}\) and the \(x_s\)'s are potentially chosen adaptively, that is \(x_s, 1 \leq s \leq T\) is allowed to depend on the filtration \(\mathcal{F}_{s-1} = \{(x_r, y_r)\}_{r=1}^{s-1}\).

For a regularization parameter \(\eta > 0\), define

\[
H_T(\eta, \theta) := \sum_{s=1}^T \mu(x_s^T \theta)x_s + \eta I
\]

We begin by defining our estimator. Let

\[
\hat{\theta}^{\text{MLE}}_{\eta,T} = \arg \max_{\theta \in \mathbb{R}^d} \sum_{s=1}^T y_s \log \mu(x_s^T \theta) + (1-y_s) \log(1-\mu(x_s^T \theta)) - \frac{\lambda}{2} \|\theta\|_2^2.
\]

Define,

\[
\hat{\theta}_T = \arg \min_{\|\theta\|_2 \leq S_*} \|g_T(\theta) - g_T(\hat{\theta}^{\text{MLE}}_{\eta,T})\|_{H_T(\eta,\theta)^{-1}}
\]

where \(g_T(\theta) = \sum_{s=1}^T \mu(x_s^T \theta)x_s + \eta \theta\). Finally, define

\[
\gamma_T(\delta) = \sqrt{\eta}(S_* + 1/2) + \frac{2}{\sqrt{\eta}} \log(1/\delta) + \frac{2d}{\sqrt{\eta}} \log(2(1 + T/d\eta)^{1/2})
\]

We recall the following lemma from (Faury et al., 2020).

Lemma 15 (Lemma 11 of (Faury et al., 2020)). On an event \(\mathcal{E}\) which is true with probability greater than \(1 - \delta\), for all \(t \geq 1\)

\[
\theta^* \in \{\theta \in \mathbb{R}^d : \|\theta\| \leq S_*, \|\theta - \hat{\theta}_T\|_{H_T(\eta,\theta)^{-1}} \leq (2 + 4S_*)\gamma_T(\delta)\}
\]

In the following we will take \(\eta = (d + \log(1/\delta))/S_* + 1/2\). Plugging this in to \(\gamma_T(\delta)\)

\[
\sqrt{\eta}(S_* + 1/2) + \frac{2}{\sqrt{\eta}} \log(1/\delta) + \frac{2d}{\sqrt{\eta}} \log\left(2\left(1 + \frac{T}{d\eta}\right)^{1/2}\right) = \sqrt{d + \log(1/\delta)\over S_* + 1/2} (S_* + 1/2) + \frac{2}{\sqrt{d + \log(1/\delta)\over S_* + 1/2}} \log(1/\delta) + \frac{2d}{\sqrt{d + \log(1/\delta)\over S_* + 1/2}} \log\left(2\left(1 + d + \log(1/\delta)\over S_* + 1/2\right)^{1/2}\right)
\]

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Algorithm 7 RAGE-GLM-2

Input: $\epsilon, \delta$, $\mathcal{X}$, $\mathcal{Z}$, $\kappa_0$, $S_*$, effective rounding procedure $\text{round}(n, \epsilon, \lambda)$, $\eta = (d + \log(1/\delta))/(S_* + 1/2)$

1: initialize $t = 1$, $Z_1 = \mathcal{Z}$, $r(\epsilon) = d^2/\epsilon$, $c = c(S_*, \epsilon) = 48\sqrt{(1+\epsilon)(2S_* + 1)^3}$

2: $\theta_0 \leftarrow \text{BurnIn}(\mathcal{X}, \kappa_0)$ \hspace{1cm} \Comment{Burn-in phase}

3: while $|Z_t| > 1$ do \hspace{1cm} \Comment{Elimination phase}

4: $f(\lambda) := \min_{z, z' \in Z_t} \|z - z'\|_{H(\lambda, \theta_0)}^{-1}$

5: $\lambda_t = \arg \min_{\lambda \in \Delta_\mathcal{X}} f(\lambda)$

6: $r_t = \left[ 2^{2t} \epsilon^2 f(\lambda_t) \left( \sqrt{d \log \epsilon^2 2^{2t}(2S_* + 1)} f(\lambda_t)/d + \sqrt{\log(t^2|Z|/\delta)} \right) \right]^2$

7: $n_t = \max\{r_t, r(\epsilon)\}$

8: $x_1, \ldots, x_{n_t} \leftarrow \text{round}(n, \epsilon, \lambda)$

9: Observe rewards $y_1, \ldots, y_{n_t} \in \{0, 1\}$

10: Compute $\hat{\theta}_t$ on the samples $\{(x_s, y_s)\}_{s=1}^{n_t}$ \Comment{(Use Eq (23))}

11: $\hat{z}_t = \arg \max_{z \in Z_t} \hat{\theta}_t^\top z$

12: $Z_{t+1} \leftarrow Z_t \setminus \{z \in Z_t : \hat{\theta}_t^\top (\hat{z}_t - z) \geq 2^{-t}\}$

13: $t \leftarrow t + 1$

14: end while

15: return $\hat{z}_t$

\[
\begin{align*}
&\leq \sqrt{d + \log(1/\delta)} \sqrt{S_* + 1/2} + 2\sqrt{S_* + 1/2} \sqrt{\log(1/\delta)} + \frac{2d \sqrt{S_* + 1/2}}{\sqrt{d}} \log(2(1 + \frac{T(2S_* + 1)}{2d})^{1/2}) \\
&= \sqrt{S_* + 1/2} \left( \sqrt{d + \log(1/\delta)} + 2\sqrt{\log(1/\delta)} + 2\sqrt{d} \log \left( 2 \left( 1 + \frac{T(2S_* + 1)}{2d} \right)^{1/2} \right) \right) \\
&\leq \sqrt{S_* + 1/2} \left( \sqrt{d} \left( 1 + 2 \log(2) + \frac{1}{2} \log \left( 1 + \frac{T(2S_* + 1)}{2d} \right) \right) \right) + 3\sqrt{\log 1/\delta} \\
&\leq 3\sqrt{S_* + 1/2} \left( \sqrt{d} \log \left( \frac{T(2S_* + 1)}{2d} \right) + \sqrt{\log 1/\delta} \right)
\end{align*}
\]

where the last line uses, $(1 + 2 \log(2) + 1/2 \log(1 + x)) \leq 3 \log(x), x \geq 2$. So as long as $T \geq 4d$, we have the following bound.

\[
\gamma_T(\delta) \leq 3\sqrt{2S_* + 1} \left[ \sqrt{d} \log \left( \frac{T(2S_* + 1)}{2d} \right) + \sqrt{\log 1/\delta} \right] := \Gamma_T(\delta)
\]

The guarantee that $T \geq 4d$ will be satisfied by the rounding procedures in the algorithm - indeed, taking $\epsilon \leq 1/2$ guarantees that the minimum number of samples we take in each round $r(\epsilon) = (d(d + 1) + 2)/\epsilon \geq 4d$.

J.2 Proof of Sample Complexity

We now provide a sample complexity for Algorithm 7. In this section, we take $\theta_t$ as defined in 23 using the samples $\{(x_s, y_s)\}_{s=1}^{n_t}$ in each round $t$. Note that in particular, we are not recycling samples between rounds. We will consider that extension in the next section.

In the regularized setting, rounding implies that,

\[
H_t(\eta, \theta^*) := H_t(\theta^*) + \eta I
\]

\[
\geq \frac{n}{1 + \epsilon} \sum_{x \in \mathcal{X}} \lambda x^\top \theta^* xx^\top + \eta I
\]

\[
\geq \frac{n}{1 + \epsilon} H(\lambda, \theta^*) + \eta I
\]

\[
\geq \frac{n}{1 + \epsilon} H(\lambda, \theta^*)
\]

where the last line uses, $(1 + 2 \log(2) + 1/2 \log(1 + x)) \leq 3 \log(x), x \geq 2$. So as long as $T \geq 4d$, we have the following bound.

\[
\gamma_T(\delta) \leq 3\sqrt{2S_* + 1} \left[ \sqrt{d} \log \left( \frac{T(2S_* + 1)}{2d} \right) + \sqrt{\log 1/\delta} \right] := \Gamma_T(\delta)
\]

The guarantee that $T \geq 4d$ will be satisfied by the rounding procedures in the algorithm - indeed, taking $\epsilon \leq 1/2$ guarantees that the minimum number of samples we take in each round $r(\epsilon) = (d(d + 1) + 2)/\epsilon \geq 4d$. 

Define

\[ \mathcal{E}_1 := \left\{ \frac{1}{4}H(\lambda_0, \theta^*) \leq H(\lambda_0, \theta_0) \leq 4H(\lambda_0, \theta^*) \right\} \]

By Lemma 10, \( \mathbb{P}(\mathcal{E}_1) \geq 1 - \delta \).

Define

\[ \mathcal{E}_2 = \cap_{t=1}^{\infty} \{ \forall z \in \mathcal{Z}_t, |(z^* - z, \hat{\theta}_t - \theta^*)| \leq 2^{-t} \} \]

**Lemma 16.** \( \mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}_1) \geq 1 - 3\delta \) and on \( \mathcal{E}_1 \cap \mathcal{E}_2 \), \( z^* \in \mathcal{Z}_t \) for all \( t \).

**Proof.** **Claim 1:** \( \mathbb{P}(\mathcal{E}_2 | \mathcal{E}_1) \geq 1 - \delta \). Assuming \( \mathcal{E}_1 \), for \( z \in \mathcal{Z}_t \), with probability greater than \( 1 - \frac{\delta}{|\mathcal{Z}|} \)

\[ |(z^* - z)^\top (\hat{\theta}_t - \theta^*)| \leq \|z^* - z\|_{H_t(\eta, \theta^*) - 1} \|\theta^* - \hat{\theta}_t\|_{H_t(\eta, \theta^*)} \]

\[ \leq (2 + 4S_*) \|z^* - z\|_{H_t(\eta, \theta^*) - 1} \Gamma_n(\delta) \] (Lemma 15)

\[ \leq 2(1 + 2S_*) \frac{1 + \epsilon}{n} \|z^* - z\|_{H_t(\eta, \theta^*) - 1} \Gamma_n(\delta) \] (Rounding Lemma 19)

\[ \leq 8(1 + 2S_*) \sqrt{\frac{1 + \epsilon}{n} \|z^* - z\|_{H_t(\eta, \theta_0) - 1} \Gamma_n(\delta)} \] (\( \mathcal{E}_1 \))

We wish for this quantity to be bounded above by \( 2^{-t} \). Plugging in \( \Gamma_n(\delta) \), it suffices to take

\[ \sqrt{n} \geq 24 \cdot 2^t (2S_* + 1)^{3/2} (1 + \epsilon) f_1 \left[ \sqrt{d \log \left( \frac{nt_1(2S_* + 1)}{2d} \right)} + \sqrt{\log(t^2 |\mathcal{Z}| / \delta)} \right] \]

where for ease of notation we have denoted \( f_1 = f(\lambda_t) \). Using Lemma 21 below, shows that it suffices to take,

\[ n_t = \left[ c^2 2^{2t} f_1 \left( \sqrt{d \log(c^2 2^{2t} (2S_* + 1) \rho_t / d)} + \sqrt{\log(t^2 |\mathcal{Z}| / \delta)} \right) \right]^2 \]

where \( c = 2 \cdot 24 \sqrt{1 + \epsilon} (2S_* + 1)^{3/2} \), which is precisely the number of samples we take in the algorithm. Union bounding over \( z \in \mathcal{Z}_t \subset \mathcal{Z} \) and \( t \geq 1 \) now gives the result.

**Claim 2:** \( \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \geq 1 - 3\delta \). Note that,

\[ \mathbb{P}(\mathcal{E}_1^c \cup \mathcal{E}_2^c) \leq \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c) \]

\[ = \mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1^c) \mathbb{P}(\mathcal{E}_1^c) + \mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1) \mathbb{P}(\mathcal{E}_1) + \mathbb{P}(\mathcal{E}_2^c) \]

\[ \leq \mathbb{P}(\mathcal{E}_2^c | \mathcal{E}_1) + 2\mathbb{P}(\mathcal{E}_1^c) \]

\[ \leq \delta + 2\delta \]

\[ \leq 3\delta \]

**Claim 3:** On \( \mathcal{E}_1 \cap \mathcal{E}_2 \), \( \max_{z \in \mathcal{Z}_t} |z^* - z, \theta^*| \leq 2 \cdot 2^{-t} \) for all \( t \geq 1 \). Identical argument to Lemma 13

\[ \square \]

**Remark.** We point out that this analysis is not particularly tight, and many constants and the dependence upon \( S_* \) can be improved upon in practice. In particular, we can trade off a smaller constant for a larger burn-in phase.

**Theorem 9** (Sample Complexity). **Algorithm 7**, returns \( z^* \) with probability greater than \( 1 - 2\delta \) in a number of samples no more than

\[ O \left( (1 + \epsilon)(2S_* + 1)^3 \log \left( \frac{1}{\Delta_{\min}} \right) \sum_{r=1}^{\log(1/\Delta_{\min})} 2^{2^t} \rho_t \left( d \log \left( \frac{(2S_* + 1) \rho_t}{\Delta_{\min}} \right) + \log(t^2 |\mathcal{Z}| / \delta) \right) + r(\epsilon) \log_2(1/\Delta_{\min}) + \kappa_0^{-1}(1 + \epsilon)d_1(d) \log(|\mathcal{X}| / \delta) \right) \]
Algorithm 8 Anytime RAGE-GLM-2

Input: $\epsilon, \delta, X, Z, \kappa_0, S_*$, effective rounding procedure $\text{round}(n, \epsilon, \lambda)$, $\eta = (d + \log(1/\delta))/(S_* + 1/2)$

1: initialize $t = 1$, $Z_1 = Z$, $r(\epsilon) = d^2/\epsilon, c(S_*, \epsilon) = 16(1 + \epsilon)(2S_* + 1)^3$

2: $\theta_0 \leftarrow \text{BurnIn}(\mathcal{X}, \kappa_0)$ $\triangleright$ Burn-in phase

3: while $|Z_t| > 1$ do $\triangleright$ Elimination phase
   
   4: $f(\lambda) := \min_{z,z' \in Z_t} \|z - z'\|_{H(\lambda, \theta_0)}^{-1}$
   
   5: $\lambda_t = \arg \min_{\lambda \in \Delta_X} f(\lambda)$
   
   6: $r_t = 2^{2t} c^2 f(\lambda_t)(16d \log(c^2 f(\lambda_t)) + \log(1/\delta))$
   
   7: $n_t = \max\{r_t, r(\epsilon)\}$
   
   8: $x_{m_{t-1} + 1}, \ldots, x_{m_{t-1} + n_t} \leftarrow \text{round}(n, \epsilon, \lambda)$
   
   9: Observe rewards $y_{m_{t-1} + 1}, \ldots, y_{m_{t-1} + n_t} \in \{0, 1\}$
   
   10: Compute $\hat{\theta}_t$ based on $\{(x_s, y_s)\}_{s=1}^{m_t}$ $\triangleright$ (Use Eq (23))
   
   11: $\hat{z}_t = \arg \max_{z \in Z_t} \hat{\theta}_t^\top z$
   
   12: $Z_{t+1} \leftarrow Z_t \setminus \{z \in Z_t : \hat{\theta}_t^\top (\hat{z}_t - z) \geq 2^{-t}\}$
   
   13: $t \leftarrow t + 1$
   
   14: $m_t = m_{t-1} + n_t$

end while

16: return $\hat{z}_t$

where $S_t = \{z \in Z : (\hat{z}_t - z)^\top \theta_s \leq 2 \cdot 2^{-t}\} \text{ and } \rho_t = \min_{\lambda \in \Delta_X} \max_{z,z' \in S_t} \|z - z'\|_{H(\lambda, \theta_0)}^{-1}$ and we assume $\epsilon \leq 1/2$.

Proof. Firstly note that $\mathbb{P}(\mathcal{E}_1 \cup \mathcal{E}_2) \leq 2\delta$. For the remainder of the proof we will assume that $\mathcal{E}_1 \cap \mathcal{E}_2$ holds.

By Lemma 16, we have that $Z_t \subset S_t$, likewise on $\mathcal{E}_1$ we have that $H(\lambda_t, \theta_0) \geq \frac{1}{2} H(\lambda_0, \theta_*)$. Thus, in each round,

\[
\max_{z,z' \in S_t} \|z - z'\|_{H(\lambda_t, \theta_0)}^{-1} \leq 4 \max_{z,z' \in S_t} \|z - z'\|_{H(\lambda_0, \theta_*)}^{-1}
\]

Denoting $\rho_t = \min_{\lambda \in \Delta_X} \max_{z,z' \in S_t} \|z - z'\|_{H(\lambda, \theta_0)}^{-1}$, we see that $f_t \leq \rho_t$. This implies that $n_t \leq 4c^2 2^{2t} \rho_t [\sqrt{d} \log(c^2 2^{2t}(2S_* + 1) \rho_t/d) + \sqrt{\log(t^2 |Z|^2 /\delta)/2}]^2 + r(\epsilon) \log(1/\Delta_{\min}) + n_0$

Thus an upper bound on our final sample complexity is given by

\[
\sum_{t=1}^{\log_2(1/\Delta_{\min})} n_t + r(\epsilon) \log(1/\Delta_{\min}) + n_0
\]

\[
\leq 8c^2 \sum_{t=1}^{\log_2(1/\Delta_{\min})} 2^{2t} \rho_t [d \log^2(c^2 2^{2t}(2S_* + 1) \rho_t/d) + \log(t^2 |Z|^2 /\delta)] + r(\epsilon) \log(1/\Delta_{\min}) + n_0
\]

\[
\leq 8c^2 \sum_{t=1}^{\log_2(1/\Delta_{\min})} 2^{2t} \rho_t [d \log^2(c^2 2^{2t}(2S_* + 1) \rho_t/d) + \log(t^2 |Z|^2 /\delta)] + r(\epsilon) \log(1/\Delta_{\min}) + n_0
\]

\[
= O\left(1 + \epsilon\right) (2S_* + 1)^3 \sum_{r=1}^{\log_2(1/\Delta_{\min})} 2^{2t} \rho_t [d \log^2((2S_* + 1) \rho_t/\Delta_{\min})
\]

\[
+ \log(t^2 |Z|^2 /\Delta_{\min})] + r(\epsilon) \log(1/\delta) + \kappa_0^{-1}(1 + \epsilon) d\gamma(d) \log(|\mathcal{X}| /\delta)
\]

\hfill \square
K Anytime RAGE-GLM-2

One of the advantages of the confidence interval presented in (Faury et al., 2020) is that it is anytime. Unlike the confidence interval of Theorem 6, which requires a fixed design, The confidence interval in Lemma 11 of (Faury et al., 2020) holds for an adaptively chosen sequence of \( x_t \)'s. In practice, recycling samples over rounds can lead to a smaller overall sample complexity. In this section we modify RAGE-GLM-2 to be anytime, without incurring any large ramifications on sample complexity compared to that of Theorem 9.

Let \( m_t \) denote the total samples taken up through the end of round \( t \) and let \( n_t \) be the number of samples taken in round \( t \). We begin by defining our estimator as \( \hat{\theta}_t \) in round \( t \) to be the MLE given in equation 23 computed with the samples \( \{(x_s, y_s)_{s=1}^m_t\} \).

Remark: We abuse notation with the indexing slightly since we are taking \( T = m_t \) and so \( \hat{\theta}_t \) in this section would be denoted \( \hat{\theta}_{m_t} \) in Section J.

We define two events. Firstly, as in the previous sections event 1,

\[
E_1 := \left\{ \frac{1}{4}H(\lambda_0, \theta^*) \leq H(\lambda_0, \hat{\theta}_0) \leq 4H(\lambda_0, \theta^*) \right\}
\]

By Lemma 10, \( \mathbb{P}(E_1) \geq 1 - \delta \).

We also let \( E_2 \) be the event that the anytime concentration inequality of 15 holds at anytime.

**Lemma 17 (Concentration).** On \( E_1, E_2 \), in round \( t \), if we take \( n_t \) samples as specified in the algorithm, then

\[
| (z^* - z)^\top (\theta - \theta^*) | \leq 2^{-t}
\]

for all \( z \in \mathbb{Z}_t \).

**Proof.** Firstly note that \( H_{m_t}(\eta, \theta) = \sum_{s=1}^{m_t} \mu(x_s^\top \theta) + \eta I \geq \sum_{s=m_t+1}^{m_t} \mu(x_s^\top \theta)x_\sigma x_s^\top + \eta I \). We denote the quantity on the right as \( H_{t}(\eta, \theta) \).

For \( z \in \mathbb{Z} \),

\[
| (z^* - z)^\top (\hat{\theta}_t - \theta^*) | \leq \| z^* - z \| H_{m_t}(\eta, \theta^*)^{-1} \| \theta^* - \hat{\theta}_t \| H_{m_t}(\eta, \theta^*)
\]

\[
\leq (2 + 4S_t) \| z^* - z \| H_{t}(\eta, \theta^*)^{-1} \beta_{m_t}(\delta)
\]

\[
\leq 2(1 + 2S_t) \sqrt{\frac{1 + \epsilon}{n}} \| z^* - z \| H_{t}(\eta, \theta^*)^{-1} \beta_{m_t}(\delta)
\]

\[
\leq 8(1 + 2S_t) \sqrt{\frac{1 + \epsilon}{n}} \| z^* - z \| H_{t}(\eta, \theta_0)^{-1} \beta_{m_t}(\delta)
\]

\[
\leq 8(1 + 2S_t) \sqrt{\frac{1 + \epsilon}{n}} \| z^* - z \| H_{t}(\eta, \theta_0)^{-1} \beta_{m_t}(\delta)
\]

We wish for this quantity to be bounded above by \( 2^{-t} \). Plugging in \( \beta_{m_t}(\delta) \), it suffices to take

\[
\sqrt{n_t} \geq 24 \cdot 2^t (2S_t + 1)^3/2 \sqrt{1 + \epsilon} f_t \left[ \sqrt{d \log \left( \frac{(n_t + m_{t-1})(2S_t + 1)}{d} \right)} + \sqrt{\log(1/\delta)} \right]
\]

Define, \( c = 2 \cdot 24(2S_t + 1)^3/2 \sqrt{(1 + \epsilon)} f_t \). Thus, using Lemma 21, we can take

\[
n_t = \left[ c^2 2^t f_t \sqrt{d \log(c^2(2S_t + 1)^2 2^{2t} f_t / d^2)} + \sqrt{\log(m_{t-1}) + \sqrt{\log(1/\delta)^2}} \right]
\]

**Theorem 10 (Sample Complexity).** Algorithm 8, returns \( z^* \) with probability greater than \( 1 - 2\delta \) in a number of samples no more than

\[
\tilde{O} \left( (2S_t + 1)^3 (1 + \epsilon) \sum_{t=1}^{\log_2(2/\Delta_{min})} 2^t \rho_t [d + \log(1/\delta)] \right) + r(\epsilon) \log(1/\delta) + \kappa_0^{-1}(1 + \epsilon) d \log(|X|/\delta)
\]

up to logarithmic factors in \( S_t \) and \( 1/\Delta_{min} \).
Lemma 18. Consider a sequence $a_t$ such that $a_t \leq 3t + c + 2a_{t-1}$ then, $a_T \leq 3T^2 + (T - 1)c + a_1$

Proof. Inducting backwards, we see that,

\[ a_t \leq 3 \sum_{i=1}^{T} t + (T - 1)c + a_1 \]
\[ \leq 3T^2 + (T - 1)c + a_1 \]

So we see that,

\[ \log(m_t) \leq O(t^2 + t(\log(Rc^2d(2S_* + 1)^2) + \log_2 \log(1/\delta))) \]

Thus,

\[ \sum_{t=1}^{T} n_t \leq \sum_{t=1}^{T} 4c^22^t \rho_t (log^2(2c^2(2S_* + 1)^22^t \rho_t/d^2) + \log^2(m_{t-1})) + \sum_{t=1}^{T} 4c^22^t \rho_t \log(1/\delta)) \]
Taking $T = \log_2(2/\Delta_{\min})$ gives the final result,

$$
\sum_{t=1}^{T} n_t \leq O \left( \sum_{t=1}^{T} 2^{2t}(2S_+ + 1)^3(1 + \epsilon)\left[ d \log(1/\Delta_{\min}) \log^2 \left( \frac{Rd(2S_+ + 1)\log(1/\Delta_{\min})}{\Delta_{\min}} \right) + \log(1/\Delta_{\min}) \right] \right)
$$

$\square$

L  Miscellaneous results

We let round($\lambda$, $n$) denote an efficient rounding procedure as explained in Chapter 12 of [Pukelsheim, 2006], or summarized in Section B of the Appendix of [Fiez et al., 2019].

**Lemma 19 (Rounding).** Assume that $\lambda \in \Delta_X$, and that we have sampled $x_1, \cdots, x_n \sim \text{round}(\lambda, n, \epsilon)$ with $n \geq r(\epsilon) = (d(d+1) + 2)/\epsilon$, and $\epsilon \leq 1$. Then, for any $\theta$, $\sum_{s=1}^{n} \hat{\mu}(x_{s}^\top \theta) x_{s} x_{s}^\top \geq \frac{n}{1 + \epsilon} \sum_{x \in X} \lambda_x \hat{\mu}(x^\top \theta) xx^\top$. This in particular implies

- For any $z$,
  $$
  \left\| x \right\|^2 (\sum_{s=1}^{n} \hat{\mu}(x_{s}^\top \theta) x_{s} x_{s}^\top)^{-1} \leq \frac{1 + \epsilon}{n} \left\| x \right\|^2 (\sum_{x \in X} \lambda_x \hat{\mu}(x^\top \theta) xx^\top)^{-1}
  $$
- $\lambda_{\min}(\sum_{s=1}^{n} \hat{\mu}(x_{s}^\top \theta) x_{s} x_{s}^\top) \geq \frac{n}{1 + \epsilon} \lambda_{\min}(\sum_{x \in X} \lambda_x \hat{\mu}(x^\top \theta) xx^\top)$

**Proof.** Let $s = (n_x)_{x \in X} \in \mathbb{N}^X$ denote the allocation returned by the rounding procedure and let $\gamma = s/n \in \Delta_X$ denote the associated fractional allocation. Now consider,

$$
\epsilon_{\gamma/\lambda} = \min_{x \in \text{supp}(\lambda)} \frac{\gamma_x}{\lambda_x} = \max\{\kappa \geq 0 : \gamma_x \geq \kappa \lambda_x \text{ for all } x \in X\}
$$

By definition of $\epsilon_{\gamma/\lambda},$

$$
\sum_{x \in X} \gamma_x \hat{\mu}(x^\top \theta) xx^\top \geq \epsilon_{\gamma/\lambda} \sum_{x \in X} \lambda_x \hat{\mu}(x^\top \theta) xx^\top
$$

By Theorem 12.7 of [Pukelsheim, 2006], $\epsilon_{\gamma/\lambda} \geq 1 - p/n$ where $p = |\text{supp}\lambda|$. When dim span $X = d$, Carathéodory’s Theorem [Vershynin, 2018], implies $p \leq d(d+1)/2 + 1$. Hence,

$$
\sum_{s=1}^{n} \hat{\mu}(x_{s}^\top \theta) x_{s} x_{s}^\top = n \sum_{x \in X} \gamma_x \hat{\mu}(x^\top \theta) xx^\top \geq n(1 - \frac{p}{n}) \sum_{x \in X} \lambda_x \hat{\mu}(x^\top \theta) xx^\top \geq \frac{n}{1 + \epsilon} \sum_{x \in X} \lambda_x \hat{\mu}(x^\top \theta) xx^\top
$$

as long as $n \geq (d(d+1) + 2)/\epsilon$. The result now follows.

$\square$

As long as $n_t \geq r(\epsilon)$, we have a guarantee that $H_t(\theta) \geq \frac{n_t}{1 + \epsilon} H(\lambda_t)$ for any $\theta$. This implies, $H_t(\theta)^{-1} \leq \frac{1 + \epsilon}{n_t} H(\lambda_t)^{-1}$.  

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This is a modification of the argument in Fiez et al. (2019).

**Lemma 20.** Let \( \theta \in \mathbb{R}^d \). Suppose \( D = \max_{x \in X} |x^\top(\theta - \theta^*)| \leq 1 \). Then, for all \( x \),

\[
\frac{1}{2D + 1} H(\lambda, \theta^*) \leq H(\lambda, \theta) \leq (2D + 1) H(\lambda, \theta^*)
\]

**Proof.** The proof is identical to Lemma 5. \qed

**Lemma 21.** Assume \( a > 0, b > 2 \), then for any \( t \geq \max[(2a)^2(\ln((2a)^2/c) + \ln b + d)^2, 2c] \) we have that \( \sqrt{t} \geq a[\log(b + t/c) + d] \)

**Proof.** Note that if \( t > 2c \), a \( \log(b + t/c) \leq a \log(b) + a \log(t/c) \), so it suffices to show \( \sqrt{t} \geq a \log(b) + a \log(t/c) + ad \) or equivalently, \( \frac{1}{\sqrt{a}} \sqrt{t} \geq \log(b) - \frac{d}{a} \log(u) \) for \( t \geq (2a)^2(\ln((2a)^2/c + \ln b + d)^2) \). However, this follows directly from Proposition 6 of (Antos et al., 2010). \qed

### M Other details

#### M.1 Lower Bound

In this section, we provide an information theoretic lower bound for any PAC-\( \delta \) algorithm. Define,

\[
\beta(a, b) = \int_0^1 (1 - t)\mu(a + t(b - a))dt
\]

and analogous to \( H(\lambda, \theta) \) we define two additional matrix valued functions,

\[
G(\lambda, \theta_1, \theta_2) = \sum_{x \in X} \lambda_x \alpha(x, \theta_1, \theta_2)xx^\top
\]

\[
K(\lambda, \theta_1, \theta_2) = \sum_{x \in X} \lambda_x \beta(x, \theta_1, \theta_2)xx^\top
\]

**Theorem 11.** Any PAC-\( \delta \) algorithm for the pure exploration logistic bandits problem has a stopping time \( \tau \) satisfying,

\[
\mathbb{E}[\tau] \geq c(\lambda)^{-1} \log \frac{1}{2.4\delta}, c(\lambda) = \max_{\lambda \in \mathbb{R}^d} \min_{z \neq z' \in \mathbb{Z}} \frac{\|\theta - \theta_z\|_{K(\lambda, \theta^*, \theta_z)}}{\|\theta^* - z\|_{G(\lambda, \theta^*, \theta_z)}}
\]

where \( \theta_z := \min_{\theta \in \mathbb{R}^d: \theta^\top(z^* - z) \leq 0} \frac{\|\theta - \theta_z\|_{K(\lambda, \theta^*, \theta_z)}}{\|\theta^* - z\|_{G(\lambda, \theta^*, \theta_z)}} \) and is given explicitly as the solution to the fixed-point equation

\[
\theta_z = \theta^* - \frac{(z^* - z)^\top \theta^* G(\lambda, \theta^*, \theta_z)^{-1}(z^* - z)}{\|z^* - z\|_{G(\lambda, \theta^*, \theta_z)}}
\]

**Proof.** Let \( C = \{ \theta \in \Theta : \exists z \in \mathbb{Z}, \theta^\top(z^* - z) \leq 0 \} \). The transportation theorem of (Kaufmann et al., 2016) implies that any algorithm that is \( \delta \)-PAC, takes at least \( T \) samples with

\[
\mathbb{E}[T] \geq \log \left( \frac{1}{2.4\delta} \right) \min_{\lambda \in \Delta z} \max_{\theta \in C} \sum_{x \in X} \frac{1}{\lambda_x K(\nu_{x, \theta} | \nu_{x, \theta})}
\]

\[
\geq \log \left( \frac{1}{2.4\delta} \right) \min_{\lambda \in \Delta z} \max_{\theta \in \mathbb{R}^d: \theta^\top(z^* - z) \leq 0} \sum_{x \in X} \frac{1}{\lambda_x K(\nu_{x, \theta} | \nu_{x, \theta})}
\]

where \( \nu_{x, \theta} \) is the distribution of arm \( x \) under the parameter vector \( \theta \), i.e. \( \nu_{x, \theta} = \text{Bernoulli}(x^\top \theta) \)

For a fixed \( z' \in \mathbb{Z} \), s.t. \( z' \neq z^* \) consider

\[
\min_{\theta \in \mathbb{R}^d: \theta^\top(z^* - z') \leq 0} \sum_{x \in X} \lambda_x K(L(\nu_{x, \theta} | \nu_{x, \theta})
\]

We have that
\[ KL(\nu_{x,\theta} | \nu_{x,\theta}) = \mu(x^T \theta^*) \log \left( \frac{\frac{e^{x^T \theta^*}}{1+e^{x^T \theta^*}}}{\frac{1}{1+e^{x^T \theta^*}}} \right) + (1 - \mu(x^T \theta^*)) \log \left( \frac{\frac{1}{1+e^{x^T \theta^*}}}{\frac{1}{1+e^{x^T \theta^*}}} \right) \]

\[ = \mu(x^T \theta^*) x^T (\theta^* - \theta) + \log \left( \frac{\frac{1}{1+e^{x^T \theta^*}}}{\frac{1}{1+e^{x^T \theta^*}}} \right) \]

\[ = \mu(x^T \theta^*) x^T (\theta^* - \theta) + \log \left( \frac{1 - \mu(z^T \theta^*)}{1 - \mu(x^T \theta)} \right) \]

\[ = \mu(x^T \theta^*) x^T (\theta^* - \theta) + (1 - \mu(x^T \theta^*)) - \log(1 - \mu(x^T \theta)) \]

Differentiating with respect to \( \theta \) gives,

\[ \nabla_{\theta} KL(\nu_{x,\theta} | \nu_{x,\theta}) = -\mu(x^T \theta^*) x + \frac{\mu(x^T \theta)}{1 - \mu(x^T \theta)} = (\mu(x^T \theta) - \mu(x^T \theta^*)) x \]

using the fact that \( \dot{\mu}(a) = \mu(a)(1 - \mu(a)) \) so this implies

\[ \nabla_{\theta} \sum_{x \in \mathcal{X}} \lambda_z KL(\nu_{x,\theta} | \nu_{x,\theta}) = \sum_{x \in \mathcal{X}} \lambda_z (\mu(x^T \theta) - \mu(x^T \theta^*)) x \]

Assuming \( \Theta = \mathbb{R}^d \) and letting \( \psi \) denote the Lagrange Multiplier corresponding to the constraint \( \theta^T (z^* - z) \leq 0 \) gives that the minimal \( \theta \) satisfies,

\[ \sum_{x \in \mathcal{X}} \lambda_z (\mu(x^T \theta) - \mu(x^T \theta^*)) x = \psi \cdot (z^* - z) \]

Now by definition, \( \mu(x^T \theta) - \mu(x^T \theta^*) = \alpha(x, \theta, \theta^*) x^T (\theta - \theta^*) \) so, this reduces to,

\[ \left( \sum_{x \in \mathcal{X}} \lambda_z \alpha(x, \theta, \theta^*) x x^T \right) (\theta - \theta^*) = \psi(z^* - z) \Rightarrow \theta = \theta^* + \psi G(\lambda, \theta, \theta^*)^{-1} (z^* - z) \]

Let \( \theta_z \) be the solution to this fixed point equation. Since we are saturating the constraint, it should be true that \( \theta_z(z^* - z) = 0 \). With this, we take an inner product with \( z^* - z \) on both sides to obtain

\[ \psi = -\frac{(z^* - z)^T \theta^*}{\|z^* - z\|^2_{G(\lambda, \theta, \theta^*)^{-1}}} \]

So finally, we see that

\[ \theta_z = \theta^* - \frac{\theta^* (z^* - z) G(\lambda, \theta_z, \theta^*)^{-1} (z^* - z)}{\|z^* - z\|^2_{G(\lambda, \theta, \theta^*)^{-1}}} \]

and

\[ \theta_z = \arg\min_{\theta \in \mathbb{R}^d, \theta^T (z^* - z') \leq 0} \sum_{x \in \mathcal{X}} \lambda_z KL(\nu_{x,\theta} | \nu_{x,\theta}) \]

Now to finish the proof, note

\[ KL(\nu_{x,\theta} | \nu_{x,\theta}) = \mu(x^T \theta^*) x^T (\theta^* - \theta) + (1 - \mu(x^T \theta^*)) - \log(1 - \mu(x^T \theta)) \]

\[ = (\theta^* - \theta) \left[ \frac{\mu(x^T \theta^*)}{(x^T (\theta^* - \theta))^2} + \log(1 - \mu(z^T \theta^*)) - \log(1 - \mu(x^T \theta)) \right] x x^T (\theta^* - \theta) \]

\[ \overset{(a)}{=} \|\theta^* - \theta\|^2_{\beta(z^T \theta, z^T \theta^*)} \]
where the last expression follows from the computation,

\[ \beta(a, b) = \int_0^1 (1 - t) \mu(a + t(b - a)) dt \]

\[ = \frac{\ln(e^{-a} + 1)}{(b - a)^2} - \frac{\ln(e^{-b} + 1)}{(b - a)^2} - \frac{1}{(e^b + 1) (b - a)} \]

\[ \square \]

In general, it is not clear how to compare our upper bound from Theorem 6 to this lower bound due to the non-explicit nature of \( G(\lambda, \theta, \theta^*) \). In the case of Gaussian linear bandits, previous work has shown an elimination scheme similar to Algorithm 6 is indeed near optimal.