Quantum corrections to Boltzmann equations

A. Jakovác
Institute of Physics, Technical University of Budapest, Budafoki út. 8
H-1111 Budapest, Hungary

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Abstract

Because of IR (pinch) singularities a resummation is necessary in non-equilibrium field theories, that can be performed by using Kadanoff–Baym equations. Taking Landau prescription correctly into account, Kadanoff–Baym equations reduce to Boltzmann equations only in a restricted kinematical range; in other cases a new equation (the former constraint equation) has to be considered. In relaxation time approximation this new equation results in the shifting and smearing of multiparticle thresholds.

1 Introduction

The application of Boltzmann equations is a generally accepted tool for studying non-equilibrium processes in different domains of physics. In field theory one can justify this method by using Kadanoff–Baym equations [1]. This method presents, however, just an indication that Boltzmann equation is consistent with field theory, but neither embeds it into field theory, nor proves that there cannot be other important contributions from different sources.

If we try to use Boltzmann equations consistently with perturbation theory, we have to investigate which is that subset of the Feynman series that is responsible for the appearance of Boltzmann equations, why is it relevant, and what is the effect of the terms left out from this subset. It is known for long times that in statistical physics one can obtain results from diagrammatic approach which is consistent with Boltzmann equations if we resum ladder diagrams [2]. Recently it has been demonstrated that ladder and extended ladder diagrams give important contribution to composite operator expectation values in scalar theories, and the results are consistent with Boltzmann equations [3].

What is that effect that renders the ladder diagrams so important? As it was proved in [4], each pairs of propagators in a ladder yields pinch singular...
contributions. In real space these diagrams give secular terms (proportional to some power of time); after resummation they result in Boltzmann equations. Boltzmann equations in linear response theory yield exponential damping. Still, in several numerical studies one observes power law damping of different physical quantities. In some systems these are consequences of non-linear, hydrodynamic effects. However, many particle (quantum) coherence can give such terms even in linear response regime. These effects clearly cannot be part of the Boltzmann solution.

In this contribution we try to show that, from field theoretical point of view, linearized Kadanoff–Baym equations represent a tool for resummation the pinch singular parts of the ladder diagrams. We will argue that for long time evolution of a quasi-conserved quantity exactly these are the diagrams that are relevant for the exponential damping, and that the solution of Boltzmann equations yields correct damping rate. We will argue furthermore that, besides the Boltzmann contribution, there can be other effects coming from multiparticle thresholds that yield exponentially damped power law time dependence. For more details see.

2 Long time behavior

First we produce the out of equilibrium state from where the non-equilibrium time dependence can be started. We can do it by starting from equilibrium and modifying the time evolution for a certain time as $H \rightarrow H + \Delta H$. We denote $\Delta H = \int \hat{F} \zeta(x)$, where $\hat{F}$ is a local operator, $\zeta(x)$ denotes its strength. Assuming small deviation from equilibrium we can use linear response in $\Delta H$, i.e. for an arbitrary operator having zero expectation value in equilibrium we can write

$$\langle O(x) \rangle = \int d^4x' G^{OF}_{RR}(x-x') \zeta(x'),$$

where $G^{OF}_{RR}(x) = i \Theta(t) \langle [O(x), \hat{F}(x')] \rangle_{eq}$. It is similar to the Kubo formula; however, if we fix only finite number of initial observables, we have a big freedom in the choice of $\Delta H$. The best we can expect is that after long times we observe universal time dependence.

Under some assumptions we can rewrite these expressions as (suppressing the spatial dependence)

$$\langle O(t) \rangle = \int \frac{dk_0}{2\pi} e^{-ik_0 t} \rho^{OF}(k_0) \zeta(k_0),$$

where $\rho^{OF}(k_0) = \text{Disc} G^{OF}_{RR}(k_0)$. Analyzing this expression for long times we obtain two possible contributions: one from saddle points (maxima), the other from thresholds. The former yields exponential, the latter yields power law time dependence.

If $O$ is conserved at tree level then $\rho^{OF}_0(k_0) \sim \delta(k_0)$. Interactions spoil the conservation, but, if it is quasi-conserved, the trace of the tree level peak should
remain. Then we expect the form
\[ \langle O(t) \rangle = Q_1 e^{-\Gamma t} + Q_2 t^{-\alpha+1} \cos(\Omega t + \phi). \] (3)

Here \( Q_1, Q_2 \) and \( \phi \) depend on the choice of \( \Delta H \), i.e., on initial conditions; \( \Gamma \) (the width of the peak at zero momentum), \( \alpha \) and \( \Omega \) (the index and position of the threshold), however, we expect to be universal quantities.

### 3 Pinch singularities and resummation

The appearance of the damping means IR divergences for an infinite subset of diagrams. This is because a broad peak at zero momentum should come from a Green’s function that behaves as \( G^{OF}(k_0) \sim (k_0 + i\Gamma)^{-1} \) near the peak. Since \( \Gamma \sim g^\# \) where \( g \) is the coupling, this behavior must show up in perturbation theory as \( (1/k_0) \sum_n (-i\Gamma/k_0)^n \). Therefore in \( n \)th order of perturbation theory we expect \( k_0^{-n-1} \) pole for small momenta.

To identify the origin of this IR behavior in the perturbative analysis we consider ladder diagrams in R/A formalism. Multiplying a retarded and an advanced propagator we find singular behavior if the external momentum vanishes; up to non-singular terms
\[ G_R(Q - \frac{K}{2})G_A(Q + \frac{K}{2}) \rightarrow i\rho(Q - \frac{K}{2}) \left( \frac{\Theta(|2q_0| - |k_0|)}{QK} + \frac{\Theta(|k_0| - |2q_0|)}{Q^2 + \frac{K^2}{4} - m^2} \right). \] (4)

applying Landau prescription. Both equations may produce IR divergence, when \( K = 0 \) or when \( Q^2 + \frac{K^2}{4} - m^2 = 0 \). In a ladder diagram with \( n \) rungs this divergence appears on the \( n \)th power. Thus for \( k = 0 \) we find the same divergence structure as we expected from the finite width.

After identifying the IR divergences we can think about their resummation. Since ladder diagrams are recursive structures we should find an equation for the non-equilibrium two-point function of the form
\[ (Q^2 + K^2/4 - m^2)\tilde{G}(Q, K) = (A_+ \ast \tilde{G})(Q, K) \quad \text{for} \quad |k_0/2| > |q_0| \]
\[ 2QK\tilde{G}(Q, K) = (A_- \ast \tilde{G})(Q, K) \quad \text{for} \quad |k_0/2| < |q_0|, \] (5)

where \( A_\pm \) are linear kernels, non-zero in the pinch singular limit. Exactly this is the form of the Kadanoff–Baym equations, so we can use them for the resummation procedure. In a very generic form we can start from the Schwinger–Dyson equations
\[ (p^2 - m^2)G(p, q) = F_1(G), \quad (q^2 - m^2)G(p, q) = F_2(G), \] (6)

perform Wigner transformation \( p, q = Q \pm K/2 \) and take their difference and sum
\[
\begin{align*}
\text{(diff)} & \quad 2QK\tilde{G} = F_1(G) - F_2(G) \\
\text{(sum)} & \quad 2(Q^2 + K^2/4 - m^2)\tilde{G} = F_1(G) + F_2(G).
\end{align*}
\] (7)
The first (diff) equation is the Boltzmann equation for $K \to 0$, and as it could be seen from the previous analysis, it can be applied for $|k_0/2| < |q_0|$. The second (sum) equation, which is plays the role of the on-shellness condition in case of the usual Boltzmann equations, for $|k_0/2| > |q_0|$ becomes a differential equation. For the free case it generates a two-particle cut, therefore it yields a power law time dependence with $\alpha = 1/2$. In the interacting case the solution is rather complicated, but as a first study we can try to adapt relaxation time approximation for it, and solve

$$\left[\frac{1}{2}(k_0 + i\Gamma_c)^2 - Q^2 - m_{eff}^2\right] G = \text{initial conditions.} \quad (8)$$

The effect of the solution can be seen on Fig. 1. The threshold is shifted

![Figure 1: a) free threshold; b) effect of the relaxation time approximation.](image)

and smeared out. The non-analyticity disappears, and the original $t^{-3/2}$ time dependence becomes $e^{-\Gamma_c t} t^{-1}$. The damping factor $\Gamma_c$ is different from the Boltzmann damping rate, so this term, in principle, can be dominating for long times.

4 Conclusions

Using Boltzmann equations in QFT: In perturbation theory IR (pinch) divergences at finite temperature make resummation necessary. The relevant diagrams are the ladders, their pinch singular part can be resummed using Boltzmann and threshold equations.

Using QFT to correct Boltzmann equations: To describe long time behavior, besides the Boltzmann equation we have to take into account the other Kadanoff–Baym equation as differential equation. Both equations have to be considered in their respective validity range in the momentum space.

In relaxation time approximation, we have a “dressed” threshold: shifted and smeared out as compared with the undressed function. Instead of power law time evolution of the collisionless approximation we find damped power law time evolution.
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