On spectral convergence of vector bundles and convergence of principal bundles

Kota Hattori

Keio University
3-14-1 Hiyoshi, Kohoku, Yokohama 223-8522, Japan
hattori@math.keio.ac.jp

Abstract

In this article we consider the continuity of the eigenvalues of the connection Laplacian of $G$-connections on vector bundles over Riemannian manifolds. To show it, we introduce the notion of the asymptotically $G$-equivariant measured Gromov-Hausdorff topology on the space of metric measure spaces with isometric $G$-actions, and apply it to the total spaces of principal $G$-bundles equipped with $G$-connections over Riemannian manifolds.

1 Introduction

For a closed connected Riemannian manifold $(X, g)$, the Laplace operator is defined by

$$\Delta f := d^* df$$

for any smooth functions $f$ on $X$, where $d^*$ is the formal adjoint of exterior derivative $d$. As one of the fundamental results of harmonic analysis, we obtain the orthonormal basis $\{f_j\}_j$ of $L^2(X)$ and eigenvalues

$$0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$$

such that $\lim_{j \to \infty} \lambda_j = \infty$ and $\Delta f_j = \lambda_j f_j$.

Here, we consider the function $\lambda_j$, which maps the isometric class of closed Riemannian manifolds to the $j$-th eigenvalue of its Laplacian. In [3], Fukaya has shown the continuity of $\lambda_j$ with respect to the measured Gromov-Hausdorff topology under the assumptions that the sectional curvatures of
Riemannian manifolds are bounded. He also conjectured that the continuity should holds if the lower bound of the Ricci curvatures are given.

His conjecture was solved by Cheeger and Colding in [2]. They defined the Laplacian on the metric measure space \((X, d, \mu)\) which is the measured Gromov-Hausdorff limit of a sequence of smooth connected closed Riemannian manifolds \(\{(X_i, g_i)\}\) with

\[
diam(X_i) \leq D, \quad \text{Ric}_{g_i} \geq \kappa g_i
\]

for some constant \(D > 0\) and \(\kappa \in \mathbb{R}\), then they showed that

\[
\lim_{i \to \infty} \lambda_j(X_i, g_i) = \lambda_j(X, d, \mu).
\]

Here, \((X_i, g_i)\) can be regarded as a metric measure space by the Riemannian distance \(d_{g_i}\) and the provability measure \(\frac{\mu_{g_i}}{\mu_{g_i}(X)}\), where \(\mu_{g_i}\) is the Riemannian measure.

On the Riemannian manifolds, the Laplace operators can be also defined for the differential forms or smooth section of vector bundle with the metric connections. In [8], Lott discussed with the eigenvalues of the Laplacian acting on differential forms on the Riemannian manifolds with bounded sectional curvatures and diameters, but may be collapsing. He also discussed with the eigenvalues of the Dirac operators in [7]. Recently, in [4], Honda showed the continuity of the eigenvalues of the Hodge Laplacian acting on the 1-forms with respect to the measured Gromov-Hausdorff topology under the assumption that the Ricci curvatures are bounded from below and above, the diameters are bounded from above and the volumes are bounded from below. He also showed the continuity of the eigenvalues of connection Laplacian acting on the tensor bundle of cotangent bundles and tangent bundles under the same assumptions.

In another direction, Kasue showed the convergence of the eigenvalues of the connection Laplacian on vector bundles over the Riemannian manifolds who have the uniform estimates of heat kernels in [5]. He considered the vector bundle \(E_\rho = P \times_\rho V\) with \(G\)-connection \(\nabla\) on a smooth Riemannian manifold \((X, g)\). Here, \(P\) is a principal \(G\)-bundle, \((\rho, V)\) is a representation of \(G\), \(P \times_\rho V\) is the associate bundle and \(\nabla = \nabla^A\) is the connection on \(E_\rho\) induced by a connection form \(A \in \Omega^1(P, g)\). Then the connection Laplacian \(\nabla^* \nabla\) acting on \(\Gamma(E)\) can be identified with the Laplacian of the certain Riemannian metric \(h = h(g, A, \sigma)\) on \(P\) determined by \(g, A\) and an bi-invariant metric \(\sigma\) on \(G\).

In this article, we also consider the continuity of the connection Laplacian on \(E_\rho\). The difference between [5] and this article is the topologies on the space of the vector bundles with connections. In this article we introduce...
the asymptotically $G$-equivariant measured Gromov-Hausdorff topology on the space of compact metric spaces with isometric $G$-actions. Then we can define the convergence of the sequence of $\{(P_i, A_i)\}_i$, where $P_i$ is a principal $G$-bundle over a Riemannian manifold $(X_i, g_i)$ and $A_i$ is a $G$-connection on $P_i$. Denote by $\lambda^\rho_{i,j}$ the $j$-th eigenvalue of $\nabla^{A_i} \nabla^{A_i}$ acting on $\Gamma(P_i \times_{\rho} V)$. The aim of this article is to show that if $\{(P_i, A_i)\}_i$ is the convergent sequence, then the limit $\lim_{i \to \infty} \lambda^\rho_{i,j}$ exists for any $j$ and $(\rho, V)$. As a consequence, we have the following theorem.

**Theorem 1.1.** Let $G$ be a compact Lie group and $(\rho, V)$ is a real unitary representation of $G$. For any $\kappa \in \mathbb{R}, D, N > 0$ and $j \in \mathbb{Z}_{\geq 0}$, there exist constants $0 \leq c_j < C_j$ depending only on $n, \kappa, D, N, j, G, \rho, V$ such that $\lim_{j \to \infty} c_j = \infty$ and the following holds. For any closed Riemannian manifold $(X, g)$ of dimension $n$ and principal $G$-bundle $\pi : P \to X$ with the $G$-connection $A$ such that

$$\dim X = n, \quad \text{Ric}_g \geq \kappa g, \quad \text{diam } X \leq D,$$

$$\| (d^{\nabla^A})^* F^A \|_{L^\infty} \leq N, \quad \| F^A \|_{L^\infty} \leq N,$$

we have

$$c_j \leq \lambda^A_j \leq C_j,$$

where $F^A \in \Omega^2(X, E^\rho)$ is the curvature form of $A$.

**Theorem 1.1** is proved as follows. To discuss with the connection Laplacian $\nabla^* \nabla$, we need the relation between $\text{Ric}_g \geq \kappa g$ and $\text{diam } X \leq D$.

$$\nabla^* \nabla : \Gamma(E^\rho) \to \Gamma(E^\rho)$$

and the Laplacian $\Delta$ of $h$ on $P$. In Section 2 we review the relation between the sections of $E^\rho$ and $G$-equivariant $V$-valued smooth functions on $P$. Next we construct the Riemannian metric $h$ on $P$ from $(X, g), A$ then show that $\nabla^* \nabla$ is related to the Laplacian of $h$ along \[\text{[5][7]}\] in Sections 3 and 4. In Section 5 we compute the Ricci curvature of $h$, and we will see that the boundedness of $\| F^A \|_{L^\infty}$ and $\| (d^{\nabla^A})^* F^A \|_{L^\infty}$ in the assumption of Theorem 1.1 is necessary to give the lower bound of the Ricci curvature of $h$. In Section 6 we introduce the notion of the asymptotically $G$-equivariant measured Gromov-Hausdorff convergences for the sequences of compact metric measure spaces with isometric $G$-actions, then show that if the sequence is precompact with respect to the measured Gromov-Hausdorff topology, then it is also precompact with respect to the asymptotically $G$-equivariant measured
Gromov-Hausdorff topology. In Sections 7 and 8, we review the results for the convergence of spectral structures on some metric measure spaces along \[2\] and apply them to our situation, then we see that these arguments are compatible with the $G$-actions.

Acknowledgments. The author would like to thank Professor Shouhei Honda for useful conversations and telling him the related works. The author was supported by Grant-in-Aid for Young Scientists (B) Grant Number 16K17598. The author was partially supported by JSPS Core-to-Core Program, “Foundation of a Global Research Cooperative Center in Mathematics focused on Number Theory and Geometry”.

2 Connections on principal bundles

Let $X$ be a smooth manifold and $\pi: P \to X$ be a principal $G$-bundle. A $G$-connection on $P$ is a differential form $A \in \Omega^1(P) \otimes \mathfrak{g}$ satisfying

$$A_u(\xi^u) = \xi, \quad R_\gamma^*A = \text{Ad}_{\gamma^{-1}}A$$

for all $u \in P$, $\xi \in \mathfrak{g}$ and $\gamma \in G$. Here, $\xi^u \in \mathcal{X}(P)$ is the vector field generated by $\xi \in \mathfrak{g}$, defined by

$$\xi^u_u := \left. \frac{d}{dt} \right|_{t=0} u \exp(t\xi).$$

The $G$-connection determines the horizontal distribution $H = \{H_u\}_{u \in P}$ by $H_u := \text{Ker} A_u \subset T_uP$. The curvature form $F^A \in \Omega^2(P) \otimes \mathfrak{g}$ is defined by

$$F^A := dA|_H = dA + \frac{1}{2}[A \wedge A].$$

2.1 Local trivialization

Let $(U, x^1, \ldots, x^n)$ be a local coordinate on $X$ and we suppose that $P|_U = \pi^{-1}(U) = U \times G$. Let $\hat{v}_u \in H_u$ be the horizontal lift of $v_x \in T_xX$. Now we have $T_{(x,\gamma)}P = T_xX \oplus T_\gamma G$, and let $e_1, \ldots, e_k \in \mathfrak{g}$ be a basis. Then $\{\partial_i = \frac{\partial}{\partial x_i}, e_\alpha^\gamma\}$ becomes a basis of $T_{(x,\gamma)}P$, where $(e_\alpha^\gamma) = (L_\gamma)_*e_\alpha$. Let $A^\alpha_i: U \times G \to \mathbb{R}$ be defined by

$$A^\alpha_i(x,\gamma)e_\alpha := A_{x,\gamma}(\partial_i).$$
Then we have
\[ A^n_i(x, \gamma)e_\alpha = \text{Ad}_{\gamma^{-1}}\{A^n_i(x, 1)e_\alpha\}, \]
\[ (\hat{\partial}_i)(x, \gamma) = (\partial_i)_x - A^n_i(x, \gamma)(e^\gamma_\alpha). \]

Now, we fix a real unitary representation \( \rho : G \to O(V) \) for a vector space \( V \) with inner product, and have
\[
\Omega^k_B(P, \rho, V) = \{ \tau \in \Omega^k(P) \otimes V; R^*_\tau = \rho(\gamma^{-1})\tau, \ i\xi \tau = 0 \}. 
\]

For the associate vector bundle \( E_\rho := P \times_\rho V \), we have the natural identification
\[
\Omega^k(X, E_\rho) \cong \Omega^k_B(P, \rho, V). 
\]

In particular, the correspondence between \( \tau \in \Gamma(X, E_\rho) \) and \( \hat{\tau} \in \Omega^0_B(P, \rho, V) \) is given by
\[
\tau(x) = u \times_\rho \hat{\tau}(u) \quad (u \in \pi^{-1}(x)). 
\]

Denote by \( \nabla^A \) the covariant derivative on \( E_\rho \), then \( \nabla \tau \) corresponds to \( d\hat{\tau}|_H \) under the identification \( \Omega^1(X, E_\rho) \cong \Omega^1_B(P, \rho, V) \). Under the direct decomposition \( T_u P = H_u \oplus \text{Ker } d\pi_u \), we have
\[
d\hat{\tau} = d\hat{\tau}|_H + d\hat{\tau}|_{\text{Ker } d\pi}. 
\]

For \( \xi \in \mathfrak{g} \) we have
\[
d\hat{\tau}_u(\xi_u) = \frac{d}{dt} \bigg|_{t=0} \hat{\tau}(u \exp(t\xi)) \\
= \frac{d}{dt} \bigg|_{t=0} \rho(\exp(-t\xi))\hat{\tau}(u) = -\rho_*(\xi)\hat{\tau}(u), 
\]
hence we obtain
\[
d\hat{\tau} = d\hat{\tau}|_H - \rho_* \circ A(\cdot)\hat{\tau}, \quad (1) 
\]

### 3 Riemannian metric on \( P \)

In this section we describe the relation between the rough Laplacian on the associate bundle \( E_\rho \) and the Laplacian on \( P \) for the certain metric along [5][7].
Fix an Ad$_G$-invariant metric $\sigma$ on $\mathfrak{g}$ and define a Riemannian metric
\[ h = h(g, A, \sigma) \] (2)
on $P$ by
\[ h_{ij} := h(\partial_i, \partial_j) = g(\partial_i, \partial_j) =: g_{ij}, \]
\[ h_{ia} = h(\partial_i, e^a_\bigcirc) = 0, \]
\[ h_{a\beta} := h(e^a_\bigcirc, e^\beta_\bigcirc) = \sigma(e_a, e_\beta) =: \sigma_{a\beta}. \]
By the decomposition (1), we have
\[ h^{-1}(d\hat{\tau}_0, d\hat{\tau}_1) = h^{-1}(d\hat{\tau}_0^H, d\hat{\tau}_1^H) + \sigma^{\alpha\beta}\langle \rho_*(e_\alpha)\hat{\tau}_0, \rho_*(e_\beta)\hat{\tau}_1 \rangle_V \]
\[ = g^{ij}\langle \nabla^A_\partial \hat{\tau}_0, \nabla^A_\partial \hat{\tau}_1 \rangle_{E_\rho} - \langle \sigma^{\alpha\beta}\rho_*(e_\beta)\rho_*(e_\alpha)\hat{\tau}_0, \hat{\tau}_1 \rangle_V. \]
Therefore by integrating on $P$ we have
\[ \int_P d^*d\hat{\tau}_0 \cdot \hat{\tau}_1 d\text{vol}_h = \int_P \langle (\nabla^A)^*\nabla^A \hat{\tau}_0, \hat{\tau}_1 \rangle_{E_\rho} d\text{vol}_h \]
\[ - \int_P \langle \sigma^{\alpha\beta}\rho_*(e_\beta)\rho_*(e_\alpha)\hat{\tau}_0, \hat{\tau}_1 \rangle_V d\text{vol}_h \]
for any $\tau_0, \tau_1 \in \Gamma(X, E_\rho)$, which gives
\[ \Delta^h\hat{\tau} = \Delta^A\tau - \sigma^{\alpha\beta}\rho_*(e_\beta)\rho_*(e_\alpha)\hat{\tau}, \]
where $\Delta^h = d^*d$ is the Laplacian of $h$ acting on the functions and $\Delta^A = (\nabla^A)^*\nabla^A$ is the rough Laplacian acting on the sections of $E_\rho$.

Here,
\[ \sigma^{\alpha\beta}\rho_*(e_\alpha)\rho_*(e_\beta): V \rightarrow V \]
is a $G$-equivariant map whose eigenvalues are real and nonpositive. Consequently, if $(\rho, V)$ is an irreducible representation, then by the following lemma we may write
\[ \sigma^{\alpha\beta}\rho_*(e_\alpha)\rho_*(e_\beta) = -\chi_{\sigma, \rho} \]
for some nonnegative number $\chi_{\sigma, \rho}$, called Casimir invariant, determined by $(\rho, V)$ and $\sigma$.

**Lemma 3.1.** Let $(\rho, V)$ be an irreducible real $G$-representation and $\Phi: V \rightarrow V$ be a $G$-equivariant linear map which has at least one real eigenvalue. Then there is $a \in \mathbb{R}$ such that $\Phi = a \cdot \text{id}_V$.

**Proof.** Let $a \in \mathbb{R}$ be an eigenvalue of $\Phi$ and $V(a) \subset V$ be the eigenspace associated with $a \in \mathbb{R}$. Since $V(a)$ is a subrepresentation of $V$, $V(a) = V$ holds since $V$ is irreducible. \qed
4 Eigenspaces

Let $X, P, G, E, \rho, V, h, A$ be as above. For $\lambda \in \mathbb{R}$ put

$$W^{E}_{\rho}(\lambda) := \{ s \in \Gamma(E_{\rho}) ; \Delta^{A}s = \lambda s \},$$

$$W^{h}(\lambda) := \{ f \in C^{\infty}(P) ; \Delta^{h}f = \lambda f \},$$

then we can see

$$(W^{h}(\lambda) \otimes V)^G = \{ f \in C^{\infty}_{B}(P, \rho, V) ; \Delta^{h}f = \lambda f \}.$$  

By the previous section we obtain an isomorphism

$$
(W^{h}(\lambda) \otimes V)^G \xrightarrow{\cong} W^{E}_{\rho}(\lambda - \chi_{\sigma, \rho}) \xrightarrow{\cong} u \times_{\rho} \hat{\tau}.
$$

5 Curvature

In this section we compute the curvature of $h(g, A, \sigma)$. Define $F^{\alpha}_{ij} \in C^{\infty}(P|_{U})$ by

$$F^{\alpha}_{ij}e_{\alpha} := F^{A}([\hat{\partial}_{i}, \hat{\partial}_{j}] \in g,$

and let $\Gamma^{k}_{ij}$ be the Christoffel symbols of $g$. By Section 3 we have

$$[\hat{\partial}_{i}, \hat{\partial}_{j}] = -F^{a}_{ij}e_{\alpha}^{a}, \quad [\hat{\partial}_{i}, e_{\beta}^{*}] = 0, \quad [e_{\alpha}^{*}, e_{\beta}^{*}] = [e_{\alpha}, e_{\beta}]^{g},$$

$$\nabla^{\hat{\partial}_{i}}\hat{\partial}_{j} = \Gamma^{k}_{ij}\hat{\partial}_{k} - \frac{1}{2}F^{a}_{ij}e_{\alpha}^{a}, \quad \nabla^{e_{\alpha}^{*}}e_{\beta}^{*} = \frac{1}{2}[e_{\alpha}, e_{\beta}]^{g},$$

$$\nabla^{\hat{\partial}_{i}}e_{\alpha}^{*} = \nabla^{e_{\alpha}^{*}}\hat{\partial}_{j} = \frac{g^{k\beta\sigma}(F_{ih}, e_{\beta})}{2} \hat{\partial}_{k} - \frac{g^{k\beta\sigma}F^{a}_{ih\alpha}e_{\alpha}^{a}}{2} \hat{\partial}_{k},$$

where $F_{ij} = F_{ij}^{\alpha}e_{\alpha}$. Moreover the 2nd Bianchi identity yields

$$0 = dF^{A}(\hat{\partial}_{i}, \hat{\partial}_{j}, \hat{\partial}_{k}) = \hat{\partial}_{i}(F_{jk}) - \hat{\partial}_{j}(F_{ik}) + \hat{\partial}_{k}(F_{ij}).$$
Now we denote by $\hat{R}$ the curvature tensor of $h$, and by $R$ that of $g$. Then we have

$$\hat{R}(\hat{\partial}_i, \hat{\partial}_j)\hat{\partial}_k = R_{ijk}\hat{\partial}_l + \frac{\hat{\partial}_k(F_{ij}^\alpha) - F_{il}^\alpha \Gamma_{jk}^l - F_{lj}^\alpha \Gamma_{ik}^l}{2} e^\alpha_\alpha$$

$$+ \frac{g^{hl}\sigma_{\alpha\beta}(2F_{ij}^\alpha F_{kh}^\beta - F_{jk}^\alpha F_{ih}^\beta - F_{ki}^\alpha F_{jh}^\beta)\hat{\partial}_l,}$$

$$\hat{R}(\hat{\partial}_i, e^\alpha_\alpha)\hat{\partial}_j = \frac{g^{hl}\sigma_{\alpha\beta}(\hat{\partial}_l(F_{jh}^\beta) - F_{jk}^\beta \Gamma_{lh}^k)}{2} e^\alpha_\alpha$$

$$- \frac{g^{kh}F_{jh}^\beta \Gamma_{ik}^l}{4}\sigma_{\alpha\beta} e^\alpha_\mu e^\beta_\mu + \frac{e^\alpha_\alpha(F_{ij}^\beta) e^\beta_\beta}{2} e^\alpha_\beta + \frac{F_{ij}^\beta[e_\alpha, e_\beta]^2}{4},$$

$$\hat{R}(\hat{\partial}_i, e^\alpha_\beta)e^\mu_\mu = \frac{[\hat{\partial}_l(F_{ij}^\beta) e^\alpha_\beta][e^\alpha_\mu, [e^\alpha_\mu, e^\alpha_\beta]]}{4}. $$

Here, we have

$$e^\alpha_\alpha(F_{ij}^\beta) e^\beta_\beta = -F_{ij}^\beta[e_\alpha, e_\beta], \quad e^\alpha_\alpha(F_{ij}^\beta) e^\beta_\beta = -F_{ij}^\beta[e_\alpha, e_\beta]^2.$$

By putting

$$ (\nabla F)^\alpha_{kj} = \hat{\partial}_k(F_{ij}^\alpha) - F_{il}^\alpha \Gamma_{jk}^l - F_{lj}^\alpha \Gamma_{ik}^l, $$

The 2nd Bianchi identity implies

$$ (\nabla F)^\alpha_{ij} + (\nabla F)^\alpha_{ji} + (\nabla F)^\alpha_{ki} = 0, $$
therefore we obtain

\[
\hat{R}(\hat{\partial}_i, \hat{\partial}_j)\hat{\partial}_k = R^l_{ijk}\hat{\partial}_l + \frac{1}{2} (\nabla F)^a_{ij} e^e_{\alpha} \\
+ \frac{g^{lh} \sigma_{\alpha\beta} (2F^a_{ij} F^\beta_{kh} - F^a_{jk} F^\beta_{ih} - F^a_{ki} F^\beta_{jh})}{4} \hat{\partial}_l,
\]

\[
\hat{R}(\hat{\partial}_i, e^e_{\alpha})\hat{\partial}_j = \frac{g^{lh} \sigma_{\alpha\beta} (\nabla F)^a_{ij}}{2} \hat{\partial}_l - \frac{g^{kh} F^\mu_{ih} \sigma_{\alpha\beta} e^e_{\mu}}{4} \\
- \frac{F^\beta_{ij} [e_\alpha, e_\beta]}{4},
\]

\[
\hat{R}(\hat{\partial}_i, e^e_{\beta})e^e_{\alpha} = -\frac{g^{lh} F^\mu_{ih} \sigma_{\alpha\beta} [e_\mu, [e_\alpha, e_\beta]]}{4} \hat{\partial}_l - \frac{g^{kh} F^\mu_{ih} \sigma_{\alpha\beta} g^{ip} F^\delta_{kp} \sigma_{\delta\alpha}}{4} \hat{\partial}_l,
\]

\[
\hat{R}(e^e_{\alpha}, e^e_{\beta})e^e_{\mu} = \frac{g^{kh} F^\mu_{ih} \sigma_{\alpha\beta} [e_\mu, e_\beta]}{4} \hat{\partial}_l + \frac{g^{kh} F^\mu_{ih} g^{ip} F^\delta_{kp} \sigma_{\delta\alpha} - \sigma_{\mu\alpha} \sigma_{\delta\beta}}{4} \hat{\partial}_l,
\]

Hence the Ricci curvature of $h$ is given by

\[
\hat{\text{Ric}}(\hat{\partial}_j, \hat{\partial}_k) = \text{Ric}_{jk} - \frac{g^{lh} \sigma_{\alpha\beta} F^a_{ki} F^\beta_{jh}}{2},
\]

\[
\hat{\text{Ric}}(\hat{\partial}_j, e^e_{\beta}) = -\frac{g^{lh} \sigma_{\beta\mu} (\nabla F)^a_{ij}}{2} \hat{\partial}_l,
\]

\[
\hat{\text{Ric}}(e^e_{\beta}, e^e_{\mu}) = \frac{g^{kh} F^\alpha_{hi} g^{ip} F^\delta_{kp} \sigma_{\delta\beta} + \sigma_{\alpha\beta} [e_\alpha, e_\beta]}{4}.
\]

Now, define $F^* F \in \Gamma(\text{Symm}_2(H^*)) \otimes \text{Symm}_2(g)$ by

\[
F^* F = g^{kl} F^a_{ik} F^\beta_{jl} \hat{\partial}^l \otimes e_\alpha \otimes e_\beta
\]
where $H^* \rightarrow P$ is the dual of the horizontal distribution $H \rightarrow P$ and $\{\hat{\partial}_i\}_i$ is the dual basis of $\{\hat{\partial}_i\}_i$. Note that $\{(d\nabla)^* F\}^\alpha_\beta_{jk} = -g^{\delta k}(\nabla F)^\alpha_{hij}$. Then we have

\[
\hat{\text{Ric}}(\hat{\partial}_j, \hat{\partial}_k) = \text{Ric}_{jk} - \frac{(F^* F)^{\alpha \beta}_{jk} \sigma_{\alpha \beta}}{2},
\]

\[
\hat{\text{Ric}}(\hat{\partial}_j, e^x_\beta) = \frac{\{(d\nabla)^* F\}^\mu_{jk} \sigma_{\beta \mu}}{2},
\]

\[
\hat{\text{Ric}}(e^x_\beta, e^x_\mu) = \frac{g_{jk}(F^* F)^{\alpha \beta}_{jk} \sigma_{\alpha \beta \sigma \delta \mu}}{4} + \frac{\sigma^{\alpha \delta \sigma}([e_\alpha, e_\beta], [e_\delta, e_\mu])}{4}.
\]

6 \quad G\text{-structures on metric spaces}

In Section 3, we have shown that $\pi: (P, h) \rightarrow (X, g)$ is a Riemannian submersion and every $G$-orbit is totally geodesically embedded in $P$ and isometric to $(G, \sigma)$. Conversely, if $(P, h)$ is a Riemannian manifold with isometric free $G$-action for a compact Lie group $G$ and every $G$-orbit is isometric to $(G, \sigma)$, then $X = P/G$ is a smooth manifold with a Riemannian metric $g$ such that $\pi: (P, h) \rightarrow (X, g)$ is a Riemannian submersion, and the horizontal distribution of $P$ defines a $G$-connection. We generalize this picture to the metric spaces.

In this article, $G$-actions on the metric spaces always mean the right actions, and the maps

\[
P \times G \rightarrow P
\]

\[
(\psi)(u, \gamma) \mapsto u\gamma
\]

are always supposed to be continuous. We denote by $\bar{u} \in P/G$ the equivalence class represented by $u \in P$.

**Proposition 6.1.** Let $G$ be a compact topological group, and $(P, d)$ be a metric space with isometric continuous right $G$-action. Then the quotient map $\pi: (P, d) \rightarrow (P/G, \bar{d})$ is a submetry, where $\bar{d}$ is a distance function on $P$ defined by $\bar{d}(\bar{u}_0, \bar{u}_1) := \inf_{g \in G} d(u_0, u_1 g)$.

**Proof.** Let $D(u_0, r) := \{u_1 \in P; d(u_1, u) \leq r\}$. We show $\pi(D(u_0, r)) = D(\bar{u}_0, r)$ for all $u_0 \in P$ and $r > 0$. Let $u_1 \in D(u_0, r)$. Then $d(\bar{u}_0, \bar{u}_1) \leq d(u_0, u_1) \leq r$, hence we have $\pi(u_1) = \bar{u}_1 \in D(\bar{u}_0, r)$. Conversely, if $\bar{u}_1 \in D(\bar{u}_0, r)$, then there exists $g \in G$ such that $d(u_0, u_1 g) = \bar{d}(\bar{u}_0, \bar{u}_1) \leq r$, consequently $\bar{u}_1 \in \pi(D(u_0, r))$. \hfill \Box

**Definition 6.2.**
(1) Let \((P', d')\) and \((P, d)\) be metric spaces. A map \(\phi : P' \to P\) is an \(\varepsilon\)-isometry if (i) \(d'(u_0, u_1) - d(\phi(u_0), \phi(u_1))\) < \(\varepsilon\) for any \(u_0, u_1 \in P'\), (ii) \(P \subset B(f(P'), \varepsilon)\).

(2) Let \(\{(P_i, d_i, \nu_i)\}_i\) be a sequence of metric measure spaces. A metric measure space \((P_{\infty}, d_{\infty}, \nu_{\infty})\) is the measured Gromov-Hausdorff limit of \(\{(P_i, d_i, \nu_i)\}_i\) if there are positive numbers \(\{\varepsilon_i\}_i\) with \(\lim_{i \to \infty} \varepsilon_i = 0\) and Borel \(\varepsilon_i\)-isometries \(\phi_i : P_i \to P_{\infty}\) for every \(i\) such that \(\phi_{i*} \nu_i \to \nu_{\infty}\) weakly as \(i \to \infty\), that is,

\[
\lim_{i \to \infty} \int_{P_{\infty}} f d\phi_{i*} \nu_i = \int_{P_{\infty}} f d\nu_{\infty}
\]

holds for any continuous function \(f : P_{\infty} \to \mathbb{R}\).

**Definition 6.3.** Let \(G\) be a compact topological group.

(1) Let \((P', d')\) and \((P, d)\) be metric spaces with isometric \(G\)-action. A map \(\phi : P' \to P\) is an \(\varepsilon\)-approximation of \(G\)-equivariant isometry if \(\phi\) is an \(\varepsilon\)-isometry satisfying \(d(\phi(u'\gamma), \phi(u')\gamma) < \varepsilon\) for any \(u' \in P'\) and \(\gamma \in G\). Moreover if \(\phi\) is a Borel map then it is called a Borel \(\varepsilon\)-approximation of \(G\)-equivariant isometry.

(2) Let \(\{(P_i, d_i, \nu_i)\}_i\) be a sequence of metric measure spaces with isometric \(G\)-action. A metric measure space \((P_{\infty}, d_{\infty}, \nu_{\infty})\) with isometric \(G\)-action is called the asymptotically \(G\)-equivariant measured Gromov-Hausdorff limit of \(\{(P_i, d_i, \nu_i)\}_i\) if there is a Borel \(\varepsilon_i\)-approximation of \(G\)-equivariant isometry \(\phi_i : P_i \to P_{\infty}\) for each \(i\) such that \(\lim_{i \to \infty} \varepsilon_i = 0\) and \(\phi_{i*} \nu_i \to \nu_{\infty}\) weakly as \(i \to \infty\).

**Remark 6.4.** In Definition 6.3 \(\phi_i : P_i \to P_{\infty}\) are not required to be \(G\)-equivariant. If all of \(\phi_i\) are \(G\)-equivariant, then we obtain another topology which is already introduced by Lott in [Z03].

We denote by

\[
(P_i, d_i, \nu_i) \xrightarrow{\text{mGH}} (P_{\infty}, d_{\infty}, \nu_{\infty}) \quad \text{or} \quad P_i \xrightarrow{\text{mGH}} P_{\infty}
\]

the pair of a sequence of metric measure spaces \(\{(P_i, d_i, \nu_i)\}_i\) and its measured Gromov-Hausdorff limit \((P_{\infty}, d_{\infty}, \nu_{\infty})\). Similarly, we write

\[
(P_i, d_i, \nu_i) \xrightarrow{G\text{-mGH}} (P_{\infty}, d_{\infty}, \nu_{\infty}) \quad \text{or} \quad P_i \xrightarrow{G\text{-mGH}} P_{\infty}
\]

if \(G\) acts on \((P_i, d_i, \nu_i)\) and \((P_{\infty}, d_{\infty}, \nu_{\infty})\), and \((P_{\infty}, d_{\infty}, \nu_{\infty})\) is the asymptotically \(G\)-equivariant measured Gromov-Hausdorff limit of \(\{(P_i, d_i, \nu_i)\}_i\).
Proposition 6.5. Let \((P, d)\) and \((P', d')\) be metric spaces with isometric \(G\)-action, and \(\phi : P' \to P\) be an \(\varepsilon\)-approximation of \(G\)-equivariant isometry. Then there exists a \(2\varepsilon\)-isometry \(\phi : P'/G \to P/G\). Moreover, suppose that \(\nu\) and \(\nu'\) are Borel measures on \(P\) and \(P'\) respectively, \(\phi\) is a Borel map, \(X\) is compact, and there is a Borel section \(s' : P'/G \to P'\), namely, Borel map \(s'\) with \(\pi' \circ s' = \text{id}_{P'/G}\). Then \(\phi\) is also a Borel map and the following holds. For any \(f \in C(X)\) and \(\varepsilon' > 0\) there exists \(\delta > 0\) depending only on \(\varepsilon'\) and \(f\) such that if \(\varepsilon \leq \delta\) then

\[
\left| \int_X f \, d\phi_\ast \nu - \int_X f \, d\nu' \right| \leq \nu'(P')\varepsilon' + \left| \int_P f \circ \pi \, d\phi_\ast \nu' - \int_P f \circ \pi \, d\nu \right|
\]

holds, where \(\bar{\nu} = \pi_\ast \nu\) and \(\bar{\nu}' = \pi_\ast \nu'\).

Proof. Put \(X := P/G\), \(X' := P'/G\). First of all we define \(\bar{\phi} : P'/G \to P/G\) such that

\[
|\bar{d}(\bar{u}_0, \bar{u}_1) - \bar{d}(\bar{\phi}(\bar{u}_0), \bar{\phi}(\bar{u}_1))| < 2\varepsilon.
\]

Fix \(s' : P'/G \to P'\) with \(\pi' \circ s' = \text{id}_{P'/G}\) and put

\[
\bar{\phi}(x) := \phi(s'(x)).
\]

Note that \(\bar{\phi}\) is a Borel map if so are \(\phi\) and \(s'\).

For \(x_0, x_1 \in P'\), take \(\gamma_0 \in G\) such that

\[
d(\phi(s'(x_0)), \phi(s'(x_1))\gamma_0) = \bar{d}(\bar{\phi}(x_0), \bar{\phi}(x_1)).
\]

Then we have

\[
\bar{d}(x_0, x_1) - \bar{d}(\bar{\phi}(x_0), \bar{\phi}(x_1)) = \bar{d}(x_0, x_1) - d(\phi(s'(x_0)), \phi(s'(x_1))\gamma_0)
\]

\[
< \bar{d}(x_0, x_1) - d(\phi(s'(x_0)), \phi(s'(x_1))\gamma_0) + \varepsilon
\]

\[
< \bar{d}(x_0, x_1) - d'(s'(x_0), s'(x_1))\gamma_0 + 2\varepsilon
\]

\[
\leq 2\varepsilon.
\]

Take \(\gamma_1 \in G\) such that \(d'(s'(x_0), s'(x_1))\gamma_1) = \bar{d}(x_0, x_1)\). Then we have

\[
d(\phi(s'(x_0)), \phi(s'(x_1))\gamma_1) = \bar{d}(\bar{\phi}(x_0), \bar{\phi}(x_1)) - d'(s'(x_0), s'(x_1))\gamma_1
\]

\[
< \bar{d}(\bar{\phi}(x_0), \bar{\phi}(x_1)) - d(\phi(s'(x_0)), \phi(s'(x_1))\gamma_1) + \varepsilon
\]

\[
< \bar{d}(\bar{\phi}(x_0), \bar{\phi}(x_1)) - d(\phi(s'(x_0)), \phi(s'(x_1))\gamma_1) + 2\varepsilon
\]

\[
\leq 2\varepsilon.
\]
Next we show $X \subset B(\tilde{\phi}(X'), 2\varepsilon)$. Let $\bar{u} \in X$, then there is $u' \in P'$ such that $d(u, \phi(u')) < \varepsilon$. Take $\gamma_2 \in G$ such that $u' = s'(\bar{u})\gamma_2$. Then we have

$$d(\bar{u}, \phi(\bar{u})) \leq d(u, \phi(s'(\bar{u}))\gamma_2) < d(u, \phi(s'(\bar{u})\gamma_2)) + \varepsilon < 2\varepsilon,$$

which implies $\bar{u} \in B(\tilde{\phi}(X'), 2\varepsilon)$.

Suppose $\nu$ and $\nu'$ are Borel measures on $P$ and $P'$ respectively and $f: X_\infty \to \mathbb{R}$ is a continuous function. Now we have

$$\left| \int_X f \, d(\tilde{\phi}_* \nu') - \int_X f \, d\nu \right| \leq \left| \int_{P'} f \circ \tilde{\phi} \circ \pi' \, d\nu' - \int_{P'} f \circ \pi \circ \phi \, d\nu' \right|$$

$$+ \left| \int_P f \circ \pi \, d\phi_* \nu' - \int_P f \circ \pi \, d\nu \right|$$

$$\leq \| f \circ \tilde{\phi} \circ \pi' - f \circ \pi \circ \phi \|_{L^\infty(P')}$$

$$+ \left| \int_P f \circ \pi \, d\phi_* \nu' - \int_P f \circ \pi \, d\nu \right|.$$ 

Since $X_\infty$ is a compact metric space, $f$ is uniformly continuous, therefore for any $\varepsilon' > 0$ there is $\delta > 0$ which depends only on $\varepsilon'$ and $f$ such that $d(x_0, x_1) < \delta$ implies $|f(x_0) - f(x_1)| < \varepsilon'$. Since

$$\bar{d}(\phi \circ \pi'(u'), \pi \circ \phi(u')) = \inf_{\gamma \in G} d(\phi(s'(\bar{u})), \phi(u')\gamma)$$

$$< \inf_{\gamma \in G} d(\phi(s'(\bar{u})), \phi(u')\gamma) + \varepsilon$$

$$< \inf_{\gamma \in G} d'(s'(\bar{u}), u'\gamma) + 2\varepsilon = 2\varepsilon,$$

hence if $\varepsilon$ is not more than $\frac{\delta}{2}$ then

$$\left| \int_X f \, d(\tilde{\phi}_* \nu') - \int_X f \, d\nu \right| \leq \nu'(P')\varepsilon' + \left| \int_P f \circ \pi \, d\phi_* \nu' - \int_P f \circ \pi \, d\nu \right|$$

holds.

As a consequence of Proposition 6.5 we obtain the following results.

**Corollary 6.6.** If $(P_i, d_i, \nu_i) \xrightarrow{G-mGH} (P_\infty, d_\infty, \nu_\infty)$, $P_\infty$ is compact and every $P_i \to P_i/G$ has a Borel sections, then $(P_i/G, \bar{d}_i, \tilde{\nu}_i) \xrightarrow{mGH} (P_\infty/G, \bar{d}_\infty, \tilde{\nu}_\infty)$.

**Proposition 6.7.** Let $(P_i, d_i) \xrightarrow{GH} (P_\infty, d_\infty)$, that is, there are $\varepsilon_i$-isometry $\phi_i: P_i \to P_\infty$ for each $i$ and $\lim_{i \to \infty} \varepsilon_i = 0$ holds. Suppose that a compact
topological group $G$ acts on every $P_i$ isometrically and $P_\infty$ is compact. Moreover assume that the family of continuous maps \( \{F_{i,u}\}_{i \in \mathbb{Z}_{\geq 0}, u \in P_i}\), where

\[
F_{i,u} : G \times G \rightarrow \mathbb{R} \quad (\gamma, \gamma') \mapsto d_i(u\gamma, u\gamma'),
\]
is equicontinuous. For any decreasing sequence \( \{\varepsilon_k\}\) with \( \lim_{k \to \infty} \varepsilon_k = 0 \), there are a subsequence \( \{(P_{i_k}, d_{i_k})\}\) and an isometric $G$-action on \( (P_\infty, d_\infty) \) such that \( \phi_{i_k} \) is an $\varepsilon_k$-approximation of $G$ equivariant isometry.

**Proof.** Put

\[ G_0 := \{ \gamma \in G ; \ u\gamma = u \text{ for all } u \in X_i \text{ and } i \in \mathbb{Z}_{\geq 0} \}, \]
then $G_0$ is a closed normal subgroup of $G$. First of all we induce the metric on the quotient group $G_0 \backslash G$ compatible with its quotient topology. For every $u \in P_i$, let $P_{i,u}$ be the metric space isometric to $P_i$ and put

\[ \mathcal{X} := \prod_{i \in \mathbb{Z}_{\geq 0}, u \in P_i} P_{i,u} \]
be the metric space whose distance function is given by

\[ d_{\mathcal{X}}((p_{i,u})_{i,u}, (q_{i,u})_{i,u}) := \sup_{i,u} d_i(p_{i,u}, q_{i,u}). \]

Define the injective map $f_\mathcal{X} : G_0 \backslash G \to \mathcal{X}$ by $f_\mathcal{X}(\gamma) := (u\gamma)_{i,u}$. Here, we denote by $\gamma$ the right coset in $G_0 \backslash G$ represented by $\gamma$ for the brevity. Then the induced metric $d_G := f_\mathcal{X}^*d_{\mathcal{X}}$ on $G_0 \backslash G$ is given by

\[ d_G(\gamma, \gamma') = \sup_{i,u} d(u\gamma, u\gamma') = \sup_{i,u} F_{i,u}(\gamma, \gamma'). \]

Denote by $\mathcal{O}$ the quotient topology on $G_0 \backslash G$ and denote by $\mathcal{O}_{d_G}$ the topology induced by $d_G$. Since $\{F_{i,u}\}_{i,u}$ is equicontinuous, $d_G : G_0 \backslash G \times G_0 \backslash G \rightarrow \mathbb{R}$ is continuous with respect to $\mathcal{O}$, hence $\mathcal{O}_{d_G}$ is weaker than or equal to $\mathcal{O}$ then one can see $(G_0 \backslash G, \mathcal{O}_{d_G})$ is compact and $f_\mathcal{X}(G_0 \backslash G)$ is closed in $\mathcal{X}$. Now we can see that $f_\mathcal{X} : (G_0 \backslash G, \mathcal{O}) \rightarrow \mathcal{X}$ is continuous by the following reason. If $x \in \mathcal{X} \setminus f_\mathcal{X}(G_0 \backslash G)$, there exists $\delta > 0$ such that $B(x, \delta) \subset \mathcal{X} \setminus f_\mathcal{X}(G_0 \backslash G)$, hence $f_\mathcal{X}^{-1}(B(x, \delta)) = \emptyset$. If $x \in f_\mathcal{X}(G_0 \backslash G)$ then there exists $\gamma_0 \in G$ such that $f_\mathcal{X}(\gamma_0) = x$ and we have $f_\mathcal{X}^{-1}(B(x, \delta)) = d_G(\gamma_0, \cdot)^{-1}((\infty, \delta))$, which is an open set. Thus we can see that $f : (G_0 \backslash G, \mathcal{O}) \rightarrow f_\mathcal{X}(G_0 \backslash G)$ is a bijective continuous map from a compact space to a Hausdorff space, hence it is a homeomorphism, which implies $\mathcal{O} = \mathcal{O}_{d_G}$. 

14
Since $P_\infty$ and $G_0 \setminus G$ are compact metric spaces, they are separable. Let
\[
\{u^{(\alpha)}\}_{\alpha=0}^{\infty} \subset P_\infty, \quad \{\gamma^{(\beta)}\}_{\beta=0}^{\infty} \subset G_0 \setminus G
\]
be countable dense subsets.

By the compactness of $P_\infty$ and $G_0 \setminus G$, we may assume that there are increasing sequences of integers $0 = K_0 < K_1 < K_2 < \cdots$ and $0 = L_0 < L_1 < L_2 < \cdots$ such that
\[
\bigcup_{\alpha=K_m}^{K_{m+1}-1} B \left( u^{(\alpha)}, \frac{1}{2m} \right) = P_\infty, \quad \bigcup_{\beta=L_m}^{L_{m+1}-1} B \left( \gamma^{(\beta)}, \frac{1}{2m} \right) = G_0 \setminus G.
\]

First of all we define $u^{(\alpha)} \cdot \gamma^{(\beta)} \in P_\infty$. Take $u^{(\alpha)}_i \in P_i$ such that $d_\infty(\phi_i(u^{(\alpha)}_i), u^{(\alpha)}) < \varepsilon_i$.

Since $P_\infty$ is compact, there is a subsequence of $\{\phi_i(u^{(\alpha)}_i \gamma^{(\beta)})\}$ converging to some points in $P_\infty$. By repeating this procedure for $(\alpha, \beta) = (0, 0), (1, 0), (0, 1), (2, 0), (1, 1), (0, 2), \ldots$ and by the diagonal argument, there is $\{i_k\}_{k=0} \subset \{i = 0, 1, 2, \ldots\}$ such that $\{\phi_{i_k}(u^{(\alpha)}_i \gamma^{(\beta)})\}_{i_k}$ converges to the limit. We put
\[
u^{(\alpha)}_i \cdot \gamma^{(\beta)} := \lim_{k \to \infty} \phi_{i_k}(u^{(\alpha)}_i \gamma^{(\beta)}),
\]
then we obtain a map
\[
\{u^{(\alpha)}\}_\alpha \times \{\gamma^{(\beta)}\}_\beta \to P_\infty, \quad (u^{(\alpha)}, \gamma^{(\beta)}) \mapsto \nu^{(\alpha)}_i \cdot \gamma^{(\beta)}.
\]

By taking a subsequence $\{i_{0,k}\}_k$ of $\{i_k\}_k$, we may suppose that
\[
d_\infty(\phi_{i_{0,k}}(u^{(\alpha)}_{i_{0,k}} \gamma^{(\beta)}), u^{(\alpha)}_i \gamma^{(\beta)}) < \frac{1}{2^k}
\]
for any $0 \leq \alpha < K_1$, $0 \leq \beta < L_1$ and $k \geq 0$. We can take a subsequence $\{i_{m,k}\}_k$ of $\{i_{m-1,k}\}_k$ inductively such that
\[
d_\infty(\phi_{i_{m,k}}(u^{(\alpha)}_{i_{m,k}} \gamma^{(\beta)}), u^{(\alpha)}_i \gamma^{(\beta)}) < \frac{1}{2^k}
\]
for any $K_m \leq \alpha < K_{m+1}$, $L_m \leq \beta < L_{m+1}$ and $k \geq 0$. By replacing $i_k$ by $i_{k,k}$, we may assume that
\[
d_\infty(\phi_{i_k}(u^{(\alpha)}_i \gamma^{(\beta)}), u^{(\alpha)}_i \gamma^{(\beta)}) < \frac{1}{2^k}
\]
for any $K_m \leq \alpha < K_{m+1}$, $L_m \leq \beta < L_{m+1}$ and $k \geq m$

Next we show the continuity of the above map. We have

$$d_\infty(u^{(\alpha)} \gamma^{(\beta)}(\beta), u^{(\alpha')} \gamma^{(\beta')}) = d_\infty\left(\lim_{k \to \infty} \phi_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}), \lim_{k \to \infty} \phi_{ik}(u^{(\alpha')}_{ik} \gamma^{(\beta')})\right)$$

$$= \lim_{k \to \infty} d_\infty(\phi_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}), \phi_{ik}(u^{(\alpha')}_{ik} \gamma^{(\beta')}))$$

$$\leq \limsup_{k \to \infty} \left( d_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}, u^{(\alpha')}_{ik} \gamma^{(\beta')}) + \varepsilon_{ik} \right)$$

$$= \limsup_{k \to \infty} d_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}, u^{(\alpha')}_{ik} \gamma^{(\beta')})$$

$$\leq \limsup_{k \to \infty} \left( d_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}, u^{(\alpha')}_{ik} \gamma^{(\beta')}) + d_{ik}(u^{(\alpha)}_{ik}, u^{(\alpha')}_{ik}) \right)$$

$$\leq d_G(\gamma^{(\beta)}, \gamma^{(\beta')}) + d_\infty(u^{(\alpha)}, u^{(\alpha')})$$

which gives the continuity of (3). Then we can extend the map to

$$P_\infty \times G \to P_\infty, \quad (u, \gamma) \mapsto u\gamma.$$  (4)

continuously. Next we show that for any $u_{ik} \in P_k$ and $\gamma \in G$ with $\phi_{ik}(u_{ik}) \to u$, we have $\phi_{ik}(u_{ik}\gamma) \to u\gamma$. Take $u^{(\alpha)}$ and $\gamma^{(\beta)}$ such that $d_\infty(u^{(\alpha)}, u)$ and $d(\gamma^{(\beta)}, \gamma)$ are small. Then one can see

$$d_\infty(\phi_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}), \phi_{ik}(u_{ik} \gamma)) < d_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}, u_{ik} \gamma) + \varepsilon_{ik}$$

$$\leq d_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}, u^{(\alpha)}_{ik} \gamma)$$

$$+ d_{ik}(u^{(\alpha)}_{ik} \gamma, u_{ik} \gamma) + \varepsilon_{ik}$$

$$\leq d_G(\gamma^{(\beta)}, \gamma) + d_{ik}(u^{(\alpha)}_{ik}, u_{ik}) + \varepsilon_{ik}$$

$$\leq d_G(\gamma^{(\beta)}, \gamma) + d_\infty(u^{(\alpha)}_{ik}, \phi_{ik}(u_{ik})) + 2\varepsilon_{ik}$$

$$\leq d_G(\gamma^{(\beta)}, \gamma) + d_\infty(u^{(\alpha)}, u)$$

$$+ d_{ik}(\phi_{ik}(u_{ik}), u) + 3\varepsilon_{ik},$$

therefore we obtain

$$d_\infty(u \gamma, \phi_{ik}(u_{ik} \gamma)) < d_\infty(u \gamma, u^{(\alpha)} \gamma^{(\beta)}) + d_\infty(u^{(\alpha)} \gamma^{(\beta)}, \phi_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)}))$$

$$= d_G(\gamma^{(\beta)}, \gamma) + d_\infty(u^{(\alpha)}, u)$$

$$+ d_{ik}(\phi_{ik}(u_{ik}), u) + 3\varepsilon_{ik}$$

$$\leq d_\infty(u^{(\alpha)} \gamma^{(\beta)}, \phi_{ik}(u^{(\alpha)}_{ik} \gamma^{(\beta)})) + 2d_G(\gamma^{(\beta)}, \gamma)$$

$$+ 2d_\infty(u^{(\alpha)}, u) + d_\infty(\phi_{ik}(u_{ik}), u) + 3\varepsilon_{ik}.$$  (5)
Now, choose \( \alpha, \beta \) such that \( K_k \leq \alpha < K_{k+1}, \) \( L_k \leq \beta < L_{k+1} \) and \( d_\infty(u^{(\alpha)}, u) < \frac{1}{2k}, \) \( d_G(\gamma^{(\beta)}, \gamma) < \frac{1}{2k} \). Then by (5) we have
\[
\d( u_\gamma, \phi_{i_k}(u_{i_k}\gamma) ) \leq \frac{5}{2k} + 3\epsilon_{i_k} + d_\infty(\phi_{i_k}(u_{i_k}), u), \tag{6}
\]
which implies \( \lim_{k \to \infty} \phi_{i_k}(u_{i_k}\gamma) = u_\gamma \).

Next we have to show \( (u\gamma)\gamma' = u(\gamma'\gamma) \) for any \( u \in P_\infty \) and \( \gamma, \gamma' \in G \). Take \( u_{i_k} \in P_k \) such that \( \phi_{i_k}(u_{i_k}) \to u \) as \( k \to \infty \). Then we have \( \phi_{i_k}(u_{i_k}\gamma) \to u_\gamma \), hence \( \phi_{i_k}(u_{i_k}\gamma\gamma') \to (u\gamma)\gamma' \). Obviously, \( \phi_{i_k}(u_{i_k}\gamma\gamma') \to u(\gamma'\gamma) \) also holds, therefore we obtain \( (u\gamma)\gamma' = u(\gamma'\gamma) \).

One can see that the action on \( P_\infty \) is isometric since
\[
d_\infty(u_0\gamma, u_1\gamma) = \lim_{k \to \infty} d_\infty(\phi_{i_k}(u_{0,i_k}\gamma), \phi_{i_k}(u_{1,i_k}\gamma))
= \lim_{k \to \infty} d_{i_k}(u_{0,i_k}\gamma, u_{1,i_k}\gamma)
= \lim_{k \to \infty} d_{i_k}(u_{0,i_k}, u_{1,i_k})
= \lim_{k \to \infty} d_\infty(\phi_{i_k}(u_{0,i_k}), \phi_{i_k}(u_{1,i_k})) = d_\infty(u_0, u_1),
\]
where \( \phi_{i_k}(u_{0,i_k}) \to u_0, \phi_{i_k}(u_{1,i_k}) \to u_1 \).

By (6), we have
\[
d_\infty(\phi_{i_k}(u'\gamma), \phi_{i_k}(u'\gamma)) \leq \frac{5}{2k} + 3\epsilon_{i_k},
\]
for some \( \beta \) with \( d_G(\gamma(\beta), \gamma) < \frac{1}{2k} \), which implies that \( \phi_{i_k} \) is an \( \frac{5}{2k} + 3\epsilon_{i_k} \)-approximation of \( G \)-equivariant isometry.

**Remark 6.8.** Suppose \( P_i \) is compact. The following map
\[
P_i \times G \times G \quad \mapsto \quad \mathbb{R}
\]
\[
(u, \gamma, \gamma') \quad \mapsto \quad d_i(u\gamma, u\gamma'),
\]
is continuous, hence uniformly continuous. Then \( \{F_{i,u}\}_u \) is always equicontinuous for any fixed \( i \).

As a consequence of Proposition 6.7, we also obtain the following.

**Proposition 6.9.** Let \((P_i, d_i)\) and \((P_\infty, d_\infty)\) be metric spaces with isometric \( G \)-actions satisfying the assumption of Proposition 6.7 and let \((P_i, d_i, i_1) \xrightarrow{mGH} (P_\infty, d_\infty, \nu_\infty)\) as \( i \to \infty \). Then there exists a subsequence \( \{(P_{i_k}, d_{i_k}, \nu_{i_k})\}_k \) such that \((P_{i_k}, d_{i_k}, \nu_{i_k}) \xrightarrow{G-mGH} (P_\infty, d_\infty, \nu_\infty)\) as \( k \to \infty \).
Any smooth closed Riemannian manifold \((X, g)\) can be regarded as a metric measure space by the Riemannian distance \(d_g\) and the measure \(\mu_g\), where \(\mu_g\) is the Riemannian measure. For a principal \(G\)-bundle \(\pi: P \to X\) and a \(G\)-connection \(A\) on \(P\), let \(|F^A|^2(x) := (F^A(x), F^A(x))_{\sigma x \otimes \sigma x}\), where \(\sigma\) is a \(\text{Ad}_G\)-invariant metric on \(G\). Similarly we can define \(|(d^{\nabla^A})^*F^A|^2(x)\).

**Theorem 6.10.** Let \(\{(X_i, g_i)\}_{i=0}^\infty\) be a sequence of closed Riemannian manifolds with

\[
\dim X_i = n, \; \text{Ric}_{g_i} \geq \kappa g_i, \; \text{diam} \; X_i \leq D
\]

for some constants \(n, \kappa, D\) independent of \(i\), \(G\) be a compact Lie group and \(\pi_i: P_i \to X_i\) be principal \(G\)-bundle with \(G\)-connection \(A_i\) satisfying

\[
\sup_{x \in X_i} |(d^{\nabla^{A_i}})^*F^{A_i}|(x) < N, \quad \sup_{x \in X_i} |F^{A_i}|(x) < N
\]

for a constant \(N > 0\) independent of \(i\). Then there exists a subsequence \(\{i_k\}_k\) and a metric measure space \((P_\infty, d_\infty, \nu_\infty)\) with the isometric \(G\)-action such that

\[
(P_{i_k}, h(g_{i_k}, A_{i_k}, \sigma)) \xrightarrow{G-\text{mGH}} (P_\infty, d_\infty, \nu_\infty), \quad (X_{i_k}, g_{i_k}) \xrightarrow{\text{mGH}} (P_\infty/G, \bar{d}_\infty, \bar{\nu}_\infty)
\]

where \(h(g_i, A_i, \sigma)\) is the metric on \(P_i\) defined by (2).

**Proof.** By the assumption, the Ricci curvatures of \(h(g_i, A_i, \sigma)\) are bounded below by the constant independent of \(i\), hence, \(\{(P_i, h(g_i, A_i, \sigma))\}_i\) is precompact with respect to the measured Gromov-Hausdorff topology. Moreover, the functions \(F_{i,u}\) in the assumption of Proposition 6.7 are equal to the distance function induced by \(\sigma\), accordingly \(\{F_{i,u}\}_{i,u}\) is equicontinuous. Therefore, by Proposition 6.7 there exists a convergent subsequence of \(\{(P_i, h(g_i, A_i, \sigma))\}_i\) with respect to the asymptotically \(G\)-equivariant measured Gromov-Hausdorff topology. Finally, to apply Corollary 6.6 it suffices to show that all principal \(G\)-bundles \(\pi: P \to X\) over a compact smooth manifold has a Borel section. Since \(X\) is compact, there is a finite open covering \(\{U_\alpha\}_{\alpha=1}^N\) of \(X\) such that \(P|_{U_\alpha} \to U_\alpha\) are trivial bundles. Put \(U_0^\alpha := U_\alpha\), \(U_\alpha^\prime := X \setminus U_\alpha\) and \(V_\sigma := \bigcap_{\alpha=1}^N U_\alpha^{(\sigma)}\) for a map \(\sigma: \{1, \ldots, N\} \to \{0, 1\}\). Then we have \(X = \bigsqcup_\sigma V_\sigma\) and every \(P|_{V_\sigma} \to V_\sigma\) has a continuous section \(s_\sigma: V_\sigma \to P|_{V_\sigma}\). Since \(V_\sigma\) are Borel sets, we have the Borel section of \(P \to X\) by gluing \(\{s_\sigma\}_\sigma\).
7 Convergence of eigenfunctions

Any Riemannian manifold \((X, g)\) can be canonically regarded as the metric measure space \((X, d_g, \mu_g)\), where \(d_g\) is the Riemannian distance of \(g\) and \(\mu_g\) is the Riemannian measure. Denote by \(\mathcal{M}(n, \kappa, D)\) the set consisting of isometric classes of metric measure spaces coming from closed Riemannian manifolds \((X, g)\) with \(\dim X = n, \text{Ric}_g \geq \kappa g, \text{diam} X \leq D\), and let \(\overline{\mathcal{M}(n, \kappa, D)}\) be the closure with respect to the measured Gromov-Hausdorff distance. Here, we review the properties of the Laplacian \(\Delta_\nu\) defined for \((P, d, \nu) \in \mathcal{M}(n, \kappa, D)\), which was introduced in [2]. In [2], the self-adjoint operator \(\Delta_\nu : \mathcal{D} \to L^2(P)\) is defined on a dense subspace \(\mathcal{D} \subset L^2(P)\). \(f \in \mathcal{D}\) is called an eigenfunction if \(\Delta_\nu f = \lambda f\) holds for some \(\lambda \in \mathbb{R}\), and \(\lambda\) is called an eigenvalue.

**Theorem 7.1 ([2]).**

1. Let \((P, d, \nu) \in \overline{\mathcal{M}(n, \kappa, D)}\). Then there exist eigenvalues \(0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots\) and eigenfunctions \(f_j\) with \(\Delta_\nu f_j = \lambda_j f_j\) such that \(\lim_{i \to \infty} \lambda_j = \infty, \int_P f_j f_k d\nu = \delta_{jk}\) and \(L^2(P) = \text{span}\{f_1, f_2, \ldots\}\). Moreover, all of \(f_j\) are Lipschitz continuous.

2. Let \((P, d, \nu) \in \mathcal{M}(n, \kappa, D)\). Then \(\Delta_\nu\) coincides with the ordinary Laplacian defined by Riemannian metrics.

The value \(\lambda_j\) in (1) of Theorem 7.1 is called the \(j\)-th eigenvalue of \(\Delta_\nu\). Since it is determined uniquely for each \((P, d, \nu) \in \mathcal{M}(n, \kappa, D)\), one obtain a function

\[
\lambda_j : \overline{\mathcal{M}(n, \kappa, D)} \to \mathbb{R}
\]

\[\psi\]

\((P, d, \nu) \mapsto \text{the } j\text{-th eigenvalue of } \Delta_\nu\)

Cheeger and Colding showed the continuity of \(\lambda_j\) as follows.

**Theorem 7.2 ([2]).**

1. Let \(\lambda_j : \overline{\mathcal{M}(n, \kappa, D)} \to \mathbb{R}\) be as above. Then \(\lambda_j\) is continuous.

2. Let \((P_i, d_i, \nu_i) \in \overline{\mathcal{M}(n, \kappa, D)}\) for \(i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}\),

\[
(P_i, d_i, \nu_i) \overset{\text{mGH}}{\to} (P_\infty, d_\infty, \nu_\infty),
\]

19
and $\phi_i, \varepsilon_i$ be as in Definition 6.2. Put $\lambda_{j,\infty} := \lambda_j(P_\infty, d_\infty, \nu_\infty)$ and let $\{f_j,\infty\}_j$ be the complete orthonormal system of $L^2(P_\infty)$ such that $\Delta_{\nu_\infty} f_{j,\infty} = \lambda_{j,\infty} f_{j,\infty}$. Then for all $i$ there exist complete orthonormal systems $\{f_{j,i}\}_j$ of $L^2(P_i)$ such that $\Delta_{\nu_i} f_{j,i} = \lambda_{j,i} f_{j,i}$ and $\lim_{i \to \infty} \|f_{j,i} - f_{j,\infty}\|_{GH} = 0$ for all $j$, where

$$\|f_{j,i} - f_{j,\infty}\|_{GH} := \sup_{u \in P_i} |f_{j,i}(u) - f_{j,\infty} \circ \phi_i(u)|.$$ 

**Remark 7.3.** Put $\lambda_{j,i} := \lambda_j(P_i, d_i, \nu_i)$. In (2) of Theorem 7.2, $\{\lambda_{j,i}\}_j = \{\lambda_{j,i}\}_j$ holds. However, $\lambda_{j,i} = \lambda_{j,i}$ does not hold in general, even if $i$ is sufficiently large, since the eigenvalues of $\Delta_{\nu_\infty}$ may have multiplicity.

### 7.1 Convergence of spectral structures

In [6], Kuwae and Shioya introduced the notion of spectral structures for the Laplacian which enabled us to treat the convergence of eigenvalues in the systematic way. In this subsection we review the framework developed in [6].

Let $H_i$ be Hilbert spaces over $K = \mathbb{R}$ or $\mathbb{C}$ for $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, $C \subset H_\infty$ be a dense subspace and $\Phi_i : C \to H_i$ be linear operators which satisfy

$$\lim_{i \to \infty} \|\Phi_i(u)\|_{H_i} = \|u\|_{H_\infty}$$

for any $u \in C$.

**Definition 7.4 ([6]).** Let $u_i \in H_i$ for $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

1. A sequence $\{u_i\}_i$ converges to $u_\infty$ strongly as $i \to \infty$ if there exists a sequence $\{\tilde{u}_k\}_k \subset H_\infty$ tending to $u_\infty$ such that

$$\lim_{k \to \infty} \limsup_{i \to \infty} \|\Phi_i(\tilde{u}_k) - u_i\|_{H_i} = 0.$$

2. A sequence $\{u_i\}_i$ converges to $u_\infty$ weakly as $i \to \infty$ if

$$\lim_{i \to \infty} \langle u_i, v_i \rangle_{H_i} = 0$$

holds for any $(v_i)_i \in \prod_{i \in \mathbb{Z} \cup \{\infty\}} H_i$ such that $v_i \to v_\infty$ strongly.

For a Hilbert space $H$, let $A : \mathcal{D}(A) \to H$ be a self-adjoint linear operator on $H$, where $\mathcal{D}(A)$ is the domain of $A$, and suppose $\mathcal{E}$ is given by $\mathcal{E}(u) := \langle Au, u \rangle_H$ for $u \in \mathcal{D}(A)$ and $\mathcal{E}(u) := \infty$ for $u \in H \setminus \mathcal{D}(A)$. Moreover we assume that $\mathcal{E}$ is closed, namely, $\mathcal{D}(A)$ is complete with respect to the norm
defined by $\|u\|_E := \sqrt{\|u\|^2_{H} + \mathcal{E}(u)}$. The spectral structure $\Sigma$ generated by $A$ is defined by

$$\Sigma := (A, \mathcal{E}, E, \{T_t\}_{t \geq 0}, \{R_{\zeta}\}_{\zeta \in \rho(A)}),$$

where $E$ is the spectral measure of $A$, $T_t := e^{-tA}$, $R_{\zeta} = (\zeta - A)^{-1}$ and $\rho(A)$ is the resolvent set of $A$.

**Definition 7.5** ([6]). A sequence of closed quadratic forms $\{\mathcal{E}_i : H_i \to \mathbb{R}\}_i$ compactly converges to $\mathcal{E}_\infty : H_\infty \to \mathbb{R}$ as $i \to \infty$ if

1. $\mathcal{E}_\infty(u_\infty) \leq \liminf_{i \to \infty} \mathcal{E}_i(u_i)$ for any $\{u_i\}_i$ with $u_i \to u_\infty$ weakly,
2. for any $u_\infty \in H_\infty$ there exists $\{u_i\}_i$ strongly converging to $u_\infty$ such that $\mathcal{E}_\infty(u_\infty) = \lim_{i \to \infty} \mathcal{E}_i(u_i)$.
3. for any $\{u_i\}_i$ with $\limsup_{i \to \infty} (\|u_i\|^2_{H_i} + \mathcal{E}_i(u_i)) < \infty$, there exists a strongly convergent subsequence.

**Definition 7.6** ([6]). Let $A_i$ be a self-adjoint nonnegative operator on $H_i$ and $\Sigma_i$ be the spectral structure generated by $A_i$. Then $\{\Sigma_i\}_i$ compactly converges to $\Sigma_\infty$ as $i \to \infty$ if $E_i \to E_\infty$ compactly as $i \to \infty$.

The authors of [6] have shown that the compact convergence of $\{\mathcal{E}_i\}_i$ is equivalent to the certain convergence of the other materials consisting of $\Sigma_i$. See Section 2.6 of [6] for the details.

The definitions of the notions in Definition 7.4 and 7.5 depend on the choice of $\{\Phi_i\}_i$ with (7), however, we can replace it with other $\{\hat{\Phi}_i : \hat{C} \to H_i\}_i$ by the following lemma.

**Lemma 7.7.** Let $\hat{C} \subset C$ be dense subspaces of $H_\infty$. If $\Phi_i : C \to H_i$ and $\hat{\Phi}_i : \hat{C} \to H_i$ satisfy (7) and $\lim_{i \to \infty} \|\Phi_i(u) - \hat{\Phi}_i(u)\|_{H_i} = 0$ for any $u \in \hat{C}$, then $u_i \to u_\infty$ strongly with respect to $\{\Phi_i\}_i$ iff $u_i \to u_\infty$ strongly with respect to $\{\hat{\Phi}_i\}_i$.

**Proof.** Suppose $u_i \to u_\infty$ strongly with respect to $\{\hat{\Phi}_i\}_i$, that is, there is a sequence $\{\hat{u}_k\}_k \subset \hat{C}$ tending to $u_\infty$ such that

$$\lim_{k \to \infty} \limsup_{i \to \infty} \|\hat{\Phi}_i(\hat{u}_k) - u_i\|_{H_i} = 0.$$

Then we have

$$\limsup_{i \to \infty} \|\Phi_i(u) - u_i\|_{H_i} \leq \limsup_{i \to \infty} \|\hat{\Phi}_i(\hat{u}_k) - u_i\|_{H_i}$$

$$+ \limsup_{i \to \infty} \|\Phi_i(u) - \hat{\Phi}_i(\hat{u}_k)\|_{H_i}$$

$$= \limsup_{i \to \infty} \|\hat{\Phi}_i(\hat{u}_k) - u_i\|_{H_i}.$$
for all \( \hat{u}_k \), hence \( u_i \to u_\infty \) strongly with respect to \( \{\Phi_i\}_i \). The converse follows by the similar argument since the definition of the strong convergence of \( \{u_i\} \) does not depend on the choice of \( \{\hat{u}_k\}_k \) tending to \( u_\infty \).

Now let \((P_i, d_i, \nu_i) \in \mathcal{M}(n, \kappa, D)\) and suppose that \(\{(P_i, d_i, \nu_i)\}_i\) measured Gromov-Hausdorff converges to \((P_\infty, d_\infty, \nu_\infty)\). Then there are Borel \( \varepsilon_i \)-isometries \( \phi_i: P_i \to P_\infty \) such that \( \varepsilon_i \to 0 \) and \( \phi_i \nu_i \to \nu_\infty \) as \( i \to \infty \). Put \( H_i := L^2(P_i, \nu_i), \ C := C(X_\infty) = \{f: X_\infty \to \mathbb{R}; f \text{ is continuous}\}, \Phi_i(f) := f \circ \phi_i \) and \( \Sigma_i \) be the spectral structure generated by \( \Delta \nu_i \).

**Theorem 7.8** ([6]). If \((P_i, d_i, \nu_i) \xrightarrow{\text{mGH}} (P_\infty, d_\infty, \nu_\infty)\), then \( \Sigma_i \to \Sigma_\infty \) compactly.

Suppose that the sequence \((P_i, d_i, \nu_i)\) have isometric \( G \)-actions, then \( G \) acts on \( H_i = L^2(P_i, \nu_i) \) isometrically. In the following subsections we are going to see that we can replace \( \Phi_i \) by \( G \)-equivariant maps \( \hat{\Phi}_i \) which satisfies the assumptions of Lemma 7.7.

### 7.2 The eigenfunctions on metric measure spaces with isometric \( G \)-actions

In this subsection we assume that \( G \) is a compact Lie group, \((P_i, d_i, \nu_i) \in \mathcal{M}(n, \kappa, D)\) and \((P_\infty, d_\infty, \nu_\infty) \in \mathcal{M}(n, \kappa, D)\) have isometric \( G \)-actions and

\[
(P_i, d_i, \nu_i) \xrightarrow{G-mGH} (P_\infty, d_\infty, \nu_\infty).
\]

Let \( \phi_i, \varepsilon_i \) be as in Definition 6.3.

For any \( \gamma \in G \), we have \((R_\gamma)_* \nu_i = \nu_i\) since \( \nu_i \) is the Riemannian measure and \( G \) acts on \((P_i, d_i)\) isometrically. Since we have

\[
\left| \int_{P_\infty} f d(R_\gamma \circ \phi_i)_* \nu_i - \int_{P_\infty} f d\phi_i \nu_i \right| \leq \sup_{u' \in P_i} |f(\phi_i(u' \gamma)) - f(\phi_i(u' \gamma))| \nu_i(P_i),
\]

for any continuous function \( f: P_\infty \to \mathbb{R} \) and the continuous functions on compact metric spaces are uniformly continuous, then the weak limit of \( \{(R_{\gamma_i} \circ \phi_i)_* \nu_i\}_i \) and \( \{(\phi_i \circ R_\gamma)_* \nu_i\}_i \) should coincides, consequently we have

\[
\int_{P_\infty} f dR_{\gamma \nu_\infty} = \int_{P_\infty} f d\nu_\infty
\]

for any continuous function \( f \) and \( \gamma \in G \). By [1], \( \nu_\infty \) and \( R_{\gamma \nu_\infty} \) are Radon measures, hence we can see

\[
R_{\gamma \nu_\infty} = \nu_\infty
\]
for any $i$ replacing $W$ preserve $\Delta_{\nu,f} = \lambda f$.  
Then by Theorem 7.1, each $W^i(\lambda)$ has finite dimension and we have the orthogonal decomposition

$$L^2(P_i) = \bigoplus_{\lambda} W^i(\lambda).$$

Similarly as in Section 4, for $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the left $G$-actions on $L^2(P_i)$ preserve $W^i(\lambda)$ since $G$ acts on $P_i$ preserving $d_i, \nu_i$ and the structure of $\nu_i$-rectifiable space. Consequently, we obtain the real unitary representation $G \to O(W^i(\lambda))$.

Fix the eigenvalue $\lambda$ of $\Delta_{\nu,\infty}$, then we may suppose

$$\lambda_{j_0} < \lambda = \lambda_{j_0+1,\infty} = \lambda_{j_0+2,\infty} = \cdots = \lambda_{j_0+j_1,\infty} < \lambda_{j_0+j_1+1,\infty}$$
for some $j_0$, where $j_1 := \dim_\mathbb{R} W^\infty(\lambda)$.

We choose an orthonormal basis $\{f_{j_0+1,\infty}, f_{j_0+2,\infty}, \ldots, f_{j_0+j_1,\infty}\}$ of $W^\infty(\lambda)$ and by extending it we obtain the complete orthonormal system $\{f_{j,\infty}\}_{j}$ of $L^2(P_\infty)$ such that $\Delta_{\nu,\infty} f_{j,\infty} = \lambda_{j,\infty} f_{j,\infty}$.

**Lemma 7.9.** Let $(P_i, d_i, \nu_i), (P_\infty, d_\infty, \nu_\infty), \{f_{j,\infty}\}$, $\lambda$, $j_0$ and $j_1$ be as above. Let $\Delta_{j,i}, f_{j,i}$ be as in the conclusion of Theorem 7.2. Then there exists $\delta_0 > 0$ and $i_\lambda \in \mathbb{Z}_{\geq 0}$ such that $\Delta_{j,i} < \lambda - \delta_0$ holds for any $j \leq j_0$ and $i \geq i_\lambda$, $\Delta_{j,i} > \lambda + \delta_0$ holds for any $j > j_0 + j_1$ and $i \geq i_\lambda$, and $|\lambda_{j,i} - \lambda| < \delta_0$ holds for any $j_0 + 1 \leq j \leq j_0 + j_1$ and $i \geq i_\lambda$.

**Proof.** Put

$$\delta_0 := \frac{1}{2} \min \{\lambda_{j_0+j_1+1,\infty} - \lambda, \lambda - \lambda_{j_0,\infty}\} > 0.$$ 

By Theorem 7.2 $\lim_{r \to \infty} \lambda_{j,i} = \lambda_{j,\infty}$ holds, hence there exist $i_\lambda$ such that $\Delta_{j,i} < \lambda - \delta_0$ for any $j \leq j_0$ and $i \geq i_\lambda$, moreover we may also suppose $|\lambda_{j,i} - \lambda| < \delta_0$ for any $j_0 + 1 \leq j \leq j_0 + j_1$ and $i \geq i_\lambda$.

Next we put $\lambda_{j,i} = \lambda_j(P_i, d_i, \nu_i)$. By the continuity of $\lambda_{j_0+j_1+1}$, and by replacing $i_\lambda$ larger if it is necessary, we may suppose

$$\lambda_{j_0+j_1+1,i} > \lambda_{j_0+j_1+1,i} - \delta_0 \geq \lambda + \delta_0$$
for any $i \geq i_\lambda$. Then combining with the above argument one can see

$$\{\lambda_{j,i}; 0 \leq j \leq j_0 + j_1\} = \{\lambda_{j,i}; 0 \leq j \leq j_0 + j_1\},$$
$$\{\lambda_{j,i}; j \geq j_0 + j_1 + 1\} = \{\lambda_{j,i}; j \geq j_0 + j_1 + 1\},$$
hence one can obtain the assertion.  \[\square\]
Fix a sufficiently large $i_\lambda$ such that Lemma 7.9 holds. Note that $i_\lambda$ may also depend on $(P_\infty, d_\infty, \nu_\infty)$. Define a linear map

$$\Psi_i : W^\infty(\lambda) \to \bigoplus_{j=1}^{j_l} W^j(\lambda_{j_0+j,i})$$

(8)

by $\Psi_i(f_{j_0+j,i}) := f_{j_0+j,i}$. From now on we fix $i \geq i_\lambda$ arbitrarily, then $\Psi_i$ is an isomorphism by Lemma 7.9 and we write

$$f'_{j} := f_{j_0+j, i}, \quad f_{j} := f_{j_0+j, \infty}.$$

Put

$$R_\gamma f_{j} = \sum_{l=1}^{j_l} A_{jl}(\gamma) f_l,$$

$$R_\gamma f'_{j} = \sum_{l=1}^{j_l} B_{jl}(\gamma) f'_l$$

for some $A_{jl}(\gamma), B_{jl}(\gamma) \in \mathbb{R}$, then $(A_{jl}(\gamma))_{jl}, (B_{jl}(\gamma))_{jl}$ are orthogonal matrices. The aim of this subsection is to show that $\sup_{j, i} |A_{jl}(\gamma) - B_{jl}(\gamma)| \to 0$ as $i \to \infty$.

We have

$$A_{jl}(\gamma) = \langle R_\gamma f_{j}, f_{l} \rangle_{L^2} = \int_{P_\infty} R_\gamma f_{j} \cdot f_{l} \, d\nu_\infty,$$

$$B_{jl}(\gamma) = \langle R_\gamma f'_{j}, f'_{l} \rangle_{L^2} = \int_{P_{\lambda}} R_\gamma f'_{j} \cdot f'_{l} \, d\nu_{\lambda},$$

and

$$|A_{jl}(\gamma) - B_{jl}(\gamma)| \leq \left| \int_{P_\infty} R_\gamma f_{j} f_{l} \, d\nu_\infty - \int_{P_\infty} R_\gamma f_{j} f_{l} \, d\phi_{l_1} \nu_{l_1} \right|$$

$$+ \left| \int_{P_{\lambda}} \phi_{l_1}^*(R_\gamma f_{j}, f_{l_1}) \, d\nu_{l_1} - \int_{P_{\lambda}} R_\gamma f'_{j} f'_{l_1} \, d\nu_{l_1} \right|.$$  

The second term of the right-hand-side is bounded by

$$\left| \int_{P_{\lambda}} \{ \phi_{l_1}^* R_\gamma^* f_{j} - R_\gamma^* f'_{j} \} \phi_{l_1}^* f_{l_1} \, d\nu_{l_1} \right| + \left| \int_{P_{\lambda}} R_\gamma^* f'_{j} \left( \phi_{l_1}^* f_{l_1} - f'_{l_1} \right) \, d\nu_{l_1} \right|$$

$$\leq \left| \int_{P_{\lambda}} \{ \phi_{l_1}^* R_\gamma^* f_{j} - R_\gamma^* \phi_{l_1}^* f_{j} \} \phi_{l_1}^* f_{l_1} \, d\nu_{l_1} \right|$$

$$+ \| \phi_{l_1}^* f_{j} - f'_{j} \|_{L^2} \cdot \left( \int_{P_\infty} f_{l_1}^2 \, d\phi_{l_1} \nu_{l_1} \right)^{\frac{1}{2}} + \| f'_{j} \|_{L^2} \| \phi_{l_1}^* f_{l_1} - f'_{l_1} \|_{L^2}. \quad (9)$$

24
Now we have
\[
\left| \int_{P_i} \{ \phi_i^* R_\gamma f_j - R_\gamma^* \phi_i^* f_j \} \phi_i^* f_i \, d\nu_i \right|
\leq \sup_{u_i \in P_i} |f_j(\phi_i(u_i)\gamma) - f_j(\phi_i(u_i)\gamma)| \sqrt{\nu_i(P_i)} \left( \int_{P_i} f_i^2 \, d\phi_i \nu_i \right)^{\frac{1}{2}}.
\]
By (1) of Theorem 7.1 we can take a constant \( C > 0 \) depending only on \((P_\infty, d_\infty, \nu_\infty)\) and \( \lambda \) such that
\[
|f_j(u) - f_j(u')| \leq Cd_\infty(u, u')
\]
holds for all \( u, u' \in P_\infty \) and \( j = 1, \ldots, j_1 \), then we obtain
\[
\sup_{u \in P_i} |f_j(\phi_i(u)\gamma) - f_j(\phi_i(u)\gamma)| \leq Cd_\infty(\phi_i(u)\gamma, \phi_i(u)\gamma) < C\varepsilon_i,
\]
which implies
\[
\left| \int_{P_i} \{ \phi_i^* R_\gamma f_j - R_\gamma^* \phi_i^* f_j \} \phi_i^* f_i \, d\nu_i \right| \leq C\varepsilon_i \sqrt{\nu_i(P_i)} \left( \int_{P_i} f_i^2 \, d\phi_i \nu_i \right)^{\frac{1}{2}}. \tag{11}
\]
Here, we have applied that \( \phi_i \) is \( \varepsilon_i \)-approximation of \( G \)-equivariant isometry and the \( G \)-action preserves \( \nu_i \). Since \( \phi_i \nu_i \) converges to \( \nu_\infty \) weakly, we can replace \( i_\lambda \) larger enough such that
\[
\left| \int_{P_\infty} f_i^2 \, d\phi_i \nu_i - \int_{P_\infty} f_i^2 \, d\nu_\infty \right| \leq 1, \tag{12}
\]
\[
|\nu_i(P_i) - \nu_\infty(P_\infty)| \leq 1 \tag{13}
\]
holds for any \( i \geq i_\lambda \) and \( f \in W^\infty(\lambda) \) with \( \|f\|_{L^2} = 1 \). By \( (12)(13) \) we obtain
\[
\|\phi_i^* f_j - f_j^*\|_{L^2} \cdot \left( \int_{P_\infty} f_i^2 \, d\phi_i \nu_i \right)^{\frac{1}{2}} + \|f_j^* f_i - f_i^* f_j\|_{L^2}
\leq \sqrt{2(1 + \nu_\infty(P_\infty))} \|f_j^* - f_j\|_{L^2} + \sqrt{2 + \nu_\infty(P_\infty)} \|f_i^* - f_i\|_{L^2} + \nu_\infty(P_\infty) \sup_{1 \leq j \leq j_1} \|f_j^* - f_j\|_{L^2}.
\]
then by combining \( (11) \), we have
\[
|A_{jl}(\gamma) - B_{jl}(\gamma)| \leq \left| \int_{P_\infty} R_\gamma^* f_j f_i \, d\nu_\infty - \int_{P_\infty} R_\gamma^* f_i f_j \, d\phi_i \nu_i \right|
+ \sqrt{2(1 + \nu_\infty(P_\infty))} \left( \sup_{1 \leq j \leq j_1} \|f_j^* - f_j\|_{L^2} + C\varepsilon_i \right). \tag{14}
\]
Next we estimate the first term. By the weak convergence of the measures, the first term converges to 0 for each \( j, l \) and \( \gamma \in G \) as \( i \to \infty \), however, we need the uniform estimate with respect to \( \gamma \).

**Lemma 7.10.** Let \( G \) be a compact Lie group equipped with the distance \( d_G \). There is a constant \( C > 0 \) depending on \((P_\infty, d_\infty, \nu_\infty)\) and \( \lambda \) such that the following holds. For any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for any \( \delta \)-covering \( G(\delta) \) of \( G \), we have

\[
\left| \int_{P_\infty} R_\gamma f_j f_l d\nu_\infty - \int_{P_\infty} R_{\gamma_0} f_j f_l d\phi_i \nu_i \right| \leq \sup_{\gamma_0 \in G(\delta)} \left| \int_{P_\infty} R_{\gamma_0} f_j f_l d\nu_\infty - \int_{P_\infty} R_{\gamma_0}^* f_j f_l d\phi_i \nu_i \right| + C \varepsilon
\]

for any \( \gamma \in G \) and \( j, l = 1, \ldots, j_1 \).

**Proof.** By applying (12)-(13), we have

\[
\left| \int_{P_\infty} R_\gamma f_j f_l d\nu_\infty - \int_{P_\infty} R_{\gamma_0} f_j f_l d\phi_i \nu_i \right| \leq \left| \int_{P_\infty} R_{\gamma_0} f_j f_l d\nu_\infty - \int_{P_\infty} R_{\gamma_0}^* f_j f_l d\phi_i \nu_i \right| + \left| \int_{P_\infty} (R_\gamma f_j - R_{\gamma_0}^* f_j) f_l d\nu_\infty \right| + \left| \int_{P_\infty} (R_\gamma^* f_j - R_{\gamma_0} f_j) f_l d\phi_i \nu_i \right|
\]

\[
\leq \left( \int_{P_\infty} R_{\gamma_0} f_j f_l d\nu_\infty - \int_{P_\infty} R_{\gamma_0}^* f_j f_l d\phi_i \nu_i \right) + 2 \sqrt{1 + \nu_\infty(P_\infty)} \sup_{P_\infty} |R_\gamma^* f_j - R_{\gamma_0}^* f_j|.
\]

By (10), we obtain

\[
\sup_{P_\infty} |R_\gamma^* f_j - R_{\gamma_0}^* f_j| \leq C \sup_{u \in P_\infty} d_\infty(u_\gamma, u_{\gamma_0}).
\]

Define a continuous function \( F : G \times G \to \mathbb{R} \) by

\[
F(\gamma, \gamma_0) := \sup_{u \in P_\infty} d_\infty(u_\gamma, u_{\gamma_0}).
\]

Since \( G \) is compact metric space, \( F \) is uniformly continuous. Therefore, for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d_G(\gamma, \gamma_0) < \delta \) implies

\[
|F(\gamma, \gamma_0) - F(\gamma_0, \gamma_0)| < \varepsilon
\]
for any \( \gamma, \gamma_0 \in G \). Since \( F(\gamma_0, \gamma_0) = 0 \), we obtain

\[
\sup_{u \in P_\infty} d_\infty(u\gamma, u\gamma_0) < \varepsilon
\]

for any \( \gamma, \gamma_0 \) with \( d_G(\gamma, \gamma_0) < \delta \).

Now we apply Lemma 7.10 to (14). Take \( \varepsilon > 0 \) arbitrarily, and take \( \delta > 0 \) as in Lemma 7.10 and fix a \( \delta \)-covering \( G(\delta) \subset G \) with \( \# G(\delta) < \infty \).

Then by the weak convergence of \( \nu_i \) and (2) of Theorem 7.2, we can take \( i_1 = i(\varepsilon; G(\delta), P_\infty, d_\infty, \nu_\infty, \lambda) > 0 \) such that

\[
\sup_{1 \leq j \leq j_1} \| f_j' - f_j \|_{GH} < \varepsilon,
\]

\[
\sup_{j, l} \sup_{\gamma_0 \in G(\delta)} \left| \int_{P_\infty} R_{\gamma_0}^* f_j f_l d\nu_\infty - \int_{P_\infty} R_{\gamma_0}^* f_j f_l d\phi_i \nu_l \right| < \varepsilon
\]

for any \( i \geq i_1 \), therefore we obtain the following result.

**Proposition 7.11.**

\[
\lim_{i \to \infty} \sup_{\gamma \in G} \sum_{j, l} |A_{jl}(\gamma) - B_{jl}(\gamma)|^2 = 0.
\]

### 7.3 Existence of the \( G \)-equivariant map

**Proposition 7.12.** Let \((\tau, W)\) and \((\tau', W')\) be a real unitary representation of a compact Lie group \( G \) with \( \dim W = \dim W' = j_1 \), and \( \langle , \rangle_W, \langle , \rangle_{W'} \) be the inner product on \( W \) and \( W' \) respectively. Let \( \Psi : W \to W' \) be a linear isomorphism preserving the inner products which satisfies

\[
\sup_{\gamma \in G} \sum_{j, l} \left| \langle (\tau(\gamma)f_j, f_l)_W - (\tau'(\gamma)\Psi(f_j), \Psi(f_l))_{W'} \rangle \right|^2 \leq \varepsilon_0
\]

for \( 0 < \varepsilon_0 < 1 \) and some orthonormal basis \( f_1, \ldots, f_{j_1} \in W \). Then there is a \( G \)-equivariant isomorphism \( \hat{\Phi} : W \to W' \) preserving the inner products such that

\[
\sup_{w \neq 0} \frac{\| \Psi(w) - \hat{\Phi}(w) \|_{W'}}{\| w \|_W} \leq \frac{2\sqrt{\varepsilon_0}}{1 - \sqrt{\varepsilon_0}}
\]
Proof. Let $\Psi: W \to W'$ satisfy the assumptions. Denote by $\mu$ one of the right-invariant measures on $G$. Define a $G$-equivariant map $\Psi_0: W \to W'$ by

$$\Psi_0(w) := \frac{1}{\mu(G)} \int_G \tau'(\gamma^{-1})\Psi(\tau(\gamma)w)d\mu.$$  

Then we have

$$\|\Psi(w) - \Psi_0(w)\|^2_{W'} = \frac{1}{\mu(G)^2} \left|\int_G \{\Psi(w) - \tau'(\gamma^{-1})\Psi(\tau(\gamma)w)\}d\mu\right|^2_{W'} = \frac{1}{\mu(G)} \int_G \|\Psi(w) - \tau'(\gamma^{-1})\Psi(\tau(\gamma)w)\|^2_{W'}, d\mu = \frac{1}{\mu(G)} \int_G \|\tau'(\gamma)\Psi(w) - \Psi(\tau(\gamma)w)\|^2_{W'}, d\mu.$$  

By putting $w = \sum_j w^j f_j \neq 0$, we have

$$\|\tau'(\gamma)\Psi(w) - \Psi(\tau(\gamma)w)\|^2_{W'}$$

$$= \left\|\sum_j w^j \{\tau'(\gamma)\Psi(f_j) - \Psi(\tau(\gamma)f_j)\}\right\|^2_{W'} = \sum_l \left(\sum_j w^j \{\langle \tau'(\gamma)\Psi(f_j), \Psi(f_l)\rangle - \langle \tau(\gamma)f_j, f_l\rangle\}\right)^2$$

$$\leq \sum_l \|w\|^2_{W'} \sum_j \{\langle \tau'(\gamma)\Psi(f_j), \Psi(f_l)\rangle - \langle \tau(\gamma)f_j, f_l\rangle\}^2$$

$$\leq \varepsilon_0\|w\|^2_{W'},$$

hence we obtain

$$\frac{\|\Psi(w) - \Psi_0(w)\|_{W'}}{\|w\|_W} \leq \sqrt{\varepsilon_0}. \quad (15)$$

Since $\varepsilon_0 < 1$, then the kernel of $\Psi_0$ should be trivial, hence $\Psi_0$ is isomorphic. Let $^t\Psi_0: W' \to W$ be the adjoint of $\Psi_0$. Note that $^t\Psi_0\Psi_0: W \to W$ is a $G$-equivariant isomorphism whose eigenvalues are positive. Then $W$ has the decomposition $W = \bigoplus_i V(a_i)$, where $V(a_i)$ is the eigenspace of $^t\Psi_0\Psi_0$ associate with the eigenvalue $a_i > 0$. Here, each $V(a_i)$ is a $G$-subspace. Let $\sqrt{^t\Psi_0\Psi_0}: W \to W$ be defined by

$$\sqrt{^t\Psi_0\Psi_0}_{|V(a_i)} := \sqrt{a_i} \cdot \text{id}_{V(a_i)},$$

hence we obtain

$$\frac{\|\Psi(w) - \Psi_0(w)\|_{W'}}{\|w\|_W} \leq \sqrt{\varepsilon_0}. \quad (15)$$
then $\sqrt{\Psi_0} \Psi_0$ is also a $G$-equivariant isomorphism. Now (15) gives

$$||\Psi_0(w)||_{W'} - ||w||_W| \leq \sqrt{\varepsilon_0} ||w||_W,$$

accordingly we can see

$$(1 - \sqrt{\varepsilon_0})^2 \leq \frac{\langle t \Psi_0 \Psi_0(w), w \rangle_W}{||w||^2_W} \leq (1 + \sqrt{\varepsilon_0})^2$$

for $w \neq 0$, which gives

$$1 - \sqrt{\varepsilon_0} \leq a_i \leq 1 + \sqrt{\varepsilon_0}.$$ 

Now, define $\hat{\Phi}: W \rightarrow W'$ by $\hat{\Phi} := \Psi_0 \circ \sqrt{\Psi_0} \Psi_0^{-1}$. Then $\hat{\Phi}$ is obviously $G$-equivariant, and one can see that it preserves inner products. Moreover, we have

$$\frac{||\hat{\Phi}(w) - \Psi_0(w)||_{W'}}{||w||_W} \leq \frac{\sqrt{\varepsilon_0}(1 + \sqrt{\varepsilon_0})}{1 - \sqrt{\varepsilon_0}},$$

hence we obtain

$$\frac{||\hat{\Phi}(w) - \Psi(w)||_{W'}}{||w||_W} \leq \frac{\sqrt{\varepsilon_0}(1 + \sqrt{\varepsilon_0})}{1 - \sqrt{\varepsilon_0}} = \frac{2\sqrt{\varepsilon_0}}{1 - \sqrt{\varepsilon_0}}.$$ 

Combining Propositions 7.11 and 7.12 we have the followings.

**Proposition 7.13.** Let $G$ be a compact Lie group, $(P_i, d_i, \nu_i) \in M(n, \kappa, D)$ and $(P_\infty, d_\infty, \nu_\infty) \in M(n, \kappa, D)$ have isometric $G$-actions and

$$(P_i, d_i, \nu_i) \overset{G\cdot mGH}{\longrightarrow} (P_\infty, d_\infty, \nu_\infty).$$

Let $\phi_i, \varepsilon_i$ be as in Definition 6.3, put $H_i := L^2(P_i, \nu_i), C := C(X_\infty), \Phi_i(f) := f \circ \phi_i$. Then there exists a subspace $\hat{C} \subset \hat{C}$ such that $\hat{C}$ is dense in $H_\infty$, $G \cdot \hat{C} \subset \hat{C}$ holds and there are linear operators $\hat{\Phi}_i: \hat{C} \rightarrow H_i$ such that $\lim_{i \rightarrow \infty} ||\Phi_i(u) - \hat{\Phi}_i(u)||_{_L^\infty} = 0$ for any $u \in \hat{C}$. In particular, $u_i \rightarrow u_\infty$ strongly with respect to $\{\Phi_i\}_i$ iff $u_i \rightarrow u_\infty$ strongly with respect to $\{\hat{\Phi}_i\}_i$ by Lemma 7.7.

**Proof.** Let $\check{C} := \bigoplus \lambda W^\infty(\lambda)$. Then Theorem 7.1 implies that $\check{C}$ is dense in $H_\infty$ and $\check{C} \subset C$. Take $i_\lambda$ such that Lemma 7.9 holds for each eigenvalue $\lambda$, then replace $i_\lambda$ by the larger one defined by

$$\max\{i_{\lambda'}; \lambda' \text{ is an eigenvalue of } \Delta_\nu \text{ with } \lambda' \leq \lambda\}.$$ 

29
Put
\[ W^i[a, b] := \bigoplus_{a \leq \lambda \leq b} W^i(\lambda). \]

Take the linear isomorphisms defined by (8) for each \( \lambda' \leq \lambda \) and \( i \geq i_\lambda \), then we obtain the linear isomorphisms
\[ \Psi_i : W^\infty[0, \lambda] \to W^i[0, \lambda + \delta_i] \]

preserving the \( L^2 \)-inner products for a constant \( \delta_i > 0 \). By the continuity of \( \lambda_j : \mathcal{M}(n, \kappa, D) \to \mathbb{R} \), we can take \( \{\delta_i\}_{i \geq i_\lambda} \) such that \( \delta_i \to 0 \) as \( i \to \infty \) and \( \Psi_i(W^\infty(\lambda')) = W^i[\lambda' - \delta_i, \lambda' + \delta_i] \) hold for any eigenvalues \( \lambda' \leq \lambda \). Fix an orthonormal basis \( f_1, \ldots, f_j \) of \( W^\infty(\lambda') \) and put
\[ \varepsilon_i := \sup_{\lambda' \leq \lambda} \sup_{\gamma \in G} \sum_{j,l} |\langle R^*_\gamma f_j, f_l \rangle_{L^2} - \langle R^*_\gamma \Psi_i(f_j), \Psi_i(f_l) \rangle_{L^2}|^2. \]

Then Proposition 7.11 gives \( \lim_{i \to \infty} \varepsilon_i = 0 \). Consequently, by Proposition 7.12 we obtain a \( G \)-equivariant isomorphism \( \hat{\Phi}_i : W^\infty[0, \lambda] \to W^i[0, \lambda + \delta_i] \) which preserves the \( L^2 \) inner products and satisfies
\[ \sup_{f \in W^\infty(\lambda') \setminus \{0\}} \frac{\|\hat{\Phi}_i(f) - \Psi_i(f)\|_{L^2}}{\|f\|_{L^2}} \leq \frac{2\sqrt{\varepsilon_i}}{1 - \sqrt{\varepsilon_i}}, \tag{16} \]

and \( \hat{\Phi}_i(W^\infty(\lambda')) = W^i[\lambda' - \delta_i, \lambda' + \delta_i] \) for any \( \lambda' \leq \lambda \). Put
\[ \hat{\Phi}_i(f_j) = \sum_l \xi^l_j \Psi_i(f_l) \]

for some constants \( \xi^l_j \in \mathbb{R} \). By applying (16) for \( f = f_j \), we can see that
\[ |\xi^l_j - \delta^l_j| \leq \frac{2\sqrt{\varepsilon_i}}{1 - \sqrt{\varepsilon_i}}, \]

hence we have
\[ \|\hat{\Phi}_i(f_j) - \Psi_i(f_j)\|_{L^\infty} \leq \sum_l |\xi^l_j - \delta^l_j| \|\Psi_i(f_l)\|_{L^\infty} \]
\[ \leq \sum_l \frac{2\sqrt{\varepsilon_i}}{1 - \sqrt{\varepsilon_i}} (\|\Psi_i(f_l) - f_l\|_{GH} + \|f_l\|_{L^\infty}) \]
\[ \leq \frac{2\sqrt{\varepsilon_i} \dim W^\infty[0, \lambda]}{1 - \sqrt{\varepsilon_i}} \sup_{\|f\|_{L^2} = 1} (\|\Psi_i(f) - f\|_{GH} + \|f\|_{L^\infty}). \]
Now, by replacing $i_\lambda$ larger if necessary, we may suppose $\sup_{f \in W^\infty[0, \lambda] \setminus \{0\}} \|f\|_{L^2} = 1$ for any $i \geq i_\lambda$. Moreover, there is a constant $C > 0$ depending only on $\lambda$ and $(P_\infty, d_\infty, \nu_\infty)$ such that
\[
\sup_{f \in W^\infty[0, \lambda] \setminus \{0\}} \frac{\|f\|_{L^\infty}}{\|f\|_{L^2}} \leq C,
\]
hence we can show
\[
\|\hat{\Phi}_i(f_j) - \Psi_i(f_j)\|_{L^\infty} \leq \frac{2(1 + C)\sqrt{\epsilon_i} \dim W^\infty[0, \lambda]}{1 - \sqrt{\epsilon_i}},
\]
which gives
\[
\lim_{i \to \infty} \|\hat{\Phi}_i(f) - \Psi_i(f)\|_{L^\infty} = 0
\]
for any $f \in W^\infty[0, \lambda]$.

Since we also have $\lim_{i \to \infty} \|\Phi_i(f) - \Psi_i(f)\|_{L^\infty} = 0$ by (2) of Theorem 7.2, we have $\lim_{i \to \infty} \|\Phi_i(f) - \hat{\Phi}_i(f)\|_{L^\infty} = 0$ for any $f \in W^\infty[0, \lambda]$. Now, we extend $\hat{\Phi}_i$ to $\hat{\Phi}_i : \hat{\mathcal{C}} \to H_i$

by putting $\hat{\Phi}_i|_{W^\infty(\lambda)} = 0$ if $i < i_\lambda$. Then it is easy to see that $\hat{\Phi}_i$ are $G$-equivariant isometries and satisfy $\lim_{i \to \infty} \|\Phi_i(f) - \hat{\Phi}_i(f)\|_{L^\infty} = 0$ for any $f \in \hat{\mathcal{C}}$.

8 Spectral structures with group actions

Let $H_i, C, \Phi_i : C \to H_i$ be as in Section 7.1 and $\Sigma_i$ be the spectral structures generated by $A_i : \mathcal{D}(A_i) \to H_i$. In this section we observe some spectral structures induced from $\Sigma_i$.

8.1 Tensor products

Let $V$ be a real vector space of dimension $k$ with a positive definite inner product. Then $H_i \otimes V$ are Hilbert spaces whose inner products are given by
\[
\left\langle \sum_{\alpha} u^\alpha \otimes e_\alpha, \sum_{\alpha} v^\alpha \otimes e_\alpha \right\rangle_{H_i \otimes V} = \sum_{\alpha} \langle u^\alpha, v^\alpha \rangle_{H_i},
\]
where $e_1, \ldots, e_k \in V$ is an orthonormal basis and $u^\alpha, v^\alpha \in H_i$. Then $C \otimes V$ is dense in $H_{\infty} \otimes V$ and we define $\Phi^V_i : C \otimes V \to H_i \otimes V$ by $\Phi^V_i(\sum_{\alpha} u^\alpha \otimes e_\alpha) := \sum_{\alpha} \Phi_i(u^\alpha) \otimes e_\alpha$. Then we can check the followings easily.
Proposition 8.1. Let $u^\alpha_i \in H_i$.

(1) $\sum_\alpha u^\alpha_i \otimes e_\alpha \to \sum_\alpha u^\alpha_\infty \otimes e_\alpha$ strongly iff $u^\alpha_i \to u^\alpha_\infty$ strongly for all $\alpha$.

(2) $\sum_\alpha u^\alpha_i \otimes e_\alpha \to \sum_\alpha u^\alpha_\infty \otimes e_\alpha$ weakly iff $u^\alpha_i \to u^\alpha_\infty$ weakly for all $\alpha$.

Let $A^V_i : \mathcal{D}(A_i) \otimes V \to H_i \otimes V$ be defined by

$$A^V_i \left( \sum_\alpha u^\alpha \otimes e_\alpha \right) := \sum_\alpha A_i(u^\alpha) \otimes e_\alpha$$

and put $E^V_i(u) := \langle A^V_i(u), u \rangle_{H_i}$, then we have $E^V_i \left( \sum_\alpha u^\alpha \otimes e_\alpha \right) = \sum_\alpha E_i(u^\alpha)$.

Proposition 8.2.

(1) If $E_i$ is closed, then $E^V_i$ is closed.

(2) If $E_i \to E_\infty$ compactly, then $E^V_i \to E^V_\infty$ compactly.

Proof. (1) is obvious. We show (2). Suppose $E_i \to E_\infty$. If $\sum_\alpha u^\alpha_i \otimes e_\alpha \to \sum_\alpha u^\alpha_\infty \otimes e_\alpha$ weakly, then by Proposition 8.2 we can see $u^\alpha_i \to u^\alpha_\infty$ weakly for any $\alpha$ and

$$E^V_i \left( \sum_\alpha u^\alpha_\infty \otimes e_\alpha \right) \leq \sum_\alpha \liminf_{i \to \infty} E_i(u^\alpha_i)$$

Next we take $\sum_\alpha u^\alpha_\infty \otimes e_\alpha \in H_\infty \otimes V$ arbitrarily. By the assumption there are $u^\alpha_i \in H_i$ such that $u^\alpha_i \to u^\alpha_\infty$ strongly and $E_\infty(u^\alpha_\infty) = \lim_{i \to \infty} E_i(u^\alpha_i)$. Then $\sum_\alpha u^\alpha_i \otimes e_\alpha \to \sum_\alpha u^\alpha_\infty \otimes e_\alpha$ strongly by Proposition 8.2 and we obtain

$$E^V_\infty \left( \sum_\alpha u^\alpha_\infty \otimes e_\alpha \right) = \sum_\alpha E_\infty(u^\alpha_\infty) = \sum_\alpha \lim_{i \to \infty} E_i(u^\alpha_i)$$

Finally, we take $\{ \sum_\alpha u^\alpha_i \otimes e_\alpha \}_i$ such that

$$\limsup_{i \to \infty} \left( \| \sum_\alpha u^\alpha_i \otimes e_\alpha \|^2_{H_i} + E^V_i \left( \sum_\alpha u^\alpha_i \otimes e_\alpha \right) \right) < \infty.$$
Proposition 8.3. Let $\Sigma_i$. 

Proposition 8.5. Let $\pi$.

Then by Proposition 8.2 we have a subsequence $\{i_j\} \subset \{i\}$ such that $\{u_{i_j}\}_j$ strongly converge for all $\alpha$. Then $\{\sum \alpha u_{i_j} \otimes e_{\alpha}\}_j$ strongly converges to some points. \qed

As a consequence, we obtain the following.

**Proposition 8.3.** Let $\Sigma_i^V$ be the spectral structure on $H_i^V$ generated by $A_i^V$. If $\Sigma_i \to \Sigma_{\infty}$ compactly, then $\Sigma_i^V \to \Sigma_{\infty}^V$ compactly.

### 8.2 Closed subspaces

Let $K_i \subset H_i$ be closed subspaces. In this subsection we assume the followings.

(i) For $C' := C \cap K_{\infty}$ and $C'' := C \cap K_{\infty}^\perp$, we have the decomposition $C = C' \oplus C''$.

(ii) $\Phi_i(C') \subset K_i$ and $\Phi_i(C'') \subset K_i^\perp$.

Moreover, denote by $\pi_{K_i^\perp}: H_i \to K_i$ and $\pi_{K_i^\perp}$ the orthogonal projections.

**Proposition 8.4.** Let $u_i \in H_i$ and $u_i \to u_{\infty}$ strongly with respect to $\{\Phi_i\}_i$. Then $\pi_{K_i}(u_i) \to \pi_{K_{\infty}}(u_{\infty})$ strongly with respect to $\{\Phi_i|_{C'}\}_i$.

**Proof.** Suppose $\{\tilde{u}_k\}_k \in C$ converges to $u_{\infty}$ and satisfies

$$\lim_{k \to \infty} \limsup_{i \to \infty} \|\Phi_i(\tilde{u}_k) - u_i\|_{H_i} = 0.$$ 

Take $\{	ilde{u}_k\}_k \subset C'$ and $\{\tilde{u}_k''\}_k \subset C''$ such that $\tilde{u}_k = \tilde{u}_k' + \tilde{u}_k''$. Then

$$\|\Phi_i(\tilde{u}_k) - u_i\|^2_{H_i} = \|\Phi_i(\tilde{u}_k') - \pi_{K_i}(u_i)\|^2_{H_i} + \|\Phi_i(\tilde{u}_k'') - \pi_{K_i^\perp}(u_i)\|^2_{H_i}$$

holds, which gives

$$\lim_{k \to \infty} \limsup_{i \to \infty} \|\Phi_i(\tilde{u}_k') - \pi_{K_i}(u_i)\|_{H_i} = 0.$$ 

Since $\tilde{u}_k' \to \pi_{K_{\infty}}(u_{\infty})$, we have the assertion. \qed

**Proposition 8.5.** Let $u_i \in K_i$.

(1) $u_i \to u_{\infty}$ strongly with respect to $\{\Phi_i|_{C'}\}_i$ iff $u_i \to u_{\infty}$ strongly with respect to $\{\Phi_i\}_i$. 

33
Proposition 8.8.

(2) \( u_i \rightarrow u_\infty \) weakly with respect to \( \{ \Phi_i|_{C'} \} \) if \( u_i \rightarrow u_\infty \) weakly with respect to \( \{ \Phi_i \} \).

Proof. Suppose \( u_i \rightarrow u_\infty \) strongly with respect to \( \{ \Phi_i \} \). Then there is \( \tilde{u}_k \in C \) converging to \( u_\infty \) as \( k \rightarrow \infty \) such that

\[
\lim_{k \rightarrow \infty} \limsup_{i \rightarrow \infty} \| \Phi_i(\tilde{u}_k) - u_i \|_{H_i} = 0.
\]

Since \( \{ \pi_{K_i}\tilde{u}_k \}_{k \in C'} \subset C' \) converges to \( u_\infty \), we can show that \( u_i \rightarrow u_\infty \) strongly with respect to \( \{ \Phi_i|_{C'} \} \). The converse is trivial.

Next we show (2). Suppose \( u_i \rightarrow u_\infty \) weakly with respect to \( \{ \Phi_i|_{C'} \} \). Then it suffices to show that \( \lim_{i \rightarrow \infty} \langle u_i, v_i \rangle \) for any strongly converging sequence \( v_i \in H_i \). Since \( \pi_{K_i}(v_i) \rightarrow \pi_{K_\infty}(v_\infty) \) strongly with respect to \( \{ \Phi_i \} \) by the previous proposition, then \( \pi_{K_i}(v_i) \rightarrow \pi_{K_\infty}(v_\infty) \) strongly with respect to \( \{ \Phi_i|_{C'} \} \) by (1). Then we can deduce that

\[
\lim_{i \rightarrow \infty} \langle u_i, v_i \rangle = \lim_{i \rightarrow \infty} \langle u_i, \pi_{K_i}(v_i) \rangle = \langle u_\infty, \pi_{K_\infty}(v_\infty) \rangle.
\]

The converse is trivial. \( \square \)

Assume that \( A_i \) preserves the decomposition \( H_i = K_i \oplus K_i^\perp \) in the following sense;

(iii) \( D(A_i) = D(A_i) \cap K_i \oplus D(A_i) \cap K_i^\perp \),
(iv) \( A_i(D(A_i) \cap K_i) \subset K_i \) and \( A_i(D(A_i) \cap K_i^\perp) \subset K_i^\perp \).

We can see that \( \mathcal{E}_i|_{K_i} \) is closed quadratic form on \( K_i \) since \( \mathcal{E}_i \) is closed. Moreover, we have the followings by combining Propositions 8.4 and 8.5.

**Proposition 8.6.** If \( \mathcal{E}_i \rightarrow \mathcal{E}_\infty \) compactly, then \( \mathcal{E}_i|_{K_i} \rightarrow \mathcal{E}_\infty|_{K_\infty} \) compactly.

**Proposition 8.7.** Let \( \Sigma_i|_{K_i} \) be the spectral structure on \( K_i \) generated by \( A_i|_{D(A_i) \cap K_i} \). If \( \Sigma_i \rightarrow \Sigma_\infty \) compactly, then \( \Sigma_i|_{K_i} \rightarrow \Sigma_\infty|_{K_\infty} \) compactly.

### 8.3 \( G \)-invariant subspaces

Now let \( G \) be a compact topological group and \( (\rho, V) \) be a unitary representation of \( G \). Suppose \( G \) acts isometrically on each \( H_i \), \( G \cdot C \subset C \) and \( \Phi_i \) are all \( G \)-equivariant. Then \( (C \otimes V)^G \) is dense in \( (H_\infty \otimes V)^G \) and \( \Phi_i \) and \( \Sigma_i \) induce \( \Phi_i^\rho := \Phi_i^V|_{(C \otimes V)^G} : (C \otimes V)^G \rightarrow (H_i \otimes V)^G \) and \( \Sigma_i^\rho := \Sigma_i^V|_{(H_i \otimes V)^G} \). By Propositions 8.3 and 8.7 we have the following result.

**Proposition 8.8.** If \( \Sigma_i \rightarrow \Sigma_\infty \) compactly, then \( \Sigma_i^\rho \rightarrow \Sigma_\infty^\rho \) compactly.
9 Main result

Combining Propositions 7.13 and 8.8 we have the following result.

**Theorem 9.1.** Let $G$ be a compact Lie group, $(P_i, d_i, \nu_i) \in \mathcal{M}(n, \kappa, D)$ and $(P_{\infty}, d_{\infty}, \nu_{\infty}) \in \mathcal{M}(n, \kappa, D)$ have isometric $G$-actions and

$$(P_i, d_i, \nu_i) \xrightarrow{G\text{-}mGH} (P_{\infty}, d_{\infty}, \nu_{\infty}).$$

Let $\Sigma_i$ be the spectral structures generated by $\Delta_{\nu_i}$ and $(\rho, V)$ be a finite dimensional unitary representation of $G$. Then $\Sigma^\rho_i \to \Sigma^\rho_{\infty}$ compactly as $i \to \infty$.

**Theorem 9.2.** Let $G$ be a compact Lie group, $(P_i, d_i, \nu_i) \in \mathcal{M}(n, \kappa, D)$ have isometric $G$-action for each $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ and

$$(P_i, d_i, \nu_i) \xrightarrow{G\text{-}mGH} (P_{\infty}, d_{\infty}, \nu_{\infty}).$$

Denote by $\lambda^\rho_{i,j}$ the $j$-th eigenvalue of $\Delta_{\nu_i}: (D(\Delta_{\nu_i}) \otimes V)^G \to (L^2(\pi)^{\otimes^n} \otimes V)^G$ and let $\{f_{\infty,j}\}_j$ be the orthonormal system of $(L^2(\pi)^{\otimes^n} \otimes V)^G$ such that $\Delta_{\nu_i} f_{i,j} = \lambda^\rho_{i,j} f_{i,j}$ and $\lim_{i \to \infty} \|f_{i,j} - f_{\infty,j}\|_{GH} = 0$.

**Proof.** Let $\hat{\Phi}_i$ be as in Proposition 7.13 and $\hat{\Phi}_i^\rho$ be as in Section 8.3. By putting $f_{i,j} := \hat{\Phi}_i^\rho(f_{\infty,j})$ we have the assertion by Proposition 7.13. \qed

Let $P$ be a principal $G$-bundle, $A$ be a $G$-connection on $P$, $(\rho, V)$ be a real unitary irreducible representation of $G$. Put $E^\rho = P \times_{\rho} V$ and let $\nabla = \nabla^A$ is induced from $A$, and $\lambda_j^\nabla$ be the $j$-th eigenvalue of the rough Laplacian $\nabla^* \nabla$.

As a consequence of the above theorem, we have the following result.

**Theorem 9.3.** Let $G$ be a compact Lie group. For any $\kappa \in \mathbb{R}$, $D, N > 0$, there exist constants $0 \leq c_j < C_j$ depending only on $j, n, \kappa, D, N, G, \rho, V$ such that $\lim_{j \to \infty} c_j = \infty$ and the following holds. For any closed Riemannian manifold $(X, g)$ of dimension $n$ and principal $G$-bundle $\pi: P \to X$ with the $G$-connection $A$ such that

$$\dim X = n, \quad \text{Ric}_g \geq \kappa g, \quad \text{diam} X \leq D,$$

$$\|(d^{\nabla^A})^* F^A\|_{L^\infty} \leq N, \quad \|F^A\|_{L^\infty} \leq N,$$

we have

$$c_j \leq \lambda_j^{\nabla^A} \leq C_j.$$
Proof. For fixed \( n, \kappa, D, N \), let \( \mathcal{M}_G(n, \kappa, D, N, \sigma) \) be the space consists of isometric classes of the Riemannian manifold \((P, h(g, A, \sigma))\) satisfying the assumptions. By Theorem 6.10, \( \mathcal{M}_G(n, \kappa, D, N, \sigma) \) is precompact with respect to the asymptotically \( G \)-equivariant measured Gromov-Hausdorff topology. Since the function which maps \((P, h(g, A, \sigma))\) to \( \lambda_j^\nabla = \lambda_j^\rho - \chi_{\sigma, \rho} \) is continuous by Section 3 and Theorem 9.2, it has the minimum \( c_j \) and maximum \( C_j \). Next we show that \( c_j \to \infty \) as \( j \to \infty \). Suppose not. Then there are a constant \( B > 0 \), an increasing sequence \( 1 \leq j_1 < j_2 < j_3 < \cdots \), principal bundles \( P_k \to X_k \) with \( G \)-connections \( A_k \) such that \( \lambda_{j_k}^{\nabla A_k} \leq B \) for every \( k \). Then by the precompactness of \( \mathcal{M}_G(n, \kappa, D, N, \sigma) \), there exists the limit \((P, d, \nu)\) of some subsequences of \( \{P_k\}_k \). On \((P, d, \nu)\), one can check that \( \lambda_j^\rho \leq B + \chi_{\rho, \sigma} \) for all \( j \), hence we have the contradiction.

References

[1] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. I. *J. Differential Geom.*, 46(3):406–480, 1997.
[2] Jeff Cheeger and Tobias H. Colding. On the structure of spaces with Ricci curvature bounded below. III. *J. Differential Geom.*, 54(1):37–74, 2000.
[3] Kenji Fukaya. Collapsing of Riemannian manifolds and eigenvalues of Laplace operator. *Invent. Math.*, 87(3):517–547, 1987.
[4] Shouhei Honda. Spectral convergence under bounded Ricci curvature. *J. Funct. Anal.*, 273(5):1577–1662, 2017.
[5] Atsushi Kasue. Spectral convergence of Riemannian vector bundles. *Sci. Rep. Kanazawa Univ.*, 55:25–49, 2011.
[6] Kazuhiro Kuwae and Takashi Shioya. Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry. *Comm. Anal. Geom.*, 11(4):599–673, 2003.
[7] John Lott. Collapsing and Dirac-type operators. In *Proceedings of the Euroconference on Partial Differential Equations and their Applications to Geometry and Physics (Castelvecchio Pascoli, 2000)*, volume 91, pages 175–196, 2002.
[8] John Lott. Collapsing and the differential form Laplacian: the case of a smooth limit space. *Duke Math. J.*, 114(2):267–306, 2002.