Topology Changes by Quantum Tunneling in Four Dimensions

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Abstract

We investigate topology-changing processes in 4-dimensional quantum gravity with a negative cosmological constant. By playing the “gluing-polytope game” in hyperbolic geometry, we explicitly construct an instanton-like solution without singularity. Because of cusps, this solution is non-compact but has a finite volume. Then we evaluate a topology change amplitude in the WKB approximation in terms of the volume of this solution.

Topology change may occur in quantum gravity though it would not happen in physically restricted classical spacetimes [1]. In 3-dimensional spacetime with a negative cosmological constant, Fujiwara, Higuchi, Hosoya, Mishima and one of the present authors(M. S.) [2] demonstrated that the topology change can occur due to the quantum tunneling effect by constructing the explicit examples of the solutions. According to Gibbons and Hartle [3], the quantum tunneling spacetime is semi-classically approximated by a Riemannian manifold

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with totally geodesic boundaries. In Ref. [2] such manifolds are constructed from regular truncated polyhedra embedded in a hyperbolic 3-space. In this paper we extend their procedure to a 4-dimensional spacetime and construct a solution of the Einstein equation. Using 4-dimensional regular truncated polytope embedded in a hyperbolic 4-space [4], we construct a 4-dimensional spacetime solution corresponding to the topology change by a quantum tunneling effect.

Gibbons and Hartle [3] proposed that the tunneling process is described by a Riemannian manifold which has the boundaries $\Sigma_i$ and $\Sigma_f$ in the WKB approximation. The topology change is characterized by the difference of topologies of the initial spatial hypersurface $\Sigma_i$ and the final spatial hypersurface $\Sigma_f$. For quantum tunneling in the semi-classical picture, these spatial hypersurfaces are required to have vanishing extrinsic curvatures. We call these boundary hypersurfaces with vanishing extrinsic curvature as totally geodesic boundaries [3].

When we suppose that the spacetime is homogeneous and the Weyl tensor vanishes everywhere, the Riemannian manifold becomes locally isometric to any one of the cases of $S^4$ (4-sphere), $R^4$ (4-plane) or $H^4$ (4-hyperboloid). In Ref. [3] it is also stated that if we require that the $\Sigma_i$ and $\Sigma_f$ are disconnected, the spacetime should at some points violate an energy condition, which demands

$$R_{\mu\nu}V^\mu V^\nu > 0$$

for all vector $V^\mu$. Therefore we can exclude $S^4$ from our considerations of topology changing manifold because the curvature is positive. Since the variety of hyperbolic manifolds (Riemannian manifolds locally isometric to $H^4$) is very rich, we shall consider the vacuum case with a negative cosmological constant. Then the following question arises;

Can we construct a hyperbolic 4-manifold with totally geodesic boundaries $\Sigma_i$ and $\Sigma_f$ which have different topologies?

In this case it is noted that from the Gauss-Codazzi equation, the vanishing extrinsic curvature makes $\Sigma_i$ and $\Sigma_f$ also have a hyperbolic structure (locally isometric to $H^3$). Any
4-manifold with hyperbolic structure is the quotient manifold of a hyperbolic 4-space with a discrete subgroup of its isometry group $SO(4,1)$. The fundamental region of this quotient 4-manifolds is a 4-polytope embedded into $H^4$. The boundaries of the fundamental region are identified with each other and the fundamental region forms a 4-manifold. Following the procedure of 3-dimensional case [2], we determine the fundamental region and the identifications of its boundary faces in the hyperbolic geometry. Our steps to construct the 4-dimensional solution are following:

**Step (1):** Decide 4-polytopes which we shall use.

**Step (2):** Give them a hyperbolic structure and truncate its vertices.

**Step (3):** Find identifications of faces on 3-boundaries made by the truncation so as to form 3-manifolds.

**Step (4):** Find the identifications of polyhedra bounding the 4-polytope, which induces the identifications found in the step (3) on the 3-boundaries.

The step (3) is trivial in 3-dimensional case. While all 2-manifolds with hyperbolic structure are classified as Riemann surfaces, we know only a small number of hyperbolic 3-manifolds. This step makes our trial non-trivial. In the step (1), we use a more or less systematic way to decide 4-polytopes which will be explained in a separate paper [7]. Here we decide to use twelve 8-cell’s which are 4-polytopes bounded by eight congruent hexahedra. The development of an 8-cell on 3-space is shown in Fig[1]. Gluing faces in four dimensions according to the arrows depicted in Fig[1] we get a 4-dimensional polytope surrounded by these eight hexahedra, which has sixteen vertices.
FIG. 1. The development of an 8-cell. Gluing the faces of the hexahedra along the arrows in four dimensions, we get the 8-cell of a 4-dimensional polytope.

To give a hyperbolic structure to these twelve 8-cell’s, we embed them into a hyperbolic 4-space $H^4$. Here we shall use an n-dimensional projective model of a hyperbolic n-space [4], where a totally geodesic hypersurface is a hyperplane in the sense of Euclidean geometry. The n-projective model is a model on an open n-disk

$$D^n = \{ x^i \in \mathbb{R}^n | x^i x_i < 1 \},$$

(2)

in which a metric is

$$ds^2 = \frac{1}{1-r^2} \left( \frac{dr^2}{1-r^2} + r^2 d\Omega^2_{n-1} \right).$$

(3)

When $r$ goes to 1, one approaches to a sphere at infinity $\partial D^n$. This metric gives a constant sectional curvature $-1$. In this model all gluing procedures of hyperplanes are executed by isometries $SO(n, 1)$ of the hyperbolic n-space.

The size of the embedded 8-cell is determined so that its dicellular angles (the angle between two adjacent hexahedra in 4-dimensions) becomes $\pi/3$ (the angle decreases as the size increases in the hyperbolic geometry). It is noted that in this size every vertex is out of the sphere at infinity $\partial D^4$ and edges of the 8-cell are tangent to the sphere. Then, each
hexahedron of the 8-cell is embedded into an induced 3-projective model (sub-model of the 4-projective model) as shown in Fig.2. As it shows, every vertex is out of the sphere at infinity \( \partial D^3 \) and all edges are tangent to the sphere.

![Diagram of the hexahedron](image)

**FIG. 2.** The shaded sphere is a sphere at infinity of a 3-projective model. Each edge of the hexahedra is tangent to the sphere at \( s, t, u,... \). The sphere is cut by planes through \( s, t, u,... \). Along these planes we truncate the vertices of the hexahedron.

In the same way as the 3-dimensional case [2], we truncate each vertex of the 8-cell. Let us pay attention to the four hexahedra having a vertex in common. A remarkable property of the hyperbolic space guarantees the existence of a unique 3-hyperplane which is perpendicular to all of the four hexahedra. We cut the sixteen vertices of the 8-cell along these 3-hyperplanes to get a regular truncated 8-cell embedded completely in the 4-projective model. These truncations of 8-cell induce truncations of each hexahedron bounding the 8-cell. The resultant hexahedron is shown in Fig.2. On the section of the open 3-disk a triangle appears and its three vertices are on the sphere at infinity \( \partial D^3 \). It is noted that the triangles share vertices with neighboring triangles and every original edge of the hexahedron is completely truncated off by two adjacent truncations. Since a single 8-cell has sixteen vertices, the truncated polytope contains sixteen boundary components made by the
truncation. Because four hexahedra share one vertex in an 8-cell and the triangles made by the truncation form a tetrahedron, we see that the boundary is a tetrahedron whose vertices are on the sphere at infinity (see Fig. 3). The dihedral angle of the tetrahedron is $\pi/3$ due to the regular truncation. An easy calculation tells us that such tetrahedra have a finite volume though they are non-compact.

We label the faces of a tetrahedra by the cells which the face belongs to, and the vertices of a tetrahedra by the nearest neighbor vertices of the 8-cell's.

For the step (3), we consider a 3-manifold on the boundary, which should be constructed from several tetrahedra with a dihedral angle $\pi/3$ and their vertices on the sphere at infinity $\partial D^3$ but have a finite volume. In this paper, we construct a suitable manifold $M_b$ from such twelve tetrahedra, which is a non-compact but complete smooth 3-manifold with a finite volume. The faces and vertices of the twelve tetrahedra are labeled as depicted in Fig. 4. It should be noticed that the six primed tetrahedra are the mirror reflections of the other unprimed six tetrahedra. We call these primed tetrahedra as right-handed and the unprimed as left-handed. The following pairs of the faces of right-handed tetrahedra and the faces of the left-handed tetrahedra are glued so that each labeled vertex matches.
\(A_1 - A'_1, B_1 - B'_3, C_1 - C'_2, D_1 - D'_4\)
\(A_2 - A'_2, B_2 - B'_1, C_2 - C'_4, D_2 - D'_3\)
\(A_3 - A'_3, B_3 - B'_2, C_3 - C'_5, D_3 - D'_6\)
\(A_4 - A'_4, B_4 - B'_6, C_4 - C'_1, D_4 - D'_5\)
\(A_5 - A'_5, B_5 - B'_4, C_5 - C'_6, D_5 - D'_1\)
\(A_6 - A'_6, B_6 - B'_5, C_6 - C'_3, D_6 - D'_2\)

For instance, \(A_1\) is matched with \(A'_1\). All the vertices \(p_1, p_2, p_3\) of \(A_1\) are identified with the vertices \(p_1, p_2, p_3\) of \(A'_1\), respectively. Since the dihedral angle is \(\pi/3\), a neighborhood of each point on edge is a 3-ball (This can be checked by exhaustion in the way executed in Ref. [2]).

\[i = 1 \sim 6\]

FIG. 4. We consider six left-handed tetrahedra and six right-handed tetrahedra. Twelve tetrahedra form \(\mathcal{M}_b\).

On the other hand, the vertices of the tetrahedra are on the sphere at infinity \(\partial D^3\). Hence these vertices form four cusps \(\mathbb{E}\) corresponding to the vertices \(p_1, p_2, p_3, p_4\). Then \(\mathcal{M}_b\) is non-compact. We, however, are not disappointed by this non-compactness since the manifold has a finite volume and is complete and smooth. There is no singularity. If there were no cusps, we should also check the consistency of identifications additionally on each vertex about solid angles. This consistency check makes the searching of solutions much involved. In fact, we know much more cusped 3-manifolds than compact 3-manifolds. Hence, admitting the cusp in our manifold we get the following simplest example of topology change solutions.
In the step (4) we expect that there are appropriate identifications among cells (hexahedra) of twelve 8-cell’s, which induce identifications on the $16 \times 12$ tetrahedra given by the truncation of the vertices of the twelve 8-cell’s (possessing sixteen vertices) so as to form sixteen $M_b$’s. We find such identifications of the cells, as follows. We divide the twelve 8-cell’s into six left-handed ones and six right-handed ones with prime (Fig.5) following the twelve tetrahedra composing $M_b$ (see Fig.4). Each vertex and cell (hexahedra) of the 8-cell’s are labeled as shown in Fig.5. One identifies the pairs of the cells (hexahedra) of the left-handed 8-cell’s and that of the right-handed 8-cell’s so that the labeled vertices ($a \sim p$) match as shown below. The pairs are determined following the gluing of the tetrahedra composing $M_b$ (5).

$$
1_1 - 1'_1 \quad 4_1 - 4'_1 \quad 3_1 - 3'_1 \quad 2_1 - 2'_1 \\
1_2 - 1'_2 \quad 4_2 - 4'_2 \quad 3_2 - 3'_2 \quad 2_2 - 2'_2 \\
1_3 - 1'_3 \quad 4_3 - 4'_2 \quad 3_3 - 3'_3 \quad 2_3 - 2'_3 \\
1_4 - 1'_4 \quad 4_4 - 4'_6 \quad 3_4 - 3'_4 \quad 2_4 - 2'_4 \\
1_5 - 1'_5 \quad 4_5 - 4'_1 \quad 3_5 - 3'_5 \quad 2_5 - 2'_5 \\
1_6 - 1'_6 \quad 4_6 - 4'_5 \quad 3_6 - 3'_6 \quad 2_6 - 2'_6 \\
5_1 - 5'_1 \quad 8_1 - 8'_3 \quad 7_1 - 7'_2 \quad 6_1 - 6'_4 \\
5_2 - 5'_2 \quad 8_2 - 8'_1 \quad 7_2 - 7'_4 \quad 6_2 - 6'_3 \\
5_3 - 5'_3 \quad 8_3 - 8'_2 \quad 7_3 - 7'_3 \quad 6_3 - 6'_5 \\
5_4 - 5'_4 \quad 8_4 - 8'_4 \quad 7_4 - 7'_1 \quad 6_4 - 6'_6 \\
5_5 - 5'_5 \quad 8_5 - 8'_5 \quad 7_5 - 7'_5 \quad 6_5 - 6'_5 \\
5_6 - 5'_6 \quad 8_6 - 8'_5 \quad 7_6 - 7'_3 \quad 6_6 - 6'_2
$$

Of course these identifications are orientation preserving isometry transformation because of reflection symmetry between the primed 8-cell’s and the others. The resultant space is orientable.
FIG. 5. There are two types of 8-cell's. Upper ones are with left-handed tetrahedra. Lower ones (with prime) are with right-handed tetrahedra. The corresponding cells (for example, $1_1$ and $1'_1$, $4_1$ and $4'_3$ ...) are identified.

By these identifications, twelve vertices with the same name (for example, twelve vertices named $(a)$ in each 8-cell in Fig.5) are all identified. Then corresponding twelve tetrahedra made by the truncation of these twelve vertices with the same name are glued. The gluing of the faces of the tetrahedra is determined by the gluing of the cells. For example, labeling the faces of the tetrahedron named $(a)$ by the index of the cell which the face belongs to, and the vertices of the tetrahedron $(a)$ by the index of the nearest neighbour vertices of the 8-cell's, these twelve tetrahedra are equivalent to the tetrahedra composing $\mathcal{M}_b$ (see Fig.3 and Fig.4). Comparing the two gluings (4) and (5), we find that the twelve tetrahedra $(a)$ form $\mathcal{M}_b$ since the both gluings are done so that the labeled vertices of the tetrahedra match.

It is easy to check that the tetrahedra made by the truncations of the vertices with the other names $(b) \sim (p)$ also form $\mathcal{M}_b$. Then, the 4-space constructed from the 8-cell has sixteen totally geodesic boundaries $\mathcal{M}_b$'s, since a regular truncation guarantees that the
boundary given by the truncation is a totally geodesic smooth manifold \([2]\).

To check that this 4-space is complete smooth 4-manifold, we consider the neighborhood of faces, edges and vertices. In 4-dimensions, when we turn around each face completely the total angle has to be \(2\pi\) by consistency. On the boundary 3-hypersurface, however, this amounts to checking the \(2\pi\) turn around the edges (\(\alpha, \beta, \gamma\ldots\) in Fig.2) of the boundary. This consistency is guaranteed because the boundary is an already checked manifold \(M_b\). The remaining vertex after the regular truncation (\(s, t, u\ldots\) in Fig.2) causes no problem since they form 4-cusps at infinity. The edges are located only on the boundaries which form 3-manifold after the gluing (\(\alpha, \beta, \gamma\ldots\) in Fig.2). As mentioned above, this does not bring any trouble. Hence this space is a hyperbolic complete smooth 4-manifold with totally geodesic 3-boundaries possessing a hyperbolic structure. The boundaries are sixteen \(M_b\)'s.

From Gibbons and Hartle \([3]\) and Fujiwara, Higuchi, Hosoya, Mishima and one of the present authors (M. S.) \([2]\) we can see that this manifold is regarded as an instanton causing topology change by quantum tunneling, for example, ‘from nothing to sixteen \(M_b\)’s’, ‘from one \(M_b\) to fifteen \(M_b\)’s’ or ‘from two \(M_b\)’s to fourteen \(M_b\)’s’, and so on (see Fig.3). It is also worthy of notice that by plumbing them we can get infinite series of topology change solutions as examplified in Fig.3.
FIG. 6. The Riemannian manifold with eight boundaries is regarded as the topology change solution ‘from nothing to sixteen $M_b$’s’, ‘from one $M_b$ to fifteen $M_b$’s’ or ‘from two $M_b$’s to fourteen $M_b$’s’, and so on. Furthermore, by plumbing of the solution we get various types of topology change solutions.

The topology change shown in this paper is the first explicit example of a topology change in four dimensions by quantum tunneling effect, which cannot be reduced to a lower-dimensional spacetime. Brill [8] investigated the topology changing 4-spacetime which is essentially a two dimensional topology change. Though the manifold does not satisfy the ‘no boundary’ boundary condition rigorously, we assume that the amplitude can be formally described by the Hawking’s Riemannian path integral as

$$T(h_i, h_f) = \sum_{M_R} \int \mathcal{D}g \exp(-S_E[g]),$$  \hspace{1cm} (6)

where $h_i$ and $h_f$ are the 3-dimensional metrics on the initial spatial hypersurface $\Sigma_i$ and the final spatial hypersurface $\Sigma_f$, respectively. $S_E$ is the Euclidean action,

$$S_E = -\frac{1}{16\pi G} \int (R - 2\Lambda) \sqrt{g} d^4x.$$ \hspace{1cm} (7)

The path integral is over smooth 4-metric $g$ on the Riemannian manifold $M_R$ which has appropriate boundaries $\Sigma_i$ and $\Sigma_f$ by assumption. For the present case, it turns out that $M_R$ is a 4-manifold bounded by sixteen $M_b$’s. Then we can use the obtained solution to evaluate the path integral (6) in the WKB approximation. Since our solution has a constant negative curvature $R = 4\Lambda$, the classical action $\bar{S}_E$ is given by

$$\bar{S}_E = \frac{1}{8\pi G} \frac{V}{|\Lambda|},$$ \hspace{1cm} (8)

where $V$ is a numerical value representing the volume of $M_R$ in the case of $\Lambda = -1$. Though our manifold has cusps, the volume is finite. The calculation of the volume will be shown in forthcoming paper [7].

People might be disturbed by the existence of the cusps. However, we can argue that it physically causes no trouble since the cusps are at infinity and we cannot see. We see only
the pattern of spatial periodicity \cite{4}. If we take the position that the cusp is not allowed, the construction becomes more difficult. In this case we may use a computational calculation \cite{7}.

We would like to thank Professor A. Hosoya and Professor S. Kojima for helpful discussions. One of the authors (M. S.) thanks the Japan Society for the Promotion of Science for financial support. This work was supported in part by the Japanese Grant-in-Aid for Scientific Research Fund of the Ministry of Education, Science and Culture.
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