Coordinate Descent Without Coordinates: Tangent Subspace Descent on Riemannian Manifolds

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Abstract

Coordinate descent is a well-studied variant of the gradient descent algorithm which offers significant per-iteration computational savings for large-scale problems defined on Euclidean domains. On the other hand, optimization on a general manifold domain can capture a variety of constraints encountered in many problems, but are difficult to model with the usual Euclidean geometry. In this paper, we present an extension of coordinate descent to general manifold domains, and provide a convergence rate analysis for geodesically convex and non-convex smooth objective functions. Our key insight is to draw an analogy between coordinates in Euclidean space and tangent subspaces of a manifold, hence our method is called tangent subspace descent. We study the effect that non-trivial curvature of manifolds have on the convergence rate and its computational properties. We find that on a product manifold, we can achieve per-iteration computational savings analogous to the Euclidean setting. On such product manifolds, we further extend our method to handle non-smooth composite objectives, and provide the corresponding convergence rates.

1 Introduction

The basic gradient descent algorithm for unconstrained optimization, arguably one of the most fundamental first-order optimization techniques, was first proposed as early as 1847 by Cauchy [8]. However, in the past decade or so there has been a huge amount of research more generally on first-order algorithms. This is mainly due to the prevalence of large-scale problems in machine learning and data science, for which second-order algorithms are intractable.

A partial list of the new developments include: projected gradient/mirror descent [4, 25, 26], primal-dual methods such as augmented Lagrangian [7, 24], (Bregman) proximal gradient methods [3, 23, 34], stochastic first-order methods and their variance-reduced modifications for handling finite-sum objective functions (which are common in machine learning settings) [11, 18, 27] as well as accelerated methods [5, 28, 29].

The majority of these new developments considered domains endowed with the natural Euclidean geometry. In contrast, there have also been some work on extending these algorithms to more general spaces endowed with non-Euclidean geometry, such as curved manifolds [1, 2, 35]. The utility of such extensions is substantiated by examples of problems which are non-convex in the Euclidean geometry, yet satisfy a generalized definition of convexity when recast in a diffeo-geometric form.
While some convergence results to stationary points were known for optimization on manifolds, to the best of our knowledge, Zhang and Sra [37] provided the first global convergence rate for a generalization of the (projected) gradient descent algorithm for geodesically convex functions on manifolds. Consequently, there has been a flurry of activity providing convergence analysis of extensions for many of the aforementioned first-order algorithms to manifolds [9, 13, 17, 21, 22, 32, 38, 39, 40].

The coordinate descent method [6, 14, 30, 31] is one of the most popular first-order algorithms due to its ability to handle a large number of variables efficiently. However, its extension to the manifold setting is not obvious. The notion of a ‘coordinate system’ is much less straightforward for manifolds. Consider, for example, the unit sphere embedded in 3-dimensional Euclidean space. While each point is described by three coordinates, we know that the sphere can be locally parametrized with two variables, which suggests that it is essentially a two-dimensional object. Thus, rather than using the three natural coordinates, it may be better to exploit the relationship between the sphere and the plane in some fashion. In fact, much of the theory of manifolds is aimed at giving so-called ‘coordinate-free definitions’ for common concepts in Euclidean space, so that these mix-ups become inconsequential. Nevertheless, coordinate descent has proven to be a useful algorithm in the Euclidean setting for large-scale problems, thus our goal in this paper is to investigate how this naturally generalizes to the manifold setting, and whether this enjoys the same computational savings or not.

1.1 Contribution and Outline

Our key insight is as follows. A manifold’s ‘dimension’ is defined to be the dimension of the Euclidean space to which it is locally diffeomorphic to, which is also the same as the dimension of the tangent space at each point (i.e., the space of ‘directions’ which do not leave the manifold). Motivated by this observation, in this paper, we rigorously define an extension of coordinate descent to the manifold setting by thinking of coordinates as basis vectors of the tangent space. With this interpretation, we give a natural generalization of coordinate descent, which we call tangent subspace descent.

We provide convergence analyses for both geodesically convex and non-convex objectives in Section 3, giving global convergence rates for the former, and convergence rates to a stationary point for the latter. We discuss the computational savings obtainable from tangent subspace descent in Section 3.4. We find that, in the most general case, there is little computational benefit. However, for an important subclass of problems on product manifolds, tangent subspace descent enjoys the same kind of savings as coordinate descent does in the Euclidean case; we present this in Section 3.5. On this subclass, we also consider a composite optimization problem in Section 4, where the objective is the sum of a smooth term and a decomposable non-smooth term. We generalize block proximal algorithms to product manifold domains, and provide convergence rates for both geodesically convex and non-convex settings.

Our analysis for smooth objectives is partially inspired by Frongillo and Reid [14], who considered randomized (not necessarily orthogonal) subspace descent in the Euclidean setting. We extend this analysis to cyclic and randomized subspace descent on manifolds. To our knowledge, the only papers which have considered coordinate descent in a non-Euclidean setting are Gao et al. [15], Shalit and Chechik [33]. Both of these examine specifically the Stiefel manifold of orthogonal matrices. The algorithm by Gao et al. [15] updates a single column of the matrix at each step, but is not a gradient algorithm; instead, it iteratively minimizes the objective over each column while preserving the orthogonality constraints. The algorithm by Shalit and Chechik [33] is a specializa-
tion of our proposed one to the Stiefel manifold, using a particular type of geometry, and modifying a single matrix entry at each step.

Our work, in contrast, examines a more general setting for any manifold, with any geometry, and any (valid) specification of the tangent subspaces. (In fact, our tangent subspaces are even permitted to overlap.) Our analysis also gives global convergence guarantees for geodesically convex objectives, in addition to convergence to stationary points for non-convex objectives. Finally, we consider block proximal algorithms for non-smooth objectives on product manifolds, which has not been considered previously.

2 Problem Definition and Preliminaries

For an integer $k \in \mathbb{N}$, we denote $[k] := \{1, \ldots, k\}$. We consider the following general optimization problem

$$f^* = \min_x \{f(x) : x \in M\} \tag{1}$$

where $M$ is a $n$-dimensional connected smooth manifold: a connected, Hausdorff topological space with a countable base of open sets $\{U\}$ each of which is homeomorphic to an open set of $\mathbb{R}^n$. These sets are required to be smoothly compatible, i.e. if $U, V \subseteq M$ are basis elements with defining homeomorphisms $\phi : U \to \bar{U} \subseteq \mathbb{R}^n$ and $\psi : V \to \bar{V} \subseteq \mathbb{R}^n$ then $\phi \circ \psi^{-1}$ is smooth (infinitely differentiable) and smoothly invertible as a map between open sets of Euclidean spaces provided $U \cap V$ is non-empty. The pair $(U, \phi)$ is called a chart. We briefly list the concepts needed in the paper, and defer full details to Absil et al. [1], Lee [19, 20], Vishnoi [36].

Suppose $f : M \to N$ is a function between manifolds $M, N$. Then $f$ is continuous (resp. $k$-times differentiable) if for every $x \in M$ there exist charts $(U, \phi)$ on $M$ and $(V, \psi)$ on $N$ such that $x \in U$, $f(x) \in V$, $F(U) \subseteq V$, and $\phi \circ F \circ \psi^{-1}$ is a continuous (resp. $k$-times differentiable) map between Euclidean spaces. It is called smooth if $\phi \circ F \circ \psi^{-1}$ is infinitely differentiable. We let $C^k(M, N)$ denote the set of $k$-times differentiable functions between $M$ and $N$. In the special case of $N = \mathbb{R}$ we will simply write $C^k(M)$.

An $n$-dimensional vector space, $T_xM$, called a tangent space, associates to each $x \in M$. Each $v \in T_xM$ is a linear functional on $C^\infty(M)$ that obeys the product rule

$$v(fg) = g(x)v(f) + f(x)v(g)$$

for $f, g \in C^\infty(M)$. For $f \in C^\infty(M)$ and $v \in T_xM$, $v(f) \in \mathbb{R}$ is analogous to the directional derivative of $f$ in the direction $v$ at the point $x$. Intuitively, $T_xM$ is the set of ‘valid’ directions in which we can take a directional derivative. (As a counter-example, if $M$ is a linear subspace of $\mathbb{R}^n$, an invalid direction would be one that is orthogonal to $M$.) The reason why we think of $v \in T_xM$ as an abstract linear functional instead of vectors in $\mathbb{R}^n$ is because generally a manifold need not be described as a topological subspace of $\mathbb{R}^n$.

Each $v \in T_xM$ can be associated with a smooth curve $\gamma : I \to M$ (in the sense of maps between manifolds $I$ and $M$) where $I$ is an interval in $\mathbb{R}$, $0 \in I$, and $\gamma(0) = x$. For a function $f : M \to \mathbb{R}$, we compute $v(f) := (f \circ \gamma)'(0)$. Scaling $v$ by a amounts to reparametrizing the curve to \( \tilde{\gamma} : t \mapsto \gamma(at) \).

Let $\gamma_1, \gamma_2$ be curves which give $v_1, v_2 \in T_xM$. A curve which gives $v_1 + v_2$ is constructed as follows. Fix some chart $(U, \psi)$ with $x \in U$ and define vectors $u_1, u_2 \in \mathbb{R}^n$ via $u_k^i = (\psi^i \circ \gamma_k)'(0)$ for $i \in [n]$, $k = 1, 2$. Then define $\gamma(t) = \psi^{-1}(\psi(x) + t(u_1 + u_2))$. Consequently, this formulation shows that tangent vectors, though defined as linear functionals on $C^\infty(M)$ have a well-defined action on functions in $C^1(M)$. Given a smooth curve $\gamma : I \to \mathbb{R}$, we will sometimes denote $\gamma'(t) \in T_{\gamma(t)}M$ as the tangent vector defined by the action $f \mapsto (f \circ \gamma)'(t)$. 

3
We denote the tangent bundle $TM = \{(x,v) : x \in M, v \in T_x M\}$, which is itself a smooth manifold. Often, we write $(x,v) \in TM$ simply as $v$, where the base point $x$ is understood from context. A vector field is a map $V : M \to TM$ satisfying $V(x) \in T_x M$ for each $x \in M$. We let $\mathcal{X}(M)$ denote the set of smooth vector fields on $M$. The differential of a function $F \in C^1(M,N)$ at a point $x \in M$ is a linear operator $dF_x : T_x M \to T_{F(x)} N$ defined as follows: for any $v \in T_x M$, $dF_x[v] \in T_{F(x)} N$ is the tangent vector which computes $dF_x[v](f) = v(f \circ F)$ for any $f \in C^\infty(M)$.

We assume that $M$ has a Riemannian metric, i.e., a collection of inner products $\langle \cdot, \cdot \rangle_x$ on each $T_x M$ which is smoothly varying in the sense that for any two smooth vector fields $V, W : M \to TM$, $x \mapsto \langle V(x), W(x) \rangle_x$ is a smooth function on $M$. We say that $M$ is a Riemannian manifold. Let $\| \cdot \|_x$ be the norm induced by $\langle \cdot, \cdot \rangle_x$ on $T_x M$. A Riemannian metric makes each $T_x M$ into a Hilbert space. A function $f \in C^1(M)$ can be thought of as a linear functional on $T_x M$ also, since $\langle \alpha v + \beta v', f \rangle = \alpha \langle v, f \rangle + \beta \langle v', f \rangle$. By the Riesz representation theorem, there exists some $v_{f,x} \in T_x M$ such that $f(v) = \langle v_{f,x}, v \rangle_x$. We call this vector $v_{f,x}$ the gradient of $f$ at $x$, denoted $\nabla f(x)$.

The length of a piecewise smooth curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ is defined as

$$L(\gamma) = \int_0^1 \| \gamma'(t) \|_{\gamma(t)} dt.$$ 

The Riemannian distance $d(x,y)$ between points $x$ and $y$ is the infimum of the lengths of all piecewise smooth curves joining $x$ and $y$. The manifold $M$ is a metric space when endowed with the Riemannian distance. If $M$ is complete and connected under this metric then the distance can be attained by some smooth curve. We say that a curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ is a geodesic between $x$ and $y$ if $L(\gamma) = d(x,y)$. The exponential map at a point $x \in M$ is denoted $\exp_x : T_x M \to M$ gives us a way to generalize the operation $x + v$ in Euclidean space, where $v \in T_x M$. We have $\exp_x(0) = x$ for any $x \in M$.

The exponential map gives rise to geodesics: the curve $t \mapsto \gamma(t) := \exp_x(tv)$ is always a geodesic between $x$ and $\exp_x(v)$. We assume that $\exp_x(v)$ is well defined for all $x \in M, v \in T_x M$, that is, $M$ is geodesically complete. The Hopf-Rinow theorem states that geodesic completeness and completeness under the Riemannian distance are equivalent. Thus, this is a decidedly mild assumption. We define a set-valued inverse exponential map $\exp_x^{-1} : M \ni y \mapsto T_y M$ as $\exp_x^{-1}(y) = \arg \min_{v \in T_x M} \{ \| v \|_x : \exp_x(v) = y \}$. Furthermore, we have that for any $v \in \exp_x^{-1}(y)$, $d(x,y) = \| v \|_x$, and for general $v \in T_x M$, we have $\| v \|_x \geq d(x,\exp_x(v))$.

For any smooth curve $\gamma : I \to M$ defined on an interval $I \subset \mathbb{R}$, there is a well-defined collection of invertible linear maps called parallel transport operators along $\gamma$, denoted $\Gamma^{t_1}_{\gamma,t_0} : T_{\gamma(t_0)} M \to T_{\gamma(t_1)} M$, for $t_0, t_1 \in I$. These maps satisfy $(\Gamma^{t_1}_{\gamma,t_0})^{-1} = \Gamma^{t_0}_{\gamma,t_1}$ and $\Gamma^{t_0}_{\gamma,t_0} = \text{id}_{T_{\gamma(t_0)} M}$. Furthermore, a key property is that they preserve the Riemannian metric, i.e., $\langle v, v' \rangle_{\gamma(t_0)} = \langle \Gamma^{t_1}_{\gamma,t_0} v, \Gamma^{t_1}_{\gamma,t_0} v' \rangle_{\gamma(t_1)}$ for any $t_0, t_1 \in I, v, v' \in T_{\gamma(t_0)} M$. In this paper, we will only be interested in parallel transport operators along geodesic curves. When $y = \exp_x(v)$, we use the shorthand $\Gamma_{x}^{y} : T_x M \to T_y M$ to denote $\Gamma_{\gamma,0}^{t_1}$. Note that while $\Gamma_{x}^{y}$ depends on $v$, we will suppress this in the notation, since the dependence will be implied from $y = \exp_x(v)$, which will be clear from context. We will also use the notation $\Gamma_{x}^{y}$ when there is a unique geodesic curve between $x$ and $y$, i.e., when $\exp_x^{-1}(y) = \{ v \}$ is a singleton. When $y = \exp_x(v)$ is not clear from context, the fact that $\exp_x^{-1}(y) = \{ v \}$ will be made clear. Note that if $y = \exp_x(v)$, then $x = \exp_{y}(-\Gamma_{y}^{x} v)$ and $(\Gamma_{y}^{x})^{-1} = \Gamma_{y}^{x}$.

In this paper, we are interested in two function classes on geodesically complete Riemannian
manifolds:
\[
f : M \to \mathbb{R} \text{ s.t. } \forall (x,v) \in TM, y = \text{Exp}_x(v), \quad \|\Gamma^y_x \nabla f(x) - \nabla f(y)\|_y \leq L_f d(x,y).
\]
\[
f : M \to \mathbb{R} \text{ s.t. } \forall x,y \in M, v \in \text{Exp}^{-1}_x(y), f(y) \geq f(x) + \langle \nabla f(x), v \rangle_x.
\]
(2)
(3)

We say that a function \(f\) satisfying (2) is \(L_f\)-smooth, and a function satisfying (3) is geodesically convex, or g-convex. Note that \(\|\Gamma^y_x \nabla f(x) - \nabla f(y)\|_y = \|\nabla f(x) - \Gamma^x_y \nabla f(y)\|_x\), and that \(L_f\)-smoothness of \(f\) implies the following upper bound:
\[
\forall x \in M, v \in T_x M, \quad f(\text{Exp}_x(v)) \leq f(x) + \langle \nabla f(x), v \rangle_x + \frac{L_f}{2} \|v\|_x^2.
\]
(4)

In this paper, we use the term smoothness of the function to mean both infinite differentiability and Lipschitz continuity of the gradient. Which definition we mean will be clear from context. When \(f\) is non-differentiable, then \(\nabla f(x)\) in (3) should be replaced with any \(v \in \partial f(x)\), where \(\partial f(x) := \{v \in T_x M : f(y) \geq f(x) + \langle v, w \rangle_x \forall y \in M, w \in \text{Exp}^{-1}_x(y)\}\) is the subdifferential. Furthermore, Ferreira and Oliveira [12, Theorem 2.3] show that g-convexity is equivalent to the following: for any \(x, y \in M, v \in \text{Exp}^{-1}_x(y)\), and \(\beta \in [0, 1]\), \(f(\text{Exp}_x(\beta v)) \leq (1 - \beta)f(x) + \beta f(y)\).

3 Tangent Subspace Descent on General Riemannian Manifolds

To begin our study of algorithms for (1), we present some simple lemmas which will be used to derive convergence rates. We first verify that the necessary first-order stationarity condition \(\nabla f(x^*) = 0\) for minimizers of (1) still holds on Riemannian manifolds. In this case, we say that \(x^*\) is a stationary point of \(f\).

**Lemma 3.1.** Let \(f\) be differentiable on \(M\), and suppose that \(f(x^*) = \min_{x \in M} f(x)\). Then \(\nabla f(x^*) = 0\). Furthermore, when \(f\) is g-convex, then \(\nabla f(x^*) = 0\) only if \(x^*\) minimizes (1).

**Proof.** Choose any infinitely differentiable curve \(\gamma : I \to M\) such that \(\gamma(0) = x^*\). We know that this defines an element \(v \in T_{x^*} M\) with the action \(v(f) = (f \circ \gamma)'(0)\). Furthermore, we know that \(f \circ \gamma : I \to \mathbb{R}\) has a minimum at \(t = 0\) since \((f \circ \gamma)(0) = f(\gamma(0)) = f(x^*) = f^*\). Then \(v(f) = (f \circ \gamma)'(0) = 0\) by the usual first-order condition.

This holds for any curve, hence \(v(f) = \langle \nabla f(x^*), v \rangle_{x^*} = 0\) for all \(v \in T_{x^*} M\), so it must be the case that \(\nabla f(x^*) = 0\).

When \(f\) is g-convex, we have \(f(x) \geq f(x^*) + \langle \nabla f(x^*), v \rangle_{x^*}\) for any \(v \in \text{Exp}^{-1}_{x^*}(x)\), so if \(\nabla f(x^*) = 0\), then \(f(x) \geq f(x^*)\) for any \(x \in M\), hence \(x^*\) solves (1). \(\square\)

The convergence analyses of (non-accelerated) gradient-based methods for smooth convex optimization problems is often based on establishing the following so-called ‘sufficient decrease’ relation between iterates:
\[
f(x^t) - f(x^{t+1}) \geq \eta \|\nabla f(x^t)\|_{x^t}^2.
\]
(5)

Often, (5) is derived only from smoothness of \(f\) rather than convexity. Thus, when \(f\) only satisfies smoothness, we can get a stationarity guarantee when this sufficient decrease condition is met. In the manifold setting, we need to ensure that we are using the right norm for each gradient according to the Riemannian metric.
Lemma 3.2. Let \( \{x^t\}_{t \in \mathbb{N}} \subset M \) be a sequence such that (5) holds. Then

\[
\| \nabla f(x^t) \|_{x^t} = o(1/\sqrt{t})
\]

\[
\min_{s \in [t]} \| \nabla f(x^s) \|_{x^s} \leq \sqrt{\frac{f(x^1) - f^*}{\eta t}}.
\]

Proof. Summing the inequality, the right hand side telescopes to give

\[
\eta \sum_{s \in [t]} \| \nabla f(x^s) \|_{x^s}^2 \leq f(x^1) - f(x^{t+1}) \leq f(x^1) - f^*.
\]

Thus, the infinite sum \( \sum_{t \geq 1} \| \nabla f(x^t) \|_{x^t}^2 \) is bounded, hence we must have \( \| \nabla f(x^t) \|_{x^t} = o(1/t) \). The second result follows from \( \eta t \min_{s \in [t]} \| \nabla f(x^s) \|_{x^s}^2 \leq \eta \sum_{s \in [t]} \| \nabla f(x^s) \|_{x^s}^2 \).

If, in addition, \( x^t \to x^* \in M \), then we can infer that \( x^* \) is a stationary point.

Lemma 3.3. Let \( \{x^t\}_{t \in \mathbb{N}} \subset M \), \( x^* \in M \) be such that \( \| \nabla f(x^t) \|_{x^t} \to 0 \) and \( d(x^t, x^*) \to 0 \). Then \( \nabla f(x^*) = 0 \).

Proof. The proof of this relies on a more general result discussed in Section 4, and is provided in Appendix B.

In the geodesically convex case, the sufficient decrease condition can be used to derive a relation between function values of iterates, from which we derive convergence rates.

Lemma 3.4. Suppose \( f \) is g-convex and we have a sequence \( \{x^t\}_{t \in \mathbb{N}} \subset M \) such that (5) holds. Then

\[
f(x^t) - f(x^{t+1}) \geq \frac{\eta}{d(x^t, x^*)^2} (f(x^t) - f^*)^2.
\]

Proof. Since \( f \) is geodesically convex,

\[
0 \leq f(x^t) - f(x^*) \leq -\langle \nabla f(x^t), \text{Exp}_{x^t}^{-1}(x^*) \rangle_{x^t}
\]

\[
\leq \| \nabla f(x^t) \|_{x^t} \cdot \| \text{Exp}_{x^t}^{-1}(x^*) \|_{x^t}
\]

\[
= \| \nabla f(x^t) \|_{x^t} \cdot d(x^t, x^*).
\]

Combining this with (5) gives us the result.

Under the mild assumption that the level set \( X^1 = \{ x \in M : f(x) \leq f(x^1) \} \) is bounded, we have the following result.

Lemma 3.5. Let \( f : M \to \mathbb{R} \) be a g-convex function, and \( \{x^t\}_{t \in [n]} \subset M \) be a sequence such that (5) holds. Furthermore, suppose that the set \( \{ x \in M : f(x) \leq f(x^1) \} \) has a diameter of \( R \), i.e., \( d(x, x^t) \leq R \) whenever \( f(x), f(x^t) \leq f(x^1) \). Then

\[
f(x^{t+1}) - f^* \leq \frac{R^2 (f(x^1) - f^*)}{R^2 + (f(x^1) - f^*) \eta t}.
\]
Proof. Lemma 3.4 gives
\[ f(x^t) - f(x^{t+1}) \geq \frac{\eta}{d(x^t, x^*)^2}(f(x^t) - f^*)^2. \]

Write \( A_t = f(x^t) - f^* \geq 0 \). We can write the above inequality as \( A_t - A_{t+1} \geq \eta_t A_t^2 \) after defining \( \eta_t := \eta/d(x^t, x^*)^2 \). We also know that since \( f(x^t) - f(x^{t+1}) \geq \eta \cdot \|\nabla f(x^t)\|^2_{x^t} \), we have \( A_t \geq A_{t+1} \). Furthermore, \( x^t, x^* \in \{x \in M : f(x) \leq f(x^1)\} \), so \( \eta_t \geq \eta/R^2 \). By redefining \( \eta \rightarrow \eta/R^2 \), we can write \( A_t - A_{t+1} \geq \eta A_t^2 \). Lemma 3.6 below applies to get the result. \( \square \)

Lemma 3.6. Let \( \{A_t\}_{t \in \mathbb{N}} \subset \mathbb{R} \) be a monotonically decreasing sequence such that \( A_t - A_{t+1} \geq \eta A_t^2 \). Then
\[ A_{t+1} \leq \frac{A_t}{1 + A_t \eta}. \]

Proof. If \( A_{t+1} = 0 \), then we trivially have the result, so suppose \( A_{t+1} > 0 \). Divide both sides by \( A_t A_{t+1} \) to get
\[ \frac{1}{A_{t+1}} - \frac{1}{A_t} \geq \eta \frac{A_t}{A_{t+1}} \geq \eta. \]

Summing this up, the left hand side telescopes, hence \( \frac{1}{A_{t+1}} - \frac{1}{A_1} \geq t \eta \implies A_{t+1} \leq \frac{A_1}{1 + A_1 t \eta} \). \( \square \)

We now propose two schemes, described in Algorithms 1 and 2, to obtain the sufficient decrease condition (5) via the exponential map on a manifold \( M \).

3.1 Deterministic sufficient decrease via cyclic descent

Fix any point \( x \in M \). We give a cyclic descent scheme to find a point \( y \in M \) such that
\[ f(x) - f(y) \geq \eta \|\nabla f(x)\|^2_x, \]
for some \( \eta > 0 \). The key idea is the following. Let \( x^0 = x \). We first decompose \( T_{x^0}M \) into \( m \) (not necessarily orthogonal) subspaces \( \{V^{0,k} : k \in [m]\} \) so that
\[ T_{x^0}M = \text{Span}\left(\{V^{0,k} : k \in [m]\}\right). \]

Denote \( U^{0,k} \) as the orthogonal projection operator onto the subspace \( V^{0,k} \) (with orthogonality defined by the Riemannian inner product \( \langle \cdot, \cdot \rangle_{x^0} \)). We assume that for any \( v \in V^{0,1} \)
\[ f(\text{Exp}_{x^0}(v)) \leq f(x^0) + \langle \nabla f(x^0), v \rangle_{x^0} + \frac{L_1}{2} \|v\|^2_{x^0}. \]
(This holds, e.g., if we have (2) with \( L_1 = L_f \).) We can write the right hand side as \( f(x^0) + \frac{1}{2} \left\| \sqrt{L_1} v + \frac{1}{L_1} U^{0,1} \nabla f(x^0) \right\|^2_{x^0} - \frac{1}{2L_1} \|U^{0,1} \nabla f(x^0)\|^2_{x^0} \), since \( \langle U^{0,1} \nabla f(x^0), v \rangle_{x^0} = \langle \nabla f(x^0), v \rangle_x \) as \( U^{0,1} \) is an orthogonal projection. Thus, if we choose \( v = -\frac{1}{L_1} U^{0,1} \nabla f(x^0) \), this minimizes the right hand side, resulting in
\[ f(x^0) - f\left( \text{Exp}_{x^0} \left( -\frac{1}{L_1} U^{0,1} \nabla f(x^0) \right) \right) \geq f(x^0) - f \left( \text{Exp}_{x^0} \left( \frac{1}{L_1} U^{0,1} \nabla f(x^0) \right) \right) \geq \frac{1}{2L_1} \|U^{0,1} \nabla f(x^0)\|^2_{x^0}. \]
We denote \( x^1 := \text{Exp}_{x^0} \left( -\frac{1}{L_1} U^{0,1} \nabla f(x^0) \right) \).

Following the same pattern, we repeat this scheme to iteratively generate points \( x^1, \ldots, x^m \): for \( i \in [m] \)

\[ \text{Exp}_{x^{i-1}} \left( -\frac{1}{L_1} U^{0,1} \nabla f(x^{i-1}) \right) \]
• Obtain a decomposition of $T_{x^{i-1}}M$ into subspaces $\{V^{i-1,k}\}_{k\in[m]}$ such that $\text{Span} \left( \{V^{i-1,k}\}_{k\in[m]} \right) = T_{x^{i-1}}M$, and for any $i \in [m]$, $v \in V^{i-1,i}$, $f(\text{Exp}_{x^{i-1}}(v)) \leq f(x^{i-1}) + \langle \nabla f(x^{i-1}), v \rangle_{x^{i-1}} + \frac{L_i}{2} \|v\|^2_{x^{i-1}}$.

Set

$$x^i := \text{Exp}_{x^{i-1}} \left( -\frac{1}{L_i}U^{i-1,i}\nabla f(x^{i-1}) \right).$$

We now have a series of bounds for $i \in [m]$: $f(x^{i-1}) - f(x^i) \geq \frac{1}{2L_i} \|U^{i-1,i}\nabla f(x^{i-1})\|^2_{x^{i-1}}$.

Recalling $x = x^0$ and denoting $y := x^m$, we get via summing these bounds $f(x) - f(y) \geq \sum_{i \in [m]} \frac{1}{2L_i} \|U^{i-1,i}\nabla f(x^{i-1})\|^2_{x^{i-1}}$.

However, our original goal was to obtain $f(x) - f(y) \geq \eta \|\nabla f(x)\|^2_2$. Due to the subspace decomposition, we will use a different norm described in Lemma 3.7. Furthermore, the right condition on the subspace decompositions described in Lemma 3.8 will give us a way to show sufficient decrease.

**Lemma 3.7.** For $i \in [m]$, suppose we have subspaces $\{V^{i-1,k}\}_{k\in[m]}$ such that $\text{Span} \left( \{V^{i-1,k}\}_{k\in[m]} \right) = T_{x^{i-1}}M$. Then the following are norms on $T_{x^{i-1}}M$:

$$v \mapsto \|v\|_{x^{i-1},U^{i-1}} := \sqrt{\sum_{k \in [m]} \|U^{i-1,k}v\|^2_{x^{i-1}}}.$$

**Proof.** Absolute homogeneity and the triangle inequality are inherited from the 2-norm in $\mathbb{R}^m$. If $v \neq 0$, then since $\text{Span} \left( \{V^{i-1,k}\}_{k\in[m]} \right) = T_{x^{i-1}}M$, there exists some $k$ such that $U^{i-1,k}v \neq 0$, hence $\|v\|_{x^{i-1},U^{i-1}} \neq 0$. \(\square\)

For convenience, we denote $\|\cdot\|_{x,U} := \|\cdot\|_{x^0,U^0}$. We now proceed to derive the right relation between the subspaces $V_k^i$. We will start by making the subspaces isometric to each other in the natural sense.

**Lemma 3.8.** For $i \in [m]$, suppose that there exist linear isometries $P_{i-1}^0 : T_{x^{i-1}}M \to T_{x^0}M$ (in the Riemannian metric) such that $P_{i-1}^0 V^{i-1,k} = V^{0,k}$. Then for any $v \in T_{x^{i-1}}M$, $P_{i-1}^0 U^{i-1,k}v = U^{0,k} P_{i-1}^0 v$. Furthermore, $P_{i-1}^0$ is an isometry of norms $\|\cdot\|_{x^{i-1},U^{i-1}} \to \|\cdot\|_{x,U}$.

**Proof.** Let $v \in T_{x^{i-1}}M$ and $U^{i-1,k}_{\perp}$ be the projection onto the orthogonal complement of $U^{i-1,k}$. Since $v = U^{i-1,k}_{\perp}v + U^{i-1,k}v$ and $P_{i-1}^0$ is a linear map, we have $P_{i-1}^0 v = P_{i-1}^0 U^{i-1,k}_{\perp}v + P_{i-1}^0 U^{i-1,k}v$. But notice now that by assumption $P_{i-1}^0 U^{i-1,k}_{\perp}v \in V^{0,k}$, and since $P_{i-1}^0$ is an isometry, any vector $v'$ orthogonal to $U^{i-1,k}_{\perp}v$ also has $(P_{i-1}^0 v', P_{i-1}^0 U^{i-1,k}_{\perp})_x = 0$. Thus $P_{i-1}^0 U^{i-1,k}_{\perp}v$ must be in the orthogonal complement of $V^{0,k}$, hence $U^{0,k} P_{i-1}^0 U^{i-1,k}_{\perp}v = 0$. This implies $U^{0,k} P_{i-1}^0 v = P_{i-1}^0 U^{i-1,k}v$.\(\square\)
Observe that for any \( v \in T_{x^{-1}} M \),
\[
\|P_{i-1}^0 v\|_{x,U}^2 = \sum_{k \in [m]} \|U^{0,k} P_{i-1}^0 v\|_{x,U}^2 = \sum_{k \in [m]} \|P_{i-1}^0 U^{0,k} v\|_{x,U}^2 = \sum_{k \in [m]} \|U^{i-1,k} v\|_{x^{i-1},U^{i-1}}^2 = \|v\|_{x^{i-1},U^{i-1}}^2.
\]

Under the condition in Lemma 3.8 and an additional one to be shown later, we get the following sufficient decrease condition.

**Lemma 3.9.** Let \( P_{i-1}^0 : T_{x^{-1}} M \to T_x M, \ i \in [m], \) be a collection of linear isometries such that \( P_{i-1}^0 V^{-1,k} = V^{0,k} \) for each \( i,k \in [m] \). Furthermore, suppose that the iterates satisfy
\[
\|U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 V f(x^{-1})\|_x^2 \leq C \sum_{j \in [i]} \|U^{j-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2.
\]
Then
\[
f(x) - f(y) \geq \frac{1}{4L_{\text{max}}(1 + Cm)} \| \nabla f(x) \|_{x,U}^2.
\]

**Proof.** We know that
\[
f(x) - f(y) \geq \sum_{i \in [m]} \frac{1}{2L_i} \| U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2 \geq \frac{1}{2L_{\text{max}}} \sum_{i \in [m]} \| U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2
\]
\[
\| \nabla f(x)\|_{x,U}^2 = \sum_{i \in [m]} \| U^{0,i} \nabla f(x)\|_x^2.
\]
For every \( i = 1, \ldots, m \),
\[
\| U^{0,i} \nabla f(x)\|_x^2 = \| U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 \nabla f(x^{-1}) + U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_x^2 \\
\leq \left( \| U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_x + \| U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_{x^{-1}} \right)^2 \\
\leq 2 \| U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_x^2 + 2 \| U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_{x^{-1}}^2 \\
= 2 \| U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_x^2 + 2 \| P_{i-1}^0 U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2 \\
= 2 \| U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 \nabla f(x^{-1})\|_x^2 + 2 \| U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2 \\
\leq 2 \| U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2 + 2C \sum_{j \in [i]} \| U^{j-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2.
\]
Summing this over \( i = 1, \ldots, m \) we have
\[
\| \nabla f(x)\|_{x,U}^2 = \sum_{i \in [m]} \| U^{0,i} \nabla f(x)\|_x^2 \leq 2 \sum_{i \in [m]} (1 + (m - i)C) \| U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2 \\
\leq 2 (1 + Cm) \sum_{i \in [m]} \| U^{i-1,i} \nabla f(x^{-1})\|_{x^{-1}}^2.
\]
\[\square\]
We now further refine the isometries $P^0_{i-1}$ to ensure $\|U^{0,i} \nabla f(x) - P^0_{i-1} U^{i-1,i} \nabla f(x^{i-1})\|_x^2 \leq C \sum_{j \in [i]} \|U^{j-1,j} \nabla f(x^{j-1})\|_{x^{-1}}^2$. Specifically, we set the isometries to be $P^0_i = P^0_{i-1} \Gamma_{x^i}^{x^{i-1}}$ for $i \in [m]$, where $P^0_0 = \text{id}_{T_x M}$.

**Lemma 3.10.** If $P^0_0 = \text{id}_{T_x M}$, $P^0_i = P^0_{i-1} \Gamma_{x^i}^{x^{i-1}}$ for $k \in [m]$, and (2) holds, then

$$\|U^{0,i} \nabla f(x) - U^{0,i} P^0_{i-1} \nabla f(x^{i-1})\|_x^2 \leq \frac{L^2_j (m - 1)}{L^2_{\text{min}}} \sum_{j \in [i-1]} \|U^{j-1,j} \nabla f(x^{j-1})\|_{x^{-1}}^2.$$ 

**Proof.** Since $U^{0,i}$ is a projection onto a subspace $V^{0,i}$,

$$\|\nabla f(x) - P^0_{i-1} \nabla f(x^{i-1})\|_x^2 \geq \|U^{0,i} \nabla f(x) - U^{0,i} P^0_{i-1} \nabla f(x^{i-1})\|_x^2,$$

Thus, it suffices to examine $\|\nabla f(x) - P^0_{i-1} \nabla f(x^{i-1})\|_x^2$ to get the bound. For convenience, write $\nabla f(x) = P^0_0 \nabla f(x^0)$ where $P^0_0 : T_x M \rightarrow T_x M$ is the identity map. Then for $i > 1$,

$$\|\nabla f(x) - P^0_{i-1} \nabla f(x^{i-1})\|_x^2 = \left( \sum_{j \in [i-1]} \|P^0_{j-1} \nabla f(x^{j-1}) - P^0_j \nabla f(x^j)\|_x \right)^2 \leq \left( \sum_{j \in [i-1]} \|P^0_{j-1} \nabla f(x^{j-1}) - P^0_j \nabla f(x^j)\|_x \right)^2 \leq \left( \sum_{j \in [i-1]} \|\nabla f(x^{j-1}) - \Gamma_{x^j}^{x^{j-1}} \nabla f(x^j)\|_{x^{-1}} \right)^2 \leq (i - 1) \sum_{j \in [i]} \|\nabla f(x^{j-1}) - \Gamma_{x^j}^{x^{j-1}} \nabla f(x^j)\|_{x^{-1}}^2 \leq L^2_j (i - 1) \sum_{j \in [i-1]} d(x^{j-1}, x^j)^2 \leq L^2_j (i - 1) \sum_{j \in [i-1]} \frac{1}{L^2_j} \|U^{j-1,j} \nabla f(x^{j-1})\|_{x^{-1}}^2 \leq \frac{L^2_j (m - 1)}{L^2_{\text{min}}} \sum_{j \in [i-1]} \|U^{j-1,j} \nabla f(x^{j-1})\|_{x^{-1}}^2.$$

In the fourth inequality, we used the fact that $x^j = \text{Exp}_{x^{j-1}} \left( -\frac{1}{L^2_j} U^{j-1,j} \nabla f(x^{j-1}) \right).$
Algorithm 1: Cyclic sufficient decrease method

Data: initial point \( x \in M \), decomposition \( T_x M = \text{Span}(\{V^0,k\}_{k \in [m]}) \).

Result: New point \( y \in M \), linear isometry \( P : T_x M \to T_y M \)

Denote \( x^0 = x, P^0_0 = \text{id}_{T_x M} \) and \( V^0,k \) to be projection matrices onto \( V^0,k \) for \( k \in [m] \);

for \( i = 1, \ldots, m \) do

Update \( V^{i-1,i} = P^{i-1}_0 V^0,i \subseteq T_{x^{i-1}} M \), \( U_i \) is the projection onto \( V^{i-1,i} \);

Obtain (or compute) \( L^2 \geq \max_{v \in T_{x^{i-1}} M} \frac{f(\exp_{x^{i-1}}(U_i v)) - f(x^{i-1}) - \langle \nabla f(x^{i-1}), U_i v \rangle}{\|U_i v\|^2_{x^{i-1}}} \);

Compute new point \( x^i = \exp_{x^{i-1}} \left( \frac{1}{L_i} U_i \nabla f(x^{i-1}) \right) \);

Update \( P^i_0 = \Gamma_{x^{i-1}}^{x^i} P^{i-1}_0 \);

end

Denote \( y = x^m, P = P^{0,m} \);

The conditions in the above lemma give an exact specification on how to choose the isometries and subspace decomposition at each iteration:

\[
P^0_0 = \text{id}_{T_x M} \quad P^0_{i-1} = \Gamma_{x^0}^{x^i} \cdots \Gamma_{x^{i-2}}^{x^i} \quad i \geq 2 \quad V^{i-1,i} = (P^0_{i-1})^{-1} V^0,i = \Gamma_{x^{i-1}}^{x^i} \cdots \Gamma_{x^0}^{x^i} V^0,i \quad i \geq 2.
\]

Note that each iteration we do not need to compute \( V^{i-1,k} \) if \( k \neq i \).

Algorithm 1 describes the scheme explicitly and Proposition 3.11 gives the guarantee.

**Proposition 3.11.** Suppose that \( f \) is a \( L_f \)-smooth function (such that (2) holds). Let \( x \in M \), subspaces \( \{V_k\}_{k \in [m]} \) and \( y \in M \) be inputs and output of Algorithm 1 respectively. Let \( \| \cdot \|_{x,U} \) be the norm induced by the subspaces \( \{V_k\}_{k \in [m]} \) (as in Lemma 3.7). Then

\[
f(x) - f(y) \geq \frac{L^2_{\min}}{4L_{\max}} \left( \frac{L^2_{\min}}{L^2_{\min} + L^2_f m(m - 1)} \right) \| \nabla f(x) \|_{x,U}^2.
\]

**Proof.** This is a consequence of the above lemmas. \( \square \)

Compared to the sufficient decrease result for in Euclidean space (see, e.g., Beck and Tetruashvili [6]) where the coefficient is \( O(L^2_{\min}/(L_{\max} L^2_f m)) \), Proposition 3.11 incurs an extra \( m - 1 \) factor in the denominator. This is due to the fact that we are on a Riemannian manifold which may have non-trivial curvature in general, which prevents transitivity of parallel transport operators, i.e., \( \Gamma_{T^y R}^z \neq \Gamma_{T^x R}^z \). A well-known example where this fails is the unit sphere in \( \mathbb{R}^n \). In contrast, on the whole of \( \mathbb{R}^n \), the tangent space \( T_x \mathbb{R}^n \) can always be identified with \( \mathbb{R}^n \), and parallel transport in this setting is simply the identity map. In Section 3.5 we will explore more carefully the impact of this, as well as situations where we may circumvent this phenomenon.
Algorithm 2: Randomized sufficient decrease method

Data: initial point $x \in M$, decomposition $T_xM = \text{Span}(\{V_k\}_{k \in [m]}).

Result: New point $y \in M$

Denote $U_k$ to be projection matrices onto $V_k$ for $k \in [m]$;

for $k \in [m]$ do
    Obtain (or compute) $L_k \geq \max_{v \in T_xM} \frac{f(\text{Exp}_x(U_kv)) - f(x) - \langle \nabla f(x), U_kv \rangle_x}{\|U_kv\|_x^2};$
end

Draw $i \in [m]$ randomly with probability $p_i = \frac{L_i}{\sum_{k \in [m]} L_k};$

Compute $y = \text{Exp}_x\left(-\frac{1}{L_i}U_i\nabla f(x)\right);$

3.2 Randomized sufficient decrease

We give an alternative randomized procedure in Algorithm 2 for finding a new point $y$ given a point $x \in M$ with sufficient decrease in expectation.

We know that

$$f(x) - f\left(\text{Exp}_x\left(-\frac{1}{L_i}U_i\nabla f(x)\right)\right) \geq \frac{1}{2L_i}\|U_i\nabla f(x)\|_x^2.$$

Thus, if we randomly choose $i \in [m]$ according to Algorithm 2, we get the following result.

**Proposition 3.12.** Let $x \in M$, subspaces $\{V_i\}_{i \in [m]}$ and $y \in [m]$ be inputs and output of Algorithm 2 respectively. Let $\| \cdot \|_{x,U}$ be the norm induced by the subspace decomposition $\{V_i\}_{i \in [m]}$ (as in Lemma 3.7). Then

$$f(x) - \mathbb{E}[f(y) | x] \geq \frac{1}{2\sum_{i \in [m]} L_i}\|\nabla f(x)\|_{x,U}^2.$$

**Proof.** This is a simple computation for the right hand side:

$$\sum_{i \in [m]} p_i \left(\frac{1}{2L_i}\|U_i\nabla f(x)\|_x^2\right) = \frac{1}{2\sum_{i \in [m]} L_i} \sum_{i \in [m]} \|U_i\nabla f(x)\|_x^2 = \frac{1}{2\sum_{i \in [m]} L_i}\|\nabla f(x)\|_{x,U}^2.$$

In contrast to Proposition 3.11, Proposition 3.12 incurs no extra factors when compared with its Euclidean counterpart, i.e., using non-orthogonal subspaces on a Riemannian metric does not impact the rate. The only change is that we use a different norm, but this also occurs in Euclidean space.

3.3 Subspace descent algorithms

We now combine our findings in Sections 3.1 and 3.2 with Lemmas 3.2, 3.3 and 3.5 to derive tangent subspace descent algorithms together with their convergence guarantees. The basic construction is simple: starting at some $x^1 \in M$, obtain $x^{t+1}$ from $x^t$ by using either Algorithm 1 or Algorithm 2. The only additional thing we need to specify in each of these is the choice of subspace decomposition $\{V_k\}_{k \in [m]}$ at each iteration. We would like to relate the new norms obtained from the subspace decomposition $\| \cdot \|_{x^t,U^t}$ at each step back to the natural norm from the Riemannian metric $\| \cdot \|_{x^t}$ at
The tangent subspace descent algorithms are described in Algorithms 3 and 4. We now present their convergence analysis, assuming that at each iteration, the numbers \( \{L_i\}_{i \in [m]} \) required at each iteration are provided for us, and are the same across the iterations. In the basic case, we can always take \( L_i = L_f \), but if more information is known about \( f \), we may be able to give improvements; we discuss the plausibility of this in Section 3.4.

**Lemma 3.13.** Let \( x, y \) be two distinct points in \( M \), and let \( P : T_x M \to T_y M \) be a linear isometry of the Riemannian inner product, i.e., \( \langle P v, P v' \rangle_y = \langle v, v' \rangle_x \) for any \( v, v' \in T_x M \). Let \( \{V^x_k\}_{k \in [m]} \) be subspaces of \( T_x M \) with orthogonal projection operators \( \{U^x_k\}_{k \in [m]} \) such that \( \text{Span}(\{V^x_k\}_{k \in [m]}) = T_x M \), and define a norm \( v \mapsto \|v\|_{x,U} := \sqrt{\sum_{k \in [m]} \|U^x_k v\|_x^2} \). Furthermore, define a subspace decomposition of \( T_y M \) by \( \{V^y_k = PV^x_k\}_{k \in [m]} \) with orthogonal projections \( \{U^y_k\}_{k \in [m]} \) and a norm \( w \mapsto \|w\|_{y,U} := \sqrt{\sum_{k \in [m]} \|U^y_k w\|_y^2} \).
\[
\sqrt{\sum_{k \in [m]} \| U_k^y w \|_2^2}. \text{ Then we have}
\]
\[
\max_{v \in T_x M : \| v \|_{x, U} = 1} \| v \|_x = \max_{w \in T_y M : \| w \|_{y, U} = 1} \| w \|_y < \infty.
\]

**Proof.** Lemma 3.8 shows that \( P \) is an isometry between norms, so \( \| P v \|_{y, U} = \| v \|_{x, U} \). Also, since \( P \) is an isometry and \( T_x M, T_y M \) are \( n \)-dimensional vector spaces, \( P \) must also be a bijection, so for any \( w \in T_y M \), there exists some \( v \in T_x M \) such that \( P v = w \). Therefore
\[
\max_{w \in T_y M : \| w \|_{y, U} = 1} \| w \|_y = \max_{v \in T_x M : \| P v \|_{y, U} = 1} \| P v \|_y = \max_{v \in T_x M : \| v \|_{x, U} = 1} \| v \|_x.
\]
Finiteness follows since all norms are equivalent on finite-dimensional vector spaces. \( \square \)

**Theorem 3.14.** Let \( f : M \to \mathbb{R} \) be a \( L_f \)-smooth function (satisfying (2) and (4)). Suppose that the sequence \( \{ x^t \}_{t \in \mathbb{N}} \) is generated from Algorithm 3. Define \( C := \frac{4L_{\max}(L_{\min}^2 + L_{\min}^4 m(m-1))}{L_{\min}} \) and \( S := \max_{v \in T_x^1 M : \| v \|_{x, U, t} = 1} \| v \|_{x, U} \).

(a) We have
\[
\| \nabla f(x^t) \|_{x^t} = o(1/\sqrt{t}), \quad \min_{x \in [t]} \| \nabla f(x^t) \|_{x^t} \leq \sqrt{\frac{CS^2 (f(x^t) - f^*)}{t}} = O \left( \frac{m}{\sqrt{t}} \right).
\]

(b) Any limit point \( x^* \in M \) of \( \{ x^t \}_{t \in \mathbb{N}} \) is a stationary point \( \nabla f(x^*) = 0 \).

(c) If in addition \( f \) is \( g \)-convex and the diameter of \( \{ x \in M : f(x) \leq f(x^1) \} \) is \( R \), then
\[
f(x^{t+1}) - f^* \leq \frac{R^2 CS^2 (f(x^1) - f^*)}{R^2 CS^2 + (f(x^1) - f^*) t} = O \left( \frac{m^2}{t} \right).
\]

(d) If the initial subspaces used in Algorithm 3 \( \{ V_k^1 \}_{k \in [m]} \) are orthogonal to each other, then the same bounds hold in (a) and (c) but with \( S := 1 \).

**Proof.** (a) We get the sufficient decrease condition \( f(x^t) - f(x^{t+1}) \geq \frac{1}{C} \| \nabla f(x^t) \|_{x^t, U}^2 \) from Proposition 3.11, where \( \| \nabla f(x^t) \|_{x^t, U} \) is the norm defined from the subspace decomposition at iteration \( t \). Note that \( S \| \nabla f(x^t) \|_{x^t, U} \geq \| \nabla f(x^t) \|_{x^t} \), where the bound holds uniformly over all \( t \) by Lemma 3.13, so \( f(x^t) - f(x^{t+1}) \geq \frac{1}{CS^2} \| \nabla f(x^t) \|_{x^t}^2 \). We now apply Lemma 3.2 to get the result.

(b) The reasoning from part (a) tells us that \( \| \nabla f(x^t) \|_{x^t} \to 0 \), then we apply Lemma 3.3 to show that any limit point \( x^* \) is a stationary point.

(c) We get the sufficient decrease condition \( f(x^t) - f(x^{t+1}) \geq \frac{1}{C} \| \nabla f(x^t) \|_{x^t, U}^2 \) from Proposition 3.11. We also get \( f(x^t) - f(x^*) \leq \langle \nabla f(x^t), \exp_{x^t}^{-1}(x^*) \rangle_{x^t} \leq \| \nabla f(x^t) \|_{x^t} \| \exp_{x^t}^{-1}(x^*) \|_{x^t} \leq S \| \nabla f(x^t) \|_{x^t, U} d(x^t, x^*) \leq \sqrt{CS} d(x^t, x^*) \sqrt{f(x^t) - f(x^{t+1})} \). We can thus proceed as in Lemma 3.5 to get the result.

(d) We have \( S = 1 \) since \( \| v \|_{x^t, U} = \| v \|_{x^t} \) due to orthogonality of the subspaces. \( \square \)
The analysis for the randomized case is similar, but requires more detail, since the sequence of iterates \( \{x^t\}_{t \in \mathbb{N}} \) is now a random sequence (over the randomness introduced in Algorithm 2), and hence the sufficient decrease guarantee \( f(x^t) - E[f(x^{t+1}) \mid x^t] \geq \text{const} \cdot \| \nabla f(x_t) \|^2_{x_t} \) of Proposition 3.12 must be interpreted as an inequality between random variables \( \| \nabla f(x^t) \|_{x_t}, f(x^t), E[f(x^t) \mid x^t] \). Nevertheless, we can show analogous performance guarantees to Theorem 3.14 but with expectations, as well as almost sure convergence of \( \| \nabla f(x^t) \|_{x_t} \to 0 \) and hence almost surely, any limit point of our random sequence is a stationary point. To do the latter, we first prove the following auxiliary result on sequences of random variables.

**Proposition 3.15.** Let \( \{a_t, b_t\}_{t \in \mathbb{N}} \) be integrable sequences of random variables adapted to a filtration \( \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F} \) of some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \), which almost surely satisfy for all \( t \in \mathbb{N} \)

\[
0 \leq b_t \leq a_t - E[a_{t+1} \mid \mathcal{F}_t] \\
\alpha_t \geq \alpha^* \in \mathbb{R}.
\]

Then \( b_t \to 0 \) almost surely.

**Proof.** First, observe that by taking expectation, and using the law of total expectation, for each \( t \in \mathbb{N} \), we have \( E[b_t] \leq E[a_t] - E[a_{t+1}] \). Since the right hand side telescopes, we have

\[
\sum_{t \in \mathbb{T}} E[b_t] \leq E[a_1] - E[a_{T+1}] \leq E[a_1] - a^*,
\]

where the second inequality follows since \( a_{t+1} \geq a^* \) almost surely.

Now fix some \( \epsilon > 0 \). By Markov’s inequality, \( E[b_t]/\epsilon \geq \mathbb{P}[b_t \geq \epsilon] \), therefore substituting this into the previous inequality and taking \( T \to \infty \) gives

\[
\sum_{t=1}^{\infty} \mathbb{P}[b_t \geq \epsilon] \leq \frac{E[a_1] - a^*}{\epsilon} < \infty.
\]

By the Borel-Cantelli lemma, we know that \( \mathbb{P}[b_t \geq \epsilon \text{ infinitely often}] = 0 \).

In fact, this holds for any \( \epsilon > 0 \). Let \( \{\epsilon_n\}_{n \in \mathbb{N}} \) be a sequence such that \( \epsilon_n \to 0 \). Then by the union bound

\[
\mathbb{P} \left[ \exists n \text{ s.t. } b_t \geq \epsilon_n \text{ infinitely often} \right] \leq \sum_{n=1}^{\infty} \mathbb{P}[b_t \geq \epsilon_n \text{ infinitely often}] = 0.
\]

Finally, we observe that \( \mathbb{P}[b_t \to 0] = 1 - \mathbb{P} \left[ \exists n \text{ s.t. } b_t \geq \epsilon_n \text{ infinitely often} \right] = 1 .
\]

**Theorem 3.16.** Let \( f : M \to \mathbb{R} \) be a \( L_f \)-smooth function (satisfying (2) and (4)). Suppose that the sequence \( \{x^t\}_{t \in \mathbb{N}} \) is generated from Algorithm 3. Define \( C := 2 \sum_{i \in [m]} L_i \) and \( S := \max_{\nu \in \mathcal{T}_1^J M} \| \nu \|_{x_1, U_1 = 1} \| \nu \|_{x^1} \).

(a) We have

\[
E \left[ \| \nabla f(x^{t+1}) \|_{x^{t+1}} \right] = o(1/\sqrt{t}), \quad \min_{s \in [t]} E \left[ \| \nabla f(x^s) \|_{x^s} \right] \leq \sqrt{\frac{CS^2(f(x^1) - f^*)}{t}} \leq O \left( \sqrt{m/t} \right).
\]
(b) Almost surely, \(\|\nabla f(x^t)\|_{x^t} \to 0\) and any limit point \(x^* \in M\) of \(\{x^t\}_{t \in \mathbb{N}}\) is a stationary point \(\nabla f(x^*) = 0\).

(c) If in addition \(f\) is \(g\)-convex and the diameter of \(\{x \in M : f(x) \leq f(x^1)\}\) is \(R\), then

\[
\mathbb{E}[f(x^{t+1})] - f^* \leq \frac{R^2 CS^2(f(x^1) - f^*)}{R^2 CS^2 + (f(x^1) - f^*)^2} = O\left(\frac{m}{t}\right).
\]

Proof. (a) The sufficient decrease condition of Proposition 4.6 is \(f(x^t) - \mathbb{E}[f(x^{t+1}) | x^t] \geq \frac{1}{2\epsilon^2}\|\nabla f(x^t)\|_{x^t}^2\). We have by Lemma 3.13 that \(S^2\|\nabla f(x^t)\|_{x^t}^2 \geq \|\nabla f(x^t)\|_{x^t}^2\). Plugging this in, take expectations of the sufficient decrease condition in Proposition 3.12 to get \(\frac{1}{CS^2}\mathbb{E}[\|\nabla f(x^t)\|_{x^t}^2] \leq \mathbb{E}[f(x^t)] - \mathbb{E}[f(x^{t+1})]\). Then the result follows by an analogous proof to Lemma 3.2.

(b) Almost sure convergence of \(\|\nabla f(x^t)\|_{x^t} \to 0\) follows by taking \(\mathcal{F}_t\) to be the filtration generated by \(\{x^1, \ldots, x^t\}\), \(a_t = f(x_t)\) and \(b_t = \frac{1}{CS^2}\|\nabla f(x^t)\|_{x^t}^2\). Then \(a_t \geq f^*\) and integrability of \(a_t\) holds since \(a_t = f(x^t) \leq f(x^1)\), which holds since by construction \(f(x^t)\) is monotonically decreasing. Proposition 3.12 gives \(b_t \leq a_t - \mathbb{E}[a_t | \mathcal{F}_t]\), which also shows integrability of \(b_t\). Then apply Proposition 3.15 to deduce \(\|\nabla f(x^t)\|_{x^t} \to 0\) almost surely. Lemma 3.3 now applies to deduce that any limit point is a stationary point.

(c) Proposition 3.12 and Lemma 3.13 gives \(\frac{1}{CS^2}\|\nabla f(x^t)\|_{x^t}^2 \leq f(x^t) - \mathbb{E}[f(x^{t+1}) | x^t]\). Since \(f\) is \(g\)-convex, we have for any minimizer \(x^*\), \(f(x^*) - f^* \leq \mathbb{E}[\|\nabla f(x^t)\|_{x^t}^2]\). This gives \(\frac{1}{CS^2 R^2}(f(x^t) - f^*)^2 \leq f(x^t) - \mathbb{E}[f(x^{t+1}) | x^t]\). By Jensen’s inequality, the expectation of the left hand side is \(\geq \frac{1}{CS^2 R^2} (\mathbb{E}[f(x^t)] - f^*)^2\), hence \(\frac{1}{CS^2 R^2} (\mathbb{E}[f(x^t)] - f^*)^2 \leq \mathbb{E}[f(x^t)] - \mathbb{E}[f(x^{t+1})]\). Applying Lemma 3.6 gives the result.

3.4 Computational considerations

We now discuss the per-iteration computational complexity of Algorithms 3 and 4, as compared to the standard Riemannian gradient algorithm given by \(x^{t+1} = \text{Exp}_{x^t}\left(\frac{-1}{t} \nabla f(x^t)\right)\). In the Euclidean setting, the coordinate descent algorithm saves on the per-iteration computational complexity of gradient descent in two ways: only partial derivatives of a subset of coordinates need to be computed, and only this subset of coordinates need to be updated each iteration.

For both Algorithms 3 and Algorithm 4, we say that each gradient update step is one iteration; thus in Algorithm 3 there are \(m\) iterations between \(x^t\) and \(x^{t+1}\), whereas there is a single iteration between \(x^t\) and \(x^{t+1}\) in Algorithm 4. In each iteration of Algorithms 3 or 4, we need only compute the projection of \(\nabla f(x)\) onto a certain subspace, rather than the whole gradient \(\nabla f(x)\). Depending on the manifold, this may give some computational savings. On the other hand, in each iteration we still have to compute the exponential map, as well as parallel transport operators, which are often computational bottlenecks. On certain matrix manifolds, these may involve computing matrix inversions and/or matrix exponentials.

While Algorithms 3 and 4 are interesting from a theoretical perspective, since they show the extent to which the Euclidean coordinate descent algorithm can be generalized to Riemannian manifolds as well as the limitations imposed by non-trivial curvature of general manifolds, from a computational point of view they do not give us the intended computational savings that coordinate descent gives to gradient descent in the Euclidean setting. This is mainly due to the fact that
the exponential map is not decomposable in general, even when we use an orthogonal subspace decomposition. On the other hand, in Section 3.5, we describe the class of product manifolds, where the exponential map is decomposable when we choose the right subspace decomposition, thus allowing us to realize the per-iteration computational savings analogous to the Euclidean case. It turns out that on product manifolds, we can also remove the extraneous \(m - 1\) factor that turns up in the cyclic descent convergence rate analysis.

Another difference between gradient and coordinate descent in the Euclidean setting is the introduction of the partial smoothness parameters \(L_i\), which may be significantly lower than the total smoothness parameter \(L_f\). Since we are assuming \(f\) satisfies (2), (4), we can take each \(L_i = L_f\) for the Riemannian setting. In principle, \(L_i\) can be taken as some constant which satisfies

\[
f(\text{Exp}_{x^t}(v)) \leq f(x^t) + \langle \nabla f(x^t), v \rangle_{x^t} + \frac{L_i}{2} \|v\|_{x^t}^2
\]

for each \(v \in V^t_i\), where \(\{V^t_i\}_{i \in [m]}\) is the subspace decomposition of \(T_{x^i}M\) at iteration \(t\). However, since \(V^t_i\) in general depends on the sequence of iterations \(x^1, \ldots, x^i\), even if some constant \(L_i < L_f\) did exist, it does not seem feasible to check for it a priori to running the algorithm. On product manifolds, however, this is less daunting. We describe this in the next section.

### 3.5 Efficiency Improvements with Product and Flat Structure

The convergence rates of cyclic tangent subspace descent (Theorem 3.14(a,c,d)) suffer from an extra factor of \(m\) relative to the known cyclic block coordinate descent rates in \(\mathbb{R}^n\) [6]. As mentioned, this is due to the fact that on a general manifold, the parallel transport operator is not necessarily transitive, i.e., \(\Gamma^x_z \Gamma^z_y \neq \Gamma^x_y\). In this section, we explore two structural conditions on our manifold which allow us to eliminate this extra factor: it is a product of manifolds \(M = M_1 \times \ldots \times M_m\) endowed with the natural product metric, or when \(M\) is flat near the stationary points. While it is not true in general that every manifold decomposes into a product, there exists important applications, such as tensor PCA, which possesses this structure. Flatness near a point \(x\) means that there exists a neighbourhood of \(x\) which is isometric to \(\mathbb{R}^n\), and thus endows the neighbourhood with some convenient properties of Euclidean spaces. Consequently, if our sequence of points eventually enters this neighbourhood, then we can use these properties to improve the local convergence rate.

In this section, we will explain these conditions and explain at a high level how each of these conditions yield decompositions of the parallel transport and exponential maps crucial to the disposal of the additional factor of \(m\). We begin by re-analysing Algorithm 1 in the setting where the subspace decomposition \(\{V^0,k\}_{k \in [m]}\) and the isometries \(P^0_{i-1} : T_{x^{i-1}}M \to T_{x^0}M\) satisfy two generic properties:

\[
P^0_{i-1} = \Gamma^0_{x^0} \ldots \Gamma^{x^{i-2}}_{x^{i-1}} = \Gamma^{x^0}_{x^i-1}
\]

\[
x^i = \text{Exp}_{x^0} \left( \sum_{j \in [i-1]} \left( -\frac{1}{L_j} P^0_{j-1} U^{j-1} \nabla f(x^{j-1}) \right) \right)
\]

(6) and (7) remove the additional \(m\) factor.

**Lemma 3.17.** Let the sequence \(x^0 = x, x^1, \ldots, x^{m-1}, x^m = y\) be generated by Algorithm 1, and
that (6) and (7) hold. Then
\[
\| U^{0,i} \nabla f(x) - U^{0,i} P_{i-1}^0 \nabla f(x^{i-1}) \|_2^2 \leq \frac{L_f^2}{\mu_{\min}} \sum_{j \in [i-1]} \| U^{j-1,j} \nabla f(x^{j-1}) \|_{x_{j-1}}^2.
\]

Proof. Following the reasoning of Lemma 3.10, it suffices to bound \( \| \nabla f(x^0) - P_{i-1}^0 \nabla f(x^{i-1}) \|_2^2 \): for any \( v \in \text{Exp}_{x^0}(x^i) \),
\[
\| \nabla f(x^0) - P_{i-1}^0 \nabla f(x^{i-1}) \|_2^2 = \| \nabla f(x^0) - \Gamma_{x^{i-1}}^0 \nabla f(x^{i-1}) \|_2^2 \leq L_f^2 \| v \|_{x^0}^2 \\
\leq L_f^2 \sum_{j \in [i-1]} \left( -\frac{P_{i-1}^0 U^{j,j-1} \nabla f(x^{j-1})}{L_f} \right) \| \nabla f(x^{j-1}) \|_2^0 \\
= L_f^2 \sum_{j \in [i-1]} \frac{1}{L_f} \| U^{j,j-1} \nabla f(x^{j-1}) \|_2^0 \leq \frac{L_f^2}{\mu_{\min}} \sum_{j \in [i-1]} \| U^{j,j-1} \nabla f(x^{j-1}) \|_{x^0}^2.
\]

The first inequality follows since \( P_{i-1}^0 = \Gamma_{x^{i-1}}^0 \) by (6), the smoothness assumption on \( f \), and the fact that \( d(x^0, x^{i-1}) = \| v \|_{x^0}^2 \) for any \( v \in \text{Exp}_{x^0}(x^{i-1}) \). The second inequality follows from (7).

The same reasoning as in Theorem 3.14 can be applied to get rates comparable to the Euclidean setting. We now discuss product and flat structure, which we can show will satisfy (6) and (7).

For a (Cartesian) product of manifolds \( M = M_1 \times \ldots \times M_m \). The tangent space naturally decomposes as \( T_{(x_1, \ldots, x_m)} M = \bigoplus_{i \in [m]} T_{x_i} M_i := T_{x_1} M_1 \times \ldots \times T_{x_m} M_m \), where the Cartesian product of the vector spaces \( T_{x_i} M_i \) is endowed with the usual inherited vector space structure: \( \alpha(v_1, \ldots, v_m) + \beta(w_1, \ldots, w_m) = (\alpha v_1 + \beta w_1, \ldots, \alpha v_m + \beta w_m) \). Also, \( M \) is given the product metric
\[
v = (v_1, \ldots, v_m), w = (w_1, \ldots, w_m) \in T_x M, \quad (v, w)_x := \sum_{i \in [m]} \langle v_i, w_i \rangle_{x_i}.
\]

We will see that if in Algorithm 1 we choose \( V^{0,k} := T_{x^0} M_k \) for \( k \in [m] \), where \( x^0 = (x^0_1, \ldots, x^0_m) \), then we can ensure that (6) and (7) hold. This is unsurprising as the standard convergence analysis of block coordinate descent [6] exploits the fact that we can decompose \( \mathbb{R}^n \) into a product of Euclidean subspaces corresponding to the blocks. Securing the above properties is a nearly trivial exercise using properties of the Levi-Civita connection, a key construct in Riemannian geometry. However, such a proof requires a more detailed elaboration on connections, so we defer it to Appendix A.

We apply Lemma 3.17 to obtain the following theorem concerning non-asymptotic convergence rates for product manifolds.

**Theorem 3.18.** Let \( M = M_1 \times \ldots \times M_m \) endowed with the product structure and product metric, and \( f : M \to \mathbb{R} \) be \( L_f \)-smooth satisfying (2) and (4). Suppose \( \{ x^t \}_{t \in \mathbb{N}} \subset M \) be generated from
Algorithm 3 with \( V^i_k := T^{(x^i)_k} M_k \) for \( i \in [m] \). Then for each iteration \( t \in \mathbb{N} \) and \( k \in [m] \), \( V^i_t = T^{(x^i)_k} M_k \). Let \( C := \frac{4L_{max}(L_{min}^2 + L_{min}^2 m)}{L_{min}} \). Then

\[
\min_{x \in [t]} \| \nabla f(x^t) \|_{x^t} \leq \sqrt{\frac{C(f(x^1) - f^*)}{t}} = O \left( \sqrt{\frac{m}{t}} \right).
\]

If \( f \) is further assumed to be \( g \)-convex then

\[
f(x^{t+1}) - f^* \leq \frac{R^2 C(f(x^1) - f^*)}{R^2 C + (f(x^1) - f^*) t} = O \left( \frac{m}{t} \right)
\]

where the diameter of \( \{ x \in M : f(x) \leq f(x^1) \} \) is \( R \).

**Proof.** At iteration \( t \), the \( P^t \) output from Algorithm 1 is \( P^t = \Gamma^{x_{t,m-1}}_{x_{t,m-1}} \ldots \Gamma^{x_{t,0}}_{x_{t,0}} = \Gamma^{x^t}_{x^t} \) as equation (18) from Lemma A.2 implies that (6) holds. That (7) holds follows from (16) of Lemma A.2. Then \( V^{t+1} = \Gamma^{x^t}_{x^t} T^{(x^t)_k} M_k = T^{(x^{t+1})}_k M_k \), where the second inequality follows due to (17) of Lemma A.1, which tells us that the parallel transport decomposes. The rates follow from using Lemmas 3.17, with a similar proof to Theorem 3.14.

We now return to the discussion on per-iteration complexity in Section 3.4. On a product manifold \( M_1 \times \ldots \times M_m \), the exponential map at a point \( x = (x_1, \ldots, x_m) \in M \) has the convenient decomposition

\[
\text{Exp}_x(v_1, \ldots, v_m) = (\text{Exp}^1_{x_1}(v_1), \ldots, \text{Exp}^m_{x_m}(v_m))
\]

for \( v_i \in T_{x_i} M_i \) and \( i \in [m] \), and where \( \text{Exp}^i \) is the exponential map on \( M_i \). Thus, each iteration of Algorithm 3 with the product setup need only compute one gradient update \( \text{Exp}^i \) each iteration. This will provide significant improvements on the per-iteration complexity compared to using the total exponential map.

Furthermore, in Section 3.4 we discussed that on a general manifold, the possibility that \( L_i < L_f \) is difficult to check in general. On the other hand, on a product manifold, we can give a similar condition to (2) that can give improved \( L_i \), which is analogous to the block Lipschitz continuous gradients in the Euclidean setting. Given \( x = (x_1, \ldots, x_m) \), we denote by \( (x_{-i}, y_i) \in M_1 \times \ldots \times M_m \) to be the point with all components the same as \( x \) except that \( x_i \) is replaced by \( y_i \).

**Proposition 3.19.** Suppose \( f \in C^1(M_1 \times \ldots \times M_m) \), and \( \nabla_i f(x) \) is the projection of \( \nabla f(x) \) onto \( T_{x_i} M_i \). Suppose \( \nabla f \) is Lipschitz continuous on \( M_i \) in the sense that

\[
\| \nabla_{\text{Exp}^i_{x_i}}(v) \|_{x_i} - \nabla_i f(x) \|_{x} \leq L_i d_i(\text{Exp}^i_{x_i}(v), x_i),
\]

for all \( x = (x_{-i}, x_i) \in M_1 \times \ldots \times M_m \), and \( v \in T_{x_i} M \). Then

\[
f(x_{-i}, \text{Exp}^i_{x_i}(v)) \leq f(x) + \langle \nabla_i f(x), v \rangle_{x_i} + \frac{L_i}{2} \| v \|_{x_i}^2,
\]

for all \( x = (x_{-i}, x_i) \in M_1 \times \ldots \times M_m \), and \( v \in T_{x_i} M \).
Proof. Pick \( x \in M \) and \( v \in T_x M \). To simplify notation, let \( \gamma(t) = \text{Exp}^i_x(tv) \) and \( y = (x - i, \text{Exp}^i_x(v)) = (x - i, \gamma(1)) \). A straightforward computation using the fundamental theorem of calculus for line integrals [19, Theorem 11.39] shows

\[
f(y) - f(x) - \langle \nabla f_i(x), v \rangle_{x_i} = \int_0^1 df_{\gamma(t)}(\gamma(t)) dt - \langle \nabla_i f(x), v \rangle_{x_i} \\
= \int_0^1 \langle \nabla_i f(\gamma(t)), \gamma'(t) \rangle_{\gamma(t)} dt - \langle \nabla_i f(x), v \rangle_{x_i} \\
= \int_0^1 \langle \nabla_i f(\gamma(t)), \Gamma_{\gamma(t)}^{\gamma(t)}v \rangle_{\gamma(t)} dt - \langle \nabla_i f(x), v \rangle_{x_i} \\
= \int_0^1 \langle \Gamma_{\gamma(t)}^{\gamma(t)} \nabla_i f(\gamma(t)) - \nabla_i f(x), v \rangle_{x_i} dt \\
\leq \int_0^1 \langle \Gamma_{\gamma(t)}^{\gamma(t)} \nabla_i f(\gamma(t)) - \nabla_i f(x) \rangle \| v \|_{x_i} dt \\
\leq \int_0^1 v \| v \|_{x_i}^2 dt = \frac{L_i}{2} \| v \|_{x_i}^2
\]

where we use the definition of the gradient in the second equality, the definition of \( \gamma \) in the third equation, the fact that parallel transport is an isometry in the fourth equality, Cauchy-Schwarz in the first inequality, and the Lipschitz condition in the final inequality. \( \square \)

Another condition which gives us (6) and (7) is that the manifold is flatness near stationary points. Formally, this means there is an open neighborhood \( U \subseteq M \) of \( \{ x \in M : \| \nabla f(x) \|_x = 0 \} \), an open set \( \bar{U} \subseteq \mathbb{R}^n \) with a Riemannian metric inherited from \( \mathbb{R}^n \), and an isometry \( \varphi : U \to \bar{U} \subseteq \mathbb{R}^m \), that is, a diffeomorphism for which \( d\varphi_x : T_x M \to \mathbb{R}^n \) is an isometry of inner product spaces for each \( x \in U \). Lee [20, Theorem 7.10] tells us that this assumption is equivalent to there being no curvature on \( U \). If Algorithm 1 outputs \( \{ x^t \}_{t \in [m]} \cup \{ 0 \} \subset U \) and the geodesic between \( x^t \) and \( x^{t+1} \) lies inside \( U \), then (6) and (7) will hold. We defer the proof of this to Appendix A. This is much stronger than the product manifold setting, since it holds regardless of selection of initial subspace decomposition of \( \{ V^{0,k} \}_{k \in [m]} \). First we show that if \( \| \nabla f(x) \|_x \) is sufficiently small, then \( x \in U \).

**Lemma 3.20.** Suppose that \( Z \subset M \) is closed and bounded, and that there exists a neighbourhood \( U \supseteq \{ x \in M : \| \nabla f(x) \|_x = 0 \} \cap Z \). There exists \( r > 0 \) such that \( \{ x \in M : \| \nabla f(x) \|_x \leq r \} \cap Z \subset U \).

**Proof.** Suppose that this is not true. Then for each \( t \in \mathbb{N} \), there exists \( x^t \in Z \) such that \( \| \nabla f(x^t) \|_{x^t} \leq 1/t \) but \( x^t \notin U \). Since \( \{ x^t \}_{t \in \mathbb{N}} \) belong to the compact set \( Z \), by passing to a convergent subsequence we can assume \( x^t \) converges, so there exists \( x \in Z \) such that \( d(x^t, x) \to 0 \). Now \( \| \nabla f(x) \|_x = 0 \) by Lemma 3.3, so \( x \in U \). But then since \( U \) is open, for sufficiently large \( t \), \( x^t \in U \), which is a contradiction. \( \square \)

We can now show that, asymptotically, tangent subspace descent with flatness near stationary points matches the Euclidean rate of coordinate descent.

**Theorem 3.21.** Let \( \{ x^t \}_{t \in \mathbb{N}} \) be the sequence output by Algorithm 3. Suppose that \( Z := \{ x \in M : f(x) \leq f(x^1) \} \) is bounded and that \( M \) is flat near \( \{ x \in M : \| \nabla f(x) \|_x = 0 \} \cap Z \). Let
\[ C := 4L_{\max} \left( 1 + \frac{L_i^2}{L_{\min}} \right). \] Then there exists \( T \in \mathbb{N} \) such that \( t > T \) implies
\[ \min_{T \leq s \leq t} \| \nabla f(x^s) \|_{x^s} \leq \sqrt{\frac{C(f(x^t) - f^*)}{t - T}} = O \left( \sqrt{\frac{m}{t}} \right). \]

If \( f \) is further assumed to be \( g \)-convex then
\[ f(x^{t+1}) - f^* \leq \frac{R^2C(f(x^t) - f^*)}{R^2C + (f(x^t) - f^*)(t - T)} = O(m/t) \]
where \( R \) is the diameter of \( \{ x \in M : f(x) \leq f(x^t) \} \).

**Proof.** Let \( U \subseteq M \) be the neighborhood of \( \{ x \in M : \| \nabla f(x) \|_{x} = 0 \} \cap Z \) that is isometric to a subset of \( \mathbb{R}^n \) per the flatness condition. By Theorem 3.14(a), we know that \( \| \nabla f(x^t) \|_{x^t} \to 0 \), so by Lemma 3.20 there exists \( T \in \mathbb{N} \) such that \( t > T \) implies \( x_t \in U \). We wish to apply Lemma 3.17 similar to Theorem 3.14. To do this, we must show that conditions (6) and (7) hold between iterates \( x^t \) and \( x^{t+1} \) for \( t > T \). We do this by showing that the geodesic curve between each gradient update from Algorithm 1 (used to compute \( x^{t+1} \) from \( x^t \)) lies in \( U \).

First, since \( Z := \{ x \in M : f(x) \leq f(x^t) \} \) is closed and bounded, hence compact, Lemma 3.20 states that for some \( t > 0, S_r \cap Z \subset U \), where \( S_r := \{ x \in M : \| \nabla f(x) \|_{x} \leq r \} \). Now since \( S_r \cap Z \) are both closed by continuity of \( f \) and \( \nabla f \) and the norm, \( S_r \cap Z \) is also compact. Furthermore, it is disjoint from the closed set \( U^c := \{ x : x \not\in U \} \), so they are positively separated (on a metric space, a compact set can be positively separated from a disjoint closed set), i.e., there exists \( \epsilon > 0 \) such that for any \( x \in S_r \cap Z \) and \( x' \in U^c \), \( d(x, x') > \epsilon \). In addition, since \( \| \nabla f(x^t) \|_{x^t} \to 0 \), we know that \( x^t \in S_r \cap Z \) for sufficiently large \( t \).

Let \( x^{t,i}, i \in [m] \cup \{0\} \) be the intermediate points generated by Algorithm 1 at iteration \( t \), with \( x^{t,0} = x^t \) and \( x^{t,m} = x^{t+1} \). Following the reasoning from the start of Section 3.1, we know that
\[ f(x^t) - f(x^{t+1}) \geq \sum_{i \in [m]} \frac{1}{2L_i} \| U^{t,i-1} \nabla f(x^{t,i-1}) \|_{x^{t,i-1}}^2, \]
where \( U^{t,i-1} \) is the projection operator onto the appropriate subspace at step \( i \) of iteration \( t \). Summing this up, we see that the left hand side is bounded, and the right hand side is an infinite sum, so the terms in the sum
\[ \sum_{i \in [m]} \frac{1}{2L_i} \| U^{t,i-1} \nabla f(x^{t,i-1}) \|_{x^{t,i-1}}^2 \to 0. \]

Now, since \( x^{t,i} = \text{Exp}_{x^{t,i-1}} \left( \frac{-1}{L_i} U^{t,i-1} \nabla f(x^{t,i-1}) \right) \), we have \( d(x^{t,i-1}, x^{t,i}) \leq \frac{1}{L_i} \| U^{t,i-1} \nabla f(x^{t,i-1}) \|_{x^{t,i-1}}, \) hence
\[ \frac{L_{\min}}{2\sqrt{m}} \max_{i \in [m]} d(x^t, x^{t,i})^2 \leq \frac{L_{\min}}{2} \sum_{i \in [m]} d(x^{t,i-1}, x^{t,i})^2 \leq \frac{1}{2} \sum_{i \in [m]} L_i d(x^{t,i-1}, x^{t,i})^2 \to 0. \]

In particular, this implies that for \( t \) sufficiently large, \( \max_{i \in [m]} d(x^t, x^{t,i}) < \epsilon \). But this means that for \( t \) sufficiently large, \( x^{t,i} \in U \) for all \( i \in [m] \).
Now fix $i \in [m]$ and consider a geodesic curve between $x^{t,i-1}$ and $x^{t,i}$ defined by $\gamma(s) := \text{Exp}_{x^{t,i-1}}(-sU^{t,i-1}\nabla f(x^{t,i-1}))$ for $s \in [0, 1/L_i]$. Note that

$$f(x^{t,i-1}) - f(\gamma(s)) \geq -\frac{L_i s}{2} \left( s - \frac{2}{L_i} \right) \|U^{t,i-1}\nabla f(x^{t,i-1})\|^2_{x^{t,i-1}} \geq 0$$

for all $s \in [0, 1/L_i]$. Since $f(x^t) \leq f(x^1)$, we have $f(\gamma(s)) \leq f(x^{t,i-1}) \leq f(x^1)$, so $\gamma(s) \in Z$ for all $s \in [0, 1/L_i]$. Now since $\nabla f(\gamma(s)) = \nabla f(\gamma(s)) - \Gamma^{t,i}_{x^{t,i-1}} \nabla f(x^t) = \Gamma^{t,i}_{x^{t,i-1}} \nabla f(x^t)$, so

$$\|\nabla f(\gamma(s))\|_{\gamma(s)} \leq \|\nabla f(\gamma(s)) - \Gamma^{x^{t,i}}_{x^{t,i-1}} \nabla f(x^t)\|_{\gamma(s)} + \|\Gamma^{x^{t,i}}_{x^{t,i-1}} \nabla f(x^t)\|_{\gamma(s)}$$

$$\leq L_f d(\gamma(s), x^t) + \|\nabla f(x^t)\|_{x^t}$$

$$\leq L_f \left( d(\gamma(s), x^{t,i-1}) + d(x^{t,i-1}, x^t) \right) + \|\nabla f(x^t)\|_{x^t}$$

$$\leq L_f \left( \frac{1}{L_i} \|U^{t,i-1}\nabla f(x^{t,i-1})\|^2_{x^{t,i-1}} + d(x^{t,i-1}, x^t) \right) + \|\nabla f(x^t)\|_{x^t}$$

$$\leq L_f \left( \sum_{i \in [m]} \left( \frac{1}{L_i} \|U^{t,i-1}\nabla f(x^{t,i-1})\|^2_{x^{t,i-1}} + d(x^{t,i-1}, x^t)^2 \right) + \|\nabla f(x^t)\|_{x^t} \right).$$

We know that the two terms above converge to 0 as $t \to \infty$, thus, for sufficiently large $t$ and any $i \in [m]$, the geodesic $\gamma(s)$ between $x^{t,i-1}$ and $x^{t,i}$ is always in $S_r$, hence the curves lie in $U$. □

4 Randomized Proximal Block Descent with Product Structure

Consider a product manifold $M = M_1 \times \ldots \times M_m$. Recall that at $x = (x_1, \ldots, x_m) \in M$, the tangent space decomposes as $T_x M = \bigoplus_{i \in [m]} T_{x_i} M_i$. We examine the problem

$$F^* = \min_{x \in M} \{ F(x) := f(x) + g(x) \}, \quad f(x) = f(x_1, \ldots, x_m), \quad g(x) = \sum_{i \in [m]} g_i(x_i). \quad (8)$$

We consider a composite optimization setting: $f$ is an $L_f$-smooth function, but each $g_i$ is possibly non-smooth. In the Euclidean setting, problems of the form (8) are solved via proximal gradient algorithms, i.e., those which compute

$$x^{t+1} = \arg\min_{x \in \mathbb{R}^n} \left\{ f(x^t) + \langle \nabla f(x^t), x - x^t \rangle + \frac{L_f}{2} \|x - x^t\|^2 + g(x) \right\}.$$

Chen et al. [9], Huang and Wei [17] have generalized proximal gradient algorithms to manifolds. In the Euclidean setting, block proximal gradient algorithms are often employed when the non-smooth function $g$ has a decomposable structure, e.g., in $\ell_1$-regularized learning problems. These compute the next point via a proximal operator

$$x^{t+1}_i = \arg\min_{x_i} \left\{ f(x^t) + \langle \nabla_i f(x^t), x_i - x_i^t \rangle + \frac{L_i}{2} \|x_i - x_i^t\|^2 + g_i(x_i) \right\} \quad (9)$$

for some block $i$ while leaving other blocks unchanged. In this section, we generalize block proximal gradient algorithms to the manifold setting.
We denote $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m)$ and $f(x_{-i}, y_i)$ to be $f(y)$ where $y_j = x_j$ for all $j \neq i$. Our first task is to generalize the Euclidean update (9) to the Euclidean setting appropriately. There is a general pattern for doing this. First, differences of points $y_i - x_i$ are interpreted as directions, hence must now live in a tangent space, specifically, $y_i - x_i \to \text{Exp}_{x_i}^{-1}(y_i) \in T_{x_i}M_i$. This means that $y_i = x_i + (y_i - x_i) \to \text{Exp}_{x_i}(\text{Exp}_{x_i}^{-1}(y_i))$. Rewriting the objective of (9) slightly to $f(x^t) + \langle \nabla_i f(x^t), x_i \rangle + \frac{L_i}{2} \|x_i\|^2 + g_i(x_i^t + x_i)$, we can translate this to the manifold setting, writing $v = \text{Exp}_{x_i}^{-1}(y_i)$, and define an analogous block proximal operator

$$\text{Prox}_i^k(x) := \arg\min_{v \in T_{x_i}M_i} \left\{ f(x) + \langle \nabla_i f(x), v \rangle_{x_i} + \frac{L_i}{2} \|v\|^2_{x_i} + g_i(\text{Exp}_{x_i}(v)) \right\}. \quad (10)$$

Here, $\nabla_i f(x) \in T_{x_i}M_i$ is be the projection of $\nabla f(x)$ onto the subspace $T_{x_i}M_i$, or equivalently, it is the gradient of the function $f_i^k : M \to \mathbb{R}$ defined by $f_i^k(x_i) = f(x_{-i}, x_i)$. For analysis purposes, we also define the full proximal operator

$$\text{Prox}^k(x) := \arg\min_{v \in T_xM} \left\{ f(x) + \langle \nabla f(x), v \rangle_x + \frac{L}{2} \|v\|^2_x + g(\text{Exp}_x(v)) \right\}. \quad (11)$$

Note that due to the product structure of $M$, it is easy to see that

$$\text{Prox}^k(x) = (\text{Prox}_1^k(x), \ldots, \text{Prox}_m^k(x)).$$

We assume that we have block-wise smoothness upper bounds:

$$\forall i \in [m], \ x_i \in M_i, \ v \in T_{x_i}M_i, \ f(x_{-i}, \text{Exp}_{x_i}(v)) \leq f(x) + \langle \nabla_i f(x), v \rangle_{x_i} + \frac{L_i}{2} \|v\|^2_{x_i}. \quad (12)$$

We analyse a randomized block proximal scheme, described in Algorithm 5; the analysis of the cyclic version remains open. Note that Algorithm 5 is a modified version of Algorithm 4, where we no longer need to update the subspace decompositions since they will remain constant due to the product structure of $M$.

### 4.1 Geodesically convex setting

As in Section 3, the first step is to derive a sufficient decrease condition between iterates. In fact, we will first consider the $g$-convex setting, and prove directly the lower bound with the optimality gap, allowing us to immediately use Lemma 3.6.
Proposition 4.1. Let \( f : M \to \mathbb{R} \) be a function satisfying (12) with \( L_i = \bar{L} < L \). Let \( i \in [m] \) be chosen randomly with probability \( p_i = 1/m \), and given \( x \in M \), define \( y \in M \) via \( y_i = \text{Exp}_x(\text{Prox}_i^f(x)) \), \( y_{-i} = x_{-i} \). If \( f, g \) are \( g \)-convex and for some minimizer \( x^* \) of (8), \( \max\{d(x, x^*), \sqrt{(F(x) - F^*)/L}\} \leq R \), then

\[
F(x) - \mathbb{E}[F(y) \mid x] \geq \frac{(F(x) - F^*)^2}{2mLR^2}.
\]

Proof. For convenience, denote \( v_i = \text{Prox}_i^f(x) \) and \( v = \text{Prox}^L(x) \). Since probabilities are \( 1/m \), we have for any choice of \( w = (w_1, \ldots, w_m) \in T_x M \), such that \( w = \text{Exp}_x^{-1}(\text{Exp}_x(w)) \) (i.e., it is the shortest tangent vector \( w \) which maps to \( x' = \text{Exp}_x(w) \in M \)), we have

\[
\mathbb{E}[f(y) + g(y) \mid x] = \frac{1}{m} \sum_{i \in [m]} \left( f(x - i, \text{Exp}_x, (v_i)) + \sum_{j \neq i} g_j(x_j) + g_i(\text{Exp}_x(\text{Prox}_i^f(x))) \right)
\]

\[
\leq \frac{1}{m} \sum_{i \in [m]} \left( \sum_{j \neq i} g_j(x_j) + f(x) + \langle \nabla_i f(x), v_i \rangle_{x_i} + \frac{L}{2} \left\| v_i \right\|^2_{x_i} + g_i(\text{Exp}_x(v_i)) \right)
\]

\[
= \left( 1 - \frac{1}{m} \right) (f(x) + g(x)) + \frac{1}{m} \left( f(x) + \langle \nabla f(x), v \rangle_x + \frac{L}{2} \left\| v \right\|^2_x + \sum_{i \in [m]} g_i(\text{Exp}_x(v_i)) \right)
\]

\[
\leq \left( 1 - \frac{1}{m} \right) (f(x) + g(x)) + \frac{1}{m} \left( f(\text{Exp}_x(w)) + \frac{L}{2} \left\| w \right\|^2_x + \sum_{i \in [m]} g_i(\text{Exp}_x(w_i)) \right)
\]

\[
= \left( 1 - \frac{1}{m} \right) F(x) + \frac{1}{m} \left( F(\text{Exp}_x(w)) + \frac{L}{2} \left\| w \right\|^2_x \right).
\]

The first inequality is from using (12), the second inequality is from replacing \( v \) with \( w \) since \( v \) solves (11), and the third inequality follows from \( g \)-convexity of \( f \).

We now choose \( w \) as follows. Let \( x^* \) be some minimizer for (8), and \( \tilde{w} \in \text{Exp}_x^{-1}(x^*) \); note that \( d(x, x^*) = \|w\|_x \). Let \( \gamma_{x, \tilde{w}} : [0, 1] \to M \) be the geodesic between \( x \) and \( x^* \) defined from the exponential map, so that \( \gamma_{x, \tilde{w}}(0) = x \) and \( \gamma_{x, \tilde{w}}(1) = x^* \). Then let \( \beta' = \arg\min_{\beta \in [0, 1]} \left\{ F(\gamma_{x, \tilde{w}}(\beta)) + \frac{L\beta^2}{2} \right\} \).

We set \( w = \beta' \tilde{w} \), which gives

\[
\mathbb{E}[f(y) + g(y) \mid x] \leq \left( 1 - \frac{1}{m} \right) F(x) + \frac{1}{m} \left( F(\text{Exp}_x(w)) + \frac{L}{2} \left\| w \right\|^2_x \right)
\]

\[
= \left( 1 - \frac{1}{m} \right) F(x) + \frac{1}{m} \min_{\beta \in [0, 1]} \left\{ F(\gamma_{x, \tilde{w}}(\beta)) + \frac{L\beta^2}{2} \right\}
\]

\[
\leq \left( 1 - \frac{1}{m} \right) F(x) + \frac{1}{m} \min_{\beta \in [0, 1]} \left\{ (1 - \beta)F(x) + \beta F(x^*) + \frac{L\beta^2}{2} d(x, x^*)^2 \right\}
\]

\[
\leq \left( 1 - \frac{1}{m} \right) F(x) + \frac{1}{m} \min_{\beta \in [0, 1]} \left\{ F(x) + \beta(F(x^*) - F(x)) + \frac{L\beta^2}{2} R^2 \right\}
\]

24
= \left(1 - \frac{1}{m}\right) F(x) + \frac{1}{m} \left(F(x) - \frac{(F(x) - F^*)^2}{2LR^2}\right).

The second inequality holds since \( F \) is \( g \)-convex. Note that the global minimizer of the right hand side of the third inequality is guaranteed to be in \([0,1]\) by the choice of \( R \). Rearranging this gives us the result. \(\square\)

Proposition 4.1 immediately gives us a convergence rate in the \( g \)-convex setting.

**Theorem 4.2.** Let \( f : M \rightarrow \mathbb{R} \) be a function satisfying (12) with \( L_i = \bar{L} \). Let \( \{x^t\}_{t\in\mathbb{N}} \subset M \) be a sequence computed using Algorithm 5 and any \( L > \bar{L} \). If \( f, g \) are \( g \)-convex, \( \{x \in M : F(x) \leq F(x^1)\} \) has diameter bounded by \( R \), \( \sqrt{(F(x^1) - F^*)}/L \leq R \), and \( C = \frac{1}{2mL^2} \), then

\[
E[F(x^{t+1})] - F^* \leq \frac{F(x^1) - F^*}{1 + C(F(x^1) - F^*)t} = O(m/t).
\]

**Proof.** Denote \( x^t = x \) and fix \( i \in [m] \). We always have

\[
f(x) + g(x) \geq \sum_{j \neq i} g_j(x_j) + f(x) + \langle \nabla_i f(x), \text{Prox}^L_i(x) \rangle_{x_i} + \frac{L}{2} \| \text{Prox}^L_i(x) \|_{x_i}^2 + g_i(\text{Exp}_{x_i}(\text{Prox}^L_i(x))),
\]

since \( v = 0 \) is always a feasible solution to (10) with objective \( f(x) + g_i(x_i) \). Now substituting \( f(x_{i-1}, \text{Exp}_{x_i}(\text{Prox}^L_i(x))) \leq f(x) + \langle \nabla_i f(x), \text{Prox}^L_i(x) \rangle_{x_i} + \frac{L}{2} \| \text{Prox}^L_i(x) \|_{x_i}^2 \), we have

\[
f(x) + g(x) \geq f(x_{i-1}, \text{Exp}_{x_i}(\text{Prox}^L_i(x))) + \sum_{j \neq i} g_j(x_j) + g_i(\text{Exp}_{x_i}(\text{Prox}^L_i(x))).
\]

Observe that \( f(x_{i-1}, \text{Exp}_{x_i}(\text{Prox}^L_i(x))) + \sum_{j \neq i} g_j(x_j) + g_i(\text{Exp}_{x_i}(\text{Prox}^L_i(x))) = f(y) + g(y) \) if \( i \) is chosen. Taking expectations gives us that \( E[F(x^{t+1}) | x^t] \leq F(x^t) \), hence \( E[F(x^t)] \geq E[F(x^{t+1})] \).

Defining \( A_t = E[F(x^t)] \) and taking the expectation of the inequality from Proposition 4.1 gives \( A_t - A_{t+1} \geq C A^2_t \), so applying Lemma 3.6 gives the result. \(\square\)

### 4.2 Non-convex setting

For the non-convex setting, we wish to show almost sure convergence to a stationary point, as well as a quantifiable bound on some certificate of stationarity, similar to Theorem 3.16. Previously when \( g \) was not present, we took the certificate of stationarity to be the first-order condition \( \nabla f(x) = 0 \). This no longer works for (8), but instead the appropriate condition is \( \text{Prox}^L(x) = 0 \).

We first explore why \( \text{Prox}^L(x) = 0 \) is a necessary condition for optimality of (8). Since (8) now involves a potentially non-differentiable function \( g \), we must first introduce a notion of subgradients for \( g \). The definition of this assumes that \( g \) is locally Lipschitz, so we will assume that from now on. Let \( (\phi, U) \) be a chart at \( x \). For any \( v \in T_xM \), define the **generalized directional derivative of** \( h : M \rightarrow \mathbb{R} \) in the direction \( v \) as

\[
h^\circ(x; v) := \limsup_{y \rightarrow \phi(x), t \downarrow 0} \frac{(h \circ \phi^{-1})(\phi(y) + td\phi_x[v]) - (h \circ \phi^{-1})(y)}{t}.
\]

(Recall that \( d\phi_x[v] \) is the differential of the map \( \phi : U \rightarrow \mathbb{R}^n \), which is simply a vector in \( \mathbb{R}^n \).) Note that since \( \phi \) is a diffeomorphism, \( \{d\phi_x[v] : v \in T_xM\} = \mathbb{R}^n \), so this is actually equivalent to
the Euclidean generalized Clarke directional derivative of the function $h \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ in the direction $d\phi_x[v] \in \mathbb{R}^n$. The subdifferential of $h$ at $x$ is

$$\partial h(x) := \{w \in T_xM : h^\circ(x;v) \geq \langle w, v \rangle_x \ \forall v \in T_xM\}.$$ 

A vector $v \in \partial h(x)$ is called a subgradient of $h$ at $x$. If $h$ is differentiable, then $\partial h(x) = \{\nabla h(x)\}$. We refer to Hosseini and Pouryayevali [16] for a detailed introduction to these generalized gradient concepts.

**Remark 4.1.** Note that our definition of subgradients differs from the one in Hosseini and Pouryayevali [16] slightly. However, Lemmas 3.3, 4.4 and 4.8(b) are consequences or re-statements of results from Hosseini and Pouryayevali [16]. In Appendix B, we explain how to rigorously modify the definition from Hosseini and Pouryayevali [16] so that these lemmas are valid.

We have the following necessary condition for optimality.

**Lemma 4.3.** If $x^\ast$ is optimal for $\min_{x \in M} h(x)$, then $0 \in \partial h(x^\ast)$.

**Proof.** Let $(\phi, U)$ be a chart at $x^\ast$. Since $\phi$ is a diffeomorphism, $d\phi_{x^\ast} : T_{x^\ast}M \to \mathbb{R}^n$ is an isomorphism [19, Proposition 3.6(d)] and hence $d\phi_{x^\ast}[0] = 0$. We consider the function $h \circ \phi^{-1} : \phi(U) \to \mathbb{R}$. It must be the case that $\phi(x^\ast) \in \phi(U) \subseteq \mathbb{R}^n$ is the minimizer of $h \circ \phi^{-1}$. Since $h^\circ(x^\ast;v)$ is precisely the Clarke directional derivative of $h \circ \phi^{-1}$ at $\phi(x)$ in direction $D\phi_{x^\ast}[v]$ as a function in Euclidean space. Thus, by the optimality condition on the Clarke subdifferential in Euclidean space [10, Theorem 1], $0 \in \partial (h \circ \phi^{-1})(\phi(x))$, which means that $h^\circ(x^\ast;v) = (h \circ \phi^{-1})^\circ(\phi(x^\ast);d\phi_x[v]) \geq 0$ for all $v \in T_{x^\ast}M$. Therefore $0 \in \partial h(x^\ast)$. □

We can use Lemma 4.3 to derive the optimality condition for the proximal mapping (11), but first, we must understand what the subdifferential of $g \circ \text{Exp} : T_xM \to \mathbb{R}$ is. To do that, we use the chain rule from Hosseini and Pouryayevali [16, Proposition 3.4] (see also Remark 4.1). Given $x, y \in M$ and a linear operator $A : T_xM \to T_yM$, the adjoint operator $A^* : T_yM \to T_xM$ is the unique linear operator which satisfies

$$\forall v \in T_xM, w \in T_yM, \quad \langle A^*(w), v \rangle_x = \langle w, A(v) \rangle_y.$$

**Lemma 4.4** (Hosseini and Pouryayevali [16, Proposition 3.4]; see Remark 4.1). Let $M, N$ be smooth manifolds and $H : M \to N$, $h : N \to \mathbb{R}$ with $H$ being a smooth map. Then for any $x \in M$ such that $h$ is Lipschitz on a neighbourhood of $H(x)$ and $dH_x : T_xM \to T_{H(x)}N$ is onto,

$$\partial (h \circ H)(x) = \{dH_x^*[w] : w \in \partial h(H(x))\} \subseteq T_xM.$$

For the function $g \circ \text{Exp} : T_xM \to \mathbb{R}$ appearing in (11), we have

$$\partial (g \circ \text{Exp}_x)(v) = \{d(\text{Exp}_x)^*[w] : w \in \partial g(\text{Exp}_x(v))\} \subseteq T_xM. \quad (13)$$

Note that technically, $d(\text{Exp}_x)v : T_v(T_xM) \to T_{\text{Exp}_x(v)}M$, hence $\partial (g \circ \text{Exp}_x)(v)$ should be a subset of $T_v(T_xM)$, but we can identify $T_v(T_xM) \cong T_xM$, so $d(\text{Exp}_x)v : T_xM \to T_{\text{Exp}_x(v)}M$ and $\partial (g \circ \text{Exp}_x)(v) \subseteq T_xM$. We explain this in more detail in Appendix B.5. We can derive the following optimality condition for the proximal operator.

**Proposition 4.5.** If $\text{Prox}^L(x) = 0$, then $x$ is a stationary point for (8), i.e., $0 \in \nabla f(x) + \partial g(x)$.  

26
Proof. The gradients of \( v' \mapsto \langle \nabla f(x), v' \rangle \) and \( v' \mapsto \|v'\|^2 \) at \( v \in T_xM \) with \( \nabla f(x), 2v \in T_xM \) respectively. We defer the verification of this to Appendix B.5.

Denote \( v = \text{Prox}^L(x) \) and \( \bar{x} = \text{Exp}_x(v) \). Using the chain rule (13) together with Lemma 4.3, the optimality condition for (2) is: there exists some \( \bar{w} \in \partial g(\bar{x}) \) such that

\[
0 = \nabla f(x) + Lv + d(\text{Exp}_x)^*v[\bar{w}] \in T_xM.
\]

If \( v = 0 \), then \( \bar{x} = x \), and furthermore, we know that \( d(\text{Exp}_x)_0 = \text{id}_{T_xM} \) by Lee [20, Proposition 5.19(d)], so the adjoint \( d(\text{Exp}_x)^* = \text{id}_{T_xM} \) also, hence \( 0 = \nabla f(x) + \bar{w} \in \nabla f(x) + \partial g(x) = \partial F(x) \). \( \square \)

Thus, \( \| \text{Prox}^L(x) \|_x \) can be used as a certificate to verify stationarity of \( x \). We give analogous performance guarantees on \( \| \text{Prox}^L(x^t) \|_{x^t} \) as in Theorem 3.16 via an analogous sufficient decrease condition involving the proximal operator.

**Proposition 4.6.** Let \( f : M \to \mathbb{R} \) be a function satisfying (12), and \( L > \max_{i \in [m]} L_i \). Let \( i \in [m] \) be chosen randomly with probability \( p_i = 1/(L - L_i) \), and given \( x \in M \), define \( y \in M \) via \( y_i = \text{Exp}_{x_i}(\text{Prox}^L_i(x)) \), \( y_{-i} = x_{-i} \). Then

\[
F(x) - \mathbb{E}[F(y) \mid x] \geq \frac{1}{2 \sum_{i \in [m]} 1/(L - L_i)} \sum_{i \in [m]} \| \text{Prox}^L_i(x) \|_{x_i}^2 \geq \frac{1}{2 \sum_{i \in [m]} 1/(L - L_i)} \| \text{Prox}^L(x) \|_x^2.
\]

Proof. Fixing \( i \in [m] \), we always have

\[
f(x) + g(x) \geq \sum_{j \neq i} g_j(x_j) + f(x) + (\nabla_i f(x), \text{Prox}^L_i(x))_{x_i} + \frac{L}{2} \| \text{Prox}^L_i(x) \|_{x_i}^2 + g_i(\text{Exp}_{x_i}(\text{Prox}^L_i(x))),
\]

since \( v = 0 \) is always a feasible solution to (10) with objective \( f(x) + g_i(x_i) \). Now substituting \( f(x_{-i}, \text{Exp}_{x_i}(\text{Prox}^L_i(x))) \leq f(x) + (\nabla_i f(x), \text{Prox}^L_i(x))_{x_i} + \frac{L}{2} \| \text{Prox}^L_i(x) \|_{x_i}^2 \), we have

\[
f(x) + g(x) \geq f(x_{-i}, \text{Exp}_{x_i}(\text{Prox}^L_i(x))) + \sum_{j \neq i} g_j(x_j) + g_i(\text{Exp}_{x_i}(\text{Prox}^L_i(x))) + \frac{L - L_i}{2} \| \text{Prox}^L_i(x) \|_{x_i}^2.
\]

Observe that \( f(x_{-i}, \text{Exp}_{x_i}(\text{Prox}^L_i(x))) + \sum_{j \neq i} g_j(x_j) + g_i(\text{Exp}_{x_i}(\text{Prox}^L_i(x))) = f(y) + g(y) \) if \( i \) is chosen. Taking expectations gives us the result. \( \square \)

**Theorem 4.7.** Let \( f : M \to \mathbb{R} \) be a function satisfying (12), and \( L > \max_{i \in [m]} L_i \). Suppose that the sequence \( \{x^t\}_{t \in \mathbb{N}} \) is generated from Algorithm 5. Define \( C := \left( \frac{1}{2 \sum_{i \in [m]} 1/(L - L_i)} \right)^{-1} \).

(a) We have

\[
\mathbb{E} \left[ \| \text{Prox}^L(x^{t+1}) \|_{x^{t+1}} \right] = o(1/\sqrt{t}), \quad \min_{s \in [t]} \mathbb{E} \left[ \| \text{Prox}^L(x^s) \|_{x^s} \right] \leq \sqrt{\frac{C(F(x^1) - F^*)}{t}} = O\left( \sqrt{m/t} \right).
\]

(b) Almost surely, \( \| \text{Prox}^L(x^t) \|_{x^t} \to 0 \).

Proof. (a) Take expectations of the sufficient decrease condition in Proposition 4.6 to get \( \frac{1}{t} \mathbb{E} \left[ \| \text{Prox}^L(x^t) \|_{x^t}^2 \right] \leq \mathbb{E}[f(x^t)] - \mathbb{E}[f(x^{t+1})] \). Then the result follows by an analogous proof to Lemma 3.2.

27
Theorem 4.9. (b) For every \(x \in x(t)\), it is well-defined for \(t \in [0, T]\) and is converging to \((x^t, v^t)\) as \(t \to \infty\). Let \(\{x^t\}_{t \in T}\) be the filtration generated by \(\{x^t, v^t\}_{t \in T}\). Then \(\{x^t\}_{t \in T}\) is well-defined and integrability of \(x^t\) is only well-defined when \(x = \exp_{x}^t(u^t)\) or when \(\exp_{x}^t(x) = \{\}.\) Thus for sufficiently large \(t\), we can make use of this definition in Lemma 4.8, which gives some useful properties for the generalized directional derivative and the subdifferential.

Lemma 4.8 (Hosseini and Pourayevali [16, Theorems 2.5(b), 2.13]; see Remark 4.1). Let \(h : M \to \mathbb{R}\) that is locally Lipschitz on some neighbourhood \(U\) of \(x, i.e., for \(x^t, v^t \in U, |h(x^t) - h(x^0)| \leq Gd(x^t, x^0)\).

(a) Let \(TM|_U = \{(x^t, v^t) : x^t \in U, v^t \in T_x M\}\). The function \(h^o : TM|_U \to \mathbb{R}\) defined by \((x^t, v^t) \mapsto h^o(x^t; v^t)\) is upper semi-continuous, i.e., if we have \(\{(x^t, v^t)\}_{t \in T} \subset TM|_U\) such that \(d(x^t, x) \to 0, \|\Gamma^x_{x^t} v^t - v\|_x \to 0, \) then

\[
\limsup_{t \to \infty} h^o(x^t; v^t) \leq h^o(x; v).
\]

(b) For every \(v \in T_x M\) and \(\epsilon > 0\), there exists some \(\delta_{v, \epsilon} > 0\) such that if \(d(x^t, x) < \delta_{v, \epsilon}\) and \(x \in \partial h(x^t)\), there exists \(w \in \partial h(x)\) such that \(|\langle \Gamma^x_{x^t} v - w, v \rangle_x | < \epsilon\).

Theorem 4.9. Under the same conditions, let \(\{x^t\}_{t \in T} \subset M\) be the random sequence generated in Theorem 4.7. If \(x^*\) is a limit point of this sequence, then \(0 \in \partial F(x^*)\) almost surely.

Proof. By passing to the convergent subsequence, we can without loss of generality assume that \(x^t \to x^*\). Let \(v^t = \text{Prox}^t(x^t), \bar{x}^t = \exp_{x^t}(v^t)\) and \(x = x^*\). We also know that \(\|v^t\|_{x^t} \to 0\) almost surely by Theorem 4.7(b). From the optimality condition for (11), there exists \(\bar{w}^t \in \partial g(x^t)\) such that for all \(t\),

\[
0 = \nabla f(x^t) + L v^t + d(E_{x^t})^* [\bar{w}^t].
\]

We show that \(-\nabla f(x) \in \partial g(x)\). Fix some \(w \in T_x M\). Then, since \(d(x^t, x) \to 0\), the parallel transport \(\Gamma^x_{x^t}\) is well-defined for \(t\) sufficiently large [20, Theorem 6.17]. Thus for sufficiently large \(t\),

\[
\langle \nabla f(x), w \rangle_x = \langle \Gamma^x_{x^t} \nabla f(x^t) + L v^t, w \rangle_x - \langle \Gamma^x_{x^t} \nabla f(x^t), w \rangle_x
\]

\[
= \langle \Gamma^x_{x^t} \nabla f(x^t) - \nabla f(x), w \rangle_x + L \langle \Gamma^x_{x^t} v^t, w \rangle_x - \langle \nabla f(x^t) + L v^t, \Gamma^x_{x^t} w \rangle_x
\]

\[
= \langle \Gamma^x_{x^t} \nabla f(x^t) - \nabla f(x), w \rangle_x + L \langle \Gamma^x_{x^t} v^t, w \rangle_x + \langle d(E_{x^t})^* [\bar{w}^t], \Gamma^x_{x^t} w \rangle_x.
\]
\[
\begin{align*}
&= (\Gamma^x_x \nabla f(x') - \nabla f(x), w)_x + L(\Gamma^x_x v', w)_x + \left< \bar{w}^t, d(\text{Exp}_{x^t})_{x^t} \left[ \Gamma^x_x w \right] \right>_{x^t} \\
&\leq (\Gamma^x_x \nabla f(x') - \nabla f(x), w)_x + L(\Gamma^x_x v', w)_x + g^x \left< \bar{x}^t; d(\text{Exp}_{x^t})_{x^t} \left[ \Gamma^x_x w \right] \right>.
\end{align*}
\]

The second equality is from the fact that \( \Gamma^x_x = (\Gamma^x_x)^{-1} \) is an isometry and applying it to both arguments of the second inner product term. The third equality is from using the optimality condition on (11). The fourth equality is from the definition of the adjoint. The inequality is from the definition of \( \bar{w}^t \in \partial g(\bar{x}^t) \). We examine each term separately.

For the first term, note that \( \partial f(x^t) = \{ \nabla f(x') \} \), \( \partial f(x) = \{ \nabla f(x) \} \), and for any \( \epsilon > 0 \), \( w \in T_x M \), we will eventually have \( d(x^t, x) < \epsilon \). Therefore by Lemma 4.8(b), \( |(\Gamma^x_x \nabla f(x') - f(x), w)_x| < \epsilon \) eventually, which means that for any \( w \), \( (\Gamma^x_x \nabla f(x') - \nabla f(x), w)_x \to 0 \).

Using Cauchy-Schwarz, the second term is upper and lower bounded by

\[
-L\|v^t\|_{x^t}\|w\|_x \leq L\|\Gamma^x_x v', w\|_x \leq L\|v^t\|_{x^t}\|w\|_x.
\]

These bounds \( \to 0 \) as \( t \to \infty \) since \( \|v^t\|_{x^t} \to 0 \).

Since \( \bar{x}^t = \text{Exp}_{x^t}(v^t) \), \( d(x^t, \bar{x}^t) \leq \|v^t\|_{x^t} \to 0 \), and \( d(x^t, x) \to 0 \), we have \( d(\bar{x}^t, x) \to 0 \) also. Now \( T : (x', v', w') \to d(\text{Exp}_{x'})_{v'}[w'] \) (where \( v', w' \in T_x M \)) is a smooth map by Absil et al. [1, Definition 8.1.1 and Chapter 8.1.2], since it is a special type of map known as a vector transport obtained as the differential of the exponential map retraction. Also, \( \|\Gamma^x_x \Gamma^x_x w - w\|_x = 0 \), \( d(\text{Exp}_{x'})_{0}[w] = w \) [20, Proposition 5.19(d)], and we know that \( x' \to x, v' \to 0 \), so

\[
\limsup_{t \to \infty} g^x \left( \bar{x}^t; d(\text{Exp}_{x^t})_{x^t} \Gamma^x_x w \right) \leq g^x(x; w).
\]

Together, this implies that taking appropriate limits, we have

\[
\langle -\nabla f(x), w \rangle_x \leq g^x(x; w)
\]

for any arbitrary \( w \in T_x M \). Therefore \( -\nabla f(x) \in \partial g(x) \), i.e., \( 0 \in \partial F(x) \).

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A  The Levi-Civita Connection and Decomposability of the Parallel Transport Operator and Exponential Map

To properly define geodesics, the exponential map so that we may derive their properties crucial to our analysis we must first define a metric connection. A function $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a Levi-Civita connection compatible with the Riemannian metric $g$, written $(X,Y) \mapsto \Delta_X Y$ if it is linear respectively over $C^\infty(M)$ in $X$ and $\mathbb{R}$ in $Y$, and satisfies the two product rules and symmetry conditions

$$\nabla_X (Y, Z) = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle$$

$$\nabla_X (f Y) = f \nabla_X Y + Y \nabla_X f$$

$$(\nabla_X Y - \nabla_Y X)f = X(Yf) - Y(Xf)$$

for all $f \in C^\infty(M)$, and $X,Y,Z \in \mathfrak{X}(M)$. The Fundamental Theorem of Riemannian Geometry [20, Theorem 5.10] guarantees every Riemannian manifold has a unique Levi-Civita connection.

Given a smooth curve $\gamma : I \to M$, $I$ denotes an interval in $\mathbb{R}$, we define $\mathfrak{X}(\gamma)$ to be the set of smooth functions $V : I \to TM$ such that $V(t) \in T_{\gamma(t)} M$. Note $\mathfrak{X}(M) \subseteq \mathfrak{X}(\gamma)$. Connections allow us to differentiate vector fields along curves. The covariant differentiation operator along $\gamma$, $D_t : \mathfrak{X}(\gamma) \to \mathfrak{X}(\gamma)$, satisfies the following properties for all $X,Y \in \mathfrak{X}(\gamma)$:

- **Product rule:**
  $$D_t(fX) = f'X + fD_tX$$
  for $f \in C^\infty(I)$.

- **Linearity over $\mathbb{R}$:**
  $$D_t(aX + bY) = aD_tX + bD_tY$$

- If $X$ extends to a vector field $\tilde{X} \in \mathfrak{X}(M)$ then $D_t X(t) = \nabla_{\gamma'}(t) \tilde{X}$

We call $D_t X$ the covariant derivative of $X$ along $\gamma$.

We can now formalize geodesics as constant speed curves. The acceleration of a curve $\gamma : [0,1] \to M$ is defined to be $D_t \gamma'$: the covariant derivative of its tangent vector along the curve itself. A geodesic is a curve $\gamma : [0,1] \to M$ with zero acceleration. The Hopf-Rinow theorem [20, Theorem 6.19] and the existence and uniqueness theorem for geodesics [20, Theorem 4.27] tell us that on a metrically complete Riemannian manifold, we can always find a unique geodesic $\gamma : [0,1] \to M$ such that $\gamma'(0) = v$ and $\gamma(0) = x$ for any $(x,v) \in TM$. The exponential map, $\text{Exp} : TM \to M$, is defined to be the rule $\text{Exp}_x(v) = \gamma_v(1)$ for $(x,v) \in TM$. The exponential map and the unique geodesic $\gamma$ corresponding to $(x,v) \in TM$ are related by the rule $\text{Exp}_{\gamma'(t)}(tv) = \gamma_v(t)$ [20, Proposition 5.19].

Given a piecewise smooth curve $\gamma : [0,1] \to M$, and $v \in T_{\gamma(t)} M$ for some $t \in I$, there exists a unique $V \in \mathfrak{X}(\gamma)$ that solves the equation

$$D_t V \equiv 0, \quad D_{t_0} V = v$$

[20, Corollary 4.33] provided we treat the term $D_t V \equiv 0$ piecewise. Thus, we can define the parallel transport map from $\gamma(t)$ to $\gamma(s)$, $\Gamma_{\gamma(s)}^{\gamma(t)} : T_{\gamma(t)} M \to T_{\gamma(s)} M$ by the rule $v \mapsto V(s)$ where $V$ solves (14). Intuitively, the vector field $V$ is always pointing in same direction as and has the same magnitude.
as $v$. In symbolic terms, this means that the parallel transport is a unitary transformation between $T_{\gamma(t)}M$ and $T_{\gamma(s)}M$.

The decomposibility of the parallel transport and exponential map in the cases of product manifolds and manifolds flat near the optimum, as embodied in properties (6) and (7) of subsection 3.5, undergird the improved rates presented in subsection 3.5. We now prove that these properties hold in each of these cases. The crux of these proofs is the following lemma which describes the Levi-Civita connection in each of these settings. It follows from a trivially straightforward verification of the properties of the Levi-Civita connection so we omit the proof.

**Lemma A.1.** Let $M = M_1 \times \ldots \times M_m$ be a Riemannian product manifold with a Levi-Civita connection $\nabla$, and $\nabla^i$ denotes the Levi-Civita connection on $M_i$. As every $X,Y \in \mathfrak{X}(M)$ are given by $X = \sum_{i=1}^{m} X_i$, $Y = \sum_{i=1}^{m} Y_i$ for $X_i,Y_i \in \mathfrak{X}(M_i)$, $i \in [m]$, the connection $\nabla$ is expressed in terms of $\nabla^i$ as

$$\nabla_Y X = \sum_{i=1}^{m} \nabla^i_{Y_i} X_i$$

Before stating the key lemma for product manifolds, we introduce some convenient direct sum notation form linear algebra. For vector spaces (over $\mathbb{R}$) $V_1, \ldots, V_m$ the direct sum is defined as $V_1 \oplus \ldots \oplus V_m = \bigoplus_{i \in [m]} V_i := \{(v_1, \ldots, v_m) : v_i \in V_i\}$ endowed with the vector space structure $\alpha(v_1, \ldots, v_m) + \beta(w_1, \ldots, w_m) = (\alpha v_1 + \beta w_1, \ldots, \alpha v_m + \beta w_m)$. Thus, given $x = (x_1, \ldots, x_m) \in M_1 \times \ldots \times M_m$ we can write $T_x M = \bigoplus_{i \in [m]} T_{x_i}M_i$. Furthermore, given linear operators $A_i : V_i \to V_i$, the direct sum operator $\bigoplus_{i \in [m]} A_i$ is defined by the action $\left(\bigoplus_{i \in [m]} A_i\right)(v_1, \ldots, v_m) = (A_1v_1, \ldots, A_mv_m)$. For convenience, given some index set $\mathcal{I} \subseteq [m]$, we will write $(v_i)_{i \in \mathcal{I}}$ to be the vector $(w_1, \ldots, w_m)$ where $w_i = v_i$ if $i \in \mathcal{I}$ and $w_i = 0_i$ if $i \notin \mathcal{I}$, where $0_i$ is the zero vector of $V_i$. If $\mathcal{I} = \emptyset$, then $(v_i)_{i \in \emptyset}$ is the zero vector of $\bigoplus_{i \in [m]} V_i$.

**Lemma A.2.** Let $x = (x_1, \ldots, x_m) \in M := M_1 \times \ldots \times M_m$ and $(v_1, \ldots, v_m) \in T_x M$. Then the exponential map at $x$ has the form

$$\text{Exp}_x (v_1, \ldots, v_m) = (\text{Exp}_{x_1}^1(v_1), \ldots, \text{Exp}_{x_m}^m(v_m))$$

(15)

where $\text{Exp}^i$ is the exponential map on $M_i$, $i \in [m]$. In particular, this implies

$$\text{Exp}_x ((v_i)_{i \in [k]}) = (\text{Exp}_{x_1}^1(v_1), \ldots, \text{Exp}_{x_k}^k(v_k), x_{k+1}, \ldots, x_m) = \text{Exp}_{\text{Exp}_x((v_i)_{i \in [k-1]})} ((v_i)_{i \in [k]})$$

(16)

for $1 \leq k \leq m$. The parallel transport along the curve $t \mapsto \text{Exp}_x(tv)$ is

$$\Gamma_{x}(tv) = (\Gamma_{1}^1_{x} (tv_1) \oplus \ldots \oplus (\Gamma_{m}^m_{x} (tv_m))$$

(17)

with $\Gamma_i$ being the parallel transport in $M_i$ along the curve $t \mapsto \text{Exp}_{x_i}^i(tv_i)$. In particular, this implies

$$\Gamma_{x} (v_i)_{i \in [k]} = \Gamma_{x}^{\text{Exp}_x(i \in [k])} \Gamma_{x}^{\text{Exp}_x(i \in [k-1])} \ldots \Gamma_{x}^{\text{Exp}_x(i \in [1])}$$

(18)

**Proof.** We first prove (15). Let $\gamma_i(t)$ denote the $i$-th component of $\text{Exp}_x t(v_i)_{i \in [m]}$. By Lemma A.1, the defining differential equation system for the exponential map on $M$ decomposes into

$$D^t_0 \gamma_i \equiv 0, \quad \gamma_i(0) = x, \quad i \in [m].$$
where $D_i^t$ is the covariant derivative along the curve $γ_i$. For each $i ∈ [m]$, we recover the defining differential equation for $\text{Exp}^i_{x_i}(tv_i)$: an equation with a unique solution. Thus, $γ_i(t) = \text{Exp}_{x_i}(tv_i)$. A nearly identical proof using the system of equations \((14)\), establishes \((17)\).

Equation \((15)\) readily follows from \((16)\) by setting $v_ℓ = 0_ℓ$ for $ℓ = k + 1, \ldots, m$. To establish that \((18)\) follows from \((17)\), observe that for $1 ≤ ℓ ≤ m$

$$\Gamma_x^{\text{Exp}_x((v_i)_{i ∈ [ℓ−1]})} = \left(\bigoplus_{i ∈ [ℓ−1]} (Γ_\ell)^{\text{Exp}_x^i(v_i)}\right) ⊕ \text{id}_\bigoplus_{i = ℓ}^m T_{x_i} M_i$$

$$\Gamma_{\text{Exp}_x((v_i)_{i ∈ [ℓ−1]})}^\text{Exp}_x((v_i)_{i ∈ [ℓ−1]}) = \text{id}_\bigoplus_{i ∈ [ℓ−1]} T_{\text{Exp}_x(v_i)} M_i ⊕ (Γ_\ell)^{\text{Exp}_x^i(v_i)} ⊕ \text{id}_\bigoplus_{i = ℓ+1}^m T_{x_i} M_i$$

so $\Gamma_x^{\text{Exp}_x((v_i)_{i ∈ [ℓ−1]})} = \Gamma_x^{\text{Exp}_x((v_i)_{i ∈ [ℓ−1]})}^{\text{Exp}_x((v_i)_{i ∈ [ℓ−1]})} \Gamma_x^{\text{Exp}_x((v_i)_{i ∈ [ℓ−1]})}$, and the rest follows from induction. □

We now consider flat open sets of $M$.

**Lemma A.3.** Let $U ⊆ M$ be an open set that is isometric to an open set $\tilde{U} ⊆ \mathbb{R}^{\dim M}$. If $γ_1, γ_2 : [0, 1] → U$ share the same start and endpoints then the parallel transport operator between the two points along each of the curves is the same.

**Proof.** If $γ : [0, 1] → \tilde{U}$ is a piecewise smooth curve then, using the coordinate representation of $T\tilde{U}$ its parallel transport operator satisfies

$$\Gamma_γ^t(0) \left(\sum_{i ∈ [n]} v_i \frac{∂}{∂x_i}|_{\gamma(0)}\right) = \sum_{i ∈ [n]} v_i \frac{∂}{∂x_i}|_{\gamma(t)}$$

Thus, the parallel transport between two points does not depend upon the curve in $\tilde{U}$ connecting them. The claim holds if $U = \tilde{U}$. The naturality of the Levi-Civita connection [20, Proposition 5.13] and the preservation of the parallel transport applied piecewise [20, Proposition 4.38] clinch the more general case when $U$ is isometric to $\tilde{U}$.

□

**Lemma A.4.** Let $U ⊆ M$ be an open set that is isometric to an open set $\tilde{U} ⊆ \mathbb{R}^{\dim M}$. If $x_0, \ldots, x_k ∈ U$ and $v_i ∈ T_{x_{i−1}}$ satisfy the recursion

$$x_i = \text{Exp}_{x_{i−1}}(v_i)$$

and $\text{Exp}_{x_{i−1}}(tv_i) ∈ U$ for all $t ∈ [0, 1]$ and $i = 1, \ldots, k$ then

$$x_k = \text{Exp}_{x_0} \left(\sum_{i = 1}^k \Gamma_{x_{i−1}}^x(v_i)\right)$$

**Proof.** We first prove the claim on $\tilde{U}$. As $\text{Exp}_{x_{i−1}}(tv_i) ∈ U$ for all $t ∈ [0, 1]$ and $i = 1, \ldots, k$, we have that $\text{Exp}_{x_{i−1}}(tv_i) = tv_i + x_{i−1}$ for all $t ∈ [0, 1]$, where we regard $T_{\tilde{U}}$ as $\tilde{U} × \mathbb{R}^n$. Consequently, $\text{Exp}_{x_0} \left(\sum_{i = 1}^k v_i\right) = x_{k−1}$ establishing the claim in $\tilde{U}$. The naturality of the Levi-Civita connection [20, Proposition 5.13] and the preservation of the parallel transport applied piecewise [20, Proposition 4.38] clinch the more general case when $U$ is isometric to $\tilde{U}$.

□
\section*{B Dual tangent spaces, subdifferentials and tangent spaces of tangent spaces}

In this paper we defined subgradients \( v \in \partial h(x) \) as members of the tangent space \( T_x M \). Actually, it is more natural to define subgradients as members of the dual space \( T^*_x M = \{ w^* : T_x M \to \mathbb{R} \text{ s.t. } w^* \text{ is linear} \} \), the space of linear functionals on \( T_x M \). This is the view taken in Hosseini and Pouryayevali \[16\]. However, since we have an inner product \( \langle \cdot, \cdot \rangle_x \) on \( T_x M \) from the Riemannian metric, we can identify \( w^* \in T^*_x M \) with an element \( w \in T_x M \) such that \( w^*(v) = \langle w, v \rangle_x \) for all \( v \in T_x M \) by the Riesz representation theorem. This is an isomorphism between \( T^*_x M \) and \( T_x M \) (a bijective linear operator), and is equivalent to the sharp operator defined in Lee \[20\, \text{Chapter 2}\], hence we denote it \((\cdot)^\sharp : T^*_x M \to T_x M \). Its inverse is called the flat operator, \((\cdot)^\flat : T_x M \to T^*_x M \), i.e., \( w^\flat(\cdot) = \langle w, \cdot \rangle_x \). 

In this paper, we make use of some results from Hosseini and Pouryayevali \[16\], where they are originally stated in terms of the dual space. However, we state and use these results after applying the canonical transformation back to the tangent space. We check that the re-statement of these results is valid.

\subsection*{B.1 Definition of the subdifferential}

The definition of the subdifferential in Hosseini and Pouryayevali \[16\] is
\[
\partial h^*(x) := \{ w^* \in T^*_x M : h^0(x; v) \geq w^*(v) \forall v \in T_x M \},
\]

compared to \( \partial h(x) := \{ w \in T_x M : h^0(x; v) \geq \langle w, v \rangle_x \forall v \in T_x M \} \).

It is clear that if \( w^* \in \partial h^*(x) \), then for any \( v \in T_x M \), \( \langle (w^*)^\sharp, v \rangle_x = w^*(v) \leq h^0(x; v) \), so \( (w^*)^\sharp \in \partial h(x) \). Furthermore, if \( w \in \partial h(x) \), then \( w^\flat(v) = \langle w, v \rangle_x \leq h^\circ(x; v) \), so \( w^\flat \in \partial h^*(x) \). Thus \( \partial h(x) = \{ (w^*)^\sharp : w^* \in \partial h^*(x) \} \), and this is a one-to-one correspondence.

\subsection*{B.2 Definition of the adjoint operator and proof of Lemma 4.4}

The definition of adjoint of a linear operator \( A : T_x M \to T_y M \) is defined in Hosseini and Pouryayevali \[16\] as the unique linear operator \( A^* : T^*_y M \to T^*_x M \) which satisfies: for any \( v \in T_x M \), \( w^* \in T^*_x M \), we have \( A^*(w^*)(v) = w^*(L(v)) \). In contrast, we defined the adjoint of \( A \) as \( A^* : T_y M \to T^*_x M \) satisfying: for all \( v \in T_x M \), \( w \in T_y M \), \( \langle A^*(w), v \rangle_x = \langle w, A(v) \rangle_y \). However, \( A^* \) can be obtained from \( A^\flat \) by \( A^*(w) = (A^\flat(w))^\sharp \) for all \( w \in T_y M \). To see this, note that \( \langle A^*(w), v \rangle_x = \langle (A^\flat(w))^\sharp, v \rangle_x = A^\flat(w)^\flat(v) = w^\flat(L(v)) = \langle w, A(v) \rangle_y \). By a similar argument, \( A^\flat \) can be obtained from \( A^\flat \) by \( A^\flat(w)^\flat = (A^\flat(w^\flat))^\flat \). Thus, the two notions of the adjoints can be reconciled via the sharp and flat operators.

Hosseini and Pouryayevali \[16, \text{Theorem 3.4}\] state that for \( H : M \to N \) and \( h : N \to \mathbb{R} \),
\[
\partial(h \circ H)^*(x) = \left\{ dH_x^*(w^*) : w^* \in \partial h^*(H(x)) \right\} \subseteq T^*_y M,
\]

where \( dH^*: T^*_y M \to T^*_x M \) is the adjoint. By definition we have \( (dH^*_x(w^*))^\sharp = dH^*_x((w^*)^\sharp) \), but we also know that \( \partial h(H(x)) = \{ (w^*)^\sharp : w^* \in \partial h^*(H(x)) \} \), which proves Lemma 4.4.
B.3 Proof of Lemma 4.8(b)
Hosseini and Pouryayevali [16, Theorem 2.13] state that if \( h : M \to \mathbb{R} \) is Lipschitz on some ball of \( x \), then for every \( v \in T_x M \) and \( \epsilon > 0 \), there exists some \( \delta_{v, \epsilon} > 0 \) such that if \( d(x', x) < \delta \) and \( u^* \in \partial h^*(x') \), there exists \( w^* \in \partial h^*(x) \) such that \( \left| \langle \left( \Gamma_{x^*}^x \right)^* u^* - w^* \rangle(t) \right| < \epsilon \). Applying the sharp operator to \( \Gamma_{x^*}^x u^* - w^* \), we get \( \left( \Gamma_{x^*}^x u^* - w^* \right)^2 = \left( \Gamma_{x^*}^x u^* \right)^2 - (w^*)^2 \). Let \( w = (w^*)^2 \in \partial h(x) \) and \( u = (u^*)^2 \in \partial h(x') \), and note that by definition \( \left( \Gamma_{x^*}^x u^* \right)^2 = (\Gamma_{x^*}^x u^*)^* u^* \). Also, \( (\Gamma_{x^*}^x)^* = \Gamma_{x^*}^x \), i.e., the adjoint of the parallel transport is its inverse. To see this, observe that since the parallel transport preserves inner products, for any \( v \in T_x M, w \in T_x M, \langle v, \Gamma_{x^*}^x w \rangle_{x'} = \langle \Gamma_{x^*}^x v, \Gamma_{x^*}^x w \rangle_x = \langle \Gamma_{x^*}^x v, w \rangle_x \), so because adjoints are unique, \( (\Gamma_{x^*}^x)^* = \Gamma_{x^*}^x \). Therefore \( \left( \Gamma_{x^*}^x u^* - w^* \right)^2 = \Gamma_{x^*}^x u - w \), which means that

\[
|\langle \Gamma_{x^*}^x u - w, v \rangle_x| = \left| \left( \Gamma_{x^*}^x u^* - w^* \right) \langle t \rangle \right| < \epsilon.
\]

In summary, for every \( v \in T_x M \) and \( \epsilon > 0 \), there exists some \( \delta_{v, \epsilon} > 0 \) such that if \( d(x', x) < \delta \) and \( u \in \partial h(x') \), there exists \( w \in \partial h(x) \) such that \( |\langle \Gamma_{x^*}^x u - w, v \rangle_x| < \epsilon \).

B.4 Proof of Lemma 3.3
Denote \( x^* = x \). Note that \( \partial f(x') = \{ \nabla f(x') \}, \partial f(x) = \{ \nabla f(x) \} \), and for any \( \epsilon > 0, w \in T_x M \), we will eventually have \( d(x', x) < \epsilon \). Therefore by Lemma 4.8(b), \( |\langle \Gamma_{x^*}^x \nabla f(x') - \nabla f(x), w \rangle_x| < \epsilon \), which means that for any \( w \),

\[
0 = \lim_{t \to \infty} \langle \Gamma_{x^*}^x \nabla f(x'), w \rangle = \langle \nabla f(x), w \rangle_x.
\]

Picking \( w = \nabla f(x) \) completes the proof.

B.5 Justification for (13) and the proof of Proposition 4.5
For any \( v \in T_x M \), we show that any \( \tilde{v} \in T_{\gamma}(T_x M) \) can be identified with a unique vector in \( T_x M \), and vice versa. To do this, we must understand the action of \( \tilde{v} \) on functions \( h: T_x M \to \mathbb{R} \). By the definition of tangent vectors, there will exist some smooth curve \( \gamma(t): (-1,1) \to T_x M \) such that \( \gamma(0) = v \) and

\[
\tilde{v}(h) = (h \circ \gamma)'(0) = \lim_{t \to 0} \frac{h(\gamma(t)) - h(v)}{t}
\]

for any differentiable \( h : T_x M \to \mathbb{R} \).

Since \( T_x M \cong \mathbb{R}^n \), we can choose a basis \( \{b_1, \ldots, b_n\} \subset T_x M \) and write \( \gamma(t) = \sum_{i=1}^{n} \gamma_i(t)b_i \) where \( \gamma_i : (-1,1) \to \mathbb{R} \) are smooth functions in the usual sense. By Taylor’s theorem, we can write \( \gamma_i(t) = \gamma_i(0) + \gamma_i'(0)t + \gamma_i''(0)t^2 \) where \( \gamma_i''(t) \to 0 \) as \( t \to 0 \). Define \( \bar{v}^1 = \sum_{i=1}^{n} \gamma_i'(0)b_i \) and \( \bar{v}^2(t) = \sum_{i=1}^{n} \gamma_i''(t)b_i \). Then \( \gamma(t) = v + tv^1 + t^2v^2(t) \). We claim that computing \( \tilde{v}(h) \) can be done independently of \( v^2(t) \).

Let \( \{b_1, \ldots, b_n\} \) be an orthonormal basis for \( T_x M \) and consider the isometry \( H : \mathbb{R}^n \to T_x M \) defined by \( H(r_1, \ldots, r_n) = \sum_{i=1}^{n} b_i r_i \). This is an isometry because \( r^T r' = \langle H(r), H(r') \rangle_x \). Denote \( \tilde{v} = (\gamma_1(0), \ldots, \gamma_n(0)) \in \mathbb{R}^n, \tilde{v}_1 = (\gamma_1'(0), \ldots, \gamma_n'(0)) \in \mathbb{R}^n, \tilde{v}_2(t) = (\gamma_1''(t), \ldots, \gamma_n''(t)) \in \mathbb{R}^n, \gamma(t) = \gamma_1(0) + \gamma_1'(0)t + \gamma_1''(0)t^2 \). Then \( \gamma(t) = H(\gamma(t)) \), and \( \tilde{v}(h) = \lim_{t \to 0} \frac{h(H(\gamma(t))) - h(\tilde{v})(t)}{t} = \nabla h(\tilde{v}) \). where \( \nabla (h \circ H)(\tilde{v}) \in \mathbb{R}^n \) is the gradient of \( h \circ H : \mathbb{R}^n \to \mathbb{R} \), and is computed depending only on \( \tilde{v} \), which follows from basic calculus results in Euclidean space. But now

\[
\tilde{v}(h) = \nabla (h \circ H)(\tilde{v})^\top \tilde{v}_1 = \langle H(\nabla (h \circ H)(\tilde{v})), H(\tilde{v}_1) \rangle_x = (H(\nabla (h \circ H)(\tilde{v})), \tilde{v}_1)_x.
\]
This shows that \( \bar{v} \) can be described only with \( v \) and \( v^1 \in T_x M \). Thus any \( \bar{v} \in T_v(T_x M) \) is fully determined by some \( v^1 \in T_x M \) and \( v \) itself. The converse also holds: given \( v^1 \in T_x M \), define \( \bar{v}(t) = \sec(t, v + tv^1 - h(v))/t \). Thus we can identify \( T_x M \) and \( T_v(T_x M) \).

For the proof of Proposition 4.5, we verify that the gradients of \( v' \mapsto \langle \nabla f(x), v' \rangle_x \) and \( v' \mapsto \|v'\|^2_x \), as functions \( T_x M \to \mathbb{R} \), are indeed \( \nabla f(x) \in T_x M \) and \( 2v \in T_x M \) respectively. For \( \bar{v} \in T_v(T_x M) \) with associated curve \( \gamma(t) = v + tv^1 + v^2(t)^2 \), we can directly compute \( \bar{v}(h) = (h \circ \gamma)'(0) \) for \( h \) corresponding to the two functions respectively:

\[
(h \circ \gamma)(t) = \langle \nabla f(x), v + tv^1 + t^2v^2(t) \rangle_x, \quad (h \circ \gamma)'(0) = \langle \nabla f(x), v^1 \rangle_x
\]
\[
(h \circ \gamma)(t) = \|v + tv^1 + t^2v^2(t)\|^2_x, \quad (h \circ \gamma)'(0) = 2\langle v, v^1 \rangle_x.
\]

This shows that the directional derivatives can be computed as the inner product of \( v^1 \in T_x M \) with \( \nabla f(x) \) and \( 2v \) respectively, so we can identify the gradient of the two functions at \( v \in T_x M \) with \( \nabla f(x) \) and \( 2v \) respectively.