A TRIPLE MODE ROBUST SLIDING MODE CONTROLLER FOR A NONLINEAR SYSTEM WITH MEASUREMENT NOISE AND UNCERTAINTY

Nasim Ullah∗
College of Engineering, Department of Electrical Engineering, TAIF University KSA
TAIF, KSA
Ahmad Aziz Al-Ahmadi
College of Engineering, Department of Electrical Engineering, TAIF University KSA
TAIF, KSA
(Communicated by Ying Yang)

Abstract. This research work proposes a novel triple mode sliding mode controller for a nonlinear system with measurement noise and uncertainty. The proposed control has the following goals (1) it ensures the transient and steady state robustness of the system in closed loop (2) it reduces chattering in the control signal with measurement noise. Fuzzy system is used to tune the appropriate order of the fractional operators for the proposed control system. Depending on the tuned range of the fractional operators, the proposed controller can operate effectively in the following three modes (1) classical sliding mode (SMC) (2) fractional order sliding mode (FSMC) (3) Integral sliding mode control (ISMC). With the noisy feedback, the performance of the classical SMC and SMC with boundary layer degrades significantly while ISMC shows better performance. However ISMC exhibits large transient overshoots. The proposed control method optimally selects the appropriate mode of the controller to ensure performance (transient and steady state) and suppresses the effect of noisy feedback. The proposed scheme is derived for the permanent magnet synchronous motor’s (PMSM) speed regulation problem which is subject to uncertainties, measurement noise and un-modeled dynamics as a case study. The effectiveness of proposed scheme is verified using numerical simulations.

1. Introduction. Sliding mode control (SMC) is a robust control method that is applied to both linear and nonlinear systems [1]. There are many advantages associated with the classical SMC such as its simplicity in the design and its insensitivity to the matched disturbance. A major drawback of the classical SMC is the presence of high frequency chattering in the control signal which makes it unsuitable for practical applications [2]. To minimize the chattering phenomenon, there are several techniques that have been proposed in the literature. Out of these, the most common techniques include the disturbance observer based sliding mode control [3], adaptive fuzzy sliding mode [4] and boundary layer design method [5]. In a situation where the plant dynamics are not well known the disturbance observer based method is not very efficient. Similarly adaptive fuzzy method is not effective

2020 Mathematics Subject Classification. Primary: 58F15, 58F17; Secondary: 53C35.
Key words and phrases. Sliding mode control, fractional calculus, robustness, fuzzy system.
∗ Corresponding author: NasimUllah.
with noisy feedback. The widely utilized method is the boundary layer design in which the discontinuous function is replaced by a continuous function. Using this method, the chattering phenomenon is minimized significantly but the steady state error does not converge to zero [5]. Boundary layer method is also not very effective with noisy feedback signal where instead of removing the chattering phenomena it is elevated. A novel boundary layer design method has been introduced to remove the chattering phenomena with noisy feedback [6]. Low pass filter is applied to the output of the control signal for removal of noise [6]. This design is successful but at the cost of increase in the order of the plant to be controlled. With the increase in the order of the plant, the dynamics of the feedback system become very complex especially in the case when the order of the plant is originally greater than two. Yao proposed adaptive robust control methods for a nonlinear electro-hydraulic servo system using Lyapunov function [7-8]. To address the chattering and steady state error of the system, Bartolini proposed second order sliding mode control method [9]. In the second order SMC, due to the introduction of the integral action, the degree of the system is virtually raised, so the steady state error is zero but at the cost of degradation of transient performance. Seshagiri and Khalil presented conditional integral sliding mode control method to recover transient performance. The main idea was to apply integral action inside the boundary layer [5]. All the above cited literature addresses the improvement of transient performance and removal of noise/chattering independently, however the combined effect has never been exploited. Fractional order control is an emerging area which is becoming popular in the scientific community. Fractional order systems offer several advantages such as high degree of freedom, flexibility to adjust the desired performance and robustness to noise. The design idea of fractional order controller was first developed by Oustaloup. The first robust fractional order controller CRONE (Commande Robuste d’Ordre Non Entier) was developed in 1996 by Oustaloup [10]. Later on different variants of the fractional order controllers were proposed for many novel applications. Delavari proposed adaptive fractional order proportional, integral and derivative (PID) control for the robot manipulator [11]. In order to achieve finite time convergence of the states, several researchers have proposed fractional order robust control methods for nonlinear systems. In [12], a fractional order sliding mode control method has been proposed for the velocity loop of the permanent magnet synchronous motor (PMSM). Similarly Dadras and Momeni proposed a fractional order terminal sliding mode control for the dynamical system with uncertainties [13-15]. Fractional order control combines the fractional differentiation and fractional integrals in the derived controllers [16]. The expressions for numerical simulations and implementation of the fractional operator are explained in [17-18]. From the control performance point of view, fractional order SMC offers more flexibility to adjust the desired performance due to its additional design parameters [19]. Hence, the fractional order controllers can be optimized for the excellent dynamic response. The stability of the fractional order systems is derived in [20-21]. Moreover, the authors in [22] proposed fractional order Lyapunov theorem for deriving the fractional-order controllers. Apart from the general theory of robust control system, some researchers have proposed artificial intelligence (AI) based hybrid methods such as Fuzzy sliding mode control, adaptive fuzzy control for uncertain nonlinear systems and fuzzy tuned control methods [23]. Fuzzy logic system can be incorporated either as estimator for the unknown smooth and bounded functions or as a supplementary controller to tune the parameters of the main control system.
Several authors have proposed adaptive tracking controllers by utilizing the universal function approximation tools such as fuzzy logic systems [27-29]. All the above cited literature addresses the improvement of transient performance and removal of noise/chattering independently, however the combined effect has never been exploited. As compared to the cited literature, the main contribution of this proposal is highlighted as following:

1: This work addresses the combined effect of noisy feedback and transient performance of a nonlinear controller. In the prior cited work, the effect of noisy feedback and improvement of transient performance of a system has been studied independently. In this work both the improvement of transient performance and effects of noisy feedback on system performance are analyzed together.

2: The proposed controller ensures smooth transition from one mode to another.

3: The stability of the tipple mode controller is proved using Lyapunov theorem.

Based on the above literature survey, this work presents a novel triple mode sliding mode control method for a nonlinear system with measurement noise and uncertainties. First a novel fractional order sliding mode manifold is chosen and based on it, a generalized control system is derived. A fuzzy system controller is used to update the appropriate order of the fractional operator in the sliding manifold and the corresponding controller based on the dynamics of the sliding surface. Depending on the selection of the fractional operator, the proposed controller adjusts itself in the appropriate mode to compensate the noisy feedback and ensure transient performance of the system.

The rest of the paper is organized as following. In section 2, mathematical preliminaries are defined. Section 3 shows the derivation of the classical SMC and integral SMC for an uncertain dynamic system. In section 4, main results about the proposed control scheme are discussed. Finally the simulation results are presented in section 5.

2. Basics Of fractional order mathematics. This section gives a brief description of fractional order mathematics.

Definition 2.1. The Riemann–Liouville fractional order integration and derivative of a function \( f(t) \) are expressed as shown in Eq. 1 and Eq. 2 [38-39].

\[
\begin{align*}
t_0 I_t^\alpha f(t) &= D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} f(\tau) (t-\tau)^{-\alpha} d\tau \\
t_0 D_t^\alpha f(t) &= \frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} f(\tau) (t-\tau)^{\alpha-m-1} d\tau
\end{align*}
\]

Here \( \Gamma(.) \) represents the gamma function, \( m \in \mathbb{N} \) and \( m-1 < \alpha \leq m \).

From Eq. 1 and Eq. 2 the following relation holds: \( t_0 D_t^\alpha (t_0 I_t^\alpha f(t)) = f(t) \)

Definition 2.2. The Caputo fractional order derivative of a function \( f(t) \) is given by [38-39].
Lemma 2.8. [16]: The fractional integral operator 
\[ F^{\delta} \] control input and then \[ G \] variation of the system, term representing the uncertain parameters, un-modeled dynamics or structural

Definition 2.4. [13-15]: For analytic function \( f(t) \), its fractional derivative \( D^\alpha f(t) \) is also analytically function of \( t \). For \( \alpha = n \), the operation \( D^\alpha f(t) \) represents classical differentiation of integer order \( n \).

Definition 2.5. [13-15]: The additive index law \( D^\alpha (t^\beta f(t)) = t^\beta (D^\alpha f(t)) \) is true under the assumption that \( f(t) \) is reasonably constrained.

Definition 2.6. [13-15]: The fractional order derivative commutes with integer order derivative as following: \( D^\alpha (t^\beta f(t)) = t^\beta (D^\alpha f(t)) \) is true under the assumption that \( f(t) \) is reasonably constrained.

Lemma 2.7. [16]: If integral of fractional derivative \( D^\alpha f(t) \) exists, then
\[ D^\alpha (\int_0^t f(\tau) d\tau) = \int_0^t D^\alpha f(\tau) d\tau \]

Lemma 2.8. [16]: If integral of fractional derivative \( D^\alpha f(t) \) exists, then
\[ \| D^\alpha f(t) \|_p \leq K \| f(t) \|_p ; \quad 1 \leq p \leq \infty ; \quad 1 \leq K \leq \infty \]

3. Non-integer SMC and ISMC. This section is focused on the derivation of integer order robust controllers for a generalized second order uncertain system.

Consider the second order nonlinear system of the following form:
\[
\begin{cases}
\dot{x}_1 = x_2 \\
\dot{x}_2 = F(X,t) + G(X,t)u + \delta(X,t) \\
y = x_1
\end{cases}
\]

Here \( X(t) = [x_1 \ x_2]^T \in \mathbb{R}^n \), and it represents the system states, \( u \in \mathbb{R}^b \) is the control input and \( \delta(X,t) \in \mathbb{R}^c \) is the external disturbance. The unknown function \( F(X,t) \) is expressed as \( F(X,t) = F_n(X,t) + \Delta F(X,t) \) with \( \Delta F(X,t) \) is the uncertain term representing the uncertain parameters, un-modeled dynamics or structural variation of the system, \( G(X,t) = G_n(X,t) + \Delta G(X,t) \), with \( G_n(X,t) > 0 \) is
the classical SMC and integral SMC, the sliding surfaces are defined as:

\[
\dot{x}_1 = x_2 \\
\dot{x}_2 = F_n(X, t) + G_n(X, t)u + D(X, u, t) \\
y = x_1
\]

(7)

Here the nominal state matrices are defined as following:

\[
F \in \mathbb{R}^{2 \times 2}
\]

and \(G_n(X, t) = \begin{bmatrix} 0 & 1 \\ P\psi & 0 \end{bmatrix}\). In the above expressions \(B, P\) and \(\psi_I\) are defined in the results and simulation section. In Eq. 7, the unknown term is expressed as:

\[
D(X, u, t) = \Delta G(X, t) + \Delta F(X, t) + \delta(X, t)
\]

Assumption 1: The uncertainty term \(D(X, u, t)\) is assumed to satisfy the inequality:

\[
||D(X, u, t)||_2 \leq \chi [30].
\]

Here \(\chi \geq 0\) represents a known scalar function. To derive the classical SMC and integral SMC, the sliding surfaces are defined as :

\[
\begin{cases}
S_1 = C_1 e \\
\dot{S}_2 = C_1 e + C_2 \int e \\
\dot{S}_1 = C_1 \dot{e} \\
\dot{S}_2 = C_1 \dot{e} + C_2 e
\end{cases}
\]

(8)

In Eq. 8, the error terms are defined as: \(e = x_2 - x_{2r}\), \(\dot{e} = \dot{x}_2 - \dot{x}_{2r}\), sliding surface \(S_1\) is a proportional type, \(S_2\) is PI type, \(C_1\) and \(C_2\) represents the positive constants that are chosen such that the resultant polynomials of \(S_1\) and \(S_2\) are Hurwitz. From Eq. 7 and 8, the dynamics of the proportional sliding surface is written as following:

\[
\dot{S}_1 = C_1 \dot{e} = C_1 (F_n(X, t) + G_n(X, t)u + D(X, u, t) - \dot{x}_{2r})
\]

(9)

From Eq. 9 the classical sliding mode control expression is derived as shown in Eq. 10.

\[
u = [C_1 G_n(X, t)]^{-1} (-C_1 F_n(X, t) + C_1 \dot{x}_{2r} - [\xi_1 + \xi_2 + K] \text{sign}(S_1))
\]

(10)

Here \(\xi_1, \xi_2\) represent the upper bound on the uncertain terms \(\Delta G(X, t), \Delta F(X, t)\). The constant control gain is chosen as: \(K > \chi\) such that the expression in Eq. 9 is Hurwitz. More precisely the sum of all constant terms should be greater than the upper bound of the total uncertainty term i.e. \([\xi_1 + \xi_2 + K] > \chi\). To overcome the chattering problem, a boundary layer of \(\phi\) width is selected and the discontinuous function \(\text{sign}(.)\) is replaced with a continuous function \(\text{sat}(.)\) such that

\[
\text{sat}(\frac{S_1}{\phi}) = \begin{cases}
\frac{S_1}{\phi} & \text{if } |\frac{S_1}{\phi}| < 1 \\
\text{sign}(\frac{S_1}{\phi}) & \text{if } |\frac{S_1}{\phi}| \geq 1
\end{cases}
\]

(11)

By replacing \(\text{sign}(.)\) with \(\text{sat}(.)\) function, chattering phenomenon will be minimized but at the cost of loss of robustness. Secondly with noisy feedback, chattering phenomena is enhanced. To address the robustness issue, Integral action is required. Using Eq. 7 and 8, the integral sliding surface dynamics are written as

\[
\dot{S}_2 = C_1 \dot{e} + C_2 e = C_1 (F_n(X, t) + G_n(X, t)u + D(X, u, t) - \dot{x}_{2r}) + C_2 e
\]

(12)
Using Eq. 12 the integral control law is formulated as following:

\[ u = [G_n(X, t)]^{-1} \left( -F_n(X, t) + \dot{x}_{2r} - \frac{C_2}{C_1} e - \frac{[\xi_1 + \xi_2 + K]}{C_1} sat(\frac{\xi_1}{\beta}) \right) \]  (13)

Eq. (13) ensures the steady state response of the system due to the presence of the integral action but the transient response is degraded [5]. To overcome the problems mentioned in the above analysis, a novel fuzzy fractional order sliding mode control system is proposed and the main results are presented in the next section.

4. Main results. In this section a systematic approach is used to derive the fuzzy tuned fractional order sliding mode controller for an uncertain dynamic system. To derive the proposed scheme a novel fractional order derivative integral (DI) type sliding manifold is defined as

\[
\begin{align*}
S_3 &= C_1 D^\beta e + C_2 D^{-\alpha} e \\
\dot{S}_3 &= C_1 D^\beta \dot{e} + C_2 D^{1-\alpha} e
\end{align*}
\]  (14)

Here \( \alpha \) and \( \beta \) represent the orders of the fractional operators used in Eq. 14. For the proposed fractional order sliding manifold given in Eq. 4, the range of the fractional operators are given as \( 0 \leq \alpha \leq 1 \) and \( 0 \leq \beta < \frac{\alpha}{2} \). The constants \( C_1 \) and \( C_2 \) are already defined in Eq. 8. Using Eq. 7 and 14, the proposed control law is formulated as

\[ \dot{S}_3 = C_1 D^\beta (F_n(X, t) + G_n(X, t) u + D(X, u, t) - \dot{x}_{2r}) + C_2 D^{1-\alpha} e \]  (15)

By letting \( \dot{S}_3 = -[\xi_1 + \xi_2 + K] sign(S_3) \) and dividing Eq. 15 by a factor of \( D^\beta \), the resultant control law is expressed as following:

\[ u = [G_n(X, t)]^{-1} \left( -F_n(X, t) + \dot{x}_{2r} - \frac{C_2}{C_1} D^{1-\alpha-\beta} e - \frac{[\xi_1 + \xi_2 + K]}{C_1} D^{-\beta} sat(\frac{\xi_1}{\beta}) \right) \]  (16)

The order of \( \beta \) is restricted to \( \frac{\alpha}{2} \) because large values of means large integral term would appear around the saturation function given in the control law of the Eq. (16). Hence it will eliminate the steady state error but at the same time degrades the transient response. The order of \( \beta \) is selected such that as to have minimum chattering in the control signal with small transient error.

**Theorem 4.1.** Using fractional order sliding surface of Eq. (14) and the control law of Eq. (16), the state errors of the system (7) converge to zero in finite time. Moreover the fractional order sliding surface of Eq. (14) and control law of Eq. (16) reduce to classical sliding mode control if fractional operators vector is set to zero i.e. \([\alpha \beta] = [0 \ 0] \) and it is equivalent to integral SMC control if \([\alpha \beta] = [1 \ 0] \) . For the fractional orders of \([\alpha \beta]\) other than the above two combinations, the proposed method is equivalent to fractional order integral SMC.

**Proof.** To prove the stability of the system, the following three cases are discussed.

**Case 1:** When \([\alpha \beta] = [0 \ 0] \), then using Definition. 2 \( D^0 e = e \) and \( D^1 e = \dot{e} \), so Eq. 14 is expressed in the following form:

\[
\begin{align*}
S_3 &= (C_1 + C_2) e \\
\dot{S}_3 &= (C_1 + C_2) \dot{e}
\end{align*}
\]  (17)

Similarly by using Definition 2, Eq. 16 is re-written as

\[ u = G_n(X, t)^{-1} \left( -F_n(X, t) + \dot{x}_{2r} - \frac{C_2}{C_1} \dot{e} - \frac{[\xi_1 + \xi_2 + K]}{C_1} sat(\frac{\xi_1}{\beta}) \right) \]  (18)
Replacing $\dot{e}$ and combining the nominal dynamics of Eq. 7 and Eq. (18) yields

$$u = G_n(X,t)^{-1} \left( \frac{-F_n(X,t) + \dot{x}_{2r} - \frac{C_2}{C_1} (F_n(X,t) + G_n(X,t)u - \dot{x}_{2r})}{-\frac{[\xi_1 + \xi_2 + K]}{C_1} \text{sat}(\frac{S_3}{\phi})} \right) \quad (19)$$

By rearranging Eq. (18), the modified expression is written as

$$uG_n(X,t) \frac{C_1 + C_2}{C_1} = \left( -F_n(X,t)(\frac{C_1 + C_2}{C_1}) + \dot{x}_{2r} - \frac{C_2}{C_1} \text{sat}(\frac{S_3}{\phi}) \right) \quad (20)$$

From Eq. 20, the final form of the control law is written as following:

$$u = G_n(X,t)^{-1} \left( -F_n(X,t) + \dot{x}_{2r} - \frac{[\xi_1 + \xi_2 + K]}{C_1 + C_2} \text{sat}(\frac{S_3}{\phi}) \right) \quad (21)$$

Eq. (17) is equivalent to the classical proportional type sliding manifold and the control law of Eq. (21) is similar to the classical sliding mode control law of eq. (25). To prove the stability of the closed loop, the Lyapunov candidate function is expressed as:

$$\left\{ V_p = \frac{1}{2} S_3^2 ; \dot{V}_p = S_3 \dot{S}_3 \right\} \quad (22)$$

By combining Eq. (7), (17), (21) and Eq. (22), one obtains

$$\dot{V}_p = S_3 \dot{S}_3 = (C_1 + C_2) S_3 \dot{e} = (C_1 + C_2) S_3 (\dot{x}_2 - \dot{x}_{2r}) = (C_1 + C_2) S_3 (F_n(X,t) + G_n(X,t) \dot{x}_{2r}) - \text{sat}(\frac{S_3}{\phi}) + D(X,t)$$

$$= -[\xi_1 + \xi_2 + K] S_3 \text{sat}(\frac{S_3}{\phi}) + S_3 D(X,u,t)$$

Using Assumption 1 and by choosing $[\xi_1 + \xi_2 + K] > \chi$, it is easy to show that $\dot{V}_p \leq 0$.

**Case 2:** When $[\alpha \beta] = [1 \ 0]$, then Eq. 14 can be expressed in the following form:

$$\begin{align*}
S_3 &= C_1 \dot{e} + C_2 \dot{D}^{-1} e \\
\dot{S}_3 &= C_1 \dot{e} + C_2 \dot{D}^0 e
\end{align*} \quad (24)$$

Similarly the control law of Eq. 16 is simplified as following:

$$u = [G_n(X,t)]^{-1} \left( -F_n(X,t) + \dot{x}_{2r} - \frac{C_2}{C_1} e - \frac{[\xi_1 + \xi_2 + K]}{C_1} \text{sat}(\frac{S_3}{\phi}) \right) \quad (25)$$

For case 2, the deduced sliding manifold (24) is equivalent to the PI type expression of Eq. (8) and the control law of Eq. (25) is equivalent to the integer type control law of Eq. (13). To prove the stability of the closed loop, the Lyapunov candidate function is expressed as following:

$$\left\{ V_{PI} = \frac{1}{2} S_3^2 ; \dot{V}_{PI} = S_3 \dot{S}_3 \right\} \quad (26)$$
By combining Eq. (7), (24), (25) and Eq. (26), one obtains
\[
\dot{V}_{pI} = S_3 \dot{S}_3 \\
= S_3(C_1 \dot{e} + C_2 e) = S_3(C_1(\dot{x}_2 - \dot{x}_2r) + C_2 e) = S_3(C_1(F_n(X, t) \\
+ G_n(X, t) \left\{ \left[ G_n(X, t) \right]^{-1} \left( -F_n(X, t) + \dot{x}_2r \right. \right. \\
- \frac{C_2}{C_1} e \left. \left. \right) - \frac{[\xi_1 + \xi_2 + K]}{c_1} \right] + D(X, t) - \dot{x}_2r + C_2 e) \\
\dot{V}_{pI} = -[\xi_1 + \xi_2 + K] S_3(sat(\frac{S_2}{\varphi})) + S_3 D(X, u, t) \\
\]  
(27)

Using Assumption 1 and by choosing \([\xi_1 + \xi_2 + K] > \chi\) it is easy to show that \(\dot{V}_{pI} \leq 0\).

**Case 3:** When \([0 < (\alpha + \beta) < 1]\) Then Eq. 14 represents the fractional order DI type sliding manifold and Eq. 16 is equivalent to a fractional order integral sliding mode control. For case 3, to prove the closed loop stability; the Lyapunov candidate function is defined as
\[
\begin{cases}
V = \frac{1}{2} S_3^2 \\
\dot{V} = S_3 \dot{S}_3
\end{cases}
\]  
(28)

By solving Eq. 15, 16 and 28 one obtains the following expression:
\[
\dot{V} = S_3 \left\{ C_1 D^\beta \left( \left[ G_n(X, t) \right]^{-1} \left( -F_n(X, t) + \dot{x}_2r - \frac{C_2}{C_1} D^{1-a-\beta} e \right. \right. \\
+ D(X, u, t) - \dot{x}_2r \right) \\
+ C_2 D^{1-a} e
\right\} \\
\]  
(29)

By simplifying the expression of Eq. 29 one obtains
\[
\dot{V} = S_3 \left\{ C_1 D^\beta \left( -\frac{C_2}{C_1} D^{1-a-\beta} e + D(X, u, t) - \frac{[\xi_1 + \xi_2 + K]}{c_1} D^{-\beta} sat(\frac{S_2}{\varphi}) \right) \\
+ C_2 D^{1-a} e \right\} \\
\]  
(30)

Using Definition 2 & 4: \(D^\beta, D^{1-a-\beta} = D^{1-a}\) and \(D^{-\beta} = D^0 = 1\), Then Eq. 30 is expressed as
\[
\dot{V} = S_3 \left\{ -C_2 D^{1-a} e - [\xi_1 + \xi_2 + K] sat(\frac{S_2}{\varphi}) + C_2 D^{1-a} e + C_1 D^\beta D(X, u, t) \right\} \\
\]  
(31)

Expression in Eq.31 is simplified as following:
\[
\dot{V} = -[\xi_1 + \xi_2 + K] S_3 sat(\frac{S_2}{\varphi}) + S_3 C_1 D^\beta D(X, u, t) \\
\]  
(32)

**Assumption 2:** Using Assumption 1, since the total uncertainty term \(D(X, u, t)\) is upper bounded so the following inequality holds; \(D^\beta D(X, u, t) \leq \varpi\). Here \(\varpi\) represents the upper bound of the fractional derivative of the uncertainty term.

Using Assumption 1 and Assumption 2, the first derivative of the Lyapunov function in Eq. 32 is expressed as \(\dot{V} \leq 0\) such that the following inequality holds; \([\xi_1 + \xi_2 + K] > \varpi\). From Eq. 32, the reaching condition of sliding surface is satisfied i.e. \(S_3 = 0\). To prove convergence property, Eq.14 is written as;
\[
D^\beta e = -C_1^{-1} C_2 D^{-\alpha} e \\
\]  
(33)
Multiply Eq. 33 by $D^{-\beta}$, one obtains
\[ D^{-\beta}(D^\beta e) = -C_1^{-1}C_2D^{-\alpha-\beta}e \] (34)

Using Lemma 1, Eq. 34 is expanded as;
\[ e - [t, D_t^\beta-1(e)]_{t=t_r} \frac{(t-t_r)^{\beta-1}}{\Gamma(\beta)} = -C_1^{-1}C_2D^{-\alpha-\beta}e \] (35)

At $t = t_r$, left hand side of Eq. 35 under fractional integration is equal to zero
\[ [t, D_t^\beta-1(e)]_{t=t_r} \frac{(t-t_r)^{\beta-1}}{\Gamma(\beta)} = 0 \] (36)

Using Eq. 36 and 35, one obtains
\[ e = -C_1^{-1}C_2D^{-\alpha-\beta}e \] (37)

Now using definition 2 and 4: one can express the error term as: $e = D^{-2}D^2e$, then Eq. 37 is expressed as
\[ D^{-2}(D^2e) = -C_1^{-1}C_2D^{-\alpha-\beta}e \] (38)

Using Lemma 1, Eq. 38 is written in the following form
\[ e(t) - [t, D_t^{\beta-1}e]_{t=t_r} \frac{(t-t_r)^{\beta-1}}{2} - e(t_r) = -C_1^{-1}C_2D^{-\alpha-\beta}e(t) \] (39)

Using Lemma 2, the right hand side of Eq. 39 is formulated as
\[ -C_1^{-1}C_2D^{-\alpha-\beta}e = -C_1^{-1}C_2Q||e(t)|| \] (40)

Here $Q$ is a positive constant. By combining Eq. 39 and 40 one obtains;
\[ ||e(t) - [t, D_t^{\beta-1}e]_{t=t_r} \frac{(t-t_r)^{\beta-1}}{2}|| - ||e(t_r)|| \leq -C_1^{-1}C_2Q||e(t)|| \] (41)

Since the tracking error $e(t = t_s) = 0$ then it is necessary to prove that $t_r \leq t_s < \infty$. Eq. 41 is re-written as
\[ ||[t, D_t^{\beta-1}e]_{t=t_s} (t_s-t_r)|| \leq 2||e(t_r)|| \] (42)

Simplifying Eq. 42 yields;
\[ t_r \leq \frac{2||e(t_r)||}{||\dot{e}|||_{t=t_r}} + t_s \] (43)

From Eq. 43, it is clear that the tracking errors of the closed loop system converge in finite time. This proves last part of Theorem 1.

4.1. Fractional operator tuning using fuzzy logic system. Apart from the mathematical proof of Theorem 1, it is crucial to design an online adaptive system that would map and optimally update the two fractional operator $\alpha$ and $\beta$ in the control expression of Eq. 16. In order for the proposed Theorem 1 to remain valid, the nonlinear mapping of the two fractional operators $\alpha$ and $\beta$ with respect to the sliding manifold $|S_3|$ is shown in Figure 1a and 1b. Since the mapping is nonlinear hence the ordinary adaptive laws are not an effective way for the online updating of the fractional operators [27]. To overcome these limitations, fuzzy logic system is utilized [28-29].

As shown in Figure 1a and 1b, the input to the fuzzy system is the absolute of sliding surface. The ideal curve of fractional operator $\alpha$ and $|S_3|$ is divided into three regions $A$, $B$ and $C$. The first region $A$ represents the start of the cycle when the error surface is large. In this region the integral action is either zero or very low since the sliding surface $|S_3|$ is large and outside the boundary layer $\phi$. In the second region $B$, the sliding surface is decreasing but greater than the boundary
layer width $\varphi$. In region B, the idea is to increase the order of fractional operator $\alpha$ very smoothly. The effects of the region B include the improved performance in terms of the rise and settling times. The third region C represents the dynamics of the sliding manifold when it is inside the boundary layer. In this region the order of fractional operator $\alpha$ is always one and hence a full integral action is applied inside the boundary layer. Fuzzy sets for the input and output variables are shown in Fig. 2a, 2b, 2c and Fig. 2d. Mathematically the operator $\beta$ is defined as

$$
\beta = \begin{cases} 
0 & \text{if } \alpha = 1 \text{ or } \alpha = 0 \\
\eta \alpha & \text{if } 0 < \alpha \leq 0.5 \\
\eta \alpha - 0.4 & \text{if } 0.5 < \alpha < 1 
\end{cases}
$$

(44)

Here $\eta$ represents the constant of proportionality. Output of the fuzzy controller is computed by if-Then rules of the form as given:

Rule (i): IF $x_i$ is $F(x_i)$ Then $y_i$ is $F(y_i)$

5. Results and discussions. PMSM motor is widely used in the industrial applications due to its low cost and simplicity. Compared with the brushed DC motors, absence of brushes and slip rings in PMSM motors make it less noisy. Secondly, the maintenance cost of the PMSM motors is lower and it can operate over a long period. Rotors of both the BLDC and PMSM motors are of the permanent magnet type; however, they have different types of back emf waveforms. A PMSM motor with sinusoidal back emf is easily driven by the sinusoidal excitation waveform and
Figure 2. (a): Fuzzy sets of input variable $|S_3|$(b): Fuzzy sets of output variable $\alpha$(c): Fuzzy sets of output variable $\beta$(d): Variation of $\beta$ with $\alpha$
hence torque ripples are significantly reduced. The mathematics model of a PMSM can be described as follows:

\[
\begin{aligned}
\dot{x}_1 &= x_2 \\
J \dot{x}_2 &= -Bx_2 + P\psi_f u + D(X, u, t)
\end{aligned}
\]  

(45)

Here the state variables \(x_1\) and \(x_2\) represent the angle and velocity of the PMSM servo system, \(u\) is the control effort, \(D(X, u, t)\) represents lumped uncertainty including the parametric uncertainty and external disturbances. \(B\) is the viscous friction coefficient, \(J\) is the motor moment inertia constant, \(P\) is number of pole pairs and \(\psi_f\) is the flux linkage of the permanent magnet. The nominal values of the system parameters are defined as \(B = 0.244 N.m.s.rad^{-1}\), \(J = 0.004 Kg.m^2\), \(P = 4\) and \(\psi_f = 0.514 Wb\). The controller parameters are selected as following: \(F_n = B/J = 61\), \(G_n = P\psi_f/J = 514\), \(C_1 = 0.5\), \(C_2 = 0.15\), \(\eta = 0.8\), \(\varphi = 0.01K + \xi_1 + \xi_2 = 15\), \(\beta = 0 \rightarrow 0.4\), \(K_I = 0.2\), \(K_P = 0.8\), \(D(X, u, t) = T_L + \Delta F + \Delta G = 2Nm + 3x_1^2 + 5x_2, T_L(\text{max}) = 2.5Nm\) and \(\alpha = 0 \rightarrow 1\). In the numerical simulations, PI current controller is used to regulate the inner current loop.

5.1. Speed regulation simulations and analysis using conventional control methods. The reference command of the speed controller is \(w_r(t) = 1000 RPM\). The tracking performance of the classical SMC is shown in Fig. 3(a). The feedback signal is assumed as noise free and the uncertainty term is zero, i.e. \(D(X, u, t) = 0\). Classical SMC ensures good regulation performance but the control signal suffers from high frequency chattering which is shown in Fig. 3(b). The linear sliding manifold converges to zero asymptotically as shown in Fig. 3(c).

![Figure 3. (a) speed regulation, (b) control signal, (c) sliding surface](image-url)

Fig. 4 shows zoomed view of the speed tracking, control signal and sliding manifold. From the numerical results presented in Fig 4(a), 4(b) and 4(c) it is clear that both the control signal as well as sliding surface suffers from high frequency oscillations which means that the method is not suitable for practical applications.

In Fig. 5, the comparative analysis of the speed regulation for the PMSM is presented using classical SMC, SMC with boundary layer and integral SMC. For
this case the lumped disturbance term is set to $D(X, u, t) = T_L + \Delta F + \Delta G = 2Nm + 3x_1^2 + 5x_2$. The tracking error comparisons are given in Fig. 5(a). From the numerical results presented it is concluded that under the action of both the classical SMC and SMC with boundary layer the system’s transient performance is ensured while in case of ISMC there is a large transient overshoot which is undesirable. Similarly the overshoot is dominant in the control signal and sliding surface simulations presented in Fig. 5(b) and Fig. 5(c). The lumped uncertainty compensation and chattering phenomena comparisons are given in Fig. 6. From Fig. 6(a) it is concluded that the system under the SMC with boundary layer show steady state error, while in the case of ISMC the steady state error eventually become zero. Control signal comparisons are shown in Fig. 6(b). Since ISMC also utilize boundary layer so the control signal is chattering free and it is comparable to the boundary layer SMC. Similar observations are found in the sliding surface simulations shown in Fig. 6(c). Although ISMC performs better in the context of the steady state error and chattering minimization but at the same time system’s transient response degrades. Fig. 7 and 8 compares the speed regulation performance of the system using classical SMC, SMC with boundary layer and ISMC. The system is subject to the uncertainty term expressed as following: $D(X, u, t) = T_L + \Delta F + \Delta G = 2Nm + 3x_1^2 + 5x_2$ and white noise. For this, the noise power is chosen as 0.001 and initial seed is set to 100. Fig. 7(a) shows the transient performance of the system under different control schemes. Fig. 7(b) and 7(c) shows the control signals and sliding surface comparisons under all three variants of the controllers. For in depth analysis and to show the effect of noise on the speed regulation performance in the steady state, the enlarged view of the simulations is presented in Fig. 8. From Fig. 8(a), it is concluded that the tracking performance under classical SMC and boundary layer SMC degrades with noisy feedback while ISMC shows good performance.
In case of ISMC, the integrator acts as low pass filter for random noise so the PI sliding surface is robust to noise as compared to the other two methods, thus it gives better control performance. The control signal comparison is given in Fig. 8(b). In case of ISMC, the control signal contains less chattering and the same is true for sliding surfaces as shown in Fig. 8(c).

5.2. Speed regulation simulations and analysis using proposed control.
From the above analysis it is concluded that the system with uncertainty and noisy feedback shows good performance under the action of ISMC as compared to classical SMC and SMC with boundary layer. However ISMC has two limitations. First the transient performance of the system degrades and secondly chattering phenomena still exists with noisy feedback.

To show the effectiveness of the proposed control scheme under the uncertainty and noisy feedback, the reference command is chosen as $w_r(t) = 1000\, \text{RPM}$. The parameters of measurement noise are selected as following: noise power= 0.001, initial seed = 100. The lumped uncertainty term is expressed as $D(X, u, t) = T_L + \Delta F + \Delta G = 2Nm + 3x_1^2 + 5x_2$. The uncertainty is applied at $t = 20\, \text{sec}$.

Fig. 9 shows the comparison of the speed tracking performance using the proposed method and ISMC. From Fig. 9(a), it is concluded that the proposed method offers smaller overshoot in the transient time as compared to the ISMC. As shown in Fig. 9(b), under the proposed control scheme, the tracking error and sliding surface converge in shorter time. Under the action of the proposed control scheme, the system’s settling time is 1 sec with 5 percent overshoot while in case of the ISMC the setting time is 5 sec with 20 percent overshoot. The control signal comparisons are shown in Fig. 9(c). From the results it is concluded that the overshoot in the
Figure 6. Enlarged view of (a) speed error, (b) control signal, (c) sliding surface with $D(X,u,t)$

Figure 7. (a) speed error, (b) control signal, (c) sliding surface with $D(X,u,t)$ and measurement noise
control signal also reduces significantly in the transient time. Fig. 10 shows the uncertainty rejection performance and chattering minimization phenomena of the proposed method. From Fig. 10(a) it is clear that the uncertainty rejection ability of the proposed method is comparable to that of ISMC, while from Fig. 10(b) and (c) it is clear that with proposed control method the chattering phenomenon is significantly minimized. Figure 11 shows the adaptation of the fractional orders
Figure 10. (a) Speed error (b) Control signal (c) Sliding surface with $D(X, u, t)$ and measurement noise

Figure 11. (a) Adaptation of $\alpha$ (b) Adaptation of $\beta$

From the presented results it is clear that the proposed fuzzy controller ensures the nonlinear mapping of $\alpha$ and $\beta$ appropriately. The adaptation results closely resemble to the nonlinear mapping set as benchmark shown in Fig. 1a and 1b.
6. Conclusion. In this article a fuzzy tipple mode sliding mode control is proposed for the speed regulation of PMSM motor and its stability is guaranteed using Lyapunov method. With noisy feedback the performance of the classical SMC and SMC with boundary layer degrades significantly while ISMC shows better performance. However using ISMC large transient overshoots are introduced in the speed regulation, thus it leads to large transient errors. The proposed method ensured good transient and steady state performance. Moreover chattering phenomena has minimized due to the inclusion of the fractional dynamics. Numerical simulations confirms the superiority of the proposed method over integer order methods.

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Received December 2019; revised January 2020.

E-mail address: nasimullah@tu.edu.sa
E-mail address: aziz@tu.edu.sa