On the non-relativistic structure of the AdS/CFT superalgebras

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Abstract
The property of the conformal algebra to contain the Schrödinger algebra in one less space dimension is extended to the supersymmetric case. More precisely, we determine the non-relativistic counterpart of any field theory admissible superconformal algebra. Even if each type of superalgebra provides a different solution, its basis decomposition into two copies of the super Schrödinger algebra, differing only by their super Heisenberg part, remains valid in all cases, so generalizing a feature already observed in the non-supersymmetric conformal case.

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1. Introduction

Soon after the discovery of the Schrödinger algebra, defined as the maximal kinematical symmetry of the Schrödinger equation [1] (see also [2]), it was realized that such an algebra, defined in \( d \) space and 1 time dimensions, is embedded in the conformal algebra acting on the Minkowski space with \((d+1)\) space and 1 time dimensions [3]. Actually, using a well-adapted basis for the conformal algebra, two conjugated Schrödinger algebras \( \tilde{S}ch(d-1, 1) \), with a common part constituted by the rotations and conformal transformations, can be recognized in \( SO(d+1, 2) \) in a way analogous to the decomposition of the conformal algebra into two conjugate Poincaré ones. We recall that in the last case, the conformal algebra can be seen as the direct sum of the Lorentz algebra plus a dilatation, to which, on one side, the \((d+1)\) translations and, on the other side, the \((d+1)\) special conformal transformations can be added, each set constituting a Poincaré-like algebra.
In the last few years, there have been a lot of activities in studying, within the framework of AdS/CFT correspondence, the string theory embedding of a spacetime with non-relativistic Schrödinger symmetry [4]. Indeed, it is at first tempting, due to the difficulty of solving the general problem, to look for a limit—actually a non-relativistic one—of the AdS superstring. This idea has led to the study of some gravity backgrounds, candidates to be gravity dual of conformal quantum mechanical systems. However, even more interestingly, such an approach is particularly useful in the context of several condensed matter systems [5, 6]. Indeed, the microscopic description of many condensed matter systems in the vicinity of a quantum critical point is certainly non-relativistic, but exhibits conformal invariance [7]. Let us emphasize, at this point, that the treatment of strongly anisotropic critical systems with the help of Schrödinger invariance had already been proposed, about 15 years ago, in [8].

Such an interest in the Schrödinger invariance naturally raises the question of its supersymmetric extension, and more precisely the embedding of such a supersymmetric analog in the superconformal algebras. But then, it is natural to ask whether an intrinsic definition of the super Schrödinger algebra can be given, the Schrödinger algebra having been introduced as the maximal kinematical invariance algebra of the Schrödinger equation. A first answer to this question has been given in [9] where the non-relativistic spin-1/2 particle action is constructed. In about the same period, an extended superconformal Galilean algebra, which slightly differs from the one of [9], was proposed in the context of the non-relativistic limit of $N = 2$ supersymmetric Chern–Simons matter systems, and therefore in the special case of $d = 2$ space dimensions, see [10]. Such a difference was clarified by the authors of [11] who performed a more direct algebraic construction of supersymmetric extensions of the Schrödinger algebra; we will come back to this approach in section 3. At this point, it is interesting to note that Schrödinger supersymmetries have more recently been considered in different contexts. Among them, let us mention [12] where it has been shown that there are infinite classes of supergravity solutions of type IIB exhibiting such a type of symmetry, and also [13] where a representation of an $N = 2$ superalgebra was constructed in the context of many-body quantum mechanics.

The list of admissible superconformal algebras has been known for a long time, and can be easily selected from the simple superalgebras in the Nahm classification of manifest supersymmetries [14]. They are the simple unitary superalgebras $SU(2, 2/N)$, the orthosymplectic ones $Osp(8^*/N)$, with $N$ even, and $Osp(M/4, R)$ and the real form commonly denoted$^4$ as $F(4; 2)$ with the bosonic part $SU(2) \oplus SO(5, 2)$ of the exceptional superalgebra $F(4)$. One obviously recovers in $pSU(2, 2/4)$ the supersymmetry algebra of the 10-dimensional type 2 supergravity, and in $Osp(8/4)$ the one of the 11-dimensional M-theory with corresponding spacetime manifolds $AdS_5 \times S^5$ and $AdS_7 \times S^4$, respectively, while $Osp(8/4, R)$ is related to the M-theory with spacetime manifold $AdS_4 \times S^7$. Finally, the exceptional solution has also been considered and the $AdS(6)/CFT(5)$ correspondence investigated for an $F(4; 2)$ supergravity theory [16].

The purpose of this paper is to determine, for each admissible superconformal symmetry, its associated super Schrödinger algebra, a natural extension of the Schrödinger part contained in the conformal algebra. Special care will be taken to present each algebra in an explicit form, that is as a semi-direct sum of a reductive part, i.e. a direct sum of simple (super) algebras, eventually with $U(1)$ factors, acting on a super Heisenberg part. Moreover, it will be shown that each such superconformal algebra contains a couple of super Schrödinger algebras, a natural extension of the couple of Schrödinger algebras embedded in its bosonic conformal

$^4$ The four real forms of $F(4)$ are often denoted as $F(4; p)$, with $p = 0, 1, 2, 3$, the complete bosonic part of $F(4; p)$ being $SO(2, R) \oplus SO(p, 7 - p)$ for $p = 0, 3$ and $SU(2) \oplus SO(p, 7 - p)$ for $p = 1, 2$. We note at this point a misprint in table 3.75 of [15].
part. At this point, we must mention [17], where the problem of finding a super Schrödinger algebra in each of the three superalgebras $pSU(2, 2/4)$, $Osp(8^* / 4)$ and $Osp(6, 2/4)$ has recently been considered using adequate projections on the spinorial fermionic sectors. We agree with the presented results, but would like to insist on the more general aspects of our approach, which, as just mentioned above, propose a complete as possible characterization of super Schrödinger algebras and also includes the $F(4, 2)$ case.

The plan of our paper is as follows. We start, in section 2, with a reminder on the Schrödinger algebra showing up as a subalgebra of the conformal one in one more space dimension, and some comments on the special position of this subalgebra inside the conformal symmetry. We comment, in section 3, on the super Schrödinger symmetries already discovered in physical situations and the attempt to recognize them in a supersymplectic framework, as proposed in [11]; our method of determination of the super Schrödinger algebras inside superconformal ones is also rapidly summarized. A separate section is devoted to the construction of the Schrödinger counterpart for each family of superconformal algebras: that is the unitary $SU(2, 2/N)$ for $N$ different from 4, and $pSU(2, 2/4)$ superalgebras with conformal symmetry in $3 + 1$ dimensions (section 4); the orthosymplectic $Osp(N/4, R)$ in $2 + 1$ dimensions (section 5), the orthosymplectic $Osp(6, 2/2N)$ in $5 + 1$ dimensions (section 6), and finally, the exceptional $F(4, 2)$ algebra in $5 + 1$ spacetime dimensions (section 7). A comparison between the different types of obtained Schrödinger superalgebras (section 8) concludes the paper.

2. Schrödinger algebra inside conformal algebra

As first remarked in [3], an adequate choice of basis for the conformal algebra acting on the Minkowski space $M(d, 1)$ allows us to identify—up to an isomorphism—the Schrödinger algebra $\mathfrak{Sch}(d – 1)$, i.e. in $(d – 1)$ space dimensions and in 1 time dimension, as a subalgebra of $SO(d + 1, 2)$. Actually, two copies of $\mathfrak{Sch}(d – 1)$, one conjugate to the other and sharing a semi-simple common part, can be used for reconstructing, up to one-dimensional generator, the conformal algebra. We hereafter explicate the (most common) case $d = 3$, the generalization to any $d$ being straightforward. So, let us consider the $SO(4, 2)$ algebra generated by the 15 skew-symmetric elements $M_{\mu \nu}$ ($\mu, \nu = 0, 1, 2, 3, 4$) such that

$$[M_{\mu \nu}, M_{\alpha \beta}] = g_{\mu \sigma} M_{\nu \beta} + g_{\nu \sigma} M_{\mu \beta} - g_{\nu \alpha} M_{\mu \sigma} - g_{\mu \alpha} M_{\nu \sigma}. \quad (1)$$

The metric $g_{\mu \nu}$ satisfies

$$g_{\mu \nu} = 0, \quad \mu \neq \nu, \quad g_{\mu \mu} = 1, \quad \mu = 0, 1, 2, 3, 4. \quad (2)$$

We note that the $M_{\mu \nu}$, where $\mu, \nu = 0, 1, 2, 3$, can be seen as the generators of the Lorentz algebra. Then, the combinations

$$p_\mu = M_{0 \mu} + M_{\mu 0} \quad \text{and} \quad c_\mu = M_{0 \mu} - M_{\mu 0}, \quad \mu = 0, 1, 2, 3, \quad (3)$$

correspond to the translations and special conformal transformations, each set forming, with $SO(3, 1)$, a Poincaré-like algebra, and these two-dimensional subalgebras being conjugate in $SO(4, 2)$. As a last generator, we have $M_{0 4}$, acting as a dilatation on the $p_\mu$s and $c_\mu$s and commuting with the Lorentz part. One can first select, in (one of) the Poincaré parts, the two-dimensional extended Galilean algebra by performing a change of basis, already used around the 1970s in the infinite momentum formalism. Let us define

$$H = \frac{1}{2} (p_0 - p_3), \quad M = p_0 + p_3, \quad J_3 = M_{12}, \quad (4)$$

$$K_a = -(M_{0a} + M_{3a}), \quad p_a = p_a, \quad a = 1, 2, \quad (5)$$

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satisfying the non-vanishing commutation relations:

\[ [J_3, P_a] = \epsilon_{3ab} P_b, \quad [H, K_a] = -K_a, \quad a, b = 1, 2, \]  
\[ [J_3, K_a] = \epsilon_{3ab} K_b, \quad [K_a, P_b] = -\delta_{ab} M. \]  

To this Galilean algebra, the two generators

\[ C = \frac{1}{2}(c_0 + c_3) \quad \text{and} \quad D = M_{04} - M_{03} \]  

can be added which, together with \( H \), defined in equation (4), close into an algebra isomorphic to \( SU(1, 1) \equiv SO(2, 1) \),

\[ [C, H] = D, \quad [D, C] = 2C, \quad [D, H] = -2H. \]  

We note that this \( SU(1, 1) \) algebra commutes with \( J_3 \) and \( M \), and acts on the \( H(2) \) Heisenberg part generated by \( P_a \), \( K_a \) and \( M \) as follows:

\[ [D, P_a] = -P_a \quad [C, P_a] = K_a \quad [D, K_a] = K_a. \]  

The five-dimensional \( H(2) \) Heisenberg together with the \( SO(2) \) algebra generated by \( J_3 \) and the \( SU(1, 1) \) defined in equation (9) span a nine-dimensional \( SO(4, 2) \) subalgebra which is isomorphic to the (extended) Schrödinger algebra \( \tilde{Sch}(2) \) in 2 space and 1 time dimensions. It can be conveniently seen as the semi-direct sum of the \( SO(2) \) rotation algebra and the \( SU(1, 1) \) conformal part acting on the \( H(2) \) Heisenberg algebra:

\[ \tilde{Sch}(2) = [SO(2) \oplus SU(1, 1)] \triangleright H(2). \]  

Such an inclusion among algebras can be formally illustrated by rewriting the d’Alembert equation

\[ p^2 |0\rangle = 0, \quad p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 \]  

with \( |0\rangle \) being the state of a relativistic massless particle in the momentum space

\[ (H - \frac{p^2}{M}) |0\rangle = 0 \]  

where \( p_\perp = (p_1, p_2) \) for states such that \( M^2 |0\rangle \neq 0 \). We recognize in equation (13) the Schrödinger equation for a free massive non-relativistic particle in one less space dimension and invariant under \( \tilde{Sch}(2) \). Now in the same way a \( \tilde{Gal}(2) \) Galilean algebra can be extracted from the Poincaré algebra generated by \( M_{\mu\nu} \) and \( p_\mu \); a \( \tilde{Gal}^\circ(2) \) algebra can obviously be obtained from the set \( \{M_{\mu\nu}, c_\mu\} \). The \( \tilde{Gal}(2) \) algebra is conjugate to \( \tilde{Gal}(2) \) in \( SO(4, 2) \) and is simply constructed by adding to the previously introduced rotation \( J_3 \) and time-translation \( H \) generators the elements

\[ p_a^* = M_{0a} - M_{3a}, \quad K_a^* = c_a, \quad M^* = c_0 - c_3, \quad a = 1, 2. \]  

The five generators of equation (14) form a Heisenberg algebra that we denote \( H^*(2) \). Then, keeping the same \( SU(1, 1) \) conformal algebra generated by \( H, C \) and \( D \), one gets a second \( (2 + 1) \)-dimensional Schrödinger algebra \( \tilde{Sch}^\circ(2) \):

\[ \tilde{Sch}^\circ(2) = [SO(2) \oplus SU(1, 1)] \triangleright H^*(2) \]  

with \( \tilde{Sch}(2) \) and \( \tilde{Sch}^\circ(2) \), conjugate one to the other, differing therefore by their Heisenberg part. Finally, as a vector space (v.s.), one can decompose the algebra \( SO(4, 2) \) as follows:

\[ SO(4, 2) =_{v.s.} SO(2) \oplus SO(2, 1) \oplus H(2) \oplus H^*(2) \oplus SO(1, 1) \]

\footnote{We denote by \( \triangleright \) the semi-direct sum.}
with an $SO(1, 1)$ generator
\[ \Lambda = -(M_{04} - M_{03}) \] (17)
itself acting as a scale transformation on the Heisenberg algebra elements:
\[ [\Lambda, P_a] = P_a, \quad [\Lambda, P_a^*] = -P_a^*; \]
\[ [\Lambda, K_a] = K_a, \quad [\Lambda, K_a^*] = -K_a^*; \] (18)
in a way completely analogous to the dilatation $M_{04}$ acting on the translations $P_a$ and special conformal transformations $C_\mu$. Let us note that the three generators $\Lambda$, $M$ and $M^*$ close into an $SU(1, 1) \equiv SO(2, 1)$ algebra, see equation (18) and
\[ [M, M^*] = -4\Lambda. \] (19)
This $SU(1, 1)$ algebra commutes with the conformal $SU(1, 1)$ defined above in equation (9), as well as with the rotation $SO(2)$. So, we recognize the embedding
\[ SO(4, 2) \supset SO(2) \oplus SO(2, 2) = SO(2) \oplus SO(2, 1) \oplus SO(2, 1). \] (20)
The generalization to $d \neq 3$ of the decomposition (20) is straightforward and, in particular, equations (15) and (20) become, respectively,
\[ \tilde{\text{Sch}}(d - 1) = (SO(d - 1) \oplus SO(2, 1)) \triangleright H(d - 1) \] (21)
\[ SO(d + 1, 2) \supset SO(d - 1) \oplus SO(2, 2) = SO(d - 1) \oplus SO(2, 1) \oplus SO(2, 1). \] (22)
Considering the algebras as vector spaces, we can write, generalizing equation (16),
\[ SO(d + 1, 2) \cong_{v.s.} SO(d - 1) \oplus SO(2, 1) \oplus H(d - 1) \oplus H^*(d - 1) \oplus SO(1, 1). \] (23)
As a last remark, let us note that the Schrödinger algebra $\tilde{\text{Sch}}(d - 1)$ can be seen as the stabilizer of the $M$ generator, i.e. the set of elements commuting with $M$, in the conformal $SO(d + 1, 2)$ algebra.

3. Schrödinger algebra and supersymmetry

Supersymmetric extensions of the Schrödinger algebra have been considered in different ways, leading to two types of (super) symmetry algebras. Superalgebras of the first family—which we will call the `orthosymplectic type'—hold for any (integer) $N$ supersymmetries and $d$ space dimensions, while those of the second one—which we will call the 'unitary type'—work for any $N$, but only in the case of $d = 2$ space dimensions. In both cases, the time dimension is 1. First exhibited in the study of the supersymmetric harmonic oscillator [18], in [9] the orthosymplectic type appeared as the symmetry of the non-relativistic spin-1/2 action. Later the unitary type was found by the authors of [10] as the symmetry of the $(d = 2)$ non-relativistic Chern–Simons matter systems extending in this way the Schrödinger symmetry discovered before in [19] for this theory. Actually an elegant way to recognize, in the same framework, these two types of symmetry algebras has been proposed in [11], where a geometrical symplectic approach is used. Indeed, the structure of the (extended) Schrödinger algebra $\tilde{\text{Sch}}(d)$ may suggest to consider $\tilde{\text{Sch}}(d)$ as a subalgebra of the (central extended) algebra of the inhomogeneous symplectic transformations $\tilde{\text{Sp}}(2d)$, i.e. the semi-direct sum of the symplectic algebra $Sp(2d, R)$ acting on the Heisenberg part $H(d)$:
\[ \tilde{\text{Sp}}(2d) \equiv Sp(2d, R) \triangleright H(d). \] (24)

\[ \text{In our notation } Sp(2d, R) \text{ is the symplectic algebra of rank } d. \]
As shown in [11], the natural supersymmetric extension \( \widetilde{Osp}(N/2d) \) is defined as

\[
\widetilde{Osp}(N/2d) \equiv Osp(N/2d, R) \triangleright SH(d/N)
\]

(25)

where \( Osp(N/2d, R) \) is a natural supersymmetric extension of \( Sp(2d, R) \), and \( SH(d/N) \) is defined as the super Heisenberg algebra obtained from \( H(d) \) by adding the (fermionic) super-translations generators \( \Xi_a^j \) \( (a = 1, \ldots, d; \ j = 1, \ldots, N) \), which commute with all the \( H(d) \) generators:

\[
[\xi_a, \Xi_b^j] = [K_a, \Xi_b^j] = [M, \Xi_b^j] = 0
\]

(26)

and satisfy

\[
[\Xi_a^i, \Xi_b^j] = \delta^i_j \delta_{ab} M \quad a, b = 1, \ldots, d; \ i, j = 1, \ldots, N.
\]

(27)

The next and final step stands in looking for supersymmetric extensions of \( \widetilde{Sch}(d) \) in \( \widetilde{Osp}(N/2d) \), via the canonical contact structure on \( R^{2d+1} \) suitably extended with the generators \( \xi^j (j = 1, \ldots, N) \) of the Grassmann algebra \( \wedge R^N \). As a result, two families of Schrödinger superalgebras have been proposed, each of them valid for any positive (or null) integer value of \( N \).

- **Algebras of the ‘orthosymplectic type’**

\[
[SO(d) \oplus Osp(N/2, R)] \triangleright SH(d/N)
\]

(28)

valid for any value of the positive integer \( d \). \( SO(d) \) is the rotation algebra acting on the translations, super-translations and Galilean boosts. The \( Sp(2, R) \) algebra commuting with \( SO(N) \) in the bosonic sector of \( Osp(N/2, R) \) is actually the ‘conformal’ part, generated by \( H, C \) and \( D \), of the Schrödinger algebra defined in section 2. It is this kind of Schrödinger superalgebra which shows up in [9] for \( N = 1 \), and in [18] for \( N = 2 \).

- For the special value \( d = 2 \), another kind of superalgebra has been detected. From the set of commutation and anti-commutation relations given in [11], one can recognize the algebra

\[
[SO(2) \oplus SU(N/1, 1)] \triangleright SH(2/N)
\]

(29)

where the \( R \)-symmetry standing in the \( SU(N/1, 1) \) bosonic sector is now an \( SU(N) \) algebra. It might be useful to note the doubling in the number of fermionic generators in going from \( Osp(N/2, R) \) to \( SU(N/1, 1) \). At this point let us mention that \( SO(2) \oplus SU(N/1, 1) \) is determined as a subalgebra\(^7\) of \( Osp(2N/2, R) \). It is a superalgebra of this kind which has been discovered in [10]. We will, rather naturally, denote such superalgebras as Schrödinger superalgebras of the unitary type.

Let us emphasize that such orthosymplectic, equation (28), as well as unitary, equation (29), structures will show up as super counterparts of the superconformal \( Osp(N/4, R) \) and \( SU(2, 2/N) \) algebras\(^8\), respectively; see sections 4 and 5 below. But slightly different configurations will be detected in \( Osp(6, 2/2N) \) and \( F(4; 2) \) superalgebras, as explicated in sections 6 and 7.

Of course, the method for determining the symmetries used in [11] is not adapted to our problem. As remarked in the previous section, at the bosonic level, the Schrödinger algebra is simply the stabilizer in the conformal algebra of the mass generator \( M \). It is this property that we plan to extend to the supersymmetric case. More precisely, our way of proceeding will consist, given an admissible superconformal algebra, of first characterizing the \( M \) generator in the bosonic part, and then determining its stabilizer in the whole superalgebra.

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7 See also [20] for a study of the maximal embedding: \( Osp(2m/2n, R) \triangleright SU(m/p, q) \oplus U(1) \) with \( p + q = n; \ p, q \geq 0 \).

8 As can be seen in section 4, the case \( N = 4 \) presents a peculiarity, then starting from \( pSU(2, 2/4) \) the Schrödinger symmetry is simply \( SU(4/1, 1) \triangleright SH(2/4) \), i.e. the extra \( SO(2) \) of equation (29) is no longer present.
4. The cases of $SU(2, 2/N)$ with $N \neq 4$ and $pSU(2, 2/4)$

Any element $M$ of $SU(2, 2/N)$ can be written in the matrix form, see [20], as
\[(M^*)^t L_{4,N} + L_{4,N} M = 0 \iff M = -L_{4,N}^{-1} (M^*)^t L_{4,N}\]
(30)
where $M \in GL(4/N)$ is given by the following matrix:
\[M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}.\]
(31)
* denotes the complex conjugation and the superscript ‘st’ denotes the supertransposition:
\[M^{st} \equiv \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}.\]
(32)
Here, $A$ is a $4 \times 4$ matrix, $D$ is an $N \times N$ matrix, and $B$ and $C$ are, respectively, the $4 \times N$ and $N \times 4$ matrices. $A$ and $B$ are even grading, and $C$ and $D$ are odd grading, with the condition
\[
\text{tr } A = \text{tr } D,
\]
(33)
and $L_{4,N}$ is defined by
\[L_{4,N} \equiv \begin{pmatrix} 1_{2,2} & 0 \\ 0 & -i1_N \end{pmatrix}\]
(34)
where $1_{2,2}$ is
\[1_{2,2} \equiv \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}\]
(35)
with $1_N$ being the identity matrix in $N$ dimensions. From equation (30)
\[B = i1_{2,2} C^\dagger \iff C = iB^\dagger 1_{2,2}\]
(36)
and the $A$ and $D$ matrices satisfy
\[A = -1_{2,2} A^\dagger 1_{2,2}, \quad D = -D^\dagger\]
(37)
where $\dagger$ stands for the Hermitian conjugation. The 15 generators in $A$ split into two sets of compact and non-compact generators which are, respectively, given by $(j, k = 1, 2, 3)$
\[
\begin{pmatrix} i\sigma_j & 0 \\ 0 & -i\sigma_j \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & i\sigma_j \end{pmatrix}, \quad \begin{pmatrix} i1_2 & 0 \\ 0 & -i1_2 \end{pmatrix}
\]
(38)
\[
\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i\sigma_k \\ -i\sigma_k & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i1_2 \\ -i1_2 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}
\]
(39)
One recognizes the maximal compact subalgebra $SU(2) \oplus SU(2) \oplus U(1)$ in equation (38), the Lorentz algebra $SO(3, 1)$ generated by the rotations $J_j$ and boosts $B_k$:
\[J_j = \begin{pmatrix} i\sigma_j & 0 \\ 0 & -i\sigma_j \end{pmatrix}\]
(40)
\[B_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}.\]
(41)
Finally, the dilatation $d$, the translations $p_\mu$ and the special conformal transformations $c_\mu$ ($\mu = 0, 1, 2, 3$) can be chosen as
\[ d = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \]  

(42)

\[ p_j = i \begin{pmatrix} \sigma_j & \sigma_j \\ -\sigma_j & -\sigma_j \end{pmatrix} \quad p_0 = i \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]  

(43)

\[ c_j = i \begin{pmatrix} \sigma_j & -\sigma_j \\ \sigma_j & -\sigma_j \end{pmatrix} \quad c_0 = i \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \]  

(44)

Now, it is useful to introduce the matrices \( E_{mn} \) with entries \( e_{kl} = \delta_{mk}\delta_{nl} \), which satisfy, from the property 
\[ [E_{mn}, E_{kl}] = \delta_{nk} E_{ml} \]  

(45)

Then, the generators of the \( D \) part can be chosen as
\[ E_{4+p,4+q} \quad p, q = 1, \ldots, N, \quad p \neq q, \]
\[ i(E_{4+p,4+q} + E_{4+q,4+p}), \]
\[ i(E_{4+r,4+rr} - E_{5+rr,5+rr}), \quad r = 1, \ldots, N - 1. \]  

(46)

There is one last generator in the ‘block-diagonal’ part of \( A + D \) that one can choose, in order to ensure the supertrace condition, equation (33), as
\[ X_N = i \left( N \sum_{i=1}^{4} E_i + 4 \sum_{p=1}^{N} E_{4+p,4+p} \right). \]  

(47)

In the case \( N = 4 \), this diagonal generator obviously becomes a multiple of the identity and then has to be eliminated. Considering the quotient of \( SU(2,2/4) \) by the one-dimensional ideal generated by \( X_4 \), one naturally gets \( pSU(2,2/4) \).

A basis for the ‘anti-diagonal’ \( B \oplus C \) can also be commonly obtained in terms of \( E_{il} \) matrices: \( (a = 1, 2; \quad p = 1, \ldots, N) \):
\[ E_{a,4+p} + iE_{4+p,a}, \quad iE_{a,4+p} + E_{4+p,a}, \]
\[ E_{a+2,4+p} - iE_{4+p,a+2}, \quad iE_{a+2,4+p} - E_{4+p,a+2}. \]  

(48)

In order to identify the elements of the Schrödinger algebra in the notations of [3], as explicated in section 2, we introduce the following generators:
\[ J = \frac{i}{2} (\sigma_3) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \]  

(49)

\[ H = \frac{1}{2} (p_0 - p_3) = \frac{i}{2} (1 + \sigma_3) \otimes \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \]  

(50)

\[ C = \frac{1}{2} (c_0 + c_3) = \frac{i}{2} (1 + \sigma_3) \otimes \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \]  

(51)

\[ D = \frac{1}{2} (1 + \sigma_3) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \]  

(52)

Here, \( J \) is the generator of an \( SO(2) \) algebra, and \( H, C \) and \( D \) span an \( SU(1,1) \) algebra:
\[ [C, H] = -4D, \quad [D, C] = 2C, \quad [D, H] = -2H. \]  

(53)
Let us define

\[ K_1 = \frac{1}{2} \begin{pmatrix} i\sigma_2 & i\sigma_1 \\ i\sigma_1 & -i\sigma_2 \end{pmatrix}, \quad K_2 = -\frac{1}{2} \begin{pmatrix} i\sigma_1 & -i\sigma_2 \\ -i\sigma_2 & i\sigma_1 \end{pmatrix}, \] (54)

\[ P_1 = \frac{i}{2} \sigma_1 \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad P_2 = \frac{i}{2} \sigma_2 \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \] (55)

The \( K_a, P_a \) \((a, b = 1, 2)\) define a two-dimensional Heisenberg algebra \( H(2) \):

\[ [K_a P_b] = \delta_{ab} M, \quad [M, P_a] = [M, K_a] = 0 \] (56)

where \( M = \frac{1}{2} (p_0 + p_3) = \frac{i}{2} (1 - \sigma_3) \otimes \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \) (57)

We also have the commutations relations \((a, b = 1, 2)\)

\[ [\Lambda, P_a] = -2 P_a, \quad [\Lambda, K_a] = -2 K_a, \] (58)

\[ [J_3, P_a] = \varepsilon_{3ab} P_b, \quad [J_3, K_a] = -\varepsilon_{3ab} K_b \] (59)

where \( \Lambda = \frac{i}{2} (1 - \sigma_3) \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \) (60)

The second two-dimensional Heisenberg algebra \( H^*(2) \) is spanned by

\[ K_1^* = \frac{i}{2} \sigma_1 \otimes \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad K_2^* = \frac{i}{2} \sigma_2 \otimes \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \] (61)

\[ P_1^* = \frac{1}{2} \begin{pmatrix} i\sigma_2 & -i\sigma_1 \\ -i\sigma_1 & i\sigma_2 \end{pmatrix}, \quad P_2^* = -\frac{1}{2} \begin{pmatrix} i\sigma_1 & \sigma_2 \\ \sigma_2 & i\sigma_1 \end{pmatrix}, \] (62)

\[ M^* = \frac{i}{2} (1 - \sigma_3) \otimes \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}. \] (63)

\[ [K_a^*, P_b^*] = \delta_{ab} M^*, \quad [M^*, P_a^*] = [M, K_a^*] = 0. \] (64)

Note that the elements \( M, M^* \) and \( \Lambda \) generate a (second) \( SU(1, 1) \) algebra, which commutes with the \( SU(1, 1) \) generated by \( D, C \) and \( H \). Their commutation relations read

\[ [\Lambda, M] = -2M, \quad [\Lambda, M^*] = 2M^*, \quad [M, M^*] = -4\Lambda, \] (65)

the dilatation \( \Lambda \) acting on \( H_2 \) and \( H_2^* \) as \((a, b = 1, 2)\)

\[ [\Lambda, P_a] = -P_a, \quad [\Lambda, P_a^*] = P_a^*, \quad [\Lambda, K_a] = -K_a, \quad [\Lambda, K_a^*] = K_a^*. \] (66)

Finally, the commutation relations between \( H(2) \) and \( H^*(2) \) generators

\[ [K_a K_b^*] = \delta_{ab} C, \quad [P_a, P_b^*] = \delta_{ab} H, \quad [P_a, K_b^*] = -2\varepsilon_{ab3} J_3 + \delta_{ab}(D + \Lambda) \] (67)

provide the \( SO(2) \oplus SU(1, 1) \oplus SO(1, 1) \) algebra generated by \( J, H, C, D \) and \( \Lambda \), respectively.
Let us emphasize that the $SO(2) \oplus SU(1, 1)$ part is the common bosonic algebra of the two Schrödinger algebras $\mathcal{S}ch(2)$ and $\mathcal{S}ch^*(2)$ of, respectively, equations (11) and (15).

Now, let us turn our attention to the fermionic generators of the $SU(2, 2/N)$ superalgebra. It is straightforward to note that the $4N$ generators, which we can group into the four $N$-vectors of the $SU(N)$ part ($p = 1, \ldots, N$),

\[
\begin{align*}
Q^1_p &= E_{1,4+p} + iE_{4+p,1} \\
Q^2_p &= iE_{1,4+p} + E_{4+p,1} \\
S^1_p &= E_{3,4+p} - iE_{4+p,3} \\
S^2_p &= iE_{3,4+p} - E_{4+p,3}
\end{align*}
\]

(68)

commute simultaneously with $M$ and $M^\ast$. We remark also that each couple $(Q^1_p, Q^2_p)$ and $(S^1_p, S^2_p)$ form a doublet under the $J$-rotation. There are $2N$ more generators commuting with $M$ (but not with $M^\ast$) that we can also consider as two $SU(N)$ vectors or $NJ$-$SO(2)$ doubles:

\[
\begin{align*}
\Xi^1_p &= (E_{4+p,2} + E_{4+p,4}) + i(E_{2,4+p} - E_{4,4+p}) \\
\Xi^2_p &= (E_{2,4+p} - E_{4,4+p}) + i(E_{4+p,2} + E_{4+p,4})
\end{align*}
\]

(69)

and finally, $2N$ generators commuting with $M^\ast$ (but not with $M$) again classified in the representation $(2, N)$ of $SO(2) \oplus SU(N)$:

\[
\begin{align*}
\Xi^{(1)}_p &= (E_{4+p,2} - E_{4+p,4}) + i(E_{2,4+p} + E_{4,4+p}) \\
\Xi^{(2)}_p &= (E_{2,4+p} + E_{4,4+p}) + i(E_{4+p,2} - E_{4+p,4}).
\end{align*}
\]

(70)

The anti-commutation relations between the $Q^a_p$ and $S^b_q$ ($a, b = 1, 2; p, q = 1, \ldots, N$) can be summarized as follows:

\[
\begin{align*}
\{ Q^a_p, Q^b_q \} &= \{-S^1_p, S^2_q \} = -(E_{4+p,4+q} - E_{4+q,4+p}) \\
\{ Q^a_p, S^1_q \} &= \{ Q^b_p, S^2_q \} = \delta_{pq} H + C/2 \\
\{ Q^a_p, S^2_q \} &= \{-Q^b_p, S^1_q \} = -\delta_{pq} D \\
\{ Q^a_p, Q^b_q \} &= \{ Q^b_p, Q^a_q \} = J - H/2 \pm 1/2N X_N + Z_{N;p} \\
\{ S^1_p, S^1_q \} &= \{ S^2_p, S^2_q \} = -(Q^1_p, Q^1_q) \quad (p \neq q) \\
\{ S^1_p, S^2_q \} &= \{ S^2_p, S^1_q \} = -(Q^1_p, Q^2_q) \quad (p \neq q)
\end{align*}
\]

(71)

where

\[
\begin{align*}
Z_{N;p} &= -\frac{2i}{N}(E_{55} - E_{66}) - \frac{4i}{N}(E_{66} - E_{77}) - \cdots - \frac{2(p - 1)i}{N}(E_{3p,3p} - E_{4p,4p}) \\
&+ \frac{2(N - p)i}{N}(E_{4p,4p} - E_{5p,5p}) + \frac{2(N - p - 1)i}{N}(E_{5p,5p} - E_{6p,6p}) \\
&+ \cdots + \frac{2i}{N}(E_{3N,3N} - E_{4N,4N}).
\end{align*}
\]

(72)

Note that, for $N = 4$, $X_4$ disappears on the rhs of the above anti-commutation relations when considering $pSU(2, 2/4)$. We note that all these anti-commutation relations close into the elements of $H$, $C$ and $D$ of $SU(1, 1)$, the elements of $SU(N)$ and the $SO(2)$ generator $(J + X)_{2N}$ which commutes with $SU(1, 1)$ as well as $SU(N)$. Adding to this reductive algebra $SU(1, 1) \oplus SU(N) \oplus SO(2)$, the $4N$ elements $Q^a_p, S^b_q$ with $a = 1, 2$, and $p = 1, \ldots, N$, one obtains a realization of the $SU(1, 1/N)$ superalgebra.

At this point, let us consider more precisely the action of $J$ and $X_0$ on the fermionic sector; hence, one obtains

\[
\begin{align*}
[J, Q^a_p] &= \frac{1}{2} \epsilon^{ab}_p Q^b_p, \\
[J, S^a_p] &= \frac{1}{2} \epsilon^{ab}_p S^b_p.
\end{align*}
\]

(73)
with \( \epsilon^b_a \) being the 2 \times 2 anti-symmetric tensor with \( \epsilon^1_1 = -\epsilon^2_1 = 1 \), and
\[
[X_N, Q^a_p] = (N - 4) \epsilon^a_b Q^b_p, \quad [X_N, S^a_p] = (N - 4) \epsilon^a_b S^b_p, \quad [X_N, \Xi^a_p] = -(N - 4) \epsilon^a_b \Xi^b_p.
\] (74)

It follows that the rotation generator \( J \) in the bosonic Schrödinger algebra appears, for \( N \neq 4 \), as a combination of the two compact generators \((J + \frac{X}{2N})\) in \( SU(1, 1/N) \) and \((J - \frac{X}{2N})\) in \( SU(1, 1/N) \), the last element commuting with \( SU(1, 1/N) \), with which it therefore constitutes the direct sum \( SO(2) \oplus SU(1, 1/N) \).

Now, adding to the Heisenberg algebra \( H(2) \), generated by \( P_a, K_a \) (\( a = 1, 2 \)) and \( M \), the fermionic generators \( \Xi^a_p \) (\( p = 1, \ldots, N \)), one obtains the super Heisenberg algebra \( SH(2/N) \).

In the same way, the elements \( P_a^*, K_a^*, \Xi^a_p \) and \( M^* \) generate the super Heisenberg algebra \( SH^*(2/N) \). One easily verifies the action of \( SU(1, 1/N) \) on \( SH(2/N) \) and, similarly, on \( SH^*(2/N) \).

Let us summarize our results.

- The super Schrödinger algebra in \( SU(2, 2/N) \) with \( N \neq 4 \), extension of the Schrödinger algebra in \( SU(2, 2) \) and stabilizer of the bosonic generator \( M \), see equation (57), shows up as the semi-direct sum of the above-defined algebra \( SO(2) \oplus SU(1, 1/N) \) on the super Heisenberg algebra \( SH(2/N) \), that is,
\[
[SO(2) \oplus SU(1, 1/N)] \triangleright SH(2/N).
\] (75)

Another copy, conjugate in \( SU(2, 2/N) \), of the super Schrödinger algebra is provided by
\[
[SO(2) \oplus SU(1, 1/N)] \triangleright SH^*(2/N)
\] (76)
and a ‘schematic’ decomposition of the \( SU(2, 2/N) \), \( N \neq 4 \), superalgebra is as follows:
\[
SU(2, 2/N) =_{vs.} SH(2/N) \triangleright [SO(2) \oplus SU(1, 1/N)] \oplus SO(1, 1) \triangleright SH^*(2/N),
\] (77)
the last \( SO(1, 1) \) in the above expressions being generated by \( \Lambda \), defined in equation (60). We recall that \( \Lambda, M \) and \( M^* \) are close into an \( SU(1, 1) \), see equation (65).

- In the case \( N = 4 \), the relevant superconformal algebra is \( pSU(2, 2/4) \) and the above \( SO(2) \) is no longer present in the super Schrödinger sector, which then reduces to
\[
SU(1, 1/4) \triangleright SH(2/4)
\] (78)
and also to
\[
SU(1, 1/4) \triangleright SH^*(2/4),
\] (79)
providing the vector space decomposition
\[
pSU(2, 2/4) =_{vs.} SH(2/4) \triangleright [SU(1, 1/4) \oplus SO(1, 1)] \triangleright SH^*(2/4).
\] (80)

5. The case of \( Osp(N/4, R) \)

Any element \( M \) of \( Osp(N/4, R) \) can be written in matrix representation on \( R \) as
\[
M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\] (81)
with \( A \) and \( D \) being, respectively, the \( N \times N \) and \( 4 \times 4 \) matrices (even grading parts) and \( B \) and \( C \), respectively, the \( N \times 4 \) and \( 4 \times N \) matrices (odd grading parts), with the following condition, see [20]:
\[
M^{st}K + KM = 0 \iff M = -K^{-1}(M)^{st}K.
\] (82)
Here, ‘st’ denotes the supertransposition defined in equation (32) and $K$ stands for the matrix

$$K = \begin{pmatrix} 1_S & 0 \\ 0 & J_4 \end{pmatrix}$$

(83)

with $J_4$ being

$$J_4 = \begin{pmatrix} 1_N \\ 0 \\ 0 \\ J_4 \end{pmatrix}$$

(84)

One easily gets

$$A = -A', \quad B = -C'J_4, \quad C = -J_4B', \quad D = J_4D'J_4$$

(85)

from which an explicit basis for $Osp(N/4, R)$ can be derived with the $A$ part, or $SO(N)$ algebra, generated by

$$A_{ij} = E_{ij} - E_{ji} = -A_{ji} \quad (i, j = 1, 2, \ldots, N)$$

(86)

and the $D$ part, or $Sp(4, R) \cong SO(3, 2)$ algebra, generated by

$$E_{N+1,N+1} - E_{N+3,N+3}, \quad E_{N+2,N+2} - E_{N+4,N+4}, \quad E_{N+1,N+2} - E_{N+3,N+4}, \quad E_{N+2,N+1} - E_{N+4,N+3}, \quad E_{N+1,N+3} + E_{N+2,N+4}, \quad E_{N+3,N+1} - E_{N+4,N+2},$$

$$E_{N+3,N+2} + E_{N+4,N+1}, \quad E_{N+4,N+3} + E_{N+2,N+3},$$

(87)

and finally, the $B$ and $C$ or the fermionic part spanned by $(i = 1, 2, \ldots, N)$

$$E_{i,N+1} + E_{N+3,i}, \quad E_{i,N+2} + E_{N+4,i}, \quad E_{i,N+3} - E_{N+1,i}, \quad E_{i,N+4} - E_{N+2,i}. \quad (88)$$

Since the algebraic results which follow are similar whatever the value of the positive integer $N$, in the following we will choose $N = 8$, which is the relevant value in the present AdS $\times S^7$ M-theory. First focusing our attention to the $Sp(4, R) \cong SO(3, 2)$ algebra, we can recognize the subalgebras

$$SO(2, 2) \cong SO(2, 1) \oplus SO(2, 1)$$

(89)

with the basis

$$\{E_{9,11}, E_{11,9}, E_{9,9} - E_{11,11}\} \oplus \{E_{10,12}, E_{12,10}, E_{10,10} - E_{12,12}\}. \quad (90)$$

We take the first three generators to generate the $H, C$ and $D$ elements and the other three to get the $M, M^*$ and $\Lambda$ elements. Then, the two $d = 1$ Heisenberg algebras show up with $H(1)$ generated by

$$P = E_{9,12} + E_{10,11}, \quad K = E_{10,9} - E_{11,12}, \quad M = 2E_{10,12}, \quad (91)$$

and $H^*(1)$ generated by

$$P^* = E_{9,10} - E_{12,11}, \quad K^* = E_{12,9} + E_{11,10}, \quad M^* = 2E_{12,10}. \quad (92)$$

Considering the fermionic part, one can check that each of the 16 generators $(i = 1, 2, \ldots, 8)$

$$Q_i = E_{i,9} + E_{11,i}, \quad S_i = E_{i,11} - E_{9,i} \quad (93)$$

commutes both with $M$ and $M^*$. Moreover, the eight generators

$$\Xi_i = E_{i,12} - E_{10,i}$$

(94)

commute with $M$, but not with $M^*$, while the eight generators

$$\Xi_i^* = E_{i,10} + E_{12,i}$$

(95)
commute with $M^*$, but not with $M$.

It is a simple exercise to verify some more commutation relations and to deduce that the Heisenberg algebra $H(1)$ can be extended to $SH(1/8)$ by adding to it the elements $\Xi_i$. In the same way, $H^*(1)$ can be extended to $SH^*(1/8)$ by adding the $\Xi_i^*$ elements.

As could be expected, the anti-commutation relations between the $Q_i$ and $S_j$ ($i, j = 1, 2, \ldots, 8$) provide the generators of the $SO(8)$ as well as $Sp(2, R)$ algebras generated by $H, C$ and $D$:

$$\{Q_i, S_j\} = \delta_{ij}(E_{11,11} - E_{9,9}) - A_{ij}$$

$$\{Q_i, Q_j\} = 2\delta_{ij}E_{11,9}$$

$$\{S_i, S_j\} = -2\delta_{ij}E_{9,11}.$$  (96)

Moreover, one can prove that the $Q'_i$s and $S'_i$s form the fermionic part of the $Osp(8/2, R)$ superalgebra, the bosonic part being $SO(8) \oplus Sp(2, R)$. The super Schrödinger algebra in $Osp(8/4, R)$ can finally be seen as the semi-direct sum of $Osp(8/2, R)$ acting on $SH(1/8)$:

$$Osp(8/2, R) \triangleright SH(1/8)$$  (97)

or on $SH^*(1/8)$:

$$Osp(8/2, R) \triangleright SH^*(1/8)$$  (98)

leading to the ‘schematic decomposition’ of $Osp(8/4, R)$:

$$Osp(8/4, R) \cong_{vs} SH(1/8) \prec [Osp(8/2, R) \oplus SO(1, 1)] \triangleright SH^*(1/8)$$  (99)

with $SO(1, 1)$ being generated by

$$\Lambda = E_{10,10} - E_{12,12}.$$  (100)

This decomposition holds for any non-negative value of the integer $N$ :

$$Osp(N/4, R) \cong_{vs} SH(1/N) \prec [Osp(N/2, R) \oplus SO(1, 1)] \triangleright SH^*(1/N)$$  (101)

in which the super Schrödinger algebra reads

$$Osp(N/2, R) \triangleright SH(2/N).$$  (102)

6. The case of $Osp(6, 2/2N)$

This case presents the peculiarity that the $R$-symmetry is described by a (compact form of) symplectic algebra. As we will see, this leads to results slightly different from the ones obtained when the $R$-symmetry is a unitary or orthogonal algebra. Now, using again the definition of $[20]$, any element of $Osp(6, 2/2N)$ can be written as an $(8 + 2N) \times (8 + 2N)$ matrix of $Osp(8/2N)$, see equation (82), with the extra condition:

$$M = -\tilde{K}^{-1}(M^*)^a\tilde{K}$$  (103)

where $\tilde{K}$ is the matrix:

$$\tilde{K} = \begin{pmatrix} J_8 & 0 \\ 0 & 1_{2N} \end{pmatrix}.$$  (104)

The other symbols in equation (103) are already defined in sections 4 and 5.

Let us, for the time being, develop our computation for the case $Osp(6, 2/4)$. We will come back to the general case $Osp(6, 2/2N)$ at the end of this section. Then, any element $M$ of the considered superalgebra can be represented by the $12 \times 12$ matrix:

$$M = \begin{pmatrix} \hat{A} & B \\ C & D \end{pmatrix}.$$  (105)
Equation (103) implies for the $8 \times 8$ matrix $\hat{A}$, for the $4 \times 4$ matrix $D$ and for the matrices $B$ and $C$ the relations

$$\hat{A} = J_8 \hat{A}^* J_8 = -\hat{A}', \quad D = J_8 D' J_8 = -D'^*, \quad B = J_8 B' J_8, \quad C = -J_4 B'. \quad (106)$$

In this framework, the $SO(6, 2)$ algebra, corresponding to the matrices $\hat{A}$ in equation (105), can be conveniently described in the following basis ($j = 1, 2, \ldots, 6; k = 1, 2, \ldots, 10$):

$$\hat{A}_j = \begin{pmatrix} A_j & 0 \\ 0 & A_j \end{pmatrix}, \quad \hat{S}_k = \begin{pmatrix} 0 & S_k \\ -S_k & 0 \end{pmatrix} \quad (107)$$

for the compact part, and

$$\hat{B}_j = \begin{pmatrix} iA_j & 0 \\ 0 & -iA_j \end{pmatrix}, \quad \hat{C}_j = \begin{pmatrix} 0 & iA_j \\ iA_j & 0 \end{pmatrix} \quad (108)$$

for the non-compact part, with $A$ being $4 \times 4$ real antisymmetric matrices defined as follows:

$$A_1 = (E_{12} - E_{21}) + (E_{34} - E_{43})$$
$$A_2 = -(E_{23} - E_{32}) - (E_{14} - E_{41})$$
$$A_3 = -(E_{13} - E_{31}) + (E_{24} - E_{42})$$
$$A_4 = (E_{12} - E_{21}) - (E_{34} - E_{43})$$
$$A_5 = (E_{23} - E_{32}) - (E_{14} - E_{41})$$
$$A_6 = -(E_{13} - E_{31}) - (E_{24} - E_{42}) \quad (109)$$

with $\{A_1, A_2, A_3\}$ and $\{A_4, A_5, A_6\}$ generating two commuting $SO(3)$. $S$ are $4 \times 4$ real asymmetric matrices defined as follows:

$$S_1 = (E_{23} + E_{32}) - (E_{14} + E_{41})$$
$$S_2 = (E_{12} + E_{21}) + (E_{34} + E_{43})$$
$$S_3 = (E_{11} - E_{22} + E_{33} - E_{44})$$
$$S_4 = (E_{11} - E_{33} + E_{22} - E_{44})$$
$$S_5 = (E_{23} + E_{32}) + (E_{14} + E_{41})$$
$$S_6 = (E_{13} + E_{31}) - (E_{24} + E_{42})$$
$$S_7 = (E_{13} + E_{31} + E_{24} + E_{42})$$
$$S_8 = (E_{11} - E_{22} - E_{33} + E_{44})$$
$$S_9 = (E_{12} + E_{21}) - (E_{34} + E_{43})$$
$$S_{10} = (E_{11} + E_{22} + E_{33} + E_{44}). \quad (110)$$

Since $SO(6, 2)$ stands for the conformal algebra in $5 + 1$ dimensions, i.e. 5 space and 1 time dimensions, the Schrödinger counterpart we are looking for will act on a $(4 + 1)$-dimensional space. Considering the embedding

$$SO(6, 2) \supset SO(4) \oplus SO(2, 2) \cong SO(4) \oplus SO(2, 1) \oplus SO(2, 1). \quad (111)$$

The ‘rotation’ algebra $SO(4)$, acting on the translation $P_a$ and momenta $K_a$ ($a = 1, 2, 3, 4$), can be chosen as

$$SO(4) \equiv \{\hat{A}_1, \hat{A}_2, \hat{A}_3, \hat{S}_1, \hat{S}_2, \hat{S}_3\}. \quad (112)$$
The two commuting $SO(2, 1)$ algebras can be chosen as
\[
\{\hat{A}_6 - \hat{S}_{10}, \hat{B}_4 + \hat{C}_5, \hat{B}_5 - \hat{C}_4\} \oplus \{\hat{A}_6 + \hat{S}_{10}, \hat{B}_4 - \hat{C}_5, \hat{B}_5 + \hat{C}_4\}. \tag{113}
\]

The first $SO(2, 1)$, which we will denote $SO(2, 1)_c$, will be chosen to contain the $H, C$ and $D$ generators. Due to the commuting relations
\[
[\hat{A}_6 - \hat{S}_{10}, \hat{B}_4 + \hat{C}_5] = -4(\hat{B}_5 - \hat{C}_4), \quad [\hat{A}_6 - \hat{S}_{10}, \hat{B}_5 - \hat{C}_4] = 4(\hat{B}_4 + \hat{C}_3),
\]
\[
[\hat{B}_4 + \hat{C}_5, \hat{B}_5 - \hat{C}_4] = -4(\hat{A}_6 - \hat{S}_{10}), \tag{114}
\]
one can make the identification
\[
D = \frac{1}{\sqrt{2}}(\hat{A}_6 - \hat{S}_{10}), \quad H = \frac{1}{\sqrt{2}}(\hat{B}_4 + \hat{C}_5), \quad C = \frac{1}{\sqrt{2}}(\hat{B}_5 - \hat{C}_4). \tag{115}
\]

Then, the second $SO(2, 1)$ will contain the generators $M, M^*$ and $\Lambda$ and we will choose
\[
M = (\hat{A}_6 + \hat{S}_{10}) + (\hat{B}_5 + \hat{C}_4), \tag{116}
\]
\[
M^* = (\hat{A}_6 + \hat{S}_{10}) - (\hat{B}_5 + \hat{C}_4), \tag{117}
\]
\[
\Lambda = -4(\hat{B}_4 - \hat{C}_5). \tag{118}
\]

As the final step, we have to determine the four translations $P_a$ and the four momenta $K_a$ $(a = 1, 2, 3, 4)$. From what we already know, the space translation $p_5$ and time translation $p_0$ now appear, in the conformal algebra, as
\[
p_5 = \hat{A}_6 + \hat{C}_4, \quad p_0 = \hat{S}_{10} + \hat{B}_5. \tag{119}
\]

By the action on $p_5$ of the $SO(5)$ rotation algebra, constructed from the above-defined $SO(4)$, equation (112), by addition of the four generators $[\hat{A}_6, \hat{S}_4, \hat{S}_5, \hat{S}_6]$, one can generate the $p_a \equiv P_a$ translations and, finally, the momenta $K_a$, which have to satisfy the condition
\[
[K_a, P_b] = \delta_{ab} M \quad (a, b = 1, 2, 3, 4). \tag{120}
\]

We obtain
\[
P_1 = -\frac{1}{\sqrt{2}}(\hat{S}_4 - \hat{B}_1), \quad K_1 = -\frac{1}{\sqrt{2}}(\hat{S}_4 + \hat{C}_1),
\]
\[
P_2 = \frac{1}{\sqrt{2}}(\hat{S}_8 + \hat{B}_2), \quad K_2 = -\frac{1}{\sqrt{2}}(\hat{S}_6 + \hat{C}_2),
\]
\[
P_3 = -\frac{1}{\sqrt{2}}(\hat{S}_9 - \hat{B}_3), \quad K_3 = \frac{1}{\sqrt{2}}(\hat{S}_5 - \hat{C}_3),
\]
\[
P_4 = \frac{1}{\sqrt{2}}(\hat{A}_4 - \hat{C}_6), \quad K_4 = \frac{1}{\sqrt{2}}(\hat{A}_3 - \hat{B}_6). \tag{121}
\]

The second Heisenberg algebra $H^*(4)$ is easily obtained by first operating a change of sign in front of the $B$ and $\hat{C}$ (that is replacing $i$ by $-i$ in the non-compact generators), appearing in the above combination, equation (121), and then denoting $K^*_a$ the so-transformed $P_a$, and $P^*_a$ the so-transformed $-K_a$. More precisely, we have
\[
P^*_1 = \frac{1}{\sqrt{2}}(\hat{S}_4 - \hat{C}_1), \quad K^*_1 = -\frac{1}{\sqrt{2}}(\hat{S}_7 + \hat{B}_1),
\]
\[
P^*_2 = \frac{1}{\sqrt{2}}(\hat{S}_6 - \hat{C}_2), \quad K^*_2 = \frac{1}{\sqrt{2}}(\hat{S}_8 - \hat{B}_2),
\]
\[
P^*_3 = -\frac{1}{\sqrt{2}}(\hat{S}_5 + \hat{C}_3), \quad K^*_3 = -\frac{1}{\sqrt{2}}(\hat{S}_9 + \hat{B}_3),
\]
\[
P^*_4 = -\frac{1}{\sqrt{2}}(\hat{A}_5 + \hat{B}_6), \quad K^*_4 = \frac{1}{\sqrt{2}}(\hat{A}_4 + \hat{C}_6). \tag{122}
\]

As expected, one gets
\[
[K^*_a, P^*_b] = \delta_{ab} M^*. \tag{123}
\]
Let us close our study of the bosonic part by briefly studying the algebra $Sp(4)$. A natural basis for the $4 \times 4$ matrix $D$, satisfying equation (106), is provided by the ten generators:

$\begin{align*}
  i(E_{11} - E_{33}), & \quad i(E_{22} - E_{44}), & \quad i(E_{13} + E_{31}), & \quad i(E_{24} + E_{42}), \\
  i(E_{12} + E_{21} - E_{34} - E_{43}), & \quad i(E_{14} + E_{41} + E_{23} + E_{32}),
\end{align*}$

(124)

Now, we turn our attention to the fermionic part. A basis for the 32 fermionic generators, satisfying equation (106), can be chosen as $(i, j = 1, 2, 3, 4)$

$\begin{align*}
  F_{i,9} &= E_{i,9} + E_{9,4i} - E_{4i,11} + E_{i,11}, & \quad F_{i,10} &= E_{i,10} + E_{10,4i} - E_{4i,12} + E_{i,12}, \\
  F_{i,11} &= E_{i,11} + E_{11,4i} + E_{4i,9} - E_{9,j}, & \quad F_{i,12} &= E_{i,12} + E_{12,4i} + E_{4i,10} - E_{10,j},
\end{align*}$

and

$\begin{align*}
  G_{j,9} &= i(E_{j,9} - E_{9,4j} - E_{4j,11} - E_{j,11}), & \quad G_{j,10} &= i(E_{j,10} - E_{10,4j} + E_{4j,12} + E_{j,12}), \\
  G_{j,11} &= i(E_{j,11} - E_{11,4j} - E_{4j,9} - E_{9,j}), & \quad G_{j,12} &= i(E_{j,12} - E_{12,4j} + E_{4j,10} - E_{10,j}).
\end{align*}$

(125)

(126)

The anti-commutation relations of the above-defined fermionic generators are reported in the appendix. It is straightforward, but a tedious exercise, to determine the fermionic combinations which commute with $M$ and $M^*$. It appears that 16 fermions commute with $M$ and $M^*$ simultaneously; they can be chosen as

$\begin{align*}
  Q_1 &= F_{1,9} - F_{3,11}, & \quad Q_2 &= F_{2,9} - F_{3,11}, \\
  Q_3 &= F_{3,9} + F_{1,11}, & \quad Q_4 &= F_{4,9} + F_{2,11}, \\
  Q_5 &= F_{1,10} - F_{3,12}, & \quad Q_6 &= F_{2,10} - F_{4,12}, \\
  Q_7 &= F_{3,10} + F_{1,12}, & \quad Q_8 &= F_{4,10} + F_{2,12}, \\
  S_1 &= G_{1,9} + G_{3,11}, & \quad S_2 &= G_{2,9} + G_{4,11}, \\
  S_3 &= G_{3,9} - G_{1,11}, & \quad S_4 &= G_{4,9} - G_{2,11}, \\
  S_5 &= G_{1,10} + G_{3,12}, & \quad S_6 &= G_{2,10} + G_{4,12}, \\
  S_7 &= G_{3,10} - G_{1,12}, & \quad S_8 &= G_{4,10} - G_{2,12}.
\end{align*}$

(127)

(128)

One can recognize the representations $(1/2, 1/2, 4)$ of the algebra $SO(2, 1)_c \oplus SO(3) \oplus Sp(4)$ algebra, with $SO(2, 1)_c$ being the ‘conformal’ one given by equation (115), the $SO(3)$ is generated by $\{A_1 + S_1, A_2 - S_2, A_3 - S_3\}$ and we denote it as $SO(3)_c$, and $Sp(4)$ is the $R$-symmetry of our problem given by equation (124). Performing all the anti-commutation relations among the $Q_\mu$ and the $S_\mu$ ($\mu = 1, 2, \ldots, 8$), one gets back the whole above-defined semi-simple algebra, so proving that we have a realization of the $Osp(4^* / 4)$ superalgebra. It is important to remark that the ‘rotation’ $SO(4)$ algebra defined by equation (112) is the semi-direct sum of the above-defined $SO(3)_c$ and of a second $SO(3)$, denoted $SO(3)_s$, generated by $\{A_1 - S_1, A_2 + S_2, A_3 + S_3\}$. One can check that $SO(3)_s$ does not act on $Q_\mu$ and $S_\mu$, but effectively acts on $\Xi_\mu$ and $\Xi^*_\mu$ which will respectively be added to the $H(4)$ and $H^*(4)$ Heisenberg algebras to constitute the $SH(4)$ and $SH^*(4)$ super Heisenberg ones, and which
can be defined as follows:
\[ \Xi_1 = (F_{1,0} + F_{3,11}) - (G_{2,9} - G_{4,11}), \quad \Xi_2 = (F_{2,9} + F_{4,11}) + (G_{1,9} - G_{3,11}), \]
\[ \Xi_3 = (F_{3,9} - F_{1,11}) + (G_{4,9} + G_{2,11}), \quad \Xi_4 = (F_{4,9} - F_{2,11}) - (G_{3,9} + G_{1,11}), \]
\[ \Xi_5 = (F_{1,10} + F_{3,12}) - (G_{2,10} - G_{4,12}), \quad \Xi_6 = (F_{2,10} + F_{4,12}) + (G_{1,10} - G_{3,12}), \]
\[ \Xi_7 = (F_{3,10} - F_{1,12}) + (G_{4,10} + G_{2,12}), \quad \Xi_8 = (F_{4,10} - F_{2,12}) - (G_{3,10} + G_{1,12}), \]
\[ \Xi_1^* = (F_{1,9} + F_{3,11}) + (G_{2,9} - G_{4,11}), \quad \Xi_2^* = (F_{2,9} + F_{4,11}) - (G_{1,9} - G_{3,11}), \]
\[ \Xi_3^* = (F_{3,9} - F_{1,11}) - (G_{4,9} + G_{2,11}), \quad \Xi_4^* = (F_{4,9} - F_{2,11}) + (G_{3,9} + G_{1,11}), \]
\[ \Xi_5^* = (F_{1,10} + F_{3,12}) + (G_{2,10} - G_{4,12}), \quad \Xi_6^* = (F_{2,10} + F_{4,12}) - (G_{1,10} - G_{3,12}), \]
\[ \Xi_7^* = (F_{3,10} - F_{1,12}) - (G_{4,10} + G_{2,12}), \quad \Xi_8^* = (F_{4,10} - F_{2,12}) + (G_{3,10} + G_{1,12}). \]

Note that fermions \( \Xi \)'s are obtained from the \( \Xi \)'s given by equation (129) by complex conjugation, i.e. replacing \( G_{j,s} \) by \( -G_{j,s} \). The \( \Xi \)'s, as well as the \( \Xi \)'s, stand in the representation \( (1/2, 4) \) of the algebra \( SO(3)_+ \oplus Sp(4) \). It can also be noted that \( SO(3)_+ \) acts trivially on the \( \Xi \) and \( \Xi^* \).

As a conclusion, we have determined two (conjugate) super Schrödinger algebras in the Lie superalgebra \( Osp(6, 2/4) \), the first one being
\[ \bar{\mathcal{S}}ch(4/4)_{\text{symp}} = [SO(3)_- \oplus Osp(4^*/4)] \triangleright SH(4/4) \]
and the second one differing from the first simply by the super Heisenberg part, \( SH^*(4/4) \) replacing \( SH(4/4) \).

As in the previous section, we formally represent the \( Osp(6, 2/4) \) superalgebra as
\[ Osp(6, 2/4) \Rightarrow_{v.s.} SH(4/4) \triangleleft [Osp(4^*/4) \oplus SO(3)_- \oplus SO(1, 1)] \triangleright SH^*(4/4), \]
keeping in mind that the generators \( M \) in \( SH(4/4) \) and \( M^* \) in \( SH^*(4/4) \) close under commutation relations into the dilatation generator \( \Lambda \), represented in the \( O(1, 1) \) part of the decomposition equation (132). We also recall that the bosonic part of \( Osp(4^*/4) \) is \( SO(2, 1)_+ \oplus SO(3)_+ \oplus Sp(4) \), where \( Sp(4) \) is compact, \( SO(2, 1) \), is generated by \( H, C \) and \( D \), and \( SO(3)_+ \) together with \( SO(3)_- \) forms the ‘rotation’ algebra acting on the \( P_{m,s} \) and \( K_{m,s} \) (respectively, \( P_{m,s}^* \) and \( K_{m,s}^* \)) that is
\[ SO(4) = SO(3)_+ \oplus SO(3)_-. \]
The generalization to \( Osp(6, 2/2N) \) is straightforward, the compact \( Sp(4) \) algebra being replaced by \( Sp(2N) \), \( N \) positive integer, and one gets
\[ \bar{\mathcal{S}}ch(2N/4)_{\text{symp}} = [SO(3)_- \oplus Osp(4^*/2N)] \triangleright SH(4/2N) \]
\[ Osp(6, 2/N) \Rightarrow_{v.s.} SH(4/2N) \triangleleft [Osp(4^*/2N) \oplus SO(3)_- \oplus SO(1, 1)] \triangleright SH^*(4/2N). \]

7. The case of \( F(4; 2) \)

As could be expected, the super Schrödinger algebra which can be extracted from \( F(4; 2) \) exhibits some exceptional features. In particular, the supersymmetric extension of the Heisenberg part \( H(3) \), arising from \( SO(5, 2) \), will not be obtained by adding a triplet but a
quadruplet of fermions, in other words a spinorial representation, namely labeled by \( j = 3/2, \) of the associated rotation group \( SO(3). \)

In order to determine the real form of the \( F(4) \) superalgebra we are interested in, that is the one with \( SO(5, 2) \oplus SU(2) \) as bosonic part, let us first start by considering the \( F(4) \) superalgebra defined on the complex field \( \mathbb{C} \) \cite{15}. Then, its bosonic part \( SL(2) \oplus SO(7) \) can be generated by the elements \( T_i \ (i = 1, 2, 3) \) and \( M_{pq} = -M_{qp} \ (p, q = 1, \ldots, 7), \) respectively, satisfying

\[
[T_i, T_j] = i\varepsilon_{ijk} T_k, \quad [T_i, M_{pq}] = 0, \quad (136)
\]

\[
[M_{pq}, M_{rs}] = \delta_{qr} M_{pr} + \delta_{ps} M_{qr} - \delta_{pr} M_{qs} - \delta_{qs} M_{pr}, \quad (137)
\]

while the 16 generators of its fermionic part stand in the representation \((2, 8) \equiv (1/2, 1/2, 1/2, 1/2) \) of \( SL(2) \oplus SO(7). \) We denote by \( F_{\alpha \mu}, \alpha = \pm \) and \( \mu = (\pm, \pm, \pm) \) a basis of generators satisfying the relations

\[
[T_i, F_{\alpha \mu}] = \frac{1}{2} \sigma_i^{\alpha \beta} F_{\beta \mu}, \quad [M_{pq}, F_{\alpha \mu}] = \frac{i}{4} (\Gamma_p \Gamma_q)_{\alpha \beta} F_{\beta \mu}, \quad (138)
\]

\[
[F_{\alpha \mu}, F_{\beta \nu}] = 2 C_{\alpha \beta}^{(8)} (\sigma^{(2)})_{\alpha \mu} T_i + \frac{i}{4} C_{\alpha \beta}^{(2)} (\sigma^{(8)})_{\mu \nu} M_{pq}. \quad (139)
\]

The \( \sigma^i \) are the usual Pauli matrices. The eight-dimensional matrices \( \Gamma_p \) form a Clifford algebra

\[
\{\Gamma_p, \Gamma_q\} = 2 \delta_{pq} \quad (140)
\]

and are chosen as

\[
\Gamma_1 = \sigma_1 \otimes \sigma_1, \quad \Gamma_2 = \sigma_1 \otimes \sigma_1 \otimes \sigma_2, \quad \Gamma_3 = \sigma_1 \otimes \sigma_1 \otimes \sigma_3, \quad \Gamma_4 = \sigma_1 \otimes \sigma_2 \otimes 1, \quad \Gamma_5 = \sigma_1 \otimes \sigma_3 \otimes 1, \quad \Gamma_6 = \sigma_2 \otimes 1 \otimes 1, \quad \Gamma_7 = \sigma_3 \otimes 1 \otimes 1. \quad (141)
\]

We note that \( \Gamma_p = (-1)^{p+1} \Gamma_p \) and \( \Gamma_p \Gamma_q \) generates the Lie algebra \( SO(7). \) Finally, \( C^{(2)} \) and \( C^{(8)} \) are, respectively, the \( 2 \times 2 \) and \( 8 \times 8 \) real charge conjugation matrices given by

\[
C^{(2)} = i \sigma_2, \quad C^{(8)} = \Gamma_1 \Gamma_3 \Gamma_4 \Gamma_7 = (i \sigma_2) \otimes \sigma_3 \otimes (i \sigma_2). \quad (142)
\]

Following \cite{21}, the \( F(4) \) real form with \( SO(5, 2) \oplus SU(2) \) as bosonic part is made from the elements \( X + C_0(X), \) with \( X \) belonging to the (complex) \( F(4) \) superalgebra and \( C_0 \) being the semi-involutive semi-morphism of \( F(4) \) acting on the bosonic part as follows \((X^\dagger = \overline{X}^\dagger)\):

\[
C_0(X) = -X'^\dagger, \quad X \in SL(2), \quad (143)
\]

\[
C_0(X) = \tau(X), \quad X \in SO(7), \quad (144)
\]

with \( \tau \) acting on the orthogonal algebra generated by \( \Gamma_{p \Gamma_q} \ (1 \leq p \leq q \leq 7) \) as

\[
\tau(\Gamma_{p \Gamma_q}) = \Gamma_4 \overline{\Gamma_p \Gamma_q \Gamma_4 \Gamma_4 \Gamma_4}^{-1} \quad (144)
\]

and acting on the fermionic part as follows:

\[
C_0(\mathbf{v} \otimes \mathbf{x}) = i J \mathbf{v} \otimes \Gamma_4 \mathbf{x} \quad (145)
\]

where \( \mathbf{v} \otimes \mathbf{x} \) is the most general element of the representation \((1/2) \otimes (1/2, 1/2, 1/2, 1/2) \) and \( J \) acts on the states \( F(\pm) \) of the \( SL(2) \) two-dimensional representation as

\[
J F(\pm) = \pm F(\pm). \quad (146)
\]
In this framework, a basis of the $SO(5, 2)$ algebra shows up, made by the 11 elements of the maximal compact subalgebra $SO(5) \oplus SO(2)$,

$$\{M_{13}, M_{14}, M_{15}, M_{17}, M_{34}, M_{35}, M_{37}, M_{45}, M_{57}\} \cong SO(5) \oplus SO(2)$$

(147)

and, by the ten non-compact generators,

$$i[M_{12}, M_{16}, M_{23}, M_{24}, M_{25}, M_{27}, M_{36}, M_{46}, M_{56}, M_{67}]$$

(148)

while for the $SU(2)$ remaining algebra, the $R$-symmetry of our problem, one gets the generators $iT_i$ ($i = 1, 2, 3$).

As for the fermionic part of $F(4; 2)$, it can be seen as the direct sum of two $(1/2, 1/2)$ representations of the $SO(3) \oplus SU(2)$ algebra, where the considered $SO(3)$ algebra is generated by

$$SO(3) \equiv \{M_{13}, M_{14}, M_{34}\}$$

(149)

and the $SU(2)$ part by the $iT_j$ generators. This $SO(3)$ algebra will be chosen (see below) as the rotation algebra acting on $P_i$ and $K_j$ ($j = 1, 2, 3$) elements of our Heisenberg algebra: that is why we prefer to keep the notations $SO(3)$, instead of $SU(2)$, for this algebra, although it acts on spinorial representations. A basis for these two representation reads

$$F(\pm; +, +, +) - F(-; -,-, +), \quad i(F(\pm; +, +, +) + F(-; -,-, +)),$$

$$F(\pm; +, +, -) - F(-; -,-, -), \quad i(F(\pm; +, +, -) + F(-; -,-, -)),$$

$$F(-, +, +, +) + F(+; -, - ,+), \quad i(F(-, +, +, +) - F(+; -, - ,+)),$$

$$F(-, +, +, -) + F(+; -, -, -), \quad i(F(-, +, +, -) - F(+; -, -, -)),$$

for the first one, and

$$F(\pm; +, -, +) + F(-; -, +, +), \quad i(F(\pm; +, -, +) - F(-; -, +, +)),$$

$$F(\pm; +, -, -) + F(-; -, -, -), \quad i(F(\pm; +, -, -) - F(-; -, -, -)),$$

$$F(-, +, - ,+ ) + F(+; -, +, +), \quad i(F(-, +, - ,+ ) - F(+; -, +, +)),$$

$$F(-, +, - ,-) + F(+; -, -, -), \quad i(F(-, +, - ,-) - F(+; -, -, -)),$$

for the second one.

Coming back to the bosonic part, it is convenient to take as the Lorentz algebra in the conformal $SO(5, 2)$ one, the $SO(4, 1)$ algebra generated by the elements $M_{\mu
v}$ and $M_{\mu\nu}$, with $\mu, \nu = 1, 3, 4, 5$, and to consider the embedding

$$SO(5, 2) \supset SO(3) \oplus SO(2, 2) \cong SO(3) \oplus SO(2, 1) \oplus SO(2, 1)$$

(152)

with the ‘rotation’ $SO(3)$ algebra being given by equation (149) and the two $SO(2, 1)$ by

$$SO(2, 1) \equiv \{i(M_{25} + M_{76}), i(M_{56} + M_{72}), M_{57} + M_{26}\}$$

(153)

$$SO(2, 1) \equiv \{i(M_{25} - M_{76}), i(M_{56} + M_{72}), M_{57} - M_{26}\}$$

(154)

Taking $SO(2, 1)$ as the ‘non-relativistic conformal algebra’, we define

$$H \equiv (M_{57} - M_{26}) - i(M_{25} - M_{76})$$

(155)

$$C \equiv -(M_{57} - M_{26}) - i(M_{25} - M_{76})$$

(156)

$$D \equiv i(M_{25} - M_{56})$$

(157)

which satisfy the commutation relations

$$[D, C] = 2C, \quad [D, H] = -2H, \quad [H, C] = -4D,$$
while the $SO(2, 1)$ will contain the elements

\[ M = (M_{57} + M_{26}) - i(M_{25} + M_{76}) \]  
\[ M^* = (M_{57} + M_{26}) + i(M_{25} + M_{76}) \]  
\[ \Lambda = i(M_{27} + M_{65}) \]

satisfying

\[ [\Lambda, M] = 2M, \quad [\Lambda, M^*] = -2M^*, \quad [M, M^*] = 4\Lambda. \]

Then the three-dimensional Heisenberg algebra $H(3)$ contains, in addition to $M$, the elements

\[ P_1 = iM_{21} + M_{71}, \quad P_2 = iM_{23} + M_{73}, \quad P_3 = iM_{24} + M_{74}, \]
\[ K_1 = iM_{61} + M_{51}, \quad K_2 = iM_{63} + M_{53}, \quad K_3 = iM_{64} + M_{54}, \]

and the three-dimensional Heisenberg algebra $H^*(3)$ will be constituted by $M^*$ and

\[ P_1^* = iM_{61} - M_{51}, \quad P_2^* = iM_{63} - M_{53}, \quad P_3^* = iM_{64} - M_{54}, \]
\[ K_1^* = M_{71} - iM_{21}, \quad K_2^* = M_{73} - iM_{23}, \quad K_3^* = M_{74} - iM_{24}, \]

such that

\[ [P_a, K_b] = -\delta_{ab}M, \quad [P_a^*, K_b^*] = -\delta_{ab}M^*(a, b = 1, 2, 3). \]

Now, from the expressions given by equation (161), one can deduce which fermionic generators commute with $M$ or $M^*$ or both.

- There are eight fermions commuting with $M$ and $M^*$:

\[ Q_1 \equiv (F(+: +, +, +) - F(\:\:\:\: -\:\:\:\: -\:\:\:\: -)) + (F(\:\:\:\: -, +, +) + F(\:\:\:\: +, -, -)) \]
\[ Q_2 \equiv (F(\:\:\:\: +, +, +) - F(\:\:\:\: -\:\:\:\: -, -, -)) - (F(\:\:\:\: +, -, +) + F(\:\:\:\: -, +, -)) \]
\[ Q_3 \equiv (F(\:\:\:\: -, +, +) - F(\:\:\:\: +, -, +)) - (F(\:\:\:\: +, +, -) + F(\:\:\:\: -, -, -)) \]
\[ Q_4 \equiv (F(\:\:\:\: +, +, +) + F(\:\:\:\: -\:\:\:\: -, -, -)) + (F(\:\:\:\: +, -, -) - F(\:\:\:\: -, +, -)) \]
\[ S_1 \equiv i[(F(\:\:\:\: +, +, +) + F(\:\:\:\: -, -, -)) - (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, -, -))] \]
\[ S_2 \equiv i[(F(\:\:\:\: +, +, +) + F(\:\:\:\: -, -, -)) - (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, -, -))] \]
\[ S_3 \equiv i[(F(\:\:\:\: +, +, +) + F(\:\:\:\: -, -, -)) - (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, -, -))] \]
\[ S_4 \equiv i[(F(\:\:\:\: +, +, +) + F(\:\:\:\: -, -, -)) - (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, -, -))]. \]

- There are four fermions commuting with $M$ but not with $M^*$:

\[ \Xi_1 \equiv (F(\:\:\:\: +, +, +) - F(\:\:\:\: -, -, -)) - (F(\:\:\:\: +, +, -) + F(\:\:\:\: +, -, -)) \]
\[ \Xi_2 \equiv (F(\:\:\:\: +, +, +) - F(\:\:\:\: -, -, -)) + (F(\:\:\:\: +, +, -) + F(\:\:\:\: +, -, -)) \]
\[ \Xi_3 \equiv i[(F(\:\:\:\: +, +, +) + F(\:\:\:\: -, -, -)) + (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, -, -))] \]
\[ \Xi_4 \equiv i[(F(\:\:\:\: +, +, +) + F(\:\:\:\: -, -, -)) + (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, -, -))]. \]

- Finally, there are four fermions commuting with $M^*$ but not with $M$:

\[ \Xi'_1 \equiv (F(\:\:\:\: -, +, -) - F(\:\:\:\: -, +, +)) + (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, +, -)) \]
\[ \Xi'_2 \equiv (F(\:\:\:\: -, +, -) + F(\:\:\:\: -, +, +)) - (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, +, -)) \]
\[ \Xi'_3 \equiv i[(F(\:\:\:\: -, +, -) + F(\:\:\:\: -, +, +)) - (F(\:\:\:\: +, +, -) - F(\:\:\:\: +, +, -))] \]
\[ \Xi'_4 \equiv i[(F(\:\:\:\: -, +, -) + F(\:\:\:\: -, +, +)) + (F(\:\:\:\: +, +, -) + F(\:\:\:\: +, +, -))]. \]
We note that the $\Xi_\mu^a$s as well as $\Xi_\mu^a$s ($\mu = 1, 2, 3, 4$) transform as the representation $(3/2)$ under the rotation algebra $SO(3)$ given by equation (149), and the transformation $(1/2) \oplus (1/2)$ of the $R$-symmetry algebra $SU(2)$, respectively, while the ‘conformal algebra’ $SO(2, 1)$, generated by the elements given by equations (155)–(156)–(157), acts trivially on each of these elements. The commutation and anti-commutation relations,

$$[P_a, \Xi_\mu] = [K_a, \Xi_\mu] = 0, \quad [\Xi_\mu, \Xi_\nu] = 4\delta_{\mu\nu}M, \quad a = 1, 2, 3; \mu, \nu = 1, 2, 3, 4,$$

and

$$[P_a^*, \Xi_\mu^*] = [K_a^*, \Xi_\mu^*] = 0, \quad [\Xi_\mu^*, \Xi_\nu^*] = 4\delta_{\mu\nu}M^*, \quad a = 1, 2, 3; \mu, \nu = 1, 2, 3, 4,$$

allow us to consider the superalgebra generated by $\{P_a, K_a, \Xi_\mu\}$ and $\{P_a^*, K_a^*, \Xi_\mu^*\}$, respectively, as the supersymmetric extensions of the $H(3)$ and $H^*(3)$ Heisenberg algebras. Due to the presence of the spinorial representation $(3/2)$ in the fermionic sector, we will denote them, respectively, as ‘spinorial super Heisenberg algebras’ $SH(3/2)$ and $SH^*(3/2)$.

Finally, considering the anti-commutation relations of the eight fermions $Q_{\mu}$ and $S_{\mu}$, which commute with $M$ and $M^*$, we can reconstruct the $SO(3) \oplus SU(2) \oplus SO(2, 1)$ direct sum. Moreover, while the rotation $SO(3)$ as well as the $R$-symmetry $SU(2)$ algebras split the fermionic sector into two four-dimensional representations, namely $\{Q_1, Q_2, S_1, S_2\}$ and $\{Q_3, Q_4, S_3, S_4\}$, the conformal algebra $SO(2, 1)'$ decomposes it into four two-dimensional representations, generated by $\{Q_1, Q_3\}, \{Q_2, Q_4\}, \{S_1, S_3\}$ and $\{S_2, S_4\}$.

Therefore, we recognize the $Osp(4/2, R)$ algebra built from this set of 17 elements. One can directly check that this $Osp(4/2, R)$ transforms the $SH(3/2)_\text{spin}$ and $SH^*(3/2)_\text{spin}$ super Heisenberg into themselves leading to the conclusion that the super Schrödinger algebra in $F(4, 2)$ is the semi-direct sum

$$\tilde{Sch}(3/2)_{FE} \equiv Osp(4/2, R) \triangleright SH(3/2)_{\text{spin}},$$

and that $F(4, 2)$ can be formally decomposed as

$$F(4, 2) = _{\text{vs.}}SH(3/2)_{\text{spin}} \lt [Osp(4/2, R) \otimes O(1, 1)] \triangleright SH^*(3/2)_{\text{spin}},$$

the $SO(1, 1)$ algebra commuting with $Osp(4/2, R)$ being generated by the generator $\Lambda$, see equation (161).

### 8. Conclusion

The supersymmetric extension of the Schrödinger algebra has been constructed in each admissible superconformal algebra. Although the obtained symmetries present some differences according to the type of considered superconformal algebra, they always show up as the semi-direct sum of a unitary or orthosymplectic superalgebra (to which is sometimes added an $SO(2)$ or $SO(3)$ factor) acting on a super Heisenberg part, see table 1:
One may note that the super Heisenberg part constructed from the exceptional $F(4; 2)$ presents a difference with respect to the other cases, its fermionic sector transforming as a spinorial representation of the corresponding $SO(3)$ rotation algebra. This is, of course, a consequence of the spinorial character of the fermionic $F(4; 2)$ sector and it is already known that this exceptional superalgebra gives peculiar results [16]. We also remark that the super Schrödinger algebras arising from $Osp(6, 2/2N)$ differ from those extracted in the other unitary and orthosymplectic ones for an extra $SO(3)$, part of $SO(4)$ rotation symmetry of the problem, and added to the $Osp(4^*/2N)$ factor. Let us at this point mention that this case is the only one with a $R$-symmetry of the symplectic type. Finally, let us emphasize the property of any super Schrödinger algebra to occupy a particular position in its corresponding superconformal algebra, where an adequate basis decomposition provides two copies of the super Schrödinger symmetry, differing by their super Heisenberg parts.

We hope that this study and the characterization of the super Schrödinger algebras that we have obtained will help to develop properties of such symmetries, for example the theory of their representations\(^9\), and mainly to better understand the physics behind their structures.

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Appendix. Anti-commutation relations of the fermionic generators of $Osp(6, 2/4)$

The anti-commutation relations of the fermionic generators \{F; G\} defined by equations (125) and (126) are

\[
\begin{align*}
\{F_{i,9}, F_{j,9}\} &= \{F_{i,11}, F_{j,11}\} = (E_{i,4j} - E_{4i,j}) + (E_{i,4i} - E_{4i,i}) + 2\delta_{ij}(E_{11,9} - E_{9,11}) & (A.1) \\
\{F_{i,9}, F_{j,10}\} &= \{F_{i,11}, F_{j,12}\} = \delta_{ij}(E_{11,10} - E_{10,11} - E_{9,12} + E_{12,9}) & (A.2) \\
\{F_{i,9}, F_{j,11}\} &= \{F_{i,10}, F_{j,12}\} = -(E_{i,j} - E_{j,i}) + (E_{4i,j} - E_{4j,i}) & (A.3) \\
\{F_{i,9}, F_{j,12}\} &= -(F_{i,10}, F_{j,11}) = \delta_{ij}(E_{9,10} - E_{10,9} + E_{11,12} - E_{12,11}) & (A.4) \\
\{F_{i,10}, F_{j,10}\} &= \{F_{i,12}, F_{j,12}\} = (E_{i,4j} - E_{4i,j}) + (E_{j,4i} - E_{4i,i}) + 2\delta_{ij}(E_{12,10} - E_{10,12}) & (A.5) \\
\{G_{i,9}, G_{j,9}\} &= \{G_{i,11}, G_{j,11}\} = E_{i,4j} + E_{4i,j} - E_{4i,j} - E_{i,4i} + 2\delta_{ij}(E_{9,11} - E_{11,9}) & (A.6) \\
\{G_{i,9}, G_{j,10}\} &= \{G_{i,11}, G_{j,12}\} = \delta_{ij}(E_{10,11} - E_{11,10} + E_{9,12} - E_{12,9}) & (A.7) \\
\{G_{i,9}, G_{j,11}\} &= \{G_{i,10}, G_{j,12}\} = (E_{i,j} - E_{j,i}) + (E_{4i,j} - E_{4j,i}) & (A.8) \\
\{G_{i,9}, G_{j,12}\} &= -(G_{i,10}, G_{j,11}) = \delta_{ij}(-E_{9,10} + E_{10,9} - E_{11,12} + E_{12,11}) & (A.9) \\
\{G_{i,10}, G_{j,10}\} &= \{G_{i,12}, G_{j,12}\} = E_{i,4j} + E_{4i,j} - E_{4i,j} - E_{i,4i} + 2\delta_{ij}(E_{10,12} - E_{12,10}) & (A.10) \\
\{F_{i,9}, G_{j,9}\} &= i[(E_{i,4j} - E_{4i,j}) - (E_{i,4j} - E_{4i,j}) + 2\delta_{ij}(E_{9,11} + E_{11,9})] & (A.11)
\end{align*}
\]

\(^9\) Representations of the Schrödinger group have been constructed in [22].
\[
\{ F_{i,9}, G_{j,10} \} = \{ F_{i,10}, G_{j,9} \} = -\{ F_{i,11}, G_{j,12} \} = i\delta_{ij}(E_{9,12} + E_{12,9} + E_{10,11} + E_{11,10}) \tag{A.12}
\]
\[
\{ F_{i,9}, G_{j,11} \} = i[-(E_{i,j} - E_{j,i}) + (E_{4+i,4+j} - E_{4+j,4+i}) - 2\delta_{ij}(E_{9,9} - E_{11,11})] \tag{A.13}
\]
\[
\{ F_{i,9}, G_{j,12} \} = \{ F_{i,10}, G_{j,11} \} = \{ F_{i,11}, G_{j,10} \} = \{ F_{i,12}, G_{j,9} \} = i\delta_{ij}(E_{9,10} - E_{10,9} + E_{11,12} + E_{12,11}) \tag{A.14}
\]
\[
\{ F_{i,10}, G_{j,10} \} = i[(E_{i,4+i} - E_{4+i,j}) - (E_{i,4+j} - E_{4+j,i}) + 2\delta_{ij}(E_{10,12} + E_{12,10})] \tag{A.15}
\]
\[
\{ F_{i,10}, G_{j,12} \} = i[-(E_{i,j} - E_{j,i}) + (E_{4+i,4+j} - E_{4+j,4+i}) + 2\delta_{ij}(E_{12,12} - E_{10,10})] \tag{A.16}
\]
\[
\{ F_{i,11}, G_{j,9} \} = i[(E_{i,j} - E_{j,i}) - (E_{4+i,4+j} - E_{4+j,4+i}) - 2\delta_{ij}(E_{9,9} - E_{11,11})] \tag{A.17}
\]
\[
\{ F_{i,11}, G_{j,11} \} = i[-(E_{i,4+j} - E_{4+j,i}) - (E_{4+i,j} - E_{j,4+i}) - 2\delta_{ij}(E_{9,11} + E_{11,9})] \tag{A.18}
\]
\[
\{ F_{i,11}, G_{j,10} \} = \{ F_{i,12}, G_{j,9} \} = i[-(E_{9,10} - E_{10,9} + E_{11,12} + E_{12,11})] \tag{A.19}
\]
\[
\{ F_{i,12}, G_{j,10} \} = i[(E_{i,j} - E_{j,i}) - (E_{4+i,4+j} - E_{4+j,4+i}) - 2\delta_{ij}(E_{10,10} - E_{12,12})] \tag{A.20}
\]
\[
\{ F_{i,12}, G_{j,12} \} = i[-(E_{i,4+j} - E_{4+j,i}) - (E_{4+i,j} - E_{j,4+i}) - 2\delta_{ij}(E_{10,12} + E_{12,10})]. \tag{A.21}
\]

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