Abstract. Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics founded by Friedman and developed extensively by Simpson and others. The aim of RM is to find the minimal axioms needed to prove a theorem of ordinary, i.e. non-set-theoretic, mathematics. As suggested by the title, this paper deals with the study of the topological notions of dimension and paracompactness, inside Kohlenbach’s higher-order RM. As to splittings, there are some examples in RM of theorems $A, B, C$ such that $A \leftrightarrow (B \land C)$, i.e. $A$ can be split into two independent (fairly natural) parts $B$ and $C$, and the aforementioned topological notions give rise to a number of splittings involving highly natural $A, B, C$. Nonetheless, the higher-order picture is markedly different from the second-one: in terms of comprehension axioms, the proof in higher-order RM of e.g. the paracompactness of the unit interval requires full second-order arithmetic, while the second-order/countable version of paracompactness of the unit interval is provable in the base theory $\text{RCA}_0$. We obtain similarly ‘exceptional’ results for the Urysohn identity, the Lindelöf lemma, and partitions of unity.

1. Introduction

Reverse Mathematics (RM hereafter) is a program in the foundations of mathematics initiated around 1975 by Friedman ([15],[16]) and developed extensively by Simpson ([33]) and others. We refer to [33] for a basic introduction to RM and to [11],[13] for an (updated) overview of RM. We will assume basic familiarity with RM, the associated ‘Big Five’ systems and the ‘RM zoo’ ([12]). We do introduce Kohlenbach’s higher-order RM in some detail Section 2.1.

Topology studies those properties of space that are invariant under continuous deformations. The modern subject was started by Poincaré’s Analysis Situs at the end of 19th century, and rapid breakthroughs were established by Brouwer in a two-year period starting 1910, as discussed in [8, p. 168]. We generally base ourselves on the standard monograph by Munkres ([31]).

Now, the RM of topology has been studied inside the framework of second-order arithmetic in e.g. [32],[33]. This approach makes heavy use of coding to represent uncountable objects via countable approximations. Hunter develops the higher-order RM of topology in [19], and points out some potential problems with the aforementioned coding practice. Hunter’s observations constitute our starting point and motivation: working in higher-order RM, we study the RM of notions like dimension and paracompactness motivated as follows: the former is among the most...
basic/fundamental notions of topology, while the latter has already been studied in
second-order RM, e.g. in the context of metrisation theorems ([34,43]).

As it turns out, the picture we obtain in higher-order RM is completely different
from the well-known picture in second-order RM. For instance, in terms of comprehen-
sion axioms, the proof in higher-order RM of the paracompactness of the unit
interval requires full second-order arithmetic by Theorem 3.11, while the second-
order/countable version of paracompactness of the unit interval is provable in the
base theory $\text{RCA}_0$ of second-order RM by [33, II.7.2]. Furthermore, the Urysohn
identity connects various notions of dimension, and a proof of this identity for $[0,1]$
similarly requires (comprehension axioms as strong as) full second-order arithmetic.
We also study the Lindelöf lemma and partitions of unity.

The aforementioned major difference between second-order and higher-order RM
begs the question as to how robust the results in this paper are. For instance,
do our theorems depend on the exact definition of cover? What happens if we
adopt a more general definition? We show in Sections 3.1 and 3.4 that our results
indeed boast a lot of robustness, and in particular that they do not depend on the
definition of cover, even in the absence of the axiom of (countable) choice. The
latter feature is important in view of the topological ‘disasters’ (See e.g. [20]) that
manifest themselves in the absence of the axiom of (countable) choice.

We also obtain a number of highly natural splittings, where the latter is defined as
follows. As discussed in e.g. [18, §6.4], there are (some) theorems $A, B, C$ in the RM
zoo such that $A \leftrightarrow (B \land C)$, i.e. $A$ can be split into two independent (fairly natural)
parts $B$ and $C$ (over $\text{RCA}_0$). It is fair to say that there are only few natural examples
of splittings in second-order RM, though such claims are invariably subjective in
nature. A large number of splittings in higher-order RM may be found in [40].

Finally, like in [36,37], statements of the form ‘a proof of this theorem requires
full second-order arithmetic’ should be interpreted in reference to the usual scale of
comprehension axioms that is part of the Gödel hierarchy (See Appendix A for the
latter). The previous statement thus (merely) expresses that there is no proof of this
theorem using comprehension axioms restricted to a sub-class, like e.g. $\Pi_1^1$-formulas
(with only first and second-order parameters). An intuitive visual clarification may
be found in Figure 1 where the statement the unit interval is paracompact is shown
to be independent of the medium range of the Gödel hierarchy. Similarly, when
we say ‘provable without the axiom of choice’, we ignore the use of the very weak
instances of the latter included in the base theory of higher-order RM.

In conclusion, it goes without saying that our results highlight a major
difference between second- and higher-order arithmetic, and the associated development of
RM. We leave it the reader to draw conclusions from this observation.

2. Preliminaries

2.1. Higher-order Reverse Mathematics. We sketch Kohlenbach’s higher-order
Reverse Mathematics as introduced in [23]. In contrast to ‘classical’ RM, higher-
order RM makes use of the much richer language of higher-order arithmetic.

As suggested by its name, higher-order arithmetic extends second-order arith-
metic. Indeed, while the latter is restricted to numbers and sets of numbers, higher-
order arithmetic also has sets of sets of numbers, sets of sets of sets of numbers,
et cetera. To formalise this idea, we introduce the collection of all finite types $T$, defined by the two clauses:

(i) $0 \in T$ and (ii) If $\sigma, \tau \in T$ then $(\sigma \to \tau) \in T$,

where 0 is the type of natural numbers, and $\sigma \to \tau$ is the type of mappings from objects of type $\sigma$ to objects of type $\tau$. In this way, $1 \equiv 0 \to 0$ is the type of functions from numbers to numbers, and where $n + 1 \equiv n \to 0$. Viewing sets as given by characteristic functions, we note that $\mathbb{Z}_2$ only includes objects of type 0 and 1.

The language $L_\omega$ includes variables $x^\rho, y^\rho, z^\rho, \ldots$ of any finite type $\rho \in T$. Types may be omitted when they can be inferred from context. The constants of $L_\omega$ includes the type 0 objects 0, may be omitted when they can be inferred from context. The constants of $L_\omega$ includes the recursor constant $\tau$ defined by the two clauses:

- From numbers to numbers, and where $n = 0$.
- Equality at higher types is defined in terms of `=` as follows: for any objects $x^\tau, y^\tau$, we have

$$[x =_\tau y \equiv (\forall z^1 \ldots z^k)(xz_1 \ldots z_k =_0 yz_1 \ldots z_k)], \quad (2.1)$$

if the type $\tau$ is composed as $\tau = (\tau_1 \to \ldots \to \tau_k \to 0)$. Furthermore, $L_\omega$ also includes the recursor constant $R_\sigma$ for any $\sigma \in T$, which allows for iteration on type $\sigma$-objects as in the special case $RCA_2$. Formulas and terms are defined as usual.

**Definition 2.1.** The base theory $RCA_0^\omega$ consists of the following axioms:

1. Basic axioms expressing that 0, 1, $<_0$, $+_0$, $\times_0$ form an ordered semi-ring with equality $=_0$.
2. Basic axioms defining the well-known $\Pi$ and $\Sigma$ combinators (aka $K$ and $S$ in $\lambda$-calculus), which allow for the definition of $\lambda$-abstraction.
3. The defining axiom of the recursor constant $R_0$: For $m^0$ and $f^1$:

$$R_0(f, m, 0) := m \text{ and } R_0(f, m, n + 1) := f(R_0(f, m, n)). \quad (2.2)$$

4. The axiom of extensionality: for all $\rho, \tau \in T$, we have:

$$(\forall x^\rho, y^\rho, \varphi^\rho)(x =_\rho y \to \varphi(x) =_\tau \varphi(y)]. \quad (\exists_{\rho, \tau})$$

5. The induction axiom for quantifier-free formulas of $L_\omega$.
6. $QF-AC^{1,0}$: The quantifier-free axiom of choice as in Definition 2.2.

**Definition 2.2.** The axiom $QF-AC$ consists of the following for all $\sigma, \tau \in T$:

$$(\forall x^\sigma)(\exists y^\tau)A(x, y) \to (\exists Y^\sigma)(\forall y^\tau)A(x, Y(x)), \quad (QF-AC^{\sigma, \tau})$$

for any quantifier-free formula $A$ in the language of $L_\omega$.

As discussed in [23], $RCA_0^\omega$ and $RCA_0$ prove the same sentences ‘up to language’ as the latter is set-based and the former function-based. Recursion as in (2.2) is called primitive recursion; the class of functionals obtained from $R_\rho$ for all $\rho \in T$ is called Gödel’s system $T$ of all (higher-order) primitive recursive functionals.

We use the usual notations for natural, rational, and real numbers, and the associated functions, as introduced in [23, p. 288-289].

**Definition 2.3** (Real numbers and related notions in $RCA_0^\omega$).

1. Natural numbers correspond to type zero objects, and we use ‘$n^0$’ and ‘$n \in \mathbb{N}$’ interchangeably. Rational numbers are defined as signed quotients of natural numbers, and ‘$q \in \mathbb{Q}$’ and ‘$<_{\mathbb{Q}}$’ have their usual meaning.

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1To be absolutely clear, variables (of any finite type) are allowed in quantifier-free formulas of the language $L_\omega$: only quantifiers are banned.
(2) Real numbers are coded by fast-converging Cauchy sequences \(q(\cdot) : \mathbb{N} \to \mathbb{Q}\), i.e. such that \((\forall n, n')((q_n - q_{n+1}) < q \frac{1}{n^2})\). We use Kohlenbach’s ‘hat function’ from [23, p. 289] to guarantee that every \(f^1\) defines a real number.

(3) We write ‘\(x \in \mathbb{R}\)’ to express that \(x^1 := q(1)\) represents a real as in the previous item and write \([x](k) := q_k\) for the \(k\)-th approximation of \(x\).

(4) Two reals \(x, y\) represented by \(q(\cdot)\) and \(r(\cdot)\) are equal, denoted \(x =_\mathbb{R} y\), if 
\[
(\forall n)((q_n - r_n) < \frac{1}{n^2}).
\]
Inequality ‘\(<_\mathbb{R}\)’ is defined similarly. We sometimes omit the subscript ‘\(\mathbb{R}\)’ if it is clear from context.

(5) Functions \(F : \mathbb{R} \to \mathbb{R}\) are represented by \(\Phi^{1 \rightarrow 1}\) mapping equal reals to equal reals, i.e. \((\forall x, y \in \mathbb{R})(x =_\mathbb{R} y \to \Phi(x) =_\mathbb{R} \Phi(y))\).

(6) The relation ‘\(x \leq_\mathbb{R} y\)’ is defined as in (2.1) but with ‘\(<_0\)’ instead of ‘\(=\)’.

Binary sequences are denoted ‘\(f^1, g^1 \leq_1 1\)’, but also ‘\(f, g \in C\)’ or ‘\(f, g \in 2^{\mathbb{N}}\)’.

Finally, we mention the ECF-interpretation, of which the technical definition may be found in [37, p. 138, 2.6]. Intuitively speaking, the ECF-interpretation \([A]_{ECF}\) of a formula \(A \in \mathbb{L}_o\) is just \(A\) with all variables of type two and higher replaced by countable representations of continuous functionals. The ECF-interpretation connects \(RCA_0\) and \(RCA_0\) (see [23, Prop. 3.1]) in that if \(RCA_0\) proves \(A\), then \(RCA_0\) proves \([A]_{ECF}\), again ‘up to language’, as \(RCA_0\) is formulated using sets, and \([A]_{ECF}\) is formulated using types, namely only using type zero and one objects.

### 2.2. Some axioms of higher-order arithmetic

We introduce some functionals which constitute the counterparts of \(Z_2\), and some of the Big Five systems, in higher-order RM. We use the formulation of these functionals as in [23].

First of all, \(ACA_0\) is readily derived from the following ‘Turing jump’ functional:
\[
(\exists f^2 \leq_2 1)(\forall f^1)[(\exists n)(f(n) = 0) \leftrightarrow \varphi(f) = 0],
\]
and \(ACA_0^w \equiv RCA_0^w + (\exists^2)\) proves the same \(\Pi^1_2\)-sentences as \(ACA_0\) by [39, Theorem 2.2]. This functional is discontinuous at \(f = 11\ldots\), and \((\exists^2)\) is equivalent to the existence of \(F : \mathbb{R} \to \mathbb{R}\) such that \(F(x) = 1\) if \(x > 0\) and 0 otherwise ([23, 33]).

Secondly, full second-order arithmetic \(Z_2^\omega\) is readily derived from the sentence:
\[
(\exists Y^3 \leq_3 1)(\forall Y^2)[(\exists f^1)Y(f) = 0 \leftrightarrow E(Y) = 0],
\]
and we define \(Z_2^\omega \equiv RCA_0^w + (\exists^3)\). The (unique) functional from \((\exists^3)\) is also called ‘\(^3^3\)’, and we will use a similar convention for other functionals.

Thirdly, the comprehension for Cantor space functional, introduced in [37], yields a conservative extension of \(WKL_0\) by [23, Prop. 3.15]:
\[
(\exists \kappa^3 \leq_3 1)(\forall Y^2)[\kappa_0(Y) = 0 \leftrightarrow (\exists f \in C)(Y(f) > 0)],
\]
Fourth, recall that the Heine-Borel theorem (aka Cousin’s lemma) states the existence of a finite sub-cover for an open cover of a compact space. Now, a functional \(\Psi : \mathbb{R} \to \mathbb{R}^+\) gives rise to the canonical cover \(\bigcup_{x \in I} I_x^\psi\) for \(I = [0, 1]\), where \(I_x^\psi\) is the open interval \((x - \Psi(x), x + \Psi(x))\). Hence, the uncountable cover \(\bigcup_{x \in I} I_x^\psi\) has a finite sub-cover by the Heine-Borel theorem; in symbols:
\[
(\forall \Psi : \mathbb{R} \to \mathbb{R}^+)\exists(y_1, \ldots, y_k) (\forall x \in I)(\exists i \leq k)(x \in I_y^\psi).
\]
(HBU)

There is also the highly similar Lindelöf lemma stating the existence of a countable sub-cover of possibly non-compact spaces. We restrict ourselves to \(\mathbb{R}\) as follows.
\[
(\forall \Psi : \mathbb{R} \to \mathbb{R}^+)\exists(\exists^0)(\forall x \in \mathbb{R})(\exists n^0)(x \in I_{\Phi(n)}^\psi),
\]
(LIN)
By the results in [36,37], \( Z_\omega^2 \) proves HBU, but \( \Pi^1_1\text{-CA}_0^\omega \) cannot (for \( k \geq 1 \)). The same holds for LIN, if we add QF-AC\(^0,1\), while the latter implies HBU \( \leftrightarrow \text{ [WKL + LIN]} \).

The importance/naturalness of HBU and LIN is discussed in Section 3.1.

Furthermore, since Cantor space (denoted \( C \) or \( 2^\omega \)) is homeomorphic to a closed subset of \([0,1]\), the former inherits the same property. In particular, for any \( G^2 \), the corresponding ‘canonical cover’ of \( 2^\omega \) is \( \bigcup_{f \in 2^\omega} [I^f G(f)] \) where \([\sigma^0] \) is the set of all binary extensions of \( \sigma \). By compactness, there is a finite sequence \( \{f_0, \ldots, f_n\} \) such that the set of \( \bigcup_{i \leq n} [I^f G(f_i)] \) still covers \( 2^\omega \). By [36, Theorem 3.3], HBU is equivalent to the same compactness property for \( C \), as follows:

\[
(\forall G^2)(\exists (f_1, \ldots, f_k))((\forall f^1 \leq 1)(\exists i \leq k)(f \in [I^f G(f_i)])) \quad \text{(HBU)}
\]

We now introduce the specification \( SCF(\Theta) \) for a functional \( \Theta^{2 \rightarrow 1^\omega} \) which computes such a finite sub-cover. We refer to such a functional \( \Theta \) as a realiser for the compactness of Cantor space, and simplify its type to ‘3’ to improve readability.

\[
(\forall G^2)(\forall f^1 \leq 1)(\exists g \in \Theta(G))(f \in [I^f G(g)]) \quad \text{(SCF(\Theta))}
\]

Clearly, there is no unique \( \Theta \) as in \( SCF(\Theta) \) (just add more binary sequences to \( \Theta(G) \)); nonetheless, we have in the past referred to any \( \Theta \) satisfying \( SCF(\Theta) \) as ‘the’ special fan functional \( \Theta \), and we will continue this abuse of language.

3. Reverse Mathematics of Topology

We study the RM of theorems of topology pertaining to the following notions: (topological) dimension and the Urysohn identity (Section 3.2) and paracompactness (Section 3.3). We introduce a suitable notion of cover (Section 3.1) and show (Section 3.4) that our aforementioned results are independent of the definition of cover, without making use of the axiom of choice. We discuss similar results for the Lindelöf lemma and partitions of unity (Section 3.4).

3.1. Introduction: topology in higher-order arithmetic. We discuss how higher-order arithmetic can accommodate the central topological notion of cover. In particular, we introduce a generalisation of the notion of cover used in [36,37] and shall show in Section 3.4 that the new notion yields covering lemmas equivalent to the original, without a need for the axiom of countable choice.

First of all, early covering lemmas, like the Cousin and Lindelöf lemmas, did not make use of the (general) notion of cover. Indeed, Cousin and Lindelöf talk about (uncountable) covers of \( \mathbb{R}^2 \) and \( \mathbb{R}^n \) as follows (resp. in 1895 and 1903):

- we suppose that to each point of \( S \) corresponds a circle of non-zero finite radius and with this point as centre ([6, p. 22])
- for every point \( P \), let us construct a sphere \( S_P \) with \( P \) as the centre and a variable radius \( \rho_P \) ([25, p. 698])

To stay close to the original formulation by Cousin and Lindelöf, we introduced in [36,37] the notion of ‘canonical’ open covers \( \bigcup_{x \in I} I_x^\Psi \) of \( I = [0,1] \) generated by \( \Psi : I \rightarrow \mathbb{R}^+ \) and where \( I_x^\Psi = (x - \Psi(x), x + \Psi(x)) \). Unfortunately, such covers always involve points that are covered by arbitrarily many intervals; this property makes such covers unsuitable for e.g. the study of topological dimension, in which the (minimal) number of intervals covering a point is central.

Secondly, the previous observation shows that we have to generalise our notion of canonical cover, and we shall do this by considering \( \psi : I \rightarrow \mathbb{R} \). i.e. we allow empty
In this way, we say that ‘∪_{x \in I} \psi_x' covers [0, 1]' if (∀x ∈ I)(∃y ∈ I)(x ∈ I)^y_x. This notion of cover gives rise to the following version of the Heine-Borel theorem.

\((\forall \psi : I \to \mathbb{R}) [I \subset \bigcup_{x \in I} \psi_x \rightarrow (\exists y_1, \ldots, y_k \in I)(I \subset \bigcup_{i \leq k} I)^y_{x_i}]. \) (HBT)

We establish in Section 3.4 that our ‘new’ notion of cover is quite robust by showing that (i) HBU ↔ HBT over RCA^ω_0 + QF-AC, i.e., the new notion of cover is not a real departure from the old one, and (ii) the previous equivalence can also be proved without the axiom of choice. Item (ii) should be viewed in the light of the topological ‘disasters’ (See e.g. [20]) that apparently happen in the absence of the axiom of (countable) choice. We also show that any notion of cover definable in Z^2_2 inherits the aforementioned ‘nice’ properties. Thus, we may conclude that our results boast a lot of robustness, and in particular that they do not depend on the definition of cover, even in the absence of the axiom of (countable) choice.

Finally, we discuss the mathematical naturalness of HBU and (3^2).

**Remark 3.1.** Dirichlet already discusses the characteristic function of the rationals, which is essentially $\exists^{2^2}$, around 1829 in [11], while Riemann defines a function with countably many discontinuities via a series in his Habilitationsschrift ([22, p. 115]). Furthermore, the Cousin lemma from [6, p. 22], which is essentially HBU, dates back about 135 years. As shown in [56], (3^2) and HBU are essential for the development of the gauge integral ([3]). This integral was introduced by Denjoy ([7]), in a different, more complicated form, around the same time as the Lebesgue integral; the reformulation of Denjoy’s integral by Henstock and Kurzweil in Riemann-esque terms (See [3, p. 15]), provides a direct and elegant formalisation of the Feynman path integral ([5,28,30]) and financial mathematics ([29,30]).

**3.2. The notion of dimension.** The notion of dimension of basic spaces like [0, 1] or $\mathbb{R}^n$ is intuitively clear to most mathematicians, but finding a formal definition of dimension that does not depend on the topology is a non-trivial problem.

We introduce three notions of dimension: the topological dimension dim $X$ and the small and large inductive dimensions ind $X$ and Ind $X$. We study the RM properties of the Urysohn identity ([11, p. 272]) which expresses that these dimension are equal for a large class of spaces, including separable metric spaces.

First of all, the covering dimension, later generalised to the topological dimension, goes back to Lebesgue. Indeed, Munkres writes the following:

We shall define, for an arbitrary topological space $X$, a notion of topological dimension. It is the “covering dimension” originally defined by Lebesgue. ([31, p. 305])

The following definition of topological dimension may be found in Munkres’ seminal monograph [31, p. 161], and in [11, p. 274], [13, Ex. 1.7.E and Prop. 3.2.2].

**Definition 3.2.** (Order) A collection $\mathcal{A}$ of subsets of the space $X$ is said to have order $m + 1$ if some point of $A$ lies in $m + 1$ elements of $\mathcal{A}$, and no point of $X$ lies in more than $m + 1$ elements of $A$.

The collected works of Pincherle contain a footnote by the editors (See [38, p. 67]) which states that the associated Teorema (published in 1882) corresponds to the Heine-Borel theorem. Moreover, Weierstrass proves the Heine-Borel theorem (without explicitly formulating it) in 1880 in [50, p. 204]. A detailed motivation for these claims may be found in [20, p. 96-97].
Definition 3.3. [Refinement] Given a collection \( A \) of subsets of \( X \), a collection \( B \) is said to refine \( A \), or to be a refinement of \( A \) if for each element \( B \in B \) there is an element \( A \in A \) such that \( A \subseteq B \).

Definition 3.4. [Topological dimension] A space \( X \) is said to be finite-dimensional if there is \( m \in \mathbb{N} \) such that for every open covering \( A \) of \( X \), there is an open covering \( B \) of \( X \) that refines \( A \) and has order at most \( m + 1 \). The topological dimension of \( X \) is the smallest value of \( m \) for which this statement holds; we denote it by \( \dim X \).

In the context of \( \text{RCA}_0^\omega \), we say that \( '\phi : I \to \mathbb{R} \) is a refinement of \( \psi : I \to \mathbb{R} \) if \((\forall x \in I)(\exists y \in I)(I_x^\phi \subseteq I_y^\psi)\). With this definition in place, statements like \( ' \)the topological dimension of \([0, 1] \) is at most 1', denoted \( '\dim ([0, 1]) \leq 1' \), makes perfect sense in \( \text{RCA}_0^\omega \). Such a statement turns out to be quite hard to prove, as full second-order arithmetic is needed to prove HBT by Theorem 3.15.

Theorem 3.5. The system \( \text{ACA}_0^\omega + \text{QF-AC}^{1,1} + \left[\dim ([0, 1]) \leq 1 \right] \) proves HBT.

Proof. Let \( \psi : I \to \mathbb{R} \) be such that \( \bigcup_{x \in I} I_x^\psi \) covers \([0, 1] \), and let \( \phi : I \to \mathbb{R} \) be the associated refinement of \( \phi \) at most 1. Since the innermost formula is \( x \in \mathbb{N} \) (with parameters), we may apply \( \text{QF-AC}^{1,1} \) to \((\forall x \in I)(\exists y \in I)(y \in I_x^\psi)\) to obtain \( \Xi^{1,1} \) such that \( \Xi(x) \) provides such \( y \). Define \( \zeta^{0,1} \) as follows: \( \zeta(0) := \Xi(0) + \phi(\Xi(0)) \) and \( \zeta(n + 1) := \Xi(\zeta(n)) + \phi(\Xi(\zeta(n))) \). Now consider the following formula:

\[
(\exists x \in I)(\forall n \in \mathbb{N})(\zeta(n) <_R x) \tag{3.1}
\]

If \( (3.1) \) is false, take \( x = 1 \) and note that if \( \zeta(n_0) \geq_R 1 \), the finite sequence \( I^\phi_{\Xi(0)}, I^\phi_{\Xi(1)}, I^\phi_{\Xi(\zeta(n_0 + 1))}, \ldots, I^\phi_{\Xi(\zeta(n_1 + 1))} \) yield a finite sub-cover of \( \bigcup_{x \in I} I_x^\psi \). In this case, we apply \( \text{QF-AC}^{1,1} \) (using also \( (\exists^2) \)) to \((\forall x \in I)(\exists y \in I)(I_x^\psi \subseteq I_y^\psi)\) to go from a finite sub-cover of \( \bigcup_{x \in I} I_x^\psi \) to a finite sub-cover of \( \bigcup_{x \in I} I_x^\psi \), and HBT follows.

If \( (3.1) \) is true, let \( x_0 \in I \) be the least \( x \in I \) such that \( \zeta(n) <_R x \). Since \( \zeta \) is \( \Pi_1^0 \), we can use \( \exists^2 \) and the usual interval-halving technique to find \( x_0 \); alternatively, use the monotone convergence theorem \( \Xi^{3,1} \) (III.2.2.), provable in \( \text{ACA}_0 \). However, \( I^\phi_{\Xi(x_0)} \) covers \( x_0 \), and thus for \( n_1 \) large enough, \( \zeta(n) \) for \( n \geq n_1 \) will all be in the former interval, by the leastness of \( x_0 \). But then there are points of order 3 in the (by definition non-empty) intersection of \( I^\phi_{\Xi(\zeta(n_1))} \) and \( I^\phi_{\Xi(\zeta(n_1 + 1))} \), as this intersection is also inside \( I^\phi_{\Xi(x_0)} \). This observation contradicts the assumption \( \dim([0, 1]) \leq 1 \), and hence \( (3.1) \) must be false, and we are done.

The previous theorem has a number of corollaries. First of all, we obtain an equivalence over a weak base theory; we believe the components of the left-hand side to be independent\(^4\), i.e. that a proper ‘splitting’ of HBT is achieved.

Corollary 3.6. \( \text{RCA}_0^\omega + \text{QF-AC}^{1,1} \) proves that \( (\text{WKL} + [\dim(I) = 1]) \leftrightarrow \text{HBT} \).

Proof. For the forward direction, in case \( (\exists^2) \), the proof of the theorem goes through. In case \( \neg (\exists^2) \), all \( F : \mathbb{R} \to \mathbb{R} \) are continuous, while all \( F^2 \) are continuous on Baire space, and hence uniformly continuous (and thus bounded) on

\(^4\)Firstly, \( \text{WKL} \) cannot imply the other component, as \( \text{RCA}_0^\omega + (\exists^2) + \text{QF-AC} \) does not imply HBU by the results in [33, §6]. Secondly, \( \dim(I) = 1 \) seems to be consistent with recursive mathematics by the proof of [4, Theorem 6.1, p. 69], i.e. the former cannot imply \( \text{WKL} \).
Cantor space by WKL (See [19, Prop. 3.7 and 3.12] and [24, Prop. 4.10]). Now consider the following statement, which (only) holds since \( \psi : I \to \mathbb{R} \) is continuous:

\[
(\forall f \in C)(\exists q \in I \cap \mathbb{Q})(\exists n \in \mathbb{N})(r(f) \in I^0_q \land \psi(q) \geq \frac{1}{2^n}),
\]

where \( r(f) \) is \( \sum_{n=0}^{\infty} \frac{f(n)}{2^n} \) for binary \( f \), and where the underlined formula is \( \Sigma^0_1 \). Applying QF-AC\(^{1,0} \) to (3.2), there is \( \Xi^2 \) such that \( n \leq \Xi(f) \) in (3.2). Since \( \Xi \) is bounded on \( C \), there is \( N_0 \in \mathbb{N} \) such that

\[
\Xi(f) \leq N_0 \quad \text{for all } f 
\]

which immediately implies that \( \cup_{x \in I} I^0_x \) has a finite sub-cover (generated by rationals), and the latter may be found by applying QF-AC\(^{1,0} \) to (3.3) and iterating the choice function at most \( 2^{N_0+1} \) times. Since \( \frac{1}{2^n} \) has an obvious binary representation, we do not need to convert arbitrary \( x \in I \) to binary. We obtain HBT in each case, and \( (\exists^2) \lor \neg(\exists^2) \) finishes the proof.

For the reverse direction, note that HBT \( \to \) HBU \( \to \) WKL. To prove \( \dim(I) = 1 \), the finite sub-cover provided by HBT is readily converted to a refinement of order 1 using \( \exists^2 \), as the latter functional can decide equality between real numbers. Now, in case \( \neg(\exists^2) \), obtain [20] in the same way as above, and let \( \Xi \) be a choice function that provides \( \Xi(f) = q \). Define \( \zeta \) as follows: \( \zeta(0) := \Xi(00\ldots) + \frac{1}{2^{N_0}} \) and \( \zeta(n+1) := \Xi(\zeta(n)) + \frac{1}{2^{N_0}} \). For \( n > 2^{N_0+1} \), this function readily yields a finite open cover of \( I \) that is also a refinement of the cover generated by \( \psi \). Since all points are rationals, we can refine this cover to have order 1, and \( (\exists^2) \lor \neg(\exists^2) \) finishes the proof.

For future reference, we note that the proof actually establishes \( \text{RCA}^0_\omega \vdash (\exists^2) \lor \neg(\exists^2) \rightarrow [\text{WKL} + \dim(I) = 1] \leftrightarrow \text{HBT} \), i.e. the axiom of choice is not used.

It is a natural question (posed before by Hirschfeldt; see [27, §6.1]) whether the axiom of choice is really necessary in the previous (and below) theorems. We answer this question in the negative in Section 4.1.3.

Next, in order to prove the next corollary concerning Urysohn’s identity, we introduce the notion of inductive definition as in [15, §1.1.1].

**Definition 3.7.** [Inductive dimension] We inductively define the smallest inductive dimension \( \text{ind} \) for a topological space \( X \) as follows.

- \( \text{(d1)} \) For the empty set \( \emptyset \), we define \( \text{ind} \emptyset = \text{Ind} \emptyset = -1 \);
- \( \text{(d2)} \) \( \text{ind} \) \( X \leq n \), where \( n = 0, 1, \ldots \), if for every point \( x \in X \) and each neighbourhood \( V \subset X \) of the point \( x \) there exists an open set \( U \subset X \) such that \( x \in U \subset V \) and \( \text{ind}(\partial U) < n - 1 \);
- \( \text{(d3)} \) \( \text{ind} \) \( X = n \) if \( \text{ind} \) \( X \leq n \) and \( \text{ind} \) \( X > n - 1 \), i.e., the inequality \( \text{ind} \) \( X < n - 1 \) does not hold;
- \( \text{(d4)} \) \( \text{ind} \) \( X = \infty \) if \( \text{ind} \) \( X > n \) for \( n = -1, 0, 1, \ldots \).

The **large inductive dimension** \( \text{Ind} \) \( X \) is obtained by replacing (d2) by:

- \( \text{(d2')} \) \( \text{Ind} \) \( X < n \), where \( n = 0, 1, \ldots \), if for every closed set \( A \subset X \) and each open set \( V \subset X \) which contains the set \( A \) there exists an open set \( U \subset V \) such that \( A \subset U \subset V \) and \( \text{Ind}(\partial U) < n - 1 \).

If \( X \) is Euclidean space, \( V \) is generally chosen to be a ball centred at \( x \).

In light of Definition 3.7 the (small and large) inductive dimension of real numbers, or the unit interval, makes sense in \( \text{RCA}^0_\omega \), and is respectively 0 and 1. Moreover, the Urysohn identity is the statement that \( \dim X = \text{ind} X = \text{Ind} X \), and holds
for a large class of spaces $X$; this identity constitutes one of the main problems in dimension theory, according to [1, p. 274], while it is called the the fundamental theorem of dimension theory in [13].

**Corollary 3.8.** The system $\text{RCA}_0^\omega + \text{QF-AC}^{1,1}$ proves that HBT is equivalent to: the conjunction of WKL and Urysohn’s identity for the unit interval.

**Proof.** Immediate from Corollary 3.6. □

### 3.3. Paracompactness

The notion of paracompactness was introduced in 1944 by Dieudonné in [9] and plays an important role in the characterisation of metrisable spaces via e.g. Smirnov’s metrisation theorem ([31, p. 261]). The fact that every metric space is paracompact is Stone’s theorem (See [17, 45] and [31, p. 252]).

Our interest in paracompactness stems in part from its occurrence in classical RM (See e.g. [33, 34, 43]), as detailed in Remark 3.18. The aim of this section is to show that there is a huge difference in logical and computational hardness between the ‘second-order/countable’ version of paracompactness, and the ‘actual’ definition. Indeed, the fact that the unit interval is paracompact implies HBT; moreover, the latter can be ‘split’ into the former plus WKL by Corollary 3.12.

Munkres states the following definition of paracompactness in [31, p. 253].

**Definition 3.9.** [Locally finite] A collection $A$ of subsets of a space $X$ is locally finite if any $x \in X$ has a neighbourhood that intersects only finitely many $A \in A$.

**Definition 3.10.** [Paracompact] A space $X$ is paracompact if every open covering $A$ of $X$ has a locally finite open refinement $B$ that covers $X$.

With these definitions, the statement that the unit interval is paracompact, makes sense in $\text{RCA}_0^\omega$. By Stone’s theorem, a metric space is paracompact, but this fact is not provable in ZF alone (See [17]). Similarly, Stone’s theorem for the unit interval is not provable in any system $\Pi^1_1-\text{CA}_0^\omega$ by the following theorem.

**Theorem 3.11.** The system $\text{ACA}_0^\omega + \text{QF-AC}^{1,1} + \text{‘}[0,1] \text{ is paracompact’}$ proves HBT.

**Proof.** We use the proof of Theorem 3.5 with minor modification. Let $\psi : I \to \mathbb{R}$ be such that $\bigcup_{x \in I} I_x^\psi$ covers $[0,1]$, and let $\phi : I \to \mathbb{R}$ be a locally finite refinement. Assume (3.1), where let $x_0 \in I$, $\zeta^{\phi_{\psi}}$, $\Xi^{\psi}_1$ are as in the aforementioned proof. Clearly, any neighbourhood of $x_0$ will contain all intervals $I_{\psi}^{\phi_{\psi}(n)}$ for $n$ large enough. This observation contradicts the assumption that $[0,1]$ is paracompact, and hence (3.1) must be false, implying HBT as in the proof of Theorem 3.5. □

The following corollary is proved in the same way as Corollary 3.6; the left-hand side constitutes a proper ‘splitting’ of HBT, as the ECF-translation of ‘[0,1] is paracompact’ is essentially the statement that $[0,1]$ is countably paracompact, and the latter is provable in $\text{RCA}_0$ by [43] II.7.2.

**Corollary 3.12.** $\text{RCA}_0^\omega + \text{QF-AC}^{1,1}$ proves [WKL + ‘[0,1] is paracompact’] $\leftrightarrow$ HBT.

Another interpretation of the previous corollary is as follows: by the results in [49], the notion of compactness is equivalent to ‘paracompact plus pseudo-compact’ for a large class of spaces, and pseudo-compactness essentially expresses that continuous functions are bounded on the space at hand, i.e. the pseudo-compactness of $[0,1]$ is equivalent to WKL by [43] IV.2.3 and [24] Prop. 4.10].
The following remark highlights the difference between ‘actual’ and ‘second-order/countable’ paracompactness. It also suggests formulating Corollary 3.14.

**Remark 3.13 (Paracompactness in second-order RM).** Simpson proves in [43 II.7.2] that over RCA₀, complete separable metric spaces are countably paracompact, and Mummert in [34 Lemma 4.11] defines a realiser for paracompactness as in [43 II.7.2] inside ACA₀. This realiser plays a crucial role in the proof of Mummert’s metrisation theorem, called ‘MFMT’, inside Π₁²-CA₀ (See [34 §4]). Note that Π₁²-CA₀ occurs elsewhere in the RM of topology ([32, 33]). By Theorem 3.11, the (higher-order) statement the unit interval is paracompact is equivalent to HBT, and hence not provable in ∪ₖΠ₁²-CA₀, i.e. there is a huge difference in strength between ‘second-order/countable’ and ‘actual’ paracompactness. In fact, the logical hardness of the aforementioned statement dwarfs Π₁²-CA₀ from the RM of topology.

Let us call Ω₁⁻→¹ a ‘realiser for the paracompactness of [0, 1]’ if Ω(ψ)(1) : I → R yields a locally finite open refinement of the cover associated to ψ : I → R, and if $(∀x ∈ I)(I_x^{Ω(ψ)(1)} ⊆ I_Ω(ψ)(2)(x))$, (3.4)
i.e. the refining cover is ‘effectively’ included in the original one, just like in [34][43].

**Corollary 3.14.** A realiser Ω₁⁻→¹ for the paracompactness of [0, 1], together with Feferman’s µ, computes Θ such that SCF(Θ) via a term of Gödel’s T.

**Proof.** Immediate from the proof of Theorems 3.5 and 3.11. Note that Ξ is the identity function in case we consider covers generated by Ψ : I → R⁺ as in HBU. Furthermore, a realiser for HBU computes a realiser for HBUc, i.e. the special fan functional, via a term in Gödel’s T, as discussed in [36 §3.1] □

As it turns out, the condition (3.4) for a realiser for paracompactness has already been considered, namely as follows.

all proofs of Stone’s Theorem (known to the authors) actually prove a stronger conclusion which implies AC. It is based on an idea from [. . .]. Let us call a refinement V of U effective if there is a function a : V → U such that V ⊂ a(V) for all V ∈ V. ([17 p. 1217])

As it turns out, the notion of ‘effectively paracompact’ is intimately connected to the Lindelöf lemma, as discussed in Section 3.3.3.

**3.4. Covers in higher-order arithmetic.** In Section 3.1, we introduced a generalisation of the notion of cover used in [36][37]. In this section, we show that the new notion yields covering lemmas equivalent to the original, even in the absence of the axiom of choice. We also show that any notion of cover definable in second-order arithmetic inherits these ‘nice’ properties. We treat the Heine-Borel theorem, the Lindelöf lemma, as well as theorems pertaining to partitions of unity.

3.4.1. The Heine-Borel theorem. We prove HBT ↔ HBU with and without the axiom of choice in the base theory. In this way, our new notion of cover does not really change the Heine-Borel theorem.

**Theorem 3.15.** The system RCA₀ω + QF-AC¹¹ proves HBU ↔ HBT.

---

4The notion of ‘countably paracompact’ is well-known from Dowker’s theorem (See e.g. [21 p. 172]), but Simpson and Mummert do not use the qualifier ‘countable’ in [31][43].
Proof. The reverse direction is immediate. For the forward direction, in case \((3^2)\), we obtain HBU \(\iff\) WKL and proceed as in the proof of Corollary 3.6. In case of \((3^2)\), let \(\psi\) be as in HBT and consider \((\forall x \in I)(\exists y \in I)(x \in I_y^\psi)\). Since the innermost formula is \(\Sigma_1^1\), we may apply QF-AC\(^1\) to obtain \(\exists \Xi\) such that \((\forall x \in I)(x \in I_x^\psi)\). Since \(3^2\) provides a functional that converts real numbers in \(I\) to a unique binary representation, we may assume that \(\Xi\) is extensional on the reals. Now define \(\Psi : I \to \mathbb{R}^+ \by \Psi(x) := \min([x - (\Xi(x) - \psi(\Xi(x))]), [x - (\Xi(x) + \psi(\Xi(x)))])\), and note that \(I_x^\Psi \subseteq I_x^\Xi\). Applying HBU, we obtain a finite sub-cover of \(\cup_{x \in I} I_x^\psi\), say generated by \(y_1, \ldots, y_k \in I\), and \(\cup_{i \leq k} I_{(y_i)}^\psi\) is then a finite sub-cover of \(\cup_{x \in I} I_x^\psi\). \(\square\)

Recall that HBU is provable in \(\mathbb{Z}^2\) by \cite{bt} §4, i.e. without the axiom of choice. While the use of QF-AC\(^1\) in HBU \(\iff\) HBT seems essential, it is in fact not, by the following theorem. Note that IND is the induction axiom for all formulas in the language of RCA\(^0\); the base theory is not stronger than Peano arithmetic.

**Theorem 3.16.** The system RCA\(^0\) + IND + \((\kappa_3^0)\) proves HBU \(\iff\) HBT

**Proof.** The reverse direction is immediate. For the forward direction, in case \((3^2)\), we obtain HBU \(\iff\) WKL and proceed as in the proof of Corollary 3.6. In case of \((3^2)\), let \(\psi\) be as in HBT and note that \((\forall x \in I)(\exists y \in I)(x \in I_y^\psi)\) implies:

\[
(\forall x \in I)(\exists n \in \mathbb{N})(\exists y \in I)((x - \frac{1}{n}, x + \frac{1}{n}) \subseteq I_y^\psi),
\]

where the underlined formula is decidable thanks to \((3^2) \equiv [(3^1) + (\kappa_3^0)]\). Hence, applying QF-AC\(^0\) to \((3.5)\), we obtain \(\exists \Psi : I \to \mathbb{R}^+\) such that \(\cup_{x \in I} I_x^\psi\) is a canonical cover of \(I\). Applying HBU, we obtain a finite sub-cover of \(\cup_{x \in I} I_x^\psi\), say generated by \(x_1, \ldots, x_k \in I\). By definition, we have \((\forall x \in I)(\exists y \in I)(I_x^\psi \subseteq I_y^\psi)\), and

\[
(\forall w^{1^\omega})(\exists v^{1^\omega})(\forall i < |w|)(I_{w(i)}^\psi \subseteq I_{v(i)}^\psi)
\]

follows from IND by induction on \(|w|\). Applying \((3.6)\) for \(w = \langle x_1, \ldots, x_k \rangle\), we obtain a finite sub-cover for \(\cup_{x \in I} I_x^\psi\). The law of excluded middle finishes the proof. \(\square\)

As to open questions, we do not know if the base theory proves HBT outright or not. Similarly, we do not know if RCA\(^0\) + \((\kappa_3^0)\) proves WKL or not.

In conclusion, we mention two important observations that stem from the above.

First of all, it is easy to see that the first two proofs go through for the Heine-Borel theorem for \([0, 1]\) based on any ‘reasonable’ notion of cover. Indeed, as long as the formulas \(\langle x \in U_y \rangle\) and \(\langle [a, b] \subseteq U_x \rangle\) for the new notion of cover \(\cup_{x \in I} U_x\) of \(I\) are decidable in \(\mathbb{Z}^2\), the above proofs go through (assuming \((\kappa_3^0)\)). Since \(\mathbb{Z}^2\) can decide if \(Y : \mathbb{R} \to \{0, 1\}\) represents an open subset of \(\mathbb{R}\) (using the textbook definition of open set), this notion of ‘reasonable’ seems quite reasonable.

Secondly, emulating the proof of Theorem 3.16, we observe that the above results go through in the base theory with \((\kappa_3^0) + \text{IND}\) instead of QF-AC\(^1\). These include Theorem 3.5, Corollary 3.6, Corollary 3.8, Theorem 3.11, and Corollary 3.12. Thus, these results do not require the axiom of choice.

3.4.2. The Lindelöf lemma. We show that the Lindelöf lemma does not depend on the definition of cover, similar to the case of the Heine-Borel theorem. On one hand, since \([\text{LIN} + \text{WKL}] \iff \text{HBU}\), one expects such results. On the other hand, as
shown in [37 §5], the strength of the Lindelöf lemma is highly dependent on the exact formulation, but this dependence is not problematic for our context.

We introduce the notion of cover used in [37 §5], as follows. We consider \( \psi : I \rightarrow \mathbb{R}^2 \) and covers \( \bigcup_{x \in I} J^\psi_x \) in which the interval \( J^\psi_x := (\psi(x)(1), \psi(x)(2)) \) is potentially empty but \( \forall x \in I) (\exists y \in I) (x \in J^\psi_y) \). This notion of cover yields a ‘strong’ version of the Lindelöf lemma, as follows.

\[
(\forall \psi : \mathbb{R} \rightarrow \mathbb{R}^2) \left[ \mathbb{R} \subseteq \bigcup_{x \in \mathbb{R}} J^\psi_x \rightarrow (\exists f : \mathbb{N} \rightarrow \mathbb{R})(\mathbb{R} \subseteq \bigcup_{n \in \mathbb{N}} J^\psi_{f(n)}) \right].
\]

(LIL)

Similar to the proof of [36 Theorem 3.13], one proves that HBT ↔ [WKL + LIL] over RCA^0_0 + QF-AC^{1,1}. We first prove that the Lindelöf lemma LIL is equivalent to LIN from [36 §3]. We believe that LIN does not imply countable choice QF-AC^{0,1}.

**Theorem 3.17.** The system RCA^0_0 + QF-AC^{1,1} proves LIN ↔ LIL.

**Proof.** Similar to the proof of Theorem 3.15, the reverse direction is immediate, while in case of \( \neg(\exists^2) \) each principle is provable in RCA^0_0 using the sub-cover consisting of all rationals. In case of \( \exists^2 \), let \( \psi \) be as in LIL and consider \( (\forall x \in \mathbb{R})(\exists y \in \mathbb{R})(x \in J^\psi_y) \). Since the innermost formula is \( \Sigma^0_1 \), we may apply QF-AC^{1,1} to obtain \( \exists ! \) such that \( (\forall x \in \mathbb{R})(x \in J^\psi_{\Xi(x)}) \). Since \( \exists^2 \) provides a functional that converts real numbers to a binary representation, we may assume that \( \Xi \) is extensional on the reals. Now define \( \Psi : I \rightarrow \mathbb{R}^+ \) by \( \Psi(x) := \min \{|x - \psi(\Xi(x))(1)|, |x - \psi(\Xi(x))(2)|\} \), and note that \( I^\Psi \subseteq J^\psi_{\Xi(x)} \). Applying LIN, we obtain a countable sub-cover of \( \bigcup_{x \in I} I^\psi_x \), say generated by \( \Phi^{0,1} \), and \( \cup_{i \in \mathbb{N}} J^\psi_{\Xi(\Phi(i))} \) is a countable sub-cover of \( \bigcup_{x \in I} I^\psi_x \).

For completeness, we also mention the following corollary.

**Corollary 3.18.** RCA^0_0 + QF-AC^{1,1} proves LIL ↔ dim(\mathbb{R}) ≤ 1 ↔ \mathbb{R} is paracompact.

**Proof.** We only prove the equivalence between LIL and the paracompactness of \( \mathbb{R} \). By Theorem 3.17, it suffices to prove LIN. In case \( \neg(\exists^2) \), the latter is provable outright, as all \( \mathbb{R} \rightarrow \mathbb{R} \)-functions are continuous, and then the rationals provide a countable sub-cover for any open cover as in LIN. Similarly, paracompactness reduces to countable paracompactness, and the latter is provable in RCA^0_0 by [13 II.7.2]. In case of \( \exists^2 \), the paracompactness of \( \mathbb{R} \) (and hence \( I \) with minor modification) implies HBT by Theorem 3.11, and the aforementioned result HBT ↔ [WKL + LIL] over RCA^0_0 + QF-AC^{1,1} finishes the forward direction. The reverse direction is straightforward as \( \exists^2 \) decides inequalities between reals, and hence can easily refine the countable sub-cover provided by LIL.

As it turns out, we can avoid the use of QF-AC^{1,1} as follows

**Theorem 3.19.** The system RCA^0_0 + (\kappa^3_0) + IND proves [LIN + QF-AC^{0,1}] ↔ LIL.

**Proof.** For the forward implication, in case \( \neg(\exists^2) \), the rationals provides a countable sub-cover, as all functions on the reals are continuous by [23 Prop. 3.7]. In case of \( \exists^2 \), fix \( \psi : \mathbb{R} \rightarrow \mathbb{R}^2 \) as in LIL and formulate a version of [3.5] as follows:

\[
(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(\exists y \in \mathbb{R})((x - \frac{1}{2^n}, x + \frac{1}{2^n}) \subseteq J^\psi_y),
\]

(3.7)

\(^5\)The countable sub-cover in the Lindelöf lemma can be given by a sequence of reals generating the intervals (strong version), or just a sequence of intervals (weak version). The strong version implies QF-AC^{1,1} and hence is unprovable in ZF, while the weak version is provable in ZF. 


The underlined formula is again decidable thanks to $\exists^3$, and $QF$-$AC^{1,0}$ yields a functional $\Psi: \mathbb{R} \to \mathbb{R}^+$ such that the canonical cover also $\bigcup_{x \in \mathbb{R}} I^\ast_x$ covers $\mathbb{R}$. Applying $\text{LIN}$ to (3.7), we obtain a functional $\Phi^{0\to 1}$ and the following version of (3.9):

$$\quad \forall n \in \mathbb{N} \exists i \leq n \left( I^0_{\psi_{(i)}} \subseteq I^0_{v_{(i)}} \right). \quad (3.8)$$

Applying $QF$-$AC^{0,1}$ to (3.8), we obtain $\text{LIL}$, and this direction is done.

For the reverse implication, note that $\text{LIL} \to QF$-$AC^{0,1}$ follows from $\text{[37] Theorem 5.3}$, because the base theory $\text{RCA}_0^\omega + (\kappa_0^3)$ allows us to generalise the class of covers, as discussed in $\text{[37] Remark 5.9}$. With that, we are done. $\blacksquare$

We believe that the previous splitting is proper. The following corollary to the theorem is proved in the same way.

**Corollary 3.20.** The system $\text{RCA}_0^\omega + (\kappa_0^3) + \text{IND}$ proves

$$\left[ (\dim(\mathbb{R}) \leq 1) + QF$-$AC^{0,1} \right] \leftrightarrow \left[ (\mathbb{R} \text{ is paracompact}) + QF$-$AC^{0,1} \right] \leftrightarrow \text{LIL.} \quad (3.9)$$

In conclusion, it is easy to see that the proofs of this section go through for the Lindelöf lemma for $\mathbb{R}$ based on any ‘reasonable’ notion of cover. Indeed, as long as the formulas ‘$x \in U_y$’ and ‘$[a, b] \subseteq U_x$’ for the new notion of cover $\bigcup_{x \in \mathbb{R}} U_x$ of $\mathbb{R}$ are decidable in $\mathbb{Z}^\omega_2$, the above proofs go through (assuming $(\kappa_0^3)$). Since $\mathbb{Z}^\omega_2$ can decide if $Y: \mathbb{R} \to \{0, 1\}$ represents an open subset of $\mathbb{R}$ (using the textbook definition of open set), this notion of ‘reasonable’ again seems quite reasonable.

### 3.4.3. Partitions of unity

The notion of partition of unity was introduced in 1937 by Dieudonné in $[10]$ and this notion is equivalent to paracompactness in a rather general setting by $[14]$ Theorem 5.1.9. We study partitions of unity in this section motivated as follows: on one hand, Simpson proves the existence of partitions of unity for complete separable spaces in the proof of $[43]$ II.7.2, i.e. this notion has been studied in RM. On the other hand, despite the equivalence, partitions of unity have nicer RM properties than paracompactness: we obtain versions of Theorems $3.16$ and $3.17$ where the base theory is conservative over $\text{RCA}_0^\omega$, while there is also a nice connection to the Lindelöf lemma.

The definition of partition of unity is as follows in Munkres $[31]$ p. 258

**Definition 3.21.** Let $\{U_\alpha\}_{\alpha \in J}$ be an indexed open covering of $X$. An indexed family of continuous functions $\phi_\alpha: X \to \{0, 1\}$ is said to be a partition of unity on $X$, dominated by $\{U_\alpha\}$, if:

1. $\text{support}(\phi_\alpha) \subseteq U_\alpha$ for each $\alpha \in J$.
2. The indexed family $\{\text{support}(\phi_\alpha)\}_{\alpha \in J}$ is locally finite.
3. $\sum_{\alpha \in J} \phi_\alpha(x) = 1$ for each $x \in X$.

where $\text{support}(f)$ is the closure of the open set $\{x \in X: f(x) \neq 0\}$.

---

$^6$In $\text{LIN}$, any $x \in \mathbb{R}$ is covered by $I^\ast_x$, while in $\text{LIL}$ any $x \in \mathbb{R}$ is covered by $J^y_x$ for some $y \in \mathbb{R}$. In the former case, we ‘know’ which interval covers the point, while in the latter case, we only know that it exists. We believe this (seemingly minor) difference determines whether one can obtain $QF$-$AC^{0,1}$ (like in the case of $\text{LIL}$) or not (in the case of $\text{LIN}$, we conjecture). Indeed, applying $QF$-$AC^{1,0}$ to the conclusion of $\text{LIL}$, we obtain a functional that provides for any $x \in \mathbb{R}$, an interval $J^y_x$ covering $x$; i.e. $\text{LIL}$ clearly exhibits ‘axiom of choice’ behaviour, while $\text{LIN}$ does not.

$^7$Munkres uses ‘dominated by’ in $[31]$ instead of Engelsking’s ‘subordinate to’ in $[14]$. 

Note that the second item implies that the sum in the third one makes sense.

With these definitions in place, PUNI(I) is the statement that for any cover generated by \( \psi : I \to \mathbb{R} \), there is a partition of unity of \( I \) dominated by \( \cup_{x \in I} \phi^x \).

**Theorem 3.22.** The system RCA^0_0 + \((\kappa^3_0)\) proves [WKL + PUNI(I)] \(\iff\) HBT.

**Proof.** In case of \( \neg(\exists^2) \), the equivalence is easy: all \( \mathbb{R} \to \mathbb{R} \)-functions are continuous and PUNI(I) is provable as in the proof of [43 II.7.2], while HBT follows from WKL as in the proof of Corollary 3.6. In case of \( (\exists^2) \), the reverse implication is also straightforward: the finite sub-cover provided by HBT is readily refined, and the existence of a partition of unity for a finite cover follows from [43 II.7.1].

Finally, for the forward direction assuming \( (\exists^2) \), let \( \psi : I \to \mathbb{R} \) be as in HBT and obtain \( \phi : I^2 \to \mathbb{R} \) as in PUNI(I), i.e. for \( U_x := \text{support}(\phi(x, \cdot)) \), the open cover \( \cup_{x \in I} U_x \) of \( I \) is locally finite and satisfies \( U_x \subset I^\psi_x \). Now consider:

\[
(\forall x \in I)(\exists n \in \mathbb{N})(\exists y \in I)((x - \frac{1}{2n}, x + \frac{1}{2n}) \subseteq U_y),
\]

(3.10)

Applying QF-AC^{1,0} to (3.10), since \( \exists^2 \) is given, we obtain \( \Psi : I \to \mathbb{R}^+ \) such that \( I^\Psi_x \subset U_x \) for all \( x \in I \). Now repeat the proof of Theorem 3.11 for \( \Psi \) in place of \( \phi \), which yields \( y_1, \ldots, y_k \in I \) such \( \cup_{i \leq k} I^\Psi_{y_i} \) is a finite sub-cover of \( I \). Note that in the previous ‘repeated proof’, we do not need the choice function \( \Xi \) (from the proof of Theorem 3.6), as \( I^\Psi_{y_i} \) covers \( x \) for any \( x \in I \). Since \( I^\Psi_{y_i} \subset U_{y_i} \), \( \cup_{i \leq k} U_y \) is a finite sub-cover of \( \cup_{x \in I} U_x \), and since \( U_x \subset I^\psi_x \), \( \cup_{i \leq k} I^\psi_{y_i} \) is a finite sub-cover as required by HBT, and we are done.

\[ \square \]

**Corollary 3.23.** The system RCA^0_0 + \((\kappa^3_0)\) + PUNI(I) proves HBU \(\iff\) HBT.

Note that previous base theory in the corollary (and hence the theorem) is conservative over RCA^0_0 by [23 Prop. 3.12] and the proof of [43 II.7.2].

Finally, we obtain a theorem that brings together a number of different strands from this paper, including *effective paracompactness*, first discussed at the end of Section 3.3. In the context of RCA^0_0, we say that \( \phi : \mathbb{R} \to \mathbb{R} \) is an *effective* refinement of \( \psi : \mathbb{R} \to \mathbb{R} \) if \( (\exists \xi : \mathbb{R} \to \mathbb{R})(\forall x \in \mathbb{R})(I^\xi_x \subseteq I^\psi_{\xi(x)}) \). Effective paracompactness expresses the existence of an effective refinement for any open cover. Moreover, PUNI(\( \mathbb{R} \)) is the statement that for any cover generated by \( \psi : \mathbb{R} \to \mathbb{R}^2 \), there is a partition of unity \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) dominated by \( \cup_{x \in \mathbb{R}} I^\psi_x \).

**Theorem 3.24.** The system RCA^0_0 + \((\kappa^3_0)\) proves the following

\[
PUNI(\mathbb{R}) \iff LIL \iff \mathbb{R} \text{ is effectively paracompact}.
\]

**Proof.** We first prove the first equivalence. In case of \( \neg(\exists^2) \), the equivalence is easy: all \( \mathbb{R} \to \mathbb{R} \)-functions are continuous and PUNI(\( \mathbb{R} \)) is provable as in the proof of [43 II.7.2], while LIL follows by taking the countable sub-cover given by the rationals. In case of \( (\exists^2) \), the reverse implication is also straightforward: the countable sub-cover provided by LIL is readily refined, and the existence of a partition of unity for a countable cover follows from [43 II.7.1].

Finally, for the forward direction assuming \( (\exists^2) \), let \( \psi : \mathbb{R} \to \mathbb{R}^2 \) be as in LIL and obtain \( \phi : \mathbb{R}^2 \to \mathbb{R}^2 \) as in PUNI(\( \mathbb{R} \)), i.e. for \( U_x := \text{support}(\phi(x, \cdot)) \), the open cover \( \cup_{x \in \mathbb{R}} U_x \) of \( \mathbb{R} \) is locally finite and satisfies \( U_x \subset I^\psi_x \). Now consider:

\[
(\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(\exists y \in \mathbb{R})((x - \frac{1}{2n}, x + \frac{1}{2n}) \subseteq U_y),
\]

(3.11)
Applying $\text{QF-AC}^{1,0}$ to (3.11), since $\exists^3$ is given, we obtain $\Psi : \mathbb{R} \to \mathbb{R}^+$ such that $I^\Psi_x \subset U_x$ for all $x \in \mathbb{R}$. Now repeat the proof of Theorem 3.11 for $\Psi$ in place of $\phi$ and $\mathbb{R}$ instead of $I$. Then instead of (3.1), we obtain

\[ (\forall x \in \mathbb{R})(\exists n \in \mathbb{N})(\zeta(n) \geq \mathbb{R}|x|). \tag{3.12} \]

Note that in the aforementioned ‘repeated proof’, we do not need the choice function $\Xi$ (from the proof of Theorem 3.5), as $I^\Psi_x$ covers $x$ for any $x \in \mathbb{R}$. Applying $\text{QF-AC}^{1,0}$ to (3.12), we obtain $\Phi^0 \to 1$ such that $\bigcup_{n \in \mathbb{N}} I^{\Psi}_{\Phi(n)}$ is a countable sub-cover of the canonical cover generated by $\Psi$. Since $I^\Psi_x \subset U_x$, $\bigcup_{n \in \mathbb{N}} I^{\Psi}_{\Phi(n)}$ is a countable sub-cover of $\bigcup_{x \in \mathbb{R}} U_x$, and since $U_x \subset I^\psi_x$, $\bigcup_{n \in \mathbb{N}} I^{\Psi}_{\Phi(n)}$ is a countable sub-cover as required by LIL. For the second equivalence, we note that the previous proof only makes use of the fact that $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is effectively paracompact (via the identity function). Thus, the second equivalence follows in the same way, and we are done. \qed

4. Conclusion

We have studied the higher-order RM of topology, the notions of *dimension* and *paracompactness* in particular. Basic theorems regarding the latter turn out to be equivalent to the *Heine-Borel theorem* for uncountable covers, i.e. the former are extremely hard to prove (in terms of comprehension axioms). A number of nice splittings was obtained, and we have shown that these results do not depend on the exact definition of cover, even in the absence of the axiom of choice. Finally, we obtained similar results for the *Lindelöf lemma*. We refer to Figure 11 for a visual depiction of our results vis-a-vis the *Gödel hierarchy*.

Regarding future work, the following two topics come to mind. Firstly, there are a number of notions weaker than paracompactness, and it is an interesting question if there are *natural* such notions that yield equivalences with HBT or weaker theorems. Secondly, in light of Remark 3.13 it seems interesting to study metrisation theorems in higher-order RM. We expect that such theorems go far beyond $\Pi^1_2$-$\text{CA}_0$, which features in the second-order RM of topology.

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**Appendix A. The Gödel Hierarchy**

The *Gödel hierarchy* is a collection of logical systems ordered via consistency strength, or essentially equivalent: ordered via inclusion\(^8\). This hierarchy is claimed to capture most systems that are ‘natural’ or have ‘foundational import’, as follows.

*It is striking that a great many foundational theories are linearly ordered by $\prec$. Of course it is possible to construct pairs of artificial theories which are incomparable under $\prec$. However, this is not the case for the “natural” or non-artificial theories which are usually regarded as significant in the foundations of mathematics.*

---

\(^8\)Simpson states in [42] p. 112] that inclusion and consistency strength yield the same hierarchy as depicted in [42 Table 1], i.e. one gets the ‘same’ Gödel hierarchy.
Arguably, the Gödel hierarchy is a central object of study in mathematical logic, as also stated by Simpson in [42, p. 112]. However, the above results imply that e.g. HBT and basic topological theorems dealing with dimension and paracompactness, do not fit the Gödel hierarchy. The same holds for basic properties of the gauge integral, including many covering lemmas (See [36]), as well as for so-called uniform theorems (See [37]) in which the objects claimed to exist depend on few of the parameters of the theorem. In particular, the aforementioned theorems yield a branch that is completely independent of the medium range of the Gödel hierarchy (with the latter based on inclusion ^8), as depicted in the following figure (where we assume the ordering based on inclusion):

![Figure 1. The Gödel hierarchy with a side-branch for the medium range](image)

Some remarks on the technical details concerning Figure 1 are as follows.

1. Note that we use a non-essential modification of the Gödel hierarchy, namely involving systems of higher-order arithmetic, like e.g. ACA^ω_0 instead of ACA_0; these systems are (at least) Π^1_2-conservative over the associated second-order system (See e.g. [39, Theorem 2.2]).

2. The system Z^ω_2 is placed between the medium and strong range, as the combination of the recursor R^2 from Gödel’s T and ^3 yields a system stronger than Z^ω_2. The system Π^1_2-CA^ω_0 does not change in the same way.
While $HBT$ clearly implies $WKL$, the paracompactness of the unit interval does not (by the ECF-translation); this is symbolised by the dashed line.

(4) While $HBT$ and similar statements are hard to prove (in terms of comprehension axioms), these theorems (must) have weak first-order strength in light of their derivability in intuitionistic topology (See e.g. [46,48]).

Finally, in light of the equivalences involving the gauge integral, uniform theorems, and the Cousin lemma (and hence $HBT$) from [36,37], we observe a serious challenge to the ‘Big Five’ classification from RM, the linear nature of the Gödel hierarchy, as well as Feferman’s claim that the mathematics necessary for the development of physics can be formalised in relatively weak logical systems (See [36, p. 24]).

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