Entanglement-assisted concatenated quantum codes

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Abstract

Entanglement-assisted concatenated quantum codes (EACQCs), constructed by concatenating two quantum codes, are proposed. These EACQCs show several advantages over the standard concatenated quantum codes (CQCs). Several families of EACQCs that, unlike standard CQCs, can beat the nondegenerate Hamming bound for entanglement-assisted quantum error correction codes (EAQECCs) are derived. Further, a number of EACQCs with better parameters than the best known standard quantum error correction codes (QECCs) and EAQECCs are also derived. In particular, several catalytic EACQCs with better parameters than the best known QECCs of the same length and net transmission are constructed. Furthermore, each catalytic EACQC consumes only one or two ebits. It is also shown that EACQCs make entanglement-assisted quantum communication possible even if the ebits are noisy. Finally, it is shown that EACQCs can outperform CQCs in entanglement fidelity over depolarizing channels if the ebits are less noisy than the qubits. Moreover, the threshold error probability of EACQCs is larger than that of CQCs when the error probability of ebits is sufficiently lower than that of qubits. Therefore EACQCs are not only competitive in quantum communication but also applicable in fault-tolerant quantum computation.

1 Introduction

Quantum error correction codes (QECCs) are necessary to realize quantum communications and to make fault-tolerant quantum computers\textsuperscript{[11, 37]}. The stabilizer formalism provides a useful way to construct QECCs from classical codes, but certain orthogonal constraints are required\textsuperscript{[8]}. The entanglement-assisted (EA) quantum error correction code (EAQECC)\textsuperscript{[5, 26, 6]} generalizes the stabilizer code. By presharing some entangled states between the sender (Alice) and the receiver (Bob), EAQECCs can be constructed from any classical linear codes without the orthogonal constraint. Therefore the construction could be greatly simplified. As an important physical resource, entanglement can boost the classical information capacity of quantum channels\textsuperscript{[2, 22, 23, 25, 28, 27]}. Recently, it has been shown that EAQECCs can violate the nondegenerate quantum Hamming bound\textsuperscript{[34]} or the quantum Singleton bound\textsuperscript{[20]}.

Compared to standard QECCs, EAQECCs must establish some amount of entanglement before transmission. This preshared entanglement is the price to be paid for enhanced communication capability. In a sense, we need to consider the net transmission of EAQECCs, i.e., the number of qubits transmitted minus that of ebits preshared. Further, it is difficult to preserve too many noiseless ebits in EAQECCs at present. Thus we have to use as few ebits as possible to conduct the communication, e.g., one or
two ebits are preferable [29, 24, 16, 47]. In addition, EAQECCs with positive net transmission and little entanglement can lead to catalytic quantum codes [5, 6], which are applicable to fault-tolerant quantum computation (FTQC). In Ref. [5], a table of best known EAQECCs of length up to 10 was established through computer search or algebraic methods. Several EAQECCs in Ref. [5] have larger minimum distances than the best known standard QECCs of the same length and net transmission. However, for larger code lengths, the efficient construction of EAQECCs with better parameters than standard QECCs is still unknown.

In classical coding theory, concatenated code (CC), originally proposed by Forney in 1960s [15], provide a useful way of constructing long codes from short ones. CCs can achieve very large coding gains with reasonable encoding and decoding complexity [33]. Moreover, CCs can have large minimum distances since the distances of the component codes are multiplied. As a result, CCs have been widely used in many digital communication systems, e.g., the NASA standard for the Voyager program [10], and the compact disc (CD) [35]. Similarly in QECCs, the concatenated quantum codes (CQCs), introduced by Knill and Laflamme in 1996 [30], are also effective for constructing good quantum codes. Particularly, it has been shown that CCs are of great importance in realizing FTQC [15, 11, 9].

Moreover, there exists a specific phenomenon in QECCs, called error degeneracy, which distinguishes quantum codes from classical ones in essence. It is widely believed that degenerate codes can correct more quantum errors than nondegenerate ones. Indeed, there are some open problems concerning whether degenerate codes can violate the nondegenerate quantum Hamming bound [41] or can improve the quantum channel capacity [12, 40]. Many CQCs have been shown to be degenerate even if the component codes are nondegenerate, e.g., Shor’s [9, 1, 3] code and the [25, 1, 9] CQC [18, 19]. If we introduce extra entanglement to CQCs, it is possible to improve the error degeneracy performance of CQCs.

In this article, we generalize the idea of concatenation to EAQECCs, and propose entanglement-assisted concatenated quantum codes (EACQCs). We show that EACQCs can beat the nondegenerate quantum Hamming bound while standard CQCs cannot. Several families of degenerate EACQCs that can surpass the nondegenerate Hamming bound for EAQECCs, are constructed. The same conclusion could be reached for the asymmetric error models, in which the phase-flip errors (Z-errors) happen more frequently than the bit-flip errors (X-errors) [22, 14]. Furthermore, we derive a number of EACQCs with better parameters than the best known QECCs and EAQECCs. In particular, we see that many EACQCs have positive net transmission and each of them consumes only one or two ebits. Thus they give rise to catalytic EACQCs with little entanglement and better parameters than the best known QECCs. Further, we show that the EACQC scheme makes EA quantum communication possible even if the ebits are noisy. We compute the entanglement fidelity (EF) of the [[15, 1, 9:10]] EACQC by using Bowen’s [[3, 1, 3;2]] EAQECC [4] or the [[3, 1, 3;2]] EA repetition code [3, 6] as the inner code. The outer code is the standard [[5, 1, 3]] stabilizer code. We show that the [[15, 1, 9:10]] EACQC performs much better than the [[25, 1, 9]] CQC over depolarizing channels if the ebits suffering lower error rate than the qubits. Moreover, we compute the error probability threshold of EACQCs and we show that EACQCs have much higher thresholds than CQCs when the error rate of ebits is sufficiently lower than that of qubits.

## 2 Preliminaries

Let \( q = 2^m \) \( (m \geq 1) \) be an integer and denote by \( GF(q) \) the extension field of the binary field \( GF(2) \). Let \( C \) be the field of complex numbers, and let \( V_n = (C^q)^\otimes n = C^{q^n} \) be the \( q^n \)-dimensional Hilbert space, where \( n \) is a positive integer. Define two error operators on \( C^q \) by \( X(a)|\psi\rangle = |a + \psi\rangle \) and \( Z(b)|\psi\rangle = (-1)^{tr(ba)}|\psi\rangle \), where \( a \in GF(q), b \in GF(q) \), and \( “tr” \) denotes the trace operator from \( GF(q) \) to \( GF(2) \). For a vector \( u = (u_1, \ldots, u_n) \in GF(q)^n \), denote by \( X(u) = X(u_1) \otimes \cdots \otimes X(u_n) \) and \( Z(u) = Z(u_1) \otimes \cdots \otimes Z(u_n) \).

Let \( \Xi_n = \{ X(a)Z(b) | a, b \in GF(q)^n \} \) and let \( G_n = \{ (-1)^n X(a)Z(b) | a, b \in GF(q)^n, u \in GF(2) \} \) be the group generated by \( \Xi_n \). For the operator \( e = (-1)^n X(a)Z(b) \in G_n \), the weight of \( e \) is defined by \( wt_Q(e) = |\{ 1 \leq i \leq n : (a_i, b_i) \neq (0, 0) \}| \). The definition of quantum stabilizer codes is given below.

**Definition 1** A stabilizer code \( Q \) is a \( q^k \)-dimensional \( (k \geq 0) \) subspace of \( V_n \) such that \( Q = \bigcap_{t \in T} \{ |\phi\rangle \in V_n : e|\phi\rangle = |\phi\rangle \} \), where \( T \) is a subgroup of \( G_n \). \( Q = [n, k, d]_q \) has minimum distance \( d \) if it can detect all
errors $e \in \mathcal{G}_n$ of weight $w_{\mathcal{Q}}(e)$ up to $d - 1$. Further, $Q$ is called nondegenerate if every stabilizer in $T$ has weight larger than or equal to $d$, otherwise it is called degenerate.

A concatenated quantum code (CQC) is derived from an inner code and an outer code. In general, the component codes of QCQCs can be chosen as stabilizer codes or non stabilizer codes. In this article, it suffices to consider only the case of stabilizer codes. Let the inner and outer codes be $Q_I = [[n_1, k_1, d_1]]$ and $Q_O = [[n_2, k_2, d_2]]$, respectively. Then we can derive a CQC [21] with parameters $Q_C = [[n_1n_2, k_2k_2, d_C \geq d_1d_2]]$.

An EAQECC with parameters $Q_e = [[n, k, d; c]]_q$ can encode $k$ qudits into $n$ qudits by consuming $c$ pairs of maximally entangled states between Alice and Bob. It should be noted that EAQECCs can be constructed from arbitrary classical linear codes directly. The Calderbank-Shor-Steane (CSS) framework [45, 8] provides a useful way to construct both QECCs and EAQECCs from classical linear codes.

**Lemma 1 ([5])** Denote by $C_1 = [n, k_1, d_1]_q$ and $C_2 = [n, k_2, d_2]_q$ two linear codes over $GF(q)$. There exists an EAQECC with parameters $Q_e = [[n, k_1 + k_2 - n + c, d_e \geq \min\{d_1, d_2; c\}]]_q$, where $c = \text{rank}(H_1H_2^T)$.

EAQECCs can also be constructed by using the Hermitian construction [8, 5, 46] as follows.

**Lemma 2 ([5])** Let $C = [n, k, d]_q^2$ be a linear code over $GF(q^2)$. There exists an EAQECC with parameters $Q_e = [[n, 2k - n + c, d_e \geq d; c]]_q$, where $c = \text{rank}(HH^1)$, and $H^1$ is the conjugate transpose of $H$ over $GF(q^2)$.

We organize the main results of our study in the following order. Firstly, we present the construction of the entanglement-assisted concatenated quantum codes from two component quantum codes. Secondly, we construct several families of EACQCs violating the nondegenerate Hamming bound for EAQECCs. Thirdly, we derive a number of EACQCs with better parameters than the best known QECCs and EAQECCs. At last, we show that EACQCs can correct errors in the ebits. It is shown that EACQCs can outperform CQCs in entanglement fidelity and have higher error probability thresholds than CQCs.

## 3 Entanglement-Assisted Concatenated Quantum Codes

We generalize CQCs to EACQCs by concatenating two quantum codes which can be chosen as either standard QECCs or EAQECCs. In this article, sometimes we represent an $[[n, k, d]]_q$ QECC as an $[[n, k, d; 0]]_q$ EAQECC so that we can unify the representation of QECCs and EAQECCs. Let the inner code be $Q_I = [[n_1, k_1, d_1; c_1]]$, which requires $c_1$ ebits. Denote by $k^*_I \equiv k_1 - c_1$ the net transmission of $Q_I$. Let the outer code be $Q_O = [[n_2, k_2, d_2; c_2]]$, which can either be binary or nonbinary and that depends on $k_1$. $Q_O$ uses $c_2$ edits, or equivalently, $c_2k_1$ ebits. Denote by $k^*_O \equiv k_2 - c_2$ the net transmission of $Q_O$. Notice that, for classical linear codes and quantum codes over the binary field $GF(2)$, we usually neglect the index in the code parameters if there is no ambiguity.

We have the following result about EACQCs.

**Theorem 1** Let $Q_I = [[n_1, k_1, d_1; c_1]]$ be the inner code, and let $Q_O = [[n_2, k_2, d_2; c_2]]$, be the outer code. There exists an EACQC $Q_e$ with parameters

$$Q_e = [[n_1n_2, k_1k_2, d_e \geq d_1d_2; c_e]],$$

where $c_e = c_1n_2 + c_2k_1$ is the number of ebits. The net transmission is $k^*_e = k_1k_2 - c_e$.

**Proof:** Based on the idea of code concatenation, we simply concatenate the inner code $Q_I$ with the outer code $Q_O$ to derive the EACQC [15, 30, 21]. First we encode the information state $|\mu\rangle$ by using the outer code $Q_O$, i.e.,

$$|\mu\rangle \mapsto |\psi\rangle_O = (U_O \otimes \tilde{T}_{B_0})|\mu\rangle \otimes |0\rangle^{\otimes(n_2 - k_2 - c_2)k_1} \otimes |\Psi^+_A}_{AB}^{k^*_1},$$

where there are $c_2k_1$ Bell states, $|\Psi^+_A}_{AB}^{k^*_1}$, preshared between Alice and Bob during the outer encoding. The outer encoding operation $U_O$ is applied to the qubits in Alice’s side. Bob’s halves of ebits are preshared and they do not need to be encoded.
Let the outer code be encoded. It is easy to see that the number of ebits used during the whole inner encoding is applied to the qubits in Alice’s side while Bob’s halves of ebits do not need to be encoded. For each subblock \(|\nu_i\rangle(1 \leq i \leq n_2)\), we do the inner encoding as follows:

\[
|\nu_i\rangle \mapsto |\psi_i\rangle_I = (U_I \otimes I_B)|\nu_i\rangle \otimes |0\rangle^{n_1-k_1-c_1} \otimes |\Phi_+^{\Omega}_{AB}\rangle.
\]

\(|\Phi_+^{\Omega}_{AB}\rangle\) are \(c_1\) Bell states preshared between Alice and Bob during each inner encoding. The inner encoding operation \(U_I\) is applied to the qubits in Alice’s side while Bob’s halves of ebits do not need to be encoded. It is easy to see that the number of ebits used during the whole inner encoding is \(c_1 n_2\). The encoding process of EACQCs is given in Fig. 1.

The numbers of ebits used during the outer and the inner encoding are equal to \(ck_1\) and \(c_1 n_2\), respectively. Therefore the total number of ebits is equal to \(c = c_1 n_2 + c_2k_1\). It is easy to see that the dimension of the EACQC \(Q_c\) is equal to \(2^{ck_1}k_2\). Similar to the principle of code concatenation in [15 30 21], the minimum distance of \(Q_c\) is at least \(d_1 d_2\). Therefore we can obtain an EAQECC with parameters \(Q_e = n_1 n_2, k_1 k_2, d_e \geq d_1 d_2 + c_e\).

It is easy to see that if the inner and outer codes are both standard QECCs, then the EACQC is a standard CQC. Moreover, we can use different inner codes in EACQCs. Let \(Q_{I_i} = [n_{I_i}, k_{I_i}, d_{I_i}; c_{I_i}]\) \((1 \leq i \leq n_2)\) be \(n_2\) inner codes. For simplicity, we let \(k_1 \equiv k_{I_1} = \ldots = k_{I_{n_2}}\) and let \(d_1 \equiv d_{I_1} = \ldots = d_{I_{n_2}}\). Let the outer code be \(Q_O = [n_2, k_2, d_2; c_2]\). Then we can derive an EACQC with parameters

\[
Q'_e = \left[(\sum_{i=1}^{n_2} n_{I_i}, k_1 k_2, d_e' \geq d_1 d_2; c_e')\right],
\]

where \(c_e' = \sum_{i=1}^{n_2} c_{I_i} + c_2 k_1\). The net transmission is \(k_1 k_2 - c_e'\).

## 4 EACQCs Beating the Nondegenerate Quantum Hamming Bound

Firstly, let us review the nondegenerate quantum Hamming bound for EAQECCs [31].
Lemma 3 ([31]) For a binary nondegenerate $Q_e = [[n, k, d; c]]$ EAQECC, it must satisfy
\[
\sum_{i=0}^{\lfloor d/k \rfloor} 3^i \binom{n}{i} \leq 2^{n+c-k}.
\]
Taking the limit as $n \to \infty$, this yields the asymptotic bound on the rate $k/n$:
\[
\frac{k}{n} \leq 1 + \frac{c}{n} - \frac{\delta}{2} \log_2 3 - H_2 \left( \frac{\delta}{2} \right),
\]
where $\delta = d/n$, and $H_2(x) = -x \log_2 x - (1-x) \log_2 (1-x)$ is the binary entropy function.

To the best of our knowledge, there do not exist CQCs that violate the quantum Hamming bounds [31]. However, the situation changes significantly in the entanglement-assisted scenarios. We could easily construct several families of EACQCs that violate the nondegenerate Hamming bound in Lemma 3 ([31]). We give an explicit example to illustrate the construction of EACQCs. Let $Q_I = [[5, 1, 3; 0]]$ be the inner code and let $Q_O = [[3, 1, 3; 2]]$ be the outer code in Ref. [5]. Then we can derive an EACQC with parameters $\mathcal{Q}_e = [[15, 1; 9; 2]]$ by Theorem 1. This code can beat the nondegenerate Hamming bound for EAQECCs, respectively. These EACQC codes can do so. If we encode one of the qubits of the outer encoding by using the $[[4, 1, 3; 1]]$ EAQECC, then we derive a $[[14, 1, 9; 3]]$ EAQC.

Proof: The proof is given in Appendix A. □

We give an explicit example to illustrate the construction of EACQCs. Let $Q_I = [[5, 1, 3; 0]]$ be the inner code and let $Q_O = [[3, 1, 3; 2]]$ be the outer code in Ref. [5]. Then we can derive an EACQC with parameters $\mathcal{Q}_e = [[15, 1; 9; 2]]$ by Theorem 1. This code can beat the nondegenerate Hamming bound for EAQECCs, respectively. Notice that $Q_I = [[5, 1, 3; 0]]$ and $Q_O = [[3, 1, 3; 2]]$ are both nondegenerate codes [8, 5, 31] while $\mathcal{Q}_e = [[15, 1; 9; 2]]$ is degenerate. Also notice that $Q_I$ and $Q_O$ cannot beat the nondegenerate Hamming bound in Eq. (6), but their EACQC $\mathcal{Q}_e = [[15, 1; 9; 2]]$ can do so. If we encode one of the qubits of the outer encoding by using the $[[4, 1, 3; 1]]$ EAQECC, then we derive a $[[14, 1, 9; 3]]$ EAQC. This code can also beat the nondegenerate Hamming bound for EAQECCs.

For the asymmetric channel models, we present the construction of EACQCs that can beat the nondegenerate Hamming bound for asymmetric EAQECCs. Let $d_X$ and $d_Z$ be two positive integers. From [17], an asymmetric EAQECC $Q_A = [[n, k, d_Z/d_X; c]]_q$ can detect any $X$-error of weight up to $d_X - 1$ and any $Z$-error of weight up to $d_Z - 1$, simultaneously. The number of edits is $c$. One could further obtain nondegenerate Hamming bounds for asymmetric EAQECCs [17, 31].

Lemma 4 ([17]) A binary nondegenerate asymmetric EAQECC $[[n, k, d_Z/d_X; c]]$ must satisfy
\[
\sum_{i=0}^{\lfloor d_X/k \rfloor} \binom{n}{i} \sum_{j=0}^{\lfloor d_Z/k \rfloor} \binom{n}{j} \leq 2^{n+c-k}.
\]

Let $Q_I = [[n_1, 1, n_1/1; 0]]$ be a binary asymmetric EAQECC which is used as the inner code. Let $Q_O = [[n_2, 1, d_2/d_1; n_2 - 1]]$ be the outer code, where $d_2 = n_2 - 1$ for even $n_2 \geq 2$, or $d_2 = n_2$ for odd $n_2 \geq 3$. We concatenate $Q_I$ with $Q_O$ according to Fig. 1. Then we have the following result about asymmetric EACQCs.
Corollary 1 There exists a family of asymmetric EACQCs with parameters
\[ \mathcal{Q}_A = [[n_1 n_2, 1, n_1 d_2 / d_2; n_2 - 1]], \]
where \( n_1 \geq 2 \) is an integer, \( d_2 = n_2 - 1 \) for even \( n_2 \geq 2 \), or \( d_2 = n_2 \) for odd \( n_2 \geq 3 \).

For any integer \( n_1 \geq 2 \) and any odd \( n_2 \geq 3 \), \( \mathcal{Q}_A \) in Corollary 1 can beat the nondegenerate Hamming bound for asymmetric EAQECCs in Lemma 2. For any integer \( n_1 \geq 2 \) and any even \( n_2 \geq 8 \), \( \mathcal{Q}_A \) can also beat the nondegenerate Hamming bound. Let \( n_1 = 2 \) and \( n_2 = 3 \). We can derive an asymmetric EACQC with parameters \( \mathcal{Q}_A = [[6, 1, 6 / 3; 2]] \). It is the shortest length EAQECC that can beat the nondegenerate Hamming bound known to date.

5 EACQCs beating existing QECCs and EAQECCs

Similar to classical coding theory, constructing quantum codes with parameters better than the best known results is one central topic in quantum coding theory. It is even more attractive since degenerate quantum codes have large potentials to outperform any nondegenerate quantum code. Indeed, a number of best known QECCs in [19] have been shown to be degenerate.

As argued in Ref. [5], we say that an EAQECC \([n, k, d] \) is better than a QECC \([n, k] \) if the net transmission \((k_1 - c)\) is larger than \( k \). Ref. [19] collects a list of classical linear codes and QECCs with best parameters currently known. According to the construction of EAQECCs in Lemma 2, a quaternary code in Ref. [19] corresponds to a best known nondegenerate EAQECCs. In general, it is not difficult to construct nondegenerate EAQECCs with positive net transmissions better than the best known QECCs based on Ref. [19]. Therefore we should focus on constructing (degenerate) EAQECCs with positive net transmissions that can beat the best known nondegenerate EAQECCs. This is an important topic concerning that whether degeneracy can help to improve the classical coding limit in EAQECCs.

We give two explicit constructions to show that EACQCs can beat the best known QECCs and EAQECCs. According to [36, [14], there exists a cyclic maximum-distance-separable (MDS) code with parameters \([17, 9, 9]_1 \). From Lemma 2, we can derive an entanglement-assisted quantum maximum-distance-separable (EAQMDS) code with parameters \([[17, 5, 9; 4]] \). Let \( Q_I = [[4, 2, 2]] \) be the inner code and let \( Q_O = [[17, 5, 9; 4]]_4 \) be the outer code. Then we can derive an EACQC with parameters \( \mathcal{Q}_A = [[68, 10, 18; 8]] \). Compared with the best known \( Q = [[68, 2, 16]] \) QECC in [19], the EACQC \( \mathcal{Q}_A \) has a larger minimum distance while maintaining the same length and net transmission. \( \mathcal{Q}_A \) also has a larger minimum distance than the best known nondegenerate \([[68, 10, 16; 8]] \) EAQECC from [12] of the same length and net transmission. Let \( Q_O = [[65, 17, 33; 16]]_8 \) be an EAQMDS code constructed from a cyclic MDS code \([[65, 33, 33]] \) in [36] and let \( Q_I = [[8, 3, 3; 0]] \). Then we can derive an EAQC with parameters \( \mathcal{Q}_A = [[520, 51, 99; 48]] \) by using \( Q_O = [[65, 17, 33; 16]]_8 \) and \( Q_I = [[8, 3, 3; 0]] \) as the outer and inner codes, respectively. This EACQC is better than the asymptotic Gilbert-Varshamov (GV) bound for EAQECCs in Ref. [31]. In Appendix B Table 1 and Table 2, we list more constructions of EACQCs with parameters better than the best known QECCs and EAQECCs.

In practice, we prefer to use as few as possible ebits to do the entanglement-assisted communication since storing a large number of noiseless ebits is quite difficult. Let \( Q_I = [[5, 1, 3; 0]] \) be the inner code and let \( Q_O = [[3, 2, 2; 1]] \) be the outer code, then we can derive a \([[15, 2, 6; 1]] \) EACQC. This code has larger minimum distance than the best known standard \([[15, 1, 5]] \) QECC in [19]. By using the MAGMA software [3], we know that there exists a nondegenerate \([[15, 8, 6; 7]] \) EAQECC. This code has the same minimum distance and net transmission with the \([[15, 2, 6; 1]] \) EACQC. However, the EACQC consumes only one ebit and thus it is more practical. In Appendix B Table 3, we list a number of EACQCs with parameters better than the best known QECCs and EAQECCs, and each EACQC consumes only one ebit.

In [13], several families of \( q \)-ary EAQMDS codes with distances larger than \( q + 1 \) and consuming very few edits were constructed. We use EAQMDS codes in [13] as the outer code to construct EACQCs that consume very few ebits. We give an example to illustrate the construction. Let \( Q_I = [[4, 2, 2; 0]] \) be the inner code and let \( Q_O = [[17, 4, 8; 1]] \) EAQMDS code in [13] be the outer code. Then we can derive
an EACQC with parameters $Q_e = [[68, 8, 16; 2]]$. This code has a larger minimum distance than the best known [[68, 6, 14]] QECC in [19] of the same length and net transmission. It also has a larger minimum distance than the best known nondegenerate [[68, 19, 15; 13]] EAQECC in [19] of the same length and net transmission. In Appendix B Table 4, we list a number of EACQCs with better parameters than the best known QECCs and EAQECCs in [19], and each code consumes only a few ebits.

6 Performance of EACQCs with Noisy Ebits

In this section, we evaluate the performance of EACQCs with noisy ebits. We compute the entanglement fidelity and the error probability threshold of EACQCs and make comparisons with the standard CQCs. For a quantum channel, the use of a QECC should improve the entanglement fidelity when the error probability is below a specific value, that we call it the threshold. In practical applications, QECCs with sufficiently high thresholds are needed. We will show that EACQCs can outperform CQCs in entanglement fidelity if the ebits are less noisy than the qubits. Further, we will show that the threshold of EACQCs is much higher than that of CQCs when the error probability of ebits is sufficiently lower than that of qubits.

During the process of entanglement-assisted quantum communication, the preshared ebits of Bob need to be stored faultlessly and EAQECCs can only correct errors on the transmitted qubits. However, noise in Bob’s ebits may be inevitable in practical applications [33, 3], and maintaining a large number of noiseless ebits is extremely difficult. In this article, we use EACQCs to correct errors in ebits. In the EACQC scheme, suppose that we use an EAQECC $Q_T$ as the inner code and use a standard stabilizer code $Q_O$ as the outer code. We show that the outer code $Q_O$ can not only correct errors on the physical qubits but also can correct errors on the ebits. We construct two EACQCs and show that they can outperform CQCs in entanglement fidelity when the error probability of ebits is lower than that of qubits. We construct a $Q_B = [[15, 1, 9; 10]]^B$ EACQC by using the [[5, 1, 3]] stabilizer code as the outer code and Bowen’s [[3, 1, 3; 2]] EAQECC [4] as the inner code. Alternately, we use the [[5, 1, 3]] stabilizer code as the outer code, and the [[3, 1, 3; 2]] EA repetition code as the inner code to construct another $Q_R = [[15, 1, 9; 10]]^R$ EACQC with the same parameters. Recall that the standard $Q_C = [[25, 1, 9]]$ CQC is the concatenation of the [[5, 1, 3]] stabilizer code. It is known that Bowen’s [[3, 1, 3; 2]] EAQECC is equivalent to the [[5, 1, 3]] stabilizer code and they have the same stabilizers. Thus the $Q_B = [[15, 1, 9; 10]]^B$ EACQC is equivalent to the $Q_C = [[25, 1, 9]]$ CQC. Then the $Q_B = [[15, 1, 9; 10]]^B$ EACQC has the same error correction ability with the $Q_C = [[25, 1, 9]]$ CQC. Nevertheless, we show that EACQCs can outperform CQCs in entanglement fidelity if the error probability of ebits is lower than that of qubits.

The detailed entanglement fidelity computation of the two EACQCs and the CQC was put in Appendix C. The EFs of the two EACQCs and the CQC were plotted in Fig. 2. We compare the EF of EACQCs with that of the [[25, 1, 9]] CQC. If $p_a = p_b$, the EF of the $Q_B = [[15, 1, 9; 10]]^B$ EACQC is equal to that of the [[25, 1, 9]] CQC. When $p_b = 0.5p_a$, the EF of the $Q_B = [[15, 1, 9; 10]]^B$ and the $Q_R = [[15, 1, 9; 10]]^R$ EACQCs can outperform that of the [[25, 1, 9]] CQC. As $p_b$ becomes even lower than $p_a$, e.g., $p_b = 0.1p_a, 0.01p_a$, the EF of $Q_B = [[15, 1, 9; 10]]^B$ and $Q_R = [[15, 1, 9; 10]]^R$ performs much better than that of the [[25, 1, 9]] CQC. Moreover, $Q_B = [[15, 1, 9; 10]]^B$ performs better than $Q_R = [[15, 1, 9; 10]]^R$ when $p_b = p_a$. While $p_b = 0.1p_a, 0.01p_a$, $Q_B = [[15, 1, 9; 10]]^B$ performs much better than $Q_R = [[15, 1, 9; 10]]^R$ and the [[25, 1, 9]] CQC.

We compare the error probability threshold of the two EACQCs with that of the CQC. For the [[5, 1, 3]] stabilizer code and the [[25, 1, 9]] CQC, the thresholds are $p > 0.09$ and $p > 0.18$, respectively. Thus the CQC scheme can improve the error probability threshold. For the EACQCs, when $p_b = 0.5p_a$, the thresholds of $Q_B = [[15, 1, 9; 10]]^B$ and $Q_R = [[15, 1, 9; 10]]^R$ are $p > 0.25$ and $p > 0.14$, respectively. While $p_b$ becomes sufficiently lower, e.g., $p_b = 0.01p_a$, the thresholds of $Q_B = [[15, 1, 9; 10]]^B$ and $Q_R = [[15, 1, 9; 10]]^R$ are $p > 0.41$ and $p > 0.47$, respectively. Therefore the EACQC scheme can greatly improve the error probability threshold when the error probability of ebits is much lower than that of qubits.
7 Conclusions and Discussions

In this article, we have proposed the construction of entanglement-assisted concatenated quantum codes by concatenating an inner code with an outer code. We not only have generalized the idea of concatenation to EAQECCs but also have shown that EACQCs can outperform many existed results. We have further shown that EACQCs can beat the nondegenerate Hamming bound for EAQECCs while the standard CQCs cannot do so. We have derived many EACQCs with larger minimum distances than the best known QECCs and EAQECCs in [19] of the same length and net transmission. In addition, we have constructed several catalytic EACQCs with little entanglement and better parameters than the best known QECCs and EAQECCs. We have also constructed a family of asymmetric EACQCs that can beat the nondegenerate Hamming bound for asymmetric EAQECCs. Finally, we have computed the entanglement fidelity of two EACQCs and compared them with the \([25, 1, 9]\) CQC. We have shown that EACQCs can outperform CQCs in entanglement fidelity when the ebits are less noisy than qubits. In particular, we have shown that EACQCs have much higher error thresholds than CQCs when the error probability of ebits is sufficiently lower than that of qubits. These properties of EACQCs make them very competitive with standard CQCs for both quantum communication and fault-tolerant quantum computation.

Figure 2: Entanglement fidelity of EACQCs and CQCs for \(p_b = p_a, 0.5p_a, 0.1p_a, 0.01p_a\).
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A Proof of Theorem 2

We need the following estimation of a sum of binomial coefficient.

**Lemma 5** ([36]) Let \( n \geq 2 \) be an integer. For \( 1 < i < n \), there is

\[
\frac{1}{\sqrt{8n\alpha(1-\alpha)}}n^{H_2(\alpha)} \leq \binom{n}{i} \leq \frac{1}{\sqrt{2\pi n\alpha(1-\alpha)}}n^{H_2(\alpha)},
\]

(10)

where \( \alpha = i/n \), and \( H_2(x) = -x\log_2 x - (1-x)\log_2(1-x) \) is the binary entropy function.

**Theorem 2** There exist the following four families of EACQCs with parameters

(I) \( \mathcal{Q}_{e_1} = \left[ \left[ 5n_2, 1, d_{e_1} \geq 3n_2; n_2 - 1 \right] \right] \), where \( n_2 \geq 3 \) is odd.

(II) \( \mathcal{Q}_{e_2} = \left[ \left[ 5n_2, 1, d_{e_2} \geq 3n_2 - 3; n_2 - 1 \right] \right] \), where \( n_2 \geq 10 \) is even.

(III) \( \mathcal{Q}_{e_3} = \left[ \left[ 4n_2, 1, d_{e_3} \geq 3n_2 - 3; 2n_2 - 1 \right] \right] \), where \( n_2 \geq 11 \) is odd.

(IV) \( \tilde{\mathcal{Q}}_{e_2} = \left[ \left[ 4n_2, 1, \tilde{d}_{e_2} \geq 3n_2 - 3; 2n_2 - 1 \right] \right] \), where \( n_2 \geq 32 \) is even.

EACQCs in (I) – (IV) can beat the nondegenerate quantum Hamming bound for EAQECCs, respectively.

**Proof:** We now elaborate the constructions of these EACQCs and their properties. The EACQCs in (I) – (IV) were all constructed according to Theorem 1.

The EACQC \( \mathcal{Q}_{e_1} \) in (I) was constructed by choosing the inner code \( Q_I = \left[ \left[ 5, 1, 3; 0 \right] \right] \) and the outer code \( Q_O = \left[ \left[ n_2, 1, n_2; n_2 - 1 \right] \right] \) in Ref. [32], where \( n_2 \geq 3 \) is odd. The EACQC \( \mathcal{Q}_{e_2} \) (II) was constructed similarly by choosing the inner code \( Q_I = \left[ \left[ 5, 1, 3; 0 \right] \right] \) and the outer code \( Q_O = \left[ \left[ n_2, 1, n_2 - 1; n_2 - 1 \right] \right] \) in Ref. [32], where \( n_2 \geq 4 \) is even. While letting the inner and outer codes be \( Q_I = \left[ \left[ 4, 1, 3; 1 \right] \right] \) and \( Q_O = \left[ \left[ n_2, 1, n_2; n_2 - 1 \right] \right] \), respectively, where \( n_2 \geq 3 \) is odd, the EACQC \( \mathcal{Q}_{e_3} \) in (III) was derived. The EACQC \( \mathcal{Q}_{e_4} \) in (IV) was derived by choosing the inner and outer codes as \( Q_I = \left[ \left[ 4, 1, 3; 1 \right] \right] \) and \( Q_O = \left[ \left[ n_2, 1, n_2 - 1; n_2 - 1 \right] \right] \), respectively, where \( n_2 \geq 4 \) is even.
We need to prove that EACQCs in (I) – (IV) can beat the nondegenerate Hamming bound for EAQECCs. For the $E_{e_1} = [[5n_2, 1, d_{e_1} \geq 3n_2; n_2 - 1]]$ EACQC in (I), let $n_2 = 2m_2 + 1 (m_2 \geq 1)$. According to Lemma S1, we have
\[
\frac{3n_2 - 1}{3} \left( \binom{5n_2}{i} \right) \geq 3^{3m_2 + 1} \left( \frac{10m_2 + 5}{3m_2 + 1} \right) \geq \frac{3^{3m_2 + 1}}{\sqrt{8(10m_2 + 5)\alpha_1(1 - \alpha_1)}} 2^{10m_2 + 1} \geq 2^{12m_2 + 1} = 2^{6n_2 - 2} \tag{11}
\]
for all $m_2 \geq 1$, where $\alpha_1 = (3m_2 + 1)/(10m_2 + 1)$. Thus the EACQC $E_{e_1}$ in (I) can beat the nondegenerate Hamming bound for EAQECCs. For EACQCs in (II) – (IV), we can get similar results based on Lemma S1.

In addition, it is easy to verify that the relative distance of each EACQC in (I) – (IV) violates the asymptotic bound of nondegenerate Hamming bound for EAQECCs as the code length goes to infinity. □

## B EACQCs Beating the Best Known QECCs and EAQECCs

In [13, 38], entanglement-assisted quantum maximum-distance-separable (EAQMDS) codes achieving the Singleton bound were constructed. We have the following general result about the construction of EAQMDS codes.

**Lemma 6** For any $q + 1 \leq n \leq q^2 + 1$ and any integer $2 \leq d < n/2 - 1$, there exist EAQMDS codes $Q = [[n, k, d; c]]_q$ with positive net transmission, i.e., $k > c$, where $n + c = k + 2d - 2$, and $0 \leq c \leq n$.

**Proof:** It is known that there exist $q^2$-ary classical MDS codes $C = [n, k_1, d]_{q^2}$ of length $q + 1 \leq n \leq q^2 + 1$ [44], where $d = n - k_1 + 1$. Let the parity-check matrix of $C$ be $H$ and denote by $c = \text{rank}(HH^\dagger)$ the rank of $HH^\dagger$. According to the Hermitian construction in [5, 46], there exist EAQMDS codes with parameters $Q_e = [[n, 2k_1 - n + c, d; c]]_q$. If $2 \leq d < n/2 - 1$, then we have $k = 2k_1 - n_1 + c > c$. □

In Appendix B, Table 1 and Table 2, we list a number of EACQCs constructed by using EAQMDS codes in Lemma 6 as the outer codes.
Table 1: EAQECCs with better parameters than the best known QECCs and EAQECCs of the same length and net transmission. The notation of the inner codes, for example, “15 × [4, 2, 2; 0] + 2 × [5, 2, 2; 0]”, means that the inner codes are a mix of fifteen [4, 2, 2; 0] codes and two [5, 2, 2; 0] codes. The outer codes are EAQMDS codes in Lemma S1. The parameters $k_2$, $k_4$ and $K_2^*$ stand for the net transmissions of the $[[n_2, k_2^*, d_2]]_{k_4}$ outer code, the $[[n_3, k_3^*, d_3]]_{k_4}$ EAQECC and the $[[N_2, K_2^*, D_2]]$ EAQECC, respectively. EAQECCs in the last column are obtained from the best known quaternary codes in Ref. [19]. For the QECCs and the EAQECCs missing explicit constructions in Ref. [19], their parameters are in **bold**.

| $[[n_3, k_3^*, d_3]]$ | $[[n_2, k_2^*, d_2]]_{k_4}$ Outer Codes | $[[n, k, d]]$ QECCs from Ref. [19] | $[[N, K_2^*, D_2]]$ EAQECCs from Ref. [19] |
|----------------------|------------------------------------------|---------------------------------|---------------------------------|
| 17 × [4, 2, 2; 0]    | [17, 1, 4; 16]                           | [68 × 2; 4 ≥ 18]               | [68 × 2; 16]                   |
| 16 × [4, 2, 2; 0] + [5, 2, 2; 0] | [17, 1, 4]                              | [69 × 2; 4 ≥ 18]               | [69 × 2; 17]                   |
| 15 × [4, 2, 2; 0] + [2 × [5, 2, 2; 0]] | [17, 1, 4]                              | [70 × 2; 4 ≥ 18]               | [70 × 2; 17]                   |
| 14 × [4, 2, 2; 0] + 3 × [5, 2, 2; 0] | [17, 1, 4]                              | [71 × 2; 4 ≥ 18]               | [71 × 1, 18]                  |
| 12 × [4, 2, 2; 0] + 5 × [5, 2, 2; 0] | [17, 1, 4]                              | [73 × 2; 4 ≥ 18]               | [73 × 1, 18]                  |
| 11 × [4, 2, 2; 0] + 6 × [5, 2, 2; 0] | [17, 1, 4]                              | [74 × 2; 4 ≥ 18]               | [74 × 1, 17]                  |
| 10 × [4, 2, 2; 0] + 7 × [5, 2, 2; 0] | [17, 1, 4]                              | [75 × 2; 4 ≥ 18]               | [75 × 1, 18]                  |
| 9 × [4, 2, 2; 0] + 8 × [5, 2, 2; 0] | [17, 1, 4]                              | [76 × 1, 18]                   | [76 × 0, 18]                  |
| 9 × [4, 2, 2; 0] + [5, 2, 2; 0] + 3 × [3, 2, 2; 1] | [17, 1, 4]                              | [74 × 2; 4 ≥ 18]               | [74 × 1, 18]                  |
| 8 × [4, 2, 2; 0] + [5, 2, 2; 0] + 4 × [3, 2, 2; 1] | [17, 1, 4]                              | [75 × 2; 4 ≥ 18]               | [75 × 1, 18]                  |
| 8 × [4, 2, 2; 0] + [5, 2, 2; 0] + 5 × [3, 2, 2; 1] | [17, 1, 4]                              | [76 × 2; 4 ≥ 18]               | [76 × 1, 18]                  |
| 7 × [4, 2, 2; 0] + 6 × [5, 2, 2; 0] | [17, 1, 4]                              | [77 × 2; 4 ≥ 18]               | [77 × 1, 18]                  |
| 7 × [4, 2, 2; 0] + 7 × [5, 2, 2; 0] | [17, 1, 4]                              | [78 × 2; 4 ≥ 18]               | [78 × 1, 18]                  |
| 7 × [4, 2, 2; 0] + 8 × [5, 2, 2; 0] | [17, 1, 4]                              | [79 × 2; 4 ≥ 18]               | [79 × 1, 18]                  |
| 7 × [4, 2, 2; 0] + 9 × [5, 2, 2; 0] | [17, 1, 4]                              | [80 × 2; 4 ≥ 18]               | [80 × 1, 18]                  |
| 6 × [4, 2, 2; 0] + 7 × [5, 2, 2; 0] | [17, 1, 4]                              | [81 × 2; 4 ≥ 18]               | [81 × 1, 18]                  |
| 6 × [4, 2, 2; 0] + 8 × [5, 2, 2; 0] | [17, 1, 4]                              | [82 × 2; 4 ≥ 18]               | [82 × 1, 18]                  |
| 6 × [4, 2, 2; 0] + 9 × [5, 2, 2; 0] | [17, 1, 4]                              | [83 × 2; 4 ≥ 18]               | [83 × 1, 18]                  |
| 5 × [4, 2, 2; 0] + 10 × [5, 2, 2; 0] | [17, 1, 4]                              | [84 × 2; 4 ≥ 18]               | [84 × 1, 18]                  |
Table 2: EACQCs with better parameters than the best known QECCs and EAQECCs of the same length and net transmission. The notation of the inner codes, for example, “14 × [10, 2.4, 0] + 2 × [11, 2.4, 0]”, means that the inner codes are a mix of fourteen [10, 2.4, 0] codes and two [11, 2.4, 0] codes. The outer codes are EAQMDS codes in Lemma S2. The parameters \( k^*, K^* \) stand for the net transmissions of the \([n_k, k^*, d_j]\) outer code, the \([n_k, k^*, d_j]\) EACQC and the \([N_*, K^*, D_n]\) EAQECC. EACQCs and EAQECCs in the last column are obtained from the best known quaternary codes in Ref. [19]. For the QECCs and the EAQECCs missing explicit constructions in Ref. [19], their parameters are in bold type.

Table 3: EAQECCs with better parameters than the best known QECCs and EAQECCs of the same length and net transmission. The notation of the inner codes, for example, “2 × [29, 1.11] + [30, 1.11]”, means that the inner codes are a mix of two [29, 1.11] codes and a [30, 1.11] code. EAQECCs in the last column are obtained from the best known quaternary codes in Ref. [19], and the number of ebits is computed with MAGMA. For the QECCs missing explicit constructions in Ref. [19], their parameters are in bold type. Since we cannot compute the number of ebits for the best known quaternary codes missing explicit constructions in Ref. [19], we leave some EAQECCs empty in the last column.

Table 4: EAQECCs with better parameters than the best known QECCs and EAQECCs of the same length and net transmission. The outer codes are EAQMDS codes in Ref. [19]. EAQECCs in the last column are obtained from the best known quaternary codes in Ref. [19], and the number of ebits is computed with MAGMA. For the QECCs missing explicit constructions in Ref. [19], their parameters are in bold type. Since we cannot compute the number of ebits for the best known quaternary codes missing explicit constructions in Ref. [19], we leave some EAQECCs empty in the last column.
C The Entanglement Fidelity of EACQCs

We use the entanglement fidelity (EF) as the figure of merit for the performance of different quantum codes [43]. Let $\mathbb{H}$ be a finite dimensional Hilbert space. Let $|\varphi\rangle \in \mathbb{H} \otimes \mathbb{H}_R$ be a purification of a mixed state $\rho = \text{Tr}_{\mathbb{H}_R} |\varphi\rangle \langle \varphi|$, where $\mathbb{H}_R$ is a reference system. The entanglement fidelity of $\rho$ and $\Upsilon$ is defined as

$$F_c(\rho, \Upsilon) = \langle \varphi | (I_{\mathbb{H}_R} \otimes \Upsilon) (|\varphi\rangle \langle \varphi|) |\varphi\rangle,$$  \hspace{1cm} (12)

where $I_{\mathbb{H}_R}$ is the identity operator and $\Upsilon$ is a quantum map. Suppose that we can write the quantum map $\Upsilon$ in terms of Kraus operators, i.e., $\Upsilon = \sum_i \kappa_i$, where $\kappa_i A_i = I$, then the EF can be expressed by a useful computational formula as follows:

$$F_c(\rho, \Upsilon) = \sum_i |\text{Tr}(\rho A_i)|^2.$$ \hspace{1cm} (13)

As a special case of EF, the channel fidelity [43, 33] is defined as

$$F_c(\rho) = \frac{1}{(\dim \mathbb{H})^2} \sum_i |\text{Tr}(A_i)|^2.$$ \hspace{1cm} (14)

For the depolarizing channel, the channel fidelity is equal to the probability of correctable errors after quantum error correction (QEC) and recovery [33]. Denote the encoding and recovery operations in a QEC process by $\mathcal{E}$ and $\mathcal{R}$, respectively. For a single qubit state $\rho_0$, the encoding $\mathcal{E}$ takes it to an encoded state $\rho(0)$, i.e., $\mathcal{E} : \rho_0 \rightarrow \rho(0)$. The state $\rho(0)$ is sent through the quantum channel $\Upsilon$ and the received state is $\rho(t) = \Upsilon(\rho(0))$. At the receiver, $\rho(t)$ is recovered by the decoding and recovery operation $\mathcal{R}$ and the final logical state is $\rho_f = \mathcal{R}(\rho(t))$. For the entire QEC process, the operation

$$\mathcal{W} = \mathcal{R} \circ \Upsilon \circ \mathcal{E} : \rho_0 \rightarrow \rho_f$$ \hspace{1cm} (15)

is called the effective channel, which characterizes the effective dynamics of the encoded information arising from the physical dynamics of $\Upsilon$ [59].

Suppose that we can write the quantum channel $\Upsilon$ on $N$ physical qubits as independent noise $\Upsilon^{(1)}$ on each single qubit, i.e.,

$$\Upsilon = \Upsilon^{(1)} \otimes \cdots \otimes \Upsilon^{(1)} = \Upsilon^{(1) \otimes N}.$$ \hspace{1cm} (16)

Let $Q$ be an $N$ physical qubit QEC with encoding operation and recovery operation given by $\mathcal{E}$ and $\mathcal{R}$, respectively. Define the following coding map:

$$\Omega^C : \Upsilon_0^{(1)} \rightarrow \Upsilon_f^{(1)} = \mathcal{R} \circ \Upsilon_0^{(1) \otimes N} \circ \mathcal{E},$$ \hspace{1cm} (17)

which takes the single qubit noise $\Upsilon_0^{(1)}$ to the effective channel $\Upsilon_f^{(1)}$. If $\Upsilon_0^{(1)}$ is a depolarizing channel and $Q$ is a stabilizer code, then $\Omega^C$ takes $\Upsilon_0^{(1)}$ to a depolarizing channel $\Upsilon_f^{(1)}$.

The channel fidelity of the [[5, 1, 3]] stabilizer code over the depolarizing channel [33, 59] is given by $F_c[[5,1,3]] = 1 - 45p^2/8 + 75p^3/8 - 45p^4/8 + 9p^5/8$, where $p$ is the depolarizing probability. From [39], we know that the [[5, 1, 3]] stabilizer code takes a depolarizing channel $\Upsilon_0^{(1)}$ to a depolarizing channel $\Upsilon_f^{(1)}$ under the coding map $\Omega^C$. The error probability of $\Upsilon_f^{(1)}$ is equal to $1 - F_c[[5,1,3]]$. According to [7], the entanglement fidelity of the [[5, 1, 3]] code is given by $F_e[[5,1,3]] = 1 - 10p^2 + 20p^3 - 15p^4 + 4p^5$. Then we can derive the entanglement fidelity of the $Q_9^{[25,1,9]}$ CQC as follows:

$$F_e[[25,1,9]] = 1 - 10p^2 + 20p^3 - 15p^4 + 4p^5,$$ \hspace{1cm} (18)

where $p_C = 1 - F_c[[5,1,3]]$.

Denote the communication channel between Alice and Bob by $N_A$. Denote the channel model of storing Bob’s bits by $N_B$. Suppose that both $N_A$ and $N_B$ are depolarizing channels, i.e.,

$$\phi \mapsto \left( 1 - \frac{3p}{4} \right) \phi + \frac{p}{4} X \phi X + \frac{p}{4} Y \phi Y + \frac{p}{4} Z \phi Z,$$ \hspace{1cm} (19)
where $\phi$ is a single quantum state, $0 \leq p \leq 1$ is the depolarizing probability, and $\{X,Y,Z\}$ are the Pauli operators. For a $D_c = [[n,k,d;c]]$ EAQECC with $n$ physical qubits and $c$ ebits, denote

$$\mu_i = \left(1 - \frac{3p_a}{4}\right)^{n-i} \left(\frac{p_a}{4}\right)^i,$$  \hspace{1cm} (20)$$

$$\nu_j = \left(1 - \frac{3p_b}{4}\right)^{c-j} \left(\frac{p_b}{4}\right)^j,$$  \hspace{1cm} (21)$$

where $0 \leq i \leq n$, $0 \leq j \leq c$, and, $p_a$ and $p_b$ are the depolarizing probabilities of $N_A$ and $N_B$, respectively.

It is known that the channel fidelity of Bowen’s $[[3,1,3;2]]$ EAQECC is given by

$$F_{c[[3,1,3;2]]} = \mu_0\nu_0 + 9\mu_1\nu_0 + 6\mu_3\nu_0 + 6\mu_0\nu_1 + 36\mu_2\nu_1 + 54\mu_3\nu_1 + 18\mu_1\nu_2 + 81\mu_2\nu_2 + 45\mu_3\nu_2.$$  \hspace{1cm} (22)$$

We use Bowen’s $[[3,1,3;2]]$ EAQECC as the inner code, and use the $[[5,1,3]]$ stabilizer code as the outer code to construct a $D_B = [[15,1,9;10]]B$ EACQC. The inner $[[3,1,3;2]]$ EAQECC takes a depolarizing channel to a depolarizing channel under the coding map $\Omega_C$ in Eq. \ref{eq:17}. Then the entanglement fidelity of $D_B$ is given by

$$F_{c[[15,1,9;10]]}^B = 1 - 10p_B^2 + 20p_B^3 - 15p_B^4 + 4p_B^5,$$  \hspace{1cm} (23)$$

where $p_B = 1 - F_{c[[3,1,3;2]]}$.  

Alternately, we use the $[[3,1,3;2]]$ EA repetition code as the inner code, and use the $[[5,1,3]]$ stabilizer code as the outer code to construct another $D_R = [[15,1,9;10]]R$ EACQC. It is known that the channel fidelity of the $[[3,1,3;2]]$ EA repetition code is given by

$$F_{c[[3,1,3;2]]}^R = \mu_0\nu_0 + 9\mu_1\nu_0 + 6\mu_2\nu_0 + 18\mu_1\nu_1 + 38\mu_2\nu_1 + 40\mu_3\nu_1 + 18\mu_1\nu_2 + 55\mu_2\nu_2 + 71\mu_3\nu_2.$$  \hspace{1cm} (24)$$

Then the entanglement fidelity of $D_R$ is given by

$$F_{c[[15,1,9;10]]}^R = 1 - 10p_R^2 + 20p_R^3 - 15p_R^4 + 4p_R^5,$$  \hspace{1cm} (25)$$

where $p_R = 1 - F_{c[[3,1,3;2]]}^R$.  
