Homology of the \( MSU \) Spectrum
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Abstract—We give a complete proof of the Novikov isomorphism \( \Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_2, y_3, \ldots], \) \( \deg y_i = 2i \), where \( \Omega^{SU} \) is the \( SU \)-bordism ring. The proof uses the Adams spectral sequence and a description of the comodule structure of \( H_* (MSU; \mathbb{F}_p) \) over the dual Steenrod algebra \( \mathbb{A}_p^* \) with odd prime \( p \), which was also missing in the literature.

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1. INTRODUCTION

Bordism and cobordism theories were actively developed in the 1950s–1960s. Most of the leading topologists of the time have contributed to this development. The idea of bordism was first explicitly formulated by Pontryagin [21], who related the theory of framed bordism to the stable homotopy groups of spheres using the concept of transversality. Key results of bordism theory were obtained in the works of Thom [27], Averbukh [4], Rokhlin [23], Milnor [13], Novikov [17, 18], Wall [29] and Atiyah [3].

Topologists have quickly realised the potential of the Adams spectral sequence [1] for calculations in bordism theory. It culminated in the description of the complex (or unitary) bordism ring \( \Omega^U \) in the works of Milnor [13] and Novikov [17, 18]. The ring \( \Omega^U \) was shown to be isomorphic to a graded integral polynomial ring \( \mathbb{Z}[a_i : i \geq 1] \) on infinitely many generators, with one generator in every even degree, \( \deg a_i = 2i \). This result has since found numerous applications in algebraic topology and beyond.

In Novikov’s 1967 paper [19] a brand new approach to cobordism and stable homotopy theory was proposed, based on the application of the Adams–Novikov spectral sequence and formal group law techniques. This approach was further developed in the context of bordism of manifolds with singularities in the works of Mironov [15], Botvinnik [5] and Vershinin [28]. The Adams–Novikov spectral sequence has also become the main computational tool for stable homotopy groups of spheres [22].

As an illustration of his approach, Novikov outlined a complete description of the additive torsion and the multiplicative structure of the \( SU \)-bordism ring \( \Omega^{SU} \), which provided a systematic view on the earlier geometric calculations with this ring. A modernised exposition of this description is given in the survey paper by Limonchenko, Panov and Chernykh [10]; it includes the geometric results by Wall [30], Conner and Floyd [8] and Stong [25], the calculations with the Adams–Novikov spectral sequence and the details of the arguments missing in Novikov’s paper [19]. A full description of the \( SU \)-bordism ring \( \Omega^{SU} \) relies substantially on the calculation of \( \Omega^{SU} \) with 2 inverted, namely, on proving the ring isomorphism

\[
\Omega^{SU} \otimes_\mathbb{Z} \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_2, y_3, \ldots], \quad \deg y_i = 2i.
\]

This result first appeared in Novikov’s work [18] with only a sketch of the proof: it was stated that the result can be proved using the Adams spectral sequence in a way similar to Novikov’s calculation.
of the complex bordism ring $\Omega^U$. Although the result has been considered known since the 1960s, its full proof has been missing in the literature, and it was not included in the survey [10] either.

The main goal of this work is to give a complete proof of the above isomorphism using the original methods of the Adams spectral sequence. It is presented as Theorem 4.1. While filling in the details in Novikov’s sketch, we faced technical problems that seemed to be unknown before. For example, the comodule structure of $H_*(MSU; \mathbb{F}_p)$ over the dual Steenrod algebra $\mathfrak{A}_p^*$ with odd prime $p$ has not been satisfactorily described in the literature. This calculation is one of the main results of the paper (Theorem 3.9). We also included a description of the related Hurewicz homomorphism $\pi_*(MSU) \to H_*(MSU)$ and forgetful map $\pi_*(MSU) \to \pi_*(MU)$ in appropriate generators (with 2 inverted) in Corollary 5.3, as well as a result on the divisibility of the characteristic numbers of $SU$-manifolds (Theorem 5.2).

The structure of the paper is as follows. In Section 2, we fix notation and recall necessary information on bordism, cohomology operations and the Adams spectral sequence. In Section 3, we describe $H_*(MSU; \mathbb{F}_p)$ as an $\mathfrak{A}_p^*$-comodule. Then in Section 4, using the Adams spectral sequences, we prove the isomorphism $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_2, y_3, \ldots]$. Finally, in Section 5 we describe the Hurewicz homomorphism and compute the Milnor genus $s_n(y_n)$.

2. PRELIMINARIES

In this section we recall necessary facts and notation. To fully explore the topics listed below, we recommend the sources [7, 14, 16, 22, 24–26].

Let $G$ denote either the unitary group $U$ or the special unitary group $SU$. Denote by $MG$ the corresponding Thom spectrum, i.e., the spectrum whose spaces are Thom spaces of the universal vector $G(n)$-bundles.

By the fundamental theorem of Pontryagin and Thom, the homotopy groups of the Thom spectrum $MG$ are isomorphic to the bordism ring of manifolds with a $G$-structure on the stable normal bundle.

The main technical tool for computing $\pi_*(MG)$ will be the Adams spectral sequence. We first recall some basic facts about the Steenrod operations.

Let $X$ be a topological space and $k$ a non-negative integer. There are natural cohomology operations called Steenrod operations,

$$Sq^k : H^n(X; \mathbb{F}_2) \to H^{n+k}(X; \mathbb{F}_2)$$

and, for an odd prime $p$,

$$P^k : H^n(X; \mathbb{F}_p) \to H^{n+2n(p-1)}(X; \mathbb{F}_p).$$

These operations are uniquely defined by the following properties:

1. $Sq^k$ is $\mathbb{F}_2$-linear, and $P^k$ is $\mathbb{F}_p$-linear;
2. $Sq^0 = 1$ and $P^0 = 1$;
3. $Sq^1$ is the Bockstein homomorphism;
4. if $k > \deg x$, then $Sq^k(x) = 0$; and if $2k > \deg z$, then $P^k(z) = 0$;
5. if $k = \deg x$, then $Sq^k(x) = x^2$; and if $2k = \deg z$, then $P^k(z) = z^p$;
6. (Cartan formula) $Sq^k(xy) = \sum_{i=0}^k Sq^i(x) Sq^{k-i}(y)$ and $P^k(zw) = \sum_{i=0}^k P^i(z) P^{k-i}(w)$;
7. (Adem relations) if $0 < l < 2k$, then

$$Sq^l Sq^k = \sum_{i=0}^{\lfloor l/2 \rfloor} \binom{k-i-1}{l-2i} Sq^{k+l-i} Sq^i;$$
if $0 < l < pk$, then

$$P^l P^k = \sum_{i=0}^{\lfloor l/p \rfloor} (-1)^{l+i} \left( \frac{(p-1)(k-i)-1}{l-pi} \right) P^{k+l-i} P^i,$$

and if $0 < l \leq pk$, then

$$P^l \beta P^k = \sum_{i=0}^{\lfloor l/p \rfloor} (-1)^{l+i} \left( \frac{(p-1)(k-i)-1}{l-pi} \right) \beta P^{k+l-i} P^i + \sum_{i=0}^{\lfloor (l-1)/p \rfloor} (-1)^{l+i+1} \left( \frac{(p-1)(k-i)-1}{l-pi} \right) P^{k+l-i} \beta P^i,$$

where $\beta : H^n(X; \mathbb{F}_p) \to H^{n+1}(X; \mathbb{F}_p)$ is the mod $p$ Bockstein homomorphism.

For a prime $p$, define the mod $p$ Steenrod algebra $\mathfrak{A}_p$ to be the free $\mathbb{F}_p$-algebra generated by $\text{Sq}^k$, $k \geq 1$, if $p = 2$, or by $\beta$ and $P^k$, $k \geq 1$, if $p$ is odd, modulo the Adem relations. By the Cartan–Serre theorem, the admissible monomials form an additive basis of $\mathfrak{A}_p$ for any prime $p$. Recall that a monomial is said to be admissible if the Adem relations cannot be applied to it. Namely, if $p = 2$, then a monomial $\text{Sq}^i \text{Sq}^{i_2} \ldots \in \mathfrak{A}_2$ is said to be admissible if $i_k \geq 2i_{k+1}$. If $p$ is an odd prime, then $\beta^{i_1} P^{i_1} \beta^{i_2} P^{i_2} \ldots \in \mathfrak{A}_p$, $\varepsilon_k = 0$ or 1, is admissible if $i_k \geq \varepsilon_{k+1} + pi_{k+1}$.

The Steenrod algebra is a Hopf algebra with a coproduct induced by the homotopy multiplication map of the Eilenberg–MacLane spectra

$$H\mathbb{F}_p \wedge H\mathbb{F}_p \xrightarrow{\mu} H\mathbb{F}_p.$$

The following theorems describes the structure of its dual Hopf algebra $\mathfrak{A}_p^*$.

**Theorem 2.1** (Milnor [12]). For $p = 2$, let $\xi_n \in (\mathfrak{A}_2^*)_{2^{n-1}}$ be the dual basis element with respect to the Cartan–Serre basis of admissible monomials,

$$\xi_n = (\text{Sq}^{2^{n-1}} \text{Sq}^{2^{n-2}} \ldots \text{Sq}^2 \text{Sq}^1)^*.$$

For odd $p$, let $\xi_n \in (\mathfrak{A}_p^*)_{2^{p^{n-1}}}$ and $\tau_n \in (\mathfrak{A}_p^*)_{2^{p^{n-1}}}$ be the dual basis elements with respect to the Cartan–Serre basis of admissible monomials,

$$\xi_n = (P^{p^{n-1}} P^{p^{n-2}} \ldots P^p P^1)^*, \quad \tau_n = (P^{p^{n-1}} P^{p^{n-2}} \ldots P^p P^1 \beta)^*.$$

Then for $p = 2$, we have an isomorphism of algebras

$$\mathfrak{A}_2^* \cong \mathbb{F}_2[\xi_1, \xi_2, \ldots],$$

and for odd $p$, we have an isomorphism of algebras

$$\mathfrak{A}_p^* \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \ldots].$$

Let $\xi_0 = 1$. The coproduct on $\mathfrak{A}_p^*$ is given by

$$\Delta(\xi_n) = \sum_{k=0}^{n} \xi_{n-k}^p \otimes \xi_k \quad \text{for all primes } p,$$

$$\Delta(\tau_n) = \tau_n \otimes 1 + \sum_{k=0}^{n} \xi_{n-k}^p \otimes \tau_k \quad \text{for odd primes } p.$$
Theorem 2.2 (Brown, Davis, Peterson [6, Theorem 1.2]). Let \( \xi = 1 + \xi_1 + \xi_2 + \ldots \). If \( R = (r_1,r_2,\ldots) \) is a finite sequence of non-negative integers, let \( \xi^R = \xi_1^{r_1} \xi_2^{r_2} \ldots \in \mathbb{A}_p^*, e(R) = \sum_{i \geq 1} r_i, n(R) = \sum_{i \geq 1} r_i(p^i - 1) \) and \( \xi^R_{r_1,r_2,\ldots} \) be the multinomial coefficient. Then for each integer \( k \),
\[
\overline{\xi^k} = \sum_{R} \left( \frac{c(n(R) + k + 1)}{e(R)} \right) \xi^R \quad \text{(Hopf conjugate of the } k\text{-th power of } \xi),
\]
where \( c(m) = p^i - m \), with \( i \) the smallest integer such that \( p^i - m \) is positive. Here and further, \( \overline{\psi} \in \mathbb{A}_p^* \) is the Hopf conjugate of \( \psi \in \mathbb{A}_p^* \).

For an abelian group \( G \), let
\[
p^\infty G = \bigcap_{r=1}^{\infty} p^r G
\]
be the subgroup of elements divisible by \( p^r \) for any \( r \geq 1 \), and let
\[
G_p = G/p^\infty G.
\]
If \( G \) is finitely generated, then the latter equals \( G \) modulo torsion of order prime to \( p \).

Theorem 2.3 (Adams spectral sequence). Let \( p \) be a prime, and let \( X \) be a ring spectrum of finite type. Then there is a natural multiplicative spectral sequence
\[
E_2^{s,t} \cong \text{Ext}_{\mathbb{A}_p^*}(H^*(X; \mathbb{F}_p), \mathbb{F}_p) \Rightarrow \pi_t^{-s}(X)_p
\]
where
\[
d_r: E_r^{s,t} \to E_r^{s+r+1,t-r}.
\]
Here \( s \) is an Ext degree and \( t \) is a cohomological degree.

The second page of the mod \( p \) Adams spectral sequence can be rewritten as
\[
E_2^{s,t} = \text{Cotor}_{\mathbb{A}_p^*}(\mathbb{F}_p, H_*(X; \mathbb{F}_p))_t
\]
(see [22, Appendix A1.2]).

An element \( x \) in an \( \mathbb{A}_p^* \)-comodule is said to be primitive if the structure map sends it to \( 1 \otimes x \). Recall that \( H_*(MU; \mathbb{Z}) \) is isomorphic to \( \mathbb{Z}[b_1, b_2, \ldots] \), where \( b_i = j_*((c_i^1)^*) \) and \( j: CP^\infty \to MU \) is the canonical map. The following theorem describes the \( \mathbb{A}_p^* \)-coalgebra structure of \( H_*(MU; \mathbb{F}_p) \).

Theorem 2.4 [26, Ch. 20]. Denote by \( PH_*(MU; \mathbb{F}_p) \) the subalgebra of primitive elements of \( H_*(MU; \mathbb{F}_p) \). Then
\[
PH_*(MU; \mathbb{F}_p) = \mathbb{F}_p[x_k \mid k \geq 1, k \neq p^i - 1], \quad \deg x_k = 2k.
\]
Furthermore,
\[
H_*(MU; \mathbb{F}_p) \cong \mathbb{A}_p^* \otimes_{\mathbb{F}_p} PH_*(MU; \mathbb{F}_p)
\]
as \( \mathbb{F}_p \)-algebras and \( \mathbb{A}_p^* \)-comodules, where
\[
\mathbb{A}_p^* = (\mathbb{A}_p/\langle \beta \rangle)^* \cong \begin{cases} \mathbb{F}_2[\xi_1, \xi_2^2, \ldots] & \text{if } p = 2, \\ \mathbb{F}_p[\xi_1, \xi_2, \ldots] & \text{if } p \text{ is odd.} \end{cases}
\]

3. HOMOLOGY OF THE THOM SPECTRUM MSU

We begin with collecting necessary information about the homology of the classifying space \( BSU \).

Lemma 3.1 [2, Lemma 2.4]. There is an algebra isomorphism
\[
H_*(BSU; \mathbb{Z}) \cong \mathbb{Z}[y_2, y_3, \ldots]
\]
for suitable \( y_i \in H_{2i}(BSU; \mathbb{Z}) \), \( i = 2, 3, \ldots \).
Remark 3.2. Lemma 3.1 can be easily generalised to oriented spectra. Namely, $E_*(BSU) \cong \pi_*(E)[Y_2, Y_3, \ldots]$ for any complex oriented spectrum $E$.

Lemma 3.3. The homology $H_*(BSU; \mathbb{Z})$ is a subalgebra of $H_*(BU; \mathbb{Z})$.

Proof. Consider the canonical fibration $BSU \xrightarrow{f_*} BU \xrightarrow{det} \mathbb{C}P^\infty$. The latter map corresponds to the first Chern class $c_1 \in H^2(BU; \mathbb{Z}) \cong [BU, \mathbb{C}P^\infty]$. Since the maps in the fibration are maps of $H$-spaces, they induce algebra homomorphisms in homology. Note that $f_*: H_*(BSU; \mathbb{Z}) \to H_*(BU; \mathbb{Z})$ is injective. Therefore, $H_*(BSU; \mathbb{Z}) \cong \text{Im } f_*$ is a subalgebra of $H_*(BU; \mathbb{Z})$. \[\Box\]

Lemma 3.4. The $p$-th power of any element of $H_*(BU; \mathbb{F}_p)$ lies in $H_*(BSU; \mathbb{F}_p)$.

Proof. Consider the cohomology $H^*(BU; \mathbb{Z})$ as a $\mathbb{Z}[c_1]$-module. Dualising the action map $\mathbb{Z}[c_1] \otimes H^*(BU; \mathbb{Z}) \to H^*(BU; \mathbb{Z})$, we obtain a coaction

$$\psi: H_*(BU; \mathbb{Z}) \to \Gamma_{\mathbb{Z}}[\beta] \otimes_{\mathbb{Z}} H_*(BU; \mathbb{Z}),$$

where $\Gamma_{\mathbb{Z}}[\beta]$ is a divided polynomial algebra. The subcoalgebra $P_B H_*(BU; \mathbb{Z})$ of primitive elements is isomorphic to $\text{Im } f_*$ of $H_*(BSU; \mathbb{Z})$. In particular, since for any non-constant $x \in \Gamma_{\mathbb{Z}}[\beta] \otimes \mathbb{F}_p$ its $p$th power is zero, the $p$th power of any element of $H_*(BU; \mathbb{F}_p)$ actually lies in $H_*(BSU; \mathbb{F}_p)$. \[\Box\]

Recall that the Thom isomorphism $\Phi: H_*(BSU; \mathbb{Z}) \to H_*(MSU; \mathbb{Z})$ is an algebra isomorphism, which implies

$$H_*(MSU; \mathbb{Z}) \cong \mathbb{Z}[Y_2, Y_3, \ldots].$$

It follows that the $p$th power of any element of $H_*(MU; \mathbb{F}_p)$ lies in $H_*(MSU; \mathbb{F}_p)$.

The following theorem describes specific polynomial generators of $H_*(MSU; \mathbb{F}_p)$ which are compatible with the inclusion $f_*: H_*(MSU; \mathbb{F}_p) \to H_*(MU; \mathbb{F}_p)$.

Theorem 3.5 (cf. [20]). Let $p$ be an odd prime. There are elements $z_n \in H_{2n}(MU; \mathbb{F}_p)$, $n \geq 1$, such that the following holds:

(i) $H_*(MU; \mathbb{F}_p) \cong \mathbb{F}_p[z_1, z_2, \ldots]$;

(ii) the composite

$$G: H_*(MU; \mathbb{F}_p) \xrightarrow{\rho} \mathfrak{A}_p' \otimes_{\mathbb{F}_p} H_*(MU; \mathbb{F}_p) \xrightarrow{1 \otimes \pi} \mathfrak{A}_p' \otimes_{\mathbb{F}_p} H_*(MU; \mathbb{F}_p)/(z_i, i = p^i - 1),$$

where $\rho$ is the left coaction map and $\pi$ is the canonical projection, is an isomorphism of $\mathbb{F}_p$-algebras and $\mathfrak{A}_p'$-comodules. Here the structure of an $\mathfrak{A}_p'$-comodule of the latter algebra is given by the coaction on the first tensor factor;

(iii) $G(z_{p^t-1}) = -\xi_t \otimes 1$, $t \geq 1$, where $\xi_t$ is the Hopf conjugate of $\xi_t$, and $G(z_n) = 1 \otimes z_n$, $n \neq p^t - 1$;

(iv) if $Y_n \in H_{2n}(MU; \mathbb{F}_p)$, $n \geq 2$, are defined by

$$Y_n = \begin{cases} z_{n/p}^p & \text{for } n = p^t, \\ z_n & \text{otherwise,} \end{cases}$$

then $H_*(MSU; \mathbb{F}_p) \cong \mathbb{F}_p[Y_2, Y_3, \ldots] \subset H_*(MU; \mathbb{F}_p)$.

Proof. If $n \neq p^t, p^t - 1$, let $z_n$ be the Hurewicz image of the $n$-dimensional $SU$-manifold $M_n$, where $\{M_n\}_{n \neq p^t, p^t - 1}$ are polynomial generators of $\pi_*(MU) \otimes_{\mathbb{Z}} \mathbb{F}_p$ in the given degrees. Such manifolds are described, for example, in [25, pp. 240–242]. Note that $z_n$ is primitive as it is the Hurewicz image.

If $n = p^t$, take $z_n = x_{p^t} \in H_{2p^t}(MU; \mathbb{F}_p)$, where $x_{p^t}$ is from Theorem 2.4. The elements $z_n$ are primitive by the very definition.
Finally, if \( n = p^t - 1 \), let \( z_{p^t - 1} = \Phi(e_{p^t - 1}^*) \), \( t \geq 1 \), where \( e_{p^t - 1}^* \in H_*(BU; \mathbb{F}_p) \) is the dual of \( e_{p^t - 1} \) with respect to the monomial basis in the \( e_i \). Clearly, \( z_{p^t - 1} \in H_*(MSU; \mathbb{F}_p) \) for all \( t \geq 1 \). To compute the coaction of \( z_{p^t - 1} \), we use the following theorem.

**Theorem 3.6** (Brown, Davis, Peterson [6, Theorem 1.1]). Let \( \xi = 1 + \xi_1 + \xi_2 + \ldots \) and \( C^* = 1 + \Phi(c_1^*) + \Phi(c_2^*) + \ldots \). Then for the right action \( H^*(MU; \mathbb{F}_p) \otimes_{\mathbb{F}_p} A_p \to H^*(MU; \mathbb{F}_p) \), the dual right coaction \( \Delta \) maps \( C^* \) to

\[
\Delta(C^*) = -1 \otimes \xi^{-1} + \sum_{i \geq 1} \Phi(c_i^*) \otimes \xi^{i-1}.
\]

Recall that \( \overline{\psi} \in A_p^* \) is the Hopf conjugate of \( \psi \in A^*_p \).

**Corollary 3.7.** \( \Delta(z_{p^t - 1}) = -1 \otimes \xi_t + \sum_{s=0}^{t-1} z_{p^t - s - 1}^* \otimes \xi_s \). Equivalently, \( \rho(z_{p^t - 1}) = -\xi_t \otimes 1 + \sum_{s=0}^{t-1} \xi_s \otimes z_{p^t - s - 1}^* \).

**Proof.** For \( \zeta \in A_p^* \), denote by \( \zeta_n \) the component of degree \( n \). Since \( \Phi(c_{p^t - 1}^*) = z_{p^t - 1} \), the formula from Theorem 3.6 gives

\[
\Delta(z_{p^t - 1}) = -1 \otimes (\xi^{-1})_{2(p^t - 1)} + \sum_{i \geq 1} \Phi(c_i^*) \otimes (\xi^{i-1})_{2(p^t - i - 1)}.
\] (3.1)

First, it follows easily from Theorem 2.2 that \( (\xi^{-1})_{2(p^t - 1)} = \xi_t \).

By Theorem 2.2,

\[
(\xi^{i-1})_{2(p^t - i - 1)} = \sum_{n(R)=p^t-i-1} \binom{c((p^t - i - 1) + (i - 1) + 1)}{e(R)} \binom{e(R)}{r_1, r_2, \ldots} \xi^R
\]

\[
= \sum_{n(R)=p^t-i-1} \binom{1}{e(R)} \binom{e(R)}{r_1, r_2, \ldots} \xi^R = \begin{cases} 0 & \text{if } i \neq p^t - p^s, \\ \xi_s & \text{if } i = p^t - p^s. \end{cases}
\]

The last identity holds because \( \binom{1}{e(R)} = 0 \) unless \( R = (0, \ldots, 0, 1, 0, \ldots) \), in which case \( n(R) = p^s - 1, s \geq 1 \).

Finally, \( \Phi(c_{pk}^*) = \Phi(c_k^*)^p \); hence \( \Phi(c_{p^t - p^s}) = z_{p^t - s}^* \). Substituting these expressions into (3.1), we obtain

\[
\Delta(z_{p^t - 1}) = -1 \otimes \xi_t + \sum_{s=0}^{t-1} z_{p^t - s - 1}^* \otimes \xi_s,
\]

as needed. \( \square \)

We resume the proof of Theorem 3.5. Since \( \rho \) is multiplicative, it follows from the corollary above that \( z_{p^t - 1} \) is indecomposable in \( H_*(MU; \mathbb{F}_p) \). The elements \( z_i, i \neq p^t - 1 \), are indecomposable in \( H_*(MU; \mathbb{F}_p) \) by their construction. This proves assertion (i). Assertions (ii) and (iii) are clear. If \( n \neq p^t \), then \( z_n \) actually lies in the image of \( H_*(MSU; \mathbb{F}_p) \). Therefore, they are polynomial generators of \( H_*(MSU; \mathbb{F}_p) \). By Lemma 3.4, \( z_{p^t - 1}^0, t \geq 1 \), lies in \( H_*(MSU; \mathbb{F}_p) \). And finally, by [2, Lemma 2.1], \( z_{p^t - 1}^0 \in H_{p^t}(MSU; \mathbb{F}_p) \), \( t \geq 1 \), are polynomial generators. \( \square \)

**Corollary 3.8.** There are polynomial generators \( Y_n \in H_{2n}(MSU; \mathbb{Z}) \), \( n \neq p^t \), such that \( f_*(Y_n) = e_{2n} \) modulo decomposables.

We summarise the results above in the following description of the \( A_p^* \)-coalgebra structure of \( H_*(MSU; \mathbb{F}_p) \); it is similar to Theorem 2.4.

**Theorem 3.9.** Let \( p \) be an odd prime. There are elements \( Y_n \in H_{2n}(MSU; \mathbb{F}_p) \) such that

\[ H_*(MSU; \mathbb{F}_p) \cong \mathbb{F}_p[Y_2, Y_3, \ldots] . \]
The left coaction \( \rho: H_*(MSU; \mathbb{F}_p) \to \mathfrak{A}_p^* \otimes_{\mathbb{F}_p} H_*(MSU; \mathbb{F}_p) \) is given by

\[
Y_n \mapsto \begin{cases} 
-7t \otimes 1 + \sum_{s=0}^{t-1} \xi_s \otimes Y_{p^s-s-1} & \text{if } n = p^t - 1, \\
1 \otimes Y_n & \text{otherwise.}
\end{cases}
\]

In particular, the subalgebra \( PH_*(MSU; \mathbb{F}_p) \) of primitive elements under this coaction is isomorphic to

\[ \mathbb{F}_p[Y_n, n \neq p^t - 1, n \geq 2], \quad \deg Y_n = 2n. \]

Furthermore, the composite

\[
H_*(MSU; \mathbb{F}_p) \xrightarrow{\rho} \mathfrak{A}_p^* \otimes_{\mathbb{F}_p} H_*(MSU; \mathbb{F}_p)
\]

is an isomorphism of \( \mathbb{F}_p \)-algebras and \( \mathfrak{A}_p^* \)-comodules. Moreover, this composite is given by

\[
Y_n \mapsto \begin{cases} 
-7t \otimes 1 & \text{if } n = p^t - 1, \\
1 \otimes Y_n & \text{otherwise.}
\end{cases}
\]

4. NOVIKOV’S THEOREM

In this section we compute \( \pi_*(MSU) \otimes \mathbb{Z}[\frac{1}{2}] \) using the modern adaptation of Novikov’s original proof (see [18]).

**Theorem 4.1** (Novikov [18]). There are elements \( y_n \in \pi_{2n}(MSU) \otimes \mathbb{Z}[\frac{1}{2}], n = 2, 3, \ldots \), such that

\[ \pi_*(MSU) \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_2, y_3, \ldots]. \]

**Proof.** For an odd prime \( p \), consider the mod \( p \) Adams spectral sequence for \( \pi_*(MSU) \) with the second term

\[ E_2^{s,t} \cong \text{Cotor}_{\mathfrak{A}_p}^*(F_p, H_*(MSU; F_p))_s \]

By Theorem 3.9, we have \( H_*(MSU; F_p) \cong \mathfrak{A}_p' \otimes PH_*(MSU; F_p) \), where \( \mathfrak{A}_p' = F_p[\xi_1, \xi_2, \ldots] \) and \( PH_*(MSU; F_p) = F_p[Y_n, n \neq p^t - 1, n \geq 2] \), \( \deg Y_n = 2n \), is the subalgebra of primitive elements.

By the change-of-rings theorem (see [11]),

\[
\text{Cotor}_{\mathfrak{A}_p}^*(F_p, H_*(MSU; F_p))_* = \text{Cotor}_{\mathfrak{A}_p'}^*(F_p, \mathfrak{A}_p' \otimes_{F_p} PH_*(MSU; F_p))_*,
\]

\[
= \text{Cotor}_{\mathfrak{A}_p'}^*(F_p, F_p)_{\otimes F_p} PH_*(MSU; F_p)
\]

By Theorem 2.1, \( \mathfrak{A}_p^*/\mathfrak{A}_p' \overset{\text{def}}{=} \mathfrak{A}_p^*/(\mathfrak{A}_p' \cdot \mathfrak{A}_p') = \Lambda_{F_p}[\tau_0, \tau_1, \ldots] \), \( \deg \tau_t = 2p^t - 1 \), is an exterior algebra of finite type. Therefore, \( \text{Cotor}_{\mathfrak{A}_p'}^*(F_p, F_p)_* \) is a polynomial algebra,

\[
\text{Cotor}_{\mathfrak{A}_p'}^*(F_p, F_p)_* = F_p[q_0, q_1, \ldots], \quad q_t \in \text{Cotor}_{\mathfrak{A}_p}^*(F_p, F_p)_{2p^t-1}.
\]

Hence, for odd prime \( p \) the second term of the Adams spectral sequence has the form

\[
E_2^{s,t} = F_p[q_0, q_1, \ldots] \otimes_{F_p} PH_*(MSU; F_p)
\]

\[
= F_p[q_0, q_1, \ldots] \otimes_{F_p} F_p[Y_n, n \neq p^t - 1, n \geq 2]
\]

\[
= F_p[q_0] \otimes_{F_p} F_p[m_2, m_3, \ldots],
\]
where \( q_0 \in E_2^{1,1} \) and
\[
m_n = \begin{cases} 
q_t \in E_2^{1,2n+1} & \text{if } n = p^t - 1 \text{ for some } t > 0, \\
Y_n \in E_2^{0,2n} & \text{otherwise}.
\end{cases}
\]

Note that \( E_2^{s,t} \) is zero whenever \( t - s \) is odd. Therefore, since the differentials reduce \( t - s \) by 1, \( d_r = 0 \) for every \( r = 2, 3, \ldots \) and \( E_2^{\ast,\ast} = E_\infty^{\ast,\ast} \). One can easily show that
\[
q_0 \in E_2^{1,1} = \text{Cotor}_{\mathbb{Z}_p}^1(\mathbb{F}_p, \mathbb{F}_p)_1
\]
is represented by \( p \in \pi_0(\text{MSU}) \cong \mathbb{Z} \). Thus, for every odd prime \( p \) there is an isomorphism of abelian groups
\[
\pi_\ast(\text{MSU}) \cong \mathbb{Z}[m_2, m_3, \ldots].
\]

Next we show that there is a ring isomorphism
\[
\pi_\ast(\text{MSU}) \otimes_{\mathbb{Z}} \mathbb{F}_p \cong \mathbb{F}_p[P_2, P_3, \ldots]
\]
for some \( P_i = P_i(p) \in \pi_{2i}(\text{MSU}) \).

Let \( P_i \in \pi_{2i}(\text{MSU}) \), \( i = 2, 3, \ldots \), be elements corresponding to \( m_i \in E_\infty^{0,0} \) in the mod \( p \) Adams spectral sequence for \( \text{MSU} \). Note that our choice depends on \( p \). Since \( m^a, \alpha = \{\alpha_1, \ldots, \alpha_k, 0, \ldots\} \in \bigoplus_{i=0}^\infty \mathbb{Z} \), are linearly independent in the Adams spectral sequence, it follows that the monomials \( P^\alpha = P_1^{\alpha_1} \cdots P_k^{\alpha_k} \) are linearly independent. We have to prove that the monomials \( P^\alpha \) generate \( \pi_\ast(\text{MSU}) \otimes_{\mathbb{Z}} \mathbb{F}_p \).

Lemma 20.28 in [26] states that for every \( q \in \mathbb{Z}_{\geq 0} \) there is \( t \in \mathbb{Z}_{> 0} \) such that a filtration term \( F^{t, q+t} \) is trivial mod \( p \), i.e., \( F^{t, q+t} \otimes_{\mathbb{Z}} \mathbb{F}_p = 0 \). Now using downward induction over \( s \) starting at \( s = t \), we show that the monomials \( P^\alpha \) of degree \( g \) and of filtration at least \( s \) span \( F^{s, q+s} \otimes_{\mathbb{Z}} \mathbb{F}_p \). Let \( \{m^a, \alpha \in \bigoplus_{i=0}^\infty \mathbb{Z}\} \) be monomials in \( m_i \) in \( E_\infty^{s-1, q+s-1} \). Then any \( x \in F^{s-1, q+s-1} \otimes_{\mathbb{Z}} \mathbb{F}_p \) can be written as \( x = \sum a_i P^{\alpha_i} \) mod \( F^{s, q+s} \), because the projection \( F^{s-1, q+s-1} \to E_\infty^{s-1, q+s-1} \) sends \( P^{\alpha_i} \) to \( m^{\alpha_i} \), \( i = 1, \ldots, k \). Now we can use the induction hypothesis to prove that the monomials \( P^\alpha \) generate \( \pi_\ast(\text{MSU}) \otimes_{\mathbb{Z}} \mathbb{F}_p \).

Finally, Theorem 4.1 follows from the fact that if a graded ring of finite type \( R \) is isomorphic to \( \mathbb{F}_p[P_2, P_3, \ldots] \), \( \deg P_i = 2i \), modulo an odd prime \( p \), then its localisation \( R[\frac{1}{2}] \) is isomorphic to \( \mathbb{Z}[\frac{1}{2}][y_2, y_3, \ldots] \), \( \deg y_i = 2i \). \( \Box \)

5. MULTIPlicative Generators

In this section we compute the characteristic numbers of the polynomial generators of the ring \( \pi_\ast(\text{MSU}) \otimes \mathbb{Z}[\frac{1}{2}] \). We start from the description of the mod \( p \) Hurewicz homomorphism in terms of the mod \( p \) Adams spectral sequence.

**Proposition 5.1.** Let \( X \) be a spectrum of finite type. Then the mod \( p \) Hurewicz homomorphism
\[
h: \pi_\ast(X) \to H_\ast(X; \mathbb{F}_p)
\]
coincides with the composite map
\[
\pi_\ast(X) \to E_\infty^{0,\ast} \rightarrow E_2^{0,\ast} = PH_\ast(X; \mathbb{F}_p) \hookrightarrow H_\ast(X; \mathbb{F}_p),
\]
where \( E_\infty^{0,\ast} \) and \( E_2^{0,\ast} \) are terms of the mod \( p \) Adams spectral sequence.

**Proof.** Note that the map
\[
h': X \simeq \bigwedge^\infty_S X \xrightarrow{\eta H_\ast \wedge_1} H \mathbb{F}_p \wedge X
\]

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induces the mod $p$ Hurewicz homomorphism in homotopy groups. Thus, we have a map of mod $p$ Adams spectral sequences $(h_1')_r: E_2^{r,*} \to E_2^{r,*}$, $r \geq 2$, for $\pi_*(X)$ and $\pi_*(HF_p \wedge X) \cong H_*(X; F_p)$ induced by $h'$. Since $H_*(HF_p \wedge X; F_p) \cong \mathfrak{A}_p^* \otimes F_p H_*(X; F_p)$, it follows that

$$E_2^{n,*} = \text{Cotor}_{\mathfrak{A}_p^*}^n(F_p, H_*(HF_p \wedge X; F_p)).$$

and

$$E_2^{0,*} = \text{Cotor}_{\mathfrak{A}_p^*}^0(F_p, H_*(HF_p \wedge X; F_p) \otimes F_p \Delta_{\mathfrak{A}_p^*} H_*(HF_p \wedge X; F_p) = H_*(X; F_p),$$

where $\Delta_{\mathfrak{A}_p^*}$ is a cotensor product over $\mathfrak{A}_p^*$.

The proposition now follows from the decomposition of the Hurewicz homomorphism,

$$\pi_*(X) \xrightarrow{h} H_*(X; F_p)$$

and the fact that $(h'_1) = (h'_2)$ equals the composition $E_2^{0,*} \cong PH_*(X; F_p) \to H_*(X; F_p). \quad \square$

For any positive integer $n$, we define

$$\lambda_n = \begin{cases} p & \text{if } n + 1 = p^t \text{ for some positive } t \text{ and prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

**Theorem 5.2** (Novikov [18]). Let $h: \pi_*(MSU) \otimes \mathbb{Z}[\frac{1}{2}] \to H_*(MSU; \mathbb{Z}[\frac{1}{2}])$ be the Hurewicz homomorphism and let $f: MSU \to MU$ be the canonical map. Then there exist polynomial generators $y_n \in \pi_{2n}(MSU) \otimes \mathbb{Z}[\frac{1}{2}], n \geq 2$, such that

$$f_* h(y_n) = \pm \lambda_n \lambda_{n-1} b_n$$

modulo decomposable elements in $H_{2n}(MU; \mathbb{Z}[\frac{1}{2}])$, where $b_n \in H_{2n}(MU)$ is a canonical polynomial generator.

Therefore, $y_n \in \pi_{2n}(MSU) \otimes \mathbb{Z}[\frac{1}{2}]$ can be taken as a polynomial generator if and only if

$$s_n(y_n) = \pm \lambda_n \lambda_{n-1}$$

up to a power of 2.

**Proof.** Case 1: $n \neq p^t - 1$ for an odd prime $p$. By Proposition 5.1, $y_n \in \pi_*(MSU) \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial generator if and only if $h(y_n) \in PH_*(MSU; F_p) \cong F_p[Y_n | n \neq p^t - 1, n \geq 2]$, a polynomial generator; in particular, it is a polynomial generator of $H_*(MSU; F_p)$. Since this is true for every odd prime $p$, $h(y_n)$ is a polynomial generator of $H_*(MSU; \mathbb{Z}[\frac{1}{2}])$. Thus, $h(y_n) = \pm Y_n$ modulo decomposables.

Suppose that $n \neq p^t$. Then $f_* h(y_n) = \pm b_n$ modulo decomposables by Corollary 3.8. Hence, $f_* h(y_n) = \pm b_n$ modulo decomposables, as needed.

Now suppose $n = p^t$. Then $f_* (Y_n) = \pm pb_n$ modulo decomposables by [9, Theorem 3.3]. Therefore, $f_* h(y_n) = \pm pb_n = \pm \lambda_n \lambda_{n-1} b_n$ modulo decomposables, as claimed.

Case 2: $n = p^t - 1$ for an odd prime $p$. Let $y_n \in \pi_{2n}(MSU) \otimes \mathbb{Z}[\frac{1}{2}]$ be a polynomial generator. Then $h(y_n)$ is decomposable in $PH_*(MSU; F_p) \cong F_p[Y_k | k \neq p^t - 1, k \geq 2]$ since there are no indecomposables in degree $2(p^t - 1)$. Therefore, $f_* h(y_n) \in H_{2n}(MU; \mathbb{Z})$ is divisible by $p$ modulo decomposables. An argument similar to the one in the case $n \neq p^t - 1$ implies that $h(y_n)$ is a
polynomial generator of $PH_\ast(MSU; \mathbb{F}_p)$ for any prime $q \neq p$. Hence, $f_\ast h(y_n)$ is divisible by $p$ and is not divisible by any other prime $q$ modulo decomposables. Therefore, $f_\ast h(y_n) = \pm p^k b_n$ modulo decomposables.

It remains to prove that $k = 1$ above. In other words, we need to prove that $h(y_n)$ is not divisible by $p^2$ modulo decomposables. This can be done either algebraically as below, or by providing explicit examples of $SU$-manifolds with an appropriate Milnor genus (see [10, Pt. II]).

Consider the map $h': MSU \xrightarrow{H_\ast(MSU; \mathbb{F}_p) \to HZ \wedge MSU}$ that induces the Hurewicz homomorphism in homotopy groups. The second term of the mod $p$ Adams spectral sequence for $HZ \wedge MSU$ is

$$E^2_{2,t} = \operatorname{Cotor}^\ast_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(HZ \wedge MSU; \mathbb{F}_p))_t \cong \operatorname{Cotor}^\ast_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(HZ; \mathbb{F}_p))_t \otimes_{\mathbb{F}_p} H_\ast(MSU; \mathbb{F}_p).$$

Recall that $H_\ast(HZ; \mathbb{F}_p) \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \tau_2, \ldots]$. Also, by Theorem 2.1, we have $\mathfrak{A}_p \cong \mathbb{F}_p[\xi_1, \xi_2, \ldots] \otimes_{\mathbb{F}_p} \Lambda_{\mathbb{F}_p}[\tau_0, \tau_1, \tau_2, \ldots]$. It follows that $\operatorname{Cotor}^\ast_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(HZ; \mathbb{F}_p)) \cong \mathbb{F}_p[q_0]$, where $q_0 \in \operatorname{Cotor}^2_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(HZ; \mathbb{F}_p))_0$ represents $p \in \pi_0(HZ)$. Therefore, $E^2_{2,t} = 0$ when $t - s$ is odd. As before, this implies that the spectral sequence degenerates in the second term, i.e., $E^2_{2,t} = E^\infty_{2,t}$.

The map $h'$ induces the following Cotor map:

$$h': \operatorname{Cotor}^\ast_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(MSU; \mathbb{F}_p))_t \rightarrow \operatorname{Cotor}^\ast_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(HZ \wedge MSU; \mathbb{F}_p))_t \cong \mathbb{F}_p[q_0] \otimes_{\mathbb{F}_p} H_\ast(MSU; \mathbb{F}_p). \quad (5.1)$$

In order to describe this map, we consider the following diagram of cobar complexes:

$$\begin{array}{ccc}
C^\ast_{\mathfrak{A}_p}(H_\ast(MSU; \mathbb{F}_p)) & \xrightarrow{\sim \text{quasi}} & C^\ast_{\mathfrak{A}_p/\mathfrak{A}_p}(PH_\ast(MSU; \mathbb{F}_p)) \\
\hbar & \downarrow & \\
C^\ast_{\mathfrak{A}_p[q_0]}(H_\ast(MSU; \mathbb{F}_p))
\end{array} \quad (5.2)$$

In homology, the map $\hbar$ induces the map $h'_s$ in (5.1).

Consider the class $m_n = q_t \in \operatorname{Cotor}^1_{\mathfrak{A}_p}(\mathbb{F}_p, H_\ast(MSU; \mathbb{F}_p))_{2p^t-1}$ defined in the proof of Theorem 4.1. By the cobar construction, it is represented by $[\bar{\tau}_t] \in C^\ast_{\mathfrak{A}_p/\mathfrak{A}_p}(PH_\ast(MSU; \mathbb{F}_p))$. The element

$$Q_t = -\sum_{i=0}^t [\bar{\tau}_i] \xi_{t-i}^p \in C^\ast_{\mathfrak{A}_p}(H_\ast(MSU; \mathbb{F}_p))$$

is a cycle that is mapped to $- [\bar{\tau}_t]$ under the horizontal arrow in the diagram (5.2). Indeed, recall that the conjugation is a coalgebra antihomomorphism and the coproduct on $\mathfrak{A}_p$ is given in Theorem 2.1. Therefore, we have

$$dQ_t = -\sum_{i=0}^t [\bar{\tau}_i] \xi_{t-i}^p + \sum_{i=0}^t [\bar{\tau}_i] \xi_{t-i}^p - \sum_{i=0}^t \sum_{k=0}^i [\bar{\tau}_k] \xi_{t-k}^p \xi_{t-i}^p - \sum_{i=0}^t \sum_{k=0}^i [\bar{\tau}_k] \xi_{t-k}^p \xi_{t-i-k}^p$$

$$= \sum_{a+b+c \leq t} [\bar{\tau}_a] \xi_{t-c}^{p^c} - \sum_{a+b+c \leq t} [\bar{\tau}_a] \xi_{t-c}^{p^c} = 0.$$
Now we have $\tilde{h}(Q_t) = [\tau_0] t \in C_{A[\tau_0]}^1(H_*(MSU; \mathbb{F}_p))$. Hence, $(h')_*(q_t) = q_0 t \in \mathbb{F}_{2p}^{1,2p^{2p - 1}}$. Since $q_0$ represents the multiplication by $p$, the element $h(y_n) \in H_{2n}(MSU; \mathbb{Z}^{[\frac{1}{p}]}_2)$ is divisible precisely by $p$ modulo decomposables. □

**Corollary 5.3.** There are polynomial generators

$$y_i \in \pi_2(MSU) \otimes \mathbb{Z}[\frac{1}{2}], \quad x_i \in \pi_2(MU) \otimes \mathbb{Z}[\frac{1}{2}], \quad Y_i \in H_{2i}(MSU; \mathbb{Z}^{[\frac{1}{2}]}) \quad X_i \in H_{2i}(MU; \mathbb{Z}^{[\frac{1}{2}]}),$$

such that the canonical maps are given as in the diagram

$$\pi_*(MSU) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{y_n \mapsto \lambda_{n-1} x_n} \pi_*(MU) \otimes \mathbb{Z}[\frac{1}{2}]$$

$$\downarrow_{y_n \mapsto \lambda_n Y_n} \quad \downarrow_{x_n \mapsto \lambda_n X_n}$$

$$H_*(MSU; \mathbb{Z}^{[\frac{1}{2}]}) \xrightarrow{Y_n \mapsto \lambda_{n-1} X_n} H_*(MU; \mathbb{Z}^{[\frac{1}{2}]})$$

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