Abstract
The question of whether all problems in NP class are also in P class is generally considered one of the most important open questions in mathematics and theoretical computer science as it has far-reaching consequences to other problems in mathematics, computer science, biology, philosophy and cryptography. There are intensive research on proving ‘NP not equal to P’ and ‘NP equals to P’. However, none of the ‘proved’ results is commonly accepted by the research community up to date. In this paper, motivated by approximability of traveling salesman problem (TSP) in polynomial time, we aim to provide a new perspective: showing that NP=P from polynomial time approximation-bounded solutions of TSP in Euclidean space.

Keywords: NP problems, P Problems, NP vs P, Exact Solution in Polynomial Time

1. Introduction

P versus NP problem is one of seven Millennium Prize Problems in mathematics that were stated by the Clay Mathematics Institute [1] in 2000. As of May 2016, six of the problems remain unsolved. The official statement of P versus NP problem was given by Stephen Cook [2]. We say that a problem A in NP is NP-complete when, for every other problem B in NP, B is easier than A, i.e., B < A. In computational complexity theory, Karp’s 21 NP-complete problems are a set of computational problems which are NP-complete. In his 1972 paper [9], Richard Karp used Stephen Cook’s 1971 theorem that the Boolean satisfiability problem is NP-complete (also called the Cook-Levin theorem) to show that there is a polynomial time many-one reduction from the Boolean satisfiability problem (BSP) to each of 21 combinatorial and graph theoretical computational problems, thereby showing that they are all NP-complete. This was one of the first demonstrations that many natural computational problems occurring throughout computer science are computationally intractable, and it drove interest in the study of NP-completeness and the P versus NP problem.

Simply speaking, P problems mean that the class of problems can be solved exactly in polynomial time while NP (non-deterministic polynomial) problem stands for a class of problems which can not be solved in polynomial time. Intuitively, NP problem is the set
of all decision problems for which the instances where the answer is “yes” have efficiently verifiable proofs of the fact that the answer is indeed “yes”. More precisely, these proofs have to be verifiable in polynomial time by a deterministic Turing machine. In an equivalent formal definition, NP problems is the set of decision problems where the “yes”-instances can be accepted in polynomial time by a non-deterministic Turing machine [18]. NP problems has far-reaching consequences to other problems in mathematics, biology, philosophy and cryptography.

The complexity class P is contained in NP, and NP contains many important problems. The hardest of which are NP-complete problems, whose solutions are sufficient to deal with any other NP problems in polynomial time. The most important open question in complexity theory, is the P versus NP problem which asks whether polynomial time algorithms actually exist for NP-complete problems and all NP problems. The important thing is that Karp showed that if any of them have efficient polynomial time algorithms, then they all do. Many of these problems arise from real-world optimization problems including Sub Set Sum Problem (SSP), Traveling Salesman Problem (TSP), Bin Packing Problem (BPP), Hamiltonian Cycle Problem (HCP), and Chromatic Number Problem (CNP) etc.. Researchers later extend Karp’s techniques to show hundreds, if not thousands of natural problems, are NP-complete.

It is widely believed that this is not the case in 2002 [4]. In 2012, 10 years later, the same poll was repeated [5]. The number of researchers who answered was 126 (83%) believed the answer to be no, 12 (9%) believed the answer is yes, 5 (3%) believed the question may be independent of the currently accepted axioms and therefore is impossible to prove or disprove, 8 (5%) said either don’t know or don’t care or don’t want the answer to be yes nor the problem to be resolved. On the Web site [19], Prof. Gerhard Woeginger provides the unofficial archivist of about 116 claims for the NP vs P problem from 1986 to April 2016, among them, 49 (42%) believed the answer to be no, 62 (53%) believed the answer is yes, the other 5 (5%) think Undecidable, or Unprovable or Unknow. About nine of papers in the list ‘established’ NP=P by designing algorithms for variants of the TSP [Cook, 2012], though none of them is commonly accepted yet by the research community.

There are intensive research on proving ‘NP not equal to P’ and ‘NP equals to P’. However, none of the ‘proved’ results is commonly accepted by the research community yet up to date. There do exist some problems previously believed to be in NP class but recently proved to be in P class, such as PRIME problem [20]. In this paper, we aim to provide a new perspective: showing that NP=P from solutions of traveling salesman problems in Euclidean space.

The remaining sections are organized as follows. TSP is discussed in Section 2. Approximation bounded algorithm for TSP is proposed in Section 3. Our main results are provided in Section 4. Finally we conclude in Section 5.
2. TSP Formulation in Euclidean Space

The TSP is one of most researched problems in combination optimization because of its importance in both academic need and real world applications. For surveys of the TSP and its applications, the reader is referred to [Cook, 2012] and references therein. We consider the \( n \)-node TSP defined in Euclidean space. This can be represented on a complete graph \( G = (V, E) \) where \( V \) is the set of nodes and \( E \) is the set of edges. The cost of an edge \((u, v)\) is the Euclidean distance \((c_{uv})\) between \( u \) and \( v \). Let the edge cost matrix be \( C[c_{ij}] \) which satisfies the triangle inequality.

**Definition 1.** Symmetric TSP (STSP) is TSP in Euclidean distance (called ESTSP) and the edge cost matrix \( C \) is symmetric. **Definition 2.** Asymmetric TSP (ATSP) is TSP in Euclidean distance (called EATSP) and the edge cost matrix \( C \) is asymmetric.

**Definition 3.** \( \triangle \)STSP is a STSP whose edge costs are non-negative and satisfies the triangle inequality, i.e., for any three distinct nodes (not necessary neighboring) \((i, j, k)\), \((c_{ij}+c_{jk}) \geq c_{ik}\). The STSP is also called the metric TSP.

**Definition 4.** TSP tour. Given a graph \( G \) in 2-dimensional Euclidean distance and its distance matrix \( C \) where \( c_{ij} \) denote the distance between node \( i \) and \( j \) (for both symmetric and asymmetric). A tour \( T \) with \(|V|\) nodes has length

\[
L = \sum_{k=0}^{\mid V \mid-1} c_{T(k),T(k+1)}
\]  

(1)

In 1977, Papadimitriou [12] firstly proved that the Euclidean TSP is NP-complete by reduction of the Exact Cover Problem to the ETSP.

2.1. Polynomial Time Approximation-Bounded Solutions to ESTSP

The following results are obtained from William J. Cook’s book [3], Chapter 1, where all problems are solved to optimality by different tools except for 1000,000 city problem and 1,904,711 city problem, for which optima are not known yet.

In Table 1, Nagata’s tour for 1000,00-city Mona Lisa tour is known to be at most 0.0003% longer than an optimal solution; The tour by LKH [8] for 1,904,711-city of length 7,515,790,345 meters was known to be no more than 0.0476% longer than an optimal tour.

**Definition 5. Concorde Algorithm [4]:** Concorde is a computer code for the STSP and some related network optimization problems. The code is written in the ANSI C programming language. Concorde’s TSP solver has been used to obtain the optimal solutions to the full set of 110 TSPLIB instances, the largest having 85,900 cities. Executable versions of Concorde with qsopt for Linux and Solaris are available [4]. Hans Mittelmann has created a NEOS Server (http://neos-server.org) for Concorde, allowing users to solve TSP instances online.

**Definition 6. LKH algorithm [11]:** LKH is an effective implementation of the Lin-Kernighan heuristic [10] for solving the traveling salesman problem. Computational experiments have shown that LKH is highly effective. LKH has produced optimal solutions for all
Table 1: TSP Records Variation By Years [3]

| # Nodes | Year (Solved) | Description         | Authors                              |
|---------|---------------|---------------------|--------------------------------------|
| 48      | 1954          | USA cities          | Dantzig et al. (by hand)             |
| 64      | 1971          | random nodes        | Micheal Held, Richard Karp           |
| 80      | 1975          | random nodes        | Panagiotis Miliotis                  |
| 120     | 1977          | Germany cities      | Martin Grotschel, Manfred Padberg   |
| 318     | 1987          | cities              | Martin Grotschel, Manfred Padberg   |
| 532     | 1987          | USA cities          | Martin Grotschel, Manfred Padberg   |
| 666     | 1987          | World cities        | Martin Grotschel, Manfred Padberg   |
| 1002    | 1987          | cities              | Martin Grotschel, Manfred Padberg   |
| 2392    | 1987          | cities              | Martin Grotschel, Manfred Padberg   |
| 3038    | 1992          | cities              | Concorde                             |
| 13509   | 1998          | USA cities          | Concorde                             |
| 15112   | 2001          | cities              | Concorde                             |
| 24978   | 2004          | Sweden cities       | Concorde                             |
| 85900   | 2006          | cities              | Concorde, LKH [11]                   |
| 100000  | 2009*         | Japan               | Yuchi Nagata                         |
| 1904711 | 2010*         | World TSP Challenge | LKH [11]                             |

Solved problems they have been able to obtain; including a 85,900-city instance (at the time of writing, the largest nontrivial instance solved to optimality). Furthermore, the algorithm has improved the best known solutions for a series of large-scale instances with unknown optima, among these a 1,904,711-city instance (called World TSP).

Both Concorde [4] and LKH [8] solve all 110 TSPLIB instances [14] to optimums.

Table 2: 5 Longest Running Time TSPLIB Instances Solved Exactly by LKH [8]

| Name   | #Nodes | Running Time (Seconds) |
|--------|--------|------------------------|
| fl1577 | 1577   | 10975                  |
| fnl4461| 4661   | 10973                  |
| u1817  | 1817   | 2529                   |
| pcb3038| 3038   | 3237                   |
| pla7397| 7397   | 130220                 |

Table 2 shows that LKH results for 5 STSP TSPLIB instances which are top 5 longest running time instances for LKH solved in 1998 and running times are measured in seconds on a 300 MHz G3 Power Macintosh.
2.2. Polynomial Time Exact Solutions To ATSP

Table 3 shows that exact LKH results for ATSP instances in TSPLIB, where running times are measured in seconds on a 300 MHz G3 Power Macintosh.

Table 3: ATSP Instances Solved Exactly by LKH [8]

| Name    | TotalRuns | RunningTime |
|---------|-----------|-------------|
| br17    | 100       | 0.0         |
| ft53    | 100       | 0.0         |
| ft70    | 100       | 0.0         |
| ftv33   | 100       | 0.0         |
| ftv35   | 100       | 1.0         |
| ftv38   | 100       | 1.0         |
| ftv44   | 100       | 0.0         |
| ftv47   | 100       | 0.0         |
| ftv55   | 100       | 0.0         |
| ftv64   | 100       | 0.0         |
| ftv70   | 100       | 0.0         |
| ftv170  | 100       | 10          |
| kro124p | 100       | 1           |
| p43     | 100       | 5           |
| rbg323  | 100       | 85          |
| rbg358  | 100       | 83          |
| rbg403  | 100       | 113         |
| rbg443  | 100       | 126         |
| rbg48p  | 100       | 0.0         |

Also the results of running LKH on 126 asymmetric traveling salesman instances with known optimal solutions from the work by David Soler [Soler, 2008], show that LKH found all optimal solutions for all 126 instances with average running time below 10 seconds per run (running time per run measured in seconds on a 2.93 GHz Intel Core i7 iMac).

2.3. On the approximability of metric TSP

On the approximability of metric TSP, there is a well-known theorem as follows.

Papadimitriou-Vempala Theorem. In [13], Papadimitriou and Vempala (let called Papadimitriou-Vempala Theorem) proved that, unless NP=P, there can be no polynomial-time C-approximation algorithm for the metric TSP with C less than 1.0045, i.e., less than 0.45%.

However as we already see that LKH can solve all of TSPLIB instances to optimality in polynomial time. Even for the largest instances, 1,904,711-city problem for which optima are not known yet, but the tour by LKH for 1,904,711-city was known to be no more than
0.0476\% longer than an optimal tour by Concorde bound [3]. This means that LKH solution is below 1.0045 approximation. Therefore we think that Papadimitriou-Vempala Theorem needs reevaluation. We further propose an algorithm called ITGBC which can obtain approximation ratio less than 1.0045 for metric TSP.

We firstly propose a Generalized Beta (GB) distribution [17]. The probability density function (pdf) of GB is defined as

\[
f(x, \alpha, \beta, A, B) = \frac{(x - A)^{\alpha-1}(B - x)^{\beta-1}}{Beta(\alpha, \beta)}
\]  

where \(Beta(\alpha, \beta)\) is the beta function

\[
Beta(\alpha, \beta) = \int_0^1 t^{\alpha-1}(1 - t)^{\beta-1}dt,
\]

\(A\) and \(B\) is the lower bound and upper bound respectively, \(\alpha > 0, \beta > 0\). For TSP, \(A\) and \(B\) represents the minimum and maximum tour length respectively.

**Definition 6.** \(k\)-opt. is a local search with \(k\)-exchange neighborhoods and the most widely used heuristic method for the TSP. \(k\)-opt is a tour improvement algorithm, where in each step \(k\) links of the current tour are replaced by \(k\) links in such a way that a shorter tour is achieved (see [Helsgaun 2009] for detailed introduction).

Next, we introduce our algorithm, Iterative Truncated Generalized Beta distribution Based on Christofides Algorithm (ITGBC). ITGBC algorithm performs in seven steps:

- (1). Finding the minimum spanning tree \(MST\) of the input graph \(G\) representation of metric TSP;
- (2). Taking \(G\) restricted to vertices of odd degrees in \(MST\) as the subgraph \(G^*\); This graph has an even number of nodes and is complete;
- (3). Finding a minimum weight matching \(M^*\) on \(G^*\);
- (4). Uniting the edges of \(M^*\) with those of the \(MST\) to create a graph \(H\) with all vertices having even degrees;
- (5). Creating a Eulerian tour on \(H\) and reduce it to a feasible solution using the triangle inequality, a short cut is a contraction of two edges \((i, j)\) and \((j, k)\) to a single edge \((i, k)\);
- (6). Applying Christofides algorithm [21] to a ESTSP forms a truncated GB (TGB) for the probability density function of optimal tour lengths, with expectation (average)
value at most 1.5OPT-$\epsilon$, where $\epsilon$ is a very small value; Applying $k$-opt to the result of Christofides algorithm forms another TGB for probability density function of optimal tour lengths:

- (7). Iteratively apply this approach, taking the expectation value of $(K-1)$-th iteration as the upper bound ($\hat{b}_K = \frac{\mu_{K-1}}{B-A}$) of the $K$-th iteration, we have the expectation value after $K$ iterations ($K \geq 2$), denoted as $\mu^K_t$ proved in [17],

$$\mu^K_t = A + (B - A) \frac{B_2(0, b^K, \alpha + 1, \beta)}{B_2(0, b^K, \alpha, \beta)}$$

$$= A + (B - A) g(\hat{b}_K)$$

$$\leq (1 + \frac{1}{2} (\frac{\alpha + 1}{\alpha + 2})^{-1}) A$$

(4)

$$B_2(0, t, \alpha, \beta) = \int_0^t x^{\alpha-1} (1 - x)^{\beta-1} dt$$

(5)

**Theorem 1.** ITGBC algorithm is $(1+1/2(\frac{\alpha+1}{\alpha+2})^{K-1})$-approximation where $K$ is the number of iterations in ITGBC, $\alpha$ is the shape parameter of TGB and can be determined once ETSP instance is given. In [17], the present author proved Theorem 1. One can see that as $K$ increases, the approximation ratio $(1+1/2(\frac{\alpha+1}{\alpha+2})^{K-1})$ can be less than 1.0045. Actually when $K > (1+\frac{\log 0.009}{\log(1-1/(\alpha+2))})$, the approximation ratio will be less than 1.0045.

**Theorem 2.** The computational complexity of ITGBC algorithm is of $O(max(n^3, n^{2.2}, K(k^3 + k\sqrt{n})))$.

**Proof:** In [8], a method with computational complexity of $O(k^3 + k\sqrt{n})$ is introduced for $k$-opt. Since ITGBC applies Christofides algorithm firstly which has computational complexity of $O(n^3)$ [21], and then applies $k$-opt with $K$ iterations in LKH which has computational complexity of $O(K(k^3 + k\sqrt{n}))$, the computational complexity of LKH is estimated to be $O(n^{2.2})$ [8][11], so altogether the computational complexity of ITGBC is of $O(max(n^3, n^{2.2}, K(k^3 + k\sqrt{n})))$.

2.4. Our Main Results

**Observation 1.** For all known metric TSPs, there exist polynomial time algorithms such as LKH and ITGBC to solve them in polynomial time with approximation ratios less than 1.0045.

**Theorem 3.** Metric TSP is one of NP class problem [12], which can be solved by algorithms such as LKH and ITGBC in polynomial time $C$-approximation with $C$ less than 1.0045. According to Papadimitriou-Vempala Theorem, this happens only when NP=P. Therefore we have NP=P.
3. Conclusions and Future Work

In this paper, we provide a new perspective for P versus NP problem, by proposing an algorithm ITGBC in polynomial time with approximation bounded solutions for TSP. One can see from Table 1 that, the scale (the number of nodes) of TSP is increased as year increasing; one of reasons for the TSP to become harder is because of the scale becomes larger and larger. For TSPLIB instances with node number less than 5000, LKH can solve them to optimality in less than a few hours or shorter. These mean that LKH can provide exact or approximation-bounded solutions to practical TSPs.

How about other NP problems? Can they also be solved in similar way? According to Karp’s result [9] that if any of NP problems have efficient algorithms, then they all do. Hopefully our proposed approach can shine light on other NP problems.

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