Weights with both absolutely continuous and discrete components: Asymptotics via the Riemann-Hilbert approach

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Abstract

We study the uniform asymptotics for the orthogonal polynomials with respect to weights composed of both absolutely continuous measure and discrete measure, by taking a special class of the sieved Pollazek Polynomials as an example. The Plancherel-Rotach type asymptotics of the sieved Pollazek Polynomials are obtained in the whole complex plane. The Riemann-Hilbert method is applied to derive the results. A main feature of the treatment is the appearance of a new band consisting of two adjacent intervals, one of which is a portion of the support of the absolutely continuous measure, the other is the discrete band.

Keywords: Uniform asymptotics; discrete orthogonal polynomials; Riemann-Hilbert approach; sieved Pollaczek polynomials; Airy function.

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1 Introduction

The method of Deift and Zhou has found various applications in the asymptotic studies of orthogonal polynomials. The first few examples, published in 1999, are polynomials with absolutely continuous weights; see, e.g., Deift \textit{et al}. \cite{10}. The powerful method is based on the Riemann-Hilbert problem (RHP) formulation of the orthogonal polynomials observed by Fokas, Its and Kitaev \cite{12}. A crucial idea is a deformation of the contours associated with the factorization of the oscillating jump matrices. Technique difficulties usually lie in the construction of the parametrices at critical points.

In 2007, Baik \textit{et al}. \cite{3} developed a general method for the asymptotics of discrete orthogonal polynomials by using the Riemann-Hilbert approach. The starting point of their investigation is the interpolation problem (IP) for discrete orthogonal polynomials, introduced in Borodin and Boyarchenko \cite{7}. A key step is to turn the IP into a RHP, and then the Deift-Zhou method for oscillating RHP may apply. For the case when all nodes are real, the real line is divided into intervals termed void, saturated region, or band, associated with the equilibrium measure; see \cite{3,5} when the nodes are regularly distributed. In general case, an open interval is called a saturated region, if the ratio of the density of

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the polynomial zeros and the density of the nodes is 1, called void if the ration is 0, and band otherwise.

Since then, much attention has been attracted to this topic. For example, in an attempt to achieve global asymptotics, with global referring to the domains of uniformity, Wong and coworkers considered cases with finite nodes [9, 15, 16], and infinite nodes [17, 21] regularly distributed. Very recently, the present authors [23] have studied the uniform asymptotics for discrete orthogonal polynomials on infinite nodes with an accumulation point, the mass showing a singular behavior there.

A major modification to the method has been made by Bleher and Liechty [5, 6] in the treatment of the band-saturated region endpoints. The example they take is a system of discrete orthogonal polynomials with respect to a varying exponential weight on a regular infinite lattice. Here regular infinite lattice means that the infinite nodes are equally spaced.

Other than the discrete orthogonal polynomials and those with absolutely continuous weights, there are interesting mixed-type cases that the orthogonal measures are supported on both intervals and discrete nodes. Examples can be found in Askey and Ismail [2, Ch. 6], and in Ismail [13, p. 156], where the random walk polynomials are shown having this feature. A natural question arises here: How could the Riemann-Hilbert approach be used to handle such problems?

In this paper, we illustrate the method by taking as an example a class of sieved Pollaczek polynomials. A significant fact is that the corresponding orthogonal measure consists of an absolutely continuous part on $[-1, 1]$, and a discrete part having infinite many mass points with the endpoint 1 as an accumulation point; see Charris and Ismail [8] and Wang and Zhao [20].

It is known that the sieved Pollaczek polynomials $p_n(z)$ possess the orthogonal relation
\[
\int_{-1}^{1} p_n(x)p_m(x)w_c(x)dx + \sum_{k=1}^{\infty} p_n(x_k)p_m(x_k)w_d(x_k) = \delta_{n,m}, \quad n, m = 0, 1, 2, \cdots \tag{1.1}
\]
where the absolutely continuous weight
\[
w_c(x) = \frac{2\sin \theta}{\pi} \exp \left\{ \frac{(\pi - 2\theta)b}{\sin \theta} \right\} \left[ \Gamma \left( 1 + \frac{ib}{\sin \theta} \right) \right]^2 = \frac{2b \exp \left\{ \frac{(\pi - 2\theta)b}{\sin \theta} \right\}}{\sinh \frac{b\pi}{\sin \theta}} \tag{1.2}
\]
for $\theta = \arccos x \in (0, \pi)$ while $x \in (-1, 1)$, and the mass
\[
w_d(x_k) = \frac{4b^3}{k^3 \sqrt{1 + b^2/k^2}} \left( \sqrt{1 + b^2/k^2} - b/k \right)^2 \tag{1.3}
\]
with nodes $x_k = \sqrt{1 + b^2/k^2}$, $k = 1, 2, \cdots$; see [8, 13, 20] for detailed determination of the orthogonal measure. It is easily seen that $w_d(x_k) \sim 4b^3e^{-2b/k^3}$ as $k \to \infty$, which indicates a singularity of the discrete weight at the accumulation point $x = 1$.

The polynomials $p_n(z) = p_n(z; b)$ can also be defined by the three-term recurrence relation
\[
p_{n+1}(z) + p_{n-1}(z) + \frac{2b}{n+1}p_n(z) = 2zp_n(z), \quad n = 1, 2, \cdots \tag{1.4}
\]
with initial values $p_0(z) = 1$ and $p_1(z) = 2z - 2b$; see Charris and Ismail [8], Ismail [13], and Wang and Zhao [20] and the references therein. It is seen from (1.4) and the corresponding initial conditions that
\[
p_n(z; -b) = (-1)^n p_n(-z; b). \tag{1.5}
\]
Hence, without loss of generality, we may assume that \( b > 0 \).

Methods other than the Riemann-Hilbert approach may be applied to obtain asymptotics of the sieved Pollaczek polynomials; cf., e.g., Szegö [18] and Wong and Zhao [22]. Indeed, in an earlier work, Wang and Zhao [20] have considered the uniform asymptotic expansions for the polynomials on the real line, in particular at the turning points \( x = -1 + b/n \) and \( x = 1 + b/n \) and the endpoints \( \pm 1 \), by using an integral method. The expansion in an \( O(1/n) \) neighborhood of \(-1 + b/n\) is in terms of the Airy function, while at \( 1 + b/n \), where the polynomials oscillate on both sides, we need a combination of the Airy functions to describe the behavior. The asymptotics of the extreme zeros are also obtained in [20]. However, the derivation is limited to the real line, and, rigorously, there are gaps between the intervals of uniformity. Further study is desirable for such polynomials.

The main purpose of the present investigation is to derive uniform asymptotic approximations on the whole complex plane for the orthogonal polynomials with both absolutely continuous measure and discrete measure, using the Riemann-Hilbert approach and taking the sieved Pollaczek polynomials as an example.

To this aim, first we formulate the mixed Riemann-Hilbert and interpolation problem for the polynomials. Then, we convert the problem into a Riemann-Hilbert problem by using the notion of band and saturated region of Baik et al. [3], and the treatment of the band-saturated region endpoints by Bleher and Liechty [5, 6]. The Deift-Zhou nonlinear steepest descent method for oscillating Riemann-Hilbert problems plays a central part from then on. The main idea here is the oscillating contour consists of the interval of the absolutely continuous measure, joined by the discrete band.

Technically, there are several facts in the analysis worth mentioning. Several auxiliary functions \( d_E, d_I \) and \( \chi \) are introduced at an early stage \( Y \to U \), to simplify the jump conditions, and to clarify the construction of the outer parametrix for \( N \). The \( g \)-function is supported on an infinite interval, so that the contours are finite later in the RHP for \( T \). In the parametrix for \( N \), we bring in extra singularities to fit the matching conditions on the boundaries of the shrinking neighborhoods of the MRS numbers \( \alpha \) and \( \beta \), in which the local parametries are constructed. The phase condition on the band \((\alpha, \beta)\) plays a role in the determination of the equilibrium measure, and very careful estimates of the \( \phi \)-functions are also needed since the domains of local parametries are shrinking.

The paper is arranged as follows. In Section 2, we state the main asymptotic approximations in regions covering the upper half plane. In Section 3 as the starting point of our analysis, we formulate the Riemann-Hilbert and interpolation problem (RHP and IP) for the sieved Pollaczek polynomials. In Section 4, by removing the poles at the nodes, the problem is turned into a RHP for a matrix function \( U(z) \). In Section 5 we calculate the MRS numbers and bring in auxiliary functions such as the \( g \)-function and \( \phi \)-functions. The nonlinear steepest descent analysis is carried in Section 6. The proof of the main asymptotic theorem is provided in Section 7 by using the analysis in previous sections. Several asymptotic quantities are calculated in Section 8 and a comparison of results is made with the known ones in [20].

2 Main results: Uniform asymptotic approximations

We derive asymptotic approximations for the orthonormal sieved Pollaczek polynomials \( p_n(z) \) in overlapped domains covering the whole complex plane, based on the Riemann-
Hilbert analysis carried out. In view of the symmetry with respect to the real line, we need only to work on the upper half-plane.

To describe the results, we introduce several constants and auxiliary functions.

The soft edges are located at the MRS numbers $\alpha = \alpha_n$ and $\beta = \beta_n$. A detailed analysis of these numbers will be carried out in Section 5. We will see that $\alpha = -1 + \frac{b}{n} + O \left( \frac{1}{n^2} \right)$ and $\beta = 1 + \frac{b}{n} + O \left( \frac{1}{n^2} \right)$ as $n \to \infty$.

Next, we define the functions

$$\gamma(z) = \frac{b}{\sqrt{z^2 - 1}}$$

and

$$\varrho(z) = \left( \frac{z - \beta}{z - \alpha} \right)^{1/4},$$

analytic respectively in $\mathbb{C} \setminus [-1, 1]$ and $\mathbb{C} \setminus [\alpha, \beta]$, where the branches are chosen such that $\arg(z - \alpha)$, $\arg(z - \beta)$, $\arg(z \pm 1) \in (-\pi, \pi)$; cf. Section 4 and Section 6 below. We also need the scalar function

$$\phi_0(z) = \frac{e^{\pm \pi i} \Gamma(1 - \gamma(z))}{\varphi(z) \Gamma(1 + \gamma(z))} e^{-2\gamma(z) + 2\gamma(z) \ln(\gamma(z)) \mp \pi i (\gamma(z) + 1/2)} \text{ for } \pm \Im z > 0,$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$ is analytic in $\mathbb{C} \setminus [-1, 1]$ such that $\varphi(z) \approx 2z$ for large $z$, and the logarithmic function takes real values for positive variables. It is worth noting that the boundary values of $\phi_0(z)$ on the upper and lower edges of $(-1, 1)$ are purely imaginary. Frequent use will also be made of the following scaled variables and the meromorphic functions, namely,

$$\tau_\alpha = \frac{z - \alpha}{\alpha + 1}, \quad \tau_\beta = \frac{z - \beta}{\beta - 1}, \quad f_s(\tau) = \frac{1}{12b\tau} + \frac{5}{48b\tau^2}, \quad \text{and} \quad f_r(\tau) = \frac{5\sqrt{2/\beta b^{3/2}}}{72\tau^2 \xi_0(\tau)} - f_s(\tau),$$

where $\xi_0(\tau) = \sum_{k=0}^{\infty} (-1)^k \sqrt{b/2} B(k + 2, 1/2) \tau^k$, with $B(\xi, \eta)$ being the Beta function.

To describe the behavior at the soft edges, we use conformal mappings

$$\lambda_\alpha(z) = e^{-4\pi i/3} \left( \frac{3}{2} \right)^{2/3} n^{1/3} \phi_\alpha^{2/3}(z), \quad \text{and} \quad \lambda_\beta(z) = \left( \frac{3}{2} \right)^{2/3} n^{1/3} (-\phi_\beta(z))^{2/3},$$

respectively in neighborhoods of $\alpha$ and $\beta$, and constructed in terms of the $\phi$-functions defined in Section 5 and analyzed in detail in Section 6. Briefly, the function $\phi_\alpha(z)$ is analytic in $\mathbb{C} \setminus \{(\infty, -1] \cup [\alpha, \infty)\}$ such that $n \phi_\alpha(z) = i c_\alpha(n) n^{3/2} (1 + O(\tau_\alpha))$ for small $\tau_\alpha$, with $\arg \tau_\alpha \in (0, 2\pi)$ and $c_\alpha(n) \sim \frac{2\sqrt{3}}{3} \sqrt{n}$ for large $n$. While $\phi_\beta(z)$ is analytic in $\mathbb{C} \setminus (-\infty, \beta]$, such that $n \phi_\beta(z) = -c_\beta(n) n^{3/2} (1 + O(\tau_\beta))$, with $\arg \tau_\beta \in (-\pi, \pi)$ for small $\tau_\beta$, again $c_\beta(n) \sim \frac{2\sqrt{3}}{3} \sqrt{n}$ for large $n$. One easily obtains $\lambda_\alpha(z) \sim -(2b)^{1/3} \tau_\alpha$ and $\lambda_\beta(z) \sim (2b)^{1/3} \tau_\beta$ for large $n$, respectively in neighborhoods of $\tau_\alpha = 0$ and $\tau_\beta = 0$.

Now we are in a position to state the uniform asymptotic expansions for the orthonormal polynomials $p_n(z)$ as $n \to \infty$.

**Theorem 1.** For $r \in (0, 1)$, the following holds (see Figure 7 for the regions):

(i) For $z \in A_r$,

$$p_n(z) = \frac{1}{2\sqrt{b\pi}} e^{-n\phi_\beta(z)} D(z) \left( z + \sqrt{z^2 - 1} \right)^{b/\sqrt{z^2 - 1}} \sin \left( \frac{b\pi}{\sqrt{z^2 - 1}} \right) \times \left[ e + e^{-1} + e f_s(\tau_\beta) - e^{-1} f_s(\tau_\alpha) \right] \left( 1 + O \left( \frac{1}{n^{1/2}} \right) \right),$$

(2.2)
Figure 1: Regions of uniform asymptotic approximations, where $M$ is a real constant such that $M > x_1 = \sqrt{1 + b^2}$.

where $D(z) = \exp \left\{ \frac{\sqrt{2}}{2} \left( 1 - \frac{1}{D(z)} \right) \frac{\sqrt{z - \alpha}}{\sqrt{z + 1 - \sqrt{z - \alpha}}} + O(1/\sqrt{n}) \right\}$; cf. (6.27), and $\phi_\beta(z)$ is a function analytic in $\mathbb{C} \setminus (-\infty, \beta]$, as given in (5.17).

(ii) For $z \in B_r$,

$$p_n(z) = \frac{(-1)^n}{2\sqrt{b\pi}}(z + \sqrt{z^2 - 1})^{b/\sqrt{z^2 - 1}}(1 - e^{-2\pi b/\sqrt{z^2 - 1}})^{1/2}$$

$$\times \left[ \left( 1 + f_\beta(\tau_\beta) + O \left( n^{-1/2} \right) \right) q e^{-\pi i/4} \cos \Theta_B + \left( 1 - f_\alpha(\tau_\alpha) + O \left( n^{-1/2} \right) \right) q^{-1} e^{\pi i/4} \sin \Theta_B \right], \quad (2.3)$$

where $\Theta_B = \text{in} \phi_\alpha(z) + (i \ln \phi_0(z))/2 - i \ln D(z) + \pi/4$ with $D(z)$ the same as that in (2.2), $\phi_0(z)$ is defined in (2.1) and $\phi_\alpha(z)$ is defined as (5.15).

(iii) For $z \in C_r$,

$$p_n(z) = \frac{e^b}{2\sqrt{b\pi}} \left[ \left( 1 + f_\beta(\tau_\beta) + O \left( n^{-1/2} \right) \right) q e^{-\pi i/4} \cos \Theta_C + \left( 1 + O \left( n^{-1/2} \right) \right) q^{-1} e^{\pi i/4} \sin \Theta_C \right], \quad (2.4)$$

where $\Theta_C = \text{in} \phi_\beta(z) + b\pi/\sqrt{z^2 - 1} + \pi/4 + O(1/n)$.

(iv) For $z \in D_{1,r} \cup D_{2,r} (|\tau_\beta| < r)$,

$$p_n(z) = \frac{e^b}{2\sqrt{b\pi}} \left\{ n^{1/12} A_1(z)\{\lambda_\beta(z)\}^{1/4} q^{-1} \left[ 1 + O \left( n^{-1/2} \right) \right] 
\right.
+n^{1/12} A_2(z)\{\lambda_\beta(z)\}^{-1/4} q \left[ 1 - f_\beta(\tau_\beta) + O \left( n^{-1/2} \right) \right] \right\}, \quad (2.5)$$

where the pair of functions

$$A_1(z) = - \cos \left( b\pi/\sqrt{z^2 - 1} \right) \text{Ai}(n^{1/3}\lambda_\beta(z)) + \sin \left( b\pi/\sqrt{z^2 - 1} \right) \text{Bi}(n^{1/3}\lambda_\beta(z)),
$$

$$A_2(z) = - \cos \left( b\pi/\sqrt{z^2 - 1} \right) \text{Ai}'(n^{1/3}\lambda_\beta(z)) + \sin \left( b\pi/\sqrt{z^2 - 1} \right) \text{Bi}'(n^{1/3}\lambda_\beta(z)).$$
(v) For \( z \in E_r \),
\[
\begin{align*}
  p_n(z) &= \frac{1}{2\sqrt{b\pi}}(z + \sqrt{z^2 - 1})^{b/\sqrt{z^2 - 1}} \left\{ \left[ \frac{1}{2} + \epsilon f_s(\tau_\beta) - \epsilon^{-1} f_s(\tau_\alpha) \right] \\
&\quad \times \sin\left(\frac{b\pi}{\sqrt{z^2 - 1}}\right)e^{-n\phi_\beta(z)} \left(1 + O\left(n^{-1/2}\right)\right) + O(n^{1/4}e^{\Re(n\phi_\beta(z) - i\pi/\sqrt{z^2 - 1})}) \right\}.
\end{align*}
\]

\[(2.6)\]

(vi) For \( z \in F_{1,r} \cup F_{2,r} \ (|\tau_\alpha| < r) \),
\[
\begin{align*}
  p_n(z) &= \frac{(-1)^n}{2\sqrt{b}}e^{-b + \frac{\ln n}{2\pi n}} \left\{ n^{1/12}\{\lambda_\alpha(z)\}^{1/4} \left[ 1 + O\left(n^{-1/2}\right) \right] \\
&\quad - n^{-1/12}\{\lambda_\alpha(z)\}^{-1/4} e^{-1/6b} + f_r(\tau_\alpha) + O\left(n^{-1/2}\right) \right\}.
\end{align*}
\]

\[(2.7)\]

Later in Section 8 the results will be compared with those of Wang and Zhao [20], obtained earlier via integral methods.

## 3 RHP and IP for the sieved Pollaczek polynomials

We begin with the following mixed RHP and IP formulation. The formulation has been given in Wang [19]. For an earlier RHP version, see [12], while an IP version can be found in [7]. The composite RHP and IP as is as follows:

\(Y_a\) \( Y(z) \) is analytic in \( \mathbb{C} \setminus (-1, 1) \cup \mathcal{X} \), \( \mathcal{X} = \{ x_k : x_k = \sqrt{1 + \frac{b^2}{k^2}}, k = 1, 2, \ldots \} \).

\(Y_b\) \( Y(z) \) satisfies the jump condition

\[
Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w_c(x) \\ 0 & 1 \end{pmatrix}, \quad x \in (-1, 1),
\]

where \( w_c(x) \) is the absolutely continuous weight defined in (1.2) for \( x \in (-1, 1) \).

\(Y_c\) \( Y(z) \) has simple poles at the nodes \( x_k = \sqrt{1 + \frac{b^2}{k^2}} \), and satisfies the residue condition

\[
\text{Res}_{z=x_k} Y(z) = \lim_{z \to x_k} Y(z) \begin{pmatrix} 0 & -w_d(x_k)/2\pi i \\ 0 & 0 \end{pmatrix} \quad \text{for} \quad k = 1, 2, \ldots,
\]

where the discrete weight is

\[
w_d(x) = 4x^{-1}(x^2 - 1)^{3/2}(x - \sqrt{x^2 - 1})^{2b} \quad \text{as} \quad k \to \infty.
\]

\(Y_d\) The asymptotic behavior of \( Y(z) \) at infinity is

\[
Y(z) = \left( I + O\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}, \quad \text{as} \quad z \to \infty.
\]

\[(3.4)\]
(Y_e) $Y(z)$ has the following behavior at $\pm 1$.

$$Y(z) = \begin{pmatrix} O(1) & O(\ln |z|) \\ O(1) & O(\ln |z|) \end{pmatrix}, \quad \text{as } z \to \pm 1. \quad (3.5)$$

It is readily verified that the unique solution to the RHP for $Y$ is

$$Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i} \int_{-1}^{1} \frac{\pi_n(x)w_c(x)dx}{x - z} + \frac{1}{2\pi i} \sum_{k=1}^{\infty} \frac{\pi_n(x_k)w_d(x_k)}{x_k - z} \\ -2\pi i \gamma_{n-1}^{2} \pi_{n-1}(z) & -\gamma_{n-1}^{2} \int_{-1}^{1} \frac{\pi_{n-1}(x)w_c(x)dx}{x - z} - \gamma_{n-1}^{2} \sum_{k=1}^{\infty} \frac{\pi_{n-1}(x_k)w_d(x_k)}{x_k - z} \end{pmatrix}, \quad (3.6)$$

where $\pi_n(z)$ are the monic sieved Pollazeck polynomials, and $p_n(z) = \gamma_n \pi_n(z)$ are the corresponding orthonormal polynomials.

4 Removing the poles of $Y(z)$: RHP for $U(z)$

We use the notion of band and saturated regions of Baik et al. [3] to treat the present composite weight. In later sections, we will show that the real interval $(-1, \alpha)$ is the void, $(\beta, x_1)$ is the saturated region, and $(\alpha, \beta)$ is the band: part of it belongs to the absolutely continuous support, the other part corresponds to the accumulating nodes. It will be shown that $\alpha \approx -1 + b/n$ and $\beta \approx 1 + b/n$ for large polynomial degree $n$.

Applying the ideas of Bleher and Liechty [5, 6], we may define

$$D_{\pm}^u(z) = \begin{pmatrix} 1 & \gamma'(z)w_d(z) \gamma^{-1}(z) \\ 0 & 1 \end{pmatrix}, \quad \pm \text{Im } z \geq 0, \quad (4.1)$$

$$D_{\pm}^l(z) = \pm \begin{pmatrix} \frac{e^{\pm i\pi \gamma(z)}}{2\pi i \sin(\pi \gamma(z))} & 0 \\ \frac{1}{\gamma'(z)w_d(z)} & \frac{2\pi i \sin(\pi \gamma(z))}{e^{\pm i\pi \gamma(z)}} \end{pmatrix}, \quad \pm \text{Im } z \geq 0, \quad (4.2)$$

where $\gamma(z) = \frac{b}{\sqrt{z^2 - 1}}$ and $w_d(z) = 4z^{-1}(z^2 - 1)^{3/2}(z - \sqrt{z^2 - 1})^{2b - 1}$, branches are chosen such that $\gamma(z)$ and $w_d(z)$ are analytic respectively in $\mathbb{C}\backslash[-1,1]$ and $\mathbb{C}\backslash(-\infty,1]$, and both functions are positive for $z \in (1, +\infty)$.

As in Bleher and Liechty [6], we introduce

$$U^u(z) = Y(z) \begin{cases} D_{+}^u(z), & \text{Im } z \geq 0, \\ D_{-}^u(z), & \text{Im } z \leq 0, \end{cases} \quad (4.3)$$

and

$$U^l(z) = Y(z) \begin{cases} D_{+}^l(z), & \text{Im } z \geq 0, \\ D_{-}^l(z), & \text{Im } z \leq 0. \end{cases} \quad (4.4)$$

It is readily verified that both functions $U^l(z)$ and $U^u(z)$ are analytic in $\{z | \text{Re } z > 1, \text{ Im } z \geq 0\}$, and in $\{z | \text{Re } z > 1, \text{ Im } z \leq 0\}$, that is, all simple poles of $Y(z)$ at $z = x_k = \sqrt{1 + b^2/k^2}$, $k = 1, 2, \cdots$, have been removed.
We introduce several scalar auxiliary functions.

\[
d_E(z) = \frac{\Gamma(1 - \gamma(z))}{\sqrt{2\pi}} e^{-\gamma(z) + \left(\gamma(z) - \frac{1}{2}\right) \ln(-\gamma(z))}, \quad z \in \mathbb{C} \setminus [-1, \infty),
\]

\[
d_I(z) = \frac{\sqrt{2\pi}}{\Gamma(\gamma(z))} e^{-\gamma(z) + \left(\gamma(z) - \frac{1}{2}\right) \ln(\gamma(z))}, \quad z \in \mathbb{C} \setminus (-\infty, 1],
\]

and

\[
\chi(z) = 2\sqrt{\pi} b \left(e^{\mp i\varphi(z)}\right)^{-\frac{1}{2}} \quad \text{as} \quad \pm \text{Im} z > 0, \quad z \in \mathbb{C} \setminus [-1, \infty),
\]

where \(\varphi(z) = z + \sqrt{z^2 - 1}\) is analytic in \(\mathbb{C} \setminus [-1, 1]\) such that \(\varphi(z) \approx 2z\) for large \(z\), and the branches of the logarithms are chosen such that

\[
\arg(-\gamma(z)) = \pm \pi + \arg(\gamma(z)) = \pm \pi - \frac{1}{2}(\arg(z - 1) + \arg(z + 1)), \quad \pm \text{Im} z > 0.
\]

We note that \(d_E(z)\) and \(d_I(z)\) approaches 1 as \(z \to 1\) respectively from \(\text{Re} z \leq 1\) and \(\text{Re} z \geq 1\), and that \(d_E(z) \chi(z) = 1 + O(\ln |z|/z)\) as \(z \to \infty\). The functions \(d_E(z)\) and \(d_I(z)\) will simplify the jumps, and \(\chi(z)\) will normalize the behavior at infinity, of the RHPs for \(U, T\) and \(S\) in later sections, while the behavior at \(z = 1\) retains the original form. Similar auxiliary functions have been used in Wang and Wong [21] and Lin and Wong [15].

Define a matrix-valued function \(U(z)\) as

\[
U(z) = \begin{cases} 
U^I(z) \{d_I(z) \chi(z)\}^{\sigma_3}, & z \in (\beta, M) \times (0, \pm \varepsilon), \\
U^u(z) \{d_I(z) \chi(z)\}^{\sigma_3}, & z \in (1, \beta) \times (0, \pm \varepsilon), \\
Y(z) \{d_E(z) \chi(z)\}^{\sigma_3}, & \text{otherwise},
\end{cases}
\]

where \(M\) is a constant such that \(M > x_1 = \sqrt{1 + b^2}, \varepsilon = \delta/\sqrt{n}\) for a small positive \(\delta\) independent of \(z\), and \(\beta = \beta_n\) in (4.8) is one of the MRS numbers to be determined. It will be shown later that \(\beta \approx 1 + \frac{b}{n}\) for large \(n\).

It is readily verified that \(U(z)\) solves the following RHP:

\((U_a)\) \(U(z)\) is analytic in \(\mathbb{C} \setminus \Sigma_U\), where \(\Sigma_U\) is illustrated in Figure 2

\((U_b)\) \(U(z)\) satisfies the jump condition

\[
U_+(z) = U_-(z) J_U(z), \quad z \in \Sigma_U,
\]

\[8\]
where the jump on the real axis is

\[ J_U(x) = \begin{cases} 
\begin{pmatrix} (\phi_0)_+ (x) & r(x) \\ 0 & (\phi_0)_- (x) \end{pmatrix}, & x \in (-1, 1), \\
(0, 1) & x \in (1, \beta), \\
\begin{pmatrix} e^{2\pi i \gamma(x)} & 0 \\
\phi(x) \gamma(x) w_d(x) & e^{-2\pi i \gamma(x)} \end{pmatrix}, & x \in (\beta, M), \\
\begin{pmatrix} 0 & 0 \\
0 & e^{-2\pi i \gamma(x)} \end{pmatrix}, & x \in (M, \infty),
\end{cases} \tag{4.10} \]

where \( \phi_0(z) \) is the scalar function defined in \( \text{(2.1)} \). We note that for \( x \in (-1, 1) \), both \(|(\phi_0)_\pm (x)| = 1\), and such that \( (\phi_0)_+ (x) (\phi_0)_- (x) = 1 \), and

\[ r(x) = \begin{cases} 
\frac{w_i(x)}{4\pi b} \left( 1 - \exp \left( \frac{-2\pi b}{\sqrt{1-x^2}} \right) \right) = \frac{1}{\pi} \exp \left( \frac{-2b \arccos x}{\sqrt{1-x^2}} \right), & x \in (-1, 1), \\
\frac{-\gamma'(x) w_d(x) \phi(x)}{4\pi b \chi(z) / d^2}, & x \in (1, \beta).
\end{cases} \tag{4.11} \]

can actually be extended to a continuous function in \((-1, \infty)\). While the jumps on the off-real-axis contours are

\[ J_U(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
\pm d^2(z) \gamma(z) \chi(z) / d(z) & 1 \end{pmatrix}, & z \in (\beta, M) \pm i \varepsilon, \\
\begin{pmatrix} \pm 2i \sin(\pi \gamma(z)) / e^\pm 2i \pi \gamma(z) & \gamma'(z) w_d(z) e^\mp \pi \gamma(z) \\
0 & \mp 2i \sin(\pi \gamma(z)) / e^\pm 2i \pi \gamma(z) \end{pmatrix}, & z \in (1, \beta) \pm i \varepsilon, \\
\begin{pmatrix} \pm 2i \sin(\pi \gamma(z)) / e^\pm 2i \pi \gamma(z) & \gamma'(z) w_d(z) e^\mp \pi \gamma(z) \\
d^2(z) \chi(z) / \gamma'(z) w_d(z) & 1 \end{pmatrix}, & z \in (\beta \pm i(0, \varepsilon)).
\end{cases} \tag{4.12} \]

\((U_c)\) The asymptotic behavior of \( U(z) \) at infinity is

\[ U(z) = \left( I + O \left( \frac{\ln z}{z} \right) \right) \begin{pmatrix} z^n & 0 \\
0 & z^{-n} \end{pmatrix}, \quad \text{as } z \to \infty. \tag{4.13} \]

\((U_d)\) \( U(z) \) has the following behavior

\[ U(z) = O \left( \ln |z| \mp 1 \right), \quad \text{as } z \to \pm 1. \tag{4.14} \]

5 MRS numbers and auxiliary functions

Assume that \( \psi(x) dx \) is the equilibrium measure, supported on \((\alpha, \infty)\). We consider the following \( g \)-function, to be used in the transformation \( \ref{4.1} \) below.

\[ g(z) = \int_{\alpha}^\infty \ln(z - x) \psi(x) dx, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{5.1} \]
in which the branch of the logarithm is chosen such that \( \arg(z - x) \in (-\pi, \pi) \), \( \psi(x) \) is to be determined for \( x \in (\alpha, \beta) \), and we take

\[
\psi(x) = -\frac{\gamma'(x)}{n} = \frac{1}{n} b x (x^2 - 1)^{-3/2} \quad \text{for} \quad x \in (\beta, \infty),
\]

understanding that \( -\gamma'(x) = bx(x^2 - 1)^{-3/2} \) is the limit density of the nodes \( x_k = \sqrt{1 + \frac{b^2}{k^2}} \), \( k = 1, 2, \ldots \).

The \( g \)-function can be determined by a phase condition of the form

\[
g_+(x) + g_-(x) - l + \frac{1}{n} \ln r(x) = 0, \quad x \in (\alpha, \beta),
\]

where \( l \) is the Lagrange multiplier independent of \( x \).

It is readily seen that \( r'(x)/r(x) \) is an infinitely smooth function in \((-1, 1) \cup (1, \infty)\), such that

\[
\left\{ \begin{array}{c}
\frac{r'(x)}{r(x)} = \frac{\sqrt{b \pi}}{\sqrt{2(1 + x)^{3/2}}} + O\left(\frac{1}{\sqrt{1 + x}}\right) \quad \text{as} \quad x \to -1^+,

\frac{2b}{3} + O(x - 1) \quad \text{as} \quad x \to 1^-,

O\left(\frac{1}{\sqrt{x - 1}}\right) \quad \text{as} \quad x \to 1^+.
\end{array} \right.
\]

Denoting

\[
G(z) = \frac{1}{\pi i} \int_\alpha^\infty \frac{\psi(x)dx}{x - z} \quad \text{for} \quad z \in \mathbb{C} \setminus (\alpha, \infty),
\]

we see that \( G(z) = -\frac{1}{\pi i} g'(z) \) in their common domain of analyticity, and that \( G(z) \) solves the scalar RHP

\[
\left\{ \begin{array}{c}
G_+(x) + G_-(x) = \frac{1}{n \pi i} \frac{r'(x)}{r(x)} \quad \text{for} \quad x \in (\alpha, \beta),

G_+(x) - G_-(x) = \frac{2}{n} b x (x^2 - 1)^{-3/2} \quad \text{for} \quad x \in (\beta, \infty)
\end{array} \right.
\]

and

\[
G(z) = -\frac{1}{\pi i} \left( \frac{1}{z} + O\left(\frac{1}{z^2}\right) \right) \quad \text{as} \quad z \to \infty.
\]

Such a solution can be expressed as

\[
G(z) = \frac{\sqrt{(z - \alpha)(z - \beta)}}{2n\pi i} \left[ \int_\alpha^\beta \frac{-r'(x)/r(x)}{\pi \sqrt{(x - \alpha)(\beta - x)} x - z} \ dx + \int_\beta^\infty \frac{2bx(x^2 - 1)^{-3/2}}{\sqrt{(x - \alpha)(x - \beta)} x - z} \ dx \right],
\]

subject to the condition \( [\mathbf{5.7}] \) at infinity, which now takes the form

\[
\left\{ \begin{array}{c}
\int_\alpha^\beta \frac{r'(x)/r(x)}{\pi \sqrt{(x - \alpha)(\beta - x)}} \ dx - \int_\beta^\infty \frac{2bx(x^2 - 1)^{-3/2}}{\sqrt{(x - \alpha)(x - \beta)}} \ dx = 0,

\int_\alpha^\beta \frac{x r'(x)/r(x)}{\pi \sqrt{(x - \alpha)(\beta - x)}} \ dx - \int_\beta^\infty \frac{2bx^2(x^2 - 1)^{-3/2}}{\sqrt{(x - \alpha)(x - \beta)}} \ dx = -2n.
\end{array} \right.
\]

To determine the MRS numbers \( \alpha \) and \( \beta \), we include some details in what follows.
In view of (5.4), from (5.9) we have
\[
\begin{align*}
\frac{b}{2} \int_{\alpha}^{\beta} \frac{dx}{(x+1)^{3/2} \sqrt{x - \alpha}} - \frac{b}{2} \int_{\beta}^{\infty} \frac{dx}{(x-1)^{3/2} \sqrt{x - \beta}} &= O(1), \\
-\frac{b}{2} \int_{\alpha}^{\beta} \frac{dx}{(x+1)^{3/2} \sqrt{x - \alpha}} + \frac{b}{2} \int_{\beta}^{\infty} \frac{dx}{(x-1)^{3/2} \sqrt{x - \beta}} &= -2n + O(1).
\end{align*}
\]
Noticing that the indefinite integrals
\[
\int \frac{dx}{(x+1)^{3/2} \sqrt{x - \alpha}} = \frac{2}{\alpha + 1} \sqrt{x - \alpha} + \frac{2}{\beta - 1} \sqrt{x - 1}
\]
up to an arbitrary constant, the equations in (5.9) can be written in the form
\[
\frac{b}{\alpha + 1} - \frac{b}{\beta - 1} = O(1), \quad \text{and} \quad -\frac{b}{\alpha + 1} - \frac{b}{\beta - 1} = -2n + O(1)
\]
for large \(n\). Hence we have
\[
\alpha = -1 + \frac{b}{n} + O \left( \frac{1}{n^2} \right), \quad \text{and} \quad \beta = 1 + \frac{b}{n} + O \left( \frac{1}{n^2} \right), \quad \text{as} \; n \to \infty. \tag{5.10}
\]
Refinements can be obtained by using a similar argument as in [26], or in [25].

The equilibrium measure can be expressed as\(\psi(x) = \Re G_+(x)\) for \(x \in (\alpha, \beta)\). \[\tag{5.11}\]

We proceed to define certain \(\phi\)-functions. To this aim, we seek a function \(\nu_\alpha(z)\), analytic in \(\mathbb{C} \setminus \{(-\infty, -1] \cup [\alpha, +\infty)\}\), such that
\[
(\nu_\alpha)_\pm(x) = \pm \pi i \psi(x), \quad x \in (\alpha, 1). \tag{5.12}
\]
From (5.25), and using the Sokhotski-Plemelj formula, we have
\[
G_\pm(x) = \pm \psi(x) + \frac{1}{\pi i} \text{p.v.} \int_{\alpha}^{+\infty} \frac{\psi(t)dt}{t-x}, \quad x \in (\alpha, \infty),
\]
in which the boundary values share the same imaginary part. Further more, in view of (5.6), it is readily seen that
\[
\text{Im} G_+(x) = \text{Im} G_-(x) = -\frac{1}{2n \pi} \frac{r'(x)}{r(x)} = \frac{b}{n \pi} \frac{x \arccos x - \sqrt{1-x^2}}{(1-x^2)^{3/2}} \quad \text{for} \; x \in (\alpha, 1);
\]
cf. (1.2) and (4.11). It is worth noting that \(G(z)\) is analytic in \(\mathbb{C} \setminus [\alpha, \infty)\), and \(-\frac{r'(x)}{r(x)}\) can be analytically extended from \(x \in (-1, 1)\) to
\[
h_\alpha(z) = -\frac{2b}{1-z^2} \left[ \sqrt{1-z^2} + i z \ln(z + i \sqrt{1-z^2}) \right] \left(1-z^2\right)^{3/2} \quad \text{for} \; z \in \mathbb{C} \setminus (-\infty, -1], \tag{5.13}
\]
where branches are chosen such that \( \text{arg}(z + 1) \in (-\pi, \pi) \) and \( \text{arg}(1 - z) \in (-\pi, \pi) \). Therefore, we chose

\[
\nu_\alpha(z) = \pi i G(z) - \frac{b}{n} \frac{\sqrt{1 - z^2} + i z \ln(z + i \sqrt{1 - z^2})}{(1 - z^2)^{3/2}}, \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [\alpha, +\infty)\}. \tag{5.14}
\]

Consequently, we can define a \( \phi \)-function

\[
\phi_\alpha(z) = \int_{\alpha}^{z} \nu_\alpha(\zeta) d\zeta \quad \text{for} \quad z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [\alpha, +\infty)\}. \tag{5.15}
\]

Now we turn to the other critical point \( x = \beta \approx 1 + \frac{b}{n} \). Similar to the above derivation, we introduce

\[
\nu_\beta(z) = \pi i \left\{ G(z) \pm \frac{1}{n} \gamma'(z) \right\} - \frac{1}{2n} \frac{r'(z)}{r(z)} \quad \text{for} \quad \pm \text{Im} \ z > 0, \tag{5.16}
\]

where \( r(z) = \frac{-\gamma'(z)w_d(z) \varphi(z)}{d_{\beta}^{2}(z)} \), cf. (4.11), with \( -\gamma'(z)w_d(z) = 4b \left( z - \sqrt{z^2 - 1} \right)^{2b} \) being analytically extended to \( \mathbb{C} \setminus (-\infty, -1] \), chosen such that \( \text{arg}(z \pm 1) \in (-\pi, \pi) \), and real positive for \( z = x \in (1, +\infty) \).

Therefore, \( r(z) \), and hence \( r'(z)/r(z) \), are analytic in \( \mathbb{C} \setminus (-\infty, 1] \). While \( \gamma(z) = \frac{b}{\sqrt{z^2 - 1}} \) is analytic in \( \mathbb{C} \setminus [-1, 1] \), and positive for \( z > 1 \). It is readily seen that \( \nu_\beta(z) \) is analytic in \( \mathbb{C} \setminus (-\infty, \beta] \), such that

\[
(n \nu_\beta)_\pm(x) = \pm \pi i \left\{ \psi(x) + \frac{1}{n} \gamma'(x) \right\} \quad \text{for} \quad x \in (1, \beta). \]

Here, use has been made of the phase condition (5.3). Upon these we define another \( \phi \)-function

\[
\phi_\beta(z) = \int_{\beta}^{z} \nu_\beta(\zeta) d\zeta \quad \text{for} \quad z \in \mathbb{C} \setminus (-\infty, \beta]. \tag{5.17}
\]

We derive connections between these auxiliary functions. Indeed, substituting (5.14) into (5.15) yields

\[
\phi_\alpha(z) + g(z) + \frac{ib \ln(z + i \sqrt{1 - z^2})}{n \sqrt{1 - z^2}} - \frac{1}{2} - \frac{\ln \pi}{2n} + \pi i = 0, \quad \mp \text{Im} \ z > 0. \tag{5.18}
\]

Here use has been made of the representation (5.1) and the phase condition (5.3). The constant \( l \) is the same as in the phase condition (5.3), and can be determined from (5.18) as

\[
l = 2 \int_{\alpha}^{\infty} \ln(x - \alpha) \psi(x) dx + \frac{2b \arccos \alpha}{n \sqrt{1 - \alpha^2}} - \frac{\ln \pi}{n};
\]

see (8.3) for an asymptotic approximation of \( l \).

Similarly, we have

\[
\phi_\beta(z) + g(z) \mp \frac{\pi i}{n} \gamma(z) + \frac{1}{2n} \ln r(z) - \frac{l}{2} = 0, \quad \pm \text{Im} \ z > 0 \tag{5.19}
\]

for the same \( l \), where \( r(z) \) is as defined in (5.16), and \( \ln r(z) \) is analytic in \( \mathbb{C} \setminus (-\infty, 1] \) such that \( \text{arg} r(z) = 0 \) for \( z = x \in (1, +\infty) \).
6 Nonlinear steepest descent analysis

The procedure consists of a series of invertible transformations \( U(z) \rightarrow T(z) \rightarrow S(z) \rightarrow R(z) \). The pioneering work in this respect includes Deift and Zhou et al. \cite{11,10}; see also Bleher and Its \cite{4}. To accomplish the transformations, auxiliary functions are analyzed and parametrices are constructed at critical points \( z = \alpha \) and \( z = \beta \), as well as in a region around the infinity.

6.1 The first transformation \( U \rightarrow T \)

The first transformation \( U \rightarrow T \) is the following normalization at infinity

\[
U(z) = e^{i2nl\sigma_3}T(z)e^{i(nU(z)-\frac{l}{2})}\sigma_3, \quad z \in \mathbb{C}\setminus \Sigma_U, \tag{6.1}
\]

where \( l \) is a constant; cf. \( \Sigma_U \), \( \Sigma_U \) is illustrated in Figure 2, \( \sigma_3 \) is one of the Pauli matrices, defined as

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.2}
\]

It is readily verified that \( T(z) \) solves the following RHP

\((T_a)\) \( T(z) \) is analytic in \( \mathbb{C}\setminus \Sigma_U \), with \( \Sigma_U \) illustrated in Figure 2.

\((T_b)\) \( T(z) \) satisfies the jump condition

\[
T_+(z) = T_-(z)J_T(z), \quad z \in \Sigma_U, \tag{6.3}
\]

where the jumps are

\[
J_T(x) = \begin{cases} 
\begin{pmatrix} \frac{1}{(\phi_0)_-(x)} & e^{-2n\phi_0(x)} \\
0 & 1/(\phi_0)_+(x) \end{pmatrix}, & x \in (-1, \alpha), \\
\begin{pmatrix} e^{-2n(\phi_0)_-(x)} & 1 \\
0 & e^{-2n(\phi_0)_+(x)} \end{pmatrix}, & x \in (\alpha, 1), \\
\begin{pmatrix} -e^{-2\pi i\gamma(x)-2n(\phi_\beta)_-(x)} & 1 \\
0 & -e^{2\pi i\gamma(x)-2n(\phi_\beta)_+(x)} \end{pmatrix}, & x \in (1, \beta), \\
\begin{pmatrix} 1 & 0 \\
-e^{2n\phi_\beta(x)} & 1 \end{pmatrix}, & x \in (\beta, M), 
\end{cases} \tag{6.4}
\]

and

\[
J_T(z) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
\frac{2n\phi_\beta(z)+\pi\gamma(z)}{2i\sin\pi\gamma(z)} & 1 \end{pmatrix}, & z \in (\beta, M) \pm i\varepsilon, \\
\begin{pmatrix} \frac{\pm 2i\sin(\pi\gamma(z))}{e^{\pm\pi\gamma(z)}} & \pm e^{-2n\phi_\beta(z)} \\
\frac{\pm 2i\sin(\pi\gamma(z))}{e^{\pm\pi\gamma(z)}} & 1 \end{pmatrix}, & z \in (1, \beta) \pm i\varepsilon, \\
\begin{pmatrix} \frac{\pm 2i\sin(\pi\gamma(z))}{e^{\pm\pi\gamma(z)}} & \pm e^{-2n\phi_\beta(z)} \\
\frac{\pm 2i\sin(\pi\gamma(z))}{e^{\pm\pi\gamma(z)}} & 1 \end{pmatrix}, & z \in \beta \pm i(0, \varepsilon). 
\end{cases} \tag{6.5}
\]
(\(T_c\)) The asymptotic behavior of \(T(z)\) at infinity is
\[
T(z) = I + O\left(1/z\right), \quad \text{as } z \to \infty. \quad (6.6)
\]

(\(T_d\)) \(T(z)\) has the following behavior
\[
T(z) = O\left(\ln|z|\right), \quad \text{as } z \to \pm 1. \quad (6.7)
\]

To verify the jump conditions, we may use the phase condition \((5.3)\), the relations \((5.18)\) \& \((5.19)\) between the \(g\)-function and the \(\phi\)-functions, \& the following representations derived from them:

\[
g_+(x) - g_-(x) = \begin{cases} 2\pi i, & x \in (-\infty, \alpha), \\ 2\pi i - 2(\phi_\alpha)(x) = 2\pi i + 2(\phi_\alpha)(x), & x \in (\alpha, 1), \\ \frac{2\pi i}{\ln|z|} - 2(\phi_\alpha)(x) = \frac{2\pi i}{\ln|z|} + 2(\phi_\alpha)(x), & x \in (1, \beta), \\ \frac{2\pi i}{\ln|z|}, & x \in (\beta, \infty). \end{cases} \quad (6.8)
\]

6.2 The second transformation \(T \to S\)

Now we are in a position to apply the second transformation \(T \to S\), associated with factorizations of the jump matrices. The transformation is defined explicitly as
\[
S(z) = \begin{cases} I & \text{for } z \in \Omega_{\pm}; \text{ see Figure \ref{fig:jump}} \\ \begin{pmatrix} 1 & 0 \\ \mp \phi_0(x)e^{2n\phi_\alpha(x)} & 1 \end{pmatrix} & \text{for } z \in (1, \beta) \times (0, \pm i\varepsilon), \\ \begin{pmatrix} 1 & 0 \\ \pm e^{2n\phi_\beta(x)} & 1 \end{pmatrix} & \text{for } z \in (\beta, M), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{otherwise.} \\ \end{cases} \quad (6.9)
\]

We see that \(S(z)\) solves the following RHP

(\(S_a\)) \(S(z)\) is analytic in \(\mathbb{C} \setminus \Sigma_S\), where the oriented contour \(\Sigma_S\) is illustrated in Figure \ref{fig:jump}.

(\(S_b\)) The jump conditions are
\[
S_+(z) = S_-(z)J_S(z), \quad z \in \Sigma_S, \quad (6.10)
\]

with jumps on real axis
\[
J_S(x) = \begin{cases} \begin{pmatrix} 1 & e^{-2n\phi_\alpha(x)} \\ -\phi_\alpha(x) & 1 \end{pmatrix}, & x \in (-1, \alpha), \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & x \in (\alpha, 1) \cup (1, \beta), \\ \begin{pmatrix} 1 & 0 \\ -e^{2n\phi_\beta(x)} & 1 \end{pmatrix}, & x \in (\beta, M). \end{cases} \quad (6.11)
\]
Figure 3: The contour $\Sigma$ for $S(z)$. We denote by $\Omega_+$ the domain bounded by $\Sigma_3$ and the real axis such that $\alpha < \text{Re} \, z < 1$, and by $\Omega_-$ the symmetric part bounded by $\Sigma_1$ and $(\alpha, 1)$.

and jumps on the other contours

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \phi_0(z)e^{2n\phi_\alpha(z)} & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 \\ -\phi_0(z)e^{2n\phi_\alpha(z)} & 1 \end{pmatrix}, & z \in \Sigma_3, \\ \begin{pmatrix} 1 & 0 \\ \frac{2n\phi_\beta(z)\pm\pi\gamma(z)}{2\sin\pi\gamma(z)} & 1 \end{pmatrix}, & z \in (\beta, M) \pm i\varepsilon, \\ \begin{pmatrix} 1 & 0 \\ \frac{e^{-2n\phi_\beta(z)}}{2\sin\pi\gamma(z)} & 1 \end{pmatrix}, & z \in (1, \beta) \pm i\varepsilon, \\ \begin{pmatrix} 1 & 0 \\ \pm e^{-2n\phi_\beta(z)} & 1 \end{pmatrix}, & z \in 1 \pm i(0, \varepsilon), \\ \begin{pmatrix} 1 & 0 \\ \pm e^{-2n\phi_\beta(z)} & 1 \end{pmatrix}, & z \in \beta \pm i(0, \varepsilon). \end{cases}$$

$(S_c)$ The behavior at infinity is

$$S(z) = I + O(1/z), \quad z \to \infty. \quad (6.13)$$

$(S_d)$ $S(z)$ has the following behavior

$$S(z) = O(\ln |z \mp 1|), \quad z \to \pm 1. \quad (6.14)$$

### 6.3 The functions $\phi_\alpha$ and $\phi_\beta$

Let us take a closer look at $\phi_\alpha(z)$ in a neighborhood of $z = \alpha$; cf. (5.15).

**Lemma 1.** The function $\phi_\alpha(z)$ possesses the following convergent expansion

$$n\phi_\alpha(z) = ic_\alpha(n)\tau_\alpha^{3/2} \sum_{k=0}^{\infty} c_{\alpha,k}(n)\tau_\alpha^k, \quad \tau_\alpha = \frac{z - \alpha}{\alpha + 1}, \quad \text{arg} \, \tau_\alpha \in (0, 2\pi), \quad |\tau_\alpha| < 1. \quad (6.15)$$
where $c_{0,0} = 1$, and $c_{a}(n) = \frac{\sqrt{2a-1}}{a\sqrt{n}}(1 + \alpha)^{3/2}A(\alpha) \approx 2^{1/3}\sqrt{n}$ for large $n$. It also holds the following asymptotic approximation for large $n$

$$n\phi_\alpha(z) = i\sqrt{n}n^{3/2} \left( \xi_0(\tau_\alpha) + O\left(1/\sqrt{n}\right) \right) \quad \text{for } |\tau_\alpha| = r, \ r \in (0, 1),$$

(6.16)

where $\xi_0(\tau)$ is an analytic function in $|\tau| < 1$, depending only on $\tau$, explicitly expressed in terms of the Beta function as $\xi_0(\tau) = \sum_{k=0}^{\infty}(-1)^k \sqrt{b/2} \ B(k + 2, 1/2)\tau^k$.

**Proof:** First, since $h_\alpha(z)$ is analytic in $\mathbb{C} \setminus (-\infty, -1]$, such that $h_\alpha(x) = -r'(x)/r(x)$ for $x \in (-1, 1)$; cf. (5.13), from (5.8) we can write

$$\pi G(z) = \frac{\sqrt{(z - \alpha)(z - \beta)}}{2n} \left[ h_\alpha(z) \int_\alpha^\beta \frac{1}{\pi \sqrt{(x - \alpha)(\beta - x)(x - z)}} \, dx + A(z) \right],$$

where $A(z)$, given in (6.19) below, is real for $z \in (-1, \alpha)$, and is analytic in $\mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$, to which $\alpha$ belongs. To justify the analyticity of $A(z)$, one may use the equality

$$\int_\alpha^\beta \frac{1}{\pi \sqrt{(x - \alpha)(\beta - x)(x - z)}} \, dx = -\frac{1}{\sqrt{(z - \alpha)(z - \beta)}} \quad \text{for } z \notin [\alpha, \beta],$$

(6.17)

where branches are chosen such that $\arg(z - \alpha) \in (-\pi, \pi)$ and $\arg(z - \beta) \in (-\pi, \pi)$.

Substituting all above into (5.14), we have

$$\phi_\alpha'(z) = \nu_\alpha(z) = \frac{\sqrt{(z - \alpha)(z - \beta)}}{2n}A(z) = i\sqrt{z - \alpha}A_\alpha(z)$$

(6.18)

in a neighborhood of $z = \alpha$, with $\arg(z - \alpha) \in (0, 2\pi)$, and $A_\alpha(z)$ is analytic at $z = \alpha$, being real for $z \in (-1, \alpha)$. A similar discussion can be found in Zhou and Zhao [26].

Hence, we accordingly derive (6.15) from (6.15). We include some details for the above approximation. Recalling that

$$A(z) = \int_\alpha^\beta \frac{-h_\alpha(z) - \frac{r'(x)}{r(x)}}{x - z} \, dx + \int_\alpha^\beta \frac{2bx(x^2 - 1)^{-3/2}}{\sqrt{(x - \alpha)(x - \beta)}} \, dx$$

(6.19)

for $z \in \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$; cf. (5.8). By careful estimation similar to that in Section 6 for large $n$, we have

$$A(\alpha) \sim \int_\alpha^0 \frac{h_\alpha(x) - h_\alpha(\alpha)}{x - \alpha} \, dx \sim \frac{1}{2\pi} \int_\alpha^0 \frac{[h_\alpha(x) - h_\alpha(\alpha)]}{(x - \alpha)^{3/2}} \, dx,$$

with an error $O\left(n^{3/2}\right)$. Here use has been made of (5.4) and the fact that $\alpha \sim -1 + b/n$ for large $n$. Now integrating by parts once, we further have

$$A(\alpha) \sim \frac{\sqrt{2}}{\pi} \int_\alpha^0 \frac{h_\alpha'(x)dx}{\sqrt{x - \alpha}} \sim \frac{3b}{2} \int_\alpha^0 \frac{dx}{(1 + x)^{5/2}\sqrt{x - \alpha}} \sim 2b(1 + \alpha)^{-2},$$

each time with an error $O\left(n^{3/2}\right)$. Here we have taken into account the fact that

$$h_\alpha'(x) = \frac{3\pi b}{2\sqrt{2}}(1 + x)^{-5/2} + O\left((1 + x)^{-2}\right) \quad \text{as } x \to -1^+;$$

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cf. (5.13) and (5.4). Hence we obtain the above approximation for \( c_\alpha(n) \).

Now we turn to the evaluation of the leading behavior of \( n^\phi_\alpha(z) \) for large \( n \) and mild \( \tau = \frac{\pi - \alpha}{1 + \alpha} \). To this aim, we may go further to approximate \( A^{(k)}(\alpha) \) for \( \alpha = 1, 2, \cdots \). Similar to the approximating of \( A(\alpha) \), we have

\[
A^{(k)}(\alpha) \sim \frac{k!}{\sqrt{2 \pi}} \int_{\alpha}^{0} \frac{h_\alpha(x) - h_\alpha'(\alpha)(x - \alpha) - \cdots - h_\alpha^{(k)}(\alpha)(x - \alpha)^k}{(x - \alpha)^{k + \frac{1}{2}}} dx.
\]

Integrating by parts \( k + 1 \) times, we have

\[
\frac{A^{(k)}(\alpha)}{k!} \sim \frac{\Gamma\left(\frac{3}{2}\right)}{\sqrt{2 \pi}} \int_{\alpha}^{0} \frac{h_\alpha^{(k+1)}(x)dx}{\sqrt{x - \alpha}} \sim b(-1)^k \left( k + \frac{3}{2} \right) \int_{\alpha}^{0} \frac{dx}{(1 + x)^{k + \frac{3}{2}}} \sqrt{x - \alpha}.
\]

Here use has been made of the fact that

\[
h_\alpha^{(k+1)}(x) \sim (-1)^k \frac{\pi^{\frac{3}{2}}}{\sqrt{2}} \cdots \left( k + \frac{3}{2} \right) (1 + x)^{-k - \frac{5}{2}} \text{ as } x \to -1^+;
\]

cf. (5.13). Extending the integral interval to \([\alpha, \infty)\), one obtains

\[
\frac{A^{(k)}(\alpha)}{k!} \sim (-1)^k \frac{b \left( k + \frac{3}{2} \right) B(k + 2, \frac{1}{2})}{(1 + \alpha)^{k + 2}} \text{ for } k = 0, 1, 2, \cdots.
\]

Expanding \( A(z) \) into a Maclaurin series in \( \tau = \frac{\pi - \alpha}{1 + \alpha} \), substituting all above to (6.18) and integrating it, we obtain (6.19), thus completing the proof.

Now we see from (6.15) that in a neighborhood of \( z = \alpha \) cutting alone \([\alpha, \infty)\), \( e^{2n\phi_\alpha(z)} \) is exponentially small for \( \arg(z - \alpha) \in (0, 2\pi/3) \cup (4\pi/3, 2\pi) \), and is exponentially large for \( \arg(z - \alpha) \in (2\pi/3, 4\pi/3) \), so long as \( |\tau| = \left| \frac{\pi - \alpha}{\alpha + 1} \right| \) lies in compact subsets of \((0, 1)\).

We proceed to show that the jumps \( J_S \) for \( S \) off the real line are of the form \( I \) plus exponentially small terms; cf (6.12). This may be roughly explained by using the Cauchy-Riemann condition. For example, in view of (5.15) and (5.17), we have negative derivatives

\[
\frac{\partial \text{Re} \phi_\alpha}{\partial y} = -\pi \psi(x), \quad \frac{\partial \text{Re} \left\{ \phi_\beta - \frac{\pi i}{n} \right\}}{\partial y} = -\pi \psi(x), \quad \text{and} \quad -\frac{\partial \text{Re} \phi_\beta}{\partial y} = -\pi \left[ \frac{-\gamma(x)}{n} - \psi(x) \right],
\]

respectively on the upper edge of \( x \in (\alpha, 1), x \in (1, \infty) \), and \( x \in (1, \beta) \).

Since the contours depend on \( n \), it is necessary to estimate the exponentials carefully. An approach is to check the derivatives such as \( \frac{\partial \text{Re} \phi_\alpha}{\partial y} \) in neighborhoods of real segments, as carried out in [26] and [25]. To show that the jumps (6.12) off the real line are of the form \( I \) plus exponentially small terms, we use an alternative straightforward way here. Indeed, substituting (8.2) into (6.18), we obtain

\[
\phi_\alpha(z) = \frac{(z - \alpha)(z - \beta)}{n} \int_{\beta}^{\infty} \frac{-\gamma(x)}{\sqrt{(x - \alpha)(x - \beta)}} \frac{dx}{x - z} - \ln \left( \frac{z - \alpha^{\beta} + \sqrt{(z - \alpha)(z - \beta)}}{\sqrt{2}} \right) - \frac{\ln \pi r(x)}{2n\pi} \int_{\alpha}^{\beta} \frac{dx}{\sqrt{(x - \alpha)(\beta - x)}} x - z - \frac{i b \ln(z + i \sqrt{1 - z^2})}{n \sqrt{1 - z^2}} \pm \pi i. \quad (6.20)
\]
We take part of $\Sigma_3$ (cf. Figure [3]) as an example to show $e^{2n\phi_\alpha(z)}$ is exponentially small there. Along the slope segment $\arg(z-\alpha) = \theta_+$ and $\alpha+r(1+\alpha) \leq \text{Re} z \leq \alpha + \frac{\delta}{\tan \theta_+}$, we may pick the dominant contribution from the logarithm in (6.20), so that $\text{Re}\{2n\phi_\alpha(z)\} \leq -Cn\sqrt{|z-\alpha|}$. Other exponential terms can be estimated similarly using the explicit representation.

Analysis can also be carried out for $\phi_\beta(z)$ in an $O(1/n)$ neighborhood of $z = \beta$. A new variable used here is $\tau_\beta = \frac{z-\beta}{\beta - 1}$. Indeed, a combination of (5.8) and (6.17) with (5.16) gives

$$\phi'_\beta(z) = \nu_\beta(z) = \frac{\sqrt{(z-\alpha)(z-\beta)}}{2n} B(z),$$  

where

$$B(z) = \int_{\alpha}^{\beta} \frac{r'(z)}{r(z)} \frac{dx}{x - z} \frac{2\gamma'(x)}{\pi\sqrt{(x-\alpha)(x-\beta)}} - \int_{\beta}^{\infty} \frac{2\gamma'(x)}{\sqrt{(x-\alpha)(x-\beta)}} \frac{dx}{x - z} \pm \frac{2\pi i\gamma'(z)}{\sqrt{(z-\alpha)(z-\beta)}}$$

for $\pm\text{Im} z > 0$, with $r(x)$ given in (4.11) for real $x$, $r(z)$ is the function appeared in (5.16), analytic in $\mathbb{C} \setminus (-\infty, 1]$, and $\gamma(z) = \frac{b}{\sqrt{z^2 - 1}}$, analytic in $\mathbb{C} \setminus [-1, 1]$. It is readily verified that $B(z)$ is analytic in $\mathbb{C} \setminus (-\infty, 1]$.

For large $n$, the leading order contribution to $B(\beta)$ comes from the last two terms in (6.22). Actually we have

$$B(\beta) \sim \sqrt{2} \int_{\beta}^{\infty} \frac{\gamma'(\beta) - \gamma'(x)}{(x-\beta)^{3/2}} dx = \int_{\beta}^{\infty} -2\beta^3 \gamma''(x) dx \sim \int_{\beta}^{\infty} -\frac{2b}{(x-1)^{3/2} \sqrt{x-\beta}} dx = -2b (\beta - 1)^{3/2}.$$  

Hence in analog to Lemma [1] we have

**Lemma 2.** The function $\phi_\beta(z)$ possesses the following convergent expansion

$$n\phi_\beta(z) = -c_\beta(n) \tau_\beta^{3/2} \sum_{k=0}^{\infty} c_{\beta,k}(n) \tau_\beta^k, \quad \tau_\beta = \frac{z-\beta}{\beta - 1}, \quad \text{arg} \tau_\beta \in (-\pi, \pi), \quad |\tau_\beta| < 1,$$

where $c_{\beta,0} = 1$, and $c_\beta(n) \sim \frac{2\sqrt{n}}{\sqrt{3}}\sqrt{n}$ for large $n$. Also, similar to (6.16), it holds

$$n\phi_\beta(z) = -\sqrt{n} \tau_\beta^{3/2} (\xi_0(\tau_\beta) + O(1/\sqrt{n})) \text{ for } |\tau_\beta| = r, \quad r \in (0, 1), \quad n \to \infty,$$

where $\xi_0(\tau)$ is the same function as in (6.16), analytic in the unit disc.  

### 6.4 Parametrix for the outside region

For fixed $z \in \Sigma_S$, with the possible exceptions of $(-1, \alpha)$ and $(\alpha, \beta)$, it can be verified that $J_S(z)$ is $I$, plus an exponentially small term, for large-$n$. Hence, we have the limiting RH problem:

$(N_a)$ $N(z)$ is analytic in $\mathbb{C} \setminus [-1, \beta]$.

$(N_b)$ The jump condition is

$$N_+(x) = N_-(x) \begin{cases} \left( \begin{array}{cc} (\phi_0)_+(x) & 0 \\ 0 & (\phi_0)_-(x) \end{array} \right), & x \in (-1, \alpha), \\ \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right), & x \in (\alpha, \beta), \end{cases}$$

where $\phi_0(z)$ is defined in (2.1), such that $(\phi_0)_+(x)(\phi_0)_-(x) = 1$ for $x \in (-1, \alpha)$.  

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(Nc) The behavior at infinity is
\[ N(z) = I + O(1/z), \quad z \to \infty. \]  
(6.26)

A solution can be constructed explicitly as
\[ N(z) = D(\infty)^{-\sigma_3} \left( 1 + \frac{f_s(\tau_0)}{2} - \frac{i f_s(\tau_0)}{2} \right) \left( 1 - \frac{f_s(\tau_0)}{2} - \frac{i f_s(\tau_0)}{2} \right) N_0(z)D(z)^{\sigma_3}, \]
(6.27)
where \( f_s(\tau) = \frac{1}{120} \tau + \frac{5}{48\pi} \tau, \tau_0 = \frac{\pi - \alpha}{1 + \alpha}, \tau_\beta = \frac{\pi - \beta}{\beta}, D(z) \) is the Szegő function, namely, a function analytic and non-vanishing in \( \{ \mathbb{C} \setminus [-1, \beta] \} \cup \{ \infty \}, \) such that \( D_+(x)D_-(x) = 1 \) for \( x \in (\alpha, \beta) \) and \( D_+(x)/D_-(x) = (\phi_0)_+(x) \) for \( x \in (-1, \alpha); \) see [14] for a discussion of the Szegő functions, and [24] for a relevant construction. In the present case, we have
\[
D(z) = \exp \left( -\frac{\sqrt{(z - \alpha)(z - \beta)}}{2\pi i} \int_{-1}^{\alpha} \frac{\ln \left( (\phi_0)_+(x) \right)}{\sqrt{(\alpha - x)(\beta - x)}} \ dx \right),
\]
and
\[
N_0(z) = \left( \frac{g(z) + \sqrt{z}}{2} \  \frac{g(z) - \sqrt{z}}{2} \right) \ \text{with} \ g(z) = \left( \frac{z - \beta}{z - \alpha} \right)^{\frac{1}{2}},
\]
where \( \arg(z - \alpha) \in (-\pi, \pi) \) and \( \arg(z - \beta) \in (-\pi, \pi). \) Thus for large \( n, \)
\[
D(\infty) = \exp \left( \frac{1}{2\pi i} \int_{-1}^{\alpha} \frac{\ln \left( (\phi_0)_+(x) \right) dx}{\sqrt{(\alpha - x)(\beta - x)}} \right) \sim \exp \left( \frac{1}{2\pi} \int_{-1}^{\alpha} \frac{\sqrt{x + 1}}{\sqrt{\alpha - x}} \ dx \right) = 1 + O \left( \frac{1}{n} \right).
\]

It is worth noting that the extra rational factors on the left are needed to accomplish the matching conditions below for \( P^{(\alpha)} \) and \( P^{(\beta)}. \) (see [25] (4.15)), and [25]).

6.5 Local parametrix at \( z = \alpha \)

The limiting RHP \( N(z) \) fails to approximate \( S(z) \) at \( z = \alpha \) and \( z = \beta \) since the jump \( J_S(z) \) is not close to \( I \) as \( z \to \alpha. \) We need to construct the following parametrix at a neighborhood \( |\tau| < r, \) denoted by \( U_r(\alpha), \) of \( z = \alpha. \) Here we have used a re-scaled variable \( \tau = \frac{\pi - \alpha}{4\pi}. \)

(P\(_a^{(\alpha)}\)) \( P^{(\alpha)}(z) \) is analytic in \( U_r(\alpha) \setminus \Sigma_S; \) cf. Figures 3 and 4 for \( \Sigma_S. \)

(P\(_b^{(\alpha)}\)) \( P^{(\alpha)}(z) \) satisfies the same jump conditions as \( S(z) \) in \( U_r(\alpha), \) namely,
\[
P^{(\alpha)}(z) = \begin{cases} 
\begin{pmatrix} 
1 & e^{-2n(\phi_0)(z)} \\
0 & 1 
\end{pmatrix}, & z \in (-\alpha, \alpha) \cap U_r(\alpha), \\
\begin{pmatrix} 
0 & 1 \\
-1 & 0 
\end{pmatrix}, & z \in (\alpha, 1) \cap U_r(\alpha), \\
\begin{pmatrix} 
1 & 0 \\
\phi_0(z)e^{2n(\phi_0)(z)} & 1 
\end{pmatrix}, & z \in \Sigma_1 \cap U_r(\alpha), \\
\begin{pmatrix} 
1 & 0 \\
-\phi_0(z)e^{-2n(\phi_0)(z)} & 1 
\end{pmatrix}, & z \in \Sigma_3 \cap U_r(\alpha).
\end{cases}
\]
(6.28)
Figure 4: The contours for \( P^{(\alpha)}(z) \) (left), and the contours and sectors for the model RHP \( \Psi(s) \) (right), connected by the conformal mapping \( s = \lambda_\alpha(z) \). We still denote by \( \Sigma_1 \) and \( \Sigma_3 \) the \( s \)-images (right) of the oriented \( z \)-contours (left).

\[ (P^{(\alpha)}_c) \quad \text{The matching condition holds} \]

\[ P^{(\alpha)}(z) = \left( I + O \left( \frac{1}{n} \right) \right) N(z), \quad z \in \partial U_\alpha(\alpha). \]  

(6.29)

A solution to the RHP has been constructed in, e.g., [10], see also [26], expressed in terms of the following matrix-valued function

\[ \Psi(s) = \begin{cases} 
(\text{Ai}(s) & \text{Ai}(\omega^2 s) \left( e^{-\frac{\pi i}{6} \sigma_3}, \quad s \in I; \\
\text{Ai}'(s) & \omega^2 \text{Ai}'(\omega^2 s) \end{cases} 
\]

\[ \begin{cases} 
(\text{Ai}(s) & \omega^2 \text{Ai}'(\omega^2 s) e^{-\frac{\pi i}{6} \sigma_3} (1 \ 0) , \quad s \in II; \\
\text{Ai}'(s) & \text{Ai}(\omega s) \end{cases} 
\]

\[ \begin{cases} 
(\text{Ai}(s) & -\omega^2 \text{Ai}(\omega s) e^{-\frac{\pi i}{6} \sigma_3} (1 \ 1) , \quad s \in III; \\
\text{Ai}'(s) & -\text{Ai}'(\omega s) \end{cases} 
\]

\[ \begin{cases} 
(\text{Ai}(s) & -\omega^2 \text{Ai}(\omega s) e^{-\frac{\pi i}{6} \sigma_3} , \quad s \in IV; \\
\text{Ai}'(s) & -\text{Ai}'(\omega s) \end{cases} 
\]

(6.30)

cf. [10] (7.9)], where \( \omega = e^{\frac{2\pi i}{3}} \), and the sectors \( I - IV \) are illustrated in Figure 4. We note that with the orientation of \( \Sigma_k \) indicated, \( \Psi(s) \) possesses the constant jumps

\[ J_{\Psi}(s) = \begin{cases} 
\begin{pmatrix} 1 & 0 \\
-1 & 1 \end{pmatrix} , \quad s \in \Sigma_1, \\
\begin{pmatrix} 0 & -1 \\
1 & 0 \end{pmatrix} , \quad s \in \Sigma_2, \\
\begin{pmatrix} 1 & 0 \\
1 & 1 \end{pmatrix} , \quad s \in \Sigma_3, \\
\begin{pmatrix} 1 & -1 \\
0 & 1 \end{pmatrix} , \quad s \in \Sigma_4. 
\end{cases} \]
To verify the jump conditions, use has been made of the fact that
\[ \text{Ai}(s) + \omega \text{Ai}(\omega s) + \omega^2 \text{Ai}(\omega^2 s) = 0 \quad \text{for} \quad s \in \mathbb{C}. \quad (6.31) \]

Resuming the re-scaled variable \( \tau = \tau_\alpha = \frac{s - \alpha}{1 + \alpha} \), we see from (6.15) that
\[ \lambda_\alpha(z) = e^{-4\pi i/3} \left( \frac{3}{2} \right)^{2/3} n^{1/3} \phi_\alpha^{2/3}(z) \]
defines a conformal mapping from an \( O(1) \) neighborhood of \( \tau_\alpha = 0 \) to an \( O(1) \) neighborhood of \( \lambda_\alpha = 0 \). We can then write down the parametrix as
\[ P(\alpha)(z) = E(z) \Psi \left( n^{1/3} \lambda_\alpha(z) \right) e^{-\frac{\pi}{2} i \sigma_3} e^{n \phi_\alpha(z) \lambda_\alpha(z)} \left( \phi_0(z) \right)^{\frac{3}{2} \sigma_3}, \quad z \in U_r(\alpha), \quad (6.33) \]

where \( \phi_0^\frac{3}{2} \) is determined such that \( (\phi_0^\frac{3}{2})_\pm(x) = e^{\pm \frac{1}{2} i \Theta(x)} \), with \( (\phi_0)_\pm(x) = e^{\pm i \Theta(x)} \) for \( x \in (-1,1) \) and \( \Theta(x) \) being real, \( E(z) \) is an analytic function in \( z \in U_r(\alpha) \), as can be determined by the matching condition (6.29), and by expanding \( \Psi \) in (6.33) for large \( n \).

Indeed, as \( \tau_\alpha = \frac{s - \alpha}{1 + \alpha} \sim O(1) \), \( s := n^{1/3} \lambda_\alpha(z) \sim O(n^{1/3}) \) large, and \( \zeta = \frac{s}{3} \sim n^{3/2} = n \phi_\alpha(z) \sim e^{\frac{3}{2} \sqrt{n} \tau_{\alpha/3}^3} \zeta_0(\tau_\alpha) \) for large \( n \). Substituting in the asymptotic approximation for \( \Psi(s) \) in (6.30) gives
\[ \Psi(s) e^{-\frac{3}{2} i \sigma_3} e^{n \phi_\alpha(z) \lambda_\alpha} \sim \frac{1}{2 \sqrt{\pi}} e^{-\frac{3}{2} i \sigma_3} \sum_{k=0}^{\infty} \frac{\Psi_k}{s^k} M_\alpha; \quad (6.34) \]

see [1] (10.4.7), (10.4.59), (10.4.61)], see also [25 Sec. 5.1] for a detailed discussion. In (6.34), the coefficients are given as
\[ M_\alpha = \left( \begin{array}{cc} -i & -1 \\ i & -1 \end{array} \right), \quad \Psi_{2k} = \left( \begin{array}{cc} c_{2k} & 0 \\ 0 & d_{2k} \end{array} \right), \quad \Psi_{2k+1} = \left( \begin{array}{cc} 0 & c_{2k+1} \\ d_{2k+1} & 0 \end{array} \right) \quad \text{for} \quad k = 0, 1, 2, \cdots, \]

with \( c_0 = d_0 = 1, c_k = \frac{\Gamma(3k+\frac{1}{2})}{(3k+\frac{1}{2})!}, \quad d_k = \frac{6k+1}{6k+2} c_k \) for \( k = 1, 2, \cdots \), which shares the same jump condition as \( N(z) \). Also, we need to go further to analyze the piece-wise analytic function \( \phi_0(z) \). From (2.1), and using Stirling’s formula (see [1] (6.1.37)], we obtain
\[ \phi_0(z) = 1 \pm \frac{ib_\alpha \sqrt{\tau + 1}}{\sqrt{n}} + O \left( \frac{1}{n} \right), \quad \pm \text{Im} z > 0 \]

for \( \tau_\alpha = \frac{s - \alpha}{1 + \alpha} = O(1) \) and \( n \) large, where \( b_\alpha = \sqrt{2b_\alpha} \left( 1 - \frac{1}{6b_\alpha} \right) \), and \( \arg(\tau + 1) \in (-\pi, \pi) \).

For \( \tau_\alpha = \frac{s - \alpha}{1 + \alpha} = O(1) \), we can rewrite \( N(z) \) as
\[ N(z) = \left( I + O \left( \frac{1}{n} \right) \right) M_\alpha^{-1} \left( \begin{array}{cc} f_s(\tau_\alpha) & 0 \\ 0 & 1 \end{array} \right) M_\alpha M_\alpha^{-1} \phi(\zeta)^{\sigma_3} M_\alpha D(z)^{\sigma_3}. \]

Then, we can determine the analytic function in \( z \in U_r(\alpha) \) as
\[ E(z) = 2 \sqrt{\pi} e^{\frac{3}{2} i} M_\alpha^{-1} \left( \begin{array}{cc} 1 & -1 + \frac{1}{6b_\alpha} - f_s(\tau_\alpha) \\ 0 & 1 \end{array} \right) \left( \frac{1}{4} \right)^{\sigma_3}, \quad (6.35) \]
where \( f_r(\tau) = f(\tau) - f_s(\tau) \) and \( f_s(\tau) = \frac{1}{2\pi i} + \frac{1}{4\pi i} \) is the singular part of \( f(\tau) = \frac{\sqrt{2/\tau}}{\sqrt{\zeta \alpha(\tau)}} \); cf. (6.16). The matching condition (6.29) is readily verified. In the verification, use may be made of the facts that \( M_{\alpha} - M_{\alpha}^{-1} = -\sigma_1 \) and that for \( |\tau| = O(1) \), \( \ln \phi_0(z) = \pm 2c\sqrt{z + 1} + O(1/n) \) respectively as \( \pm \text{Im } z > 0 \), with \( c = \frac{b_0}{2\sqrt{b}} = \frac{i}{\sqrt{2}} (1 - \frac{1}{\alpha}) \). Hence

\[
\phi_0^\pm(z) = \exp \left( \pm c\sqrt{z + 1} \pm \frac{\alpha}{\pi} \int_{-1}^{1} \frac{\sqrt{x + 1}}{\sqrt{\alpha - x} x} dx \right) = e^{\pm c\sqrt{z} - \alpha}
\]

for \( \pm \text{Im } z > 0 \), up to an error of order \( O(1/n) \), as follows from the identity

\[
-\frac{1}{\pi} \int_{1}^{\alpha} \frac{\sqrt{x + 1}}{\alpha - x} dx = \sqrt{\frac{z + 1}{z - \alpha}} - 1, \quad z \in \mathbb{C}, \quad \arg(z + 1), \quad \arg(z - \alpha) \in (-\pi, \pi).
\]

6.6 Local parametrix at \( z = \beta \)

We proceed to construct a parametrix at a neighborhood \( U_r(\beta) \) of \( z = \beta \), described as \( |\tau| < r \) for \( \tau = \frac{z - \beta}{\beta - 1} \). We note that \( \beta \) is the band-saturated region endpoint. The parametrix is formulated as

\[
(P_\alpha^{(\beta)}) \quad P^{(\beta)}(z) \text{ is analytic in } U_r(\beta) \setminus \Sigma_S; \text{ cf. Figure 3 for } \Sigma_S.
\]

\[
(P_b^{(\beta)}) \quad P^{(\beta)}(z) \text{ satisfies the same jump conditions as } S(z) \text{ in } U_r(\beta), \text{ namely,}
\]

\[
P^{(\beta)}(z) = P^{(\beta)}_+(z) = \begin{cases} 
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, & z \in (\beta, M) \cap U_r(\beta), \\
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}, & z \in (1, \beta) \cap U_r(\beta), \\
\begin{pmatrix}
1 & e^{-2n\phi_\beta(z)} \\
0 & 1
\end{pmatrix}, & z \in \{\beta + i(0, \varepsilon)\} \cap U_r(\beta), \\
\begin{pmatrix}
1 & e^{-2n\phi_\beta(z)} \\
0 & 1
\end{pmatrix}, & z \in \{\beta - i(0, \varepsilon)\} \cap U_r(\beta).
\end{cases}
\]

\[
(P_c^{(\beta)}) \text{ The matching condition holds}
\]

\[
P^{(\beta)}(z) = \left( I + O \left( \frac{1}{n} \right) \right) N(z), \quad z \in \partial U_r(\beta).
\]

The RHP has also been solved in earlier literature. Indeed, as in Bleher and Liechty \[6\] (11.6)-(11.7), we use the matrix function

\[
\Psi_\beta(s) = \begin{cases} 
\begin{pmatrix}
\omega^2 \text{Ai}(\omega s) & -\text{Ai}(s) \\
\omega \text{Ai}'(\omega s) & -\text{Ai}'(s)
\end{pmatrix} & \text{for } \arg s \in (0, \pi/2), \\
\begin{pmatrix}
\omega^2 \text{Ai}(\omega s) & \omega \text{Ai}(\omega s) \\
\omega \text{Ai}'(\omega s) & \omega^2 \text{Ai}'(\omega s)
\end{pmatrix} & \text{for } \arg s \in (\pi/2, \pi), \\
\begin{pmatrix}
\omega \text{Ai}(\omega s) & -\omega^2 \text{Ai}(\omega s) \\
\omega^2 \text{Ai}'(\omega s) & -\omega \text{Ai}'(\omega s)
\end{pmatrix} & \text{for } \arg s \in (-\pi, -\pi/2), \\
\begin{pmatrix}
\omega \text{Ai}(\omega s) & \text{Ai}(s) \\
\omega^2 \text{Ai}'(\omega s) & \text{Ai}'(s)
\end{pmatrix} & \text{for } \arg s \in (-\pi/2, 0).
\end{cases}
\]
It is readily verified that $\Psi_\beta(s)$ possesses constant jumps

$$
(\Psi_\beta)_+(s) = (\Psi_\beta)_-(s)
$$

where orientation is taken from left to right on the real line, and down to up along the imaginary axis.

Now we introduce a conformal mapping

$$
\lambda_\beta(z) = \left(\frac{3}{2}\right)^{2/3} n^{1/3} (-\phi_\beta(z))^{2/3}
$$

from an $O(1)$ neighborhood of $\tau_\beta = \frac{z-\beta}{\beta-1}$ of the origin to an $O(1)$ neighborhood of $\lambda_\beta = 0$. To verify the fact, we may need a convergent series expansion of the form

$$
n\phi_\beta(z) \sim -c_\beta(n)^{3/2}, \quad \arg\tau_\beta \in (-\pi, \pi),
$$

where $\tau_\beta = \frac{z-\beta}{\beta-1}$, and $c_\beta(n) \sim \frac{2\sqrt{3}}{3}\sqrt{n}$ for large $n$.

The parametrix can be represented as

$$
P^{(\beta)}(z) = E_\beta(z)\Psi_\beta \left(n^{1/3}\lambda_\beta(z)\right) e^{n\phi_\beta(z)\sigma_3} e^{\frac{i}{2}3\sigma_3}, \quad \pm \text{Im } z > 0, \quad z \in U_r(\beta),
$$

where $E_\beta(z)$ is an analytic function in $z \in U_r(\beta)$. A straightforward verification shows that, for $s = n^{1/3}\lambda_\beta(z)$,

$$
\zeta = \frac{2}{3} s^{3/2} = -n\phi_\beta(z), \quad \text{and} \quad \zeta = -n\phi_\beta(z) \sim \sqrt{n\tau_\beta^{3/2}} \xi_0(\tau_\beta) \text{ for mild } \tau_\beta \text{ and large } n,
$$

$$
\Psi_\beta(s) e^{n\phi_\beta(z)\sigma_3} e^{\frac{i}{2}3\sigma_3} \sim s^{-\frac{4}{3}\sigma_3} \frac{2\sqrt{\pi}}{\sum_{k=0}^{\infty} \frac{\Psi_k}{\zeta^k}} M_\beta, \quad \arg z \in (-\pi, \pi);
$$

see [1] (10.4.7), (10.4.59), (10.4.61)], where $M_\beta = \left(\begin{array}{cc} 1 & i \\ 1 & -i \end{array}\right)$, and the coefficients $\Psi_k$ are given in $[6.31]$.

Denoting $f(\tau) = \frac{c_{1/2}}{\tau^{3/2} \xi_0(\tau)}$, and $f_s(\tau) = \frac{1}{126} \frac{1}{\tau^2} + \frac{5}{48} \frac{1}{\tau^2}$ being the singular part of it, we rewrite $N(z)$ as

$$
N(z) = \left(I + O\left(\frac{1}{n}\right)\right) M_\beta^{-1} \left(\begin{array}{cc} 1 & f_s(\tau_\beta) \\ 0 & 1 \end{array}\right) M_\beta^{-1} \phi(z)^{-\sigma_3} M_\beta.
$$

Thus, we may choose

$$
E_\beta(z) = 2\sqrt{\pi} M_\beta^{-1} \left(\begin{array}{cc} 1 & -f_r(\tau_\beta) \\ 0 & 1 \end{array}\right) \left(\frac{s^3}{\phi(z)}\right)^{\sigma_3}, \quad z \in U_r(\beta),
$$

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where again \( f_r(\tau) = f(\tau) - f_s(\tau) \) is the regular part of \( f(\tau) \). With the analytic factor so chosen, it is readily verified that the matching condition is satisfied.

For later use, we write down the well-known formulas

\[
\omega \text{Ai}(\omega s) = -\frac{1}{2} (\text{Ai}(s) - \text{Bi}(s)) \quad \text{and} \quad \omega^2 \text{Ai}(\omega^2 s) = -\frac{1}{2} (\text{Ai}(s) + \text{Bi}(s)) \quad \text{for} \quad s \in \mathbb{C}. \tag{6.44}
\]

### 6.7 The final transformation \( S \rightarrow R \)

We bring in the final transformation by defining

\[
R(z) = \begin{cases} 
S(z) N^{-1}(z), & z \in \mathbb{C} \setminus (U_r(\beta) \cup U_r(\alpha) \cup \Sigma_S), \\
S(z) (P^{(\beta)})^{-1}(z), & z \in U_r(\beta) \setminus \Sigma_S, \\
S(z) (P^{(\alpha)})^{-1}(z), & z \in U_r(\alpha) \setminus \Sigma_S.
\end{cases} \tag{6.45}
\]

We note that the jumps along contours emanating from \( z = 1 \) is of no significance; cf. (6.10), (6.25), and the discussion in Section 6.3. On the remaining contours illustrated in Figure 5, we have the jump \( J_R(z) = O(1/n) \) uniformly for large \( n \). Hence we conclude that \( R(z) = I + O(1/n) \).

### 7 Proof of Theorem \[\text{I}\]

We prove the theorem case by case, tracing back to the transformations \( Y \rightarrow U \rightarrow T \rightarrow S \rightarrow R \). The regions are illustrated in Figure 1.

From (3.6) it is known that \( \pi_n(z) = Y_{11}(z) \), the \( (1,1) \) entry of \( Y(z) \). Also, it is readily seen from (1.4) and the initial condition that the leading coefficient of the orthonormal polynomial \( p_n(z) \) is \( \gamma_n = 2^n \), namely

\[
p_n(z) = \gamma_n \pi_n(z) = 2^n \pi_n(z). \tag{7.1}
\]

For \( z \in A_r \), we have

\[
Y(z) = e^{2n l \sigma_3} R(z) N(z) e^{n g(z) - \frac{1}{2} l \sigma_3} \{ d_E(z) \chi(z) \}^{-\sigma_3}.
\]

Accordingly,

\[
\pi_n(z) = e^{ng(z)} \{ d_E(z) \chi(z) \}^{-1} (R_{11} N_{11} + R_{12} N_{21}).
\]
In view of the fact that \( R(z) = I + O(1/n) \) as \( n \to \infty \), from \((4.5), (4.7), (5.19), (6.27)\) and \((7.1)\), we obtain \((2.2)\).

For \( z \in B_r \), it is easily seen that

\[
Y(z) = e^{\frac{1}{2}nl\sigma_3} R(z) N(z) \begin{pmatrix} 1 \\ e^{2n\phi_\alpha(z)} \\ 0 \\ 1 \end{pmatrix} e^{n(g(z) - \frac{1}{2})\sigma_3} \{ d_E(z) \chi(z) \}^{-\sigma_3}.
\]

From \((7.2)\) we have

\[
\pi_n(z) = e^{n_0(z)} \{ d_E(z) \chi(z) \}^{-1} \left[ (R_{11} N_{11} + R_{12} N_{21}) + (R_{11} N_{12} + R_{12} N_{22}) \phi_0(z) e^{2n\phi_\alpha(z)} \right]
\]

\[
= e^{n_0(z)} e^{n^0_\phi(z) \frac{1}{2}} \{ d_E(z) \chi(z) \}^{-1} \left[ R_{11}(N_{11} e^{-n_0(z)} \phi_0^\frac{1}{2}) + N_{12} e^{n_0(z)} \phi_0^\frac{1}{2} \right]
\]

\[
+ R_{12}(N_{21} e^{-n_0(z)} \phi_0^\frac{1}{2}) + N_{22} e^{n_0(z)} \phi_0^\frac{1}{2} \right].
\]

Then \((2.3)\) follows from \((2.1), (4.5), (4.7), (5.18), (6.27), (7.1)\), and the fact that \( f_s(\tau_\beta) \) and \( f_s(\tau_\alpha) \) are bounded.

For \( z \in C_r \), we have

\[
Y(z) = e^{\frac{1}{2}nl\sigma_3} R(z) N(z) \begin{pmatrix} 1 \\ -e^{2n\phi_\beta(z) - 2\pi i\gamma(z)} \\ 0 \\ 1 \end{pmatrix} e^{n(g(z) - \frac{1}{2})\sigma_3} \{ d_I(z) \chi(z) \}^{-\sigma_3} \{ D_+ \}^{-1}(z).
\]

Recalling the relation between \( g(z) \) and \( \phi_\beta(z) \), the definitions of \( r(z) \) and \( D_+ \), we obtain

\[
\pi_n(z) = e^{\frac{1}{2}nl - \frac{1}{2} \pi i} (-\gamma'(z) w_d(z))^{-\frac{1}{2}} \left[ (R_{11} N_{11} + R_{12} N_{21}) e^{-n_0(z) + \pi i\gamma(z)} \right]
\]

\[
- (R_{11} N_{12} + R_{12} N_{22}) e^{n_0(z) - \pi i\gamma(z)} \right].
\]

Therefore \((2.4)\) follows from \((6.27), (7.1)\), the fact that \(-\gamma'(z) w_d(z) = 4b e^{-2b} (1 + O(1/n))\), and that both \( f_s(\tau_\alpha) \) and \( f_s(\tau_\beta) \) are bounded.

For \( z \in D_{1,r} \cup D_{2,r} \), the series of transformations again applies, for \( z \in D_{1,r} \), we have

\[
Y(z) = e^{\frac{1}{2}nl\sigma_3} R(z) P^{(\beta)}(z) \begin{pmatrix} 1 \\ -e^{2n\phi_\beta(z) - 2\pi i\gamma(z)} \\ 0 \\ 1 \end{pmatrix} e^{n(g(z) - \frac{1}{2})\sigma_3} \{ d_I(z) \chi(z) \}^{-\sigma_3} \{ D_+ \}^{-1}(z).
\]

We can also rewrite \( \Psi_\beta(s) \) in \((6.38)\) as

\[
\Psi_\beta(s) = \begin{pmatrix} \text{Ai}(s) & \text{Bi}(s) \\
\text{Ai}'(s) & \text{Bi}'(s) \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.
\]

Combining the definitions of \( d_I(z) \) in \((4.6), \chi(z) \) in \((4.7), D_+ \) in \((4.1)\) and the relation between \( g(z) \) and \( \phi_\beta(z) \) yields

\[
\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ e^{-2\pi i\gamma} \end{pmatrix} e^{\frac{1}{2}nl\sigma_3} e^{2n\phi_\beta(z) \sigma_3} e^{n(g(z) - \frac{1}{2})\sigma_3} \{ d_I(z) \chi(z) \}^{-\sigma_3} \{ D_+ \}^{-1}(z)
\]

\[
= \begin{pmatrix} -\cos(\pi\gamma) & * \\
\sin(\pi\gamma) & * \end{pmatrix} (-\gamma'(z) w_d(z))^{-\frac{1}{2} \sigma_3}.
\]

Substituting \((7.6)\) into \((7.5)\) gives

\[
\pi_n(z) = \sqrt{\pi} e^{\frac{1}{2}nl} (-\gamma'(z) w_d(z))^{-\frac{1}{2}} \left\{ R_{11}(z) \left[ s^\frac{1}{2} g^{-1} A_1(z) + s^{-\frac{1}{2}} (f_r(\tau) + 1) A_2(z) \right]
\]

\[
+ R_{12}(z) \left[ -is^\frac{1}{2} g^{-1} A_1(z) + s^{-\frac{1}{2}} (f_r(\tau) + i) A_2(z) \right] \right\}.
\]

\[
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\]
Then (2.5) follows from (7.1) and the fact that \(-\gamma'(z)w_d(z) = 4b e^{-2b(1 + O(1/n))}\). The case \(z \in D_{2,r}\) can be treated similarly.

For \(z \in E_r\), it is readily seen that

\[
Y(z) = e^{\frac{1}{2} n \sigma_3} R(z) N(z) e^{n(g(z) - \frac{1}{2} l)\sigma_3} \{d_{\ell}(z) \chi(z)\}^{-\sigma_3} (D_{+}^l)^{-1}(z). \tag{7.8}
\]

Picking up the \((1,1)\)-entry gives

\[
\pi_n(z) = e^{\frac{1}{2} n l - \frac{1}{2} \pi i} (-\gamma'(z)w_d(z))^{-\frac{1}{2}} [2i \sin(\pi \gamma(z)) e^{-n \phi_\beta(z)} (R_{11} N_{11} + R_{12} N_{21})
- (R_{11} N_{12} + R_{12} N_{22}) e^{n \phi_\beta(z) - \pi \gamma(z)}]. \tag{7.9}
\]

Now a combination of the fact that \(D^{-1}(z)\) is bounded with (6.27) and (7.1) justifies (2.6).

For \(z \in F_{1,r} \cup F_{2,r}\), first for \(z \in F_{1,r}\), we have

\[
Y(z) = e^{\frac{1}{2} n \sigma_3} R(z) P^{(\alpha)}(z) e^{n(g(z) - \frac{1}{2} l)\sigma_3} \{d_{E}(z) \chi(z)\}^{-\sigma_3}. \tag{7.10}
\]

Then we have

\[
\pi_n(z) = [R_{11}(z) (P^{(\alpha)})_{11}(z) + R_{12}(z) (P^{(\alpha)})_{21}(z)] e^{n g(z)} \{d_{E}(z) \chi(z)\}^{-1}. \tag{7.11}
\]

Therefore, substituting \(P^{(\alpha)}(z)\) in (6.33), the relation (5.18), \(\{d_{E}(z) \chi(z)\}^{-1} (\phi_0)^{1/2} = 1 + O(1/n)\) and the asymptotic behavior of \(R(z)\) into the representation, and in view of (7.1), we obtain (2.7). The case of \(z \in F_{2,r}\) is similar to that of \(z \in F_{1,r}\).

## 8 Asymptotic quantities and comparison with known results

We evaluate several asymptotic quantities and approximations from the Riemann-Hilbert analysis and Theorem 1 and compare them with the results of Wang and Zhao [20] obtained earlier via integral methods.

### 8.1 Evaluation of the Lagrange multiplier \(\ell\)

To calculate the constant \(\ell = l(n)\) in (5.3), and (5.18)-(5.19), we use the relations

\[
\begin{align*}
g_+(x) + g_-(x) &= l + \frac{1}{n} \ln r(x), \quad x \in (\alpha, \beta), \\
g_+(x) - g_-(x) &= \frac{2\pi i}{n}, \quad x \in (-\infty, \alpha), \\
g_+(x) - g_-(x) &= \frac{2\pi i \gamma(x)}{n}, \quad x \in (\beta, \infty);
\end{align*} \tag{8.1}
\]

cf. (5.3) and (6.8). It is also seen that \(g(z) \sqrt{(z - \alpha)(z - \beta)} \to 0\) as \(z \to \alpha\) and \(z \to \beta\), and that \(g(z) = \ln z + O(z^{-1} \ln z)\) as \(z \to \infty\).

An alternative representation is obtained by solving the above scalar RHP, namely,

\[
g(z) = \ln \frac{2z - \alpha - \beta + 2 \sqrt{(z - \alpha)(z - \beta)}}{\beta - \alpha} + \frac{\sqrt{(z - \alpha)(z - \beta)}}{2n\pi} \int_\alpha^\beta \frac{\ln r(x)}{\sqrt{(x - \alpha)(\beta - x)}} \frac{dx}{x - z} + \frac{\sqrt{(z - \alpha)(z - \beta)}}{n} \int_\beta^\infty \frac{\gamma(x)}{\sqrt{(x - \alpha)(x - \beta)}} \frac{dx}{x - z} + \frac{l}{2}. \tag{8.2}
\]
Let $z \to \infty$, we find that

$$
\frac{l}{2} = \ln \frac{\beta - \alpha}{4} - \frac{\ln \pi}{2n} + \frac{1}{2n\pi} \int_\alpha^\beta \frac{\ln(\pi r(x))}{\sqrt{(x - \alpha)(\beta - x)}} \, dx + \frac{1}{n} \int_\beta^\infty \frac{\gamma(x)}{\sqrt{(x - \alpha)(x - \beta)}} \, dx
$$

(8.3)

To pick up the contribution from the first integral, we apply the Cauchy integral theorem to the function $\frac{-ib \ln(z + \sqrt{z^2 - 1})}{\sqrt{z^2 - 1}(z - \beta)}$ on the cut-plane $\mathbb{C} \setminus (-\infty, \beta]$. As a result, we have

$$
\int_{-1}^1 \frac{2b \arccos z}{\sqrt{1 - z^2}} \, dx = \int_{-\infty}^1 \frac{2b\pi dx}{\sqrt{x^2 - 1}\sqrt{(\alpha - x)(\beta - x)}} - \int_1^\beta \frac{2b \ln(x + \sqrt{x^2 - 1})}{\sqrt{x^2 - 1}\sqrt{(x - \alpha)(\beta - x)}} \, dx
$$

Recalling that $\ln(\pi r(x)) = -\frac{2b \arccos x}{\sqrt{1 - x^2}}$ for $x \in (\alpha, 1)$, and substituting the above equality of integrals into (8.3), we obtain

$$
nl = -2n \ln 2 - \ln \pi + O(1/\sqrt{n}).
$$

(8.4)

Here, use has been made of the fact that $\alpha \sim -1 + \frac{b}{n}$, $\beta \sim 1 + \frac{b}{n}$, the contribution of the integrals over the narrow intervals $[-1, \alpha]$ and $[1, \beta]$ are negligible, and the leading terms of the integrals on $[\beta, +\infty)$ and $[-\infty, -1]$ can be evaluated and are canceled with each other. For example, since $\alpha \sim -1 + \frac{b}{n}$, up to a factor $\left(1 + O\left(\frac{1}{n}\right)\right)$, we have

$$
\int_{-\infty}^\infty \frac{dx}{\sqrt{x^2 - 1}\sqrt{(x - \alpha)(x - \beta)}} \sim \int_{-\infty}^\infty \frac{dx}{(x + 1)\sqrt{(x - 1)(x - \beta)}} = \frac{1}{\sqrt{2(\beta + 1)}} \ln \frac{\sqrt{\beta + 1} + 1}{\sqrt{\beta + 1} - 1}.
$$

To deal with the second integral, a change of variable $t = \sqrt{\frac{x - \beta}{x - 1}}$ may be used.

An alternative way to find $l$ is to use a certain Szegő function, similar to [26 (6.6)].

### 8.2 Asymptotics of the leading coefficients $\gamma_n$, a consistency check

Write $Y(z) = (I + Y_1/z + \cdots) z^{\sigma_3}$, with $Y_1$ independent of $z$. Then $\gamma_{n-1}^2 = -\frac{1}{2\pi i} (Y_1)_{21}$; cf. (3.6), where $\gamma_{n-1}$ is the leading coefficient for the orthonormal polynomial $p_{n-1}(z)$. Tracing back the series of transformations $Y \to U \to T \to S$ in the last section, we have

$$
Y(z) = e^{\frac{1}{2}nl\sigma_3} S(z) e^{n(g(z) - \frac{1}{2}b)\sigma_3} \{d_E(z)\chi(z)\}^{-\sigma_3}
$$

(8.5)

for $z$ in a neighborhood of infinity.

Elementary calculation gives

$$
d_E(z)\chi(z) = 1 - b\frac{\ln(-z)}{z} + O\left(\frac{1}{n}\right) \quad as \quad z \to \infty;
$$

(8.6)

cf. (4.5) and (4.7). Also, we have

$$
g(z) = \ln z - b\frac{\ln(-z)}{z} + O\left(\frac{1}{z}\right) \quad as \quad z \to \infty,
$$

(8.7)

which follows from the fact that

$$
G(z) = \frac{1}{\pi iz} - b\frac{\ln(-z)}{n\pi i z^2} + O\left(z^{-2}\right),
$$

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as can be seen from (5.8). In the previous formulas, the logarithms take principal branches, namely, \( \arg t \in (-\pi, \pi) \) for \( \ln t \).

We may set \( S(z) = I + S_1/z + \cdots \). Then, substituting (8.6) and (8.7) into (8.5) yields

\[
\gamma_{n-1}^2 = \frac{i}{2\pi} e^{-nl} (S_1)_{21} \sim \frac{i}{2\pi} e^{-nl} (N_1)_{21},
\]

where \( N(z) = I + N_1/z + \cdots \). Taking (6.27) into account, it is readily seen that \( (N_1)_{21} = -\frac{i}{2} + O(1/n) \) for large \( n \). Now we put together the asymptotic approximation (8.4) of the constant \( l \), we obtain

\[
\gamma_{n-1} \sim 2^{n-1}
\]

for large \( n \), which agrees with the fact that \( \gamma_n = 2^n \), as can be easily verified from (1.3) and the initial conditions attached.

### 8.3 Comparison with [20]: The behavior at \( z = \alpha \) and \( z = \beta \)

In Wang and Zhao [20], using integral methods, uniform asymptotic approximations have been obtained in an interval of width \( O(1/n) \) to which \( \alpha \) belongs. More precisely, denoting \( z = -\cos \theta' = -\cos(t/\sqrt{n}) \), the leading term of the asymptotic expansion

\[
p_n(z) = \frac{(-1)^n e^{a(b^{1/3} - b)}}{\sqrt{2b}} \left( \frac{B^2(t)}{2b - t^2} \right)^{1/4} \left\{ \text{Ai}(n^{1/3} B^2) n^{1/3} (1 + O(n^{-1/2})) \right. \\
- \left. \text{Ai}'(n^{1/3} B^2) O(n^{-1/3}) \right\}, \quad n \to \infty
\]

holds uniformly for \( \delta \leq t \leq M \), with \( \delta \) and \( M \) being positive constants, where \( B^2(t) \) is an analytic function at \( t = \sqrt{2b} \) with \( B^2(t) = -2(2b)^{-1/6}(t - \sqrt{2b}) + \cdots \); see [20] (5.9).

We are in a position to compare with (8.9), starting by showing that \( B^2 \) serves as a conformal mapping at \( z = \alpha \), just as \( \lambda_\alpha \) does; cf. (6.32). Indeed, for \( z - \alpha = (1 + \alpha) \tau \), the parameters are connected as \( t^2 - 2b = 2\tau + O(1/n) \), that is, \( t - \sqrt{2b} = O(\tau) + O(1/n) \).

Several facts are readily observed:

\[
\lambda_\alpha(z) = -(2b)^{1/3} \tau (1 + O(\tau))(1 + O(1/\sqrt{n})), \\
\sqrt{2(z + 1)} = \theta' + O(1/n), \\
g(z) = 2^{1/4} b^{-1/4} n^{1/4} (-\tau)^{-1/4} (1 + O(1/n)).
\]

Substituting the approximations into (2.7), we have

\[
p_n(z) = \frac{(-1)^n e^{-b + \pi b/\theta'}}{\sqrt{2b}} \left\{ n^{1/3}(2b)^{-1/2} \text{Ai}(n^{1/3} \lambda_\alpha(z))(1 + O(\tau))(1 + O(n^{-1/2})) \right. \\
- \left. \text{Ai}'(n^{1/3} \lambda_\alpha(z)) O(n^{-1/3}) \right\},
\]

which agrees with (8.9).

In Wang and Zhao [20], a full uniform asymptotic expansion at \( z = \beta \) is derived. For \( z = \cos(t/\sqrt{n}) \), the leading behavior of the asymptotic expansion, in terms of the variables in this paper, reads

\[
p_n(z) = \frac{e^b}{\sqrt{2b}} \left( \frac{B^2(t)}{t^2 + 2b} \right)^{1/4} \bar{A}(n^{1/3} B^2) n^{1/3} (1 + O(n^{-1/2})) - \bar{A}'(n^{1/3} B^2) O(n^{-1/3})
\]
as \( n \to \infty \), uniformly for \( t \in [i[-M, -\delta]\) with positive constants \( M \) and \( \delta \) satisfying \( \delta < \sqrt{2b} < M \), where \( \tilde{A}(s) = -\cos(\pi \gamma(z))Ai(s) + \sin(\pi \gamma(z))Bi(s) \) and \( B^2(t) \) are analytic function of \( t \) at \( t = -i\sqrt{2b} \) with \( B^2(t) = 2(2b)^{-1/6}(|t| - \sqrt{2b}) + \cdots \).

For \( z - \beta = (\beta - 1)^{\tau} \), we see that \( |t| - \sqrt{2b} = O(\tau) + O(1/n) \). Quantities are related as

\[
\lambda_\beta(z) = (2b)^{1/3}\tau(1 + O(\tau))(1 + O(1/\sqrt{n})), \quad \varphi^{-1}(z) = 2^{\tau/2}n^{\tau/4}n^{-\tau/4}(1 + O(1/n)).
\]

Hence, with the above arguments, we see that (2.5) agrees with (8.10), with \( A_1 \) and \( A_2 \) in (2.5) correspond to \( \tilde{A} \) and \( \tilde{A}' \) in (8.10).

In Szegö [18], an observation was made to the Pollaczek polynomials that they show a singular behavior as compared with the classical polynomials. It is worth noting that the sieved Pollaczek polynomials also show such a singular behavior in some aspects, such as the orders in \( n \) of the behavior at the endpoints, the Toeplitz minima, the behavior of \( p_n(z) \) for \( z \) not on the orthogonal support or within \((-1, 1)\), and the large-\( n \) behavior of the smallest zeros.

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References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] R. Askey and M. Ismail, Recurrence relations, continued fractions, and orthogonal polynomials, *Mem. Amer. Math. Soc.* 49 (1984), no. 300.

[3] J. Baik, T. Kriecherbauer, K.T.-R. McLaughlin and P.D. Miller, Discrete orthogonal polynomials, asymptotics and applications, *Ann. Math. Studies*, 164, Princeton University Press, Princeton and Oxford, 2007.

[4] P. Bleher and A. Its, Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model, *Ann Math*, 150 (1999), 185-266.

[5] P. Bleher and K. Liechty, Exact solution of the six-vertex model with domain wall boundary conditions: antiferroelectric phase, *Comm. Pure Appl. Math.*, 63 (2010), 779-829.
[6] P. Bleher and K. Liechty, Uniform asymptotics for discrete orthogonal polynomials with respect to varying exponential weights on a regular infinite lattice, *Int. Math. Res. Not.*, **2011** (2011), 342-386.

[7] A. Borodin and D. Boyarchenko, Distribution of the first particle in discrete orthogonal polynomial ensembles, *Comm. Math. Phys.*, **234** (2003), 287-338.

[8] J.A. Charris and M.E.H. Ismail, On sieved orthogonal polynomials. V: Sieved Pollaczek polynomials, *SIAM J. Math. Anal.*, **18** (1987), 1177-1218.

[9] D. Dan and R. Wong, Global asymptotics of Krawtchouk polynomials-a Riemann-Hilbert approach, *Chin. Ann. Math. Ser. B*, **28** (2007), 1-34.

[10] P. Deift, T. Kriecherbauer, K.T.-R. McLaughlin, S. Venakides and X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.*, **52** (1999), 1491-1552.

[11] P. Deift and X. Zhou, A steepest descent method for oscillatory Riemann-Hilbert problems, Asymptotic for the MKdV equation, *Ann. Math.*, **137** (1993), 295-368.

[12] A.S. Fokas, A.R. Its and A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.*, **147** (1992), 395-430.

[13] M.E.H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, 2005.

[14] A.B.J. Kuijlaars, K.T.-R. McLaughlin, W. Van Assche and M. Vanlessen, The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$, *Adv. Math.*, **188** (2004), 337-398.

[15] Y. Lin and R. Wong, Global asymptotics of the Hahn polynomials, *Anal. Appl.*, **11** (2013), 1350018, 47 pp.

[16] Y. Lin and R. Wong, Global asymptotics of the discrete Chebyshev polynomials, *Asymptot. Anal.*, **82** (2013), 39-64.

[17] C.-H. Ou and R. Wong, The Riemann-Hilbert approach to global asymptotics of discrete orthogonal polynomials with infinite nodes, *Anal. Appl.*, **8** (2010), 247-286.

[18] G. Szegő, *Orthogonal Polynomials*, Fourth Edition, American Mathematical Society, Providence, Rhode Island, 1975.

[19] H.-Y. Wang, *A research on a class of orthogonal polynomials defined via a difference equation*, Ph. D. thesis, Sun Yat-sen University, 2010.

[20] H.-Y. Wang and Y.-Q. Zhao, Uniform asymptotics and zeros of a system of orthogonal polynomials defined via a difference equation, *J. Math. Anal. Appl.*, **369** (2010), 453-472.

[21] X.-S. Wang and R. Wong, Global asymptotics of the Meixner polynomials, *Asymptot. Anal.*, **75** (2011), 211-231.
[22] R. Wong and Y.-Q. Zhao, Asymptotics of orthogonal polynomials via the Riemann-Hilbert approach, *Acta Math. Sci. Ser. B*, 29 (2009), 1005-1034.

[23] X.-B. Wu, Y. Lin, S.-X. Xu and Y.-Q. Zhao, Uniform asymptotics for discrete orthogonal polynomials on infinite nodes with an accumulation point, [arXiv:1410.3948](http://arxiv.org/abs/1410.3948).

[24] S.-X. Xu, D. Dai and Y.-Q. Zhao, Critical edge behavior and the Bessel to Airy transition in the singularly perturbed Laguerre unitary ensemble, *Comm. Math. Phys.*, 332 (2014), 1257-1296.

[25] J.-R. Zhou, S.-X. Xu, and Y.-Q. Zhao, Uniform asymptotics of a system of Szegö class polynomials via the Riemann-Hilbert approach, *Anal. Appl.*, 9(2011), 447-480.

[26] J.-R. Zhou and Y.-Q. Zhao, Uniform asymptotics of the Pollaczek polynomials via the Riemann-Hilbert approach, *Proc. R. Soc. Lond. Ser. A.*, 464 (2008), 2091-2112.