Measure Preserving Diffeomorphisms of the Torus are Unclassifiable

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1 Introduction

The isomorphism problem in ergodic theory was formulated by von Neumann in 1932 in his pioneering paper [18]. The problem has been solved for some classes of transformations that have special properties. Halmos and von Neumann [11] used the unitary operators defined by Koopman to completely characterize ergodic MPTs with pure point spectrum. They showed that these are exactly the transformations that can be realized as translations on compact groups. Another notable success in solving this problem was the classification of Bernoulli shifts using the notion of entropy introduced by Kolmogorov. Many properties of ergodic MPTs were introduced and studied over the years in connection with this problem.

Starting in the late 1990’s a different type of result began to appear: anti-classification results that demonstrate in a rigorous way that classification is not possible. This type of theorem requires a precise definition of what a classification is. Informally a classification is a method of determining isomorphism between transformations perhaps by computing (in a liberal sense) other invariants for which equivalence is easy to determine.

The key words here are method and computing. For negative theorems, the more liberal a notion one takes for these words, the stronger the theorem. One natural notion is the Borel/non-Borel distinction. Saying a set $X$ or function $f$ is Borel is a loose way of saying that membership in $X$ or the computation of $f$ can be done using a countable (possibly transfinite) protocol whose basic input is membership in open sets. Say that $X$ or $f$ is not Borel is saying that determining membership in $X$ or computing $f$ cannot be done with any countable amount of resources.

In the context of classification problems, saying that an equivalence relation $E$ on a space $X$ is not Borel is saying that there is no countable amount of information and no countable, potentially transfinite, protocol for determining, for arbitrary $x, y \in X$ whether $xEy$. Any such method must inherently use uncountable resources.\footnote{Many well known classification theorems have as immediate corollaries that the resulting equivalence relation is Borel. An example of this is the Spectral Theorem, which has a consequence that the relation of Unitary Conjugacy for normal operators is a Borel equivalence relation.}

An example of a positive theorem in the context of ergodic theory is due to Halmos ([10]) who showed that the collection of ergodic measure

\footnote{Two measure preserving transformations (abbreviated to ‘MPTs’ in the paper) $T$ and $S$ are isomorphic if there is an invertible measurable mapping between the corresponding measure spaces which commutes with the actions of $T$ and $S$.}
preserving transformations is a dense \(\mathcal{G}_\delta\) set in the space of all measure preserving transformations of \(([0,1], \lambda)\) endowed with the weak topology. Moreover he showed that the set of weakly mixing transformations is also a dense \(\mathcal{G}_\delta\).\(^3\)

The first anti-classification result in the area is due to Beleznay and Foreman \([2]\) who showed that the class of measure distal transformations used in early ergodic theoretic proofs of Szemeredi’s theorem is not a Borel set. Later Hjorth \([12]\) introduced the notion of turbulence and showed that there is no Borel way of attaching algebraic invariants to ergodic transformations that completely determine isomorphism. Foreman and Weiss \([9]\) improved this results by showing that the conjugacy action of the measure preserving transformations is turbulent–hence no generic class can have a complete set of algebraic invariants.

In considering the isomorphism relation as a collection \(I\) of pairs \((S,T)\) of measure preserving transformations, Hjorth showed that \(I\) is not a Borel set. However the pairs of transformations he used to demonstrate this were inherently non-ergodic, leaving open the essential problem:

Is isomorphism of ergodic measure preserving transformations Borel?

This question was answered in the negative by Foreman, Rudolph and Weiss in \([8]\). This answer can be interpreted as saying that determining isomorphism between ergodic transformations is inaccessible to countable methods that use countable amounts of information.

In the same foundational paper from 1932 von Neumann expressed the likelihood that any abstract MPT is isomorphic to a continuous MPT and perhaps even to a differential one. This brief remark eventually gave rise to one of the yet outstanding problems in smooth dynamics, namely:

Does every ergodic MPT with finite entropy have a smooth model? \(^4\)

By a smooth model is meant an isomorphic copy of the MPT which is given by smooth diffeomorphism of a compact manifold preserving a measure equivalent to the volume element. Soon after entropy was introduced A. G. Kushnirenko showed that such a diffeomorphism must have finite entropy,\(^3\)

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\(^3\)Relatively straightforward arguments show that the set of strongly mixing transformation is a first category \(\Pi^0_2\) set. See \([6]\).

\(^4\)In \([18]\) on page 590, “Vermutlich kann sogar zu jeder allgemeinen Strömung eine isomorphe stetige Strömung gefunden werden [footnote 13], vielleicht sogar eine stetig differenziierbare, oder gar eine mechanische. Footnote 13: Der Verfasser hofft, hierfür demnächst einen Beweis anzugeben.”
and up to now this is the only restriction that is known. This paper is
the culmination of a series whose purpose is to show that the variety of
ergodic transformations that have smooth models is rich enough so that the
abstract isomorphism relation when restricted to these smooth systems is
as complicated as it is in general. We show that even when restricting to
diffeomorphisms of the 2-torus that preserve Lebesgue measure this is the
case. In this paper we prove that:

**Theorem 1** If $M$ is either the torus $\mathbb{T}^2$, the disk $D$ or the annulus then
the measure-isomorphism relation among pairs $(S, T)$ of measure preserving
$C^\infty$-diffeomorphisms of $M$ is not a Borel set with respect to the $C^\infty$-topology.

Thus the classification problem is impossible even for diffeomorphisms of
compact surfaces.

How does one prove a result such as Theorem 1? The main tool is
the idea of a reduction (Section 3.5 contains background and basic results
referred to here.). A function $f : X \to Y$ reduces $A$ to $B$ if and only for all
$x \in X$:

$$x \in A \text{ if and only if } f(x) \in B.$$  

If $X$ and $Y$ are completely metrizable spaces and $f$ is a Borel function then
$f$ is a method of reducing the question of membership in $A$ to membership
in $B$. Thus if $A$ is not Borel then $B$ cannot be either.

In the current context, the $C^\infty$ topology on the smooth transformations
refines the weak topology. Thus, by Halmos’ result quoted earlier, on the
torus (disc etc.), the ergodic transformations are still a $G_\delta$-set. (However the
famous KAM theory shows that the ergodic transformations are no longer
dense.) In particular the $C^\infty$ topology induces a completely metrizable
topology on the measure preserving diffeomorphisms of $\mathbb{T}^2$.

If $X$ is perfect and completely metrizable, a set $A \subseteq X$ is analytic if and
only if $A$ is the continuous image of a Borel set. $A$ is complete analytic if and
only if every analytic set can be reduced to $A$. It is a classical fact that
complete analytic sets are not Borel.

Here is a well-known canonical example of a complete analytic set. The
underlying space $X$ will be the space $\mathcal{T}rees$ and $A$ is the collection of ill-
founded trees; those that have infinite branches. A precise statement of the
main result of the paper:

**Theorem 2** There is a continuous function $F^s : \mathcal{T}rees \to Diff^\infty(\mathbb{T}^2, \lambda)$ such
that for $T \in \mathcal{T}rees$, if $T = F^s(T)$:
1. $\mathcal{T}$ has an infinite branch if and only if $T \equiv T^{-1}$, and

2. $\mathcal{T}$ has two distinct infinite branches if and only if $C(T) \neq \{T^n : n \in \mathbb{Z}\}$.

**Corollary 3**

- $\{T \in \text{Diff}^\infty(\mathbb{T}^2, \lambda) : T \equiv T^{-1}\}$ is complete analytic.
- $\{T \in \text{Diff}^\infty(\mathbb{T}^2, \lambda) : C(T) \neq \{T^n : n \in \mathbb{Z}\}\}$ is complete analytic, where $C(T)$ is the centralizer of $T$ in the group of abstract measure preserving transformations.

Since the map $\iota(T) = (T, T^{-1})$ is a continuous mapping of $\text{Diff}^\infty(\mathbb{T}^2, \lambda)$ to $\text{Diff}^\infty(\mathbb{T}^2, \lambda) \times \text{Diff}^\infty(\mathbb{T}^2, \lambda)$ and reduces $\{T : T \equiv T^{-1}\}$ to $\{(S, T) : S \equiv T\}$, it follows that:

**Corollary 4** $\{(S, T) : S$ and $T$ are ergodic diffeomorphisms of $\mathbb{T}^2$ and are isomorphic$\}$ is a complete analytic set and hence not Borel.

More fine-grained information is now known and will be published elsewhere. For example, Foreman, in unpublished work, showed that the problem of “isomorphism of countable graphs” is Borel reducible to the isomorphism problem for ergodic measure preserving transformations. The techniques of this paper can be used to make this concrete and show that the isomorphism problem for countable graphs can be reduced to measure theoretic isomorphism for diffeomorphism of the torus or the disk.

Here are two problems that remain open:

**Problem 1** In contrast to [9], where the authors were able to show that the equivalence relation of isomorphism on abstract ergodic measure preserving transformations is *turbulent*, this remains open for ergodic diffeomorphisms of a compact manifold.

**Problem 2** The problem of classifying diffeomorphisms of compact surfaces up to topological conjugacy remains open. We suggest that techniques similar to those in this paper may show that this equivalence relation is also not Borel.

We owe a substantial debt to everyone who has helped us with this project. These include Eli Glasner, Anton Gorodetski, Alekto Kechrís, Anatole Katok and Jean-Paul Thouvenot. We particularly want to acknowledge the contribution of Dan Rudolph, who helped pioneer these ideas and was a co-author in [8], contributing techniques fundamental to this paper.
2 An outline of the argument

This section gives an outline of the argument for Theorem 2. It uses the main results from our earlier papers: *A Symbolic Representation of Anosov-Katok Systems* ([4]) and *From Odometers to Circular Systems: A Global Structure Theorem* ([5]) which we briefly summarize. In *A Symbolic Representation of Anosov-Katok Systems* we gave a new symbolic representation for circle rotations by certain Liouvillean irrational \( \alpha \)'s and used this representation to define a class of systems that were extensions of the rotation by \( \alpha \) and were called strongly uniform **Circular Systems**. We then adapted the classic Anosov-Katok method to show that these circular systems could be represented as measure preserving diffeomorphisms of the 2-torus.\(^5\)

In *From Odometers to Circular Systems* . . . a class of systems extending an odometer was defined and we called these **Odometer Based** systems. We showed there that there is a functorial isomorphism \( F \) between the category of Odometer Based systems and the category of Circular Systems where, in each case, the morphisms are synchronous and anti-synchronous factor maps. The map \( F \) takes strongly uniform systems to strongly uniform systems.

Since the main construction in [8] uses exactly these Odometer Based systems this map enables us to adapt that construction to the smooth setting. However in order to prove our main result we still have to take into account isomorphisms of Circular Systems that are synchronous or anti-synchronous. It is to deal with this difficulty that we analyze what we call the displacement function.

To each \( \alpha \) arising as a rotation factor of a circular system we define the **displacement function** (Section 6.1) and use it to associate the set of **central values**, a subgroup of the unit circle. We then prove five facts about circular transformations \( T \):

1. (Theorem 83) If \( \beta \) is central then there is an \( \phi \in \{ T^n : n \in \mathbb{Z} \} \) such that the rotation factor of \( \phi \) is rotation by \( \beta \).

2. (Theorem 89) If \( T \) is built from sufficiently random words,\(^6\) and \( \phi \in C(T) \), then the canonical rotation factor of \( \phi \) is rotation by a central value.

3. Putting these two facts together it follows that if there is a \( \phi \in C(T) \) and \( \phi \notin \{ T^n : n \in \mathbb{Z} \} \), then there is a synchronous \( \psi \in C(T) \) such that

\(^5\)In a forthcoming paper we show how to drop the “strongly uniform” assumption.

\(^6\)i.e. \( T \) satisfies the *Timing Assumptions*.

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\[ \psi \notin \{T^n : n \in \mathbb{Z}\}. \]

4. (Theorem 91) The analogous results about isomorphisms \( \phi \) between \( T \) and \( T^{-1} \) is proved, allowing us to conclude that if \( T \) is isomorphic to \( T^{-1} \) then there is an anti-synchronous isomorphism between \( T \) and \( T^{-1} \).

5. The previous two items are the content of Theorem 92, which says that for \( T \) satisfying the Timing Assumptions we need only worry about synchronous and anti-synchronous isomorphisms.

In [8] a continuous function from the space of Trees to the strongly uniform odometer based transformations is constructed that:

- reduces the set of ill-founded trees to the transformations \( T \) that are isomorphic to their inverses by a synchronous isomorphism and

- reduces the set of trees with two infinite branches to the transformations \( T \) whose centralizer is different from the powers of \( T \).

Moreover, in the second case, there is a synchronous element of the centralizer with a specific piece of evidence that it is not the identity (it moves a \( Q_1 \)-equivalence class).

This allows us to compose the categorical isomorphism \( F \) (which is concerned with synchronous and anti-synchronous isomorphisms) with \( F \) to conclude that \( F \circ F \):

- reduces the set of ill-founded trees to collection of smoothly realizable transformations \( T \) that are isomorphic to their inverses and

- reduces the set of trees with two infinite branches to the smoothly realizable transformations \( T \) whose centralizer is different from the closure of the powers of \( T \).

Continuously realizing the circular systems (as in [4]) completes the proof that:

- The collection of ergodic measure preserving diffeomorphisms \( T \) of the torus that are isomorphic to their inverses is complete analytic. Consequently the set of pairs \((S, T)\) of ergodic conjugate measure preserving diffeomorphisms is a complete analytic set.
- The collection of ergodic measure preserving diffeomorphisms $T$ whose centralizer is different from the closure of the powers of $T$ is complete analytic.

In the next two sections we review the basic facts in ergodic theory and descriptive set theory that we use, define odometer based and circular systems and review their properties and the facts shown in [4] and [5].

The analysis of the displacement function and the associated central values, which are a subgroup of the circle canonically associated to the Liouvillean $\alpha$, is carried out in sections 5-7. Finally the proof of the main theorems are given in section 8 modulo certain properties which impose some additional conditions on the parameters of the construction in [8]. These are verified in section 9 and in section 10 we spell out the dependencies between the various parameters and show that they can be realized.

3 Preliminaries

This section reviews some basic facts and establishes notational conventions. There are many sources of background information on this including any standard text such as [17], [19] or [16]. Facts that are not standard and are simply cited here are proved in [5], [4] and [8].

3.1 Measure Spaces

We will call separable non-atomic probability spaces standard measure spaces and denote them $(X, B, \mu)$ where $B$ is the Boolean algebra of measurable subsets of $X$ and $\mu$ is a countably additive, non-atomic measure defined on $B$. Maharam and von Neumann proved that every standard measure space is isomorphic to $([0, 1], B, \lambda)$ where $\lambda$ is Lebesgue measure and $B$ is the algebra of Lebesgue measurable sets.

If $(X, B, \mu)$ and $(Y, C, \nu)$ are measure spaces, an isomorphism between $X$ and $Y$ is a bijection $\phi : X \to Y$ such that $\phi$ is measure preserving and both $\phi$ and $\phi^{-1}$ are measurable. We will ignore sets of measure zero when discussing isomorphisms; i.e. we allow the domain and range of $\phi$ to be subsets of $X$ and $Y$ of measure one.

A measure preserving system is an object $(X, B, \mu, T)$ where $T : X \to X$ is a measure isomorphism. A factor map between two measure preserving systems $(X, B, \mu, T)$ and $(Y, C, \nu, S)$ is a measurable, measure preserving function $\phi : X \to Y$ such that $S \circ \phi = \phi \circ T$. A factor map is an isomorphism between systems iff $\phi$ is a measure isomorphism.
3.2 Presentations of Measure Preserving Systems

Measure preserving systems occur naturally in many guises with diverse topologies. As far as is known, the Borel/non-Borel distinction for dynamical properties is the same in each of these presentations and many of the presentations have the same generic classes. (See the forthcoming paper [7] which gives a precise condition for this.)

Here is a review the properties of the types of presentations relevant to this paper, which are: abstract invertible preserving systems, smooth transformations preserving volume elements and symbolic systems.

3.2.1 Abstract Measure Preserving systems

As noted in section 3.1 every standard measure space is isomorphic to the unit interval with Lebesgue measure. Hence every invertible measure preserving transformation of a standard measure space is isomorphic to an invertible Lebesgue measure preserving transformation on the unit interval.

In accordance with the conventions of [6] we denote the group of measure preserving transformations of $[0,1]$ by \( \text{MPT} \). Two measure preserving transformations are identified if they are equal on sets of full measure.

Two measure preserving transformations are isomorphic if and only if they are conjugate in \( \text{MPT} \) and we will use isomorphic and conjugate as synonyms. However some caution is order. If \((M, \mu)\) is a manifold, \( T : M \to M \) is a smooth measure preserving transformation and \( \phi \) is an arbitrary measure preserving transformation from \( M \) to \( M \), then \( \phi T \phi^{-1} \) is unlikely to be smooth—the equivalence relation of isomorphism of diffeomorphisms is not given by an action of the group of measure preserving transformations in an obvious way.

Given a measure space \((X, \mu)\) and a measure preserving transformation \( T : X \to X \), define the centralizer of \( T \) to be the collection of measure preserving \( S : X \to X \) such that \( ST = TS \). This group is denoted \( C(T) \). Note that this is the centralizer in the group of measure preserving transformations. In the case that \( X \) is a manifold and \( T \) is a diffeomorphism, \( C(T) \) differs from the centralizer of \( T \) inside the group of diffeomorphisms.

To each invertible measure preserving transformation \( T \in \text{MPT} \), associate a unitary operator \( U_T : L^2([0,1]) \to L^2([0,1]) \) by defining \( U(f) = f \circ T \). In this way \( \text{MPT} \) can be identified with a closed subgroup of the unitary operators on \( L^2([0,1]) \) with respect to the weak operator topology\(^8\) on the

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\(^7\)Recently several authors have adopted the notation \( \text{Aut}(\mu) \) for the same space.

\(^8\)Which coincides with the strong operator topology in this case.
space of unitary transformations. This makes \( \text{MPT} \) into a Polish group. We will call this the \textit{weak topology} on \( \text{MPT} \) (See [10]). Halmos ([10]) showed that the ergodic transformations, which we denote \( \mathcal{E} \), is a dense \( \mathcal{G}_\delta \) set in \( \text{MPT} \). In particular the weak topology makes \( \mathcal{E} \) into a Polish subspace of \( \text{MPT} \).

There is another topology on the collection of measure preserving transformations of \( X \) to \( Y \) for measure spaces \( X \) and \( Y \). If \( S, T : X \to Y \) are measure preserving transformations, the \textit{uniform distance} between \( S \) and \( T \) is defined to be:

\[
d_U(S, T) = \mu\{x : Sx \neq Tx\}.
\]

This topology refines the weak topology and is a complete, but not a separable topology.

### 3.2.2 Diffeomorphisms

Let \( M \) be a \( C^k \)-smooth compact finite dimensional manifold and \( \mu \) be a standard measure on \( M \) determined by a smooth volume element. For each \( k \) there is a Polish topology on the \( k \)-times differentiable homeomorphisms of \( M \), the \( C^k \)-topology. The \( C^\infty \)-topology is the coarsest topology refining the \( C^k \)-topology for each \( k \in \mathbb{N} \). It is also a Polish topology and a sequence of \( C^\infty \)-diffeomorphisms converges in the \( C^\infty \)-topology if and only if it converges in the \( C^k \)-topology for each \( k \in \mathbb{N} \).

It is a classical open problem whether every finite entropy transformation is isomorphic to a measure preserving diffeomorphism of a compact manifold. However it is well-known that the genericity properties of the \( C^k \)-topologies are quite different from the genericity properties of \( \text{MPT} \).

The collection of \( \mu \)-preserving diffeomorphisms forms a closed nowhere dense set in the \( C^k \)-topology on the \( C^k \)-diffeomorphisms, and as such inherits a Polish topology.\(^9\) We will denote this space by \( \text{Diff}^k(M, \mu) \).

Viewing \( M \) as an abstract measure space one can also consider the space of abstract \( \mu \)-preserving transformations on \( M \) with the weak topology. In [3] it is shown that the collection of a.e.-equivalence classes of smooth transformations form a \( \Pi^0_3 \)-set in \( \text{MPT}(M) \), and hence the collection has the Property of Baire.

\(^9\)One can also consider the space of measure preserving homeomorphisms with the \( \| \|_{\infty} \) topology, which behaves in some ways similarly.
3.2.3 Symbolic Systems

Let $\Sigma$ be a countable or finite alphabet endowed with the discrete topology. Then $\Sigma^\mathbb{Z}$ can be given the product topology, which makes it into a separable, totally disconnected space that is compact if $\Sigma$ is finite.

**Notation:** If $u = \langle \sigma_0, \ldots, \sigma_{n-1} \rangle \in \Sigma^{<\infty}$ is a finite sequence of elements of $\Sigma$, then we denote the cylinder set based at $k$ in $\Sigma^\mathbb{Z}$ by writing $\langle u \rangle_k$. Explicitly: $\langle u \rangle_k = \{ f \in \Sigma^\mathbb{Z} : f | [k, k+n) = u \}$. The collection of cylinder sets form a base for the product topology on $\Sigma^\mathbb{Z}$.

Let $u, v$ be finite sequences of elements of $\Sigma$ having length $q$. Given intervals $I$ and $J$ in $\mathbb{Z}$ of length $q$ we can view $u$ and $v$ as functions having domain $I$ and $J$ respectively. We will say that $u$ and $v$ are located at $I$ and $J$. We will say that $u$ is shifted by $k$ relative to $v$ iff $I$ is the shift of the interval $J$ by $k$. We say that $u$ is the $k$-shift of $v$ iff $u$ and $v$ are the same words and $I$ is the shift of the interval $j$ by $k$.

The shift map:

$$ sh : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z} $$

defined by setting $sh(f)(n) = f(n + 1)$ is a homeomorphism. If $\mu$ is a shift-invariant Borel measure then the resulting measure preserving system $(\Sigma^\mathbb{Z}, \mathcal{B}, \mu, sh)$ is called a symbolic system. The closed support of $\mu$ is a shift-invariant closed subset of $\Sigma^\mathbb{Z}$ called a symbolic shift or sub-shift.

Symbolic shifts are often described intrinsically by giving a collection of words that constitute a clopen basis for the support of an invariant measure. Fix a language $\Sigma$, and a sequence of collections of words $\langle W_n : n \in \mathbb{N} \rangle$ with the properties that:

1. for each $n$ all of the words in $W_n$ have the same length $q_n$,
2. each $w \in W_n$ occurs at least once as a subword of every $w' \in W_{n+1}$,
3. there is a summable sequence $\langle \epsilon_n : n \in \mathbb{N} \rangle$ of positive numbers such that for each $n$, every word $w \in W_{n+1}$ can be uniquely parsed into segments

$$ u_0w_0u_1w_1 \ldots w_lu_l+1 $$

such that each $w_i \in W_n$, $u_i \in \Sigma^{<\mathbb{N}}$ and for this parsing

$$ \sum_{i} |u_i| \frac{q_{n+1}}{q_{n+1}} < \epsilon_{n+1}. $$
The segments $u_i$ in condition 1 are called the *spacer* or *boundary* portions of $w$.

**Definition 5** A sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ satisfying properties 1.)-3.) will be called a construction sequence.

Associated with a construction sequence is a symbolic shift defined as follows. Let $\mathcal{K}$ be the collection of $x \in \Sigma^\mathbb{Z}$ such that every finite contiguous subword of $x$ occurs inside some $w \in \mathcal{W}_n$. Then $\mathcal{K}$ is a closed shift-invariant subset of $\Sigma^\mathbb{Z}$ that is compact if $\Sigma$ is finite.

The symbolic shifts built from construction sequences coincide with transformations built by *cut-and-stack* constructions.

**Notation:** For a word $w \in \Sigma^{< \mathbb{N}}$ we will write $|w|$ for the length of $w$.

To unambiguously parse elements of $\mathcal{K}$, construction sequences need to consist of uniquely readable words.

**Definition 6** Let $\Sigma$ be a language and $\mathcal{W}$ be a collection of finite words in $\Sigma$. Then $\mathcal{W}$ is uniquely readable iff whenever $u, v, w \in \mathcal{W}$ and $uv = pws$ then either $p$ or $s$ is the empty word.

Here is a natural set set of measure one for the relevant measures:

**Definition 7** Suppose that $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ is a construction sequence for a symbolic system $\mathcal{K}$ with each $\mathcal{W}_n$ uniquely readable. Let $S$ be the collection $x \in \mathcal{K}$ such that there are sequences of natural numbers $\langle a_m : m \in \mathbb{N} \rangle$, $\langle b_m : m \in \mathbb{N} \rangle$ going to infinity such that for all $m$ there is an $n, x \upharpoonright [-a_m, b_m) \in \mathcal{W}_n$.

Note that $S$ is a dense shift-invariant $G_\delta$ set.

**Lemma 8** [4] Fix a construction sequence $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ for a symbolic system $\mathcal{K}$ in a finite language. Then:

1. $\mathcal{K}$ is the smallest shift-invariant closed subset of $\Sigma^\mathbb{Z}$ such that for all $n$, and $w \in \mathcal{W}_n$, $\mathcal{K}$ has non-empty intersection with the basic open interval $\langle w \rangle \subset \Sigma^\mathbb{Z}$.

2. Suppose that there is a unique invariant measure $\nu$ on $S \subseteq \mathcal{K}$, then $\nu$ is ergodic.

3. (See [5]) If $\nu$ is an invariant measure on $\mathcal{K}$ concentrating on $S$, then for $\nu$-almost every $s$ there is an $N$ for all $n > N$, there are $a_n \leq 0 < b_n$ such that $s \upharpoonright [a_n, b_n) \in \mathcal{W}_n$. 

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Example 9 Let \( \langle W_n : n \in \mathbb{N} \rangle \) be a construction sequence. Then \( \langle W_n : n \in \mathbb{N} \rangle \) is uniform if there is a summable sequence of positive numbers \( \langle \epsilon_n : n \in \mathbb{N} \rangle \) and \( \langle d_n : n \in \mathbb{N} \rangle \), where \( d_n : W_n \to (0,1) \) such that for each \( n \) all words \( w \in W_n \) and \( w' \in W_{n+1} \) if \( f(w,w') \) is the number of \( i \) such that \( w_i = w'_i \)

\[
\left| \frac{f(w,w')}{q_{n+1}/q_n} - d_n(w) \right| < \frac{\epsilon_{n+1}}{q_n}, \tag{3}
\]

It is shown in [4] that uniform construction sequences are uniquely ergodic. A special case of uniformity is strong uniformity: when each \( w \in W_n \) occurs exactly the same number of times in each \( w' \in W_{n+1} \). This property holds for the circular systems considered in [4] and that are used for the proof of the main theorem of this paper (Theorem 2).

3.2.4 Locations

Let \( \langle W_n : n \in \mathbb{N} \rangle \) be a uniquely readable construction sequence and let \( \nu \) be a shift invariant measure on \( S \). For \( s \in S \) and each \( n \), either \( s(0) \) lies in a well-defined subword of \( s \) belonging to \( W_n \) or in a spacer of a subword of \( s \) belonging to some \( W_{n+k} \). By Lemma 8 for \( \nu \)-almost all \( x \) and for all large enough \( n \) there is a unique \( k \) with \( 0 \leq k < q_n \) such that \( s \upharpoonright [-k,q_n-k) \in W_n \).

Definition 10 Let \( s \in S \) and suppose that for some \( 0 \leq k < q_n, s \upharpoonright [-k,q_n-k) \in W_n \). Define \( r_n(s) \) to be the unique \( k \) with with this property. We will call the interval \( [-k,q_n-k) \) the principal \( n \)-block of \( s \), and \( s \upharpoonright [-k,q_n-k) \) its principal \( n \)-subword. The sequence of \( r_n \)'s will be called the location sequence of \( s \).

Thus \( r_n(s) = k \) is saying that \( s(0) \) is the \( k \)th symbol in the principal \( n \)-subword of \( s \) containing 0. We can view the principal \( n \)-subword of \( s \) as being located on an interval \( I \) inside the principal \( n+1 \)-subword. Counting from the beginning of the principal \( n+1 \)-subword, the \( r_{n+1}(s) \) position is located at the \( r_n(s) \) position in \( I \).

Remark 11 It follows immediately from the definitions that if \( r_n(s) \) is well defined and \( n \leq m \), the \( r_m(s) \)th position of the word occurring in the principal \( m \)-block of \( s \) is in the \( r_n(s) \)th position inside the principal \( n \)-block of \( s \).

Lemma 12 [5] Suppose that \( s,s' \in S \) and \( \langle r_n(s) : n \geq N \rangle = \langle r_n(s') : n \geq N \rangle \) and for all \( n \geq N, s \) and \( s' \) have the same principal \( n \)-subwords. Then \( s = s' \).
Thus an element of $s$ is determined by knowing any tail of the sequence $\langle r_n(s) : n \geq N \rangle$ together with a tail of the principal subwords of $s$.

**Remark 13** Here are some consequences of Lemma 12:

1. Given a sequence $\langle u_n : M \leq n \rangle$ with $u_n \in W_n$, if we specify which occurrence of $u_n$ in $u_{n+1}$ is the principal occurrence, then $\langle u_n : M \leq n \rangle$ determines an $s \in K$ completely up to a shift $k$ with $|k| \leq q_M$.

2. A sequence $\langle r_n : N \leq n \rangle$ and sequence of words $w_n \in W_n$ comes from an infinite word $s \in S$ if both $r_n$ and $q_n - r_n$ go to infinity and that the $r_{n+1}$ position in $w_{n+1}$ is in the $r_n$ position in a subword of $w_{n+1}$ identical to $w_n$.

Caveat: just because $\langle r_n : N \leq n \rangle$ is the location sequence of some $s \in S$ and $\langle w_n : N \leq n \rangle$ is the sequence of principal subwords of some $s' \in S$, it does not follow that there is an $x \in S$ with location sequence $\langle r_n : N \leq n \rangle$ and sequence of subwords $\langle w_n : N \leq n \rangle$.

3. If $x, y \in S$ have the same principal $n$-subwords and $r_n(y) = r_n(x) + 1$ for all large enough $n$, then $y = sh(x)$.

### 3.2.5 A note on inverses of symbolic shifts

We define operators we label rev(), and apply them in several contexts

**Definition 14** If $x$ is in $K$, define the reverse of $x$ by setting $\text{rev}(x)(k) = x(-k)$. For $A \subseteq K$, define:

$$\text{rev}(A) = \{ \text{rev}(x) : x \in A \}.$$ 

If $w$ is a word, let $\text{rev}(w)$ to be the reverse of $w$. Explicitly viewing $w$ as sitting on an interval, we take $\text{rev}(w)$ to sit on the same interval. Thus if $w : [a_n, b_n) \rightarrow \Sigma$ is the word then $\text{rev}(w) : [a_n, b_n) \rightarrow \Sigma$ and $\text{rev}(w)(i) = w((a_n + b_n) - (i + 1))$. If $W$ is a collection of words, $\text{rev}(W)$ is the collection of reverses of the words in $W$.

If $(K, sh)$ is an arbitrary symbolic shift then its inverse is $(K, sh^{-1})$. It will be convenient to have all of the shifts go in the same direction, thus:

**Proposition 15** The map $\phi$ sending $x$ to $\text{rev}(x)$ is a canonical isomorphism between $(K, sh^{-1})$ and $(\text{rev}(K), sh)$.

The notation $L^{-1}$ stands for the system $(L, sh^{-1})$ and $\text{rev}(L)$ for the system $(\text{rev}(L), sh)$. 


3.3 Generic Points

Let $T$ be a measure preserving transformation from $(X, \mu)$ to $(X, \mu)$. Then a point $x \in X$ is generic for $T$ if and only if for all $f \in C(X)$,

$$\lim_{N \to \infty} \left( \frac{1}{N} \sum_{0}^{N-1} f(T^n(x)) \right) = \int_X f(x) d\mu(x).$$ (4)

The Ergodic Theorem tells us that for a given $f$ and ergodic $T$ equation 4 holds for a set of $\mu$-measure one. Intersecting over a countable dense set gives a set of $\mu$-measure one of generic points. For symbolic systems $K \subseteq \Sigma^\mathbb{Z}$ the generic points $x$ as those $x$ such that the $\mu$-measure of all basic open intervals $\langle u \rangle_0$ is equal to the density of $k$ such that $u$ occurs in $x$ at $k$.

3.4 Stationary Codes and $\bar{d}$-Distance

In this section we briefly review a standard idea, that of a stationary code. A reader unfamiliar with this material who is interested in the proofs of the facts cited here should see [17]

**Definition 16** Suppose that $\Sigma$ is a countable language. A code of length $2N + 1$ is a function $\Lambda : \Sigma^{-N,N} \to \Sigma$ (where $[-N, N]$ is the interval of integers starting at $-N$ and ending at $N$.)

Given a code $\Lambda$ and an $s \in \Sigma^\mathbb{Z}$ the stationary code determined by $\Lambda$ is $\bar{\Lambda}(s)$ where:

$$\bar{\Lambda}(s)(k) = \Lambda(s \upharpoonright [k - N, k + N]).$$

Let $(\Sigma^\mathbb{Z}, \mathcal{B}, \nu, sh)$ be a symbolic system. Given two codes $\Lambda_0$ and $\Lambda_1$ (not necessarily of the same length), define $D = \{ s \in \Sigma^\mathbb{Z} : \bar{\Lambda}_0(s)(0) \neq \bar{\Lambda}_1(s)(0) \}$ and $d(\Lambda_0, \Lambda_1) = \nu(D)$. Then $d$ is a semi-metric on the collection of codes. The following is a consequence of the Borel-Cantelli lemma.

**Lemma 17** Let Suppose that $\Lambda_i : i \in \mathbb{N}$ is a sequence of codes such that $\sum_i d(\Lambda_i, \Lambda_{i+1}) < \infty$. Then there is a shift-invariant Borel map $S : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z}$ such that for $\nu$-almost all $s$, $\lim_{i \to \infty} \bar{\Lambda}_i(s) = S(s)$.

A shift-invariant Borel map $S : \Sigma^\mathbb{Z} \to \Sigma^\mathbb{Z}$, determines a factor $(\Sigma^\mathbb{Z}, \mathcal{B}, \mu, sh)$ of $(\Sigma^\mathbb{Z}, \mathcal{B}, \nu, sh)$ by setting $\mu = S^* \nu$ (i.e. $\mu(A) = \nu \circ S^{-1}(A)$). Hence a convergent sequence of stationary codes determines a factor of $(\Sigma^\mathbb{Z}, \mathcal{B}, \nu, sh)$.

Let $\Lambda_0$ and $\Lambda_1$ be codes. Define $d(\Lambda_0(s), \Lambda_1(s))$ to be

$$\lim_{N \to \infty} \frac{|\{ k \in [-N, N] : \bar{\Lambda}_0(s)(k) \neq \bar{\Lambda}_1(s)(k) \}|}{2N + 1}$$

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More generally define the $\bar{d}$ metric on $\Sigma^{[a,b]}$ by setting

$$\bar{d}_{[a,b]}(x, y) = \frac{|\{k \in [a, b) : x(k) \neq y(k)\}|}{b - a}.$$ 

For $x, y \in \Sigma^\mathbb{Z}$, we set

$$\bar{d}(x, y) = \lim_{N \to \infty} \bar{d}_{[-N,N]}(x \upharpoonright [-N,N], y \upharpoonright [-N,N])$$

provided this limit exists.

To compute distances between codes we will use the following application of the ergodic theorem.

**Lemma 18** Let $\Lambda_0$ and $\Lambda_1$ be codes. Then for almost all $s \in S$:

$$d(\Lambda_0, \Lambda_1) = \bar{d}(\bar{\Lambda}_0(s), \bar{\Lambda}_1(s))$$

The next proposition is used to study alleged isomorphisms between measure preserving transformations. We again refer the reader to [17] for a proof.

**Proposition 19** Suppose that $\mathbb{K}$ and $\mathbb{L}$ are symbolic systems and $\phi : \mathbb{K} \to \mathbb{L}$ is a factor map. Let $\epsilon > 0$. Then there is a code $\Lambda$ such that for almost all $s \in \mathbb{K}$,

$$\bar{d}(\bar{\Lambda}(s), \phi(s)) < \epsilon.$$  \hspace{1cm} (5)

To show that equation 5 cannot hold (and hence show that $\mathbb{L}$ is not a factor of $\mathbb{K}$) we will want to view $\bar{\Lambda}(s)$ as limits of $\Lambda$-images of large blocks of the form $s \upharpoonright [a, b]$ with $a < 0 < b$. There is an ambiguity in doing this: if the code $\Lambda$ has length $2N + 1$ it does not make sense to apply it to $s \upharpoonright [k - N, k + N]$ for $k \in [a, a + 2N]$ or $k \in [b - 2N, b]$. However if $b - a$ is quite large with respect to $N$, then filling in the values for $\Lambda(s \upharpoonright [k - N, k + N])$ arbitrarily as $k$ ranges over these initial and final intervals makes a negligible difference to the $\bar{d}$-distances of the result. In particular if $\bar{d}(\bar{\Lambda}(s), \phi(s)) < \epsilon$ then for all large enough $a, b \in \mathbb{N}$, we have

$$\bar{d}_{[-a,b]}(\bar{\Lambda}(s \upharpoonright [-a, b]), \phi(s) \upharpoonright [-a, b]) < \epsilon,$$

no matter how we fill in the ambiguous portion.

The general phenomenon of ambiguity or disagreement at the beginning and end of large intervals is referred to by the phrase *end effects*. Because the end effects are usually negligible on large intervals we will often neglect them when computing $\bar{d}$ distances.

The next proposition is standard:
Proposition 20 Suppose that \((\Sigma^Z, B, \nu, \text{sh})\) is an ergodic symbolic system and \(\langle T_n : n \in \mathbb{N} \rangle\) is a sequence of functions from \(\Sigma^Z \to \Sigma^Z\) that commute with the shift. Then the following are equivalent:

1. The sequence \(\langle T_n \rangle\) converges to \(S\) in the weak topology.
2. \(\nu(\{s : T_n(s)(0) \neq S(s)(0)\}) \to 0\).
3. For \(\nu\)-almost all \(s\), \(\bar{d}(T_n(s), S(s)) \to 0\).
4. For some \(\nu\)-generic \(s\), for all \(\gamma > 0\) we can find an \(n\) such that for all large enough \(a, b\), the distance \(\bar{d}(T_n(s) \upharpoonright [-a, b], S(s) \upharpoonright [-a, b]) < \gamma\).

We finish with a remark that we will use in several places:

Remark 21 If \(w_1\) and \(w_2\) are words in a language \(\Sigma\) defined on an interval \(I\) and \(J \subset I\) with \(\frac{|J|}{|I|} \geq \delta\), then \(\bar{d}_I(w_1, w_2) \geq \delta \bar{d}_J(w_1, w_2)\).

3.5 Descriptive Set Theory Basics

Let \(X\) and \(Y\) be Polish spaces and \(A \subseteq X, B \subseteq Y\). A function \(f : X \to Y\) reduces \(A\) to \(B\) if and only if for all \(x \in X\):

\[ x \in A \text{ if and only if } f(x) \in B. \]

For this definition to have content there must be some definability restriction on \(f\). The relevant restrictions for this paper are \(f\) is a Borel function (i.e. the inverse image of an open set is Borel) and \(f\) is a continuous function (i.e. the inverse image of an open set is open). If \(B\) is Borel and \(f\) is a Borel reduction, then \(A\) is clearly Borel. Taking the contrapositive, if \(A\) is not Borel then \(B\) is not. If \(A\) is Borel (resp. continuously) reducible to \(B\) we will write \(A \preceq_B B\) (resp. \(A \preceq_c B\)). Both \(\preceq_B\) and \(\preceq_c\) are clearly pre-partial-orderings.\(^{10}\)

If \(S\) is a collection of sets and \(B \in S\), then \(B\) is complete for Borel (resp. continuous) reductions if and only if every \(A \in S\) is Borel (resp. continuously) reducible to \(B\). Being complete is interpreted as being at least as complicated as each set in \(S\).

For this to be useful there must be examples of sets that are not Borel. If \(X\) is a Polish space and \(B \subseteq X\), then \(B\) is analytic (\(\Sigma^1\)) if and only if it the continuous image of a Borel subset of a Polish space. This is equivalent

\(^{10}\)The reader should be aware that this is a different notion than the notion of a reduction of equivalence relations.
to there there being a Polish space $Y$ and a Borel set $C \subseteq X \times Y$ such that $B$ is the projection to the $X$-axis of $C$.

Correcting a famous mistake of Lebesgue, Suslin proved that there are analytic sets that are not Borel. This paper uses a canonical example of such a set. Let $\langle \sigma_n : n \in \mathbb{N} \rangle$ be an enumeration of $\mathbb{N}^\mathbb{N}$, the finite sequences of natural numbers. Using this enumeration subsets $S \subseteq \mathbb{N}^\mathbb{N}$ can be identified with functions $\chi_S : \mathbb{N} \to \{0, 1\}$.

A tree is a set $T \subseteq \mathbb{N}^\mathbb{N}$ such that if $\tau \in T$ and $\sigma$ is an initial segment of $\tau$, then $\sigma \in T$. Then the set $\{\chi_T : T$ is a tree$\}$ is a closed subset of $\{0, 1\}^\mathbb{N}$, hence a Polish space with the induced topology. We call the resulting space $\Trees$.

Because the topology on the space of trees is the “finite information” topology, inherited from the product topology on $\{0, 1\}^\mathbb{N}$, the following characterizes of continuous maps defined on $\Trees$.

**Proposition 22** Let $Y$ be a topological space and $f : \Trees \to Y$. Then $f$ is continuous if and only if for all open $O \subseteq Y$ and all $T$ with $f(T) \in O$ there is an $M \in \mathbb{N}$ for all $T' \in \Trees$:

$$
\text{if } T \cap \{\sigma_n : n \leq M\} = T' \cap \{\sigma_n : n \leq M\}, \text{ then } f(T') \in O.
$$

An infinite branch through $T$ is a function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}, f \upharpoonright \{0, 1, 2, \ldots n-1\} \in T$. A tree $T$ is *ill-founded* if and only if it has an infinite branch.

The following theorem is classical; proofs can be found in [14], [15] and many other places.

**Fact 23** Let $\Trees$ be the space of trees. Then:

1. The collection of ill-founded trees is a complete analytic subset of $\Trees$.

2. The collection of trees that have at least two distinct infinite branches is a complete analytic subset of $\Trees$.

The main results of this paper (Theorem 2) are proved by reducing the sets mentioned in Theorem 23 to conjugate pairs of diffeomorphisms and concluding that the sets of conjugate pairs is complete analytic—so not Borel.
4 Odometer and Circular Systems

Two types of symbolic shifts play central roles for the proofs of the main theorem, the odometer based and the circular systems. Most of the material in this section appears in [5] in more detail and is reviewed here without proof.

4.1 Odometer Based Systems

We now define the class of Odometer Based Systems. In a sequel to this paper, we prove that these are exactly the finite entropy transformations that have non-trivial odometer factors. We recall the definition of an odometer transformation. Let $\langle k_n : n \in \mathbb{N} \rangle$ be a sequence of natural numbers greater than or equal to 2. Let

$$O = \prod_{n=0}^{\infty} \mathbb{Z}/k_n \mathbb{Z}$$

be the $\langle k_n \rangle$-adic integers. Then $O$ naturally has a compact abelian group structure and hence carries a Haar measure $\mu$. $O$ becomes a measure preserving system $O$ by defining $T : O \rightarrow O$ to be addition by 1 in the $\langle k_n \rangle$-adic integers. Concretely, this is the map that “adds one to $\mathbb{Z}/k_0 \mathbb{Z}$ and carries right”. Then $T$ is an invertible transformation that preserves the Haar measure $\mu$ on $O$. Let $K_n = k_0 k_1 k_2 \ldots k_{n-1}$.

The following results are standard:

**Lemma 24** Let $O$ be an odometer system. Then:

1. $O$ is ergodic.
2. The map $x \mapsto -x$ is an isomorphism between $(O, \mathcal{B}, \mu, T)$ and $(O, \mathcal{B}, \mu, T^{-1})$.
3. Odometer maps are transformations with discrete spectrum and the eigenvalues of the associated linear operator are the $K_n$ roots of unity $(n > 0)$.

Any natural number $a$ can be uniquely written as:

$$a = a_0 + a_1 k_0 + a_2 (k_0 k_1) + \ldots + a_j (k_0 k_1 k_2 \ldots k_{j-1})$$

for some sequence of natural numbers $a_0, a_1, \ldots a_j$ with $0 \leq a_j < k_j$.

**Lemma 25** Suppose that $\langle r_n : n \in \mathbb{N} \rangle$ is a sequence of natural numbers with $0 \leq r_n < k_0 k_1 \ldots k_{n-1}$ and $r_n \equiv r_{n+1} \mod (K_n)$. Then there is a unique element $x \in O$ such that $r_n = x(0) + x(1) k_0 + \ldots + x(n) (k_0 k_1 \ldots k_{n-1})$ for each $n$. 

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We now define the collection of symbolic systems that have odometer maps as their timing mechanism. This timing mechanism can be used to parse typical elements of the symbolic system.

**Definition 26** Let \( \{W_n : n \in \mathbb{N}\} \) be a uniquely readable construction sequence with the properties that \( W_0 = \Sigma \) and for all \( n, W_{n+1} \subseteq (W_n)^{k_n} \) for some \( k_n \). The associated symbolic system will be called an odometer based system.

Thus odometer based systems are those built from construction sequences \( \{W_n : n \in \mathbb{N}\} \) such that the words in \( W_{n+1} \) are concatenations of words in \( W_n \) of a fixed length \( k_n \). The words in \( W_n \) all have length \( K_n \) and the words \( u_i \) in equation 1 are all the empty words.

Equivalently, an odometer based transformation is one that can be built by a cut-and-stack construction using no spacers. An easy consequence of the definition is that for odometer based systems, for all \( s \in S \) and for all \( n \in \mathbb{N} \), \( r_n(s) \) exists.

The next lemma justifies the terminology.

**Lemma 27** Let \( \mathbb{K} \) be an odometer based system with each \( W_{n+1} \subseteq (W_n)^{k_n} \). Then there is a canonical factor map

\[ \pi : S \rightarrow O, \]

where \( O \) is the odometer system determined by \( \{k_n : n \in \mathbb{N}\} \).

\[
\vdash \text{For each } s \in S, \text{ for all } n, r_n(s) \text{ is defined and both } r_n \text{ and } k_n - r_n \text{ go to infinity. By Lemma 25, the sequence } \{r_n(s) : n \in \mathbb{N}\} \text{ defines a unique element } \pi(s) \text{ in } O. \text{ It is easily checked that } \pi \text{ intertwines } sh \text{ and } T. \]

Heuristically, the odometer transformation \( O \) parses the sequences \( s \) in \( S \subseteq \mathbb{K} \) by indicating where the words constituting \( s \) begin and end. Shifting \( s \) by one unit shifts this parcing by one. We can understand elements of \( S \) as being an element of the odometer with words in \( W_n \) filled in inductively.

The following remark is useful when studying the canonical factor of the inverse of an odometer based system.

**Remark 28** If \( \pi : L \rightarrow O \) is the canonical factor map, then the function \( \pi : L \rightarrow O \) is also factor map from \( (L, sh^{-1}) \) to \( O^{-1} \) (i.e. \( O \) with the operation “\( -1 \)”). If \( \{W_n : n \in \mathbb{N}\} \) is the construction sequence for \( L \), then

\[ S \] is defined in Definition 7.
\langle \text{rev}(W_n) : n \in \mathbb{N} \rangle \text{ is a construction sequence for } \text{rev}(L). \text{ If } \phi : L^{-1} \to \text{rev}(L) \text{ is the canonical isomorphism given by Proposition 15, then Lemma 24 tells us that the projection of } \phi \text{ to a map } \phi^x : O \to O \text{ is given by } x \mapsto -x.

The following is proved in [5]:

**Proposition 29** Let \( K \) be an odometer based system and suppose that \( \nu \) is a shift invariant measure. Then \( \nu \) concentrates on \( S \).

### 4.2 Circular Systems

We now define circular systems. In [4] it is shown that the circular systems give symbolic characterizations of the smooth diffeomorphisms defined by the Anosov-Katok method of conjugacies.

These systems are called circular because they are related to the behavior of rotations by a convergent sequence of rationals \( \alpha_n = p_n/q_n \). The rational rotation by \( p/q \) permutes the \( 1/q \) intervals of the circle cyclically in a manner that the interval \([i/q, (i+1)/q)\) occurs in position \( j_i \equiv p^{-1}i \mod q \).

The operation \( C \) which we are about to describe models the relationship between rotations by \( p/q \) and \( p'/q' \) when \( q' \) is very close to \( q \).

Let \( k, l, p, q \) be positive natural numbers with \( p < q \) relatively prime. Set

\[
j_i \equiv q \cdot (p)^{-1}i
\]

with \( j_i < q \). It is easy to verify that:

\[
q - j_i = j_q - i
\]

Let \( \Sigma \) be a non-empty set and \( w_0, \ldots, w_{k-1} \) be words in \( \Sigma \cup \{b, e\} \) (assume that neither \( b \) nor \( e \) belong to \( \Sigma \)). Define:

\[
C(w_0, w_1, w_2, \ldots, w_{k-1}) = \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i}).
\]

**Remark 30**

- Suppose that each \( w_i \) has length \( q \), then the length of \( C(w_0, w_1, \ldots, w_{k-1}) \) is \( k l q^2 \).

---

\( ^{12} \)We assume that \( p \) and \( q \) are relatively prime and the exponent \(-1\) is the multiplicative inverse of \( p \mod q \).

\( ^{13} \)We use \( \prod \) and powers for repeated concatenation of words.
• For each occurrence of an $e$ in $C(w_0, \ldots w_{k-1})$ there is an occurrence of $b$ to the left of it.

• Suppose that $n < m$ and $b$ occurs at $n$ and $e$ occurs at $m$ and neither occurrence is in a $w_i$. Then there must be some $w_i$ occurring between $n$ and $m$.

• Words constructed with $C$ are uniquely readable.

The $C$ operation is used to build a collection of symbolic shifts. Circular systems will be defined using a sequence of natural number parameters $k_n$ and $l_n$ that is fundamental to the version of the Anosov-Katok construction presented in [13].

Fix an arbitrary sequence of positive natural numbers $\langle k_n : n \in \mathbb{N} \rangle$. Let $\langle l_n : n \in \mathbb{N} \rangle$ be an increasing sequence of natural numbers such that

**Numerical Requirement 1** $l_0 > 20$ and $1/l_{n-1} > \sum_{k=n}^{\infty} 1/l_k$.

From the $k_n$ and $l_n$ we define sequences of numbers: $\langle p_n, q_n, \alpha_n : n \in \mathbb{N} \rangle$. Begin by letting $p_0 = 0$ and $q_0 = 1$ and inductively set

$$q_{n+1} = k_n l_n q_n^2$$

(9)

(thus $q_1 = k_0 l_0$) and take

$$p_{n+1} = p_n q_n k_n l_n + 1.$$  

(10)

Then clearly $p_{n+1}$ is relatively prime to $q_{n+1}$.$^{14}$

Setting $\alpha_n = p_n / q_n$, then it is easy to check that there is an irrational $\alpha$ such that the sequence $\alpha_n$ converges rapidly to $\alpha$.

**Definition 31** A sequence of integers $\langle k_n, l_n : n \in \mathbb{N} \rangle$ such that $k_n \geq 2$, $\sum 1/l_n < \infty$ will be called a circular coefficient sequence.

Let $\Sigma$ be a non-empty finite or countable alphabet. Build collections of words $W_n$ in $\Sigma \cup \{b, e\}$ by induction as follows:

• Fix a circular coefficient sequence $\langle k_n, l_n : n \in \mathbb{N} \rangle$.

• Set $W_0 = \Sigma$.

---

$^{14}$ $p_n$ and $q_n$ being relatively prime for $n \geq 1$, allows us to define the integer $j_i$ in equation 6. For $q_0 = 1$, $\mathbb{Z}/q_0\mathbb{Z}$ has one element, $[0]$, so we set $p_0^{-1} = p_0 = 0$. 

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• Having built \( W_n \) choose a set \( P_{n+1} \subseteq (W_n)^k \) and form \( W_{n+1} \) by taking all words of the form \( C(w_0, w_1 \ldots w_{k_n-1}) \) with \( (w_0, \ldots w_{k_n-1}) \in P_{n+1} \).\(^{15}\)

We will call the elements of \( P_{n+1} \) prewords. The \( C \) operator automatically creates uniquely readable words, however we will need a stronger unique readability assumption for our definition of circular systems.

**Strong Unique Readability Assumption:** Let \( n \in \mathbb{N} \), and view \( W_n \) as a collection \( \Lambda_n \) of letters. Then each element of \( P_{n+1} \) can be viewed as a word with letters in \( \Lambda_n \). In the alphabet \( \Lambda_n \), each \( w \in P_{n+1} \) is uniquely readable.

**Definition 32** A construction sequence \( \langle W_n : n \in \mathbb{N} \rangle \) will be called circular if it is built in this manner using the \( C \)-operators, a circular coefficient sequence and each \( P_{n+1} \) satisfies the strong unique readability assumption.

**Definition 33** A symbolic shift \( K \) built from a circular construction sequence will be called a circular system.

**Notation:** we will often write \( K^c \) and \( \langle W_n^c : n \in \mathbb{N} \rangle \) to emphasize that we are building circular systems and circular construction sequences. Circular words will often be denoted \( w^c \) for emphasis.

The following definition helps understand the structure of words constructed by \( C \).

**Definition 34** Suppose that \( w = C(w_0, w_1 \ldots w_{k-1}) \). Then \( w \) consists of blocks of \( w_i \) repeated \( l - 1 \) times, together with some \( b \)'s and \( e \)'s that are not in the \( w_i \)'s. The interior of \( w \) is the portion of \( w \) in the \( w_i \)'s. The remainder of \( w \) consists of blocks of the form \( b^{q - j_i} \) and \( e^{j_i} \). We call this portion the boundary of \( w \).

In a block of the form \( w_j^{l-1} \) the first and last occurrences of \( w_j \) will be called the boundary occurrences of the block \( w_j^{l-1} \). The other occurrences will be the interior occurrences.

While the boundary consists of sections of \( w \) made up of \( b \)'s and \( e \)'s, not all \( b \)'s and \( e \)'s occurring in \( w \) are in the boundary, as they may be part of a power \( w_i^{l-1} \).

The boundary of \( w \) constitutes a small portion of the word:

\(^{15}\)Passing from \( W_n \) to \( W_{n+1} \), use \( C \) with parameters \( k = k_n, l = l_n, p = p_n \) and \( q = q_n \) and take \( j_i = (p_n)^{-1}i \mod q_n \). By Remark 30, the length of each of the words in \( W_{n+1} \) is \( q_{n+1} \).
Lemma 35  The proportion of the word $w$ written in equation 8 that belongs to its boundary is $1/l$. Moreover the proportion of the word that is within $q$ letters of boundary of $w$ is $3/l$.

\[\text{⊢}\]

The lengths of the words is $klq^2$. The boundary portions are $q * k * q$ long. The number of letters within $q$ letters of the boundary is $q * k * 3 * q$.

Remark 36 To foreshadow future arguments we illustrate how Lemma 35 can be used with Remark 21. Let $v_0, . . . v_{k-1}$ and $w_0, . . . w_{k-1}$ be sequences of words all of the same length in a language $\Sigma$. The boundary portions of $C(v_0, . . . v_{k-1})$ and $C(w_0, . . . w_{k-1})$ occur in the same positions and by Lemma 35 have proportion $1/l$ of the length. Since all of the $v_i$'s and $w_i$'s have the same length and the same multiplicity in the circular words we see:

\[
\bar{d}(C(v_0, . . . v_{k-1}), C(w_0, . . . w_{k-1})) \geq (1 - 1/l) \bar{d}(v_0v_1v_2 . . . v_{k-1}, w_0w_1 . . . w_{k-1})
\]

where $v_0v_1v_2 . . . v_{k-1}$ and $w_0w_1 . . . w_{k-1}$ are the concatenations of the various words.\(^\text{16}\)

For proofs of the next lemma see [4] (Lemma 20) and [5].

Lemma 37  Let $\mathbb{K}$ be a circular system and $\nu$ be a shift-invariant measure on $\mathbb{K}$. Then the following are equivalent:

1. $\nu$ has no atoms.

2. $\nu$ concentrates on the collection of $s \in \mathbb{K}$ such that $\{i : s(i) \notin \{b,e\}\}$ is unbounded in both $\mathbb{Z}^-$ and $\mathbb{Z}^+$.

3. $\nu$ concentrates on $S$.

4. If $\mathbb{K}$ is a uniform circular system (Example 9), then there is a unique invariant measure concentrating on $S$.

Moreover there are only two ergodic invariant measures with atoms: the one concentrating on the constant sequence $\vec{b}$ and the one concentrating on $\vec{e}$.

Remark 38 If $\mathbb{K}^c$ is circular and $s \in \mathbb{K}^c$ has a principal $n$-subword and $m > n$, then $s$ has a principal $m$-subword.

\(^{16}\)Equality holds, a fact we won’t use.
4.3 An Explicit Description of \((\mathbb{K}^c)^{-1}\).

The symbolic system \(\mathbb{K}^c\) is built by an operation \(C\) applied to collections of words. The system \((\mathbb{K}^c)^{-1}\) is built by a similar operation applied to the reverse collections of words. In analogy to equation 8, we define \(C^r\) as follows:

**Definition 39** Suppose that \(w_0, w_1, \ldots, w_{k-1}\) are words in a language \(\Sigma\). Given coefficients \(p, q, k, l\) with \(p\) and \(q\) relatively prime, let \(j_i \equiv q (p^{-1})i\) with \(0 \leq j_i < q\). Define

\[
C^r(w_0, w_1, w_2, \ldots, w_{k-1}) = q^{-1} \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (e^{q-j_i} (w_{k-j-1}^{l-1} b^{j_i+1}).
\]

(11)

From equation 8, a \(w \in \mathcal{W}_{n+1}\) is of the form \(C(w_0, \ldots, w_{k_{n-1}})\):

\[
w = \prod_{i=0}^{q-1} \prod_{j=0}^{k-1} (b^{q-j_i} w_j^{l-1} e^{j_i}).
\]

(12)

where \(q = q_n, k = k_n, l = l_n\) and \(j_i \equiv q_n (p_n)^{-1}i\) with \(0 \leq j_i < q_n\). By examining this formula we see that

\[
\text{rev}(w) = \prod_{i=1}^{q} \prod_{j=1}^{k} e^{q_j-i} \text{rev}(w_{k-j})^{l-1} b^{jq-i}.
\]

Applying the identity in formula 7, we see that this can be rewritten as

\[
\text{rev}(w) = \prod_{i=1}^{q} \prod_{j=1}^{k} (e^{q-j_i} \text{rev}(w_{k-j})^{l-1} b^{j_i}).
\]

(13)

Thus

\[
\text{rev}(w) = C^r(\text{rev}(w_0), \text{rev}(w_1), \ldots, \text{rev}(w_{k-1})).
\]

(14)

In particular if \(\langle \mathcal{W}_n^c : n \in \mathbb{N} \rangle\) is a construction sequence of a circular system \(\mathbb{K}^c\), then \(\text{rev}(\mathcal{W}_n^c)\) is the collection:

\[
\{C^r(\text{rev}(w_0), \text{rev}(w_1), \ldots, \text{rev}(w_{k_{n-1}})) : w_0 w_1 \ldots w_{k_{n-1}} \in P_n\}
\]

and is a construction sequence of \(\text{rev}(\mathbb{K}^c)\).

\[\text{We take } j_q = 0.\]
4.4 Understanding the Words

The words used to form circular transformations have quite specific combinatorial properties. Fix a sequence \( \langle W_n^c : n \in \mathbb{N} \rangle \) defining a circular system. Each \( u \in W_{n+1}^c \) has three subcales:

**Subscale 0**, the scale of the individual powers of \( w \in W_n^c \) of the form \( w^{l-1} \); We call each such occurrence of a \( w^{l-1} \) a 0-subsection.

**Subscale 1**, the scale of each term in the product \( \prod_{j=0}^{k-1}(b^{q-j_i}w_j^{l-1}e^{j_i}) \) that has the form \( (b^{q-j_i}w_j^{l-1}e^{j_i}) \); We call these terms 1-subsections.

**Subscale 2**, the scale of each term of \( \prod_{i=0}^{q-1} \prod_{j=0}^{k-1}(b^{q-j_i}w_j^{l-1}e^{j_i}) \) that has the form \( \prod_{j=0}^{k-1}(b^{q-j_i}w_j^{l-1}e^{j_i}) \); We call these terms 2-subsections.

**Summary**

| Whole Word: | \( \prod_{i=0}^{q-1} \prod_{j=0}^{k-1}(b^{q-j_i}w_j^{l-1}e^{j_i}) \) |
|-------------|-------------------------------------------------|
| 2-subsection: | \( \prod_{j=0}^{k-1}(b^{q-j_i}w_j^{l-1}e^{j_i}) \) |
| 1-subsection: | \( (b^{q-j_i}w_j^{l-1}e^{j_i}) \) |
| 0-subsection: | \( w_j^{l-1} \) |

For \( m \leq n \), we will discuss “\( m \)-subwords” of a word \( w \). These will be subwords that lie in \( W_m^c \), the \( m \)th stage of the construction sequence. We will use “\( m \)-block” to mean the location of the \( m \)-subword.

**Lemma 40** Let \( w = C(w_0, \ldots w_{k_n-1}) \) for some \( n \) and \( q = q_n, k = k_n, l = l_n \). View \( w : \{0, 1, 2, \ldots, klq^2 - 1\} \to \Sigma \cup \{b, e\} \).

1. If \( m_0 \) and \( m_1 \) are such that \( w(m_0) \) and \( w(m_1) \) are at the beginning of \( n \)-subwords in the same 2-subsection, then \( m_0 \equiv_q m_1 \).

2. If \( m_0 \) and \( m_1 \) are such that \( w(m_0) \) is the beginning of an \( n \)-subword occurring in a 2-subsection \( \prod_{j=0}^{k-1}(b^{q-j_i}w_j^{l-1}e^{j_i}) \) and \( w(m_1) \) is the beginning of an \( n \)-subword occurring in the next 2-subsection \( \prod_{j=0}^{k-1}(b^{q-j_i+1}w_j^{l-1}e^{j_i+1}) \) then \( m_1 - m_0 \equiv_q -j_1 \).

\( \vdash \) To see the first point, the indices of the beginnings of \( n \)-subwords in the same 2-subsection differ by multiples of \( q \) coming from powers of a \( w_j \) and intervals of \( w \) of the form \( b^{q-j_i}e^{j_i} \).
To see the second point, let $u$ and $v$ be consecutive 2-subsections. In view of the first point it suffices to consider the last $n$-subword of $u$ and the first $n$-subword of $v$. These sit on either side of an interval of the form $e^{j_i b^{q-j_i+1}}$. Since $j_i + q - j_{i+1} \equiv_q (p)^{-1} i - p^{-1} (i + 1) \equiv_q -p^{-1} \equiv_q -j_1$, we see that $m_0 - m_1 = q + j_i + q - j_{i+1} \equiv_q -j_1$.

Assume that $u \in W_{n+1}^c$ and $v \in W_{n+1}^c \cup \text{rev}(W_{n+1}^c)$ and $v$ is shifted with respect to $u$. On the overlap of $u$ and $v$, the 2-subsections of $u$ split each 2-subsection of $v$ into either one or two pieces. Since the 2-subsections all have the same length, the number of pieces in the splitting and the size of each piece is constant across the overlap except perhaps at the two ends of the overlap. If $u$ splits a 2-subsection of $v$ into two pieces, then we call the leftmost piece of the pair the even piece and the rightmost the odd piece.

If $v$ is shifted only slightly, it can happen that either the even piece or the odd piece does not contain a 1-subsection. In this case we will say that split is trivial on the left or trivial on the right.

**Lemma 41** Suppose that the 2-subsections of $u$ divide the 2-subsections of $v$ into two non-trivial pieces. Then

1. the boundary portion of $u$ occurring between each consecutive pair of 2-subsections of $u$ completely overlaps at most one $n$-subword of $v$

2. there are two numbers $s$ and $t$ such that the positions of the 0-subsections of $v$ in even pieces are shifted relative to the 0-subsections of $u$ by $s$ and the positions of the 0-subsections of $v$ in odd pieces are shifted relative to the 0-subsections of $u$ by $t$. Moreover $s \equiv_q t - j_1$.

This follows easily from Lemma 40.

In the case where the split is trivial Lemma 41 holds with just one coefficient, $s$ or $t$. A special case Lemma 41 that we will use is:

**Lemma 42** Suppose that the 2-subsections of $u$ divide the 2-subsections of $v$ into two pieces and that for some occurrence of a $n$-subword in an even (resp. odd) piece is lined up with an occurrence of some $n$-subword in $u$. Then every occurrence of a $n$-subword in an even (resp. odd) piece of $v$ is either:

a.) lined up with some $n$-subword of $u$ or

b.) lined up with a section of a 2-subsection that has the form $e^{j_i b^{q-j_i}}$.

Moreover, no $n$-subword in an odd (resp. even) piece of $v$ is lined up with a $n$-subword in $u$. 

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4.5 Full Measure Sets for Circular Systems

Fix a sequence \( \langle \varepsilon_n : n \in \mathbb{N} \rangle \) such that

**Numerical Requirement 2** \( \langle \varepsilon_n : n \in \mathbb{N} \rangle \) is a decreasing sequence of numbers in \([0, 1)\) such that \( \varepsilon_N > \sum_{n>N} \varepsilon_n \).

It follows from the proof of Lemma 35, that if the sequence \( \langle l_n : n \in \mathbb{N} \rangle \) grows fast enough then the proportion of boundaries that occur in words of \( W_n^c \) is always summable, independently of the way we choose the prewords to build \( W_n^c \). Recall the set \( S \subseteq \mathbb{K} \) given in Definition 7, where \( \mathbb{K} \) is the symbolic shift defined from a construction sequence.

**Definition 43** We define some sets that a typical generic point for a circular system eventually avoids. Let:

1. \( E_n \) be the collection of \( s \in S \) such that either \( s \) does not have a principal \( n \)-block or \( s(0) \) is in the boundary of the principal \( n \)-block of \( s \),

2. \( E_0^n = \{ s : s(0) \) is in the first or last \( \varepsilon_n l_n \) copies of \( w \) in a power of the form \( w^{l_n-1} \) where \( w \in W_n^c \} \),

3. \( E_1^n = \{ s : s(0) \) is in the first or last \( \varepsilon_n k_n \) 1-subsections of the 2-subsection in which it is located. \} ,

4. \( E_2^n = \{ s : s(0) \) is in the first or last \( \varepsilon_n q_n \) 2-subsections of its principal \( n + 1 \)-block \} .

**Lemma 44** Assume that \( \sum_n 1/l_n \) is finite. Let \( \nu \) be a shift-invariant measure on \( S \subseteq \mathbb{K}^c \), where \( \mathbb{K}^c \) is a circular system. Then:

1. \[ \sum_n \nu(E_n) < \infty. \]

   Assume that \( \langle \varepsilon_n \rangle \) is a summable sequence, then for \( i = 0, 1, 2 \):

2. \[ \sum_n \nu(E_i^n) < \infty. \]

In particular we see:

**Corollary 45** For \( \nu \)-almost all \( s \) there is an \( N = N(s) \) such that for all \( n > N \),
1. \( s(0) \) is in the interior of its principal \( n \)-block,

2. \( s \notin E^i_n \).

   In particular, for almost all \( s \) and all large enough \( n \):

3. if \( s \upharpoonright \lbrack \lceil -r_n(s) \rceil, r_n(s) + q_n \rceil = w \), then
   \( s \upharpoonright \lbrack -r_n(s) - q_n, -r_n(s) \rceil = s \upharpoonright \lbrack r_n(s) + q_n, r_n + 2q_n \rceil = w \).

4. \( s(0) \) is not in a string of the form \( w^{l_{n-1}}_0 \) or \( w^{l_{n-1}}_{k_{n-1}} \).

\[ \vdash \text{Borel-Cantelli Lemma.} \quad \dashv \]

The elements \( s \) of \( S \) such that some shift \( sh^k(s) \) fails one of the conclusions 1.)-4.) of Corollary 45 form a measure zero set. Consequently we work on those elements of \( S \) whose whole orbit satisfies the conclusions of Corollary 45. Note, however that the \( N(sh^k(s)) \) depends on the shift \( k \).

**Definition 46** We will call \( n \) mature for \( s \) (or say that \( s \) is mature at stage \( n \)) iff \( n \) is so large that \( s \notin E_m \cup \bigcup_{0 \leq i \leq 2} E^i_m \) for all \( m \geq n \).

**Definition 47** We will use the symbol \( \partial_n \) in multiple equivalent ways. If \( s \in S \) or \( s \in W^\circ_n \) we define \( \partial_n = \partial_n(s) \) to be the collection of \( i \) such that \( sh^i(s)(0) \) is in the boundary portion of an \( n \)-subword of \( s \). This is well-defined by our unique readability lemma. In the spatial context we will say that \( s \in \partial_n \) if \( s(0) \) is the boundary of an \( n \)-subword of \( s \).

For \( s \in S \)
\[ \partial_n(s) \subseteq \bigcup \{ \lbrack l, l + q_n \rceil : l \in \text{dom}(s) \text{ and } s \upharpoonright \lbrack l, l + q_n \rceil \in W_n \} \text{.} \]

An integer, \( i \in \partial_n(s) \subseteq \mathbb{Z} \) iff \( sh^i(s) \), viewed as an element of \( K^c \), belongs to the \( n \)-boundary, \( \partial_n \).

In what follows we will be considering a generic point \( s \) and all of its shifts. The next lemma says that if \( s \) is mature at stage \( n \), then we can detect locally those \( i \) for which the \( i \)-shifts of \( s \) are mature.

**Lemma 48** Suppose that \( s \in S \), \( n \) is mature for \( s \) and \( n < m \).

1. Suppose that \( i \in [-r_m(s), q_m - r_m(s)) \). Then \( n \) is mature for \( sh^i(s) \) iff
   
   (a) \( i \notin \bigcup_{n \leq k \leq m} \partial_k \) and
(b) $sh^i(s) \notin \bigcup_{n \leq k < m} (E_k^1 \cup E_k^2 \cup E_k^3)$.

2. For all but at most $(\sum_{n < k \leq m} 1/l_k) + (\sum_{n \leq k < m} 6\varepsilon_k q_{k+1})/q_m$ portion of the $i \in [r_m(s), q_m - r_m(s))$, the point $sh^i(s)$ is mature for $n$.

In particular, if $\varepsilon_{n-1} > \sup_m (1/q_m) \sum_{n \leq k < m} 6\varepsilon_k q_{k+1}, 1/l_n - 1 > \sum_{k=n}^{\infty} 1/l_k$ and $n$ is mature for $s$, the proportion of $i \in [r_m(s), q_m - r_m(s))$ for which the $i$-shift of $s$ is not mature for $n$ is less than $1/l_{n-1} + \varepsilon_{n-1}$.

**Numerical Requirement 3** For all $n$,

$$\varepsilon_{n-1} > \sup_m (1/q_m) \sum_{n \leq k < m} 3\varepsilon_k q_{k+1}$$

A very similar statement is the following:

**Lemma 49** Suppose that $s \in S$ and $s$ has a principal $n$-block. Then $n$ is mature provided that $s \notin \bigcup_{n \leq m} E_m^0 \cup E_m^1 \cup E_m^2$. In particular, if $n$ is mature for $s$ and $s$ is not in a boundary portion of its principal $n-1$-block or in $E_{n-1}^0 \cup E_{n-1}^1 \cup E_{n-1}^2$, then $n-1$ is mature for $s$.

### 4.6 The Circle Factor

Let $\langle k_n, l_n : n \in \mathbb{N} \rangle$ be a circular coefficient sequence and $\langle p_n, q_n : n \in \mathbb{N} \rangle$ be the associated sequence defined by formulas 9 and 10. Let $\alpha_n = p_n/q_n$ and $\alpha = \lim \alpha_n$.

For $q$ a natural number at least one, let $\mathcal{I}_q$ be the partition of $[0,1)$ with atoms $\langle [i/q, (i+1)/q) : 0 \leq i < q \rangle$, and refer to $[i/q, (i+1)/q)$ as $I_i^q$. Since $p_n$ and $q_n$ are relatively prime, the rotation $\mathcal{R}_{\alpha_n}$ enumerates the partition $\mathcal{I}_{q_n}$ starting with $I_0^{q_n}$. Thus $\mathcal{I}_{q_n}$ has two natural orderings—the usual geometric ordering and the dynamical ordering given by the order that $\mathcal{R}_{\alpha_n}$ enumerates $\mathcal{I}_{q_n}$. $I_i^q$ is the $i^{th}$ interval in the dynamical ordering.

**Definition 50** For $x \in [0,1)$ we will write $D_n(x) = j$ if $x$ belongs to the $j^{th}$ interval in the dynamical ordering of $\mathcal{I}_{q_n}$. Equivalently $D_n(x) = j$ if $x \in I_{j p_n}^{q_n}$.

**Informal motivating remarks:** Following [4], for each stage $n$, we have a periodic approximation $\tau_n$ to $\mathbb{R}^c$ consisting of towers $\mathcal{T}$ of height $q_n$ whose levels are subintervals of $[0,1)$. This approximation refines the periodic

\[18\] If $i > q$ then $I_i^q$ refers to $I_{i'}^q$ where $i' < q$ and $i' \equiv i \mod q$. 

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permutation of $I_{q_n}$ determined by $R_{\alpha_n}$. If $s$ is mature then $s$ lies is the $r^{th}$ level of $I_{q_n}$ in the dynamical ordering. Passing from $\tau_n$ to $\tau_{n+1}$ the mature points remain in the same levels of the $n$-towers as they are spread into the $n+1$-towers in $\tau_{n+1}$. The towers of $\tau_{n+1}$ can be viewed as cut-and-stack constructions–filling in boundary points between cut $n$-towers. The fillers are taken from portions of the $n$-towers.

With this view each mature point remains in the same interval of $I_{q_n}$ when viewed in $\tau_{n+1}$. Moreover if $s \in J \subseteq I \subseteq I_{q_n}$, then $R_{\alpha_{n+1}}J \subseteq R_{\alpha_n}I$.

Thus the $n+1$-tower picture for $R_{\alpha_{n+1}}$ has contiguous sequences of levels of length $q_n$ that are sublevels of the $n$-tower and the action of $R_{\alpha_n}$ and $R_{\alpha_{n+1}}$ agree on these levels.

**Definition 51** Let $\Sigma_0 = \{\ast\}$. We define a circular construction sequence such that each $W_n$ has a unique element as follows:

1. $W_0 = \{\ast\}$ and
2. If $W_n = \{w_n\}$ then $W_{n+1} = \{C(w_n, w_n, \ldots, w_n)\}$.

Let $\mathcal{K}$ be the resulting circular system.

It is easy to check that $\mathcal{K}$ has unique non-atomic measure since the unique $n$-word, $w_n$, occurs exactly $k_n(l_n - 1)q_n$ many times in $w_{n+1}$. Clearly this measure is ergodic.

Let $\mathbb{K}^c$ be an arbitrary circular system with coefficients $\langle k_n, l_n : n \in \mathbb{N} \rangle$. Then $\mathbb{K}^c$ has a canonical factor isomorphic to $\mathcal{K}$. This canonical factor plays a role for circular systems analogous to the role odometer transformations play for odometer based systems.

To see $\mathcal{K}$ is a factor of $\mathbb{K}^c$, we define the following function:

$$\pi(x)(i) = \begin{cases} x(i) & \text{if } x(i) \in \{b, e\} \\ * & \text{otherwise} \end{cases} \quad (15)$$

**Notation:** We write $w_n^\alpha$ for the unique element of $W_n$ in the construction sequence for $\mathcal{K}$. Then $w_n^\alpha$ lies in the principal $n$-block of the projection to $\mathcal{K}$ of any $s \in \mathbb{K}^c$ for which $n$ is mature.

**Theorem 52** ([4], Theorem 43.) Let $\nu$ be the unique non-atomic shift-invariant measure on $\mathcal{K}$. Then

$$(\mathcal{K}, \mathcal{B}, \nu, sh) \cong (S^1, \mathcal{D}, \lambda, R_\alpha)$$
where \( R_\alpha \) is the rotation of the unit circle by \( \alpha \) and \( \mathcal{B}, \mathcal{D} \) are the \( \sigma \)-algebras of measurable sets.

The isomorphism \( \phi_0 \) in Theorem 52 is constructed as a limit of functions \( \rho_n \), where \( \rho_n \) is defined by setting

\[
\rho_n(s) = \frac{i}{q_n} \quad (16)
\]

iff \( I^n_i \) is the \( r_n(s) \)th interval in the dynamical ordering.\(^{19}\) Equivalently, since the \( r_n(s) \)th interval in the geometric ordering is \( I^n_{\rho_n} \):

\[
i \equiv p_n r_n(s) \mod q_n \quad (17)
\]

The following follows from Proposition 44 in [4].

**Proposition 53** Suppose that \( n \) is mature for \( s \), then

\[
r_n(s) = D_n(\phi_0(s))
\]

**Remark 54** It will be helpful to understand \( \phi_0^{-1} \) explicitly. To each point \( x \) in the range of \( \phi_0, s = \phi_0^{-1}(x) \in S \). By Lemma 12, to determine \( s \) it suffices to know \( \langle r_n(s) : n \geq N \rangle \) for some \( N \) as well as the sequence \( \langle w_n : n \geq N \rangle \) of principal subwords of \( s \). Since we are working with \( K \), the only choice for \( w_n \) is \( w_n = w^n_\alpha \). For mature \( n \), Lemma 53 tells us that \( r_n(s) = D_n(x) \). Thus \( s \) is the unique element of \( S \) with the property that \( \langle r_n(s) : n \in \mathbb{N} \rangle \) agrees with \( \langle D_n(x) : n \in \mathbb{N} \rangle \) for all large \( n \).

We isolate the following fact for later use:

**Lemma 55** Suppose that \( \phi_0(s) = x \) and \( n < m \) are mature for \( s \). Then if \( I \) and \( J \) are the \( D_n(x) \)th and \( D_m(x) \)th intervals in the dynamical orderings of \( \mathcal{I}^n \) and \( \mathcal{I}^m \), then \( J \subseteq I \).

The natural way of representing the complex unit circle as an abelian group is multiplicatively: the rotation by \( 2\pi\alpha \) radians is multiplication by \( e^{2\pi i\alpha} \). It is often more convenient to identify the unit circle with \([0, 1)\). In doing so, multiplication by \( e^{2\pi i\alpha} \) corresponds to “mod one” addition and the complex conjugate \( \bar{z} \) corresponds to \( -z \).

The following result is standard:

\(^{19}\)Thus \( r_n \) and \( \rho_n \) both have the same subset of \( S \) as their domain and contain the same information. They map to different places \( r_n : S \rightarrow \mathbb{N} \), whereas \( \rho_n : S \rightarrow [0, 1) \) and is the left endpoint of the \( r^n_i \)th interval in the dynamical ordering.
Proposition 56 Suppose that $T : S^1 \to S^1$ is an invertible measure preserving transformation that commutes with $R_\alpha$. Identifying $S^1$ with $[0, 1)$ there is a $\beta$ such that for almost all $x \in S^1$

$$T(x) = x + \beta \mod 1. \quad (18)$$

(In other words for some $\beta$, $T = R_\beta$ almost everywhere.) It follows that if $T$ is an isomorphism between $R_\alpha$ and $R_\alpha^{-1}$, then $T(x) = -x + \beta \mod 1$.

4.7 The Natural Map

In our proof we use a specific isomorphism $\natural : (K, sh) \to (\text{rev}(K), sh)$ that will serve as a benchmark for understanding of potential maps $\phi : K^c \to \text{rev}(K^c)$. If we view $R_\alpha$ as a rotation of the unit circle by $\alpha$ radians one can view the transformation $\natural$ we define as a symbolic analogue of complex conjugation $z \mapsto \bar{z}$ on the unit circle, which is an isomorphism between $R_\alpha$ and $R_{-\alpha}$. Indeed, by Theorem 52, $K \cong R_\alpha$ and so $\text{rev}(K) \cong R_{-\alpha}$. Copying $\natural$ over to a map on the unit circle gives an isomorphism $\phi$ between $R_\alpha$ and $R_{-\alpha}$. Written additively, Proposition 56 says that such an isomorphism must be of the form

$$\phi(z) = -z + \beta$$

for some $\beta$. It follows immediately from this characterization that $\natural$ is an involution.\(^{20}\)

We now define $\natural$ by giving a sequence of codes $\langle \Lambda_n : n \in \mathbb{N} \rangle$ that converge to an isomorphism from $K$ to $\text{rev}(K)$ (see [5] for more details). The $\Lambda_n$ will be shifting and reversing words. The amount of shift is determined by the Anosov-Katok coefficients $p_n, q_n$ defined in equations 10 and 9. Since $p_n$ and $q_n$ are relatively prime we can define $(p_n)^{-1}$ in $\mathbb{Z}/q_n\mathbb{Z}$. For the following definition we view $(p_n)^{-1}$ as an natural number with $0 \leq (p_n)^{-1} < q_n$.

Let $A_0 = 0$ and inductively

$$A_{n+1} = A_n - (p_n)^{-1}. \quad (19)$$

It is easy to check that

$$|A_{n+1}| < 2q_n \quad (20)$$

\(^{20}\)The particular $\beta$ given by $\natural$ is determined by the specific variation of the definition one uses—indeed any central value can occur as a $\beta$. (See section 7 for the definition and use of central values.)
Define a stationary code \( \Lambda_n \) with domain \( S \) that approximates elements of \( \text{rev}(K) \) by defining
\[
\Lambda_n(s) = \begin{cases} 
sh^{A_n + 2r_n(s) - (q_n - 1)}(\text{rev}(s))(0) & \text{if } r_n(s) \text{ is defined} \\
b & \text{otherwise}
\end{cases}
\] (21)

We will use:

**Theorem 57** ([5]) The sequence of stationary codes \( \langle \Lambda_n : n \in \mathbb{N} \rangle \) converges to a shift invariant function \( \natural : K \to (\{\ast\} \cup \{b, e\})^\mathbb{Z} \) that induces an isomorphism \( \natural \) from \( K \) to \( \text{rev}(K) \).

Remark 78 of [5] implies that the convergence is prompt: for a typical \( s \) and all large enough \( n \), \( \natural(s) \) agrees with \( \bar{\Lambda}_n(s) \) on the principal \( n \)-block of \( s \).

### 4.8 Categories and the Functor \( \mathcal{F} \).

Fix a circular coefficient sequence \( \langle k_n, l_n : n \in \mathbb{N} \rangle \). Let \( \Sigma \) be a language and \( \langle W_n : n \in \mathbb{N} \rangle \) be a construction sequence for an odometer based system with coefficients \( \langle k_n : n \in \mathbb{N} \rangle \). Then for each \( n \) the operation \( C_n \) is well-defined. We define a construction sequence \( \langle W_n^c : n \in \mathbb{N} \rangle \) and bijections \( c_n : W_n \to W_n^c \) by induction as follows:

1. Let \( W_0^c = \Sigma \) and \( c_0 \) be the identity map.
2. Suppose that \( W_n, W_n^c \) and \( c_n \) have already been defined.

\[
W_{n+1}^c = \{ C_n(c_n(w_0), c_n(w_1), \ldots c_n(w_{k_n-1})) : w_0 w_1 \ldots w_{k_n-1} \in W_{n+1} \}.
\]

Define the map \( c_{n+1} \) by setting
\[
c_{n+1}(w_0 w_1 \ldots w_{k_n-1}) = C_n(c_n(w_0), c_n(w_1), \ldots c_n(w_{k_n-1})).
\]

We note in case 2 the prewords are:
\[
P_{n+1} = \{ c_n(w_0)c_n(w_1) \ldots c_n(w_{k_n-1}) : w_0 w_1 \ldots w_{k_n-1} \in W_{n+1} \}.
\]

**Remark 58** Some useful facts are:

- It follows from Lemma 35 and Numerical Requirement 1 that if \( \langle W_n : n \in \mathbb{N} \rangle \) is an odometer based construction sequence, then \( \langle W_n^c : n \in \mathbb{N} \rangle \) is a construction sequence; i.e. the spacer proportions are summable.
• If each \( w \in \mathcal{W}_n \) occurs exactly the same number of times in every element of \( \mathcal{W}_{n+1} \), then \( \langle \mathcal{W}^c_n : n \in \mathbb{N} \rangle \) is uniform.

• Odometer words in \( \mathcal{W}_n \) have length \( K_n \); the length of the circular words in \( \mathcal{W}^c_n \) is \( q_n \).

Definition 59 Define a map \( F \) from the set of odometer based subshifts to circular subshifts as follows. Suppose that \( \mathcal{K} \) is built from a construction sequence \( \langle \mathcal{W}_n : n \in \mathbb{N} \rangle \). Define

\[
F(\mathcal{K}) = \mathcal{K}^c
\]

where \( \mathcal{K}^c \) has construction sequence \( \langle \mathcal{W}^c_n : n \in \mathbb{N} \rangle \).

The map \( F \) is one to one by the unique readability of words in \( \mathcal{W} \). Suppose that \( \mathcal{K}^c \) is a circular system with coefficients \( \langle k_n, l_n : n \in \mathbb{N} \rangle \). We can recursively build functions \( c_{n-1} \) from words in \( \Sigma \cup \{ b, e \} \) to words in \( \Sigma \). The result is an odometer based system \( \langle \mathcal{W}_n : n \in \mathbb{N} \rangle \) with coefficients \( \langle k_n : n \in \mathbb{N} \rangle \). If \( \mathcal{K} \) is the resulting odometer based system then \( F(\mathcal{K}) = \mathcal{K}^c \). Thus \( F \) is a bijection.

If \( \mathcal{K} \) is an odometer based system, denote the odometer base by \( \mathcal{K}^\pi \) and let \( \pi : \mathcal{K} \to \mathcal{K}^\pi \) be the canonical factor map. If \( \mathcal{K}^c \) is a circular system, let \( (\mathcal{K}^c)^\pi \) be the rotation factor \( \mathcal{K} \) and \( \pi : \mathcal{K}^c \to \mathcal{K} \) be the canonical factor map. For both odometer based and circular systems the underlying canonical factors serve as a timing mechanism. This motives the following.

Definition 60 We define synchronous and anti-synchronous joinings.

1. Let \( \mathcal{K} \) and \( \mathcal{L} \) be odometer based systems with the same coefficient sequence, and \( \rho \) a joining between \( \mathcal{K} \) and \( \mathcal{L}^{\pm 1} \). Then \( \rho \) is synchronous if \( \rho \) joins \( \mathcal{K} \) and \( \mathcal{L} \) and the projection of \( \rho \) to a joining on \( \mathcal{K}^\pi \times \mathcal{L}^\pi \) is the graph joining determined by the identity map (the diagonal joining of the odometer factors); \( \rho \) is anti-synchronous if \( \rho \) is a joining of \( \mathcal{K} \) with \( \mathcal{L}^{-1} \) and its projection to \( \mathcal{K}^\pi \times (\mathcal{L}^{-1})^\pi \) is the graph joining determined by the map \( x \mapsto -x \).

2. Let \( \mathcal{K}^c \) and \( \mathcal{L}^c \) be circular systems with the same coefficient sequence and \( \rho \) a joining between \( \mathcal{K}^c \) and \( (\mathcal{L}^c)^{\pm 1} \). Then \( \rho \) is synchronous if \( \rho \) joins \( \mathcal{K}^c \) and \( \mathcal{L}^c \) and the projection to a joining of \( \mathcal{K}^c \times \mathcal{L}^c \) is the graph joining determined by the identity map of \( \mathcal{K} \) with \( \mathcal{L} \), the underlying rotations; \( \rho \) is anti-synchronous if \( \rho \) is a joining of \( \mathcal{K}^c \) with \( (\mathcal{L}^c)^{-1} \) and projects to the graph joining determined by \( \text{rev()} \circ \sharp \) on \( \mathcal{K} \times \mathcal{L}^{-1} \).
The **Categories** Let $\mathcal{O}B$ be the category whose objects are ergodic odometer based systems with coefficients $\langle k_n : n \in \mathbb{N} \rangle$. The morphisms between objects $\mathcal{K}$ and $\mathcal{L}$ will be synchronous graph joinings of $\mathcal{K}$ and $\mathcal{L}$ or anti-synchronous graph joinings of $\mathcal{K}$ and $\mathcal{L}^{-1}$. We call this the *category of odometer based systems*.

Let $\mathcal{C}B$ be the category whose objects consists of all ergodic circular systems with coefficients $\langle k_n, l_n : n \in \mathbb{N} \rangle$. The morphisms between objects $\mathcal{K}^c$ and $\mathcal{L}^c$ will be synchronous graph joinings of $\mathcal{K}^c$ and $\mathcal{L}^c$ or anti-synchronous graph joinings of $\mathcal{K}^c$ and $(\mathcal{L}^c)^{-1}$. We call this the *category of circular systems*.

The main theorem of [5] is the following:

**Theorem 61** For a fixed circular coefficient sequence $\langle k_n, l_n : n \in \mathbb{N} \rangle$ the categories $\mathcal{O}B$ and $\mathcal{C}B$ are isomorphic by a function $F$ that takes synchronous joinings to synchronous joinings, anti-synchronous joinings to anti-synchronous joinings, isomorphisms to isomorphisms and weakly mixing extensions to weakly mixing extensions.

It is also easy to verify that the map $\langle W_n : n \in \mathbb{N} \rangle \mapsto \langle W_n^c : n \in \mathbb{N} \rangle$ takes uniform construction sequences to uniform construction sequences and strongly uniform construction sequences to strongly uniform construction sequences.

**Remark 62** Were we to be completely precise we would take objects in $\mathcal{O}B$ to be presentations of odometer based systems by construction sequences $\langle W_n : n \in \mathbb{N} \rangle$ without spacers and the objects in $\mathcal{C}B$ to be presentations by circular construction sequences. This subtlety does not cause problems in the sequel so we ignore it.

### 4.9 Propagating Equivalence Relations and Actions

In [8] we built a continuous function from the space of trees to odometer based transformations. At the same time equivalence relations $\langle Q^n_s : M(s) \leq n, s \in \mathbb{N} \rangle$ were defined where:

1. $M$ is a monotone, strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$,
2. $Q^n_s$ is an equivalence relation on $W_n \cup \text{rev}(W_n)$,
3. $Q_0^0$ is the trivial equivalence relation with one equivalence class on $W_0 = \Sigma$. 

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Along with these equivalence relations came canonically generated groups \( \langle G_n : M(s) \leq n, s \in \mathbb{N} \rangle \) that were direct sums of copies of \( \mathbb{Z}_2 \). Each group \( G_n \) acted on \( \mathcal{W}_n / Q_n \cup \text{rev}(\mathcal{W}_n) / Q_n \) in a manner that made even parity group elements preserve the sets \( \mathcal{W}_n / Q_n \) and \( \text{rev}(\mathcal{W}_n) / Q_n \) and the odd parity group elements send elements of \( \mathcal{W}_n / Q_n \) to \( \text{rev}(\mathcal{W}_n) / Q_n \). We now give two descriptions of how the corresponding equivalence relations propagate in \( \langle \mathcal{W}_n : n \in \mathbb{N} \rangle \). They will be used in section 7.2.1 to state the timing assumptions and in section 9.2 which gives the construction specifications from [8].

Both descriptions initialize the equivalence relations in the same way:

If \( n = M(s) \), \( w_0, w_1 \in \mathcal{W}_n \) we put \( (c_n(w_0), c_n(w_1)) \) in the relation \( (Q_n)^c \) if and only if \( (w_0, w_1) \in Q_n \).

**Version 1: purely circular description.** We define how to propagate the equivalence relations and group actions staying inside the setting of circular systems.

**Definition 63** Given an equivalence relation \( Q_n \) on \( \mathcal{W}_n \) we can propagate it to an equivalence relation \( Q_{n+1} \) on \( \mathcal{W}_{n+1} \) by setting \( C(w_0, \ldots, w_{k_n-1}) \) equivalent to \( C(w'_0, \ldots, w'_{k_n-1}) \) if and only if for all \( i, w_i \) is \( Q^\alpha \)-equivalent to \( w'_i \). We call \( Q_{n+1} \) the propagation of \( Q_n \). If \( Q \) is an equivalence relation on \( \mathcal{W}^c \) we define \( \text{rev}(Q) \) to be the natural equivalence relation on \( \text{rev}(\mathcal{W}^c) \) determined by \( Q \).

We can clearly identify \( \text{rev}(\mathcal{W}_{n+1}^c) / \text{rev}(Q_{n+1}^c) \) with the collection of sequences of the form

\[
C^c(\text{rev}(w_0)/\text{rev}(Q^n), \text{rev}(w_1)/\text{rev}(Q^n), \ldots, \text{rev}(w_{k_n-1})/\text{rev}(Q^n))
\]
as \( w_0 w_1 \ldots w_{k_n-1} \) ranges over the elements of \( P_n \). To simplify notation we will err slightly by using \( Q^c \) to mean both \( Q^n \) and \( \text{rev}(Q^n) \).

We can now build \( (Q_{n+1}^c)^c \) by supposing that \( (Q_n^c)^c \) has already been defined and letting \( (Q_{n+1}^c)^c \) be the propagation of \( (Q_n^c)^c \) given in definition 63.21

Suppose that \( G \) is a finite sum of copies of \( \mathbb{Z}_2 \) with a distinguished set of generators and \( G \) acts on \( \mathcal{W}_n^c / Q^n \cup \text{rev}(\mathcal{W}_n^c / Q^n) \) in such a way that each generator takes an element of \( \mathcal{W}_n^c / Q^n \) to an element of \( \text{rev}(\mathcal{W}_n^c / Q^n) \). We can propagate this action to an action of \( G \) on \( \mathcal{W}_{n+1}^c / Q_{n+1}^c \cup \text{rev}(\mathcal{W}_{n+1}^c / Q_{n+1}^c) \) to get an action we call the (circular) skew-diagonal action.

---

21We point out that this definition makes \( (Q_n^c)^c \) into the trivial equivalence relation with one class.
To define the skew-diagonal action it suffices to specify it on the canonical generators of $G$. This is done by setting
\[ gC(w_0, w_1 \ldots w_{k-1}) =_{def} C'(gw_0, gw_1 \ldots gw_{k-1}) \]
whenever $g$ is a canonical generator of $G$. We note that the skew-diagonal action has the property that the canonical generators take elements of $\mathcal{W}^{c}_{n+1}/\mathcal{Q}^{n+1}$ to elements of $\text{rev}(\mathcal{W}^{c}_{n+1}/\mathcal{Q}^{n+1})$. It follows that the even parity elements of $G$ leave the sets $\mathcal{W}^{c}_{n+1}/\mathcal{Q}^{n+1}$ and $\text{rev}(\mathcal{W}^{c}_{n+1}/\mathcal{Q}^{n+1})$ invariant and odd parity elements of $G$ take $\mathcal{W}^{c}_{n+1}/\mathcal{Q}^{n+1}$ to elements of $\text{rev}(\mathcal{W}^{c}_{n+1}/\mathcal{Q}^{n+1})$ and vice versa.

Version 2: in terms of $F$.

The following properties are immediate consequences of the word specifications Q4-A9 in [8]:

S1 For $n \geq M(s) + 1$, $\mathcal{Q}_s^n$ is the product equivalence relation of $\mathcal{Q}_s^{M(s)}$. Hence we can view $\mathcal{W}_n/\mathcal{Q}_s^n$ as sequences of elements of $\mathcal{W}_{M(s)}/\mathcal{Q}_s^{M(s)}$ and similarly for $\text{rev}(\mathcal{W}_n/\mathcal{Q}_s^n)$.

S2 $G^n_s$ acts freely on $\mathcal{W}_n/\mathcal{Q}_s^n \cup \text{rev}(\mathcal{W}_n/\mathcal{Q}_s^n)$

S3 The canonical generators of $G^{M(s)}_s$ send elements of $\mathcal{W}_{M(s)}/\mathcal{Q}_s^{M(s)}$ to elements of $\text{rev}(\mathcal{W}_{M(s)}/\mathcal{Q}_s^{M(s)})$ and vice versa.

S4 If $M(s) \leq n$ and we view $G^{n+1}_s = G^n_s \oplus H$ then the action of $G^n_s$ on $\mathcal{W}_n/\mathcal{Q}_s^n \cup \text{rev}(\mathcal{W}_n/\mathcal{Q}_s^n)$ is extended to an action on $\mathcal{W}_{n+1}/\mathcal{Q}_s^{n+1} \cup \text{rev}(\mathcal{W}_{n+1}/\mathcal{Q}_s^{n+1})$ by the skew diagonal action.\(^{22}\) If $H$ is non-trivial then its canonical generator maps $\mathcal{W}_{n+1}/\mathcal{Q}_s^{n+1}$ to $\text{rev}(\mathcal{W}_{n+1}/\mathcal{Q}_s^{n+1})$.

The following lemma is straightforward:

**Lemma 64** Suppose that $\langle \mathcal{W}_n : n \in \mathbb{N} \rangle$ and $\langle \mathcal{Q}_s^n : n \geq M(s) \rangle$ satisfy items S1-S4 above. Let $\langle \mathcal{W}_n^c : n \in \mathbb{N} \rangle$ and $\langle (\mathcal{Q}_s^n)^c : n \geq M(s) \rangle$ be as in definition 63. Then for all $n \geq M(s)$ and $w_0, w_1 \in \mathcal{W}_n$ we have:

- $(w_0, w_1) \in \mathcal{Q}_s^n$ if and only if $w_0, w_1 \in \mathcal{W}_n$.

\(^{22}\)If $G$ is a group of involutions with a distinguished collection of free generators, then we define the skew diagonal action on $n$-sequences $x_0x_1x_2 \ldots x_{n-1} \in \mathcal{X}$ by setting
\[ g(x_0x_1x_2 \ldots x_{n-1}) = gx_{n-1}gx_{n-2} \ldots gx_0 \]
where $g$ is a canonical generator.
\( (c_n(w_0), c_n(w_1)) \in (\mathcal{Q}_s^n)^c \).

Thus as an alternate definition of \((\mathcal{Q}_s^n)^c\), for all \(n, w_0, w_1 \in \mathcal{W}_n\) we put \((c_n(w_0), c_n(w_1))\) in the relation \((\mathcal{Q}_s^n)^c\) if and only if \((w_0, w_1) \in \mathcal{Q}_s^n\).

Lemma 64 can now be used to extend the group actions. Suppose that \(M(s) \leq n\), so that the action of \(G_s^n\) is defined on \(\mathcal{W}_n/\mathcal{Q}_s^n \cup \text{rev}(\mathcal{W}_n)/\mathcal{Q}_s^n\). Then we can define an action of \(G_s^n\) on \((\mathcal{W}_n)^c/(\mathcal{Q}_s^n)^c \cup \text{rev}(\mathcal{W}_n)^c/(\mathcal{Q}_s^n)^c\). Given \(w_0' = c_n(w_0)\) and \(w_1' = c_n(w_1)\) and \(g \in G_s^n\) we put \(g \cdot [w_0]' = [w_1]'\) if and only if \(g \cdot [w_0] = [w_1]\).

It is straightforward to see that with this definition, the action of \(G_s^{n+1}\) extends the skew-diagonal action of \(G_s^n\) on \((\mathcal{W}_n^{c+1})/\mathcal{Q}_s^{n+1}\).

The conclusion we draw of this is that there are both intrinsic (i.e. version 1) and extrinsic (version 2) methods of defining the equivalence relations and the group actions and they coincide. Moreover the functor \(\mathcal{F}\) takes the sequence of factors of \(\mathcal{K}\) defined by the \(\mathcal{Q}_s^{M(s)}\)'s to the sequence of factors of \(\mathcal{K}^c\) defined by the \((\mathcal{Q}_s^{M(s)})^c\)'s.

## 5 Overview of Rotations, \(\Delta\) and Central values

Let \(\mathcal{K}\) be a rotation factor of a circular system with coefficient sequence \(\langle k_n, l_n : n \in \mathbb{N} \rangle\). In this section we develop an understanding of how automorphisms of \(\mathcal{K}\) affect the parsing of elements of \(\mathcal{K}\).

Let \((\mathbb{K}^c, \mu^c)\) and \((\mathbb{L}^c, \nu^c)\) be two circular systems with that share a given circular coefficient sequence. Any isomorphism between \(\mathbb{K}^c\) and \((\mathbb{L}^c)^{\pm 1}\) induces a unitary isomorphism between \(L^2((\mathbb{L}^c)^{\pm 1})\) and \(L^2(\mathbb{K}^c)\), and this isomorphism sends eigenfunctions for \(n\alpha\) to eigenfunctions for \(n\alpha\). Thus every isomorphism has to send the canonical factor \(\mathcal{K}_{\alpha}\) of \(\mathbb{K}^c\) to the canonical factor \(\mathcal{K}_{\alpha}^{\pm 1}\) of \((\mathbb{L}^c)^{\pm 1}\). Explicitly: suppose that \(\phi : \mathbb{K}^c \rightarrow (\mathbb{L}^c)^{\pm 1}\) is an isomorphism. If \(U_\phi : L^2((\mathbb{L}^c)^{\pm 1}) \rightarrow L^2(\mathbb{K}^c)\), then \(U_\phi\) takes the space generated by eigenfunctions of \(U_{sh}\) in \(L^2((\mathbb{L}^c)^{\pm 1})\) with eigenvalues \(\{\alpha^n : n \in \mathbb{Z}\}\) to the space generated by corresponding eigenfunctions in \(L^2(\mathbb{K}^c)\). Consequently there is a measure preserving transformation \(\phi^\pi\) making the following diagram commute:

\[
\begin{array}{ccc}
\mathbb{K}^c & \xrightarrow{\phi} & (\mathbb{L}^c)^{\pm 1} \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathcal{K}_{\alpha} & \xrightarrow{\phi^\pi} & \mathcal{K}_{\alpha}^{\pm 1}
\end{array}
\] \hspace{1cm} (22)
By Theorem 52, $K_\alpha$ is isomorphic to $R_\alpha$ on the unit circle. Hence (using additive notation) $\phi^\pi$ must be isomorphic to a transformation defined on the unit circle of the form $x \mapsto z + \beta$ for some $\beta$, where $z$ is either $x$ or $-x$, depending on whether $\phi^\pi$ maps to $K_\alpha$ or $K^{-1}_\alpha$. Since $\zeta : K_\alpha \to K^{-1}_\alpha$ is an isomorphism, if $\phi$ maps to $(L^n)^{-1}$, $\zeta(x)$ can serve as a benchmark for defining $\beta$, rather than the map $x \mapsto -x$. Explicitly: the $\beta$ associated to $\phi$ is the number making $\phi^\pi(x) = \zeta(x + \beta)$; equivalently, $\zeta^{-1} \circ \phi^\pi(x) = x + \beta$.

**Definition 65** Suppose that $\phi : K^c \to (K^c)^{\pm 1}$ is an isomorphism. We call the rotation $S_\beta$ described in the previous paragraph the rotation associated with $\phi$.

We record the following facts:

**Lemma 66** Let $K^c$ be a circular system. Then

1. The set of $\beta$ associated with automorphisms of $K^c$ form a group.
2. If $\phi : K^c \to (K^c)^{-1}$ and $\psi : K^c \to K^c$ are isomorphisms where $\phi^\pi = \zeta \circ S_\beta$ and $\psi^\pi = S_\gamma$, then $(\phi \circ \psi)^\pi = \zeta \circ S_\delta$ where $\delta = \beta + \gamma$.

⊤

It is easy to check that

• If $\phi, \psi$ are isomorphisms from $K^c$ to $K^c$ with $\phi^\pi = S_\beta$ and $\psi^\pi = S_\gamma$, then $(\phi \circ \psi)$ is also an isomorphism from $K^c$ to $K^c$ and $(\phi \circ \psi)^\pi = S_\delta$, where $\delta = \beta + \gamma$.

• If $\phi$ is an isomorphism from $K^c$ to $K^c$, and $\phi^\pi = S_\beta$, then $(\phi^{-1})^\pi = S_{-\beta}$. The second assertion is similar.

Let $\phi_0 : (K, sh) \to (S^1, R_\alpha)$ be the isomorphism given by Theorem 52. Given a rotation $R_\beta$, let

$S_\beta = def \phi_0^{-1} R_\beta \phi_0$

and set

$S_\beta = \bigcap_{n \in \mathbb{Z}} S_\beta^n(S)$

This can be described independently of $S_\beta$ as:

$\{s \in S : \text{for all } n \in \mathbb{Z}, \phi_0(s) \in (\phi_0[S] + n\beta)\}$.
It is clear that \( \nu(S_\beta) = 1 \).

Define a sequence of functions \( \langle d^n : n \in \mathbb{N} \rangle \). Each
\[
d^n : S_\beta \to \{0, 1, 2, \ldots, q_n - 1\}.
\]
For \( s \in S_\beta \) and \( t = S_\beta(s) \) we have \( t \in S_\beta \) and \( \phi_0(t) = R_\beta \phi_0(s) \). All large enough \( n \) are mature for \( t \), and \( t \) is determined by a tail segment of \( \langle r_n(t) : n \in \mathbb{N} \rangle \).

**Definition 67** If \( n \) is mature for \( s \) and \( t = S_\beta(s) \), let
\[
d^n(s) \equiv_{q_n} r_n(t) - r_n(s),
\]
and \( d^n(s) = 0 \) otherwise.

From the definition of \( r_n \), \( \phi_0(s) + \beta \) belongs to the \((r_n(s) + d^n(s))^{th}\) interval in the dynamical ordering of \( \mathcal{I}_{q_n} \).\(^{23}\)

Fix an \( n \) and suppose that \( \beta \) is not a multiple of \( 1/q_n \). Then the interval \([\beta, \beta + 1/q_n)\) intersects two geometrically consecutive intervals of the form \([i/q_n, (i + 1)/q_n)\).

**Lemma 68** Suppose that \( n \) is mature for \( s \) and \( S_\beta(s) \). Then \( d^n(s) \) belongs to \( \{D_n(\beta), D_n(\beta + 1/q_n)\} \). Thus there are only two possible values for \( d^n(s) \) and these values differ by \( j_1 \).

Suppose that \( \beta \in [i/q_n, (i + 1)/q_n) \) and \( \gamma = (i + 1)/q_n - \beta \). Then \( D_n(\beta) = j_i \). We claim that, relative to those \( s \) for which \( n \) is mature for \( s \) and \( S_\beta(s) \), \( d^n \) is constant on \( \phi_{0}^{-1}(\bigcup_{j < q_n} [j/q_n, j/q_n + \gamma)) \) and on \( \phi_{0}^{-1}(\bigcup_{j < q_n} [(j + 1)/q_n - \gamma, (j + 1)/q_n)) \), where it takes values \( D_n(\beta) \) and \( D_n(\beta + \frac{1}{q_n}) \) respectively (see figure 1).

We show that \( d^n \) is constant on the first set. Suppose that \( n \) is mature for \( s, S_\beta(s) \) and \( \phi_0(s) = x \) belongs to the interval \([0, \gamma)\). Then \( x + \beta \in [i/q_n, (i + 1)/q_n) \). Hence \( r_n(S_\beta(s)) = j_i = D_n(\beta) \). Since \( r_n(s) = 0 \) we know that \( d^n(s) = j_i \). Now suppose that \( t \in \phi_{0}^{-1}(\bigcup_{j < q_n} [j/q_n, j/q_n + \gamma)) \) and \( n \) is mature for \( t \) and \( S_\beta(t) \). Let \( k = r_n(t) \). Then \( \phi_0(t) = x + kp_n/q_n \) for some \( x \in [0, 1/q_n - \gamma) \). So \( \phi_0(t) + \beta \in [(i + kp_n)/q, (i + 1 + kp_n)/q) \). Hence
\[
r_n(S_\beta(t)) = (p_n)^{-1}(i + kp_n) = j_i + k.
\]

\(^{23}\)More accurately: if \( j < q_n \) and \( j \equiv_{q_n} r_n(s) + d^n(s) \), then \( \phi_0(s) + \beta \) belongs to the \( j^{th}\) interval in the dynamical ordering of \( \mathcal{I}_{q_n} \).
Hence

\[ d^n(t) = r_n(S_\beta(t)) - r_n(t) \]  \hfill (26)
\[ = j_i + k - k \]  \hfill (27)
\[ = j_i. \]  \hfill (28)

If \( t \in \phi_0^{-1}(\bigcup_{j<q_n}[(j+1)/q_n - \gamma, (j+1)/q_n)) \) the proof is parallel.

Finally \( \beta \) and \( \beta + \frac{1}{q_n} \) fall into consecutive intervals of \( I^{qn} \) in the geometric ordering, and hence \( D_n(\beta + \frac{1}{q_n}) = D_n(\beta) + j_1 \).

Define \( d^n_L \) and \( d^n_R \) by setting \( d^n_L = D_n(\beta) \) and \( d^n_R = D_n(\beta + \frac{1}{q_n}) \). Let

\[ L_n = \{ s : r_n(s) + d^n_L \equiv q_n r_n(S_\beta(s)) \} \]

and

\[ R_n = \{ s : r_n(s) + d^n_R \equiv q_n r_n(S_\beta(s)) \} \]

We refer to \( L_n \) and \( R_n \) as the left lane and right lane respectively.

![Figure 1: Left lane and Right lane of the \( q_n \)-tower](image)

**Lemma 69** The measure of \( L_n \) is \([q_n \beta] - q_n \beta\) and the measure of \( R_n \) is \( q_n \beta - [q_n \beta] \).

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In the proof of Lemma 68, we showed that $L_n$ is \( \phi_0^{-1}(\bigcup_{j \leq q_n} [j/q_n, j/q_n + \gamma]) \) and $R_n$ is \( \phi_0^{-1}(\bigcup_{j < q_n} [j/q_n, j/q_n + \gamma]) \), where \( \gamma = (i + 1)/q_n - \beta \) and \( \beta \in [i/q_n, (i + 1)/q_n) \). Hence the measure of the left lane is \( q_n \gamma \). But \( q_n \gamma = [q_n \beta] - q_n \beta \). The argument for the right lane is similar.

\[ \neg \]

**Notation:** Let \( \beta_n = [q_n \beta] - q_n \beta \), the measure of the left lane at stage \( n \).

Restating the discussion:

**Lemma 70** For almost all \( s \) for all \( n \) that are mature for \( s \), \( S_{\beta}(s)(0) = s(i) \) where \( i \equiv_{q_n} d^n_L \) if \( s \in L_n \) and \( i \equiv_{q_n} d^n_R \) if \( s \in R_n \).

\[ \neg \]

Assume that \( n \) is mature for \( s \). Then on its principal \( n \)-block, \( s \) agrees with \( w_n^u \).\(^{24}\) The values \( s(0) \) and \( S_{\beta}(s)(0) \) are the \( r_n(s)^{th} \) and the \( r_n(S_{\beta}(s))^{th} \) values of the word \( w_n^u \). From equation 23, \( r_n(S_{\beta}(s)) = r_n(s) + d^n(s) \). Hence \( S_{\beta}(s)(0) = s(d^n(s)) \), and the lemma follows.

\[ \neg \]

The items in the following lemma are essentially Remark 11 and Lemma 55 in a different context.

**Lemma 71** For almost all \( s \) and for \( n < m \) that are mature for \( s \) and \( S_{\beta}(s) \)

1. If \( i \equiv_{q_n} r_n(s) + d^n(s) \) and \( j \equiv_{q_m} r_m(s) + d^m(s) \) then the \( j^{th} \) place the principal \( m \)-block of \( S_{\beta}(s) \) is in the \( i^{th} \) place of the principal \( n \)-block of \( S_{\beta}(s) \).

2. Let \( I \) be the \( r_n(s) + d^n(s)^{th} \) interval of \( T^n_m \) and \( J \) the \( r_m(s) + d^m(s)^{th} \) interval of \( T^m_n \) in the dynamical orderings. Then \( J \subseteq I \).

\[ \neg \]

This follows from Remark 54 and Lemma 55. To see this note that \( r_n(S_{\beta}(s)) \equiv_{q_n} r_n(s) + d^n(s) \); i.e. \( S_{\beta}(s)(0) \) is in the \( i^{th} \) place of the principal \( n \)-block of \( s \) where \( i \equiv_{q_n} r_n(s) + d^n(s) \).

\[ \neg \]

Thus typical points in \( R_n \) and \( L_n \) are those in which the \( n \)-block of \( S_{\beta}(x) \) containing 0 is the shift of the block of \( x \) containing 0 by \( d^n_R \) and \( d^n_L \) respectively.

**Lemma 72** Suppose that \( n \) is mature for \( sh^k(s) \) and \( sh^k(S_{\beta}(s)) \) for all positions \( k \) in the principal \( n \)-subword of \( s \). Then \( d^n \) is constant on \( \{sh^k(s) : k \text{ is in an even overlap}\} \) and on \( \{sh^k(s) : k \text{ is in an odd overlap}\} \).

\(^{24}\)Recall \( w_n^u \) is the notation we use for the unique member of the \( n^{th} \) element \( W_n^u \) of the construction sequence for \( K_a \).
Lemma 41, shows that on an overlap of shifted $n$-blocks, the relative positions of the 1-subsections were constant on the even overlaps (resp. odd overlaps) of the 2-subsections.

We now need a lemma we will use later for “nesting” arguments in Section 6.3. It says that the measure of the set of $s \in S$ with $d^m(s) = d^m_L$ or $d^m(s) = d^m_R$ can be closely computed as a density in every scale bigger than $n$.

**Lemma 73** Let $n < m \in \mathbb{N}$ be natural numbers. Then $\{0, 1, 2, \ldots, q_m - 1\} = P_L^n \cup U \cup P_R^n_{25}$ such that for almost every $s$ for which $n$ is mature:

1. If $r_n(s) \in P_L^n$, then $s \in L_n$,
2. If $r_n(s) \in P_R^n$ then $s \in R_n$,
3. $|U| \leq 2q_n$,
4. $|\frac{|P_L^n|}{q_m} - \beta_m| < \frac{2q_n}{q_m}$ and
5. $|\frac{|P_R^n|}{q_m} - (1 - \beta_m)| < \frac{2q_n}{q_m}$.

As in Lemma 68, let $\gamma = \frac{(i+1)}{q_n} - \beta$, where $i = p_nD_n(\beta)$ (See figure 1). The partition $I_{q_m}$ splits each interval $I \in I_{q_n}$ into $\frac{q_m}{q_n}$ subintervals. Let $U$ be the indices of the $I_{q_m}$ intervals that lie over or under $\gamma$ and $\gamma + \frac{1}{q_m}$. Explicitly: suppose that $\gamma \in I^m_{i_0}$ and $\gamma + \frac{1}{q_m} \in I^m_{i_1}$. Let

$$U = \{i : \text{ for some } 0 \leq j < q_n, I^m_i = R^j_n I^m_{i_0}\} \cup \{i : \text{ for some } 0 \leq j < q_n, I^m_i = R^j_n I^m_{i_1}\}.$$ 

Then $|U| = 2q_n$, and if $i \notin U$, then either:

$$I^m_i \subseteq \bigcup_{j<q_n} [j/q_n, j/q_n + \gamma) \quad (29)$$

$$I^m_i \subseteq \bigcup_{j<q_n} [(j+1)/q_n - \gamma, (j+1)/q_n) \quad (30)$$

For $i \notin U$, put $i \in P^m_L$ if it satisfies equation (29) and $i \in P^m_R$ if it satisfies equation (30). It follows that for almost all $s$, if $n$ is mature for $s$ and $r_n(s) \in$

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25Properly speaking the $P^m_R$ and $P^m_L$ notation should indicate $m$ as well. Without any contextual indication of what $m$ is we take $m = n + 1$. 45
$P^n_L$, then $d^n(s) = d^n_R$ and similarly for $P^n_R$. Since $P^n_R \cup P^n_L \cup U$ is a partition of $q_m$ and $|U| \leq 2q_n$, the lemma follows.

---

**Lemma 74** Let $f \in \{0, 1\}^\mathbb{N}$ and $s$ be a typical member of $S_\beta$.

1. Let $\beta^*_n = \frac{p_n D_n(\beta) + f(i)}{q_n}$. Then $(R_{\beta^*_n} : n \in \mathbb{N})$ converges to $R_\beta$ in the $C^\infty$-topology.

As a result, in the language of symbolic systems:

2. Let $A_n = D_n(\beta + \frac{f(i)}{q_n})$ and $T$ be the shift map on $\mathcal{K}_\alpha$. Then $A_n$ is either $d^n_L$ or $d^n_R$, depending on the value of $f$ and for almost every $s \in S$, $\lim_{n \to \infty} T A_n s = S_\beta(s)$.

3. With $A_n$ as in item 2 and $\mathbb{K}^c$ an arbitrary circular system with the given coefficient sequence $(k_n, l_n : n \in \mathbb{N})$, define $a_n$ and $b_n$ to be the left and right endpoints of the principal $n$-block of $T^{A_n}(s)$. Then for almost all $s$, $\lim_{n \to \infty} a_n = -\infty$ and $\lim_{n \to \infty} b_n = \infty$.

---

The first item follows because $|\beta^*_n - \beta| < 2/q_n$. Hence $\beta^*_n$ converges rapidly to $\beta$. The second item follows from the first via the isomorphism $\phi^{-1}_0$. The third item follows because $S_\beta(s) \in S$ and $T^{A_n}(s)$ converges to $S_\beta(s)$ topologically. Hence for all $n$ there is an $N$ such that for all $m \geq N$, the principal $n$-block of $T^{A_m}(s)$ is the same as the principal $n$-block of $S_\beta(s)$. Since the principal $m$-block of $T^{A_m}$ contains the principal $n$-block of $S_\beta(s)$ and $S_\beta(s) \in S$, item three follows.

If $a_n$ and $b_n$ are as in item 3, then:

$$a_n = -r_n(s) + A_n \quad \text{and} \quad b_n = q_n - r_n(s) + A_n. \quad (31)$$

---

6 **The Displacement Function**

If $\phi : \mathbb{K}^c \to (\mathbb{L}^c)^\pm$ is an isomorphism, where $\mathbb{K}^c$ and $\mathbb{L}^c$ are built from the same coefficient sequence, then we can canonically associate the rotation $R_\beta$ that intertwines the projections to the canonical circle factors $\mathbb{K}$ and $\mathbb{L}^{\pm 1}$.\(^{26}\)

In this section we define a function $\Delta$ from $S^1$ to the extended positive real numbers that will eventually be shown to have the properties that

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\(^{26}\)We use $\mathbb{L}$ for the notation for the rotation factor of a circular system $\mathbb{L}^c$. 

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\[ \Delta(\beta) < \infty \] implies that there is an element of the centralizer of \( \mathbb{K}^c \) having \( R_\beta \) as its associated rotation.

- if \( \mathbb{K}^c \) is built suitably randomly, then every element of the centralizer of \( \mathbb{K}^c \), or isomorphism from \( \mathbb{K}^c \) to \((\mathbb{K}^c)^{-1}\) has rotation factor \( \beta \) with \( \Delta(\beta) < \infty \).

Fix \( \beta \) for the rest of this section, and let \( T : K_\alpha \to K_\alpha \) be the shift map. The next lemma says that the principal \( n \)-blocks of \( T^{d^n(s)}(s) \) and \( S_\beta(s) \) are exactly aligned.

**Lemma 75** Let \( s,s^* \in K_\alpha \) be typical and \( n < m \) be mature for both. Define \( t^* = T^{d^m(s) - d^n(s)}(s^*) \). Then \( t^*(0) \) is in the same position of its principal \( n \)-block as \( s^*(0) \) is in \( s^* \)'s principal \( n \)-block. In particular, \( T^{d^m(s) - d^n(s)}(s^*) \) has its \( 0 \) in a position inside some copy of \( w_n^\alpha \).

First note that by Corollary 45 for \( n^* \geq n \), the principal \( n^* \)-blocks of \( s \) and \( s^* \) repeat several times on either side of the location of 0 in \( s \) and \( s^* \).

Consider the point \( s' = T^{d^n(s)}(s) \). Then \( s'(0) \) is in the \( (r_n(s) + d^m(s))^\text{th} \) place in its principal \( n \)-block. By Lemma 71, \( s'(0) \) is in the \( (r_n(s) + d^m(s))\)th place in its principal \( n \)-block. Since \( t = T^{d^m(s) - d^n(s)}(s') \), the point \( t \) has its 0 in the \( r_n(s)\)th place of its principal \( n \)-block. But this is where \( s(0) \) is.

The principal \( n,m \)-blocks repeat on both sides of \( s^*(0) \). The argument just given for \( s \) depends only the ordering of the levels of the \( m \)-tower—and these repeat cyclically both at level \( n \) and \( m \). Since this structure is the same for \( s^* \), we get the conclusion.

For a particular \( s \), the sequence of shifts \( T^{d^n(s)}(s) \) converges to \( S_\beta(s) \). Lemma 75 tells us that this happens promptly: for mature \( n \), \( T^{d^n(s)}(s) \) has it’s \( 0^{\text{th}} \) place in the same position of its principal \( n \)-block as \( S_\beta(s) \) does.

We now consider the location of 0 in the principal \( n+1 \)-block of the point \( T^{d^{n+1}(s) - d^n(s)}(s) \) relative to the position of 0 in the principal \( n+1 \)-block of \( s \). For some \( j_0 \) and \( j_1 \) the principal \( n \)-block of \( T^{d^{n+1}(s) - d^n(s)}(s) \) arises from the \( j_0^{\text{th}} \) argument of \( \mathcal{C}(w_n^\alpha, \ldots w_n^\alpha) \) and the principal \( n \)-block of \( s(0) \) is in a position coming from the \( j_1^{\text{st}} \) argument.

**Definition 76** With \( j_0 \) and \( j_1 \) as just described, we will say that the \( j_0^{\text{th}} \) argument of \( \mathcal{C}(w_n^\alpha, \ldots w_n^\alpha) \) \( \beta \)-matches the \( j_1^{\text{st}} \) argument. Say that \( s \) is well-\( \beta \)-matched at stage \( n \) if \( s \) is mature at \( n \) and \( j_0 = j_1 \). If \( n \) is mature for \( s \) and \( j_0 \neq j_1 \), then \( s \) is ill-\( \beta \)-matched.

\[ \text{See the motivating remarks in section 4.6.} \]
Lemma 77  Let \( s, s^* \in S \) and suppose that \( n \) is mature for \( \pi(s), \pi(s^*), S_\beta(\pi(s)) \) and \( S_\beta(\pi(s^*)) \) and that \( \pi(s) \) is well-\( \beta \)-matched at stage \( n \). Let \( A_n = d^n(s) \) and \( A_{n+1} = d^{n+1}(s) \). Then:

1. \( r_n(T^{A_n}s^*) = r_n(T^{A_{n+1}}s^*) \) and
2. if \( I \) is the interval \([−r_n(T^{A_n}s^*), q_n − r_n(T^{A_n}s^*)] \subseteq \mathbb{Z} \), then \( (T^{A_n}s^* \upharpoonright I) = (T^{A_{n+1}}s^* \upharpoonright I) \)

Lemma 75 asserts that 0 is in the same place in the principal \( n \)-block of \( T^{A_{n+1}−A_n}(\pi(s^*))0 \) as 0 is in the principal \( n \)-block of \( \pi(s^*) \). Since \( n \) is mature for \( s^* \), the principal \( n \)-block of \( s^* \) is repeated on either side of \( s^*(0) \). Since \( n \) is mature for \( S_\beta(s^*) \), the principal \( n \)-block of \( T^{A_{n+1}}s^* \) is repeated at least twice on either side of \( T^{A_{n+1}}s^*(0) \). It follows that 0 is in the same place in the principal \( n \)-block of \( T^{A_n}(T^{A_{n+1}−A_n}(s^*)) \) as 0 is in the principal \( n \)-block of \( T^{A_n}(s^*)(0) \). This proves the first assertion.

A repetition of this argument shows the second assertion as well, using the fact that \( s \) is well-\( \beta \)-matched. Indeed the definition of well-\( \beta \)-matched implies that the principal \( n \)-words of \( T^{A_{n+1}−A_n}s \) and \( s \) are identical. Applying \( T^{A_n} \) to both, and using the fact that the principal \( n \)-words repeat one sees that the principal \( n \)-words \( T^{A_{n+1}−A_n}s \) and \( T^{A_n}s \) are identical. Since the issue of alignment only involves \( \pi(s) \), item 2 holds for all \( s^* \) with \( \pi(s) = \pi(s^*) \). Moreover, arguing as in the last paragraph using the repetition of the principal \( n \)-blocks, shifting by an \( l < q_n \) does not change this.

6.1 The Definition of \( \Delta \)

Let \((X, B, \mu, T) = (\mathbb{K}^c, B, \nu, sh)\) be circular system. Define

\[
\Delta_n(\beta) = \nu(\{s : s \text{ is ill-}\beta\text{-matched at stage } n\})
\]

and set

\[
\Delta(\beta) = \sum_n \Delta_n(\beta).
\]

Definition 78 The number \( \beta \in S^1 \) is a central value iff \( \Delta(\beta) < \infty \).

We note that \( \Delta(\beta) \) is defined using the block structure of the \( W_n \) and hence is determined by \( \beta \) together with the sequences \( \langle k_n \rangle \) and \( \langle l_n \rangle \). Thus for \( \beta \in S^1 \) the property of being central depends only on the circular coefficient.
sequence \( (k_n, l_n : n \in \mathbb{N}) \), rather than on the particular circular system \( \mathbb{K}^c \). In section 7.1, we show that if \( \Delta(\beta) \) is finite then there is an element \( T^* \) in the weak closure of \( \{ T^n : n \in \mathbb{Z} \} \) such that \( (T^*)^\pi = S_\beta \). In particular \( \beta \) is the rotation factor of an element of the centralizer. That fact does not rely on the results of the rest of this section.

6.2 Deconstructing \( \Delta(\beta) \)

Fix a \( \beta \). Let \( \langle \varepsilon_n : n \in \mathbb{N} \rangle \) be the summable sequence defined in section 4.5. Recall that Numerical Requirement 2 says: \( \varepsilon_N > \sum_{n>N} \varepsilon_n \).

For a typical \( s \) and mature \( n \) there are four ways that \( s \) can occur ill-\( \beta \)-matched:

1. \( d^n(s) = d^n_L \) or \( d^n_R \) and
2. \( d^{n+1}(s) = d^{n+1}_L \) or \( d^{n+1}_R \)

Call these possibilities \( P_{LL}, P_{LR}, P_{RL}, P_{RR} \).

**Lemma 79** Let \( n, m \in \mathbb{N} \) with \( n + 1 < m \). There is a partition \( \{ P^{n,m}_{hd_1,hd_2} : \) \( \) \( hd_1,hd_2 \in \{ L,R \} \} \cup \{ U \} \) of the set \( \{ 0, 1, \ldots q_m - 1 \} \) such that for \( s \in S \), if \( n \) is mature for \( s \) then

1. \( r_m(s) \in P^{n,m}_{hd_1,hd_2} \) implies \( (d^n(s), d^{n+1}(s)) = (d^n_{hd_1}, d^{n+1}_{hd_2}) \)
2. \( |U| \leq 2q_n + 2q_{n+1} \).

This follows immediately from Lemma 73 by holding \( m \) fixed and applying the lemma successively to \( n \) and \( n + 1 \). Except for a set \( U = \text{def} U_n \cup U_{n+1} \) that has at most \( 2q_n + 2q_{n+1} \) elements, every point in \( \{ 0, 1, \ldots q_m - 1 \} \) belongs to some \( P^{n}_i \cap P^{n+1}_j \).

The levels of the \( q_m \)-tower reflect the construction of \( w^\alpha_m \) from \( n \)-words with \( n < m \). If \( s \) and \( S_\beta(s) \) are mature at stage \( n < m \), then the locations of \( s(0) \) and \( T^{A_{n+1}-A_n}(s)(0) \) in their principal \( m \)-block and the pair \( (d^n(s), d^{n+1}(s)) \) determine whether \( s \) is ill-\( \beta \)-matched or not. For particular choices of \( hd_1,hd_2 \in \{ L,R \} \) either: all typical \( s \) in \( P_{hd_1,hd_2} \) with \( n \) mature for \( s \) and \( S_\beta(s) \) are well-\( \beta \)-matched or none are.

In the next section we will fix a particular choice of \( hd_1 \) and \( hd_2 \). For now let \( n, hd_1 \) and \( hd_2 \) be such that all \( n \)-mature \( s \) in configuration \( P_{hd_1,hd_2} \)
are ill-β-matched. Let

\[ \mathcal{Y}_n = \{ s : s \text{ is ill-β-matched at stage } n \text{ and in configuration } P_{hd_1, hd_2} \}. \]  

(35)

We need to localize the sets \( \mathcal{Y}_n \). The next lemma tells us that they are uniformly close to open sets:

**Proposition 80** Let \( n, m \in \mathbb{N} \) with \( n + 1 < m \). Then there is a set \( d^{n,m} \subseteq \{0, 1, \ldots q_m - 1\} \) such that if \( s \in S \), \( n \) is mature for \( s \) and \( r_m(s) + k \in d^{n,m} \), then

1. \( n \) is mature for \( sh^k(s) \),
2. \( d^n(sh^k(s)) = d^n_{hd_1} \) and \( d^{n+1}(sk^k(s)) = d^{n+1}_{hd_2} \).
3. \( sh^k(s) \in \mathcal{Y}_n \) and

\[
\left| \frac{|d^{n,m}|}{q_m} - \nu(\mathcal{Y}_n) \right| < 2 \left( \frac{q_n + q_{n+1}}{q_m} \right) + \frac{1}{l_{n-1}} + \varepsilon_{n-1}.
\]

\[ \vdash \] Let \( s \) be an arbitrary point in \( S \) that is mature for \( n \). Take \( d^{n,m} \) to be those numbers of the form \( r_m(s) + k \) such that \( sh^k(s) \) has its zero point in \( P_{hd_1, hd_2} \) and \( n \) is mature for \( sh^k(s) \). By Lemma 48, the collection of \( k \) such that \( sh^k(s) \) is not mature for \( n \) has density at most \( \frac{1}{l_{n-1}} + \varepsilon_{n-1} \).

\[ \neg \]

### 6.3 Red Zones

Suppose that \( \beta \) is not central, i.e. that \( \Delta(\beta) = \infty \). Then for some fixed choice of \( \{hd_1, hd_2\} \in \{L, R\} \),

\[
\sum_n \nu(\{ s : s \text{ is ill-β-matched at stage } n \text{ and in configuration } P_{hd_1, hd_2} \})
\]

is infinite. Fix such an \( hd_1, hd_2 \). Then for all \( n, \mathcal{Y}_n \) is well defined, and moreover there is a set \( G \) such that if \( n < m \) belong to \( G \) then \( n + 2 < m \) and

\[
\sum_{n \in G} \nu(\mathcal{Y}_n) = \infty.
\]

(36)

Let \( s \) be a point in \( K_{\alpha} \) such that all of the shifts of \( s \) and \( S_\beta(s) \) are generic with respect to basic open sets, the \( E_i \)'s, \( \mathcal{Y}_n \), \( P_{hd_1, hd_2} \) and the sets \( L_n, R_n \).

\[ ^{28} \text{We use the symbol } \mathcal{Y}_n \text{ to indicate the misaligned points at stage } n. \]
For large enough \( M \), we use \( s \) and \( \bigcup_{n \in G} \Psi_n \) to identify a subset of the interval \( [-r_M(s), q_M - r_M(s)] \) of density bigger than \( 3/4 \).

For the moment we assume that \( s \in \Psi_n \) and \( n \) is mature for \( s \). In defining \( \Psi_n \), the choice that \( (d^n(s), d^{n+1}(s)) = (hd_1, hd_2) \) together with \( s(0) \), give us the relative locations of the overlap of the principal \( n+1 \)-blocks of \( s \) and \( S_\beta(s) \).

Let \( u \) be the principal \( n+1 \)-block of \( s \) and \( v \) be the principal \( n+1 \)-block of \( S_\beta(s) \) and assume that they are in the position determined by \( d^n(s) \). By Lemma 41, on the overlap the 2-subsections of \( v \) split the 2-subsections of \( u \) into either one or two pieces, and the positions of all of the even pieces are shifted by the same amount relative to the 2-subsections of \( v \) and similarly for the odd pieces.

We analyze the case where \( s(0) \) occurs in an \( n+1 \)-block where the 2-subsections are split into two pieces. If they are only split into one piece (i.e. they aren’t split) the analysis is similar and easier. Without loss of generality we will assume that \( s(0) \) occurs in an even overlap.

Since neither \( s(0) \), nor \( S_\beta(0) \) occur in the first or last \( \varepsilon_n k_n \) 1-subsections of the principal 2-subsection that contains them, we know that the overlaps of the principal 2-subsections of \( s(0) \) and \( S_\beta(s)(0) \) contain at least \( \varepsilon_n k_n \) 1-subsections. The subsections of the form \( w_j^{l_n-1} \) of each 1-subsection of \( s \) in this overlap are split into at most three pieces, powers of the form \( w_i^{s_0^n j}, w_i^r \) and \( w_i^{s_1^n j} \) where \( 0 \leq r \leq 2, l_n - (s_0^n + s_1^n) \leq 3 \) and the middle power \( w_i^r \) crosses a boundary section of \( S_\beta(s) \). The powers \( s_0^n \) and \( s_1^n \) are constant on the overlap of the 2-subsections, constant in all of the even pieces of the overlap of the 2-subsections of the principal \( n+1 \)-block, and are determined by \( (hd_1, hd_2) \). Moreover, \( s_0^n > \varepsilon_n k_n \). Again, without loss of generality we assume that \( s(0) \) is in the left overlap corresponding to the power \( s_0^n \).

**Observation:** There is a number \( j_0 \) between 0 and \( k_n - 1 \) determined by the pair \( (d^n(s), d^{n+1}(s)) \) such that the even piece of a 2-subsection that contains \( s(0) \) is of the form \( \prod_{j \leq j_0} b^{l_n - j_i} w_j^{l_i - 1} e_j \), except that the last 1-subsection may be truncated. Moreover, since \( d^{n+1}(sh^k(s)) \) is constant for \( k \) in the principal \( n+1 \)-block of \( s \), if

\[
t = k_n - j_0,
\]

then \( t \neq 0 \) and for all \( j < j_0 \) the powers \( w_j^{s_0^n} \) are well-\( \beta \)-matched with \( w_j^{s_0^n + t} \) except for portions of the first and last power.

In particular, if \( k \) is such that the 0 position of \( sh^k(s) \) lies in the interior of initial power \( w_j^{s_0^n} \) in an even overlap and \( j < j_0 \), then \( sh^k(s) \in \Psi_n \) because it is lined up with \( w_{j+t} \).
Lemma 81 Suppose that $s, S_\beta(s)$ and all of their translates are generic, and that $s$ is mature at $n$. Suppose that $m > n + 2$. Then there is a set $B_n \subseteq \{0, \ldots, q_m - 1\}$ such that if $k \in [-r_m(s), q_m - r_m(s))$ and $r_m(s) + k \in B_n$, then:

1. $sh^k(s)$ has its zero located in $B_n$,
2. $n$ is mature for $sh^k(s)$,
3. $sh^k(s) \notin \psi_n$,
4. There is a $j_0 > \varepsilon_n k_n$ and a $t \neq 0$ such that $B_n$ is:

(a) a union of sets, each of the form $\bigcup_{j < j_0} U_j \subseteq \{0, 1, 2, \ldots, q_m - 1\}$,
(b) each set $\bigcup_{j < j_0} U_j$ is a subset of a position of an occurrence of an $n + 1$-subword $C(u_0, u_1, \ldots, u_{k_n - 1})$ of $w_m$ (with $u_i = w_m^\alpha$),
(c) each $U_j$ is a collection of positions of $u_j^{s_0}$ such that $u_j^{s_0}$ is $\beta$-matched with $u_j^{s_0 + t}$, except perhaps for the first or last copy of $u_j$ in $u_j^{s_0}$

and

\[ |B_n - \nu(\psi_n)| < 2 \left( \frac{q_n + q_{n+1}}{q_m} \right) + \frac{1}{l_n - 1} + \varepsilon_{n-1}. \]

The first statement is immediate. Let $d^{n,m}$ be as in Proposition 80. If $k \in d^{n,m}$ then, as in the discussion before the statement of Lemma 81, $sh^k(s)(0)$ occurs in the position of a power $u_j^{s_0}$, where $u$ is the principal $n$-block of $sh^k(s)$ and $u_j^{s_0}$ occurs on the left overlap of 1-subsections of the principal $n + 1$-block of $sh^k(s)$.

As in the observation before this lemma, to each $k \in d^{n,m}$ we can associate a set $\bigcup_{j < j_0} U_j$ containing $k$ by taking all of the positions of the powers $u_j^{s_0}$ in the even overlap determined $sh^k(s)(0)$. Let $B_n$ be the union of all of the collections $\bigcup_{j < j_0} U_j$ as $k$ ranges over $d^{n,m}$.

Assertion 4(c) follows from the observation and the fact that $d^n$ and $d^{n+1}$ are constant (and equal to $d^{n,\beta}_d$ and $d^{n+1,\beta}_d$) on $d^{n,m}$.

We show that if $k' \in B_n$ then $n$ is mature for $sh^{k'}(s)$ and that $sh^{k'}(s) \notin \psi_n$. The maturity of $n$ follows immediately from the maturity of $s$ and the fact
that the location of 0 \( s \) is in an \( n \)-subword of its principal \( n + 1 \)-block.

That \( s \) follows from the fact that \( u_j \) is \( \beta \)-matched with \( u_j \), and \( t \neq 0 \).

To finish, note that

\[
d^{m,m} \subseteq \bigcup_{j=j_0} U_j \subseteq \mathcal{V}_n.
\]

Hence

\[
\frac{|d^{m,m}|}{q_n} \leq \frac{\big| \bigcup_{j=j_0} U_j \big|}{q_n} \leq \nu(\mathcal{V}_n).
\]

Thus Lemma 81 follows from Lemma 80.

We now define the red zones corresponding to \( \beta \). Recall that if \( n < m \in G \) then \( n + 2 < m \) and \( \sum_{n \in G} \nu(\mathcal{V}_n) = \infty \). For \( n < m \) consecutive elements of \( G \), define

\[
\delta_n = 4 \left( \frac{q_n}{q_{n+1}} \right) + \frac{1}{l_{n-1}} + \varepsilon_{n-1}
\]

Then we see that:

- \( \sum_{n \in G} \delta_n < \infty \), so
- \( \sum_{n \in G} (\nu(\mathcal{V}_n) - \delta_n) = \infty \)

and if \( B_n \) is the set defined in Lemma 81, then \( \nu(\mathcal{V}_n) - \delta_n \leq \frac{|B_n|}{q_m} \leq \nu(\mathcal{V}_n) \).

**Lemma 82** Let \( N \) be a natural number and \( \delta > 0 \). Suppose that \( s, S_\beta(s) \) and all of their translates are generic, and that \( s \) is mature at \( N \). Then there is a sequence of natural numbers \( \langle n_i : 1 \leq i \leq i^* \rangle \), an \( M \) and sets \( R_i \subseteq \{0, 1, 2 \ldots q_M - 1\} \), for \( 1 \leq i \leq i^* \), such that

1. \( N < n_1 \) and \( n_i + 2 < n_{i+1} < M \),
2. \( R_i \) is disjoint from \( R_j \) for \( i \neq j \),
3. \( R_i \) is a union of blocks of the form \( B_{n_i} \), described in condition 4 in Lemma 81 inside \( n_{i+1} \)-subwords of \( \omega_M^\alpha \)
4. if \( k \in R_i \), then \( s \) is in \( \mathcal{V}_{n_i} \),
5. the density of \( \bigcup_i R_i \) in \( \{0, 1, \ldots q_M - 1\} \) is at least \( 1 - \delta \).
We can assume that \( N \) is so large that \( \bigcup_{n \geq N} \partial_n \) has measure less than \( \delta/100 \) and \( 1/l_N + \varepsilon_N < \delta/100 \). From the definition of \( G \) we can find a collection \( \langle \sigma_i : i \leq i^* \rangle \) of consecutive elements of \( G \) so that
\[
\prod_{1 \leq i \leq i^*} (1 - \nu(\mathcal{F}_{\sigma_i}) + \delta_{\sigma_i}) < \delta/100.
\]
Choose an \( M > n_{i^*} + 2 \), and for notation purposes set \( n_{i^*+1} = M \).

Define sets \( R_i \) and \( I_i \) by reverse induction from \( i = i^* \) to \( i = 1 \) with the following properties:

i. \( \{0, 1 \ldots q_M - 1\} \setminus ((\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j)) \) consists of entire locations of words \( w_{n_i}^* \) in \( w_M^n \),

ii. \( R_i \subset \{0, 1 \ldots q_M - 1\} \setminus ((\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j)) \) and has relative density at least \( \nu(\mathcal{F}_{\sigma_i}) - \delta_{\sigma_i} \),

iii. the set \( I_i \subset \bigcup_{j=n_i+1}^{n_i+1} \partial_j \cap \{0, 1 \ldots q_M - 1\} \) and hence,

iv. \( I_i \) has density less than or equal \( 1/l_{n_i} \) in \( \{0, 1 \ldots q_M - 1\} \)

To start, apply Lemma 81 with \( m = n_{i^*+1} \), to get a set \( B_{n_{i^*}} \subset \{0, 1 \ldots q_M - 1\} \) of density at least \( \nu(\mathcal{F}_{\sigma_{i^*}}) - \delta_{\sigma_{i^*}} \) satisfying conditions 3-4 of the lemma we are proving. Set \( R_{i^*} = B_{n_{i^*}} \). Let \( I_{i^*} = \bigcup_{j=n_{i^*}+1}^{M} \partial_j \cap \{0, 1 \ldots q_M - 1\} \).

Suppose that we have defined \( \langle R_j : i^* \geq j > i \rangle \) and \( \langle I_j : i^* \geq j > i \rangle \) satisfying the induction hypothesis (i-iv).

Apply Lemma 81 again to get a set \( B = B_{n_i} \subset \{0, 1 \ldots q_{n_i+1} - 1\} \). Inside each copy \( \{k, k+1, \ldots, k+q_{n_i+1} - 1\} \) corresponding to a location in \( w_M^n \) of a \( w_{n_{i+1}}^\alpha \) in the complement of \((\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j))\), we have a translated copy of \( B, k + B \). Let \( R_i \) be the union of the sets \( k + B \) where \( k \) runs over the locations the words \( w_{n_{i+1}}^\alpha \) in the complement of \((\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j))\).

Then the density of \( R_i \) relative to
\[
\{0, 1, \ldots q_M - 1\} \setminus ((\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j))
\]
is at least \( \nu(\mathcal{F}_{\sigma_i}) - \delta_{\sigma_i} \). It follows from conclusion 3 of lemma 81 that \( R_i \) is a union of blocks of length \( q_{\alpha_{n_i}} - 1 \) corresponding to positions of \( w_{n_i}^\alpha \) in \( w_{n_i}^\alpha \).

Since \( R_i \) consists of a union of the entire locations of words \( w_{n_i}^\alpha \),
\[
\{0, 1, \ldots q_M - 1\} \setminus \left( (\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j) \cup R_i \right)
\]

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consists of the entire blocks of locations of $w_{n_i}^\alpha$ together with elements of \( \bigcup_{j=n_i+1}^{n_{i+1}} \partial_j \). The latter set has density less than or equal to \( \frac{1}{l_{n_i}} \). Let
\[
I_i = \left( \{0,1,\ldots,q_M-1\} \cap \bigcup_{j=n_i+1}^{n_{i+1}} \partial_j \right) \setminus \left( \bigcup_{i^* \geq j > i} I_j \right) \setminus \left( \bigcup_{i^* \geq j > i} R_j \right) \cup R_i.
\]

It remains is to calculate the density of \( \bigcup_{1 \leq i \leq i^*} R_i \). At each step in the induction, we remove a portion of density at least \( \nu(\psi_{n_i}) - \delta_{n_i} \) from \( \{0,1,\ldots,q_M-1\} \setminus (\bigcup_{i^* \geq j > i} I_j) \cup (\bigcup_{i^* \geq j > i} R_j) \). Let \( \partial = \bigcup_{1 \leq i \leq M} \partial_{n_i} \). Then the density of the union of the \( R_i \)'s is at least
\[
1 - \prod_{i^* \geq 1} (1 - \psi_{n_i}) - |\partial|/q_m,
\]
which is at least \( 1 - \delta \).

7 The Centralizer and Central Values

In the first part of this section we show that every central value is rotation factor of an element of the closure of the powers of \( T \) and hence an element of the centralizer.

The second part shows a converse: if \( \mathbb{K}^c \) is built sufficiently randomly then the rotation factor of every element of the centralizer is a rotation by a central value.

We note in passing that every circular system is rigid: if \( s \) is mature for \( n \), then \( T^{q_n(l_n-2)}(s) \) has the same principal \( n \)-block as \( s \) does. Thus the identity in \( C(T) \) is not isolated. It follows that \( \{T^n : n \in \mathbb{Z}\} \) is a perfect Polish monothetic group.

7.1 Building Elements of the Centralizer

If \( \Delta(\beta) \) is finite, then the Borel-Cantelli lemma implies that for \( \nu \)-almost every \( s \), there is an \( n_0 \) such that for all \( n \geq n_0 \), \( s \) is well-\( \beta \)-matched at stage \( n \). As a consequence, certain sequences of translations converge. Precisely:

**Theorem 83** Suppose that \( \mathbb{K}^c \) is a uniform circular system with coefficient sequence \( \langle k_n, l_n : n \in \mathbb{N} \rangle \). Let \( T \) be the shift map on \( \mathbb{K}^c \) and \( \beta \in [0,1) \) be a number such that \( \Delta(\beta) < \infty \). Then there is a sequence of integers \( \langle A_n : n \in \mathbb{N} \rangle \) such that \( \langle T^{A_n} : n \in \mathbb{N} \rangle \) converges pointwise almost everywhere
to a $T^* \in C(T)$ with $(T^*)^\pi = S_\beta$. In particular there is a sequence $\langle A_n : n \in \mathbb{N} \rangle$ such that $\langle T^{A_n} : n \in \mathbb{N} \rangle$ converges in the weak topology to a $T^*$ with $(T^*)^\pi = S_\beta$.

**Corollary 84** If $\beta$ is central, then there is a $\phi \in \{ T^n : n \in \mathbb{Z} \}$ such that $\phi^\pi = S_\beta$.

Let $T$ be the tree of finite sequences $\pi \in \{ L,R \}^{<\infty}$. Choose an $n_0$ such that $G = \{ s : n_0 \text{ is mature for } s \text{ and for all } m \geq n_0, s \text{ is well-}\beta\text{-matched at stage } m \}$ has positive measure. By the König Infinity Lemma there is a function $f : \{ m : m \geq n_0 \} \to \{ L,R \}$ such that for all $m \geq n_0$, $\{ s \in G : d^\pi(s) = d^\pi_f(n) \}$ for all $n$ with $n_0 \leq n \leq m$ has positive measure. Let $A_n = d^m_f(n)$.

By Lemma 74, item 3 it follows that for a typical $s$ the left and right endpoints of the principal $n$-blocks of $T^{A_n}s$ go to negative and positive infinity respectively. Let $s^*$ be a typical element of $S$; e.g. $\pi(s^*)$ and $S_\beta(\pi(s^*))$ both belong to $S^\pi$, large enough $n$ are mature for $s^*$ and for all large $n$, $\pi(s^*)$ is well-\beta-matched at stage $n$. Then for all large $n$, the left and right endpoints of the principal $n$-block of $T^{A_n}s$ and $T^{A_{n+1}}s$ are the same. If $s^*$ is well-\beta-matched at stage $n$, then the words constituting principal $n$-block of $T^{A_n}s$ and $T^{A_{n+1}}s$ are the same. It follows that for typical $s^* \in S$, the sequence $T^{A_n}s^*$ converges in the product topology on $(\Sigma \cup \{ b,e \})^\mathbb{Z}$.

We now show that the map $s \mapsto \lim T^{A_n}s$ is one-to-one. If $s \neq s'$, then either $\pi(s) \neq \pi(s')$ or there is an $N$ such that for all $n \geq N$ the principal $n$-blocks of $s$ and $s'$ differ. We can assume that this $N$ is so large that $n$ is mature and well-\beta-matched for $\pi(s)$, $\pi(s')$.

If $\pi(s) \neq \pi(s')$, then $S_\beta(\pi(s)) \neq S_\beta(\pi(s'))$. Hence the limits of $T^{A_n}s$ and $T^{A_n}s'$ differ. So assume that $\pi(s) = \pi(s')$. Then, since $T^{A_n}$ is a translation by at most $q_n - 1$ and $n$ is mature for all parties (so the principal $n$-blocks of $T^{A_n}s$ and $T^{A_n}s'$ repeat) we know that the principal $n$-blocks of $T^{A_n}s$ and $T^{A_n}s'$ differ. But for all $m > n$, the principal $n$-blocks of $T^{A_m}s$ agree with the principal $n$-blocks of $T^{A_n}s$ (and similarly for $s'$). Hence the limit map is one-to-one.

We need to see that for almost all $s$, $\lim_{n \to \infty} T^{A_n}s$ belongs to $\mathbb{K}^r$. By definition of $\mathbb{K}^r$ this is equivalent to showing that for almost all $s$ if $I \subseteq \mathbb{Z}$ is an interval, then $\lim_{n \to \infty} T^{A_n}s \mid I$ is a subword of some $w \in \mathcal{W}_m^c$ for some $m$. However, by Lemma 77, for almost all $s$ we can find an $n$ so large that:

1. $I \subseteq [-r_n(s), q_n - r_n(s)]$,  

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2. $T^{A_n}s$ and $\lim_{n \to \infty} T^{A_n}s$ agree on the location of the principal $n$-block of containing $I$, and

3. $T^{A_n}s$ and $\lim_{n \to \infty} T^{A_n}s$ agree on what word lies on the principal $n$-block.

Since the principal $n$-block of $T^{A_n}s$ belongs to $W^c_n$, we are done.

Summarizing, if $T^* = \lim_{n \to \infty} T^{A_n}s$, then for almost all $s$, $T^*s$ is defined and belongs to $S$. Moreover $T^*$ is one-to-one and commutes with the shift map.

Define a measure $\nu^*$ on $S$ by setting $\nu^*(A) = \nu((T^*)^{-1}A)$. Then $\nu^*$ is a non-atomic, shift invariant measure on $S$. By Lemma 37, we must have $\nu^* = \nu$. In particular we have shown that $T^* : K^c \to K^c$ is an invertible measure preserving transformation belonging to $\{T^n : n \in \mathbb{Z}\}$, with $(T^*)^\pi = S_\beta$.

We make the following remark without proof as it is not needed in the sequel:

**Remark 85** Suppose that $K^c$ satisfies the hypothesis of Theorem 83 and $\beta$ is a central value. Then for any sequence of natural numbers $\langle A_n : n \in \mathbb{N} \rangle$ such that $A_n \alpha$ converges to $\beta$ sufficiently fast, the sequence $\langle T^{A_n} : n \in \mathbb{N} \rangle$ converges to a $T^* \in C(T)$ with $(T^*)^\pi = S_\beta$.

### 7.2 Characterizing Central Values

In this section we prove a converse of Corollary 84. We show that if $K^c$ is a circular system built from sufficiently random collections of words and $\phi$ is an isomorphism between $K^c$ and $K^c$ then $\phi^\pi = S_\beta$ for some central $\beta$ and if $\phi$ is an isomorphism between $K^c$ and $(K^c)^{-1}$ then $\phi^\pi$ is of the form $\sharp \circ S_\beta$ for some central $\beta$.

#### 7.2.1 The Timing Assumptions

We now describe the randomness assumptions about the words in the $W^c_n$’s that will allows us to assert that that the rotations associated with elements of the centralizer of $K^c$ or isomorphisms between $K^c$ and $(K^c)^{-1}$ arise from central $\beta$’s. In the last part of the paper we explain why these properties are consistent with the randomness assumptions used in [8] and how to build words with both collections of specifications.
Recall from Definition 33, that to specify a circular system with coefficient sequence \( \langle k_n, l_n : n \in \mathbb{N} \rangle \) it suffices to inductively specify collections of prewords \( P_{n+1} \subseteq (W_n^c)^{k_n} \), and define \( W_{n+1}^c \) as the collection of words:

\[
\{ C(w_0, \ldots w_{k_n-1}) : w_0 w_1 \ldots w_{k_n-1} \in P_{n+1} \}.
\]

In the construction, there will be an equivalence relation \( Q_1 \) on \( W_1^c \) that is lifted from an analogous equivalence relation on the first step of the odometer construction \( W_1 \). We describe its properties here; in section 9 we explain how it is built. Let \( \langle Q_n^1 : n \in \mathbb{N} \rangle \) be the sequence of propagations of \( Q_1 \). As the construction progresses there are groups \( G_n^1 \) acting freely on the set of \( Q_n^1 \) equivalence classes of words in \( W_n^c \). Each \( G_n^1 \) is a finite sum of copies of \( \mathbb{Z}_2 \). Inductively, \( G_n^{n+1} = G_n^1 \) or \( G_n^{n+1} = G_n^1 \oplus \mathbb{Z}_2 \). The action of \( G_n^1 \) on \( W_{n+1}^c \) arising from the \( G_n^{n+1} \) action via the inclusion map of \( G_n^1 \) into \( G_n^{n+1} \) is the skew-diagonal action. We will write \( [w]_1 \) for the \( Q_n^1 \)-equivalence class of a \( w \in W_n^c \) and \( G_n^1 [w]_1 \) for the orbit of \( [w]_1 \) under \( G_n^1 \). If \( w \in W_{n+1}^c \) and \( C \in W_n^c/Q_n^1 \) then we say that \( C \) occurs at \( t \) if there is a \( v \in W_n^c \) sitting on the interval \( [t, t + q_n) \) inside \( w \) and \( C = [v]_1 \).

**Notation:** As an aid to tracking corresponding variables script letters are used for sets and non-script Roman letters for the corresponding cardinalities. For example we will use \( Q_n \) for an equivalence relation and \( Q_n \) for the number of classes in that equivalence relation.

Here are the the assumptions used to prove the converse to Corollary 84. The first three axioms follow immediately from the definitions in section 4.9.

**T1** The equivalence relation \( Q_{n+1}^1 \) is the equivalence relation on \( W_{n+1}^c \) propagated from \( Q_n^1 \).

**T2** \( G_n^1 \) acts freely on \( W_n/Q_n^1 \cup \text{rev}(W_n/Q_n^1) \)

**T3** The canonical generators of \( G_n^1 \) send elements of \( W_n^c/Q_n^1 \) to elements of \( \text{rev}(W_n^c/Q_n^1) \) and vice versa.

The next axiom states that the \( Q_n^1 \) classes are widely separated from each other.

**T4** There is a \( \gamma \) such that \( 0 < \gamma < 1/4 \) such that for each pair \( w_0, w_1 \in \)
\[ W^c_n \cup \text{rev}(W^c_n) \text{ and each } j \geq \frac{q_n}{2} \text{ if } [w_0]_1 \neq [w_1]_1, \text{ then:} \]
\[
\bar{d}(w_0 \upharpoonright [0, j], w_1 \upharpoonright [0, j]) \geq \gamma, \\
\bar{d}(w_0 \upharpoonright [q_n - j, q_n], w_1 \upharpoonright [q_n - j, q_n]) \geq \gamma
\]
and
\[
\bar{d}(w_0 \upharpoonright [0, j], w_1 \upharpoonright [q_n - j, q_n]) \geq \gamma.
\]

Remark 86 In the axioms T5 – T7 we write \(|x_n| \approx \frac{1}{y_n}\) to mean that \(||x_n| - \frac{1}{y_n}| < \mu_n\) where \(\mu_n \ll \min(\varepsilon_n, 1/Q_1^n)\).

Numerical Requirement 4 \(\mu_n\) is chosen small relative to \(\min(\varepsilon_n, 1/Q_1^n)\).

In the next assumption we count the occurrences of particular \(n\)-word \(v\) that are lined up in an \(n + 1\)-preword \(w_0\) with the occurrences of a particular \(Q_1^n\)-class in the shift of another \(n + 1\)-preword \(w_1\) or its reverse. The shift (by \(t\) \(n\)-subwords), must be non-zero and be such that there is a non-trivial overlap after the shift.

T5 Let \(w_0, w_1\) be prewords in \(P_{n+1}\), and \(w'_1\) be either \(w_1\) or \(\text{rev}(w_1)\). Write \(w_0 = v_0v_1 \ldots v_{k_n-1}\) and \(w'_1 = u_0u_1 \ldots u_{k_n-1}\), with \(u_i, v_j \in W^c_n \cup \text{rev}(W^c_n)\). Let \(C \in W^c_n/Q_1^n\) or \(C \in \text{rev}(W^c_n)/Q_1^n\) according to whether \(w'_1 = w_1\) or \(w'_1 = \text{rev}(w_1)\). For all integers \(t\) with \(1 \leq t \leq (1 - \varepsilon_n)(k_n)\), \(v \in W^c_n\):

T5a (This is comparing \(w_0\) with \(sh^{\varepsilon_n}(w'_1)\).) Let
\[
J(v) = \{k < k_n - t : v = v_k\}.
\]

Then
\[
\frac{|\{k \in J(v) : u_{t+k} \in C\}|}{|J(v)|} \approx \frac{1}{Q_1^n}.
\]

T5b (This is comparing \(sh^{\varepsilon_n}(w_0)\) with \(w'_1\).) Let
\[
J(v) = \{k : t \leq k \leq k_n - 1 \text{ and } v = v_k\}.
\]

Then
\[
\frac{|\{k \in J(v) : u_{t-k} \in C\}|}{|J(v)|} \approx \frac{1}{Q_1^n}.
\]
T6 Suppose that \( w_0 w_1 \ldots w_{k_n-1}, w'_0 w'_1 \ldots w'_{k_n-1} \in P_{n+1} \) are prewords, \( 1 \leq t \leq (1 - \varepsilon_n) k_n \) and \( \varepsilon_n k_n \leq j_0 \leq k_n - t \). Let 

\[ S = \{ k < j_0 : \text{for some } g \in G^n_1, g[w_k]_1 = [w'_{k+t}]_1 \}. \]

Then:

\[ \frac{|S|}{j_0} \approx \frac{|G^n_1|}{Q^n_1}. \]

T7 Let \( w_0, w_1 \) be prewords in \( P_{n+1} \), and \( w'_1 \) be either \( w_1 \) or \( \text{rev}(w_1) \). Suppose that \( [w'_1]_1 \notin G^n_1[w_0]_1 \). Write \( w_0 = v_0 v_1 \ldots v_{k_n-1} \) and \( w'_1 = u_0 u_1 \ldots u_{k_n-1} \), with \( u_i, v_j \in W^n_c \cup \text{rev}(W^n_c) \). Let \( C \in W^n_c/Q^n_1 \) or \( C \in \text{rev}(W^n_c)/Q^n_1 \) according to whether \( w'_1 = w_1 \) or \( w_1 = \text{rev}(w_1) \). Then for all \( v \in W^n_c \) if

\[ J(v) = \{ t : v_t = v \} \]

then

\[ \frac{|\{ t \in J(v) : u_t \in C \}|}{|J(v)|} \approx \frac{1}{Q^n_1} \] (38)

**Definition 87** We will call the collection of axioms T1-T7 the timing assumptions for a construction sequence and an equivalence relation \( Q^n_1 \).

7.2.2 Codes and \( \bar{d} \)-Distance

We now prove some lemmas about \( \bar{d} \).

**Lemma 88** Let \( w_0 \in W^n_{c+1}, w_1 \in W^n_{c+1} \cup \text{rev}(W^n_{c+1}) \) and \( [w_0]_1 \notin G^n_1[w_1]_1 \). Let \( r > 1000 \) and \( J_0, J_1 \) be intervals in \( \mathbb{Z} \) of length \( r \cdot q_{n+1} \). Let \( I \) be the intersection of the two intervals. Put \( w'_0 \) on \( J_0 \) and \( w'_1 \) on \( J_1 \) and suppose that all but (possibly) the first or last copies of \( w_0 \) are included in \( I \). Let \( \bar{\Lambda} \) be a stationary code such that the length of \( \Lambda \) is less than \( q_n/10000 \). Then:

\[ \bar{d}([\bar{\Lambda}[w'_0] \mid I], [w'_1] \mid I) > \frac{1}{50} (1 - \frac{1}{Q^n_1}) \gamma. \] (39)

Since the length of the code \( \Lambda \) is much smaller than \( q_n \) and \( r > 10000 \), the end effects of \( \Lambda \) are limited to the first and last copies of \( w_0 \) and thus affect at most \( (1/5000) \) proportion of \( \bar{d}([\bar{\Lambda}[w'_0] \mid I], [w'_1] \mid I) \). Removing the portion of \( I \) across from the first or last copy of \( w_0 \) leaves a segment of \( I \) of proportion at least 4999/5000.

---

29Basic notation and facts about stationary codes are reviewed in section 3.4.
For all of the copies of \( w_0 \), except perhaps at most one at the end of \( J_0 \), there is a corresponding copy of \( w_1 \) that overlaps \( w_0 \) in a section of at least \( q_{n+1}/2 \). Discard the portions of \( I \) arising from copies of \( w_0 \) not overlapping the corresponding copies of \( w_1 \). After the first two removals we have a portion of \( I \) of proportion at least \((1/2)(4999/5000)\).

Because \( w_0 \) and \( w_1 \) have the same lengths, the relative alignment between any two corresponding copies of \( w_0 \) and \( w_1 \) in the powers \( w_0^r \) and \( w_1^r \) are the same. In particular, the “even overlaps” and “odd overlaps” are the same in each remaining portion of the corresponding copies of \( w_0 \) and \( w_1 \).

By Lemma 41, there are \( s, t < q_n \) such that on the even overlaps all of the \( n \)-subwords of \( sh^s(w_0^r) \) are either lined up with an \( n \)-subword of \( w_1^r \) or with a boundary section of \( w_1 \), and all of the \( n \)-subwords of \( w_0 \) in an odd overlap are lined up with an \( n \)-subword or a boundary section of \( w_1^r \) by \( sh^s(w_0^r) \).

Either the even overlaps or the odd overlaps contain at least \( 1/2 \) of the \( n \)-subwords that are not across from boundary portions of \( w_1 \). Assume that \( 1/2 \) of the \( n \)-subwords lie in even overlaps and discard the portion of \( I \) on the odd overlaps. (If more than \( 1/2 \) of the \( n \)-subwords are in odd overlaps we would focus on those.)

Suppose that \((w_0^*)^r = sh^s(w_0^r)\) on the even overlaps. Our notation will refer to any particular copy of \( w_0 \) in \((w_0^*)^r\) as \( w_0^* \). Then, except for \( W_n \)-words that get lined up with a boundary section of \( w_1 \), every \( n \)-subword of \((w_0^*)^r\) coming from an even overlap of \( (w_0^*)^r \) gets lined up with an \( n \)-subword of \((w_1^*)^r\). Write \( w_0 = C(v_1, v_2, \ldots v_{k_n-1}) \) and \( w_1 = C(u_1, u_2, \ldots u_{k_n-1}) \) (or, respectively, \( w_1 = C'(rev(u_1), rev(u_2), \ldots rev(u_{k_n-1})) \)). Then each \( n \)-subword of \( w_0^* \) coming from an even overlap is of the form \( v_i \) for some \( i \). There is a \( t \) such that for all \( i \) if \( v_i \) occurs in any copy of \( w_0^* \) and comes from an even overlap then either:

a.) \( v_i \) is lined up with \( u_{i+t} \) (respectively \( rev(u_{k_n-(i+t)}-1) \)) or
b.) \( v_i \) is lined up with a boundary portion of \( w_1 \) or
c.) \( v_i \) is lined up with \( u_{i+t+1} \) (respectively \( rev(u_{k_n-(i+t+1)}-1) \)).

On copies of \( v_i \) coming from even overlaps of 2-subsections the powers of \( v_i \) in alternatives a.) and c.) are constant. Since the even overlaps of the 2-subwords has size at least half of the lengths of the 2-subwords, it follows that \( 0 \leq t \leq k_n/2 \).

Since all of \( v_1^{l_{n-1}} \) satisfies a.), b.), or c.), after discarding the \( v_i \)’s in case b.) half of the remaining \( v_i \)’s satisfy a.) or c.). Keeping the larger
alternative and discard the other. What is left after all of the trimming has size at least:

\[(4999/5000)(1/2)(1/2)(1 - 2\partial_{n+1}) > 1/10\]

proportion of \(I\).

For some \(t\) what remains consists of \(n\)-subwords \(v_i\) in even overlaps of \((w_0)^r\) that, after being shifted by \(s\) to be subwords of \((w_0^*)^r\), are aligned with occurrences of \(n\)-subwords of \((w_1)^r\) of the form \(u_{i+t}\) \((\text{rev}(u_{k_n-(i+t)-1})\) respectively. For the rest of this proof of Lemma 88 we will call these the *good occurrences* of \(n\)-subwords.

**Claim:** Suppose that \(v \in \mathcal{W}_n^c\) and let

\[J^*(v) = \{y \in I : y \text{ is at the beginning of a good occurrence of } v \text{ in } (w_0^*)^r\}\]

Let \(C \in \mathcal{W}_n^c/\mathcal{Q}_1^n\) or \(C \in \text{rev}(\mathcal{W}_n^c)/\mathcal{Q}_1^n\) depending on whether \(w_1 \in \mathcal{W}_n^{c+1}\) or \(w_1 \in \text{rev}(\mathcal{W}_n^{c+1})\). Then

\[
\left| \frac{|\{y \in J^*(v) : \text{some element of } C \text{ occurs at } y \text{ in } w_1\}|}{|J^*(v)|} - \frac{1}{Q_1^n} \right| \leq \frac{2}{l_n + \mu_n}.
\]  

We prove the claim. We have two cases:

**Case 1:** \(t = 0\).

In this case we have a *trivial split* in the language of section 4.4. The overlap of the 2-subsections contains the whole of the two subsections except for a portion of one 1-subsection. Since \([w_0]_1 \notin G_1^n[w_1]_1\) we can apply axiom T7 to the words \(w_0\) and \(w_1\). The claim follows from inequality 38, which is the preword version of formula 40, after taking into account the boundary and the words at the ends of the blocks of \((w_0^*)^r\) and the truncated 1-subsections.

**Case 2:** \(t \neq 0\).

In this case the split is non-trivial. Because the even overlaps are at least as big as the odd overlaps of 2-subsections, the even overlap looks like:

\[
\prod_{j=0}^{t^*} (b^{j-i}v^{l-1}e^{j})
\]
but with a portion of its last 1-subsection possibly truncated. In particular
it has an initial segment of the form
\[ \prod_{j=0}^{t^*-1} (b^j t - j v^j - 1 e^j) \]
where \( t^* \geq k_n/2 \).

It follows from the timing assumption \( T5 \) that if \( J' = \{ y \in J(v) : \) some element of \( C \) occurs across from a word starting at \( y \) in the first \( t^* - 1 \)
1-subsections\} then
\[
| |J'| - |J(v)| - \frac{1}{Q_n^v} | < \mu_n.
\]

Any variation between the quantity in formula 40 and the estimate in \( T5 \) is due to the portion of the last 1-subsection of the even overlaps. This
takes up a proportion of the remaining even overlap less than or equal to
\( 1/t^* \). This proves the Claim.\(^3\)

We now shift \((w^*_0)^r\) back to be \( w^*_0 \) and consider \( s \). There is an \( l' \geq l/2 - 1 \geq l/3 \) such that all of the good occurrences of a \( v \in W^c_n \) in \((w^*_0)^r\) are
in a power \( v^r \). Depending on whether \( s \leq q_n/2 \) or \( s > q_n/2 \), for each good
occurrence of a \( v_j \) in \((w^*_0)^r\) either:

a.) there are at least \( l' - 1 \) powers of \( v_j \) in the corresponding occurrence
in \( w_0 \) such that their left overlap with \( u_{j+t} \) has length at least \( q_n/2 \)
or

b.) there are at least \( l' - 1 \) powers of \( v_j \) in the corresponding occurrence
in \( w_0 \) such that their right overlap with \( u_{j+t} \) has length at least \( q_n/2 \)

Without loss of generality we assume alternative a.). Suppose that the
overlap has length \( o \) in all of the good occurrences. Then the left side of \( v_j \)
overlaps the right side of \( u_{j+t} \) by at least \( q_n/2 \).

By axiom \( T4 \), if \( v \in W^c_n \),
\[ \tilde{d}(\tilde{\Lambda}(v \upharpoonright [0, o]), u_{j+t} \upharpoonright [q_n - o - 1, q_n]) < \gamma/2 \]
and
\[ \tilde{d}(\tilde{\Lambda}(v \upharpoonright [0, o]), u_{j'+t} \upharpoonright [q_n - o - 1, q_n]) < \gamma/2 \]

\(^3\)The axiom \( T5b \) takes care of the case where the relevant overlaps is odd.
then \([u_{j+t}]_1 = [u_{j'+t}]_1\). It follows that if we fix a \(v \in \mathcal{W}_n^c\) and let 
\[J(v) = \{j : v_j = v\}\]
then
\[
\frac{|\{j \in J(v) : \bar{d}(c(v_j \upharpoonright [0,o), u_{j+t} \upharpoonright [q_n - o - 1, q_n - 1]) < \gamma/2\}|}{|J(v)|}
\]
is less than \(\frac{1}{Q_n^1} + \mu_n\).

Since at least \(1/30\) proportion of \(I\) consists of left halves of good occurrences of the various \(v\)'s belonging to \(\mathcal{W}_n^c\) it follows that
\[
\bar{d}(\bar{\Lambda}|w_0^r \upharpoonright I], w_1^r) \geq \frac{1}{30}(1 - \frac{1}{Q_n^m} - \mu_n)(\gamma/2).
\](41)
The lemma follows. \(\dashv\)

### 7.2.3 Elements of the Centralizer

In this section we prove the theorem linking central values to elements of the centralizer of \(\mathbb{K}^c\).

**Numerical Requirement 5** \(\sum \frac{|G_n^r|}{Q_n^1} < \infty\).

**Theorem 89** Suppose that \((\mathbb{K}^c, \mathcal{B}, \nu, sh)\) is a circular system built from a circular construction sequence satisfying the timing assumptions. Let \(\phi : \mathbb{K}^c \to \mathbb{K}^c\) be an automorphism of \((\mathbb{K}^c, \mathcal{B}, \nu, sh)\). Then \(\phi^\sigma = \mathcal{S}_\beta\) for some central value \(\beta\).

\(\vdash\) Fix a \(\phi\) and suppose that \(\phi^\sigma = \mathcal{S}_\beta\). We must show that \(\beta\) is central. Suppose not. The idea of the proof is to choose a stationary code \(\bar{\Lambda}^r\) well approximating \(\phi\) and an \(N\) such such for all \(M > N\), passing over the principal \(M\)-block of most \(s \in \mathbb{K}^c\) with \(\bar{\Lambda}^r\) gives a string very close to \(\phi(s)\) in \(\bar{d}\)-distance. Consider an \(s\) where \(\bar{\Lambda}^r\) codes well on this principal \(M\)-block.

We use Lemma 82 to build a red zone corresponding to \(M\). Using Lemma 88, we will see that \(\bar{\Lambda}^r\) cannot code well on the red zone. Since the red zone takes up the vast majority of the principal \(M\)-block, \(\bar{\Lambda}^r\) cannot code well on the principal \(M\)-block, yielding a contradiction. In more detail:
Let $\gamma$ be as in Axiom T4. By Proposition 19 there is an code $\Lambda^*$ such that for almost all $s \in \mathbb{K}^c$,

$$d(\Lambda^*(s), \phi(s)) < 10^{-9}\gamma.$$ 

By the Ergodic theorem there is a $N_0$ so large that for a set $E \subseteq \mathbb{K}^c$ of measure $7/8$ for all $s \in E$ and all $N > N_0$, $s$ is mature for $N$ and if $B$ is the principal $N$-block of $s$ then

$$d(\Lambda^*(s \upharpoonright B), \phi(s) \upharpoonright B) < 10^{-9}\gamma.$$  \quad (42)

Let $s \in E$. Choose an $N > N_0$ such that the code length of $\Lambda^*$ is much smaller than $q_N$, $\frac{1}{Q_1} < 10^{-9}$ and $l_N > 10^{12}$. Apply Lemma 82, with $\delta = 10^{-9}$ to find an $M$ and $\langle R_i : i < i^* \rangle$ satisfying the conclusions of Lemma 82. Since $\bigcup_{i < i^*} R_i \subseteq q_M$ we view $\bigcup_{i < i^*} R_i$ as a subset of the principal $M$-block of $s$.

Each $R_i$ is a union of collections of locations of the form $\bigcup_{j < j_0} U_j$, with each $U_j$ consisting of the locations of $u_j^{s_{n_i}}$ for $j \in [0, j_0)$.\footnote{$s_{n_i}$ is as in condition 4.c) of Lemma 81.} Moreover there is a $t$ such that each power $u_j^{s_{n_i}}$ is $\beta$-matched with a $v_j^{s_{n_i} + t}$ in $\phi(s)$ for some $t \neq 0$.

Because $j_0 > \varepsilon_n k_n$ axiom T6 applies and thus for at least $(1 - |G_1^{n_i}| + \mu_n)$ proportion of $\{u_0, u_1, \ldots u_{j_0-1}\}$, $u_j$ and $v_j+t$ are in different $G_1^{n_i}$-orbits. In Lemma 88, inequality 39 implies that if $u_i$ and $v_{i+t}$ are in different $G_1^{n_i}$ orbits then, restricted to the overlaps of the locations of all of the $u_j^{s_{n_i}}$ and $v_j^{s_{n_i} + t}$, the $d$ distance between $\Lambda^*(s) \upharpoonright U_j$ and $\phi(s) \upharpoonright U_j$ is at least $\frac{1}{50}(1 - \frac{1}{Q_1^{n_i}})\gamma$.

Since the first and last powers of $u_j$ in $u_j^{s_{n_i}}$’s take up $2/s_0^{n_i}$ of $u_j^{s_{n_i}}$ and $s_0^{n_i} \geq l_n/2 - 2$, we know that

$$d(\Lambda^*(s) \upharpoonright U_j, \phi(s) \upharpoonright U_j) \geq (1 - 10^{-11}) \frac{1}{50}(1 - \frac{1}{Q_1^{n_i}})\gamma$$

Because the proportion of $j$’s for which $u_j$ and $v_j+t$ are in different $G_1^{n_i}$-orbits is at least $(1 - \frac{|G_1^{n_i}|}{Q_1^{n_i}} + \mu_n)$ it follows that

$$d(\Lambda^*(s) \upharpoonright \bigcup_{j < j_0} U_j, \phi(s) \upharpoonright \bigcup_{j < j_0} U_j)$$
is at least 

\[
(1 - \frac{|G_{n_i}|}{Q_{1_i}} + \mu_n)(1 - 10^{-11}) \frac{1}{500}(1 - \frac{1}{Q_{1_i}^n})\gamma.
\]

This in turn is at least \(\gamma/1000\). Since \(R_i\) is a union of sets of the form \(\bigcup_{j<j_i} U_j\):

\[
\bar{d}(\Lambda\ast(s) \upharpoonright R_i, \phi(s) \upharpoonright R_i) \geq \gamma/1000.
\]

Since \(\bigcup_{i<i'} R_i\) has density at least \(1 - 10^{-9}\) if \(B\) is the principal \(M\)-block of \(s\):

\[
\bar{d}(\Lambda\ast(s \upharpoonright B), \phi(s) \upharpoonright B) > \gamma/10^4.
\]

However this contradicts the inequality 42.

\[\triangleright\]

**Corollary 90** Let \(\mathbb{K}^c\) be a circular system built from a circular construction sequence satisfying the timing assumptions. Then \(\beta\) is a central value if and only if there is a \(\phi \in \{T^n : n \in \mathbb{N}\}\) with \(\phi^\pi = S_\beta\). It follows that for each construction sequence \(\langle k_n, l_n : n \in \mathbb{N}\rangle\) satisfying the Numerical Requirements collected in Section 10, the central values form a subgroup of the unit circle.

\[\triangleright\]

Theorem 83 says that if \(\beta\) is central, then there is a \(\phi \in \{T^n : n \in \mathbb{N}\}\) with \(\phi^\pi = S_\beta\). Theorem 89 is the converse. To see the last statement, we prove in Section 9 that for every coefficient sequence satisfying the Numerical Requirements, we can find a circular construction sequence satisfying the timing assumptions.

\[\triangleright\]

7.2.4 Isomorphisms Between \(\mathbb{K}^c\) and \((\mathbb{K}^c)^{-1}\)

We now prove a theorem closely related to Theorem 89

**Theorem 91** Suppose that \((\mathbb{K}^c, \mathcal{B}, \nu, sh)\) is a circular system built from a circular construction sequence satisfying the timing assumptions. Suppose that \(\phi : (\mathbb{K}^c, \mathcal{B}, \nu, sh) \rightarrow ((\mathbb{K}^c)^{-1}, \mathcal{B}, \nu, sh)\) is an isomorphism. Then \(\phi^\pi = \sharp \circ S_\beta\) for some central value \(\beta\).

\[\triangleright\]

We concentrate here on the differences with the proof of Theorem 89. The general outline is the same: Fix a \(\phi\). Then there is a unique \(\beta\) such that \(\phi^\pi = \sharp \circ S_\beta\). Suppose that \(\beta\) is not central. Choose a stationary code \(\Lambda\ast\) that well approximates \(\phi\) in terms of \(\bar{d}\) distance (say within \(\gamma/10^{10}\)), and derive a contradiction by choosing a large \(M\) and getting lower bounds for \(\bar{d}\) distance along the principal \(M\)-block of a generic \(s\).
This is done by first comparing a typical \( s \) with \( S_\beta(s) \). As in Theorem 89, a definite fraction of a large principal \( M \)-block of \( s \) is misaligned with \( S_\beta(s) \). But most of the \( n \)-blocks of \( S_\beta(s) \) are aligned with reversed \( n \)-blocks of \( \dual(S_\beta(s)) \) that have been shifted by a very small amount. This can be quantified by looking at the codes \( \tilde{\Lambda}_n \) for large \( n \), which agree with \( \dual(S_\beta(\pi(s))) \) on the \( M \)-block of \( S_\beta(s^\pi) \).

Here are more details. Recall \( \dual \) is the limit of a particular sequence of stationary codes \( \langle \bar{\Lambda}_n : n \in \mathbb{N} \rangle \). The proof of Theorem 57 showed that for almost all \( s^\pi \in \mathcal{K} \) for all large enough \( n \) the principal \( n \)-blocks of \( \bar{\Lambda}_n(s^\pi) \) and \( \bar{\Lambda}_{n+1}(s^\pi) \) agree. Fix a generic \( s \in \mathcal{K}_c \) and a large \( N \) such that:

1. the code \( \overline{\Lambda}^\pi \) codes \( \phi \) well on the principal \( n \)-block of \( s \) for all \( n \geq N \),
2. for all \( n \geq N \) the principal \( n \)-blocks of \( \bar{\Lambda}_n(S_\beta(\pi(s))) \) and \( \bar{\Lambda}_{n+1}(S_\beta(\pi(s))) \) agree,
3. \( s \) is mature at \( N \),
4. the length of \( \Lambda \) is very small relative to \( N \) and
5. \( l_N \) is very large.

Comparing \( \pi(s) \) and \( S_\beta(\pi(s)) \), Lemma 82 gives us an \( M > N \) and a red zone in the principal \( M \)-block \( s \). We assume that the red zones take up at least \( 1 - 10^{-100} \) proportion of the principal \( M \)-block and have the form given in Lemma 82.

We will derive a contradiction by showing that \( \overline{\Lambda}^\pi \) cannot code well. This is done by considering the blocks of \( \phi(s) \) that are lined up with the red zones of the principal \( M \)-block of \( s \) and using Lemma 88 to see that \( \overline{\Lambda}^\pi \) cannot code well on these sections. This is possible because the mismatched \( n \)-blocks of \( S_\beta(\pi(s)) \) are lined up closely with the \( n \)-blocks of \( \dual(S_\beta(\pi(s))) = \phi^\pi(s) \). In more detail:

Use Lemma 82, to choose red zones \( \langle R_i : i < i^* \rangle \) that take up a \( 1 - 10^{-9} \) proportion of the principal \( M \)-block of \( s \).32

The boundary portions of \( n \)-words with \( n < M + 1 \) take up at most \( 2/l_M \) proportion of the overlap of the principal \( M \)-blocks of \( s \) and \( \phi(s) \). Since this proportion is so small, Remark 21 allows us to completely ignore blocks corresponding to \( n_i \)-words in \( s \) that are lined up with boundary in \( \phi(s) \) and vice versa.

---

32 We use the notation in Lemma 82 and Theorem 57.
We now examine the how $\mathcal{S}_\beta(\pi(s))$ compares with $\mathcal{S}_\beta(\pi(s))$. Temporarily denote $\mathcal{S}_\beta(\pi(s))$ by $s'$. By the choice of $s$, for all $n \in [N, M]$ the alignments of the principal $n$-blocks of $\Lambda_n(s')$ and $\Lambda_M(s')$ agree.

The red zones of $s^\pi$ line up blocks of the form $u^s_{j+t}$ occurring in $s'$ that are shifted by $d^u(s)$ (so $t \neq 0$). Except for those blocks that line up with boundary portions of $\mathcal{S}(s')$ these blocks are lined up with blocks of the form $v^s_{k_n-(j+t)-1}$ in $\mathcal{S}(s')$. In particular the blocks of powers of $v^s_{j+t}$ are lined up with a very small shift of $\text{rev}(v^s_{k_n-(j+t)-1})$ in $\mathcal{S}(s')$.

Thus vast majority of blocks $U_j$ that are positions of $u^s_{j+t}$ in $s^\pi$ are lined up with a shift by less than $q_{n_i}$ of a block of $\mathcal{S}(s')$ in a position of $v^s_{k_n-(j+t)-1}$ in $\mathcal{S}(s')$. Consider $s$ and $\phi(s)$. Suppose that $u_j$ are the $n_i$-words of $s$ corresponding to the $U_j$ and $v_{k_n-(j+t)-1}$ are the $n_i$-words of $\phi(s)$ across from them. By axiom $T_{5a}$, at most $\frac{1}{Q_1} + \mu_{n_i}$ of the $j < j_0$ happen to have $[u_j] \in G^s_{n_i}[v_{k_n-(j+t)-1}]$. At least $1 - \frac{1}{Q_1} + \mu_{n_i}$ proportion of the powers of $u_j$ the $\bar{d}$-distance between $\Lambda^\pi$ and $\phi$ is at least $\frac{1}{500}(1 - \frac{1}{Q_1})\gamma$.

It follows that on $R_i$ the $\bar{d}$-distance is at least $\gamma/1000$. If we choose $\bigcup_{i \leq i^*} R_i$ to have density at least $1 - 10^{-9}$ and let $B$ be the principal $M$-block of $s$ then (as in Theorem 89)

$$\bar{d}(\Lambda^\pi(s \mid B), \phi(s) \mid B) > \gamma/10^4,$$

a contradiction.

\[ \square \]

### 7.3 Synchronous and Anti-synchronous Isomorphisms

View a circular system $(\mathbb{K}, B, \nu, sh)$ as an element $T$ of the space $MPT$ endowed with the weak topology.

**Theorem 92** Suppose that $\mathbb{K}$ is a circular system satisfying the timing assumptions. Then:

1. If there is an isomorphism $\phi : \mathbb{K} \to \mathbb{K}$ such that $\phi \notin \{T^n : n \in \mathbb{Z}\}$, then there is an isomorphism $\psi : \mathbb{K} \to \mathbb{K}$ such that $\psi \notin \{T^n : n \in \mathbb{Z}\}$ and $\psi^\pi$ is the identity map.

\[ \text{See the qualitative discussion of $\mathcal{S}$ that occurs after its definition in [5].} \]
2. If there is an isomorphism $\phi : \mathbb{K}^c \to (\mathbb{K}^c)^{-1}$ then there is an isomorphism $\psi : \mathbb{K}^c \to (\mathbb{K}^c)^{-1}$ such that $\psi^n = \sharp$.

$\therefore$ To see the first assertion, let $\phi : \mathbb{K}^c \to \mathbb{K}^c$ be an isomorphism with $\phi \notin \{T^n : n \in \mathbb{Z}\}$. Then by Theorem 89, $\phi^n = S_\beta$ for a central $\beta$. Corollary 90 implies that there is a $\theta \in \{T^n : n \in \mathbb{N}\}$ such that $\theta^n = S_\beta$. Then $\phi \circ \theta : \mathbb{K}^c \to \mathbb{K}^c$ is an isomorphism such that $(\phi \circ \theta)^n$ is the identity map. Since $\{T^n : n \in \mathbb{N}\}$ is a group, $\phi \circ \theta \notin \{T^n : n \in \mathbb{N}\}$.

The proof of the second assertion is very similar. Suppose that $\phi : \mathbb{K}^c \to (\mathbb{K}^c)^{-1}$ is an isomorphism. Then, by Theorem 91, $\phi^n = \sharp \circ S_\beta$ for a central $\beta$. Let $\theta \in \{T^n : n \in \mathbb{N}\}$ be such that $\theta^n = S_\beta$. Then $\phi \circ \theta$ is an isomorphism between $\mathbb{K}^c$ and $(\mathbb{K}^c)^{-1}$ with $(\phi \circ \theta)^n = \sharp$.

8 The Proof of the Main Theorem

In this section we prove the main theorem of this paper, Theorem 2. By Fact 23, it suffices to prove the following:

**Theorem 93** There is a continuous function $F^s : \mathcal{Trees} \to \text{Diff}^\infty(\mathbb{T}^2, \lambda)$ such that for $\mathcal{T} \in \mathcal{Trees}$, if $T = F^s(\mathcal{T})$:

1. $\mathcal{T}$ has an infinite branch if and only if $T \cong T^{-1}$,

2. $\mathcal{T}$ has two distinct infinite branches if and only if

$$C(T) \neq \{T^n : n \in \mathbb{Z}\}.$$  

We split the proof of this theorem into three parts. In the first we define $F^s$ and show that it is a reduction. In the second part we show that $F^s$ is continuous. The proofs of first two parts combine the properties of the constructions in [4] and [5] with the timing assumptions introduced in Section 7.2.1. These induce randomness requirements on the word constructions beyond those in [8].

The third part of the proof describes how to perform the word construction from [8] with these additional requirements. We present the third part of the proof separately in Section 9.

We begin by defining $F^s$. The main result of [8] relied on the construction of a continuous function $F : \mathcal{Trees} \to \text{MPT}$ such that for all $\mathcal{T} \in \mathcal{Trees}$, if $S = F(\mathcal{T})$ then:

**Fact 1** $\mathcal{T}$ has an infinite branch if and only if $S \cong S^{-1}$,
Fact 2 \( \mathcal{T} \) has two distinct infinite branches if and only if
\[ C(S) \neq \{ S^n : n \in \mathbb{Z} \}. \]

Fact 3 The function \( F \) took values in the strongly uniform odometer based
transformations and for \( S \) in the range of \( F \), \( S \cong S^{-1} \) if and only if
there is an anti-synchronous isomorphism \( \phi \) between \( S \) and \( S^{-1} \).

Fact 4 \((8), \) Corollary 40, page 1565) If \( S \) is in the range of \( F \) and
\( C(S) \neq \{ S^n : n \in \mathbb{Z} \} \) then there is a synchronous \( \phi \in C(S) \) such that for
some \( n \), non-identity element \( g \in G_1^n \) and all generic \( s \in \mathbb{K} \) and all
large enough \( m \), if \( u \) and \( v \) are the principal \( m \)-subwords of \( s \) and \( \phi(s) \)
respectively then:
\[ [v]_1 = g[u]_1. \]

Fact 5 (Equations 1 and 2 on pages 1546 and 1547 of \(8\)) For all \( n_0 \) there
is an \( M \) such that if \( \mathcal{T} \) and \( \mathcal{T}' \) are trees and\(^{34}\)
\[ \mathcal{T} \cap \{ \sigma_n : n \leq M \} = \mathcal{T}' \cap \{ \sigma_n : n \leq M \} \]
then the first \( n_0 \)-steps of the construction sequences for \( F(\mathcal{T}) \) are
equal to the first \( n_0 \)-steps of the construction sequence for \( F(\mathcal{T}') \); i.e.
\[ \langle W_k(\mathcal{T}) : k < n_0 \rangle = \langle W_k(\mathcal{T}') : k < n_0 \rangle. \]

Fact 6 The construction sequence for \( F(\mathcal{T}) \) satisfies the specifications given
in \(8\). In Section 9.2, these specifications are augmented by the addi-
tion of J10.1 and J11.1. In Section 9.3 we argue that if \( \langle W_n : n \in \mathbb{N} \rangle \)
is a construction sequence for an odometer based system that satisfies
the augmented specifications, then the associated circular construction
sequence \( \langle W_n^c : n \in \mathbb{N} \rangle \) satisfies the timing assumptions.
Moreover the construction sequence for \( F(\mathcal{T}) \) is strongly uniform and
hence the construction sequence for \( F \circ F(\mathcal{T}) \) is strongly uniform.

Fact 7 Construction sequences satisfying the augmented specifications are
easily built using the techniques of \(8\) with no essential changes; con-
sequently we can assume that the the construction sequences for \( F(\mathcal{T}) \)
satisfy the augmented specifications.

In \(4\) (Theorem 60) it shown that if \( \langle W_n^c : n \in \mathbb{N} \rangle \) is a strongly uniform
circular construction sequence with coefficients \( \langle k_n, l_n : n \in \mathbb{N} \rangle \), where \( l_n :
\( n \in \mathbb{N} \) grows fast enough and \( |W^c_n| \) goes to infinity then there is a smooth measure preserving diffeomorphism \( T \in \text{Diff}^\infty(T^2, \lambda) \) measure theoretically isomorphic to \( \mathbb{K}^c \). This gives a map \( R \) from circular systems with fast growing coefficients to \( \text{Diff}^\infty(T^2, \lambda) \).

If \( \mathcal{F} \) is the canonical functor from odometer systems to circular systems we define

\[
F^s = R \circ \mathcal{F} \circ F
\]

8.1 \( F^s \) is a Reduction

Because \( R \) preserves isomorphism, to show that \( F^s = R \circ \mathcal{F} \circ F \) is a reduction, it is suffices to show that \( \mathcal{F} \circ F \) is a reduction. Let \( S \) be the transformation corresponding to the system \( \mathbb{K} = F(T) \) and \( T \) the transformation corresponding to \( \mathbb{K}^c = F \circ F(T) \).

\textbf{Item 1 of Theorem 93:} Suppose that \( T \) is a tree and \( T \) has an infinite branch. By Facts 1 and 3, there is an anti-synchronous isomorphism \( \phi: \mathbb{K} \to \mathbb{K}^{-1} \). By Theorem 105 of [5], if \( \mathbb{K}^c = \mathcal{F}(\mathbb{K}) \), there is an isomorphism \( \phi^c: \mathbb{K}^c \to (\mathbb{K}^c)^{-1} \).

Now suppose that \( F^s(T) \cong (F^s(T))^{-1} \), then \( \mathbb{K}^c \cong (\mathbb{K}^c)^{-1} \). By Fact 6, the construction sequence \( \langle W^c_n : n \in \mathbb{N} \rangle \) for \( F^s(T) \) satisfies the timing assumptions. By Theorem 92, there is an anti-synchronous isomorphism \( \phi^c: \mathbb{K}^c \to (\mathbb{K}^c)^{-1} \). Again by Theorem 105 of [5], there is an isomorphism between \( \mathbb{K} \) and \( \mathbb{K}^{-1} \). By [8], \( T \) has an infinite branch.

\textbf{Item 2 of Theorem 93:} Suppose that \( T \) has at least two infinite branches. Then the centralizer of \( S = F(T) \) is not equal to the powers of \( S \). By Fact
4, we can find a synchronous \( \phi \in C(S) \setminus \{ S^n : n \in \mathbb{Z} \} \). Let \( \psi = \mathcal{F}(\phi) \), then \( \psi \) is synchronous. We claim that \( \psi \notin \{ T^n : n \in \mathbb{Z} \} \). Using Fact 4, and lifting the group action of \( G^\mathbb{Z}_n \) and the equivalence relation \( Q^\mathbb{Z}_n \), we see that for all generic \( s^c \in \mathbb{K}^c \), and all large enough \( m \), if \( u^c \) and \( v^c \) are the principal \( m \)-subwords of \( s^c \) and \( \psi(s^c) \), respectively, then:

\[
[v^c]_1 = g[u^c]_1
\]

for some \( g \neq e \). In particular, \([v^c]_1 \neq [u^c]_1\).

By the timing assumption \( \text{T4} \), there is a \( \gamma > 0 \) such that for all large \( m \) and all shifts \( A \) with \( |A| \) of size less than \( q_m/2 \), we have

\[
\bar{d}(T^A(u^c), v^c) > \gamma. \tag{43}
\]

Suppose that \( \psi \in \{ T^n : n \in \mathbb{Z} \} \). Then, by Proposition 20, we can find an \( A \in \mathbb{Z} \) and a generic \( s^c \) such that

\[
\bar{d}(T^A(s^c), \psi(s^c)) < \gamma/2. \tag{44}
\]

But inequality 44 and the Ergodic Theorem imply that for large enough \( m \gg A \) if \( u^c \) and \( v^c \) are the principal \( m \)-blocks of \( s^c \) and \( \psi(s^c) \) then

\[
\bar{d}(T^A(u^c), v^c) < \gamma,
\]

contradicting inequality 43.

Now suppose that there is a \( \psi \in C(T) \) such that \( \psi \notin \{ T^n : n \in \mathbb{Z} \} \). Then by Theorem 92, there is such a \( \psi \) that is synchronous. In particular, for all \( n \), \( \psi \neq T^n \). Thus if \( S \) is the transformation corresponding to \( F(T) \), \( F^{-1}(\psi) \) belongs to the centralizer of \( S \) and is not a power of \( S \).

### 8.2 \( F^s \) is Continuous

Fix a metric \( d \) on \( \text{Diff}^{\infty}(\mathbb{T}^2, \lambda) \) yielding the \( C^\infty \)-topology. For each circular system \( T \), let \( \langle P^T_n : n \in \mathbb{N} \rangle \) be the sequence of collections of prewords used to construct \( T \). By Proposition 61 of [4], if we are given \( T = F^s(T) \) and a \( C^\infty \)-neighborhood \( B \) of \( T \), we can find a large enough \( M \), for all \( S \in \text{range}(R) \) if \( \langle P^S_n : n \leq M \rangle = \langle P^T_n : n \leq M \rangle \), then \( S \in B \). For all odometer based transformations, the sequence \( \langle W_n : n \leq M \rangle \) determines \( \langle P_n : n \leq M \rangle \). and hence for all \( T' \), if the first \( M \) members of the construction sequence for \( F(T') \) are the same as the first \( M \) members of the construction sequence for \( F(T) \), then \( F(T') \in U \). By Fact 5, there is a basic open interval \( V \subseteq \mathcal{Trees} \) that contains \( T \) and is such that the first \( M \) members of the construction sequence are the same for all \( T' \in V \). It follows that for all \( T' \in V \), \( F^s(T') \in U \).
8.3 Numerical Requirements Arising from Smooth Realizations

The construction of $R$ depends on various estimates that put lower bounds on the growth of the coefficient sequences. We now list these numerical requirements. The claims in this subsection presuppose a knowledge of [4].

The map $R$ depends on various smoothed versions $h^s_n$ of the permutations $h_n$ of the unit interval arising from $\langle W_n : n \in \mathbb{N} \rangle$. To solve this problem, we fix in advance such approximations, making sure that each approximation $h^s_n$ agrees sufficiently well with $h_n$ as to not disturb the other estimates.

This introduces various numerical constraints on the growth of the $l_n$’s. The diffeomorphism $T$ is built as a limit of periodic approximations $T_n$. To make the sequence of $T_n$’s converge at each stage $l_n$ must be chosen sufficiently large. Thus the growth rate of $l_n$ depends on $\langle k_m, s_m, h_m : m \leq n \rangle$, $\langle l_m : m < n \rangle$, $s_{n+1}, h_{n+1}$. Since there are only finitely many possibilities for $\langle h_m : m \leq n \rangle$’s corresponding to a given sequence $\langle k_m : m \leq n \rangle$, $\langle s_m : m \leq n + 1 \rangle$ we can find one growth rate that is sufficiently fast to work for all choices of $h_m$’s.

**Numerical Requirement 6** $l_n$ is big enough relative to a lower bound determined by $\langle k_m, s_m : m \leq n \rangle$, $\langle l_m : m < n \rangle$, $s_{n+1}$ to make the periodic approximations to the diffeomorphism converge.

The argument for the ergodicity of the diffeomorphism formally required that:

**Numerical Requirement 7** $s_n$ goes to infinity as $n$ goes to infinity, $s_{n+1}$ is a multiple of $s_n$

The next requirement makes it possible to choose $s_{n+1}$ and then, by making $k_n$ sufficiently large, construct $s_{n+1}$ sufficiently random words using elements of $W_n$.

**Numerical Requirement 8** $s_{n+1} \leq s_n^{k_n}$

9 The Specifications

In this section we describe how the timing assumptions are related to the specifications given in [8], show that they are compatible and describe how to construct odometer words so that both sets of assumptions hold. This completes the proof of Theorem 93, subject to the verification that all of
the Numerical Requirements we have introduced are consistent with the numerical requirements of [8]. We take this up in section 10. We will assume that the reader is familiar with sections 7 and 8 of [8].

| Timing Assumption | Specification |
|-------------------|---------------|
| T1                | Q5            |
| T2                | Q7            |
| T3                | A8            |
| T4                | New           |
| T5                | J10           |
| T6                | J10           |
| T7                | J11           |

Figure 3: The Specifications related to the Timing Assumptions

9.1 Corresponding Specifications and Notation

Figure 3 gives a table that links the Timing Assumptions we use in this paper to the corresponding Specification in [8]:

Specification T4 doesn’t directly correspond to one of the Specifications, but (as we will show) holds naturally in the circular words lifted from an odometer construction satisfying the specifications.

Notation shift. In this paper we have adopted the notation used in [1], which conflicts with the notation in [8], accordingly we provide a table for translating between the two. In the table, NEW means the notation used in this paper, OLD means the notation used in [8].

An equivalent description of the numbers we are calling $k_n$ in this paper is that they are the number of words in $W^c_n$ concatenated to form elements of $P_{n+1}$. The number $k_n$ is equal to the number $K_{n+1}/K_n$ and $l_{n+1}/l_n$ in the old notation of [8].

Numerical Requirement 9 In the current construction we have two summable sequences: $\langle \epsilon_n : n \in \mathbb{N} \rangle$ and $\langle \varepsilon_n : n \in \mathbb{N} \rangle$. We use the lunate “$\epsilon_n$” notation for the specifications from [8] and the classical “$\varepsilon_n$” notation (“varepsilon” in LaTex) for the numerical requirements relating to circular systems and their realizations as diffeomorphisms. A requirement for the construction is that

$$\epsilon_n < \varepsilon_n.$$

It follows that the $\epsilon_n$ are summable. We also assume that the $\epsilon_n$’s are decreasing and $\epsilon_0 < 1/40$. 

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| NEW     | OLD     | Description                                                                 |
|---------|---------|-----------------------------------------------------------------------------|
| $s_n$   | $W_n$   | $s_n$ is the number of words in $W_n^c$                                     |
| $k_n$   | $l_{n+1}/l_n$ | The number of words concatenated to make $W_{n+1}$ from $W_n$          |
| $e(n)$  | $k(n)$  | Controls the number of $Q_{s+1}$ classes in each $Q_s$ class             |
| $\gamma$ | $s_1$  | The separation between $Q^n_s$ classes                                     |
| $K_n$   | $l_n$   | $K_n$ is this paper’s notation for the lengths of the odometer based words in $W_n$, $l_n$ was the notation for the lengths of the words in [8] |
| $q_n$   | $l_n$   | The lengths of the circular words in current paper vs. odometer based words in [8]. The new $q_n$ refers to the lengths of the words in $W_n^c$. |
| $l_n$   | no analogue | Coefficient needed to grow fast for smooth transformations |

Figure 4: Notation Shift

9.2 Augmenting the Specifications from [8]

The paper [8] constructs a reduction $F$ from the space of trees to the odometer based systems. The system $K = F(T)$ was built according to a list of specifications which we reproduce here in order to show how to modify them to imply the timing assumptions used in the proofs of Theorems 92 and 93 and verify that they are consistent. The specifications directly relevant to the timing assumptions are $J10$ and $J11$. The others, which describe the scaffolding for the construction, are only relevant in that they set the stage for the application of the functor $F$ defined in section 4.

To fully explain the specifications we need to remind the reader of some of the definitions in [8]. There was a fixed enumeration of the finite sequences of natural numbers, $⟨σ_n : n ∈ N⟩$, with the property that if $σ$ is an initial segment of $τ$ then $σ$ is enumerated before $τ$. Let $T$ be a tree whose elements are $⟨σ_{n_i} : i ∈ N⟩$. Here are the specifications for the construction sequence $W = W(T)$ used to build $F(T)$.

There is a sequence of groups $G^n_s$ built as follows. For all $n$, $G^0_0$ is the trivial group ($e$) and if we let

$$X^n_s = \{σ_{n_i} : i ≤ n \text{ and } σ_{n_i} \text{ has length } s\}$$
then

\[ G_s^n = \sum_{\sigma \in X_s^n} (\mathbb{Z}_2)_\sigma \]

i.e. \( G_s^n \) is a direct sum of copies of \( \mathbb{Z}_2 \) indexed by elements of \( X_s^n \). There are canonical homomorphisms from \( G_{s+1}^n \) to \( G_s^n \) that send a generator of \( G_{s+1}^n \) corresponding to a sequence of the form \( \tau \sim j \) to the generator of \( G_s^n \) corresponding to \( \tau \).

The sequence \( \langle W_n : n \in \mathbb{N} \rangle \), equivalence relations \( Q_s^n \) and the group actions of \( G_s^n \) are constructed inductively. The words in \( W_n \) are sequences of elements of \( \Sigma = \{0, 1\} \). To start \( W_0 = \{0, 1\} \) and \( Q_0^n \) is the trivial equivalence relation with one class. The collection of words \( W_n \) is built when the \( n^{th} \) element of \( T \) is considered. We will say that words in \( W_n \) have even parity and words in \( \text{rev}(W_n) \) have odd parity.

E1. Any pair \( w_1, w_2 \) of words in \( W_n \) have the same length.

E2. Every word in \( W_{n+1} \) is built by concatenating words in \( W_n \). Every word in \( W_n \) occurs in each word of \( W_{n+1} \) exactly \( p_n^2 \) times, where \( p_n \) is a large prime number chosen when the \( n^{th} \) element of \( T \) is considered.

E3. (Unique Readability) If \( w \in W_{n+1} \) and

\[ w = pw_1 \ldots w_ks \]

where each \( w_i \in W_n \) and \( p \) or \( s \) are sequences of 0’s and 1’s that have length less than that of any word in \( W_n \), then both \( p \) and \( e \) are the empty word. If \( w, w' \in W_{n+1} \) and \( w = w_1w_2 \ldots w_ks \) and \( w' = w'_1w'_2 \ldots w'_{k_n} \) with \( w_i, w'_i \in W_n \), and \( k = [k_n/2] + 1 \) we have \( wkw_{k+1} \ldots w_{k_n} \neq w'_1w'_2 \ldots w'_{k_n-[k]-1} \); i.e. the first half of \( w' \) is not equal to the second half of \( w \).

We let \( s(n) \) be the length of the longest sequence among the first \( n \) sequences in \( T \) and if \( T = \langle \sigma_{n_i} : i \in \mathbb{N} \rangle \) then \( M(s) \) is the least \( i \) such that \( \sigma_{n_i} \) has length \( s \).

The equivalence relations \( Q_s^n \) on \( W_n \) are defined for all \( s \leq s(n) \). The equivalence relation \( Q_0^n \) on \( W_0 \) is the trivial equivalence relation with one class.

Q4. Suppose that \( n = M(s) \). Then any two words in the same \( Q_s^n \) equivalence class agree on an initial segment of proportion least \( (1 - \epsilon_n) \).
Q5. For \( n \geq M(s) + 1 \), \( Q^n_s \) is the product equivalence relation of \( Q^M_s \).

Hence we can view \( W_n/Q^n_s \) as sequences of elements of \( W_{M(s)}/Q^{M(s)}_s \) and similarly for \( \text{rev}(W_n)/Q^n_s \).

Q6. \( Q^n_{s+1} \) refines \( Q^n_s \) and each \( Q^n_s \) class contains \( 2^{\epsilon(n)} \) many \( Q^n_{s+1} \) classes, where \( \epsilon \) is a strictly increasing function. The speed of growth of \( \epsilon \) is discussed in section 10.

A7. \( G^n_s \) acts freely on \( W_n/Q^n_s \cup \text{rev}(W_n/Q^n_s) \) and the \( G^n_s \) action is subordinate to the \( G^n_{s-1} \) action via the natural homomorphism \( \rho_{s,s-1} \) from \( G^n_s \) to \( G^n_{s-1} \).

A8. The canonical generators of \( G^M_s \) send elements of \( W_{M(s)}/Q^M_s \) to elements of \( \text{rev}(W_{M(s)})/Q^M_s \) and vice versa.

A9. If \( M(s) \leq n \) and we view \( G^{n+1}_s = G^n_s \oplus H \) then the action of \( G^n_s \) on \( W_n/Q^n_s \cup \text{rev}(W_n/Q^n_s) \) is extended to an action on \( W_{n+1}/Q^{n+1}_s \cup \text{rev}(W_{n+1}/Q^{n+1}_s) \) by the skew diagonal action. If \( H \) is non-trivial then \( H = \mathbb{Z}_2 \) and its canonical generator maps \( W_{n+1}/Q^{n+1}_s \) to \( \text{rev}(W_{n+1}/Q^{n+1}_s) \).

Suppose that \( u \) and \( v \) are elements of \( W_{n+1} \cup \text{rev}(W_{n+1}) \) and \( (u',v') \) an ordered pair from \( W_n \cup \text{rev}(W_n) \). Suppose that \( u \) and \( v \) are in positions shifted relative to each other by \( t \) units. Then an occurrence of \( (u',v') \) in \( (sh^t(u),v) \) is a \( t' \) such that \( u' \) occurs in \( u \) starting at \( t + t' \) and \( v \) starting at \( t' \). We have changed the variables used in the statement of J10 in [8] to conform to the notation described in section 9.1. We use \( Q^n_s \) for the number of classes of \( Q^n_s \) and \( C^n_s \) for the number of elements of each \( Q^n_s \) class.

To satisfy the timing assumptions we need to strengthen specifications J10 and J11 to deal with \( d \)-distance on initial and tail segments and on words that are shifted. The spirit of specification J10 is that pairs of \( n \)-words \( (u',v') \) occur randomly in the overlap of \( u \) and \( v \) when \( u \) is shifted by a suitable multiple \( t \) of the lengths of \( n \)-words. J10.1 says the same thing relative to non-trivial initial segments of the overlap of the shift of \( u \) and \( v \).

The specification J11 says that if \( [u]_s \) is in the \( G^n_s \)-orbit of \( [v]_s \) and \( s \) is maximal with this property, then the occurrences of \( (u',v') \) are approximately conditionally random. More explicitly, suppose that \( g[u]_s = [v]_s \), and we are given \( u' \in W_n \). Then there are \( Q^n_s \) many pairs of \( Q^n_s \)-classes \( ([u^*_s],[v^*_s]) \) with \( g[u^*_s] = [v^*_s] \), and so \( ([u^*_s],[v^*_s]) \) should occur randomly \( 1/Q^n_s \) proportion of the time. There are \( C^n_s \) many elements of \( W_n \) in the \( Q^n_s \)-classes, and conditional on \( g[u]_n = [v']_n \), the chances of such a pair
(u', v') randomly matching is $1/(C^n_s)^2$. The specification J11.1 strengthens this (but only for $Q^n_0$, which is the trivial equivalence relation and $G^n_0 = (v)$) by asking that this holds over any non-trivial interval of length $j_0 K_n$ at the beginning or end of an $n + 1$-word.

Here are the joining specifications as given in [8]:

J10. Let $u$ and $v$ be elements of $W_{n+1} \cup \text{rev}(W_{n+1})$. Let $1 \leq t < (1 - \epsilon_n)(k_n)$ be an integer and $K_n$ be the length of the words in $W_n$. Then for each pair $u', v' \in W_n \cup \text{rev}(W_n)$ such that $u'$ has the same parity as $u$ and $v'$ has the same parity as $v$, let $r(u', v')$ be the number of occurrences of $(u', v')$ in $(sh^{j_0 K_n}(u), v)$ on their overlap. Then

$$\left| \frac{r(u', v')}{k_n - t} - \frac{1}{s^2_n} \right| < \epsilon_n.$$  

J11. Suppose that $u \in W_{n+1}$ and $v \in W_{n+1} \cup \text{rev}(W_{n+1})$. We let $s = s(u, v)$ be the maximal $i$ such that there is a $g \in G^n_i$ such that $g[u] = [v]$. Let $g = g(u, v)$ be the unique $g$ with this property and $(u', v') \in W_n \times (W_n \cup \text{rev}(W_n))$ be such that $g[u] = [v]$. Let $r(u', v')$ be the number of occurrences of $(u', v')$ in $(u, v)$. Then:

$$\left| \frac{r(u', v')}{k_n} - \frac{1}{Q^n_s} \left( \frac{1}{C^n_s} \right)^2 \right| < \epsilon_n.$$  

To prove the timing assumption T6, we need to strengthen J10 to say:

J10.1 Let $u$ and $v$ be elements of $W_{n+1} \cup \text{rev}(W_{n+1})$. Let $1 \leq t < (1 - \epsilon_n)(k_n)$. Let $j_0$ be a number between $\epsilon_n k_n$ and $k_n - t$. Then for each pair $u', v' \in W_n \cup \text{rev}(W_n)$ such that $u'$ has the same parity as $u$ and $v'$ has the same parity as $v$, let $r(u', v')$ be the number of $j < j_0$ such that $(u', v')$ occurs in $(sh^{j K_n}(u), v)$ in the $(j K_n)^{th}$ position in their overlap. Then

$$\left| \frac{r(u', v')}{j_0} - \frac{1}{s^2_n} \right| < \epsilon_n.$$  

To satisfy the timing assumptions we need specification J11 as well as a strengthening of a special case:

J11.1 Suppose that $u \in W_{n+1}$ and $v \in W_{n+1} \cup \text{rev}(W_{n+1})$ and $[u] \notin G^n_1[v]$. Let $j_0$ be a number between $\epsilon_n k_n$ and $k_n$. Suppose that $I$ is either an initial or a tail segment of the interval $\{0, 1, \ldots K_n - 1 \}$ having length
Then for each pair $u', v' \in W_n \cup \text{rev}(W_n)$ such that $u'$ has the same parity as $u$ and $v'$ has the same parity as $v$, let $r(u', v')$ be the number of occurrences of $(u', v')$ in $(u \restriction I, v \restriction I)$. Then:

$$\left| \frac{r(u', v')}{j_0} - \frac{1}{s_n^2} \right| < \epsilon_n.$$ 

We have augmented the specifications in [8] with J10.1 and J11.1. Formally we must argue that it is possible to build a construction sequence satisfying the additional specifications. This means constructing $s_{n+1}$ many pseudo-random words. However, this is routine using the techniques in [8] provided we take $k_n$ large enough. This leads to a numerical requirement:

**Numerical Requirement 10** $k_n$ is chosen sufficiently large relative to a lower bound determined by $s_{n+1}, \epsilon_n$.

### 9.3 Verifying the Timing Assumptions

In this section we prove that the augmented specifications E1-J11.1 imply the timing assumptions, introduced in Section 7.2.1. The first three timing assumptions $T1-T3$ follow easily from the results in section 4.9 together with specifications Q5, Q7 and A8.

The following remark is easy and illustrates the idea behind the demonstrations of T4-T7.

**Remark 94** Suppose that $\mathcal{L}$ is an alphabet with $s$ symbols in it and $\mathcal{C} \subset \mathcal{L}$ with $|\mathcal{C}| = C$. For $u, w$ words in $\mathcal{L}$ of the same length and $x, y \in \mathcal{L}$, set $r(x, y)$ to be the number of occurrences of $(x, y)$ in $(u, w)$, $r(x, \mathcal{C})$ to be the number of occurrences of some element of $\mathcal{C}$ in $w$ opposite an occurrence of $x$ in $u$ and $f(x)$ to be the number of occurrences of $x$ in $u$. Then for all $\epsilon > 0$ there is a $\delta = \delta(\epsilon, s)$ such that whenever $u, v$ are two words in $\mathcal{L}$ of the same length $\ell$, if for all $x, y \in \mathcal{L}$,

$$\left| \frac{r(x, y)}{\ell} - \frac{1}{s^2} \right| < \delta$$

then for all $x$:

$$\left| \frac{r(x, \mathcal{C})}{f(x)} - \frac{C}{s} \right| < \epsilon$$
Because $f(x) = \sum_y r(x, y)$, by taking $\delta$ sufficiently small we can arrange that
\[
\frac{f(x)}{\ell} \approx \frac{1}{s},
\]
and the approximation improves as $\delta$ gets smaller. Simplemindedly:
\[
\frac{r(x, y)}{f(x)} = \frac{r(x, y)}{\ell} \frac{\ell}{f(x)} \approx \frac{1}{s^2} s \approx \frac{1}{s},
\]
Since $r(x, C) = \sum_{y \in C} r(x, y)$ we see that
\[
\frac{r(x, C)}{f(x)} \approx \frac{C}{s}.
\]
As we take $\delta$ smaller the final approximation improves.

We now establish the timing assumptions T4-T7.

**Assumption T5:** Assume that specification J10 holds for sufficiently small $\epsilon_n$. In T5, the number $f(x)$ is $|J(v)|$ and $C$ is the cardinality of any equivalence class of $Q_1^n$ and $s = s_n$. Since each class of $Q_1^n$ has the same number of elements, $\frac{s}{C}$ is equal to the number of classes: $\frac{s}{C} = Q_1^n$. Thus $\frac{C}{s} = \frac{1}{Q_1^n}$ and T5 follows.

**Assumption T6:** We can write the set $S$ as:
\[
S = \bigcup_{v \in W_n} \bigcup_{g \in G_1^n} \{ k < j_0 : v = w_k and w_{k+t} \in g[v]_1 \}.
\]
which can be written in turn as:
\[
S = \bigcup_{v \in W_n} \bigcup_{g \in G_1^n} \bigcup_{v' \in g[v]_1} \{ k < j_0 : v = w_k and w_{k+t} = v' \}.
\]
Thus, using J10.1, we can estimate the size of $S$ as
\[
|S| \approx s_n |G_1^n| C_1^n \left( \frac{j_0}{s_n^2} \right).
\]
Since $C_1^n = s_n/Q_1^n$ we can simplify this to $\frac{|G_1^n|}{Q_1^n} j_0$. The assumption T6 follows.
**Assumption T7:** Under the assumption that $[w'_1]_1 \notin G^n_1[w_0]_1$, $s = 0$ and $Q^n_0$ is the trivial equivalence relation. The estimate in $J_{11}$ simplifies to:

$$\left| \frac{r(u', v')}{k_n} - \frac{1}{s_n^2} \right| < \epsilon_n. \quad (45)$$

To apply Remark 94, we set $L = W_n$ and $x = v$ which makes $|J(v)| = f(x)$, in the language of the remark. With this notation equation 45 is the hypothesis of Remark 94. The conclusion of the remark is that

$$\frac{|\{t \in J(v) : C \text{ occurs at } t \text{ in } [u'_1][u'_2] \ldots [u'_{k_n-1}]\}|}{|J(v)|} \approx \frac{C^n_1}{s_n}. \quad (46)$$

Since $\frac{C^n_1}{s_n} = \frac{1}{Q^n_1}$, assumption T7 follows.

In using Remark 94 to verify the timing assumptions the number $\epsilon_n$ is playing the role of $\delta$ and $\mu_n$ is playing the role of $\epsilon$. This introduces a numerical requirement that

**Numerical Requirement 11**

$\epsilon_n$ is small relative to $\mu_n$.

**Assumption T4:** T4 is the hardest timing assumption to verify. We motivate the proof by remarking that if $u, v$ are long mutually random words in a language $L$ that has $s$ letters, then $\bar{d}(u, v) \approx 1 - 1/s^2$. Thus $u$ and $v$ are far apart. Specifications $J_{10.1}$ and $J_{11.1}$ imply that most $(u, v)$ and their relative shifts are nearly mutually random. We use this to establish that $w_0$ and $w_1$ are distant in $\bar{d}$.

**Numerical Requirement 12** $\epsilon_0 k_0 > 20$, the $\epsilon_n k_n$’s are increasing and $\sum 1/\epsilon_n k_n$ is finite.

Let

$$\gamma_1 = (1 - 1/4 - \epsilon_0)(1 - 1/\epsilon_0 k_0)(1 - 1/l_0).$$

For $n \geq 2$, set:

$$\gamma_n = \gamma_1 \prod_{0 < m < n} (1 - 10(1/k_m \epsilon_m + 1/l_m + 1/Q^n_1 + \epsilon_{m-1}))$$

and finally

$$\gamma = \gamma_1 \prod_{0 < m} (1 - 10(1/k_m \epsilon_m + 1/l_m + 1/Q^n_1 + \epsilon_{m-1})).$$
Assumption T4 says that if $w_0, w_1 \in W_n^c \cup \text{rev}(W_n^c)$ are not $Q_n^1$-equivalent, then the overlaps of sufficiently long initial segments, or sufficiently long tail segments or of a sufficiently long initial segment with a tail segment of $w_0$ and $w_1$ are at least $\gamma$ distant in $\bar{d}$. In T4 sufficiently long means at least half of the length of the word. We prove something stronger by induction on $n$:

**Proposition 95** Let $n \geq 0$ and $w_0, w_1 \in W_{n+1}^c \cup \text{rev}(W_{n+1}^c)$ with $[w_0]_1 \neq [w_1]_1$. Let $I$ be an initial segment and $T$ be a tail segment of of $\{0, 1, \ldots, q_{n+1} - 1\}$ of the same length $\ell > \epsilon_{n,q_{n+1}}$. Then we have:

\[
\bar{d}(w_0 \upharpoonright I, w_1 \upharpoonright I) \geq \gamma_{n+1} \tag{47}
\]
\[
\bar{d}(w_0 \upharpoonright T, w_1 \upharpoonright T) \geq \gamma_{n+1} \tag{48}
\]
\[
\bar{d}(w_0 \upharpoonright I, w_1 \upharpoonright T) \geq \gamma_{n+1}. \tag{49}
\]

We will consider the situation where $w_0, w_1 \in W_{n+1}^c$. The situation where they both belong to $\text{rev}(W_{n+1}^c)$ follows, and the argument in the case where $w_0, w_1$ have different parities is a small variation of the basic argument.

The strategy for the proof is to consider $n+1$-words $w_0$ and $w_1$ and gradually eliminate small portions of $I$ and $T$ so that we are left with only segments of $n$-words that align in $w_0$ and $w_1$ in such a way that they have large $d$-distance. The remaining portion of the $w_0$ and $w_1$ are far apart and they constitute most of the word. By Remark 21, we get an estimate on the distance of $w_0$ and $w_1$.

Suppose that

\[
w_0 = \mathcal{C}(u_0, u_1, \ldots, u_{k_n-1})
\]
\[
w_1 = \mathcal{C}(v_0, v_1, \ldots, v_{k_n-1}),
\]

and let $u'_i = c_n^{-1}(u_i), v'_i = c_n^{-1}(v_i)$.

A general initial segment $w \upharpoonright I$ of a word $w \in W_{n+1}$ has the following form with $q = q_n, k = k_n, l = l_n$. For some $0 \leq i_0 \leq q_n, 0 \leq j_0 \leq k_n$:

\[
\prod_{i < i_0}(\prod_{j < k} b^{q-j_i} w_j^{l-1} e^{j_i}) \ast (\prod_{j < j_0} b^{q-j_0} w_j^{l-1} e^{j_0}) \ast (b^{q*} w_j^{l*} \ast w_e^{j*})
\]

where $w^*$ is a possibly empty, possibly incomplete $n$-word, $0 \leq j^* < j_{i_0}, 0 \leq l^* \leq l - 1, 0 \leq q^* \leq q - j_{i_0}$. This is a block of complete 2-subsections, followed by a block of complete 1-subsections, followed by a possibly empty, incomplete 1-subsection.
Similarly a general tail segment $w \upharpoonright T$ as the following form:

$$(b^q w^* w^*_j e^{j^*}) \ast \left( \prod_{j_0 \leq j < k} b^{q - j_0} w^{l - 1}_j e^{j_0} \right) \ast \left( \prod_{i_0 < q \leq j < k} b^{q - j} w^{l - 1}_j e^{j_0} \right)$$

**Initial Segments:** We now argue for inequality 47. Since $\epsilon_n q_{n+1} = (\epsilon_n k_n l_n q_n) \ast q_n$, one of the following holds:

1. There are no complete 2-subsections and $j_0 > \epsilon_n k_n q_n$
2. There is at least one complete 2-subsection and $j_0 > \epsilon_n k_n$
3. There is at least one complete 2-subsection and $j_0 \leq \epsilon_n k_n$.

In the first case we can eliminate the partial 1-subsection at the end and we are left with a concatenation of at least $\epsilon_n k_n q_n$ complete 1-subsections. In eliminating the incomplete 1-subsection we eliminate at most a section of length $l_n q_n - 1$, which is of proportion less than $1/\epsilon_n k_n$ of $I$. Similarly in the second case we can eliminate the incomplete 1-subsection at the end by removing proportion less than $1/\epsilon_n k_n$ of $I$. In the final case by removing both the final incomplete 1-subsection and $(\prod_{j < j_0} b^{q - j_0} w^{l - 1}_j e^{j_0})$ we eliminate less than $1/\epsilon_n k_n$ proportion of $I$.

In all three cases, we are left an $I_0$ such that $w_0 \upharpoonright I_0$ and $w_1 \upharpoonright I_0$ are made up of a possibly empty initial segment of complete 2-subsections followed either by no complete 1-subsections or at least $\epsilon_n k_n$ complete 1-subsections. We now delete the boundary portions of $w_0 \upharpoonright I_0$, which are aligned with the boundary portions of $w_1 \upharpoonright I_0$. These have proportion $1/l_n$ in each complete 1-subsection–hence proportion $1/l_n$ of $I_0$. Let $I_1$ be the remaining portion of $I$. Then $I_1$ contains proportion at least $(1 - 1/\epsilon_n k_n)(1 - 1/l_n)$ of $I$.

**Case 1:** $[w_0]_1 \notin G^n_1[w_1]_1$.

Let $u'$ be the concatenation of $(u'_0, u'_1 \ldots u'_{k_n-1})$, and $v'$ similarly the concatenation of the $v'_i$'s. Then $u', v' \in W_{n+1}$ and $[u']_1 \notin G^n_1[v']_1$. Let $u, v \in W_n$ and $I^*$ be an initial or final segment of $\{0, 1, \ldots, k_n - 1\}$ of length at least $\epsilon_n k_n$. Let $(u^*, v^*)$ be the concatenations of $\{u'_i \mid i \in I^*\}$ and $\{v'_i \mid i \in I^*\}$. By J11.1, we see that the number $r(u, v)$ of occurrences of $(u, v)$ in $(u^*, v^*)$ satisfies:

$$\frac{r(u, v)}{|I^*|} \approx \left( \frac{1}{s_n} \right)^2$$  \hspace{1cm} (50)

\[35\text{We note that because } G^n_0 = \langle \epsilon \rangle, \text{ if } n = 0 \text{ we are in Case 1.}\]
Fix such an $I^*$ and let $C$ be a $Q_1^n$-class. Then $C$ has $C_1^n$ elements. It follows from equation 50 that the number of occurrences of a pair $(u, v)$ in $(u^*, v^*)$ with $u, v \in C$ takes proportion of $|I^*|$ approximately
\[
\frac{(C_1^n)^2}{s_2^n} = \left( \frac{1}{Q_1^n} \right)^2
\]

Since there are $Q_1^n$ many classes $C$ that need to be considered we see that the number of pairs $u'_i$ and $v'_i$ with $[u'_i]_1 = [v'_i]_1$ is approximately
\[
(1/Q_1^n)|I^*|.
\]

Thus by taking $\epsilon_n$ small enough\(^{36}\) we can assume that
\[
\frac{|\{i \in I^*: [u'_i]_1 = [v'_i]_1\}|}{|I^*|}
\]
is within $1/Q_1^n$ of
\[
\frac{1}{Q_1^n}.
\]

The locations in $w_0 \upharpoonright I_1$ are made up of powers $u_i^{l_i-1}$. These fall into two categories, those locations occurring in whole 2-subsections and those occurring in the final product of 1-subsections. Applying the previous reasoning separately to the whole 2-subsections and the relatively long 1-subsection at the end of $I$, we see that the proportion of $u_i$ occurring in $w_0 \upharpoonright I_1$ across from a $v_i$ in $w_1 \upharpoonright I_1$ that is $Q_1^n$ equivalent is also extremely close to $1/Q_1^n$.

If $n = 0$, then specification J11.1 implies that
\[
\left| \bar{d}(u^*, v^*) - \frac{3}{4} \right| < \epsilon_0.
\]

So $\bar{d}(w_0 \upharpoonright I_1, w_1 \upharpoonright I_1) > (1 - 1/4 - \epsilon_0)$ and hence
\[
\bar{d}(w_0 \upharpoonright I, w_1 \upharpoonright I) > \gamma_1.
\]

In general, the induction hypothesis yields that $Q_1^n$-inequivalent words have $\bar{d}$-distance at least $\gamma_n$-apart. Thus on $I_1$:
\[
\bar{d}(w_0 \upharpoonright I_1, w_1 \upharpoonright I_1) > (1 - 2/Q_1^n)\gamma_n.
\]

\(^{36}\)We record this as a numerical requirement at the end of the proof.
Allowing for agreement on boundary portions and applying Remark 21 we see that
\[
\bar{d}(w_0 | I, w_1 | I) \geq \left(1 - 2 \left(\frac{1}{Q_n^i} + \frac{1}{\epsilon_n k_n} + \frac{1}{l_n} \right)\right) \gamma_n > \gamma_{n+1}.
\]

Case 2: \([w_0]_1 \in G_1^n[w_1]_1\).
In this case \(n \neq 0\). Let \(g \in G_1^n\) with \(g[w_1]_1 = [w_0]_1\). Since \([w_0]_1 \neq [w_1]_1\), \(g\) is not the identity. Since \(G_1^n\) acts diagonally, for all \(i\) with \(u_i\) intersecting the interval \(I_1\), we have \([u_i]_1 = g[v_i]_1\). In particular, \([u_i]_1 \neq [v_i]_1\).
Hence \(\bar{d}(w_0 | I_1, w_1 | I_1) \geq \gamma_n\), and thus
\[
\bar{d}(w_0 | I, w_1 | I) \geq \left(1 - 2 \left(\frac{1}{\epsilon_n k_n} + \frac{1}{l_n} \right)\right) \gamma_n > \gamma_{n+1}.
\]

Tail Segments: The argument for tail segments (inequality 48) follows the argument for initial segments, except that we delete small parts of the beginning of \(T\), instead of the end of \(I\).

Tail Segments compared to initial segments: To show inequality 49, we proceed by induction, considering \(w_0, w_1 \in W_{n+1}^c\). In the comparing two initial segments or two tail segments, not only did the 2 and 1-subsections line up, but the \(n\)-subwords did as well. When comparing initial segments with tail segments, the \(n\)-subwords may be shifted, causing additional complications. The proof proceeds as in the easier cases, eliminating small sections of \(I\) (or equivalently \(T\)) a bit at a time until we are left with \(n\)-words. The alignment of these \(n\)-words allows us to apply the induction hypothesis and conclude that the vast majority of \(I\) and \(T\) have \(d\)-distance at least \(\gamma_n\).

a.) Of the 2-subsections of \(w_0\) that intersect \(I\), at most one is not a subset of \(I\) (namely the last one), and a similar statement holds for the first 2-subsection intersecting \(w_1 | T\).

b.) Each 2-subsection of \(w_0 | I\) overlaps one or two 2-subsections of \(w_1 | T\). An overlap of a 2-subsection of \(w_0 | I\) with a 2-subsection of \(w_1 | T\) that has proportion bigger than \(\epsilon_n\) of the 2-subsection implies that the overlap contains at least \(\epsilon_n k_n\) complete 1-subsections.

1. Among the complete 2-subsections of \(w_0 | I\), delete overlaps of proportion less than \(\epsilon_n\).
2. Delete the possible partial 2-subsection at the end of \( w_0 \upharpoonright I \) if it contains less than \( \epsilon_n k_n \) complete 1-subsections.

The proportion of \( I \) that has been deleted is less than \( 2 \epsilon_n \).

c.) It could be that some of the portions of the remaining 2-subsections start or end with incomplete 1-subsections; i.e. not a whole word of the form \( b^{q_n-j_i} v_{j_i}^{l_n} \). Delete these incomplete sections. This leaves initial or tail segments of 2-subsections of the form \( \prod_{j<k_n} b^{q_n-j_i} v_{j_i}^{l_n-1} e^{j_i} \) that consist of at least \( \epsilon_n k_n \) whole 1-subsections. This trimming removes at most \( 1/k_n \epsilon_n \) proportion of \( I \).

d.) We also remove the boundary sections of \( w_0 \upharpoonright I \). This removes at most \( 1/l_n \) of what remains of \( I \) at this stage.

e.) We are left with a portion \( I' \subset I \) such that \( w_0 \upharpoonright I' \) consisting entirely of 0-subsections. These are blocks of the form \( u_j^{l-1} \), where \( u_j \in W_n^c \). Each individual \( n \)-word \( u_i \) can occur opposite a portion of \( w_1 \upharpoonright T \) in various ways. These are:

i. \( u_i \) might occur exactly opposite a \( v_{i+t} \) or

ii. \( u_i \) might span portions of two copies of \( v_{i+t} \) in a power \( v_{i+t}^{l-1} \). The two copies have the form \( v_{i+t} v_{i+t} \), or

iii. \( u_i \) might overlap a portion of the boundary of \( w_1 \). This can happen in two ways: boundary inside a 2-subsection (i.e. boundary of the form \( e^{j_i} b^{q_n-j_i} \)) and boundary between consecutive 2-subsections (i.e. boundary of the form \( e^{j_i} b^{q_n-j_i+1} \)). In each \( u_i^{l-1} \) there are at most 3 copies of \( u_i \) overlapping boundary in \( w_0 \). Hence by removing proportion at most \( 4/l_n \) we are left with a portion of \( w_0 \upharpoonright I \) consisting of powers of \( u_j \)'s that do not overlap any boundary in \( w_1 \).

f.) After the deletions described in a.)-e.) the remaining portions of \( w_0 \upharpoonright I \) consists of blocks of powers of \( u_i \)'s in initial segments of 2-subsections:

\[
\begin{align*}
u_0 u_0 \ldots u_0 & \star u_0 \# u_1 u_1 \ldots u_1 & \star u_1 \# \ldots \\
u_k & \star u_k \star u_k \ldots u_k &
\end{align*}
\]

and in tail segments of 2-subsections:

\[
\begin{align*}
u_j u_j \ldots u_j & \star u_j \# u_{j+1} u_{j+1} \ldots u_{j+1} & \star u_{j+1} \# \ldots \\
u_{k_n-1} & \ldots u_{k_n-1} \star u_{k_n-1} \ldots u_{k_n-1} &
\end{align*}
\]

\[37\] This is what happens in the case that \( n = 0 \).
where ∗’s stand for u’s deleted opposite boundary of \( w_1 \) and #’s stand for the boundary of \( w_0 \) that has been deleted. An important point for us is that in each block \( k \geq \epsilon_n k_n \) and \( k_n - j - 1 \geq \epsilon_n k_n \).

We now turn to the u\(_j\)’s in situation (e.ii). The v\(_{i+t}\)’s split u\(_i\) into two pieces. By deleting a portion of the individual u\(_j\)’s of size less than \( \epsilon_n - 1 q_n \) we can assume that all of the overlap of u\(_j\)’s is in sections of length at least \( \epsilon_n - 1 q_n \). By doing this for all u\(_j\)’s we remove a parts of the remaining elements of \( w_0 \) of proportion at most \( \epsilon_{n-1} \).

g.) We now look more carefully at the two types of blocks of words described in item f.). The case (e.i) is similar and easier than the case (e.ii) so we omit it. Along the blocks described in f.) the initial portions of u\(_i\) are lined up with v\(_{i+t}\) and the second portions are lined up with v\(_{i+t+1}\). Critically, the t is constant along the block.

According to whether \( t = 0 \) or not, we apply specifications J11.1 (as in Case 1 of the Initial Segments argument) and J10 to see that at most proportion \( 2/Q_1^n \) of the u\(_i\)’s in a segment of the forms in f.) are lined up with v\(_{i+t}\) are \( Q_1^n \)-equivalent. Hence we can make a final deletion of proportion at most \( 2/Q_1^n \) to get a portion \( I^* \subseteq I \) consisting of relatively long pieces of \( W_n^c \)-words in \( w_0 \upharpoonright I' \) overlapping \( W_n^c \)-words in \( w_1 \upharpoonright T \) that lie in different \( Q_1^n \)-equivalence classes.

We now finish the argument using Remark 21. After all of the deletions we are left with \( I^* \) having at least \( (1 - (2\epsilon_n + 1/\epsilon_n k_n + 5/l_n + \epsilon_{n-1} + 2/Q_1^n)) \)-proportion of I and \( w_0 \upharpoonright I^* \) consists of relatively long pieces of \( W_n^c \) words that are overlapping portions of \( W_n^c \) words in \( w_1 \upharpoonright T \) that lie in different \( W_1^n \)-classes.

By the induction hypothesis each of the pieces of n-words in \( w_0 \upharpoonright I^* \) of \( \bar{d} \)-distance at least \( \gamma_n \) from the corresponding portion of \( w_1 \). Consequently:

\[
\bar{d}(w_0 \upharpoonright I, w_1 \upharpoonright T) \geq \gamma_n > (1 - (2\epsilon_n + 1/\epsilon_n k_n + 5/l_n + \epsilon_{n-1} + 2/Q_1^n))\gamma_{n+1}
\]

thus finishing the proof of Proposition 95. \( \dashv \)

Since assumption T4 is an immediate corollary of Proposition 95 we have finished verifying the timing assumptions.

**Numerical Requirement 13** The numbers \( \epsilon_n = \epsilon(Q_1^n) \) should be small enough that estimates in 52 hold.
We note in passing that inequality 49 holds even if \( w_0 = w_1 \) provided that the choice of initial and tail segment misalign corresponding 1-subsections.

We have proved:

**Theorem 96** Suppose that \( \mathbb{K}_c \) is a system in the range of \( F^s \) with construction sequence \( \langle W_c^n : n \in \mathbb{N} \rangle \). Then \( \langle W_c^n : n \in \mathbb{N} \rangle \) satisfies the timing assumptions.

10 The Consistency of the Numerical Requirements

During the course of this construction we have accumulated numerical conditions about growth and decay rates of several sequences. The majority of the numerical constants are not inductively determined–they are given immediately by knowing a small portion of the tree \( T \). We call these *exogenous* requirements. Other sequences of numbers depend on previous choices for the numbers–hence are determined recursively. In this section we list the the recursive requirements, explicate their interdependencies and resolve their consistency.

Some of the conditions are easy to satisfy, as they don’t refer to other sequences. Thus Numerical Requirement 1 (that \( \sum_1^n 1/l_n < \infty \)) can be satisfied once and for all by assuming that \( l_n > 20 \cdot 2^n \). Others are trickier, in that they depend on the growth rates of the other sequences. For example, in defining the sequence of \( k_n \)'s we require that \( k_n \) be large relative to the choice of \( s_{n+1} \). We call the former conditions *absolute* and the latter *dependent*. The dependent conditions introduce the risk of circular or inconsistent growth and decay rate conditions.

Our approach here is to gather all of the conditions in this paper and its predecessors and classify them as absolute or dependent. We label them A or D accordingly. This process allows us to make a diagram of the dependent conditions to verify that there are no circularities. The lack of a cycle in the diagram gives a clear method of recursively satisfying all of the numerical conditions.

Due to an overabundance of numerical parameters we were forced into some awkward notational choices. As noted before we have two types of epsilons: the *lunate* \( \epsilon_n \), often used for set membership and the *classical* \( \varepsilon_n \). They play similar but slightly different roles. The lunate epsilons come from construction requirements related to the original construction in [8], the classical epsilons come from requirements related to circular systems and realizing them as smooth systems. As is to be expected there is interaction
between the two. This occurs via the intermediary numbers we called $\mu_n$’s in Numerical Requirements 4 and 11.

10.1 The Numerical Requirements Collected.

In this section we collect the relevant numerical requirements. Specifically, in constructing $F^s(T)$ we are presented with $T$ as a subsequence $\langle \sigma_n : i \in \mathbb{N} \rangle$ of a fixed enumeration of $\mathbb{N}^{<\mathbb{N}}$.

In the statements of the specifications in [8] we built a sequence $W_n$ just in case $\sigma_n \in T$. If $\langle W_n : n \in \mathbb{N} \rangle$ is the construction sequence for $F^s(T)$, then $\langle W_j : j < i \rangle$ is determined by $\langle \sigma_{n_j} : j < i \rangle$. The specifications there discussed “successive” (or “consecutive”) elements of $T$. These are $\sigma_m$ and $\sigma_n$ that belong to $T$, but have no $\sigma_j \in T$ with $j \in (m, n)$. Hence the next collection of words constructed for $T$ after $W_m$ is $W_n$. This led the notation for construction sequences to look like: $\langle W_{n_i} : i \in \mathbb{N} \rangle$.

To avoid double subscripts, in our new notation the construction sequence for $F^s(T)$ will be denoted $\langle W_n(T) : n \in \mathbb{N} \rangle$ (or simply $\langle W_n : n \in \mathbb{N} \rangle$ if $T$ is clear). We construct $W_k(T)$ at the stage we discover the $k^{th}$ element of $T \cap \{\sigma_m : m \in \mathbb{N}\}$; what we call $W_k(T)$ in the new notation would be $W_{n_k}$ in the older notation. When $T$ is clear we can simply write $W_k$ for $W_k(T)$.

We begin with the inductive requirements from [8].

**Inherited Numerical Requirements**

We have changed the notation from [8] as described in Section 9.1 (figure 4). The number of elements of $W_m$ is denoted $s_m$; the numbers $Q^m_s$ and $C^m_s$ denote the number of classes and sizes of each class of $Q^m_s$ respectively. In [8] we have sequences $\langle \epsilon_n : n \in \mathbb{N} \rangle$, $\langle s_n, k_n, e(n), p_n : n \in \mathbb{N} \rangle$

**Inherited Requirement 1** $\langle \epsilon_n : n \in \mathbb{N} \rangle$ is summable.

**Inherited Requirement 2** $2^{e(n)}$ the number of $Q^m_{s+1}$ classes inside each $Q_{s}^m$ class. The numbers $e(n)$ will be chosen to grow fast enough that

$$2^n2^{-e(n+1)} < \epsilon_n$$ (53)

If $s$ is the maximal length of an element of $T \cap \{\sigma_m : m \leq n\}$ and $|T \cap \{\sigma_m : m \leq n\}| = i_0$ then then we set $C_{s}^{i_0} = 2^{e(i_0)}$ as well. This forces $s_m, Q^m_s$ and $C^m_s$ all to be powers of 2.

**Inherited Requirement 3** If $T = \langle \sigma_{n_i} : i \in \mathbb{N} \rangle$ then

$$2\epsilon_i \sigma_i^2 < \epsilon_{i-1}$$ (54)
Inherited Requirement 4

\[ \epsilon_i k_i s_{i-1}^{-2} \to \infty \text{ as } i \to \infty \quad (55) \]

Inherited Requirement 5

\[ \prod_{n \in \mathbb{N}} (1 - \epsilon_n) > 0 \quad (56) \]

Since this is equivalent to the summability of the \( \epsilon_n \)-sequence, it is redundant and we will ignore in the rest of this paper.

Inherited Requirement 6 There will be prime numbers \( p_i \) such that \( k_i = p_i^2 s_{i-1} \). The \( p_i \)'s grow fast enough to allow the probabilistic arguments in [8] involving \( k_n \) to go through.

Inherited Requirement 7 \( s_n \) is a power of 2

Inherited Requirement 8 The construction of \( F(T) \) requires that if \( T = \langle \sigma_{i_n} : n \in \mathbb{N} \rangle \) then \( \epsilon_n < 2^{-i_n} \).

Numerical Requirements introduced in this paper:

Numerical Requirement 1 \( l_0 > 20 \) and \( 1/l_{n-1} > \sum_{k=n}^{1} 1/l_k \).

Numerical Requirement 2 \( \langle \epsilon_n : n \in \mathbb{N} \rangle \) is a sequence of numbers in \([0, 1)\) such that \( \epsilon_N > 4 \sum_{n>N} \epsilon_n \).

Numerical Requirement 3 For all \( n \),

\[ \epsilon_{n-1} > \text{sup}_m (1/q_m) \sum_{n \leq k < m} 3 \epsilon_k q_{k+1} \]

Numerical Requirement 4 \( \mu_n \) is chosen small relative to \( \min(\epsilon_n, 1/Q_1^n) \).

Numerical Requirement 5 \( \sum \frac{|G_n|}{Q_1^n} < \infty \).

Numerical Requirement 6 \( l_n \) is big enough relative to a lower bound determined by \( \langle k_m, s_m : m \leq n \rangle \), \( \langle l_m : m < n \rangle \) and \( s_{n+1} \) to make the periodic approximations to the diffeomorphism converge.

Numerical Requirement 7 \( s_n \) goes to infinity as \( n \) goes to infinity and \( s_{n+1} \) is a power of \( s_n \).
Numerical Requirement 8 $s_{n+1} \leq s^{k_n}$.

Numerical Requirement 9 The $\epsilon_n$’s are decreasing, $\epsilon_0 < 1/40$ and $\epsilon_n < \epsilon_n$.

Numerical Requirement 10 $k_n$ is chosen sufficiently large relative to a lower bound determined by $s_{n+1}, \epsilon_n$.

Numerical Requirement 11 $\epsilon_n$ is small relative to $\mu_n$.

Numerical Requirement 12 $\epsilon_0 k_0 > 20$, the $\epsilon_n k_n$’s are increasing and $\sum 1/\epsilon_n k_n < \infty$.

Numerical Requirement 13 The numbers $\epsilon_n$ should be small enough, as a function of $Q_n$, that estimate 52 hold.

10.2 Resolution

A list of parameters, their first appearances and their constraints

We classify the constraints on a given sequence according to whether the refer to other sequences or not. Requirements that inductively refer to the same sequence are straightforwardly consistent. Those that refer to other sequences risk the possibility of being circular and thus inconsistent. We refer to the former as absolute conditions and the latter as dependent conditions.

1. The sequence $\langle k_n : n \in \mathbb{N} \rangle$.

   Absolute conditions:

   A1 The sum $\sum_n 1/k_n$ is finite.

   Dependent conditions:

   D1 Numerical Requirement 10, $k_n$ depends on $s_{n+1}, \epsilon_n$.
   D2 Inherited Requirement 6. By taking $p_n$ prime and large enough we can arrange that $k_n$ depends on $s_n$.
   D3 From Inherited Requirement 4, equation 55 requires that $\epsilon_n k_n s_{n-2}$ goes to $\infty$ as $n$ goes to $\infty$. This can be satisfied by choosing $k_n$ large enough as a function of $\epsilon_n, s_{n-1}$.

   We note that equation 55 implies that $\sum 1/\epsilon_n k_n$ is finite.
D4 Numerical Requirement 12 says that $\epsilon_0 k_0 > 20$ and the $\epsilon_n k_n$’s are increasing and $\sum 1/\epsilon_n k_n$ is finite. As noted the last condition follows from D3. The other parts of Numerical Requirement 12 are satisfied by taking $k_n$ large relative to $\epsilon_n$.

D5 Numerical Requirement 8 implies that $k_n$ is large enough that $s_{n+1} \leq s_n^{k_n}$. This implies that $k_n$ is large relative to $s_{n+1}$.

From D1-D5, we see that $k_n$ is dependent on the choices of $\langle k_m, l_m : m < n \rangle$, $\langle s_m : m \leq n + 1 \rangle$, and $\epsilon_n$.

2. The sequence $\langle l_n : n \in \mathbb{N} \rangle$.

Absolute conditions

A2 Numerical Requirement 1 says that $1/l_n > \sum_{k=n+1}^{\infty} 1/k$. We also require that $l_n > 20 \cdot 2^n$, an exogenous requirement.

Dependent conditions

D6 By Numerical Requirement 6, $l_n$ is bigger than a number determined by $\langle k_m, s_m : m \leq n \rangle$, $\langle l_m : m < n \rangle$ and $s_{n+1}$.

Thus $l_n$ depends on $\langle k_m, s_m : m \leq n \rangle$, $\langle l_n : m < n \rangle$ and $s_{n+1}$.

3. The sequence $\langle s_n : n \in \mathbb{N} \rangle$.

Absolute conditions

A3 Inherited Requirement 7 says that $s_n$ is a power of 2.

Numerical Requirement 7 says that:

A4 The sequence $s_n$ goes to infinity.

A5 $s_{n+1}$ is a multiple of $s_n$.

Dependent conditions

D7 The function $e(n) : \mathbb{N} \rightarrow \mathbb{N}$ referred to in equation 53 gives the number of $Q_{n+1}^s$ classes inside each $Q_n^s$ class. It has the dependent requirement that $2^n e(n) < \epsilon_{n-1}$. Moreover $s_n$ is a power of $2^{e(n)}$ exogenously determined by the first $n$ elements of the tree $T$.

The result is that $s_{n+1}$ depends on the first $n + 1$ elements of $T$ $\langle k_m, s_m, l_m : m < n \rangle$, $s_n$, and $\epsilon_n$.

\[\text{It is important to observe that the choice of } s_{n+1} \text{ does not depend on } k_n \text{ or } l_n.\]
4. **The sequence** $\langle \epsilon_n : n \in \mathbb{N} \rangle$.

**Absolute conditions**

**A6** Numerical Requirement 9 and Inherited Requirement 1 say that the $\langle \epsilon_n : n \in \mathbb{N} \rangle$ is decreasing and summable and $\epsilon_0 < 1/40$.

**A7** Inherited Requirement 8 says that if $\mathcal{T} = \langle \sigma_n : n \in \mathbb{N} \rangle$ then $\epsilon_n < 2^{-i_n}$

**Dependent conditions**

**D8** Numerical Requirement 9 says $\epsilon_n < \varepsilon_n$.

**D9** Equation 54 of Inherited Requirement 3 says $2\epsilon_n s_n^2 < \epsilon_{n-1}$

**D10** Numerical Requirement 11 says that $\epsilon_n$ must be small enough relative to $\mu_n$.

**D11** Numerical Requirement 13 says that $\epsilon_n$ is small as a function of $Q_n^1$.

The result is that $\epsilon_n$ depends exogenously on the first $n$ elements of $\mathcal{T}$, and on $Q_n^1, s_n, \varepsilon_n, \epsilon_{n-1}$ and $\mu_n$.

5. **The sequence** $\langle \varepsilon_n : n \in \mathbb{N} \rangle$.

**Absolute conditions**

**A8** Because the sequence $\langle q_n : n \in \mathbb{N} \rangle$ is increasing Numerical Requirement 3 is satisfied if $\varepsilon_{n-1} > 4 \sum_{k \geq n} \varepsilon_n$, a rephrasing of Numerical Requirement 2. This is an absolute condition and implies that $\langle \varepsilon_n : n \in \mathbb{N} \rangle$ is decreasing and summable.

**Dependent conditions**

**D12** $\langle k_n \varepsilon_n : n \in \mathbb{N} \rangle$ goes to infinity. This already follows from the fact that $\epsilon_n < \varepsilon_n$ and item D4.

Since item D12 follows from item D4, all of the requirements on $\langle \varepsilon_n : n \in \mathbb{N} \rangle$ are absolute or follow from previously resolved dependencies.

6. **The sequence** $\langle Q_n^1 : n \in \mathbb{N} \rangle$.

Recall $Q_n^1$ is the number of equivalence classes in $Q_n^1$. We require:

**Absolute conditions**
A9 Since $|G^n_1|$ is determined exogenously by $\mathcal{T}$ the requirement that
\[ \sum_{Q_1^n} |G^n_1| < \infty \]
is an absolute condition. It follows that $\sum 1/Q_1^n < \infty$.

Dependent conditions
None

7. The sequence $\langle \mu_n : n \in \mathbb{N} \rangle$.
This sequence gives the required pseudo-randomness in the timing assumptions.

Absolute conditions
None.

Dependent conditions

D13 Numerical Requirement 4 requires that $\mu_n$ be very small relative to $\varepsilon_n$ and $1/Q_1^n$.

The recursive dependencies of the various coefficients are summarized in Figure 5, in which an arrow from a coefficient to another coefficient shows that the latter is dependent on the former. Here is the order the the coefficients can be chosen consistently.

Assume:

The coefficient sequences $\langle k_m, l_m, Q_1^m, \mu_m, \varepsilon_m, \varepsilon_m : m < n \rangle$ and $s_n$ have been chosen. The first $n$ sequences on the tree $\mathcal{T}$ are known.

To do:

Choose $k_n, l_n, Q_1^n, \mu_n, \varepsilon_n$ and $s_{n+1}$. Each requirement is to choose the corresponding variable large enough or small enough where these adjectives are determined by the dependencies outlined above.

Figure 5 gives an order to consistently choose the next elements on the sequences; Choose the successor coefficients in the following order:

$Q_1^n, \varepsilon_n, \mu_n, \varepsilon_n, s_{n+1}, k_n, l_n$. 

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Figure 5: Order of choice of Numerical parameters dependency diagram.
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