MH-Octant-CPV entanglement, vacuum alignment of $A_4$ flavour symmetry, and $0\nu\beta\beta$ decay in an inverse seesaw model

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Abstract

Neutrino experiments have measured some of the oscillation parameters with unprecedented accuracy, yet parameters such as the lightest neutrino mass, octant of the atmospheric angle, leptonic CP-violating phases, neutrino mass hierarchy, nature and absolute masses of the neutrinos remain to be fixed with precision, though some data is available on them also. Long baseline neutrino experiments (LBLs) have the potential to measure these, but are inflicted with the problem of octant degeneracy [1],[2], and in the measurements of experiments like NO$\nu$A, the MH sensitivity depends upon the value of CP-violating phase $\delta_{CP}$ (also called MH-Octant-CPV entanglement). At the same time, the flavour structure of fermions is still not understood. In this work, we aim to address all these issues, in an inverse seesaw (ISS) model within the framework of $A_4$ flavour symmetry. We find the unknown parameters up to a tolerance of $<10^{-5}$ in numerical computation. Keeping in view that a recent global analysis has shown a preference for normal hierarchy and higher octant of $\theta_{23}$, we discuss our results in the context of this and find that we can pinpoint the VEV alignment of $A_4$ triplet flavon to be (1,-1,-1). Finally, we compare our very precise prediction on $m_{ee}$ and lightest neutrino mass with the latest bounds and sensitivities of the neutrinoless double-beta ($0\nu\beta\beta$) decay experiments and our prediction lies within the allowed region. We emphasize that when MH-Octant-CPV entanglement is resolved by more precise measurements of future experiments, then through our results, it is possible to precisely pinpoint the favourable VEV alignment of the triplet flavon field of $A_4$ symmetry (or vice-versa - if VEV alignment is known, then this entanglement can be resolved). The model being a low-scale seesaw model, is interesting as it is testable in the ongoing and future neutrino experiments, and is highly predictive too.

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I. INTRODUCTION

The unfolding of neutrino properties after the advent of neutrino oscillation has been a breakthrough achievement for theoretical physicists in the 20th century. Since then, the neutrino parameters have been measured to an unprecedented accuracy by various neutrino experiments with overwhelming evidence in support of non-zero neutrino masses and hence mixings. However, there are still some open issues such as the nature and mass hierarchy (MH) of the neutrinos, absolute mass of neutrinos, octant of the atmospheric mixing angle $\theta_{23}$ and the leptonic CP-violating (CPV) phases, that need to be addressed by theory and experiments alike. Some recent experiments have provided precise information on neutrino oscillation parameters, such as Double ChooZ [3], Daya-Bay [4], RENO [5], and T2K [6].

The canonical seesaw mechanisms like type I and type II seesaw models can have heavy right-handed (RH) neutrinos in the range of $(\sim 10^{10} - 10^{14})$ GeV [7]. This questions the testability of the high scale seesaw models at the ongoing collider experiments. However, in the low-scale seesaw scenarios like an inverse seesaw, the heavy particles or the seesaw breaking scale can be brought down to TeV ($10^{12}$ eV) range which makes the low-scale seesaw based models testable and hence more appealing. The smallness of the neutrino mass can be naturally explained by the lepton number violating (LNV) Majorana mass terms in the case of ISS and LNV mass scale between the left-handed neutrinos ($\nu_L$) and the fermionic singlet (S) via Yukawa coupling [8] (this is called the Gerardus’t Hooft’s naturalness criterion (‘t Hooft, 1980) [9]). Also, the probing of these Majorana particles at the TeV scale in the ongoing collider experiments adds a novel signature to a new, theoretically motivated physics. Hence in this work, we chose to consider an ISS model with $A_4$ flavour symmetry.

LBL experiments like NO$\nu$A, T2K etc have the potential to determine MH and CPV phase because of their long baselines. In this work, we have attempted to address the issue of parameter entanglement (MH-Octant-CPV) which inflict the measurements of LBLs and use this information to throw light on completely unknown dynamics of $A_4$ flavour symmetry, in particular the favoured VEV alignment of the triplet flavon field that breaks this symmetry. This can be possible when future measurements can fix these unknown
light neutrino parameters (MH, Octant, CPV) precisely, i.e., when MH-Octant-CPV entanglement is completely resolved. We construct the Lagrangian of the ISS model based on the simplest non-abelian discrete group $A_4$ (for flavour symmetry) with additional symmetries like $U(1)_X$, $Z_4$, $Z_5$, where $U(1)_X$ is a global symmetry which is broken softly (to avoid the occurrence of Goldstone bosons). The Lagrangian is constructed up to dimension 6 with mass term $\mu$, as we need to make sure that the mass scale of the $\mu$ is very very small than the rest of the elements in the matrix given in Eq. (4). These symmetries are needed to avoid unwanted terms in the Lagrangian. Next, we compare the neutrino mass matrix obtained in our model, with the phenomenological one, which gives us a set of simultaneous equations for the flavon fields. By solving these equations, we obtain the correlation plots among various neutrino parameters, i.e., the mixing angles ($\theta_{12}, \theta_{13}, \theta_{23}$) and the neutrino mass differences ($\Delta m^2_{21}, \Delta m^2_{31}$) with the unknown parameters like $m_{\text{lightest}}$, Dirac leptonic CPV phase $\delta_{\text{CP}}$, Majorana phases ($\alpha$ and $\beta$), as a measure of prediction and testability of the model. For this, we impose the constraint that our results lie within the $3\sigma$ ranges of their latest global best fit data which are summarized in Table (I). Several possible cases of VEV alignments of the flavour triplet flavon field are considered in the analysis. This generated data not only helps us to find the correlation pattern among various neutrino parameters but also motivates us to study its phenomenological implications, like (MH-Octant-CPV) entanglement and effective mass of neutrino that can be measured in neutrinoless double beta decay ($0\nu\beta\beta$). Observations of $0\nu\beta\beta$ will establish the Majorana nature of neutrinos, and would also open a new window for new physics beyond SM (standard model) by providing the first direct evidence for violation of lepton number. This discussion motivates us to take up the current work, as we have presented new and precise results on resolving MH-Octant-CPV entanglement, preferred VEV of $A_4$ symmetry, lightest neutrino mass and $m_{ee}$.

We would like to mention that similar works on neutrino masses for different alignments of various flavon fields under $A_4$ flavour symmetry have been reported in [10]-[13]. However, the MH-Octant-CPV entanglement problem hasn’t been studied in those works. Neutrino models with low scale type II seesaw have recently been studied in [14, 15], wherein [15], they have interestingly incorporated $Z_4$ symmetry to describe low-scale seesaw with inelastic sub-GeV dark matter. A variety of low scale inverse and linear seesaw based
models under the framework of $A_4$ symmetry have also been studied in [16]-[18] wherein [16] inverse and linear seesaw models are presented using $SU(2)_L \times Z_3 \times A_4$, while in [18] a linear seesaw scheme is presented for $A_4 \times Z_4 \times Z_3 \times U(1)_X$ symmetry, $U(1)_X$ being an extra global symmetry. In [20] they have discussed models for Dirac neutrinos for type I and inverse seesaw model with $A_4 \times Z_4 \times Z_3$ symmetry, Also in [21], an $A_4$ LRSM was discussed. For several other discrete symmetry-based models one can also refer to references [22] - [28]. Some recent works on resolving parameter degeneracy in various baseline experiments are discussed in [29] - [36]. The octant degeneracy resolution in JUNO can be found in [37], and in [38]-[39] also, octant degeneracy in LBL experiment (DUNE, USA) was discussed. A recent discussion on 2σ tension in T2K and NOνA can be found in [19], [40] -[44], wherein [40], NSI is introduced to explain the new physics due to the discrepancy in T2K and NOνA experiments. However, the idea of parameter entanglement regarding the information on flavour symmetry has not been discussed earlier. Moreover, most of them have not mentioned clearly that up to what tolerance they have computed their results - and this is important in the context of precision measurements.

However, in a very recent global analysis [19], they have presented some preferences for NH (normal hierarchy) and HO (higher octant), and we have discussed our results with reference to this analysis too, and found that the VEV $(1,-1,-1)/(-1,1,1)$ of the $A_4$ triplet flavon field is preferred. Such information on pinpointing VEV of flavour symmetry has not been presented earlier. This novel information is then used to predict very precise values of $m_{\text{lightest}}$ and $m_{ee}$ (of $0\nu\beta\beta$ decay) in our model, which can be tested in future.
The paper has been organized as follows. In section II, we briefly review the (MH-Octant-CPV) entanglement. The implication on neutrinoless double beta decay from the study of neutrino oscillation parameters is discussed briefly in Section III. Section IV contains our model on inverse seesaw mechanism with $A_4$ symmetry. We present the numerical method of our analysis and results in Section V. A discussion on our results that highlight the novel information on the resolution of (MH-Octant-CPV) entanglement and preferred VEV alignment of the triplet flavon field present in our model and neutrinoless double beta decay has been presented in Section VI. We conclude in Section VII.

II. MH-OCTANT-CPV ENTANGLEMENT

The discovery of neutrino oscillation at the Super-Kamiokande experiment (SK) [45] and Sudbury Neutrino Observatory (SNO) [46], [47] gave us strong evidence that neutrinos are particles beyond the Standard Model that are massive in nature, and that there is the mixing of flavours while neutrinos travel. Long baseline experiments are sensitive to appearance and disappearance channels, and for example, the probability in the latter can be expressed as

$$P_{\mu\mu} = 1 - \sin^2 2\theta_{23} \sin^2 \frac{1.27 \Delta m_{31}^2 L}{E} + 4 \sin^2 \theta_{13} \sin^2 \theta_{23} \cos 2\theta_{23} \sin^2 \frac{1.27 \Delta m_{31}^2 L}{E}$$

(1)

where $\Delta m_{ij}^2 = m_i^2 - m_j^2$ and $\frac{\Delta_{21}^2}{\Delta_{31}^2}$ is assumed to be small. LBL experiments such as T2K and NO\(\nu\)A cannot differentiate the octant of the atmospheric mixing angle, as it is clear from Eq. (1) that the probability is sensitive to $\sin^2 2\theta_{23}$ and hence $P(\theta_{23}) = P(\pi/2 - \theta_{23})$. This is called octant degeneracy as one cannot differentiate if $\theta_{23} > \pi/4$ or $\theta_{23} < \pi/4$ [1], [2] (or in other words it cannot differentiate if $\sin^2 2\theta_{23} > 0.5$ or $\sin^2 2\theta_{23} < 0.5$ [19]) from experimental measurements. These experiments are also sensitive to the measurement of CP-violating phase, $\delta_{CP}$. Recent measurements have shown that $\delta_{CP} \approx 0.8\pi$ from NO\(\nu\)A analysis while it disfavours the region $\delta_{CP} \approx 1.5\pi$ which coincides with the T2K best fit values [19]. Hence, there is ambiguity in CPV phase measurements at T2K and NO\(\nu\)A. It is also observed that the mass-hierarchy (MH) of neutrinos, CPV phase $\delta_{CP}$ and octant of $\theta_{23}$ are entangled in LBL experiments measurements - for example, NO\(\nu\)A sensitivity on MH depends on the value of $\delta_{CP}$ [30], and degeneracy is also present in $(\theta_{23} - \delta_{CP})$ plane [38], [45]. The atmospheric neutrino results from Super-Kamiokande [48] and Deep Core (Ice Cube) exper-
iments [49], [50] when combined with the data of the long baseline experiments [40], [41], [42], then substantially more preference for normal hierarchy is observed. However, as the measurements of $\delta_{CP}$ in T2K and NO$\nu$A are inflicted with discrepancy as discussed above, no clearcut preference for a particular MH is seen in their data. Hence the measurement of $\delta_{CP}$ is quite crucial as it will directly affect the measurement of other neutrino oscillation parameters and their corresponding MH. However in a recent work [30], it was shown that the data of T2K, NO$\nu$A and JUNO can be combined to enhance the measurement of $\delta_{CP}$ and resolve MH and octant degeneracy problem too. Earlier, one of us had shown that octant degeneracy can be resolved by combining results of LBL experiment (DUNE, USA) and reactor experiments [38], and there we argued that since reactor experiments can determine the angle $\theta_{13}$ with precision, it helps in pinpointing the true octant of angle $\theta_{23}$.

From the above discussion, it is clear that the neutrino oscillation experiments face the problem of parameter entanglement - measurement of one parameter depends on that of another, and it is still a challenge before theorists as well as experimentalists alike. Many proposals have been given to resolve these ambiguities [36], [38] and [39]. Suppose in future, this parameter entanglement is resolved, then, we propose in this work, that we can pinpoint the VEV alignment of the triplet flavon field. It may be noted that at present, no information about flavour symmetries is available, and hence findings of this work command importance.

III. IMPLICATIONS FOR NEUTRINOLESS DOUBLE BETA DECAY

This study can also provide us with an interesting and complementary piece of information on the bounds of effective neutrino mass, $m_{\beta\beta}$ that can be measured in the neutrinoless double beta ($0\nu\beta\beta$) decay experiments. In neutrinoless double-beta decays, no neutrinos are emitted (emitted neutrinos are immediately reabsorbed) due to the Majorana nature of the neutrinos. The effective mass of light neutrino that can be measured in $0\nu\beta\beta$ experiments is given by ( [51], [52])

$$m_{ee}^{\nu} = |c_{12}^2 c_{13}^2 m_1 + s_{12}^2 c_{13}^2 m_2 e^{2i\alpha} + s_{13}^2 m_3 e^{2i\beta}|,$$  

(2)
where $c$ and $s$ stands for sine and cosine of the mixing angles. Eq. (2) depends on the light neutrino oscillation parameters, the values of which are obtained in our analysis (constrained in their allowed $3\sigma$ range) and given as input in Eq. (2). The half-life of the isotope $T_{1/2}^{0\nu}(\mathcal{N})$ which is involved in the $0\nu\beta\beta$ decay is constrained by its decay amplitude [19], [53]. This lifetime of isotopes can constrain the bounds on the $m_{ee}$ from the $0\nu\beta\beta$ events as [19]

$$T_{1/2}^{0\nu}(\mathcal{N}) = \frac{m_e^2}{G_{0\nu}^N|M_{0\nu}^N|^2 m_{\beta\beta}^2}$$

(3)

where $m_e$ is the electron mass, $G_{0\nu}^N$ and $M_{0\nu}^N$ represents the phase space factor and the nuclear matrix element respectively, both depending on the isotope involved in the decay.

The strongest bounds on half-life, $T_{1/2}^{0\nu}(\mathcal{N})$ set by GERDA [54], CUORE [55] and KamLAND-Zen [56] are $T_{1/2}^{0\nu} > 9 \times 10^{25}$ yr for $^{76}$Ge, $T_{1/2}^{0\nu} > 3.2 \times 10^{25}$ yr for $^{130}$Te and $T_{1/2}^{0\nu} > 1.07 \times 10^{26}$ yr for $^{136}$Xe respectively at 90% confidence level. The upper limit of $m_{\beta\beta}$ from GERDA experiment is $m_{0\nu\beta\beta} < 104$-228 meV [54], $m_{0\nu\beta\beta} < 75$-350 meV by CUORE [55] experiment, and $m_{0\nu\beta\beta} < 61$-165 meV by KamLAND-Zen experiment [56]. However from the recent updates from the nEXO experiment, the exclusion band of the $m_{\beta\beta}$ falls between (5.7-17.7) meV [57] that arises from the range of nuclear matrix elements, with EDF (energy density function) and QRPA (Quasi-particle Random Phase Approximation). The particle physicists aim to increase sensitivity for $m_{0\nu\beta\beta}$ of 20 meV or more in the next generation $0\nu\beta\beta$ experiments, such as LEGEND, SuperNEMO, CUPID, SNO+, KamLAND2-Zen, nEXO, NEXT, PANDAX-III [58, 59]. In later sections, we compute $m_{0\nu\beta\beta}$ allowed in our model and find that it agrees well with its current best fit values.

IV. THE ISS MODEL WITH $A_4 \times Z_4 \times Z_5 \times U(1)_X$ SYMMETRY

We now present our model - the ISS model [60, 61], with $A_4$ flavour symmetry, where besides the SM particles, we have included right-handed neutrino $N$ and three other singlet fermions $S_{i=1,2,3}$ (Sterile neutrinos). The neutrino mass matrix that is obtained from the Lagrangian of ISS mechanism in the basis ($\nu_L^c, N, S$), after spontaneous symmetry breaking takes the form as follows [62]:

\[ \text{\ldots} \]
\[ M_\nu = \begin{bmatrix} 0 & M_D & 0 \\ M_D^T & 0 & M \\ 0 & M^T & \mu \end{bmatrix}. \] (4)

The effective light neutrino mass matrix formula with the mass hierarchy of \( M_D, M \gg \mu \) is given by,

\[ m_\nu = M_D (M^T)^{-1} \mu M^{-1} M_D^T. \] (5)

The lepton number being only an approximate symmetry is broken by the \( \mu \) term [63]. The flavons that are needed to describe the flavour structure of the particles are - \((\phi_T, \phi_S, \eta, \xi, \tau, \rho, \rho', \rho'')\), which carry quantum numbers of the flavour group \( A_4 \) and discrete groups \( Z_4 \) and \( Z_5 \), to define the flavour structure of the lepton mixing. An extra global symmetry \( U(1)_X \) has been added that can explain the flavour structure and smallness of the \( \mu \) term at leading order of dimension 6 with the help of the light pseudo-Goldstone boson called Majoron [64] (that is associated with the spontaneous breaking of the global symmetry). The smallness of \( \mu \) term is a necessary condition (for Eq. (5) to be viable), which is generated dynamically, such that the \( \mu \) term can be of keV scale and the seesaw breaking scale of order \( \sim 1 \text{ TeV} \). The matter content of our model is shown in Table (II).

| L H | e_R | \( \nu_R \) | \( \tau_R \) | N S | \( \Phi_T \) | \( \Phi_s \) | \( \eta \) | \( \xi \) | \( \tau \) | \( \rho \) | \( \rho' \) | \( \rho'' \) |
|-----|-----|----------|----------|-----|--------|--------|------|-----|-----|------|------|------|-------|
| \( A_4 \) | 3 1 1 | 1' | 1' | 3 3 | 3 3 | 1 1' | 1'' | 1 1 1 |
| \( Z_4 \) | 1 1 i i i i i | i i i i i i i i | i -i -i -i i i i |
| \( Z_5 \) | 1 1 \( \omega \) \( \omega \) \( \omega \) \( \omega^2 \) | \( \omega \) \( \omega \) \( \omega \) \( \omega^2 \) | \( \omega \) \( \omega \) \( \omega \) \( \omega^2 \) |
| \( U(1)_X \) | -1 0 -1 -1 -1 | -1 1 0 -1 -1 -1 -1 0 -4 -3 |

TABLE II: Transformation of the fields in our model under \( A_4 \times Z_4 \times Z_5 \times U(1)_X \) symmetry for neutrino mass model realizing inverse seesaw mechanism.

With the above content of the fields, the gauge and discrete symmetries assumed in the model allow the Lagrangian for the charged leptons of the form :

\[ \mathcal{L}_{c.l.} \supset \frac{y_e}{\Lambda} (\bar{L}_T \Phi_T^\dagger) H e_R + \frac{y_\mu}{\Lambda} (\bar{L}_T \Phi_T^\dagger)' H \mu_R + \frac{y_\tau}{\Lambda} (\bar{L}_T \Phi_T^\dagger)'' H \tau_R. \] (6)
Hence, the charged lepton mass matrix takes the form:

\[
M_l = \frac{v^\dagger_T v_h}{\Lambda} \begin{bmatrix}
y_e & 0 & 0 \\
0 & y_\mu & 0 \\
0 & 0 & y_\tau
\end{bmatrix},
\] (7)

to the leading order, where \(\Lambda\) represents the cut-off scale of the theory and \(y_e\), \(y_\mu\) and \(y_\tau\) are the respective coupling constants. Terms within the first parenthesis describe the product of two \(A_4\) triplets, which further contract with \(A_4\) singlets 1, 1\(''\) and 1\('\) corresponding to \(e_R, \mu_R\) and \(\tau_R\) fields respectively to constitute an \(A_4\) singlet. \(A_4\) multiplication rules can be summarized as:

\[
1' \times 1' = 1'', \\
1' \times 1'' = 1', \\
1'' \times 1'' = 1'\,',
\]
and \(3 \times 3 = 1 + 1' + 1'' + 3_A + 3_S\) [65]. We choose the VEV of \(\phi_T\) as \(\langle \phi_T \rangle = v_T (1, 0, 0)\) [66] so that the charged lepton mass matrix turns out to be diagonal in the leading order and can be written as \(M_l = v_h \frac{v^\dagger_T}{\Lambda} \text{diag}(y_e, y_\mu, y_\tau)\).

The relevant part of the neutrino sector of the Lagrangian is shown below:

\[
\mathcal{L}_Y \supset Y_D \frac{\bar{L} H N \rho^\dagger}{\Lambda} + Y_M N S \rho^\dagger + Y_\mu S S [\rho' \rho'' (\Phi_s + \eta + \xi + \tau) / \Lambda^2] + \text{h.c.},
\] (8)

where \(Y_D, Y_M, Y_\mu\) are the dimensionless coupling constants. The scalars take non-zero VEVs as \(\langle H \rangle = v_h, \langle \eta \rangle = v_\eta, \langle \rho \rangle = v_\rho, \langle \rho' \rangle = v_\rho', \langle \rho'' \rangle = v_\rho'', \langle \xi \rangle = v_\xi, \langle \tau \rangle = v_\tau, \langle \Phi_s \rangle = v_s(\Phi_a, \Phi_b, \Phi_c)\). The invariance of the scalar potential under \(A_4 \times Z_4 \times Z_5\) with the extra global symmetry \(U(1)_X\) is shown rigorously in Appendix C. Thus we get the 3 \(\times\) 3 mass matrices as follows

\[
M_D = \frac{Y_D v_h v^\dagger}{\Lambda} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
\] (9)

\[
M = \frac{Y_M \rho^\dagger}{\Lambda} \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix},
\] (10)
and,

\[ \mu = \frac{Y_{\mu}^{\dagger} v_{\rho} v_{\rho}^{\dagger}}{\Lambda^2} \left( \begin{array}{ccc}
 v_\eta + 2v_s \phi_a & v_\xi - v_s \phi_c & v_\tau - v_s \phi_b \\
v_\xi - v_s \phi_c & v_\tau + 2v_s \phi_b & v_\eta - v_s \phi_a \\
v_\tau - v_s \phi_b & v_\eta - v_s \phi_a & v_\xi + 2v_s \phi_c
\end{array} \right). \quad (11) \]

Putting the values of these matrices in the Eq. (5) we get (with \( M_D, M \gg \mu \)),

\[ \Rightarrow m_\nu = F \left( \begin{array}{ccc}
 v_\eta + 2v_s \phi_a & v_\xi - v_s \phi_c & v_\tau - v_s \phi_b \\
v_\xi - v_s \phi_c & v_\tau + 2v_s \phi_b & v_\eta - v_s \phi_a \\
v_\tau - v_s \phi_b & v_\eta - v_s \phi_a & v_\xi + 2v_s \phi_c
\end{array} \right), \quad (12) \]

where, \( F = \frac{Y_D^2 Y_{\mu}^{\dagger} v_{\rho}^2 v_{\rho}^{\dagger}}{Y_M^2 \Lambda^4} \), a constant that depends on various couplings, scales and VEVs of the model.

V. NUMERICAL ANALYSIS AND RESULTS

Above we obtained the light neutrino mass matrix for the ISS model in (12). The diagonalizing matrices of neutrino and charged lepton mass matrices \( U_\nu, U_\ell \) can be obtained from the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) leptonic mixing matrix, as

\[ U = U_\ell^{\dagger} U_\nu. \quad (13) \]

The parametrisation of the PMNS mixing matrix can be done as:

\[ U = \begin{pmatrix}
 c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\
 -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\
 s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13}
\end{pmatrix} U_{\text{Maj}}, \quad (14) \]

where \( c_{ij} = \cos \theta_{ij}, s_{ij} = \sin \theta_{ij} \) and \( \delta_{CP} \) is the Dirac CPV phase. \( U_{\text{Maj}} = \text{diag}(1, e^{i\alpha}, e^{i(\beta+\delta_{CP})}) \) is the diagonal matrix with the Majorana CP phases \( \alpha, \beta \). The light neutrino mass matrix can be constructed from the parametrized PMNS mixing matrix, \( U \) as

\[ m_\nu = U m_\nu^{\text{diag}} U^T, \quad (15) \]

where \( m_\nu^{\text{diag}} = \text{diag}(m_1, m_2, m_3) \) is the diagonal light neutrino mass matrix. The three neutrino mass eigenvalues for the NH are given as \( m_\nu^{\text{diag}} = \)
\( \text{diag}(m_1, \sqrt{m_2^2 + \Delta m_{21}^2}, \sqrt{m_1^2 + \Delta m_{31}^2}) \), and for the IH, they can be given as
\[ m_{\nu}^{\text{diag}} = \text{diag}(\sqrt{m_2^2 + \Delta m_{23}^2 - \Delta m_{21}^2}, \sqrt{m_2^2 + \Delta m_{23}^2}, m_3) \).

After comparing the Eq.(12) with Eq.(15), the unknown values of the flavons’ VEVs considered to break the flavor symmetry can be found in terms of light neutrino oscillation parameters. As the light neutrino mass matrix is complex symmetric, we get six complex equations from the above matrix equation. We then scan our solutions with possible VEV alignments of the triplet flavon scalar, and compute four experimentally undetermined neutrino parameters (\( m_{\text{lightest}} \), one Dirac CP phase \( \delta_{CP} \) and two Majorana phases \( \alpha \) and \( \beta \)) by solving these equations of flavon VEVs. For our numerical analysis, we consider 26 possible cases of VEV alignments which are shown in Table (III). Our computational analysis is similar to the works of [11], [12] and [13]. Our results have also been summarized in Table (IV) and in Figs (1-4) as correlation plots among the unknown light neutrino parameters. In Fig (5), we have presented the combined experimental fit for neutrinoless double beta decay, taken from Ref. [67], so that we may it may give a better visualization for comparison of our corresponding results. Latest global fit results from Refs. [19], [68] have been summarized in Table (V), for comparison with our results.

VI. DISCUSSION ON RESULTS

To understand the CP violation in weak interactions, we need to measure the CPV phase parameter in the leptonic sector. From the recent data of T2K [69] and NO\( \nu \)A [43] experiments, it is seen that both the experiments prefer normal hierarchy (NH), without any new physics (for standard oscillation picture). However, a slight tension is seen at the 2\( \sigma \) level, where T2K prefers \( \delta \sim 3\pi/2 \) for NH which is excluded by NO\( \nu \)A at 90\% confidence level. NO\( \nu \)A, in general, does not have any strong preference for any particular value of the CPV phase and has its best fit value around \( \delta \sim \pi \) for NH. This discrepancy is perhaps due to the configuration of baselines and the effect of matter density as neutrinos in NO\( \nu \)A experience a much stronger matter effect, and hope that this discrepancy can be alleviated from the robust data obtained by forthcoming neutrino experiments. A slight preference of IH over NH (without new physics) is seen when the recent data of T2K and NO\( \nu \)A are combined [40], though data from Super Kamiokande still prefer NH over IH [41]. However,
FIG. 1: Correlation plots for neutrino oscillation parameters for our ISS model (normal hierarchy) with allowed triplet flavon VEV alignment $(1, -1, -1)/(-1, 1, 1)$.

FIG. 2: Correlation plots for neutrino oscillation parameters for our ISS model (normal hierarchy) with allowed triplet flavon VEV alignment $(0, -1, -1)/(0, 1, 1)$. 
TABLE III: Summary of our result after solving the triplet flavon equations of different VEV alignments for all the inverse seesaw mechanism. The green sign, X indicate ruled out cases, where no output is obtained for the tolerance level of $10^{-5}$.

| SL. NO. | VEV         | ISS | SL. NO. | VEV         | ISS |
|---------|-------------|-----|---------|-------------|-----|
|         |             | NH  |         |             | IH  |
| 1       | (1,0,0)     | X   | 14      | (-1,1,-1)   | X   |
| 2       | (0,1,0)     | X   | 15      | (-1,-1,1)   | X   |
| 3       | (0,0,1)     | X   | 16      | (1,1,-1)    | X   |
| 4       | (1,1,0)     | X   | 17      | (1,-1,1)    | X   |
| 5       | (1,0,1)     | X   | 18      | (-1,1,1)    | allowed |
| 6       | (0,1,1)     | allowed | 19 | (-1,-1,0)   | X   |
| 7       | (1,-1,0)    | X   | 20      | (-1,0,-1)   | X   |
| 8       | (1,0,-1)    | X   | 21      | (0,-1,1)    | allowed |
| 9       | (-1,0,1)    | X   | 22      | (-1,0,0)    | X   |
| 10      | (-1,1,0)    | X   | 23      | (0,-1,0)    | X   |
| 11      | (0,-1,1)    | allowed | 24 | (0,0,-1)    | X   |
| 12      | (0,1,-1)    | allowed | 25 | (1,1,1)     | X   |
| 13      | (1,-1,-1)   | allowed | 26 | (-1,-1,-1) | X   |

TABLE IV: Ranges of parameters obtained in our analysis, with triplet flavon VEV (0,-1,1)NH / (0,1,1)NH, VEV (-1,1,1)NH / (1,-1,1)NH, and VEV (0,-1,1)IH / (0,1,-1)IH.

after combining the new experimental data from T2K, NOvA, Super-K [70], it is seen that the NH is preferred over the IH (please see [40] - [42]), and this feature is also observed in our analysis. From Table (III), it is seen that out of 26 cases of triplet flavon VEV alignments for both NH and IH, four of them, i.e., (0,1,1), (0,-1,-1), (1,-1,-1) and (-1,1,1) favour NH while only two of them, i.e., (0,-1,1) and (0,1,-1) favour IH, that allows light neutrino oscillation parameters within the allowed 3σ range of their current best fit global
values. The number of allowed cases are very few as we are taking only those high-precision solutions whose tolerance is $< 10^{-5}$ with $10^6$ random points being generated (for each case).

Results in the correlation plots in Figs. (1-4) show that our model can predict the values of all the unknown neutrino parameters, such that the known ones lie in the range allowed currently by experiments. Our model has been able to predict the value of the lightest neutrino mass, which can be tested in future. Next we discuss these results with reference to (MH-Octant-CPV) entanglement [30], [38], [39]. Suppose, in future, the experiments are able to pinpoint very precisely, the neutrino mass hierarchy- let’s say it is NH. Then, we can say that two cases - A (with triplet flavon VEV (0,1,1) and with Lower octant of $\theta_{23}$), and B (with triplet flavon VEV (-1,1,1) and with Higher octant of $\theta_{23}$) could be possibly true. Next, to pinpoint the exact model (i.e. to resolve this octant degeneracy), we can take guidance from measurements of octant of $\theta_{23}$. If the Octant is measured to be lower Octant (LO), then case A would be true, while if it turns out to be higher Octant (HO), then case B would be true. This way, the preferred VEV alignment due to flavour symmetry can be pinpointed. On the other hand, if the MH turns out to be inverted, then
FIG. 4: Correlation plots of lightest neutrino mass vs effective neutrino mass of neutrinoless double beta decay for allowed triplet flavon VEV alignments for our model.

FIG. 5: Updated predictions on $m_{\beta\beta}$ from oscillations as a function of the lightest neutrino mass in the two cases of NH and IH [67]. The shaded areas correspond to the $3\sigma$ regions due to error propagation of the uncertainties on the oscillation parameters. Our predictions of Fig. (4) lie in the allowed ranges of this plot.
case C (triplet flavon VEV (1,-1,1) with HO would be the true case. So, with the help of future precision measurements on MH and octant of $\theta_{23}$, our study can pinpoint the favoured VEV alignment of the triplet flavon. Also, from the plot in ($\delta_{CP} - \theta_{23}$) plane, it is seen that once the octant of $\theta_{23}$ is fixed, the value of the CPV phase can be fixed too. Hence, we conclude that the VEV of the triplet flavon field can be fixed once the parameter entanglement is resolved by experiments. This is a novel result of this work - i.e., the preferred direction along which flavon field vacuum stabilizes can be pinpointed once this entanglement among parameters is fixed, and vice-versa (if the preferred VEV of flavon field is known, it can help us resolve the parameter entanglement). Though parameter degeneracies and entanglement are not resolved completely, however, it would not be wrong to say that neutrino oscillation parameters are currently measured with very good precision. For the sake of completeness, we have summarised the results from a recent global analysis of neutrino data [19], in Table (V), as they are relevant in the context of our work. In this work, they have considered data from T2K, NO$\nu$A as LBLs, and have presented global analysis on $0\nu\beta\beta$ decay too. The results from nEXO [68] have also been included for the results of $0\nu\beta\beta$ decay in Table (V).

From a careful comparison of our results in Fig.(4) with the latest analysis on results of neutrinoless double beta decay experiments, we find that the the mass hierarchy and range of masses of lightest neutrino mass and the effective mass obtained for the allowed cases of triplet flavon VEVs from our model agree very strongly with that obtained from experiments such as GERDA [54], CUORE [55] and KamLAND-Zen [56] for $^{76}$Ge, for $^{130}$Te and for $^{136}$Xe respectively at 90% confidence level. To be precise, all our predicted values shown in Fig (4) lie in the allowed region of Fig. (5). Moreover, if we compare our results in Fig (1-3) with Table (V), we find that for the their preferences for NH and HO (higher octant) indicates that the VEV (1,-1,-1)/(-1,1,1) of the $A_4$ triplet flavon field is favoured (case B in Table (IV)) . Thus, we have been able to pinpoint the VEV alignment of $A_4$ symmetry. Next, we use this novel information to predict very precise values of $m_{\text{lightest}}$ and $m_{ee}$ in our model, which corresponds to upper right panel (NH, HO) in Fig. (4) - i.e., $m_{\text{lightest}} = (0.00243 - 0.00366) \text{ eV}$, and $m_{ee} = (8.24 \times 10^{-6} - 0.00094) \text{ eV}$, which falls in the allowed region of Fig. (5). Hence, our results of Fig (1-4) show coherence with reference to that of recent experimentally allowed regions, and so our model is testable with new
| Atmospheric angle (\(\theta_{23}\)) | CPV phase (\(\delta_{CP}\)) | Mass ordering MO (sign of \(\Delta m_{21}^2\)) | Results from 0\(\nu_\beta\beta\) Experiments |
|---------------------------------|--------------------------|------------------------------------------|---------------------------------------------|
| LBLs- two degenerate solutions for both the Octants (LO and HO) | NO\(\nu\)A - preference for \(\delta_{CP} = 0.8\pi\), disfavouring a region around best fit of T2K \(\delta_{CP} = 1.5\pi\) | Independent analysis of both T2K and NO\(\nu\)A does not show any preference for Mass ordering | \(m_{\beta\beta} \leq 104 - 228\) meV by GERDA |
| Combination of all Acc. LBL + Reactor | Combination of LBL+reactor-CP conserving value \(\delta_{CP} = 0\) is disfavoured, while other CP conserving values \(\delta_{CP} = \pi\) is still allowed | All LBL data favour IO -as a consequence of tension in T2K and NO\(\nu\)A data | \(m_{\beta\beta} \leq 75 - 350\) meV by CUORE |
| Combination of all Acc. LBL + atm data (SK) - shifts the best fit HO | Combination of all LBL + reactor data favour NO Atmospheric SK data favour NO, whole combination of LBL+SK favour NO | | \(m_{\beta\beta} \leq 61 - 165\) meV by KamLand Zen |
| Best fit - \(\sin^2\theta_{23} = 0.574(0.578)\) for NO(IO) | Best fit - \(\delta_{CP} = 1.08\pi(1.58\pi)\) for NO(IO) | Best fit is for NO . A small tension in IO | nEXO 1. Sensitivity at 90% CL \(m_{\beta\beta} \leq 4.7 - 26\) meV for NH. \(m_{\beta\beta} < 15\) meV for IH. |
| | | | 2. Discovery potential at (3\(\sigma\)): \(m_{\beta\beta} \leq 5.0 - 26\) meV for NH. \(m_{\beta\beta} < 15\) meV for IH. |

**TABLE V:** Summary of latest global fit values of neutrino parameters taken from [19], [68].

predictions, from this point of view too.

**VII. CONCLUSION**

To conclude, in this work, we presented new ideas about how the resolution of (MH-CPV-Octant) entanglement present in the measurements of LBL experiments can be used
to get novel information on the VEV alignment of triplet flavon field of \( A_4 \) flavour symmetry and neutrinoless double beta decay. We considered the inverse seesaw model with \( A_4 \) symmetry and some other discrete symmetries. Our model has predicted values of unknown parameters like \( m_{\text{lightest}} \), and CPV phases within their allowed 3\( \sigma \) ranges, up to the tolerance of \(< 10^{-5} \). Also the correlation between the lightest neutrino mass and the effective 0\( \nu \beta \beta \) mass obtained from our model agree very well with their current experimental 3\( \sigma \) bounds and sensitivities, and corresponding MH. We further note that if future experimental measurements can resolve the parameter entanglement, then the ideas presented in this work can help towards the model building of the low scale seesaw models along with the theoretical understanding of the flavour symmetries. By comparing our results with a very recent global analysis (which seem to favor NH and HO), we pinpointed the favoured VEV alignment of \( A_4 \) triple flavon to be \((1,-1,-1)/(-1,1,1)\), as well as very precise value of \( m_{\text{lightest}} \) and \( m_{ee} \), as shown in Fig. (4) and Table (IV). The current experimental challenges for three flavour neutrino mixing are - the neutrino mass hierarchy, absolute neutrino mass scale (or min of \( m_j \)), CP-violating phase in the lepton sector, parameter entanglement of neutrino oscillation parameters and flavour structure of particles - and results of our work can throw new light on almost all of them, with high precision of \(< 10^{-5} \). Thus, this study can provide important guidelines on all these important issues, which are still not resolved in neutrino physics (with flavour symmetry), and we hope that future experiments would be able to support or refute the ideas presented here.

**Appendix A: \( A_4 \) product rules**

In order to describe the quark and lepton masses, mixing and CP violation, a discrete non-abelian group of even permutations of four objects called \( A_4 \) flavour symmetry is considered. It is also the symmetry group of a tetrahedron. This group consist of four irreducible representations: three one-dimensional and one three dimensional whose conventional notations are \( 1, 1', 1'' \) and \( 3 \) respectively[71]. Their product rules are given as

\[
1 \otimes 1 = 1
\]

\[
1' \otimes 1' = 1''
\]

\[
1' \otimes 1'' = 1
\]
\[ 1'' \otimes 1'' = 1' \]

\[ 3 \otimes 3 = 1 \otimes 1' \otimes 1'' \otimes 3_a \otimes 3_s \]

where \( a \) and \( s \) in the subscript refers to anti-symmetric and symmetric parts respectively. If \((a_1, b_1, c_1)\) and \((a_2, b_2, c_2)\) are two triplets, then their direct product can be decomposed into the direct sum as follows

\[ 1 = a_1 a_2 + b_1 c_2 + c_1 b_2 \]
\[ 1' = c_1 c_2 + a_1 b_2 + b_1 a_2 \]
\[ 1'' = b_1 b_2 + c_1 a_2 + a_1 c_2 \]
\[ 3_s = (2a_1 a_2 - b_1 c_2 - c_1 b_2, 2c_1 c_2 - a_1 b_2 - b_1 a_2, 2b_1 b_2 - a_1 c_2 - c_1 a_2) \]
\[ 3_a = (b_1 c_2 - c_1 b_2, a_1 b_2 - b_1 a_2, c_1 a_2 - a_1 c_2) \]

Appendix B: Equations for \( A_4 \) Flavon VEVs in our model

\[
\Phi_a = \frac{F'}{3}(e^{2i\alpha}c_{13}^2m_{21}^2s_{12} - c_{13}c_{23}^2m_{32}s_{23}e^{2i(\beta + \delta)} + c_{13}^2c_{12}^2m_{1} + m_{31}s_{13}e^{2i(\beta + \delta) - 2i\delta} - m_1(s_{12}s_{23} - c_{12}c_{23}^ie^{i\delta}s_{13})(-c_{23}s_{12} - c_{12}e^{i\delta}s_{13}s_{23}))
\]

(B1)

\[
\Phi_b = \frac{F'}{3}(c_{13}c_{23}^2m_{31}s_{23}e^{2i(\beta + \delta)} - c_{13}c_{23}m_{3}s_{13}e^{2i(\beta + \delta) - i\delta} - e^{2i\alpha}c_{13}^2m_{21}s_{12}(-c_{12}s_{23} + c_{23}(-e^{i\delta}s_{12}s_{13}) - c_{12}c_{13}m_1(s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13}))
\]

(B2)

\[
\Phi_c = \frac{F'}{3}(c_{13}c_{23}^2m_{3}s_{13}s_{23}e^{2i(\beta + \delta) - i\delta} + e^{2i\alpha}m_2(-c_{12}s_{23} + c_{23}(-e^{i\delta}s_{12}s_{13})^2 + m_1(s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13})^2)
\]

(B3)

\[
\eta = \frac{F'}{3}(e^{2i\alpha}c_{13}^2m_{21}s_{12}^2 + c_{13}^2c_{12}^2m_{1} + m_{3}s_{13}s_{23}e^{2i(\beta + \delta) - 2i\delta} + 2(e^{2i\alpha}m_2(-c_{12}s_{23} + c_{23}(-e^{i\delta})s_{12}s_{13})(c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23}) + m_1(s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13})(-c_{23}s_{12} - c_{12}e^{i\delta}s_{13}s_{23}))
\]

(B4)
After the minimization of the triplet scalar potential $C_2$, the possible solutions that we get are:

\[
\tau = \frac{F'}{3}\left(c_{13}^2m_3s_{23}^2e^{2i(\beta+\delta)} + m_1(-c_{23}s_{12} - c_{12}e^{i\delta}s_{13}s_{23})^2 \right) \\
2(e^{2i\alpha}c_{13}m_2s_{12}(-c_{12}s_{23} + c_{23}(-e^{i\delta})s_{12}s_{13}) + c_{13}c_{23}m_3s_{13}e^{2i(\beta+\delta)-i\delta}
+c_{12}c_{13}m_1(s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13})) + e^{2i\alpha}m_2(c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23})^2 \right)
\]

\[
\xi = \frac{F'}{3}(c_{13}^2c_{23}m_3s_{23}e^{2i(\beta+\delta)} + e^{2i\alpha}m_2(-c_{12}s_{23} + c_{23}(-e^{i\delta})s_{12}s_{13})^2
2(e^{2i\alpha}c_{13}m_2s_{12}(c_{12}c_{23} - e^{i\delta}s_{12}s_{13}s_{23}) + c_{13}m_3s_{13}s_{23}e^{2i(\beta+\delta)-i\delta} \\
c_{12}c_{13}m_1(-c_{23}s_{12} - c_{12}e^{i\delta}s_{13}s_{23})) + m_1(s_{12}s_{23} - c_{12}c_{23}e^{i\delta}s_{13})^2 \right)
\]

where, $F' = \frac{Y_M^2}{Y_D^2\mu} \left[ \frac{\Lambda^4}{v_h^2v_\rho v_\rho'} \right]$. Here $v_s$ is absorbed in $\Phi_a, \Phi_b, \Phi_c$.

Appendix C: Minimisation of potential in our model and solutions of VEV alignments

The scalar potential for the most general renormalizable case containing all the flavon fields which are invariant under $A_4 \times Z_4 \times Z_5$ that also respects $U(1)_X$ global symmetry can be written as:

\[
V = V(H) + V(\phi_T) + V(\phi_S) + V(\eta) + V(\xi) + V(\tau) + V(\rho) + V(\rho') + V(\rho'')
+ V(H, \phi_T, \phi_S, \eta, \xi, \tau, \rho, \rho', \rho'') + V(\phi_T, \phi_S, \eta, \xi, \tau, \rho, \rho', \rho'') + V_{ex}(H, \phi_T, \phi_S, \eta, \xi, \tau, \rho, \rho', \rho'')
\]

where,

\[
V(H) = \mu_H^2H^\dagger H + \lambda_H(H^\dagger H)(H^\dagger H)
\]

\[
V(\phi_s) = -\mu_s^2[\phi_a^\dagger \phi_a + \phi_b^\dagger \phi_b + \phi_c^\dagger \phi_c] + \lambda_s[\phi_a^\dagger \phi_a + \phi_b^\dagger \phi_b + \phi_c^\dagger \phi_c]^2
+ (\phi_a^\dagger \phi_b + \phi_a^\dagger \phi_c + \phi_b^\dagger \phi_a)(\phi_c^\dagger \phi_c
+ \phi_a^\dagger \phi_b + \phi_b^\dagger \phi_a) + (2\phi_a^\dagger \phi_a - \phi_b^\dagger \phi_c + \phi_c^\dagger \phi_b)^2 + 2(2\phi_c^\dagger \phi_c - \phi_b^\dagger \phi_b - \phi_b^\dagger \phi_b)(2\phi_b^\dagger \phi_b - \phi_a^\dagger \phi_c - \phi_c^\dagger \phi_a)
\]

The potential $V_{ex}(H, \phi_T, \phi_S, \eta, \xi, \tau, \rho, \rho', \rho'')$ is responsible for the breaking of $U(1)_X$ global symmetry explicitly. This scalar potential includes several free parameters that naturally allows the required VEV alignments of the flavons $\langle \phi_s \rangle = v_s(\phi_a, \phi_b, \phi_c), \langle \phi_T \rangle = v_T(1, 0, 0), \langle H \rangle = v_h, \langle \eta \rangle = v_\eta, \langle \rho' \rangle = v_\rho', \langle \rho'' \rangle = v_\rho'', \langle \xi \rangle = v_\xi, \langle \tau \rangle = v_\tau$. After the minimization of the triplet scalar potential $C_2$, the possible solutions that we get are:
1. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, 0, 0)\)
2. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, 1, -1) \ast \left(-\frac{0.242536\mu_s}{\sqrt{\lambda_s}}\right)\)
3. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, -1, 1) \ast \left(-\frac{0.242536\mu_s}{\sqrt{\lambda_s}}\right)\)
4. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, \frac{(0.210042 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.210042 - 0.121268i)\mu_s}{\sqrt{\lambda_s}})\)
5. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, \frac{(0.210042 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.210042 - 0.121268i)\mu_s}{\sqrt{\lambda_s}})\)
6. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, \frac{(0.210042 - 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.210042 + 0.121268i)\mu_s}{\sqrt{\lambda_s}})\)
7. \((\phi_a, \phi_b, \phi_c) \rightarrow (0, \frac{(0.210042 - 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.210042 + 0.121268i)\mu_s}{\sqrt{\lambda_s}})\)
8. \((\phi_a, \phi_b, \phi_c) \rightarrow (1, 1, 1) \ast \left(-\frac{0.288675\mu_s}{\sqrt{\lambda_s}}\right)\)
9. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(-\frac{0.288675\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 - 0.25i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 - 0.25i)\mu_s}{\sqrt{\lambda_s}}\right)\)
10. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(-\frac{0.288675\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 - 0.25i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 - 0.25i)\mu_s}{\sqrt{\lambda_s}}\right)\)
11. \((\phi_a, \phi_b, \phi_c) \rightarrow (1, 1, 1) \ast \left(-\frac{0.288675\mu_s}{\sqrt{\lambda_s}}\right)\)
12. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(\frac{0.288675\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 + 0.25i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 - 0.25i)\mu_s}{\sqrt{\lambda_s}}\right)\)
13. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(\frac{0.288675\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 + 0.25i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.144338 - 0.25i)\mu_s}{\sqrt{\lambda_s}}\right)\)
14. \((\phi_a, \phi_b, \phi_c) \rightarrow (1, 0, 0) \ast \left(-\frac{0.316228\mu_s}{\sqrt{\lambda_s}}\right)\)
15. \((\phi_a, \phi_b, \phi_c) \rightarrow (1, 2, 2) \ast \left(-\frac{0.105409\mu_s}{\sqrt{\lambda_s}}\right)\)
16. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(-\frac{0.105409\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 - 0.182574i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 + 0.182574i)\mu_s}{\sqrt{\lambda_s}}\right)\)
17. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(-\frac{0.105409\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 + 0.182574i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 - 0.182574i)\mu_s}{\sqrt{\lambda_s}}\right)\)
18. \((\phi_a, \phi_b, \phi_c) \rightarrow (1, -2, -2) \ast \left(-\frac{0.105409\mu_s}{\sqrt{\lambda_s}}\right)\)
19. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(\frac{0.105409\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 - 0.182574i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 + 0.182574i)\mu_s}{\sqrt{\lambda_s}}\right)\)
20. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(\frac{0.105409\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 + 0.182574i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.105409 - 0.182574i)\mu_s}{\sqrt{\lambda_s}}\right)\)
21. \((\phi_a, \phi_b, \phi_c) \rightarrow (1, 0, 0) \ast \left(-\frac{0.316228\mu_s}{\sqrt{\lambda_s}}\right)\)
22. \((\phi_a, \phi_b, \phi_c) \rightarrow (2, -1, -1) \ast \left(-\frac{0.140028\mu_s}{\sqrt{\lambda_s}}\right)\)
23. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(-\frac{0.280056\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 - 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}\right)\)
24. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(-\frac{0.280056\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 - 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}\right)\)
25. \((\phi_a, \phi_b, \phi_c) \rightarrow (2, -1, -1) \ast \left(-\frac{0.140028\mu_s}{\sqrt{\lambda_s}}\right)\)
26. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(\frac{0.280056\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 - 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}\right)\)
27. \((\phi_a, \phi_b, \phi_c) \rightarrow \left(\frac{0.280056\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}, \frac{(0.070014 + 0.121268i)\mu_s}{\sqrt{\lambda_s}}\right).
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