Nonstationary torsion problems for the finite cylinder partly coupled with rigid base

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Abstract. The axisymmetric dynamic problems of determining the stress state in the vicinity of the delamination in a finite cylinder partly coupled with the rigid base were solved. In contrast to the traditional solution methods based on the use of the integral Laplace transform, the proposed method consists in the difference approximation only of the time derivative. As a result, the original problem is replaced by a sequence of homogeneous boundary value problems for the Helmholtz equation that reduce to the Fredholm integral equation of the second kind. The numerical solution found made it possible to obtain an approximate formula for calculating the stress intensity factor.

1. Introduction

A large number of machine elements and structures are cylindrical. The stress state of finite and infinite cylindrical bodies under static loading has been studied sufficiently enough [1–4], but works with the analysis of the stress state under dynamic loading conditions is much smaller. As a rule, the researchers study cylinders of infinite length [5–7] or finite cylinders under harmonic loading [8–10].

The complexity of theoretical studies of dynamic problems is due to the necessity of using the Laplace integral transform in time with subsequent numerical inversion. However, this task is not only mathematically complicated but incorrect. Mixed numerical-experimental methods allow avoiding these difficulties [11, 12]. But these methods are characterized by the disadvantages associated with the need to carry out experiments for each particular sample. This complicates the study of the impact of cylinder and defects geometric dimensions on stress intensity factor (SIF) values.

In this paper, we used a modified method of finite difference in time [13]. With the help of this method, this paper solves the problem of determining SIF in the vicinity of the delamination in a finite cylinder under torsional loading. So far, such a problem has been considered only in a stationary formulation [14], for a harmonic moment [8–10] or cylinders with cracks [15, 16].

2. Problems formulations

An isotropic finite elastic cylinder with height \(a\) and radius \(r_0\) is considered. The cylinder is related to the cylindrical system of coordinates, whose center coincides with the center of the lower base, and the \(Oz\) axis with the cylinder axis.

The top end of the cylinder is joined with a rigid plate with thickness \(d\) and radius \(r_0\). The plate suffers the action of a torsional moment \(M(t)\). The bottom end of the cylinder \(z = 0\) is
Figure 1. Cylinder with delamination: a) — circular, b) — ring-shaped

coupled with a rigid base. In the coupling area there is partial delamination in the form of a
circle (figure 1a) or a ring (figure 1b). Both the cylinder side surface and delamination area are
considered to be free of stresses.

Under these conditions, the cylinder is in a state of axisymmetric torsional deformation
and only the angular displacement \( \tilde{w}(r, z, t) \) will be different from 0. Next, to formulate the
initial boundary-value problem, it is expedient to pass on to dimensionless quantities using the
formulas:

\[
\tilde{w}(r, z, t) = \frac{r}{r_0} w(\eta, \zeta, \tau), \quad \eta = \frac{a}{r_0}, \quad l = \frac{c}{a}, \quad \beta = \frac{b}{r_0},
\]

\[
r = r_0 \eta, \quad 0 \leq \eta \leq 1, \quad z = a \zeta, \quad 0 \leq \zeta \leq 1, \quad t = \frac{r_0 \tau}{c_2}, \quad \tau \in (0, +\infty), \quad c_2^2 = \frac{G}{\rho},
\]

where \( \rho, G \) is the density and shear moduli for the cylinder material.

Then, dimensionless displacement will satisfy the equation:

\[
D_{\eta\zeta} w = \frac{\partial^2 w}{\partial \tau^2}, \quad D_{\eta\zeta} = \frac{\partial^2}{\partial \eta^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} - \frac{1}{\eta^2} + \frac{1}{\gamma^2} \frac{\partial^2}{\partial \zeta^2},
\]  

(1)

Equation (1) is considered as having zero initial conditions.

On the lateral surface of the cylinder, there must be fulfilled the equality:

\[
\tau_{\eta\zeta}(1, \zeta, \tau) = 0, \quad 0 \leq \zeta \leq 1, \quad \tau \in [0, +\infty).
\]  

(2)

Dynamic loading of the top end of the cylinder is carried out through a rigid plate joined to
it, which is under the action of torque. In this case, the equality holds:

\[
w(\eta, 1, \tau) = \eta \alpha(\tau), \quad 0 \leq \eta \leq 1, \quad \tau \in [0, +\infty),
\]  

(3)

where \( \alpha(\tau) \) is unknown angle of rotation of the plate, determined from the equation of its motion

\[
\frac{\pi \gamma m_0}{2} \ddot{\alpha}(\tau) = M_0(\tau) - M_R(\tau), \quad \alpha(0) = 0, \quad \dot{\alpha}(0) = 0,
\]
where \( m_0 \) is ratio of the plate mass to cylinder mass, \( M_0 \) and \( M_R \) are dimensionless moments.

The boundary conditions at the bottom end of the cylinder are formulated depending on the form of delamination. With circular delamination (figure 1a), the following conditions are fulfilled:

\[
\tau_{\varphi \zeta}(\eta, 0, \tau) = 0, \quad 0 \leq \eta \leq \beta, \quad w(\eta, 0, \tau) = 0, \quad \beta \leq \eta \leq 1, \quad \tau \in [0, +\infty). \tag{4}
\]

If there is external ring-shaped delamination (figure 1b), the boundary conditions have the form:

\[
w(\eta, 0, \tau) = 0, \quad 0 \leq \eta \leq \beta, \quad \tau_{\varphi \zeta}(\eta, 0, \tau) = 0, \quad \beta \leq \eta \leq 1, \quad \tau \in [0, +\infty). \tag{5}
\]

To avoid difficulties arising in problems with mixed boundary conditions, instead of conditions (4), (5) we consider the following boundary conditions:

for circular delamination:

\[
w(\eta, 0, \tau) = \begin{cases} \chi_1(\eta, \tau), & 0 \leq \eta \leq \beta, \\ 0, & \beta < \eta \leq 1, \end{cases} \quad \tau \in [0, +\infty) \tag{6}
\]

and for ring-shaped delamination

\[
\tau_{\varphi \zeta}(\eta, 0, \tau) = \begin{cases} \chi_2(\eta, \tau), & 0 \leq \eta \leq \beta, \\ 0, & \beta < \eta \leq 1, \end{cases} \quad \tau \in [0, +\infty), \tag{7}
\]

where \( \chi_1(\eta, \tau) \) and \( \chi_2(\eta, \tau) \) are unknown functions, ensuring the fulfillment of the first equalities from (4) and (5).

To solve the formulated initial-boundary problems (1)–(3), (6) and (1)–(3), (7) we apply a method based on the difference approximation of time derivatives, detailed in [13]. For this purpose, we create a time grid:

\[
\tau_k = \sum_{\nu=1}^{k} h_{\nu}, \quad h_{\nu} = \tau_k - \tau_{k-1} \quad (\tau_0 = 0), \quad k = 1, 2, 3, \ldots, \quad h_i \neq h_j, \quad i \neq j.
\]

We introduce to the designation \( w(\eta, \zeta, \tau_k) = w_k(\eta, \zeta) \) and use the left difference for approximation time derivatives.

Then, from initial conditions and equation (1), we find the following differential equations

\[
w_0 = 0, \quad D_{\eta \zeta}w_1 = \frac{w_1}{h_1^2}, \quad D_{\eta \zeta}w_k - \frac{w_k}{h_k^2} = \frac{k_2}{k_k h_{k-1}} - \frac{k_{k-1}}{h_k} \left( \frac{1}{h_k} + \frac{1}{h_{k-1}} \right), \quad k = 2, 3, \ldots
\]

Next, according to [13], we write angular displacement, angle of rotation of the plate and stress in the form of a linear combination of new unknown functions:

\[
w_k = \sum_{\nu=1}^{k} C_{k\nu} U_{\nu}, \quad \alpha_k = \sum_{\nu=1}^{k} C_{k\nu} A_{\nu}, \quad \tau_{\varphi \zeta k} = \sum_{\nu=1}^{k} C_{k\nu} \tau_{\varphi \zeta \nu}, \quad \tau_{\varphi \zeta k} = \sum_{\nu=1}^{k} C_{k\nu} \tau_{\varphi \zeta \nu}. \tag{8}
\]

It was shown in [13], that if we choose the coefficients in formulas (8) according to the formulas

\[
C_{kk} = 1, \quad k = 1, 2, 3, \ldots, \quad C_{k,k-1} = \frac{h_{k-1}}{h_{k-1} - h_k}, \quad k = 2, 3, \ldots,
\]

\[
C_{k,\nu} = \frac{h_{\nu}^2}{h_k^2 - h_{\nu}^2} \left[ \frac{h_k}{h_{k-1}} C_{k-2,\nu} - \left( 1 + \frac{h_k}{h_{k-1}} \right) C_{k-1,\nu} \right], \quad k = 3, 4, \ldots; \quad \nu = 1, 2, \ldots, k - 2,
\]

\[\sum_{\nu=1}^{k} C_{k\nu} \tau_{\varphi \zeta \nu} = 0, \quad k = 1, 2, 3, \ldots,
\]
then the functions $U_\nu$ satisfy the homogeneous Helmholtz equations:

$$D\eta\kappa U_\nu - \kappa^2 U_\nu = 0, \quad \nu = 1, 2, \ldots, \quad \kappa_\nu = h_\nu^{-1}. \quad (9)$$

The boundary conditions on the lateral surfaces and top end of the cylinder with respect to these functions can be written as follows:

$$\tau_{\varphi\eta}(1, \zeta) = 0, \quad 0 < \zeta < 1, \quad U_\nu(\eta, 1) = \eta A_\nu, \quad 0 < \eta < 1. \quad (10)$$

$A_\nu$ is determined from the equation:

$$\pi \gamma m_0 \kappa^2 A_\nu = 2(\mu_0 - \mu_R\nu), \quad \mu_R\nu(\tau) = 2\pi \int_0^1 \eta^2 \tau_{\varphi\zeta}(\eta, 1) \, d\eta, \quad (11)$$

where $\mu_0\nu$ can be found from the recurrence relation.

And the boundary conditions on the bottom end of the cylinder with circular delamination will take the form:

$$U_\nu(\eta, 0) = \begin{cases} \chi_{1\nu}(\eta), & 0 \leq \eta \leq \beta, \\ 0, & \beta < \eta \leq 1, \end{cases} \quad \chi_1(\eta, \tau_k) = \sum_{\nu=1}^{k} C_{k\nu} \chi_{1\nu}(\eta). \quad (12)$$

In the case of ring-shaped delamination:

$$\tau_{\varphi\zeta}(\eta, 0) = \begin{cases} \chi_{2\nu}(\eta), & 0 \leq \eta \leq \beta, \\ 0, & \beta < \eta \leq 1, \end{cases} \quad \chi_2(\eta, \tau_k) = \sum_{\nu=1}^{k} C_{k\nu} \chi_{2\nu}(\eta). \quad (13)$$

3. Solution to the problem for the cylinder with circular delamination

We represent the solution to the boundary-value problem (9), (10), (12) as the sum:

$$U_\nu(\eta, \zeta) = U^0_\nu(\eta, \zeta) + U^*_\nu(\eta, \zeta). \quad (14)$$

The first term is the solution to the problem in the absence of crack and it is given by the formula:

$$U^0_\nu(\eta, \zeta) = A_\nu \eta \frac{\sinh(\gamma \zeta \kappa_\nu)}{\sinh(\gamma \kappa_\nu)}. \quad \text{(15)}$$

The second term is the solution that satisfies the zero conditions on the top end and lateral surface of the cylinder. The condition on the bottom end has the form (12), and on the delamination, equality should be satisfied:

$$\tau_{\varphi\zeta^0}(\eta, 0) = -\tau_{\varphi\zeta^*(\eta, 0), \quad 0 < \eta < \beta. \quad (16)$$

The solution to this boundary value problem is constructed by the integral transform method, analogous to papers [8, 15] and it has the form:

$$U^*_\nu(\eta, \zeta) = \int_{0}^{\beta} \xi \chi_{1\nu}(\xi) F^-(\xi, \eta, \zeta) \, d\xi, \quad \text{(17)}$$

where

$$F^-(\xi, \eta, \zeta) = -2 \gamma \sum_{j=1}^{\infty} \frac{\lambda_j^\gamma}{\gamma} B_{j\nu}^\gamma \sin(\lambda_j^\gamma \zeta),$$

$$B_{j\nu}^\gamma = g_{j\nu}(\xi, \eta) + \frac{K_2(q_{j\nu})}{I_2(q_{j\nu})} I_1(q_{j\nu} \xi) I_1(q_{j\nu} \eta), \quad q_{j\nu} = \sqrt{(\lambda_j^\gamma)^{2\gamma-2} + \kappa_\nu^2}, \quad \lambda_j^\gamma = \pi j,$$

$$g_{j\nu}(\xi, \eta) = -\int_{0}^{\infty} \frac{u}{u^2 + (q_{j\nu}^\gamma)^2} J_1(\eta u) J_1(\xi u) \, du = \begin{cases} I_1(q_{j\nu} \eta) K_1(q_{j\nu} \xi), & 0 \leq \xi < \eta, \\ I_1(q_{j\nu} \xi) K_1(q_{j\nu} \eta), & \eta < \xi \leq 1. \end{cases}$$
The solution (16) contains the unknown function $\chi_{1\nu}(\xi)$. To determine it, we use the condition on the delamination (15) and obtain the integral equation

$$\int_0^\beta \xi \chi_{1\nu}(\xi) F_0^{-}(\xi, \eta) \, d\xi = -\gamma \kappa_{\nu} A_{\nu} \frac{\eta}{\sinh(\gamma \kappa_{\nu})},$$

where $F_0^{-}(\xi, \eta) = \partial F^{-}(\xi, \eta, \zeta) / \partial \zeta |_{\zeta = 0}$.

To solve this equation, we reduce it to the Fredholm equation of the second kind according to the known method \[8, 15\].

To do this, we integrate equation (17) by parts and introduce the new unknown function

$$\psi_{\nu}(\xi) = \frac{1}{\xi} \frac{d}{d\xi} [\xi \chi_{1\nu}(\xi)], \quad \psi_{\nu}(\xi) = -\frac{2}{\pi} \frac{d}{d\xi} \int_0^\beta \frac{\tau \varphi_{\nu}(\tau)}{\sqrt{\tau^2 - \xi^2}} \, d\tau.$$

And then we apply the operator

$$D_2[f] = \frac{d}{dx} \int_0^x y \, dy \int_0^y f(\eta) \, d\eta.$$

Due to these transformations, the introduction of designations

$$\lambda = u \kappa_{\nu}, \quad \sqrt{u^2 + 1} = p, \quad \tau = \beta y, \quad \varphi_{\nu}(\tau) = \varphi_{\nu}(\beta y) = \beta g_{\nu}(y), \quad y \in [0, 1],$$

and even extension of the function $g_{\nu}(y)$ on the interval $[-1, 1]$, equation (17) is reduced to the Fredholm integral equation of the second kind:

$$g_{\nu}(s) - \frac{2\beta}{\pi \gamma} \int_{-1}^1 g_{\nu}(y) [B(y, s) + Q(y) - Q(y - s)] \, dy - g_{\nu}(0) = -\frac{2 \beta \kappa_{\nu} A_{\nu}}{\sinh(\gamma \kappa_{\nu})} s^2,$$

where $B(Y)$ and $Q(Y)$ are represented as uniformly convergent series and proper integrals.

Equation (19) contains unknown $A_{\nu}$, to determine it, one should use equation (11).

An approximate solution to equation (19), as in \[8, 9, 15, 16\], is sought in the form of an interpolation polynomial.

We approximate its integrals according to the quadrature Gauss-Legendre formula and obtain a system of linear algebraic equations with respect to the values of the unknown function in the interpolation nodes. After solving the system, the unknown function is approximated by the interpolation polynomial.

$$g_{\nu}(y) \approx g_{\nu}^I(y) = \sum_{m=1}^n g_{\nu m} \frac{P_n(y)}{(y - y_m) P'_n(y_m)}, \quad g_{\nu m} = g_{\nu}(y_m), \quad m = 1, 2, 3, \ldots, n,$$

where $P_n(y)$ is the $n$th Legendre polynomial, and $y_m$ is the polynomial root.

4. Solution to the problem for the cylinder with ring-shaped delamination

The solution to the boundary value problem (9), (10), (13) we also represent in the form (14) where the second term also satisfies zero conditions on the lateral surface and the top end of the cylinder. The condition on the bottom end has the form (13), and on the delamination, the equality should be satisfied:

$$\tau_{\nu}^s(\eta, 0) = \chi_{2\nu}(\eta) - \tau_{\nu}^0(\eta, 0), \quad \beta < \eta < 1.$$
The solution to this boundary value problem is also constructed by the integral transform method, analogous to papers [9, 16] and it has the form:

\[ U^\ast_\nu(\eta, \zeta) = \int_0^\beta \xi \chi_\nu(\xi) F^+(\xi, \eta, \zeta) d\xi + A_\nu \eta \frac{\sinh[\gamma \kappa_\nu (1 - \zeta)]}{\cosh(\gamma \kappa_\nu) \sinh(\gamma \kappa_\nu)}, \]

where

\[
F^+(\xi, \eta, \zeta) = \frac{2}{\gamma} \sum_{j=1}^{\infty} B^+_j(\eta, \zeta) \cos(\lambda_j^+ \xi), \quad q_j^+ = \sqrt{(\lambda_j^+)^2 - 2 + \kappa_\nu^2}, \quad \lambda_j^+ = \frac{\pi(2j - 1)}{2},
\]

\[
B^+_j(\eta, \zeta) = g^+_j(\eta, \zeta) + \frac{K_2(q_j^+)}{I_2(q_j^+)} I_1(q_j^+ \xi) I_1(q_j^+ \eta),
\]

\[
g^+_j(\eta, \zeta) = -\int_0^\infty \frac{u}{u^2 + (q_j^+)^2} J_1(\eta u) J_1(\xi u) du = \begin{cases} I_1(q_j^+ \eta) K_1(q_j^+ \xi), & 0 \leq \xi < \eta, \\ I_1(q_j^+ \xi) K_1(q_j^+ \eta), & \eta < \xi \leq 1. \end{cases}
\]

This solution contains the unknown function \( \chi_{2\nu}(\xi) \). To determine it, we use the condition:

\[ U^\ast_\nu(\eta, 0) = 0, \quad \eta \in [0, \beta]. \]

From this equality we obtain the integral equation for an unknown function:

\[ \int_0^\beta \xi \chi_{2\nu}(\xi) F^+_0(\xi, \eta) d\xi = \frac{A_\nu \eta}{\cosh(\gamma \kappa_\nu)}, \quad (20) \]

where \( F^+_0(\xi, \eta, \zeta) = F^+(\xi, \eta, 0) \).

Next, we reduce equation (20) to the Fredholm equation of the second kind according to the known method [9, 16], which consists in introducing the new unknown function

\[
\varphi_\nu(\tau) = \int^\beta\!_\tau \frac{\tau \chi_\nu(\xi)}{\sqrt{\xi^2 - \tau^2}} d\xi, \quad \chi_\nu(\xi) = -\frac{2}{\pi} \frac{d}{d\xi} \int^\beta\!_0 \frac{\varphi_\nu(\tau)}{\sqrt{\tau^2 - \xi^2}} d\tau, \quad \varphi(\tau) \equiv 0, \quad \tau > \beta
\]

and applying the operator

\[
D_1[f] = \frac{d^2}{dx^2} \int_0^x \frac{y dy}{\sqrt{x^2 - y^2}} \int_0^y f(\eta) d\eta.
\]

Due to these transformations, the introduction of designations (18) end odd extension of the function \( g_\nu(y) \) on the interval \([-1, 1]\), equation (20) is reduced to the Fredholm integral equation of the second kind:

\[ g_\nu(s) + \frac{1}{2\pi} \int_{-1}^1 g_\nu(y) [A(y - s) + Q(y - s)] dy = \frac{A_\nu s}{\cosh(\gamma \kappa_\nu)}, \quad (21) \]

where \( A(Y) \) and \( Q(Y) \) are represented as uniformly convergent series and proper integrals.

The method for obtaining an approximate solution to equation (21) is similar to that for equation (19).
Figure 2. Time dependence of dimensionless SIF for $M_0(\tau) = H(\tau)$: a) cylinder with circular delamination, b) cylinder with ring-shaped delamination

5. Results of numerical analysis
For the criteria of destruction, an important role is played by SIF, which is determined by the formula:

$$K(t_k) = \lim_{r \to b+0} \sqrt{r-b\tau_{\varphi k}(r,0)}, \quad K_k = \lim_{r_{\varphi k} \to b-0} \sqrt{b-r_{\varphi k} \tau_{\varphi k}(r,0)}.$$ 

The dimensionless SIF value can be obtained from the formulas:

$$K(\tau_k) = \frac{K(t_k)}{G \sqrt{b}}, \quad K(\tau_k) = \sum_{\nu=1}^{k} C_{\nu} K_{\nu}, \quad K_{\nu} = -\frac{1}{\sqrt{2\pi \gamma}} g_{\nu}(1), \quad K_{\nu} = \frac{\sqrt{2}}{\pi} g_{\nu}(1). \quad (22)$$

By formulas (22), there was performed a numerical study of the dependence of SIF on the dimensionless time $\tau = c_2 t / r_0$. The time grid nodes were condensed near the point $\tau = 0$. During these calculations, it was considered that dimensionless thickness of the plate is $\delta = d/a = 0.1$, the dimensionless radius of the circular delamination and the dimensionless internal radius of the ring-shaped delamination are $\beta = b/r_0 = 0.5$.

The influence of the type of load and the mass of the plate on the time dependence of the SIF was studied. The results of the calculations are shown in figures 2 and 3 in the form of graphs of time dependencies of dimensionless SIFs. During these calculations, it was considered that dimensionless cylinder height is $\gamma = a/r_0 = 2$. The charts have been constructed for the case of the action of a suddenly applied torsional load (figure 2), and the case of specifying the torsional load by a suddenly applied moment of the unit length (figure 3). The curves 1–3 correspond to different values of the relative plate density: $\bar{\rho} = \rho_{\text{plate}}/\rho_{\text{cyl}}$: 0.25, 1, 4.

The influence of the geometric parameters of the cylinder and delamination on the time dependence of the SIF was also studied. Calculations were made for the case of a suddenly applied torsional load at the relative plate density $\bar{\rho} = \rho_{\text{plate}}/\rho_{\text{cyl}} = 1$. In figure 4, the curves 1–4 are constructed for the values of the relative cylinder height $\gamma = a/r_0$: 0.5, 1, 2, 4 with the relative radius of delamination $\beta = b/r_0 = 0.5$. And in figure 5, the curves 1–3 are constructed for the values of the relative radius of the delamination $\beta = b/r_0$: 0.5, 0.75, 0.95 with the relative height of the cylinder $\gamma = a/r_0 = 2$.

From the graphs in figure 2 it can be seen that in all considered types of loading, during the transient process, the maximum SIF values are observed. When a sudden constant load is applied, this maximum is 2–2.5 times higher than the static value of SIF. Hence, it is most likely that the destruction of the cylinder will occur during the transient period.

Also the analysis of the graphs in figures 2 and 3, it can be concluded that an increase in the mass of the plate leads to an increase in the time until the SIF reaches its maximum value.
**Figure 3.** Time dependence of dimensionless SIF for $M_0(\tau) = H(\tau) - H(\tau - 1)$: a) cylinder with circular delamination, b) cylinder with ring-shaped delamination

**Figure 4.** Influence of the relative height of the cylinder on dimensionless SIF value: a) cylinder with circular delamination, b) cylinder with ring-shaped delamination

**Figure 5.** Influence of the relative delamination radius on dimensionless SIF value: a) cylinder with circular delamination, b) cylinder with ring-shaped delamination
However, in the case of a suddenly applied torsional load, it practically does not affect the maximum value itself (figure 2). In the case of a suddenly applied moment of unit length (figure 3), a decrease in the maximum SIF value is observed. It can be explained by the fact that during the action of the torque, SIF does not have time to reach its maximum.

In figure 4 it can be seen that an increase in the height of the cylinder leads to a decrease in the maximum SIF value in the case of circular delamination and vice versa in the case of ring-shaped delamination.

The graphs in figure 5 show that an increase in the area of the delamination leads to an increase in the maximum of SIF.

**Conclusion**

The article proposes a method for solving the problem of determining the stress-strain state of an elastic finite cylindrical body partly coupled with the rigid base that is under torsional loading. This technique is based on the differential approximation of the time derivative and use of a time grid with specially selected nodes. Numerical results demonstrate the effectiveness of such an approach when investigating the transient processes that occur immediately after load application.

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