A NOTE ON A DEGREE SUM CONDITION FOR LONG CYCLES IN GRAPHS

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Abstract. We conjecture that a 2-connected graph $G$ of order $n$, in which $d(x) + d(y) \geq n - k$ for every pair of non-adjacent vertices $x$ and $y$, contains a cycle of length $n - k$ ($k < n/2$), unless $G$ is bipartite and $n - k$ is odd. This generalizes to long cycles a well-known degree sum condition for hamiltonicity of Ore. The conjecture is shown to hold for $k = 1$.

1. Introduction

The subject of this note is the following conjecture, in which we generalize to long cycles a well-known degree sum condition for hamiltonicity of Ore [4]. All graphs considered are finite, undirected, with no loops or multiple edges.

Conjecture 1.1. Let $G$ be a 2-connected graph of order $n \geq 3$, $n \neq 5, 7$, and let $k < n/2$ be an integer. If $d(x) + d(y) \geq n - k$ for every pair of non-adjacent vertices $x$ and $y$, then $G$ contains a cycle of length $n - k$, unless $G$ is bipartite and $n - k \equiv 1 \pmod{2}$.

Remark 1.2. The conjecture is sharp. First of all, a quick look at $C_5$ and $C_7$ ensures that the assumption $|G| \neq 5, 7$ is necessary. Secondly, it is easy to see that without the 2-connectedness assumption, there could be no long cycles at all. Consider, for instance, a graph $G$ obtained from disjoint cliques $H_1 = K_{\lfloor n/2 \rfloor}$ and $H_2 = K_{\lceil n/2 \rceil}$ by joining a single vertex $x_0$ of $H_2$ with every vertex of $H_1$. Finally, the bound for the degree sum of non-adjacent vertices is best possible, as shown in the example below.

Example 1.3. Let $G$ be a graph obtained from the complete bipartite graph $K_{(n-k-1)/2,(n+k+1)/2}$ by joining all the vertices in the smaller colour class. Then $d(x) + d(y) \geq n - k - 1$ for every pair of non-adjacent vertices $x$ and $y$, and $G$ contains no cycle of length greater than $n - k - 1$.

Our main result is the following theorem that implies Conjecture 1.1 for $k = 1$, as shown in Section 2. The proof of Theorem 1.4 is given in the last section.

Theorem 1.4. Let $G$ be a 2-connected graph of order $n \geq 3$, in which $d(x) + d(y) \geq n - 1$ for every pair of non-adjacent vertices $x$ and $y$.

Key words and phrases. Hamilton cycle, long cycle, degree sum condition, Ore-type condition.
(i) If \( n \) is even, then \( G \) is hamiltonian.
(ii) If \( n \) is odd, then \( G \) contains a cycle of length at least \( n - 1 \).
Moreover, \( G \) is not hamiltonian only if the minimal degree of its \( n \)-closure, \( \text{Cl}_n(G) \), equals \( (n - 1)/2 \). In this case, \( \text{Cl}_n(G) \) is a maximal non-hamiltonian graph.

Recall that the \( n \)-closure \( \text{Cl}_n(G) \) of \( G \) is a graph obtained from \( G \) by successively joining all pairs \((x, y)\) of non-adjacent vertices satisfying \( d(x) + d(y) \geq n \).

2. Long cycles in graphs

Proposition 2.1. Conjecture 1.1 holds for \( k = 1 \).

For the proof, we will need the following result of [3]:

Theorem 2.2 (Haggkvist-Faudree-Schelp). Let \( G \) be a hamiltonian graph on \( n \) vertices. If \( G \) contains more than \( \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 1 \) edges, then \( G \) is pancyclic or bipartite.

Proof of Proposition 2.1. By Theorem 1.4, we may assume that \( G \) is hamiltonian. Suppose first that \( G \) is a 2-connected non-bipartite hamiltonian graph of order \( n \), in which \( d(x) + d(y) \geq n - 1 \) whenever \( xy \notin E(G) \).

Consider a vertex \( x \) of minimal degree \( d(x) = \delta(G) \) in \( G \). Write \( \delta = \delta(G) \). Then \( G \) has precisely \( n - 1 - \delta \) vertices non-adjacent to \( x \), each of degree at least \( n - 1 - \delta \).
The remaining \( \delta + 1 \) vertices are of degree at least \( \delta \) each, hence

\[
\|G\| \geq \frac{1}{2}[(\delta + 1)\delta + (n - 1 - \delta)^2].
\]

As \( \delta \geq 2 \), one immediately verifies that

\[
\frac{1}{2}[(\delta + 1)\delta + (n - 1 - \delta)^2] > \frac{(n - 1)^2}{4} + 1,
\]
whenever \( n \neq 5 \).

It remains to consider the case of \( G \) a bipartite 2-connected hamiltonian graph of order \( n \). But then \( n \) must be even, for otherwise \( G \) would contain an odd cycle.
Thus \( n - 1 \equiv 1 \pmod{2} \), which completes the proof. \( \square \)

For convenience, let us finally recall two well-known results, that we shall need in the proof of Theorem 1.4:

Theorem 2.3 (Dirac [2]). Let \( G \) be a graph of order \( n \geq 3 \) and minimal degree \( \delta(G) \geq n/2 \). Then \( G \) is hamiltonian.

Theorem 2.4 (Bondy-Chvatal [1]). Let \( G \) be a graph of order \( n \) and suppose that there is a pair of non-adjacent vertices \( x \) and \( y \) of \( G \) such that \( d(x) + d(y) \geq n \). Then \( G \) is hamiltonian if and only if \( G + xy \) is hamiltonian.

Corollary 2.5. A graph \( G \) is hamiltonian if and only if its \( n \)-closure \( \text{Cl}_n(G) \) is so.
3. Proof of Theorem 1.4

Proof of part (i). Suppose there exists an even integer \( n \geq 4 \) for which the assertion of the theorem does not hold. Let \( G \) be a maximal non-hamiltonian 2-connected graph of order \( n \), in which \( d(x) + d(y) \geq n - 1 \) whenever \( xy \notin E(G) \).

By maximality of \( G \), \( G + xy \) is hamiltonian for every pair of non-adjacent vertices \( x, y \in V(G) \). Hence, by Theorem 2.4 we must have

\[
(*) \quad d(x) + d(y) = n - 1 \quad \text{whenever} \quad xy \notin E(G).
\]

The minimal degree \( \delta(G) \) of \( G \) satisfies inequality \( \delta(G) < n/2 \), by Theorem 2.3, hence, in particular, \( n - 1 - \delta(G) \geq \delta(G) + 1 \).

Pick \( x \in V(G) \) with \( d(x) = \delta(G) \). There are precisely \( n - 1 - \delta(G) \) vertices in \( G \) non-adjacent to \( x \), each of degree \( n - 1 - \delta(G) \), by \((*)\). Put \( V = \{ v \in V(G) : xv \notin E(G) \} \). Pick \( y \in V \). As \( d(y) = n - 1 - \delta(G) \), there are precisely \( \delta(G) \) vertices in \( G \) non-adjacent to \( y \), each of degree \( \delta(G) \), by \((*)\) again. Put \( U = \{ u \in V(G) : uy \notin E(G) \} \). Then \( |U| = \delta(G), \ |V| = n - 1 - \delta(G), \ and \ U \cap V = \emptyset, \) because vertices in \( U \) are of degree \( \delta(G) \) and those in \( V \) are of degree \( n - 1 - \delta(G) > \delta(G) \). It follows that there exists a vertex \( z \) in \( G \) such that \( V(G) = U \cup V \cup \{ z \} \) is a partition of the vertex set of \( G \).

We will now show that \( d(z) = n - 1 \): Observe first that \( d(z) > \delta(G) \). Indeed, if \( d(z) = \delta(G) \), then by \((*)\), \( z \) is adjacent to every vertex in \( U \), as \( 2\delta(G) < n - 1 \). But \( z \) is also adjacent to \( y \), as \( z \notin U \), hence \( d(z) \geq |U| + 1 = \delta(G) + 1 \); a contradiction. Consequently, \( z \) is adjacent to every vertex in \( V \), by \((*)\) again, as \( d(z) + (n - 1 - \delta(G)) > n - 1 \). Hence \( d(z) \geq |V| = n - 1 - \delta(G) \). On the other hand, \( z \) is adjacent to \( x \), as \( z \notin V \), which yields \( d(z) \geq |V| + 1 = n - \delta(G) \). This last inequality paired with \((*)\) implies that \( z \) is adjacent to every other vertex in \( G \), as required.

Next observe that \( u_1 u_2 \in E(G) \) for every pair of vertices \( u_1, u_2 \) in \( U \), as \( d(u_1) + d(u_2) = 2\delta(G) < n - 1 \). It follows that \( N(u) \supset U \cup \{ z \} \setminus \{ u \}, \) and hence, by comparing cardinalities, \( N(u) = U \cup \{ z \} \setminus \{ u \} \) for every \( u \in U \).

Similarly, \( v_1 v_2 \in E(G) \) for every pair \( v_1, v_2 \) in \( V \), hence \( N(v) = V \cup \{ z \} \setminus \{ v \} \) for every \( v \in V \). Therefore \( G = G_1 \cup G_2 \), where \( G_1 \) is a complete graph of order \( \delta(G) + 1 \) spanned on the vertices of \( U \cup \{ z \} \), and \( G_2 \) is a complete graph of order \( n - \delta(G) \) spanned on \( V \cup \{ z \} \). Then \( z \) is a cutvertex, contradicting the assumption that \( G \) be 2-connected.

Proof of part (ii). Suppose there exists a 2-connected graph of odd order \( n \geq 3 \), in which \( d(x) + d(y) \geq n - 1 \) for every pair of non-adjacent vertices \( x \) and \( y \), that does not contain neither a Hamilton cycle nor a cycle of length \( n - 1 \). Let \( G \) be maximal such a graph of order \( n \). By maximality of \( G \), \( G + xy \) contains a cycle of length at least \( n - 1 \) whenever \( xy \notin E(G) \). Hence \( G \) contains a path of length at least \( n - 2 \) between any two of its non-adjacent vertices.

Pick a pair of non-adjacent vertices \( x \) and \( y \). By a theorem of Pósa, \( G \) contains a Hamilton \( x - y \) path \( P \), and hence, by Theorem 2.4, the sum \( d(x) + d(y) \) actually equals \( n - 1 \). Write \( P = u_1 u_2 \ldots u_n \), where \( u_1 = x \) and \( u_n = y \).

Put \( I_x = \{ i : xu_{i+1} \in E(G), 1 \leq i \leq n - 1 \} \) and \( I_y = \{ i : uy \in E(G), 1 \leq i \leq n - 1 \} \). If \( I_x \cap I_y \neq \emptyset \), say \( i_0 \in I_x \cap I_y \), then \( G \) contains a Hamilton cycle

\[
u_1 u_{i_0+1} u_{i_0+2} \ldots u_n u_{i_0} u_{i_0-1} \ldots u_2 u_1.
\]
We may thus assume that $I_x \cap I_y = \emptyset$. Then, for every $1 \leq i \leq n - 1$, either $u_i$ is adjacent to $y$ or else $u_{i+1}$ is adjacent to $x$, because $|I_x| + |I_y| = d(x) + d(y) = n - 1$. Let $d = d(y)$ and let $v_1, \ldots, v_d = y$ be the vertices that lie on $P$ next to the (respective) neighbours of $y$.

If there exists $j < d$ such that $v_j \notin N(y)$, then $v_j = u_{i_0}$ for some $i_0 \in I_x$. It follows that $u_{i_0+1}$ is adjacent to $x$, and $G$ contains a cycle of length $n - 1$ of the form

$$u_1 u_{i_0+1} u_{i_0+2} \ldots u_n u_{i_0-1} u_{i_0-2} \ldots u_2 u_1.$$ 

Therefore we can assume that

$$(i) \quad v_1, \ldots, v_{d-1} \text{ are all adjacent to } y.$$ 

Let $z$ denote the furthermost neighbour of $y$ on $P$. It follows from $(i)$ that all the vertices between $z$ and $y$ on $P$ are adjacent to $y$, and hence $z = u_{n-d}$.

Suppose $N(v_j) \subset \{z, v_1, \ldots, v_d\}$ for $j \leq d$. Then $N(u_i) \subset \{u_1, \ldots, u_{n-d-1}, z\}$ for $i \leq n - d - 1$. Consequently, $d(u_i) \leq n - d - 1$, $d(v_j) \leq d$, and $u_i v_j \notin E(G)$ for $i \leq n - d - 1$ and $j \leq d$. But then $d(u_i) + d(v_j) \geq n - 1$ yields

$$d(u_i) = n - d - 1 \quad \text{and} \quad d(v_j) = d \quad \text{for} \quad i = 1, \ldots, n - d - 1, \quad j = 1, \ldots, d.$$ 

Therefore, as in the proof of part $(i)$, we get that $G = G_1 \cup G_2$, where $G_1$ is a complete graph of order $n - d$ spanned on the vertices $\{u_1, \ldots, u_{n-d-1}, z\}$ and $G_2$ is a complete graph of order $d + 1$ on $\{z, v_1, \ldots, v_d\}$. Then $z$ is a cutvertex contradicting our assumptions on $G$.

It remains to consider the case of some $v_{j_0}$ being adjacent to $u_{i_0}$, where $i_0 \leq n - d - 1$. But then again $G$ contains a Hamilton cycle

$$u_1 \ldots u_{i_0} v_{j_0} \ldots v_d v_{j_0-1} \ldots u_{i_0+1} u_1.$$ 

For the proof of the last assertion of Theorem 1.4, suppose that $n = 2k + 1$ is odd and $G$ is a non-hamiltonian 2-connected graph on $n$ vertices, satisfying $d(x) + d(y) \geq n - 1$ for every pair of non-adjacent $x$ and $y$. Then the $n$-closure of $G$, $G^* = Cl_n(G)$ is not hamiltonian either, by Theorem 2.1 and we have equality

$$d_{G^*}(x) + d_{G^*}(y) = n - 1 \quad \text{whenever} \quad xy \notin E(G^*).$$ 

Now, if $\delta(G^*) < k = \frac{2n - 1}{2}$, then $n - 1 - \delta(G^*) > \delta(G^*)$ and one can repeat the proof of part $(i)$ to show that $G^*$ contains a Hamilton cycle, which contradicts the assumptions on $G$.

Thus $\delta(G^*) = \frac{2n - 1}{2}$. Moreover, $d_{G^*}(x) + d_{G^*}(y) = n - 1 = 2k$ for $xy \notin E(G^*)$ implies that $d_{G^*}(x) = k$ or $d_{G^*}(x) = n - 1$ for every vertex $x$.

Suppose $G^*$ is not maximal among the non-hamiltonian 2-connected graphs on $n$ vertices. Then $G^*$ has a pair of non-adjacent vertices $x$ and $y$ such that $G^* + xy$ is contained in a maximal non-hamiltonian graph $H$. By maximality of $H$, $H + uv$ contains a Hamilton cycle for every $uv \notin E(H)$, so Theorem 2.2 implies that $d_H(u) + d_H(v) = n - 1$ for every $uv \notin E(H)$.

Notice that $d_{G^*}(x) = k$, as $d_{G^*}(x) < n - 1$. Then $d_H(x) \geq k + 1$ and hence, for every $v$ non-adjacent to $x$ in $G^*$, $d_H(x) + d_H(v) \geq d_{G^*}(x) + 1 + d_{G^*}(v) > n - 1$, implying $xv \in E(H)$. Therefore $H$ is obtained from $G$ by increasing degrees of at least $x$ and all its non-neighbours in $G^*$, that is, at least $1 + (n - 1 - k) = k + 1$ vertices. But then $H$ contains at least $k + 1$ vertices of degree $n - 1$, which
means that $\delta(H) \geq k + 1 = \frac{n+1}{2}$, and hence $H$ is hamiltonian by Theorem 2.3; a contradiction.

\[ \square \]

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