A STRONGER VERSION OF CLASSICAL BORSUK-ULAM THEOREM

JUN WANG* AND XUEZHI ZHAO**

Abstract. The classical Borsuk-Ulam theorem states that for any continuous map $f : S^m \to \mathbb{R}^m$, there is a pair of antipodal points having the same image. Being a generalization of the classical Borsuk-Ulam theorem, Yang-Bourgin theorem tells us that for any continuous map $f : S^m \to \mathbb{R}^d$, if $m \geq d$, then there exists at least one pair of antipodal points having the same image. In this paper, we shall prove that there is also another pair of non-antipodal points having the same image for such a map $f$. This gives a stronger version of the classical Borsuk-Ulam theorem and Yang-Bourgin theorem. Our main tool is the ideal-valued index of $G$-space defined by E. Fadell and S. Husseini. Actually, by using this index we also obtain some sufficient conditions to guarantee the existence of self-coincidence of maps from $S^m$ to $\mathbb{R}^d$.

1. Introduction

The classical Borsuk-Ulam theorem was conjectured by St. Ulam [3, footnote, page 178] and was proved by K. Borsuk [3] in 1933. This theorem is a great theorem because that there are several different equivalent versions, many different proofs, a host of extensions and generalizations, and numerous interesting applications (see [11]). For the better part of the past century, many mathematicians have been contributing to generalizing and extending the Borsuk-Ulam theorem in various ways (see [5, 8, 9, 15, 16]). One of the significant generalization is Yang-Bourgin theorem proved by C.-T. Yang [18, 19] and (independently) Bourgin D.G. [4]. Another significant generalization is to consider more general free involution pairs $(X, \tau)$ (see [7, 12, 14, 18]), which means that given any continuous map $f : X \to Y$ between topological spaces with a free action of involution $\tau : X \to X$ ($\tau^2 = 1$), does there

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exists a pair of points $x_1, x_2 \in X$ satisfying $\tau(x_1) = (x_2)$ such that $f(x_1) = f(x_2)$? This general case replaces the pair of antipodal points in classical Borsuk-Ulam theorem by an orbit of some point under the free action of the involution $\tau$, actually, from a special $\mathbb{Z}_2$-action to a general $\mathbb{Z}_2$-action.

It is natural to ask that does there exists other pair of points having the same image in addition to an orbit of some point under the special $\mathbb{Z}_2$-action or the general $\mathbb{Z}_2$-action. Basing on this problem, we obtain a stronger version of classical Borsuk-Ulam theorem in this paper.

**Theorem:**

Let $f : S^m \to \mathbb{R}^d$ be a continuous map. If the map $f$ and the values of $\{m, d\}$ satisfy one of the following conditions,

1. $m > d$,
2. $f(-p) = -f(p)$, $m = d$ is odd,

then there exist at least a pair of points $p_1, p_2 \in S^m, p_1 \neq \pm p_2$ such that $f(p_1) = f(p_2)$, and a pair of points $p_0, -p_0 \in S^m$ such that $f(p_0) = f(-p_0)$.

In particular let $d = m$, then the above theorem gives a stronger conclusion than classical Borsuk-Ulam theorem for all the odd dimensional spheres. The above theorem also gives a stronger conclusion than Yang-Bourgin theorem. That means, for the continuous map, if the map $f$ and the values of $\{m, d\}$ satisfy the assumptions of the above theorem, then there is another pair of distinct points $\{p_1, p_2\}$ such that $f(p_1) = f(p_2)$ in addition to the pair of antipodal points obtained in Borsuk-Ulam theorem and Yang-Bourgin theorem. The calculations of the ideal-valued indices (see [6] or Section 2 for definition) of generalized configuration spaces $W_{k,n}(S^m)$ (see [17] for more details) have enormous effects in the proof of our results.

Our paper is organized as follows. In Section 2, we shall review the definition and some properties of the ideal-valued index. In Section 3, we shall calculate the ideal-valued indices of generalized configuration spaces $W_{k,n}(S^m)$. Through that, we obtain some sufficient conditions in Section 4 to guarantee the existence of self-coincidence of maps from sphere to Euclidean space, and give the proof of the stronger version of Borsuk-Ulam theorem.

2. The ideal-valued index

In this section, we shall give a brief account of the definition of ideal-valued index and its related properties. More details and applications can be found in [2, 6, 10].
Definition 2.1. (E. Fadell and S. Husseini [6]) Let $X$ be a paracompact Hausdorff space admitting an action of a compact Lie group $G$, and $R$ be a commutative ring. The ideal-valued index of the space $X$, which is denoted by $\text{Index}^G(X; R)$, is defined to be the kernel

$$\ker(c^* : H^*(BG; R) \to H^*(EG \times_G X; R))$$

of the ring homomorphism $c^*$, where $c : EG \times_G X \to BG$ is the map induced by the projective map $EG \times X \to EG$.

Remark 2.2. It is worth mention that the ideal-valued index of $X$ is related to the action of $G$ on $X$.

If $f : X \to Y$ is a $G$-equivariant map, then we have the following commutative diagram

$$
\begin{array}{ccc}
H^*(BG; R) & \xrightarrow{\cong} & H^*(BG; R) \\
\downarrow c_2^* & & \downarrow c_1^* \\
H^*(EG \times_G Y; R) & \xrightarrow{f^*} & H^*(EG \times_G X; R).
\end{array}
$$

Thus we obtain that

Proposition 2.3. Let $X$ and $Y$ be two $G$-spaces, $f : X \to Y$ be a $G$-equivariant map, then

$$\text{Index}^G(Y; R) \subset \text{Index}^G(X; R).$$

As a special case, let $G = \mathbb{Z}_2$, $R = \mathbb{Z}_2$, then $\text{Index}^\mathbb{Z}_2(X; \mathbb{Z}_2)$, which is the ideal-valued index of $X$, shall be an ideal of the cohomology ring

$$H^*(BG; R) = H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \cong \mathbb{Z}_2[\xi], \quad \dim \xi = 1,$$

and it is easy to obtain the following proposition.

Proposition 2.4. Consider the $\mathbb{Z}_2$-action on the sphere $S^m$ which is induced by the antipodal map, then $\text{Index}^\mathbb{Z}_2(S^m; \mathbb{Z}_2)$ is equal to the ideal $(\xi^{m+1})$ in $\mathbb{Z}_2[\xi]$ generated by $\xi^{m+1}$. 
3. The ideal-valued indices of generalized configuration spaces

In this section, we shall calculate the ideal-valued indices of some generalized configuration spaces, which will be used into the proof of our main result.

Recall that the generalized configuration space of sphere \( S^m \) is defined by
\[
W_{k,n}(S^m) = \{ (p_1, ..., p_n) | p_i \in S^m, 1 \leq i \leq n; \text{ for any } k\text{-elements subset } \{i_1, ..., i_k\} \subset \{1, ..., n\}, p_{i_1}, ..., p_{i_k} \text{ are linearly independent} \}.
\]
It is easy to obtain that the Stiefel manifold \( V_{m+1,n} \) is a subset of the generalized configuration space \( W_{k,n}(S^m) \), and \( V_{m+1,n} \) is a deformation retract of \( W_{n,n}(S^m) \) (more details can be seen in [17]).

Unless otherwise specified, the actions of \( \mathbb{Z}_2 \) on generalized configuration spaces \( W_{k,n}(S^m) \) and Stiefel manifolds \( V_{m,n} \) are defined by
\[
\mathbb{Z}_2 \times W_{k,n}(S^m) \to W_{k,n}(S^m),
\]
\[
(\tau, x_1, ..., x_n) \mapsto (-x_1, ..., -x_n),
\]
and
\[
\mathbb{Z}_2 \times V_{m,n} \to V_{m,n},
\]
\[
(\tau, y_1, ..., y_n) \mapsto (-y_1, ..., -y_n),
\]
where \( \tau \) is the non-trivial element of \( \mathbb{Z}_2 \).

By the relation between generalized configuration spaces and Stiefel manifolds, we obtain the following lemma.

**Lemma 3.1.** Let \( \nu_2(m+1) \) be the exponent of the highest power of 2 dividing \( m+1 \) (it means that \( \nu_2(m+1) = \max\{u \mid 2^u \mid (m+1)\} \)).

If \( m \geq n \) and \( \nu_2(m+1) \geq -\lceil -\log_2(n) \rceil \), then \( \text{Index}_{\mathbb{Z}_2}(W_{k,n}(S^m); \mathbb{Z}_2) \) is equal to the ideal \( (\xi^{m+1}) \) in \( \mathbb{Z}_2[\xi] \) generated by \( \xi^{m+1} \).

**Proof.** Consider two \( \mathbb{Z}_2 \)-equivariant maps \( i \) and \( j \) between generalized configuration spaces defined by
\[
i : W_{k+1,n}(S^m) \to W_{k,n}(S^m),
\]
\[
(x_1, ..., x_n) \mapsto (x_1, ..., x_n),
\]
\[
j : W_{k,n}(S^m) \to W_{k,n-1}(S^m),
\]
\[
(x_1, ..., x_n) \mapsto (x_1, ..., x_{n-1}).
\]
By Proposition 2.3, we obtain

\[ \text{Index}^{Z_2}(W_{k,n}(S^m); \mathbb{Z}_2) \subset \text{Index}^{Z_2}(W_{k+1,n}(S^m); \mathbb{Z}_2), \]

\[ \text{Index}^{Z_2}(W_{k,n-1}(S^m); \mathbb{Z}_2) \subset \text{Index}^{Z_2}(W_{k,n}(S^m); \mathbb{Z}_2). \]

It was shown that Stiefel manifold \( V_{m+1,k} \) is a deformation retract of \( W_{k,k}(S^m) \) when \( k \leq m+1 \) (see [17]), so

\[ \text{Index}^{Z_2}(V_{m+1,k}; \mathbb{Z}_2) = \text{Index}^{Z_2}(W_{k,k}(S^m); \mathbb{Z}_2), k \leq m+1. \]

Then we have a sequence

\[ \text{Index}^{Z_2}(V_{m+1,k}; \mathbb{Z}_2) = \text{Index}^{Z_2}(W_{k,k}(S^m); \mathbb{Z}_2) \]
\[ \subset \text{Index}^{Z_2}(W_{k,k+1}(S^m); \mathbb{Z}_2) \subset \cdots \subset \text{Index}^{Z_2}(W_{k,n-1}(S^m); \mathbb{Z}_2) \]
\[ \subset \text{Index}^{Z_2}(W_{k,n}(S^m); \mathbb{Z}_2) \subset \cdots \subset \text{Index}^{Z_2}(W_{n,n}(S^m); \mathbb{Z}_2) \]
\[ = \text{Index}^{Z_2}(V_{m+1,n}; \mathbb{Z}_2). \]

In [13, Theorem 4], it was proved that if \( \ell \leq m, \nu_2(m+1) \geq \lceil -\log_2(\ell) \rceil \), then \( \text{Index}^{Z_2}(V_{m+1,\ell}; \mathbb{Z}_2) \) is equal to the ideal \((\xi^{m+1})\) in \( \mathbb{Z}_2[\xi]\) generated by \( \xi^{m+1} \). It is easily seen that \( k \leq n \) by the definition of generalized configuration space, thus \( \lceil -\log_2(k) \rceil \leq \lceil -\log_2(n) \rceil \). The above discussion implies that if \( m \geq n, \nu_2(m+1) \geq \lceil -\log_2(n) \rceil \), then

\[ \text{Index}^{Z_2}(V_{m+1,k}; \mathbb{Z}_2) = (\xi^{m+1}) = \text{Index}^{Z_2}(V_{m+1,n}; \mathbb{Z}_2). \]

Thus by the sequence

\[ \text{Index}^{Z_2}(V_{m+1,k}; \mathbb{Z}_2) \subset \text{Index}^{Z_2}(W_{k,n}(S^m); \mathbb{Z}_2) \subset \text{Index}^{Z_2}(V_{m+1,n}; \mathbb{Z}_2), \]

we obtain the ideal-valued indices of \( W_{k,n}(S^m) \) when \( m \geq n, \nu_2(m+1) \geq \lceil -\log_2(n) \rceil \), and the proof is completed. \( \square \)

**Lemma 3.2.**

\[ \text{Index}^{Z_2}(W_{m,m+1}(S^m); \mathbb{Z}_2) = \text{Index}^{Z_2}(W_{m,m}(S^m); \mathbb{Z}_2) \]

**Proof.** For generalized configuration spaces \( W_{m,m+1}(S^m) \) and \( W_{m,m}(S^m) \), define a map \( f : W_{m,m+1}(S^m) \to W_{m,m}(S^m) \) as \( f(x_1, \ldots, x_{m+1}) = (x_1, \ldots, x_m) \), and a map \( g : W_{m,m}(S^m) \to W_{m,m+1}(S^m) \) as \( g(x_1, \ldots, x_m) = (x_1, \ldots, x_m, \frac{x_1 + \cdots + x_m}{\|x_1 + \cdots + x_m\|}). \)
It is not hard to verify that the maps $f$ and $g$ both are $\mathbb{Z}_2$-equivariant. Thus we obtain that

$$\text{Index}^{\mathbb{Z}_2}(W_{m,m+1}(S^m); \mathbb{Z}_2) \subset \text{Index}^{\mathbb{Z}_2}(W_{m,m}(S^m); \mathbb{Z}_2),$$

$$\text{Index}^{\mathbb{Z}_2}(W_{m,m+1}(S^m); \mathbb{Z}_2) \supset \text{Index}^{\mathbb{Z}_2}(W_{m,m}(S^m); \mathbb{Z}_2),$$

then $\text{Index}^{\mathbb{Z}_2}(W_{m,m+1}(S^m); \mathbb{Z}_2) = \text{Index}^{\mathbb{Z}_2}(W_{m,m}(S^m); \mathbb{Z}_2).$ \hfill $\square$

**Lemma 3.3.** $\text{Index}^{\mathbb{Z}_2}(W_{m,m+1}(S^m); \mathbb{Z}_2)$ is equal to the ideal $(\xi^{m+1})$ in $\mathbb{Z}_2[\xi]$ generated by $\xi^{m+1}$, if and only if $m + 1$ is a power of 2.

*Proof.* For the ideal-valued indices of the generalized configuration spaces $W_{m,m}(S^m)$, we have the relation

$$\text{Index}^{\mathbb{Z}_2}(W_{m,m}(S^m); \mathbb{Z}_2) = \text{Index}^{\mathbb{Z}_2}(V_{m+1,m}; \mathbb{Z}_2) = \text{Index}^{\mathbb{Z}_2}(SO(m+1); \mathbb{Z}_2).$$

In [13, Corollary 1], it was proved that $\text{Index}^{\mathbb{Z}_2}(SO(m+1); \mathbb{Z}_2)$ is equal to the ideal $(\xi^{m+1})$ in $\mathbb{Z}_2[\xi]$ generated by $\xi^{m+1}$, if and only if $m + 1$ is a power of 2. Thus by Lemma 3.2, we obtain the ideal-valued indices of $W_{m,m+1}(S^m)$ for some $m$. \hfill $\square$

Denote the two connected components of generalized configuration space $W_{m+1,m+1}(S^m)$ by $W_{m+1,m+1}^+(S^m)$ and $W_{m+1,m+1}^{-}(S^m)$, and we get that $\text{Index}^{\mathbb{Z}_2}(W_{m+1,m+1}^+(S^m); \mathbb{Z}_2) = \text{Index}^{\mathbb{Z}_2}(W_{m+1,m+1}^-(S^m); \mathbb{Z}_2)$, and we get that $\text{Index}^{\mathbb{Z}_2}(W_{m,m+1}(S^m); \mathbb{Z}_2) = \text{Index}^{\mathbb{Z}_2}(W_{m,m}(S^m); \mathbb{Z}_2)$. Thus it is easy to obtain that

**Corollary 3.4.** $\text{Index}^{\mathbb{Z}_2}(W_{m+1,m+1}^+(S^m); \mathbb{Z}_2)$ is equal to the ideal $(\xi^{m+1})$ in $\mathbb{Z}_2[\xi]$ generated by $\xi^{m+1}$, if and only if $m + 1$ is a power of 2.

### 4. The proof of main results

In this section, we shall give some sufficient conditions to guarantee the existence of self-coincidence of maps from $S^m$ to $\mathbb{R}^d$, and the proof of our stronger version of the classical Borsuk-Ulam theorem.

**Theorem 4.1.** Let $f: S^m \to \mathbb{R}^d$ be a continuous odd map which means $f(-p) = -f(p)$, and let $\nu_2(m + 1)$ be the exponent of the highest power of 2 dividing $m + 1$ (it means that $\nu_2(m + 1) = \max\{u \mid 2^u \mid (m + 1)\}$).

If $m \geq \max\{n, (n-1)d\}$ and $\nu_2(m + 1) \geq -\lfloor -\log_2(n) \rfloor$, then there is at least an element $(p_1, \ldots, p_n) \in W_{k,n}(S^m)$ such that $f(p_1) = \cdots = f(p_n)$. 


Proof. Let $\varphi : \mathbb{R}^d \Delta_n \to \tilde{S}^{(n-1)d-1}$ be defined by
\[
\varphi(x_1, \ldots, x_n) = \left( x_1 - \frac{x_1 + \cdots + x_n}{n}, \ldots, x_n - \frac{x_1 + \cdots + x_n}{n} \right) \times \left( x_1^2 + \cdots + x_n^2 - \frac{(x_1 + \cdots + x_n)^2}{n} \right)^{1/2},
\]
where $\Delta_n^d = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}^d, x_1 = \cdots = x_n \}$ and $\tilde{S}^{(n-1)d-1}$ is the $((n-1)d-1)$-dimension sphere $\{(y_1, \ldots, y_n) \mid y_i \in \mathbb{R}^d, y_1 + \cdots + y_n = 0, |y_1|^2 + \cdots + |y_n|^2 = 1\}$.

For the continuous map $f : S^m \to \mathbb{R}^d$, define $\tilde{f} : W_{k,n}(S^m) \to \mathbb{R}^d$ as $\tilde{f}(p_1, \ldots, p_n) = (f(p_1), \ldots, f(p_n))$. If there does not exist $(p_1, \ldots, p_n) \in W_{k,n}(S^m)$ such that $f(p_1) = \cdots = f(p_n)$, then $\tilde{f}(W_{k,n}(S^m)) \subset \mathbb{R}^d - \Delta_n^d$ and we can construct a map
\[
g = \varphi \circ \tilde{f} : W_{k,n}(S^m) \to \tilde{S}^{(n-1)d-1}.
\]
It is not hard to verify that the map $g$ is $\mathbb{Z}_2$-equivariant since $f$ is an odd map $(f(-p) = -f(p))$. Thus by Proposition 2.3, we obtain that
\[
\text{Index}_{\mathbb{Z}_2}(\tilde{S}^{(n-1)d-1}; \mathbb{Z}_2) \subset \text{Index}_{\mathbb{Z}_2}(W_{k,n}(S^m); \mathbb{Z}_2).
\]

For the ideal-valued index of the sphere, we have that $\text{Index}_{\mathbb{Z}_2}(\tilde{S}^{(n-1)d-1}; \mathbb{Z}_2) = (\xi^{(n-1)d}) \subset \mathbb{Z}_2[\xi]$ by Proposition 2.4. And Lemma 3.1 tells us that if $n \leq m$ and $\nu_2(m+1) \geq -[-\log_2(n)]$, then $\text{Index}_{\mathbb{Z}_2}(W_{k,n}(S^m); \mathbb{Z}_2) = (\xi^{m+1}) \subset \mathbb{Z}_2[\xi]$.

Then by the formula (4.2), it is obtained that
\[
(\xi^{(n-1)d}) \subset (\xi^{m+1}),
\]
thus $m+1 \leq (n-1)d$ and it is in contradiction to the assumption of the theorem. □

Given different values of $\{m, d, k, n\}$ in Theorem 4.1, we get more interesting results. In particular, let $k = n = 2$, we get the following lemma.

Lemma 4.2. Let $f : S^m \to \mathbb{R}^d$ be a continuous odd map which means $f(-p) = -f(p)$. If $m$ is odd, $m \geq \max\{2, d\}$, then there is at least a pair of points $p_1, p_2 \in S^m, p_1 \neq \pm p_2$ such that $f(p_1) = f(p_2)$.

Through the ideal-valued indices of $W_{m,m+1}(S^m)$ (see Lemma 3.3) and $W_{m+1,m+1}^+(S^m)$ (see Corollary 3.4), we also obtain the following lemma by the analogous arguments as in the proof of Theorem 4.1.
Lemma 4.3. Let $f : S^m \to \mathbb{R}$ be a continuous odd map which means $f(-p) = -f(p)$. If $m + 1$ is a power of 2, then there is at least an element $(p_1, \ldots, p_{m+1}) \in W_{k,m+1}(S^m)$ such that $f(p_1) = \cdots = f(p_{m+1})$.

Then by Lemma 4.2 and Lemma 4.3, we obtain the following theorem.

Theorem 4.4. Let $f : S^m \to \mathbb{R}^d$ be a continuous odd map which means $f(-p) = -f(p)$. If $m$ is odd, and $m \geq d$, then there exist at least a pair of points $p_1, p_2 \in S^m, p_1 \neq \pm p_2$ such that $f(p_1) = f(p_2)$.

Moreover, for the case of $m > d$, the map $f : S^m \to \mathbb{R}^d$ is not need to be odd.

Theorem 4.5. Let $f : S^m \to \mathbb{R}^d$ be a continuous map. If $m > d$, then there exist at least a pair of points $p_1, p_2 \in S^m, p_1 \neq \pm p_2$ such that $f(p_1) = f(p_2)$.

Proof. Consider the action $Z_2 \times W_{2,2}(S^m) \to W_{2,2}(S^m)$ defined by $(\tau, p_1, p_2) = (p_2, p_1)$, where $\tau$ is the non-trivial element of $Z_2$. In [1, Theorem 4.8], it is proved the ideal-valued index of $V_{m+1,2}$ under the action of permuting the orthogonal vectors is equal to the ideal $(\xi^m)$ in $Z_2[\xi]$ generated by $\xi^m$. Thus the ideal-valued index of $W_{2,2}(S^m)$ under the action of permuting the vectors is equal to the ideal $(\xi^m)$ in $Z_2[\xi]$ generated by $\xi^m$.

For the continuous map $f : S^m \to \mathbb{R}^d$, define $\tilde{f} : W_{2,2}(S^m) \to \mathbb{R}^d$ as $\tilde{f}(p_1, p_2) = f(p_1) - f(p_2)$. If there does not exist $(p_1, p_2) \in W_{2,2}(S^m)$ such that $f(p_1) = f(p_2)$, then $\tilde{f}(W_{2,2}(S^m)) \subset \mathbb{R}^d - \{0\}$ and we can construct a map

$$g = \phi \cdot \tilde{f} : W_{2,2}(S^m) \to S^{d-1}$$

where $\phi : \mathbb{R}^d - \{0\} \to S^{d-1}$ is defined by $\phi(x) = \frac{x}{|x|}$. Considering the action of $Z_2$ on $S^{d-1}$ induced by the antipodal map, it is not hard to verify that the map $g$ is $Z_2$-equivariant. Thus by Proposition 2.3, we obtain that

$$\text{Index}^{Z_2}(S^{d-1}; Z_2) \subset \text{Index}^{Z_2}(W_{2,2}(S^m); Z_2).$$

Then it is obtained that

$$(\xi^d) \subset (\xi^m) \subset Z_2[\xi],$$

thus $m \leq d$, and it is in contradiction to the assumption of the theorem. Hence the proof is completed. □

By Theorem 4.4, Theorem 4.5 and Yang-Bourgin theorem, we obtain the main result of this paper, the stronger version of Borsuk-Ulam theorem.
Theorem A. Let \( f : S^m \to \mathbb{R}^d \) be a continuous map. If the map \( f \) and the values of \( \{m, d\} \) satisfy one of the following conditions,

- (1) \( m > d \),
- (2) \( f(-p) = -f(p), \ m = d \) is odd,

then there exist at least a pair of points \( p_1, p_2 \in S^m \), \( p_1 \neq \pm p_2 \) such that \( f(p_1) = f(p_2) \), and a pair of points \( p_0, -p_0 \in S^m \) such that \( f(p_0) = f(-p_0) \).

Actually, for some values of \( \{k, m, n\} \), the generalized configuration spaces \( W_{k,n}(S^m) \) and the Stiefel manifolds \( V_{m+1,n} \) both have the same ideal-valued indices, then through that we obtain many corollaries of the above results. For instance, the following corollary is obtained by Lemma 4.3.

**Corollary 4.6.** Let \( f : S^m \to \mathbb{R} \) be a continuous odd map which means \( f(-p) = -f(p) \). If \( m + 1 \) is a power of 2, then there are mutually orthogonal vectors \( p_1, ..., p_{m+1} \in S^m \) such that \( f(p_1) = \cdots = f(p_{m+1}) \).

Actually, it is a special case of Kukutani-Yamabe-Yujobô theorem which states that for any continuous function \( f : S^m \to \mathbb{R} \), there are mutually orthogonal vectors \( p_1, ..., p_{m+1} \in S^m \) such that \( f(p_1) = \cdots = f(p_{m+1}) \).

By the definition of generalized configuration spaces, it is understood that the Stiefel manifold \( V_{m+1,n} \) is a subset of the generalized configuration space \( W_{k,n}(S^m) \), then Corollary 4.6 and Lemma 4.3 suggest an interesting question as follows.

**Question 4.7.** For any continuous map \( f : S^m \to \mathbb{R}^d \), does there exist an element \( (p_1, ..., p_n) \in W_{k,n}(S^m) \setminus V_{m+1,n} \) such that \( f(p_1) = \cdots = f(p_n) \)?

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* School of Mathematical Sciences, Hebei Normal University, Shijiazhuang, Hebei, China.

Email address: * wjun@hebtu.edu.cn

** School of Mathematical Sciences, Capital Normal University, Beijing, China.

Email address: ** zhaoxve@cnu.edu.cn