Observability through matrix-weighted graph

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Abstract

Observability of an array of identical LTI systems with incommensurable output matrices is studied, where an array is called observable when identically zero relative outputs imply synchronized solutions for the individual systems. It is shown that the observability of an array is equivalent to the connectivity of its interconnection graph, whose edges are assigned matrix weights. The interconnection graph is studied by means of a collection of simpler graphs, each of which is associated to an eigenvalue of the system matrix of individual dynamics. It is reported that the interconnection graph is connected if and only if no member of this collection is disconnected. Moreover, to better understand the relative behavior of distant units, pairwise observability which concerns with the synchronization of a certain pair of individual systems in the array is studied. This milder version of observability is shown to be closely related to certain connectivity properties of the interconnection graph as well. Pairwise observability is also analyzed using the circuit theoretic tool effective conductance. The observability of a certain pair of units is proved to be equivalent to the nonsingularity of the (matrix-valued) effective conductance between the associated pair of nodes of a resistive network (with matrix-valued parameters) whose node admittance matrix is the Laplacian of the array’s interconnection graph.

1 Introduction

Observability is one of the central concepts in systems theory which, for LTI systems, can be expressed in many seemingly different yet mathematically equivalent forms [7]. One alternative is the following. A pair \([C, A]\) is observable if \(C(x_1(t) - x_2(t)) \equiv 0\) implies \(x_1(t) \equiv x_2(t)\), where \(x_i(t)\) denote the solutions of two identical systems \(\dot{x}_i = Ax_i\), \(i = 1, 2\). Admittedly, this appears to be an uneconomical definition, for the implication therein employs two systems where one would have sufficed. The overuse, however, has a relative advantage: it points in an interesting direction of generalization. Namely, for a pair \([C_{ij}]_{i,j=1}^q, A\) the below condition suggests itself as a natural extension.

\[
C_{ij}(x_i(t) - x_j(t)) \equiv 0 \quad \text{for all} \ (i, j) \implies x_i(t) \equiv x_j(t) \quad \text{for all} \ (i, j)
\]  

(1)

where \(x_i(t)\) are the solutions of \(q\) identical systems \(\dot{x}_i = Ax_i\), \(i = 1, 2, \ldots, q\). Aside from its theoretical allure, this particular choice of generalization is not without practical motivation; the condition \(\textbf{(1)}\) happens to be both necessary and sufficient for synchronization of certain arrays of oscillators. We give two examples in the sequel.

Coupled electrical oscillators. Consider the individual oscillator in Fig.\(\textbf{1}\) where \(p\) linear inductors with inductances \(m_1, m_2, \ldots, m_p > 0\) are connected by linear capacitors with capacitances \(k_1, k_2, \ldots, k_{p+1} > 0\). The node voltages are denoted by \(z^{[\ell]} \in \mathbb{R}\). Letting \(z = [z^{[1]}] z^{[2]} \cdots z^{[p]}]^T\) the model of this system reads

\[
K z + M^{-1} z = 0 \quad \text{where} \quad M = \text{diag}(m_1, m_2, \ldots, m_p)
\]

and

\[
K = \begin{bmatrix}
    k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\
    -k_2 & k_2 + k_3 & -k_3 & \cdots & 0 \\
    0 & -k_3 & k_3 + k_4 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & 0 & \cdots & k_p + k_{p+1}
\end{bmatrix}.
\]

Let now an array be constructed by coupling \(q\) identical oscillators in the arrangement shown in Fig.\(\textbf{2}\). If we let \(z_i \in \mathbb{R}^p\) denote the node voltage vector for the \(i\)th oscillator and \(b_{ij}^{[\ell]} = b_{ji}^{[\ell]} \geq 0\) be the

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conductance of the resistor connecting the \( \ell \)th nodes of the oscillators \( i \) and \( j \), we obtain \( K \ddot{z}_i + M^{-1} \dot{z}_i + \sum_{j=1}^q B_{ij} (\dot{z}_i - \dot{z}_j) = 0 \), where \( B_{ij} = \text{diag}(b_{ij1}, b_{ij2}, \ldots, b_{ijp}) \). Denoting by \( x_i = [z_i^T \dot{z}_i^T]^T \) the state of the \( i \)th system we can then rewrite the coupled dynamics as

\[
\dot{x}_i = \begin{bmatrix}
0 & I_p \\
-K^{-1}M^{-1} & 0
\end{bmatrix} x_i + \sum_{j=1}^q
\begin{bmatrix}
0 & 0 \\
0 & K^{-1}B_{ij}
\end{bmatrix}
(x_j - x_i), \quad i = 1, 2, \ldots, q.
\]  

(2)

To understand the collective behavior of these coupled oscillators one can employ the Lyapunov function

\[
V(x_1, x_2, \ldots, x_q) = \frac{1}{2} \sum_{i=1}^q x_i^T \begin{bmatrix}
M^{-1} & 0 \\
0 & K
\end{bmatrix} x_i.
\]  

(3)

In particular, combining (2) and (3) we reach

\[
\frac{d}{dt} V(x_1(t), x_2(t), \ldots, x_q(t)) = -\sum_{i>j} (x_i(t) - x_j(t))^T \begin{bmatrix}
0 & 0 \\
0 & B_{ij}
\end{bmatrix} (x_i(t) - x_j(t)).
\]  

(4)

Note that the righthand side is negative semidefinite because all \( B_{ij} \) are positive semidefinite. Hence the solutions \( x_1(t), x_2(t), \ldots, x_q(t) \) remain bounded. Now, for

\[
A = \begin{bmatrix}
0 & I_p \\
-K^{-1}M^{-1} & 0
\end{bmatrix} \quad \text{and} \quad C_{ij} = \begin{bmatrix}
0 & 0 \\
0 & B_{ij}
\end{bmatrix}
\]

suppose that the condition (1) holds. Then, and only then, (4) yields by Krasovskii-LaSalle invariance principle [9] that the oscillators synchronize, i.e., \( \|x_i(t) - x_j(t)\| \to 0 \) for all \( (i, j) \) and all initial conditions \( x_1(0), x_2(0), \ldots, x_q(0) \).

Coupled mechanical oscillators. Our second example employs the mechanical system shown in Fig. 3 where \( p \) masses are connected by linear springs. Such chains are used to model the interaction of atoms in a crystal [3]. Let \( z^{[\ell]} \in \mathbb{R} \) be the displacement of the mass \( m_\ell > 0 \) from the equilibrium. The spring
constants are denoted by \( k_1, k_2, \ldots, k_{p+1} > 0 \). Letting \( z = [z[1]^T \ z[2]^T \ \ldots \ z[p]^T]^T \) the model of this oscillator reads \( M\ddot{z} + Kz = 0 \), where the matrices \( M, K \) are borrowed from the previous example.

Let now an array be formed by coupling \( q \) replicas of this oscillator in the arrangement shown in Fig. 4. If we let \( z_i \in \mathbb{R}^p \) denote the displacement vector for the \( i \)th oscillator and \( b_{ij}^{[q]} = b_{ji}^{[q]} \geq 0 \) represent the viscous friction (damping) between the \( q \)th masses of the oscillators \( i \) and \( j \), we can write \( M\ddot{z}_i + Kz_i + \sum_{j=1}^{q} B_{ij}(z_i - z_j) = 0 \) with \( B_{ij} = \text{diag}(b_{ij}^{[1]}, b_{ij}^{[2]}, \ldots, b_{ij}^{[p]}) \). The Lyapunov approach previously adopted for the synchronization analysis of the coupled electrical oscillators is valid here, too. The outcome is the same. Namely, under the condition (1), this time with

\[
A = \begin{bmatrix}
0 & I_p \\
-M^{-1}K & 0
\end{bmatrix}
\quad \text{and} \quad
C_{ij} = \begin{bmatrix}
0 & 0 \\
0 & B_{ij}
\end{bmatrix},
\]

the mechanical oscillators synchronize. Having motivated the condition (1) in the context of synchronization, we will next try to explain its relation to certain existing assumptions.

Synchronization of linear systems is a broad area of research, where one of the main goals of the researcher is to unearth conditions under which the solutions of coupled units converge to a common trajectory. Different sets of assumptions have led to a rich collection of results, bringing our understanding on the subject closer to complete; see, for instance, [12, 18, 19, 16, 6, 11]. Despite their differences in degree and direction of generality, all these works share two assumptions in common: (i) the graph describing the interconnection contains a spanning tree and (ii) the individual system is observable (detectable). We intend to emphasize in this paper that these two separate assumptions, the former on connectivity and the latter on observability, dissolve inseparably in the condition (1). In particular, for an array represented by the pair \([C_{ij}]_{i,j=1}^q, A\), it is in general not meaningful to search for a spanning tree because the interconnection graph will be matrix-weighted, whereas a tree is well defined for a scalar-weighted graph only. As for the second assumption, requiring the individual systems to be observable also falls prey to ambiguity since there is not a single output matrix for each system; instead every system is coupled to each of its neighbors through a different matrix \( C_{ij} \). It is true that separation is possible in the special case \( C_{ij} = w_{ij}C \) with \( C \in \mathbb{C}^{m \times n} \) and \( w_{ij} \in \mathbb{R} \). In this much-studied scenario, where the output matrices are commensurable, the scalar weights \( w_{ij} \) are used to construct the interconnection graph, which can be checked to contain a spanning tree; and the pair \([C, A]\) can separately be checked for observability. However, in general, the condition (1) in its entirety is what we have to deal with, which requires that we work with the matrix-weighted graphs. We will explain how these matrix-valued weights emerge soon. But first, let us review the scarce literature on observability over networks.

Observability over networks, motivated in general by synchronization (consensus) of coupled systems, is largely an unexplored area of research. Among the few works is [8], where the observability of sensor networks is studied by means of equitable partitions of graphs. This tool is employed also in [14]. The observability of path and cycle graphs is studied in [13] and of grid graphs in [12]. Recently, the networks

![Figure 3: Mechanical oscillator.](attachment:image3.png)

![Figure 4: Array of coupled mechanical oscillators.](attachment:image4.png)
whose individual systems’ dynamics are allowed to be nonidentical is covered in [20]. Each of these investigations covers a different case, yet they all consider interconnections that can be described by graphs with scalar-weighted edges. At this point our work is located relatively far from the reported results. In particular, to the best of our knowledge, observability over matrix-weighted graphs has not yet been studied in detail.

In the first half of this paper we report conditions on the array $[C_{ij}]_{ij=1}^q, A$ that imply observability in the sense of (1). To this end, we construct a graph $\Gamma$ (with q vertices) where to each pair of vertices $(v_i, v_j)$ we assign a weight that is a Hermitian positive semidefinite matrix, whose null space is the unobservable subspace corresponding to the individual pair $[C_{ij}, A]$. We reveal that the array $[C_{ij}]_{ij=1}^q, A$ is observable if and only if the interconnection graph $\Gamma$ is connected. Also, we notice that for each distinct eigenvalue of $A$ there exists a graph (we call it an eigengraph) and the observability of $[C_{ij}]_{ij=1}^q, A$ is ensured if no eigengraph is disconnected. For our analysis we define the connectivity of a graph through a certain spectral property of its Laplacian. We note that the connectivity of a matrix-weighted graph cannot in general be characterized by the standard tools of graph theory such as path and tree. The reason is that the meaning or function of an edge (out of which paths and trees are constructed) becomes equivocal when one has to allow semidefinite weights.

In the second half of the paper we focus on the so-called $(k, \ell)$-observability of $[C_{ij}]_{ij=1}^q, A$. Namely, for a given pair of indices $(k, \ell)$, we search for conditions under which $x_k(t) = x_\ell(t)$ provided that $C_{ij}(x_i(t) - x_j(t)) \equiv 0$ for all $(i, j)$. To this end we define $(k, \ell)$-connectivity of a matrix-weighted graph through its Laplacian. We show the expected equivalence between the $(k, \ell)$-observability of the array $[C_{ij}]_{ij=1}^q, A$ and the $(k, \ell)$-connectivity of the interconnection graph $\Gamma$ as well as the unexpected lack of equivalence between the $(k, \ell)$-observability of the array and the $(k, \ell)$-connectivity of its eigengraphs. Moreover, we present the interesting interchangeability between the $(k, \ell)$-observability of an array and the nonsingularity of the matrix $\Gamma_{kl}$, where $\Gamma_{kl}$ is the (matrix-valued) effective conductance between the nodes $k$ and $\ell$ of a resistive network (with matrix-valued parameters) whose node admittance matrix is the Laplacian of the array’s interconnection graph $\Gamma$. From a graph-theoretic point of view the nonsingularity of the effective conductance $\Gamma_{kl}$ may be interpreted to indicate that the pair of vertices $(v_k, v_\ell)$ of the matrix-weighted graph $\Gamma$ are connected. This therefore allows one to study connectivity of vertices without employing paths; which is potentially useful since defining a path, as mentioned above, is problematic for matrix-weighted graphs.

One may ask why our formulation is in terms of effective conductance instead of the commoner effective resistance, e.g., [17]. The reason is that the conductances we work with are matrix-valued and not necessarily invertible. That is, since resistance is the inverse of conductance, we would have run into certain difficulties had we chosen to employ effective resistance instead. Potential applications of generalized electrical circuits with matrix-valued parameters seem to have so far gone unnoticed by the control theorists. Notable exceptions are the works [2] on the problem of estimation over networks.

2 Preliminaries and notation

In this section we provide the formal definitions for the observability of an array and the connectivity of an $n$-graph through its Laplacian matrix. (The reader should be warned that the term $n$-graph has appeared in the literature in different meanings. In this paper it means a weighted graph, where each pair of vertices is assigned an $n$-by-$n$ matrix.)

A pair $[(C_{ij})_{ij=1}^q, A]$ is meant to represent the array of $q$ identical systems

$$
\begin{align*}
\dot{x}_i & = Ax_i \\
y_{ij} & = C_{ij}(x_i - x_j), \\ i, j & = 1, 2, \ldots, q
\end{align*}
$$

(5a)

where $x_i \in \mathbb{C}^n$ is the state of the $i$th system with $A \in \mathbb{C}^{n \times n}$ and $y_{ij} \in \mathbb{C}^{m_{ij}}$ is the $ij$th relative output with $C_{ij} \in \mathbb{C}^{m_{ij} \times n}$. We let $C_{ii} = 0$. In our paper we will solely be studying the case $y_{ij}(t) \equiv 0$ for all $(i, j)$. Hence we suppose $C_{ij} = C_{ji}$ without loss of generality. The generality is not lost because if $C_{ij} \neq C_{ji}$ then we can always redefine $C_{ij}^{new} = C_{ij}^{new} := [C_{ij}^{T} C_{ji}^{T}]^T$; and then $y_{ij}^{new}(t) \equiv 0$ for all $(i, j)$ if and only if $y_{ij}(t) \equiv 0$ for all $(i, j)$. The ordered collection $(C_{ij})_{ij=1}^q$ will sometimes be compactly written as $(C_{ij})$ when there is no risk of ambiguity.

4
For each \((i, j)\) we denote by \(W_{ij}\) the observability matrix of the individual pair \([C_{ij}, A]\). Namely,

\[
W_{ij} = \begin{bmatrix}
C_{ij} \\
C_{ij}A \\
\vdots \\
C_{ij}A^{n-1}
\end{bmatrix}.
\]

The associated unobservable subspace is denoted by \(U_{ij} \subset \mathbb{C}^n\). Recall that \(U_{ij} = \text{null } W_{ij}\) and that \(U_{ij}\) is invariant under \(A\). In particular, \(x_i(0) - x_j(0) \in U_{ij}\) implies \(x_i(t) - x_j(t) \in U_{ij}\) for all \(t\) since \(x_i(t) - x_j(t) = e^{\lambda t}(x_i(0) - x_j(0))\). By \(\mu_1, \mu_2, \ldots, \mu_n (1 \leq m \leq n)\) we denote the distinct eigenvalues of \(A\). By \(V_\sigma \in \mathbb{C}^{n \times n}\), \(\sigma = 1, 2, \ldots, m\), we denote a full column rank matrix satisfying range \(V_\sigma = \text{null } [A - \mu_\sigma I_n]\), where \(I_n\) is the \(n\)-by-\(n\) identity matrix. Note that the columns of \(V_\sigma\) are the linearly independent eigenvectors of \(A\) corresponding to the eigenvalue \(\mu_\sigma\). In particular, we have \(AV_\sigma = \mu_\sigma V_\sigma\).

The below definition is what this paper is all about.

**Definition 1** An array \([[C_{ij}, A]]\) is said to be observable if

\[
y_{ij}(t) \equiv 0 \text{ for all } (i, j) \implies x_{ij}(t) \equiv x_{ij}(t) \text{ for all } (i, j)
\]

for all initial conditions \(x_{ij}(0), x_{ij}(0), \ldots, x_{ij}(0)\).

An \(n\)-graph \(\Gamma = (V, w)\) has a finite set of vertices \(V = \{v_1, v_2, \ldots, v_q\}\) and a weight function \(w : V \times V \to \mathbb{C}^{n \times n}\) with the properties

- \(w(v, v) = 0\),
- \(w(u, v) = w(v, u)\),
- \(w(u, v) = w(u, v)^* \geq 0\),

where \(w(u, v)^*\) indicates the conjugate transpose of \(w(u, v)\). Let \(G_{ij} = w(v_i, v_j)\). The \(nq\)-by-\(nq\) matrix

\[
lap \Gamma = \begin{bmatrix}
G_{11} & -G_{12} & \cdots & -G_{1q} \\
-G_{21} & G_{22} & \cdots & -G_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
-G_{q1} & -G_{q2} & \cdots & G_{qq}
\end{bmatrix}
\]

is called the Laplacian of \(\Gamma\). Let \(L = \lap \Gamma\). By construction the Laplacian is Hermitian, i.e., \(L^* = L\), and enjoys some other desirable properties. Let \(\xi = [x^T_1 x^T_2 \cdots x^T_q]^T\) with \(x_i \in \mathbb{C}^n\) and define the synchronization subspace as \(S_n = \{\xi \in (\mathbb{C}^n)^q : x_i = x_j \text{ for all } (i, j)\}\). We see that null \(L \supset S_n\). Also, since we can write \(\xi^* \in \xi = \sum_{i>j}(x_i-x_j)^*G^{ij}(x_i-x_j) \geq 0\), the Laplacian is positive semidefinite. Therefore all its eigenvalues are real and nonnegative, thanks to which the ordering \(\lambda_1(L) \leq \lambda_2(L) \leq \ldots \leq \lambda_{nq}(L)\) is not meaningless. In the sequel, \(\lambda_k(L)\) denotes the \(k\)th smallest eigenvalue of \(L\).

We denote by \(\Gamma(H_{ij})_{i,j=1}^{n}\) (or by \(\Gamma(H_{ij})\) when there is no risk of confusion) the \(n\)-graph of a collection \((H_{ij})_{i,j=1}^{n}\) with \(H_{ji} = H_{ij} \in \mathbb{C}^{m_{ij} \times n}\) and \(H_{ii} = 0\). The graph \(\Gamma(H_{ij})\) has the vertex set \(V = \{v_1, v_2, \ldots, v_q\}\) and its weight function \(w\) is such that \(w(v_i, v_j) = H_{ij}^*H_{ij}\). Regarding the array \([\text{5}]\) two graph constructions are particularly important. One of them is the \(n\)-graph \(\Gamma(W_{ij})_{i,j=1}^{n}\) which we call the interconnection graph. The other is the \(n\)-graph \(\Gamma(C_{ij}V_\sigma)_{i,j=1}^{n}\), called the eigengraph corresponding to the eigenvalue \(\mu_\sigma\).

In graph theory \([4]\), connectivity (in the classical sense) is characterized by means of adjacency. A connected graph is said to have a path between each pair of its vertices, where a path is a sequence of adjacent vertices. For \(1\)-graphs the definition of adjacency is unequivocal: a pair of vertices \((u, v)\) are adjacent if \(w(u, v) \geq 0\) and nonadjacent if \(w(u, v) = 0\). (Adjacent vertices are said to have an edge between them.) For \(n\)-graphs \((n \geq 2)\) however, since we have the in-between semidefinite case \(w(u, v) \geq 0\), how to define adjacency and, in turn, connectivity becomes a matter of choice. For our purposes in this paper we (inevitably) abandon the concept of adjacency altogether and define connectivity of a graph through its Laplacian. Recall that a \(1\)-graph \(\Gamma\) is connected if and only if \(\lambda_2(\lap \Gamma) > 0\). Since this is an equivalence result it can replace the definition of connectivity for \(1\)-graphs. This substitute turns out to be much easier to generalize than the standard definition that uses paths.
**Definition 2** An n-graph $\Gamma$ is said to be connected if $\lambda_{n+1}(\text{lap } \Gamma) > 0$.

The next three facts will find frequent use later in the paper.

**Lemma 1** An n-graph $\Gamma$ is connected if and only if null $\text{lap } \Gamma = S_n$.

**Proof.** Let $L = \text{lap } \Gamma$ denote the Laplacian. Suppose null $L = S_n$. By definition $\dim S_n = n$. Therefore $L$ has $n$ linearly independent eigenvectors whose eigenvalues are zero. Since $L^* = L$ this means that $L$ has exactly $n$ eigenvalues at the origin. That all the eigenvalues of $L$ are nonnegative then yields $\lambda_{n+1}(L) > 0$. To show the other direction this time we begin by letting $\lambda_{n+1}(L) > 0$. That is, $L$ has at most $n$ eigenvalues at the origin. The property $L^* = L$ then implies that $L$ has at most $n$ eigenvectors whose eigenvalues are zero. In other words, $\dim \text{null } L \leq n$. This implies, in the light of the facts null $L \supseteq S_n$ and $\dim S_n = n$, that null $L = S_n$. $\blacksquare$

**Lemma 2** Consider the solutions $x_i(t)$ of the array (5). Let $\xi(t) = [x_1(t)^T \ x_2(t)^T \ \cdots \ x_n(t)^T]^T$ and $L = \text{lap } \Gamma(W_{ij})$. We have

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) \iff L\xi(t) \equiv 0 \iff L\xi(0) = 0.$$ 

**Proof.** Let $\zeta = [\rho_1^T \ \rho_2^T \ \cdots \ \rho_n^T]^T$ with $\rho_i \in \mathbb{C}^n$. Observe $\sum_{i>j} ||W_{ij}(\rho_i - \rho_j)||^2 = \sum_{i>j}(\rho_i - \rho_j)^*W_{ij}(\rho_i - \rho_j) = \zeta^*L\zeta$. Since $L^* = L \geq 0$ we also have $\zeta^*L\zeta = 0$ if and only if $L\zeta = 0$. Recalling $U_{ij} = \text{null } W_{ij}$ we can now write

$$\rho_i - \rho_j \in U_{ij} \text{ for all } (i, j) \iff W_{ij}(\rho_i - \rho_j) = 0 \text{ for all } (i, j) \iff \sum_{i>j} ||W_{ij}(\rho_i - \rho_j)||^2 = 0 \iff \sum_{i>j} ||W_{ij}(\rho_i - \rho_j)||^2 = 0 \iff \zeta^*L\zeta = 0 \iff L\zeta = 0.$$ 

Therefore

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) \iff x_i(0) - x_j(0) \in U_{ij} \text{ for all } (i, j) \iff L\xi(0) = 0.$$ 

Recall that $x_i(0) - x_j(0) \in U_{ij}$ implies $x_i(t) - x_j(t) \in U_{ij}$ for all $t$. Hence

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) \iff x_i(0) - x_j(0) \in U_{ij} \text{ for all } (i, j) \iff x_i(t) - x_j(t) \in U_{ij} \text{ for all } (i, j) \text{ and } t \iff L\xi(t) \equiv 0$$

which completes the proof. $\blacksquare$

**Lemma 3** Let $\sigma \in \{1, 2, \ldots, m\}$ and $L = \text{lap } \Gamma(W_{ij})$. The graph $\Gamma(C_{ij}V_\sigma)$ is not connected if and only if there exists a vector $\zeta \notin S_n$ satisfying $\zeta \in \text{range } [I_q \otimes V_\sigma] \cap \text{null } L$.

**Proof.** Given $\sigma$, let us suppose that $\Gamma(C_{ij}V_\sigma)$ is not connected. Since null $L_{\sigma} \supseteq S_{n_{\sigma}}$, by Lemma 1 there exists $\eta \in (\mathbb{C}^{n_{\sigma}})^q$ that satisfies $\eta \notin S_{n_{\sigma}}$ and $L_{\sigma}\eta = 0$, where $L_{\sigma} = \text{lap } \Gamma(C_{ij}V_\sigma)$. Let us employ the partition $\eta = [z_1^T \ z_2^T \ \cdots \ z_q^T]^T$ with $z_i \in \mathbb{C}^{n_{\sigma}}$ and define $\zeta \in (\mathbb{C}^n)^q$ as $\zeta = [I_q \otimes V_\sigma]\eta$. That is, $\zeta = [(V_{\sigma}z_1)^T \ (V_{\sigma}z_2)^T \ \cdots \ (V_{\sigma}z_q)^T]^T$. Clearly, $\zeta \in \text{range } [I_q \otimes V_\sigma]$. Since $V_{\sigma}$ is full column rank, $\eta \notin S_{n_{\sigma}}$ yields $\zeta \notin S_n$. Lastly, we have to establish $\zeta \in \text{null } L$. Recall $AV_{\sigma} = \mu_{\sigma}V_{\sigma}$. Therefore

$$V_{\sigma}W_{ij}^*W_{ij}V_{\sigma} = V_{\sigma}^* \left( C_{ij}^*C_{ij} + A^*C_{ij}^*C_{ij}A + \cdots + A^{(n-1)*}C_{ij}^*C_{ij}A^{n-1} \right) V_{\sigma} = V_{\sigma}^* \left( C_{ij}^*C_{ij} + \mu_{\sigma}C_{ij}^*C_{ij}\mu_{\sigma} + \cdots + \mu_{\sigma}^{(n-1)*}C_{ij}^*C_{ij}\mu_{\sigma}^{n-1} \right) V_{\sigma} = \left( 1 + |\mu_{\sigma}|^2 + \cdots + |\mu_{\sigma}|^{2(n-1)} \right) V_{\sigma}^*C_{ij}^*C_{ij}V_{\sigma}.$$
Now we can proceed as
\[
\zeta^* L\zeta = \sum_{i>j}(V_\sigma z_i - V_\sigma z_j)^* W_{ij}^* W_{ij} (V_\sigma z_i - V_\sigma z_j)
\]
\[
= \sum_{i>j}(z_i - z_j)^* V_\sigma^* W_{ij}^* V_\sigma (z_i - z_j)
\]
\[
= (1 + |\mu_\sigma|^2 + \cdots + |\mu_\sigma|^{2(n-1)}) \sum_{i>j}(z_i - z_j)^* V_\sigma^* C_{ij}^* C_{ij} V_\sigma (z_i - z_j)
\]
\[
= \left(1 + |\mu_\sigma|^2 + \cdots + |\mu_\sigma|^{2(n-1)}\right) \eta^* L_\sigma \eta
\]
\[
= 0.
\]
Since $L$ is Hermitian positive semidefinite, $\zeta^* L\zeta = 0$ implies $L\zeta = 0$, i.e., $\zeta \in \text{null } L$.

To show the other direction, suppose that there exists $\zeta \notin S_n$ satisfying $\zeta \in \text{range } [I_q \otimes V_\sigma]$ and $L\zeta = 0$. Since $\zeta \in \text{range } [I_q \otimes V_\sigma]$ we can find $\eta = [z_1^T \ z_2^T \ \cdots \ z_q^T]^T$ with $z_i \in \mathbb{C}^{n_\sigma}$ satisfying $\zeta = [I_q \otimes V_\sigma] \eta$. And since $V_\sigma$ is full column rank $\zeta \notin S_n$ implies $\eta \notin S_{n_\sigma}$. Now we turn the same wheels as in the first part, but in the opposite direction.

\[
\eta^* L_\sigma \eta = \sum_{i>j}(z_i - z_j)^* V_\sigma^* C_{ij}^* C_{ij} V_\sigma (z_i - z_j)
\]
\[
= \left(1 + |\mu_\sigma|^2 + \cdots + |\mu_\sigma|^{2(n-1)}\right)^{-1} \sum_{i>j}(z_i - z_j)^* V_\sigma^* W_{ij}^* W_{ij} V_\sigma (z_i - z_j)
\]
\[
= \left(1 + |\mu_\sigma|^2 + \cdots + |\mu_\sigma|^{2(n-1)}\right)^{-1} (V_\sigma z_i - V_\sigma z_j)^* W_{ij}^* W_{ij} (V_\sigma z_i - V_\sigma z_j)
\]
\[
= \left(1 + |\mu_\sigma|^2 + \cdots + |\mu_\sigma|^{2(n-1)}\right)^{-1} \zeta^* L\zeta
\]
\[
= 0.
\]
Since $L_\sigma$ is Hermitian positive semidefinite, $\eta^* L_\sigma \eta = 0$ implies $L_\sigma \eta = 0$, i.e., $\eta \in \text{null } L_\sigma$. This allows us to assert null $L_\sigma \neq S_{n_\sigma}$ because $\eta \notin S_{n_\sigma}$. Then by Lemma 1 the graph $\Gamma(C_{ij} V_\sigma)$ is not connected.

3 Observability and connectivity

In this section we establish the equivalence between observability and connectivity. Then we present a corollary on an interesting special case followed by a relevant numerical example. We end the section with a theorem on detectability. Below is our main result.

**Theorem 1** The following are equivalent.

1. The array $[(C_{ij}), A]$ is observable.
2. The interconnection graph $\Gamma(W_{ij})$ is connected.
3. All the eigengraphs $\Gamma(C_{ij} V_1), \Gamma(C_{ij} V_2), \ldots, \Gamma(C_{ij} V_m)$ are connected.

**Proof.** $1 \implies 2$. Suppose that $\Gamma(W_{ij})$ is not connected. Hence null $L \neq S_n$ by Lemma 1 where $L = \text{lap } \Gamma(W_{ij})$. Since null $L \supset S_n$ there must exist a vector $\zeta \in (\mathbb{C}^n)^g$ that satisfies both $\zeta \notin S_n$ and $L\zeta = 0$. Choose the initial conditions $x_1(0)$ of the systems (10) so as to satisfy $[x_1(0)^T \ x_2(0)^T \ \cdots \ x_q(0)^T]^T = \zeta$. Then by Lemma 2 we have $y_{ij}(t) \equiv 0$ for all $(i, j)$. However there exists at least one pair $(k, \ell)$ for which $x_k(t) \neq x_\ell(t)$ because $[x_1(0)^T \ x_2(0)^T \ \cdots \ x_q(0)^T]^T \notin S_n$. Hence the array $[(C_{ij}), A]$ cannot be observable.

$2 \implies 3$. Suppose that $\Gamma(C_{ij} V_\sigma)$ is not connected for some $\sigma \in \{1, 2, \ldots, m\}$. Then by Lemma 3 there exists $\zeta \in (\mathbb{C}^n)^g$ that satisfies $\zeta \notin S_n$ and $L\zeta = 0$. That is, null $L \neq S_n$. This implies by Lemma 1 that $\Gamma(W_{ij})$ is not connected.

$3 \implies 1$. Suppose that the array $[(C_{ij}), A]$ is not observable. Then we can find some initial conditions $x_1(0), x_2(0), \ldots, x_q(0)$ for which the solutions $x_i(t)$ of the systems (10) yield $y_{ij}(t) \equiv 0$ for all $(i, j)$ and $x_k(t) \neq x_\ell(t)$ for some $(k, \ell)$. 


Let \( \xi(t) = [x_1(t)^T \; x_2(t)^T \; \cdots \; x_q(t)^T]^T \). By Lemma 2, we have \( L\xi(0) = 0 \) because \( y_{ij}(t) \equiv 0 \) for all \( (i, j) \). We also have \( x_k(0) \neq x_i(0) \) because \( x_k(t) \neq x_i(t) \). Hence \( \xi(0) \notin S_n \). Combining \( L\xi(0) = 0 \) and \( \xi(0) \notin S_n \) (in the light of \( \text{null } L \supset S_n \)) implies that \( \text{null } L \) is a strict superset of \( S_n \). Let \( \dim \text{null } L = n + p \). (Note that \( p \geq 1 \).) Let \( S \in \mathbb{C}^{(nq) \times n} \) and \( U \in \mathbb{C}^{(nq) \times p} \) be two full column rank matrices satisfying \( \text{range } S = S_n \) and \( \text{range } [S \; U] = \text{null } L \). Recall that the unobservable subspaces \( U_{ij} = \text{null } W_{ij} \) are invariant with respect to the matrix \( A \). As a consequence \( \text{null } L \) is invariant with respect to the matrix \( [I_q \otimes A] \). To see that let \( \zeta = [\rho_1^T \; \rho_2^T \; \cdots \; \rho_q^T]^T \) with \( \rho_i \in \mathbb{C}^n \). We can write

\[
\zeta \in \text{null } L \implies \sum_{i>j} \|W_{ij}(\rho_i - \rho_j)\|^2 = \zeta^*L\zeta = 0
\]

\[
\implies (\rho_i - \rho_j) \in \text{null } W_{ij} \text{ for all } (i, j)
\]

\[
\implies A(\rho_i - \rho_j) \in \text{null } W_{ij} \text{ for all } (i, j)
\]

\[
\implies \sum_{i>j} \|W_{ij}(A\rho_i - A\rho_j)\|^2 = \zeta^*[I_q \otimes A]^*L[I_q \otimes A]\zeta = 0
\]

\[
\implies [I_q \otimes A]\zeta \in \text{null } L
\]

where for the last implication we use the fact that \( L \) is Hermitian positive semidefinite. Now, due to invariance, there have to exist matrices \( \Omega \in \mathbb{C}^{n \times p} \) and \( \Lambda \in \mathbb{C}^{p \times p} \) that satisfy

\[
[I_q \otimes A]U = S\Omega + U\Lambda.
\]

Let \( f \in \mathbb{C}^p \) be an eigenvector of \( \Lambda \) with eigenvalue \( \lambda \in \mathbb{C} \), i.e., \( Af = \lambda f \). Also, let \( \zeta_1 = Uf \) and \( \zeta_2 = S\Omega f \). Note that \( \zeta_1 \notin S_n \) and \( \zeta_2 \in S_n \). Now we can write

\[
([I_q \otimes A] - \lambda I_{nq})\zeta_1 = [I_q \otimes A]Uf - \lambda Uf = [I_q \otimes A]Uf - U\lambda f = ([I_q \otimes A]U - U\Lambda)f = S\Omega f = \zeta_2.
\]

Let us employ the partitions \( \zeta_1 = [\bar{\rho}_1^T \; \bar{\rho}_2^T \; \cdots \; \bar{\rho}_q^T]^T \) with \( \bar{\rho}_i \in \mathbb{C}^n \) and \( \zeta_2 = [\bar{\rho}^T \; \bar{\rho}^T \; \cdots \; \bar{\rho}^T]^T \) with \( \bar{\rho} \in \mathbb{C}^n \). Then \( ([I_q \otimes A] - \lambda I_{nq})\zeta_1 = \zeta_2 \) yields \( (A - \lambda I_n)\bar{\rho}_i = \bar{\rho} \) for all \( i \in \{1, 2, \ldots, q\} \). Choose an arbitrary index \( a \in \{1, 2, \ldots, q\} \) and define \( \zeta_3 = [\bar{\rho}_a^T \; \bar{\rho}_a^T \; \cdots \; \bar{\rho}_a^T]^T \) and \( \zeta_4 = \zeta_1 - \zeta_3 \). Note that \( \zeta_3 \in S_n \) and \( \zeta_4 \notin S_n \). Moreover, since both \( \zeta_1 \) and \( \zeta_3 \) belong to \( \text{null } L \), we have \( L\zeta_4 = 0 \). Now observe

\[
([I_q \otimes A] - \lambda I_{nq})\zeta_4 = ([I_q \otimes A] - \lambda I_{nq})(\zeta_1 - \zeta_3) = \zeta_2 - ([I_q \otimes A] - \lambda I_{nq})\zeta_3 = \begin{bmatrix} \bar{\rho} \\ \vdots \\ \bar{\rho} \end{bmatrix} - \begin{bmatrix} (A - \lambda I_n)\bar{\rho}_a \\ \vdots \\ (A - \lambda I_n)\bar{\rho}_a \end{bmatrix} = 0.
\]

Let \( [\bar{\rho}_1^T \; \bar{\rho}_2^T \; \cdots \; \bar{\rho}_q^T]^T = \zeta_4 \). We can write

\[
\begin{bmatrix} (A - \lambda I_n)\bar{\rho}_1 \\ \vdots \\ (A - \lambda I_n)\bar{\rho}_q \end{bmatrix} = ([I_q \otimes A] - \lambda I_{nq})\zeta_4 = 0.
\]

Therefore \( \lambda = \mu_\sigma \) for some \( \sigma \in \{1, 2, \ldots, m\} \) and every nonzero \( \bar{\rho}_i \) is an eigenvector of \( A \). In particular, for each \( \bar{\rho}_i \) there uniquely exists \( z_i \in \mathbb{C}^{n_\sigma} \) such that \( \bar{\rho}_i = V_\sigma z_i \). By stacking these \( z_i \) into \( \eta = [z_1^T \; \cdots \; z_q^T]^T \) we have \( \zeta_4 = [I_q \otimes V_\sigma]\eta \). Recall that we have already obtained \( \zeta_4 \notin S_n \) and \( L\zeta_4 = 0 \). Hence Lemma 3 assures us that the eigengraph \( \Gamma(C_{ij} V_\sigma) \) is not connected. 

Theorem 4 has an interesting implication concerning 1-graphs. Let \( d_A(s) \) and \( m_A(s) \) respectively denote the characteristic polynomial and the minimal polynomial of the matrix \( A \). Note that if each \( V_\sigma \) consists of a single column, i.e., each eigenvector \( \mu_\sigma \) has a unique (up to a scaling) eigenvector, then all the eigengraphs \( \Gamma(C_{ij} V_1), \Gamma(C_{ij} V_2), \ldots, \Gamma(C_{ij} V_m) \) become 1-graphs. A sufficient condition for this is that the eigenvalues \( \mu_1, \mu_2, \ldots, \mu_m \) are all simple, i.e., \( m = n \). More generally:
Corollary 1 Suppose \( m_A(s) = d_A(s) \). Then the array \([\{C_{ij}\}, A]\) is observable if and only if all the 1-graphs \( \Gamma(C_{ij}V_1), \Gamma(C_{ij}V_2), \ldots, \Gamma(C_{ij}V_m) \) are connected.

An example. Consider the array \([\{C_{ij}\}_{i,j=1}^4, A]\) with

\[
A = \begin{bmatrix}
0 & 1 & -7 & -14 & 21 & 31 \\
1 & 1 & 1 & 3 & -7 & -11 \\
3 & 6 & -28 & -43 & 7 & 5 \\
-2 & -4 & 18 & 28 & -7 & -7 \\
-2 & -4 & -2 & 1 & -32 & -49 \\
1 & 2 & 3 & 2 & 20 & 31 \\
\end{bmatrix}
\]

and

\[
C_{12} = \begin{bmatrix} 2 & 3 & 8 & 12 & 6 & 10 \end{bmatrix}, \quad C_{23} = \begin{bmatrix} 2 & 3 & 4 & 6 & 6 & 9 \end{bmatrix}, \quad C_{34} = \begin{bmatrix} 4 & 6 & 6 & 10 & 6 & 9 \end{bmatrix}, \quad C_{41} = \begin{bmatrix} 1 & 2 & 6 & 9 & 4 & 7 \end{bmatrix}.
\]

The matrices \( C_{13} \) and \( C_{24} \) are zero. (Recall \( C_{ij} = C_{ji} \) and \( C_{ii} = 0 \).) The characteristic polynomial of \( A \) reads \( d_A(s) = s^6 - s^2 = s^2(s - 1)(s + 1)(s + j) \). That is, \( A \) has \( m = 5 \) distinct eigenvalues: \( \mu_1 = 0, \mu_2 = 1, \mu_3 = -1, \mu_{4,5} = \pm j \). The eigenvalue at the origin is repeated, yet \( \dim \text{null } A = 1 \). Therefore there is a single eigenvector corresponding to \( \mu_1 \). Consequently we have \( m_A(s) = d_A(s) \). The matrices (or, in this case, vectors) \( V_c \) corresponding to the eigenvalues \( \mu_c \) are given below.

\[
V_1 = \begin{bmatrix} -5 \\ 2 \\ -4 \\ 3 \\ 5 \\ -3 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ -1 \\ -5 \\ 3 \\ -4 \\ 3 \end{bmatrix}, \quad V_3 = \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \quad V_{4,5} = \begin{bmatrix} -17 \\ 4 \\ -19 \\ 14 \\ 22 \\ -13 \end{bmatrix} + j \begin{bmatrix} 0 \\ 1 \\ 8 \\ 5 \\ -3 \\ 1 \end{bmatrix}.
\]

Now let us determine whether the array \([\{C_{ij}\}, A]\) is observable or not. Thanks to Corollary 1 we can do this by checking the connectivities of the 1-graphs \( \Gamma(C_{ij}V_1), \Gamma(C_{ij}V_2), \ldots, \Gamma(C_{ij}V_5) \). A pleasant thing about a 1-graph is that its connectivity can be read from its visual representation. In this universal picture every vertex is represented by a dot and a line (called an edge) is drawn connecting a pair of vertices \((v_i, v_j)\) if the value of the weight function (which is scalar for a 1-graph) is positive, i.e., \( w(v_i, v_j) > 0 \). Then the graph is connected if we can reach from any dot to any other dot by tracing the lines. Our 1-graphs are shown in Fig. 5. Clearly, all of them are connected. Hence we conclude that the array \([\{C_{ij}\}, A]\) is observable. Note that for some nonzero \( C_{ij} \) certain edges \((v_i, v_j)\) are missing. For instance,

\[
\Gamma(C_{ij}V_1) \quad \Gamma(C_{ij}V_2) \quad \Gamma(C_{ij}V_3) \quad \Gamma(C_{ij}V_4) \quad \Gamma(C_{ij}V_5)
\]

Figure 5: Connected eigengraphs.

for the graph \( \Gamma(C_{ij}V_1) \) the edge \((v_1, v_2)\) is absent despite \( C_{12} \neq 0 \). The reason is that \( \mu_1 = 0 \) is an unobservable eigenvalue for the pair \([C_{12}, A]\). In particular, we have \( C_{12}V_1 = 0 \), i.e., the eigenvector \( V_1 \) belongs to the null space of \( C_{12} \). Interestingly, the intersection of the graphs in Fig. 5 yields an empty set of edges. This is because there is not a single pair \([C_{ij}, A]\) that is observable. Still, that does not prevent the array \([\{C_{ij}\}, A]\) from being observable.

A few words on a practical issue. As is well known, sometimes the system designer may have to settle upon less than observability, where \( y_{ij}(t) \equiv 0 \) need not imply the desired \( x_i(t) \equiv x_j(t) \), but rather guarantees only \( \|x_i(t) - x_j(t)\| \to 0 \) as \( t \to \infty \). This suggests to slacken Definition 1 a bit.
Definition 3  An array \([C_{ij}], A\) is said to be detectable if
\[ y_{ij}(t) \equiv 0 \text{ for all } (i, j) \implies \|x_i(t) - x_j(t)\| \to 0 \text{ for all } (i, j) \]
for all initial conditions \(x_1(0), x_2(0), \ldots, x_q(0)\).

Since (for a continuous-time system) the terms in the solution related to the eigenvalues of \(A\) that are on the open left half-plane will die out eventually, detectability is assured if the eigenvalues on the closed right half-plane are observable. More formally:

Theorem 2  The array \([C_{ij}], A\) is detectable if and only if all the eigengraphs \(\Gamma(C_{ij}V_\sigma)\) with \(\text{Re}\mu_\sigma \geq 0\) are connected.

Proof. Con. \(\implies\) Det. Suppose that \([C_{ij}], A\) is not detectable. Then we can find some initial conditions \(x_1(0), x_2(0), \ldots, x_q(0)\) for which the solutions \(x_i(t)\) of the systems (5) yield
\[ y_{ij}(t) \equiv 0 \text{ for all } (i, j) \text{ and } \|x_i(t) - x_j(t)\| \not\to 0 \]
for some pair \((k, \ell)\). Let \(\xi(t) = [x_1(t)^T \ x_2(t)^T \ \cdots \ x_q(t)^T]^T\). Note that each \(x_i(t)\) solves \(\dot{x}_i = Ax_i\). Also, recall that \(\mu_1, \mu_2, \ldots, \mu_m\) denote the distinct eigenvalues of \(A\). Therefore \(\xi(t)\) solves \(\dot{\xi} = [I_q \otimes A]\xi\) and enjoys the structure
\[ \xi(t) = p_1(t)e^{\mu_1 t} + p_2(t)e^{\mu_2 t} + \cdots + p_m(t)e^{\mu_m t} \]
for some polynomials \(p_1(t), p_2(t), \ldots, p_m(t)\) whose coefficients are vectors in \((\mathbb{C}^n)^q\). Let \(L = \text{Lap}(\Gamma(W_{ij}))\) and observe \([(e_k - e_\ell) \otimes I_q]^T\xi(t) = x_k(t) - x_\ell(t)\). By Lemma 2 we have \(L\xi(t) \equiv 0\) because \(y_{ij}(t) \equiv 0\) for all \((i, j)\). We also have \([(e_k - e_\ell) \otimes I_q]^T\xi(t) \not\to 0\) because \(\|x_k(t) - x_\ell(t)\| \not\to 0\). Let us here make a few observations. Since \(\mu_1, \mu_2, \ldots, \mu_m\) are distinct, the collection of mappings \(\{t \mapsto p_\sigma(t)e^{\mu_\sigma t} : p_\sigma(t) \neq 0, \sigma = 1, 2, \ldots, m\}\) are linearly independent. Therefore \(L\xi(t) \equiv 0\) implies
\[ Lp_\sigma(t)e^{\mu_\sigma t} = 0 \] (6)
for all \(\sigma\). Moreover, \([(e_k - e_\ell) \otimes I_q]^T\xi(t) \not\to 0\) implies
\[ [(e_k - e_\ell) \otimes I_q]^T p_\sigma(t)e^{\mu_\sigma t} \not\to 0 \] (7)
for some \(\sigma\). Finally, \(\dot{\xi}(t) = [I_q \otimes A]\xi(t)\) implies
\[ \frac{d}{dt}\{p_\sigma(t)e^{\mu_\sigma t}\} = [I_q \otimes A] p_\sigma(t)e^{\mu_\sigma t} \] (8)
for all \(\sigma\). Let us now fix an index \(\sigma \in \{1, 2, \ldots, m\}\) that satisfies (7). Clearly, we have \(\text{Re}\mu_\sigma \geq 0\). Let \(p_\sigma(t) = \vartheta_0 t^r + \cdots + \vartheta_1 t + \vartheta_0\) with \(\vartheta_0, \vartheta_1, \ldots, \vartheta_r \in (\mathbb{C}^n)^q\) and \(\vartheta_r \neq 0\). By (5) we can write \(p_\sigma(t)e^{\mu_\sigma t} = [I_q \otimes e^{At}]\vartheta_0\). This implies
\[ [(e_k - e_\ell) \otimes I_q]^T \vartheta_0 \neq 0 \]
for otherwise \([(e_k - e_\ell) \otimes I_q]^T \vartheta_0 = 0\) we would have had
\[ [(e_k - e_\ell) \otimes I_q]^T p_\sigma(t)e^{\mu_\sigma t} = [(e_k - e_\ell) \otimes I_q]^T[I_q \otimes e^{At}]\vartheta_0 \]
\[ = [(e_k - e_\ell)^T e^{At}]\vartheta_0 \]
\[ = e^{At}[(e_k - e_\ell) \otimes I_q]^T \vartheta_0 \]
\[ = 0 \]
which contradicts (7). Note that the mappings \(t \mapsto \vartheta_0 e^{\mu_\sigma t}, t \mapsto \vartheta_1 t e^{\mu_\sigma t}, \ldots, t \mapsto \vartheta_r t^r e^{\mu_\sigma t}\) are linearly independent. Therefore (8) yields \(L\vartheta_\nu e^{\mu_\nu t} \equiv 0\) for all \(\nu \in \{0, 1, \ldots, r\}\). Consequently, \(L\vartheta_\nu = 0\) for all \(\nu \in \{0, 1, \ldots, r\}\).

Since \(t \mapsto (\vartheta_0 t^r + \cdots + \vartheta_1 t + \vartheta_0)e^{\mu_\sigma t}\) is a solution of \(\dot{\xi} = [I_q \otimes A]\xi\) we have the following chain
\[ ([I_q \otimes A] - \mu_\sigma I_q)\vartheta_0 = \vartheta_1 \]
\[ ([I_q \otimes A] - \mu_\sigma I_q)\vartheta_1 = 2\vartheta_2 \]
\[ \vdots \]
\[ ([I_q \otimes A] - \mu_\sigma I_q)\vartheta_{r-1} = r\vartheta_r \]
\[ ([I_q \otimes A] - \mu_\sigma I_q)\vartheta_r = 0. \]
Let us now fix an index \( \nu \in \{0, 1, \ldots, r\} \) that satisfies \( \vartheta_{\nu} \notin S_\nu \) and \( ([I_\nu \otimes A] - \mu_\sigma I_{nq}) \vartheta_{\nu} \in S_\nu \). Such \( \nu \) should exist because \( \vartheta_0 \notin S_\nu \) (thanks to \( (e_k - e_\ell) \otimes I_n^T \vartheta_0 \neq 0 \)) and \( ([I_\nu \otimes A] - \mu_\sigma I_{nq}) \vartheta_\tau = 0 \in S_\nu \). Let \( \zeta_1 = [\hat{\rho}_1^T \hat{\rho}_2^T \cdots \hat{\rho}_q^T]^T = \vartheta_{\nu} \) with \( \hat{\rho}_i \in \mathbb{C}^n \) and \( \zeta_2 = [\hat{\rho}_1^T \hat{\rho}_2^T \cdots \hat{\rho}_q^T]^T = ([I_\nu \otimes A] - \mu_\sigma I_{nq}) \vartheta_{\nu} \) with \( \hat{\rho} \in \mathbb{C}^n \). Then \( ([I_\nu \otimes A] - \mu_\sigma I_{nq}) \zeta_1 = \zeta_2 \) yields \( (A - \mu_\sigma I_n) \hat{\rho}_i = \hat{\rho} \) for all \( i \in \{1, 2, \ldots, q\} \). Choose an arbitrary index \( a \in \{1, 2, \ldots, q\} \) and define \( \zeta_3 = [\hat{\rho}_a^T \hat{\rho}_1^T \cdots \hat{\rho}_{a-1}^T \hat{\rho}_{a+1}^T \cdots \hat{\rho}_q^T]^T \) and \( \zeta_4 = \zeta_1 - \zeta_3 \). Note that \( \zeta_3 \in S_\nu \) and \( \zeta_4 \notin S_\nu \). Moreover, since both \( \zeta_1 \) and \( \zeta_3 \) belong to null \( L \), we have \( L\zeta_4 = 0 \). Observe

\[
([I_\nu \otimes A] - \mu_\sigma I_{nq}) \zeta_4 = ([I_\nu \otimes A] - \mu_\sigma I_{nq}) (\zeta_1 - \zeta_3) = \zeta_2 - ([I_\nu \otimes A] - \mu_\sigma I_{nq}) \zeta_4 = \begin{bmatrix} \hat{\rho} \\ \vdots \\ \hat{\rho} \end{bmatrix} - \begin{bmatrix} \begin{bmatrix} A - \mu_\sigma I_n \end{bmatrix} \hat{\rho}_a \\ \vdots \\ \begin{bmatrix} A - \mu_\sigma I_n \end{bmatrix} \hat{\rho}_a \end{bmatrix} = 0.
\]

Let \( [\hat{\rho}_1^T \hat{\rho}_2^T \cdots \hat{\rho}_q^T]^T = \zeta_4 \). We can write

\[
\begin{bmatrix} (A - \mu_\sigma I_n) \hat{\rho}_1 \\ \vdots \\ (A - \mu_\sigma I_n) \hat{\rho}_q \end{bmatrix} = ([I_\nu \otimes A] - \mu_\sigma I_{nq}) \zeta_4 = 0.
\]

Therefore every nonzero \( \hat{\rho}_i \) is an eigenvector of \( A \) with eigenvalue \( \mu_\sigma \). In particular, for each \( \rho_i \) there uniquely exists \( z_i \in \mathbb{C}^{n_e} \) such that \( \rho_i = V_{\nu} z_i \). By stacking these \( z_i \) into \( \eta = [z_1^T z_2^T \cdots z_q^T]^T \) we have \( \zeta_4 = ([I_\nu \otimes V_\sigma] \eta \). Recall that we have already obtained \( \zeta_4 \notin S_\nu \) and \( L\zeta_4 = 0 \). Hence Lemma \( 3 \) assures us that the eigengraph \( \Gamma(C_{ij} V_\sigma) \) is not connected.

**Det. \( \implies \text{Con.} \)** Suppose that for some \( \sigma \in \{1, 2, \ldots, m\} \) the graph \( \Gamma(C_{ij} V_\sigma) \) is not connected and \( \text{Re} \mu_\sigma \geq 0 \). By Lemma \( 3 \) we know that we can find \( \zeta \in (\mathbb{C}^n)^q \) which can be written as \( \zeta = ([I_\nu \otimes V_\sigma] \eta \) for some \( \eta \in (\mathbb{C}^{n_e})^q \) while satisfying \( L\zeta = 0 \) and \( \zeta \notin S_\nu \). Choose the initial conditions \( x_i(0) \) of the systems \( 5 \) so as to satisfy \( [x_1(0)^T x_2(0)^T \cdots x_q(0)^T]^T = \zeta \). Note that

\[
\begin{bmatrix} A x_1(0) \\ \vdots \\ A x_q(0) \end{bmatrix} = ([I_\nu \otimes A] \zeta = [I_\nu \otimes A] [I_\nu \otimes V_\sigma] \eta = [I_\nu \otimes AV_\sigma] \eta = [I_\nu \otimes \mu_\sigma V_\sigma] \eta = \mu_\sigma x_1(0) = \begin{bmatrix} \mu_\sigma x_1(0) \\ \vdots \\ \mu_\sigma x_q(0) \end{bmatrix}.
\]

That is, each nonzero \( x_i(0) \) is an eigenvector of \( A \) with eigenvalue \( \mu_\sigma \). Therefore we have \( x_i(t) = x_i(0)e^{\mu_\sigma t} \) for all \( i \in \{1, 2, \ldots, q\} \). Since \( L\zeta = 0 \) the solutions \( x_i(t) \) yield by Lemma \( 2 \) \( y_{ij}(t) \equiv 0 \) for all \( (i, j) \). However, there exists some pair \( (k, \ell) \) for which \( x_k(0) \neq x_\ell(0) \) because \( [x_1(0)^T x_2(0)^T \cdots x_q(0)^T]^T = \zeta \notin S_\nu \). Moreover, we can write

\[
||x_k(t) - x_\ell(t)|| = ||x_k(0)e^{\mu_\sigma t} - x_\ell(0)e^{\mu_\sigma t}|| = ||e^{\mu_\sigma t}|| \cdot ||x_k(0) - x_\ell(0)|| \neq 0
\]

because \( \text{Re} \mu_\sigma \geq 0 \). Therefore the array \( [(C_{ij}), A] \) cannot be detectable.

### 4 Pairwise observability and effective conductance

Hitherto, regarding the array \( 5 \), we have focused solely on the total synchronization, i.e., \( x_i(t) \equiv x_j(t) \) for all pairs \( (i, j) \). Henceforth we consider partial synchronization, where certain pairs of systems are possibly out of synchrony. The reason that this problem is worth tackling is threefold. First, it puts the previous analysis on a more complete footing. Second, for applications where total synchronization is desired but not achieved, it is meaningful to want to determine how far off we are from the goal. Third, it is not difficult to imagine situations where total synchronization is not desired for reasons of security. For instance, in a secure communication scenario there may be an array with many identical systems among which there is a pair of distant units that are desired to synchronize without synchronizing with their immediate neighbors. These motivate the following definition.
Definition 4 An array \([\{C_{ij}\}, A]\) is said to be \((k, \ell)\)-observable if

\[
y_{ij}(t) = 0 \text{ for all } (i, j) \implies x_k(t) = x_\ell(t)
\]

for all initial conditions \(x_1(0), x_2(0), \ldots, x_q(0)\).

And the sister definition reads:

Definition 5 An array \([\{C_{ij}\}, A]\) is said to be \((k, \ell)\)-detectable if

\[
y_{ij}(t) = 0 \text{ for all } (i, j) \implies \|x_k(t) - x_\ell(t)\| \to 0
\]

for all initial conditions \(x_1(0), x_2(0), \ldots, x_q(0)\).

As before, where we studied the observability of an array by means of the connectivity of interconnection graph, we will again approach the problem from graphic angle. Recall that a 1-graph is connected when the null space of its Laplacian is spanned by the vector of all ones, which led us to the generalization stated in Lemma \([\mathbf{H}]\). Likewise, a pair of vertices \((v_k, v_\ell)\) of a 1-graph is connected if any vector that belongs to the null space of the Laplacian is with identical \(k\)th and \(\ell\)th entries. We now obtain the natural generalization of pairwise connectivity for \(n\)-graphs. Let \(\Gamma\) be an \(n\)-graph with \(q\) vertices. Let \(e_i \in \mathbb{C}^q\) be the unit vector with \(i\)th entry one and the remaining entries zero. For \(n = 1\) the connectedness of the pair \((v_k, v_\ell)\) is equivalent to the condition \((e_k - e_\ell) \in (\text{null lap } \Gamma)\). This suggests:

Definition 6 An \(n\)-graph \(\Gamma\) is said to be \((k, \ell)\)-connected if range\([(e_k - e_\ell) \otimes I_n]\) \(\subset (\text{null lap } \Gamma)^\perp\).

It may seem reasonable to expect that Theorem \([\mathbf{I}]\) and Theorem \([\mathbf{II}]\) of the previous section can be effortlessly converted into “pairwise” statements by simply replacing the words observable, detectable, connected therein with \((k, \ell)\)-observable, \((k, \ell)\)-detectable, \((k, \ell)\)-connected, respectively. Surprisingly enough this is not the case; certain associations disappear in the pairwise domain. In particular, an array that is not \((k, \ell)\)-observable may still have all its eigengraphs \((k, \ell)\)-connected. We now proceed by establishing the remaining links. Then we provide evidence (counterexample) for the missing implications.

Theorem 3 The array \([\{C_{ij}\}, A]\) is \((k, \ell)\)-observable if and only if the \(n\)-graph \(\Gamma(W_{ij})\) is \((k, \ell)\)-connected.

Proof. Con. \(\implies\) Obs. Suppose that \(\Gamma(W_{ij})\) is \((k, \ell)\)-connected. Consider the solutions \(x_i(t)\) of the systems \([3]\). Let \(L = \text{lap } \Gamma(W_{ij})\) be the Laplacian and \(\xi(t) = [x_1^T(t) x_2^T(t) \cdots x_q^T(t)]^T\). Since \(\Gamma(W_{ij})\) is \((k, \ell)\)-connected we have range\([(e_k - e_\ell) \otimes I_n]\) \(\subset (\text{null } L)^\perp = \text{range } L\) because \(L\) is Hermitian. Therefore we can find a matrix \(R \in \mathbb{C}^{(q+\ell) \times n}\) such that \(LR = (e_k - e_\ell) \otimes I_n\). Then by Lemma \([\mathbf{II}]\) we can write

\[
y_{ij}(t) \equiv 0 \text{ for all } (i, j) \implies L\xi(t) \equiv 0
\]

\[
\implies R^*L\xi(t) \equiv 0
\]

\[
\implies [(e_k - e_\ell) \otimes I_n]^T \xi(t) \equiv 0
\]

\[
\implies x_k(t) \equiv x_\ell(t).
\]

Hence the array \([\{C_{ij}\}, A]\) is \((k, \ell)\)-observable.

Obs. \(\implies\) Con. Suppose that \(\Gamma(W_{ij})\) is not \((k, \ell)\)-connected. Then there must exist a vector \(\zeta \in (\mathbb{C}^n)^q\) that satisfies both \(L\zeta = 0\) and \([(e_k - e_\ell) \otimes I_n]^T \zeta \neq 0\). Choose the initial conditions \(x_i(0)\) of the systems \([3]\) so as to satisfy \([x_1(0)^T x_2(0)^T \cdots x_q(0)^T]^T = \zeta\). Then by Lemma \([\mathbf{II}]\) we have \(y_{ij}(t) \equiv 0\) for all \((i, j)\). However, \(x_k(t) \neq x_\ell(t)\) because \([(e_k - e_\ell) \otimes I_n]^T [x_1(0)^T x_2(0)^T \cdots x_q(0)^T]^T \neq 0\). I.e., the array \([\{C_{ij}\}, A]\) cannot be \((k, \ell)\)-observable.

The proofs of the next two theorems are almost identical. We therefore prove only the latter.

Theorem 4 If the array \([\{C_{ij}\}, A]\) is \((k, \ell)\)-observable then all the eigengraphs \(\Gamma(C_{ij}V_x)\) are \((k, \ell)\)-connected.

Theorem 5 If the array \([\{C_{ij}\}, A]\) is \((k, \ell)\)-detectable then all the eigengraphs \(\Gamma(C_{ij}V_\sigma)\) with \(\text{Re } \mu_\sigma \geq 0\) are \((k, \ell)\)-connected.
Proof. Suppose that for some \( \sigma \in \{1, 2, \ldots, m\} \) the graph \( \Gamma(C_{ij}V_\sigma) \) is not \((k, \ell)\)-connected and \( \Re \mu_\sigma \geq 0 \). Then there exists \( \eta \in (\mathbb{C}^n_\sigma)^q \) that satisfies \([e_k - e_\ell] \otimes I_{n_\sigma}]^T \eta \neq 0 \) and \( L_\sigma \eta = 0 \), where \( L_\sigma = \text{lap} \Gamma(C_{ij}V_\sigma) \). Let \( \zeta \in (\mathbb{C}^n_\sigma)^q \) be defined as \( \zeta = [I_q \otimes V_\sigma] \eta \). We can write
\[
[(e_k - e_\ell) \otimes I_{n_\sigma}]^T \zeta = [(e_k - e_\ell) \otimes I_{n_\sigma}]^T [I_q \otimes V_\sigma] \eta = V_\sigma [(e_k - e_\ell) \otimes I_{n_\sigma}]^T \eta \neq 0
\]
because \([e_k - e_\ell] \otimes I_{n_\sigma}]^T \eta \neq 0 \) and the matrix \( V_\sigma \) is full column rank. Let \( L = \text{lap} \Gamma(W_{ij}) \). Recall that we have \( \zeta^T L \zeta = (1 + |\mu_\sigma|^2 + \cdots + |\mu_\sigma|^{2(n-1)}) \eta^T L_\sigma \eta \) (see the proof of Lemma 3). Hence \( L_\sigma \eta = 0 \) implies \( L \zeta = 0 \) because \( L \) is Hermitian positive semidefinite. Let us choose now the initial conditions \( x_i(0) \) of the systems \( \dot{x}_i = A x_i \) so as to satisfy \([x_1(0)^T x_2(0)^T \cdots x_q(0)^T]^T = \zeta \). Recall that each nonzero \( x_i(0) \) has to be an eigenvector of \( A \) with eigenvalue \( \mu_\sigma \) because \( \zeta = [I_q \otimes V_\sigma] \eta \) (see the proof of Theorem 2). Therefore we have \( x_i(t) = x_i(0)e^{\mu_\sigma t} \) for all \( i \). Since \( L \zeta = 0 \) Lemma 2 gives us \( y_{ij} (t) \equiv 0 \) for all \( (i, j) \). However, we have \( x_k(0) \neq x_\ell(0) \) because \([e_k - e_\ell] \otimes I_{n_\sigma}]^T \zeta \neq 0 \). Moreover, we can write
\[
\|x_k(t) - x_\ell(t)\| = \|x_k(0)e^{\mu_\sigma t} - x_\ell(0)e^{\mu_\sigma t}\| = \|e^{\mu_\sigma t}\| \cdot \|x_k(0) - x_\ell(0)\| \neq 0
\]
because \( \Re \mu_\sigma \geq 0 \). Therefore the array \([C_{ij}], A\] cannot be \((k, \ell)\)-detectable. \( \Box \)

As mentioned earlier, an array that is not \((k, \ell)\)-observable may still have all its eigengraphs \((k, \ell)\)-connected. This we find counterintuitive. Hence a counterexample here is appropriate.

A counterexample. Consider the pair \([C_{ij}]_{i,j=1, A}\) with
\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]
and
\[
C_{12} = \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad C_{23} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad C_{31} = \begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
\]

(Recall \( C_{ij} = C_{ji} \) and \( C_{ii} = 0 \).) Clearly, \( A \) has a single \((m = 1)\) distinct eigenvalue \( \mu_1 = 0 \). The corresponding matrix \( V_1 \) satisfying \( V_1 = \text{null} [A - \mu_1 I_4] \) reads
\[
V_1 = \begin{bmatrix}
1 \\
0 \\
0 \\
0
\end{bmatrix}.
\]

Now consider the solutions
\[
x_1(t) = \begin{bmatrix}
t \\
1 \\
0 \\
0
\end{bmatrix}, \quad x_2(t) = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}, \quad x_3(t) = \begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix}.
\]

It is not difficult to check that these \( x_i(t) \) satisfy \( \dot{x}_i = Ax_i \) as well as \( C_{ij} (x_i(t) - x_j(t)) \equiv 0 \) for all \((i, j)\). Noting \( x_2(t) \neq x_3(t) \) we conclude therefore that the array is not \((2, 3)\)-observable. Now let us see what the eigengraphs say on the matter. In fact, the 2-graph \( \Gamma(C_{ij}V_1) \) is the only eigengraph of the array. The associated Laplacian can be computed to equal
\[
\text{lap} \Gamma(C_{ij}V_1) = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]
whose null space is spanned by the columns of the matrix $N$ given below

$$N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$  

Observe $[(e_2 - e_3) \otimes I_2]^T N = 0$. Hence range $[(e_2 - e_3) \otimes I_2] \subset (\text{null lap } \Gamma(C_{ij} V_1))^\perp$. That is, the eigengraph $\Gamma(C_{ij} V_1)$ is $(2, 3)$-connected despite the fact that the array $[(C_{ij}), A]$ is not $(2, 3)$-observable.

To better understand $(k, \ell)$-connectivity we now direct our attention to circuit theory, which has been a fruitful source of ideas for graph theory. A significant example has been reported in [10] where the effective resistance between two nodes of a resistive network is shown to be a meaningful tool to measure distance between two vertices of a graph. This work inspires us to employ effective conductance over an $n$-graph, which we will eventually show to be closely related to pairwise observability. Let us however first remember the definition of effective conductance for a 1-graph, which will be our starting point for generalization. Let $\Gamma$ be a 1-graph with $q$ vertices and weight function $w$. Then lap $\Gamma$ equals the node admittance matrix of a resistive network $N$ with $q$ nodes, where the resistor connecting the nodes $i$ and $j$ has the conductance value $w(v_i, v_j) = g_{ij}$ (in mhos). For the network $N$ the effective conductance $\gamma_{k\ell}$ between the nodes $k$ and $\ell$ is equal to the value of the current (in amps) leaving a 1-volt voltage source connected to the node $k$ while the node $\ell$ is grounded, see Fig. 6. Regarding the case depicted in Fig. 6 observe that the effective conductance $\gamma_{14}$ satisfies the following equation

$$\text{lap } \Gamma \begin{bmatrix} 1 \\ x_2 \\ x_3 \\ 0 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \gamma_{14},$$

where $x_i \in \mathbb{R}$ denote the appropriate node voltages. Imitating this equation yields:

Figure 6: The effective conductance between the first and fourth nodes equals the current $\gamma_{14}$.

**Definition 7** Given an $n$-graph $\Gamma$ with set of vertices $V = \{v_1, v_2, \ldots, v_q\}$, the effective conductance $\Gamma_{k\ell} \in \mathbb{C}^{n \times n}$ associated to the pair of distinct vertices $(v_k, v_\ell)$ satisfies

$$\text{lap } \Gamma \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} = (e_k - e_\ell) \otimes \Gamma_{k\ell} \quad \text{subject to} \quad X_k = I_n \text{ and } X_\ell = 0$$

for some $X_i \in \mathbb{C}^{n \times n}$.
It is not evident that this definition is unambiguous. Hence we have to make sure that effective conductance always exists, preferably uniquely. To be able to do this we need to introduce some notation. Let \( e_{k\ell} \in \mathbb{C}^{q(q-2)} \) denote the matrix obtained from the identity matrix \( I_q \) by removing the \( k \)th and \( \ell \)th columns, i.e., \( e_{k\ell} = [e_1 \cdots e_{k-1} e_{k+1} \cdots e_{\ell-1} e_{\ell+1} \cdots e_q] \). Furthermore, for \( L = \text{lap} \Gamma \), where \( \Gamma \) is an \( n \)-graph with \( q \) vertices we adopt the following shortcuts.

\[
L_{k\ell,k\ell} = [e_{k\ell} \otimes I_n]^T L [e_{k\ell} \otimes I_n],
L_{k\ell,k} = [e_{k\ell} \otimes I_n]^T L [e_{k} \otimes I_n],
L_{k,k\ell} = [e_{k} \otimes I_n]^T L [e_{k\ell} \otimes I_n],
L_{k,\ell} = [e_{k} \otimes I_n]^T L [e_{\ell} \otimes I_n].
\]

Lastly, \((L_{k\ell,k\ell})^+\) indicates the pseudo-inverse of \( L_{k\ell,k\ell} \) and \( 1_q \in \mathbb{C}^q \) is the vector of all ones.

**Proposition 1** Let \( \Gamma \) be an \( n \)-graph with set of vertices \( V = \{v_1, v_2, \ldots, v_q\} \) and \( L = \text{lap} \Gamma \). For each pair of (distinct) vertices \((v_k, v_\ell)\) the effective conductance \( \Gamma_{k\ell} \) uniquely exists and satisfies \( \Gamma_{k\ell} = \Gamma_{\ell k} = \Gamma^*_{k\ell} \geq 0 \). In particular,

\[
\Gamma_{k\ell} = L_{k\ell,k} - L_{k\ell,k\ell} (L_{k\ell,k\ell})^+ L_{k\ell,k}.
\]

**Proof.** Let \((v_k, v_\ell)\) be given. For \( q = 2 \) the result follows trivially, since \( \Gamma_{12} = w(v_1, v_2) \), where \( w \) is the weight function associated to \( \Gamma \). In the sequel we will consider the case \( q \geq 3 \).

**Existence.** We first show that \((9)\) can always be solved. For \( E \in \mathbb{C}^{n(q-2) \times n} \) consider the equation

\[
L_{k\ell,k\ell}E + L_{k\ell,k} = 0. \tag{10}
\]

Note that a solution \( E \) exists for \((10)\) if range \( L_{k\ell,k\ell} \supset \text{range} \ L_{k\ell,k} \), which is equivalent to null \( L_{k\ell,k\ell}^* \subset \text{null} \ L_{k\ell,k} \) since \( (L_{k\ell,k\ell})^* = L_{k\ell,k\ell} \) and \( (L_{k\ell,k})^* = L_{k,k\ell} \). Let us now take an arbitrary \( \eta \in \mathbb{C}^{n(q-2)} \) satisfying \( L_{k\ell,k\ell}\eta = 0 \) and define \( \zeta = [e_{k\ell} \otimes I_n]\eta \). We can write

\[
\zeta^* L \zeta = \eta^* [e_{k\ell} \otimes I_n]^T L [e_{k\ell} \otimes I_n] \eta = \eta^* L_{k\ell,k\ell} \eta = 0,
\]

which implies \( L \zeta = 0 \) since \( L \) is Hermitian positive semidefinite. Observe

\[
L_{k,k\ell} \eta = [e_k \otimes I_n]^T L [e_{k\ell} \otimes I_n] \eta = [e_k \otimes I_n]^T L \zeta = 0.
\]

Therefore null \( L_{k\ell,k\ell} \subset \text{null} \ L_{k,k\ell} \) and we can indeed find \( E \) satisfying \((10)\).

Choose now some \( E \) satisfying \((10)\) and define

\[
\Gamma_{k\ell} = L_{k,k\ell}E + L_{k,k\ell}. \tag{11}
\]

Observe that we have \( [1_q \otimes I_n]^T L = 0 \) since null \( L \supset S_n \) and \( L^* = L \). Also, \( e_k + e_\ell = 1_q - e_{k\ell} 1_{q-2} \). We can now write using \((10)\) and \((11)\)

\[
L_{k,k\ell}E + L_{k,k} = [e_\ell \otimes I_n]^T (L [e_{k\ell} \otimes I_n]E + L [e_k \otimes I_n])
= [(1_q - e_{k\ell} 1_{q-2} - e_k) \otimes I_n]^T (L [e_{k\ell} \otimes I_n]E + L [e_k \otimes I_n])
= [1_q \otimes I_n]^T L ([e_{k\ell} \otimes I_n]E + [e_k \otimes I_n])
= -[1_{q-2} \otimes I_n]^T L_{k\ell,k\ell} -[1_{q-2} \otimes I_n]^T L_{k\ell,k} + [1_{q-2} \otimes I_n]^T L_{k,k\ell} -[1_{q-2} \otimes I_n]^T L_{k,k} + [1_q \otimes I_n]^T L_{k\ell,k\ell} -[1_q \otimes I_n]^T L_{k\ell,k} + [1_q \otimes I_n]^T L_{k,k\ell} -[1_q \otimes I_n]^T L_{k,k} = -\Gamma_{k\ell}. \tag{12}
\]
Let us define $\Xi = [X_1^T X_2^T \cdots X_q^T]^T$ with $X_i \in \mathbb{C}^{n \times n}$ as $\Xi = [e_{k\ell} \otimes I_n]E + [e_k \otimes I_n]$. We claim that this choice $\Xi$ and $\Gamma_{kl}$ defined in (11) together satisfy (9), i.e.,

$$L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl} = 0 \quad \text{subject to} \quad X_k = I_n \text{ and } X_\ell = 0.$$

Note that $X_k = [e_k \otimes I_n]^T \Xi = [e_k \otimes I_n]^T ([e_{k\ell} \otimes I_n]E + [e_k \otimes I_n]) = I_n$ since $e_{k\ell}^T e_{k\ell} = 0$ and $e_k^T e_k = 1$. Likewise, $X_\ell = [e_\ell \otimes I_n]^T \Xi = 0$ since $e_\ell^T e_\ell = 0$ and $e_{k\ell}^T e_{k\ell} = 0$. To establish $L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl} = 0$ it suffices that we show $[e_i \otimes I_n]^T (L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) = 0$ for all $i \in \{1, 2, \ldots, q\}$. Let $i \neq k, \ell$. Then we can write $[e_i \otimes I_n]^T = [e_i \otimes I_n]^T [e_{k\ell} \otimes I_n] [e_{k\ell} \otimes I_n]^T$. Hence, using (10) and $e_k^T (e_\ell - e_k) = 0$ we obtain

$$[e_i \otimes I_n]^T (L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) = [e_i \otimes I_n]^T [e_{k\ell} \otimes I_n] [(e_{k\ell} \otimes I_n)^T (L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl})$$

$$= [e_i \otimes I_n]^T [e_{k\ell} \otimes I_n] [(L_{k\ell,k\ell} E + L_{k\ell,k\ell})$$

$$= 0.$$ 

Let $i = k$. Using (11) and $e_k^T (e_\ell - e_k) = -1$ we can write

$$[e_k \otimes I_n]^T (L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) = L_{k,k\ell} E + L_{kk,k\ell} - \Gamma_{kl} = 0.$$

Finally, let $i = \ell$. Using (12) and $e_\ell^T (e_\ell - e_k) = 1$ we reach

$$[e_\ell \otimes I_n]^T (L\Xi + (e_\ell - e_k) \otimes \Gamma_{kl}) = L_{k\ell,k\ell} E + L_{\ell\ell,k\ell} + \Gamma_{kl} = 0.$$

**Hermitian positive semidefiniteness.** Let $[X_1^T X_2^T \cdots X_q^T]^T = \Xi$ with $X_i \in \mathbb{C}^{n \times n}$ and $\Gamma_{kl} \in \mathbb{C}^{n \times n}$ satisfy (9). Since $X_k = I_n$ and $X_\ell = 0$ we have $\Xi^* [(e_k - e_\ell) \otimes \Gamma_{kl}] = \Gamma_{kl}$. We can write

$$\Gamma_{kl} = \Xi^* [(e_k - e_\ell) \otimes \Gamma_{kl}]$$

$$= \Xi^* L\Xi.$$

Since $L$ is Hermitian positive semidefinite so is $\Xi^* L\Xi$.

**Uniqueness.** Let $[X_1^T X_2^T \cdots X_q^T]^T = \Xi$ with $X_i \in \mathbb{C}^{n \times n}$ and $\Gamma_{kl} \in \mathbb{C}^{n \times n}$ satisfy (9). Suppose that $\Gamma_{kl}$ is not unique. Then we can find $[\Xi_1^T \Xi_2^T \cdots \Xi_q^T]^T = \Xi$ and $\Gamma_{kl} \neq \Gamma_{kl}$ that also satisfy (9). Recall $X_k = \tilde{X}_k = I_n$ and $X_\ell = \tilde{X}_\ell = 0$. Using $L^* = L$ and $\Gamma_{kl}^* = \Gamma_{kl}$ one can generate the following contradiction

$$\tilde{\Gamma}_{kl} = \Xi^*[ (e_k - e_\ell) \otimes \tilde{\Gamma}_{kl}]$$

$$= (L\Xi)^* \Xi = [(e_k - e_\ell) \otimes \Gamma_{kl}]^* \Xi = \Gamma_{kl}^*$$

Thanks to uniqueness we can combine (10) and (11) to write the expression for effective conductance

$$\Gamma_{kl} = L_{k,k\ell} - L_{k\ell,k\ell} (L_{k\ell,k\ell})^+ L_{k\ell,k\ell}.$$

**Reciprocity.** Let $[X_1^T X_2^T \cdots X_q^T]^T = \Xi$ with $X_i \in \mathbb{C}^{n \times n}$ and $\Gamma_{kl} \in \mathbb{C}^{n \times n}$ satisfy (9). Define $\tilde{X}_i = I_n - X_i$ for $i = 1, 2, \ldots, q$. Note that $\tilde{X}_k = I_n$ and $\tilde{X}_\ell = 0$ because $X_\ell = 0$ and $X_k = I_n$. Let $\tilde{\Xi} = [\tilde{X}_1^T \tilde{X}_2^T \cdots \tilde{X}_q^T]^T$. Recall that we have $L[1_q \otimes I_n] = 0$. We can write

$$L\tilde{\Xi} = L[1_q \otimes I_n] - \Xi) = -L\Xi = (e_\ell - e_k) \otimes \Gamma_{kl}$$

which implies $\Gamma_{\ell k} = \Gamma_{k\ell}$.

Effective conductance turns out to be a definite indicator of pairwise connectivity. In particular, it allows us to improve Theorem (9).

**Theorem 6** The following are equivalent ($k \neq \ell$).

1. The array $[(C_{ij}), A]$ is $(k, \ell)$-observable.
2. The interconnection graph $\Gamma(W_{ij})$ is $(k, \ell)$-connected.
3. The effective conductance $\Gamma_{k\ell}(W_{ij})$ is full rank.
Proof. 1 $\iff$ 2. By Theorem \( \text{K} \).

$2 \implies 3$. Let us employ the shortcuts $\Gamma_{kl} = \Gamma_{kl}(W_{ij})$ and $L = \text{lap}(\Gamma_{ij})$. Suppose that $\Gamma(W_{ij})$ is $(k, \ell)$-connected. Then range $[(e_k - e_\ell) \otimes I_n] \subset (\text{null } L)^{\perp} = \text{range } L$ which implies that we can find matrices $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_q \in \mathbb{C}^{n \times n}$ such that

$$L \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} = (e_k - e_\ell) \otimes I_n.$$ 

Define $\hat{X}_i = \hat{X}_i - \hat{X}_\ell$ for $i = 1, 2, \ldots, q$. Note that $\hat{X}_\ell = 0$ and we can write

$$L \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} = L \left( \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} - [I_q \otimes I_n] \hat{X}_\ell \right) = (e_k - e_\ell) \otimes I_n$$

since $L[I_q \otimes I_n] = 0$. Recalling that $L$ is Hermitian positive semidefinite we proceed as follows

$$n = \text{rank } [(e_k - e_\ell) \otimes I_n] = \text{rank } \left( L \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} \right) = \text{rank } \left( \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix}^* L \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} \right).$$

Then we write (recalling $\hat{X}_\ell = 0$)

$$\begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix}^* L \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} = \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix}^* [(e_k - e_\ell) \otimes I_n] = \hat{X}_k^{-1}.$$ 

Combining (14) and (15) we deduce that $\hat{X}_k$ is nonsingular. This allows us to define $X_i = \hat{X}_i \hat{X}_k^{-1}$ for $i = 1, 2, \ldots, q$. Note that $X_k = I_n$ and $X_\ell = 0$. Revisiting (13) we can write

$$L \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} = L \begin{bmatrix} \hat{X}_1 \\ \vdots \\ \hat{X}_q \end{bmatrix} \hat{X}_k^{-1} = [(e_k - e_\ell) \otimes I_n] \hat{X}_k^{-1} = (e_k - e_\ell) \otimes \hat{X}_k^{-1}$$

which implies (since $X_k = I_n$ and $X_\ell = 0$) that $\Gamma_{kl} = \hat{X}_k^{-1}$. Clearly, $\Gamma_{kl}$ is full rank.

$3 \implies 2$. Suppose $\text{rank } \Gamma_{kl} = n$. Then $\Gamma_{kl}^{-1}$ exists. Recall that $\Gamma_{kl}$ satisfies (9) for some $X_1$, $X_1$, $\ldots$, $X_q \in \mathbb{C}^{n \times n}$. We can write

$$L \begin{bmatrix} X_1 \Gamma_{kl}^{-1} \\ \vdots \\ X_q \Gamma_{kl}^{-1} \end{bmatrix} = L \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} \Gamma_{kl}^{-1} = [(e_k - e_\ell) \otimes \Gamma_{kl}] \Gamma_{kl}^{-1} = (e_k - e_\ell) \otimes I_n.$$ 

Hence range $[(e_k - e_\ell) \otimes I_n] \subset \text{range } L = (\text{null } L)^{\perp}$. That is, $\Gamma(W_{ij})$ is $(k, \ell)$-connected.

Finally, we present a result on detectability, which can be considered as an extension of the well-known PBH test.

Theorem 7 The array $[(C_{ij}), A]$ is $(k, \ell)$-detectable $(k \neq \ell)$ if and only if

$$\text{rank } \begin{bmatrix} A - I_n \\ \Gamma_{kl}(W_{ij}) \end{bmatrix} = n \quad \text{for all } \text{Re } \lambda \geq 0.$$ 

Proof. Suppose that $[(C_{ij}), A]$ is not $(k, \ell)$-detectable. Then we can find some initial conditions $x_1(0), x_2(0), \ldots, x_q(0)$ for which the solutions $x_i(t)$ of the systems $[5]$ yield

$$y_{ij}(t) \equiv 0 \text{ for all } (i, j) \text{ and } ||x_k(t) - x_\ell(t)|| \not\to 0.$$ 

Earlier we have discovered (in the proof of Theorem 2) that (17) implies the existences of an eigenvalue $\mu_\sigma$ in the closed left-half plane ($\text{Re } \mu_\sigma \geq 0$) and vectors $\vartheta_0, \vartheta_1, \ldots, \vartheta_r \in \mathbb{C}^n$ satisfying:
where $L = \text{lap} \Gamma(W_{ij})$. Let us now fix an index $\nu \in \{0, 1, \ldots, r\}$ that satisfies $((e_k - e_\ell) \otimes I_n)^T \vartheta_\nu \neq 0$ and $[(e_k - e_\ell) \otimes I_n]^T ((I_q \otimes A) - \mu_\sigma I_{nq}) \vartheta_\nu = 0$. Such $\nu$ should exist because $[(e_k - e_\ell) \otimes I_n]^T \vartheta_0 \neq 0$ and $((I_q \otimes A) - \mu_\sigma I_{nq}) \vartheta_r = 0 \in \text{null} [(e_k - e_\ell) \otimes I_n]^T$. Let $[\rho_1^T, \rho_2^T, \ldots, \rho_q^T]^T = \vartheta_\nu$ with $\rho_i \in \mathbb{C}^n$. Then let $\mu_i = \bar{\rho}_i - \bar{\rho}_\ell$ for $i = 1, 2, \ldots, q$ and $\zeta_i = [\rho_1^T, \rho_2^T, \ldots, \rho_q^T]^T$. Observe that $\mu_k = \bar{\rho}_k - \bar{\rho}_\ell = [(e_k - e_\ell) \otimes I_n]^T \vartheta_\nu \neq 0$ and $\mu_\ell = 0$. Moreover, we can write
\[
[A - \mu_\sigma I_n] \rho_k = \left[A - \mu_\sigma I_n \right] ((e_k - e_\ell) \otimes I_n)^T \vartheta_\nu
\]
\[
= [(e_k - e_\ell) \otimes I_n]^T ((I_q \otimes A) - \mu_\sigma I_{nq}) \vartheta_\nu
\]
\[
= 0 .
\]
Hence $\mu_k$ is an eigenvector of $A$ with eigenvalue $\mu_\sigma$. Lastly, $L \zeta_1 = L \vartheta_\nu - [\rho_1^T, \rho_2^T, \ldots, \rho_q^T]^T = 0$ because $L \vartheta_\nu = 0$ and $[\rho_1^T, \rho_2^T, \ldots, \rho_q^T]^T \in S_n \subset \text{null} L$. Let now $\Gamma_{kl} = \Gamma_{kl}(W_{ij})$ be the effective conductance satisfying (9) for some $X_1, X_2, \ldots, X_q \in \mathbb{C}^{n \times n}$. Also, note that $\zeta_1^* [(e_k - e_\ell) \otimes I_n] = \rho_k^* - \rho_\ell^* = \rho_k$. By (9) we can now write
\[
\rho_k \Gamma_{kl} \rho_k = \zeta_1^* [(e_k - e_\ell) \otimes I_n] \Gamma_{kl} \rho_k
\]
\[
= \zeta_1^* [(e_k - e_\ell) \otimes I_n] \Gamma_{kl} \rho_k
\]
\[
= \zeta_1^* \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} \rho_k
\]
\[
= 0
\]
since $\zeta_1^* L = (L \zeta_1)^* = 0$. By Proposition we can write $\Gamma_{kl} \rho_k = 0$. Therefore $\rho_k^* \Gamma_{kl} \rho_k = 0$ for all $k$. Since $[A - \mu_\sigma I_n] \rho_k = 0$ we can write
\[
\begin{bmatrix} A - \mu_\sigma I_n \\ \Gamma_{kl} \end{bmatrix} \rho_k = 0
\]
which implies that (10) fails.

To show the other direction this time suppose that (10) fails. This implies that we can find an eigenvalue $\mu_\sigma$ on the closed right half-plane $(\text{Re} \mu_\sigma \geq 0)$ and an eigenvector $\rho \in \mathbb{C}^n$ satisfying $\Gamma_{kl} \rho = 0$ and $[A - \mu_\sigma I_n] \rho = 0$. Let $X_1, X_2, \ldots, X_q \in \mathbb{C}^{n \times n}$ satisfy (9) and define $\zeta_2 = [\rho_1^T, \rho_2^T, \ldots, \rho_q^T]^T$ where $\bar{\rho}_i = X_i \bar{\rho}$. Note that $\bar{\rho}_k = \rho$ and $\bar{\rho}_\ell = 0$ because $X_k = I_n$ and $X_\ell = 0$. By (9) we can now write
\[
L \zeta_2 = L \begin{bmatrix} X_1 \\ \vdots \\ X_q \end{bmatrix} \rho = [(e_k - e_\ell) \otimes I_n] \rho = (e_k - e_\ell) \otimes (\Gamma_{kl} \rho) = 0 .
\]
Let the initial conditions $x_i(0)$ of the systems satisfy $[x_1(0)^T, x_2(0)^T, \ldots, x_q(0)^T]^T = \zeta_2$. Since $x_k(0) = \bar{\rho}_k = \rho$ is an eigenvector for the eigenvalue $\mu_\sigma$ we have $x_k(t) = \rho e^{\mu_\sigma t}$. Also, $x_\ell(0) = \bar{\rho}_\ell = 0$ implies $x_\ell(t) \equiv 0$. Now, since $L \zeta_2 = 0$ Lemma gives us $y_i(t) \equiv 0$ for all $(i, j)$. Moreover, we have
\[
\|x_k(t) - x_\ell(t)\| = \|\rho e^{\mu_\sigma t} - 0\| = |e^{\mu_\sigma t}| \cdot \|\rho\| \neq 0
\]
because $\text{Re} \mu_\sigma \geq 0$. Therefore the array $[[C_{ij}], A]$ cannot be $(k, \ell)$-detectable.

5 Conclusion

In this paper we studied the observability of an array of LTI systems with identical individual dynamics, where an array was called observable when identically zero relative outputs implied identical solutions for the individual systems. In our setup the relative output for each pair of units admitted a (possibly)
different matrix. This incommensurability of the output matrices made it necessary to study the observability of the array via the connectivity of a matrix-weighted interconnection graph instead of the usual scalar-weighted topologies. In the first part of the paper we established the equivalence between the observability of an array and the connectivity of its interconnection graph. In addition we showed that the observability of an array could be studied also through the connectivity of the so-called eigengraphs, each of which corresponded to a particular eigenvalue of the system matrix of the individual dynamics.

In the second part we investigated the pairwise observability of an array, where an array was called \((k, \ell)\)-observable when identically zero relative outputs implied identical solutions for the \(k\)th and \(\ell\)th individual systems. There too we addressed the problem from the graph connectivity point of view. Our findings were partially parallel to those in the first part. In particular, we obtained the equivalence between the \((k, \ell)\)-observability of an array and the \((k, \ell)\)-connectivity of its interconnection graph. However, in contrast with the first part, the \((k, \ell)\)-observability of an array was not in general guaranteed by the \((k, \ell)\)-connectivity of its eigengraphs. Moreover, we showed that pairwise observability could be studied via (matrix-valued) effective conductance, which was obtained from the interconnection graph by treating its Laplacian as the node admittance matrix of some resistive network, where nodes were connected by resistors with matrix-valued conductances. We found that an array was \((k, \ell)\)-observable if and only if the associated effective conductance was full rank.

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