On Coloring Properties of Graph Powers

Hossein Hajiabolhassan
*Department of Mathematical Sciences
Shahid Beheshti University, G.C.,
P.O. Box 19839-63113, Tehran, Iran
hhaji@sbu.ac.ir

Ali Taherkhani
Department of Mathematics
Institute for Advanced Studies in Basic Sciences
P.O. Box 45195-1159, Zanjan 45195, Iran
ali.taherkhani@iasbs.ac.ir

Abstract

This paper studies some coloring properties of graph powers. We show that
\( \chi_c(G^{2r+1}_{2s+1}) = (2s+1) \chi_c(G) \) provided that \( \chi_c(G^{2r+1}_{2s+1}) < 4 \). As a consequence,
one can see that if \( \frac{2r+1}{2s+1} \leq \frac{\chi_c(G)}{3(\chi_c(G)-2)} \), then \( \chi_c(G^{2r+1}_{2s+1}) = \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1} \).

In particular, \( \chi_c(K_{3n+1}^{\frac{1}{3}}) = \frac{9n+3}{3n+2} \) and \( K_{3n+1}^{\frac{1}{3}} \) has no subgraph with circular chromatic number equal to \( \frac{6n+1}{2n+1} \). This provides a negative answer to a question asked in [Xuding Zhu, Circular chromatic number: a survey, Discrete Math., 229(1-3):371–410, 2001]. Also, we present an upper bound for the fractional chromatic number of subdivision graphs. Precisely, we show that \( \chi_f(G^{2r+1}_{2s+1}) \leq \frac{(2s+1)\chi_f(G)}{s\chi_f(G)+1} \). Finally, we investigate the nth multichromatic number of subdivision graphs.

Keywords: graph homomorphism, circular coloring, fractional chromatic number, multichromatic number.

Subject classification: 05C

1 Introduction

It was shown in [7] that one can compute the fractional chromatic number of \( M(G) \) in terms of that of \( G \), where \( M(G) \) stands for the Mycielskian of \( G \). There are a few interesting and similar results for the circular chromatic number. Hence, it is of interest to find a map or a functor \( F \) from the category of graphs to itself such that, for any graph \( G \), it is possible to determine the exact value of the circular chromatic number of \( F(G) \) in terms of that of \( G \). In this paper, we show that graph powers can be considered as such functors (graph powers preserve the graph homomorphism).

In Section 1, we set up notation and terminology. Section 2 establishes the tight relation between the circular chromatic number and graph powers. In fact, we show that it is possible to determine the circular chromatic number of \( G^{2r+1}_{2s+1} \) in terms of that of \( G \) provided that \( \frac{2r+1}{2s+1} \) is sufficiently small. In Section 3, we investigate the fractional chromatic number and the nth multichromatic number of subdivision graphs.

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Throughout this paper we consider finite simple graphs which have no loops and multiple edges. For a given graph $G$, the notation $\text{og}(G)$ stands for the odd girth of $G$. We denote by $[m]$ the set $\{1, 2, \ldots, m\}$. Let $G$ and $H$ be two graphs. A homomorphism from $G$ to $H$ is a mapping $f : V(G) \rightarrow V(H)$ such that $f(u)f(v) \in E(H)$ whenever $uv \in V(G)$. We write $G \rightarrow H$ if there exists a homomorphism from $G$ to $H$. Two graphs $G$ and $H$ are homomorphically equivalent if $G \rightarrow H$ and $H \rightarrow G$ and it is indicated by the symbol $G \leftrightarrow H$.

Let $d$ and $n$ be positive integers, where $n \geq 2d$. The circular complete graph $K_{\frac{n}{d}}$ has the vertex set $\{0, 1, \ldots, n-1\}$ in which $ij$ is an edge if and only if $d \leq |i-j| \leq n-d$. An $(n, d)$–coloring of graph $G$ is a homomorphism from $G$ to the circular complete graph $K_{\frac{n}{d}}$. The circular chromatic number $\chi_c(G)$ of $G$ is defined as

$$\chi_c(G) = \inf \left\{ \frac{n}{d} \mid G \text{ admits an } (n, d) - \text{coloring} \right\}.$$ 

Two kinds of graph powers were introduced in [2, 3]. Especially, it was illustrated that there is a tight relationship between graph powers and the circular chromatic number. Also, the connection between graph homomorphism and graph powers has been studied in [2, 3, 12].

For a graph $G$, let $G^k$ be the $k$th power of $G$, which is obtained on the vertex set $V(G)$, by connecting any two vertices $u$ and $v$ for which there exists a walk of length $k$ between $u$ and $v$ in $G$. Also, assume that $G^{\frac{k}{d}}$ is the graph obtained by replacing each edge of $G$ with the path $P_{s+1}$. Set $G^{\frac{k}{d}} = (G^{\frac{k}{d}})^r$. This power, called fractional power as a functor, preserves the graph homomorphism. In this terminology, we have the following lemma.

**Lemma A.** [3] Let $r$ and $s$ be positive integers and $G$ be a graph. Then

$$G \rightarrow H \quad \Rightarrow \quad G^{\frac{k}{d}} \rightarrow H^{\frac{k}{d}}.$$ 

**Lemma B.** [3] Let $r$, $s$, $p$, and $q$ be non-negative integers and $G$ be a graph. Then

$$(G^g)^{\frac{2r+1}{2q+1}} \rightarrow G^{\frac{2r+1}{2q+1}} \rightarrow G^{\frac{2r+1}{2q+1}}.$$ 

For a given graph $G$ with $v \in V(G)$, set

$$N_i(v) = \{u \mid \text{there is a walk of length } i \text{ joining } u \text{ and } v\}.$$ 

For two subsets $A$ and $B$ of the vertex set of a graph $G$, we write $A \triangleright B$ if every vertex of $A$ is joined to every vertex of $B$. Also, for any non-negative integer $s$, define the graph $G^{\frac{1}{2+1}}$ as follows.

$$V(G^{\frac{1}{2+1}}) = \{ (A_1, \ldots, A_{s+1}) \mid A_i \subseteq V(G), |A_1| = 1, \emptyset \neq A_i \subseteq N_{i-1}(A_1), i \leq s+1 \}.$$ 

Two vertices $(A_1, \ldots, A_{s+1})$ and $(B_1, \ldots, B_{s+1})$ are adjacent in $G^{\frac{1}{2+1}}$ if for any $1 \leq i \leq s$ and $1 \leq j \leq s+1$, $A_i \subseteq B_{i+1}$, $B_i \subseteq A_{i+1}$, and $A_j \triangleright B_j$. Here is the
definition of dual power as a functor as follows. Let \( r \) and \( s \) be non-negative integers. For any graph \( G \) define the graph \( G^{\frac{2r+1}{2s+1}} \) as follows

\[
G^{\frac{2r+1}{2s+1}} = \left( G^{\frac{s+1}{r+1}} \right)^{2r+1}.
\]

These powers, in sense of graph homomorphism, inherit several properties from power in numbers.

**Lemma C.** \cite{3} Let \( r, p, \) and \( q \) be non-negative integers. For any graph \( G \) we have

a) \( G^{\frac{(2r+1)(2p+1)}{(2r+1)(2q+1)}} \leftrightarrow G^{\frac{2p+1}{2q+1}}. \)

b) \( G^{\frac{(2r+1)(2p+1)}{(2r+1)(2q+1)}} \leftrightarrow G^{\frac{2p+1}{2q+1}}. \)

It was proved in \cite{3} that these two powers are dual of each other as follows.

**Theorem A.** \cite{3} Let \( G \) and \( H \) be two graphs. Also, assume that \( \frac{2r+1}{2s+1} < \log(G) \) and \( 2s+1 < \log(H^{\frac{2r+1}{2s+1}}) \). We have

\[
G^{\frac{2r+1}{2s+1}} \rightarrow H \iff G \rightarrow H^{\frac{2s+1}{2r+1}}.
\]

Now, we consider the parameter \( \theta_i(G) \) which in some sense measures the homomorphism capabilities of \( G \).

**Definition 1.** Assume that \( G \) is a non-bipartite graph. Also, let \( i \geq -\chi(G) + 3 \) be an integer. The \( i \)th power thickness of \( G \) is defined as follows.

\[
\theta_i(G) = \sup \left\{ \frac{2r+1}{2s+1} \mid \chi(G^{\frac{2r+1}{2s+1}}) \leq \chi(G) + i, \frac{2r+1}{2s+1} < \log(G) \right\}.
\]

For simplicity, when \( i = 0 \), the parameter is called the power thickness of \( G \) and is denoted by \( \theta(G) \). Also, when \( i = \chi(G) - 3 \), we set \( \theta_{\chi(G)-3}(G) = \mu(G) \). \♠

**Lemma D.** \cite{3} Let \( G \) and \( H \) be two non-bipartite graphs with \( \chi(G) = \chi(H) - j, \ j \geq 0 \). If \( G \rightarrow H \) and \( i + j \geq -\chi(G) + 3 \), then

\[
\theta_{i+j}(G) \geq \theta_i(H).
\]

It is interesting that \( \mu(G) \) is computed in terms of circular chromatic number. Hence, \( \theta_i(G) \)'s can be considered as a generalization of circular chromatic number.

**Theorem B.** \cite{3} Let \( G \) be a non-bipartite graph. Then

\[
\mu(G) = \frac{\chi_c(G)}{3(\chi_c(G) - 2)}.
\]
Some properties of graph powers and its close relationship to the circular chromatic number of non-bipartite graphs have been studied in [3]. In particular, an equivalent definition of the circular chromatic number in terms of graph powers was introduced as follows.

**Theorem C.** [3] Let $G$ be a non-bipartite graph with chromatic number $\chi(G)$.

a) If $0 < \frac{2r+1}{2s+1} \leq \frac{\chi(G)}{3(\chi(G) - 2)}$, then $\chi(G^{2r+1}) = 3$. Furthermore, $\chi(G) \neq \chi_c(G)$ if and only if there exists a rational number $\frac{2r+1}{2s+1} > \frac{\chi(G)}{3(\chi(G) - 2)}$ for which $\chi(G^{2r+1}) = 3$.

b) $\chi_c(G) = \inf\{\frac{2n+1}{n-1} | \chi(G^{2n+1}) = 3, n > t > 0\}$.

Here, we show that if $2r + 1 < \log(K_n^d)$, then $K_n^{2r+1}$ is isomorphic to a circular complete graph.

**Lemma 1.** Let $n$ and $d$ be positive integers, where $n > 2d$.

a) If $r$ is a non-negative integer and $\frac{n}{d} < \frac{2r+1}{r}$, then $K_n^{2r+1} \simeq K_{\frac{n}{(2r+1)d-rn}}$.

b) If $s$ is a nonnegative integer, then $K_n^s \leftrightarrow K_{\frac{2r+1}{sn+d}}$.

**Proof.** Let $t \leq r$ be a non-negative integer. If $i$ is an arbitrary vertex of $K_n^d$, it is not hard to check that $N_{2r+1}(i) = \{i+(2t+1)d-tn, i+(2t+1)d-tn+1, \ldots, i-(2t+1)d+tn+1\}$, where the summation is modulo $n$. Therefore, $K_n^{2r+1}$ is isomorphic to the circular complete graph $K_{\frac{n}{(2r+1)d-rn}}$. The next part is an immediate consequence of part (a).

Now, we introduce an upper bound for the circular chromatic number of graph powers.

**Theorem 1.** Let $r$ and $s$ be non-negative integers and $G$ be a non-bipartite graph with circular chromatic number $\chi_c(G)$. If $\frac{2r+1}{2s+1} < \frac{\chi_c(G)}{\chi_c(G) - 2}$, then

$$\chi_c(G^{2r+1}) \leq \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G) + 2r+1}.$$

**Proof.** Let $\chi_c(G) = \frac{n}{d}$. It is easy to see that if $\frac{2r+1}{2s+1} < \frac{n}{d-2}$, then $\frac{(2s+1)n}{sn+d} < \frac{2r+1}{r}$.

$$G \rightarrow K_n^d \implies G^{\frac{2r+1}{2s+1}} \rightarrow (K_n^d)^{\frac{2r+1}{2s+1}} \quad \text{(By Lemma A)}$$

$$\implies G^{\frac{2r+1}{2s+1}} \rightarrow (K_{\frac{2s+1}{sn+d}}^{\frac{2r+1}{2s+1}}) \quad \text{(By Lemma I b)}$$
\[ G^{\frac{2r+1}{3r+1}} \longrightarrow K^{\frac{2r+1}{(2r+1)n}}_{\frac{3r+1}{3n+4}} \quad \text{(By Lemmas B and C)} \]
\[ G^{\frac{2r+1}{2r+1}} \longrightarrow K^{\frac{(2r+1)n}{(2r+1)(s+r)n-(2r+1)n}}_{s+r+1} \quad \text{(By Lemma 1(a))} \]
\[ \chi_c(G^{\frac{2r+1}{3r+1}}) \leq \frac{(2s+1)n}{(s-r)n+2r+1} \]
\[ \chi_c(G^{\frac{2r+1}{2r+1}}) \leq \frac{(2s+1)^n \chi_c(G)}{(s-r)^n \chi_c(G)+(2r+1)} \]

\[ \Box \]

Tardif [12] has shown that the cube root, in sense of dual power, of any circular complete graph with circular chromatic number less than 4, is homomorphically equivalent to a circular complete graph.

**Lemma E.** [12] *Let* \( n \) *and* \( d \) *be positive integers, where* \( n > 2d \). *If* \( \frac{n}{d} < 4 \), *then* \( K^{\frac{1}{3}}_{\frac{n}{n+d}} \leftarrow K^{\frac{3n}{n+d}}_n \).

Here is a generalization of Lemma E.

**Lemma 2.** *Let* \( n \) *and* \( d \) *be positive integers, where* \( n > 2d \). *If* \( \frac{n}{d} < 4 \), *then*

\[ K^{\frac{1}{3}}_{\frac{n}{n+d}} \leftarrow K^{\frac{2r+1}{3r+1}}_{\frac{3n}{3n+4}} \]

**Proof.** Theorem A implies that \( K^{\frac{2r+1}{(2r+1)n}}_{\frac{3r+1}{3n+4}} \longrightarrow K^{\frac{1}{3}}_{\frac{n}{n+d}} \) if and only if \( K^{\frac{2r+1}{(2r+1)n}}_{\frac{3r+1}{3n+4}} \longrightarrow K^{\frac{n}{d}}_n \). On the other hand, Lemma 1(b) shows that the circular complete graphs \( K^{\frac{2r+1}{(2r+1)n}}_{\frac{3r+1}{3n+4}} \) and \( K^{\frac{n}{d}}_n \) are homomorphically equivalent. Conversely, it is sufficient to prove that

\[ \chi_c(K^{\frac{1}{3}}_{\frac{n}{n+d}}) \leq \frac{(2r+1)n}{rn+d} \]

Take a rational number \( \frac{2k+1}{3} \) such that \( \frac{1}{2r+1} \leq \frac{2k+1}{3} < \frac{1}{2} \). It is easy to see that

\[ K^{\frac{1}{3}}_{\frac{n}{n+d}} \longrightarrow (K^{\frac{1}{3}}_{\frac{1}{3}})^{\frac{2k+1}{3}} \]

If \( G \) is non-bipartite graph, Theorem A and Lemma C yield that \( G^{\frac{1}{3}}_{\frac{1}{3^{i-1}}} \longrightarrow (G^{\frac{1}{3}}_{\frac{1}{3^{i-1}}})^{\frac{2k+1}{3}} \). Since \( \frac{n}{d} < 4 \), by induction on \( i \) and Lemma E we have \( K^{\frac{1}{3}}_{\frac{n}{n+d}} \leftarrow K^{\frac{2k+1}{3}}_{\frac{3n}{3n+4}} \). Therefore, there is a homomorphism from \( K^{\frac{2k+1}{3}}_{\frac{3n}{3n+4}} \) to \( K^{\frac{2k+1}{3}}_{\frac{3n}{3n+4}} \). By
Lemma (a), two graphs $K_{\frac{3^k-1}{2}n+1}$ and $K_{\frac{3^k}{2}(\frac{3^k-1}{2}n+1)-k3^k}$ are homomorphically equivalent. Hence,

$$\chi_c(K_{\frac{3^k+1}{2}}) \leq \frac{3^i n}{(2k+1)d + (\frac{3^i-1}{2} - k)n}.$$ 

Since the set of parameters $\{\frac{2k+1}{d} | k \geq 1, i \geq 1\}$ is dense in the interval $(0, +\infty)$,

$$\chi_c(K_{\frac{2r+1}{2}}) \leq \inf \left\{ \frac{3^i n}{(2k+1)d + (\frac{3^i-1}{2} - k)n} \mid \frac{1}{2r+1} \leq \frac{2k+1}{3^i} < \frac{1}{2} \right\}.$$ 

This infimum is equal to $\frac{(2r+1)n}{r n + d}$, as desired. ■

Here, we determine the circular chromatic number of some graph powers.

**Theorem 2.** Let $G$ be a non-bipartite graph with circular chromatic number $\chi_c(G)$. Also, assume that $r$ and $s$ are non-negative integers. Then we have $\chi_c(G^{\frac{2s+1}{2}}) = \frac{(2s+1)\chi_c(G)}{s\chi_c(G)+1}$. Moreover, If $\chi_c(G^{\frac{2s+1}{2}}) < 4$, then

$$\chi_c(G^{\frac{2s+1}{2}}) = \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1}.$$ 

**Proof.** Note that, in view of Theorem (A) $G \rightarrow K_{\frac{2s+1}{s\chi_c(G)+1}}$ if and only if $G \rightarrow K_{\frac{(2s+1)\chi_c(G)}{s\chi_c(G)+1}}$. On the other hand, by using Lemma (a), two graphs $K_{\frac{2s+1}{2s+1}(\frac{2s+1}{s\chi_c(G)+1})}$ and $K_{\frac{2s+1}{2s+1}(\frac{2s+1}{s\chi_c(G)+1})}$ are homomorphically equivalent. Consequently, $\chi_c(G^{\frac{2s+1}{2}}) < \frac{2s+1}{s}$.

Let $\frac{n}{d} < \frac{2s+1}{s}$.

$$\chi_c(G^{\frac{2s+1}{2}}) \leq \frac{n}{d} \iff G^{\frac{2s+1}{2}} \rightarrow K_{\frac{n}{d}}$$

$$\iff G \rightarrow K_{\frac{2s+1}{n}}$$ (By Theorem (A))

$$\iff G \rightarrow K_{\frac{n}{(2s+1)d-an}}$$ (By Lemma (a))

$$\iff \chi_c(G) \leq \chi_c(K_{\frac{n}{(2s+1)d-an}})$$

$$\iff \frac{(2s+1)\chi_c(G)}{s\chi_c(G)+1} \leq \frac{n}{d}$$

To prove the next part, it suffices to show that for any $2 \leq \frac{n}{d} < 4$, $\chi_c(G^{\frac{2s+1}{2}}) \leq \frac{n}{d}$ is equivalent to $\frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1} \leq \frac{n}{d}$. Assume that $\chi_c(G^{\frac{2s+1}{2}}) \leq \frac{n}{d} < 4$. 


\[ \chi_c(G^{\frac{2r+1}{2s+1}}) \leq \frac{n}{d} \iff G^{\frac{2r+1}{2s+1}} \rightarrow K_{\frac{n}{d}} \]

\[ \iff G^{\frac{2r+1}{2s+1}} \rightarrow K_{\frac{n}{d}}^{\frac{1}{d}} \quad \text{(By Theorem A)} \]

\[ \iff G^{\frac{2r+1}{2s+1}} \rightarrow K_{\frac{(2r+1)n}{r+\frac{n}{d}}} \quad \text{(By Lemma 2)} \]

\[ \iff \chi_c(G^{\frac{2r+1}{2s+1}}) \leq \frac{(2r+1)n}{r+\frac{n}{d}} \]

\[ \iff \frac{(2s+1)\chi_c(G)}{s\chi_c(G)+1} \leq \frac{(2r+1)\frac{n}{d}}{r+\frac{n}{d}+1} \]

\[ \iff \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1} \leq \frac{n}{d}. \]

**Corollary 1.** Let \( r \) and \( s \) be non-negative integers and \( G \) be a non-bipartite graph. If \( \frac{2r+1}{2s+1} \leq \frac{\chi_c(G)}{3(\chi_c(G)-2)} \), then \( \chi_c(G^{\frac{2r+1}{2s+1}}) = \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1} \).

**Proof.** Since \( \frac{2r+1}{2s+1} \leq \frac{\chi_c(G)}{3(\chi_c(G)-2)} \), Theorem 2 implies that \( \chi_c(G^{\frac{2r+1}{2s+1}}) \leq 3. \) Now, by the previous theorem, we have \( \chi_c(G^{\frac{2r+1}{2s+1}}) = \frac{(2s+1)\chi_c(G)}{(s-r)\chi_c(G)+2r+1} \).

**Corollary 2.** Let \( r \) and \( s \) be non-negative integers and \( G \) be a non-bipartite graph such that \( \chi_c(G^{\frac{2r+1}{2s+1}}) < 4. \) Then we have

\[ \mu(G^{\frac{2r+1}{2s+1}}) = \frac{2s+1}{2r+1} \mu(G) = \frac{2s+1}{3(2r+1)} \chi_c(G). \]

Given a rational number \( \frac{n}{d} \), a rational number \( \frac{n'}{d'} \) is unavoidable by \( \frac{n}{d} \) if every graph \( G \) with \( \chi_c(G) = \frac{n}{d} \) contains a subgraph \( H \) with \( \chi_c(H) = \frac{n'}{d'} \). It is known [4] if \( m \) is an integer and \( m < \frac{n}{d} \), then \( m \) is unavoidable by \( \frac{n}{d} \).

Suppose \((n, d) = 1\), i.e., \( n \) and \( d \) are coprime. Let \( n' \) and \( d' \) be the unique integers such that \( 0 < n' < n \) and \( nd' - n'd = 1 \). We call \( \frac{n'}{d'} \) the lower parent of \( \frac{n}{d} \), and denote it by \( F(\frac{n}{d}) \). The following question was posed in [13, 14].

**Question A.** [13, 14] Is true that for every rational \( \frac{n}{d} > 2 \), \( F(\frac{n}{d}) \) is unavoidable by \( \frac{n}{d} \)?

Here, we give a negative answer to the aforementioned question.

**Corollary 3.** Let \( k \) be a positive integer. Then there exists a graph \( G \) with \( \chi_c(G) = \frac{9k+3}{3k+2} \) such that \( G \) does not contain any subgraph with circular chromatic number equal to \( \frac{9k+1}{3k+2} \).

**Proof.** Let \( n = 9k + 3 \), \( d = 3k + 2 \), \( n' = 6k + 1 \), and \( d' = 2k + 1 \). Obviously, \( nd' - n'd = 1 \). By Theorem 2 we have \( \chi_c(K_{\frac{4}{3k+1}}) = \frac{9k+3}{3k+2} \). Suppose that \( e \in \)
$E(K^\frac{1}{2}_{3k+1})$. It is readily seen that there exists a homomorphism from $K^\frac{1}{2}_{3k+1} \setminus e$ to $K^\frac{1}{2}_{3k}$.

Hence, if $H$ is a proper subgraph of $K^\frac{1}{2}_{3k+1}$, then $\chi_c(H) \leq \chi_c(K^\frac{1}{2}_{3k}) = \frac{9k}{3k+1} < \frac{6k+1}{2k+1}$.

Therefore, $G$ contains no subgraph with circular chromatic number $\frac{n^2}{d}$.

It should be noted that one can introduce more rational numbers such that their lower parents are not unavoidable. For instance, we show that $\frac{15n+7}{6n+4}$ is not unavoidable by $\frac{18n+9}{6n+4}$. To see this, for $d \geq 2$ and $n \geq 3$, define the graph $H_d(K_n)$ as follows. Let $G_1, \ldots, G_d$ be $d$ graphs such that each of them is isomorphic to the complete graph $K_n$. Assume that $v_i w_i \in E(G_i)$ for any $1 \leq i \leq d$. The graph $H_d(K_n)$ obtained from the disjoint union of $G_1 \cup \cdots \cup G_d$ by identifying the vertices $w_i$ with $v_{i+1}$ for any $1 \leq i \leq d - 1$, deleting the edges $v_i w_i$ for any $1 \leq i \leq d$, and by adding the edge $v_1 w_d$. In fact, it is a simple matter to check that $H_d(K_n)$ follows by applying Hajós construction to the complete graphs $G_1, \ldots, G_d$. Hence, $\chi(H_d(K_n)) = n$ and the graph $H_d(K_n)$ is a critical graph, i.e., $\chi(H_d(K_n) \setminus e) = n - 1$ for any $e \in E(H_d(K_n))$.

Now, we show that $\chi_c(H_d(K_n)) = \frac{d(n-1)+1}{d}$. To see this, assume that $V(G_i) = \{v_1, u_2, \ldots, u_{n(n-1)+1}, w_i\}$. Define a coloring $c : V(H_d(K_n)) \rightarrow \{1, 2, \ldots, dn - d + 1\}$ as follows. For any $1 \leq i \leq d$ and $2 \leq j \leq n - 1$, set $c(u_{ij}) = (j - 1)d + i$, $c(w_i) = i$, and $c(v_1) = d(n-1)+1$. It is easy to check that $c$ is a $(d(n-1)+1, d)$ coloring of $H_d(K_n)$.

On the other hand, it is straightforward to check that the independence number of $H_d(K_n)$ is equal to $d$. Consequently, $\chi_c(H_d(K_n)) = \frac{dn-d+1}{d}$.

The graph $H_2(K^\frac{1}{2}_{3n+2})$ has circular chromatic number $\frac{18n+9}{6n+5}$. It is readily seen that there is a homomorphism from $H_2(K^\frac{1}{2}_{3n+2}) \setminus e$ to $K^\frac{1}{2}_{3n+1}$. Hence, if $H$ is a proper subgraph of $H_2(K^\frac{1}{2}_{3n+2})$, then $\chi_c(H) \leq \chi_c(K^\frac{1}{2}_{3n+1}) = \frac{9n+3}{3n+2} < \frac{15n+7}{5n+4}$. Therefore, $H_2(K^\frac{1}{2}_{3n+2})^\frac{1}{2}$ contains no subgraph with circular chromatic number $\frac{15n+7}{5n+4}$.

Let $\zeta(G)$ be the minimum number of vertices of $G$, necessary to be deleted, in order to reduce the chromatic number of the graph.

**Question B.** Let $\chi_c(G) = \frac{n}{d}$, where $(n, d) = 1$ and $n = (\chi(G) - 1)d + r$. Is it true that $\zeta(G) \geq r$?

When $G$ is a critical graph, we have $\zeta(G) = 1$. If the aforementioned question is true, then for every critical graph $G$ with $\chi(G) = n$, its circular chromatic number is equal to $\frac{dn-d+1}{d}$ for an appropriate $d$. It is worth noting that $H_d(K_n)$ is a critical graph with $\chi_c(H_d(K_n)) = \frac{dn-d+1}{d}$.

### 3 Fractional and Multichromatic Number

As usual, we denote by $[m]$ the set $\{1, 2, \ldots, m\}$, and denote by $\binom{[m]}{n}$ the collection of all $n$-subsets of $[m]$. The **Kneser graph** $KG(m, n)$ (resp. the **generalized Kneser graph** $KG(m, n, s)$) is the graph on the vertex set $\binom{[m]}{n}$, in which two distinct vertices $A$ and $B$ are adjacent if and only if $A \cap B = \varnothing$ (resp. $|A \cap B| \leq s$). It was conjectured by
Kneser [6] in 1955, and proved by Lovász [8] in 1978, that \( \chi(KG(m, n)) = m - 2n + 2 \). The fractional chromatic number is defined as a generalization of the chromatic number as follows

\[
\chi_f(G) = \inf\{ \frac{m}{n} | G \rightarrow KG(m, n) \}.
\]

An \( n \)-tuple coloring of graph \( G \) with \( m \) colors assigns to each vertex of \( G \), an \( n \)-subset of \([m]\) so that adjacent vertices receive disjoint sets. Equivalently, \( G \) has an \( n \)-tuple coloring with \( m \) colors if there exists a homomorphism from \( G \) to \( KG(m, n) \). The \( n \)-th multichromatic number of \( G \), denoted by \( \chi_n(G) \), is the smallest \( m \) such that \( G \) has a \( n \)-tuple coloring with \( m \) colors. These colorings were first studied in the early 1970s and the readers are referred to [5, 10, 11] for more information.

**Theorem D.** [9] Suppose that \( m \) and \( n \) are positive integers with \( m > 2n \). Then the following two conditions on non-negative integers \( k \) and \( l \) are equivalent.

- For any two (not necessarily distinct) vertices \( A \) and \( B \) of \( KG(m, n) \) with \( |A \cap B| = k \), there is a walk of length exactly \( l \) in \( KG(m, n) \) beginning at \( A \) and ending at \( B \).
- \( l \) is even and \( k \geq n - \frac{l}{2}(m - 2n) \), or \( l \) is odd and \( k \leq \frac{l - 1}{2}(m - 2n) \).

In view of Theorem [2], we have \( \chi_c(G^{\frac{k}{l+k}}) = \frac{(2s+1)\chi_c(G)}{s\chi_f(G) + 1} \). Here, we present a tight upper bound for the fractional chromatic number of subdivision graphs.

**Theorem 3.** Let \( G \) be a non-bipartite graph and \( s \) be a non-negative integer. Then

\[
\chi_f(G^{\frac{1}{s+1}}) \leq \frac{(2s + 1)\chi_f(G)}{s\chi_f(G) + 1}.
\]

**Proof.** Let \( f \) be a homomorphism from \( G \) to \( KG(m, n) \). We claim that there is a homomorphism from \( G \) to the generalized Kneser graph \( KG((2s+1)m, sm + n, (m - 2n)s) \). To see this, for every vertex \( v \in V(G) \), define \( g(v) \) as follows

\[
\bigcup_{i \in f(v)} \{(i-1)(2s+1)+1, \ldots, (i-1)(2s+1)+s\} \cup \bigcup_{i \notin f(v)} \{(i-1)(2s+1)+s+1, \ldots, i(2s+1)\}.
\]

It is easy to see that, for any vertex \( v \in V(G) \), \( |g(v)| = sm + n \). Also, if \( u \) and \( v \) are two adjacent vertices in \( G \), then \( |g(u) \cap g(v)| = (m - 2n)s \). Now, in view of Theorem [2], we have

\[
KG((2s+1)m, sm + n, (m - 2n)s) \leftrightarrow KG((2s+1)m, sm + n)^{2s+1}.
\]

Let \( G \rightarrow KG(m, n) \) and \( \chi_f(G) = \frac{m}{n} \). By the previous discussion, there is a homomorphism from \( G \) to \( KG((2s+1)m, sm + n, (m - 2n)s) \). Now, Theorem [1] implies that \( G^{\frac{1}{s+1}} \rightarrow KG((2s+1)m, sm + n) \). Hence, \( \chi_f(G^{\frac{1}{s+1}}) \leq \frac{(2s+1)\chi_f(G)}{s\chi_f(G) + 1} \).

Equality does not always hold in Theorem [3]. For instance, consider the graph \( R_{10}^\frac{1}{3} \). We know that the third power of the Petersen graph \( P^3 \) is isomorphic to \( K_{10} \).
Hence, in view of Lemma [B] there exists a homomorphism from $K_{10}^{\frac{1}{2}}$ to the Petersen graph. Consequently, $\chi_f(K_{10}^{\frac{1}{2}}) \leq \frac{6}{2}$ which is less than $\frac{10}{11}$.

It is simple to see that there exists a homomorphism from $G_{2n+1}^{\frac{1}{2}s+1}$ to $C_{2n+1}$. On the other hand, the odd cycle $C_{2n+1}$ is an induced subgraph of the Kneser graph $KG(2n+1, n)$. Therefore, if $G$ is a non-bipartite graph and $s \geq n$, then $\chi_n(G_{2n+1}^{\frac{1}{2}s+1}) = 2n + 1$.

**Theorem 4.** Let $G$ be a non-bipartite graph. If $i, n$ and $s$ are positive integers such that $is = n - 1$, then

$$\chi_n(G_{2n+1}^{\frac{1}{2}s+1}) \leq 2n + i \iff \chi(G) \leq \binom{2n+i}{n}.$$

**Proof.**

$$\chi_n(G_{2n+1}^{\frac{1}{2}s+1}) \leq 2n + i \iff G_{2n+1}^{\frac{1}{2}s+1} \rightarrow KG(2n + i, n)$$

$$\iff G \rightarrow KG(2n + i, n)^{2s+1} \quad \text{(By Theorem [A])}$$

$$\iff G \rightarrow KG(2n + i, n, is) \quad \text{(By Theorem [D])}$$

$$\iff G \rightarrow KG(2n + i, n, n - 1)$$

$$\iff G \rightarrow K_{\binom{2n+i}{n}}$$

$$\iff \chi(G) \leq \binom{2n+i}{n}.$$

We know that $\chi_2(G_{2n+1}^{\frac{1}{2}s+1}) = 5$ whenever $s \geq 2$. The following corollary, which is an immediate consequence of the aforementioned theorem, determines the other cases.

**Corollary 4.** Let $G$ be a non-bipartite graph. If $\chi(G) \leq 10$, then $\chi_2(G_{2n+1}^{\frac{1}{2}s+1}) = 5$. Otherwise, $\chi_2(G_{2n+1}^{\frac{1}{2}s+1}) = 6$.

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