Subshifts of Quasi-Finite Type

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Abstract

We introduce subshifts of quasi-finite type as a generalization of the well-known subshifts of finite type. This generalization is much less rigid and therefore contains the symbolic dynamics of many non-uniform systems, e.g., piecewise monotonic maps of the interval with positive entropy. Yet many properties remain: existence of finitely many ergodic invariant probabilities of maximum entropy; lots of periodic points; meromorphic extension of the Artin-Mazur zeta function.

1 Introduction

Jacques Hadamard [12] founded symbolic dynamics in 1898 when he realized that the dynamics of the geodesic flow on surfaces of negative curvature can be represented by very simple subsets of $A^Z$ ($A$ being some finite subset). Namely, these subsets are defined by excluding a finite number of words. Such subsets are now called subshifts of finite type (or S.F.T.). They have been thoroughly studied (see, e.g., [17]) and the result of Hadamard has been generalized to all uniformly hyperbolic systems (see, e.g., [21]). However, S.F.T. are much too rigid to provide a description of more general dynamics (for instance, there are only countably many topological conjugacy classes of S.F.T.).

Therefore a key problem is to enlarge S.F.T. to accommodate wide classes of non-uniform dynamics and yet keep most of the basic features of S.F.T.

In this paper, we provide a solution by introducing a new class of subshifts, which we call subshifts of quasi-finite type. They include the symbolic dynamics of a large class of non-uniform dynamical systems: piecewise monotonic maps [15] with positive entropy and more generally entropy-expanding maps [7] satisfying a technical assumption.

Whereas subshifts of finite type are described by finitely many constraints, we allow a slowly growing number of constraints of a given length, ”slow growth” meaning with a rate strictly less than the topological entropy.
We prove that these subshifts of quasi-finite type remarkably have the same basic properties as S.F.T. at least with respect to ”complexity”:

- they have finitely many ergodic invariant probability measures maximizing entropy;
- they have lots of periodic points;
- their Artin-Mazur zeta functions have meromorphic extensions.

This paper can be considered as yet another illustration of the following principle [7]: complexity bounds imply semi-uniform hyperbolicity.

1.1 Definitions

We consider a subshift, i.e., a closed $\sigma$-invariant subset $\Sigma \subset A^\mathbb{Z}$ ($A$ is some finite set, $\sigma$ denotes the left-shift on $A^\mathbb{Z}$). The one-sided version of $(\Sigma, \sigma)$ is $(\Sigma_+, \sigma_+)$ with $\Sigma_+ := \{A_0 A_1 A_2 \ldots : A \in \Sigma \} \subset A^\mathbb{Z}$ and $\sigma_+$ the left-shift on these one-sided sequences.

It is customary to consider follower sets [17]: if $A_{-n} \ldots A_0$ is some finite word on the alphabet $A$, then

$$\text{Fol}(A_{-n} \ldots A_0) := \{B_0 B_1 B_2 \ldots : B \in \Sigma \text{ and } B_{-n} \ldots B_0 = A_{-n} \ldots A_0 \} \subset [A_0]_+ \subset \Sigma_+.$$ 

By convention, the follower of the empty word is the whole of $\Sigma_+$. We can also write:

$$\text{Fol}(A_{-n} \ldots A_0) = \sigma_+^n([A_{-n} \ldots A_0]_+) \subset \Sigma_+$$

where $[A_{-n} \ldots A_0]_+$ denotes the cylinder in $\Sigma_+$.

**Definition 1** A left constraint is a finite word $A_{-n} \ldots A_0$, $n \geq 0$, such that $\emptyset \neq \text{Fol}(A_{-n} \ldots A_0) \subsetneq \text{Fol}(A_{-n+1} \ldots A_0)$.

The set of left constraints with length $n$ is denoted by $C(\Sigma, n)$ (or just $C(n)$).

The **left constraint entropy** is the quantity:

$$h_C(\Sigma) := \limsup_{n \to \infty} \frac{1}{n} \log^+ \#C(\Sigma, n).$$

**Definition 2** The **symmetric constraint entropy** is:

$$h_{SC}(\Sigma) = \min(h_C(\Sigma), h_{C(\bar{\Sigma})}).$$

where $\bar{\Sigma} := \{(A_n)_{n \in \mathbb{Z}} : (A_{-n})_{n \in \mathbb{Z}} \in \Sigma \}$.

Recall that the **topological entropy** is:

$$h_{top}(\Sigma) = \limsup_{n \to \infty} \frac{1}{n} \log \#C(\Sigma, n).$$

where $C(\Sigma, n) := \{A_0 \ldots A_{n-1} \in A^n : [A_0 \ldots A_{n-1}] \Sigma \neq \emptyset \}$ where $[A_0 \ldots A_{n-1}]\Sigma$ (or simply $[A_0 \ldots A_{n-1}]$) denotes a cylinder in $\Sigma$.

\(^1\text{Observe that } \text{Fol}(A_{-n} \ldots A_0) = \emptyset \text{ if and only if the given word does not appear in } \Sigma.\)
We are at least in position to define the main object of this paper:

**Definition 3** Σ is a subshift of quasi-finite type (or Q.F.T.) iff:

\[ h_{SC}(\Sigma) < h_{top}(\Sigma). \]

### 1.2 Examples and relatives

**Lemma 1** All S.F.T. and sofic shifts with non-zero topological entropy are Q.F.T. More precisely, \( h_{SC}(\Sigma) = 0 \) if \( \Sigma \) is a S.F.T. (but \( h_{SC}(\Sigma) > 0 \) is possible for sofic \( \Sigma \)).

On the other hand, many symbolic dynamics which are not S.F.T. or sofic are Q.F.T.:

A **piecewise monotonic map** \cite{18} is a map \( f : I \rightarrow I \) on some compact interval \( I \) such that there is a finite partition of \( I \) into subintervals on each of which the restriction of \( f \) is continuous and strictly monotonic. The natural partition \( P \) is the collection of maximum open intervals on which \( f \) is continuous and strictly monotonic. The symbolic dynamics is:

\[ \Sigma(f) := \{ A \in P^\mathbb{Z} : \forall n \in \mathbb{Z} \forall k \geq 0 < A_n \ldots A_{n+k} > \neq \emptyset \} \]

where the notation \( < A_0 \ldots A_k > := A_0 \cap f^{-1}A_1 \cap \ldots \cap f^{-k}A_k \subset I \) stands for the **geometric cylinders**.

A **multi-dimensional β-transformation** \cite{5} is a map \( T : [0,1)^d \rightarrow [0,1)^d \), with \( d \geq 1 \) and \( T(x) = B.x \mod \mathbb{Z}^d \) where \( B \) is an expanding\(^2\) affine map of \( \mathbb{R}^d \). The natural partition \( P \) is the finite collection of maximum open subsets of \( (0,1)^d \) on which \( T(x) - B.x \) is constant. The symbolic dynamics is defined as above. According to \cite{5}, this is a special case of connected piecewise entropy-expanding map.

A **piecewise entropy-expanding map** is \((X, P, f)\) with (see \cite{6}):

- \( X \) is a compact subset of some Euclidean space;
- \( P \) is a finite collection of pairwise disjoint open subset of \( X \);
- \( f : \bigcup_{A \in P} P \rightarrow X \) is such that each restriction \( f : A \rightarrow f(A) \) can be extended to a homeomorphism between neighborhoods of \( A \) and \( f(A) \);
- the fundamental inequality:

\[
h_B(X, P, f) := \limsup_{n \rightarrow \infty} \frac{1}{n} \log \# \left\{ A \in P^n : < A > \cap \bigcup_{B \in P} \partial f(B) \neq \emptyset \right\} < h_{top}(\Sigma(f))
\]

\(^2\)i.e., \( \exists \lambda > 1 \forall x, y \in \mathbb{R}^d \|B.x - B.y\| \geq \lambda \|x - y\| .\)
Its symbolic dynamics $\Sigma(f)$ is again defined in the same way.

Notice that entropy-expanding does not imply expanding.

$(X,P,f)$ is said to be **connected** if every $P$-cylinder is connected.

**Lemma 2** The symbolic dynamics of the following dynamical systems are Q.F.T.:

1. piecewise monotonic maps with positive topological entropy.
2. connected piecewise entropy-expanding maps (hence, in particular, multidimensional $\beta$-transformations).

This implies immediately:

**Corollary 1** The entropy of a Q.F.T. can take any value in $(0, \infty)$. In particular, there are uncountably many conjugacy classes of Q.F.T., in contrast to the case of S.F.T.

There is an important weakening of Q.F.T.:

**Definition 4** A left constraint $A_{-n} \ldots A_0$, $n \geq 0$, is **extendable** if there exist a sequence $B \in A^{-\infty}$ with $B_{-m} \ldots B_0 = A_{-n} \ldots A_0$ and infinitely many integers $m \geq 0$ such that: $B_{-m} \ldots B_0$ is again a left constraint.3

The set of extendable left constraints with length $n$ is denoted by $C^*(\Sigma, n)$ (or just $C^*(n)$).

The **extendable left constraint entropy** is

$$h_C^*(\Sigma) := \limsup_{n \to \infty} \frac{1}{n} \log \# C^*(\Sigma, n).$$

Subshifts $\Sigma$ with $h_{SC}^*(\Sigma) : = \min(h_C^*(\Sigma), h_C^*(\Sigma)) < h_{\text{top}}(\Sigma)$ are called **weak-Q.F.T.**

Weak-Q.F.T. are definitely not as nice as Q.F.T.:

**Lemma 3** There exists a weak-Q.F.T. with countably infinitely many maximum measures.

The following qualitative generalization of sofic shifts is a special case of weak-Q.F.T.:

**Definition 5** $\Sigma$ is **eventually Markovian on the left** iff for each $A \in \Sigma$ there exists an integer $N$ such that:

$$\forall n \geq N \quad \text{Fol}(A_{-n} \ldots A_0) = \text{Fol}(A_{-N} \ldots A_0).$$

**Lemma 4** If $\Sigma$ is eventually Markovian on the left, then $h_C^*(\Sigma) = 0$.

In particular, all sofic shifts have $h_C^*(\Sigma) = 0$ (compare with $h_C(\Sigma)$) and are weak-Q.F.T. (if they have positive topological entropy).

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3Remark that this is strictly stronger than requiring that $A_{-n} \ldots A_0$ is a suffix of infinitely many left constraints.
We exhibit some facts that show that the refinements of our definitions (extendability condition, symmetry) do enlarge the class of subshifts under consideration.

**Lemma 5** There are subshifts $\Sigma$ such that $h_C(\bar{\Sigma}) \neq h_C(\Sigma)$ and even such that:

$$h_C(\bar{\Sigma}) < h_C(\Sigma) = h_{\text{top}}(\Sigma).$$

The same is true for $h^*_C$.

**Lemma 6** Obviously, we have:

$$0 \leq h^*_C(\Sigma) \leq h_C(\Sigma) \leq h_{\text{top}}(\Sigma)$$

and there exist subshifts for which all these inequalities are strict. In particular, there are weak-Q.F.T. which are not Q.F.T.

We now compare Q.F.T. to previously studied notions:

**Coded systems with synchronizing words** of Blanchard and Hansel [4] have a significant intersection with weak Q.F.T.:

**Lemma 7** Any coded system such that the set of sequences not containing a synchronizing word has topological entropy $< h_{\text{top}}(\Sigma)$ is a weak-Q.F.T.

All coded systems are topologically transitive\(^4\), in contrast to Q.F.T. Hence we have trivial examples of Q.F.T. which are not coded. Anne Bertrand [3] characterized (one-dimensional) $\beta$-transformation with symbolic dynamics which are coded systems with synchronizing words. In particular, not all of them have this property. Hence:

**Lemma 8** There are topologically transitive Q.F.T. which are not coded systems with synchronizing words.

We note the condition introduced by B.M. Gurevich [11] which involves explicitly the speed with which finite order Markov topological chains approximate the subshift. It seems unrelated to Q.F.T.

We have the following relationships of $h^*_C(\Sigma)$ and $h_C(\Sigma)$ with known characteristics:

**Lemma 9** The boundary capacity defined by Keller [15]:

$$\text{cap}(\Sigma) := \limsup_{n \to \infty} \frac{1}{n} \log \sup_{r \geq 1, w \in \mathcal{L}(\Sigma, r)} \sup \#\{a \in \mathcal{L}(\Sigma, n) : [a]_+ \cap \sigma^r_+[w] \neq \emptyset \}
\quad \text{and} \quad [a]_+ \cap (\Sigma_+ \setminus \sigma^r_+[w]) \neq \emptyset \}$$

satisfies $\text{cap}(\Sigma) \geq h_C(\Sigma)$ and the inequality may be strict.

\(^4\)i.e., there exists $A \in \Sigma$ such that $\{\sigma^n A : n \geq 0\}$ is dense in $\Sigma$. 5
Lemma 10  The entropy of minimal forbidden words $h_M(\Sigma)$ considered by Béal and others [1]:

$$h_M(\Sigma) := \limsup_{n \to \infty} \frac{1}{n} \# \{ A_1 \ldots A_n : [A_1 \ldots A_n] = \emptyset \text{ and } [A_1 \ldots A_{n-1}] \neq \emptyset \}$$

satisfies $h_M(\Sigma) \leq h_C(\Sigma)$, the inequality being strict in some cases.

On the other hand, both $h_M(\Sigma) > h^*_C(\Sigma)$ and $h_M(\Sigma) < h^*_C(\Sigma)$ occur. Moreover, there exists a subshift $\Sigma$ with $h_M(\Sigma) = 0$ with $h^*_C(\Sigma) = h_{top}(\Sigma) > 0$.\footnote{I would be interested by an example with $h_M(\Sigma) < h_{top}(\Sigma)$ and infinitely many maximal measures.}

Finally we turn to the natural-looking follower entropy:

$$h_{Fol}(\Sigma) := \limsup_{n \to \infty} \frac{1}{n} \log \# \{ Fol(w) : w \in L(\Sigma, n) \}.$$  

It is natural to ask whether this gives rise to a reasonable variant of Q.F.T. This is not the case:

Lemma 11  There exist subshifts with $h_{Fol}(\Sigma) = 0 < h_{top}(\Sigma)$ which have uncountably many maximum measures.\footnote{i.e., ergodic invariant probability measures with maximum entropy.}

1.3 Basic properties

Proposition 1  $h_C(\Sigma)$ and $h^*_C(\Sigma)$ (and therefore their symmetric versions) are invariants of topological conjugacy. On the other hand, $h_C(\Sigma)$ does not necessarily decrease under factor maps also the Q.F.T. and weak-Q.F.T. properties are not preserved under extensions or factors.

Question: Can $h^*_C(\Sigma)$ also increase under factor maps?

Lemma 12  For unions:

$$h_C(\Sigma_1 \cup \Sigma_2) = \max(h_C(\Sigma_1), h_C(\Sigma_2))$$

and for products:

$$h_{top}(X \times Y) - h_C(X \times Y) = \min(h_{top}(X) - h_C(X), h_{top}(Y) - h_C(Y))$$

In particular, if $X$ and $Y$ are Q.F.T., then $X \times Y$ is also a Q.F.T.

These properties are also true for $h^*_C$.

In particular, a product of an arbitrary subshift with a subshift of zero-entropy is never Q.F.T or even weak-Q.F.T.

We shall see the following topological properties:

Lemma 13  A Q.F.T. is not necessarily topologically transitive. A weak-Q.F.T. always contains periodic points hence it is never topologically minimal.
2 Main results

Theorem 1 Let $\Sigma$ be a Q.F.T. Then,

1. $\Sigma$ admits a finite number of maximum measures, each one of which is Bernoulli (up to a period$^7$); 
2. the Artin-Mazur zeta function:
   \[ \zeta(z) := \exp - \sum_{n \geq 1} \frac{z^n}{n} \#\{x \in \Sigma : \sigma^n x = x\} \]
   extends from a holomorphic function on $|z| < e^{-h_{\text{top}}(\Sigma)}$ to a meromorphic function on $|z| < e^{-h_{\text{sc}}(\Sigma)}$. Moreover, $|z| = e^{-h_{\text{sc}}(\Sigma)}$ is the natural boundary$^8$ of meromorphic extension for some Q.F.T.$^9$
3. the number of periodic points satisfies:
   \[ 0 < \limsup_{n \to \infty} \frac{\#\{x \in \Sigma : \sigma^n x = x\}}{e^{nh_{\text{top}}(\Sigma)}} < \infty. \]

If $\Sigma$ is only weak-Q.F.T. then the following properties remain true:

1. there are at most a countable number of maximum measures (but there can be infinitely many of them — see Lemma$^3$); 
2. the zeta function may fail to have a meromorphic extension (see also Lemma$^3$). 
3. $\limsup_{n \to \infty} (1/n) \log \#\{x \in \Sigma : \sigma^n x = x\} = h_{\text{top}}(\Sigma)$.

We recall that a Markov shift is defined as follows. Given a countable oriented graph $G$, the associated Markov shift $\Sigma(G)$ is the set of all paths on $G$:

\[ \Sigma(G) := \{g \in G^\mathbb{Z} : \forall n \in \mathbb{Z} \; g_n \to g_{n+1} \; \text{in} \; G\} \]

together with the left-shift $\sigma$. Observe that if we require the graph to be finite, then this reduces to S.F.T.

The theorem above will follow from the following structure theorem:

Theorem 2 Let $\Sigma \subset \mathcal{A}^\mathbb{Z}$ be a weak-Q.F.T. Then there is a countable oriented graph $\mathcal{G}$ and a map $\pi$ from the set of vertices of $\mathcal{G}$ to $\mathcal{A}$ such that the induced map $\pi : \Sigma(\mathcal{G}) \to \Sigma$:

- is well-defined and satisfies $\pi \circ \sigma = \sigma \circ \pi$;

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$^7$\(\mu\) is Bernoulli up to a period $p \geq 1$ iff there exists a subset $X$ such that $\sigma^p(X) = X$, $\Sigma$ is the disjoint union $\bigcup_{k=0}^{p-1} \sigma^k X$ (up to a negligible subset) and $(\sigma^p, \mu | X)$ is Bernoulli.

$^8$\emph{i.e.}, $\zeta$ cannot be extended meromorphically to a connected set $U \supset \{ |z| < e^{-h_{\text{sc}}(\Sigma)} \}$.

$^9$This is known for the symbolic dynamics of $\beta$-transformations for Lesbegue-a.e. $\beta > 1$. 

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• is one-to-one between $\Sigma(\mathcal{G})$ and $\Sigma \setminus \mathcal{X}$ with:
  1. $\limsup_{n \to \infty} \frac{1}{n} \log^+ \#\mathcal{X} \cap \{ x : \sigma^n x = x \} \leq h_{SC}(\Sigma);$
  2. $h(\sigma, \mu) \leq h_{SC}(\Sigma)$ for all invariant probability measures $\mu$ with $\mu(\mathcal{X}) = 1$.

Given any $h > h_{SC}(\Sigma)$, there are only finitely many irreducible parts of $\mathcal{G}$ with entropy larger than $h$.

In the language of [7],

**Corollary 2** A weak-Q.F.T. is entropy-conjugate to a Markov shift.

**Remark.** This theory can easily accomodate weights, i.e., one can introduce a potential function $\psi : \Sigma \to \mathbb{R}$ and define constraint pressure $P_c(\Sigma)$ and consider subshifts with $P_c(\Sigma) < P_{\text{top}}(\Sigma)$, the usual topological pressure w.r.t. the fixed potential. Then all the above results hold (or rather their weighted counterparts), with the possible exception of the extendability of the zeta-function (the corresponding result for Markov shifts has not been proved as far as I know).

**Some questions**

• Could we prove the above theorem by direct methods, i.e., without using Markov diagrams? (This is possible in a geometric setting by using induction on a Markov rectangle.)

• It would be more elegant to have a single construction instead of breaking the left/right symmetry.

• The above theorem ensures the existence of a “good” presentation for any Q.F.T. On the other hand, what is the set of presentations of QFT?

• Does topologically mixing implies uniqueness of the maximum measure for a Q.F.T.?

• Is entropy a complete invariant w.r.t. almost topological conjugacy within topologically mixing Q.F.T.?

• Can one state and prove a ”disjointness” property of ”irreducible” Q.F.T. from zero-entropy systems?

• Does this result (or an analogue based on Yoccoz puzzle) extend to not necessarily connected entropy-expanding maps?

• Can it be applied to non-uniformly hyperbolic dynamics (by opposition to the non-uniformly expanding examples given here)?

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\[\text{We have some preliminary results} \ [8] \ \text{for partially hyperbolic diffeomorphisms with} \ \dim E^u = 1.\]
Outline of the paper

We first relate Q.F.T. with other classes of dynamical systems (section 3) before proving some basic properties (section 4). The rest of the paper is devoted to the proof of the theorems. We first introduce the complete Markov diagram and prove that it is conjugate with a subset of the Q.F.T. (section 5). Then we control measures and periodic points supported in the complement of this subset (section 6). We bound entropy at infinity by $h_C(\Sigma)$ (section 7). We finally prove both theorems (section 8).

In an appendix, we prove and analyze a weaker construction involving Hofbauer's Markov diagram instead of the complete one.

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3 Construction of examples and comparisons

Proof of Lemma 1: If $\Sigma$ is an S.F.T. then the left constraints are only the trivial ones: the one-letter words, so that $h_C(\Sigma) = 0$.

Consider the sofic subshift over the alphabet $\{0, 1, 2\}$ defined by the condition "only an even number of non-zero symbols may appear between two 0". Its left constraint entropy is non-zero (it is log 2).

Let $\Sigma$ be a sofic subshift with non-zero entropy. We assume that it is irreducible (the general case follows easily). There exists a synchronizing word $v$, i.e., such that Fol($uvw$) = Fol($vw$) for all words $u, w$ [17, 3.3.16]. Therefore no left constraint can contain an occurrence of $v$ anywhere except at its very beginning. But forbidding a word from a sofic subshift strictly decreases entropy [17, 4.4.9]. This proves that $h_C(\Sigma) < h_{\text{top}}(\Sigma)$.

Proof of Lemma 2: In the two examples, we have the same situation:

1. there is a natural partition $P$ defining a symbolic dynamics $\Sigma$ by
   \[ \Sigma := \{ A \in P^Z : \forall n \in Z \forall k \geq 0 < A_n \ldots A_{n+k} > \neq \emptyset \}; \]

2. all geometric cylinders are connected because they are intervals or because they are convex;

3. the number of $n$-cylinders which meet the boundary of the image of an element of $P$ is bounded by $C \exp nH$ with $H < h_{\text{top}}(\Sigma)$. 

□
Let us check condition 3:

- for the case of a piecewise monotonic map, there is a finite number of boundary points, say $N$, and only twice as much $n$-cylinders can touch these points, hence one can take $H = 0$;

- for the case of a piecewise entropy-expanding map this is part of the definition.

We conclude the proof of the Lemma by showing that $h_C(\Sigma) \leq H$.

Take $A_{-n} \ldots A_0 \in C(n+1)$. By definition,

$$\emptyset \neq \text{Fol}(A_{-n} \ldots A_0) \subset \text{Fol}(A_{-n+1} \ldots A_0)$$

This implies

$$\emptyset \neq f^{n-1}(f(A_{-n}) \cap < A_{-n+1} \ldots A_0 >) \subset f^{n-1} < A_{-n+1} \ldots A_0 >$$

Hence, $f(A_{-n})$ meets but does not cover $< A_{-n+1} \ldots A_0 >$. This last set is connected. Hence it must meet the boundary of $f(A_{-n})$. Thus $\#C(n+1) \leq \#P \times Ce^{Hn}$. This concludes the proof of the Lemma.

**Proof of Corollary**

The entropy of a piecewise monotonic map is well-known to take any nonnegative value and the entropy of the symbolic dynamics is equal to it.

**Proof of Lemma**

We consider the subshift $\Sigma \subset \{0, a, b\}^\mathbb{Z}$ defined by the following exclusions. For all $k, \ell$ distinct positive integers, we have, using a well-known notation\(^\text{11}\)

- $(a|b)^0(0|b)$ and $(a|b)^0(a|b)$ cannot both appear in the same sequence;

- $(0|b)a^\ell(0|b)$ cannot appear to the right of $(a|b)^0(a|b)$.

As we have excluded only finite words, $\Sigma$ is closed and indeed a subshift.

It is easy to see that the invariant measures on $\Sigma$ are supported by $\bigcup_{k \geq 0} \Sigma_n'$ defined as follows. $\Sigma_0 = \{a, b\}^\mathbb{Z}$, $\Sigma_n$ is obtained from $\Sigma_0$ by replacing every instance of $ba^n b$ by $b0^n b$ and $\Sigma_n'$ is obtained by taking all sequences in $\Sigma_n$ and, for each couple $(n, m) \in (\mathbb{Z} \cup \{-\infty, +\infty\})^2$, setting to 0 all symbols with index $\leq n$ or $> m$. Therefore, $\Sigma$ has infinitely many maximum measures.

On the other hand, there is no extendable left constraint except $O^n, n \geq 1$, so that $h_C(\Sigma) = 0$. Thus $\Sigma$ is weak-Q.F.T. but this does not imply finiteness.
Proof of Lemma 4: If there is an extendable left constraint, one can build an infinite sequence \( A \in \Sigma \) such that: \( A_{-n} \ldots A_0 \) is a left constraint for infinitely many \( n \). But this means that
\[
\text{Fol}(A_{-n} \ldots A_0)
\]
decreases infinitely many times. Therefore we have found a sequence in \( \Sigma \) which is not eventually Markovian on the left. \( \square \)

Proof of Lemma 5: Fix some large integer \( N \geq 1 \) and consider the bi-infinite sequences obtained by concatenating blocks of the following form:
\[
(\{\mid\} \mid 1 \mid 2 \ldots |N\}^n (a|b)^k \quad (n \geq 1)
\]
under the constraint that matching parenthesis are of the same type (i.e., \{ with \}, etc.). Taking the closure (which only adds sequences of the form \((1|2|\ldots |N\}^\infty (a|b)^\infty, (1|2|\ldots |N\}^\infty \) and \((a|b)^\infty\), we obtain a subshift \( \Sigma \).

The left constraints of \( \Sigma \) are the blocks (we omit the trivial, one-letter words as we shall do without further notice in the sequel):
- \((1|2|\ldots |N\}^n (a|b)^k \) for \( 0 \leq k < n < \infty \);
- \((\{\mid\} \mid 1 \mid 2 \ldots |N\}^n (a|b)^k \) for \( 0 \leq k < n < \infty \);
- \(B_1 B_2 \ldots B_{r-1} B_r \) where each \( B_i \) is a block from eq. (1). \( B'_r \) is a prefix of such a block and \( B_1 \) contains an opening parenthesis which is not matched in \( B_1 \ldots B'_r \).

Hence \( h^*_C(\Sigma) = h_C(\Sigma) = \log N \).

Symmetrically, \( h^*_C(\Sigma) = h_C(\Sigma) = \frac{1}{2} \log N \). We see that left and right quantities are distinct.

Moreover, it is easily seen that \( h^{\top}(\Sigma) = \log N \), \( N \) being large. Thus, \( \Sigma \) is a Q.F.T. with \( h^{SC}(\Sigma) = h_C(\Sigma) < h^{\top}(\Sigma) \) but \( h_C(\Sigma) = h^{\top}(\Sigma) \). \( \square \)

Proof of Lemma 6: The inequalities are obvious as \( C^*(n) \leq C(n) \leq L(n) \). We describe an example where the inequalities are strict:

Take the product \( \Sigma_1 \) of the 2-shift together with a sturmian system (symbolic dynamics of a rotation by an irrational angle \( \alpha \) w.r.t. the partition \( \{[0, 1-\alpha], [1-\alpha, 1]\} \), see, e.g., [2]). Then \( C^*(\Sigma_1, n) = C(\Sigma_1, n) = L(\Sigma_1, n) \) and \( h^*_C(\Sigma_1) = h_C(\Sigma_1) = \log 2 \).

\( \Sigma_2 \) will be the product of the usual even-shift with the full 3-shift, i.e., the subshift of \((\{0,1\} \times \{a,b,c\})^\mathbb{Z} \) defined by forbidding the words\(^\star\):  
\[
(0,)(1,*)^{2n+1}(0,*)
\]
It is a sofic subshift hence (cf. Lemma 4) \( h^*_C(\Sigma_2) = 0 \). We compute:
\[
C(\Sigma_2, n) = \{(0, A_0)(1, A_1)(1, A_2) \ldots (1, A_{n-1}) : A_0 \ldots A_{n-1} \in \{a, b, c\}^n \}
\]
\(^\star\) The stars stand for any of the three symbols \( a, b, c \).
Hence $h_C(\Sigma_2) = \log 3 < h_{\text{top}}(\Sigma_2)$. Taking $\Sigma = \Sigma_1 \cup \Sigma_2$ and recalling Lemma 12 we obtain:

$$0 < h^*_C(\Sigma) = \log 2 < h_C(\Sigma) = \log 3 < h_{\text{top}}(\Sigma).$$

□

**Proof of Lemma 7** The proof is the same as the proof of Lemma 1. □

**Proof of Lemma 8** Obvious from the remarks above the statement of the Lemma. □

**Proof of Lemma 9** We first prove $h_C(\Sigma) \leq \text{cap}(\Sigma)$. Let $A_0 \ldots A_n$ be a left constraint. Therefore one can find a finite word $A_{n+1} \ldots A_{p}$ such that:

$[A_0 \ldots A_p] = \emptyset$ but $[A_1 \ldots A_p] \neq \emptyset$.

This implies:

$[A_1 \ldots A_n]_+ \cap (\Sigma_+ \setminus \sigma_+[A_0]_+) \neq \emptyset$

whereas it is obvious that:

$[A_1 \ldots A_n]_+ \cap \sigma_+[A_0]_+ \neq \emptyset$.

Hence, $A_1 \ldots A_n$ is a word that gets counted in Keller’s boundary capacity. This implies the claimed inequality.

We show that the inequality can be strict. Consider the subshift defined by concatenating the following blocks:

- $(1|2)^n$, for any $n \geq 1$;
- $0^n w w$, for any $n \geq 1$ and $w$ of the form $(1|2)^n$.

The left constraints are the blocks of the form $0^n(1|2)^k$ and $(1|2)0^n(1|2)^k$ with $0 \leq k < n$. Hence, $h_C(\Sigma) = \log 2/2$.

Then, for all $w$ of the form $0(1|2)^n$, $n \geq 0$:

$$[w]_+ \cap \sigma^n_+([0^{n+1}]_+) \neq \emptyset \quad \text{and} \quad [w]_+ \cap (\Sigma_+ \setminus \sigma^n_+([0^{n+1}]_+) \neq \emptyset.$$ 

so that $\text{cap}(\Sigma) = \log 2 > h_C(\Sigma)$. □

**Proof of Lemma 10** If $A_0 \ldots A_n$ is a minimal forbidden word, then $A_0 \ldots A_{n-1}$ is certainly a left constraint since it cannot be followed by $A_n$, whereas $A_1 \ldots A_n$ is allowed. Thus $\# C(n)$ is at least the number of forbidden word of length $n + 1$ divided by $\# A$. This proves that $h_M(\Sigma) \leq h_C(\Sigma)$. 

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We give an example where this inequality is strict. Consider the subshift \( \Sigma_1 \) over \( \mathcal{A}_1 = \{0, 1, 2, a, b\} \) defined by the concatenations of the following finite sequences:

\[
(0|1|2)^n(a|b)^k \quad \forall n \geq 1 \forall 1 \leq k \leq n^2.
\]

The minimal forbidden words are:

\[
(a|b)(0|1|2)^n(a|b)^{n^2+1} \quad \forall n \geq 1.
\]

Hence, \( h_M(\Sigma_1) = \log 2 \). On the other hand, the left constraints are:

\[
(a|b)(0|1|2)^n(a|b)^k \quad \forall n \geq 1 \forall 0 \leq k \leq n^2
\]

together with the same without the first symbol \( a \) or \( b \).

Hence, \( h_C(\Sigma_1) = \log 3 \). Finally, all sequences are eventually Markovian hence \( h^*(\Sigma_1) = 0 \). Thus, we have:

\[
0 = h^*(\Sigma_1) < h_M(\Sigma_1) < h_C(\Sigma_1).
\]

We now exhibit another subshift \( \Sigma_2 \) with \( h^*_C(\Sigma_2) > h_M(\Sigma_2) \). \( \Sigma_2 \) will be obtained by the concatenations of blocks of the same structure as above but we introduce new, long-range restrictions to create many extendable left constraints.

We proceed as follows. First, we restrict the blocks to \( n \geq 1000 \). Then we consider:

- blocks of the form \( B(n) := (0|1|2)^n(a|b)^{n^2} \) with \( n \geq 1000 \) and even to be an “opening parenthesis” of type \( \lfloor n/2 \rfloor \);
- similar blocks but with \( n \) odd to be a “closing parenthesis” of type \( \lfloor n/2 \rfloor \);
- all other blocks (i.e., all blocks with \( k \leq n^2 \)) to be “absorbing”.

The restriction is that two matching parenthesis must be of the same type unless there is one absorbing block between them.

Thus among the left constraints are all the blocks of the form:

\[
(a|b)B(n_1)B(n_2)\ldots B(n_r)(0|1|2)^n(a|b)^k
\]

for all \( n \geq 1000 \) and \( 1 \leq k < n^2 \) with \( n_1, \ldots, n_r \) (\( r \geq 1 \)) positive integers with the restriction that \( B(n_1) \) is an opening parenthesis which is not matched and all the matchings between \( B(n_2), \ldots, B(n_r) \) are between parenthesis of the same type. It follows that \( h^*_C(\Sigma_2) = \log 3 \).

On the other hand, the minimal forbidden words can be split into:

- the same as for \( \Sigma_1 \);
- \( (a|b)(0|1|2)^n(a|b) \) with \( n < 1000 \);
- \( B(n_1)B(n_2)\ldots B(n_r) \) with \( B(n_1) \) and \( B(n_r) \) matching parenthesis of distinct types — the point here is that only blocks with \( n_r^2 \)-blocks of \( a, b \) can appear.
It follows that $h_M(\Sigma_2)$ may only be slightly larger than $\log 2$. Hence we have:

$$\log 2 \approx h_M(\Sigma_2) < h^*_C(\Sigma_2) = h_C(\Sigma_2) = \log 3.$$ 

\[ \square \]

**Proof of Lemma**

We build a subshift over $\{0, a, b\}$. Let $X = (X_n)_{n \in \mathbb{N}}$ be a sequence of finite sets of finite words. Define $\Sigma_X \subset \{0, a, b\}^\mathbb{Z}$ as the set of sequences such that for all $n = 1, 2, \ldots$, no word in $X_n$ appears to the right of any occurrence of $(a|b)0^n(a|b)$. It is easy to check that $\Sigma_X$ is indeed a subshift (i.e., it is closed).

Observe that all $\sigma$-invariant probability measures of $\Sigma_X$ live on:

$$\bigcup_{S \subset \mathbb{N}} \Sigma_X, S$$

where $\Sigma_X, S$ is the S.F.T. defined by excluding the words $0^n$ for all $n \notin S$ as well as the words in $\bigcup_{n \in S} X_n$.

The left constraints of $\Sigma_X$ are the (legal) words of the following form:

- $(a|b)0^n w$ where $w$ starts with $a$ or $b$ and does not contain a word of the form $(a|b)0^n(a|b)$;
- $0^n w$ where $w$ starts with $a$ or $b$ and does not contain $0^n$.

We set:

$$X_n = \{(0|b)a^n(0|b)\} \quad \forall n \geq 1.$$ 

We have $\Sigma_{X, \emptyset} = \{a, b\}^\mathbb{Z}$ and each $\Sigma_{X, S} \subset \{a, b\}^\mathbb{Z}$ is obtained from $\{a, b\}^\mathbb{Z}$ by substituting $b0^n b$ for all blocks $ba^n b$, for $n \in \mathbb{N}$. Hence for all $S \subset \mathbb{N}$

$$h_{\text{top}}(\Sigma_{X, S}) = \log 2 = h_{\text{top}}(\Sigma).$$

We see that there are **uncountably many maximum measures**, one on each S.F.T. $\Sigma_{X, S}$, $S \subset \mathbb{N}$.

We have $h^*_C(\Sigma) = h_C(\Sigma) = h_{\text{top}}(\Sigma)$. Indeed, it follows from the main theorem (or can be easily checked from the above description of left constraints and the observation that any minimal left constraint can be extended by inserting longer and longer runs of $0$, so that $h^*_C(\Sigma) = h_C(\Sigma)$).

We claim that $h_{\text{Fol}}(\Sigma) = 0$. The follower set of a word $w$ is described by giving the set of distinct lengths of the $0$-blocks bounded by letters in $w$ (say $1 \leq \ell_1 < \ell_2 < \ldots < \ell_r$ for some $0 \leq r < n$) together with the lengths $0 \leq \ell_+, \ell_- \leq n$ of the runs of zeroes that begin and end $w$.

We see that $n \geq \sum_{i=1}^r \ell_i \geq \sum_{i=1}^r i \geq r^2/2$. Hence $r \leq 2\sqrt{n}$. Therefore the number of distinct follower sets defined by words of length $n$ is bounded by:

$$(n+1)^2 C\sqrt{n}.$$ 

This proves the claim. \[ \square \]
4 Proofs of basic properties

Proof of Proposition 1 We first observe that the Q.F.T. and weak-Q.F.T. properties are not preserved under extensions or factor maps, already for trivial reasons:

Indeed, take a Q.F.T. $\Sigma_1$ and a subshift $\Sigma_2$ with the same entropy which is not a Q.F.T. Let $\pi|\Sigma_1 = \Id$ whereas $\pi(\Sigma_2)$ is a fixed point. Then consider $\pi : \Sigma_1 \cup \Sigma_2 \to \Sigma_1 \cup \{0\} : \Sigma_1 \cup \{0\}$ is a Q.F.T. with an extension, $\Sigma_1 \cup \Sigma_2$ which is not.

Take now a Q.F.T. $\Sigma_1$ and a subshift $\Sigma_2$ with a strictly smaller entropy which is not a Q.F.T. Let $\pi|\Sigma_2 = \Id$ whereas $\pi(\Sigma_1)$ is a fixed point 0. Then consider $\pi : \Sigma_1 \cup \Sigma_2 \to \Sigma_2 \cup \{0\} : \Sigma_1 \cup \Sigma_2$ is a Q.F.T. with an image $\{0\} \cup \Sigma_2$ which is not.

Now, we have seen that there are sofic subshifts $\Sigma$ with $h_C(\Sigma) > 0$ whereas of course their S.F.T. extension $\Sigma_0$ has $h_C(\Sigma_0) = 0$ so that the left constraint entropy does not always decrease under factor maps.

We finally turn to the invariance of $h_C(\Sigma)$. Let $h : \Sigma' \to \Sigma$ be the conjugacy. We have:

$$(h(x))_i = H(x_{i-L} \ldots x_{i+L}) \quad \forall x \in \Sigma' \forall i \in \mathbb{Z}$$

for some integer $L \geq 0$ and some map $H : \mathcal{A}^{2L+1} \to \mathcal{A}$. Similarly, there is a map $H : \mathcal{A}^{2L+1} \to \mathcal{A}$ for $h^{-1}$ (maybe after increasing $L$).

For all $n$ large enough, we shall construct a map $\psi : \mathcal{C}(\Sigma, n) \to \bigcup_{k=0}^{n} C(\Sigma')$ which is at most $\#\mathcal{A}^{2L}$ to 1. This will clearly imply $h_C(\Sigma) \leq h_C(\Sigma')$ (notice that this would not work if $h_C(\Sigma)$ were defined using lim inf instead of lim sup).

Thus we take $A_{-n} \ldots A_0$ a left constraint of $\Sigma$. We observe that there exist $L, R \in \Sigma$ such that for some $0 \leq k < \infty$:

$L_{-n} \ldots L_0 = A_{-n} \ldots A_0$, $R_{-n+1} \ldots R_0 = A_{-n+1} \ldots A_0$, and $[A_{-n} R_{-n+1} \ldots R_k] = \emptyset$. (2)

Let $L' = h^{-1}(L)$ and $R' = h^{-1}(R)$. Observe that

$L'_{-n+L+1} \ldots L'_{-L} = R'_{-n+L+1} \ldots R'_{-L}$.

Claim. There is $0 \leq \ell \leq 2L$ such that $L'_{-n+L-\ell} \ldots L'_{-L}$ is a left constraint.

The claim will give the map $\psi$ discussed above and therefore the inequality for $h_C(\Sigma)$.

We prove the claim by contradiction. We first observe that $\Fol(L'_{-n+L-\ell} \ldots L'_{-L})$ is evidently non-empty for all $\ell \geq 0$. Hence, if the claim is false, it means that:

$$\Fol(L'_{-n-L} \ldots L'_{-L}) = \Fol(L'_{-n+L+1} \ldots L'_{-L}) = \Fol(R'_{-n+L+1} \ldots R'_{-L}) \ni \Fol(R'_{-n+L+1} \ldots R'_{-L+1})$$

Hence $[L'_{-n-L} \ldots L'_{-n+L} R'_{-n+L+1} \ldots R'_{k+L}] \neq \emptyset$. Applying $h$, we find that $[L_{-n} R_{-n+1} \ldots R_k] \neq \emptyset$, a contradiction. The claim is proved.
We now turn to $h^*_\mathcal{C}(\Sigma)$.
Let $A_{-n} \ldots A_0$, $n \geq 0$, be an extendable left constraint. We can find $L \in \Sigma$ with $L_{-n} \ldots L_0 = A_{-n} \ldots A_0$ such that for infinitely many integers $m \geq 0$, there exist $R^{(m)} \in \Sigma$ and $k^{(m)} \geq 0$ such that:

$$R^{(m)}_{-m+1} \ldots R^{(m)}_0 = L_{-m+1} \ldots L_0 \text{ and } [L_{-m} \ldots L_0 R^{(m)}_1 \ldots R^{(m)}_{k^{(m)}}] = \emptyset.$$ 

Applying the previous argument we obtain for each value of $m$, a left constraint $L'_{-m+L-\ell(m)} \ldots L'_{-n+L}$ with $0 \leq \ell(m) \leq 2L$. Hence we see that $L'_{-m+L-\ell} \ldots L'_{-n+L}$ is indeed an extendable left constraint. □

Proof of Lemma 12: Let $X$, resp. $Y$, be a subshift over the alphabet $A$, resp. $B$. We claim that

$$\text{Fol}(A_{-n}, B_{-n}) \ldots (A_0, B_0) = \text{Fol}(A_{-n} \ldots A_0) \times \text{Fol}(B_{-n} \ldots B_0).$$

Indeed, observe that:

$$(A_{-n}, B_{-n}) \ldots (A_0, B_0) \in \mathcal{C}(X \times Y, n) \iff A_{-n} \ldots A_0 \in \mathcal{C}(X, n) \text{ or } B_{-n} \ldots B_0 \in \mathcal{C}(Y, n).$$

(3)

Thus,

$$\mathcal{C}(X \times Y, n) = \mathcal{C}(X, n) \times \mathcal{L}(Y, n) \cup \mathcal{L}(X, n) \times \mathcal{C}(Y, n)$$

so that $h_\mathcal{C}(X \times Y) = \max(h_\mathcal{C}(X) + h_{\text{top}}(Y), h_{\text{top}}(X) + h_\mathcal{C}(Y))$. This gives the result for $h_\mathcal{C}$.

But it is obvious that the equivalence (3) is also valid for extendable left constraints. This concludes the proof of the Lemma. □

5 Partial conjugacy

We shall build a conjugacy with the following system:

Definition 6 The complete Markov diagram of $\Sigma$ is the graph $\mathcal{D}$ the vertices of which are the left constraints and the arrows: $A_{-n} \ldots A_0 \rightarrow B_{-m} \ldots B_0 B_1$ if and only if: $m \leq n$ and

$$B_{-m} \ldots B_0 = A_{-m} \ldots A_0 \text{ and } \text{Fol}(A_{-n} \ldots A_0 B_1) = \text{Fol}(B_{-m} \ldots B_0)$$

The corresponding Markov shift is denoted by $\hat{\Sigma}$.

Remark. This is a variant of Hofbauer’s Markov diagram. However, it is necessary to use this variant to exploit the bound on $h^*_\mathcal{C}(\Sigma)$. See the Appendix.
Partial isomorphism

The natural projection \( \pi : \hat{\Sigma} \rightarrow \Sigma \) is defined by \( (\pi(\alpha))_n = A \) iff the finite word \( \alpha_n \) ends in \( A \).

**Lemma 14** \( \pi : \hat{\Sigma} \rightarrow \Sigma \) is well-defined.

**Proof:** Let \( \alpha \in \hat{\Sigma} \) and set \( A = \pi(\alpha) \). We have to prove that for all \( n \in \mathbb{Z}, \ p \geq 0 \),

\[
[A_n \ldots A_{n+p}] \neq \emptyset.
\]

But it follows from the definition of the arrows of \( D \) and an immediate induction that:

\[
\text{Fol}(\alpha_{n+p}) = \sigma^p_x (\text{Fol}(\alpha_n) \cap [A_n \ldots A_{n+p}])
\]

As \( \text{Fol}(\alpha_{n+p}) \neq \emptyset \), the lemma is proved. \( \Box \)

The conjugacy will be restricted to a set \( \Sigma_M \subset \Sigma \). Recall Definition 5 of an eventually Markovian.

**Definition 7** \( A \in \Sigma \) is completely Markovian iff \( \sigma^n A \) is eventually Markovian for all \( n \in \mathbb{Z} \).

The set of completely Markovian sequences is denoted by \( \Sigma_M \).

**Proposition 2** The restriction \( \pi : \hat{\Sigma} \rightarrow \Sigma_M \) is a conjugacy.

**Proof:** We define a partial inverse \( i : \Sigma_M \rightarrow \hat{\Sigma} \) to \( \pi \) by the formula:

\[
i(A) = \alpha \text{ with } \alpha_n = A_{n-\ell} \ldots A_n
\]

where \( \ell = \ell(A,n) \) is the minimum integer such that, for all \( k \geq \ell \),

\[
\text{Fol}(A_{n-k} \ldots A_n) = \text{Fol}(A_{n-\ell} \ldots A_n).
\]

We check that for all \( A \in \Sigma_M \) \( i(A) \) is a well-defined element of \( \hat{\Sigma} \):

- As \( \ell \) is chosen minimum, \( A_{n-\ell} \ldots A_n \) is indeed a left constraint, hence a vertex of \( D \);
- Taking \( L \geq \max(\ell(n), \ell(n-1) + 1) \), we have:

\[
\text{Fol}(A_{n-\ell(n)} \ldots A_n) = \text{Fol}(A_{n-L} \ldots A_{n-1}A_n)
\]

\[
= \sigma_+ (\text{Fol}(A_{n-L} \ldots A_{n-1})) \cap A_n
\]

\[
= \text{Fol}(A_{n-1-\ell(n-1)} \ldots A_{n-1}A_n),
\]

hence \( (i(A))_{n-1} \rightarrow (i(A))_n \) is an arrow of \( D \).
It is clear that $\pi \circ i = \text{Id}_{\Sigma_M}$.

It remains to see that $\pi(\Sigma) \subset \Sigma_M$. Let $\alpha \in \hat{\Sigma}$ and $A = \pi(\alpha)$. We have to prove that $\sigma^n A$ is eventually Markovian for all $n \in \mathbb{Z}$. We consider the case $n = 0$, the general case being exactly the same.

Let $m$ be the length of the left constraint $\alpha_0$. $\alpha_{-m}$ is some left constraint $C_{-p} \ldots C_{-1}A_{-m}$ for some $p$. We prove by induction that:

$$\alpha_{-m+k} \text{ is a suffix of } C_{-p} \ldots C_{-1}A_{-m} \ldots A_{-m+k}.$$  

Indeed it is true for $k = 0$ and the definition of $D$ ensures that $\alpha_{-m+k+1} \text{ is a suffix of } \alpha_{-m+k}A_{-m+k+1}$.

Therefore, $\alpha_0$ is the suffix of length $m$ of $C_{-p} \ldots C_{-1}A_{-m} \ldots A_0$. Hence $\alpha_0 = A_{-m} \ldots A_0$. By the same token,

$$\text{Fol}(A_{-q} \ldots A_0) = \text{Fol}(A_{-m} \ldots A_0)$$

for all $q \geq m$. This proves that $A$ is eventually Markovian and concludes the proof of the proposition. □

### 6 Control of the non-Markovian part

We prove the simpler statement for periodic points first:

**Lemma 15** The periodic orbits in $\Sigma \setminus \Sigma_M$ satisfy:

$$\limsup_{n \to \infty} \frac{1}{n} \log \# \{x \in \Sigma \setminus \Sigma_M : \sigma^n x = x\} \leq h^*_C(\Sigma).$$

**Proof:** Let $X_n := \{A \in \Sigma \setminus \Sigma_M : \sigma^n A = A\}$ and take $A \in X_n$. As $A$ is not eventually Markovian, we have, that for infinitely many $k \geq 0$, $A_{-k} \ldots A_0 \in C^*(\Sigma, k+1)$. Hence, for such a $k$, $A_{-k} \ldots A_{-k+n-1} \in C^*(\Sigma, n)$. $A$ being $n$-periodic, this means that $\#X_n \leq n \times \#C^*(\Sigma, n)$. We turn to the measures:

**Proposition 3** Let $\mu$ be a $\sigma$-invariant probability measure with $\mu(\Sigma \setminus \Sigma_M) = 1$. Then

$$h(\mu, \sigma) \leq h^*_C(\Sigma).$$

**Remark.** The above estimate is sharp in that the inequality can be an equality: take the union of 3-shift and of the product of the 2-shift with the symbolic dynamics of an irrational rotation. Then there is a measure on $\Sigma_M$ with entropy $\log 2 = h^*_C(\Sigma) < h_{\text{top}}(\Sigma) = \log 3$.  

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Proof: We fix \( \mu \) as above and bound its entropy. We denote by \( N \) the set of sequences which are not eventually Markovian. We first claim that \( \mu(N) = 1 \).

Indeed, if \( A \) is not eventually Markovian, then

\[
\text{Fol}(A_{-n} \ldots A_0) \subset \neq \text{Fol}(A_{-n+1} \ldots A_0)
\]

for infinitely many \( n \geq 0 \). But eq. (5) is equivalent to:

\[
\sigma_+(A_{-n}) \nsubseteq [A_{-n+1} \ldots A_0]_+
\]

This last condition obviously implies:

\[
\sigma_+(A_{-n}) \nsubseteq [A_{-n+1} \ldots A_{-1}]_+.
\]

Thus, \( \sigma^{-1}A \) is also not eventually Markovian. Therefore \( \sigma^{-1}(N) \subset N \). We have \( \mu(\sigma^{-1}(N)) = \mu(N) \) by the \( \sigma \)-invariance of \( \mu \). Thus, \( \mu(N \Delta \sigma^{-1}(N)) = 0 \). We conclude that

\[
\Sigma \setminus \Sigma_M = \bigcup_{n \in \mathbb{Z}} \sigma^n(N) = N
\]

up to \( \mu \)-negligible sets, hence by ergodicity, \( \mu(N) = 1 \) as claimed. This argument is due to Hofbauer.

We bound the entropy of \( \mu \) by bounding the minimal number of \( n \)-cylinders whose union has measure \( > 1/2 \) (see, e.g., [20]). Let \( \epsilon > 0 \).

Let \( K_0 < \infty \) be such that \( \#\mathcal{C}^*(n) \leq e^{(h_{\Sigma}(\Sigma)+\epsilon)n} \) for all \( n \geq K_0 \). We also assume \( C_{n/K_0} \leq e^{\epsilon n} \) for all large \( n \).

Let \( n(A) = \min\{k \geq K_0 : A_{-k} \ldots A_0 \in \mathcal{C}^*(k+1)\} \). As \( \mu(N) = 1 \), \( n(\cdot) < \infty \) \( \mu \)-a.e.

There exists \( N_0 < \infty \) such that \( n(\cdot) > N_0 \) on a set of measure \( < \epsilon / \log \#A \).

By Birkhoff’s ergodic theorem, there exist an integer \( M_0 < \infty \) and a measurable set \( G_0 \subset \Sigma \) with \( \mu(G_0) > 1/2 \) such that for all \( A \in G_0 \), all \( n \geq M_0 \),

\[
\frac{1}{n} \#\{0 \leq k < n : n(\sigma^k A) > N_0\} < \frac{\epsilon}{\log \#A}.
\]

We may and do assume that \( M_0 \geq N_0 \log \#A / \epsilon \).

It is easy to see that for any \( n \geq M_0 \), any \( A \in G_0 \), \( A_{0} \ldots A_{n-1} \) can be decomposed into:

- segments belonging to some \( \mathcal{C}^*(\ell) \) with \( \ell \geq K_0 \);
- an initial segment of length at most \( N_0 \);
- at most \( \epsilon n / \log \#A \) left-overs.

Thus, the number of \( n \)-cylinders meeting \( G_0 \) is bounded by:

\[
C_{n/K_0} e^{(h_{\Sigma}(\Sigma)+\epsilon)n} \#A^{N_0+\epsilon n / \log \#A} \leq e^{(h_{\Sigma}(\Sigma)+4\epsilon)n}
\]

for all large \( n \).

As \( \epsilon > 0 \) was arbitrary, this proves that \( h(\mu, \sigma) \leq h_{\Sigma}^*(\Sigma) \).
7 Entropy at infinity

**Proposition 4** For any $\epsilon > 0$, there exist a number $\delta > 0$ and a finite subset $D_0 \subset D$ such that any ergodic, $\sigma$-invariant probability measure $\mu$ on $\Sigma$ such that $\mu(\bigcup_{D \in D_0} |D|) < \delta$ satisfies: $h(\mu, \sigma) \leq h_C(\Sigma) + \epsilon$.

**Proof:** Let $\epsilon > 0$. Let $K_0 < \infty$ be such that for all $n \geq K_0$, $\#C(n) \leq e^{(h_C(\Sigma) + \epsilon)n}$. We assume $K_0$ to be large enough so that $C_n^{2\delta n} \leq e^{en}$ for all large $n$.

Let $D_0 := \bigcup_{n \leq K_0} C(n)$. Let $0 < \delta < \epsilon / (K_0 \log A)$ be such that $C_n^{2\delta n} \leq e^{en}$ for all large $n$.

Let $\hat{\mu}$ be as above. We bound its entropy as in the proof of Proposition 3 by finding an upper bound for the number of $n$-cylinders $A_0 \ldots A_{n-1}$ of the form $A = \pi(\alpha)$ with:

\[ \frac{1}{n} \# \{ 0 \leq k < n : \ell(\alpha_k) < K_0 \} \leq \delta \]

and $\ell(\alpha_0) < L_0$ for some large $L_0$, possibly depending on $\hat{\mu}$.

We cut $A_0 \ldots A_{n-1}$ into maximal segments according to whether $\ell(\alpha_k) < K_0$ or not. There are at most $2\delta n$ cutting points. Hence at most $C_n^{2\delta n} \leq e^{en}$ choices of positions.

Each interval below level $K_0$ is described by giving directly the symbols involved. There are at most $\#A^{\delta n} \leq e^{en}$ choices.

Each interval $A_m \ldots A_m+k-1$ above level $K_0$ is in turn divided into subintervals as follows. We start from the end setting $n_0 = m + k$ and, inductively, $n_i+1 = n_i - \ell(\alpha_k)$. We stop at the smallest $i = i_s$ such that $n_i \leq m$. Thus, there are at most $e^{(h_C(\Sigma) + \epsilon)(m + k - n_s)}$ choices of symbols.

We have to find a lower bound for $n_{i_s}$. Observe that $\ell(\alpha_{n_{i_s} - 1}) \leq \ell(\alpha_m) + n_{i_s} - m$.

If $m > 0$, then $\ell(\alpha_m) = K_0$ and $n_{i_s} \geq m - K_0$. The number of choices of symbols for the interval $A_{m+1} \ldots A_{m+k}$ is bounded by $e^{(h_C(\Sigma) + \epsilon)(k + K_0)}$.

If $m = 0$, then $\ell(\alpha_0) \leq L_0$ and the number of choices of symbol is bounded by $e^{(h_C(\Sigma) + \epsilon)(k + L_0)}$.

We notice that there are at most $\delta n + 1$ such intervals.

Taking product, we find a total number choices for $A_0 \ldots A_{n-1}$ bounded by

\[ e^n e^{(h_C(\Sigma) + \epsilon)L_0} e^{h_C(\Sigma) + \epsilon} n e^{(h_C(\Sigma) + \epsilon)K_0(\delta n + 1)} \leq C e^{2\delta n e^{(h_C(\Sigma) + \epsilon)(1 + \delta)n}} \]

with $C = e^{(h_C(\Sigma) + \epsilon)(L_0 + K_0)}$. Thus,

\[ h(\hat{\mu}, \sigma) \leq (h_C(\Sigma) + 4\epsilon). \]

But $\epsilon > 0$ was arbitrary. \qed

8 Proof of the Theorems

We may and do assume that $h_C(\Sigma) = h_{SC}(\Sigma)$ (or $h^*_C(\Sigma) = h^*_{SC}(\Sigma)$ depending on the case). Otherwise replace $\Sigma$ by $\Sigma$.  

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8.1 Structure theorem

We collect the previous results that imply the structure theorem.

The countable oriented graph \( G \) of the statement is the complete Markov diagram. \( \pi : \Sigma(G) \to \Sigma \) is induced by the natural projection \( G \to A \). \( \mathcal{X} \) is \( \Sigma \setminus \Sigma_M \). Then the conjugacy between \( \Sigma(G) \) and \( \Sigma \setminus \mathcal{X} \) is given by Proposition 2. The control on \( \mathcal{X} \) follows from Proposition 3 (for measures) and Lemma 15 (for periodic points). The finiteness at infinity follows from Proposition 4.

This concludes the proof of the structure theorem.

8.2 Main theorem

We deduce Theorem 11 from the structure theorem, Theorem 2. To begin with, we consider the case of a weak-Q.F.T.

The first point follows from the conjugacy (up to measures of entropy \( \leq h_{SC}(\Sigma) \)) using the following result of Gurevich [10]: on each irreducible Markov shift there is at most one maximum measure and obviously there are at most countably irreducible Markov subshifts, as the graph itself is countable. The maximum measures are Bernoulli by Proposition 2 of section 5 of [14] which shows that Markovian measures are weak Bernoulli and therefore Bernoulli by Ornstein’s isomorphism theorem.

Let us prove the estimate on the number of periodic points. The upper bound follows from the definition of topological entropy. We establish the lower bound. First observe that \( \Sigma \), as a subshift on a finite alphabet is expansive and therefore admits a maximum measure (see [9]).

Using the conjugacy of the structure theorem and the variational principle of Gurevich (see [16] for background on Markov shifts), we see that \( G \) has an irreducible subgraph defining a Markov shift which carries a probability measure with entropy equal to its Gurevich entropy equal to \( h_{top}(\Sigma) \). This implies that the the number \( \ell_n \) of loops of length \( n \) at a given vertex in this irreducible subgraph satisfies \( \lim sup_{n \to \infty} \frac{1}{n} \log \ell_n = h_{top}(\Sigma) \) according to Vere-Jones.

But these loops define periodic points for \( \Sigma(G) \) hence for \( \Sigma \) using the embedding \( \pi \). This proves the estimate on periodic points.

We now turn to the case of a Q.F.T.

The finite number of maximum measures follows from Proposition 4.

Deferring the proof on the meromorphy of the zeta function we recall how to deduce the estimate on periodic points from it (one usually finds more delicate estimates, see, e.g., page 101 of [19]).

There are \( 1 \leq t \leq s < \infty, \kappa < 1 \), complex numbers \( \lambda_1, \ldots, \lambda_s \) with moduli \( e^{h_{top}(\Sigma)} \), and positive integers \( q_1, \ldots, q_s \) such that:

\[
\zeta(z) = \psi(z) \prod_{i=1}^{s} (1 - \lambda_i^{-1}z)^{-q_i}
\]

where \( \psi(z) \) is holomorphic and non-zero on \(|z| < \rho\) for some \( \rho > e^{-h_{top}(\Sigma)}/\kappa \).
We compute the logarithmic derivative of each side:

\[ \frac{\zeta'(z)}{\zeta(z)} = -\sum_{n \geq 0} \#\{ x \in \Sigma : \sigma^{n+1}x = x \} z^n \]

and

\[ \frac{\psi'(z)}{\psi(z)} - \sum_{i=1}^{s} \frac{\lambda_i^{-1} q_i}{1 - \lambda_i^{-1} z} = \sum_{n \geq 0} \left( \phi_n - \sum_{i=1}^{s} \lambda_i^{-n-1} q_i \right) z^n \]

with \(|\phi_n| \leq C \kappa e^{nh_{\text{top}}(\Sigma)}\) as \(\psi'(z)/\psi(z)\) is analytic on \(|z| < \rho\). It follows that:

\[ \left| e^{-(n-1)h_{\text{top}}(\Sigma)} \#\{ x \in \Sigma : \sigma^n x = x \} - \sum_{i=1}^{s} e^{n\sqrt{-1}\theta_i} q_i \right| \leq C \kappa^n \]

where \(e^{\sqrt{-1}\theta_i} = \lambda_i/|\lambda_i|\). But, as \(q_i \neq 0\), it follows that

\[ 0 < \limsup_{n \to \infty} \sum_{i=1}^{s} e^{n\sqrt{-1}\theta_i} q_i < \infty \]

The claim on the number of periodic points follows.

It remains to prove the analyticity properties of the zeta function.

**The zeta function**

Lemma 15 immediately implies that \(\zeta_\Sigma(z)\) is a holomorphic function over the disk \(|z| < e^{-h_{\text{top}}(\Sigma)}\). Hence it is enough to prove the result for \(\zeta_{\Sigma_M}(z)\).

Observe that, \(\pi\) being a conjugacy between \(\hat{\Sigma}\) and \(\Sigma_M\), it defines a bijection between periodic points of \(\hat{\Sigma}\) and of \(\Sigma_M\) and this bijection of course preserves minimum period. Hence \(\zeta_{\Sigma_M}(z) = \zeta_{\hat{\Sigma}}(z)\). We study this last function adapting the proof of Hofbauer and Keller from [13].

The Markov diagram \(D\) defines a countable matrix \(K : D \times D \to \{0,1\}\) according to: \(K(i,j) = 1 \iff i \to j\). We observe that:

\[ \zeta_{\hat{\Sigma}}(z) = \exp -\sum_{n \geq 1} \frac{z^n}{n} \text{Tr} K^n \]

where \(\text{Tr} K^n := \sum_{i \in D} (K^n)(i,i)\). We observe that for each \(n\), \(\text{Tr} K^n < \infty\). In fact it is bounded by \(\#\mathcal{L}(\Sigma,n)\) which is at most of the order of \(e^{nh_{\text{top}}(\Sigma)}\). This proves the analyticity claim.

For convenience we assume some identification of \(D\) with \(N\). Let \(n \geq 1\) be some integer. Write \(K = (A \cup B)\) where \(A\) is a \(n \times n\)-submatrix.

Let \(k > n\) be some other integer. Write \(\tilde{K}\) for the finite matrix obtained by truncation of \(K\) to the indices \((i,j)\) with \(\text{max}(i,j) \leq k\). Write \(\tilde{K} = (A \cup B)\).
Claim. A. Given $\epsilon > 0$, the spectral radius of $\tilde{B}$, $\rho(\tilde{B})$, is bounded by $e^{hc(\Sigma)+\epsilon}$ as soon as $n \geq n_0(\epsilon)$.

Indeed, each coefficient $(\tilde{B}^m)_{ij}$ is bounded by the number of paths of length $m$ on the subset of $D$ corresponding to the integers $\geq n$ and starting at $i$ and ending at $j$. Therefore (cf. the proof of Proposition 4) $\|\tilde{B}^m\|$ grows at most like $C_{n_0}(i)e^{(h_c(\Sigma)+\epsilon)m}$ if $n$ is large enough. This proves the claim.

Lemma 16 [13, Lemma 2] If $L$ is a finite matrix and if $L = \left( \begin{array}{cc} L_{11} & L_{12} \\ L_{21} & L_{22} \end{array} \right)$ is a block decomposition with $L_{22}$ invertible, then:

$$\det L = \det L_{22} \det(L_{11} - L_{12}L_{22}^{-1}L_{21})$$

We apply this Lemma to $I - z\tilde{K} = \left( \begin{array}{cc} I - zA & -z\tilde{U} \\ -z\tilde{V} & I - z\tilde{B} \end{array} \right)$ ($I$ denotes each time the identity matrix of the required dimensions). This is possible because, by the previous claim, for all $|z| < e^{-h_c(\Sigma)}$, $I - z\tilde{B}$ is invertible. Thus,

$$\det(I - z\tilde{K}) = \exp \det(I - z\tilde{B}) \times \det(I - zA - z^2\tilde{U}(I - z\tilde{B})^{-1}\tilde{V}).$$

Hence

$$\exp \left( - \sum_{m \geq 1} \frac{z^m}{m} \text{Tr} \tilde{K}^m \right) = \exp \left( - \sum_{m \geq 1} \frac{z^m}{m} \text{Tr} \tilde{B}^m \right) \times \det(I - zA - z^2\tilde{U}(I - z\tilde{B})^{-1}\tilde{V}).$$

Claim. B. We have uniform convergence on all compact subsets of $|z| < e^{-h_{\text{top}}(\Sigma)}$ when $k \to \infty$, of

$$\exp \left( \sum_{m \geq 1} \frac{z^m}{m} \text{Tr} \tilde{K}^m \right) \to \exp \left( \sum_{m \geq 1} \frac{z^m}{m} \text{Tr} K^m \right) = \zeta_\Sigma(z)^{-1}.$$

The same is true for $|z| < e^{-h_c(\Sigma)}$ for $B$ instead of $K$. Call $B_n(z)$ the resulting analytic function.

The claim follows from routine arguments if

$$0 \leq \text{Tr} \tilde{K}^m \leq \text{Tr} K^m \leq C_{\epsilon}e^{(h_{\text{top}}(\Sigma)+\epsilon)m}$$

and, (for $m \geq m(\epsilon)$)

$$0 \leq \text{Tr} \tilde{B}^m \leq \text{Tr} B^m \leq C_{\epsilon}e^{(h_c(\Sigma)+\epsilon)m}$$

with constants $C_{\epsilon}$ independent of $k$.

These inequalities requires a little care since they a priori involve infinitely many coefficients, in contrast to Claim A.
For the first inequality, it is enough to remark that
\[
\text{Tr}K^m \leq \# \{ x \in \Sigma : \sigma^m x = x \}
\]
using that \( \pi \) embeds \( \Sigma \) into \( \Sigma \). The inequality follows from the definition of the entropy of \( \Sigma \).

We turn to \( \text{Tr}B^m \). It is obviously bounded by the number of closed paths of length \( m \) which stay above \( n \) in \( D \). Observe also that, by Proposition 2, it is enough to count the projections on \( \Sigma \) of these loops.

Take one such loop. It determines \( \alpha \in \Sigma \) with \( \sigma^m \alpha = \alpha \). \( \alpha \) projects to \( A \in \Sigma \) with \( \sigma^m A = A \). Set \( i_0 = m \) and define recursively \( i_{s+1} = i_s - \ell(\alpha_{i_s-1}) \) for \( s \geq 0 \) (\( \ell(\gamma) \) is the length of the finite word \( \gamma \)). Let \( S \) be the smallest integer such that \( i_S \leq 0 \). We have cut the sequence \( A_{i_0} \ldots A_{m-1} \) into left constraints with length \( \geq L(n) := \min \{ \ell(\gamma) : \gamma \in D \text{ with } \gamma \geq n \} \).

Consider first the case where \( i_S \leq m \). Recall that any prefix of a left constraint is a left constraint. Hence \( A_{i_0} \ldots A_{i_S+1} \in C(m) \). Such loops of length \( m \) are therefore in numbers bounded by \( m C e^{m(h_c(\Sigma)+\epsilon)} \), as obviously \( \lim_{n \to \infty} L(n) = \infty \).

Now assume that \( -m < i_S \leq 0 \), so that \( 0 < i_S + m \leq m \). Set \( t := 1 \) if \( i_S = 0 \) or \( t := \min \{ s : i_s < i_S + m \} \) otherwise. We consider the cutting of \( A_{i_0} \ldots A_{i_S+m-1} \) into

\[
A_{i_0} \ldots A_{i_S-1}, \ldots, A_{i_{t-1}}, A_t, \ldots A_{i_S+m-1}
\]

where each block is a left constraint and each has a length at least \( L(n) \), except possibly the last. It is now easy to bound the number of such loops by \( C e^{m(h_c(\Sigma)+\epsilon)} \) (cf. the proof of Proposition 4). This concludes the proof of eq. 4.

**Claim. C.** We have uniform convergence on all compact subsets of \( |z| < e^{-(h_c(\Sigma)+\epsilon)} \) when \( k \to \infty \) of the \( n \times n \)-matrices

\[
I - zA - z^2 \tilde{U}(I - zB)^{-1}V \to I - zA - z^2 \tilde{U}(I - zB)^{-1}V. =: D_n(z)
\]

Indeed,

\[
\tilde{U}(I - zB)^{-1}V = \sum_{m \geq 0} z^m \tilde{U}B^m V
\]

and the coefficient \( (\tilde{U}B^m V)_{i,j} \) is the number of paths of length \( m+2 \) going from \( i \) to \( j \) with \( i, j \leq n \) staying above \( n \) and below \( k \). As \( D \) has finite outdegree, a path of length \( m \) starting from \( i \leq n \) cannot go above some integer \( k_1(n,m) \). Hence, \( \tilde{U}B^m V = UB^m V \) as soon as \( k \geq k_1(n,m) \).

Moreover, once again for the same reasons as in the proof of Claim A,

\[
0 \leq (\tilde{U}B^m V)_{i,j} \leq C e^{m(h_c(\Sigma)+\epsilon)}
\]

The claim C follows immediately.
To conclude, we see by Claims B and C that $\zeta(z)^{-1} = B_n(z) \det D_n(z)$ on $|z| < e^{-h_{\text{top}}(\Sigma)}$. But the right hand side has an obvious holomorphic extension to $|z| < e^{-(h_C(\Sigma)+\epsilon)}$ by Claim C.

Finally, by letting $n \to \infty$, we obtain the result on the full disk $|z| < e^{-h}$. This proves the claimed properties of the zeta function and concludes the proof of the Theorem.

Appendix: Hofbauer’s Markov shift

**Definition 8** The Hofbauer Markov diagram of $\Sigma$ is the graph $D^*$ the vertices of which are the follower sets and the arrows:

\[ F \to G \iff \exists A \in \mathcal{A} \ G = \sigma_+(F) \cap [A]. \]

The corresponding Markov shift is denoted by $\hat{\Sigma}^*$.

The natural projection $\pi : \hat{\Sigma}^* \to \Sigma$ is defined by $(\pi(\alpha))_n = A$ iff $\alpha_n \subset [A]$. $\pi$ does not define an isomorphism of the whole of $\hat{\Sigma}^*$, but we have to take a subset:

**Definition 9** $\alpha \in \hat{\Sigma}^*$ is explicitly Markovian if for all $n \in \mathbb{Z}$, there exists $\ell = \ell(n) \geq 0$ such that

\[ \alpha_n = \text{Fol}(A_{n-\ell} \ldots A_n) \]

where $A := \pi(\alpha)$. We write $\hat{\Sigma}_M^*$ for the set of explicitly Markovian sequences of $\hat{\Sigma}^*$.

**Proposition 5** The restriction $\pi : \hat{\Sigma}_M^* \to \Sigma_M$ is a conjugacy.

The proof of this proposition is the same as Proposition 2

**Proposition 6** Let $\hat{\mu}$ be a $\sigma$-invariant probability measure on $\hat{\Sigma}^*$ with $\hat{\mu}(\hat{\Sigma}^* \setminus \hat{\Sigma}_M^*) = 1$. Then

\[ h(\hat{\mu}, \sigma) \leq h_C(\Sigma). \]

**Example.** We prove that the above inequality can be reached and that the entropy of the measure can exceed $h^*_{\text{top}}(\Sigma) < h_C(\Sigma)$.

Let $\Sigma$ be the sofic subshift over the alphabet $\{0, 1, 2\}$ defined by the condition: between two zeroes there are an even number of non-zero symbols. The Hofbauer’s Markov diagram contains the following vertices:

1. Fol(0);
2. Fol(01), Fol(02);
3. Fol(011), Fol(022);
4. Fol(1), Fol(2).
The arrows are the following:

- Fol(0) points to all the vertices of type 2;
- each vertex of type 2 points to all the vertices of type 3;
- each vertex of type 3 points to Fol(0) as well as to all the vertices of type 2;
- each vertex of type 4 points to all vertices of type 4 as well as to Fol(0).

It is easy to check:

- $h^{∗}_{C}(Σ) = 0$ ($Σ$ is sofic);
- $h_{C}(Σ) = \log 2$ ($C(n)$ is the set of words $0\{1,2\}^{n-1}$);
- the non-explicitly Markovian sequences are exactly the paths living on vertices of type 2 and 3 only.

**Proof of the proposition:** We fix $μ$ as above and bound its entropy. Without losing generality, we assume $ˆμ$ to be ergodic. We denote by $ˆN$ the set of sequences in $ˆΣ$ which are not explicitely Markovian. We first claim that:

$ˆΣ \setminus ˆΣ_M = ˆN$ up to a $ˆμ$-negligible set. Indeed, if $α$ is explicitely Markovian, then

$α_0 = \text{Fol}(A_{−n} \ldots A_0)$

for all $n \geq 0$ (where $A = \pi(A)$). Applying $σ_{+}(-\cdot) \cap [A_1]$ to each side of this inclusion

$α_1 = \text{Fol}(A_{−n} \ldots A_0 A_1)$.

Hence $σ_{−1}(N) \subseteq ˆN$. As in the proof of the previous proposition, the claim follows.

Let $μ = π_{+} ˆμ$. We remark that $h(ˆμ, σ) = h(μ, σ)$ as $π : ˆΣ \to Σ$ is countable-to-one. We bound the entropy of $μ$ by bounding the minimal number of $n$-cylinders whose union has measure $> 1/2$ (see, e.g., [20]). Let $ε > 0$.

Let

$ℓ(α) = \min\{ℓ \geq 0 : \exists B_{−ℓ} \ldots B_0 \text{ s.t. } α_0 = \text{Fol}(B_{−ℓ} \ldots B_0)\}$.

$ℓ(α) < ∞$ everywhere. Hence, one can find $L_0 < ∞$ such that $\{α \in ˆΣ : ℓ(α) < L_0\}$ has positive $ˆμ$-measure.

Let $K_0 < ∞$ be such that $#C^∗(n) \leq e^{(h^{∗}_{C}(Σ)+ε)n}$ for all $n \geq K_0$. We assume $K_0 \geq L_0/ε$.

Let $n(α) = \min\{k \geq K_0 : ℓ(σ_{−k}α) < L_0\}$. This is well-defined $μ$-a.e. by ergodicity. There exists $N_0 < ∞$ such that $n(\cdot) > N_0$ on a set of $μ$-measure $< ε/\log #A$. 
By Birkhoff’s ergodic theorem, there exist an integer $M_0 < \infty$ and a measurable set $\hat{G}_0 \subset \Sigma$ with $\mu(\hat{G}_0) > 1/2$ such that for all $\alpha \in \hat{G}_0$, all $n \geq M_0$,
\[
\frac{1}{n} \# \{0 \leq k < n : n(\sigma^k \alpha) > N_0\} < \frac{\epsilon}{\log \#A}.
\]
We may assume that $M_0 \geq N_0 \log \#A/\epsilon$.

It is easy to see that for any $n \geq M_0$, any $\alpha \in G_0$, $A_0 \ldots A_{n-1}$ (recall $A = \pi(\alpha)$) can be decomposed into:

- segments of the form $A_{m-k} \ldots A_m$ with $k := n(\sigma^m \alpha) < N_0$;
- an initial segment of length at most $N_0$;
- at most $\epsilon n/\log \#A$ left-overs.

Let us show that the segments of the first type are essentially left constraints. Fix one such segment $w = A_{m-k} \ldots A_m$ with $k := n(\sigma^m \alpha)$.

\[
\alpha_m = \text{Fol}(B_{-p} \ldots B_0 A_{m-k+1} \ldots A_m) \subset \not\subset \text{Fol}(A_{m-k} \ldots A_m)
\]

Hence, $\text{Fol}(B_{-p} \ldots B_0)$ meets but does not include $[A_{m-k} \ldots A_m]$. Let $C \in [A_{m-k} \ldots A_m] \setminus \text{Fol}(B_{-p} \ldots B_0)$ so that $B_{-p} \ldots B_{-1} C_0 C_1 \ldots \notin \Sigma_+$. Hence, if we let $q$ be the smallest integer such that $B_{-q} \ldots B_{-1} C_0 C_1 \ldots \notin \Sigma_+$, we get that:

\[
B_{-q} \ldots B_{-1} C_0 \ldots C_k \in C(k + q)
\]

Notice that $q \leq \epsilon n$.

Thus, the number of $n$-cylinders meeting $\pi(\hat{G}_0)$ (which has $\mu$-measure $> 1/2$) is bounded by:

\[
C_n^{2n/K_0} e^{(h_C(\Sigma) + \epsilon)(1+\epsilon)n} \#A N_0 + \epsilon n/\log \#A \leq e^{(h_C(\Sigma) + 4\epsilon)(1+\epsilon)n}.
\]

As $\epsilon > 0$ was arbitrary, this proves that $h(\mu, \sigma) \leq h_C(\Sigma)$.

\[\square\]

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