The Equivalence Principle as a Stepping Stone from Special to General Relativity: A Socratic Dialog

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Abstract

In this paper we show how the student can be led to an understanding of the connection between special relativity and general relativity by considering the time dilation effect of clocks placed on the surface of the Earth. This paper is written as a Socratic dialog between a lecturer Sam and a student Kim.
I. SETTING THE SCENE

Sam is in the office and has just finished reading Plato’s *Meno* in which Socrates uses a self-discovery technique to teach a boy Pythagoras’ theorem. Sam is inspired by this dialog and is pondering its applicability to lecturing undergraduate physics when a tap on the door breaks that chain of thought. Kim enters the room looking bleary eyed and pale. “Been out celebrating the last lecture of the year” Sam surmises, little knowing that other things have kept Kim awake.

II. THE DIALOG

Kim: Your lectures on special relativity fascinated me, and when I got home I wondered if I could construct a simple experiment to prove or disprove time dilation, the aspect of special relativity that interests me the most. While lying in bed before dozing off I realised that a clock placed at the equator should run slower than a clock placed at the pole. So I did a little calculation and found that special relativity predicts that a clock on the equator runs slower by about 100 nanoseconds per day with respect to a clock at the pole. While this effect is not large it is certainly measurable with modern atomic clocks. So I went onto the internet to see if I could find any reference to such an experiment and to my surprise I couldn’t.

I was starting to get so frustrated that I couldn’t sleep. I glanced at the clock (3am). I thought to myself “How accurate is my clock? I should check it against internet time.” Then it occurred to me that the world timing standard organisations must mention a latitude effect on local clock accuracies. So I got onto the internet again and checked *The Bureau International des Poids et Mesures (BIPM)* as they calculate the international atomic time (TAI). BIPM calculate TAI from atomic clocks located in more than 30 countries around the world. I was sure that I must find something about the latitude effect on their web site. After spending hours trawling through the site and then other sites on the web, I came up with nothing. There was a discussion of the relativistic effect of placing clocks at high altitudes, but nothing about latitude. In my despair I gave up and collapsed into a fitful sleep.

I came to see you today in the hope that you could cure my insomnia.
Sam: You are in good company in thinking that clocks at the equator and the pole should tick at different rates. Einstein himself predicted as much in his famous 1905 paper on the special theory of relativity. Luckily for physics the effect was not measurable with the instruments of the day as Einstein’s prediction would have failed to match experiment.

Let us return to your findings:

1. According to the special theory of relativity a clock located at the equator should run slower than one at the pole

2. All clocks located at sea-level on the Earth’s surface tick at the same rate, regardless of latitude

To help you understand how both apparently contradictory statements can be true I will ask you a question.

If the Earth was a rotating perfect fluid and we could ignore the gravitational effects of the Sun and the Moon what shape would it be?

Kim: Well, I don’t see how this is relevant, but I would answer your question by drawing a free-body diagram. Can I use your blackboard?
Now let me see . . . consider a test mass placed on the surface of the Earth. We know that the forces acting on the test mass are the outward pointing force due to the difference in pressure and the inward pointing force due to gravity. If the test mass is in hydrostatic equilibrium then the pressure force must be perpendicular to the surface and the sum of the gravitational and pressure forces is the centripetal force, which is perpendicular to the axis of rotation.\[4\] Hmmmm . . . you would have a complicated integral equation to solve because the direction of the gravitational force vector would depend on the distribution of mass, furthermore the pressure gradient would be perpendicular to the surface we are trying to calculate. It seems to be a complicated problem and, to be honest, I am not sure that I could solve it.

Sam: It is a difficult problem and one whose solutions involve hyperbolic and elliptic functions. Chandrasekhar has devoted a whole book to the subject.\[5\] Before we travel that arduous mathematical road let us see if we can use some physics to help us. Taking our model of Earth as a rotating perfect fluid, is the Earth an equipotential surface?

Kim: (Thinks . . .) Yes.

Sam: Why?

Kim: Because if it wasn’t the sea water would feel a force $\vec{F} = -m\nabla \Phi$ and would move until $\nabla \Phi = 0$ everywhere on the surface.

Sam: So if I told you what the Earth’s gravitational field is could you tell me the shape of the Earth?

Kim: Yes, I think I could.

Sam: How?

Kim: If you told me that the Earth’s gravitational field is $\Phi_g(r, \theta)$, where $r$ is the distance from the centre and $\theta$ is the colatitude\[6\] then I could calculate the effective potential felt by an observer co-rotating with the Earth by including the centripetal force:

$$\Phi_{ep} = \Phi_g(r, \theta) - \frac{1}{2} \omega^2 r^2 \sin^2 \theta$$

$\theta \in [0, \pi]$\[1\]

$\theta = 0$ at the north pole, $\pi/2$ at equator and $\pi$ at the south pole
where \( \omega \) is the Earth’s rotation rate and \( r \) is the distance from the centre to the Earth’s surface. The second term on the right hand side of Eq. (1) is the so called “centrifugal potential”. Now we have already argued that a co-rotating observer on the surface of the Earth feels no change in effective potential regardless of their latitude, i.e., \( \Phi_{ep} \) is constant. Furthermore, you have told me that we know what the Earth’s gravitational field \( \Phi_g(r, \theta) \) is, so all I need to do is rearrange equation Eq. (1) and voilà we have an expression for the shape of the Earth’s surface. Mind you, as \( \Phi_g(r, \theta) \) may be a complicated function; I am not sure that I can find an analytic expression for \( r \) anyway.

All this is very interesting, but I don’t see how it answers my question about why clocks tick at the same rate on the Earth’s surface.

Sam: Patience, we are coming to that. First let us investigate the discovery you have made, namely the shape of the Earth. Let me see, I know I have it in here somewhere . . .

*Sam flicks through some notes in the filing cabinet*

Ah here it is. Despite the Earth’s complicated shape with mountains and valleys its gravitational field can be modelled to a fractional accuracy of \( 10^{-14} \) by:

\[
\Phi_g(r, \theta) = \frac{-GM_e}{r} - \frac{J_2GM_ea^2(1 - 3\cos^2 \theta)}{2r^3},
\]

(2)

where

- \( GM_e = 3.98600442 \times 10^{14}\text{m}^3\text{s}^{-2} \) is the product of the gravitational constant and the mass of the Earth\(^7\)

- \( J_2 = 1.082636 \times 10^{-3} \) is a measure of the Earth’s equatorial bulge and is related to the Legendre polynomials\(^9\)

- \( a = 6378137\text{m} \) is the Earth’s equatorial radius\(^10\)

To evaluate your equation for the Earth’s surface (which incidentally is called the Geoid) you will need an accurate value of the Earth’s rotation rate.

*Sam shuffles through some files . . .

Yes here it is\(^11\):

\[
\omega = 7.292116 \times 10^{-5}\text{rad s}^{-1}.
\]
Now your Geoid equation is going to be a bit tricky to solve analytically so instead of doing that let us see if we are on the right track. The easiest thing for us to do is to check that your equation for the Earth’s effective potential $\Phi_{ep}$ is the same at the equator and the pole.

- $\Phi_{ep}$ at the pole: The Earth’s mean polar radius is $\bar{c} = 6356.76 \pm 0.07$ km

\[
\Phi_{ep}(r = \bar{c}, \theta = 0) = -\frac{GM_e}{\bar{c}} + \frac{J_2 GM_e a^2}{\bar{c}^3} = -6.2637 \times 10^7 m^2 s^{-2} \tag{3}
\]

- $\Phi_{ep}$ at the equator: The Earth’s mean equatorial radius is $\bar{a} = 6378.1 \pm 0.2$ km

\[
\Phi_{ep}(r = \bar{a}, \theta = \pi/2) = -\frac{GM_e}{\bar{a}} - \frac{J_2 GM_e a^2}{2\bar{a}^3} - \frac{1}{2} \omega^2 \bar{a}^2 = -6.2637 \times 10^7 m^2 s^{-2} \tag{4}
\]

Look the two values for $\Phi_{ep}$ are the same!

What have you shown?

Kim: We have shown that the Earth is indeed an equipotential surface with respect to an observer sitting on the surface. But Sam, this has nothing to do with the question I originally asked you!

Sam: Doesn’t it? What did you ask me again?

Kim: I asked you why all clocks tick at the same rate on the surface of the Earth when special relativity predicts that they should run slower at the equator than at the pole.

Sam: Kim do you remember how we derived Einstein’s famous formula $E = mc^2$?

Kim: Yes, and to be honest I was a little disappointed with it. Once we learnt that a constant speed of light lead to the Lorentz transformations, the rest was just algebra.

Sam: Remind me of the algebra.

Kim: We got to the point that we realised that the proper time interval, $d\tau$ must be defined as

\[
c^2 d\tau^2 = c^2 dt^2 - d\bar{x}^2, \tag{5}
\]
with $dt$ and $d\vec{x}$ the coordinate time and space interval respectively. Then we simply multiplied equation Eq.(5) by $\frac{m^2c^2}{dt^2}$ to get

$$m^2c^4 = m^2c^4 \left(\frac{dt}{d\tau}\right)^2 - m^2\vec{u}^2c^2$$

equating $\vec{u}$ with $\frac{d\vec{x}}{d\tau}$

$$= m^2c^4\gamma^2 - \vec{p}^2c^2$$

since $\gamma = \frac{dt}{d\tau}$ and $\vec{p} = m\vec{u}$

$$= E^2 - \vec{p}^2c^2$$

since relativistic kinetic energy is $mc\gamma$.

(6)

So if $\vec{p} = 0$, then $E = mc^2$, like I said, just algebra.

Sam: Hmm, yes indeed. Suppose you are floating in a room with no windows or doors. All of a sudden, you feel a force that throws you against the wall. If their were two possible forces, gravitational or centrifugal, are you able to determine which force you are feeling?

Kim: I don’t see how.

Sam: And what would you (sitting in this closed room) say your time dilation was with respect to an observer who was not feeling the centrifugal or gravitational force?

Kim: I think I see what you are getting at. I can’t say whether the force is gravitational or centrifugal, so I must treat their effects as the same. If I knew the force was centrifugal, I would say that my time dilation with respect to a stationary observer depends only on my velocity $v$, i.e., $\gamma = \sqrt{1-v^2/c^2}$. As I don’t know where the energy to thrust me against the wall has come from, to be consistent, I must say that the time dilation depends only on the effective potential, which is the sum of the gravitational and centripetal potentials.

Sam: Excellent! The idea that you can’t know if the force is a uniform gravitational force, or a combination of uniform forces, is called the equivalence principle. What does it tell you about clocks on the surface of the Earth?

Kim: Yes, yes, of course. According to somebody standing anywhere on the surface of the Earth, all their energy is effective potential energy $\Phi_{ep}$. The rate at which their clock ticks depends only on this effective potential. We already showed that the effective potential over the surface of the Earth is constant. So all clocks on the surface of the Earth tick at the same rate. Eureka, I can sleep again!
Sam: Yes, you can sleep well indeed because you have just discovered one of the fundamental arguments that led to the development of the general theory of relativity. Before you go, let me clarify one point. To determine the time dilation, you used the effective potential which came from newtonian arguments about gravitational and centrifugal forces. According to general relativity the newtonian effective potential is an approximation to the relativistic effective potential. This does not change your conclusion in any way, the effective potential is still constant, it just means that in general relativity we have a slightly different version of $\Phi_{ep}$ (see Appendix A). Having said that, you should note that for the Earth, the newtonian and relativistic effective potentials are almost identical. To learn precisely what the difference is, you will have to take my general relativity course, unless you continue to derive general relativity by yourself!

After exchanging pleasantries, Kim leaves for the long cycle home.

Kim reflects that the thought experiment involving a person in a windowless room who didn’t know if the force they felt was gravitational or centrifugal was very similar to the arguments about absolute and relative motion that they learnt in their special relativity course.

Sam contemplates this conversation with Kim and wonders if it should be entered into next year’s general relativity lecture notes.

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1 Meno by Plato Translated with an introduction by Benjamin Jowett [http://etext.library.adelaide.edu.au/p/p71mo/index.html](http://etext.library.adelaide.edu.au/p/p71mo/index.html)

2 The Bureau International des Poids et Mesures (BIPM) [http://www.bipm.fr/](http://www.bipm.fr/)

3 Alex Harvey and Engelbert Schucking, A Small Puzzle from 1905, Physics Today, March 2005
Colatitude is \( \pi/2 \) minus latitude. It is the coordinate of choice for most references on planetary gravitational potentials.

This is the total mass of the Earth including the atmosphere. As we are calculating the gravitational potential at sea level we should subtract the mass of the atmosphere but we do not need to do this as the mass of the atmosphere is roughly one millionth of the total mass of the Earth.

This value of \( a \) is defined to fit the ellipsoidal Earth model WGS84 see for more details.

for more details.
APPENDIX A: GENERAL RELATIVISTIC CORRECTIONS TO THE EFFECTIVE POTENTIAL

According to the general theory of relativity the proper time interval \(d\tau\) for a clock in a weak gravitational field (such as the earth’s) is given by

\[-c^2 d\tau^2 \approx - \left( 1 + 2 \frac{\Phi_g}{c^2} \right) c^2 dt^2 + \left( 1 - 2 \frac{\Phi_g}{c^2} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,
\]

where \(\Phi_g\) is given by Eq.(2) and \(\frac{\Phi_g}{c^2} \ll 1\). For a clock sitting on the surface of the earth \(dr = d\theta = 0\) and \(d\phi = \omega dt\) so the proper time interval is

\[d\tau = dt \sqrt{1 + 2 \frac{\Phi_g}{c^2} \frac{r^2 \omega^2}{c^2} \sin^2 \theta - \frac{2\Phi_{ep}/c^2}{c^2}} \tag{A1}\]

where \(\Phi_{ep}\) is the newtonian gravitational potential (see Eq.(1)). The time dilation effect is obtained by rearranging Eq.(A1):

\[\frac{dt}{d\tau} = \frac{1}{\sqrt{1 + 2 \frac{\Phi_{ep}}{c^2}}} \tag{A2}\]

We have shown in this paper that the weak equivalence principle effectively states that time dilation can be calculated in terms of the effective potential only, i.e.,

\[\frac{dt}{d\tau} = 1 - \frac{\Phi_{ep}^{GR}}{c^2} \tag{A3}\]

where \(\Phi_{ep}^{GR}\) is the relativistic effective potential. The relativistic effective potential can be determined in terms of the newtonian potential by expanding Eq.(A2) and equating it with Eq.(A3):

\[\frac{\Phi_{ep}^{GR}}{c^2} = \frac{\Phi_{ep}}{c^2} - \frac{3}{2} \frac{\Phi_{ep}^2}{c^4} + O \left( \frac{\Phi_{ep}^3}{c^6} \right) \approx \frac{\Phi_{ep}}{c^2} - \frac{3}{2} \frac{\Phi_{ep}^2}{c^4} ; \text{ if } \frac{\Phi_{ep}}{c^2} \ll 1 \tag{A4}\]

Comparing Eq.(A1) with Eq.(1), using the values for \(\Phi_{ep}\) as calculated in Eqs.(3) and (4) we see that the relativistic effective potential differs from the newtonian effective potential to a fractional accuracy of \(10^{-11}\).