Regular Moebius transformations of the space of quaternions

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Abstract Quaternionic Moebius transformations have been investigated for more than 100 years and their properties have been characterized in detail. In recent years G. Gentili and D. C. Struppa introduced a new notion of regular function of a quaternionic variable, which is developing into a quite rich theory. Several properties of regular quaternionic functions are analogous to those of holomorphic functions of one complex variable, although the diversity of the non-commutative setting introduces new phenomena. Unfortunately, the (classical) quaternionic Moebius transformations are not regular. However, in this paper we are able to construct a different class of Moebius-type transformations that are indeed regular. This construction requires several steps: we first find an analog to the Casorati-Weierstrass theorem and use it to prove that the group $\text{Aut}(\mathbb{H})$ of biregular functions on $\mathbb{H}$ coincides with the group of regular affine transformations. We then show that each regular injective function from $\mathbb{H} = \mathbb{H} \cup \{\infty\}$ to itself belongs to a special class of transformations, called regular fractional transformations. Among these, we focus on the ones which map the unit ball $B = \{q \in \mathbb{H} : |q| < 1\}$ onto itself, called regular Moebius transformations. We study their basic properties and we are able to characterize them as the only regular bijections from $B$ to itself.

Keywords Function of one quaternionic variable · Hypercomplex analysis · Quaternionic Moebius transformation · Quaternionic linear fractional transformation · Quaternionic affine transformation · Casorati-Weierstrass theorem

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1 Introduction

Denote by $\mathbb{H}$ the real algebra of quaternions, obtained by endowing $\mathbb{R}^4$ with the following multiplicative operation: if $1, i, j, k$ denotes the standard basis, define

\[ i^2 = j^2 = k^2 = -1, \]

\[ ij = -ji = k, jk = -kj = i, ki = -ik = j, \]

let $1$ be the neutral element and extend the operation by linearity and distributivity to all quaternions $q = x_0 + x_1i + x_2j + x_3k$.

Over the last century, there have been several attempts to identify a class of quaternionic functions serving as the holomorphic functions do in the complex case. The best known is due to Fueter [11–13], who considered solutions of the equation $\frac{\partial f}{\partial \bar{q}} = 0$, where

\[ \frac{\partial}{\partial \bar{q}} = \frac{1}{4}\left( \frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \right). \]

Fueter proved analogs of Cauchy’s theorem and Cauchy’s integral formula for this class of functions (see [27] for an excellent survey). This gave rise to a rich theory that still produces new results and has been extended to other Clifford algebras with the notion of monogenic function (see [3,4] and references therein). Despite the success of this theory, room was left for alternative notions of regularity for quaternionic functions: consider, for instance, the fact that the identity function and the powers $q \mapsto q^2, q \mapsto q^3, \ldots$ fail to comply with Fueter’s definition. An interesting class of functions containing all polynomials of the type $a_0 + qa_1 + \cdots + q^n a_n$ has been defined by Gentili and Struppa in [18,19] on the basis of a notion of regularity, inspired by Cullen’s work [10]. If $\mathcal{S} = \{ q \in \mathbb{H} : q^2 = -1 \}$ denotes the 2-sphere of quaternionic imaginary units and if, for all $I \in \mathcal{S}$, we let $L_I = \mathbb{R} + IR \simeq \mathbb{C}$, their definition can be stated as follows.

**Definition 1.1** Let $\Omega$ be a domain in $\mathbb{H}$. A real differentiable function $f : \Omega \to \mathbb{H}$ is said to be Cullen-regular if, for all $I \in \mathcal{S}$, its restriction $f_I = f|_{\Omega_I}$ to $\Omega_I = \Omega \cap L_I$ is holomorphic, i.e. the function $\bar{\partial}_I f : \Omega_I \to \mathbb{H}$ defined by

\[ \bar{\partial}_I f(x + Iy) = \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy) \]

vanishes identically.

The set of Cullen-regular functions and Fueter’s class do not include each other, as shown by the aforementioned polynomial examples and by the fact that the function $x_0 + ix_1 + jx_2 + kx_3 \mapsto x_0 + ix_1$ solves Fueter’s equation, but it is not Cullen-regular. After identifying $\mathbb{H} = (\mathbb{R} + IR) + (\mathbb{R} + IR)j$ with $\mathbb{C}^2$, the same examples prove that Cullen-regularity does not imply nor is implied by holomorphicity in two complex variables.

The properties of Cullen-regular functions proven in [19] and in the subsequent papers [7,9,14–17,20,22,26] recall holomorphic functions of one complex variable rather than Fueter’s theory. For instance, the zero-sets of Fueter’s functions can have real dimension zero, one, two, or four while the zero-set of a Cullen-regular function consists of isolated points and isolated 2-spheres of a special type. An even more interesting fact is that Fueter’s functions are not open in general, while there is an analog of the open mapping theorem for Cullen-regular functions (these results are surveyed in Sect. 2). Let us also mention that the study of Cullen-regular functions allowed the construction of new theories of functional calculus in non commutative settings (see [5,6,8]). Finally, Cullen-regular functions also have