HEAT KERNEL BOUNDS FOR NONLOCAL OPERATORS WITH SINGULAR KERNELS

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ABSTRACT. We prove sharp two-sided bounds of the fundamental solution for integro-differential operators of order $\alpha \in (0, 2)$ that generate a $d$-dimensional Markov process. The corresponding Dirichlet form is comparable to that of $d$ independent copies of one-dimensional jump processes, i.e., the jumping measure is singular with respect to the $d$-dimensional Lebesgue measure.

1. INTRODUCTION

Heat kernel bounds play an important role in the study of Markov processes and differential operators. In the theory of partial differential equations, corresponding two-sided Gaussian bounds are known as Aronson bounds. Given uniformly elliptic coefficients $(a_{ij})$, it is shown in [Aro68] that the fundamental solution $\Gamma(t, x; s, y)$ of the operator $u \mapsto \partial_t u - \partial_i (a_{ij} \partial_j u)$ satisfies for all $t, s > 0$ and $x, y \in \mathbb{R}^d$ the two-sided estimate

\begin{equation}
(1.1) \quad g_1(t - s, x - y) \leq \Gamma(t, x; s, y) \leq g_2(t - s, x - y),
\end{equation}

where $g_j(t, x) = a_j t^{d/2} \exp(-b_j |x|^2)$ and $a_j, b_j$ are some positive constants. Up to multiplicative constants, the fundamental solution of the Laplacian $-\Delta$ bounds the fundamental solution of any uniformly elliptic operator in divergence form from above and below. One main feature of the result is that no further regularity of $(a_{ij})$ as a function on $\mathbb{R}^d$ is required. Using a more probabilistic language, estimate (1.1) says that the heat kernel of a non-degenerate diffusion is controlled from above and below by the heat kernel of the Brownian Motion.

A similar result for certain integro-differential operators resp. Markov jump processes is obtained in [BL02] and [CK03]. Let $K(t, x; s, y)$ denote the fundamental solution of the operator

\begin{equation}
(1.2) \quad u \mapsto \partial_t u - \text{p.v.} \int_{\mathbb{R}^d} (u(y) - u(x)) J(x, y) \, dy,
\end{equation}

where $J(x, y)$ is symmetric and satisfies for some $\alpha \in (0, 2)$ and $c_1, c_2 > 0$ the relation $c_1 |x - y|^{-d-\alpha} \leq J(x, y) \leq c_2 |x - y|^{-d-\alpha}$ for all $x \neq y$. Then, analogously to (1.1), the authors establish for all $t, s > 0$ and $x, y \in \mathbb{R}^d$ the two-sided estimate

\begin{equation}
(1.3) \quad k_1(t - s, x - y) \leq K(t, x; s, y) \leq k_2(t - s, x - y),
\end{equation}

where $k_j(t, x) = c_j t^{d/2} \left(1 + \frac{t}{|x|^{d/\alpha}}\right)^{\frac{d+\alpha}{\alpha}}$ and $c_j$ denotes some positive constant. Note that the functions $k_1, k_2$ are known to be comparable with the heat kernel of the isotropic...
\(\alpha\)-stable process. As in the case of a diffusion, it turns out that the heat kernel of a non-degenerate jump process behaves like the heat kernel of the corresponding translation-invariant model process. That is, up to multiplicative constants, the fundamental solution of the fractional Laplacian \((-\Delta)^{\alpha/2}\) bounds the fundamental solution of corresponding non-degenerate integro-differential operator of the form (1.2).

In other words, pointwise heat kernel bounds are robust under bounded multiplicative changes of the coefficients. This statement can be seen as the result obtained in [Aro68] for the Brownian Motion and confirmed in [BL02, CK03] for non-degenerate isotropic Lévy stable processes. The main aim of the present work is to show that the robustness result extends to non-degenerate non-isotropic Lévy stable processes.

Let us introduce the main objects of our study. Consider a Markov jump process \(Z_t = (Z_1^t, \ldots, Z_d^t)\), where the coordinate processes \(Z_1^t, \ldots, Z_d^t\) are independent one-dimensional symmetric stable processes of index \(\alpha \in (0, 2)\). The infinitesimal generator of the corresponding semigroup of the process \(Z\) is the integro-differential operator \(L = -(\partial_{11})^{\alpha/2} - (\partial_{22})^{\alpha/2} - \ldots - (\partial_{dd})^{\alpha/2}\), whose symbol resp. multiplier is given by \(\sum_{i=1}^d |\xi_i|^\alpha\). The process \(Z\) resp. its generator are not to be mixed up with the isotropic \(\alpha\)-stable process resp. the fractional Laplace operator \(-\Delta)^{\alpha/2}\), whose symbol is given by \(|\xi|^\alpha\). In this work we show that, up to multiplicative constants, the fundamental solution of the operator \(L\) bounds the fundamental solution of a corresponding non-degenerate integro-differential operator with bounded measurable coefficients. In a more probabilistic fashion: We consider a \(d\)-dimensional pure jump Markov process \(X\) in \(\mathbb{R}^d\) whose jump kernel is comparable to that one of the process \(Z\). We show that the heat kernels of \(Z\) and \(X\) satisfy the same sharp two-sided estimates.

Let us be more precise. Given \(\alpha \in (0, 2)\), let \(\nu\) be a measure on the Borel sets of \(\mathbb{R}^d\) defined by

\[
\nu(dh) = \sum_{i=1}^d |h_i|^{-1-\alpha} dh_i \prod_{j \neq i} \delta_{\{0\}}(dh_j).
\]

Then \(\nu\) is a non-degenerate \(\alpha\)-stable Lévy measure. Its corresponding process is the process \((Z_t)\) up to a multiplicative constant. The measure \(\nu\) charges only sets that have a nonempty intersection with one of the coordinate axes. For \(u \in C_c^\infty(\mathbb{R}^d)\), the corresponding generator \(L\) can be written as follows

\[
Lu(x) = \text{p.v.} \int_{\mathbb{R}^d} (u(x + h) - u(x)) \nu(dh) \quad (x \in \mathbb{R}^d)
\]

and one easily computes for \(\xi \in \mathbb{R}^d\)

\[
\mathcal{F}(-Lu)(\xi) = c_\alpha \left( \sum_{i=1}^d |\xi_i|^\alpha \right) \mathcal{F}(u)(\xi) = c_\alpha \mathcal{F}\left( (-\partial_{11})^{\alpha/2} u + (-\partial_{22})^{\alpha/2} u + \ldots + (-\partial_{dd})^{\alpha/2} u \right)(\xi)
\]
for some constant depending only on $\alpha$. The corresponding Dirichlet form $(\mathcal{E}^{\alpha}, D^{\alpha})$ on $L^2(\mathbb{R}^d)$ is given by

$$D^{\alpha} = \{ u \in L^2(\mathbb{R}^d) | \mathcal{E}^{\alpha}(u,u) < \infty \},$$

$$\mathcal{E}^{\alpha}(u,v) = \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \int_{\mathbb{R}} (u(x + e^i \tau) - u(x)) (v(x + e^i \tau) - v(x)) \frac{d\tau}{|\tau|^{1+\alpha}} \right) dx$$

$$= \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \int_{\mathbb{R}} (u(x + e^i \tau) - u(x)) (v(x + e^i \tau) - v(x)) J^\alpha(x,x + e^i \tau) d\tau \right) dx,$$

where $J^\alpha(x,y) = |y^i - x^i|^{-1-\alpha}$ if, for some $i$, $x^i \neq y^i$ and $x^j = y^j$ for every $j \neq i$. Note that there is no need to specify all values $J^\alpha(x,y)$ with $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$. For simplicity, we set $J^\alpha(x,y) = 0$ if $x^i \neq y^i$ for more than one index $i$.

Now we can explain our main result in detail. Let $\alpha \in (0,2)$ and $\Lambda \geq 1$ be given. Assume $J : \mathbb{R}^d \times \mathbb{R}^d \setminus \text{diag}$ is a non-negative function satisfying for all $x \neq y$

$$\Lambda^{-1} J^\alpha(x,y) \leq J(x,y) \leq \Lambda J^\alpha(x,y). \quad (1.4)$$

Set

$$D = \{ u \in L^2(\mathbb{R}^d) | \mathcal{E}(u,u) < \infty \},$$

$$\mathcal{E}(u,v) = \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \int_{\mathbb{R}} (u(x + e^i \tau) - u(x)) (v(x + e^i \tau) - v(x)) J(x,x + e^i \tau) d\tau \right) dx.$$

Let $C^1_c(\mathbb{R}^d)$ be the space of $C^1(\mathbb{R}^d)$ functions with compact support, and $\overline{C^1_c(\mathbb{R}^d)}^\mathcal{E}_1$ be the closure of $C^1_c(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$ with respect to the metric $(\mathcal{E}_1(\cdot,\cdot))^{1/2}$, where $\mathcal{E}_1(u,v) := \mathcal{E}(u,u)^{1/2}$. $(\mathcal{E}, D)$ is a regular (symmetric) Dirichlet form on $L^2(\mathbb{R}^d)$ where $D = \overline{C^1_c(\mathbb{R}^d)}^\mathcal{E}_1$. Moreover, the corresponding Hunt process $X$ has the Hölder continuous transition density $p_t(x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$, see [Xu13].

Here is the main result of the present work.

**Theorem 1.1.** There exists $C \geq 1$ such that for any $t > 0$, $x, y \in \mathbb{R}^d$

$$C^{-1} t^{-d/\alpha} \prod_{i=1}^{d} \left( 1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha} \leq p_t(x,y) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left( 1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha}.$$

The lower bound on $p_t(x,y)$ has already been established in [Xu13] together with some non-optimal upper bound.

**Theorem 1.2.** [Xu13, Theorem 3.14 and Theorem 4.21] There exists $C \geq 1$ such that for any $t > 0$, $x, y \in \mathbb{R}^d$

$$C^{-1} t^{-d/\alpha} \prod_{i=1}^{d} \left( 1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{1+\alpha} \leq p_t(x,y) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left( 1 \wedge \frac{t^{1/\alpha}}{|x^i - y^i|} \right)^{\alpha/3}.$$

It has been an open problem to establish the matching upper bound. Our result solves this problem and, together with the lower bound from [Xu13], we obtain the desired two-sided heat kernel estimates.

Let us explain the novelty of our paper. In general, obtaining the off-diagonal heat kernel upper bound requires hard work because one has to sum up all the possible trajectories of the process moving from $x$ to $y$ in time $t$. For diffusion processes, the so-called
Davies method (resp. its extension by Carlen-Kusuoka-Stroock) is a very useful analytical method to derive the Gaussian upper bound. For jump processes, if the jumping kernel is comparable to a radially symmetric kernel (namely an isotropic case), then the so-called Meyer’s decomposition that decomposes the jumping kernel into small jumps and large jumps works well. However, for heat kernel estimates in a non-isotropic setting, there is no useful method known so far. In fact, within the framework of non-isotropic stable-like processes, the present work is the first one to establish robustness of heat kernel estimates in a non-isotropic setting. Our method is a self-improvement method of the off-diagonal upper bound. The idea of this method comes from [BGK09], however [BGK09] treats only the isotropic case and our method is much more involved in order to take care of the non-isotropy. We note that the proof of the upper bound of Theorem 1.2 by [Xu13] uses the Davies method; it is an interesting question whether one can prove the optimal estimate with this method or not.

Let us formulate a conjecture that, based on the aforementioned results, looks promising. **Conjecture:** Let $Z$ be a non-degenerate $\alpha$-stable process in $\mathbb{R}^d$ with Lévy measure $\nu$. Let $X$ be a symmetric Markov process whose Dirichlet form has a symmetric jump intensity $J(x,dy)$ that is comparable to the one of $Z$, i.e., $J(x,dy) \approx \nu(x-dy)$. Then the heat kernel of $X$ is comparable to the one of $Z$.

The conjecture is proved in [CK03] in the case $\nu(dh) = |h|^{-d-\alpha}dh$. The present work establishes the conjecture in the non-isotropic singular case

$$\nu(dh) = \sum_{i=1}^{d} |h_i|^{-1-\alpha}dh_i \prod_{j \neq i} \delta_{\{0\}}(dh_j)$$

where $\alpha \in (0,2)$. Both cases are limit cases for non-degenerate $\alpha$-stable Lévy measures. Hence, it is plausible to expect the assertion of the conjecture to be true.

We have already mentioned some related results from the literature. Let us mention some further results related to systems of jump processes driven by stable processes resp. to nonlocal operators with singular jump intensities. These works mainly address regularity questions, which is a closely related topic. The weak Harnack inequality and Hölder regularity estimates for solutions to parabolic equations driven by the Dirichlet form $\mathcal{E}$ under assumption (1.4) have been established in [KS14]. In the elliptic setting, a general approach to the weak Harnack inequality for singular and non-singular cases is developed in [DK20].

Systems of Markov jump processes of the form

$$dY_t^i = \sum_{j=1}^{d} A_{ij}(Y_{t-})dZ_t^j$$

with one-dimensional independent symmetric $\alpha_j$-stable components $Z_t^j$, $j = 1, \ldots, d$, are studied in several works. In the case $\alpha = \alpha_j$ for all $j$, [BC06] establishes unique weak solvability via the martingale problem. Hölder regularity of corresponding harmonic functions is provided in [BC10]. Of course, some conditions on the matrix-valued function $A$ need to be imposed. In [KRS20] the authors prove the strong Feller property for the corresponding stochastic jump process. They show that for any fixed $\gamma \in (0,\alpha)$ the semigroup $(P_t)$ of the process $(X_t)$ satisfies

$$|P_t f(x) - P_t f(y)| \leq c t^{-\frac{\gamma}{2}} |x-y|^\gamma \|f\|_\infty$$
for bounded Borel functions $f$ and $x,y \in \mathbb{R}^d$. The regularity results of [BC10] are extended in [Cha20] to the case where the index of stability $\alpha_j$ is different for different components $Z_j$. Existence and uniqueness for weak solutions in this case is proved in [Cha19]. However, for the uniqueness the article assumes the matrix $A_{ij}$ to be diagonal. Proving uniqueness under natural assumptions in this case seems to be a challenging problem.

In a recent paper [KR18], two-sided heat kernel estimates similar to ours are obtained when the matrix $A_{ij}$ is diagonal and the diagonal elements are bounded and Hölder continuous. They use a method based on Levi’s freezing coefficient argument. We note that their method does not seem to work in our case.

Another interesting open question in this context is the Feller property which is established in [KR20] under the assumption $\min(\alpha_j) \geq 2/3 \max(\alpha_j)$. This condition is not required in the study of Hölder regularity of weak solutions to nonlocal equations stemming from symmetric Dirichlet form as in [KS14, DK20], cf. [CK20].

The paper is organized as follows: In Section 2 we present the main strategy of our proof and provide some auxiliary results. In Subsection 2.1 we formulate three lemmas, which imply our main result. Section 3 is devoted to the proof of these lemmas. In Section 4 we provide the proof of our main auxiliary result, which is Proposition 3.3. Finally, we provide the proof of an important but less innovative auxiliary result (Lemma 2.4) in an appendix.

2. Auxiliary results and strategy of the main proof

In this section we present some auxiliary result and discuss the strategy of our proof. In Subsection 2.1 we explain three lemmas, which are the main building blocks of our proof. In Subsection 2.2 we provide a few auxiliary results. First of all, let us explain the notation that we are using.

**Notation:** As is usual, $\mathbb{N}_0$ denotes the non-negative integers including Zero. For two non-negative functions $f$ and $g$, the notation $f \asymp g$ means that there are positive constants $c_1$ and $c_2$ such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for $f$ and $g$. For $a, b \in \mathbb{R}$, we use $a \wedge b$ for $\min\{a, b\}$ and $a \vee b$ for $\max\{a, b\}$. Given any sequence $(a_n)$ of real numbers and $n_1, n_2 \in \mathbb{N}_0$, we set $\prod_{n=n_1}^{n_2} a_n$ as equal to 1 if $n_1 > n_2$.

2.1. Strategy of the main proof. In this subsection, we present the main strategy of the proof of Theorem 1.1. As explained in the introduction, due to [Xu13] we only need to establish the upper bound, i.e., we will prove the following result:

**Theorem 2.1.** Assume $\alpha \in (0, 2)$. There is a positive constant $C$ such that for all $x, y \in \mathbb{R}^d$, $t > 0$ the following estimate holds:

$$p_t(x,y) \leq Ct^{-d/\alpha} \prod_{i=1}^{d} \left(\frac{t}{|x^i - y^i|^\alpha} \wedge 1\right)^{1+\alpha^{-1}}.$$  \hfill (2.1)

In the remaining part of this section, we explain the skeleton of the proof of Theorem 2.1. We are able to reduce the proof to three auxiliary lemmas, which we approach in the following section. For any $q > 0$ and $l \in \{1, \ldots, d-1\}$, we consider the following conditions.
\((H^0_q)\) There exists a positive constant \(C_0 = C(q, \Lambda, d, \alpha)\) such that for all \(t > 0, x, y \in \mathbb{R}^d\),
\[
p_t(x, y) \leq C_0 t^{-d/\alpha} \prod_{i=1}^{d-l} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^q.
\] (2.2)

\((H^1_q)\) There exists a positive constant \(C_1 = C(l, q, \Lambda, d, \alpha)\) such that for all \(t > 0\) and all \(x, y \in \mathbb{R}^d\) with \(|x^1 - y^1| \leq \ldots \leq |x^d - y^d|\) the following holds true:
\[
p_t(x, y) \leq C_1 t^{-d/\alpha} \prod_{i=1}^{d-l} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^q \prod_{i=d-l+1}^{d} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^{1+\alpha^{-1}}.
\] (2.3)

**Remark 2.2.** Note that the constant \(C_1\) from (2.3) depends on the jumping kernel \(J\) only through the constant \(\Lambda\), i.e., different choices of \(J\) lead to the same estimate as long as (1.4) remains true.

Let us make some further observations.

**Remark 2.3.**

1. The assertion of Theorem 2.1 is equivalent to \((H^1_{1+\alpha^{-1}})\) for any \(l \in \{0, \ldots, d-1\}\).
2. Note that \((H^1_q)\) gets stronger as \(q\) increases, that is, for \(q < q'\), \((H^1_q)\) implies \((H^1_{q'})\).
3. For \(q \in [0, 1 + \frac{1}{\alpha}]\) and \(l \leq l'\), \((H^1_q)\) implies \((H^1_{q'})\).

The above three observations can be established easily. The following lemma shows that (2.3) implies a much stronger result due to Remark 2.2.

**Lemma 2.4.** Assume condition \((H^1_q)\) holds true for some \(q \in [0, 1 + \alpha^{-1}]\). Then with the same constant \(C_1 = C(l, q, \Lambda, d, \alpha)\) for every \(t > 0\), all \(x, y \in \mathbb{R}^d\), and every permutation \(\sigma\) of the indices \(1, \ldots, d\) that satisfies \(|x^{\sigma(1)} - y^{\sigma(1)}| \leq \ldots \leq |x^{\sigma(d)} - y^{\sigma(d)}|\) the following holds true:
\[
p_t(x, y) \leq C_1 t^{-d/\alpha} \prod_{i=1}^{d-l} \left( \frac{t}{|x^{\sigma(i)} - y^{\sigma(i)}|^\alpha} \wedge 1 \right)^q \prod_{i=d-l+1}^{d} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^{1+\alpha^{-1}} (2.4)
\]
\[
\leq C_1 t^{-d/\alpha} \prod_{i=1}^{d-l} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^q \prod_{i=d-l+1}^{d} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^{1+\alpha^{-1}}. (2.5)
\]

**Remark.** Lemma 2.4 reduces the complexity of our iterative argument. In particular, it proves that condition \((H^1_q)\) with \(q \in [0, 1 + \alpha^{-1}]\) implies condition \((H^1_q)\) with \(q \in [0, 1 + \alpha^{-1}]\), which is defined as follows:

\((H^1_q)\) Given \(q > 0\) and \(l \in \{1, \ldots, d-1\}\) there exists a positive constant \(C_l = C(l, q, \Lambda, d, \alpha)\) such that for all \(t > 0\) and all \(x, y \in \mathbb{R}^d\)
\[
p_t(x, y) \leq C_l t^{-d/\alpha} \prod_{i=1}^{d-l} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^q \prod_{i=d-l+1}^{d} \left( \frac{t}{|x^i - y^i|^\alpha} \wedge 1 \right)^{1+\alpha^{-1}}. (2.6)
\]

Lemma 2.4 is trivial if \(l = d\) or \(l = 0\). We provide the proof of Lemma 2.4 in the appendix.

Next, let us explain in detail how the main proof makes use of the condition \((H^1_q)\). Set
\[
\lambda_l := \frac{1}{2} \left( \sum_{i=1}^{d-l-1} (1 + \alpha^{-1})^i \right)^{-1} \quad \text{for} \quad l \in \{0, 1, \ldots, d-2\}, \quad \text{and} \quad \lambda_{d-1} := 1.
\]
Note that \( \frac{1}{2} \left( \frac{\alpha}{1+\alpha} \right)^{2(d-l-1)} \leq \lambda_l \leq \frac{\alpha}{2(1+\alpha)} \) for \( l \in \{0,1,\ldots,d-2\} \), where the first inequality follows from \( \lambda_l \geq \frac{1}{2} \left( (d-l-1) \left( \frac{\alpha+1}{\alpha} \right)^{d-l-1} \right) ^{-1} \geq \frac{1}{2} \left( \frac{\alpha+1}{\alpha} \right)^{-2(d-l-1)} \), which makes use of \( \frac{\alpha+1}{\alpha} > \frac{3}{2} \). Our main aim is to prove assertion \( H_{q+\lambda}^{d-1} \). It will be the last assertion in a sequence of assertions which are proved subsequently in the following order:

\[
(H_0^0) \hookrightarrow (H_{N_0}^0) \hookrightarrow (H_{N_0+1}^0) \hookrightarrow \ldots \hookrightarrow (H_{N_0+l}^0) \\
(H_1^0) \hookrightarrow (H_{N_1}^1) \hookrightarrow \ldots \hookrightarrow (H_{N_1+l}^1) \\
\vdots \\
(H_{d-1}^0) \hookrightarrow (H_{N_{d-1}}^d) \hookrightarrow \ldots \hookrightarrow (H_{N_{d-1}+l}^{d-1})
\]

where \( N_l := \left\lfloor 1 + \frac{\alpha-1}{\lambda_l} \right\rfloor \) for \( l \in \{0,\ldots,d-1\} \). Note \( N_{d-2} = \left\lfloor 1 + \frac{2}{\alpha} + \frac{1}{\alpha^2} \right\rfloor \geq 2 \) and \( N_0 = \left\lfloor 1 + \frac{2}{\alpha} \sum_{i=1}^{d-1} (1 + \alpha^{-1})^i \right\rfloor \geq \frac{2}{\alpha} \sum_{i=1}^{d-1} (1 + \alpha^{-1})^i \geq \left( \frac{3}{2} \right)^{d-1} \).

The above scheme will be established with the help of the following implications. Note that they make use of Lemma 2.4.

**Lemma 2.5.** Assume condition \( H_q^l \) holds true for some \( l \in \{0,\ldots,d-2\} \) and \( q < \alpha^{-1} \). Then \( H_{q+\lambda}^{l+1} \) holds true.

**Lemma 2.6.** Assume condition \( H_q^l \) holds true for some \( l \in \{0,\ldots,d-2\} \) and \( q > \alpha^{-1} \). Then \( H_0^{l+1} \) holds true.

**Lemma 2.7.**

(i) Assume condition \( H_q^{d-1} \) holds true for some \( q < \alpha^{-1} \). Then \( H_{q+\lambda}^{d-1} \) holds true.

(ii) Assume condition \( H_q^{d-1} \) holds true for some \( q > \alpha^{-1} \). Then \( H_{1+\alpha^{-1}}^{d-1} \) holds true.

Note that the case \( q = \alpha^{-1} \) is left open here. In this case one can apply Remark 2.3 (2) and obtain the corresponding conclusion for any \( q < \alpha^{-1} \). Assertion (i) of Lemma 2.7 and assertion of Lemma 2.5 can be seen as one implication \( H_q^l \Rightarrow H_{q+\lambda}^l \) being true for every \( l \in \{0,\ldots,d-1\} \). However, we decide to split the assertion into two cases. As we will see, the proof of Lemma 2.7 is much simpler than the one of Lemma 2.5. However, both rely on our main technical result, Proposition 3.3.

Altogether we have shown that Theorem 2.1 follows once we have established Lemma 2.5, Lemma 2.6 and Lemma 2.7.

2.2. Auxiliary results. In this subsection we provide several auxiliary results.

Let us explain the connection between the kernel \( J \) and the corresponding stochastic process \( X \). The function \( J \) is called the jumping kernel of \( X \) and describes the intensity of jumps of the process \( X \). The formal relation is given by the following Lévy system formula, which can be found in [CK08, Appendix A].

**Lemma 2.8.** For any \( x \in \mathbb{R} \), stopping time \( S \) (with respect to the filtration of \( X \)), and non-negative measurable function \( f \) on \( \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \) with \( f(s,y,y) = 0 \) for all \( y \in \mathbb{R} \) and \( s \geq 0 \), we have

\[
\mathbb{E}^x \left[ \sum_{s \leq S} f(s,X_{s-},X_s) \right] = \mathbb{E}^x \left[ \int_0^S \left( \sum_{i=1}^d \int \mathbb{E}^x \left[ f(s,X_s,X_{s+e_i}h)J(X_s,X_s+e_ih)dh \right] ds \right) \right].
\]
Next, let us introduce some results which will be used in the proofs.

**Proposition 2.9.** [Xu13, Proposition 3.12 (b)] There is a positive constant $C$ such that
\[ p_t(x, y) \leq Ct^{-d/\alpha} \]
for all $t > 0$ and $x, y \in \mathbb{R}^d$. Moreover, $p$ is a continuous function in $t, x, y$.

For any open set $U \subset \mathbb{R}^d$, let $\tau_U := \inf\{t > 0 : X_t \notin U\}$ be the first exit time of the process $X$ from $U$.

**Proposition 2.10.** [Xu13, Proposition 4.4] There is a positive constant $C$ such that
\[ \mathbb{P}^x(\tau_{B(x, R)} < t) \leq CtR^{-\alpha} \]
for all $t > 0$, $R > 0$ and $x \in \mathbb{R}^d$.

For any non-negative Borel functions $f$ on $\mathbb{R}^d$ and for any $t > 0$, $x \in \mathbb{R}^d$, let $\{P_t\}_{t \geq 0}$ be the transition semigroup of $X$ defined by
\[ P_tF(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^d} p_t(x, y)f(y)dy. \]

For any two non-negative measurable functions $f, g$ on $\mathbb{R}^d$, set
\[ (f, g) = \int_{\mathbb{R}^d} f(x)g(x)dx. \]

**Lemma 2.11.** [BGK09, Lemma 2.1] Let $U$ and $V$ be two disjoint non-empty open subsets of $\mathbb{R}^d$ and $f, g$ be non-negative Borel functions on $\mathbb{R}^d$. Let $\tau = \tau_U$ and $\tau' = \tau_V$ be the first exit times from $U$ and $V$, respectively. Then, for all $a, b, t > 0$ such that $a + b = t$, we have
\[ (P_tf, g) \leq \left( \mathbb{E}^x[1_{\{\tau \leq a\}}P_{t-\tau}f(X_{\tau})], g \right) + \left( \mathbb{E}^x[1_{\{\tau' \leq b\}}P_{t-\tau'}g(X_{\tau'})], f \right). \] (2.8)

3. **Proof of Lemma 2.7, Lemma 2.6 and Lemma 2.5.**

The aim of this section is to provide the proofs of Lemma 2.7, Lemma 2.6 and Lemma 2.5. The proofs rely on an involved technical result, Proposition 3.3. In Subsection 3.1 we apply this result and derive the three lemmas. In Section 4 we give the proof of Proposition 3.3.

Recall that $\alpha \in (0, 2)$ and $t > 0$. Given two points $x_0, y_0 \in \mathbb{R}^d$ and $t > 0$, we need to specify their relative position.

**Definition 3.1.** Let $x_0, y_0 \in \mathbb{R}^d$ satisfy $|x_0^i - y_0^i| \leq |x_0^{i+1} - y_0^{i+1}|$ for every $i \in \{1, \ldots, d-1\}$. Let $t > 0$, set $\rho := t^{1/\alpha}$. For $i \in \{1, \ldots, d\}$ define $\theta_i \in \mathbb{Z}$ and $R_i > 0$ such that
\[ \frac{52^i \rho}{\rho} \leq \frac{|x_0^i - y_0^i|}{\rho} < \frac{10^i \rho}{\rho} \quad \text{and} \quad R_i = 2^{\theta_i} \rho. \] (3.1)

Then $\theta_i \leq \theta_{i+1}$ and $R_i \leq R_{i+1}$. We say that condition $R(i_0)$ holds if
\[ \theta_1 \leq \ldots \leq \theta_{i_0 - 1} \leq 0 < 1 \leq \theta_{i_0} \leq \ldots \leq \theta_d. \]

We say that condition $R(d + 1)$ holds if $\theta_1 \leq \ldots \leq \theta_d \leq 0 < 1$.

In the proof of $(H^0_{q_1})$ and $(H^1_{q_1})$ we need to consider arbitrary tuples $(x_0, y_0)$. In specific geometric situations, the implications Lemma 2.7, Lemma 2.6 and Lemma 2.5 follow directly.
Lemma 3.2. Let \( t > 0 \) and \( x_0, y_0 \) be two points in \( \mathbb{R}^d \) satisfying condition \( \mathcal{R}(i_0) \) for \( i_0 \in \{ d - l + 1, \ldots, d + 1 \} \). Assume that \( (H^l_i) \) holds for some \( l \in \{ 0, \ldots, d - 1 \} \) and \( q \geq 0 \). Then
\[
p_l(x_0, y_0) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left( \frac{t}{|x_0 - y_0|^\alpha} \right)^{1+\alpha^{-1}} \tag{3.2}
\]
for some constant \( C > 0 \) independent of \( t \) and \( x_0, y_0 \).

Proof. (i) The case \( \mathcal{R}(d + 1) \) (i.e. \( |x_0^d - y_0^d| < \frac{10}{4} t^{1/\alpha} \)) is simple because, in this case Proposition 2.9 implies \( p_l(x_0, y_0) \leq c t^{-d/\alpha} \). Estimate (3.2) follows.

(ii) Assume, condition \( \mathcal{R}(i_0) \) holds true for some \( i_0 \in \{ d - l + 1, \ldots, d \} \). Then for any \( j \leq d - l \leq i_0 - 1 \)
\[
|x_0^j - y_0^j| < \frac{10}{4} \rho^j \rho \leq \frac{10}{4} \rho \rho \leq \frac{10}{4} \rho \implies \left( \frac{t}{|x_0^j - y_0^j|^\alpha} \right)^{1+\alpha^{-1}} \approx 1.
\]
Hence, \( (H^l_q) \) implies
\[
p_l(x_0, y_0) \leq C t^{-d/\alpha} \prod_{i=1}^{d} \left( \frac{t}{|x_0 - y_0|^\alpha} \right)^{1+\alpha^{-1}}
\]
\[
x \leq t^{-d/\alpha} \prod_{i=1}^{d} \left( \frac{t}{|x_0 - y_0|^\alpha} \right)^{1+\alpha^{-1}}.
\]

Remark. Lemma 3.2 allows us in the arguments below to restrict ourselves to the case \( \mathcal{R}(i_0) \) for \( i_0 \in \{ 1, \ldots, d - l \} \).

Here is our main technical result.

Proposition 3.3. Let \( \alpha \in (0, 2) \). Assume that \( (H^l_q) \) holds true for some \( l \in \{ 0, 1, \ldots, d - 1 \} \) and \( q \in [0, 1 + \alpha^{-1}] \). Let \( t > 0 \), set \( \rho = t^{1/\alpha} \). Consider \( x_0, y_0 \in \mathbb{R}^d \) satisfying the condition \( \mathcal{R}(i_0) \) for some \( i_0 \in \{ d - l + 1, \ldots, d \} \). Let \( f \) be a non-negative Borel function on \( \mathbb{R}^d \) supported in \( B(y_0, \frac{2}{5}) \). For \( j_0 \in \{ i_0, \ldots, d - l \} \) define an exit time \( \tau \) by \( \tau = \tau_{B(x_0, R_j_0/8)} \), where \( R_j = \rho^j \rho \) as in (3.1). Then there exists \( C_{3.3} > 0 \) independent of \( x_0, y_0 \) and \( t \) such that for every \( x \in B(x_0, \frac{2}{5}) \),
\[
\mathbb{E}^x \left[ 1_{\{ \tau \leq t/2 \}} P_{t - \tau} f(X_\tau) \right]
\]
\[
\leq C_{3.3} t^{-d/\alpha} \| f \|_1 \prod_{j=0}^{d+l} \left( \frac{t}{|x_0^j - y_0^j|^\alpha} \right)^{q} \prod_{j=d-l+1}^{d} \left( \frac{t}{|x_0^j - y_0^j|^\alpha} \right)^{1+\alpha^{-1}} \tag{3.3}
\]
\[
\times \left\{ \begin{array}{ll}
\left( \frac{t}{|x_0^j - y_0^j|^\alpha} \right)^{1+q} & \text{if } q < \alpha^{-1} \\
\left( \frac{t}{|x_0^j - y_0^j|^\alpha} \right)^{1+\alpha^{-1}} & \text{if } q > \alpha^{-1}.
\end{array} \right.
\]
We postpone the proof of Proposition 3.3 until Section 4.

3.1. Proof of Lemma 2.7, Lemma 2.6 and Lemma 2.5.

In this subsection we explain how to apply Proposition 3.3 and derive Lemma 2.7, Lemma 2.6 and Lemma 2.5. This proves Theorem 2.1.

Before we provide the actual proofs, let us explain how the estimate (3.3) is used in our approach. Consider non-negative Borel functions \( f, g \) on \( \mathbb{R}^d \) supported in \( B(y_0, \frac{2}{5}) \) and \( B(x_0, \frac{2}{5}) \), respectively. We apply Lemma 2.11 with functions \( f, g \), subsets \( U := \)
Proof of Lemma 2.7. As mentioned in Remark 2.3, \( (H_q^{d-1}) \) implies \( (H_{q+1}^{d-1}) \) for \( q > 1 + \alpha^{-1} \). Thus when proving (ii) we may limit ourselves to the case \( q \in (\frac{1}{\alpha}, 1 + \frac{1}{\alpha}) \) for the rest of the proof. Hence, for (ii) we assume \( (H_q^{d-1}) \) for \( q \in (\frac{1}{\alpha}, 1 + \frac{1}{\alpha}) \). Because of Lemma 3.2, for any \( t > 0 \) we only need to consider the case where \( x_0, y_0 \) satisfy condition \( R(1) \). Recall that \( \rho = t^{1/\alpha} \) and \( R_1 = 2^{\theta_1} \rho \). Applying Proposition 3.3 with \( \iota_0 = \iota_1 = 1 \), for \( x \in B(x_0, t) \) and \( \tau = \tau_{B(x_0, \infty)} \), we obtain

\[
\mathbb{E}^{x} \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{d,3} t^{-d/\alpha} \|f\|_1 \prod_{j=2}^{d} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+\alpha^{-1}} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+q} \quad \text{if } q < \alpha^{-1},
\]

and by (3.4),

\[
\left( \mathbb{E}^{x} \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] , g \right) \leq c t^{-d/\alpha} \|f\|_1 \|g\|_1 \prod_{j=2}^{d} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+\alpha^{-1}} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+q} \quad \text{if } q < \alpha^{-1},
\]

Similarly we obtain the second term of right hand side of (2.8) and therefore,

\[
(P_t f, g) \leq c t^{-d/\alpha} \|f\|_1 \|g\|_1 \prod_{j=2}^{d} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+\alpha^{-1}} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+q} \quad \text{if } q < \alpha^{-1},
\]

Since \( P_t f(x) = \int_{\mathbb{R}^d} p_t(x,y) f(y) dy \) and \( p \) is a continuous function, we obtain for \( t > 0 \) and \( x_0, y_0 \) satisfying the condition \( R(1) \) the following estimate

\[
p_t(x_0, y_0) \leq c t^{-d/\alpha} \prod_{j=2}^{d} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+\alpha^{-1}} \left( \frac{t}{|x_0-y_0|^{\alpha}} \right)^{1+q} \quad \text{if } q < \alpha^{-1},
\]

This proves Lemma 2.7. 

Proof of Lemma 2.6. Let \( l \in \{0, 1, \ldots, d-2\} \), \( t > 0 \) and \( x_0, y_0 \) satisfy the condition \( R(i_0) \) for some \( i_0 \in \{1, \ldots, d-1\} \). As noted in Remark 2.3 \( (H_q^{l}) \) implies \( (H_{q+\alpha}^{l}) \) for \( q > 1 + \alpha^{-1} \). Thus, we limit ourselves to the case \( q \in (\alpha^{-1}, 1 + \alpha^{-1}) \) for the rest of the proof. Assume \( (H_q^{l}) \) for \( q \in (\alpha^{-1}, 1 + \alpha^{-1}) \). Recall that \( \rho = t^{1/\alpha} \) and \( R_1 = 2^{\theta_1} \rho \). By
Proposition 3.3 with $j_0 = d - l$, for $x \in B(x_0, \frac{\rho}{8})$ and $\tau = \tau_{B(x_0, \frac{\rho}{4d-4})}$, we obtain

$$E^x \left[ \mathbb{1}_{\{t \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{3.3} t^{-d/\alpha} \|f\|_1 \prod_{j=d-l}^d \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{1+\alpha^{-1}}.$$  

Similarly to the proof of Lemma 2.7, we obtain for $t > 0$ and for a.e. $(x, y) \in B(x_0, \frac{\rho}{8}) \times B(y_0, \frac{\rho}{8})$,

$$p_t(x, y) \leq ct^{-d/\alpha} \prod_{j=d-l}^d \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{1+\alpha^{-1}},$$

and therefore for $t > 0$ and $x_0, y_0$ satisfying the condition $\mathcal{R}(i_0)$,

$$p_t(x_0, y_0) \leq ct^{-d/\alpha} \prod_{i=d-l}^d \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \right)^{1+\alpha^{-1}}. \quad (3.6)$$

This implies $(H_0^{l+1})$ for $l \in \{0, 1, \ldots, d-2\}$ by Lemma 3.2 and hence we have proved Lemma 2.6.

The proof of Lemma 2.5 is more complicated.

Proof of Lemma 2.5. Let $l \in \{0, 1, \ldots, d-2\}$, $t > 0$ and $x_0, y_0$ satisfy the condition $\mathcal{R}(i_0)$ for some $i_0 \in \{1, \ldots, d-l\}$. Assume $(H_0^l)$ for $0 \leq q < \alpha^{-1}$. Recall that $\rho = t^{1/\alpha}$ and $R_i = 2^{\theta_i} \rho$. By Proposition 3.3 with $j_0 \in \{i_0, \ldots, d-l\}$, we obtain that for $x \in B(x_0, \frac{\rho}{8})$ and $\tau = \tau_{B(x_0, \frac{\rho}{4d-4})}$

$$E^x \left[ \mathbb{1}_{\{t \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \leq C_{3.3} t^{-d/\alpha} \|f\|_1 G_{j_0}(l)$$

where

$$G_{j_0}(l) := \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{1+q} \prod_{j=j_0+1}^{d-l} \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^q \prod_{j=d-l+1}^d \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{1+\alpha^{-1}}.$$  

Similarly to the proof of Lemma 2.7, we obtain for $t > 0$ and for a.e. $(x, y) \in B(x_0, \frac{\rho}{8}) \times B(y_0, \frac{\rho}{8})$

$$p_t(x, y) \leq ct^{-d/\alpha} G_{j_0}(l) \quad \text{for } j_0 \in \{i_0, \ldots, d-l\},$$

and hence for $t > 0$ and $x_0, y_0$ satisfying the condition $\mathcal{R}(i_0)$,

$$p_t(x_0, y_0) \leq ct^{-d/\alpha} \left( G_{i_0}(l) \wedge G_{i_0+1}(l) \wedge \ldots \wedge G_{d-l}(l) \right). \quad (3.7)$$

Given $l \in \{0, 1, \ldots, d-2\}$, in order to obtain $\lambda_l$ in Lemma 2.5, we first define $\lambda_l^0$ inductively for $i_0 \in \{1, \ldots, d-l\}$. First, let $i_0 = d-l$ and $j_0 = d-l$. By (3.7), for $t > 0$ and $x_0, y_0$ satisfying $\mathcal{R}(d-l)$, we obtain that

$$p_t(x_0, y_0) \leq ct^{-d/\alpha} \left( \frac{t}{|x_0^d - y_0^d|^{\alpha}} \right)^{1+q} \prod_{j=d-l+1}^d \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{1+\alpha^{-1}}$$

$$\times ct^{-d/\alpha} \prod_{j=1}^{d-l} \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{\lambda_l^0} \prod_{j=d-l+1}^d \left( \frac{t}{|x_0^j - y_0^j|^{\alpha}} \right)^{1+\alpha^{-1}}. \quad (3.8)$$

where $\lambda_l^0 := 1$. If $i_0 \in \{1, \ldots, d-l-1\}$, we set

$$\lambda_l^{i_0} := \frac{1}{2} \left( \sum_{i=1}^{d-l-i_0} (1 + \alpha^{-1})^i \right)^{-1} \leq \frac{1}{2}.$$
For $a, b \in \mathbb{N}_0$, define $\Theta(a, b) := \sum_{j=a}^{b} \theta_j$ if $a \leq b$, and otherwise $\Theta(a, b) := 0$. If $0 < \lambda_i^{l_0} \leq \frac{\theta_{j_0} - \Theta(i_0, j_0 - 1)}{\Theta(i_0, d_l - 1)}$ for some $j_0 \in \{i_0, \ldots, d_l - 1\}$, since $\frac{t}{t_j^{l_0}} = 2^{-\alpha \theta_j}$, we obtain

$$
\left( \frac{t}{t_j^{l_0}} \right)^{1+q} \prod_{j=j_0+1}^{d_l} \left( \frac{t}{t_j^{l_0}} \right)^q = 2^{-\theta_{j_0}(1+q)} 2^{-\Theta(j_0+1, d_l-1) \alpha q}
$$

$$
\leq 2^{-\alpha \Theta(i_0, d_l-1)(q+\lambda_i^{l_0})} = \prod_{j=i_0}^{d_l} \left( \frac{t}{t_j^{l_0}} \right)^{q+\lambda_i^{l_0}}.
$$

(3.9)

On the other hand, if $\lambda_i^{l_0} > \max_{j_0 \in \{i_0, \ldots, d_l-1\}} \left( \frac{\theta_{j_0} - \Theta(i_0, j_0 - 1)}{\Theta(i_0, d_l - 1)} \right)$, since

$$
\lambda_i^{l_0} > \frac{\theta_{i_0+1}}{\Theta(i_0, d_l - 1)} - \lambda_i^{l_0} q
$$

$$
\implies (1+q)\lambda_i^{l_0} \geq \frac{\theta_{i_0+1}}{\Theta(i_0, d_l - 1)},
$$

$$
\lambda_i^{l_0} > \frac{\theta_{i_0+2}}{\Theta(i_0, d_l - 1)} - (1+(1+q))\lambda_i^{l_0} q
$$

$$
\implies (1+q)^2\lambda_i^{l_0} \geq \frac{\theta_{i_0+2}}{\Theta(i_0, d_l - 1)},
$$

$$
\vdots
$$

$$
\lambda_j^{l_0} > \frac{\theta_{d_l-1}}{\Theta(i_0, d_l - 1)} - \left( \sum_{i=1}^{d_l-1-i_0} (1+q)^i \right) \lambda_j^{l_0} q
$$

$$
\implies (1+q)^{d_l-1-i_0}\lambda_j^{l_0} \geq \frac{\theta_{d_l-1}}{\Theta(i_0, d_l - 1)},
$$

for $q < \alpha^{-1}$, we have that

$$
\frac{\Theta(i_0, d_l-1)}{\Theta(i_0, d_l-1)} (1+q) \leq \lambda_i^{l_0} \sum_{i=1}^{d_l-1-i_0} (1+q)^i \leq \lambda_i^{l_0} \sum_{i=1}^{d_l-1-i_0} (1+\alpha^{-1})^i \leq \frac{1}{2}.
$$

These observations imply that

$$
\frac{\theta_{d_l-1}}{\Theta(i_0, d_l-1)} (1+q) = 1 - \frac{\Theta(i_0, d_l-1)}{\Theta(i_0, d_l-1)} (1+q) \geq \frac{1}{2} \geq \lambda_i^{l_0},
$$

and hence

$$
\left( \frac{t}{t_j^{l_0}} \right)^{1+q} = 2^{-\alpha \theta_{d_l-1}(1+q)} \leq 2^{-\alpha \Theta(i_0, d_l-1)(q+\lambda_i^{l_0})} = \prod_{j=i_0}^{d_l} \left( \frac{t}{t_j^{l_0}} \right)^{q+\lambda_i^{l_0}}.
$$

(3.10)

Since $R_i \asymp |x_0^i - y_0^i|$ (see (3.1)), (3.9)–(3.10) yield that

$$
(G_{i_0}(l) \wedge G_{i_0+1}(l) \wedge \ldots \wedge G_{d_l-1}(l)) \leq c \prod_{j=i_0}^{d_l} \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \right)^{q+\lambda_i^{l_0}} \prod_{i=d_l-1-i_0}^{d_l} \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \right)^{1+\alpha^{-1}}.
$$

Combining the above inequality with (3.7), for any $t > 0$ and $x_0, y_0$ satisfying $\mathcal{R}(i_0)$, $i_0 \in \{1, \ldots, d_l - 1\}$, we obtain

$$
p_t(x_0, y_0) \leq c t^{-d/\alpha} \prod_{j=i_0}^{d_l} \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \right)^{q+\lambda_i^{l_0}} \prod_{i=d_l-1-i_0}^{d_l} \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \right)^{1+\alpha^{-1}}
$$

$$
\asymp t^{-d/\alpha} \prod_{i=1}^{d_l} \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \wedge 1 \right)^{q+\lambda_i^{l_0}} \prod_{i=d_l-1-i_0}^{d_l} \left( \frac{t}{|x_0^i - y_0^i|^{\alpha}} \wedge 1 \right)^{1+\alpha^{-1}},
$$

(3.11)
where \( \lambda_i^{\alpha_0} = \frac{1}{2} \left( \sum_{i=1}^{d-l-\alpha_0} (1 + \alpha^{-1})^i \right)^{-1} \). Therefore by (3.8) and (3.11) in connection with Lemma 3.2, we have \( H_l^{i+\lambda_i} \) with \( \lambda_i := \min_{i \in \{1,2,\ldots,d-l\}} \lambda_i^{\alpha_0} = \frac{1}{2} \left( \sum_{i=1}^{d-l-\alpha_0} (1 + \alpha^{-1})^i \right)^{-1} \) for \( l \in \{0,1,\ldots,d-2\} \). Hence the proof of Lemma 2.5 is complete. \( \blacksquare \)

Using Proposition 3.3 we have established Lemma 2.7, Lemma 2.6 and Lemma 2.5. This completes the proof of Theorem 2.1.

4. PROOF OF PROPOSITION 3.3

In this section, we present the proof of our main technical result, Proposition 3.3. The proof is based on several auxiliary observations and computations.

We first introduce a decomposition of \( \mathbb{R}^d \) given by sets \( D_k \subset \mathbb{R}^d \). Later, in the main proof we fix \( y_0 \in \mathbb{R}^d \) and \( \rho > 0 \), and work with sets \( A_k = y_0 + \rho D_k \).

**Definition 4.1.**

1. Define \( D_0 = \bigcup_{i=1}^{d} \{ |x_i| < 1 \} \cup (-2,2)^d \).
2. Given \( k \in \mathbb{N}, \gamma \in \mathbb{N}_0^d \) with \( \sum_{i=1}^{d} \gamma_i = k \) and \( \epsilon \in \{-1,1\}^d \), define a box (hyper-rectangle) \( D_k^{\gamma,\epsilon} \) by
   \[
   D_k^{\gamma,\epsilon} = \epsilon_1[2^{\gamma_1},2^{\gamma_1+1}) \times \epsilon_2[2^{\gamma_2},2^{\gamma_2+1}) \times \cdots \times \epsilon_d[2^{\gamma_d},2^{\gamma_d+1}) .
   \]
3. Given \( k \in \mathbb{N} \) and \( \gamma \in \mathbb{N}_0^d \) with \( \sum_{i=1}^{d} \gamma_i = k \), define \( D_k^{\gamma} = \bigcup_{\epsilon \in \{-1,1\}^d} D_k^{\gamma,\epsilon} \).

Next, we define shifted boxes with center \( y_0 \). For \( y_0 \in \mathbb{R}^d \), \( t > 0 \) and \( \rho = t^{1/\alpha} \) let \( A_0 := y_0 + \rho D_0 \). Given \( k \in \mathbb{N}, \gamma \in \mathbb{N}_0^d \) with \( \sum_{i=1}^{d} \gamma_i = k \) and \( \epsilon \in \{-1,1\}^d \), we define
\[
A_{k,\gamma,\epsilon} = y_0 + \rho D_k^{\gamma,\epsilon} ,
\]
\[
A_{k,\gamma} = y_0 + \rho D_k^{\gamma} ,
\]
\[
A_k = y_0 + \rho D_k .
\]

Let us collect some useful properties of the boxes resp. the corresponding decomposition. We formulate the results for the sets \( D_k \) but they imply corresponding results for the sets \( A_k \) such as \( A_k \cap A_l = \emptyset \) for \( k \neq l \) and \( \bigcup_{k=0}^{\infty} A_k = \mathbb{R}^d \).

**Lemma 4.2.** Let \( k \in \mathbb{N}_0, \gamma \in \mathbb{N}_0^d \) and \( \epsilon \in \{-1,1\}^d \).

1. Given \( k \in \mathbb{N}, \gamma \) with \( \sum_{i=1}^{d} \gamma_i = k \), there are \( 2^d \) sets of the form \( D_k^{\gamma,\epsilon} \), and \( |D_k^{\gamma,\epsilon}| = \prod_{i=1}^{d} 2^{\gamma_i} = 2^k \).
2. Given \( k \in \mathbb{N}, \epsilon \in \{-1,1\}^d \), there are \( \binom{d+k-1}{d-1} \) sets \( D_k^{\gamma,\epsilon} \) with \( \sum_{i=1}^{d} \gamma_i = k \).

Thus, the set \( D_k \) consist of \( 2^d \binom{d+k-1}{d-1} \) disjoint boxes.
3. \( D_k \cap D_l = \emptyset \) if \( k \neq l \) and \( \bigcup_{k=0}^{\infty} D_k = \mathbb{R}^d \).
Since the proof is elementary, we omit it.

The proof of Proposition 3.3 is based on several technical observations. We assume $l \in \{0, 1, \ldots, d - 1\}$ and $i_0 \in \{1, \ldots, d - l\}$. We assume $t > 0$ and $x_0, y_0 \in \mathbb{R}^d$ such that the condition $\mathcal{R}(i_0)$ is satisfied. We set $\rho = t^{1/\alpha}$. The main idea is to use a decomposition of the left-hand side in (3.3) according to the following:

$$
\mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_{\tau}) \right] = \sum_{k=0}^{\infty} \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_{\tau} \in A_k\}} P_{t-\tau} f(X_{\tau}) \right],
$$

where $x \in B(x_0, \frac{\rho}{8})$, $s(j_0) = \frac{R_{j_0}}{s}$, and $\tau = \tau_{B(x_0, s(j_0))}$. This approach requires a careful tracking of the random points $X_{\tau}$. We will need some refinements of the decomposition (4.1). To this end, set

$$
\Phi(k) := \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_{\tau} \in A_k\}} P_{t-\tau} f(X_{\tau}) \right] \quad \text{for } k \in \mathbb{N}_0,
$$

$$
\Phi^i(k) := \mathbb{E}^x \left[ \mathbb{1}_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_{\tau} \in A^i_k\}} P_{t-\tau} f(X_{\tau}) \right] \quad \text{for } k \in \mathbb{N}_0, \ i \in \{1, 2, \ldots, d\},
$$

**Figure 1.** The sets $A_k$ and $B_{s_0}(x_0) := B(x_0, s(j_0))$
where, for $k \in \mathbb{N}_0$ and $i \in \{1, 2, \ldots, d\}$, we set

\[ A_k^i := A_k \cap \bigcup_{u \in B(x_0, s(j_0))} \{ u + he_i \mid h \in \mathbb{R} \}. \]

The set $A_k^i$ contains all possible points in $A_k$ that the process $X$ can jump to when it leaves the ball $B(x_0, s(j_0))$ by a jump in direction $i$. Note that the set $A_k^i$ depends on $j_0$ resp. $s(j_0)$, too. The decomposition (4.1) now can be written in the following way:

\[ \mathbb{E}^x \left[ \mathbf{1}_{\{ \tau \leq t/2 \}} P_{t-\tau} f(X_\tau) \right] = \sum_{k=0}^{\infty} \Phi(k) = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{d} \Phi^i(k) \right) + \Phi(0) = \sum_{k=1}^{\infty} \Phi^d(k) + \sum_{k=1}^{\infty} \Phi^{d-1}(k) + \ldots + \sum_{k=1}^{\infty} \Phi^1(k) + \Phi(0) \]  

(4.4)

Next, let us quantify the position of the random point $X_\tau$, which the process jumps to when leaving the ball $B(x_0, s(j_0))$. Recall that for $i = 1, 2, \ldots, d$, in Definition 3.1, we have set $R_i = 2^\theta_i \rho$, which implies

\[ \frac{5}{4} R_i \leq |x_0^i - y_0^i| < \frac{10}{4} R_i. \]

**Lemma 4.3.** Recall that $x_0, y_0 \in \mathbb{R}^d$ satisfy condition $\mathcal{R}(i_0)$ for some $i_0$ and that $s(j_0) = \frac{R_{j_0}}{8}$ for $j_0 \in \{i_0, \ldots, d\}$. The following holds true:

\[ \bigcup_{u \in B(x_0, s(j_0))} \{ u + he_i \mid h \in \mathbb{R} \} \subset y_0 + \left( \bigotimes_{j=1}^{j_0-1} \mathcal{J}_{\theta_j} \times \mathbb{R} \times \bigotimes_{j=j_0}^{d} \mathcal{I}_{\theta_j} \right) \quad \text{if } i < j_0, \]

\[ \bigcup_{u \in B(x_0, s(j_0))} \{ u + he_i \mid h \in \mathbb{R} \} \subset y_0 + \left( \bigotimes_{j=1}^{j_0-1} \mathcal{J}_{\theta_j} \times \mathcal{I}_{\theta_j} \times \mathbb{R} \times \bigotimes_{j=j_0}^{d} \mathcal{I}_{\theta_j} \right) \quad \text{if } i \geq j_0, \]

where $\mathcal{I}_{\theta_j} := \pm[2^\theta_j \rho, 2^{\theta_j+2} \rho)$, $\mathcal{J}_{\theta_j} := \pm[0, 2^{\theta_j+2} \rho)$, and $\theta_j, \ldots, \theta_d \in \mathbb{N}$.

**Proof.** The set on the left-hand side describes all possible points that the jump process can jump to when leaving the set $B(x_0, s(j_0))$ by a jump in the $i$-th coordinate direction. In the coordinate direction $k$ for $k \leq j_0$ it might happen that the ball $B(x_0, s(j_0))$ intersects a coordinate axis.

Take $z \in \bigcup_{u \in B(x_0, s(j_0))} \{ u + he_i \mid h \in \mathbb{R} \}$ for $i \in \{1, \ldots, d\}$. For any $j \in \{j_0, \ldots, d\}\{i\}$, since $R_j \geq R_{j_0}$ and $u^j = z^j$, it holds that

\[ R_j \leq \frac{5}{4} R_j - s(j_0) \leq |x_0^j - y_0^j| - |x_0^j - z^j| \leq |z^j - y_0^j| \leq |x_0^j - y_0^j| + |x_0^j - z^j| \leq \frac{10}{4} R_j + s(j_0) < 4R_j, \]

and therefore $|z^j - y_0^j| \leq [2^\theta_j \rho, 2^{\theta_j+2} \rho)$ with $\theta_j \in \mathbb{N}$. For any $j \in \{1, \ldots, j_0 - 1\}\{i\}$, since $R_j < R_{j_0}$ and $u^j = z^j$,

\[ 0 \leq |z^j - y_0^j| \leq |x_0^j - y_0^j| + |x_0^j - z^j| \leq \frac{10}{4} R_j + s(j_0) < 4R_{j_0}, \]

and $|z^j - y_0^j| \in [0, 2^{\theta_{j_0}+2} \rho)$ with $\theta_{j_0} \in \mathbb{N}$. The proof of the lemma is complete.

**Remark 4.4.** Lemma 4.3 implies the following observation for $k \in \mathbb{N}_0$ and $i \in \{1, \ldots, d\}$:

\[ A_k^i \neq \emptyset \quad \implies \quad \begin{cases} k = 0 \text{ or } k \geq \sum_{j \in \{j_0, \ldots, d\}} \theta_j & \text{if } i < j_0, \\ k = 0 \text{ or } k \geq \sum_{j \in \{j_0, \ldots, d\}\{i\}} \theta_j & \text{if } i \geq j_0. \end{cases} \]  

(4.5)
More precisely, for $z \in A_k^z$, $k \geq 1$, there exists $\gamma \in N_0^d$ such that $z \in A_{k,\gamma}$, so that $|z^j - y_0^j| \in [2^{j-\rho}, 2^{j+\rho} \| \rho \) \cap [2^{j-\rho}, 2^{j+1} \| \rho \) \) for $j \in \{j_0, \ldots, d\}\{i\}$ by Lemma 4.3. Since $\theta_j, \gamma_j \in \mathbb{Z}$, $\gamma_j$ is one of $\theta_j$ or $\theta_j + 1$ for $j \in \{j_0, \ldots, d\}\{i\}$. Therefore
\[
k = \sum_{j \in \{1, \ldots, d\}} \gamma_j \geq \sum_{j \in \{j_0, \ldots, d\}} \gamma_j \geq \sum_{j \in \{j_0, \ldots, d\}} \theta_j \quad \text{if } i < j_0,
\]
and $k \geq \sum_{j \in \{j_0, \ldots, d\}\{i\}} \gamma_j \geq \sum_{j \in \{j_0, \ldots, d\}\{i\}} \theta_j \quad \text{if } i \geq j_0.

**Lemma 4.5.** (i) For all $k \in \mathbb{N}$, $i \in \{1, \ldots, d\}$ and $z \in A_k^z$ there is $\gamma \in N_0^d$ with $\sum_{i=0}^d \gamma_i = k$ such that for all $y \in B(y_0, \frac{\rho}{8})$ and $j \in \{1, \ldots, d\}$
\[
|z^j - y^j| \in [2^{j-\rho}, 2^{j+\rho} \| \rho \).
\]
(ii) For all $k \in N_0$, $i \in \{1, \ldots, d\}$, $z \in A_k^z$, $y \in B(y_0, \frac{\rho}{8})$ and all $j \in \{1, \ldots, d\}$
\[
|z^j - y^j| \in [0, 2^{j_0+3} \rho) \quad \text{if } j \in \{1, \ldots, j_0 - 1\}\{i\},
\]
\[
|z^j - y^j| \in [2^{j_0-1} \rho, 2^{j_0+3} \rho) \quad \text{if } j \in \{j_0, \ldots, d\}\{i\}.
\]

**Proof.** (i) Let $k \in \mathbb{N}$, $i \in \{1, \ldots, d\}$ and $z \in A_k^z$. Choose $\gamma \in N_0^d$ such that $z \in A_{k,\gamma}$. Then for all $y \in B(y_0, \frac{\rho}{8})$ and $j \in \{1, \ldots, d\}$,
\[
2^{\gamma_i-1} \rho \leq 2^{\gamma_i} \rho / 8 \leq |z^j - y_0^j| - |y_0^j - y^j| \leq |z^j - y^j| \\
\leq |z^j - y_0^j| + |y_0^j - y^j| \leq 2^{\gamma_i+1} \rho + 8 \rho \leq 2^{\gamma_i+2} \rho,
\]
which proves (4.6). (ii) Let $k \in N_0$, $i \in \{1, \ldots, d\}$, $z \in A_k^z$, $y \in B(y_0, \frac{\rho}{8})$ and $j \in \{1, \ldots, d\}$. If $j \in \{1, \ldots, j_0 - 1\}\{i\}$, by Lemma 4.3
\[
0 \leq |z^j - y^j| \leq |z^j - y_0^j| + |y_0^j - y^j| \leq 2^{j_0+2} \rho + 8 \rho \leq 2^{j_0+3} \rho.
\]
If $j \in \{j_0, \ldots, d\}\{i\}$, again by Lemma 4.3,
\[
2^{\gamma_i-1} \rho \leq 2^{\gamma_i} \rho / 8 \leq |z^j - y_0^j| - |y_0^j - y^j| \leq |z^j - y^j| \\
\leq |z^j - y_0^j| + |y_0^j - y^j| \leq 2^{\gamma_i+2} \rho + 8 \rho \leq 2^{\gamma_i+3} \rho.
\]
This proves (4.8) and (4.7).

Using Remark 4.4, we can refine the decomposition (4.4) further. For $a, b \in \mathbb{N}$, we set
\[
\Theta(a, b) := \sum_{j=a}^{b} \theta_j \quad \text{and} \quad \Upsilon(a, b) := \sum_{j=a}^{b} \gamma_j.
\]
Note that we will make use of the abbreviation $\Upsilon(a, b)$ only much further below. The following decomposition will be the main starting point in our proof:
\[
\mathbb{E}^x \left[ \mathbb{I}_{\{\tau \leq \tau/2\}} P_{t-\tau} f(X_t) \right] \]
\[
= \sum_{k=\Theta(j_0, d) - \theta_d}^{\infty} \Phi^d(k) + \sum_{k=\Theta(j_0, d) - \theta_{d-1}}^{\infty} \Phi^{d-1}(k) + \ldots + \sum_{k=\Theta(j_0, d) - \theta_0}^{\infty} \Phi^0(k)
\]
\[
+ \sum_{k=\Theta(j_0, d)}^{\infty} \left( \Phi^{j_0-1}(k) + \ldots + \Phi^1(k) \right) + \Phi(0)
\]
\[
= \sum_{i=j_0}^{d} S^i + \sum_{i=1}^{j_0-1} T^i + \Phi(0)
\]

(4.10)
where \( S^i := \sum_{k=\Theta(j_0,d)}^{\infty} \Phi^i(k) \) and \( T^i := \sum_{k=\Theta(j_0,d)}^{\infty} \Phi^i(k) \).

Before we continue, let us recall the setting that we are going to use for the rest of Section 4:

We assume \( l \in \{0,1,\ldots,d-1\} \), \( i_0 \in \{1,\ldots,d-l\} \), \( j_0 \in \{i_0,\ldots,d-l\} \), \( x_0, y_0 \in \mathbb{R}^d \) satisfying \( R(i_0) \), \( s(j_0) = R_{i_0}^0/8 \), \( \tau = \tau_{B(x_0,s(j_0))} \).

Note that, because of Proposition 2.10, for \( x \in B(x_0, \xi/8) \) and \( \tau = \tau_{B(x_0,s(j_0))} \),

\[
\mathbb{P}^x(\tau \leq t/2, X_\tau \in A_k^{e_i}) \leq \mathbb{P}^x(\tau \leq t/2) \leq ctR_{\alpha}^{-\alpha}.
\]

(4.11)

Our aim is to estimate \( \Phi(0), S^i, i \in \{j_0,\ldots,d\} \) and \( T^i, i \in \{1,\ldots,j_0-1\} \).

Given \( l \in \{0,\ldots,d-1\} \), consider \( j_0 \in \{i_0,\ldots,d-l\} \) and let

\[
F_{j_0}(l) := \prod_{j=j_0}^{d-l} \left( \frac{t}{R_j^0} \right)^q \prod_{j=d-l+1}^{d} \left( \frac{t}{R_j^0} \right)^{1+\alpha-1}.
\]

(4.12)

Since \( \frac{t}{R_j^0} = 2^{-\theta_j \alpha} \), note that \( F_{j_0}(l) = 2^{-\Theta(j_0,d-l)\alpha q} \cdot 2^{-\Theta(d-l+1,d)\alpha(1+\alpha-1)} \).

Estimates of \( \Phi(0) \). We will apply the decomposition \( \Phi(0) = \sum_{i=1}^{d} \Phi^i(0) \). Let us fix \( i \in \{1,\ldots,d\} \). Our first goal is the derivation of an upper bound for

\[
P_{t-\tau} f(z) = \int_{B(y_0, \xi/8)} p_{t-\tau}(z,y) f(y) dy
\]

(4.13)

for \( z \in A_0^{e_i}, t/2 \leq t-\tau \leq t \) and \( f \) is a non-negative Borel function on \( \mathbb{R}^d \) supported in \( B(y_0, \xi/8) \). The goal is achieved once we have proved (4.20), (4.21) and (4.22).

Fix \( y \in B(y_0, \rho/8) \). Note that trivially \( \left( \frac{t}{|z-y|^\alpha} \right)^{\alpha} \leq 1 \) for any index \( j \). In the following we make use of (4.7) and (4.8), i.e.,

\[
\begin{cases}
|z^j - y^j| \in [0, 2\theta_{j_0}^i + 3 \rho) & \text{if } j \in \{1,\ldots,j_0-1\} \backslash \{i\}, \\
|z^j - y^j| \in [2\theta_{j-1}^i \rho, 2\theta_{j+3}^i \rho) & \text{if } j \in \{j_0,\ldots,d\} \backslash \{i\}.
\end{cases}
\]

Set \( r_j := |z^j - y^j| \) for \( j \in \{1,\ldots,d\} \). We will estimate \( p_{t-\tau}(z,y) \) in dependence of the size of \( r_i \). Given \( r_i \) we choose \( m \in \{0,\ldots,\theta_i\} \) as follows:

1. We choose \( m = 0 \) if \( r_i \leq \rho \), which implies \( r_i \leq r_{d-1} \).
2. We choose \( m = \theta_i \) if \( r_i > \rho 2^\theta_i \).
3. We choose \( m \in \{1,\ldots,\theta_i-1\} \) such that \( \rho 2^{m-1} < r_i \leq \rho 2^{m+1} \) if \( \rho < r_i \leq \rho 2^{\theta_i} \).

We consider different cases for the size of the index \( i \).

Case 1: \( d-l < i \). Condition \( (H^i_d) \) (to be precise \( (H^i_d)^I \)) yields for \( t/2 \leq t-\tau \leq t \)

\[
p_{t-\tau}(z,y) \leq c t^{-d/\alpha} \prod_{j \in \{1,\ldots,d-l-1\}} \left( \frac{t}{r_j^0} \right)^q \prod_{j \in \{d-l+1,\ldots,d\} \backslash \{i\}} \left( \frac{t}{r_j^0} \right)^{1+\alpha-1} \left( \frac{t}{(r_{d-l} \wedge r_i)^{\alpha}} \right)^{1+\alpha-1}.
\]

(4.14)

We want to estimate the term on the right-hand side in (4.14) from above. The case \( m = 0 \) corresponds to the case \( r_i \leq r_{d-1} \). The case \( m = \theta_i \) corresponds to \( r_i > \rho 2^\theta_i \), which implies \( 2^3 r_i > r_{d-1} \) in this case. The above choice of \( m \in \{1,\ldots,\theta_i-1\} \) corresponds to \( \rho 2^m < r_i \leq \rho 2^{m+1} \). Thus, for \( m < \theta_{d-1} \) we have \( r_{d-l} \geq \rho 2^{\theta_{d-1} - 1} \geq \rho 2^m \geq \frac{1}{2} r_i \), and for
Remark. We will apply the aforementioned upper bounds for \(m \geq \theta_{d-l}\) we have \(r_{d-l} < \rho^{2\theta_{d-l}+3} \leq 8\rho^m \leq 8r_i\). We use these inequalities to estimate \(r_{d-l} \land r_i\) and \(r_{d-l} \lor r_i\) in (4.14) and obtain

\[
p_{t-\tau}(z, y) \leq ct^{-d/\alpha} \begin{cases} 2^{-\left(m + \sum_{j=0}^{d-1}\theta_j\right)\alpha q_2\left(\theta_i - \left(\sum_{j=d-i+1}^{d}\theta_j\right)\alpha (1+\alpha^{-1})\right)} & \text{if } 0 \leq m < \theta_{d-l}, \\ 2^{-\left(\sum_{j=d-i+1}^{d}\theta_j\right)\alpha (1+\alpha^{-1})} & \text{if } \theta_{d-l} \leq m \leq \theta_i, \end{cases}
\]

where one might need to change the constant \(c\) appropriately. Equivalently, we write

\[
p_{t-\tau}(z, y) \leq ct^{-d/\alpha} 2^{-\left(\sum_{j=d-i+1}^{d}\theta_j\right)\alpha (1+\alpha^{-1})} \begin{cases} 2^{-\alpha(mq+\theta_{d-i}(1+\alpha^{-1}))} & \text{if } 0 \leq m < \theta_{d-l}, \\ 2^{-\alpha(\theta_{d-i}q+m(1+\alpha^{-1}))} & \text{if } \theta_{d-l} \leq m \leq \theta_i, \end{cases}
\]

(4.15)

Case 2: \(j_0 \leq i \leq d-l\). Here we deduce from \((H^d_\theta)\) (to be precise \((H^d_\theta)\)) for \(t/2 \leq t-\tau \leq t\) the inequality

\[
p_{t-\tau}(z, y) \leq ct^{-d/\alpha} \prod_{j \in \{1, \ldots, d-l\} \setminus \{i\}} \left(\frac{t}{r_j} \land 1\right)^q \prod_{j \in \{d-l+2, \ldots, d\}} \left(\frac{t}{r_j} \land 1\right)^{1+\alpha^{-1}} \left(\frac{t}{r_{d-l+1} \land r_i} \land 1\right)^q \left(\frac{t}{r_{d-l+1} \lor r_i} \land 1\right)^{1+\alpha^{-1}}.
\]

(4.17)

Again, we estimate \(p_{t-\tau}(z, y)\) from above in dependence of the size of \(r_i\) resp. the choice of \(m\). We obtain

\[
p_{t-\tau}(z, y) \leq ct^{-d/\alpha} 2^{-\alpha q(\theta_i + m + \sum_{j=0}^{d-i}\theta_j)} 2^{-\alpha(1+\alpha^{-1})\sum_{j=d-i+1}^{d}\theta_j} \text{ if } 0 \leq m \leq \theta_i.
\]

(4.18)

Case 3: \(i < j_0\). In this case we again use (4.17) in order to obtain

\[
p_{t-\tau}(z, y) \leq ct^{-d/\alpha} 2^{-\sum_{j=0}^{d-i}\theta_j\alpha q_2^{-\sum_{j=d-i+1}^{d}\theta_j\alpha (1+\alpha^{-1})}} \text{ if } 0 \leq m \leq \theta_i.
\]

(4.19)

Remark. We will apply the aforementioned upper bounds for \(p_{t-\tau}(z, y)\) in the three cases for different choices of \(m \in \{0, \ldots, \theta_i\}\). Note that these bounds are comparable for different choices \(m_1, m_2 \in \{0, \ldots, \theta_i\}\) with \(|m_2 - m_1| \leq 6\) or any other fixed number. This observation will used without further mentioning.

We can now approach our first goal, the upper bound of \(P_{t-\tau}f(z)\) in (4.13) for \(t/2 \leq t-\tau \leq t\) and \(z \in A^e_0\). Let us consider four different cases:

Case (i): \(i < j_0\). The case is simple because (4.19) implies

\[
P_{t-\tau}f(z) \leq ct^{-d/\alpha}\|f\|_1F_{j_0}(l).
\]

(4.20)

Case (ii): \(i \geq j_0\) and \(|z^i - y_0^i| < \frac{1}{2}\rho\). In this case \(r_i = |z_i - y_i| \leq |z^i - y_0^i| + |y_0^i - y_i| \leq \rho\) for \(y \in B(y_0, \rho/8)\) and one can apply the case \(m = 0\) in (4.15) and (4.18). We obtain

\[
P_{t-\tau}f(z) \leq ct^{-d/\alpha}\|f\|_1F_{j_0}(l) \begin{cases} 2^{\theta_i\alpha q_2(-\theta_d-l+\theta_i)\alpha (1+\alpha^{-1})} & \text{if } d-l < i, \\ 2^{\theta_i\alpha q_2} & \text{if } j_0 \leq i < d-l. \end{cases}
\]

(4.21)
Case (iii): \( i \geq j_0, \frac{1}{2} \rho \leq |z^i - y_0^i| < 2^{\theta_i + 1} \rho \). Given \( y \in B(y_0, \rho/8) \) and \( r_i = |z^i - y^i| \) there is \( m \in \{1, \ldots, \theta_i - 1\} \) such that \( 2^{m-3} \rho \leq r_i < 2^{m+3} \rho \). We apply the corresponding case in (4.15) and (4.18) together with the remark above. We deduce

\[
P_{t-\tau} f(z) = \int_{B(y_0, \frac{\rho}{8})} p_{t-\tau}(z, y) f(y) dy
\]

(4.22)

\[
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0}(l) \begin{cases} 2^{-(m-\theta_d-l)\alpha} 2^{-(\theta_d-l-\theta_i)\alpha(1+\alpha^{-1})} & \text{if } d - l < i \text{ and } m < \theta_d - l, \\ 2^{-(\theta_i)\alpha(1+\alpha^{-1})} & \text{if } d - l < i \text{ and } \theta_d - l \leq m < \theta_i, \\ 2^{-(\theta_i)\alpha} & \text{if } j_0 \leq i \leq d - l \text{ and } 1 \leq m < \theta_i. \end{cases}
\]

Case (iv): \( i \geq j_0, |z^i - y_0^i| \geq 2^{\theta_i + 1} \rho \). In this case \( r_i = |z^i - y^i| \geq |z^i - y_0^i| - |y_0^i - y^i| \geq 2^{\theta_i + 1} \rho - \frac{1}{8} \rho \geq 2^{\theta_i} \rho \) for \( y \in B(y_0, \rho/8) \). Thus we can apply the case \( m = \theta_i \) and conclude (4.20) from (4.15) and (4.18).

Finally, we can estimate \( \Phi(0) = \sum_{i=1}^d \Phi^i(0) \), which is an important step in the estimate of \( \mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} P_{t-\tau} f(X_\tau) \right] \) in (4.10). Let us fix \( i \in \{1, \ldots, d\} \) and estimate \( \Phi^i(0) \). The ideas will later be used in an analogous manner in order to estimate \( \Phi^i(k) \) for \( k \geq 1 \).

For \( i < j_0 \), Case (i) and (4.11) above implies

\[
\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_\tau \in A^i_0\}} P_{t-\tau} f(X_\tau) \right] \leq \sup_{z \in A^i_0} P_{t-\tau} f(z) \cdot \mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_\tau \in A^i_0\}} \right]
\]

(4.23)

Recall \( F_{d-l-1}(l) = \prod_{j=d-l+1}^d \left( \frac{t}{R_j^2} \right)^{1+\alpha^{-1}} \) because of \( \prod_{j=d-l+1}^d \ldots = 1 \) by definition.

Now, let us assume \( i \geq j_0 \).

First, we observe that due to Case (iv) and (4.11) above, we have

\[
\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_\tau \in A^i_0\}} 1_{|X_{i+1}^i - y_0^i| \geq 2^{\theta_i + 1} \rho} P_{t-\tau} f(X_\tau) \right]
\]

\[
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0}(l) 2^{-\theta_j \alpha} \leq c t^{-d/\alpha} \|f\|_1 F_{j_0+1}(l) 2^{-\theta_j \alpha(1+\gamma)}. \]  

(4.24)

For the remaining two cases we need some auxiliary estimate. The idea is to decompose the event \( X_\tau \in A^i_0 \) into several disjoint events in dependence on the position of \( X^i_{i-1} \), i.e., the point just before the jump out of \( B(x_0, s(j_0)) \). We define \( I^i_{0,0} := \{ \ell \in \mathbb{R} : |\ell - y_0^i| \in [0, 2\rho) \} \) and \( I^i_{0,m} := \{ \ell \in \mathbb{R} : |\ell - y_0^i| \in [2^{m-1}\rho, 2^{m+1}\rho) \} \) for \( m \in \{1, \ldots, \theta_i - 1\} \). Then, by the Lévy system formula

\[
\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} 1_{\{X_\tau \in A^i_0\}, 2^{m-1} \rho \leq |X^i_{i-1} - y_0^i| < 2^{m+1} \rho} \right]
\]

\[
\leq \mathbb{E}^x \left[ \int_{I^i_{0,0}} \int_{I^i_{0,m}} \frac{1}{|X^i_{i-1} - \ell|^{1+\alpha}} d\ell ds \right] \leq \frac{ct}{R_i^{1+\alpha}} 2^m \rho \quad \text{(4.25)}
\]

The estimate (4.25) follow easily by considering \( w \in B(x_0, s(j_0)) \) and \( z \in \mathbb{R}^d \) satisfying \( |z^i - y_0^i| \in [2^{\gamma_i} \rho, 2^{\gamma_i+1} \rho) \) with \( \gamma_i < \theta_i \) or \( |z^i - y_0^i| \in [0, 2\rho) \cup [2^{m-1}\rho, 2^{m+1}\rho) \) for some \( m < \theta_i \). Then for \( i \geq j_0 \),

\[
|w^i - z^i| \geq |x_0^i - y_0^i| - |w^i - x_0^i| - |z^i - y_0^i| \geq 5R_i/4 - R_i/8 - 2^{\theta_i} \rho = R_i/8,
\]

which implies the second inequality of (4.25).
Now, we can proceed with Case (ii). By (4.21) and (4.25), we obtain
\[
\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_r \in A_i^x, |X_{r-l} - y_0| \leq \rho/2\} P_{t-\tau} f(X_r)} \right] \\
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0}(l) \begin{cases} 
2^{-\theta_0 \alpha(q-1-\alpha^{-1})} & \text{if } d-l < i, \text{ and } m < \theta_d - l, \\
2^{-\alpha} & \text{if } j_0 \leq i \leq d-l \text{ and } 1 \leq m \leq \theta_i,
\end{cases}
\]
\[
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0+1}(l) \begin{cases} 
2^{-\theta_0 \alpha(q-1-\alpha^{-1})} & \text{if } d-l < i, \text{ and } m \leq \theta_i, \\
2^{-\theta_0 \alpha(1+q)} \text{ if } j_0 \leq i \leq d-l \text{ and } 1 \leq m \leq \theta_i, \\
2^{-\theta_0 \alpha(q-1+\alpha^{-1})} & \text{if } d-l < i, \text{ and } m \leq \theta_i, \\
2^{-\theta_0 \alpha(1+q)} \text{ if } j_0 \leq i \leq d-l \text{ and } 1 \leq m \leq \theta_i.
\end{cases}
\]
(4.26)
The first inequality holds in the case $d-l < i$ since $q \leq 1 + \alpha^{-1}$ and $\theta_0 \leq \theta_d - l$. Note that $j_0 \in \{i_0, \ldots, d-l\}$ implies $\theta_j \leq \theta_d - l$. The last inequality holds in the case $j_0 \leq i \leq d-l$ because $q \leq 1 + \alpha^{-1}$ and $\theta_i \geq \theta_j$.

It remains to consider Case (iii). By (4.22) and (4.25),
\[
\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_r \in A_i^x, 2^{m-1} \rho \leq |X_{r-l} - y_0| < 2^{m+1} \rho\} P_{t-\tau} f(X_r)} \right] \\
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0}(l) \begin{cases} 
2^{-\theta_0 \alpha(q-1-\alpha^{-1})} & \text{if } d-l < i, \text{ and } m < \theta_d - l, \\
2^{-\alpha} & \text{if } j_0 \leq d-l \text{ and } 1 \leq m < \theta_i,
\end{cases}
\]
\[
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0+1}(l) \begin{cases} 
2^{-\theta_0 \alpha(q-1-\alpha^{-1})} & \text{if } d-l < i, \text{ and } m < \theta_i, \\
2^{-\theta_0 \alpha(1+q)} \text{ if } j_0 \leq i \leq d-l \text{ and } 1 \leq m \leq \theta_i, \\
2^{-\theta_0 \alpha(q-1+\alpha^{-1})} & \text{if } d-l < i, \text{ and } m < \theta_i, \\
2^{-\theta_0 \alpha(1+q)} \text{ if } j_0 \leq i \leq d-l \text{ and } 1 \leq m \leq \theta_i.
\end{cases}
\]
Since $2^{-\theta_0 \alpha(q-1-\alpha^{-1})} \leq 1$ for $q > \alpha^{-1}$ and $2^{-\theta_0 \alpha(q-1+\alpha^{-1})} \leq 2^{-\theta_0 \alpha(q-1+\alpha^{-1})}$ for $q < \alpha^{-1}$ and $m < \theta_i \wedge \theta_d - l$,
\[
\mathbb{E}^x \left[ 1_{\{\tau \leq t/2\}} \mathbb{1}_{\{X_r \in A_i^x, 2^{m-1} \rho \leq |X_{r-l} - y_0| < 2^{m+1} \rho\} P_{t-\tau} f(X_r)} \right] \\
\leq c t^{-d/\alpha} \|f\|_1 F_{j_0+1}(l) \begin{cases} 
2^{-\theta_0 \alpha(1+q)} & \text{if } q < \alpha^{-1}, \text{ and } m < \theta_i, \\
2^{-\theta_0 \alpha(1+\alpha^{-1})} & \text{if } q > \alpha^{-1}.
\end{cases}
\]
(4.27)

Therefore, by (4.23) – (4.27), we obtain
\[
\Phi(0) = \sum_{i=1}^d \Phi^i(0) \leq c t^{-d/\alpha} \|f\|_1 F_{j_0+1}(l) \begin{cases} 
2^{-\theta_0 \alpha(1+q)} & \text{if } q < \alpha^{-1}, \text{ and } m < \theta_i, \\
2^{-\theta_0 \alpha(1+\alpha^{-1})} & \text{if } q > \alpha^{-1}.
\end{cases}
\]
(4.28)

**Estimates of $S^i$** := $\sum_{k=\Theta(i_0,d)}^{\Theta(i_0,d)} \Phi^i(k)$ for $i \in \{j_0, \ldots, d\}$. In the sequel we will make use of the following. Let $x \in B(x_0, \frac{\rho}{8})$ and $i \geq j_0$. Let us consider two cases. (1) For $k \geq 1$ and $\gamma_i < \theta_i$, we set $I_k := \{\ell \in \mathbb{R} : |\ell - y_0| \in [2^\gamma \rho, 2^{\gamma+1} \rho)\}$ with $\gamma_i$ as in Definition 4.1. In this case,
\[
\mathbb{E}^x \left[ \int_0^{t/2} \int_{I_k} \frac{1}{|X_s - \ell|^{1+\alpha}} ds d\ell \right] \leq \frac{ct}{R_i^{1+\alpha}} 2^{\gamma_i} \rho.
\]
(4.29)
The proof of (4.29) is analogous to the one of (4.25). (2) For $k \geq 1$ and $\gamma_i \geq \theta_i$, we will use (4.11).

Let $l \in \{0, 1, \ldots, d-1\}$, $i_0 \in \{1, \ldots, d-l\}$ and $j_0 \in \{i_0, \ldots, d-l\}$. We follow the strategy of the estimate of $\Phi(0)$. For $z \in A_i^x$ and $y \in B(y_0, \rho/8)$, let $r_j := |z^j - y^j|$ for $j \in \{1, \ldots, d\}$. Note that $2^{\gamma_j-1} \rho \leq r_j < 2^{\gamma_j+2} \rho$ for $j \in \{1, \ldots, j_0 - 1\} \cup \{i\}$ and
$2^\theta_i - 1 \rho \leq r_j < 2^\theta_{i+3}\rho$ for $j \in \{j_0, \ldots, d\} \setminus \{i\}$ by (4.6) and (4.8). Hence $(H^l_q)$ (to be precise $(H^l_q)')$ yields for $t/2 \leq t - \tau \leq t$ and $i \leq d - l$,

$$p_{t-\tau}(z, y) \leq ct^{-d/\alpha} \prod_{j \in \{1,2,\ldots,d-l\}} \left( \frac{t}{r_j^\alpha} \wedge 1 \right)^i \prod_{j \in \{d-l+1,\ldots,d\}} \left( \frac{t}{r_j^\alpha} \wedge 1 \right)^{1+\alpha^{-1}}$$

$$\leq ct^{-d/\alpha} \prod_{j \in \{1,2,\ldots,d-l\}} \left( \frac{t}{r_j^\alpha} \right)^i \prod_{j \in \{d-l+1,\ldots,d\}} \left( \frac{t}{r_j^\alpha} \right)^{1+\alpha^{-1}}$$

$$\leq ct^{-d/\alpha} 2^{-\sum_{j=1}^{i-1} \gamma_j + \gamma_i} 2^{-\left(\sum_{j=1}^{d-i} \theta_j - \theta_i\right) \alpha q - \left(\sum_{j=d-i+1}^{d} \theta_j\right) \alpha (1+\alpha^{-1})}.$$  

We note that $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d$ is determined once we fix $A_k^c$ and $B(y_0, \rho/8)$. Note that it is independent of the choice of the elements $z$ and $y$.

The case $i > d - l$ can be dealt with similarly. Altogether, by (4.6)–(4.8), $(H^l_q)$ yields the following estimate for $t/2 \leq t - \tau \leq t$ and $z \in A_k^c \cap A_k^d$:

$$P_{t-\tau}f(z) = \int_{B(y_0, \frac{\rho}{8})} p_{t-\tau}(z, y) f(y) dy$$

$$\leq ct^{-d/\alpha} \|f\|_1 \left\{ \begin{array}{ll}
2^{-\left(\sum_{j=1}^{i-1} \gamma_j + \gamma_i\right) \alpha q} & \text{if } i \leq d - l, \\
2^{-\left(\sum_{j=1}^{d-i} \theta_j - \theta_i\right) \alpha q} + 2^{-\left(\sum_{j=d-i+1}^{d} \theta_j\right) \alpha (1+\alpha^{-1})} & \text{if } i > d - l \\
\end{array} \right. \quad \text{(4.30)}$$

$$\leq ct^{-d/\alpha} \|f\|_1 \cdot F_{j_0}(l) \left\{ \begin{array}{ll}
2^{-\left(\sum_{j=1}^{i-1} \gamma_j + \gamma_i\right) \alpha q} & \text{if } \gamma_i < \theta_i, \\
2^{-\left(\sum_{j=1}^{d-i} \theta_j - \theta_i\right) \alpha q} & \text{if } \gamma_i \geq \theta_i \\
\end{array} \right. \quad \text{(4.31)}$$

where $F_{j_0}(l)$ is defined in (4.12). Note that we have used $\Upsilon(a, b)$ as defined in (4.9). Here the last inequality is due to the fact $- (\gamma_i - \theta_i) \alpha q < - (\gamma_i - \theta_i) \alpha (1 + \alpha^{-1})$ for $\gamma_i < \theta_i$ and the opposite inequality holds for $\gamma_i \geq \theta_i$ (note that $0 \leq q \leq 1 + \alpha^{-1}$).

With regard to the definition of $A_k$, recall $k = \sum_{j=1}^{d} \gamma_j$. For $z \in A_k^c$ such that $z \in A_k^c \cap \mathbb{N}_0^d$ with $\gamma_j = \theta_j$ or $\theta_j + 1$ for $j \in \{j_0, \ldots, d\} \setminus \{i\}$ (see, Remark 4.4). Therefore,

$$\Upsilon(1, j_0 - 1) + \Theta(j_0, d) - \theta_i + \gamma_i \leq k \leq \Upsilon(1, j_0 - 1) + \Theta(j_0, d) - \theta_i + \gamma_i + d. \quad \text{(4.32)}$$

Now decompose $S^j$ as follows:

$$S^j = \sum_{k=\Theta(j_0, d) - \theta_i}^{\infty} \Phi^j(k) 1_{\{\gamma_i < \theta_i\}} + \sum_{k=\Theta(j_0, d) - \theta_i}^{\infty} \Phi^j(k) 1_{\{\gamma_i \geq \theta_i\}} =: I + II.$$  

By (2.7), (4.29) and (4.31),

$$I = \sum_{k=\Theta(j_0, d) - \theta_i}^{\infty} \mathbb{E}^z \left[ 1_{\left\{ \tau \leq t/2 \right\}} 1_{\left\{ X_r \in A^c_k \right\}} P_{t-\tau}f(X_r) 1_{\{\gamma_i < \theta_i\}} \right]$$

$$\leq ct^{-d/\alpha} \|f\|_1 F_{j_0}(l) \sum_{k=\Theta(j_0, d) - \theta_i}^{\infty} 2^{-\left(\sum_{j=1}^{i-1} \gamma_j + \gamma_i\right) \alpha q} \cdot \mathbb{E}^z \left[ \int_0^{t/2} \int_{B_Y} 1_{\{\gamma_i < \theta_i\}} dz^1 d\Gamma \right]$$

$$\leq ct^{-d/\alpha} \|f\|_1 F_{j_0}(l) \sum_{k=\Theta(j_0, d) - \theta_i}^{\infty} 2^{-\left(\sum_{j=1}^{d-i} \theta_j - \theta_i\right) \alpha q} 2^{\gamma_i - \theta_i \alpha (1 + \alpha^{-1})} 1_{\{\gamma_i < \theta_i\}}.$$  

Note that $\Upsilon(1, j_0 - 1) + \gamma_i - \theta_i \geq k - \Theta(j_0, d) - d$ by the second inequality of (4.32) and $\gamma_i - \theta_i \leq k - \Upsilon(1, j_0 - 1) - \Theta(j_0, d) \leq k - \Theta(j_0, d)$ by the first inequality of (4.32). So
when \( \gamma_i < \theta_i \),

\[
2^{-(\Upsilon(1, j_0 - 1) + \gamma_i - \theta_i)} 2^{\alpha(q - \frac{d}{\alpha})} 2^{-\theta_i} \leq \begin{cases} 
2^{d_{\alpha q}} \cdot 2^{-(k - \Theta(j_0, d))(\alpha q - 1)} 2^{-\theta_i} \\
2^{d_{\alpha q}} \cdot 2^{-(k - \Theta(j_0, d))\alpha q - \theta_i}.
\end{cases}
\]

Therefore,

\[
I \leq c t^{-d/\alpha} \| f \|_1 F_{j_0}(l) \left( \sum_{k=\Theta(j_0, d) - \theta_i}^{\Theta(j_0, d) - 1} 2^{-(\Upsilon(1, j_0 - 1) + \gamma_i - \theta_i)} 2^{-\theta_i} + \sum_{k=\Theta(j_0, d)}^{\infty} 2^{-(k - \Theta(j_0, d))\alpha q - \theta_i} \right)
\]

\[
\leq c t^{-d/\alpha} \| f \|_1 F_{j_0}(l) \left( 2^{-\theta_i} \left( \begin{array}{c}
\frac{2^{-\theta_i}}{d_{\alpha q}} \cdot 2^{-\theta_i} \left( q < \alpha^{-1} \right), \\
2^{-\theta_i(\alpha - q + 1)} + 2^{-\theta_i} \left( q > \alpha^{-1}. \right)\end{array}\right) \right)
\]

(4.33)

For II, applying (4.11), (4.31) and (4.32), we have that

\[
II \leq c t^{-d/\alpha} \| f \|_1 F_{j_0}(l) \sum_{k=\Theta(j_0, d)}^{\infty} 2^{-(\Upsilon(1, j_0 - 1) + \gamma_i - \theta_i)} 2^{-\theta_i} \leq c t^{-d/\alpha} \| f \|_1 F_{j_0}(l) \cdot 2^{-\theta_i}. \quad (4.34)
\]

Since \( j \to \theta_j \) is increasing and \( q \leq 1 + \alpha^{-1} \), (4.33)–(4.34) imply that for any \( i \in \{ j_0, \ldots, d \} \),

\[
S^i \leq c t^{-d/\alpha} \| f \|_1 F_{j_0}(l) \left( \begin{array}{c}
2^{-\theta_i} + 2^{-\theta_i} \left( q < \alpha^{-1} \right), \\
2^{-\theta_i} \left( q > \alpha^{-1} \right) \end{array}\right) \leq c t^{-d/\alpha} \| f \|_1 F_{j_0}(l) \left( \begin{array}{c}
2^{-\theta_i(1+q)} \left( q < \alpha^{-1} \right), \\
2^{-\theta_i(1+\alpha)} \left( q > \alpha^{-1} \right) \end{array}\right)
\]

(4.35)

**Estimates of \( \mathcal{T}^i \) for \( i \in \{ 1, 2, \ldots, j_0 - 1 \} \).** Let \( l \in \{ 0, 1, \ldots, d - 1 \}, i_0 \in \{ 1, \ldots, d - l \} \) and \( j_0 \in \{ i_0, \ldots, d - l \} \). Analogous to the proof of (4.30), we apply (4.6)–(4.8) together with \( (H^i) \) (to be precise \( (H^i) \)) in order to prove that for \( t/2 \leq t - \tau \leq t \) and \( z \in A_{\kappa}^i \),

\[
P_{t-\tau} f(z) = \int_{B_{\Theta(j_0, d)}} p_{t-\tau}(z, y) f(y)dy \leq c t^{-d/\alpha} \| f \|_1 2^{-(\Upsilon(1, j_0 - 1) + \Theta(j_0, d - l))\alpha q - \Theta(d, d + 1, d)\alpha(1 + \alpha^{-1})} \]

\[
= c t^{-d/\alpha} \| f \|_1 2^{-(\Upsilon(1, j_0 - 1))\alpha q} \cdot F_{j_0}(l)
\]

(4.36)

where \( F_{j_0}(l) \) is defined in (4.12). Regarding the definition of \( A_k \), recall \( k = \sum_{j=1}^d \gamma_j \). For \( z \in A_{\kappa}^i \), there exists \( \gamma \in \mathbb{N}_0^d \) such that \( z \in A_{k, \gamma} \) with \( \gamma_j = \theta_j \) or \( \theta_j + 1 \) for \( j \in \{ j_0, \ldots, d \} \) (see, Remark 4.4), hence

\[
\Upsilon(1, j_0 - 1) + \Theta(j_0, d) \leq k \leq \Upsilon(1, j_0 - 1) + \Theta(j_0, d) + d.
\]

(4.37)
Therefore, by (4.11) and (4.36) with (4.37),
\[ T^i = \sum_{k \geq \Theta(j_0, d)} \mathbb{E}^x \left[ \mathbbm{1}_{\{\tau \leq t/2\}} \mathbbm{1}_{\{X_r \in A_k\}} P_{t-\tau} f(X_r) \right] \]
\[ \leq ct^{-d/\alpha}\|f\|_1 F_{j_0}(l) \sum_{k \geq \Theta(j_0, d)} 2^{-Y(1, j_0, -1)\alpha q} \cdot \mathbb{P}^x(\tau \leq t/2, X_\tau \in A_k) \]
\[ \leq ct^{-d/\alpha}\|f\|_1 F_{j_0}(l) \sum_{k \geq \Theta(j_0, d)} 2^{-Y(1, j_0, -1)\alpha q} 2^{-\theta j_0 \alpha} \]
\[ \leq ct^{-d/\alpha}\|f\|_1 F_{j_0}(l) \sum_{k \geq \Theta(j_0, d)} 2^{-(k-\Theta(j_0, d)-d)\alpha q} 2^{-\theta j_0 \alpha} \]
\[ \leq ct^{-d/\alpha}\|f\|_1 F_{j_0}(l) 2^{-\theta j_0 \alpha} = ct^{-d/\alpha}\|f\|_1 F_{j_0+1}(l) \cdot 2^{-\theta j_0 \alpha(1+q)}. \quad (4.38) \]

**Conclusion.** Finally, we use the estimates (4.28), (4.35) and (4.38) in the representation (4.10). Since \( \frac{t}{R} = 2^{-\theta \alpha} \) for \( i \in \{1, \ldots, d\} \) by (3.1) and
\[ F_{j_0+1}(l) = \prod_{j=j_0+1}^{j_0} \left( \frac{t}{R} \right)^q \prod_{j=d-l+1}^{d} \left( \frac{t}{R} \right)^{1+\alpha-1}, \]
we obtain the upper bound of (4.10) as follows:
\[ \mathbb{E}^x \left[ \mathbbm{1}_{\{\tau \leq t/2\}} P_{t-\tau} f(X_r) \right] = \sum_{i=1}^{j_0} T^i + \sum_{i=j_0}^{d} S^i + \Phi(0) \]
\[ \leq ct^{-d/\alpha}\|f\|_1 \prod_{j=j_0+1}^{d-l} \left( \frac{t}{R} \right)^q \prod_{j=d-l+1}^{d} \left( \frac{t}{R} \right)^{1+\alpha-1} \cdot \left( \frac{t}{R} \right)^{1+q} \text{ if } q < \alpha^{-1} \]
\[ \left( \frac{t}{R} \right)^{1+\alpha-1} \text{ if } q > \alpha^{-1}. \quad (4.39) \]
This proves Proposition 3.3 since \( R_i \approx |x_0 - y_0| \) for \( i \in \{1, \ldots, d\} \), cf. (3.1).

5. Appendix

In this section we provide the proof of the auxiliary result Lemma 2.4. Its proof makes use of a simple algebraic observation, which we provide first.

**Lemma 5.1.** Assume \( 0 \leq q \leq a, l \in \{0, \ldots, d\} \) and \( z^i, \ldots, z^d > 0 \). Let \( \sigma : \{1, \ldots, d\} \rightarrow \{1, \ldots, d\} \) denote a permutation satisfying \( z^{\sigma(i)} \geq z^{\sigma(i+1)} \) for every \( i \in \{1, \ldots, d-1\} \). Then
\[ \prod_{i=1}^{d-l} \left( z^{\sigma(i)} \right)^q \prod_{i=d-l+1}^{d} \left( z^{\sigma(i)} \right)^a \leq \prod_{i=1}^{d-l} \left( z^i \right)^q \prod_{i=d-l+1}^{d} \left( z^i \right)^a. \quad (5.1) \]

**Proof.** We prove (5.1) by induction for \( d \). First, consider the case \( d = 2 \). If \( z^1 > z^2 \), then \(\sigma\) is identity and (5.1) is trivial, so consider the case \( z^1 \leq z^2 \), in which case \( \sigma = (1 2) \). When \( l = 0 \) or \( l = 2 \), (5.1) trivially holds with equality, so consider the case \( l = 1 \). In this case, (5.1) is \( (z^2)^q (z^1)^a \leq (z^1)^q (z^2)^a \), which is equivalent to \( (z^1)^{a-q} \leq (z^2)^{a-q} \). This inequality holds true because the function \( x \rightarrow x^{a-q} \) is monotone for \( x > 0 \). We have proved (5.1) for \( d = 2 \). (Note that, by the same proof the above holds for transpositions of any pair of natural numbers \( i < j \) with \( z^{\sigma(i)} \geq z^{\sigma(j)} \).)

Now assume (5.1) holds for \( d \leq m \) and let us prove it for \( d = m+1 \). Let \( \sigma : \{1, \ldots, m+1\} \rightarrow \{1, \ldots, m+1\} \) denote a permutation satisfying \( z^{\sigma(i)} \geq z^{\sigma(i+1)} \) for all \( i \in \{1, \ldots, m\} \). Set \( k = \sigma(1) \). We may assume \( k \neq 1 \) since otherwise (5.1) holds by the induction
hypothesis. Let \( \hat{\sigma} : \{1, \ldots, k-1, k+1, \ldots, m+1 \} \to \{1, \ldots, k-1, k+1, \ldots, m+1 \} \) be such that \( \hat{\sigma}(i) = \sigma(i+1) \) for \( 1 \leq i \leq k-1 \) and \( \hat{\sigma}(i) = \sigma(i) \) for \( k+1 \leq i \leq m+1 \) if \( k \leq m \). Also, let \( \tau_i = (k - i, k) = (\sigma(1) - i, \sigma(1)) \) for \( 1 \leq i \leq k-1 \). Then it holds that \( z^{\sigma(i)} \geq z^{\sigma(j)} \) for \( i < j \), \( z^{\sigma(k-i)} = z^k \geq z^{k-i} = z^{\tau_i} \) for \( 1 \leq i \leq k-1 \), and that
\[
\sigma = \hat{\sigma} \circ \tau_{k-1} \circ \cdots \circ \tau_1,
\]
where the order of operations is from the right to the left. Hence, using the induction hypothesis repeatedly, we obtain (5.1) for \( d = m+1 \). \( \square \)

Proof of Lemma 2.4. The proof consists of two parts. Let us first show the inequality (2.5). Assume \( t > 0 \) and \( x, y \in \mathbb{R}^d \). Let \( \sigma : \{1, \ldots, d\} \to \{1, \ldots, d\} \) denote a permutation such that \( |x^{\sigma(i)} - y^{\sigma(i)}| \leq |x^{\sigma(i+1)} - y^{\sigma(i+1)}| \) for every \( i \in \{1, \ldots, d-1\} \). Assume that (2.4) holds true. We apply Lemma 5.1 with \( z^i = \frac{t}{|x^{\sigma(i)} - y^{\sigma(i)}|} \) and \( a = 1 + \alpha^{-1} \). Then (2.5) follows from Lemma 5.1.

The second task is to show that \( (H^d_\sigma) \) implies (2.4). Remark 2.2 will be essential in this step. Recall that the jump kernel \( J \) satisfies (1.4) and the corresponding heat kernel is denoted by \( P_t(x, y) \). Assume \( t > 0 \) and \( x_0, y_0 \in \mathbb{R}^d \). We want to show (2.4) for \( t > 0 \) and \( x_0, y_0 \). Let \( \sigma : \{1, \ldots, d\} \to \{1, \ldots, d\} \) denote a permutation such that \( |x^{\sigma(i)}_0 - y^{\sigma(i)}_0| \leq |x^{\sigma(i+1)}_0 - y^{\sigma(i+1)}_0| \) for every \( i \in \{1, \ldots, d-1\} \). For \( x \in \mathbb{R}^d \), allowing abuse of notation denote by \( \sigma(x) \) the point \( (x^{\sigma(1)}, x^{\sigma(2)}, \ldots, x^{\sigma(d)}) \).

Define a new jump kernel \( J^\sigma \) by \( J^\sigma(x, y) = J(\sigma(x), \sigma(y)) \) and a new bilinear form \( E^\sigma \) by
\[
E^\sigma(u, v) = \int_{\mathbb{R}^d} \left( \sum_{i=1}^{d} \int_{\mathbb{R}} (u(x + e^i \tau) - u(x))(v(x + e^i \tau) - v(x)) J^\sigma(x, x + e^i \tau) d\tau \right) dx.
\]
The domain \( D = \{ u \in L^2(\mathbb{R}^d) \mid E(u, u) < \infty \} \) stays unchanged. For \( \lambda > 0 \) we set \( E^\lambda(u, v) = E(u, v) + \lambda(u, v) \) and define \( E^\lambda \) analogously. Then for all \( f, g \in D \)
\[
E^\lambda(f, g) = E(f \circ \sigma^{-1}, g \circ \sigma^{-1}).
\]
Note that \( (E^\sigma, D) \) is a regular Dirichlet form just like \( (E, D) \). Let us denote the process corresponding to \( (E^\sigma, D) \) by \( X^\sigma \), the semigroup by \( P^\sigma \) and the corresponding heat kernel by \( p^\sigma_t(x, y) \). Recall that we intend to show that \( (H^d_\sigma) \) implies (2.4). The variables \( t > 0 \) and \( x_0, y_0 \) have been fixed at he beginning of the proof. Let us assume that we can show
\[
p_t(x_0, y_0) = p^\sigma_t(\sigma(x_0), \sigma(y_0)). \tag{5.2}
\]
Because of Remark 2.2 and the fact that the tuple \( (\sigma(x_0), \sigma(y_0)) \) satisfies the required ordering we can apply (2.3) to \( p^\sigma_t \) and the points \( \sigma(x_0), \sigma(y_0) \). Thus, the desired estimate in (2.4) for \( x_0, y_0 \) would follow. Hence, it is sufficient to prove (5.2).

In order to show (5.2) it is sufficient to prove for every non-negative function \( f \) and every \( x \in \mathbb{R}^d \)
\[
P^\sigma_t f(x) = P_t(f \circ \sigma)(\sigma^{-1}(x)). \tag{5.3}
\]
Condition (5.3) implies for every \( x \in \mathbb{R}^d \) and every \( f \geq 0 \)
\[
\int p^\sigma_t(x, y)f(y)dy = P^\sigma_t f(x) = P_t(f \circ \sigma)(\sigma^{-1}(x)) = \int p_t(\sigma^{-1}(x), z)f(\sigma(z))dz = \int p_t(\sigma^{-1}(x), \sigma^{-1}(y))f(y)dy,
\]
which proves (5.2). In order to prove (5.3) we introduce for $\lambda > 0$ the Green operators $G_\lambda, G_\lambda^\sigma$ in the usual way:

\[ G_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x) dt, \quad G_\lambda^\sigma f(x) = \int_0^\infty e^{-\lambda t} P_t^\sigma f(x) dt. \]

By the uniqueness of the Laplace transform (note that we know $P_t f$ and $P_t^\sigma f$ are continuous in this case) it is sufficient to prove for every non-negative function $f$ and every $x \in \mathbb{R}^d$

\[ G_\lambda^\sigma f(x) = G_\lambda (f \circ \sigma) (\sigma^{-1}(x)). \tag{5.4} \]

Let $\phi, f$ be non-negative functions and $\lambda > 0$. Applying the reproducing property for $\mathcal{E}_\lambda$, i.e., the identity $\int uv \, dx = \mathcal{E}_\lambda (G_\lambda u, v)$ resp. the analogous identity for $\mathcal{E}_\lambda^\sigma$, we obtain

\[
\begin{align*}
\int \phi(x) G_\lambda (f \circ \sigma) (\sigma^{-1}(x)) dx &= \mathcal{E}_\lambda^\sigma (G_\lambda^\sigma \phi, G_\lambda (f \circ \sigma) (\sigma^{-1}(\cdot))) \\
&= \mathcal{E}_\lambda (G_\lambda^\sigma \phi (\sigma^{-1}(\cdot)), G_\lambda (f \circ \sigma) (\sigma^{-1} \circ \sigma^{-1}(\cdot))) \\
&= \int G_\lambda^\sigma \phi (\sigma^{-1}(x)) (f \circ \sigma) (\sigma^{-1}(\sigma^{-1}(x))) dx \\
&= \int G_\lambda^\sigma \phi (z) f(z) \, dz = \int \phi(x) G_\lambda^\sigma f(x) \, dx,
\end{align*}
\]

which proves (5.4). Note that the main ingredient in this part of the proof is the rotational invariant of the Lebesgue measure. The proof is complete.

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