INFINITE LOTTERIES, SPINNERS, AND THE
APPLICABILITY OF HYPERREALS

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Abstract. We analyze recent criticisms of the use of hyperreal probabilities as expressed by Pruss, Easwaran, Parker, and Williamson. We show that the alleged arbitrariness of hyperreal fields can be avoided by working in the Kanovei–Shelah model or in saturated models. We argue that some of the objections to hyperreal probabilities arise from hidden biases that favor Archimedean models. We discuss the advantage of the hyperreals over transferless fields with infinitesimals. In [18] we analyze two underdetermination theorems by Pruss and show that they hinge upon parasitic external hyperreal-valued measures, whereas internal hyperfinite measures are not underdetermined.

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Date: August 27, 2020.
2010 Mathematics Subject Classification. Primary 03H05; Secondary 03H10, 00A30, 60A05, 26E30, 01A65.
Key words and phrases. Infinitesimals; hyperreals; hyperfinite measures; internal entities; probability; regularity; axiom of choice; saturated models; underdetermination; non-Archimedean fields.
1. Introduction

Since Abraham Robinson introduced his framework for infinitesimal analysis in the 1960s (see [68] and [69]), a sizeable literature has developed in connection with the applicability of said infinitesimal analysis in probability theory, physics, and other fields. Despite the growing body of literature featuring such applications, the recent years have seen a vigorous debate concerning the applicability of Robinson’s framework in the sciences, with a number of advocates and also a number of critics. The latter include Easwaran, Elga, Parker, Pruss, Towsner, and Williamson. Recent additions to the literature are Easwaran–Towsner ([28], 2018), Pruss ([65], 2018), and Parker ([62], 2019). Easwaran and Towsner call into question the applicability of Robinson’s framework to the description of physical phenomena. A rebuttal appears in Bottazzi et al. ([17], 2019). The present article focuses mainly on the critiques as formulated by Parker, Pruss, and Williamson. These authors have questioned the applicability of hyperreal models in probability.

In the present text and in the sequel article [18], we analyze a claim by Alexander Pruss (AP) that hyperreal models are underdetermined, in the sense that, given a model, allegedly “there is no rational reason to choose a particular infinitesimal member of an extension to be a value for the probability” ([65, Section 3.1]) of a single event. To buttress his claim, AP exhibits measures assigning a different infinitesimal value to the event. We argue, however, that all of AP’s additional measures are parasitic in the sense of Clendinnen [21]. More specifically, AP ignores a key property of entities such as functions and measures in Robinson’s framework, namely the property of being internal. The importance of internality stems from the fact that Robinson’s transfer principle only applies to internal entities. Meanwhile, we prove that all

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1Applications to physics (Albeverio et al. ([1], 1986), Faris ([31], 2006), Van den Berg and Neves ([51], 2007), Loeb and Wolff ([57], 2015)), to probability theory (Nelson [61], 1987), to stochastic analysis (Capiński and Cutland [20], 1995), to canards (Diener and Diener [24], 1995), to mathematical economics and theoretical ecology (Campillo and Lobry [19], 2012), to error analysis (Dinis and van den Berg [25], 2019), to Markov processes (Duanmu et al. [26], 2021).

2Clendinnen points out the possibility that “All members of any set of empirically equivalent but logically distinct theories might be parasitic on one key theory. That is each of the other members of the set might only be able to be formulated by utilizing the formulation of the key theory. If this situation obtained, [differential underdetermination] would not hold: for the predictions which could be made by using any one of the set of theories would nevertheless require the selection of the single key theory. So the making of these predictions would depend on the selection of a unique theory” [21, p. 76].
of AP’s additional measures are external. Thus, if one considers only internal hyperfinite measures, no underdetermination occurs.

AP also claims that certain transferless ordered fields properly extending the reals, such as the Levi-Civita field or the surreal numbers, may have advantages over hyperreal fields in probabilistic modeling. However, we show in [18] that probabilities developed over such fields are less expressive than real-valued probabilities, and inferior to probabilities developed over hyperreal fields.

In more detail, AP ([65], 2018) attacks Robinson’s framework for mathematics with infinitesimals, claiming that its applications in probability cannot have any physical meaning. AP’s critique is more sophisticated than that of Easwaran–Towsner, in that he acknowledges at the outset that some commonly voiced objections to hyperreal numbers are unconvincing. Nevertheless, AP claims that hyperreal probabilities are underdetermined, namely that there is no rational reason to assign a particular infinitesimal probability to non-empty events that classically have probability zero. His argument hinges upon the following:

- examples of uniform processes that allegedly do not allow for a uniquely defined infinitesimal probability for singletons, and
- a pair of theorems asserting that for every hyperreal-valued probability measure there exist uncountably many others that induce the same decision-theoretic preferences.

We will argue that the underdetermination claim is baseless, by addressing each of these critiques. The first critique is addressed in Section 2 of the present article, whereas his pair of theorems are analyzed in detail in the sequel article [18].

We note that the underdetermination attack is different from a previous attack against hyperreal probabilities developed by AP in ([64], 2014). In that paper, AP sought to argue that infinitesimals are “too small” to give plausible probabilities of individual outcomes in a countably infinite lottery. The 2014 attack was countered by Benci et al. ([12], 2018, Section 4.5, pp. 531–534). The more recent Prussian charge is unrelated to the

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3 For a discussion of the notions of internal and external entities see Section 3.1.

4 See Sections 2.1 and 2.2 for technical details.

5 Throughout the paper probability measures are assumed to be finitely additive. Notice that finitely additive probability measures include also σ-additive probability measures, but some infinite sample spaces admit finitely additive probability measures that are not σ-additive.

6 Ironically, Reeder has criticized hyperreal infinitesimals for allegedly being too big in ([67], 2017). Reeder’s claim is refuted by Bottazzi ([15], 2019).
“smallness” argument of [64]. Instead, AP argues that it is impossible uniquely to assign an infinitesimal as the probability of an event. A similarity between [64] and [65] is that in both texts AP fails to take into account the crucial distinction between internal and external sets and functions in Robinson’s framework. Meanwhile, the analysis in Benci et al. [12] is insufficient to address the underdetermination claim.

AP asserts that such underdetermination is a feature of infinitesimal probabilities generally, but his examples and theorems mainly focus on hyperreal-valued probability functions. Moreover, he suggests that other non-Archimedean extensions of the real field could be more suitable for the development of infinitesimal probabilities. Significantly, he makes no attempt to present a model of his uniform processes in such alternative frameworks with infinitesimals. We will address AP’s claim in Section 3 and in [18, Section 4].

In Section 2.2 we point out some common hidden assumptions in mathematical modeling of physical phenomena, and analyze some common biases against Robinson’s framework. Such biases include the following: the insistence on the use of the natural numbers as the only possible model for the time scale of processes that “go on for the rest of time” (Section 2.3) and the claim that uncountably many hypernatural numbers are not suitable for the representation of such discrete processes (Section 2.4).

In Section 3 we highlight the significance of the transfer principle of Robinson’s framework. AP suggests that measures taking values in non-Archimedean fields other than hyperreal fields may be more suitable for the development of infinitesimal probabilities. We show that this suggestion overlooks the significance of the transfer principle.

A common flaw of criticisms of infinitesimal models, as pursued by opponents of Robinson’s framework, is the assumption that certain properties of the Archimedean accounts for infinite processes must also be satisfied by every non-Archimedean probability that represents it. However this assumption is unjustifiable; see [18, Section 2]. Moreover, in [18] we show that there are appropriate hyperfinite representations of the process that are not underdetermined.

The issues with the article by AP could be classified along the following lines:

(P1) (philosophical) (a) AP naively assumes that hyperreal models must mimick the properties of Archimedean ones; see Sections 2.3, 2.4 and [18, Section 2]. (b) AP fails to establish the relevance of the parasitic external measures he introduces to buttress his underdetermination charge; see [18, Section 3.4].
(P2) (historical) AP ignores the Klein–Fraenkel criteria for the utility of a theory of infinitesimals; see Section 3.3.

(P3) (consistency) While at the outset AP admits that the arbitrariness claim against the hyperreals is mathematically incoherent, he lapses into it later in his article; see Section 3.4.

(P4) (mathematical) AP makes inappropriate choices in hyperreal modeling. Adopting more appropriate choices dissolves AP’s argument against hyperreal modeling; see [18, Section 2.4].

In the present article we address items (P1), (P2), and (P3), whereas in [18] we address items (P1) and (P4).

2. ON MATHEMATICAL REPRESENTATION OF PHYSICAL PROCESSES

AP opens his analysis by articulating some “intuitions” based on scenarios that can be represented by both Archimedean and non-Archimedean probability measures (before going on to discuss his underdetermination theorems that we will analyze in [18, Section 3]). Such scenarios can be grouped into the following categories:

- some infinite processes, such as coin tosses [65, Sections 3.2 and 4.2], or the estimate of the value of a utility one can have “every day for eternity” [65, Section 3.5];
- a pair of uniform processes over a single sample space, exemplified by the motion of two spinners (rotating pointers) [65, Sections 3.2 and 3.3] and by a pair of uniform lotteries over \( \mathbb{N} \) [65, Section 4.1].

We will discuss the details of AP’s representation of the two spinners in [18, Section 2]. Here we will comment more generally on the issue of mathematical representation of physical processes. We argue that certain physical processes may admit distinct mathematical models. For such processes, there does not exist a unique, well-defined model that would represent them in a way resembling anything like an isomorphism. Such a perspective is accepted even by mathematicians and philosophers who adopt a responsible variety of mathematical realism, as discussed in Section 2.3 (see also Bottazzi et al. [17], 2019, Section 1). We further argue that hyperreal models can be used on par

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The example of an infinite collection of coin tosses has already been used to attack hyperreal-valued probability measures by Williamson [33], Easwaran [27], Parker [62] and other authors. Rebuttals of this argument can be found for instance in Weintraub ([32], 2008), Bascelli et al. ([10], 2014), Hofweber ([12], 2014), and Howson ([13], 2019).
with Archimedean models based upon the Cantor–Dedekind representations of the continuum. This issue is also dealt with by Herzberg ([40], 2007, Section 4).

As acknowledged by AP, a common objection to the use of hyperreal fields in such representations, namely that such fields are arbitrarily specified, can be countered in several ways. Such an objection can be countered either by working in the Kanovei–Shelah definable hyperreal field [50] or by using suitable saturated hyperreal fields; see Section 2.1.

In Sections 2.3 and 2.4 we address further hidden assumptions in Archimedean mathematical descriptions of the infinite processes represented by coin tosses from the aforementioned point of view that rejects the postulation of a unique mathematical representation. In particular, we argue that some commonly voiced objections to the use of hyperreals in the representation of scenarios involving an infinity of events stem from such hidden assumptions concerning mathematical representation of physical events.

2.1. Are hyperreal fields arbitrary? AP acknowledges at the outset the failure of the commonly voiced objection of arbitrariness (and even that of ineffability), namely that one cannot specify a particular hyperreal extension of \( \mathbb{R} \) (see Section 2.2 for details). Such an objection was voiced by Alain Connes[8] and others. The objection is specifically refuted by the Kanovei–Shelah definable hyperreal field [50]; see also [41]. For a rebuttal of the Bishop–Connes critique see Katz–Leichtnam ([51], 2013), Kanovei et al. ([48], 2015), and Sanders ([72], 2020).

Furthermore, AP acknowledges that the arbitrariness objection can also be refuted by working with a hyperreal field defined up to an isomorphism, and that for suitable cardinals \( \kappa \), there is a unique-up-to-isomorphism \( \kappa \)-saturated hyperreal field of cardinality \( \kappa \). We will source such “concessions” by AP in Section 2.2.

2.2. Pruss admits failure of arbitrariness/ineffability claims. AP mentions a worry that

\[
\text{the choice of a hyperreal extension appears to be not only arbitrary but ineffable …} \quad \text{— we cannot successfully refer to a particular extension, and so a particular extension cannot reflect our credences … \[65\}, \text{Section 1]}
\]

[8]See ([22], 2004, p. 14) where Connes describes Robinson’s framework as “some sort of chimera.”

[9]More precisely, the condition is that an infinite cardinal \( \kappa \) should either be inaccessible or satisfy \( 2^\kappa = \kappa^+ \) (that is, the continuum hypothesis holds at \( \kappa \)). For details see Keisler ([53], 1994, Section 11) and further references therein.
However, he immediately acknowledges that “the ineffability argument does not apply to all extensions of the reals, and even as restricted to the hyperreals it is unsuccessful” (ibid.; emphasis added).

Thus AP acknowledges at the outset that the commonly voiced objection of arbitrariness (and even that of ineffability), namely the claim that one cannot specify a particular hyperreal extension of $\mathbb{R}$, is unsuccessful, and specifically refuted by the Kanovei–Shelah definable hyperreal field (see [50], [41]):

[B]y leveraging the idea that even when it is difficult to specify a particular ultrafilter, one can specify sets of ultrafilters, Kanovei and Shelah (2004) explicitly defined a particular free ultrafilter on a particular infinite set$^{10}$... Furthermore, Kanovei and Shelah used their construction to make an explicitly specified extension of the reals (an iteration of the hyperreal extension using this ultrafilter) having further desirable properties. [65, Section 2]

Moreover, AP acknowledges that the arbitrariness objection can also be refuted by working with a hyperreal field defined up to an isomorphism:

[W]e can specify a set of hyperreals up to isomorphism. For some cardinals $\kappa$, there is a unique-up-to-isomorphism $\kappa$-saturated non-standard real line of cardinality $\kappa$... And there might be some non-arbitrary way to choose the cardinal $\kappa$, perhaps a way matching the particular problem under discussion. [65, Section 2]

2.3. **Shift-invariance hypothesis.** Various scenarios involving infinite processes have been discussed by AP and other authors, including Easwaran, Parker, and Williamson. Such discussions often exhibit a bias in favor of Archimedean models, which feature

- a countable infinity of events, and
- events that are ordered in time (rather than simultaneous).

A typical process that is modeled with such hidden assumptions is the outcome of an infinite amount of coin tosses. A number of arguments against hyperreal probabilities for infinite coin tosses hinge upon the events $H(n)$ that AP defines as follows:

$^{10}$This characterisation by AP of the Kanovei–Shelah technique contains a mathematical inaccuracy. The technique does not exploit an ultrafilter on an infinite set. Rather, it exploits a maximal filter in a particular algebra of subsets of a certain infinite set (the algebra in question does not contain all subsets!).
starting with day \( n \), it’s all heads for the rest of time.

AP models such a process by a sequence of tosses labeled by \( \mathbb{N} \), and makes the following assumption:

**Shift-invariance hypothesis:** Events \( H(n) \) and \( H(m) \) are isomorphic for all \( m, n \in \mathbb{N} \) (the term *shift-invariance hypothesis* is ours). Meanwhile, alternative models of these infinite processes can be obtained with hyperfinite techniques, as discussed for instance by Benci et al. [11, pp. 44–46]; see also Nelson ([61], 1987) and Albeverio et al. ([1], 1986). These models show that the shift-invariance hypothesis is spurious, since it does not hold in a hyperfinite representation of the infinite collection of coin tosses.

The shift-invariance hypothesis is often justified by an appeal to the “physical structure” of the infinite process. Thus, Williamson writes:

“A fair coin will be tossed infinitely many times at one second intervals” in ([83], 2007) on page 174. By the middle of page 175, he is ready to claim an ability to

map the constituent single-toss events of \( H(1 \ldots) \) one-one onto the constituent single-toss events of \( H(2 \ldots) \) in a natural way that preserves the physical structure of the set-up just by mapping each toss to its successor.

(ibid., p. 175; emphasis added)

Williamson appears to be taking for granted a “physical structure” possessing a considerable supply of physical seconds.

The same assumption appears in the more recent text by Parker ([62], 2019). Parker implicitly assumes that a countable sequence of coin tosses is physically feasible, and bases his *Isomorphism Principle* ([62, p. 4] on such an assumption.

Bascelli et al. ([10], 2014) analyzed similar biases in favor of modeling based upon a countable infinity of time-ordered events in Easwaran ([27], 2014). Namely, an assumption of a countable time-ordered model already involves a full-fledged idealisation lacking a referent.

What Williamson and Parker fail to recognize is that, even from the viewpoint of a responsible variety of mathematical realism, a mathematical description of a physical event typically involves some level of idealisation and introduces some spurious properties (as already argued

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\[1\] Howson argued that the events \( H(n) \) and \( H(m) \) are not equiprobable whenever \( n \neq m \) even in the Archimedean model where the sample space is the \( \sigma \)-algebra generated by the cylinder sets in \( \{0, 1\}^\mathbb{N} \) ([13], 2019, Section 3). A similar observation was made by Benci et al. [12, pp. 21–22]. Both arguments have been addressed by Parker [62].
for instance in [17, Section 1]). As a consequence of such idealisation, it is not possible to claim that a physical process has a unique well-defined sample space, or that one particular sample space provides the only correct mathematical description of a physical process. For more details, we refer to [17, Section 1.4] and to references therein.

In the coinflip case, it is obvious that it is physically impossible to flip a coin, as Parker would have it, “infinitely many times, at times $t_0 + n$ seconds for $n = 0, 1, 2, \ldots$” [62, p. 8]. Nevertheless, it is possible to model this situation as a sequence of coin tosses over $\mathbb{N}$, or with other notions of number, as already mentioned.

Furthermore, starting with a physical intuition of something going on “for the rest of time,” there are several possibilities of formalizing such an intuition mathematically. One way is to interpret time increments as ranging over the traditional $\mathbb{N}$. An alternative way of modeling such an intuition would be to postulate that the “time” in question comes to an end rather than goes on indefinitely, given the likelihood of physical armageddon expected by some modern theories in astrophysics. If so, then finite and hyperfinite modeling, which postulate such a final moment, are arguably more faithful to physical intuition than modeling by $\mathbb{N}$. In this sense, an assumption that an intuition of “for the rest of time” necessarily refers to $\mathbb{N}$, involves circular reasoning, as the conclusion is built into the premise. Attempting to base “intuitive reasons” against infinitesimal probabilities on such an idealized model, as AP does in [65, Section 3.2], begs the question as to why one assumes precisely such a model rather than, say, a hyperfinite number of simultaneous coin flips. Indeed, the model chosen by Easwaran, Parker, Pruss, and Williamson predetermines the outcome of their analyses. The shift-invariance hypothesis is analyzed further in Section 2.5.

2.4. Cardinality objection. AP puts forth the following objection to the use of hypernatural numbers:

[I]t turns out that for any positive infinite number $M$ in $\mathbb{R}$, if $\mathbb{R}$ has a collection of hypernaturals, then there will be uncountably many (in external cardinality) hypernaturals between 1 and $M$ (Pruss 2014, Appendix).

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12Or finite nonstandard number of flips in Nelson’s framework; see Section 3.2.
13AP’s parenthetical comment referring to external cardinality indicates that he is aware of the distinction between internal and external entities (in this case, cardinality). Six years prior to the online publication of [65], he referred to internal cardinality in his posting [63]. However, AP fails to take into account the distinction between internal and external hyperreal probabilities, as we will show in [18], Section 2.5.
And so the countable number of future days that we’ve imagined is not what is counted by $M$: instead, $M$ counts the number of members of the uncountable set \{1, 2, \ldots, M\} of hypernaturals. (Pruss [65], 2018, Section 3.5)

What AP is claiming is that if one is interested in countably many “future days” (i.e., trials), hypernaturals do not provide an accurate model by cardinality considerations. However, his cardinality objection is not valid, for the following reason. Skolem [80] already developed elementary (in the sense of PA) extensions $\mathbb{N}_{Sk}$ of $\mathbb{N}$ in the 1930s. Skolem’s precedent was clearly acknowledged by Robinson ([69], 1966, pp. vii, 88, 278), who noted that “Skolem’s method foreshadows the ultrapower construction” (op. cit., p. 88). Being built out of equivalence classes of (definable) sequences of natural numbers, $\mathbb{N}_{Sk}$ naturally embeds in $^*\mathbb{R}$ (for details see Kanovei et al. [49], 2013). If one’s interest is in countable structures only, one can proceed as follows:

1. construct a countable extension $\mathbb{N}_{Sk}$ of $\mathbb{N}$ following Skolem;
2. form the field of fractions $F$ of $\mathbb{N}_{Sk}$;
3. $F$ is then an ordered field properly including $\mathbb{Q}$.

In particular, there will be only countably many numbers in such an extension $F$, and hence countably many numbers in the set \{1, 2, \ldots, M\}. An identical rebuttal applies to AP’s rejection of a nonstandard solution to the paradox of Thomson’s lamp in ([66], 2018, p. 41).

2.5. Standard model of the naturals and coinflipping. In this section we will examine the relation of the so-called standard model of arithmetic to modeling infinite processes such as infinite lotteries, coin flips, etc. It is possible to disassociate the issue of scientific modeling (in physics, probability, etc.) from the issue of putative existence of a standard model (a.k.a. the intended interpretation) of $\mathbb{N}$. Even modulo such an $\mathbb{N}$ along the Cantor–Dedekind lines (not along the Nelson lines), one can question the Pruss–Williamson (PW) assumption that $\mathbb{N}$ can be embedded in physical time. PW make no effort to justify the assumption, which is surprising for publications in venues such as Analysis and Synthese.

In the following, we adopt the analysis of Kuhlemann ([55], 2018). When PW speak of performing a trial every second (or day) from now to eternity and of the “physical structure” of the process, they may be referring to either metalanguage natural numbers or the object language natural numbers.
Logicians make a distinction between, on the one hand, metalanguage naturals, and, on the other, the object language naturals (e.g., numbers in the putative standard model a.k.a. intended interpretation). Thus, Simpson denotes metanaturals by $\omega$ to distinguish them from $\mathbb{N}$ [29, pp. 9–10]. At best, metanaturals can be related to as a sorites-type subcollection (of the object language naturals $\mathbb{N}$) which does not exist as a set, blocking implementation of a set of trials indexed by metanaturals (see [25, p. 255] for a related model of the sorites paradox).

If PW mean to refer to metanaturals, the analysis above would undermine the shift-invariance hypothesis (see Section 2.3) and the PW attempt to identify $H(1)$ with $H(2)$ “naturally” \footnote{Williamson actually uses the term natural in reference to applying the shift to physical processes. AP actually speaks of “[t]he countable number of future days” [65, Section 3.5].}

If, on the other hand, PW are referring to the object language naturals, then they are already making an assumption favoring one type of idealisation over another, so that their conclusion is built into their premise. $\mathbb{N}$ is not naturally built into intuitions and thought experiments involving lotteries and coinflips (though it may be built into the type of undergraduate mathematical training that PW received).

3. ON THE STRENGTH OF THEORIES WITH INFINITESIMALS

We will define the notions of internal and external objects in Section 3.1 and present the transfer principle of Robinson’s framework in Section 3.2. These notions play a major role in mathematical modeling with hyperreal numbers. The significance of the transfer principle is also related to the historical development of mathematical theories with infinitesimals, as discussed in Section 3.3. Thus these notions will be central to our discussion of the infinitesimal models of the uniform processes proposed by AP (see [18], Section 2), and of AP’s pair of theorems (see [18], Section 3).

We will also evaluate AP’s claim that transferless non-Archimedean extensions of the real numbers might be more suitable for the development of infinitesimal probabilities. In Sections 3.3 and 3.4 we will elaborate on some consequences of the absence of transfer in the surreal numbers and the Levi-Civita field. What this entails for their applicability or otherwise to probability theory is discussed in detail in [18], Section 4.
3.1. **Constructing hyperreal fields.** It is well known that fields \( ^\ast \mathbb{R} \) of hyperreal numbers can be obtained by the so-called ultrapower construction. In this approach, one sets \( ^\ast \mathbb{R} = \mathbb{R}^N / \mathcal{U} \), where \( \mathcal{U} \) is a nonprincipal ultrafilter over \( \mathbb{N} \). The operations and relations on \( ^\ast \mathbb{R} \) are defined from the quotient structure. For instance, given \( x = [x_n] \) and \( y = [y_n] \), we set \( x + y = [x_n + y_n] \) and \( x \cdot y = [x_n \cdot y_n] \). We have \( x < y \) if and only if \( \{ n \in \mathbb{N} : x_n < y_n \} \in \mathcal{U} \).

Let \( \mathcal{P} = \mathcal{P}(\mathbb{R}) \) be the power set of \( \mathbb{R} \). Then the star transform produces the object \( ^\ast \mathcal{P} \). An internal subset \( A \subseteq \mathbb{R} \) of \( ^\ast \mathbb{R} \) is by definition a member of \( ^\ast \mathcal{P} \). More concretely, in the ultrapower construction an internal subset \( A \subseteq ^\ast \mathbb{R} \) is represented by a sequence \( (A_n) \) of subsets \( A_n \subseteq \mathbb{R} \). Here an element \( [x_n] \in ^\ast \mathbb{R} \) belongs to \( A = [A_n] \) if and only if \( \{ n \in \mathbb{N} : x_n \in A_n \} \in \mathcal{U} \). A subset of \( ^\ast \mathbb{R} \) which is not internal is called external.

More generally, in the context of the star transform from the superstructure over \( \mathbb{R} \) to the superstructure over \( ^\ast \mathbb{R} \), a set \( A \) of the latter is internal if and only if it is a member of \( ^\ast Z \) for some \( Z \) in the superstructure over \( \mathbb{R} \). For further properties of the ultrapower construction of hyperreal numbers and the superstructures, see Fletcher et al. ([32], 2017) and Goldblatt ([36], 1998).

3.2. **The transfer principle of Robinson’s framework.** Kanovei et al. describe the transfer principle of Robinson’s framework as a type of theorem that, depending on the context, asserts that rules, laws or procedures valid for a certain number system, still apply (i.e., are “transferred”) to an extended number system. ([47], 2018, p. 113)

The transfer principle asserts that the internal objects of Robinson’s framework satisfy all the first-order properties of the corresponding classical objects.

The simplest examples of transfer involve the extension of sets and functions via the \( ^\ast \) map. For instance, a continuous function \( f: \mathbb{R} \to \mathbb{R} \) is a function that satisfies the formula

\[
\forall x_0 \in \mathbb{R} \forall \varepsilon \in \mathbb{R}, \varepsilon > 0 \exists \delta \in \mathbb{R}, \delta > 0 \forall x \in \mathbb{R} \ (|x - x_0| < \delta \to |f(x) - f(x_0)| < \varepsilon).
\]

The function \( f \) is extended to a function \( ^\ast f: ^\ast \mathbb{R} \to ^\ast \mathbb{R} \) that satisfies

\[
\forall x_0 \in ^\ast \mathbb{R} \forall \varepsilon \in ^\ast \mathbb{R}, \varepsilon > 0 \exists \delta \in ^\ast \mathbb{R}, \delta > 0 \forall x \in ^\ast \mathbb{R} \ (|x - x_0| < \delta \to |^\ast f(x) - ^\ast f(x_0)| < \varepsilon).
\]
(an internal function satisfying this formula is sometimes called "*continuous"). Moreover, "*f" satisfies the Intermediate Value Theorem, the Mean Value Theorem, and every other first-order property of f.

We now turn to properties of sets under extension. For instance, the Archimedean property of \( \mathbb{R} \) is expressed by the formula

\[
\forall x, y \in \mathbb{R} \ ( (0 < x \land x < y) \rightarrow \exists n \in \mathbb{N} \ (y < nx) ) \tag{3.1}
\]

An application of the transfer principle to the above formula yields

\[
\forall x, y \in *\mathbb{R} \ ( (0 < x \land x < y) \rightarrow \exists n \in *\mathbb{N} \ (y < nx) ) \tag{3.2}
\]

The latter formula needs to be distinguished from the former, since, as is well known, a ring extension of \( \mathbb{R} \) with infinitesimal elements is non-Archimedean.

The first-order properties are preserved also by certain sets and functions that are not of the form "*X" for some classical X. A relevant example is given by a hyperfinite set, i.e., a set that can be put in an (internal) one-to-one correspondence with a set of hypernatural numbers of the form \( \{ x \in *\mathbb{N} : x \leq H \} \). Hyperfinite sets being internal, the transfer principle ensures that hyperfinite sets have the same first-order properties as finite sets. As a consequence, hyperfinite sets can be routinely applied to a wide variety of problems. For a discussion of selected applications, we refer to Arkeryd et al. (1997).

Katz–Sherry (2013) suggest that the transfer principle can be interpreted as a formalisation of the law of continuity of Leibniz; see also Sherry–Katz (2014). Such a connection was first mentioned by Robinson (1966, p. 266). Robinson’s historical chapter 10 has occasioned a reappraisal of the legacy in infinitesimal analysis of pioneers like Fermat, Gregory, Leibniz, Euler, and Cauchy.

We emphasize that the transfer principle applies only to internal entities of Robinson’s framework. Note that external entities do not exist in Nelson’s framework Internal Set Theory (1977). For this reason, attempted arguments from first principles that do not take into account Nelson’s framework are not actually based on first principles as they are.

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The fundamental difference between the two formulas is that, in (3.2), the variable \( n \) can take infinite hypernatural values.

In Nelson’s framework, set theory is enriched by a one-place predicate \( \text{st} \). The formula \( \text{st}(x) \) asserts that an entity \( x \) is standard. The ZFC axioms are enriched by the addition of further axioms governing the interaction of the new predicate with the axioms of traditional set theory. Infinitesimals, say \( \varepsilon \), are found within the ordinary real line, and satisfy \( 0 < |\varepsilon| < r \) for all standard \( x \in \mathbb{R}^+ \). It is shown in (1977) that the new theory is conservative with respect to ZFC. A related system was developed independently by Hrbacek (1983). For further details, see Fletcher et al. (2017) and Hrbacek–Katz (2020).
claimed to be, but rather involve an unspoken commitment to a specific set-theoretic framework (for instance, Zermelo–Fraenkel set theory plus the Axiom of Choice) expressed in the $\in$-language. This is done at the expense of other possible foundational frameworks. Thus, from the point of view of Internal Set Theory, internal probability measures are no less underdetermined than the traditional ones.

3.3. Usefulness of infinitesimals: Klein and Fraenkel. During the opening decades of the 20th century, both Felix Klein (1908) and Abraham Fraenkel (1928) formulated a pair of criteria to gauge the success of theories with infinitesimals. These criteria are

1. the availability and provability (by infinitesimal techniques) of the Mean Value Theorem, and
2. the introduction of the definite integral in terms of infinitesimal increments.

For a detailed discussion, see Kanovei et al. (2018). Klein and Fraenkel both observed that the infinitesimal theories available at the time (including the Levi-Civita field) did not enable a satisfactory treatment of these topics. When Robinson introduced his framework for analysis with infinitesimals, Fraenkel related to Robinson’s framework as an important accomplishment that finally solved the old problem of developing a usable non-Archimedean field. Thanks to the transfer principle, in Robinson’s framework it is possible to prove the Mean Value Theorem (MVT) and to define the Riemann integral of a continuous function by means of hyperfinite summation of infinitesimal terms.

Notice that the MVT and the definition of an integral, and in general the development of a calculus on non-Archimedean structures that extends the real calculus, require an extension of real functions. Costin et al. observe that

A longstanding aim has been to develop analysis on [the surreal numbers] as a powerful extension of ordinary analysis on the reals. This entails finding a natural way of extending important functions $f$ from the reals to the reals to functions $f^*$ from the surreals to the surreals, and naturally defining integration on the $f^*$. [Abstract]

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17 More specifically, the unspoken commitment typically involved is to a set theory in the $\in$-language rather than a set theory in the $\in$-st-language; see note 16.

18 In recent decades, there has been a deplorable attempt by Mehrtens (1990), Gray (2008), and others to discredit Klein both mathematically and politically. A rebuttal appears in Bair et al. (2017).
In Robinson’s framework, such an extension is provided by the \( * \) map. Meanwhile, it is still an open problem to define well-behaved extensions of real functions to the surreals or to the Levi-Civita field. For the surreal numbers, the problem is caused by the necessity to define functions from the \textit{simplicity hierarchical structure} of the surreal number tree.\(^{19}\)

Meanwhile, every real continuous function admits a canonical extension to a continuous function on the Levi-Civita field. However, this extension does not preserve many properties of the original real continuous functions, such as an Intermediate Value Theorem or a Mean Value Theorem.\(^{20}\)

Moreover, the surreal field and the Levi-Civita field satisfy only some restricted versions of the Klein–Fraenkel criteria. Namely, in the surreal numbers it is not possible to prove the MVT, and it is still an open problem to define an integral (see for instance Costin et al. \(^{23}\) and Fornasiero \(^{34}\)). Meanwhile, for the Levi-Civita field, Shamseddine showed that the MVT is valid only for analytic functions.\(^{73}\) Consequently, this theorem fails for instance for the extension of non-analytic real continuous functions. The Levi-Civita field does have a notion of integral in dimensions 1, 2 and 3, but the integral is not defined in terms of sums of infinitesimal contributions; see Berz–Shamseddine (\(^{13}\), 2003), Shamseddine (\(^{75}\), 2012), Shamseddine–Flynn (\(^{77}\), 2018). In addition, it turns out that the extensions of continuous but non-analytic real functions are not measurable\(^{21}\) (Bottazzi \(^{16}\)).

When Klein and Fraenkel formulated their criteria for the evaluation of non-Archimedean extensions of the reals (see \(^{47}\)), a number of such non-Archimedean options were available, including the Levi-Civita field. It is those ordered fields that Klein and Fraenkel were referring to when they expressed disappointment with the (then) current rate of progress, when even the Mean Value Theorem was not yet provable using infinitesimal analysis. AP’s critique of Robinson’s framework fails to take into account the fact that at present, Robinson’s is the only theory of infinitesimals that meets the Klein–Fraenkel

\(^{19}\) For instance, no surreal extension of the sine function usable in ordinary mathematics is as yet available; see e.g., Kanovei’s remark at \url{https://mathoverflow.net/a/307114}.

\(^{20}\) In this context, Bottazzi suggested an analogy between these extensions and external functions of Robinson’s framework \(^{14}\).

\(^{21}\) Recall that every real continuous function is Lebesgue-measurable and, if it is defined over a closed interval, it also has a well-defined Riemann integral. The failure of measurability for the extensions of non-analytic real functions is a significant limitation for the measure theory on the Levi-Civita field and it is also a blatant failure of transfer for this field.
criteria of utility. These criteria are prerequisites for a measure or probability theory.

3.4. **Infinitesimals without transfer.** The importance of the transfer principle in non-Archimedean extensions of the real numbers can be better appreciated if one considers what happens when this principle is not available. We will refer to such non-Archimedean fields as *transferless*.

Thus, in Henle’s *non-nonstandard analysis* [39] what is available is a weak form of transfer that applies only to equations, inequalities and their conjunctions, but not to their disjunction. As a consequence, some properties of ordered fields fail, and Henle’s extension is only a partially ordered ring with zero divisors.\(^{22}\)

Similarly, in the Levi-Civita field the absence of a transfer principle makes it necessary to prove individually many theorems of the calculus, such as the Intermediate Value Theorem and the Mean Value Theorem for analytic functions. For a more detailed discussion, see Shamseddine–Berz ([76], 2010) and Shamseddine ([73], 2011). In addition, currently it is possible to extend only analytic real functions to the Levi-Civita field in a way that preserves their first-order properties, as shown by Bottazzi ([14], 2018).

In other transferless fields the situation might be even more difficult; we are not aware of any research towards establishing some (even limited) forms of transfer in such settings. Nevertheless, AP suggests that there are “multiple methods” of developing infinitesimal probabilities in transferless fields:

> Hyperreals are not the only way to get infinitesimals. There are multiple methods that do not make use of anything like the arbitrary choice of an ultrafilter\(^{23}\) [65, Section 2]

AP argues that, as a consequence,

> The friend of infinitesimal probabilities has a real hope of non-arbitrarily specifying a particular field of infinitesimals. (ibid.)

\(^{22}\)In some cases, working with number systems lacking the habitual properties can lead the author into error. Thus, Laugwitz pointed out that Henle’s article lapses into using denominators when working with a ring. Laugwitz goes on to invite the reader to “rewrite the relevant passages” [50].

\(^{23}\)Here AP seems to lapse into the arbitrariness charge against Robinson’s framework discussed in Section 2.1. Notice that this passage comes after AP’s admission that it is possible to uniquely specify some particular hyperreal fields.
Moreover, he claims that certain transferless fields, namely the surreals, fields of Laurent series or the Levi-Civita field, have some advantages over hyperreal fields. Thus AP writes:

\[ T \]he surreals have the advantage of being exhaustively large, large enough that they escape the cardinality arguments against regularity of Pruss (2013a). The fields of formal Laurent series and the Levi-Civita field, on the other hand, have the advantage of being elegantly small. (On the other hand, the Kanovei–Shelah field, while mathematically fascinating, probably has little going for it in this context.\[^24\] [65, Section 2]

It can be argued that the main advantage of the field of Laurent series and the Levi-Civita field is that of being definable from the natural numbers in a choice-free manner. If one works in the von Neumann–Bernays–Gödel set theory with global choice, then the surreal numbers are also uniquely specified. It is also true that many hyperreal fields obtained as ultrapowers of \( \mathbb{R} \) are not definable in a choice-free manner; however, as AP acknowledges, suitable \( \kappa \)-saturated fields of hyperreal numbers are uniquely specified up to an isomorphism.\[^25\]

However, the issue at hand is not whether a non-Archimedean field is definable from the natural numbers without additional parameters. The real issue is the applicability of such a field. In this regard, the Levi-Civita field has some limited applications in automatic derivation \[^76\] and in the description of physical phenomena \[^33\], while there is a vast literature of applications of hyperreal fields.\[^26\]

That such applications are possible is due in particular to the transfer principle of Robinson’s framework. Significantly, the principle is not discussed by AP in \[^65\]. Instead, AP expresses enthusiasm about the surreal numbers, Laurent series, and Levi-Civita fields, but fails to explain their shortcoming, namely lack of transfer. Significantly, AP does not develop a model either for the infinite coin tosses or for his spinners in any of these transferless structures. Thus, his claim that

\[^24\]AP’s parenthetical claim that the KS model has little advantage over the transferless fields mentioned is questionable, since the KS model can be used just as well as any traditional \( \mathbb{R}^N/\mathcal{U} \) model (see Section 3.1), particularly with the saturation improvement provided by Kanovei–Shelah \[^50\]. In particular, the KS model has the advantage of the transfer principle.

\[^25\]See Section 2.1.

\[^26\]Some relevant examples can be found in note 1.

\[^27\]It is a matter of public record that AP is aware of the transfer principle, since he mentions it both in \[^65\] Appendix and in \[^66\] p. 41].
these non-Archimedean fields may be suitable for the development of an infinitesimal probability is baseless.

Indeed, in [18, Section 4] we argue that attempts to develop infinitesimal probabilities over the surreal numbers or over the Levi-Civita field encounter a number of difficulties.

For instance, in order to accomplish anything with the surreals one would have first to import the transfer principle via an identification of maximal class-size surreals with maximal class-size hyperreals, as mentioned by Ehrlich [30, Theorem 20]. Without such an identification and without the transfer principle of Robinson’s framework, it is currently not possible to develop a measure theory on the surreal numbers (for more details, see [18, Section 4.1])

The Levi-Civita field does obey a type of transfer principle, albeit limited to a particular extension of real analytic functions; see Bottazzi (14, 2018).

One could also work directly with the measure-theoretic tools available in the various theories to define non-Archimedean probability measures; however, there seem to be some difficulties.

Consider for instance the case of the Levi-Civita field, where a uniform measure is currently under development by Shamseddine and Berz [13], Shamseddine [75], Shamseddine and Flynn [77] and Bottazzi [16]. So far this uniform measure is not able to accommodate more than locally analytic probability functions, and has no notion of hyperfiniteness comparable to that of Robinson’s framework. This example is discussed further in [18, Section 4.2].

4. Conclusion

We have examined some commonly voiced objections to the use of hyperreal numbers in mathematical modeling. Many of these objections are based upon naive assumptions regarding the possibility of uniquely specifying some hyperreal fields, and upon examples of infinite processes that allegedly do not allow for a uniquely defined infinitesimal probability for singletons.

The first objection, namely that it is allegedly not possible to specify a hyperreal field in a unique way, is refuted by the Kanovei–Shelah definable hyperreal field and by the fact that, for suitable infinite cardinals $\kappa$, there is a unique-up-to-isomorphism $\kappa$-saturated hyperreal

\footnote{It should be noted that the omnific surnaturals do not satisfy the axioms of Peano Arithmetic; e.g., there exist surnaturals $p, q$ such that $p^2 = 2q^2$. For further details see [59], [71], [46].}
field of cardinality \( \kappa \). Note that this rebuttal is accepted also by some detractors of Robinson’s framework for analysis with infinitesimals.

With regard to objections based upon the analysis of certain infinite processes, we have observed that physical processes often admit distinct mathematical models, so that there does not exist a unique, well-defined model that would represent them in a way resembling anything like an isomorphism. Thus we have shown that, for some commonly used models e.g., of infinite coin tosses, some objections to the use of Robinson’s framework stem from hidden and unnecessary assumptions that predetermine the choice of an Archimedean model. Dropping such hidden assumptions enables alternative models of these infinite processes, obtained via hyperfinite techniques.

Moreover, we have started addressing the claim by Pruss that transferless extensions of the real numbers (such as the surreal field, the Levi-Civita field, or the field of Laurent series) might be more suitable for the development of infinitesimal probabilities. The proposal of working with such non-Archimedean fields ignores both the Klein–Fraenkel criteria for gauging the applicability of theories with infinitesimals, and the power and utility of the transfer principle. In [18] we show that, due to these limitations, the measure theory on such transferless fields is less expressive than the hyperfinite counting measures.

Pruss claims that “[w]hatever you can do with hyperreals, you can do with surreals” and that “[t]he fields of formal Laurent series and the Levi–Civita field . . . have the advantage of being elegantly small” [65]. However, in [18] we will see that, by his own Theorem 1, these fields suffer from underdetermination due to the possibility of rescaling the infinitesimal part of the probability, and do not possess a notion of internality that enables one to escape such underdetermination in Robinson’s framework. Prussian Theorem 1, when properly analyzed, boomerangs to undercut his own underdetermination thesis. Prussian Theorem 2 is similarly Boomerang 2 due to the existence of nontrivial automorphisms for all such transferless fields.

Arguments based on the non-effectiveness of ultrafilters are not limited to the work of Pruss; see e.g., Easwaran–Towsner [28]. In spite of an initial plausibility of such arguments against Robinson’s framework, the arguments dissolve upon closer inspection, and even tend to prove the opposite of what their authors intended. One can well wonder

\[ \text{\[18\] See Ehrlich (1994, pp. 253) for the existence of nontrivial automorphisms of the surreals as an ordered field, and Shamseddine (2011, p. 224, Remark 3.17) for such existence for all ordered extensions of } \mathbb{R}. \]
why such arguments *ad ultrafiltrum* don’t succeed. A recent development suggests a possible reason. It turns out that the main body of the applicable part of Robinson’s framework admits a formalisation that requires modest foundational means not exceeding those required for traditional non-infinitesimal methods in ordinary mathematics; see Hrbacek–Katz [45]. Thus the alleged non-effectiveness is simply not there to begin with.

**Acknowledgments**

We are grateful to Karel Hrbacek, Vladimir Kanovei, Karl Kuhlmann, and David Sherry for insightful comments on earlier versions that helped improve our article, and to anonymous referees for constructive criticism. The influence of Hilton Kramer (1928–2012) is obvious.

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