A cross-validation check in the covariance analysis of isospin sensitive observables from heavy ion collision

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Abstract. This paper focuses on two problems. The first is related to the consistency checks of the complete correlation coefficients between two groups of variables obtained in [1]. The first group collects parameters that describe properties of nuclear matter referring to the transport model involved. The second one is a group of observables adopted in the heavy ion collision experiments. The second problem concerns the method of determining the values of pure correlations between some variables in heavy ion collision. The application of this method for the analysis of the correlations of the isospin sensitive variables is pointed out. 

Key words. Heavy ion collisions–Statistics–covariance analysis–Symmetry energy

1 Introduction

Attempts at explaining physical phenomena related to such diverse objects as atomic nuclei and neutron stars depend critically on proper description and understanding of the equation of state (EoS) of asymmetric nuclear matter. Heavy ion collisions (HICs) offer the opportunity to explore this EoS under conditions, which depending on the energy of the beam and the isotopic composition, make it possible to reproduce and study the nuclear matter for various ranges of density \( \rho \) and neutron-proton asymmetry \( \delta = (\rho_n - \rho_p)/(\rho_n + \rho_p) \), where \( \rho_n \) and \( \rho_p \) are neutron and proton densities. So far, significant constraints have been obtained on the symmetric part of the nuclear matter EoS [2]. However, constraints on its isospin dependent part are subject to the severe uncertainty, especially in the high density limit. Researches on asymmetric nuclear matter focus on developing theoretical and experimental methods to overcome difficulties associated with the fact that the symmetry energy being encoded in the isospin dependent part of the EoS is not an observable. Hence the overriding problem is to determine for given physical conditions optimal observables that would allow one to impose reliable constraints on parameters, which describe asymmetric nuclear matter. In particular the estimators of parameters that enable to describe the symmetry energy dependence on density include, among others, the symmetry energy coefficient \( S_0 \) and the slope of the symmetry energy \( L \). As it will be shown, one of the most important experiments that permit extraction of information about correlations of the estimators of these parameters are HICs. However, interpretations of data collected in measurements carried out in these experiments are not straightforward. They are affected by various types of uncertainties in the transport model involved.

This paper bases on the results reported in [1]. The covariance analysis presented there permitted to estimate the complete linear correlation coefficients between the variables, which are called the force parameters and the group of observables adopted in the HIC experiments to extract information on the nuclear matter EoS. The force parameters are adopted in the transport model (i.e., the ImQMD-Sky model [1]) wildly used in the interpretation of HIC data. In the present, paper the formal relation between complete correlation coefficients obtained in [1] and partial correlation coefficients [3,4] for the force parameters is used (Eq. 1) to perform an out-of-sample check for the mentioned complete correlation coefficients. This is a kind of the cross-validation check for assessing if the complete correlation coefficients, obtained from the one set of HIC data, are consistent with the complete correlation coefficients obtained from another independent data set (see Section II).

It was shown [5,6] that in the ImQMD-Sky transport model, involving the effective Skyrme interactions, as the input variables the parameters \( \{\rho_0, E_0, K_0, S_0, L, m_s^*, m_v^*\} \) can be used. These parameters that mean successively: \( E_0 \) - the binding energy of nuclear matter, \( K_0 \) its incompressibility, \( S_0 \equiv S(\rho_0) \) - the symmetry energy coefficient, \( L \) slope of the symmetry energy, \( m_s^* \) and \( m_v^* \) the isoscalar and isovector effective masses, determine properties of nuclear

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matter at saturation density \( \rho_0 \). Analysis of the selected
Skyrme forces, on the basis of which EoSs of asymmetric
nuclear matter were obtained, had been described in \[2\].
After obtaining the EoS, it becomes possible to derive the
formula for the symmetry energy \( S(\rho) \) as a function of
density \( \rho \).

Referring these results to the parameters describing nu-
clear matter, it becomes attainable to determine, among
others, \( S(\rho_0) \), \( L \) and \( K_0 \) for the saturation density \( \rho_0 \).
Expressions defining \( S(\rho) \), \( L \) and \( K \) at arbitrary value of \( \rho \)
depend on various sets of coefficients and therefore, one
should expect a different degree of correlations between
these quantities. Some of them may be physical.

An example that compiles data on selected 142 Skyrme
interactions meeting strictly defined physical constraints
is given in \[2\]. In this case, taking into account the con-
straint associated with the reproduction of the binding
energy, very weak correlation between \( S_0 \) and \( L \) was ob-
tained. This led to the conclusion that given the symmetry
energy dependence on density, it could be possible to ex-
tract correlations between \( L \) and \( S(\rho_0) \) only if symmetry
energy sensitive constraints are imposed \[2\].

Thus, it is reasonable to present a different approach to
study the existence of nontrivial correlations between pa-
rameters that describe properties of nuclear matter. The
analysis presented underneath gives, along with the above
cross-validation check on the consistency of the results on the complete correlation coefficients obtained
from various HIC experiments \[1\], a method for deter-
ing the existence of correlations between the force parame-
ters. In this respect only the partial correlation coefficients
give the pure nature of these correlations \[3,4\].

2 Basic formulas and consistency conditions

Zhang et al. \[1\] performed the covariance analysis between
two classes of variables. Firstly, the variables \( A_1 \equiv K_0, A_2 \equiv S_0 \) and \( A_3 \equiv L \) form, among others, the group \( A \) of
the force parameters. Secondly, the observables: 1) single
\( n/p \) ratio \( B_1 \equiv C_1 - R_2(n/p) = \frac{Y_1(n)}{Y_1(p)} \), 2) double \( n/p \) ratio
\( B_2 \equiv C_1 - DR(n/p) = C_1 - R_2(n/p) - C_1 - R_1(n/p) = C_1 - R_21(n/n)/C_1 - R_21(p/p), 3) \) isoscaling ratios \( B_3 \equiv C_1 - R_21(n/n) = \frac{Y_2(n)}{\sqrt{Y_1(n)}} \), 4) \( B_4 \equiv C_1 - R_21(p/p) = \frac{Y_2(p)}{\sqrt{Y_1(p)}} \)
and 5) the isospin ratio \( B_5 \equiv R_{\text{def}} \) form the group \( B \) of five isospin sensitive observables. In the above
expressions \( C_1 \) denotes the coalescence invariant nucleon
yield spectra, \( Y_i(n) \) and \( Y_i(p) \) are integrated \( C_1 \) neutron
and proton yields from reaction \( i \). In the analysis per-
formed in \[1\] the nucleon yield observables were obtained for the the reactions \( ^{124}_{24} \text{Sn} + ^{124}_{24} \text{Sn} \) and \( ^{112}_{24} \text{Sn} + ^{112}_{24} \text{Sn} \)
for 50 MeV and 120 MeV per nucleon, \([\text{MeV/u}]\). Additionally,
in the case of \( R_{\text{def}} \) the mixed reaction \( ^{124}_{24} \text{Sn} + ^{112}_{24} \text{Sn} \)
was considered. Following this notation the out-of-sample
check for the complete, partial and multiple linear corre-
lation coefficients between the variables \( K_0, S_0 \) and \( L \) is
below presented. The analysis concerns experimental data,
discussed in \[1\], for two essential energy values 50 MeV/u
and 120 MeV/u.

The complete correlation coefficients \( r_{A_iA_j|B_a} \) between
the variables \( A_i, i = 1, 2, 3, \) and \( B_a, a = 1, 2, ..., 5 \), were ob-
tained in \[1\]. Below, on this basis only, the possible infor-
mation on the complete, partial and multiple correlation
coefficients between the variables \( A_i \) and \( A_j, i, j = 1, 2, 3, \)
\( i \neq j \), is extracted. Every partial correlation coefficient
between the variables \( A_i \) and \( A_j, i, j = 1, 2, 3, i \neq j \), with the variable \( B_a, a = 1, 2, ..., 5 \), which is under control, is
given as follows \[3,4\]:

\[
r_{A_iA_j|B_a} = \frac{r_{A_iA_j} - r_{A_iB_a} r_{A_jB_a}}{\sqrt{1 - r_{A_iB_a}^2} \sqrt{1 - r_{A_jB_a}^2}}. \tag{1}
\]

The influence of \( B_a \) on the correlation of \( A_i \) and \( A_j \) is removed in \( r_{A_iA_j|B_a} \) by the regression adjustment of \( A_i \)
with respect to \( B_a \) and separately by the regression ad-
justment of \( A_j \) with respect to \( B_a \) \[3,4\].

If one assumes that the equalities:

\[
r_{A_iA_j|B_a} = r_{A_iA_j|B_a} \quad \text{for } a \neq b, \quad a, b = 1, 2, ..., 5 \tag{2}
\]

hold then it follows from Eq. \(1\) that the value of the com-
plete correlation coefficients \( r_{A_iA_j} \) \( i, j = 1, 2, 3 \),
\( i \neq j \), are equal to:

\[
r_{A_iA_j} = \frac{r_{A_iB_a} r_{A_jB_a} - r_{A_iB_a} r_{A_jB_a}}{\sqrt{1 - r_{A_iB_a}^2} \sqrt{1 - r_{A_jB_a}^2}} \times \frac{1}{\sqrt{1 - r_{A_iB_a}^2} \sqrt{1 - r_{A_jB_a}^2}}^{-1} \tag{3}
\]

Thus, three complete correlation coefficients \( r_{A_iA_j} \) \( = (r_{SKr_{VLK}, r_{SSL}}) \) are obtained.

The consistency conditions. If the correlations coefficients
\( r_{A_iA_j} \) and \( r_{A_iA_j|B_a} \) are calculated from one sample
obtained in the experiment "a" then by the theorems
\[3,4\], they have to lie in the range \((-1, 1)\). Yet, if \( r_{A_iA_j} \)
is obtained from \( r_{A_iB_a} \) via the comparison \( r_{A_iA_j|B_a} =
\tag{3} \)
\( r_{A_iA_j} \), \( a \neq b \), i.e., from different samples "a" and "b",
then the requirement \( r_{A_iA_j} \in (-1, 1) \) may fail and some inconsist-
ency can appear. Then \( r_{A_iA_j} \in (-1, 1) \) stands only for the consistency condition, giving the out-
of-sample check for the results obtained from the consid-
ered different collision experiments. This may be called
the different samples results consistency problem.
The sources of the inconsistency can be diverse. Firstly,
the statistical one, different samples results inconsistency
can appear as the result of the fact that the correlations
\( r_{A_iB_a} \) prescribed to the observables \( B_a, a = 1, 2, ..., 5 \), were
(\text{via the physical model}) calculated \[1\] for different, finite
accuracy experiments "a". Secondly, the theoretical in-
consistency can appear when these correlations \( r_{A_iB_a} \) are cal-
culated via different theoretical models with emphasis on the
transport model involved.

Now, after checking all \( r_{A_iA_j} \) (Eq. \(3\)) consistency
conditions \( r_{A_iA_j} \in (-1, 1), i, j = 1, 2, 3, i \neq j \), three partial
correlation coefficients \( r_{A_i|A_j|A_k} = \frac{r_{A_iA_j} - r_{A_iA_k}r_{A_jA_k}}{\sqrt{1 - r^2_{A_iA_j}}\sqrt{1 - r^2_{A_iA_k}}} \). \( r_{A_iA_j|A_k} \) can be calculated as follows [34]:

\[
r_{A_i|A_j|A_k} = \frac{r_{A_iA_j} - r_{A_iA_k}r_{A_jA_k}}{\sqrt{1 - r^2_{A_iA_j}}\sqrt{1 - r^2_{A_iA_k}}}. \tag{4}
\]

Similarly, only these pairs of the experiments "a" and "b", \( a \neq b \), are accepted for which the consistency conditions \( r_{A_i|A_j|A_k} \in (-1,1) \) are fulfilled. Due to Eq. (3) the conditions \( r_{A_i|A_j|A_k} \in (-1,1), \), \( i, j, k = 1, 2, 3, i \neq j \neq k \), give three intervals for \( r_{SK} \), \( r_{SL} \) and \( r_{KL} \), and it has to be checked if (self-consistently) the values \( r_{SK} \), \( r_{SL} \) and \( r_{KL} \) belong to them.

Finally, three different multiple correlation coefficients \( r_{A_i|A_j|A_k} \), \( i, j, k = 1, 2, 3, i \neq j \neq k \), which characterise the liner correlation of \( A_i \) on both \( A_j \) and \( A_k \) can be calculated:

\[
r_{A_i|A_j|A_k} = \sqrt{1 - (1 - r^2_{A_iA_j})(1 - r^2_{A_iA_k})}. \tag{5}
\]

The consistency condition requires the acceptance of only these cases for which \( 0 \leq r_{A_i|A_j|A_k} \leq 1 \). Yet, if both complete and partial correlation coefficients fulfill the consistency conditions then from Eq. (5) it follows that every multiple correlation coefficient also fulfills it.

3 Results of the numerical analysis and conclusions

3.1 Consistency analysis

In what follows the isospin observables \( B_a, a = 1, 2, ..., 5 \) are used to identify the experiment "a". Taking into account all consistency conditions for both the complete and partial correlation coefficients, the numerical analysis based on Eq. (3) and Eq. (4) gives the following results:

- for the energy 50 MeV/u results of the experiments: \( B_1-B_2, B_1-B_3, B_1-B_4, B_1-B_5, B_2-B_4 \) are inconsistent
- for the energy 120 MeV/u results of the experiments: \( B_1-B_3, B_1-B_5, B_2-B_4, B_3-B_5 \) are inconsistent.

From this it can be inferred that the statistical inconsistency (see Section 2) for \( B_1-B_2 \) and \( B_1-B_3 \) vanishes with the increase of the energy. The results of the experiments \( B_1-B_4, B_1-B_5 \) and \( B_2-B_4 \) are inconsistent for both energy values, and therefore, by this analysis alone, one cannot claim if these results have mainly the statistical inconsistency or whether some theoretical inconsistencies are present.

Now, according to [1], one should expect better accuracy of the measurements with the energy increase. Therefore, it is possible that the results of the experiments \( B_1-B_5 \), \( B_3-B_5 \), whose inconsistency has been detected only for 120 MeV/u, reflect mainly the theoretical inconsistency (see Section 2). This suggests that the results for the experiments \( B_1-B_3 \) reveal also the theoretical inconsistency (seen for both energies). Therefore, the above analysis suggests caution when comparing results from experiments that involve as the isospin sensitive observable the diffusion of the nucleons in the neck region during the nuclear collisions quantified by the isospin transport ratios \( B_5 \equiv R_{diff} \), with those inferred from the analysis of the remaining \( B_a, a = 1, 2, 3, 4 \), observables.

3.2 Correlation analysis results

For energy 50 MeV/u (Figure 1) the symmetry energy coefficient \( S_0 \) depends moderately or weakly on the incompressibility of the pure neutron matter \( K_0 \) and the slope of the symmetry energy \( L \). For example for the latter case \( r_{SL|K} \in (-0.45, 0.34) \).

When the energy increases to 120 MeV/u (Figure 2) then the linear dependance of \( S_0 \) on \( L \) becomes much stronger, i.e., \( r_{SL|K} \in (-0.92, -0.71) \), except the results connected with \( B_5 \equiv R_{diff} \) that for \( B_5-B_2 \) gives \( r_{SL|K} = 0.17 \). The other partial correlations are as follows: for 50 MeV/u, \( r_{SK|L} \in (-0.59, 0.37) \) and \( r_{LK|S} \in (-0.22, 0.81) \). For 120 MeV/u, \( r_{LK|S} \in (0.02, 0.46) \) and \( r_{SK|L} \in (0.33, 0.56) \), with the exception of the negative value result for \( B_2-B_5 \), for which \( r_{SK|L} = -0.76 \). Once again there is a problem with \( B_5 \). Having obtained the complete and partial correlation coefficients, the value of the multiple correlation coefficient \( r_{SL|K} \) has been obtained from Eq. (5). It increases with energy, mainly as the result of the behaviour of \( r_{SL|K} \). For 50 MeV/u, \( r_{SL|K} \) lies in the region (0.13, 0.65) and for 120 MeV/u in the region (0.77, 0.98). Thus, with the increase of energy the linear correlation of \( S_0 \) with both \( L \) and \( K_0 \) together becomes stronger. As the scatter of these correlation coefficients decreases with energy, the value of \( r_{SL|K} \) stabilises, which most likely is connected with the increase of the accuracy of the experiments with the energy [1].
3.3 Final conclusions

The conclusions are twofold. Firstly, it has been shown that along with the increase of energy per nucleon there are classes of observables for which the correlation coefficients for the force parameters are very close in value. This indicates the existence of strong $r_{S,L,K}$ correlation for these classes of observables. Secondly, an out-of-sample validity check has been performed. This refers to the extent to which a given result obtained in $\mathbb{I}$ for a particular observable $B_a$ can be considered consistent with the results obtained from another isospin sensitive observable $B_b$, $a \neq b$. The performed analysis points to the existence of some inconsistencies between results got with the use of different observables. The consistency tests reveal, for the higher value of energy, groups of observables for which the calculated correlation coefficients reach very close values, with the exception of $B_5$ (see Figure 2). The predicted and measured isospin sensitive observables $B_a$, ($a = 1, \ldots 5$) depend crucially on many conditions with the density dependence of the symmetry energy being the most important one $\mathbb{I}$. The relevant factor is connected with the fact that these observables probe the isospin dependent part of the EoS at different physical conditions of which the density and the value of isospin asymmetry $\delta$ play the key role. This applies in particular to the parameter $B_5 \equiv R_{\text{diff}}$ denoting isospin diffusion and sheds new light on the $B_5$ modelling problem.

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Authors contributions

All the authors were involved in the preparation of the manuscript. All the authors have read and approved the final manuscript.