ENTROPY FOR A-COUPLED-EXPANDING MAPS AND CHAOS

CHOL-GYUN RI AND HYON-HUI JU AND XIAOQUN WU

Abstract. The concept of "A-coupled-expanding" map for a transition matrix $A$ has been studied as one of the most important criteria of chaos in the past years. In this paper, the lower bound of the topological entropy for strictly $A$-coupled-expanding maps is studied as a criterion for chaos in the sense of Li-Yorke, which is less conservative and more generalized than the latest result is presented. Furthermore, some conditions for $A$-coupled-expanding maps excluding the strictness to be factors of subshifts of finite type are derived. In addition, the topological entropy of partition-$A$-coupled-expanding map, which is put forward in this paper, is further estimated on compact metric spaces. Particularly, the topological entropy for partition-$A$-coupled-expanding circle maps is given, with that for the Kasner map being calculated for illustration and verification.

Chaos; Topological entropy; Coupled-expanding map; Transition matrix; Topological semi-conjugacy

1. Introduction

The term "chaos" was first introduced into mathematics by Li and Yorke [Li & Yorke, 1975]. Since then, various definitions of chaos have been proposed. However, in general, they do not coincide, and none of them can be considered as a perfect definition of chaos. Most of the definitions are based on the feature of long-term unpredictability of chaotic behaviors due to sensitivity to initial conditions.

The relations between those definitions have been widely studied, as well as other measures of complexity, such as the topological entropy, conjugacy or semi-conjugacy to symbolic dynamical systems, the Lyapunov exponent, the Hausdorff dimension, and so on.

The notion of "coupled-expansion" as a criterion for chaos was originated from the terminology of turbulence [Block & Coppel, 1986, Block & Coppel, 1992] and was considered as an important property of one-dimensional dynamical system. Specifically, a continuous map $f : I \to I$, where $I$ is the unit interval, is said to be turbulent if there exist closed non degenerated subintervals $J, K$ of $I$ with pairwise disjoint interiors such that $J \cup K \subset f(J) \cap f(K)$.

Furthermore, the map $f$ is said to be strictly turbulent if the subintervals $J$ and $K$ can be chosen disjoint. In fact, the same concept was studied in one-dimensional dynamical system by Misiurewicz [Misiurewicz, 1979,
Misiurewicz, 1980], who called this property as “horseshoe”, since it is similar to the Smale’s horseshoe effect [Smale, 1967]. Let $f : I \rightarrow I$ be an interval map and $J_1, \ldots, J_n$ be chosen non degenerate subintervals with pairwise disjoint interiors such that $J_1 \cup \cdots \cup J_n \subset f(J_i)$ for $i = 1, \ldots, n$. Then $(J_1, \ldots, J_n)$ is called an $n$-horseshoe, or simply a horseshoe if $n \geq 2$.

In recent years, the study of chaos in the setting of general topological dynamics has attracted wide attention. In 2006, Shi and Yu captured the essential meanings of the concept of turbulence for continuous interval maps and extended it to maps in general metric spaces [Shi & Yu, 2006a], where the maps were still called turbulent. Since the term turbulence is well-established in fluid mechanics, they changed the term “turbulence” to the “coupled-expansion” [Shi & Yu, 2006b], which is more intuitive in reflecting the conditions that the map satisfies. In 2009, Shi, Ju and Chen extended the concept of ”coupled-expansion” to a more general one – ”$A$-coupled-expansion” for the transition matrix $A$ [Shi et al., 2009], which contains ”coupled-expansion” as a special case when each entry of the matrix $A$ equals to 1. In these papers, several criteria of chaos induced by strictly coupled-expanding maps or $A$-coupled-expanding maps have been established in metric spaces, essentially with the compactness, by using the conjugacy to symbolic dynamical systems. Some extended criteria of chaos based on a result in [Shi et al., 2009] in the context of the complete metric space, can also be found in [Zhang & Shi, 2010].

Actually, the essential ideas of these notions include ”horseshoe” and ”turbulence”. Thus, ”Coupled-expansion” and ”$A$-coupled-expansion” in metric spaces are based on domain splitting and it is inclined to use conjugacy or semi-conjugacy to symbolic dynamical systems, that is, shift or subshift of finite type. This is one of the common and useful ways to study chaos or complex behaviors. Therefore, there are various other results, which are more or less similar.

In 2001, Kennedy and York [Kennedy & Yorke, 2001] studied topological horseshoes and proved that a continuous map in a compact invariant set of a metric space could be topologically semi-conjugate to a symbolic dynamical system under some hypotheses, called the horseshoe hypotheses. Particularly, in 2001, Fu and Lu et al. [Fu et al., 2001] (Definition 7.1, Theorem 7.2) presented the concept of ”distillation” and conditions for conjugacy to the subshift of finite type, which are akin to the results in [Shi et al., 2009] but in the Hausdorff space.

As mentioned above, the $A$-coupled-expanding property has been regarded as an important criterion for chaos. In addition, the positive topological entropy, which measures the complexity of a system, is also known as one of the main indices to characterize chaos. Therefore, it is natural to consider the topological entropy for $A$-coupled-expanding maps and the relationships between various concepts of chaos. As to this topic, the topological entropy for a $p$-horseshoe map (that is, a $p$-coupled expanding map) on the interval was considered in [Ruette, 2003]. In [Miyazawa, 2002], the
necessary and sufficient condition for a continuous map from a circle into itself to have a positive topological entropy was presented, and it was proved that the circle map is chaotic in the sense of Devaney if and only if its topological entropy is positive.

In this paper, the topological entropy of A-coupled-expanding maps on compact metric spaces is studied. The lower bound of the topological entropy for strictly A-coupled-expanding maps on compact metric spaces is obtained and a less conservative and more generalized criterion for chaos in the sense of Li-Yorke is further presented. A new concept, the partition-A-coupled-expanding map, is defined excluding the usual strictness condition, and its topological entropy is estimated. Particularly, the topological entropy for the partition-A-coupled-expanding circle map is given. As an illustrative example, the topological entropy for the Kasner map is calculated, which is one of the most important tools in the research on Cosmological models of Bianchi type in the big-bang singular limit.

The paper is organized as follows. In Section 2, some basic concepts and lemmas are briefly introduced. In Section 3, the topological entropy for the strictly A-coupled-expanding maps on compact metric spaces is discussed and a criterion for chaos in the sense of Li-Yorke which is less conservative and more generalized than the result of [Shi et al., 2009] is presented. In Section 4, some conditions for A-coupled-expanding maps to be a factor of subshifts of finite type and the topological entropy for A-coupled-expanding circle maps excluding the strictness condition are discussed. The topological entropy and a chaotic property of the Kasner map are further presented as an illustrative example of our theoretical results for demonstration. A brief conclusion is drawn in Section 5.

2. Preliminaries

2.1. Basic Definitions. A topological dynamical system is a pair \((X, T)\), where \(X\) is a compact metric space with a metric \(d\) and \(T\) is a surjective continuous map from \(X\) to itself.

Among the various definitions of chaos, the first and well-known one appeared in mathematics is chaos in the sense of Li and Yorke. It is based on the idea of [Li & Yorke, 1975], however, was formalized afterwards.

**Definition 2.1.** A set \(S \subset X\) is called a scrambled set if, for any two distinct points \(x, y \in S\),

\[
\liminf_{n \to \infty} d(T^n(x), T^n(y)) = 0 \quad \text{and} \quad \limsup_{n \to \infty} d(T^n(x), T^n(y)) > 0.
\]

The system \((X, T)\) is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set \(S\).

**Definition 2.2.** [Robinson, 1995] A \(p \times p\) matrix \(A = (a_{ij})_{1 \leq i, j \leq p}\) is said to be transition if \(a_{ij} = 0\) or 1 for all \(i, j\); \(\sum_{j=1}^{p} a_{ij} \geq 1\) for all \(i\); and \(\sum_{i=1}^{p} a_{ij} \geq 1\) for all \(j\).
Definition 2.3. [Brin & Stuck, 2003] A $p \times p$ matrix $A = (a_{ij})_{1 \leq i, j \leq p}$ is said to be nonnegative (positive) if any entry of $A$ is nonnegative (positive). The matrix $A$ is said to be irreducible if for any pair $(i, j), 1 \leq i, j \leq p$, there exists a positive integer $k$ such that $a_{ij}^k > 0$, where $A^k = (a_{ij}^k)_{1 \leq i, j \leq p}$. $A$ is said to be primitive (or eventually positive [Shi et al., 2009]) if $A^k$ is positive for some positive integer $k$.

Definition 2.4. [Shi et al., 2009] Let $(X, d)$ be a metric space, $T : D \subset X \to X$ a map, and $A = (a_{ij})_{1 \leq i, j \leq p}$ a transition matrix, where $p \geq 2$. If there exist $p$ nonempty subsets $\Lambda_i (1 \leq i \leq p)$ of $D$ with pairwise disjoint interiors such that

$$T(\Lambda_i) \supset \bigcup_{j \neq i} \Lambda_j, 1 \leq i \leq p$$

then $T$ is said to be $A$-coupled-expanding in $\Lambda_i, 1 \leq i \leq p$. Further, the map $T$ is said to be strictly $A$-coupled-expanding in $\Lambda_i, 1 \leq i \leq p$, if $d(\Lambda_i, \Lambda_j) > 0$ for all $1 \leq i \neq j \leq p$.

Remark 2.1. The dynamical system $(X, T)$ is said to be $A$-coupled-expanding if $T$ is $A$-coupled-expanding.

Remark 2.2. In the special case that all entries of $A$ are ones, the (strict) $A$-coupled-expansion is (strict) $p$-coupled-expansion or, (strict) $p$-coupled-expansion.

Definition 2.5. [Pollicott & Yuri, 1998] Let $(X_1, T_1), (X_2, T_2)$ be topological dynamical systems. If there exists a homeomorphism $h : X_1 \to X_2$ such that $h \circ T_1 = T_2 \circ h$, then $T_1$ is said to be topologically conjugate to $T_2$. If $h$ is continuous and surjective, but not necessarily invertible, and $h \circ T_1 = T_2 \circ h$, then $T_1$ is said to be topologically semi-conjugate to $T_2$, and $T_2$ is said to be a factor of $T_1$.

2.2. Topological Entropy. Adler, Konheim and McAndrew defined topological entropy for dynamical systems on compact metric spaces [Adler et al. (1965)], a key quantifier for complicated dynamical behaviors that plays an important role in the classification of dynamical systems. In the following the definition is briefly introduced. More details can be found in [Pollicott & Yuri, 1998] and [Walters, 1982].

Suppose $(X, T)$ be a topological dynamical system as addressed above. Let $\alpha = \{A_1\}$ and $\beta = \{B_1\}$ be covers of $X$, and define the refinement as $\alpha \vee \beta = \{A_i \cap B_j : A_i \cap B_j \neq \phi\}$. Moreover, if $\alpha^r = \{A_1^r, \cdots, A_n^r\} (r = 1, \cdots, k)$ are covers of $X$, define their refinement as

$$\bigvee_{r=1}^k \alpha^r = \{A_{i_1}^1 \cap A_{i_2}^2 \cap \cdots \cap A_{i_k}^k : i_j \in \{1, \cdots, N_r\}, j = 1, \cdots, k\}.$$


Denote \( T^{-1} \alpha = \{ T^{-1} A_1, \cdots, T^{-1} A_n \} \). More generally, one has
\[
\bigvee_{i=0}^{k-1} T^{-i} \alpha = \alpha \lor \cdots \lor T^{-(k-1)} \alpha = \{ A_{i_0} \cap T^{-1} A_{i_1} \cap \cdots \cap T^{-(k-1)} A_{i_{k-1}} : 1 \leq i_0, \cdots, i_{k-1} \leq n \}.
\]

Let \( N(\alpha) \) be the smallest number of sets that can be used as a subcover) of \( \alpha \).

**Definition 2.6.** The topological entropy for \( T \) relative to a cover \( \alpha \) is defined by
\[
h_{\text{top}}(\alpha, T) = \limsup_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right).
\]

**Remark 2.3.** The sequence \( \left( \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) \right)_{n \geq 1} \) is sub-additive, therefore
\[
h_{\text{top}}(\alpha, T) = \lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right) = \inf_{n \geq 1} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right).
\]

**Definition 2.7.** The topological entropy for the topological dynamical system \( (X, T) \) is defined by
\[
h_{\text{top}}(X, T) = \sup \{ h_{\text{top}}(\alpha, T) : \alpha \text{ is a finite open cover of } X \}.
\]

### 2.3. Transition matrix and graph

Define the norm \( \| \cdot \| \) of matrix \( A = (a_{ij})_{1 \leq i, j \leq p} \) by \( \| A \| = \sum_{1 \leq i, j \leq p} |a_{ij}| \). It is known that \( \limsup_{n \to \infty} \| A^n \|^{\frac{1}{n}} = \sup \{ |\lambda| : \lambda \text{ is the eigenvalue of } A \} \).

**Theorem 1** (Perron-Frobenius). Let \( A \) be a non-negative irreducible square matrix. Then there exists an eigenvalue \( \lambda \) of \( A \) with the following properties:
(i) \( \lambda > 0 \), (ii) \( \lambda \) is a simple root of the characteristic polynomial, (iii) \( \lambda \) has a positive eigenvector, (iv) if \( \mu \) is any other eigenvalue of \( A \), then \( |\mu| \leq \lambda \), (v) if \( k \) is the number of eigenvalues of modulus \( |\lambda| \), then the spectrum of \( A \) (with multiplicity) is invariant under the rotation of the complex plane by angle \( 2\pi/k \).

Let \( A = (a_{ij})_{1 \leq i, j \leq p} \) be a transition matrix and \( \Gamma_A \) be a directed graph associated to \( A \). In other words, \( \Gamma_A \) is the directed graph with vertices \( \{1, \cdots, p\} \) such that there is an edge \( i \to j \) if and only if \( a_{ij} = 1 \).

**Definition 2.8.** The eigenvalue \( \lambda \) of a matrix is said to be maximal if for any other eigenvalue \( \mu \), \( |\mu| \leq \lambda \). The maximal eigenvalue of matrix \( A \) is denoted by \( \lambda_A \).

**Definition 2.9.** A cycle is said to be full in the graph if it has all vertices of the graph.
Definition 2.10. The graph is said to be unified if there is a full cycle in it.

Lemma 1. A transition matrix $A$ is irreducible if and only if there is a full cycle in $\Gamma_A$.

Proof. It is obvious from the definition. □

Lemma 2. Let $A = (a_{ij})_{1 \leq i,j \leq p}$ and $B = (b_{ij})_{1 \leq i,j \leq p}$ be two transition matrices. If $A \leq B$ then $\lambda_A \leq \lambda_B$, where $A \leq B$ means $a_{ij} \leq b_{ij}$ for $1 \leq i,j \leq p$.

Proof. From Theorem 8.1.18 of [Horn & Johnson, 1985], one gets $\rho(A) \leq \rho(B)$, where $\rho(A)$ and $\rho(B)$ are respectively the spectrum radius of $A$ and $B$. Since $A$ and $B$ are nonnegative matrices, from Theorem 8.3.1 in [Horn & Johnson, 1985], $\rho(A)$ and $\rho(B)$ are respectively the eigenvalue of $A$ and $B$. Therefore, $\lambda_A \leq \lambda_B$. □

Lemma 3. If $A$ is a transition matrix, then its maximal eigenvalue $\lambda_A \geq 1$.

Proof. This can be concluded directly from Theorem 8.1.22 of [Horn & Johnson, 1985]. □

3. Entropy for strictly A-coupled-expanding maps on compact metric spaces and chaos

Lemma 4. Let $(X,T)$ be a topological dynamical system and $A$ a transition matrix. If the map $T : X \to X$ is strictly $A$-coupled-expanding in $\Lambda_1, \cdots, \Lambda_p \subset X$, then $T$ is also strictly $A$-coupled-expanding in $\overline{\Lambda}_1, \cdots, \overline{\Lambda}_p$, where $\overline{\cdot}$ is the closure of the set $\cdot$.

Proof. From the conditions, one obtains $\overline{\Lambda}_i \cap \overline{\Lambda}_j = \emptyset, 1 \leq i \neq j \leq p$, and $T(\Lambda_i) \supset \bigcup_{a_{ij}=1}^{j} \overline{\Lambda}_j, 1 \leq i \leq p$. Furthermore, since $\overline{\Lambda}_i$ is compact and $T$ is continuous, one has

$$T(\overline{\Lambda}_i) \supset \bigcup_{a_{ij}=1}^{j} \overline{\Lambda}_j = \bigcup_{a_{ij}=1}^{j} \overline{\Lambda}_j, 1 \leq i \leq p.$$ 

This completes the proof. □

Lemma 5. Let $(X,T)$ be a topological dynamical system. If the map $T : X \to X$ is strictly $p$-coupled-expanding in $\Lambda_1, \cdots, \Lambda_p \subset X$, then $h_{top}(X,T) \geq \log p$, where $p \geq 2$.

Proof. From Lemma 4, one can suppose that $\Lambda_1, \cdots, \Lambda_p$ are closed without loss of generality. From $\Lambda_i \cap \Lambda_j \neq \emptyset, 1 \leq i \neq j \leq p$, there exist open sets $U_1, \cdots, U_p$ in $X$ such that $\Lambda_i \subset U_i, U_i \cap U_j = \emptyset, 1 \leq i \neq j \leq p$. 


Let $U_{p+1} = X \setminus \bigcup_{i=1}^{p} \Lambda_i$. Then $U_{p+1}$ is open and $\mathcal{U} = \{U_1, \ldots, U_p, U_{p+1}\}$ is a finite open cover of $X$, where $U_{p+1} \cap \Lambda_i = \emptyset$ for $1 \leq i \leq p$.

For any $k \geq 0$ and any $(i_0 \cdots i_k) \in \{1, \ldots, p\}^{k+1}$, define
\[
\Lambda_{i_0 \cdots i_k} = \Lambda_{i_0} \cap T^{-1}(\Lambda_{i_1}) \cap \cdots \cap T^{-k}(\Lambda_{i_k}).
\]

Therefore, $T^k(\Lambda_{i_0 \cdots i_k}) = \Lambda_{i_k}(1 \leq i_0, \ldots, i_k \leq p)$. This is obvious when $k = 0$. While if $k > 0$, one has
\[
T^k(\Lambda_{i_0 \cdots i_k}) = T \circ T^{k-1}(\Lambda_{i_0 \cdots i_{k-1}} \cap T^{-k}(\Lambda_{i_k}))
\]
\[
= T \circ T^{k-1}\left(\Lambda_{i_0 \cdots i_{k-1}} \cap T^{-(k-1)}(T^{-1}(\Lambda_{i_k}))\right)
\]
\[
= T(\Lambda_{i_{k-1}} \cap T^{-1}(\Lambda_{i_k})) = T(\Lambda_{i_{k-1}}) \cap \Lambda_{i_k} = \Lambda_{i_k}.
\]

Therefore, $\Lambda_{i_0 \cdots i_k}$ is non-empty and closed for any $k(k \geq 0)$ and $(i_0 \cdots i_k) \in \{1, \ldots, p\}^{k+1}$.

If $(i_0 \cdots i_{n-1}) \neq (j_0 \cdots j_{n-1})$, there exists some $k(0 \leq k \leq n-1)$ such that $i_k \neq j_k$. Since $T^k(\Lambda_{i_0 \cdots i_{n-1}}) \subset \Lambda_{i_k} \subset U_{i_k}$, $T^k(\Lambda_{j_0 \cdots j_{n-1}}) \subset \Lambda_{j_k} \subset U_{j_k}$, and $U_{i_k} \cap U_{j_k} = \emptyset$, the set $U_{i_k} \cap T^{-1}\left(U_{i_1}\right) \cap \cdots \cap T^{-(n-1)}\left(U_{i_{n-1}}\right)$ includes $\Lambda_{i_0 \cdots i_k}$ but $\Lambda_{j_0 \cdots j_{k-1}}$. Thus, $U_{i_k} \cap T^{-1}\left(U_{i_1}\right) \cap \cdots \cap T^{-(n-1)}\left(U_{i_{n-1}}\right)$ and $U_{j_0} \cap T^{-1}\left(U_{j_1}\right) \cap \cdots \cap T^{-(n-1)}\left(U_{j_{n-1}}\right)$ are different. Therefore, $N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) \geq \log p$. Hence,
\[
h_{\text{top}}(X, T) \geq h_{\text{top}}(U, T) = \limsup_{n \to \infty} \frac{1}{n} \log N\left(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}\right) \geq \log p.
\]

**Lemma 6.** Let $(X, T)$ be a topological dynamical system and $A$ a transition matrix. If the map $T : X \to X$ is strictly $A$-coupled-expanding in $\Lambda_1, \ldots, \Lambda_p \subset X$, then $h_{\text{top}}(X, T) \geq \log \lambda_A$.

**Proof.** From Lemma 4, one can obtain that $\Lambda_1, \ldots, \Lambda_p$ are closed. Let $\lambda_A$ be the maximal eigenvalue of some $A_k$, which is a unified subgraph of $\Gamma_A$.

Denote $A_k = (a_{ij})_{1 \leq i, j \leq m}$ and $A^n_k = (a^n_{ij})_{1 \leq i, j \leq m}$. Then
\[
\limsup_{n \to \infty} \left\|A^n_k\right\|^\frac{1}{n} = \limsup_{n \to \infty} \left\|\sum_{1 \leq i, j \leq m} a^n_{ij}\right\|^\frac{1}{n} = \lambda_A.
\]

Since
\[
\limsup_{n \to \infty} \left(\sum_{1 \leq i, j \leq m} a^n_{ij}\right)^\frac{1}{n} = \max_{1 \leq i, j \leq m} \limsup_{n \to \infty} (a^n_{ij})^\frac{1}{n},
\]
there exist $i$ and $j$ $(1 \leq i, j \leq m)$ such that
\[
\limsup_{n \to \infty} (a^n_{ij})^\frac{1}{n} = \lambda_A,
\]
where $a^n_{ij}$ means the number of paths with length $n$ from vertex $i$ to vertex $j$ in the graph $\Gamma_{A_k}$. Since $\Gamma_{A_k}$ is unified, there is an $n_0 \geq 0$ such that
These cycles by

\[
\begin{array}{c}
  i \\
  i_{l+1}^{n+n_0-1} \\
  i_l^{n+n_0-2} \\
  \vdots \\
  i_1^n \\
  i_1
\end{array}
\]

1 ≤ \( l \leq a^{n+n_0}_{ii} \).

Since \( \Gamma_A \) is a graph on the set of vertices \( \{1, \cdots, p\} \) such that \( i \to j \) if and only if \( T(A_i) \supset A_j \), define

\[
C_l = A_i \cap T^{-1}(A^{l+1}) \cap T^{-2}(A^{l+2}) \cap \cdots \cap T^{-(n+n_0-1)}(A^{l+n+n_0-1}) , \quad 1 ≤ l ≤ a^{n+n_0}_{ii} ,
\]

then \( C_l \neq \phi \), and if \( l_1 \neq l_2 \) then \( C_{l_1} \cap C_{l_2} = \phi \). Furthermore, \( T^{n+n_0}(A_l) = a^{n+n_0}_{ii} \). Thus \( T^{n+n_0} \) is strictly \( a^{n+n_0}_{ii} \)-coupled-expanding. From

Lemma 5 \( h_{top}(X, T^n) ≥ \log a^{n+n_0}_{ii} \). Therefore, \( h_{top}(X, T) = \frac{1}{n} h_{top}(X, T^n) ≥ \frac{1}{n} \log a^{n+n_0}_{ii} \) for all \( n \in \mathbb{N} \). Since \( \lim \sup_{n \to \infty} (a^{n+n_0}_{ii})^\frac{1}{n} ≥ \lambda_A \), one has \( h_{top}(X, T) ≥ \log \lambda_A \). □

**Remark 3.1.** It was recently proved that a continuous strictly \( A \)-coupled-expanding map on compact subsets of a metric space is topologically semi-conjugate to \( \sigma_A \) (Theorem 3.1 of [?]). Lemma 6 can be resulted immediately from it using Proposition 4.1 of this paper.

**Lemma 7.** Let \( A = (a_{ij})_{1 ≤ i, j ≤ p} \) be an irreducible transition matrix. Then \( \lambda_A > 1 \) if and only if there exists some \( i_0 (1 ≤ i_0 ≤ p) \) such that \( \sum_{j=1}^{p} a_{i_0 j} ≥ 2 \).

In particular, \( \lambda_A ≥ 2^\frac{i}{p} \).

**Proof.** From Lemma 1, it is known \( \Gamma_A \) has a full cycle. Firstly, suppose that there exists an \( i_0 (1 ≤ i_0 ≤ p) \) such that \( \sum_{j=1}^{p} a_{i_0 j} ≥ 2 \). Then vertex \( i_0 \) is bifurcating, which means that it has 2 or more outgoing edges. \( \sum_{j=1}^{p} a_{i_0 j} \) means the number of \( p \)-paths from vertex \( i \), where \( p \)-paths are paths of length \( p \).

Since \( \Gamma_A \) has \( p \) vertices and matrix \( A \) is transition, for any \( i \), at least one of the paths from vertex \( i \) arrives at the bifurcation vertex \( i_0 \) for at least \( p - 1 \) times. Therefore, the number of \( p \)-paths is no less than 2. Thus, \( \sum_{j=1}^{p} a_{i_0 j} ≥ 2 \).

A \( 2p \)-path from any vertex \( i \) can be considered as a \( p \)-path which is from the end of a \( p \)-path passing the same vertex \( i \). Therefore, the number of \( 2p \)-paths is no less than \( 2^2 \). That is, \( \sum_{j=1}^{p} a_{ij}^{2p} ≥ 2^2 \).
Similarly, \( \sum_{j=1}^{p} a_{ij}^{np} \geq 2^n \). Hence, \( \|A^{np}\| \geq 2^n p \). Thus,

\[
\lambda_A = \limsup_{n \to \infty} \frac{\|A^{np}\|^{1/p}}{p} \geq \limsup_{n \to \infty} \left(2^n \right)^{1/p} = 2^{\frac{1}{p}}.
\]

Finally, suppose that \( \sum_{j=1}^{p} a_{ij} = 1 \) for any \( i \). Then \( \Gamma_A \) is a simple cycle passing all vertices, with each vertex having only one outgoing edge and one incoming edge. So, \( \|A^n\| = \sum_{j=1}^{p} a_{ij}^n = p \). Hence,

\[
\lambda_A = \limsup_{n \to \infty} \frac{\|A^n\|^{1/p}}{p} = 1.
\]

This completes the proof. \( \square \)

In [Blanchard et al., 2002], it was proved that if the topological entropy for \((X, T)\) is positive, then there exists a scrambled Cantor set. Particularly, \( T \) is chaotic in the sense of Li-Yorke. Here, by using Lemma 6 and Lemma 7, one obtains the following criterion for chaos.

**Theorem 2.** Let \((X, T)\) be a topological dynamical system and \( A \) an irreducible matrix, which has a row with the row sum being no less than 2. If \((X, T)\) is strictly \( A \)-coupled-expanding, then \( h_{top}(X, T) \geq \frac{\log 2}{p} \) and \((X, T)\) is chaotic in the sense of Li-Yorke.

**Remark 3.2.** As a criterion for chaos in the sense of Li-Yorke, the conditions in this theorem are less conservative than that in Theorem 5.1 in [Shi et al., 2009].

**Remark 3.3.** In Theorem 3.1 of [Zhang et al., 2012], it was proved that if a dynamical system on metric space is strictly \( A \)-coupled-expanding, then there is a subsystem which is topologically semi-conjugate to the finite symbolic dynamical system, furthermore, if the entropy of the finite symbolic dynamical system is positive then the topological dynamical system is chaotic in the sense of Li-Yorke. They put emphasis on semi-conjugacy of the dynamical system to the finite symbolic dynamical system, while Theorem 2 of this paper puts emphasis on proof of the positivity of entropy for the topological dynamical system. The positivity of entropy cannot derived from Theorem 3.1 of [Zhang et al., 2012].

4. **Topological entropy for \( A \)-coupled-expanding circle maps and chaos**

In this section we consider some conditions for \( A \)-coupled-expanding maps to be factors of subshifts of finite type, and the topological entropy for \( A \)-coupled-expanding circle maps. To compute the upper bound of topological entropy for a map, the map must be considered as a full-system, not only as a sub-system. Unfortunately, the \( A \)-coupled-expanding map
as a full-system generally does not satisfy the condition of strictness. For example, the Kasner map, which is one of the important tools in the research on Cosmological models of Bianchi type in the big-bang singular limit, is a ”partition”-A-coupled-expanding map excluding the strictness. In [Ruette, 2003], Ruette considered the lower bound of topological entropy for the p-coupled-expanding map from the interval to itself. In this section, we obtain some conditions for A-coupled-expanding maps excluding the strictness to be factors of subshifts of finite type, as well as the upper and lower bounds of topological entropy for ”partition”-A-coupled-expanding circle maps excluding the strictness.

4.1. Some conditions for A-coupled-expanding maps to be factors of subshifts of finite type.

Lemma 8. Let \((X, T)\) be a topological dynamical system and \(A\) a transition matrix. If \(T\) is A-coupled-expanding in compact sets \(\Lambda_1, \cdots, \Lambda_p \subset X\) such that \(\bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n})\) is singleton for any \(\alpha = (a_0a_1\cdots) \in \Sigma^+_A\), then there exists a nonempty compact invariant subset \(\Lambda\) such that \((\Sigma^+_A, \sigma_A)\) is topologically semi-conjugate to \((\Lambda, T|_{\Lambda})\), that is, the sub-system \((\Lambda, T|_{\Lambda})\) is a factor of \((\Sigma^+_A, \sigma_A)\).

Proof. Let

\[ \Lambda := \bigcup_{\alpha = (a_0a_1\cdots) \in \Sigma^+_A} \left( \bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n}) \right). \]

Obviously \(\Lambda \neq \emptyset\). Define \(\pi : \Sigma^+_A \to \Lambda\) as follows:

\[ \pi(\alpha) := \bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n}), \alpha = (a_0a_1\cdots) \in \Sigma^+_A. \]

Then \(\pi\) is a surjection and \(T(\Lambda) \subset \Lambda\). To show that \(\pi\) is continuous, choose any \(\alpha = (a_0a_1\cdots) \in \Sigma^+_A\). Since \(\bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n})\) is singleton, one has

\[ \forall \varepsilon \geq 0, \exists N_\varepsilon \in N : d \left( \bigcap_{n=0}^{N_\varepsilon} T^{-n}(\Lambda_{a_n}) \right) \leq \varepsilon, \]

where \(d(\bullet)\) denotes the diameter of \(\bullet\). Let \(\delta = \frac{1}{2^{N_\varepsilon+1}}\), then

\[ \beta \in \Sigma^+_A, d_{\Sigma^+_A}(\alpha, \beta) \leq \delta \Rightarrow a_n = b_n, 0 \leq n \leq N_\varepsilon. \]

Therefore \(d(\pi(\alpha), \pi(\beta)) \leq \varepsilon\), that is, \(\pi\) is continuous. Thus, from the compactness of \(\Sigma^+_A\) and continuity of \(\pi\), one obtains that \(\Lambda = \pi(\Sigma^+_A)\) is compact.

Next, for any \(\alpha = (a_0a_1\cdots) \in \Sigma^+_A\),

\[ \pi \circ \sigma_A(\alpha) = \pi(\sigma_A(a_0a_1\cdots)) = \pi(a_1a_2\cdots) = \bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n}) = T \circ \pi(\alpha), \]

that is, \(\pi \circ \sigma_A = T \circ \pi\) holds. The proof is thus completed. \(\square\)
Remark 4.1. Lemma 8 presents a condition for a subsystem of a given system to be a factor of a subshift of finite type. However, the condition for full systems themselves to be factors of subshifts of finite type requires estimating the upper bound of the topological entropy for the systems, which is discussed in the following theorems.

Theorem 3. Let \((X, T)\) be a topological dynamical system and \(A\) a transition matrix. Let \(T\) be \(A\)-coupled expanding in sets \(\Lambda_1, \cdots, \Lambda_p \subset X\), where \(\bigcup_{j=1}^p \Lambda_j = X\) (it is called partition-\(A\)-coupled-expanding). If \(T\) satisfies the following conditions:

(iii) For any \(\alpha = (a_0a_1\cdots) \in \Sigma^+_A\), \(\bigcap_{n=0}^\infty T^{-n}(\Lambda_{a_n})\) is a singleton,

(iiiii) \(T(\Lambda_i) = \bigcup_{a_{ij}=1} \Lambda_j, 1 \leq i \leq p\),

then \((\Sigma^+_A, \sigma_A)\) is topologically semi-conjugate to \((X, T)\), that is, the full system \((X, T)\) is a factor of \((\Sigma^+_A, \sigma_A)\).

Proof. It is sufficient to prove that the set

\[
\Lambda = \bigcup_{\alpha = (a_0a_1\cdots) \in \Sigma^+_A} \left( \bigcap_{n=0}^\infty T^{-n}(\Lambda_{a_n}) \right)
\]

considered in Lemma 8 is equal to the set \(X\). In fact, condition (ii) implies that

\[
\forall x \in X, \exists \alpha = (a_0a_1\cdots) \in \Sigma^+_A : T^n(x) \in \Lambda_{a_n}.
\]

That is, \(X \subset \Lambda\). Therefore, one has \(\Lambda = X\). \(\Box\)

Remark 4.2. Condition (ii) in Theorem 3 is the necessary and sufficient condition for \(X\) to be equal to

\[
\Lambda = \bigcup_{\alpha = (a_0a_1\cdots) \in \Sigma^+_A} \left( \bigcap_{n=0}^\infty T^{-n}(\Lambda_{a_n}) \right).
\]

Lemma 9. Let \(A\) be a transition matrix, \((X, T)\) be a topological dynamical system, and \(T\) be \(A\)-coupled-expanding in compact sets \(\Lambda_1, \cdots, \Lambda_p \subset X\). If for any \(\alpha = (a_0a_1\cdots) \in \Sigma^+_A\), a set \(\bigcap_{n=0}^\infty T^{-n}(\text{int}\Lambda_{a_n})\) is singleton, then there exists a nonempty compact invariant set \(\Lambda\) such that \((\Sigma^+_A, \sigma_A)\) is topologically semi-conjugate to \((\Lambda, T|_\Lambda)\) (where \(\text{int}\Lambda_{a_n}\) represents the interior of \(\Lambda_{a_n}\)).

Proof. Let

\[
\Lambda = \bigcup_{\alpha = (a_0a_1\cdots) \in \Sigma^+_A} \left( \bigcap_{n=0}^\infty T^{-n}(\text{int}\Lambda_{a_n}) \right)
\]

and

\[
\pi(\alpha) = \bigcap_{n=0}^\infty T^{-n}(\text{int}\Lambda_{a_n}), \alpha = (a_0a_1\cdots) \in \Sigma^+_A.
\]
This lemma can be proved in the same way as that of Lemma 8. □

**Remark 4.3.** Lemma 9 can be conveniently applied to various systems in differentiable spaces.

**Remark 4.4.** The condition of Lemma 9 is weaker than that of Lemma 8. In fact,
\[ \bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n}) \supset \bigcap_{n=0}^{\infty} T^{-n}(int\Lambda_{a_n}). \]
However, the inverse does not hold in general.

**Theorem 4.** Let \( A \) be a transition matrix and \( (X, T) \) a topological dynamical system. Suppose that \( T \) is partition-\( A \)-coupled-expanding in sets \( \Lambda_1, \ldots, \Lambda_p \subset X \) such that \( \bigcup_{i=1}^{p} int\Lambda_i = X \). If \( T \) satisfies the following conditions:

(iii) For any \( \alpha = (a_0a_1 \cdots) \in \Sigma_A^+ \), \( \bigcap_{n=0}^{\infty} T^{-n}(int\Lambda_{a_n}) \) is singleton,

(iiiii) \( int\Lambda_i \subset \bigcup_{a_j=1}^{j} T^{-1}(int\Lambda_j) \),

then \( (\Sigma_A^+, \sigma_A) \) is topologically semi-conjugate to \( (X, T) \), that is, the full system \( (X, T) \) is a factor of \( (\Sigma_A^+, \sigma_A) \).

**Proof.** Denote
\[ \Lambda = \bigcup_{\alpha=(a_0a_1\cdots) \in \Sigma_A^+} \left( \bigcap_{n=0}^{\infty} T^{-n}(int\Lambda_{a_n}) \right). \]
Since
\[ \bigcup_{i=1}^{p} int\Lambda_i = X, \]
it follows that
\[ \forall x \in X, \exists a_0 \in \{1, \cdots, p\} : x \in int\Lambda_{a_0}. \]
Next, from condition (ii) and by reduction, one can conclude that
\[ \exists a_n \in \{1, \cdots, p\}(\langle A \rangle_{a_n-1,n} = 1) : x \in T^{-n}(int\Lambda_{a_n}), n \geq 1. \]
Let \( \alpha = (a_0a_1\cdots) \), it is obvious that \( \alpha \in \Sigma_A^+ \) and \( x \in \Lambda \). Therefore \( \Lambda = X \). This completes the proof. □

4.2. **Entropy for partition-A-coupled-expanding circle maps.** In this subsection, the topological entropy for partition-A-coupled-expanding circle maps is investigated based on above results.

**Proposition 1.** [Pollicott & Yuri, 1998] Let \( (X_1, T_1) \) and \( (X_2, T_2) \) be topological dynamical systems. If system \( (X_1, T_1) \) is topologically semi-conjugate to system \( (X_2, T_2) \), then \( h_{top}(X_1, T_1) \geq h_{top}(X_2, T_2) \).
Theorem 5. Let $X$ be an interval or a circle, $(X, T)$ a topological dynamical system and $A$ a transition matrix. If $T$ is a partition-$A$-coupled-expanding map in sets $\Lambda_1, \cdots, \Lambda_p \subset X$, satisfying the following conditions:

(iii) For any $\alpha = (a_0a_1 \cdots) \in \Sigma_A^+, \bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{a_n})$ is singleton,

(iiiii) $T(\Lambda_i) = \bigcup_{j} \Lambda_j, 1 \leq i \leq p$,

then

$$h_{top}(X, T) = \log \lambda_A.$$ 

Proof. On one hand, since $(\Sigma_A^+, \sigma_A)$ is topologically semi-conjugate to $(X, T)$, as can be obtained from Theorem 3, one has

$$h_{top}(X, T) \leq h_{top}(\Sigma_A^+, \sigma_A) = \log \lambda_A$$

from Proposition 1.

On the other hand, by using the method similar to that used in the proof of Proposition 4.2.16 of [Ruette, 2003], for an interval map, one can conclude that

$$h_{top}(X, T) \geq \log \lambda_A.$$ 

Therefore, $h_{top}(X, T) = \log \lambda_A$. This completes the proof. \qed

Theorem 6. Let $S^1$ be a unit circle, $A$ a transition matrix and $T : S^1 \to S^1$ a partition-$A$-coupled-expanding map in closed arcs of the circles $\Lambda_1, \cdots, \Lambda_p \subset S^1$. If $T$ satisfies following conditions:

(iii) For some $r > 1$ and any arc $V \subset \Lambda_i, d(T(V)) \geq rd(V), 1 \leq i \leq p,$

where $d(\bullet)$ denotes the natural length of $\bullet$,

(iiiiii) $\bigcup_i T(\partial \Lambda_i) \subset \bigcup_i \partial \Lambda_i$,

then $(S^1, T)$ is a factor of $(\Sigma_A^+, \sigma_A)$, and $h_{top}(S^1, T) = \log \lambda_A$.

Proof. For $N \geq 0$ define a set $D_N$ as

$$D_N = \bigcap_{n=0}^{N} T^{-n}(\Lambda_{a_n}).$$

It is obvious that $D_0 \supset D_1 \supset \cdots$. Since $T$ is continuous and $\Lambda_{a_n}$ is compact for any $n$, $D_N \neq \emptyset$ is also compact. Therefore, for any $N$

$$\bigcap_{n=0}^{N} D_n \neq \emptyset.$$ 

From condition (i),

$$T(\Lambda_{a_0} \cap T^{-1}(\Lambda_{a_1})) = \Lambda_{a_1},$$

$$d(\Lambda_{a_0} \cap T^{-1}(\Lambda_{a_1})) \leq \frac{1}{r} d(\Lambda_{a_i}) \leq \frac{1}{r} d(S^1), i = 1, 2$$
\[ d(D_N) \leq \frac{1}{r^N}d(S^1). \]

It follows that \( d(D_N) \to 0(N \to \infty) \) and \( \bigcap_{n=0}^{\infty} T^{-n}(\Lambda_{\alpha_n}) \) consists of one point, which satisfies Condition (i) of Theorem 5. Furthermore, Condition (ii) of this theorem satisfies that of Theorem 5. Therefore, the conclusion can be drawn. Thus the proof is completed. \( \Box \)

In [Miyazawa, 2002], it was proved that Devaney’s chaos is equivalent to having positive entropy for a continuous circle map. Therefore, one can get the following criterion as a sufficient condition for a circle map to be chaotic based on Theorem 5, 6 and Lemma 7.

**Proposition 2.** Let \( A \) be an irreducible matrix, which has a row with the row-sum no less than 2. If \( T : S_1 \to S_1 \) is a continuous \( A \)-coupled-expanding map satisfying the assumptions of Theorem 5 or Theorem 6, then \( T \) is chaotic in the sense of Devaney as well as Li-Yorke.

### 4.3. A numerical example.

#### 4.3.1. Background of the Kasner map in Cosmological models of Bianchi type IX.

Cosmological models of Bianchi type yield spatially homogeneous, anisotropic solutions \( g_{\alpha\beta} \) of the Einstein field equations,

\[ R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = T_{\alpha\beta}. \]

Here \( R_{\alpha\beta} \) denotes the Ricci curvature tensor and \( R \) the scalar curvature of the Lorentzian metric \( g_{\alpha\beta} \), whereas \( T_{\alpha\beta} \) denotes the stress energy tensor.

This problem can be reduced to a five-dimensional system of ordinary differential equations in expansion-normalized variables representing the spatial homogeneity by a three-dimensional Lie algebra. For unimodal Lie algebras, Bianchi class A, the reduced equations are [Rendall, 1997] [Ringström, 2001]

\[
\begin{align*}
N'_1 &= (q - 4\Sigma_+)N_1, \\
N'_2 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, \\
N'_3 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, \\
\Sigma'_+ &= -(2 - q)\Sigma_+ - 3S_+, \\
\Sigma'_- &= -(2 - q)\Sigma_- - 3S_-,
\end{align*}
\]

(1)

where

\[
\begin{align*}
q &= \frac{1}{2}(3\gamma - 2)\Omega + 2(\Sigma_+^2 + \Sigma_-^2), \\
S_+ &= \frac{1}{2}[(N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)], \\
S_- &= \sqrt{3}(N_3 - N_2)(N_1 - N_2 - N_3).
\end{align*}
\]

(2)

Here, the superscript ‘’ denotes the derivative with respect to time \( \tau \), \( N_i(i = 1, 2, 3) \) are the spatial curvature variables, \( \Sigma_+ \) and \( \Sigma_- \) are the shear variables, \( q \) is the deceleration parameter, \( \Omega \) is the density parameter, and \( \frac{2}{3} < \gamma \leq \frac{3}{2} \).
2) describes the uniformly distributed matter. The Hamiltonian constraint is

\[ \Omega + \Sigma_+^2 + \Sigma_-^2 + \frac{3}{4}(N_1^2 + N_2^2 + N_3^2 - 2(N_1N_1 + N_2N_2N_3 + N_3N_1)) = 1. \tag{3} \]

The Bianchi type to which a solution of (1) corresponds depends on the values of \( N_1, N_2 \) and \( N_3 \). If all the three are zeros the Bianchi type is \( I \). If precisely one is non-zero then it is type \( II \). If precisely two are nonzero it is either type \( VI_0 \) (signs opposite) or type \( VII_0 \) (signs equal). If all three are non-zero it is either type \( IX \) (all signs equal) or type \( VIII \) (one sign different from the other two).

The invariant set of (1) with \( \Omega = 0 \) corresponds to the vacuum model. The set of equilibria of Bianchi type \( I \) becomes \( \{(N_1, N_2, N_3, \Sigma_+, \Sigma_-, \Omega) : N_1 = N_2 = N_3 = 0, \Sigma_+^2 + \Sigma_-^2 = 1, \Omega = 0\} \). The set \( K = \{(\Sigma_+, \Sigma_-) : \Sigma_+^2 + \Sigma_-^2 = 1\} \) on the \( \Sigma_\pm \)-plane is called the Kasner circle. There are three special points on the Kasner circle \( K \) with coordinates \((-1, 0), \left(\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right)\), which divide the circle \( K \) into three equal parts. They are denoted by \( T_1, T_2 \) and \( T_3 \) [Rendall, 1997], as shown Fig.1. The \( \alpha \)-limit set of a solution of type \( II \) with \( N_1 \neq 0 \) lies on the longer one of the two open arcs with endpoints \( T_2 \) and \( T_3 \), while the \( \omega \)-limit set lies on the shorter one of the arcs. The points \( T_1, T_2 \) and \( T_3 \) are permuted cyclically by the threefold symmetry, which shows what happens for type \( II \) solution with \( N_2 \neq 0 \) or \( N_3 \neq 0 \). In other words, the trajectories of Bianchi type \( II \) vacuum solutions consist of heteroclinic orbits to equilibria on the Kasner circle, of which projections onto the \( \Sigma_\pm \)-plane yield straight lines through the point \( (\Sigma_+, \Sigma_-) = (2, 0) \) in the case of \( N_1 \neq 0 \). The projections of the other cases of Bianchi type \( II (N_2 \neq 0 \) or \( N_3 \neq 0) \) are given similarly.

Therefore, if \( x \) is a point in \( K \setminus \{T_1, T_2, T_3\} \), then there is a point \( y \) in \( K \) and vacuum type \( II \) heteroclinic orbit with \( x \) being an \( \omega \)-limit point and \( y \) being an \( \alpha \)-limit point. Furthermore, if \( x \) lies on the shorter of the two open arcs with endpoints \( T_2 \) and \( T_3 \), then \( y \) lies on the longer of these arcs while \( x \) and \( y \) lie on the straight lines through the point \( (\Sigma_+, \Sigma_-) = (2, 0) \) in the case of \( N_1 \neq 0 \). The same results can be obtained for the other two cases, \( N_2 \neq 0 \) or \( N_3 \neq 0 \), whereas the straight line goes through the point \((-1, -\sqrt{3})\) and \((-1, +\sqrt{3})\) respectively.

The Kasner map \( \varphi : K \to K \) maps each \( x \) in \( K \setminus \{T_1, T_2, T_3\} \) to \( y \) in \( K \), and maps \( T_i \) to \( T_i \) (\( i = 1, 2, 3 \)).

The \( \alpha \)-limit of system (1) corresponds to the initial singularity (big-bang singularity) of the cosmological model. Belinskii, Khalatnikov and Lifshitz [Belinskii et al., 1982] and Misner [Misner, 1969] conjectured that the dynamics of the Bianchi type \( IX \) type models in this limit follows the Kasner map. In [Ringström, 2001], it was proved that at least for Bianchi type \( IX \) solutions, the Bianchi attractor formed by the union of the Kasner circle and its heteroclinic orbits is indeed an attractor for trajectories to generic initial data under the time-reversed flow.
4.3.2. *Topological Entropy for the Kasner map.* Here we will consider the entropy and chaotic property of the Kasner map using the above results.

The chaotic dynamics of the Bianchi type IX cosmological models in the big-bang singular limit has been widely discussed in the last few decades. The transient behavior of the IX models towards the initial singularity can be described by sequences of anisotropic Kasner states, that is, Bianchi type I vacuum solutions. These sequences are determined by a discrete map, Kasner map, which implies an oscillatory anisotropic behavior. In the first work by Barrow [Barrow, 1982], the chaotic property of this discrete map represented by the Gauss map, was studied. But the Gauss map itself corresponds to a specific time slicing and the ambiguity of time was manifest in this research area [Rendall, 1997]. However, the Kasner map represented by the above mentioned form as shown in Fig. 1 has been discussed, which doesn’t depend on any time slicing [Rendall, 1997]. Therefore, it is meaningful to consider the chaotic property for this map, especially in the circumstance of focus on the research interest in chaos for cosmological models in the big-bang singular limit.

We will firstly show that this type of the Kasner map is $A$-coupled-expanding for a primitive matrix $A$ excluding the strictness and compute its topological entropy, and discuss its chaotic properties in the sense of Li-Yorke and Devaney.

For convenience, the polar co-ordinate system $(r, \theta)$ is introduced in the $(\Sigma_+, \Sigma_-)$-plane, thus the Kasner circle $K$ can be denoted as

$$K = \{(1, \theta) \mid \theta \in [0, 2\pi]\}.$$ 

The co-ordinates of the special points $T_1$, $T_2$, $T_3$ are respectively $(1, \pi/3)$, $(1, 5\pi/3)$, and $(1, \pi)$. Let $\Lambda_1$ be the shorter of the arcs between $T_2$ and $T_3$ on the circle, $\Lambda_2$ and $\Lambda_3$ be the shorter of the arcs between $T_1$ and $T_3$, $T_1$ and $T_2$, respectively. In particular,

$$\Lambda_1 = \{(1, \theta) \mid \theta \in [0, \pi/3] \cup [5\pi/3, 2\pi]\},$$

$$\Lambda_2 = \{(1, \theta) \mid \theta \in [\pi, 5\pi/3]\},$$

$$\Lambda_3 = \{(1, \theta) \mid \theta \in [\pi/3, \pi]\},$$

as shown in Fig. 4.3.2.

For an eventually positive $3 \times 3$ matrix

$$A_0 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

one can see that the Kasner map $\varphi$ is an $A_0$-coupled-expanding map in $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$. 
In fact, it is easy to see that \( \Lambda_1, \Lambda_2 \) and \( \Lambda_3 \) are closed subsets with disjoint interiors in \( K \) and satisfy
\[
\varphi(\Lambda_1) = \Lambda_2 \cup \Lambda_3, \\
\varphi(\Lambda_2) = \Lambda_3 \cup \Lambda_1, \\
\varphi(\Lambda_3) = \Lambda_1 \cup \Lambda_2.
\]

For convenience, one can denote points on \( K \) by one parameter \( \theta \in [0, 2\pi]/(\mod 2\pi) \) and define the natural metric on the Kasner circle \( K \) as
\[
d(\theta_1, \theta_2) = \min\{|\theta_1 - \theta_2|, 2\pi - |\theta_1 - \theta_2|\}, \quad \theta_1, \theta_2 \in [0, 2\pi)
\]

The Kasner map \( \varphi : K \to K \) can be expressed by a map \( \Phi : [0, 2\pi]/\sim \to [0, 2\pi]/\sim \), where \( \sim \) indicates that 0 and \( 2\pi \) are identical. Then one has the following lemma.

**Lemma 10.** The map \( \Phi \) satisfies

(iii) \( \Phi(\theta) \in C^1([0, 2\pi]/\sim) \),

(iiiii) \( |\Phi'(\theta)| \geq 1 \) for any \( \theta \in K \).

The equal sign of (ii) holds only for the special points \( T_1, T_2 \) and \( T_3 \), that is, for \( \theta = \pi, \pi/3 \) and \( 5\pi/3 \).

It is sufficient to consider for \( \theta \in [0, \pi/3] \) by symmetry of the Kasner map. This lemma can thus be easily proved by using the fact that the map \( \Phi \) for \( \theta \in [0, \pi/3] \) can be described by
\[
\Phi(\theta) = \pi - \theta - 2 \arctan \frac{\sin \theta}{2 - \cos \theta} : \theta \in [0, \pi/3].
\]

**Proposition 3.** The Kasner map is chaotic in the sense of Devaney as well as Li-Yorke. Moreover the Kasner map is a factor of the subshift \( \sigma_{A_0} \) and its topological entropy is \( \log 2 \).
Proof. $A_0$ is an irreducible matrix with row-sum 2, and it is obvious that $\Phi$ is a partition-$A_0$-coupled expanding map satisfying Condition (ii) of Theorem 5. From (4), it can be confirmed that condition 1 of Theorem 5 is also satisfied. Therefore, the Kasner map is chaotic in the sense of Devaney as well as Li-Yorke from Proposition 2. From Theorem 5, one can see that the second statement obviously holds since the largest eigenvalue of $A_0$ is 2. □

Let $h : \Sigma^+_3(A_0) \to K$:

$$h(\alpha) = \bigcap_{n=0}^{\infty} \Phi^{-n}(\text{int} \Lambda_{\alpha_n}), \quad \alpha = (\alpha_n) \in \Sigma^+_3(A_0).$$

One can see that the map $h$ is surjective and continuous, and it satisfies $h \circ \sigma_{A_0} = \varphi \circ h$. That is, $h$ is a topologically semi-conjugate map for which the Kasner map is a factor of the subshift $\sigma_{A_0} : \Sigma^+_3(A_0) \to \Sigma^+_3(A_0)$. Therefore, one can easily obtain the following proposition.

**Proposition 4.** Let $h : \Sigma^+_3(A_0) \to K$ be the topologically semi-conjugate map for which the Kasner map $\varphi$ is a factor of $\sigma_{A_0}$ as mentioned above, then one has

(iii) For any $y \in K$, $y$ has exactly one or two pre-images in $\Sigma^+_3(A_0)$, i.e. $h^{-1}(y)$ consists of either one or two points,

(iiiii) The set of points $y \in K$ with $h^{-1}(y)$ consisting of more than one point is contained in the countable set $\bigcup_{n=1}^{\infty} h^{-n}(\{T_1, T_2, T_3\})$.

**Remark 4.5.** As mentioned above, the Kasner map (actually the BKL map) represented by the Gauss map has chaotic properties and its topological entropy is also $\log 2$ even though the Gauss map itself corresponds to a specific time slicing and ambiguity of time is manifest in cosmological model area [Cornish & Levin, 1997]. Here we have proved that the Kasner map represented as in [Rendall, 1997], which doesn’t depend on any time slicing, is chaotic in the sense of Devaney as well as Li-Yorke. Moreover we have found that this map is a factor of a subshift and its topological entropy is also $\log 2$. Proposition 4.4 shows clearly the relation between the Kasner map and the subshift $\sigma_{A_0}$.

5. Conclusion

In this paper, the topological entropy for strictly $A$-coupled-expanding maps on compact metric spaces has been studied as a criterion for chaos in the sense of Li-Yorke has been presented, which is less conservative and more generalized than the latest result. Some conditions for $A$-coupled-expanding maps excluding the strictness to be factors of subshifts of finite type have been obtained, along with the topological entropy for $A$-coupled-expanding circle maps excluding the strictness. In particular, the topological entropy of our proposed "partition-$A$-coupled-expanding" map has been estimated and illustrated with the Kasner map.
Acknowledgment

This work is supported by the Chinese National Natural Foundation under Grant No. 61174028. Also it is with immense gratitude that I acknowledge anonymous reviewers.

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