Cosmologies with a Chaplygin gas have recently been explored with the objective of explaining the transition from a dust dominated epoch towards an accelerating expansion stage. We consider the hypothesis that the transition to the accelerated period involves a quantum mechanical process. Three physically admissible cases are possible. In particular, we identify a minisuperspace configuration with two Lorentzian sectors, separated by a classically forbidden region. The Hartle-Hawking and Vilenkin wave functions are computed, together with the transition amplitudes towards the accelerating epoch. Furthermore, it is found that for specific initial conditions, the parameters characterizing the generalized Chaplygin gas become related through an expression involving an integer $n$. We also introduce a phenomenological association between some brane-world scenarios and a FRW minisuperspace cosmology with a generalized Chaplygin gas. The aim is to promote a discussion and subsequent research on the quantum creation of brane cosmologies from such a perspective. Results suggest that the brane tension would become related with generalized Chaplygin gas parameters through another expression involving an integer.

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I. INTRODUCTION

Mounting evidence from supernova Ia (SNIa) observations [1] and the more recent cosmic microwave background radiation (CMBR) data (see, e.g., Ref. [2] and [3]) is suggesting that the expansion of our universe seems to be in an accelerated state. This has been designated as the “dark energy” effect [4]. A possible candidate responsible for this evolution is the usual vacuum energy represented by a cosmological constant, $\Lambda$, providing a negative pressure [5, 6]. However, this idea has to deal with the fact that the observational value of $\Lambda$ is 120 orders of magnitude smaller than that established from field theory methods [5, 6]. Moreover, it is a special “cosmic coincidence” that the energy density associated with the cosmological constant has a value near to the density of other matter types exactly today. A plausible alternative is to consider a dynamical vacuum energy (also called “quintessence”) [7], involving one or two scalar fields, where some scenarios are associated with potentials justified from supergravity theories [8]. Nevertheless, when quintessence models confront the cosmic coincidence issue, they face fine-tuning problems and no satisfactory solution has yet been found.

Recently, an interesting proposal has been made [9], describing a transition from a universe filled with dust-like matter to an accelerating expanding stage: the scenario of the Chaplygin gas applied to cosmology, which was later generalized in Ref. [10, 11]. The generalized Chaplygin gas model is described by a perfect fluid obeying an exotic equation of state [11]

$$p = -\frac{A}{\rho^\alpha},$$

where $A$ is a positive constant and $0 < \alpha \leq 1$. The standard Chaplygin gas [9] corresponds to $\alpha = 1$. Inserting this equation into the relativistic energy conservation equation, leads to an energy density evolving as

$$\rho(a) = \left( A + \frac{B}{a^3(1+\alpha)} \right)^{\frac{1}{1+\alpha}},$$

where $B$ is a constant. This model interpolates between a universe dominated by dust and a DeSitter stage, through a phase described by an equation of state of the form $p = \alpha \rho$. It is curious to mention that the Chaplygin gas was originally introduced in aerodynamics [12].

Quite a few publications devoted to Chaplygin gas cosmological models have already appeared in the literature [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. Recent reviews can be found in [25, 26] and a dynamical system analysis regarding the stability of FRW cosmologies with a generalized Chaplygin gas was presented in Ref. [27].

Within an observational context, it is worth mentioning the analysis of compatibility at the level of CMBR...
Chaplygin gas can be obtained in the context of a general spacetime \cite{18}. In particular conditions, a generalized written as a generalized Born-Infeld action \cite{10,11,29}, description of a complex scalar field whose action can be time models have also received increased attention. In indications have been drawn when confronting generalized dark matter scenarios \cite{17}. Overall, some favourable power spectrum of large scale structure \cite{15,16} and uni-

We also focused on an additional aim: a discussion on the quantum creation of a brane-world cosmology. This perspective is suggested by the phenomenological context that some brane-worlds acquire within generalized Chaplygin gas cosmologies. This allows us to make use of important similarities between the quantum cosmology of a brane-world model and a FRW quantum cosmology with a Chaplygin gas content, which we will point in detail herein. While the former may require some non-standard tools (see e.g., Ref. \cite{59}), the latter imports techniques from well known quantum cosmological developments \cite{48,50,51,52}. In more detail, an analysis can be made exploring the minisuperspace resemblances between particular brane-worlds, generalized Chaplygin gas cosmological models and FRW minisuperspace scenarios with a complex scalar field $\phi$, with $\phi \equiv e^{i\theta} \phi$ (where $\theta$ constitutes a cyclical coordinate, determining that the conjugate momentum $p_\theta$ corresponds to a conserved charge). We will present herein some interesting results which are retrieved when including additional quantum corrections to the WKB description.

Accordingly, this paper is organized as follows. In Sec. II we present our minisuperspace model, pointing to the possible relation between a generalized Chaplygin gas cosmology and some brane-world properties. The corresponding Hartle-Hawking and Vilenkin states are obtained in Sec. III. Interesting quantization consequences are described in Sec. IV, and possible extensions beyond the saddle-point approximation are discussed in Sec. V. Sec. VI constitutes a summary of the results herein presented. An appendix is also included. It conveys the calculations of turning points, which are pertinent for the minisuperspace case (investigated throughout sections II to V), whose interesting similarities with some brane-world cosmologies are discussed in this paper.
II. MINISUPERSPACE MODEL

Let us take the case of a closed \((k = 1)\) FRW model in the presence of a positive cosmological constant \(\Lambda > 0\) with a generalized Chaplygin gas. The minisuperspace Lagrangian has the form (see, e.g., ref. \[53, 54\])

\[
L = -\frac{3\pi}{4G} \left( \dot{a}^2 a - N a \right) - 2\pi a^3 N \frac{\Lambda}{8\pi G} - 2\pi a^3 N \rho. \tag{3}
\]

After determining the corresponding canonical momentum to \(a\) and therefore the Hamiltonian constraint,

\[
-\frac{G}{3\pi} \dot{a}^2 + \frac{3\pi}{4G} V(a) = 0, \tag{4}
\]

its quantum description can be represented by the Wheeler-DeWitt equation, written as

\[
\left[ -\frac{G}{3\pi} \frac{d^2}{da^2} + \frac{3\pi}{4G} V(a) \right] \Psi(a) = 0, \tag{5}
\]

where \(\Psi(a)\) is the wave function characterizing the quantum state of this FRW universe. The effective minisuperspace potential \(V(a)\) is given by

\[
V(a) = a^2 - \lambda a^4 - \bar{\rho}(a) a^2, \tag{6}
\]

where \(\lambda = \frac{2\pi}{3}\),

\[
\bar{\rho}(a) = \left( \tilde{A} + \frac{\tilde{B}}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}}, \tag{7}
\]

together with \(0 < \alpha \leq 1\), \(\tilde{A} = \tilde{A} \left( \frac{8\pi G}{3} \right)^{\alpha+1}, \tilde{B} = \tilde{B} \left( \frac{8\pi G}{3} \right)^{-\alpha+1}\). As mentioned, for \(\alpha = 1\) we recover the usual Chaplygin gas case. We will consider a semiclassical WKB (Wentzel, Kramers, Brillouin) description, where the wave function of the universe can be approximated by

\[
\Psi = C_1 \psi_1 + C_2 \psi_2, \tag{8}
\]

where

\[
\psi_i = \exp \left[ -\frac{1}{2} S_i^{(i)}(a) \right], \tag{9}
\]

with \(i = 1, 2\) and \(S_0^{(i)}(a)\) satisfying the Hamilton-Jacobi equation

\[
\left( \frac{dS_0^{(i)}(a)}{da} \right)^2 = \frac{9\pi^2}{4} V(a). \tag{10}
\]

In particular, \(\psi_1\) and \(\psi_2\) constitute outgoing or ingoing modes in the classically allowed regions, decreasing or increasing functions in the classically forbidden regions, respectively, and \(C_1, C_2\) are constants. The WKB validity condition is given by \(G \left| \frac{dV(a)}{da} \right| \ll |V(a)|^{3/2}\).

In order to retrieve the wave functions in Eq. (8) in an analytical form, a saddle-point formulation will be employed. Therefore, it is necessary to determine the possible turning points for the potential \(V(a)\). Analytically also requires we employ the following approximation (where we henceforth drop the overbars on \(a\))

\[
\left( A + \frac{B}{a^{3(1+\alpha)}} \right)^{\frac{1}{1+\alpha}} \approx \frac{B^{\frac{1}{1+\alpha}}}{a^3} \left[ 1 + \frac{1}{2} \frac{1}{1+\alpha} \left( \frac{1}{1+\alpha} - 1 \right) \frac{A^2}{B^2} a^{6(\alpha+1)} + \ldots \right]. \tag{11}
\]

and if we take

\[
a \ll \left[ (\alpha + 1) \frac{B}{A} \right]^{-\frac{1}{3(\alpha+1)}}, \tag{12}
\]

we can write the potential \(V(a)\) as

\[
V_0(a) \approx a^2 - \lambda a^4 - aB^{\frac{1}{1+\alpha}}, \tag{13}
\]

where we neglected the second as well as higher order terms in Eq. (11) as \(0 < \alpha \leq 1\) (see Eq. (6) and (11)). We can thus identify a suitable range of values for the parameters \(A, B, \alpha\), and the scale factor \(a\) where Eq. (13) is valid, corresponding to the analysis presented in this paper. As far as the next order of approximation is concerned, we will point in section V how the term proportional to \(a^{4+3\alpha}\) in \(V(a)\) can be dealt with in a perturbative manner when \(A \ll 1\).

In the linear approximation provided by Eq. (13) we can identify three physically different cases \[60, 61\]. A plot for \(V_0(a)\) regarding the second and third cases is given in Fig. 1 (see appendix A for more details):

1. When \(\lambda B^2 > \frac{4}{9}\), there is one real root and pair of complex conjugated roots for the equation \(f(a) = a^3 - a/\lambda + B/\lambda = 0\). The only value for which \(V_0(a)\) vanishes is \(a = 0\) and \(V_0(a) < 0\) for \(a > 0\).

2. In this case, we have \(\tilde{B} = \frac{2}{1/\sqrt{\lambda}}\). It is found that \(V_0(a)\) vanishes for \(a = 0\) and there is a positive root for \(f(a) = 0\) when \(a = 1/\sqrt{\lambda}\). The other root is at \(a = -2/\sqrt{\lambda}\). We further have that \(V_0(a) < 0\) for \(0 < a < 1/\sqrt{\lambda}\) and \(a > 1/\sqrt{\lambda}\).

3. In the final case, we have \(\lambda B^2 < \frac{4}{9}\) and three real roots can be found such that two are positive: \(a_1 > a_2 > 0 > a_3\). Moreover, \(V_0(a) < 0\) for \(0 < a < a_3\) and \(a > a_1\), but \(V_0(a) > 0\).

1 See Sec. VI for a discussion on the presence of \(\Lambda\): it makes our model analytically feasible and can represent the lowest order effect of the effective cosmological constant determined by the generalized Chaplygin gas.

2 We are ignoring factor ordering aspects related with \(\pi_a^2\), with \(\pi_a\) being the conjugate momentum to \(a\).
by a classically forbidden region (there are two Lorentzian zones ($a < a_1$) and Hartle-Hawking [57] boundary conditions were then pictured in Fig. 1. Generalized tunnelling (Vilenkin) [58] different sections in the potential $a = \phi$ parameter. For arbitrary constant values of $a$, the form $a = \phi$ implies that $3^{(a+1)}$ the linear approximation is valid if there are values of $a$ such that $a_1 < a < a_{\text{max}}$. This leads to $\left(\frac{\Lambda B^2}{9}\right)^{3(a+1)} = \frac{B}{a}$, which together with $\Lambda B^2 < \frac{4}{9}$ implies that $3^{(a+1)}AB^2 < 1$.

Before proceeding to obtain quantum solutions for the Wheeler-DeWitt equation (see Eq. [13] and [19]), it is of relevance to point the following. Quantum cosmological scenarios with a physical configuration similar to the third case above described (where $\Lambda B^2 < \frac{4}{9}$), can be found in the literature but in somewhat differently related contexts.

One such situation is present in Ref. [55, 56] when a FRW minisuperspace with a complex scalar field of the form $\phi = e^{i\theta} \varphi$, with a mass term proportional to $\varphi^2$ and a conserved charge (i.e., conjugate momentum) associated with $\pi_\theta$ is considered. This produces a minisuperspace potential $U(a, \varphi; \pi_\theta)$ where $\pi_\theta$ is a constant parameter. For arbitrary constant values of $\varphi$ we obtain different sections in the potential $U$, inducing a similar dependence on $a$ as in $V_0(a)$ given by Eq. [19] and depicted in Fig. [11]. Generalized tunnelling (Vilenkin) [55] and Hartle-Hawking [57] boundary conditions were then introduced in Ref. [55, 56]. The reason that the physical model in Ref. [55, 56] relates to the above Chaplygin gas Hamiltonian formulation could be traced to the fact that the Chaplygin gas FRW action can be retrieved from the action of FRW models with the complex scalar field $\phi = e^{i\theta} \varphi$, under specific conditions [6, 11, 11].

The association between generalized Chaplygin gas cosmologies and minisuperspace models with complex scalar fields can be brought towards a recent and promising domain: that of brane cosmology [36, 37]. In fact, the complex scalar field employed in Ref. [10, 11] to obtain a generalized Chaplygin gas can be identified within a Born-Infeld action, which arise form the imbedding of the (3+1)-D brane into the (4+1)-D spacetime [18]. Moreover, it can also be seen to correspond to a “perturbed” $d$–brane in a $(d+1,1)$ spacetime [18]. Furthermore actions of the Born-Infeld have recently been the subject of wide interest (see ref. [29] and references therein). This comes from the result that the effective action for the open string ending on $d$–branes can be written in a Born-Infeld form. In that respect, Born-Infeld cosmological solutions could assist us in the understanding of some brane-world dynamics regarding the universe evolution.

The above context regarding brane-worlds can be further specified. In fact, FRW models with a Chaplygin gas may constitute a phenomenological tool to study some brane-world models: another physical situation with two Lorentzian sectors separated by a classically forbidden Euclidean region can be found in the Hamiltonian treatment of a DeSitter $(3 + 1)$–brane imbedded in a $(4 + 1)$–Minkowski background [38]. The presence of a classically disconnected epoch (as in Fig. 1 for $a < a_3$) has been identified as a property of brane-world quantum cosmology [39]. These remarks thus suggest the view that FRW models with a generalized Chaplygin gas content would enable us to investigate specific aspects of particular brane-world models. In particular through an Hamiltonian formulation, where symmetries and conserved quantities, possibly related with physical properties of brane cosmologies, will be of relevance. In Sec. V, where we consider the next order of approximation in Eq. [10] and [19] (going beyond the Hamilton-Jacobi equation [10]), we will point a further correspondence between brane-world parameters and a generalized Chaplygin gas.

![FIG. 1: This figure shows the behavior of the potential $V_0(a)$ as function of $a$. The dashed line corresponds to the case $\Lambda B^2 = 4/9$. In this case, $V_0(a)$ vanishes at $a = 1/\sqrt{\Lambda}$ and there are two Lorentzian zones corresponding to $a < 1/\sqrt{\Lambda}$ and $1/\sqrt{\Lambda} < a$. The solid line corresponds to $\Lambda B^2 < 4/9$. The potential vanishes at $a = a_1$ and $a = a_3$. In this case there are two Lorentzian zones ($a < a_3$ and $a_1 < a$) separated by a classically forbidden region ($a_3 < a < a_1$).](image)

III. HARTLE-HAWKING AND VILENKIN WAVE FUNCTIONS

We begin by addressing the second case ($\tilde{B} = \frac{\sqrt{\Lambda}}{2\sqrt{a}}$): we have two classically allowed Lorentzian regions separated by the point $a = 1/\sqrt{\Lambda}$ where $V_0(a)$ vanishes. Let us first analyze the region where $a > 1/\sqrt{\Lambda}$. Besides having to satisfy the range present in Eq. [13] it is also...
necessary to have
\[
\frac{1}{\sqrt{A}} \ll \left( (\alpha + 1) \frac{B}{A} \right)^{\alpha + 1},
\]
\[14\]
(in order to have values of \(a\) such that the linear approximation is still valid) which leads to
\[
B^4 \ll A \left( \frac{2}{3} \right)^{\alpha + 1}.
\]
\[15\]
The wave function is then expressed in the form of Eq. 8 and 9 where
\[
S_0(a) = i \frac{3\pi}{2} \int_{-\infty}^{a} \sqrt{-V_0(a)} da,
\]
\[16\]
Introducing the Lorentzian conformal time \(\eta\) using
\[
\frac{dS_0}{da} = i \frac{3\pi}{2} \frac{da}{d\eta},
\]
\[17\]
we get the following equation for \(a(\eta)\)
\[
\frac{da}{d\eta} = \sqrt{-V_0(a)},
\]
\[18\]
from where it can be found that
\[
a(\eta) = \frac{1}{\sqrt{A}} \left( \frac{3}{\cosh(\eta) - 2} + 1 \right),
\]
\[19\]
with \(\eta \in ]-\infty, -\cosh^{-1}(2)[\), noticing that when \(\eta \to -\infty\) then \(a \to \frac{1}{\sqrt{A}}\) and when \(\eta \to -\cosh^{-1}(2)\) then \(a \to +\infty\). For the wave function it is obtained that
\[
\int_{-\infty}^{a} \sqrt{-V_0(a)} da = \frac{1}{3} \left[ a \left( a + \frac{2}{\sqrt{A}} \right) \right]^{\frac{3}{2}}
\]
\[
- \frac{1}{\sqrt{A}} \left( a + \frac{1}{\sqrt{A}} \right) \sqrt{a \left( a + \frac{2}{\sqrt{A}} \right)}
\]
\[
+ \frac{1}{\Lambda^2} \ln \left[ \sqrt{a \left( a + \frac{2}{\sqrt{A}} \right)} + \left( a - \frac{1}{\sqrt{A}} \right) \right]
\]
\[
\sqrt{3} - \ln \left[ \frac{(2 + \sqrt{3})}{\sqrt{A}} \right]
\]
\[
+ \frac{3}{\Lambda^2}.
\]
\[20\]
Employing the same method for the region where \(a < 1/\sqrt{A}\) it is found \[60\], \[61\] that
\[
a(\eta) = \frac{1}{\sqrt{A}} \left( \frac{3}{\cosh(\eta) + 2} + 1 \right),
\]
\[21\]
with \(\eta \in [0, +\infty[\), noticing that when \(\eta \to +\infty\) then \(a \to \frac{1}{\sqrt{A}}\) and when \(\eta \to 0\) then \(a \to 0\). For the wave function it can be concluded that \(\int_{-\infty}^{a} \sqrt{-V_0(a)} da\) is equal to minus the expression in Eq. \[20\].

In the next paragraphs we will retrieve for the third case \(\Lambda B^2 < \frac{1}{2}\) the Hartle-Hawking \[57\] and tunneling (Vilenkin) \[52\] wave functions (see also Ref. \[48\], \[50\], \[51\], \[52\] for more details). In order to obtain the solutions \(S_0(a)\) of Eq. \[10\] it is necessary to determine the analytical expressions
\[
\int \sqrt{|V_0(a)|} da \equiv \frac{\sqrt{\Lambda}}{3} \int I(a) da.
\]
\[22\]
This integral will of course depend on the specific chosen region, e.g., between \(0 < a < a_3\) for the first classically allowed Lorentzian region. Let us determine the values for the integral in \[22\] in the following. After some lengthy calculations, it is found that \[60\], \[61\]
\[
\int_{a_1}^{a_3} I(a) da = \frac{a_3}{3} - \beta a_2^2 a_1 a_3 - \gamma a_1 (a_3 - a_1) F(v, q) + \beta a_1 a_3 \left[ (a_3 - a_1) F(v, q) \right]
\]
\[
+ \frac{(a_1 + a_3)^2}{2a_3 + a_1} \left[ (a_3 - a_1) F(v, q) \right]
\]
\[
+ (2a_1 + a_3) F(v, q),
\]
\[23\]
where \(\beta \equiv \frac{(a_1^2 + a_2^2 + a_3^2)}{8}, \gamma \equiv \left( \sqrt{a_1 (2a_3 + a_1)} \right)^{-1}, q \equiv \sqrt{\frac{a_2 (2a_1 + a_3)}{a_1 (2a_3 + a_1)}}\), \(v \equiv \arcsin \left( \frac{2a_1 + a_3}{a_1 (2a_3 + a_1)} \right), \) together with
\[
\int_{a}^{a_3} I(a) da = -\frac{a_3}{3} - \beta a_2^2 a_3 - a_1 \gamma a_3 \left[ (a_3 - a_1) F(v, q) \right]
\]
\[
+ \beta a_1 a_3 \left[ (a_3 - a_1) F(v, q) \right]
\]
\[
+ \frac{(a_1 + a_3)^2}{2a_1 + a_3} \left[ (a_3 - a_1) F(v, q) \right]
\]
\[
- \frac{a_3}{3} F(v, q) + \beta a_1 a_3 \left[ (a_3 - a_1) F(v, q) \right],
\]
\[24\]
with \(\chi \equiv \arcsin \left( \frac{a_1 (a_3 - a)}{a_3 (a_1 - a)} \right), \) as well as \(\int_{a_1}^{a_3} I(a) da = \)
\[
\lim_{a \to a_1} \int_{a_3}^{a} I(a) \, da = \frac{a I}{3} + \beta a^2 (a - a_3) I + \gamma \left\{ -a_1 a_3 (a_1 + a_3) \Pi \left( \frac{a_1 - a_3}{a_1}, r \right) + 6 \beta^2 F(\delta, r) - \beta a_1 \left( a_1 - a_3 \right) F(\delta, r) + a_3 \Pi(\delta, 1, r) \right\} \times \left[ a_3 \Pi(\delta, r^2, r) - (2a_3 + a_1) F(\delta, r) \right] \right\}, \\
\text{(25)}
\]

with \( \delta \equiv \arcsin \sqrt{\frac{a_1(a-a_3)}{(a_1-a_3)a}} \) and \( r \equiv \sqrt{\frac{a_1-a_3}{2a_3+a_1a_3}} \).

The functions \( F \) and \( \Pi \) are elliptic integrals of the first and third kind, respectively (see, e.g., Ref. 60 for more details). When calculating the expression \( \lim_{a \to a_1} \int_{a_3}^{a} I(a) \, da \) it is found that it is well defined except for the terms corresponding to \( a^2(a-a_3) + \frac{1}{2} \int_{a_3}^{a} \frac{a^2(a-a_3)}{(a_1-a_3)a} \, da \), as each of these expressions diverges when \( a \to a_1 \). However, it can be seen that these divergences cancel each other \( 61 \) and therefore the integral in \( \text{(25)} \) is well defined and can be used to determine the transition amplitudes for the FRW universe with a generalized Chaplygin gas to tunnel from a Lorentzian region between \( 0 < a < a_3 \) to another where \( a > a_1 \) (see Fig. 1).

Following Ref. 61, the no-boundary (Hartle-Hawking) wave function representing the region under the potential barrier \( V_0(a) \) corresponds to the increasing modes

\[
\Psi \propto \exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} \, da \right], \\
\text{(26)}
\]

where we are using \( V_0(a) \sim -\lambda a(a-a_1)(a-a_3)(a+a_1+a_3) \). In more precise terms we obtain 52

\[
\Psi(a) = C \cos \left[ \frac{3\pi}{2G} \int_{a}^{a_3} \sqrt{-V_0(a)} \, da + \frac{\pi}{4} \right], \quad (a < a_3)
\]

\[
\Psi(a) = C \exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a} \sqrt{V_0(a)} \, da \right], \quad (a_3 < a < a_1)
\]

\[
\Psi(a) = 2C \cos \left[ \frac{3\pi}{2G} \int_{a_1}^{a} \sqrt{-V_0(a)} \, da - \frac{\pi}{4} \right], \quad (a_1 < a),
\]

\[
\text{(27)} \quad \text{(28)} \quad \text{(29)}
\]

where \( C \equiv C \exp \left[ \frac{3\pi}{2G} \int_{a_1}^{a_3} \sqrt{V_0(a)} \, da \right] \) and \( C \) is an arbitrary constant.

As far as the tunnelling (Vilenkin) wave function is concerned, it contains only outgoing modes and we find

\[
\Psi(a) = C \left\{ \exp \left[ -\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} \, da \right] \times \cos \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-V_0(a)} \, da + \frac{\pi}{4} \right] + 4i \exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} \, da \right] \times \cos \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-V_0(a)} \, da - \frac{\pi}{4} \right] \right\}, \quad (a < a_3)
\]

\[
\Psi(a) = C \left\{ \exp \left[ -\frac{3\pi}{2G} \int_{a}^{a_1} \sqrt{V_0(a)} \, da \right] \times \cos \left[ \frac{3\pi}{2G} \int_{a}^{a_1} \sqrt{-V_0(a)} \, da + \frac{\pi}{4} \right] + 2i \exp \left[ \frac{3\pi}{2G} \int_{a}^{a_1} \sqrt{V_0(a)} \, da \right] \times \cos \left[ \frac{3\pi}{2G} \int_{a}^{a_1} \sqrt{-V_0(a)} \, da - \frac{\pi}{4} \right] \right\}, \quad (a_3 < a < a_1)
\]

\[
\Psi(a) = 2Ce^{i\pi/4} \exp \left[ -i \frac{3\pi}{2G} \int_{a_1}^{a} \sqrt{-V_0(a)} \, da \right], \quad (a_1 < a).
\]

\[
\text{(30)} \quad \text{(31)} \quad \text{(32)}
\]

Having established explicitly the analytical form for the Hartle-Hawking and tunnelling (Vilenkin) wave functions in Eq. \( \text{(27)-(29)} \) and Eq. \( \text{(30)-(32)} \), respectively, we can now analyze the transition amplitude \( A \), for the universe to evolve from the classically allowed Lorentzian region between \( 0 < a < a_3 \) to another where \( a > a_1 \) (see figure 1).3 This transition amplitude is given by

\[
A = \exp \left( \mp 2I \right)
\]

\[
\text{(33)}
\]

where the – and + sign correspond, respectively, to the tunnelling (Vilenkin) and Hartle-Hawking wave functions, and we use the expression

\[
I = \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{\lambda a(a_1-a)(a+a_1+a_3)(a-a_3)} \, da.
\]

\[
\text{(34)}
\]

We henceforth consider the new dimensionless quantity \( \lambda IG \) which will be analyzed as function of the parameter

\[
X \equiv \sqrt{\frac{\lambda}{3}} \sqrt{\Lambda}
\]

\[
\text{(35)}
\]

that is also dimensionless. When \( X = 0 \) we recover the well known closed FRW-DeSitter quantum cosmology case. From Eq. \( \text{(35)} \) it can be seen that \( \lambda IG \) decreases when \( X \) varies from 0 to 2/3 and hence the transition amplitude, \( A \), for the tunnelling (Vilenkin) boundary condition increases when \( X \) increases (for a fixed value of the cosmological constant), while for the Hartle-Hawking boundary conditions \( A \) decreases.

\[
3 \text{ For a discussion on the general meaning of tunneling universes, see, e.g., Ref. 52.}
\]
In terms of the generalized Chaplygin gas parameters $B$ and $\alpha$, we have that when $\alpha$ increases from 0 to 1 and $B$ is constant and smaller that 1, $X$ also increases and consequently $\lambda \tilde{G}$ decreases, which for a fixed value of $\Lambda$ determines that $\tilde{I}$ also decreases. This then implies that the transition amplitude for the tunnelling (Vilenkin) boundary condition increases but the transition amplitude associated to the no-boundary proposal decreases. The opposite conclusion is reached when $\alpha$ increases and $B$ is constant and larger than 1. Taking now fixed values of $\alpha$ and $\Lambda$, if $B$ increases then $X$ increases. As a consequence, we conclude that similarly to the previous situation (for $\alpha$ increasing when $B$ is constant and smaller than 1) also the transition amplitude for the tunnelling (Vilenkin) boundary condition increases but the transition amplitude associated to the Hartle-Hawking proposal decreases. In Fig. 2 we plot how $\lambda \tilde{I}G$ varies as function of $X$.

![Graph showing $\lambda \tilde{I}G$ vs $X$]

**Fig. 2**: This figure shows the behavior of $\lambda \tilde{I}G$, where $\tilde{I}$ is defined in Eq. (34), in terms of the dimensionless parameter $X$ defined in Eq. (35). For $X = 0$ we recover the closed FRW-DeSitter case, while the case $X = 2/3$ corresponds to $\Lambda B^2 = 4/9$. In this last case there is no tunnelling between the two Lorentzian regions.

### IV. QUANTIZATION CONDITION INVOLVING $B$ AND $\alpha$

Let us now establish how the quantum states obtained in the previous section may determine which sets of values for $B$ and $\alpha$ are allowed.

In the case of a closed FRW universe with a positive cosmological constant and a generalized Chaplygin gas, it can be seen that a singularity is present at $a = 0$ as the Ricci scalar diverges. This divergence in $R$ is present whether we consider the approximation introduced in Sec. II [see Eq. (11)-(13)] or the the full expression of the energy density of the Chaplygin gas. In fact, for the latter we have

$$R = 4 \left\{ \Lambda + 2\pi G \left[ A + \frac{B}{a^{3(1+\alpha)}} \right]^{-\frac{\alpha}{1+\alpha}} \left[ 4A + \frac{B}{a^{3(1+\alpha)}} \right] \right\}. \quad (36)$$

As it can be shown, for small values of the scale factor, $R$ diverges. Furthermore, for small values of the scale factor, $R$ in Eq. (36) will have the same behavior as that we would obtain from the approximations in Sec. II. Such divergence could be avoided if some other material content dominates the energy density of the universe at that stage (for example some effective scalar field), but we disregard this possibility in the present work.

The presence of a curvature divergence becomes more complicated in the FRW scenario with a generalized Chaplygin gas: the divergence is present in a Lorentzian (classically allowed) region (see Ref. 32 as well as 53 54). A suggestion has been advanced, designated as DeWitt’s argument 32 63 64, in that $\Psi(a = 0) = 0$ is imposed. The interpretation is that the divergence is not replaced by an Euclidean conic-singularity-free pole, but instead is neutralized by making it quantum mechanically inaccessible to wave packets.

Before proceeding, let us also mention the following. Taking $\lambda = 0$ in Eq. (6), it can be shown (see Sec. VI for more details) that $V(a)$ for $A \ll 1$ is essentially as represented in Fig. 1 for $V_0(a)$ with $\lambda \neq 0$ (within the mentioned approximations). Hence, the validity of the results presented in this section concerning $B$ and $\alpha$ are not restricted to our approximations. The physical consequences will hold in a wider scenario, albeit with some modifications in the mathematical expressions.

Assuming therefore that the wave function vanishes at the origin, i.e., $a = 0 \Rightarrow \Psi(a = 0) = 0$, from Eq. (20), (27) it is obtained that in the case of the Hartle-Hawking wave function we will have

$$\frac{3\pi}{2G} \int_0^{a_3} \sqrt{-V_0(a)} da - \frac{\pi}{4} = n\pi, \quad n \in \mathbb{Z}. \quad (37)$$

It can be further seen that when $B^{\frac{1}{1+\alpha}}$ is small

$$\int_0^{a_3} \sqrt{-V_0(a)} da \approx \int_0^{B^{\frac{1}{1+\alpha}}} \sqrt{-a^2 + aB^{\frac{1}{1+\alpha}}} = \frac{\pi}{8} B^{\frac{1}{1+\alpha}}. \quad (38)$$

From Eq. (37) and (38) it follows that

$$\frac{3\pi}{4G} B^{\frac{1}{1+\alpha}} - 1 \approx 4n, \quad (39)$$

with $n \in \mathbb{Z}$.

---

4 See Sec. VI for a discussion on this issue.
Recalling that the Vilenkin wave function can be written as
\[
\Psi(a) = C \exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \right]
\times \cos \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{\nu_0(a)} da + \frac{\pi}{4} \right] + 4iC \exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \right] \times \cos \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{\nu_0(a)} da - \frac{\pi}{4} \right],
\]
for the case of \(0 < a < a_3\) and assuming again that \(\Psi(a = 0) = 0\), it now results that
\[
\left\{ \begin{array}{l}
\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-V_0(a)} da + \frac{\pi}{4} = \frac{\pi}{2} + k\pi, \\
\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-\nu_0(a)} da - \frac{\pi}{4} = \frac{\pi}{2} + \tilde{k}\pi,
\end{array} \right.
\]
where \(k, \tilde{k} \in \mathbb{Z}\). Both conditions in (41) cannot generally hold simultaneously. However, there is an exception. When \(\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-V_0(a)} da \gg 1\), it is enough that the last condition holds. This statement can be proven as follows. Let us take
\[
\cos \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-V_0(a)} da + \frac{\pi}{4} \right] + 4i \exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \right] \times 
\cos \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{-V_0(a)} da - \frac{\pi}{4} \right] = 0.
\]
If we now define
\[
\Upsilon = \exp \left[ \frac{3\pi}{G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \right],
\]
we can write
\[
\left\{ \begin{array}{l}
\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da = \frac{\pi}{2} + \tilde{n}\pi, \\
\sin \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \right] = \frac{1 - \cos 3\tilde{n}\pi}{1 + 4\tilde{n}\pi},
\end{array} \right.
\]
where \(\tilde{n} \in \mathbb{Z}\). These equalities represent (41) and then cannot both hold unless \(\Upsilon \to 0\) or \(\Upsilon \gg 1\). As we are considering the possibility that the universe tunnels from one Lorentzian region \((a < a_3)\) to another Lorentzian region \((a_1 < a)\), the parameter \(\Upsilon\) is greater that the unity. Consequently, the case \(\Upsilon \to 0\) is physically impossible, while the case \(\Upsilon \gg 1\) is admissible. In this situation, we have, with \(\tilde{n} = 2\tilde{k} + 1\), that
\[
\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da = \frac{3\pi}{4} + \tilde{k}\pi,
\]
which is equivalent to the second equality in expression (41).

Finally, we can address the question when is the condition \(\frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \gg 1\) fulfilled. To this aim we write
\[
\exp \left[ \frac{3\pi}{2G} \int_{a_3}^{a_1} \sqrt{V_0(a)} da \right] = \exp \left[ \frac{3\pi}{2} \right].
\]
We know the behavior of \(\tilde{\lambda}G\) in terms of the parameter \(X\) [defined in Eq. (35)]: \(\tilde{\lambda}G\) will take its largest values when \(X\) is small. So in order that \(\exp\left(3\pi/2\right) \gg 1\), it is necessary that \(\tilde{\lambda}G\) gets its largest values and \(\lambda\) be small, i.e., \(\lambda\) has to be small. Hence, what we have is that the condition \(\Psi(a = 0) = 0\) applied to the Vilenkin wave function implies the universe to be close to a DeSitter model and to have a small cosmological constant. In this case we obtain that
\[
\frac{3\pi}{4G} B \frac{\tilde{\lambda}}{\tilde{\kappa}} \approx 3 + 4\tilde{k}.
\]

V. BEYOND THE HAMILTON-JACOBI EQUATION

In this section we will consider the next order of approximation in Eq. (11), i.e., bringing the terms \(a^{4+3\alpha}(1 + \alpha)\) into the minisuperspace potential. This will be done perturbatively\(^5\) with the parameter \(A\) satisfying \(A \ll 1\), allowing a quantum cosmological analysis beyond the WKB formulation in Eq. (12)-(10). Furthermore, it will allow us to include explicitly in the analysis a parameter that can be related to the brane tension in some brane world cosmologies (see Ref. 11).

Let us consider again the Wheeler-DeWitt equation (1) and employ the following expansion where we take \(A \ll 1\):
\[
\Psi(a) = \exp \left[ -\frac{1}{G} \left( S_0(a) + AS_1(a) + A^2 S_2(a) + \cdots \right) \right].
\]
Using the expansion in Eq. (11) up to the term in \(a^{4+3\alpha}\) we then obtain in powers of \(A\)
\[
\begin{align*}
- \frac{1}{3\pi G} \left[ \frac{dS_0(a)}{da} + A \frac{dS_1(a)}{da} + A^2 \frac{d^2 S_2(a)}{da^2} \right]^2 \Psi(a) \\
- \frac{1}{3\pi} \left[ \frac{d^2 S_0(a)}{da^2} + A \frac{d^2 S_1(a)}{da^2} + A^2 \frac{d^2 S_2(a)}{da^2} \right] \Psi(a) \\
+ \frac{3\pi}{4G} \left[ V_0(a) + AV_1(a) + A^2 V_2(a) \right] \Psi(a) + \cdots = 0,
\end{align*}
\]
where
\[
\begin{align*}
V_0(a) &= a^2 - \lambda a^4 - aB^{1+\alpha}, \\
V_1(a) &= -\frac{a^{4+3\alpha}}{2 + \alpha} B^{1+\alpha},
\end{align*}
\]
with \(V_2(a)\) following in a similar manner (but will not be necessary at this level).

\(^5\) See Ref. 51 and 52 for a formal description regarding the expansion of a wave function of the universe in terms of a perturbative parameter.
In the WKB approximation where $\frac{d^2 S_0(a)}{da^2}$ is neglected, we get at the $A^0$ and $A^1$ level, respectively, the equations

\begin{equation}
\left( \frac{dS_0(a)}{da} \right)^2 = \frac{9\pi^2}{4} V_0(a),
\end{equation}

\begin{equation}
\frac{2}{3\pi G} \frac{dS_0(a)}{da} \frac{dS_1(a)}{da} = - \frac{1}{3\pi} \frac{d^2 S_1(a)}{da^2} + \frac{3\pi}{4G} V_1(a) = 0,
\end{equation}

where the former is the Hamilton-Jacobi equation (obtained in Sec. II). If we further assume that $S_1(a)$ is slowly varying, then Eq. \((52)\) can be approximated by

\begin{equation}
\frac{2}{3\pi G} \frac{dS_0(a)}{da} \frac{dS_1(a)}{da} + \frac{3\pi}{4G} V_1(a) \approx 0.
\end{equation}

If we consider that the turning points are exclusively determined by the approximate potential $V_0$ and that $V_1$ is a correction when $A$ is small, then it can be possible to obtain (by an approximate analytical expression) a quantification rule involving $\alpha, B$ as well as $A$. In order to get analytical expressions for this case, we should integrate the expression

\begin{equation}
J = \int_0^{a_1} \frac{V_1(a)}{\sqrt{-V_0(a)}} da,
\end{equation}

as implied from Eq. \((51)\) and \((53)\). By means of an approximation similar to the one given in Eq. \((53)\), namely with $B^{1/(1+\alpha)}$ being small (i.e., $\Lambda B^2$ small), then $J$ can be approximated as follows

\begin{equation}
J \approx \int_0^{\frac{a_1}{\alpha}} a^{\frac{9}{2}+3\alpha} \left( B^{\frac{1}{1+\alpha}} - a \right)^{-\frac{1}{2}} da.
\end{equation}

This expression can be written analytically as

\begin{equation}
J \approx B^{1+3\alpha} B \left( \frac{9}{2} + 3\alpha, \frac{1}{2} \right),
\end{equation}

where $B$ is the beta function (see Ref. \([63, 65]\)). Quantification rules follow when using wave functions in the region $0 < a < a_1$ for the two boundary conditions that we have considered. We will not pursue this issue further in this paper, as it will be addressed in a future work.

Therefore and with suitable choices of boundary conditions we can then obtain an explicit form for $S_1(a)$ and hence for

\begin{equation}
\Psi(a) \approx \exp \left( \frac{1}{G} (S_0(a) + AS_1(a)) \right). \tag{57}
\end{equation}

Within this procedure we can thus include explicitly the parameter $A$ in the wave function of the universe. This may be of relevance: a connection between the Chaplygin gas ($\alpha = 1$) and brane-world models is that the brane tension is $\sqrt{\Lambda}$ \([10]\). In the case of a generalized Chaplygin gas the “generalized” tension would be $A^{1/\pi}$.

VI. CONCLUSIONS AND DISCUSSIONS

As a summary, let us point the main results conveyed in this paper. Our purpose was twofold.

First, it was our aim to present a quantum mechanical description of a closed FRW model with a positive cosmological constant and a generalized Chaplygin gas. The physical scenario we are contemplating is that, underlying a classical universe, there was a wave function vestige (i.e., a rapidly oscillating state with a particularly small amplitude) whose physical implications would emerge within the epoch of cosmological dust dominance. Such specific remnant component of the original wave function of the universe had been present during that period of the universe evolution. It can be interpreted as a robust component (with respect to decoherence) in the sense introduced in Ref. \([10, 11, 13]\). In other words, after the universe reached a dust stage, the transition towards an accelerating expansion would be associated to a quantum mechanical transition for that specific wave function remnant component of the original state. In this context, we obtained the Hamiltonian and the corresponding Wheeler-DeWitt equation associated with such transition. In particular, we established a range for the variables and parameters of the model where a linear approximation for the effective minisuperspace potential could be used.

The quantum mechanical amplitudes for the transitions from a dust stage towards an expanding accelerating phase were obtained either in the form of a Hartle-Hawking or Vilenkin wave function. We established how these transition amplitudes vary according to the allowed values for $\alpha$ and $B$: when either $\alpha$ or $B$ increase, the Vilenkin state (generally, see Sec. III) allowed the probability for the universe to tunnel towards the accelerated stage to increase, while the opposite situation occurred for the Hartle-Hawking wave function.

Subsequently, we investigated whether this quantum mechanical formulation for the transition from a dust stage towards an exponentially accelerated phase determined any physical consequences regarding the Chaply-
gin gas cosmological scenario. We then found that the Hartle-Hawking and Vilenkin wave functions could imply that the parameters $\alpha$ and $B$ would be related and restricted to a quantization condition\(^6\) characterized by the presence of an integer $n$.

This result is obtained as we impose the condition $\Psi(a = 0) = 0$ corresponding to the presence of a divergence in the curvature. The divergence is generic, i.e., it is not a consequence of our linear approximation. The presence of such divergence in a classically allowed region can be dealt following, e.g., the approach suggest in Ref. \[37, 55, 56\] (also designated as DeWitt’s argument): the divergence is not replaced by an Euclidean conic-singularity-free pole, being neutralized by making it quantum mechanically inaccessible. Let us remember that, e.g., the tunneling (Vilenkin) boundary condition was thought of for a problem of (semi-classical) creation of the universe, in analogy with the tunneling process in quantum mechanics. Note, however, that the singularity in our model is located in a classical region, before tunneling. We can consider waves proceeding from a classical region, with a probability to exit from it towards the other classical region (see figure 1).

Nevertheless\(^7\), it is only a sufficient but not necessary that the wave function vanishes at the origin in order to avoid singularities. On the one hand, both the Hartle-Hawking and Vilenkin wave functions can be taken generally as non-zero at $a = 0$ (being a minimum (Hartle-Hawking) or a Green’s function (Vilenkin)), increasing or decreasing through the barrier. The singularity could be smoothed by means of some non-trivial measure in the inner product. On the other hand, quantum mechanical situations with singularities occur and are dealt with, as with the solution of the Dirac equation for the ground state of the hydrogen atom which is singular when $r \to 0$. We have, however, herein chosen to follow the approach presented in Ref. \[36, 55, 56\] to address the transition from a dust phase towards an accelerating expansion stage by means of a quantum mechanical tunneling mediating the evolution from an initial Lorentzian (classically allowed) region to the current accelerating epoch.

Second, it was also our objective to introduce a discussion (and promote subsequent research) on the interesting similarities between the quantum cosmology of a specific brane-world models \[36\] and FRW quantum cosmologies with a Chaplygin gas content. This was based on the feature that FRW models with a Chaplygin gas may constitute a phenomenological tool to investigate brane-world models: both frameworks share a physical minisuperspace where two Lorentzian sectors are separated by a classically forbidden Euclidean region\[55, 56\]. This suggests a view that FRW models with a generalized Chaplygin gas content may enable us to investigate specific aspects within brane-worlds. E.g., the quantum cosmological creation of the universe, by means of an Hamiltonian formulation, taking into account the presence of potential symmetries and conserved charges that could be related with physical quantitites in brane-world physics \[32, 33, 40, 41, 42, 43, 44, 45\].

This association between FRW cosmologies with a generalized Chaplygin gas and some brane-world cosmologies is strengthen when considering the next order of approximation in Eq. \[45\] and \[46\] (i.e., going beyond the Hamilton-Jacobi equation \[41\]) see Eq. \[47\] - \[48\]. This order of approximation was described through of a perturbative expansion in the parameter $A$ present in the Chaplygin gas energy density. Let us remind that a connection between the Chaplygin gas ($\alpha = 1$) and brane-world models is that the brane tension is $\sqrt{A}$ \[10\]. For the case of a generalized Chaplygin gas the “generalized” tension would be $A^{\lambda/3}$. A subsequent quantization condition involving $A$ as well as $\alpha$ and $B$ would be retrieved, depending on the boundary condition (Hartle-Hawking or Vilenkin) chosen.

Let us now discuss the presence of the positive cosmological constant, $\Lambda$, in our minisuperspace scenario. It may seem that its purpose is merely to induce the tunneling towards a DeSitter classical stage, as Eq. \[43\] suggests. Nevertheless, it is worthy to point the following. If we take $\lambda = 0$ in Eq. \[25\] and depict $V(\alpha)$ for different values of $\alpha$ and with $A \ll 1$, we get a physical situation as described in Fig. 4 (see Fig. 4 for a comparison), where

![Graph of V(\alpha)](image)

FIG. 3: This figure shows the behavior of the potential $V(\alpha)$, when the cosmological constant, $\Lambda$, is zero and the constant $A$ is very small. The physical situation described in this case is similar to the one in Fig. 1. i.e. two classically allowed regions disconnected and a Euclidean one.

\(^6\) Let us mention that, in a different physical context, other quantization conditions (bearing formal similarities) were obtained in ref. \[62\] when requiring normalization of the quantum states.

\(^7\) The authors are grateful to C. Kiefer and A. Vilenkin for their comments on this issue.
this paper with positive cosmological constant, $\Lambda$, is analytically feasible, providing a similar minisuperspace dynamics through Eq. (13). The presence of $\lambda$ (pertinent for larger values of $a$) may be interpreted as the lowest order effect of the effective cosmological constant behavior induced in the generalized Chaplygin gas scenario. This provides explicit analytical expressions for the transition amplitudes (from an initial Lorentzian (classically allowed) region to the current accelerating epoch through a tunneling process) to be computed. Improved expressions could be considered by taking next orders of approximation, beyond the Hamilton-Jacobi Eq. (14), as indicated in Sec. V.

Another issue that is important to mention is the following. In Eq. (8) we have a superposition of states, that may be considered from the point of view of decoherence (see Ref. [47] for an extensive and thorough review). In generic quantum cosmology, a full wave function could take the form $\Psi = \sum_{n} C(n)e^{iS(n)H_{0}}\chi(n)$ where the $S_{0}^{(n)}$ are solutions to the Hamilton-Jacobi equation (treating the gravitational part) and the states $\chi(n)$ are matter states obeying $i\nabla S_{0}^{(n)}\nabla\chi(n) = H_{0}(n)\chi(n)$, where $H_{0}(n)$ is the corresponding matter Hamiltonian (see Ref. [17]). Such expansion for $\Psi$ has some relevant non-classical features. On the one hand, it is a superposition of many WKB states. Only if those different branches are dynamically independent of each other we recover a background space-time without quantum mechanical interferences. On the other hand, restricting to a single component in $\Psi$ may not yet mean a classical space-time as the WKB function may spread over all configuration space. A decoherence process is needed thus prior to identify a classical behavior.

Concerning our case study, we focused our attention in the transition amplitudes for the universe to proceed from a classical initial region, through a quantum mechanical tunneling sector, towards another classical region where an accelerating stage takes place. We are taking that different WKB components will decohere as discussed in [47] and considering solely a robust component (with respect to decoherence, in the sense introduced in Ref. [40,47,49]) of the original wave function. This component would induce a transition from a dust dominating stage toward an accelerated epoch as described in sections I and II. A discussion on decoherence issues in our model (or other models where the Euclidean (classically forbidden) region contour is generically a closed curve) may be the subject of future research work.

Finally, it may be of interest to indicate additional possible subsequent directions of research to explore. An interesting option is to investigate whether other braneworld models (see, e.g. Ref. [35,36,41,42,43,44,45]) can admit a phenomenological description in terms of a generalized Chaplygin gas minisuperspace cosmology. This could be investigated by means of suitable canonical transformations at the Hamiltonian formulation. Another possibility is to pursue the analysis of the next orders of approximation (beyond the Hamilton-Jacobi equation as in section V) in a broader perspective [51,52]. This would mean extracting the corresponding functional Schrodinger equation from the Wheeler-DeWitt equation, together with quantum gravitational corrections to it (see Ref. [52] for more details). Moreover, the quantum cosmology of FRW models with a generalized Chaplygin gas could be investigated from the following perspective: taking into account the Chaplygin gas influence not as an additional term in the superpotential depending on $a$, but as a couple of independent conjugate dynamical variables, which would imply a modification of the Wheeler-DeWitt equation. The guidelines of Ref. [28] could be of importance towards such Hamiltonian formulation. This will bring changes as the dimension of the configuration space will increase and the hyperbolic nature of the kinetic term has to be considered.

\section*{APPENDIX A: POSITIVE ROOTS IN CASE $\Lambda B^2 < 4/9$}

Let us write $V_{0}(a)$ as

$$V_{0}(a) = -\lambda a f(a)$$  \hspace{1cm} (A1)

with

$$f(a) = a^{3} + b_{2}a^{2} + b_{1}a + b_{0}a^{0},$$  \hspace{1cm} (A2)

where $b_{2} = 0$, $b_{1} = -\lambda^{-1}$, $b_{0} = \frac{\tilde{B}}{\Lambda}$. Using Ref. [61], we define

$$q = -\Lambda^{-1}, \quad r = \frac{3\tilde{B}}{2\Lambda} \Rightarrow q^{3} + r^{2} = -\Lambda^{-3} + \frac{9\tilde{B}^{2}}{4\Lambda^{2}}$$ \hspace{1cm} (A3)

and then obtain for the case $q^{3} + r^{2} < 0$, i.e., $\Lambda B^2 < \frac{4}{9}$, the following roots for $f(a) = 0$:

$$a_{1} = 2|x|^{\frac{1}{3}} \cos \left(\frac{\theta}{3}\right),$$ \hspace{1cm} (A4)

$$a_{2} = -2|x|^{\frac{1}{3}} \cos \left(\frac{\theta}{3} - \frac{\pi}{3}\right),$$ \hspace{1cm} (A5)

$$a_{3} = -2|x|^{\frac{1}{3}} \sin \left(\frac{\pi}{6} - \frac{\theta}{3}\right).$$ \hspace{1cm} (A6)

In the last expression $|x| = \Lambda^{-\frac{3}{2}}$ and $\tan \theta = -\sqrt[3]{\frac{4 - 9B^{2}}{27B^{2}}}.\lambda$. Moreover, we have that $a_{1} - a_{3} > 0$, together with

$$a_{1} > a_{3} > 0 > a_{2}.$$ \hspace{1cm} (A7)

Therefore, $V_{0}(a)$ has three real roots determining two Lorentzian regions and one classically forbidden region for the range of $a \geq 0$. 

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