Neutral atoms confined in (quasi) periodic optical potentials and manipulated by lasers provide us with one of the most promising avenues to implement a quantum computer or to perform quantum simulations[1]. For example, a Bose–Einstein Condensate can be loaded in an optical lattice achieving almost unit occupation per lattice site through a superfluid–Mott insulator quantum phase transition[2]. A universal set of quantum gates can then be implemented by individual laser manipulation and inducing cold collisions between the atoms[3]. In several remarkable experiments, all these phenomena have been observed[4, 5].

At the moment, quantum computation with atoms in optical lattices is hindered by three major obstacles: (1) Lack of addressability; (2) Presence of defects; (3) Uncontrolled number of atoms. The first obstacle is due to the fact that the separation between atoms is of the order of an optical wavelength (that of the laser which creates the confining potential) so that in order to address them with a laser one has to focus it close to (or even beyond) the diffraction limit. A possible way to circumvent this problem consists of using optical superlattices[6], or other optical micro traps[7], in which the separation between atoms increases. Quantum gates in these set-ups, however, may become harder than in the standard optical lattice. The second obstacle occurs due to the fact that there always exist sites that have either no atom or more than one. A single defect will unavoidably spoil any quantum computation, and may also have important consequences in quantum simulations. In present experiments one can estimate that the number of defects is relatively high[8]. This last obstacle can be to a very large extent overcome by a filtering process[9], where the lattice sites in which there is more than one atom are emptied until a single atom remains there. Alternatively, one can define collective qubits independent of the number of atoms per sites[10]. Both procedures should avoid situations in which there exist a defect with no atom present. Finally, the number of atoms which form the quantum computer must be well defined since, otherwise, when performing quantum gates the rest of the atoms will act as an environment.

In this paper we introduce a novel method of performing quantum computations in optical lattices (or, more generally, periodic potentials) which circumvents the above mentioned obstacles. One of the fundamental ideas of our method is to use defects (which are delocalized in the lattice) in order to mark the atoms that build the quantum computer and to break the translational symmetry in order to obtain addressability. Note that we do not know where the defects are, but their only presence (wherever they are) is sufficient for our purposes. On the other hand, the defects allow us to build “pointer” sites, also delocalized, which will be used to perform a universal set of quantum gates. Note also that since there will be several defects in the atomic sample, we will have several quantum computers running in parallel, randomly distributed all over the optical lattice. This situation resembles the ensemble quantum computation set-up[11], and in fact some of the ideas developed there can be directly incorporated in our method to make it more efficient. As we will show, the method alone suffers, parallel to what happens in ensemble quantum computation, from the scalability problem. Here we will also present a method to overcome it and to make the present proposal scalable. Note that even though our method is developed for atoms in optical lattices, some of its ideas may also apply to very different implementations where similar obstacles are present.

We consider a three dimensional periodic potential in which atoms are loaded. The atoms have three internal states, \(|a\), \(|b\) and \(|p\)\). The first two will later on store the qubit, whereas the third one will be used by the pointer. We will consider each 1D lattice independently, i.e. we assume that tunnelling is switched off for all times along the \(y\) and \(z\) directions. Thus, we can effectively reduce the system to a set of 1D periodic lattices. We will use a second quantization description of the states; that is, for each lattice site \(k\) we will write a state \(|m_k, m_k', n_k\rangle\), where \(m_k, m_k'\), and \(n_k\) are natural numbers that indicate the occupation number of levels \(|a\), \(|b\) and \(|p\), respectively. Thus, a typical state of one 1D lattice will be

\[
\otimes_k |m_k, m_k', n_k\rangle_k. \tag{1}
\]
We will assume that four kind of basic operations are available. These operations act in exactly the same way in each lattice site, since we do not assume that the sites can be individually addressed. On the other hand, they are based on physical processes which have been demonstrated in the current experimental set-ups:

(i) Particle transfers in between internal states. We will consider two kinds: (i.1) Those in which an integer number of particles is transferred between states \( a \) and \( p \). For example,

\[
U_{m,n}^{m+x,n-x} : |m, 0, n⟩ \leftrightarrow |m + x, 0, n - x⟩,
\]

where \( x \) is an integer. Note that for the unitary operator which describes this process at each site holds

\[
U_{m,n}^{m+x,n-x} = U_{m,n}^{m,n}U_{m,n}^{m+x,n-x}.
\]

Another example that we will use later on will be \( W : |1, 0, 1⟩ \leftrightarrow |0, 1, 1⟩ \). These operations can be carried out using the blockade mechanisms which is present due to atom-atom interactions \( \hat{b} \). (1.2) Those which generate superpositions. For example, \( V = \exp(-iH_{ab}π/8) \), which acts only on the \( a \) and \( b \) levels, with \( H_{ab} = \hat{a} \hat{b} + \hat{b} \hat{a} \), where \( \hat{a} \) and \( \hat{b} \) are the annihilation operators for particles in states \( |a⟩ \) and \( |b⟩ \), respectively. These operations can be easily carried out using laser or rf fields.

(ii) Collisional shifts: They are due to the interactions between particles in the states \( a \) and \( p \). For example, the unitary operation \( \mathcal{C}(ϕ) : |1, 0, 1⟩ \leftrightarrow e^{iϕ}|1, 0, 1⟩ \) can be obtained by waiting the appropriate time \( \hat{S}_x \).

(iii) Lattice shifts: We denote by \( \hat{S}_x \) the operations which shift the pointer states \( x \) steps to the right. For example, \( \hat{S}_{-1} \) transforms the state in Eq. (1) to \( \otimes_k |m_k, m_k', n_k⟩ \). They can be realized by changing the intensity and polarization of the lasers \( \hat{S}_x \).

(iv) Emptying sites: All atoms in internal states \( a \) or \( p \) are thrown away. This can be done, for example, by switching off the lattice potential for the corresponding internal state. We will denote them by \( E_a \) or \( E_p \) and they transform the state in Eq. (1) into \( \otimes_k |m_k, m_k', 0⟩ \), and \( \otimes_k |m_k, 0, n_k⟩ \), respectively.

Initially, all atoms are in the internal state \( |a⟩ \), distributed along the lattice according to some probability distribution, i.e. the state will be a mixture of states in the form of Eq. (1) with \( m_k = n_k = 0 \). Thus, the goal is to show how with these random states and with the operations which are available in the lab, and that do not require addressing, we can perform arbitrary quantum computations. This will be achieved in two steps. First there will be a preparation step, and then a computation step. At the end we will show how to include an additional step to make the system scalable.

In the preparation stage of our protocol, only states \( a \), and \( p \) will be occupied. Thus, we will simplify our notation denoting \( |m, n⟩ \) := \( |m, 0, n⟩ \). Moreover, the states that we will use now will be product states, i.e. of the form

\[
|m_1, n_1⟩ \otimes |m_2, n_2⟩ \ldots \otimes |m_N, n_N⟩
\]

where we have not included the subscript \( k \) to simplify the notation. This step starts out by reducing all occupation numbers \( \geq 2 \) to two (Fig. 1). This is done by applying the operation \( U_{x, 2}^{2, x} \) first and then \( E_p \), and then repeating those actions for \( x > 2 \) (up to some value of \( x \) in which we are confident that no site with this number of particles is present).

The next step is to “format” the lattice. We produce several areas, randomly distributed across the lattice, with exactly \( n \) neighboring sites having a single atom in \( a \) and one site at the right edge with two atoms, one in \( p \) and the other in \( a \) (see Fig. 1). In order to accomplish this, we have to keep only the areas in which initially there are \( n \) neighboring two-atom sites and a one-atom site at the edge. The rest of the atoms are thrown away, and then we manipulate the remaining atoms to obtain the desired states. The sites in which initially there was a single atom that has survived will now contain the “pointers” (the extra atom in level \( p \)). This atom will then be used to perform the quantum gates.

First, we change the internal states of the 1-atom sites from \( a \) to \( p \). These atoms are now called the “pointers”. They will be essential to create the quantum computers in the lattice. Each of those atoms mark the position where we try to establish one of those quantum computers. We want that such a pointer survives during the following protocol if it has on its left at least \( n \) sites with exactly 2-atoms in each. We thus proceed as follows. We shift the pointer one lattice site to the left. We transfer the pointer atom to the state \( a \) if there are two atoms in that site by applying \( U_{2, 1}^{3, 0} \). By emptying the internal states \( p \) we delete all pointers which fail to have a 2-atom site next to their starting position. Then we raise the pointer again by \( U_{2, 1}^{3, 1} \). By repeating this procedure for the next \( n - 1 \) sites we delete all pointers that fail to have \( n \) 2-atom sites on the left of their starting position. Note that this also implies that every pointer in one of the \( n \) sites on the right of each surviving pointer is deleted. This means that every pointer

![FIG. 1: First, the sites with more than 2 atoms are depopulated. The “format” step produces sites with 2 atoms in levels a and p, surrounded by a “reserved area” to their left which contains exactly n sites with a single atom.](image-url)
can act on its own “reserved” sites, i.e. there are no overlapping reserved areas. Having the pointer and the reserved $n$ 2-atom sites, we can effectively address single sites of this reserved space. This enables us to reduce the number of atoms in each site in this reserved area to one and afterwards to empty the remaining sites that are not reserved by any pointer. In terms of the operations described above, the protocols is given by a sequence of the following operators: 1) $U_{1,0}^{0,1}$, 2) $S_{-1}$, $U_{2,1}^{3,0}$, $E_p$, $U_{2,1}^{3,0}$, and then repeat this whole step $n - 1$ times; 3) $U_{1,2}^{2,1}$; 4) $S_{1}$, $U_{2,1}^{3,1}$, $E_p$, $U_{1,2}^{3,0}$, and then repeat this whole step $n - 1$ times; 5) $S_{1}$, $U_{2,0}^{2,2}$, $E_p$, $U_{2,0}^{1,1}$.

The randomly distributed quantum computers consist of $n$ sites, all of them with a single atom in state $|a\rangle$, and the pointer atom in state $|p\rangle$ in the rightmost site (see Fig. 1). The first atoms store a qubit each, with states $|\downarrow\rangle = |1, 0, 0\rangle$ and $|\uparrow\rangle = |0, 1, 0\rangle$, whereas the pointer atom in state $|0, 0, 1\rangle$ carry the quantum gates.

Now we show how to carry out a universal set of quantum gates using the operations defined above. The idea is to move the pointer atom to the sites which participate in the quantum gate and then apply the appropriate operations. The set is composed of $\mathbb{E}_2$: (a) control-$\pi$ phase gate on two arbitrary qubits: We first move the pointer to the first site, and apply the operator $U_{1,0}^{2,0}$. Then we move it to the other site and we wait until the collisional shift operation $C(\pi)$ is applied. Finally, we move the pointer back to the first site and apply again $U_{1,0}^{2,0}$. It is simple to show that this will only add a $\pi$ phase if both qubits are in the state $|\uparrow\rangle$. Note that after the first step the pointer atom is sometimes transferred into the $a$ level. Moving the pointers to the second qubit and back acts like the identity operator in those cases; (b) Phase-gate $\varphi$ on an arbitrary qubit (see Fig. 2): We bring the pointer to the corresponding site and wait for the appropriate collisional shift operation $C(\varphi)$; (c) Hadamard gate on an arbitrary qubit: We first bring the pointer to the site. Then we apply the following sequence of operations: $V$, $C(\pi), V^\dagger$, and then $C(\pi/2)$.

Measurement on an arbitrary qubit in the $|\downarrow\rangle, |\uparrow\rangle$ basis can be performed as follows. We promote the corresponding atom to the pointer level provided it is in state $|a\rangle$ i.e., if the qubit is in the state $|\downarrow\rangle$. For the measurement, we count the numbers of atoms in the pointer level (by analyzing the fluorescence coming from that level) and drop them afterwards. Note that this occurs in the same way as in usual ensemble quantum computation $\mathbb{E}_2$, in which we get the global information about all quantum computers. To save the original pointer from being emptied we need an extra resting-site, with one atom in the ground state (for example, the rightmost qubit can be reserved for this purpose). In summary, we: 1) move the pointer to the corresponding site and apply $U_{1,1}^{0,2}$; 2) move pointer to the resting-site and apply $U_{1,1}^{2,0}$ and $U_{1,2}^{2,1}$; 3) count atoms in pointer level and apply $E_p$; 4) apply $U_{2,0}^{1,1}$. The measured qubit-site is emptied, iff the qubit was found in state $|\downarrow\rangle$, while the pointer and the resting-qubit survived unchanged. We can continue by moving the pointer back to the target qubit, applying $W$ and then repeating above protocol. The number of atoms counted in the pointer-level this time is equal to the number of qubits with measured in state $|\uparrow\rangle$. Alternatively, we can leave out this step and relate the number of qubits found in $|\downarrow\rangle$ to the total number of quantum computers in the lattice. This number can be estimated either by the statistics of the starting distribution or by measuring the number of pointers/atoms at the end of the computation.

So far we have shown how to build a quantum register of $n$ qubits, for any arbitrary integer $n$, and how to perform quantum computations. Note that in order to prepare the initial state it is necessary that there are areas in the lattice which have no defects, i.e. no empty sites nor 1 atom sites. If the number of such defects well inside the lattice is larger than the number of 1D lattices, then the probability of ending up with at least one quantum computer will decrease exponentially with $n$, and thus the method proposed here will not be scalable. In detail, if we assume that every site of the lattice is filled independently with zero, one or two atoms, according to the probabilities distribution $p_0, p_1, p_2$, then the expected number of quantum computer of length $n$ in a 1D lattice can be estimated by $N p_1 p_2^n = N p_1 (1 - p_0 - p_1)^n$, where the length $N$ of the lattice has to be much larger than $n$. This quantity decreases exponentially with $n$ which makes the proposed method not scalable.

In the following we show how to boost the probability of creating a quantum computer in the lattice by correcting the defects, i.e., making $p_0$ and $p_1$ arbitrarily small. Having this possibility, we change the probabilities to $p_0 = 0$ and $p_1 = 1/n$. The resulting expected number of quantum computers in a lattice of size $N$ is then given by $N/n (1 - 1/n)^n$, which goes to $N/(n \cdot e) \sim 1/n$ for large
n, i.e. our method becomes scalable. The procedure of correcting the defects will also be useful if one would like to perform quantum simulations with large spin chains.

The main idea of the protocol is to first fill all sites which are empty with one atom coming from a different site, which is overpopulated. Then, the sites with one atom are filled with another atom also coming from overpopulated sites (see Fig. 3). Thus, we have to assume that there are as many overpopulated sites as defects, an achievable requirement for sufficiently high densities.

First, we reduce all occupation number $> 4$ to four $|1\rangle$. Then, the protocol starts out by promoting two atoms to the state $p$ whenever a site has four atoms. Then we shift the lattice corresponding to level $p$ by a random amount $x$ and try to deposit one of such atoms in an empty site. The remaining atom in the $p$ level is thrown away. Note, that for every corrected defect we lose one atom in this protocol. Losing atoms while correcting defects is unavoidable, since it is the only way to reduce the entropy of the state in our setup. We proceed in the same vein until we make the probability of having sites with no atoms vanishingly small. In more detail, we apply the following sequence of operations several times: $U_{4,0}$, $S_x$, $U_{0,2}$, $S_{-x}$, $U_{2,0}$, $E_p$. With this we will have filled the empty sites. Now, we can do the same but replacing $U_{4,0}^{2,2}$ and $U_{0,2}^{1,1}$ by $U_{x,0}^{2,2}$ and $U_{1,2}^{2,1}$, so that sites with a single atom get double occupation. For a finite lattice of length $N$ there are only $N$ different possibilities for the $x$, so the protocol requires at most $N$ repetitions.

We still need some defects to act as pointers. So we either do not fill up all the one-atom defects or we have to create new defects. The latter can be done by first reducing all occupation numbers to two and then applying a unitary operation that changes $|2,0,0\rangle$ to the superposition state $\sqrt{\varepsilon}|1,1,0\rangle + \sqrt{1-\varepsilon}|2,0,0\rangle$ followed by $E_p$. With probability $\varepsilon$ a one-atom-site defect is created out of a 2–atom site.

We have shown that it is possible to perform quantum computations in optical lattices in the presence of lattice defects and without the necessity to address single lattice sites, nor to specify the total number of atoms in the lattice. In practice, a very high degree of control is required, something which is being achieved in current experiments with optical lattices. The ideas presented here not only apply to the field of quantum computation but they can also be used to prepare and manipulate the states in the lattice, and to build some prescribed atomic patterns. Furthermore, all these methods can be generalized in a straightforward way to 2–dimensional or 3–dimensional lattices. Finally, note that some of the protocols given here require a large number of steps, something which is experimentally demanding. We are currently using the ideas of quantum compression in order to develop new efficient methods for loading the lattices.

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