Generalized Constructive Tree Weights

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Abstract

The Loop Vertex Expansion (LVE) is a quantum field theory (QFT) method which explicitly computes the Borel sum of Feynman perturbation series. This LVE relies in a crucial way on symmetric tree weights which define a measure on the set of spanning trees of any connected graph. In this paper we generalize this method by defining new tree weights. They depend on the choice of a partition of a set of vertices of the graph, and when the partition is non-trivial, they are no longer symmetric under permutation of vertices. Nevertheless we prove they have the required positivity property to lead to a convergent LVE; in fact we formulate this positivity property precisely for the first time. Our generalized tree weights are inspired by the Brydges-Battle-Federbush work on cluster expansions and could be particularly suited to the computation of connected functions in QFT. Several concrete examples are explicitly given.

Key words: Trees, Feynman graphs, Combinatorics, Constructive quantum field theory.
1 Introduction

The fundamental step in quantum field theory (QFT) is to compute the logarithm of a functional integral. This comes from a fundamental theorem of enumerative combinatorics, stating the logarithm counts the connected objects. The main advantage of the perturbative expansion of a QFT into a sum of Feynman amplitudes is to perform this computation explicitly: the logarithm of the functional integral is simply the same sum of Feynman amplitudes restricted to connected graphs. The main disadvantage is that the perturbative series indexed by Feynman graphs typically diverges. Constructive theory is the right compromise, which allows both to compute logarithms, hence connected quantities, but through convergent series. However it has the reputation to be a difficult technical subject.

Perturbative QFT writes quantities of interest (free energies or connected functions) as sums of amplitudes of connected graphs

\[ S = \sum_G A_G. \]  

(1)

However such a formula (obtained by expanding in a power series the exponential of the interaction in Feynman functional integral and then "illegally" commuting the power series and the functional integration) is not a valid definition since usually, even with cutoffs, even in zero dimension (!) we have

\[ \sum_G |A_G| = \infty. \]  

(2)

This divergence, known since [1], is due to the more-than-exponential growth of the number of graphs with many vertices. We can say that Feynman graphs proliferate too fast. More precisely the power series in the coupling constant \( \lambda \) corresponding to (1) has zero radius of convergence. Nevertheless for the stable Bosonic models which have been rigorously built by constructive field theory, the constructive answer is always the Borel sum of the perturbative series (see [4] and references therein). Hence the perturbative expansion,

\footnote{The main feature of QFT is the renormalization group, which is made of a sequence of such fundamental steps, one for each scale.}

\footnote{This can be proved easily for \( \phi^4 \), the Euclidean Bosonic QFT with quartic interaction in dimension \( d \), with fixed ultraviolet cutoff, where the series behaves as \( \sum_n (-\lambda)^n K^n n! \). It is expected to remain true also for the renormalized series without cutoff; this has been proved in the super-renormalizable cases \( d = 2, 3 \) [2, 3].}
although divergent, contains all the information of the theory; but it should be reshuffled into a convergent sum\footnote{Although this may sound like a technical question, convergence is such a critical issue in QFT that new technical tools for its solution might lead, like Feynman graphs themselves, to new physical insights.}

The key point for the success of constructive theory is that trees do not proliferate as fast as graphs, and they are sufficient to show connectivity, hence to compute logarithms. This central fact is not emphasized as such in the classical constructive literature\cite{6}. It is usually obscured by the historic tools which constructive theory borrowed from statistical mechanics, namely lattice cluster and Mayer expansions.

The loop vertex expansion (LVE)\cite{7} is a relatively recent simple constructive technique which precisely reshuffles ordinary perturbative expansion into a convergent expansion using canonical combinatoric tools rather than non-canonical lattices. Initially introduced to analyze matrix models with quartic interactions, it has been extended to many other stable interactions in\cite{8} and shown compatible with direct space decay estimates\cite{9} and with renormalization in simple super-renormalizable cases\cite{10,11}. It has also recently been used in the context of group field theory\cite{12} and improved\cite{13} to organize the $1/N$ expansion\cite{14,15,16} for random tensors models\cite{17,18,19}, a promising approach to random geometry and quantum gravity in more than two dimensions\cite{20,21}.

The combinatoric core of the LVE has been reformulated in a more transparent way in\cite{24}. The basic idea is to define a set of positive weights\[ w(G,T) \]
which are associated to any pair made of a connected graph $G$ and a spanning tree $T \subset G$. They are normalized so as to form a probability measure on the spanning trees of $G$:

$$\sum_{T \subset G} w(G,T) = 1. \quad (3)$$

To compute constructively instead of perturbatively a QFT quantity $S$ one should use equation (3) to introduce a sum over trees for each graph, and then simply exchange the order of summation between graphs and their spanning trees

$$S = \sum_G A_G = \sum_G \sum_{T \subset G} w(G,T)A_G = \sum_T A_T, \quad A_T = \sum_{G \supset T} w(G,T)A_G. \quad (4)$$
The constructive "miracle" is that if one uses the right graphs and right weights, then for stable Bosonic interactions with cutoffs we get

\[ \sum_T |A_T| < +\infty, \] (5)

which means that \( S \) is now well defined; furthermore the result is the desired one, namely the Borel sum of the ordinary perturbation expansion.

The three main tasks are then to identify the most general class of right weights to use for (5) to hold, then the most general class of interactions leading to right graphs so that (5) holds, and finally to extend the formalism to include renormalization, hence to treat QFTs without cutoffs.

The essential discovery of the LVE [7] is that even in the case of the simplest stable \( \phi^4 \) quartic interaction the "right graphs" to use for (5) to hold are not the ordinary Feynman graphs, but the Feynman graphs of the so-called intermediate field representation of the theory. This representation decomposes ordinary interactions of order higher than three into three-body interactions. This result, initially limited to quartic interactions, has been extended to include all even monomials in [8].

Consider from now on a positive interaction such as \( \phi^4 \) and its intermediate field perturbation expansion. Not every probability measure \( w(G,T) \) on the spanning trees of the corresponding graphs leads to a constructive reshuffling, namely one for which (5) holds. For instance the trivial, equally-distributed weights \( 1/\chi(G) \), where \( \chi(G) \), the complexity of \( G \), is the number of its spanning trees, form obviously such a probability measure, but there is no reason to think they lead to a constructive reshuffling.

The LVE weights used in [7] are defined in terms of the Taylor forest formula of [22, 23], because this formula has a particular positivity property. This is why they lead to a convergent LVE, for which (5) holds.

These LVE weights are fully symmetric under action of the permutation group of the vertices of the graph, hence from now on we call denote them \( w_s(G,T) \). In [24] these symmetric weights were identified with the percentage of Hepp sectors [25] in which the tree \( T \) is leading, in the sense of Kruskal "greedy algorithm" [26] (see below).

However, historically, non-symmetric forest formulas for constructive cluster expansions were discovered before the symmetric Taylor forest formula (see [27], [28] or [29]). It is therefore clear that the symmetric weights are not the only ones with constructive positivity. In this paper we explore this issue in detail. First we define precisely this property of constructive positivity.
Then we identify a very general class of weights, associated to any non-trivial partition of the vertices of a graph, which have this constructive property\textsuperscript{4}. This is the main result of this paper.

Examples are then given: in particular weights for rooted and multi-rooted graphs (which just correspond to particular cases of vertex partitions, where one considers one or more singletons and the rest of vertices are grouped into a single remaining block). are defined and compared to the symmetric weights. Although symmetric weights look the most natural for the constructive expansion of quantities such as the free energy of a QFT, non-symmetric rooted weights could be also useful, in particular to compute Schwinger functions with external arguments. Indeed the latter correspond to particular marked vertices bearing cilia in the LVE \cite{13}, and there is no reason to think that the optimal LVE in that case should be so symmetric as to mix these particular external marked vertices with the ordinary internal vertices.

\section{Prerequisites}

Consider a fixed set $V$ of vertices. We associate weights $w(G, T)$ (also called amplitudes in the QFT context) to any pair $(G, T)$, where $G$ is a labeled connected graph $G$ with vertex set $V$ and edge set $E$, and $T$ is a spanning tree of $G$ (where by spanning tree we mean an acyclic maximal subset of $E$, hence of cardinality $|V| - 1$). Self-loops (a. k. a. tadpoles in the QFT language) and multiple edges in $G$ are allowed, since they are a fundamental feature of QFT. From now on we omit the word ”spanning”, since throughout this paper the trees considered are always spanning for a related graph $G \supset T$.

We write $\sum_{T \subseteq G} w(G, T)$ to indicate summation over the finite family of trees of a fixed $G$ of such weights $w(G, T)$. We can also consider trees $T$ as particular labeled connected graph themselves with vertex set $V$. There is an infinite family of graphs obtained by adding an arbitrary number $L$ of edges between the vertices of $T$, since self-loops and multiple edges are allowed. Such graphs have $|E(G)| = |V| - 1 + |L|$ edges and nullity $L$ (that is $L$ independent loops, since we deal with connected graphs, as already stated above). In that case we write $W(T) = \sum_{G \supset T} w(G, T)$ to indicate summation over the infinite family of such $G$’s\textsuperscript{5}. The corresponding series may of course

\textsuperscript{4}This class may even be the most general one with constructive positivity but we do not know how to prove or disprove this last point.

\textsuperscript{5}This family could possibly be enlarged or restricted by additional ”Feynman rules”
be divergent or convergent depending on the exact weights considered.

A complete ordering of the $|E(G)|$ edges of $G$ is called a Hepp sector in QFT terminology [25]. The set of such orderings, $S(G)$, has $|E(G)|!$ elements. For any such Hepp sector $\sigma \in S(G)$, Kruskal greedy algorithm defines a particular tree $T(\sigma)$, which minimizes $\sum_{\ell \in T} \sigma(\ell)$ over all trees of $G$. We call it for short the leading tree for $\sigma$. Let us briefly explain how the algorithm works. The algorithm simply picks the first edge $\ell_1$ in $\sigma$ which is not a self-loop. The algorithm then picks the next edge $\ell_2$ in $\sigma$ that does not add a cycle to the (disconnected) graph with vertex set $V$ and edge set $\ell_1$ and so on [26]. Another way to look at it is through a deletion-contraction recursion: following the ordering of the sector $\sigma$, every edge is either deleted if it is a self-loop or contracted if it is not. The set of contracted edges is exactly the leading tree for $\sigma$.

Remark that this leading tree $T(\sigma)$ has been considered intensively in the context of perturbative and constructive renormalization in QFT [4], as it plays an essential role to get sharp bounds on renormalized quantities.

Remark also that given any Hepp sector $\sigma$ the (unordered) tree $T(\sigma)$ comes naturally equipped with an induced ordering $\tau$ (the order in which the edges of $T(\sigma)$ are picked by Kruskal’s algorithm). The corresponding ordered tree will be denoted as $T_\tau$.

**Definition 2.1.** A probability measure on trees is a set of positive weights $w(G, T)$ for any labeled connected graph $G$ and tree $T \subset G$ such that

$$\sum_{T \subset G} w(G, T) = 1.$$ (6)

The measure and the weights $w$ are called rational if all $w(G, T) \in \mathbb{Q}$ and they are called symmetric if $w(G, T) = w(G^\nu, T^\nu)$, where $\nu$ is any permutation of $V$, hence any relabeling of the vertices of $G$ and $T$. The measure and the weights are called constructive if there exists a $T$-dependent probability

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6The category of Feynman graphs to consider for the constructive applications below has in fact slightly more structure, since Feynman graphs have also labeled half-edges, according to Wick theorem. It provides them in particular with a canonical ciliated ribbon structure, by labeling the half-edges starting from the cilium. These additional (important) subtleties are not considered here, as the half-edge labeling will play no role in this paper.
measure \((\Omega_T, \Sigma_T, \mu_T)\) and a \((T, u)\) dependent real positive symmetric matrix \(X^T_{v,v'}(u)\) for any \(u \in \Omega_T\), with diagonal \(X^T_{v,v}(u) = 1\) for any \(u\), such that

\[
w(G, T) = \int_{\Omega_T} d\mu_T(u) \prod_{\ell \not\in T} X^T_{i(\ell), j(\ell)}(u)
\]

where \(v(\ell)\) and \(v'(\ell)\) denote, by a slight abuse of notation, the two vertices that the edge \(\ell\) hooks to.

Note that the order of the two vertices \(v(\ell)\) and \(v'(\ell)\) above plays no rôle, since the matrix \(X\) is symmetric. From a QFT perspective, this comes from the fact that one can endow the internal edges of a Feynman graph with any orientation.

This constructive property is exactly what allows, in the case of stable Bosonic interactions such as \(\phi^4\), to rewrite any tree amplitude \(A_T\) of the loop vertex expansion in (4) as an integral over \(\Omega_T\) for the measure \(d\mu_T\) of a functional integral over a positive Gaussian measure of covariance \(X^T_{ij}(u)\) of a well-bounded integrand. Hence it is exactly the property necessary for inequality (5) to hold. For non-constructive weights, there is no such functional integral representation.

Let us recall here the definition of the symmetric weights \(w_s(G, T)\) (see again [24]):

**Definition 2.2.** The symmetric weights \(w_s(G, T)\) are the percentage of Hepp sectors for which the tree \(T\) is a leading tree

\[
w_s(G, T) = \frac{1}{|E(G)|!!} \sum_{\sigma \in S(G)} \chi(T(\sigma) = T)
\]

where \(\chi(T(\sigma) = T)\) is 1 if the leading tree for the Hepp sector \(\sigma\) is the tree \(T\), and 0 otherwise.

Normalization and rationality of these weights are obvious. We then recall the main result of [24]:

**Theorem 2.1.** The symmetric weights \(w_s(G, T)\) are constructive.

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\(^7\)In all concrete examples \(\Omega_T\) is a topological space, and the sigma-algebra \(\Sigma_T\) is its Borel sigma-algebra and will play no further role.
Indeed, identity \([7]\) holds for \(\Omega_T = [0, 1]^{E(T)}\) with Lebesgue measure and for the matrix \(X^T_{ij}(u)\) which is 1 for \(i = j\) and which is for \(i \neq j\) the infinum over the \(u_\ell\) parameters of the lines \(\ell\) in \(P^T_{ij}\), where \(P^T_{ij}\) is the unique path in \(T\) between vertices \(i\) and \(j\) \([22]\). The fact that this matrix is symmetric positive for any \(u\) is then a well-known property of this infinum function and of the Taylor forest formula \([22, 23, 7]\).

### 3 Partition Tree Weights

Consider again a fixed vertex set \(V\).

A partition of \(V\) into \(k\) non empty disjoint subsets \(V = V_1 \cup \cdots \cup V_k\) is called **trivial** if \(k = 1\) and **non-trivial** if \(k \geq 2\). The subsets of the partition are also called **blocks** in what follows. From now on we suppose we made a choice of a fixed such partition \(\Pi\). Our goal is to define, for any graph \(G\) with vertex set \(V\) an associated rational constructive measure for the trees of \(G\).

An edge \(\ell \in G\) with ends \(i\) and \(j\) is called **trans-block** for the partition \(\Pi\) if the vertices \(i\) and \(j\) belong to two distinct blocks \(V_{k(i)}\) and \(V_{k(j)}\) of \(\Pi\). Remark that a self-loop is never trans-block, for any partition.

Given the graph \(G\) and a trans-block edge \(\ell \in G\) with ends \(i\) and \(j\), we can consider the contracted graph \(G/\ell\) in which the vertices \(i\) and \(j\) are replaced by a single contracted vertex \(\hat{ij}\) and the edge \(\ell\) is removed. This contracted graph is naturally equipped with a **contracted partition** \(\Pi/\ell\) defined by the blocks \(V_1, \ldots, V_{k(i)} - \{i\}, \ldots, V_{k(j)} - \{j\}, \ldots, V_k, V_{k+1} = \{\hat{ij}\}\) any empty block being omitted. Hence it is a partition into \(k'\) blocks, with \(k - 1 \leq k' \leq k + 1\). Remark that this reduced partition has always at least a singleton block, namely \(V_{k+1}\). Remark also that it can be trivial only if the graph \(G\) has exactly two vertices; indeed the equation \(k' = 1\) implies that the initial partition was solely made of \(V_{k(i)} = \{i\}\) and \(V_{k(j)} = \{j\}\).

Iterating this construction we arrive at the definition of a trans-block ordered forest:

**Definition 3.1.** An ordered forest \(\mathcal{F} = \{\ell_1, \cdots, \ell_p\}\) \(p \leq |V| - 1\) is called **trans-block** for the partition \(\Pi\) if the edge \(\ell_1\) is trans-block for the partition \(\Pi\), the edge \(\ell_2\) is trans-block for the contracted graph \(G/\ell_1\) and its contracted partition \(\Pi/\ell_1\) and so on until the last edge \(\ell_p\) of \(\mathcal{F}\) which is trans-block for the contracted graph \(G/\ell_1/\ell_2/\cdots/\ell_{p-1}\) and the contracted partition \(\Pi/\ell_1/\ell_2/\cdots/\ell_{p-1}\).
The sequence of graphs $\{G_0 = G, G_1 = G/\ell_1, \cdots, G_p = G/\ell_1/\cdots/\ell_p\}$ and the sequence of partitions $\{\Pi_0 = \Pi, \Pi_1 = \Pi/\ell_1, \cdots, \Pi_p = \Pi_{p-1}/\ell_p\}$ is noted $\mathcal{S}(G, \Pi, \mathcal{F})$.

Remark that the sequence $\mathcal{S}(G, \Pi, \mathcal{F})$ indeed depends on the ordering of the forest in a critical way.

The maximal trans-block ordered forests are the trans-block ordered trees:

**Lemma 3.1.** For any ordered trans-block forest $\mathcal{F}$, the last partition in $\mathcal{S}(G, \Pi, \mathcal{F})$ is trivial (i.e. made of a single block) if and only if $\mathcal{F}$ is a tree, i.e. has exactly $|V| - 1$ edges.

**Proof:** Consider a trans-block ordered forest $\mathcal{F} = \{\ell_1, \cdots, \ell_p\}$ and the sequence of $p + 1$ reduced graphs $G_0 = G, G_1, \cdots, G_p$. The number of vertices decreases by exactly one in each step of this sequence, so the last graph has a single vertex, hence we reach a trivial partition if and only if $p = |V| - 1$, hence if and only if $\mathcal{F}$ is a (trans-block) tree.

**Definition 3.2.** An ordering $\tau$ of a given tree $T$ is called admissible for the partition $\Pi$ if the ordered tree $T_\tau$ is trans-block for the respective partition. The set of such admissible orderings for a given tree $T$ is denoted by $A^\Pi(T)$.

The set of admissible orderings is never empty if the respective partition $\Pi$ is non trivial. Any admissible ordering $\tau$ of $T$ defines a sequence of contracted graphs and partitions $\mathcal{S}(G, \Pi, T_\tau)$. Do not confuse orderings $\tau$ and the Hepp sectors for the full graph $G$ considered in the previous section. Remark however that the orderings $\tau$ can be considered as Hepp sectors for the tree $T$.

The partition weight $w^\Pi(G, T)$ will be defined in formula (11) below as a sum over all admissible orderings $\tau \in A^\Pi(T)$ of certain finite dimensional simple integrals. Their definition requires first that we define the so-called contact indices. These indices are defined for any pair of vertices $(v, v')$ of $G$ (including the case $v = v'$) and any ordered trans-block tree $T_\tau$:

**Definition 3.3 (Contact Indices).** Consider an ordered trans-block tree $T_\tau = \{\ell_1, \cdots, \ell_{|V| - 1}\}$ and its associated sequence of reduced graphs and partitions $\mathcal{S}(G, \Pi, T_\tau) = \{G_p, \Pi_p\}$ with $0 \leq p \leq |V| - 1$. We define the first contact index $i^\Pi_{T_\tau}(v, v')$ as the smallest value of $p$ such that the two vertices $v$ and $v'$ belong to different blocks for $\Pi_p$, and the second contact index $j^\Pi_{T_\tau}(v, v')$ as the smallest value of $p$ for which $v$ and $v'$ are collapsed into a single
reduced vertex in $G_p$. If $v = v'$ we set by convention: $i^\Pi_{T_r}(v,v') = -1$ and $j^\Pi_{T_r}(v,v') = 0$.

Let us make the following remark. If the two vertices $v$ and $v'$ belong to distinct blocks of the partition $\Pi$, we have therefore $i^\Pi_{T_r}(v,v') = 0$.

**Lemma 3.2.** The two contact indices obey $i^\Pi_{T_r}(v,v') < j^\Pi_{T_r}(v,v')$.

**Proof:** This follows directly from the definition above.

**Definition 3.4 (Contact Matrices).** For any graph with vertex set $V$, any given vertex set partition $\Pi$, any given ordered tree $T_r$ and $u = \{u_1, \ldots, u_{|V|-1}\}$ in $[0,1]^{|V|-1}$, we define the $|V|$ by $|V|$-dependent real symmetric matrix, called the contact matrix, $X^{\Pi,T_r}(u)$ by the following formula:

$$X^{\Pi,T_r}_{v,v'}(u) := \prod_{i^\Pi_{T_r}(v,v') < k \leq j^\Pi_{T_r}(v,v')} u_k, \quad \forall v,v' \in V. \quad (9)$$

Moreover, we define the tree edge factor, by the following formula:

$$Y^\Pi_{T_r}(u) := \prod_{i^\Pi_{T_r}(v,v') < k \leq j^\Pi_{T_r}(v,v')} u_k, \quad \forall \ell \in T. \quad (10)$$

Note that in (10) we have again denoted by $v(\ell)$ and $v'(\ell)$ (by the same slight abuse of notation) the two vertices that the (tree) edge $\ell$ hooks to.

Remark that $X^{\Pi,T_r}(\sigma)(u) = 1$, since by convention an empty product is one, hence the matrix $X^{\Pi,T_r}(u)$ has diagonal entries all equal to one.

We are now finally in position to define the partition weights associated to an admissible sector $\sigma$.

**Definition 3.5 (Partition Tree Weights).** For a set $V$ and a partition $\Pi$ fixed, the partition weights $w^\Pi(G,T)$ associated to any tree $T$ of the graph $G$ (whose vertex set is $V$) are defined as

$$w^\Pi(G,T) := \sum_{\tau \in A^\Pi(T)} \int_0^1 du_1 \cdots \int_0^1 du_{|V|-1} \prod_{\ell \in T} Y^\Pi_{\ell}(u) \prod_{\ell \notin T} X^{\Pi,T_r}_{v(\ell),v'(\ell)}(u) \quad (11)$$

where $v(\ell)$ and $v'(\ell)$ are the two vertices that the edge $\ell$ hooks to.
Each partition tree weight $w^\Pi(G, T)$ being obviously a sum over $\mathcal{A}^\Pi(T)$ of positive rational numbers is a positive rational number. We first prove a lemma stating the normalization of these weights, for a given graph and a given partition of its vertex set.

Lemma 3.3. We have

$$\sum_{T \subseteq G} w^\Pi(G, T) = 1. \quad (12)$$

Proof: The key idea is to Taylor expand the function $1 = \prod_{\ell \in G} x_\ell \big|_{x_\ell = 1 \forall \ell}$ according to a $\Pi$-dependent recursion with $|V| - 1$ steps which, at each step $i$, uses an elementary first order Taylor expansion with integral remainder

$$f(1) = f(0) + \int_0^1 f'(u_i) du_i, \quad i = 1, \ldots, |V| - 1. \quad (13)$$

This recursion “builds” step by step a sum over all trans-block ordered trees.

Let $F_0(x_1, \ldots, x_{|E|}) := x_1 \ldots x_{|E|}$. The first step of the induction requires first to multiply each variable $x$ corresponding to a trans-block edge for the partition $\Pi_0$ with a dummy variable $u_1$. This amounts to consider the function

$$F_0(u_1 x_1, \ldots, u_1 x_{k_0}, x_{k_0+1}, \ldots, x_{|E|}) \big|_{x_i = 1, \forall i}, \quad (14)$$

where, without any loss of generality we have placed on the first $k_0$ positions the $k_0$ variables associated to the trans-block edges for the partition $\Pi_0$. Note that since $\Pi_0$ is non-trivial and $G$ is connected, one has $k_0 > 0$. Using the multi-variable version of the Taylor expansion (13) leads to

$$1 = F(0, \ldots, 0, 1, \ldots, 1)$$

$$+ \sum_{\ell_1} \int_0^1 du_1 \frac{\partial}{\partial x_{\ell_1}} F_0(u_1 x_1, \ldots, u_1 x_{k_0}, x_{k_0+1}, \ldots, x_{|E|}) \big|_{x_i = 1, \forall i}, \quad (15)$$

where the sum over $\ell_1$ runs over the $k_0$ trans-block edges for the partition $\Pi_0$. Note that the first term on the RHS of equation (15) vanishes (because of the definition of the function $F_0$ and because $k_0 \neq 0$, as already noticed above). For the sake of simplicity, let us fix the trans-block edge in the sum (15). It will be the first edge in the recursive construction of our ordered tree. In the RHS of equation (15) the derivative leads to

$$1 = \sum_{\ell_1} \int_0^1 du_1 u_1^{k_0-1} F_1. \quad (16)$$
In the function $F_1$ we fix definitely $x_{\ell_1} = 1$, so that $F_1$ no longer depends on it. We can now proceed with the next induction, which is identical but for the graph $G_1 = G/\ell_1$ and the partition $\Pi_1 = \Pi/\ell_1$.

At a generic step $n$, this expansion leads to

$$1 = \sum_{\ell_1, \ldots, \ell_n} \int_0^1 du_1 \ldots du_n u_1^{k_0-1} \ldots u_n^{k_n-1} F_n \quad (17)$$

where the exponent $k_j$ represents the number of trans-block edges for the partition $\Pi_j$ in our sequence of partitions. As above, the factor $F_n$ does not depend on the variables associated to the contracted edges $\ell_1, \ldots, \ell_n$. This process continues until there are no more trans-block edges for the respective partition; this happens at the $(|V| - 1)$th step. The corresponding factor $F_{|V| - 1}$ is then no longer interpolated hence equal to 1. The expansion therefore results in a sum over all trees and all their admissible sectors, renaming dummy variables $u_1 = \sum_{T, \tau \in \mathcal{A}_{\Pi}(T)} Y_{\Pi,T}^{\Pi,0,0} \prod_{\ell \in T} X_{\Pi,0,0}^{\Pi,0,0} = u_1^{k_0-1} \ldots u_{|V| - 1}^{k_{|V| - 2}}. \quad (18)$

Let us now identify, for a fixed ordered tree $T_r$ in this sum, the integrand $u_1^{k_0-1} \ldots u_{|V| - 1}^{k_{|V| - 2}}$ with the two products appearing in (11):

$$\prod_{\ell \in T_r} Y_{\Pi,0,0}^{\Pi,0,0} \prod_{\ell \in T_r \setminus \{i\}} X_{\Pi,0,0}^{\Pi,0,0} = u_1^{k_0-1} \ldots u_{|V| - 1}^{k_{|V| - 2}}. \quad (19)$$

Indeed within the LHS of (19) - which is, by construction, a product of various powers of the (tree) edge variables $u_k$ ($k = 1, \ldots, |V| - 1$) - the variable $u_1$ appears as many times as there are edges $\ell$ with contact indexes $i = 0$ and $j \geq 1$. This is nothing but the number of edges which are trans-block for the partition $\Pi_0$ minus 1, the minus 1 correction coming from the fact that, for $\ell_1$, the contact indices are 0 and 1, and the factor $u_1$ is not taken into account, by definition, within the tree factor $Y_{\ell_1}^{\Pi,0,0}$. The analogous reasoning holds for $u_k$ ($k = 2, \ldots, |V| - 1$) and this concludes the proof.

Let us now rewrite formula (11) as

$$w_{\Pi}(G, T) = \sum_{\tau \in \mathcal{A}_{\Pi}(T)} w_{\Pi}(G, T^\tau), \quad (20)$$

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\[ \]
where we define:

\[
\omega^\Pi(G,T) := \int_0^1 du_1 \cdots \int_0^1 du_{|V|-1} \left[ \prod_{t \in T} Y_{t}^{\Pi,T}(u) \right] \left[ \prod_{t \in T} X_{v(t),v'(t)}^{\Pi,T}(u) \right].
\]  

(21)

One has:

**Corollary 3.1.** For a given graph \(G\), vertex partition \(\Pi\) and ordered tree \(T^\tau\), the weights defined in \((21)\) can be written as

\[
\omega^\Pi(G,T^\tau) = |V| - 2 \prod_{i=0}^{k_i-1} \frac{1}{k_i},
\]

(22)

where \(k_i\) give the number of trans-block edges for the partition \(\Pi_i\) in the corresponding partition sequence.

**Proof:** This is a direct consequence of equation (17).

Let us mention here that a somehow similar use of the Taylor formula with integral remainder was made in [30], within the cluster expansion framework. The use of the integral remainder is one possible way of not dealing with the whole perturbative series, since this series most often diverges in QFT.

Let us now prove the following lemma:

**Lemma 3.4.** The symmetric matrix \(X_{v,v'}^{\Pi,T}(u)\) is positive semi-definite.

**Proof:** We prove the positivity for any fixed ordered tree \(T^\tau\) again by a recursion based upon the sequence of graphs \(G_0, \ldots, G_{|V|-1}\) and the associated partitions \(\Pi_0, \ldots, \Pi_{|V|-1}\).

We define first for any \(n\) by \(n\) matrix \(X\) and any partition \(\Pi\) of \([1, \ldots, n]\), the projected matrix \(X^\Pi\) which has elements \(X^\Pi_{ij} = X_{ij}\) if \(i\) and \(j\) belong to the same block of \(\Pi\) and 0 otherwise. We then remark that for any symmetric positive \(X\), \(X^\Pi\) is positive and for any real \(u \in [0, 1]\) the interpolated matrix \(X(u) = uX + (1-u)X^\Pi\) is also positive, as barycentric combination of two positive matrices.

Let us now define the positive \(|V|\) by \(|V|\) matrix \(X_0\), which has matrix elements all equal to one.
We then define:

\[ X_1(u_1) = u_1X_0 + (1 - u_1)X_0^{\Pi_0} \]
\[ X_2(u_1, u_2) = u_2X_1(u_1) + (1 - u_2)X_1^{\Pi_1}(u_1) \]

\[ \ldots \]
\[ X_{|V|-1}(u_1, \ldots, u_{|V|-1}) = u_{|V|-1}X_{|V|-2}(u_1, \ldots, u_{|V|-2}) + (1 - u_{|V|-1})X u_{|V|-2}^{\Pi_{|V|-2}}(u_1, \ldots, u_{|V|-2}). \quad (23) \]

Note that all these matrices are positive, again as barycentric combinations of two positive matrices. This mechanism thus leads to a “final” \(|V|\) by \(|V|\) matrix \(X_{|V|-1}\) for which the matrix element corresponding to the entry \((v, v')\) \((v, v' = 1, \ldots, |V|)\) is a product of variables \(u_j\) \((j = 1, \ldots, |V| - 1)\).

Each such variable \(u_j\) is present in this product if and only if the edge (or the edges, since multi-edges are allowed) connecting the vertices \(v\) and \(v'\) in \(G_{j-1}\) is a trans-block edge for the corresponding partition \(\Pi_{j-1}\).

We have thus “constructed” the matrix given by the formula (9), since the combinatorial definition of the contact indices lead to the same product of the tree variables \(u_j\). This concludes the proof. \(\square\)

The main result of this paper is the following theorem:

**Theorem 3.1.** For any set \(V\) and partition \(\Pi\) the partition weights \(w^\Pi(G, T)\) on the trees \(T\) of any graph \(G\) with vertex set \(V\) form a rational constructive probability measure.

**Proof:** The claim follows from Lemmas 3.3 and 3.4. Indeed the set \(\Omega_T\) is the disjoint union for all admissible \(\tau\)'s of a distinct copy of \([0, 1]^{|V|-1}\) with measure

\[ d\mu_{T, \tau} = \prod_{\ell \in T} Y^{\Pi, T, \tau}(u) \prod_{k=1}^{|V|-1} du_k. \quad (24) \]

Remark that the normalization of the full measure \(d\mu_T\) is nothing but equation (12) for the particular case when \(G = T\) concludes the proof. \(\square\)

Before ending this section, let us mention that an important particular case of our results can be obtained when the vertex partition is made of a singleton plus a single block containing all the remaining vertices. As already mentioned in the introduction, the mechanism exposed in this paper leads to constructive tree weights for a *rooted graph* (the singleton being the root of
the graph). An example of such tree weights for a rooted graph is given in subsection 4.2 below. This is then generalized for multi-rooted graphs, which correspond to vertex partitions made of several singletons and a remaining block containing the rest of the vertices. An example of a double-rooted graph is analyzed in detail in subsection 4.3 below.

4 Examples

4.1 Symmetric weights - complete partition

Symmetric weights correspond to the symmetric partition $\Pi^s$ of $V$ into $|V|$ singletons. Let us check directly that this is indeed the case, namely that $w_s(G, T)$, as defined in (8) is equal to $w^{\Pi^s}(G, T)$ defined by formula (22):

Lemma 4.1. The symmetric weights $w_s(G, T)$ are the partition weights for the partition $\Pi^s$ of $V$ into $|V|$ singletons:

$$w_s(G, T) = \sum_{\sigma | T(\sigma) = T} 1/|E(G)|! = \sum_{\tau} \prod_{i=0}^{d \tau - 2} \frac{1}{k_i} = w^{\Pi^s}(G, T), \quad (25)$$

where the sum over $\tau$ is performed over all the orderings of T and $k_i$ is the number of trans-block edges for the partition $\Pi^s$ in the partition sequence corresponding to $T_\tau$, starting from the all-singletons partition $\Pi^s$.

Proof: Remark that in the symmetric case every sector is admissible, hence there is no restriction on the sum over $\tau$. We work by induction on the number of vertices of $G$ in (25). Let us start with an initial general graph $G = G_0$, and suppose it has a certain set $L_0$ of tadpole edges. Consider the graph $G'_0 = G_0 - L_0$ with all tadpoles of $G$ deleted. Since the weights $w_s(G, T)$ cannot depend on the position of the tadpoles edges in the Hepp sector $\sigma$, we have $w_s(G_0, T) = w_s(G'_0, T)$. In $G'_0$, which has no tadpoles, all edges are trans-block at first step (since $\Pi_0 = \Pi^s$ is made of singletons).
Therefore the factor $k_0$ in (22) is $k_0 = |E(G'_0)|$. We can write
\[
\begin{align*}
  w_s(G, T) &= \sum_{\ell_1=\sigma(1)\in T} \frac{1}{k_0} \sum_{\sigma_1|T(\sigma_1)=T-\ell_1} \frac{1}{(k_0-1)!} \\
  &= \sum_{\ell_1=\sigma(1)\in T} \frac{1}{k_0} w_s(G_1, T_1) \\
  &= \sum_{\ell_1=\sigma(1)\in T} \frac{1}{k_0} w^{\Pi_s}(G_1, T_1) = w^{\Pi_s}(G, T). 
\end{align*}
\]
where $\ell_1$ is the first edge of $T_{\tau}$; $G_1$ is obtained by contracting the edge $\ell_1$ in $G'_0$. $T_1$ is obtained by contracting the first edge $\ell_1$ in $T_{\tau}$ and the sum over $\sigma_1$ runs over the Hepp sector of $G_1$. Since $G_1$ has one vertex less than $G_0$ we used the induction hypothesis in the last line of (26) to conclude.

Consider the particular example of the graph of Fig. 2. The symmetric weights are:
\[
\begin{align*}
  w_s(G, T_{125}) &= w_s(G, T_{126}) = w_s(G, T_{156}) = w_s(G, T_{256}) = 1/15, \\
  w_s(G, T_{135}) &= w_s(G, T_{136}) = w_s(G, T_{235}) = w_s(G, T_{236}) = w_s(G, T_{145}) \\
  &= w_s(G, T_{146}) = w_s(G, T_{245}) = w_s(G, T_{246}) = 11/120. 
\end{align*}
\]
These weights were computed in [24] (note the different labeling we use here with respect to the one of [24]).

### 4.2 One singleton partition - rooted graph

The next case we deal with is the one when the partition is made of a certain number $p$ of singletons plus a single block with all other remaining vertices. As already mentioned above, when $p = 1$, the partition corresponds to work on a rooted graph. We obtain weights related to the Brydges-Battle-Federbush constructive QFT approach (see [27, 28, 29]). When the number $p$ of singletons is at least two, the weights correspond to multi-rooted weights (see the example in subsection 4.3).

Consider the graph of Figure 1. Let us find the tree weights in the case of root at $v_1$. This correspond to the partition
\[
\Pi_1 = \{\{v_1\}, \{v_2, v_3\}\}.
\]
Figure 1: An example of a three vertex graph.

There are six admissible ordered trees, namely $T_{12}, T_{13}, T_{14}, T_{21}, T_{23}$ and finally $T_{24}$. The weights are

$$w^{\Pi_1}(G, T_{12}) = w^{\Pi_1}(G, T_{21}) = \int_0^1 du_1 du_2 u_1(u_2)^2 = 1/6$$  \hspace{1cm} (29)

since $i(l_1) = 0, j(l_1) = 1; i(l_2) = 0, j(l_2) = 2; i(l_3) = 1, j(l_3) = 2; i(l_4) = 1, j(l_4) = 2$, where, from now on, we have simplify the notations for the contact indices (since an edge is identified with a pair of (non-necessarily distinct) vertices of the graph).

Similarly, one has

$$w^{\Pi_1}(G, T_{13}) = w^{\Pi_1}(G, T_{14}) = w^{\Pi_1}(G, T_{23}) = w^{\Pi_1}(G, T_{24}) = \int_0^1 du_1 du_2 (u_1 u_2). (u_2) = 1/6$$  \hspace{1cm} (30)

since $i(l_1) = 0, j(l_1) = 1; i(l_2) = 0, j(l_2) = 2; i(l_3) = 1, j(l_3) = 2; i(l_4) = 1, j(l_4) = 2$.

We can check that $\sum_{T \subset G} w^{\Pi_1}(G, T) = 1$.

Let us now count the factors in the case of the root at $v_2$. This corresponds, in the formalism of this paper, to consider the partition:

$$\Pi_2 = [\{v_2\}, \{v_1, v_3\}]$$  \hspace{1cm} (31)

As above, there are seven admissible ordered trees, namely $T_{12}, T_{13}, T_{14}, T_{31}, T_{32}, T_{41}$ and $T_{42}$. The associated weights compute to:

$$w^{\Pi_2}(G, T_{12}) = \int_0^1 du_1 du_2 [1.1.(u_1 u_2). (u_1 u_2)] = 1/9$$  \hspace{1cm} (32)
since \( i(l_1) = 0, j(l_1) = 1; i(l_2) = 1, j(l_2) = 2; i(l_3) = 0, j(l_3) = 2; i(l_4) = 0, j(l_4) = 2 \), and

\[
w^{\Pi_2}(G, T_{13}) = w^{\Pi_2}(G, T_{14}) = \int_0^1 du_1 du_2 [1. u_2. u_1] = 1/9 \tag{33}
\]
since \( i(l_1) = 0, j(l_1) = 1; i(l_2) = 1, j(l_2) = 2; i(l_3) = 0, j(l_3) = 2; i(l_4) = 0, j(l_4) = 2 \). and

\[
w^{\Pi_2}(G, T_{31}) = w^{\Pi_2}(G, T_{41}) = \int_0^1 du_1 du_2 u_1 u_2.1 = 1/6 \tag{34}
\]
since \( i(l_1, T_{31}) = 0, j(l_1, T_{31}) = 2; i(l_2, T_{31}) = 1, j(l_2, T_{31}) = 2; i(l_3, T_{31}) = 0, j(l_3, T_{31}) = 1; i(l_4, T_{31}) = 0, j(l_4, T_{31}) = 1 \). and

\[
w^{\Pi_2}(G, T_{32}) = w^{\Pi_2}(G, T_{42}) = \int_0^1 du_1 du_2 u_1 u_2.1 = 1/6 \tag{35}
\]
since \( i(l_1, T_{32}) = 0, j(l_1, T_{32}) = 2; i(l_2, T_{32}) = 1, j(l_2, T_{32}) = 2; i(l_3, T_{32}) = 0, j(l_3, T_{32}) = 1; i(l_4, T_{32}) = 0, j(l_4, T_{32}) = 1 \). As expected, we have again:

\[
\sum_{T \subseteq G} w^{\Pi_2}(G, T) = 1.
\]

### 4.3 Two singleton partition - multi-rooted graph

We consider the graph of Fig. 2 with four vertices and six edges, for the partition

\[
\Pi = \{\{v_1\}; \{v_2\}; \{v_3, v_4\}\}. \tag{36}
\]

This corresponds to considering a graph with two roots, the first at the vertex \( v_1 \) and the second at the vertex \( v_2 \). This graph has twelve trees:

\[
T_{125}, T_{135}, T_{145}, T_{235}, T_{245},
T_{236}, T_{246}, T_{256},
T_{126}, T_{136}, T_{146}, T_{156}. \tag{37}
\]

The first five trees in this list (the trees in the first line of (37) above) can be each endowed with six admissible orders (each order is for these trees admissible). The next three trees in the list (the trees in the second line of (37)) can be endowed with only four admissible orders, while the last four
trees (the trees in the last line of (37)) can be endowed with three admissible orders. This makes up for a total of fifty-four ordered trans-block trees to consider for this graph.

Let us explicitly consider the first of these ordered trees, namely the \((l_1, l_2, l_5)\) ordered tree. For the three tree lines, the contact indices are: \(i(l_1) = 0, j(l_1) = 1, i(l_2) = 0, j(l_2) = 2\) and \(i(l_5) = 0, j(l_5) = 3\). For the remaining three loop lines, the contact indices are: \(i(l_3) = 0, j(l_3) = 2, i(l_4) = 0, j(l_4) = 2\) and finally \(i(l_6) = 0, j(l_6) = 1\). This leads to the contribution:

\[
\int_0^1 du_1 du_2 du_3 u_1^4 u_2^3 u_3 = \frac{1}{40}.
\] (38)

The other five admissible orders for this tree lead to the weights \(1/80, 1/50, 1/100, 1/100\) and finally, again \(1/100\). Thus, for the total of six admissible orders that one can endow this tree with, one obtains a total weight of \(7/80\).

After a tedious but straightforward computation, we obtain all forty-eight admissible order contributions and find the complete list of all tree weights for this partition:

\[
\begin{align*}
    w^\Pi(G, T_{135}) &= w^\Pi(G, T_{145}) = 47/400, \\
    w^\Pi(G, T_{235}) &= w^\Pi(G, T_{245}) = 11/100, \\
    w^\Pi(G, T_{236}) &= w^\Pi(G, T_{246}) = 2/25, \\
    w^\Pi(G, T_{136}) &= w^\Pi(G, T_{146}) = 3/40, \\
    w^\Pi(G, T_{256}) &= 1/20, \\
    w^\Pi(G, T_{126}) &= 11/200, \\
    w^\Pi(G, T_{125}) &= 7/80, \\
    w^\Pi(G, T_{156}) &= 17/400.
\end{align*}
\] (39)

Note that these tree weights are different from the symmetric weights of (27). Finally, one can check that \(\sum_{T \in G} w^\Pi(G, T) = 1\), as expected.
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