Parameterized Complexity of Fair Vertex Evaluation Problems

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Abstract

A prototypical graph problem is centered around a graph theoretic property for a set of vertices and a solution is a set of vertices for which the desired property holds. The task is to decide whether, in the given graph, there exists a solution of certain quality, where we use size as a quality measure. In this work we are changing the measure to the fair measure [Lin&Sahni: Fair edge deletion problems. IEEE Trans. Comput. 89]. The measure is $k$ if the number of solution neighbors does not exceed $k$ for any vertex in the graph. One possible way to study graph problems is by defining the property in a certain logic. For a given objective an evaluation problem is to find a set (of vertices) that simultaneously minimizes the assumed measure and satisfies an appropriate formula.

In the presented paper we show that there is an FPT algorithm for the MSO FAIR VERTEX EVALUATION problem for formulas with one free variable parameterized by the twin cover number of the input graph. Here, the free variable corresponds to the solution sought. One may define an extended variant of MSO FAIR VERTEX EVALUATION for formulas with $\ell$ free variables; here we measure maximum number of neighbors in each of the $\ell$ sets. However, such variant is $\text{W}[1]$-hard even on graphs with twin cover one. Furthermore, we study the FAIR VERTEX COVER (FAIR VC) problem. FAIR VC is among the simplest problems with respect to the demanded property (i.e., the rest forms an edgeless graph). On the negative side, FAIR VC is $\text{W}[1]$-hard when parameterized by both treedepth and feedback vertex set of the input graph. On the positive side, we provide an FPT algorithm for the parameter modular width.

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Introduction

Undoubtedly, Courcelle’s Theorem [7] for graph properties expressible in the monadic second order logic (MSO) on graphs of bounded treewidth plays a prime role among model checking algorithms. In particular, Courcelle’s Theorem gives for an MSO sentence $\varphi$ an algorithm that given an $n$-vertex graph $G$ with treewidth $k$ decides whether $\varphi$ holds in $G$ in time $f(k, |\varphi|)n$, where $f$ is some computable function. In terms of parameterized complexity, a natural complexity framework for analysis of model checking, such an algorithm is called fixed parameter tractable (the problem belongs to the class FPT). We refer the reader to monographs [11, 12] on parameterized complexity theory and algorithm design. Of course, this is not the only FPT model checking algorithm – an algorithm for (existential counting) modal logic model checking on graphs of bounded treewidth [38], MSO model checking on graphs of bounded neighborhood diversity [29], or MSO model checking on graph of bounded shrubdepth [19] are known. First order logic (FO) model checking received quite some attention as well and algorithms for graphs with bounded degree [39], nowhere dense graphs [22], and some dense graph classes [16] were given.

Kernelization is one of the most popular techniques for designing FPT algorithms. Kernelization is a preprocessing routine running in polynomial time that reduces the input instance to an equivalent one whose size and parameter value can be upper-bounded in terms of the input parameter value. It should be noted that kernelization is not really common for model checking algorithms. However, there are a few notable exceptions – the result of Lampis [29] was, up to our knowledge, the first result of this kind and was followed by a recent result of Gajarský and Hliněný [15]. It is worth noting that it was possible to use the kernel of [29] to extend his model checking algorithm for so called fair objectives and their generalizations [33, 25]. In this work we continue this line of research.

There are several possible research directions once a model checking algorithm is known. One possibility is to broaden the result either by enlarging the class of graphs it works for [8] or by extending the expressive power of the concerned logic, e.g. by introducing a new predicate [27]. Another obvious possibility is to find the exact complexity of the general model checking problem by providing better algorithms (e.g. for subclasses [29]) and/or lower-bounds for the problem [14, 30]. Finally, one may instead of deciding a sentence turn an attention to finding a set satisfying a formula with a free variable that is optimal with respect to some objective function [2, 9, 20]. In this work we take the last presented approach – we are extending previous work on MSO model checking for so called fair objectives.

When extending a model checking result to incorporate an objective function or a predicate, we face two substantial difficulties. On the one hand, we are trying to introduce as strong extension as possible, while on the other, we try not to worsen the running time too much. Of course, these two are in a clash. One evident possibility is to sacrifice the running time and obtain an XP algorithm, that is an algorithm running in time $f(|\varphi|)G^{\varphi(k)}$. As an example there is an XP algorithm for the Fair MSO$_2$ Vertex Evaluation problems parameterized by treewidth (and the formula) by Kolman et al. [26]. An orthogonal extension is due to Szeider [40] for so called local cardinality constraints (MSO-LCC) who gave an XP algorithm parameterized by treewidth. If we decide to keep the FPT running time, such result are not possible as we give W[1]-hardness result for these here (this was already known [33]). However, Ganian and Obdržálek [20] study CardMSO and provide an FPT algorithm parameterized by neighborhood diversity. Recently, Masařík and Toufar [34] studied fair objective cost and provide an FPT algorithm for the Fair MSO$_1$ Vertex Evaluation problem parameterized by neighborhood diversity. See [25] for comprehensive
discussion of various extensions of MSO.

Before we give the formal definition of the problems we are dealing with, we first introduce (fair) deletion problems, motivate their study, and relate them to model checking. Deletion problems are standard reformulation of some classical problems in combinatorial optimization introduced by Yannakakis [43]. For a graph property \( \pi \) we formulate a vertex deletion problem. That means, given a graph \( G = (V, E) \), find the smallest possible set of vertices \( W \) such that \( G \setminus W \) satisfies property \( \pi \). Many classical problems can be formulated in this way such as Minimum Vertex Cover (here the task is to obtain an edgeless graph) or Minimum Feedback Vertex Set (now the graph has to be a forest). Of course, the complexity is determined by the desired property \( \pi \) but most of these problems are NP-complete [42, 1, 28].

**Fair Objective.** Fair deletion problems are such modifications where instead of minimizing the size of the deleted set we change the objective in the following way (introduced by Lin and Sahni [32]). The Fair Vertex Deletion problem is defined as follows. For a given graph \( G = (V, E) \) and a property \( \pi \), the task is to find a set \( W \subseteq V \) which minimizes the maximum number of neighbors in the set \( W \) over all vertices, such that the property \( \pi \) holds in \( G \setminus W \). Intuitively, the solution should not be concentrated in a neighborhood of any vertex. In order to classify (fair) vertex deletion problems we study the associated decision version, that is, we are interested in finding a set \( W \) of size at most \( k \), for a given number \( k \). Note that this can introduce only polynomial slowdown as the value of our objective is bounded by 0 from below and by the number of vertices of the input graph from above. Since we are about to use a formula with a free variable \( X \) to express the desired property \( \pi \), we naturally use \( X \) to represent the set of deleted vertices in the formula.

The fair cost of a set \( W \subseteq V \) is defined as \( \max_{v \in V} |N(v) \cap W| \). We refer to the function that assigns each set \( W \subseteq V \) its fair cost as the fair objective function. Here, we give a formal definition of the computational problem when the property \( \pi \) is defined in some logic \( L \).

| FAIR L VERTEX DELETION |
|------------------------|
| **Input:** An undirected graph \( G = (V, E) \) and a positive integer \( k \). |
| **Fixed:** An \( L \) sentence \( \varphi \). |
| **Question:** Is there a set \( W \subseteq V \) such that \( G \setminus W \models \varphi \) with fair cost at most \( k \)? |

With respect to fair objective we mainly focus on formulas with one free variable. Let \( \varphi(X) \) be a formula with one free variable and let \( G = (V, E) \) be a graph. For a set \( W \subseteq V \) by \( \varphi(W) \) we mean that we substitute \( W \) for \( X \) in \( \varphi \).

| FAIR L VERTEX EVALUATION\(^4\) |
|-----------------------------|
| **Input:** An undirected graph \( G = (V, E) \) and a positive integer \( k \). |
| **Fixed:** An \( L \) formula \( \varphi(X) \) with one free variable. |
| **Question:** Is there a set \( W \subseteq V \) such that \( G \models \varphi(W) \) having fair cost at most \( k \)? |

We would like to point out one crucial difference between deletion and evaluation problems, namely that in evaluation problem we have access to the variable that represents the solution. This on the one hand enables evaluation problems to impose some conditions on the solution (think of e.g. connectivity), while on the other hand the logic \( L \) must be able to handle the variable representing the solution. Thus, FAIR FO EDGE EVALUATION does not make sense, since one cannot introduce (and thus use) an edge set variable in FO. Observe that FAIR FO

\(^4\) This problem is called GENERALIZED FAIR L VERTEX-DELETION in [34]. However, we believe that evaluation is a more suitable expression and coincides with standard terminology in logic.
Edge Deletion is correctly defined, since we first delete a set of edges and only then we check that an FO sentence holds.

Since classical Courcelle’s theorem [7] have an exponential tower dependence on the quantifier depth of the MSO formula, it could be interesting to find a more efficient algorithm. However, Frick and Grohe [14] proved that this dependence is unavoidable, unless P = NP. On the other hand, there are classes where MSO model checking can be done in asymptotically faster time, e.g. the neighborhood diversity admits a single-exponential dependence on the quantifier depth of the formula [29]. For a recent survey of algorithmic metatheorems consult the article by Grohe and Kreutzer [21]. Fair Vertex L Evaluation problems have many potential applications. We want to turn a particular attention to the Fair VC problem which, besides its natural connection with the Vertex Cover problem, has interesting properties. For example we can think about classical vertex cover as having several crossroads (vertices) and roads (edges) that we need to monitor. However, people could get uneasy if they will see too many policemen from one single crossroad. In contrast, if the vertex cover has low fair cost, it covers all roads while keeping a low number of policemen in the neighborhood of every single crossroad.

1.1 Our Results

We give a metatheorem for graphs having bounded twin cover. Twin cover (introduced by Ganian [17]) is one possible generalization of vertex cover number. Here we measure the number of vertices needed to cover all edges that are not twin-edges; an edge \{u, v\} is a twin-edge if \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \). Ganian introduced twin cover in hope that it should be possible to extend algorithms designed for parameterization by vertex cover number to a broader class of graphs.

Theorem 1. The Fair MSO₁ Vertex Evaluation problem parameterized by twin cover number admits an FPT algorithm.

We would like to point out here that in order to obtain this result we have to reprove the original result of Ganian [18] for MSO₁ model checking on graphs of bounded twin cover. For this we extend arguments given by Lampis [30] in the design of an FPT algorithm MSO₁ model checking for graphs of bounded neighborhood diversity. We do this in order to obtain more insight into the structure of the graph on which model checking is performed (its size is bounded by a function of the parameter). This in turn allows us to find a solution minimizing the fair cost and satisfying the fixed MSO₁ formula. The result by Ganian in version [17] is based on the fact that graphs of bounded twin cover have bounded shrubdepth and so MSO₁ model checking algorithm on shrubdepth ([19]) can be used.

When proving hardness results it is convenient to show the hardness result for a concrete problem that is expressible by an MSO₁ formula, yet as simple as possible. Therefore, we introduce a key problem for Fair Vertex Deletion – the Fair VC problem.

| Fair Vertex Cover (Fair VC) |
|-----------------------------|
| **Input:** An undirected graph \( G = (V, E) \), and a positive integer \( k \). |
| **Question:** Is there a set \( W \subseteq V \) such that \( G \setminus W \) is an independent set and for every vertex \( v \) of \( G \), it holds that \( |N(v) \cap W| \leq k \)? |

We begin the description of our results with the hardness result and then we move our attention to positive results.
Table 1 The table summarize related (with a citation) and all the presented (with a reference) results on the studied parameters. Green cell denote FPT results and red cell represents hardness result. Logic L in metatheorems is specified by a logic used in the respective theorem. Symbol L∅ denotes any logic that can express an edgeless graph. Symbol * denotes implied results from previous research and symbol ✓ denotes new implied results. Question mark (?) indicates that the complexity is unknown. The Fair Edge L Deletion problem is delimited from Vertex problems since there are no obvious relations in between them. The discussion about it is postponed to Section 5.

| Problem          | vc | fvs + td | tc | nd | cvd | mw |
|------------------|----|----------|----|----|-----|----|
| Fair VC          | ✓  | ✓        | ✓  | ✓  | ✓   | ✓  |
| FV L Del         | MSO2 * | L0 ✓  | MSO1 ✓ | MSO1 * | ✓   | MSO1 | ✓ |
| FV L Eval        | MSO2 [34] | L0 ✓  | MSO1 T1 | MSO1 * | ✓   | ✓   | ✓ |
| ℓ-FV L Eval      | MSO1 * | L0 ✓  | MSO1 T4 | MSO1 [25] | MSO1 ✓ | MSO1 ✓ |
| FE L Del         | MSO2 [34] | FO [34] | ✓   | ✓  | ✓   | ✓  |

Theorem 2. The Fair VC problem parameterized by treedepth td(G) and feedback vertex set fvs(G) combined is W[1]-hard. Moreover, if there exists an algorithm running in time \( f(w)n^{o(\sqrt{w})} \), where \( w := \text{td}(G) + \text{fvs}(G) \), then Exponential Time Hypothesis fails.

This simpler problem allows us to derive hardness results with respect to the same parameterization and improves the lower bound under Exponential Time Hypothesis [23]; previously \( n^{o(w^{1/2})} \) given by Masařík and Toufar [34]. The hardness result highly relies on the structure of sparse graphs and refutes existence of a metatheorem for essentially every logic we can find. On the other hand we show that Fair VC can be solved efficiently in dense graph models:

Theorem 3. The Fair VC problem parameterized by modular width mw(G) admits an FPT algorithm with running time \( 2^{\text{mw}(G)} \text{mw}(G)n^3 \), where \( n \) is the number of vertices in \( G \).

We point out that the Fair VC problem is (rather trivially) AND-compositional and thus it does not admit a polynomial kernel for parameterization by modular width (see Lemma 14). Moreover, an analogue to Theorem 3 cannot hold for parameterization by shrubdepth of the input graph. This is a consequence of Theorem 2 and [19, Proposition 3.4].

One may naturally generalize the Fair L Vertex Evaluation problem in such a way that the formula \( \varphi \) has \( \ell \) free variables; let \( \varphi(X_1, \ldots, X_\ell) \) be the formula and \( W_1, \ldots, W_\ell \) the corresponding sets in \( G \). Denote the problem \( \ell \)-Fair L Vertex Evaluation. Now, the fair cost of \( W_1, \ldots, W_\ell \) is \( \max_{v \in V} \max_{i \in [\ell]} |N(v) \cap W_i| \). In [25] this is done in more general and an FPT algorithm is presented therein in context of MSO\(_{\text{lin}}\); there it is possible to specify both lower- and upper-bound for each vertex and each set \( W_i \). However, here we show that such generalization is not possible for twin cover.

Theorem 4. The \( \ell \)-Fair L Vertex Evaluation problem is W[1]-hard for parameter \( \ell \) even on graphs with twin cover of size one. Moreover, if there exists an algorithm running in time \( f(\ell)n^{O(\ell/\log \ell)} \), then Exponential Time Hypothesis fails.

For an overview of the results please refer to Table 1 and to Figure 1 for the classes hierarchy.

1.2 Preliminaries

We denote the set \( \{1, \ldots, n\} \) by \( [n] \). We deal with simple undirected graphs, for further standard notation we refer to monographs: graph theory [35], parameterized complexity [11]. We
Figure 1  Hierarchy of graph parameters with depicted complexity of the Fair L Vertex Evaluation problem. An arrow indicates that a graph parameter upper-bounds the other. Thus, hardness results are implied in direction of arrows and FPT algorithms are implied in the reverse direction. Green color indicates FPT results for MSO$_2$, orange are FPT for MSO$_1$, blue are open, and red are hardness results. We denote treewidth by $tw$, shrubdepth by $sd$, and clique width by $cw$. We refer to book [11] for definitions. Other parameters and their respective shorthands are defined in Subsection 1.2.

denote the neighborhood of vertex $v$ by $N(v)$ and by $N[v]$ we denote the closed neighborhood of vertex $v$, i.e., $N(v) \cup \{v\}$.

A parameter closely related to twin cover is the cluster vertex deletion ($cvd(G)$), that is the smallest number of vertices one has to delete from a graph in order to get a collection of (disjoint) cliques. Treedepth of a graph $G$ ($td(G)$) is the minimum height of a rooted forest whose transitive closure contains the graph $G$ [36]. Feedback vertex set ($fvs(G)$) is the minimum number of vertices of a graph $G$ whose removal leaves a graph without cycles. The modular width of a graph $G$ ($mw(G)$), is the smallest positive integer $r$ such that $G$ can be obtained from an algebraic expression of width at most $r$, defined as follows. The width of expression $A$ is the maximum number of operands used by any occurrence of substitution operation $A$, where $A$ is an algebraic expression that uses the following operations:

1. Create an isolated vertex.
2. The substitution operation with respect to a template graph $T$ with vertex set $[r]$ and graphs $G_1, \ldots, G_r$ created by algebraic expression. The substitution operation, denoted by $T(G_1, \ldots, G_r)$, results in the graph on vertex set $V = V_1 \cup \cdots \cup V_r$ and edge set $E = E_1 \cup \cdots \cup E_r \cup \bigcup_{(i,j) \in E(T)} \{\{u,v\} : u \in V_i, v \in V_j\}$, where $G_i = (V_i, E_i)$ for all $i \in [r]$.

An algebraic expression of width $mw(G)$ can be computed in linear time [41]. A more restrictive way how to create a graph is to allow only one substitution operation with respect to a template graph $T$ such that every $G_i$ is either a clique or an edgeless graph. The smallest integer $r$ such that the graph $G$ can be created in such a way is the neighborhood diversity of a graph $G$ ($nd(G)$).

2 Metatheorem for Fair Evaluation

First we show an FPT algorithm for the fair MSO$_1$ vertex evaluation problem parameterized by the twin cover number as it is stated in Theorem 1. We give a more detailed statement that implies the promised result straightforwardly. We split the proof in two parts. First we show an algorithm for MSO$_1$ model checking parameterized by twin cover of the graph (Proposition 7). In the second part we prove that we can compute the fair cost of the smaller graphs (Proposition 11) and so derive the promised result.

Overview of the Algorithm. For the model checking algorithm, we closely follow the approach of Lampis [30]. The idea is that if there is a large set of vertices with the same closed neighborhood, then some of them are irrelevant, i.e., we can delete them without affecting the validity of the given formula $\varphi$. For graphs of bounded neighborhood diversity using this rule alone can reduce the number of vertices below a bound that depends on $nd(G)$ and $|\varphi|$ only, thus providing an FPT model checking algorithm. For the graphs of bounded
twin cover, this approach can be used to reduce the size of all cliques, yet their number can still be large. We take the approach one step further and describe deletion of irrelevant cliques in a similar manner; these rules together yield a model checking algorithm for graphs of bounded twin cover.

The reduction rules also lead to a notion of shape of a set \( W \subseteq V \). The motivation behind shapes is to partition all subsets of \( V \) such that if two sets \( W, W' \) have the same shape, then \( G \models \varphi(W) \) if and only if \( G \models \varphi(W') \). This allows us to consider only one set of each shape for the purposes of model checking. Since the number of all distinct shapes is bounded by some function of parameters, we can essentially brute force through all possible shapes.

A final ingredient is an algorithm that for given shape outputs a subset of vertices with this shape that minimizes the fair cost. This algorithm uses ILP techniques, in particular minimizing quasiconvex function subject to linear constraints.

**Notation.** In what follows \( G \) is a graph and \( K \) is its twin cover of size \( k \). An \( \text{MSO}_1 \) formula \( \varphi \) contains \( q_S \) set quantifiers and \( q_v \) element (vertex) quantifiers. Given a twin cover \( K \) and \( A \subseteq K \), we say that \( A \) is the cover set of a set \( S \subseteq V \setminus K \) if every \( v \in S \) has \( N(v) \cap K = A \). Note that, by definition of twin cover, every vertex in a clique outside the cover has the same cover set and also that set \( A \) is the cover set of a disjoint union of cliques. We say that two cliques have the same type if they have the same size and the same cover set. Clearly, if the cover set is fixed, two cliques agree on type if and only if their sizes are the same. We define a labeled graph, that is, a graph and a collection of labels on the vertices. Those labels represent an assignment of a formula. We say that two cliques have the same labeled type if all of them have the same size, the same cover set and the same labels on vertices.

**2.1 Model checking**

We give a reformulated combination of Lemma 5 and Theorem 4 by Lampis [29].

▶ **Proposition 5 ([29, Lemma 5 and Theorem 4]).** Let \( \varphi \) be a fixed \( \text{MSO}_1 \) formula and let \( G \) be a labeled graph. If there is a set \( S \) of more than \( 2^{q_S} q_v \) vertices having the same closed neighborhood and the same labels, then for any \( v \in S \) we have \( G \models \varphi \) if and only if \( G \setminus v \models \varphi \).

In particular, if \( G \) is a graph with just one label, then for any clique \( C \) where each vertex has exactly the same closed neighborhood in \( G \) the following holds. Either there is a vertex \( v \in C \) such that \( G \models \varphi \) if and only if \( G \setminus v \models \varphi \) or the size of \( C \) is bounded by

\[
2^{q_S+1} q_v.
\]

Proposition 5 bounds the size of a maximum clique in \( G \setminus K \) because we can apply it repeatedly for each clique that is bigger than the threshold. Now, we need to bound the number of cliques of each type. For this we establish the following technical lemma.

▶ **Lemma 6 ( ).** Let \( G \) be a labeled graph with twin cover \( K \) and with the size of maximum clique in \( G \setminus K \) bounded by \( \ell \). Let \( \varphi \) be a fixed \( \text{MSO}_1 \) formula with \( q_v \) element quantifiers and \( q_S \) set quantifiers. If there are strictly more than

\[
\alpha(q_S, q_v) = 2^{q_S}(q_v + 1)
\]

cliques of the same labeled type \( T \) then there exists a clique \( C \) of the labeled type \( T \) such that \( G \models \varphi \) if and only if \( G \setminus C \models \varphi \).

. We prove the lemma by induction on \( q_S + q_v \). Without loss of generality all the quantifiers are assumed to be existential.
The base case of the induction is a quantifier free formula. If there is at least one labeled type with at least two cliques $C_1, C_2$ then the following holds. If $G \models \varphi$ then $G \setminus C_1 \models \varphi$ clearly holds as well since clique $C_2$ has the same cover set and the same labels and a quantifier free formula can only examine the labeled vertices. If $(G \setminus C_1) \models \varphi$ and since $C_1, C_2$ have the same labels the same size and the same cover set so $G \models \varphi$.

For the induction case we consider the first quantified variable in $\varphi$ and we split the proof whether it is set or vertex variable. Suppose there is at least one labeled type which contains strictly more than $\alpha(q_S, q_v)$ cliques.

If it is a set variable then we try all possible assignments of the variable to cliques of the chosen labeled type. There are at most $2^k$ of possible assignments to single clique and so from cliques of one labeled type emerge at most $2^k$ different labeled (sub)types of cliques. We can compute that at least one of them has strictly more than $\alpha(q_S - 1, q_v)$:

$$\left\lceil \frac{\alpha(q_S, q_v) + 1}{2^k} \right\rceil \geq \left\lceil 2^k(q_S - 1)(q_v + 1) + \frac{1}{2^k} \right\rceil \geq \alpha(q_S - 1, q_v) + 1.$$  

So, by the induction assumption we know that there is a clique $C$ in the newly created labeled (sub)type of the promised properties and so of the larger labeled type.

If it is a vertex variable then only one more different labeled type can be created and importantly at most one single clique may contain the new label. We can compute:

$$\alpha(q_S, q_v) + 1 - 1 \geq 2^{q_S}q_v + 2^{q_S} \geq \alpha(q_S, q_v - 1) + 1.$$  

The argument follows from the induction assumption by the same reasoning as in the previous case.

From this we can derive a model checking algorithm.

**Proposition 7 (Model checking on graphs of bounded twin cover).** Let $G$ be a graph with twin cover $K$ of size $k$ and the size of the maximum clique in $G \setminus K$ bounded by $2^\ell q_v$ and $\varphi$ is an MSO$_1$ sentence then either there exists a clique $C \in G \setminus K$ such that $G \models \varphi$ if and only if $G \setminus C \models \varphi$ or the size of $G$ is bounded by

$$k + (q_v + 1)2^{2k + 2q_S + 2^\ell q_v}q_v = 2^{O(k + 2^{1/2}q_S q_v)}.$$  

There are $k$ vertices in the cover and $2^k \ell$ types of cliques and each of them (by Lemma 6) is repeated at most $\alpha(q_S, q_v) = 2^{q_S}(q_v + 1)$ otherwise one clique of that type cannot be distinguished by formula $\varphi$. The maximal size of the clique is $\ell = 2^\ell q_v$ from Proposition 5 and this gives us the desired bound.

$$k + 2^k \ell^2 \alpha(q_S, q_v) = k + 2^k \ell^2 2^{q_S q_v} + (q_v + 1) = k + 2^k(2^{q_S q_v})^2 2^{q_S q_v} (q_v + 1) = k + (q_v + 1)q_v^2 2^{k + 2q_S + 2^{1/2}q_S q_v}.$$  

\[\square\]

### 2.2 Finding a Fair Solution

In the upcoming proof we follow the ideas of Masařík and Toufar [34]. They define, for a given formula $\varphi(X)$, a so called shape of a set $W \subseteq V$ in $G$.

The idea behind a shape is that in order to do the model checking we have deleted some vertices from $G$ that cannot change the outcome of $\varphi(X)$, however, we have to derive a
solution of minimal cost in the whole graph $G$. Thus shape characterizes a set under which $\varphi(X)$ holds and we have to be able to find a set $W \subseteq V(G)$ for which $\varphi(W)$ holds and $W$ minimizes the fair cost among all set having this shape.

**Shape.** Let $G = (V, E)$ be a graph, $\varphi(X)$ an MSO formula, $K \subseteq V$ a twin cover of $G$, $A \subseteq K$, and let $\ell = 2^{q_0+1}q_0$ and $\alpha = 2^{q_0}(q_0 + 1)$. Let $C$ be the collection of all cliques in $G$ such that $A$ is their cover set. We define an $A$-shape. An $A$-shape of size $\ell$ is a two dimensional table $S_A$ of dimension $(\ell + 2) \times (\ell + 2)$ indexed by $[0, 1, \ldots, \ell + 1] \times [0, 1, \ldots, \ell + 1]$. Each entry $S_A(i, j) \in [0, \ldots, \alpha + 1]$. The interpretation of $S_A(i, j)$ is the minimum of $\alpha + 1$ and of the number of cliques $C$ with $N(C) = A$ such that

$$\min(\alpha + 1, |C \cap W|) = i \text{ and } \min(\alpha + 1, |C \setminus W|) = j.$$  

Finally, the shape of $X$ in $G$ is a union of $A$-shapes for all $A \subseteq K$.

A solution for $C$ with cover set $A$ can be formally described by a function $\text{sol}: C \rightarrow \mathbb{N} \times \mathbb{N}$. The solution $\text{sol}$ is valid if for every $C \in C$ with $\text{sol}(C) = (i, j)$ either $i + j = |C|$ or $|C| \geq \ell$, $i = \ell + 1$ (or equivalently $j = \ell + 1$), and $i + j < |C|$. For an illustration of a valid assignment please refer to Figure 2. We say that a valid solution is compatible with the shape $S_A$ if $S(i, j) = |\text{sol}^{-1}(i, j)|$, whenever $S(i, j) \leq \alpha$ and $|\text{sol}^{-1}(i, j)| \geq \alpha + 1$ if $S(i, j) = \alpha + 1$. The $A$-shape $S_A$ is said to be valid if there exists a valid solution for $S_A$. Note that such a solution does not exists if the shape specifies too many (or too few) cliques of certain sizes. The shape $S$ is valid if all its $A$-shapes are valid.

The following lemma is a key observation about shapes.

> **Lemma 8.** Let $\varphi$ be a fixed MSO$_1$ formula with one free variable, $G$ a graph and $W, W'$ two subsets of vertices having the same shape. Then $G \models \varphi(W)$ if and only if $G \models \varphi(W')$.

The proof follows using Proposition 5 and Lemma 6. Indeed, if we take the graph $G$ with one label corresponding to set $W$ and apply the reduction rules given by Proposition 5 and Lemma 6 and repeat the same process with $W'$, we obtain two isomorphic labeled graphs.

Lemma 8 allows us to say that a formula with one free variable holds under a shape, since it is irrelevant which subset of vertices of this particular shape is assigned to the free variable. Also note that deciding whether the formula holds under the shape can be done in FPT time by simply picking arbitrary assignment of the given shape and running a model checking algorithm. Lemma 10 computes a solution of minimal cost for an $A$-shape. We do this by reducing the task to integer linear programming (ILP) while using non-linear objective:

> **Theorem 9** ([37] – simplified). Integer linear programming with objective minimization of a quasiconvex function in dimension $p$ is FPT with respect to $p$ and space exponential in $L$ the length of encoding of the ILP instance.

> **Lemma 10.** Let $G = (V, E)$ be a graph, $K$ be a twin cover of $G$, and $\emptyset \neq A \subseteq K$. There is an algorithm that given an $A$-shape $S_A$ of size $\ell$ computes a valid solution for $S_A$ of minimal cost in time $f(|K|, \ell) \cdot |G|^{O(1)}$ or reports that $S_A$ is not valid.

**Proof.** Let $C$ be the collection of all cliques such that $A$ is their cover set. We split the task of finding a minimal solution to $S_A$ into two independent parts depending on the size of cliques assigned in the phase.

The first phase is for cliques in $C$ with sizes at most $\ell$. Observe that these can be assigned deterministically in a greedy way. This is because no cell of $S_A$ is shared by two sizes and we can see that if there are more cells with value $\alpha$ on the corresponding diagonal we can always prefer the top one as this minimizes the cost (see Figure 3). However, this is not
We are about to minimize the following objective
\[
\sum_{q \in Q} x_{i,j}^q \geq |S_A(i,j)| \quad \forall 1 \leq i,j \leq \ell + 1
\]
\[
\sum_{1 \leq i,j \leq \ell+1} x_{i,j}^q = |C_q| \quad \forall q \in Q
\]
\[
x_{i,j}^q \geq 0 \quad \forall 1 \leq i,j \leq \ell + 1, \forall q \in Q
\]
We are about to minimize the following objective
\[
\sum_{1 \leq i,j \leq \ell+1} \sum_{1 \leq q \leq 2\ell} j \cdot x_{i,j}^q + (\ell + 1)\cdot x_{\ell+1,\ell+1}^\text{max} + g \left( x_{\ell+1,1}^\text{max}, \ldots, x_{\ell+1,\ell}^\text{max} \right),
\]
where \(g: \mathbb{N}^\ell \to \mathbb{N}\) is a function that has access to sizes of all cliques in \(C_{\text{max}}\) and computes the minimum possible assignment. We now claim that the function \(g\) is a separable convex function in variables \(x_{\ell+1,1}^\text{max}, \ldots, x_{\ell+1,\ell}^\text{max}\). The result then follows from Theorem 9 as the number of integral variables is \(O(\ell^2)\).

To see this, observe that the value of \(g\left(x_{\ell+1,1}^\text{max}, \ldots, x_{\ell+1,\ell}^\text{max}\right)\) is equal to sum of sizes of cliques “assigned to the last row” minus \(\sum_{i=1}^{\ell} x_{\ell+1,i}^\text{max}\). We order cliques in \(C_{\text{max}}\) according

---

**Figure 2** Example of a \(7 \times 7\) \(A\)-shape.
All cliques of size 3 will be assigned to yellow (light gray) fields, while cliques of size 8 will be assigned to orange (darker gray) fields.

**Figure 3** An example of uncertainty in computation of objective function. The value in the last row depends on the size of the clique we are assigning to those cells. The value in the cell is how much we pay for any compatible clique assigned to this cell.

possible for larger cliques as they may in general share some cells of \(S_A\) and thus we defer them to the second phase.

Now observe that the most important vertices for computing the cost are the vertices constituting the set \(A\). To see this just note that all other vertices see only \(A\) and their neighborhood (a clique) which is at least as large as for the vertices in \(A\). It follows that we should only care about the number of selected vertices such that \(A\) is their cover set. Thus if the size of all cliques in \(C\) is bounded in terms of \(k\) we are done. Alas, this is not the case.

We split the set \(C\) into sets \(C_1, \ldots, C_{2\ell}\), and \(C_{\text{max}}\). A clique \(C \in C\) belongs to \(C_{\ell+1}\) if \(1 \leq |C| \leq 2\ell\) and belongs to \(C_{\text{max}}\) otherwise. Note that cliques from \(C_{\text{max}}\) may be assigned only to cells having at least one index \(\ell + 1\). As mentioned we are about to design an ILP with a non-linear objective function. This ILP has variables \(x_{i,j}^q\) that express the number of cliques from the set \(C_q\) assigned to the cell \((i,j)\) of \(S_A\) (that is \(1 \leq i,j \leq \ell + 1\) and \(q \in Q = \{1, \ldots, 2\ell\} \cup \{\text{max}\}\)). The obvious conditions are the following (the \(\geq\) symbol translates to \(=\) if \(S(i,j) \leq \alpha\) while it translates to \(\geq\) if \(S(i,j) = \alpha + 1\)).
to their size, that is, $|C_1| \leq |C_2| \leq \cdots \leq |C_t|$, where $t$ denotes the size of $c_{\text{max}}$. Now, $g\left(x_{\text{max}}^{\ell+1}, \ldots, x_{\text{max}}^{|L|}\right) = g^i\left(\sum_{i=1}^\ell x_{\text{max}}^{\ell+1,j}\right) - \sum_{i=1}^\ell x_{\text{max}}^{\ell+1,i}$. This finishes the proof, since $x^i$ is a convex function; see Remark in Section 2.2.1 for details.

Now we are ready to prove the main result of this section. It essentially follows by exhaustive search among all possible shapes $S$ such that $\varphi$ is true under $S$ and application of Lemma 10.

**Proposition 11.** Let $G = (V, E)$ be a graph with twin cover $K$ of size $k$. For a fixed MSO$_1$ formula $\varphi(X)$ with one free variable that represents the set to be deleted it is possible to find a set $W \subseteq V$ such that

- $\varphi(W)$ holds in $G$ and
- the cost of $W$ is minimized among all subset of $V$ satisfying $\varphi(X)$ in time $f(k, |\varphi|)|V|^{O(1)}$ for some computable function $f$.

**Proof of Proposition 11.** We proceed as follows. For every possible selection of $K \cap W$ we generate all possible shapes and check whether $\varphi(X)$ evaluates to true under shape $S$ and if so, we compute $W$ for $S$ having the minimal fair-cost. Finally, we return the set $X$ minimizing the cost.

We stress that Lemma 6 applies only in cases where the cover set of the cliques at hand is not empty. Thus, these cliques are exceptional and we have to handle them separately. Note that in this case the objective is different – we want to minimize the maximum number of selected vertices in any clique. It is not hard, however, to do this via standard tricks.

In order to summarize the running time we have

- at most $2^k$ possible selections for $W \cap K$ and
- at most $2^k \cdot (\alpha + 1)^{\ell+1}$ possible shapes (for each such selection).

For every shape $S$ we label the graph according to $S$. Afterwards, apply Lemma 6 and proposition 5 exhaustively on the labeled graph to resolve whether $\varphi(X)$ evaluates to true under $S$. If $\varphi(X)$ was evaluated to true, then using Lemma 10 on every possible $A$ we obtain a set $W_A$ minimizing the cost for vertices in $A$ and put $W = \bigcup_{A \subseteq K} W_A$. Observe that this union gives optimal cost for the selected shape $S$. Finally we return the set $W$ minimizing the cost for any true evaluated shape. Clearly this routine runs in FPT time with respect to $k$ and $|\varphi|$ as parameters.

**2.2.1 Polynomial Space Version of Proposition 11**

We now argue that it is possible to implement Lemma 10 in polynomial space via reducing it to integer linear programming. We note that similar application of this technique is recently presented by Bodereck et al. [4].

**Theorem 12 (Lenstra & Frank, Tardós [31, 13]).** There is an algorithm that given an ILP with $p$ variables finds an optimal solution to it using $O(p^2 5^p \text{poly}(L))$ arithmetic operations and space polynomial in $L$, where $L$ is the bitsize of the ILP.

**Remark.** Let $G = (V, E)$ be a graph, $K$ be a twin cover of $G$, and $A \subseteq K$. There is an algorithm that given an $A$-shape $S_A$ of size $\ell$ computes a valid solution for $S_A$ of minimal cost in time $f(|K|, \ell) \cdot |G|^{O(1)}$ and space polynomial in $|G|$ or reports that $S_A$ is not valid.

**Proof.** Here the only difference is that we have to rewrite the function $g(x_{\text{max}}^{\ell+1}, \ldots, x_{\text{max}}^{|L|})$ it using new variable $y$ (representing its value) into constraints of ILP and thus obtain a linear objective.
We do this by adding a variable \( y \geq 0 \) with constraint \( y = \sum_{j=1}^{t} x_{i_j} \) and work with univariate function \( g(y) \) instead. We order cliques in \( C_{\text{max}} \) according to their size, that is, \( |C_1| \leq |C_2| \leq \cdots \leq |C_t| \), where \( t \) denotes the size of \( C_{\text{max}} \). By \( c_i \), we denote the sum \( \sum_{j=1}^{t} |C_{i_j}| \) and note that \( g(y) = c_y \). Finally we introduce a variable \( g_y \) representing the value of \( g(y) \) and add constraints

\[
g_y \geq (y - i)c_i \quad \forall 1 \leq i \leq t.
\]

Our result then follows as \( t \leq |V| \) and thus we add at most \( |V| \) new constraints. ▶

3 The Fair VC problem

3.1 Hardness for Treedepth and Feedback Vertec Set

We begin with several simple observations about the fair objective value \( k \) in the FAIR VC problem.

Observation 13. Let \( G = (V,E) \) be a graph and \( U \subseteq V \) be a vertex cover of \( G \) with fair objective \( k \), that is, \( \forall v \in V \) it holds that \( |N(v) \cap U| \leq k \). If \( v \in V \) has \( \deg(v) \geq k \), then \( v \in U \).

Note that we can use Observation 13 to enforce a vertex \( v \) to be a part of the fair vertex cover by attaching \( k + 1 \) degree 1 vertices to \( v \). Observe further that we may adjust (lower) the global budget \( k \) for individual vertex \( v \) by attaching vertices to \( v \) and then attaching \( k \) new leaves to the newly added vertices. To this end if the above operations are applied to a graph \( G \) resulting in a graph \( G' \), then \( \text{td}(G') \leq \text{td}(G) + 2 \) and \( \text{fvs}(G') = \text{fvs}(G) \).

We observe substantial connection between FAIR VC and TARGET SET SELECTION (TSS) problems. As a result, our reduction given here is, in certain sense, dual to the one given by Chopin et al. [6] for the TSS problem. The archetypal \( \mathcal{W}[1] \)-hard problem is the \( \ell \)-MULTICOLORED CLIQUE problem [11]:

| **Input:** | An \( \ell \)-partite graph \( G = (V_1 \cup \cdots \cup V_\ell, E) \), where \( V_c \) is an independent set for every \( c \in [\ell] \) and they are pairwise disjoint. |
| **Parameter:** | \( \ell \) |
| **Question:** | Is there a clique of the size \( \ell \) in \( G' \)? |

Proof of Theorem 2. Let \( G = (V_1 \cup \cdots \cup V_\ell, E) \) be an instance of the \( \ell \)-MULTICOLORED CLIQUE problem and let \( n = |V_i| \) for all \( i \in [\ell] \). We denote by \( E_{(a,b)} \) the set of edges between \( V_a \) and \( V_b \) and by \( m = |E_{(a,b)}| \). We will describe graph \( H = (W,F) \) that together with \( k = \max(m - 1, 2n) \) will form an equivalent instance of the FAIR VC problem. The reduction has the following properties:

- \( |W| = \text{poly}(n,k) \) and \( |F| = \text{poly}(m,k) \),
- \( \text{td}(H) = O(\ell^2) \), and \( \text{fvs}(H) = O(\ell^2) \).

For an overview of the reduction please refer to Figure 4. There are three types of gadgets in our reduction, namely the vertex selection gadget, the edge selection gadget, and the incidence check gadget. We start by enumerating the vertices in each color class by numbers from \([n]\) and edges by numbers in \([m]\). Throughout the proof \( a,b \) are distinct numbers from \([\ell]\).

The \( V_a \) selection gadget consists of \( n \) choice vertices (representing the color class \( V_a \)), a special vertex called guard, and a group of \( n^2 \) enumeration vertices. The guard vertex is
connected to all choice vertices, it is enforced to be a part of the fair vertex cover, and its budget is lowered so that at most \( n - 1 \) choice vertices can be in any fair vertex cover. The \( i \)-th choice vertex is connected to \( n \) enumeration vertices. For each choice vertex there are \( n \) such vertices and so these are private for vertex \( i \). We further divide these vertices into two parts – the lower part consists of \( i \) vertices and the upper part consists of \( n - i \) vertices.

The \( E_{\{a,b\}} \) selection gadget consists of \( m \) choice vertices, a special vertex called guard, and a group of \( 2nm \) enumeration vertices and is constructed analogously to the vertex selection gadget. If the \( q \)-th edge of \( E_{\{a,b\}} \) connects \( i \)-th vertex in \( V_a \) and \( j \)-th vertex in \( V_b \), there are (private) \( 2n \) enumeration vertices are connected to the \( q \)-th choice vertex. These are split into four groups – lower \( a \)-part consisting of \( i \) vertices, upper \( a \)-part consisting of \( n - i \) vertices, and similarly lower and upper \( b \)-parts.

The \( ab \)-incidence check gadget consist of two vertices \( c^1_{ab} \) and \( c^2_{ab} \). Both \( c^1_{ab} \) and \( c^2_{ab} \) are enforced to be a part of the solution and with lowered budget in a way that at most \( n \) vertices in the neighborhood of each of them can be part of any fair vertex cover. The vertex \( c^2_{ab} \) is connected to every lower part vertex in the selection gadget for \( V_a \) and to every upper \( a \)-part vertex in the selection gadget for \( E_{\{a,b\}} \). The vertex \( c^2_{ab} \) is connected to every upper part vertex in the selection gadget for \( V_a \) and to every lower \( a \)-part vertex in the selection gadget for \( E_{\{a,b\}} \).

This finishes the construction of \( H \). Now observe that if we remove vertices \( c^1_{ab}, c^2_{ab} \) from \( H \), then each component of the resulting graph is a tree (rooted in its guard vertex) of depth at most 3. It follows that \( td(H) = O(\ell^2) \) and \( fvs(G) = O(\ell^2) \), as so is the size of the removed set of vertices. We finish the proof by showing that the two instances are equivalent.

Suppose \( (G, \ell) \) is a yes-instance which is witnessed by a set \( K \subseteq V_1 \times \cdots \times V_\ell \). We now construct a vertex cover \( C_K \) of \( H \) having \( |N(w) \cap C_K| \leq k \) for all \( w \in W \). The set \( C_K \) contains the following:

- all enforced vertices (including all guard and check vertices),
if \(v \in V_a \cap K\) is the \(i\)-th vertex of \(V_a\), then all selection vertices of \(V_a\) but the vertex \(i\) are in \(C_K\) and lower and upper enumeration vertices of \(i\) are in \(C_K\),

if \(v \in V_a \cap K\) and \(w \in V_b \cap K\) are adjacent through \(q\)-th edge of \(E_{(a,b)}\), then all selection vertices of \(E_{(a,b)}\) but the vertex \(q\) are in \(C_K\) and \(q\)'s enumeration vertices are in \(C_K\).

For the other direction we prove that a vertex cover \(C\) in \(H\) fulfills \(|N(w) \cap C| \leq k\) for all \(w \in W\) if it corresponds to a clique in \(G\). For the other direction suppose that there is a vertex cover \(C\) in \(H\) such that \(|N(w) \cap C| \leq k\) for all \(w \in W\). Recall that \(C\) has to contain all enforced vertices. This implies that at least 1 choice vertex for \(V_a\) is not in \(U\); we will show that exactly 1 such choice vertex is in \(U\). The same holds for the edge choice vertices. To see this suppose for contradiction that 2 choice vertices for \(V_a\) (vertex \(i\) and \(j\)) are not in \(U\). Because \(U\) is a vertex cover of \(H\) it follows that their enumeration vertices must belong to \(U\). But now take vertices \(c_{ab}^1, c_{ab}^2\). In their neighborhood \(U\) have at least 3\(n\) vertices (2\(n\) from the \(V_a\)'s numeration part and \(n\) from the \(a\)-numeration part of \(E_{(a,b)}\)). This is absurd as these vertices (due to the lowered budget) can have at most 2\(n\) vertices in their neighborhood and thus at least one of them exceeds its budget. Thus the selection gadgets actually encode some selection of vertices \(v_{ab}\) and edges \(e_{ab}\). To finish the proof we have to observe that both \(c_{ab}^1\) and \(c_{ab}^2\) have at most (in fact, exactly) \(n\) neighbors in \(U\) if and only if the vertex \(v_{ab}\) is incident to the edge \(e_{ab}\). If this holds for all possible combinations of \(a, b\), then we have selected a clique in the graph \(G\).

It remains to discuss the ETH based lower-bound. This follows straightforwardly from our reduction and the result of Chen et al. [5] who proved that there is no \(f(k)n^{o(\ell)}\) algorithm for \(\ell\)-MULTICOLORED CLIQUE, unless ETH fails. Since we have \(nd(G) = O(\ell^2)\) in our reduction, the lower-bound follows.

### 3.2 FPT algorithm for Modular Width

Since the algebraic expression \(A\) of width \(n(w(G))\) can be computed in linear time [41], we can assume that we have \(A\) on the input. We construct the rooted ordered tree \(T\) corresponding to \(A\). Each node \(t \in T\) is assigned a graph \(G^t \subseteq G\) that is the graph constructed by the subexpression of \(A\) rooted at \(t\). That is, Suppose we are performing substitution operation at node \(t\) with respect to template graph \(T\) and graphs \(G_1, \ldots, G_r\). Denote the resulting graph \(G^t\) and denote by \(n_t\) the size of \(V(G_i)\).

The edges in \(G^t\) between two vertices of \(G_i\) will be referred to as old edges, the edges between \(G_i\) and \(G_j\) for \(i \neq j\) (i.e., edges introduced by the template operation) will be referred to as new edges.

The computation will be carried out recursively from the bottom of the tree \(T\).

We first describe the structure of all vertex covers \(C\) in \(G^t\). First observe that if \(ij \in E(T)\) then at least one of \(V(G_i), V(G_j)\) must be a subset of \(C\); otherwise there would be a new edge not covered by \(C\). From this we can see that the set \(C_T = \{i : V(G_i) \subseteq C\}\) is a vertex cover of the template graph \(T\). We call the \(C_T\) the type of the vertex cover \(C\). Furthermore, every set \(C \cap V(G_i)\) must be a vertex cover of \(G_i\) (otherwise there would be an old edge uncovered by \(C\)).

We now describe the fair cost of the cover \(C\) in terms of fair costs and sizes of the sets \(C \cap V(G_i)\). Denote by \(c_i\) the size \(|C \cap V(G_i)|\) and by \(f_i\) the fair cost of \(C \cap V(G_i)\) in \(G_i\). The fair cost of \(C\) in \(W \subseteq V(G)\) is defined as \(\max_{v \in W} |C \cap N(v)|\). For \(i \in [r]\) the fair cost of \(C\) in \(V(G_i)\) can be written as \(f_i + \sum_{j: ij \in E(T)} c_j\). Clearly, fair cost of \(C\) is the maximum of the last expression over \(i \in [r]\).
If we know the type $C_T$ of the cover $C$ this can be simplified based on whether $i$ lies in $C_T$. If $i \in C_T$ then $f_i$ is $\Delta(G_i)$ (the maximal degree of $G_i$). If on the other hand $i \notin C_T$ then all its neighbors $j$ are in $C_T$ and in this case $c_j = n_j$. Combining those observations, we have:

$$\text{fair cost of } C \text{ in } G_i = \begin{cases} \Delta(G_i) + \sum_{j \notin C_T : i \in E(T)} c_j + \sum_{j \in C_T : i \in E(T)} n_j & i \in C_T, \\ f_i + \sum_{j \notin E(T)} n_j & i \notin C_T. \end{cases}$$

For each node $t$ of the tree $T$ we keep an $|V(G^t)|$ table $\text{Tab}^t$ of integer values from $[n] \cup \infty$. The value at position $\text{Tab}^t[p]$ is the smallest size of a cover in $G^t$ of fair cost $p$ or $\infty$ if such cover do not exists.

The computation of $\text{Tab}$ in leaves of $T$ is trivial. We describe how to compute value $\text{Tab}^t[p]$ given that we know $\text{Tab}$ in all children of $t$. It is enough to determine whether there exists a vertex cover $C$ of $G^t$ of fixed type; we can simply iterate over all types as there are at most $2^{|V(T)}$ of them.

Fix a type $C_T$. The cover $C$ of type $C_T$ and fair cost at most $p$ and $\text{Tab}^t[p] \neq \infty$ exists if and only if for every $i \notin C_T$ there is a vertex cover $C_i$ of $G_i$ of fair cost $p_i$ such that $\text{Tab}_i[p_i] \neq \infty$. Moreover, we require that the values $p_i$ satisfy the following inequalities:

\begin{align*}
    p &\geq \Delta(G_i) + \sum_{j \notin C_T : i \in E(T)} \text{Tab}_j[p_i] + \sum_{j \in C_T : i \in E(T)} n_j & \forall i \in C_T, \quad (1) \\
    p &\geq p_i + \sum_{j \notin E(T)} n_j & \forall i \notin C_T, \quad (2) \\
    \text{Tab}^t[p] &\geq \sum_{j \notin C_T} \text{Tab}_j[p_i] + \sum_{j \in C_T} n_j. \quad (3)
\end{align*}

First, for every $i \notin C_T$ we set $p_i$ to the highest possible value without violating the inequality (2), that is $p_i := p - \sum_{j \notin E(T)} n_j$. Note that this is always a safe choice; $p_i$ does not appear anywhere else in the constraints and choosing the highest possible value give us greatest freedom due to the monotonicity of the table. Clearly, if any such $p_i$ is negative we know that given constraints cannot be satisfied and there is no vertex cover $C$ of given type and fair cost.

If for any $i$ $\text{Tab}_i[p_i] = \infty$, then there is no vertex cover $C_i$ in $G_i$ of fair cost $p_i$. This means that we cannot find cover $C$ of given type and fair cost so we set $\text{Tab}_i[p_i] = \infty$. We check whether inequalities (1) holds. If not, set $\text{Tab}_i[p_i] = \infty$. Otherwise we set $\text{Tab}^t[p]$ be equal to the expression in (3) on the right side. We claim that there is a vertex cover $C$ of given type and fair cost; we can set $C = \bigcup_{i \in C_t} V(G_i) \cup \bigcup_{i \in C_T} C_i$, where $C_i$ is any vertex cover of $G_i$ of fair cost $p_i$ and size $\text{Tab}_i[p_i]$ (this is guaranteed to exist because $\text{Tab}_i[p_i]$ was not $\infty$. It is straightforward to check that $C$ has required properties. Moreover, from our choice of values $p_i$ it follows that a vertex cover of type $C_T$, fair cost $p$ and size $\text{Tab}^t[p]$ exists if and only if the described procedure finds values of $p_i$. By iterating over all types $C_T$ we can fill the value $\text{Tab}^t[p]$ as required.

To complete the description of the algorithm, it is enough to look whether there is not $\infty$ value in $\text{Tab}_{\text{root}}[k]$, where $k$ was the desired fair cost.

The running time is $n$ for the induction over expression $A$ times $2^r$ different type of covers in any single node times filling the table of size at most $\mathcal{O}(n)$ times $nr$ for determining the correct values $p_i$ and checking other inequalities for every $i \in [r]$. This altogether yields a $2^r n r^3$ time algorithm.

**Lemma 14**. The FAIR VC problem parameterized by the modular width of the input graph does not admit a polynomial kernel, unless $\text{NP} \subseteq \text{coNP} / \text{poly}$. 

![Image](image-url)
First we observe that modular width is trivially compositional, that is, for any two graphs $G_1, G_2$ it holds that $\text{mw}(G_1 \cup G_2) = \max(\text{mw}(G_1), \text{mw}(G_2))$, where $\cup$ denotes the disjoint union. Indeed this follows from the fact that disjoin union is one of the operations not affecting modular width. Now, it remains to show that $\text{Fair VC}$ is AND-compositional, see [11, Chapter 15]; the rest then follows from the framework of Bodlaender et al. [3]. To this end, observe that if a graph $G$ is not connected, then $U$ is a vertex cover in $G$ if and only if $U \cap C$ is a vertex cover in $G[C]$ for every connected component $C$ of $G$. ▷

4 Further Hardness Results

We use the Unary $\ell$-Bin Packing problem as the starting point of our hardness reduction. Unary $\ell$-Bin Packing is $\text{W[1]}$-hard for parameter $\ell$ the number of bins to be used [24]. Here, the item sizes are encoded in unary and the task is to assign $n$ items to $\ell$ bins such that the sum of sizes of items assigned to any bin does not exceed its capacity $B$. Formally, Unary $\ell$-Bin Packing is defined as follows.

| Unary $\ell$-Bin Packing | Parameter: $\ell$ |
|--------------------------|------------------|
| **Input:** | Positive integers $\ell$ and $B$ and item sizes $s_1, \ldots, s_n$ encoded in unary. |
| **Question:** | Is there a packing of all items into at most $\ell$ bins of size $B$? |

**Proof of Theorem 4.** We construct a formula $\varphi(X_1, \ldots, X_\ell)$ and an instance $(G, k)$ of $\text{Fair Vertex MSO Evaluation}$ with $k = B$ from an instance of Unary $\ell$-Bin Packing as follows. The graph $G$ is formed by $n$ disjoint cliques and a universal vertex $u$. Cliques in $G$ represent the items by their respective sizes, that is, there is a clique with $s_i$ vertices for every $i \in [n]$; denote the clique representing item $i$ by $C_i$. This finishes the description of the graph; now we turn our attention to the formula. Free variables $X_1, \ldots, X_\ell$ are going to represent an assignment of items to bins. Note that it is possible to recognize the universal vertex $u$ by the following FO formula, since $u$ is the only vertex of $G$ satisfying it:

$$\text{univ}(v) := (\forall w \in V)((w \neq v) \rightarrow (wv \in E)).$$

We fix $u$ for the rest of the description of $\varphi(X_1, \ldots, X_\ell)$; this can easily be done by attaching $(\exists u \in V)(\text{univ}(u))$ to it. Let $p(v)$ be a predicate. For $Q \in \{\exists, \forall\}$ we use the following $Qv \in (V \setminus \{u\})(\psi(v))$ as a short form of the expression

$$(Qv \in V)((v \neq u) \rightarrow \psi(v)).$$

Note that this can be straightforwardly extended for more quantifiers.

In order to represent the bin choice we need to ensure two conditions. First, every item is packed, that is, every non-universal vertex must belong to some $X_j$. Second, every item is fully packed inside (at least) one bin, that is, vertices belonging to the same clique agree on $X_j$ membership. To do so we first introduce the following predicates (representing these conditions):

$$\text{cover}(X_1, \ldots, X_\ell) := (\forall v \in V \setminus \{u\}) \left( \bigvee_{j=1}^{\ell} v \in X_j \right)$$

and

$$\text{same}(X_1, \ldots, X_\ell) := (\forall v, w \in V \setminus \{u\}) \left( (vw \in E) \rightarrow \bigwedge_{j=1}^{\ell} (v \in X_j \Leftrightarrow w \in X_j) \right).$$
The construction of the new instance is finished by letting
\[ \varphi(X_1, \ldots, X_\ell) = (\exists u \in V)(\text{univ}(u)) \land \text{cover}(X_1, \ldots, X_\ell) \land \text{same}(X_1, \ldots, X_\ell). \]

It remains to argue that the instances are indeed equivalent. Note that the bin capacity/fair cost is essentially checked only for \( u \), since the fair cost of any vertex cannot exceed its degree. The degree of any other vertex (not \( u \)) does not exceed \( \max_{i \in [n]} s_i \) and this is always upper-bounded by \( B \).

Let \((B, \ell, s_1, \ldots, s_\ell)\) be a Yes-instance of 3-UNARY BIN PACKING. This is witnessed by a mapping \( \sigma: [n] \to [\ell] \) assigning items to bins. Now, we put
\[ W_j = \bigcup_{i \in [n]}: \sigma(i) = j C_i. \]

Observe that \( |N(u) \cap W_j| = \sum_{i \in [n], \sigma(i) = j} s_i \leq B \) for every \( j \in [\ell] \), since \( \sigma \) represented a valid assignment. Furthermore, \( \varphi(W_1, \ldots, W_\ell) \) holds, since every clique vertex is covered and vertices of a clique are always in the same \( W_j \).

For the opposite direction assume we have graph \( G = (V, E) \) and that there exist sets \( W_1, \ldots, W_\ell \subseteq V \) such that \( \varphi(W_1, \ldots, W_\ell) \) holds in \( G \). Recall that \( u \) is the universal vertex of \( G \). First, we have that \( V = \{u\} \cup W_1 \cup \cdots \cup W_\ell \), since the predicate \( \text{cover}(W_1, \ldots, W_\ell) \) holds if and only if every vertex in a clique \( C_i \) is in some \( W_j \). Furthermore, since the predicate \( \text{same}(W_1, \ldots, W_\ell) \) holds, we have that \( C_i \cap W_j \) is either an empty set or \( C_i \) for every \( i \in [n] \) and \( j \in [\ell] \). This allows us to construct an assignment \( \sigma: [n] \to [\ell] \) by setting \( \sigma(i) = j \), where \( j \in [\ell] \) is the smallest number such that \( W_j \cap C_i = C_i \) holds. Now, we have that \( \sum_{i \in [n], \sigma(i) = j} s_i \leq |W_j| \leq B \) by the fair objective. This finishes the proof, since we have constructed a valid assignment.

**Proof of Theorem 17.** We construct a sentence \( \varphi \), a graph \( G \), and \( k \) forming an instance of FAIR EDGE FO DELETION with \( k = B \) from an instance of Unary \( \ell \)-Bin Packing as follows. Each item is represented by clique on \( 3B \) vertices; denote the clique associated with \( i \)-th item by \( C_i \). In addition, there are \( \ell \) vertices \( v_1, \ldots, v_\ell \) (representing bins) and \( \ell \) guard vertices \( g_1, \ldots, g_\ell \). For each \( j \in [\ell] \) we connect \( v_j \) with exactly \( s_j \) vertices in \( C_i \) (call these vertices special) and with \( g_j \). This finishes the description of \( G \).

The sentence \( \varphi \) is constructed using auxiliary predicates which we describe first. A predicate \( \text{guard}(v) \) is used to recognize the guard vertices
\[ \text{guard}(v) := (\exists u \in V)((u \in E) \land (\forall w \in W \setminus \{u, v\})(vw \notin E)). \]

\[ \text{bin}(v) := (\exists u)(wu \in E \land \text{guard}(u)). \]

\[ \text{item-edge} := (\forall v, w \in V: v \neq w)\left( (\exists u \in V)(\neg \text{bin}(u) \land \text{vu} \in E \land wu \in E) \right) \rightarrow (vw \in E). \]

Recall that we set the fair set to \( B \) and thus the solution \( F \) deletes at most \( B \) edges incident to any vertex. Note that a vertex \( v \) must be a guard vertex in order to fulfill the guard\( (v) \), since the degree of every other vertex in \( G \) is at least \( 3B \). Suppose that the predicate item-edge holds in \( G \setminus F \). We claim that \( vw \in E \) for any two vertices \( v, w \in C_i \), since \( |C_i| = 3B \) it follows that \( v \) and \( w \) have at least \( B - 2 \) common neighbors in \( C_i \). Thus, the edge \( vw \) cannot be deleted, as \( v \) and \( w \) have a common (non-bin) neighbor (provided \( B \geq 3 \)). We define an auxiliary sentence
\[ \psi = (\exists v_1, \ldots, v_\ell \in V)\left( \bigwedge_{j \in [\ell]} \text{bin}(v_j) \land \bigwedge_{j \neq j' \in [\ell]} v_j \neq v_{j'} \right) \land \text{item-edge} \]
and, since \( \psi \) implicitly assures existence of \( \ell \) (different) guards, conclude the following claim.
Claim 15. Let $F$ be a set of edges in $G$. We have that $G \setminus F \models \psi$ if and only if $F$ contains only edges between bin and special vertices.

The next predicate we introduce is the notable($v$), defined as follows

$$\text{notable}(v) := \neg \text{guard}(v) \land ((\exists u \in V)(\text{bin}(u) \land uv \in E)).$$

Claim 16.
1. If $G \setminus F \models \text{notable}(v)$, then $v$ is a special vertex.
2. $G \models \text{notable}(v)$ if and only if $v$ is a special vertex.
3. If $G \setminus F \not\models \text{notable}(v)$ for a special vertex $v \in C_i$, then $N_{G \setminus F}(v) \subseteq C_i$.

Proof: A vertex $v$ is notable in a graph if notable($v$) holds in that graph. The second part follows immediately from the construction of $G$, since the only vertices attached to the bin vertex $v_j$ are the special vertices and its guard vertex $g_j$ (which is not notable). On the other hand, the set $F$ may contain edges $ev_j$ for all $j \in [\ell]$. Clearly, such a former special vertex is not notable. Finally, if $v$ is special and $G \setminus F \not\models \text{notable}(v)$, then $v$ lost all of its edges to bin vertices $v_1, \ldots, v_k$.

The goal is to delete an edge set $F$ that describes a valid assignment of items into bins. We need to ensure two conditions. First, every special vertex is not a neighbor of some bin vertex $v_j$. Second, if a special vertex $v \in C_i$ is not a neighbor of a bin vertex $v_j$, then all special vertices in $C_i$ are not neighbors of $v_i$. The two conditions correspond to the following two predicates:

$$\text{cover} := (\forall v \in V)(\exists u \in V)(\text{bin}(u) \land uv \not\in E)$$

$$\text{same} := (\forall v, w \in V)\left(\text{notable}(v) \land \text{notable}(w) \land vw \in E \rightarrow (\forall u \in V : \text{bin}(u))(uw \in E \iff vw \in E)\right)$$

Now, we are ready to give the sentence $\varphi$ that describes the problem:

$$\varphi = \psi \land \text{same} \land \text{cover}.$$ 

This finishes the description of the reduction. We are left with validating that the two instances are equivalent.

Suppose we were given a Yes-instance of $\ell$-UNARY BIN PACKING and let $\sigma : [n] \rightarrow [\ell]$ be the assignment of items to bins witnessing this fact. The set $F$ contains an edge $uv_j$ for a special vertex $u \in C_i$ and a bin vertex $v_j$ whenever $\sigma(i) = j$. Clearly, we have $\{|e \in F : e \ni v_j\}| \leq \sum_{i \in [n], \sigma(i) = j} s_i \leq B = k$ for every vertex $v_j$ with $j \in [\ell]$, while every other vertex has at most one incident edge in $F$. Now, we have to verify that $G \setminus F \models \varphi$. We have $G \setminus F \models \text{same}$, since $\sigma$ is an assignment. Finally, $G \setminus F \models \psi$, since our $F$ fulfills the condition of Claim 15.

Suppose now that we have a set $F$ of edges of $G$ with fair cost $B$ such that $G \setminus F \models \varphi$. By Claim 15 we have that $F$ contains only edges between special vertices in $G$ and bin vertices $v_1, \ldots, v_k$. We partition the set of special vertices into sets $N$ and $R$; we put a special vertex $v$ in $N$ if $G \setminus F \models \text{notable}(v)$ and we put it in $R$ otherwise. Note that $|R| \leq B$, since a vertex in $R$ contributes to the fair cost of every bin vertex $v_j$. By Claim 16 and the fact that $G \setminus F \models \text{same}$ we have that a bin vertex $v_j$ is either completely attached or non-attached to $C_i \cap N$ for every $j \in [\ell]$ and every $i \in [n]$. Furthermore, there are no edges between a vertex in $R$ and a bin vertex $v_j$ in $G \setminus F$ for every $j \in [\ell]$. We define the assignment $\sigma : [n] \rightarrow [\ell]$ by
defining $\sigma(i)$ to be the smallest integer such that there are no edges between $v_{\sigma(i)}$ and $C_i$ in $G \setminus F$. Since $\sigma$ is an assignment by $G \setminus F \models \text{cover}$, it remains for verify that the capacity condition is fulfilled. For that we have

$$\sum_{i \in [n], \sigma(i) = j} s_i \leq |\{e \in F : v_j \in e\}| \leq \sum_{i \in [n], \sigma(i) = j} |N \cap C_i| + |R| \leq k = B.$$

We conclude that $\sigma$ is a valid assignment and the theorem follows.

\[\] 5 Conclusions

One can define edge deletion problems in a similar way as vertex deletion problems. Recall, in dense graph classes one cannot obtain an MSO$_2$ model checking algorithm running in FPT-time [30]. That is a reason why evaluation problems does not make sense in this context.

FAIR L EDGE DELETION

Input: An undirected graph $G = (V, E)$, an L sentence $\varphi$, and a positive integer $k$.

Question: Is there a set $F \subseteq E$ such that $G \setminus F \models \varphi$ and for every vertex $v$ of $G$, it holds that $|\{e \in F : e \ni v\}| \leq k$?

Minimizing the fair cost arises naturally for edge problems in many situations as well, e.g. in defective coloring [10]. A graph $G = (V, E)$ is $(k, d)$-colorable if every vertex can be assigned a color from the set $[k]$ in such a way that every vertex has at most $d$ neighbors of the same color. This problem can be reformulated in terms of fair deletion; we aim to find a set of edges $F$ such that graph $G' = (V, F)$ has maximum degree $d$ and $G \setminus F$ can be partitioned into $k$ independent sets.

The crucial open problem is to resolve parameterized complexity of the Fair FO Edge Deletion problems for parameterization by neighborhood diversity and twin cover. There is a big difference between vertex and edge deletion problems – in our hardness reduction we used a deletion to an edgeless graph but a fair edge cost in this case equals to the maximum degree of the former graph (an thus is computable in polynomial time). To motivate the study we prove the following hardness result for parameterization by the cluster vertex deletion number. Observe that for any graph its cluster vertex deletion number is upper-bounded by the size of its twin cover.

\[\] Theorem 17 (,). The Fair FO Edge Deletion problem is W[1]-hard when parameterized by the cluster vertex deletion number of the input graph.

Another open problem is to resolve parameterized complexity of the Fair MSO$_1$ Vertex Evaluation problems with respect to graph parameters generalizing neighborhood diversity or twin cover (e.g. modular width or cluster vertex deletion number). Finally, we would like to ask whether there is an NP-hard fair vertex deletion problem that admits an FPT algorithm for parameterization by treedepth (and feedback vertex set) of the input graph.

It seems that a more careful analysis of the model-checking algorithm may yield (again) sufficient insight into the structure of the fair solution and thus lead to an FPT algorithm for this wider class of graphs. Though, it remains open whether this is possible and there is an FPT algorithm for Fair MSO Vertex Evaluation for this parameterization or not.

MSO with Local Linear Constraints. Previously, an FPT algorithm for evaluation of a fair objective was given for parameter neighborhood diversity [34]. That algorithm was extended [25] to a so called local linear constraints again for a fixed formula $\varphi(\cdot)$ with one free variable that is defined as follows. Every vertex $v \in V(G)$ is accompanied with two positive
integers $\ell(v), u(v)$, the lower and the upper bound, and the task is to find a set $X$ that not only $G \models \varphi(X)$ but for each $v \in V(G)$ it holds that $\ell(v) \leq |N(v) \cap X| \leq u(v)$. Note that this is a generalization as fair objective of value $t$ can be tested in this framework by setting $\ell(v) = 0$ and $u(v) = t$ for every $v \in V(G)$. Is this extension possible for parameterization by the twin cover number?

To support this question we note that in the proof of Lemma 10 the minimal size of the neighborhood of $B$ for a shape in cover set of $B$ is computed. It is not hard to see that by a similar argument we can compute the maximal size of the neighborhood of $B$ for a shape in cover set of $B$. Furthermore, lower and upper bounds for vertices in a clique can be assumed to be nearly the same – each differ by at most 1 \cite{25}. Thus, Lemma 10 gives only that if at least one of the computed bounds for a vertex $v$ in the twin cover is within $\ell(v)$ and $u(v)$, then there is a solution with desired properties. However, if on the other hand it happens that both $\ell(v), u(v)$ are in between the computed values, we do not know whether or not any of the desired values is attainable. To see that not all values in the thus computed range are attainable one can construct a formula that for twin cliques up to a certain size checks that the number of selected vertices is even. Then, if the input graph contains only cliques up to this size no twin cover vertex has odd number of neighbors in the set $X$ (provided the cover vertices form an independent set).

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