A closed-formula solution to the color-trace decomposition problem

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In these notes we present a closed-formula solution to the problem of decomposing traces of Lie algebra generators into symmetrized traces and structure constants. The solution is written in terms of Solomon idempotents and exploits a projection derived by Solomon in his work on the Poincaré-Birkhoff-Witt theorem.

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1. Introduction

The purpose of these notes is to present a closed-formula solution to one of the problems addressed in [1] via computer algebra [2]. Given a simple Lie algebra whose generators $T^a$ satisfy

$$[T^a, T^b] = i f^{abc} T^c,$$

$$\text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}, \quad (1.1)$$

where $f^{abc}$ denote the totally anti-symmetric structure constants and $\delta^{ab}$ is the Kronecker delta, the problem consists in expressing traces $\text{Tr}(T^{a_1} \cdots T^{a_n})$ of products of Lie algebra generators $T^a$ (color factors) in terms of symmetrized traces

$$d^{a_1 \cdots a_n} := \frac{1}{n!} \sum_{\sigma \in S_n} \text{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) \quad (1.2)$$

and structure constants $f^{abc}$. For example,

$$\text{Tr}(T^{a_1} T^{a_2} T^{a_3}) = d^{a_1 a_2 a_3} + \frac{i}{4} f^{a_1 a_2 a_3}.$$

(1.3)

These decompositions have important applications in the evaluation of loop amplitudes in perturbative field and string theories, as they allow an efficient handling of their associated color structures in a manner described in [1].

While in [1] an algorithm was obtained to generate these decompositions using computer algebra, we will see here that this color trace decomposition problem admits an elegant closed-formula solution using a result known in the free Lie algebra literature. The formula involves the so-called Solomon idempotent or first Eulerian idempotent [3,4,5,6,7] and its first few cases are given by (to avoid cluttering we write $j$ instead of $a_j$)

$$\text{Tr}(T^{1} T^{2}) = d^{12},$$

$$\text{Tr}(T^{1} T^{2} T^{3}) = d^{123} + d^{1a} E^{23}_a,$$

$$\text{Tr}(T^{1} T^{2} T^{3} T^{4}) = d^{1234} + d^{12a} E^{34}_a + d^{13a} E^{24}_a + d^{14a} E^{23}_a + d^{1a} E^{234}_a,$$

$$\text{Tr}(T^{1} T^{2} T^{3} T^{4} T^{5}) = d^{12345} + d^{123a} E^{45}_a + d^{124a} E^{35}_a + d^{135a} E^{25}_a + d^{13a} E^{245}_a + d^{14a} E^{235}_a + d^{15a} E^{234}_a$$

$$+ d^{145a} E^{234}_a + d^{12a} E^{345}_a + d^{13a} E^{245}_a + d^{14a} E^{235}_a + d^{15a} E^{234}_a$$

$$+ d^{1ab} (E^{23} E^{45}_b + E^{24} E^{35}_b + E^{25} E^{34}_b) + d^{1a} E^{2345}_a,$$

where $E^{12 \cdots n}_a$ denote the expansion coefficients of the Solomon idempotent with respect to the Lie algebra generators $E(T^{1} \cdots T^{n}) = E^{1 \cdots n}_a T^a$. We refer to subsection 2.1 for more precise definitions, and for now just point out that these coefficients can be explicitly
computed as polynomials in the structure constants \( f^{abc} \), using results from [7] (see also [8]). For instance, this yields the following solution to the color trace decomposition problem up to \( n = 5 \),

\[
\begin{align*}
\text{Tr}(T^1T^2) &= d^{12} = \frac{1}{2} \delta^{12} \\
\text{Tr}(T^1T^2T^3) &= d^{123} + \frac{i}{4} f^{123} \\
\text{Tr}(T^1T^2T^3T^4) &= d^{1234} - \frac{1}{6} f^{23a} f^{a41} + \frac{1}{12} f^{24a} f^{a31} \\
&\quad + \frac{i}{2} d^{12a} f^{a34} + \frac{i}{2} d^{13a} f^{a24} + \frac{i}{2} d^{14a} f^{a23} \\
\text{Tr}(T^1T^2T^3T^4T^5) &= d^{12345} + \frac{i}{24} \left( -3 f^{23a} f^{a4b} f^{b51} + f^{23a} f^{a5b} f^{b41} + f^{24a} f^{a3b} f^{b51} \\
&\quad + f^{24a} f^{a5b} f^{b31} + f^{25a} f^{a3b} f^{b41} - f^{25a} f^{a4b} f^{b31} \right) \\
&\quad - \frac{1}{4} d^{1ab} \left( f^{23a} f^{45b} + f^{24a} f^{35b} + f^{25a} f^{34b} \right) \\
&\quad + d^{12a} \left( -\frac{1}{3} f^{34b} f^{b5a} + \frac{1}{6} f^{35b} f^{b4a} \right) + d^{13a} \left( -\frac{1}{3} f^{24b} f^{b5a} + \frac{1}{6} f^{25b} f^{b4a} \right) \\
&\quad + d^{14a} \left( -\frac{1}{3} f^{23b} f^{b5a} + \frac{1}{6} f^{25b} f^{b3a} \right) + d^{15a} \left( -\frac{1}{3} f^{23b} f^{b4a} + \frac{1}{6} f^{24b} f^{b3a} \right) \\
&\quad + \frac{i}{2} \left( d^{123a} f^{a45} + d^{124a} f^{a35} + d^{125a} f^{a34} + d^{134a} f^{a25} + d^{135a} f^{a24} + d^{145a} f^{a23} \right),
\end{align*}
\]

recovering computations from [9].

Our solution to the color trace decomposition problem shall depend on a projection formula due to Solomon [3], and related to the Poincaré-Birkhoff-Witt Theorem (see [10]). Recall that in particular the latter implies that a product of generators \( T^{p_1} \cdots T^{p_n} \) can be expanded as a linear combination of symmetrized products of Lie monomials in the generators \( T^{p_1}, \ldots, T^{p_n} \) (for instance \( T^1T^2 = \frac{1}{2}(T^1T^2 + T^2T^1) + \frac{1}{2}[T^1, T^2] \)). Solomon’s formula provides such an expansion explicitly in terms of the first Eulerian idempotent, and from this (and the usual cyclic properties of the trace) we shall deduce the following compact formula (1.6), containing (1.4) as particular cases. Given a word \( P = p_1 \cdots p_n \), we denote by \( T^P := T^{p_1} \cdots T^{p_n} \) and by \( \overline{\delta}_k(P) = \sum_{(P)} P_{(1)} \otimes \cdots \otimes P_{(k)} \) the \( k \)-th (reduced)

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1 The expansions in [9] use a different basis of color factors. That particular basis follows from expanding the Eulerian idempotents \( E(x_1, \ldots, x_n) \) (2.7) in terms of the right-to-left free Lie algebra basis \( r(\ldots, x_n) \) rather than the left-to-right \( \ell(x_1, \ldots) \) as chosen in this work.
the deshuffle map applied to $P$, using Sweedler’s notation (see the subsection below for more precise definitions): then our formula reads

$$\text{Tr}(T^1T^P) = \sum_{k \geq 1} \sum_{(P)} \frac{1}{k!} d^{1a_1a_2\ldots a_k} \epsilon_{a_1}^{-1} T_{a_2}^{P(1)} T_{a_2}^{P(2)} \ldots T_{a_k}^{P(k)},$$

(1.6)

(where the second summation runs over the set of $k$-deshuffles $P(1) \otimes \ldots \otimes P(k)$ of $P$).

After expanding the $E^{i_1\ldots i_k}$ as polynomials in the structure constants, we finally obtain the following closed formula solution for the color trace decomposition problem

$$\text{Tr}(T^0T^1 \ldots T^n) = \sum_{S_a \ni \sigma_1 \ldots \sigma_k} i^{n-k} C_{\sigma_1} \cdot \ldots \cdot C_{\sigma_k} d^{0a_1 \ldots a_k} F_{a_1}^{\sigma_1} \ldots F_{a_k}^{\sigma_k}.$$

(1.7)

The latter formula deserves some explanations. First of all, the sum runs over the set of permutations $\sigma \in S_n$, which are identified with the corresponding words $\sigma(1) \ldots \sigma(n)$. Then $\sigma = \sigma_1 \ldots \sigma_k$ denotes the standard factorization of $\sigma$, i.e., the unique factorization of $\sigma$ as the concatenation product of subwords $\sigma_1, \ldots, \sigma_k$ such that $\sigma_1 > \ldots > \sigma_k$ in the lexicographical order and for all $1 \leq j \leq k$ the first letter in $\sigma_j$ is the minimum among its letters. A few examples are given by,

$$1432 = (1432), \quad 2134 = (2)(134), \quad 54132 = (5)(4)(132), \quad 42671835 = (4)(267)(1835).$$

(1.8)

Finally, given a word $P = p_1 \ldots p_i$ we denote by $C_P := (-1)^{d_P} |P|^{(d_P)}$ (where $d_P$ is the number of descents in $P$, once again we refer to the subsection below for more details) and by $F_a^1 := \delta_{1a}$, $F_a^{12} := f_{12a}$, $F_a^{123} := f_{12b}f_{3a}$, $F_a^{1234} := f_{12c}f_{c3b}f_{4a}$, in general,

$$F_a^{P} = f_{a_1}^{p_1} \ldots f_{a_i}^{p_i} := f_{p_1p_2c_1}^{c_1} f_{p_2p_3c_2}^{c_2} \ldots f_{c_{i-2}p_i}^{c_{i-2}}.$$

(1.9)

To better understand the above formula (1.7) the reader might check that for $n \leq 4$ it precisely recovers (1.5) (after the obvious shift of indices). For instance, for $n = 4$, $\sigma = 2413 \in S_4$, the standard factorization is $\sigma = \sigma_1\sigma_2 = (24)(13)$ and the corresponding term in (1.7) is $i^2 C_{24} C_{13} d_{0ab} f_{a}^{24} f_{b}^{13} = -\frac{1}{4} d_{0ab} f_{24a} f_{13b}$. As further examples, for $\sigma = 3142 = \sigma_1\sigma_2 = (3)(142)$ we get the term $i^2 C_3 C_{142} d_{0ab} f_{a}^{3} f_{b}^{142} = \frac{1}{6} d_{03b} f_{14c} f_{c2b}$, for $\sigma = 4231 = \sigma_1\sigma_2\sigma_3 = (4)(23)(1)$ the one $i C_4 C_{23} C_1 d_{0abc} f_{a}^{4} f_{b}^{23} f_{c}^{1} = \frac{i}{2} d_{014b} f_{b23}$ and for $\sigma = 1432 = \sigma_1$ the one $i^3 C_{1432} d_{0a} f_{a}^{1432} = -\frac{i}{12} d_{0a} f_{14c} f_{c3b} f_{b2a} = -\frac{i}{24} d_{02b} f_{b3c} f_{c14}$. It is important to observe that the output of (1.7) is already written down in a basis of color factors, that is, no linear relations among its terms can be deduced using only the Jacobi identities (2.4).
1.1. Notation on words

In this paper the labels in indices such as $a_2$ will be interpreted as *letters* from the alphabet of natural numbers \{1, 2, 3, \ldots\} and denoted by lower-case letters (e.g. $j = 2$). *Words* composed of such letters will be denoted by capital letters such as $P = 13245$. The length of the word $P$ is denoted $|P|$ and it is given by the number of its letters. Given a word $P$, a descent in $P$ is a pair of consecutive letters $P = \cdots p_j p_{j+1} \cdots$ such that $p_j > p_{j+1}$, and the *descent number* $d_P$ of $P$ is the number of descents in it. Furthermore, given a word $P$ we shall denote by $C_P$ the number

$$C_P := \frac{(-1)^{d_P}}{|P|(|P|-1)}.$$  

(1.10)

For instance for $X = 25316$ we have $d_X = 2$ and $C_X = \frac{1}{5} = \frac{1}{30}$, while for $Y = 351642$ we have $d_Y = 3$ and $C_Y = \frac{-1}{6} = \frac{-1}{60}$.

The shuffle product between two words is given by [11]

$$\emptyset \shuffle A = A \shuffle \emptyset = A, \quad A \shuffle B \equiv a_1(a_2 \ldots a_n \shuffle B) + b_1(b_2 \ldots b_m \shuffle A),$$  

(1.11)

and it gives rise to all possible ways of interleaving the letters of $A$ and $B$ without changing their original orderings within $A$ and $B$. For example $12 \shuffle 34 = 1234 + 1324 + 1342 + 3124 + 3142$. The deconcatenation of a word $P$ into two factors is denoted by $P = XY$ and it corresponds to all possible ways of splitting the word $P$ into two words $X$ and $Y$. For example, if $P = 123$ then $P = XY$ gives rise to the pair of words $(X, Y) = (\emptyset, 123), (1, 23), (12, 3), (123, \emptyset)$. The generalization to $P = X_1 X_2 \ldots X_k$ is straightforward. Finally, the scalar product between two words $X$ and $Y$ is given by

$$\langle X, Y \rangle \equiv \begin{cases} 1, & \text{if } X = Y; \\ 0, & \text{otherwise}. \end{cases}$$  

(1.12)

The deshuffle map $\delta_k(P)$ is defined inductively as

$$\delta_k(i) = \underbrace{i \otimes \emptyset \otimes \ldots \otimes \emptyset}_{k \text{ times}} + \emptyset \otimes i \otimes \ldots \otimes \emptyset + \cdots + \emptyset \otimes \emptyset \otimes \ldots \otimes i$$  

(1.13)

$$\delta_k(i_1 \ldots i_n) = \delta_k(i_1) \cdots \delta_k(i_n),$$

or, equivalently, in terms of the shuffle product

$$\delta_k(P) = \sum_{X_1, \ldots, X_k} \langle P, X_1 \shuffle \ldots \shuffle X_k \rangle X_1 \otimes \ldots \otimes X_k.$$  

(1.14)
The reduced deshuffle map \( \tilde{\delta}_k(P) \) is obtained from \( \delta_k(P) \) by removing those terms which contain the empty word \( \emptyset \) as a tensor factor. For example, we have \( \delta_1(P) = P, \delta_2(12) = \emptyset \otimes 12 + 1 \otimes 2 + 2 \otimes 1 + 12 \otimes \emptyset, \tilde{\delta}_2(12) = 1 \otimes 2 + 2 \otimes 1, \) \( \tilde{\delta}_3(12) = 0 \) and

\[
\begin{align*}
\delta_2(123) &= \emptyset \otimes 123 + 1 \otimes 23 + 2 \otimes 13 + 3 \otimes 12 + 12 \otimes 3 + 13 \otimes 2 + 23 \otimes 1 + 123 \otimes \emptyset, \\
\tilde{\delta}_2(123) &= 1 \otimes 23 + 2 \otimes 13 + 3 \otimes 12 + 12 \otimes 3 + 13 \otimes 2 + 23 \otimes 1, \\
\tilde{\delta}_3(123) &= 1 \otimes 2 \otimes 3 + 1 \otimes 3 \otimes 2 + 2 \otimes 1 \otimes 3 + 2 \otimes 3 \otimes 1 + 3 \otimes 1 \otimes 2 + 3 \otimes 2 \otimes 1.
\end{align*}
\]

We shall also adopt Sweedler’s notation and write \( \delta_k(P) = \sum_{(P)} P(1) \otimes \ldots \otimes P(k) \)

We denote by \( \min(P) \) the minimum among the letters in \( P \). We shall always deal with \textit{multilinear words}, i.e., words with no repeated letters. Given such a word \( P \), its \textit{standard factorization} \( P = P_1 \cdots P_k \) is defined iteratively as follows. We put \( j = \min(P) \) and consider the unique factorization \( P = P' j P'' \): if \( P' = \emptyset \) we say that \( P \) is a \textit{(multilinear) Lyndon word} and we define its standard factorization to be \( P = P_1 \), otherwise we take \( P' = P_1 \cdots P_{k-1} \) the standard factorization of \( P' \), \( P_k := jP'' \) and the standard factorization of \( P \) is \( P = P_1 \cdots P_k \). Notice that by construction all the factors \( P_1, \ldots, P_k \) in the standard factorization of \( P \) are Lyndon words, and \( P_1 > \cdots > P_k \) in the lexicographical order; furthermore, the standard factorization is the only factorization of \( P \) satisfying both these properties\(^2\). For instance, the standard factorization of \( X = 56427138 \) is \( X = X_1 X_2 X_3 X_4 = (56)(4)(27)(138) \), and the standard factorization of \( Y = 37528416 \) is \( Y = Y_1 Y_2 Y_3 = (375)(284)(16) \).

2. \textbf{The color trace decomposition problem}

Let us consider a simple Lie algebra whose generators \( T^a \) satisfy\(^3\)

\[
[T^a, T^b] = i f^{abc} T^c, \quad \text{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab},
\]

where \( f^{abc} \) denote the totally anti-symmetric structure constants and \( \delta^{ab} \) is the Kronecker delta. The symmetrized trace of Lie algebra generators is defined by

\[
d^{12\ldots n} \equiv \text{Str}(T^1 T^2 \ldots T^n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{Tr}(T^{\sigma(1)} T^{\sigma(2)} \ldots T^{\sigma(n)}),
\]

\(^2\) In fact, this is a special case (the multilinear case) of the more general fact that any word admits a standard factorization into a non-increasing product of Lyndon words, see [10].

\(^3\) In [1] the left-hand side is written in terms of a representation-dependent normalization \( I_{2R} \) as \( \text{Tr}(T^a T^b)_{I_{2R}} = I_{2R} \delta^{ab} \). For convenience we use \( I_{2R} = \frac{1}{2} \) throughout this paper.
where the sum is over all $n!$ elements of $S_n$ and we abbreviate the customary index of the Lie algebra generator $a_j$ simply by $j$. Due to the cyclicity of the trace we have from (2.2) and (2.1) that $d^{12} = \frac{1}{2}\delta^{12}$.

As discussed in [1], one is interested in decomposing the trace $\text{Tr}(T^1 \ldots T^n)$ in terms of symmetrized traces and structure constants leading to an expansion of the form

$$\text{Tr}(T^1 T^2 \ldots T^n) = d^{12 \ldots n} + \sum_n (f d + f f d + \cdots + f f \cdots f), \quad (2.3)$$

which can always be done in a systematic manner. Using the algorithm implemented in the color package of FORM [2] and rewriting the results in the color basis\(^4\) to be described below gives the formulas in (1.5), with similar expansions at higher multiplicities (see e.g. the appendix B of [9]). These have been written in the basis of color factors chosen in [9], in which the letter 1 is always in the symmetrized trace factor $d^{1 \ldots}$. The remaining factors of structure constants will either have contracted indices such as $d^{12a} f^{45b} f^{b3a}$ and $d^{1ab} f^{24a} f^{35b}$ or will encompass all labels from 1 to $n$ when no factor of $d^{1 \ldots}$ is present. In both these cases we rearrange the labels in such a way that the minimum and maximum labels are at the extremities\(^5\) (where we consider a contracted index to be maximum). This leads to basis elements such as $f^{13a} f^{a2b} f^{b45}$ or $d^{12a} f^{34b} f^{b5a}$ and can be achieved using the Jacobi identities [1]

$$f^{a[i j f^{k}]}ab = 0, \quad d^{a(i_1 i_2 \ldots i_{n-1} f^{i_n})}ab = 0. \quad (2.4)$$

As explained in [1], the decomposition (2.3) can always be done using the following argument: starting from the trivial identity

$$\text{Tr}(T^1 \ldots T^n) = \text{Tr}(T^1 \ldots T^n) - \text{STr}(T^1 \ldots T^n) + d^{12 \ldots n} \quad (2.5)$$

one uses the commutation relation (2.1) to move the generators in each one of the $n!$ terms in $\text{STr}(T^1 \ldots T^n)$ to be in the same order as they appear in $\text{Tr}(T^1 \ldots T^n)$. Doing this for all $n!$ terms in $-\text{Str}(T^1 \ldots T^n)$ cancels the term $\text{Tr}(T^1 \ldots T^n)$ in the right-hand side of (2.5) while generating lower-order terms containing structure constants as a result of the commutation relation (2.1) and leading to (2.3).

Before discussing the general solution to decomposing traces of color factors we briefly review the definition of the Solomon idempotent.

\(^4\) The results given by the color package are not written in a basis of color factors.

\(^5\) This choice is inspired by the del Duca–Dixon–Maltoni (DDM) basis [12].
2.1. The Solomon idempotent

The Solomon idempotent appeared for the first time in the work of Solomon [3], who also noted its connection with Eulerian numbers. Hence the name Eulerian idempotent is also commonly attributed to it\footnote{Its characterization as a Lie idempotent was made by Reutenauer in [5] (see also [6]) but this aspect will not play a role in these notes.}. The Solomon idempotent appears in several different contexts in the mathematical literature such as in representations of the symmetric group [4,5], in free Lie algebras [10], in Hochschild homology [13,14] and more recently it has been used in connection with the Magnus series expansion solution to differential equations [8].

In order to define the Solomon idempotent, first recall the definition of the descent number $d_\sigma$ of the permutation $\sigma$,

$$d_\sigma \equiv |\{1 \leq i \leq n - 1 | \sigma(i) > \sigma(i + 1)\}|. \quad (2.6)$$

For example, the permutation 43512 has two descents (at the first and third positions) so $d_{43512} = |\{1, 3\}| = 2$. In addition, we define left-to-right nested commutators recursively by $\ell(i_1, i_2, \ldots, i_n) \equiv [\ell(i_1, i_2, \ldots, i_{n-1}), i_n]$, where $[i, j] = ij - ji$. For example $\ell(1, 2, 3, 4) = [[[1, 2], 3], 4]$. It is well known that Lie polynomials with $n$ letters can be written in terms of the $(n-1)!$ dimensional Dynkin basis $\ell(1, \sigma(2), \sigma(3), \ldots, \sigma(n))$.

The Solomon idempotent in the Dynkin basis of Lie polynomials is given by (this is shown in [7], see also [8])

$$E(x_1 x_2 \cdots x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \frac{(-1)^{d_\sigma}}{(d_\sigma)!} \ell(x_1, x_{\sigma(2)}, \ldots, x_{\sigma(n)}), \quad (2.7)$$

where $x_1, \ldots, x_n$ are non-commutative indeterminates. Using the notation from (1.10), we might also write

$$E(x_1 x_2 \cdots x_n) = \sum_{\sigma \in S_n} C_\sigma \ell(x_1, x_{\sigma(2)}, \ldots, x_{\sigma(n)}). \quad (2.8)$$

For instance for $n \leq 4$, defining $E^{12\ldots n} \equiv E(T^1 T^2 \cdots T^n)$ for the Solomon idempotent written with Lie-algebra generators $T^j$, formula (2.7) yields

$$E^1 = T^1 \quad (2.9)$$
In view of (2.1) we define

\[ E^P := E^a T^a, \]  

and note that, with the exception of \( E^a_1 = \delta^1_a \), all expansion coefficients \( E^a_P \) are polynomials in the structure constants. From \( E^{12} = \frac{1}{2} [T^1, T^2] = \frac{i}{2} f^{12a} T^a \) we get

\[ E^{12} = \frac{i}{2} f^{12a}. \]  

And similarly,

\[ E^{123}_a = -\frac{1}{3} f^{12j} f^{j3a} + \frac{1}{6} f^{13j} f^{j2a}, \]  

\[ E^{1234}_a = -\frac{i}{4} f^{12j} f^{j3k} f^{k4a} + \frac{i}{12} f^{12j} f^{j4k} f^{k3a} + \frac{i}{12} f^{13j} f^{j2k} f^{k4a} + \frac{i}{12} f^{13j} f^{j3k} f^{k2a} + \frac{i}{12} f^{14j} f^{j2k} f^{k3a} - \frac{i}{12} f^{14j} f^{j3k} f^{k2a}. \]

In general, using the notations (1.9) and (1.10) from the introduction, we may rewrite (2.8) as

\[ E^{1...n}_a = \sum_{\sigma \in S_n \atop \sigma(1)=1} i^{n-1} C_\sigma F^\sigma_a. \]  

A brief inspection of the expansions in (1.5) and (2.12) reveals that the Solomon idempotent captures the coefficients of the various terms in (1.5). This will be demonstrated below for the general case.

2.2. Trace decomposition from Solomon’s projection

In order to obtain a closed formula that solves the color trace decomposition problem we recall the projection obtained by Solomon in [3]

\[ T^P = \sum_{k \geq 1} \sum_{X_1, X_2, \ldots, X_k} \frac{1}{k!} (P, X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k) E^{X_1} E^{X_2} \ldots E^{X_k} \]  


where $T^P \equiv T^{P_1}T^{P_2}\ldots T^{P_n}$ for a word $P = p_1p_2\ldots p_n$. The multiplicity-two instance of (2.14) corresponds to the well-known decomposition into a symmetric and antisymmetric combination (recall that $E^i = T^i$)

$$T^1T^2 = E^{12} + \frac{1}{2}(E^1E^2 + E^2E^1) = \frac{1}{2}[T^1, T^2] + \frac{1}{2}(T^1T^2 + T^2T^1).$$

(2.15)

But already at multiplicity three

$$T^1T^2T^3 = E^{123} + \frac{1}{2}(E^{12}T^3 + E^{13}T^2 + E^{23}T^1 + T^1E^{23} + T^2E^{13} + T^3E^{12})$$

$$+ \frac{1}{3!}(T^1T^2T^3 + T^1T^3T^2 + T^2T^1T^3 + T^2T^3T^1 + T^3T^1T^2 + T^3T^2T^1).$$

(2.16)

it is far from obvious that plugging in the expansions of the Solomon idempotents from (2.9) into the right-hand side recovers the monomial $T^1T^2T^3$ in the left-hand side.

As one can see from the above examples, the formula (2.14) projects the product $T^1T^2\ldots T^n$ into its totally symmetric component $\frac{1}{n!}T^{(1)T^2\ldots T^n} := \frac{1}{n!} \sum_{\sigma \in S_n} T^{\sigma(1)}\ldots T^{\sigma(n)}$ plus lower-order terms containing Eulerian idempotents. After taking the trace on both sides of Solomon’s projection (2.14), the totally symmetric component is mapped to the symmetrized trace while the lower order terms are mapped to sums of symmetrized traces multiplied by linear combinations of structure constants as dictated by the Eulerian idempotents. This is the solution to the color trace decomposition problem.

To see this more explicitly, we use the definition (2.10) to rewrite (2.14) as

$$T^P = \sum_{k \geq 1} \sum_{X_1, X_2, \ldots, X_k} \frac{1}{k!} \langle P, X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k \rangle E_{a_1}^{X_1} E_{a_2}^{X_2} \ldots E_{a_k}^{X_k} T^{a_1}T^{a_2}\ldots T^{a_k}$$

(2.17)

where we used that the shuffle product is commutative to obtain the symmetrized product of the algebra generators by ordering the sum according to $X_1 > X_2 > \cdots > X_k$ and defined $\tau^{a_1\ldots a_k} = \frac{1}{k!}T^{(a_1\ldots a_k)}$. Therefore multiplying (2.17) by $T^1$ from the left, taking the trace on both sides and using that $\text{Tr}(T^{1}\tau^{a_1\ldots a_k}) = d^{1a_1\ldots a_k}$ leads to

$$\text{Tr}(T^1P) = \sum_{k \geq 1} \sum_{X_1, X_2, \ldots, X_k} \langle P, X_1 \sqcup X_2 \sqcup \ldots \sqcup X_k \rangle E_{a_1}^{X_1} E_{a_2}^{X_2} \ldots E_{a_k}^{X_k} d^{1a_1a_2\ldots a_k}$$

(2.18)

Alternatively, lifting the ordering restriction in the sum while compensating the overcount with $\frac{1}{k!}$ and using (1.14) leads to formula (1.6) from the introduction

$$\text{Tr}(T^1P) = \sum_{k \geq 1} \sum_{(P)} \frac{1}{k!} d^{1a_1a_2\ldots a_k} E_{a_1}^{P(1)} E_{a_2}^{P(2)} \ldots E_{a_k}^{P(k)},$$

(2.19)
concluding its proof.

Applying (2.19) for traces with up to six generators yields:

\[
\begin{align*}
\text{Tr}(T^1 T^2) &= d^{12} \\
\text{Tr}(T^1 T^2 T^3) &= d^{123} + d^{1a} E_a^{23} \\
\text{Tr}(T^1 T^2 T^3 T^4) &= d^{1234} + d^{12a} E_a^{34} + d^{13a} E_a^{24} + d^{14a} E_a^{23} + d^{1a} E_a^{234} \\
\text{Tr}(T^1 T^2 \cdots T^5) &= d^{12345} + d^{123a} E_a^{45} + d^{124a} E_a^{35} + d^{125a} E_a^{34} + d^{134a} E_a^{25} + d^{135a} E_a^{24} \\
&\quad + d^{145a} E_a^{23} + d^{12a} E_a^{345} + d^{13a} E_a^{245} + d^{14a} E_a^{235} + d^{15a} E_a^{234} \\
&\quad + d^{1ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) + d^{1a} E_a^{2345} \\
\text{Tr}(T^1 T^2 \cdots T^6) &= d^{123456} + d^{123a} E_a^{56} + d^{1235a} E_a^{46} + d^{1236a} E_a^{45} + d^{1245a} E_a^{36} + d^{1246a} E_a^{35} \\
&\quad + d^{1256a} E_a^{34} + d^{1345a} E_a^{26} + d^{1346a} E_a^{25} + d^{1356a} E_a^{24} + d^{1456a} E_a^{23} \\
&\quad + d^{12a} E_a^{3456} + d^{12a} E_a^{56} + d^{13a} E_a^{345} + d^{14a} E_a^{235} + d^{15a} E_a^{234} \\
&\quad + d^{1ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) + d^{1a} E_a^{2345} \\
&\quad + d^{12ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) + d^{13ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) \\
&\quad + d^{14ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) + d^{15ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) \\
&\quad + d^{16ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) + d^{12} E_a^{3456} + d^{13} E_a^{2456} + d^{14} E_a^{2356} + d^{15} E_a^{2346} + d^{16} E_a^{2345} \\
&\quad + d^{1ab} (E_a^{23} E_b^{45} + E_a^{24} E_b^{35} + E_a^{25} E_b^{34}) + E_a^{35} E_b^{245} + E_a^{36} E_b^{245} + E_a^{45} E_b^{236} + E_a^{46} E_b^{235} + E_a^{56} E_b^{234} \\
&\quad + d^{1a} E_a^{23456}
\end{align*}
\]

It is not difficult to see that the total number of terms generated by the formula (2.19) for \( n = 3, 4, 5, 6, 7, 8, 9 \ldots \) is equal to the Bell numbers 2, 5, 15, 52, 203, 877, 4140, \ldots, respectively.

Finally, in order to obtain formula (1.7) we look again at equation (2.18), with the letter 1 replaced by 0 and the word \( P \) replaced by \( 1 \cdots n \). For a fixed deshuffle \( X_1 \otimes \cdots \otimes X_k \) of \( P \), we expand the coefficients \( E_{a_1}^{X_1}, \ldots, E_{a_k}^{X_k} \) in terms of structure constants according to (2.8). Notice that for \( 1 \leq j \leq k \) we have \( X_j = i_1 \cdots i_{|X_j|} \) with \( i_1 < \cdots < i_{|X_j|} \): by
(2.13) we get \( E_{a_j}^{X_j} = \sum_{\sigma_j} i^{|X_j| - 1} C_{\sigma_j} F_{a_j}^{\sigma_j} \), where the sum runs over the words \( \sigma_j \) obtained by permuting the last \(|X_j| - 1\) letters of \( X_j \) while keeping the first one fixed. We obtain

\[
\text{Tr}(T^{01...n}) = \sum_{k \geq 1} \sum_{X_1 > ... > X_k} \langle 1...n, X_1 \sqcup ... \sqcup X_k \rangle \sum_{\sigma_1, ..., \sigma_k} i^{n-k} C_{\sigma_1} ... C_{\sigma_k} F_{a_1}^{\sigma_1} ... F_{a_k}^{\sigma_k} d^{a_1 a_2 ... a_k},
\]

where the last sum runs over the words \( \sigma_1, ..., \sigma_k \) obtained from \( X_1, ..., X_k \) as above. Notice that each of the words \( \sigma_1, ..., \sigma_k \) has \( \min(\sigma_j) = \min(X_j) \) as its first letter (in the terminology of subsection 1.1, all the \( \sigma_1, ..., \sigma_k \) are Lyndon words), and \( \sigma_1 > ... > \sigma_k \) in the lexicographical order (since \( X_1 > ... > X_k \)). Therefore, taking the permutation \( \sigma \in S_n \) associated with the word \( \sigma_1 ... \sigma_k \), the standard factorization of \( \sigma \) as a word is precisely \( \sigma = \sigma_1 \cdots \sigma_k \). In the other direction, given \( \sigma \in S_n \) with standard factorization \( \sigma = \sigma_1 \cdots \sigma_k \), we define \( X_1, ..., X_k \) by rewriting the letters of \( \sigma_1, ..., \sigma_k \) in increasing order: then \( X_1 \otimes ... \otimes X_k \) is a deshuffle of \( 1...n \) with \( X_1 > ... > X_k \), and the words \( \sigma_1, ..., \sigma_k \) are obtained from \( X_1, ..., X_k \) as required in the last summation of (2.21). This establishes a bijective correspondence between the terms of (2.21) and (1.7), showing that the two formulas are equivalent and concluding the proof of the latter.

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