Optimal $1 \rightarrow M$ universal quantum cloning via spin networks

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We present a scheme that transforms 1 qubit to $M$ identical copies with optimal fidelity via free dynamical evolution of spin star networks. We show that the Heisenberg XXZ coupling can fulfill the challenge. The initial state of the copying machine and the parameters of the spin Hamiltonian are discussed in detail. Furthermore we have proposed a feasible method to prepare the initial state of the copying machine.

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One of the most fundamental differences between classical and quantum information is the no-cloning theorem [1,2]. It states that accurate cloning of any arbitrary quantum state is impossible. Nevertheless it doesn’t forbid one to clone quantum states approximately. In the early work of Bužek and Hillery [3], an optimal $1 \rightarrow 2$ universal quantum cloning machine (UQCM) was proposed where the copying process is input-state independent. And quantum cloning machine for equatorial qubits which are so called phase-covariant cloning (PCC) machine was proposed by Bruß et al. [4]. Optimal fidelity and optimal quantum cloning transformation of general $N$ to $M$ ($M > N$) are presented in [5–8]. It was also shown a UQCM can be realized by a network consisting of quantum gates [3].

Several approaches have been made to realize the unitary transformations leading to the cloning process experimentally [10–13]. However, most of these schemes are based on quantum logic gates and post-selection methods, which need time modulations. Recently, quantum computation via spin networks based on Heisenberg couplings was presented [14–27]. One achieve is that with Heisenberg chains, high fidelity quantum state transfer can be achieved [15–24]. The most attracting feature of this approach is that it needn’t time modulation for the qubits couplings. Once the initial states and the evolution Hamiltonian is determined, the system can faithfully implement designated computation task through free dynamical evolution. The whole computational evolution does not involve any external controlling, which provides relatively longer decoherence time for the system. Schemes for PCC via spin networks was proposed in the work of De Chiara et al. [25, 26]. Chen et al. [27] further improved the $1 \rightarrow M$ PCC case to an optimal level. However the optimal UQCM via a spin network is still a challenge.

In this paper, we show that by properly introducing the ancilla qubits, designing the spin exchange interactions, and choosing the initial state of the cloning machine, optimal $1 \rightarrow M$ UQCM can be realized via the free evolution of a spin star network Hamiltonian. Moreover a scheme on preparing the initial state of the cloning machine have been proposed.

The spin network involved in our scheme forms a star configuration (See Fig.1(1)). The central qubit (input state) is labeled $I$, the $M$ target qubits labeled $T$, and the $M − 2$ ancillas labeled $A$. We start with the conventional Heisenberg XXZ coupling Hamiltonian without an externally applied magnetic field.

$$ H = \frac{J_1}{2} \sum_{i=1}^{M} (\sigma_i^x \sigma_i^x + \sigma_i^y \sigma_i^y + \lambda_1 \sigma_i^z \sigma_i^z) + \frac{J_2}{2} \sum_{i=1}^{M-2} (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \lambda_2 \sigma_i^z \sigma_{i+1}^z), \quad (1) $$

where $\sigma_i^{x,y,z}$, $\sigma_{i+1}^{x,y,z}$, $\sigma_{A_i}^{x,y,z}$ are Pauli matrices of the input particle, the target qubit, and the ancilla qubits respectively (we introduce $M−2$ ancilla qubits), $J_1$ and $J_2$ are the exchange spin coupling coefficients between the input qubit with the target qubits and the ancilla qubits respectively, $\lambda_1$ and $\lambda_2$ are the anisotropy parameters (when $\lambda = 0$, the Hamiltonian reduces to $XX$ model while $\lambda = 1$ it corresponds to Heisenberg model).

Following with Gisin and Massar [3] we suppose the unitary transformation for optimal $1 \rightarrow M$ cloning take the form:

$$ U_{1,M} \ket{\uparrow}_I \otimes \ket{R} = \sum_{i=0}^{M-1} \gamma_i |S(M, M-i)\rangle_T \otimes |R_i\rangle, \quad (2) $$

$$ U_{1,M} \ket{\downarrow}_I \otimes \ket{R} = \sum_{i=0}^{M-1} \gamma_{M-1-i} \times |S(M, M-1-i)\rangle_T \otimes |R_i\rangle, \quad (3) $$

$$ \gamma_i = \sqrt{\frac{2(M-i)}{M(M+1)}}, $$

where $U_{1,M} = e^{-iHt_0}$ ($t_0$ is the evolution time) denotes the free evolution of the spin system, $|R\rangle$ denotes the initial state of the copying machine and $M$ blank copies. $|S(M,i)\rangle_T$ is the normalized symmetry state of the $M$ target qubits with $i$ spins up, $|R_i\rangle$ are orthogonal normalized sates of the ancilla qubits (here include the input
We choose the initial state $|R\rangle$ as follows:

$$|R\rangle = C \sum_{i=1}^{M-1} \sqrt{i(M-i)} |a_i\rangle,$$

where $C = \sqrt{\frac{6}{(M-1)M(M+1)}}$ is the normalization factor, $|S(M,i)\rangle_T \otimes |S(M-2,M-1-i)\rangle_A$ is the normalized symmetry state of the $M-2$ ancilla qubits. Noticing $|R\rangle$ is invariant under the spin flipping operation, we will show later by spin flipping both sides of Eq. (2), Eq. (3) is automatically satisfied.

First we discuss the conditions to satisfy Eq. (2). Instead of studying all the states in the Hilbert space of the Hamiltonian (1), we would rather to introduce a two dimensional subspace $\mathcal{H}_{ab}$ (we use $\psi_{ab}$ to note states in this subspace, $\psi_{a}^{\perp}$ to note states orthogonal to this subspace), which is spanned by two basic normalized orthogonal states $|a\rangle$, and $|b\rangle$

$$|a\rangle = |\uparrow\rangle \otimes |R\rangle,$$

$$|b\rangle = \frac{\sqrt{2C}}{2} |\downarrow\rangle \otimes \left( \sum_{j=1}^{M-1} \sqrt{j(j+1)} |b_j\rangle \right).$$

Notice $|a\rangle$ is our initial state for Eq. (2), and we will show that some linear combination of these two states has the same form of the right hand side of Eq. (2). We find if the parameters of the spin Hamiltonian (1) obey the following relations

$$J_1 = J_2 = \mathcal{J}, \quad \lambda_1 = -\lambda_2 = \lambda.$$

the subspace we choosing is closed, i.e., $\langle \psi_{ab} | H | \psi_{ab}^{\perp} \rangle = 0$. Then it is convenient for us to calculate the free evolution of the system in this two dimensional subspace. It is useful to rewrite the Hamiltonian (1) with the Ladder operators. Using the relations (7) the Hamiltonian take the form:

$$s_1^+ = (\sigma_1^x + i\sigma_1^y)/2, \quad J_1^+ = \sum_T s_T^+, \quad J_a^+ = \sum_A s_A^+,$$

$$s_3^z = \sigma_3^z/2, \quad J_3 = \sum_T \sigma_T^z/2, \quad J_3 = \sum_A \sigma_A^z/2,$$

$$H = \mathcal{J}(s_T^z(J_T^z - J_a^z) + s_A^z(J_T^z - J_a^z) + 2\lambda s_T^z(J_T^z + J_a^z)).$$

With this representation of the spin Hamiltonian it is easy for us to calculate $H$ act upon our basis.

$$H|a\rangle = \sqrt{2}\mathcal{J}|b\rangle,$$

$$H|b\rangle = -\mathcal{J}\lambda|b\rangle + \sqrt{2}\mathcal{J}|a\rangle.$$
where $|S(M-1, i)⟩_{A⊗I}$ denotes the normalized symmetry state in the direct product space of the input qubit and ancilla qubits. The state $|a(t_0)⟩$ is exactly the same form as the righthand side of Eq. (2). The orthogonal normalized states $|R_i⟩$ take the form:

$$|R_i⟩ = -ie^{i\mathcal{J}t}|S(M-1, i)⟩_{A⊗I}.$$  

To go further, we introduce the spin flipping operator,

$$P = P^{-1} = \sigma^z_I (\prod_i \sigma^z_{T_i}) (\prod_j \sigma^z_{A_j}).$$

This unitary operation flips all the spins in our consideration. It is easy to see that the Heisenberg XXZ spin Hamiltonian (1) is invariant under such operation, i.e., $PH^{-1}P^{-1} = H$. The initial state for Eq. (3) is $|⟩⟩_I ⊗ |R⟩ = P |a⟩ = |a_r⟩$, after having evolved for $t$

$$|a_r(t)⟩ = U_{1,M}(t)|a_r⟩ = P \hat{U}_{1,M}(t)|a⟩ = P |a(t)⟩.$$  

When the evolution time $t = t_0$,

$$|a_r(t_0)⟩ = P |a(t_0)⟩ = -ie^{i\mathcal{J}t} \sum_{i=0}^{M-1} \gamma_i |S(M, i)⟩_T ⊗ |S(M-1, M-1-i)⟩_{A⊗I}.$$  

It is exactly the same form as the righthand side of Eq. (3). The above calculation shows that we can find such conditions (17)(8) satisfying Eq. (2) simultaneously, i.e., the optimal cloning can be fulfilled under such conditions.

One interesting thing is that through the beginning to the end of this free evolution the fidelity of a single copy to the input is independent of the input state (a universal cloning). Suppose the input state is: $|\text{input}⟩_I = |α⟩ |↓⟩_I + |β⟩ |↑⟩_I$. After having evolved for $t$, the state of the system takes the form: $|t⟩ = |α|a(t)⟩ + |β|a_r(t)⟩$. The reduced density matrix of a single copy at $t$ can be calculated directly,

$$\rho = \frac{\cos^2\varphi}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sin^2\varphi}{3M} \times \begin{pmatrix} α^2(1+2M) + β^2(M-1) & β^2α(M+2) \\ α^2β(M+2) & β^2(1+2M) + α^2(M-1) \end{pmatrix}.$$  

The fidelity of this copy is

$$F = \frac{1}{2} \cos^2\varphi + \frac{2M+1}{3M} \sin^2\varphi.$$  

$F$ is only a function of the rescaled time $\varphi (\varphi = \sqrt{3}t)$.

So the whole cloning process is input state independent. When $t = 0$ the fidelity is $1/2$, and when $t = t_0$ the fidelity reaches its optimal bound $(2M+1)/3M$ (see Fig.2).

One shortcoming of quantum cloning based on logic gates is the circuit becomes more complicated as $M$ increases. As a result, when $M$ is large it may be difficult for one to go through the copying process before the state having been decoherenced. However, the evolution time of our scheme is $t_0 = \sqrt{3π/(12J)}$, which is independent of $M$. This is an advantage to fulfill large $M$ cloning.

The problem now is how to prepare the initial state (4). For $M = 2$ ($M = 3$), $|R⟩$ is two (four) particle symmetry state. But for $M > 3$, $|R⟩$ is not simply a symmetry state. Interestingly, we find that $|R⟩$ is exactly the ground state of some spin Hamiltonian. And it is feasible for one to prepare it by just cooling the system. Such Hamiltonian is consisted of two parts

$$H' = H'_0 + H'_1.$$  

$H'_0$ is the part with Heisenberg XXZ coupling ($\lambda = -1$) between the target qubits and the ancilla qubits (Fig.1(2)),

$$H'_0 = J'(J^+_tJ^-_t + J^+_A J^-_A - 2J^+_tJ^-_A),$$

where $J'$ is the spin coupling coefficient, $J_t$ and $J_A$ are the total angular momentum operators of the target qubits and the ancilla qubits respectively. $H'_1$ is the part with Ising coupling between all the qubits,

$$H'_1 = \Delta \sum_{k<l}^{M} \sigma^x_{T_k} \sigma^x_{T_l} + \sum_{k=1}^{M-2} \sigma^z_{A_j} \sigma^z_{A_k} + \sum_{i=1}^{M} \sum_{j=1}^{M-2} \sigma^z_{T_i} \sigma^z_{A_j},$$

where $\Delta$ is the coupling coefficient. These two parts are commute, $[H'_0, H'_1] = 0$. We find $|R⟩$ is an eigenvector of
$H'_0$ and $H'_1$ simultaneously

$$H'_0 | R \rangle = \frac{J'(M^2 - 4)}{2} | R \rangle,$$

$$H'_1 | R \rangle = - \frac{\Delta(M - 1)}{2} | R \rangle.$$

To prove $| R \rangle$ is the ground state we solve the spectrum of $H'$. We introduce the unitary operator $Q_T$ to act on the target qubits (it is equivalence to introduce $Q_A$ acting on the ancilla qubits),

$$Q_T = Q_T^{-1} = \prod_{i=1}^{M} \sigma^z_{i_T}.$$

This unitary operation transforms $H'_0$ to the Heisenberg Hamiltonian and leaves $H'_1$ unchange,

$$Q_T H'_0 Q_T^{-1} = -\mathcal{J}'(J_T^z J_A^z - J_T J_A^2 + 2J_T^z J_A^2)$$

$$Q_T H'_1 Q_T^{-1} = H'_1$$

So the spectrum of $H'$ is

$$E' = -\mathcal{J}'(\Delta J_T + j_T - j_T^2) + \Delta j_T^2 - \Delta(M - 1)$$

where $j$ ($j_T^z$) is the total (z component) angular momentum quantum number of the transformed Hamiltonian.

If we choose $\mathcal{J}' < 0$, and $\Delta > 0$, the nondegenerate ground state energy of $H'$ is $\mathcal{J}'(M^2 - 4)/2 - \Delta(M - 1)/2$ ($j = 1, j_T = M/2, j_A = (M - 2)/2, j_T^z = 0$), which is just the eigenvalue of $| R \rangle$. So far, we have proved $| R \rangle$ is the ground state of $H'$. Thus the initial state of the copying machine can be prepared by cooling the system. No measurement is involved in this implementation, and also we needn’t any time modulation of the Hamiltonian.

Throughout this paper, optimal UQCM that produce $M$ copies out of a single input via free evolution of spin star networks has been discussed. We have proved for arbitrary $M$ the unitary evolution can be fulfilled in a two dimensional subspace. Using this character we find the analytical solutions for the optimal $1 \rightarrow M$ universal cloning process. Through this process the fidelity keeps input state independent, and it reaches the optimal bound at $t = \sqrt{3\pi /6}\mathcal{J}$, which is independent of $M$. Also we have studied the initial state of the copying machine in detail, and find it is exactly the ground state of some spin Hamiltonian (only quadratic terms are involved). Thus, the preparation of the initial state can be accomplished by cooling such systems. No measurement and time modulation is involved here. Therefore our result opens up a promising prospect towards robust optimal UQCM. Such a prospect is relevant for several experimental systems [28, 29].

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