CONVERGENCE OF THE WEAK KÄHLER-RICCI FLOW ON MANIFOLDS OF GENERAL TYPE

TAT DAT TÔ

Abstract. We study the Kähler-Ricci flow on compact Kähler manifolds whose canonical bundle is big. We show that the normalized Kähler-Ricci flow has long time existence in the viscosity sense, is continuous in a Zariski open set, and converges to the unique singular Kähler-Einstein metric in the canonical class. The key ingredient is a viscosity theory for degenerate complex Monge-Ampère flows in big classes that we develop, extending and refining the approach of Eyssidieux-Guedj-Zeriahi.

Introduction

Let \((X, \omega)\) be a compact Kähler manifold of general type, i.e the canonical bundle \(K_X\) is big. We study the normalized Kähler-Ricci flow on \(X\):

\[
\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) - \omega_t, \quad \omega|_{t=0} = \omega_0.
\]

(0.1)

Let \(T\) be the maximal existence time of the smooth flow. It is known that \(T = \infty\) if and only if the canonical bundle \(K_X\) is nef, and in this case the normalized Kähler-Ricci flow converges to a singular Kähler-Einstein metric on \(K_X\) (cf. \([Tsu88, TZ06]\)). When \(K_X\) is not nef, the flow has a finite time singularity \((T < \infty)\). The limit class of the flow is

\[
\{\alpha_T\} = \lim_{t \to T} \{\omega(t)\} = e^{-T}\{\omega_0\} + (1 - e^{-T})c_1(K_X).
\]

The class \(\alpha_T\) is big and nef. For \(t > T\), \(\alpha_t := \{\omega(t)\}\) is still big but no longer nef, thus we can not continue the flow in the classical sense (we refer to \([SW13, Tos18]\) for more details about the Kähler-Ricci flow).

It was asked by Feldman-Ilmanen-Knopf \([FIK03, Question 8, Section 10]\) whether one can define and construct weak solutions of Kähler-Ricci flow after the maximal existence time. In \([ST12, ST17]\), Song and Tian have succeeded in repairing some finite time singularities, defining weak solutions in the sense of pluripotential theory, by using strong algebraic results from the Minimal Model Program and by changing the underlying manifolds. In \([BT12]\), Boucksom and Tsuji have tried to run the weak normalized Kähler-Ricci on projective varieties beyond the maximal time using the discretization of the Kähler-Ricci flow and algebraic tools. They have proposed the following:

Conjecture.\([BT12, Conjecture 1, page 208]\) Let \(X\) be a compact Kähler manifold with pseudoeffective canonical bundle and \(\omega_0\) be a Kähler form on \(X\). Then there exists a family of closed semipositive current \(\omega(t)\) on \(X\) such that
\[ \left\{ \omega(t) \right\} = e^{-t} \{ \omega_0 \} + \left( 1 - e^{-t} \right) c_1(K_X) \text{ and } \omega(0) = \omega_0, \]

(2) for any \( T > 0 \), there exists a nonempty Zariski open subset \( U(T) \) such that \( \omega(t) \) is a Kähler form on \( U(T), \forall t \in [0, T) \).

(3) on \( U(T) \times [0, T) \), \( \omega(t) \) satisfies the normalized Kähler-Ricci flow \((0.1)\).

In this note we give an answer to the question of Feldman-Ilmanen-Knopf and study the conjecture of Boucksom-Tsuji. We moreover show that the weak normalized Kähler-Ricci flow converges to the unique singular Kähler-Einstein metric in \( K_X \) constructed in \([BEGZ10, EGZ09]\). Our method is based on a viscosity approach; It does not use any deep algebraic technology and allows us to keep working on the same underlying manifold. Precisely, we have the following theorem:

**Theorem A.** Let \((X, \omega_0)\) be a compact Kähler manifold with \( K_X \) is big. Fix \( \theta \) a smooth \((1, 1)\)-form in \( c_1(K_X) \). Then the Kähler-Ricci flow starting from \( \hat{\omega} \in \{ \omega_0 \} \)

\[ \frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) - \omega_t \]

admits a unique viscosity solution \( \omega_t = e^{-t} \omega_0 + (1 - e^t) \theta + dd^c \varphi_t \) for all time.

Moreover the flow converges, exponentially fast in \( \text{Amp}(K_X) \) (see Definition 1.4), to the unique singular Kähler-Einstein metric in the canonical class \( K_X \). And

- for \( 0 < t < T_{max} \), the function \( x \mapsto \varphi_t(x) \) identifies with the smooth solution in \([TZ06]\), where \( T_{max} \) is the maximal existence time of the smooth flow,
- for \( t \geq T_{max} \) the flow \( (\varphi_t) \) is continuous in \([T_{max}, +\infty) \times \text{Amp}(K_X)\).

A compact Kähler manifold with \( K_X \) big turns out to be projective, by a classical result of Moishezon (cf. \([Moi67]\)). We actually prove a more general convergence result (see Theorem 4.4), valid in the Kähler setting.

Since the Kähler-Ricci flow can rewritten as a parabolic complex Monge-Ampère equation, a key ingredient of our approach is to construct weak solutions for degenerate complex Monge-Ampère flows

\[ (\theta_t + dd^c \varphi_t)^n = e^{\varphi_t + F(t, x, \varphi_t)} \mu \text{ on } X_T := [0, T) \times X, \]

(0.2)

where

- \( (\theta_t)_{t \in [0, T]} \) be a continuous family of smooth closed \((1, 1)\)-forms such that \( \alpha_t = \{ \theta_t \} \) is big,
- \( F(t, x, r) \) is a continuous in \([0, T) \times X\) and non decreasing in \( r \),
- \( \mu(x) = f(x) dV \geq 0 \) is a continuous volume form on \( X \).

Degenerate complex elliptic Monge-Ampère equations on compact Kähler manifold have recently been studied intensively using tools from pluripotential theory following the pioneering work of Bedford and Taylor in the local case (cf. \([BT76, BT82, Kol98, GZ05, GZ07, BEGZ10]\)). A pluripotential theory for the parabolic side only developed recently \([GLZ18a, GLZ18b]\).

A complementary viscosity approach for complex Monge-Ampère equations has been developed in \([EGZ11, EGZ17, EGZ15a, HL09, Wan12]\). The similar theory for the parabolic case has been developed in \([EGZ15b]\) on complex domains (see also \([DLT19]\)
for its extension) and in [EGZ16, EGZ18] on compact Kähler manifolds. It is very interesting to compare the viscosity and pluripotential notions (we refer the reader to [GLZ] for more details).

For complex Monge-Ampère flows, both theories have been developed when the involved class \((\alpha_t)\) is big and semipositive. For further applications, we need to extend these theories in the general case where \((\alpha_t)\) is not necessarily semipositive.

In the first part of the note, we therefore establish a viscosity theory for degenerate complex Monge-Ampère flows where the involved classes \((\alpha_t)\) are big, not necessarily semipositive, extending the results in [EGZ16, EGZ18]. We refer the reader to Section 2.1 for the adapted definition of viscosity subsolution (resp. supersolution). The first result is a general viscosity comparison principle as follows.

**Theorem B.** Assume that \(\theta_t \geq \theta\) for a smooth \((1,1)\)-form \(\theta\) in some fixed big class \(\alpha\). Let \(\varphi\) (resp. \(\psi\)) be a viscosity subsolution (resp. supersolution) to (0.2). Then for any \((t,x) \in [0,T) \times \text{Amp}(\alpha)\)

\[
(\varphi - \psi)(t,x) \leq \max\{ \sup_{\{0\} \times X} (\varphi - \psi)^n, 0\}.
\]

This comparison principle not only generalizes previous results to the case of big cohomology classes but also refines the Kähler case, since our assumption is weaker than these previous works (the authors needed some conditions on either \(\partial_t \varphi\) or \(\theta_t\), cf. [EGZ16, Theorem 2.1], [EGZ18, Theorem 4.2]). In the present work, we exploit the concavity of \(\log \det\) to overcome the difficulties in [EGZ16, EGZ18]. Moreover, Theorem B can be extended to adapt the involved classes of the Kähler-Ricci flow (we refer the reader to Corollary 2.11).

As a first application of the comparison principle, we study the Cauchy problem on a compact Kähler manifold \((X,\omega)\)

\[
(CP_1) \begin{cases}
(\theta + dd^c \varphi_t)^n = e^{\varphi_t + \varphi_t} \mu \\
\varphi(0,x) = \varphi_0,
\end{cases}
\]

where

- \(\theta\) is a smooth \((1,1)\)-form in a fixed big class,
- \(\varphi_0\) is an \(\theta\)-psh function with minimal singularities which is continuous in \(\text{Amp}(\{\theta\})\)
- \(\mu = fdV > 0\) is a continuous volume form on \(X\).

There exists a unique solution to the static (elliptic) equation

\[
(\theta + dd^c \varphi)^n = e^\varphi \mu
\]

by [BEGZ10].

We first prove the existence of viscosity subsolutions and supersolutions to \((CP_1)\) and construct barriers at each point \(\{0\} \times \text{Amp}(\{\theta\})\). We then use the Perron method to show the existence of a unique viscosity solution:

**Theorem C.** The exists a unique viscosity solution to \((CP_1)\) in \([0,\infty) \times \text{Amp}(\{\theta\})\). Moreover, the flow asymptotically recovers the solution of the elliptic Monge-Ampère equation (0.3).
In the second part, we apply our techniques and study the normalized Kähler-Ricci flow on compact Kähler manifolds whose the canonical bundle is \textit{big}, i.e \textit{manifolds of general type}. We first use the viscosity theory above to construct the weak flow for all time. We then prove the convergence result of the flow, finishing the proof of Theorem A. In particular, Theorem C is used to construct a viscosity supersolution which gives a uniform upper bound to the potential of the flow in \text{Amp}(K_X).

The paper is organized as follows. In Section 1 we recall some notations of the viscosity theory on compact Kähler manifolds. In Section 2 we define the viscosity sub/super solutions for Complex Monge-Ampère flows on big classes, and prove Theorem B. As a first application of Theorem B, we prove Theorem C in Section 3. Finally we prove the existence and convergence of the normalized Kähler-Ricci flow on compact Kähler manifolds of general type in Section 4.

\textbf{Acknowledgement.} The author is grateful to Vincent Guedj for support, suggestions and encouragement. We also would like to thank Sébastien Boucksom, Hoang Son Do, Henri Guenancia and Ahmed Zeriahi for fruitful discussions. We would like to thank Hoang-Chinh Lu for very useful discussions, suggestions and his encouragement to write down a missing argument in the proof of Theorem 4.3. The author would like to thank the referees for very useful comments and suggestions. This work is supported partially by the project ANR GRACK.

1. Preliminary

1.1. Monge-Ampère operator in big cohomology classes.

1.1.1. Big cohomology classes. Let \((X, \omega)\) be a compact Kähler manifold and let \(\alpha \in H^{1,1}(X, \mathbb{R})\) be a real \((1, 1)\)-cohomology class.

\textbf{Definition 1.1.} The class \(\alpha\) is \textit{pseudo-effective} if it can be represented by a closed positive \((1, 1)\)-current \(T\). Moreover \(\alpha\) is called \textit{big} if the \((1, 1)\)-current \(T\) can be chosen to be \textit{strictly positive}, i.e \(T\) dominates some smooth positive form on \(X\).

\textbf{Definition 1.2.} The class \(\alpha\) is \textit{nef} if it lies in the closure of the Kähler cone, i.e the convex cone containing all Kähler classes.

Let \(T, T'\) be two positive closed current in \(\alpha\) with the local potentials \(\varphi, \varphi'\) respectively. We say that \(T\) is less singular than \(T'\) if \(\varphi' \leq \varphi + O(1)\). In addition, \(T\) is said to have \textit{minimal singularities} if it is less singular than any other positive current in \(\alpha\).

One important example for such a current is the following. We first pick \(\theta \in \alpha\) a smooth representative, then the upper envelope

\[ V_\theta := \sup\{\varphi; \varphi \in PSH(X, \theta) \text{ and } \sup_X \varphi \leq 0\} \]

yields a current \(\theta + dd^c V_\theta\) with minimal singularities (see [BD12] for the regularity of this current).

\textbf{Definition 1.3.} A positive closed current \(T\) has \textit{analytic singularities} if it can be locally written \(T = dd^c u\), with

\[ u = \frac{c}{2} \log \sum |f_j|^2 + v, \]
where \( c > 0 \), \( v \) is smooth and the \( f'j \)'s are holomorphic functions.

**Definition 1.4.** If \( \alpha \) is a big class, we denote \( \text{Amp}(\alpha) \) the ample locus of \( \alpha \), i.e. the set of all \( x \in X \) for which there exists a Kähler current in \( \alpha \) with analytic singularities which is smooth in a neighborhood of \( x \).

By definition the ample locus is a Zariski open subset and it is non-empty by Demailly’s regularization result [Dem92]. It follows from [Bou04] that there exists a strictly positive current \( T = \theta + dd^c \psi \in \alpha \) with analytic singularities such that

\[
\text{Amp}(\alpha) = X \setminus \text{Sing} T, \quad \text{and} \quad T \geq C \omega,
\]

for some \( C > 0 \).

**Lemma 1.5.** [Bou04] There exists a \( \theta \)-psh function \( \rho \) with such that

1. \( \theta + dd^c \rho \geq \epsilon \omega_X \), for some \( \epsilon > 0 \),
2. \( \rho \) is smooth in \( \text{Amp}(\alpha) \) and \( \text{Amp}(\alpha) = \{ \rho = -\infty \} \),
3. \( \rho \leq V_\theta \),
4. \( \rho(z) - V_\theta(z) \to -\infty \) as \( z \to \partial \text{Amp}(\alpha) \).

1.1.2. Non-pluripolar product. Let \( X \) be an \( n \)-dimensional complex manifold. Let \( u_1, \ldots, u_p \) be psh functions on \( X \). Denote

\[
\mathcal{O}_k := \cap_{j=1}^p \{ u_j > -k \},
\]

**Definition 1.6.** If \( u_1, \ldots, u_p \) are psh functions on \( X \), we say that the non-pluripolar product \( \langle \Lambda_{j=1}^p dd^c u_j \rangle \) is well-defined on \( X \) if for each compact subset \( K \) of \( X \) we have

\[
\sup_k \int_{K \cap \mathcal{O}_k} \omega^{n-p} \cap \Lambda_{j=1}^p dd^c \max \{ u_j, -k \} < \infty. \tag{1.1}
\]

Now let \((X, \omega)\) be a compact Kähler manifold and \( \alpha \) be a big cohomology class on \( X \). Given a \( \theta \)-psh function \( \phi \), we can define its non-pluripolar Monge-Ampère by \( \text{MA}(\phi) := \langle (\theta + dd^c \phi)^n \rangle \). Then we have

\[
\int_X \langle (\theta + dd^c \phi)^n \rangle \leq \text{vol}(\alpha).
\]

We say that the function \( \phi \) such that the equality holds has full Monge-Ampère mass. In particular, all \( \theta \)-psh functions with minimal singularities have full Monge-Ampère mass.

**Notation.** From now we denote the non-pluripolar Monge-Ampère product \( (\theta + dd^c \phi)^n \) instead of \( \langle (\theta + dd^c \phi)^n \rangle \).

1.2. Monge-Ampère equation in big cohomology classes.

1.2.1. Pluripotential approach. Fix \( \alpha \) a big class on \( X \) and \( \theta \) a smooth \((1, 1)\) form representing \( \alpha \). We consider the following Monge-Ampère equation

\[
(\theta + dd^c \phi)^n = fdV \tag{1.2}
\]

where \( f \in L^p(X) \) for some \( p > 1 \). Then we have the following theorem of existence of solution due to [BEGZ10, Theorem 4.1]
Theorem 1.7. There exists a unique solution \( \varphi \in \text{PSH}(X, \theta) \) to the equation \((1.2)\) satisfying \( \sup_X \varphi = 0 \). Moreover, there exists a constant \( M \) only depending on \( \theta, dV, p \) such that
\[
\varphi \geq V_\theta - M\|f\|_{L^p}^{1/n}.
\]

In that same paper [BEGZ10], the authors also established the existence of solutions to the equation
\[
(\theta + dd^c \varphi)^n = e^\varphi \mu \quad (1.3)
\]
with \( \mu \) is a smooth volume form. This implied the existence of a unique singular Kähler-Einstein metric on \( K_X \). More precisely, we have

Theorem 1.8. [BEGZ10] Let \( \mu = fdV \) be a volume form with \( f \in L^p(X) \) for some \( p > 1 \). Then there exists a unique \( \theta \)-psh function \( \varphi \) such that
\[
(\theta + dd^c \varphi)^n = e^\varphi \mu. \quad (1.4)
\]
Furthermore, \( \varphi \) has minimal singularities.

We refer the reader to [GZ17, PS12] for more details about complex Monge–Ampère equations.

1.2.2. Viscosity approach. The pluripotential theory gives us the existence of \( \theta \)-psh solution with minimal singularities to the equation \((1.4)\). In [EGZ15a], the authors developed a viscosity theory for the complex Monge–Ampère equation \((1.4)\) in which \( \mu \) is continuous. They proved the existence of a unique viscosity solution to \((1.4)\) which is \textit{continuous} in \( \text{Amp}(\alpha) \). Furthermore, this is exactly the pluripotential solution.

We recall here the basic results in [EGZ15a] on the viscosity approach to the equation:
\[
(\text{MA}_\mu) \quad (\theta + dd^c \varphi)^n = e^\varphi \mu,
\]
where \( \theta \) is a smooth \((1,1)\) form representing \( \alpha \). Denote \( \Omega = \text{Amp}(\alpha) \).

Definition 1.9. (Test functions) Let \( \varphi : X \to \mathbb{R} \) be any function and \( x_0 \) a given point such that \( \varphi(x_0) \) is finite. An upper test function (resp. a lower test function) for \( \varphi \) at \( x_0 \) is a \( C^2 \)- function \( q \) in a neighborhood of \( x_0 \) such that \( \varphi(x_0) = q(t_0, x_0) \) and \( \varphi \leq q \) (resp. \( \varphi \geq q \)) in a neighborhood of \( x_0 \).

Definition 1.10. A function \( \varphi : X \to \mathbb{R} \cup \{-\infty\} \) is a viscosity subsolution of \((\text{MA}_\mu)\) on \( X \) if
- \( \varphi : \Omega \to \mathbb{R} \) is upper semi-continuous,
- \( \varphi \leq V_\theta + C \) on \( X \), for some constant \( C \) and \( \varphi \neq -\infty \),
- for any point \( x_0 \in \Omega \) and any upper test function \( q \) for \( \varphi \) at \( x_0 \), we have
  \[
  (\theta(x_0) + dd^c q(x_0))^n \geq e^{\varphi(x_0)} \mu(x_0).
  \]

Definition 1.11. A function \( \psi : X \to \mathbb{R} \cup \{-\infty, +\infty\} \) is a viscosity supersolution of \((\text{MA}_\mu)\) on \( X \) if
- \( \psi : \Omega \to \mathbb{R} \) is lower semi-continuous,
- \( \psi \geq V_\theta - C \) on \( X \), for some \( C > 0 \) and \( \varphi \neq +\infty \),
for any point $x_0 \in \Omega$ and any lower test function $q$ for $\psi$ at $x_0$, we have

$$(\theta(x_0) + dd^c q(x_0))^n_{\omega^+} \leq e^{\theta(x_0)} \mu(x_0).$$

Here $\omega^+ = \omega$ if $(1,1)$-form $\omega$ is semipositive and $\omega^+ = 0$ otherwise.

**Definition 1.12.** A viscosity solution of $(MA_\mu)$ is a function that is both a viscosity subsolution and viscosity supersolution. In particular, a viscosity solution has the same singularities as $V_{\theta}$.

**Theorem 1.13.** Let $\alpha$ be a big cohomology class. Let $\varphi$ (resp. $\psi$) be a viscosity subsolution (resp. supersolution) of $(MA_\mu)$, then

$$\varphi \leq \psi \text{ in } \text{Amp}(\alpha).$$

We sketch a proof following [GZ17, Theorem 13.11] in which the cohomology class $\alpha$ is semipositive and big. This is slightly different from the proof in [EGZ15a].

**Proof.** Since $\alpha = \{\theta\}$ is big, there exists a $\theta$-psh function $\rho \leq \phi$ satisfying

$$\theta + dd^c \rho \geq C\omega.$$ 

Moreover, it follows from Lemma 1.5 that one can find $\rho$ to be smooth in the ample locus $\Omega = \text{Amp}(\alpha)$ such that $(\rho(x) - V_{\theta}) \to -\infty$ as $x \to \partial \Omega$.

Now, fix $\lambda \in (0,1)$ and set $\tilde{\varphi} = (1 - \lambda)\varphi + \lambda \rho$. By the definitions of sub/super-viscosity solutions $\varphi - \psi$ is bounded from above on $X$, hence $\tilde{\varphi} - \psi$ is also bounded from above on $X$, so we can extend it as an usc function $(\tilde{\varphi} - \psi)^*$ on $X$. Since $(\tilde{\varphi} - \psi) = (1 - \lambda)(\varphi - V_{\theta}) - (\psi - V_{\theta}) + \lambda(\rho - V_{\theta})$ is usc in $\Omega$ and tends to $-\infty$ as $x \to \partial \Omega$, the maximum of $(\tilde{\varphi} - \psi)^*$ is achieved at some point $x_0$ in $\Omega$,

$$\sup_{x \in X} (\tilde{\varphi} - \psi)^* = \tilde{\varphi}(x_0) - \psi(x_0).$$

We now need prove that $\tilde{\varphi}(x_0) \leq \psi(x_0)$. The proof of this claim is similar as in [GZ17, Theorem 13.11]. Finally we have $\tilde{\psi} \leq \psi$, and letting $\lambda \to 0$ we get the required inequality. \qed

As a corollary of the comparison principle we have:

**Theorem 1.14.** [EGZ15a] Let $\alpha$ be a a big cohomology class and $\mu > 0$ is a continuous density. Then there exists a unique pluripotential solution $\varphi$ of $(MA_\mu)$ on $X$, such that

1. $\varphi$ is a $\theta$-psh function with minimal singularities,
2. $\varphi$ is a viscosity solution in $\text{Amp}(\alpha)$ hence continuous here,
3. Its lower semicontinuous regularization $\varphi_*$ is a viscosity supersolution.

2. Degenerate complex Monge-Ampère flows in big classes

In this section we define viscosity solutions to degenerate complex Monge-Ampère flows in big cohomology classes following [CIL92, EGZ11, EGZ15a, EGZ15b, EGZ16, EGZ18]. Our goal is to establish a general comparison principle for viscosity subsolutions and supersolution to the Monge-Ampère flows in big cohomology classes. This extends some results in [EGZ16, EGZ18] where the authors studied the case when the involved cohomology classes are semipositive and big. In particular, we do not assume any condition on either the derivative of subsolution or the involved form $\theta_t$. 

2.1. Viscosity subsolutions and supersolutions. Let $X$ be a $n$-dimensional compact Kähler manifold. Fix $\alpha$ is a big cohomology class and $\theta$ is a smooth $(1,1)$-form representing $\alpha$. Denote by $\Omega = \text{Amp}(\alpha)$ the ample locus of $\alpha$. We extend some definitions from the viscosity theory for complex Monge-Ampère flows developed in [EGZ16, EGZ18].

We consider the following degenerate complex Monge-Ampère flow

$$ (\theta_t + dd^c \phi_t)^n = e^{\phi_t + F(t, x, \phi_t)} \mu \quad (CMAF), $$

where

- $F(t, x, r)$ is a continuous in $X_T = [0, T) \times X$ and non decreasing in $r$.
- $\mu(t, x) \geq 0$ is a family of bounded continuous volume forms on $X$.
- $\theta_t(x)$ is a family of smooth $(1,1)$-forms representing big cohomology classes $\alpha_t$ such that $\text{Amp}(\alpha_t) \supset \Omega$.
- $\phi : [0, T) \times X \to \mathbb{R}$ is the unknown function with $\phi_t(\cdot) := \phi(t, \cdot)$.

**Definition 2.1.** (Test functions) Let $\varphi : X_T \to \mathbb{R}$ be any function and $(t_0, x_0)$ a given point such that $\varphi(t_0, x_0)$ is finite. An upper test function (resp. a lower test function) for $\varphi$ at $(t_0, x_0)$ is a $C^{1,2}$- function $q$ in a neighborhood of $(t_0, x_0)$ such that $\varphi(t_0, x_0) = q(t_0, x_0)$ and $\varphi \leq q$ (resp. $\varphi \geq q$) in a neighborhood of $(t_0, x_0)$.

**Definition 2.2.** A function $\varphi : X_T \to \mathbb{R} \cup \{-\infty\}$ is a viscosity subsolution of (CMAF) on $X_T$ if

- $\varphi : \Omega_T \to \mathbb{R}$ is upper semi-continuous,
- $\varphi(t, x) \leq V_{\theta_t}(x) + C, \forall (t, x) \in X_T$ for some $C$ possibly depending on $T$.
- for any point $(t_0, x_0) \in \Omega_T$ and any upper test function $q$ for $\varphi$ at $(t_0, x_0)$, we have

  $$ (\theta_0(x_0) + dd^c q(t_0, x_0))^n \geq e^{q(t_0, x_0) + F(t_0, x_0, q(t_0, x_0))} \mu(t_0, x_0). $$

**Definition 2.3.** A function $\psi : X_T \to \mathbb{R} \cup \{-\infty, +\infty\}$ is a viscosity supersolution of (CMAF) on $X_T$ if

- $\psi : \Omega_T \to \mathbb{R}$ is lower semi-continuous,
- $\psi(t, x) \geq V_{\theta_t}(x) - C, \forall (t, x) \in X_T$, for some $C > 0$ possibly depending on $T$.
- for any point $(t_0, x_0) \in \Omega_T$ and any lower test function $q$ for $\psi$ at $(t_0, x_0)$, we have

  $$ (\theta_0(x_0) + dd^c q(t_0, x_0))^n \leq e^{q(t_0, x_0) + F(t_0, x_0, q(t_0, x_0))} \mu(t_0, x_0), $$

where $\omega_+ = \omega$ if the $(1,1)$-form $\omega$ is semipositive and $\omega_+ = 0$ otherwise.

**Notation 2.4.** From now, we also write $\varphi_t$ for a function $\varphi$ depending on $t$, i.e $\varphi_t(x) = \varphi(t, x)$, and $\partial_t \varphi_t$ or $\partial_t \varphi$ for its derivative in the time variable.

**Definition 2.5.** A viscosity solution of (CMAF) on $X_T$ is a function that is both a viscosity subsolution and a viscosity supersolution. In particular, a viscosity solution is continuous in $\Omega_T$ and have the same singularities with $V_{\theta_t}$ on $X_T$.

**Lemma 2.6.** Let $u$ is a $\theta_t$-psh with minimal singularities such that

- $u$ is continuous in $\Omega_T$. 

• $u$ admits a continuous partial $\partial_t u$ with respect to $t$.
• for any $t \in (0, T)$ the restriction of $u_t$ of $u$ to $X_t = \{t\} \times X$ satisfies

$$\left(\theta_t + dd^c u_t\right)^n \geq e^{\theta_t} F(t, x, u_t) \mu$$

in the pluripotential sense on $\Omega_t$.

Then $u$ is a subsolution of $(CMAF)$ in $\Omega_T$.

Proof. Suppose $q$ is a test function of $u$ at $(t_0, x_0) \in [0, T) \times \text{Amp}(\alpha)$. Then we have $q(t_0, x)$ is also a test function to $u_{t_0}(x)$ at $x_0$. Moreover, by the hypothesis, $u_{t_0}(x)$ satisfies

$$\left(\theta_{t_0} + dd^c u_{t_0}\right)^n(x) \geq \nu(x)$$

in the pluripotential sense, where $\nu = e^{(\theta u_{t_0}(t_0, x_0) + F(t_0, x, u_{t_0}(t_0, x_0)) \mu(t_0, x_0}$ is a volume form with continuous density in $\text{Amp}(\alpha)$. Therefore, by [EGZ11, Theorem 1.9], we get

$$\left(\theta + dd^c q\right)^n(t_0, x_0) \geq \nu(x_0) = e^{(\theta u_{t_0}(t_0, x_0) + u(t_0, x_0) + F(t_0, x_0, u(t_0, x_0)) \mu(t_0, x_0)$$

$\geq e^{(\theta q(t_0, x_0) + F(t_0, x_0, q(t_0, x_0))} \mu(t_0, x_0).$

Hence $u$ is a viscosity subsolution of $(CMAF)$. $\square$

Lemma 2.7. Let $v : \Omega_T \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a lsc function satisfying

• The restriction $v_t$ of $v$ to $\Omega_t = \{t\} \times \Omega$ is $\theta_t$-psh function with minimal singularities.
• $v$ admits a continuous partial derivative $\partial_t v$ with respect to $t$.
• there exists a function $w$ on $X$ such that $w$ is continuous on $\Omega$ and $\partial_t v_t + F(t, x, v) \geq w$. Moreover, $w$ satisfies

$$\left(\theta + dd^c v_t\right) \leq e^w \mu$$

in the pluripotential sense on $\Omega_t$.

Then $v$ is a viscosity supersolution of $(CMAF)$ in $\Omega_T$.

Proof. Fix $(t_0, x_0) \in \Omega_T$. By the hypothesis, we have

$$\left(\theta_{t_0} + dd^c v_{t_0}\right)^n(x) \leq \nu(x),$$

where $\nu := e^w \mu(t_0, x)$ is a volume form with continuous density in $\Omega$. It follows from [EGZ11, Lemma 4.7] that $v_{t_0}$ is a viscosity supersolution of the equation $(\theta + dd^c u)^n(x) = \nu(x)$ in $\Omega$.

Now suppose $q$ is a lower test function of $v$ at $(t_0, z_0) \in \Omega_T$. Then $q(t_0, x_0)$ is also a lower test function for $v_{t_0}(x)$ at $x_0$. Therefore

$$\left(\theta_{t_0} + dd^c q\right)^n(t_0, x_0) \leq e^{w(x_0)} \nu(x_0)$$

$$\leq e^{(\theta v)(t_0, x_0) + F(t_0, x_0, v_{t_0})} \mu(t_0, x_0).$$

Hence $v$ is a viscosity supersolution of $(CMAF)$ in $\Omega_T$. $\square$

We show that subsolutions to parabolic Monge-Ampère flows are plurisubharmonic in space variable.
**Proposition 2.8.** Let $\varphi$ be a viscosity subsolution of (CMAF). For each $t \in (0, T)$ we have $\varphi_t \in PSH(\Omega, \theta_t)$.

**Proof.** Observe first that the problem is local. For any $x_0 \in \Omega$, we can choose a small neighborhood $U$ of $x_0$ such that $\theta_t = dd^c h_t$ for all $(t, x) \in (t - \varepsilon, t + \varepsilon) \times U$ for some $\varepsilon > 0$ sufficiently small. We then infer that $u = h + \varphi$ is a viscosity subsolution of the local equation

\[(dd^c u_t)^n = e^{\hat{u} + \bar{F}(t, z, u_t)} \mu\]  

on $(t - \varepsilon, t + \varepsilon) \times U$, where $\bar{F}(t, z, u_t) = F(t, z, u_t - h_t) - \dot{h}_t$. It follows from [EGZ15a, Corollary 3.7] that $u_{t_0}$ is plurisubharmonic on $U$ for any $t_0 \in (t - \varepsilon, t + \varepsilon)$. Therefore we have $\varphi_t \in PSH(\Omega, \theta_t)$ as required. \hfill \Box

### 2.2. A useful local comparison principle.

We recall here a useful lemma due to [EGZ18, Corollary 3.9] for the local equation

\[(dd^c u_t)^n = e^{\bar{u} + \bar{F}(t, z, u_t)} \mu(t, z), \quad (MAF)_{F, \mu}\]  

with the initial condition $u(0, z) = u_0$ a continuous psh function in $D \subset \mathbb{C}^n$.

**Lemma 2.9.** Assume that $\mu(t, z) \geq 0$ be a continuous family of volume forms on some domain $D \subset \mathbb{C}^n$. Let $u : [0, T) \times D \rightarrow \mathbb{R}$ be a viscosity subsolution to the local equation $(MAF)_{F, \mu}$ and let $v : [0, T) \times D \rightarrow \mathbb{R}$ be a supersolution to the local equation $(MAF)_{G, \mu}$. Assume that

- the function $u_t - v_t$ achieves a local maximum at some $(t_0, z_0) \in (0, T) \times D$.
- there exits a constant $C_1 > 0$ such that $z \mapsto u(t, z) - 2C_1|z|^2$ is a plurisubharmonic near $z_0$ and $t$ near $t_0$.

If either $\mu(t_0, z_0) > 0$ or $\mu = \mu(z)$, then

\[F(t_0, z_0, u(t_0, z_0)) \leq G(t_0, z_0, v(t_0, z_0)). \]  

### 2.3. Comparison principle for complex Monge-Ampère flows.

Let $(X, \omega)$ be a compact Kähler manifold. Fix $\alpha$ is a big cohomology class and $\theta$ is a smooth $(1, 1)$-form representing $\alpha$. We consider the following degenerate complex Monge-Ampère flow

\[ (\theta_t + dd^c \phi_t)^n = e^{\hat{\phi}_t + \bar{F}(t, x, \phi_t)} \mu \quad (CMAF), \]  

where

- $F(t, x, r)$ is a continuous in $X_T = [0, T) \times X$ and non decreasing in $r$.
- $\mu = fdV \geq 0$ is a continuous volume form on $X$.
- $\theta_t(x)$ is a family of smooth $(1, 1)$-forms representing big cohomology classes $\alpha_t$ such that $\text{Amp}(\alpha_t) \supset \Omega := \text{Amp}(\alpha)$.
- $\phi : [0, T) \times X \rightarrow \mathbb{R}$ is the unknown function with $\phi_t(\cdot) := \phi(t, \cdot)$.

We now prove the following comparison principle extending the one in [EGZ16, EGZ18] where the class $\{\theta_t\}$ is semipositive and big. In particular, we exploit the concavity of log det avoiding the difficulties from the time derivative of the subsolution.
Convergence of the Weak Kähler-Ricci Flow

**Theorem 2.10.** Assume that \( \theta_t \geq \theta \) for a smooth \((1,1)\)-form \( \theta \) in some fixed big class \( \alpha \). Let \( \varphi \) (resp. \( \psi \)) be a viscosity subsolution (resp. a supersolution) to \( \text{CMAF} \). Then for any \((t, x) \in [0, T) \times \text{Amp}(\alpha)\)

\[
(\varphi - \psi)(t, z) \leq \max \left\{ \sup_{(0) \times X} (\varphi - \psi)^*, 0 \right\}
\]

**Proof.** Since \( \alpha = \{\theta\} \) is big, by Lemma 1.5, there exists a \( \theta \)-psh function \( \rho \leq \varphi \) satisfying

\[
\theta + dd^c \rho \geq C \omega,
\]

\( \rho \) is smooth in the ample locus \( \text{Amp}(\alpha) \) and \((\rho(x) - V_\theta) \to -\infty\) as \( x \to \partial \Omega \). Since \( \theta_t \geq \theta \), we have \( V_\theta \leq V_{\theta_t} \), so \((\rho(x) - V_{\theta_t}) \to -\infty\) as \( x \to \partial \Omega \).

Now fix \( \delta > 0 \), \( \lambda \in (0, 1) \) and set

\[
\varphi_\lambda(t, x) := (1 - \lambda)\varphi(t, x) + \lambda \rho(x) - \frac{\delta}{T-t} - A \lambda t \leq \varphi,
\]

where \( A \) will be chosen hereafter. Then \( \varphi_\lambda \) is a strictly \( \theta_t \)-psh function since \( \theta + dd^c \varphi_t \geq \lambda(\theta + dd^c \rho) \geq C \omega \). We now prove that \( \varphi_\lambda \) satisfies

\[
(\theta_t + dd^c \varphi_\lambda)^n \geq e^{\partial_t \varphi_t + (1-\lambda)F(t,x,\varphi_t) + n \lambda \log C + A + \frac{\delta}{(T-t)} \mu}
\]

in the viscosity sense. Indeed, let \( q \) be an upper test for \( \varphi_\lambda \) at \((t_0, z_0)\) then

\[
\tilde{q} = (1-\lambda)^{-1} \left( q - \lambda \rho(x) + \frac{\delta}{T-t} + A \lambda t \right)
\]

is an upper test for \( \varphi \) at \((t_0, z_0)\). If \( \mu(z_0) = 0 \) then we have already \((\theta_t + dd^c \tilde{q})^n \geq 0 \) at \((t_0, z_0)\) hence \((\theta_t + dd^c \tilde{q})^n \geq 0 = \mu(z_0) \). Now assume that \( \mu(z_0) \neq 0 \). Using the concavity of \( \log \text{det} \) and \( \theta_t + dd^c \rho \geq \theta + dd^c \rho \geq C \omega_X \), we have at \((t_0, z_0)\)

\[
\log \frac{(\theta_t + dd^c q)^n}{\mu} = \log \frac{(1-\lambda)(\theta_t + dd^c \tilde{q}) + \lambda(\theta + dd^c \rho))^n}{\mu} \geq (1-\lambda) \log \frac{(\theta_t + dd^c \tilde{q})^n}{\mu} + \lambda \log \frac{(\theta_t + dd^c \rho)^n}{\mu} \geq (1-\lambda) \log \frac{(\theta_t + dd^c \tilde{q})^n}{\mu} + n \lambda \log C.
\]

It follows from the definition of viscosity subsolution and the inequality above that at \((t_0, z_0)\)

\[
\partial_t q = (1-\lambda)\partial_t \tilde{q} + A \lambda + \frac{\delta}{(T-t)^2} \leq (1-\lambda) \left( \log \frac{(\theta_t + dd^c \tilde{q})^n}{\mu} - F(t_0, z_0, \tilde{q}(t_0, z_0)) \right) - A \lambda - \frac{\delta}{(T-t)^2} \leq \log \frac{(\theta_t + dd^c q)^n}{\mu} - (1-\lambda)F(t_0, z_0, q(t_0, z_0)) - n \lambda \log C - A \lambda - \frac{\delta}{(T-t)^2} \leq \log \frac{(\theta_t + dd^c q)^n}{\mu} - (1-\lambda)F(t_0, z_0, q(t_0, z_0)) - n \lambda \log C - A \lambda - \frac{\delta}{(T-t)^2}.
\]
where the last inequality comes from the fact that $F$ is non-decreasing in the third variable and $\varphi_\lambda \leq \varphi$. This implies (2.3) as required.

By the definition of viscosity sub/super solutions, $\varphi_\lambda - \psi \leq \varphi - \psi$ is bounded from above, we can extend it as an usc function $(\varphi_\lambda - \psi)^*$ on $X$. Moreover

$$\varphi_\lambda - \psi = (1 - \lambda)(\varphi - V_{\theta_1}) - (\psi - V_{\theta_1}) + \lambda(\rho - V_{\theta_1}) - \frac{\delta}{T - t} - A\lambda t,$$

(2.4)

hence $\varphi_\lambda - \psi \to -\infty$ as either $x \to \partial \Omega$ or $t \to T$. It follows that there exists $(t_0, x_0) \in [0, T) \times \Omega$ such that

$$\sup_{X_T} (\varphi_\lambda - \psi)^* = \varphi_\lambda(t_0, x_0) - \psi(t_0, x_0).$$

The idea is to localize near $x_0$ and use Lemma 2.9. We choose the complex coordinates $z = (z_1, \ldots, z_n)$ near $x_0$ defining a biholomorphism identifying a closed neighborhood $B$ of $x_0$ to the closed complex ball $B_3 := B(0, 3) \subset \mathbb{C}^n$ of radius 3, sending $x_0$ to the origin in $\mathbb{C}^n$. Let $h_t$ be a smooth local potential for $\theta_t$ in $B_2$, i.e. $dd^c h_t = \theta_t$ in $B_3$.

We have

$$\max_{[0, T) \times (B_2 \setminus B_3)} (\varphi_\lambda - \psi) = \varphi_\lambda(t_0, x_0) - \psi(t_0, x_0).$$

If $t_0 = 0$, we are done. Otherwise, assume $t_0 \in (0, T)$, we now prove that $\varphi_\lambda \leq \psi$ in $[0, T] \times \Omega$.

Now $u(t, z) = \varphi_\lambda \circ z^{-1} + h_t \circ z^{-1}$ is upper semi-continuous in $[0, T) \times B_2$ and strictly psh in $B_2$ since $dd^c u_t \geq \lambda C \omega X$. It follows from (2.3) that

$$(\theta_t + dd^c \varphi_\lambda)^n \geq e^{\frac{\partial_t \varphi_\lambda + (1 - \lambda)F(t, x, \varphi_\lambda) + n\lambda \log C + A + \frac{\delta}{(T - t)^2} \mu}{(r - t)^2}}$$

in $(t - r, t_0 + r) \times B_2$. Therefore

$$(dd^c u_t)^n \geq e^{\frac{\partial_t u + \tilde{F}(t, z, u_t)}{\beta}} \bar{\mu}, \text{ in } B_2,$$

where $\tilde{\mu} = z_* \mu \geq 0$ is a continuous volume form and

$$\tilde{F}(t, z, s) = (1 - \lambda)F(t, z, s - h(t, z)) + n\lambda \log C + \frac{\delta}{(T - t)^2} + A\lambda - \partial h_t(z).$$

Similarly, we have $v = \psi \circ z^{-1} + h_t \circ z^{-1}$ is lower semi-continuous in $[0, T) \times B_2$ and satisfies

$$(dd^c v)^n \leq e^{\partial_t \tilde{v} + \tilde{G}(t, z, v(t, z))} \tilde{\mu}, \text{ in } B_2,$$

where

$$\tilde{G}(t, z, s) := F(t, z, s - h_t(z)) - \partial_t h_t(z).$$

By our assumption we have

$$\max_{[0, T) \times B_2} u - v = u(t_0, 0) - v(t_0, 0).$$

It follows from Lemma 2.9 that

$$F(t_0, 0, u(t_0, 0)) \leq \tilde{G}(t_0, 0, v(t_0, 0)).$$
\[
F(t_0, x_0, \varphi_\lambda) + \lambda(A - F(t_0, x_0, \varphi_\lambda) + n \log C) + \frac{\delta}{(T-t)^2} \leq F(t_0, x_0, \psi(t_0, x_0)).
\]

Choosing \(A = -n \log C + \sup_{X,T} F(t, x, \varphi)\), we have
\[
F(t_0, \varphi_\lambda(t_0, x_0)) + \frac{\delta}{(T-t)^2} \leq F(t_0, x_0, \psi(t_0, x_0)).
\]

This implies that \(\varphi_\lambda(t_0, x_0) \leq \psi(t_0, x_0)\), hence \(\varphi_\lambda \leq \psi\) in \([0, T) \times X\). Letting \(\lambda \to 0\) and \(\delta \to 0\) we get \(\varphi \leq \psi\) in \([0, T) \times \Omega\), we thus conclude that \(\varphi \leq \psi\) in \([0, T) \times \Omega\) as required.

**Corollary 2.11.** With the same assumption above, but replacing the condition \(\theta_i \geq \theta\) by \(\theta_i \geq g(t)\theta\) for some smooth positive function \(g : [0, T] \to \mathbb{R}\) with \(g' > 0\). Then if \(\varphi(0, x) \leq \psi(0, x), \forall x \in X\), we have \(\varphi(t, x) \leq \psi(t, x), \forall (t, x) \in \Omega\).

**Proof.** Denote \(\omega_t(z) = g(t)^{-1} \theta_t\). Since \(\varphi\) is a subsolution to \((CP)\), we have
\[
(\omega_t + dd^c \tilde{\varphi}_t)^n \geq e^{g(t)\theta_t + F(t, z, \tilde{\varphi})} \mu,
\]
in the sense of viscosity, where \(\tilde{\varphi}(t, x) := g(t)^{-1} \varphi(t, x)\) and \(\tilde{F}(t, z, s) = F(t, z, s) + g'(t)s\).

We change the time variable:
\[
\chi(t, x) = \omega(h(t), x) \quad \text{and} \quad \phi(t, x) := \tilde{\varphi}(h(t), x) = \frac{1}{g(h(t))} \varphi(h(t), x),
\]
where \(h\) will be chosen hereafter. Then
\[
\partial_t \phi(t, x) = (\partial_t \tilde{\varphi})(h(t), x) h'(t),
\]
hence
\[
(\chi(t, x) + dd^c \phi(t, x))^n \geq e^{\frac{g(h(t))}{t}} (\partial_t \phi + \tilde{F}(t, x, \phi)) \mu. \quad (2.5)
\]
We choose \(h\) such that \(h'(t) = g(h(t))\) and \(h(0) = 0\), hence \(\phi(t, x)\) is a subsolution of
\[
(\chi_t + dd^c \phi)^n = e^{\partial_t \phi + \tilde{F}(t, x, \phi)} \mu. \quad (2.6)
\]
Similarly, we have \(\hat{\psi}(t, x) = \frac{1}{g(h(t))} \psi(h(t), x)\) is a supersolution of \((2.6)\). Since \(\chi \geq \theta\), Theorem 2.10 thus implies the desired inequality. \(\square\)

### 2.4. Viscosity solutions to Cauchy problem for complex Monge-Ampère flows.
Let \(X\) be a \(n\)-dimensional compact Kähler manifold and \(\alpha\) be a fixed big class on \(X\). We have a general Cauchy problem on \(X_T = [0, T) \times X\)

\[
(CP) \quad \begin{cases}
(\theta_t + dd^c \varphi_t)^n = e^{\varphi_t + F(t, x, \varphi_t)} \mu(t, z), \\
\varphi(0, x) = \varphi_0(x) \quad x \in X
\end{cases},
\]

where

- \(F(t, x, r)\) is a continuous in \([0, T) \times X\) and non decreasing in \(r\).
- \(\mu(t, x) \geq 0\) is a family of bounded continuous volume forms on \(X\),
• \( \theta_t(x) \) is a family of smooth \((1,1)\)-forms representing big cohomology classes \( \alpha_t \) such that \( \text{Amp}(\alpha_t) \supset \Omega := \text{Amp}(\alpha) \).
• \( \varphi_0 \) is a given \( \theta_0 \)-psh function.

**Definition 2.12.** A *subsolution* to \((CP)\) is a viscosity subsolution \( u \) to the flow

\[
(\theta_t + dd^c \varphi_t)^n = e^{\varphi_t + F(t,x,\varphi_t)} \mu,
\]

on \( X_T \), satisfying that \( u(0,x) \leq \varphi_0(x) \) for all \( x \in X \).

A *supersolution* to \((CP)\) is a viscosity supersolution \( v \) to the flow

\[
(\theta_t + dd^c \varphi_t)^n = e^{\varphi_t + F(t,x,\varphi_t)} \mu,
\]

on \( X_T \), satisfying \( v(0, x) \geq \varphi_0(x) \) for all \( x \in X \).

**Definition 2.13.** A function \( u \) on \( X \) is a viscosity solution to the Cauchy problem \((CP)\) if it is both a subsolution and a supersolution for \((CP)\).

3. **Cauchy problem in a big cohomology class**

Let \((X, \omega)\) be a Kähler manifold. Fix \( \alpha \) is a big cohomology class on \( X \) and \( \theta \) is a smooth \((1,1)\)-form representing \( \alpha \). In this section we consider the following Cauchy problem

\[
\begin{aligned}
\left\{(\theta + dd^c \varphi)^n = e^{\varphi + F(t,x,\varphi)} \mu \text{ in } [0, T) \times X, \\
\varphi(0, x) = \varphi_0(x), \quad (0, x) \in \{0\} \times X,
\right. 
\end{aligned}
\]

where \( \mu = \mu(x) > 0 \) is a positive continuous volume form and \( \varphi_0 \) is a given \( \theta \)-psh function on \( X \) with minimal singularities which is continuous in \( \Omega := \text{Amp}(\alpha) \). We first have the following lemma which is useful to construct subsolutions.

**Lemma 3.1.** Let \( f : \mathbb{R}^+ \to \mathbb{R} \) be the smooth solution of the ODE

\[
f'(t) + f(t) = C \log(1 - Be^{-t}) \quad \text{with} \quad f(0) = 0,
\]

for some \( B, C > 0 \). There exists \( A > 0 \) such that for all \( t \geq 0 \), \(-A(t+1)e^{-t} \leq f(t) \leq 0 \).

3.1. **Existence of viscosity sub/super-solutions.**

**Lemma 3.2.** \((CP_1)\) admits a viscosity subsolution.

**Proof.** It flows from [Bou04] that there exists a \( \theta \)-psh function \( \rho \) with analytic singularities satisfying

\[
\theta + dd^c \rho \geq \epsilon \omega \quad \text{and} \quad \sup_x \rho = 0.
\]

Set

\[
u(t, x) := e^{-t}\varphi_0 + (1 - e^{-t})\rho - At + f(t)
\]
such that the function $f$ satisfies the ODE: $f'(t) + f(t) = n \log(1 - e^{-t})$ and $f(0) = 0$. Then $u$ is continuous on $[0, T] \times \Omega$ and $u(0, x) = \varphi_0(x)$ and

$$(\theta + dd^c u)^n = \left( e^{-t}(\theta + dd^c \varphi_0) + (1 - e^{-t})(\theta + dd^c \rho) \right)^n$$

$$\geq (1 - e^{-t})^n \theta^n + e^{n \log(1 - e^{-t})} \epsilon \omega^n$$

$$= e^{\partial_t u + u - \rho + A + \log(\epsilon C)} \mu,$$

$$\geq e^{\partial_t u + u + A + \log(\epsilon C)} \mu,$$

where $C^n \mu \leq \omega^n$. By choosing $A = - \log(\epsilon C)$, we infer that $(\theta + dd^c u)^n \geq 0$ in the pluripotential sense. Lemma 2.6 implies that $u$ is a viscosity subsolution of $(CP_1)$. $\square$

**Lemma 3.3.** There exists a viscosity supersolution of $(CP_1)$.

**Proof.** We can assume that $\theta \leq \omega$ for some Kähler form $\omega$ on $\mathcal{X}$. Let $\phi$ be the unique continuous $\omega$-psh function satisfying

$$(\omega + dd^c \phi)^n = c \mu, \quad \sup_{\mathcal{X}} \phi = 0.$$  

where

$$c = \frac{\int_{\mathcal{X}} \omega^n}{\int_{\mathcal{X}} \mu} > 0.$$  

Set $v = \phi + C$ with $C > 0$ such that $\varphi_0 \leq v$ and $\log c \leq v$. Then we have

$$(\theta + dd^c v)^n \leq (\omega + dd^c \phi)^n = c \mu \leq e^{\partial_t v + v} \mu.$$  

(3.2)

It follows from Lemma 2.7 that $v$ is a supersolution to $(CP_1)$. $\square$

**3.2. Barrier construction.**

**Definition 3.4.** Fix $(0, x_0) \in \{0\} \times \Omega$ and $\varepsilon > 0$.

- An upper semi-continuous function $u : \Omega_T \to \mathbb{R}$ is an $\varepsilon$-subbarrier to the $(CP_1)$ at the boundary point $(0, x_0)$, if $u$ is subsolution to $(CP_1)$ in $[0, T) \times \mathcal{X}$, and

$$u_\varepsilon(0, x_0) \geq \varphi_0(x_0) - \varepsilon.$$  

When $\varepsilon = 0$, $u$ is called a subbarrier.

- A lower semi-continuous $v : \Omega_T \to \mathbb{R}$ is an $\varepsilon$-supperbarrier to the $(CP_1)$ at the boundary point $(0, x_0)$, if $v$ is a supersolution to the $(CP_1)$ in $[0, T) \times \mathcal{X}$, and

$$v_\varepsilon(0, x_0) \leq \varphi_0(x_0) + \varepsilon.$$  

When $\varepsilon = 0$, $v$ is called a superbarrier.

**Proposition 3.5.** Fix $\varepsilon > 0$, there exist an $\varepsilon$-subbarrier and an $\varepsilon$-supperbarrier to the Cauchy problem $(CP_1)$ in $[0, T) \times \text{Amp}(\alpha)$.

**Proof.** It is straightforward that the subsolution $u$ constructed in Lemma 3.2 is a subbarrier, so $\varepsilon$-subbarrier for all $\varepsilon > 0$.

We now find an $\varepsilon$-supperbarrier. Assume that $\theta \leq \omega$ for some Kähler form $\omega$ on $\mathcal{X}$, then $\varphi_0$ is also a $\omega$-psh function. Suppose $h_j$ be a sequence of smooth function decrease
to \( \varphi_0 \). Denote \( \varphi_j = P(h_j) := \sup_X \{ \phi \in PSH(X, \omega) \mid \phi \leq h_j \} \) the envelope of \( h_j \) then \( \varphi_j \downarrow \varphi \) and

\[
(\omega + dd^c \varphi_j)^n = 1_{\{\varphi_j = h_j\}}(\omega + dd^c h_j)^n = 1_{\{\varphi_j = h_j\}} f_j \mu,
\]

where \( f_j \geq 0 \) is bounded. Thus for any \( \varepsilon \) sufficiently small and any \( x_0 \in \text{Amp}(\alpha) \), there exists a function \( \varphi_j \) such that

\[
\varphi_0(x_0) \leq \varphi_j(x_0) \leq \varphi_0(x_0) + \varepsilon.
\]

Define

\[
v_j(t, x) := \varphi_j + Bt,
\]

for a positive constant \( B \) will be chosen hereafter, then

\[
(\theta + dd^c v_j)^n = (\theta + dd^c \varphi_j)^n
\]

\[
= 1_{\{\varphi_j = h_j\}}(\theta + dd^c h_j)^n
\]

\[
= 1_{\{\varphi_j = h_j\}} f_j \mu
\]

where \( f_j \geq 0 \) is bounded.

Since \( h_j \) is smooth on \( X \), there exists a constant \( C_j > 0 \) such that

\[
1_{\{\varphi_j = h_j\}} f_j e^{-\varphi_j} = 1_{\{\varphi_j = h_j\}} f_j e^{-h_j} \leq C_j.
\]

We now choose \( B > 0 \) such that \( e^B \geq C_j \), hence

\[
(\theta + dd^c v_j)^n = 1_{\{\varphi_j = h_j\}} f_j \mu
\]

\[
= 1_{\{\varphi_j = h_j\}} f_j e^{-\varphi_j e^{\varphi_j} \mu}
\]

\[
\leq C_j e^{-B e^{v_j + v_{j}} \mu}
\]

\[
\leq e^{v_j + v_{j}} \mu.
\]

It follows from Lemma 2.7 that \( v_j(t, x) \) is an \( \varepsilon \)-supperbarrier to \((CP_1)\) at \((0, x_0) \in \{0\} \times \text{Amp}(\alpha)\), so is

\[
v_{\varepsilon} := \inf \{v_j(t, x), v + A_2 t\},
\]

where \( v \) is the supersolution to \((CP_1)\) in Lemma 3.3. □

### 3.3. The Perron envelope.

Consider the upper envelope

\[
\varphi := \sup \{w, w \text{ is a subsolution of } (CP_1), u \leq w \leq v\},
\]

where \( u \) and \( v \) are the viscosity sub/super-solution from Lemma 3.2 and 3.3.

**Theorem 3.6.** The upper envelope \( \varphi \) is the unique viscosity solution to \((CP_1)\) in \([0, T) \times \Omega\).

**Proof.** Let \( \varphi^* \) (resp. \( \varphi_* \)) be the upper (resp. lower) semi-continuous envelope for \( \varphi \) in \([0, T) \times \Omega\), and set \( \varphi^*(t, x) = \varphi_*(t, x) = \varphi(t, x) \) in \([0, T) \times (X \setminus \Omega)\). Observe that \( \varphi^* \) (resp. \( \varphi_* \)) is a subsolution (resp. supersolution) to the complex Monge-Ampère flow

\[
(\theta + dd^c \phi)^n = e^{\partial_t \phi + \phi} \mu
\]

on \((0, T) \times \Omega\). We now show that they are also subsolution and supersolution respectively to the Cauchy problem \((CP_1)\).
We first have $\varphi \geq u$ on $\Omega_T$. Since $u$ is continuous in $[0, T) \times \Omega$, $\varphi_0(x_0) \geq \varphi_0(x_0)$ for any $x_0 \in \operatorname{Amp}(\alpha)$. This shows that $\varphi_0$ is a supersolution to the Cauchy problem $(CP_1)$.

We prove now that $\varphi^*(0, \cdot) \leq \varphi_0$ in $\Omega$. Fix $\varepsilon > 0$, by Proposition 3.5 there exists an $\varepsilon$-supperbarrier $v_{\varepsilon}$ to $(CP_1)$ at any point $(0, x_0)$ with $x_0 \in \Omega$. It follows from the comparison principle (Theorem 2.10) that

$$\varphi(t, x) \leq v_{\varepsilon}(t, x) \quad \text{in } [0, T) \times \Omega,$$

together with $x_0 \in \Omega$, hence $\varphi^*$ is a viscosity subsolution to the Cauchy problem $(CP_1)$.

The comparison principle (Theorem 2.10) therefore implies that $\varphi^* = \varphi_0 = \varphi$ in $[0, T) \times \Omega$.

Finally, the uniqueness of viscosity solution in $[0, T) \times \Omega$ is deduced by the comparison principle (Theorem 2.10).

3.4. **Long term behavior of the flow.** It follows from [BEGZ10] and [EGZ15a] that the Monge-Ampère equation

$$(\theta + dd^c\varphi)^n = e^\varphi \mu \quad (3.3)$$

have a unique pluripotential solution which is a viscosity solution in $\operatorname{Amp}(\alpha)$. In this section, we will prove that the solution of the Monge-Ampère flow

$$(\theta + dd^c\varphi_t)^n = e^{\varphi_t + \varepsilon t} \mu \quad (3.4)$$

converges to the solution of (3.3) as $t \to +\infty$.

**Theorem 3.7.** The solution $\varphi_t$ of the complex Monge-Ampère flow 3.4 starting at $\varphi_0$ converges, exponentially fast in $\operatorname{Amp}(\alpha)$, as $t \to +\infty$, to the solution of the degenerate elliptic Monge-Ampère equation (3.3).

**Proof.** Let $\phi$ be the unique pluripotential solution to the equation (3.3). Set

$$u(t, x) := e^{-t} \varphi_0 + (1 - e^{-t}) \phi + f(t),$$

where $f(t) = O(te^{-t})$ is the unique solution of the ODE $f'(t) + f(t) = n \log(1 - e^{-t})$ and $f(0) = 0$. We now have $u(0, x) = \varphi_0(x)$ and

$$(\theta + dd^c u)^n = \left(e^{-t}(\theta + dd^c \varphi_0) + (1 - e^{-t})(\theta + dd^c \phi)\right)^n \geq \left(1 - e^{-t}(\theta + dd^c \phi)^night)^n = e^{\partial_{\bar{z}} u + u} \mu.$$

Lemma 2.6 implies that $u$ is a subsolution of $(CP_1)$ in $[0, T) \times \operatorname{Amp}(\alpha)$. It follows from the comparison principle (Theorem 2.10) we get $u \leq \varphi$ in $[0, T) \times \operatorname{Amp}(\alpha)$. Therefore

$$\varphi_t - \phi \geq e^{-t}(\varphi_0 - \phi) + f(t) \quad \text{in } [0, T) \times \operatorname{Amp}(\alpha).$$

In addition, we also have

$$v(t, x) = Ae^{-t} + \phi,$$
where $A > 0$ satisfies $|\varphi_0 - \phi| \leq A$, is a supersolution of $(CP_t)$ in $[0, T) \times \text{Amp}(\alpha)$. By the comparison principle (Theorem 2.10), we obtain $v(t, x) \geq \varphi_t$ in $[0, T) \times \text{Amp}(\alpha)$, hence

$$\varphi_t - \phi \leq Ae^{-t}$$

in $[0, T) \times \text{Amp}(\alpha)$.

All together yields $|\varphi_t - \phi| = O(te^{-t})$ in $[0, T] \times \text{Amp}(\alpha)$. Letting $t \to +\infty$ we obtain

$$\varphi_t \to \phi$$

in $\Omega$. □

4. The Kähler-Ricci flow on manifolds of general type

Let $X$ be a Kähler manifold with the canonical bundle $K_X$ is big but not nef. Fix $\alpha_0$ a Kähler class on $X$ and $\omega_0$ is a Kähler form representing $\alpha_0$. A (classical) solution of the normalized Kähler-Ricci flow on $X$ starting at $\hat{\omega} \in \alpha_0$ is a family of Kähler forms $(\omega_t)$ solving

$$\frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) - \omega_t, \quad \omega_{t|t=0} = \hat{\omega}. \quad (4.1)$$

Note that $\omega_t \in \alpha_t := e^{-t}\alpha_0 + (1 - e^{-t})c_1(K_X)$. Moreover it follow from [TZ06] (see also [Cao85, Tsu88] for special cases) that the normalized Kähler-Ricci flow with initial metric $\omega_0$ has a unique smooth solution on on $[0, T)$ with

$$T := \sup\{t > 0\mid e^{-t}\alpha_0 + (1 - e^{-t})c_1(K_X) > 0\}.$$

Assuming that $\alpha_T$ is big, Tosatti and Collins [CT15] proved that as $t \to T^-$ the metric $\omega(t)$ develop singularities precisely on the Zariski closed set $X \setminus \text{Amp}(K_X)$.

Now at $T$, $\alpha_T$ is big and nef. However, for $t > T$, $\alpha_t$ is still big but no longer nef, thus we can not continue the flow in the classical sense. In [FIK03] the authors asked whether one can define and construct weak solutions of Kähler-Ricci flow after the maximal existence time for smooth solutions. In [BT12], Boucksom and Tsuji have tried to run the normalized Kähler-Ricci on projective varieties in a weak sense beyond the maximal time using the discretization of the Kähler-Ricci flow and algebraic geometry tools. They have proposed a conjecture (cf. [BT12, Conjecture 1, page 208]) with respect to this direction.

In this section we answer the question of a Feldman-Ilmanen-Knopf and give an analytic approach to the conjecture of Boucksom and Tsuji using the viscosity theory established in Section 2. Moreover we show that the weak flow exists for all time and converges to the singular Kähler-Einstein metric contructed in [BEGZ10].

4.1. Existence and uniqueness of extended flow. Now let $\theta$ be a smooth closed $(1, 1)$-form representing $c_1(K_X)$ and set

$$\theta_t := e^{-t}\omega_0 + (1 - e^{-t})\theta.$$
Let $dV$ be a smooth volume form on $X$, then $-\text{Ric}(dV) \in c_1(K_X)$. Therefore there exists $f \in C^\infty(X)$ such that $\theta = \text{Ric}(\mu)$ with $\mu = e^f dV$. Then the normalized Kähler-Ricci flow (4.1) can be written as the complex Monge-Ampère flow

\[(CP_2) \quad \begin{cases} (\theta_t + \ddc \varphi_t)^n = e^{\varphi_t + \varphi_0} \\ \varphi(0, x) = \varphi_0 \end{cases},\]

where $\hat{\omega} = \omega_0 + \ddc \varphi_0$.

**Lemma 4.1.** For any $T > 0$, there exists a subsolution and a supersolution to the Cauchy problem \((CP_2)\) in $[0, T] \times \text{Amp}(K_X)$.

**Proof.** It follows from [Bou04] that there exists a $\theta$-psh function $\rho$ with analytic singularities satisfying

\[\theta + \ddc \rho \geq \epsilon \omega_0 \quad \text{and} \quad \sup_X \rho = 0.\]

We consider

\[u(t, x) := e^{-t} \varphi_0 + (1 - e^{-t}) \rho + f(t) - At,\]

where $f(t)$ is the unique solution of $f'(t) + f(t) = n \log(1 - e^{-t})$ and $f(0) = 0$. Then $u(0, x) = \varphi_0(x)$ and $u$ is a $\theta_t$-psh function since

\[ \theta_t + \ddc u = e^{-t} \omega_0 + (1 - e^{-t}) \theta + \ddc u = e^{-t} (\omega_0 + \ddc \varphi_0) + (1 - e^{-t})(\theta + \ddc \rho) \geq 0.\]

Therefore Then $u$ is continuous on $[0, T) \times \Omega$ and $u(0, x) = \varphi_0(x)$ and

\[(\theta_t + \ddc u)^n = (e^{-t}(\omega_0 + \ddc \varphi_0) + (1 - e^{-t})(\theta + \ddc \rho))^n \geq (1 - e^{-t})^n (\theta + \ddc \rho)^n \geq e^{n \log(1 - e^{-t})} e^n \omega^n = e^{\theta u + u - \rho + A + \log(eC)} \mu, \]

\[\geq e^{\theta u + u + A + \log(eC)} \mu,\]

where $C^n \mu \leq \omega^n$. By choosing $A = -\log(eC)$, we infer that $(\theta_t + \ddc u)^n \geq e^{\theta u + u + \mu}$ in the pluripotential sense. Lemma 2.6 thus implies that $u$ is a subsolution to $(CP_2)$ in $[0, T) \times \text{Amp}(K_X)$.

For supersolution, we suppose that $\theta \leq A \omega_0$ for some $A > 0$. Denote $\psi$ is the unique solution of the complex Monge-Ampère equation

\[(\omega_0 + \ddc \psi)^n = c \mu, \quad \min_X \psi = 0, \quad \text{with} \quad c = \frac{\int_X \omega_0^n}{\int_X \Omega}.\]

Then we set $v = a(t) \psi + Bt$, where $a(t) = e^{-t} + A(1 - e^{-t})$ and $g$ will be chosen hereafter. We have

\[(\theta_t + \ddc v)^n \leq (a(t) \omega_0 + \ddc v)^n = e^{\log c + n \log a(t) \Omega} \]

By choosing

\[B = \log c + n \max_{[0, T]} \log a(t),\]
we have \( \log c + n \log a(t) \leq \dot{v} + v \), Lemma 2.7 thus implies that \( v \) is a supersolution to \((CP_2)\).

**Lemma 4.2.** Fix \( \varepsilon > 0 \). There exist an \( \varepsilon \)-subbarrier and an \( \varepsilon \)-supperbarrier to the Cauchy problem \((CP_2)\).

**Proof.** Observe that for any \( \varepsilon > 0 \), the subsolution \( u(t, x) \) in Lemma 4.1 is also a \( \varepsilon \)-subsolution since \( u(0, x) = \varphi_0 \).

For the \( \varepsilon \)-supperbarrier at \((t_0, x_0) \in \{t_0\} \times \text{Amp}(K_X)\), we use the same argument in Proposition 3.5. Approximate \( \varphi_0 \) by a decreasing sequence \((h_j)\) of smooth functions. Denote \( \varphi_j = P(h_j) \) the envelope of \( h_j \) then \( \varphi_j \searrow \varphi_0 \) and

\[
(\omega_0 + dd^c \varphi_j)^n = \mathbb{1}_{\{\varphi_j = h_j\}}(\omega_0 + dd^c h_j)^n = \mathbb{1}_{\{\varphi_j = h_j\}} f_j \mu,
\]

where \( f_j \geq 0 \) is bounded. Thus there exists a function \( \varphi_j \) such that

\[
\varphi_0 \leq \varphi_j \leq \varphi_0 + \varepsilon.
\]

Define \( v(t, x) = a(t) \varphi_j + Bt \) with \( a(t) \) as in Lemma 4.1, then

\[
(\theta_t + dd^c v)^n \leq (a(t)\omega_0 + dd^c v)^n \leq \mathbb{1}_{\{\varphi_j = h_j\}} f_j e^{n \log a(t)} \mu \leq \mathbb{1}_{\{\varphi_j = h_j\}} f_j e^{(A-1)\varphi_j + n \log a(t)} - B e^\theta_v \mu.
\]

Since \( h_j \) is smooth on \( X \) and \( f_j \) is bounded on \( X \), there is a constant \( C_j \) such that

\[
\mathbb{1}_{\{\varphi_j = h_j\}} f_j e^{(A-1)h_j} \leq e^{C_j}.
\]

By choosing \( B = \max_{[0,T]} n \log a(t) + C_j \), we get

\[
(\theta_t + dd^c v)^n \leq e^{\theta_v + v} \mu.
\]

Since \( \varphi_j \) is continuous on \( X \), so is \( \dot{v} + v \), Lemma 2.7 infers that \( v \) is a \( \varepsilon \)-supperbarrier to the Cauchy problem \((CP_2)\) in \([0,T) \times \text{Amp}(K_X)\). \( \square \)

We now obtain the solution of \((CP_2)\) using the Perron envelope as in Theorem 3.6.

**Theorem 4.3.** For any \( T > 0 \), there exists a unique viscosity solution to the Cauchy problem \((CP_2)\) on \([0,T) \times X\). As consequence, the normalized Kähler-Ricci flow exists for all time in the viscosity sense.

**Proof.** We consider the upper envelope

\[
\varphi := \sup \{ w \mid w \text{ is a subsolution of } (CP_2), u \leq w \leq v \},
\]

where \( u \) (reps. \( v \)) is the subsolution (reps. supersolution) of \((CP_2)\) constructed above. Let \( \varphi^* \) (resp. \( \varphi_* \)) be the upper (resp. lower) semi-continuous envelope for \( \varphi \) in \([0,T) \times \text{Amp}(K_X)\), and set \( \varphi^*(t, x) = \varphi_*(t, x) = \varphi(t, x), \forall (t, x) \in [0,T) \times (X \setminus \text{Amp}(K_X))\).

Then we have \( \varphi^*(x) \) is a viscosity subsolution to the equation

\[
(\theta_t + dd^c \varphi)^n = e^{\partial_\varphi^* + \varphi \cdot \mu} \tag{4.2}
\]

on \([0,T) \times X\). In addition, it follows from the bump construction (cf. [CIL92, EGZ11]) that \( \varphi_* \) satisfies the viscosity inequality in Definition 2.3.
To see that $\varphi_*$ is a supersolution of (4.2), we need to prove further that $\varphi(t, x) \geq V_{\theta_t}(x) - C(t), \forall t \in [0, T]$ for some time-dependent constant $C(t)$ (cf. Definition 2.3). It is sufficient to find a subsolution $\psi_t$ of $(CP_2)$ such that $\psi_t \geq V_{\theta_t}(x) - C(t), \forall t \in [0, T]$. This is straightforward in Theorem 3.6 when $\theta_t$ is independent of $t$, but not trivial when $\theta_t$ depends on $t$. We would like to thank Hoang-Chinh Lu for pointing out this missing argument in our last version. We now give such supersolution in our case when $\theta_t = e^{-t}\omega_0 + (1 - e^{-t})\theta$.

We first remark that
\[
\chi_t := \frac{\theta_t}{1 - e^{-t}} = \theta + \frac{\omega_0}{e^t - 1}
\]
is decreasing in $t$.

Let $\phi_t(x)$ be the unique elliptic viscosity solution of
\[
(\chi_t + dd^c \phi_t)^n = e^{\phi_t} \mu,
\]
for each $t \in [0, T)$. Since $t \mapsto \chi_t$ is decreasing, for $s \leq t$ we have
\[
(\chi_s + dd^c \phi_t)^n \geq (\chi_s + dd^c \phi_t)^n = e^{\phi_s} \mu.
\]
Therefore $\phi_t$ is a viscosity subsolution to $(\chi_s + dd^c \phi_s)^n = e^{\phi_s} \mu$. It follows from the comparison principle (cf. Theorem 1.13) that $\phi_t \leq \phi_s$ in $\text{Amp}(\{\chi_s\}) \supset \text{Amp}(K_X)$. Hence $t \mapsto \phi_t$ is decreasing on $[0, T) \times \text{Amp}(K_X)$.

Set $\psi_t := (1 - e^{-t})\phi_t + f(t) + A$, where $A = \min\{\inf_X \varphi_0, 0\}$, and $f$ satisfies the ODE: $f'(t) + f(t) = n \log(1 - e^{-t})$ and $f(0) = 0$. Then we have
\[
\partial_t \psi_t + \psi_t = (1 - e^{-t})\partial_t \phi_t + e^{-t} \phi_t + (1 - e^{-t}) \phi_t + n \log(1 - e^{-t}) + A
\]
\[
= (1 - e^{-t})\partial_t \phi_t + \phi_t + n \log(1 - e^{-t}) + A
\]
Since $t \mapsto \phi_t$ is decreasing on $[0, T) \times \text{Amp}(K_X)$ and $A \leq 0$, we infer that
\[
(\theta_t + dd^c \psi_t)^n = (1 - e^{-t})^n(\chi_t + dd^c \phi_t)^n
\]
\[
= e^{\phi_t + n \log(1 - e^{-t})} \mu
\]
\[
\geq e^{(1 - e^{-t})\partial_t \phi_t + \phi_t + n \log(1 - e^{-t}) + A} \mu
\]
in viscosity sense. This follows that $\psi_t$ is a viscosity subsolution to $(CP_2)$. Since $\phi_t \geq V_{\chi_t} - C(t)$ for $t \in [0, T)$, we have $\psi_t \geq V_{\theta_t} - C'(t)$ for $t \in [0, T)$ as required.

Finally, as in the proof of Theorem 3.6, we use the $\varepsilon$-sub/supper-barriers constructed above to prove that $\varphi^*$ (resp. $\varphi_*$) is the subsolution (resp. supersolution) to $(CP_2)$. Then the comparison principle (Corollary 2.11) implies that $\varphi^* = \varphi_* = \varphi$ in $[0, T) \times \text{Amp}(K_X)$, hence $\varphi$ is a viscosity solution to $(CP_2)$. The uniqueness again follows from the comparison principle (Corollary 2.11).
4.2. Convergence of the weak normalized Kähler-Ricci flow. We now study the long-time behavior of the normalized Kähler-Ricci flow on compact Kähler manifolds of general type. Precisely we prove that the normalized Kähler-Ricci flow continuously deforms any initial Kähler form $\omega_0$ towards the unique singular Kähler-Einstein metric $\omega_{KE} = \theta + \ddc \varphi_{KE}$ in the canonical class $K_X$, with

$$\left(\theta + \ddc \varphi_{KE}\right)^n = e^{\varphi_{KE}} \mu$$

(4.4)

(we refer to [EGZ09] and [BEGZ10] for the construction of $\omega_{KE}$).

**Theorem 4.4.** The viscosity solution of the Monge-Ampère flow

$$\begin{cases}
(\theta_t + \ddc \varphi_t)^n = e^{\varphi_t + \varphi_t} \mu \\
\varphi(0, x) = \varphi_0
\end{cases}$$

converges, as $t \to +\infty$, locally uniformly on $\text{Amp}(K_X)$ to the unique solution $\varphi_{KE}$ of the Monge-Ampère equation

$$\left(\theta + \ddc \varphi_{KE}\right)^n = e^{\varphi_{KE}} \mu.$$

**Proof.** Set

$$u(t, x) = e^{-t} \varphi_0 + (1 - e^{-t}) \varphi_{KE} + f(t)$$

where $f(t) = O(te^{-t})$ is the unique solution of the ODE $f'(t) + f(t) = n \log(1 - e^{-t})$ and $f(0) = 0$ (cf. Lemma 3.1). Then $u$ is a $\theta_t$-psh function and

$$\begin{align*}
(\theta_t + \ddc u)^n &= \left(e^{-t}(\omega_0 + \ddc \varphi_0) + (1 - e^{-t})(\theta + \ddc \varphi_{KE})\right)^n \\
&\geq (1 - e^{-t})^n (\theta + \ddc \varphi_{KE})^n \\
&\geq e^{n \log(1 - e^{-t}) + \varphi_{KE}} \mu \\
&= e^{\varphi + \varphi} \mu.
\end{align*}$$

Lemma 2.6 implies that $u$ is a viscosity subsolution. It follows from Theorem 2.10 that

$$\varphi_t \geq u \quad \text{on } [0, +\infty) \times \text{Amp}(K_X),$$

hence

$$\varphi_t - \varphi_{KE} \geq e^{-t}(\varphi_0 - \varphi_{KE}) + f(t),$$

(4.5)

on $[0, +\infty) \times \text{Amp}(K_X)$ with $f(t) = O(te^{-t})$.

For the upper bound of $\varphi_t - \varphi_{KE}$ we need to use the following lemma

**Lemma 4.5.** There exists a unique viscosity solution $\phi_t$ for the following flow

$$\begin{cases}
\left(a(t) \theta + \ddc \phi_t\right)^n = e^{\phi_t + \phi_t} \mu \\
\phi(0, x) = \phi_0
\end{cases}$$

(4.7)

for any $\phi_0 \in PSH(X, (1 + A) \theta) \cap C^0(\text{Amp}(K_X))$ with minimal singularities, where $a(t) = 1 + Ae^{-t}$ with $A \geq 0$. Moreover, the flow converges to $\varphi_{KE}$, locally uniformly on $\text{Amp}(K_X)$ as $t \to +\infty$. 

Proof. Observe that
\[ (a(t)\theta + dd^c\phi_t)^n = a^n(t)(\theta + dd^c a(t)^{-1}\phi_t)^n. \] (4.8)

By setting \( \tilde{\phi}_t = a(t)^{-1}\phi_t - b(t) \), with \( b(t) = a(t)^{-1}v(t) \) where \( v = O(te^{-t}) \) is the unique solution of the ODE
\[ v' + v = n\log a(t) \quad \text{and} \quad v(0) = 0. \]

We now can rewrite (4.7) to the flow
\[
\begin{cases}
(\theta + dd^c\tilde{\phi}_t)^n = e^{a(t)\theta t + \phi_0} \\
\tilde{\phi}(0, x) = \phi_0
\end{cases} \tag{4.9}
\]

Finally, by changing of the time variable \( \psi(t, x) = \tilde{\phi}(h(t), x) \) where \( h(t) \) is the unique solution of the ODE
\[ h'(t) = a(h(t)) \quad \text{and} \quad h(0) = 0. \]

Then the equation (4.7) can be rewritten as
\[
\begin{cases}
(\theta + dd^c\psi_t)^n = e^{\theta_t + \psi_t} \\
\psi(0, x) = \phi_0
\end{cases} \tag{4.10}
\]

which is the flow we studied in Section 3. Since \( b(t) \to 0 \) and \( h(t) \to +\infty \) as \( t \to \infty \) the convergence is followed from Theorem 3.7. \( \square \)

Since \( K_X \) is big there exists a \( \theta \)-psh function \( \rho \) with analytic singularities satisfying
\[ \theta + dd^c\rho \geq C^{-1}\omega_0 \quad \text{and} \quad \sup_X \rho = 0, \] (4.11)
for some \( C > 0 \) (cf. [Bou04]). We can assume further that \( C \geq 1 \).

Now set \( u(t, x) = \varphi_t - Be^{-t} + Ce^{-t}\rho \), and \( a(t) = 1 + (C - 1)e^{-t} \). Using (4.11) we have
\[
a(t)\theta + dd^c u = Ce^{-t}\theta + (1 - e^{-t})\theta + dd^c u \\
\geq e^{-t}(\omega_0 - Cdd^c\rho) + (1 - e^{-t})\theta + dd^c u \\
= \theta_t + dd^c \varphi_t,
\]
hence
\[ (a(t)\theta + dd^c u)^n \geq e^{\partial_u + v}\mu \]
in the viscosity sense. Fix \( \phi_0 \) a \( C\theta \)-psh function with minimal singularities, then \( \phi_0 - C\rho \) is bounded from below. Therefore we can choose \( B > 0 \) such that \( \phi_0 - C\rho \geq \varphi_0 - B \). This implies that \( u \) is a subsolution of the Cauchy problem (4.7). Since the flow (4.7) can be written as the flow (4.10) after changing of time variable, the comparison principle also holds for the flow (4.7). Therefore we get
\[ u \leq \varphi_t \]
on \([0, \infty) \times \Amp(K_X)\). Combining with (4.6) and Lemma 4.5, we imply that \( \varphi_t \) converges to \( \varphi_{KE} \) on \( \Amp(K_X) \). \( \square \)
References

[BT76] E. Bedford, B. A. Taylor, The Dirichlet problem for a complex Monge-Ampère equation, Invent. Math. 37, (1976) no. 1, 1-44.

[BT82] E. Bedford, B. A. Taylor, A new capacity for plurisubharmonic functions, Acta Math 149, (1982) no. 1, 1-40.

[BD12] R. Berman, J.-P. Demailly, Regularity of plurisubharmonic upper envelopes in big cohomology classes, in Perspectives in analysis, geometry, and topology, 39–66, Progr. Math., 296, Birkhäuser/Springer, New York, 2012

[Bou04] S. Boucksom, Divisorial Zariski decompositions on compact complex manifolds, Ann. Sci. ENS (4) 37, no. 1, 45-76 (2004).

[BEGZ10] S. Boucksom, P. Eyssidieux, V. Guedj, and A. Zeriahi, Monge-Ampère equations in big cohomology classes, Acta Math. 205 (2010), 199–262.

[BT12] S. Boucksom, H. Tsuji, Semipositivity of relative canonical bundles via Kähler-Ricci flows, RIMS Kôkyûroku, no. 1783 (2012) Potential theory and fiber spaces, 200–215.

[Cao85] H. Cao, Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds, Invent. Math. 81 (1985), 359–372.

[CT15] V. Tosatti and T. Collins, Kähler currents and null loci, Invent. Math. 202 (2015), no.3, 1167-1198.

[CIL92] M. Crandall, H. Ishii, and P. L. Lions, User’s guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992), 1–67.

[Dem92] J-P. Demailly, Regularization of closed positive currents and intersection theory J. Algebraic Geom. 1 (1992), no. 3, 361–409.

[DLT19] H. S. Do, G. Le, and T. D. Tô, Viscosity solutions to parabolic complex Monge-Ampère equations, Preprint (2019).

[DDT19] S. Dinew, H. S. Do and T. D. Tô, A viscosity approach to the Dirichlet problem for degenerate complex Hessian type equations, Analysis & PDE, 12 (2019), No. 2, 505–535.

[EGZ09] P. Eyssidieux, V. Guedj, and A. Zeriahi, Singular Kähler-Einstein metrics, J. Amer. Math. Soc. 22 (2009), 369-378.

[EGZ11] P. Eyssidieux, V. Guedj, and A. Zeriahi, Viscosity solutions to degenerate Complex Monge-Ampère Equations, Comm. Pure Appl. Math. 64 (2011), no. 8, 1059–1094.

[EGZ17] P. Eyssidieux, V. Guedj, and A. Zeriahi, Corrigendum: Viscosity solutions to complex Monge-Ampère equations, Comm. Pure Appl. Math. 70 (2017), no. 5, 815–821.

[EGZ15a] P. Eyssidieux, V. Guedj, and A. Zeriahi, Continuous approximation of quasipseudoharmonic functions, Cont. Math. 644 (2015), 67–78.

[EGZ15b] P. Eyssidieux, V. Guedj, and A. Zeriahi, Weak solutions to degenerate complex Monge-Ampère flows I, Math. Ann. 362 (2015), 931–963.

[EGZ16] P. Eyssidieux, V. Guedj, and A. Zeriahi, Weak solutions to degenerate complex Monge-Ampère flows II, Adv. Math. (2016), 37–80.

[EGZ18] P. Eyssidieux, V. Guedj and A. Zeriahi, Convergence of weak Kähler-Ricci Flows on minimal models of positive Kodaira dimension, Comm. Math. Phys. 357 (2018), no. 3, 1179–1214.

[FIK03] M. Feldman, T. Ilmanen, and D. Knopf, Rotationally Symmetric Shrinking and Expanding Gradient Kähler-Ricci Solitons, J. Differential Geom. Vol 65, Number 2 (2003), 169-209.

[GLZ18a] V. Guedj, C.H. Lu, A. Zeriahi, The pluripotential Cauchy-Dirichlet problem for complex Monge-Ampère flows, arXiv:1810.02122 (2018)

[GLZ18b] V. Guedj, C.H. Lu, A. Zeriahi, Pluripotential Kähler-Ricci flows, arXiv:1810.02121 (2018)

[GLZ] V. Guedj, C.H. Lu, A. Zeriahi, Viscosity vs Pluripotential solutions to complex Monge-Ampère flows, Preprint (2019)

[GZ05] V. Guedj and A. Zeriahi, Intrinsic capacities on compact Kähler manifolds, J. Geom. Anal. 15 (2005), no. 4, 607–639.

[GZ07] V. Guedj and A. Zeriahi, The weighted monge-ampère energy of quasipseudoharmonic functions, J. Funct. An. 250 (2007), 442–482.
[GZ17] V. Guedj and A. Zeriahi, *Degenerate complex Monge-Ampère equations*, EMS Tracts in Mathematics, 26. European Mathematical Society (EMS), Zürich, 2017. xxiv+472 pp. ISBN: 978-3-03719-167-5

[HL09] F. R. Harvey and H.B. Lawson, *Dirichlet duality and the nonlinear Dirichlet problem*, Commun. Pure Appl. Math. 62 (2009), no. 3, 396–443.

[Kol98] S. Kołodziej, *The complex Monge-Ampère equation*, Acta Math. 180 (1998), no. 1, 69–117.

[ST12] J. Song and G. Tian, *Canonical measures and Kähler-Ricci flow*, J. Amer. Math. Soc. 25 (2012), no. 2, 303–353.

[ST17] J. Song and G. Tian, *The Kähler-Ricci flow through singularities*, Invent. Math. 207 (2017), no. 2, 519–595

[SW13] J. Song and B. Weinkove, An introduction to the Kähler-Ricci flow. In *An introduction to the Kähler-Ricci flow*, 89–188, Lecture Notes in Math., 2086, Springer, Cham, 2013.

[Tsu88] H. Tsuji, *Existence and degeneration of kähler-einstein metrics on minimal algebraic varieties of general type*, Math. Ann. 281 (1988), no. 1, 123–133.

[TZ06] G. Tian and Z. Zhang, *On the Kähler-Ricci flow of projective manifolds of general type*, Chin. Ann. Math. 27 (2006), no. 2, 179–192.

[Tos18] V. Tosatti, *KAWA lecture notes on the Kähler-Ricci flow*, Ann. Fac. Sci. Toulouse Math. 27 (2018), no. 2, 285–376.

[Moi67] B. Moishezon, *On n-dimensional compact varieties with n algebraically independent meromorphic functions I*, Amer. Math. Soc. Transl. Ser. 2, 63, 51-93, (1967)

[PS12] D. H. Phong, J. Song, J. Sturm, *Complex Monge Ampère Equations*, Surveys in Differential Geometry, vol. 17, 327-411 (2012).

[Wan12] Y. Wang, *A viscosity approach to the Dirichlet problem for complex Monge–Ampère equations*, Math. Z. 272 (2012), no. 1, 497–513.

**Ecole Nationale de l’Aviation Civile, Université de Toulouse, 7, Avenue Edouard Belin, FR-31055 Toulouse Cedex 04, France**

**Institut Mathématiques de Toulouse, Université de Toulouse, CNRS, UPS, 31062 Toulouse Cedex 09, France (Associated Researcher).**

*Email address: tat-dat.to@enac.fr, tat-dat.to@math.univ-toulouse.fr*