Geometrical and dynamical properties of Lorenz type system

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Abstract. A new topological invariant (Lorenz-manuscript) leading to the existence of uncountable set of topologically various attractors is proposed. A new definition of the hyperbolic properties of the Lorenz system close to singular hyperbolicity is introduced. This definition gives the opportunity to prove that small non-autonomous perturbations do not lead to the appearance of the stable solutions.

1. Introduction
In 1963 Lorenz [1-2] carried out numerical research of the system
\[
\begin{align*}
\dot{x} &= \sigma(y-x), \\
\dot{y} &= \rho x - y - xz, \\
\dot{z} &= -\beta z + xy,
\end{align*}
\]
with \(\sigma = 10, \beta = \frac{8}{3}, \rho = 28\). Lorenz observed that the system demonstrated the sensitive dependence of initial conditions. The small difference in initial conditions cause solutions which look completely different after a while. What is the reason of the phenomenon? Is it the result of approximation mistakes or intrinsic properties of the system?

Afraimovich, Bykov and Shil’nikov [3, 8], as well as, Guckenheimer and Williams [4-7] introduced two geometrical models of the Lorenz flow, which reflect the behavior of trajectories of the Lorenz system with classical parameter values. Here we follow the model of Guckenheimer and Williams. This model is called “the geometrical Lorenz flow”. This flow \(\Phi_\tau\) is defined in some region \(U \subset \mathbb{R}^3\). \(U\) is invariant under the flow \(\Phi_\tau\) for all \(\tau \geq 0\). \(\Phi_\tau\) satisfies the following properties:
1. The flow has an equilibrium point \(O\). Let \(X\) denote the vector field associated with the flow. The derivative \(DX(O)\) has eigenvalues \(\lambda_1 < \lambda_2 < 0 < \lambda_3\) such that \(\lambda_2 + \lambda_3 > 0\). Hence the point \(O\) has two-dimensional stable manifold \(W^s(O)\) and one-dimensional unstable manifold \(W^u(O)\) consisting of two trajectories (see figure 1).
2. There is a two-dimensional domain \(D\) transversal to the lines of the flow, \(W'(O)\) crosses \(D\) on some curve \(S\). Trajectories of points \(z \in D\setminus S\) come back on domain \(D\) and trajectories of points \(z \in S\) tend to \(O\). The first return map \(P: D\setminus S \to D\) is schematically represented in figure 2.
3. The map \(P\) is hyperbolic (roughly speaking \(P\) contracts in the horizontal and expands in the vertical direction). More precisely, there can be introduced the coordinate system on \(D\) such that
The line $S$ is defined by the equation $y = 0$. There exists constants $\theta_a > 0$ and $\theta_s > 0$ such that
- if $l$ is graph of the function $x = h(y)$, $|h'(y)| < \theta_a$ and $l \cap S = \emptyset$ then $P(l)$ is the similar curve. That is, $P(l)$ is graph of the function $x = H(y)$, $|H'(y)| < \theta_a$.
- if $w$ is graph of the function $y = g(x)$ and $|g'(x)| < \theta_s$ then $P^{-1}(w \cap T(D))$ is graph of the function $y = G(x)$, $|G'(y)| < \theta_s$.

As a consequence we have that in the domain $D$ there exists stable foliation invariant under $P$, which consists of approximately horizontal lines [3].

4. The factorization along the leaves of the stable foliation transforms the domain $D$ factorized onto some interval $I = [0,1]$, the map $P$ is projected to the map of interval $f : I \to I$. Let us denote $s$ as the projection of the line $S$ on $I$. The map $f$ satisfies the conditions:

4.1. $f$ is monotonous on the intervals $[0, s]$ and $(s, 1]$. \( \lim_{x \to s, x < 0} f(x) = p_1 \), \( \lim_{x \to s, x > 0} f(x) = p_2 \).

4.2. $f\big|_{[p_1, p_2]}$ is locally eventually onto, that is, for each interval $J \subset [p_1, p_2]$ there exists $n \geq 1$ such that $f^n J = [p_1, p_2]$ (see figure3).

The flow satisfying properties 1-4 is called the geometrical Lorenz flow.

**Remark.** For the real Lorenz system the conditions 1-4 were confirmed by many authors by means of numeric simulations. However proof of these properties was obtained by W. Tucker recently [9].

**The Lorenz attractor**

There are many definitions of the term "attractor". To be precise, we give the definition of the Lorenz system attractor.

Let $\Phi_r$ be the Lorenz flow. The attractor of the flow is the set $\Lambda = \bigcap_{t \geq 0} \Phi_t(U)$. In fact, the attractor $\Lambda$ is the closure of the set of trajectories that intersect the domain $D$ infinitely many times in the past and in the future. If conditions 1-4 are satisfied, then periodic trajectories are dense in $\Lambda$ and the flow $\Phi_r\big|_{\Lambda}$ is topologically transitive [3-5].
2. The problem of homeomorphism of the Lorenz attractors

R Williams [6,7] introduced the following model of the Lorenz system attractor. He assumed that there is one-dimensional invariant stable foliation in the area $U$ transversal to the trajectories of the flow $\Phi_t$. The result of the area $U$ factorization on this foliation has in general very a complicated structure. In addition, Williams assumed that there is a neighborhood $U_1$ of the attractor $\Lambda$, such that $U_1$ factorized by the stable foliation is the branched manifold (we denote it by $L$), and $\Phi_t$ is projected to the semiflow $\varphi_t$, $\tau \geq 0$ on $L$ (see figure 4).

Let $L$ be a smooth branched manifold of one of three kinds: orientable (a), semiorientable (b) and nonorientable (c) (see figure 4). The horizontal interval $I$ contains a line of branching, which is indicated by a heavy line.

The fixed point $O$ of the flow $\Phi_t$ is projected to the saddle fixed point of the semiflow $\varphi_t$ (we also denote this point by $O$). Let $W^u(O)$, $W^s(O)$ are accordingly unstable and stable manifolds of the point $O$ (considered as the fixed point of the semiflow $\varphi_t$). The line $W^s(O)$ has two sides: left $W^s_l$ and right $W^s_r$.

Denote $\hat{L} = \lim(L, \varphi_t, \tau \in R^+)$. The elements $\hat{x} \in \hat{L}$ are the branches of the negative semitrajectories of $\varphi_t$, namely

$$\hat{L} = \{ \hat{x} = (x_\theta)_{\theta \geq 0}; \varphi_t x_\theta = x_{\theta - \tau}, \text{ for all } \theta > \tau \geq 0 \}.$$

There is flow $\hat{\varphi}_t$ naturally defined on $\hat{L}$ by the such a way:

if $\tau > 0$ then $\hat{\varphi}_t \hat{x} = (y_\theta)_{\theta \geq 0}$, where $y_\theta = \varphi_t x_\theta$;

if $\tau < 0$ then $\hat{\varphi}_t \hat{x} = (y_\theta)_{\theta \geq 0}$, where $y_\theta = x_{\theta - \tau}$.

The pair $(\hat{L}, \hat{\varphi}_t)$ is called the geometrical Lorenz attractor. The inverse limit $\lim(L, \varphi_t, \tau \in R^+)$ is homeomorphic to $\Lambda$, the flow $\hat{\varphi}_t$ is topologically equivalent to $\Phi_t$.

R Williams investigated the geometrical structure of $\hat{L}$. He used kneading sequences. The kneading sequences characterize the order in which two lines $W^s_l$ and $W^s_r$ go around the left and the right holes in $L$. More precisely a kneading sequence is the sequence of “0” and “1”, where “0” corresponds to detour of the left hole of $L$, “1” - to detour of the right hole. R. Williams proved that the pair of the kneading sequences corresponding to $W^s_l$ and $W^s_r$ is a conditional topological invariant. It means that
if \( \hat{L}_1 \) and \( \hat{L}_2 \) are homeomorphic and this homeomorphism is sufficiently close to identity then their pairs of kneading sequences coincide [7].

In [10] a new topological invariant is constructed. This invariant distinguishes an uncountable set of nonhomeomorphic attractors \( \hat{L} \). Let’s give preliminary definitions.

Denote \( \hat{O} \) the fixed point of the flow \( \hat{\phi}_\tau, \hat{O} = (x_0)_0, \ x_0 = 0 \) for all \( \theta \geq 0 \).

Denote \( \hat{W}^u(\hat{O}) \) the unstable manifold of the point \( \hat{O} \).
\[ \hat{W}^u(\hat{O}) = \{ \hat{z} = (z_0)_{0 \theta} ; z_0 \to O \text{ if } \theta \to \infty \} \]. All the points of the line \( \hat{W}^u(\hat{O}) \) differ from other points of \( \hat{L} \).

**Proposition [6,7], [10].** Each point \( \hat{z} \in \hat{W}^u(\hat{O}) \) lays in the spine of a Cantor book \( K \subset \hat{L} \), whereas no other points of \( \hat{L} \) lie in such a set.

The Cantor book is the Cartesian product of the Cantor cone and interval (see figure 5). If we remove the "special" line \( \hat{W}^u(\hat{O}) \) from \( \hat{L} \) then \( \hat{L} \) breaks up to the uncountable set of layers - arc components of \( \hat{L} \setminus \hat{W}^u(\hat{O}) \). Each arc component of \( \hat{L} \setminus \hat{W}^u(\hat{O}) \) is a two-dimensional layer or sometimes a one-dimensional layer.

Let \( \hat{z} \in \hat{L} \setminus \hat{W}^u(\hat{O}) \). The corresponding negative semitrajectory \( \hat{z} = (z_0)_{0 \theta} \) crosses the line \( I \) infinity many times. Denote \( x_1, x_2, x_3, ..., x_n, ... \) consecutive points of crossing. Let \( \alpha(\hat{z}) = \alpha_1 \alpha_2 \alpha_3, ... \) be a sequence consisting of zeros and units such that \( \alpha_n = 0 \) if \( x_n \) lays on \( I \) to the left of \( s \), and \( \alpha_n = 1 \) if \( x_n \) lays on \( I \) to the right of \( s \). Here the point \( s \in W^s(\hat{O}) \cap I \) (positive trajectory of \( s \) does not return to \( I \)) as in figure 4. We call \( \alpha(\hat{z}) \) the symbolic prehistory of the point \( \hat{z} \).

We call two sequences from zero and units \( \alpha \) and \( \gamma \) to be equivalent \( (\alpha \sim \gamma) \) if there are numbers \( m \) and \( n \) such that for all \( k \geq 0 \) \( \alpha_{n+k} = \gamma_{m+k} \).

We have \( \alpha(\hat{z}_1) \sim \alpha(\hat{z}_2) \) in the only case when points \( \hat{z}_1 \) and \( \hat{z}_2 \) belong to the same arc component of \( \hat{L} \setminus \hat{W}^u(\hat{O}) \).

Denote \( A = \text{clos} \hat{W}^u(\hat{O}) \setminus \hat{W}^u(\hat{O}) \). Let \( \Omega \) be a set of all possible sequences \( \alpha(\hat{z}), \hat{z} \in A \) and \( \hat{\Omega} = \Omega \) to be the set of equivalence classes. There is a one-to-one correspondence between \( \hat{\Omega} \) and the set of arc components containing trajectories of points of the set \( A \). Let’s call the trajectory of point \( \hat{z} \in A \) as a “marked trajectory”. Denote \( \{ \hat{\phi}_\tau(\hat{z}) \}_{\theta \geq 0} = l(\hat{z}) \) (the full trajectory of point \( \hat{z} \)).

If \( H \) is homeomorphism \( \hat{L}_1 \to \hat{L}_2 \) then \( H(\hat{W}^u(\hat{O}_1)) = \hat{W}^u(\hat{O}_2) \), and each arc component of the set \( \hat{L}_1 \setminus \hat{W}^u(\hat{O}_1) \) is mapped accordingly in an arc component of the set \( \hat{L}_2 \setminus \hat{W}^u(\hat{O}_2) \), \( H(A_1) = A_2 \).

For each \( \gamma \in \hat{\Omega} \) denote \( C(\gamma) \) as the set of the marked trajectories belonging to the arc component \( \gamma \):
\[ C(\gamma) = \{ l(\hat{z}) : \hat{z} \in A, \alpha(\hat{z}) \in \gamma \} \].

Let \( \chi = \{ \text{card} C(\gamma) \}_{\gamma \in \hat{\Omega}} \).
Theorem 1 [10].

1. If $\hat{L}_1$ and $\hat{L}_2$ are homeomorphic then $\chi_1 = \chi_2$.

2. For any increasing sequence of natural numbers $n_1, n_2, n_3, ...$, there is a pair $\hat{L}, \hat{\phi}_t$ such that $\chi = \{n_0, c, n_1, n_2, n_3, ...\}$

Here $n_0$ is the cardinal of countable set, $c$ is the cardinal of continuum.

Geometrical sense of the invariant $\chi$ is as follows. The set $\hat{L}$ looks like a book with the spine $\hat{W}^s(\hat{O})$ and pages with marked trajectories. The invariant $\chi$ is the set of cardinals, where each cardinal is a quantity of marked lines on a page.

**Definition.** The invariant $\chi$ is called $L$-manuscript.

**Remark.** The invariant $\chi$ is not full. It is possible to construct an uncountable set of nonhomeomorphic attractors having the same set $\chi = \{n_0, c, n_1, n_2, n_3, ...\}$.

**Theorem 1.1.** For any infinite sequence of natural numbers $k_1, k_2, k_3, ...$ there exists a pair $\hat{L}_i, \hat{\phi}_t$ such that $\chi = \{n_0, c, n_1, n_2, n_3, ...\}$, and for all $i \geq 1$ the number of arc components of the set $\hat{L} \setminus \hat{W}^s(0)$, containing exactly $n_i$ trajectories from $A$ is $k_i$.

Theorem 1 is extended on a case of branched manifolds and semiflows, which are the suspensions of an expanding map of interval with several points of discontinuity. Let $f$ be a map of the interval $I = [0,1]$ with $n$ points of discontinuity $s_1, s_2, s_3, ..., s_n$.

The map $f$ is monotonous and expanding on the intervals $(s_{i-1}, s_i), [0, s_i), (s_i, 1]$. Hence, there exists limits $\lim_{x \to s_i^-} f(x) = p_i^-$, $\lim_{x \to s_i^+} f(x) = p_i^+$. Consider a rectangle $\Pi = [0,1] \times [0,1]$ with the sections along the lines $s_k \times [0,1]$ (see figure 6). The points of the left and right coast of the section are denoted accordingly $s_k^- \times x$ and $s_k^+ \times x$. Let there be a vertical flow on $\Pi$ with saddle fixed points $O_k = s_k \times 0.5$, $k=1, 2, ..., n$. Identify points in a rectangle $\Pi$ as follows: if $x \times t(x) \times 0$ then $x$ is not the point of discontinuity; $s_k^- \times 1 \sim p_k^- \times 0$, $s_k^+ \times 1 \sim p_k^+ \times 0$. $\Pi$ is transformed to the two-dimensional branched manifold $L$ (see figure 7). The flow on $\Pi$ generates a semiflow $\phi_u$, $u \in R^+$ on $L$ with saddle fixed points $O_k$.

Denote $\hat{L} = \lim_{\tau \to \infty}(L, \phi_t, \tau \in R^+)$, the flow $\hat{\phi}_t$ is naturally defined on the set $\hat{L}$. We call the pair $(\hat{L}, \hat{\phi}_t)$ the generalized geometrical Lorenz attractor.
Let \( W^u(\hat{O}_i), W^u(\hat{O}_2), \ldots, W^u(\hat{O}_n) \) be unstable manifolds of the fixed points. Let \( A_i \) be the set of limit points for \( W^u(\hat{O}_i), i = 1, \ldots, n \) belonging to \( \hat{L} \setminus \bigcup_{k=1}^{n} W^u(\hat{O}_k) \).

Each point \( \hat{z} \in \hat{L} \setminus \bigcup_{k=1}^{n} W^u(\hat{O}_k) \) has symbolic prehistory \( \alpha(\hat{z}) = \alpha_1 \alpha_2 \alpha_3 \ldots \) consisting of \( n+1 \) symbols "0", "1", "\ldots", "n" (points of discontinuity divide \( I \) into \( n+1 \) parts). Consider \( \Omega = \{ \alpha(\hat{z}), \hat{z} \in \bigcup_{k=1}^{n} A_k \} \) and \( \hat{\Omega} = \Omega \sim \) where equivalence \( \sim \) is defined above.

For each \( \beta \in \hat{\Omega} \) denote \( C(\beta) = (\bigcup_{z \in \hat{\Omega}}: \alpha(\hat{z}) \in \beta, \hat{z} \in A_k) \).

Denote \( \mathcal{X} = \{ \text{card}C(\beta), \text{card}C(\beta), \ldots, \text{card}C(\beta) \}_{\beta \in \hat{\Omega}} \).

Let \( \Theta = \{ \Theta_0, c, 0, 1, 2, 3, \ldots, n, \ldots \} \) be the set of cardinals.

**Theorem 2** [10]: Let \( (L_1, \varphi_1) \) and \( (L_2, \varphi_2) \) be two branched manifolds with semiflows, which are the suspensions of expanding maps of interval with \( n \) points of discontinuity. Let \( \hat{L}_1 \) and \( \hat{L}_2 \) be inverse limits. If \( \hat{L}_1 \) and \( \hat{L}_2 \) are homeomorphic then

1. \( \chi_1 = \chi_2 \) within transposition, that is, there exists a transposition \( (i_1, i_2, \ldots, i_n) \) such that \( \chi_1 = \{ \text{card}C(\beta), \text{card}C(\beta), \ldots, \text{card}C(\beta) \}_{\beta \in \hat{\Omega}} = \{ \text{card}C(\gamma), \text{card}C(\gamma), \ldots, \text{card}C(\gamma) \}_{\beta \in \hat{\Omega}} = \chi_2 \)

2. For any subset \( \Theta^* \subseteq \Theta^* = \Theta \times \Theta \times \ldots \times \Theta \) containing \( (c, c, c, \ldots, c) \) there is a pair \( \hat{L}, \hat{\varphi} \), such that \( \chi = \Theta^* \).

**3. Perturbations of the Lorenz system**

A perturbation of the Lorenz system can be defined as a system

\[
\begin{align*}
\hat{x} &= \sigma(y - x) + h_1, \\
\hat{y} &= \rho x - y - xz + h_2, \\
\hat{z} &= -\beta z + xy + h_3,
\end{align*}
\]

where \( h_1, h_2 \) and \( h_3 \) are small functions of the variables \( x, y, z \). We shall say that such perturbations are *autonomous*. In view of discontinuities the Poincare map is not structurally stable; therefore the flow is not structurally stable either. However, many properties (and in particular, the hyperbolic structure) survive perturbations.

A *non-autonomous perturbation* (or a time depended perturbation) is a system (1) in which the functions \( h_1, h_2 \) and \( h_3 \) depend on the variables \( x, y, z \) and time \( t \). We measure the smallness of perturbations in terms of the norm.
We assume that the second-order derivatives are bounded.

The supply of robust properties for non-autonomous perturbations is much scantier than for autonomous ones. In particular, non-autonomous perturbations destroy the hyperbolic structure. For this reason non-autonomous perturbations are poorly investigated in hyperbolic theory.

In the present paper we consider small non-autonomous perturbations of systems of Lorenz type. We give a sketch of proof of the following proposition.

**Theorem 3.** Small non-autonomous perturbations of Lorenz type system do not have stable solutions.

The full proof is given in [11].

Denote by \( x \) points \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), and by \( X(x) \) a vector field with components \( X_1(x) \), \( X_2(x) \) and \( X_3(x) \) in a bounded subdomain \( U \) of \( \mathbb{R}^3 \). We call \( X(x) \) the velocity vector. We denote by \( \Phi_t \) the local flow generated by the system of equations \( \dot{x} = X(x) \). We assume that \( U \) is \( \Phi_t \)-invariant, that is, \( \Phi_t(U) \subset U \) for \( t > 0 \). We see that the positive semitrajectory of each point \( x \in U \) lies entirely in \( U \) and \( \Phi_t \) is semiflow in \( U \).

We denote by \( D(x) \) the matrix of partial derivatives

\[
D(x) = \begin{bmatrix}
\frac{\partial X_1}{\partial x_1} \\
\frac{\partial X_1}{\partial x_2} \\
\frac{\partial X_1}{\partial x_3}
\end{bmatrix},
\]

and by \( d\Phi_t \) the differential of the map \( \Phi_t \). To underline that we consider the differential at a single point \( x \) we shall also use the notation \( d\Phi_t(x) \). If \( f(x) \) is a continuous function that is differentiable along the trajectories, is defined in \( U \), and takes values in some set, then we denote by \( \dot{f} \) or \( \frac{df}{dt} \) its derivative along the vector field \( X(x) \).

We denote \( T_xU \) the tangent space of \( U \) at \( x \). Its dual space is denoted by \( T_x^*U \). We shall regard elements of the tangent space as column vectors and elements of its dual as row vectors. For \( q \in T_xU \), \( v \in T_xU \) one defines the scalar product \( (q,v) \). Vectors \( q \in T_x^*U \) and \( v \in T_xU \) are orthogonal if \( (q,v) = 0 \).

It can be slightly confusing that the tangent space at all points in \( \mathbb{R}^3 \) can be identified; moreover, the tangent space and its dual can also be identified. In the present paper we distinguish between the tangent space at distinct points and we distinguish between a tangent space and its dual.

In the spaces \( T_xU \) and \( T_x^*U \) one can consider the usual Euclidean norm and the corresponding scalar product and angle measurement. We denote the length of vector by \( |v| \) or \( |q| \).

We call a function \( q(x) \) taking values in \( T_x^*U \) a vector field (here we bear in mind that elements of \( T_x^*U \) are column vectors).

We shall consider left and right eigenvectors of an arbitrary matrix \( A \). A right eigenvector is a column vector defined by the equation \( AV = \lambda V \); a left eigenvector is a row vector defined by the equation \( qA = \lambda q \). As is well known, if \( q \) and \( v \) are left and right eigenvectors corresponding to distinct eigenvalues \( \lambda_1, \lambda_2 \) then \( (q,v) = 0 \). This is known as the orthogonality relation. We regard left eigenvectors as elements of the space \( T_x^*U \).
The equation in variations and the conjugate equation in variations

In what follows we consider the well-known equation in variations. If \( x(t) \) is a trajectory then the equation in variations along \( x(t) \) has the form \( \dot{v} = D(x(t))v \). The vector \( v(t) \) in this equation is represented as a vector-column, that is, an element of \( T_x U \). Besides the equation in variations we shall consider the conjugate equation in variations along the trajectory \( x(t) \), which has the form \( \dot{q} = qD(x(t)) \). Here \( q \) is viewed as a row vector, that is, an element of \( T^*_x U \).

One treats the equation in variations and the conjugate equation as follows. We have a standard identification of the space \( T_x U \) with \( 3 \times R^3 \). In accordance with this identification, we write an equation with solution \( v(t) \in R^3 \). After that for each \( t \) we identify \( v(t) \) with a vector in \( T_{x(t)} U \).

As is well known, the differential \( d\Phi_t \) of the flow \( \Phi_t \) relates to the equation in variations: if \( v(t) \) is a solution of the equation in variations along a trajectory \( x(t) \), then \( d\Phi_t v(0) = v(t) \).

We associate with the conjugate equation in variations an operator \( d\Phi_t^* \), the conjugate differential. The operator \( d\Phi_t^* \) is defined in the space \( T^*_x U \) and takes values in \( T^*_{x(t)} U \), so that if \( q(t) \) is a solution of the conjugate equation in variations along a trajectory \( x(t) \), \( x(0) = x \) then \( q(0)d\Phi_t^* = q(t) \).

We express the action of \( d\Phi_t^* \) as a right action since this notation agrees with the standard definition of matrix product (the product of a row vector and a \( 3 \times 3 \)-matrix).

Definition of a singularly hyperbolic structure

We present the modification of the definition of a singularly hyperbolic system for a class of systems more general than that of Lorenz type. Namely, let \( U \) be a bounded open subset of \( 3 \times R^3 \) and \( X(x) \) a smooth vector field in it, which generates a local flow \( \Phi_t \). Let \( \Lambda \) be the set of points having the following property: the full trajectory through the point lies in \( U \). If the positive semitrajectory of each \( x \in U \) lies entirely in \( U \), then \( \Lambda \) is an attractor.

We do not assume \( X(x) \) to be defined outside \( U \).

Definition. A cone in the Euclidean space with the axis space \( Q \) and of span \( \alpha \) is the set of vectors \( v \) such that \( \angle(v,Q) < \alpha \) (we denote by \( \angle(v,Q) \) is the angle between the vector \( v \) and the space \( Q \)).

Definition. One has on \( U \) a singularly hyperbolic structure and the system is said to be singularly hyperbolic if the following condition holds.

Condition \((H^*U)\). For each point \( x \in U \) there exists a cone \( K^*_x \subset T^*_x U \) with one-dimensional axis space \( Q^*_x \); the cone family is invariant in the following sense: there exists \( t_i > 0 \) such that \( \text{clos}(K^*_x d\Phi_t) \subset \text{int}(K^*_{\Phi_t(x)}) \) for \( t > t_i \). In addition, there exist constants \( c_1 > 0, \gamma_1 > 0, c_2 > 0 \) and \( \gamma_2 > 0 \) such that

a) for each point \( x \in U \), each vector \( q \in K^*_x \), and each \( t > 0 \) such that \( \Phi_t \) is well defined one has the inequality \( |qd\Phi_t^*| > c_1 e^{\gamma_1 t} |q| \);

b) for each point \( x \in U \), all vectors \( v_i \) and \( v_2 \) orthogonal to some vector \( q \in K^*_x \), and each \( t > 0 \) such that \( \Phi_t \) is well defined one has the inequality \( S(d\Phi_t v_1, d\Phi_t v_2) > c_2 e^{\gamma_2 t} S(v_1, v_2) \), where \( S(v_1, v_2) \) (respectively, \( S(d\Phi_t v_1, d\Phi_t v_2) \)) is
the area of the parallelogram spanned by the vectors $v_1$ and $v_2$ (respectively, $d\Phi_1v_1$, $d\Phi_2v_2$).

Note, that if vectors $v_1$ and $v_2$ are orthogonal to $q$, then the vectors $v_1(t) = d\Phi_1v_1$ and $v_2(t) = d\Phi_2v_2$ are orthogonal to $q(t) = qd\Phi_1$ for each $t$.

In [9] was proven that the Lorenz system induces the first return mapping which is a uniformly hyperbolic mapping with singularities. In [11] is proven the following theorem:

**Theorem 4.** Suppose that there exists a surface $P$ such that the first return mapping induced on $P$ by the flow $\Phi_t$ is hyperbolic mapping with singularities. Then the flow $\Phi_t$ is singularly hyperbolic.

Finally, there is not difficult to prove that the property of differential equation to be the singularly hyperbolic is stable under small non-autonomous perturbations.

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